19. Numerical Integration Formulas

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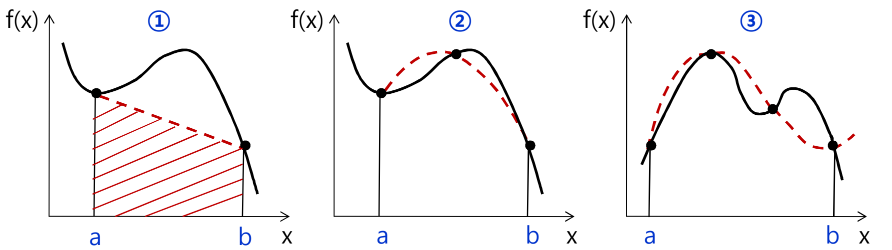
Newton-Cotes integration formulas : 복잡한 함수나 표로 되어 있는 data들을 적분하기 쉬운 다항식으로 대신하여 적분하는 방법

$$I = \int_a^b f(x)\,dx \cong \int_a^b \underbrace{f_n(x)}_{n\text{차 interpolating polynomial}}\,dx$$

① **Trapezoidal rule :** 직선으로 근사하여 사다리꼴 면적 계산

② **Simpson's 1/3 rule :** 3개의 점으로 2차 다항식으로 근사하여 적분

③ **Simpson's 3/8 rule :** 4개의 점으로 3차 다항식으로 근사하여 적분



① Trapezoidal(사다리꼴의) Rule

$$I = \int_a^b f_n(x)\,dx$$

$$= \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

(a, f(a)), (b, f(b))
두 점을 지나는 직선

$$\boxed{I = (b - a)\frac{f(a) + f(b)}{2}}$$

면적 (윗변+밑변) X 높이 X 1/2

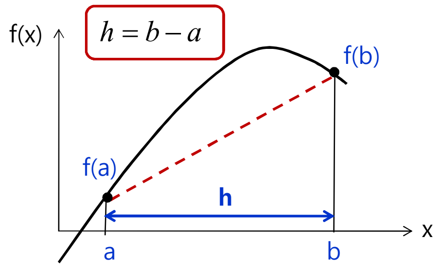
- 오차

$$\boxed{E_t = -\frac{1}{12} f''(\xi) h^3}$$

$$= -\frac{1}{12} f''(\xi)(b - a)^3$$

(ξ : a 와 b 사이의 어떤 점)

☞ 오차가 2차 도함수에 비례



$$f(x) = \underbrace{f(a) + \frac{f(b) - f(a)}{b - a}(x - a)}_{\text{Newton's 1차 다항식}} + R_1 \quad \text{절단오차}$$

$$R_1 = \frac{f''(\xi)}{2!}(x - a)(x - b)$$

Taylor series 것과 비슷

$$E_t = \int_a^b R_1\,dx = \int_a^b \frac{f''(\xi)}{2!}(x - a)(x - b)\,dx$$

$$= \frac{f''(\xi)}{2!}\int_0^1 \alpha h \cdot (\alpha - 1)h \cdot h\,d\alpha$$

$\frac{x - a}{h} = \alpha$ 로 치환하여 적분, $h = b - a$

$$= \frac{f''(\xi)}{2!} h^3 \int_0^1 \alpha(\alpha - 1)\,d\alpha$$

$$= \frac{f''(\xi)}{2!} h^3 \left[\frac{\alpha^3}{3} - \frac{\alpha^2}{2} \right]_0^1$$

$$= -\frac{1}{12} h^3 f''(\xi)$$

Ex) Use the single of the trapezoidal rule to numerically integrate

$f(x) = 400x^5 - 900x^4 + 675x^3 - 200x^2 + 25x + 0.2$ from a=0 to b=0.8.

Estimate the error. (참 적분 값은 1.640533)

$$\left.\begin{array}{l} f(0) = 0.2 \\ f(0.8) = 0.232 \end{array}\right] \rightarrow \quad I = (0.8-0)\frac{0.2+0.232}{2} = 0.1728$$

참 오차 : $E_t = 1.640533 - 0.1728 = \underline{1.467733}$

- 실제 상황에선 참 값을 모르므로 근사오차 $E_a = -\frac{1}{12}\bar{f}''(x)(b-a)^3$ 필요

$f''(x) = 8000x^3 - 10800x^2 + 4050x - 400$

$$\bar{f}''(x) = \frac{\int_0^{0.8} 8000x^3 - 10800x^2 + 4050x - 400dx}{0.8-0} = -60$$

근사 오차 : $E_a = -\frac{1}{12}(-60)(0.8)^3 = \underline{2.56}$

근사오차라서 $\bar{f}''(x)$ 와 $f''(\xi)$ 일치는 불필요

Composite Trapezoidal Rule : 사다리꼴 공식의 정확도 높이기 위해

구간을 같은 간격의 여러 조각(segment)으로 나누어 적분

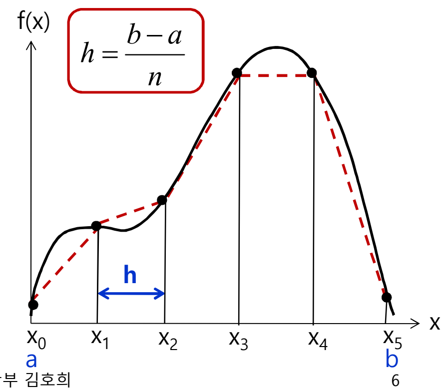
$$I = \int_{x_0}^{x_1} f_n(x)\,dx + \int_{x_1}^{x_2} f_n(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f_n(x)\,dx$$

(n+1)개 점
n개 구간

$$= h\frac{f(x_0)+f(x_1)}{2} + h\frac{f(x_1)+f(x_2)}{2} + \cdots + h\frac{f(x_{n-1})+f(x_n)}{2}$$

$$\boxed{= \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]}$$

$$\boxed{h = \frac{b-a}{n}}$$



- 오차

각 segment 들의 오차의 합이므로

$$E_t = -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i)$$

(ξ_i : 각 segment 의 임의의 점)

- 근사오차

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

$$\bar{f}'' = \frac{1}{n}\sum_{i=1}^{n} f''(\xi_i)$$

☞ segment 많을 수록 오차 적다 (n 이 2배가 되면 오차는 ¼로 감소)

Ex) Use the two-segment trapezoidal rule to estimate the integral of

$f(x) = 400x^5 - 900x^4 + 675x^3 - 200x^2 + 25x + 0.2$ from a=0 to b=0.8.

Estimate the error.   (참 적분 값은 1.640533)

n=2 이므로 h=0.4

$$\left[\begin{array}{l} f(0) = 0.2 \\ f(0.4) = 2.456 \\ f(0.8) = 0.232 \end{array}\right] \rightarrow I = (0.4)\frac{0.2 + \boxed{2}(2.456) + 0.232}{2} = 1.0688$$

$$E_t = 1.640533 - 1.0688 = 0.57173$$

$$E_a = -\frac{(0.8)^3}{12(2)^2}\underbrace{(-60)}_{\bar{f}''} = 0.64$$

② **Simpson's 1/3 rule :** 정확한 적분 값 얻으려 점 연결에 고차 다항식 사용, 세 점 $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$을 지나는 2차 Lagrange 다항식

$$f_n(x) = \frac{(x-x_1)}{(x_0-x_1)}\frac{(x-x_2)}{(x_0-x_2)}f(x_0) + \frac{(x-x_0)}{(x_1-x_0)}\frac{(x-x_2)}{(x_1-x_2)}f(x_1) + \frac{(x-x_0)}{(x_2-x_0)}\frac{(x-x_1)}{(x_2-x_1)}f(x_2)$$

$$I = \int_{x_0}^{x_2} f_n(x)\,dx = \boxed{\frac{h}{3}}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$?

x_1은 x_0~x_2 의 중간

(Slide 9)

$$\boxed{= (b-a)\frac{\left[f(x_0) + 4f(x_1) + f(x_2)\right]}{6}}$$

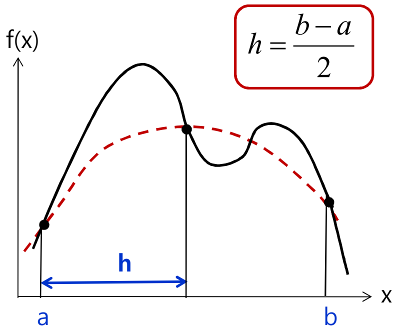
$$\boxed{h = \frac{b-a}{2}}$$

- 오차

$$\boxed{E_t = -\frac{1}{90}h^5 f^{(4)}(\xi)}$$?

(Slide 10~11)

$$= -\frac{(b-a)^5}{2880}f^{(4)}(\xi)$$



☞오차가 4차 도함수에 비례

$$I=\int_{x_0}^{x_2}\overset{①}{\frac{(x-x_1)}{(x_0-x_1)}\frac{(x-x_2)}{(x_0-x_2)}}f(x_0)+\overset{②}{\frac{(x-x_0)}{(x_1-x_0)}\frac{(x-x_2)}{(x_1-x_2)}}f(x_1)+\overset{③}{\frac{(x-x_0)}{(x_2-x_0)}\frac{(x-x_1)}{(x_2-x_1)}}f(x_2)\,dx$$

① $$\int_{x_0}^{x_2}\frac{(x-x_1)}{(x_0-x_1)}\frac{(x-x_2)}{(x_0-x_2)}f(x_0)dx=\frac{f(x_0)}{2h^2}\int_{-h}^{h}u(u-h)du=\frac{f(x_0)}{2h^2}\left[\frac{u^3}{3}\right]_{-h}^{h}=\frac{h}{3}f(x_0)$$

$x-x_1 = u$ 로 치환

② $$\int_{x_0}^{x_2}\frac{(x-x_0)}{(x_1-x_0)}\frac{(x-x_2)}{(x_1-x_2)}f(x_1)dx=\frac{-f(x_1)}{h^2}\int_{0}^{2h}u(u-2h)du=\frac{-f(x_1)}{h^2}\left[\frac{u^3}{3}-hu^2\right]_{0}^{2h}$$

$$=\frac{-f(x_1)}{h^2}\left[\frac{8h^3}{3}-4h^3\right]=\frac{-f(x_1)}{h^2}\left(\frac{-4h^3}{3}\right)=\frac{4h}{3}f(x_1)$$

$x-x_0 = u$ 로 치환

③ $$\int_{x_0}^{x_2}\frac{(x-x_0)}{(x_2-x_0)}\frac{(x-x_1)}{(x_2-x_1)}f(x_2)dx=\frac{f(x_2)}{2h^2}\int_{0}^{2h}u(u-h)du=\frac{f(x_2)}{2h^2}\left[\frac{u^3}{3}-h\frac{u^2}{2}\right]_{0}^{2h}$$

$$=\frac{f(x_2)}{2h^2}\left[\frac{8h^3}{3}-2h^3\right]=\frac{f(x_2)}{2h^2}\left(\frac{2h^3}{3}\right)=\frac{h}{3}f(x_2)$$

$x-x_0 = u$ 로 치환

$$f(x)=\underbrace{f(x_0)+f[x_1,x_0](x-x_0)+f[x_2,x_1,x_0](x-x_0)(x-x_1)}_{\text{Newton's 2차 다항식}}+\underset{\text{절단오차}}{R_2}$$

$$R_2=f[x_3,x_2,x_1,x_0](x-x_0)(x-x_1)(x-x_2)+\frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$f[x_1,x_0]=\frac{f(x_1)-f(x_0)}{h}=\frac{\Delta f(x_0)}{h}$$

$$f[x_2,x_1,x_0]=\frac{f(x_2)-2f(x_1)+f(x_0)}{2!\,h^2}=\frac{\Delta^2 f(x_0)}{2!\,h^2}$$

$$f[x_3,x_2,x_1,x_0]=\frac{\Delta^3 f(x_0)}{3!\,h^3}$$

계속

$$E_t = \int_{x_0}^{x_2} \left[\frac{\Delta^3 f(x_0)}{3!\,h^3}(x-x_0)(x-x_1)(x-x_2) + \frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3) \right] dx$$

$\frac{x-x_0}{h} = \alpha$ 로 치환, $dx = h d\alpha$

$$= \int_0^2 \left[\frac{\Delta^3 f(x_0)}{3!\,h^3}\alpha h \cdot (\alpha-1)h \cdot (\alpha-2)h + \frac{f^{(4)}(\xi)}{4!}\alpha h \cdot (\alpha-1)h \cdot (\alpha-2)h \cdot (\alpha-3)h \right] h d\alpha$$

$$= \frac{\Delta^3 f(x_0)}{3!} h \underbrace{\int_0^2 \alpha(\alpha-1)(\alpha-2)\, d\alpha}_{0} + \frac{f^{(4)}(\xi)}{4!} h^5 \int_0^2 \alpha(\alpha-1)(\alpha-2)(\alpha-3)\, d\alpha$$

$$= \frac{f^{(4)}(\xi)}{4!} h^5 \left(-\frac{4}{15} \right)$$

$$= -\frac{1}{90} h^5 f^{(4)}(\xi)$$

Ex) Use the single application of Simpson's 1/3 rule to estimate the integral of $f(x) = 400x^5 - 900x^4 + 675x^3 - 200x^2 + 25x + 0.2$ from a=0 to b=0.8. Estimate the error. (참 적분 값은 1.640533)

$f(0) = 0.2$

$f(0.4) = 2.456$

$f(0.8) = 0.232$

$$I = (0.8)\frac{0.2 + \boxed{4}(2.456) + 0.232}{6} = 1.367467$$

$E_t = 1.640533 - 1.367467 = 0.2730667$ $\qquad E_a = -\frac{(0.8)^5}{2880}(-2400) = 0.2730667$

$f^{(4)}(x) = 48000x - 21600$

$$\bar{f}^{(4)}(x) = \frac{\int_0^{0.8}(48000x - 21600)dx}{0.8 - 0} = -2400$$

일반적으로, 근사오차라서 $\bar{f}^{(4)}(x)$ 과 $f^{(4)}(\xi)$ 가 일치하지 않음

☞사다리꼴 공식보다 더 정확

Composite Simpson's 1/3 Rule: 구간을 여러 조각(segment)으로

$$I=\int_{x_0}^{x_2} f(x)\,dx+\int_{x_2}^{x_4} f(x)\,dx+\cdots+\int_{x_{n-2}}^{x_n} f(x)\,dx$$

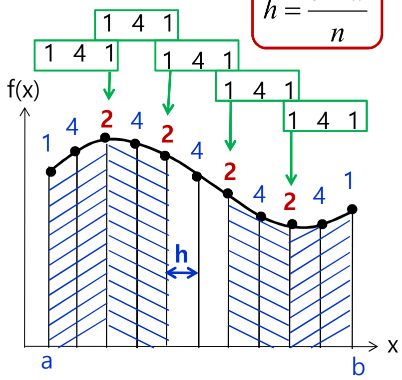
같은 간격의
even개 segments
(odd 개 점)

$$=2h\frac{f(x_0)+4f(x_1)+f(x_2)}{6}+2h\frac{f(x_2)+4f(x_3)+f(x_4)}{6}$$

$$+\cdots+2h\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}$$

$$h=\frac{b-a}{n}$$

$$\boxed{=\frac{h}{3}\left[f(x_0)+4\sum_{i=1,3,5}^{n-1}f(x_i)+2\sum_{j=2,4,6}^{n-2}f(x_j)+f(x_n)\right]}$$



- 근사오차

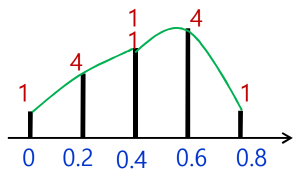
$$E_a=-\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}$$

$$\bar{f}^{(4)}=\frac{1}{n/2}\sum_{i=1}^{n/2}f^{(4)}(\xi_i)$$

Ex) Use the composite of Simpson's 1/3 rule with n=4 to estimate the integral of $f(x)=400x^5-900x^4+675x^3-200x^2+25x+0.2$ from a=0 to b=0.8. Estimate the error. (참 적분 값은 1.640533)

$f(0)=0.2$ $\quad f(0.2)=1.288$
$f(0.4)=2.456$ $\quad f(0.6)=3.464$
$f(0.8)=0.232$

→ n=4



$$I=(0.4)\frac{0.2+\boxed{4}(1.288+3.464)+\boxed{2}(2.456)+0.232}{6}=1.623467$$

$$E_t=1.640533-1.623467=\underline{0.017067} \qquad E_a=-\frac{(0.8)^5}{180(4)^4}\underbrace{(-2400)}_{\bar{f}^{(4)}}=\underline{0.017067}$$

일반적으로, 근사오차라서 $\bar{f}^{(4)}(x)$ 과 $f^{(4)}(\xi)$ 가 일치 하지는 않음

☞ single of Simpson's 1/3 rule 보다 오차 더 감소

③ **Simpson's 3/8 rule :** 네 점을 지나는 3차 Lagrange 다항식으로 interpolating 하여 적분, 등 간격의 odd 개 segments 일 때 유용

$$I=\int_{x_0}^{x_3} f_n(x)\,dx=\boxed{\frac{3h}{8}}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right]$$

3차 Lagrange 다항식

$$=(b-a)\frac{\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right]}{8}$$

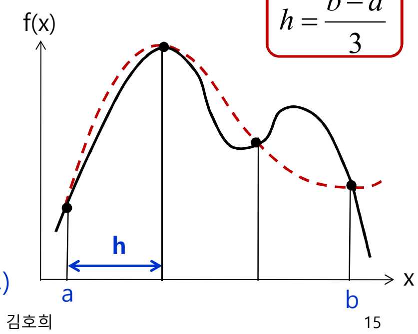
- 오차

$$E_t=-\frac{3}{80}h^5 f^{(4)}(\xi)$$

?

$$=-\frac{(b-a)^5}{6480}f^{(4)}(\xi)$$

$$h=\frac{b-a}{3}$$



☞ 3/8 공식이 1/3공식보다 조금 더 정확 하지만 1/3 공식이 더 선호 (1/3 공식은 세 점으로 같은 정확도이므로)

Ex) a) Use Simpson's 3/8 rule to integrate

$$f(x)=400x^5-900x^4+675x^3-200x^2+25x+0.2$$ from a=0 to b=0.8.

$$\left[\begin{array}{ll} f(0)=0.2 & f(0.2667)=1.432724 \\ f(0.5333)=3.487177 & f(0.8)=0.232 \end{array}\right]$$

(참 적분 값은 1.640533)

$$\rightarrow\quad I=(0.8)\frac{0.2+\boxed{3}(1.432724+3.487177)+0.232}{8}=1.51970$$

b) Use it in conjunction with Simpson's 1/3 rule to integrate the same function for five segments.

$$\left[\begin{array}{lll} f(0)=0.2 & f(0.16)=1.296919 & f(0.32)=1.743393 \\ f(0.48)=3.186015 & f(0.64)=3.181929 & f(0.8)=0.232 \end{array}\right]$$

$$I=\underbrace{(0.32)\frac{0.2+\boxed{4}(1.296919)+1.743393}{6}}_{\text{Simpson's 1/3 적분}}+\underbrace{(0.48)\frac{1.743393+\boxed{3}(3.186015+3.181929)+0.232}{8}}_{\text{Simpson's 3/8 적분}}$$

$=1.645077$ ☞ 등 간격의 odd 개 segments 일 때는 1/3과 3/8 을 결합

```
>> x = [0 .12 .22 .32 .36 .4 .44 .54 .64 .7 .8];
>> y = 0.2+25*x-200*x.^2+675*x.^3-900*x.^4+400*x.^5;
>> trapz(x,y)
ans =
    1.5948

>> cumtrapz(x,y)
ans =
  Columns 1 through 5
         0    0.0906    0.2213    0.3738    0.4501
  Columns 6 through 10
    0.5407    0.6467    0.9642    1.2987    1.4651
  Column 11
    1.5948
```

사다리꼴 적분 계산하는 built-in 함수

누적(cumulative) 적분 계산하는 built-in 함수

Newton-Cotes integration formulas

	I	E_t	h
Trapezoidal	$(b-a)\frac{f(a)+f(b)}{2}$	$-\frac{1}{12}f''(\xi)h^3$	$b-a$
Simpson's 1/3	$(b-a)\frac{f(x_0)+4f(x_1)+f(x_2)}{6}$	$-\frac{1}{90}h^5 f^{(4)}(\xi)$	$\frac{b-a}{2}$
Simpson's 3/8	$(b-a)\frac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}$	$-\frac{3}{80}h^5 f^{(4)}(\xi)$	$\frac{b-a}{3}$

예) $\int_{-1}^{2} x^3 + 2x^2 + x + 1\, dx = \frac{57}{4}$

Simpson's 1/3 : $3\frac{1+4 \cdot \frac{17}{8}+19}{6} = \frac{57}{4}$

Simpson's 3/8 : $3\frac{1+3 \cdot 1+3 \cdot 5+19}{8} = \frac{57}{4}$