An integration by parts then implies the irrotational property of $\vec{E}$ ,

$$\vec{\nabla} \times \vec{E} = 0 \quad .$$

The advantage to this functional form of the energy, (10.10), is seen by again considering finite variations (called $\delta\vec{D}$). If $\vec{D}$ is the correct physical solution, then $E(\vec{D})$ is an absolute minimum,

$$E(\vec{D} + \delta\vec{D}) = E(\vec{D}) + \int (d\vec{r}) \frac{(\delta\vec{D})^2}{8\pi\varepsilon} \geq E(\vec{D}) \quad , \tag{10.12}$$

(since the linear term in $\delta\vec{D}$ is zero due to the stationary principle). Therefore, the correct energy is the minimum value of (10.10) while an arbitrary $\vec{D}$ [compatible with (10.9)] will give an upper bound to $E$ . These bounds, (10.8) and (10.12), are useful for finding approximate solutions when exact solutions are difficult or impossible to obtain.

## 10-2. Force on Dielectrics

As a simple application, consider the effect of a small change in the dielectric constant, $\delta\varepsilon$ ,

$$\varepsilon(\vec{r}) \to \varepsilon(\vec{r}) + \delta\varepsilon(\vec{r}) \quad .$$

The first order change in the energy, from the form (10.7), is

$$\delta E(\phi) = - \int (d\vec{r}) \frac{\delta\varepsilon(\vec{\nabla}\phi)^2}{8\pi} = - \int (d\vec{r}) \frac{\delta\varepsilon E^2}{8\pi} \quad . \tag{10.13}$$

The potential $\phi$ is an implicit function of $\varepsilon$ but the first order variation so induced vanishes by the stationary principle. Equivalently, from the second form, (10.10), the first order variation in the energy is

$$\delta E(\vec{D}) = -\int (d\vec{r})\, \frac{1}{8\pi} \left(\frac{\vec{D}}{\varepsilon}\right)^2 \delta\varepsilon = -\int (d\vec{r})\, \frac{\delta\varepsilon\, E^2}{8\pi} \quad , \tag{10.14}$$

in agreement with (10.13).

An example is provided by the infinitesimal displacement of an uncharged inhomogeneous dielectric. If the material is displaced by $\delta\vec{r}$ , the new dielectric constant at $\vec{r}$ is the old dielectric constant at $\vec{r} - \delta\vec{r}$ :

$$\varepsilon(\vec{r}) \rightarrow \varepsilon(\vec{r}) + \delta\varepsilon(\vec{r}) = \varepsilon(\vec{r}-\delta\vec{r}) = \varepsilon(\vec{r}) - \delta\vec{r}\cdot\vec{\nabla}\varepsilon(\vec{r}) \quad ,$$

or

$$\delta\varepsilon(\vec{r}) = -\delta\vec{r}\cdot\vec{\nabla}\varepsilon(\vec{r}) \quad . \tag{10.15}$$

Then, the change in energy is

$$\delta E = \delta\vec{r}\cdot\int (d\vec{r})(\vec{\nabla}\varepsilon)\, \frac{E^2}{8\pi} = -\vec{F}\cdot\delta\vec{r} \quad . \tag{10.16}$$

We therefore identify the force, $\vec{F}$ , on the dielectric due to the inhomogeneity of the medium to be

$$\vec{F} = -\int (d\vec{r})\, \frac{E^2}{8\pi}\, \vec{\nabla}\varepsilon \quad . \tag{10.17}$$

This same result, (10.17), can also be derived from the stress tensor, (7.13), with $\vec{H} = 0$ ,

$$\overleftrightarrow{T} = \overleftrightarrow{1}\, \frac{\varepsilon E^2}{8\pi} - \frac{\varepsilon \vec{E}\,\vec{E}}{4\pi} \quad . \tag{10.18}$$

Since the stress tensor describes the outward flow of momentum per unit area, the total force <u>on</u> a body bounded by a closed surface $S$ is

$$\vec{F} = -\int_S d\vec{S}\cdot\overleftrightarrow{T} = -\int (d\vec{r})\, \vec{\nabla}\cdot\overleftrightarrow{T} \quad . \tag{10.19}$$

The divergence of the stress tensor is

$$\vec{\nabla}\cdot\overleftrightarrow{T} = (\vec{\nabla}\varepsilon)\,\frac{E^2}{8\pi} + \frac{1}{4\pi}\,D_i\,\vec{\nabla}E_i - \frac{\vec{\nabla}\cdot\vec{D}}{4\pi}\,\vec{E} - \frac{(\vec{D}\cdot\vec{\nabla})\vec{E}}{4\pi} \quad . \tag{10.20}$$

Since for electrostatics,

$$D_i\vec{\nabla}E_i - (\vec{D}\cdot\vec{\nabla})\vec{E} = \vec{D}\times(\vec{\nabla}\times\vec{E}) = 0 \quad ,$$

$\frac{\partial \varepsilon}{\partial r} = F$

$\delta\varepsilon = \delta r \cdot F$

and when no free charge is present,

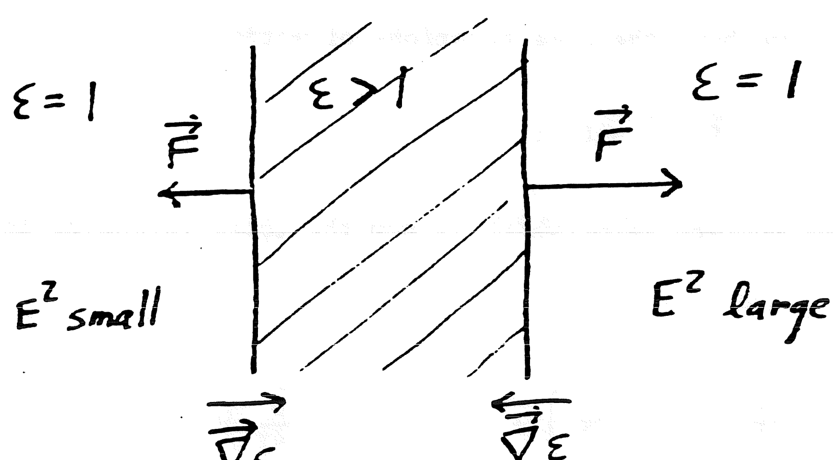
$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho = 0 \quad ,$$

the divergence reduces to

$$\vec{\nabla}\cdot\overleftrightarrow{T} = (\vec{\nabla}\varepsilon)\,\frac{E^2}{8\pi}$$

so that the force calculated from (10.19) is identical with (10.17).

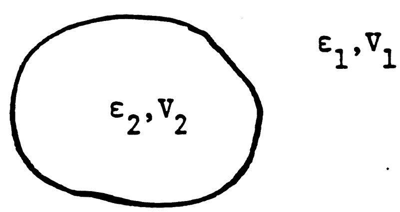
As an application of the above result, consider a slab of dielectric material, with $\varepsilon$ = constant, immersed in an inhomogeneous electric field. The gradient of $\varepsilon$ arises from the discontinuity of the dielectric constant between vacuum and medium. The situation might be described by the diagram below.




If $E^2$ is small on the left and large on the right, the dielectric material will be pulled to the right, that is, into the region of strong field. [Note that this depends on $\varepsilon > 1$ so that the directions of $\vec{\nabla}\varepsilon$ are as shown in the diagram.]

10-3. Boundary Conditions

An arrangement of objects with different values of $\varepsilon$ that are in contact with each other provides the simplest example of an inhomogeneous dielectric constant. Imagine we have two volumes, $V_1$ and $V_2$, with dielectric constants $\varepsilon_1$ and $\varepsilon_2$, respectively, sharing a common surface, as shown in the figure.




Because of this discontinuity of $\varepsilon$ on the surface, when the energy expression (10.4) is varied and expressed in the form (10.5), there is an additional contribution from the surface term previously omitted. In the interior of both $V_1$ and $V_2$, the same arguments as those given in Subsection 10-1 still apply, so that we have the same equations of motion,

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \ ; \quad \vec{\nabla}\times\vec{E} = 0 \ .$$

The additional surface term, which is now the total variation in the energy, is

$$\delta E = \int_{V_1} (d\vec{r})\, \vec{\nabla}\cdot\left(\frac{\vec{D}}{4\pi}\,\delta\phi\right) + \int_{V_2} (d\vec{r})\, \vec{\nabla}\cdot\left(\frac{\vec{D}}{4\pi}\,\delta\phi\right)$$

$$= \int_S dS \left[ \vec{n}_1 \cdot \frac{\vec{D}_1}{4\pi} \delta\phi_1 + \vec{n}_2 \cdot \frac{\vec{D}_2}{4\pi} \delta\phi_2 \right] , \tag{10.22}$$

where $S$ is the common surface and $\vec{n}_1$ and $\vec{n}_2$ are the oppositely directed <u>outward</u> normals for $V_1$ and $V_2$, respectively:

$$\vec{n}_1 = -\vec{n}_2 . \tag{10.23}$$

What must be the connection between $\phi_1$ and $\phi_2$ and between $\vec{D}_1$ and $\vec{D}_2$ ? We insist that $\phi$ must be continuous across the boundary, in order that $\vec{E}(= -\vec{\nabla}\phi)$ exist there, so that the boundary condition that $\phi$ satisfies is

$$\phi_1 = \phi_2 , \qquad \delta\phi_1 = \delta\phi_2 . \tag{10.24}$$

The variation of the energy, (10.22), at the stationary point is therefore

$$\delta E = \int_S dS \frac{1}{4\pi} [\vec{n}_2 \cdot (\vec{D}_2 - \vec{D}_1)] \delta\phi = 0 \tag{10.25}$$

or, since $\delta\phi$ is arbitrary,

$$\vec{n}_2 \cdot (\vec{D}_2 - \vec{D}_1) = 0 . \tag{10.26}$$

The normal component of $\vec{D}$ is continuous across the boundary between the two media, because of our implicit assumption that there is no free surface charge density ($\sigma = 0$). If $\sigma \neq 0$, there is one further term in the variation of the energy (10.4),

$$\int_S dS \, \sigma \, \delta\phi .$$

The resulting generalization of (10.25) is

$$\delta E = \int_S dS \, \frac{1}{4\pi} \, [\vec{n}_2 \cdot (\vec{D}_2 - \vec{D}_1) + 4\pi\sigma] \, \delta\phi = 0 \quad , \tag{10.27}$$

so correspondingly

$$\vec{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = 4\pi\sigma \tag{10.28}$$

is the general boundary condition on the normal component of $\vec{D}$ . We may regard (10.28) as the surface version of the volume statement

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho \quad .$$

Likewise, there is a surface statement corresponding to $\vec{\nabla} \times \vec{E} = 0$ . This follows from the continuity of $\phi$ , (10.24), which implies that of the tangential derivative (that is, the component of the gradient parallel to the surface):

$$(\vec{n}_2 \times \vec{\nabla})(\phi_1 - \phi_2) = 0$$

or

$$\vec{n}_2 \times (\vec{E}_1 - \vec{E}_2) = 0 \quad . \tag{10.29}$$

Equation (10.29) states that the tangential component of the electric field is continuous.

As a special application of the above discussion of dielectric interfaces, we consider the surface of a conductor. Inside the conductor the current density is linearly related to the electric field [see (5.9)]. Since in the static situation there is no flow of charge, $\vec{E} = 0$ everywhere inside the conductor. The continuity of the tangential component of $\vec{E}$ , (10.29), then implies

$$\vec{n} \times \vec{E} = 0 \tag{10.30}$$

on the surface of the conductor. Moreover, since $\vec{E}$ and $\vec{D}$ vanish inside, the other boundary condition, (10.28), implies

$$\vec{n} \cdot \vec{D} = 4\pi\sigma \tag{10.31}$$

just outside the conductor, where $\vec{n}$ is the outward normal to the surface, and again $\sigma$ is the surface charge density.

## XI. Introduction to Green's Functions

In a dielectric medium the differential equation satisfied by the scalar potential $\phi$ in electrostatics is

$$-\vec{\nabla}\cdot[\epsilon\vec{\nabla}\phi] = 4\pi\rho \quad . \tag{11.1}$$

Our task is to find the solution for $\phi$ , for a given charge distribution. Since (11.1) is a linear differential equation relating $\rho$ and $\phi$ , the solution is given as a linear integral expression:

$$\phi(\vec{r}) = \int (d\vec{r}\,')\; G(\vec{r},\vec{r}\,')\; \rho(\vec{r}\,') \quad , \tag{11.2}$$

where we have introduced Green's function, $G(\vec{r},\vec{r}\,')$ . It is evident that this Green's function is the potential at $\vec{r}$ arising from a unit point charge at $\vec{r}\,'$ , so that it satisfies the differential equation

$$-\vec{\nabla}\cdot[\epsilon(\vec{r})\; \vec{\nabla}G(\vec{r},\vec{r}\,')] = 4\pi\delta(\vec{r}-\vec{r}\,') \quad . \tag{11.3}$$

Equation (11.2) expresses the fact that the potential due to a charge distribution is simply the sum of the contributions of each of the charges. Once we have Green's function, the solution for any charge distribution is a matter of integration.

An important property of Green's function is that it satisfies the so-called reciprocity condition,

$$G(\vec{r},\vec{r}\,') = G(\vec{r}\,',\vec{r}) \quad . \tag{11.4}$$

A first derivation of this follows from a consideration of the energy of the system. A convenient expression for the energy can be derived from (10.7) by making the replacement

$\lambda \to \lambda + \delta\lambda$
$\delta E =$

$$\phi \to \lambda\phi$$

where $\phi$ is the physical solution (which can be recovered by letting $\lambda = 1$) and $\lambda$ is a constant. The energy functional, (10.7), evaluated at $\lambda\phi$ is

$$E(\lambda\phi) = \lambda \int (d\vec{r})\ \rho\phi - \lambda^2 \int (d\vec{r})\ \frac{\varepsilon(\vec{\nabla}\phi)^2}{8\pi} \ , \tag{11.5}$$

which is required to be stationary under variations in $\lambda$ about $\lambda = 1$. That is, the derivative of the energy with respect to $\lambda$ at $\lambda = 1$ is zero:

$$\frac{\partial}{\partial\lambda} E(\lambda\phi) = 0 \quad \text{at} \quad \lambda = 1 \ ,$$

or

$$0 = \int (d\vec{r})\ \rho\phi - 2 \int (d\vec{r})\ \frac{\varepsilon(\vec{\nabla}\phi)^2}{8\pi} \ . \tag{11.6}$$

Therefore, the energy of the physical configuration can be written in two alternative forms:

$$E = \int (d\vec{r})\ \varepsilon\, \frac{\vec{E}^2}{8\pi} = \frac{1}{2} \int (d\vec{r})\ \rho\phi \ . \tag{11.7}$$

If we use the second of these expressions [as well as (11.2)]

$$\begin{aligned} E &= \frac{1}{2} \int (d\vec{r})\ \rho(\vec{r})\ \phi(\vec{r}) \\ &= \frac{1}{2} \int (d\vec{r})(d\vec{r}\,')\ \rho(\vec{r})\ G(\vec{r},\vec{r}\,')\ \rho(\vec{r}\,') \\ &= \frac{1}{2} \int (d\vec{r})(d\vec{r}\,')\ \rho(\vec{r})\ G(\vec{r}\,',\vec{r})\ \rho(\vec{r}\,') \end{aligned} \tag{11.8}$$

(where the last form has used $\vec{r} \leftrightarrow \vec{r}\,'$), we see that only the symmetrical part of $G(\vec{r},\vec{r}\,')$ contributes to the energy. This proves (11.4) because the energy completely determines the potential function of an arbitrary charge distribution, and thereby, Green's function. The latter remark follows from the stationary property of the energy functional, as applied to a variation of the charge distribution:

$$\delta E = \int (d\vec{r})\ \delta\rho(\vec{r})\ \phi(\vec{r}) \quad .$$

Another proof of (11.4) can be achieved through the use of the differential equation directly. The equations satisfied by the Green's function due to point charges at $\vec{r}\,'$ and $\vec{r}\,''$ , respectively, are

$$-\vec{\nabla}\cdot[\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,')] = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{11.9a}$$

$$-\vec{\nabla}\cdot[\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,'')] = 4\pi\delta(\vec{r}-\vec{r}\,'') \quad . \tag{11.9b}$$

We now multiply (11.9a) by $G(\vec{r},\vec{r}\,'')$ and (11.9b) by $G(\vec{r},\vec{r}\,')$ , subtract the resulting equations, and then integrate over all space. These manipulations lead to

$$\begin{aligned} &4\pi[G(\vec{r}\,',\vec{r}\,'') - G(\vec{r}\,'',\vec{r}\,')] \\ &\quad = \int (d\vec{r})\ \{G(\vec{r},\vec{r}\,')\ \vec{\nabla}\cdot[\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,'')] - G(\vec{r},\vec{r}\,'')\ \vec{\nabla}\cdot[\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,')]\} \\ &\quad = \int (d\vec{r})\ \{\vec{\nabla}\cdot[G(\vec{r},\vec{r}\,')\ \varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,'') - G(\vec{r},\vec{r}\,'')\ \varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,')] \\ &\qquad - [\vec{\nabla}G(\vec{r},\vec{r}\,')]\cdot\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,'') + [\vec{\nabla}G(\vec{r},\vec{r}\,'')]\cdot\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,')\} = 0 \quad , \end{aligned} \tag{11.10}$$

where the surface term at infinity vanishes for a localized charge distribution. Thus we have again proved the reciprocity relation, (11.4).

Lecture 11

## XII. Electrostatics in Free Space

The simplest electrostatic situation is for the vacuum, $\varepsilon = 1$ . The differential equation for the potential is then Poisson's equation,

$$-\nabla^2 \phi(\vec{r}) = 4\pi\rho(\vec{r}) \quad , \tag{12.1}$$

so that the corresponding Green's function equation is

ind. $-\epsilon\nabla^2 =$ [illegible]

$$-\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad . \tag{12.2}$$

The solution to (12.2) is the well-known Coulomb potential, since it is the potential at $\vec{r}$ produced by a unit point source at $\vec{r}\,'$ :

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} \quad . \tag{12.3}$$

We will now derive (12.3) from the differential equation (12.2). In order to do this, we require an integral representation for the delta function. We recall that the latter is defined by the properties

$$\delta(x-x') = 0 \quad \text{if} \quad x \neq x' \quad , \tag{12.4a}$$

while it is so singular at $x = x'$ that

$$\int_{-\infty}^{\infty} dx \; \delta(x-x') = 1 \quad . \tag{12.4b}$$

Now note that the elementary integral $(\varepsilon > 0)$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}\, e^{-\varepsilon|k|}$$

$$= \int_{0}^{\infty} \frac{dk}{\pi} \cos k(x-x')\, e^{-\varepsilon k}$$

$$= \frac{1}{\pi} \frac{\varepsilon}{(x-x')^2 + \varepsilon^2} = Re\, \frac{i}{\pi} \frac{1}{(x-x') \mp i\varepsilon}$$

becomes, in the limit $\varepsilon \to 0$ ,

$$\frac{1}{\pi} \frac{\varepsilon}{(x-x')^2 + \varepsilon^2} \to \begin{cases} 0\ , & x \neq x'\ , \\ \infty\ , & x = x'\ , \end{cases} \tag{12.5}$$

while

$$\int_{-\infty}^{\infty} d(x-x')\, \frac{1}{\pi} \frac{\varepsilon}{(x-x')^2 + \varepsilon^2} = 1 \quad , \tag{12.6}$$

independent of the non-zero value of $\varepsilon$ . Therefore, an integral representation for the delta function is

$$\delta(x-x') = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}\, e^{-\varepsilon|k|} \quad , \tag{12.7a}$$

or, formally,

$$\delta(x-x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')} \quad , \tag{12.7b}$$

with the reference to $\varepsilon$ and its limiting procedure being left implicit. The corresponding representation for the three-dimensional delta function is

$$\delta(\vec{r}-\vec{r}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$= \int \frac{(d\vec{k})}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \quad . \tag{12.8}$$

We now employ the above representation of the delta function to solve (12.2) for the Green's function. This can be very simply accomplished if we note that

$$\vec{\nabla} e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} = i\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \quad , \tag{12.9}$$

or, effectively,

$$\vec{\nabla} \to i\vec{k} \quad ,$$

so that we can read off the solution to (12.2),

$$G(\vec{r},\vec{r}\,') = 4\pi \int \frac{(d\vec{k})}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{k^2} \quad . \tag{12.10}$$

We can verify that this is in fact the known result, (12.3), by using spherical coordinates for $\vec{k}$, with $\vec{r}-\vec{r}\,'$ pointing along the z-axis,

$$\vec{k}\cdot(\vec{r}-\vec{r}\,') = kR\cos\theta \quad ,$$

$$R = |\vec{r}-\vec{r}\,'| \quad ,$$

$$(d\vec{k}) = k^2 dk\; 2\pi d(\cos\theta) \quad ,$$

so that

$$G(\vec{r},\vec{r}\,') = 4\pi \int \frac{k^2 dk\; 2\pi d(\cos\theta)}{8\pi^3} \frac{e^{ikR\cos\theta}}{k^2}$$

$$= \frac{1}{\pi} \int_0^\infty dk\, \frac{1}{ikR} \left(e^{ikR} - e^{-ikR}\right)$$

$$= \frac{2}{\pi}\frac{1}{R}\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{1}{R} \quad . \tag{12.11}$$

In the above, we have treated all three directions of space on an equal footing. However, we need not do this. We can separate out one direction, say that of $z$, and treat it differently. The reason this is useful is because there are geometries in which physically interesting quantities vary in only a single direction. Singling out the $z$ direction, we can write the Green's function, (12.10), as

$$G(\vec{r},\vec{r}\,') = 4\pi \int \frac{dk_x\, dk_y}{(2\pi)^2}\, e^{i[k_x(x-x') + k_y(y-y')]} \times \int \frac{dk_z}{2\pi} \frac{e^{ik_z(z-z')}}{k_x^2 + k_y^2 + k_z^2} \quad . \tag{12.12}$$

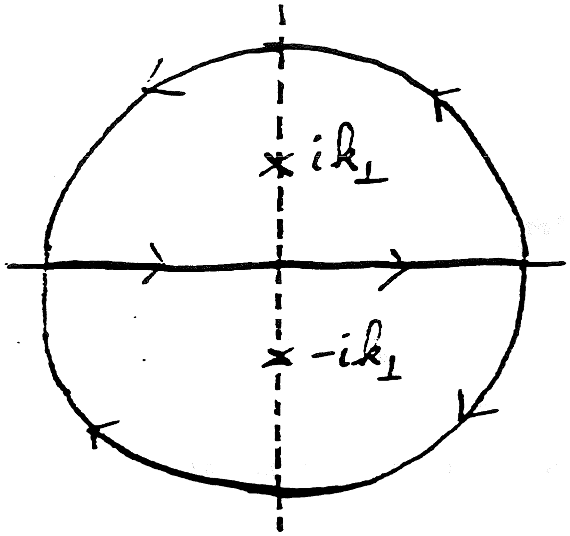
Adopting the nomenclature that the two-dimensional space of $x$ and $y$ is transverse $(\perp)$ to the selected direction, we write the first part of (12.12) as

$$\int \frac{(d\vec{k}_\perp)}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \quad . \tag{12.13}$$

The remaining integration over $k_z$ $(k_\perp^2 = k_x^2 + k_y^2$ and $k_\perp \geq 0)$

$$\int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{e^{ik_z(z-z')}}{k_\perp^2 + k_z^2} = \frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|} \quad , \tag{12.14}$$

is performed by doing a contour integration as indicated:




$$z-z' > 0: \quad 2\pi i \frac{1}{2\pi} \frac{1}{2ik_\perp} e^{-k_\perp(z-z')} ,$$

$$z-z' < 0: \quad -2\pi i \frac{1}{2\pi} \frac{1}{-2ik_\perp} e^{k_\perp(z-z')} .$$

We have therefore recast Green's function into the form

$$G(\vec{r},\vec{r}\,') = 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} g(z,z',k_\perp) \quad , \tag{12.15}$$

where, for free space, the "reduced" Green's function is

$$g(z,z',k_\perp) = \frac{1}{2k_\perp} e^{-k_\perp|z-z'|} \quad . \tag{12.16}$$

The form (12.15) applies to any problem which is translationally invariant in $x$ and $y$ but not necessarily in $z$. The representation is particularly adapted to the situation in which the dielectric constant is only a function of $z$. For the case at hand, where the Green's function satisfies (12.2), we can easily derive the differential equation for $g(z,z',k_\perp)$ , as follows:

$$-\nabla^2 G = 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \left[k_\perp^2 - \frac{\partial^2}{\partial z^2}\right] g(z,z',k_\perp)$$

$$= 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \delta(z-z') \quad , \qquad \leftarrow \text{from 12.10}$$

which implies

$$\left[ k_\perp^2 - \frac{\partial^2}{\partial z^2} \right] g(z,z',k_\perp) = \delta(z-z') \quad . \tag{12.17}$$

This equation is solved by noting that for $z \neq z'$ ,

$$\left[ k_\perp^2 - \frac{\partial^2}{\partial z^2} \right] g(z,z',k_\perp) = 0 \quad , \tag{12.18}$$

$g(z,z',k_\perp) = \frac{1}{2k_\perp} e^{-k_\perp|z-z'|}$

while the discontinuity in the derivative of $g$ at $z = z'$ is

$$- \frac{\partial}{\partial z} g(z,z',k_\perp) \Big|_{z'-0}^{z'+0} = 1 \quad . \tag{12.19}$$

Imposing boundedness at infinity yields the solution (12.16), of course.

# XIII. Half-Infinite Dielectric Slab

## 13-1. Green's Function

We now apply the above representation, (12.15), to find the Green's function for the simplest situation involving a non-uniform dielectric constant, that of a half infinite dielectric slab, which possesses the required translational invariance in $x$ and $y$ :

$$\varepsilon = \text{constant} \quad \Big| \quad \varepsilon = 1 \qquad .$$

$$\xrightarrow{\quad} z \qquad z=0$$

The Green's function equation, (11.3),

$$-\vec{\nabla}\cdot[\varepsilon(z)\ \vec{\nabla}G(\vec{r},\vec{r}\,')] = 4\pi\delta(\vec{r}-\vec{r}\,') \quad ,$$

becomes, in the two regions,

$$z > 0 : \quad -\nabla^2 G = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{13.1a}$$

$$z < 0 : \quad -\varepsilon\nabla^2 G = 4\pi\delta(\vec{r}-\vec{r}\,') \quad . \tag{13.1b}$$

To solve these equations, we must impose appropriate boundary conditions. Across the interface between the vacuum and the medium, the normal component of the displacement vector must be continuous [see (10.26)],

$$\left(-\frac{\partial}{\partial z}\right) G\Big|_{z=+0} = \varepsilon\left(-\frac{\partial}{\partial z}\right) G\Big|_{z=-0} \quad . \tag{13.2}$$

Of course, $G$ must be continuous across the interface because it represents the potential of a point charge. The reduced Green's function, $g(z,z',k_\perp)$ , introduced in (12.15), satisfies the differential equations [see (12.17)]

$$z > 0 : \quad \left( - \frac{\partial^2}{\partial z^2} + k_\perp^2 \right) g = \delta(z-z') \quad , \tag{13.3a}$$

$$z < 0 : \quad \varepsilon \left( - \frac{\partial^2}{\partial z^2} + k_\perp^2 \right) g = \delta(z-z') \quad , \tag{13.3b}$$

subject to the boundary conditions

$$g \Big|_{-0} = g \Big|_{+0} \quad , \tag{13.4a}$$

$$\varepsilon \frac{\partial}{\partial z} g \Big|_{-0} = \frac{\partial}{\partial z} g \Big|_{+0} \quad . \tag{13.4b}$$

In the following we will solve this problem by assuming that $z' > 0$ (that is, the unit charge lies in the vacuum and not in the dielectric). (For the converse situation see Problem ~~14.~~ 7.2) The solutions to (13.3) and (13.4) can be expressed in terms of the solutions, $e^{k_\perp z}$ and $e^{-k_\perp z}$, of the corresponding homogeneous equation. The forms of the solution in the three regions are as follows:

# of boundary conditions
~~3~~ 4
2 each at z=z' and z=0

$$z < 0 : \qquad g = A\, e^{k_\perp z} \quad , \tag{13.5a}$$

$$z > z' : \qquad g = B\, e^{-k_\perp z} \quad , \tag{13.5b}$$

$$0 < z < z' : \quad g = C\, e^{k_\perp z} + D\, e^{-k_\perp z} \quad , \tag{13.5c}$$

where the single exponentials in (13.5a) and (13.5b) are required by the boundary condition that $g$ remain finite for $|z| \to \infty$ . The boundary conditions at $z = 0$ , (13.4), require

$$A = C + D \quad , \tag{13.6a}$$

$$\varepsilon k_\perp A = k_\perp (C-D) \quad , \tag{13.6b}$$

from which we infer

$$C = \frac{\varepsilon+1}{2} A \ , \quad D = -\frac{\varepsilon-1}{2} A \ . \tag{13.6c}$$

Just as in the situation mentioned at the end of the last section [see (12.19)], the singularity in the differential equation requires, at $z = z'$ ,

$$g \Big|_{z=z'-0}^{z=z'+0} = 0 \ , \tag{13.7a}$$

$$-\frac{\partial}{\partial z} g \Big|_{z=z'-0}^{z=z'+0} = 1 \ , \tag{13.8a}$$

which imply, explicitly,

$$B e^{-k_\perp z'} = C e^{k_\perp z'} + D e^{-k_\perp z'} \ , \tag{13.7b}$$

$$k_\perp B e^{-k_\perp z'} + k_\perp (C e^{k_\perp z'} - D e^{-k_\perp z'}) = 1 \ . \tag{13.8b}$$

The solution to this system of equations, (13.6c), (13.7b), and (13.8b), is

$$A = \frac{2}{\varepsilon+1} \frac{1}{2k_\perp} e^{-k_\perp z'} \ , \qquad B = -\frac{\varepsilon-1}{\varepsilon+1} \frac{1}{2k_\perp} e^{-k_\perp z'} + \frac{1}{2k_\perp} e^{k_\perp z'} \ ,$$

$$C = \frac{1}{2k_\perp} e^{-k_\perp z'} \ , \qquad D = -\frac{\varepsilon-1}{\varepsilon+1} \frac{1}{2k_\perp} e^{-k_\perp z'} \ . \tag{13.9}$$

Inserting these coefficients back into (13.5), we find for the reduced Green's function, $g$ , in the two media,

$$z > 0 : \quad g = \frac{1}{2k_\perp} \left[ e^{-k_\perp |z-z'|} - \frac{\varepsilon-1}{\varepsilon+1} e^{-k_\perp (z+z')} \right] \ , \tag{13.10a}$$

$$z < 0 : \quad g = \frac{1}{2k_\perp}\frac{2}{\varepsilon+1} e^{-k_\perp(z'-z)}$$

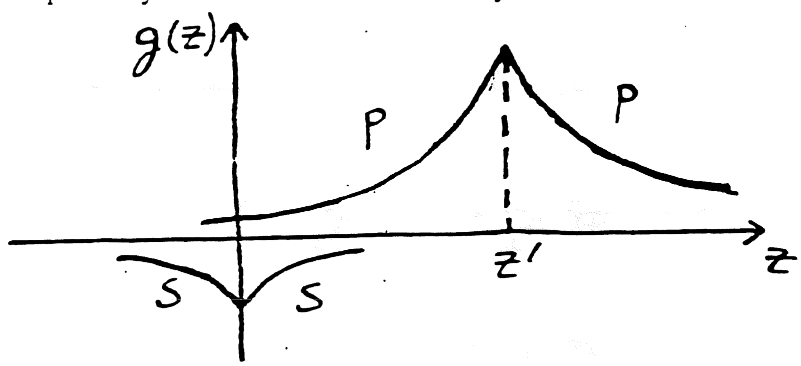
$$= \frac{1}{2k_\perp}\left[ e^{-k_\perp|z-z'|} - \frac{\varepsilon-1}{\varepsilon+1} e^{-k_\perp(z'-z)} \right] . \tag{13.10b}$$

Of course, if we set $\varepsilon = 1$ , we recover the vacuum result, (12.16), as we must.

It is helpful to analyze the Green's function we have found in terms of primary and secondary fields. The primary field results from the point charge at $z'$ and so is represented by the Green's function (12.16). The secondary field is due to the bound charge built up on the interface and is given by

$$g_S = -\frac{\varepsilon-1}{\varepsilon+1}\frac{1}{2k_\perp} e^{-k_\perp z'} e^{-k_\perp|z|} . \tag{13.11}$$

The situation is illustrated by the following picture for $\varepsilon > 1$ (P for primary and S for secondary).




$g = g_P + g_S$

In order to easily identify the full Green's function, we note that we can write (13.10) as a sum of exponentials of the form appearing in (12.16). That is, we write

$$z > 0 : \quad g = \frac{1}{2k_\perp}\left[ e^{-k_\perp|z-z'|} - \frac{\varepsilon-1}{\varepsilon+1} e^{-k_\perp|z-\bar{z}'|} \right] , \tag{13.12a}$$

$$z < 0 : \quad g = \frac{1}{2k_\perp} \frac{2}{\varepsilon+1} e^{-k_\perp |z-z'|} , \tag{13.12b}$$

where we have introduced $\bar{z}'$ , the image point of $z'$ , defined by

$$\bar{z}' = -z' . \tag{13.13}$$

In the two regions, we can interpret (13.12) as follows. For $z > 0$ , the Green's function appears to describe the potential due to two point charges, one, of magnitude unity, at $\vec{r}' = (x',y',z')$, and another, the image charge, of magnitude $-\frac{\varepsilon-1}{\varepsilon+1}$ , at the image point $\bar{\vec{r}}' = (x',y',\bar{z}')$ . For $z < 0$ , only one point charge appears, of magnitude $\frac{2}{\varepsilon+1}$ , located at $\vec{r}'$ . In either medium, the total effective charge is the same. With this interpretation, the full Green's function may be written down immediately,

$$z > 0 : \quad G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{\varepsilon-1}{\varepsilon+1} \frac{1}{|\vec{r}-\bar{\vec{r}}'|} , \tag{13.14a}$$

$$z < 0 : \quad G(\vec{r},\vec{r}') = \frac{2}{\varepsilon+1} \frac{1}{|\vec{r}-\vec{r}'|} . \tag{13.14b}$$

Lecture 12

### 13-2. Force between Charge and Dielectric

Having calculated Green's function, we can now consider the energy of interaction between charges and the dielectric. The total energy, (11.8),

$$E = \frac{1}{2} \int (d\vec{r})(d\vec{r}')\, \rho(\vec{r})\, G(\vec{r},\vec{r}')\, \rho(\vec{r}') , \tag{13.15}$$

includes the mutual interactions of the charges. We are not interested in this but rather in the <u>change</u> of the energy due to the introduction of the dielectric. To calculate this change, we let $G_0$ be Green's function in vacuum, while $G$ is Green's function in the presence of the dielectric, as found above. Therefore, the interaction energy between the charge distribution and the dielectric is

$$E \text{ (due to dielectric)} = \frac{1}{2}\int (d\vec{r})(d\vec{r}\,')\, \rho(\vec{r})\, [G(\vec{r},\vec{r}\,') - G_o(\vec{r},\vec{r}\,')]\, \rho(\vec{r}\,') \quad . \tag{13.16}$$

Evaluating this for a point charge at position $\vec{r}_o$, with $z_o > 0$,

$$\rho(\vec{r}) = e\delta(\vec{r}-\vec{r}_o) \quad ,$$

and making use of (13.14a), we find the energy of interaction to be

$$E = \frac{e^2}{2}[G(\vec{r},\vec{r}\,') - G_o(\vec{r},\vec{r}\,')]_{\vec{r},\vec{r}\,'=\vec{r}_o} = -\frac{e^2}{2}\frac{\varepsilon-1}{\varepsilon+1}\frac{1}{2z_o} \quad . \tag{13.17}$$

Is this a physically meaningful result? If $\varepsilon \geq 1$, $E < 0$ so that there is a force of attraction pulling the dielectric toward the charge and into the region of higher fields, in agreement with the earlier discussion of Subsection 10-2. The magnitude of this force is

$$F = -\frac{\partial E}{\partial(-z_o)} = \frac{\varepsilon-1}{\varepsilon+1}\frac{e^2}{4z_o^2} = \frac{\left|-\frac{\varepsilon-1}{\varepsilon+1}e\right||e|}{(2z_o)^2} \quad , \tag{13.18}$$

which can be interpreted as the force between the charge and the image charge.

The field point of view, as opposed to that of action-at-a-distance, provides an alternate derivation of this result. To calculate the force on the dielectric, we calculate the normal component of the flow of momentum into the dielectric. In terms of the stress tensor, this force is

$$F = -\int dxdy\, T_{zz} \tag{13.19}$$

where the integration is over a surface just outside the dielectric, at $z = +0$. Correspondingly, we use the vacuum form of $T_{zz}$, (3.11),

$$T_{zz} = -\frac{1}{8\pi}\left(E_z^2 - E_x^2 - E_y^2\right) . \tag{13.20}$$

Since Green's function is the potential of a unit point charge, the electric field is

$$\begin{aligned} \vec{E}(\vec{r}) &= -\vec{\nabla} e\, G(\vec{r},\vec{r}_0) \\ &= e\left[\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} - \frac{\varepsilon-1}{\varepsilon+1}\,\frac{\vec{r}-\bar{\vec{r}}_0}{|\vec{r}-\bar{\vec{r}}_0|^3}\right] . \end{aligned} \tag{13.21}$$

If we adopt a coordinate system such that

$$\vec{r}_0 = (0,0,z_0) , \quad \vec{r} = (x,y,0) = (\vec{r}_\perp,0) ,$$

we find

$$\vec{E}_\perp = \frac{2e}{\varepsilon+1}\,\frac{\vec{r}_\perp}{(x^2+y^2+z_0^2)^{3/2}}$$

$$E_z = -\frac{2\varepsilon e}{\varepsilon+1}\,\frac{z_0}{(x^2+y^2+z_0^2)^{3/2}} .$$

Using polar coordinates on the surface,

$$r_\perp^2 = x^2 + y^2 = \rho^2 ,$$

$$dxdy = 2\pi\rho d\rho ,$$

we easily evaluate the force to be

$$F = \frac{e^2}{8\pi}\int_0^\infty 2\pi\ \rho d\rho \left[\frac{4\varepsilon^2}{(\varepsilon+1)^2}\frac{z_o^{\ 2}}{(\rho^2+z_o^{\ 2})^3} - \frac{4}{(\varepsilon+1)^2}\frac{\rho^2}{(\rho^2+z_o^{\ 2})^3}\right]$$

$$= \frac{\varepsilon-1}{\varepsilon+1}\frac{e^2}{4z_o^{\ 2}} \quad , \tag{13.22}$$

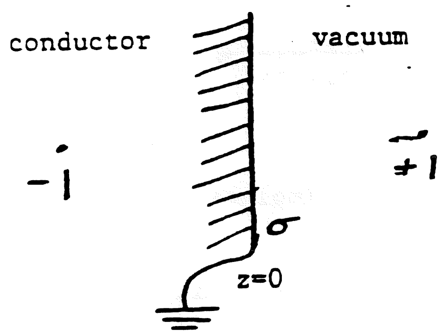
in agreement with (13.18).

## XIV. Infinite Conducting Plate

In the limit that $\varepsilon \to \infty$ , (13.14) becomes

$$G(\vec{r},\vec{r}\,') = \begin{cases} \dfrac{1}{|\vec{r}-\vec{r}\,'|} - \dfrac{1}{|\vec{r}-\bar{\vec{r}}\,'|} , & z > 0 , \\ \\ 0 , & z < 0 , \end{cases} \tag{14.1}$$

which is obviously Green's function for a grounded conductor. For $z > 0$ , Green's function can be interpreted as the potential of a unit point charge at $\vec{r}\,' = (0,0,z')$ and an image charge of strength $-1$ at $\bar{\vec{r}}\,' = (0,0,-z')$ .




For such a unit point charge, we now calculate the surface charge density, $\sigma$ , induced on the conductor. We know, from (10.31), that

$$4\pi\sigma = E_z\Big|_{z=+0} = -\frac{\partial}{\partial z} G\Big|_{z=+0} . \tag{14.2}$$

We have two forms for Green's function:

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{1}{|\vec{r}-\bar{\vec{r}}\,'|}$$

$$= 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \frac{1}{2k_\perp} \left[ e^{-k_\perp|z-z'|} - e^{-k_\perp(z+z')} \right] . \tag{14.3}$$

From the first form, we learn that $(\rho^2 = x^2 + y^2)$

$$4\pi\sigma = -\frac{2z'}{(\rho^2+z'^2)^{3/2}} \quad , \tag{14.4}$$

while from the second form

$$4\pi\sigma = -4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp}\, e^{-k_\perp z'} \quad . \tag{14.5}$$

The equivalence of (14.4) and (14.5) will be exploited in the next Section. For now, let us check that in both cases the total induced charge on the surface of the conductor is $-1$ , the strength of the image charge. The first form, (14.4), yields for the total charge

$$Q = \int_0^\infty 2\pi\, \rho d\rho \left( -\frac{1}{2\pi} \right) \frac{z'}{(\rho^2+z'^2)^{3/2}} = -1 \quad , \tag{14.6}$$

while the second form, (14.5), implies

$$\begin{aligned} Q &= -\int dxdy \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\, e^{ik_x(x-x')}\, e^{ik_y(y-y')}\, e^{-\sqrt{k_x^2+k_y^2}\,z'} \\ &= -\int \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\, 2\pi\,\delta(k_x)\, 2\pi\,\delta(k_y)\, e^{-\sqrt{k_x^2+k_y^2}\,z'} \\ &= -1 \quad . \end{aligned} \tag{14.7}$$

In deriving (14.7), we have used the fact that [cf. (12.7b)], for example,

$$\int_{-\infty}^{\infty} d(x-x')\, e^{ik_x(x-x')} = 2\pi\,\delta(k_x) \quad . \tag{14.8}$$

## XV. Bessel Functions

Useful mathematical identities can be obtained if we solve a physical problem by using different representations for the Green's function. In particular, through the consideration of situations where physical quantities vary only in a single direction, we learn of the properties of the important class of functions called Bessel functions. An illustration of this was encountered in the last section, where we obtained two forms for the surface charge density, (14.4) and (14.5), the identity of which may be written as (for $z > 0$)

$$-\frac{2z}{(\rho^2+z^2)^{3/2}} = -4\pi \int \frac{k\,dk\,d\phi}{(2\pi)^2}\, e^{ik\rho\cos\phi}\, e^{-kz} \quad , \tag{15.1}$$

where $\rho = |(\vec{r}-\vec{r}\,')_\perp|$ and we have introduced polar coordinates for $\vec{k}_\perp$. The $\phi$ integral here is defined as the Bessel function of zeroth order, $J_0$,

$$\int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{ik\rho\cos\phi} = J_0(k\rho) \quad , \tag{15.2}$$

in terms of which (15.1) may be expressed as the following relation satisfied by $J_0$.

$$\frac{z}{(\rho^2+z^2)^{3/2}} = \int_0^\infty k\,dk\, J_0(k\rho)\, e^{-kz} \quad . \tag{15.3}$$

Since this result was obtained by equating $z$ derivatives [cf. (14.2)], it may be immediately integrated to yield

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty dk\, J_0(k\rho)\, e^{-k|z|} \quad . \tag{15.4}$$

(The integration constant vanishes since both sides go to zero as $|z| \to \infty$ .) Note that (15.4) may be directly derived by equating the two Green's functions (12.15,16) and (12.3). Here we recognize two different forms for the potential of a unit point charge at the origin, which satisfies Laplace's equation,

$$\nabla^2 \phi = 0 \quad , \tag{15.5}$$

except at the origin. Actually, the integrand of the right hand side of (15.4) is also a solution of Laplace's equation, that is

$$\nabla^2 [J_0(k\rho)\, e^{-k|z|}] = 0 \quad , \tag{15.6}$$

for $\rho \neq 0,\ z \neq 0$ . To prove this, it is convenient to express the Laplacian in cylindrical coordinates,

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \quad , \tag{15.7}$$

leading to the reduced form for (15.6) ,

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + k^2\right] J_0(k\rho) = 0 \quad , \tag{15.8}$$

or, with $t = k\rho$ ,

$$\left[\frac{1}{t}\frac{d}{dt}\left(t\frac{d}{dt}\right) + 1\right] J_0(t) = 0 \quad . \tag{15.9}$$

Inserting here the integral representation of the Bessel function, (15.2), and multiplying by $t$ , we have

$$\int_0^{2\pi} \frac{d\phi}{2\pi}\, [-t\cos^2\phi + i\cos\phi + t]\, e^{it\cos\phi}$$

$$= \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{d}{d\phi} [i \sin\phi \, e^{it \cos\phi}] = 0 \quad , \tag{15.10}$$

thereby proving (15.6), and establishing the equation satisfied by the Bessel function of order zero, (15.9).

Looking back at (12.15) and (12.16), we see that the basic solution to Laplace's equation that we have here encountered is

$$e^{i\vec{k}_\perp \cdot \vec{r}_\perp} e^{-k_\perp |z|} \quad . \tag{15.11}$$

We integrated this over $\phi$ to obtain $J_0(k\rho)\, e^{-k|z|}$, but even without such integration this satisfies Laplace's equation for $z \neq 0$. This is obvious since we have the effective replacements

$$\nabla_\perp^2 \to -k_\perp^2 \, , \quad \left(\frac{\partial}{\partial z}\right)^2 \to k_\perp^2 \quad .$$

These basic solutions, unlike those involving $J_0$, are not cylindrically symmetric. Using polar coordinates,

$$\vec{k}_\perp \cdot \vec{r}_\perp = k\rho \cos\phi \quad , \tag{15.12}$$

and defining

$$k\rho = t \quad , \tag{15.13}$$

$$e^{i\phi} = \frac{1}{i} u \quad , \tag{15.14}$$

we employ the first exponential in (15.11) as a generating function for Bessel functions of integer order, $J_m$,

$$e^{ik\rho\cos\phi} = e^{\frac{1}{2}t\left(u-\frac{1}{u}\right)} = \sum_{m=-\infty}^{\infty} u^m J_m(t) \quad . \tag{15.15}$$

Since the generating function is invariant under the substitution

$$u \to -\frac{1}{u} \quad ,$$

that is,

$$\sum_{-\infty}^{\infty} u^m J_m(t) = \sum_{-\infty}^{\infty} (-1)^m u^{-m} J_m(t) = \sum_{-\infty}^{\infty} (-1)^m u^m J_{-m}(t) \quad ,$$

we learn that Bessel functions of positive and negative orders are related by

$$J_{-m}(t) = (-1)^m J_m(t) \quad . \tag{15.16}$$

By writing (15.15) in terms of $\phi$ ,

$$e^{ik\rho\cos\phi} = \sum_{-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho) \quad , \tag{15.17}$$

and using the orthogonality condition

$$\int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{-im'\phi}\, e^{im\phi} = \delta_{mm'} \quad , \tag{15.18}$$

we obtain an integral representation for $J_m$ ,

$$i^m J_m(k\rho) = \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{i(k\rho\cos\phi - m\phi)} \quad , \tag{15.19}$$

$m \to -m$

$i^{-m} J_{-m}(k\rho) = -(\quad) = -i^m J_m$

which contains the result for $J_0$ , (15.2). Since this is the Fourier coefficient of $e^{ik\rho\cos\phi}$, we see that another set of solutions to Laplace's equation is

$$e^{im\phi}\, J_m(k\rho)\, e^{-k|z|} \quad . \tag{15.20}$$

The Laplacian, (15.7), acting on this solution yields the differential equation satisfied by $J_m$,

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) - \frac{m^2}{\rho^2} + k^2\right] J_m(k\rho) = 0 \quad , \tag{15.21}$$

or

$$\left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{m^2}{t^2} + 1\right] J_m(t) = 0 \quad . \tag{15.22}$$

## Lecture 13

With an eye towards developing further properties of Bessel functions, we make some additional remarks about delta functions. We first recall the integral representation for the three-dimensional delta function, (12.8),

$$\int \frac{(d\vec{k})}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} = \delta(\vec{r}-\vec{r}\,') \quad ,$$

which has evident analogs in two and one dimensions. The defining property of the delta function,

$$f(\vec{r}) = \int (d\vec{r}\,')\, \delta(\vec{r}-\vec{r}\,')\, f(\vec{r}\,') \quad ,$$

becomes, in this representation,

$$f(\vec{r}) = \int \frac{(d\vec{k})}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \int (d\vec{r}\,') \, e^{-i\vec{k}\cdot\vec{r}\,'} f(\vec{r}\,') \quad . \tag{15.23}$$

This equation has the form of a <u>completeness statement</u>, that is, any function can be constructed from a linear combination of basic functions. Here these functions are $e^{i\vec{k}\cdot\vec{r}}$, with the expansion coefficients as indicated in (15.23).

The two-dimensional analog of (12.8) is

$$\int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} = \delta((\vec{r}-\vec{r}\,')_\perp) = \frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\phi-\phi') \quad , \tag{15.24}$$

where we have introduced polar coordinates $\vec{r}_\perp(\rho,\phi)$ and $\vec{r}_\perp{}'(\rho',\phi')$ . Correspondingly, if we use polar coordinates for $\vec{k}_\perp(k,\alpha)$ , (15.24) becomes

$$\int \frac{k\,dk\,d\alpha}{(2\pi)^2} e^{ik\rho\cos(\phi-\alpha)} e^{-ik\rho'\cos(\phi'-\alpha)} = \frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\phi-\phi') \quad . \tag{15.25}$$

We next expand the exponentials here by use of the generating function expression, (15.17) [with $\phi \to \phi-\alpha$] together with its complex conjugate [with $\phi \to \phi'-\alpha$], and perform the $\alpha$ integration by means of (15.18). The result of these operations is the identity

$$\int_0^\infty \frac{k\,dk}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho)\, e^{-im\phi'} J_m(k\rho') = \frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\phi-\phi') \quad . \tag{15.26}$$

Again this is a completeness statement, this time for the functions

$e^{im\phi} J_m(k\rho)$ :

$$f(\rho,\phi) = \int_0^\infty \frac{k\,dk}{2\pi} \sum_{-\infty}^{\infty} e^{im\phi} J_m(k\rho) \int \rho'd\rho'\,d\phi'\, e^{-im\phi'} J_m(k\rho')\, f(\rho',\phi') \quad . \tag{15.27}$$

We may now easily isolate the individual $\rho$ and $\phi$ dependencies of (15.26).

If we multiply (15.26) by $e^{-im(\phi-\phi')}$ and integrate over $\phi$, we select a particular value of $m$, according to (15.18), so that we obtain

$$\int_0^\infty kdk\, J_m(k\rho)\, J_m(k\rho') = \frac{1}{\rho}\,\delta(\rho-\rho') \quad . \tag{15.28a}$$

This states that $J_m(k\rho)$ is a complete set of functions of $\rho$ for <u>any</u> $m$ :

$$f(\rho) = \int_0^\infty kdk\, J_m(k\rho) \int_0^\infty \rho' d\rho'\, J_m(k\rho')\, f(\rho') \quad . \tag{15.28b}$$

Putting this information back into (15.26), we determine the completeness relation for the functions $e^{im\phi}$ :

$$\sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\, e^{im(\phi-\phi')} = \delta(\phi-\phi') \quad , \tag{15.29a}$$

and correspondingly

$$f(\phi) = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^{2\pi} \frac{d\phi'}{2\pi}\, e^{-im\phi'}\, f(\phi') \quad , \tag{15.29b}$$

which is the Fourier series expansion for $f(\phi)$ .

A further property of the Bessel functions may be obtained by expanding $e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}$ using (15.17), and integrating over $\alpha$ [as we did in going from (15.25) to (15.26)]:

$$\int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp} = \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho)\, e^{-im\phi'} J_m(k\rho') \quad . \tag{15.30}$$

On the other hand, from

$$\vec{k}_\perp \cdot (\vec{r}-\vec{r}\,')_\perp = k|(\vec{r}-\vec{r}\,')_\perp| \cos(\alpha-\Phi)$$

where $\Phi$ is the polar angle for the vector $(\vec{r}-\vec{r}\,')_\perp$, the integral on the left-hand side of (15.30) [with the shift of variable $\alpha \to \alpha + \Phi$] is immediately identified as the Bessel function of zeroth order, (15.2),

$$\int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{ik|(\vec{r}-\vec{r}\,')_\perp| \cos\alpha} = J_0(k|(\vec{r}-\vec{r}\,')_\perp|) \quad , \tag{15.31}$$

where

$$|(\vec{r}-\vec{r}\,')_\perp| = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi-\phi')} \quad . \tag{15.32}$$

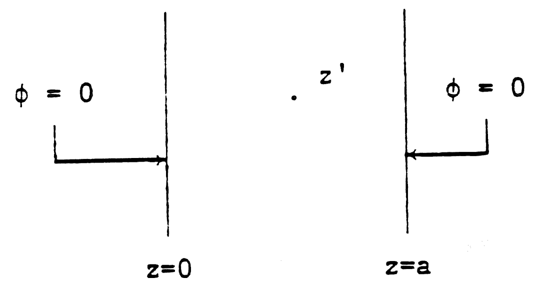
Therefore we have derived the <u>addition theorem</u> for the Bessel functions of integer order:

$$\begin{aligned} &J_0[k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi-\phi')}] \\ &\quad = \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho)\, e^{-im\phi'} J_m(k\rho') \quad . \end{aligned} \tag{15.33}$$

# XVI. Parallel Conducting Plates

## 16-1. Reduced Green's Function

Having developed some mathematical machinery, let us now turn to another essentially one-dimensional problem, that of the potential due to a point charge between two parallel grounded conducting plates, as illustrated in the figure.




The Green's function is defined by the differential equation

$$-\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{16.1a}$$

together with the boundary conditions

$$G = 0 \quad \text{at} \quad z = 0,a \quad . \tag{16.1b}$$

Since the geometry depends only on the z coordinate, the Green's function can be written in the form (12.15), where the reduced Green's function $g(z,z',k_\perp)$ satisfies (12.17). As in (13.5) the solutions are still of the form $e^{\pm kz}$ (with $k = |\vec{k}_\perp|$ ); the linear combinations that satisfy the boundary conditions are

$$z < z' : \quad g = A \sinh kz \quad , \tag{16.2a}$$

$$z > z' : \quad g = B \sinh k(a-z) \quad . \tag{16.2b}$$

The constants $A$ and $B$ are to be determined by the conditions on $g$ in the neighborhood of $z = z'$ [recall (13.7a) and (13.8a)],

$g$ continuous,

$$-\frac{\partial}{\partial z} g \Big|_{z=z'-0}^{z=z'+0} = 1 \quad ,$$

which leads to the equations

$$A \sinh kz' = B \sinh k(a-z') \quad , \tag{16.3a}$$

$$kB \cosh k(a-z') + kA \cosh kz' = 1 \quad . \tag{16.3b}$$

It is convenient to satisfy (16.3a) by letting

$$A = C \sinh k(a-z') \quad ,$$
$$B = C \sinh kz' \quad , \tag{16.4}$$

which, when substituted into (16.3b) yields

$$kC \sinh ka = 1 \quad . \tag{16.5}$$

The reduced Green's function is thus found to be

$$g(z,z',k) = \frac{\sinh kz_< \sinh k(a-z_>)}{k \sinh ka} \quad , \tag{16.6}$$

where $z_>(z_<)$ is the greater (lesser) of $z$ and $z'$. Note that the reciprocity condition, (11.4), is satisfied because $g(z,z',k)$ is symmetrical.

## 16-2. Induced Charge

One application of this result lies in calculating the charge densities on the conducting plates induced by a unit point charge at $z'$. According to (10.31), these charge densities are

$$z = 0 : \quad 4\pi\sigma = E_z = -\frac{\partial}{\partial z} G \quad , \tag{16.7a}$$

$$z = a : \quad 4\pi\sigma = -E_z = \frac{\partial}{\partial z} G \quad . \tag{16.7b}$$

As in Section XIV, it is simplest to calculate the total charge induced on each plate:

$$Q(z=0) = -\frac{1}{4\pi} 4\pi \int dxdy \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp \cdot (\vec{r}-\vec{r}\,')_\perp} \frac{\partial}{\partial z} g(z,z',k)\Big|_{z=0} \quad , \tag{16.8a}$$

$$Q(z=a) = \frac{1}{4\pi} 4\pi \int dxdy \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp \cdot (\vec{r}-\vec{r}\,')_\perp} \frac{\partial}{\partial z} g(z,z',k)\Big|_{z=a} \quad . \tag{16.8b}$$

We have seen such integrals previously [see (14.7) and (14.8)]. The spatial integrations over $x$ and $y$ yield $(2\pi)^2 \delta(\vec{k}_\perp)$ while the subsequent $\vec{k}_\perp$ integration sets $k = 0$ so that

$$g(z,z',0) = \frac{1}{a} z_<(a-z_>) \quad . \tag{16.9}$$

The total induced charges are therefore

$$Q(z=0) = -\left(1 - \frac{z'}{a}\right) \quad , \tag{16.10a}$$

$$Q(z=a) = -\frac{z'}{a} \quad , \tag{16.10b}$$

with the total induced charge on both plates being $-1$, of course.

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## 16-3. Eigenfunction Expansion

We now wish to investigate the properties of $g(z,z',k)$, (16.6), considered as a function of a complex variable $k$. First notice that $g$ is even in $k$ and is finite at $k = 0$. The behavior of $g$ when the real part of $k$ is large and positive is

$$g \sim \frac{\frac{1}{2} e^{kz_<} \frac{1}{2} e^{k(a-z_>)}}{k \frac{1}{2} e^{ka}} = \frac{1}{2k} e^{-k(z_>-z_<)} = \frac{1}{2k} e^{-k|z-z'|} .$$

This limiting form is the reduced Green's function for empty space, (12.16), which is evidently bounded (in fact, goes to zero) as $\mathrm{Re}\, k \to \infty$. The singularities of $g(z,z',k)$ occur on the imaginary axis where

$$\sinh ka = 0 , \tag{16.11}$$

that is, at the points where

$$ka = in\pi , \qquad n = \pm 1, \pm 2, \ldots . \tag{16.12}$$

The behavior of $g$ in the neighborhood of this singularity is

$$g \sim \frac{i \sin \frac{n\pi}{a} z_< \; i \sin \frac{n\pi}{a} (a-z_>)}{i \frac{n\pi}{a} \left( k - i \frac{n\pi}{a} \right) a \cos n\pi}$$

$$= \frac{1}{n\pi} \frac{\sin \frac{n\pi}{a} z \sin \frac{n\pi}{a} z'}{ik + \frac{n\pi}{a}} , \tag{16.13}$$

that is, $g$ possesses a simple pole there. Therefore, since apart from these poles, $g$ is a bounded, analytic function of $k$, it is expressible entirely as a sum over the pole contributions given in (16.13),

$$g(z,z',k) = \sum_{n\neq 0} \frac{1}{n\pi} \frac{\sin\frac{n\pi}{a}z \sin\frac{n\pi}{a}z'}{\frac{n\pi}{a} + ik} \quad .$$

Combining together the contributions from $n$ and $-n$, we obtain an alternative representation for the reduced Green's function,

$$g(z,z',k) = \sum_{n=1}^{\infty} \frac{2}{a} \frac{\sin\frac{n\pi}{a}z \sin\frac{n\pi}{a}z'}{k^2 + \left(\frac{n\pi}{a}\right)^2} \quad . \tag{16.14}$$

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To verify explicitly that (16.14) is the Green's function, we must check that it is a solution of the differential equation,

$$\left(-\frac{\partial^2}{\partial z^2} + k^2\right) g(z,z',k) = \delta(z-z') \quad , \tag{16.15}$$

satisfying the boundary conditions,

$$g = 0 \quad \text{at} \quad z = 0,a \quad .$$

The latter conditions are trivially satisfied by (16.14), while the differential equation (16.15) will be confirmed if

$$\frac{2}{a}\sum_{n=1}^{\infty} \sin\frac{n\pi}{a}z \sin\frac{n\pi}{a}z' = \delta(z-z') \quad . \tag{16.16}$$

The correctness of this follows from the completeness relation (15.29):

$$\frac{2}{a}\sum_{n=1}^{\infty} \sin\frac{n\pi}{a}z \sin\frac{n\pi}{a}z' = \frac{1}{a}\sum_{n=-\infty}^{\infty} \sin\frac{n\pi}{a}z \sin\frac{n\pi}{a}z'$$

$$= \frac{1}{2a}\sum_{-\infty}^{\infty}\left[\cos\frac{n\pi}{a}(z-z') - \cos\frac{n\pi}{a}(z+z')\right]$$

$$= \frac{1}{2a} \sum_{-\infty}^{\infty} \left[ e^{i \frac{n\pi}{a}(z-z')} - e^{i \frac{n\pi}{a}(z+z')} \right]$$

$$= \frac{\pi}{a} \left[ \delta\left( \frac{\pi}{a}(z-z') \right) - \delta\left( \frac{\pi}{a}(z+z') \right) \right] = \delta(z-z') \;, \qquad 0 < z,z' < a \;,$$

where we have used the elementary fact that

$$\lambda\delta(\lambda x) = \delta(x) \;, \quad \lambda > 0 \;. \tag{16.17}$$

Notice that (16.16) is a discrete analogue of the continuum representation of the delta function, (12.7b),

$$\int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} = \delta(z-z') \;.$$

We have here another completeness statement, that any function, $f(z)$, defined on the interval $0$ to $a$, and vanishing on the boundaries, can be constructed in terms of $\sin \frac{n\pi}{a} z$.

## 16-4. Green's Function

An eigenfunction expansion for Green's function, $G(\vec{r},\vec{r}')$, can now be obtained by substituting the corresponding form for the reduced Green's function, (16.14), into (12.15). The Bessel function of zeroth order, $J_0$, (15.31), is introduced upon performing the angular integration associated with $\vec{k}_\perp$, so that Green's function becomes

$$G(\vec{r},\vec{r}') = 2 \int_0^{\infty} kdk\, J_0(kD)\, \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{a} z \sin \frac{n\pi}{a} z'}{k^2 + \left(\frac{n\pi}{a}\right)^2} \;, \tag{16.18}$$

where

$$D = |(\vec{r}-\vec{r}')_{\perp}| = [\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi-\phi')]^{1/2} \quad .$$

A complete eigenfunction decomposition of $G$ can be obtained by using the addition theorem (15.33). In (16.18), we encounter a new type of Bessel function, $K_0$, the modified Bessel function of zeroth order, defined by the integral representation

$$K_0(\kappa\rho) = \int_0^\infty kdk \frac{J_0(k\rho)}{k^2+\kappa^2} \quad . \tag{16.19}$$

In terms of this new function, the Green's function in the region between two parallel plates is

$$G(\vec{r},\vec{r}') = \frac{4}{a} \sum_{n=1}^{\infty} \sin\frac{n\pi}{a} z \sin\frac{n\pi}{a} z' K_0\left(\frac{n\pi}{a} D\right) \quad . \tag{16.20}$$

This form becomes particularly simple in the limit when $D >> a$. The modified Bessel function, $K_0$, can be shown to approach the asymptotic expression [see problem 24]

$$K_0(t) \sim \sqrt{\frac{\pi}{2t}}\, e^{-t} \quad , \qquad t \to \infty \quad . \tag{16.21}$$

Due to this exponentially decreasing behavior, in the limit when the transverse distance between the points is large compared to the separation of the plates, the first term of (16.20) dominates:

$$D >> a \; , \qquad G \sim \sqrt{\frac{8}{Da}} \sin\frac{\pi}{a} z \sin\frac{\pi}{a} z' \, e^{-\frac{\pi}{a} D} \quad . \tag{16.22}$$

Notice that we can do the same thing for the Coulomb Green's function (12.12), which applies to free space without conductors or dielectrics:

$$G(\vec{r},\vec{r}\,') = 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{e^{ik_z(z-z')}}{k_\perp^2 + k_z^2}$$

$$= \frac{2}{\pi} \int kdk\, J_0(kD) \int_0^\infty dk_z \frac{\cos k_z(z-z')}{k^2 + k_z^2}$$

$$= \frac{2}{\pi} \int_0^\infty dk_z \cos k_z(z-z')\, K_0(k_z D) \quad . \tag{16.23}$$

In particular, if in (16.23) we take $\vec{r}\,' = 0$ and use $\rho$ to denote $|\vec{r}_\perp|$, we learn that [cf. (15.4)]

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty dk\, J_0(k\rho)\, e^{-k|z|} = \frac{2}{\pi} \int_0^\infty dk \cos kz\, K_0(k\rho) \quad . \tag{16.24}$$

Observe that the $J_0$ form has exponential damping in the $z$ direction but oscillations in the $\rho$ direction, while the $K_0$ form exhibits the reverse behavior, damping in $\rho$ and oscillations in $z$. Since $1/r$ is a solution to Laplace's equation, for $r \neq 0$, it is impossible to get damping in all directions (since then the second derivatives would be of the same sign and could not sum to zero).

# XVII. Spherical Harmonics

## 17-1. Solutions to Laplace's Equation

The fundamental solution to Laplace's equation in unbounded space is $1/r$ ,

$$\nabla^2 \frac{1}{r} = 0 \quad , \qquad r > 0 \quad . \tag{17.1}$$

In terms of this solution, we can generate a large number of others. For example, taking $\vec{a}$ to be a constant vector,

$$\nabla^2 \, (\vec{a}\cdot\vec{\nabla}) \, \frac{1}{r} = 0 \quad , \tag{17.2}$$

$\nabla^2 \; (\vec{a}\cdot\vec{\nabla})\, f(r) =$

$\partial_i \, A_m B_m$

$\partial_i \, A_m B_m +$

$B_m \partial_i A_m$

$\nabla(\nabla\cdot$

$\nabla\cdot(\nabla(\vec{A}\cdot\vec{B})) =$

$\nabla\cdot$

we find

$$\vec{a}\cdot\vec{\nabla}\,\frac{1}{r} = -\,\frac{\vec{a}\cdot\vec{r}}{r^3} \tag{17.3}$$

$\partial_i^2 \; (\vec{A}\cdot\vec{B})$

$= \partial_i^2 \, A_m B_m$

$= A_m \partial_i^2 B_m$

is also a solution for $r \neq 0$ . Continuing this operation, we see that

$$(\vec{a}_1\cdot\vec{\nabla})(\vec{a}_2\cdot\vec{\nabla})\,\frac{1}{r} = \frac{3(\vec{a}_1\cdot\vec{r})(\vec{a}_2\cdot\vec{r}) - (\vec{a}_1\cdot\vec{a}_2)r^2}{r^5} \tag{17.4}$$

is yet a third solution. This process can be repeated an indefinite number of times, to yield the following solution to Laplace's equation,

$$[\vec{a}_1\cdot\vec{\nabla}\;\vec{a}_2\cdot\vec{\nabla}\;\ldots\;\vec{a}_\ell\cdot\vec{\nabla}]\,\frac{1}{r} = \frac{1}{r^{2\ell+1}}\,f_\ell(\vec{r}) \quad , \tag{17.5}$$

where $f_\ell(\vec{r})$ is a homogeneous function of $\vec{r}$ of degree $\ell$ . We also observe that $f_\ell(\vec{r})$ itself is a solution to Laplace's equation. [This follows from the inversion theorem, that if $\phi(\vec{r})$ is a solution to Laplace's equation, so is $\frac{1}{r}\,\phi\left(\frac{\vec{r}}{r^2}\right)$ . See problem 22.]

Thus, our attention is directed to solutions that are homogeneous polynomials of degree $\ell$ . The above construction provides examples for $\ell = 0,1,2$ . (Here we denote $(x,y,z)$ by $(x_1,x_2,x_3)$.)

| $\ell$ | $f_\ell$ | number of independent solutions |
|---|---|---|
| 0 | 1 | 1 |
| 1 | $x_1,x_2,x_3$ | 3 |
| 2 | $3x_m x_n - \delta_{mn} r^2$ | 5 |

Why are there only five independent solutions for $\ell = 2$ ? A symmetrical tensor has six independent components but because of the constraint that the tensor satisfies Laplace's equation, it must be traceless, leaving but five independent components.

The general polynomial of degree $\ell$ can be constructed from the monomials

$$x_1^{k_1}\, x_2^{k_2}\, x_3^{k_3} \;, \qquad k_1 + k_2 + k_3 = \ell \;.$$

How many of these monomials are there? To answer this, we first ask the analogous question in two dimensions: how many monomials of the form

$$x_1^{k_1}\, x_2^{k_2}$$

are there with $k_1 + k_2 = n$ ? The answer to this question is simple since if $k_1$ goes from 0 to $n$ , $k_2$ must go from $n$ to 0 , giving $n+1$ possibilities

Thus to answer our three-dimensional question, we first assign a definite value to $k_3$,

$$k_1 + k_2 = \ell - k_3 \quad .$$

The number of monomials with this value of $k_3$ is

$$\ell - k_3 + 1 \quad ,$$

so the number of homogeneous polynomials of degree $\ell$ is

$$\sum_{k_3=0}^{\ell} (\ell - k_3 + 1) = \frac{1}{2}(\ell+1)(\ell+2) \quad .$$

From this set of polynomials, we wish to find those combinations which are solutions to Laplace's equation. Since $\nabla^2$ acting on a homogeneous polynomial of degree $\ell$ produces a homogeneous polynomial of degree $\ell-2$, of which there are

$$\frac{1}{2}(\ell-2+1)(\ell-2+2) = \frac{1}{2}\ell(\ell-1)$$

independent ones, there are $\frac{1}{2}\ell(\ell-1)$ restrictions on the polynomials, that is, the number of independent solutions to Laplace's equation of degree $\ell$ is

$$\frac{1}{2}(\ell+1)(\ell+2) - \frac{1}{2}\ell(\ell-1) = 2\ell+1 \quad .$$

For the cases $\ell = 0,1,2$, this agrees with what we found above. The solutions we find in this way are called <u>solid harmonics</u>, $Y_\ell(\vec{r})$. To emphasize the fact that they are homogeneous polynomials of degree $\ell$, the solid harmonic may be written in terms of a surface (or spherical) harmonic, $Y_\ell\left(\frac{\vec{r}}{r}\right)$ :

$$Y_\ell(\vec{r}) = r^\ell Y_\ell\left(\frac{\vec{r}}{r}\right) \tag{17.6a}$$

$$\to \frac{1}{r^{\ell+1}} Y_\ell\left(\frac{\vec{r}}{r}\right) \tag{17.6b}$$

where the latter form, also a solid harmonic, results from inversion and is the solution constructed in (17.5).

## 17-2. Spherical Harmonics

Our next task is to devise a way to systematically and conveniently generate the spherical harmonics as functions of the spherical angles $\theta$ and $\phi$. We first ask under what condition is the polynomial,

$$(\vec{a}\cdot\vec{r})^\ell , \tag{17.7}$$

a solution to Laplace's equation? Since

$$\vec{\nabla}(\vec{a}\cdot\vec{r})^\ell = \ell(\vec{a}\cdot\vec{r})^{\ell-1}\,\vec{a} ,$$

we see that the Laplacian acting on this polynomial is

$$\nabla^2(\vec{a}\cdot\vec{r})^\ell = \ell(\ell-1)(\vec{a}\cdot\vec{r})^{\ell-2}\,\vec{a}^2 ,$$

which is Laplace's equation if $\vec{a}^2$ is zero (necessarily, $\vec{a}$ must then be complex). A convenient way to write $\vec{a}^2$ is

$$\vec{a}^2 = (a_1 - ia_2)(a_1 + ia_2) + a_3^2 , \tag{17.8}$$

suggesting that the condition that $\vec{a}^2$ be zero can be automatically satisfied if we write

$$a_1 + ia_2 = 2\xi_-^{\,2} \ ,$$

$$a_1 - ia_2 = -2\xi_+^{\,2} \ ,$$

$$a_3 = 2\xi_+\xi_- \ , \qquad (17.9)$$

where $\xi_\pm$ are two arbitrary complex numbers. Then we have

$$\vec{a}\cdot\frac{\vec{r}}{r} = \frac{1}{2}(a_1-ia_2)\frac{x+iy}{r} + \frac{1}{2}(a_1+ia_2)\frac{x-iy}{r} + a_3\frac{z}{r}$$

$$= -\xi_+^{\,2}\sin\theta\, e^{i\phi} + \xi_-^{\,2}\sin\theta\, e^{-i\phi} + 2\xi_+\xi_-\cos\theta \ , \qquad (17.10)$$

and the polynomial (17.7),

$$(\vec{a}\cdot\vec{r})^\ell = r^\ell\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^\ell \ ,$$

can be rewritten in terms of $\xi_\pm$ as

$$\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^\ell = \left(\frac{\xi_+^{\,2}\, e^{i\phi}}{\sin\theta}\right)^\ell \left[\left(\frac{\xi_-}{\xi_+}\sin\theta\, e^{-i\phi}\right)^2 + 2\frac{\xi_-}{\xi_+}\sin\theta\, e^{-i\phi}\cos\theta - \sin^2\theta\right]^\ell$$

$$= \left(\frac{\xi_+^{\,2}\, e^{i\phi}}{\sin\theta}\right)^\ell \left[\left(\frac{\xi_-}{\xi_+}\sin\theta\, e^{-i\phi} + \cos\theta\right)^2 - 1\right]^\ell$$

$$= \frac{\xi_+^{\,2\ell}\, e^{i\phi\ell}}{\sin^\ell\theta}\sum_{m=-\ell}^{\ell}\frac{\left(\frac{\xi_-}{\xi_+}\sin\theta\, e^{-i\phi}\right)^{\ell-m}}{(\ell-m)!}\left[\frac{d}{d(\cos\theta)}\right]^{\ell-m}(\cos^2\theta-1)^\ell \ ,$$

where, in the last line, we have employed a convenient form of a Taylor expansion. In this way, we have constructed, from polynomials of degree $\ell$, $2\ell+1$ independent functions which are solutions to Laplace's equation, the coefficients of the powers of $\xi_\pm$ in the expansion

$$\left[\vec{a}\cdot\frac{\vec{r}}{r}\right]^{\ell} = 2^{\ell}\ell! \sum_{m=-\ell}^{\ell} \frac{\xi_+^{\ell+m}\,\xi_-^{\ell-m}}{\sqrt{(\ell+m)!\,(\ell-m)!}} \sqrt{\frac{(\ell+m)!}{(\ell-m)!}}\,(\sin\theta)^{-m}\,e^{im\phi}$$

$$\times \left[\frac{d}{d(\cos\theta)}\right]^{\ell-m} \frac{(\cos^2\theta-1)^{\ell}}{2^{\ell}\ell!} \ . \qquad (17.12)$$

Lecture 15

All we need is a normalization factor in order to define the spherical harmonics. Employing the notation for the monomials for $\theta \to \pi \to (-1)^{\ell}\dfrac{\xi_+^{\ell}\xi_-^{\ell}}{\ell!}$

$$\frac{\xi_+^{\ell+m}\,\xi_-^{\ell-m}}{\sqrt{(\ell+m)!\,(\ell-m)!}} = \psi_{\ell m} \ , \qquad \underset{\theta\to 0}{\psi_{\ell m} Y_{\ell m}} \to \frac{1}{2^{\ell}\ell!}\left[2\xi_+\xi_-\right]^{\ell} \qquad (17.13)$$

$$= \xi_+^{\ell}\,\xi_-^{\ell}/\ell!$$

we obtain the generating function for the spherical harmonics, $Y_{\ell m}(\theta,\phi)$ ,

$$\frac{(\vec{a}\cdot\vec{r})^{\ell}}{2^{\ell}\ell} = r^{\ell}\sum_{m=-\ell}^{\ell}\psi_{\ell m}\sqrt{\frac{4\pi}{2\ell+1}}\,Y_{\ell m}(\theta,\phi) \ , \qquad (17.14)$$

where, according to (17.12),

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell+m)!}{(\ell-m)!}}\,e^{im\phi}\,(\sin\theta)^{-m}\left[\frac{d}{d(\cos\theta)}\right]^{\ell-m}\frac{(\cos^2\theta-1)^{\ell}}{2^{\ell}\ell!} \ , \qquad (17.15)$$

in which $-\ell \le m \le \ell$ , with $\ell = 0,1,2,\ldots$ . An alternative form can be derived by noting that the left hand side of (17.14) is unaltered by the transformation [see (17.10)]

$$\xi_+ \leftrightarrow \xi_- \ , \quad \theta \to -\theta \ , \quad \phi \to -\phi \ ,$$

which implies that the spherical harmonics must remain unchanged under the substitutions

$$m \to -m \ , \quad \theta \to -\theta \ , \quad \phi \to -\phi \ .$$

In this way, we learn that

$$Y_{\ell m}(\theta,\phi) = Y_{\ell,-m}(-\theta,-\phi) \quad , \tag{17.16}$$

or, using the explicit form (17.15), we obtain the alternate version

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \, e^{im\phi} \, (-\sin\theta)^{m} \left[\frac{d}{d(\cos\theta)}\right]^{\ell+m} \frac{(\cos^2\theta-1)^{\ell}}{2^{\ell}\ell!} \ . \tag{17.17}$$

Sometimes it is convenient to separate $Y_{\ell m}$ into its $\theta$ and $\phi$ dependences,

$$Y_{\ell m}(\theta,\phi) = \frac{e^{im\phi}}{\sqrt{2\pi}} \, \Theta_{\ell m}(\theta) \quad , \tag{17.18}$$

where

$$\Theta_{\ell m}(\theta) = \sqrt{\frac{2\ell+1}{2}} \sqrt{\frac{(\ell\pm m)!}{(\ell\mp m)!}} \, (\pm\sin\theta)^{\mp m} \left[\frac{d}{d(\cos\theta)}\right]^{\ell\mp m} \frac{(\cos^2\theta-1)^{\ell}}{2^{\ell}\ell!} \ . \tag{17.19}$$

## 17-3. Orthonormality Condition

The particular factors that occur in the definition of $Y_{\ell m}$ are such as to make the spherical harmonics an orthonormal set of functions. To see this, consider the product of two generating functions, with parameters $\vec{a}$ and $\vec{a}^{\,*}$, integrated over all angles:

$$\int d\Omega \left(\vec{a}^{\,*} \cdot \frac{\vec{r}}{r}\right)^{\ell} \left(\vec{a} \cdot \frac{\vec{r}}{r}\right)^{\ell'} \tag{17.20}$$

with

$$d\Omega = \sin\theta \; d\theta \; d\phi \; . \tag{17.21}$$

This integral can only contain rotationally invariant combinations, that is, it has to be a function of scalars constructed from $\vec{a}$ and $\vec{a}^*$. Since $\vec{a}^2 = \vec{a}^{*2} = 0$, the only such scalar is $\vec{a}^* \cdot \vec{a}$. Therefore, there must be an equal number of factors of $\vec{a}$ and $\vec{a}^*$, which means that the integral (17.20) is zero unless $\ell = \ell'$ ; we have

$$\int d\Omega \left( \vec{a}^* \cdot \frac{\vec{r}}{r} \right)^{\ell} \left( \vec{a} \cdot \frac{\vec{r}}{r} \right)^{\ell'} = \delta_{\ell\ell'} \, C_\ell (\vec{a}^* \cdot \vec{a})^{\ell} \; . \tag{17.22}$$

To calculate $C_\ell$ we consider a particular form of $\vec{a}$ :

$$\vec{a} = (1, i, 0) \; , \quad \vec{a}^* = (1, -i, 0) \; .$$

The quantities appearing in (17.22) are then

$$\vec{a}^* \cdot \frac{\vec{r}}{r} = \sin\theta \, e^{-i\phi} \; , \quad \vec{a} \cdot \frac{\vec{r}}{r} = \sin\theta \, e^{i\phi} \; ,$$

$$\vec{a}^* \cdot \vec{a} = 2 \; ,$$

implying that the integral, (17.22), for $\ell = \ell'$ , is

$$\begin{aligned} C_\ell 2^\ell &= \int_0^\pi \sin\theta \; d\theta \int_0^{2\pi} d\phi (\sin\theta \; e^{-i\phi})^\ell \; (\sin\theta \; e^{i\phi})^\ell \\ &= 2\pi \int_{-1}^{1} d(\cos\theta) \; (1-\cos^2\theta)^\ell \\ &= 4\pi \int_0^1 dx (1-x^2)^\ell \\ &= 4\pi \, \frac{[2^\ell \ell!]^2}{(2\ell+1)!} \; . \end{aligned} \tag{17.23}$$

The final integral in (17.23) is evaluated as follows. Defining

$$I_\ell = \int_0^1 dx(1-x^2)^\ell \ , \quad I_0 = 1 \ ,$$

we integrate by parts, for $\ell > 0$, to derive the recursion formula

$$I_\ell = 2\ell \, I_{\ell-1} - 2\ell \, I_\ell \ ,$$

which implies that

$$\begin{aligned} I_\ell &= \frac{2\ell}{2\ell+1} I_{\ell-1} \\ &= \frac{2\ell}{2\ell+1}\,\frac{2\ell-2}{2\ell-1}\,\frac{2\ell-4}{2\ell-3} \cdots I_0 \\ &= \frac{[2^\ell \ell!]^2}{(2\ell+1)!} \ , \end{aligned}$$

the last form being valid for $\ell \geq 0$. Alternatively, we could evaluate $I_\ell$ in terms of the beta function, $B(m,n)$,

$$\begin{aligned} I_\ell &= \frac{1}{2}\int_{-1}^{1} dx(1-x)^\ell \, (1+x)^\ell \\ &= 2^{2\ell}\int_0^1 dt \; t^\ell (1-t)^\ell \qquad \left[ t = \frac{1-x}{2} \right] \\ &= 2^{2\ell} \, B(\ell+1, \ \ell+1) \\ &= 2^{2\ell} \, \frac{\ell! \ \ell!}{(2\ell+1)!} \end{aligned}$$

where we have noted that for integer $m$ and $n$,

$$B(m,n) = \frac{(m-1)! \ (n-1)!}{(m+n-1)!} \ .$$

We have therefore learned that

$$\int d\Omega \frac{(\vec{a}^*\cdot\vec{r}/r)^{\ell}}{2^{\ell}\ell!} \frac{(\vec{a}\cdot\vec{r}/r)^{\ell'}}{2^{\ell'}\ell'!} = \delta_{\ell\ell'}\, 4\pi \frac{1}{2^{\ell}(2\ell+1)!} (\vec{a}^*\cdot\vec{a})^{\ell}$$

$$= \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell'}^{\ell'} \psi^*_{\ell m} \psi_{\ell' m'} \sqrt{\frac{4\pi}{2\ell+1}} \sqrt{\frac{4\pi}{2\ell'+1}} \int d\Omega\, Y^*_{\ell m}(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi) \quad , \tag{17.24}$$

where we have used the generating function (17.14). What we now must do is extract the coefficient of $\psi^*_{\ell m} \psi_{\ell' m'}$ from $(\vec{a}^*\cdot\vec{a})^{\ell}$, which is achieved as follows:

$$\frac{(\vec{a}^*\cdot\vec{a})^{\ell}}{2^{\ell}(2\ell+1)!} = \frac{(\xi_+{}^*\xi_+ + \xi_-{}^*\xi_-)^{2\ell}}{(2\ell+1)!}$$

$$= \frac{1}{(2\ell+1)!} \sum_{m=-\ell}^{\ell} \frac{(2\ell)!}{(\ell+m)!\,(\ell-m)!} (\xi_+{}^*\xi_+)^{\ell+m} (\xi_-{}^*\xi_-)^{\ell-m}$$

$$= \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \psi^*_{\ell m} \psi_{\ell m} \tag{17.25}$$

where we have used (17.9), (17.13), and the binomial expansion. Comparing this with (17.24), we obtain the orthonormality condition for the spherical harmonics:

$$\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\, \delta_{mm'} \quad . \tag{17.26}$$

When $Y_{\ell m}$ is separated as in (17.18), the orthonormality condition reads

$$\int_0^{\pi} \sin\theta\, d\theta\, \Theta_{\ell m}(\theta)\, \Theta_{\ell' m}(\theta) = \delta_{\ell\ell'} \quad . \tag{17.27}$$

## 17-4. Legendre Polynomials

A few special cases of $\Theta_{\ell m}$ can be easily extracted from (17.19):

$$\Theta_{\ell\ell}(\theta) = \sqrt{\frac{(2\ell+1)!}{2}}\,\frac{(-1)^{\ell}\,(\sin\theta)^{\ell}}{2^{\ell}\ell!} \quad , \tag{17.28}$$

$$\Theta_{\ell,-\ell}(\theta) = \sqrt{\frac{(2\ell+1)!}{2}}\,\frac{(\sin\theta)^{\ell}}{2^{\ell}\ell!} \quad , \tag{17.29}$$

$$\Theta_{\ell,0}(\theta) = \sqrt{\frac{2\ell+1}{2}}\left[\frac{d}{d(\cos\theta)}\right]^{\ell}\frac{(\cos^2\theta-1)^{\ell}}{2^{\ell}\ell!}$$

$$= \sqrt{\frac{2\ell+1}{2}}\,P_{\ell}(\cos\theta) \quad . \tag{17.30}$$

Occurring in (17.30) is the Legendre polynomial of order $\ell$ ,

$$P_{\ell}(\cos\theta) = \left[\frac{d}{d(\cos\theta)}\right]^{\ell}\frac{(\cos^2\theta-1)^{\ell}}{2^{\ell}\ell!} \tag{17.31}$$

$\xi \sin\theta^{\mp m} \frac{d^{\mp m}}{d(\cos\theta)} P_\ell(\cos\theta) = \Theta_{\ell m}$

which is so normalized that

$$P_{\ell}(1) = 1 \quad . \tag{17.32}$$

According to (17.27), the Legendre polynomials satisfy the following orthonormality condition: see 17.30

$$\int_{-1}^{1} d(\cos\theta)\; P_{\ell}(\cos\theta)\; P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'} \quad . \tag{17.33}$$

## XVIII. Coulomb Potential and Spherical Harmonics

The motivation for constructing the solid harmonics was that they formed, in terms of homogeneous functions, a particular set of solutions to Laplace's equation. Since these harmonics are functions of the spherical angles $\theta$ and $\phi$, Laplace's equation should be expressed in spherical coordinates, where the Laplacian has the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] . \tag{18.1}$$

Thus, since the solid harmonics, (17.6),

$$Y_{\ell m}(\vec{r}) = \left\{\begin{matrix} r^{\ell} \\ r^{-\ell-1}\end{matrix}\right\} Y_{\ell m}(\theta,\phi) , \tag{18.2}$$

are solutions to

$$\nabla^2 Y_{\ell m}(\vec{r}) = 0 , \qquad r \neq 0 , \tag{18.3}$$

and since

$$\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\left\{\begin{matrix} r^{\ell} \\ r^{-\ell-1}\end{matrix}\right\} = \ell(\ell+1)\left\{\begin{matrix} r^{\ell} \\ r^{-\ell-1}\end{matrix}\right\} , \tag{18.4}$$

the differential equation satisfied by the spherical harmonics, $Y_{\ell m}$, is

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \ell(\ell+1)\right] Y_{\ell m}(\theta,\phi) = 0 . \tag{18.5}$$

When the $\theta$ and $\phi$ dependence of $Y_{\ell m}$ is separated as in (17.18), the differential equation for $\Theta_{\ell m}$ is

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]\Theta_{\ell m}(\theta) = 0 \quad . \tag{18.6}$$

The fundamental solution of Laplace's equation is the Coulomb potential, (12.3), for $\vec{r} \neq \vec{r}\,'$ , which, written in spherical coordinates, is

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} \quad , \tag{18.7}$$

where $\gamma$ is the angle between $\vec{r}$ and $\vec{r}\,'$ , explicitly,

$$\cos\gamma = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\,\cos(\phi-\phi') \quad . \tag{18.8}$$

We now expand (18.7) as

$$\frac{1}{\sqrt{r_>^2 + r_<^2 - 2r_>r_<\cos\gamma}} = \frac{1}{r_>\sqrt{1 - 2\frac{r_<}{r_>}\cos\gamma + \left(\frac{r_<}{r_>}\right)^2}}$$

$$= \sum_{\ell=0}^{\infty}\left(\frac{r_<^{\ell}}{r_>^{\ell+1}}\right)\text{(Polynomial of degree } \ell \text{ in } \cos\gamma) \quad , \tag{18.9}$$

where $r_>(r_<)$ is the greater (lesser) of $r$ and $r'$ . The polynomial of degree $\ell$ appearing here is a solution to (18.5), and so must be a linear combination of $Y_{\ell m}(\theta,\phi)$'s , $-\ell \leq m \leq \ell$ . On the other hand, as we will show below, this is just the Legendre polynomial in $\cos\gamma$, (17.31), that is

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\gamma) \quad . \tag{18.10}$$

For $\gamma = 0$ , this expansion is trivially

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$\rho = 30$

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r h

$$\frac{1}{r_> - r_<} = \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}}$$

which supplies the normalization condition

$$P_\ell(1) = 1 \quad , \tag{18.11}$$

as is required for the Legendre polynomials [see (17.32)].

We now wish to expand the Coulomb potential

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} \quad , \tag{18.12}$$

which satisfies the inhomogeneous Green's function equation

$$-\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{18.13}$$

in terms of the solutions to the homogeneous Laplace's equation, (18.3). In spherical coordinates, the delta function is

$$\delta(\vec{r}-\vec{r}\,') = \frac{1}{r^2}\,\delta(r-r')\,\frac{1}{\sin\theta}\,\delta(\theta-\theta')\;\delta(\phi-\phi') \quad , \tag{18.14}$$

while the Laplacian is given by (18.1). For $r < r'$ , (18.10) shows that the solution to (18.13) can be expanded in powers of $r$ ,

$$r < r' : \qquad G = \sum_{\ell m} r^\ell Y_{\ell m}(\theta,\phi)\, A_{\ell m} \quad , \tag{18.15}$$

while for $r > r'$ , the expansion is in terms of powers of $1/r$ ,

$$r > r' : \qquad G = \sum_{\ell m} r^{-\ell-1}\, Y_{\ell m}(\theta,\phi)\, B_{\ell m} \quad . \tag{18.16}$$

The expansion coefficients, $A_{\ell m}$ and $B_{\ell m}$ , depending on $r'$, $\theta'$ and $\phi'$, are to be determined by the conditions on Green's function near the source:

$$G \text{ is continuous at } r = r' \; ; \tag{18.17a}$$

and

$$\left[ -r^2 \frac{\partial}{\partial r} G \right]_{r'-0}^{r'+0} = 4\pi \frac{1}{\sin\theta} \delta(\theta-\theta') \delta(\phi-\phi') \quad . \tag{18.17b}$$

?

These two conditions imply, respectively,

$$r'^{\ell} A_{\ell m} = \frac{1}{r'^{\ell+1}} B_{\ell m} \quad , \tag{18.18a}$$

$$\sum_{\ell m} \left[ (\ell+1) \frac{1}{r'^{\ell}} B_{\ell m} Y_{\ell m} + \ell r'^{\ell+1} A_{\ell m} Y_{\ell m} \right] = 4\pi \frac{1}{\sin\theta} \delta(\theta-\theta') \delta(\phi-\phi') \quad . \tag{18.18b}$$

If we write

$$A_{\ell m} = r'^{-\ell-1} C_{\ell m} \; , \quad B_{\ell m} = r'^{\ell} C_{\ell m} \quad , \tag{18.19}$$

(18.18a) is satisfied automatically, while (18.18b) reads

$$\sum_{\ell m} (2\ell+1) C_{\ell m} Y_{\ell m}(\theta,\phi) = 4\pi \frac{1}{\sin\theta} \delta(\theta-\theta') \delta(\phi-\phi') \quad . \tag{18.20}$$

The use of the orthonormality condition, (17.26), now yields

$$C_{\ell m} = \frac{4\pi}{2\ell+1} Y^{*}_{\ell m}(\theta',\phi') \quad . \tag{18.21}$$

By substituting this into (18.20) we obtain the completeness statement for the spherical harmonics,

$$\sum_{\ell m} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi') = \frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\phi-\phi') \quad , \tag{18.22}$$

which allows us to expand any function of $\theta$ and $\phi$ in terms of spherical harmonics. We therefore have obtained such an expansion for the Green's function

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell m} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi') \quad . \tag{18.23}$$

Comparing this with the alternative representation, (18.10), we obtain the relation

Addition theorem

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi') \quad . \tag{18.24}$$

We must finally show that this function of $\cos\gamma$ actually is the Legendre polynomial, (17.31). This is easily done by considering a particular coordinate system, where

$$\theta' = 0 \implies \gamma = \theta \quad .$$

From (17.19), we learn that

$$Y_{\ell m}(\theta',\phi') \propto (\sin\theta')^{|m|} \quad ,$$

implying [see (17.30)]

$$Y_{\ell m}(0,\phi') = \delta_{m0}\sqrt{\frac{2\ell+1}{4\pi}} \quad ,$$

so that only the $m = 0$ term contributes to the right-hand side of (18.24), which is therefore, by (17.30),

$$\frac{4\pi}{2\ell+1}\sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell(\cos\theta)\sqrt{\frac{2\ell+1}{4\pi}} = P_\ell(\cos\theta) \quad .$$

Thus we have proved that the function of $\cos\gamma$ occurring in (18.10) is indeed Legendre's polynomial. The relation (18.24) is called the addition theorem for spherical harmonics.

Lecture 16

## XIX. Multipoles

In terms of the above discussion of spherical harmonics, we now make a general analysis of the potential, due to a given charge distribution, $\rho(\vec{r}')$, outside of that distribution.


The potential is given by (1.2), or

$$\phi(\vec{r}) = \int (d\vec{r}')\, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \quad , \tag{19.1}$$

where, for convenience, we will choose the origin to lie within the charge distribution. If $r$ is large compared to the characteristic dimensions of the charge distribution, we may expand the Coulomb potential as follows:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} - \vec{r}'\cdot\vec{\nabla}\,\frac{1}{r} + \frac{1}{2}\,(\vec{r}'\cdot\vec{\nabla})^2\,\frac{1}{r} + \ldots$$

$= \sum_{n=0}^{\infty} \frac{1}{n!} (-x \cdot \nabla)^n$

$$= \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \frac{1}{2}\,\frac{1}{r^5}\,\vec{r}\cdot[3\vec{r}'\vec{r}' - \overleftrightarrow{1}r'^2]\cdot\vec{r} + \ldots \quad , \tag{19.2}$$

so that the potential, in its leading behavior for large distances, has the form

$$\phi(\vec{r}) = \frac{e}{r} + \frac{\vec{r}\cdot\vec{d}}{r^3} + \frac{1}{2}\,\frac{1}{r^5}\,\vec{r}\cdot\overleftrightarrow{q}\cdot\vec{r} + \ldots \quad . \tag{19.3}$$

Occurring here are the first three moments of the charge distribution,

$$e = \int (d\vec{r}')\, \rho(\vec{r}') \quad , \tag{19.4a}$$