Physics 222 A. 12/4.

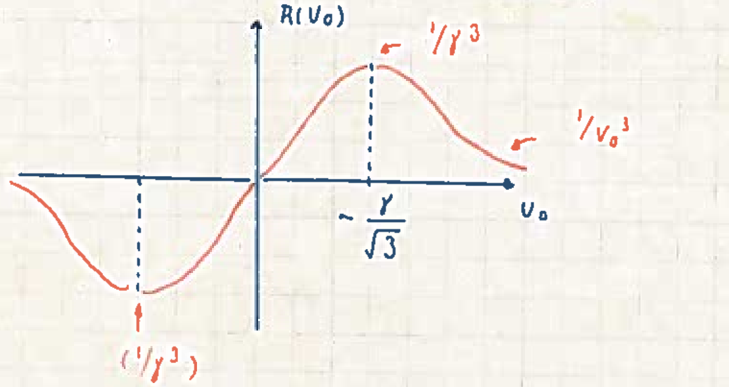
Today

Landau Damping.

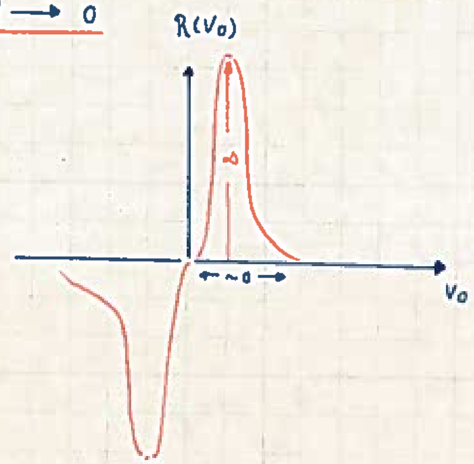
- $m \dfrac{\partial \langle v \rangle}{\partial t} = \underbrace{(-e)}_{q} E_0 \, e^{-\gamma t} \cos kx$

- $\langle v^{(2)}(\infty) \rangle = - \dfrac{k \, \mathcal{E}^2 \, k v_0 \cdot (2)}{\left[\gamma^2 + (kv_0)^2 \right]^2}$, $\boxed{\mathcal{E} \equiv \dfrac{qE_0}{2m}}$

- $\boxed{\langle \Delta p^{(2)}(\infty) \rangle = - \dfrac{q^2 E_0^2 k}{2m} \cdot \left(\dfrac{k v_0}{\left[\gamma^2 + (kv_0)^2 \right]^2} \right)}$ ← $R(v_0)$



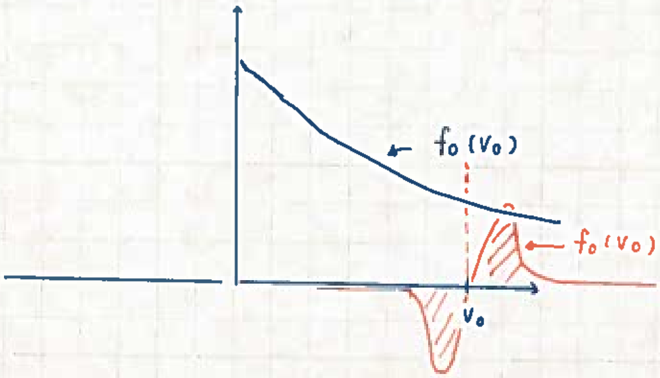
- limit of $\gamma \longrightarrow 0$



- we need total momentum change.

$$\Delta p = \int_{-\infty}^{\infty} f_0(v_0) \, \langle p^{(2)}(\infty, v_0) \rangle \, dv_0$$

$$\boxed{\langle p^{(2)}(\infty, v_0) \rangle = - \frac{q^2 E_0^2 k}{2m} \, \frac{k v_0}{\left[\gamma^2 + (kv_0)^2 \right]^2}}$$



- limit of $\gamma \to 0$.
  - * expand $f_0'(v_0)$

$$\boxed{f_0(v_0) \cong \underset{\uparrow}{f_0(0)} + \frac{\partial f}{\partial v_0} v_0 + O(v_0^2)}$$

$0 = \frac{\omega}{k}$

- $f_0(0)$ term vanishes identically, because $\langle p^{(2)}(\infty, v_0)\rangle$ is anti-symmetric in v_0.

(*) $\longrightarrow$
$$\boxed{\Delta p^{(2)}_{TOTAL} = -\frac{q^2 E_0^2}{2m} k \left(\frac{\partial f_0}{\partial v_0}\right) \int_{-\infty}^{\infty} \frac{v_0 k v_0}{\left[\gamma^2 + (k v_0)^2\right]^2} dv_0 \quad \left(\frac{\partial f_0}{\partial v_0}\right)}$$

- The above expression does not apply for a cold beam ...
- An correction to momentum should contain $\boxed{\text{odd}}$ derivative of f_0 only.
- Scaled version of (*) is.

$$\boxed{\Delta P^{(2)}_{TOT} = -\frac{q_0^2 E_0^2}{2mk\gamma}\left(\frac{\partial f}{\partial v_0}\right) \int_{-\infty}^{\infty} du \frac{u^2}{(1+u^2)^2}}$$

$\pi/2$

- $$\boxed{\Delta P^{(2)}_{TOT} = -\frac{\pi}{4} \frac{q^2 E^2}{mk\gamma} \frac{\partial f_0}{\partial v_0}}$$

- $\Delta P^{(2)}_{TOT}$ increases.....
- if $\frac{\partial f}{\partial v_0} = 0 \longrightarrow \Delta P^{(2)}_{TOT} = 0 \longrightarrow$ no damping
- by analytic continuation, if

$$\boxed{\frac{\partial f}{\partial v_0} > 0 \longrightarrow \text{instability.}}$$

- Momentum is conserved !! $\longrightarrow$ Important !!

$$\boxed{\text{Wave Momentum} + \text{Particle Momentum} = \text{Const}}$$

- Wave Momentum = Energy / phase velocity.

  e.g. photons. $\dfrac{\hbar\omega}{\frac{\omega}{k}} = \hbar k$

• Wave momentum.

$$\frac{\frac{\partial}{\partial \omega}(\epsilon\omega)\frac{|E_0|^2}{16\pi}}{\omega/k}$$

$E(t) \sim E_0 \cos(\omega t)$

• conservation says.

$$\boxed{\underbrace{P_{wave}(+\infty)}_{=0} - P_{wave}(0) + \Delta P^{(2)}_{TOT}(\infty) = 0}$$

$$P_{wave}(0) = \Delta P^{(2)}_{TOT}(\infty) \;!$$

$$\frac{1}{\left(\frac{\omega}{k}\right)} \frac{\partial(\epsilon\omega)}{\partial\omega} \frac{E_0^2}{16\pi} = -\frac{\pi}{4} \frac{q^2 E^2}{m k \gamma} \frac{\partial f_0}{\partial v_0}$$

$$\longrightarrow \boxed{\gamma = -\pi \frac{\omega_p^2}{k^2} \frac{1}{\left(\frac{1}{\omega}\right)\frac{\partial}{\partial\omega}(\epsilon\omega)} \left[\frac{1}{n_0}\frac{\partial f}{\partial v_0}\right]_{v_0}}$$

$$\cdot\ \gamma = -\pi \frac{\cancel{\omega_p^2}}{k} \left(\frac{4\pi q^2}{m}\right) \cdot \frac{1}{k^2} \cdot \frac{1}{\left(\frac{1}{\omega}\right)\frac{\partial(\epsilon\omega)}{\partial\omega}} \cdot \frac{\partial f}{\partial v_0}$$

$$\cdot\ \frac{\partial(\epsilon\omega)}{\partial\omega} = \cancel{\epsilon} + \underbrace{\omega\frac{\partial\epsilon}{\partial\omega}}_{\simeq 2.} \qquad \epsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\boxed{\frac{\gamma}{\omega_p} = -\frac{\pi}{2} \frac{\omega_p^2}{k^2} \frac{1}{n_0} \left(\frac{\partial f}{\partial v_0}\right)_{v = \frac{\omega}{k}}}$$

• What about energy conservation. ?

$$(\Delta KE)_{LAB} = \frac{1}{2} m \left(v_f^2 - v_0^2\right)$$

$$\boxed{\cdot\ v_f = v_0 + \Delta v}$$

$$v_f^2 - v_0^2 = \cancel{v_0^2} + 2 v_0 \Delta v + \Delta v^2 - \cancel{v_0^2}$$

$$(\Delta KE)_{LAB} \simeq \frac{1}{2} m \left[2\left(\frac{\omega}{k}\right)\Delta v + \Delta v^2\right]$$

in linear theory, $\Delta v \ll \left(\frac{\omega}{k}\right)$

In linear theory.

$$(\Delta KE)_{LAB} = \frac{\omega}{k} m \Delta v$$

- Energy conservation.

$$\boxed{\Delta \text{ Wave energy} = \Delta \text{ Particle energy}}$$

$$\frac{\partial}{\partial \omega}(\epsilon \omega) \frac{|E_0|^2}{16\pi} = \frac{\omega}{k} m \Delta v.$$

$$\longrightarrow \boxed{\frac{\text{Wave Energy}}{\omega/k} = m \Delta \dot{v}}$$ Momentum Conservation

11-13 Langmuir Waves

the differential eqn: $\frac{\partial^2}{\partial t^2}\tilde{n} - 3\bar{v}_e^2 \nabla^2 \tilde{n} + \omega_p^2 \tilde{n} = 0$

for plane waves $\tilde{n} \sim e^{i(\underline{k}\cdot\underline{r}-\omega t)}$

then $\omega^2 = \omega_p^2 + \underbrace{3k^2\bar{v}_e^2}_{}$ Bohm-Gross

must be small for this to be valid.

OR $\omega = \sqrt{\omega_p^2 + 3k^2\bar{v}_e^2}$ exact prediction of Bohm-Gross

BUT we will find from kinetic theory a more complicated expression - expanded to lowest order; we find small k Bohm-Gross.

$$\omega \simeq \omega_p\left(1 + \frac{3}{2}\frac{k^2\bar{v}_e^2}{\omega_p^2}\right)$$

Langmuir waves arise from finite pressure on the cold plasma oscillations: So there is a finite group velocity

$$2\omega\frac{\partial\omega}{\partial k} = 2\cdot 3k\bar{v}_e^2 \quad \text{so}$$

$$v_g \equiv \frac{\partial\omega}{\partial k} = 3k\frac{\bar{v}_e^2}{\omega_p} = 3\bar{v}_e\left(\frac{k}{k_0}\right) \qquad \frac{k}{k_0} \sim 0.1$$

so $v_g \ll \bar{v}_e$

OR: $\nabla^2 \tilde{\underline{E}} + k_0^2 \epsilon(\omega) \tilde{\underline{E}} = 0$ $\qquad k_0 \equiv \frac{\omega}{c}$ the vacuum wavenumber

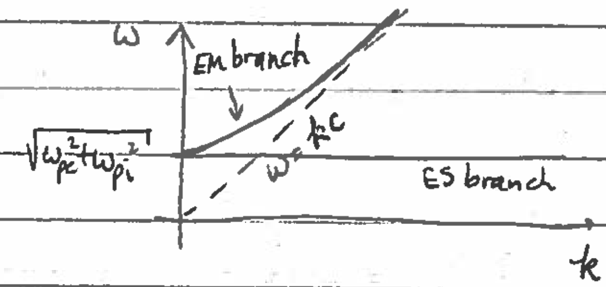
Now, if we also Fourier analyse in space

$$\tilde{\underline{E}} = \tilde{\underline{E}}_0 e^{i \underline{k} \cdot \underline{r}} \quad \text{then}$$

$$\left(-k^2 + k_0^2 \epsilon(\omega)\right) \tilde{E} = 0$$

$$\boxed{k^2 = k_0^2 \epsilon(\omega)}$$ for EM waves - dispersion rel'n.

explicitly $\qquad \omega^2 = (\omega_{pe}^2 + \omega_{pi}^2) + k^2 c^2$

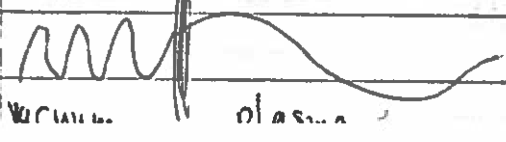


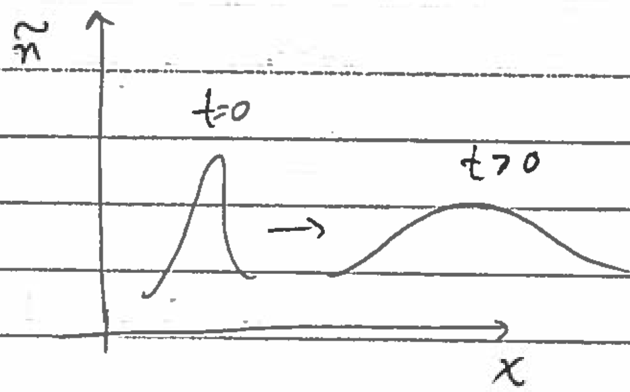
In unmagnetized plasma, there are two well separated modes.
In a magnetized plasma, this is impossible - there are always a mixture of modes because the medium is anisotropic

Since $\epsilon(\omega) \leq 1$ $\quad$ k in plasma is less than in vacuum

$$v_g = \frac{\partial \omega}{\partial k} = c^2 \frac{k}{\omega} \Rightarrow v_g v_{ph} = c^2$$

$v_g < v_{g\,vacuum}$. $\quad$ ie $\left.\begin{array}{l} v_g < c \\ v_{ph} > c \end{array}\right\}$ for vacuum.





because $v_g(k)$, as the pulse moves, it also spreads.

Modulational description

- a powerful technique to solve realistic problems i.e., not able to generate a disturbance with only one wavenumber.

→ project out the large, trivial feature of ω_{pe} oscillation!

let $\tilde{n}(r,t) = A(r,t)\, e^{-i\omega_p t}$ — A is a complex amplitude

and approximate $\left| \frac{1}{A} \frac{\partial}{\partial t} A \right| \ll \omega_p$ — ie, a slowly varying amplitude.

$$\frac{\partial}{\partial t}\tilde{n} = -i\omega_p \tilde{n} + \frac{\partial}{\partial t} A\, e^{-i\omega_p t}$$

$$\frac{\partial^2}{\partial t^2}\tilde{n} = -i\omega_p \left(-i\omega_p \tilde{n} + \frac{\partial}{\partial t} A\, e^{-i\omega_p t} \right) + \frac{\partial^2}{\partial t^2} A\, e^{-i\omega_p t}$$
$$-i\omega_p \frac{\partial}{\partial t} A\, e^{-i\omega_p t}$$

$$= -\omega_p^2 \tilde{n} - 2i\omega_p e^{-i\omega_p t} \frac{\partial}{\partial t} A + \underbrace{e^{-i\omega_p t} \frac{\partial^2}{\partial t^2} A}$$

neglect, since it is smaller than term #2.

So, the dif. eq. becomes

$$\underbrace{-\omega_p^2 \tilde{n} - 2i\omega_p e^{-i\omega_p t} \frac{\partial}{\partial t} A - 3\bar{v}_e^2 \nabla^2 \tilde{n} + \omega_p^2 \tilde{n}}_{\text{cancel}} = 0$$

And we're left with the Schrodinger equation and the $e^{-i\omega_p t}$ factors out,:

$$2i\omega_p \frac{\partial}{\partial t} A + 3\bar{v}_e^2 \nabla^2 A = 0$$

$\omega_p +$

Can include nonlinearities: $\omega_p^2 \to (\omega_p^2)_0 + \alpha |A|^2$
ie, amplitude dependent, and this generate solitons, wave collapse, etc.

Low frequency waves in the moment description (with p)

electron response: $m \frac{d}{dt} \underline{v}_e = e\nabla\phi - \frac{1}{n_e} \nabla p_e$

low $\omega \to 0$ inertial term is negligible! and pressure has to balance the fields.

We don't know what p_e is, so we model it by a "perfect gas"

$$p_e = n_e T_e \qquad (\gamma = 1 \text{ for low frequencies})$$

and T_e is constant (over the space + time scales we're looking at)

We have

$$0 \simeq e\nabla\phi - \nabla(T_e \ln n_e)$$

this gives $\quad n_e = n_e^0 \, e^{e\phi/T_e} \qquad$ Boltzmann

Disturbance moves slowly, so the electrons can respond adiabatically! Hence, you get a Boltzmann factor.

Consistency check –

we said inertia ≪ pressure $\qquad m\omega\tilde{v}_e \ll \frac{T_e}{n_0} k \tilde{n}_e$

but $\tilde{v}_e$ is related to $\tilde{n}_e$ through continuity: (not independent quantities)

$$\omega \tilde{n}_e = n_0 k \tilde{v}_e$$

so, combining, we have – $\quad \left(\frac{\omega}{k}\right)^2 \ll \frac{T_e}{m} = v_e^2$

So, the Boltzmann factor approx only applies to waves that have very small phase velocities compared to electron thermal vel.

Ion response — $M \frac{d}{dt} \underline{v}_i = -q \nabla \phi - \frac{1}{n_i} \nabla p_i$

now large mass M obviates low frequency, so that the inertial term is large - disturbance passes very fast.

Hence ∇p_i is negligible!

linearize $\frac{\partial}{\partial t} \tilde{\underline{v}}_i = -\frac{q}{M} \nabla \phi$

The same consistency check flips the inequality, and we require

$\frac{\omega}{k} \gg \bar{v}_i$ ie, the ions are slow

We also have the ion continuity:

$$\frac{\partial}{\partial t} \tilde{n}_i + \nabla \cdot (n_{0i} \tilde{\underline{v}}_i) = 0$$

apply $\frac{\partial}{\partial t}$ and use F=ma

$$\frac{\partial^2}{\partial t^2} \tilde{n}_i + \nabla \cdot \left(-n_{0i} \frac{q}{M} \nabla \phi \right) = 0$$

To eliminate $\tilde{n}_i$, we need Poisson's equation

$$\nabla^2 \phi = 4\pi \left[e \tilde{n}_e - q \tilde{n}_i \right]$$ $\tilde{n}_e(\phi)$ is known!

$$\tilde{n}_i = -\frac{1}{4\pi q} \nabla^2 \phi + \frac{e}{q} \tilde{n}_e$$

$$\frac{\partial^2}{\partial t^2}\left\{\frac{-1}{4\pi q}\nabla^2\phi + \frac{e}{q}\tilde{n}_e\right\} - \nabla\cdot\left\{\frac{n_{oi}q}{M}\nabla\phi\right\} = 0$$

for uniform medium, we consolidate

$$\frac{\partial^2}{\partial t^2}\left(\frac{e}{q}\tilde{n}_e\right) - \left\{\frac{1}{4\pi q}\frac{\partial^2}{\partial t^2} + \frac{n_{oi}q}{M}\right\}\nabla^2\phi = 0$$

$$\frac{\partial^2}{\partial t^2}\left(4\pi e\tilde{n}_e\right) - \left(\frac{\partial^2}{\partial t^2} + \omega_{pi}^2\right)\nabla^2\phi = 0 \qquad n_e = n_e^0 e^{e\phi/T_e}$$

A formidable nonlinear diff eq — can describe shocks, solitons, etc. - lest describe the ~~etc.~~ linear response:

$$e\phi \ll T_e \qquad e^{e\phi/T_e} \simeq 1 + e\phi/T_e$$

so $$n_e \cong n_e^0 + n_e^0\frac{e\phi}{T_e} \equiv n_e^0 + \tilde{n}_e$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{4\pi n_e^0 e^2}{T_e}\phi\right) - \left(\frac{\partial^2}{\partial t^2} + \omega_{pi}^2\right)\nabla^2\phi = 0$$

now, low (verily low) freq: $\frac{\partial^2}{\partial t^2} \ll \omega_{pi}^2$

we recover old-fashioned wave eqn:

$$\frac{\partial^2}{\partial t^2}\phi - \frac{Z T_e}{M}\nabla^2\phi = 0$$

assuming quasi-neutral
$n_i^0 q = n_e^0 e$
and $q = Ze$

$$c_s \equiv \sqrt{\frac{ZT_e}{M}} \quad \text{- ion-acoustic speed}$$

$$= \sqrt{\frac{ZT_e}{T_i}}\,\bar{v}_i$$

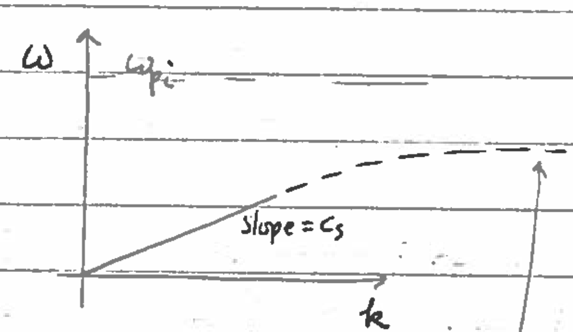
we approx earlier that $\frac{\omega}{k} \gg \bar{v}_i$, OR $T_e \gg T_i$

Ion acoustic waves require $\boxed{Z\frac{T_e}{T_i} \sim 10}$ typical of discharges.

For plane waves $\phi \sim e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

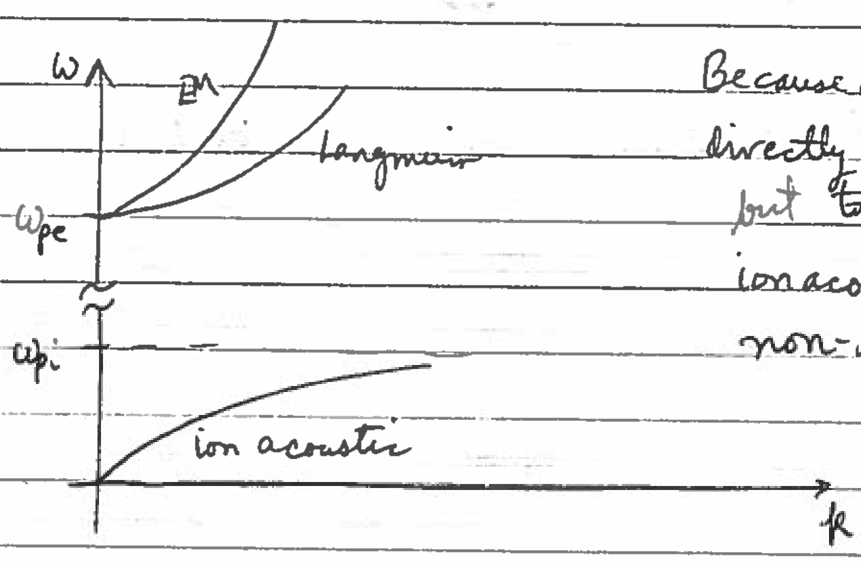
$$\Rightarrow \quad \omega^2 = k^2 c_s^2$$

non-dispersive $\quad v_g = c_s$



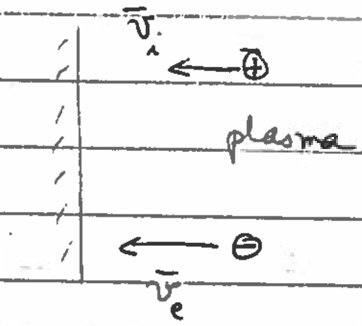
keeping extra term. (heavily damped)

SUMMARY



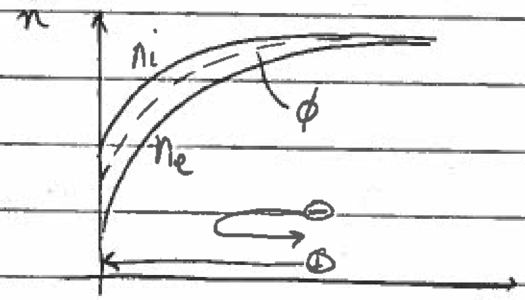
Because of degeneracy, EM can directly pump Langmuir waves, but to cascade to low f, ionacoustic, you need a non-linear process.

Sheath around a planar wall



e^- outrun ions, charging the wall, $^{(-)}$ developing potential that slows further e^- from reaching wall.

$\Rightarrow$ non-neutrality develops



there's a simple exercise to calculate wall potential ϕ_w relative to interior "floating potential"

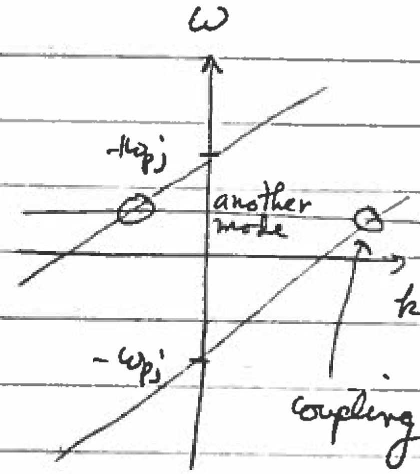
$$\phi_w = \frac{eT_e}{2}\left[\ln\frac{M}{m} + \ln\sqrt{\frac{T_e}{T_i}}\right]$$

The potential created to keep the net current zero!

for hydrogen $\frac{\phi_w}{eT_e} \approx 3.8$

and this occurs on a scale of a Debye length.

11-20 Beam Modes



$\omega = \underline{k} \cdot \underline{v}_0 \pm \omega_{pj}$ ← plasma frequency of stream

They are stable by themselves, but can become unstable if there is another mode to couple to. I.e., there can be feedback

beam mode → collective mode → field modulates beam

feedback - possibility of instability

Instabilities

absolute : fix $\underline{k}$, what happens as fn. of time?

grows everywhere $\sim e^{\gamma t}$
easiest to calculate, but not always relevant.

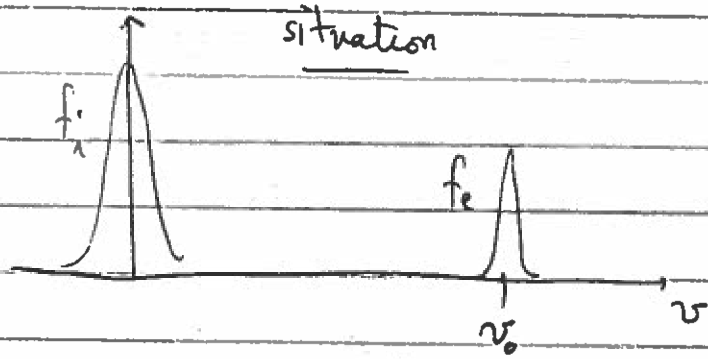
convective : Source at fixed ω - grows in space $\sim e^{k_I z}$
no feedback required → more robust

gradients, boundaries do not ruin the loop as in absolute.

Buneman (Electron-Ion two stream) Instability (1959)

Phys Rev 115, 503

situation

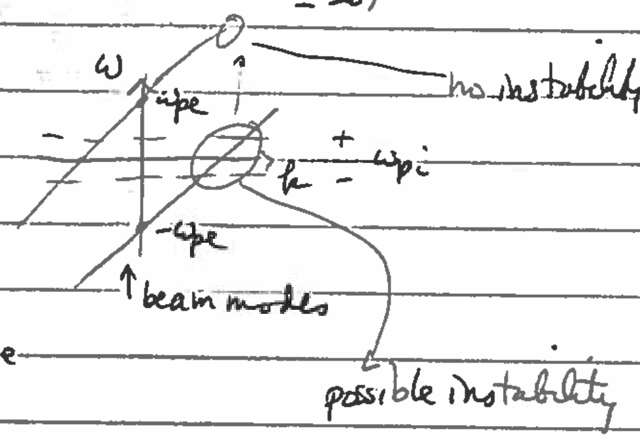


The entire electron population streams past the ions

$v_0 \gg \bar{v}_e, \bar{v}_i$

The dielectric is: $\epsilon(k,\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega - \underline{k}\cdot\underline{v}_0)^2}$

without electrons: $\omega = \pm\omega_{pi}$



without ions: $\omega = \underline{k}\cdot\underline{v}_0 \pm \omega_{pe}$

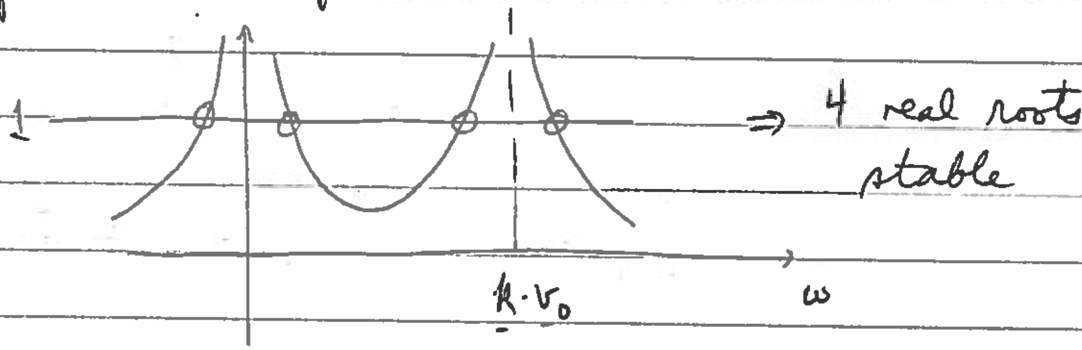
The doppler shift can be significant (must be) to interact with the ions: $k v_0 - \omega_{pe} \sim \omega_{pi}$

Since only one branch interacts with the ions, we can formulate the problem ignoring the other branch.

How do we know there's an instability? Plot $\epsilon = 0$!
i.e., plot $\omega(k)$, the solution to $\epsilon = 0$. And if
$\mathrm{Im}\,\omega = 0 \Rightarrow$ stable
$\mathrm{Im}\,\omega > 0 \Rightarrow$ growth [convention $E \sim e^{-i\omega t}$]

The solution is $1 = \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{(\omega - \underline{k}\cdot\underline{v}_0)^2}$ [4th order eqn]

Instead of solving for $\omega(\underline{k})$, simply plot both sides of the above eqn.



But, if you lower $\underline{k}\cdot\underline{v}_0$, the peaks will move closer together, and the minimum lifts above 1. Hence, only two real roots, and you get imaginary roots $\Rightarrow$ instability!

Marginal Stability

double real root

$1 = \chi(\underline{k}, \omega)$ $\Rightarrow$ when minimum barely touches, then it's marginally stable.

$\frac{\partial \chi}{\partial \omega} = 0$ and $\chi \geq 1$ But this won't give you growth rate

Resonant Mode Analysis

- keep only the strongly coupled roots
  ie, there will be a region of unstable $\underline{k}$, the rest will be stable.

rewriting $$1 - \frac{\omega_{pe}^2}{(\omega - \underline{k}\cdot\underline{v}_0)^2} = \frac{\omega_{pi}^2}{\omega^2}$$

$$(\omega - \underline{k}\cdot\underline{v}_0)^2 - \omega_{pe}^2 = \frac{\omega_{pi}^2}{\omega^2}(\omega - \underline{k}\cdot\underline{v}_0)^2$$

difference of two squares

$$= \left[(\omega - \underline{k}\cdot\underline{v}_0) - \omega_{pe}\right]\left[\omega - \underline{k}\cdot\underline{v}_0 + \omega_{pe}\right]$$

"idler"

strongly interacting root
ie $\underline{k}\cdot\underline{v}_0 - \omega_{pe}$ is small

here, let
$\underline{k}\cdot\underline{v}_0 = \omega_{pe}$ and $\omega \sim 0$

here we make the assumptions as well.

$$= (-2\omega_{pe})\left(\omega - (\underline{k}\cdot\underline{v}_0 - \omega_{pe})\right) = \frac{\omega_{pi}^2}{\omega^2}(-\omega_{pe})^2$$

rewriting $$\boxed{\omega^2\left(\omega - [\underline{k}\cdot\underline{v}_0 - \omega_{pe}]\right) = \frac{1}{2}\omega_{pi}^2\,\omega_{pe}}$$ resonant mode dispersion relation

Now, scale the variables by ω_s:

$$\omega_s \equiv \left(\frac{\omega_{pi}^2\,\omega_{pe}}{2}\right)^{1/3}$$ this is the natural scale.

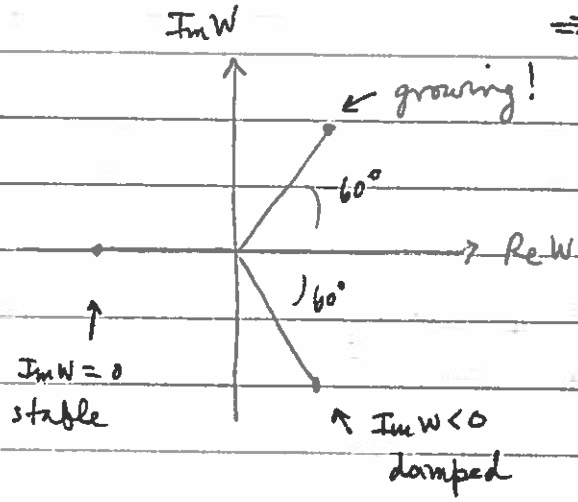
let $\omega \equiv W\omega_s$ $\quad y = \dfrac{\underline{k}\cdot\underline{v}_0 - \omega_{pe}}{\omega_s}$

$$\boxed{W^2(W - y) = -1}$$ universal dispersion relation — no parameters

$\underline{y=0}$ $\quad (\underline{k}\cdot v_0 = \omega_{pe})$ $\qquad W^3 = -1 = e^{i(2n+1)\pi}$

$$\Rightarrow \quad W = e^{i\frac{2n+1}{3}\pi}$$

$$= e^{i\frac{\pi}{3}}, \quad e^{i\pi}, \quad e^{i\frac{5\pi}{3}}$$



$$\mathrm{Re}\,\omega = \omega_s \cos \pi/3$$

Im ω > Re ω $\Rightarrow$ purely growing instability

$$\mathrm{Im}\,\omega = \omega_s \sin \pi/3$$

To find fastest growing mode (maximize; ie take derivative and set = 0)

$$W^2 \left[W - y \right] = -1$$

$$\frac{d}{dy}: \quad 2W \frac{dW}{dy}(W-y) + W^2 \left(\frac{dW}{dy} - 1 \right) = 0$$

$$\frac{dW}{dy}\left(2W^2 - 2Wy + W^2 \right) = W^2$$

$$\frac{dW}{dy} \left(= \frac{W}{3W - 2y} \right.$$

Max growth rate implies $\frac{d}{dy} \operatorname{Im} W = 0$ so we need to rationalize!

$$\frac{dW}{dy} = \frac{W_r + i W_I}{3W_r - 2y + i 3W_I} =$$

$$\frac{dW_I}{dy} = \frac{W_I(3W_r - 2y) - 3W_r W_I}{(3W_r - 2y)^2 + (3W_I)^2} = 0$$

only the numerator is important:

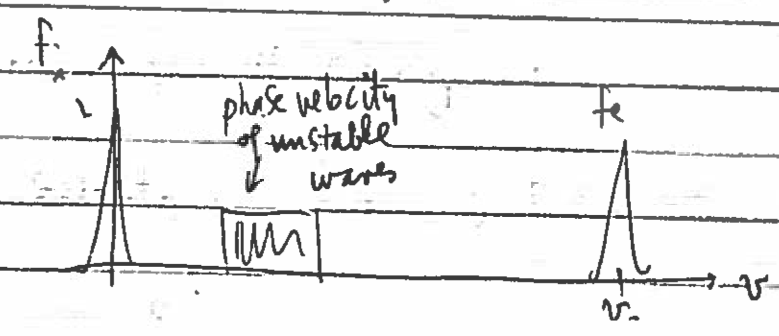
$$\text{so} \quad -2y W_I = 0 \quad \text{or} \quad \boxed{y = 0}$$ max growth rate

We also want to know the band around the region $y=0$ which will be unstable.

Phase Velocity of fastest growing mode.

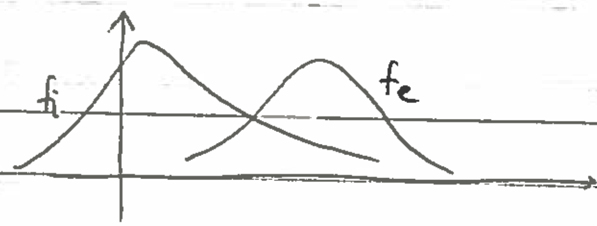
$$v_p = \frac{\operatorname{Re} \omega}{k} = \frac{\omega_s \cos \frac{\pi}{3}}{\omega_{pe}/v_0} = \left(\frac{\omega_{pi}^2 \omega_{pe}}{2}\right)^{1/3} \frac{v_0 \cos \pi/3}{\omega_{pe}}$$

$$= \left(\frac{m}{M}\right)^{1/3} \frac{1}{2^{4/3}} v_0 \sim \frac{1}{10} v_0$$ much slower than the stream.



at $t=0$

later in time:



the waves accelerated the ions, and slowed down the electrons

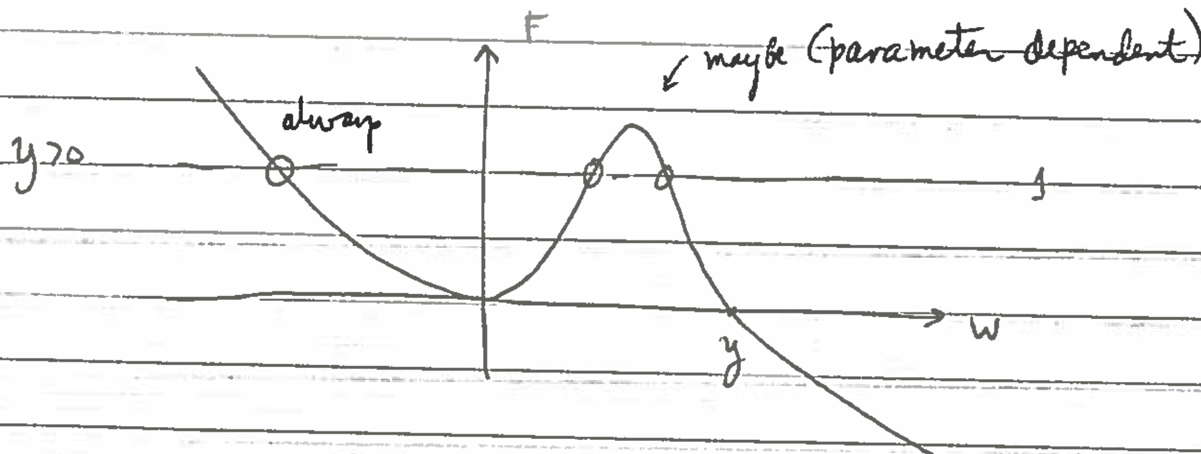
This results in an "anamolous" resistivity, because initialy, the electrons are moving so fast they are collisionless.

It used to be thought that this was a path to fusion, but the instability not only heats ions, but sends them to the wall ⇒ confinement time too short.
Still around in fast pinches, etc.

Width of instability

$$W^2 (W - y) = -1 \qquad F(W) \equiv W^2 (y - W)$$

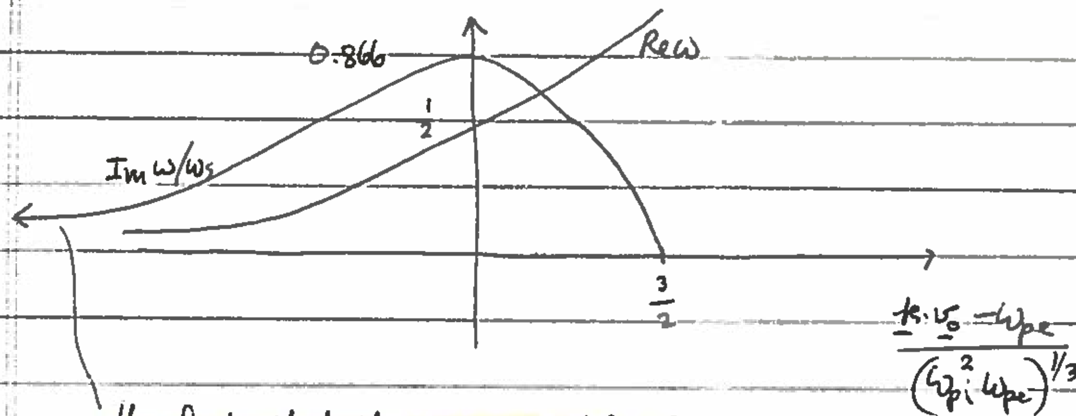
roots are $F = 1$



again, play the same game:

Stability: $\dfrac{dF}{dW} = 0$; $F \geq 1$ ⇒ $k \cdot v_0 - \omega_{pe} \geq \frac{3}{2} \left(\omega_{pi}^2 \omega_{pe}\right)^{1/3}$

$\gamma < 0$ only one crossing $\Rightarrow$ always unstable



the fact that this is unstable for
all negative values will be truncated
when you include the other beam root!

I.e., resonant mode analysis is no longer applicable.

(MONDAY - 10/14)

Physics 222A
Homework Assignment #1
Due: Wednesday, October 9, 1991

G. J. Morales
Fall 1991

This homework assignment is meant to be a survey of important results in basic mechanics and E & M which have a significant impact on the behavior of plasmas.

Prob (13.1) $\sigma = \infty$

1. Derive the differential cross section for scattering between two classical point particles having charges and masses (q_1, m_1) and (q_2, m_2), respectively. (This is the Rutherford cross section.) ½

(13.2)

2. Calculate the energy transferred $\Delta E(b)$ by a heavy ion of mass M, charge q, and asymptotic initial velocity v, to a free electron of charge e, mass m, which is initially at rest. Assume M >> m and a nonrelativistic treatment to express ΔE as a function of the impact parameter b.

½ (13.2) (???) Jackson

3. Calculate the amount of energy transferred $\Delta E(b)$ by a heavy ion of charge q, mass M, and asymptotic initial velocity v to a harmonically bound electron of charge e and mass m. Assume a nonrelativistic treatment with M >> m and a damped harmonic oscillator model for the bound electron. Express answer in terms of impact parameter b, oscillator frequency ω_o, and particle data. ½

½ (13.9?) Jackson (long)

4. Derive an expression for the energy lost per unit length (dE/dx) by a charged particle of charge q, mass M, due to Cerenkov radiation in an arbitrary medium having a scalar dielectric function $\varepsilon(\omega)$. a) Express result in terms of an integral over frequencies; b) give an approximate result for a cold plasma, i.e., $\varepsilon(\omega) = 1 - (\omega_p/\omega)^2$, where ω_p is the electron plasma frequency and ω the frequency of the radiated signal. (WED) ½

(????)

5. A nonrelativistic ion of charge q_1, mass M_1, and asymptotic initial velocity v, makes a head-on collision with a fixed ion of charge q_2. Calculate the total amount of energy radiated in this process. (ASK)

(???)

6. Calculate the time-averaged power radiated per unit solid angle ($d\overline{P}/d\Omega$) by a nonrelativistic charge q executing simple harmonic motion at frequency ω_o in one dimension. Calculate to lowest order in $\beta \equiv v/c$, the time-averaged power, per unit solid angle, radiated at the second harmonic, i.e., $2\omega_o$. (ASK) → gets 2nd harmonics for v/c

JDJ (13.7)

A charged particle of charge q and mass M moves with constant velocity v along the z axis of a medium having a scalar dielectric $\varepsilon(\omega)$. The coordinates are chosen such that the particle is at z = 0 at t = 0. Calculate the potential at frequency ω generated at an arbitrary position $\underline{r}$ by the particle, i.e., find $\phi(\omega,\underline{r})$.

$v = \frac{M}{m} v'$

more energy while near the equator, or while near its turning points? Why? Make quantitative statements if you can.

14.5 As in Problem 14.2*a* a charge e moves in simple harmonic motion along the z axis, $z(t') = a\cos(\omega_0 t')$.

(*a*) Show that the instantaneous power radiated per unit solid angle is:

$$\frac{dP(t')}{d\Omega} = \frac{e^2 c\beta^4}{4\pi a^2}\frac{\sin^2\theta\cos^2(\omega t')}{(1+\beta\cos\theta\sin\omega_0 t')^5}$$

where $\beta = a\omega_0/c$.

(*b*) By performing a time averaging, show that the average power per unit solid angle is:

$$\frac{dP}{d\Omega} = \frac{e^2 c\beta^4}{32\pi a^2}\left[\frac{4+\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^{7/2}}\right]\sin^2\theta$$

(*c*) Make rough sketches of the angular distribution for nonrelativistic and relativistic motion.

14.6 Show explicitly by use of the Poisson sum formula or other means that, if the motion of a radiating particle repeats itself with periodicity T, the continuous frequency spectrum becomes a discrete spectrum containing frequencies that are integral multiples of the fundamental. Show that a general expression for the power radiated per unit solid angle in each multiple m of the fundamental frequency $\omega_0 = 2\pi/T$ is:

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^4 m^2}{(2\pi c)^3}\left|\int_0^{2\pi/\omega_0} \mathbf{v}(t)\times\mathbf{n}\exp\left[im\omega_0\left(t-\frac{\mathbf{n}\cdot\mathbf{x}(t)}{c}\right)\right]dt\right|^2$$

14.7 (*a*) Show that for the simple harmonic motion of a charge discussed in Problem 14.5 the average power radiated per unit solid angle in the mth harmonic is:

$$\frac{dP_m}{d\Omega} = \frac{e^2 c\beta^2}{2\pi a^2}m^2\tan^2\theta J_m^2(m\beta\cos\theta) \quad \longrightarrow P^{(2m)}$$

(*b*) Show that in the nonrelativistic limit the total power radiated is all in the fundamental and has the value:

$$P \simeq \frac{2}{3}\frac{e^2}{c^3}\omega_0^4\overline{a^2}$$

where $\overline{a^2}$ is the mean square amplitude of oscillation.

14.8 A particle of charge e moves in a circular path of radius R in the x-y plane with constant angular velocity ω_0.

(*a*) Show that the exact expression for the angular distribution of power radiated into the mth multiple of ω_0 is:

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^4 R^2}{2\pi c^3}m^2\left[\left(\frac{dJ_m(m\beta\sin\theta)}{d(m\beta\sin\theta)}\right)^2 + \frac{\cot^2\theta}{\beta^2}J_m^2(m\beta\sin\theta)\right]$$

where $\beta = \omega_0 R/c$, and $J_m(x)$ is the Bessel function of order m.

(*b*) Assume nonrelativistic motion and obtain an approximate result for $dP_m/d\Omega$. Show that the results of Problem 14.2*b* are obtained in this limit.

(*c*) Assume extreme relativistic motion and obtain the results found in the text for a relativistic particle in instantaneously circular motion. (Watson, pp. 79, 249, may be of assistance to you.)

```
(d6)                                    output

(c7) d1;

                                          1
(d7)                              ----------------
                                   3        2
                                  r   sqrt(r   - b r)

(c8) integrate(d7,r,INF,b);

Is  b  positive, negative, or zero?

positive;
gc:[*list:1230{52%}; fixnum:51{2%}; ut:67%]
gc:[*list:1230{52%}; fixnum:51{2%}; ut:67%]
gc:[list:1230{53%}; *fixnum:61{2%}; ut:48%]
gc:[list:1230{53%}; *fixnum:71{2%}; ut:33%]

                                         16
(d8)                                  - -----
                                            3
                                        15 b

(c9) describe(integrate);
```

INTEGRATE(exp, var) integrates exp with respect to var or returns an integral expression (the noun form) if it cannot perform the integration (see note 1 below). Roughly speaking three stages are used:

(1) INTEGRATE sees if the integrand is of the form F(G(X))*DIFF(G(X),X) by testing whether the derivative of some subexpression (i.e. G(X) in the above case) divides the integrand. If so it looks up F in a table of integrals and substitutes G(X) for X in the integral of F. This may make use of gradients in taking the derivative. (If an unknown function appears in the integrand it must be eliminated in this stage or else INTEGRATE will return the noun form of the integrand.)

(2) INTEGRATE tries to match the integrand to a form for which a specific method can be used, e.g. trigonometric substitutions.

(3) If the first two stages fail it uses the Risch algorithm.

Functional relationships must be explicitly represented in order for INTEGRATE to work properly. INTEGRATE is not affected by DEPENDENCIES set up with the DEPENDS command.

INTEGRATE(exp, var, low, high) finds the definite integral of exp with respect to var from low to high or returns the noun form if it cannot perform the integration. The limits should not contain var. Several methods are used, including direct substitution in the indefinite integral and contour integration. Improper integrals may use the names INF for positive infinity and MINF for negative infinity. If an integral "form" is desired for manipulation (for example, an integral which cannot be computed until some numbers are substituted for some parameters), the noun form 'INTEGRATE may be used and this will display with an integral sign. (See Note 1 below.)

The function LDEFINT uses LIMIT to evaluate the integral at the lower and upper limits.

Sometimes during integration the user may be asked what the sign of an expression is. Suitable responses are POS;, ZERO;, or NEG;.

```
(C1) INTEGRATE(SIN(X)**3,X);
```

```
                                3
                           COS  (X)
(D1)                       ------- - COS(X)
                              3

(C2) INTEGRATE(X**A/(X+1)**(5/2),X,0,INF);
IS  A + 1  POSITIVE, NEGATIVE, OR ZERO?

POS;
IS  2 A - 3  POSITIVE, NEGATIVE, OR ZERO?

NEG;
                                      3
(D2)                  BETA(A + 1, - - A)
                                      2

(C3) GRADEF(Q(X),SIN(X**2));
(D3)                                 Q(X)

(C4) DIFF(LOG(Q(R(X))),X);

                                d           2
                              (-- R(X)) SIN(R (X))
                               dX
(D4)                          --------------------
                                    Q(R(X))

(C5) INTEGRATE(%,X);
(D5)                              LOG(Q(R(X)))
```

Note 1) The fact that MACSYMA does not perform certain integrals does not always imply that the integral does not exist in closed form. In the example below the integration call returns the noun form but the integral can be found fairly easily. For example, one can compute the roots of X^3+X+1 = 0 to rewrite the integrand in the form 1/((X-A)*(X-B)*(X-C)) where A, B and C are the roots. MACSYMA will integrate this equivalent form although the integral is quite complicated.

```
(C6) INTEGRATE(1/(X^3+X+1),X);

                              /
                              [      1
(D6)                          I ---------- dX
                              ]  3
                              / X  + X + 1

(d9)                                describe done

(c10) quit();
```