Optimization Problems in Correlated Networks

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***Abstract*—Solving the shortest path routing problem and the min-cut problem in various kinds of networks are major concerns of achieving high performance and robust communication networks. Those problems have been heavily studied in deterministic and independent networks both in their original formulations as well as several constrained variants.**

However, in the real-world networks, the link weights (e.g., delay, bandwidth, failure probability) are often correlated due to spatial or temporal reasons, and these correlated link weights together behave in a different manner and are not always additive. Meanwhile, the link weights are sometimes stochastic (or uncertain) instead of deterministic because of inaccurate Network State Information (NSI). In this paper, we first propose two correlated link weight models, namely (i) deterministic correlated model and (ii) (log-concave) stochastic correlated model. Subsequently, we study the shortest path problem and the min-cut problem under these two correlated models. We prove that these two problems are NP-hard under the deterministic correlated model, and even cannot be approximated to arbitrary degree in polynomial time. In particular, these two problems are shown polynomial-time solvable under the (constrained) nodal/pairwise deterministic correlated model. On the other hand, we show that both of these two problems can be solved by convex optimization formulations under the (log-concave) stochastic correlated model.

***Index Terms*—Shortest path, min-cut, correlated networks, convex optimization.**

I. Introduction

The shortest path problem and the min-cut problem have been extensively existed in various kinds of networks' routing application (e.g., transportation network, optical network, etc.) and the robust network design application, respectively. Fortunately, both of these two problems are polynomial-time solvable. Via a shortest path from the source to the destination when the link weight is additive, a traffic request can be accommodated in the most efficient way (e.g., the minimum delay path, the most reliable path, energy-aware path). On the other hand, the min-cut problem arises from the application of the network reliability, network throughput/flow analysis, etc.

However, often correlations or (inter-)dependencies exist among link weights. For example, in overlay [1] or multilayer (e.g., IP-over-WDM) networks, the abstract links in the logical layer are mapped to different physical links in the physical layer. In this context, two or more abstract links which contain the same physical links may have correlated latencies [2], bandwidth usage [3] or geographically failures [4]. Another example is interdependent networks [5], where for instance the electricity network and Internet network are coupled and inter-connected, and one node or link failure in one network may cause failures of nodes or links in the other network. A similar case is reflected by optical Shared-Risk Link Group (SRLG) networks [6], where the fiber links which are deployed in the same duct will fail simultaneously, if their duct fails. The (inter-)dependencies in interdependent and SRLG networks can also be seen as correlations, so we use the term correlations throughout this paper and study relevant problems in the so-called correlated networks. Our key contributions are as follows:

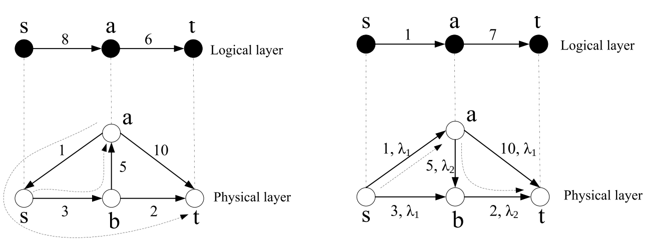
- We propose two correlated link weight models, namely *deterministic correlated model* and *stochastic correlated model*.
- We study the shortest path problem and the min-cut problem under the deterministic correlated model, where we prove both of them are NP-hard and even cannot be approximated in polynomial time.
- On the other hand, we also show that both the shortest path problem and the min-cut problem are polynomial-time solvable under a (constrained) nodal/pairwise deterministic correlated model.
- To solve both two problems under the proposed correlated models, we propose exact brute-force algorithms under the deterministic correlated model, and develop convex optimization formulations under the stochastic correlated model.

The remainder of this paper is organized as follows. Section II introduces two proposed correlated link weight models. In Section III and Section IV, we study the shortest path problem and min-cut problem for the proposed models and devise algorithms to solve them exactly, respectively. An overview of the related work is presented in Section V and we conclude in Section VI.

II. Correlated Link Weight Models

A network having node and link weights can be transformed to a directed network with only link weights, as done in [7]. Therefore, we assume nodes have uncorrelated weight 0 in this paper and only consider correlated link weights. Throughout this paper, we use the deterministic correlated model to represent the deterministic correlated link weight model. And similarly refer to the stochastic correlated model.

Let us first consider an example of multilayer networks shown in Fig. 1 to illustrate the importance of link weight correlations. In multilayer networks, usually only the links in the logical layer can be known. More specifically, in Fig. 1(a), the cost is labeled on each link in the physical layer, and the links in the logical layer are mapped to the links in the physical layer with the minimum costs. Suppose we want to find a minimum cost path from s to t in Fig. 1(a). Since we are only aware of the links in the logical layer, we can find that the path cost is $8+6=14$. However, the optimal solution is path s-b-t with cost 5 in the physical layer. The reason is that (s,a) and (a,t) in the logical layer share the same link (s,b) which leads to a greater result, whereas in fact these two logical links have a "decreasing" joint cost than their original sum value. Fig. 1(b) follows the similar meaning with Fig. 1(a), except that each link in the physical layer has one additional wavelength number. In the absence of wavelength conversion, it is required that the lightpath occupies the same wavelength on all links it traverses, which is referred to as the wavelength-continuity constraint in WDM-enabled networks. Now, suppose we want to find a minimum cost lightpath from s to t, which obeys the wavelength-continuity constraint. Clearly, if we are only aware of the links in the logical layer, the result is path s-a-t with cost 8. However, since this path is mapped to the path s-a-b-t in the physical layer, it violates the wavelength-continuity constraint. Actually, the optimal solution is path s-a-t in the physical layer via wavelength λ_1. Its cost is 11, which has an "increasing " joint cost than the sum of those two logical links' costs. The above example reveals that even though we know each individual link's weight in the logical layer in multilayer network, it is still not enough to find the solution. To tackle it, we also require the "correlations" among link weights, which we will further formulate in the following.



(a) Decreasing joint cost (b) Increasing joint cost

Fig. 1: A multilayer networks example.

A. Deterministic Correlated Model

Without loss of generality, we use $w(l)$ to represent the weight of link l. For simplicity, in this paper we call $w(l)$ as the cost of l, although the other metrics such as delay, energy, etc. can also be used. In the deterministic correlated model, for any two links l_i and l_j, their joint total cost is represented by $w(l_i) \oplus w(l_j)$, where the operator $\oplus$ indicates the joint total cost of the links and it may be not equal to the $+$ operator when they are *correlated*. In this sense, the use of one link may influence the cost of another in this model. For example, in Fig. 4 where the cost is shown above each link, it is assumed that only link (s,a) and (b,t) are correlated with joint cost of 11 for simplicity, and all the other links have uncorrelated costs. We can see that in path s-b-t, the cost of link (b,t) is 10 since another link (s,b) in this path is uncorrelated with it. Therefore this path's cost is equal to the sum of these two links' cost, which is 18. However, in path s-a-b-t, the cost of link (b,t) should be calculated together with another link (s,a) with a joint cost 11, since they are correlated and both appear in this path. Therefore, this path's total cost is equal to $11+4=15$, which is not equal to the sum of its belonged link's cost $(6+4+10=20)$.

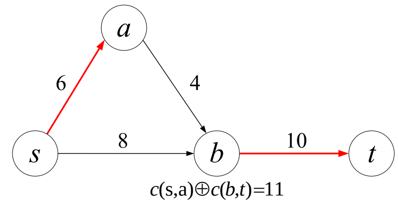


Fig. 2: An example of the deterministic correlated model.

Equivalently, we could formulate $w(l_i) \oplus w(l_j) = \rho_{i,j} \cdot (w(l_i)+w(l_j))$, where $\rho_{i,j}$ stands for the correlation coefficient between links l_i and l_j, and its value varies in $(0,\infty)$, since we do not consider negative costs. When $\rho_{i,j}$ is equal to 1, it indicates that l_i and l_j are uncorrelated, when $\rho_{i,j}$ is greater than 1, it indicates that l_i and l_j have an increasing correlation, which is the case of Fig. 1(b), otherwise we say l_i and l_j have a decreasing correlation, which is the case of Fig. 1(a).

Analogously, for given $m > 1$ links $l_1, l_2, ..., l_m$ in the deterministic correlated model, their joint total cost can be expressed as follows:

$$w(l_1) \oplus w(l_2) \cdots \oplus w(l_m) = \rho_{1,2,...,m} \cdot (w(l_1)+w(l_2)+\cdots+w(l_m)) \tag{2.1}$$

Similarly, if the link l's weight is multiplicative (e.g., failure probability), then by using $-log(w(l))$ to represent its weight value, Eq. (2.1) also applies.

In probability theory, given are two random variables X and Y with respective expected values μ_X and μ_Y, and respective standard deviations σ_X and σ_Y, their *linear* correlation coefficient $\rho(X,Y)$ is defined as:

$$\rho(X,Y) = \frac{Cov[X,Y]}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y} \tag{2.2}$$

where $Cov[X,Y]$ represents the covariance of X and Y.

However, the *linear* correlation coefficient in probability theory is different from and cannot be transformed to the one defined in the deterministic correlated model because: the variances of X and Y in Eq. (2.2) must be nonzero and finite. However, in the deterministic correlated model, for any link l, when none of its correlated links simultaneously happen with it, its cost value is fixed/deterministic with variance of 0; when l happens together with other correlated links (say m), then l and m together behave in a joint way with joint cost which may be different from their original sum. In this context, the variance of link l cannot be determined/calculated.

B. Stochastic Correlated Model

For completeness, we first present the standard definitions [8] of a convex set and a convex function. We define $\mathcal{R}^n$ to be the set of n-dimensional real vectors.

Definition 1: Convex Set. A subset C of $\mathcal{R}^n$ is called convex if $\alpha x + (1-\alpha)y \in C, \forall x, y \in C, \forall \alpha \in [0,1]$.

Definition 2: Convex Function. Let C be a convex subset of $\mathcal{R}^n$. We say that a function f: $C \to R$ is convex if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$, $\forall x, y \in C, \forall \alpha \in [0,1]$.

A function f is said to be concave if and only if $-f$ is convex. If the second derivative of f is always nonnegative, then f is convex. Linear functions can be regarded to be convex or concave

In convex analysis, a nonnegative function f: $\mathcal{R}^n \to \mathcal{R}$ is logarithmically concave or log-concave if its domain is a convex set, and if it satisfies the following inequality:

$$f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{1-\theta} \tag{2.3}$$

for all $x, y \in$ dom f and $0 < \theta < 1$. If f is strictly positive, then Eq. (2.3) can be rewritten as:

$$\log f(\theta x + (1-\theta)y) \geq \theta \log f(x) + (1-\theta)\log f(y) \tag{2.4}$$

In many real-life networks, the link weight which can represent delay, cost is usually uncertain because of inaccurate Network State Information (NSI) [9]. Papagiannaki *et al.* [10] measure that the queuing delay distribution can be approximated by a Weibull distribution. Since the Cumulative Density Function (CDF) of Weibull distribution is log-concave and the CDFs of many common distributions (e.g., Exponential distribution, Uniform distribution, etc.) are log-concave according to [11], [12], we make a mild (general) assumption that the link l's weight follows a log-concave distribution. This assumption has also been widely used in the formulation of stochastic link weights (e.g., see [9], [13]).

Before introducing the stochastic correlated model, let us first define the Correlated Group (CG):

Definition 3: Given is a network $G(\mathcal{N}, \mathcal{L})$ where $\mathcal{N}$ represents a set of N links and $\mathcal{L}$ denotes a set of L links. A Correlated Group (CG) is a subset of links $L_{CG} \subseteq \mathcal{L}$, and $\forall l \in L_{CG}$, $\exists l' \in L_{CG} \backslash \{l\}$, such that l and l' are correlated ($\rho_{l,l'} \neq 1$).
Accordingly, the Maximum Correlated Group (MCG) is defined as a CG with the maximum number of correlated links. If a link l is uncorrelated/independent with all the other links, then we say $\{l\}$ is a single element MCG. Suppose there are Ω Maximum Correlated Groups (MCGs), and there are $m_i > 0$ links (denoted as l_1^i, l_2^i,...,$l_{m_i}^i$) in i-th MCG, where $1 \leq i \leq \Omega$. Each link l has an upper bound of allocating cost $w_l^{\max}$. In i-th MCG, a multivariate m_i dimensional log-concave Cumulative Density Function $CDF_i(x_1, x_2, ..., x_{m_i})$ is given to allocate cost x_1, x_2,..., x_{m_i} for links l_1^i, l_2^i,..., $l_{m_i}^i$, respectively.

Therefore, if the possible cost allocated on link l ranges from 0 to $w_l^{\max}$ ($0 < w_l^{\max}$), then the probability of allocating a cost value out of this range is 0. Hence, we have $CDF_i(w_l^{\max}) = 1$ for a single element MCG i, and $CDF_j(w_{l_1^j}^{\max}, w_{l_2^j}^{\max}, ..., w_{l_{m_j}^j}^{\max}) = 1$ for a multi-element MCG j.

To better illustrate it, we take the log-concave multivariate Normal distribution for example. Fig. 3 shows a 2-dimensional multivariate Normal distribution, where both two variables are in the range of $[0, 4]$ with the mean of 2, and the two variables' covariance matrix is $\begin{bmatrix} 0.9 & 0.4 \\ 0.4 & 0.3 \end{bmatrix}$. Similarly to Eq. (2.2), the correlation matrix (composed of linear correlation coefficients) can be derived from the covariance matrix [14] and the variables' standard variances in the multivariate Normal distribution. However, we do not explicitly use *linear* correlation coefficient in the stochastic correlated model, since we will later prove that via the log-concave property of this model, the shortest path problem under this model can be solved by a convex optimization formulation.

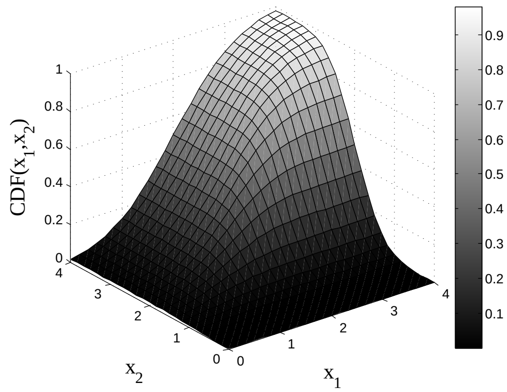


Fig. 3: A 2-dimensional multivariate Normal distribution.

III. Shortest Path in Correlated Networks

A. Shortest Path under the Deterministic Correlated Model

Definition 4: Given is a directed network $G(\mathcal{N}, \mathcal{L})$, and each link $l \in \mathcal{L}$ has a cost $w(l)$, following the deterministic correlated model. The Shortest Path under the Deterministic Correlated Model (SPDCM) problem is to find a path from the source s to the destination t with minimum cost.

In conventional deterministic networks with each link associated with one non-negative weight, a subpath of a shortest path is also the shortest. We refer to this property of the shortest path as the dominance of the subpath. However, this is not the case in the networks with deterministic correlated link weights, which means a non-dominated path may also composite an optimal solution. For instance, in the same example of Fig. 4, we can see that although subpath s-b has a smaller cost than subpath s-a-b, path s-a-b-t (instead of path s-b-t) has the minimum cost. Therefore, a shortest path cannot always find an optimal solution in the SPDCM problem. In the following, we will study the complexity of the SPDCM problem.

Theorem 1: The SPDCM problem is NP-hard.

Proof: When the correlation coefficient is equal to 1, the SPDCM problem can be solved by a conventional shortest path algorithm in polynomial time, we therefore prove in the following that, the SPDCM problem is NP-hard for both increasing correlation case and decreasing correlation case.

Increasing Correlation:

When the correlation coefficient is greater than 1, we make a reduction from and into the forbidden pairs shortest path problem, which is known NP-hard [15]. In a given network and given set of node pairs ζ, the forbidden pairs shortest path problem looks for the shortest path between s and t such that at most one node from each pair in the set ζ lies on this path. Let us consider a network with deterministic correlated link weights, where two nodes i and j form a forbidden pair, their costs are correlated such that $w(i,.) \oplus w(j,.) = +\infty$, where $(i,.)$ and $(j,.)$ represent any link which contains an end node of i and j for simplicity, respectively. In all the other cases, the link costs are uncorrelated and finite. Since $w(i,.) \oplus w(j,.) = +\infty$, if the two forbidden nodes appear in the same path then the cost of this path will be $+\infty$, so it will never be the minimum cost path. Now, the SPDCM problem is equivalent to the forbidden pairs shortest path problem.

Decreasing Correlation:

When the correlation coefficient is less than 1, we make a reduction of the SPDCM problem to the Minimum Color Single-Path (MCSiP) problem, which is proved NP-hard [16]. Given a network $G(\mathcal{N}, \mathcal{L})$, and given the set of colors $C = \{c_1, c_2, ..., c_g\}$ where g is the total number of colors in G, and given the color set $\{c_l\}$ associated for each link $l \in \mathcal{L}$, the Minimum Color Single-Path (MCSiP) problem is to find one path from source node s to destination node t such that it uses the least amount of colors.

Assume each color c_i is associated with cost 1, where $1 \leq i \leq z$. We further assume that $w(l_1) \oplus w(l_2) \oplus \cdots \oplus w(l_m) = x$, where x is the total number of distinct colors belonging to these m links. Hence, the SPDCM problem is equal to the MCSiP problem. The proof is therefore complete. ∎

Theorem 2: The SPDCM problem cannot be approximated to arbitrary degree in polynomial time, unless P=NP.

Proof: We prove it by contradiction under both increasing correlation case and decreasing correlation case.

Increasing Correlation:

Assume such a polynomial-time approximation algorithm exits and can find a path with a cost at most $\alpha \cdot opt$, where $\alpha > 1$ is an approximation ratio. For a pair of forbidden nodes i and j, we further assume $w(i,.) \oplus w(j,.) > \alpha * opt$. Therefore, if such an approximation algorithm can find a path ψ with cost at most $\alpha * opt$ from s to t, then i and j cannot be simultaneously traversed by this path ψ, which means that the forbidden pairs shortest path problem can be solved in polynomial time. However, the forbidden pairs shortest path problem is NP-hard, which results in a contradiction.

Decreasing Correlation:

We first introduce the Disjoint Connecting Path problem [17]. Given a directed network $G(\mathcal{N}, \mathcal{L})$, a collection of disjoint node pairs (s_1, t_1), (s_2, t_2), ..., (s_z, t_z), does G contain z mutually link-disjoint paths, one connecting s_i and t_i for each i, $1 \leq i \leq z$. This problem is NP-hard when $z \geq 2$. Assume such a polynomial-time approximation algorithm exits and can find a path with a cost at most $\alpha \cdot opt$, where $\alpha > 1$ is an approximation ratio. Assuming all the links in the network have the link weight of 1, and link (u, v) and any $m > 0$ links in $\mathcal{L} \backslash \{(u, v)\}$ are correlated, with a total cost of $\frac{1}{\beta} \cdot m$. Moreover, any two or more links in $\mathcal{L} \backslash \{(u, v)\}$ are assumed to be uncorrelated/independent.

The minimum value of a shortest path is 1 if link (u, v) is not traversed according to this assumption, i.e., it traverses only one link from s to t. However, the optimal solution which traverses link (u, v) has a total cost of $opt = \frac{1}{\beta} \cdot c$, where c is the the sum of minimum hops from s to u and from v to t. For any given α, let $\frac{\beta}{c} > \alpha$, then $1 > \alpha \cdot \frac{1}{\beta} \cdot c$, which means $1 > \alpha \cdot opt$. It means that the shortest path without traversing link (u, v) has a total cost of greater than $\alpha \cdot opt$. To find a path with cost at most $\alpha \cdot opt$, the polynomial-time algorithm must find a path which traverses link (u, v). It indicates that the algorithm can in polynomial time find two link-disjoint paths from s to u, and from v to t, respectively. However, the *Disjoint Connecting Path* problem is NP-hard, which results in a contradiction. ∎

Next, we study the performance of a conventional shortest path algorithm running on a graph with each link associated with an "uncorrelated" weight value.

Theorem 3: When all the correlation coefficients are greater than 1, by assigning each link l with cost $w(l)$, then a conventional shortest path ψ has a total cost at most $\frac{\rho_{\max}}{\rho_{opt}} \cdot opt$, where $\rho_{\max}$ and ρ_{opt} are the largest correlation coefficient and optimal solution's correlation coefficient, respectively, and opt is the cost of the optimal solution.

Proof: Let $U(\psi) = \sum_{l \in \psi} w(l)$, and denote $C(\psi) = \rho_u \cdot U(\psi) = \rho_u \cdot \sum_{l \in \psi} w(l)$ as the total joint cost of path ψ considering their correlation, where ρ_u indicates the correlation coefficient of path ψ. On one hand, a conventional shortest path ψ should satisfy $U(\psi) \leq \frac{opt}{\rho_{opt}}$. On the other hand, $C(\psi) \leq \rho_{\max} \cdot U(\psi)$ considering ρ_{max} is the largest correlation coefficient. Hence, $C(\psi) \leq \rho_{\max} \cdot U(\psi) \leq \frac{\rho_{\max}}{\rho_{opt}} \cdot opt$. ∎

Theorem 4: When all the correlation coefficients are less than 1, by assigning each link l with cost $w(l)$, then a conventional shortest path ψ has a cost at most $\frac{1}{\rho_{\min}} \cdot opt$, where $\rho_{\min}$ is the smallest correlation coefficient among all the correlation coefficients.

Proof: Let $V(\psi) = \sum_{l \in \psi} w(l)$, and denote $C(\psi) = \rho_u \cdot \sum_{l \in \psi} w(l)$ as the total joint cost of path ψ considering their correlation. Since all the correlations are decreasing ($\rho < 1$), we have $C(\psi) \leq V(\psi)$. On the other hand, $\rho_{\min} \cdot V(\psi) \leq opt$ considering that $\rho_{\min}$ is the smallest correlation coefficient. In all, $C(\psi) \leq V(\psi) \leq \frac{1}{\rho_{\min}} \cdot opt$. ∎

By collecting Theorems 3 and 4, we have Corollary 5.

Corollary 5: In a network with links following the deterministic correlated model, by assigning each link l with cost $w(l)$, a conventional shortest path can have cost at most $max(\frac{\rho_{\max}}{\rho_{opt}}, \frac{1}{\rho_{\min}}) \cdot opt$.

Corollary 5 reveals that a conventional shortest path may have a performance as worse as possible, since either $\frac{\rho_{\max}}{\rho_{opt}}$ can be infinitely large or ρ_{min} can be infinitely small. It also verifies the correctness of Theorem 2, which states that the

SPDCM problem cannot be approximated to any arbitrary in polynomial time.

B. An Exact Algorithm to Solve the SPDCM problem

In conventional deterministic networks with each link associated with one non-negative weight, a subpath of a shortest path is also the shortest. We refer to this property of the shortest path as the dominance of the subpath. However, this is not the case in correlated networks, which means a non-dominated path may also composite an optimal solution. For instance, in Fig. 4 where the cost is shown above each link, it is assumed that only link (s, a) and (b, t) are correlated with joint cost of 11. We can see that although subpath s-b has a smaller cost than subpath s-a-b, path s-a-b-t (instead of path s-b-t) has the minimum cost.

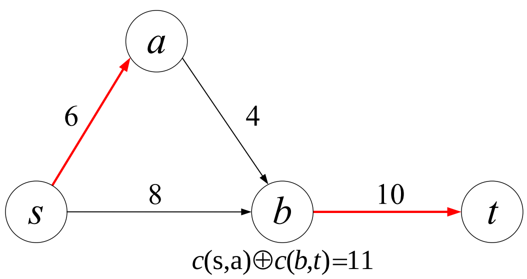


Fig. 4: Non-dominance of the SPDCM problem, where the source and destination are s and t, respectively.

To solve the SPDCM problem exactly, we can apply a modified Dijkstra's algorithm by letting each node store as many subpaths as possible, which is similar to the exact algorithm for solving multi-constrained routing problem [18]. Since each node can store as many subpaths as possible, its running time is exponential. We start with some notations used in the algorithm:

$sus[u][h]$: the parent node of node u for its stored h-th subpath from s to it.
$dist[u][h]$: the cost value stored at node u for its stored h-th subpath from s to it.
$counter[u]$: the number of stored subpaths of node u.
$u[m]$: node u's stored m-th subpath from s to it.
$adj(u)$: the set of adjacent nodes of node u.

The pseudo-code of the exact algorithm is given in Algorithm 1.

C. Shortest Path under the Nodal Deterministic Correlated Model

In some real-world networks (e.g., SRLG networks), the links which are spatially (geographically) close to each other are usually correlated whereas the links which are located spatially far from each other are usually uncorrelated. We make a further confined assumption, which is that only the links sharing the same node can be correlated. We name this kind of correlation as nodal correlation or pairwise correlation.

Algorithm 1 $SPDCM(G, s, t)$

$Q \leftarrow s$, $P \leftarrow \emptyset$, $dist[s][1] \leftarrow 0$, $dist[i][h] \leftarrow +\infty$, $sus[i][h] \leftarrow i$, $counter[s] \leftarrow 1$, $counter[i] \leftarrow 0$, $\forall i \in \mathcal{N} \backslash \{s\}$.
While $Q \neq \emptyset$
    $u[m] \leftarrow$ Extract-min(Q)
    If $(u == t)$ **do**
        Insert (P, $u[m]$)
    Else
        Foreach $v \in adj(u)$ **do**
            $counter(v) = counter(v) + 1$
            Calculate the total cost of subpath $u[m] \rightarrow v$ and assign it to $dist[v][counter(v)]$
            $sus[v][counter(v)] \leftarrow u$
            Insert $(Q, v, counter(v))$
return min(P).

Although the SPDCM problem is NP-hard, we will show that, by traversing the original graph to an auxiliary graph, the Shortest Path under the Nodal Deterministic Correlated Model (SPNDCM) problem is polynomial-time solvable. Since when (a, b) and (a, c) are correlated, the path can only choose at most one link if looping is not allowed, we only need to consider the case when (a, b) and (b, c) are correlated. To start, we first define if (a, b) and (b, c) are correlated, then a and b are called the *correlated node*, which is represented by C_n, otherwise the node is called *uncorrelated node*, which is denoted by U_n. Subsequently, on basis of original graph $G(\mathcal{N}, \mathcal{L})$, the auxiliary graph $G^A(\mathcal{N}^A, \mathcal{L}^A)$ can be done like this:

1) For any two links $(u, v) \in \mathcal{L}$ and $(v, w) \in \mathcal{L}$ that are correlated in G, create new nodes u_v, v^{uw} and v_w in G^A if they do not exist.
2) For any node $a \in \mathcal{N}$ and if it is an *uncorrelated node* (in U_n), create node a_a in G^A.
3) For any two correlated links (u, v) and (v, w) in G, create links (u_v, v^{uw}), (v^{uw}, v_w) and (v_w, w_w) in $\mathcal{G}^A$. Assign the links (u_v, v^{uw}) and (v_w, w_r) with the weights of $w(u, v)$ and $w(v, w)$, respectively, and the link (v^{uw}, v_w) with the weight of $(\rho_{(u,v)(v,w)} - 1) \cdot (w(u, v) + w(v, w))$, where $\rho_{(u,v)(v,w)}$ is the correlation coefficient of links (u, v) and (v, w).
4) For each link $(a, b) \in \mathcal{L}$ such that both node a and node b are not *correlated node*, create the link (a_a, b_b) also in G^A with the link weight of $w(a, b)$.
5) For each link $(a, b) \in \mathcal{L}$ such that $a \in U_n$ and $b \in C_n$, draw links (a_a, b_r) in G^A, where $r \in \mathcal{N}$ and $b_r \in G^A$.
6) For each link $(a, b) \in \mathcal{L}$ such that $a \in C_n$ and $b \in U_n$, draw links (a^{rp}, b_b) in G^A, where $r, p \in \mathcal{N}$ and $a^{rp} \in \mathcal{N}^A$.

The idea of the auxiliary graph is that if two links (u, v) and (v, w) are correlated, we create four corresponding nodes u_v, v^{uw}, v_w, w_w and then draw three links (u_v, v^{uw}),(v^{uw}, v_w) and (v_w, w_w). We use (u_v, v^{uw}) and (v_w, w_w) to indicate

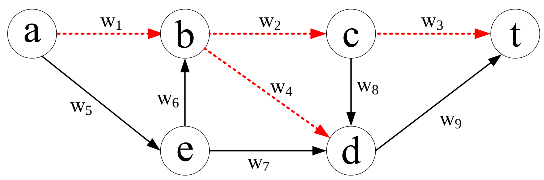


Fig. 5: An example network with dotted links following the nodal deterministic correlated model.

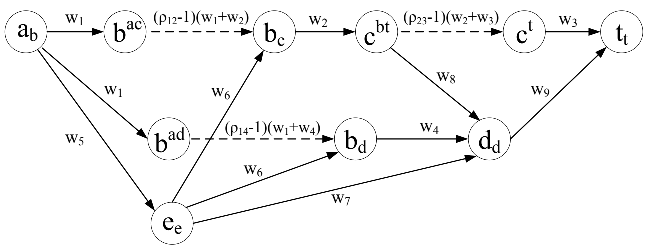


Fig. 6: Auxiliary graph of Fig. 5 for the SPDCM problem under the nodal deterministic correlated model.

(regard) their uncorrelated value, respectively, if only one of these two links is traversed, and (v^{uw}, v_w) to represent the correlated loss (decreasing correlation)/gain (increasing correlation), respectively, if they are traversed simultaneously. For instance, in Fig. 5 where the link weight is labeled above each link, assuming links (a, b), (b, c), (c, t) and (b, d) are nodal correlated, then Fig. 6 is its corresponding auxiliary graph with the assigned weight shown on each link. In particular, since (a, b), (b, c) and (a, b), (b, d) may have different correlation coefficients, in Fig. 6 we use (b^{ac}, b_c) and (b^{ad}, b_d) to represent their correlation value, respectively. Meanwhile, when there is a link from *uncorrelated node* to *correlated node* in the original graph, e.g., (e, b) in Fig. 5, we draw links (e_e, b_c) and (e_e, b_d) to reflect it (Step 5); When there is a link from *correlated node* to *uncorrelated node* in the original graph, e.g., (c, d) in Fig. 5, we draw link (c^{bt}, d_d) to represent it (Step 6). Considering that there are at most $N(N-1)$ nodal links in a graph, the original graph can be transferred to the auxiliary graph in polynomial time.

Consequently, running shortest path algorithm on the auxiliary graph can find a minimum cost path under the nodal deterministic correlated model. Our auxiliary graph can deal with both decreasing and increasing correlation case. Considering that $(\rho_{(u,v)(v,w)} - 1) \cdot (w(u, v) + w(v, w)) < 0$ under decreasing correlation case, and Dijkstra's algorithm cannot handle the negative link weight case, we could for instance run Bellman-Ford's algorithm on the auxiliary graph.

D. Shortest Path under the Stochastic Correlated Model

The Shortest Path under the Stochastic Correlated Model (SPSCM) problem is defined as following:

Definition 5: The Shortest Path under the Stochastic Correlated Model (SPSCM) problem: In a given directed graph $\mathcal{G}(\mathcal{N}, \mathcal{L})$ where the link costs follow the stochastic correlated model, it is assumed that there are in total Ω Maximum Correlated Groups (MCGs). The SPSCM problem is to find a path from the source s to the destination t such that its total cost is minimized and the probability to realize this value is no less than P_s.

We present a convex optimization model to solve the SPSCM problem. A convex optimization problem is a problem in which the objective function is either a maximization of a concave function or a minimization of a convex function, and its constraints are all convex [8]. Convex optimization problems can usually be solved quickly and accurately with convex optimization solvers. To that end, let us first introduce how to develop a Linear Programming (LP) formulation to solve the shortest path problem in deterministic networks in below:

Objective:

$$\max \ d_t \tag{3.1}$$

Constraints:

$$d_s = 0 \tag{3.2}$$

$$d_v - d_u \leq w(u, v) \quad \forall (u, v) \in \mathcal{L} \tag{3.3}$$

where d_u is a value between 0 and 1. Similarly, the SPSCM problem can be solved by the following convex formulation:

Objective:

$$\max \ d_t \tag{3.4}$$

Constraints:

$$d_s = 0 \tag{3.5}$$

$$d_v - d_u \leq f(u, v) \quad \forall (u, v) \in \mathcal{L} \tag{3.6}$$

$$\sum_{i \in \Omega} -\log\left(CDF_i(f(l_1^i), f(l_2^i), ..., f(l_{m_i}^i))\right) \leq -\log(P_s) \tag{3.7}$$

where the variable $f(u, v), f(l_1^i), f(l_2^i), ..., f(l_{m_i}^i)$ indicate the allocated possible cost by link (u, v), l_1^i, l_2^i,..., $l_{m_i}^i$, respectively. Constraint Eq. (3.7) ensures that the total probability of realizing the total cost is no more than P_s. In Eq. (3.7), for each MCG we apply the multi-dimensional CDF functions to calculate the probability of realizing possible cost. Since the multi-dimensional CDF function is log-concave, $-\log\left(CDF_i(f(l_1^i), f(l_2^i), ..., f(l_{m_i}^i))\right)$ is convex, and by summing all the MCGs' CDFs together, it remains convex, which indicates that Eq. (3.7) is convex. The other constraints are also convex, which prove that the above formulation is a convex optimization model.

E. Extensions

1) Widest Path under the Deterministic Correlated Model: The Widest Path in Deterministic Networks (WPDN) problem is to find a path from s to t such that the minimum link weight among all its traversed links is maximized. This problem is defined when the link weight is a bottleneck metric such as bandwidth. For example, the bandwidth of a path is equal to minimum link's bandwidth among all its traversed links. The WPDN problem is polynomial-time solvable, and we can solve it like this: First, we order all the link weights in the network in an increasing order. After that, each round we prune a lowest weight link in the graph and run Depth First Search (DFS) or Breadth First Search (BFS) algorithm to find a path from s to t. The algorithm will end if there is no path existing from s to t and return the pruned weight value of the previous round.

In the Widest Path under the Deterministic Correlated Model (WPDCM) problem, if $m > 1$ correlated links in a path have a joint weight value W, then for each link the maximum average/amortized weight is $\frac{W}{m}$. For instance, if a path traverses three correlated links which have a joint weight value of 15 and passes another uncorrelated link with weight of 6, then this path has a "width" value of 5. The reason is that the maximum (average/amortized) weight for each of these three correlated links is $15/3 = 5$, and this value is less than another uncorrelated link's (6).

However, the WPDCM problem is still NP-hard and cannot be approximated into arbitrary degree. The proof follows analogously from Theorem 1 and 2, and therefore we omit it.

IV. Min-Cut in Correlated Networks

A. Min-Cut under the Deterministic Correlated Model

Definition 6: The Min-Cut under the Deterministic Correlated Model (MCDCM) problem: Given is a network $G(\mathcal{N}, \mathcal{L})$, and each link $l \in \mathcal{L}$ is associated with a cost $w(l)$. It is assumed that two or more link costs are correlated under the deterministic correlated model. Given a source s and a target t, to find a cut $\mathcal{C}$ which partitions G into two disjoint subsets X ($X \in \mathcal{N}$) and $\mathcal{N} - X$ such that s and t are in different subsets and the cost of the cut C is the minimum.

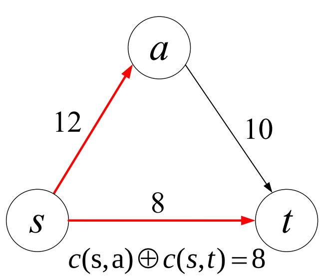


Fig. 7: An example to illustrate that the maximum flow is not equal to min-cut in the correlated networks.

The min-cut value is not equal to the maximum flow value under the deterministic correlated model. For example, in Fig. 7 assume links (s, a) and (s, t) are correlated with the joint cost of 8. In this example, the maximum flow from s to t is s-a-t with the maximum flow of 10, while the min-cut which is composed of links (s, a) and (s, t) has cost of 8. We further formalize non-polynomial solvability of the MCDCM problem in Theorem 6.

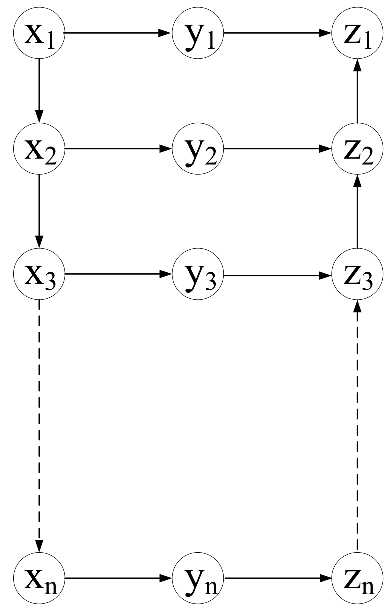


Fig. 8: NP-hardness of the MCDCM problem.

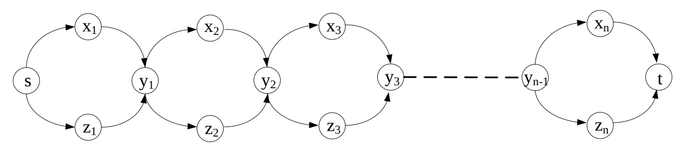


Fig. 9: A reduction of the MCDCM problem to the SPDCM problem.

Theorem 6: The MCDCM problem is NP-hard.

Proof: In Fig. 8, we assume that the links in the form of (x_i, x_{i+1}) and (z_i, z_{i+1}) have infinite uncorrelated cost and the link costs in the form of (x_i, y_i) and (y_i, z_i) follow the deterministic correlated model, where $1 \leq i \leq n$. We want to find a min-cut to separate x_1 and z_1. Based on Fig. 8, we first derive Fig. 9 with the same nodes except that we add one more node s. We regard $s = y_0$, and $t = y_n$. The link weight in Fig. 9 is set like this: (y_{i-1}, x_i) and (y_{i-1}, z_i) have 0 uncorrelated cost, while (x_i, y_i) and (z_i, y_i) have the same (correlated) costs with (x_i, y_i) and (y_i, z_i) in Fig. 8, respectively, where $1 \leq i \leq n$. In Fig. 9, we want to solve the SPDCM problem from the source s to the destination t.

Since we want to find a min-cut that separates x_1 and z_1, any cut in the form of (x_i, y_i) and (y_i, z_i), where $1 \leq i \leq n$, is not the optimal solution. The reason is that this kind of cut only separates y_i and other nodes, but not x_1 and z_1. Moreover, considering the link in the form of (x_j, x_{j+1}) or (y_j, y_{j+1}) have infinite costs, it cannot be in the optimal solution. Based on above analysis, any feasible cut C should contain one link of either (x_i, y_i) or (y_i, z_i), for all $1 \leq i \leq n$. We prove in the following that the MCDCM problem in Fig. 8 can be reduced into the SPDCM problem in Fig. 9 in polynomial time.

The SPDCM problem to the MCDCM problem: Considering an optimal solution of the SPDCM problem, and denote R_{SPDCM} as the set of links in the optimal solution of the SPDCM problem. Because R_{SPDCM} has the minimum cost, then let $C_{MCDCM} = R_{SPDCM} \backslash \{(y_i, x_{i+1}), (y_i, z_{i+1})\}$ and then $\forall (z_i, y_i) \in C_{MCDCM}$, change it to (y_i, z_i) in C_{MCDCM}. Since the links (y_i, x_{i+1}) and (y_i, z_{i+1}) have 0 cost, C_{MCDCM} has also the minimum cost (the same with R_{SPDCM}). Therefore solving the SPDCM problem yields a solution of the MCDCM problem.

The MCDCM problem to the SPDCM problem: An optimal solution of the MCDCM problem should be composed of either (x_i, y_i) or (y_i, z_i), where $1 \leq i \leq n$. Denote C_{MCDCM} as the set of links in the optimal solution of the MCDCM problem. Let $R_{SPDCM} = C_{MCDCM}$ and then $\forall (y_i, z_i) \in R_{SPDCM}$, change it to (z_i, y_i) in R_{SPDCM}. Because C_{MCDCM} has the minimum cost value and the links in the form of (y_{i-1}, x_i) or (y_{i-1}, z_i) has 0 cost, R_{SPDCM} together with (y_{i-1}, x_i) if $(x_i, y_i) \in R_{SPDCM}$ or (y_{i-1}, z_i) if $(z_i, y_i) \in R_{SPDCM}$, can form a path from s to t with the minimum cost. Hence, a solution of the MCDCM problem can also solve the SPDCM problem. ■

Theorem 7: The MCDCM problem cannot be approximated to arbitrary degree in polynomial time, unless P=NP.

Proof: Since the MCDCM problem can be reduced into the SPCDM problem in Theorem 6, and the SPDCM problem cannot be approximated to arbitrary degree in polynomial time according to Theorem 2, the proof is therefore complete. ■

Theorem 8: By assigning each link l with the cost $w(l)$, running a conventional min-cut algorithm will return a cut with total cost at most $\max(\frac{\rho_{\max}}{\rho_{opt}}, \frac{1}{\rho_{\min}}) \cdot opt$.

Proof: The proof follows analogously from Theorem 5, and we therefore have omitted it. ■

Since the MCDCM problem is NP-hard and even does not admit polynomial-time approximation algorithm, we propose a brute-force algorithm with exponential running time to solve it exactly. The idea is that we start with two sets A and B, with s in A and t in B without loss of generality. Then we have $N-2$ nodes left, and there are $\binom{N-2}{0}+\binom{N-2}{1}+\cdots+\binom{N-2}{N-2} = O(2^N)$ combinations to assign these $N-2$ nodes to sets A and B. For each combination assignment, it corresponds to a cut to separate A and B, and we calculate the total cost value of the edges which connect both A and B as the cut value. At last, we select one cut which has the minimum cost as the optimal solution.

B. Min-Cut under the SRLG-like Correlated Model

As we stated above, the SRLG network and the overlay/multilayer network are typical practical correlated networks, but the deterministic correlated model studied in Section II-A is too general to reflect them. For instance, in SRLG optical networks, each link is associated several SRLG events with their respective failure occurring probabilities. Hence, the total failure occurring probabilities (represented by P_{SRLG}) of two correlated links which have at least one SRLG in common will be equal to the product of the failure occurring probabilities of all the distinct SRLG events that belong to these two links. Let us denote $P_{l_1} = P_{s_1} \cdot P_s$ and $P_{l_2} = P_{s_2} \cdot P_s$ as the failure probability of these two links, respectively, where P_s denotes the overlapped SRLGs' failure occurring probability between l_1 and l_2, and P_{s_1} (P_{s_2}) is the non-overlapping SRLGs' failure probability of l_1 (l_2). Then we can have $P_{l_1} \cdot P_{l_2} < P_{SRLG} = P_{s_1} \cdot P_{s_2} \cdot P_s < min(P_{l_1}, P_{l_2})$. By taking the $-log$ on this inequality, we have:

$$\max(-log(P_1), -log(P_2)) < -log(P_{SRLG}) < (-log(P_{l_1})) + (-log(P_{l_2}))$$

Therefore, we define the SRLG-like correlated model as follows:

Definition 7: The SRLG-like correlated model: Suppose l_1, l_2,...,l_m ($1 < m \leq L$) form a Correlated Group (CG), then $w(l_1) \oplus w(l_2) \cdots \oplus w(l_j)$ is greater than the sum of any no more than $j-1$ links' cost, but smaller than $w(l_1)+w(l_2)\cdots+c(l_j)$, where $1 < j \leq m$.

Unfortunately, the Min-Cut under the SRLG-like correlated model problem is still NP-hard, since it can be reduced to the MCSiP problem via Figs. 8 and 9.

In the Nodal SRLG-like correlated model, it is assumed that only the links share the same node follows the SRLG-like correlated model. As section III-C shows that the Shortest Path under the Nodal Deterministic Correlation Model problem is polynomial-time solvable, we address the Min-Cut under the Nodal SRLG-like correlated model (MC-NSRLG) problem in the following. In general, the MC-NSRLG problem is still NP-hard. The reason is that the MC-NSRLG problem in Fig. 10, where the links in the form of (x_1, y_i) are assumed to be correlated, can be reduced to the SPDCM problem in Fig. 9, where the links in the form of (x_i, y_i) are assumed correlated (the same with (x_1, y_i) in Fig. 10) and the other links are assumed to have uncorrelated 0 cost.

Even this, we found that the MC-NSRLG problem is polynomial-time solvable when (1) only the nodal links in the form of (u, v) and (v, w) follow the SRLG-like correlated model and/or (2) for any node $u \in \mathcal{N}$, at most two nodal links (u, v) and (u, x) follow the SRLG-like correlated model. To prove case (1), let us first study the following theorem:

Theorem 9: Any two links in the form of (u, v) and (v, w) will never both appear in the optimal solution of the MC-NSRLG problem in case (1).

Proof: Suppose s and t are separated by a min-cut C such that s is in the subset A and t is in the subset B. We prove it by contradiction and thus we assume (u, v) and (v, w) are both in the min-cut C. Since C is the min-cut which separates s and t, then node u should be in subset A, otherwise if node u is in the subset B, there is no need to use (u, v) and (v, w) as the cut links since their existence does not affect the

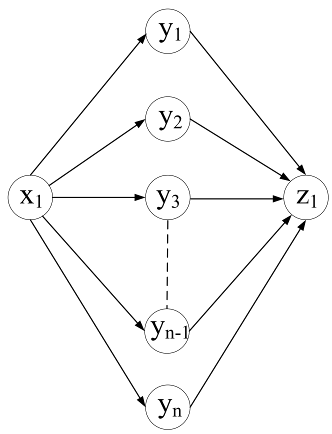


Fig. 10: NP-hardness of the MC-NSRLG problem.

connectness between A and B. In this sense, node v is in the subset B, otherwise if node v is also in A, there is no need to cut link (u, v). Based on this analysis, if w in A, then (v, w) is not necessarily the edge in the cut C, since link (v, w) does not affect the connectness from A to B. This results in a contradiction; If node w is in B, link (v, w) is also not necessarily the edge in the optimal min-cut, since node v and w are in the same subsets, which results in a contradiction. ■

Based on Theorem 9, we can use the conventional Linear Programming (LP) which solves min-cut problem in deterministic networks to solve the MC-NSRLG problem under case (1). According to [19], the LP is shown as follows.

Objective:

$$\min \sum_{(u,v)\in\mathcal{L}'} c(u,v)\cdot h_{u,v} \tag{4.1}$$

Constraints:

$$h_{s,t} \geq 1 \tag{4.2}$$

$$h_{u,v} + h_{v,w} \geq h_{u,w},\ \forall u,v,w \in \mathcal{N}: u \neq v \neq w \tag{4.3}$$

$$h_{u,v} \geq 0,\ \forall u,v \in \mathcal{N}: u \neq v \tag{4.4}$$

where $c(u, v)$ stands for the link weight (i.e., capacity) of link (u, v) in the deterministic network and $h_{u,v}$ is an indicator denoting whether (u, v) belongs to the cut.

Under case (2), the MC-NSRLG problem can be solved by running the above LP (Eqs. (4.1)-(4.4)) on an auxiliary graph $G^U(\mathcal{N}^U, \mathcal{L}^U)$ in polynomial time. The auxiliary graph can be derived from the original graph G as follows:

1) For each pair of two links $(u, v) \in \mathcal{L}$ and $(u, x) \in \mathcal{L}$ that are correlated in G, create a new node u', and draw link (u', u) with the weight of $\rho_{vx}(w(u, v)+w(u, x))$ to represent the joint cost of links (u, v) and (u, x), where ρ_{vx} represents the correlation coefficients between (u, v) and (u, x).
2) For any link $(a, u) \in L$ and $(u, b) \in \mathcal{L}\backslash\{v, x\}$ such that (u, v) and (u, x) are correlated in G, draw link (a, u') and (u', b) in G^U with the weight of $w(a, u)$ and $w(u, b)$, respectively.
3) For each pair of two links $(v, u) \in \mathcal{L}$ and $(x, u) \in \mathcal{L}$ that are correlated in G, create a new node u', and draw link (u, u') with the weight of $\rho_{vx}(w(v, u)+w(x, u))$ to represent the total cost of links (v, u) and (x, u), where ρ_{vx} represents the correlation coefficients between (v, u) and (x, u).
4) For any link $(a, u) \in \mathcal{L}\backslash\{v, x\}$ and $(u, b) \in \mathcal{L}$ such that (v, u) and (x, u) are correlated in G, draw link (a, u') and (u', b) in G^U with the weight of $w(a, u)$ and $w(u, b)$, respectively.
5) For the other links $(c, d) \in \mathcal{L}$ in G except for above four cases, create link (c, d) also in G^U with the same weight.

The proposed auxiliary graph here shares similarities with the auxiliary graph in Fig. 6. As an example, Fig. 12 is an auxiliary graph of the original graph shown in Fig. 11. Moreover, we mention that the proposed auxiliary graph also applies when there are $m > 2$ links starting from the same node follow the SRLG-like correlated model, but as long as one correlated link "encounters failure", the other $m-1$ links simultaneously "fail" (e.g., SRLG networks, inter-dependent networks).

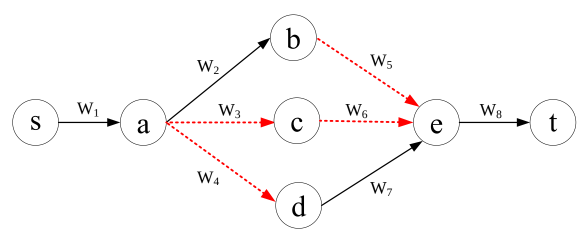


Fig. 11: An example network with dotted links following the SRLG-like correlated model.

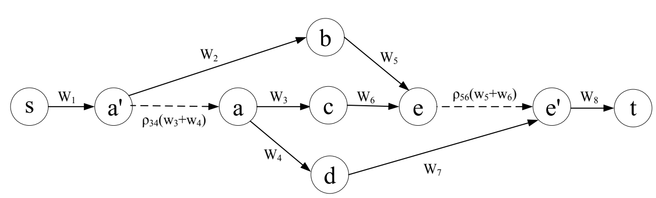


Fig. 12: Auxiliary graph of Fig. 11 for the MC-NSRLG problem.

C. Min-Cut under the Stochastic Correlated Model

The Min-Cut under Stochastic Correlated Model problem can be defined as follows:

Definition 8: The Min-Cut under Stochastic Correlated Model (MCSCM) problem: In a given directed graph $\mathcal{G}(\mathcal{N},\mathcal{L})$ with link costs following the stochastic correlated model, the MCSCM problem is defined to find a cut $\mathcal{C}$ which partition G into two disjoint subsets X ($X \in \mathcal{N}$) and $\mathcal{N} - X$ such that:

- s and t are in different subsets.
- the allocated cost of the cut C is the minimum.
- the total probability of realizing the cost value is no less than P_c.

We propose a convex optimization formulation to solve it. Based on Eq. (4.1)-Eq. (4.4), the MCDCM problem under the correlated stochastic model can be solved by the following convex optimization formulation:

Objective:

$$\min \sum_{(u,v)\in\mathcal{L}} f(u,v)\cdot h_{u,v} \tag{4.5}$$

Constraints:

$$\sum_{i\in\Omega} -\log\left(CDF_i(f(l_1^i), f(l_2^i), ..., f(l_{m_i}^i))\right) \le -\log(P_c) \tag{4.6}$$

$$0 \le f(u,v) \le c_{(u,v)}^{\max} \quad \forall (u,v)\in\mathcal{L} \tag{4.7}$$

$$h_{s,t} \ge 1 \tag{4.8}$$

$$h_{u,v} + h_{v,w} \ge h_{u,w},\ \forall u,v,w \in \mathcal{N} : u \ne v \ne w \tag{4.9}$$

$$h_{u,v} \ge 0, \quad \forall u,v \in \mathcal{N} : u \ne v \tag{4.10}$$

where $f(u,v), f(l_1^i), f(l_2^i), ..., f(l_{m_i}^i)$ indicate the allocated possible cost by link (u,v), l_1^i, l_2^i,..., $l_{m_i}^i$, respectively. In particular, Eq. (4.6) ensures that the probability of realizing the sum of allocated cost of the min-cut is no less than P_c. More specifically, for each MCG we apply the multi-dimensional CDF functions to calculate the probability of realizing possible cost. Since the CDF function is log-concave, then $-\log\left(CDF_i(f(l_1^i), f(l_2^i), ..., f(l_{m_i}^i))\right)$ is convex, by summing all the MCGs together, it remains convex, which indicates that Eq. (4.6) is convex.

It remains to show that Eq. (4.5) is convex. In general, the product of two convex functions is not always convex, however, according to [8, pp. 119], one special case is: "If functions f and g are convex, both nondecreasing (or nonincreasing), and positive (nonnegative) functions on an interval, then $f \cdot g$ is convex." Therefore, for each $(u,v)\in\mathcal{L}$, $f(u,v)\cdot h_{u,v}$ is convex.

V. Related Work

A. Routing with correlated link weights

In a network with each link having multiple additive link weight metrics (e.g., delay, cost, jitter, etc.), the Quality of Service (QoS) routing problem is to find a path that satisfies a given metric vector constraint. Kuipers and Van Mieghem [20] study the QoS routing problem when $m > 1$ link weights form a linear function or form a concave function with a given metric vector constraint. Another common correlation about the link failures is the Shared-Risk Link Groups (SRLGs). Sometimes one SRLG can also be represented by one color in different literature, but they share the same meaning in terms of reliability. In this context, Yuan *et al.* [16] prove that the Minimum Color Single-Path (MCSiP) problem is NP-hard. Besides, Yuan *et al.* also prove that finding two link-disjoint paths with total minimum distinct amount of colors or least amount of coupled/overlapped colors are NP-hard. Stefanakos [21] proves that the MCSiP problem cannot be approximated with a ratio of $c \log N$ for some positive constant $c > 0$, even in chains. Stefanakos [21] studies the Minimum Color Congestion Routing problem, which is to accommodate a set of requests such that the maximum number of paths that contain the same color is minimized. A constant factor approximation algorithm is proposed to solve it by Stefanakos [21]. Zhu and Jue [22] prove that the problem of finding a spanning tree with the least amount of distinct SRLGs is NP-hard. They propose a constant factor approximation algorithm to solve this problem. Lee *et al.* [23] propose a probabilistic SRLG framework to model correlated link failures and develop an Integer Nonlinear Programming Programming (INLP) to find one unprotected path or two link-disjoint paths with the lowest failure probability.

There are also some literature dealing with correlated routing problems in stochastic networks [24], [25]. More specifically, in [24], only two possible states are assumed, which are congested and uncongested, and each state corresponds to a cost value. A probability matrix $P_{a,b}^{u,v,w}$, which represents the probability that if (u,v) is in state a then (v,w) is in state b, is given. Two similar link weight models called link-based congestion model and node-based congestion model are also proposed in [25]. Based on these models, they [24], [25] define and solve the least expected routing problem, which is to find a path from the source to the destination with the minimum expected costs. However, in [24], [25] there are only two possible states for each link and only the correlation of the adjacent links is known/assumed. On the other hand, we assume a more general (and different) stochastic correlated model, where as long as the links (not necessarily adjacent) are correlated, their joint CDF for allocating costs is known.

B. Min-Cut in Conventional Networks

The (s,t) Min-Cut problem refers to partition the network into two disjoint subsets such that nodes s and t are in different subsets and the total weight of the cut edges are minimized. This problem can be solved by finding the maximum flow from s to t according to [26]. There are also a lot of work on the Min-Cut problem with no specified node pairs (s, t). A summary and comparison of the existing classical polynomial-time algorithms to solve the Min-Cut problem can be found in [27]. The fastest algorithm to solve the Min-Cut problem which has a time complexity of $O(L \log^3 N)$ is proposed by Karger [28]. Accordingly, the Min-Cut problem can be tackled by solving at most $\binom{N}{2}$ times of the (s, t) Min-Cut problem.

C. Constrained Maximum Flow

As a dual of the min-cut problem, the maximum flow problem in conventional networks is polynomial-time solvable [26]. However, this problem becomes NP-hard if some constraints are imposed on links. Suppose that the negative disjunctive constraints indicate that a certain set of links cannot be used simultaneously for the optimal solution, while the positive disjunctive constraints force at least one of a certain set of links to be present in the optimal solution. Pferschy and Schauer [29] prove that the maximum flow problem in both negative and positive disjunctive constraints are NP-hard and do not admit Polynomial-Time Approximation Scheme (PTAS). Here, the disjunctive constraint corresponds to the correlated link weights in this chapter, so the maximum flow problem in correlated networks is also NP-hard and does not admit PTAS. Assuming the link's bandwidth and delay follows a log-concave distribution, Kuipers *et al.* [13] propose a polynomial-time convex optimization formulation to find the maximum flow in the so-called stochastic networks. When the delay constraint is imposed on each path, the maximum flow problem is NP-hard. To solve it, they [13] propose an approximation algorithm and a tunable heuristic algorithm.

VI. Conclusion

In this paper, we have studied the shortest path problem and the min-cut problem under two proposed link weight models, namely (i) deterministic correlated (link weight) model and (ii) (log-concave) stochastic correlated (link weight) model. We have proved that these two problems are NP-hard under the deterministic correlated model, and cannot be approximated to arbitrary degree, unless P=NP. Subsequently, we have proposed exact algorithms to solve them, respectively. In particular, we have shown that both of them are polynomial-time solvable under a (constrained) nodal/pairwise deterministic correlated model. On the other hand, we have shown that these two problems under the stochastic correlated model can be solved by convex optimization formulations.

References

[1] W. Cui, I. Stoica, and R. H. Katz, "Backup path allocation based on a correlated link failure probability model in overlay networks," in *Proc. of IEEE ICNP'02*, 2002, pp. 236–245.

[2] S. Savage, A. Collins, E. Hoffman, J. Snell, and T. Anderson, "The end-to-end effects of internet path selection," in *ACM SIGCOMM Computer Communication Review*, vol. 29, no. 4. ACM, 1999, pp. 289–299.

[3] D. Kostić, A. Rodriguez, J. Albrecht, and A. Vahdat, "Bullet: High bandwidth data dissemination using an overlay mesh," in *ACM SIGOPS Operating Systems Review*, vol. 37, no. 5, 2003, pp. 282–297.

[4] K. Kim and N. Venkatasubramanian, "Assessing the impact of geographically correlated failures on overlay-based data dissemination," in *Proc. of IEEE GLOBECOM'10*, 2010, pp. 1–5.

[5] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, "Catastrophic cascade of failures in interdependent networks," *Nature*, vol. 464, no. 7291, pp. 1025–1028, Apr. 2010.

[6] J. Strand, A. L. Chiu, and R. Tkach, "Issues for routing in the optical layer," *IEEE Communications Magazine*, vol. 39, pp. 81–87, 2001.

[7] G. Dantzig and D. R. Fulkerson, "On the max flow min cut theorem of networks," *Linear inequalities and related systems*, vol. 38, pp. 225–231, 2003.

[8] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York, NY, USA: Cambridge University Press, 2004.

[9] D. H. Lorenz and A. Orda, "Qos routing in networks with uncertain parameters," *IEEE/ACM Transactions on Networking*, vol. 6, no. 6, pp. 768–778, 1998.

[10] K. Papagiannaki, S. Moon, C. Fraleigh, P. Thiran, F. Tobagi, and C. Diot, "Analysis of measured single-hop delay from an operational backbone network," in *Proc. of IEEE INFOCOM'02*, vol. 2, 2002, pp. 535–544.

[11] M. Bagnoli and T. Bergstrom, "Log-concave probability and its applications," *Economic Theory*, vol. 26, no. 2, pp. 445–469, 2005.

[12] G. R. Mohtashami Borzadaran and H. A. Mohtashami Borzadaran, "log-concavity property for some well-known distributions," *Surveys in Mathematics and its Applications*, vol. 6, pp. 203–219, December 2011.

[13] F. A. Kuipers, S. Yang, S. Trajanovski, and A. Orda, "Constrained maxmum flow in stochastic networks," in *Proc. of IEEE ICNP'14*, North Carolina, October 2014.

[14] P. Van Mieghem, *Performance Analysis of Complex Networks and Systems*. Cambridge University Press, 2014.

[15] H. Gabow, S. N. Maheshwari, and L. Osterweil, "On two problems in the generation of program test paths," *IEEE Transactions on Software Engineering*, vol. SE-2, no. 3, pp. 227–231, 1976.

[16] S. Yuan, S. Varma, and J. Jue, "Minimum-color path problems for reliability in mesh networks," in *Proc. of IEEE INFOCOM'05*, vol. 4, 2005, pp. 2658–2669.

[17] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*. New York, NY, USA: W. H. Freeman & Co., 1979.

[18] P. Van Mieghem and F. Kuipers, "Concepts of exact quality of service algorithms," *IEEE/ACM Transactions on Networking*, vol. 12, no. 5, pp. 851–864, 2004.

[19] A. Tamir, "Polynomial formulations of min-cut problems," *Manuscript, Department of Statistic and Operations Research, Tel Aviv University, Israel*, 1994.

[20] F. A. Kuipers and P. Van Mieghem, "The impact of correlated link weights on qos routing," in *Proc. of IEEE INFOCOM'03*, vol. 2, 2003, pp. 1425–1434.

[21] S. Stefanakos, "Reliable routings in networks with generalized link failure events," *IEEE/ACM Transactions on Networking*, vol. 16, no. 6, pp. 1331–1339, 2008.

[22] Y. Zhu and J. P. Jue, "Reliable collective communications with weighted srlgs in optical networks," *IEEE/ACM Transactions on Networking*, vol. 20, no. 3, pp. 851–863, 2012.

[23] H.-W. Lee, E. Modiano, and K. Lee, "Diverse routing in networks with probabilistic failures," *IEEE/ACM Transactions on Networking*, vol. 18, no. 6, pp. 1895–1907, 2010.

[24] S. T. Waller and A. K. Ziliaskopoulos, "On the online shortest path problem with limited arc cost dependencies," *Networks*, vol. 40, no. 4, pp. 216–227, 2002.

[25] Y. Fan, R. Kalaba, and J. Moore, "Shortest paths in stochastic networks with correlated link costs," *Computers & Mathematics with Applications*, vol. 49, no. 9, pp. 1549–1564, 2005.

[26] L. R. Ford and D. R. Fulkerson, "Maximal flow through a network," *Canadian Journal of Mathematics*, vol. 8, no. 3, pp. 399–404, 1956.

[27] C. S. Chekuri, A. V. Goldberg, D. R. Karger, M. S. Levine, and C. Stein, "Experimental study of minimum cut algorithms," in *Proceedings of the eighth annual ACM-SIAM symposium on Discrete algorithms*. Society for Industrial and Applied Mathematics, 1997, pp. 324–333.

[28] D. R. Karger, "Minimum cuts in near-linear time," *Journal of the ACM (JACM)*, vol. 47, no. 1, pp. 46–76, 2000.

[29] U. Pferschy and J. Schauer, "The maximum flow problem with disjunctive constraints," *Journal of Combinatorial Optimization*, vol. 26, no. 1, pp. 109–119, 2013.