VII. Linearized Waves in a Uniform, Magnetized Plasma

With the exception of a very brief discussion of one-fluid MHD in Chap. IV, all of our considerations so far have been confined to a plasma with no external, applied magnetic field. This enabled us to become familiar with the basic concepts and techniques of plasma physics, including the collective, self consistent aspects which constitute its most essential feature, free of the considerable algebraic complexities associated with the magnetic case. All of these concepts and techniques carry over, mutatis mutandis, to the magnetized plasma, the only difference being that in presence of a uniform, constant magnetic field $\underline{B}_0$ the trajectories for the equilibrium plasma (i.e., without waves or other perturbing fields) are no longer straight lines but, instead, helices in configuration space.

For the unmagnetized plasma without external magnetic (or electric) fields, the trajectory functions ($\underline{X}$, $\underline{V}$) discussed in Chap. VI, are given simply by

$$\begin{aligned} \underline{V}\,(t';\,\underline{x},\,\underline{v},\,t) &= \underline{v} \\ X\,(t';\,\underline{x},\,\underline{v},\,t) &= \underline{x} + \underline{v}\,(t' - t) \end{aligned} \tag{7.1}$$

In a uniform external magnetic field, the equations of motion for $\underline{X}$ and $\underline{V}$ are

$$\frac{d\underline{X}}{dt'} = \underline{V}; \qquad \frac{d\underline{V}}{dt'} = \frac{q}{mc}\,\underline{V} \times \underline{B}_0 = \underline{V} \times \underline{\Omega} \tag{7.2}$$

with

$$\underline{\Omega} = q\underline{B}_o/mc = \Omega\underline{b}; \qquad \underline{b} = \underline{B}_0/B_0 \tag{7.3}$$

Note that $\Omega \equiv qB_0/mc$ is a signed cyclotron frequency, positive for ions and negative for electrons. The solution of (2) is

$$\underline{V}\,(t';\underline{x},\,\underline{v},\,t) = \underline{v}_\perp \cos\Omega\,(t'-t) + \underline{v}_\perp \times \underline{b}\sin\Omega\,(t'-t) + \underline{v}_\parallel$$
$$= \underline{v} + \Delta\underline{V}$$
$$\underline{X}\,(t';\,\underline{x},\,\underline{v},\,t) = \underline{x} + \underline{v}_\parallel\,(t'-t) + \Omega^{-1}\{\underline{v}_\perp \sin\Omega\,(t'-t) + \underline{v}_\perp \times b\,[1-\cos\Omega\,(t'-t)]\} = \underline{x} + \Delta\underline{X} \qquad (7.4)$$

where we decompose $\underline{v}$ into parallel and perpendicular components in the usual way,

$$\underline{v}_\parallel = \underline{b}\,\underline{b}\cdot\underline{v} \qquad\qquad \underline{v}_\perp = \underline{v} - \underline{v}_\parallel = \underline{b} - (\underline{v}\times\underline{b})$$

Note that $\Delta\underline{V}$ and $\Delta\underline{X}$, which are defined implicitly by (7.4), depend on t and t' only through the combination $\tau = t - t'$.

The complexity of (7.4), as compared with (7.1), is a good measure of the additional difficulties which the magnetic field brings. Considered as a dielectric medium, the plasma becomes anistropic, since the response to external electromagnetic fields will clearly depend upon their orientation relative to $\underline{B}_0$. For short times, $[\Omega\,(t'-t)] << 1$, it is clear that (7.4) reduces to (7.1), so that, as is to be expected, magnetic field effects will be negligible for phenomena which are fast on the time scale of Ω^{-1}, e.g., if the growth or decay rates of waves exceed Ω.

From (7.4) it follows that the motion of a single particle consists of circular rotation, at frequency Ω, about a "guiding center" which moves with velocity $v_\parallel$ along $\underline{B}_0$. The sense of rotation is left-handed for positive charges ($\Omega > 0$), right-handed for negative charges ($\Omega < 0$). The radius of the circle is the cyclotron radius, $r_c = v_\perp/\omega_c$, where we introduce the notation $\omega_c = |\Omega|$ for the magnitude of the cyclotron frequency.

This simple physical picture is most apparent if we introduce cylindrical coordinates $(v_\perp, v_z, \theta)$ in velocity space with the symmetry axis, which we take to be the z-axis, parallel to $\underline{B}_o$. We have

$$v_x = v_\perp \cos\phi \qquad\qquad v_y = v_\perp \sin\phi$$
$$v_z = (v^2 - v_\perp^2)^{1/2} \qquad\qquad v_\perp = (v_x^2 + v_y^2)^{1/2}$$

If $\underline{v}$ is specified at time t by $(v_\perp, v_z, \phi)$ and at time t' by $(V_\perp, V_z, \Phi)$, then (7.4) is equivalent to

$$V_\perp = v_\perp \quad V_z = v_z \quad \Phi = \phi + \Omega\tau \tag{7.5}$$

with $\tau = t - t'$. In the course of time, only the azimuth angle Φ changes, V_z and $V_\perp$ remaining constant.

A similar representation is also convenient for the spatial position. We define the "guiding center" (g.c.) position at time t as

$$\underline{x}_g \equiv (x - r_c \cos\theta, y - r_c \sin\theta, z)$$

with

$$\theta \equiv \phi + \pi/2 \qquad\qquad r_c = v_\perp/|\Omega|$$

Similarly, at time t' we define the g.c. position as

$$\underline{X}_g = (X - r_c \cos\Theta', Y - r_c \sin\Theta', Z_g)$$

where

$$\Theta \equiv \theta + \Omega\tau = \Phi + \pi/2 \qquad\qquad Z_g \equiv Z - v_{\parallel}\tau.$$

and the arguments of X_g and X are understood to be the usual set $(t'; \underline{x}, \underline{v}, t)$

It follows from (7.4) that the parts of $\underline{x}_g$ and $\underline{X}_g$ normal to $\underline{B}_o$ are equal, i.e., that

$$\underline{x}_g - \underline{X}_g = \underline{v}_{\parallel}\tau$$

and that the position of time t' is

$$\underline{X} = \underline{X}_g + r_c\ (\cos\Theta\ ,\ \sin\Theta\ ,\ 0) \qquad (7.6)$$

Thus, the guiding center moves along a field line with velocity $v_{\parallel}$ and the particle revolves around it with angular frequency Ω. The relations among the various quantities are shown in Fig. 7.1.

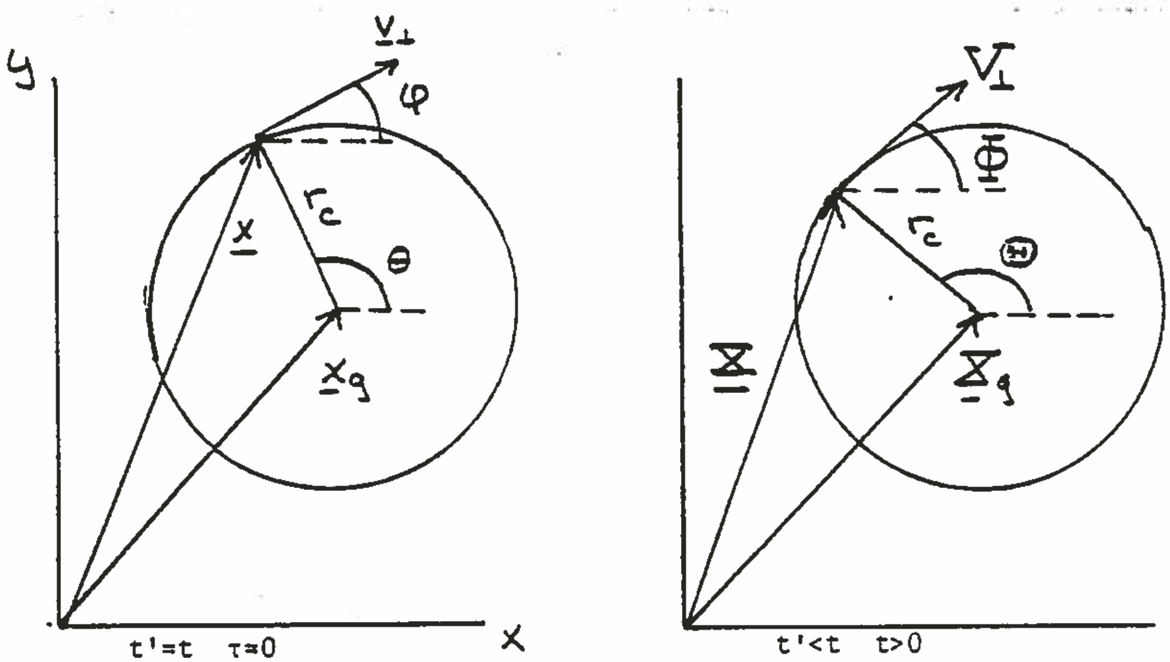

Fig. 7.1. Kinematic variables describing the cyclotron gyration of a positively charged particle in a uniform magnetic field $\underline{B}_o = B_o\,\hat{\underline{z}}$. This projection on a plane normal to $\underline{B}_o$ does not show the uniform motion of the guiding center along the z-axis with velocity $v_{\parallel}$.

A good guide to the new physics associated with the presence of the magnetic field is provided by a study of the linearized equations, i.e., the properties of small amplitude waves in a uniform, infinite homogeneous plasma. The Fourier-Laplace transform of Maxwell's equations for the linearized fields $\underline{E}$ $(\underline{k},\underline{\omega})$, $\underline{B}$ $(\underline{k},\underline{\omega})$ have the form

$$ic\underline{k} \times \underline{E} = i\omega\underline{B}$$

$$ic\underline{k} \times \underline{B} = 4\pi\underline{j} - i\omega\underline{E} \tag{7.7}$$

where we have assumed that $\underline{E} = \underline{B} = 0$ at $t = 0$. (See prob. 7.1.) The current density, $\underline{j}$, is the sum of the external current density, $\underline{j}_e$, and the current $\underline{j}_p$ induced in the plasma by the fields $\underline{E}$ and $\underline{B}$. When we linearize the dynamical equations for the plasma, be they fluid or kinetic, we find in general that, aside from initial value contributions, $\underline{j}_p$ is proportional to $\underline{E}$ and $\underline{B}$. Since it follows from (7.7) that $\underline{B} = c(\underline{k} \times \underline{E})/\omega$, we can simply write

$$\underline{j}_p = \underset{\sim}{\sigma} \cdot \underline{E} \, .$$

Of course, the conductivity tensor $\underset{\sim}{\sigma}$ must be calculated from the plasma equations of motion; in fact, that calculation constitutes a major portion of this chapter. In general, of course, $\underline{j}_p$ may contain terms associated with the initial perturbations in the plasma, and the Fourier transformed Maxwell equations should allow for initial values of $\underline{E}$ and $\underline{B}$, i.e., (7.7) should have additional terms $B(\underline{k},\, t = 0)$ and $-\underline{E}(\underline{k}, t = 0)$ on the right hand sides of the first equations, respectively. Taking account of these possibilities we should really have

$$\underline{J} = \underline{j}_p + \underline{j}_e + \underline{j}_o = \underset{\sim}{\sigma} \cdot \underline{E} + \underline{J}; \qquad \underline{J} = \underline{j}_e + \underline{j}_o$$

where $\underline{j}_o$ represents initial value contributions whose exact form is seldom of practical interest (see prob. 1.) Substituting this expression for j into (7.7) and eliminating $\underline{B}$ gives an equation for $\underline{E}$:

$$\underset{\sim}{M} \cdot \underline{E} \equiv (c/\omega)^2\, \underline{k} \times (\underline{k} \times \underline{E}) + \underset{\sim}{\varepsilon} \cdot \underline{E} = \frac{-4\pi i}{\omega}\, \underline{J} \tag{7.8}$$

where we define the dielectric tensor as

$$\underset{\sim}{\varepsilon} \equiv \underset{\sim}{1} + 4\pi i \underset{\sim}{\sigma}/\omega$$

Equation (7.8) (or its $\underline{x}$, t version) is sometime called the <u>vector wave equation</u> for $\underline{E}$. For given sources $\underline{J}$ we can solve (7.8) for $\underline{E}(k,\omega)$:

$$\underline{E}\,(k,\omega) = M^{-1}\,(4\pi i/\omega)\underline{J}\quad(-4\pi i/\omega)M^{-1}\cdot\ \underline{J} \tag{7.9}$$

where $\underset{\sim}{M}^{-1}$ is the inverse of the 3 x 3 matrix defined by (7.8); from E we immediately find $\underline{B} = c\underline{k} \times E/\omega$.

Just as in the unmagnetized case, the long time behavior of $\underline{E}(\underline{k},t)$ is determined by the singularities of $\underset{\sim}{M}^{-1}$ and $\underline{J}$ in the complex ω plane. The latter will be specific to the external sources or initial conditions which excite the waves, but the former will be general, i.e., independent of the type of excitation and determined entirely by the unperturbed state of the system as described by $f_o(v)$. Singularities of $\underset{\sim}{M}^{-1}$ occur at the zeros of $M = \det \underset{\sim}{M}$ (or at branch cuts, etc. of M). The roots of M are found from the dispersion relation

$$M = \det \underset{\sim}{M} = 0$$
$$\underset{\sim}{M} \equiv \underset{\sim}{\varepsilon} + \underline{N}\,\underline{N} - N^2\underset{\sim}{1} \tag{7.10}$$

where $\underline{N} = \underline{k}c/\omega$ is the vector index of refraction i.e., a vector whose magnitude is the usual index of refraction (c divided by phase velocity) and whose direction is along the phase velocity of the wave.

Our first task is to find $\underset{\sim}{\varepsilon}(\underline{k},\omega)$. Substituting it into the dispersion relation (7.10) and solving for ω as a function of $\underline{k}$ gives the frequencies, real or complex, which characterize that part of the long time response to an initial perturbation associated with zeros of M. Of course, just as in Chap. III, this dispersion relation can be interpreted in two different ways. When we consider the homogeneous equations of motion of the system (Maxwell's equation plus some plasma dynamic equations, with no external sources or initial conditions), (7.10) gives the "normal modes", i.e., the values of ω, for given $\underline{k}$, for which the homogeneous equations can heve consistent, non-trivial solutions. This is the Simple Minded Plane Wave Substitution approach. If, instead, we take the more physically realistic view, recognizing that there will always be external sources and/or initial conditions, represented by $\underline{J}$, then we can say that the solutions of (7.10) provide information concerning the long time response of the system to an initial perturbation (ω as a function of real $\underline{k}$) and also the asymptotic spatial propagation characteristics for steady state (localized) excitation (k as a function of $\underline{k}/k$ and ω).

While it is a straighforward matter to compute $\underset{\sim}{\varepsilon}(\underline{k},\omega)$ from the Vlasov equation, the result is so complicated that we need a simpler guide to the essential physics. This is provided by the $\underset{\sim}{\varepsilon}$ for a cold, two-fluid plasma; while the latter is, of course, a limiting case of the Vlasov plasma, the derivation for this special case is so much simpler that it is worth discussing separately. We therefore begin with a brief derivation of $\underset{\sim}{\varepsilon}$ for the cold two-fluid plasma and then consider the $\underset{\sim}{\varepsilon}$ resulting from a kinetic (Vlasov equation) treatment. With these results in hand, we explore, in section B, solutions to the dispersion equation (7.9) for specific cases of interest.

A. The Dielectric Tensor, $\underset{\approx}{\epsilon}$

1) The Cold, Two Fluid $\underset{\approx}{\epsilon}$

We follow the method of "Simple-Minded Plane Wave Substitution" discussed in Chapter III, since we know how to incorporate the effects of external fields or initial value conditions through $\underline{J}$. For each species we have, from the continuity and momentum equations,

$$\omega n = n_o \underline{k} \cdot \underline{u}$$

$$- i\omega \underline{u} = \frac{q}{m} \underline{E} + \underline{u} \times \underline{\Omega} \qquad (7.11)$$

Choosing the z-axis along $\underline{B}_o$, we introduce circularly polarized coordinated

$$u_\pm = \frac{u_x \pm iu_y}{\sqrt{2}}, \qquad E_\pm = \frac{E_x \pm iE_y}{\sqrt{2}}, \text{ etc.} \qquad (7.12)$$

from which follows (for any two vectors)

$$\underline{a} \cdot \underline{b} = a_+ b_- + a_- b_+ + a_z b_z$$

A wave with only $E_+(E_-) \neq 0$ corresponds to left-hand (right-hand) circular polarization relative to $\underline{B}_0$. No te that this is a different convention from the one used in optics, where the designation of a circularly polarized wave as left or right handed is usually referred to the direction of $\underline{k}$. In a plasma the most important question is whether a circularly polarized wave has an $\underline{E}$ vector which rotates in the same sense as the gyration of a particle about $\underline{B}_o$, independent of the direction of $\underline{k}$; positive (negative) particles can resonate only with a wave whose polarization includes a circular component which is left-handed (right-handed) in the sense defined here, i.e., one which had $E_+ \neq 0$ ($E_- \neq 0$).

From (7.12) we have

$$(\underline{b} \times \underline{u})_{\pm} = \pm i \underline{u}_{\pm} \tag{7.13}$$

so (7.11) gives

$$u_{\pm} = \frac{i(q/m)E_{\pm}}{(\omega \pm \Omega)} \tag{7.14}$$

$$u_z = (iq/m\omega)E_z$$

i.e., as we might expect, the equation of motion is diagonal in terms of circularly polarized variables. The contribution of species α to the current density is

$$n_o q_\alpha \underline{u}_\alpha = \underset{\sim}{\sigma}_\alpha \cdot \underline{E} \tag{7.15}$$

with

$$\underset{\sim}{\sigma}_\alpha = i\left(\frac{\omega_{p\alpha}^2}{4\pi}\right) \cdot \begin{pmatrix} \frac{1}{\omega - \Omega_\alpha} & 0 & 0 \\ 0 & \frac{1}{\omega + \Omega_\alpha} & 0 \\ 0 & 0 & \frac{1}{\omega} \end{pmatrix} \tag{7.16}$$

if we understand the rows and columns of $\underset{\sim}{\sigma}$ to be labelled by (+, -, z). In these same circularly polarized coordinates, we also have

$$\underset{\sim}{\varepsilon} = 1 + (4\pi i/\omega)\underset{\sim}{\sigma} = 1 - \sum_\alpha (\omega_{p\alpha}^2/\omega)\begin{pmatrix} \frac{1}{\omega - \Omega_\alpha} & 0 & 0 \\ 0 & \frac{1}{\omega + \Omega_\alpha} & 0 \\ 0 & 0 & \frac{1}{\omega} \end{pmatrix} \tag{7.17}$$

$$= \begin{pmatrix} L & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & P \end{pmatrix}$$

where

$$L(\omega) = R(-\omega) = 1 - \sum \frac{\omega_{p\alpha}{}^2}{\omega(\omega-\Omega_\alpha)}$$

$$P(\omega) = 1 - \sum \omega_p{}^2/\omega^2 \qquad (7.18)$$

For a two species, neutral, electron-ion plasma we have

$$L(\omega) = 1 - \omega_p{}^2/(\omega + \omega_{ce})\ (\omega - \omega_{ci})$$

$$R(\omega) = 1 - \omega_p{}^2/(\omega - \omega_{ce})\ (\omega + \omega_{ci}) \qquad (7.19)$$

$$P = 1 - \omega_p{}^2/\omega^2$$

where

$$\omega_p{}^2 = \omega_{pe}{}^2 + \omega_{pi} \doteq \omega_{pe}{}^2$$

$$\omega_{ce} \equiv eB_o/mc = -\Omega_e \qquad (7.20)$$

$$\omega_{ci} \equiv eB_o/Mc = \Omega_i$$

Although $\underset{\sim}{\varepsilon}$ is a diagonal in the circular polarization representation, the remainder of $\underset{\sim}{M}$ is diagonal in this representation only for propagation along $\underline{B}_o$. In xyz coordinates with z along the magnetic field $\underline{B}_o$ and y perpendicular to the plane of $\underline{k}$ and $\underline{B}_o$ we have in general

$$\underline{N}\,\underline{N} - \underline{N}^2\underline{\underline{1}} = -N^2 \begin{pmatrix} \cos^2\theta & 0 & -\sin\theta\cos\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix} \qquad (7.21)$$

where θ is the angle between $\underline{k}$ and $\underline{B}_o$ (cf Fig. 7.2):

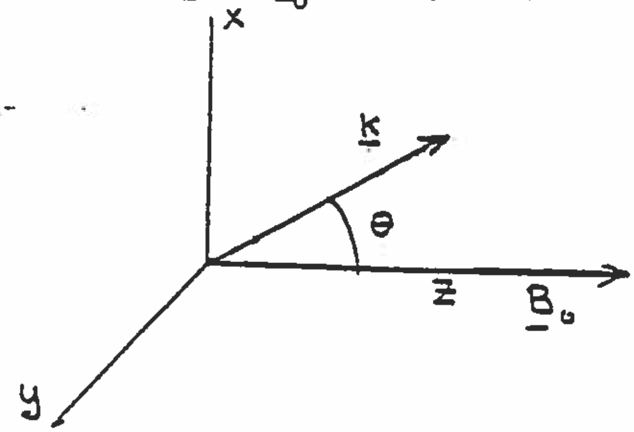

Fig. 7.2 Conventional choice of axes for wave propagation in a uniform, unhomogeneous magnetized plasma.

For $k \,||\, \underline{B}_o$, $\theta = 0$, this matrix is diagonal in both the (xyz) and (+ - z) representations. Then $\underset{\sim}{M}$ will be diagonal in the circular representation, so for propagation parallel to $\underline{B}_o$ that representation will be the most convenient one. For propagation oblique to $\underline{B}_o$ or perpendicular to $\underline{B}_o$, it is, in general, easiest to work with the xyz representation. In problems involving propagation through weakly inhomogeneous plasmas (e.g., the ionosphere), where a WKB analysis is commonly used, the direction of the density gradient is sometimes chosen as one coordinate axis, say the z-axis, resulting in expressions for $\underset{\sim}{\varepsilon}$ which differ from those given here just by an othogonal transformation. In such problems, it is often appropriate to take ω and the x and y components of $\underline{k}$ as fixed, and solve for k_z, which, in a WKB sense, corresponds to $-i\partial/\partial z$.

Since

$$(\underset{\sim}{\sigma} \cdot \underline{E})_x = [(\underset{\sim}{\sigma} \cdot \underline{E})_+ + (\underset{\sim}{\sigma} \cdot \underline{E})_-]2^{-1/2} =$$

$$[(\sigma_{++} + \sigma_{--})E_x + i(\sigma_{++} - \sigma_{--})\, E_y]/2$$

we have

$$\sigma_{xx} = (\sigma_{++} + \sigma_{--})/2; \; \sigma_{xy} = i(\sigma_{++} - \sigma_{--})/2; \text{ etc.}$$

Thus, in rectangular representation, with rows and columns labeled by x,y,z, we have

$$\underset{\sim}{\sigma}\,_\alpha = \left(\frac{\omega_{p\alpha}{}^2}{4\pi i}\right) \begin{pmatrix} \frac{\omega}{\Omega_\alpha{}^2-\omega^2} & \frac{-i\Omega_\alpha}{\Omega_\alpha{}^2-\omega^2} & 0 \\ \frac{i\Omega_\alpha}{\Omega_\alpha{}^2-\omega^2} & \frac{\omega}{\Omega_\alpha{}^2-\omega^2} & 0 \\ 0 & 0 & -\frac{1}{\omega} \end{pmatrix}$$

Similarly, from (7.17), we obtain the rectangular representations for $\underset{\sim}{E}$ and $\underset{\sim}{M}$:

$$\underset{\sim}{\varepsilon} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

and

$$\underset{\sim}{M} \equiv \underset{\sim}{\varepsilon} + \underline{N}\,\underline{N} - N^2\underset{\sim}{1} = \begin{pmatrix} S - N^2\cos^2\theta & -iD & N^2 \sin\theta\cos\theta \\ iD & S-N^2 & 0 \\ N^2\sin\theta\cos\theta & 0 & P - N^2\sin^2\theta \end{pmatrix} \quad (7.22)$$

where

$$S = \frac{L + R}{2} \qquad\qquad D = \frac{R - L}{2} \tag{7.23}$$

The dispersion relation,

$$M = \det \underset{\sim}{M} = 0 \tag{7.24}$$

is a biquadratic in N (the N^6 terms dropping out):

$$A\,N^4 - B\,N^2 + C = 0 \tag{7.25}$$

with

$$A = S\sin^2\theta + P\cos^2\theta$$

$$B = PS(1 + \cos^2\theta) + RL\sin^2\theta$$

$$C = (S^2 - D^2)P = RLP$$

Thus, for given θ there are, in general, two solutions for N^2:

$$N^2 = (B \pm F)/2A$$

$$F = \sqrt{B^2 - 4AC} = [\sin^4\theta\ (PS - RL)^2 + 4P^2D^2\cos^2\theta]^{1/2} \tag{7.26}$$

Since (7.25) is a homogeneous quadratic form in $\sin\theta$ and $\cos\theta$, we can solve for $\tan\theta$:

$$\tan^2\theta = \frac{P(N^2 - R)\ (N^2 - L)}{(SN^2 - RL)\ (P - N^2)}$$

We defer a discussion of the physical significance of these solutions to Section B, and proceed now to derive the $\underset{\sim}{\varepsilon}$ corresponding to the

Vlasov equation, so that when considering the waves we can take into account thermal and kinetic connections to the cold fluid results.

For later reference, we note that (7.11) can be solved for $\underline{u}$ without introducing the circular polarization representation. We have

$$- i\omega\underline{u} = \underline{F} + \underline{u} \times \underline{\Omega}, \qquad \underline{F} \equiv q\,\underline{E}/m$$

The cross and dot products with $\underline{\Omega}$ give

$$- i\omega\underline{u} \times \underline{\Omega} = \underline{F} \times \underline{\Omega} + \underline{u} \cdot \underline{\Omega}\,\underline{\Omega} - \Omega^2\underline{u}$$

$$- i\omega\underline{u} \cdot \underline{\Omega} = \underline{F} \cdot \underline{\Omega}$$

Eliminating $\underline{u} \cdot \underline{\Omega}$ and $\underline{u} \times \underline{\Omega}$ from these three equations we have

$$\underline{u} = \frac{i}{\omega}\underline{F}_{\parallel} + \frac{i\omega}{\omega^2 - \Omega^2}\left[\underline{F}_{\perp} + \frac{i}{\omega}\underline{F}_{\perp} \times \underline{\Omega}\right] \tag{7.27}$$

with

$$\underline{F} \equiv q\underline{E}/m = \underline{F}_{\parallel} + \underline{F}_{\perp}$$

$$\underline{F}_{\perp} = \underline{b} \times (\underline{F} \times \underline{b}) \qquad \underline{F}_{\parallel} = \underline{F} - \underline{F}_{\perp} = \underline{b}F_{\parallel}$$

For $\Omega = 0$, or, more generally, $\omega >> \Omega$, this reduces to the result $\underline{u} = i\underline{F}/\omega$ found in Chapter 3. Naturally, the expression for $\underset{\sim}{\sigma}$ in rectangular representation can be derived directly from (7.27).

In general, as (7.27) shows, an electric field, $\underline{E}_{\perp}$, induces a flow parallel to $\underline{E}_{\perp}$ and another perpendicular to $\underline{E}_{\perp}$. The latter term, the $\underline{F}_{\perp} \times \underline{\Omega}$ in (7.27), arises directly from the basic gyrotropic character of charged particle motion in a magnetic field and is the source of the well-known "Hall effect." If ω is of order Ω, the normal term and the Hall term are of comparable magnitude, but if $\omega \ll \Omega$ then the Hall term dominates. It is associated with the $\underline{E} \times \underline{B}$ drift to be discussed in a later chapter. For $\omega \ll \Omega$, the velocity normal to $\underline{B}$ is

$$\underline{u}_{\perp} = \underline{F}_{\perp} \times \underline{\Omega}/\Omega^2$$

from which follows

$$\underline{F}_{\perp} = \underline{\Omega} \times \underline{u}_{\perp}$$

If there is a current $\underline{j}_{\perp} = nq\underline{u}_{\perp}$ normal to $\underline{B}$, there must be an electric field $E_{\perp} = (m/nq^2)\ \underline{\Omega} \times \underline{j}_{\perp} = (\underline{B} \times \underline{j})/nqc$; the observation of this field and the associated potential drop in a direction normal to both $\underline{j}$ and $\underline{B}$ is the classic Hall effect.

2) The Vlasov $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$

In this section, we shall make use of a number of properties of the Bessel functions J_n, I_n of integral order. While the identities can be found in standard mathematical tables and hardbooks, we give, in Appendix 7A, concise derivations of the identities used in this chapter; all of them can be easily obtained from the generating function for the "Bessel coefficients" J_n:

$$\exp[z(t - t^{-1})/2] = \sum_{-\infty}^{\infty} J_n(z)t^n$$

a. Heuristic Preview

Since this section entails considerable formal complexity, it is helpful to consider first what new effects we can anticipate as compared to either the cold fluid treatment of magnetized plasmas or the Vlasov theory of unmagnetized plasmas. Thermal motion of particles <u>along</u> the field will have the general effect of smearing out the singularities at $\omega = \Omega$, since a wave with frequency ω and parallel wave number k_z appears, to a particle with velocity v_z along $\underline{B}_o$, to have frequency $\tilde{\omega} = \omega - k_z v_z$. It is this Doppler shifted frequency which replaces ω in the resonant demoninators and when we integrate $(\omega - k_z v_z - \Omega)^{-1}$ over v_z we will obtain a Z function, appropriate to the $f_o(v_z)$ involved. Thus, we expect that singular terms like $(\omega - \Omega)^{-1}$ in the cold plasma $\underset{\sim}{\varepsilon}$ will be replaced by $Z[(\omega - \Omega)/k_z a_z]$.

In addition, the inclusion of thermal motion in the plane perpendicular to $\underline{B}_o$ gives rise to effects not encountered in unmagnetized plasmas. To illustrate these, consider the response of a single particle to an electric field, perpendicular to $\underline{B}_o$,

$$\underline{E} = E_1 \hat{\underline{x}} \exp[i(k_\perp x + k_z z - \omega t)]$$

The particle equations of motion

$$\ddot{x} = \dot{v}_x = (q/m)E_x + \Omega v_y$$

$$\dot{v}_y = \Omega v_x = -\Omega \dot{x}$$

$$\ddot{z} = 0$$

have the exact integral (corresponding to conservation of the canonical momentum p_y)

$$v_y + \Omega x = \text{const.}$$

It follows that

$$\ddot{x} + \Omega^2 x = (q/m)\ E_x$$

where we have chosen $x = 0$ at the point where $v_y = 0$. In absence of the electric field, this gives the usual gyration about the magnetic field,

$$x = r_o\ \sin\Omega t \qquad y = r_o \cos\Omega t \qquad z = v_z t$$

where r_o, v_z are constants and we have taken $t = 0$ when $x = 0$. (For a typical particle in a thermal distribution, v_z and Ω_o will be of order of the thermal velocity, a.) To find the response to $\underline{E}$, we set

$$x = r_o\ \sin\Omega t + x_1$$

which gives

$$\ddot{x}_1 + \Omega^2 x_1 = (q/m)E_1 \exp[ik_\perp x_o \sin\Omega t + i(k_z v_z - \Omega)t + ik_\perp x_1] + \text{c. c.}$$

For small E, we can neglect the $k_\perp x_1$ term in the exponent. Using the identity (7A.3) we have

$$\exp(ik_\perp x_o \sin\Omega t) = \sum_n J_n (k_\perp x_o)e^{in\Omega t}$$

so in the case where $|k_\perp x_1| << 1$ the equation for x_1 can be written

$$\left.\begin{aligned} &\ddot{x}_1 + \Omega^2 x_1 = \sum_n c_n e^{-i[\omega - k_z v_z - n\Omega]t} + \text{c.c.} \\ &\text{with} \\ &c_n = (q/m)E_1 J_n (k_\perp x_o) \end{aligned}\right\} \qquad (7.28)$$

The general solution of (7.28) is a superposition of the general solution of the homogeneous equation, plus a particular solution of the inhomogeneous equation,

$$x_1 = A\exp(-i\Omega t) + \sum c_n(\Omega^2 - \omega_n^2)^{-1} \cdot \exp(-i\omega_n t) + \text{c.c.}$$

where

$$\omega_n \equiv \omega - n\Omega - k_z v_z$$

However, there will be a secular response ($x_1 \propto t$) if $\omega_n = \Omega$ for some n, i.e., if

$$\omega - k_z v_z = (n+1)\Omega$$

For n = 0, this is just the usual Doppler shifted cyclotron resonance, but for $n \neq 1$ we have a new phenomenon, namely a resonance when the Doppler shifted frequency $\tilde{\omega}$ matches a harmonic of the cyclotron frequency. This new effect is clearly a consequence of the thermal motion across the magnetic field (i.e., of thermal gyration of particles around their guiding centers) and we shall, naturally, find a similar effect when, in the next section, we compute the perturbation in distribution function produced by an electromagnetic wave.

From (7.28) we see that the resonance (7.29) will be proportional to $J_n(k_\perp r_o)$ so the importance of the harmonic terms increases with $k_\perp r_o$. When we presently integrate over all initial velocities to find $\rho(\underline{\kappa},\omega)$ and $\underline{j}(\underline{\kappa},\omega)$, we can see from (7.28) that we will encounter integrals over $\underline{v}$ of terms of the form

$$J_n(k_\perp r_o)[\omega - k_z v_z - (n+1)\Omega]^{-1}$$

If there is no thermal motion along $\underline{B}_o$ ($v_z = 0$) or if $\underline{k}$ is exactly perpendicular to $\underline{B}_o$ ($k_z = 0$) we will get resonant denominators $[\omega - (n+1)\Omega]$ in ρ and $\underline{J}$ and hence in $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$. In all other cases the integration over v_z will yield Z functions of argument $[\omega - (n+1)\Omega]/k_z a_z$.

With this heuristic indication of the effects to be expected, we now turn to the formal calculation of $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$.

b. Integration Over Unperturbed Orbits

Instead of solving the linearized, Fourier-Laplace transformed Vlasov equation, as we have done for the unmagnetized plasma, we shall use the method of orbit integration, discussed in Chapter VI. The algebra involved is about the same but the integration over orbits will be useful when we consider inhomogeneous magnetized plasmas and the homogeneous case provides a good introduction to the technique. We include in the "unperturbed" field only the constant, uniform external magnetic field, $\underline{B}_o$, while $\underline{E}$ and $\underline{B}_1 = \underline{B} - \underline{B}_o$ are put on the right hand side of the Vlasov equation:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{q}{mc} (\underline{v} \times \underline{B}_o) \cdot \frac{\partial f}{\partial \underline{v}} = h(\underline{x}, \underline{v}, t) \equiv - \frac{q}{m} (\underline{E} + \frac{\underline{v} \times \underline{B}_1}{c}) \cdot \frac{\partial f}{\partial \underline{v}} \qquad (7.30)$$

Using the abbreviated notation $r = (\underline{x}, \underline{v})$, $R = (\underline{X}, V)$ we then have, as in (6.8),

$$f(r,t) = f[R\ (0;\ r,\ t),\ 0] + \int_o^t dt' h[r(t';\ r,\ t)] \qquad (7.31)$$

where $\underline{X}$ and $\underline{V}$ are the unperturbed trajectory functions given by (7.4) and (7.5). Of course, (7.29), with h defined by (7.28), is not an explicit solution of the Vlasov equation, but rather an integral equation formulation of the Vlasov equation. To obtain a solution, we therefore shall presently introduce the approximations appropriate to linearization about a spatially uniform, time-dependent solution of the "unperturbed" Vlasov equation, $f_o(\underline{v})$.

A solution of the unperturbed Vlasov equation must satisfy

$$\frac{\partial f_o}{\partial t} + \underline{v} \cdot \frac{\partial f_o}{\partial \underline{x}} + \underline{v} \times \underline{\Omega} \cdot \frac{\partial f_o}{\partial \underline{v}} = 0 \qquad (7.32)$$

If f_o is to be independent of $\underline{x}$ and t, then

$$\underline{v} \times \underline{\Omega} \cdot \frac{\partial f_o}{\partial \underline{v}} = - \frac{\partial f_o}{\partial \phi} = 0$$

It follows that

$$f_o(\underline{v}) = f_o\ (v_{\parallel},\ v_{\perp}) \qquad (7.33)$$

i.e., f_o must be independent of ϕ, although it may be an arbitrary function of v" and $v_\perp$. If, in (7.31), we choose f at t = 0 to be of this character, i.e., choose f at t = 0 to be independent of $\underline{x}$ and of ϕ, then the first term in (7.31) is just

$$f(\underline{X},\ \underline{V},\ 0) = f_o\ (\underline{V}) = f_o(V_{\shortparallel},\ V_\perp) = f_o\ (v_{\shortparallel},\ v_\perp) \qquad (7.34)$$

the last equality being a consequence of (7.5).

Setting $f(\underline{x},\ \underline{v},\ t) = f_o\ (\underline{v}) + f_1\ (\underline{x},\ \underline{v},\ t)$ and neglecting terms of second order in $\underline{E}$, $\underline{B}_1$, we have from (7.31)

$$f_1(\underline{x},\ \underline{v},\ t) = \frac{q}{m} \int_o^t dt'\ \underline{\mathcal{E}}\ [\underline{X},\ \underline{V},\ t'] \cdot \frac{\partial f_o}{\partial \underline{V}}\ (\underline{V}) \qquad (7.35)$$

with $\underline{X} = \underline{X}\ (t';\ \underline{x},\ \underline{v},\ t)$, etc. Since we wish to find $\underset{\sim}{\rho}(k,\ \omega)$ and $\underset{\sim}{\mathcal{E}}(\underline{k},\omega)$, we take the Fourier-Laplace transform of (7.36):

$$f_1(\underline{k},\ \underline{v},\ \omega) = -\frac{q}{m} \int_o^\infty dt \int d\underline{x}\ \exp[-i(\underline{k}\ \underline{x}\omega t)] \int_o^t dt'\ \underline{\mathcal{E}}\ [\underline{X},\ \underline{V},\ t'] \cdot \frac{\partial f_o}{\partial V} \qquad (7.36)$$

The $\underline{X}$ and t' integrations can be carried out if we take advantage of two basic properties of the trajectory functions, $\underline{X}$ and $\underline{V}$. We recall from (7.4) that ΔX and ΔV 1) are independent of $\underline{x}$, depending only on $\underline{v}$, t and t' and 2) depend upon t and t' only through the difference $\tau = t - t'$ and not upon t or t' individually. The first property makes the integration over $\underline{X}$ trivial and the second property allows us to carry out the t' integration if we simply interchange the orders of the t and t' integrations and use τ as integration variable in place of t:

$$f_1\ (\underline{k},\ \underline{v},\ \omega) = -\frac{q}{m} \int_o^\infty dt' \int_{t'}^\infty dt\ e^{i(\underline{k}\ \cdot \Delta X\ +\ \omega t)} \cdot \underline{\mathcal{E}}\ (\underline{k},\ \underline{V},\ t') \cdot \frac{\partial f_o}{\partial \underline{V}} =$$

$$= -\frac{q}{m}\int_0^\infty dt' \int_0^\infty d\tau \exp[i(\underline{k}\cdot\Delta\underline{X} + \omega\tau + \omega t')]\, \mathcal{E}(\underline{k}, \underline{V}, t') \cdot \partial f_o/\partial\underline{V} =$$

$$= -\frac{q}{m}\int_0^\infty d\tau \exp\,[i\underline{k}\cdot\Delta\underline{X}\,(\tau,v) + i\omega\tau]\, \mathcal{E}\,[\underline{k}, \underline{V}\,(\tau, \underline{v}), \omega]\, \frac{\partial f_o}{\partial\underline{V}} \qquad (7.37)$$

Finally, we introduce $\phi' = \Omega t$ as integration variable. Since all factors in the integrand of (7.37) save for $e^{i\omega\tau}$ are periodic in ϕ', with period 2π, we can simplify (7.37) using the identity, valid for any periodic function $H(s) = H(s + 2\pi)$,

$$\int_0^\infty dse^{i\nu s}\, H(s) = \sum_{n=0}^{\infty} e^{2\pi i\nu n} \int_0^{2\pi} dse^{i\nu s}\, H(s) = -\,Q(\nu) \int_0^{2\pi} dse^{i\nu s}\, H(s) \quad (7.38)$$

where $Q(\nu) \equiv [\exp(2\pi i\nu)\ -1]^{-1}$.

As in section 1, we chose the y axis perpendicular to both $\underline{k}$ and $\underline{B}_o$, so it follows from (7.5) that

$$\omega\tau + \underline{k}\cdot\Delta X = (\omega - k_{\shortparallel}v_{\shortparallel})\,\tau + \left(\frac{k_\perp v_\perp}{\Omega}\right)[-\cos\phi\sin\phi' + \sin\phi\,(1-\cos\phi')] =$$

$$= \nu\phi' + \kappa\,[\sin\phi - \sin\,(\phi' + \phi)] \qquad (7.39)$$

where

$$\nu \equiv (\omega - k_{\shortparallel}v_{\shortparallel}/\Omega) \qquad (7.40)$$

is the Doppler-shifted frequency, normalized to Ω, and

$$\kappa = k_\perp v_\perp/\Omega \qquad (7.41)$$

is 2π times the ratio of cyclotron radius to perpendicular wavelength. Using (7.38) and (7.39), we obtain from (7.37)

$$f_1(\underline{k}, \underline{v}, \omega) = Q(\nu)\int_0^{2\pi} d\phi' \exp\{i\nu\phi' + i\kappa\,[\sin\phi - \sin(\phi + \phi')]\}g(\phi) \qquad (7.42)$$

with

$$g(\phi) \equiv \frac{q}{m} \underset{\sim}{\mathcal{E}} [\underline{k}, \underline{V}, (\tau, \underline{v}), \delta] \frac{\partial f_o}{\partial \underline{V}} \tag{7.43}$$

According to (7.31), $\underline{V}$ $(\tau, \underline{v})$ is just $\underline{v}$ with ϕ replaced by $\Phi = \phi + \Omega\tau = \phi + \phi'$, so we may compute

$$g(\phi) = \frac{q}{m} \underset{\sim}{\mathcal{E}} (\underline{k}, \underline{v}, \omega) \cdot \frac{\partial f_o}{\partial \underline{v}} \tag{7.44}$$

and then replace ϕ with $(\phi' + \phi)$. Using the Maxwell equation $\underline{k} \times \underline{E} = (\omega \underline{B}_1/c)$ we have

$$\begin{aligned}(B_o/c) g(\phi) &= [\underline{E} + \underline{v} \times (\underline{k} \times \underline{E})/\omega] \cdot (\frac{\partial f_o}{\partial v_z} \hat{\underline{z}} + \frac{\partial f_o}{\partial v_\perp} \frac{\underline{v}_\perp}{v_\perp})] = \\ &= E_z (\frac{\partial f_o}{\partial v_z} - G \frac{\underline{k} \cdot \underline{v}_\perp}{\omega v_\perp}) + \frac{\underline{E} \cdot \underline{v}_\perp}{v_\perp} (\frac{\partial f_o}{\partial v_\perp} + \frac{k_z G}{\omega})\end{aligned} \tag{7.45}$$

where

$$G \equiv v_\perp \, \partial f_o / \partial v_z - v_z \, \partial f_o / \partial v_\perp \tag{7.46}$$

measures the anistropy of f_o, vanishing if f_o is a function of $v^2 = v_z{}^2 + v_\perp{}^2$. Finally, then, (7.42) gives

$$\begin{aligned} f_1 \; (\underline{k}, \underline{v}, \omega) &= \frac{c}{B_o} \, Q(\nu) \int_o^2 d\phi' \exp[i\nu\phi' + i\kappa\sin\phi - i\kappa\sin(\phi' + \phi)] \\ &[E_z \; (\frac{\partial f_o}{\partial v_z} - G \frac{\underline{k} \cdot \underline{V}_\perp}{\omega v_\perp}) + \frac{\underline{E} \cdot \underline{V}_\perp}{v_\perp} (\frac{\partial f_o}{\partial v_\perp} + \frac{k_z G}{\omega})] \end{aligned} \tag{7.37}$$

and we can compute

$$\underline{j} = \int d\underline{v} \; nq\underline{v}f = \underset{\sim}{\sigma} \cdot \underline{E} \tag{7.38}$$

From (7.49) we find

$$\underset{\sim}{g} = \int d\underline{v} \frac{\omega_p^2}{4\pi\Omega} Q(\nu) \int_0^{2\pi} 2\pi d\phi' \exp[i\nu\phi' + i\kappa\sin\phi - $$
$$-i\kappa\sin(\phi' + \phi)] \underset{\sim}{S}(\kappa,\nu,\phi,\phi') \tag{7.49}$$

with

$$\underset{\sim}{S} = \begin{matrix} \underset{\sim}{S}_\perp & \underset{\sim}{S}_{.z} \\ \underset{\sim}{S}_{z.} & \underset{\sim}{S}_{zz} \end{matrix} \tag{7.50}$$

and

$$\underset{\sim}{S}_\perp = v_\perp \frac{\partial f_o}{\partial v_\perp} + \frac{k_z G}{\omega} \begin{matrix} \cos\phi \cos(\phi' + \phi) & \cos\phi \sin(\phi' + \phi) \\ \sin\phi \cos(\phi' + \phi) & \sin\phi \sin(\phi' + \phi) \end{matrix}$$

$$S_{.z} = v_\perp \quad \frac{\partial f_o}{\partial v_z} - \frac{k_\perp G}{\omega} \quad \cos(\phi' + \phi)[\cos\phi, \sin\phi] \tag{7.51}$$

$$S_{z.} = \frac{\partial f_o}{\partial v_\perp} + \frac{k_z G}{\omega} \quad v_z \; [\cos(\phi' + \phi), \sin(\phi' + \phi)]$$

$$S_{zz} = v_z \quad \frac{\partial f_o}{\partial v_z} - \frac{k_\perp G}{\omega} \quad \cos(\phi' + \phi)$$

the • subscript standing for x and y. Note that <u>all</u> of the ϕ and ϕ' dependence in the integrand of (7.44) is explicitly shown in (7.49) and (7.51); all other factors, such as f_o, its derivatives and G are <u>independent of ϕ and ϕ'</u>.

c. <u>Details of the Integration</u>

An essential point is now that of the four integrations in (7.49), namely v_z, $v_\perp$, ϕ and ϕ', the last two can be done once and for all, <u>independent of f_o</u>. To organize this integration, we define, for any function, $F(\phi, \phi')$.

$$\langle F\rangle \equiv (2\pi)^{-1} \int_0^{2\pi} d\phi' \int_0^{2\pi} d\phi\ F(\phi,\phi') \exp\{i[\nu\phi' - $$
$$-\kappa\sin(\phi' + \phi) + \kappa\sin\phi]\} \tag{7.52}$$

Then, as shown in Appendix 6.2, all of the integrals $\langle\sin\phi\rangle$, $\langle\cos\phi\rangle$, etc., required for (7.49) can be expressed in terms of a single function of κ and ν,

$$A(\kappa,\nu) \equiv \langle 1\rangle = (2\pi)^{-1} \int_0^{2\pi} d\phi' \int_0^{2} d\phi\ \exp[i\nu\phi' + $$
$$+ i\kappa\sin\phi - i\kappa\sin\ (\phi' + \phi)] \tag{7.53}$$

and its κ derivatives, $A' = \partial A/\partial\kappa$ and $A' = \partial^2 A/\partial\kappa^2$. Using the identity (7A.3)

$$e^{ix\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta}$$

we can write

$$A(\kappa,\nu) = \int_0^{2\pi} d\phi e^{i\nu\phi} \sum_n J_n{}^2(\kappa)\ e^{in\phi}$$

Integrating on ϕ then gives an infinite series representation for A,

$$A(\kappa,\nu) = \sum_n \frac{J_n{}^2\ (\kappa)\ (e^{2\pi i\nu}-1)}{i(\nu - n)} = \sum_n \frac{J_n^2(\kappa)}{i(\nu-n)Q(\nu)} \tag{7.54}$$

Carrying out the sum over n, using (A7.19), gives a closed form expression for this series,

$$A(\kappa,\nu) = \int_0^{2\pi} d\phi e^{i\nu\phi}\ J_o(2\kappa\sin\tfrac{\phi}{2}) \tag{7.55}$$

The expressions for the elements of $\langle S\rangle$ can be written most compactly by introducing the function

$$C(\kappa,\nu) = A + i/\nu Q(\nu) =$$
$$= \int_0^{2\pi} d\phi\ e^{i\nu\phi}\ \{J_o[2\kappa\ \sin\ (\phi/2) - 1\} \tag{7.56}$$

and, in place of A", the function

$$D(\kappa,\nu) = (1/2)\ [A(\kappa,\nu + 1) + A(\kappa,\ \nu-1)] = (A'' + A'/\kappa + 2A)/2 =$$

$$= \int_0^{2\pi} d\phi \quad e^{i\nu\phi} \cos\phi\ J_o[2\kappa \sin(\phi/2)] \qquad (7.57)$$

Using the explicit expressions (7B.15) for the integrals $\langle\sin\phi\rangle$, $\langle\cos\phi\rangle$, etc., we obtain from (7.51) the result

$$(\nu/\kappa)^2 C \qquad\qquad (i\nu/2\kappa)A'$$

$$\langle S_\perp\rangle = \frac{\partial f_o}{\partial v_\perp} + \frac{k_z G}{\omega} \quad v_\perp$$

$$-(i\nu/2\kappa)A' \qquad\qquad D - (\nu/\kappa)^2 C$$

$$\langle S_{.z}\rangle = \frac{\partial f_o}{\partial v_z} - \frac{\nu k_\perp G}{\kappa\omega} \quad v_\perp\ (\nu C/\kappa,\ iA'/2)$$

$$\langle S_{z.}\rangle = \frac{\partial f_o}{\partial v_\perp} + \frac{k_z G}{\omega} \quad v_z\ \frac{\nu C}{\kappa},\ \frac{iA}{2}$$

$$\langle S_{zz}\rangle = A\,\frac{\partial f_o}{\partial v_z} - \frac{\nu C k_\perp G}{\kappa\omega} \quad v_z$$

The final simplication arises from the fact that in the first three equations of (7.58) the factors in square brackets on the right side are all the same. In fact, from the definition (7.42) of G it follows that

$$R \equiv \frac{\partial f_o}{\partial v_z} - \frac{\nu k_\perp G}{\kappa\omega} \quad v_\perp = \frac{\partial f_o}{\partial v_\perp} + \frac{k_z G}{\omega} \quad v_z = v_\perp \frac{\partial f_o}{\partial v_z} - \frac{\tilde{\omega}}{\omega} \qquad (7.59)$$

Thus we have finally

$$S_\perp = R(v_\perp/v_z) \begin{matrix} (\nu/\kappa)^2 C & (i\nu/2\kappa)A' \\ -i\nu/2\kappa A' & D - (\nu/\kappa)^2 C \end{matrix}$$

$$<S_{.z}> = R(\nu C/\kappa,\ -\ iA'/2)$$

$$<S_{z.}> = R(\nu C/\kappa,\ iA'/2) \tag{7.60}$$

$$<S_{zz}> = RAv_z/v_\perp - i(v_z/v_\perp)\ (\Omega/\omega)G/Q$$

In this form, the useful symmetries

$$S_{xy} = -S_{yx} \qquad S_{yz} = -S_{zy} \qquad S_{xz} = S_{zx} \tag{7.61}$$

are apparent. These reduce the number of different elements in $\underset{\sim}{\sigma}$ to 6.

Using the expressions (7.60) for $<\underset{\sim}{S}>$, we need only multiply by $Q(\nu)$ and integrate over v_z and $v_\perp$ to find $\underset{\sim}{\sigma}$. Since $<\underset{\sim}{S}>$ is independent of ϕ, we can include that integration, when convenient, and simply write (7.49) as

$$\underset{\sim}{\sigma} \ = \ \int d\underline{v}\ (\omega_p^{\ 2}/4\pi\Omega)\ Q(\nu)\ <\underset{\sim}{S}> \tag{7.62}$$

with $<\underset{\sim}{S}>$ given by (7.60).

In (7.60) and (7.62) we have the <u>most general expressions possible</u> for $\underset{\sim}{\sigma}$ without specifying f_o. From $\underset{\sim}{\sigma}$ we can compute $\underset{\sim}{\varepsilon} = \underset{\sim}{1} + 4\pi i\underset{\sim}{\sigma}/\omega$ and use this in the associated dispersion equation, (7.9). In general, this leads to complicated algebra but for the special case of electrostatic waves ($\underline{k} \times \underline{E} = 0$, $\underline{E} = \hat{\underline{k}}E$) the procedure can be simplified. The same techniques used here to compute $\underline{j}$ can be used to find the contribution

$$\rho_\alpha = \int d\underline{v} n_\alpha q_\alpha f_\alpha(v) = -(ik/4\pi)\chi_\alpha E$$

which a particular species makes to the charge density. We obtain (dropping the species index),

$$\chi = (i\omega_p^2/k^2\Omega)\int d\underline{v}\, Q(\nu) \quad [k_z A\partial f_o/\partial v_z + \\ + k_\perp(\nu/\kappa)C\, \partial f_o/\partial v_\perp] \tag{7.63}$$

the associated dispersion relation being, as usual,

$$\varepsilon_\ell = 1 + \sum \chi_\alpha = 0 \tag{7.64}$$

This is the same result as we would obtain from (7.9), since,

$$\varepsilon_\ell = \hat{\underline{k}} \cdot \underset{\sim}{\varepsilon} \cdot \hat{\underline{k}}$$

Of course, we cannot legislate in advance the polarization of the waves. For given $\underline{k}$, ω and $\underline{J}$, both the direction and magnitude if $\underline{E}$ are determined by (7.8) and in general we will find that $\underline{E} \times \underline{k} \neq 0$. However, as we shall seem there are interesting cases where $\underline{E} \times \underline{k}$ is small, if not zero, and certainly any $\underline{k}$, ω which leads to $\underline{E} \times \underline{k} \doteq 0$ must satisfy $\varepsilon_\ell\ (\underline{k},\omega) = 0$.

d. Explicit Expressions for Specific Choices of f_o

It now remains to calculate $\underset{\sim}{\sigma}$ for given f_o. A common, fairly general choice for f_o is the bi-Maxwellian with streaming along the magnetic field,

$$f_o(\underline{v}) = \exp\, (-v_z - V)^2/a_{\parallel}^2 - v_\perp^2/a_\perp^2]/\pi^{3/2}a_{\parallel}a_\perp^2 \tag{7.65}$$

This allows different "temperatures" parallel and perpendicular to $\underline{B}_o$. A situation which can arise when either the parallel or perpendicular degree of freedom is heated (or cooled) at a rate faster than the collisional energy equilibrium rate between the two degrees of freedom. Using the expansion (7.54) in (7.62) leads to a sum over n of integrals like

$$\int d\underline{v}\; f_o(\underline{v})\; J_n^{\;2}(\kappa)/(\nu-n) =$$

$$= \frac{2\Omega}{\pi^{1/2} a_{\shortparallel} a_\perp^{\;2}} \int_{\infty}^{\infty} dv_{\shortparallel}\, \frac{e^{-(v_{\shortparallel}-V)^2/a_{\shortparallel}^{\;2}}}{(\omega-n\Omega)} \int^{\infty} dv_\perp v_\perp e^{-v_\perp^{\;2}/a_\perp^{\;2}}.\;\; J_n^{\;2}(k_\perp v_\perp/\;\;) =$$

$$= (\Omega/k_{\shortparallel}a_{\shortparallel})\; Z\; \left(\frac{\omega-k_{\shortparallel}V-n}{k_{\shortparallel}a_{\shortparallel}}\right)\; e^{-b^2}\, I_n(b^2) \qquad (7.66)$$

where

$$b^2 = (k_\perp a_\perp/\Omega)^2/2 \qquad (7.67)$$

and we have introduced the plasma dispersion function Z, (5.42), and used the identity (cf Appendix 7.1, Eq. (27).)

$$\int_0^{\infty} dt\; te^{-p^2t^2}\; J_n^{\;2}(at) = e^{-b^2}\; I_n(b^2)/2p^2 \qquad (7.68)$$

$$b^2 = a^2/2p^2 = (k_\perp/\Omega)^2/2a_\perp^{\;2}$$

The factor

$$\Lambda_n(b) \equiv e^{-b^2}\; I_n(b^2) \qquad (7.69)$$

gives rise to what are called finite cyclotron radius effects. This refers to the fact that for a vert strong magnetic field, when $r_c \to 0$ and the particles may be thought of as moving along magnetic field lines, $b \to 0$ so

$$\Lambda_o \to 0 \qquad\qquad \Lambda_o \to b^2/2 \qquad (7.70)$$

the other Λ_n vanishing as higher powers of b. If we keep only $n = 0, \pm 1$ terms in (7.62), then $a_\perp^{\;2}$ drops out and the only thermal effects which remain arise from thermal motion along $\underline{B}_o$, which enters via the Z functions for $n = 0 \pm 1$. In the limit $k_{\shortparallel}a_{\shortparallel} \to 0$, the Z functions with $n = \pm 1$ give the cold plasma cyclotron resonances, since for $\omega \mp \Omega \neq 0$

$$\lim_{k_{\shortparallel}a_{\shortparallel} \to 0} \left[\frac{\Omega}{k_{\shortparallel}a_{\shortparallel}} \; Z\left(\frac{\omega \mp \Omega}{k_{\shortparallel}a_{\shortparallel}}\right)\right] = \frac{-\Omega}{\omega \mp \Omega} \tag{7.71}$$

However, it is clear that this corresponds to a singular case, either propagation precisely perpendicular to $\underline{B}_o$ or zero parallel thermal velocity. In practice, $k_{\shortparallel}a_{\shortparallel}$ will always be finite, so that right at the "resonance," $\omega \mp \Omega = 0$, we will actually have, in place of the singular limit (7.71), the finite result

$$\lim_{\omega \mp \Omega \to 0} \left[Z\left(\frac{\omega \mp \Omega}{k_{\shortparallel}a_{\shortparallel}}\right)\right] = i\pi^{1/2} \tag{7.72}$$

Effects associated with non-vanishing b or "finite r_c" come in through the terms of order b^2 in $(\Lambda_o - 1)$, the terms of order b^4 in $(\Lambda_1 - b^2/2)$, and the terms of order b^{2n} in Λ_n, $n>1$.

In Table 1 we give an explicit representation of $\underline{\varepsilon}$ for the fairly general case of an anisotropic, streaming Maxwellian and in Table 2 a special case of this, namely an isotropic Maxwellian.

Before proceeding to apply our expressions for $\underset{\sim}{\varepsilon}$ to the study of linearized waves in a magnetized plasma, we add some remarks concerning our mathematical treatment of the Doppler-shifted cyclotron harmonic resonances at $\omega - k_{\shortparallel}v_{\shortparallel} = n\Omega$.

1) The factor $Q(\nu) = \exp(2\pi i\nu)-1]^{-1}$ in (7.42) and subsequent equations is singular at each of the resonances, i.e., at $\nu = \omega/\Omega = n$. In fact, we have $Q(\nu) = (2\pi i)^{-1} \sum_n (\nu - n)^{-1}$.

2) $A(\kappa,\nu)$, as given by (7.53) or (7.55) is finite and well behaved for all ν and κ. It is, as we see from (7.54), only the quality AQ which has singularities, at integer values of ν.

3) In calculating $\underset{\sim}{\sigma}$ or χ, we have a choice of using an expression like (7.53) or (7.55) for A, in which case all the cyclotron resonances

are in Q, or of using an expression like (7.54)

$$AQ = i\sum J_n^{\,2}(\nu - n)^{-1}$$

in which case $\underset{\sim}{g}$ or χ is written as an explicit sum of resonant terms. In general, the latter is more convenient for calculations in which a single resonance dominates the behavior, whereas the former provides a way of including all of the resonances if ω is near any single one.

B. Solutions of the Dispersion Relation

The simplest solutions of (7.9) and (7.10) are those which correspond to propagation parallel to the magnetic field ($\underline{k} \parallel B_o$, $\theta = 0$) or perpendicular to the field ($\underline{k} \perp \underline{B}_o$, $\theta = \pi/2$), so we discuss those first. In both cases, we shall begin with the cold, two-fluid version of $\underset{\sim}{\varepsilon}$, (7.17). It is then easy to solve (7.10) or (7.24) for $N = kc/\omega$ as a function of ω for given θ, as shown in (7.26). Of particular interest are the values of ω for which N is singular (conventionally called "resonances", since they occur when the wave frequency coincides with a natural frequency of some single particle motion, such as Ω) and the values for which N vanishes (conventionally called "cut-offs", since they typically represent a boundary between a frequency range corresponding to propagation of a wave and one corresponding to evanescense). We shall then consider the modifications resulting from the kinetic equation (Vlasov) expression for $\underset{\sim}{\varepsilon}$.

For general values of θ, the analysis becomes rather tedious and we shall only give some particular results.

1. Propagation Along the Magnetic Field

When $\underline{k}$ is parallel to $\underline{B}_o$, $\theta = 0$, the circular polarization representation for $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$ is advantageous, since both $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{M}$ are diagonal.

a) Cold Plasma Approximation

$$\underset{\sim}{M} = \underset{\sim}{\varepsilon} - N^2 \underset{\sim}{1} + \underline{N}\,\underline{N} = \begin{pmatrix} L - N^2 & & \\ & R - N^2 & \\ & & P \end{pmatrix} \tag{7.73}$$

so the dispersion equation has roots

$$N^2 = L \qquad N^2 = R \qquad P = 1 - \omega_p{}^2/\omega^2 = 0 \tag{7.74}$$

corresponding, respectively, to left-hand polarization ($E_+ \neq 0$; $E_- = E_z = 0$); right hand polarization ($E_- \neq 0$; $E_+ = E_z = 0$); and longitudinal polarization, $\underline{E} = E\,\hat{\underline{k}}$ The third of these represents nothing new: when $\underline{E}$ is parallel to $\underline{B}_o$, the particle motion, being likewise parallel to $\underline{B}_o$, is unaffected by the magnetic field, so we have just the simple electrostatic dispersion relation, $\omega = \omega_p$, which we encountered in Chaps II and III for an unmagnetized, cold plasma.

The other two roots represent transverse waves and have, clearly, resonances at $\omega = \omega_{ci}$ for $N^2 = L$ and at $\omega = \omega_{ce}$ for $N^2 = R$. (It suffices to consider only $\omega > 0$; negative ω solutions provide no additional information.) These resonances are, of course, associated with the ion and electron cyclotron resonances which occur when the sense of the circular wave polarization coincides with that of the particle gyration. There are also two cut-offs ($N^2 = 0$), since

$$L(\omega) = R(-\omega) = 1 - \frac{\omega_p{}^2}{(\omega-\omega_{ci})\,(\omega+\omega_{ce})} = 0$$

gives

$$\omega^2 + (\omega_{ce}-\omega_{ci})\omega - \omega_p{}^2 - \omega_{ce}\omega_{ci} = 0 \tag{7.75}$$

whose solutions are $\omega = \omega_L$ and $\omega = -\omega_R$,

$$\omega_{\binom{L}{R}} = (1/2)\{[(\omega_{ce} + \omega_{ci})^2 + 4\omega_p^{\ 2}]^{1/2} \mp (\omega_{ce} - \omega_{ci})\} \tag{7.76}$$

Thus, for $\omega > 0$, we have a cut-off frequency for each polarization:

$$N^2 = L = 0 \quad \text{at} \quad \omega = \omega_L$$

$$N^2 = R = 0 \quad \text{at} \quad \omega = \omega_R$$

If we rationalize the denominator in (7.75), L becomes a quotient of quadratic functions of ω. We know where the zeros occur, so we can write

$$L(\omega) = R(-\omega) = \frac{(\omega - \omega_L)\ (\omega + \omega_R)}{(\omega - \omega_{ci})\ (\omega + \omega_{ce})} \tag{7.77}$$

Since

$$L(0) = R(0) = 1 + \frac{\omega_p^{\ 2}}{\omega_{ci}\omega_{ce}} = 1 + \frac{c^2}{C_A^{\ 2}} , \tag{7.78}$$

we see that in the low frequency limit, $\omega << \omega_{ci}$, we have $n^2 = (kc/\omega)^2 \to 1 + c^2/c_A^{\ 2}$, i.e, the phase velocity for both transverse polarizations approaches the Alfven speed, c_A, discussed in Chap. IV, provided $c_A << c$. (The 1 added to $(c/c_A)^2$ comes from the displacement current, which is usually negligible in the MHD limit. For atomic weight A, $c_A/c < 1$ means $B < (nA)^{1/2}/7.4$ with B in gauss, n in cm^{-3}.) In the high frequency limit, $\omega >> \omega_{ce}$, both L and R approach 1, so we get just transverse electromagnetic waves, $\omega = kc$, unaffected by the plasma. In between these limits we have a resonance and a cut-off. Using these properties of N as a function of ω, it is easy to make a qualitatively correct sketch of N^2 vs ω provided we also know the location of any maxima or minima. Since

$$dL/d\omega = \omega_p^2 \frac{(2\omega + \omega_{ce} - \omega_{ci})}{(\omega - \omega_{ci})^2 (\omega + \omega_{ce})^2}$$

vanishes only at $\omega = (\omega_{ci} - \omega_{ce})/2$, we see that for $\omega > 0$ the $N^2 = L$ branch has no extrema, while the $N^2 = R$ branch has one, namely a minimum at

$$\omega = \frac{\omega_{ce} - \omega_{ci}}{2} \doteq \omega_{ce}/2$$

where

$$N^2 = 1 + 4\omega_p^2/\omega_{ce}^2$$

To summarize:

1) For $\omega \to 0$, $N^2 \to 1 + c^2/c_A^2$ for both L and R modes.
2) For $\omega \to \infty$, $N^2 \to 1$ for both L and R modes.
3) The L(R) mode has a cut-off at $\omega = \omega_L$ $(\omega = \omega_R)$ and a resonance at $\omega = \omega_{ci}$ $(\omega = \omega_{ce})$.
4) $dN^2/d\omega > 0$ for the L mode, $0 \leqslant \omega \leqslant \infty$, while $dN^2/d\omega = 0$ at $\omega = \omega_{ce}/2$ for the R mode.

These facts suffice to define the principal topological features of the dispersion diagram for each mode, as shown in Fig. 7.2

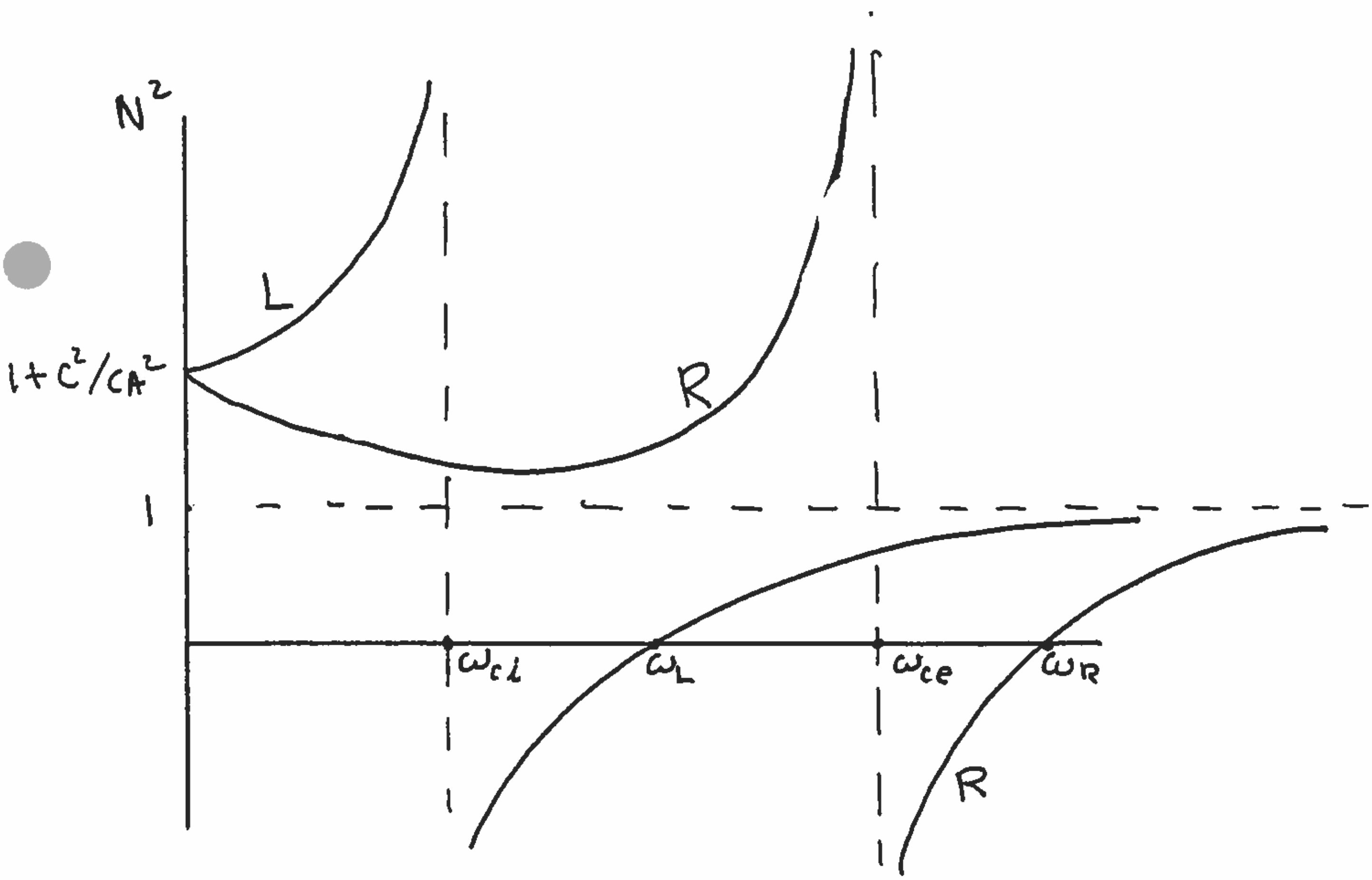

Fig. 7.2 N^2 vs ω dispersion diagram for propagation of transverse waves along the magnetic field.

The regions of ω where $N^2 < 0$ correspond to evanescent waves. Thus, there are no propagating, left-hand (right-hand) polarized waves for $\omega_{ci} < \omega < \omega_L$ ($\omega_{ce} < \omega < \omega_R$).

The values of the cut-off frequency relative to the cyclotron and plasma frequencies depends on the density and magnetic field. This is illustrated in Fig. 7.3 where ω_L and ω_R are plotted as a function of ω_{pe}/ω_{ce}. (The hybrid frequencies ω_{LH} and ω_{UH} also shown in Fig. 7.3 will be introduced in Section 2 below). Convenient expressions for ω_{pe}/ω_{ce} and for other useful dimensionless ratios are as follows:

$$\omega_{pe}/\omega_{ce} = n^{1/2}/312B_o$$

$$\omega_{pe}/(\omega_{ce}\ \omega_{ce})^{1/2} = \omega_{pi}/\omega_{ci} = c/cA = (\omega_{pe}/\omega_{ce})(AM/m)^{1/2} = \qquad (7.79)$$

$$= (nA)^{1/2}/7.4B$$

$$\omega_{pe}/\omega_{ci} = (\omega_{pe}/\omega_{ce})(AM/m) = 5.8An^{1/2}B_o$$

where A is the atomic weight of the ionic species M/m = 1837, n is in cm^{-3} and B_o is in gauss. Setting each of these ratios equal to 1 gives curves in an (n, B_o) parameter space which divide that space into high density, medium density, low density and very low density regimes, as shown in Fig. 7.4. No matter what the value of ω_{ce}/ω_{pe}, we always have

$$\omega_L > \omega_{ci} \qquad\qquad \omega_R > \omega_{ce}$$

but ω_L may be greater or less than ω_{ce}, as shown in Fig. 7.3.

From Fig. 7.2, we can immediately dedeuce the topology of a k^2 or k vs ω dispersion diagram: where $N^2 \to$ constant (e.g., for $\omega \to 0, \infty$) in Fig. 7.2 we will have $k \propto \omega$, while cut-offs and resonances remain unchanged.

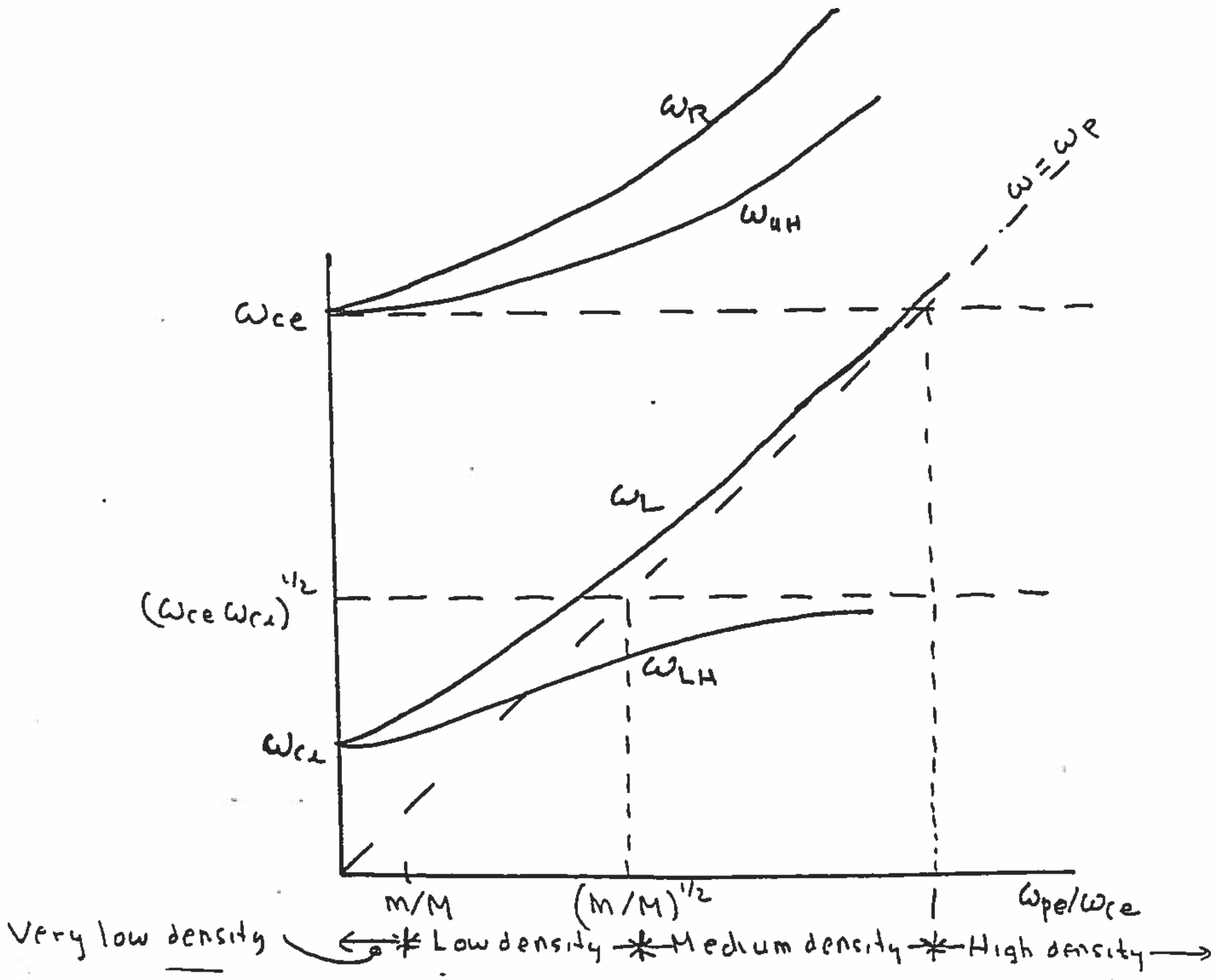

Fig. 7.3 The cut-off and resonant frequencies for a cold hydrogen plasma as a function of ω_{pe}/ω_{ce}.

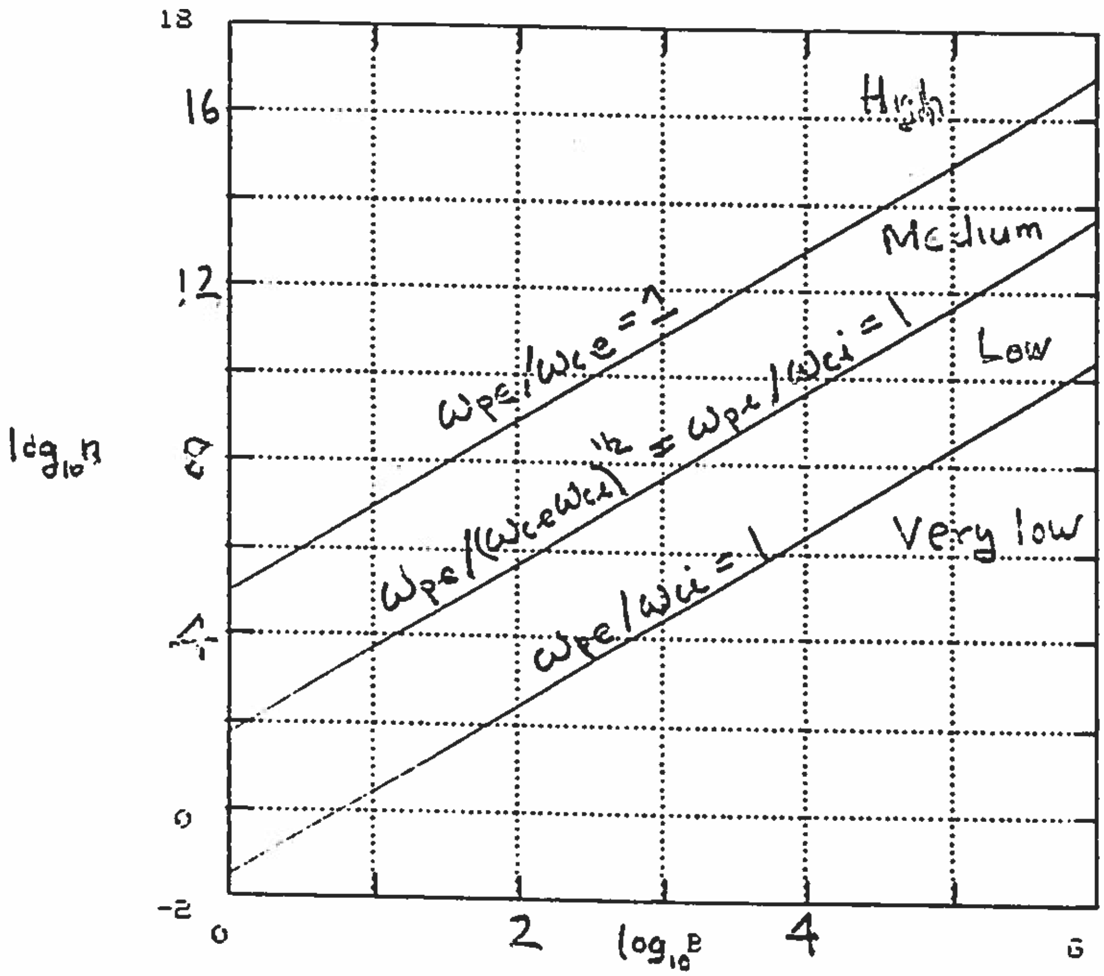

Fig. 7.4 Characterization of the (n^-, B_o) parameter space in terms of the commonly occuring dimensionless frequency ratios for hydrogen, A = 1. For other values of the weight, A, the ordinate should be labelled $\log_{10}(nA)$ for the middle curve and $\log_{10}(nA^2)$ for the lowest curve.

If, finally we rotate the k vs ω diagram by 90° we obtain ω vs κ dispersion diagram shown in Fig. 7.5, where

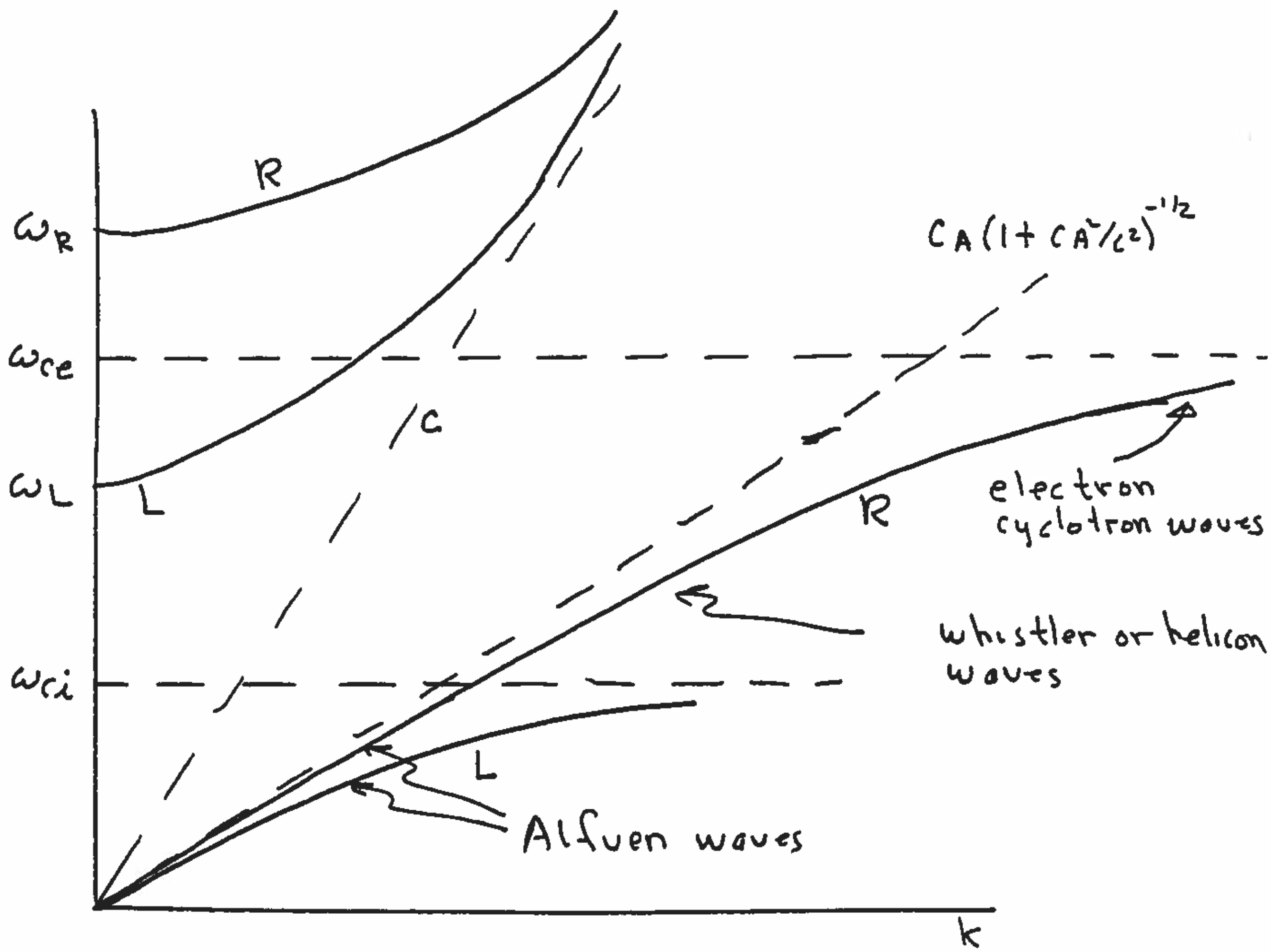

Fig. 7.5. ω vs k dispersion diagram for transverse waves propagating along the magnetic field.

In Fig. 7.5., we have identified the various portions of the modes with the names commonly applied to them. (In both Fig. 7.1 and 7.5, a straight line through the point $\omega = \omega_p$, parallel to the k or N^2 axis, could be added to represent the longitudinal wave.)

As we see from Fig. 7.5, at low frequencies ($\omega \ll \omega_{ci}$) we recover the Alfven waves which we derived from one fluid MHD in Chap. IV. In that approximation there is a degeneracy between the two possible polarizations in the plane perpendicular to $\underline{B}_o$, but in the two fluid picture, where the cyclotron resonances are taken into account, this degeneracy is resolved and we see that the normal modes are just the circularly polarized waves and that these have different phase velocities when $k \neq 0$. Since the direction of ion gyration about the magnetic field has the same sense as that of the electric field in the left-hand polarized wave, the Alfven wave having that polarization has a resonance ($k \to \infty$) as ω approaches ω_{ci}. The right-hand polarized wave, on the other hand, shows no unusual behavior at ω_{ci} but has, of course, a resonance at ω_{ce}, since its electric field rotates in the same sense as the gyration of electrons.

The term "whistler" is applied to the right-hand polarized branch because such waves have long been observed in the ionosphere, where they frequently occur at frequencies in the audio range and sound like a whistle of descending pitch. They are thought to originate in lightning strokes or other inospheric excitations near one end of a $\underline{B}$ line of the earth's magnetic field and to propagate along the field (guided, probably, by ducts of depressed density) to the other end of the field line. Since $k = N\omega/c$, we have for the group velocity

$$v_g = d\omega/dk = c(N'\omega + N)^{-1}$$

In the frequency range $0 < \omega < \omega_{ce}/2$, we see from Fig. 5.2 that N is a smoothly decreasing function of ω, with $N' < 0$. In this region both N and $N'\omega$ will be decreasing functions of ω, so v_g will increase with ω. For a pulsed excitation, like a lightning discharge, which contains a broad spectrum of frequencies, the higher frequencies in the range $0 < \omega < \omega_{ce}/2$ will arrive first, giving rise to a sound whose pitch drops with time. Of course, observations in the region above $\omega_{ce}/2$ will show the opposite effect. There, v_g is a decreasing function of ω and frequencies above $\omega_{ce}/2$. Thus, strictly speaking, if we think of a "whistler" as a signal with a pitch which descends with time, then the term would properly apply only to the low frequency region, $\omega < \omega_{ce}/2$. In fact, it is generally used for the whole branch, save for the portion very near ω_{ce}, where, as we shall see, wave-particle resonance effects become important.

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ach

$f_o/\partial \underline{v}$

1

which are of interest are those which give an f_1, and hence $\underline{j}$, proportional to $\underline{E}$ and thus lead to an expression for $\underset{\sim}{\sigma}$. Thus, it is a particular solution of (7.80), as it is called in the theory of differential equations, which we need, and this is easily found by substituting the ansatz

$$f_1 = \sum_\tau h_\tau e^{-i\tau\phi} \tag{7.83}$$

into (7.80) where the summation is over $\tau = \pm 1, 0$. This gives

$$h_\tau = -ig_\tau(\tilde{\omega} - \tau\Omega)^{-1}$$

and hence the current density has the form

$$\underline{j} = \int d\underline{v}\ nq\underline{v}h\tau e^{i\tau\phi}$$

Since (7.82) and (7.83) show that $h_\tau \propto E_\tau$ we have

$$j_z = \int d\underline{v}\ nqv_z h_o = \sigma_z E_z$$

$$J_\pm = 2^{-1/2} \int d\underline{v}\ nqv_\perp\ e^{\pm i\phi} \sum h_\tau e^{-i\tau\phi} =$$

$$= 2^{-1/2} \int d\underline{v}\ nqv_\perp h_\perp = \sigma_\pm\ E_\pm$$

Thus, $\underset{\sim}{\sigma}$ is diagonal in circular coordinates, just as in the cold fluid approximation. Specifically, we find

$$\sigma_\pm = \int d\underline{v}(\omega_p{}^2/8\pi)v_\perp\ [(\tilde{\omega}/\omega)\ \partial f_o/\partial v_\perp +$$

$$+ (k_\perp v_\perp/\omega)\ \partial f_o/\partial v_z](\tilde{\omega} \mp \Omega)^{-1}$$

$$\sigma_z = \int d\underline{v}\ (\omega_p{}^2/4\pi\tilde{\omega})\ v_z\ \partial f_o/\partial v_z$$

Of course, $\underset{\sim}{\varepsilon} = 1 + 4\pi i\underset{\sim}{\sigma}/\omega$ is then also diagonal

$$\underset{\approx}{\varepsilon}_c = 1 + \frac{4\pi i}{\omega}\underset{\approx}{\sigma}_c = \begin{pmatrix} \varepsilon_+ & & \\ & \varepsilon_- & \\ & & \varepsilon_z \end{pmatrix} \tag{7.84}$$

where

$$\varepsilon_\pm = 1 + \int d\underline{v}\ (\omega_p{}^2/2\omega^2)\ [\tilde{\omega}\ \partial f_o/\partial v_\perp + (kv_\perp\ \partial f_o/\partial v_z]v_\perp(\tilde{\omega} \mp \Omega)^{-1} \tag{7.85}$$

$$\varepsilon_z = 1 + \int dv\ (\omega_p{}^2/\omega\tilde{\omega})\ v_z \partial f_o/\partial v_z \tag{7.86}$$

are just the Vlasov generalizations of L, R and P. The matrix $\underset{\approx}{M}$ which gives the dispersion relation is also diagonal:

$$M = \det \underset{\approx}{M} = \det (\underset{\approx}{\varepsilon} - N^2\underset{\approx}{1} + \underline{N}\underline{N}) =$$

$$= \det \begin{pmatrix} \varepsilon_+ - N^2 & 0 & 0 \\ 0 & \varepsilon_- - N^2 & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} = (N^2 - \varepsilon_+)\ (N^2 - \varepsilon_-)\ \varepsilon_z \tag{7.87}$$

so we immediately obtain three modes, just as for the cold plasma case:

$N^2 = \varepsilon_+$	transverse wave with left-hand polarization ($E_+ \neq 0$)	
$N^2 = \varepsilon_-$	transverse wave with right-hand polarization ($E_- \neq 0$)	(7.88)
$\varepsilon_z = 0$	longitudinal wave ($\underline{E} = E\hat{\underline{k}}$).	

The expression for ε_z is just the same as for longitudinal waves in an unmagnetized plasma:

$$\begin{aligned}\varepsilon_z &= 1 + \int d\underline{v}(\omega_p{}^2/\omega)[(\omega - \tilde{\omega})/k\tilde{\omega}]\partial f_o/\partial v_z = \\ &= 1 - \int dv_z(\omega_p{}^2/k^2)(\partial f_o/\partial v_z)(v_z-\omega/k)^{-1}\end{aligned} \tag{7.89}$$

This result is clear on physical ground: the mode corresponding to $\varepsilon_z = 0$ has longitudinal polarization $\underline{k} \times \underline{E} = 0$ so there will be no first order magnetic field. Since the changes in velocity due to $\underline{E}$ are parallel to $\underline{B}_o$, the external magnetic field can have no effect.

In the low temperature limit, the expressions (7.85) reduce to the cold plasma results, since

$$\varepsilon_+ \to L \qquad\qquad \varepsilon_- \to R \qquad\qquad \varepsilon_z \to P$$

when $f_o(\underline{v}) \to \delta(\underline{v})$. However if there is any thermal motion along the magnetic field, the resonance at $\omega = \Omega$ is broadened by Doppler shifting and in place of singular terms like the $(\omega \mp \Omega)^{-1}$ in L or R we have terms like $Z[(\omega \mp \Omega)/ka]$ which are, of course, perfectly finite at $\omega = \pm\Omega$. The inclusion of thermal or kinetic effects has some other important physical consequences:

1) The various waves discussed in the cold two fluid approximation -- Alfven waves, whistlers, etc. -- which have a component of E perpendicular to $\underline{B}_o$ will be damped due to wave-particle resonance. However, in place of the usual condition that the wave phase velocity match the particle speed, $\omega_r/k \simeq v$, where ω_r is the real part of the frequency, we have instead the condition (associated with the vanishing of the real part of the denominator in ε_+ or ε_- as given by (7.85))

$$\omega_r - kv_{\parallel} \mp \Omega \approx 0$$

This has two simple physical interpretations:

a) A particle moving with speed $v_{\parallel}$ along the wave experiences a Doppler-shifted frequency $\omega_r - kv_{\parallel}$ and resonance occurs when this coincides with the cyclotron frequency, the velocity v_+ (for left-hand polarized waves) or v_- (for right hand polarization) then showing secular behavior, i.e., increasing linearly with time.

b) For a particle gyrating about the field with frequency Ω, a circularly polarized wave with frequency ω appears to have a frequency $\omega \mp \Omega$. Thus, the phase velocity of the wave, as experienced by the particle, is $(\omega \mp \Omega)/k$ and there is resonance (i.e., secular time behavior) when this matches the parallel particle velocity, $v_{\parallel}$.

2) Relative streaming of two species, or more generally, the existence of a region in which $\partial f_o/\partial v_z > 0$, can cause one of the waves to become unstable, in close analogy with the two stream instabilities or inverse Landau damping encountered in unmagnetized plasmas.

3) Non-isotropic f_o, with different distributions for $v_{\parallel}$ and $v_{\perp}$, can cause instabilities. This can also occur for unmagnetized plasmas (the so-called Weibel instability) although without a magnetic field we must rely on some other phenomenon to define a preferred direction in velocity space.

We now give examples of these effects. For a bi-Maxwellian,

$$f_o(\underline{v}) = \exp[-v_\perp^2/a_\perp^2 - v_z^2/a_z^2]/\pi^{3/2}a_\perp^2 a_z \tag{7.90}$$

(7.85) gives

$$\begin{aligned}\varepsilon_{\pm} &= 1 - \int d\underline{v}\, \frac{\omega_p^2 v_\perp^2}{\omega^2}\, \frac{(\omega/a_\perp^2 + kv_z/a_z^2)f_o}{\bar{\omega} \mp \Omega} = \\ &= 1 - \int d\underline{v}\, (\omega_p/\omega)^2 (v_\perp/a_\perp)^2 (\omega - \Omega\, kv_z) f_o (\bar{\omega} \mp \Omega)^{-1} \\ &= 1 + \sum (\omega_p^2/\omega^2)[(\omega/ka_z)Z + (\alpha/a)Z']\end{aligned} \tag{7.91}$$

where the sum is over species,

$$\alpha = 1 - a_\perp^2/a_z^2 = (T_\parallel - T_\perp)/T_\parallel \tag{7.92}$$

is a measure of the anistopy, and the argument of the Z and Z' functions is $(\omega \mp \Omega)/ka_z$. Even for real k, ω, we know that Z and Z' are complex so we expect to find damped solutions, $\omega = \omega_r + i\gamma$ with $\gamma < 0$.

The damping will be weak if

$$|(\omega \mp \Omega)/ka_z| >> 1 \tag{7.93}$$

since there will then be very few particles which can resonate with the wave (namely, those in the tail of the Maxwellian, $v_z >> a_z$.) It follows that we can use the asymptotic forms

$$Z(s) = s^{-1} + i\pi^{1/2}e^{-s^2}, \qquad Z'(s) = -s^2 - 2i\pi^{1/2}se^{-s^2} \tag{7.94}$$

so that when (7.93) is satisfied, the dispersion relation (7.88) for the transverse waves becomes

$$N^2 = (kc/\omega)^2\ \varepsilon_{\pm} = 1 + \sum \{-\omega_p{}^2/\omega(\omega \mp \Omega) + \alpha(\omega_p ka_z)^2/2\omega^2(\omega \mp \Omega)^2$$

$$+ i\pi^{1/2}[\omega_p{}^2/ka_z\omega - \alpha\omega_p{}^2(\omega \mp \Omega)/\omega^2 ka_z]\cdot\exp[-(\omega \mp \Omega)^2/k^2a_z{}^2]\} \qquad (7.95)$$

$$= L(\pm\omega) + \sum \{(\alpha/2)(\omega_p ka_{\shortparallel}/\omega)^2(\omega \mp \Omega)^{-2} +$$

$$+ i\pi^{1/2}(\omega_p{}^2/ka_z\omega)[a_\perp{}^2/a_z{}^2 \pm \alpha\Omega/\omega]\exp[-(\omega \pm \Omega)^2/k^2az^2]\}$$

Consider first the isotropic case $a_z = a_\perp = a$, or $\alpha = 0$:

$$N^2 - [1 - \sum \omega_p{}^2/\omega(\omega \mp \Omega)] = i\pi^{1/2}\sum(\omega_p{}^2/ka\omega)\cdot\exp[-(\omega \mp \Omega)^2/k^2a^2] \qquad (7.96)$$

In the limit $a \to 0$ we of course recover the cold plasma results,

$$N^2 - L = 0 \qquad \text{or} \qquad N^2 - R = 0 \qquad (7.97)$$

So long as (7.93) is satisfied, the right side of (7.94) will be small and we can set

$$\omega = \omega_o + i\gamma,$$

where ω_o satisfies one of the equations (7.97) and $|\gamma| << \omega_o$. A perturbation solution for γ then gives

$$\gamma = -\pi^{1/2}\,\frac{\omega_p{}^2\ \exp[-(\omega_o \mp \Omega)^2/k^2{}_a{}^2]}{D(\omega_o)\ ka\omega_o} \qquad (7.98)$$

where for left hand polarization

$$D(\omega) = D_L(\omega) \equiv d[L(\omega) - N^2]/d\omega = L'(\omega) + 2N^2/\omega$$

and L is replaced by R for right hand polarization. An isotropic Maxwellian distribution function should be stable, there being no source of free energy, and indeed γ as given by (7.98) is never positive. This is clear for left hand polarization: $L(\omega)$ is a monotone increasing function, so $D_L(\omega) \geq 0$. For right hand polarization, $R(\omega)$ is a monotone increasing function only for $\omega > (\omega_{ce} - \omega_{ci})/2$, where R has a minimum, but D_R is nevertheless nonnegative (see problem 7.5). The collisionless damping, given by 7.96, which is similar to Landau damping, is generally termed cyclotron damping. Like Landau damping, it arises from wave particle resonance. If (7.93) is not satisfied, an appreciable fraction of particles can resonate with the wave, resulting in a strong damping which must be calculated numerically using the exact form, (7.91), for $\varepsilon_{\pm}$.

We now consider the anistropic case, $\alpha \neq 0$. All of the corrections to the cold plasma result ($N^2 = L$ and $N^2 = R$) given by (7.95) involve thermal effects, but the real term in α is present when $\alpha \neq 0$ even in the limit of very large $[(\omega \mp \Omega)/ka_z]$, when the imaginary terms, associated with wave particle resonance, are negligible, so that (7.96) reduces to

$$N^2 = 1 - \sum \omega_p^2/\omega(\omega \mp \Omega) + \sum \alpha(\omega_p k a_z)^2/2\omega^2(\omega \mp \Omega)^2 \tag{7.99}$$

In the low frequency limit, $\omega \ll \omega_{ci}$, this gives

$$N^2 = 1 + c^2/c_A^2 + 4\pi\,(p_\parallel - p_\perp)N^2/B$$

or

$$\omega/kc = \frac{1 - 4\pi(p_\parallel - p_\perp)/B^2}{1 + c^2/c_A^2} \tag{7.100}$$

where $p_\parallel = \sum nT_\parallel$, $p_\perp = \sum nT_\perp$ are the total pressures. If $p_\parallel$ exceeds $p_\perp$,

$$p_\parallel - p_\perp > B^2/4\pi \tag{7.101}$$

ω will be pure imaginary and there will be a purely growing perturbation (with zero real frequency). This mode, called the "firehose instability," can be obtained from the simple one-fluid MHD theory, as might be expected from the fact that no wave-particle resonance effects are involved and only the total fluid pressure components, $p_{\parallel}$ and $p_{\perp}$, appear in the final result.

2. Propagation Across the Magnetic Field

a) Cold Plasma Approximation for $\underset{\sim}{\varepsilon}$

We use the expression (7.22) for $\underset{\sim}{M}$ which reduces, in the case $\theta = \pi/2$, to

$$\underset{\sim}{M} = \begin{pmatrix} S & -iD & 0 \\ iD & S-N^2 & 0 \\ 0 & 0 & P-N^2 \end{pmatrix} \tag{7.110}$$

giving the dispersion relation

$$M = (P - N^2)\ (S^2 - D^2 - N^2 S) = 0 \tag{7.111}$$

We see that (7.111) has two roots. The first,

$$N^2 = P = 1 - \omega_p^{\,2}/\omega^2 \qquad E_z \neq 0 \tag{7.112}$$

describe a transverse wave with $\underline{E}$ parallel to $\underline{B}_o$. (Remember that "transverse" refers to the polarization, $\underline{k} \cdot \underline{E} = 0$, and is to be distinguished from the direction of propagation, which happens here to be perpendicular, $\underline{k} \cdot \underline{B}_o = 0$.) We can write (7.112) in the form

$$\omega^2 = \omega_p^{\,2} + (kc)^2 \tag{7.113}$$

which coincides with the result found in Chap. III for an unmagnetized plasma. This is to be expected, since the particle motions induced by $\underline{E}$ are parallel to B_o, and hence unaffected by it. Because it is unaltered by the magnetic field, this mode is called the ordinary mode.

The other solution of (7.111), called the extraordinary mode, is given by

$$N^2 = (S^2 - D^2)/S = RL/S \tag{7.114}$$

To determine its properties, we must examine the ω dependance of RL/S. From (7.77) we have

$$RL = (\omega^2 - \omega_R^2)\ (\omega^2 - \omega_L^2)/(\omega^2 - \omega_{ci}^2)\ (\omega^2 - \omega_{ce}^2) \tag{7.115}$$

and

$$S = (R + L)/2 = [\omega^4 - (\omega_p^2 + \omega_{ce}^2)\ \omega^2 +$$

$$+ \omega_{ce}\omega_{ci}\ (\omega_p^2 + \omega_{ce}\omega_{ci})]/(\omega^2 - \omega_{ce}^2)\ (\omega^2 - \omega_{ci}^2) \tag{7.116}$$

or

$$S = (\omega^2 - \omega_{UH}^2)\ (\omega^2 - \omega_{LH}^2)/(\omega^2 - \omega_{ci}^2)\ (\omega^2 - \omega_{ce}^2) \tag{7.117}$$

The roots of S = 0, known as the hybrid frequencies, are defined by

$$\omega_H^2 = 1/2\ \{\omega_p^2 + \omega_{ce}^2 \pm [(\omega_p^2 + \omega_{ce}^2)^2 - 4\omega_{ce}\omega_{ci}(\omega_p^2 + \omega_{ce}\omega_{ci})]^{1/2}\} \tag{7.118}$$

the upper sign giving ω_{UH}^2 and the lower sign, ω_{LH}^2. Since the second term in the radicand of (7.118) is always smaller than the first by a factor of m/M, regardless of the value of ω_p^2/ω_{ce}^2, we have

$$\omega_{UH}^2 = \omega_p^2 + \omega_{ce}^2 \tag{7.119}$$

$$\omega_{LH}^2 = \omega_{ce}\omega_{ci}(\omega_p^2 + \omega_{ce}\omega_{ci})/(\omega_p^2 + \omega_{ce}^2) =$$

$$= \omega_{ce}\omega_{ci}\,(\omega_{pi}^2 + \omega_{ci}^2)/(\omega_{pi}^2 + \omega_{ce}\omega_{ci}) =$$

$$= \omega_{ce}^2(\omega_{pi}^2 + \omega_{ci}^2)/(\omega_{pe}^2 + \omega_{ce}^2) \tag{7.120}$$

We have given several equivalent forms for ω_{LH}, some of which are more convenient than others for particular purposes. The last form may equally well be written

$$\omega_{LH}^2 = \omega_{ci}^2 + \omega_{pi}^2\,(1 + \omega_{pe}^2/\omega_{ce}^2)^{-1} \tag{7.121}$$

Although (7.121) appears to be different from (7.120) because the first term of (7.121) lacks a factor $(1 + \omega_{pe}^2/\omega_{ce}^2)^{-1}$, there is no actual discrepancy: whenever this factor is appreciably different from 1, the entire first term in (7.121) will be of order (m/M) compared to the second, since $\omega_{pi}^2/\omega_{ci}^2 = (M/m)\,(\omega_{pe}^2/\omega_{ce}^2)$.

Substituting (7.115) and (7.117) into (7.114) gives the extraordinary mode dispersion equation,

$$N^2 = (\omega^2 - \omega_L^2)\,(\omega^2 - \omega_R^2)/(\omega^2 - \omega_{LH}^2)(\omega^2 - \omega_{UH}^2) \tag{7.127}$$

We see immediately that there are resonances at the hybrid frequencies, ω_{LH} and ω_{uH}, and cut-offs at ω_L and ω_R. For $\omega \to \infty$, $N^2 \to 1$. For $\omega^2 = 0$, $N^2 = (\omega_L\omega_R/\omega_{LH}\omega_{uH})^2$. From (7.75) it follows that $\omega_L\omega_R = (\omega_{pe}{}^2 + \omega_{ce}\omega_{ci})$ while (7.116) gives $(\omega_{LH}\omega_{uH})^2 = (\omega_{ce}\omega_{ci})(\omega_p{}^2 + \omega_{ce}\omega_{ci})$. Thus, at $\omega = 0$

$$N^2 = 1 + \omega_{pe}{}^2/\omega_{ce}\omega_{ci} = 1 + c^2/c_A{}^2$$

and we recover the usual Alfven wave. (Recall from Chap. IV that there are two waves with phase velocity c_A for propagation along the field and one for propagation perpendicular to the field.)

Finally, we need to know the relative magnitudes of ω_L, ω_R, ω_{UH} and ω_{LH}. It is clear from (7.76), (7.114) and (7.120) that for $n \to 0$,

$$\omega_L,\ \omega_{LH} \to \omega_{ci};\ \omega_R,\ \omega_{UH} \to \omega_{ce}$$

while at high densities $(\omega_{pe}{}^2/\omega_{ce}{}^2 >> 1)$

$$\omega_L,\ \omega_R,\ \omega_{UH} \to \omega_p, \qquad \omega_{LH} \to (\omega_{ce}\omega_{ci})^{1/2}$$

(For moderate densities, $m/M < \omega_{pe}{}^2/\omega_{ce}{}^2 < 1$, we have $\omega_{LH} \simeq \omega_{pi}$.) At all densities, it can be shown that

$$\omega_{LH} \leqslant \omega_L \leqslant \omega_{UH} \leqslant \omega_R \tag{7.123}$$

as illustrated in Fig. 7.3

It is now an easy matter to plot N^2 vs ω, leading to the result shown in Fig. 7.6, and from this to plot ω vs k, as shown in Fig. 7.7.

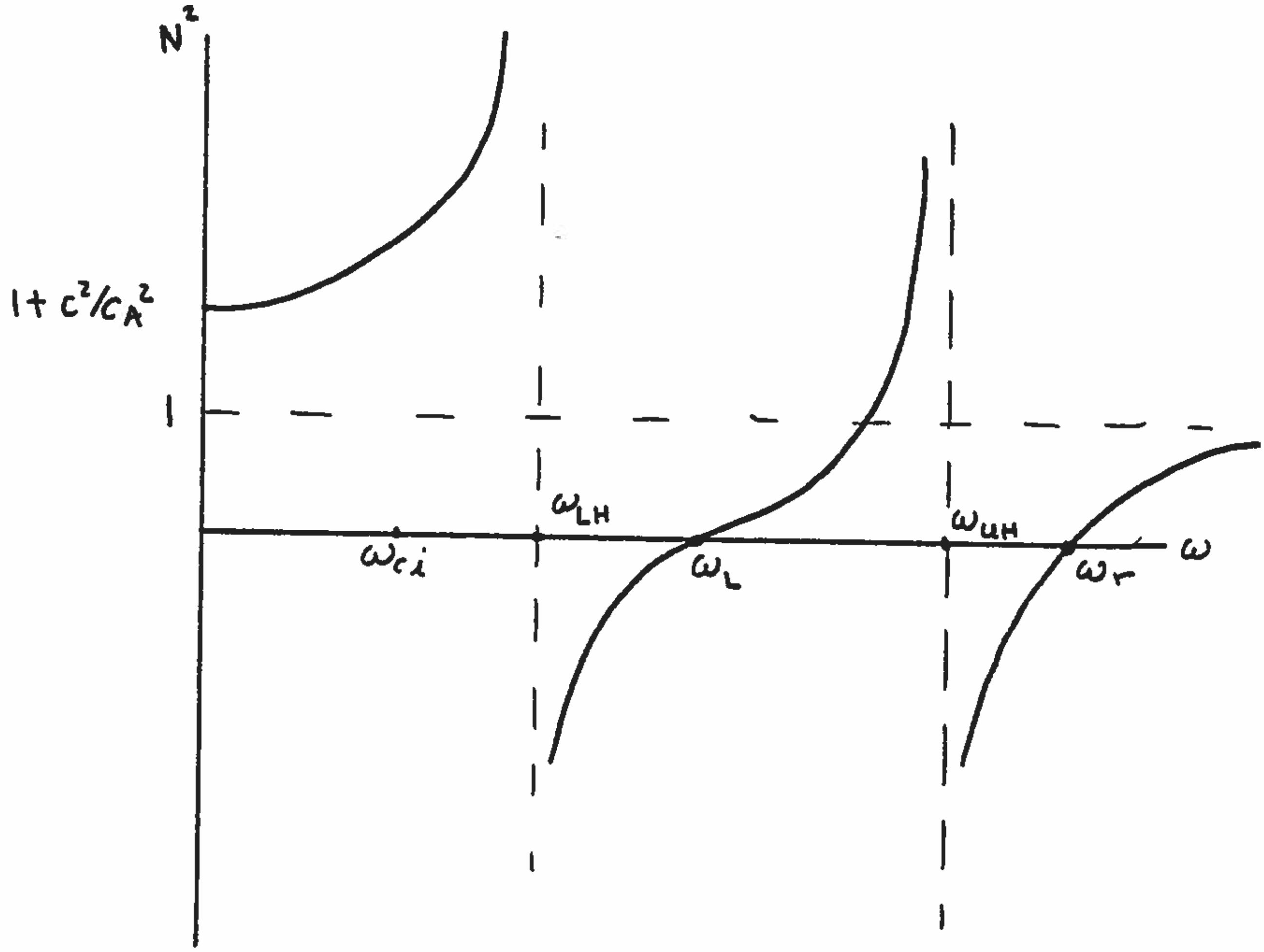


Fig. 7.6 N^2 vs ω dispersion diagram for extraordinary wave propagation across a magnetic field.

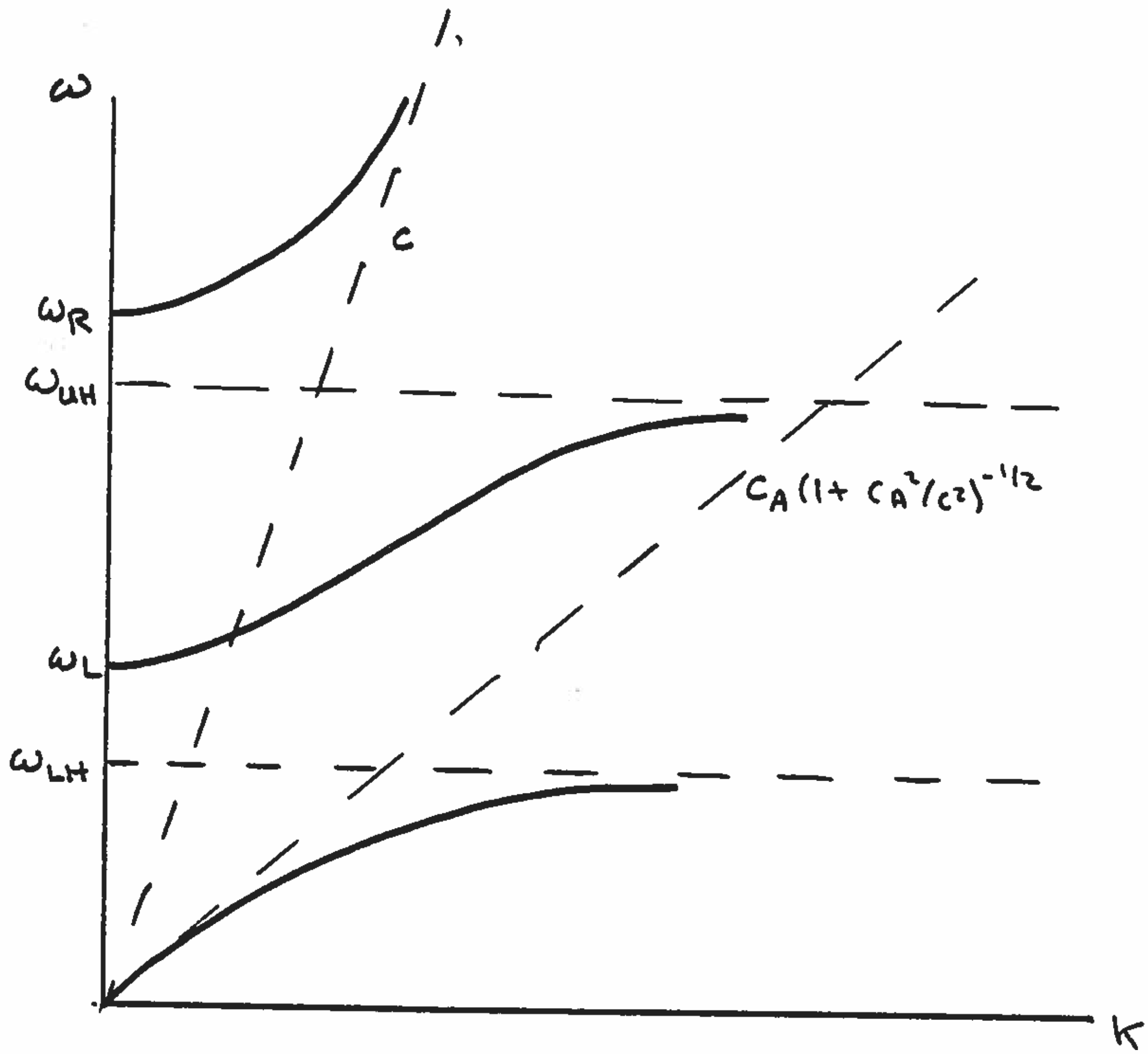


Fig. 7.7 ω vs k dispersion diagram for extraordinary wave propagation across a magnetic field.

There are several noteworthy features of Figs. 7.6 and 7.7. For long wavelengths and low frequency ($\omega << \omega_{ci}$) we recover, as expected, the Alfven wave which, according to the one fluid MHD analysis of Chap. IV, can propagate across the magnetic field. (Just as in the case of parallel propagation, the displacement current changes the phase velocity from c_A to $c_A(1 + c_A^2/c^2)^{-1/2}$.) It is at first sight surprising that, in contrast to the situation for parallel propagation, there are no resonances at either the electron or ion cyclotron frequencies: N shows no singularity at either ω_{ce} or ω_{ci}. It appears from Fig. 7.6 that these resonances are, so to speak, shifted upwards to the "hybrid" frequencies, ω_{UH} and ω_{LH}. Physical pictures for the absence of the cyclotron resonances and the appearance of the new hybrid resonances, based on elementary charge separation effects, are as follows.

Suppose an external electric field normal to $\underline{B}_o$,

$$\underline{E}_{ext} = \underline{E}_e \exp[i(kx - \omega t)] \qquad \underline{E}_e \cdot \underline{B}_o = 0 \tag{7.124}$$

is applied to the plasma, with ω being near the electron cyclotron frequency ω_{ce}. Since a linearly polarized wave like (7.124) can be decomposed into two circularly polarized components, we would expect the right hand component to resonate with the electrons. However, because of the x dependence of the field, the current $\underline{j}$ driven by $\underline{E}_{ext}$ will also vary with x, and since $\underline{j}$ will have both x and y components, there will be a non-vanishing $\nabla \cdot \underline{j}$ and hence an induced charge density. Since a plasma always tries to compensate for any charge separation, we expect that the applied field will be shielded in some manner. Exactly at resonance, $\omega = \omega_{ce}$, the induced charge would be infinite so there the shielding must be perfect. Just how this is accomplished depends on the polarization of the external field. If $N = kc/\omega >> 1$, so that inductive effects are small, we can give a very simple picture of the shielding. If $\underline{E}_e$ is in the x direction, then it tends to be cancelled by the electrostatic field in the same direction arising from the charge separation, the cancellation being exact at resonance. However, if $\underline{E}_e$ is in the y direction, the induced electrostatic field is, again, (necessarily!) in the x direction, equal in magnitude to E_e but this time shifted in phase by 90° so that the right hand circularly polarized component of the net field is exactly zero. Since it is only this component which can resonate with the gyrating electrons, there is no singularity and hence no resonance. If N is not large, then induced electric fields also play a small role, as explored in Problem 7.8, but the general picture remains the same.

A picture of the hybrid resonances can most easily be obtained by examining the single particle motion of electrons and ions in a self-consistent electric field perpendicular to $\underline{B}_o$. For a normal mode, meaning a solution of the homogeneous equations obtained by neglecting the right hand side of (7.8), we have, for $\theta = \pi/2$,

$$E_y/E_x = -M_{xx}/M_{xy} = S/iD = i(\omega^2 - \omega_{UH}^2)(\omega^2 - \omega_{LH}^2)/\omega_p^2\omega(\omega_{ce} - \omega_{ci}) \quad (7.125)$$

Therefore near either of the hybrid resonances $\underline{E}$ is nearly along $\underline{k} = k\hat{\underline{x}}$, i.e., to good approximation the field is electrostatic. If we look at single particle motions in a magnetic field and take into account the self-consistent electrostatic field, perpendicular to $\underline{B}_o$, arising from the charge density associated with the particle motion, then we find the natural oscillations of the particles are not at the cyclotron frequencies, as would be the case if we neglected the self-consistent electric field, but are, in fact, at one of the hybrid frequencies. Thus, we should expect to find resonances when a wave with $\underline{E}$ perpendicular to $\underline{B}_o$ has such a frequency.

The electron equations of motion are

$$\dot{v}_x = \frac{-e}{m} E_x - \omega_{ce} v_y \tag{7.126}$$

$$\dot{v}_y = \omega_{ce} v_x \tag{7.127}$$

giving $$\ddot{v}_x = -(e/m)\,\dot{E}_x - \omega_{ce}{}^2 v_x \tag{7.128}$$

Similarly, for the ion velocity, V_x, we have

$$\ddot{V}_x = (e/M)\dot{E}_x - \omega_{ci}{}^2 V_x \tag{7.129}$$

To determine $\underline{E}$, we observe that taking the divergence and curl of the vector wave equation

$$\nabla \times (\nabla \times \underline{E}) = \nabla(\nabla \cdot \underline{E}) - \nabla^2\underline{E} = -(\partial/\partial t)(4\pi\, \underline{j} + \dot{\underline{E}}) \tag{7.130}$$

shows that $\nabla \cdot (4\pi\underline{j} + \dot{\underline{E}}) = 0$, and that the curl of this quantity also vanishes if, as we have assumed, $\underline{E}$ is electrostatic, $\nabla \times \underline{E} = 0$. Thus, we have

$$\dot{E}_x = 4\pi j_x = 4\pi\, en_o(V_x - v_x)$$

For high frequency electronic motions, we can neglect the ion contribution to E_x, so that

$$\dot{E}_x = -\, 4\pi n_o e v_x \tag{7.131}$$

Then from (7.128) and (7.131) we have

$$\ddot{v}_x = -(\omega_p{}^2 + \omega_{ce}{}^2)v_x \tag{7.132}$$

It follows that v_x and with it v_y, x and y satisfy an oscillator equation with natural frequency $\omega_{UH} = (\omega_{ce}^2 + \omega_{pe}^2)^{1/2}$. In the low density limit, $\omega_{pe}/\omega_{ce} << 1$, the electrons feel only the restoring force of the $\underline{v} \times \underline{B}$ term, and we have the usual cyclotron motion. In the absence of a magnetic field, the restoring force arises only from the self-consistent electrostatic field, and we have the simple Langmuir oscillations at ω_{pe}. In general, both restoring forces are present, giving rise to the hybrid resonance. From (7.127) it follows that the particle orbits are elliptical in general, with a ratio of major to minor axes equal to ω_{UH}/ω_{ce}.

For low frequency, ionic motions, we must take account of the fact that the electrons can shield the self-consistent field associated with the ion charge density by moving across the magnetic field. For frequencies small compared to ω_{ce}, the left side of (7.128) can be neglected and we have the "polarization drift"

$$v_x = -(e/m\omega_{ce}^2)\dot{E}_x \qquad (7.133)$$

Substituting this into (7.131) gives

$$\dot{E}_x = 4\pi n_o eV_x - (\omega_{pe}^2/\omega_{ce}^2)\dot{E}_x = 4\pi n_o eV_x(1 + \omega_{pe}^2/\omega_{ce}^2)^{-1} \qquad (7.134)$$

Comparison of this with (7.131) shows how the electrons shield the response to low frequency fields. Using (7.134) in (7.129) we have finally

$$\dot{V}_x = -[\omega_{ci}^2 + \omega_{pi}^2 (1 + \omega_{pe}^2/\omega_{ce}^2)^{-1}]V_x \qquad (7.135)$$

Thus, the ions have a natural oscillation frequency

$$\omega_{LH} = [\omega_{ci}^2 + \omega_{pi}^2 (1 + \omega_{pe}^2/\omega_{ce}^2)^{-1}]^{1/2} \qquad (7.136)$$

an expression analogous to that for ω_{UH}, but with a denominator, $(1 + \omega_{pe}^2/\omega_{ce}^2)$ which accounts for the shielding of the electric field by the electrons when

$\omega_{pe} \gtrsim \omega_{ce}$. Of course, for low densities, $\omega_{pe} << \omega_{ce}$, this shielding is unimportant and the expression for ω_{LH} is just like that for ω_{UH}, with ion frequencies (ω_{ci}, ω_{pi}) in place of electron frequencies (ω_{ce}, ω_{pe}). (The electron drift across the magnetic field associated with the varying $\underline{E}$ field, as given by (7.133), is called the polarization drift and we shall examine it in more detail, along with other cross-field drifts, in Chap. VIII.)

b) Kinetic Effects

We now consider the effects associated with thermal motions of the plasma particles. In contrast to the case of parallel propagation, the general expressions for $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{M}$ cannot be derived in any simpler way than used for the general case (Sec. A.2), so we must simply set $\theta = \pi/2$ in (7.62) or (7.63). However, the analysis of the dispersion relation is considerably simplified if we limit ourselves to distribution functions f_o which are symmetrical in v_z, a class which includes Maxwellians and bi-Maxwellians and even allows streaming motions, provided these are symmetrical for each species. The point is as follows.

In all of the cases discussed so far, the x and y (or left and right hand circularly polarized) components of the vector wave equation (7.8) have been decoupled from the z components, i.e., even when M was not diagonal it was what we shall call "semi-diagonal," meaning that

$$M_{xz} = M_{yz} = M_{zx} = M_{zy} = 0 \qquad (7.137)$$

This was a consequence of the fact that $(\underline{NN} - N^2\underset{\sim}{1})$ is semi-diagonal for $\theta = 0$ and $\theta = \pi/2$ and that $\underset{\sim}{\varepsilon}$ itself is semi-diagonal both in the cold plasma limit (independent of $\underline{k}$) and in the kinetic case when $\theta = 0$. For $\theta \neq 0$, $\underset{\sim}{\varepsilon}$ is in general not semi-diagonal, so that all modes are coupled and the algebra is rather complicated. However, when $\theta = \pi/2$ and f_0 is symmetric in v_z,

$$f_0(v_\perp, v_z) = f_0(v_\perp, -v_z) \qquad (7.138)$$

then $\underset{\sim}{\varepsilon}$ is again semi-diagonal. (This follows directly from (7.60), since when $k_z = 0$, so that $\nu = \omega/\Omega$, the only v_z dependence in $\langle S_{.z}\rangle$ and $\langle S_{z.}\rangle$ occurs in $R = v_z(k_z G/\omega + \partial f_o/\partial v_\perp)$ and it is odd in v_z.)

When (7.138) holds, as we shall assume in this section, then, independent of the form of f_0, we have, for $\theta = \pi/2$

$$\underset{\sim}{M} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yz} & \varepsilon_{yy} - N^2 & 0 \\ 0 & 0 & \varepsilon_{zz} - N^2 \end{pmatrix} \tag{7.139}$$

so that the dispersion relation

$$M = \det M = (\varepsilon_{zz} - N^2)(\varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}\varepsilon_{yx} - N^2\varepsilon_{xx}) \tag{7.140}$$

splits into two factors and we can again, as in the cold fluid case, speak of an ordinary wave, given by

$$N^2 = \varepsilon_{zz} \tag{7.141}$$

and an extraordinary wave, which satisfies

$$N^2 = \varepsilon_{yy} - \varepsilon_{xy}\varepsilon_{yx}/\varepsilon_{xx} \tag{7.142}$$

The former is a transverse wave ($\underline{k} \cdot \underline{E} = 0$) and since the polarization is along $\underline{B}_0$, i.e., $\underline{E} = E\hat{\underline{z}}$, it is unaffected by the magnetic field and has the properties already discussed in Chapter V. Therefore, we need only concern ourselves with the changes in the extraordinary wave produced by thermal effects. Of course, in the cold plasma limit (7.141) and (7.142) just reduce to (7.112) and (7.114).

For a wave propagating exactly perpendicular to $\underline{B}_o$, there can be no wave-particle resonance effects of the kind we have discussed for unmagnetized plasmas and for waves propagating parallel to $\underline{B}_o$ since particles being restricted to helical trajectories around the magnetic field lines are not free to stream across the magnetic field and thus come into resonance with the wave. However, we saw in the heuristic discussion of Sec. A.2.a that even for $k_z = 0$ the thermal motion of particles perpendicular to $\underline{B}_o$ has the consequence that the particles experience fields at harmonics $n\Omega$ of the cyclotron frequency. (Since the amplitudes of these fields are proportional to $J_n(k_\perp r_c)$, vanishing, for $n \neq 0$ and $r_c \to 0$, these effects are sometimes referred to as "finite cyclotron radius" effects or, in an older terminology, "finite Larmor radius" (FLR) effects.) Thus, we expect the principal effect of thermal motions to be the appearance of such harmonic excitations.

Since the full analysis of (7.142) is rather complicated, it is instructive to begin with the limiting case of electrostatic waves. As noted before, we cannot, of course, mandate that waves be electrostatic, since the polarization is determined by the homogeneous equations

$$\underset{\sim}{M} \cdot \underline{E} = 0 \qquad (7.143)$$

together with the dispersion relation. However, we can look for waves whose velocity is small compared to c, so that N >> 1. It follows from (7.142) that N will be large if and only if ε_{xx} is very small, since the Vlasov $\underset{\sim}{\varepsilon}$ is always bounded. (In the cold plasma limit, of course, $\underset{\sim}{\varepsilon}$ does become singular near the cyclotron frequencies, but as we have seen, N is not large there, singularities of N in the cold fluid limit occuring only at the hybrid frequencies, which are separated from the cyclotron frequencies by factors of $(1 + \omega_{pe}^2/\omega_{ce}^2)$, $(1 + \omega_{pe}^2/\omega_{ci}^2)$, etc.). Thus, for waves with large N we can, to a good approximation, replace (7.142) by

$$\varepsilon_{xx} = 0 \qquad (7.144)$$

with the proviso, of course, that we must disregard any solution of (7.144) for which N is not large. The fact that the resulting solutions will be electrostatic can be seen from the fact that

$$\varepsilon_{xx} = \hat{\underline{k}} \cdot \varepsilon \cdot \hat{\underline{k}} = \varepsilon_{\ell} = 1 + \sum_{\alpha} \chi_{\alpha} \tag{7.145}$$

so (7.144) just corresponds to the electrostatic dispersion relation (7.64). Alternatively, we can observe that since any solution of (7.143) must satisfy

$$E_y/E_x = - \varepsilon_{xx}/\varepsilon_{xy}$$

$\underline{E}$ will be parallel to $\underline{k}$ for $\varepsilon_{xx} \to 0$.

If we consider the example of a Maxwellian f_o, then the susceptibility is given by Table 7.1 as

$$\chi = \sum_{r=-\infty}^{\infty} (k_D/k)^2 \Lambda_r [1 + (\omega/k_z a_z) Z_r] \tag{7.146}$$

where

$$Z_r = Z[(\omega - r\Omega)/k_z a] \tag{7.147}$$

In the limit $k_z \to 0$, combining r and -r terms gives

$$\begin{aligned} \chi &= \sum_{r} (k_D/k)^2 \, r\Omega\Lambda_r \, (\omega - r\Omega)^{-1} = \\ &= 2 \sum_{r>0} \Lambda_r (k_D/k)^2 [1 - \omega^2(\omega^2 - r^2\Omega^2)^{-1}] \end{aligned} \tag{7.148}$$

Since

$$\Lambda_r = \Lambda_{-r} = e^{-b} \, I_r(b) = 2 \int_o^{\infty} du u e^{-u^2} \, J_r^{\,2}(k_{\perp} a u/\Omega)$$

with $\qquad b = (k_{\perp}a/\Omega)^2/2 = (k_{\perp}r_c)^2/2$

the identity (7A.17)

$$\sum_{-\infty}^{\infty} J_r^{\,2} = 1$$

implies

implies

$$\sum_{-\infty}^{\infty} \Lambda_r = 2 \sum_{1}^{\infty} \Lambda_r + \Lambda_o = 1 \tag{7.150}$$

Using this in (7.148) we obtain, finally, for the dispersion relation for electrostatic waves propagating across the magnetic field, the relatively simple form

$$\begin{aligned} \varepsilon_\ell = 1 + \sum_\alpha \chi_\alpha = 1 + \sum_\alpha (k_{D\alpha}/k)^2 (1 - \Lambda_{o\alpha}) - \\ - 2\omega^2 \sum_{r>o} \sum_\alpha (k_{D\alpha}/k)^2 \Lambda_{r\alpha} (\omega^2 - r^2\Omega_\alpha^2)^{-1} = 0 \end{aligned} \tag{7.51}$$

1) Electron Bernstein waves

Consider first high frequency waves ($\omega >> \omega_{ci}, \omega_{pi}$) where we neglect the ion dynamics, i.e., take $m/M \to 0$. Then (7.151) becomes just

$$1 + (k/k_D)^2 - \Lambda_o = 2\omega^2 \sum_{r>o} \Lambda_r (\omega^2 - r^2\Omega^2)^{-1} \tag{7.152}$$

where all species-dependent qualities now refer to electrons. This is a convenient form of the dispersion relation and, like the two stream cold fluid dispersion relation (3.81) which it resembles, can readily be solved graphically. The right hand side is a simple function of ω, having singularities at each cyclotron harmonic, so we can immediately read off the roots from a plot like Fig. 7.8a.

In the strong field or low temperature limit, where $b = (k/k_D)^2(\omega_{pe}/\omega_{ce})^2 << 1$, we can approximate the Λ_r functions by

$$\Lambda_r \doteq 1 - b \qquad \Lambda_r \doteq (b/2)^r/r! \qquad r \neq 0. \tag{7.153}$$

It is then convenient to write (7.152) in the form (still valid for arbitrary values of b)

$$(1 - \Gamma_o) + b\ (\Omega/\omega_p)^2 = 2\omega^2 \sum_{r>0} \Gamma_r\ (\omega^2 - r^2\Omega^2)^{-1} \tag{7.154}$$

Using (7.153) we see that when b << 1, each term has a factor b; dividing this out we obtain the simpler form, valid in the limit of small cyclotron radius,

$$(\omega_{UH}/\omega_p)^2 = \omega^2(\omega^2 - \Omega^2)^{-1} + \omega^2 \sum_2^\infty (b/2)^{r-1}/r!(\omega^2 - r^2\Omega^2) \equiv C(\omega) \tag{7.155}$$

Since b is small, the first term in C will dominate except in the immediate neighborhood of a cyclotron harmonic, so C will have the form shown in Fig. 7.8b. It follows that the lowest root will occur when

$$(\omega_{UH}/\omega_p)^2 = \omega^2(\omega^2 - \omega^2)^{-1} \tag{7.156}$$

or $\omega^2 = \omega_{UH}{}^2$ which agrees, as it must, with the cold fluid limit.

We can also see from Fig. 7.8b that additional roots will occur near each harmonic $r\omega_{ce}$. An approximate value for these roots can be obtained by setting

$$\omega = r\omega_c\ (1 + \Delta) \tag{7.157}$$

The (7.155) gives

$$(\omega_{UH}/\omega_p)^2 \doteq r^2(r^2 - 1)^{-1} + (b/2)^{r-1}/2r!\Delta \tag{7.158}$$

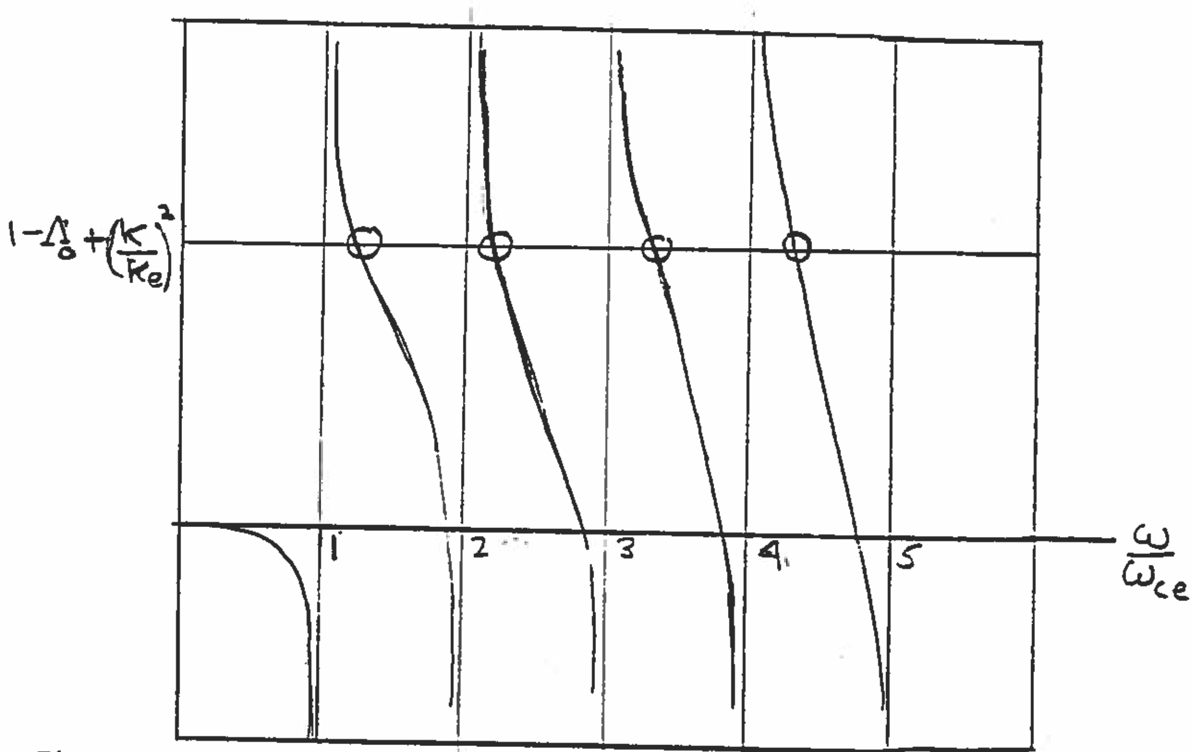


Fig. 7.8a. Plot of right hand side of (7.152) vs ω for $(\omega_{pe}/\omega_{ce})=$
$k/k_D =$, b = . Small circles indicate the roots of (7.152).

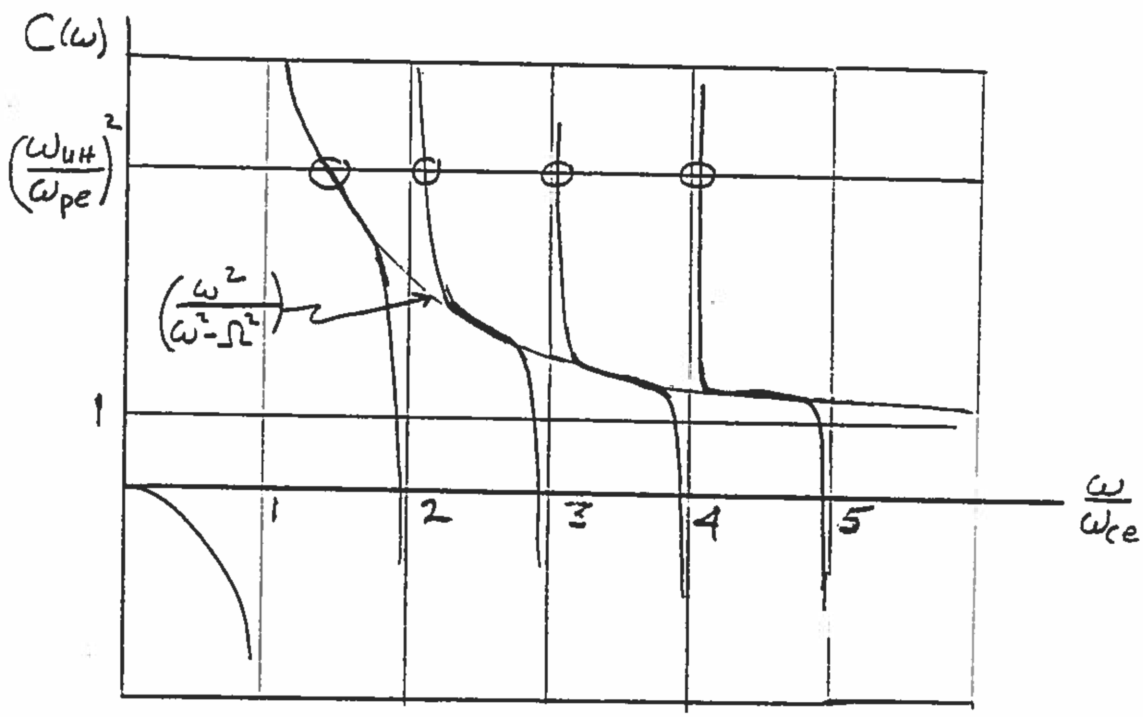


Fig. 7.8b. Plot of $C(\omega)$, defined by (7.155) for
$(k/k_D)^2 =$, b = . The small circles indicate the roots of (7.155).

$$\Delta \doteq [(b/2)^{r-1}(r^2-1)/2r!]\ \omega_p{}^2(r^2\Omega^2 - \omega_{UH}{}^2)^{-1} \tag{7.159}$$

We see that Δ can be positive or negative depending on the relative magnitudes of $r\omega_{ce}$ and ω_{UH}. In fact, for each harmonic, $r\omega_{ce}$, there is a critical density, n_r, defined by

$$\omega_{UH}{}^2 = (r\Omega)^2 \qquad \text{or} \qquad \omega_p{}^2 = (r^2 - 1)\Omega^2 \tag{7.160}$$

For density $n < n_r$, $\Delta > 0$ and there is a root just above $r\omega_{ce}$; for $n > n_r$ $\Delta < 0$ and there is a root just below $r\omega_{ce}$.

In particular, if $\omega_p{}^2 > 3\Omega^2$, then $n < n_2$ and so on $n < n_r$ for all $r \geq 2$, i.e., there are roots just above every harmonic (not including the fundamental). This corresponds to the situation depicted in Fig. 7.8b. For any density above n_2, i.e., for $\omega_p{}^2 > 3\Omega^2$, there will be some integer, $r(n)$, such that

$$r^2(n) - 1 > (\omega_p/\Omega)^2 > [r(n) - 1]^2 - 1 \tag{7.161}$$

For each $r > r(n)$ there will be roots just above r ; for each $r < r(n)$ there will be roots just below r ; and ω_{UH} will lie between $[r(n) - 1]\Omega$ and $r(n)\Omega$.

From these considerations, we can readily infer the character of the density dependence of the roots for small b: for low density there will be a root just above each harmonic $r\omega_{ce}$, $r \geq 2$ and it will make a transition to the region just below $(r + 1)\omega_{ce}$ as the density passes through the value n_r given by (7.160). Between harmonics, where the infinite sum in (7.155) is small, we we have ω approximately equal to ω_{UH}. These features are illustrated in Fig. 7.9.

The modification which occur for larger b (i.e., larger $k_\perp r_{ce}$ resulting from weaker fields, higher temperatures or shorter wavelengths) can also

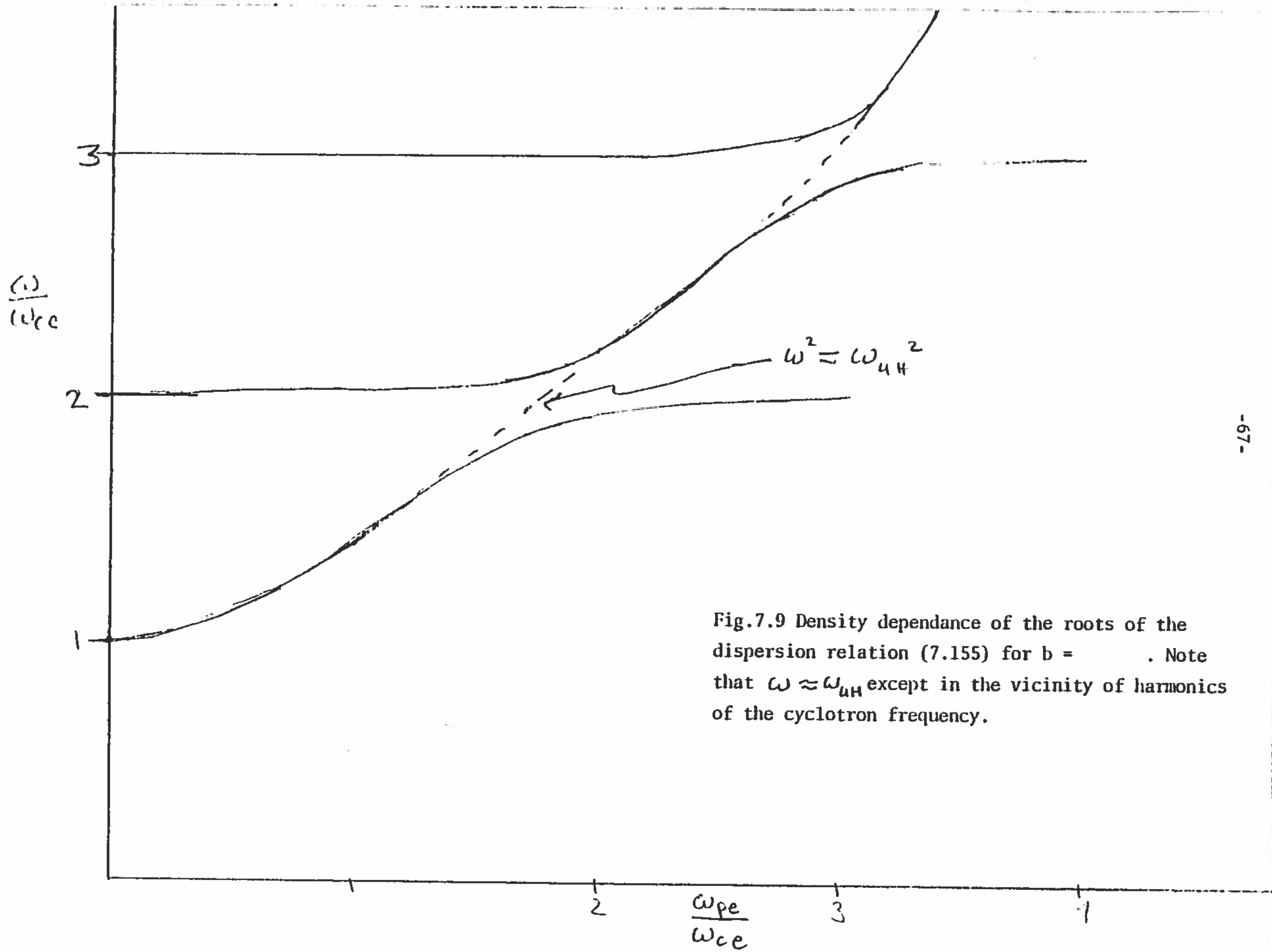


Fig.7.9 Density dependance of the roots of the dispersion relation (7.155) for b = . Note that $\omega \approx \omega_{uH}$ except in the vicinity of harmonics of the cyclotron frequency.

be inferred from (7.159): Δ increases with b, so roots just below $r\omega_{ce}$ move down; those just above $r\omega_{ce}$ move up; those near ω_{UH} may move up or down. When b becomes comparable to 1, or larger, $(b/2)^r/r!$ is no longer an adequate approximation for Λ and we must abandon (7.155) and the equations derived from it. We use instead the exact form (7.154) together with an asymptotic ($b >> 1$) expression for Λ_r. Using the asymptotic form of I_r, (7A.31), we have Λ_r

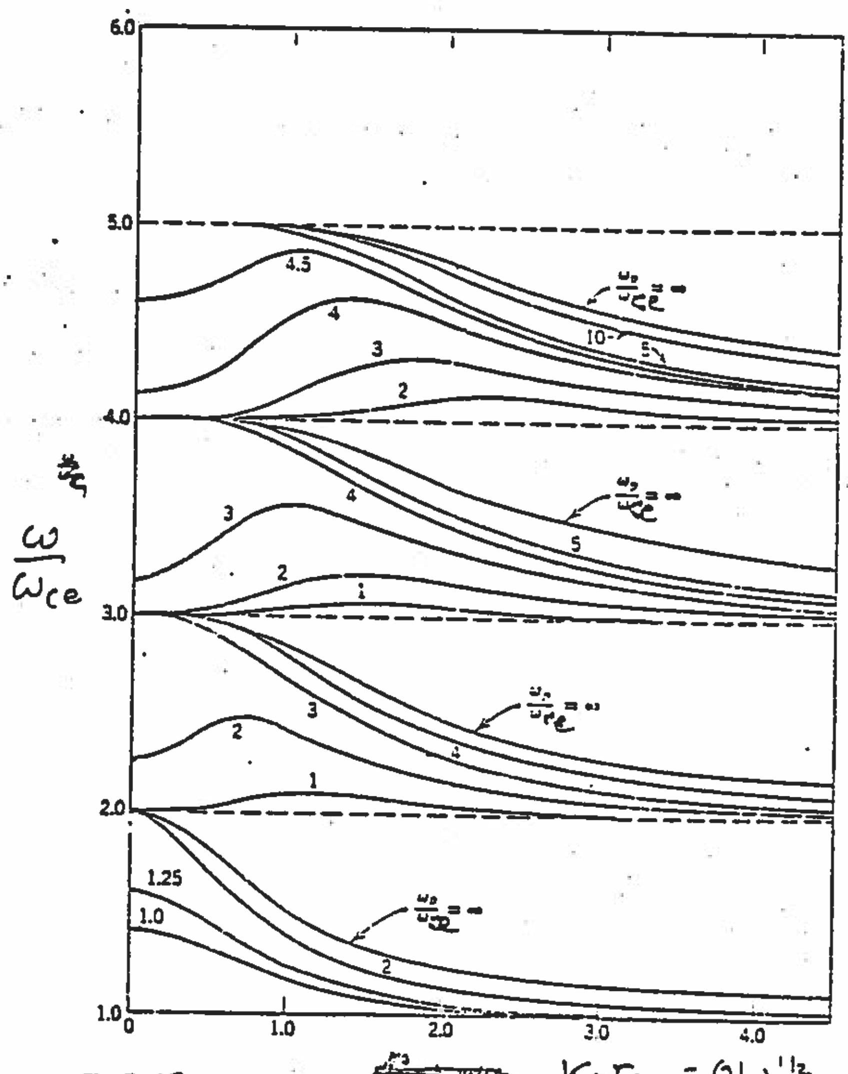
$$\Lambda_r \doteq (2\pi)^{-1/2} b^{-1/2} \exp(-r^2/2b) \tag{7.162}$$

Then (7.154) becomes, for large b,

$$(\Omega/\omega_p)^2 = [2\omega^2/(2\pi)^{1/2} b^{3/2}] \sum_r (\omega^2 - r^2\Omega^2)^{-1} \exp(-r^2/2b) \tag{7.163}$$

Plotting the right side of (7.163) vs. ω gives a diagram somewhat similar to Fig. (7.8a), showing that there are roots just <u>above</u> each harmonic, <u>regardless</u> of the density, the roots monotonically approaching the harmonics as the density increases.

From all of these considerations it follows that the variation of ω with $k_\perp r_{ce}$ for given ω_p/ω_{ce} will have the character shown in Fig. 7.10. The dispersion relation (7.154) for electrostatic waves propagating across the magnetic field was studied in detail by Ira Bernstein and such waves are commonly called electron Bernstein modes.



7.10

Fig. 7.10 Dispersion curves for longitudinal waves propagating in a thermal plasma at right angles to the magnetic field. [After Stone and Auer (1965); and Crawford and Tataronis (1965b).]

ii) Ion Bernstein Waves

If we consider low frequencies, $\omega << \omega_{ce}$, then thermal effects associated with electrons are unimportant and we can take the limit $b_e = (k_\perp r_{ce})^2/2 \to 0$. The electron susceptibility given by (7.148) then reduces to

$$\chi_e = 2(k_e/k)^2 \Lambda_o(b_e) = b_e(k_e/k)^2 = (\omega_{pe}/\omega_{ce})^2 \tag{7.164}$$

and the dispersion relation becomes

$$\varepsilon_\ell = (\omega_{UH}/\omega_{ce})^2 + (k_e/k)^2[1 - \Lambda_o - 2\omega^2 \sum_{r>0} \Lambda_r(\omega^2 - r^2\omega_{ci}{}^2)^{-1}] = 0 \tag{7.165}$$

where the Λ_r have argument $b_i = (k_\perp r_{ci})^2/2$. We multiply (7.165) by $(k/k_i)^2$ and use the identity

$$(k/k_i)^2(\omega_{UH}/\omega_{ce})^2 = b_i(\omega_{ce}/\omega_{pe})^2(\omega_{UH}/\omega_{ce})^2 =$$

$$= b_i(\omega_{ce}/\tilde{\omega}_{pi})^2$$

where we have defined an effective ion plasma frequency, $\tilde{\omega}_{pi}$:

$$\omega_{pi}{}^2 \equiv \omega_{pi}{}^2 \; (1 + \omega_{pe}{}^2/\omega_{ce}{}^2)^{-1} \tag{7.167}$$

The coefficient of $\omega_{pi}{}^2$ comes from the electron shielding of the ion motion, a phenomenon we encountered earlier in the heuristic derivation of (7.136). The result,

$$1 - \Lambda_o(b_i) + b_i(\omega_{ci}/\tilde{\omega}_{pi})^2 = 2\,\omega^2 \sum_{r>0} \Lambda_r \; (b_i)(\omega^2 - r^2\omega_{ce}{}^2)^{-1} \tag{7.168}$$

is an exact analogue of (7.154) with electron quantities replaced by ions quantities and ω_{pe} replaced by ω_{pi}. Consequently, all of the results derived for electron Bernstein waves (including Figs. 7.8 through 7.10), apply equally well to the ion modes, with appropriate changes of scale, and with $\omega_{LH} = (\tilde{\omega}_{pi}^{2} + \omega_{ci}^{2})^{1/2}$ replacing ω_{UH}.

Appendix 7.A Useful Properties of Bessel Functions

1. Definition

The Bessel functions of integral order, or "Bessel coefficients" $J_n(z)$ are defined, for complex z, as the coefficients of t^n in the Laurent series expansion of the generating function

$$G(t,\, z) \equiv \exp[z(t - t^{-1})/2] = \sum_{-\infty}^{\infty} t^n\, J_n(z) \tag{7A.1}$$

From this we now derive the elementary properties of the J_n commonly encountered in plasma physics.

2. $J_{-n}(z)$ and $J_n(-z)$

The substitutions $t \leftrightarrow -1/t$ and $t \leftrightarrow 1/t$ in (7A.1) show that

$$J_n(-z) = J_{-n}(z) = (-1)^n\, J_n(z) \tag{7A.2}$$

3. The Jacobi Identities

The substitution $t = e^{i\theta}$ in (7A.1) gives directly the invaluable relation

$$e^{iz\sin\theta} = \sum_{-\infty}^{\infty} e^{in}\;\; J_n(z) \tag{7A.3}$$

Since this is a Fourier series expansion in θ, it follows that

$$J_n(z) = (2\pi)^{-1} \int_0^{2\pi} d\theta\, (2\pi)^{-1}\, \exp[i(z\sin\theta) - n\theta)] \tag{7A.4}$$

which is Bessel's integral for J_n.

4. Power Series

Expanding the exponentials in (1) gives

$$G = \sum_{k=0}^{\infty} [(zt/2)^k/k!] \sum_{\ell=0}^{\infty} [(-z/2t)^{\ell}/\ell!]$$

$$= \sum_{\ell=0}^{\infty} [-z/a)^{\ell}/\ell!] \sum_{n=-\ell}^{\infty} [t^n(z/2)^{\ell+n}/(\ell+ n)!]$$

$$= \sum_{n=-\infty}^{\infty} (zt/2)^n \sum_{\infty} (-1)^{\ell}(z/2)^{2\ell}/\ell!\ (\ell + n)!$$

where the lower limit on ℓ is 0 if $n > 0$ and is $-n$ if $n < 0$. Thus, for $n > 0$, we have

$$J_n(z) = (z/2)^n \sum_{\ell=0}^{\infty} (-1)^{\ell}(z/2)^{2\ell}/\ell!(\ell+n)! \tag{7A.5}$$

In particular,

$$J_o(z) = 1 - (z/2)^2 + - - -$$

$$J_1(z) = (z/2)\ [1 - z^2/8 + - -]$$

$$J_2(z) = (z^2/8)\ [1 - z^2/12 + - -]$$

5. The Modified Bessel Functions

When z is purely imaginary, (7A.5) shows that $J_n(z)i^n$ is purely real. This motivates us to define the modified Bessel functions, I_n, for general z:

$$J_n(iz) = i^n\ I_n(z) \tag{7A.6}$$

If $z = +ix$, we have then from (7A.3)

$$e^{-x\sin\theta} = \sum_{-\infty}^{\infty} I_n(x)\ e^{in(\theta + \pi/2)}$$

or

$$e^{x\cos\theta} = \sum_{-\infty}^{\infty} I_n(x)e^{in\theta} \tag{7A.7}$$

which is analogous to (7A.3) while the consequent relation

$$I_n(x) = (2\pi)^{-1} \int_0^{2\pi} d\theta \, \exp[x\cos\theta - in\theta] \qquad (7A.8)$$

is analogous to (7A.4).

6. Recursion Relations

Differentiating (7A.1) with respect to t gives

$$\partial G/\partial t = (2n/z)(1 + t^{-2})G = \sum nJ_n t^{n-1}$$

or

$$(J_{n-1} + J_{n+1}) = (2n/z)_n \qquad (7A.9)$$

7. Differential Relations

Differentiating (7A.1) with respect to z gives

$$1/2(t - 1/t)G = \quad J_n{}'(z)t^n$$

or

$$J_{n-1} - J_{n+1} = 2J_n{}' \qquad (7A.10)$$

For the particular case n = 0, (7A.10) and (7A.2) give

$$J_o{}' = - J_1 \qquad (7A.11)$$

Adding (7A.9) and (7A.10) gives

$$J_{n-1} = J_n{}' + nJ_n/z = z^{-n} \frac{d}{dz} (z'J_n) \qquad (7A.12)$$

while their difference gives

$$J_{n+1} = nJ_n/z - J_n{}' = -z^n \frac{d}{dz} (z^{-n}J_n) \qquad (7A.13)$$

From (7A.9), (7A.10), and (7A.6), we have

$$I_{n-1} + I_{n+1} = 2I_n'$$

$$I_{n-1} - I_{n+1} = 2nI_n/z \tag{7A.14}$$

8. The Differential Equation

Differentiating (7A.12) gives

$$J_n'' + nJ_n'/z - nJ_n/z^2 = J_{n-1}'$$

while (7A.13) plus (7A.12) gives

$$J_{n-1}' = (n-1)J_{n-1}/z - J_n$$

$$= \frac{(n-1)}{z}[J_n' + nJ_n/z] - J_n$$

Thus

$$J_n'' + J_n'/z + (1 - n^2/z^2)J_n = 0 \tag{7A.15}$$

which, with (7A.16), gives

$$I_n'' + I_n'/z - (1 + n^2/z^2)I_n = 0 \tag{7A.16}$$

9. Addition Rules

From

$$G(t, z)\, G(1/t, z) = 1$$

we have

$$\sum_{n,m} t^{m-n} J_n J_m = \sum_k t^k J_n J_{n+k} = 1$$

from which follows

$$\sum_{-\infty}^{\infty} J_n^{\ 2} = 1 \qquad\qquad \sum_{-\infty}^{\infty} J_n J_{n+k} = 0, \ k \neq 0 \tag{7A.17}$$

Similarly from

$$G(te^{i\theta/2}, z)\, G(t^{-1}e^{i\theta/2}, z) =$$

$$= \exp\left\{ \frac{[2z\sin(\theta/2)]}{2} (s - \frac{1}{s}) \right\} \quad ; \quad s = it$$

we have

$$\sum_{n,m} t^{(n - m)} e^{i\frac{\theta}{2}(n + m)} J_n(z)\, J_m(z) =$$

$$= \sum t^k e^{i\theta(n - k/2)} J_n J_{n - k} =$$

$$= \sum s^k J_k[2z\sin(\theta/2)]$$

or

$$i^k J_k(2z\sin\frac{\theta}{2}) = \sum_{-\infty}^{\infty} J_n(z)\, J_{n - k}(z) e^{i(n - k/2)\infty} \tag{7A.18}$$

The case $k = 0$:

$$J_o(2k\sin\frac{\theta}{2}) = \sum_{-\infty}^{\infty} J_n^{\ 2}(z) e^{in\theta} \tag{7A.19}$$

is particularly useful.

10. Weber's Second Identity

Consider the integral (similar to those involved in the evaluation of σ for Maxwellian f_o):

$$I(x) \equiv \int_0^{\infty} dr \int_0^{2\pi} d\theta \int_{\infty}^{\phi} d\theta \ re^{-r^2} e^{ixr\sin(\phi - \alpha)}$$

$$e^{-ixr\sin(\theta + \alpha)} e^{-\varepsilon\theta} \tag{7A.20}$$

where $\varepsilon \to 0_+$. Doing the θ integration first,

$$\int_{\infty}^{\phi} d\theta\ e^{-xr\sin(\theta\ +\ \alpha)}e^{-\varepsilon\theta} =$$

$$= \int_{\infty}^{\phi} d\theta \sum_n J_n(xr)e^{-in(\theta\ +\ \alpha)}e^{-\varepsilon\theta} =$$

$$= \sum iJ_n(xr)e^{-in(\phi\ +\ \alpha)}/n$$

we have

$$I = i\int_0^{\infty} dr\ re^{-r^2}\int_0^{2\pi} d\phi \sum J_m(xr)e^{im(\phi\ -\ \alpha)}\cdot$$

$$\cdot \sum J_n(xr)e^{-in(\theta\ +\ \alpha)}/n =$$

$$= 2\pi i\int_0^{\infty} dr\ r\ e^{-r^2} \sum J_n{}^2\ (xr)\ e^{-2in\alpha}/a \quad . \qquad (7A.21)$$

Alternatively, we can write (7A.20) as

$$I = \int_0^{\infty} dr \int_0^{2\pi} d\phi \int_{\infty}^{0} d\psi\ r\ e^{-r^2}\ e^{-ixr[\sin(\phi-\alpha)\ -\ \sin(\phi+\psi+\alpha)]\ -\ \varepsilon\psi} \qquad (7A.22)$$

Transforming from the polar coordinates (r,ϕ) to $\xi = r\cos\phi$, $\eta = r\sin\phi$, and setting $a = \sin\alpha + \sin(\alpha+\psi)$, $b = \cos\alpha - \cos(\alpha+\psi)$, we have

$$I = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{\infty}^{0} d\psi\ e^{-(\xi^2+\eta^2)}\ e^{ix(a\xi+b\eta)-\varepsilon\psi}$$

$$= -\pi \int_0^{\infty} d\psi\ \ e^{-(a^2+b^2)x^2/4-\varepsilon\psi} \qquad (7A.23)$$

$$= -\pi \int_0^{\infty} d\psi\ e^{-x^2/2}\ e^{x^2\cos(2\alpha+\psi)/2-\varepsilon\psi}$$

or, using the Jacobi identity (7A.6),

$$I = -\pi \int_0^\infty d\psi\ e^{-x^2/2} \sum_n I_n(x^2/2)\ e^{in(\psi+2\alpha)-\epsilon\psi} \tag{7A.24}$$

$$= \pi\ i \sum_n e^{-x^2/2}\ I_n(x^2/2)\ e^{-2in\alpha}/n \tag{7A.25}$$

Comparing (7A.21) and (7A.25), we obtain

$$2 \int_0^\infty \phi r\ re^{-r^2}\ J_n^{\ 2}\ (xr) = C^{-x^2/2} I_n(x^2/2) \tag{7A.26}$$

11. Laplace Transform

From (4) we have for $p > 0$

$$L_n(p) \equiv \int_0^\infty dx\ e^{-px}\ J_n(x) =$$

$$= - \int_0^{2\pi} \frac{d\theta\ e^{-in\theta}}{2\pi(i\sin\theta - p)} =$$

$$= - \int_0^{2\pi} \frac{d\theta\ e^{-i(n-1)\theta}}{\pi(e^{2i\theta} - 2pe^{-i\theta} - 1)}$$

Introducing $z = -1/s = e^{i\theta}$ we have

$$L_n = \oint \frac{dz\ z^{-n}}{\pi i(z^2 - 2pz - 1)} = \oint \frac{dz\ z^{-n}}{\pi i(z-a_+)(z-a_-)} \tag{7A.27}$$

where we integrate over a unit circle in the z plane in the counter-clockwise direction and

$$a_\pm = p \pm (P^2+1)^{1/2}; \qquad a_+a_- = -1$$

then

$$L_n = -\frac{1}{\pi i} \oint \frac{ds\ s^n}{(s+a_+)(s+a_-)} \qquad s = z^{-1} \tag{7A.28}$$

where the integration over a unit circle in the s plane but in the clockwise direction. Since $|a_+| > 1$ and $|a_-| < 1$, there is one simple pole, at $s = -a_-$, inside the contour of integration giving

$$L_n = \frac{2(-a_-)^n}{(a_+ - a_-)} = \frac{1}{(p^2+1)^{1/2}[p+(p^2+1)^{1/2}]}$$

For the case of $J_n(\alpha x)$ we have immediately

$$\int_0^\infty dx\ e^{-pe}\ J_n(\alpha x) = \left[\frac{\alpha}{p+(p^2+\alpha^2)^{1/2}}\right]^n \frac{1}{(p^2+\alpha^2)^{1/2}} \tag{7A.29}$$

Appendix 7B Identities Used in Computing $\underset{\sim}{g}$

Nine integrals are involved in the equation of

$$<\underset{\sim}{S}> \equiv (2\pi)^{-1} \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \, \underset{\sim}{S}$$

with $\underset{\sim}{S}$ given by (7B.1). They are $<\sin\phi>$, $<\cos\phi>$, $<\sin(\phi' + \phi)>$, $<\cos(\phi + \phi)>$, $<\sin\phi \, \sin(\phi' + \phi)>$, $<\cos\phi \, \cos\phi(\phi' + \phi)>$, $<\sin\phi \, \cos\phi(\phi' + \phi)>$, $<\cos\phi \, \sin\phi(\phi' + \phi)>$, and $<1> = A(\kappa, \nu)$. To express all of these in terms of A and its derivatives, we begin with the following

<u>Lemma</u>. If f and g are periodic functions with period 2π, then

$$<f(\phi)g(\phi' + \phi) = <f(-\phi-\phi')g(\phi)> \tag{7B.1}$$

This follows immediately from the definition of < > and the fact that

$$\int_a^{a+2\pi} f(\phi)d\phi$$

is independent of a for any periodic function f.

As a first application of (7B.1) we have

$$<\cos\phi> = <\cos(\phi' + \phi)> \tag{7B.2}$$

$$<\sin\phi> = <\sin(\phi' + \phi)> \tag{7B.3}$$

Differentiating (7.53) with respect to κ and using (7B.3) gives

$$\begin{aligned} A' \equiv \partial A/\partial\kappa &= i<\sin\phi - \sin(\phi + \phi')> = \\ &= 2i<\sin\phi> \end{aligned} \tag{7B.4}$$

which expresses $<\sin\phi>$ in terms of A'. (We use a prime here to denote differentation with respect to κ.)

Most of the other integrals we need can be obtained directly from the following

<u>Lemma</u>. If F (ϕ,ϕ') is a periodic function of ϕ and ϕ' with period 2π in each variable, then

$$\langle\partial F/\partial\phi'\rangle = [2\pi Q(\nu)]^{-1}\int_0^{2\pi} d\phi F(\phi,0) + i\,\langle F[\kappa\cos(\phi'+\phi)-\nu]\rangle \qquad (7B.5)$$

The proof requires simply an integration by parts in the ϕ' integration.

Setting F = 1 in (7B.5) gives

$$0 = Q^{-1} + i\kappa\,\langle\cos(\phi'+\phi)\rangle - i\nu A \qquad (7B.6)$$

and defining

$$C = A + i/\nu Q \qquad (7B.7)$$

we have

$$\langle\cos\phi\rangle = \langle\cos(\phi'+\phi)\rangle = \nu C/\kappa \qquad (7B.8)$$

Setting F = $\sin\phi$ in (7B.5) gives

$$\langle\sin\phi\,\cos(\phi'+\phi)\rangle = (\nu/\kappa)\langle\sin\phi\rangle = (\nu/2i\kappa)\,A' \qquad (7B.9)$$

and using (7B.1) we have immediately

$$\langle\cos\phi\,\sin(\phi'+\phi)\rangle = -(\nu/2i\kappa)A' \qquad (7B.10)$$

Finally, setting F = $\cos\phi$ gives

$$\langle\cos\phi\,\cos(\phi'+\phi)\rangle = (\nu/\kappa)\,\langle\cos\phi\rangle = (\nu/\kappa)^2 C \qquad (7B.11)$$

We now have expressions for all of the integrals involved in S save for $\langle\sin\phi\,\sin(\phi'+\phi)\rangle$. To evaluate it we differentiate (7B.4) with respect to κ,

$$A'' = 2\,\langle\sin\phi\,\sin(\phi'+\phi) - \sin^2\phi\rangle$$

which gives

$$\langle \sin\phi \; \sin(\phi' + \phi)\rangle = A''/2 + A - \langle \cos^2\phi \rangle \tag{7B.11}$$

To find $\langle \cos^2\phi \rangle$ we set $F = \cos(\phi' + \phi)$ in (7B.5) to obtain

$$\langle \sin(\phi + \phi')\rangle = i\langle\, \nu \cos(\phi' + \phi) - \kappa \cos^2(\phi' + \phi)\rangle \tag{7B.12}$$

Using (7B.1) we have

$$\begin{aligned} \langle \cos^2\phi \rangle &= -(i/\kappa) \langle \sin\phi \rangle + (\nu/\kappa)\langle \cos\phi \rangle \\ &= -A'/2\kappa + (\nu/\kappa)^2 C \end{aligned} \tag{7B.13}$$

and finally

$$\langle \sin\phi \; \sin(\phi' + \phi)\rangle = A''/2 + A'/2\kappa + A - (\nu/\kappa)^2 C = D - (\nu/\kappa)^2 C \tag{7B.14}$$

where D is defined by (7.57).

We have now found explicit expressions, in terms of A and its derivatives, for all of the integrals required for evaluating $\langle \underset{\sim}{S} \rangle$. Collecting our results we have

$$\langle \cos\phi \rangle = \langle \cos(\phi' + \phi)\rangle = \nu C/\kappa$$

$$\langle \sin\phi \rangle = \langle \sin(\phi' + \phi)\rangle = A'/2i$$

$$\langle \cos\phi \; \cos(\phi' + \phi)\rangle = (\nu/\kappa)^2 C$$

$$\langle \cos\phi \; \sin(\phi' + \phi)\rangle = -\langle \sin\phi \; \cos(\phi' + \phi)\rangle = (i\nu/2\kappa)A'$$

$$\langle \sin\phi \; \sin(\phi' + \phi)\rangle = D - (\nu/\kappa)^2 C$$

$$\langle 1 \rangle = A$$

Substituting these into (7.51) gives (7.58).