CS 2420: Stack

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Outline

- This topic discusses the concept of a stack:
  - Description of an Abstract Stack
  - List applications
  - Implementation
  - Example applications
    - Parsing: XHTML, C++
    - Function calls
    - Reverse-Polish calculators
    - Robert's Rules
  - Standard Template Library

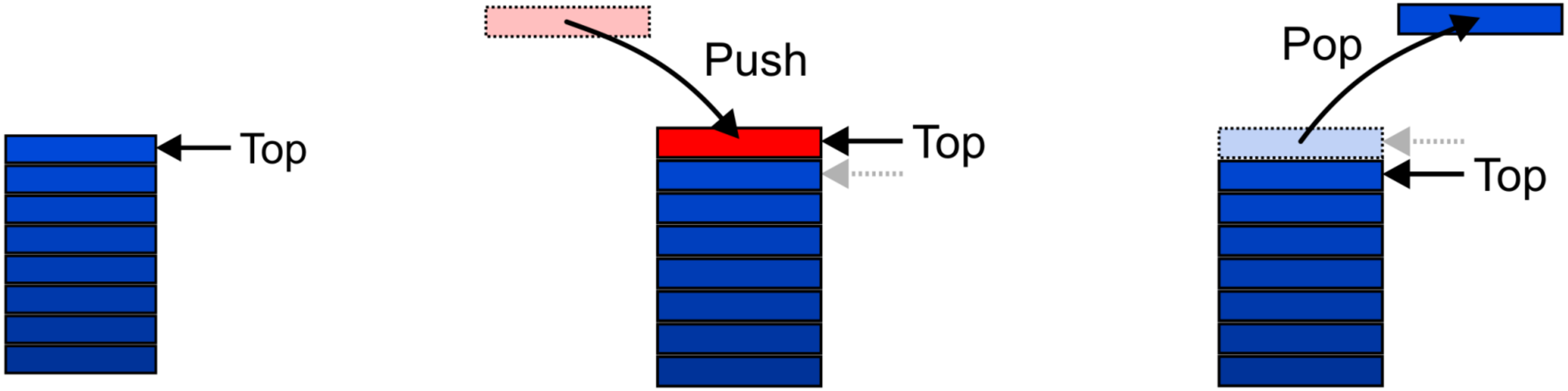
Abstract Stack

- An Abstract Stack (Stack ADT) is an abstract data type which emphasizes specific operations:
  - Uses a explicit linear ordering
  - Insertions and removals are performed individually
  - Inserted objects are *pushed onto* the stack
  - The *top* of the stack is the most recently object pushed onto the stack
  - When an object is *popped* from the stack, the current *top* is erased

Stack Definition

Also called a *last-in–first-out* (LIFO) behaviour

- Graphically, we may view these operations as follows:



There are two exceptions associated with abstract stacks:

- It is an undefined operation to call either pop or top on an empty stack

Stack Applications

- Numerous applications:
  - Parsing code:
    - Matching parenthesis
    - XML (e.g., XHTML)
  - Tracking function calls
  - Dealing with undo/redo operations
  - Reverse-Polish calculators
  - Assembly language

- The stack is a very simple data structure
  - Given any problem, if it is possible to use a stack, this significantly simplifies the solution

Stack Applications

- Problem solving:
  - Solving one problem may lead to subsequent problems
  - These problems may result in further problems
  - As problems are solved, your focus shifts back to the problem which lead to the solved problem

- Notice that function calls behave similarly:
  - A function is a collection of code which solves a problem

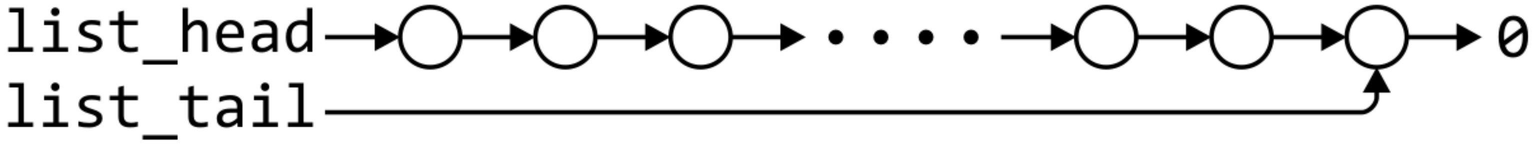
Stack Implementation

- We will look at two implementations of stacks:
  - Singly linked lists
  - One-ended arrays

- The optimal asymptotic run time of any algorithm is $\Theta(1)$
  - The run time of the algorithm is independent of the number of objects being stored in the container
  - We will always attempt to achieve this lower bound

Linked-List Implementation

- Operations at the front of a singly linked list are all $\Theta(1)$



	Front/1^{st}	Back/n^{th}
Find	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(1)$	$\Theta(1)$
Erase	$\Theta(1)$	$\Theta(n)$

- The desired behaviors of an Abstract Stack may be reproduced by performing all operations at the front

Single_list Definition

```
template <typename Type>
class Single_list {
    public:
    Single_list();
      ~Single_list();
        int size() const;
        bool empty() const;
        Type front() const;
        Type back() const;
        Single_node<Type> *head() const;
        Single_node<Type> *tail() const;
        int count( Type const & ) const;
        void push_front( Type const & );
        void push_back( Type const & );
        Type pop_front();
        int erase( Type const & );
};
```

Stack-as-List Class

- The stack class using a singly linked list has a single private member variable:

```
template <typename Type>
class Stack {
    private:
        Single_list<Type> list;
    public:
        bool empty() const;
        Type top() const;
        void push( Type const & );
        Type pop();
};
```

Stack-as-List Class

- A constructor and destructor is not needed
  - Because `list` is declared, the compiler will call the constructor of the `Single_list` class when the `Stack` is constructed

```
template <typename Type>
class Stack {
    private:
        Single_list<Type> list;
    public:
        bool empty() const;
        Type top() const;
        void push( Type const & );
        Type pop();
};
```

Stack-as-List Class

- The empty and push functions just call the appropriate functions of the `Single_list` class

```
template <typename Type>
bool Stack<Type>::empty() const {
    return list.empty();
}

template <typename Type>
void Stack<Type>::push( Type const &obj ) {
    list.push_front( obj );
}
```

Stack-as-List Class

- The top and pop functions, however, must check the boundary case:

```
template <typename Type>
Type Stack<Type>::top() const {
    if ( empty() ) {
        throw underflow();
    }

    return list.front();
}
```

```
template <typename Type>
Type Stack<Type>::pop() {
    if ( empty() ) {
        throw underflow();
    }

    return list.pop_front();
}
```

Array Implementation

- For one-ended arrays, all operations at the back are $\Theta(1)$

	Front/1^{st}	Back/n^{th}
Find	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(n)$	$\Theta(1)$
Erase	$\Theta(n)$	$\Theta(1)$

Destructor

- We need to store an array:
  - In C++, this is done by storing the address of the first entry

```
Type *array;
```

- We need additional information, including:
  - The number of objects currently in the stack

```
int stack_size;
```

  - The capacity of the array

```
int array_capacity;
```

Stack-as-Array Class

- We need to store an array (address of the first entry)

```
template <typename Type>
class Stack {
    private:
        int stack_size;
        int array_capacity;
        Type *array;
    public:
        Stack( int = 10 );
        ~Stack();
        bool empty() const;
        Type top() const;
        void push( Type const & );
        Type pop();
};
```

Constructor

- The class is only storing the address of the array
  - We must allocate memory for the array and initialize the member variables
  - The call to `new Type[array_capacity]` makes a request to the operating system for `array_capacity` objects

```
#include <algorithm>
// ...

template <typename Type>
Stack<Type>::Stack( int n ):
stack_size( 0 ),
array_capacity( std::max( 1, n ) ),
array( new Type[array_capacity] ) {
    // Empty constructor
}
```

Constructor

- Warning: in C++, the variables are initialized in the order in which they are defined:

```
template <typename Type>
Stack<Type>::Stack( int n ):
stack_size( 0 ),
array_capacity( std::max( 1, n ) ),
array( new Type[array_capacity] ) {
    // Empty constructor
}
```

```
template <typename Type>
class Stack {
    private:
        int stack_size;
        int array_capacity;
        Type *array;
    public:
        Stack( int = 10 );
        ~Stack();
        bool empty() const;
        Type top() const;
        void push( Type const & );
        Type pop();
};
```

Destructor

- The call to new in the constructor requested memory from the operating system
  - The destructor must return that memory to the operating system:

```
template <typename Type>
Stack<Type>::~Stack() {
    delete [] array;
}
```

Empty

- The stack is empty if the stack size is zero:

```
template <typename Type>
bool Stack<Type>::empty() const {
    return ( stack_size == 0 );
}
```

- The following is unnecessarily tedious:
  - The == operator evaluates to either `true` or `false`

```
if ( stack_size == 0 ) {
    return true;
} else {
    return false;
}
```

Top

- If there are n objects in the stack, the last is at index $n - 1$

```
template <typename Type>
Type Stack<Type>::top() const {
    if ( empty() ) {
        throw underflow();
    }

    return array[stack_size - 1];
}
```

Pop

- Removing an object simply involves reducing the size
  - It is invalid to assign the last entry to “0”
  - By decreasing the size, the previous top of the stack is now at the location `stack_size`

```
template <typename Type>
Type Stack<Type>::pop() {
    if ( empty() ) {
        throw underflow();
    }

    --stack_size;
    return array[stack_size];
}
```

Push

- Pushing an object onto the stack can only be performed if the array is not full

```
template <typename Type>
void Stack<Type>::push( Type const &obj ) {
    if ( stack_size == array_capacity ) {
        throw overflow();  // Best solution?????
    }

    array[stack_size] = obj;
    ++stack_size;
}
```

Exceptions

- The case where the array is full is not an exception defined in the Abstract Stack

- If the array is filled, we have five options:
  - Increase the size of the array
  - Throw an exception
  - Ignore the element being pushed
  - Replace the current top of the stack
  - Put the pushing process to “sleep” until something else removes the top of the stack

- Include a member function `bool full() const;`

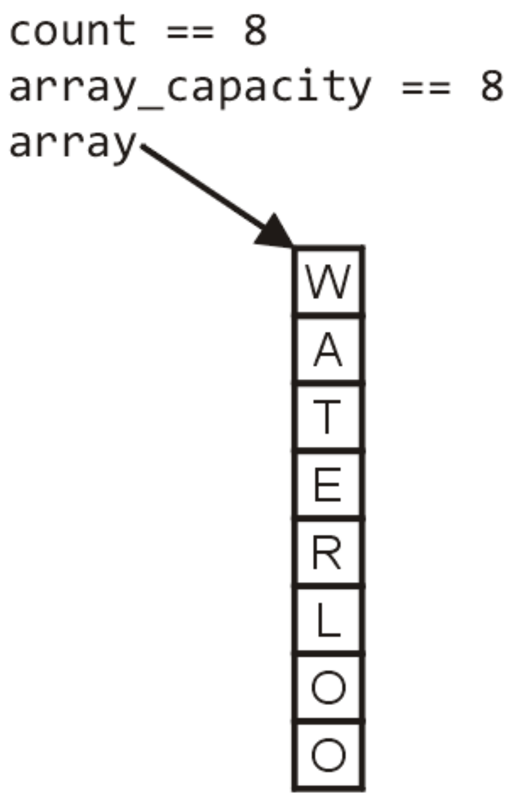
Array Capacity

- If dynamic memory is available, the best option is to increase the array capacity

- If we increase the array capacity, the question is:
  - How much?
  - By a constant? `array_capacity += c;`
  - By a multiple? `array_capacity *= c;`

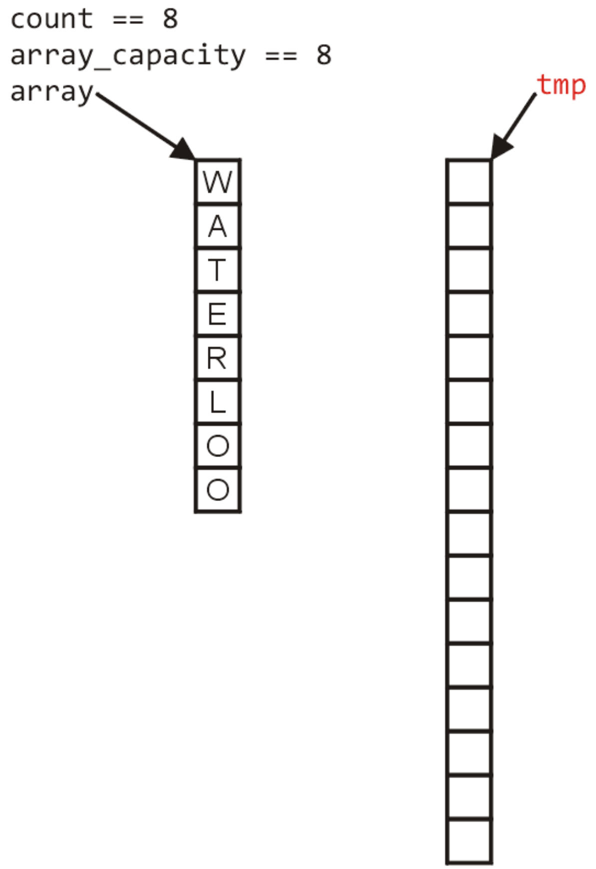
Array Capacity

- First, let us visualize what must occur to allocate new memory



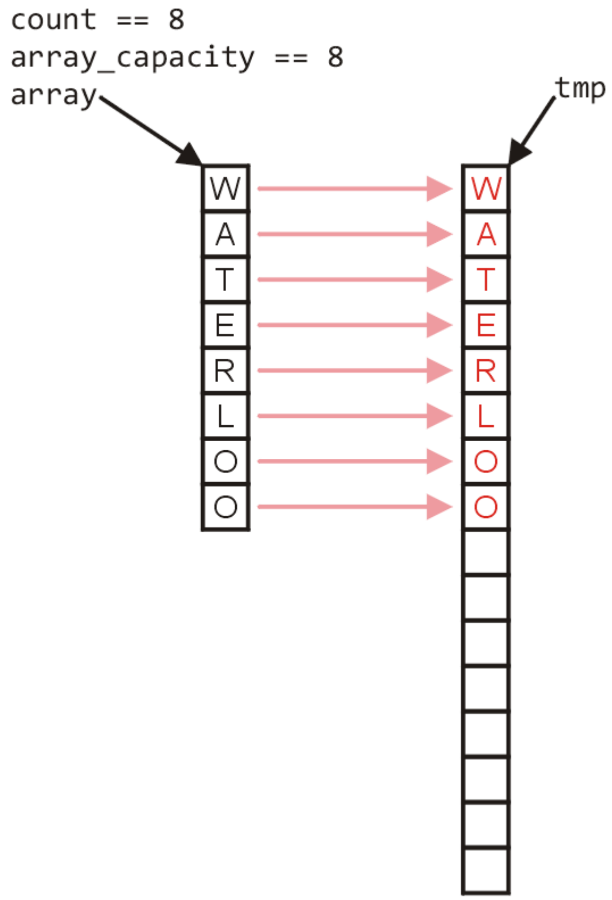
Array Capacity

- First, this requires a call to `new Type[N]` where *N* is the new capacity
  - We must have access to this so we must store the address returned by new in a local variable, say `tmp`



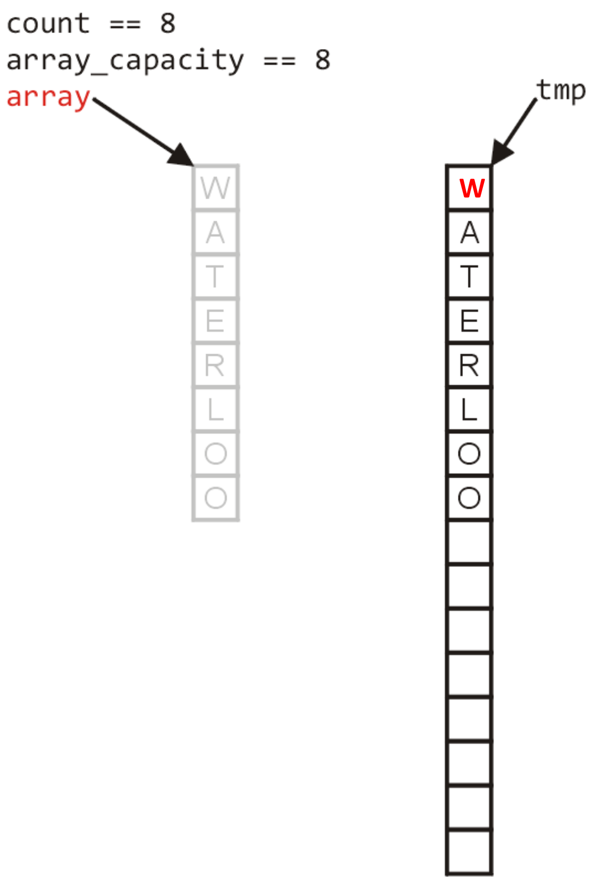
Array Capacity

- Next, the values must be copied over



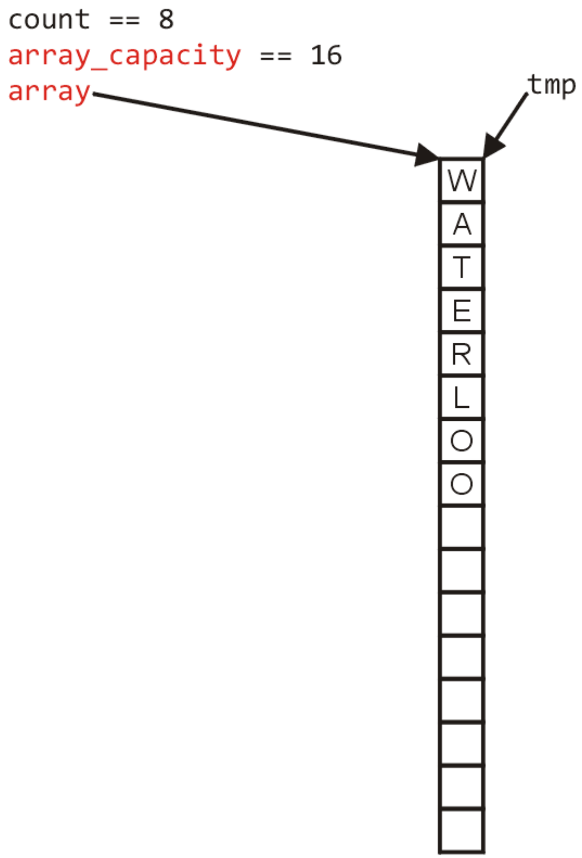
Array Capacity

- The memory for the original array must be deallocated



Array Capacity

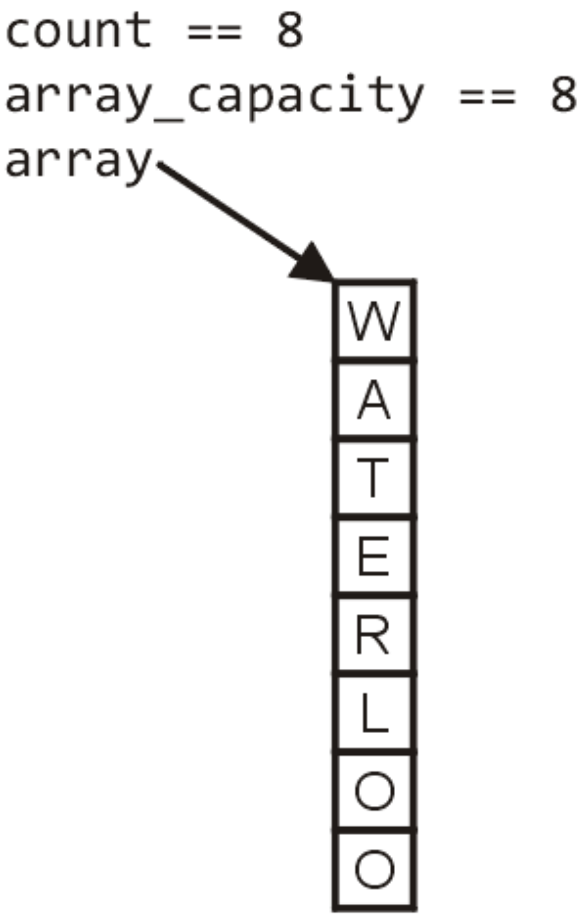
- Finally, the appropriate member variables must be reassigned



Array Capacity

The implementation:

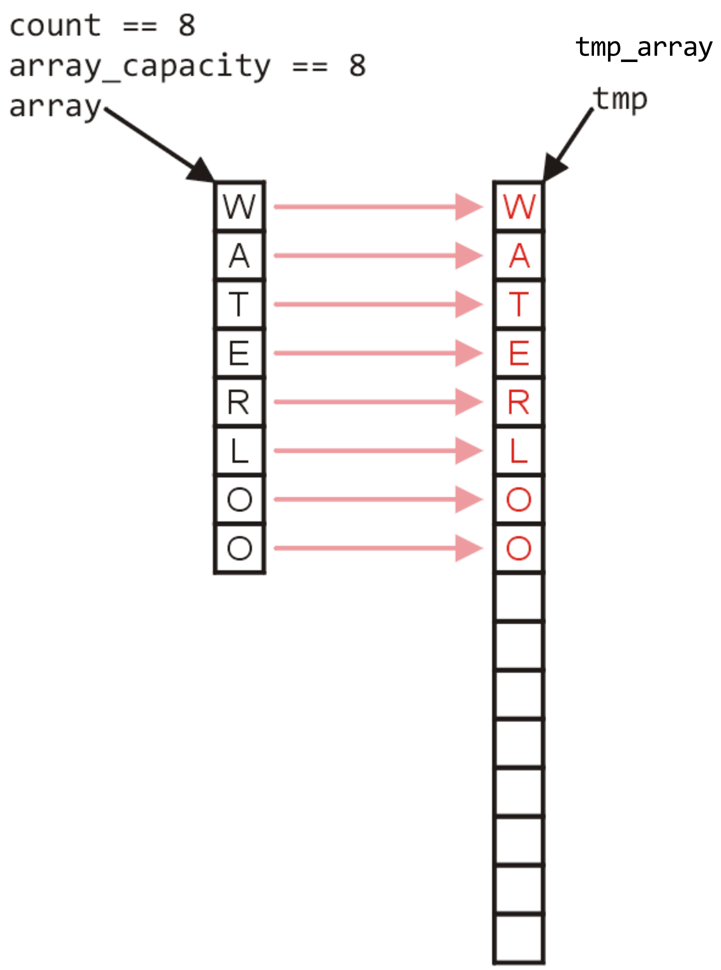
```
void double_capacity() {
    Type *tmp_array = new Type[2*array_capacity];









}
```



Array Capacity

- The implementation:

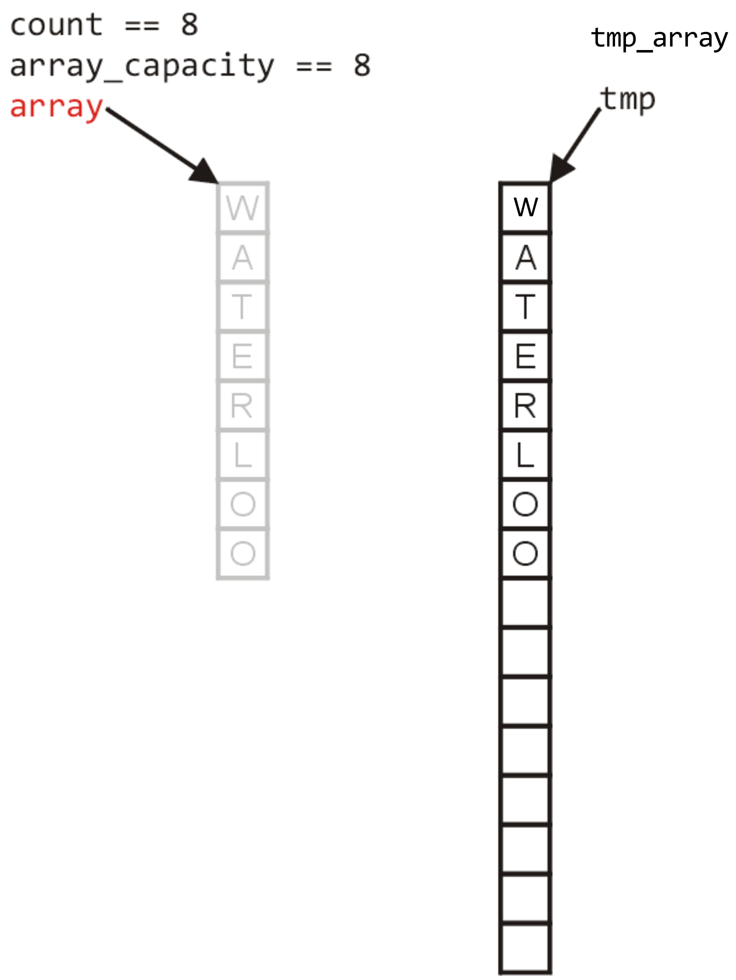
```
void double_capacity() {
    Type *tmp_array = new Type[2*array_capacity];

    for ( int i = 0; i < array_capacity; ++i ) {
        tmp_array[i] = array[i];
    }

}
```



Array Capacity

- The implementation:

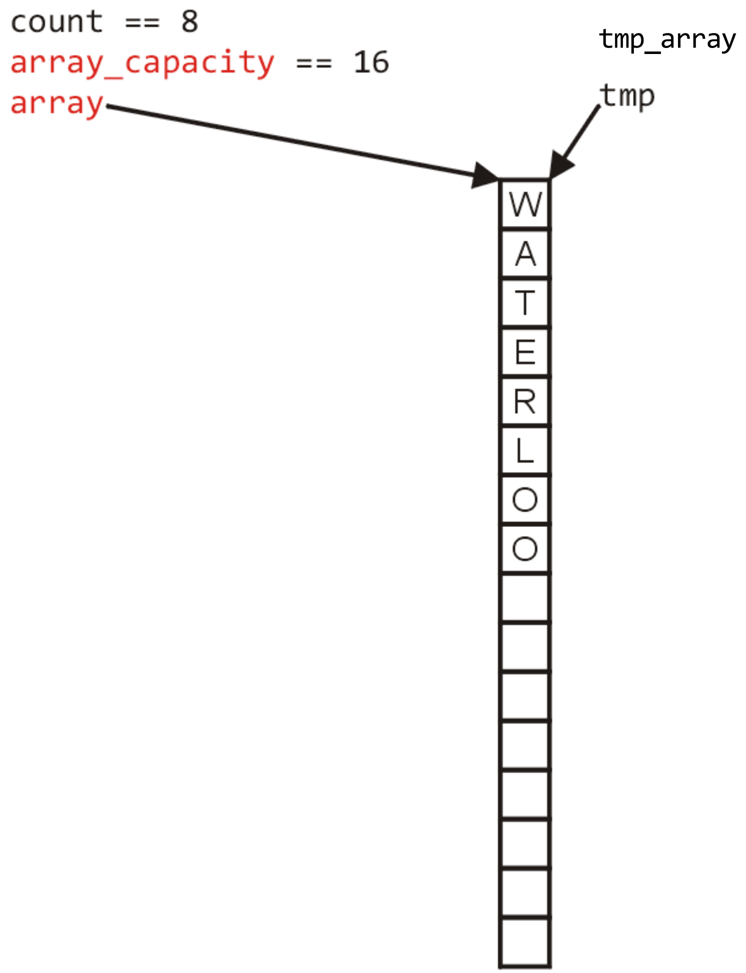
```
void double_capacity() {
    Type *tmp_array = new Type[2*array_capacity];

    for ( int i = 0; i < array_capacity; ++i ) {
        tmp_array[i] = array[i];
    }

    delete [] array;



}
```



Array Capacity

- The implementation:

```
void double_capacity() {
    Type *tmp_array = new Type[2*array_capacity];

    for ( int i = 0; i < array_capacity; ++i ) {
        tmp_array[i] = array[i];
    }

    delete [] array;
    array = tmp_array;

    array_capacity *= 2;
}
```



Array Capacity

- Back to the original question:
  - How much do we change the capacity?
  - Add a constant?
  - Multiply by a constant?

- First, we recognize that any time that we `push` onto a full stack, this requires n copies and the run time is $\Theta(n)$

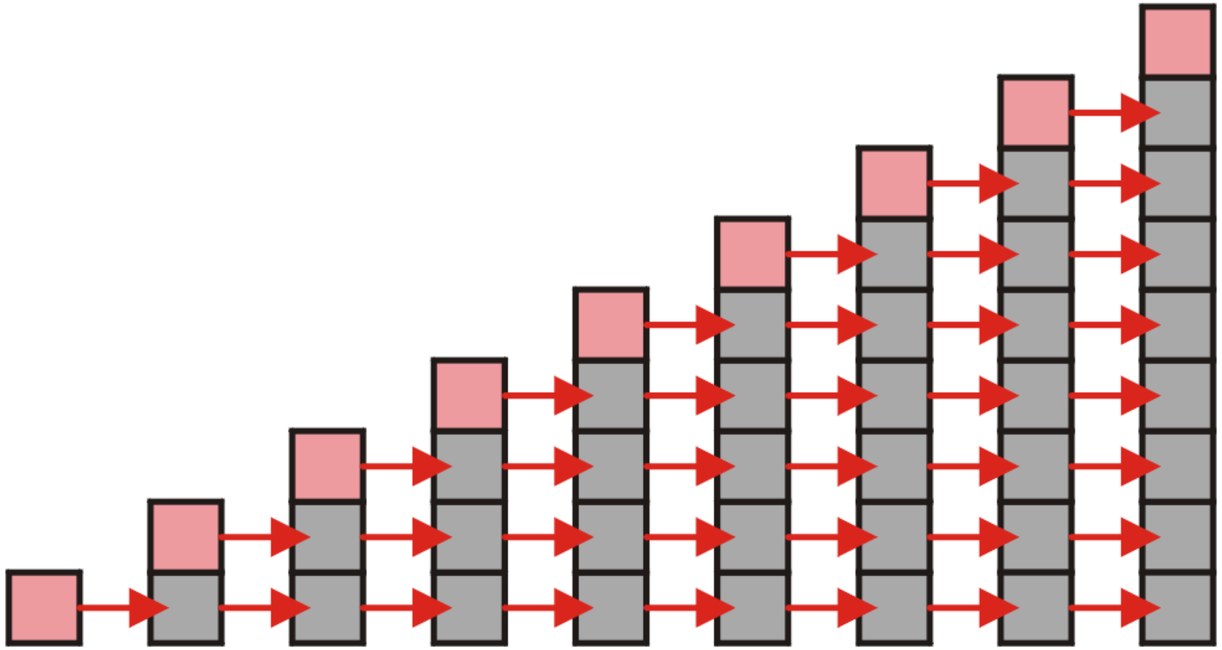
- Therefore, `push` is usually $\Theta(1)$ except when new memory is required

Array Capacity

- To state the average run time, we will introduce the concept of amortized time:
  - If n operations requires $\Theta(\mathrm{f}(n))$, we will say that an individual operation has an amortized run time of $\Theta(\mathrm{f}(n)/n)$
  - Therefore, if inserting n objects requires:
    - $\Theta(n^2)$ copies, the amortized time is $\Theta(n)$
    - $\Theta(n)$ copies, the amortized time is $\Theta(1)$

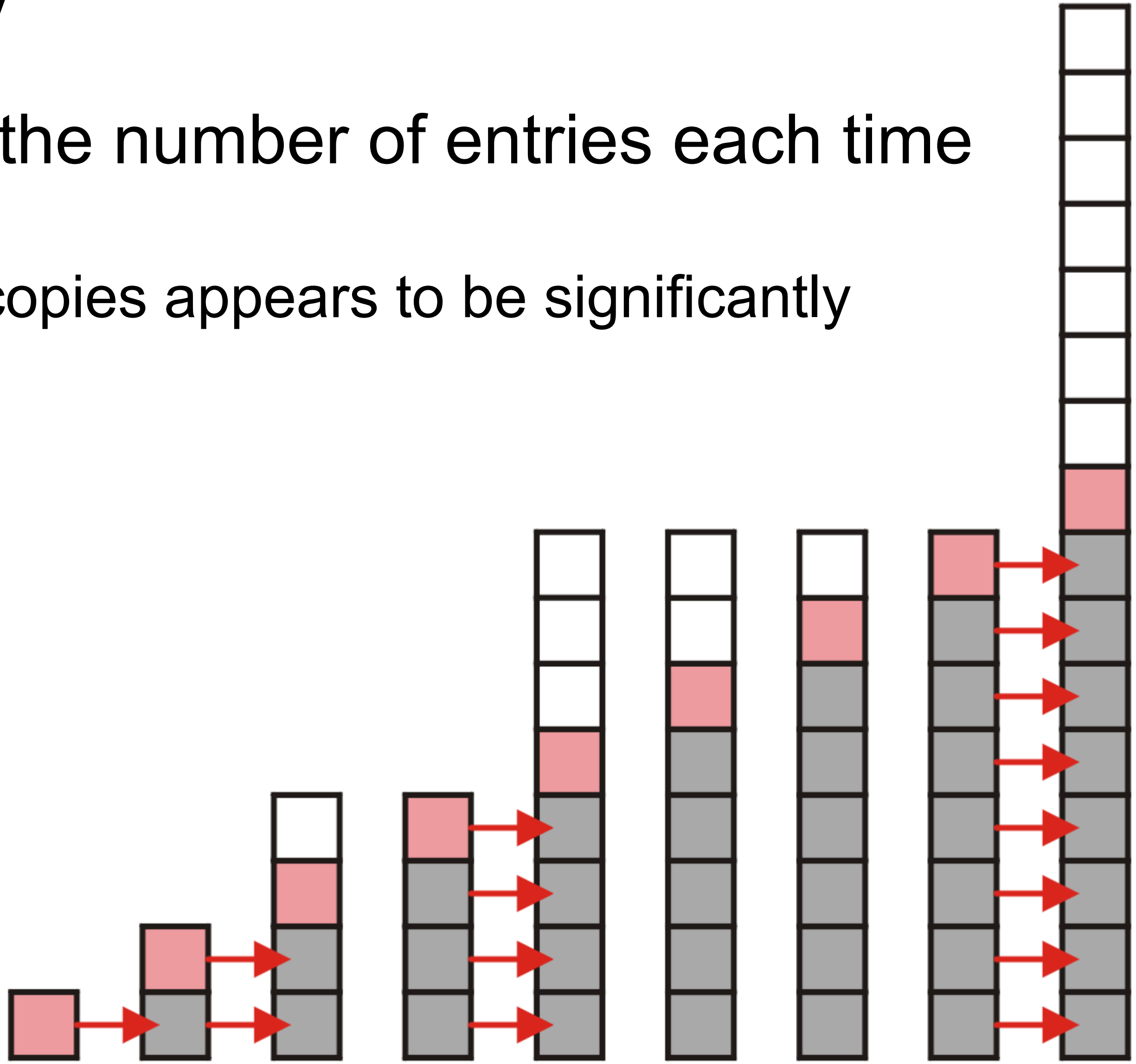
Array Capacity

- Let us consider the case of increasing the capacity by 1 each time the array is full
  - With each insertion when the array is full, this requires all entries to be copied



Array Capacity

- Suppose we double the number of entries each time the array is full
  - Now the number of copies appears to be significantly fewer



Array Capacity

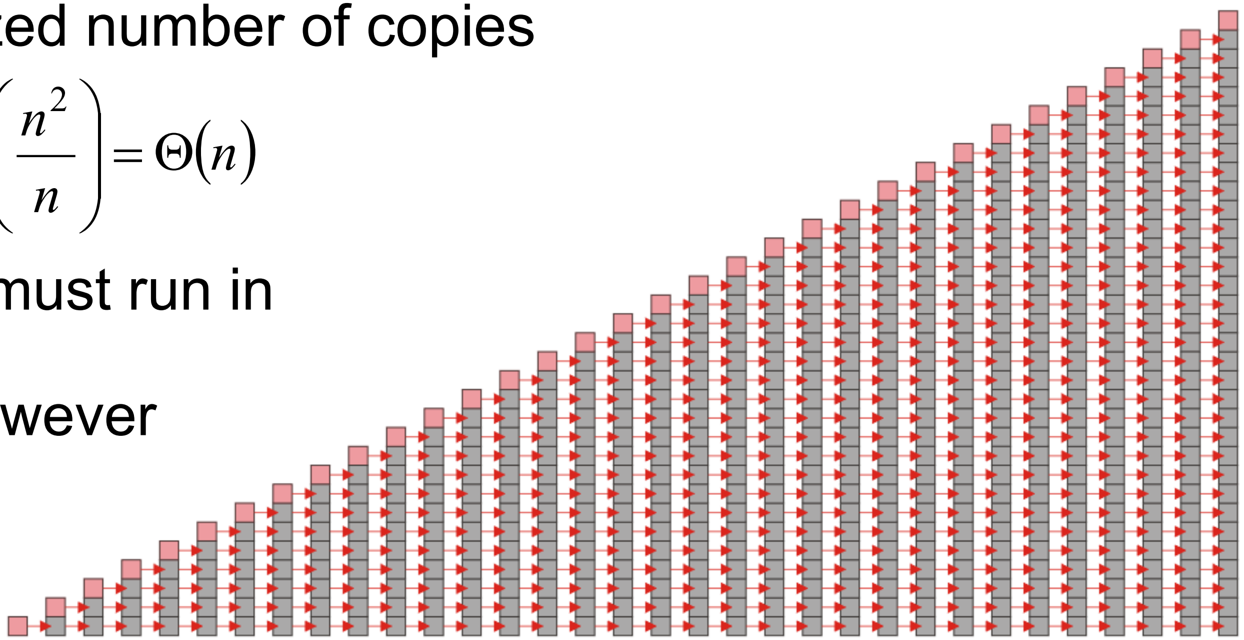
- Suppose we insert n objects
  - The pushing of the k^{th} object on the stack requires $k - 1$ copies
  - The total number of copies is now given by:

$$\sum_{k=1}^{n}(k-1) = \left(\sum_{k=1}^{n} k\right) - n = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2} = \Theta(n^2)$$

  - Therefore, the amortized number of copies is given by

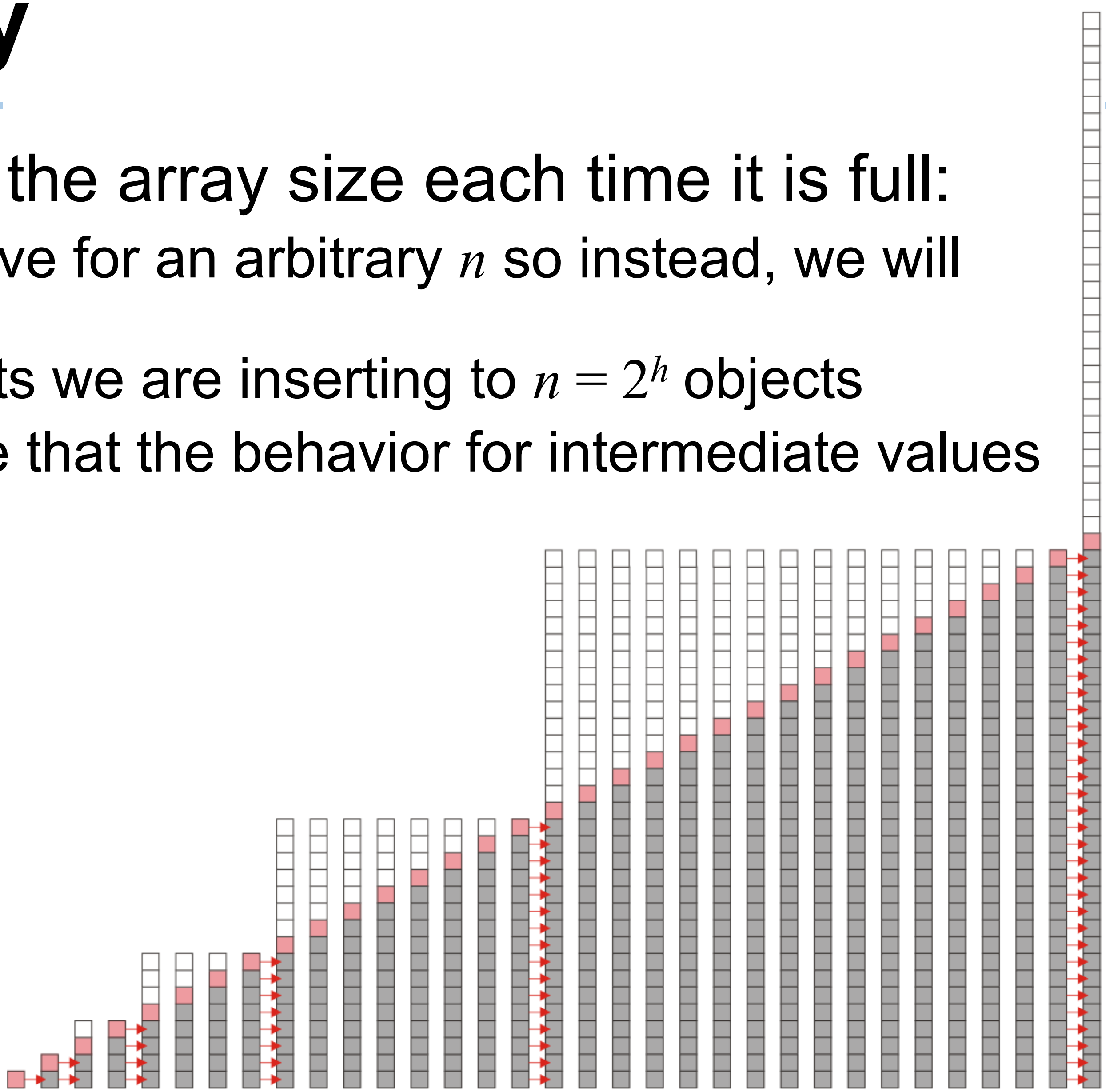
$$\Theta\left(\frac{n^2}{n}\right) = \Theta(n)$$

  - Therefore each push must run in $\Theta(n)$ time
  - The wasted space, however is $\Theta(1)$



Array Capacity

- Suppose we double the array size each time it is full:
  - This is difficult to solve for an arbitrary n so instead, we will restrict
    the number of objects we are inserting to $n = 2^h$ objects
  - We will then assume that the behavior for intermediate values of n will be similar



Array Capacity

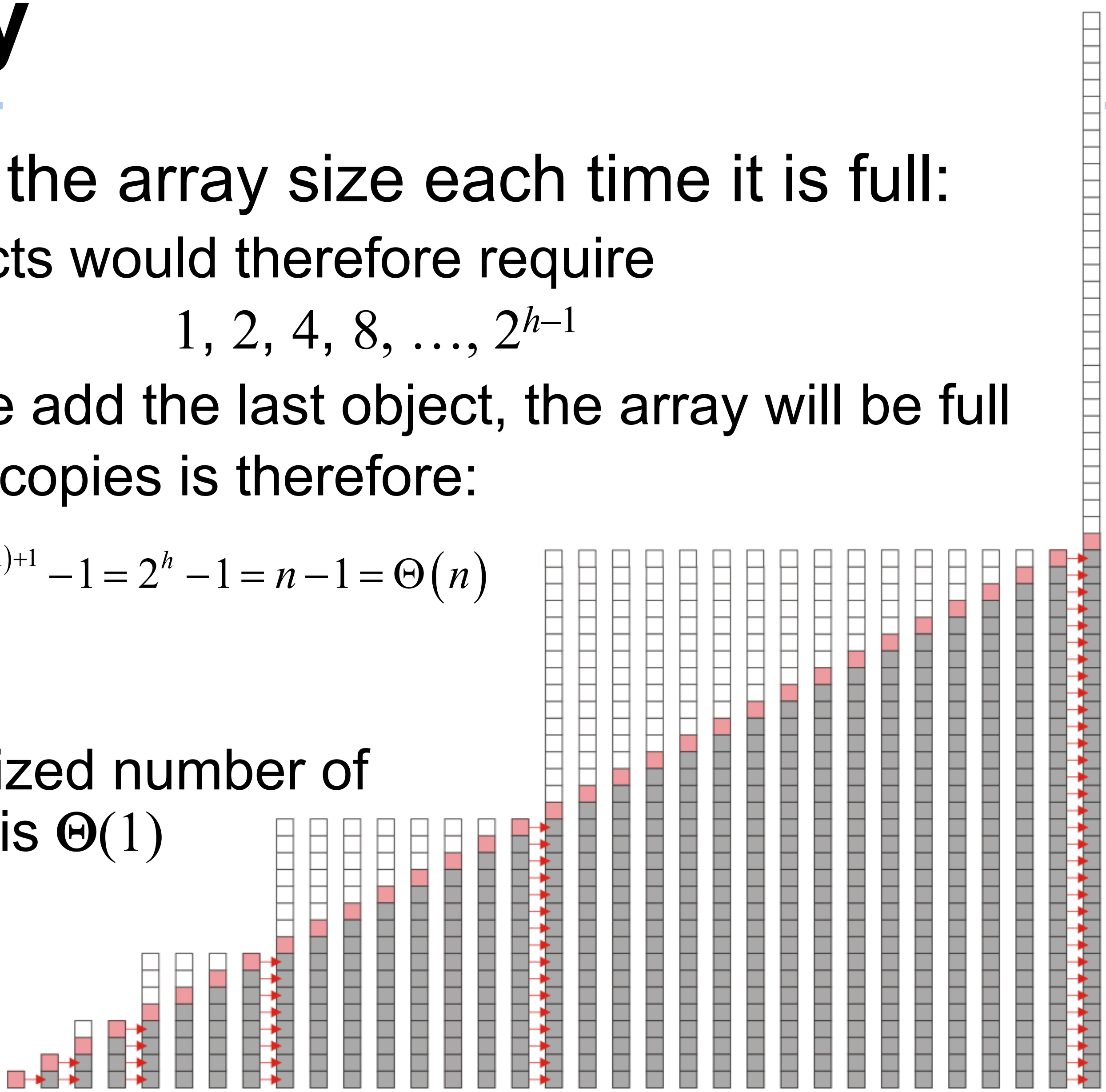
- Suppose we double the array size each time it is full:
  - Inserting $n = 2^h$ objects would therefore require

    $$1, 2, 4, 8, \ldots, 2^{h-1}$$

    copies, for once we add the last object, the array will be full
  - The total number of copies is therefore:

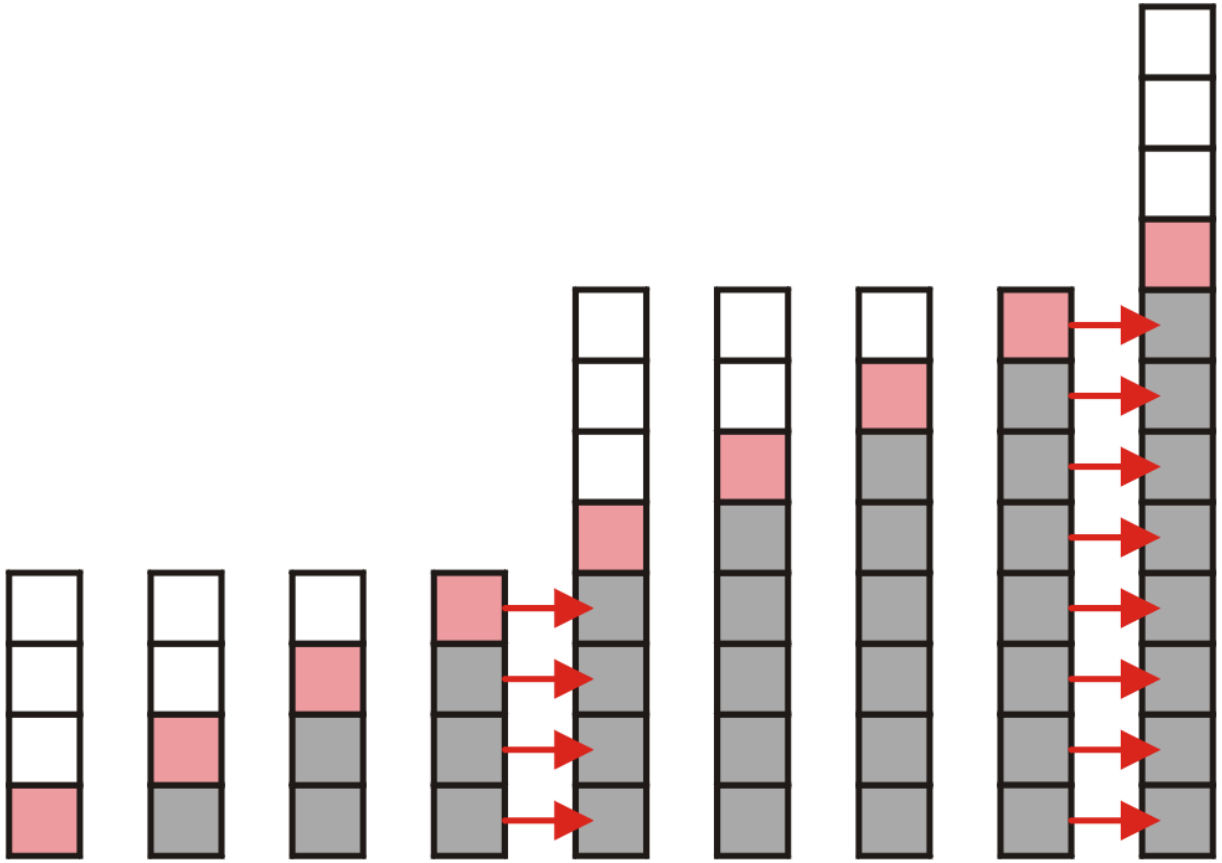
    $$\sum_{k=0}^{h-1} 2^k = 2^{(h-1)+1} - 1 = 2^h - 1 = n - 1 = \Theta(n)$$

  - Therefore the amortized number of copies per insertion is $\Theta(1)$
  - The wasted space, however is $\mathbf{O}(n)$



Array Capacity

- What if we increase the array size by a larger constant?
  - For example, increase the array size by 4, 8, 100?



Array Capacity

- Suppose we increase it by a constant m and we add $n = \ell m$ objects
  - To add n items, we will have to make

$$m, 2m, 3m, \ldots, (\ell - 1)m$$

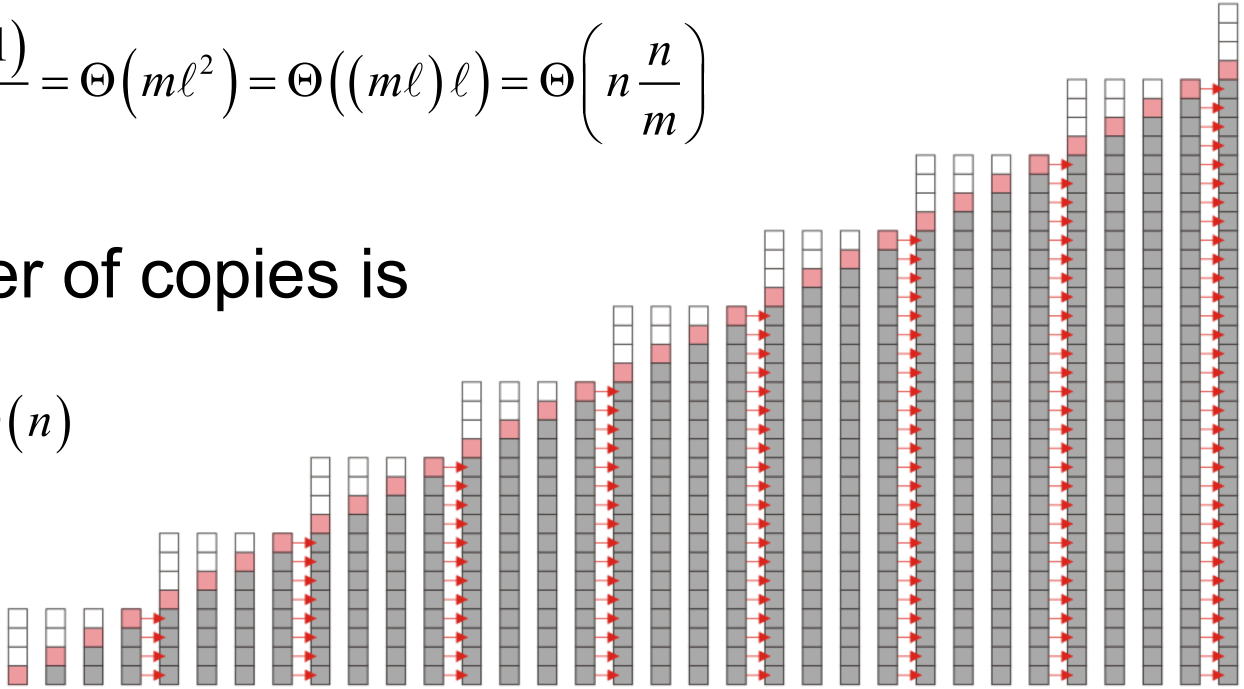
copies in total, or

$$\sum_{k=1}^{\ell-1} km = m\sum_{k=1}^{\ell-1} k = m\frac{\ell(\ell-1)}{2} = \Theta(m\ell^2) = \Theta((m\ell)\ell) = \Theta\left(n\frac{n}{m}\right)$$

The amoritized number of copies is

$$\Theta\left(\frac{n}{m}\right) = \Theta(n)$$

as m is fixed



Array Capacity

- Note the difference in worst-case amortized scenarios:

	Copies per Insertion	Unused Memory
Increase by 1	$n - 1$	0
Increase by m	n/m	$m - 1$
Increase by a factor of 2	1	n
Increase by a factor of $r > 1$	$1/(r - 1)$	$(r - 1)n$

LAB: std::stack

- C++ Standard Template Library (STL) stack
  - https://en.cppreference.com/w/cpp/container/stack

```
/* stack example */
#include <iostream>
#include <stack>

int main()
{
        std::stack<int> stk;
        stk.push(1);
        stk.push(2);
        std::cout<<stk.top();

        /* clear the stack */
        while(!stk.empty())
      stk.pop();
}
```

Reverse-Polish Notation

- Normally, mathematics is written using what we call *in-fix* notation:

$$(3 + 4) \times 5 - 6$$

- The operator is placed between to operands

  One weakness: parentheses are required

$$\begin{aligned}
(3 + 4) \times 5 - 6 &= 29 \\
3 + 4 \times 5 - 6 &= 17 \\
3 + 4 \times (5 - 6) &= -1 \\
(3 + 4) \times (5 - 6) &= -7
\end{aligned}$$

Reverse-Polish Notation

- Alternatively, we can place the operands first, followed by the operator:

$$(3 + 4) \times 5 - 6$$

$$3 \;\; 4 \;\; + \;\; 5 \;\; \times \;\; 6 \;\; -$$

- Parsing reads left-to-right and performs any operation on the last two operands:

$$3 \;\; 4 \;\; + \;\; 5 \;\; \times \;\; 6 \;\; -$$

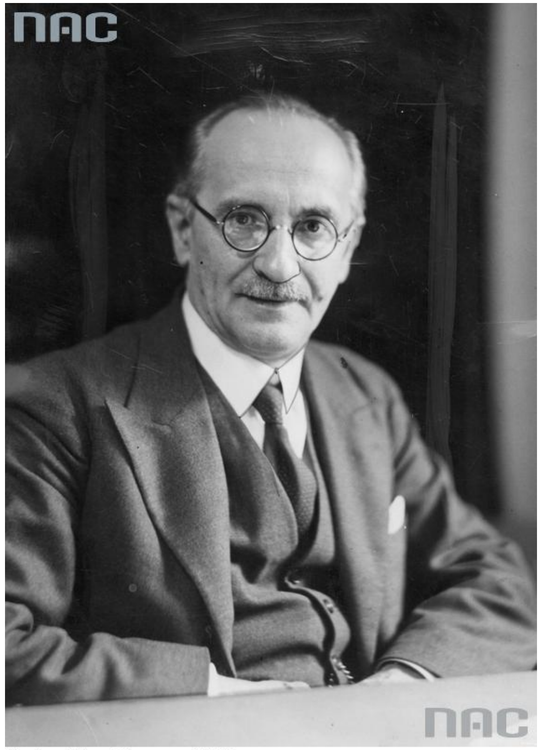
$$7 \;\; 5 \;\; \times \;\; 6 \;\; -$$

$$35 \;\; 6 \;\; -$$

$$29$$

Reverse-Polish Notation

- This is called *reverse-Polish* notation after the mathematician Jan Łukasiewicz
  - This forms the basis of the recursive stack used on all processors
- He also made significant contributions to logic and other fields
  - Electrical engineering
  - Computer-aided design
  - VLSI



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Reverse-Polish Notation

- Other examples:

$$3\ 4\ 5\ \times\ +\ 6\ -$$
$$3\ \ 20\ \ +\ 6\ -$$
$$23\ \ 6\ -$$
$$17$$

$$3\ 4\ 5\ 6\ -\ \times\ +$$
$$3\ 4\ \ {-1}\ \ \times\ +$$
$$3\ \ {-4}\ \ +$$
$$-1$$

Reverse-Polish Notation

- Benefits:
  - No ambiguity and no brackets are required
  - It is the same process used by a computer to perform computations:
    - operands must be loaded into registers before operations can be performed on them
  - Reverse-Polish can be processed using stacks

Reverse-Polish Notation

- Reverse-Polish notation is used with some programming languages
  - *e.g.*, postscript, pdf, and HP calculators

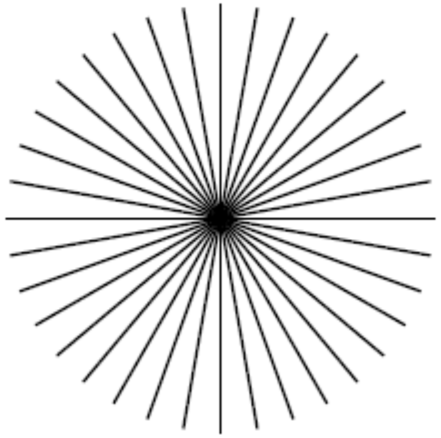
- Similar to the thought process required for writing assembly language code
  - you cannot perform an operation until you have all of the operands loaded into registers

```
MOVE.L #$2A, D1        ; Load  42 into Register D1
MOVE.L #$100, D2       ; Load 256 into Register D2
ADD D2, D1             ; Add D2 into D1
```

Reverse-Polish Notation

- A quick example of postscript:

```
0 10 360 {                  % Go from 0 to 360 degrees in 10-degree steps
    newpath                 % Start a new path
    gsave                   % Keep rotations temporary
        144 144 moveto
        rotate              % Rotate by degrees on stack from 'for'
        72 0 rlineto
        stroke
    grestore                % Get back the unrotated state
} for % Iterate over angles
```



http://www.tailrecursive.org/postscript/examples/rotate.html

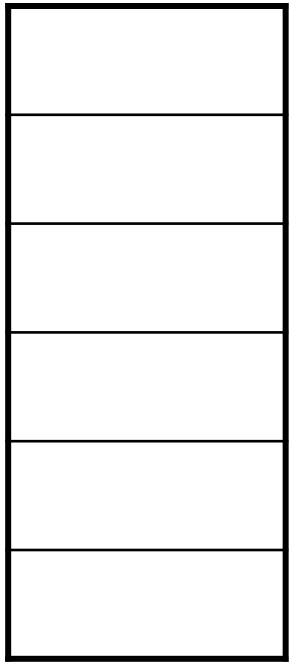
Reverse-Polish Notation

- The easiest way to parse reverse-Polish notation is to use an operand stack:
  - operands are processed by pushing them onto the stack
  - when processing an operator:
    - pop the last two items off the operand stack,
    - perform the operation, and
    - push the result back onto the stack

Reverse-Polish Notation

Evaluate the following reverse-Polish expression using a stack:

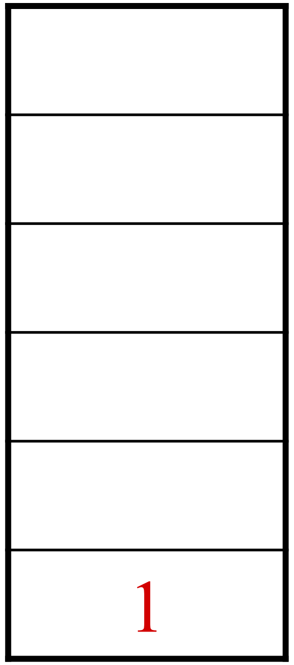
$$1\ 2\ 3\ +\ 4\ 5\ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ 9\ \times\ +$$



Reverse-Polish Notation

Push 1 onto the stack

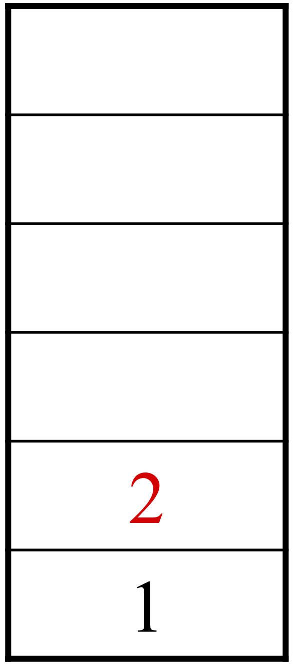
$1\ 2\ 3\ +\ 4\ 5\ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ 9\ \times\ +$



Reverse-Polish Notation

Push 1 onto the stack

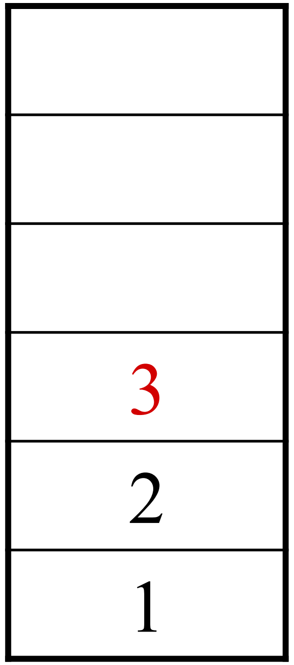
$1 \;\; 2 \;\; 3 \; + \; 4 \;\; 5 \;\; 6 \; \times \; - \; 7 \; \times \; + \; - \; 8 \;\; 9 \; \times \; +$



Reverse-Polish Notation

Push 3 onto the stack

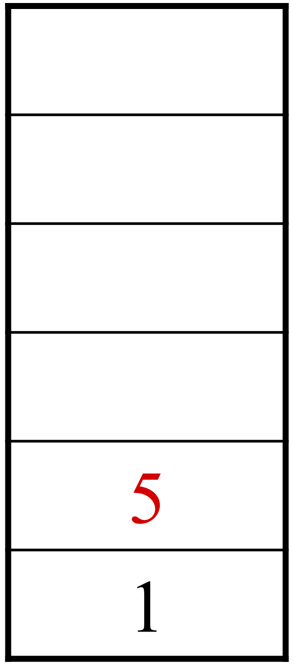
$1 \ 2 \ 3 \ + \ 4 \ 5 \ 6 \ \times \ - \ 7 \ \times \ + \ - \ 8 \ 9 \ \times \ +$



Reverse-Polish Notation

Pop 3 and 2 and push $2 + 3 = 5$

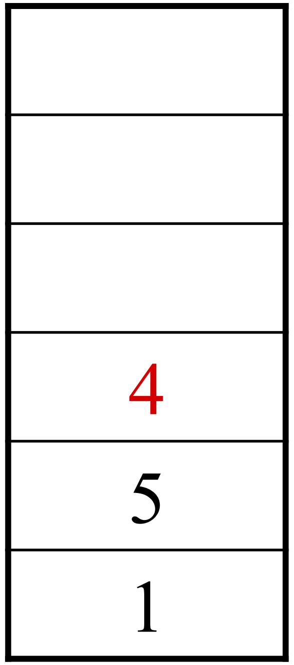
$$1\ \ 2\ \ 3\ \ +\ \ 4\ \ 5\ \ 6\ \ \times\ \ -\ \ 7\ \ \times\ \ +\ \ -\ \ 8\ \ 9\ \ \times\ \ +$$



Reverse-Polish Notation

Push 4 onto the stack

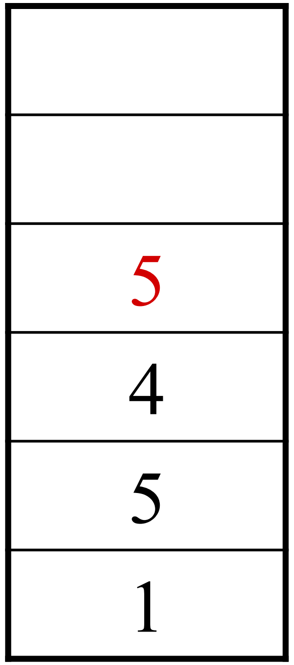
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Push 5 onto the stack

$$1\ 2\ 3\ +\ 4\ \color{red}{5}\ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ 9\ \times\ +$$



Reverse-Polish Notation

Push 6 onto the stack

$$1\ 2\ 3\ +\ 4\ 5\ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ 9\ \times\ +$$

6
5
4
5
1

Reverse-Polish Notation

Pop 6 and 5 and push $5 \times 6 = 30$

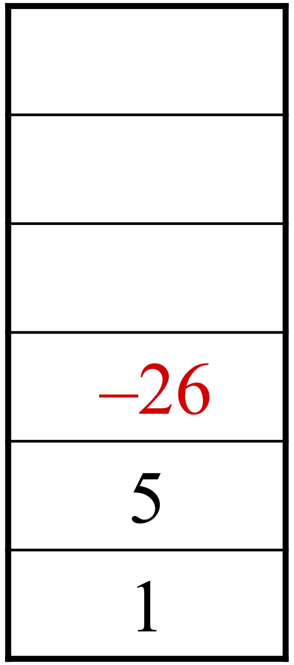
$1\ 2\ 3\ +\ 4\ 5\ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ 9\ \times\ +$

30
4
5
1

Reverse-Polish Notation

Pop 30 and 4 and push $4 - 30 = -26$

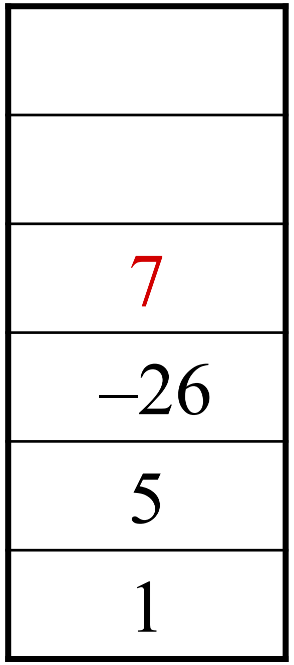
$$1\ 2\ 3\ +\ 4\ 5\ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ 9\ \times\ +$$



Reverse-Polish Notation

Push 7 onto the stack

$$1\ 2\ 3\ +\ 4\ 5\ 6\ \times\ -\ \color{red}{7}\ \times\ +\ -\ 8\ 9\ \times\ +$$



Reverse-Polish Notation

Pop 7 and –26 and push $-26 \times 7 = -182$

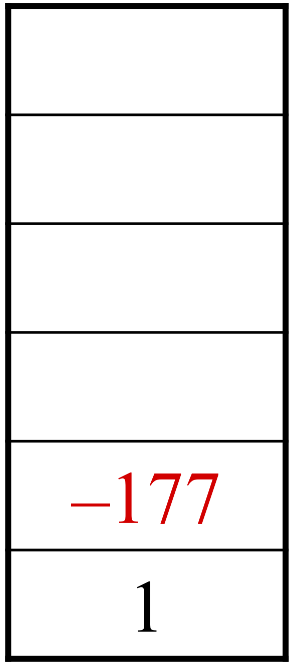
$1\ 2\ 3\ +\ 4\ 5\ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ 9\ \times\ +$

–182
5
1

Reverse-Polish Notation

Pop -182 and 5 and push $-182 + 5 = -177$

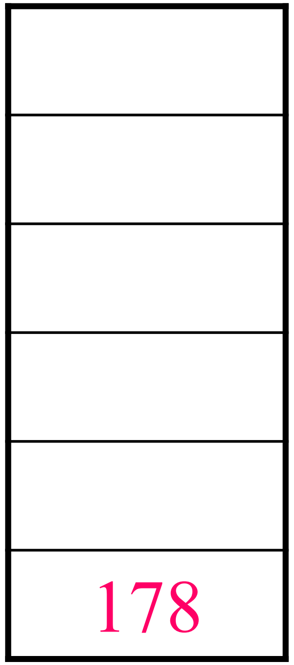
$$1 \quad 2 \quad 3 \quad + \quad 4 \quad 5 \quad 6 \quad \times \quad - \quad 7 \quad \times \quad + \quad - \quad 8 \quad 9 \quad \times \quad +$$



Reverse-Polish Notation

Pop –177 and 1 and push 1 – (–177) = 178

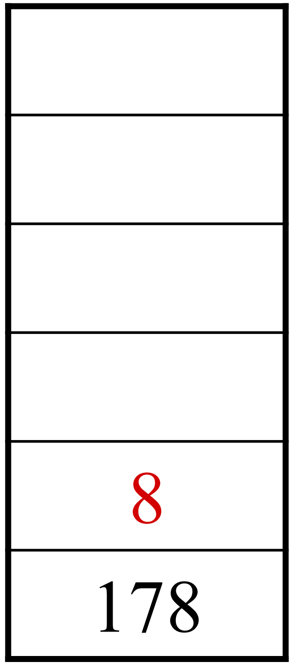
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Push 8 onto the stack

$1\ \ 2\ \ 3\ \ +\ \ 4\ \ 5\ \ 6\ \ \times\ \ -\ \ 7\ \ \times\ \ +\ \ -\ \ {\color{red}8}\ \ 9\ \ \times\ \ +$



Reverse-Polish Notation

Push 1 onto the stack

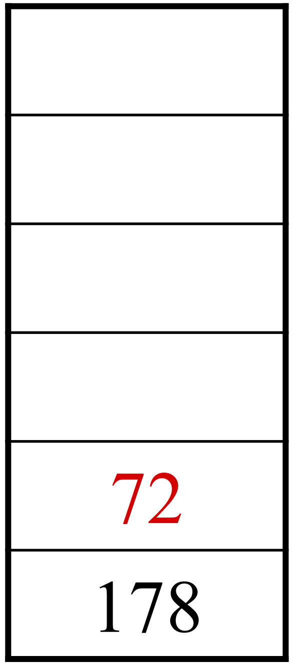
$$1\ 2\ 3\ +\ 4\ 5\ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ 9\ \times\ +$$

9
8
178

Reverse-Polish Notation

Pop 9 and 8 and push $8 \times 9 = 72$

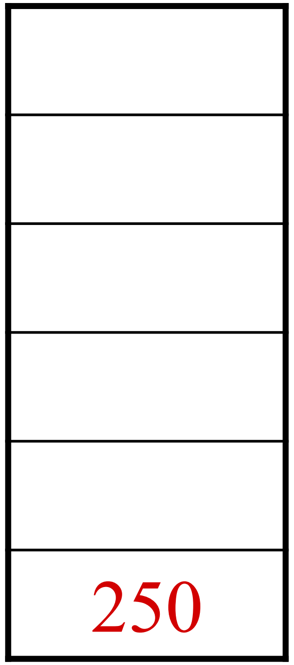
$$1 \quad 2 \quad 3 \quad + \quad 4 \quad 5 \quad 6 \quad \times \quad - \quad 7 \quad \times \quad + \quad - \quad 8 \quad 9 \quad \times \quad +$$



Reverse-Polish Notation

Pop 72 and 178 and push $178 + 72 = 250$

$$1 \;\; 2 \;\; 3 \;\; + \;\; 4 \;\; 5 \;\; 6 \;\; \times \;\; - \;\; 7 \;\; \times \;\; + \;\; - \;\; 8 \;\; 9 \;\; \times \;\; +$$



Reverse-Polish Notation

Thus

$$1\ \ 2\ \ 3\ \ +\ \ 4\ \ 5\ \ 6\ \ \times\ \ -\ \ 7\ \ \times\ \ +\ \ -\ \ 8\ \ 9\ \ \times\ \ +$$

evaluates to the value on the top: 250

The equivalent in-fix notation is

$$((1 - ((2 + 3) + ((4 - (5 \times 6)) \times 7))) + (8 \times 9))$$

We reduce the parentheses using order-of-operations:

$$1 - (2 + 3 + (4 - 5 \times 6) \times 7) + 8 \times 9$$

Reverse-Polish Notation

Incidentally,

$$1 - 2 + 3 + 4 - 5 \times 6 \times 7 + 8 \times 9 = -132$$

which has the reverse-Polish notation of

$$1 \ 2 \ - \ 3 \ + \ 4 \ + \ 5 \ 6 \ 7 \ \times \ \times \ - \ 8 \ 9 \ \times \ +$$

For comparison, the calculated expression was

$$1 \ 2 \ 3 \ + \ 4 \ 5 \ 6 \ \times \ - \ 7 \ \times \ + \ - \ 8 \ 9 \ \times \ +$$

Standard Template Library

- The Standard Template Library (STL) has a *wrapper* class stack with the following declaration:

```
template <typename T>
class stack {
    public:
        stack();                          // not quite true...
        bool empty() const;
        int size() const;
        const T & top() const;
        void push( const T & );
        void pop();
};
```

Standard Template Library

```
#include <iostream>
#include <stack>
using namespace std;
int main() {
    stack<int> istack;

    istack.push( 13 );
    istack.push( 42 );
    cout << "Top: " << istack.top() << endl;
    istack.pop();                              // no return value
    cout << "Top: " << istack.top() << endl;
    cout << "Size: " << istack.size() << endl;

    return 0;
}
```

Standard Template Library

- The reason that the `stack` class is termed a wrapper is because it uses a different container class to actually store the elements
- The `stack` class simply presents the *stack interface* with appropriately named member functions:
  - `push`, `pop`, and `top`

Summary

- The stack is the simplest of all ADTs
  - Understanding how a stack works is trivial

- The application of a stack, however, is not in the implementation, but rather:
  - Where possible, create a design which allows the use of a stack

- We looked at:
  - Parsing, function calls, and reverse Polish