field lines, since to zero order they are in the z direction (unless the $\partial B_x/\partial x$ and $\partial B_y/\partial y$ terms cancel).

Before computing in detail what will happen to particle orbits, we may look at this problem in view of what we have already learned. Suppose the particle is gyrating about B_z and at the same time moving along the B lines — i.e., in the z direction. Then it will see a magnetic field whose strength is changing in time. By our assumption that $\vec{B}$ is slowly varying in space, this will be a slow time variation provided the motion of the particle along $\vec{B}$ is not extremely fast. From our treatment of the time-varying $\vec{B}$ field we should expect that the perpendicular energy of the particle would vary in such a way as to keep the magnetic moment constant

$$W_\perp = |B|\mu = |B| \frac{W_{\perp 0}}{B_0}. \tag{107}$$

Now the particle's energy must be constant since the magnetic field does no work on it and hence there must be an equal and opposite change in the parallel energy of the particle.

$$W_\perp + W_\parallel = W = \text{constant}, \tag{108}$$

$$W_\parallel = W - |B|\mu = W_{\parallel 0} + W_{\perp 0} - |B|\mu. \tag{109}$$

Thus $|B|\mu$ acts like a potential for the motion along the lines of force. Eq. (109) may be written in differential form. For the time interval dt we have

$$dW_\parallel = m v_\parallel \, dv_\parallel = -\mu \frac{d|B|}{dz} v_\parallel \, dt \tag{110}$$

or

$$m\frac{dv_{\parallel}}{dt} = -\mu\frac{d|B|}{dz}. \tag{111}$$

This should be a familiar form. The force on a magnetic dipole is the product of the dipole moment and the field gradient. The negative sign results from the fact that the dipole is diamagnetic. Equivalently we can see that this force comes about because of the interaction of the particle's perpendicular motion with the radial $\vec{B}$ field, as shown in Fig. 27. These conclusions are actually borne out by the more detailed calculations which we shall now give. The equations of motion for our particle are

$$m\frac{d\vec{V}}{dt} = \frac{q}{c}\vec{V}\times\vec{B}. \tag{112}$$

Substituting in $\vec{B}$ from Eqs. (95), (96), and (97), and writing in component form gives

$$m\frac{dv_z}{dt} = \frac{q}{c}\left[v_x\left(\frac{\partial B_y}{\partial y}\right)_0 y - v_y\left(\frac{\partial B_x}{\partial x}\right)_0 x\right], \tag{113}$$

$$m\frac{dv_x}{dt} = \frac{q}{c}\left[v_y B_0 - v_z\left(\frac{\partial B_y}{\partial y}\right)_0 y\right], \tag{114}$$

and

$$m\frac{dv_y}{dt} = -\frac{q}{c}\left[v_x B_0 - v_z\left(\frac{\partial B_x}{\partial x}\right)_0 x\right]. \tag{115}$$

We consider a particle whose center of gyration (guiding center) is instantaneously at the origin — i.e., z = 0 . Now the zero order solutions of Eqs. (114) and (115) are

$$w = w_0 e^{-i\omega_c t} = v_x + i v_y$$

and

$$\xi = -\frac{w_0}{i\omega_c} e^{-i\omega_c t} = x + i y$$

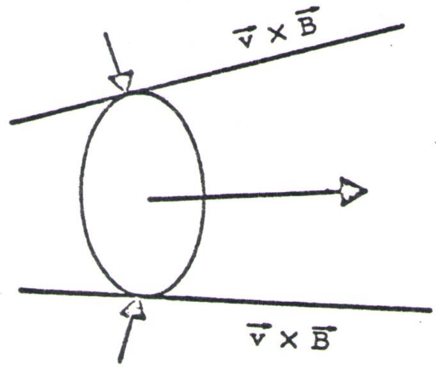


Figure 27

where
$$\omega_c = \frac{qB}{mc}.$$

We may choose γ_0 to be 0 by proper choice of phase or coordinates; also, we may choose w_o to be real. Substituting in Eq. (113) gives

$$m\frac{dv_z}{dt} = \frac{qw_0^2}{c\omega_c}\left(-\sin^2\omega_c t\left(\frac{\partial B_y}{\partial y}\right)_0 - \cos^2\omega_c t\left(\frac{\partial B_x}{\partial x}\right)_0\right). \tag{116}$$

Averaging over one orbit gives

$$m\overline{\frac{dv_z}{dt}} = \frac{q w_0^2}{2c\omega_c}\left[\left(\frac{\partial B_x}{\partial x}\right)_0 + \left(\frac{\partial B_y}{\partial y}\right)_0\right] =$$

$$= \frac{-q w_0^2}{2c\omega_c}\left(\frac{\partial B_z}{\partial z}\right)_0. \tag{117}$$

Now w_o^2 is $2W_\perp/m$, so that we may rewrite Eq. (117) in the form

$$m\overline{\frac{dv_z}{dt}} = -\frac{W_\perp}{B}\left(\frac{\partial B_z}{\partial z}\right) = -\mu\left(\frac{\partial B_z}{\partial z}\right) \tag{118}$$

Now multiplying both sides of Eq. (118) by v_z gives

$$\frac{m}{2}\overline{\frac{d}{dt}(v_z)^2} = -\mu\frac{dB_z}{dt}. \tag{119}$$

Multiplying Eq. (114) by v_x and Eq. (115) by v_y and adding gives

$$\frac{m}{2}\frac{d}{dt}(v_x^2+v_y^2) = \frac{qv_z}{c}\left[x v_y\left(\frac{\partial B_x}{\partial x}\right)_0 - y v_x\left(\frac{\partial B_y}{\partial y}\right)_0\right] \tag{120}$$

This is just $-v_z$ times Eq. (113), so that we find

$$\frac{m}{2}\frac{d(v_z^2)}{dt} = -\frac{m}{2}\frac{d(v_\perp^2)}{dt} \quad \text{where} \quad v_\perp^2 = v_x^2+v_y^2 \tag{121}$$

which is just the equation for conservation of energy. Thus we may write Eq. (119) in the form

$$\frac{dW_\perp}{dt} = \frac{m}{2}\frac{d(v_\perp^2)}{dt} = \mu \frac{dB_z}{dt} = \frac{W_\perp}{B}\frac{dB_z}{dt} \tag{122}$$

or

$$\frac{d}{dt}\left(\frac{W_\perp}{B}\right) = \frac{d\mu}{dt} = 0. \tag{123}$$

Thus the magnetic moment is constant in this spatially-varying B field as well as in a time-varying field (provided the variations are not too rapid) and our previous analysis is justified.

(2) Effects of $(\partial B_x/\partial z)$ and $(\partial B_y/\partial z)$ (Curvature of the Lines of Force)

Let us now look at terms of the form $(\partial B_x/\partial z)_o$ and $(\partial B_y/\partial z)_o$. We need consider only one of these, since by appropriate orientation of the xy plane the other can be eliminated. That is, if we choose our x axis to point in the direction $(\partial B_\perp/\partial z)$ ($B_\perp$ is the component of $\vec{B}$ perpendicular to z), then we get only a $(\partial B_x/\partial z)$ term.

Let us again look at what this implies about the shape of the lines of force. We return to Eq. (98) for the lines of force, and again to zero order in derivatives we find straight lines of force. To first order we find

$$\frac{dx_1}{dz} = \frac{\left(\frac{\partial B_x}{\partial z}\right)_o z}{B_o} \tag{124}$$

or

$$x_1 = x_o + \frac{z^2}{2B_o}\left(\frac{\partial B_x}{\partial z}\right)_o \tag{125}$$

The lines of force are curved as shown in Fig. 28. For small z we find that the curve may be regarded as a segment of a circle.

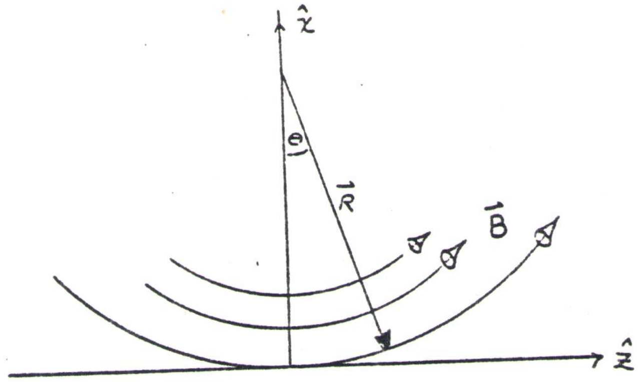


Figure 28

From this figure we have

$$\frac{B_x}{B_o} = \tan\theta \cong \theta \cong \frac{z}{R}. \tag{126}$$

Thus

$$z/R = B_x/B_o = \frac{z}{B_o}\left(\frac{\partial B_x}{\partial z}\right)_o \tag{127}$$

or

$$R = \frac{B_o}{\left(\frac{\partial B_x}{\partial z}\right)_o} \tag{128}$$

In general the vector radius of curvature $\vec{R}$ of a curve is given in terms of the unit tangent to the curve $\vec{n}_t$ (in this case $\vec{n}_t = \frac{\vec{B}}{|B|}$).

The relation is

$$\frac{\vec{R}}{|R^2|} = -(\vec{n}\cdot\vec{\nabla})\vec{n} \tag{129}$$

To solve this problem we introduce the local cylindrical coordinates such that the axis of the cylinder is perpendicular to the local plane of the field lines and so that it passes through their center of curvature, as shown in Fig. 29(a).

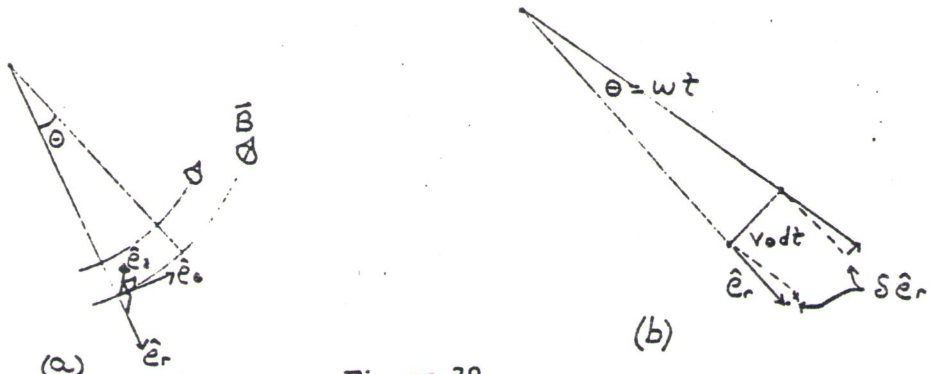


Figure 29

We choose axes so that the unit vector e_θ lies along B, the e_r unit vector is perpendicular to B and in the plane of B and points away from the center of curvature and e_z is chosen normal to θ and r so that e_r, e_θ, e_z forms a right-handed coordinate system. Locally the magnetic field has only a θ component. We have for $d\vec{v}/dt$

$$\frac{d\vec{V}}{dt} = \hat{e}_r \frac{dV_r}{dt} + V_r \frac{d\hat{e}_r}{dt} + \hat{e}_\theta \frac{dV_\theta}{dt} + V_\theta \frac{d\hat{e}_\theta}{dt} + \hat{e}_z \frac{dV_z}{dt} \tag{130}$$

and from Fig. 29(b)

$$\frac{d\hat{e}_r}{dt} = \frac{V_\theta}{r}\hat{e}_\theta, \qquad \frac{d\hat{e}_\theta}{dt} = -\frac{V_\theta}{r}\hat{e}_r. \tag{131}$$

Our equation of motion becomes

$$\begin{aligned} m\frac{d\vec{V}}{dt} &= m\left\{\hat{e}_r\left[\frac{dV_r}{dt} - \frac{V_\theta^2}{r}\right] + \hat{e}_\theta\left[\frac{dV_\theta}{dt} + \frac{V_\theta V_r}{r}\right] + \hat{e}_z\frac{dV_z}{dt}\right\} \\ &= \frac{q}{c}\vec{V}\times\vec{B} = \frac{qB_0}{c}\left(-\hat{e}_r V_z + \hat{e}_z V_r\right). \end{aligned} \tag{132}$$

The scalar equation representing e_θ terms gives

$$\frac{dV_\theta}{dt} = -\frac{V_\theta V_r}{r} \tag{133}$$

or

$$\frac{d v_\theta}{v_\theta} = -\frac{dr}{r} \quad \text{(where } v_r = \frac{dr}{dt}\text{)}. \tag{134}$$

$$v_\theta = \frac{v_0 r_0}{r}. \tag{135}$$

This just says that the angular momentum $\propto v_\theta r$ about the center of curvature of the lines is conserved and this leads to slight fluctuations of the v_θ as the particle gyrates about the B lines. The other two scalar equations are

$$m\left(\frac{dv_r}{dt} - \frac{v_\theta^2}{r}\right) = -\frac{q}{c} v_z B_0 \tag{136}$$

and

$$m\left(\frac{dv_z}{dt}\right) = \frac{q}{c} v_r B_0. \tag{137}$$

Now if we neglect the slight fluctuations in v_θ just found (these are of the order of the ratio of the Larmor radius to the radius of curvature of the lines of force and are hence small by the assumption that variations of B over regions of the size of a Larmor orbit are small), then these are the equations for the gyration of a particle about a uniform magnetic field when subjected to an external force of magnitude $+\frac{m v_\theta^2}{r}$. This is the centrifugal force which acts on the particle when it follows the curved field lines. It gives rise to a drift in the z direction equal to

$$v_{zD} = \frac{c m v_\theta^2}{r q B_0} = \frac{2 W_{\parallel} c}{R q B}. \tag{138}$$

The v_{zD} drift results in a current since it depends on q . This current, in turn, produces the centripetal force required for the circular motion.

We can write Eq. (138) in vector form

$$\vec{V}_r = \frac{2cW_{\parallel}}{q\,B^2}\,\vec{B}\times(\vec{n}\cdot\vec{\nabla})\vec{n} \tag{139}$$

(3) Effect of $\partial B_z/\partial x$ and $\partial B_z/\partial y$

These terms do not give rise to any slope or curvature of the B lines, but simply state that the strength of the magnetic field varies in the xy plane. Again we need only consider one of these terms, since we can choose a coordinate system in which the other is zero. That is, we can, say, choose the x axis so that it lies along $\vec{\nabla}_\perp B_z$ and then $\partial B_z/\partial y = 0$, $\vec{\nabla}_\perp$ means $(\vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y})$. The situation is shown in Fig. 30.

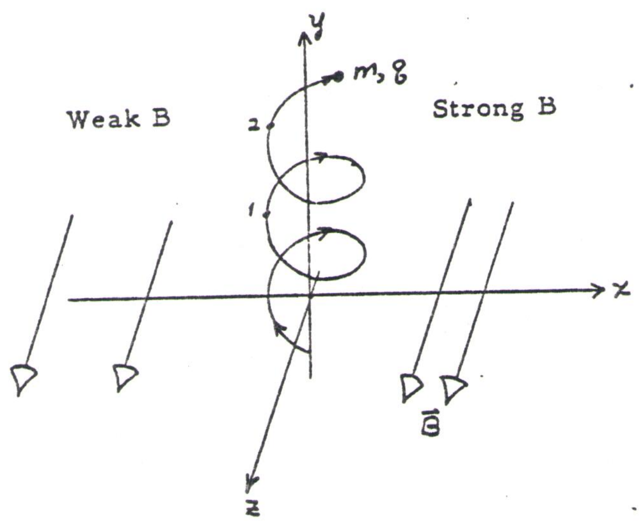


Figure 30

If the charged particle were to execute an undisplaced circular orbit, the force in the negative x direction, while the particle is in the right half orbit in the stronger magnetic field, would exceed the opposite force during the time the particle is in the left half orbit. The drift along y produces a $\vec{j} \times \vec{B}$ force along x which just compensates this.

A simple calculation of this effect starts out by noting that the average force along x must be zero. We average over one cycle of the periodic motion

$$\int_{t_1}^{t_2} F_x \, dt = 0 \tag{140}$$

where

$$F_x = \frac{e}{c} v_y B_z(x) = \frac{e}{c} v_y \left(B_o + x \left(\frac{\partial B_z}{\partial x} \right)_o \right). \tag{141}$$

Substituting Eq. (141) in Eq. (140),

$$\int_{t_1}^{t_2} B_o v_y \, dt + \int_{t_1}^{t_2} x \left(\frac{\partial B_z}{\partial x} \right)_o v_y \, dt = 0 \tag{142}$$

thus

$$\delta y = y_2 - y_1 = -\frac{1}{B_o} \left(\frac{\partial B_z}{\partial x} \right)_o \int_{t_1}^{t_2} x v_y \, dt \tag{143}$$

since B_o and $\left(\frac{\partial B_z}{\partial x} \right)_o$ are constants. Since the field changes are small by assumption, the orbits are only slightly disturbed from circular ones: we use for the integral of $x v_y \, dt$ over one period simply πa^2, where a is the Larmor radius, giving

$$\delta y = -\frac{1}{B_o} \left(\frac{\partial B_z}{\partial x} \right)_o \pi a^2 \tag{144}$$

which can be written

$$\delta y = -\frac{1}{B_o} \frac{\partial B_z}{\partial x} \left(\frac{2\pi}{\omega_c} \right) \left(\frac{m}{2} v_\perp^2 \right) \frac{c}{e B_o} \tag{145}$$

where δy is the displacement of the orbit in a time of one cycle, $2\pi/\omega_c$. Then the drift velocity is δy divided by $2\pi/\omega_c$

$$V_y = -\frac{c W_\perp}{q B_0^2} \frac{\partial B_z}{\partial x} \tag{146}$$

or, in general, since the drift is in the direction $\vec{n} \times \vec{\nabla} B$,

$$V_G = \frac{c W_\perp}{q B^2} \vec{n} \times \vec{\nabla}(\vec{B} \cdot \vec{n}). \tag{147}$$

(4) Effects of $\partial B_x/\partial y$ and $\partial B_y/\partial x$

These components of $\vec{\nabla}\vec{B}$ represent shear or twisting of the magnetic lines of force, as shown in Fig. 31.

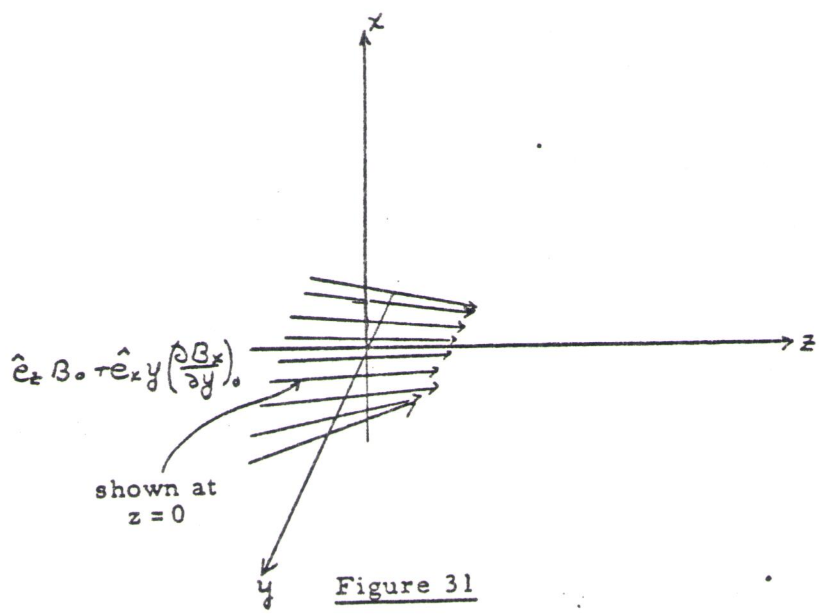


Figure 31

We can solve for their effects in the same manner as we did for $(\partial B_z/\partial x)$. They give rise only to driving terms at $2\omega_c$ in the w_1 equation, and hence result in distortions of the orbit, but give rise to no net drift of the particles.

V. Summary of Drifts and Currents

A. Drifts

• Electric $$\vec{V}_E = c\,\frac{\vec{E}\times\vec{B}}{B^2} \qquad (1)$$

Dielectric, $\dot{E}$ $$\vec{V}_{\dot{E}} = \frac{mc^2}{q B^2}\underbrace{\vec{n}\times(\dot{\vec{E}}\times\vec{n})}_{\dot{\vec{E}}_\perp} \qquad (2)$$

Curvature $$\vec{V}_R = \frac{2c\,W_\parallel}{q B^2}\left[\vec{B}\times(\vec{n}\cdot\vec{\nabla})\vec{n}\right] \qquad (3)$$

$$\vec{n} = \vec{B}/|B|$$

Gradient $$\vec{V}_G = \frac{c\,W_\perp}{q B^2}\left[\vec{n}\times\vec{\nabla}(\vec{B}\cdot\vec{n})\right] \qquad (4)$$

External Force $$\vec{V}_F = \frac{c}{q}\,\frac{\vec{F}\times\vec{B}}{B^2} \qquad (5)$$

B. Currents

Magnetization $$\vec{J}_m = c\;\vec{\nabla}\times\vec{M} \qquad (6)$$

$$M = -NW_\perp\,\frac{\vec{B}}{B^2}$$

Polarization $$\vec{J}_P = \frac{\epsilon}{4\pi}\vec{n}\times(\dot{\vec{E}}\times\vec{n}) = \frac{\epsilon}{4\pi}\dot{\vec{E}} \qquad (7)$$

$$\epsilon = \frac{4\pi N m c^2}{B^2}$$

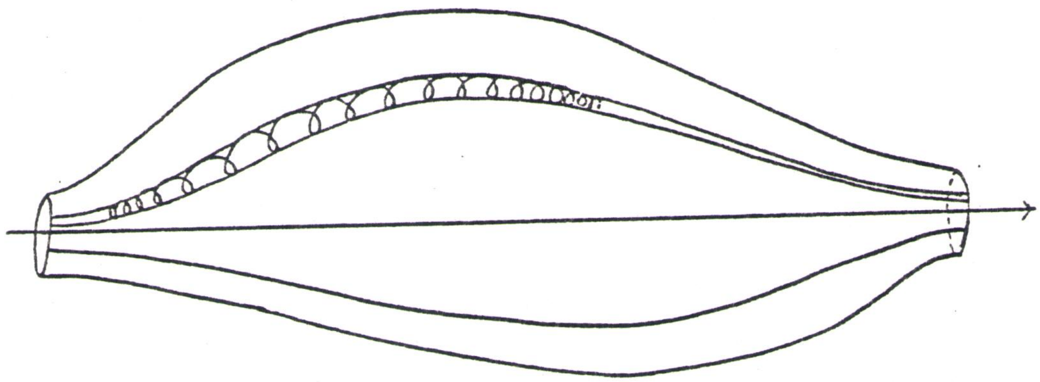
Curvature $$\vec{J}_R = \frac{2Nc\,W_\parallel}{B^2}\;\underbrace{\vec{B}\times((\vec{n}\cdot\vec{\nabla})\vec{n})}_{\vec{R}\times\vec{B}/R^2} \qquad (8)$$

Gradient $$\vec{J}_G = \frac{Nc\,W_\perp}{B^2}\;\vec{n}\times\vec{\nabla}(\vec{B}\cdot\vec{n}) \qquad (9)$$

External Force $$\vec{J}_F = Nc\;\frac{\vec{F}\times\vec{B}}{B^2} \qquad (10)$$

VI. The Longitudinal Invariant

We have already shown that the magnetic moment μ of a particle is an adiabatic invariant. Because of this property a particle may be trapped between regions of high magnetic field, as shown in Fig. 32.



The particle executes a periodic motion back and forth between the regions of large B. There is a second adiabatic invariant associated with this motion which we will now investigate.

There is an adiabatic invariant associated with any periodic motion. This invariant is the action associated with the motion

$$J = \oint p \, dq \tag{1}$$

where p is the momentum, q is the position coordinate, and the integral is to be taken over a closed orbit. For the motion considered here, the appropriate invariant is

$$J = m \int \vec{V_{\parallel}} \cdot d\vec{S} \tag{2}$$

where $\vec{v}_{\parallel}$ is the velocity parallel to the magnetic lines of force, and $\vec{ds}$ is an element of arc length along the magnetic field lines.

In our earlier work we found

$$m\frac{dv_{\parallel}}{dt} = -\mu\frac{\partial B}{\partial s} \tag{3}$$

or

$$m\frac{ds}{dt}\frac{dv_{\parallel}}{ds} = m v_{\parallel}\frac{dv_{\parallel}}{ds} = -\mu\frac{\partial B}{\partial s}. \tag{4}$$

Eq. (4) can be integrated directly to yield an energy relation for the motion

$$W = \frac{mv_{\parallel}^2}{2} + \mu B. \tag{5}$$

Thus if B is time independent, W will be conserved. However, for time-varying B, W will also vary with time.

We may solve Eq. (5) for $v_{\parallel}$ and thus have

$$v_{\parallel} = \pm\sqrt{\frac{2}{m}(W-\mu B)}. \tag{6}$$

Now let B vary slowly with time (slowly means slow compared to the period of oscillation between regions of strong magnetic field — i.e., between magnetic mirrors). We may compute the change in J during one period.

$$\Delta J = \sqrt{\frac{m}{2}}\left\{\oint\frac{\Delta W-\mu\Delta B}{\pm\sqrt{W-\mu B}}ds\right\}. \tag{7}$$

Here ΔW and ΔB are the changes in W and B during one period. The integral is understood to be evaluated at one instant of time (fixed W and B). Now ΔB is given by

$$\Delta B(s) = \frac{\partial B}{\partial s}T \tag{8}$$

where T is to be the period of one oscillation. We have for T

$$T = \oint \frac{ds}{V_{\parallel}} = \sqrt{\frac{m}{2}} \oint \frac{ds}{\pm\sqrt{W-\mu B}} \tag{9}$$

and hence $\Delta B(s)$ is given by

$$\Delta B = \frac{\partial B}{\partial t} \sqrt{\frac{m}{2}} \oint \frac{ds}{\pm\sqrt{W-\mu B}}. \tag{10}$$

The change in W is given by

$$\Delta W = \int_0^T \frac{dW}{dt} dt; \tag{11}$$

the integration is to be carried out along an orbit.

From Eq. (5) we have

$$W = \frac{m V_{\parallel}^2}{2} + \mu B = \frac{m \dot{s}^2}{2} + \mu B \tag{12}$$

and

$$\frac{dW}{dt} = m \dot{s} \ddot{s} + \mu \frac{dB}{dt} = m \dot{s} \ddot{s} + \mu \frac{\partial B}{\partial t} + \mu \frac{\partial B}{\partial s} \dot{s} \tag{13}$$

implies d/dt. But $m \ddot{s} = -\mu \frac{\partial B}{\partial s}$ and hence

$$\frac{dW}{dt} = \mu \frac{\partial B}{\partial t}. \tag{14}$$

This is quite a reasonable result, for μB is the effective potential energy and Eq. (13) states that the rate of change of the particle's energy is equal to the rate of change of the potential energy at the point where the particle is located; or, to express this result in another way, the particle does not change energy in moving through an arbitrary magnetic field, unless the field changes in time. From Eq. (14) we find for the change in ΔW

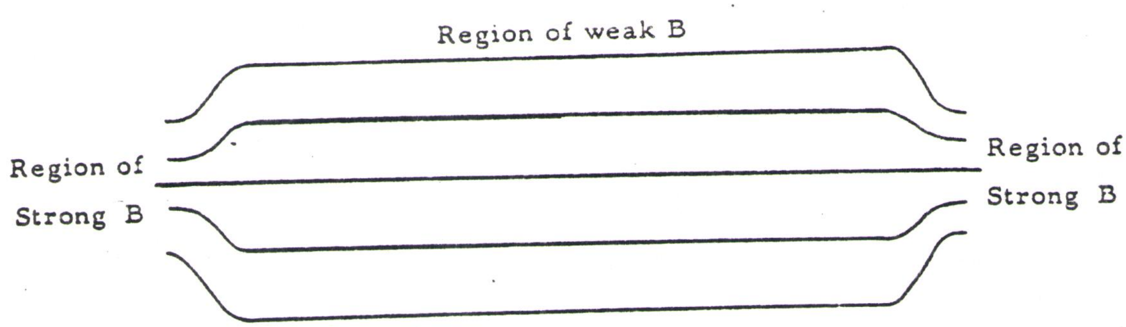


Figure 33

becomes strong is much shorter than L and so we can neglect these regions in evaluating J (we have sharp reflecting boundaries). For this situation J is given by

$$J = m\oint V_{\parallel}\, dS = 2m\, |V_{\parallel}| L \cong \sqrt{2mW}\, 2L. \tag{17}$$

Here we have assumed that μB is negligible between mirrors. Now if the distance between the mirrors changes, then W must change in order to keep J constant.

$$\frac{dJ}{dt} = 2\sqrt{2mW}\,\frac{dL}{dt} + \frac{\sqrt{2m}}{\sqrt{W}}\, L\,\frac{dW}{dt} = 0 \tag{18}$$

or

$$\frac{dW}{W} = -2\,\frac{dL}{L} \tag{19}$$

or

$$W = W_0\left(\frac{L_0}{L}\right)^2 \tag{20}$$

thus

$$V = \frac{V_0 L_0}{L}. \tag{21}$$

Thus if the mirrors move towards each other and L decreases v increases, while if they move apart v decreases. According to Eq. (20) the longitudinal energy or temperature of a gas being compressed between approaching mirrors is proportional to $1/L^2$. Now if we have an ideal gas and compress it adiabatically, then the temperature and volume are related by

$$TV^{\gamma-1} = T_0 V_0^{\gamma-1} \tag{22}$$

where T is the temperature or mean energy per particle, V is the volume and γ is $(n + 2)/n$, where n is the number of degrees of freedom involved in the compression. Here the volume is proportional to L, and only one degree of freedom is involved, the degree associated with the motion back and forth between the mirrors. Thus our adiabatic formula would lead to

$$T = T_0 \frac{L_0^2}{L^2} \tag{23}$$

in agreement with Eq. (20).

This offers one method for heating a gas. However, this means is limited because as the parallel velocity increases it becomes more and more difficult for the mirror fields to trap the particles, and ultimately they escape. Fermi proposed that such a mechanism may be responsible for the acceleration of particles up to cosmic ray energies. Particles would be trapped between magnetic fields associated with large gas clouds. If the clouds are moving towards each other the particle would gain energy until it had sufficient energy to escape. By repeated trappings and compressions, particles could gain energy. Of course if the particle were trapped between the clouds which were separating, it would lose energy. However, in such processes, on the average, particles gain energy if for no other reason than the fact that they can gain an unlimited amount of energy, but they can never lose more than they have.

VII. The Motion of Magnetic Lines of Force

It is sometimes stated that in a plasma in which particle collisions can be neglected the lines of force move with the particles. We will now look at this concept in some detail.

First, this statement is outside the original framework of Maxwell's equations, for in these equations it is not necessary to assign a persistent identity to the field lines. Second, the statement will clearly be true only in the limit of large q/m, when the excursion of the particle involved in the Larmor motion can be neglected. In this limit all drifts are negligible except the $\vec{E} \times \vec{B}$ drift provided $\vec{E} \cdot \vec{B}$ is zero. If $\vec{E} \cdot \vec{B} \neq 0$, then particles are strongly accelerated along field lines, $W_{\parallel}$ is proportional to q and curvature drifts are also important. In this limit the particle moves with the $\vec{E} \times \vec{B}$ drift velocity, and hence we wish to assign this velocity to the field lines. We shall show that we can do this when the component of $\vec{E}$ parallel to $\vec{B}$ is 0, and that the mapping of the B field, which results from this motion, (1) preserves lines of force and (2) preserves the flux through any closed curve.

Consider two particles on the same line of force at time $t = 0$ and which are separated by a small distance $\Delta \ell$.

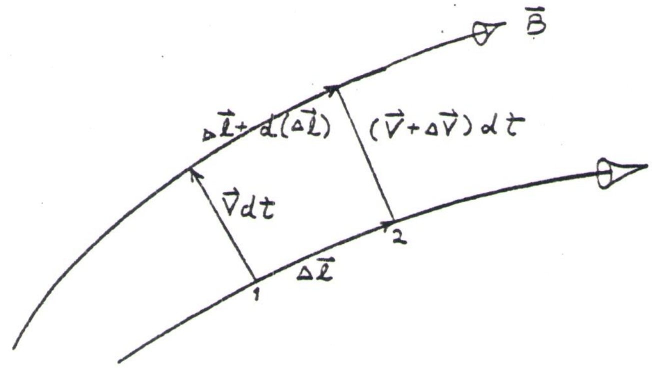


Figure 34

To show that a line of force remains a line of force we must show that $\vec{\Delta l}$ remains parallel to $\vec{B}$ under the transformation, or that

$$\frac{d}{dt}(\vec{\Delta l} \times \vec{B}) = \frac{d(\Delta\vec{l})}{dt} \times \vec{B} + \Delta\vec{l} \times \frac{d\vec{B}}{dt} = 0. \tag{1}$$

Now that we have for $d\,\vec{\Delta l}$

$$d\,\Delta\vec{l} = \underbrace{\left[\Delta\vec{l} + (\vec{V} + (\Delta\vec{l}\cdot\vec{\nabla})\,\vec{V})dt\right]}_{\text{Position of point 2 at dt.}} - \underbrace{\left[\Delta\vec{l} + \vec{V}dt\right]}_{\text{Position of point 1 at dt.}} \tag{2}$$

or

$$\frac{d\,\Delta\vec{l}}{dt} = (\vec{\Delta l}\cdot\vec{\nabla})\vec{V}. \tag{3}$$

We must now compute B at the displaced point

$$d\vec{B} = \left[\frac{\partial\vec{B}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\,\vec{B}\right]dt. \tag{4}$$

Now

$$\vec{V} = \frac{c\,\vec{E}\times\vec{B}}{B^2}, \quad \text{(assuming } \vec{E}\perp\vec{B}) \tag{5}$$

and hence

$$\vec{V}\times\vec{B} = \frac{c(\vec{E}\times\vec{B})\times\vec{B}}{B^2} = -c\vec{E} \tag{6}$$

since $\vec{E}\cdot\vec{B}$ is taken to be zero.

From Maxwell's equations

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{7}$$

and hence from Eq. (6)

$$\vec{\nabla}\times(\vec{V}\times\vec{B}) = \frac{\partial\vec{B}}{\partial t} \tag{8}$$

or

$$(\vec{B}\cdot\vec{\nabla})\vec{V}-(\vec{V}\cdot\vec{\nabla})\vec{B}-\vec{B}(\vec{\nabla}\cdot\vec{V})+\vec{V}(\cancelto{0}{\vec{\nabla}\cdot\vec{B}}) = \frac{\partial\vec{B}}{\partial t}. \tag{9}$$

Thus we find for dB/dt

$$\frac{d\vec{B}}{dt} = (\vec{B}\cdot\vec{\nabla})\vec{V}-\vec{B}(\vec{\nabla}\cdot\vec{V}). \tag{10}$$

Substituting Eqs. (10) and (3) in Eq. (1) gives

$$\frac{d(\vec{\Delta l}\times\vec{B})}{dt} = ((\vec{\Delta l}\cdot\vec{\nabla})\vec{V})\times\vec{B}+\vec{\Delta l}\times\left[(\vec{B}\cdot\vec{\nabla})\vec{V}-\vec{B}(\vec{\nabla}\cdot\vec{V})\right]. \tag{11}$$

Now $\vec{\Delta \ell}$ is a vector along the direction of $\vec{B}$. Hence we may replace $\vec{\Delta \ell}$ by $\epsilon\vec{B}$ in the above expression and we immediately see that the right-hand side of Eq. (11) is zero.

$$\frac{d}{dt}(\vec{\Delta l}\times\vec{B}) = 0. \tag{12}$$

Thus the transformation takes lines into lines.

Now to prove that V_E is flux-preserving, we must show that the flux through an area Δs which follows the motion remains constant. If $\Delta\phi$ is the flux through the area, we must show that

$$\frac{d\phi}{dt} = \frac{d}{dt}\int \vec{B}\cdot d\vec{S} = 0. \tag{13}$$

Now ϕ changes for two reasons: first because $\vec{B}$ changes, and secondly because the area changes. The change due to the changing $\vec{B}$ is given by

$$\left(\frac{\partial \phi}{\partial t}\right)_s = \int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} = -c\int (\vec{\nabla}\times\vec{E})\cdot d\vec{S}. \tag{14}$$

The change in ϕ due to the distortion in s (keeping B constant) is given by

$$\left(\frac{\partial \phi}{\partial t}\right)_B = \int_c \vec{B}\cdot(\vec{V}\times d\vec{l}) \tag{15}$$

where c is the bounding curve (see Fig. 35).

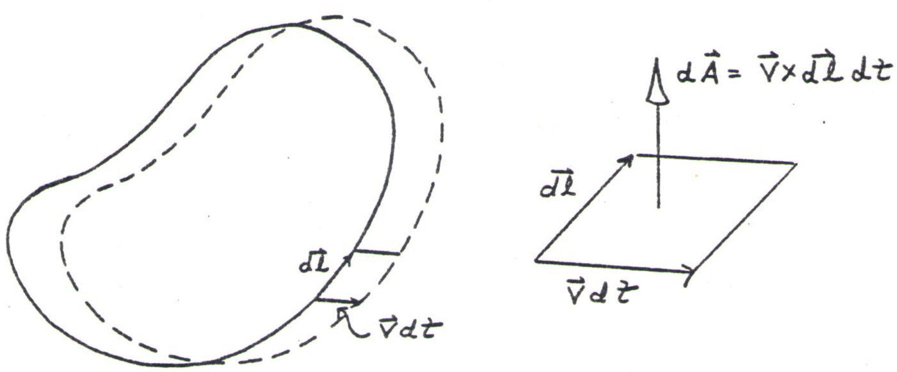


Figure 35

We may interchange the dot and cross products in Eq. (15) to obtain

$$\left(\frac{\partial \phi}{\partial t}\right)_B = -\int_c (\vec{V}\times\vec{B})\cdot d\vec{l}. \tag{16}$$

Converting Eq. (16) into a surface integral gives

$$\left(\frac{\partial \phi}{\partial t}\right)_B = -\int \vec{\nabla}\times(\vec{V}\times\vec{B})\cdot d\vec{S}. \tag{17}$$

Combining Eqs. (17) and (14) gives

$$\frac{d\phi}{dt} = \left(\frac{\partial\phi}{\partial t}\right)_s + \left(\frac{\partial\phi}{\partial t}\right)_B = -\int \vec{\nabla} \times (c\vec{E} + \vec{V} \times \vec{B}) \cdot d\vec{s}. \tag{18}$$

In order for this to hold true for every possible surface implies that the integrand must be 0 everywhere, or

$$\vec{\nabla} \times (c\vec{E} \times \vec{V} \times \vec{B}) = 0. \tag{19}$$

Now if $\vec{v}$ is given by

$$\vec{V} = \frac{c\vec{E} \times \vec{B}}{B^2},$$

then Eq. (19) becomes

$$c\,\vec{\nabla} \times \left(\cancel{\vec{E}}^{\,0} - \cancel{\vec{E}}^{\,0} + \frac{\vec{B}(\vec{B}\cdot\vec{E})}{B^2}\right) = 0 \tag{20}$$

or

$$\vec{\nabla} \times \left[\frac{\vec{B}(\vec{B}\cdot\vec{E})}{B^2}\right] = 0. \tag{21}$$

Eq. (21) is automatically satisfied if $\vec{B} \cdot \vec{E}$ is zero. We also needed this condition to show that lines went into lines. Thus we see that if $\vec{E} \cdot \vec{B}$ is zero we can assign the velocity $c\vec{E} \times \vec{B}/B^2$ to the lines and this takes lines into lines and preserves the flux through any surface. In a perfect conductor, where inertia can be neglected, $\vec{E} \cdot \vec{B}$ must be zero, for if it were not so the charges would immediately move so as to eliminate $\vec{E}$ parallel to B . To the extent to which this is true for a plasma, the plasma particles are stuck to lines of force.

VIII. Applications of Orbit Theory

A. Static, Straight B Lines, No External Force

We take $\vec{B}$ to be in the z direction. From $\vec{\nabla} \cdot \vec{B} = 0$ we have

$$\frac{\partial B_z}{\partial z} = 0. \tag{1}$$

Since $\vec{B}$ only has a z component, we also have

$$(\vec{B} \cdot \vec{\nabla}) \vec{B} = 0. \tag{2}$$

We further assume that $\vec{E}$ is zero, that the plasma is neutral, and that the particle number and energy densities are independent of z. We now sum up the currents. First the magnetization current is obtained from Eq. (6), Section V.

$$\begin{aligned} J_m &= -c\, \vec{\nabla} \times \left(\overline{N W_\perp}\, \frac{\vec{B}}{B^2}\right) = c\, \vec{n} \times \vec{\nabla}\left(\frac{\overline{N W_\perp}}{|B|}\right) \\ &= \frac{c}{|B|}\, \vec{n} \times \vec{\nabla}(\overline{N W_\perp}) - c\frac{\overline{N W_\perp}}{B^2}\, \vec{n} \times \vec{\nabla}|B|. \end{aligned} \tag{3}$$

Secondly we have the current due to a gradient in $\vec{B}$. This we obtain from Eq. (9), Section V.

$$J_G = c\, \frac{\overline{N W_\perp}}{B^2}\, \vec{n} \times \vec{\nabla}(\vec{B} \cdot \vec{n}). \tag{4}$$

Adding these two currents gives

$$\vec{J}_m + \vec{J}_G = c\, \frac{\vec{n}}{|B|} \times \vec{\nabla}(\overline{N W_\perp}) = c\, \frac{\vec{B}}{B^2} \times \vec{\nabla}(\overline{N W_\perp}). \tag{5}$$

Finally, we must use Maxwell's equations to get a self-consistent solution. From

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} \tag{6}$$

and Eq. (5) we have

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{n}|B|) = -\vec{n} \times \vec{\nabla}|B| = 4\pi \cancel{\left(\frac{c}{c}\right)}^{1} \frac{\vec{n}}{|B|} \times \vec{\nabla}(\overline{NW_\perp}) \tag{7}$$

or

$$\vec{n} \times \left\{ \vec{\nabla}|B| + \frac{4\pi}{|B|} \vec{\nabla}(\overline{NW_\perp}) \right\} = 0. \tag{8}$$

Since the term in the brackets is perpendicular to $\vec{n}$,

$$\vec{\nabla} \left(\frac{B^2}{8\pi} + \overline{NW_\perp} \right) = 0 \tag{9}$$

or

$$\left[\frac{B^2}{8\pi} + (\overline{NW_\perp}) \right] = \text{CONSTANT.} \tag{10}$$

$B^2/8\pi$ is the pressure associated with the magnetic field lines, while $\overline{NW_\perp}$ is the pressure of the plasma perpendicular to B . Eq. (10) says that the sum of these pressures is constant, or that we have pressure balance.

B. Plasma in a Gravitational Field which is Perpendicular to a Magnetic Field Whose Lines are Straight

Again we take the direction of the magnetic field to be in the z direction, and we take the gravitational field to be in the negative y direction. We assume all quantities are independent of z .

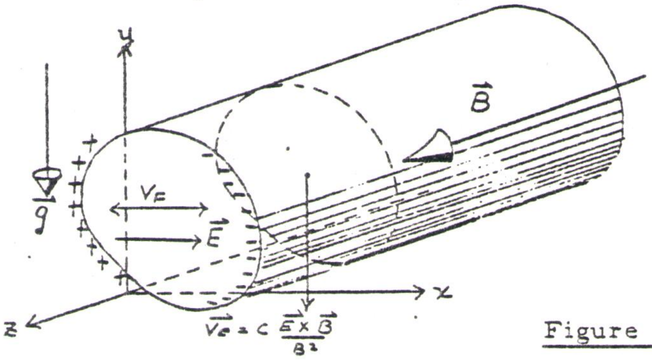


Figure 36

We shall further assume that $\overline{NW_\perp}$, the perpendicular pressure, is negligibly small compared to $B^2/8\pi$. From Eq. (10) we have

$$\frac{B^2}{8\pi} + \overline{NW_\perp} = \text{constant}$$

Hence

$$\frac{B^2}{8\pi}\left[1 + \frac{\overline{NW_\perp}}{B^2/8\pi}\right] = \frac{B^2}{8\pi}(1+\beta) = \text{constant} \tag{11}$$

or

$$B \cong \text{constant} \tag{12}$$

under our assumptions. The quantity β is the ratio of the perpendicular gas pressure to the magnetic field pressure. Under this approximation we can neglect the variations in B due to the gas pressure. We will therefore take B to be constant.

We now ask what will happen if we suddenly release such a plasma. Here an E field will develop and we must include its effects.

First, the particles have a drift due to the gravitational field which is given by Eq. (5), Section V.

$$\vec{V}_F = \frac{c\,m}{q}\frac{\vec{g}\times\vec{B}}{B^2}. \tag{13}$$

Ions and electrons move in opposite directions, so that a current is set up, given by Eq. (10), Section V; however, the resulting charge separation tends to oppose this current. The resulting $\vec{E}$ field enters in two ways: first, because $\vec{E}$ is time-dependent it gives rise to a polarization current; and second, because of $\vec{E}$ there is an $\vec{E}\times\vec{B}$ drift of the whole plasma in the -y direction. To compute the time-dependence of E we must use Maxwell's equations (making use of the result which we already found, that the plasma behaves like a dielectric). From Eqs. (4) and (5), Section IV, we have

$$\nabla \cdot \left[\frac{\dot{\vec{E}}}{4\pi} + \dot{\vec{j}} \right] = 0. \tag{14}$$

If we assume that $\vec{\nabla} \times \vec{E}$ is zero ($\vec{B}$ negligible), and also that $\vec{\nabla} \times \vec{j} = 0$ (this is reasonable because there is nothing to make currents circulate in the problem), then

$$\dot{\vec{E}} = 4\pi \dot{\vec{j}}. \tag{15}$$

Now $\vec{j}$ has two parts, one coming from $\vec{g}$ and the other from $\vec{E}$. These are obtained from Eqs. (7) and (10), Section V, and are given by

$$\vec{j}_E = \frac{c^2}{B^2} Nm \dot{\vec{E}} \tag{16}$$

and

$$\vec{j}_g = \frac{cNm}{B^2} \vec{g} \times \vec{B} = -\hat{e}_x \frac{cNm}{|B|} g \tag{17}$$

where use has been made of the geometry in writing down $\vec{j}_g$.

Thus from Eq. (15) we have

$$\dot{\vec{E}} \left(1 + \frac{4\pi c^2 Nm}{B^2} \right) - \hat{e}_x \frac{4\pi c Nm}{|B|} g = 0 \tag{18}$$

or

$$\dot{\vec{E}} = \frac{\hat{e}_x \frac{4\pi c Nm g}{|B|}}{1 + \frac{4\pi c^2 Nm}{B^2}}. \tag{19}$$

Now the E_x which results from Eq. (19) gives rise to an $\vec{E} \times \vec{B}$ drift of the whole plasma in the $-y$ direction. We have