Lecture 4

$\boxed{\text{Two qubits \& entanglement}}$

Reminder from last lecture:

$$|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle \quad - 1 \text{ qubit state}$$

If we have two qubits in states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ respectively, then state of the entire system (both qubits together) is:

$$|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \equiv |\varphi_1\rangle|\varphi_2\rangle \equiv |\varphi_1\varphi_2\rangle$$

~~in gene~~

The most general form of 2-qubit state is

basis states

$$|\xi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle =$$

$$= \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

a, b, c, d – amplitudes

$|a|^2, |b|^2, |c|^2, |d|^2$ – probabilities of measuring $00, 01, 10, 11$

$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$ – normalization

Exercise

For state $|x\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$

1) what is the amplitude of basis state $|01\rangle$?

Answer: $\frac{1}{\sqrt{2}}$

2) What is the probability of measuring '00' ?

Answer: $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

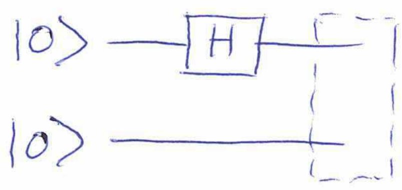
Reminder:

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \equiv |+\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \equiv |-\rangle$$

Example:

$|\psi\rangle = ?$



$$|\psi\rangle = (H|0\rangle) \otimes |0\rangle = |+\rangle \otimes |0\rangle = |+0\rangle =$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|0\rangle\right) =$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + 0\cdot|01\rangle + |10\rangle + 0\cdot|11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

This state $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$

Superposition of two states, but in <u>both</u> 2nd qubit is in state $|0\rangle$, so all the superposition is happening on the first qubit only !!!

this means, that this is a pair of single qubit states $|+\rangle$ and $|0\rangle$ and the qubits may simply not be „aware" of each other existence!

Such states are called <u>„product states"</u> as they are result of tensor product of single qubit states

<u>Reminder from previous lecture</u>

How to construct product states?

First qubit: $|\Psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

Second qubit: $|\Psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

$$|\Psi_1\Psi_2\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle = \begin{bmatrix} \alpha_1 \cdot \alpha_2 \\ \alpha_1 \cdot \beta_2 \\ \beta_1 \cdot \alpha_2 \\ \beta_1 \cdot \beta_2 \end{bmatrix}$$

so called tensor product

Not all 2-qubit states can be expressed a tensor product of single qubit states.
Let's consider the so called Bell state:

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

There are no single qubit states for which tensor product would give Bell's state.

Proof: $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

$$|\psi_1\psi_2\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_2\beta_1|10\rangle + \beta_1\beta_2|11\rangle$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}|00\rangle + 0\cdot|01\rangle + 0\cdot|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$\Downarrow$$

$$\begin{cases}\alpha_1\alpha_2 = \frac{1}{\sqrt{2}}\\ \alpha_1\beta_2 = 0\\ \alpha_2\beta_1 = 0\\ \beta_1\beta_2 = \frac{1}{\sqrt{2}}\end{cases} \Rightarrow$$ one of $\alpha_1\beta_2, \alpha_2\beta_1$ needs to be zero

$$\Downarrow$$

either $\alpha_1\alpha_2$ or $\beta_1\beta_2$ ~~wo~~ will also be zero which is not what we need

c.b.d.o.

The states which are not ~~single states~~ tensor products of single qubit states are called „entangled states"

$\Downarrow$

$$
\boxed{
\begin{tikzcd}
 & \text{Multi-qubit states} \arrow[dl] \arrow[dr] & \\
\text{product states} & & \text{entangled states}
\end{tikzcd}
}
$$

Exercise:

To which cathegory the following states belong:

$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $\rightarrow$ Answer: entangled

$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\rightarrow$ Answer: product

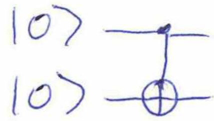
$\left(\frac{1}{\sqrt{2}} \left(|00\rangle + |01\rangle \right) \right) =$

$= |0\rangle \otimes |+\rangle$

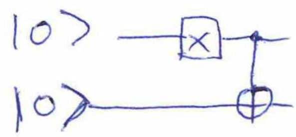
$\downarrow$

To create entangled states on quantum computer, <u>we need to use 2-qubit gates</u> (otherwise all qubits would be independent) as we start from product state $\underbrace{|000\ldots0\rangle}_{n}$

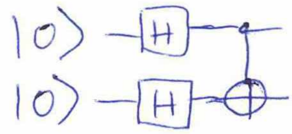
CNOT does not always create entanglement:



$\Rightarrow |00\rangle$

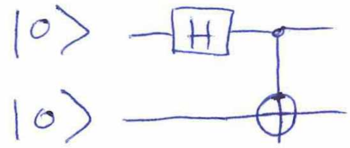


$\Rightarrow |11\rangle$



$\Rightarrow |++\rangle$ (no effect: $|++\rangle \to |++\rangle$)

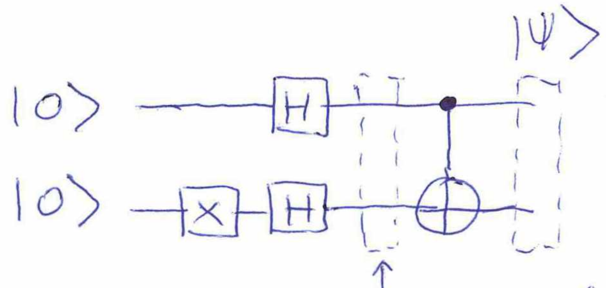
finally:



$\Rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow$ entanglement

CNOT $\equiv$ controlled not $\equiv$ CX

Now, let's examine what happens in the following circuit:



$$|+-\rangle = \frac{1}{2}\big(|0\rangle + |1\rangle\big) \otimes \big(|0\rangle - |1\rangle\big) =$$

$$= \frac{1}{2}\big(|00\rangle - |01\rangle + |10\rangle - |11\rangle\big) =$$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Now after applying CNOT gate, we get:

$$|\psi\rangle = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{2}\big(|00\rangle - |01\rangle - |10\rangle + |11\rangle\big) =$$

$$= \frac{1}{2}\Big[\underbrace{\big(|0\rangle - |1\rangle\big)}_{\text{first qubit}}\underbrace{\big(|0\rangle - |1\rangle\big)}_{\text{2nd qubit}}\Big] = |-\rangle \otimes |-\rangle = |--\rangle$$

$$\Downarrow$$

$$|+-\rangle \xrightarrow{\text{CNOT}} |--\rangle$$

$$\Downarrow$$

$\boxed{\text{CNOT has changed state of controll qubit}}$

It changed it from $|+\rangle$ to $|-\rangle$:

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$
$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$\Downarrow$

it changed sign of $|1\rangle$

$\Downarrow$

it <u>changed phase</u> of the qubit (coefficient by which $|1\rangle$ is multiplied not changing it's norm : $-1 \equiv e^{i\pi}$)

$\downarrow$

$\boxed{\text{this is called „phase kickback"}}$)

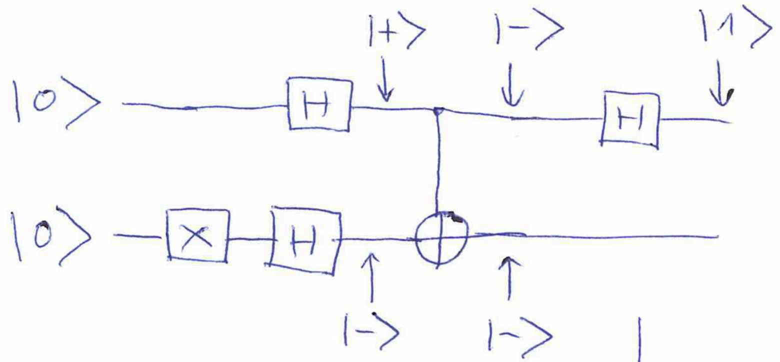
Now, this is important as:

$$H|-\rangle = |1\rangle$$
$$H|+\rangle = |0\rangle$$

$\Downarrow$

if we apply Hadamard gate to such changed qubit, the effect will be $|1\rangle$ instead of $|0\rangle$ if the phase would not be changed.

Let's do this:



now, this is exactly the Deutsch algorithm (for certain oracle form:)

$\Downarrow$

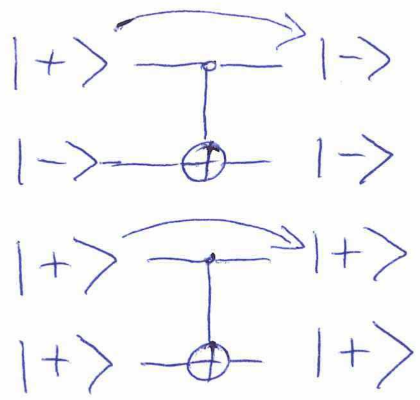
using CNOT with first qubit as controlled one, we changed it's state completely from $|0\rangle$ to $|1\rangle$!

$\Downarrow$

even if CNOT should not change control qubit, the two qubits are now one system (they are entangled)

we can provide different interpretation of CNOT gate:
Applies Z gate to control qubit if target qubit is in state $|-\rangle$ and does nothing if target is in state $|+\rangle$

$\downarrow$



Do we need entanglement in quantum computers? $\boxed{\text{YES}}$

- without entanglement, we would have independent qubits
- one qubit has two parameters: $\underline{\alpha}|0\rangle+\underline{\beta}|1\rangle$, so n qubits would have 2n parameters => it's complexity would scale linearly with number of qubits
- with entanglement n qubits gives us 2^n parameters -> which is drastically different thing!!!

↓