Unit 6. Quine-McCluskey Method

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Objectives

1. Find the prime implicants of a function by using the Quine-McCluskey method.

2. Define prime implicants and essential prime implicants

3. Given the prime implicants, find the essential prime implicants and a minimum sum-of-products expression for a function, using a prime implicant chart and using Petrick method

4. Minimize an incompletely specified function, using the Quine-McCluskey method

5. Find a minimum sum-of-products expression for a function, using the method of map-entered variables

Quine-McClusky Method

- Systematic simplification procedure which can be programmed for a digital computer.
- Function must be given as a sum of minterms
- Eliminate as many literals as possible from each term by systematically applying XY + XY' = X
- Use a prime implicant chart to select a minimum set of prime implicants

Determination of Prime Implicants

- The minterms are represented in binary notation and combined using

$$XY + XY' = X$$

- The binary notation and its algebraic equivalent

$$AB'CD' + AB'CD = AB'C$$

$$\underbrace{1\ 0}_{X}\ \underbrace{1\ 0}_{Y} + \underbrace{1\ 0}_{X}\ \underbrace{1\ 1}_{Y'} = \underbrace{1\ 0}_{X}\ 1\ \text{-}$$

-- (the dash indicates a missing variable)

$$A'BC'D + A'BCD'$$

(will not combine)

$$0\ 1\ 0\ 1 + 0\ 1\ 1\ 0$$

Determination of Prime Implicants

- The binary minterms are sorted into groups

$$f(a,b,c,d) = \sum m(0,1,2,5,6,7,8,9,10,14)$$

Is represented by the following list of minterms:

group 0	0	0000
group 1	1	0001
	2	0010
	8	1000
group 2	5	0101
	6	0110
	9	1001
	10	1010
group 3	7	0111
	14	1110

Determination of Prime Implicants

$$f(a,b,c,d)=\sum m(0,1,2,5,6,7,8,9,10,14)$$

Table 7–1

Determination of Prime Implicants

Column I			Column II		Column III	
group 0	0	0000 ✓	0, 1	000– ✓	0, 1, 8, 9	–00–
group 1	1	0001 ✓	0, 2	00–0 ✓	0, 2, 8, 10	–0–0
	2	0010 ✓	0, 8	–000 ✓	~~0, 8, 1, 9~~	~~–00–~~
	8	1000 ✓	1, 5	0–01	~~0, 8, 2, 10~~	~~–0–0~~
group 2	5	0101 ✓	1, 9	–001 ✓	2, 6, 10, 14	––10
	6	0110 ✓	2, 6	0–10 ✓	~~2, 10, 6, 14~~	~~––10~~
	9	1001 ✓	2, 10	–010 ✓		
	10	1010 ✓	8, 9	100– ✓		
group 3	7	0111 ✓	8, 10	10–0 ✓		
	14	1110 ✓	5, 7	01–1		
			6, 7	011–		
			6, 14	–110 ✓		
			10, 14	1–10 ✓		

Determination of Prime Implicants

1. Sort the binary minterms into groups according to the number of 1's in each term

2. Compare the terms in adjacent groups. Each time a term is combined with another term, it is checked off

3. Compare terms which have dashes in corresponding places and which differ by exactly one in the number of 1's

4. Keep comparing terms and forming new groups of terms and new columns until no further terms can be combined

5. Terms which have not been checked off – prime implicant

Determination of Prime Implicants

The function is equal to the sum of its prime implicants

$$f = \underset{(1,5)}{a'c'd} + \underset{(5,7)}{a'bd} + \underset{(6,7)}{a'bc} + \underset{(0,1,8,9)}{b'c'} + \underset{(0,2,8,10)}{b'd'} + \underset{(2,7,10,14)}{cd'}$$

Using the consensus theorem to eliminate redundant terms yields

$$f = a'bd + b'c' + cd'$$

Definition: Given a function F of n variables, a product term P is an implicant of F iff for every combination of values of the n variables for which P=1, F is also equal to 1.

Definition: A Prime implicant of a function F is a product term implicant which is no longer an implicant if any literal is deleted from it.

The Prime Implicant Chart

		0	1	2	5	6	7	8	9	10	14
$(0,1,8,9)$	$b'c'$	x	x					x	$\otimes$		
$(0,2,8,10)$	$b'd'$	x		x				x		x	
$(2,6,10,14)$	cd'			x		x				x	$\otimes$
$(1,5)$	$a'c'd$		x		x						
$(5,7)$	$a'bd$				x		x				
$(6,7)$	$a'bc$					x	x				

- If a given column contains only one X, the corresponding row is an EPI
- Each time a prime implicant is selected for inclusion in the minimum sum, the corresponding row & columns should be crossed out (see next slide)

Determination of Prime Implicants

The resulting chart (Table 6-3)

		0	1	2	5	6	7	8	9	10	14
$(0,1,8,9)$	$b'c'$	x	x					x	x		
$(0,2,8,10)$	$b'd'$	x		x				x		x	
$(2,6,10,14)$	cd'			x		x				x	x
$(1,5)$	$a'c'd$		x		x						
$(5,7)$	$a'bd$				x		x				
$(6,7)$	$a'bc$					x	x				

- a'bd is essential?

The resulting minimum sum of products is

$$f = b'c' + cd' + a'bd$$

Determination of Prime Implicants

Example: **cyclic prime implicants** (two more X's in every column in chart)

$$F = \sum m(0, 1, 2, 5, 6, 7)$$

Derivation of prime implicants

0	000	✓		0,1	00-
1	001	✓		0,2	0-0
2	010	✓		1,5	-01
5	101	✓		2,6	-10
6	110	✓		5,7	1-1
7	111	✓		6,7	11-

Determination of Prime Implicants

The resulting prime implicant chart (Table 6-4)

				0	1	2	5	6	7
①	→	(0,1)	$a'b'$	x	x				
		(0,2)	$a'c'$	x		x			
		(1,5)	$b'c$		x		x		
②	→	(2,6)	bc'			x		x	
③	→	(5,7)	ac				x		x
		(6,7)	ab					x	x

One solution:

$$F = a'b' + bc' + ac$$

Determination of Prime Implicants

Again starting with the other prime implicant that covers column 0.

The resulting table (Table 6-5)

			0	1	2	5	6	7
P_1	(0,1)	$a'b'$	x	x				
P_2	(0,2)	$a'c'$	x		x			
P_3	(1,5)	$b'c$		x		x		
P_4	(2,6)	bc'			x		x	
P_5	(5,7)	ac				x		x
P_6	(6,7)	ab					x	x

Finish the solution and show that

$$F = a'c' + b'c + ab.$$

Notice

- Compare the previous two solutions with Fig. 5-9
- Each minterm can be covered by two different loops vs. each minterm can be covered by two different prime implicants

Petrick’ s Method

- Technique to determine all minimum SOP solutions from a PI chart
- As the # of variables increases, the complexity of the PI chart increases significantly
- Before applying Petrick's method, all EPI's and the minterms they cover should be removed from the chart

Petrick's Method

1. Reduce the PI chart by eliminating EPI rows & corresponding columns
2. Label the rows of the reduced PI chart P_1, P_2, P_3, etc
3. Form a logic function P
4. Reduce P to a minimum SOP by multiplying out and applying $(X+Y)(X+Z)=X+YZ$ and $X+XY=X$
5. Each term represents a solution
6. Choose the term which has minimum total number of literals

Petrick's Method

- A technique for determining all minimum SOP solution from a PI chart

			0	1	2	5	6	7
P_1	(0,1)	$a'b'$	x	x				
P_2	(0,2)	$a'c'$	x		x			
P_3	(1,5)	$b'c$		x		x		
P_4	(2,6)	bc'			x		x	
P_5	(5,7)	ac				x		x
P_6	(6,7)	ab					x	x

Because we must cover all of the minterms, the following function must be true:

$$P = (P_1 + P_2)(P_1 + P_3)(P_2 + P_4)(P_3 + P_5)(P_4 + P_6)(P_5 + P_6) = 1$$

minterm0 minterm1 ....

Petrick's Method

- Reduce P to a minimum SOP

First, we multiply out, using (X+Y)(X+Z) = X+YZ and the ordinary Distributive law

$$\begin{aligned} P &= (P_1 + P_2P_3)(P_4 + P_2P_6)(P_5 + P_3P_6) \\ &= (P_1P_4 + P_1P_2P_6 + P_2P_3P_4 + P_2P_3P_6)(P_5 + P_3P_6) \\ &= P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_2P_3P_5P_6 + P_1P_3P_4P_6 \\ &\quad + P_1P_2P_3P_6 + P_2P_3P_4P_6 + P_2P_3P_6 \end{aligned}$$

Use X+XY=X to eliminate redundant terms from P

$$P = P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_1P_3P_4P_6 + P_2P_3P_6$$

- Choose *P1,P4,P5* or *P2,P3,P6* for minimum solution

$F = a'b' + bc' + ac$ or $F = a'c' + b'c + ab.$

Simplification of Incompletely Specified Functions

- How to modify Q-M when don't care terms are present?
  - To find the prime implicants, we treat X's as if they are required minterms
  - When forming the prime implicant chart, X's are not listed at the top

Simplification of Incompletely Specified Functions

Example: $F(A,B,C,D)=\sum m(2,3,7,9,11,13)+\sum d(1,10,15)$

Don't care terms are treated like required minterms…

1	0001 ✓	(1,3)	00-1 ✓	(1,3,9,11)	-0-1
2	0010 ✓	(1,9)	-001 ✓	(2,3,10,11)	-01-
3	0011 ✓	(2,3)	001- ✓	(3,7,11,15)	--11
9	1001 ✓	(2,10)	-010 ✓	(9,11,13,15)	1--1
10	1010 ✓	(3,7)	0-11 ✓		
7	0111 ✓	(3,11)	-011 ✓		
11	1011 ✓	(9,11)	10-1 ✓		
13	1101 ✓	(9,13)	1-01 ✓		
15	1111 ✓	(10,11)	101- ✓		
		(7,15)	-111 ✓		
		(11,15)	1-11 ✓		
		(13,15)	11-1 ✓		

Simplification of Incompletely Specified Functions

Don’t care columns are omitted when forming the PI chart…

	2	3	7	9	11	13
(1, 3, 9, 11)		x		x	x	
* (2, 3, 10, 11)	x	x			x	
* (3, 7, 11, 15)		x	x		x	
* (9, 11, 13, 15)				x	x	x

$$F = B'C + CD + AD$$

Replace each term in the final expression for F

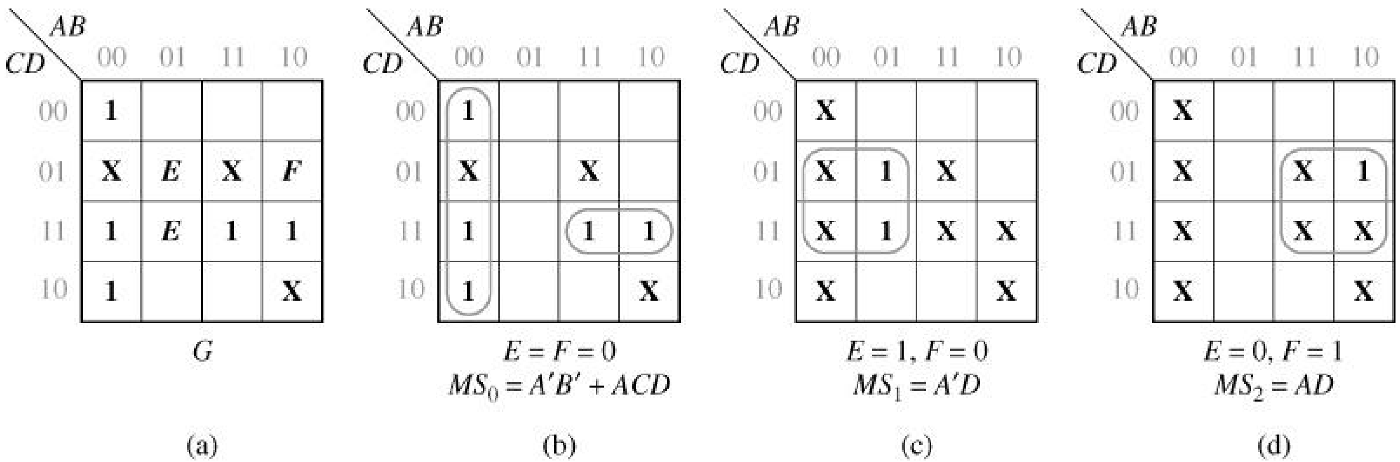
$$F = (m_2 + m_3 + m_{10} + m_{11}) + (\cancel{m_3} + m_7 + \cancel{m_{11}} + m_{15}) + (m_9 + \cancel{m_{11}} + m_{13} + \cancel{m_{15}})$$

The don’t care terms in the original truth table for F

$$for \quad ABCD = 0001,\ F = 0; \qquad for\ 1010, F = 1; \qquad for\ 1111,\ F = 1$$

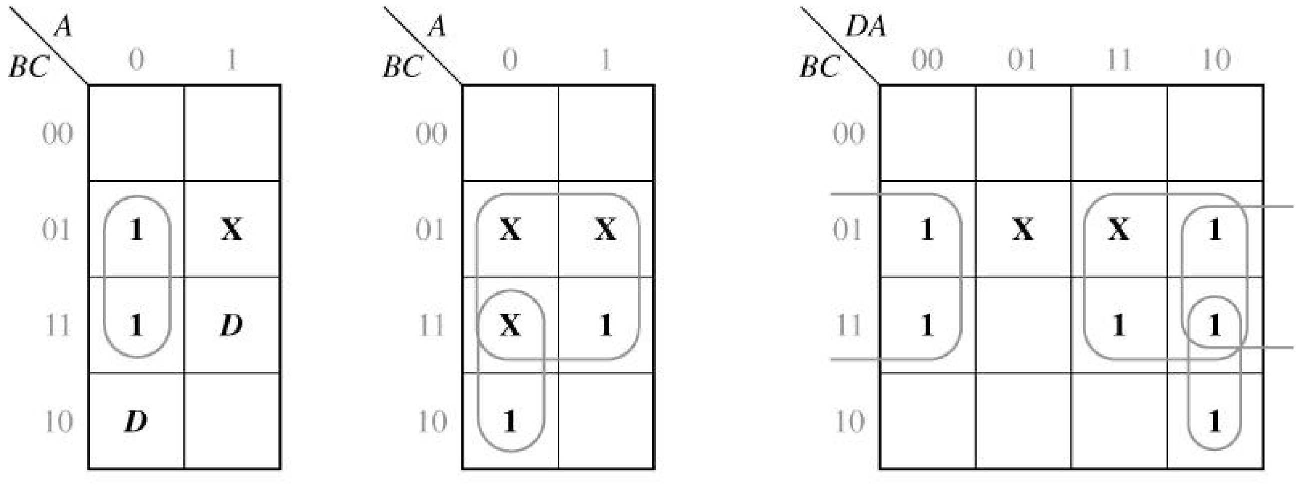
Simplification Using Map-Entered Variables

- Quine-McCluskey method : not efficient for functions which have many variables & few terms
- Extend K-map : to have variables in the map entry
- Fig 6-1 : 4 variable map with two additional variables entered in the map
- When E appears in the map
  - if E = 1, the corresponding minterm is present
  - if E = 0, the minterm is absent
- $G = m_0 + m_2 + m_3 + Em_5 + Em_7 + Fm_9 + m_{11} + m_{15}$ + (don't care terms)



Simple Example

- F = A'B'C + A'BC + A'BC'D + ABCD + (AB'C)
- Choose D as a map-entered variable. Why?
- How to solve?
  - when D = 0: Fig 6-2(a)
    F = A'C
  - when D = 1: Two 1's already covered, they are changed to X's. (Fig 6-2(b))
    F = C + A'B
  - Combining these two
    F = A'C + D(C+A'B) = A'C + CD + A'BD
  - verify using 4 variable map



(a) (b) (c)

General Method

- If a variable P_i is placed in square m_j of a map of a function F, this means that F = 1 when $P_i = 1$ and the variables are chosen so that $m_j = 1$
- $F = MS_0 + P_1MS_1 + P_2MS_2 + ...$
  where
  - MS_0 : minimum sum obtained by setting $P_1=P_2=... = 0$
  - MS_1 : minimum sum obtained by setting $P_1=1$, $P_j=0(j!=1)$, and replacing all 1's on the map with don't cares
  - MS_2 : minimum sum obtained by setting $P_2=1$, $P_j=0(j!=2)$, and replacing all 1's on the map with don't cares
  - .....

Example

- $G = m_0 + m_2 + m_3 + Em_5 + Em_7 + Fm_9 + m_{11} + m_{15}$ + (don't care terms)
- $G = MS_0 + P_1MS_1 + P_2MS_2 + \ldots$
  $= A'B'+ACD+EA'D+FAD$
- Verify the result using 6 variable map at home