Decentralized Protection Strategies against SIS Epidemics in Networks

Stojan Trajanovski, Yezekael Hayel, Eitan Altman, Huijuan Wang, and Piet Van Mieghem

Abstract—Defining an optimal protection strategy against viruses, spam propagation or any other kind of contamination process is an important feature for designing new networks and architectures. In this work, we consider decentralized optimal protection strategies when a virus is propagating over a network through a SIS epidemic process. We assume that each node in the network can fully protect itself from infection at a constant cost, or the node can use recovery software, once it is infected.

We model our system using a game theoretic framework and find pure, mixed equilibria, and the Price of Anarchy (PoA) in several network topologies. Further, we propose both a decentralized algorithm and an iterative procedure to compute a pure equilibrium in the general case of multiple communities network. Finally, we evaluate the algorithms and give numerical illustrations of all our results.

Index Terms—Game theory; Network Protection; Decentralized Protection; Virus-spread

I. Introduction

The spread of viruses in communication and computer networks or rumors in social networks can be similarly described, using epidemic models [1], [2], [3], [4]. The probability of infection over time, especially in the steady-state, in relation to the properties of the underlying network has been widely studied in the past [1], [2]. We consider the Susceptible Infected Susceptible (SIS) model, which is one of the mostly studied epidemic models [1], [5]. In the SIS model, at a specific time, the state of each node is either susceptible or infected. The recovery (curing) process of each infected node is an independent Poisson process with a recovery rate δ. Each infected node infects each of its susceptible neighbors with a rate β, which is also an independent Poisson process. Immunization [6], [7] (e.g. via antivirus software) or quarantining [8] (via modular partitioning) fully prevent nodes from being infected, while additional mechanisms, like anti-spyware software or clean-up tools, could clean the virus from an infected node. Finally, the network can be modified to increase the epidemic threshold [9].

This paper considers investment games that find appropriate protection strategies against SIS virus spread. In particular, we consider a game, in which, each node (host) is a player in the game and decides individually whether or not to invest in antivirus protection. Further, if a host does not invest in

The first two authors have contributed equally.

S. Trajanovski, H. Wang and P. Van Mieghem are with Delft University of Technology, Faculty of Electrical Engineering, Mathematics and Computer Science, P.O. Box 5031, 2600 GA Delft, The Netherlands (e-mails: {S.Trajanovski, H.Wang, P.F.A.VanMieghem}@tudelft.nl).

Y. Hayel is with University of Avignon, Avignon, France (yezekael.hayel@univ-avignon.fr).

E. Altman is with INRIA, Sophia Antipolis, France (eitan.altman@inria.fr).

antivirus protection, it remains vulnerable to the virus spread process, but can recover (e.g., by a system recovery or clean-up software). The utility or payoff of each node (player) is: (i) the investment cost, if the node decides to invest in antivirus software or else (ii) the cost of being infected, which is proportional to the infection probability in the epidemic steady-state.

Our main contributions are summarized as follows:

1) We prove that the game is a potential game by showing that it is equivalent to a congestion game. We determine a closed-form expression for the pure equilibrium for a *single community/full mesh* network. We also prove the existence and uniqueness of a mixed equilibrium.
2) We provide a measure of the equilibrium efficiency based on the Price of Anarchy (PoA) and propose a simple, fully decentralized Reinforcement Learning Algorithm (RLA) that converges to the equilibrium.
3) We extend our equilibrium analysis to *bipartite* networks, where we show that multiple equilibria are possible. At an equilibrium, the number of nodes that invest in one partition is often close to the number of nodes that invest in the other partition.
4) In *multi-communities* network, in which several communities are communicating through a single core node, we introduce the concept of parametric potential games and subsequently, show the convergence of an iterative procedure to a pure Nash equilibrium. Here, we also show that the iterative procedure can be used to find the metastable stationary infection probability of the core node that links all the communities.

The paper is organized as follows. An overview of decentralized protection strategies, epidemics and game theory is given in Section II. The SIS epidemic model is introduced in Section III. Sections IV and V describe the game models in a single community (full mesh) and bipartite network, respectively and subsequently prove game theoretic results. In Section VI, we study the potential parametric games and equilibrium properties in a *multi-communities* network. Evaluation of the RLA algorithm, its convergence to the equilibrium point and numerical illustrations are given in Section VII. We conclude in Section VIII. In order to enhance the readability, the mathematical proofs of the non-trivial prepositions and corollaries are given in Appendices A, B and C.

II. Related work

Virus spread and epidemics in networks have been thoroughly explored during the last decades [1], [2], [3], [4]. These

works involve studies of epidemic propagation, the number of infected hosts [1] over the time and stationary regimes, epidemic threshold [2] or the relation with eigenvalues, see e.g., [10]. One of the widely explored Susceptible Infected Susceptible (SIS) approximations is the N-intertwined mean-field approximation NIMFA [1], [5].

Game theoretical studies for networks problems have been conducted, in routing [11], [12], network flow [13], workload on the cloud [14] or optimal network design [15], [16], employing standard game-theoretic concepts [17], [18] such as pure Nash or mixed equilibrium. The Price of Anarchy (PoA) [19], [18] is often used as an equilibrium performance evaluation metric.

While many papers have been focused on the process of virus spread and network immunizations to suppress the spread, few epidemic studies control the curing rate via game theory. Omić *et al.* [20] tune the strength of the nodal antivirus protection i.e. how big those (different) δ should be taken. Contrarily to [20], (i) we fix the curing and infection rates, which are not part of the game, and the decision consists of a player's choice to invest in an antivirus or not; (ii) we also consider mixed strategies Nash Equilibrium and (iii) propose convergence algorithm into the equilibrium point. The goal of [20] is in finding the optimal δ_i for each player i, while this paper targets the optimal decision of taking an anti-virus that fully protects the host, because today's antivirus software packages provide accurate and up to date virus protection. The related papers on security games [21], [22], [23], [24], [25] are usually applied in non-SIS environments (e.g., (i) without considering the infection state of the neighbors and (ii) without an additional mechanism for recovery), for generalized game settings [24] or by assigning nodal weights to reflect the security level [23].

III. SIS EPIDEMICS ON NETWORKS

We start with the general case, where the network G is connected, undirected and unweighted with a large number N of nodes. The virus behaves as an SIS epidemic, where an infected node can infect each of its direct, healthy neighbors with rate β. Each node can be cured at rate δ, after which the node becomes healthy, but susceptible again to the virus. Both infection and curing process are independent Poisson processes. All nodes in the network G are prone to a virus that can re-infect the nodes multiple times.

We denote the viral probability of infection for node i at time t by $v_i(N;t)$. For each node i of the graph with N nodes, the SIS governing equation, under the standard N-Intertwined mean-field approximation (NIMFA) [26] and scaling the time as $t^* = \delta t$, is given by

$$\frac{dv_i(N;t)}{dt^*} = -v_i(N;t) + \tau(1 - v_i(N;t)) \sum_{j=1}^{N} a_{ij} v_j(N;t)$$

where $\tau = \frac{\beta}{\delta}$ is the effective spreading rate. We further confine ourselves to the stationary regime of the SIS epidemic process, meaning $\lim_{t \to \infty} \frac{dv_i(N;t)}{dt} = 0$, and we denote $v_{i\infty}(N) = \lim_{t \to \infty} v_i(N;t)$. The steady-state equations, valid for any graph [1], [26], are

$$v_{i\infty}(N) = 1 - \frac{1}{1 + \tau \sum_{j=1}^{N} a_{ij} v_{j\infty}(N)} \quad (1)$$

for $i = 1, 2, \ldots, N$. These steady-state equations only have two possible solutions: the trivial $v_{i\infty}(N) = 0$, corresponding to the exact absorbing state in SIS epidemics, and the non-trivial solution, corresponding to the metastable SIS regime.

The infection probabilities could be substantially different after some nodes decide to invest in a protection, causing those nodes not to be part of the epidemic process. Three scenarios are considered. The first one is a single community game, which could be regarded as a simple social network or a wireless and other full mesh networks (e.g., MANETs). We also study bipartite networks, often employed in the design of telecommunication networks. The main reason is that a bipartite topology provides satisfactory level of robustness after node or link failures. For instance, the topology of the Amsterdam Internet Exchange is designed as a bipartite network such that all the locations in Amsterdam are connected to two high throughput Ethernet switches. In addition, the topologies of sensor networks are also bipartite graphs. The third, we consider a topology inspired by online social networks, where agents are connected if they belong to the same community school, institution, geographical location, interest in sport or music activities. These communities may weakly overlap via a small number of nodes that belong to multiple communities.

A. Single community (full mesh) network

We first consider a single community (or full mesh) network, modeled as a complete graph K_N, where $a_{ij} = 1$ for all i and j. By symmetry, we find that $v_{i\infty}(N) = v_{j\infty}(N) = v_\infty(N)$ for any node i and j. From (1), we have [1], [5]

$$v_{i\infty}(N) = \begin{cases} 1 - \frac{1}{\tau(N-1)}, & \text{if } \tau \geq \frac{1}{N-1}, \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

for each node i in a complete graph.

B. Bipartite network

The bipartite network $K_{M,N}$ consists of two clusters with M and N nodes, such that one of the nodes within a cluster are connected, but all the nodes from different clusters are connected to each other as illustrated in Fig. 1a. Therefore, there are exactly $(M+N)$ nodes and MN links in the network.

The bipartite network also possesses an interesting property, if a node is removed from the original graph from any of the two clusters, the resulting graph is again a bipartite graph either $K_{M-1,N}$ or $K_{M,N-1}$.

The governing equations (1) reduce for $K_{N,M}$ to two equations with two unknowns - the infection probabilities $v_{m\infty}$ and $v_{n\infty}$ in the two clusters in metastable state stationary regime [1], [27], which solution is given by

$$v_\infty^m = \frac{\tau^2 MN - 1}{\tau M(\tau N + 1)} \text{ and } v_\infty^n = \frac{\tau^2 MN - 1}{\tau N(\tau M + 1)} \quad (3)$$
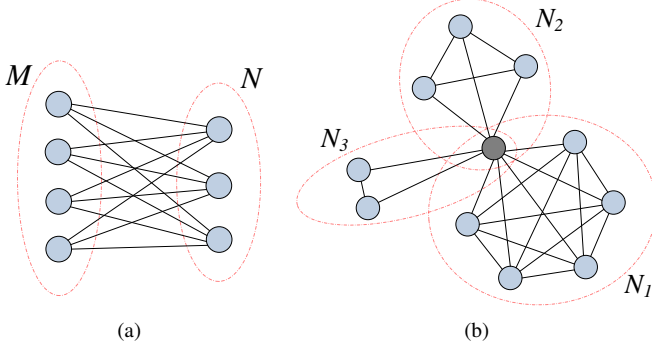
Fig. 1: (a) Example of a bipartite graph and (b) Example of a multi-communities network. The core node is given in dark gray color.

C. Multi-communities network

We consider the *multi-communities network* composed of M cliques/communities, where each community has finite size $N_m + 1$ for all $m \in \{1, \ldots, M\}$. Furthermore, we assume that each sub-network is connected to any other sub-network by a *core node*, and all communities overlap only via one core node (in dark gray in Fig. 1b). The core node does not participate into the game and cannot invest in an anti-virus protection. The total number of nodes in this network is $1 + \sum_{m=1}^{M} N_m$. This communities network has the property that after removing some nodes (those who decide to invest), the resulting graph is still a communities network. The degree of any node i, which belongs to community m, but is not the core node, is N_m (all the other nodes from his community plus the core node).

The core node functions as a bridge between all communities. In other words, the M fully connected graphs are interconnected through one (common) node. We will compute the infection probability for each none core node of each community and also for the core node. Different communities might have different virus-spread dynamic. We consider the more general in-homogeneous SIS model, see e.g., [28], with different effective infection rate τ_m for each community m. We use NIMFA and obtain for any m,

$$v_m(N_m, v_\infty) = 1 - \frac{1}{1 + \tau_m(N_m - 1)v_m + \tau_m v_\infty}. \quad (4)$$

where v_m is the infection probability for any none core node of community m and v_∞ is the infection probability for the core node.

IV. GAME MODEL ON A SINGLE COMMUNITY NETWORK

In the investment game on the complete graph K_N, each node is a player and decides individually to invest in antivirus protection. The investment cost is C, while the infection cost is H. When a node invests, it is assumed to be directly immune to the virus and not part of the epidemic process anymore. Hence, this node cannot infect other nodes nor can be infected. If a node does not invest in antivirus protection, it is prone to the epidemics and might be infected by the virus (with rate β), but also can use additional protective mechanisms, like recovery or anti-spyware software (with rate δ). The induced network topology, after the removal of nodes that decide to invest, determines the process of epidemic spread.

A. Pure strategies

The investment cost for any player is a constant C and does not depend on the action of the other players. If a player decides not to invest, his cost is a linear function of his expected infection time

$$E[T] = \frac{1}{T}\int_0^T E[X_i(N; t)]dt = \frac{1}{T}\int_0^T \Pr[X_i(N; t) = 1]dt,$$

since $E[X_i] = \Pr[X_i]$ for Bernoulli random variables.

When the time T becomes large, $E[T]$ is approximated by the infection probability $v_{i\infty}(N)$ of node i in the metastable state of the SIS process. The probability $v_{i\infty}(N)$ depends explicitly on the number of nodes that decide not to invest. In other words, there is an initial contact graph G in which all the nodes are connected and the decisions of all the nodes induce an overlay graph G_g only composed of the nodes that have decided *not* to invest.

1) Congestion Game: Due to players' decisions, we have a congestion game, because the utility of each player depends on the number of players that have decided not to invest. Each node has the choice between two actions: invest (further denoted by 1) or not (further denoted by 0). The payoff obtained by a player, in case he does not invest, depends on the number of players that choose the same action (0) not to invest. We denote by $\sigma_i \in \{0, 1\}$ the action of node i. For example, the payoff S_{i1} of a player $i \in \{1, 2, \ldots, N\}$ which decides to invest is defined by: $S_{i1} = C := S_1$, while the payoff of a player i which decides not to invest is: $S_{i0}(n) = Hv_{i\infty}(n) := S_0(n)$, where n is the number of players that do not invest. This game is a congestion game [29] as the payoff of a player depends on the number of players that choose his action. In the context of a congestion game, a (pure) Nash equilibrium is a vector of (pure) strategies, characterized by the number of nodes n^* that do not invest. We remark that several Nash equilibria lead to the same n^*. We are interested in the existence and uniqueness of this value n^*.

Definition 1. *At a Nash equilibrium, no node has an interest to change unilaterally his decision. The number n^* of nodes that do not invest at a Nash equilibrium is defined for any player i, by: $S_{i1} \leq S_{i0}(n^* + 1)$ and $S_{i0}(n^*) \leq S_{i1}$.*

Our game is symmetric as all players share the same set of payoff functions. The following important property (in Preposition 1) says that our game is not only a congestion game but also a potential game, due to the potential formula in [30, Theorem 3.1].

Proposition 1. *The game is* potential*, where $\Phi(n) = C(N - n) + H \sum_{k=2}^{n} v_\infty(k)$ is the potential function of the game.*

The existence of a potential function in a game shows the existence of pure Nash equilibrium: any minimum of the potential function Φ is a pure equilibrium. The existence also allows decentralized procedures like best response dynamics or reinforcement learning [31], [32] to converge to the pure Nash equilibrium. We can assume, for example, that an investment is valid only for a fixed amount of time and then each node pays again after expiration of his license.

Proposition 2. *For the number of nodes n^* that do not invest at equilibrium, the following inequality holds:*

$$v_\infty(n^*) \le \frac{C}{H} \le v_\infty(n^*+1).$$

Moreover, above the epidemic threshold ($\tau > \frac{1}{N-1}$), n^* is uniquely defined by:

$$n^* = \begin{cases} \min\left\{N, \lceil \frac{1}{(1-\frac{C}{H})\tau} \rceil \right\}, & \text{if } C < H \\ N, & \text{otherwise} \end{cases} \quad (5)$$

where $\lceil x \rceil$ is the closest integer greater or equal than x.

2) Performance of the equilibrium: In order to evaluate the performance of the system, considering a non-cooperative behavior of each node, we use the Price of Anarchy (PoA) metric [19]. We define the social welfare $SW(n)$ of this system, when n users do not invest, as the summation of the cost for all users:

$$SW(n) = \sum_{i=1}^{N} S_{i\sigma_i}(n) = C(N-n) + nHv_\infty(n) \quad (6)$$

We define n^{opt} such that: $n^{opt} = \arg\min_n SW(n)$, while the Price of Anarchy, considering pure strategies, is defined by:

$$PoA_p = \frac{SW(n^{opt})}{SW(n^*)} \le 1.$$

Proposition 3. *The value that minimizes the social welfare is $n^{opt} \in \{N, \lceil 1 + \frac{1}{\tau} \rceil\}$.*

The following Corollary 1 also holds:

Corollary 1. *The equilibrium value n^* is at least as large as the optimum value n^{opt}, thus $n^* \ge n^{opt}$.*

Because, we have determined n^* and n^{opt}, we can find PoA_p in an exact, but rather complex form. We can also bound PoA_p.

Corollary 2. *The Price of Anarchy PoA_p is bounded by:*

$$1 - (1 + \frac{1}{\tau})\frac{1}{N} \le PoA_p \le 1.$$

3) Decentralized Algorithm: Here, we propose a simple fully decentralized algorithm that converges to a pure Nash Equilibrium in our invest game. We consider discrete time slots and we denote by $p_i(t)$ the probability that node i invests at time slot t. Then, at each time slot t, independently, each node i decides to invest with probability $p_i(t)$. We also denote by $\bar{\sigma}(t) = (\sigma_1(t), \ldots, \sigma_N(t))$ the vector of pure actions of all the nodes at time t (for any time t, $\sigma_i(t) = 1$ if node i invests at time t, it is 0 otherwise) and $G_N(\bar{\sigma}(t))$ is the corresponding induced graph composed only of the nodes that do not to invest. The principle of our learning algorithm is the following:

1) Set an initial probability $p_i(0)$, for each user $i = 1, \ldots, N$.
2) At every time step t, each node i invests with probability $p_i(t)$. This determines his pure action $\sigma_i(t)$.
3) Each player i obtains his payoff $S_i(\sigma_i(t), \bar{\sigma}(t))$ which is equal to:

$$S_i(\sigma_i(t), \bar{\sigma}(t)) = \begin{cases} C, & \text{if } \sigma_i(t) = 1, \\ Hv_\infty(1 + \sum_{j \ne i} \sigma_j(t)), & \text{otherwise}. \end{cases}$$

4) The payoff of each node i is normalized: $\tilde{S}_i(\sigma_i(t), \bar{\sigma}(t)) = \frac{S_i(\sigma_i(t), \bar{\sigma}(t))}{C+H}$. A normalization is necessary for this algorithm [33].
5) Each node i updates its probability according to the following rule:

$$p_i(t+1) \leftarrow p_i(t) + b(t)\tilde{S}_i(\sigma_i(t), \bar{\sigma}(t))(\sigma_i(t) - p_i(t)),$$

where $b(t)$ is the learning rate.
6) Stop when a stopping criterion is met (for example, the maximum of the differences between consecutive updates is smaller than a small ϵ; else increase t by 1 and go to step 2).

Sastry *et al.* [33] have proved that as $b(t) \to 0$, for any node i, the probability sequence converges weakly to the solution of the following ordinary differential equation

$$\dot{p}_i(t) = p_i(t)(1 - p_i(t))[S_i(0, p(t)) - S_i(1, p(t))],$$

which is known in evolutionary game theory as the *replicator dynamic equation* [34]. Bournez and Cohen [35] show that any potential game possesses a Lyapunov function. Hence, the decentralized algorithm proposed converges almost surely to a pure Nash equilibrium of our game. This algorithm is simple and fully distributed, the only required information for each node is his instantaneous payoff at each time slot. Similar algorithms have been successfully used in many networking applications [36], [37] and we illustrate the behavior of our RLA for our game in Section VII.

B. Symmetric mixed strategies

We now assume that each individual decides with a probability p to invest in the anti-virus protection. Moreover, the game is symmetric and then we look for a symmetric mixed Nash equilibrium. Each individual is faced with a new game, which depends on the realization of the random choice process of all the other individuals. We denote by $\bar{S}_i(\sigma_i, p)$ the expected cost of player i choosing the pure strategy σ_i against the probability choice p of the other $N-1$ players. For any user i, we have $\bar{S}_{i\sigma_i}(p) = \sum_{n=0}^{N-1} S_{i\sigma_i}(n+1)\binom{N-1}{n}(1-p)^n p^{N-1-n}$, where by definition $S_{i0}(1) = 0$. Hence, the total expected cost of node i which invests with probability p' and when all the other nodes invest with probability p, is:

$$\bar{S}_i(p', p) = p'\bar{S}_{i1}(p) + (1 - p')\bar{S}_{i0}(p).$$

Definition 2. *At equilibrium, the indifference property p^* is solution of $\bar{S}_i(0, p^*) = \bar{S}_i(1, p^*)$.*

We now show the existence and uniqueness of a symmetric mixed equilibrium p^*.

Proposition 4. *A symmetric mixed equilibrium exists.*

Now, the equilibrium point p^* could be determined from an exact, but rather complex, non-closed expression in p:

$$\bar{S}_i(0, p) = \sum_{n=0}^{N-1} S_{i0}(n+1)\binom{N-1}{n}(1-p)^n p^{N-1-n}$$

$$= H \sum_{n=\underline{n}}^{N-1} (1 - \frac{1}{\tau n})\binom{N-1}{n}(1-p)^n p^{N-1-n}, \quad (7)$$

with $\underline{n} = \lceil \frac{1}{\tau} \rceil$ because $S_{i0}(n+1) = v_\infty(n+1)$ if $\tau \ge \frac{1}{n}$ and $S_{i0}(n+1) = 0$, otherwise.

Proposition 5. *The symmetric mixed equilibrium is unique.*

Further, it is possible to modify (7) in a complex form involving *generalized hyper-geometric function* [38], which explains the difficulty of finding a closed form for p^*.

1) Approximation: In order to get a closed form expression of the symmetric mixed strategy, we consider the following approximation: instead of considering a player faced to realize a symmetric mixed strategy of the other players and optimizing his average cost, we consider that a player is part of an average game. In this case, if player i chooses strategy 1 with probability p_i we obtained the following average approximated cost: $\hat{S}_i(p_i, p') = p_i C + (1 - p_i) H v_{i\infty}(\bar{n}(p') + 1)$, where $\bar{n}(p')$ is the average number of nodes, except node i, that decide not to invest, i.e. $\bar{n}(p') = (1 - p')(N - 1)$.

Proposition 6. *If we approximate the number of nodes that do not invest by its average, we obtain the following symmetric mixed Nash equilibrium:*

$$\hat{p}^* = \begin{cases} 1 - \frac{H}{\tau(H-C)(N-1)}, & \text{if } C < H(1 - \frac{1}{\tau(N-1)}), \\ 0, & \text{otherwise.} \end{cases}$$

If the investment cost C is higher than the curing cost H, then the equilibrium is $\hat{p}^* = 0$, because even, if a node is infected, its cost H is less than the cost C, then he would pay to be protected.

2) Performance of symmetric mixed equilibrium: The social welfare can be defined considering the mixed strategies: $SW(p) = \sum_{i=1}^{N} \bar{S}_i(p, p)$.

Further, we compute the optimal social welfare by using the approximated cost function $\hat{S}_i$:

$$\arg\min_p SW(p) \simeq \arg\min_p \left[p\hat{S}(1, p) + (1 - p)\hat{S}(0, p) \right]$$
$$= \arg\min_p \begin{cases} (C - H)p + H(1 - \frac{1}{\tau(N-1)}), & \text{if } p \in [0, 1 - \frac{1}{\tau(N-1)}) \\ pC, & \text{if } p \in [1 - \frac{1}{\tau(N-1)}, 1]. \end{cases}$$

Proposition 7. *The optimal solution of the social welfare (SW) is $N \min\{C, H\}(1 - \frac{1}{\tau(N-1)})$ and it is achieved for*

$$\hat{p}^{opt} = \begin{cases} 0, & \text{if } C > H \\ [0, 1 - \frac{1}{\tau(N-1)}], & \text{if } C = H \\ 1 - \frac{1}{\tau(N-1)}, & \text{if } C < H. \end{cases}$$

Based on Proposition 7 and $SW(p^*) = NC$, we are now able to approximate PoA_m in the case of mixed strategies:

Corollary 3. *When each node uses a mixed strategy, the Price of Anarchy PoA_m can be approximated by:*

$$PoA_m \approx \frac{\min\{C, H\}(1 - \frac{1}{\tau(N-1)})}{C}.$$

C. Comparison of strategies

In the previous sections, we have studied two different approaches for our non-cooperative investment game: the pure and the mixed strategies. These two game variants assume significantly different decision processes for each node. First, the approximation of the expected number of nodes that do not invest at equilibrium is very close to the result obtained using the potential game approach: $\hat{n} = N(1 - \hat{p}^*) \simeq n^*$. Second, we compare the social welfares obtained in each situation, and we observe that pure strategies always yields a lower social welfare compared with symmetric mixed strategies.

As stated in Corollary 3, $\bar{S}_i(0, p^*) = \bar{S}_i(1, p^*) = C$, i.e. the payoff of all the players is equal, hence $SW_m^* = CN$. On the other hand, in the proof of Corollary 2, we find $SW_p^* := SW(n^*) > CN$. Corollary 4 immediately follows.

Corollary 4. *The social welfare is smaller if all the nodes use a pure strategy (SW_p^*) compared to the case in which all the nodes use a symmetric mixed strategy (SW_m^*), i.e. $SW_p^* < SW_m^*$.*

The bound achieved in Corollary 4 is tight, because $(SW(n^*) - CN)$ is small - based on Preposition 2. This is also visualized in Fig. 2a in Section VII, where indirectly, by comparing the Price of Anarchy for different equilibria, we show the approximation leads to almost correct value for the real expected payoff.

V. GAME MODEL IN BIPARTITE NETWORK

In this section, we characterize the equilibrium points, their existence and uniqueness for bipartite network $G_{M,N}$. For simplicity, we define $k = \frac{C}{H}$.

Proposition 8. *The equilibrium pair (n^*, m^*) exists and satisfies the following inequalities:*

$$\forall m \in \mathcal{M}, \ v_\infty^{m^*}(n^*, m^*) < \frac{C}{H} \leq v_\infty^{m^*+1}(n^*, m^* + 1) \text{ and}$$
$$\forall n \in \mathcal{N}, \ v_\infty^{n^*}(n^*, m^*) < \frac{C}{H} \leq v_\infty^{n^*+1}(n^* + 1, m^*)$$

Moreover, above the epidemic threshold, the following hold:

1) *for a given n^* (m^*) there is no more than one m^* (n^*).*
2) *for any τ and $k \geq \frac{1}{2}$; or $\tau \geq \frac{(1+k)(1-2k)}{2k(1-k)}$ and $k < \frac{1}{2}$: $|n^* - m^*| \leq 1$ i.e. n^* and m^* are either equal or consecutive integers.*
3) *in general, it is possible to have multiple equilibria pairs such that $|n^* - m^*| \geq 2$ for some (n^*, m^*).*

When n and m players do not invest in anti-virus protection, from the first and the second partition, respectively, the social welfare is given by:

$$SW(n, m) = \sum_{i=1}^{N} S_{i\sigma_i}(n) = C(N - n + M - m)$$
$$+ H(nv_\infty^n(n, m) + mv_\infty^m(n, m)). \quad (8)$$

We define the optimal pair (n^{opt}, m^{opt}) as:

$$(n^{opt}, m^{opt}) = \arg\min_{(n,m)} SW(n, m).$$

and the Price of Anarchy in $K_{N,M}$: $PoA := \frac{SW(n^{opt}, m^{opt})}{SW(n^*, m^*)}$.

Proposition 9. *In $K_{N,M}$, the minimum (optimal) value of the social welfare is equal to $SW = \max\{\tau^2 MN - 1, 0\} \cdot \min\{\frac{C}{\tau^2 \max\{M,N\}}, H\frac{\tau(M+N)+2}{\tau(\tau M+1)(\tau N+1)}\}$. In particular,*

1) *if $MN \leq \frac{1}{\tau^2}$, then $SW = 0$ and $(n^{opt}, m^{opt}) = (N, M)$.*
2) *if $MN > \frac{1}{\tau^2}$, $\tau \max\{M, N\}\frac{\tau(M+N)+2}{(\tau M+1)(\tau N+1)} \geq k$ then: $SW = C\frac{\tau^2 MN - 1}{\tau^2 \max\{M,N\}}$ and $(n^{opt}, m^{opt}) = (\frac{1}{\tau^2 M}, M)$ if $M > N$; $(n^{opt}, m^{opt}) = (N, \frac{1}{\tau^2 N})$ if $M < N$ or both points for $M = N$.*

3) if $MN > \frac{1}{\tau^2}$, $\tau \max\{M,N\}\frac{\tau(M+N)+2}{(\tau M+1)(\tau N+1)} < k$ then $SW = H\frac{(\tau^2 MN-1)[\tau(M+N)+2]}{\tau(\tau M+1)(\tau N+1)}$ and $(n^{opt}, m^{opt}) = (N,M)$.

Based on the results in Prepositions 8 and 9, we can now find a tight bound for the Price of Anarchy (PoA).

Corollary 5. *The Price of Anarchy is bounded by:*

$$PoA \geq \frac{\max\{\tau^2 MN-1, 0\}\min\{\frac{1}{\tau\max\{M,N\}}, \frac{H(\tau(M+N)+2)}{C(\tau M+1)(\tau N+1)}\}}{\tau(M+N)}.$$

The only used inequality in the proof of Corollary 5 is from Preposition 8, where the last well-approaches the correspondent real value. Moreover, we have the following Corollary 6.

Corollary 6. *The lower bound in Corollary 5* $\frac{\max\{\tau^2 MN-1, 0\}\min\{\frac{1}{\tau\max\{M,N\}}, \frac{H(\tau(M+N)+2)}{C(\tau M+1)(\tau N+1)}\}}{\tau(M+N)}$ *is smaller than* $\min\{\frac{1}{2}, \frac{H}{C}\}$.

Because the bound of PoA from Preposition 5 is accurate, Corollary 6 tells us that the loss of the social welfare due to decentralized investment decision often cannot be larger than 50%.

For a bipartite graph, not much could be said about the mixed equilibrium due the fact that the bipartite network is not symmetric, and players' uniform social welfare function cannot be defined.

VI. GAME MODEL IN MULTI-COMMUNITIES NETWORK

In this section, we study the game model for the *multi-communities network*. In the single community case, we first find that pure equilibrium strategies yield better performance and second, the analysis is simpler, compared to symmetric mixed strategies; hence we decide to restrict the analysis of the multi-communities network investment game only by looking at the existence of pure Nash equilibrium. We start with the concept of parametric potential games.

A. Parametric potential games

We assume that the infection probability v_∞ of the core node is given. Further, our game with M communities is equivalent to M independent potential games. We observe that, if the infection probability v_∞ is given, then based on (4) the infection probability v_m of a none core node in community m depends only on the number of nodes N_m. Further, nodes from each community play a potential game within its own community. The approach is the following: we first compute the equilibrium in each community using the parametric potential game approach (v_∞ fixed); second, we compute the value of v_∞ by solving the fixed point equation (4).

Proposition 10. *For a given v_∞, for any community m, the infection probability v_m (is in the interval $(0,1)$ and) is*

$$v_m(N_m, v_\infty) = \frac{V(\tau_m, N_m, v_\infty)\left(1 + \sqrt{1 + \frac{4\tau_m^2 v_\infty(N_m-1)}{V(\tau_m, N_m, v_\infty)}}\right)}{2\tau_m(N_m-1)}$$

where $V(\tau_m, N_m, v_\infty) = \tau_m(N_m - 1) - \tau_m v_\infty - 1$.

Although, we have shown that the parametric solution v_m is always in the interval $(0,1)$. It remains to show, whether the resulting $v_\infty = 1 - \frac{1}{1+\sum_{m=1}^M \tau_m N_m v_m} = 1 - \frac{1}{1+\sum_{m=1}^M \frac{N_m(V(\tau_m, N_m, v_\infty)+\sqrt{V(\tau_m, N_m, v_\infty)^2+4\tau_m^2 v_\infty(N_m-1)})}{2(N_m-1)}}$ is feasible, thus satisfies $v_\infty \in (0,1)$.

Proposition 11. *The parametric solution given in Preposition 10 is feasible i.e. there exists a unique $v_\infty \in (0,1)$.*

Proposition 11 assures the existence of a pure Nash equilibrium in this multi-communities network scenario.

B. Computation of the equilibrium

We propose an iterative procedure to compute a pure Nash equilibrium of this parametric potential game.

1) Fixed an initial value for v_∞^0 (in $t = 0$).
2) Based on this value, we solve the M independent potential games using the result of Proposition 10 and we obtain the solution vector $\mathbf{n}^*(v_\infty^t) = (n_1^*(v_\infty^t), \ldots, n_M^*(v_\infty^t))$, where $n_i^*(v_\infty^t)$ is the number of nodes of group i that do not invest at equilibrium given the infection probability v_∞^t of the core node. We denote for each community m, the following parametric potential function: $\Phi_{m,v_\infty^t}(n_m) = C(N_m - n_m) + H\sum_{k=2}^{n_m} v_m(k, v_\infty^t)$. Hence, $n_m^*(v_\infty^t) = \arg\min_{n_m} \Phi_{m,v_\infty^t}(n_m)$ for all m.
3) Further, we compute the infection probability of a node from community m by the function $v_m^t(n_m^*(v_\infty^t), v_\infty^t)$ and the infection probability of the core node by: $v_\infty^{t+1} = 1 - \frac{1}{1+\sum_{m=1}^M \tau_m n_m^*(v_\infty^t)v_m^t(n_m^*(v_\infty^t), v_\infty^t)}$.
4) Stop if $|v_\infty^{t+1} - v_\infty^t| < \epsilon$ and thus $v_\infty = v_\infty^{t+1}$, otherwise increase $t \leftarrow t + 1$ and start with step 2).

VII. NUMERICAL EVALUATION

A. Single-community network

1) Performance of the decentralized system: We evaluate the performance of the decentralized system (equilibrium) compared to the centralized point of view (social optimum) via the Price of Anarchy of our system in different cases: pure and mixed strategies. We show how this metric depends on the system parameters, such as the number of nodes (decision makers), the effective epidemic spreading rate $\tau = \frac{\beta}{\delta}$ and the costs C and H.

Fig. 2a illustrates the PoA with the following costs $C = 0.4$, $H = 0.5$ and the effective spreading rate $\tau = 2/3$. We observe that when the number of nodes is relatively small ($N < 8$), using pure strategies yields a higher PoA compared to using mixed strategies. Moreover, we find that the lower bound of the pure PoA_p is very close to PoA_m, when N becomes relatively large ($N > 10$). We also observe that the approximation of the expected payoff, which induces a closed form expression of the mixed equilibrium, is very close to the exact PoA_m. We show in Fig. 2a (inset) the ratio $\frac{SW_p^*}{SW_m^*}$ of the social welfares depending on the size N of the network. Fig. 2a matches Corollary 4, i.e., the social welfare obtained using
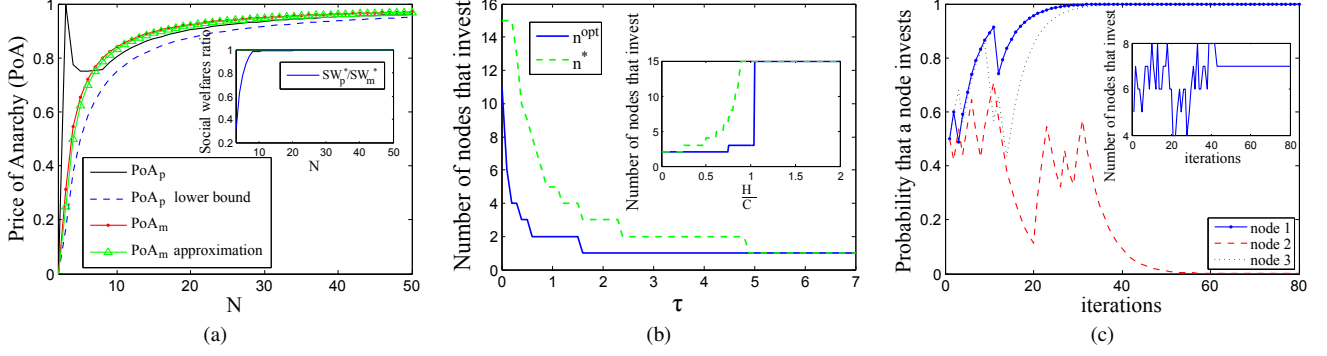
Fig. 2: (a) Price of Anarchy depending on the number of nodes N (main plot). Ratio $\frac{SW_p^*}{SW_m^*}$ of the social welfares depending on the size N of the network (inset); (b) Number of nodes which do not invest as a function of the effective spreading factor τ (main plot) or the ratio $\frac{H}{C}$ (inset) for $N = 15$; and (c) Convergence of the RLA to the pure Nash equilibrium. Probability of 3 random nodes that invest at each iteration (main plot); number of nodes that invest after each iteration (inset).

pure strategies in the game, is lower than the one obtained via symmetric mixed strategies. This difference is noticeable when the network is small but diminishes quickly (e.g., for $N = 8$ the ratio is $\frac{SW_p^*}{SW_m^*} = 0.9821$).

In Fig. 2b, we describe the number of nodes which do not invest considering the two methods: decentralized n^* (Nash equilibrium) and the centralized case n^{opt} (social welfare), depending on the effective spreading rate τ (main plot) and ratio of the costs of not investing and investing $\frac{H}{C}$ (inset). First, we observe that our result is correct, i.e., considering a decentralized point of view, the number of nodes which invest is lower than that of the centralized point of view. This result is somewhat surprising, as in general in a decentralized system, the players are more suspicious and we would think that in our setting, more nodes would invest at equilibrium compared to the central decision. Second, those numbers are exponentially decreasing with the effective spreading rate τ: the more the infection rate β dominates the curing rate δ, more nodes decide to invest in equilibrium. On the other hand, the number of nodes increases if the relative cost of investment decreases, as expected. However, the increase is faster in a decentralized system for a fixed $\frac{H}{C}$ (Fig. 2b inset).

2) Distributed Learning: We simulate the epidemic process in a full-mesh network of size $N = 15$ in which each node i uses the decentralized algorithm proposed in Section IV. The main problem is the two time scales: the update of strategies of each individual and the epidemic process. Our analysis is based on the metastable state of the infection process. When all nodes have taken their decision, the update rule of the RLA strategy depends on the infection probability for each node in the metastable state, i.e. v_∞. We assume that the strategy update time scale is large compared to the infection time scale, and then, the infection probability has enough time to converge to the metastable state. This assumption is realistic in a case in which the decision to buy an anti-virus is not very often taken compared to the propagation process of information/virus in a network. In Fig. 2c (main plot), we show the convergence of the RLA for three particular nodes. We observe that two nodes converge to the decision to invest and one not to invest.

Moreover, in Fig. 2c (inset) we plot the number of nodes that invest, at each iteration of our RLA. This number converges after few iterations to a pure equilibrium ($n^* = 8$).
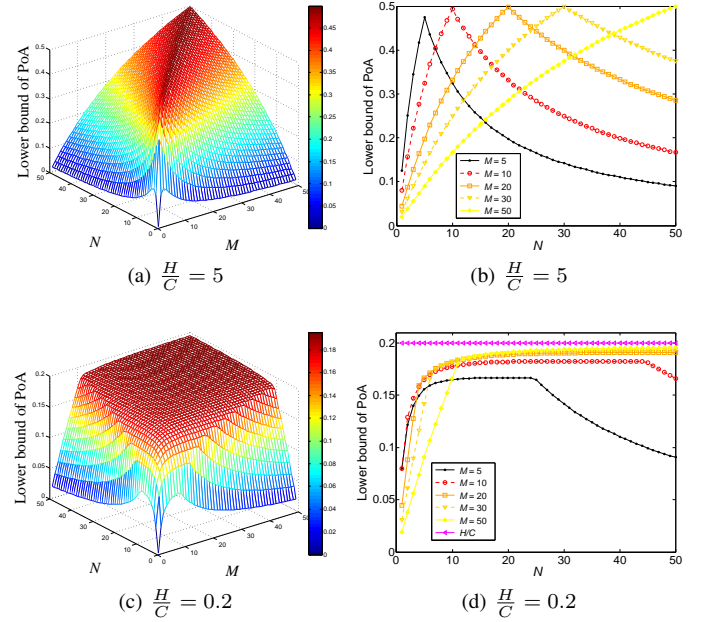

Fig. 3: The lower bound of the Price of Anarchy. (a) and (c) 3D plots as functions of M and N. (b) and (d) 2D plots as functions of N for fixed M.

B. Bipartite network

For the bipartite network, the lower bound of the Price of Anarchy (PoA) is illustrated in Fig. 3. In particular, Figs. 3a and 3c show the lower bound as a function of both M and N as a 3 dimensional plot, while Fig. 3b and 3d demonstrate the change of the lower bound of the Price of Anarchy as a function of N for several fixed values of M. All the figures confirm Corollary 6 that the lower bound of PoA is not bigger than the smaller of $\frac{1}{2}$ or $\frac{H}{C}$. In all the cases, the closer M and

N are to one another - the higher lower bound of PoA (red regions in Figs. 3a and 3c and maximum values in Figs. 3b and 3d). For fixed M and $\frac{H}{C} > \frac{1}{2}$, the lower bound is dominated by $\frac{\tau^2 MN - 1}{\tau^2(M+N)\max\{M,N\}}$ (Corollary 5), which is a function that increases in N for $N < M$, achieves its maximum (not bigger than $\frac{1}{2}$) and then decreases for $N > M$ (Fig. 3b). For fixed M and $\frac{H}{C} < \frac{1}{2}$, the lower bound is dominated by $\frac{H}{C}\frac{(\tau^2 MN - 1)(\tau(M+N)+2)}{\tau(M+N)(\tau M+1)(\tau N+1)}$ (Corollary 5), which is also a function that increases in N, achieves its maximum (not bigger than $\frac{H}{C}$) and then decreases (Fig. 3d).

C. Multi-communities network

We illustrate the results obtained in Section VI. Particularly, we present how the iterative algorithm can be used to obtain the equilibrium of the game by using the parametric potential game approach. First, we consider an example with $M = 2$ communities with $N_1 = 10$, $N_2 = 15$, $\tau_1 = 0.5$ and $\tau_2 = 1.5$. Second, we consider the following stopping criteria $\epsilon = 10^{-7}$ and we observe that the number of iterations to achieve an equilibrium is very small (8 iterations). In Fig. 4 (main plot), we observe that the number of nodes which do not invest at equilibrium is $N_1^* = 6$ (over 10 nodes) and $N_2^* = 3$ (over 15 nodes). The iterative procedure starts with $v_\infty^0 = 0.5$. The infection probability of the core node obtained at the convergence of the iterative procedure is $v_\infty = 0.8389$ (Fig. 4 inset). We consider also a more complex scenario with $M = 7$ communities and the following features $(N) = (10, 15, 12, 8, 9, 4, 15)$ and $(\tau) = (0.5, 1.5, 1, 1.2, 1.4, 0.8, 0.1)$. The iterative algorithm converges after only 10 iterations for a precision $\epsilon = 10^{-7}$ and $v_\infty^0 = 0.5$. We obtain the infection probability of the core node $v_\infty = 0.935$ and the following equilibrium $N^* = (6, 3, 4, 3, 3, 4, 15)$. For the last community, for which the effective spreading rate is the lowest, none of the nodes invests at equilibrium, whereas, for the second community, for which the effective spreading rate is the highest, the number of nodes that invest is 12 over 15 nodes.
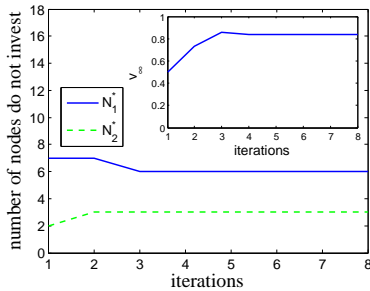

Fig. 4: The number of nodes that do not invest at the equilibrium in the main plot, while inset figure shows the infection probability of the core node v_∞.

VIII. CONCLUSIONS

In this paper, we explore the problem of finding optimal decentralized protection strategies in a network, where a node decides to invest in an anti-virus or to be prone to the virus SIS epidemic spread process. If a node (host) decides to invest, it cannot be infected, while if a node chooses not to invest, it can be infected by a virus and further spreads the virus inside the network. We study this problem from a game theoretic perspective. If a node decides to invest, the cost function of the node is the investment cost, otherwise the cost function is linearly proportional to the node's infection probability in the epidemic steady state.

We look for the optimal, fully decentralized protection strategies for each node. We show the existence of a potential structure, which allows us to prove the existence and uniqueness and derive the pure and mixed equilibrium in a *single-community* (or *mesh*) network. Moreover, we find the pure equilibrium in a *bipartite* network. We also evaluate the performance of the equilibrium by finding the Price of Anarchy (PoA). Finally, we propose a simple, fully decentralized algorithm which converges to the pure equilibrium of the game. In a *multi-communities* topology, in which several communities are communicating through a single core node, we introduce the concept of *parametric potential games* and further derive accurate and fast iterative procedure to compute the Nash equilibrium.

APPENDIX

A. Proofs of prepositions and corollaries in a single community network

Preposition 2: Let S be the set of nodes that invest C. A node $k \notin S$ must pay an infection cost equal to $Hv_\infty(n^*)$. If that node k deviates[1], then it will pay instead C. At equilibrium this user has no interest to deviate, meaning that: $C \geq Hv_\infty(n^*)$.

It remains to show that nodes in S do not gain by deviating. Each node in S pays constant cost C. When deviating, a node l, originally in S, becomes connected to those not in S, which implies that node l changes the size of the set $N \setminus S$ which becomes $n^* + 1$. The following inequality at equilibrium also applies: $C \leq Hv_\infty(n^* + 1)$.

Now, we show that n^* exists and is uniquely defined. For $C > H$, we have a trivial solution $n^* = N$, otherwise based on (2), we arrive at

$$H(1 - \frac{1}{(n^*-1)\tau}) < C \leq H(1 - \frac{1}{n^*\tau})$$

which gives $n^* = \lceil \frac{1}{(1-\frac{C}{H})\tau} \rceil$, if $\lceil \frac{1}{(1-\frac{C}{H})\tau} \rceil < N$, otherwise we have the trivial upper bound of $n^* = N$. ∎

Preposition 3: The payoff of the nodes that do not invest is non-negative since $H \geq 0$ and $v_\infty(n) \geq 0$. If $n < 1 + \frac{1}{\tau}$, $SW(n) = C(N - n)$ which decreases in n. On the other hand, if $n \geq 1 + \frac{1}{\tau}$, the derivative of (6) is $SW'(n) = H - C + \frac{H}{\tau(n-1)^2}$. Two cases can be distinguished:

---

[1] Deviation means that the node changes its action. If the multistrategy for the node is to invest, then "deviating" means that the node does not invest or *vice versa*.

1) $C < H$: the function $SW(n)$ is strictly increasing over the interval $[1 + \frac{1}{\tau}, N]$, so the minimum is achieved in $\lceil 1 + \frac{1}{\tau} \rceil$.
2) $C \geq H$: the function $SW(n)$ is increasing over the interval $[1 + \frac{1}{\tau}, 1 + \sqrt{\frac{H}{\tau(C-H)}}]$ and decreasing over $[1 + \sqrt{\frac{H}{\tau(C-H)}}, N]$, so the minimum is achieved in $\{\lceil 1 + \frac{1}{\tau} \rceil, N\}$ depending on the parameters of the system. ∎

Corollary 1: If $C \geq H$, based on the proof in Proposition 2, we have $n^* = N$, which is clearly as large as any value of $n^{opt} \leq N$. Otherwise ($C < H$), based on Preposition 2: $n^{opt} = \lceil 1 + \frac{1}{\tau} \rceil$ and $SW(n)$ is increasing. Using the definition of PoA: $SW(n^{opt}) \leq SW(n^*)$, which gives $n^{opt} \leq n^*$. ∎

Corollary 2: First, the denominator $SW(n^*)$ of PoA_p is strictly lower than CN. Indeed, using Preposition 2 into (6) we have: $SW(n^*) = C(N - n^*) + n^* H v_\infty(n^*) = CN - n^*(C - Hv_\infty(n^*)) < CN$.

In Proposition 3, we obtain $n^{opt} \in \{N, \lceil 1 + \frac{1}{\tau} \rceil\}$. If $n^{opt} = N$ then $n^* = N$ (Corollary 1) and $PoA_p = 1$. For the case $n^{opt} = \lceil 1 + \frac{1}{\tau} \rceil$, the following (based on Proposition 3) applies: (i) $C < H$: the function $SW(n)$ is strictly increasing and (ii) $C \geq H$ and $n^{opt} = \lceil 1 + \frac{1}{\tau} \rceil$, $SW(n)$ is also strictly increasing. Therefore, in both cases $n^{opt} = \lceil 1 + \frac{1}{\tau} \rceil \geq 1 + \frac{1}{\tau}$, hence $SW(n^{opt}) \geq SW(1 + \frac{1}{\tau}) = C(N - (1 + \frac{1}{\tau}))$ i.e. $PoA_p = \frac{SW(n^{opt})}{SW(n^*)} \geq 1 - (1 + \frac{1}{\tau})\frac{1}{N}$. ∎

Preposition 4: For any $p \in [0, 1]$ and any player i, we have: $\bar{S}_i(1, p) = C \sum_{n=0}^{N-1} \binom{N-1}{n}(1-p)^n p^{N-1-n} = C$. We also have: $\bar{S}_i(0, 0) = H(1 - \frac{1}{(N-1)\tau}) > 0$, and $\bar{S}_i(0, 1) = 0$.

If $C < H(1 - \frac{1}{(N-1)\tau})$ the mixed strategy p^* exists because the function $\bar{S}_i(0, p)$ is continuous. Otherwise, we have for all $p \in [0, 1]$, $\bar{S}_i(1, p) > \bar{S}_i(0, p)$, meaning that the strategy 0 is dominant irrespective of the mixed strategy of the other players. In this case, the action 0 is the equilibrium. ∎

Preposition 5: The proof relies on the monotonicity of $\bar{S}_i(0, p)$ and the fact that $\bar{S}_i(1, p) = C$ (a horizontal line), so the two curves intersect in one point. However, proving the monotonicity of $\bar{S}_i(0, p)$ is not trivial, because the well know function $H \sum_{n=\bar{n}}^{N-1} \binom{N-1}{n}(1-p)^n p^{N-1-n}$ decreases faster than $\bar{S}_i(0, p)$ on some intervals of p, but slower on other intervals. In what follows, we prove the monotonicity of $\bar{S}_i(0, p)$. For simplicity, we denote $M_n^N(p) = (1 - \frac{1}{\tau n})\binom{N-1}{n}(1-p)^n p^{N-1-n}$. Taking the first derivative in p and using the fact that $1 - \frac{1}{\tau n} < 1 - \frac{1}{\tau(n+1)}$, we obtain

$$\frac{d(M_n^N(p))}{dp} = (N-1)(1 - \frac{1}{\tau n})\left[C_n^{N-1}(p) - C_{n-1}^{N-1}(p)\right]$$
$$< (N-1)\left[(1 - \frac{1}{\tau(n+1)})C_n^{N-1}(p) - (1 - \frac{1}{\tau n})C_{n-1}^{N-1}(p)\right] \quad (9)$$

where $C_n^N(p) = \binom{N-1}{n}(1-p)^n p^{N-1-n}$ (sometimes referred to as *Bernstein Basis Polynomials*, see e.g. [39]). Summing (9) over all $n = \bar{n}, \ldots, N-1$, multiplied by H and knowing the values of the boundary term $C_{N-1}^{N-1}(p) = \binom{N-2}{N-1}(1-$

$p)^{N-1}p^{-1} = 0$ result with

$$\frac{d(\bar{S}_i(0, p))}{dp} = H \sum_{n=\bar{n}}^{N-1} \frac{d(M_n^N(p))}{dp} < (N-1)H \times$$

$$\left[\sum_{n=\bar{n}}^{N-1}(1 - \frac{1}{\tau(n+1)})C_n^{N-1}(p) - \sum_{n=\bar{n}-1}^{N-2}(1 - \frac{1}{\tau(n+1)})C_n^{N-1}(p)\right]$$

$$= (N-1)H\left[(1 - \frac{1}{\tau N})C_{N-1}^{N-1}(p) - (1 - \frac{1}{\tau \bar{n}})C_{\bar{n}-1}^{N-1}(p)\right]$$

$$= -(N-1)H(1 - \frac{1}{\tau \bar{n}})C_{\bar{n}-1}^{N-1}(p) < 0$$

i.e. $\bar{S}_i(0, p)$ is a decreasing function. ∎

Preposition 6: Using (2), the average approximated cost is: $\hat{S}_i(p_i, p') = p_i C + (1 - p_i) H v_{i\infty}(\bar{n} + 1) = p_i C + (1 - p_i) H (1 - \frac{1}{\tau(1-p')(N-1)})$. Based on Definition 2 for the equilibrium, assuming that is achieved for $p' = \hat{p}^*$, we have $\hat{S}_i(0, \hat{p}^*) = \hat{S}_i(1, \hat{p}^*)$, which gives $C = H(1 - \frac{1}{\tau(1-p^*)(N-1)})$. Finally, $\hat{p}^* = 1 - \frac{H}{\tau(H-C)(N-1)}$, if $C < H(1 - \frac{1}{\tau(N-1)})$, otherwise $\hat{p}^* = 0$ is an equilibrium, according to Preposition 4. ∎

Preposition 7: First, the function is continuous in p, because the value is the same from the left and the right side of $1 - \frac{1}{\tau(N-1)}$. If $p \in [1 - \frac{1}{\tau(N-1)}, 1]$ then it is increasing for any C and H. If $C > H$, the function is increasing on the whole interval $p \in [0, 1]$, hence $\hat{p}^{opt} = 0$ and the value is $H(1 - \frac{1}{\tau(N-1)})$. If $C = H$ the function is constant on $[0, 1 - \frac{1}{\tau(N-1)}]$, hence $\hat{p}^{opt} \in [0, 1 - \frac{1}{\tau(N-1)}]$ and the value is $H(1 - \frac{1}{\tau(N-1)})$. If $C < H$ the function is decreasing on $[0, 1 - \frac{1}{\tau(N-1)}]$, hence $\hat{p}^{opt} = 1 - \frac{1}{\tau(N-1)}$ and the value is $C(1 - \frac{1}{\tau(N-1)})$. ∎

B. Proofs of prepositions and corollaries in a bipartite network

Preposition 8: By definition, at equilibrium a user from each of the two partitions has no interest to change its pure strategy, i.e. a player that decides to invest has no interest to change its strategy. Let S be the set of nodes that invest C. A node $k \notin S$ must pay an infection cost equal to $Hv_\infty^{m^*}(n^*, m^*)$. If that node k deviates, then it will pay instead C. At equilibrium a user from each of the partitions has no interest to deviate, meaning that: $C \geq Hv_\infty^{m^*}(n^*, m^*)$ and $C \geq Hv_\infty^{n^*}(n^*, m^*)$.

It remains to show that nodes in S do not gain by deviating. These nodes pay C each. When deviating, a node l in the partition with n^* nodes, originally in S becomes connected to those not in S, which implies that node l changes the size of $N \setminus S$ which becomes $n^* + 1$. The following inequality at equilibrium also applies: $C \leq Hv_\infty^{n^*+1}(n^* + 1, m^*)$. In a similar way, considering a node in the partition with m^* nodes, it also holds $C \leq Hv_\infty^{m^*+1}(n^*, m^* + 1)$.

If $k = \frac{C}{H} > 1$ then for $m^* < M$ or $n^* < N$, we obtain a contradiction in the relations in the previous paragraph as $v_\infty^{m^*+1}(n^*, m^* + 1)$ or $v_\infty^{n^*+1}(n^* + 1, m^*)$ will be greater than 1. Hence, the only possible value is $(m^*, n^*) = (M, N)$. Based on the discussion in the previous paragraph and the exact expression in (3) we end up with the following

$$m < \frac{1}{\tau(\tau n(1-k) - k)} \leq m + 1 \qquad (10)$$

$$n < \frac{1}{\tau(\tau m(1-k) - k)} \leq n + 1 \qquad (11)$$

Hence, we have $m = \lceil \frac{1}{\tau(\tau n(1-k)-k)} \rceil - 1$ and $n = \lceil \frac{1}{\tau(\tau m(1-k)-k)} \rceil - 1$, from which for a given n, we have a unique m or *vice versa*, which proves point 1) of the preposition.

Let us assume that $k \geq \frac{1}{2}$ or $\tau \geq \frac{(1+k)(1-2k)}{2k(1-k)}$. For simplicity in the derivations we denote $A = \tau^2(1-k)$ and $B = \tau k$. From (11), we get $\frac{1}{Am-B} - 1 \leq n$, hence $\frac{A}{Am-B} - A - B \leq An - B < \frac{1}{m}$. From the last inequality, we obtain $\frac{B}{A+B} \leq m(Am - B)$. Finally,

$$m > \frac{B + \sqrt{B^2 + \frac{4AB}{A+B}}}{2A} \tag{12}$$

Further, from (11) we obtain $Am - B < \frac{1}{n}$, hence $\frac{A(B+\sqrt{B^2+\frac{4AB}{A+B}})}{2A} - B \leq Am - B < \frac{1}{n}$ i.e. $\frac{\sqrt{B^2+\frac{4AB}{A+B}}-B}{2} < \frac{1}{n} \Leftrightarrow n < \frac{2}{\sqrt{B^2+\frac{4AB}{A+B}}-B} = \frac{2(B+\sqrt{B^2+\frac{4AB}{A+B}})}{\frac{4AB}{A+B}} = \frac{(1+\frac{A}{B})(B+\sqrt{B^2+\frac{4AB}{A+B}})}{2A}$ i.e.

$$n < \frac{(1+\frac{A}{B})(B + \sqrt{B^2 + \frac{4AB}{A+B}})}{2A} \tag{13}$$

From (12) and (13), we arrive at

$$n - m < \frac{A}{B}\frac{B+\sqrt{B^2+\frac{4AB}{A+B}}}{2A} = \frac{1+\sqrt{1+\frac{4A}{B(A+B)}}}{2}$$
$$= \frac{1+\sqrt{1+\frac{4\tau^2(1-k)}{\tau^2k(\tau(1-k)+k)}}}{2} = \frac{1+\sqrt{1+\frac{4(1-k)}{k(\tau(1-k)+k)}}}{2} \tag{14}$$

The condition $k \geq \frac{1}{2}$ or $\tau \geq \frac{(1+k)(1-2k)}{2k(1-k)}$ is equivalent to $\frac{1-k}{k(\tau(1-k)+k)} < 2$, and applying this in (14), gives $n - m < \frac{1+\sqrt{1+4\cdot2}}{2} = 2$. In the same way, $m-n < 2$, hence $|n-m| \leq 1$, which completes the proof for 2). In conclusion, we find out to have limited number of possibilities to be checked $n = m$; $n = m-1$ or $n = m+1$, from which the system of (10) and (11) boils down to significantly simplified one in one variable.

For 3), we give a counter example, which is not covered by the conditions in 2). Indeed, taking $\tau = \frac{1+\sqrt{40000001}}{20000} \approx 0.316278$ and $k = \frac{\sqrt{40000001}-1}{20000} \approx 0.000316178$ give $A = 10^{-1}$ and $B = 10^{-4}$ and the system (10) and (11) will give 6 solutions: $(n^*, m^*) = \{(1,10),(2,5),(3,3),(5,2),(10,1)\}$, where not all numbers are equal or consecutive integers. $\blacksquare$

Preposition 9: $SW(n,m)$ is a function of two variables. Bellow the epidemic threshold ($mn \leq \frac{1}{\tau^2}$) and $SW(n,m) = C(N+M-n-m)$. If $MN \leq \frac{1}{\tau^2}$, then $(n^{opt}, m^{opt}) = (N,M)$ is the optimal pair and $SW = 0$. In the remaining cases ($MN > \frac{1}{\tau^2}$), because the first derivatives in both m and n give constant non-zero values, we look for the extremal points on the boundaries in m,n-plane: $mn = \frac{1}{\tau^2}$; $m = M$ or $n = N$.
1) If $m = M$, then $n = \min\{N, \frac{1}{\tau^2M}\} = \frac{1}{\tau^2M}$ and $SW = C\frac{\tau^2MN-1}{\tau^2M}$.
2) Similarly, if $n = N$, then $m = \frac{1}{\tau^2N}$ and $SW = C\frac{\tau^2MN-1}{\tau^2N}$.
3) If $mn = \frac{1}{\tau^2}$ then $\frac{1}{\tau^2M} \leq n \leq N$ and $SW(n) = C(M+N-n-\frac{1}{\tau^2n})$. $SW(n)$ increases to some point ($n = \frac{1}{\tau}$) and then starts to decrease, hence the minimum is on one of the boundaries, in the same points as 1) and 2). Finally,

we take the minimum of 1) and 2), which gives $SW = C\frac{\tau^2MN-1}{\tau^2\max\{M,N\}}$, achieved for $(n^{opt}, m^{opt}) = (\frac{1}{\tau^2M}, M)$ for $M > N$; $(n^{opt}, m^{opt}) = (N, \frac{1}{\tau^2N})$ for $M < N$ or both points for $M = N$.

Above the epidemic threshold ($mn \geq \frac{1}{\tau^2}$), we have

$$SW(n,m) = C(N+M-n-m) + H(\frac{\tau^2mn-1}{\tau(\tau n+1)} + \frac{\tau^2mn-1}{\tau(\tau m+1)})$$

Taking the first derivatives and equaling them to 0, results with

$$SW'_n(n,m) = -C + H(\frac{\tau m}{\tau m+1} + \frac{\tau m+1}{(\tau n+1)^2}) = 0$$
$$SW'_m(n,m) = -C + H(\frac{\tau n}{\tau n+1} + \frac{\tau n+1}{(\tau m+1)^2}) = 0 \tag{15}$$

Subtracting the two equations of (15) gives

$$\frac{H\tau(2+(m+n)\tau)^2(m-n)}{(1+\tau m)^2(1+\tau n)^2} = 0$$

Therefore $m = n$ is the only possibility. Going back into the first equation of (15) results with $C = H$. Hence, if $C \neq H$ there is no singular point inside the region and we should again look for the extrema on the boundaries: $mn \geq \frac{1}{\tau^2}$, $n \leq N$ and $m \leq M$.
1) if $mn = \frac{1}{\tau^2}$, then $SW(n,m) = C(N+M-n-m)$, so we again end up with the same solution as for the case bellow the epidemic threshold, considered before.
2) if $m = M$, we have $SW(n) = C(N-n) + H(\frac{\tau^2Mn-1}{\tau(\tau n+1)} + \frac{\tau^2Mn-1}{\tau(\tau M+1)})$. The first derivative is $SW'(n) = -C + H(\frac{\tau M}{\tau M+1} + \frac{\tau M+1}{(\tau n+1)^2})$ and $SW''(n) = -H\frac{\tau(\tau M+1)}{(\tau n+1)^3} < 0$. Therefore, the function could only have local maximum and we should look for the minimum on the extremal points for $\frac{1}{\tau^2M} \leq n \leq N$.

- $n = \frac{1}{\tau^2M}$ then $SW = C(N - \frac{1}{\tau^2M})$ for $(n^{opt}, m^{opt}) = (\frac{1}{\tau^2M}, M)$, which is again a boundary case exactly on the epidemic threshold and it was considered above.
- $n = N$ then $SW = H(\frac{\tau^2MN-1}{\tau(\tau N+1)} + \frac{\tau^2MN-1}{\tau(\tau M+1)}) = H\frac{(\tau^2MN-1)[\tau(M+N)+2]}{\tau(\tau M+1)(\tau N+1)}$ for $(n^{opt}, m^{opt}) = (N,M)$.

3) if $n = N$, we have similar cases as in 2).

If $C = H$ (i.e. $k = 1$) and $m = n$, then the social welfare function boils down to $SW(n) = C(N+M-2n) + 2C\frac{\tau^2n^2-1}{\tau(\tau n+1)} = C(N+M-2n) + 2C\frac{\tau n-1}{\tau} = C(N+M-\frac{2}{\tau}) = $ const i.e. SW is constant and does not depend on n or m. However, $C(N+M-\frac{2}{\tau}) \geq C\frac{\tau^2MN-1}{\tau^2M}$ for any m as the last is equivalent to $(\sqrt{M} - \frac{1}{\tau\sqrt{M}})^2 \geq 0$. In conclusion, $SW = \max\{\tau^2MN-1, 0\} \cdot \min\{\frac{C}{\tau^2\max\{M,N\}}, H\frac{\tau(M+N)+2}{\tau(\tau M+1)(\tau N+1)}\}$. $\blacksquare$

Corollary 5: First, the denominator of PoA: $SW(n^*, m^*)$ is strictly lower than $C(N+M)$. Indeed, using Proposition 8 and equation (8) we have:

$$SW(n^*, m^*) = C(N + M - n^* - m^*)$$
$$+ n^*Hv_\infty^{n^*}(n^*, m^*) + m^*Hv_\infty^{m^*}(n^*, m^*)$$
$$= C(N+M) - n^*(C - Hv_\infty^{n^*}(n^*, m^*))$$
$$- m^*(C - Hv_\infty^{m^*}(n^*, m^*)) < C(N+M)$$

which is an upper bound for $SW(n^*, m^*)$ and $SW(n^{opt}, m^{opt})$ could be determined exactly based on Preposition 9, which completes the proof. $\blacksquare$

Corollary 6: For simplicity, we denote $h(M, N, \tau) = \frac{\max\{\tau^2 MN - 1, 0\} \min\{\frac{1}{\tau \max\{M,N\}}, \frac{H(\tau(M+N)+2)}{C(\tau M+1)(\tau N+1)}\}}{\tau(M+N)}$. If $\tau^2 MN \leq 1$, then $h(M, N, \tau) = 0 \leq \min\{\frac{1}{2}, \frac{H}{C}\}$. If $\tau^2 MN > 1$, then

$$h(M, N, \tau) \leq \frac{\tau^2 MN - 1}{\tau^2 \max\{M,N\}(M+N)} = \frac{1 - \frac{1}{\tau^2 MN}}{\max\{M,N\}(\frac{1}{M} + \frac{1}{N})}$$

$$= \frac{1 - \frac{1}{\tau^2 MN}}{1 + \frac{\max\{M,N\}}{\min\{M,N\}}} < \frac{1 - 0}{1 + 1} = \frac{1}{2} \text{ and}$$

$$h(M, N, \tau) \leq \frac{H}{C} \frac{(\tau^2 MN - 1)(\tau(M+N)+2)}{\tau^2 MN(\tau M + 1)(\tau N + 1)}$$

$$\leq \frac{H}{C} \frac{\tau^2 MN(M+N) - \tau M - \tau N - 2}{\tau^2 MN(M+N) + \tau M(M+1) + \tau N(N+1)} < \frac{H}{C}.$$
∎

C. Proofs of prepositions a multi-communities network

Preposition 10: For simplicity, we denote $V(\tau_m, N_m, v_\infty) = \tau_m(N_m - 1) - \tau_m v_\infty - 1$ and $W(\tau, N_m, v_\infty) = (\tau_m(N_m - 1))^2 + (\tau_m v_\infty)^2 + 1 + 2(\tau_m v_\infty) + 2\tau_m v_\infty \tau_m (N_m - 1)$. For a given v_∞, for any none core node in community m, based on equation (4), we have an equation of degree 2, which solutions are the following infection probabilities:

$v_m(N_m, v_\infty) = \frac{V(\tau_m, N_m, v_\infty)(1 \pm \sqrt{1 + \frac{4\tau_m^2 v_\infty (N_m - 1)}{V(\tau_m, N_m, v_\infty)}})}{2\tau_m(N_m - 1)} = \frac{V(\tau_m, N_m, v_\infty) + \sqrt{W(\tau, N_m, v_\infty) - 2\tau_m(N_m - 1)}}{2\tau_m(N_m - 1)}$, where $V(\tau_m, N_m, v_\infty)$ is positive, otherwise, we would have two negative solutions. The value under the square root is greater than 1, so the solution with "-" sign is negative, hence it is not valid. Using the fact that $-2\tau_m(N_m - 1) < 2\tau_m(N_m - 1)$, we obtain $v_m(v_\infty) = \frac{V(\tau_m, N_m, v_\infty) + \sqrt{W(\tau, N_m, v_\infty) - 2\tau_m(N_m - 1)}}{2\tau_m(N_m - 1)} < \frac{V(\tau_m, N_m, v_\infty) + \sqrt{W(\tau, N_m, v_\infty) + 2\tau_m(N_m - 1)}}{2\tau_m(N_m - 1)} = 1$. Hence, the solution with sign "+" is in the interval $(0, 1)$. ∎

Preposition 11: Let us consider the function $f(x) = x - 1 + \frac{1}{1 + \sum_{j=1}^{M} \frac{N_i((\tau_i(N_i - 1) - \tau_i x - 1) + \sqrt{(\tau_i(N_i - 1) - \tau_i x - 1)^2 + 4\tau_i^2 x(N_i - 1)})}{2(N_i - 1)}}$ in the interval $[0, 1]$. We have shown earlier that $v_i > 0$, hence the denominator in $f(x)$ is always positive. The values under the square roots are clearly positive, hence $f(x)$ is a continuous function. Further, we have the following:

1) $f(0) = -1 + \frac{1}{1 + \sum_{j=1}^{M} \frac{N_i((\tau_i(N_i - 1) - 1) + \sqrt{(\tau_i(N_i - 1) - 1)^2})}{2(N_i - 1)}} = -1 + \frac{1}{1 + \sum_{j=1}^{M} N_i \frac{\tau_i(N_i - 1) - 1}{N_i - 1}}$. Physically, the metastable state does exist only over the epidemic threshold defined by $\tau_i > \frac{1}{N_i - 1}$, consequently $f(0) < 0$.

2) $f(1) = \frac{1}{1 + \sum_{j=1}^{M} N_i \frac{(\tau_i(N_i - 2) - 1) + \sqrt{(\tau_i(N_i - 2) - 1)^2 + 4\tau_i^2(N_i - 1)}}{2(N_i - 1)}}$. Clearly, $\sqrt{(\tau_i(N_i - 2) - 1)^2 + 4\tau_i^2(N_i - 1)} \geq |\tau_i(N_i - 2) - 1| \geq \pm(\tau_i(N_i - 2) - 1)$. All the members in the sum are positive, hence $f(1) > 0$.

Finally, because $f(0) < 0$, $f(1) > 0$ and $f(x)$ is a continuous function, it follows that there is a real number $v_\infty \in (0, 1)$, such that $f(v_\infty) = 0$ is satisfied.

It remains to prove that v_∞ is unique in $(0, 1)$. Indeed, assume that we have more than one solutions of v_∞ in $(0, 1)$, which means that we have at least two solutions of the governing NIMFA epidemic equations (see e.g., Eq. (3) in [5]). However, this is in a contradiction with the fact that there is one solution (given as an infinite Laurent series), according to Lemma 2 in [5]. Thus, v_∞ is also unique in $(0, 1)$, which completes the proof. ∎

REFERENCES

[1] P. Van Mieghem, J. Omić, and R. Kooij, "Virus spread in networks," *IEEE/ACM Trans. on Netw.*, vol. 17, pp. 1 –14, feb. 2009.

[2] D. Chakrabarti, Y. Wang, C. Wang, J. Leskovec, and C. Faloutsos, "Epidemic thresholds in real networks," *ACM Trans. Inf. Syst. Secur.*, vol. 10, pp. 1:1–1:26, Jan. 2008.

[3] A. Ganesh, L. Massoulie, and D. Towsley, "The effect of network topology on the spread of epidemics," in *IEEE INFOCOM*, vol. 2, pp. 1455–1466 vol. 2, 2005.

[4] J. Kephart and S. White, "Directed-graph epidemiological models of computer viruses," in *IEEE Symposium on Research in Security and Privacy*, pp. 343–359, 1991.

[5] P. Van Mieghem, "The viral conductance of a network," *Computer Communication*, vol. 35, pp. 1494–1506, July 2012.

[6] R. Cohen, S. Havlin, and D. ben Avraham, "Efficient immunization strategies for computer networks and populations," *Phys. Rev. Lett.*, vol. 91, p. 247901, Dec 2003.

[7] C. M. Schneider, T. Mihaljev, S. Havlin, and H. J. Herrmann, "Suppressing epidemics with a limited amount of immunization units," *Phys. Rev. E*, vol. 84, p. 061911, Dec 2011.

[8] J. Omić, J. Martín-Hernández, and P. Van Mieghem, "Network protection against worms and cascading failures using modularity partitioning," in *International Teletraffic Congress (ITC)*, pp. 1–8, Sept 2010.

[9] P. Van Mieghem, D. Stevanović, F. Kuipers, C. Li, R. van de Bovenkamp, D. Liu, and H. Wang, "Decreasing the spectral radius of a graph by link removals," *Phys. Rev. E*, vol. 84, p. 016101, Jul 2011.

[10] P. Van Mieghem, "Epidemic phase transition of the SIS type in networks," *EPL (Europhysics Letters)*, vol. 97, no. 4, p. 48004, 2012.

[11] A. Orda, R. Rom, and N. Shimkin, "Competitive routing in multiuser communication networks," *IEEE/ACM Trans. Netw.*, vol. 1, pp. 510–521, Oct. 1993.

[12] E. Altman, T. Jimenez, T. Basar, and N. Shimkin, "Competitive routing in networks with polynomial cost," in *IEEE INFOCOM*, vol. 3, pp. 1586–1593 vol.3, 2000.

[13] Y. A. Korilis and A. A. Lazar, "On the existence of equilibria in noncooperative optimal flow control," *JACM*, vol. 42, pp. 584–613, May 1995.

[14] A. Nahir, A. Orda, and D. Raz, "Workload factoring with the cloud: A game-theoretic perspective," in *IEEE INFOCOM*, pp. 2566–2570, 2012.

[15] A. Nahir, A. Orda, and A. Freund, "Topology design of communication networks: A game-theoretic perspective," *IEEE/ACM Trans. on Netw.*, (*to appear*).

[16] E. Gourdin, J. Omic, and P. Van Mieghem, "Optimization of network protection against virus spread," in *8th Intl. Workshop on the Design of Reliable Communication Networks (DRCN)*, pp. 86–93, 2011.

[17] E. Koutsoupias and C. Papadimitriou, "Worst-case equilibria," *Computer Science Review*, vol. 3, no. 2, pp. 65 – 69, 2009.

[18] T. Roughgarden and E. Tardos, "How bad is selfish routing?," *JACM*, vol. 49, pp. 236–259, Mar. 2002.

[19] N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani, *Algorithmic Game Theory*. Cambridge University Press, 2007.

[20] J. Omić, A. Orda, and P. Van Mieghem, "Protecting against network infections: A game theoretic perspective," in *IEEE INFOCOM*, pp. 1485–1493, 2009.

[21] J. Aspnes, K. Chang, and A. Yampolskiy, "Inoculation strategies for victims of viruses and the sum-of-squares partition problem," *Journal of Computer and System Sciences*, vol. 72, no. 6, pp. 1077 – 1093, 2006.

[22] D. Acemoglu, A. Malekian, and A. Ozdaglar, "Network security and contagion," tech. rep., MIT/LIDS intern report, 2013.

[23] L. Jiang, V. Anantharam, and J. Walrand, "Efficiency of selfish investments in network security," NetEcon, pp. 31–36, ACM, 2008.

[24] M. Lelarge and J. Bolot, "Network externalities and the deployment of security features and protocols in the internet," in *SIGMETRICS*, pp. 37–48, ACM, 2008.

[25] M. Kearns and L. E. Ortiz, "Algorithms for interdependent security games," in *In Advances in Neural Information Processing Systems*, MIT Press, 2004.

[26] P. Van Mieghem, "The N-intertwined SIS epidemic network model," *Computing*, vol. 93, no. 2-4, pp. 147–169, 2011.

[27] J. Omic, R. Kooij, and P. Van Mieghem, "Virus spread in complete bi-partite graphs," in *Bionetics*, pp. 49–56, 2007.

[28] P. Van Mieghem and J. Omić, "In-homogeneous Virus Spread in Networks," *ArXiv*, June 2013.

[29] R. Rosenthal, "A class of games possessing pure-strategy nash equilibria," *Intl. Journal of Game Theory*, vol. 2, no. 1, pp. 65–67, 1973.

[30] D. Monderer and L. S. Shapley, "Potential games," *Games and Economic Behavior*, vol. 14, no. 1, pp. 124 – 143, 1996.

[31] Y. Chen and R. Gazzale, "When does learning in games generate convergence to nash equilibria? the role of supermodularity in an experimental setting," *American Economic Review*, vol. 94, pp. 1505–1535, September 2004.

[32] R. Cominetti, E. Melo, and S. Sorin, "A payoff-based learning procedure and its application to traffic games," *Games and Economic Behavior*, vol. 70, pp. 71–83, September 2010.

[33] P. S. Sastry, V. V. Phansalkar, and M. Thathachar, "Decentralized learning of nash equilibria in multi-person stochastic games with incomplete information," *IEEE Trans. on Systems, Man and Cybernetics*, vol. 24, no. 5, pp. 769–777, 1994.

[34] J. Weibull, *Evolutionary Game Theory*. Cambridge University Press, 1995.

[35] O. Bournez and J. Cohen, "Learning equilibria in games by stochastic distributed algorithms," in *Computer and Information Sciences III*, pp. 31–38, Springer London, 2013.

[36] C. T. Pierre Coucheney and B. Gaujal., "Fair and efficient user-network association algorithm for multi-technology wireless networks.," in *IEEE INFOCOM*, 2009.

[37] Y. Xin and R. Chandramouli, "Stochastic learning solution for distributed discrete power control game in wireless data networks," *IEEE/ACM Trans. on Netw.*, vol. 16, no. 4, pp. 932–944, 2008.

[38] G. Gasper and M. Rahman, *Basic Hypergeometric Series*. Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2004.

[39] G. Lorentz, *Bernstein polynomials*. University of Toronto Press, 1953.