Unit 3. Boolean Algebra II

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Objectives

Topics introduced in this chapter

- ⇨ Apply Boolean laws and theorems to manipulation of expression
- ⇨ Simplifying
- ⇨ Finding the complement
- ⇨ Multiplying out and factoring
- ⇨ Exclusive-OR and Equivalence operation (Exclusive-NOR)
- ⇨ Consensus theorem

Multiplying Out and Factoring Expressions

To obtain a sum-of-product form ➔ Multiplying out using distributive laws

$$X(Y+Z) = XY + XZ$$

$$(X+Y)(X+Z) = X + YZ$$

Theorems for multiplying out

$$(X+Y)(X'+Z) = XZ + X'Y \qquad (3\text{-}3)$$

$$\text{If } X = 0, (3\text{-}3) \text{ reduces to } Y(1+Z) = 0 + 1 * Y \text{ or } Y = Y.$$

$$\text{If } X = 1, (3\text{-}3) \text{ reduces to } (1+Y)Z = Z + 0 * Y \text{ or } Z = Z.$$

$$\text{because the equation is valid for both } X = 0 \text{ and } X = 1, \text{ it is always valid.}$$

$$\text{The following example illustrates the use of Theorem } (3\text{-}3) \text{ for factoring:}$$

Theorems for factoring

$$AB + A'C = (A+C)(A'+B)$$

Multiplying Out and Factoring Expressions

- **Theorems for multiplying out**

$$(Q+AB')(C'D+Q')=QC'D+Q'AB'$$

- **Multiplying out using distributed laws**

$$(Q+AB')(C'D+Q')=QC'D+\boxed{QQ'+AB'C'D}+AB'Q'$$

Redundant terms

- **Multiplying out : (1) distributed laws, (2) theorem (3-3)**

$$(A+B+C')(A+B+D)(A+B+E)(A+D'+E)(A'+C)$$
$$=(A+B+C'D)(A+B+E)[AC+A'(D'+E)]$$
$$=(A+B+C'DE)(AC+A'D'+A'E)$$
$$=AC+\cancel{ABC}+A'BD'+A'BE+A'C'DE$$

What theorem was applied to eliminate ABC?

Multiplying Out and Factoring Expressions

To obtain a product-of-sum form ➔ Factoring using distributive laws

Theorems using factoring

$$\underbrace{A\overbrace{B+A'}C}=(A+C)(A'+B)$$

Example of factoring

$$
\begin{aligned}
& AC+A'BD'+A'BE+A'C'DE \\
&=\underbrace{AC}_{XZ}+\underbrace{A'}_{X'}(\underbrace{BD'+BE+C'DE}_{Y}) \\
&=(A+BD'+BE+C'DE)(A'+C) \\
&=[\underbrace{A+C'DE}_{X}+\underbrace{B}_{Y}(\underbrace{D'+E}_{Z})](A'+C) \\
&=(A+B+C'DE)(A+\cancel{C'DE}+D'+E)(A'+C) \\
&=(A+B+C')(A+B+D)(A+B+E)(A+D'+E)(A'+C)
\end{aligned}
$$

Exclusive-OR and Equivalence Operations

- **Exclusive-OR**

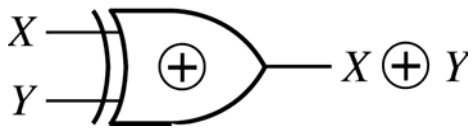
$$0 \oplus 0 = 0 \qquad 0 \oplus 1 = 1$$

$$1 \oplus 0 = 1 \qquad 1 \oplus 1 = 0$$

- **Truth Table**

XY	$X \oplus Y$
0 0	0
0 1	1
1 0	1
1 1	0

- **Symbol**



Exclusive-OR and Equivalence Operations

- **Theorems for Exclusive-OR:**

$$X \oplus Y = X'Y + XY'$$

Because $X \oplus Y = 1$ iff X is 0 and Y is 1 or X is 1 and Y is 0

$$X \oplus 0 = X$$

$$X \oplus 1 = X'$$

$$X \oplus X = 0$$

$$X \oplus X' = 1$$

$$X \oplus Y = Y \oplus X \text{ (commutative law)}$$

$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) = X \oplus Y \oplus Z \text{ (associative law)}$$

$$X(Y \oplus Z) = XY \oplus XZ \text{ (distributive law)}$$

$$(X \oplus Y)' = X \oplus Y' = X' \oplus Y = XY + X'Y'$$

Exclusive-OR and Equivalence Operations

- **Equivalence operation (Exclusive-NOR)**

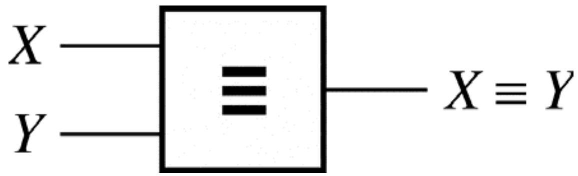
$$(0 \equiv 0) = 1 \qquad (0 \equiv 1) = 0$$

$$(1 \equiv 0) = 0 \qquad (1 \equiv 1) = 1$$

- **Truth Table**

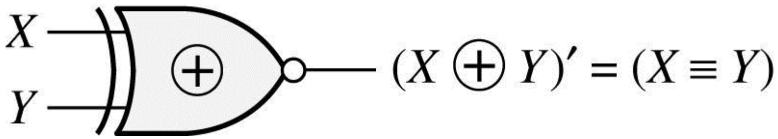
XY	$X \equiv Y$
0 0	1
0 1	0
1 0	0
1 1	1

- **Symbol**



Exclusive-OR and Equivalence Operations

- **Exclusive-NOR**



- **Example of EXOR and Equivalence**

$$F = (A'B \equiv C) + (B \oplus AC')$$

$$\begin{aligned}
F &= [(A'B)C + (A'B)'C'] + [B'(AC') + B(AC')'] \\
&= A'BC + (A + B')C' + AB'C' + B(A' + C) \\
&= B(A'C + A' + C) + C'(A + B' + AB') = B(A' + C) + C'(A + B')
\end{aligned}$$

- **Useful theorem**

$$(XY' + X'Y)' = XY + X'Y' \qquad \text{(3-19)}$$

$$\begin{aligned}
A' \oplus B \oplus C &= [A'B' + (A')'B] \oplus C \\
&= (A'B' + AB)C' + (A'B' + AB)'C && \text{(by (3-6))} \\
&= (A'B' + AB)C' + (A'B + AB')C && \text{(by (3-19))} \\
&= A'B'C' + ABC' + A'BC + AB'C
\end{aligned}$$

The Consensus Theorem

- **Consensus Theorem :** $XY + X'Z + YZ = XY + X'Z$

- **Proof :** $XY + X'Z + YZ = XY + X'Z + (X + X')YZ$

$$= (XY + XYZ) + (X'Z + X'YZ)$$

$$= XY(1+Z) + X'Z(1+Y) = XY + X'Z$$

- **Example :** $a'b' + ac + bc' + b'c + ab = a'b' + ac + bc'$

consensus

consensus

- **Dual form of consensus theorem :**

$$(X+Y)(X'+Z)(Y+Z) = (X+Y)(X'+Z)$$

- **Example :** $(a+b+c')(a+b+d')(b+c+d') = (a+b+c')(b+c+d')$

The Consensus Theorem (Cont'd)

- **Example: eliminate BCD**

$$A'C'D + A'BD + \cancel{BCD} + ABC + ACD'$$

- **Example: eliminate A'BD, ABC**

$$A'C'D + \cancel{A'BD} + BCD + \cancel{ABC} + ACD'$$

- **Example: Reducing an expression by adding a term**

$$F = ABCD + B'CDE + A'B' + BCE'$$

$$F = ABCD + B'CDE + A'B' + BCE' + \boxed{ACDE}$$

Consensus term added

- **Final expression:**

$$F = A'B' + BCE' + ACDE$$

Algebraic Simplification of Switching Expressions

- **Combining terms:** $XY + XY' = X$

  Example: $abc'd' + abcd' = abd' \quad [X = abd', Y = c]$

- **Adding terms using X + X = X**

  $$ab'c + abc + a'bc = ab'c + abc + abc + a'bc = ac + bc$$

  Example: $(a + bc)(d + e') + a'(b' + c')(d + e') = d + e'$

  $[X = d + e', Y = a + bc, Y' = a'(b' + c')]$

- **Eliminating terms:** $X + XY = X$

  Example:

  $a'b + a'bc = a'b \quad [X = a'b]$

  $a'bc' + bcd + a'bd = a'bc' + bcd \quad [X = c, Y = bd, Z = a'b]$

Algebraic Simplification of Switching Expressions

- **Eliminating literals:** $X + X'Y = X + Y$

Example:

$$
\begin{aligned}
A'B + A'B'C'D' + ABCD' &= A'(B + B'C'D') + ABCD' \\
&= A'(B + C'D') + ABCD' \\
&= B(A' + ACD') + A'C'D' \\
&= B(A' + CD') + A'C'D' \\
&= A'B + BCD' + A'C'D'
\end{aligned}
$$

- **Adding redundant terms:**

Example:

$WX + XY + X'Z' + WY'Z'$ (add WZ' by consensus theorem)

$= WX + XY + X'Z' + WY'Z' + WZ'$ (eliminate $WY'Z'$)

$= WX + XY + X'Z' + WZ'$ (eliminate WZ')

$= WX + XY + X'Z'$

Proving Validity of an Equation

Proving an equation valid

- ⇨ Construct a truth table and evaluate both sides
  - ⇨ Tedious, not elegant method
- ⇨ Manipulate one side by applying theorems until it is the same as the other side
- ⇨ Reduce both sides of the equation independently
- ⇨ Apply same operation in both sides if the operation is reversible
  - ⇨ Complement both sides etc
  - ⇨ not permissible: add terms, multiply terms

Proving Validity of an Equation

Strategy to prove equation valid

1. First reduce both sides to SOP (or POS)
2. Compare the two sides of the equation to see how they differ
3. Then try to add terms to one side of the equation that are present on the other side
4. Finally, try to eliminate terms form one side that are not present on the other

Proving Validity of an Equation

- **Prove:**

$$A'BD' + BCD + ABC' + AB'D = BC'D' + AD + A'BC$$

$$= A'BD' + BCD + ABC' + AB'D + BC'D' + A'BC + ABD$$

(add consensus of *A'BD'* and *ABC'*)
(add consensus of *A'BD'* and *BCD*)
(add consensus of *BCD* and *ABC'*)

$$= AD + A'BD' + BCD + ABC' + BC'D' + A'BC = BC'D' + AD + A'BC$$

(eliminate consensus of *BC'D'* and *AD*)
(eliminate consensus of *AD* and *A'BC*)
(eliminate consensus of *BC'D'* and *A'BC*)

Proving Validity of an Equation

- **Some of Boolean Algebra are not true for ordinary algebra**

Example: If $x + y = x + z,$ then $y = z$ — **True in ordinary algebra**

$1 + 0 = 1 + 1$ but $0 \neq 1$ — **Not True in Boolean algebra**

Example: If $xy = xz,$ then $y = z$ — **True in ordinary algebra** / **Not True in Boolean algebra**

Example: If $y = z,$ then $x + y = x + z$ — **True in ordinary algebra**

If $y = z,$ then $xy = xz$ — **True in Boolean algebra**