CS 2420: Heap Sort

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Outline

- This topic covers the simplest $\Theta(n \ln(n))$ sorting algorithm: *heap sort*

- We will:
  - define the strategy
  - analyze the run time
  - convert an unsorted list into a heap
  - cover some examples

- Bonus: may be performed in place

Heap Sort

Recall that inserting n objects into a min-heap and then taking n objects will result in them coming out in order

Strategy: given an unsorted list with n objects, place them into a heap, and take them out

In-place Implementation

Problem:

- This solution requires additional memory, that is, a min-heap of size n

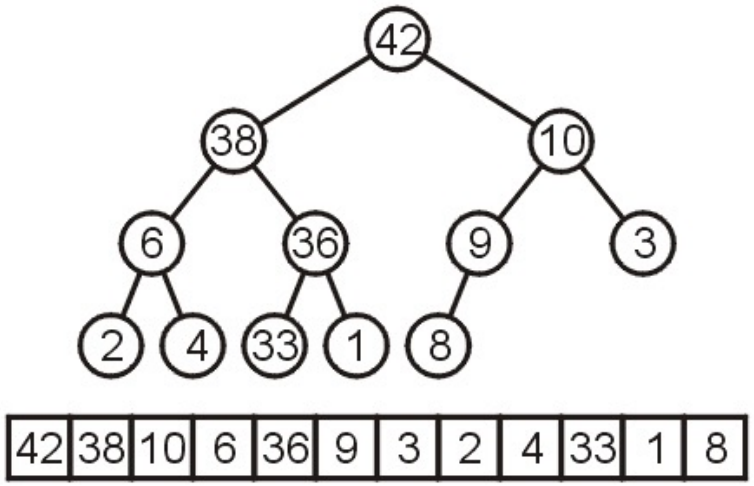
This requires $\Theta(n)$ memory and is therefore not in place

Is it possible to perform a heap sort in place, that is, require at most $\Theta(1)$ memory (a few extra variables)?

In-place Implementation

Instead of implementing a min-heap, consider a max-heap:

- A heap where the maximum element is at the top of the heap and the next to be popped

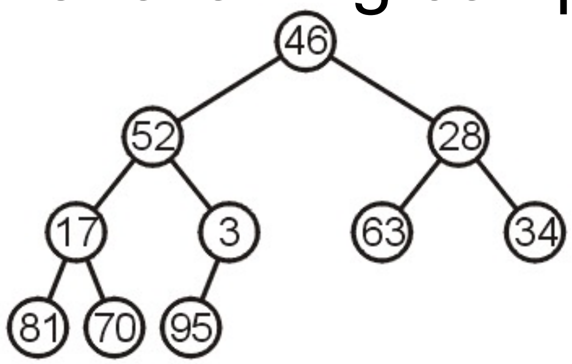


In-place Heapification

Now, consider this unsorted array:

46	52	28	17	3	63	34	81	70	95

This array represents the following complete tree:



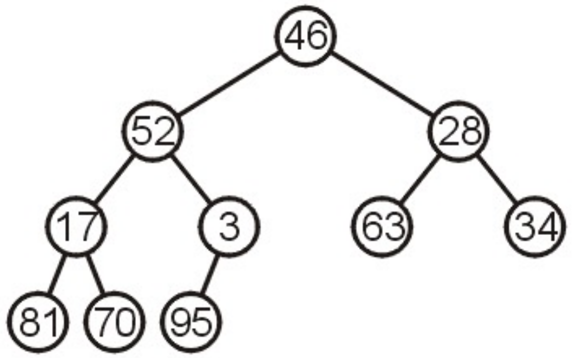
This is neither a min-heap, max-heap, or binary search tree

In-place Heapification

Now, consider this unsorted array:

46	52	28	17	3	63	34	81	70	95

Additionally, because arrays start at 0 (we started at entry 1 for binary heaps) , we need different formulas for the children and parent



The formulas are now:

Children `2*k + 1` `2*k + 2`

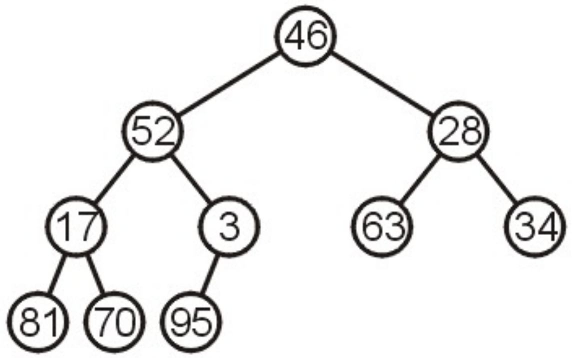
Parent `(k + 1)/2 - 1`

In-place Heapification

Can we convert this complete tree into a max heap?

Restriction:

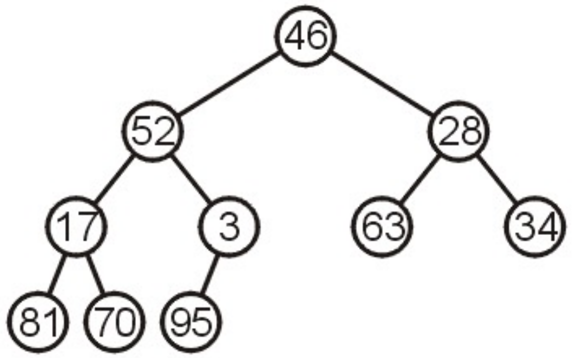
- The operation must be done in-place



In-place Heapification

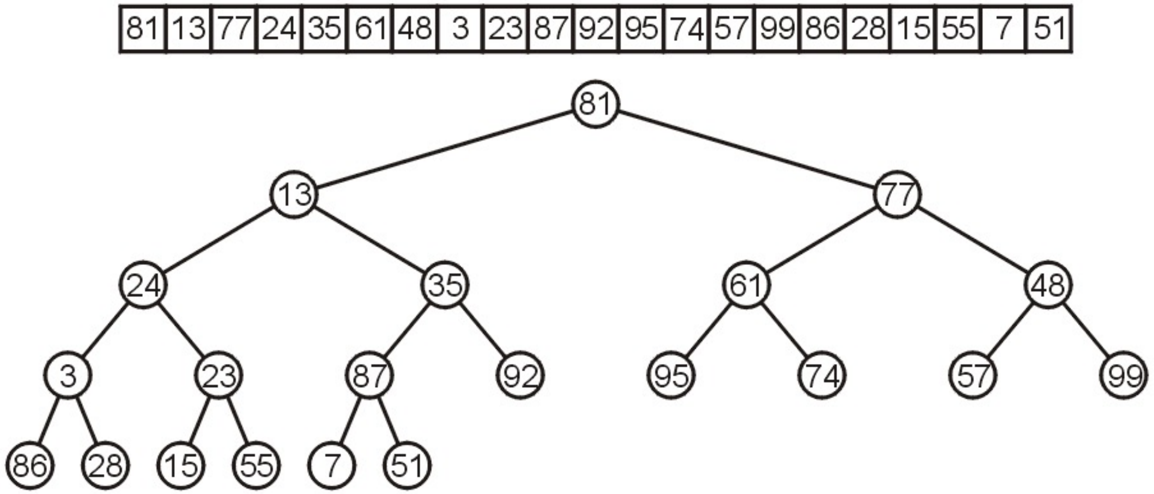
Two strategies:

- Assume 46 is a max-heap and keep inserting the next element into the existing heap (similar to the strategy for insertion sort)
- Start from the back: note that all leaf nodes are already max heaps, and then make corrections so that previous nodes also form max heaps



In-place Heapification

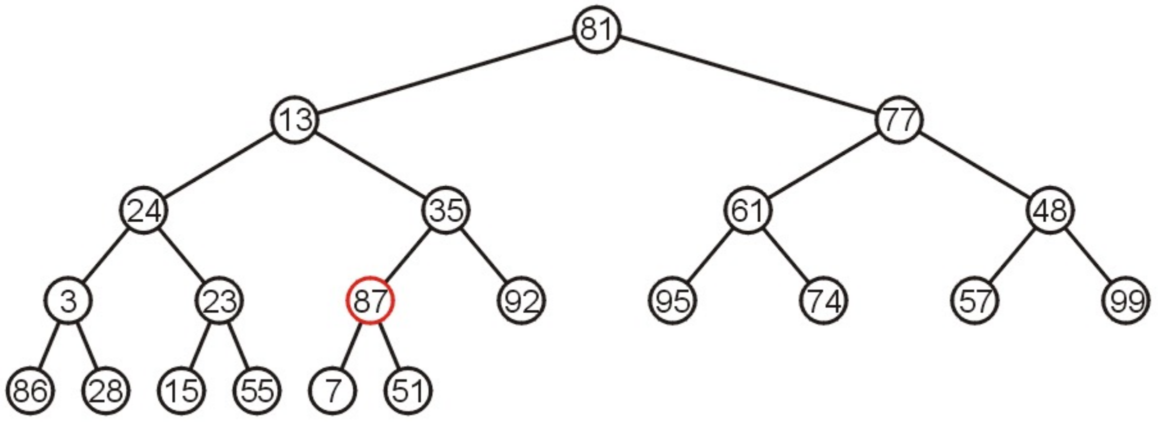
Let's work bottom-up: each leaf node is a max heap on its own



In-place Heapification

Starting at the back, we note that all leaf nodes are trivial heaps

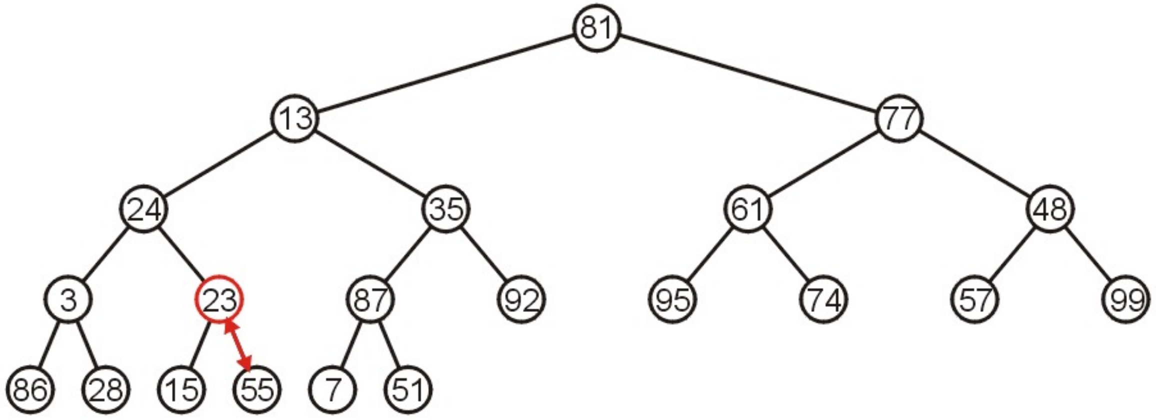
Also, the subtree with 87 as the root is a max-heap



In-place Heapification

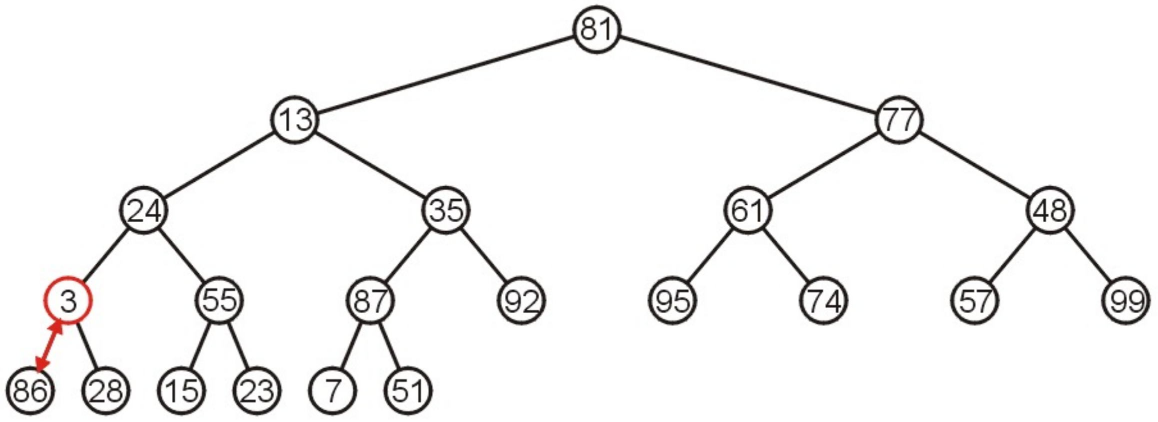
The subtree with 23 is not a max-heap, but swapping it with 55 creates a max-heap

This process is termed *percolating down*



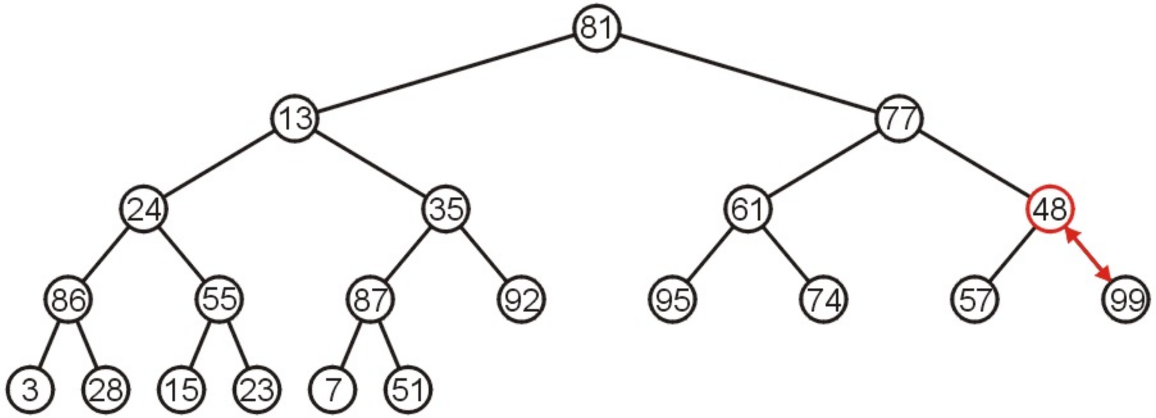
In-place Heapification

The subtree with 3 as the root is not max-heap, but we can swap 3 and the maximum of its children: 86



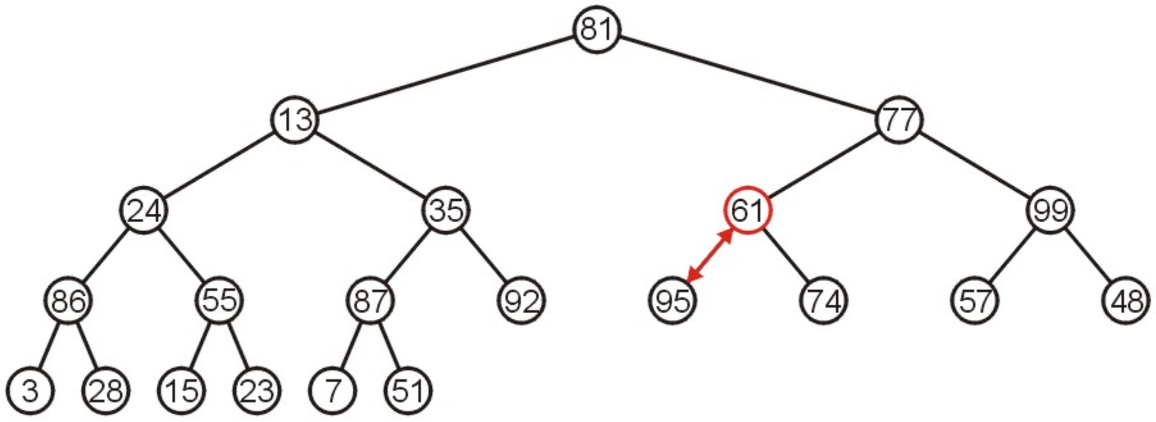
In-place Heapification

Starting with the next higher level, the subtree with root 48 can be turned into a max-heap by swapping 48 and 99



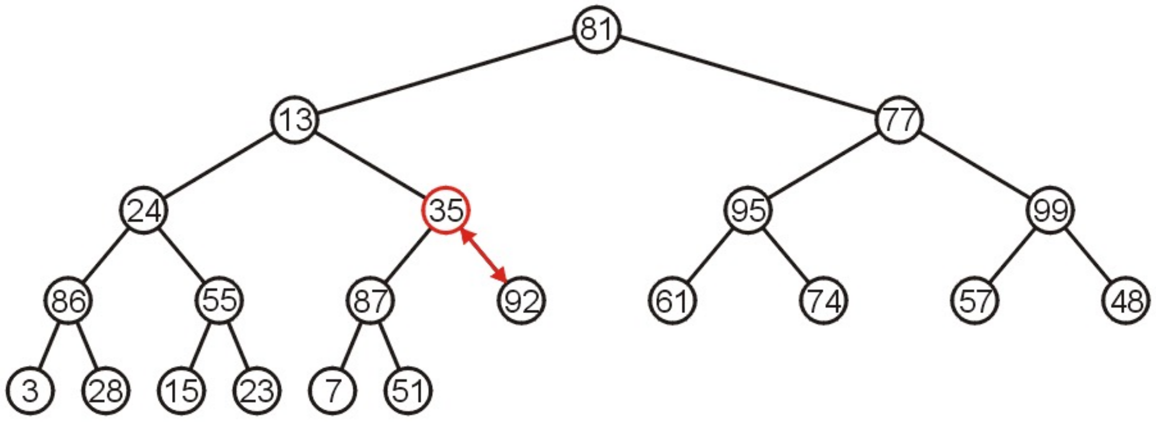
In-place Heapification

Similarly, swapping 61 and 95 creates a max-heap of the next subtree



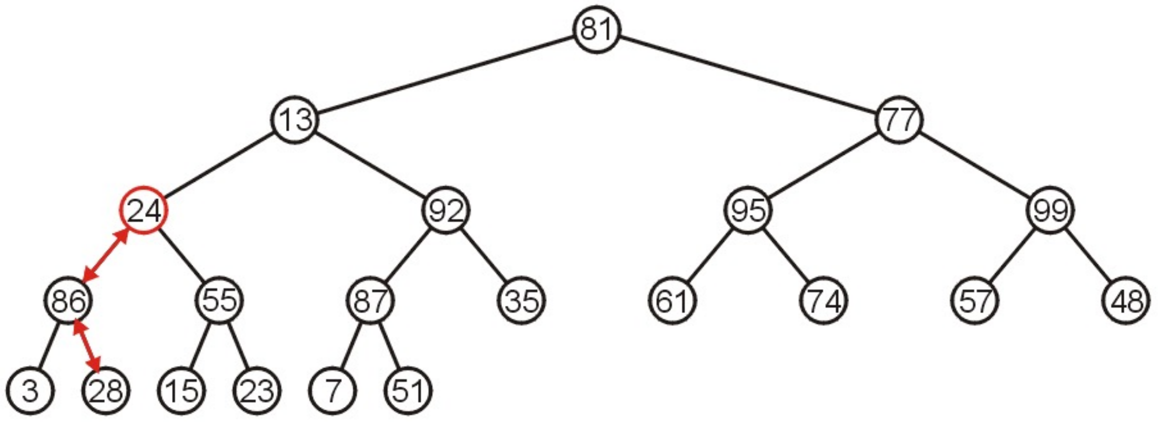
In-place Heapification

As does swapping 35 and 92



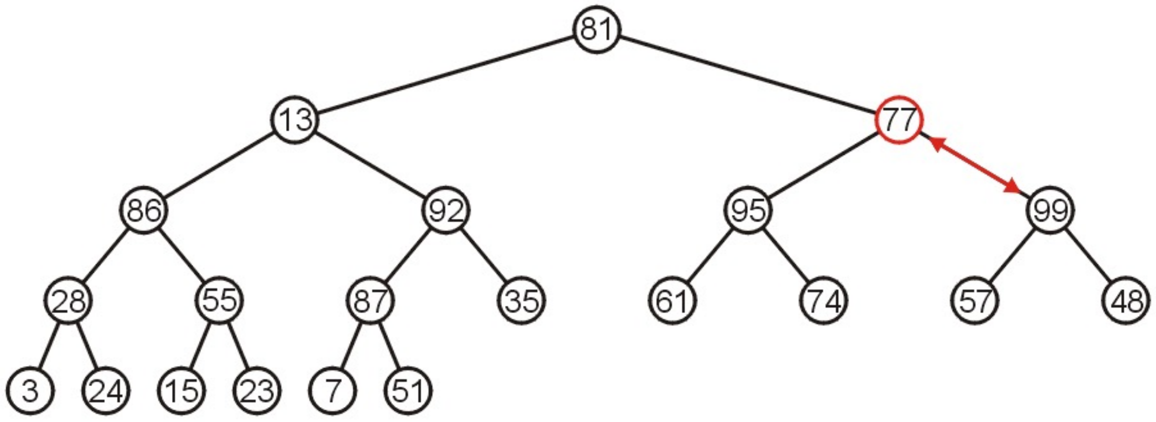
In-place Heapification

The subtree with root 24 may be converted into a max-heap by first swapping 24 and 86 and then swapping 24 and 28



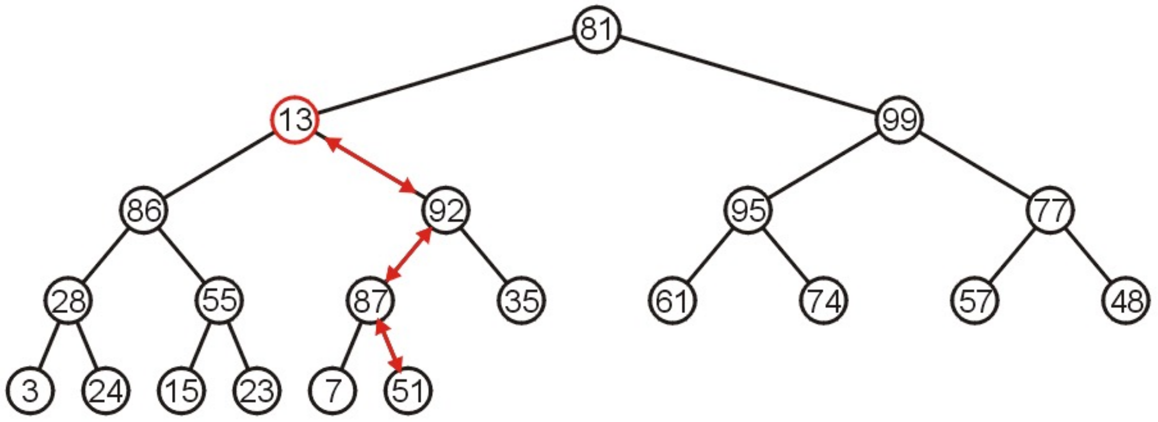
In-place Heapification

The right-most subtree of the next higher level may be turned into a max-heap by swapping 77 and 99



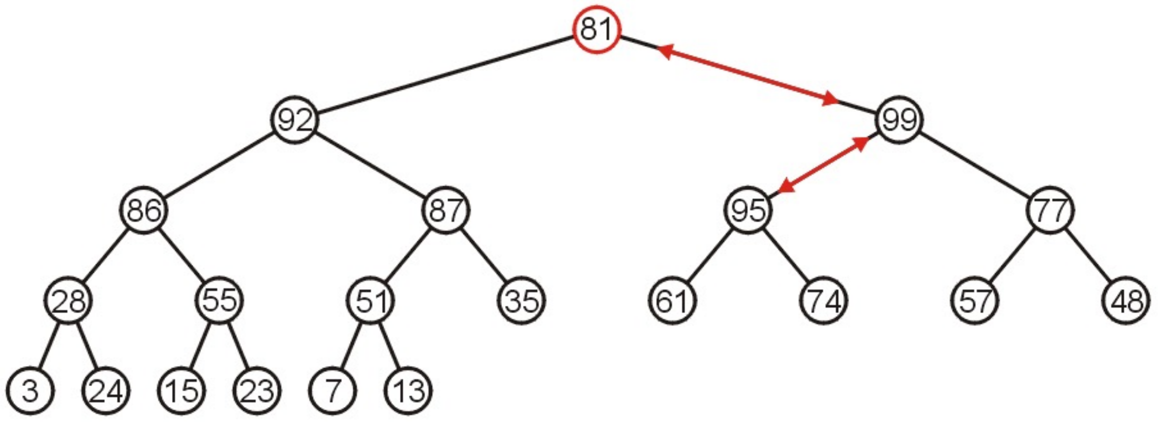
In-place Heapification

However, to turn the next subtree into a max-heap requires that 13 be percolated down to a leaf node



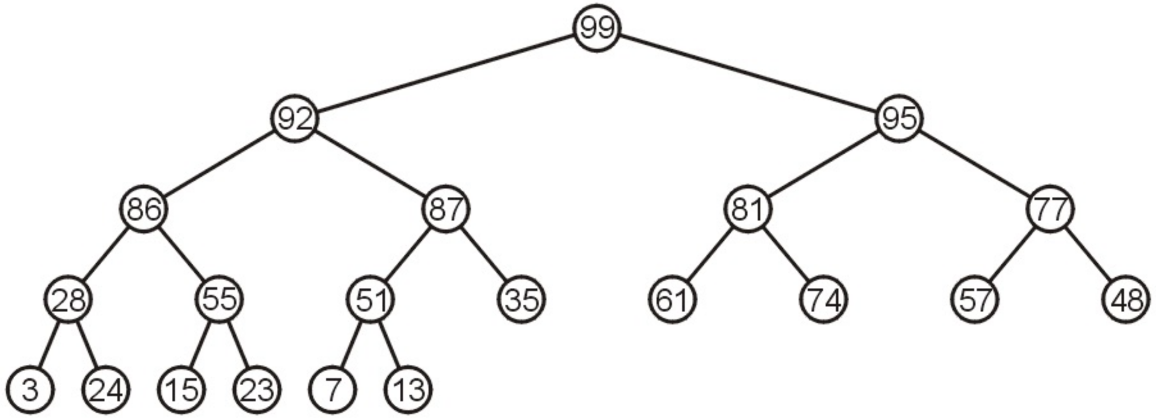
In-place Heapification

The root need only be percolated down by two levels



In-place Heapification

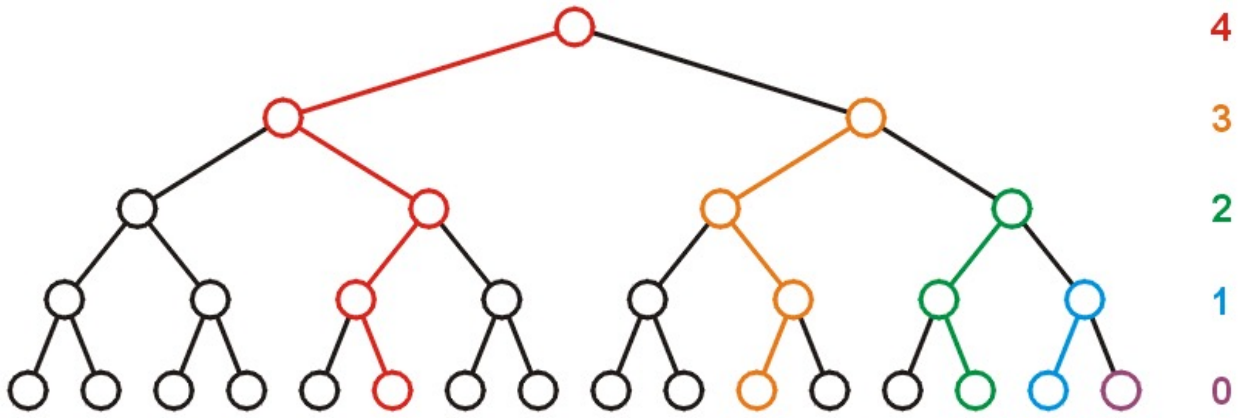
The final product is a max-heap



Run-time Analysis of Heapify

Considering a perfect tree of height h:

- The maximum number of swaps which a second-lowest level would experience is 1, the next higher level, 2, and so on



Run-time Analysis of Heapify

At depth k, there are 2^k nodes and in the worst case, all of these nodes would have to percolated down $h - k$ levels

- In the worst case, this would require a total of $2^k(h - k)$ swaps

Writing this sum mathematically, we get:

$$\sum_{k=0}^{h} 2^k (h - k) = \left(2^{h+1} - 1\right) - (h + 1)$$

Run-time Analysis of Heapify

Recall that for a perfect tree, $n = 2^{h+1} - 1$ and $h + 1 = \lg(n + 1)$, therefore

$$\sum_{k=0}^{h} 2^k (h-k) = n - \lg(n+1)$$

Each swap requires two comparisons (which child is greatest), so there is a maximum of $2n$ (or $\Theta(n)$) comparisons

Run-time Analysis of Heapify

Note that if we go the other way (treat the first entry as a max heap and then continually insert new elements into that heap, the run time is at worst

$$\begin{aligned}\sum_{k=0}^{h} 2^k k &= 2^{h+1}\left(h-1\right)+2 \\ &= \left(2^{h+1}+1\right)\left(h-1\right)-\left(h-1\right)+2 \\ &= n\left(\lg\left(n+1\right)-2\right)-\lg\left(n+1\right)+4=\Theta\left(n\ln\left(n\right)\right)\end{aligned}$$

- It is significantly better to start at the back

Example Heap Sort

Let us look at this example: we must convert the unordered array with $n = 10$ elements into a max-heap

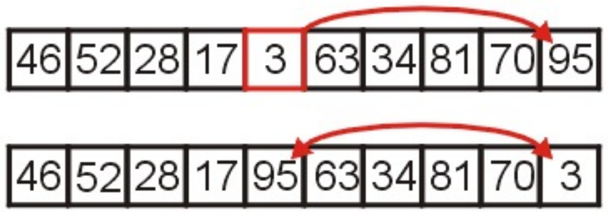
46	52	28	17	3	63	34	81	70	95

None of the leaf nodes need to be percolated down, and the first non-leaf node is in position $n/2$

Thus, we start with position $10/2 = 5$

Example Heap Sort

We compare 3 with its child and swap them



Example Heap Sort

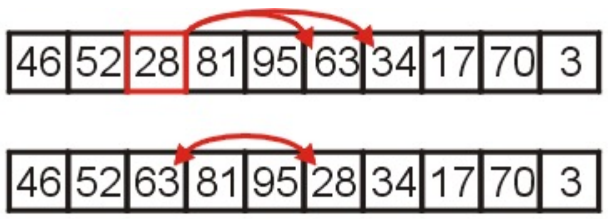
We compare 17 with its two children and swap it with the maximum child (70)

46	52	28	17	95	63	34	81	70	3

46	52	28	81	95	63	34	17	70	3

Example Heap Sort

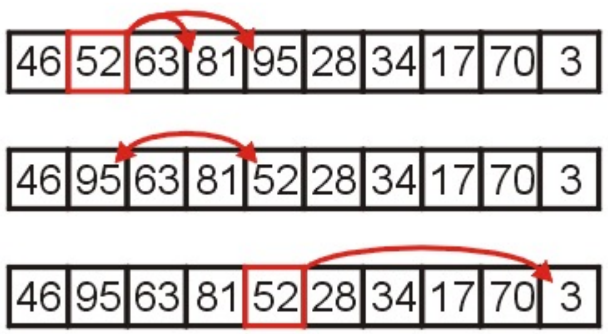
We compare 28 with its two children, 63 and 34, and swap it with the largest child



Example Heap Sort

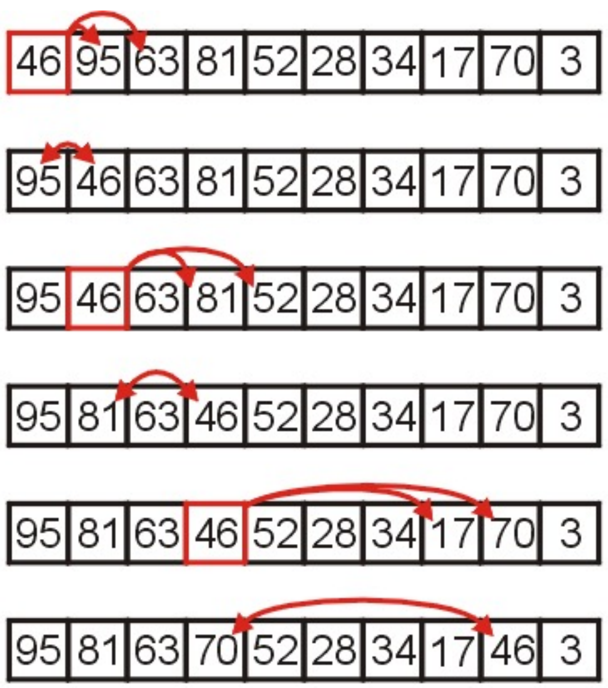
We compare 52 with its children, swap it with the largest

- Recursing, no further swaps are needed



Example Heap Sort

Finally, we swap the root with its largest child, and recurse, swapping 46 again with 81, and then again with 70



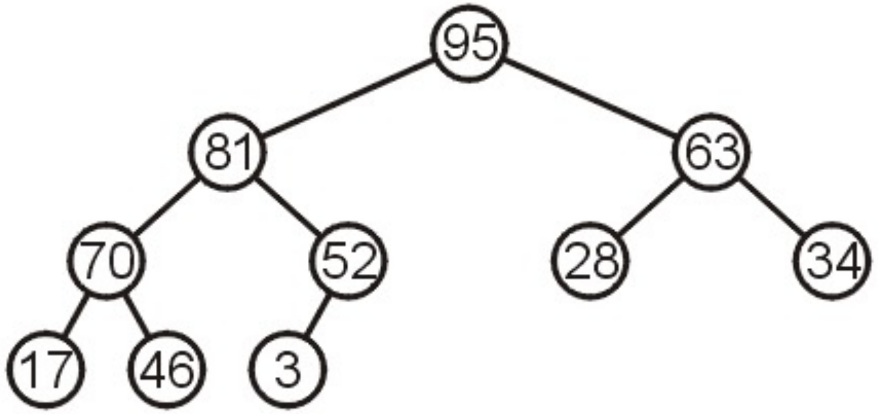
Heap Sort Example

We have now converted the unsorted array

46	52	28	17	3	63	34	81	70	95

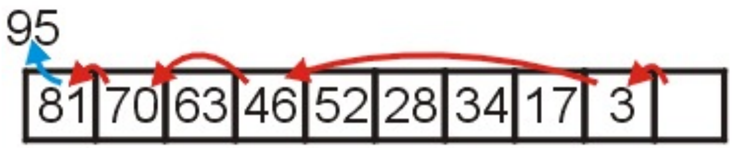
into a max-heap:

95	81	63	70	52	28	34	17	46	3

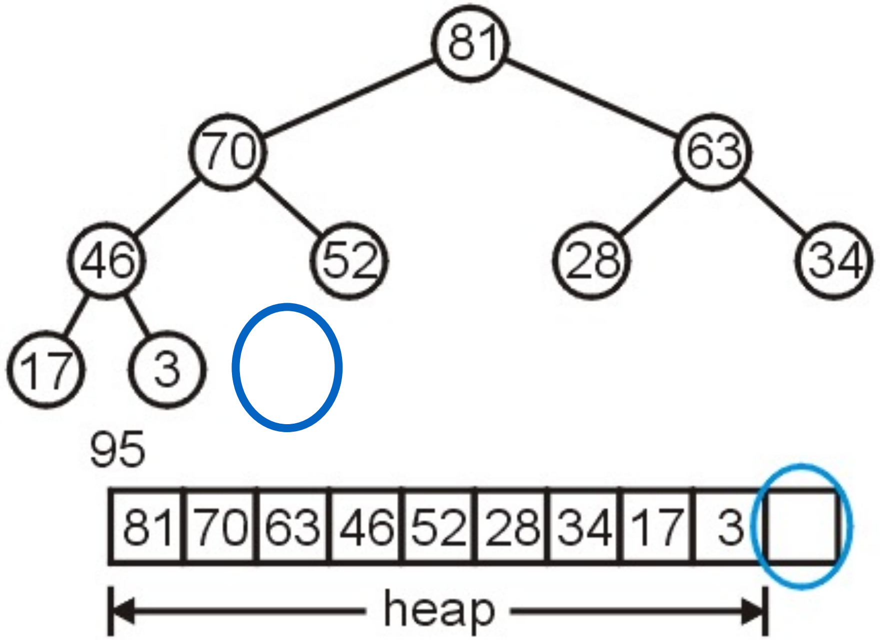


Heap Sort Example

Suppose we pop the maximum element of this heap

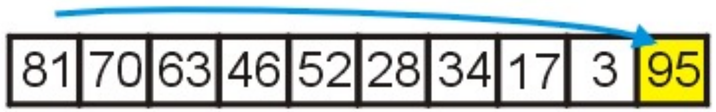


This leaves a gap at the back of the array:

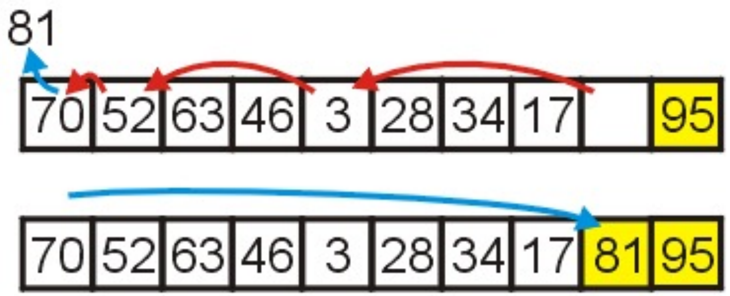


Heap Sort Example

This is the last entry in the array, so why not fill it with the largest element?



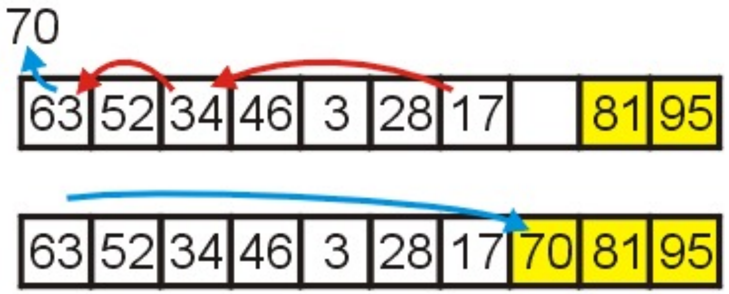
Repeat this process: pop the maximum element, and then insert it at the end of the array:



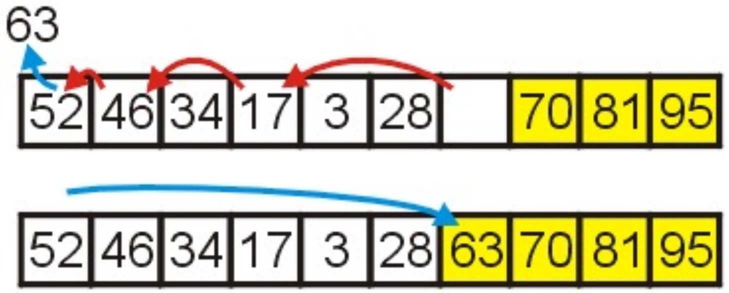
Heap Sort Example

Repeat this process

- Pop and append 70



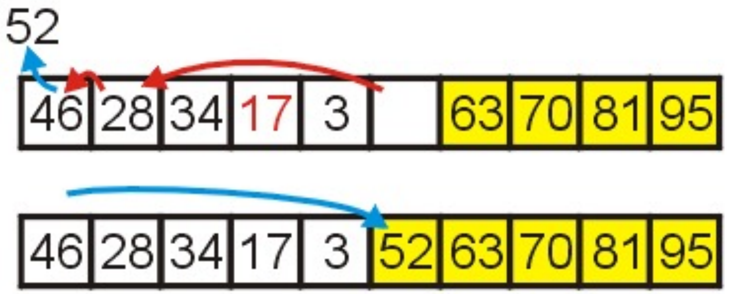
- Pop and append 63



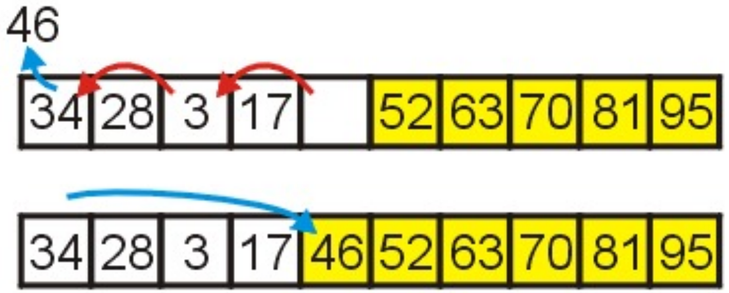
Heap Sort Example

We have the 4 largest elements in order

- Pop and append 52



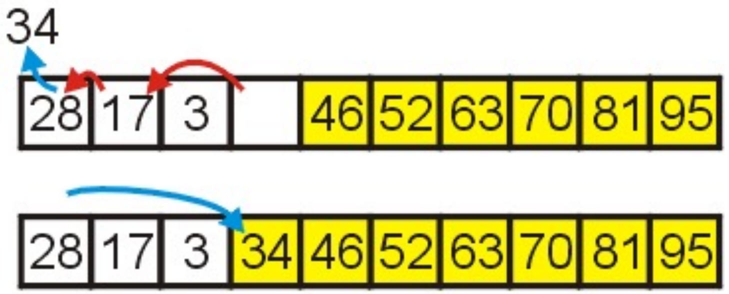
- Pop and append 46



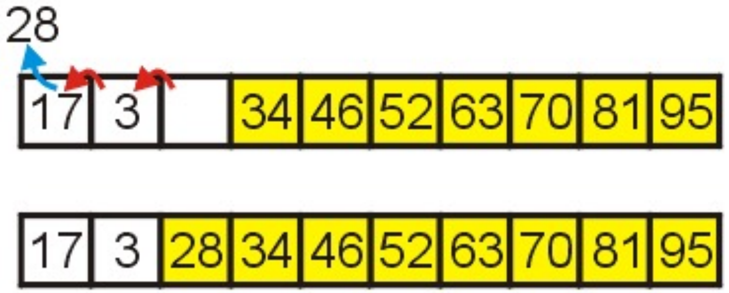
Heap Sort Example

Continuing...

- Pop and append 34

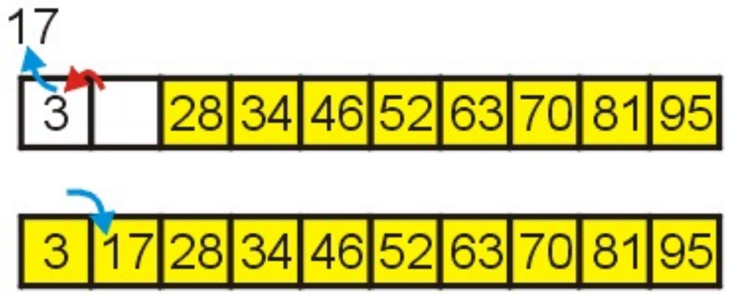


- Pop and append 28



Heap Sort Example

Finally, we can pop 17, insert it into the 2nd location, and the resulting array is sorted



Black Board Example

Sort the following 12 entries using heap sort

34, 15, 65, 59, 79, 42, 40, 80, 50, 61, 23, 46

Heap Sort

Heapification runs in $\Theta(n)$

Popping n items from a heap of size n, as we saw, runs in $\Theta(n \ln(n))$ time

- We are only making one additional copy into the blank left at the end of the array

Therefore, the total algorithm will run in $\Theta(n \ln(n))$ time

Heap Sort

There are no worst-case scenarios for heap sort

- Dequeuing from the heap will always require the same number of operations regardless of the distribution of values in the heap

There is one best case: if all the entries are identical, then the run time is $\Theta(n)$

The original order may speed up the *heapification*, however, this would only speed up an $\Theta(n)$ portion of the algorithm

Run-time Summary

The following table summarizes the run-times of heap sort

Case	Run Time	Comments
Worst	$\Theta(n \ln(n))$	No worst case
Average	$\Theta(n \ln(n))$	
Best	$\Theta(n)$	All or most entries are the same

Summary

We have seen our first in-place $\Theta(n \ln(n))$ sorting algorithm:

- Convert the unsorted list into a max-heap as complete array
- Pop the top n times and place that object into the vacancy at the end
- It requires $\Theta(1)$ additional memory—it is truly in-place

It is a nice algorithm; however, we will see two other faster $n \ln(n)$ algorithms; however:

- Merge sort requires $\Theta(n)$ additional memory
- Quick sort requires $\Theta(\ln(n))$ additional memory