X. The Quasilinear Approximation

In the preceeding chapters, we have explored the physics contained in the "Vlasov approximation", derived in Chapter II by neglecting fluctuations. More specifically, this approximation neglects correlations between fluctuations, i.e. quantities like $<\delta f(\underline{x}, \underline{v}, t)\, \delta f(\underline{x}', \underline{v}', t)>$ which appear on the right side of (2.18). In this approximation, the plasma is described by the Vlasov equation for each species plus the Maxwell equations for the ensemble-averaged fields $\underline{E}$ and $\underline{B}$.

As we showed in Chapter II, the Vlasov equation can be derived in a heuristic fashion from arguments involving particle conservation in phase space, assuming the particles move in response to "self-consistent" $\underline{E}$ and $\underline{B}$ fields but do not experience abrupt binary collisions. This suggests that the Vlasov equation in some sense describes a "collisionless" plasma, but that requires closer examination. In the first place, the "self-consistent" fields already incorporate some of the inter-particle Coulomb interaction (in fact, as we shall see, just the long range part, corresponding to impact parameters $b >> e^2/T$) so at best one could say that the Vlasov equation involves the neglect of short range collisions ($e^2/T \lesssim b << k_D^{-1}$). In addition, the long range of the Coulomb potential does not correspond to discrete, binary collisions. A particle interacts with many other particles simultaneously, so the simple view of collisions used in deriving, for example, the Boltzmann equation for a neutral gas must be modified appropriately in considering a plasma. We shall discuss this in more detail later in this chapter.

To summarize, the Vlasov approximation corresponds to neglecting second order correlations of fluctuations like $<\delta f \delta f>$. In this chapter, we examine the next level of approximation, the "Quasilinear Approximation", obtained by including second order correlations but neglecting those of higher order, like $\langle\delta\mathcal{f}\delta\mathcal{f}\delta\mathcal{f}\rangle$. We distinguish two different situations. We say that the plasma is "stable" if the velocity dependance of f is such that the tensor

$$\underset{\sim}{M} = \underset{\sim}{\varepsilon} + \underline{k}\,\underline{k} - k^2 \underset{\sim}{1}, \tag{10.1}$$

with $\underset{\sim}{\varepsilon}$ computed from f, has no roots in the upper half ω plane for any $\underline{k}$. Otherwise we say the plasma is "unstable". For a stable plasma, the inclusion of the quadratic fluctuation term on the right side of (2.18) may be described in heuristic terms as the inclusion of "collisional" effects, a very important step since it provides the basis for transport theory (diffusion, viscosity, thermal conductivity and electrical resistivity). For an unstable plasma, it leads to an important nonlinear mechanism for stabilization of the growing modes and to one of the theories for "anomalous" transport.

From a formal viewpoint, our task is simply to solve the coupled equations (2.18) and (2.19). Our approach is to first regard f as known and solve (2.19), which, with the addition of the Maxwell equations for the fluctuation fields $\delta\underline{\mathcal{E}}$, $\delta\underline{\mathcal{B}}$, is a linear integral equation for $\delta\mathcal{f}$. In fact, it turns out to be just a linearized Vlasov equation, which can be solved, using the techniques developed in Chapters III and V, for $\delta\mathcal{f}\,(\underline{x}, \underline{v}, t)$ in terms of f and $\delta\mathcal{f}(\underline{x}, \underline{v}, o)$. Substituting this $\delta\mathcal{f}$ into the right side of (2.18) then gives a kinetic equation, i.e., an equation for the single particle distribution function f (plus, of course, the ensemble averaged Maxwell equations),

$$\partial f/\partial t + \underline{v}\cdot\nabla f + (q/m)\,(\underline{E} + \underline{v} \times \underline{B}/c)\cdot(\partial f/\partial\underline{v}) = \partial f/\partial t \tag{10.2}$$

where

$$\delta f/\delta t \equiv -(q/m)\ (\partial/\partial \underline{v}) \cdot \langle(\delta \underline{\mathcal{E}} + \underline{v} \times \delta \underline{\mathcal{B}}/c)\delta f\rangle \tag{10.3}$$

represents the effects of correlations, (or, with appropriate caveats, collisions).

A. Stable Plasmas; the Collision Integral

1. Some Elementary Statistical Relations

As in Chapter II, we make use of a "statistical ensemble", i.e., an assembly of many copies of the plasma system. If one such system consists of N particles, then its state is described by a set of values for all the positions and velocities. As in Chapter VI, we let R_i designate the position and velocity of particle i, $R_i \equiv (\underline{x}_i, \underline{v}_i)$. Then the state of a single system is characterized by $R_1, R_2 \ldots R_N$, or, equivalently, by single point in a 6N-dimensional phase space, Γ. (For a relativistic plasma, where retardation effects are important in the solution of Maxwell's equations, we need also the instantaneous transverse field coordinates or the equivalent "field oscillators", but we shall restrict our discussion to the non-relativistic case.) The state of the ensemble as a whole can then be characterized by the Liouville phase space density function $D\ (R_1, R_2 \ldots R_N)$ which gives the relative numbers of systems of the ensemble having the phase space coordinates $R_1, R_2, \ldots R_N$. D is symmetric under the interchange of any two R_i corresponding to particles of the same species. (In general, D is also a function of time, but we need not indicate that explicitly.)

If D is normalized to unity

$$\int D\ dR_1 \ldots dR_N = 1 \tag{10.4}$$

then the ensemble average of any function of the R_i is obtained by simply integrating its product with D over the Γ phase space. For example, consider the ensemble averaged distribution function for species α. If the plasma has N_α particles of species α, N_β of species β etc. we may choose the first N_α coordinates to correspond to particles of species α, the next N_β to correspond to particles of species β, etc. Then we write

$$(R_1, R_2 \ldots R_N) = (R_1^\alpha, R_2^\alpha \ldots R_{N_\alpha}^\alpha ; R_1^\beta, R_2^\beta \ldots R_{N_\beta}^\beta ; \ldots R_{N_\gamma}^\gamma) \tag{10.5}$$

and

$$\begin{aligned} f_\alpha(R) &= f_\alpha(\underline{x}, \underline{v}) = \langle \mathcal{f}_\alpha(\underline{x}, \underline{v}) \rangle = \\ &= \int dR_1 \ldots dR_N \, \overline{n}_\alpha{}^{-1} \, D \sum_{i=1}^{N_\alpha} \delta[R - R_i{}^\alpha] = \\ &= V \int dR_2 \ldots dR_N \, D(R, R_2^\alpha \ldots R_{N_\alpha}^\alpha ; R_1^\beta \ldots R_{N_\beta}^\beta ; \ldots R_{N_\gamma}^\gamma) \end{aligned} \tag{10.6}$$

(Here, as in Chapter II, $\overline{n}_\alpha = N_\alpha/V$, the average density of particles of species/α. In this chapter we shall only be considering homogeneous plasmas, so we can drop the bar and henceforth write simply n_α.) As expected, $f_\alpha(R)/V$ is just the probability of finding a particle of species α at R, regardless of the location of the remaining (N-1) particles of the system. In statistical mechanics, f_α is called the <u>one-body function</u> and a hierarchy of two-body, three-body, etc. functions is defined. For instance, the two-body function is

$$\begin{aligned} f_2^{\alpha\beta}(R, R') \equiv V^2 \int & dR_2^\alpha \ldots dR_{N_\alpha}^\alpha ; dR_2^\beta \ldots dR_{N_\beta}^\beta ; \ldots dR_N \, \cdot \\ & \cdot D(R, R_2^\alpha \ldots R_{N_\alpha}^\alpha ; R', R_2^\beta \ldots R_{N_\beta}^\beta ; \ldots R_N) \end{aligned} \tag{10.7}$$

and f_3, f_4, etc. have similar definitions.

From these elementary considerations follows an important result:

$$\langle \not{f}_\alpha(R)\ \not{f}_\beta(R')\rangle = (n_\alpha\ n_\beta)^{-1} \int dR_1 --- dR_N D\ \cdot$$

$$\cdot \sum_{i=1}^{N_\alpha} \sum_{j=1}^{N_\beta} \delta(R - R_i^{(\alpha)})\ \delta(R' - R_j^{\beta}) =$$

$$= f_2^{\alpha\beta}\ (R,\ R') + n_\alpha^{\ -1}\ f_\alpha(R) \qquad (10.8)$$

For $\alpha \neq \beta$ the $f_2^{\alpha\beta}$ term arises from the $N_\alpha N_\beta$ terms obtained by letting i take on its N_α values and j its N_β values. If $\alpha = \beta$, then $f_2^{\alpha\alpha}$ comes from terms with $i \neq j$. It has a factor $(N_\alpha - 1)/N_\alpha$, but we can set that equal to 1 since we always assume $N_\alpha >> 1$. The f_α term arises from the N_α terms with $i = j$. As usual, we shall omit species indices wherever possible and write (10.8) as

$$\langle \not{f}(R)\ \not{f}(R')\rangle = f_2(R,\ R') + \bar{n}^{-1}\delta(R,\ R')\ f(R) \qquad (10.9)$$

with the convention that $\delta(R,\ R')$ represents $\delta(R - R')$ times a Kronecker delta in the suppressed species indices. From (10.8) or (10.9) and the definition (2.9) for $\delta\not{f}$, we have

$$f_2(R,\ R') = f(R)\ f(R') + \langle\delta\not{f}(R)\ \delta\not{f}(R')\rangle - n^{-1}\delta(R,\ R')\ f(R) \qquad (10.10)$$

If there are <u>no correlations</u> between plasma particles, the Liouville density function is just a product,

$$D(R_1,\ R_2 \ldots R_N) = f(R_1) \ldots f(R_N) \qquad (10.11)$$

with

$$\int dR\ f(R) = 1 \qquad (10.12)$$

and the same is therefore true of the two-body function

$$f_2(R, R') = f(R)\, f(R') \tag{10.13}$$

as well as the higher order functions, f_3, f_4, etc. It is conventional to define the intrinsic two-body correlation function $g_2(R, R')$ by writing

$$f_2(R, R') = f(R)\, f(R') + g_2(R, R') \tag{10.14}$$

Thus, g_2 measures the extent of two particle correlations. From (10.10) we have

$$g_2(R, R') = \langle \delta \mathcal{f}(R)\, \delta \mathcal{f}(R') \rangle - n^{-1}\, \delta(R, R')\, f(R) \tag{10.15}$$

the last term representing the self-correlation of a particle with itself. Non-equilibrium statistical mechanics can be developed in terms of g_2 and analogous correlation or cluster functions g_3, g_4, etc., or in terms of the averages of products of $\delta \mathcal{f}$; for plasma physics, the latter usually proves more convenient and we shall continue to follow that approach here.

NB Keep in mind that in general D and all of the distribution functions — $\mathcal{f}$, $\delta \mathcal{f}$, f, f_2, etc. — will also be functions of t, so that t is to be understood as being an additional independent variable in (10.4) through (10.15).

In our discussion in this chapter, we shall consider spatially homogeneous systems. For these, ensemble averaged quantities like f must be independent of $\underline{x}$, while quadratic quantities like $\mathcal{f}(R)\, \mathcal{f}(R')$ or $f_2\,(R, R')$ must be functions only of $(\underline{x} - \underline{x}')$. Neglecting the velocity and time dependence for the moment and letting $A(\underline{x})$, $B(\underline{x})$ denote any two real microscopic variables, with Fourier transforms,

$$A(\underline{k}) = \int d\underline{x}\ A(x) e^{-i\underline{k}\cdot\underline{x}} = A^*(-\underline{k}), \text{ etc.} \tag{10.16}$$

we see that

$$<A(\underline{x})\ B(x')> = \frac{1}{(2\pi)^6} \int d\underline{k}\ dk'\ <A(\underline{k})B(-\underline{k}')>\ e^{i(\underline{k}\cdot\underline{x} - \underline{k}'\cdot\underline{x}')} \tag{10.17}$$

can be a function of $(\underline{x}-\underline{x}')$ only if $<A(\underline{k})\ B(-\underline{k}')>$ vanishes for $\underline{k} \neq \underline{k}'$. Thus, we can write

$$<A(\underline{k})\ B^*(\underline{k}')> = (2\pi)^3 << A(\underline{k})\ B(\underline{k}) >> \delta(\underline{k}-\underline{k}') \tag{10.18}$$

where the << >> is defined by (10.18)

The $(2\pi)^3$ is included for convenience, namely so that $<<A(\underline{k})\ B^*(\underline{k})>>$ will be just the Fourier transform of the correlation function

$$C_{AB}(\underline{x}-\underline{x}') \equiv <A(\underline{x})\ B(\underline{x}')> \tag{10.19}$$

In fact, from (10.17) and (10.18) we have

$$C_{AB}(\underline{x}) = \int \frac{d\underline{k}}{(2\pi)^3} <<A(\underline{k})\ B^*(\underline{k})>>\ e^{i\underline{k}\cdot\underline{x}} \tag{10.20}$$

2. Two-Time Scale Considerations

In solving (2.18) and (2.19) we shall introduce three simplifying assumptions:

i) The basic assumption is that f and δf develop on different time scales, specifically that δf, which describes fluctuations, waves, etc. varies rapidly compared to f, which describes the (ensemble) averaged plasma properties. This two-time scale hypothesis allows us to solve (2.19) treating f as fixed, thus obtaining $\delta\mathcal{f}$ as a functional of f. Substituting $\delta\mathcal{f}$ into (2.18) then gives an equation for determining the time evolution of f. This procedure is motivated by the following physical picture. For given f, there is an equilibrium level of fluctuations in the plasma, resulting from a balance between the Cerenkov radiation of waves by individual particles in the plasma and the subsequent Landau damping of these waves. To calculate this equilibrium level, suppose that at some initial time, say t=0, there are no correlations (so that D is a product of one-body functions (10.11) and the intrinsic two-body correlation function g_2=0). Then after a time which is long on the time scale of waves and fluctuations (ω_p^{-1}, γ_{Landau}^{-1}, etc.) but short enough so that f is essentially unchanged, $\langle\delta\mathcal{f}\,\delta\mathcal{f}\rangle$ will have reached its asymptotic value. This is the equilibrium level of fluctuations for the given f and provides us with a value for $\delta f/\delta t$.

ii) We shall include only electrostatic fluctuations, i.e. take $\delta\tilde{\underline{\mathcal{E}}} = \delta\underline{\mathcal{E}}$. While the inclusion of the ($\underline{v} \times \delta\underline{\mathcal{B}}/c$) Lorentz force associated with electromagnetic fluctuations would complicate the algebra it would not alter the method of solution in any essential way. For most non-relativistic plasmas the electrostatic effects (Coulomb forces) are indeed the dominant part of the "collisional" or transport effects, but the electromagnetic terms can be included following the same approach used here for the electrostatic part.

iii) In calculating $\delta\mathcal{f}$, we shall assume that there are no external fields, i.e. that $\underline{E} = \underline{B} = 0$. (If $\underline{E} \neq 0$ we have an inherently non-equilibrium situation, with constant energy input to the plasma. The field drives currents which lead to instabilities, and while this is of great importance e.g., in connection with anomalous resistivity, here we consider only the case of stable plasmas. The case of a uniform external magnetic field, which has been studied by Rostoker, can be handled in exactly the same way as the non-magnetic case considered here, but the algebra is more complicated because of the replacement of straight line orbits by helical ones and the resulting modifications in the collisional terms are not usually very significant.

3. The Lenard-Balescu Expression for $\delta f/\delta t$

Thanks to assumptions ii and iii, the equation (2.19) for $\delta\mathcal{F}$ reduces to

$$(\partial/\partial t + \underline{v}\cdot\nabla)\delta\mathcal{F} + (q/m)\ \ \delta\underline{\mathcal{E}}\cdot\ \partial f/\partial\underline{v} = 0 \tag{10.21}$$

i.e., to a linearlized Vlasov equation. To implement the approach described above, we must solve this to find $\delta\mathcal{F}$ and $\delta\underline{\mathcal{F}}$ for given f; take the limit as $t \to \infty$; and then compute

$$\delta f/\delta t = \lim_{t\to\infty}(-q/m)(\partial/\partial\underline{v})\ \cdot\ \ \langle\delta\underline{\mathcal{E}}(\underline{x},t)\,\delta\mathcal{F}(\underline{x},\underline{v},t)\rangle \tag{10.22}$$

Our procedure is as follows:

i) From the Fourier-Laplace transform of (10.21) we compute $\delta\ (\underline{k},\underline{v},\ \omega)$ in terms of the initial fluctuations $\delta\mathcal{F}_0(k,v,) \equiv \delta\mathcal{F}(\underline{k},\underline{v},\ t = 0)$

ii) Inverting the transform and taking the limit $t \to \infty$, we find $\delta\overline{\mathcal{F}}\ \ (\underline{x},\ \underline{v}) = \lim_{t\to\infty} \delta\mathcal{F}(\underline{x},\underline{v},t)$, again in terms of $\delta\mathcal{F}_0$

iii) Substituting $\delta\overline{\mathcal{F}}$ and $\delta\overline{\mathcal{E}}$, which can be computed from it, into (10.22) gives $\partial f/\partial t$ in terms of $\langle\delta\mathcal{F}_0\delta\mathcal{F}_0\rangle$.

iv) We evaluate $\langle\delta\mathcal{F}_o\delta\mathcal{F}_o\rangle$ by assuming that initially there are no correlations, $g_2 = 0$, we can then complete the evaluation of $\partial f/\partial t$, including the use of some integral identities, to bring it to the final Lenard-Balescu form.

Taking the Fourier-Laplace transform of (10.21) as in Chapter III, we have

$$\delta\mathcal{F}(\underline{k},\ \underline{v},\ \omega) = \frac{\delta\mathcal{F}_0(k,v) - (q/m)\ \hat{\underline{k}}\cdot(\partial f/\partial\underline{v})\delta\mathcal{E}}{i(\underline{k}\cdot\underline{v} - \omega)} \tag{10.23}$$

where

$$\delta\mathcal{F}_o\ (\underline{k},\ \underline{v}) \equiv\ \delta\mathcal{F}(\underline{k},\ \underline{v},\ t = 0) \tag{10.24}$$

is the initial fluctuation.

Then Poisson's equation gives

$$\delta\underline{\mathcal{E}}(\underline{k},\omega) = \underline{k}(4\pi/ik)\int d\underline{v}\ \bar{n}q\ \partial\mathcal{F}(\underline{k},\ \underline{v},\ \omega) =$$

$$= -[4\pi\underline{k}/k^2\varepsilon(\underline{k},\omega)]\int d\underline{v}\ \bar{n}q\ \partial\mathcal{F}_0\ (\underline{k},\ \underline{v})(\underline{k}\cdot\underline{v}-\omega)^{-1} \quad (10.25)$$

where

$$\varepsilon(k,\omega) = 1 - \int d\underline{v}\ (\omega_p{}^2/k^2)\ \underline{k}\cdot(\partial f/\partial v)(\underline{k}\cdot\underline{v}-\omega)^{-1} \quad (10.26)$$

is the usual dielectric function for electrostatic waves. As in Chapter II, $\bar{n}$ denotes the average density, N/V. In this section we shall omit the bar for convenience, so the symbol n is henceforth to be understood as meaning just N/V.

It is convenient to write (10.25) in a form which expresses $\delta\mathcal{E}$ as a linear (integral) operator acting on $\delta\mathcal{F}_0$,

$$\delta\underline{\mathcal{E}}(\underline{k},\ \omega) = \int d\underline{v}\ \underline{P}\ (\underline{k},\underline{v},\omega)\ \delta\mathcal{F}_0\ (\underline{k},\underline{v}) \quad (10.27)$$

where

$$\underline{P}\ (\underline{k},\ \underline{v},\ \omega) \equiv -4\pi nq/k\varepsilon(\underline{k},\omega)(\underline{k}\cdot\underline{v}-\omega) \quad (10.28)$$

From (10.23) and (10.27) we can, similarly express $\delta\mathcal{F}(\underline{k},\ \underline{v},\ t)$ in terms of $\delta\mathcal{F}_0$,

$$\delta\mathcal{F}(\underline{k},\ \underline{v},\ \omega) = \int d\underline{v}\ Q\ (\underline{k},\ \underline{v},\ \underline{v}',\ \omega)\partial\mathcal{F}_0(\underline{k},\ \underline{v}') \quad (10.29)$$

with

$$Q(\underline{k},\underline{v},\underline{v}',\omega) = -i(\underline{k}\ \underline{v}-\omega)^{-1}\ [\delta(\underline{v},\underline{v}')-(q/m)\hat{\underline{k}}\cdot(\partial f/\partial\underline{v})\underline{P}(\underline{k},\underline{v}',\omega)] =$$

$$= -i(\underline{k}\ \underline{v}-\omega)^{-1}\ [\delta(\underline{v},\underline{v}') + 4\pi n'qq'\underline{k}\cdot(\partial f/\partial\underline{v})/k^2(\underline{k}\cdot\underline{v}'-\omega)\varepsilon(\underline{k},\omega)]$$

(10.30)

To find $\partial f/\partial t$, we must evaluate $\delta\mathcal{F}$ and $\delta\underline{\mathcal{E}}$ in the space-time domain, i.e., take the inverse Fourier and Laplace transforms, then go to the limit $t \to \infty$. Since the plasma is stable, by assumption, the poles in P and Q arising from the roots of ε will all be damped and at large times only the contributions of the $(\underline{k} \cdot \underline{v} - \omega)$ denominators will survive. Using a bar to denote the $t \to \infty$ limit we have

$$\overline{P}\ (\underline{k}, \underline{v}', t) \equiv \lim_{t\to\infty} \underline{P}\ (\underline{k}, \underline{v}', t) = \lim_{t\to\infty} (2\pi)^{-1} \int d\omega\ P\ (\underline{k}, \underline{v}', \omega) e^{-i\omega t} =$$
$$= -[4\pi n' q' i/k\varepsilon(\underline{k}, \underline{k} \cdot \underline{v}')] e^{-i\underline{k}\ \underline{v}' t} \tag{10.31}$$

In the computation of

$$\overline{Q}\ (\underline{k}, \underline{v}, \underline{v}', t) \equiv \lim_{t\to\infty} Q\ (\underline{k}, \underline{v}, \underline{v}', t) \tag{10.32}$$

the second term in the square bracket of (10.20) leads to an integral of the form

$$I = \lim_{t\to\infty} (2\pi i)^{-1} \int d\omega e^{-i\omega t}/(\omega - \underline{k} \cdot \underline{v})(\omega - \underline{k} \cdot \underline{v}')\ \varepsilon(\underline{k}, \omega) =$$
$$= -\lim_{t\to\infty} [\underline{k} \cdot (\underline{v} - \underline{v}')]^{-1}\ [e^{-i\underline{k}\cdot\underline{v}t}/\varepsilon(\underline{k}, \underline{k}\cdot\underline{v}) - e^{-i\underline{k}\cdot\underline{v}'t}/\varepsilon(\underline{k}, \underline{k}'\cdot \underline{v}')] \tag{10.33}$$

(where we have, as usual, closed the contour in the lower half ω plane). Although I appears to have a singularity when $u = \underline{k} \cdot (\underline{v} - \underline{v}') = 0$, it is actually nonsingular since the numerator is also linear in u near $u = 0$. Thus, the result is the same if we replace u in the demoninator by $u + i\eta$, where η is a small positive number. We can then write (10. 33) as the sum of two terms, each of which is nonsingular (for $\eta \neq 0$). We can further simplify (10.33) by observing that in the $\underline{k}$ integration required to invert the Fourier transform and compute $\delta\mathcal{F}(\underline{x}, \underline{v}, t)$, the term

$$\frac{e^{-i\underline{k}\cdot\underline{v}'t}}{\varepsilon(\underline{k},\ \underline{k}\cdot\underline{v}')[\underline{k}\cdot(\underline{v}-\underline{v}')+i\eta]}$$

will make no contribution in the limit $t \to \infty$. In the complex $\underline{k} \cdot \underline{v}'$ plane the only singularities are those at the roots of ε (which will all be in the lower half plane for a stable plasma) and one in the upper half plane at $\underline{k} \cdot \underline{v}' = \underline{k} \cdot \underline{v} + i\eta$. For $t > 0$, we can close the contour in the lower half plane. There is then no contribution from the pole at $\underline{k} \cdot \underline{v}' = \underline{k} \cdot \underline{v} + i\eta$, and the roots of $\varepsilon(\underline{k}, \underline{k} \cdot \underline{v}')$ give rise to a sum of damped terms which all vanish as $t \to \infty$. For large t, then, (10.33) reduces to

$$\begin{aligned} I &= -e^{-i\underline{k}\,\underline{v}t}/\varepsilon(k,\ \underline{k}\cdot\underline{v})\ [\underline{k}\cdot(\underline{v}-\underline{v}')+i\eta] \\ &= -[e^{-i\underline{k}\cdot\underline{v}t}/\varepsilon(\underline{k},\ \underline{k}\cdot\underline{v})]\ \{P[k\cdot(\underline{v}-\underline{v}')]^{-1} - \pi i\delta[\underline{k}\cdot(\underline{v}-\underline{v}')]\} \end{aligned} \tag{10.34}$$

In the integration over $\underline{k}$ required to get $\delta\mathcal{F}(\underline{x}, \underline{v}, t)$, the contribution from the principal value term in (10.34) will vanish for large t because the coefficient of the rapidly oscillating factor $\exp(-i\,\underline{k}\cdot\underline{v}t)$ is a slowly varying function of $\underline{k} \cdot \underline{v}$. (This is generally referred to as "phase mixing"). Of course, this is not true for the delta function term so that we have finally

$$I = \pi i\ \delta[\underline{k}\cdot(\underline{v}-\underline{v}')]e^{-i\underline{k}\cdot\underline{v}t}/\varepsilon(\underline{k},\ \underline{k}\cdot\underline{v}) \tag{10.35}$$

From (10.20) and (10.26) it follows that

$$\begin{aligned} \overline{Q}(\underline{k},\ \underline{v},\ \underline{v}',\ t) &= e^{-i\underline{k}\cdot\underline{v}t}\{\delta\ (\underline{v},\ \underline{v}') + \\ &+ \frac{4\pi^2 n'q'qi}{k^2 m}\ (\underline{k}\cdot\partial f/\partial\underline{v})\delta(\underline{k}\cdot(\underline{v}\ \underline{v}')/\varepsilon(\underline{k},\ \underline{k}\cdot\underline{v})\} \end{aligned} \tag{10.36}$$

Taking the Laplace transform of (10.27) and (10.29) we have

$$\delta\overline{\mathcal{E}}(\underline{k},\ t) \equiv \lim_{t\to\infty} \delta\mathcal{E}(\underline{k},\ t) = \int d\underline{v}\ \overline{P}(\underline{k},\ \underline{v}, t)\delta\mathcal{F}_0\ (\underline{k},\ \underline{v}) \tag{10.37}$$

$$\delta\overline{\mathcal{F}}(\underline{k}, \underline{v}, t) \equiv \lim_{t\to\infty} \delta\mathcal{F}(\underline{k}, \underline{v}, t) = \int d\underline{v}' \, \overline{Q} \, (\underline{k}, \underline{v}, \underline{v}', t) \, \delta\mathcal{F}_o \, (\underline{k}, \underline{v}') \tag{10.38}$$

with $\overline{P}$ and $\overline{Q}$ given by (10.31) and (10.36). From (10.22) we then find

$$\begin{aligned} \partial f/\partial t &= -(q/m)(\partial/\partial \underline{v})\cdot\int d\underline{k} \int d\underline{k}'(2\pi)^{-6} <\delta\overline{\mathcal{E}}^* \, (\underline{k}', t) \\ &\quad \delta\overline{\mathcal{F}}(\underline{k}, \underline{v}, t) \;\; \exp\,[i \, (\underline{k} - \underline{k}') \cdot \underline{x}] = \\ &= -(q/m)(\partial/\partial \underline{v}) \cdot \int d\underline{k} \cdot \int d\underline{k}' \int (2\pi)^{-6} \, \exp[i(\underline{k} - \underline{k}') \cdot \underline{x}] \, \hat{\underline{k}}' \cdot \\ &\quad \cdot \int d\underline{v}' \overline{P}(\underline{k}, \underline{v}', t)^* \cdot \int d\underline{v}'' \, \overline{Q} \, (\underline{k}, \underline{v}, \underline{v}'', t) <\delta\mathcal{F}_o^* \, (\underline{k}, v') \, \delta\mathcal{F}_o(\underline{k}', \underline{v}'')> \\ &= (q/m)(\partial/\partial \underline{v}) \cdot \int d\underline{k} (2\pi)^{-3} \int d\underline{v}' \int d\underline{v}'' \, \hat{\underline{k}} \, \overline{P}^* \, (\underline{k}, \underline{v}', t) \\ &\quad \overline{Q} \, (\underline{k}, \underline{v}, \underline{v}'', t) << \delta\mathcal{F}_o^* \, (\underline{k}, \underline{v}')\delta \,_o(\underline{k}, \underline{v}'') >> \end{aligned} \tag{10.39}$$

To complete the calculation of $\partial f/\partial t$ we must specify the initial fluctuations, $\partial\mathcal{F}_o$. We assume that initially there are no correlations, i.e., that

$$f_2(R, R', 0) = f(R, 0) \, f(R', 0) \tag{10.40}$$

or, in the notation of (10.15) and (10.16)

$$g_2(R, R', 0) = <\mathcal{F}_o(R, 0) \; \delta\mathcal{F}(R', 0)> - n^{-1}\delta(R, R')f(R, 0) = 0 \tag{10.41}$$

Then the initial level of fluctuation arises only from the self-correlations, i.e., from the particle discreteness,

$$<\delta\mathcal{F}_o(\underline{x}, \underline{v})\delta\mathcal{F}_o \, (\underline{x}', \underline{v}')>> = n^{-1} \; \delta(\underline{x} - \underline{x}') \;\; \delta(\underline{v}, \underline{v}')f(v) \tag{10.42}$$

This gives

$$\begin{aligned} <\delta\mathcal{F}_o^* \, (\underline{k}', \underline{v}) \;\; \delta\mathcal{F}_o \, (\underline{k}, \underline{v}')> &= \int d \, \underline{x} \int d\underline{x}' \;\; e^{i(\underline{k}\cdot\underline{x}-\underline{k}\cdot\underline{x})} <\delta\mathcal{F}_o \, (\underline{x}, v) \;\; \delta \,_o(\underline{x}', \underline{v}')> \\ &= (2\pi)^3\delta(\underline{k} - \underline{k}') \quad \delta \, (\underline{v}, \underline{v}') \;\; f(v)n'' \end{aligned} \tag{10.43}$$

and hence

$$<<\delta\mathcal{F}_o^* \, (\underline{k}, \underline{v}') \;\; \delta\mathcal{F}_o(\underline{k}, \underline{v}'')>> = \delta \, (\underline{v}', \underline{v}'') \;\; f(v')/n' \tag{10.44}$$

Substituting this into (10.39) gives

$$\partial f/\partial t = (q/m)(2\pi)^{-3}(\partial/\partial \underline{v}) \cdot \int d\underline{k} \int d\underline{v}' \ \hat{\underline{k}}\ \overline{P}^{*}(\underline{k}, \underline{v}', t)\ \overline{Q}\ (\underline{k}, \underline{v}, \underline{v}', t)\ f(\underline{v}')/n'$$

$$= (q/m)(2\pi)^{-3}(\partial/\partial \underline{v}) \cdot \int d\underline{k} \int d\underline{v}' \ \hat{\underline{k}}\ (4\pi q' i/k)\ f(v')$$

$$\{\delta(\underline{v}, \underline{v}')/\varepsilon^{*}(\underline{k}, \underline{k} \cdot \underline{v}) - \frac{4\pi^2 n' q q' i}{mk^2}\ \underline{k} \cdot \frac{\partial f}{\partial \underline{v}}\ \delta(\underline{k} \cdot (\underline{v} - \underline{v}')/|\varepsilon(\underline{k}, \underline{k} \cdot \underline{v})|^2\} \quad (10.45)$$

From (10.26) we have

$$\varepsilon(\underline{k}, \underline{k} \cdot \underline{v}) = 1 - \int d\underline{v}'\ (\omega_p^{\ 2}/k^2)(\underline{k} \cdot \partial f/\partial \underline{v}')[\underline{k} \cdot (\underline{v}' - \underline{v}) - i\eta]$$

$$= 1 - P \int d\underline{v}'\ (\omega_p^{\ 2}/k^2)(\underline{k} \cdot \partial f/\partial \underline{v}')[\underline{k} \cdot (\underline{v}' - v)]^{-1}$$

$$- \pi i \int d\underline{v}'\ (\omega_p^{\ 2}/k^2)(\underline{k} \cdot \partial f/\partial \underline{v}')\ \delta[\underline{k} \cdot (\underline{v} - \underline{v}')] \quad (10.46)$$

Since the real part is even in $\underline{k}$ and the imaginary part is odd, the same will be true of

$$(\varepsilon^{*})^{-1} = \varepsilon/|\varepsilon|^2 \quad (10.47)$$

It follows that in the first term in the curly brackets of (10.45), we can neglect the real part of $(\varepsilon^{*})^{-1}$, which will vanish in the $\underline{k}$ integration, and replace $1/\varepsilon^{*}$ by

$$i\text{Im}\ (\varepsilon^{*})^{-1} = -|\varepsilon|^{-2}\ \pi i \int d\underline{v}'\ (\omega_p^{\ 2}/k^2)\underline{k} \cdot (\partial f/\partial \underline{v}')\delta[\underline{k} \cdot (\underline{v} - \underline{v}')] \quad (10.48)$$

Then

$$\delta f/\delta t = (q/m)(2\pi)^{-3}(\partial/\partial \underline{v}) \cdot \int d\underline{k}\ \hat{\underline{k}}\ |\varepsilon(\underline{k}, \underline{k} \cdot \underline{v})|^{-2}$$

$$\cdot \{16\pi^3\ q/mk^3]\ \underline{k} \cdot (\partial f/\partial \underline{v}) \int d\underline{v}'\ (q')^2 f(\underline{v}')\ \delta[\underline{k} \cdot (\underline{v} - \underline{v}')] -$$

$$- [4\pi^2 q\ f(\underline{v})/k] \int d\underline{v}'\ (\omega_p^{\ 2}/k^2)\ \underline{k} \cdot (\partial f/\partial \underline{v}')\ \delta[\underline{k} \cdot (\underline{v} - \underline{v}')] \quad (10.49)$$

Collecting numerical coefficients we write this in the final form

$$\delta f/\delta t = (\partial/\partial \underline{p}) \cdot \underline{J} \tag{10.50}$$

with $\underline{p} = m\underline{v}$ and

$$\underline{J} = \int d\underline{v}'\ n'\ (f(v)\ \partial f(\underline{v}')/\partial \underline{p}' - \partial f(\underline{v}')\ f(\underline{v})/\partial \underline{p}] \cdot \underset{\sim}{K} \tag{10.51}$$

and

$$\underset{\sim}{K} = 2(qq')^2 \int d\underline{k}\ \underline{k}\ \underline{k}\ \delta\ (\underline{k} \cdot (\underline{v} - \underline{v}'))]/k^4 |\epsilon\ (\underline{k},\ \underline{k} \cdot \underline{v})|^2 \tag{10.52}$$

a result which was first derived (independently) by Lenard and by Bolescu.

Note that the same formalism can be also used to calculate other correlations of interest, e.g., $<\delta \mathcal{F}(\underline{x}, \underline{v}, t)\delta\mathcal{F}(\underline{x}', \underline{v}', t')>$, $<\delta \mathcal{E}(\underline{x}, t)\delta\mathcal{E}(\underline{x}'', t')>$, etc. We also observe that in the expression (10.39) for $\partial f/\partial t$, there would be a factor $\exp i[\underline{k} \cdot (\underline{v}' - \underline{v}'')t]$ in the integrand, causing the whole expression to phase mix to zero were it not for the $\delta(\underline{v}' - \underline{v}'')$ contained in $<<\delta\mathcal{F}_0(\underline{k}\cdot\underline{v}')\cdot \delta\mathcal{F}_0(\underline{k}, \underline{v}'')>>$. That is, $\partial f/\partial t$ arises only from the singular self-correlation term in $<\delta\mathcal{F}(R,0)\ <\delta\mathcal{F}(R',0)$; even if we had not assumed $g_2 = 0$ initially, but allowed some correlations, the result for $\partial f/\partial t$ would not be changed. Our assumption that $g_2 = 0$ at $t = 0$ is this seen to be only a convenience; any other choice for $g_2(R, R', 0)$ will give the same result provided it is not singular in $\underline{v} - \underline{v}'$.

Finally we recall that, strictly speaking, our derivation of $\partial f/\partial t$ is valid only for a homogeneous plasma, with f a function of $\underline{v}$ and t only, and that we have neglected the ensemble averaged fields $\underline{E}$ and $\underline{B}$ in computing $\delta\mathcal{F}$. Nevertheless, for plasma which are weakly inhomogeneous and for fields which are not too large, we can still use the $\partial f/\partial t$ given by (10.50) through (10.52). This is a kind of local approximation, in which at each $\underline{x}$ and t we use the instantaneous local $f(\underline{x}, \underline{v}, t)$ for computing ε and then $\underline{J}$, thus obtaining $\partial f/\partial t$ as a function of $\underline{x}$ and t, to be substituted into the kinetic equation which gives the evolution of f,

$$\partial f/\partial t + \underline{v}\ \nabla \partial f + (q/m)(\underline{E} + \underline{v} \times \underline{B}/c) \cdot (\partial f/\partial \underline{v}) = \delta f/\delta t \tag{10.53}$$