Final Exam for CS 333201

10:10 - 12:00 a.m., June 14, 2017

1. (10%) Suppose that the IQ of a randomly selected student from a university is normal with mean 100 and standard deviation 20.
   (a) (5%) Determine the probability that the IQ score of such student is more than 125.
   (b) (5%) Determine the probability that the IQ score of such student is between 95 and 105.

2. (10%) Let $X \sim N(0, \sigma^2)$. Calculate the density function of $Y = X^2$.

3. (25%) The joint pdf of X and Y is defined as follows,

$$f(x, y) = \begin{cases} 0.5, & 0 < x < 2, 0 < y < 2, and\ 0 < x + y < 2 \\ 0, & otherwise \end{cases}.$$

   (a) (5%) Find the marginal density function of $f_X(x)$.
   (b) (5%) Find the conditional density function $f_{X|Y}(x|y)$ of X given Y.
   (c) (5%) Find the expected value of the function $h(X, Y) = XY$.
   (d) (10%) Find the correlation coefficient $\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$ between X and Y.

4. (20%) Let X be a gamma random variable with parameter (γ, λ).
   (a) (10%) Derive a formula for $M_X(t)$ for $t < \lambda$.
   (b) (10%) Use $M_X(t)$ to calculate $E(X)$ and $Var(X)$.

5. (25%) In a printer queue, the waiting time for the next job is an exponential random variable with parameter λ. Let X_1 be the waiting time of the 1st job, and X_2 be the waiting time between the 1st and the 2nd job.
   (a) (10%) Find $M_{X_1}(t)$ for $t < \lambda$.
   (b) (10%) Let Y be the time to the 2nd job, that is, $Y = X_1 + X_2$. Find $M_Y(t)$ and determine $f_Y(y)$.
   (c) (5%) If jobs arrive at an average rate of 30 jobs per hour. What is the probability that the waiting time to the next job is between 2 and 4 minutes?

6. (10%) In the buffer of a switch in the Internet, the average occupancy of packets is known to be 40%. Let X be the random variable of the buffer occupancy.
   (a) (5%) Use Markov inequality to find a bound on the probability that buffer occupancy equals or exceeds 60%.
   (b) (5%) If the standard deviation of buffer occupancy is known to be 10%, use Chebyshev's inequality to find a bound on the probability that buffer occupancy will be between 20% and 60%.

Theorem 11.8 (Markov's Inequality) *Let X be a nonnegative random variable; then for any $t > 0$,*

$$P(X \geq t) \leq \frac{E(X)}{t}.$$

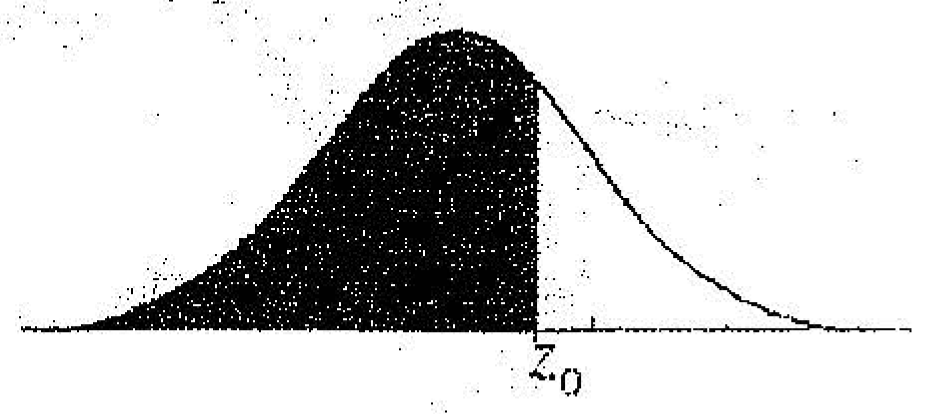
Theorem 11.9 (Chebyshev's Inequality) *If X is a random variable with expected value μ and variance σ^2, then for any $t > 0$,*

$$P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}.$$

Definition *A random variable X is called* ***normal****, with parameters* μ *and* σ, *if its density function is given by*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right], \qquad -\infty < x < \infty.$$

$$\Phi(z_0) = P(Z \le z_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_0} e^{-x^2/2}\, dx$$



z_0	0	1	2	3	4	5	6	7	8	9
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319

Definition *A continuous random variable X is called* ***exponential*** *with parameter* $\lambda > 0$ *if its density function is given by (7.2).*

$$f(t) = F'(t) = \begin{cases} \lambda e^{-\lambda t} & t \ge 0 \\ 0 & t < 0 \end{cases} \tag{7.2}$$

Definition *A random variable X with probability density function*

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{\Gamma(r)} & \text{if } x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

is said to have a ***gamma*** *distribution with parameters* (r, λ), $\lambda > 0$, $r > 0$.

$$\Gamma(r) = \int_0^{\infty} t^{r-1} e^{-t}\, dt.$$