Unit 2. Boolean Algebra

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School of Electrical Engineering

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Objectives

Topics introduced in this chapter

- ⇨ Understand the basic operations and laws of Boolean algebra
- ⇨ Relate these operations and laws to AND, OR, NOT gates and switches
- ⇨ Prove these laws using a truth table
- ⇨ Manipulation of algebraic expression using
  - ⇨ Multiplying out
  - ⇨ Factoring
  - ⇨ Simplifying
  - ⇨ Finding the complement of an expression

Introduction

- ⇨ Basic mathematics for logic design: Boolean algebra
- ⇨ Restrict to switching circuits (Two state values 0, 1) – Switching algebra
- ⇨ Boolean Variable : X, Y, … can only have two state values (0, 1)
  - ⇨ representing True(1) False (0)

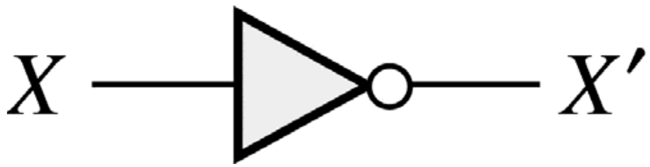
Basic Operations

- **Not (Inverter)**

$$0' = 1 \text{ and } 1' = 0$$

$$X' = 1 \text{ if } X = 0 \text{ and } X' = 0 \text{ if } X = 1$$

- **Gate Symbol**



Basic Operations

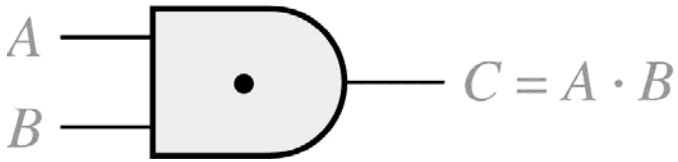
- **AND**

$$0 \cdot 0 = 0,\ \ 0 \cdot 1 = 0,\ \ 1 \cdot 0 = 0,\ \ 1 \cdot 1 = 1$$

- **Truth Table**

A B	C = A · B
0 0	0
0 1	0
1 0	0
1 1	1

- **Gate Symbol**



Basic Operations

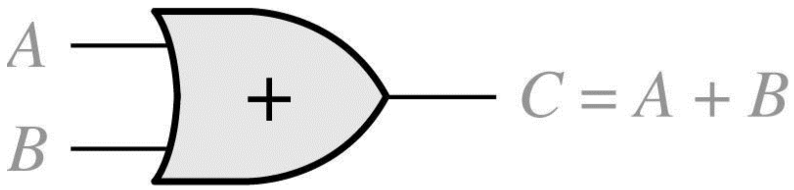
- **OR**

$$0+0=0,\ \ 0+1=1,\ 1+0=1,\ 1+1=1$$

- **Truth Table**

A B	C = A · B
0 0	0
0 1	1
1 0	1
1 1	1

- **Gate Symbol**

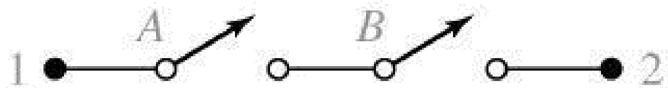


Basic Operations

Apply to Switch

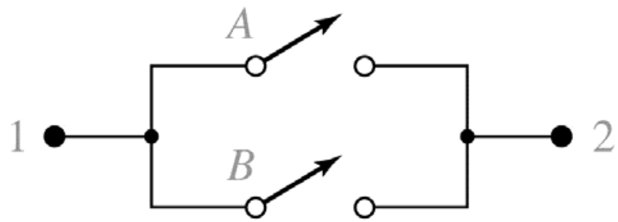
$X = 0 \rightarrow$ switch open
$X = 1 \rightarrow$ switch closed

AND T=A·B



$T = 0 \rightarrow$ open circuit between terminals 1 and 2
$T = 1 \rightarrow$ closed circuit between terminals 1 and 2

OR T=A+B

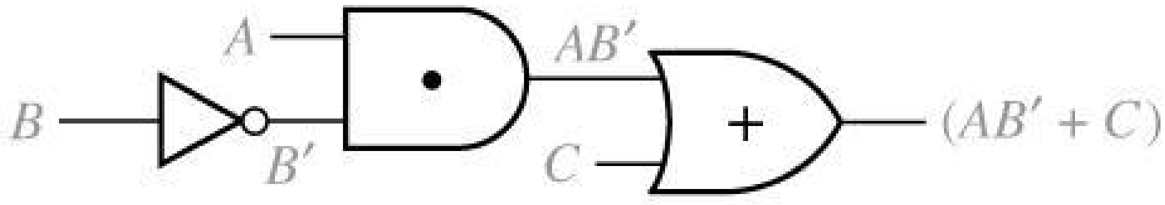


Boolean Expressions and Truth Tables

- **Logic Expression :**

$$(AB'+C)$$

- **Circuit of Logic**

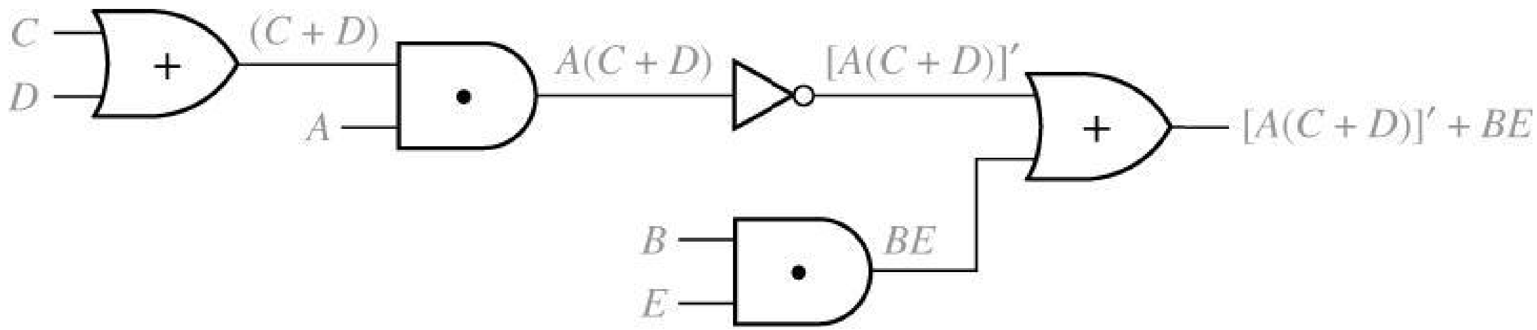


Boolean Expressions and Truth Tables

- **Logic Expression :**

$$[A(C+D)]'+BE$$

- **Circuit of Logic**

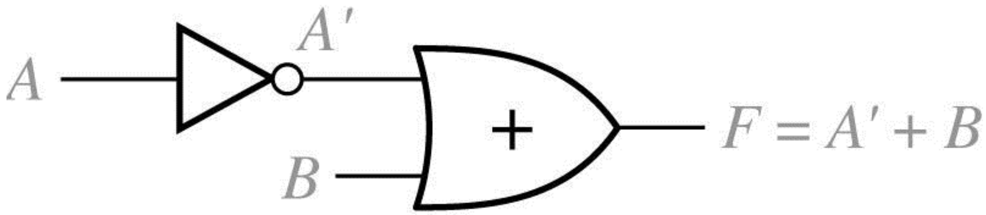


- **Logic Evaluation : A=B=C=1, D=E=0**

$$[A(C+D)]'+BE = [1(1+0)]'+1\cdot 0 = [1(1)]'+0 = 0+0 = 0$$

Boolean Expressions and Truth Tables

2-Input Circuit and Truth Table



A B	A’	F = A’ + B
0 0	1	1
0 1	1	1
1 0	0	0
1 1	0	1

Boolean Expressions and Truth Tables

- **Proof using Truth Table**

$$AB'+C = (A+C)(B'+C)$$

n variable needs $\underbrace{2x2x2x}_{\text{n times}}\cdots = 2^n$ rows

A B C	B'	AB'	AB' + C	A + C	B' + C	(A + C) (B' + C)
0 0 0	1	0	0	0	1	0
0 0 1	1	0	1	1	1	1
0 1 0	0	0	0	0	0	0
0 1 1	0	0	1	1	1	1
1 0 0	1	1	1	1	1	1
1 0 1	1	1	1	1	1	1
1 1 0	0	0	0	1	0	0
1 1 1	0	0	1	1	1	1

Basic Theorems

- **Operations with 0, 1** $X+0=X$ $X\cdot 1=X$

  $X+1=1$ $X\cdot 0=0$
- **Idempotent Laws** $X+X=X$ $X\cdot X=X$
- **Involution Laws** $(X')'=X$
- **Complementary Laws** $X+X'=1$ $X\cdot X'=0$

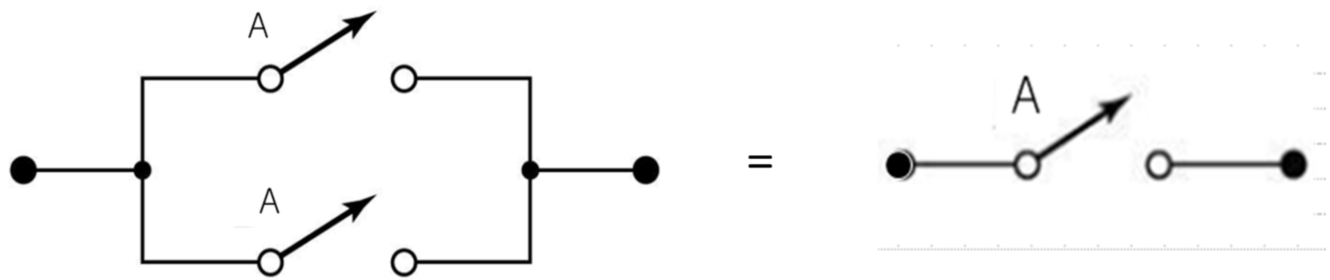
- **Proof** $X=0,\quad 0+0'=0+1,\quad \text{and if } X=1,\quad 1+1'=1+0=1$

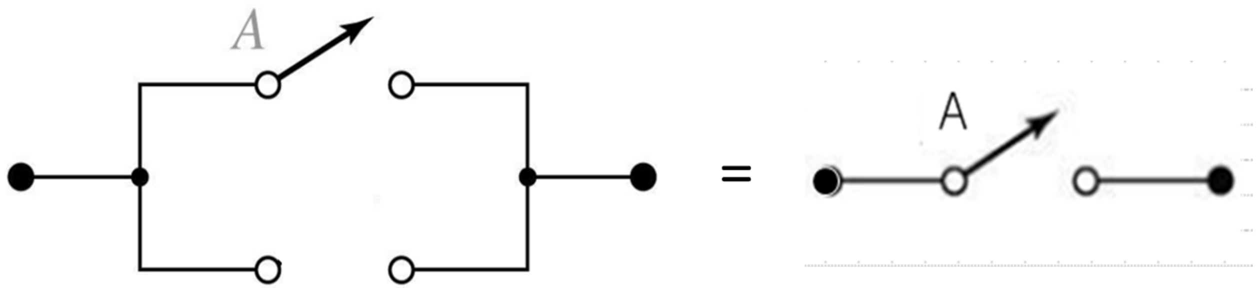
- **Examples**

$$(AB'+D)E+1=1$$

$$(AB'+D)(AB'+D)'=0$$

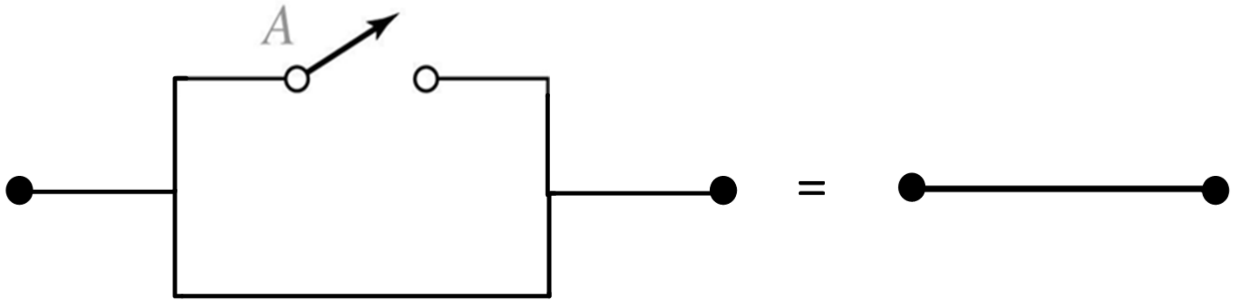
Basic Theorems with Switch Circuits



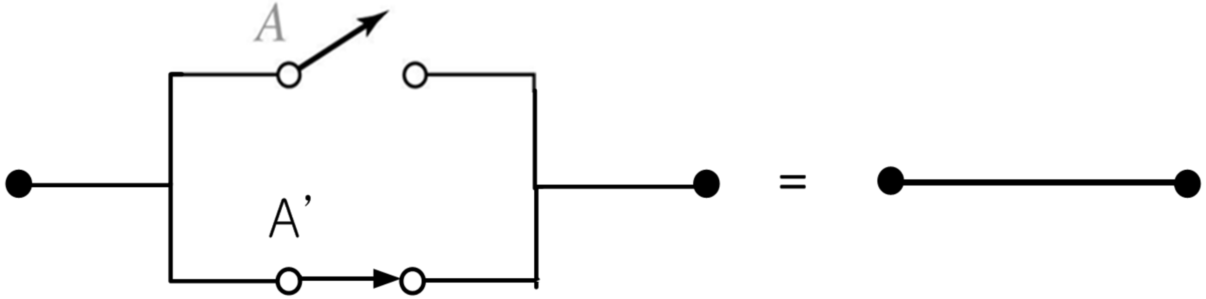


(A+0=A)

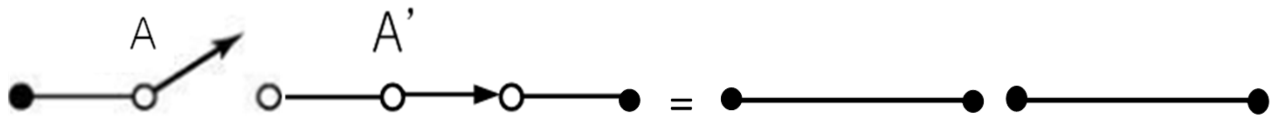
Basic Theorems with Switch Circuits



(A+1=1)



(A+A'=1)



(A A'=0)

Commutative, Associative, Distributive

- **Commutative Laws :** $XY = YX, \quad X + Y = Y + X$

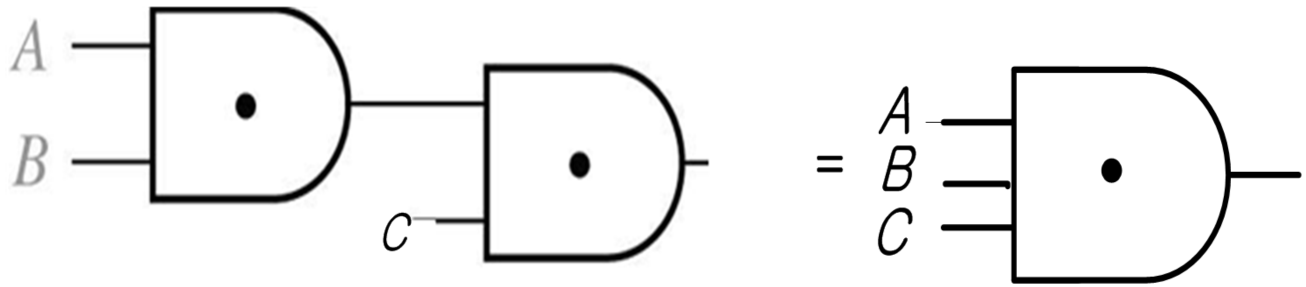
- **Associative Laws :** $(XY)Z = X(YZ) = XYZ$

  $(X+Y)+Z = X+(Y+Z) = X+Y+Z$

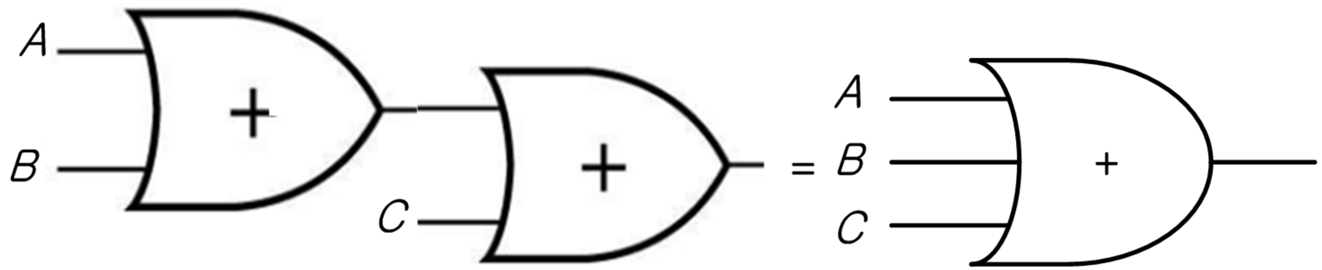
Proof of Associative Law for AND

X Y Z	XY	YZ	(XY)Z	X(YZ)
0 0 0	0	0	0	0
0 0 1	0	0	0	0
0 1 0	0	0	0	0
0 1 1	0	1	0	0
1 0 0	0	0	0	0
1 0 1	0	0	0	0
1 1 0	1	0	0	0
1 1 1	1	1	1	1

Associative Laws for AND and OR



$(AB)C = ABC$



$(A+B)+C = A+B+C$

Commutative, Associative, Distributive

- **AND :** $XYZ = 1 \text{ iff } X = Y = Z = 1$
- **OR :** $X + Y + Z = 0 \text{ iff } X = Y = Z = 0$
- **Distribute Laws :** $X(Y+Z) = XY + XZ$

$$X + YZ = (X+Y)(X+Z)$$

Valid only Boolean algebra not for ordinary algebra

Proof

$$(X+Y)(X+Z) = X(X+Z) + Y(X+Z) = XX + XZ + YX + YZ$$

$$= X + XZ + XY + YZ = X \cdot 1 + XZ + XY + YZ$$

$$= X(1 + Z + Y) + YZ = X \cdot 1 + YZ = X + YZ$$

Simplification Theorems

Useful Theorems for Simplification

$$XY + XY' = X \qquad (X+Y)(X+Y') = X$$

$$X + XY = X \qquad X(X+Y) = X$$

$$(X+Y')Y = XY \qquad XY' + Y = X + Y$$

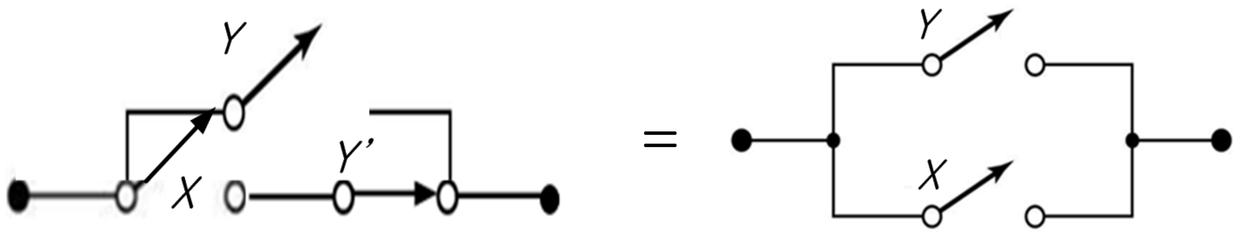
Proof

$$X + XY = X \cdot 1 + XY = X(1+Y) = X \cdot 1 = X$$

$$X(X+Y) = XX + XY = X + XY = X$$

$$Y + XY' = (Y+X)(Y+Y') = (Y+X)1 = Y + X$$

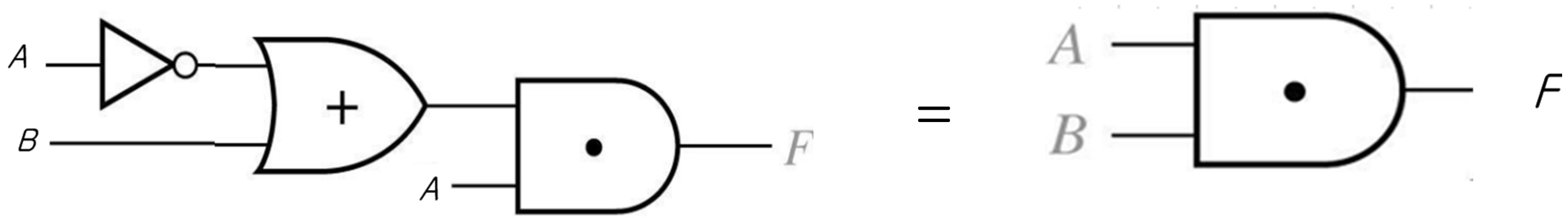
Proof with Switch



Simplification Theorems

- **Equivalent Gate Circuits**

$$F = A(A' + B) = AB$$



Multiplying Out and Factoring

To obtain a sum-of-product form ➔ Multiplying out using distributive laws

- **Sum of product form :** $AB' + CD'E + AC'E$
- **Not in sum of product form :** $(A+B)CD + EF$
- **Multiplying out and eliminating redundant terms :**

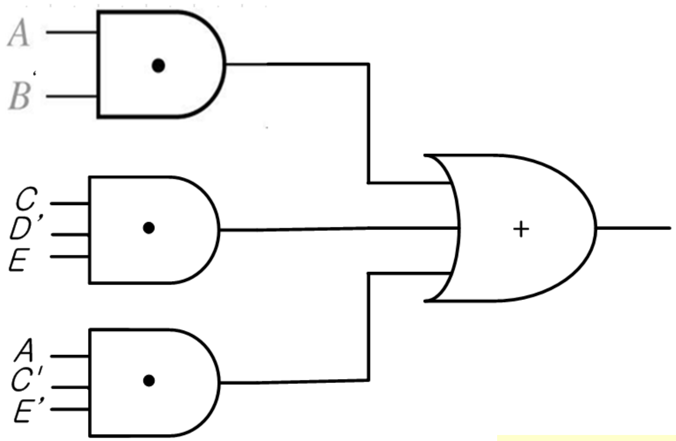
$$\begin{aligned}(A+BC)(A+D+E) &= A + AD + AE + ABC + BCD + BCE \\ &= A(1 + D + E + BC) + BCD + BCE \\ &= A + BCD + BCE\end{aligned}$$

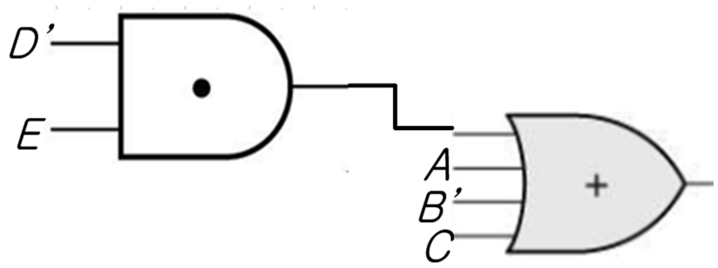
Multiplying Out and Factoring

To obtain a product of sum form → all sums are the sum of single variable

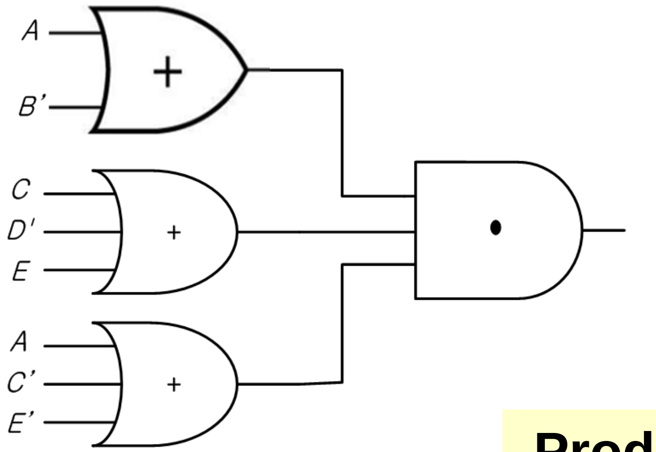
- **Product of sum form :** $(A+B')(C+D'+E)(A+C'+E')$

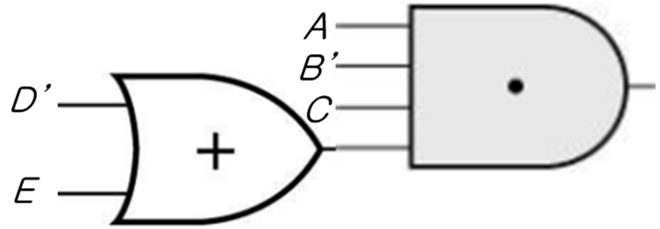
Circuits of SOP and POS Forms





Sum of product form





Product of sum form

DeMorgan's Laws

DeMorgan's Laws

$$(X+Y)' = X'Y'$$

$$(XY)' = X'+Y'$$

Proof

X Y		X' Y'	X + Y	(X + Y)'	X' Y'		XY	(XY)'	X' + Y'
0 0		1 1	0	1	1		0	1	1
0 1		1 0	1	0	0		0	1	1
1 0		0 1	1	0	0		0	1	1
1 1		0 0	1	0	0		1	0	0

DeMorgan's Laws for n variables

$$(X_1 + X_2 + X_3 + \ldots + X_n)' = X_1' X_2' X_3' \ldots X_n'$$

$$(X_1 X_2 X_3 \ldots X_n)' = X_1' + X_2' + X_3' + \ldots + X_n'$$

Example

$$(X_1 + X_2 + X_3)' = (X_1 + X_2)' X_3' = X_1' X_2' X_3'$$

DeMorgan's Laws

- **Inverse of A'B + AB'**

$$F' = (A'B + AB')' = (A'B)'(AB')' = (A + B')(A' + B)$$
$$= AA' + AB + B'A' + BB' = A'B' + AB$$

A B		A' B	A B'	F = A'B+AB'		A' B'	A B	F' = A'B' + AB
0 0		0	0	0		1	0	1
0 1		1	0	1		0	0	0
1 0		0	1	1		0	0	0
1 1		0	0	0		0	1	1

- **Dual: 'dual' is formed by replacing AND with OR, OR with AND, 0 with 1, 1 with 0**

$$(XYZ\ldots)^D = X + Y + Z + \ldots \qquad (X + Y + Z + \ldots)^D = XYZ\ldots$$

$$(AB' + C)' = (AB')'C' = (A'B)C', \quad \text{so} \quad (AB' + C)^D = (A + B')C$$