Disclaimer

This document is an exam summary and follows the given material of the lecture *Biomedical Imaging*. Its contribution is a short summary that contains the most important concepts, formulas and algorithms. Due to curriculum content updates, some content may not be relevant to future versions of the course.

I do not guarantee the accuracy or completeness, nor is this document endorsed by the instructors. Any errors that are pointed out to me are welcome. The complete LaTeX source code can be found at `https://github.com/tstreule/eth-cheat-sheets`.

Biomedical Imaging

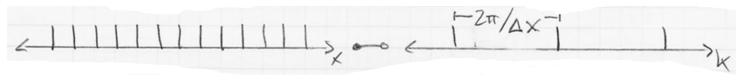
Timo Streule, tstreule@ethz.ch - 18.01.2022

1 General

1.1 SigSys Basics   SNR ↑ ≙ resolution ↓ $x\leftrightarrow t, k\leftrightarrow f$

IFFT: $f(x,y)=\frac{1}{2\pi}\iint \mathcal{F}[f](k_x,k_y)e^{j(k_xx+k_yy)}\,dk_x\,dk_y$

FFT: $\mathcal{F}[f](k_x,k_y)=\frac{1}{2\pi}\iint f(x,y)e^{-j(k_xx+k_yy)}\,dx\,dy$



Prevent **aliasing**: $\boxed{\Delta x<\frac{2\pi}{BW}}\leftrightarrow\boxed{\Delta k<\frac{2\pi}{FOV}}$ sampl. res. ↔ image size

$\mathcal{F}[f](k_m)=\underbrace{e^{-jk_mx_0}}_{\text{shift}}\sum_{n=0}^{N-1}f(x_n)e^{-j\frac{2\pi mn}{N}}$ normally: $x_0=0$

Noise: Gaussian: $P_n(\eta)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{\eta^2}{2\sigma^2}}$ FWHM $=2\sqrt{2\ln 2}$

Poisson: $P_\lambda(x)=\frac{\lambda^x}{x!}e^{-\lambda}$ (→ discrete events)

$\boxed{SNR=|S[f](x,y)|\frac{1}{\sigma}}\leftrightarrow\boxed{CNR_{AB}=|SNR_A-SNR_B|}$

2 Generation of X-rays

2.1 Basic formulas

- $c=\frac{1}{\sqrt{\mu_0\epsilon_0}}=\lambda\nu$ $E=h\nu=\frac{hc}{\lambda}$ rest mass: $mc^2=511\,eV$
- Range of X-Rays: $\lambda=10\,pm-10\,nm$
- $E_{radiation}=n\cdot E_{photon}$ with $E_{photon}=h\cdot\nu=eU_a$
- $p_{photon}=mc=h/\lambda$, $\lambda_{min}=\frac{hc}{eU_a}$

2.2 Spectrum

Photon flux vs E: Speaks by **Characteristic Radiation**. Continuous spectrum by **Bremsstrahlung**: 1st el. deflection: Spectrum $0-E_{max}$, 2nd el. defl. Not to E_{max} anymore. If material changes: $E_{max}=const$, slope changes. If E changes: slope const., intersection with E-axis @ E. Low-Energy rad. absorbed anyway → Al filter → dose ↓

2.3 Heat generation & dissipation

Efficiency: $\eta=\frac{\text{absorbed power}}{\text{incident power}}=1\%$

$P_{heat}=U_a\cdot I_a\cdot(1-\eta)=P-P_{X\text{-ray}}$

Dissipation (Infrared): Diffusion ($\propto T^4$) and Radiation ($\propto\Delta T$): λ vs. I, $\lambda_{peak}=b/T$, $b=2.9\mathbb{E}[-3]\,mK$

3 Imaging with X-rays

In log E vs. log μ plot is Compton linear, but Photo decreasing.

- **Photo effect, char. radiation** (Desired): Photon gets completely absorbed by throwing an electron out of atom → damage. **Probability** $\propto\rho\cdot Z^3/E^3$, ρ: tissue density. *Effective in* contrast agent, lead, bone
- **Compton Scattering** (→ resolution loss): Photon only gets deflected. **Probability:** $\propto\rho/E$. *Effective in* water, air, soft tissue, bone

Scattering angle ϕ: $\lambda_p-\lambda_{p'}=\frac{h}{m_c\cdot c}(1-\cos\phi)$

To only have photo effect → E ↓. Compromise with thick targets.

Spatial resolution: MTF is indicated in const vs line pairs/mm

Temporal resolution: periodic s.t. $\Delta t<1/BW_{heartbeat}$

3.1 Attenuation coefficients   efficient for small E

Mass absorption: $\mu'=\frac{A_{eff}N_A}{\text{atomic weight}}[cm^2/g]$ area to collide with

Linear attenuation: $\mu=\mu'\cdot\rho[cm^{-1}]$

Beer-Lambert's law: $I=\int_0^{E_{max}}I_0(E)e^{-\int_{-\infty}^{\infty}\mu(E,x)\,dx}\,dE$

μ homog.: $I=I_0e^{-\mu x}$ **Contrast** $\propto(\mu_1-\mu_0)d=\ln(I_0/I)$

3.2 Detector Technology

Analog photographic film: $Ag^++Br^-\underset{photon}{\longrightarrow}AgBr$

Use fluorescent screen (spacial res.↑) ⟶ 5x efficiency

Anti-scatter grid: No grid: $c_{scatter}=\frac{(I+I_S)-(I_B+I_S)}{I_B+I_S}=c\cdot\frac{1}{1+I_S/I_B}$ where $c=\frac{I-I_B}{I_B}$ is contrast with grid

Digital Detector: photofilm, excited el's → laser → PMT

DSA: Contrast Agent: Iodine: increases x-ray absorption.

4 Computed Tomography

Quantitative with **Houndsfield unit** (normed to water):

CT value $=\frac{\mu-\mu_{water}}{\mu_{water}}\cdot 1000\,HU$

Beam types: Pencil, Fan, Parallel fan, cone.

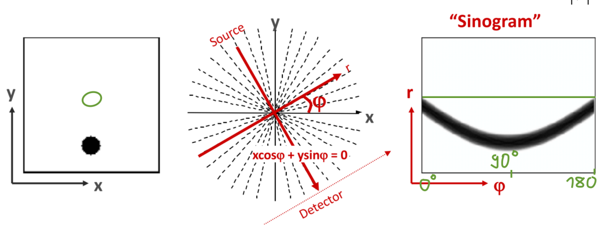
Algebraic Reconstruction: Out of a $n\times n$-Image, make a $n^2\times n^2$ equation system: $\vec P=W\vec\mu$ (P are the measurements, W are the weightings, how much a beam crossed a box/pixel)

$\vec\mu=(W^HW)^{-1}W^H\vec P$ ^{H}: hermetisch transponiert.

Projection Slice Theorem: Project a 2D function onto a line and do a 1D Fourier transform ⟺ Do a 2D Fourier transform of that 2D function and slice it through its origin (parallel to the line above).

4.1 Backprojection

When doing simple Backprojection: $PSF=\frac{1}{|r|}$



$P_\phi(r)=-R[\mu](r,\phi)=-\int\int_{-\infty}^{\infty}\mu(x,y)\delta(\underbrace{x\cos\phi+y\sin\phi-r}_{\text{line formula}})dxdy$

Math. Deduction: **1st Reconstruction formula**:

Projection $P_\phi(r)=-\int\mu(r,s)ds$

$\mathcal{F}\{P_\phi(r)\}(u,\phi)=\int P_\phi(r)e^{-jur}dr=$

$=-\iint\mu(r,s)e^{-jur}drds=$ (with $r=x\cos\phi+y\sin\phi$)

$=-\iint\mu(x,y)\exp(-jx\underbrace{u\cos\phi}_{p\hat=k_x}-jy\underbrace{u\sin\phi}_{q\hat=k_y})dxdy=$

$=-\iint\mu(x,y)e^{-jxp-jyq}dxdy$ (until here just proj. slice th.)

$\Longrightarrow\mu(x,y)=\frac{-1}{4\pi^2}\iint\mathcal{F}\{P_\phi(r)\}(p,q)e^{jxp+jyq}\,dp\,dq$

Fill k-space with $\mathcal{F}$ of the proj.s ≙ Backproj. of $\mathcal{F}$. Then do 2D-backtransformation

2nd variant: Change of integration variables:

$u(x,y)=\frac{-1}{4\pi^2}\int_0^\pi\int_{-\infty}^{\infty}\mathcal{F}\{P_\phi(u)\}e^{jur}|u|\,du\,d\phi$

Filtered Backprojection:

$\hat u(x,y)=\int_0^\pi P_\phi\cdot\mathcal{F}^{-1}\{|u|\}d\phi=\sum_{j=0}^n(p(r,\varphi_j)*h(r))d\varphi$.

4.2 Effects that occur in practice & Spiral CT

"Streak artifacts": Too few projections have been acquired. Correction: Additional low-pass ⟹ SNR rises.

"data grabbing": Convert cartesian ↔ polar coordinaates

Filter in image-domain: $|u|$ ○—● $\frac{1}{(2\pi r)^2}$

If Fourier-domain is band-limited, the filter becomes a **Ram-Lak**: $\text{sinc}(r)*\frac{1}{(2\pi r)^2}$ (don't forget: rect(r) ○—● sinc(r))

Have other filter, in order to not amplify noise: **Shepp-Logan, Cosine or Hann-Filter**, i.e. combine with low-pass.

Spiral CT: Before backprojection, do linear interpolation of data. **Pitch**=1: The neighboring slices touch, Pitch=2: slice distance = 1 slice thickness

4.3 (Quantum) Noise

Distribution of photon counts is $P_\lambda(x)$. $E\{P_\lambda\}=\lambda\equiv\bar N$, $\sigma=\sqrt\lambda$.

⟹ SNR $=\frac{\text{average}}{\text{std. deviation}}=\frac{\bar N}{\sqrt{\bar N}}=\sqrt{\bar N}$

where N: #X-rays, λ: #Photons registered.

⟹ Image SNR $=\frac{\bar I}{\sqrt{Var\{I\}}}$ where $\sigma_I\equiv\sqrt{Var\{I\}}\propto\sqrt{\frac{1}{I_A t\Delta z}}$

Pixel noise = $\sigma=\sqrt{\frac{1}{M-1}\sum_{i=1}^M(I_i-\bar I)^2}=\sqrt{1/(Q\Delta z)}$

where Q: Tube current I_A × scan time t, Δz: detector thickness

4.4 Dose considerations

Absorbed dose: 1 Gray = 1 Gy = 1 J/Kg

Equivalent dose: 1 Sievert = 1 Sv = $Q\cdot N$ Gy

with the **Quality factor Q** (mostly 1) and the **Pertinent factor N** (bone, lung, stomach: 0.12 - skin: 0.1)

Excess Relative Risk:

$RR=\frac{\text{Cancer Deaths/y}}{\text{Cancer Deaths control pop./y}}$ $ERR=\frac{RR-1}{\text{Additional dose}}$

It is higher for women and children. 2.5/Sv − 0.25/Sv

5 SPECT – Single Photon Emission CT

SNR $\propto\sqrt{\text{total \#detected gamma-rays}}$

5.1 Radioactivity in the body

$Q=-\frac{dN}{dt}=\lambda\cdot N\Longrightarrow N(t)=N_0e^{-\lambda t}$

$[Q]$ = Curie = Ci = $3.7\cdot10^{10}$Bq where 1Bq = 1 disintegration/s

Physical $t_{1/2}=\frac{\ln 2}{\lambda}$,   **biological** $t_{1/2bio}=\frac{\ln 2}{\lambda_{bio}}$

$N(t)=N_0e^{-(\lambda+\lambda_{bio})t}=N_0e^{-\lambda_{1/2eff}t}$ where $t_{1/2eff}=\frac{t_{1/2}\cdot t_{1/2bio}}{t_{1/2}+t_{1/2bio}}$

γ **photons: Absorption** or **Scattering**: Change of direction by θ. $E'_\gamma=\frac{m_e\cdot c^2}{m_ec^2/E_\gamma+1-\cos\theta}$

5.2 Interaction of γ photons with matter

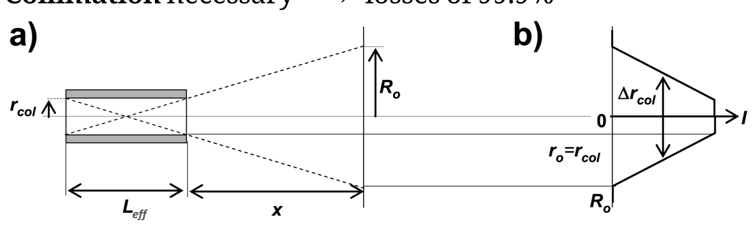
Absorption leads to ejection of orbital electron

Scattering: Change of direction by θ.

$E'_\gamma=\frac{m_e\cdot c^2}{m_ec^2/E_\gamma+1-\cos\theta}$

5.3 Anger Camera

Collimation necessary ⟹ losses of 99.9%



Bad isolation of **septa** (lamellae) ⟹ $L_{eff}=L-\frac{2}{\mu_{septa}}$

$R_0(x)=\frac{2r_{col}}{L_{eff}}(x+\frac{L_{eff}}{2})$ FWHM $=\Delta r_{col}(x)=\frac{2r_{col}}{L_{eff}}(x+L_{eff})$

Signal Path: γ-rays @140keV → Scintillation crystal → PMT → (Pos. network || pulse height analyzer) → digitizer

- Scintillation crystal efficiency: $\epsilon=1-e^{-\mu d}$. μ: attenuation coefficient of crystal, d: its thickness
- Usually many PMTs per crystal. Positioning network calculates where the event in the crystal was.
- $U_{PMT}\propto\gamma$ energy. Scattering ⟹ $E_{scatteredphotons}$ ↓. Pulse height analyzer has energy window ΔE around E_0.
- Poisson distributed $P_n(N)|\sigma^2=\mu\Longrightarrow SNR=\sqrt\mu$
- **Spatial resolution**: $R^2_{system}=R^2_{gamma}+R^2_{coll}$ R_{gamma} = uncertainty of positioning network. R_{coll} see above.

CT Version: Res.: best at edge, worst in center. ≈ 7mm

For small objects: Use **pinhole** (cone-shaped, tungsten) camera: Problem: more dose needed, "transparency" around hole

5.4 Generation of Technetium

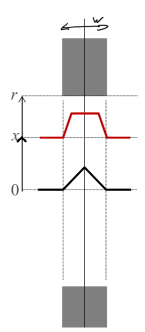
No cyclotron needed, only generator: Essentially $^{99}_{42}Mo$ (67 h ≙ $2.9\mathbb{E}[-6]\,s^{-1}$) decays into $^{99}_{43}Tc$ (6 h ≙ $3.1\mathbb{E}[-5]\,s^{-1}$)

Kinetic equations:

$\frac{d}{dt}N_{Mo}=-k_{Mo}N_{Mo}$ and $\frac{d}{dt}N_{Tc}=k_{Mo}N_{Mo}-k_{Tc}N_{Tc}$

$\Longrightarrow N_{Tc}(t)=N_{Mo}(0)\frac{k_{Mo}}{k_{Tc}-k_{Mo}}(e^{-k_{Mo}t}-e^{-k_{Tc}t})$ exp. decay

6 PET – Positron Emission Tomography



Positron range $\propto$ E.   FWHM ≈ 0.1 − 0.5mm.

Radionuclide: ^{18}F better than ^{11}C (110 vs 20 min)

Image production: Scintillator (fine grid) → PMT / avalance Diode → Electronics. Event finding in scintillator is linear. $y=\frac{S_A+S_B-S_C-S_D}{S_A+S_B+S_C+S_D}$

PSF: Trapezoid/Triangle.   FWHM $=w\frac{r+x}{2r}$

r: Ring diameter, x: distance from center, w: grid spacing of sc.

6.1 Efficiency (to actually measure concentrations)

Detection eff. $\epsilon=(1-e^{-\mu d})\cdot\Phi$ (Φ: frac. of events in E window)

Geometric eff.: $\Omega=4\pi\sin(\arctan(z/D))$ $D=2r$, z: length

Radial geometric coverage ϕ: Fraction not gap between crystals

Total sensitivity: $\eta=\epsilon^2\phi\frac{\Omega}{4\pi}$

6.2 Problems, solutions and additions

Correction table for which scintillator under PMT

Correction matrix: against non-uniform detector eff.

Attenuation correction: Measure Image in HU via X-Ray CT, → attenuation coeff. → multiply PET lines by $e^{\mu D_{ij}}$

Random coincidence: Measure Background noise, then -

Detector dead time δ: $N_{measured}=N_{true}e^{-N_{true}\delta}$

TOF (Time Of Flight of photon): $\Delta x=\frac{c\Delta t}{2}=75\,mm\hat=500\,ps$

3D image reconstr.: Use coinc. between different rings.

6.3 Quantitative PET - various effects and definitions

Partial voluming: When measuring the intensity of an object, it appears smaller at the edge of the object than it is. → Only measure in the center

Injected dose per gram of tissue

$\boxed{\%ID/g = \frac{c_t v_t}{D_{inj}} \cdot \frac{1}{m_t} \cdot 100\%}$, c_t: tissue conc., v_t: vol. of tissue ROI, m_t: mass of tissue ROI, D_{inj}: injected dose

Standardized uptake value (M: Body mass, S: surface)

$SUV = [\%ID/g] \cdot M/100$ $\quad SUV' = [\%ID/g] \cdot S/100$

Distribution volume: Volume of blood needed for M #tracer.

$\boxed{V_d = M/c_b = V_t c_t/c_b + V_b = V_t\lambda + V_b}$ $\quad \lambda = c_t/c_b$

$V_t\lambda = V_1$ = equiv blood vol. to tissue vol. where tracer is c_b: typ. conc. of that tracer in blood. $V_d > 7\ell \implies$ body.

6.4 The scatchard equation

Ligand (tracer/drug), **Receptor**. Reaction: L+R = RL

$[L], [R], [RL]$: conc. — $[R_T]$: total #receptors — $k_{\text{off}}, k_{\text{on}}$: reaction rates — $k_{\text{off}} \cdot [RL]$: #bound pairs that separate.

Equilibrium eq.: $k_{\text{off}} \cdot [RL] = k_{\text{on}} \cdot [R] \cdot [L]$

Eq. const.: $K_d = \frac{k_{\text{off}}}{k_{\text{on}}} = \frac{[R]\cdot[L]}{[RL]}$ $\quad \frac{[RL]}{[L]} = -\frac{1}{K_d} \cdot [RL] + \frac{R_T}{K_d}$

Scatchard plot: slope $= 1/K_d$, R_T: intersection with $[RL]$-axis

6.5 Kinetic models

Renking-Crone eq.: (amount of substance diffuses out of a blood capillary)

$= F \cdot E \quad E = 1 - e^{-P \cdot S/F}$ E: Efficiency, P: vascular permeability, S: capillary surface, F: blood flow

Kinetic model: (c_p: Plasma ← particular part)

$$c_p \underset{k_2}{\overset{k_1}{\rightleftarrows}} c_f \underset{k_4}{\overset{k_3}{\rightleftarrows}} c_b$$

c_f: free ligands, c_b: bound l

Kin. eq.: $\frac{\mathrm{d}c_f}{\mathrm{d}t} = k_1 c_p - (k_2 + k_3)c_f + k_4 c_b$, — $\frac{\mathrm{d}c_b}{\mathrm{d}t} = k_3 c_f - k_4 c_b$, — $k_1 = FE$, — $k_2 = k_1 c_p/c_f$, — $k_3 = k_{on}[R]$, — $k_4 = k_{off}$

What is measured: $c_t(t) = c_f(t) + c_b(t)$. $*c_p(t)$ should be in the equation. → At the end determine the rate constants.

Improvement: Look at reference section in brain without receptors. Only 2 compartment model, measure k_1 and k_2.

To measure behaviour of a drug (cold): Tracer (hot) → same receptors. Experiment with and without drug. Assumption: $c_b << c_{b,d}$. The drug changes the # total receptors from B_{max} to B'_{max}. **Receptor occupancy for drug**:

$\boxed{\text{RO}[\%] = \frac{c_{b,d}}{B_{\text{max}}} = (1 - B'_{\text{max}}/B_{\text{max}}) \cdot 100\%}$

7 MRI

7.1 Quantum basics

Spin $S_z = \hbar m \quad m = -I, -I+1, ... I \quad |S| = \hbar\sqrt{I(I+1)}$

Magnetic moment: $\boxed{\vec{\mu} = \gamma \vec{S}}$ (γ: gyromagnetic ratio)

Energy content in $\vec{B}$-**Filed**: $\boxed{E_m = -\vec{\mu} \cdot \vec{B} = \pm\frac{\hbar}{2}\gamma B_0}$

Blotzmann stat.: $\frac{n_{-1/2}}{n_{+1/2}} = \exp(-\frac{\Delta E_m}{k_B T})$ $\quad \frac{\Delta n}{n} \propto \frac{\gamma \hbar B_0}{2 k_B T}$

Larmor frequency: E gap ↔ photons: $\omega_L = \gamma B_0$

Macrosc. mag. dipolar moment: $\boxed{\vec{M}_0 = \sum \vec{\mu} = \Delta n \mu_z}$

Bloch equation ($T_2 << T_1$)

$$\frac{d}{dt}\vec{M} = \begin{pmatrix} -1/T_2 & -\gamma B_z & \gamma B_y \\ \gamma B_z & -1/T_2 & -\gamma B_x \\ -\gamma B_y & \gamma B_x & -1/T_1 \end{pmatrix} \vec{M} + \begin{pmatrix} 0 \\ 0 \\ M_0/T_1 \end{pmatrix}$$

$M_z(t) = M_0 \cos\alpha + (M_0 - M_0\cos\alpha)(1 - \exp(-t/T_1))$, $\alpha = \gamma B_1 \tau_{B1}$: tip angle

$\vec{B}$ with $|\vec{B}| = B_1 = const$ and $B_z = B_0$ spinning with ω_{RF} around the z-Axis. $\omega_{RF} = \omega_L \iff \vec{M}$ spins ↓,↑,↓ etc.

Rot. frame of reference: $\vec{B}$ stays at a slight angle and $\vec{M}$ rotates around it. Correction: $B_z = B_0 - \omega_{RF}/\gamma$

7.2 Basic setup

1. Have B_z + rot. field to push spins in x-y-plane.
2. Coils to make gradients: Maxwell (z), and two Golay pairs
3. Changing the gradient → travel k-space and sample it
4. IFFT to get the image weighted with proton density

Signal from entire obj: $S(t) = e^{j\omega_0 t}\int_{\text{obj}} \rho(\vec{r}) e^{j\vec{k}(t)\cdot\vec{r}} \mathrm{d}^3\vec{r}$

Fourier transform of $\rho(\vec{r})$: $S(\vec{k}) = e^{-j\omega_0 t}\int_{\text{obj}} \rho(\vec{r}) e^{-j\vec{k}\cdot\vec{r}} \mathrm{d}^3\vec{r}$

Slice sel.: Grad. in z-dir. and flip spins with a sinc×gauss-pulse (≈ ○—● rect, range of freq. where $\Delta B_z = 0$).

Grad. → **dephasing**. $\implies$ invert G to **rephase** spins.

7.2.1 Measuring the spin

$\boxed{\hat{U}_{\text{ind}} = j\omega I_0^{-1}\hat{\vec{\mu}} \cdot \hat{\vec{B}}^t(\vec{r}) = M_{xy} V s(\vec{r})}$

$s(\vec{r}) = j\omega I_0^{-1}(\hat{B}_x^t(\vec{r}) - j\hat{B}_y^t(\vec{r}))$ $\quad B_1^{(-)} = s/(j\omega)$

I_0: Transmit current, B^t: Field received at $\vec{r}$ when transmitting, s: **Coil sensitivity**, $\hat{\vec{\mu}} = V(M_{xy}, -jM_{xy}, 0)^T$, s of smaller coil ↑. Large distance: s of bigger coil ↑

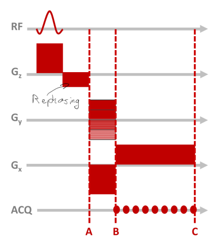
Signal $\propto \frac{dM}{dt} = j\omega_0 M_0 e^{j\omega_0 t}$ $\propto \gamma^3$

7.3 Measurement procedures (All Gradient Echo)

Echo-planar imaging (EPI): ⩶, spiral, radial

↻ θ, T_E, sample, $T_R - T_E$, again

$I \propto \rho \frac{(1-e^{-T_R/T_1})\sin\theta}{1-e^{-T_R/T_1}\cos\theta} e^{-T_E/T_2}$

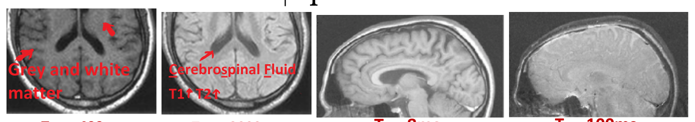


$\alpha_{\text{Ernst}} = \arccos(e^{-T_R/T_1})$

Sat. method: $\boxed{1 - \exp(-t/T_1)}$ $T_E \approx 0$, $T_R \downarrow \implies T_1$ weighted. **Inv. recovery** also T_1

Spin-echo method: $\boxed{\exp(-t/T_2)}$ T_2^* **decay (const. loc. inhomog.)**. ↻ at $T_E/2$ by 180°. $T_E \uparrow \implies T_2$ weight.

Saturation Method | Spin Echo Method



T_{Rec} = 400ms $\quad T_{Rec}$ = 2000ms $\quad T_E$ = 8ms $\quad T_E$ = 100ms

Inflow contrast: T_1 weight. (Blood with ↑ M_z during T_R)

7.4 Noise, SNR and Resolution

$\sigma^2_{\text{noise}} = 4k_B \cdot T \cdot R \cdot BW$ $\quad T$: Temp., $R = U/I$, BW: Bandwidth

If frequency indep. and uniform T (I_0 is transmit c.):

$\sigma^2_{noise} = 4k_B \cdot T \cdot BW \int \sigma(\vec{r}) \frac{|\vec{E}(\vec{r})|^2}{I_0^2} dV$

Otherwise: $\sigma^2_{\text{noise}} = 4k_B \iint T(\vec{r})\sigma(\omega, \vec{r}) \frac{|\vec{E}(\omega,\vec{r})|^2}{I_0^2} \frac{d\omega}{2\pi} dV$

$\boxed{\sigma^2_{\text{noise}} = 4k_B \cdot BW(R^{\text{eff}}_{\text{sample}} T_{\text{sample}} + R^{\text{eff}}_{\text{coil}} T_{\text{coil}} + R^{\text{eff}}_{\text{env}} T_{\text{env}})}$

(last Term usually negligable)

$\boxed{\text{SNR} = \frac{\omega B_1^{(-)}(\vec{r}) M_{xy}(\vec{r}) \Delta V}{\sqrt{4k_B BW(T_{sample}R_{sample} + T_{coil}R_{coil})}} \sqrt{N_{avg}}}$

$BW \propto$ Gradient strength $\quad \propto$ Body size $\quad \propto t^{-1}_{\text{acq}}$

$t_{scan} \propto N_{avg} \implies SNR \propto \Delta V \sqrt{t}$

SNR↑: magnetization↑ by B_0 ↑. ω ↑. T_{coil} ↓, R_{coil} ↓

Resolution limits:

In k-space, the image is multiplied with

$H(k) = \text{rect}\left(\frac{k}{2k_{\text{max}}}\right) e^{-t/T_2^*} \implies$ **PSF** : $\Delta x \geqslant \frac{\pi}{\gamma G_{\text{max}} T_2^*}$

Diffusion Area: $\langle \Delta x^2 \rangle = 6 \cdot D \cdot T_{\text{acq}}$ $\quad D = 10^{-3}\,\text{mm}^2/\text{s}$ for H_2O.

7.5 Various

T/R Switch: Block kW in transmit mode, nW in r mode.

RF Body Coil - Gradients - B_0**-Magnet (3-7T) - Shield Coils - Cryogenics @4K.**

$\boxed{1\,\text{Gauss} = 0.1\,\text{mT}}$ **Limits**: 5 Gauss: Pacemakers, Credit cards. — 50 Gauss magnetic objects

7.5.1 fMRI – functional MRI (brain imaging)

Oxygenated hemoglobin is diamagnetic, deoxy-Hb is paramagnetic. Param. disturbs the B-field $\implies$ reduces T_2^*. $S_{task} > S_{idle}$. Use echo planar imaging because it is fast. Calc **scalar product** of activation and **paradigm** (± 1)

8 Ultrasound Imaging $c_{\text{sound}} \simeq 1540\,\text{m/s}$

Frequencies of 1-50 MHz $\implies \lambda = 1....0.03$mm

Wave formula: $\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$ $\quad$ Max. p = compressional pressure= p_c, Min. p = rarefactional pressure = p_r

κ = compressibility, ρ = density, u_z = particle velocity

$c = \frac{1}{\sqrt{\kappa\rho}}, \quad p = \rho c u_z, \quad Z = p/u_z = \rho c = \sqrt{\frac{\rho}{\kappa}}, \quad I = p u_z/2$

Pressure coeff.: $r = \frac{Z_2 - Z_1}{Z_2 + Z_1}$ $\quad t = \frac{2Z_2}{Z_2 + Z_1}$

Refraction: $\theta_i = \theta_r$, $\quad \sin\theta_i/\sin\theta_t = c_i/c_t$

Rayleigh scattering by structures smaller λ → speckles/noise ($\sigma_S \propto \lambda^{-4}$). It has a char. length of $\lambda/2$:

Attenuation (due to scattering and absorption):

$p(z) = p_0 e^{-\alpha z} \approx p_0 10^{-\frac{\text{att}}{20\,\text{dB}}}$ ($[z]$=cm, att$= \alpha_0 \cdot z =$const$\cdot f \cdot z$)

Decibel notation: $\alpha_0 = 20\log\left(\frac{p_0}{p(z)}\right)\frac{1}{z} = 8.686\alpha[dB/cm]$

Transducer: Tx/Rx switch, Damping, matching layer, f_0

8.1 Beam geometry

Near field (Fresnel) zone with const. beamwidth of $2r$.

Far field (Fraunhofer) zone which starts at

$\text{NFB} \approx \frac{r^2}{\lambda}$.

Beamwidth = lateral resolution

Angle of beam: $\theta = 2\arcsin\left(\frac{0.61\lambda}{r}\right)$

Focusing with acoustic lens: Focal distance, lateral resolution, aperture dimension. Depth of focus: Over which distance is it narrow

Axial resolution: $\Delta z \geqslant \frac{\lambda}{2} = \frac{p_d c_{\text{sound}}}{2}$ $\quad \lambda$: pulse length

Range Gain: The longer signals take to come back, the ↓. To compensate: Amplify the late signals exponentially

Thermal noise: $\boxed{P_N = k_B \cdot T \cdot BW[\text{W}]}$ (BW: typ. 1MHz)

SNR in dB = Transmitted Power - power losses - P_N.

Power losses: Attenuation & reflection coefficient (factor 20)

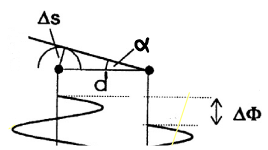
8.2 Transducer setups

Linear Array. pitch $d \approx \lambda$ and **kerf** (gap). Sweep through.

Good resolution $\implies$ small pitch $\implies$ large θ

Phased parallel operation:

$\Delta\Phi = k\Delta s = kd\sin\alpha \mod 2\pi$. Multiple solutions for $\boxed{d > \lambda/2 \rightarrow}$ **grating lobes** (weaker since not time aligned)



Receiving analog: delay elements, then sum all up.

Variable focusing (with shifting) and a combination of all

Multi-dim. arrays: 2D or 1.5D (1/2 D = "elevation").

Curved arrays (instead of flat): For small acoustic windows

Annular Array (circular sections): Simpler adjustable focus and circular symmetry = more **isotropic depiction**

8.3 Scanning modes

A-Mode (Amplitude): time (distance) vs amplitude. Usage: e.g. measuring the thickness

M-Mode (Motion): time vs depth. Amplitude with brightness. Must be tilted manually

B-Mode (Brightness): 2D spacial. Amplitude with brightness

Scanning Procedures:

• **Parallel scan** for large acoustic window. • **Sector scan** for small ac. window. • **Radial scan** for transd. in blood vessels (measuring the vessel wall). • **Compound scan** from different directions → redundancy and reduce speckles.

8.4 Measuring the blood flow

Observer at velocity v, receives $f^{\text{eff}} = f\frac{c+v}{c}$

Blood vessel flows under the receiver with angle θ rel. to the vertical to transducer: $\boxed{f_{\text{rec}} = f_i\frac{2f_i\cos\theta}{c} + \frac{f_i v^2\cos^2\theta}{c^2}}$

$f_D = f_i - f_{\text{rec}} \approx \frac{2f_i v\cos\theta}{c}$

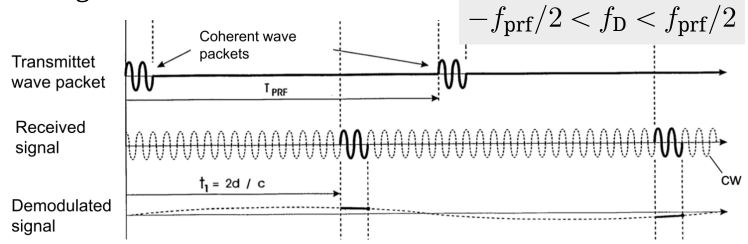
Must know θ $\implies$ do B-Mode scan first

Red blood cells $\approx 7\mu m$ wide $\implies$ Use high freq., typ. 5MHz

Cont. setup: Separated receiver/transmitter. Quadr. encoder: Mult. with cos (→ real part) and sin (→ imag part) → double sided spectrum. Neg. side is *negative flow*. Then *low-pass* + *high-pass* to remove (quasi) stationary echos.

Pulsed measurement: 1 transducer. Only receive signal during a narrow time window. Its delay = depth.

$-f_{\text{prf}}/2 < f_D < f_{\text{prf}}/2$



8.5 Appendix

$f(at)$ ○—●	$\frac{1}{\lvert a\rvert}F(s/a)$	$f(t-a)$ ○—●	$e^{-as}F(s)$
$f(t)e^{at}$ ○—●	$F(s-a)$	$f'(t)$ ○—●	$sF(s) - f(0^+)$
t^n ○—●	$n!/s^{n+1}$	$t^n f(t)$ ○—●	$(-1)^n F^{(n)}(s)$
$\sin(at)$ ○—●	$\frac{a}{s^2+a^2}$	$\cos(at)$ ○—●	$\frac{s}{s^2+a^2}$
e^{at} ○—●	$\frac{1}{s-a}$	$t^n e^{at}$ ○—●	$\frac{n!}{(s-a)^{n+1}}$

Constants:

$h = 6.626\mathbb{E}[-34]\,\text{JS} = 4.135\mathbb{E}[-15]\,\text{eV s}$, $\quad \hbar = \frac{h}{2\pi}$

$\epsilon_0 = 8.85\mathbb{E}[-5]\,\text{As/Vm}$

$\mu_0 = 4\pi\mathbb{E}[-7]\,\text{N/A}^2$

$k_B = 1.38\mathbb{E}[-23]\,\text{J/K} = 8.617\mathbb{E}[-5]\,\text{eV/K}$

$q = 1.602\mathbb{E}[-19]\,\text{C}$, $\quad m_e = 9.109\mathbb{E}[-31]\,\text{kg}$, $\quad m_p = 1.672\mathbb{E}[$

$m_e c^2 = 511\,\text{eV}$

$0°\text{C} = 273.15\,\text{K}$