17. Polynomial Interpolation

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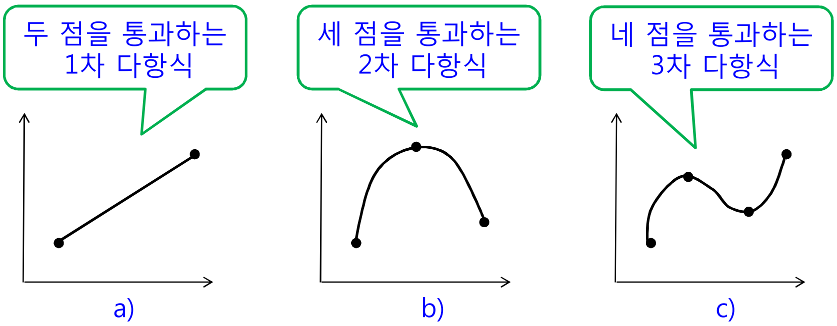
Polynomial Interpolation : 정확한 데이터들의 중간 값을 추측할 때

- n 개의 모든 점을 통과하는 (n-1)차 다항식은 유일하나

$f(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1}$ 를 표현하는 수학적 형식은 다양

① Newton Interpolating Polynomials

② Lagrange Interpolating Polynomials



Ex) Determine the coefficients of the parabola, $f(x) = p_1 x^2 + p_2 x + p_3$, that passes through the three points.

$$\begin{cases} x_1 = 300, & f(x_1) = 0.616 \\ x_2 = 400, & f(x_2) = 0.525 \\ x_3 = 500, & f(x_3) = 0.457 \end{cases}$$

→ 2차 다항식에 세 점을 각 각 대입

Vandermonde matrix →
ill-conditioned →
반올림오차에 민감
(방정식 수 많을 수록 더 심각)

$$\begin{bmatrix} 90000 & 300 & 1 \\ 160000 & 400 & 1 \\ 250000 & 500 & 1 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 0.616 \\ 0.525 \\ 0.457 \end{Bmatrix}$$

$$\boxed{\Rightarrow f(x) = 0.00000115x^2 - 0.001715x + 1.027}$$

```
>> A =[ 90000  300  1; 160000  400  1; 250000  500  1];
>> b = [0.616 0.525 0.457]';
>> p = A\b
 p =  0.00000115000000
     -0.00171500000000
      1.02700000000000
```

☞ 이런 방법은 부적합

①Newton Interpolating Polynomials : 가장 보편, 유용

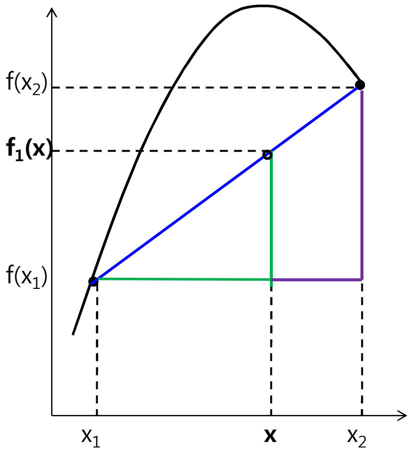
- **Linear Interpolation** : 두 점을 직선으로 연결하여 interpolation

닮은 꼴 삼각형에 의해

$$\frac{f_1(x) - f(x_1)}{(x - x_1)} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Newton's 1차 다항식

$$f_1(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$



☞ x_1~x_2 간격이 감소할수록 더 좋은 함수 근사값을 얻음

Ex) Estimate ln 2 using linear interpolation. First, perform the computation by interpolating between ln 1=0 and ln 6. Then, repeat the procedure, but use a smaller interval from ln 1 to ln 4.

(ln 2 의 참 값은 0.6931472)

- $\begin{cases} x_1 = 1, & f(x_1) = 0 \\ x_2 = 6, & f(x_2) = \ln 6 = 1.791759 \end{cases}$

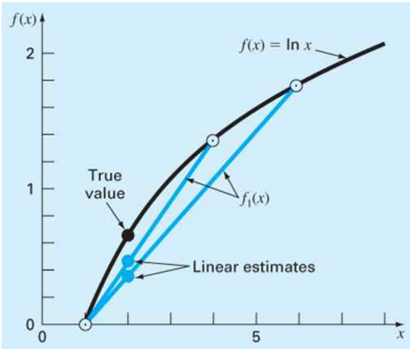
$$\rightarrow f_1(x) = 0 + \frac{1.791759 - 0}{6 - 1}(x - 1)$$

$$\rightarrow f_1(2) = 0.3583519$$

- $\begin{cases} x_1 = 1, & f(x_1) = 0 \\ x_2 = 4, & f(x_2) = \ln 4 = 1.386294 \end{cases}$

$$\rightarrow f_1(x) = 0 + \frac{1.386294 - 0}{4 - 1}(x - 1)$$

$$\rightarrow f_1(2) = 0.4620981$$



☞ 간격 감소하니 더 좋은 근사값을 얻음

- **Quadratic Interpolation** : 오차 향상을 위해 세 점 사이를 곡선 (2차 다항식)으로 연결하여 interpolation

$$f_2(x) = b_1 + b_2(x - x_1) + b_3(x - x_1)(x - x_2)$$

$(x_1, f(x_1))$ 을 대입 → $b_1 = f(x_1)$ → 위 다항식에 대입

$(x_2, f(x_2))$ 을 대입 → $b_2 = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ → 위 다항식에 대입

$(x_3, f(x_3))$ 을 대입 → $b_3 = \dfrac{\dfrac{f(x_3) - f(x_2)}{x_3 - x_2} - \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1}$

Taylor Series 와 비슷

Newton's 2차 다항식

$$f_2(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + \frac{\dfrac{f(x_3) - f(x_2)}{x_3 - x_2} - \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1}(x - x_1)(x - x_2)$$

Ex) Employ a second-order Newton polynomial to estimate ln 2 with the three points. (ln 2 의 참 값은 0.6931472)

$$\begin{cases} x_1 = 1, & f(x_1) = 0 \\ x_2 = 4, & f(x_2) = 1.386294 = \ln 4 \\ x_3 = 6, & f(x_3) = 1.791759 = \ln 6 \end{cases}$$

$$b_1 = 0$$

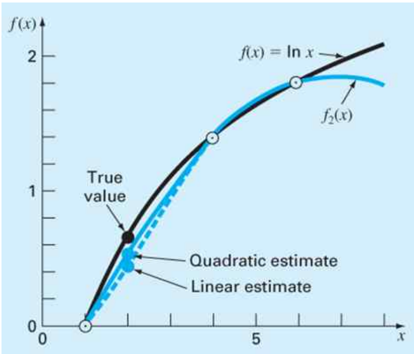
$$b_2 = \frac{1.386294 - 0}{4 - 1} = 0.4620981$$

$$b_3 = \frac{\dfrac{1.791759 - 1.386294}{6-4} - 0.4620981}{6-1}$$

$$= -0.0518731$$

$$\boxed{\begin{aligned} f_2(x) &= 0 + 0.4620981(x-1) \\ &\quad - 0.0518731(x-1)(x-4) \end{aligned}}$$

$$f_2(2) = 0.5658444$$



☞ Quadratic interpolation 이 Linear interpolation 보다 참 값에 더 근접

General form of Newton Interpolating Polynomials

- n 개의 점을 통과하는 (n-1) 차 Newton 다항식은

$$\begin{aligned} f_{n-1}(x) &= f(x_1) + f[x_2, x_1](x - x_1) + f[x_3, x_2, x_1](x - x_1)(x - x_2) \\ &\quad + \cdots + f[x_n, x_{n-1}, \cdots, x_2, x_1](x - x_1)(x - x_2)\cdots(x - x_{n-1}) \end{aligned}$$

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

$$f[x_n, x_{n-1}, \cdots, x_2, x_1] = \frac{f[x_n, x_{n-1}, \cdots, x_2] - f[x_{n-1}, x_{n-2}, \cdots, x_1]}{x_n - x_1}$$

☞ data들의 같은 간격 불필요, x 좌표 값의 오름차순 불필요

☞ higher-order difference 는 lower-order difference 로 구성(recursive)

Ex) Adding a fourth point (x_4=5, f(x_4)=1.609438) to the three points of the previous examples, estimate ln 2 with a third-order Newton polynomial.

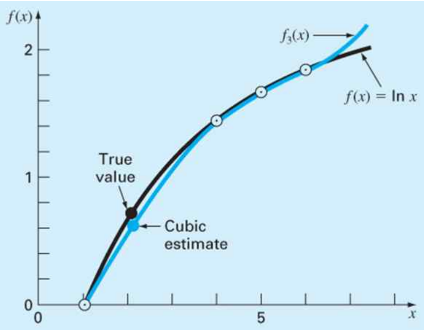
(ln 2 의 참 값은 0.6931472)

$$\begin{cases} x_1 = 1, & f(x_1) = 0 \\ x_2 = 4, & f(x_2) = 1.386294 = \ln 4 \\ x_3 = 6, & f(x_3) = 1.791759 = \ln 6 \\ x_4 = 5, & f(x_4) = 1.609438 = \ln 5 \end{cases}$$

$$f[x_3,x_2,x_1] = \frac{f[x_3,x_2]-f[x_2,x_1]}{6-1} = -0.05187311$$

$$f[x_4,x_3,x_2] = \frac{f[x_4,x_3]-f[x_3,x_2]}{5-4} = -0.02041100$$

$$b_4 = f[x_4,x_3,x_2,x_1] = \frac{f[x_4,x_3,x_2]-f[x_3,x_2,x_1]}{5-1} = 0.007865529$$



$$\boxed{\begin{aligned} f_3(x) = {} & 0 + 0.4620981(x-1) \\ & -0.05187311(x-1)(x-4) \\ & +0.007865529(x-1)(x-4)(x-6) \end{aligned}}$$

$$f_3(2) = 0.6287686$$

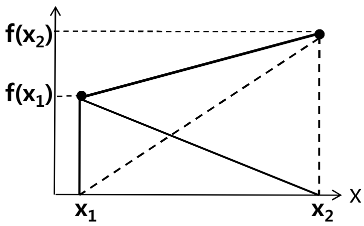
☞ Cubic interpolation 이 참 값에 더 근접

②Lagrange Interpolating Polynomials : finite divided difference 를 계산하지 않고 간단하게 만든 다항식

옆 그림에서 두 점을 지나는 직선은

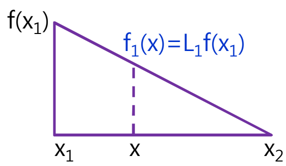
$$f_1(x) = L_1 f(x_1) + L_2 f(x_2)$$

(L : weighting factor)



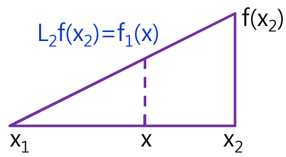
- L_2=0 → f(x_1) : L_1f(x_1) = (x_2–x_1) : (x_2–x)

$$\rightarrow \quad L_1 = \frac{x - x_2}{x_1 - x_2}$$



- L_1=0 → f(x_2) : L_2f(x_2) = (x_2–x_1) : (x–x_1)

$$\rightarrow \quad L_2 = \frac{x - x_1}{x_2 - x_1}$$



- 2 개의 점 $(x_1, f(x_1)), (x_2, f(x_2))$ 을 지나는

$$f_1(x) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

Lagrange 1차 다항식

- 3 개의 점 $(x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3))$ 을 지나는

$$f_2(x) = \frac{(x - x_3)(x - x_2)}{(x_1 - x_3)(x_1 - x_2)} f(x_1) + \frac{(x - x_3)(x - x_1)}{(x_2 - x_3)(x_2 - x_1)} f(x_2) + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

Lagrange 2차 다항식

- n 개의 점을 지나는 (n-1) 차 Lagrange 다항식은

$$f_{n-1}(x_i) = \sum_{i=1}^{n} L_i(x) f(x_i) \quad , \quad L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Ex) Use a Lagrange interpolating polynomial of the first and second order to evaluate f(x) at x=15 based on the following data:

$$\begin{cases} x_1 = 0, & f(x_1) = 3.85 \\ x_2 = 20, & f(x_2) = 0.800 \\ x_3 = 40, & f(x_3) = 0.212 \end{cases}$$

- Lagrange 1차 다항식

$$f_1(x) = \frac{x - 20}{0 - 20} 3.85 + \frac{x - 0}{20 - 0} 0.800 \quad \longrightarrow \quad \underline{f_1(15) = 1.5625}$$

- Lagrange 2차 다항식

$$f_2(x) = \frac{(x - 40)(x - 20)}{(0 - 40)(0 - 20)} 3.85 + \frac{(x - 40)(x - 0)}{(20 - 40)(20 - 0)} 0.800 + \frac{(x - 0)(x - 20)}{(40 - 0)(40 - 20)} 0.212$$

$$\longrightarrow \quad \underline{f_2(15) = 1.3316875}$$