Deutsch algorithm | 1985

Let's consider binary function

$$f: \{0,1\} \rightarrow \{0,1\}$$

There are 4 such functions:

input:	f_1	f_2	f_3	f_4
0	0	0	1	1
1	0	1	0	1

$f(0) = f(1) \Rightarrow$ function is constant

$f(0) \neq f(1) \Rightarrow$ function is „balanced"

$$\Downarrow$$

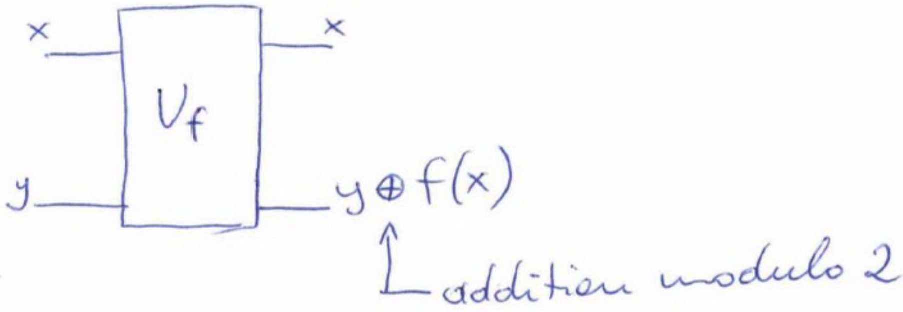
f_1, f_4 – constant

f_2, f_3 – balanced

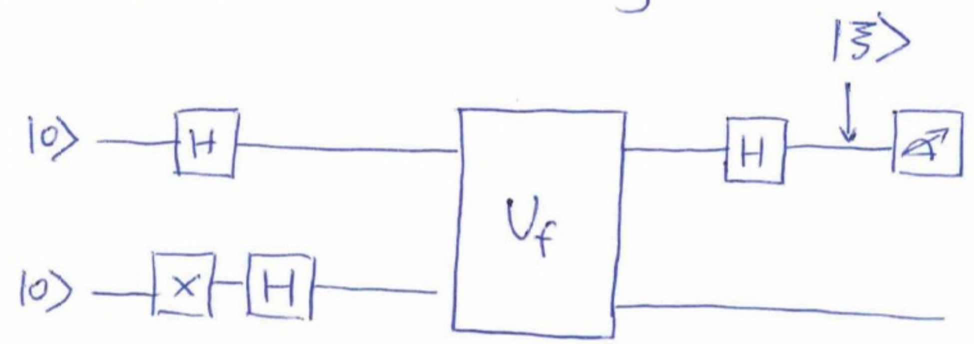
- Let's imagine we have „black box" implementing one of these functions.
- To understand if it is balanced or constant, classically we need to compute both $f(0)$ and $f(1) \Rightarrow$ query the black box <u>twice</u>
- Deutsch algorithm allows to do this on quantum computer calculating function value only <u>once</u>.
- simplest algorithm demonstrating gain from quantum computers

$\downarrow$

Black box (oracle):



Let's consider following circuit:



We can show, that state $|\xi\rangle$ of the first qubit will have the following form:

$$\boxed{|\xi\rangle = \text{const.}\left[\left((-1)^{f(0)} + (-1)^{f(1)}\right)|0\rangle + \left((-1)^{f(0)} - (-1)^{f(1)}\right)|1\rangle\right]}$$

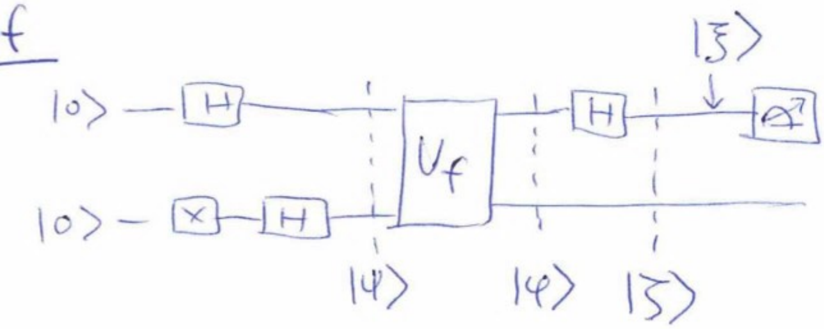
Interpretation:

- if we measure first qubit in state $|0\rangle$, then it means that $f(0)=f(1) \Rightarrow f$ is const
- if we measure first qubit in state $|1\rangle$, then it means, that $f(0)=-f(1) \Rightarrow f$ balanced

$\Downarrow$

We can measure just one time and know the answer (classically twice) oof

(D2)



$$|\psi\rangle = H|0\rangle \otimes H|1\rangle = \frac{1}{2}\big(|0\rangle + |1\rangle\big) \otimes \big(|0\rangle - |1\rangle\big) =$$

$$= \frac{1}{2}\big(|0\rangle|0\rangle + |1\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|1\rangle\big)$$

$$|\varphi\rangle = U_f|\psi\rangle = \left\{ U_f : |x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle \right\} =$$

$$= \frac{1}{2}\Big(|0\rangle|0 \oplus f(0)\rangle + |1\rangle|0 \oplus f(1)\rangle -$$

$$- |0\rangle|1 \oplus f(0)\rangle - |1\rangle|1 \oplus f(1)\rangle\Big) =$$

$$= \frac{1}{2}\Big[|0\rangle\big(|0 \oplus f(0)\rangle - |1 \oplus f(0)\rangle\big) + |1\rangle\big(|0 \oplus f(1)\rangle - |1 \oplus f(1)\rangle\big)\Big] =$$

$$= \frac{1}{2}\Big[|0\rangle\big(|f(0)\rangle - |1 \oplus f(0)\rangle\big) + |1\rangle\big(|f(1)\rangle - |1 \oplus f(1)\rangle\big)\Big] =$$

$$= \frac{1}{2}\sum_{x=0}^{1} |x\rangle\big(|f(x)\rangle - |1 \oplus f(x)\rangle\big) =$$

$$= \left\{ |f(x)\rangle - |1 \oplus f(x)\rangle = \begin{cases} |0\rangle - |1\rangle & \text{for } f(x) = 0 \\ |1\rangle - |0\rangle & \text{for } f(x) = 1 \end{cases} \right\} =$$

$$= \frac{1}{2}\sum_{x=0}^{1} |x\rangle (-1)^{f(x)}\big(|0\rangle - |1\rangle\big) =$$

$$= \frac{1}{2}\Big[(-1)^{f(0)}|0\rangle\big(|0\rangle - |1\rangle\big) + (-1)^{f(1)}|1\rangle\big(|0\rangle - |1\rangle\big)\Big] = \longrightarrow$$

$$= \frac{1}{2}\left[(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle\right]\left(|0\rangle - |1\rangle\right)$$

Now we apply the final H gate on the first qubit:

$$|\xi\rangle = \frac{1}{2}\underbrace{\left[(-1)^{f(0)}\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right) + (-1)^{f(1)}\frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right)\right]}_{\substack{\text{state of first qubit}\\ \text{(only this is of interest}\\ \text{to us)}\ |\xi\rangle}}\left(|0\rangle-|1\rangle\right)$$

$$|\xi\rangle = \text{const.}\left[(-1)^{f(0)}|0\rangle + (-1)^{f(0)}|1\rangle + \right.$$

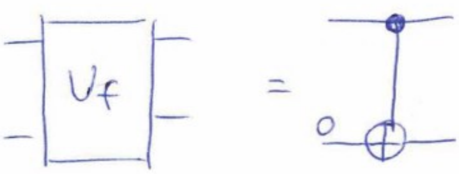
$$\left. + (-1)^{f(1)}|0\rangle - (-1)^{f(1)}|1\rangle\right] =$$

$$= \text{const.}\left[\left((-1)^{f(0)} + (-1)^{f(1)}\right)|0\rangle + \left((-1)^{f(0)} - (-1)^{f(1)}\right)|1\rangle\right]$$

cbdo.

$\square$

Examples:

A) 

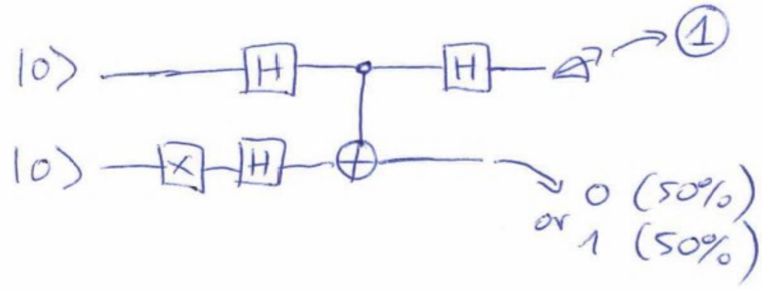
$\underline{0}0 \to 0\underline{0}$

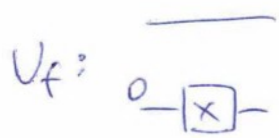
$\underline{1}0 \to 1\underline{1}$ $= f: \begin{array}{l} 0 \to 0 \\ 1 \to 1 \end{array}$

(f_2)

<u>balanced</u>

(CNOT is just adding modulo 2)

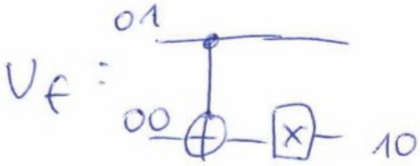


B) 

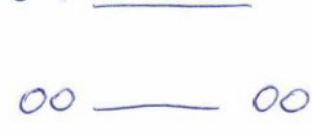
$00 \to 01$
$10 \to 11$ $\Rightarrow$ $\begin{array}{l} 0 \to 1 \\ 1 \to 1 \end{array}$

$\Downarrow$

f constant (f_4)

c) 

$\Rightarrow$ $\begin{array}{l} 0 \to 1 \\ 1 \to 0 \end{array}$ $\Rightarrow$ balanced (f_3)

d) 

$\begin{array}{l} 0 \to 0 \\ 1 \to 0 \end{array}$ $\Rightarrow$ constant (f_1)