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INTRODUCTION TO PLASMA PHYSICS

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Introduction to Plasma Physics

Professor J. M. Dawson

1994

I. Introduction

A plasma may be roughly defined as a material containing mobile charges, either negative or positive, or both and in which the electric and magnetic interactions between particles play a dominant role in the dynamics of the system. In most cases there are equal numbers of positive and negative charges so that the system as a whole is neutral. However, there are cases of interest in which the plasma is not neutral and even cases where it consists of a single type of charged particle; we will consider mainly neutral plasmas but will include a short discussion of non-neutral plasma. Because of the long range nature of electric and magnetic forces, plasmas exhibit collective motions; motions in which many particles move coherently. Because of this the physics of plasmas is extremely rich as we will see during this course.

Some examples of plasmas are:

(1) Gases which are heated to such a high temperature that some or all electrons are detached from the constituent atoms and molecules. Examples are fusion reactor plasmas, plasmas generated by strong shock waves, plasmas created by laser heated materials, etc.

(2) The gas in any discharge. Here the electrons gain sufficient energy from the applied electric field that they can ionize other atoms and molecules. The discharge can be DC or AC. Examples are lightning, fluorescent lights, neon lights, all types of laboratory discharges, and many others.

(3) Interstellar gas which is ionized by the ultraviolet light from stars. A special case is the ionization of interstellar gas by ultraviolet and x-rays from novas and supernovas. These plasmas have quite low temperatures.

(4) Plasma blowing off of stars (the solar wind, stellar winds). Plasma produced by the ionization of interstellar gas by energetic particles generated by pulsars (the Crab Nebula, for example).

(5) The ionosphere of the Earth and other planets produced by ultraviolet radiation from the Sun.

(6) The mobile electrons in metals and semiconductors. Here the perturbations caused by neighboring atoms weaken the binding of the electrons to the atoms to such an extent that some of the electrons are free to move through the material.

(7) The hot dense material in the interior of stars. This can range from the material at the center of the Sun ($T = 10^7$K, $\rho = 100$ gm/cm^3) to that in white dwarfs ($T = 10^7$K, $\rho = 10^6$ gm/cm^3) to that at the core of a supernova ($T = 2 \times 10^{10}$K, $\rho = 10^{15}$ gm/cm^3).

In addition, there are systems which contain mobile charges but which are not generally classified as plasmas. Examples of systems of this type are salt solutions, gases heated to temperatures where there are very few free electrons, and the free electrons in semiconductors where collisions are important. Systems of this type contain so many neutral molecules and collisions between the charged particles and neutrals are so dominant that many of the properties that we generally associate with plasmas (namely collective motions) are masked. However, such systems can exhibit some of the properties of plasmas and some plasma concepts are useful for such systems. A system can pass continuously from a non-plasma state to a plasma state. (For example, by gradually raising the temperature of a gas.)

From the above examples we see that the plasma state encompasses an enormous range of physical parameters ranging from temperatures of 100's of degrees Kelvin to more than 10^{10}K, and densities from a few particles per m^3 to 10^{38} per cm^3.

Plasma has been called the fourth state of matter. In spite of the fact that there is no sharp transition from a gas to the plasma state, there is much justification for this point of view. This is because plasmas behave so differently from the non-conducting gases and fluids that we are much more familiar with. The mobile charges in a plasma allow it to carry electric current. It thus interacts strongly with electric and magnetic fields. This leads to a great wealth of phenomena not exhibited by non-plasmas. The physics of plasmas is much richer than that of air and water which we are much more familiar with. Until quite recently man had little direct experience with plasma. He was of course familiar with lightning and he could see the sun and stars but he could not handle it or manipulate it; he could not touch it or feel it or control it in any way. Only in the last 100 years with the rise of the electrical industry and the electronics industry has man been able to experiment with plasma. In fact only in the last 40 or 50 years or so with the development of the controlled fusion program and the exploration of space has man started to investigate plasmas in detail and discover just how rich and complex their physics is.

To produce a plasma by heating a gas, we must heat it to such a temperature that collisions between particles are able to knock off electrons. Since first ionization energies for atoms range from roughly 4 eV for Cs to 24 eV for He, the colliding energies must be comparable. A temperature of 11,600° K corresponds to an average thermal energy of 1 eV (kT = 1 eV), therefore we see that temperatures of thousands of degrees are required.

We do not strictly require kT to equal the ionization temperature to obtain appreciable ionization. It is well known that the thermal energy distribution for molecules in a gas (taken to be classical) is a Maxwell Boltzmann distribution,

$$P(E) = A \exp(-E/kT) \quad , \tag{1}$$

where $P(E)$ is the probability of having an energy E and A is a normalizing constant. In a gas at thermal equilibrium, there are always particles with considerably more energy than the average and collisions between these particles can produce ionization. The electron density is determined by a balance between such ionization and recombination of the ions and electrons. In a low-density gas, the electrons and ions rarely approach each other and recombination can be quite slow (later we will give some estimates of what these rates are). Thus, a substantial percentage of ionization can exist even when $kT \ll E(\text{ionization})$.

For a gas in thermal equilibrium, the degree of ionization depends only on the temperature. Every process that can produce ionization can take place in reverse to give recombination. In thermal equilibrium, each process and its inverse takes place at exactly the same rate (the law of detailed balance). For most laboratory plasmas, one does not have thermal equilibrium; if nothing else, thermal equilibrium would require enormous levels of radiation. A plasma at a temperature of 10 eV (by the Stephan Boltzmann Law) would radiate 6 x 10^9 watts per square cm. of its surface. Needless to say, to maintain a plasma radiating at such a rate would require a huge input power; fortunately, low density laboratory plasmas are optically thin and radiate much more feebly than this. They are not in radiation equilibrium and often are not in equilibrium in other ways; we will examine these shortly.

For thermal equilibrium the degree of ionization is determined by the Saha equation. This equation has the form

$$(n_e \, n_i \, / n_0) \simeq (1/\lambda^3 g_i) \, \exp(-E_i/kT) \; , \qquad (2)$$

$$\lambda^2 = h^2/2m_e kT$$

where λ^2 is the square of the DeBroglie wave length for the average electron energy.

Here n_e, n_i, n_0 are the electron, ion and neutral densities, E_i, is the ionization energy, g_i , is the the number of ground state levels of the atom, m_e is the electron mass, and h is Planck's constant. A derivation of this formula is outside the scope of this course (see Kittel, Statistical Mechanics), but we will point out the physical significance of the terms.

For a singly ionized substance with overall charge neutrality, $n_e = n_i$ and Eq. (2) can be written in the form

$$(n_e / n_0) = (V_e/\lambda^3 g_i) \exp(-E_i/kT) \ . \tag{3}$$

where $V_e = 1/n_e$ is the volume per electron. Now the number of free (continuum) states for an electron in a volume V is proportional to V/λ^3; thus, V_e/λ^3 is the number of free states per electron (number of free states for a volume containing one electron). If there is one free electron in the volume V_e, then there is one ion and this ion has g_i ground states. The probability that the electron is in one of the g_i ground states is $\exp(E_i/kT)$ times larger than the probability that it is in a free state so the ratio n_e/n_o is given by Eq. (3). In this discussion we have neglected bound states other than the ground state; these are less populated than the ground state by the appropriate Boltzmann factor. Since they are less numerous than the free states, they generally can be neglected.

Besides thermal equilibrium plasmas there are many types of plasmas which are not in thermal equilibrium. Examples of these are:

(1) most electrical discharges (fluorescent lights, neon signs, Aurora, sparks, etc.).

(2) interstellar gas which is ionized by radiation from stars,

(3) the ionospheres of planets,

(4) the solar and stellar winds and many others.

For discharge-type plasmas, the electrons gain enough energy from electric fields so that they can knock other electrons from the constituent atoms of the gas. This requires the electron energies in the discharge to be roughly a few eV as ionization energies of atoms and molecules range from 3.9 eV for Cs to 24.5 eV for He (it is, of course, the energetic tails of the electron distributions that do the ionizing). Thus, the electron temperature in a florescent light is about 30,000° K; the ions on the other hand are only slightly hotter than room temperature because they are in good thermal contact with the walls of the tube and the electrons transfer energy to them slowly. It is difficult to heat the electrons to a much higher temperature because they lose so much energy in ionizing and exciting the atoms inside the discharge tube. These losses can only be overcome with very high powers and currents; any mechanism that insulates the discharge from its cold surroundings, such as magnetic plasma confinement, can greatly help in reducing power requirements.

The plasmas in gas discharges are rarely in thermal equilibrium. The mean free paths for particle encounters are often comparable to the size of the container. Thus, ions and neutral atoms, though slightly heated by collisions with hot electrons, remain pretty much at the temperature of the walls; wall temperatures are typically of the order of (1/30) eV. As we shall see later, electric fields develop naturally near the walls that tend to confine the electrons to the discharge and isolate them from the walls.

Virtually all laboratory plasmas are not in thermal equilibrium with the electromagnetic radiation field. The self-absorption lengths are so long and the rates of emission are so low that most radiation freely leaves the plasma; there is not enough room for the plasma to reach a balance between emission and absorption (the plasma is optically thin.) If this were not true the plasma in a florescent light would be 600 times brighter than the surface of the sun.

Even though the radiation levels are low compared with black body levels, a lot of energy is carried off by radiation. The electrons, in colliding with the atoms and molecules of the gas, raise their bound electrons to excited states. Many excited states give up their energy immediately as radiation. Since energetic electrons are generally more efficient at exciting and ionizing, the energetic electrons tend to be brought down to lower energies by these processes and their number is depleted. Thus, such discharges contain fewer energetic electrons than would a plasma in thermal equilibrium. The velocity distribution for electrons, $f(v)$, in a thermal equilibrium plasma is a Maxwellian

$$f(v) \propto \exp(-m_e v^2/2kT) \quad . \tag{4}$$

where v is the electron velocity, k is Boltzmann's constant, and T is the temperature. Such a distribution function is shown in Fig. 1.

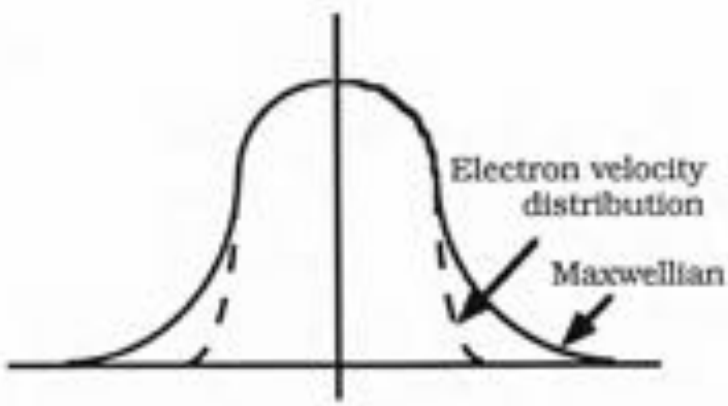


Figure 1

For a gas discharge plasma the tail dies out more rapidly; a distribution that is often used is the Druyvesteyn distribution

$$f(v) \propto \exp(-v^4/v_o^4) \tag{5}$$

where v_o is determined by the average electron energy.

As we have already said, for a plasma in thermal equilibrium, all processes satisfy the condition of detailed balance; that is every process and its inverse are exactly in balance. For a non-equlibrium plasma, this no longer holds. For example, we can have ionization taking place in the plasma and recombination taking place at the walls. If the plasma is in steady-state, then the rate of ionization must equal the rate of recombination but different processes can be involved in ionization and recombination. The distributions, the excited levels of atoms, the ionization levels, the chemical species and the radiation levels are all strongly dependent on cross-sections. It is the non-equlibrium properties of such plasmas which make them so rich in phenomenon and useful in practice (for example, lasers, plasma etching, light sources, etc). Because of the large number of processes that are involved in gas discharges and their various cross-sections, the analysis of non-equlibrium plasmas can become very complex. Most of these notes will be devoted to the case of fully ionized plasmas where the multitude of atomic processes need not be considered. However, atomic processes do play a central role in nearly all laboratory plasmas and in many natural plasmas. Therefore, we will give some attention to these processes using a semi-classical approach which gives reasonable estimates of their size and which can be used to estimate the role and importance of such processes in situations of interest.

II. Particle Interactions, Coulomb Collisions, Ionization, Excitation, Recombination, Radiation

(a) Debye Shielding

As already mentioned, in order for us to produce a plasma, particle energies (electron, ion, atomic, and photon) must be at least comparable to the ionization energy of the constituent atoms. The electrons which are detached will have similar energies (energies of 10% or even 1% might be considered similar). Furthermore, when the electron is removed from an atom, it wanders around in the space between atoms and in general is quite far away (compared to atomic dimensions) from the positive ions. The electron ion potential energies are quite small compared to the ionization energies throughout most of space so that the electron kinetic energy is much greater than the potential energy almost everywhere. For example, consider a laboratory Q machine plasma of Cs with an electron density of $n_e = 10^{12}/cm^3$ and at a temperature of 2500° K (0.22 eV). From the Saha equation, Eq. 2, we find the degree of ionization is very high. Now the mean distance between an electron and an ion is 10^{-4} cm. This corresponds to a potential of

$$e^2/r = (e^2/a_0)(a_0/r) = 27\text{eV}\ (5 \times 10^{-9}/10^{-4}) = 1.35 \times 10^{-3}\ \text{eV}\ . \tag{6}$$

Here a_0 is the Bohr radius for an electron in a hydrogen atom and 27 eV is its potential energy at that radius. The electron kinetic energy is more than two orders of magnitude larger than this potential energy.

As they move through a plasma, the free electrons are usually in regions of small potential. The effect of the electric fields on their trajectories is small except during the relatively rare close approaches, i.e., during a collision, of an electron to an ion or another electron; the same is true for ions if they have even a modest temperature (in the example above, even if the ions are at room temperature [.03 eV] their mean kinetic energy is more than 20 times the mean potential energy between two ions).

While the interactions between pairs of charged particles is generally quite small, the sum of the fields of many ions or electrons can be important. This is because of the long range nature of the Coulomb interaction which falls off as r^{-2}. If we consider the number of electrons in a spherical shell of radius r and thickness dr centered on a given electron, then this number is equal to $4\pi n_e r^2 dr$. We see that this number times r^{-2} is independent of r. Of course, if the electrons are distributed evenly over the sphere, the electric field at the center will be zero. However, if we move only a small fraction of the electrons, say 1%, from the left-hand side to the right-hand side of all the spheres, then each spherical shell will contribute as much to the electric field as every other shell. If this continues out to some large distance, a very large electric field will be produced. Of course, since the electrons (and ions) are free to move, they quickly move from regions where there are excess electrons towards regions where there are excess ions and thus attempt to neutralize regions of charge imbalance. Actually the electrons do not stop once charge balance is established; because of their motion they overshoot the charge balance state. Regions that start with an excess of electrons end up with a deficit while regions that initially had a deficit end up with an excess. The resulting electric fields stop the electron motions; the process then repeats in the reverse direction; the electron density oscillates about the neutral state. Such charge density oscillations are called plasma oscillations; the characteristic frequency for these oscillations is the so-called plasma frequency, ω_p

$$\omega_p^2 = 4\pi n_e e^2/m_e \ , \tag{7}$$

$$\omega_p \simeq 6 \times 10^4 (n_e)^{1/2} \quad ,$$

where ω_p is in sec^{-1} and n_e is in cm^{-3}.

To gain an idea of how strong space charge effects are in a plasma, let us consider an infinite homogeneous plasma and let us ask how much energy it would take to move all the electrons in a spherical region of radius r to its surface. The situation is illustrated in Fig. 2.

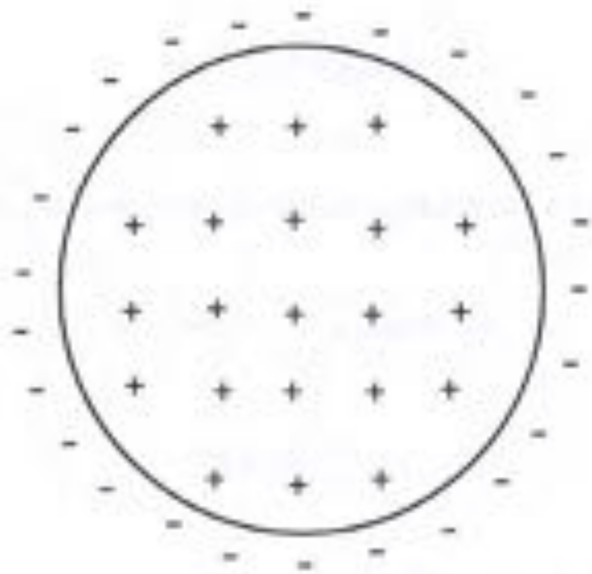


Figure2

We will hold the ions fixed at their initial uniform density. The potential energy is stored in the electric field; energy density equals $E^2/8\pi$; the total potential is given by

$$W = \int \frac{E^2}{8\pi}\, dV$$

The electric field is radial outward from the center of the sphere and is given by

$$E(\rho) = (4/3)\,\pi n_i\, Z e \rho, \qquad \rho < r \qquad (8)$$

$$0, \qquad \rho > r\,.$$

Carrying out the integration gives

$$W = (8/45)\pi^2 n_e{}^2 e^2 r^5 \tag{9}$$

Let us equate W to the electron kinetic energy in the sphere.

$$K = (4/3)\pi n_e r^3 (3/2) kT = W$$

$$r^2 = 45(kT/4\pi n_e e^2). \tag{10}$$

This is the radius of a sphere where the electron thermal energy is just enough to remove themselves from the sphere; such a motion is energetically possible but extremely unlikely. For larger spheres there is not enough energy to separate the electrons from the ions while for smaller spheres there is sufficient energy. If the sphere is one tenth this big, only one percent of the kinetic energy is required for the electrons removal. For smaller spheres we can expect that large fluctuations in electron density will take place because of the random motions of the electrons; the fluctuations will be essentially the same as those that would occur in a neutral gas ($\delta N^2 \propto N$). For larger spheres the deviations from neutrality will be greatly suppressed due to the large amount of energy that is required to create them.

The quantity λ_D given by the expression

$$\lambda_D{}^2 = kT/4\pi n_e e^2 = v_T{}^2/\omega_p{}^2 \ . \tag{11}$$

$$\omega_p^2 = 4\pi n_e e^2/m \ .$$

and

$$v_T^2 = kT/m$$

is called the Debye length (it was first introduced in the 1930's by Debye in the study of electrolytes). It is a measure of the size of a region in which appreciable deviations from charge neutrality occur in a thermal plasma. The quantity ω_p is called the plasma frequency (originally defined by Tonks and Langmuir) and it gives the basic response frequency of the electrons to applied electric fields as we shall see presently. The Debye length is the electron thermal velocity divided by the plasma frequency. A thermal electron moving through a plasma pushes other electrons away; at a distance of a Debye length or greater, the plasma electrons have time to adjust to the passing electron so as to neutralize its electric field and shield the rest of the plasma from its presence. We have made this argument for an electron. Later on in the course we will see that the Debye length is also the distance at which an ions field is shielded out, even though the ion is moving much slower. In this case the electrons move past the ion before they can respond for distances less than the Debye length.

It is illuminating to write Eq. (11) in the following form.

$$\lambda_D^2/d^2 = kTd^3/d^2 4\pi e^2 = kT/(4\pi e^2/d) = kT/4\pi\phi[d] \,, \qquad (12)$$

where $d^3 = 1/n_e$; d is the inter-particle spacing and $\phi[d]$ is the potential of an electron (or ion) evaluated at a separation d. We have already shown that in a plasma kT is much larger than $\phi[d]$. Hence, λ_D is large compared to d. Thus, in a plasma there are many particles in a Debye sphere (sphere of radius λ_D). For the Cs plasma example given, $T = 2500°$ K, and $n_c = 10^{12}$, $\lambda_D = 5d$ and the number of particles per Debye sphere is, $N_D = 600$; for a typical Fusion plasma, $T = 10^8$ K, $n_e = 10^{14}$, $\lambda_D \sim 400d$ and $N_D = 2.5 \times 10^8$; at the center of the sun, $T = 10^7$ K, $n_e = 6 \times 10^{25}$, $\lambda_D \sim 16d$.

The quantity ω_p in Eq. (11) is the frequency at which electrons can respond to a disturbance or it is the natural frequency of vibration of the electron density fluctuations. We can see this from the following simple calculation. Consider an infinite uniform plasma. Let us imagine that we displace all the electrons in a slab of plasma of thickness L (the slab is perpendicular to x and infinite in the y, z directions) by a distance δ. See Fig. 3. Then we will create two regions, one will be charged positive since the electrons have been moved out of that region and the other will be charged negative since we have moved extra electrons into that region. The total charge per unit area σ contained in these areas is $\pm\, en_e\delta$ and the electric field in the region of the slab is (the situation is like that of a parallel plate capacitor)

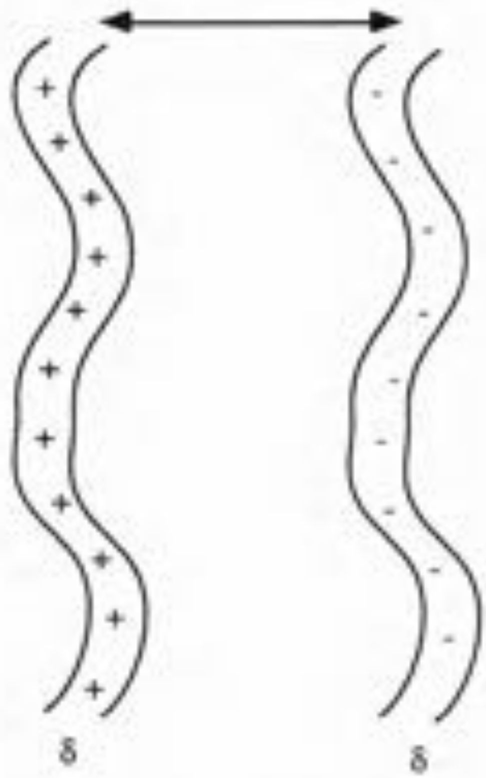


Figure 3

$$E = 4\pi\sigma = 4\pi e n_e \delta \ . \qquad (13)$$

The force on the electrons in the slab is

$$F = -eE = -4\pi e^2 n_e \delta \ . \qquad (14)$$

From Newton's laws of motion, we have

$$m_e d^2x/dt_2 = m_e d^2\delta/dt^2 = - 4\pi e^2 n_e \delta \ , \qquad (15)$$

and

$$d^2\delta/dt^2 = -(4\pi e^2 n_e/m_e)\delta = -\omega_p^2\delta\ . \qquad (16)$$

This is the equation of motion of a simple harmonic oscillator which oscillates at the plasma frequency. In numerical terms we have

$$\omega_p = 5.64 \times 10^4\ n_e^{1/2}\ , \qquad (17)$$

and

$$f_p = \omega_p/2\pi = 8.97 \times 10^3\ n_e^{1/2}\ . \qquad (18)$$

The force on the ions is of equal magnitude but oppositely directed; however, due to their large mass, their motion is relatively small and has been neglected here. We have also neglected effects associated with the random thermal motions of the electrons; these will be taken up later in the course.

The plasma frequency is a measure of the minimum response time of the electrons to an electric field, $\tau_p \sim 1/\omega_p$. For times longer than τ_p, the electrons will move in such a way as to reduce an applied electric field. Thus, a plasma is transparent to electromagnetic waves with frequencies higher than the plasma frequency; below the plasma frequency the plasma acts like a good conductor and electromagnetic waves in this region are reflected; we will study this in much more detail when we get to plasma waves.

Of course if the collision frequency is higher than the plasma frequency, such motions are impeded and electron density oscillations are quickly damped. The plasma also becomes an absorbing medium for electromagnetic waves. For hot plasmas, as we have already emphasized, the particle kinetic energy is much larger than the potential energy of interaction between pairs of particles so their deflections, as they pass one another, are small (collisions are weak). Thus, most plasmas of interest are only weakly collisional and plasma oscillations are an important aspect of their behavior.

III. <u>Collisions and Atomic Processes</u>

It is not possible to take collisions completely into account in describing a plasma. First of all there are a very large variety of collisions. Secondly, for plasmas containing atoms and molecules that have not been fully striped of their electrons, relatively few cross-sections are well known, either on the basis of theory or from experimental measurements. Recent research has greatly increased our knowledge in this area and ongoing research continually adds to it. However, the number of possibilities is so enormous that this will be a field of research for a long time to come. Despite this it is possible to make reasonable estimates of many types of collisions and these prove very useful when planing experiments or when trying to interpret experimental data.

In this section we will provide a brief treatment of the most important types of collisions involved in laboratory plasmas and we will show how to make reasonable estimates of the sizes of the various cross-sections. The intent is to provide sufficient information to enable order-of-magnitude estimates of collisional effects to be made. If such estimates indicate that particular collisional processes are important for situations of interest, then detailed cross-sections can be sought in the literature.

Data on cross-sections is given in several different units in the literature. Much recent data is given in cm^2/atom. Also in common use is the dimension $\pi a_0^2 = 0.88 \times 10^{-16}$ cm^2, where a_0 is $h^2/m_e e^2$, the Bohr radius of the hydrogen atom. Since atomic cross-sections are frequently in the region of 10^{-16} cm^2, Angstroms squared, $Å^2$, are used. In the older literature, particularly with respect to elastic collisions, cross-sections are given in terms of collision probability per cm per unit of pressure (mm Hg at 0° C). To change them into cm^2/molecule, one must divide by the number of molecules per cm^3 at a pressure of 1 mm Hg; this is 3.53×10^{16}.

If we watch a given "test particle" proceed through a region where the density of target or field particles is n, then the probable number of a given type of collision per cm of path length is simply σn, where σ is the cross-section for the particular type of interaction. The average distance traveled between collisions, or mean free path, is then $1/n\sigma$. The number of collisions per second, collision frequency, is $n\sigma v$, where v is the relative velocity of the test particle through the field particles. If the field particles are in motion or if we are concerned with the average behavior of a large number of test particles moving at different velocities, then since σ (for most processes) is a function of velocity we must average σv over all velocities. Thus, if $f(\mathbf{v})d^3v$ is the number of particles having velocities in a small volume of velocity space d^3v about a velocity $\mathbf{v}$ and our test particle has a velocity $\mathbf{v}_0$, then we may write

$$\int f(\mathbf{v})\sigma(\mathbf{v}-\mathbf{v}_0)\left|\mathbf{v}-\mathbf{v}_0\right| d^3v = n\langle\sigma v\rangle \quad . \tag{19}$$

The quantity $n\langle\sigma v\rangle$ is called the rate coefficient. If instead of one particle there are n_t test particles per cubic centimeter, we must multiply Eq. (19) by n_t to get the total number of reactions of the type described by σ. If there is more than one type of reaction, then we must make similar calculations for every process involved.

Coulomb Collisions

This is a process of great importance to us; not only is it a basic process that goes on in fully ionized plasmas but it will also provide us with the basis for estimating a wide variety of other collisional processes.

We consider the problem of a light particle (test particle) of charge ze and mass m approaching at a velocity v_0 a stationary heavy (immobile) particle (field particle) of charge Ze. The situation is shown in Fig. 4.

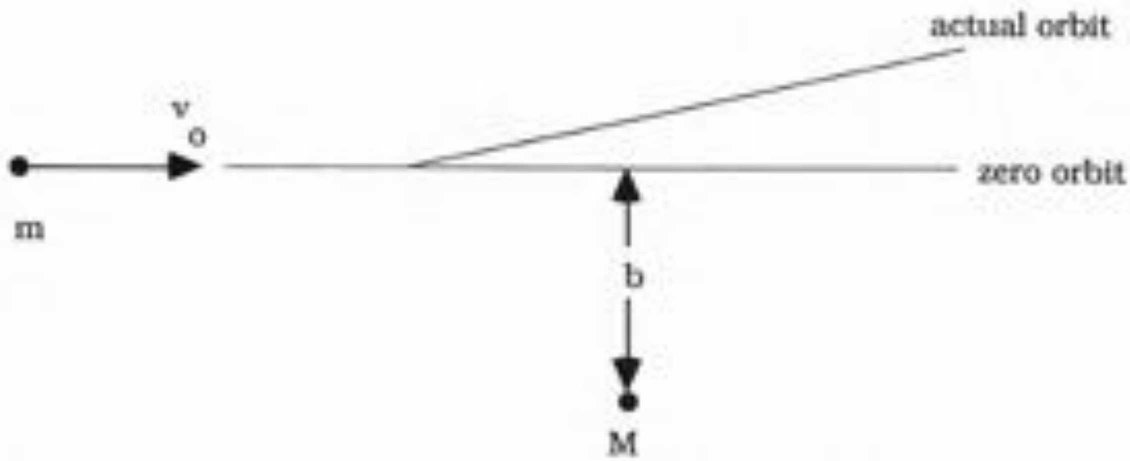


Figure 4

Coulomb Collisions

Let us assume that the deflection angle is small so that in the zero order approximation the particle follows a straight line; denote the minimum distance from this straight line orbit and the field particle by b; this is called the impact parameter. Most of the deflection is due to the force exerted during the time the particles are close together, this occurs for a time interval $\tau = 2b/v_0$ centered at the time of closest approach and its magnitude is roughly zZe^2/b^2. The momentum acquired by the test particle perpendicular to v_0 is

$$\Delta p_\perp = F_\perp \tau = (zZe^2/b^2)2b/v_0 = 2zZe^2/bv_0 \quad . \qquad (20)$$

The scattering angle is

$$\Theta = \Delta p_{\perp}/p_0 = 2zZe^2/bmv_0^2 \; . \tag{21}$$

An accurate calculation yields

$$\tan(\Theta/2) = zZe^2/bmv_0^2 \tag{22}$$

(see for example, Goldstein's "Classical Mechanics"). For small angles of deflection Eq. (21) and (22) are equal.

Note that the magnitude of the scattering angle depends on the impact parameter and the initial velocity and is the same magnitude, but of opposite sign, for like and unlike charges. The impact parameter for a 90° scattering (from Eq. 22) is

$$b_0 = zZe^2/mv_0^2 \; , \tag{23}$$

i.e., the distance at which the potential energy is twice the original kinetic energy. The scattering angle for arbitrary b may be written as

$$\tan(\Theta/2) = b_0/b \tag{24}$$

For small angle deflections, $\Theta = 2b_0/b$; since b_0 is usually quite small, there are very few large angle deflections and we can use this approximation so long as we cut it off at approximately b_0.

All particles that pass within db of b will be scattered within a corresponding $d\Theta$ of $\Theta(b)$. We can define a differential cross-section $I(\Theta,\phi,v)$, where I is the cross-section for scattering into a small solid angle $d\Omega$ about the angles Θ and ϕ; spherical polar coordinates are used. Θ is the angle of deflection with respect to the initial direction of motion of the test particle (we take this to be the z direction) and ϕ is the angle the orbit plane makes with the x, z plane (x is perpendicular to the z axis and its direction can be chosen to be the most convenient for making calculations). The total cross-section is obtained by integrating $I(\Theta,\phi,v)$ over all solid angles,

$$\sigma(v) = \int_{4\pi} I(\Theta,\phi,v)\, d\Omega \ . \qquad (25)$$

By cylindrical symmetry $I(\Theta,\phi,v)$ is independent of ϕ and we can write

$$I(\Theta,v)2\pi\sin\Theta d\Theta = 2\pi b db \ . \qquad (26)$$

Eliminating b from Eqs. (24) and (26) yields the Rutherford scattering formula

$$I(\Theta) = (zZe^2/2mv_0^2)^2 \operatorname{cosec}^4(\Theta/2) \;. \tag{27}$$

— OK

For small values of Θ this is approximately

$$I(\Theta) = 16(zZe^2/mv_0^2)^2/\Theta^4 \;. \tag{27a}$$

4

The total cross-section is obviously infinite; it is the contributions that come from small Θ which give this divergence. Small Θ come from large values of b; for distances greater than the Debye length the plasma shields out the E field of the field particle and we should not include impact parameters greater than this. We will investigate this in much more detail shortly.

Multiple Coulomb Scattering

Because of the long range of the Coulomb force, there are many more small angle scatterings than large angle scatterings ($\Theta \sim 1$). It turns out that not only are there many more small angle scatterings than large angle scatterings, but their overall effect is much more important. We will show this shortly; however, first we will compute the effect of many small angle scatterings.

We consider a group of electrons which are initially moving in the z direction through a plasma that we will consider infinite and homogeneous. To begin with we will consider only their scattering by ions that we will treat as infinitely massive. On the average there are as many deflections up as down, and as many to the right as to the left. On the average there is no deflection. This does not mean that if we look at a particular particle that it will not be deflected. Any one electron will by chance encounter more ions on one side than on another; there will be a symmetrical spread of deflections about the z axis.

For a given electron the total velocity in the x-direction, acquired after N individual scatterings, will be

$$\Delta v_x = (\Delta v_x)_1 + (\Delta v_x)_2 + (\Delta v_x)_3 + \cdots + (\Delta v_x)_N \ . \quad (28)$$

For the ensemble (collection) of all the electrons in the group, we consider the average of Δv_x. Then due to the equality of a scattering in one direction and in the opposite direction, we have

$$\langle \Delta v_x \rangle = \langle \Delta v_y \rangle = 0 \ . \quad (29)$$

From our comments above, however, it is clear that this does not mean that there is no deflection; to get a measure of this let us look at the average of Δv_x^2; this is

$$\langle \Delta v_x^2 \rangle = \langle [(\Delta v_x)_1 + (\Delta v_x)_2 + (\Delta v_x)_3 + \cdots + (\Delta v_x)_N]^2 \rangle$$
(30)

The sum will consists of two types of terms, $\langle (\Delta v_x)_n{}^2 \rangle$ and $\langle (\Delta v_x)_n (\Delta v_x)_m \rangle$. The second type of these terms averages to zero if we assume that the ions are randomly distributed and uncorrelated ~~correlated~~. Thus, we are left with only the first type of term; since the plasma is uniform and homogeneous, $<(\Delta v_x)_n{}^2>$ is the same for all n. We therefore get

$$<\Delta v_x{}^2> = N<(\Delta v_x)_1{}^2> . \tag{31}$$

Furthermore, by symmetry we have

$$<\Delta v_x{}^2> = <\Delta v_y{}^2> \equiv <\Delta v_\perp{}^2>/2 . \tag{32}$$

We also have

$$\Delta v_\perp{}^2 = v_0{}^2 \sin^2 \Theta \sim v_0{}^2 \Theta^2 . \tag{33}$$

Using Eq. (24) we have

$$\sin\Theta = 2(b_0/b)/(1 + [b_0/b]^2) . \tag{34}$$

and

$$\Delta v_\perp{}^2 = [4v_0{}^2(b/b_0)^2]/(1 + [b/b_0]^2)^2 . \tag{35}$$

The cross-section for a collision within db of b is $2\pi b db$. The number of collisions of this type per unit path length is $nI(\Theta,v)d\Omega = 2\pi nbdb$. At each collision, the change in the $\Delta v_\perp{}^2$ is given by Eq. (35). Thus, the average total change in $\Delta v_\perp{}^2$

suffered by a test electron per cm of path length traveled in the plasma is

$$d\langle \Delta v_\perp^2 \rangle/dz = \int_0^{b_m} \{[4v_0^2(b/b_0)^2]/(1 + [b/b_0]^2)^2\}2\pi nbdb \ . \qquad (36)$$

where b_m is the maximum distance at which the Coulomb potential can be applied in view of the shielding of the ion charge by the plasma ($b_m \sim \lambda_D$). Spitzer discusses this point (see L. Spitzer, Jr., Physics of Fully Ionized Gases [Interscience Publishers, New York, N. Y., 1962]; 2nd ed.) and we will look at it more deeply later on in the course.

$$d\langle \Delta v_\perp^2 \rangle/dz = 8\pi n_f v_0^2 b_0^2 \int_0^{b_m \backslash b_0} [x^3/(1 + x^2)^2]dx \qquad (37)$$

$$d\langle \Delta v_\perp^2 \rangle/dz = 4\pi n_f v_0^2 b_0^2 \left\{ \ln[1 + (b_m/b_0)^2 + 1/[1 + (b_m/b_0)^2] - 1 \right\}. \qquad (38)$$

It is customary (later on we will show that it is correct) to take for b_m the Debye length, λ_D. The quantity b_0 is the impact parameter for a 90° collision, the separation distance at which the potential energy is equal to twice the initial kinetic energy; it is generally of the order of or smaller than atomic dimensions. On the other hand, b_m is much larger than the inter-particle spacing. Assuming b_m/b_0 and $\ln(b_m/b_0) \gg 1$, Eq. (38) becomes

$$d\langle \Delta v_\perp^2 \rangle/dz = 8\pi n_f v_0^2 b_0^2 \ln(b_m/b_0). \qquad (39a)$$

Inserting the value of $b_0 (b_0 = zZe^2/mv_0^2)$ from Eq. (23) gives

$$d\langle\Delta v_\perp^2\rangle/dz = 8\pi n_t (zZe^2/m_t v_0)^2 \ln(\Lambda) \qquad (39b)$$

where Λ is b_m/b_0. Typically the term $\ln(\Lambda)$ is between 10 and 20 for conditions of interest.

We may roughly say that when $\langle\Delta v_\perp^2\rangle = v^2$ the particle has been scattered through 90°. If we use this criterion we may compare the importance of small angle scattering to that of large angle scattering. We find from this

$$\text{(mult. scattering) /(single 90° scattering)}$$
$$\sim[8\pi \cancel{v_0^2} b_0^2 \ln(b_m/b_0)]/\pi b_0^2. \qquad (40)$$

Thus, it is about two orders of magnitude more probable that a particle is scattered through 90° by multiple small angle scatterings than it is for it to happen by one large angle scattering.

If the force between particles went as r^{-m}, then for $m \geq 3$ this is no longer true and large angle scatterings dominate.

The Coulomb scattering described here is elastic (we have neglected radiation, multiple ionization processes, and excitation of partially striped atoms): kinetic energy is conserved. In the center of mass coordinate system the individual particle energies are unchanged by a collision. This is no longer true in the laboratory frame. For example, if a light particle scatters from a stationary heavy particle by 180°, the light particle looses 2m/M of its energy. We shall now look at this more completely.

Energy Transfer and Changes in Parallel Velocity for Collisions of Particles with Finite m/M Ratios

Let us now look at encounters between two particles with finite mass ratios. To start with we will take one of the particles moving (the test particle) and one to be at rest (the field particle); this of course depends on the frame of reference we use. We saw in the last section that for a hot plasma, a good approximation is to assume to lowest order that the particles move along their undisturbed orbits. We then calculate their deflection by integrating the acceleration they would feel as they follow these lowest order orbits. For the case under consideration the zero order orbit has the test particle moving along a straight line orbit and the field particle at rest. The situation is the same as shown in Fig. 4 and our small angle deflection calculation for the test particle of the last section applies.

Now from conservation of momentum we have

$$m_t \, \Delta v_{t\perp} + m_f \, \Delta v_{f\perp} = 0 \; . \qquad (41)$$

We also have from conservation of energy

$$\Delta E_f = m_f\ \Delta v_{f\perp}^2/2 = (m_t^2/m_f 2)\Delta v_{t\perp}^2\ . \qquad (42)$$

The change in the energy of the field particle must come from the energy of the test particle.

$$\Delta E_t = -m_f\ \Delta v_{f\perp}^2/2 = -(m_t^2/m_f 2)\Delta v_{t\perp}^2\ , \qquad (43)$$

where the last equality follows by making use of Eq. (41). We may sum up the loss of energy given by Eq. (43) for encounters with many field particles; we thus get

$$\langle\Delta E_t\rangle = -(m_t^2/m_f 2)\ \langle\Delta v_{t\perp}^2\rangle\ . \qquad (44)$$

Now using Eq. (39), we find for the energy lost per unit distance in the plasma

$$d\langle\Delta E_t\rangle/dz = -(m_t^2/2m_f)\,8\pi n_f v_0^2 b_0^2 \ln(b_m/b_0)\ . \qquad (45)$$

Substituting in $b_0 = zZe^2/mv_0^2$ from Eq. (23) gives

$$d\langle\Delta E_t\rangle/dz = -(m_t^2/2m_f)\,8\pi n_f (zZe^2/m_t v_0)^2 \ln(\Lambda)\ , \qquad (46)$$

where Λ is given by

$$\Lambda = b_m/(zZe^2/mv_0^2)\ , \qquad (47)$$

Since $\ln(\Lambda)$, varies only slowly with Λ, $\ln(\Lambda)$ is typically between 10 and 20. We can also find the time rate of change of $<\Delta E_t>$ by multipling equation (46) by $dz/dt = v_0$. This gives

$$d<\Delta E_t>/dt = -(m_t^2/2m_f)8\pi n_f v_0 (zZe^2/m_t v_0)^2 \ln(\Lambda) \quad . \qquad (48)$$

Next let us consider the change in the velocity parallel to z; this can also be obtained from conservation of energy. From conservation of energy in an encounter we have

$$m_t\ v_0^2/2 = m_t\ ([v_0 + \Delta v_{tz}]^2 + \Delta v_{t\perp}^2)/2 + m_f\ (\Delta v_{f\perp}^2 + \Delta v_{fz}^2)/2 \ .$$
(49)

Assuming that the Δv_z^2 terms are negligible, as can be shown at the end of the calculation, Eq. (49) leads to

$$m_t\ (2[v_0\ \Delta v_{tz}] + \Delta v_{t\perp}^2) + m_f\ \Delta v_{f\perp}^2 = 0 \ , \qquad (50)$$

or

$$\Delta v_{tz} = -(m_t\, \Delta v_{t\perp}^2 + m_f\ \Delta v_{f\perp}^2)/2m_t\ v_0 \ , \qquad (51)$$

$$<\Delta v_{tz}> = -(1 + m_t/m_f)\ (<\Delta v_{t\perp}^2>/2v_0) \quad , \qquad (52a)$$

$$d<\Delta v_{tz}>/dz = -(1 + m_t/m_f)\ (1/2v_0)\, d<\Delta v_{t\perp}^2>/dz \ . \qquad (52)$$

When $<\Delta v_{t\perp}^2> \simeq v_0^2$, a large angle deflection will have occurred. We can estimate the distance (mean free path) required for this to happen by multipling Eq. (39) by l_{MFP} and equating the result to v_0^2. The result is

$$l_{MFP,\Theta} = (m_t v_0^2)^2/(8\pi n_f (zZe^2)^2 \ln(\Lambda)). \qquad (53)$$

We may also calculate the change in energy of a test particle in passing a distance l_{MFP} through a plasma by multipling Eq. (46) by l_{MFP}; this gives

$$\Delta E_t = -(m_t^2/2m_f) v_0^2 = -E_t m_t/m_f. \qquad (54)$$

If $m_f >> m_t$, as in the case of the scattering of electrons by ions, then the loss in energy per large angle deflection is m_t/m_f of the initial energy. To loose all its energy, the test particle must undergo m_f/m_t large angle deflections.

If $m_t >> m_f$, as in the case of the scattering of ions by electrons, then the loss in energy per large angle deflection is m_t/m_f of the initial energy which is much larger than the initial energy. This means the particle (ion) is brought to rest (by the light electrons) long before it is significantly scattered. We may define a mean free path for energy loss in the same way we did for large angle scattering; i.e., we multiply Eq. (46) by $l_{MFP,E}$ and equate it to the negative of the initial energy. This gives

$$l_{MFP,E} = (m_f/m_t)(m_t v_0^2)^2/(8\pi n_f (zZe^2)^2 \ln(A)) \qquad (55)$$

$$l_{MFP,E} = (m_f/m_t)\, l_{MFP,\Theta}$$

We may also compute a mean free path for stopping of the velocity in the initial direction of motion; to do this we multiply Eq. (54) by $l_{MFP,v}$ and equate it to the negative of the initial velocity. This gives

$$l_{MFP,v} = (m_t v_0^2)^2/(\{1 + m_t/m_f\}(4\pi n_f (zZe^2)^2 \ln(A)))$$

$$l_{MFP,v} = 2 l_{MFP,\Theta}/(1 + m_t/m_f). \qquad (56)$$

If $m_t >> m_f$, the particle is brought to rest in $m_f/2m_t$ of a large angle scattering distance.

Other quantities of interest are scattering times, stopping times and energy loss times. We can roughly get these from the mean free paths by dividing by the initial velocity, $t = l_{MFP}/v_0$. Using this relationship we get the following:

$$\tau(\Theta) = m_t^2 v_0^3/(8\pi n_f (zZe^2)^2 \ln(A)). \qquad (57)$$

$$\tau(E) = (m_f/m_t) m_t^2 v_0^3/(8\pi n_f (zZe^2)^2 \ln(A)). \qquad (58)$$

$$\tau(v) = m_t^2 v_0^3/(\{1 + m_t/m_f\}(4\pi n_f (zZe^2)^2 \ln(A))). \qquad (59)$$

From these we have the following relations:

$$\tau(E) = (m_f/m_t)\tau(\Theta) \tag{60}$$

$$\tau(v) = 2\tau(\Theta)/\{1+m_t/m_f\} \ . \tag{61}$$

The formulas (53) to (61) were derived assuming the field particle was at rest. It is clear that they can usually be applied to the interaction of electrons with ions since ions generally have relatively slow motions. We can also apply them to the interaction of very energetic ions (say fusion reaction products) with ions and if the electrons are not too hot with the electrons. Of course these formulas can be applied accurately to the interactions of the energetic tails of the electron distribution with the bulk of the electrons and likewise for the energetic ion tail with the bulk of the ions. These formulas can also be roughly applied to electron-electron and ion-ion interactions in a thermal plasma by using the thermal velocity for v_0. In this case, while the field particles are moving their motion is roughly of the same order as that of the test particle and large errors will not be made.

One case where they cannot be applied is to the slowing down of most ions by electrons or to the exchange of energy between most ions and the electrons. However, we can find the rate of energy loss by the ions to the electrons by making use of the property of detailed balance; for a thermal plasma, the ions must be losing energy to the electrons at the same rate they are receiving energy from the electrons. Equation (58) gives the time it takes for a set of test particles to transfer their energy to the field particles. If we let the test particles be electrons and the field particles be ions and if we use the electron thermal velocity for v_0, then this equation gives us the time for the electrons to give their energy to the ions. By detailed balance this must also be the time the ions take to transfer their energy to the electrons.

We can also find the slowing down or stopping time for an ion by the electrons. The ion energy is proportional to their velocity squared $E_i = M_i v_i^2/2$; thus,

$$dE_i/dt = M_i\ v_i dv_i/dt$$

or

$$E_i^{-1} dE_i/dt = 2v_i^{-1} dv_i/dt\ . \tag{62}$$

The slowing down time for the ions is twice the energy loss time or twice the number given by Eq. (58). As the ions move through the electrons, they gradually slow down. For slow moving ions, we may expect the drag force to be proportional to their velocity; this implies that the stopping time is the same for all velocities (so long as the velocity is small compared to the thermal velocity of the electrons). A more rigorous treatment verifies this.

The following table summarizes the results just obtained for collision times:

SUMMARY OF COULOMB COLLISION TIMES

$$\tau_{ei}(\Theta) = m_e^2 v_T^3/(8\pi n_i (Ze^2)^2 \ln(\Lambda)). \tag{I.1}$$

$$\tau_{ei}(E) = (m_i/m_e) m_e^2 v_T^3/(8\pi n_i (Ze^2)^2 \ln(\Lambda)). \tag{I.2}$$

$$\tau_{ei}(v_T) = m_e^2 v_T^3/((1+m_e/m_i)(4\pi n_i (Ze^2)^2 \ln(\Lambda))). \tag{I.3}$$

$$\tau_{ee}(\Theta) = m_e^2 v_T^3/(8\pi n_e e^4 \ln(\Lambda)). \tag{I.4}$$

$$\tau_{ee}(E) = m_e^2 v_T^3/(8\pi n_e e^4 \ln(\Lambda)). \tag{I.5}$$

$$\tau_{ee}(v_T) = m_e^2 v_T^3/(8\pi n_e e^4 \ln(\Lambda)). \tag{I.6}$$

$$\tau_{ii}(\Theta) = m_i^2 v_T^3/(8\pi n_i (Ze^2)^2 \ln(\Lambda)). \tag{I.7}$$

$$\tau_{ii}(E) = m_i^2 v_T^3/(8\pi n_i (Ze^2)^2 \ln(\Lambda)). \tag{I.8}$$

$$\tau_{ii}(v_T) = m_i^2 v_T^3/(8\pi n_i (Ze^2)^2 \ln(\Lambda)). \tag{I.9}$$

$$\tau_{ie}(E) = (m_i/m_e) m_e^2 v_{T,e}^3/(8\pi n_e Z e^4 \ln(\Lambda)). \tag{I.10}$$

$\leftarrow Z^2$?

$$\tau_{ie}(v_{0,i}) = (m_i/m_e) m_e^2 v_{T,e}^3/(8\pi n_e Z e^4 \ln(\Lambda)). \tag{I.11}$$

$\leftarrow Z^2$?

The corresponding mean free paths are obtained from these times by multiplying them by the appropriate thermal velocities. In cases where the formula can be applied to non-thermal situations, one simply uses the appropriate velocity to obtain the corresponding mean free paths.

Electrical Conductivity of a Plasma

We can use the results we have just obtained to compute the electrical conductivity of a plasma to a good approximation. For this we use a simple fluid model for the electrons; we write Newton's equation of motion for the electrons as follows

$$m_e \, d\langle v_e \rangle/dt = -eE - m_e \langle v_e \rangle/\tau_{ei}(v_{Te}) \; . \tag{63}$$

Here $\langle v_e \rangle$ is the average velocity of the electrons through the ions, E is the electric field, and $\tau_{ei}(v_{Te})$ is the collision time associated with a thermal electron. We look at a steady state so we set $d\langle v_e \rangle/dt$ equal to zero. Then solving for $\langle v_e \rangle$, we find

$$\langle v_e \rangle = (-eE/m_e)\tau_{ei}(v_{Te}) \; . \tag{64}$$

Multiplying this equation by $-en_e$ gives us the current density

$$-en_e\langle v_e \rangle = j = (n_e e^2/m_e)\tau_{ei}(v_e)E \; . \tag{65}$$

From this we get

$$j = \sigma E = (\omega_{pe}{}^2/4\pi)\tau_{ei}(v_{Te})E \; .$$

and

$$\sigma = (\omega_{pe}^2/4\pi)\tau_{ei}(v_{Te}) \ . \qquad (66)$$

The resistivity is the reciprocal of σ so we have

$$\eta = 4\pi/[\omega_{pe}^2\tau_{ei}(v_{Te})] \ . \qquad (67)$$

If we use the $\tau_{ei}(v_T)$ obtained from Eq. (I.3) and neglect the ratio m_e/m_i, then we find for η

$$\eta = (16\pi^2 n_e Z e^4 \ln(\Lambda))/[\omega_{pe}^2 m_e^2 v_{Te}^3] \ .$$

Using $v_{Te}^2 = 3kT_e/m_e$ we get

$$\eta = (4\pi Z e^2 m_e^{1/2} \ln(\Lambda))/[3kT_e]^{3/2} \ . \qquad (68)$$

We can compare this with the result given by Spitzer (see L. Spitzer, Jr., Physics of Fully Ionized Gases [Interscience Publishers, New York, N.Y., 1962]; 2nd ed.) which was obtained by solving the Boltzmann equation for the electrons with only electron ion collisions included; he gives

$$\eta = (\pi^{3/2} Z e^2 m_e^{1/2} \ln(\Lambda))/2[2kT_e]^{3/2}. \qquad (69)$$

Numerically Spitzer's value of η is given by

$$\eta = 3\mathrm{x}10^{-3}Z\ln(\Lambda))/T_e^{3/2}\mathrm{ohm\text{-}cm} \qquad (70)$$

where T_e is in eV. The value of η that we find is 1.6 times larger than that found by Spitzer; however all the dependences are correct. We can see that we should find a larger value than Spitzer since the electron collision rate is inversely proportional to their velocity cubed; thus, higher velocity electrons will contribute more to the conductivity (and less to the resistivity) than lower velocity electrons. To get a more correct value we should replace $v_{Te} = (3kT_e/m_e)^{1/2}$ by $v_{Te} = (\langle v^3 \rangle)^{1/3}$. This gives $(\langle v^3 \rangle)^{1/3} = 2.08(kT_e/m_e)^{1/2}$, this reduces the resistivity by a factor of 1.74 which brings it into pretty good agreement with that given by Spitzer.

In the above calculation we have neglected electron-electron collisions; only electron-ion collisions are kept. This is known as a Lorentz model for the conductivity or resistivity. This model is appropriate because electron-electron collisions do not change the current directly; electron momentum is conserved for such collisions and so is the total electron current. Although they do not directly change the electron current, electron-electron collisions do affect the current indirectly. We have seen that the current is preferentially carried by the energetic electrons. These electrons collide with the lower energy bulk electrons and transfer their current to them. The lower energy electrons collide more rapidly with the ions and so dissipate the current more rapidly. Spitzer has carried out detailed calculations of the resistivity for both the Lorentz model and from the kinetic equations including electron-electron collisions. If the ionic charge is one, electron-electron collisions increase the resistivity by a factor of 1.7 (approximately 2) over that for the Lorentz model. For large ionic charges, the factor by which the resistivity is increased decreases (roughly as [1+ 1/Z], for more precise values see the Spitzer reference). Numerically Spitzer gives the following formula for the plasma resistivity in the important case of Z equal one.

$$\eta = 5.2\mathrm{x}10^{-3}\ln(\Lambda))/T_e^{3/2} \text{ ohm-cm} \qquad (71)$$

An interesting point about the resistivity is that it is independent of the density. This is because as the density goes up, the collision time decreases in proportion to the density and so the drift velocity of the electrons decreases in proportion to the density. However, the number of carriers increases in proportion to the density so that the current density, $j \propto n\langle v_e \rangle$, is independent of n. Of course for a given E, the drift velocity becomes larger and larger as the density gets lower and at some point the drift will approach the thermal velocity and the resistivity (conductivity) is modified by the drift itself; the resistivity becomes nonlinear at this point.

It is informative to compute the conductivity of some typical plasmas. For a laboratory plasma of hydrogen at 10 eV, Eq. (71) gives (setting ln(Λ) equal to 10) η = 1.7 x 10^{-3}ohm-cm. For a fusion plasma at 10^4 eV we get $\eta = 5.2 \times 10^{-8}$ ohm-cm which is much lower than that for room temperature copper ($\eta = 1.7 \times 10^{-6}$ ohm-cm). We might also try to apply this formula to metallic copper. In this case we must use the Fermi energy of the electrons instead of the temperature; this is roughly 10 eV. Since the resistivity is independent of density, this would give the same value for the resistivity that we found for the 10 eV laboratory plasma. This value is about 1000 times too large. It is interesting that for mercury the resistivity is 10^{-4} ohm-cm., which is only one order of magnitude better than our laboratory plasma. Actually for Hg the $\ln(\Lambda)$ term is only about 1 - 2 so the conductivity is about an order or magnitude larger, or the resistivity an order of magnitude smaller. We get roughly the right value of the resistivity from our plasma formula. The reason for the discrepancy for Cu is due to the high degree of order in the arrangement of the Cu atoms in the metal so that they do not act like a lot of random scatterers; for Hg this is not such a large effect. Of course one should make these calculation quantum mechanically to be correct.

Runaway Electrons

From the above calculations we have seen that the collision time and the mean free paths depend on the particles energy. The collision time of an electron goes essentially as its energy to the three halves power [see Eq. (57)]. If an electric field is applied to a plasma to drive current, the very energetic electrons will very weakly collide and will freely accelerate. They will gain energy faster than they are scattered. As they gain energy, collisions become weaker and weaker and so they become more and more collisionless; they will "run away". We can explore this phenomenon by a calculation that parallels our conductivity calculation. Consider the electrons that start with a velocity v_0. We write a fluid equation for the acceleration of these electrons

$$m_e\, d\Delta v(v_0)/dt = -eE - m_e\, \Delta v(v_0)/\tau_{ei}(v_0). \qquad (72)$$

Let us now compute the steady state $\Delta v(v_0)$ that is achieved; this is

$$\Delta v(v_0) = -\,[(eE/m_e)\tau_{ei}(v_T)][v_0^3/v_T^3] \qquad (73)$$

For $[(eE/m_e)\tau_{ei}(v_T)][v_0^2/v_T^3] \geq 1$, $\Delta v(v_0) \geq v_0$ and the collision rate is dropping faster than the electrons can be scattered, this is known as the Dreicer runaway condition. When the Dreicer condition holds, it implies that there will be a significant change in velocity and hence a large reduction in the collision rate before a collision has taken place. The collision rate will drop fast and the particle will freely accelerate, i.e., "run away".

Low Energy Electron Atom and Ion Atom Collisions

Let us consider a low energy charged particle (electron or ion) neutral atom collision. The situation is illustrated in Fig. 5 below.

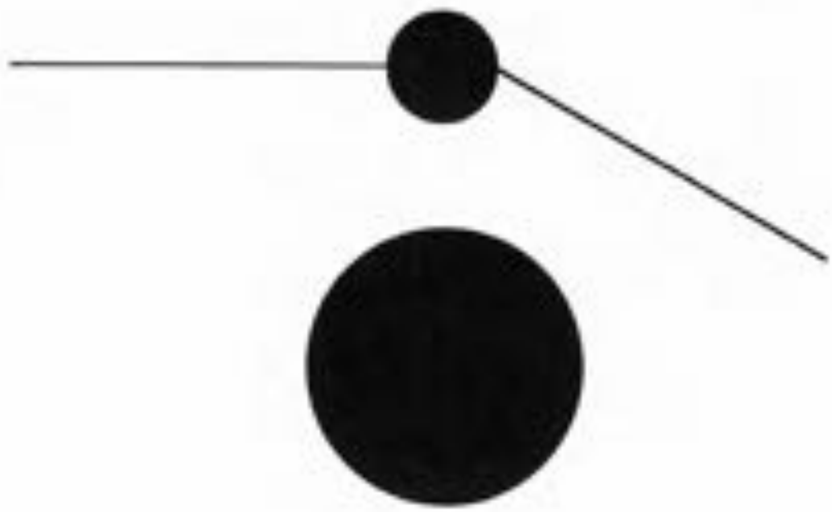


Figure 5

The electric field of the charged particle polarizes the atom as it passes. For an ion as is shown, the electrons are pulled towards it and the nucleus is pushed away. The polarization of the atom produces a dipole moment $\mathbf{P}$.

$$\mathbf{P} = \alpha\mathbf{E} = \alpha Q\mathbf{r}/r^3 \ . \tag{74}$$

Here α is the atomic polarizability, Q is the charge on the particle, $\mathbf{r}$ is the vector separation distance, and r is the magnitude of the separation. The electric field produced by the atomic dipole at the charged particle is

$$E = 2P/r^3. \tag{75}$$

This field is always in a direction so as to give attraction between the charge and the atom and is directed along the line joining them. The attractive force is

$$F = Q\,E = 2Q^2\,\alpha/r^5. \tag{76}$$

We can obtain an approximation to α by regarding the atomic electrons as being harmonically bound to the nuclei. Let Δr be the displacement of the electron cloud relative to the nucleus. Then our harmonic approximation gives us the following relation

$$d^2\Delta r/dt^2 = -\omega_a{}^2\Delta r - eE/m_e. \tag{77}$$

Assuming that the time variation of E is slow compared to the atomic response time, $\omega_a{}^{-1}$, we find for Δr

$$\Delta r = -eE/m_e\omega_a{}^2\ . \tag{78}$$

The dipole moment is given by

$$P = -e\Delta r = e^2E/m_e\omega_a{}^2\ . \tag{79}$$

This gives for the polarizability

$$\alpha = e^2/m_e\omega_a{}^2. \tag{80}$$

Now ω_a is related to the ionization energy I roughly by $h\omega_a/2\pi = I$ (this approximation can be justified from quantum mechanics). This gives

$$\alpha = (e^2/m_e)(h/2\pi I)^2 \qquad (81)$$

If we now use the relations

$$p = h/\lambda \ , \ p^2/2m_e = K = (h/\lambda)^2/2m_e \ ,$$

from quantum mechanics and $e\Phi = 2K$ ($-e\Phi$ is the potential energy) from the Virial Theorem for a system interacting by Coulomb forces, we find

$$e\Phi - K = \ I \simeq \ K = (h/\lambda)^2/2m_e \ . \qquad (82)$$

Using Eqs. (8~~2~~1) and (8~~3~~2) we get

$$\alpha = 2(e^2/I)\lambda^2)/2\pi \simeq a_A^3 \qquad (83)$$

table doesn't have αI^2

where $a_A{}^3$ is the volume of the atom, a_A is the atomic radius. The following table lists I, α, and αI^2 for the noble gases. By Eq. (82) αI^2 would be constant; we see from the table that it varies by roughly a factor of 5 in going from He to Xe. For the heavier more complex atoms, our simple one electron model is clearly too simple but it does remarkably well.

Table 1

Table I - Diameters of Some Atomic Ions, In Å

		O^{--}	2.6	S^{--}	3.5	Se--	3.5
H^-	2.5	F-	2.7	Cl^-	3.6	Br-	3.9
He	1.9	Ne	2.3	Ar	2.8	Kr	3.2
Li^+	1.6	Na^+	2.0	K^+	2.7	Rb^+	3.0
Be^{++}	0.7	Mg^{++}	1.6	Ca^{++}	2.1	Sr^{++}	2.5
B^{+++}	—	Al^{+++}	1.2	Sc^{+++}	1.7	Y^{+++}	2.1
C^{++++}	0.4	Si^{++++}	0.8	Ti^{++++}	1.3	Zr^{++++}	1.7
N^{+++++}	0.3	P^{+++++}	0.7				

Mean Polarizabilities of Inert Gases and Atomic Ions.

(The numbers denote $\bar{\alpha} \cdot 10^{24}$ cm.^{3})

		F^-	0.99	Cl^-	3.05	Br^-	4.17	I^-	2.8
He	0.202	Ne	0.392	Ar	1.629	Kr	2.46	Xe	4.00
Li^+	0.075	Na^+	0.21	K^+	0.87	Rb^+	1.81	Cs^+	2.79
		Mg^{++}	0.12			Sr^{++}	1.42		
		Al^{+++}	0.065						
		Si^{++++}	0.043						

Ionization and Excitation

The calculations of ionization and excitation cross-sections require quantum mechanics (see Mott, N.F. and H.S.W. Massey, "The Theory of Atomic Collisions", 2nd edition [Clarendon Press, Oxford, 1949], and Massey, H.S.W. and E.H.S. Burhop, "Electronic and Ionic Impact Phenomena", [Clarendon Press, Oxford, 1952]). However, it is possible to make reasonable estimates of these from a semi-classical calculations; we shall do this now.

We use the results we developed in our treatment of Coulomb collisions. We consider an energetic electron impinging on an atom and ask what the cross-section is for imparting the ionization energy or more to an atomic electron. For simplicity we will consider the atomic electrons at rest; we could improve the model by allowing them to have a distribution of energies but we will see that we can get reasonably good results without doing this. From our Coulomb scattering calculations, Eqs. (33), (34), and (42), we get for the energy transfer

$$\Delta W = 2e^4/(m_e v_0^2\, b^2) = [\Phi(b)]^2/K\ . \qquad (84)$$

The cross-section for transferring energy ΔW or more is

$$\sigma = \pi b^2 = \pi e^4/(K\ \Delta W)\ . \qquad (85)$$

To get ionization, ΔW must be the ionization energy I. Thus, we get

$$\sigma_i = \pi e^4/(K\ I) = 729 x \pi a_0^2/IK\ . \tag{86}$$

This formula predicts that the cross-section decreases as K^{-1}; it is clear that it must fail when K approaches I; we cannot get ionization when K is below I. Furthermore, when K approaches I the motion of the atomic electron becomes more and more important and should be taken into account. We have seen that to treat Coulomb collisions in a plasma we had to introduce maximum and minimum distance cut offs; these gave the $\ln\Lambda$ factor. If we include a distribution of energies for the atomic electrons and averaged it in computing energy transfers or if we were more precise and made a quantum mechanical calculation, then such a ln factor would enter the result. A ln factor that makes the cross-section go to zero at K equal I is lnK/I. In fact the following is found to be a good approximation to ionization cross-sections.

$$\sigma_i = [\pi e^4 \ln(K/I)]/KI = [729 x \pi a_0^2 \ln(K/I)]/IK$$

$$\sigma_i = [5.7x10^{-14} \ln(K/I)]/IK\ . \tag{87}$$

This expression has a maximum for K equal to eI (e is the base of natural logarithms) or roughly 27.2 ok ~~2.7~~xI. This gives a peak cross-section of 1/16 ~~1.8~~x$10^{-14}/I^2$. For atomic hydrogen this gives a peak ionization cross-section of $2x10^{-16}$ for

5×10^{-17}

Figure 6

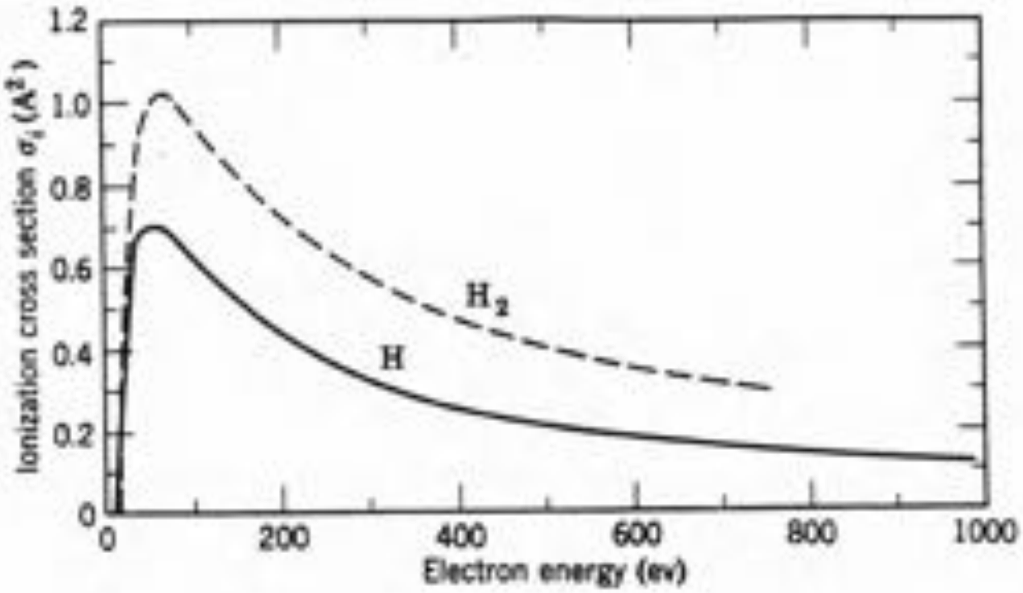


Fig.6 - Ionization cross sections σ_i for atomic and molecular hydrogen by electron impact. After W.L. Fite and R.T. Brackman, *Phys. Rev.*, **112**, 1141 (1958).

37eV electrons compared to a quantum calculation of 0.7×10^{-16} for about 50 eV electrons (see D.J. Rose and M. Clark, Jr., "Plasmas and Controlled Fusion", The M.I.T. Press, Third Printing, 1973). Figure 6 is a plot of σ_i = vs. K/I.

We can also estimate the total excitation cross-section this way. For this we need to transfer an amount of energy between the first excited level and the ionization energy. Replacing I in Eq. (87) by the first excitation energy and subtracting the cross-section for ionization (since transfer of this much energy gives ionization and not excitation) gives

$$\sigma_E = \pi e^4\{[\ln(K/E_1)]/KE_1 - [\ln(K/I)]/KI\}. \qquad (88)$$

This cannot tell us which excited states are generated, but it allows us to estimate the energy transferred to the atoms and radiated away.

Radiative Recombination

For plasma formation and maintenance, we must consider not only the ionization processes (and radiative energy loss processes) but we should also look at recombination which removes charged particles from the free state. Conceptually the simplest of these, though often not a very important process from the numerical point of view, is radiative recombination. In this case we have an electron passing an ion (shown in Fig. 7); the

no fig 7

acceleration of the electron causes it to radiate and if it radiates enough energy it will go into a bound orbit. Strictly speaking, this problem must be treated by using quantum mechanics. However, we will find that a semi-classical semi-quantum treatment will get us in the right range.

For an electron to be captured, it must radiate away at least its initial kinetic energy. The rate of radiation by a classical electron is given by

$$P = (2e^2a^2)/(3c^3). \tag{89}$$

The acceleration of an electron as it passes a charge Z at a distance ρ is

$$a = (Ze^2)/(m_e\rho^2). \tag{90}$$

Thus, the power radiated is

$$P = (2Z^2e^6)/(3m_e{}^2c^3\rho^4). \tag{91}$$

The total energy radiated is roughly

$$\Delta W = Pt - 2\rho P/v\ . \tag{92}$$

Substituting in Eq. (91) for P gives

$$\Delta W = (4Z^2e^6)/(3m_ec^3\rho^3m_ev) = m_ev_o^2/2. \tag{93}$$

One might be tempted to use the classical electron velocity in this formula; however if one does this one finds that the electron must pass so close to the nucleus that the uncertainty principal comes into play. From the uncertainty principal there is a relation between the the size of a region, ρ, into which we squeeze an electron and its momentum; this is $p\rho = h/2\pi$. This gives a velocity $v = h/2\pi m_e\rho$; we use this velocity in the denominator of the left-hand side of Eq. (93). Substituting these expressions into our expression for ΔW gives the expression for capture

$$\Delta W = (8\pi Z^2e^6)/(3m_ec^3\rho^2h) = m_ev_o^2/2. \tag{94}$$

The cross-section for capture is

$$\pi\rho^2 = (16\pi^2Z^2e^6)/(3m_e{}^2v_o{}^2c^3h). \tag{95}$$

In numerical values this is

$$\sigma = \pi\rho^2 = 1.2\mathrm{x}10^{-21}Z^2/K\ \mathrm{cm}^2\ . \tag{96}$$

To find the rate of recombination in a plasma, we must average this cross-section over the velocity distribution of the electrons; for a thermal plasma we should average it over a Maxwellian. The result for a thermal plasma is

$$\langle\sigma v\rangle = \alpha Z^2[2m/\pi kT]^{1/2}, \qquad (97)$$

$$\alpha = 1.2\times10^{-21} \; ,$$

or in numerical values

$$\langle\sigma v\rangle = 4.0 \times 10^{-14} Z^2/[kT]^{1/2}. \qquad (98)$$

Spitzer gives the result of a much more detailed Quantum Mechanical calculation; he gives

$$\langle\sigma v\rangle_{Spitzer} = 1.94 \times 10^{-13} Z^2/[kT]^{1/2}\Phi(\beta), \qquad (99)$$

where $\Phi(\beta)$ is a numerical factor that is a function of temperature and the ionic charge, $\beta = 13.5Z^2/T$. Spitzer gives the following table for $\Phi(\beta)$.

β	1	2	5	10	100	1000
$\Phi(\beta)$	0.96	1.26	1.69	2.02	3.2	4.3

The expression that we get is about four times smaller than that given by Spitzer which is not surprising given the crudeness of our calculation.

The rate of recombination is obtained from

$$dn_e/dt = - <\sigma v>n_e n_i = -n_e n_i \, 4.\overset{0}{\cancel{9}} \times 10^{-14} Z^2/[kT]^{1/2}. \qquad (100)$$

<u>Three-Body Recombination</u>

One recombination process that is important in low temperature and high density plasmas is that of three body recombination. In this case two electrons encounter a single ion at the same time; one of the electrons gives up enough energy to the other that it cannot escape the ion. This is of course a complex three-body problem and cannot be solved exactly. However, by using the work we have done on Coulomb collisions and ionization of atoms, we can get a pretty good answer that agrees favorably with the more detailed calculations that have been carried out.

Consider a low temperature plasma for which $kT << I$. We calculate the recombination rate using the law of detailed balance for thermal equilibrium and the rate of ionization of highly excited atoms.

We first compute the number of nuclei with an electron bound with energy of kT. These are the important atoms. If the binding is much larger than kT, then there are very few electrons in the plasma that can kick the electron out (ionize the atom) and ionization will be very slow. The orbiting electron is constantly loosing energy by radiation and is becoming more and more tightly bound. It therefore has effectively recombined. On the other hand if the electron is less tightly bound than kT, nearly all the electrons in the plasma are capable of ionizing the atom and it will be ionized in very short order.

It is possible to treat these highly excited atoms classically by the correspondence principal. The density of such atoms is roughly

$$n_a = n_e n_i (4\pi/3)\ r_0^3\ \underline{\{\exp[e\phi/kT]\}},\quad r_0 = e^2Z/kT. \ (101)$$

The exponential factor that is underlined has exponent 1 and to the accuracy we are working, we will treat the whole quantity as one. The factor, $n_e(4p/3)\ r_0^3$ is the number of electrons that we expect to find in a sphere of radius r_0; a very large fraction of them will be bound to the ion.

The rate of ionization of these highly excited atoms is given by

$$dn_a/dt = -\ n_e n_a <\sigma v> = -n_e^2 n_i (4\pi/3)\ r_0^3 <\sigma v>. \ (102)$$

From the section on the ionization of atoms we found

$$\sigma_i = [\pi e^4 \ln(K/I)]/KI\ ,$$

since virtually every electron has an energy around kT and can ionize the atom, we can approximate this by

$$\sigma_i = \pi e^4/T^2 \ . \tag{103}$$

We approximate the velocity by the thermal velocity, i.e., $v_T = 6 \times 10^7 T^{1/2}$ and we approximate $<\sigma v>$ by

$$<\sigma v> = (\pi e^4/T^2) v_T = 6 \times 10^7 (\pi e^4/T^{3/2}) . \tag{104}$$

Using this in Eq. (101) gives

$$dn_a/dt = - n_e n_a <\sigma v> = -6 \times 10^7 n_e n_a (\pi e^4/T^{3/2}) . \tag{105}$$

Now using Eq. (100) for n_a with the underlined exp set equal to one, we get

$$dn_a/dt = -6 \times 10^7 n_e^2 n_i (4\pi/3) \ r_0^3 \ (\pi e^4/T^{3/2}) . \tag{106}$$

Replacing r_0 by Ze^2/T, we get

$$dn_a/dt = -6 \times 10^7 n_e^2 n_i (4\pi/3) \ (Ze^2)^3 \ (\pi e^4/T^{9/2}) . \tag{107}$$

By plugging in numbers we get

$$dn_a/dt = -7 \times 10^{-26} n_e^2 n_i Z^3/T^{9/2} . \tag{108}$$

Since highly excited atoms with binding energy about T will be rapidly formed and destroyed in the plasma, we expect them to be in thermal equilibrium. Thus, their rate of destruction by electron collisions should equal their rate of formation by the inverse process of three-body recombination in accordance with the law of detailed balance. We thus have for the rate of three-body recombination into these highly excited atoms

$$dn_e/dt = -7 \times 10^{-26} n_e^2 n_i Z^3 / T^{9/2}. \qquad (109)$$

As we have already commented, for electrons that are just a little more tightly bound than this, the rate of detachment will become so slow that they have little chance of re-ionizing before the electron's radiation carries them so far down into the potential well that they cannot ionize again.

We do not expect that the rate of formation of these slightly more tightly bound states will be significantly different from that which we have just calculated. We take Eq. (108) to be roughly the three-body recombination coefficient for a plasma.

Detailed calculations, based on quantum mechanics, of the three-body recombination rate have been made by Hinnov and Herschberg (Phys. Rev. 125, 795, [1962], of course some approximations are made in this derivation also). They give the following expression for the three-body recombination rate

$$dn_e/dt = -\alpha n_e^2 n_i Z^3 / T^{9/2} = -5.6 \times 10^{-27} n_e^2 n_i Z^3 / T^{9/2}. \quad (110)$$

This expression is about a factor of eight smaller than our result and to some degree has been checked experimentally; however, all the functional dependences are the same.

Other Recombination Effects

There are a great many other effects that lead to recombination. To treat them in any detail would be a course in itself. We will, however, mention some of them and the physics behind them so that the student will be aware of them and can look them up if he has need of them.

(1) Molecular Ion Dissociative Recombination: here an electron encounters a molecular ion and recombines with it; the molecule breaks apart, the fragments carrying off the recombination energy. An example is

$$e + H_2^+ \leftrightarrow 2H \; .$$

(2) Dielectronic Recombination: here an electron encounters a partially stripped atom (an atom which has lost only a fraction of its electrons). The incoming electron encounters one of the bound electrons and transfers a sufficient amount of its energy to it so that it cannot immediately escape; an atom with two excited electrons is produced. Such an atom has sufficient energy to eject an electron and return to its initial state or possibly an excited state. However, the excited electrons are radiating and

if they have time to radiate away their energy before such auto detachment takes place, recombination will have occurred. The rate of radiation is roughly proportional to Z^6; the rate of re-encounter is inversely proportional to the volume of the electron cloud which goes roughly as Z^3 and iis nversely proportional to the relative energies of the two electrons to the three halves power, this goes roughly as Z^{-3}. Thus, the radiative loss takes over at high Z and is important for the capture of electrons by heavy impurities in fusion machines and by heavy atoms in the sun's corona.

Charge Exchange

Another very important process that plays a large role in many plasmas is that of charge exchange. In this process, an ion passes a neutral atom and an electron is transfered from the neutral to the ion. The ion is thus neutralized and an ion is left behind. This process is particularly important for a collision between an ion and an atom of the same type; in this case the electron transfer involves no change in energy; the transfer is therefore a resonant process. Some processes where charge exchange is important are the following:

(1) Charge exchange has a large effect on the flow of ion current across a magnetic field. A moving ion exchanges an electron with a stationary neutral. A moving neutral is produced which freely crosses the magnetic field; also a stationary

ion is produced which has to be accelerated from rest by existing electric fields (we will look at this in much more detail shortly).

(2) The above is an important process in star formation: two important effects are involved here. First the collapsing stellar gas must slip through the interstellar magnetic field. If it cannot do this, then the magnetic field will be compressed as the cloud contracts. Without this slip, the product BA (B is the magnetic field and A is a typical cross-sectional area) remains constant as the cloud carries the magnetic flux along with it. Since very large radial contractions are involved ($\sim 10^8$) and hence even larger area contractions are involved ($\sim 10^{16}$), even starting with very small interstellar magnetic fields, very large magnetic fields will be produced before the star forms. The resulting magnetic forces (pressure) will prevent further collapse.

The second effect is that the gas cloud must be able to get rid of its angular momentum in order to collapse. The interstellar magnetic field is anchored in the vast amount of plasma that surrounds the collapsing cloud. The ions are tied to this magnetic field (as we shall see later) and so are rotating very slowly. The spinning collapsing neutral gas cloud transfers momentum to the ions through charge exchange and the ions in turn transfer the momentum to the magnetic field (by twisting it up) which carries it out to the surrounding interstellar gas.

(3) Charge exchange is an important process for fusion reactors for a number of reasons:

(a) It is a serious source of cooling of the fusion plasma. A neutral atom can enter a fusion plasma from the wall; if it charge exchanges with a fuel atom, it will produce a fast neutral that can freely cross the confining magnetic field and escape; the initial atom from the wall remains as a cold ion in the plasma.

(b) The escaping fast neutrals do have a use; they can be collected by a suitable detector and their energy distribution can be measured. This allows one to measure the ion temperature, or even the ion velocity distribution function of the plasma. For this reason, sometimes beams of modest energy neutrals are injected into the plasma so that the escaping energetic neutrals can be analyzed.

(c) Energetic neutral beams can be used to heat plasmas to fusion energies. One starts by accelerating protons, deuterons, or tritons to energies in the range of 100 to 200 KeV; these are then passed through a neutral gas cell where the energetic ions are converted to energetic neutrals through charge exchange. The energetic neutrals cross the confining magnetic field and penetrate the fusion plasma. Some of the neutrals are ionized by encounters with the deuterons, tritons, and electrons of the plasma. Others charge exchange with the plasma ions but even in this case a relatively low energy neutral leaves the plasma and a highly energetic ion remains in the plasma. In all cases the plasma is heated.

Semi-Quantitative Calculation of Charge Exchange

Let us consider a proton approaching a neutral hydrogen atom as shown:

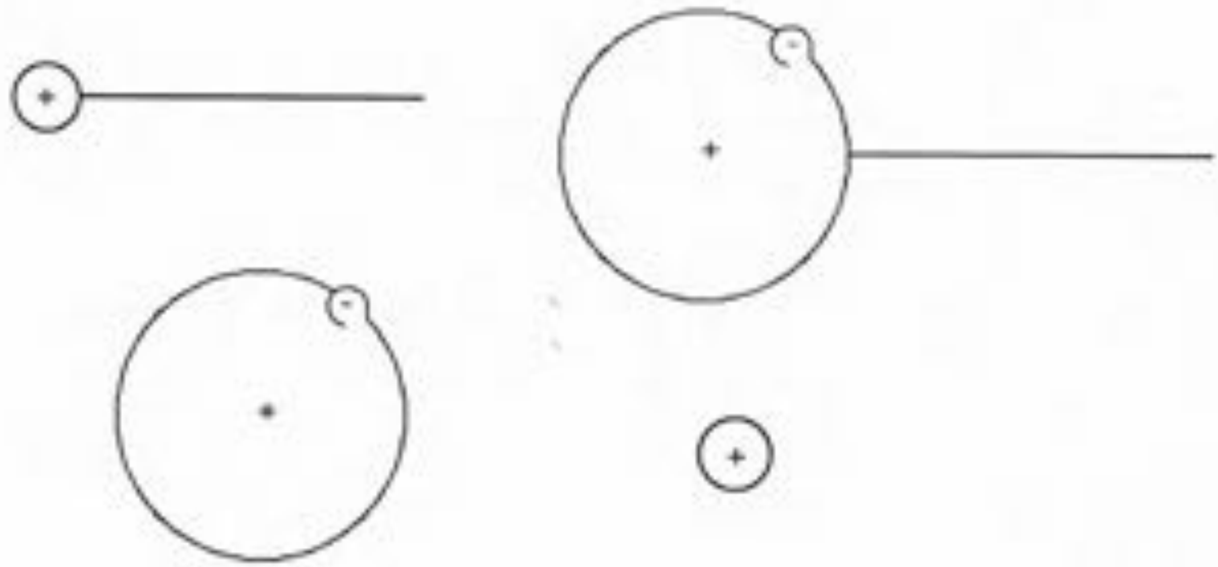


Fig. 7

There are quantum orbits for the electron orbiting each of the nuclei. To transfer from one atom to the other, the electron must pass through a region of negative kinetic energy and must tunnel through this region. This has to be treated quantum mechanically. In order to do this we will expand the wave functions of the combined system in terms of the wave functions of two isolated atoms.

During charge exchange, the ions are generally moving past each other slowly when compared to the electron orbital velocity. We can find the electronic wave function as if the protons were at rest at whatever their instantaneous separation is (the Born Oppenheimer approximation). We may write the wave function as a combination of the wave functions of the two protons. By the symmetry of the system we can write the combined wave functions as symmetric and antisymmetric combinations of the two possible atomic wave functions. Thus, we have

$$\Psi_I = \alpha_I(\Psi_1 + \Psi_2),\ \Psi_{II} = \alpha_{II}(\Psi_1 - \Psi_2). \tag{111}$$

Since charge transfer generally takes place for proton separations large compared to the size of the atoms, Ψ_1 is very small in the region where Ψ_2 is appreciable and vise versa. Now α_I and α_{II} are determined by the normalization conditions

$$\int \Psi_I \times \Psi_I^* d\tau = 1,\ \int \Psi_{II} \times \Psi_{II}^* d\tau = 1. \tag{112}$$

By Eq. (110) this is equal to

$$\alpha_I \alpha_I^* \int (\Psi_1 + \Psi_2) \times (\Psi_1 + \Psi_2)^* d\tau = 1, \tag{113}$$

and

$$\alpha_{II} \alpha_{II}^* \int (\Psi_1 - \Psi_2) \times (\Psi_1 - \Psi_2)^* d\tau = 1. \tag{114}$$

By our argument that Ψ_1 is small where Ψ_2 is appreciable and vise versa, all the cross products involving Ψ_1's and Ψ_2's make negligible contributions. The squares of the Ψ_1's and Ψ_2's integrate to 1 so that Eqs. (112) and (113) give

$$2\alpha_I \alpha_I^* = 1, \tag{115}$$

$$2\alpha_{II} \alpha_{II}^* = 1, \tag{116}$$

and by symmetry

$$\alpha_I = \alpha_{II} = 2^{-1/2}. \tag{117}$$

We can now compute the energies associated with these two states from

$$E = \int \Psi H \Psi^* d\tau, \tag{118}$$

where H is the Hamiltonian for the system.

$-H$!

$$(h/2\pi)^2\nabla^2 + e^2/|\mathbf{r}-\mathbf{R}_1| + e^2/|\mathbf{r}-\mathbf{R}_2| = H. \tag{119}$$

We get

$$E_I = \int \Psi_I H \Psi_I^* d\tau = (1/2)\int (\Psi_1 + \Psi_2) H (\Psi_1^* + \Psi_2^*) d\tau \;, \quad (12$$

and

$$E_{II} = \int \Psi_{II} H \Psi_{II}^* d\tau = (1/2)\int (\Psi_1 - \Psi_2) H (\Psi_1^* - \Psi_2^*) d\tau. \quad (12$$

There are two important integrals that contribute to the difference in energies; these are

$$I_1 = \int \Psi_1(e^2/|\mathbf{r}-\mathbf{R}_2|)\Psi_1^* d\tau, \tag{122}$$

and

$$I_2 = \int \Psi_2(e^2/|\mathbf{r}-\mathbf{R}_2|)\Psi_1^* d\tau. \tag{123}$$

Integrals with the subscripts 1 and 2 interchanged also enter but by symmetry are identical to the corresponding integrals in Eq. (121) and (122). The energies are

$$E_I = E_0 - (1/2)(I_1 - I_2). \tag{124}$$

and

$$E_{II} = E_0 - (1/2)(I_1 + I_2). \tag{125}$$

The difference in energy is

$$\Delta E = E_I - E_{II} = -I_2. \tag{126}$$

The major contribution to I_2 comes from the region around the mid-point between the two protons. To a good approximation I_2 is given by

$$\frac{\pi^{1/2}}{4^{3/4}}\left(\frac{R}{a_0}\right)^{1/2}\frac{e^2}{R}e^{-\frac{R}{a_0}}$$

$$I_2 = (8e^2/a_0)(\exp[-R/a_0]). \tag{127}$$

!

$$\left(\frac{R}{a_0}\right)^{1/2}\frac{e^2}{a_0}e^{-R/a_0}$$

where a_0 is the Bohr radius. If we denote the initial neutral atom by the subscript 1 and the initial proton by the subscript 2, then at t equal zero the wave function for the system is $\Psi_1 = 2^{-1/2}(\Psi_I + \Psi_{II})$.

$2^{1/2}$

The two wave functions Ψ_I and Ψ_{II} have different frequencies. $\omega_I = 2\pi E_I/h$ and $\omega_{II} = 2\pi E_{II}/h$. Thus, as time goes on the two wave functions will get out of phase with each other; when Ψ_{II} is π out of phase with Ψ_I, then the wave function of the system will be Ψ_2 and the electron will have transfered from the initial neutral atom to the initial proton.

To estimate when this will happen we note that the difference in frequencies of the two states is largest when the separation of the nuclei is the smallest, i.e., at the distance of closest approach. Let us call this distance R_0; the difference in frequencies of the two states at that separation is

$$\Delta\omega = (16\pi e^2/h a_0)(\exp[-R_0/a_0]) = (8\omega_0)(\exp[-R_0/a_0]), \quad (128)$$

where ω_0 is the orbital frequency associated with the ground state of hydrogen. The time the particles remain close together is roughly

$$\tau = 2R_0/V_0. \quad (129)$$

Multipling $\Delta\omega$ by t and equating it to π gives the distance R_0 at which we can expect charge exchange to take place

$$(\Delta\omega)\tau = (16\omega_0)(R_0/V_0)(\exp[-R_0/a_0]) = \pi. \quad (130)$$

The strongest dependence on R_0 is in the exponential; we may expect R_0/a_0 to be about 10; if we assume this to be true then (128) gives

$$(16a_0\omega_0)(R_0/(a_0V_0) = (16v_e/V_0)(R_0/a_0) - 160v_e/V_0, \quad (131)$$

where v_e is the orbital velocity of an electron in a Bohr hydrogen atom (velocity corresponding to 13.5eV). For protons in the range of 10eV, the right hand side of (129) is ~7200 which comes pretty close to making $R_0/a_0 = 10$. Even for very energetic ions with $V_0 = v_e$, (proton energy equal to 27,000 eV) $R_0/a_0 \sim 4$. If we take $R_0/a_0 = 10$, the charge exchange cross-section is $\sigma = 100\pi a_0^2 \sim 0.8 \times 10^{-14}$

For creating energetic neutral beams for heating fusion reactor plasmas, positively charged deuterons are accelerated to energies between 100 and 200 KeV. These high energy ions are then passed through a neutral gas cell where some of the ions are converted to neutrals through charge exchange; these neutrals can freely cross the magnetic field confining the plasma. They enter the hot plasma where they are ionized and deposit their energy in the plasma. These energetic ions also have large fusion cross-sections and significantly contribute to the fusion reaction rate. The fraction of the energetic ion beam that ends up as neutral atoms is determined by a balance between charge exchange and reionization by energetic neutral gas neutral ionizing collisions. At 100 KeV the charge exchange cross-section and the re-ionization cross-sections are about equal and about half the ion beam gets neutralized; at higher energies the charge exchange cross-section falls off while the ionization cross-section remains rather flat so that the fraction of the beam that ends up as neutral falls off.

Momentum Transfer Cross Section

The fractional energy loss of an electron of mass m scattered by an atom of mass M is [neglecting terms of order $(m/M)^2$]

$$2(1 - \cos\theta)\frac{m}{M} . \tag{10}$$

Problem: Derive Eq. (10).

The total fractional energy loss per unit length is then

$$\epsilon = \int n(1 - \cos\theta)\, \frac{2m}{M}\, I\, 2\pi \sin\theta\, d\theta \tag{11}$$

$$\epsilon = \frac{2m}{M}\sigma_m n \tag{12}$$

where

$$\sigma_m = 2\pi \int_0^\pi I(1 - \cos\theta)\sin\theta\, d\theta \tag{13}$$

and is called the momentum transfer cross section. The average fractional energy loss per collision is then Eq. (12) divided by the total number of collisions per unit length, σn

$$\frac{\Delta E}{E} = \frac{2m}{M}\frac{\sigma_m}{\sigma_c} . \tag{14}$$

In most cases the momentum transfer cross section is within 10% of the total collision cross section, σ_c, as may be seen in Fig. 8.

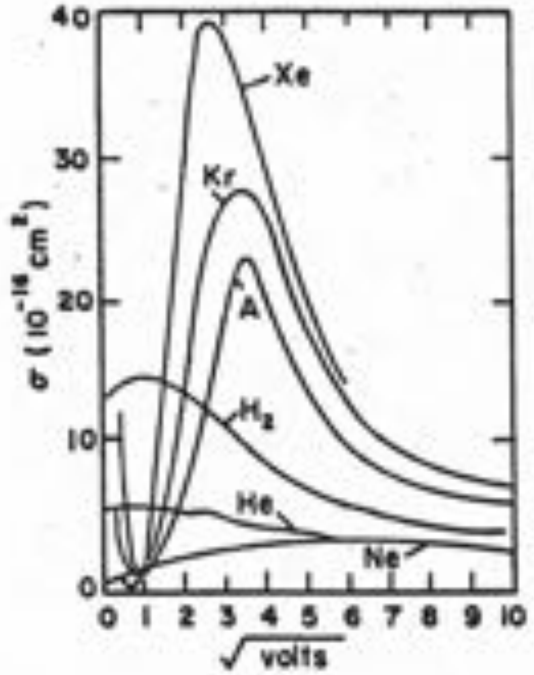


Fig. 6. Total collision cross sections in Ne, A, Kr, and Xe; R. B. Brode, Rev. Mod. Phys. $\underline{5}$, 257 (1933).

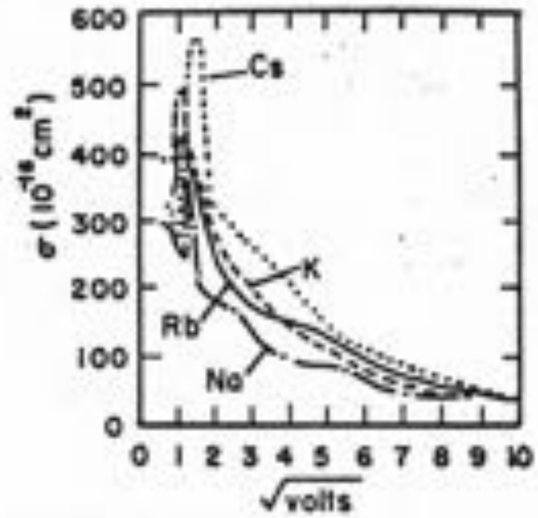


Fig. 7. Total collision cross sections in the alkali metals; R. B. Brode, Rev. Mod. Phys. $\underline{5}$, 257 (1933).

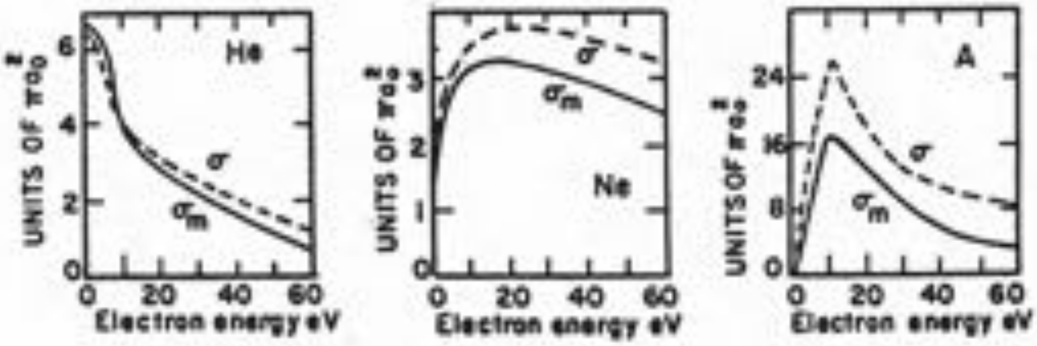


Fig. 8. Comparison of total collision cross section σ with momentum transfer cross section σ_m for He, Ne, and A. (Massey and Burhop)

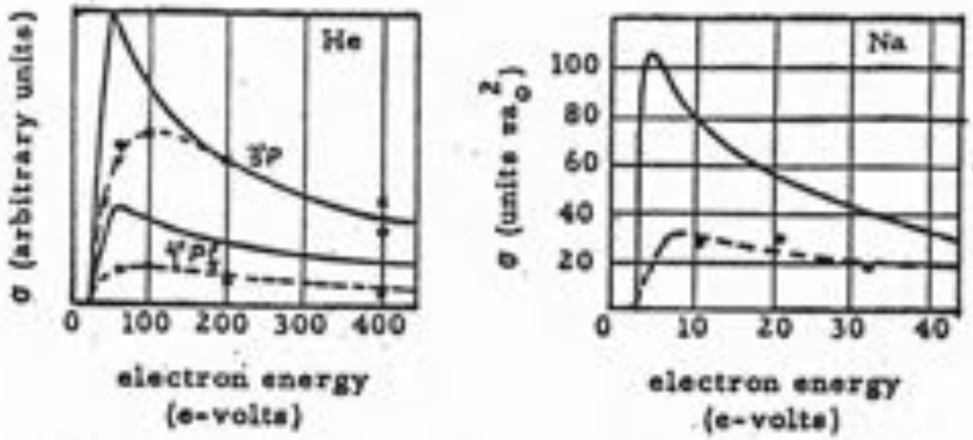


Fig. 9. Cross-sections for optically allowed transitions in He and Na.

——— calculated
------observed

[Massey and Burhop.]

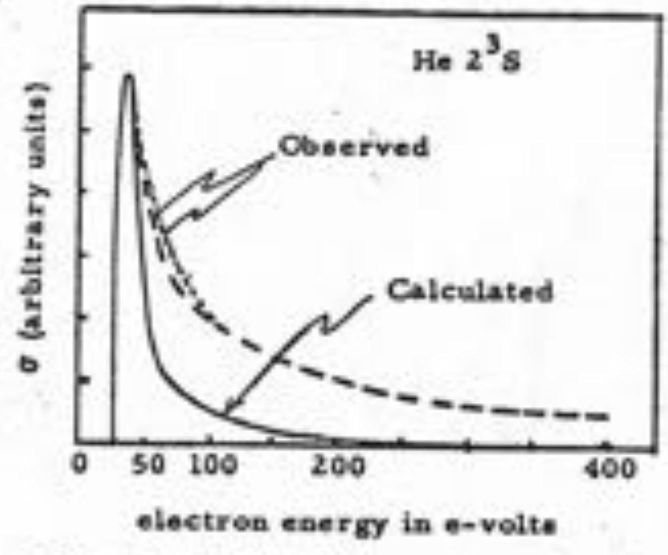


Fig. 10. Excitation cross-sections for the He 2^3S metastable level.

[Massey and Burhop.]

Diffusion and Mobility in Partially Ionized Gases

We derive here an elementary description of electron diffusion and mobility processes in slightly ionized gases where elastic collisions predominate.

We consider first the diffusion of electrons (or ions) through a neutral gas. The concentration of charged particles is so low that electron-neutral collisions dominate over Coulomb collisions, and the temperature is low enough that elastic collisions dominate. The temperature is assumed uniform and the density of electrons is a function of x only. There will then be a net flow of electrons from the region of higher to the region of lower concentration.

The flux Γ_2 crossing surface S from the right consists of electrons which on the average engaged in a collision at a distance $\bar{x}_2$ from S where the electron density is n_2. The flux from the left came from a distance $\bar{x}_1$ where the electron density was n_1. Since the background neutral gas density is constant, $\bar{x}_2 = \bar{x}_1$ and

$$n_1 = n_o - \frac{dn}{dx}\bar{x} \tag{15}$$

$$n_2 = n_o + \frac{dn}{dx}\bar{x} \tag{16}$$

This of course may not be legitimate if the density varies too rapidly — e.g., in the case of shocks.

From kinetic theory we recall that the total number of particles crossing a unit plane for a Maxwellian distribution is $\frac{1}{4}n\bar{v}$. The net flux crossing S is then

$$\Gamma = \frac{1}{4}n_1\bar{v}_1 - \frac{1}{4}n_2\bar{v}_2 \tag{17}$$

$$\Gamma = -\frac{1}{2}\frac{dn}{dx}\bar{x}\bar{v} \tag{18}$$

But from kinetic theory the mean free path is related to the mfp across a plane by

$$\bar{x} = \frac{2}{3}\lambda \tag{19}$$

$$\Gamma = -\frac{1}{3}\lambda\bar{v}\frac{\partial n}{\partial x} \tag{20}$$

$$\Gamma = -\frac{1}{3}\frac{v^2}{\nu_c}\frac{\partial n}{\partial x} = -D\frac{\partial n}{\partial x} \tag{21}$$

where the diffusion coefficient is

$$D = +\frac{1}{3}\frac{v^2}{\nu_c} \tag{22}$$

where ν_c is an effective collision frequency.

If we consider the three-dimensional case,

$$\Gamma = - D \nabla n \tag{23}$$

$$\nabla \cdot \Gamma = - \frac{\partial n}{\partial t} \tag{24}$$

$$\frac{\partial n}{\partial t} = D \nabla^2 n \quad \text{(the diffusion equation)} \tag{25}$$

If D varies with position, then we must write

$$\Gamma = - \nabla D n \ . \tag{26}$$

We can also construct a simple model of mobility. Again we consider a case where collisions with neutrals are dominant. We consider an electron in a uniform electric field E. Then the electron acquires, in addition to its thermal motion, an accelerated motion along $-E$. On the average, a fraction α of the drift momentum is lost at each collision. If the collision frequency is ν_c', the equation of motion for the average electron is

$$m \frac{dv}{dt} - m v \alpha \nu_c' = - e E \ . \tag{27}$$

We now consider a steady state so that the collisional drag of the neutrals compensates for the electric field, and define the effective collision frequency ν_c as $\alpha \nu_c'$. We can now solve for the average drift motion

$$v = \left(\frac{e}{m \nu_c}\right) E = \mu E \tag{28}$$

where μ is called the mobility. If the drift velocity is much smaller than the thermal velocity, ν_c, and thus μ, will be independent of E. If the diffusion coefficient is divided by the mobility, we obtain the Einstein relation

$$\frac{D}{\mu} = \frac{1}{3}\,\frac{v^2}{\nu_c}\left(\frac{m\,\nu_c}{e}\right) = \frac{1}{3}\,\frac{m\,v^2}{e} \tag{29}$$

which depends on kinetic energy only. If the particles are in thermal equilibrium, then

$$\frac{3}{2}\,KT = \frac{1}{2}\,m\,v^2 \tag{30}$$

and Eq. (29) becomes

$$\frac{D}{\mu} = \frac{kT}{e} \tag{31}$$

a result valid for Maxwell-Boltzmann distribution.

Since ν_c is directly proportional to p, we expect the drift velocity to be proportional to E/p so long as the drift velocity remains small compared with the thermal velocity. At high E/p this inequality is no longer satisfied and the drift velocity is no longer proportional to E/p.

This theory is obviously oversimplified. We have seen, for example, that the elastic collision frequency is a strong function of velocity. The electron drift velocity as a function of E/p is shown in Figs. 11 to 14.

Ion Mobility

Similar arguments hold for ion mobility. Once again for low E/p one expects the drift velocity to be proportional to E/p. Because of the greater mass, ion mobilities are of course far smaller than electron mobilities. The case of ions drifting through neutrals of the same kind — e.g., helium ions in helium atoms — is subject to charge exchange collision and the mobilities are smaller. (See Fig. 15.)

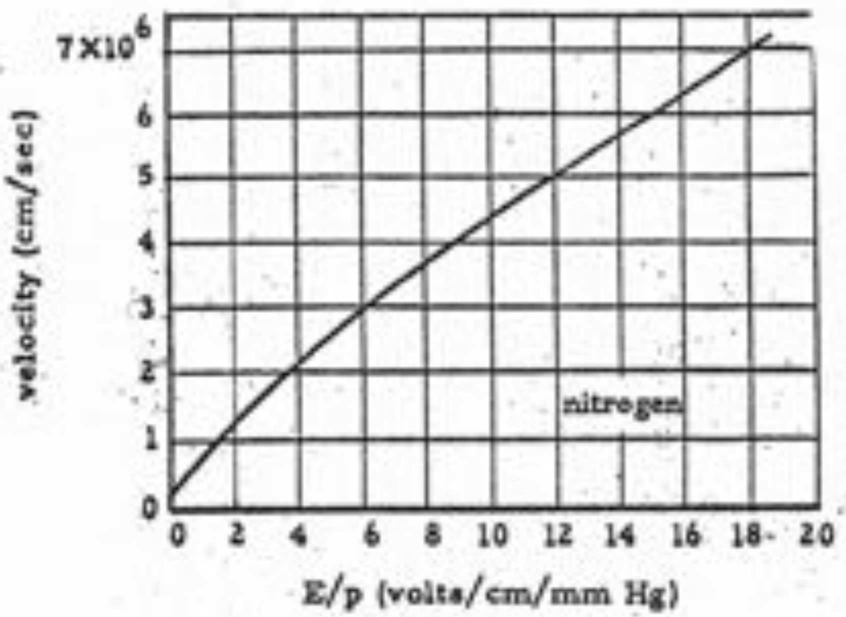


Fig. 11. Drift velocity of electrons in nitrogen as a function of E/p.

[R. A. Nielsen, Phys. Rev. 50, 950 (1936).
L. Colli and U. Facchini, Rev. Sci. Instr. 23, 39 (1952).]

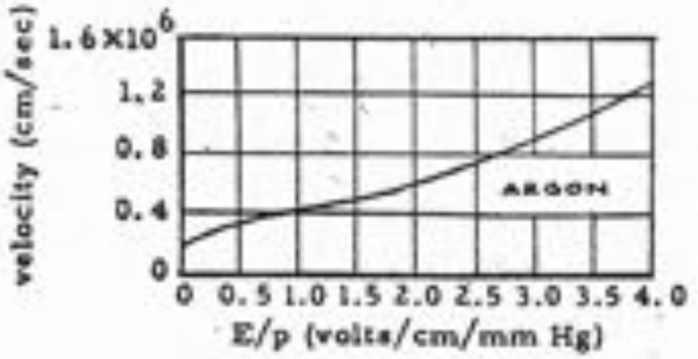


Fig. 12. Drift velocity of electrons in argon as a function of E/p.

[R. A. Nielsen, Phys. Rev. 50, 950 (1936).
L. Colli and U. Facchini, Rev. Sci. Instr. 23, 39 (1952).
J. M. Kirshner and D. S. Toffolo, J. Appl. Phys. 23, 594 (1952)]

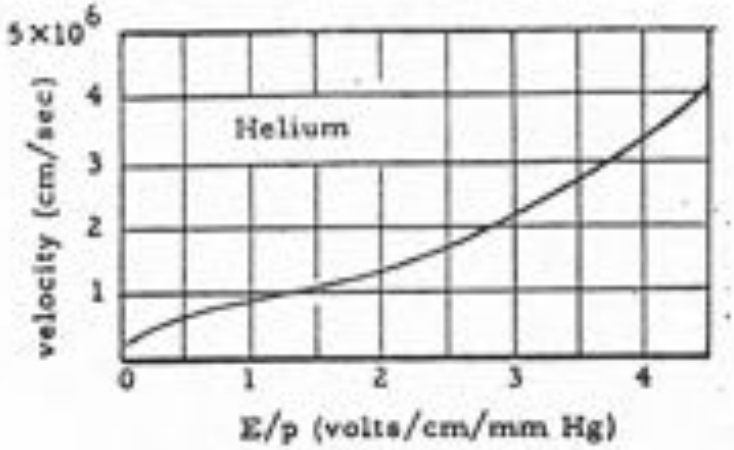


Fig. 13. Drift velocity of electrons in helium as a function of E/p.

[R. A. Nielsen, Phys. Rev. 50, 950 (1936).
J. A. Hornbeck, Phys. Rev. 83, 374 (1951).]

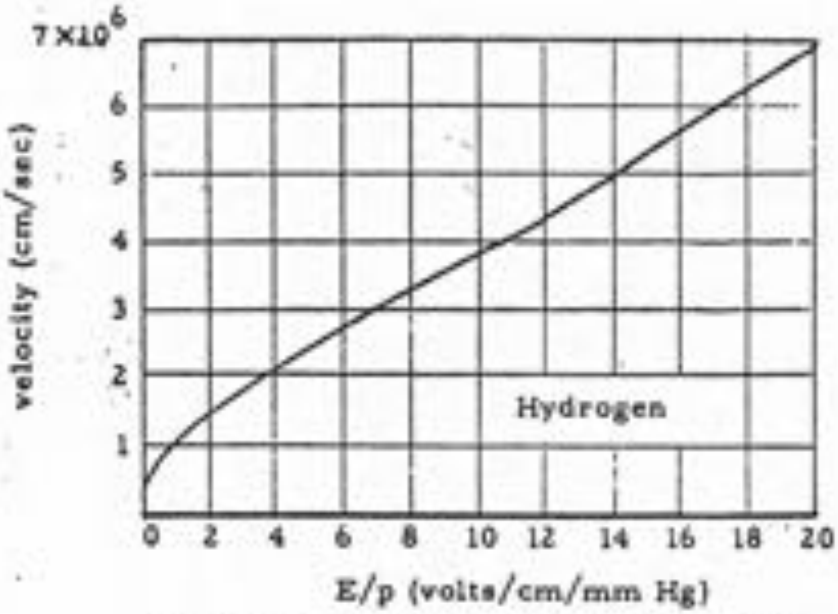


Fig. 14. Drift velocity of electrons in hydrogen as a function of E/p.

[N. E. Bradbury and R. A. Nielsen, Phys. Rev. 49, 388 (1936).]

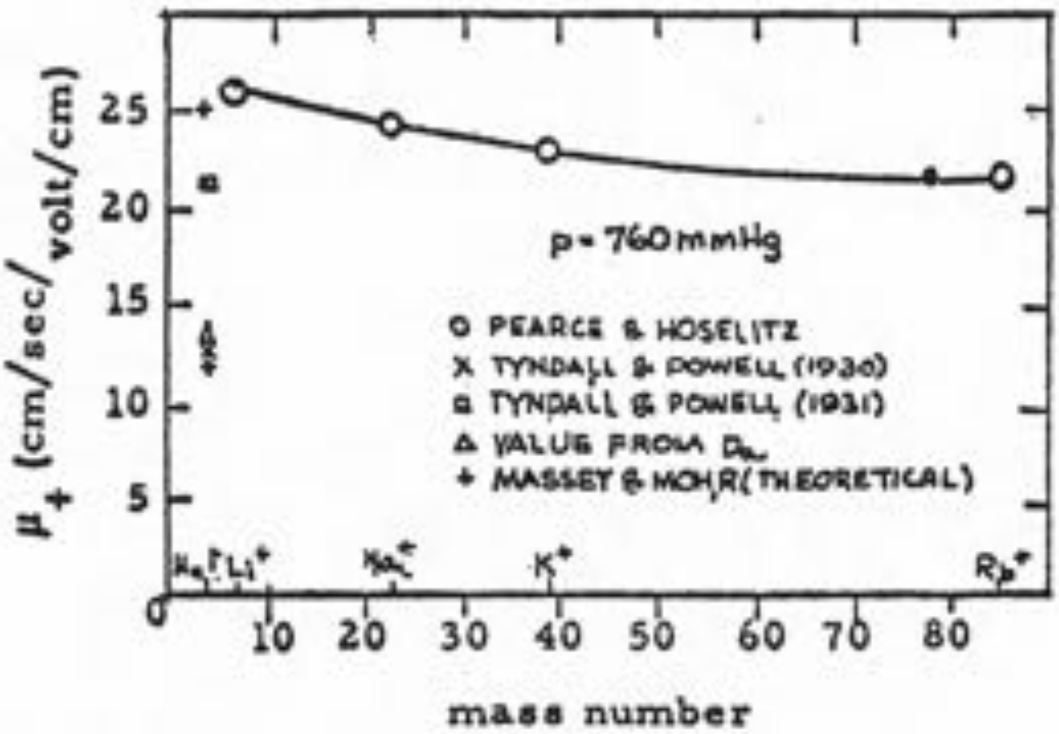


Fig. 15. Mobility of various positive ions in helium.

Ambipolar Diffusion

We consider a steady state plasma in a container. The electron and ion losses are assumed to be compensated by additional ionization processes taking place in the volume. First let us consider the loss processes before steady state is reached. If the initial electron density n_e is equal to the initial ion density n_i and $T_e \geq T_i$, the diffusion coefficient for electrons D_e being much greater than that for ions D_i, the initial loss of electrons to the wall exceeds the ion losses. This sets up an electric field toward the wall which decreases the electron loss rate and increases the ion loss rate. This process continues until the electric field has increased to a value which makes the ion and electron loss rates equal. This is the steady state referred to above. For the electrons and ions, we write for the fluxes

$$\Gamma_i = - D_i \nabla n_i + \mu_i E n_i \tag{34}$$

$$\Gamma_e = - D_e \nabla n_e - \mu_e E n_e \ . \tag{35}$$

The electric field here is the sheath field caused by the excess electron loss. There is no applied field. At steady state the total losses are equal. We assume that the fluxes of both particles are the same in all regions.

$$\Gamma_i = \Gamma_e = \Gamma_a \tag{36}$$

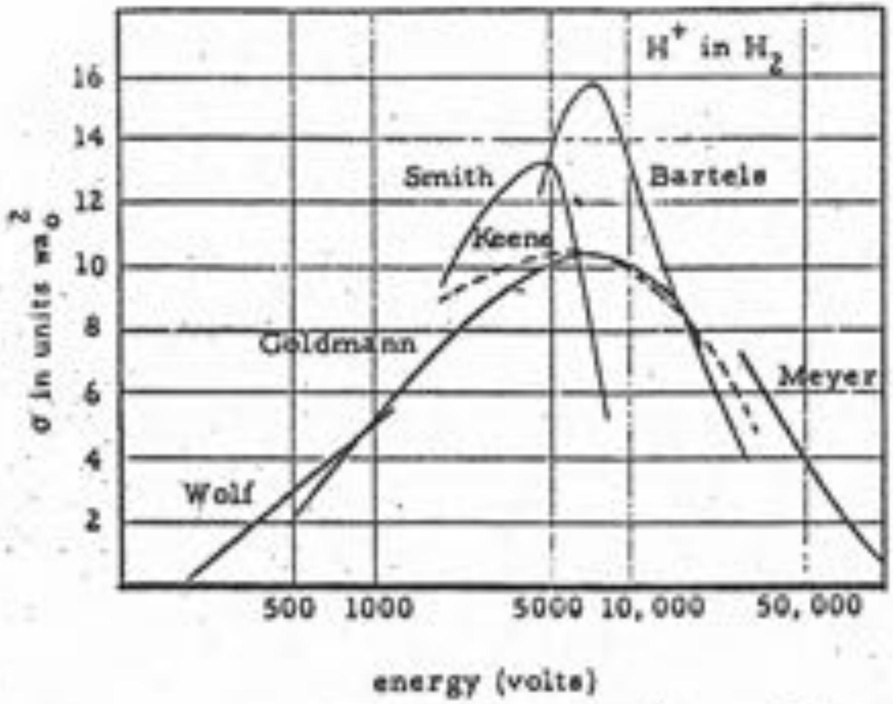


Fig. 16. Charge transfer cross-section of H^+ in H_2. [Massey and Burhop]

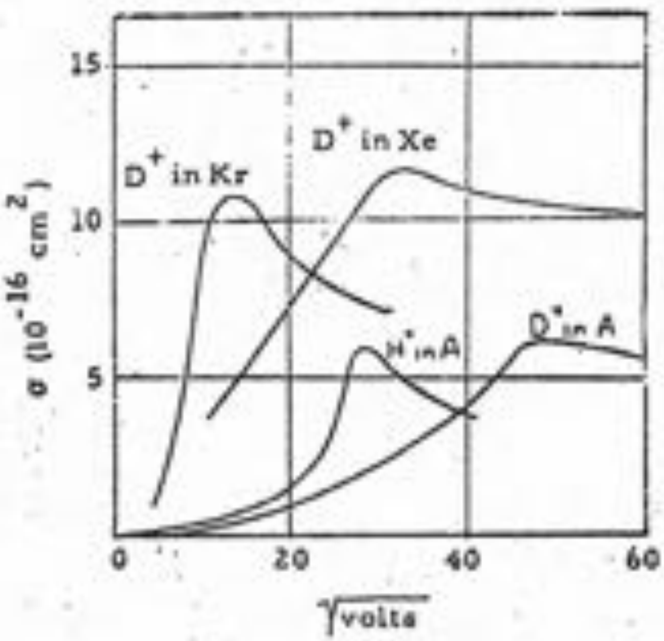


Fig. 17. Charge transfer cross-sections of H^+ in argon; D^+ in argon, krypton, and xenon.
[J. B. Hasted, Proc. Roy. Soc. (London) A212, 235 (1952), as show in Brown.]

using the subscript a to refer to ambipolar. We assume that $n_i \approx n_e = n$; that is, the ion and electron density are initially sufficiently large that the requisite electric field may be created by a slight inequality in n_i and n_e. Eqs. (34) and (35) become

$$\Gamma_a = -D_i \nabla n + \mu_i E n \tag{37}$$

$$\Gamma_a = -D_e \nabla n + \mu_e E n \ . \tag{38}$$

Eliminating E between Eqs. (37) and (38), we get

$$\Gamma_a = -D_a \nabla n \tag{39}$$

where

$$D_a = \left(\frac{D_i \mu_e + D_e \mu_i}{\mu_i + \mu_e}\right) \ . \tag{40}$$

From the Einstein relation we have

$$\frac{D_i}{\mu_i} = \frac{kT_i}{e} \ ; \qquad \frac{D_e}{\mu_e} = \frac{kT_e}{e} \ . \tag{41}$$

If we now assume that

$$\alpha T_i = T_e \tag{42}$$

where $\alpha \geq 1$,

$$\frac{D_i}{\mu_i} = \frac{1}{\alpha}\frac{D_e}{\mu_e} \ , \tag{43}$$

Using Eq. (43) to eliminate D_e from Eq. (40), and assuming $\mu_i << \mu_e$, we find

$$D_a = D_i (1 + \alpha) \ . \tag{44}$$

This is another way of saying that the loss rate of both types of charged particle is essentially determined by the loss rate of the slower.

Note that while we have given an effective diffusion coefficient, mobility is definitely involved. There is an electric field. From Eqs. (37) and (38) we can eliminate Γ_a, giving

$$E = \frac{\nabla n}{n}\left(\frac{D_i - D_e}{\mu_i + \mu_e}\right) \tag{45}$$

which under the approximations $D_i < D_e$, $\mu_i < \mu_e$ becomes

$$E = -\frac{\nabla n}{n}\frac{D_e}{\mu_e} = \frac{\nabla n}{n}\frac{kT_e}{e} \quad . \tag{46}$$

We assume that $(n_e - n_i) << n$. Let us find the conditions under which this is valid. From the Maxwell equation for the divergence of E, we have

$$\nabla \cdot E = 4\pi\, e(n_i - n_e) \sim \frac{E}{L} \,, \tag{47}$$

where L is a length characteristic of the region where $n_i \neq n_e$. From Eqs. (46) and (47),

$$E = L\, 4\pi e(n_i - n_e) = -\frac{\nabla n}{n}\frac{kT_e}{e} \,. \tag{48}$$

We approximate ∇n as n/L. In steady state, any region where there is a gradient of the density ambipolar diffusion must be taking place; therefore the electric field must exist over the same region as the density gradient. We therefore use the same L here. Thus we have

$$\frac{n_i - n_e}{n} = -\frac{kT}{4\pi n e^2 L^2} = -\frac{\lambda_D^{\,2}}{L^2} \quad . \tag{49}$$

Thus the condition that $n_i - n_e << n$ becomes one that the Debye length be small compared with the region over which the density changes.

Breakdown and the First Townsend Coefficient

We consider a neutral gas of density n_g to which is applied a uniform electric field E, say in the $-x$ direction. At some point $x = x_o$ electrons are injected at density n_o. The electric field is sufficiently high that ionizing collisions take place, so that as the electron drifts in the x direction, n is a function of x.

Since each electron produces ν_i secondary electrons per second, and there are at any time n electrons per cubic centimeter, the rate of increase of n due to ionization is just $n\nu_i$, where $\nu_i = n_g \overline{\sigma v}$. Under the influence of the electric field, the electrons drift across the boundaries. The difference between the number entering and leaving unit volume per second is $\bar{v}\nabla n$, where $\bar{v}$ is a constant since E and p, and thus the mobility are constant. Thus, for a steady state

$$\nu_i n = \bar{v}\frac{dn}{dx} \tag{50}$$

$$n = n_o e^{\alpha x} \tag{51}$$

where

$$\alpha = \frac{\nu_i}{\bar{v}} \tag{52}$$

and is called the first Townsend coefficient. $\bar{v}$ is a function of E/p. ν_i is proportional to pressure (at fixed $\bar{v}$). Thus, α/p is a function of E/p. Typical values are shown in Figs. 18 and 19.

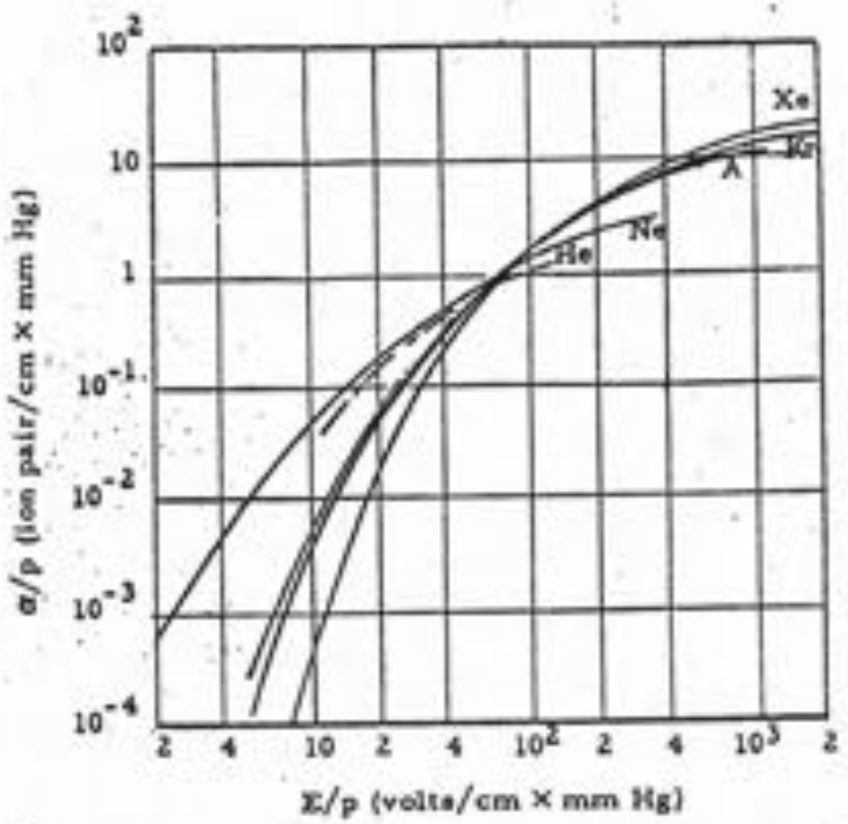


Fig.18. First Townsend ionization coefficients in noble gases.
[A. von Engel, Handbuch der Physik, Springer Verlag, Berlin (1956) Vol. 21, p. 504, as shown in Brown.]

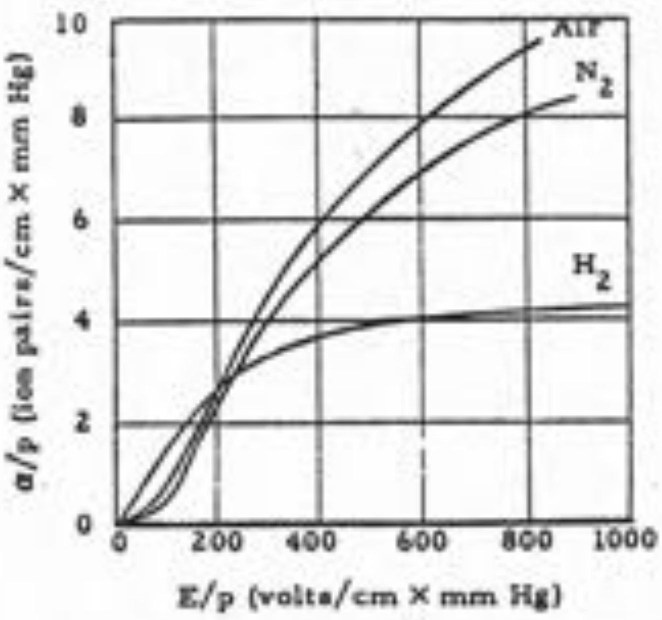


Fig.19. First Townsend ionization coefficients of air, N_2 and H_2.
[A. von Engel, Handbuch der Physik, Springer Verlag, Berlin (1956) Vol. 21, p. 504, as shown in Brown.]

This, of course, applies to dc discharges. When ac fields are applied the dominant losses may be quite different. In an ac field the particle position oscillates with a limited amplitude (in the absence of collisions). If the dimension (along E) is large compared with this amplitude, mobility losses no longer dominate. Diffusion becomes dominant. See, for example, Brown.

We have shown how an initial density of electrons n_o builds up exponentially in space. In order to effect breakdown, a feedback mechanism is needed. Consider, for example, a discharge between electrodes separated by a distance d . Ions striking the cathode may produce secondary emission as one feedback mechanism. Assuming a cascading density as in Eq. (51), the ion production per unit time in the whole length is

$$\int_o^d n(x)\,\nu_i\,dx = \nu_i \frac{n_o}{\alpha}(e^{\alpha d} - 1) \tag{53}$$

where n_o is the total number at the cathode. Then if γ_i is the ratio of secondary electron production rate to the column ion production rate, the rate of secondary electron production is

$$\Gamma_e = \gamma_i \nu_i \frac{n_o}{\alpha}(e^{\alpha d} - 1) \ . \tag{54}$$

If n_s is the number of secondaries per second, then since $\Gamma_e = n_s \bar{v}_e$

$$n_s = \gamma_i n_o (e^{\alpha d} - 1) \ . \tag{55}$$

Then is n_o' is the initiating flux (perhaps one electron),

$$n_o = \gamma_i n_o (e^{\alpha d} - 1) + n_o' \tag{56}$$

$$n(d) = \frac{n_o' e^{\alpha d}}{1 - \gamma_i(e^{\alpha d} - 1)} \tag{57}$$

If E/p is increased, α increases and the condition may be achieved that the denominator of Eq. (57) approaches zero,

$$1 - \gamma_i(e^{\alpha d} - 1) = 0 \tag{58}$$

thus effecting breakdown. An additional feedback mechanism is photo emission from the cathode. The sum of γ_i for secondary emission and photo emission is called the second Townsend coefficient. The value of the quantity γ_i depends on the nature of the electrodes and on the geometry.

If we plot the voltage required for breakdown between two electrodes against the product of pressure p times electrode separation d, the so-called Paschen curve results. Fig. 20 shows the typical form of this curve. For small pd there is very little cascading. More secondary emission is needed so a larger potential is required.

This is an approximate description of one type of breakdown. It obviously does not apply to discharges which are dominated by other factors such diffusion or negative ion formation.

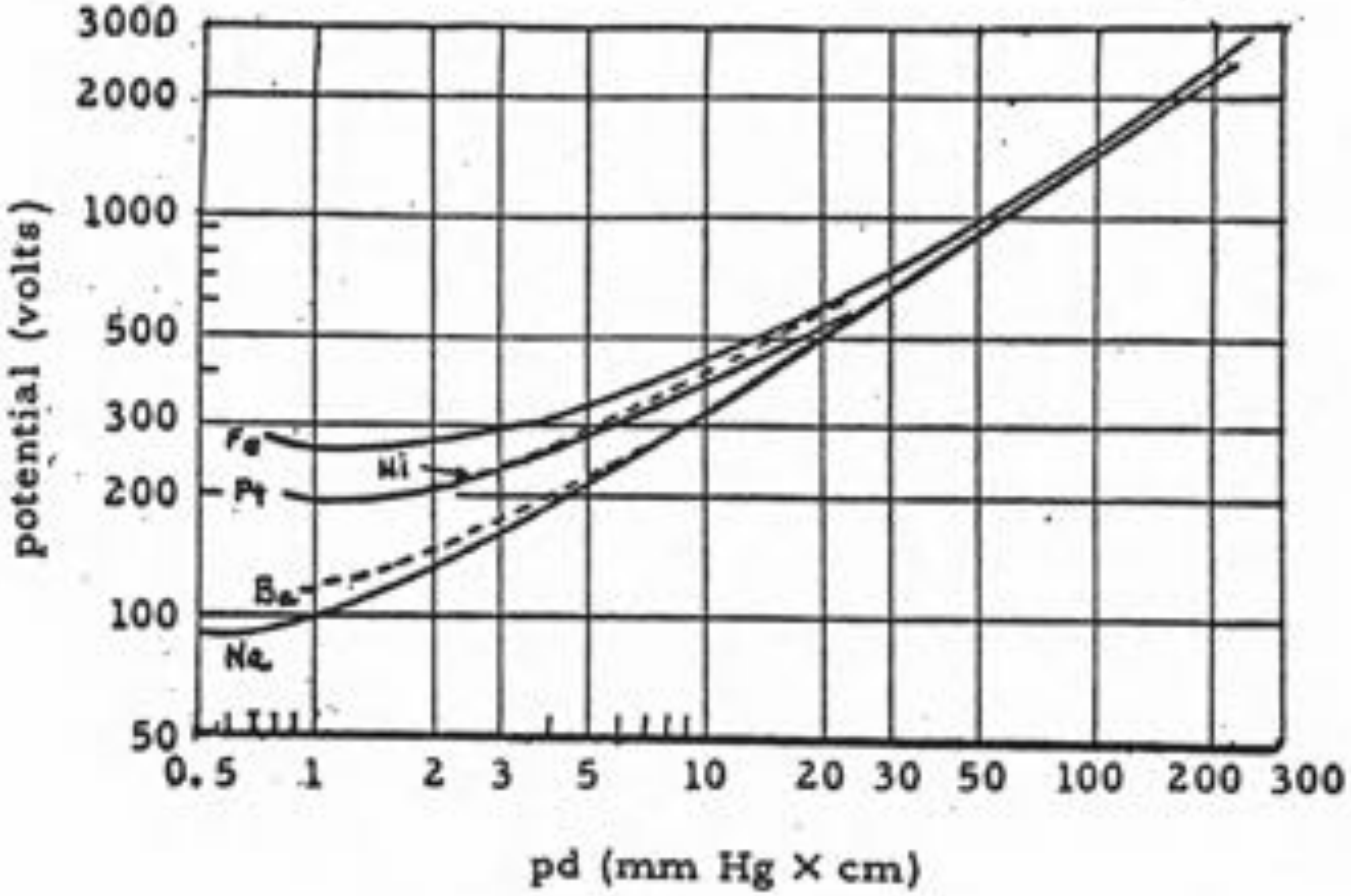


Fig. 20. Breakdown potential in argon between plates for various cathode materials.

[As shown in Brown.]

Particle Orbits

We have seen that interactions between individual pairs of charged particles are to some approximation negligible. On the other hand, when many particles are interacting the fields can become quite large and the interactions can become quite important (this lead to the plasma oscillations we saw earlier). The fields due to a large number of charged particles are rather smooth with small fluctuations superimposed on them due to the discrete nature of the particles. The small fluctuations are responsible for scattering and collisions, while the macroscopic smoothed out fields give the collective motions of the plasma (coherent motion of many particles). Thus, to the extent that we can neglect collisions, we can approximate the plasma motion by treating the system as a collection of charged particles, each one moving in the smoothed-out field of all the other particles plus, of course, any externally-applied fields. We will see that we can gain much insight into the behavior of plasmas by investigating the motions of single charged particles in arbitrary electric and magnetic fields. We can go further and make the fields consistent with the motion of all the particles. We determine the fields from Maxwell's equations and the charges and currents associated with the particle motions. The particle motions are determined from the Lorentz force acting on the particles. Maxwell's equations are:

$$\nabla \times \mathbf{E} = -(1/c)\partial \mathbf{B}/\partial t \ . \tag{132}$$

$$\nabla \times \mathbf{B} = (1/c)\partial \mathbf{E}/\partial t + 4\pi \mathbf{j}/c \ . \tag{133}$$

$$\nabla \bullet \mathbf{B} = 0 \ . \tag{134}$$

and

$$\nabla \bullet \mathbf{E} = 4\pi\rho \ . \tag{135}$$

From Eqs. (133) and (136) we get the equation of continuity

$$\partial\rho/\partial t + \nabla \bullet \mathbf{j} = 0 \ . \tag{136}$$

where

$$\rho = \Sigma_i q_i \langle n_i \rangle \ . \tag{137}$$

and

$$\mathbf{j} = \Sigma_i q_i \langle \mathbf{v}_i n_i \rangle \ . \tag{138}$$

Here ρ and $\mathbf{j}$ are the charge and current densities due to the plasma particles, i refers to the ith species of particle. The particles move according to their equation of motion:

Non-relativistic case

$$m_i d\mathbf{v}_i/dt = q_i(\mathbf{E}_i + [\mathbf{v}_i X \mathbf{B}_i]/c), \tag{139}$$

or in the relativistic case by

$$d\mathbf{p}_i/dt = q_i(\mathbf{E}_i + [\mathbf{v}_i X \mathbf{B}_i]/c), \tag{140}$$

with

$$\mathbf{p}_i = \gamma m_i \mathbf{v}_i, \quad \gamma = (1 - [v/c]^2)^{-1/2}.$$

Here the subscript i on $\mathbf{E}$ and $\mathbf{B}$ means that these quantities are to be evaluated at the position of particle i. These are the so-called Vlasov equations for a plasma. We will be studying them in some detail throughout this course.

A. Cyclotron Motion

The equation of motion of a particle is given by Eqs. (139) or (140). If we dot Eq. (139) with $\mathbf{v}_i$ then we get the energy equation

$$d(m_i \mathbf{v}_i^2/2)/dt = q_i \mathbf{v}_i \bullet \mathbf{E}_i \ . \tag{141}$$

The magnetic term drops out of this equation since the magnetic force is always perpendicular to the velocity and hence does no work. Integrating gives

$$\Delta(m_i v_i^2/2) = \int q_i \mathbf{v}_i \cdot \mathbf{E}_i \, dt = \int_{orbit} q_i \mathbf{E}_i \cdot d\mathbf{s} \; , \qquad (142)$$

where $d\mathbf{s}$ is a vector element of the orbit. If $\mathbf{E}$ is an electrostatic field (i.e., derivable from a potential, $\mathbf{E} = -\nabla\Phi$), then Eq. (142) may be written in conservation of energy form.

$$(m_i v_i^2/2) + q_i\Phi_i = \text{constant} \qquad (143)$$

Returning to Eq. (139), the solution of this equation for arbitrary $\mathbf{E}$ and $\mathbf{B}$ can be very complicated. As we often do in physics, we will build up the more complicated motions from results obtained by looking at a number of simple cases.

The simplest possible situations of course involve spatially uniform, time independent $\mathbf{E}$ and $\mathbf{B}$ fields. Let us first consider the case of a uniform $\mathbf{B}$. Since the magnetic force is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$, there is no force on the particle in the $\mathbf{B}$ direction and the velocity in that direction is constant. We need not consider this velocity any more. The velocity perpendicular to the magnetic field has a constant magnitude by Eq. (142), the particle moves in a circular orbit about the magnetic field. One can obtain the radius of this circle by balancing the centrifugal force against the the magnetic force.

$$m v_\perp^2/r = q v_\perp B/c \; , \qquad (144)$$

or

$$r = (mcv_{\perp}/qB).$$

The quantity qB/mc is the angular frequency of the particle and is called the cyclotron frequency, $\omega_c = qB/mc$. The radius is called the Larmor radius. Numerically these are given by:

$$\omega_{ce} \equiv 2\times10^{7}B\ ,$$

$$\omega_{cp} \equiv 10^{4}B\ ,$$

and

$$\omega_{c,Z,A} \equiv (Z/A)10^{4}B\ ,$$

where Z is the ionic charge and A is the atomic weight.

If one wishes to be more formal in solving for the motion, one can proceed as follows. Let the direction of B be the z direction. Then $v_{\perp}$ has x and y components and Eq.(9) can be written in the form

$$\frac{dv_x}{dt} = \omega_c v_y \qquad\qquad \omega_c = \frac{qB_z}{mc} \tag{17}$$

$$\frac{dv_y}{dt} = -\omega_c v_x \tag{18}$$

Let $w = v_x + i v_y \quad i = \sqrt{-1}$

Multiplying Eq. (18) by i and adding to Eq. (17) gives

$$\frac{dw}{dt} = -i\,\omega_c w \tag{19}$$

This immediately integrates to

$$w = w_0 e^{-i\omega_c t} \tag{20}$$

This equation says that w rotates at a constant rate in the complex w plane. If w is integrated with respect to t, one obtains the complex position z.

$$z = x + i y \tag{21}$$

$$z = + i \frac{w_0}{\omega_c} e^{-i\omega_c t} + z_0 \tag{22}$$

where z_0 is the initial position $i\frac{w_0}{\omega_c}$

$$z_0 = x_0 + i y_0 - i\frac{w_0}{\omega_c} \tag{23}$$

Equation (22) says that z is given by a vector of magnitude $\frac{|w_0|}{\omega_c}$ rotating at a constant rate about the point z_0 (Fig. 21).

We see from the definition of ω_c, Eq. (17), and from Eq. (22) that the direction of rotation about the field depends on the sign of the charge. The direction of rotation is shown in Fig. 22.

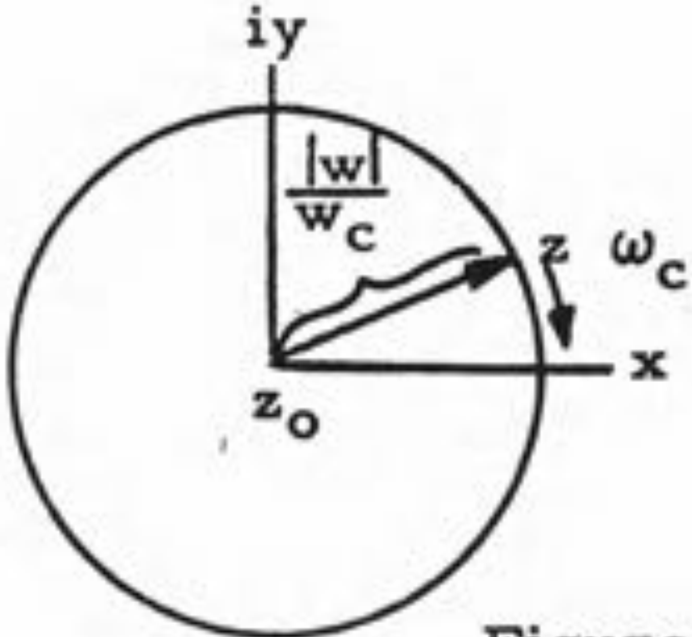


Figure 21

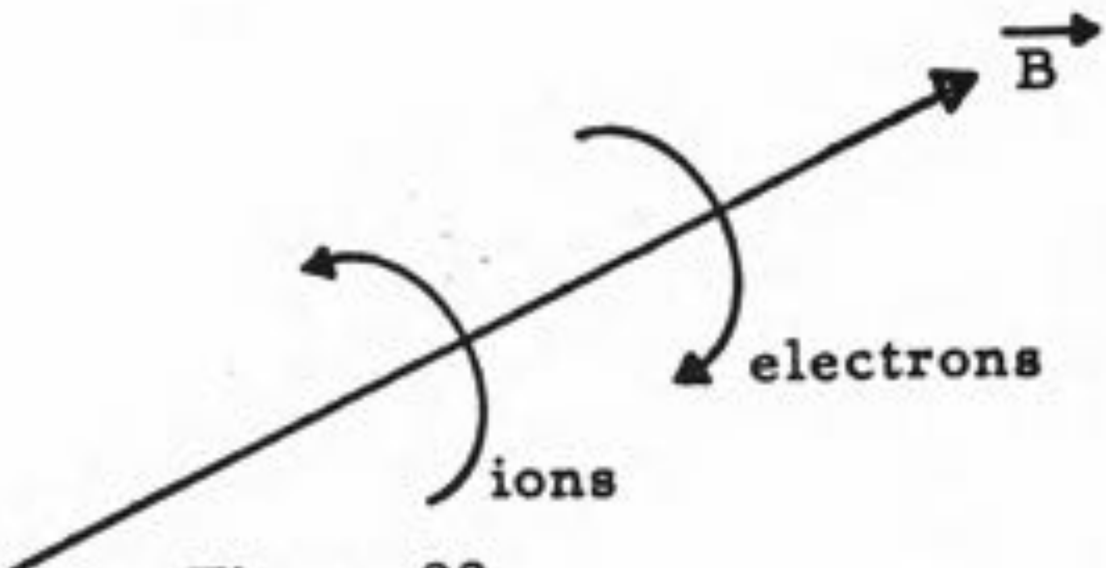


Figure 22

<u>Problem:</u> Find the cyclotron frequency for electrons and protons in a field of 10,000 gauss; in a field of .1 gauss. The latter is roughly the strength of the earth's field at 4,000 miles. Compute the radius of gyration of a 1 MeV proton in a field of .1 gauss.

B. Magnetic Moment

A current loop has a dipole moment associated with it of magnitude

$$\mu = \frac{IA}{c} \tag{24}$$

where I is the current (in esu units) and A is the area of the loop. The circular orbit of a charge in a magnetic field is the area of the loop. The circular orbit of a charge in a magnetic field on the average constitutes a current loop. The average current is the average charge per unit time which passes a point on the orbit. This is $\frac{1}{\tau} q/c$, where τ is the period. The magnetic moment is thus

$$\begin{aligned}\mu &= \frac{q}{c\tau}\pi r^2 = \frac{q\omega_c}{c\,2\pi}\pi\frac{v_\perp^2}{\omega_c^2} = \frac{\pi q}{2\pi c}\frac{v_\perp^2}{qB/mc} \\ &= mv^2/2B = W_\perp/B\end{aligned} \tag{25}$$

Here $W_\perp$ is the energy of the particle due to its perpendicular motion.

In addition to the magnitude $W_\perp/B$, the magnetic moment has a direction associated with it — the direction of the magnetic dipole with the equivalent magnetic moment. From Fig. 22 and the right-hand rule, we see that the current loop is such as to reduce the field inside of itself, and hence the magnetic moment has a direction opposite to the direction of the field.

$$\vec{\mu} = -\frac{W_\perp}{B^2}\vec{B} \tag{26}$$

C. Magnetization

In a plasma containing many particles the magnetic field produced by all the magnetic moments can be appreciable. To compute this effect we must make use of the Maxwell equation

$$\vec{\nabla} \times \vec{B} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}}{c} \tag{27}$$

First we shall assume that all quantities are time-independent, or at least vary so slowly that we can neglect the $\partial E/\partial t$ term. Second, for the time being we imagine that j is produced only by plasma particles. Equation (27) may be written in the integral form

$$\oint_c \vec{B} \cdot d\vec{\ell} = \int_s \vec{\nabla} \times \vec{B} \cdot d\vec{A} = \frac{4\pi}{c} \int_s \vec{j} \cdot d\vec{A} \tag{28}$$

where c is a curve bounding an area s, $d\vec{\ell}$ is an element of the curve c, and $d\vec{A}$ is a vector element of the surface s and has the direction of the normal to s [Eq. (28) follows from the Stokes theorem]. To compute the integral on j in Eq. (28) we must find the current normal to s or the total charge crossing s per unit time. Consider the situation shown in Fig. 23.

Let us compute the total charge crossing s per unit time. Orbits like 2, which intersect s twice, give no transfer of charge across s. On the other hand, orbits like 1 and 3, which intersect s only once, transfer an amount of charge q (the charge on the particle) every time they cross s. Thus only those orbits which loop the curve c contribute to the current through s. Now they transfer the charge q (the charge on a particle) for each revolution, or they transfer charge at the rate q/τ_c.

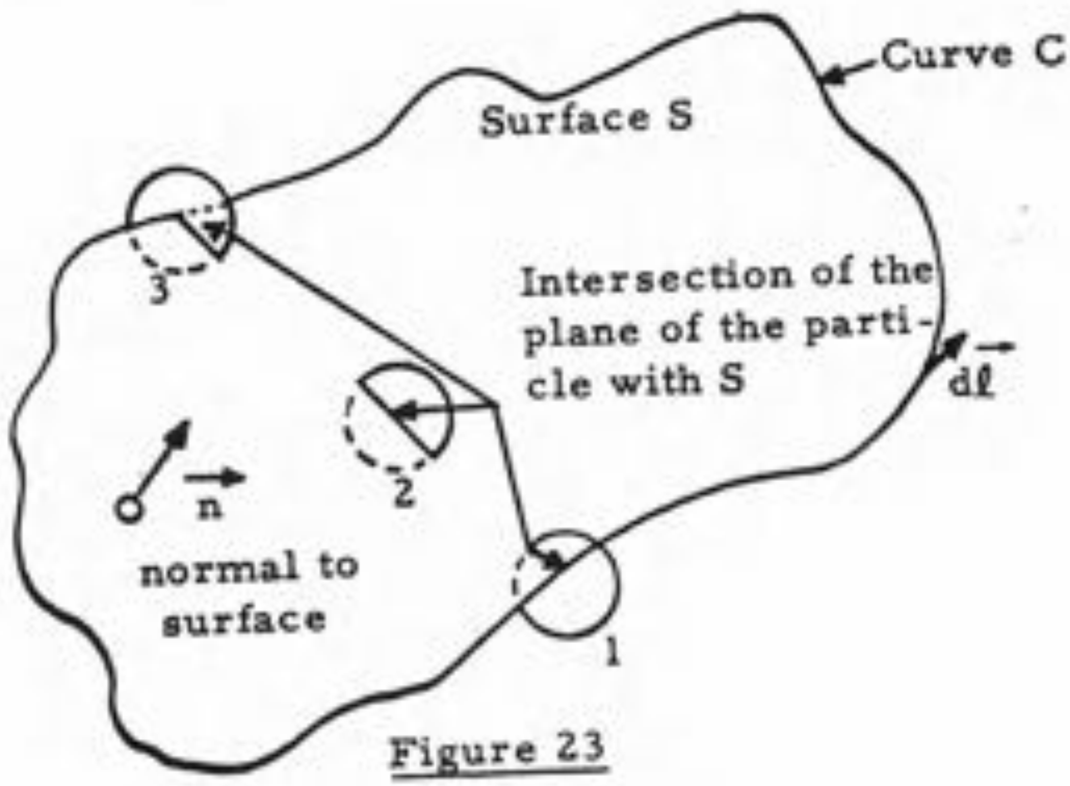

Curve C
Surface S
Intersection of the plane of the particle with S
$\vec{d\ell}$
3
2
1
$\vec{n}$
normal to surface

Figure 23

The number of orbits which loop c in a length $|d\vec{\ell}|$ is the density of such orbits times the areas of such orbits normal to $d\vec{\ell}$, times $|d\vec{\ell}|$

$N(q/\tau_c)a_n|d\vec{\ell}| = Ia_n|d\vec{\ell}|$.

Now we must take account of whether charge is transferred across the surface in the positive or negative direction. We must compute the net charge crossing in the positive direction. If the magnetic moment $\vec{\mu} = \frac{I\vec{A}}{c}$ is in the direction of $d\vec{\ell}$ then the current crosses the surface in the positive direction, while if $\vec{\mu}$ is opposite to $d\vec{\ell}$ the current crosses the surface in the negative direction (a little consideration using the right-hand rule shows this). Thus the total current crossing s is given by

$$\int_s \vec{j}\cdot d\vec{A} = \int_c NI\vec{a}\cdot d\vec{\ell} = c\int_c N\vec{\mu}\cdot d\vec{\ell} \tag{29}$$

Eq. (28) may thus be written in the form

$$\int_c \vec{B}\cdot d\vec{\ell} = 4\pi\int_c N\vec{\mu}\cdot d\vec{\ell} = \int_s \vec{\nabla}\times\vec{B}\cdot d\vec{A} = 4\pi\int_s \vec{\nabla}\times N\vec{\mu}\cdot d\vec{A} \tag{30}$$

Writing $\vec{M} = N\vec{\mu}$, we may write

$$\int_s \vec{\nabla}\times\vec{B}\cdot d\vec{A} = 4\pi\int_s \vec{\nabla}\times\vec{M}\cdot d\vec{A} \tag{31}$$

or

$$\vec{\nabla}\times(\vec{B} - 4\pi\vec{M}) = 0$$

If we do not neglect $\partial\vec{E}/\partial t$, and if there are currents other than those contributed by the plasma particles (let us denote such currents by j_e), then we can proceed in a similar manner and we obtain Eq. (32) in place of Eq. (31).

$$\vec{\nabla}\times(\vec{B} - 4\pi\vec{M}) = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi\vec{j}_e}{c} \tag{32}$$

In the classical treatment of magnetic materials, $\vec{B} - 4\pi\vec{M}$ would have been called $\vec{H}$, and Eq. (32) would then read

$$\nabla \times \vec{H} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}_e}{c} \tag{33}$$

For normal materials and small magnetic fields, the magnetization is proportional to B

$$\vec{M} = \alpha \vec{B} \tag{34}$$

so that H is also proportional to B. Here, however,

$$\vec{M} = -\frac{N W_\perp \vec{B}}{B^2} \tag{35}$$

and is proportional to $1/B^2$. Thus α, and also the magnetic permeability $[1/(1-4\pi\alpha)]$ are not constant. H is not useful; we will use only B.

We may substitute Eq. (35) in Eq. (32) and obtain

$$\nabla \times \left[\vec{B}\left(1 - 4\pi\frac{N W_\perp}{B^2}\right)\right] = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}_e}{c} \tag{36}$$

From this we see that the particles begin to make an important contribution to the field when their energy density $N W_\perp$ becomes comparable to the energy density of the magnetic field $B^2/8\pi$.

D. The Electric Field Drift

Next in simplicity to the constant magnetic field case is the situation in which we have constant electric and magnetic fields. From Eq. (9) we have for the component of the motion along the magnetic field

$$m\frac{d v_\parallel}{dt} = q E_\parallel \tag{37}$$

We can integrate this equation to obtain

$$v_{\parallel} = \frac{qE_{\parallel}}{m}t + v_{\parallel_0} \tag{38}$$

$$x_{\parallel} = \frac{qE_{\parallel}}{m}\frac{t^2}{2} + v_{\parallel_0}t + x_{\parallel_0} \tag{39}$$

Thus the particle freely accelerates along the field.

The equation of motion for the components of $\vec{v}$ perpendicular to $\vec{B}$ is

$$m\frac{d\vec{v}_{\perp}}{dt} = q\left(\vec{E}_{\perp} + \frac{\vec{v}_{\perp} \times \vec{B}}{c}\right) \tag{40}$$

Now in this equation both the electric force and the magnetic force are perpendicular to $\vec{B}$ and it is possible to balance them. If they are balanced, $d\vec{v}_{\perp}/dt$ is 0, and the particle moves with a constant perpendicular velocity. Let us equate the electric and magnetic forces

$$\vec{E}_{\perp} + \frac{\vec{v}_{\perp} \times \vec{B}}{c} = 0 \tag{41}$$

Crossing this with $\vec{B}$ gives

$$\vec{B} \times \vec{E}_{\perp} + \frac{\vec{B} \times (\vec{v}_{\perp} \times \vec{B})}{c} = \vec{B} \times \vec{E}_{\perp} + \frac{\vec{v}_{\perp} B^2}{c} = 0 \tag{42}$$

or solving for $\vec{v}_{\perp}$

$$\vec{v}_{\perp} = \frac{(\vec{E}_{\perp} \times \vec{B})c}{B^2} \tag{43}$$

Let us denote this velocity by $\vec{v}_E$ and write for $\vec{v}_{\perp}$

$$\vec{v}_{\perp} = \vec{v}_1 + \vec{v}_E \tag{44}$$

Substituting in Eq. (9) the $\vec{v}_E \times \vec{B}$ term cancels the E term and we get

$$m \frac{d\vec{v}_\perp}{dt} = q \frac{\vec{v}_\perp \times \vec{B}}{c} \tag{45}$$

This is the same equation that one gets without an E field, and hence $\vec{v}_\perp$ rotates at a constant rate. The motion of the particle is thus a drift with a uniform velocity v_E plus a rotation about the magnetic field lines. We should note that the drift velocity is independent of the charge.

Motion across a magnetic field gives rise to an E field. The transformation law for E in going from one frame to another, moving with a velocity v relative to it, is (for velocities small compared to light)

$$\vec{E}' = \vec{E} + \frac{\vec{v} \times \vec{B}}{c} \tag{46}$$

For $v = v_E$ the right side vanishes: in the frame moving with the drift velocity there is no electric field and hence the particle sees only a magnetic field and moves accordingly.

We may also view the drift in another way, which is illustrated in Fig. 24.

As a positive charge spirals about the magnetic field its energy changes due to the E field. It moves faster on the upper part of its orbit and the curvature is smaller here than on the lower part of its orbit; hence the drift. For a negative charge, on the other hand, the velocity is larger on the lower part of the orbit, but since the direction of rotation is opposite to that for a positive charge, the resultant drift is in the same direction.

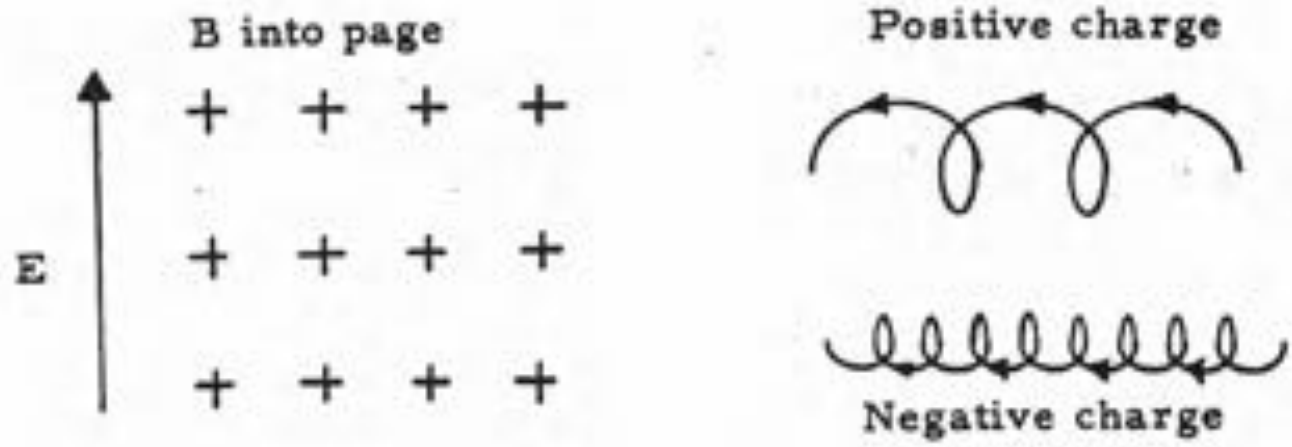


Figure 24

For a neutral plasma, since the two types of charges drift in the same direction at the same rate, there is no current. No net work is done on either type of particle, since they drift perpendicular to E .

E. The Effect of Gravity or Other External Force

If a uniform gravitational force or other external force acts on the particle in addition to the magnetic force, then the equation of motion for the particle is given by

$$m \frac{d\vec{v}}{dt} = \vec{F} + \frac{q}{c} \vec{v} \times \vec{B} \tag{47}$$

where $\vec{F}$ is the external force. We may replace $\vec{F}$ by an equivalent electric field force.

$$\vec{F} = q\vec{E}$$
$$\text{or} \tag{48}$$
$$\vec{E} = \vec{F}/q$$

and take over all our results from the previous section. The particle drifts across the magnetic field with the equivalent of an $\vec{E} \times \vec{B}$ velocity, which is now

$$\vec{v}_F = \frac{c}{q} \frac{\vec{F} \times \vec{B}}{B^2} \tag{49}$$

Superimposed upon this drift is the usual Larmor motion about the magnetic field. If $\vec{F}$ has a component parallel to $\vec{B}$, the particle accelerates in this direction at a constant rate. We see from Eq. (49) that the direction of drift (unlike that found for the E field case) depends on the sign of the charge on the particle. Thus an external force acting on a neutral cloud will cause charges of different sign to drift in opposite directions, giving rise to a current. The current gives rise to a $j \times B$ force which balances the external force.

<u>Problem:</u> Prove the statement in the last sentence of part E.

F. <u>The Effect of a Time-Varying Electric Field</u>

Let us imagine that our particle is subject to a spatially uniform B field and a spatially uniform E which has a constant direction in space, but whose magnitude varies in time. We will take the E field to be perpendicular to B since the magnetic field plays no role in the motion of a particle parallel to it. We shall further assume that the electric field is changing at a rate which is slow by comparison with the Larmor motion. That is, we assume

$$\frac{1}{E}\frac{1}{\omega_c}\frac{dE}{dt} << 1 \tag{50}$$

We will first derive what happens on simple physical grounds. Since the E field is changing slowly with time, to a first approximation the particle will move with an instantaneous $E \times B$ drift velocity plus Larmor motion. Thus $\vec{v}_\perp$ is given by

$$\vec{v}_\perp = \frac{c\,\vec{E}(t) \times \vec{B}}{B^2} + \vec{v}_{Larmor} \tag{51}$$

Now, since the drift velocity changes with time, so does the kinetic energy of the particle. Let us average this change over one Larmor period so as to remove the periodic changes already discussed. We then get

$$\frac{d}{dt}\frac{m}{2}v_\perp^2 = m\vec{v}_E \quad \frac{d\vec{v}_E}{dt} = m\frac{c^2}{B^2}\frac{dE^2/2}{dt} \tag{52}$$

The last equality follows from the fact that we assumed that E was constant in direction and perpendicular to B . Now this energy must be supplied by the electric field or

$$q v_{\parallel E} E = \frac{c^2 m}{B^2} E \frac{dE}{dt} \tag{53}$$

$$v_{\parallel E} = \frac{c^2 m}{q B^2}\frac{dE}{dt} \tag{54}$$

where $v_{\parallel E}$ denotes the component of the velocity parallel to E .

We may derive this result from still another physical argument. The sum of all the forces on a particle (including inertial forces) must be zero. Now we may treat the inertial force due to the changing $E \times B$ velocity as an external force; a gravitational force, if one wishes. Then according to our previous analysis of the drift of a particle subject to an external force, the particle will acquire a drift velocity given by

$$\vec{v}_F = c\,\frac{\vec{F}\times\vec{B}}{qB^2} \tag{55}$$

Substituting for $\vec{F}$ $- m\,dv/dt$ (the inertial force or effective gravitational force is in the opposite direction to the acceleration):

$$\vec{F} = -\frac{mc}{B^2}\frac{d\vec{E}}{dt}\times\vec{B} \tag{56}$$

and we find for $\vec{v}_{\frac{dE}{dt}} \equiv v_p$

$$\vec{v}_p = -\frac{mc^2}{qB^2}\left(\frac{d\vec{E}}{dt}\times\vec{B}\right)\times\vec{B}/B^2 \tag{57}$$

or

$$\vec{v}_p = +\frac{mc^2}{qB^2}\frac{d\vec{E}}{dt} \tag{58}$$

since by assumption E is perpendicular to B . This method has the advantage that it applies even if the direction of E is changing with time, but E must still remain perpendicular to B . This approach also applies only when E varies slowly on the scale of the Larmor frequency, for only then can we neglect the Larmor motion and the forces associated with it.

Finally, we may derive these results formally from Eq. (9). Again we write

$$\vec{v}_\perp = \vec{v}_E + \vec{v}_1 \tag{59}$$

$$\vec{v}_E = c\,\frac{\vec{E}\times\vec{B}}{B^2} \tag{60}$$

Substituting in Eq. (9) gives

$$m\left(\frac{d\vec{v}_E}{dt}+\frac{d\vec{v}_1}{dt}\right) = \frac{q}{c}\,\vec{v}_1\times\vec{B} \tag{61}$$

We set $v_1 = v_2 + v_p$, so that Eq. (61) becomes

$$m\frac{dv_E}{dt}+\frac{m\,dv_2}{dt}+\frac{m\,dv_p}{dt} = \frac{q}{c}v_2\times B+\frac{q}{c}v_p\times B \tag{62}$$

and in a manner similar to the previous case we define v_p so as to cancel the $m\frac{dv_E}{dt}$ term, i.e.,

$$v_p = \frac{mc^2}{qB^2}\frac{dE}{dt} \tag{63}$$

$$\frac{q}{c}\vec{V}_p \times \vec{B} = \frac{q\, mc^2}{c\, q\, B^2}\frac{d\vec{E}}{dt} \times \vec{B} = mc\,\frac{\frac{d\vec{E}}{dt} \times \vec{B}}{B^2} = m\,\frac{d\vec{V}_E}{dt} \tag{64}$$

Then Eq. (62) becomes

$$m\frac{d\vec{V}_p}{dt} + m\frac{d\vec{V}_2}{dt} = \frac{q}{c}\,\vec{V}_2 \times \vec{B} \tag{65}$$

If the first term is negligible, then as before $\vec{v}_2$ describes the Larmor motion in a frame moving with velocity $\vec{v}_p + \vec{v}_E$. Essentially we are utilizing a Taylor expansion (in time) of the electric field, having thus far found drifts corresponding to $\vec{E}$ and $d\vec{E}/dt$; the remaining term, $d\vec{v}_p/dt$ is a $d^2\vec{E}/dt^2$ term.

The $d\vec{v}_p/dt$ may be dropped provided that

$$\left| m\,\frac{d\vec{V}_p}{dt} \right| \ll \left| \frac{q}{c}\,\vec{V}_2 \times \vec{B} \right| \tag{66}$$

or

$$\left| \frac{mc^2}{q\, B^2}\,\frac{d^2\vec{E}}{dt^2} \right| \ll \left| \frac{q}{c}\,\vec{V}_2 \times \vec{B} \right| \quad \text{and,} \tag{67}$$

assuming that $\vec{E} \sim \vec{E}e^{i\omega t}$,

$$1 \ll \frac{q^2 B^2}{m^2 c^2}\left(\frac{B}{cE}\right)\frac{V_2}{\omega^2} \tag{68}$$

or

$$1 \ll \frac{\omega_c^2}{\omega^2}\,\frac{V_2}{V_E}. \tag{69}$$

Thus the next higher term may be dropped, unless $v_E \gg v_2$, or ω is comparable with ω_c.

Problem: By setting $\vec{v}_2 = \vec{v}_3 + \vec{v}_{\dot{p}}$, find the drift corresponding to d^2E/dt^2.

G. The Effective Dielectric Constant of a Plasma in a Magnetic Field

We have just seen that when a time-varying electric field is applied to a plasma in a magnetic field ($\vec{E} \perp \vec{B}$) particle drifts parallel to E arise, given by Eq. (58). The drifts are opposite for oppositely charged particles, so that a current arises in the plasma. The work that the $\vec{E}$ field does on this current is just what is required to get the plasma moving with the $c\,\dfrac{\vec{E}\times\vec{B}}{B^2}$ velocity.

The currents in the direction of $\vec{E}$ may be thought of as polarization currents; the plasma becomes polarized in this direction (Fig. 25).

To treat the plasma like a dielectric we divide the current into a plasma current and into currents due to external sources, in a manner analogous to the case of magnetization.

$$\vec{J} = \vec{J}_p + \vec{J}_e. \tag{70}$$

We have

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}_e + \frac{4\pi}{c}\vec{J}_p + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}. \tag{71}$$

Now if the plasma current is proportional to $\dfrac{\partial\vec{E}}{\partial t}$, as we have just found it to be when $\vec{E}$ is perpendicular to $\vec{B}$, then we can combine the last two terms on the right-hand side of Eq. (71)

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}_e + \left[\frac{4\pi c}{B^2}\sum_i n_i m_i + \frac{1}{c}\right]\frac{\partial\vec{E}}{\partial t} \tag{72}$$

(This only applies if $\vec{E}$ is perpendicular to $\vec{B}$, $\vec{J}_p = \dfrac{c^2}{B^2}\sum_i n_i m_i \dfrac{\partial\vec{E}}{\partial t}$).

We may set

$$\vec{D} = \left[\frac{4\pi\rho c^2}{B^2} + 1\right]\vec{E} = \epsilon\vec{E} \tag{73}$$

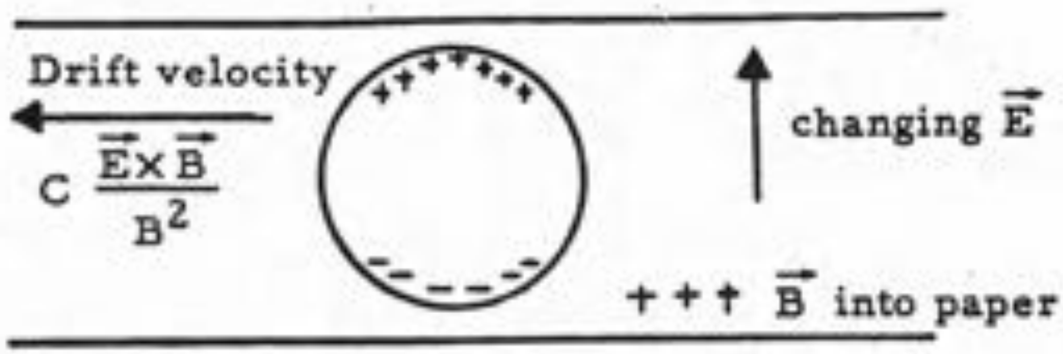

Drift velocity
$c\,\frac{\vec{E}\times\vec{B}}{B^2}$
changing $\vec{E}$
+ + + $\vec{B}$ into paper

Figure 25

and then Eq. (72) reads

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}_e + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}. \tag{74}$$

Further, we may divide the charge density into an internal and an external part.

$$\sigma = \sigma_p + \sigma_e \tag{75}$$

where σ_p has its sources in the charges within the plasma and is related to $\vec{j}_p$ by the continuity equation

$$\frac{\partial \sigma_p}{\partial t} + \vec{\nabla} \cdot \vec{J}_p = 0. \tag{76}$$

We then have

$$\vec{\nabla} \cdot \vec{E} = 4\pi(\sigma_p + \sigma_e) \tag{77}$$

and

$$\frac{\partial \sigma_p}{\partial t} + \vec{\nabla} \cdot \left[\frac{\rho c^2}{B^2} \frac{\partial \vec{E}}{\partial t} \right] = 0 \tag{78}$$

or

$$\frac{\partial}{\partial t} \left[\sigma_p + \frac{\rho c^2}{B^2} \vec{\nabla} \cdot \vec{E} \right] = 0 \tag{79}$$

or

$$\sigma_p = - \frac{\rho c^2}{B^2} \vec{\nabla} \cdot \vec{E} \tag{80}$$

If σ_p is 0 when E is 0, then from Eq. (77)

$$\nabla \cdot \left[1 + \frac{4\pi \rho c^2}{B^2} \right] \vec{E} = 4\pi \sigma_e \tag{81}$$

or

$$\vec{\nabla} \cdot \vec{D} = 4\pi \sigma_e. \tag{82}$$

Problem: Find the capacitance of a parallel plate condenser with plasma between its plates and with a magnetic field parallel to the surface of the plates. Assume that there is an insulating layer of infinitesimal thickness isolating the plates from the plasmas. Show that the energy per unit area stored in the capacitor, $\frac{1}{2} cv^2$, is stored as kinetic energy.

H. Time-Varying $\vec{B}$

Let us consider the case in which $\vec{B}$ is spatially uniform, at least in the region visited by our particles, but in which its magnitude varies with time. Because of the time variation of $\vec{B}$ there will be $\vec{E}$ fields set up. These give rise to the $\vec{E} \times \vec{B}$ and $\dot{\vec{E}} \times \vec{B}$ drifts just discussed. However, here we are not so interested in these effects as we are in the fact that $\vec{E}$ has a curl and hence will do work on a circulating charge. We will imagine that we have subtracted out the mean $\vec{E} \times \vec{B}$ drift.

Now the change in the perpendicular energy of the particles is given by

$$\delta W_{\perp} = + q \int \vec{E} \cdot d\vec{\ell}. \tag{83}$$

But now if we go around a closed orbit, then

$$\oint \vec{E} \cdot d\vec{\ell} = \int \vec{\nabla} \times \vec{E} \cdot d\vec{A} = -\int \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}. \tag{84}$$

If $\partial \vec{B}/\partial t$ is essentially constant over an orbit, then we may replace the last integral by

$$(\pm) \frac{\pi a^2}{c} \frac{\partial \vec{B}}{\partial t}$$

where a is the Larmor radius $v_{\perp}/\omega_c = a$.

The $(\pm)$ is determined by whether $\partial \vec{B}/\partial t$ is parallel or antiparallel to $d\vec{A}$. Now the direction of $d\vec{A}$ is determined by the direction of $d\vec{\ell}$ since the direction $d\vec{\ell}$ must be the direction in which the particle moves. If $\vec{B}$ is taken in the z direction, then $d\vec{A}$ is antiparallel to $\vec{B}$ for ions and parallel to $\vec{B}$ for electrons. Thus the sign is opposite to that of the charge. We thus find for $\delta W_{\perp}$

$$\delta W_\perp = \frac{|q|}{c} \pi a^2 \frac{\partial B}{\partial t} \text{ (in scalar form)} \tag{85}$$

or

$$\delta W_\perp = \frac{|q|}{c} \pi \frac{V_\perp^2 m^2 c^2}{q^2 B^2} \frac{\partial B}{\partial t} \tag{86}$$

or

$$\delta W_\perp = W_\perp \frac{2\pi mc}{|q| B} \frac{1}{B} \frac{\partial B}{\partial t} = W_\perp \frac{2\pi}{\omega_c B} \frac{\partial B}{\partial t} \tag{87}$$

thus

$$\frac{\delta W_\perp}{W_\perp} = \frac{\delta B}{B} \qquad \text{for one orbit.} \tag{88}$$

This equation may also be written in the form

$$\delta\left(\frac{W_\perp}{B}\right) = 0 \qquad \text{per orbit.} \tag{89}$$

Hence

$$\frac{W_\perp}{B} = \mu \cong \text{constant.} \tag{90}$$

The magnetic moment is thus approximately constant. It is not a strict constant since the above treatment requires that $\partial B/\partial t$ be essentially constant throughout an orbit. If B were changed instantaneously (very quickly, in the time it takes light to cross an orbit) from one value to another, then $W_\perp$ will not change since the particle will not have moved during the time B is changing. Thus $W_\perp/B$ will have changed. The magnetic moment is called an adiabatic invariant, since it is constant to a high degree of accuracy for slow variations of B. (It has been shown that the change in μ is exponentially small in the $B/\dot{B}$.)

There is a simple physical way to look at the constancy of μ. Now μ is equal to

$$\mu = \frac{W_\perp}{B} = \frac{m}{2}\frac{(a\omega_L)^2}{B}$$
$$= \frac{m}{2}\frac{a^2 e^2 B^2}{m^2 c^2 B} = \frac{e^2}{2mc^2}\, a^2 B \qquad (91)$$

where a is the Larmor radius.

Thus if μ is constant, a^2B or the flux through the orbit is constant. The orbit thus looks like a little superconducting current loop and no flux can cross it. This should not be surprising, since we have put in no mechanism for dissipating the current.

I. Spatially-Varying Magnetic Field

So far we have considered only magnetic fields which are spatially uniform over the regions visited by the particle. We wish now to consider magnetic fields which are not spatially uniform but which vary slowly with position. Here slowly means that variations of the magnetic field over a Larmor orbit are small, or

$$\frac{|\vec{\nabla}\vec{B}\cdot|a||}{B} \ll 1. \qquad (92)$$

We can then find the particle's motion as a perturbation (locally) from what it would have in a spatially-uniform field. To this end we Taylor expand $\vec{B}$ about some point $\vec{r}$; $\vec{r}$ may be a function of t .

$$\vec{B}(\vec{r}+\vec{\rho}) = \vec{B}(\vec{r}) + \vec{\rho}\cdot\vec{\nabla}\vec{B}(\vec{r}). \qquad (93)$$

We will in general choose $\vec{r}$ to be the position of the guiding center for the particle — i.e., the instantaneous center of gyration. We shall consider the various elements of the tensor $\vec{\nabla}\vec{B}$ in turn. In the case of each set of terms, we will see first the influence of the terms on the shape of the field lines, and then find the effect on the particle orbits.

$$\vec{\nabla}\vec{B} = \begin{bmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial B_x}{\partial y} & \frac{\partial B_x}{\partial z} \\ \frac{\partial B_y}{\partial x} & \frac{\partial B_y}{\partial y} & \frac{\partial B_y}{\partial z} \\ \frac{\partial B_z}{\partial x} & \frac{\partial B_z}{\partial y} & \frac{\partial B_z}{\partial z} \end{bmatrix}. \tag{94}$$

(1) <u>The Effect of Diagonal Terms (Converging and Diverging Lines of Force)</u>

First consider the effects of the diagonal terms. Since $\vec{\nabla} \cdot \vec{B} = 0$, these terms are not all independent but their sum must vanish. We will choose a coordinate system such that r is 0 and such that the magnetic field at the origin points in the z direction. We have for the local magnetic field (neglecting off-diagonal terms for the moment, since their effect will be found shortly)

$$B_z = B_0 + \left[\frac{\partial B_z}{\partial z}\right]_0 z, \tag{95}$$

$$B_y = \left[\frac{\partial B_y}{\partial y}\right]_0 y, \tag{96}$$

and

$$B_x = \left[\frac{\partial B_x}{\partial x}\right]_0 x. \tag{97}$$

First let us see what these terms imply about the lines of force. The equations for a line of force are

$$\frac{dx}{dz} = \frac{B_x}{B_z} \tag{98}$$

and

$$\frac{dy}{dz} = \frac{B_y}{B_z}. \tag{99}$$

To 0 order (x and y or derivative terms neglected) we have

$$\frac{dx}{dz} = \frac{dy}{dz} = 0. \tag{100}$$

Thus $x = x_0$, $y = y_0$, and the lines are straight and parallel to the z axis as expected, since we chose the z direction as the direction in which the major part of B points. To first order in x and y we have

$$\frac{dx}{dz} = \left(\frac{\partial B_x}{\partial x}\right)_0 \frac{x}{B_0} \tag{101}$$

and

$$\frac{dy}{dz} = \left(\frac{\partial B_y}{\partial y}\right)_0 \frac{y}{B_0} \tag{102}$$

or

$$x = \left(\frac{\partial B_x}{\partial x}\right)_0 \frac{xz}{B_0} + x_0 \tag{103}$$

and

$$y = \left(\frac{\partial B_y}{\partial y}\right)_0 \frac{yz}{B_0} + y_0. \tag{104}$$

Fig 26 missing

Thus the lines of force are tilted as shown in Fig. 26 . The lines of force are diverging or converging. Further, since $\vec{\nabla} \cdot \vec{B} = 0$, we have, to lowest order for B_z

$$\frac{\partial B_z}{\partial z} = -\left\{\left(\frac{\partial B_x}{\partial x}\right)_0 + \left(\frac{\partial B_y}{\partial y}\right)_0\right\} \tag{105}$$

or

$$B_z = B_0 - \left\{\left(\frac{\partial B_x}{\partial x}\right)_0 + \left(\frac{\partial B_y}{\partial y}\right)_0\right\} z. \tag{106}$$

Thus the strength of the main magnetic field (the B_z component) varies along the z direction or varies as one moves along the magnetic field lines, since to zero order they are in the z direction (unless the $\partial B_x/\partial x$ and $\partial B_y/\partial y$ terms cancel).

Before computing in detail what will happen to particle orbits, we may look at this problem in view of what we have already learned. Suppose the particle is gyrating about B_z and at the same time moving along the B lines — i.e., in the z direction. Then it will see a magnetic field whose strength is changing in time. By our assumption that $\vec{B}$ is slowly varying in space, this will be a slow time variation provided the motion of the particle along $\vec{B}$ is not extremely fast. From our treatment of the time-varying $\vec{B}$ field we should expect that the perpendicular energy of the particle would vary in such a way as to keep the magnetic moment constant

$$W_\perp = |B|\mu = |B| \frac{W_{\perp 0}}{B_0}. \tag{107}$$

Now the particle's energy must be constant since the magnetic field does no work on it and hence there must be an equal and opposite change in the parallel energy of the particle.

$$W_\perp + W_\parallel = W = \text{constant}, \tag{108}$$

$$W_\parallel = W - |B|\mu = W_{\parallel 0} + W_{\perp 0} - |B|\mu. \tag{109}$$

Thus $|B|\mu$ acts like a potential for the motion along the lines of force. Eq. (109) may be written in differential form. For the time interval dt we have

$$dW_\parallel = m v_\parallel \, dv_\parallel = -\mu \frac{d|B|}{dz} v_\parallel \, dt \tag{110}$$

or

$$m\frac{dv_\parallel}{dt} = -\mu\frac{d|B|}{dz}. \tag{111}$$

This should be a familiar form. The force on a magnetic dipole is the product of the dipole moment and the field gradient. The negative sign results from the fact that the dipole is diamagnetic. Equivalently we can see that this force comes about because of the interaction of the particle's perpendicular motion with the radial $\vec{B}$ field, as shown in Fig. 27. These conclusions are actually borne out by the more detailed calculations which we shall now give. The equations of motion for our particle are

$$m\frac{d\vec{V}}{dt} = \frac{q}{c}\vec{V}\times\vec{B}. \tag{112}$$

Substituting in $\vec{B}$ from Eqs. (95), (96), and (97), and writing in component form gives

$$m\frac{dv_z}{dt} = \frac{q}{c}\left[v_x\left(\frac{\partial B_y}{\partial y}\right)_0 y - v_y\left(\frac{\partial B_x}{\partial x}\right)_0 x\right], \tag{113}$$

$$m\frac{dv_x}{dt} = \frac{q}{c}\left[v_y\, B_0 - v_z\left(\frac{\partial B_y}{\partial y}\right)_0 y\right], \tag{114}$$

and

$$m\frac{dv_y}{dt} = -\frac{q}{c}\left[v_x\, B_0 - v_z\left(\frac{\partial B_x}{\partial x}\right)_0 x\right]. \tag{115}$$

We consider a particle whose center of gyration (guiding center) is instantaneously at the origin — i.e., $z = 0$. Now the zero order solutions of Eqs. (114) and (115) are

$$w = w_0 e^{-i\omega_c t} = v_x + i v_y$$

and

$$\zeta = -\frac{w_0}{i\omega_c}e^{-i\omega_c t} = x + i y$$

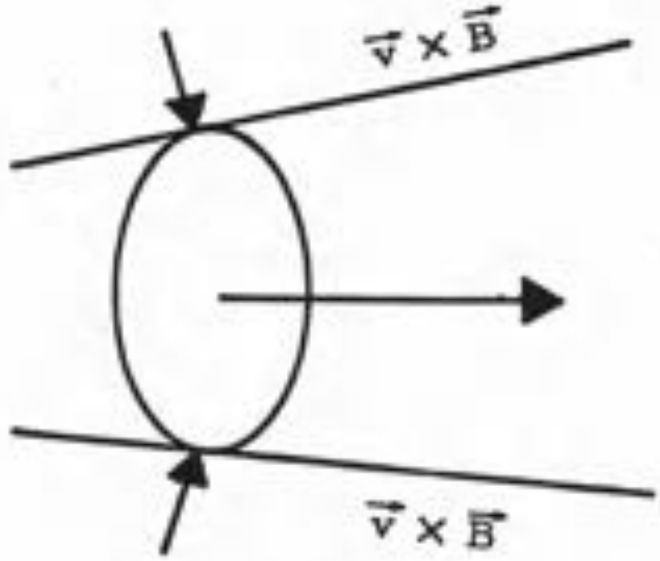


Figure 27

where $$\omega_c = \frac{qB}{mc}.$$

We may choose $\mathcal{g}_0$ to be 0 by proper choice of phase or coordinates; also, we may choose w_0 to be real. Substituting in Eq. (113) gives

$$m\frac{dv_z}{dt} = \frac{q w_0^2}{c\omega_c}\left(-\sin^2\omega_c t\left(\frac{\partial B_y}{\partial y}\right)_0 - \cos^2\omega_c t\left(\frac{\partial B_x}{\partial x}\right)_0\right). \tag{116}$$

no − sign

Averaging over one orbit gives

$$m\overline{\frac{dv_z}{dt}} = -\frac{q w_0^2}{2c\omega_c}\left[\left(\frac{\partial B_x}{\partial x}\right)_0 + \left(\frac{\partial B_y}{\partial y}\right)_0\right] =$$

$$= \frac{q w_0^2}{2c\omega_c}\left(\frac{\partial B_z}{\partial z}\right)_0. \tag{117}$$

Now w_0^2 is $2W_\perp/m$, so that we may rewrite Eq. (117) in the form

$$m\overline{\frac{dv_z}{dt}} = -\frac{W_\perp}{B}\left(\frac{\partial B_z}{\partial z}\right) = -\mu\left(\frac{\partial B_z}{\partial z}\right) \tag{118}$$

Now multiplying both sides of Eq. (118) by v_z gives

$$\frac{m}{2}\overline{\frac{d}{dt}(v_z)^2} = -\mu\frac{dB_z}{dt}. \tag{119}$$

Multiplying Eq. (114) by v_x and Eq. (115) by v_y and adding gives

$$\frac{m}{2}\frac{d}{dt}(v_x^2+v_y^2) = \frac{q v_z}{c}\left[x v_y\left(\frac{\partial B_x}{\partial x}\right)_0 - y v_x\left(\frac{\partial B_y}{\partial y}\right)_0\right] \tag{120}$$

This is just $-v_z$ times Eq. (113), so that we find

$$\frac{m}{2}\frac{d(v_z^2)}{dt} = -\frac{m}{2}\frac{d(v_\perp^2)}{dt} \quad \text{where} \quad v_\perp^2 = v_x^2+v_y^2 \tag{121}$$

which is just the equation for conservation of energy. Thus we may write Eq. (119) in the form

$$\frac{dW_\perp}{dt} = \frac{m}{2}\frac{d(v_\perp^2)}{dt} = \mu\,\frac{dB_r}{dt} = \frac{W_\perp}{B}\,\frac{dB_r}{dt} \tag{122}$$

or

$$\frac{d}{dt}\left(\frac{W_\perp}{B}\right) = \frac{d\mu}{dt} = 0. \tag{123}$$

Thus the magnetic moment is constant in this spatially-varying B field as well as in a time-varying field (provided the variations are not too rapid) and our previous analysis is justified.

(2) <u>Effects of $(\partial B_x/\partial z)$ and $(\partial B_y/\partial z)$ (Curvature of the Lines of Force)</u>

Let us now look at terms of the form $(\partial B_x/\partial z)_o$ and $(\partial B_y/\partial z)_o$. We need consider only one of these, since by appropriate orientation of the xy plane the other can be eliminated. That is, if we choose our x axis to point in the direction $(\partial B_\perp/\partial z)$ ($B_\perp$ is the component of $\vec{B}$ perpendicular to z), then we get only a $(\partial B_x/\partial z)$ term.

Let us again look at what this implies about the shape of the lines of force. We return to Eq. (98) for the lines of force, and again to zero order in derivatives we find straight lines of force. To first order we find

$$\frac{dx_1}{dz} = \frac{\left(\frac{\partial B_x}{\partial z}\right)_o z}{B_o} \tag{124}$$

or

$$x_1 = x_o + \frac{z^2}{2B_o}\left(\frac{\partial B_x}{\partial z}\right)_o \tag{125}$$

The lines of force are curved as shown in Fig. 28. For small z we find that the curve may be regarded as a segment of a circle.

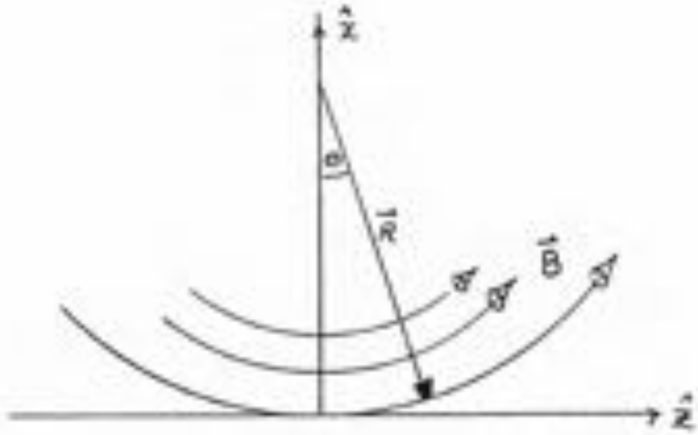


Figure 28

From this figure we have

$$\frac{B_x}{B_0} = \tan\theta \cong \theta \cong \frac{z}{R}. \tag{126}$$

Thus

$$z/R = B_x/B_0 = \frac{z}{B_0}\left(\frac{\partial B_x}{\partial z}\right)_0 \tag{127}$$

or

$$R = \frac{B_0}{\left(\frac{\partial B_x}{\partial z}\right)_0} \tag{128}$$

In general the vector radius of curvature $\vec{R}$ of a curve is given in terms of the unit tangent to the curve $\vec{n}_t$ (in this case $\vec{n}_t = \frac{\vec{B}}{|B|}$). The relation is

$$\frac{\vec{R}}{|R^2|} = -(\vec{n}\cdot\vec{\nabla})\vec{n} \tag{129}$$

To solve this problem we introduce the local cylindrical coordinates such that the axis of the cylinder is perpendicular to the local plane of the field lines and so that it passes through their center of curvature, as shown in Fig. 29(a).

Figure 29

We choose axes so that the unit vector e_θ lies along B, the e_r unit vector is perpendicular to B and in the plane of B and points away from the center of curvature and e_z is chosen normal to θ and r so that e_r, e_θ, e_z forms a right-handed coordinate system. Locally the magnetic field has only a θ component. We have for $d\vec{v}/dt$

$$\frac{d\vec{V}}{dt} = \hat{e}_r \frac{dV_r}{dt} + V_r \frac{d\hat{e}_r}{dt} + \hat{e}_\theta \frac{dV_\theta}{dt} + V_\theta \frac{d\hat{e}_\theta}{dt} + \hat{e}_z \frac{dV_z}{dt} \tag{130}$$

and from Fig. 29(b)

$$\frac{d\hat{e}_r}{dt} = \frac{V_\theta}{r}\hat{e}_\theta, \qquad \frac{d\hat{e}_\theta}{dt} = -\frac{V_\theta}{r}\hat{e}_r. \tag{131}$$

Our equation of motion becomes

$$\begin{aligned} m\frac{d\vec{V}}{dt} &= m\left\{\hat{e}_r\left[\frac{dV_r}{dt} - \frac{V_\theta^2}{r}\right] + \hat{e}_\theta\left[\frac{dV_\theta}{dt} + \frac{V_\theta V_r}{r}\right] + \hat{e}_z\frac{dV_z}{dt}\right\} \\ &= \frac{q}{c}\vec{V}\times\vec{B} = \frac{qB_\theta}{c}\left(-\hat{e}_r V_z + \hat{e}_z V_r\right). \end{aligned} \tag{132}$$

The scalar equation representing e_θ terms gives

$$\frac{dV_\theta}{dt} = -\frac{V_\theta V_r}{r} \tag{133}$$

or

$$\frac{dV_\theta}{V_\theta} = -\frac{dr}{r} \quad \text{(where } V_r = \frac{dr}{dt}\text{)}. \tag{134}$$

$$V_\theta = \frac{V_0 r_0}{r}. \tag{135}$$

This just says that the angular momentum $\propto v_\theta r$ about the center of curvature of the lines is conserved and this leads to slight fluctuations of the v_θ as the particle gyrates about the B lines. The other two scalar equations are

$$m\left(\frac{dV_r}{dt} - \frac{V_\theta^2}{r}\right) = \frac{-q}{c} V_z B_0 \tag{136}$$

and

$$m\left(\frac{dV_z}{dt}\right) = \frac{q}{c} V_r B_0. \tag{137}$$

Now if we neglect the slight fluctuations in v_θ just found (these are of the order of the ratio of the Larmor radius to the radius of curvature of the lines of force and are hence small by the assumption that variations of B over regions of the size of a Larmor orbit are small), then these are the equations for the gyration of a particle about a uniform magnetic field when subjected to an external force of magnitude $+\frac{m v_\theta^2}{r}$. This is the centrifugal force which acts on the particle when it follows the curved field lines. It gives rise to a drift in the z direction equal to

$$V_{zD} = \frac{c m V_\theta^2}{r q B_0} = \frac{2 W_{\parallel} c}{R q B}. \tag{138}$$

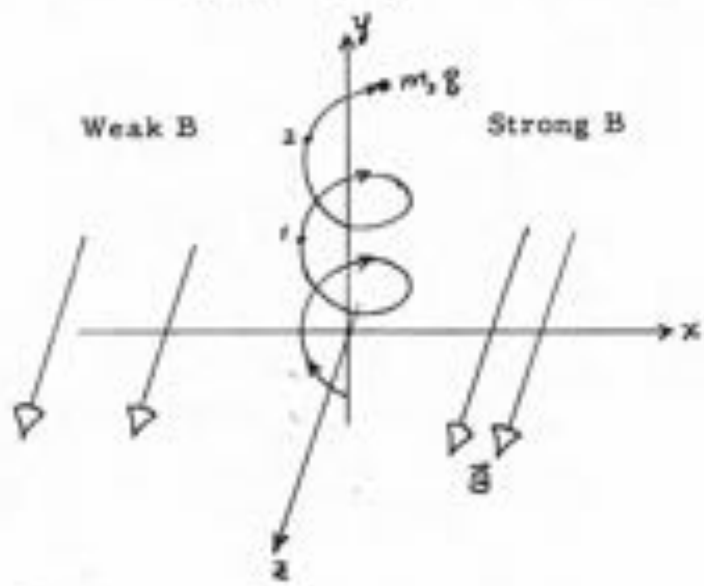
The v_{zD} drift results in a current since it depends on q. This current, in turn, produces the centripetal force required for the circular motion.

We can write Eq. (138) in vector form

$$\vec{V}_r = \frac{2 c W_{\parallel}}{q B^2} \vec{B} \times (\vec{n} \cdot \vec{\nabla}) \vec{n} \qquad (139)$$

(3) <u>Effect of $\partial B_z/\partial x$ and $\partial B_z/\partial y$</u>

These terms do not give rise to any slope or curvature of the B lines, but simply state that the strength of the magnetic field varies in the xy plane. Again we need only consider one of these terms, since we can choose a coordinate system in which the other is zero. That is, we can, say, choose the x axis so that it lies along $\vec{\nabla}_\perp B_z$ and then $\partial B_z/\partial y = 0$. $\vec{\nabla}_\perp$ means $(\vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y})$. The situation is shown in Fig. 30.



<u>Figure 30</u>

If the charged particle were to execute an undisplaced circular orbit, the force in the negative x direction, while the particle is in the right half orbit in the stronger magnetic field, would exceed the opposite force during the time the particle is in the left half orbit. The drift along y produces a $\vec{J} \times \vec{B}$ force along x which just compensates this.

A simple calculation of this effect starts out by noting that the average force along x must be zero. We average over one cycle of the periodic motion

$$\int_{t_1}^{t_2} F_x \, dt = 0 \tag{140}$$

where

$$F_x = \frac{e}{c} v_y B_z(x) = \frac{e}{c} v_y \left(B_0 + x \left(\frac{\partial B_z}{\partial x}\right)_0\right). \tag{141}$$

Substituting Eq. (141) in Eq. (140),

$$\int_{t_1}^{t_2} B_0 v_y \, dt + \int_{t_1}^{t_2} x \left(\frac{\partial B_z}{\partial x}\right)_0 v_y \, dt = 0 \tag{142}$$

thus

$$\delta y = y_2 - y_1 = -\frac{1}{B_0} \left(\frac{\partial B_z}{\partial x}\right)_0 \int_{t_1}^{t_2} x v_y \, dt \tag{143}$$

since B_0 and $\left(\frac{\partial B_z}{\partial x}\right)_0$ are constants. Since the field changes are small by assumption, the orbits are only slightly disturbed from circular ones: we use for the integral of $x v_y \, dt$ over one period simply $\mp \pi a^2$, where a is the Larmor radius, giving

− for ions
+ for electrons

$$\delta y = \pm \frac{1}{B_0} \left(\frac{\partial B_z}{\partial x}\right)_0 \pi a^2 \tag{144}$$

which can be written

$$\delta y = + \frac{1}{B_0} \frac{\partial B_z}{\partial x} \left(\frac{2\pi}{\omega_c}\right) \left(\frac{m}{2} v_\perp^2\right) \frac{c}{q B_0} \quad \text{Note sign of } q \tag{145}$$

where δy is the displacement of the orbit in a time of one cycle, $2\pi/\omega_c$. Then the drift velocity is δy divided by $2\pi/\omega_c$

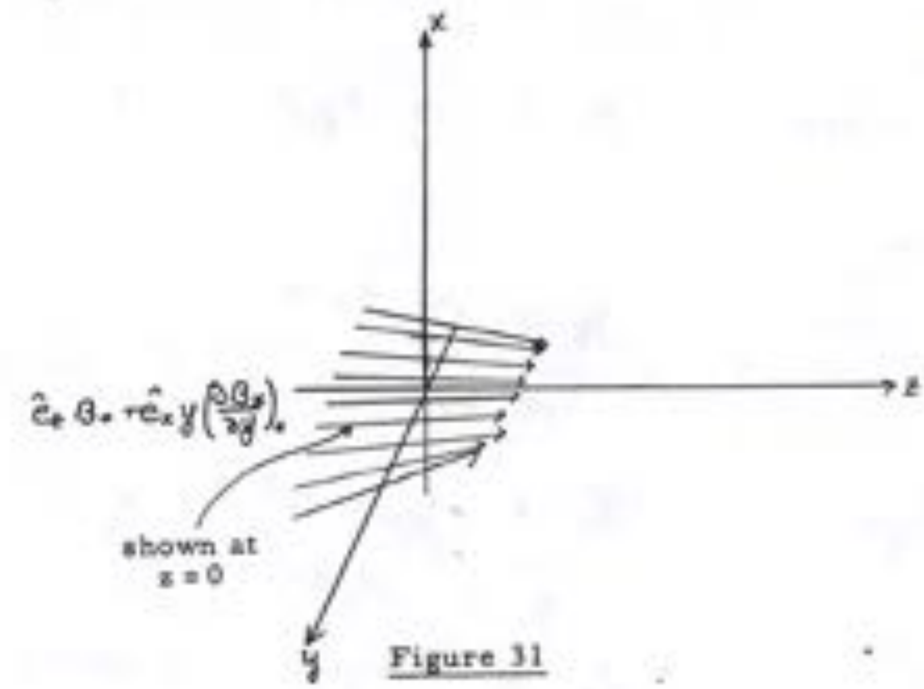
$$v_y = \mp \frac{c\, W_\perp}{e\, B_0^2} \frac{\partial B_z}{\partial x} \qquad \text{has sign of } e \tag{146}$$

or, in general, since the drift is in the direction $\vec{n} \times \vec{\nabla} B$,

$$v_D = \frac{c\, W_\perp}{e\, B^2}\, \vec{n} \times \vec{\nabla}(\vec{B} \cdot \vec{n}). \qquad \text{has sign of } e \tag{147}$$

(4) <u>Effects of $\partial B_x/\partial y$ and $\partial B_y/\partial x$</u>

These components of $\vec{\nabla}\vec{B}$ represent shear or twisting of the magnetic lines of force, as shown in Fig. 31.



<u>Figure 31</u>

We can solve for their effects in the same manner as we did for $(\partial B_z/\partial x)$. They give rise only to driving terms at $2\omega_c$ in the w_1 equation, and hence result in distortions of the orbit, but give rise to no net drift of the particles.

V. Summary of Drifts and Currents

A. Drifts

Electric $\vec{V}_E = c\,\frac{\vec{E}\times\vec{B}}{B^2}$ (1)

Dielectric, $\dot{E}$ $\vec{V}_{\dot{E}} = \frac{mc^2}{q B^3}\underbrace{\vec{n}\times(\dot{\vec{E}}\times\vec{n})}_{\dot{E}_\perp}$ (2)

Curvature $\vec{V}_R = \frac{2cW_\parallel}{q B^2}\left[\vec{B}\times(\vec{n}\cdot\vec{\nabla})\vec{n}\right]$ (3)

$$\vec{n} = \vec{B}/|B|$$

Gradient $\vec{V}_G = \frac{cW_\perp}{q B^2}\left[\vec{n}\times\vec{\nabla}(\vec{B}\cdot\vec{n})\right]$ (4)

External Force $\vec{V}_F = \frac{c}{q}\,\frac{\vec{F}\times\vec{B}}{B^2}$ (5)

B. Currents

Magnetization $\vec{J}_m = c\,\vec{\nabla}\times\vec{M}$ (6)

$$M = -NW_\perp\frac{\vec{B}}{B^2}$$

Polarization $\vec{J}_P = \frac{\epsilon}{4\pi}\vec{n}\times(\dot{\vec{E}}\times\vec{n}) = \frac{\epsilon}{4\pi}\dot{\vec{E}}_\perp$ (7)

$$\epsilon = \frac{4\pi N m c^2}{B^2}$$

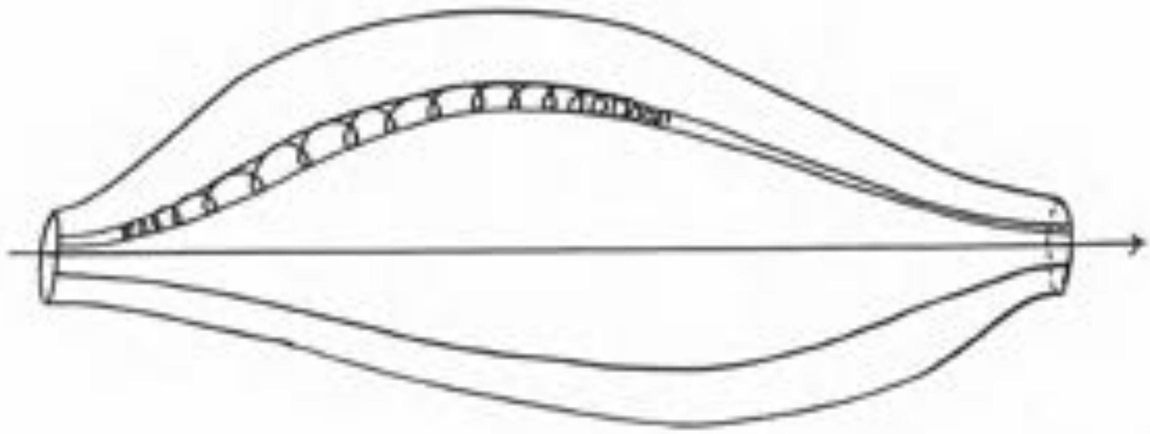
Curvature $\vec{J}_R = \frac{2NcW_\parallel}{B^2}\underbrace{\vec{B}\times((\vec{n}\cdot\vec{\nabla})\vec{n})}_{\vec{R}\times\vec{B}/R^2}$ (8)

Gradient $\vec{J}_G = \frac{NcW_\perp}{B^2}\,\vec{n}\times\vec{\nabla}(\vec{B}\cdot\vec{n})$ (9)

External Force $\vec{J}_F = Nc\,\frac{\vec{F}\times\vec{B}}{B^2}$ (10)

VI. The Longitudinal Invariant

We have already shown that the magnetic moment μ of a particle is an adiabatic invariant. Because of this property a particle may be trapped between regions of high magnetic field, as shown in Fig. 32.



The particle executes a periodic motion back and forth between the regions of large B. There is a second adiabatic invariant associated with this motion which we will now investigate.

There is an adiabatic invariant associated with any periodic motion. This invariant is the action associated with the motion

$$J = \oint p\, dq \tag{1}$$

where p is the momentum, q is the position coordinate, and the integral is to be taken over a closed orbit. For the motion considered here, the appropriate invariant is

$$J = m \int \vec{V}_{\parallel} \cdot d\vec{S} \tag{2}$$

where $\vec{v}_{\parallel}$ is the velocity parallel to the magnetic lines of force, and $\vec{ds}$ is an element of arc length along the magnetic field lines.

In our earlier work we found

$$m\frac{dv_{\parallel}}{dt} = -\mu \frac{\partial B}{\partial s} \tag{3}$$

or

$$m\frac{ds}{dt}\frac{dv_{\parallel}}{ds} = m v_{\parallel}\frac{dv_{\parallel}}{ds} = -\mu\frac{\partial B}{\partial s}. \tag{4}$$

Eq. (4) can be integrated directly to yield an energy relation for the motion

$$W = \frac{m v_{\parallel}^2}{2} + \mu B. \tag{5}$$

Thus if B is time independent, W will be conserved. However, for time-varying B, W will also vary with time.

We may solve Eq. (5) for $v_{\parallel}$ and thus have

$$v_{\parallel} = \pm\sqrt{\frac{2}{m}(W-\mu B)}. \tag{6}$$

Now let B vary slowly with time (slowly means slow compared to the period of oscillation between regions of strong magnetic field — i.e., between magnetic mirrors). We may compute the change in J during one period.

$$\Delta J = \sqrt{\frac{m}{2}}\left\{\oint \frac{\Delta W - \mu\Delta B}{2\sqrt{W-\mu B}}\, ds\right\}. \tag{7}$$

Here ΔW and ΔB are the changes in W and B during one period. The integral is understood to be evaluated at one instant of time (fixed W and B). Now ΔB is given by

$$\Delta B(s) = \frac{\partial B}{\partial t}\, T \tag{8}$$

where T is to be the period of one oscillation. We have for T

$$T = \oint \frac{ds}{v_\parallel} = \sqrt{\frac{m}{2}} \oint \frac{ds}{\sqrt{W-\mu B}} \tag{9}$$

and hence $\Delta B(s)$ is given by

$$\Delta B = \frac{\partial B}{\partial t}\sqrt{\frac{m}{2}} \oint \frac{ds}{\sqrt{W-\mu B}}. \tag{10}$$

The change in W is given by

$$\Delta W = \int_0^T \frac{dW}{dt}\,dt; \tag{11}$$

the integration is to be carried out along an orbit.

From Eq. (5) we have

$$W = \frac{m v_\parallel^2}{2} + \mu B = \frac{m\dot{s}^2}{2} + \mu B \tag{12}$$

and

$$\frac{dW}{dt} = m\dot{s}\ddot{s} + \mu\frac{dB}{dt} = m\dot{s}\ddot{s} + \mu\frac{\partial B}{\partial t} + \mu\frac{\partial B}{\partial s}\dot{s} \tag{13}$$

implies d/dt. But $m\ddot{s} = -\mu\frac{\partial B}{\partial s}$ and hence

$$\frac{dW}{dt} = \mu\frac{\partial B}{\partial t}. \tag{14}$$

This is quite a reasonable result, for μB is the effective potential energy and Eq. (13) states that the rate of change of the particle's energy is equal to the rate of change of the potential energy at the point where the particle is located; or, to express this result in another way, the particle does not change energy in moving through an arbitrary magnetic field, unless the field changes in time. From Eq. (14) we find for the change in ΔW

$$\Delta W = \int_0^T \mu \frac{\partial B}{\partial t} dt = \int \mu \frac{\partial B}{\partial t} \frac{ds}{V_\parallel} = \sqrt{\frac{m}{2}} \oint \mu \frac{\partial B}{\partial t} \frac{ds}{\pm\sqrt{W-\mu B}} \tag{15}$$

Now Eq. (15) is supposed to be taken along the actual motion of the particle, and so we should use the value of $\partial B(s,t)/\partial t$ at the time the particle visits the point s. However, if B is changing slowly with time so that $\partial B(s,t)/\partial t$ can be considered constant in time, for every s then we may replace Eq. (15) by an integral which is evaluated at one instant of time. Substituting ΔW and ΔB from Eq. (15) and Eq. (10) into Eq. (7) for ΔJ gives

$$\Delta J = \sqrt{\frac{m}{2}} \oint \frac{ds}{\pm\sqrt{W(s)-\mu B(s)}} \left[\mu \sqrt{\frac{m}{2}} \oint \frac{\frac{\partial B}{\partial t}(s')\, ds'}{\pm\sqrt{W(s')-\mu B(s')}} \right.$$

$$\left. -\mu \frac{\partial B(s)}{\partial t} \sqrt{\frac{m}{2}} \oint \frac{ds'}{\pm\sqrt{W(s')-\mu B(s')}} \right] = 0 \tag{16}$$

where the equality follows by interchanging the order of integration on one of the terms. This shows that ΔJ for one period is 0, and hence for any number of periods, provided $\partial B/\partial t$ can be considered constant and provided B does not change by a large fraction of itself during one oscillation.

An interesting application of this longitudinal invariant is provided by the situation shown in Fig. 33 (page 77). We imagine that we have two regions of strong B separated by a large distance L in which the field is very weak. We imagine that the distance over which the field

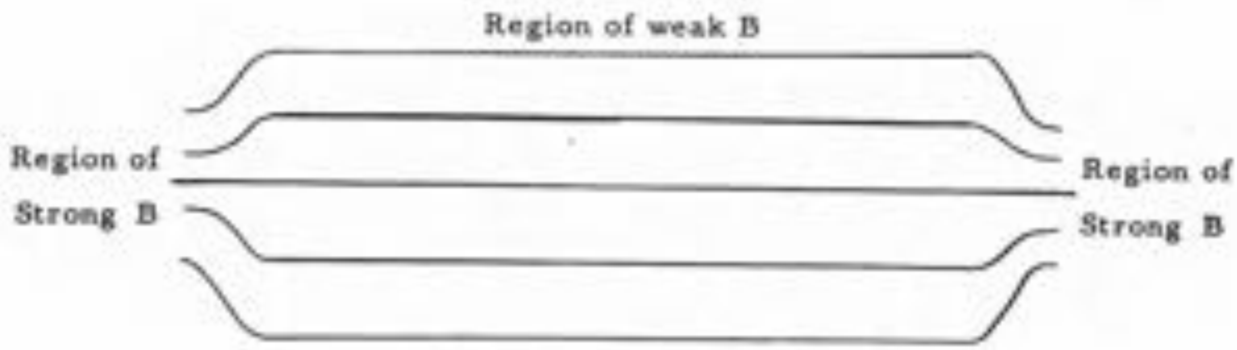


Figure 33

becomes strong is much shorter than L and so we can neglect these regions in evaluating J (we have sharp reflecting boundaries). For this situation J is given by

$$J = m\oint V_{\parallel}\, dS = 2m\, |V_{\parallel}| L \cong \sqrt{2mW}\, 2L. \tag{17}$$

Here we have assumed that μB is negligible between mirrors. Now if the distance between the mirrors changes, then W must change in order to keep J constant.

$$\frac{dJ}{dt} = 2\sqrt{2mW}\,\frac{dL}{dt} + \frac{\sqrt{2m}}{\sqrt{W}}\, L\, \frac{dW}{dt} = 0 \tag{18}$$

or

$$\frac{dW}{W} = -2\,\frac{dL}{L} \tag{19}$$

or

$$W = W_0\left(\frac{L_0}{L}\right)^2 \tag{20}$$

thus

$$V = \frac{V_0 L_0}{L}. \tag{21}$$

Thus if the mirrors move towards each other and L decreases v increases, while if they move apart v decreases. According to Eq. (20) the longitudinal energy or temperature of a gas being compressed between approaching mirrors is proportional to $1/L^2$. Now if we have an ideal gas and compress it adiabatically, then the temperature and volume are related by

$$TV^{\gamma-1} = T_0 V_0^{\gamma-1} \tag{22}$$

where T is the temperature or mean energy per particle, V is the volume and γ is $(n + 2)/n$, where n is the number of degrees of freedom involved in the compression. Here the volume is proportional to L, and only one degree of freedom is involved, the degree associated with the motion back and forth between the mirrors. Thus our adiabatic formula would lead to

$$T = T_0 \frac{L_0^2}{L^2} \tag{23}$$

in agreement with Eq. (20).

This offers one method for heating a gas. However, this means is limited because as the parallel velocity increases it becomes more and more difficult for the mirror fields to trap the particles, and ultimately they escape. Fermi proposed that such a mechanism may be responsible for the acceleration of particles up to cosmic ray energies. Particles would be trapped between magnetic fields associated with large gas clouds. If the clouds are moving towards each other the particle would gain energy until it had sufficient energy to escape. By repeated trappings and compressions, particles could gain energy. Of course if the particle were trapped between the clouds which were separating, it would lose energy. However, in such processes, on the average, particles gain energy if for no other reason than the fact that they can gain an unlimited amount of energy, but they can never lose more than they have.

VII. The Motion of Magnetic Lines of Force

It is sometimes stated that in a plasma in which particle collisions can be neglected the lines of force move with the particles. We will now look at this concept in some detail.

First, this statement is outside the original framework of Maxwell's equations, for in these equations it is not necessary to assign a persistent identity to the field lines. Second, the statement will clearly be true only in the limit of large q/m, when the excursion of the particle involved in the Larmor motion can be neglected. In this limit all drifts are negligible except the $\vec{E} \times \vec{B}$ drift provided $\vec{E} \cdot \vec{B}$ is zero. If $\vec{E} \cdot \vec{B} \neq 0$, then particles are strongly accelerated along field lines, $W_{\parallel}$ is proportional to q and curvature drifts are also important. In this limit the particle moves with the $\vec{E} \times \vec{B}$ drift velocity, and hence we wish to assign this velocity to the field lines. We shall show that we can do this when the component of $\vec{E}$ parallel to $\vec{B}$ is 0, and that the mapping of the B field, which results from this motion, (1) preserves lines of force and (2) preserves the flux through any closed curve.

Consider two particles on the same line of force at time $t = 0$ and which are separated by a small distance $\Delta \ell$.

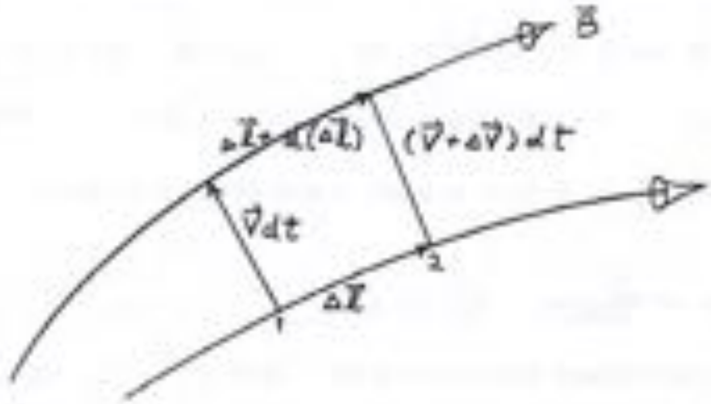


Figure 34

To show that a line of force remains a line of force we must show that $\vec{\Delta l}$ remains parallel to $\vec{B}$ under the transformation, or that

$$\frac{d}{dt}(\Delta\vec{l}\times\vec{B}) = \frac{d(\Delta\vec{l})}{dt}\times\vec{B} + \Delta\vec{l}\times\frac{d\vec{B}}{dt} = 0. \tag{1}$$

Now that we have for $d\,\vec{\Delta l}$

$$d\,\Delta\vec{l} = \underbrace{\left[\Delta\vec{l} + (\vec{V} + (\Delta\vec{l}\cdot\vec{\nabla})\vec{V})dt\right]}_{\text{Position of point 2 at dt.}} - \underbrace{\left[\Delta\vec{l} + \vec{V}dt\right]}_{\text{Position of point 1 at dt.}} \quad - \Delta l \tag{2}$$

or

$$\frac{d\,\Delta\vec{l}}{dt} = (\Delta\vec{l}\cdot\vec{\nabla})\vec{V}. \tag{3}$$

We must now compute B at the displaced point

$$d\vec{B} = \left[\frac{\partial\vec{B}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{B}\right]dt. \tag{4}$$

Now

$$\vec{V} = \frac{c\,\vec{E}\times\vec{B}}{B^2}, \qquad (\text{assuming } \vec{E}\perp\vec{B}) \tag{5}$$

and hence

$$\vec{V} \times \vec{B} = c\frac{(\vec{E}\times\vec{B})}{B^2}\times\vec{B} = -c\vec{E} \tag{6}$$

since $\vec{E}\cdot\vec{B}$ is taken to be zero.

From Maxwell's equations

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{7}$$

and hence from Eq. (6)

$$\vec{\nabla}\times(\vec{V}\times\vec{B}) = \frac{\partial\vec{B}}{\partial t} \tag{8}$$

or

$$(\vec{B}\cdot\vec{\nabla})\vec{V} - (\vec{V}\cdot\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}\cdot\vec{V}) + \vec{V}\overset{0}{\cancel{(\vec{\nabla}\cdot\vec{B})}} = \frac{\partial\vec{B}}{\partial t}. \tag{9}$$

Thus we find for dB/dt

$$\frac{d\vec{B}}{dt} = (\vec{B}\cdot\vec{\nabla})\vec{V} - \vec{B}(\vec{\nabla}\cdot\vec{V}). \tag{10}$$

Substituting Eqs. (10) and (3) in Eq. (1) gives

$$\frac{d}{dt}(\vec{\Delta l}\times\vec{B}) = ((\vec{\Delta l}\cdot\vec{\nabla})\vec{V})\times\vec{B} + \vec{\Delta l}\times\left[(\vec{B}\cdot\vec{\nabla})\vec{V} - \vec{B}(\vec{\nabla}\cdot\vec{V})\right]. \tag{11}$$

Now $\vec{\Delta l}$ is a vector along the direction of $\vec{B}$. Hence we may replace $\vec{\Delta l}$ by $\epsilon\vec{B}$ in the above expression and we immediately see that the right-hand side of Eq. (11) is zero.

$$\frac{d}{dt}(\vec{\Delta l}\times\vec{B}) = 0. \tag{12}$$

Thus the transformation takes lines into lines.

Now to prove that V_E is flux-preserving, we must show that the flux through an area Δs which follows the motion remains constant. If $\Delta\phi$ is the flux through the area, we must show that

$$\frac{d\phi}{dt} = \frac{d}{dt}\int \vec{B}\cdot d\vec{S} = 0 \tag{13}$$

Now ϕ changes for two reasons: first because $\vec{B}$ changes, and secondly because the area changes. The change due to the changing $\vec{B}$ is given by

$$\left(\frac{\partial\phi}{\partial t}\right)_s = \int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} = -c\int (\vec{\nabla}\times\vec{E})\cdot d\vec{S}. \tag{14}$$

The change in ϕ due to the distortion in s (keeping B constant) is given by

$$\left(\frac{\partial\phi}{\partial t}\right)_B = \int_c \vec{B}\cdot(\vec{v}\times d\vec{l}) \tag{15}$$

where c is the bounding curve (see Fig. 35).

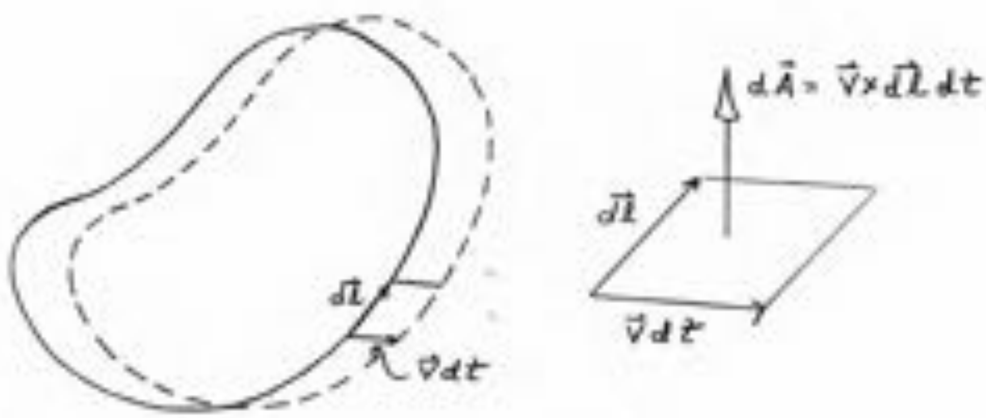


Figure 35

We may interchange the dot and cross products in Eq. (15) to obtain

$$\left(\frac{\partial\phi}{\partial t}\right)_B = -\int_c (\vec{v}\times\vec{B})\cdot d\vec{l}. \tag{16}$$

Converting Eq. (16) into a surface integral gives

$$\left(\frac{\partial\phi}{\partial t}\right)_B = -\int \vec{\nabla}\times(\vec{v}\times\vec{B})\cdot d\vec{S}. \tag{17}$$

Combining Eqs. (17) and (14) gives

$$\frac{d\phi}{dt} = \left(\frac{\partial\phi}{\partial t}\right)_s + \left(\frac{\partial\phi}{\partial t}\right)_B = -\int \vec{\nabla} \times (c\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{S}. \tag{18}$$

In order for this to hold true for every possible surface implies that the integrand must be 0 everywhere, or

$$\vec{\nabla} \times (c\vec{E} + \vec{v} \times \vec{B}) = 0 \tag{19}$$

Now if $\vec{v}$ is given by

$$\vec{v} = c\frac{\vec{E} \times \vec{B}}{B^2},$$

then Eq. (19) becomes

$$c\,\vec{\nabla} \times \left(\cancel{\vec{E}}^{\,0} - \cancel{\vec{E}}^{\,0} + \frac{\vec{B}(\vec{B} \cdot \vec{E})}{B^2}\right) = 0 \tag{20}$$

or

$$\vec{\nabla} \times \left[\frac{\vec{B}(\vec{B} \cdot \vec{E})}{B^2}\right] = 0. \tag{21}$$

Eq. (21) is automatically satisfied if $\vec{B} \cdot \vec{E}$ is zero. We also needed this condition to show that lines went into lines. Thus we see that if $\vec{E} \cdot \vec{B}$ is zero we can assign the velocity $C\vec{E} \times \vec{B}/B^2$ to the lines and this takes lines into lines and preserves the flux through any surface. In a perfect conductor, where inertia can be neglected, $\vec{E} \cdot \vec{B}$ must be zero, for if it were not so the charges would immediately move so as to eliminate $\vec{E}$ parallel to B . To the extent to which this is true for a plasma, the plasma particles are stuck to lines of force.

VIII. Applications of Orbit Theory

A. Static, Straight B Lines, No External Force

We take $\vec{B}$ to be in the z direction. From $\vec{\nabla}\cdot\vec{B} = 0$ we have

$$\frac{\partial B_z}{\partial z} = 0. \tag{1}$$

Since $\vec{B}$ only has a z component, we also have

$$(\vec{B}\cdot\vec{\nabla})\,\vec{B} = 0. \tag{2}$$

We further assume that $\vec{E}$ is zero, that the plasma is neutral, and that the particle number and energy densities are independent of z. We now sum up the currents. First the magnetization current is obtained from Eq. (6), Section V.

$$\begin{aligned} \vec{J}_m &= -c\,\vec{\nabla}\times\left(N\overline{W}_\perp \frac{\vec{B}}{B^2}\right) = c\,\vec{n}\times\vec{\nabla}\left(\frac{N\overline{W}_\perp}{|B|}\right) \\ &= \frac{c}{|B|}\,\vec{n}\times\vec{\nabla}(N\overline{W}_\perp) - c\frac{N\overline{W}_\perp}{B^2}\,\vec{n}\times\vec{\nabla}|B|. \end{aligned} \tag{3}$$

Secondly we have the current due to a gradient in $\vec{B}$. This we obtain from Eq. (9), Section V.

$$\vec{J}_G = c\,\frac{N\overline{W}_\perp}{B^2}\,\vec{n}\times\vec{\nabla}(\vec{B}\cdot\vec{n}). \tag{4}$$

Adding these two currents gives

$$\vec{J}_m + \vec{J}_G = c\,\frac{\vec{n}}{|B|}\times\vec{\nabla}(N\overline{W}_\perp) = c\,\frac{\vec{B}}{B^2}\times\vec{\nabla}(N\overline{W}_\perp). \tag{5}$$

Finally, we must use Maxwell's equations to get a self-consistent solution. From

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\,\vec{J} \tag{6}$$

and Eq. (5) we have

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{n}|B|) = -\vec{n}\times\vec{\nabla}|B| = 4\pi\cancel{\left(\frac{c}{c}\right)}\frac{\vec{n}}{|B|}\times\vec{\nabla}(\overline{NW_{\perp}}) \tag{7}$$

or

$$\vec{n}\times\left\{\vec{\nabla}|B| + \frac{4\pi}{|B|}\vec{\nabla}(\overline{NW_{\perp}})\right\} = 0. \tag{8}$$

Since the term in the brackets is perpendicular to $\vec{n}$,

$$\vec{\nabla}\left(\frac{B^2}{8\pi} + \overline{NW_{\perp}}\right) = 0 \tag{9}$$

or

$$\left[\frac{B^2}{8\pi} + (\overline{NW_{\perp}})\right] = \text{CONSTANT} \tag{10}$$

$B^2/8\pi$ is the pressure associated with the magnetic field lines, while $\overline{NW_{\perp}}$ is the pressure of the plasma perpendicular to B . Eq. (10) says that the sum of these pressures is constant, or that we have pressure balance.

B. Plasma in a Gravitational Field which is Perpendicular to a Magnetic Field Whose Lines are Straight

Again we take the direction of the magnetic field to be in the z direction, and we take the gravitational field to be in the negative y direction. We assume all quantities are independent of z .

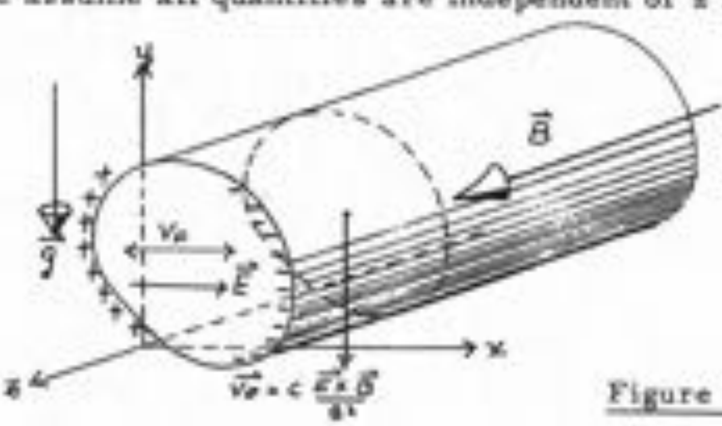


Figure 36

We shall further assume that $\overline{NW_\perp}$, the perpendicular pressure, is negligibly small compared to $B^2/8\pi$. From Eq. (10) we have

$$\frac{B^2}{8\pi} + \overline{NW_\perp} = \text{constant}$$

Hence

$$\frac{B^2}{8\pi}\left[1 + \frac{\overline{NW_\perp}}{B^2/8\pi}\right] = \frac{B^2}{8\pi}\left(1+\beta\right) = \text{constant} \tag{11}$$

or

$$B \cong \text{constant} \tag{12}$$

under our assumptions. The quantity β is the ratio of the perpendicular gas pressure to the magnetic field pressure. Under this approximation we can neglect the variations in B due to the gas pressure. We will therefore take B to be constant.

We now ask what will happen if we suddenly release such a plasma. Here an E field will develop and we must include its effects.

First, the particles have a drift due to the gravitational field which is given by Eq. (5), Section V.

$$\vec{V}_F = \frac{c\,m}{q}\,\frac{\vec{g}\times\vec{B}}{B^2}. \tag{13}$$

Ions and electrons move in opposite directions, so that a current is set up, given by Eq. (10), Section V; however, the resulting charge separation tends to oppose this current. The resulting $\vec{E}$ field enters in two ways: first, because $\vec{E}$ is time-dependent it gives rise to a polarization current; and second, because of $\vec{E}$ there is an $\vec{E}\times\vec{B}$ drift of the whole plasma in the $-y$ direction. To compute the time-dependence of E we must use Maxwell's equations (making use of the result which we already found, that the plasma behaves like a dielectric). From Eqs. (4) and (5), Section IV, we have

$$\nabla \cdot \left[\frac{\dot{\vec{E}}}{4\pi} + \dot{\vec{j}} \right] = 0. \tag{14}$$

If we assume that $\vec{\nabla} \times \vec{E}$ is zero ($\vec{B}$ negligible), and also that $\vec{\nabla} \times \vec{j} = 0$ (this is reasonable because there is nothing to make currents circulate in the problem), then

$$\dot{\vec{E}} = -4\pi \vec{j}. \tag{15}$$

Now $\vec{j}$ has two parts, one coming from $\vec{g}$ and the other from $\dot{\vec{E}}$. These are obtained from Eqs. (7) and (10), Section V, and are given by

$$\vec{j}_{\dot{E}} = \frac{c^2}{B^2} Nm\, \dot{\vec{E}} \tag{16}$$

and

$$\vec{j}_g = \frac{c\,Nm}{B^2}\, \vec{g} \times \vec{B} = -\hat{e}_x \frac{c\,Nm\,g}{|B|} \tag{17}$$

where use has been made of the geometry in writing down $\vec{j}_g$.

Thus from Eq. (15) we have

$$\dot{\vec{E}} \left(1 + \frac{4\pi c^2 Nm}{B^2} \right) - \hat{e}_x \frac{4\pi c\,Nm}{|B|} g = 0 \tag{18}$$

or

$$\dot{\vec{E}} = \frac{\hat{e}_x \frac{4\pi c\,Nm\,g}{|B|}}{1 + \frac{4\pi c^2 Nm}{B^2}}. \tag{19}$$

Now the E_x which results from Eq. (19) gives rise to an $\vec{E} \times \vec{B}$ drift of the whole plasma in the $-y$ direction. We have

$$V_y = -\frac{cE_x}{B}. \tag{20}$$

Taking the derivative of Eq. (20) and substituting $\dot{E}$ from Eq. (19) gives

$$\dot{V}_y = -\frac{4\pi c^2 Nmg}{B^2 + 4\pi c^2 Nm}. \tag{21}$$

If Nmc^2 is much larger than $B^2/4\pi$, then Eq. (21) reduces to

$$\dot{V}_y = -g. \tag{22}$$

Thus the plasma falls freely in the gravitational field just as if $\vec{B}$ were not there. The modification by the factor $\frac{1}{1 + \frac{B^2}{4\pi Nm c^2}}$ is due to the fact that not all the gravitational energy goes into kinetic energy of the plasma, but some of it must go into the $\vec{E}$ field needed to give the $\vec{E} \times \vec{B}$ drift. It is readily verified that the ratio of the electric field energy per unit volume to particle kinetic energy is $\frac{B^2}{4\pi Nm c^2}$. This results in a slight change in the effective mass of the plasma. It is not surprising that the magnetic field cannot hinder the falling of the plasma. Because of the boundaries, no sustained current can flow in the plasma, and since the only force that the magnetic field can exert on the plasma is a $j \times B$ force, there can be no magnetic force except the small one which results from the polarization current and which changes the effective mass of the plasma. (The inductance and inertia effects on the polarization currents have been neglected, and these will also give rise to slight changes in the effective mass.) Finally, it should be stated that the approximations of $\nabla \times \vec{E}$ and $\nabla \times \vec{j}$ break down near the surface of the plasma. The particles at the surface do not feel the full $\vec{E}$ field and some are scraped off. This is a difficult problem and has not been solved yet.

Problem: Show that the ratio of the electric field energy per unit volume to the particle kinetic energy (associated with the drift) per unit volume is $B^2/4\pi Nmc^2$ for the falling plasma.

C. Curved Field Lines

We now consider the case in which the field lines are circles centered on the z axis. This case applies either to the plasma in a torus or to the pinch effect. We shall use the cylindrical coordinates appropriate to the problem. Again from $\vec{\nabla}\cdot\vec{B} = 0$ we have that B does not vary in the θ direction. We also take all other quantities to be independent of z .

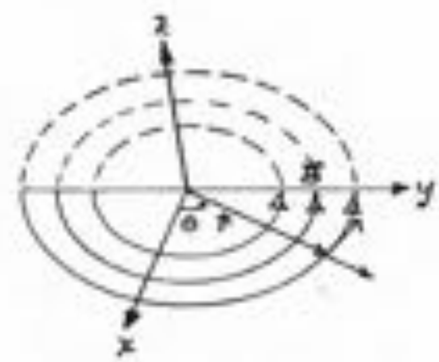


Figure 37

For the magnetization currents we have

$$\vec{j}_m = c\,\nabla\times\vec{M} \qquad (23)$$

or

$$\vec{j}_m = -c\,\vec{\nabla}\times\frac{P_\perp\vec{B}}{B^2} = -c\,\vec{\nabla}\times\frac{\hat{e}_\theta P_\perp}{B}. \qquad (24)$$

$$\vec{j}_m = -c\left[-\hat{e}_r \frac{\partial}{\partial z}\left(\frac{P_\perp}{B}\right) + \hat{e}_z \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rP_\perp}{B}\right)\right] \tag{25}$$

From the gradients we have, from Eq. (9), Section V,

$$\vec{j}_\theta = \frac{cP_\perp}{B^2}\vec{n}\times\vec{\nabla}(\vec{B}\cdot\vec{n}) \tag{26}$$

or

$$\vec{j}_\theta = \frac{cP_\perp}{B^2}\hat{e}_\theta\times\left[\hat{e}_r\frac{\partial B}{\partial r} + \hat{e}_z\frac{\partial B}{\partial z}\right] \tag{27}$$

or

$$\vec{j}_\theta = \frac{cP_\perp}{B^2}\left[-\hat{e}_z\frac{\partial B}{\partial r} + \hat{e}_r\frac{\partial B}{\partial z}\right]. \tag{28}$$

Finally, from Eq. (8), Section V, we get the curvature currents.

$$\vec{J}_R = 2c\frac{NW_\parallel}{B^2}\vec{B}\times(\vec{n}\cdot\vec{\nabla})\vec{n}$$

or

$$= \frac{cP_\parallel}{B^2}B\,\hat{e}_\theta\times(\hat{e}_\theta\cdot\vec{\nabla})\hat{e}_\theta$$

or

$$= \frac{cP_\parallel}{Br}\hat{e}_z \tag{29}$$

where $P_\parallel$ is the gas kinetic pressure parallel to the B lines.

Now from $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$ we get the following equations for B

$$\frac{1}{r}\frac{\partial}{\partial r}(rB) = \frac{4\pi}{c}j_z \tag{30}$$

and

$$\frac{\partial B}{\partial z} = -\frac{4\pi}{c}j_r. \tag{31}$$

Adding up the currents given by Eqs. (25), (28), and (29), and substituting in Eqs. (30) and (31) gives

$$\frac{1}{r}\frac{\partial}{\partial r}(rB) = -4\pi\,\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rP_\perp}{B}\right) - \frac{4\pi P_\perp}{B^2}\frac{\partial B}{\partial r} + \frac{4\pi P_\parallel}{rB} \tag{32}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}(rB) = -\frac{4\pi}{Br}\frac{\partial}{\partial r}(rP_\perp) + \frac{4\pi P_\parallel}{rB} \tag{33}$$

and

$$\frac{\partial B}{\partial z} = -4\pi\left[\frac{\partial}{\partial z}\left(\frac{P_\perp}{B}\right) + \frac{P_\perp}{B^2}\frac{\partial B}{\partial z}\right]. \tag{34}$$

Eq. (34) can be immediately reduced to

$$\frac{\partial B}{\partial z} = -\frac{4\pi}{B}\frac{\partial P_\perp}{\partial z} \tag{35}$$

or

$$\frac{B^2}{8\pi} + P_\perp = f(r) \tag{36}$$

where $f(r)$ is an arbitrary function of r.

This equation says that we have pressure balance in the z direction for each value of r. Again, $B^2/8\pi$ is the magnetic pressure and $P_\perp$ is the particle pressure perpendicular to the lines of force. Since the parallel pressure exerts no force in the z direction, the perpendicular pressure is the only force which must be balanced by the magnetic field in this direction.

Turning now to Eq. (33), we may write this equation in the form

$$\frac{B}{4\pi}\frac{\partial}{\partial r}(rB) + \frac{\partial}{\partial r}(rP_\perp) - P_\parallel = 0 \tag{37}$$

or

$$\frac{\partial}{\partial r}\left[r\left[\frac{B^2}{8\pi} + P_\perp\right]\right] + \frac{B^2}{8\pi} - P_\parallel = 0. \tag{38}$$

This equation is again the equation for the balance of forces on a little element.

$B^2/8\pi$ is the magnetic pressure and $P_\perp$ is particle pressure perpendicular to the lines. The term $B^2/8\pi$ is a term we should get if the lines of force were under a tension of magnitude $B^2/8\pi$, while the $P_\parallel$ term is equivalent to there being a compressional stress of magnitude $P_\parallel$ in the column. See Fig. 38 for the details of the forces.

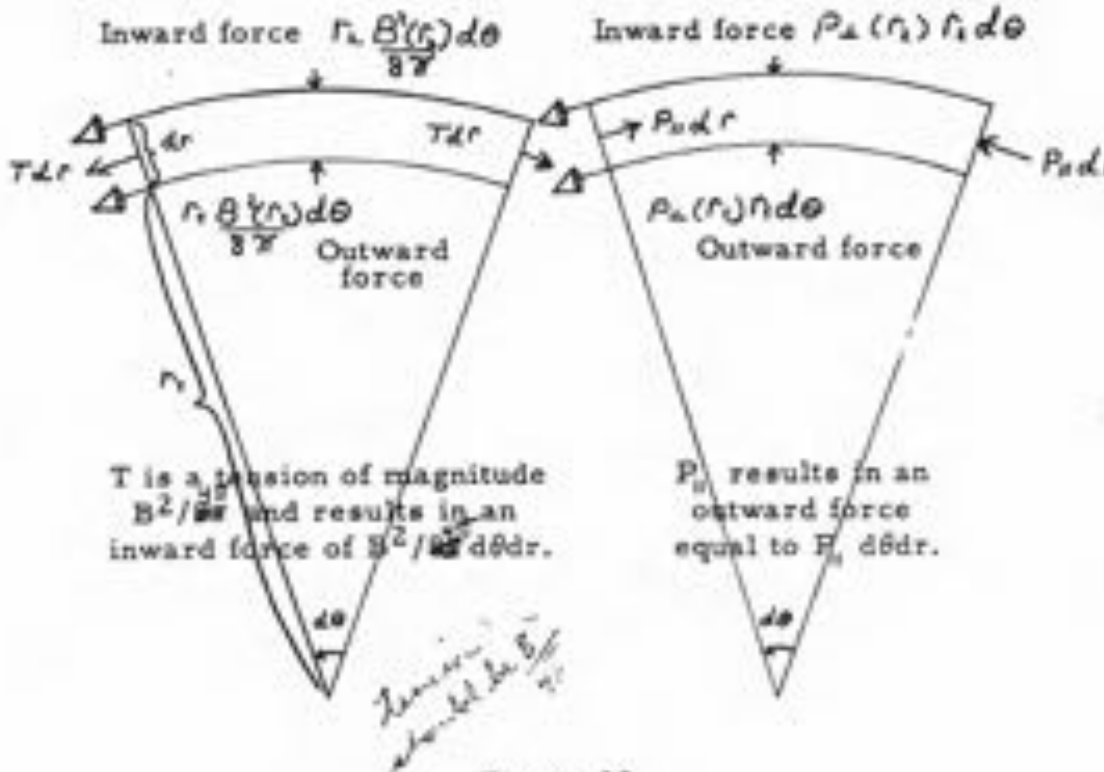


Figure 38

Later we will show that the idea of the lines exerting a pressure perpendicular to themselves, of magnitude $B^2/8\pi$, and being under tension of magnitude $B^2/4\pi$, can be formalized.

Substituting for $B^2/8\pi$ from Eq. (36) in Eq. (38) gives

$$\frac{\partial}{\partial r}\left[r\,F(r)\right] + F(r) = P_\perp + P_\parallel \tag{39}$$

Now if the plasma is to be confined in the z direction, then $P_\perp$ and $P_\parallel$ must go to zero for large z and hence they must be functions of z. However, the left-hand side of Eq. (39) is independent of z; it is only a function of r. This means we cannot have a stationary confined plasma with this type of field. The reason for this is easy to find. If we look at the particle drifts we see that electrons and ions drift in opposite directions because of both their parallel and perpendicular motions. This will lead to a charge separation and hence to an $\vec{E} \times \vec{B}/B^2$ drift. We may examine this in detail for the case in which the plasma pressure is negligible compared to the magnetic pressure. In this case we can find B from Eq. (33) by putting $P_\parallel = P_\perp = 0$. One finds

$$r B = \text{constant} \tag{40}$$

or

$$B = B_0 r_0 / r \tag{41}$$

or

$$\vec{B} = \hat{e}_\theta B_0 r_0 / r. \tag{42}$$

The field falls off as $1/r$; this is the field produced by an infinite straight wire carrying current $j = c B_0 r_0 / 2$, or the field which exists in a uniform circular toroidal solenoid. From Eqs. (25), (28), (29) and (42),

$$\vec{j}_m + \vec{j}_g = \hat{e}_r \frac{c}{B_0 r_0} \frac{\partial}{\partial z}(r P_\perp) - \hat{e}_z \frac{c}{B_0 r_0} \frac{\partial}{\partial r}(r P_\perp) \tag{43}$$

and

$$\vec{j}_\kappa = \hat{e}_z \frac{c P_\parallel}{B_0 r_0}. \tag{44}$$

In addition to these currents, there will now also be a current due to $\vec{E}$.

$$\vec{J}_e = \frac{c^2}{B^2}\rho\dot{\vec{E}} \quad \text{where } \rho = \sum_i N_i m_i. \tag{45}$$

From Maxwell's equations — see Eq. (14) — we have

$$\vec{\nabla}\cdot\left(\dot{\vec{E}} + 4\pi\vec{J}\right) = 0. \tag{46}$$

We get from Eqs. (46), (45), (44), and (43)

$$\vec{\nabla}\cdot\dot{\vec{E}}\left[1 + \frac{4\pi\rho c^2}{B^2}\right] + 4\pi\vec{\nabla}\cdot\left[\vec{J}_n + \vec{J}_G + \vec{J}_R\right] = 0 \tag{47}$$

or

$$\vec{\nabla}\cdot\dot{\vec{E}}\left[1 + \frac{4\pi\rho c^2}{B^2}\right] + \frac{4\pi c}{B_0 r_0}\frac{\partial}{\partial z}\left[P_{\parallel} + P_{\perp}\right] = 0. \tag{48}$$

If we assume that $\vec{E}$ has only a z component (strictly speaking, one should solve the above equation along with $\nabla \times E = 0$, assuming B is negligible; we could find a configuration of pressures so that our assumption was true), this equation gives

$$\dot{E}_z\left[1 + \frac{4\pi\rho c^2}{B^2}\right] + \frac{4\pi c}{B_0 r_0}\left[P_{\parallel} + P_{\perp}\right] = 0. \tag{49}$$

[Again strictly speaking, the right-hand side of Eq. (49) could be an arbitrary function of t, which must be determined from boundary conditions. This correction comes about because of fringing fields outside the plasma, which will change the effective mass. We neglect them here.]

We obtain from Eq. (49)

$$\dot{E}_z = -\frac{4\pi c}{B_0 r_0}\left[\frac{P_{\parallel} + P_{\perp}}{\left(1 + \frac{4\pi\rho c^2}{B^2}\right)}\right]. \tag{50}$$

Problem: Find the correct E if $P_\perp$ and $P_\parallel$ are uniform inside a small torus of minor radius r. In finding E, assume you can treat the torus as a cylinder. Also, find the correct rate of drop in a gravitational field of a cylindrical plasma of radius r_0 and with uniform density, when similar boundary corrections are included in that calculation.

$$\dot{\vec{E}} = -\hat{e}_r \frac{4\pi c\,(P_\parallel + P_\perp)}{B_0 r_0} \frac{1}{(1+4\pi\rho c^2 r^2/B_0^2 r_0^2)}. \tag{51}$$

This E gives rise to a drift velocity

$$\vec{V}_E = \frac{c\vec{E}\times\vec{B}}{B^2}. \tag{52}$$

Taking the time derivative of this equation gives

$$\dot{\vec{V}}_E = c\frac{\dot{\vec{E}}\times\vec{B}}{B^2} = \hat{e}_r \frac{4\pi c^2 r\,(P_\perp + P_\parallel)}{B_0^2 r_0^2\left(1+\frac{4\pi\rho c^2 r^2}{B_0^2 r_0^2}\right)}. \tag{53}$$

---

* Here we have neglected the change in B due to the motion. This is equivalent to setting $\vec{v}_e = 0$ or neglecting this term compared to B. This type of correction will be discussed in the next section; see Eq. (79).

If $\rho c^2 \gg B_0^2$,

$$\dot{\vec{V}}_E = \hat{e}_r \frac{P_\parallel + P_\perp}{\rho r}. \tag{54}$$

The column accelerates in the r direction: it is not confined by the magnetic field. The reason is, of course, similar to that involved in the plasma column falling in a gravitational field.

D. The Cylindrical Pinch

If we allow the plasma to be uniform in the z direction (no z dependence), then Eqs. (38) or (39) allow a solution where $P_\perp$ and $P_\parallel$ are functions of r alone. This solution approximates the so-called linear pinch, where we have a long column of plasma through which a large current passes. The current produces a magnetic field which confines the gas.

Problem: (a) If one adds a uniform B_z to the toroidal B_θ field, can one obtain a confined plasma in a torus? not [illegible]

(b) Verify that for the vacuum magnetic field around an infinite straight wire carrying a current that the tension of the magnetic lines balances the magnetic pressure.

IX. The General Two-Dimensional Problem (Time-Dependent)

Let us again take the B lines to be parallel, straight, and pointing in the z direction. We assume that there is no variation of any of the quantities in the z direction. We further assume that $\vec{v}_E$ is the largest velocity in the problem and we identify it with the mass velocity of the plasma. Finally, we assume that the plasma is approximately electrically neutral and denote the common number density by N. From conservation of particles we have the continuity equation

$$\frac{\partial N}{\partial t} + \vec{\nabla} \cdot (N \vec{V}_E) = 0 \tag{55}$$

or

$$\frac{1}{N} \frac{dN}{dt} = -\vec{\nabla} \cdot \vec{V}_E . \tag{56}$$

where dN/dt is the time rate of change of density moving with the particles

Second, we have Maxwell's induction equation

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}. \tag{57}$$

By making use of the relation of $\vec{v}_E$ to $\vec{E}$

$$\vec{E} = -\frac{\vec{V}_E \times \vec{B}}{c} \tag{58}$$

we can write Eq. (57) in the form

$$\vec{\nabla} \times (\vec{V}_E \times \vec{B}) = \frac{\partial \vec{B}}{\partial t}. \tag{59}$$

Recalling

$$\vec{\nabla} \times (\vec{u} \times \vec{v}) = (\vec{v}\cdot\vec{\nabla})\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{v} + \vec{u}\,(\vec{\nabla}\cdot\vec{v}) - \vec{v}\,(\vec{\nabla}\cdot\vec{u}) \tag{60}$$

we get

$$\cancelto{0}{(\vec{B}\cdot\vec{\nabla})\vec{V}_E} - (\vec{V}_E\cdot\vec{\nabla})\vec{B} + \cancelto{0}{\vec{V}_E(\vec{\nabla}\cdot\vec{B})} - \vec{B}(\vec{\nabla}\cdot\vec{V}_E) = \frac{\partial \vec{B}}{\partial t} \tag{61}$$

The first term is zero since $\vec{B}$ has only a z component and all quantities vary only in x and y direction, so that ∇_z is zero. The third term is zero from the Maxwell equation.

Now we may write $\vec{B}$ as $\hat{n}B$, where $\hat{n}$ is a unit vector in the z direction. Then Eq. (61) becomes

$$-\vec{V}_E\cdot\vec{\nabla}(B\hat{n}) - \hat{n}\,B(\vec{\nabla}\cdot\vec{V}_E) = \hat{n}\frac{\partial B}{\partial t} \tag{62}$$

or

$$\hat{n}\left[\frac{\partial B}{\partial t} + (\vec{V}_E\cdot\vec{\nabla})B + B\,\vec{\nabla}\cdot\vec{V}_E\right] = 0 \tag{63}$$

or

$$\frac{\partial B}{\partial t} + (\vec{\nabla}\cdot\vec{V}_E)B = 0 \tag{64}$$

thus

$$\frac{1}{B}\frac{dB}{dt} = \vec{\nabla}\cdot\vec{V}_E. \tag{65}$$

Again the derivative follows the motion. From Eqs. (56) and (65)

$$\frac{1}{B}\frac{dB}{dt} = \frac{1}{N}\frac{dN}{dt}. \tag{66}$$

Therefore,

$$-\frac{N}{B^2}\frac{dB}{dt} + \frac{1}{B}\frac{dN}{dt} = 0 \tag{67}$$

or

$$\frac{d}{dt}\left(\frac{N}{B}\right)_{\substack{\text{following}\\ \text{motion}}} = 0. \tag{68}$$

Eq. (68) is simply a consequence of the field being carried with the particles. If we draw a curve (in reality a cylinder) in the plasma and move the curve with the plasma, then both the number of particles inside the curve and the flux through it do not change with time. We already have shown this in the section on the motion of magnetic lines of force, and hence we could have written Eq. (68) down immediately. The above calculation gives another demonstration of this fact for the special case considered here.

We now compute the currents. The magnetization current is given by

$$\vec{J}_M = -c\,\vec{\nabla}\times\left(\frac{P_\perp \hat{n}}{B}\right) = c\,\hat{n}\times\vec{\nabla}\left(\frac{P_\perp}{B}\right) \tag{69}$$

while the gradient current is

$$j_G = \frac{cP_\perp}{B^2}\,\hat{n}\times\vec{\nabla}B. \tag{70}$$

If we also allow an external force per unit volume of the form

$$N\vec{F} = \vec{F}' \tag{71}$$

then this gives rise to a current

$$\vec{j}_P = c\,\frac{\vec{E}\times\hat{n}}{B}. \tag{72}$$

Before summing the currents, we need to modify the polarization current. We do not use Eq. (7), Section V, because that was derived on the assumption that $\vec{B}$ was constant, while here we want to allow $\vec{B}$ to be time-dependent. We have, from the equation of motion,

$$m\,\frac{d\vec{v}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right). \tag{73}$$

Writing

$$\vec{v} = \vec{v}_E + \vec{v}_1 \tag{74}$$

where

$$\vec{v}_E = c\,\frac{\vec{E}\times\vec{B}}{B^2} \qquad \text{following the particle,} \tag{75}$$

we have

$$m\left[\frac{d\vec{v}_E}{dt} + \frac{d\vec{v}_1}{dt}\right] = \frac{q}{c}(\vec{v}_1\times\vec{B}). \tag{76}$$

Again, let $\vec{v}_1 = \vec{v}_2 + \vec{v}_p$, where $\vec{v}_p$ is chosen to satisfy the equation

$$m\,\frac{d\vec{v}_E}{dt} = mc\,\frac{d}{dt}\left.\frac{\vec{E}\times\vec{B}}{B^2}\right|_{\substack{\text{following}\\ \text{the particle}}} = \frac{q}{c}\,\vec{v}_p\times\vec{B} \tag{77}$$

or (keeping in mind that d/dt means following the particle)

$$\begin{aligned}\vec{v}_p &= \frac{mc^2}{q}\,\frac{\vec{B}}{B^2}\times\frac{d}{dt}\left(\frac{\vec{E}\times\vec{B}}{B^2}\right)\\ &= \frac{mc^2}{q}\,\frac{\hat{n}}{B}\times\frac{d}{dt}\left(\frac{\vec{E}\times\hat{n}}{B}\right)\end{aligned} \tag{78}$$

or, since $\vec{E}\perp B$,

$$\vec{v}_p = \frac{mc^2}{q}\,\frac{1}{B}\,\frac{d}{dt}\left(\frac{\vec{E}}{B}\right). \tag{79}$$

Eq. (76) then becomes

$$m\frac{d\vec{v}_\perp}{dt} + m\frac{d\vec{v}_p}{dt} = \frac{q}{c}\,\vec{v}_\perp \times \vec{B}. \tag{80}$$

As in our previous arguments for a time-varying E, the term $d\vec{v}_p/dt$ can be neglected if $\vec{E}$ and $\vec{B}$ vary slowly on the time scale associated with the cyclotron frequency. Then Eq. (80) just describes the Larmor motion about the lines of force. The polarization current is obtained from Eq. (79) by adding subscript i to m and q, multiplying by $q_i N_i$ and summing over all species. We thus find for $\vec{j}_p$

$$\vec{j}_p = \frac{\rho c^2}{B}\frac{d}{dt}\left(\frac{\vec{E}}{B}\right) \qquad \text{where } \rho = \sum_i N_i m_i \tag{81}$$

Summing up the currents $\vec{j}_m$, $\vec{j}_g$, $\vec{j}_c$, and $\vec{j}_p$, and substituting in the Curl-B Maxwell equation gives

$$\nabla \times \hat{n}B = -\hat{n}\times\nabla B + B\,\nabla\times\hat{n} =$$

$$\frac{4\pi}{c}\left[\frac{c\,\hat{n}}{B}\times\nabla P_\perp + c\,\frac{\vec{F}\times\hat{n}}{B} + \frac{\rho c^2}{B}\frac{d}{dt}\left(\frac{\vec{E}}{B}\right)\right]. \tag{82}$$

Dotting both sides with $\hat{n}$ gives zero, as we should expect, since there is no motion or force parallel to $\vec{B}$. Crossing with $\hat{n}$ on the left gives

$$\nabla B = (\hat{n}\times\nabla B)\times\hat{n} =$$

$$= -\frac{4\pi}{c}\left[\frac{c}{B}\nabla P_\perp - c\,\frac{\vec{F}}{B} - \frac{\rho c^2}{B}\frac{d}{dt}\left(\frac{\hat{n}\times\vec{E}}{B}\right)\right] \tag{83}$$

or

$$-\nabla\left[\frac{B^2}{8\pi} + P_\perp\right] + \vec{F} = \rho\,\frac{d\vec{V}_E}{dt}. \tag{84}$$

Dropping the E from the v gives

$$\rho\frac{d\vec{V}}{dt} = \vec{F} - \nabla\left[P_\perp + \frac{B^2}{8\pi}\right]. \tag{85}$$

Here again we see that the magnetic field acts as if it exerts a pressure of magnitude $B^2/8\pi$.

Now we see that we have three equations — (85), (64), and (55) — for the four unknowns, v, N, B, and $P_\perp$. We need one more equation: an equation to relate $P_\perp$ to N. We can obtain this from the adiabatic invariant μ.

We have

$$P_\perp = N W_\perp = N^2 \left(\frac{W_\perp}{B}\right)\left(\frac{B}{N}\right) \tag{86}$$

Now both B/N and $W_\perp/B$ are constant following the motion, so this tells us that $P_\perp$ is proportional to N^2 or

$$P_\perp = P_{\perp 0} \left(\frac{N}{N_0}\right)^2 \tag{87}$$

or

$$\frac{P_\perp}{N^2} \propto \frac{P_\perp}{\rho^2} = \text{CONSTANT} \tag{88}$$

and thus

$$\frac{d}{dt}\left(\frac{P_\perp}{\rho^2}\right) = 0 \tag{89}$$

all following the motion.

In the section on the longitudinal invariant it was shown that under a slow compression the energy in $v_\parallel$ was related to the pressure in the same way as would be the case for a one-dimensional gas (one degree of freedom). We may show that the above law, Eq. (89), is the same law we would obtain for a two-dimensional gas (two degrees of freedom). The relation between the pressure and the density for an ideal gas undergoing an adiabatic change in volume is

$$\frac{p}{\rho^{\gamma}} = \frac{p_0}{\rho_0{}^{\gamma}}. \tag{90}$$

Again, γ is given by

$$\gamma = \frac{n+2}{n} \tag{91}$$

where n is the number of degrees of freedom involved in the process. We see that if we set γ equal to 2 in Eq.(90) we get Eq.(89). Thus the gas behaves like an ideal two-dimensional gas, as we might expect for this two-dimensional problem.

Here we might also note that the magnetic field also behaves like a gas with a γ of 2. We note that following the motion N/B or B/N is constant. Hence

$$\frac{B^2}{N^2} \propto \frac{B^2}{8\pi\rho^2} = \text{CONSTANT} \tag{92}$$

and thus the magnetic field also behaves like an ideal gas with $\gamma = 2$.

Summarizing, we have derived two-dimensional hydromagnetics from orbit theory, and the following set of equations describes the motion of the system

$$\rho \frac{d\vec{v}}{dt} = F - \nabla\left[p_\perp + \frac{B^2}{8\pi}\right], \tag{93}$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\rho\vec{v}, \tag{94}$$

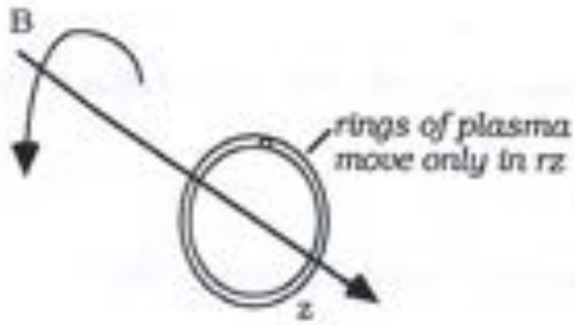
$$\frac{\partial B}{\partial t} = -\vec{\nabla}\cdot B\vec{v}, \tag{95}$$

and

$$\frac{d}{dt}\left(\frac{p_\perp}{\rho^2}\right) = 0. \tag{96}$$

The General Two Dimensional Cylindrical Time Dependent Problem

Let us consider a situation in which the **B** lines are circles centered on the z axis. We consider only motion in the r,z directions (there is no variation in the Q direction) so that the circular symmetry is preserved. We use cylindrical coordinates with only r,z variations. We are still working in the large q/m limit so that the only important velocity is the $(\mathbf{E} \times \mathbf{B})c/B^2$ velocity; the other drifts, however, contribute to the current. The situation is as shown in the Figure.



The equations for **B**, **V**, and **E**:

$$\mathbf{B} = \mathbf{e}_\Theta B(r,z) \qquad \text{IS.1}$$

$$\mathbf{V} = (\mathbf{E} \times \mathbf{B})c/B^2, \qquad \text{IS.2}$$

$$\mathbf{E} = -(\mathbf{V} \times \mathbf{B})/c. \qquad \text{IS.3}$$

$\mathbf{E}$ has only r,z components because of the assumed perfect conductivity of the plasma. We take the plasma number density to be N (equal numbers of ions and electrons with the ions having charge e). The density satisfies the continuity equation

$$\partial N/\partial t + \nabla \bullet N\mathbf{V} = 0 \qquad \text{IS.4}$$

or

$$[(dN(r,z)/dt)_{FTM}]/N(r,z) + \nabla \bullet \mathbf{V} = 0. \qquad \text{IS.5}$$

The subscript FTM means following the motion of an element of the plasma.

We also have Maxwells equations

$$\nabla \times \mathbf{E} = -(\partial \mathbf{B}/\partial t)/c \qquad \text{IS.6}$$

$$\nabla \times \mathbf{B} = (\partial \mathbf{E}/\partial t)/c + 4\pi\mathbf{j}/c \; . \qquad \text{IS.7}$$

Again we use the vector identity

$$\nabla \times (\mathbf{U} \times \mathbf{V}) = \mathbf{V}\bullet\nabla\mathbf{U} - \mathbf{U}\bullet\nabla\mathbf{V} + \mathbf{U}(\nabla\bullet\mathbf{V}) - \mathbf{V}(\nabla\bullet\mathbf{U}) \qquad \text{IS.8}$$

and replacing $\mathbf{U}$ by $\mathbf{B}$ and $\mathbf{V}$ by $\mathbf{V}$, we get

$\nabla \times (\mathbf{V} \times \mathbf{B}) = -\nabla \times (\mathbf{B} \times \mathbf{V}) = \mathbf{B}\bullet\nabla\mathbf{V} - \mathbf{V}\bullet\nabla\mathbf{B} + \mathbf{V}(\nabla\bullet\mathbf{B}) - \mathbf{B}(\nabla\bullet\mathbf{V})$. IS.9

The $\mathbf{V}(\nabla\bullet\mathbf{B})$ is zero because $\nabla\bullet\mathbf{B}$ is zero. If we write $\mathbf{B}\bullet\nabla\mathbf{V}$ in component form we get

$$\mathbf{e}_\Theta B(r,z)\bullet\{[\mathbf{e}_r(\partial/\partial r) + (\mathbf{e}_\Theta/r)(\partial/\partial\Theta) + \mathbf{e}_z(\partial/\partial z)][\mathbf{e}_r V_r + \mathbf{e}_z V_z]\} =$$
$$(B(r,z)/r)(\partial\mathbf{e}_r V_r/\partial\Theta) = [\mathbf{e}_\Theta B(r,z)][V_r/r] \qquad \text{IS.10}$$

since dotting with $\mathbf{e}_\Theta$ keeps only the Θ derivitive and $\mathbf{e}_r$ is the only quantity with a Θ dependence. Now substituting E from IS.3 in IS.6 and making use of IS.9 and IS.10 gives

$$(\partial\mathbf{e}_\Theta B(r,z)/\partial t) = [\mathbf{e}_\Theta B(r,z)][V_r/r] - \mathbf{V}\bullet\nabla[\mathbf{e}_\Theta B(r,z)] - \mathbf{e}_\Theta B(r,z)\nabla\bullet\mathbf{V}. \qquad \text{IS.11}$$

Now

$$\mathbf{V}\bullet\nabla[\mathbf{e}_\Theta B(r,z)] = \mathbf{e}_\Theta(\mathbf{V}\bullet\nabla B) + B(r,z)\mathbf{V}\bullet\nabla\,\mathbf{e}_\Theta \qquad \text{IS.12}$$

$\nabla\,\mathbf{e}_\Theta = \dfrac{-\mathbf{e}_\Theta\mathbf{e}_r}{r}$, $\mathbf{V}\bullet\nabla\,\mathbf{e}_\Theta = 0$, since $\mathbf{V}$ has no Θ component. IS.13

Therefore, IS.11 gives

$$(\partial\mathbf{e}_\Theta B(r,z)/\partial t) = -\,\mathbf{e}_\Theta B(r,z)\nabla\bullet\mathbf{V} - \mathbf{e}_\Theta\mathbf{V}\bullet\nabla[B(r,z)] + [\mathbf{e}_\Theta B(r,z)][V_r/r] \qquad \text{IS.14}$$

or

$$B(r,z)][V_r/r] - \mathbf{V}\bullet\nabla[B(r,z)] - B(r,z)\nabla\bullet\mathbf{V} = \partial B(r,z)/\partial t \qquad \text{IS.15}$$

$$\partial B(r,z)/\partial t + \nabla\bullet B(r,z)\mathbf{V} - B(r,z)][V_r/r] = 0 \qquad \text{IS.16}$$

This equation can be rewritten as

$$[(dB(r,z)/dt)_{PTM}]/B(r,z) + \nabla\bullet\mathbf{V} - V_r/r] = 0. \qquad \text{IS.17}$$

Equation IS.17 is similar to but not exactly the same as the continuity equation for N; in fact using the continuity equation, IS.5, we can write IS.17 as

$$[(dB(r,z)/dt)_{PTM}]/B(r,z) - [(dN(r,z)/dt)_{PTM}]/N(r,z) - V_r/r = 0.$$

IS.18

Now V_r/r is $(dr/dt)_{PTM}]/r$, so that IS.17 becomes

$$(dB(r,z)/dt)_{PTM}]/B(r,z)-[(dN(r,z)/dt)_{PTM}]/N(r,z)-(dr/dt)_{PTM}]/r=0$$

IS.19

Equation IS.19 can be integrated directly giving

$$\ln B(r,z) - \ln N(r,z) - \ln r = \text{constant} \qquad \text{IS.20}$$

or

$$B(r,z)/[N(r,z)r] = B(r_0,z_0)/[N(r_0,z_0)r_0] \qquad \text{IS.21}$$

Here r_0,z_0 is the r,z position that a particle starts out at $t = 0$. There is a simple physical interpretation of this result. Consider a ring of plasma of dimensions dr_0,dz_0 starting at r_0,z_0. Let it move to position r,z where its dimensions are drdz. The magnetic flux through drdz must be the same as that through dr_0dz_0; or

$$B_0dr_0dz_0 = Bdrdz = \Delta\Phi_0. \qquad \text{IS.22}$$

On the other hand, the total number of particles in the ring is conserved; this gives

$$N_0r_0dr_0dz_0 = Nrdrdz. \qquad \text{IS.23}$$

Dividing IS.22 by IS.23 gives IS.21; the magnetic field changes with the cross section area of the ring so as to keep the flux constant while the density also changes because of changes in the circumference of the plasma ring.

Problem: Consider a two-dimensional plasma, for which $\vec{F}$ is obtainable from a potential $\vec{F} = -\vec{\nabla}\phi$ and which is confined by rigid walls that are perfectly conducting so that no magnetic flux can enter or leave.

Show that the energy W is conserved

$$W = \int\left[\frac{\rho v^2}{2} + \rho\phi + P_\perp + \frac{B^2}{8\pi}\right] d\tau$$

X. Magnetoacoustic Waves

We may now apply our results on the general two-dimensional problem to a number of special cases. The first such problem we will consider is that of magnetoacoustic wave propagation in a spatially-uniform homogeneous infinite plasma.

Our Eqs. (93), (94), (95), and (96) are nonlinear in the variables n or ρ, $\vec{B}$, $P_\perp$ and $\vec{v}$. In order to obtain linear equations we assume that the wave amplitude is small. In equilibrium the plasma density ρ_o, and the pressure P_o are constant throughout the plasma, the plasma velocity $\vec{v}_o$ is zero, and the magnetic field $\vec{B}_o$ is unidirectional (along z) and has uniform strength. The amplitude of the components associated with the wave will be designated by the subscript 1 — i.e., $\vec{B}_1$, $\vec{v}_1$, n_1 or ρ_1, $P_{\perp 1}$. To demonstrate the procedure, consider Eq. (96) as applied to the equilibrium, which, carrying out the differentiation is

$$\frac{\partial P_{\perp o}}{\partial t} = 2\,\frac{P_{\perp o}}{n_o}\,\frac{\partial n_o}{\partial t}. \tag{97}$$

For the perturbed state the pressure and density become $P_{\perp o} + P_{\perp 1}$ and $n_o + n_1$, so we have

$$\frac{\partial(P_{\perp 0}+P_{\perp 1})}{\partial t} = 2\,\frac{P_{\perp 0}+P_{\perp 1}}{n_0+n_1}\,\frac{\partial(n_0+n_1)}{\partial t} \tag{98}$$

which can be written

$$\left[\cancelto{0}{\frac{\partial P_{\perp 0}}{\partial t} - 2\,\frac{P_{\perp 0}}{n_0+n_1}\,\frac{\partial n_0}{\partial t}}\right] - 2\left[\cancelto{0}{\frac{P_{\perp 1}}{n_0+n_1}\,\frac{\partial n_0}{\partial t}} + \frac{P_{\perp 1}}{n_0+n_1}\,\frac{\partial n_1}{\partial t}\right] + \left[\frac{\partial P_{\perp 1}}{\partial t} - \frac{2P_{\perp 0}}{n_0+n_1}\,\frac{\partial n_1}{\partial t}\right] = 0. \tag{99}$$

The first bracket is zero from Eq. (97) (or since P_0 and n_0 are not functions of time). The next term is zero, since n_0 is not a function of time, while the following term is neglected since it involves the product of two small quantities $P_{\perp 1}$ and n_1. So there remains, dropping n_1 as compared with n_0 in the denominator,

$$\frac{\partial P_{\perp 1}}{\partial t} - 2\,\frac{P_{\perp 0}}{n_0}\,\frac{\partial n_1}{\partial t} = 0\,. \tag{100}$$

A similar process carried out on Eqs. (93), (94), and (95) yields

$$\frac{\partial n_1}{\partial t} = n_0\,\vec{\nabla}\cdot\vec{V}_1 \tag{101}$$

$$\frac{\partial \vec{B}_1}{\partial t} = \vec{B}_0\,\vec{\nabla}\cdot\vec{V}_1 \tag{102}$$

$$\rho_0\,\frac{\partial \vec{V}_1}{\partial t} = -\vec{\nabla}\left[P_\perp + B^2/8\pi\right]_1 \tag{103}$$

Since the magnetic field behaves like a gas with $\gamma = 2$ as well as $P_\perp$, we can also write

$$\frac{\partial}{\partial t}\left[P_\perp + \frac{B^2}{8\pi}\right]_1 = \frac{2}{n_0}\left[P_{\perp 0} + \frac{B_0^2}{8\pi}\right]\frac{\partial n_1}{\partial t}. \tag{104}$$

Eq. (104) can be immediately integrated to give

$$\left[p_\perp + \frac{B^2}{8\pi}\right]_1 = \frac{2}{n_0}\left[p_{\perp 0} + \frac{B_0^2}{8\pi}\right] n_1 + \tilde{\pi}(x, y). \tag{105}$$

Here $\tilde{\pi}$ is an arbitrary function of x and y . However, since the perturbed pressure and magnetic field must be zero when n_1 is zero, $\tilde{\pi}$ must be zero.

Substituting Eq. (105) in Eq. (103) gives

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\frac{2}{n_0}\left[p_{\perp 0} + \frac{B_0^2}{8\pi}\right] \vec{\nabla} n_1. \tag{106}$$

Taking the time derivative of Eq. (101) and the divergence of Eq. (106) gives

$$\frac{\partial^2 n_1}{\partial t^2} = \frac{2}{\rho_0}\left[p_{\perp 0} + \frac{B_0^2}{8\pi}\right] \nabla^2 n_1. \tag{107}$$

Eq. (107) is a wave equation for waves propagating with velocity

$$V^2 = \frac{2}{\rho_0}\left[p_{\perp 0} + \frac{B_0^2}{8\pi}\right]. \tag{108}$$

These waves are called magnetoacoustic waves. For small magnetic fields they reduce to ordinary acoustic waves propagating at the acoustic velocity

$$V^2 = 2\,\frac{p_{\perp 0}}{\rho_0}. \tag{109}$$

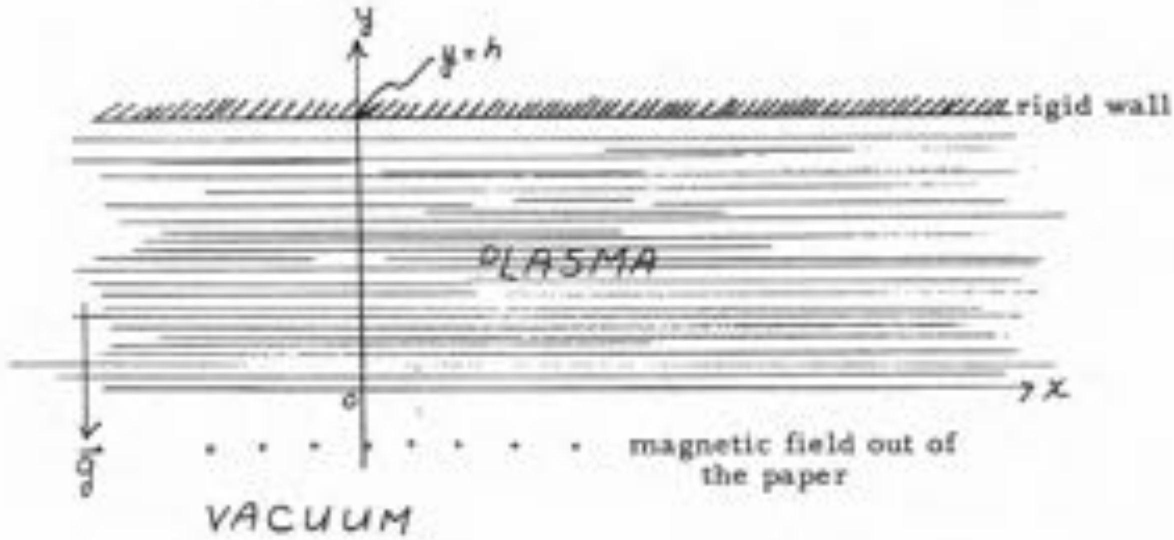
The magnetic field increases the effective pressure in the gas. These waves show no dispersion and propagate at a constant velocity. The waves are longitudinal because $\vec{v}_1$ is parallel to $\vec{\nabla} n_1$ by Eq. (106), and hence will be in the direction of $\vec{k}$ or the direction of wave propagation if we Fourier-analyze Eq. (107).

<u>Problem:</u> (1) Do the wave solutions of Eq. (107) include all possible motions of the plasma?

(2) How is it that we have lost the inertia of the magnetic field (or equivalently the energy which goes into the electric field) in this calculation? Can you include it? What effect does this have on the energy conservation calculation given on page 103?

XI. <u>The Rayleigh-Taylor Instability</u>

As a second example of the use of our two-dimensional hydromagnetic equations we will consider the problem of the Rayleigh-Taylor instability. The situation here is shown in Fig. 39.



<u>Figure 39</u>

We have a slab of plasma of thickness h supported against a gravitational field $\vec{g}$ by a magnetic field. We take the plasma to lie entirely above the x axis and to be separated from a vacuum region below the x axis by a sharp boundary. The equilibrium conditions are obtained by setting all time derivatives and velocities equal to zero in Eqs. (93) and (96).

$$0 = -\rho \vec{g} - \vec{\nabla}\left[p_\perp + \frac{B^2}{8\pi}\right]. \tag{110}$$

Rather than treat the most general problem, we will treat the specific situation in which the equilibrium plasma density and pressure are constant within the slab. Eq. (110) then reduces to

$$\rho_0 g - \frac{1}{8\pi}\frac{\partial B^2(y)}{\partial y} = 0 \quad (y > 0) \tag{111}$$

or

$$\frac{B_\rho^2(y)}{8\pi} = \frac{B_\rho^2(0)}{8\pi} - \rho_0 g y \quad (y > 0). \tag{112}$$

At the boundary (y = 0) the magnetic field strength must jump so as to balance the plasma pressure. The vacuum field is given by

$$B_v^2 = B_\rho^2(0) + 8\pi P_{\perp 0}. \tag{113}$$

The general procedure now would be to linearize Eqs. (93) to (96) for small departures from this equilibrium and to look for wave solutions to the resultant equations. Before doing this we will, however, make one further approximation. Here we are interested in the gravitational instability. For this mode the velocities and speeds of propagation are in general small compared to the velocity of a magnetoacoustic wave. This suggests that the plasma's compressibility can play only a small role in the motion and so we look for a solution assuming an incompressible plasma. We may justify this assumption a posteriori. Our set of equations then reduces to

$$\rho_0 \frac{d\vec{v}}{dt} = -\hat{g}\rho_0 g - \vec{\nabla}\left[P_{\perp 0} + \frac{B_0^2}{8\pi}\right] \tag{114}$$

and

$$\vec{\nabla}\cdot\vec{v} = 0. \tag{115}$$

The pressure equations (96) and (92) cannot be used here because ρ does not change. The incompressibility assumption assumes that the pressure is a very strong function of ρ, and hence we can have any pressure for the same density. The pressure must be determined by making it self-consistent with the motion.

In addition to Eqs. (114) and (115) we have the boundary condition that at the wall $y = h$, the normal velocity to the wall is zero

$$v_y = 0 \qquad y = h$$

The other boundary condition at the plasma vacuum interface is given in Eq. (113).

We now linearize Eqs. (114) and (115) to obtain

$$\rho_0 \frac{\partial\vec{v}}{\partial t} = -\vec{\nabla}\,\Pi_1 \tag{116}$$

and

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{117}$$

where

$$\Pi = \left[P_{\perp 0} + \frac{B_0^2}{8\pi}\right]_1 \tag{118}$$

We now look for solutions which go like

$$e^{i\omega t}. \tag{119}$$

From Eq. (116) we get

$$\vec{v} = \frac{i}{\rho_0 \omega}\,\vec{\nabla}\,\Pi_1 \tag{120}$$

and from Eqs. (117) and (120) we have

$$\nabla^2 \Pi_1 = 0. \tag{121}$$

The solutions to this equation of interest here are

$$\Pi_1 = \Pi_{1\pm}\ e^{ikx \pm ky} \tag{122}$$

or

$$\Pi_1 = (\Pi_{1+}\ e^{ky} + \Pi_{1-}\ e^{-ky})\, e^{ikx}. \tag{123}$$

From Eqs. (120) and (123) we obtain for $\vec{v}$

$$\vec{v} = \frac{1}{\rho_0 \omega} \{ \hat{y} k e^{ikx} (\Pi_{1+} e^{+ky} - \Pi_{1-} e^{-ky})$$

$$+ \hat{x}\, ik\, e^{ikx} (\Pi_{1+} e^{+ky} + \Pi_{1-} e^{-ky}) \} \tag{124}$$

The boundary condition that $v_y = 0$ at $y = h$ gives

$$\Pi_{1+} e^{kh} - \Pi_{1-} e^{-kh} = 0 \tag{125}$$

or

$$\Pi_{1-} = \Pi_{1+}\, e^{2kh}. \tag{126}$$

Hence

$$\Pi_1 = \Pi_{1+} (e^{ky} + e^{k(2h-y)})\, e^{ikx}. \tag{127}$$

Now at the plasma vacuum interface the boundary condition requires that

$$\Pi_p = \Pi_v = \frac{B_v^2}{8\pi} \tag{128}$$

since the magnetic pressure must be constant throughout this region. Before we can satisfy this condition we must find where the new boundary is.

This is obtained from the equation of motion for the boundary

$$\dot{y} = v_y \tag{129}$$

From this equation and Eqs. (124) and (126), substituting $y = 0$ to obtain the lowest order change in y, we get

$$\delta y = \frac{1}{\rho_0 \omega^2} k e^{ikx} \pi_{1+}[1 - e^{2kh}]. \tag{130}$$

Now to first order $\pi = \pi_0 + \pi_1$ at the new boundary is given by

$$\pi_v = \pi_p = \pi_0(y=0) + \pi_1(y=0) + \left.\frac{\partial \pi_0}{\partial y}\right|_{y=0} \delta y. \tag{131}$$

From Eqs. (128) and (131), and since $\pi_0 = \pi_p$, we have

$$\pi_1(y=0) + \left.\frac{\partial \pi_0}{\partial y}\right|_{y=0} \delta y = 0, \tag{132}$$

but

$$\left.\frac{\partial \pi_0}{\partial y}\right|_{y=0} \delta y = -\rho_0 g \tag{133}$$

hence

$$\pi_1(y=0) - \rho_0 g \, \delta y = 0. \tag{134}$$

Substituting the solutions we have found for π_1 and δy [Eqs. (127) and (130)] in Eq. (134) gives

$$\pi_{1+}(1 + e^{2kh}) - \frac{g}{\omega^2} k \pi_{1+}(1 - e^{2kh}) 0 \tag{135}$$

or

$$\omega^2 = g \frac{k(1 - e^{2kh})}{(1 + e^{2kh})} < 0 \quad \text{for all } k. \tag{136}$$

For large h this reduces to

$$\omega^2 = -gk. \tag{137}$$

If g is positive ($\vec{g}$ acting in the negative y direction), this gives an instability with growth rate $1/\tau = \omega$

or
$$\tau = \frac{1}{\sqrt{gk}}. \tag{138}$$

If g is negative ($\vec{g}$ acting in the positive y direction), then one gets only stable oscillations.

We may form the following physical picture of how this instability comes about. Consider the rippled plasma surface as is shown in Fig. 40.

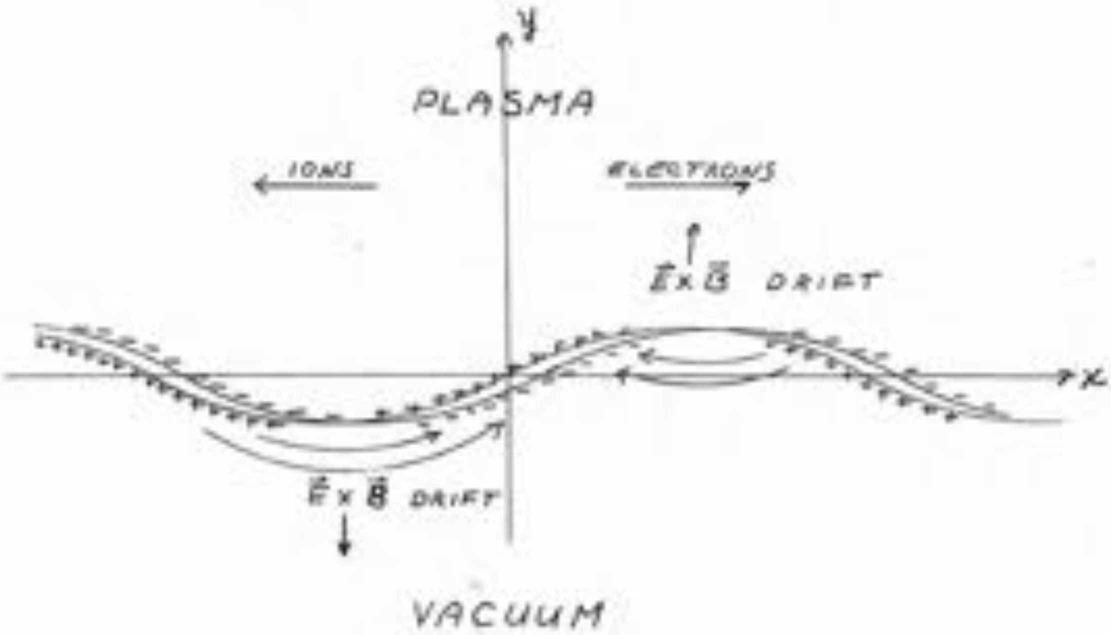


Figure 40

The gravitational field acts in the negative y direction; the magnetic field is out of the paper. Because of the gravitational field the ions drift to the left and the electrons drift to the right. If the surface is flat, no electric field develops, but if the surface is rippled the two charges are separated along the boundary by these drifts, as illustrated by the solid and dashed curves. The resultant $E \times B$ drift is in the negative y direction in the regions where the plasma has been displaced downward and is in the positive y direction where it has been displaced upward. The electric field causes the perturbation to grow.

Problem: Show that the incompressibility approximation was a good one by using π_1 to compute ρ_1 and $\dot{\rho}_1$. What is the phase relationship of all the quantities associated with the wave? Is there an $\vec{E}$ field?

XII. Alfvén Waves

As another example of the use of orbit theory we now calculate the propagation of waves parallel to $\vec{B}$ (Alfvén waves). The equilibrium situation is the same as for the case of magnetoacoustic waves; however, the propagation vector is parallel to $\vec{B}$. In this case the wave magnetic field is perpendicular to $\vec{B}$, so that we cannot use our two-dimensional hydromagnetic equation but must calculate the currents and fields directly. We assume that all quantities go like

$$e^{i(kz-\omega t)} \tag{139}$$

and that there is no variation in the x, y directions. We shall linearize about the equilibrium state, writing all quantities as $A_o + A_1$, where A_o is the equilibrium value and A_1 is the perturbation.

From $\vec{\nabla} \cdot \vec{B} = 0$ we immediately get

$$\vec{k} \cdot \vec{B}_1 = 0 \tag{140}$$

or

$$\hat{z} \cdot \vec{B}_1 = 0 \tag{141}$$

(this also implies that the magnitude of $\vec{B}$ is unchanged to first order $|B| = \sqrt{B_0^2 + B_1 \cdot B_1}$). Thus $\vec{B}_1$ has only x and y components. From Eq. (6) in section V we have, for the perturbed magnetization current,

$$\vec{j}_{M_1} = c\, i\, \vec{k} \times \vec{M}_1 = -c\, i\, \vec{k} \times \left[\left(\frac{P_\perp}{B^2} \right)_1 \vec{B}_0 + \frac{P_\perp}{B^2} \vec{B}_1 \right] \tag{142}$$

or

$$j_{M_1} = -i c \frac{P_{\perp 0}}{B_0^2} \vec{k} \times \vec{B}_1. \tag{143}$$

The lines will be bent in the motion, so there will arise a first order curvature current. This is obtained from Eq. (8) in section V and is given by

$$\vec{j}_R = \frac{c P_{\parallel 0}}{B_0} \hat{z} \times \left[(\hat{z} \cdot i \vec{k}) \frac{\vec{B}_1}{B_0} \right] \tag{144}$$

or

$$\vec{j}_R = \frac{c P_{\parallel 0}}{B_0^2}\, i\, \vec{k} \times \vec{B}_1. \tag{145}$$

Next, the first order gradient current is zero since the magnitude of $\vec{B}$ is not changed to first order

$$\vec{j}_{G_1} = 0. \tag{146}$$

And finally, we have the polarization current which is obtained from Eq. (7) in section V and is given by

$$\vec{j}_{P_1} = \frac{\rho c^2}{B_0^2} (-i \omega \vec{E}_{\perp 1}). \tag{147}$$

There is no $\vec{E}_\parallel$ since none of the currents above is along $\vec{B}$, and further, any such $\vec{E}$ would be canceled out by motion of charges along $\vec{B}$ if it were to develop.

We now substitute in Maxwell's equations to obtain

$$i\,\vec{k}\times\vec{B}_1 = -\frac{i\omega}{c}\vec{E}_1 + \frac{4\pi}{c}\left[\frac{-ic\,P_{\perp 0}}{B_0^2}\,\vec{k}\times\vec{B}_1\right]$$

$$+\frac{4\pi}{c}\left[\frac{ic\,P_{\parallel 0}}{B_0^2}\,\vec{k}\times\vec{B}_1 - \frac{i\,\omega\rho_0 c^2}{B_0^2}\,\vec{E}_1\right], \tag{148}$$

$$i\,\vec{k}\times\vec{E}_1 = i\frac{\omega}{c}\vec{B}_1, \tag{149}$$

or

$$\left[1+\frac{4\pi}{B_0^2}\left(P_{\perp 0}-P_{\parallel 0}\right)\right]\vec{k}\times\vec{B}_1 = -\frac{\omega}{c}\left[1+\frac{4\pi\rho_0 c^2}{B_0^2}\right]\vec{E}_1. \tag{150}$$

Crossing with $\vec{k}$ and substituting $\vec{k}\times\vec{E}_1$ gives

$$\left[1+\frac{4\pi}{B_0^2}\left(P_{\perp 0}-P_{\parallel 0}\right)\right]k^2\vec{B}_1 = \frac{\omega^2}{c^2}\left[1+\frac{4\pi\rho_0 c^2}{B_0^2}\right]\vec{B}_1. \tag{151}$$

This gives the dispersion relation

$$\omega^2 = c^2k^2\frac{\left[B_0^2/4\pi + \left(P_{\perp 0}-P_{\parallel 0}\right)\right]}{\left[B_0^2/4\pi + \rho_0 c^2\right]}. \tag{152}$$

For negligible ρ_0 and P this equation simply gives

$$\omega^2 = k^2c^2 \tag{153}$$

which is a light wave traveling along the magnetic lines of force.

For isotropic pressure, $P_{\parallel 0} = P_{\perp 0}$ and $\rho_0 c^2 >> B^2/4\pi$, this equation reduces to

$$\omega^2 = k^2\,\frac{B_0^2}{4\pi\rho_0\cancel{c^2}}. \tag{154}$$

These waves are known as Alfvén waves. They are transverse waves propagating along $\vec{B}$.

Finally, we note that if

$$\frac{B_o^2}{4\pi} + P_{\perp o} - P_{\parallel o} < 0 \tag{155}$$

then

$$\omega^2 < 0 \tag{156}$$

and we have an instability. The instability arises because of the motion of the particles along the curved lines of force.

Figure 41

The centrifugal force which the particle exerts on the lines of force tends to distort them further. If this force can overcome the tension in the lines of force ($B_o^2/4\pi$) the system is unstable.

The perpendicular pressure also supplies a restoring force. According to Eq.(143), $\vec{J}_m$, which is the source of this $P_\perp$ term, is in the direction $\vec{B}_1 \times \vec{k}$. This produces a $\vec{J} \times \vec{B}_o$ force which is in the direction $\vec{B}_1 \times \vec{k} \times \vec{B}_o$ which is opposite to $\vec{B}_1$.

Another argument demonstrating this effect is that work must be done against the perpendicular pressure to set up $\vec{B}_1$. We have from the constancy of μ that

$$\frac{W_\perp}{B} = \frac{W_{\perp o}}{B_o} . \tag{157}$$

Thus

$$\Delta W_\perp = \frac{W_{\perp 0}}{B_0}\,\Delta B. \tag{158}$$

But B is given by

$$B = \sqrt{B_0^2 + B_1^2} \cong B_0\left(1 + \frac{B_1^2}{2B_0^2}\right) \tag{159}$$

(there can be no second order change to $B_z = B_0$ because $\vec{\nabla} \cdot \vec{B} = \frac{\partial B}{\partial z} = 0$).

Thus ΔB and $\Delta W_\perp$ are given by

$$\Delta B = B_1^2 / 2B_0 \tag{160}$$

and

$$\Delta W_\perp = \frac{W_{\perp 0}}{2}\left(\frac{B_1}{B_0}\right)^2. \tag{161}$$

XIII. The Boltzmann Equation Approach

Up to this point we have been working with individual particle orbits. We have been able to combine the orbit calculations with Maxwell's equations for some simple cases to obtain gross plasma motions. However, we have not looked into the problem associated with the particles having a distribution of velocities or into the effects of collisions. We should like to investigate these effects. Our starting point will be the Boltzmann equation. Both limits of small and large collision rates can be treated from this equation (see appendix of Spitzer's book).

We start by defining a phase space for the particles. This space is a six-dimensional space; three of the coordinates are the position coordinates for a particle and the other three are the velocity coordinates. Given the position and velocity of a particle, we know its position in phase space, and vice versa, if we know the position of a particle in phase space we know its position and velocity. There is a point in phase space associated with every particle in the system. This set of points forms a dust or gas in phase space. As the particles move around and change their velocities, the dust moves around in the phase space. If the system contains a great many particles, then the dust will be very dense and we may treat it as a fluid. We may then define a density of points in phase space by the relation

$$f(\vec{r},\vec{v})\Delta^3 r\,\Delta^3 v = \begin{array}{l}\text{Number of particles in } \Delta^3 r\,\Delta^3 v \\ \text{centered at } r \text{ and } v\end{array} \tag{162}$$

where $\Delta^3 r$ is one element of volume in ordinary space and $\Delta^3 v$ an element of volume in velocity space. In order for this to be meaningful we must be able to choose $\Delta^3 r$ and $\Delta^3 v$ sufficiently large so that there are many particles in $\Delta^3 r\,\Delta^3 v$ and yet sufficiently small so that f does not change appreciably from one cell to the next. We assume that this is so and that f can be treated as a continuous function. Now as the particles move around, f may change with time. Since the number of particles is conserved, the changes in f must be such as to give this conservation. Let us first look at how f changes if there are no collisions between particles.

If particles flow out of a volume in phase space the density must decrease — that is, we may treat the flow of phase points like the flow of a fluid, and may write a continuity equation in this six-dimensional space. The continuity equation is

$$\frac{\partial f}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \tag{163}$$

where $\vec{J}$ is the current of phase points and $\vec{\nabla}$ is the six-dimensional gradient operator in phase space. $\vec{\nabla}$ is given by

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} + \hat{u}\frac{\partial}{\partial u} + \hat{v}\frac{\partial}{\partial v} + \hat{w}\frac{\partial}{\partial w} \tag{164}$$

where u, v, and w are the velocities in the x, y, and z directions, respectively. The current $\vec{J}$ is given by

$$\begin{aligned}\vec{J} &= \vec{V}F \\ &= \hat{x}u + \hat{y}v + \hat{z}w + \hat{u}a_x + \hat{v}a_y + \hat{w}a_z\end{aligned} \tag{165}$$

where $\vec{V}$ is the six-dimensional velocity of the phase points and a_x, a_y, and a_z are the accelerations (velocity in the u, v, w directions) in the x, y, and z directions.

If we have velocity-independent forces, then substituting these relations in Eq. (163) gives

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} + a_x\frac{\partial F}{\partial u} + a_y\frac{\partial F}{\partial v} + a_z\frac{\partial F}{\partial w} = 0 \tag{166}$$

or, vectorially,

$$\frac{\partial F}{\partial t} + \vec{v}\cdot\vec{\nabla}_r F + \vec{a}\cdot\vec{\nabla}_v F = 0 \tag{167}$$

where

$$\nabla_r = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{168}$$

and

$$\nabla_v = \hat{u}\frac{\partial}{\partial u} + \hat{v}\frac{\partial}{\partial v} + \hat{w}\frac{\partial}{\partial w}. \tag{169}$$

In general, if the forces are velocity-dependent, Eq. (163) cannot be written in the form of Eq. (166), but the accelerations must be left inside the differentiation. However, for the important case of magnetic forces one again obtains Eq. (166), even though the a's are functions of the velocity. To see this, consider

$$\vec{\nabla}_v \cdot \left[(\vec{v} \times \vec{B}) f \right] = \vec{\nabla}_v \cdot \left[\hat{u} (v B_z - w B_y) f \right.$$

$$\left. + \hat{v} (w B_x - u B_z) f + \hat{w} (u B_y - v B_x) f \right]. \tag{170}$$

For the $\hat{u}$ term the acceleration $\left[v B_z - w B_y \right]$ is independent of u and hence it can be carried to the other side of $\vec{\nabla}_v \cdot$. Thus

$$\vec{\nabla}_v \cdot \left[(\vec{v} \times \vec{B}) f \right] = (\vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f. \tag{171}$$

Hence for electromagnetic forces, Eq. (163) becomes

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_r f + \frac{q}{m} \left(\vec{E} + \vec{v} \times \vec{B} \right) \cdot \vec{\nabla}_v f = 0. \tag{172}$$

When the particles collide with each other we must add to Eq. (172) a term $\left[\frac{\partial f}{\partial t} \right]_{coll}$ which gives the change of f due to collisions. If we also add external forces $\vec{F}$ which are not electromagnetic, then Eq. (172) becomes

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_r f + \left[\frac{q}{m} \left(\vec{E} + \vec{v} \times \vec{B} \right) + \frac{\vec{F}}{m} \right] \cdot \vec{\nabla}_v f = \left[\frac{\partial f}{\partial t} \right]_{coll} \tag{173}$$

To find the change in f due to collisions, $\left[\frac{\partial f}{\partial t} \right]_{coll}$, we must look at the details of the collisional processes. We will do this later and for the time being simply carry these terms along symbolically as $\left[\frac{\partial f}{\partial t} \right]_{coll}$.

If we have more than one species of particle then there is one equation of the form of Eq. (173) for every species of particle. Also there will be a collision term for each species of particle with which a collision takes place.

XIV. Transfer Equations

A. General Equations

Let Q be some function of the velocities of the particles. For example, it might be one of the components of the momentum, or energy. The average value of Q at any spatial point is given by

$$\overline{Q(\vec{r},t)} = \frac{1}{n(\vec{r},t)} \iiint_{\text{ALL VELOCITIES}} Q\, f(x,y,z,u,v,w,t)\, du\, dv\, dw. \tag{174}$$

Although Q is not a function of $\vec{r}$ and t, and is only a function of $\vec{v}$, $\overline{Q}$ is only a function of $\vec{r}$ and t because f is a function of $\vec{r}$ and t.

We are now interested in the time rate of change of $\overline{Q}$. We have, from Eqs. (174) and (173),

$$\frac{\partial (n\overline{Q})}{\partial t} = \iiint Q \frac{\partial f}{\partial t}\, du\, dv\, dw =$$

$$-\iiint Q\left[(\vec{v}\cdot\vec{\nabla}_r) f + \left(\left[\frac{e}{m}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) + \frac{\vec{F}}{m}\right]\cdot\vec{\nabla}_v\right) f - \left(\frac{\partial f}{\partial t}\right)_{coll}\right] du\, dv\, dw. \tag{175}$$

The first term on the right is

$$\vec{\nabla}_r \cdot \iiint \vec{v}\, Q f\, d^3v = \vec{\nabla}_r \cdot n\overline{\vec{v}Q} \tag{176}$$

since $\vec{\nabla}_r$ does not operate on $\vec{v}$. The second term on the right-hand side,

$$-\iiint Q\left(\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]\cdot\vec{\nabla}_v\right) f\, d^3v, \tag{177}$$

can be integrated by parts. Consider the contribution to the integral which comes from the acceleration in the x direction, i.e.,

$$-\iiint Q\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]_x \cdot \frac{\partial f}{\partial u}\, d^3v. \tag{178}$$

Integrating by parts with respect to u, keeping v and w fixed, gives

$$-\iint Q\left[\frac{q}{m}\left[\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right]+\frac{\vec{F}}{m}\right]_x f\Bigg|_{u=-\infty}^{u=\infty} dv\,dw$$

$$+\iiint\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]_x \frac{\partial Q}{\partial u} f\, du\,dv\,dw$$

$$=\left(\frac{q}{m}\vec{E}+\frac{\vec{F}}{m}\right)_x \cdot n\,\overline{\frac{\partial Q}{\partial u}}+\frac{q}{m}\,n\,\overline{\left[\left[\frac{\vec{v}\times\vec{B}}{c}\right]_x\frac{\partial Q}{\partial u}\right]} \tag{179}$$

since, as we have seen, $\frac{\partial}{\partial u}\left(\frac{\vec{v}\times\vec{B}}{c}\right)_x = 0.$

Thus we can write Eq. (177) in the form

$$-\iiint Q\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]\cdot\vec{\nabla}_v\, f\, d^3v$$

$$=\left[\frac{q\vec{E}}{m}+\frac{\vec{F}}{m}\right]\cdot n\,\overline{\nabla_v Q}+\frac{qn}{m}\overline{\left[\left(\frac{\vec{v}\times\vec{B}}{c}\right)\cdot\nabla_v Q\right]} \tag{180}$$

the dot product $\left(\frac{\vec{v}\times\vec{B}}{c}\right)\cdot\vec{\nabla}_v Q$ being defined by comparing Eqs. (179) and (180). Then Eq. (175) takes the form

$$\frac{\partial(n\overline{Q})}{\partial t} = -\vec{\nabla}_r \cdot n\overline{(\vec{v}Q)} + \left[\frac{q}{m}\vec{E} + \frac{\vec{F}}{m}\right] \cdot n\overline{\vec{\nabla}_v Q}$$

$$+ \frac{qn}{m}\overline{\left[\left(\frac{\vec{v}\times\vec{B}}{c}\right)\cdot\nabla_v Q\right]} + \left[\frac{\partial(n\overline{Q})}{\partial t}\right]_{\text{coll.}} \tag{181}$$

B. <u>Specific Examples of Q</u>

(1) <u>$Q = 1$, The Continuity Equation</u>

As the simplest nontrivial example of Q we consider $Q = 1$. For this case, Eq. (181) becomes

$$\frac{\partial n}{\partial t} = -\vec{\nabla}_r \cdot (n\overline{\vec{v}}) \tag{182}$$

if collisions conserve particles. If particles are not conserved in a collision, then Eq. (181) takes the form

$$\frac{\partial n}{\partial t} = -\vec{\nabla}_r \cdot (n\overline{\vec{v}}) + \left(\frac{\partial n}{\partial t}\right)_{\text{coll.}} \tag{183}$$

This last case arises when we have ionization or recombination. Eq. (182) is the familiar continuity equation, while Eq. (183) is the form this equation takes when we have sources or sinks for particles.

From now on, unless otherwise stated, we shall assume that no ionization or recombination takes place.

(2) <u>$Q = \vec{v}$, Conservation of Momentum</u>

The next simplest function which Q can be is $\vec{v}$. For this case Eq. (181) becomes

$$\frac{\partial(n\overline{\vec{v}})}{\partial t} = -\nabla_r \cdot (n\overline{\vec{v}\vec{v}}) \overset{+?}{-} \left[\frac{q}{m}\vec{E} + \frac{\vec{F}}{m}\right]n$$

$$\overset{+?}{-} \frac{q}{m} n \frac{\overline{\vec{v}}\times\vec{B}}{c} + \left[\frac{\partial}{\partial t}(n\overline{\vec{v}})\right]_{\text{coll.}} \tag{184}$$

The second term involves the gradient of the momentum transfer dyadic. The physical meaning of this term is described in the next section, where we have made use of the fact that $\vec{A}\cdot\vec{\nabla}_v\vec{v} = \vec{A}$, or, in terms of dyadics

$$\vec{\nabla}_v \vec{v} = \underline{I} \tag{185}$$

where $\underline{I}$ is the unit dyadic. It should be emphasized that there is one equation of the form (184) for each species of particle. By writing

$$\sigma = \varrho n \tag{186}$$

and

$$\vec{J} = \varrho n \overline{\vec{v}} \tag{187}$$

and by noting that from conservation of momentum collisions with particles of the same species can make no contribution to

$$\left[\frac{\partial(n\overline{\vec{v}})}{\partial t}\right]_{\text{coll}}$$

we can write Eq. (184) in the form

$$m\left[\frac{\partial(n\overline{\vec{v}})}{\partial t} + \nabla_r \cdot n\overline{\vec{v}\vec{v}}\right] - \sigma\vec{E} - n\vec{F} - \frac{\vec{J}\times\vec{B}}{c} = m\left[\frac{\partial(n\overline{\vec{v}})}{\partial t}\right]_{\substack{\text{coll with}\\ \text{other species}}} \tag{188}$$

If we further write

$$\vec{v} = \vec{c} + \overline{\vec{v}} \tag{189}$$

where $\vec{C}$ is the deviation of $\vec{v}$ from its average value, then we have for $\overline{\vec{v}\,\vec{v}}$

$$\overline{\vec{v}\vec{v}} = \overline{(\vec{c}+\overline{\vec{v}})(\vec{c}+\overline{\vec{v}})} = \overline{\vec{c}\,\vec{c}} + \overline{\vec{v}}\,\overline{\vec{v}} \tag{190}$$

since $\overline{\vec{v}\,\vec{c}}$ and $\overline{\vec{c}\,\vec{v}}$ are zero.

The first term of Eq. (188) may be written

$$\frac{\partial(n\overline{\vec{v}})}{\partial t} = n\frac{\partial\overline{\vec{v}}}{\partial t} + \overline{\vec{v}}\frac{\partial n}{\partial t}. \tag{191}$$

Using the continuity equation, this becomes

$$\frac{\partial (n\overline{V})}{\partial t} = n\frac{\partial \overline{V}}{\partial t} + \overline{V}\left(-\overline{V}\cdot\vec{\nabla}_r n - n\vec{\nabla}_r\cdot\overline{V}\right). \tag{192}$$

Furthermore, the second term in Eq. (188), when Eq. (190) is applied, has the term

$$\vec{\nabla}_r\cdot n\overline{V}\,\overline{V} = \overline{V}(\overline{V}\cdot\vec{\nabla}_r n) + \overline{V}\,n(\vec{\nabla}_r\cdot\overline{V}) + n(\overline{V}\cdot\vec{\nabla}_r)\overline{V} \tag{193}$$

Using Eqs. (190), (192), and (193), Eq. (188) becomes

$$mn\left[\frac{\partial}{\partial t} + \overline{V}\cdot\vec{\nabla}_r\right]\overline{V} + \vec{\nabla}_r\cdot(nm\overline{CC}) - \sigma\vec{E}$$

$$-n\vec{F} - \frac{\vec{J}\times\vec{B}}{c} = \left[\frac{\partial \vec{P}}{\partial t}\right]_{\text{COLL. WITH OTHER SPECIES}}. \tag{194}$$

Eq. (194) is the force equation or the equation for conservation of momentum. The quantity $\underline{\underline{\Pi}}$

$$nm\overline{CC} = \underline{\underline{\Pi}} \tag{195}$$

is called the stress tensor. The right-hand side of Eq. (194) is the momentum transferred to the species under consideration, by collisions with other species of particles. If the mean velocities of the particles are small compared to their thermal velocities, then this term will be proportional to the difference in mean velocities. It can be written as a resistivity to the relative flow times the relative velocity.

The stress tensor contains not only the pressure but also the viscous stresses as well. For a general case, the stress tensor is hard to compute. However, for the case of a very high collision rate the velocity distribution will be isotropic and $\underset{\approx}{\Pi}$ will have only diagonal terms. All the diagonal terms will be equal and $\underset{\approx}{\Pi}$ can be replaced by

$$\underset{\approx}{\Pi} = p\underset{\approx}{I} \tag{196}$$

where $\underset{\approx}{I}$ is the unit dyadic and p is the scalar pressure.

When the collision rate is large, but not so large that it can be considered infinite, then the off-diagonal terms give the viscous stresses and can be written down in terms of the shear viscosity. In addition, the diagonal terms are no longer all equal, the difference arising from the bulk viscosity.

If the collision rate is small, then in general we must compute the stress tensor from the Boltzmann equation. However, there are a few cases here when $\underset{\approx}{\Pi}$ takes on a particularly simple form. If time variations are slow compared to the cyclotron frequency and spatial variations are much larger than a Larmor radius, then the velocity distribution must be symmetric in the two directions perpendicular to the magnetic field. In this case the stress tensor is again diagonal in a coordinate system with one axis along the direction of the magnetic field, and the two terms which come from the velocities perpendicular to the magnetic field are equal. That is, if we take $\vec{B}$ to be in the z direction, then $\underset{\approx}{\Pi}$ has the form

$$\underset{\approx}{\Pi} = \hat{x}\hat{x}\,\Pi_{\perp} + \hat{y}\hat{y}\,\Pi_{\perp} + \hat{z}\hat{z}\,\Pi_{\parallel}. \tag{197}$$

Here again one must, in general, solve the Boltzmann equation to find $\Pi_\perp$ and $\Pi_\parallel$. However, there are certain simple cases when they can be obtained simply. In particular, when the gradients along the field direction are small, so that variations in this direction can be neglected, then $\Pi_\perp$ can be obtained from the adiabatic law for a two-dimensional gas

$$\Pi_\perp = \Pi_{\perp 0}\, \rho^2/\rho_0^2 \tag{198}$$

as we have already seen in our treatment of the two-dimensional motions of a plasma. In certain cases $\Pi_\parallel$ is also given by an adiabatic law, that for a one-dimensional gas

$$\Pi_\parallel = \Pi_{\parallel 0}\, \rho^3/\rho_0^3. \tag{199}$$

We saw an example of this earlier when we treated the longitudinal invatiant. The adiabatic conditions are usually obtained when mixing of particles or equivalently energy flow along the lines can be neglected. This happens when the system is closed in the z direction, or for the case of wave propagation when the phase velocity along the field is much faster than the root mean square particle velocity in that direction.

When both these adiabatic laws can be employed, we say that the gas obeys a double adiabatic equation of state.

(3) <u>The Momentum Transfer Dyadic</u>

We consider a small cube with edges $\vec{e}_x\,dx$, $\vec{e}_y\,dy$, and $\vec{e}_z\,dz$ (Fig. 42). In one second all the particles within a distance v_x from face 1 will cross this area, each carrying a momentum $m\vec{v}$. The

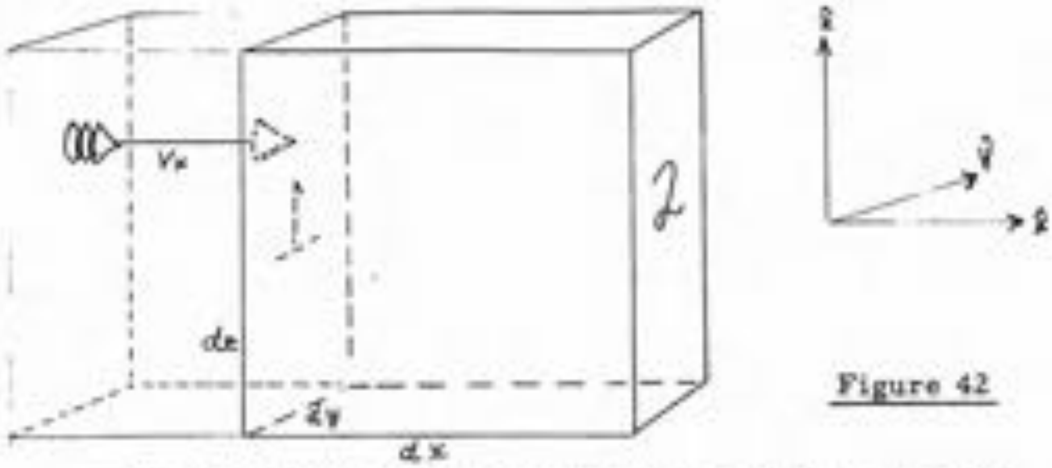


Figure 42

number of particles in this volume is $n\,v_x\,dy\,dz$. Thus the momentum entering face 1 per second is

$$m\vec{v}\,(n v_x\,dy\,dz) = mn\,v_x\vec{v}\,dy\,dz. \tag{200}$$

The momentum leaving face 2 per second is

$$mnv_x\vec{v}\,dy\,dz + \frac{\partial}{\partial x}(mnv_x\vec{v}\,dy\,dz)\,dx. \tag{201}$$

Thus the net momentum gain per second through these two surfaces is

$$-\frac{\partial}{\partial x}(mn\,v_x\vec{v}\,dx\,dy\,dz). \tag{202}$$

The other surfaces similarly contribute

$$-\frac{\partial}{\partial y}(mn\,v_y\vec{v}\,dx\,dy\,dz) - \frac{\partial}{\partial z}(mn\,v_z\vec{v}\,dx\,dy\,dz). \tag{203}$$

The increase of momentum per unit time per unit volume is just the sum of Eqs. (202) and (203), which, converting to dyadic notation, is simply

$$\vec{\nabla}_r \cdot \underline{\Pi} = \vec{\nabla}_r \cdot (mn\vec{v}\vec{v}). \tag{204}$$

(4) <u>$Q = \vec{v}\cdot\vec{v}$, Conservation of Energy</u>

Let us now look at $Q = \vec{v}\cdot\vec{v}$. This will give us an equation for the time development of the kinetic energy density. Substituting in Eq. (181) gives

$$\frac{\partial(n\overline{\vec{V}\cdot\vec{V}})}{\partial t} = -\vec{\nabla}_r\cdot(n\,\overline{\vec{V}(\vec{V}\cdot\vec{V})})$$

$$+\left[\frac{e}{m}\vec{E}+\frac{\vec{F}}{m}\right]\cdot 2n\overline{\vec{V}}+\left[\frac{\partial}{\partial t}(n\overline{\vec{V}\cdot\vec{V}})\right]_{coll.} \tag{205}$$

Here we have made use of the fact that

$$\vec{\nabla}_v(\vec{V}\cdot\vec{V}) = 2\vec{V} \tag{206}$$

and that

$$(\vec{V}\times\vec{B})\cdot\vec{V} = 0. \tag{207}$$

Again, writing

$$\vec{V} = \vec{C}+\overline{\vec{V}} \tag{208}$$

gives

$$\overline{\vec{V}\cdot\vec{V}} = \overline{\vec{C}\cdot\vec{C}}+\overline{\vec{V}}\cdot\overline{\vec{V}} \tag{209}$$

and

$$\nabla_r\cdot(n\,\overline{V(V\cdot V)}) =$$

$$\vec{\nabla}_r\cdot\left\{n\left[\overline{(\vec{C}+\overline{\vec{V}})[(\vec{C}+\overline{\vec{V}})\cdot(\vec{C}+\overline{\vec{V}})]}\right]\right\} =$$

$$\nabla_r\cdot\left\{n\,\overline{\vec{C}[\vec{C}\cdot\vec{C}]}+2n\overline{\vec{C}\vec{C}}\cdot\overline{\vec{V}}+n\,\overline{\vec{V}}\,\overline{\vec{C}\cdot\vec{C}}\right.$$

$$\left.+n\,\overline{\vec{V}}(\overline{\vec{V}}\cdot\overline{\vec{V}})+2\,\overline{\overline{\vec{V}}(\vec{C}\cdot\overline{\vec{V}})}+\overline{\vec{C}\,\overline{\vec{V}}\cdot\overline{\vec{V}}}\right\} \tag{210}$$

where the last two terms are zero, since $\overline{\vec{V}}$ is a constant which may be taken outside the average.

Substituting Eq. (210) in Eq. (205) and making use of the continuity equation (182) gives

$$n\left[\frac{\partial}{\partial t}\left(\frac{m}{2}\overline{V}\cdot\overline{V}\right)+(\overline{V}\cdot\nabla_r)\left(\frac{m}{2}\overline{V}\cdot\overline{V}\right)\right]$$

$$+n\left[\frac{\partial}{\partial t}\left(\frac{m}{2}\overline{C\cdot C}\right)+(\overline{V}\cdot\nabla_r)\left(\frac{m}{2}\overline{C\cdot C}\right)\right]$$

$$+\nabla_r\cdot\left[nm\,\overline{CC}\cdot\overline{V}+\cancel{\frac{nm}{2}\overline{V}\,\overline{C\cdot C}}\right]$$

$$+\nabla_r\cdot\left[\frac{nm}{2}\overline{C(C\cdot C)}\right]=$$

$$=\vec{E}\cdot\vec{j}+\vec{F}\cdot n\overline{V}+\left[\frac{\partial(nK)}{\partial t}\right]_{\text{COLL WITH OTHER SPECIES}} \qquad (211)$$

where we have written

$$\vec{j}=q\,n\,\overline{V} \qquad (212)$$

and K is the total mean kinetic energy per particle, and from conservation of energy K for the species of particles under consideration does not change for collisions with itself.

Problem: Verify Eq. (211).

The terms appearing in Eq. (211) have the following meanings: The first two terms on the left are the convective time derivative of the energy of mean motion; the second two terms on the left form the convective time derivative of the energy of the random motion about the mean — i.e., the heat energy. The fifth term is the divergence of the stress tensor dotted with the mean velocity. The stress tensor dotted with the mean velocity gives the rate at which the fluid stresses are doing work, so this is the divergence of this rate of doing work. The sixth term is the divergence of the flux of random energy due to the mean motion and the seventh term is the divergence of the flux of random motion due to the random motions. This seventh term is called the heat flow. The first term on the right-hand side is the rate at which the electric field does work on the current; the second term on the right-hand side is the rate at which the external force does work on the fluid. Finally, the last term on the right-hand side is the rate at which the species of particles under consideration lose energy to other types of particles.

Again, if we have large collision rates the stress tensor can be written in terms of the viscous stresses which are proportional to the gradient of $\overline{v}$, while the heat flow in this case is proportional to the temperature gradient. In general, the thermal conductivity may be a tensor, particularly for a plasma in a magnetic field, and the heat flow will be given by

$$\underset{\sim}{H} = \underset{\sim}{K} \cdot \nabla_r T. \tag{213}$$

The energy exchange between different species of particles will have two terms — one proportional to the mean relative drifts squared and the other proportional to the temperature differences (i.e., differences in mean random energy.

Here, as in the case of the momentum equation when collisions are infrequent, one cannot write simple expressions for these terms. If collisions are negligible, then we are in general forced to solve the full collisionless Boltzmann equation along with Maxwell's equations to find these quantities.

XV. The Basic Fluid Equations for a Two-Component Plasma

Let us look at the problem of a plasma composed of electrons and one species of ion with charge ze. We have just found the conservation equations for particles, momentum, and energy for a single species. We may apply these equations to our present problem. First, consider the momentum equation (194). There is one such equation for the electrons and one for the ions. We will drop the bar from these equations, since everything will be understood to stand for averages from now on.

$$m_e n_e \left(\frac{\partial \vec{v}_e}{\partial t} + \vec{v}_e \cdot \vec{\nabla}_r \vec{v}_e \right) + \vec{\nabla}_r \cdot \underset{\sim}{\Pi_e} + n_e e \vec{E} - n_e \vec{F}_e$$

$$+ n_e e \frac{\vec{v}_e \times \vec{B}}{c} = \left[\frac{\partial \vec{P}_e}{\partial t} \right]_{i.e.} \tag{214}$$

$$m_i n_i \left(\frac{\partial \vec{v}_i}{\partial t} + (\vec{v}_i \cdot \vec{\nabla}_r) \vec{v}_i \right) + \vec{\nabla}_r \cdot \underset{\sim}{\Pi_i} - n_i z e \vec{E} - n_i \vec{F}_i$$

$$- n_i z e \frac{\vec{v}_i \times \vec{B}}{c} = \left[\frac{\partial \vec{P}_i}{\partial t} \right]_{e.i} = - \left[\frac{\partial \vec{P}_e}{\partial t} \right]_{i.e} \tag{215}$$

It is possible to work with these equations directly in investigating the behavior of a plasma. This is particularly useful if only one species of particle (for example, electrons) plays a significant role in the motion. However, often the two species move together as a fluid, and it is then useful to work with two new variables — the mean velocity and the current. We therefore define the mass velocity $\vec{v}$ by

$$\vec{V} = \frac{n_i m_i \vec{V}_i + n_e m_e \vec{V}_e}{n_i m_i + n_e m_e}, \tag{216}$$

the current $\vec{J}$ by

$$\vec{J} = e\,(z n_i \vec{V}_i - n_e \vec{V}_e), \tag{217}$$

the density ρ by

$$\rho = n_i m_i + n_e m_e, \tag{218}$$

and the charge density by

$$\sigma = e\,(n_i z - n_e). \tag{219}$$

In addition to these relations, we will need the equations of continuity for both types of particles

$$\frac{\partial n_e}{\partial t} = -\vec{\nabla}_r \cdot (n_e \vec{V}_e) \tag{220}$$

and

$$\frac{\partial n_i}{\partial t} = -\vec{\nabla}_r \cdot (n_i \vec{V}_i). \tag{221}$$

By multiplying Eq. (220) by m_e and Eq. (221) by m_i and adding, we immediately obtain the general continuity equation for mass

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_r \cdot (\rho \vec{V}) = 0. \tag{222}$$

By multiplying Eq. (220) by $-e$ and Eq. (221) by ze and adding, we obtain the equation for conservation of charge

$$\frac{\partial \sigma}{\partial t} + \vec{\nabla}_r \cdot \vec{J} = 0. \tag{223}$$

These equations replace the two continuity equations (220) and (221).

Before proceeding we will make two approximations. First, we will assume that $1 >> z m_e/m_i$. Since the ratio is always less than 2000^{-1}, this is a very good approximation. Second, we shall assume that the ion and electron charge densities are equal. At the very beginning of the course we saw that the total kinetic energy of the electrons in a Debye sphere was just enough to remove them from the sphere against the resulting attractive $\vec{E}$ field. For regions much larger than this, only very small deviations from equal numbers can occur. Thus we set

$$n_e = Z n_i . \tag{224}$$

This does not mean that σ must be zero; however, it must simply be much smaller than $n_e e$. To find σ we must use Eq. (223). We can use this to check the approximation (224). With these approximations, Eqs. (216) and (217) for $\vec{v}$ and $\vec{j}$ become

$$\vec{V} = \vec{V}_i + \frac{Z m_e}{m_i} \vec{V}_e \tag{225}$$

and

$$\vec{j} = n_e e \left(\tfrac{1}{Z} \vec{V}_i - \vec{V}_e \right). \tag{226}$$

We can find $\vec{v}_i$ and $\vec{v}_e$ in terms of $\vec{v}$ and $\vec{j}$ from Eqs. (225) and (226).

$$\vec{V}_i = \vec{V} + \frac{m_e Z \vec{j}}{m_i n_e e} \tag{227}$$

and

$$\vec{V}_e = \vec{V} - \frac{\vec{j}}{n_e e}. \tag{228}$$

We shall use the form of Eq. (188) for Eqs. (214) and (215); however, splitting off the stress tensor. They have been written again below for convenience. Letting $\vec{\nabla}_r \cdot (n \vec{v} \vec{v}) = \vec{v}(\vec{v} \cdot \vec{\nabla}_r n) + \vec{v} n (\vec{\nabla}_r \cdot \vec{v}) + n(\vec{v} \cdot \vec{\nabla}_r \vec{v})$,

$$m_e \Big[\frac{\partial}{\partial t}(n_e \vec{v}_e) + \underbrace{\vec{v}_e(\vec{v}_e \cdot \vec{\nabla}_r n_e) + n_e \vec{v}_e(\vec{\nabla}_r \cdot \vec{v}_e) + n_e(\vec{v}_e \cdot \vec{\nabla}_r \vec{v}_e)}_{\vec{\nabla}_r \cdot (n_e \vec{v}_e \vec{v}_e)} \Big]$$

$$+ \vec{\nabla}_r \cdot \underline{\Pi}_e + n_e e \vec{E} - n_e \vec{F}_e + n_e e \left(\frac{\vec{v}_e \times \vec{B}}{c} \right) = \left[\frac{\partial \vec{P}_e}{\partial t} \right]_{ie} \tag{229}$$

$$m_i\left[\frac{\partial}{\partial t}(n_i\vec{v}_i) + \underbrace{\vec{v}_i(\vec{v}_i\cdot\vec{\nabla}_r n_i) + n_i\vec{v}_i(\vec{\nabla}_r\cdot\vec{v}_i) + n_i(\vec{v}_i\cdot\vec{\nabla}_r\vec{v}_i)}_{\vec{\nabla}_r\cdot(n_i\vec{v}_i\vec{v}_i)}\right]$$

$$+\ \vec{\nabla}_r\cdot\underline{\underline{\Pi}}_i - n_i z e\vec{E} - n_i\vec{F}_i - n_i z e\left(\frac{\vec{v}_i\times\vec{B}}{c}\right) = \left[\frac{\partial\vec{P}_i}{\partial t}\right]_{ei}. \tag{230}$$

We add Eqs. (229) and (230), term by term, below. With the approximations $z n_i = n_e$ and $z m_e << m_i$, adding the first term of the ion and electron equations gives

$$\frac{\partial}{\partial t}\left[m_e n_e\vec{v}_e + m_i n_i\vec{v}_i\right] = \frac{\partial}{\partial t}\left[\vec{v}(m_e n_e + m_i n_i)\right]$$

$$+\ \frac{\partial}{\partial t}\left[-\frac{n_e m_e}{n_e e}\vec{j} + \frac{n_i m_i m_e z}{m_i n_e e}\vec{j}\right]$$

$$= \frac{\partial(\rho\vec{v})}{\partial t}. \tag{231}$$

Addition of the second term of the ion and of the electron equations, when $\vec{v}_i^*$ and $\vec{v}_e^*$ are expressed in terms of $\vec{v}$ and $\vec{j}$, will involve $\vec{v}\,\vec{v}$, $\vec{j}\,\vec{j}$, and mixed ($\vec{v}\,\vec{j}$ and $\vec{j}\,\vec{v}$) terms. The $\vec{v}\,\vec{v}$ term is simply

$$\vec{\nabla}_r\cdot\left[(n_i m_i + n_e m_e)\vec{v}\,\vec{v}\right] = \vec{\nabla}_r\cdot\left[\rho\vec{v}\,\vec{v}\right]. \tag{232}$$

The $\vec{j}\,\vec{j}$ term is

$$\vec{\nabla}_r\cdot\left[n_i m_i\left(\frac{m_e z}{m_i n_e e}\right)^2\vec{j}\,\vec{j} + n_e m_e\left(\frac{1}{n_e e}\right)^2\vec{j}\,\vec{j}\right] =$$

$$\vec{\nabla}_r\cdot\left[\vec{j}\,\vec{j}\,\frac{m_e}{n_e e^2}\left(\frac{m_e z}{m_i} + 1\right)\right] \approx \vec{\nabla}_r\cdot\frac{m_e}{n_e e^2}\vec{j}\,\vec{j}. \tag{233}$$

The mixed term is

$$\vec{\nabla}_r \cdot n_e m_e\left[-\frac{1}{n_e e}(\vec{j}\vec{V}+\vec{V}\vec{j})\right] + \vec{\nabla}_r \cdot n_i m_i\left[\frac{m_e z}{m_i n_e e}(\vec{j}\vec{V}+\vec{V}\vec{j})\right]$$

$$= \nabla_r \cdot\left[-\frac{m_e}{e}(\vec{j}\vec{V}+\vec{V}\vec{j}) + \frac{m_e}{e}(\vec{j}\vec{V}+\vec{V}\vec{j})\right] = 0. \tag{234}$$

Summing all the terms gives

$$\frac{\partial}{\partial t}(\rho\vec{V}) + \vec{\nabla}_r \cdot\left[\rho\vec{V}\vec{V} + \frac{m_e}{n_e e^2}\vec{j}\vec{j}\left[1+\frac{m_e z}{m_i}\right] + \underline{\Pi}_e + \underline{\Pi}_i\right]$$

$$-\sigma\vec{E} - (n_e\vec{F}_e + n_i\vec{F}_i) - \frac{\vec{j}\times\vec{B}}{c} = 0 \tag{235}$$

or using the continuity equation and writing

$$\underline{\Pi}_T = \underline{\Pi}_e + \underline{\Pi}_i + \frac{m_e}{n_e e^2}\left[1+\frac{m_e z}{m_i}\right]\vec{j}\vec{j} \tag{236}$$

gives

$$\rho\underbrace{\left(\frac{\partial}{\partial t}+\vec{V}\cdot\vec{\nabla}_r\right)}_{d/dt}\vec{V} + \vec{\nabla}_r\cdot\underline{\Pi}_T - (n_e\vec{F}_e + n_i\vec{F}_i)$$

$$-\sigma\vec{E} - \frac{\vec{j}\times\vec{B}}{c} = 0. \tag{237}$$

The $\vec{j}\,\vec{j}$ term in the total stress tensor Π_T arises because the mean velocity for both ions and electrons is different from the mean velocity for either species and these are what were split out when we computed the stress tensors. Further, the ion-electron collision terms have canceled out because of conservation of momentum.

To obtain the current equation we multiply Eq. (229) by $-e/m_e$ and Eq. (230) by $e\,z/m_i$ and add, term by term. The first term is

$$m_i\frac{e z}{m_i}\frac{\partial}{\partial t}(n_i\vec{V}_i) - \frac{e}{m_e}m_e\frac{\partial}{\partial t}(n_e\vec{V}_e)$$

$$= \frac{\partial}{\partial t}(n_i\vec{V}_i e z - n_e\vec{V}_e e) = \frac{\partial\vec{j}}{\partial t}. \tag{238}$$

The second term is

$$e z \; \vec{\nabla}_r \cdot (n_i \vec{v}_i \vec{v}_i) - e \; \vec{\nabla}_r \cdot (n_e \vec{v}_e \vec{v}_e) \tag{239}$$

This will again involve $\vec{v}\,\vec{v}$, $\vec{j}\,\vec{j}$ and mixed terms. The $\vec{v}\,\vec{v}$ term is

$$\vec{\nabla}_r \cdot (e z n_i \vec{v}\vec{v}) - \vec{\nabla}_r \cdot (e n_e \vec{v}\vec{v}) = 0 . \tag{240}$$

The $\vec{j}\,\vec{j}$ term is

$$\vec{\nabla}_r \cdot \left[e z \; n_i \left(\frac{m_e z}{m_i n_e} e \right)^2 \vec{j}\vec{j} \right] - \vec{\nabla}_r \cdot \left[e n_e \left(\frac{1}{n_e e} \right)^2 \vec{j}\vec{j} \right]$$

$$= \vec{\nabla}_r \cdot \vec{j}\vec{j} \left(\frac{m_e^2 z^2}{m_i^2 e n_e} - \frac{1}{n_e e} \right)$$

$$= \vec{\nabla}_r \cdot \left[\vec{j}\vec{j} \left(\frac{m_e^2 z^2}{m_i^2} - 1 \right) \frac{1}{n_e e} \right] = - \vec{\nabla}_r \cdot \left(\frac{1}{n_e e} \vec{j}\vec{j} \right). \tag{241}$$

The $\vec{v}\,\vec{j}$ term is

$$\vec{\nabla}_r \cdot \left[e z n_i (\vec{v}\vec{j} + \vec{j}\vec{v}) \frac{m_e z}{m_i n_e e} \right] + \vec{\nabla}_r \cdot \left[e n_e (\vec{v}\vec{j} + \vec{j}\vec{v}) \frac{1}{n_e e} \right]$$

$$= \vec{\nabla}_r \cdot \left[\left(\frac{m_e z}{m_i} + 1 \right) (\vec{v}\vec{j} + \vec{j}\vec{v}) \right] \simeq \vec{\nabla}_r \cdot \frac{1}{n_e e} \vec{j}\vec{j} \tag{242}$$

and writing $\left[\frac{\partial \vec{P}_e}{\partial t} \right]_{ie} = \alpha \, (\vec{v}_i - \vec{v}_e)$ we get

$$\frac{\partial \vec{j}}{\partial t} - \vec{\nabla}_r \cdot \left[\frac{\vec{j}\vec{j}}{n_e e} \right] + \vec{\nabla}_r \cdot [\vec{v}\vec{j} + \vec{j}\vec{v}]$$

$$+ \vec{\nabla}_r \cdot \left[\frac{z e \, \Pi_i}{m_i} - \frac{e \, \Pi_e}{m_e} \right] - \frac{n_e e^2}{m_e} \left[1 + \frac{z m_e}{m_e} \right] \vec{E}$$

$$+ n_e e \left[\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i} \right] - \frac{n_e e^2}{m_e} \frac{\vec{v} \times \vec{B}}{c} + \frac{e}{m_e c} \vec{j} \times \vec{B} = - \frac{\alpha \, \vec{j}}{n_e m_e} \tag{243}$$

or, multiplying by $+m_e/n_e e^2$, gives

$$\frac{m_e}{n_e e^2}\frac{\partial \vec{j}}{\partial t}+\frac{m_e}{n_e e^2}\nabla_r\cdot\left\{\frac{-\vec{j}\vec{j}}{n_e e}+(\vec{v}\vec{j}+\vec{j}\vec{v})+\frac{e}{m_e}\left[\frac{Z m_e}{m_i}\underset{\sim}{\Pi_i}-\underset{\sim}{\Pi_e}\right]\right\}$$

$$-\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)+\frac{1}{n_e e c}\,\vec{j}\times\vec{B}+\frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e}-\frac{\vec{F}_i}{m_i}\right)=-\underset{\sim}{\eta}\cdot\vec{j}. \tag{244}$$

Here $\underset{\sim}{\eta}$ is the ~~conductivity~~ resistivity, and in principle it should be a tensor for a plasma in a magnetic field. If we linearize Eqs. (237) and (244) in $\vec{j}$ and $\vec{v}$ — that is, we assume j and v are small and drop all second order terms (vv, jv, jj) — then these equations become

$$\rho\frac{\partial\vec{v}}{\partial t}+\nabla_r\cdot\underset{\sim}{\Pi_r}-(n_e\vec{F}_e+n_i\vec{F}_i)-\sigma\vec{E}-\frac{\vec{j}\times\vec{B}}{c}=0 \tag{245}$$

and

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial\vec{j}}{\partial t}\right]+\frac{1}{c n_e}\nabla_r\cdot\left[\frac{Z m_e}{m_i}\underset{\sim}{\Pi_i}-\underset{\sim}{\Pi_e}\right]-\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)$$

$$+\frac{\vec{j}\times\vec{B}}{n_e e c}+\frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e}-\frac{\vec{F}_i}{m_i}\right)=-\underset{\sim}{\eta}\cdot\vec{j}. \tag{246}$$

If $\underset{\sim}{\Pi_i}$ is of the order of $\underset{\sim}{\Pi_e}$, then the ion stress tensor can be neglected in Eq. (246). If $\partial\vec{j}/\partial t$, $\vec{B}$, $\vec{F}_e$, and $\vec{F}_i$ are negligible, then Eq. (246) reduces to

$$\vec{E}=\underset{\sim}{\eta}\cdot\vec{j} \tag{247}$$

which is Ohm's law with η the resistivity. We may therefore think of Eq. (246), or more generally Eq. (244), as a generalized Ohm's law.

The terms in Eq. (245) are fairly clear — $\rho\,\partial\vec{v}/\partial t$ is the inertial term, $\nabla_r\cdot\underset{\sim}{\Pi_r}$ is the force due to the material stresses, the $\vec{F}$ terms are the external forces, and $\vec{j}\times\vec{B}$ is the magnetic force. The terms in

Eq. (246) have the following meaning. The $\partial\vec{j}/\partial t$ term is due to the inertia of the current. In most cases where the current is carried primarily by the electron, it comes from the electron inertia. The term involving the stress tensors arises because the pressure or stresses due to one species tends to accelerate that species relative to the other, creating a current. The $\vec{E} + \frac{\vec{v} \times \vec{B}}{c}$ term is the electric field as seen by an observer moving with the fluid. The $\vec{j} \times \vec{B}$ term arises because the ions and electrons carry different fractions of the current and have different masses so that they are accelerated differently, and this tends to give rise to a current or to a balancing $\vec{E}$ field. This gives rise to the Hall effect. The $\vec{F}$ terms arise from differential accelerations of the two species due to the external forces. If $\vec{F}$ arises from a gravitational field, this term cancels out. The $\eta \cdot \vec{j}$ term is, of course, the resistivity.

XVI. Summary of the Macroscopic Equations

Summing up, the macroscopic equations for the fluid are

$$\rho \frac{d\vec{v}}{dt} + \vec{\nabla} \cdot \underline{\underline{\Pi}} - (n_e \vec{F}_e + n_i \vec{F}_i) - \frac{\vec{j} \times \vec{B}}{c} = 0, \tag{248}$$

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{j}}{\partial t} + \vec{\nabla} \cdot \left[\frac{-\vec{j}\vec{j}}{n_e e} + (\vec{v}\vec{j} + \vec{j}\vec{v}) + e\left(\frac{Z \underline{\underline{\Pi}}_i}{M_i} - \frac{\underline{\underline{\Pi}}_e}{m_e} \right)\right]\right]$$
$$- \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) + \frac{\vec{j} \times \vec{B}}{n_e e c} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{M_i}\right) = -\eta \cdot \vec{j}, \tag{249}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{250}$$

and

$$\frac{\partial \sigma}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \tag{251}$$

In addition we have Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \tag{252}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}}{c}, \tag{253}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{254}$$

and

$$\vec{\nabla} \cdot \vec{E} = 4\pi\sigma. \tag{255}$$

In addition to these equations we need equations to determine the π's. We can proceed a long way, however, by making some simplifying assumptions about π. Examples of choices for π are:

(1) We assume the particle stresses are negligible and neglect π altogether.

(2) We may assume that we need use only a scalar pressure p

$$\underline{\underline{\Pi}} = p\,\underline{\underline{I}} \tag{256}$$

and that p satisfies a $\gamma = 5/3$ law.

(3) We may assume a double adiabatic law with the pressure perpendicular to the lines of force determined from a $\gamma = 2$ law and with the pressure parallel to the lines of force determined by a $\gamma = 3$ law.

(4) We may use fluid equations, including viscosity, heat conduction, exchange of energy between parallel and perpendicular degrees of freedom and between electrons and ions.

One must examine the physics of the situation under consideration to determine which, if any, of the above approximations is pertinent.

XVII. Approximations to the Equation for the Current

The first and most usual approximation we will make to Eq. (249) is that of linearization. That is, we will neglect the $\vec{j}\,\vec{j}$ and $\vec{v}\,\vec{j}$ terms so that Eq. (249) becomes

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{j}}{\partial t} + \vec{\nabla}\cdot e\left(\frac{Z\,\Pi_i}{m_i} - \frac{\Pi_e}{m_e}\right)\right] - \left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)$$

$$+ \frac{1}{n_e e c}\,\vec{j}\times\vec{B} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) = -\eta\cdot\vec{j}. \tag{257}$$

Second, in most applications the external forces are either nonexistent or negligible and can be dropped (for the case of gravitational forces they cancel). Third, if the electron and ion thermal ~~velocities~~ energies are comparable, the π_i and π_e are roughly equal and we may neglect π_i/m_i compared to π_e/m_e. With these approximations, Eq. (257) reduces to

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{j}}{\partial t} - \vec{\nabla}\cdot\frac{e\,\Pi_e}{m_e}\right] - \left[\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right]$$

$$+ \frac{1}{n_e e c}\,\vec{j}\times\vec{B} = -\eta\cdot\vec{j}. \tag{258}$$

If we solve the linearized version of Eq. (248) for $\vec{J} \times \vec{B}/c$ (also neglecting $\vec{F}$), we obtain

only holds [illegible] B [illegible]

$$\frac{\vec{J}\times\vec{B}}{c} = \rho\,\frac{\partial \vec{V}}{\partial t} + \vec{\nabla}\cdot \underline{\Pi}_T \tag{259}$$

and substituting this in Eq. (258) gives

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{J}}{\partial t} + \vec{\nabla}\cdot \frac{e}{m_e}\,\underline{\Pi}_i\right] - \frac{\rho}{n_e e}\,\frac{\partial \vec{V}}{\partial t}$$

$$-\left(\vec{E} + \frac{\vec{V}\times\vec{B}}{c}\right) = -\eta\cdot\vec{J}. \tag{260}$$

If we now look for a steady state solution and further assume that the pressure gradient is negligible, then Eq. (260) reduces to

$$\vec{E} + \vec{V}\times\vec{B}/c = \eta\cdot\vec{J}. \tag{261}$$

This says that the current is driven by the electric field seen in a frame moving with the fluid. Finally, if the conductivity of the fluid is very high, then η is negligible and Eq. (261) becomes

$$\vec{E} + \frac{\vec{V}\times\vec{B}}{c} = 0 \tag{262}$$

For a high-temperature plasma η becomes very small, as we shall see later, and this may become a good approximation provided the other assumptions made are valid. A fluid for which Eq. (262) is satisfied is said to be a perfect conductor. You will note that it satisfies our criterion that the lines of force move with the fluid (equation 19 in section VII).

XVIII. Discussion of the Relation Between Macroscopic and Microscopic Velocities

Let us consider the steady state case with isotropic pressure, comparable ion and electron pressures, zero resistivity, and small $\vec{v}$ and $\vec{j}$ (linearized equations). Eqs. (248) and (249) give

$$\vec{\nabla} p - (n_e \vec{F}_e + n_i \vec{F}_i) - \frac{\vec{j} \times \vec{B}}{c} = 0, \tag{263}$$

$$0 = -\frac{1}{n_e e} \vec{\nabla} P_e - \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) + \frac{1}{n_e e c} \vec{j} \times \vec{B} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) \tag{264}$$

where

$$p = P_e + P_i \tag{265}$$

or, substituting $\vec{j} \times \vec{B}/c$ from Eq. (263) in Eq. (264), Eq. (264) becomes

$$\frac{1}{n_e e} \vec{\nabla} P_i - \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) - \frac{\vec{F}_i}{Ze} = 0. \tag{266}$$

We should like to compare the motions given by these equations with those for individual particles. We shall consider various situations as we did for the case of particle drifts.

A. Uniform Plasma Density and $\vec{B}$ Field, with No External Force, $\vec{E}$ Perpendicular to $\vec{B}$

For this case, as we have already seen,

$$\vec{E}_\perp = - \frac{\vec{v} \times \vec{B}}{c}, \tag{267}$$

and

$$\vec{v}_\perp = \frac{c\, \vec{E}_\perp \times \vec{B}}{B^2} \tag{268}$$

This is the same drift velocity we found for individual particle motions.

B. Uniform $\vec{B}$, no $\vec{E}$ or $\vec{F}$, with a Gradient of $\vec{P}$

For this case Eq. (266) becomes

$$\vec{\nabla} P_i = n_e e \frac{\vec{v} \times \vec{B}}{c}. \tag{269}$$

Thus, $\vec{\nabla} P_i$ is perpendicular to $\vec{B}$. Also, by Eq. (263)

$$\vec{\nabla} P = \frac{\vec{j} \times \vec{B}}{c}. \tag{270}$$

$\vec{\nabla} P$ is perpendicular to $\vec{B}$. (If the ion and electron densities and temperatures are the same, then $P_i = P_e$.) Now we may solve Eq. (269) for $\vec{v}$ just as we did in the case of a uniform $\vec{E}$ field. Crossing Eq. (269) with $\vec{B}$ on the right gives

$$\vec{v} = -\frac{c}{n_e e} \left[\frac{\vec{\nabla} P_i \times \vec{B}}{B^2} \right]. \tag{271}$$

We see that there is a macroscopic velocity for the plasma in this case. However, from the particle orbit point of view the particles do not drift in a uniform B field. How does this drift arise?

The reason for the macroscopic motion is the following. The velocity $\vec{v}$ of the fluid in a little element of volume is the mean velocity of all particles within that volume (total momentum divided by the total mass). Those particles whose orbits lie entirely within the volume will not contribute to the mean velocity since they are moving up as much as down (see Fig. 43). However, near the edge of the volume there are some particles whose orbits lie only partially within the volume, as shown in Fig. 43. All those particles with their centers of gyration lying on the right side of the volume element $d\tau = dx\,dy\,dz$ contribute to a net downward velocity, while all those particles with centers lying on the

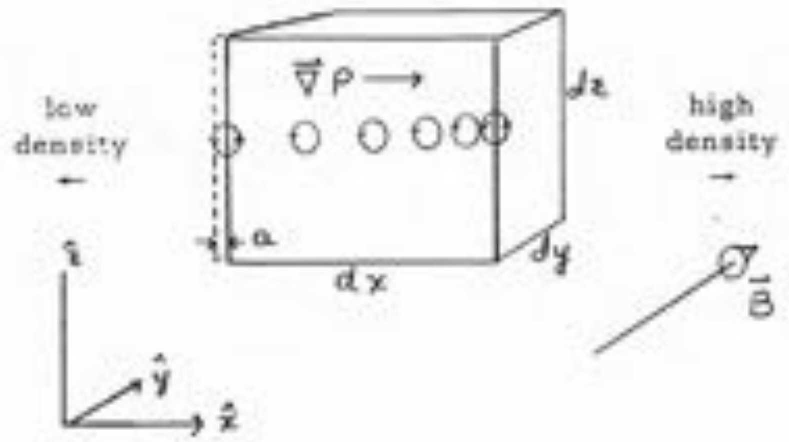


Figure 43

left-hand side of the volume contribute to a net upward velocity. If there are more particles on the right than on the left, the result will be a net downward motion of the fluid in $d\tau$. A quick estimation of the size of this effect shows that it can give Eq. (271). The downward momentum contributed by the particles on the right is

$$-m n_2 a v_\perp \, dy\, dz = -n_2 \frac{m v_\perp^2}{eB} mc \, dy\, dz \tag{272}$$

while the upward momentum contributed by the particles on the left is

$$m n_1 a v_\perp \, dy\, dz = n_1 \frac{m v_\perp^2}{eB} mc \, dy\, dz \tag{273}$$

Here a is the larmor radius and na is the number of particles per unit area within a larmor radius of the surface $d\tau$. The net downward momentum is

$$-\frac{m^2 c v_\perp^2}{eB}[n_2 - n_1]\, dy\, dz = -\frac{m^2 c v_\perp^2}{eB} \vec{\nabla} n \, dx\, dy\, dz \tag{274}$$

Dividing by the total mass of material in $d\tau$ gives

$$v_z = -\frac{m^2 c v_\perp^2}{eB} \frac{\vec{\nabla} n \, dx\, dy\, dz}{mn \, dx\, dy\, dz} = -\frac{mc v_\perp^2}{eBn} \nabla n = -\frac{c \nabla_x P_\perp}{n e B} \tag{275}$$

where it has been assumed that the temperature is independent of position and hence also v^2, so that it can be taken inside ∇ ($P_i = m_i v^2$, to order m_e/m_i only the ions contribute). We see that Eq. (275) is identical to Eq. (271) if the x direction is the same as $\vec{\nabla} P_i$.

C. Uniform Pressure, Nonuniform $\vec{B}$, no $\vec{E}$ or $\vec{F}$

For this case Eq. (266) gives

$$\frac{\vec{V} \times \vec{B}}{c} = 0$$

or

$$\vec{V}_\perp = 0 \tag{276}$$

where $\perp$ means perpendicular to $\vec{B}$.

Here we have no macroscopic velocity. On the other hand, from the particle orbit point of view the particles are drifting. How do we explain this?

Consider a small element of volume $d\tau = dx\,dy\,dz$ with $\vec{B}$ out of the paper and $\vec{\nabla} B$ in the x direction, as shown in Fig. 44.

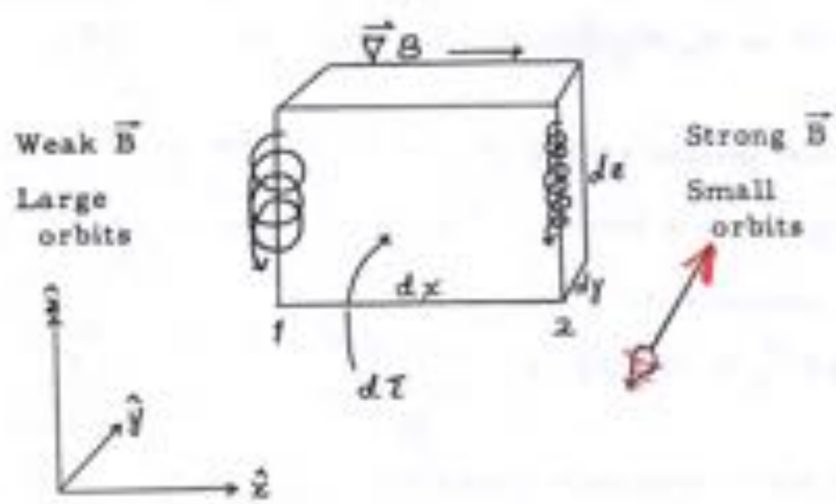


Figure 44

Now the particle orbits within the volume contribute a net momentum given by Eq. (147) section IV, which for this case is downward of magnitude

$$\frac{c m v_\perp^2 \nabla B}{e B^2} m n \, dx \, dy \, dz \tag{277}$$

The net velocities due to particles whose centers of gyration lie outside $d\tau$ but whose ~~guiding centers~~ orbits intersect $d\tau$, however, cancel this. The upward momentum contribution due to particles at face 1 is

$$+ \frac{n v_\perp^2}{e B_1} m^2 c \, dy \, dz \tag{278}$$

while the downward contribution due to particles at face 2 is

$$\frac{n v_\perp^2}{e B_2} m^2 c \, dy \, dz = \left[\frac{n v_\perp^2}{e B_1} m^2 c + \frac{\partial}{\partial x} \left(\frac{n v_\perp^2}{e B} m^2 c \right) dx \right] dy \, dz \, . \tag{279}$$

The net momentum upward due to both faces 1 and 2 is

$$\frac{n v_\perp^2 m^2 c}{e} \cdot \frac{\nabla B \, d\tau}{B^2} \tag{280}$$

which is just what is required to cancel the drift given by Eq. (277).

XIX. Diffusion of Magnetic Fields Through Matter and of Plasma Across Magnetic Fields

A. Diffusion of a Magnetic Field through a Solid Conductor

As the simplest example of the diffusion of a magnetic field through a conducting material we will consider the diffusion of a magnetic field through a solid conductor. We shall adopt the simple ohm's law

$$\vec{E} = \eta \vec{j} \tag{281}$$

as the equation which determines the current in terms of $\vec{E}$. This can be obtained from our general equation (249) by neglecting the inertial terms, the nonlinear terms, the pressure terms, the Hall term, and the external forces. We have also set the velocity equal to zero. In addition to this equation we have the Maxwell equations

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \tag{282}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j}, \tag{283}$$

and

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{284}$$

We have dropped the displacement current term ($\dot{\vec{E}}$) since here we are primarily interested in low-frequency phenomena where it is negligible. We must, however, keep the $\partial\vec{B}/\partial t$ term because it is needed to determine $\vec{E}$ which drives $\vec{j}$. In addition to these we should add

$$\vec{\nabla} \cdot \vec{j} = 0 \tag{285}$$

for if this is violated charges build up rapidly, producing large $\vec{E}$

fields which alter the current so as to prevent further buildup of charge. However, Eq.(285) is automatically satisfied because of Eq.(283). We now substitute Eq.(281) in Eq.(283), obtaining

$$\vec{\nabla} \times \vec{B} = \frac{4\pi \vec{E}}{c\eta}. \tag{286}$$

Taking the curl of Eq.(286) and making use of Eq.(282) and Eq.(284) gives

$$-\nabla^2 \vec{B} = -\frac{4\pi}{c^2\eta} \frac{\partial \vec{B}}{\partial t}. \tag{287}$$

Only $\vec{B}$'s which satisfy Eq.(284) as well as Eq.(287) are acceptable.

Now Eq.(287) is a diffusion equation with a diffusion coefficient

$$D = \frac{c^2\eta}{4\pi}. \tag{288}$$

Thus the larger the resistivity the larger is the rate of diffusion of the field through the matter.

In order to obtain a physical understanding of the meaning of this equation, consider the following simple situation. Imagine that we have a conducting slab of material which is infinite in the xy plane and has thickness 2d in the z direction (see Fig. 45).

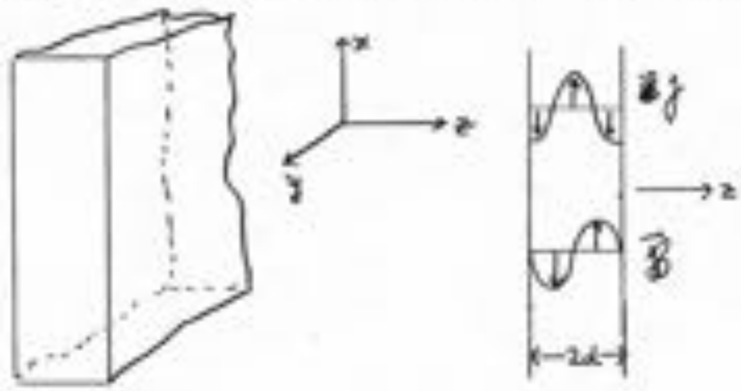


Figure 45

We take the initial $\vec{j}$ to be of the form

$$\vec{j} = \hat{x}\, j_0 \overset{\cos}{\cancel{\sin}} \frac{\pi z}{d} \tag{289}$$

and hence the initial $\vec{B}$ is given by

$$\frac{\partial B_y}{\partial z} = -\frac{4\pi}{c} j_0 \overset{\cos}{\cancel{\sin}} \frac{\pi z}{d} \tag{290}$$

or, on integration

$$B_y = -\frac{4 j_0 d}{c} \overset{\sin}{\cancel{\cos}} \frac{\pi z}{d} + B_0 . \tag{291}$$

Since we want $\vec{B}$ to vanish at $x = \pm\infty$, we choose $B_0 = 0$. Now since things are uniform in the x,y direction, there can be no variations in these directions. Further, if there are no x and z components to B initially, none will arise according to Eq. (287). Thus, in this case Eq. (287) becomes

$$\frac{c^2\eta}{4\pi} \frac{\partial^2 B_y}{\partial z^2} = \frac{\partial B_y}{\partial t} . \tag{292}$$

Writing

$$B_y = A(t) \overset{\sin}{\cancel{\cos}} \frac{\pi z}{d} \tag{293}$$

and substituting in Eq. (292) gives

$$-\frac{c^2\eta}{4\pi} \frac{\pi^2}{d^2} A(t) = \frac{\partial A}{\partial t} \tag{294}$$

or

$$A = A_0\, e^{-t/\tau} \tag{295}$$

where

$$\tau = \frac{4 d^2}{c^2 \eta \pi} = \frac{d^2}{\pi^2 D} \tag{296}$$

and from Eq. (291)

$$A_0 = -\frac{4 j_0 d}{c} . \tag{297}$$

Thus the magnetic field decays exponentially with time.

To find the physical significance of τ we compute the magnetic energy per unit area in the x y direction and the rate of dissipation of this energy by the opposition to the current offered by the resistance. The magnetic energy is given by

$$\int_{-d}^{d} \frac{B^2}{8\pi}\, dz = \int_{-d}^{d} \frac{A^2}{8\pi} \sin^2 \frac{\pi z}{d}\, dz = \frac{A^2 d}{8\pi}. \tag{298}$$

The current density in terms of A is given by

$$j = \frac{cA}{4d} \tag{299}$$

and the rate of dissipation of energy is

$$\int_{-d}^{d} \eta j^2\, dz = \eta \frac{c^2 A^2}{16 d}. \tag{300}$$

If we divide the magnetic energy by its rate of dissipation we obtain a time t

$$t = \frac{A^2 d}{8\pi} \Big/ \eta \frac{c^2 A^2}{16 d} = \frac{2 d^2}{\pi \eta c^2}. \tag{301}$$

Thus, according to Eq. (301), B^2 should decay on the time scale $t = 2d^2/\pi\eta c^2$. We can obtain the same result qualitatively from Eq. (287) if we assume the characteristic distances over which $\vec{B}$ varies is L and its decay time is τ. We then have approximately, from Eq. (287),

$$\frac{c^2 \eta B}{4\pi L^2} \cong \frac{B}{\tau} \tag{302}$$

or

$$\frac{c^2 \eta}{4\pi} \frac{B^2}{L^2} \cong B^2/\tau. \tag{303}$$

But from Eq. (283)

$$B/L \cong 4\pi j/c \tag{304}$$

or

$$\eta j^2 4\pi \cong B^2/\tau \text{ or } \eta j^2 \cong \frac{B^2}{4\pi\tau}. \tag{305}$$

The left-hand side of Eq. (305) is the rate of dissipation of energy by the current and the right-hand side is the rate of decay of magnetic energy.

(In the above example the magnetic field diffuses out of the slab into the vacuum region.)

If we had kept the $\dot{\vec{E}}$ term we would see the $\vec{B}$ field propagate away from the slab at the speed of light upon emerging from the slab. If we had kept this term, Eq. (287) would become

$$\nabla^2 \vec{B} = \frac{4\pi}{c^2\eta} \frac{\partial \vec{B}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}. \tag{306}$$

The flux between planes $z = \pm\infty$, or more exactly, between $z = \pm ct + d$ is conserved, provided there is no magnetic field outside the slab initially. However, the magnetic energy density which is proportional to B^2, decays in time.

Problem: Prove the above statement.

B. Diffusion of a Plasma Across a Magnetic Field

As the simplest example of the diffusion of a plasma across a magnetic field we will consider the case of a uniform circular plasma cylinder in a very strong field so that the gas pressure is negligible compared to the magnetic pressure. We can thus neglect the effect of the plasma on the magnetic field. We shall take the field to be uniform and to point in the z direction. We shall also assume that the diffusion is slow so that the inertial forces can be neglected. The nonlinear terms and external forces will be dropped. We shall assume that we can use a scalar pressure. Eqs. (248) and (249) thus become

$$\vec{\nabla} p - \frac{\vec{j} \times \vec{B}}{c} = 0 \tag{307}$$

and

$$\vec{E} + \frac{\vec{v} \times \vec{B}}{c} - \frac{\vec{j} \times \vec{B}}{nec} = \eta \vec{j}. \tag{308}$$

From Eq. (307), $\vec{\nabla} P$ is perpendicular to $\vec{B}$. There can be no pressure variations along $\vec{B}$. Solving Eq. (307) for the part of $\vec{J}$ which is perpendicular to $\vec{B}$ gives

$$\vec{j}_{\perp} = -c \frac{\vec{\nabla} p \times \vec{B}}{B^2}. \tag{309}$$

One may readily verify that $\vec{\nabla} \cdot \vec{j}_{\perp}$ is zero, so no accumulation of charge takes place. Thus no other currents are required — i.e., $\vec{J}_{\parallel}$ to $\vec{B}$ — and we will assume that none exist. We may also drop $\vec{E}$ because $\vec{\nabla} \times \vec{E}$ is zero since $\dot{\vec{B}}$ is, and the only $\vec{E}$ fields that can exist are electrostatic fields arising from charges. These need not exist, and we shall assume there are none.*

---

* Such charges may exist and in principle they should be determined from Eqs. (307) and (308) by requiring $\vec{\nabla} \cdot \vec{J} = 0$; i.e., no accumulation of charge must take place. This in general requires a $\vec{J}_{\parallel}$, and hence an $\vec{E}_{\parallel}$ by Eq. (308). This $\vec{E}$, along with $\vec{\nabla} \times \vec{E} = 0$, suffice to determine $\vec{E}$. If one cannot find a $\vec{J}_{\parallel}$ which makes $\vec{\nabla} \cdot \vec{J} = 0$, then one must take account of the fact that charges are accumulating and that a time-dependent $\vec{E}$ exists, hence an acceleration of the plasma and also a polarization current. This was the case of the plasma column dropping in a magnetic field. Since no $\vec{J}_{\parallel}$ is needed here, we ignore it.

One other source of $\vec{E}$ field exists here. That is, an electrostatic field with E_x and E_y components which is the same in all z planes. Such a field would exist if the plasma were rotating. Such a field would have to be put in by introducing charges initially, for it will not arise by itself because $\vec{\nabla} \cdot \vec{J} = 0$. Again we ignore it.

Substituting Eqs. (307) and (309) in (308) then gives

$$\frac{\vec{v} \times \vec{B}}{c} - \frac{\vec{\nabla} P}{n e} = - \eta c \, \frac{\vec{\nabla} P \times \vec{B}}{B^2} \tag{310}$$

Here we are treating η as a scalar. Crossing Eq. (310) on the right with $\vec{B}$ and solving for $\vec{v}_\perp$ gives

$$\vec{v}_\perp = - c \, \frac{\vec{\nabla} P \times \vec{B}}{n e B^2} - \frac{\eta c^2 \vec{\nabla} P}{B^2} \tag{311}$$

The first term on the right is a velocity perpendicular to both $\vec{\nabla} P$ and $\vec{B}$, and hence perpendicular to the density gradients if we assume a uniform temperature. This motion lies in the surfaces of constant pressure and leads to no loss of plasma.

The second term on the right-hand side is antiparallel to the pressure gradient and hence to the density gradient if the temperature is uniform. The motion carries the plasma from regions of higher density to regions of lower density.

The following argument shows the physical cause of this diffusion. The rate of dissipation of energy by the current per unit volume is just ηj^2.

By Eq. (309) this is

$$\eta j^2 = c^2 \eta \, |\vec{\nabla} P|^2 / B^2 \tag{312}$$

This energy must be supplied by the work done by the pressure of the gas as it expands. For an element of convecting fluid which is being pushed outward by gas pressure inside, from following a fluid element we get

$$\frac{dP}{dt} = 0 = \frac{\partial P}{\partial t} + \vec{v} \cdot \vec{\nabla} P. \tag{313}$$

At a given position

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{V} P. \tag{314}$$

But since pressure is thermal energy per unit volume, then the work done on the convecting fluid is $\vec{V} \cdot \vec{\nabla} P$. Equating this with Eq. (312) gives

$$\vec{V}_{vp} = c^2 \eta \frac{\vec{\nabla} P}{B^2} \tag{315}$$

which is the same as Eq. (311).

A second way to view this diffusion is the following. The ions and electrons must drift through each other to provide the current required by pressure balance. Because of collisions there is a drag of one species on the other, the drags being equal and opposite in direction if we consider singly-charged ions, or proportional to z if they are multiply-charged. The force per electron is (by equation 308) (the equivalent E in $\hat{E}\eta j$)

$$-e\eta \vec{j} = \vec{F}_e \tag{316}$$

and per ion

$$Ze\eta \vec{j} = \vec{F}_i \tag{317}$$

If we view these as external forces to each species, then these forces give rise to drifts. The drifts are in the same direction and have the same magnitude for each species. The drift velocity is

$$\vec{V}_D = \frac{c}{q} \frac{\vec{F} \times \vec{B}}{B^2} = -c\eta \frac{\vec{J} \times \vec{B}}{B^2} \tag{318}$$

or

$$= c^2 \eta \, \vec{\nabla} P / B^2 \tag{319}$$

The diffusion of a plasma given in Eq. (319), across the field, is obviously related to the diffusion of a magnetic field across a plasma, as described by Eqs. (287) and (302). Let us write Eq. (302) in a somewhat different way

$$\frac{L}{T} = \frac{c^2 \eta}{4\pi L} \sim V_B. \tag{320}$$

We compare this with that part of Eq. (319) that describes the outward diffusion of the plasma, which we will approximate

$$V_\perp = -\frac{\eta c^2}{B^2}\frac{P}{L}. \tag{321}$$

Then

$$\frac{V_\perp}{V_B} \sim \frac{P}{B^2/4\pi}. \tag{322}$$

XX. Waves in a Plasma

To investigate the propagation of waves in a plasma we shall use the macroscopic equations for ions and electrons, Eqs. (214) and (215), along with the appropriate continuity equations and Maxwell's equations, Eqs. (252) to (255). We do this rather than use the fluid equations just derived because it will give us an insight into the role played by each species of particle. Also, the procedure used is readily generalized to include more than two species of particles. We shall neglect external forces, and we shall use the linearized versions of these equations assuming no velocity or electric field in the equilibrium. We also take the plasma to be infinite and homogeneous in equilibrium. The equations to be used are the following:

$$N_e m_e \frac{\partial \vec{v}_e}{\partial t} + \nabla \cdot \underline{\Pi}_e + N_e e \vec{E} + \frac{N_e e \vec{v} \times \vec{B}_0}{c} = \left(\frac{\partial \vec{A}}{\partial t}\right)_{i.e.}, \tag{323}$$

$$N_i m_i \frac{\partial \vec{v}_i}{\partial t} + \vec{\nabla} \cdot \underline{\underline{\pi}}_i - N_i z e \vec{E} - \frac{N_i z e \, \vec{v}_i \times \vec{B}_0}{c} = -\left(\frac{\partial \vec{P}}{\partial t}\right)_{ie}, \quad (324)$$

$$\frac{\partial n_e}{\partial t} + N_e \vec{\nabla} \cdot \vec{v}_e = 0, \quad (325)$$

$$\frac{\partial n_i}{\partial t} + N_i \vec{\nabla} \cdot \vec{v}_i = 0, \quad (326)$$

$$\sigma = n_i z e - n_e e, \quad (327)$$

$$\vec{j} = N_i z e \vec{v}_i - N_e e \vec{v}_e, \quad (328)$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \quad (329)$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{j}, \quad (330)$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \sigma, \quad (331)$$

and

$$\vec{\nabla} \cdot \vec{B} = 0. \quad (332)$$

Here N_e, N_i are the equilibrium number densities for ions and electrons and to preserve charge neutrality $N_e = z N_i$; B_o is the equilibrium magnetic field. The perturbed densities, velocities, and stress tensors are n_e, n_i, $\vec{v}_e$, $\vec{v}_i$, $\underline{\underline{\pi}}_e$, $\underline{\underline{\pi}}_i$, while $\left(\frac{\partial \vec{P}}{\partial t}\right)_{ei}$ is the rate of change of the electron momentum due to collisions with ions (a first order quantity). The perturbed electric and magnetic fields are denoted by $\vec{E}$ and $\vec{B}$. To proceed we also need equations to determine the $\underline{\underline{\pi}}_e$'s. To start out, we shall assume they are negligible.

A. Waves in a Cold Plasma

1. No Magnetic Field in the Equilibrium and No Collisions

For this case Eqs. (323) and (324) reduce to

$$N_e m_e \frac{\partial v_e}{\partial t} + N_e e E = 0 \tag{333}$$

and

$$N_i M_i \frac{\partial v_i}{\partial t} - z N_i e E = 0, \tag{334}$$

Combining Eqs. (333) and (334) so as to obtain the time derivative of the current gives

$$\frac{\partial \vec{J}}{\partial t} = + \left[\frac{N_e e^2}{m_e} + \frac{N_i z^2 e^2}{m_i}\right]\vec{E} \cancel{= 0}. \tag{335}$$

Taking the time derivative of Eq. (330) and substituting it into the curl of Eq. (329) gives

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi}{c^2}\frac{\partial \vec{J}}{\partial t} \tag{336}$$

or by making use of vector identities and Eq. (335)

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \frac{4\pi}{c^2}\left[\frac{N_e e^2}{m_e} + \frac{N_i z^2 e^2}{m_i}\right]\vec{E} = 0. \tag{337}$$

If we set

$$4\pi\left[\frac{N_e e^2}{m_e} + \frac{N_i z^2 e^2}{m_i}\right] = \omega_p^2, \tag{338}$$

the electron plasma frequency to order m_e/m_i, then Eq. (337) becomes

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \frac{\omega_p^2}{c^2}\vec{E} = 0. \tag{339}$$

a. Transverse Waves

Let us first consider the case $\vec{\nabla}\cdot\vec{E} = 0$. We see from Eq. (335) that if $\vec{\nabla}\cdot\vec{E}$ is zero then the time derivative of $\vec{\nabla}\cdot\vec{j}$ is zero and hence the second time derivative of σ is zero by $\frac{\partial\sigma}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$. Since $\vec{\nabla}\cdot\vec{E} = 4\pi\sigma$, the second time derivative of $\vec{\nabla}\cdot\vec{E}$ is zero. Thus, if $\vec{\nabla}\cdot\vec{E}$ and $\vec{\nabla}\cdot\vec{j}$ are zero initially, they will remain zero forever, so this is a consistent approximation.

Eq. (339) now reduces to

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \frac{\omega_p^2}{c^2}\vec{E} = 0. \tag{340}$$

If we Fourier-analyze $\vec{E}$ in time and space — i.e., write

$$\vec{E} = \vec{E}_1\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}, \tag{341}$$

Eq. (340) gives

$$\left[-k^2 + \frac{\omega^2 - \omega_p^2}{c^2}\right]\vec{E}_1 = 0. \tag{342}$$

The condition $\vec{\nabla}\cdot\vec{E} = 0$ gives

$$\vec{k}\cdot\vec{E}_1 = 0. \tag{343}$$

Thus these waves are transverse. Eq. (342) gives the dispersion relation

$$k^2c^2 = \omega^2 - \omega_p^2. \tag{344}$$

We see from Eq. (344) that if ω^2 is less than ω_p^2, k^2 is negative. This means that k will be imaginary and hence E will become exponentially large or exponentially small as one goes to ∞ in either the $+\vec{k}$ or $-\vec{k}$ direction. The first is not allowed by energy consideration; the second corresponds to a ~~damped~~ wave going as $e^{-\vec{k}\cdot\vec{r}}$, where

$$k = \frac{\sqrt{\omega_p^2 - \omega^2}}{c}. \tag{345}$$

We may compute the phase velocity of the waves given by Eq. (344). It is

$$V_p^2 = \frac{\omega^2}{k^2} = c^2 \Big/ \left(1 - \frac{\omega_p^2}{\omega^2}\right). \tag{346}$$

The index of refraction $n' = c/v_p$ is

$$n = \left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2}. \tag{347}$$

We see from Eqs. (346) or (347) that for $\omega > \omega_p$ the phase velocity is always greater than c.

Another quantity which is of interest is the group velocity V_g which is given by

$$V_g = \frac{d\omega}{dk}. \tag{348}$$

From Eq. (344) we have

$$2\, k c^2 dk = 2\omega d\omega \tag{349}$$

or

$$\frac{d\omega}{dk} = \frac{k}{\omega} c^2 = \frac{c^2}{V_p} \tag{350}$$

thus

$$V_g = \frac{d\omega}{dk} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}. \tag{351}$$

Thus the group velocity is always less than the velocity of light. The group velocity is usually the velocity of propagation of energy, unless one is in a region of strong anomalous dispersion which usually is associated with strong absorption, or for optically active media, strong stimulated emission.

One other quantity of interest for these waves is the energy density

$$W = \frac{E^2}{8\pi} + \frac{B^2}{8\pi} + \rho\frac{V^2}{2}. \tag{352}$$

From Eq. (329) we have

$$i\frac{\omega}{c}\vec{B} = \vec{k}\times\vec{E}. \tag{353}$$

The magnetic field is ~~90° out of~~ in phase and perpendicular to the electric field. We get

$$\frac{B^2}{8\pi} = \frac{c^2k^2}{\omega^2}\frac{E^2}{8\pi} \tag{354}$$

while from Eq. (333) we have

$$i\omega N_e m_e \vec{V}_e = + N_e e\vec{E}. \tag{355}$$

The electron velocity is also 90° out of phase with the electric field, therefore

$$\vec{V}_e = \mp \frac{ie}{m_e\omega}\vec{E} \tag{356}$$

while from Eq. (334) we have, for the ions,

$$\vec{V}_i = + \frac{iZe\vec{E}}{m_i\omega}. \tag{357}$$

Hence the peak particle kinetic energy

$$\frac{N_e m_e V_e^2 + N_i m_i V_i^2}{2} = \left[\frac{N_e e^2}{m_e} + \frac{N_i Z^2 e^2}{m_i}\right]\frac{E^2}{2\omega^2}. \tag{358}$$

The peak energy density of the magnetic field and particles ~~(which are in phase)~~ is then

$$W_{K+B} = \frac{E^2}{8\pi}\left[\frac{c^2k^2 + \omega_p^2}{\omega^2}\right] = \frac{E^2}{8\pi}. \tag{359}$$

~~and the energy at one time is entirely in the electric field; 90° later it is in the magnetic field and particle kinetic energy.~~

In summary, we have found that waves similar to electromagnetic waves in a vacuum are propagated in a plasma for $\omega > \omega_p$. In the limit as the plasma density goes to zero, the waves go over to the usual electromagnetic waves. For a wave incident on a plasma such that $\omega < \omega_p$, the wave is attenuated.

Problem: Show that the energy flux given by Poynting's Theorem $\frac{c}{4\pi}(\vec{E}\times\vec{B})$ divided by the energy density is equal to the group velocity. This can be thought of as the velocity of propagation of energy.

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b. Longitudinal Oscillations

Let us now consider the case where $\vec{\nabla}\times\vec{E} = \vec{k}\times\vec{E} = 0$; i.e., $\vec{k}$ is parallel to $\vec{E}$. Then Eq. (336) gives, for this case,

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial j}{\partial t} = 0. \tag{360}$$

Fourier-analyzing Eq. (360) and Eq. (335) gives

$$-\omega^2\vec{E} - 4\pi i\omega\vec{j} = 0 \tag{361}$$

and

$$-i\omega\vec{j} - \frac{\omega_p^2}{4\pi}\vec{E} = 0. \tag{362}$$

Eliminating $i\omega\vec{j}$ between Eqs. (361) and (362) gives

$$\left(-\omega^2 + 4\pi\frac{\omega_p^2}{4\pi}\right)\vec{E} = 0 \tag{363}$$

so that the dispersion relation is simply

$$\omega^2 = \omega_p^2. \tag{364}$$

Since $\vec{\nabla}\cdot\vec{E} \neq 0$, there is also an oscillation of charge density that goes as $e^{\pm i\omega_p t}$. From this we see that the component of $\vec{E}_k$ parallel to $\vec{k}$ is determined by σ_k or the longitudinal part of $\vec{E}$ is determined by σ_k. All k's oscillate at this frequency. Thus there is no dependence of ω on k.

The phase velocity of the waves is thus proportional to 1/k and is given by

$$V_p = \frac{\omega}{k} = \frac{\omega_p}{k} \tag{365}$$

and the group velocity is zero.

$$V_g = \frac{d\omega}{dk} = 0. \tag{366}$$

These longitudinal waves do not propagate.

c. <u>The Effect of Collisions</u>

(1) <u>Transverse Waves</u>

We may easily include the effects of electron-ion collisions on the propagation of transverse and longitudinal waves through an infinite homogeneous plasma containing no zero order magnetic field and for which the thermal motions are unimportant. We do this by simply adding a term, $-R(v_e - v_i)$, to the right-hand side of Eq. (333), and $R(v_e - v_i)$ to the right-hand side of Eq. (334). Equivalently, we may substitute

$$\vec{E} - \eta \vec{J} \quad \text{for} \quad \vec{E} \tag{367}$$

in the right-hand side of Eq. (335). Here η is the usual resistivity. Eq. (335) thus becomes

$$\frac{\partial \vec{J}}{\partial t} - \frac{\omega_p^2}{4\pi} \vec{E} = -\frac{\omega_p^2}{4\pi} \eta \vec{J}. \tag{368}$$

Here, as with η equal to zero, we may divide the modes into transverse and longitudinal oscillations. First let us look at transverse oscillations with $\vec{\nabla} \cdot \vec{E} = 0$. Fourier-analyzing Eqs. (336) and (368) gives

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\vec{E} - \frac{4\pi i\omega}{c^2}\vec{j} = 0, \tag{369}$$

$$\vec{k}\cdot\vec{E} = 0, \tag{370}$$

$$-i\omega\vec{j} - \frac{\omega_p^2}{4\pi}\vec{E} = -\frac{\omega_p^2}{4\pi}\eta\vec{j} \tag{371}$$

where everything goes like $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$. Solving Eq. (371) for $\vec{j}$ in terms of $\vec{E}$ and substituting in Eq. (369) gives

$$\left[k^2 - \frac{\omega^2}{c^2} + \frac{4\pi i\omega\omega_p^2}{c^2(4\pi i\omega - \omega_p^2\eta)}\right]\vec{E} = 0 \tag{372}$$

from which we obtain the dispersion relation

$$c^2k^2 = \omega^2 - \frac{4\pi i\omega\omega_p^2}{(4\pi i\omega - \omega_p^2\eta)} \tag{373}$$

or

$$c^2k^2 = \omega^2 - \omega_p^2\left[\frac{(4\pi\omega)^2 - 4\pi\omega i\eta\omega_p^2}{(4\pi\omega)^2 + \omega_p^4\eta^2}\right]. \tag{374}$$

From Eq. (374) we see that k^2 has a positive imaginary part, as shown in Fig. 46. The root k with the positive real part has a positive imaginary part, while the root with the negative real part has a negative imaginary part, as shown.

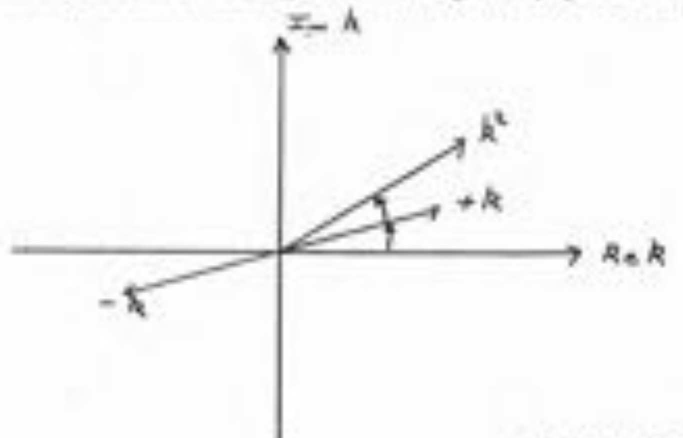


Figure 46

In either case the waves die out as one goes in the $\vec{r}$ direction parallel to k .

Problems: (1) Simplify equation for large and small ω .

(2) If Eq. (374) is solved for ω for given k , k real, show that the imaginary part of ω is always negative.

(2) Longitudinal Waves

Here, as in the case of transverse waves, we may proceed as we did in the case of no resistance. We noted there that $\vec{\nabla} \times \vec{E}$ is zero for this type of oscillation, so that Eq. (336) takes the form

$$\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial \vec{J}}{\partial t} = 0. \tag{375}$$

Eq. (375), along with Eq. (368) for the current, determines these oscillations. Fourier-analyzing Eqs. (375) and (368) gives

$$-\omega^2 \vec{E} - 4\pi i \omega \vec{J} = 0 \tag{376}$$

and

$$\vec{J} = \frac{\omega_p^2 \vec{E}}{(\omega_p^2 \eta - 4\pi i \omega)}. \tag{377}$$

Substitution of Eq. (377) into Eq. (376) gives

$$\left[\omega^2 + \frac{\omega_p^2\, 4\pi i \omega}{(\omega_p^2 \eta - 4\pi i \omega)}\right] \vec{E} = 0 \tag{378}$$

or

$$4\pi i \omega (\omega^2 - \omega_p^2) - \omega^2 \omega_p^2 \eta = 0, \tag{379}$$

$$4\pi i \omega \left[\omega^2 - \omega_p^2 + \frac{i}{4\pi}\omega \omega_p^2 \eta\right] = 0, \tag{380}$$

gives

$$\omega = \frac{1}{2}\left[\frac{i\,\omega_p^2 \eta}{4\pi} \pm \sqrt{\frac{-\omega_p^4 \eta^2}{(4\pi)^2} + 4\omega_p^2}\right] \tag{381}$$

or

$$\omega = \pm\,\omega_p \left(1 - \frac{\omega_p^2 \eta^2}{4^3 \pi^2}\right)^{1/2} - \frac{i\,\omega_p^2 \eta}{8\pi}. \qquad (382)$$

Thus for $\omega_p \eta << 1$, $\omega \sim \omega_p$ and the imaginary part is small compared with the real part. In addition, we must impose the condition that

$$\vec{\nabla} \times \vec{E} = 0 \qquad (383)$$

or

$$\vec{k} \times \vec{E} = 0. \qquad (384)$$

2. Waves in an Infinite Homogeneous Cold Plasma Containing a Uniform Magnetic Field

a. Basic Equations

We now turn to the problems of the propagation of waves through a cold plasma containing a uniform magnetic field. We shall start by neglecting collisions. We obtain our basic equations from Eqs. (323) to (332) by dropping the stress tensor and the collision term. These basic equations are

$$N_e m_e \frac{\partial \vec{v}_e}{\partial t} + N_e e\left[\vec{E} + \frac{\vec{v}_e \times \vec{B}_o}{c}\right] = 0 \qquad (385)$$

and

$$N_i m_i \frac{\partial \vec{v}_i}{\partial t} - N_i Ze\left[\vec{E} + \frac{\vec{v}_i \times \vec{B}}{c}\right] = 0, \qquad (386)$$

plus the equations of continuity and Maxwell's equations.

We shall take $\vec{B}_0$ to be in the z direction. Again, taking the curl of Eq. (252) and substituting into it the time derivative of Eq. (253) gives

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi}{c^2}\frac{\partial \vec{J}}{\partial t}. \qquad (387)$$

Again we look for solutions which go like

$$e^{i(\vec{k}\cdot\vec{r}-\omega t)} . \tag{388}$$

Substituting this form into Eqs. (385) and (386) we obtain

$$-i\omega N_e m_e \vec{v}_e + N_e e \frac{\vec{v}_e \times \vec{B}_o}{c} = -N_e e \vec{E} \tag{389}$$

$$-i\omega N_i m_i \vec{v}_i - N_i z e \frac{\vec{v}_i \times \vec{B}_o}{c} = N_i z e \vec{E}. \tag{390}$$

Writing Eqs. (389) and (390) in component form gives

$$w_e = -\frac{ie}{m_e \omega} E_z, \tag{391}$$

$$w_i = \frac{iZe}{m_i \omega} E_z, \tag{392}$$

$$-i\omega u_e + \omega_{ce} v_e = -\frac{eE_x}{m_e}, \tag{393}$$

$$-i\omega v_e - \omega_{ce} u_e = \frac{-eE_y}{m_e}, \tag{394}$$

$$-i\omega u_i - \omega_{ci} v_i = \frac{ZeE_x}{m_i}, \tag{395}$$

$$-i\omega v_i + \omega_{ci} u_i = \frac{ZeE_y}{m_i} \tag{396}$$

where u, v, and w are the x, y, z components of the velocity and

$$\omega_c = |q| \frac{B_o}{mc}. \tag{397}$$

Solving for u_e, v_e, u_i, and v_i gives

$$u_e = \frac{-e}{m_e} \frac{(i\omega E_x + \omega_{ce} E_y)}{\omega^2 - \omega_{ce}^2}, \tag{398}$$

$$v_e = \frac{-e}{m_e} \frac{(-\omega_{ce} E_x + i\omega E_y)}{\omega^2 - \omega_{ce}^2}, \tag{399}$$

$$u_i = \frac{Ze}{m_i} \frac{(i\omega E_x - \omega_{ci} E_y)}{\omega^2 - \omega_{ci}^2}, \tag{400}$$

and

$$v_i = \frac{Ze}{m_i}\frac{(\omega_{ci}E_x + i\omega E_y)}{\omega^2-\omega_{ci}^2} \tag{401}$$

Fourier-analyzing Eq. (387) and writing it in component form gives

$$\left(k^2-k_x^2-\frac{\omega^2}{c^2}\right)E_x - k_xk_yE_y - k_xk_zE_z = \frac{4\pi i\omega}{c^2}j_x, \tag{402}$$

$$-k_yk_xE_x + \left(k^2-k_y^2-\frac{\omega^2}{c^2}\right)E_y - k_yk_zE_z = \frac{4\pi i\omega}{c^2}j_y, \tag{403}$$

and

$$-k_zk_xE_x - k_zk_yE_y + \left(k^2-k_z^2-\frac{\omega^2}{c^2}\right)E_z = \frac{4\pi i\omega}{c^2}j_z. \tag{404}$$

Summing electron and ion contributions to j, using Eqs. (391), (392), and (398) to (401) for the v's, and inserting these in Eqs. (402) to (404) gives

$$\begin{aligned}&\left[k^2-k_x^2-\frac{\omega^2}{c^2}+\frac{\omega_{pe}^2\omega^2}{c^2(\omega^2-\omega_{ce}^2)}+\frac{\omega_{pi}^2\omega^2}{c^2(\omega^2-\omega_{ci}^2)}\right]E_x\\ &-\left[k_xk_y+\frac{i\omega\,\omega_{ce}\,\omega_{pe}^2}{c^2(\omega^2-\omega_{ce}^2)}-\frac{i\omega\,\omega_{ci}\,\omega_{pi}^2}{c^2(\omega^2-\omega_{ci}^2)}\right]E_y\\ &-k_xk_zE_z=0,\end{aligned} \tag{405}$$

$$\begin{aligned}&-\left[k_xk_y-\frac{i\omega}{c^2}\frac{\omega_{pe}^2\omega_{ce}}{(\omega^2-\omega_{ce}^2)}+\frac{i\omega\,\omega_{pi}^2\omega_{ci}}{c^2(\omega^2-\omega_{ci}^2)}\right]E_x\\ &+\left[k^2-k_y^2-\frac{\omega^2}{c^2}+\frac{\omega_{pe}^2\omega^2}{c^2(\omega^2-\omega_{ce}^2)}+\frac{\omega_{pi}^2\omega^2}{c^2(\omega^2-\omega_{ci}^2)}\right]E_y\\ &-k_yk_zE_z=0,\end{aligned} \tag{406}$$

and

$$-k_zk_xE_x - k_zk_yE_y + \left[k^2-k_z^2-\frac{\omega^2}{c^2}+\frac{\omega_{pe}^2+\omega_{pi}^2}{c^2}\right]E_z = 0. \tag{407}$$

We can write these three equations in tensor notation as

$$\overline{\overline{K}} \cdot \vec{E} = 0 \quad = \quad \underset{\sim}{K} \cdot \vec{E} = 0 \tag{408}$$

where K is the tensor

$$\begin{pmatrix} \left[k^2 - k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2 \omega^2}{c^2(\omega^2 - \omega_{ce}^2)} + \frac{\omega_{pi}^2 \omega^2}{c^2(\omega^2 - \omega_{ci}^2)}\right] & \left[\frac{i\,\omega\,\omega_{ci}\,\omega_{pi}^2}{c^2(\omega^2 - \omega_{ci}^2)} - \frac{i\,\omega\,\omega_{ce}\,\omega_{pe}^2}{c^2(\omega^2 - \omega_{ce}^2)} - k_x k_y\right] & -k_x k_z \\ \left[-\frac{i\,\omega\,\omega_{ci}\,\omega_{pi}^2}{c^2(\omega^2 - \omega_{ci}^2)} + \frac{i\,\omega\,\omega_{ce}\,\omega_{pe}^2}{c^2(\omega^2 - \omega_{ce}^2)} - k_x k_y\right] & \left[k^2 - k_y^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2 \omega^2}{c^2(\omega^2 - \omega_{ce}^2)} + \frac{\omega_{pi}^2 \omega^2}{c^2(\omega^2 - \omega_{ci}^2)}\right] & -k_y k_z \\ -k_x k_z & -k_z k_y & \left[k^2 - k_z^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2 + \omega_{pi}^2}{c^2}\right] \end{pmatrix}$$

and where

$$\omega_{pe}^2 = \frac{4\pi N_e e^2}{m_e} \quad \text{and} \quad \omega_{pi}^2 = \frac{4\pi N_i Z^2 e^2}{m_i}. \tag{409}$$

In order to have a nontrivial solution to Eq. (408) ($\vec{E} \neq 0$) the determinant of K must be zero. Before proceeding we may note that because of cylindrical symmetry about the magnetic field direction or z direction we can choose the xz plane to be the plane determined by $\vec{k}$ and $\vec{B}$. Hence, without loss of generality we may choose $k_y = 0$.

b. <u>Propagation Parallel to B_0</u>

We shall begin our investigation of the roots of

$$\left| \underset{\sim}{K} \right| = 0 \tag{410}$$

by looking at waves propagating in the z direction, $k_x = k_y = 0$. Our dispersion relation is then

$$\begin{vmatrix} A & iB & 0 \\ -iB & A & 0 \\ 0 & 0 & C \end{vmatrix} = 0 = (A+B)(A-B)C \tag{411}$$

where

$$A = k^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2\omega^2}{c^2(\omega^2-\omega_{ce}^2)} + \frac{\omega_{pi}^2\omega^2}{c^2(\omega^2-\omega_{ci}^2)} \tag{412}$$

$$B = -\frac{\omega}{c^2}\left[\frac{\omega_{ce}\omega_{pe}^2}{(\omega^2-\omega_{ce}^2)} - \frac{\omega_{ci}\omega_{pi}^2}{(\omega^2-\omega_{ci}^2)}\right] \tag{413}$$

$$C = \left[-\frac{\omega^2}{c^2} + \frac{\omega_{pe}^2+\omega_{pi}^2}{c^2}\right]. \tag{414}$$

The three solutions of Eq. (411) are obviously

$$A - B = 0, \; A + B = 0, \; C = 0. \tag{415}$$

The root $C = 0$ gives

$$\omega^2 = \omega_{pe}^2 + \omega_{pi}^2 \equiv \omega_p^2 \tag{416}$$

and when this condition ($C = 0$) is inserted in Eq. (408) we find that

$$E_x = E_y = 0, \; E_z \quad \text{is arbitrary.} \tag{417}$$

This oscillation has only an E_z associated with it. Since there is no motion across the field lines, the magnetic field does not influence the motion and the oscillation is the same as we obtained in the absence of a magnetic field. The root $A + B = 0$ applied to Eq. (408) gives

$$A E_x - iA E_y = 0, \; E_x = iE_y \text{ and } E_z = 0, \tag{418}$$

while $A - B = 0$ similarly gives

$$E_x = -iE_y, \; E_z = 0 \tag{419}$$

Thus there is no E_z and E_x and E_y are 90° out of phase with each other. These are transverse waves which propagate along $\overline{B}$ and are right and left circularly polarized. Handedness is here defined with respect to $\overline{k}$. The positive sign always corresponds to rotation about B_o in the negative θ direction. The plus sign corresponds to a left-hand polarized wave for propagation in the $+z$ direction, $k > 0$, and a right-hand polarized wave for propagation in the $-z$ direction, $k < 0$. The reverse is true for the minus sign.

OK

Let us write out the two roots $A \pm B = 0$.

$$\omega^2 - k^2c^2 = \frac{\omega_{pe}^2\omega^2}{\omega^2-\omega_{ce}^2} + \frac{\omega_{pi}^2\omega^2}{\omega^2-\omega_{ci}^2} \pm \omega\left[\frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2} - \frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2}\right]$$

$$= \frac{\omega_{pe}^2\omega(\omega \mp \omega_{ce})}{\omega^2-\omega_{ce}^2} + \frac{\omega_{pi}^2\omega(\omega \pm \omega_{ci})}{\omega^2-\omega_{ci}^2}$$

$$= \frac{\omega_{pe}^2\omega}{\omega \pm \omega_{ce}} + \frac{\omega_{pi}^2\omega}{\omega \mp \omega_{ci}} \qquad (420)$$

Now note that formal substitution of $-\omega$ for $+\omega$ leaves the left side unchanged and merely interchanges $\pm$ with $\mp$. That is, we may drop out the lower sign and merely look for solutions of Eq. (420) for $+$ and $-\omega$. The solutions for $+\omega$ corresponding to the upper sign are for $A + B = 0$, i.e., $E_x = iE_y$, and for $-\omega$ to $A - B = 0$, i.e., $E_x = -iE_y$. So Eq. (420) is now simply

$$k^2c^2 - \omega^2 = \frac{\omega_{pi}^2\omega}{\omega_{ci}-\omega} - \frac{\omega_{pe}^2\omega}{\omega_{ce}+\omega}. \qquad (421)$$

Let us look at the dispersion relation, Eq. (421), in various limits. First let us assume that ω is much smaller than ω_{pi} and ω_{ci}.

Then we can write Eq. (421) in the form

$$\omega^2 - k^2c^2 = \frac{\omega_{pe}^2\,\omega(\omega - \omega_{ce})}{(\omega^2 - \omega_{ce}^2)} + \frac{\omega_{pi}^2\,\omega(\omega + \omega_{ci})}{(\omega^2 - \omega_{ci}^2)}. \tag{422}$$

Dropping the ω^2 in the denominator and rewriting

$$k^2c^2 - \omega^2 = \frac{\omega_{pe}^2\omega^2}{\omega_{ce}^2} + \frac{\omega_{pi}^2\omega^2}{\omega_{ci}^2} - \left[\frac{\omega_{pe}^2\omega_{ce}}{\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega_{ci}^2}\right]\omega. \tag{423}$$

The last two terms drop out

$$\omega\left[\frac{\omega_{pi}^2}{\omega_{ci}} - \frac{\omega_{pe}^2}{\omega_{ce}}\right] = \omega\left[\frac{4\pi N_i Z^2 e^2/m_i}{ZeB/m_i c} - \frac{4\pi N_e e^2/m_e}{eB/m_e c}\right] = 0 \tag{424}$$

so we have for Eq. (423)

$$k^2c^2 - \omega^2 = \omega^2\left[\frac{4\pi N_e e^2/m_e}{e^2B^2/m_e^2c^2} + \frac{4\pi N_i Z^2 e^2/m_i}{e^2B^2Z^2/m_i^2c^2}\right] \tag{425}$$

or

$$k^2c^2 - \omega^2 = \frac{4\pi\omega^2c^2}{B^2}\left[N_e m_e + N_i m_i\right] \tag{426}$$

or

$$k^2 = \frac{\omega^2}{c^2}\left[1 + \frac{4\pi\rho c^2}{B^2}\right]. \tag{427}$$

The quantity in the brackets is the familiar low-frequency dielectric constant we saw back at Eq. (100) and the wave is just the Alfvén wave found earlier (equation (370)) for zero pressure. Since these waves are nondispersive, the group velocity is equal to the phase velocity. Further, both polarizations propagate at the same velocity. This nondispersive property holds only in the limit of $\omega^2 << \omega_{pi}^2$ and $\omega^2 << \omega_{ci}^2$.

Let us return now to Eq. (421). We plot the right and left-hand sides vs ω for a fixed value of k^2. Such a plot is shown in Fig. 47.

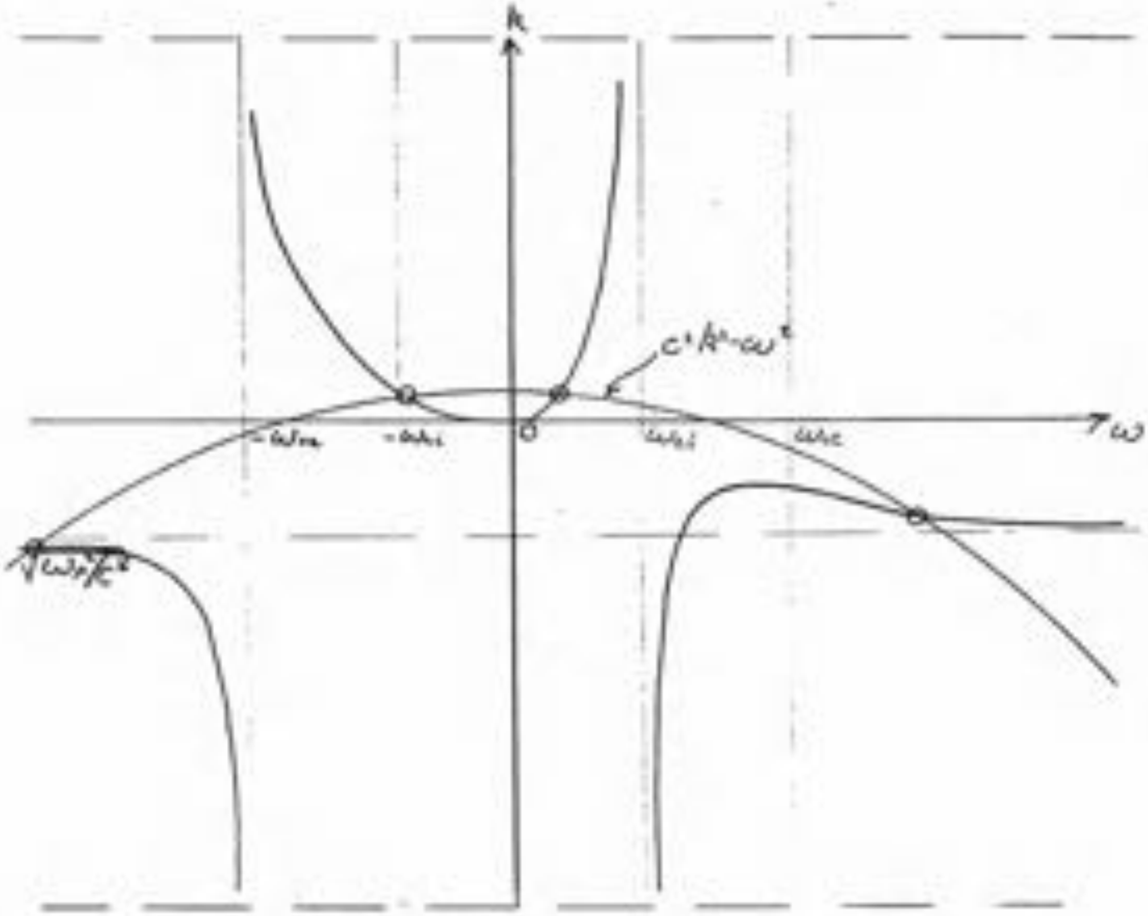


Figure 47

The first thing we note is that Eq. (420) has a singularity at $\omega = \omega_{ci}$ and $\omega = -\omega_{ce}$. Since positive ω corresponds to rotation of $\vec{E}$ about $\vec{B}$ in the negative θ direction, the direction in which ions rotate, it is natural that there should be a singular behavior at $\omega = +\omega_{ci}$. Likewise, since negative ω always corresponds to rotation of $\vec{E}$ about $\vec{B}$ in the positive θ direction, which is the direction in which electrons rotate, it is natural that there should be a singularity at $\omega = -\omega_{ce}$.

Now every place the $c^2k^2 - \omega^2$ curve crosses the curve for the right-hand side of Eq. (420) we obtain a root of the dispersion relation. We note that there are four such crossings for every value of k^2. Thus for a given value of k there are four values of ω which satisfy the dispersion relation. We note that Eq. (420) would give a 4th degree polynomial if it were rationalized, and so this is exactly the number of roots we should find. We may also note that this is the proper number from the point of view of degrees of freedom. The plasma consists of electrons and ions. There is one degree of freedom of the plasma for a given k and polarization per species, which in this case is 2. For each degree of freedom we require two constants to specify its state exactly — a displacement and a velocity. Thus the amplitudes of the four modes found here give four constants which the displacement and velocity of the electrons and ions could be written in terms of.

Now we may look at what happens as we vary k^2 from 0 to ∞. We note that for small k or long wavelength the curve for the right-hand side is parabolic near $\omega = 0$ and that the intersections with $k^2c^2 - \omega^2$ in this region will give $k^2 \alpha \omega^2$. These are just the Alfvén modes we already have found. The other two roots for small k^2c^2 are at

$$\omega = \frac{-(\omega_{ce} - \omega_{ci}) \pm \sqrt{(\omega_{ce} + \omega_{ci})^2 + 4(\omega_{pe}^2 + \omega_{pi}^2)}}{2} \tag{428}$$

If the magnetic field is weak so that ω_{ci} and ω_{ce} are negligible, then this gives $\omega = \omega_p$. These are, however, transverse waves at the plasma frequency.

As k^2 is increased now, one of the Alfvén modes goes to a wave whose frequencies are near to the ion cyclotron frequency, while the other goes into a wave whose frequencies are near the electron cyclotron frequency. Near the cyclotron frequencies there are a great many k's or waves for which the frequency of the waves are nearly the same — the appropriate cyclotron frequency. Situations like this, where ω stays finite and k goes to ∞, are called resonances. We see that for these waves the group velocity is very small

$$V_g = \frac{d\omega}{dk} \tag{429}$$

since ω changes only very slightly while k changes by a very large amount. This is not surprising because these waves correspond to the motion of little elements of the electron or ion fluid circulating at their cyclotron frequency. Since these particles are not moving there is no propagation of these waves. What propagation does take place comes about because a gyrating particle induces motion in its neighbors, but this is a relatively weak effect.

Finally we note that as k^2 goes to ∞, ω is large compared with ω_{ce} and ω_{ci}; the other two roots move out to

$$\omega^2 = c^2k^2 + \omega_p^2. \tag{430}$$

This is the same mode we would have found if the magnetic field were not there.

We have just looked at the roots for ω when k^2 is given. It is also of interest to ask, given an ω, what are the roots for k. Again we make a plot, Fig. 48, which is similar to Fig. 47, but in this case we include curves for several values of k^2.

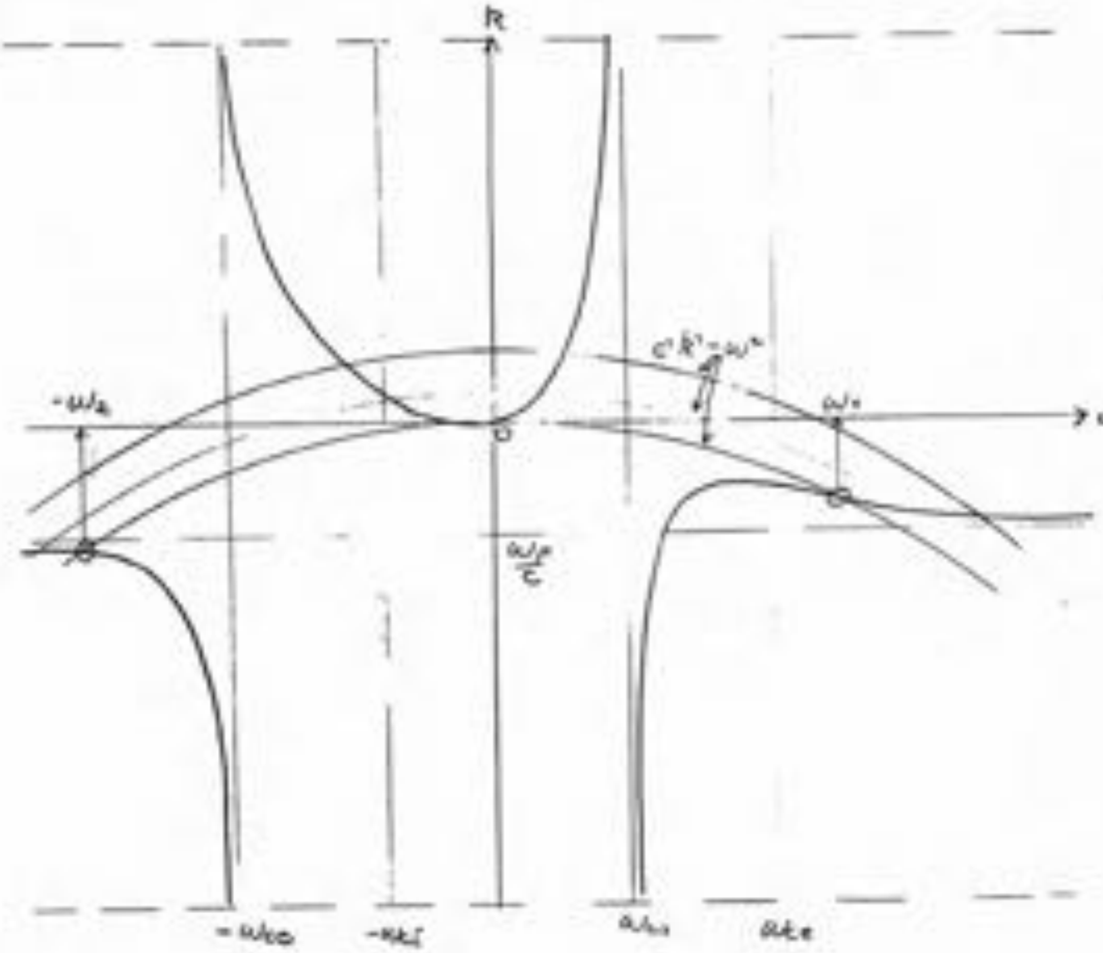


Figure 48

As we change k^2 from plus zero to plus infinity, the point at which the parabola intersects the axis moves from zero to ∞ and all parabolas lie above the one $-\omega^2$. Thus we see that there are no intersections of these curves with the right-hand side for ω between ω_{ci} and ω_1 and $-\omega_{ce}$ and $-\omega_2$. Thus if the frequency lies in either of the regions

$$\omega_{ci} \le \omega \le \omega_1 \tag{431}$$

and

$$-\omega_2 \le \omega \le -\omega_{ce} \tag{432}$$

there is no ~~real~~ positive root for k^2. In fact, we see that we get roots in these regions only if k^2 is negative since these parabolas then lie below $-\omega^2$ and go to $-\infty$ as k^2 goes to $-\infty$. This means that k is purely imaginary. Hence the waves do not propagate for ω lying in either of these regions. Such waves die out in either the plus z or minus z direction, and become exponentially large in the other direction. We can have these waves only if the plasma is bounded in the z direction. Then a wave incident upon the plasma surface, with frequency lying in one of these regions, will give rise to one of these waves which will die out exponentially as one goes further and further into the plasma. Since there is no dissipation here, the wave is reflected at the surface and is simply nonpropagating in the plasma.

c. Propagation Perpendicular to $\vec{B}_0$

We shall now look at propagation perpendicular to $\vec{B}_0$.

We therefore set $k_z = 0$ and $k = k_x$, $k_y = 0$. From Eq. (409) our dispersion relation is now

$$0 = \begin{vmatrix} \left[-\frac{\omega^2}{c^2} + \frac{\omega_{pe}^2\omega^2}{c^2(\omega^2-\omega_{ce}^2)} + \frac{\omega_{pi}^2\omega^2}{c^2(\omega^2-\omega_{ci}^2)}\right] & -\frac{i\omega}{c^2}\left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right] & 0 \\ \frac{i\omega}{c^2}\left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right] & \left[k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2\omega^2}{c^2(\omega^2-\omega_{ce}^2)} + \frac{\omega_{pi}^2\omega^2}{c^2(\omega^2-\omega_{ci}^2)}\right] & 0 \\ 0 & 0 & \left[k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2+\omega_{pi}^2}{c^2}\right] \end{vmatrix}. \tag{433}$$

Again we see that the determinant splits and we get roots for

$$k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2+\omega_{pi}^2}{c^2} = 0 \tag{434}$$

and for

$$\begin{aligned} 0 = &\left[c^2k_x^2 - \omega^2\left\{1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right\}\right]\left[1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right] \\ &+ \left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right]^2. \end{aligned} \tag{435}$$

The modes obtained from Eq. (434) have only an E_z. The E_x and E_y components of the electric field are zero, as can be shown by substituting the solution for ω in $\underset{\sim}{K} \cdot \vec{E} = 0$. Since the $\vec{E}$ field is along z and hence the motion is along z, the magnetic field plays no role in this mode. It is the same transverse mode that we obtain for a uniform plasma without a magnetic field. It propagates only if

$$\omega^2 > \omega_{pe}^2 + \omega_{pi}^2 \tag{436}$$

If Eq. (436) is not satisfied, k_x^2 is negative and the wave does not propagate.

Returning now to Eq. (435), let us first look at the case of small ω. To order ω^2 this equation reduces to

$$c^2 k_x^2 - \omega^2\left[1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} + \frac{\omega_{pi}^2}{\omega_{ci}^2}\right] = 0 \tag{437}$$

or

$$c^2 k^2 = \omega^2\left[1 + \frac{4\pi\rho c^2}{B^2}\right]. \tag{438}$$

These are transverse hydromagnetic waves propagating perpendicular to the magnetic field. They are what we called magnetoacoustic waves earlier. The dispersion relation agrees with Eq. (108) if we put P_{1o} equal to zero and if we neglect 1 compared to $\frac{4\pi\rho c^2}{B^2}$.

To investigate the general case we again resort to a graphical technique. First we solve Eq. (435) for $c^2 k_x^2 - \omega^2$.

$$c^2 k_x^2 - \omega^2 = -\frac{\left(\frac{\omega_{pe}^2 \omega_{ce}}{\omega^2 - \omega_{ce}^2} - \frac{\omega_{pi}^2 \omega_{ci}}{\omega^2 - \omega_{ci}^2}\right)^2}{\left[1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2}\right]} + \omega^2\left[\frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2}\right] \tag{439}$$

Since all terms in ω are quadratic, the plot will be symmetric. A plot of the left- and right-hand sides of Eq. (439) are shown in Fig. 49 (page 179). First we observe that the right-hand side of Eq. (439) has no singularities at $\omega = \pm\omega_{ci}$ or $\omega = \pm\omega_{ce}$. At these values of ω the singularities cancel out. There are, however, singularities where

$$1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2} = 0 \tag{440}$$

or

$$\omega^4 - \omega^2\left[\omega_{ce}^2 + \omega_{ci}^2 + \omega_{pe}^2 + \omega_{pi}^2\right] + \left[\omega_{ce}^2\omega_{ci}^2 + \omega_{pe}^2\omega_{ci}^2 + \omega_{pi}^2\omega_{ce}^2\right] = 0 \tag{441}$$

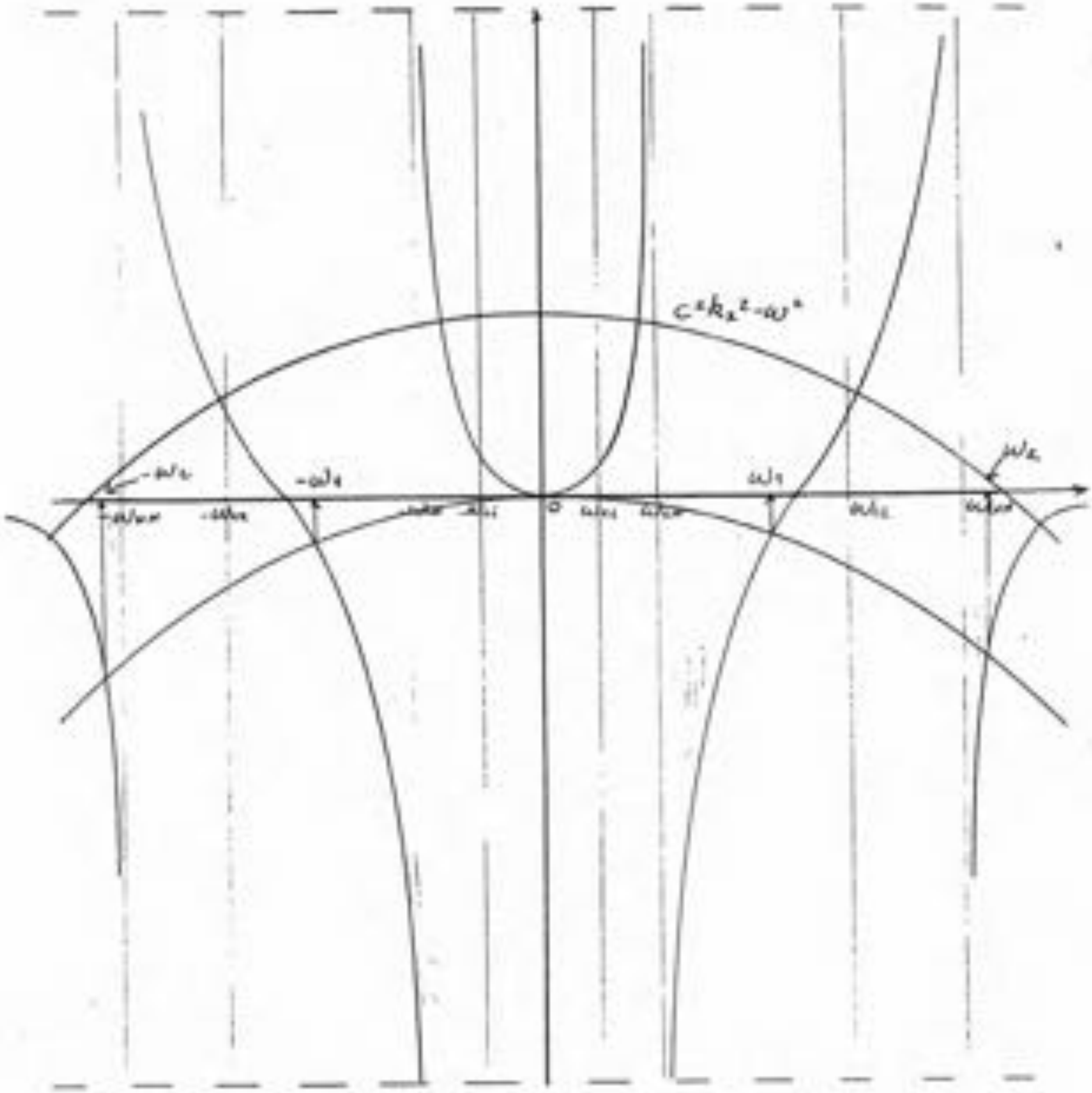


Figure 49

Eq. (441) has, ~~two~~ four roots, one in the vicinity of $\omega^2 = \omega_{ci}^2$ and the other in the vicinity of $\omega^2 = \omega_{ce}^2$. Since $\omega_{ce}^2 >> \omega_{ci}^2$, the lower mode can be found approximately by neglecting ω^2 in $\omega_{pe}^2/\omega^2 - \omega_{ce}^2$. We thus have

$$(\omega^2 - \omega_{ki}^2)\left(1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}\right) - \omega_A^2 = 0 \qquad (442)$$

or

$$\omega^2 = \frac{\omega_{pi}^2}{1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}} + \omega_{ci}^2. \tag{443}$$

For $\omega_{pe} > \omega_{ce}$ this becomes $\omega^2 \sim \omega_{ci}\,\omega_{ce} + \omega_{ci}^2 \sim \omega_{ci}\,\omega_{ce}$.

If we assume that $\omega_{ce}^2 >> \omega_{pi}^2$, then we can approximately find the high-frequency root by neglecting $\omega_{pi}^2/\omega^2 - \omega_{ci}^2$. The upper root is thus given approximately by

$$\omega^2 = \omega_{pe}^2 + \omega_{ce}^2. \tag{444}$$

For this upper mode the ions cannot respond and the electrons are primarily responsible for the oscillation. Both the magnetic field and the electric field produced by charge separation contribute to the restoring force. Thus the natural frequency is not the cyclotron frequency or the plasma frequency, but the combination of them given by Eq. (444).

For the lower frequency mode the natural frequency is no longer the ion cyclotron frequency but depends also on the ion plasma frequency and the electron plasma and cyclotron frequencies. The ion plasma frequency enters because of charge separation due to the ion motion. The electrons are prevented from following because of the magnetic field. However, because of the $\vec{E}$ drift they execute in the x direction they do tend to move with the ions in this direction and this reduces the charge separation. This may also be thought of as a dielectric effect. At these low frequencies the electrons behave like a dielectric medium just as the total plasma did for frequencies low compared to ω_{ci}. The effective dielectric constant is

$$\epsilon = 1 + \frac{4\pi N_e m_e c^2}{B_0^2}. \tag{445}$$

The two resonant frequencies found above are known as the upper and lower hybrid frequencies.

For ω lying in the regions

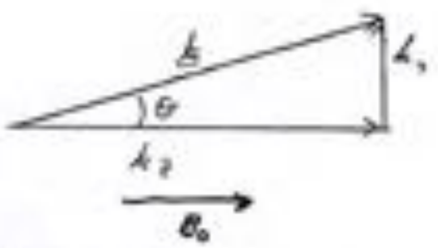
$$|\omega_{LH}| < |\omega| < |\omega_1|, \quad |\omega_{UH}| < |\omega| < |\omega_2| \tag{446}$$

(ω_1 and ω_2 are shown in Fig. 49), there is no positive value of k_x^2 which satisfies the dispersion relation. However, there are negative k^2's which will satisfy it and these represent non-propagating waves.

The modes just found are elliptically polarized in the $y\,x$ plane. They contain both longitudinal and transverse components of $\vec{E}$.

Propagation at an Arbitrary Angle

Use the magnitude of $\underline{k}$ and the angle with respect to B



$k_x = k \sin\theta,\ k_z = k\cos\theta,\ k_y = 0$

The equations are

$$\left(k^2\cos^2\theta - \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right]\right)E_x - \frac{i\omega}{c^2}\left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right]E_y - k^2\sin\theta\cos\theta E_z = 0$$

$$\frac{i\omega}{c^2}\left[-\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right]E_x + \left[k^2 - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right)\right]E_y = 0 -$$

$$- k^2\sin\theta\cos\theta E_x + \left[k^2\sin^2\theta - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2 + \omega_{pi}^2}{c^2}\right]E_z = 0$$

Let us first look at the low frequency limit

$$\begin{vmatrix} k^2c^2\cos^2\theta - \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right) & 0 & -c^2k^2\sin\theta\cos\theta \\ 0 & c^2k^2 - \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right) & 0 \\ -c^2k^2\sin\theta\cos\theta & 0 & c^2k^2\sin^2\theta - \omega^2 + \omega_{pe}^2 + \omega_{pi}^2 \end{vmatrix} = 0$$

Again the determinant splits into two factors

$$k^2c^2 - \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right) = 0$$

$$\left[k^2c^2\cos^2\theta - \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right)\right]\left[k^2c^2\sin^2\theta - \omega^2 + \omega_{pe}^2 + \omega_{pi}^2\right]$$

$$-k^4c^4\sin^2\theta\cos^2\theta = 0$$

$\cancel{k^2c^2\sin^2\theta\cos^2\theta = 0}$

The first wave has only an E_y component and propagates at an arbitrary angle to B

$$\frac{k^2c^2}{\left(1 + \frac{4\pi\rho c^2}{B^2}\right)} = \omega^2 \qquad\qquad k^2 V_A{}^2 = \omega^2$$

The motion is $\underline{E} \times \underline{B}$ and therefore has only an x component. It is a mixture of a transverse and a longitudinal mode. If pressure times had been included, the dispersion would have been modified from the alfven waves. For the second case

$$k^2c^2\left[(\omega^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2)\cos^2\theta + \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right)\sin^2\theta\right] -$$

$$-\ \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right)(\omega^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2) = 0$$

Since $\omega_{pe}{}^2$ is generally quite large, we get approximately

$$k^2c^2\cos^2\theta - \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right) = 0$$

$$k^2 V_A{}^2\cos^2\theta = \omega^2$$

$$k_z{}^2 V_A{}^2 = \omega^2$$

$$V_g = \nabla_{\underline{k}}\omega = \underline{e}_x \frac{\partial\omega}{\partial k_x} + \underline{e}_z \frac{\partial\omega}{\partial k_z} = \underline{e}_z V_A$$

The waves propagate only along z

$$\omega_{pe} \to \infty \quad \text{requires} \quad E_z = 0$$

E_x arbitrary $E_y = 0$

motion which is in the $\underset{\sim}{E} \times \underset{\sim}{B}$ direction is only in the y direction. Each xz plane oscillates independently of every other xz plane.

If we do not consider $\omega_{pe}{}^2$ to be infinitely large then

$$k^2c^2 = \frac{\omega^2(1 + \frac{4\pi\rho c^2}{B^2})\ (\omega^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2)}{(\omega^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2)\cos^2\theta + \omega^2(1 + \frac{4\pi\rho c^2}{B^2})\sin^2\theta}$$

Get a resonance when, ie $k \to \infty$

$$(\omega^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2)\cos^2\theta + \omega^2(1 + \frac{4\pi\rho c^2}{B^2})\sin^2\theta = 0$$

$$\text{or}\quad \cot^2\theta = \frac{\omega^2(1 + \frac{4\pi\rho c^2}{B^2})}{\omega_{pe}{}^2 + \omega_{pi}{}^2 - \omega^2} \doteq \frac{\omega^2}{\omega_{pe}{}^2}\ (1 + \frac{4\pi\rho c^2}{B^2}) \ll 1$$

$$\theta = \frac{\pi}{2} - \delta\theta$$

$$\delta\theta^2 = \frac{\omega^2}{\omega_{pe}{}^2}(1 + \frac{4\pi\rho c^2}{B^2})$$

$\omega_{ce} \to \infty$, ions infinitely heavy

$$(k^2\cos^2\theta - \frac{\omega^2}{c^2})E_x \quad 0E_y \quad -k\sin\theta\cos\theta E_z = 0$$

$$0\ E_x \quad (k^2 - \frac{\omega^2}{c^2})E_y \quad 0E_z = 0$$

$$-k^2\sin\theta\cos\theta E_x \qquad 0 \qquad (k^2\sin^2\theta\frac{\omega^2}{c^2} + \frac{\omega_{pe}^2}{c^2})E_z = 0$$

$$(k^2 - \frac{\omega^2}{c^2})E_y = 0$$

$$k^2c^2 = \omega^2 \qquad E_z \text{ arbitrary} \qquad E_y = 0$$

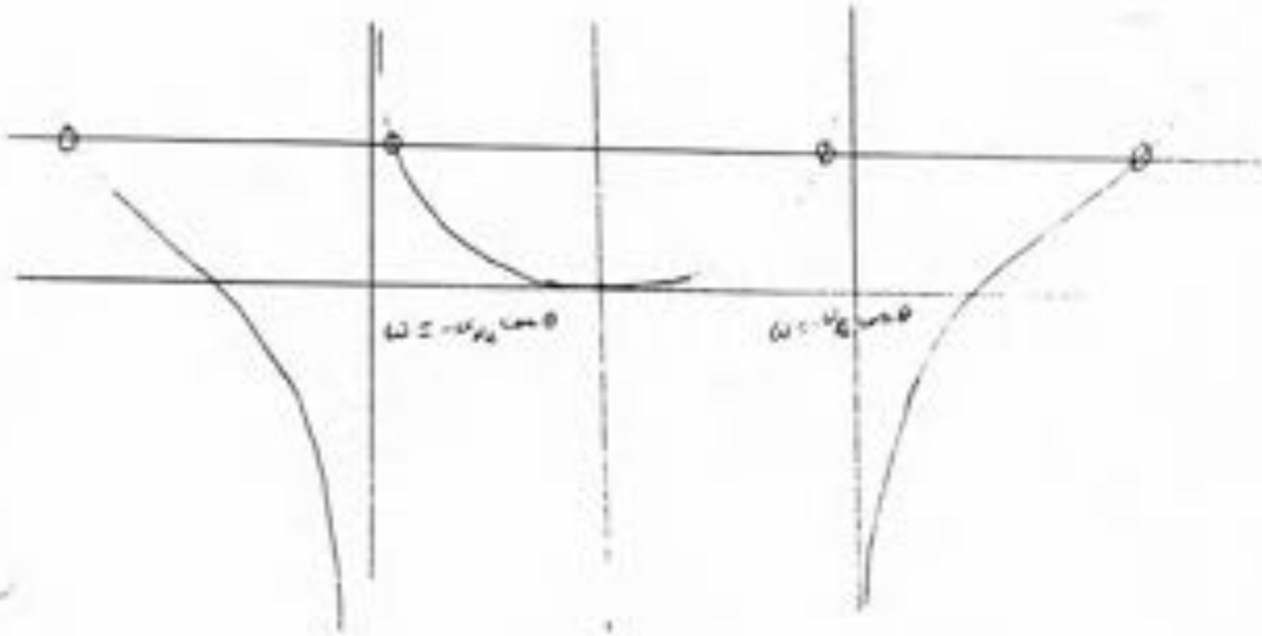
$$(c^2k^2\cos^2\theta - \omega^2)\ (c^2k^2\sin^2\theta - \omega^2 + \omega_{pe}^2) - k^4c^4\sin^2\theta\cos^2\theta = 0$$

$$-k^2c^2\ \omega^2 + k^2c^2(\omega_{pe}^2\cos^2\theta) + \omega^2(\omega^2 - \omega_{pe}^2) = 0$$

$$k^2c^2 = \frac{\omega^2(\omega^2 - \omega_{pe}^2)}{\omega^2 - \omega_{pe}^2\cos^2\theta} = \frac{\omega^2(1 - \frac{\omega_{pe}^2}{\omega^2})}{1 - \frac{\omega_{pe}^2}{\omega^2}\cos^2\theta}$$

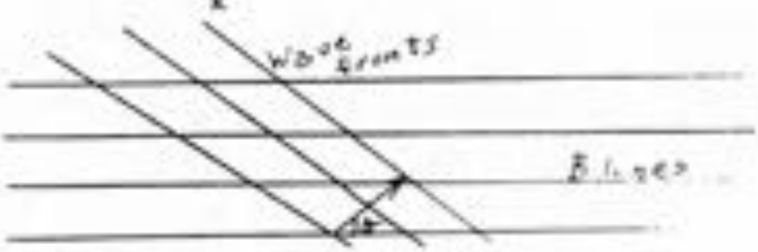
$$k^2c^2 = \frac{\omega^2(1 - \frac{\omega_{pe}^2}{\omega^2})}{1 - \frac{\omega_{pe}^2}{\omega^2}\cos^2\theta}$$

Plot the left and right hand sides



Resonance at $\omega = \omega_{pe}\cos\theta$

$\cos\theta = \frac{k_z}{k}$



This mode is electrostatic but only the component of E || to B is effective so the effective restoring force is $E\,\frac{k_z}{k}$ and the effective inertia is

$\frac{m}{\cos\theta} = \frac{mk}{k_z}$ $\quad \frac{mk}{k_z}\,Z = \frac{-eEk_z}{k}$

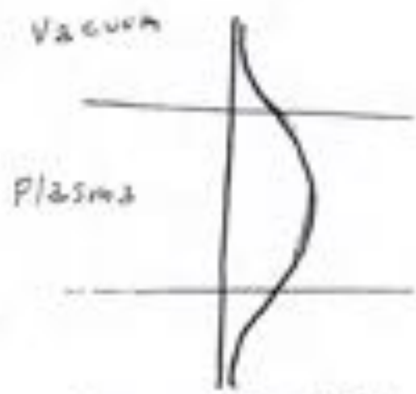
$$\ddot{z} = -\omega_p^2 \frac{k_z^2}{k^2} z$$

For the ~~other~~ this root

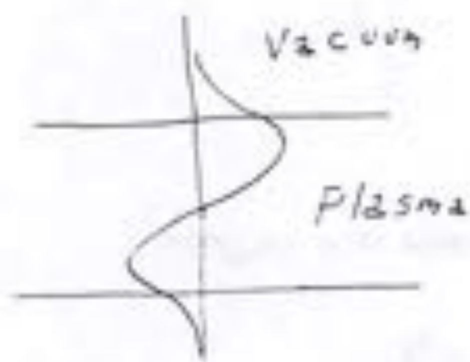
$$k^2c^2 > \omega^2$$

$$\frac{\omega^2}{k^2} = V_p < c^2$$

Therefore, the phase velocity is less than the velocity of light. This wave will remain in the plasma slab



Fundamental Mode



Ion cyclotron waves, zero mass electron, zero mass electrons implies $E_z = 0$. Also, we can neglect $\frac{\omega_{pe}^2 \omega_{ce}}{-\omega_{ce}^2}$

compared to $\frac{\omega_{pi}^2 \omega_{ci}}{\omega^2 - \omega_{ci}}$ since the last term is near a resonance

$$\begin{vmatrix} k^2\cos^2\theta - \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{pi}^{\;2}}{\omega^2 - \omega_{ci}^{\;2}}\right] & -\frac{i\omega\,\omega_{pi}^{\;2}\,\omega_{ci}}{\omega^2 - \omega_{ci}^{\;2}} \\ \frac{i\omega\omega_{pi}^{\;2}\omega_{ci}}{c(\omega^2 - \omega_{ci}^{\;2})} & k^2 - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{pi}^{\;2}}{\omega^2 - \omega_{ci}^{\;2}}\right) \end{vmatrix} = 0$$

$$[c^2k^2\cos^2\theta - \omega_{ci}^{\;2}\left(1 - \frac{\omega_{pi}^{\;2}}{\omega^2 - \omega_{ci}^{\;2}}\right)]\,[k^2 - \omega_{ci}^{\;2}\left(1 - \frac{\omega_{pi}^{\;2}}{\omega^2 - \omega_{ci}^{\;2}}\right)] -$$

$$-\frac{\omega_{ci}^{\;4}\,\omega_{pi}^{\;4}}{(\omega^2 - \omega_{ci}^{\;2})^2} = 0$$

$$k^4c^4\cos^2\theta - k^2c^2\omega_{ci}^{\;2}\left(1 - \frac{\omega_{pi}^{\;2}}{\omega^2 - \omega_{ci}^{\;2}}\right)(1 + \cos^2\theta) + \omega_{ci}^{\;4}$$

$$+\frac{4\omega_{ci}^{\;4}\,\omega_{pi}^{\;4}}{\omega^2 - \omega_{ci}^{\;2}} = 0$$

$$\frac{\omega_{ci}^{\;4}\,\omega_{pi}^{\;4}}{\omega^2 - \omega_{ci}^{\;2}}\left[4 + \frac{k^2c^2}{\omega_{ci}^{\;2}\,\omega_{pi}^{\;2}}\,(1 + \cos^2\theta)\right] = k^2c^2\,\omega_{ci}^{\;2}\,(1 + \cos^2\theta)$$

$$- k^4c^4\cos^2\theta - \omega_{ci}^{\;4}$$

$$\omega - \omega_{ci} = \frac{\frac{\omega_{ci}^3}{2}\,\omega_{pi}^4\,[4 + \frac{k^2c^2}{\omega_{ci}^2\,\omega_{pi}^2}\,(1 + \cos^2\theta)]}{k^2c^2\,\omega_{ci}^2\,(1 + \cos^2\theta) - k^4c^4\cos^2\theta - \omega_{ci}^2}$$

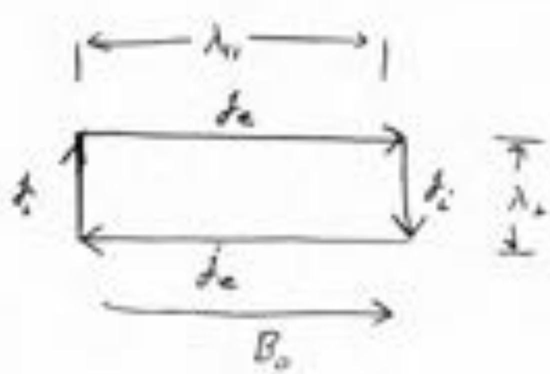
In general, $k^2c^2 \gg \omega_{ci}^2$

$$\omega = \omega_{ci} - \frac{\frac{\omega_{ci}\,\omega_{pi}^2}{2}\,(1 + \cos^2\theta)}{k^2c^2\cos^2\theta}$$

$$\omega = \omega_{ci} - \frac{\omega_{ci}\,\omega_{pi}^2}{2k^2c^2}\,\frac{(1 + \cos^2\theta)}{\cos^2\theta} = \omega_{ci}\,(1 - \frac{\omega_{pi}^2}{2k^2c^2\cos^2\theta} - \frac{\omega_{pi}^2}{2k^2c^2})$$

agrees with Stix.

Can get this answer from a simple model in which ion current flows across B and electron current flows along B to close the circuit. E_z must be zero because of zero mass electrons. The ion motions are produced by a combined space charge field and an inductive field which add up to give $E_z = 0$.



B. Thermal Effects and Landau Damping

1. Use of a Pressure - No Magnetic Field

The problem of the effects of the thermal motions of particles on the propagation of waves through a plasma is one of vast scope and we cannot hope to treat it adequately here. However, we can get some idea of how they affect wave propagation by looking at their effects on the simplest case — wave propagation in an infinite homogeneous plasma containing no magnetic field.

Let us first adopt a fluid approach to this problem and use an adiabatic law to relate the pressure to the density,

$$\frac{p}{n^{\gamma}} = \frac{p_0}{n_0^{\gamma_0}} \tag{695}$$ *

The linearized equations of motion for the electrons are

$$N_e m_e \frac{\partial \vec{v}_e}{\partial t} = -N_e e \vec{E} - \vec{\nabla} p_e \tag{696}$$

and

$$\frac{\partial n_e}{\partial t} + N_e \vec{\nabla} \cdot \vec{v}_e = 0 \tag{697}$$

while those for the ions are

$$N_i m_i \frac{\partial \vec{v}_i}{\partial t} = -N_i Z e \vec{E} - \vec{\nabla} p_i \tag{698}$$

and

$$\frac{\partial n_i}{\partial t} + N_i \vec{\nabla} \cdot \vec{v}_i = 0. \tag{699}$$

Using our pressure law we have

$$p = p_0 \left(\frac{n}{n_0}\right)^{\gamma}, \tag{700}$$

$$\delta p = \gamma \frac{p_0}{n_0} \delta n, \tag{701}$$

$$\vec{\nabla} p = \frac{\gamma p_0}{n_0} \vec{\nabla} n, \quad \text{or} \tag{702}$$

---

* Note: Equation numbers (457) through (694) have been omitted.

$$\vec{\nabla} p_e = \frac{\gamma P_{0e}}{N_e} \vec{\nabla} n_e \qquad (703)$$

and

$$\vec{\nabla} p_i = \frac{\gamma P_{0i}}{N_i} \vec{\nabla} n_i \qquad (704)$$

[Small p is the perturbation in pressure.]

Now from Eqs. (336), (327), and (328) we have

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2 \vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial \vec{J}}{\partial t} = 0 \qquad (705)$$

where also

$$\sigma = n_i Z e - n_e e, \qquad (706)$$

$$\vec{J} = N_i Z e \vec{v}_i - N_e e \vec{v}_e, \qquad (707)$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\sigma, \qquad (708)$$

and

$$\frac{\partial \sigma}{\partial t} = -\vec{\nabla}\cdot\vec{J}. \qquad (709)$$

First let us look at transverse waves. Since $\vec{J}$ is perpendicular to $\vec{k}$, we assume that $\vec{\nabla}\cdot\vec{E} = 0$. By taking the divergence of Eq. (696) we obtain

$$N_e m_e \frac{\partial}{\partial t} \vec{\nabla}\cdot\vec{v}_e = -\nabla^2 p_e. \qquad (710)$$

Similarly from (698)

$$N_i m_i \frac{\partial}{\partial t} \vec{\nabla}\cdot\vec{v}_i = -\nabla^2 p_i. \qquad (711)$$

Now if p_e and p_i are zero, then the time derivatives of $\vec{\nabla}\cdot\vec{v}_e$ and $\vec{\nabla}\cdot\vec{v}_i$ are zero and hence by the continuity equations for n_e, n_i, and σ, the second derivatives of these quantities are zero. Hence if they and their first derivatives are zero, they will remain zero for a short interval of time. But, if these are zero the second time derivatives of P_e, P_i and σ are also zero, and hence they do not develop in time. Thus we get a consistent solution by setting

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{712}$$

$$n_e = n_i = 0, \tag{713}$$

and

$$p_e = p_i = 0. \tag{714}$$

Thus the pressure plays no role in the propagation of these modes and we get the same solution that we obtained earlier in Eq. (344).

2. Phase Mixing and Landau Damping

a. A Model For Landau Damping

We just investigated the effects of the thermal motions of the particles of a plasma by using a pressure. If collisions are very frequent then this approach is justified. However, in a hot plasma collisional effects are often weak, particularly when dealing with the propagation of high-frequency waves. To find the thermal effects here we must return to the Boltzmann equation with the collision term set equal to zero. However, we must keep the self-consistent fields. Before doing this we shall investigate a simplified model which, however, makes much of the physics of what we shall find clear.

Our model is the following. We consider a number of streams of electrons flowing through each other and through a fixed neutralizing background. We take the streams to be infinite in extent and we take them all to be flowing in the x direction. Each stream has a different x velocity and in the undisturbed state all the electrons within a stream move with a uniform velocity (no random motion within a stream). We assume that the only force acting on the electrons is that due to the self-consistent electric field. We look for longitudinal waves propagating in the x direction (E parallel to x). The linearized equations of motion for the σ th beam are

$$\frac{\partial v_\sigma}{\partial t} + V_\sigma \frac{\partial v_\sigma}{\partial x} = -\frac{eE}{m_\sigma} \tag{715}$$

$$\frac{\partial n_\sigma}{\partial t} + V_\sigma \frac{\partial n_\sigma}{\partial t} + N_\circ \frac{\partial v_\sigma}{\partial x} = 0 \tag{716}$$

Here v_σ and n_σ are the perturbations in the velocity and number density of the particles in the σ th beam, while V_σ and N_σ are the corresponding unperturbed quantities. The electric field $\vec{E}$ is determined from Poisson's equation,

$$\frac{\partial E}{\partial x} = -4\pi e \sum_\sigma n_\sigma, \tag{717}$$

We now look for wave solutions where all quantitied vary like

$$e^{i(\omega t - kx)}. \tag{718}$$

Substituting into Eqs. (715) - (717) gives

$$i(\omega - kV_\sigma)v_\sigma = -\frac{eE}{m} \tag{719}$$

$$i(\omega - kV_\sigma)n_\sigma + ikN_\sigma v_\sigma = 0, \tag{720}$$

and

$$-ikE = -4\pi e \sum_\sigma n_\sigma. \tag{721}$$

Solving Eq. (719) for v_σ in terms of E gives

$$v_\sigma = \frac{ieE}{m(\omega - kV_\sigma)}. \tag{722}$$

Substituting Eq. (722) into Eq. (720) and solving for n_σ gives

$$n_\sigma = \frac{-ikeEN_\sigma}{m(\omega - kV_\sigma)^2} \tag{723}$$

Finally, substituting Eq. (723) into Eq. (721) gives

$$E = \frac{4\pi e^2}{m} E \sum_\sigma \frac{N_\sigma}{(\omega - kV_\sigma)^2} \tag{724}$$

or

$$1 = \frac{4\pi e^2}{m} \sum_\sigma \frac{N_\sigma}{(\omega - kV_\sigma)^2}. \tag{725}$$

This is the dispersion relation which ω and k must satisfy. If we plot the left and right-hand sides of Eq. (725) against ω for fixed k, we get a diagram like that shown in Fig. 51.

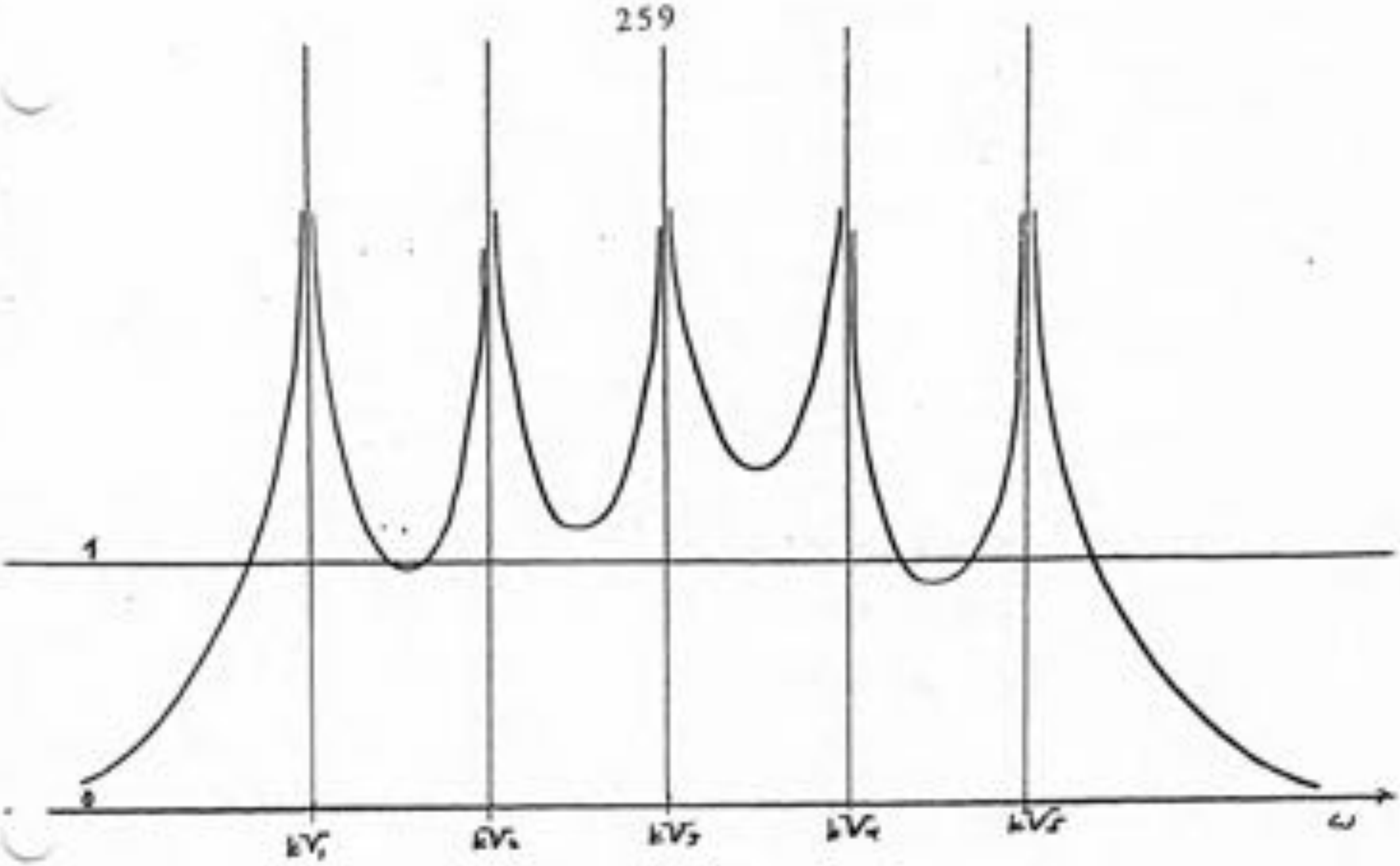


Figure 51

The sum on the right-hand side becomes infinite every time ω takes on the value of one of the kV_σ . Each of the points at which the curve of the right-hand side of Eq. (725) crosses 1 is a real root of Eq. (725). These crossings give all the real roots of Eq. (725). However, in general there are also complex roots or ω's which solve Eq. (725). There are, in fact, twice as many roots to Eq. (725) as there are streams. This is most quickly seen by writing Eq. (725) in polynomial form. One gets a polynomial of degree 2 M (M = number of beams).

At this point the amplitude of the modes is arbitrary. Since the equations are linear, any multiple of a solution is also a solution. We shall specify the solution we are considering by setting

$$E = \frac{4\pi e i}{k}. \tag{726}$$

Then any solution can be obtained by multiplying by an arbitrary constant.

Each of the ω's which satisfies the dispersion relation gives a possible mode of oscillation of the system for the given k. The system has one longitudinal degree of freedom per beam for fixed k. It takes two constants to specify the state of a beam for fixed k, the amplitudes of n_k and v_k. Thus to specify the state of M beams requires 2 M constants. The 2 M amplitudes of the normal modes we have found supply just this number of constants, so we expect this to be a complete set of normal modes.

The modes we have just found satisfy an orthogonality relation. Since orthogonality between the various k's is standard, we shall concern ourselves only with the orthogonality between modes with various ω's for fixed k. Let ω and ω' be two solutions of the dispersion relation, Eq. (725), for fixed k. Let $n_\sigma, n_\sigma', v_\sigma$, and v_σ' be the corresponding values of the n_σ's and v_σ's. Now return to Eqs. (719) - (721) and eliminate v_σ from Eq. (719) by using Eq. (720) and E from Eq. (719) by using Eq. (721).

Thus we obtain

$$(\omega - kV_\sigma)^2 n_\sigma = \frac{4\pi e^2}{m} N_\sigma \sum_\mu n_\mu . \tag{727}$$

Now multiply Eq. (727) by the normalized perturbed number density

$$n_\sigma'/N_\sigma \tag{728}$$

and Eq. (727) with prime quantities by

$$n_\sigma/N_\sigma . \tag{729}$$

Subtracting the second of these results from the first gives

$$\left[(\omega - kV_\sigma)^2 - (\omega' - kV_\sigma)^2\right] \frac{n_\sigma n_\sigma'}{N_\sigma} = \frac{4\pi e^2}{m} \sum_\mu (n_\sigma' n_\mu - n_\mu' n_\sigma) . \tag{730}$$

If we now sum this over σ the right-hand side gives zero, as can be seen by interchanging the dummy variables σ and μ . We thus obtain

$$0 = (\omega - \omega') \sum_\sigma \left[\omega + \omega' - 2kV_\sigma\right] \frac{n_\sigma n_\sigma'}{N_\sigma} . \tag{731}$$

If $\omega \neq \omega'$, then

$$\sum_\sigma \left[\omega + \omega' - 2kV_\sigma\right] \frac{n_\sigma n_\sigma'}{N_\sigma} = 0 \tag{732}$$

while if $\omega = \omega'$ then Eq. (732) need not be true and we may set

$$\sum_\sigma (\omega - kV_\sigma) \frac{n_\sigma^2}{N_\sigma} = H(\omega, k), \tag{733}$$

or from Eqs. (726) and (723) we have

$$n_\sigma(\omega, k) = \frac{4\pi e^2}{m} \frac{N_\sigma}{(\omega - kV_\sigma)^2} . \tag{734}$$

Substituting this in Eq. (733) gives

$$H(\omega, k) = \left[\frac{4\pi e^2}{m}\right]^2 \sum_\sigma \frac{N_\sigma}{(\omega - kV_\sigma)^3}. \tag{735}$$

We shall also find the expression for $\nu_\sigma(\omega, k)$ useful. This is obtained by substituting Eq. (734) into Eq. (720)

$$\nu_\sigma(\omega, k) = \frac{4\pi e^2}{mk}\frac{1}{\omega - kV_\sigma}. \tag{736}$$

We may use Eqs. (732), (733), (735), and (736) to solve the general initial value problem. Again, since the Fourier analysis in k is straightforward, we restrict ourselves to a single k. Let the amplitudes of the k^{th} Fourier components of n_σ and ν_σ at t = 0 be $n_\sigma(k)$ and $\nu_\sigma(k)$. The n_σ's and ν_σ's may be expanded in terms of the normal modes so that we may write

$$n_\sigma(k, t) = \sum_\omega C(\omega, k)\, n_\sigma(\omega, k)\, e^{i\omega t}, \tag{737}$$

$$\nu_\sigma(k, t) = \sum_\omega C(\omega, k)\, \nu_\sigma(\omega, k)\, e^{i\omega t}, \tag{738}$$

and

$$E(k, t) = \sum_\omega \frac{i4\pi e}{k} C(\omega, k)\, e^{i\omega t}, \tag{739}$$

or using Eqs. (734) and (736), Eqs. (737) and (738) become

$$n_\sigma(k, x, t) = \frac{4\pi e^2}{m}\sum_\omega \frac{C(\omega, k)}{(\omega - kV_\sigma)^2} N_\sigma e^{i(\omega t - kx)} \tag{740}$$

$$\nu_\sigma(k, x, t) = \frac{4\pi e^2}{mk}\sum_\omega \frac{C(\omega, k)}{(\omega - kV_\sigma)^2} e^{i(\omega t - kx)}. \tag{741}$$

Here the sum is over all ω's which satisfy the dispersion relation, Eq. (725), for the given k. For time $t = 0$, Eqs. (740) and (741) become

$$n_\sigma(k) = \frac{4\pi e^2}{m} \sum_\omega \frac{C(\omega,k)\, N_\sigma}{(\omega - kV_\sigma)^2}, \tag{742}$$

$$v_\sigma(k) = \frac{4\pi e^2}{mk} \sum_\omega \frac{C(\omega,k)}{\omega - kV_\sigma}. \tag{743}$$

If Eq. (742) is multiplied by $(\omega' - kV_\sigma)\frac{n_\sigma}{N_\sigma}(\omega',k)$ or

$$\frac{4\pi e^2}{m} \frac{(\omega' - kV_\sigma)}{(\omega' - kV_\sigma)^2} = (\omega' - kV_\sigma)\, \frac{n_\sigma(\omega',k)}{N_\sigma} \tag{744}$$

and if Eq. (743) is multiplied by $k\, n_\sigma(\omega',k)$ or

$$\frac{4\pi e^2}{m} \frac{k N_\sigma}{(\omega' - kV_\sigma)^2} = k\, n_\sigma(\omega',k) \tag{745}$$

and if the two expressions are added and summed over σ, we obtain

$$\begin{aligned} &\frac{4\pi e^2}{m} \sum_\sigma \left\{ \frac{N_\sigma(k)}{(\omega' - kV_\sigma)} + \frac{k\, v_\sigma(k)}{(\omega' - kV_\sigma)^2} \right\} \\ &= \left(\frac{4\pi e^2}{m}\right)^2 \sum_\sigma \sum_\omega \frac{C(\omega,k)\,(\omega + \omega' - 2kV_\sigma)\, N_\sigma(\omega,k)}{(\omega - kV_\sigma)^2\,(\omega' - kV_\sigma)^2} \\ &= 2\, H(\omega',k)\, C(\omega',k) \end{aligned} \tag{746}$$

or

$$C(\omega',k) = \frac{\frac{4\pi e^2}{m} \sum_\sigma \left\{ \frac{n_\sigma(k)}{\omega' - kV_\sigma} + \frac{k\, v_\sigma(k)}{(\omega' - kV_\sigma)^2} \right\}}{2\, H(\omega',k)} \tag{747}$$

Thus we have found the C's in terms of the initial state of the beams.

Phase Mixing

On the basis of what we have just done we may form the following physical picture of how an initial disturbance will develop in time. In general an initial perturbation will contain all possible modes. The amplitude of each mode will depend on the details of the initial perturbation. These modes will start out more or less in phase, so as to add up to the initial disturbance. However, all the modes have different frequencies, so that as time goes on they will get out of phase with each other. Thus as time goes on they will no longer add coherently, and so all coherent effects will die out. The electric field is the sum total of the electric fields due to all the waves, and hence is a coherent effect. Thus as time goes on it will die out or appear damped. This effect is called phase mixing and the damping is often called Landau damping. If there are instabilities with appreciable growth rates, then we must qualify the above statement because the growing E^* field due to the unstable modes will ultimately dominate the picture.

These arguments can be developed quantitatively by treating the initial value problem in the limit of a great many beams (strictly speaking, an infinite number of beams which approximates a continuous distribution). Rather than carry out this rather tedious limiting process we will return to the beginning and proceed in the more conventional way by using Laplace transform methods on the Vlasov equation. The conventional method has the advantage that the mathematics is more mechanical, while the above gives us a better insight into the physics of what is going on.

b. Conventional Treatment of Landau Damping

We now take up the more conventional treatment of Landau damping. We shall still restrict ourselves to an infinite homogeneous plasma with a fixed uniform background of ions. We shall look at longitudinal waves propagating in the x direction. We shall assume that magnetic effects are negligible and that the electric field can be determined from Poisson's equation. Since the motions perpendicular to the x direction play no role in the oscillations, we may neglect them and reduce the problem to a one-dimensional one. Our basic equations are the linearized collisionless Boltzmann equation for the electron and Poisson's equation.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f_0}{\partial v} = 0, \tag{748}$$

$$\frac{\partial E}{\partial x} = -4\pi e\int f\,dv. \tag{749}$$

Again we Fourier-analyze in x space, and Eqs. (748) and (749) become

$$\frac{\partial f_k}{\partial t} + ikvf_k - \frac{eE_k}{m}\frac{\partial f_0}{\partial v} = 0, \tag{750}$$

$$ikE_k = -4\pi e\int f_k\,dv. \tag{751}$$

We shall drop the subscript k from now on and understand that we are talking about a single k component.

We now Laplace transform these equations. The Laplace transform of f(t) is defined by

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt. \tag{752}$$

$$s = \sigma + i\omega \qquad \sigma \geq 0 \tag{753}$$

$Re\,\sigma$ must be sufficiently large so that Eq. (752) converges. The function f(t) is given in terms of its Laplace transform by

$$f(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\,e^{st}\,ds \tag{754}$$

The Laplace transform of $\partial f(t)/\partial t$ is given by

$$\begin{aligned} \mathcal{L}[f'(t)] &= \int_0^\infty f'(t)\,e^{-st}\,dt \\ &= f(t)\,e^{-st}\Big|_0^\infty + \int_0^\infty s f(t)\,e^{-st}\,dt \\ &= -f(0) + s\mathcal{L}[f(t)]. \end{aligned} \tag{755}$$

Laplace transforming Eq. (750) and Eq. (751) gives

$$(s+ikv)\,F(s,v) - \frac{eE(s)}{m}\frac{\partial f_0}{\partial v} = f(0,v), \tag{756}$$

$$ikE(s) = -4\pi e\int_{-\infty}^{\infty} F(s,v)\,dv. \tag{757}$$

Solving Eq. (756) for F(s,v) gives

$$F(s,v) = \frac{f(0,v)}{(s+ikv)} + \frac{e}{m}E(s)\,\frac{\partial f_0(v)/\partial v}{s+ikv}. \tag{758}$$

Substituting into Eq. (757) gives

$$E(s)\left[1+\frac{4\pi e^2}{imk}\int_{-\infty}^{\infty}\frac{(\partial f_0(v)/\partial v)\,dv}{s+ikv}\right] = -\frac{4\pi e}{ik}\int_{-\infty}^{\infty}\frac{f(0,v)\,dv}{s+ikv} \tag{759}$$

or

$$E(s) = \frac{\dfrac{i4\pi e}{k}\displaystyle\int_{-\infty}^{\infty}\frac{f(0,v)\,dv}{s+ikv}}{\left[1-\dfrac{i4\pi e^2}{mk}\displaystyle\int_{-\infty}^{\infty}\frac{(\partial f_0/\partial v)\,dv}{s+ikv}\right]}. \tag{760}$$

The perturbed number density will also be a quantity of interest. This is given by

$$n(s) = -\frac{ikE(s)}{4\pi e} = \frac{\displaystyle\int_{-\infty}^{\infty}\frac{f(0,v)\,dv}{s+ikv}}{\left[1-\dfrac{i4\pi e^2}{mk}\displaystyle\int_{-\infty}^{\infty}\frac{(\partial f_0/\partial v)\,dv}{s+ikv}\right]}. \tag{761}$$

To find E(t) ~~of~~ or n(t) we must invert these Laplace transforms. We have

$$E(t) = \frac{2e}{k}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{-\infty}^{\infty}\frac{f(0,v)\,dv\,e^{st}\,ds}{(s+ikv)\left[1-\dfrac{4\pi e^2 i}{mk}\displaystyle\int_{-\infty}^{\infty}\frac{(\partial f_0/\partial v')\,dv'}{s+ikv'}\right]}. \tag{762}$$

If we assume that we may invert the order of the v and s integration, this expression becomes

$$E(t) = \frac{Qe}{k}\int_{-\infty}^{\infty} dv\, f(0,v) \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{st}\, ds}{(s+ikv)\left\{1 - \frac{4\pi e^2 i}{mk}\int_{-\infty}^{\infty} \frac{(\partial f_0/\partial v')\, dv'}{s+ikv'}\right\}}. \tag{763}$$

Now the contour of integration for s is that shown in Fig. 52.

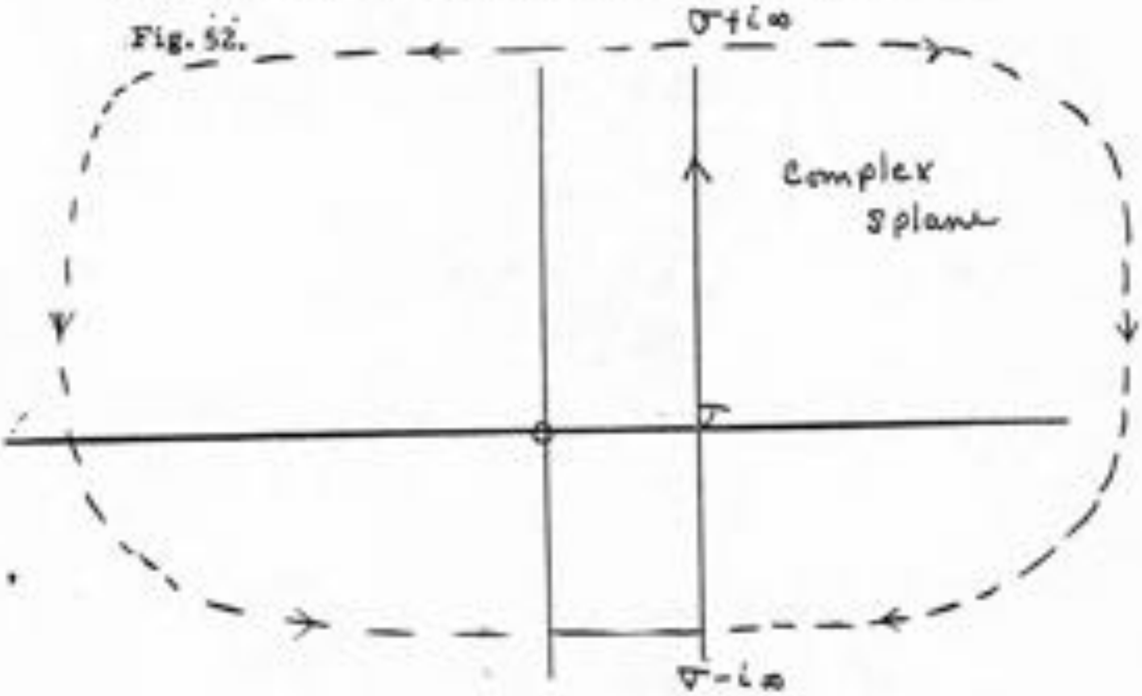


Figure 52

For t less than zero, we may close the contour by the large semicircle to the right. The integral along the semicircle gives nothing because e^{st} vanishes along it. We thus get $2\pi i$ times the sum of the residues of the poles of the integrand inside this contour. However, $s + ikv$ has no zeros inside this contour, and

$$\left\{ 1 - \frac{4\pi e^2 i}{mk} \int_{-\infty}^{\infty} \frac{(\partial f_0/\partial v')\, dv'}{s + ikv'} \right\} \tag{764}$$

also must have no zeros in this region for the following reason. If it vanished for a value of s in this region, say for

$$S = \sigma_0 + i\omega_0$$

then there would be a motion for the plasma for which the E field had the time development

$$E(t) = E\, e^{(\sigma_0 + i\omega_0)t}. \tag{765}$$

But if σ_0 is greater than σ, then the Laplace transform of E, obtained from Eq. (752), would not exist. Hence σ must be larger than $\mathrm{Re}\, S$ for any root of Eq. (764).

Now for $t > 0$ we should like to close the contour on the large semicircle to the left, for e^{St} vanishes on this semicircle. For

$$\frac{1}{s + ikv} \tag{766}$$

this is all right. However, for

$$D(s,k) = \left\{ 1 - \frac{4\pi e^2 i}{mk} \int_{-\infty}^{\infty} \frac{(\partial f_0/\partial v')\, dv'}{s + ikv'} \right\}. \tag{767}$$

This is not all right, because we have defined Eq. (767) only for Re S > maximum value of Re S for S, a root of Eq. (767).

To close the contour as desired we must define this function for all values of s (in particular, $Re\,S<0$) and in such a way that $D(s,k)$ is analytic if we are to apply the residue theorem. Thus we must analytically continue $D(s,k)$ into half plane $Re\,S<0$. Now for $Re\,S$ greater than zero, $D(s,k)$ is perfectly analytic and no problem arises (the integral is a sum of analytic function). However, when S takes on a real value the integrand becomes singular and we must define how the integrand is to be taken. If $D(s,k)$ is to be analytic, then $D(t,k)$ must be continuous. This will be true provided the poles of the integrand always stay on the same side of the contour of integration for v'. If they were to jump across the contour, then the value of the integral would jump by 2πi times the residue at the pole. Thus as S approaches the real axis we must distort the v' contour to go under $v' = i\,S/k$ (taking k positive for convenience).

Explanation incorrect

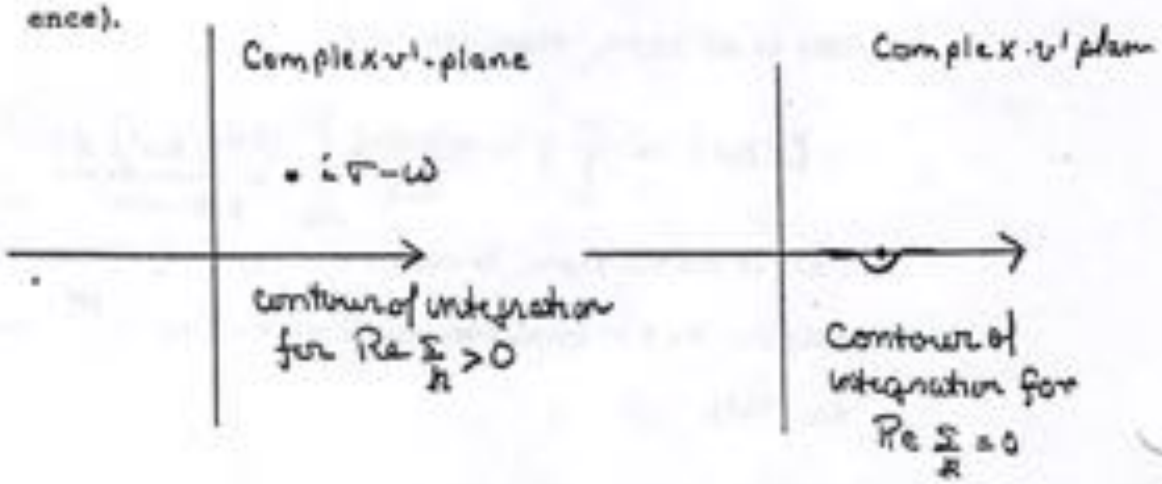


Figure 53

Now as we let $\mathrm{Re}\, S < 0$ we must always loop the v' contour under iS/k to maintain the continuity of $D(s, k)$.

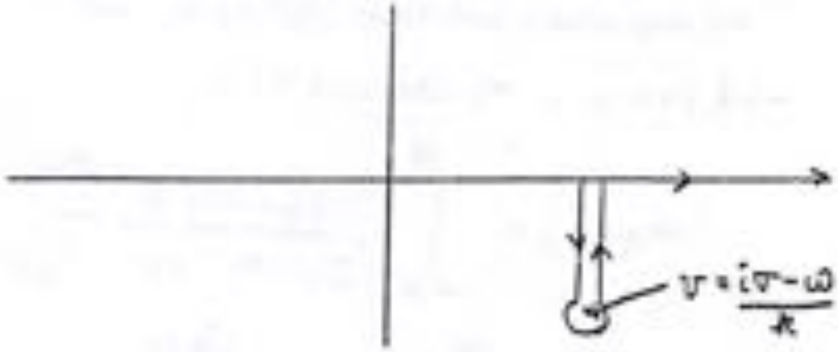


Figure 54

Thus for $\mathrm{Re}\, S < 0$, $D(s, k)$ is defined by

$$D(s,k) = 1 - \frac{4\pi e^2 L}{mk} \int_{-\infty}^{\infty} \frac{(\partial f_0/\partial v')\, dv}{s + ikv'} + \frac{8\pi^2 e^2 i L}{mk^2} f_0'(is/k)$$

$$\text{Along real axis;} \qquad \mathrm{Re}\, s < 0. \tag{768}$$

With this definition we may close the S contour on the large semicircle with $\mathrm{Re}\, S < 0$ and Eq. (763) becomes

$$E_k(t) = \frac{4\pi i e}{k} \int_{-\infty}^{\infty} \frac{f_k(0,v)\, dv\, e^{-ikvt}}{D(-ikv, k)}$$

$$+ \frac{4\pi i e}{k} \sum_\ell e^{s_\ell t} \mathrm{Res} \frac{1}{D(s_\ell, k)} \int_{-\infty}^{\infty} \frac{f_k(0,v)\, dv}{s_\ell + ikv} \tag{769}$$

where the fact that E and f refer to a single Fourier component has been restored explicitly, where the S_ℓ are the zeros of $D(s,k)$ and $\mathrm{Res}\, \frac{1}{D(s_\ell, k)}$ is the residue of D^{-1} at these roots.

We may obtain n(t) from E(t) by multiplying by $-ik/4\pi e$. We thus have for n(t)

$$n_k(t) = \int_{-\infty}^{\infty} \frac{f_k(0,v)\,dv\,e^{-ikvt}}{D(-ikv,v)}$$

no i

$$+ \sum_i \int_{-\infty}^{\infty} \frac{f_k(0,v)\,dv\,e^{is_i t}}{s_i + ikv} \,\mathrm{Res}\, \frac{1}{D(s_i,k)}. \tag{770}$$

Let us first look at Eq. (770) in the limit of e going to zero. In this limit D goes to one by Eq. (767), hence D has no zeros and the sum in Eq. (770) goes out. Eq. (770) then becomes

$$n_k(t) = \int_{-\infty}^{\infty} f_k(0,v)\,e^{-ikvt}\,dv. \tag{771}$$

This is exactly the formula we would get for a gas of noninteracting particles. To see this more clearly we include the x dependence in Eq. (771).

$$n_k(x,t) = \int_{-\infty}^{\infty} f_k(0,v)\,e^{ik(x-vt)}\,dv. \tag{772}$$

This equation says that a disturbance in the particles traveling with velocity v, over which is at x_0 at $t = 0$ will be found at position x such that

$$x - vt = x_0 \tag{773}$$

or

$$x = x_0 + vt.$$

Thus it is simply carried along by the particles. We see from this that the term

$$\frac{4\pi i e}{k}\int_{-\infty}^{\infty}\frac{f_k(0,v)\,e^{-ikvt}\,dv}{D(-ikv,v)} \tag{774}$$

appearing in Eq. (769) for E gives the E field produced by the free streaming of the particles. The appearance of $D(-ikv,v)$ in the denominator gives the modification of the individual particle field due to all the other particles or plasma. D is the dielectric constant for the plasma. The sum appearing in Eq. (769) gives the electric field due to collective motions of the particles or to plasma oscillation. The time dependence of these modes is

$$e^{i\Omega t}$$

and thus is determined by the zeros of the dielectric constant D. The integral over $\int f_k(0,v)$ simply determines the amplitude of each mode. Thus the important thing here is the zero of D which we shall now investigate.

We have for D

$$D(s,k)=\left\{1-\frac{4\pi e^2 i}{mk}\int_{-\infty}^{\infty}\frac{(\partial f_0/\partial v)\,dv}{s+ikv}\right\} \tag{775}$$

(below all zeros of $s+ikv$)

Let us restrict ourselves to a Maxwellian distribution

$$f_0=\frac{n_0}{\sqrt{2\pi}}\,\frac{e^{-\frac{v^2}{2v_0^2}}}{v_0} \tag{776}$$

We shall also write $s=i\omega+\gamma$

Then we have for D

$$D = 1 - \frac{\omega_p^2}{k} \int_{-\infty}^{\infty} \frac{- v\, e^{-\frac{v^2}{2u_0^2}}\, dv}{\sqrt{2\pi}\, u_0^3\, (\omega - i\gamma + kv)} \qquad (777)$$

(below all zeros of $\omega - i\gamma + kv$).

Let us first look for zeros of D for k small. We expect the oscillations of the plasma to be near the plasma frequency so we will assume that ω remains finite as k goes to zero. Then ω/k will be much larger than the thermal velocity u_0 as k goes to zero.

$$D = 1 + \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{v\, e^{-\frac{v^2}{2u_0^2}}\, dv}{\sqrt{2\pi}\, u_0^3\, (v + \frac{\omega}{k} - i\frac{\gamma}{k})} \qquad (778)$$

(below all zeros of $v + \frac{\omega}{k} - i\frac{\gamma}{k}$).

Now since ω/k is large we may expand the denominator to obtain (writing $-iS = \omega - i\gamma$)

$$D \simeq 1 + \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{v\, e^{-v^2/2u_0^2}\, dv}{\sqrt{2\pi}\, u_0^3} \frac{1}{(\frac{-iS}{k})} \left(1 + \frac{kv}{iS} - \frac{k^2v^2}{S^2} + \frac{ik^3v^3}{S^3} \right) \qquad (779)$$

or

-?

$$D \simeq 1 + \frac{\omega_p^2}{k^2 u_0^2} \left[\frac{k^2 u_0^2}{S^2} + \frac{3k^4 u_0^4}{S^4} \ldots \right]. \qquad (780)$$

For small k the solution to this equation is roughly

$$S = i\omega_p \qquad (781)$$

or the oscillations are approximately at the plasma frequency. For k small but not quite negligible, we can substitute the solution, Eq. (781), into the s^4 term of Eq. (780) and find a corrected value of . ?

$$s^2 = -(\omega_p^2 + 3k^2 u_0^2) \tag{782}$$

or

$$\omega^2 = \omega_p^2 + 3k^2 u_0^2. \tag{783}$$

This is the dispersion relation first obtained by Bohm and Gross and is sometimes called the Bohm and Gross dispersion relation.

The roots we have found for ω are real. However, if ω is real then the integrand of Eq. (778) has a singularity on the real axis and we have not treated this properly. The approximation we make in expanding the denominator of Eq. (778) breaks down when v becomes comparable to ω/k. However, the integrand is very small for such values of v, because of the $e^{-v^2/2u_0^2}$ dependence. The only place where we make a serious error is right near the singularity. We may add the contribution from this region to D to obtain a corrected dispersion relation. We shall assume that σ is very small. Then we may ignore the variation of $v\,e^{-v^2/2u_0^2}$ over the region where the singularity makes a contribution.

$$\frac{\omega}{k} - |\sigma| < v < \frac{\omega}{k} + |\sigma|. \tag{784}$$

Now we may evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dv}{v + \frac{\omega}{k} - i\frac{\gamma}{k}} \tag{785}$$

and we find it is equal to πi for $\frac{\gamma}{k} > 0$ and $-\pi i$ for $\gamma/k < 0$. However, if $\gamma/k < 0$ we must loop the contour under the pole, as already discussed, so that for $\gamma/k < 0$ we must add $2\pi i$ to the value $-\pi i$, and hence we get πi for both γ/k greater than or less than 0. Thus the continuity of D is preserved, as it must be. We then find for D

$$D \doteq 1 - \frac{\omega_p^2}{\omega^2}\left[1 + \frac{3k^2 u_0^2}{\omega^2}\right] + \frac{\pi i \omega_p^2}{k^2} \frac{\frac{\omega}{k}\, e^{-\frac{\omega^2}{2k^2 u_0^2}}}{\sqrt{2\pi}\, u_0^3} = 0. \tag{786}$$

If we substitute the solution for ω obtained from Eq. (783) into the imaginary part of D we can again solve approximately for ω and we obtain

$$\omega^2 = \omega_p^2 + 3k^2 u_0^2 + i\sqrt{\frac{\pi}{2}}\, \frac{\omega_p^2}{k^2} \frac{\omega^3}{u_0^3}\, e^{\frac{-\omega^2}{2k^2 u_0^2}} \tag{787}$$

or

$$\omega = \sqrt{\omega_p^2 + 3k^2 u_0^2}\left(1 + i\sqrt{\frac{\pi}{2}}\; \frac{\omega_p^2}{2} \frac{(\omega_p^2 + 3k^2 u_0^2)}{k^2 u_0^3}\, e^{-\frac{\omega_p^2 + 3k^2 u_0^2}{2k^2 u_0^2}} \cdots \right. \tag{788}$$

The imaginary part of ω is such as to give damping. One finds no growing solutions. We should expect this since a Maxwell distribution is a thermal equilibrium distribution and should be stable.

One might think that the zeros of D give normal modes of the plasma, and this is in a sense true. They give time dependences for E and n which are pure exponentials. However, if one returns to the Vlasov equations and looks for solutions which have this time dependence, one will find no such pure solutions. To find such solutions we would require

$$D = 1 - \frac{4\pi e^2}{mk} \int_{\substack{-\infty \\ \text{real axis}}}^{\infty} \frac{\partial f_0/\partial v}{s + ikv} \, dv = 0. \qquad (789)$$

This is the expression for D only for the real part of S > 0. However, the roots we found had Re S < 0, so that we cannot excite them by themselves. The change in the definition of D for Re S < 0 was required by analyticity and was forced on us by the Laplace transform.

If one looks at f one finds that it contains terms which go like e^{ikvt} as well as terms of the form e^{st}, so a pure e^{st} dependence for all quantities does not exist.

Finally, these modes are not the modes found in the beam analysis. The beam modes contain all the individual particle motions as well as any collective motions. The beam modes are true normal modes, whereas the Landau-damped modes are not.

c. An Energy Treatment of Landau Damping

We may also derive expression (786) for Landau damping from the law of conservation of energy. Again we use the one-dimensional Vlasov equation and include only the longitudinal electric forces.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f}{\partial v} = 0 \tag{790}$$

and

$$\frac{\partial E}{\partial x} = -4\pi e\int f\,dv. \tag{791}$$

If we take the time derivative of Eq. (791) and make use of Eq. (790) we have

$$\frac{\partial^2 E}{\partial t\,\partial x} = 4\pi e\int\left[v\frac{\partial f}{\partial x}dv - \frac{eE}{m}\frac{\partial f}{\partial v}\right]dv \tag{792}$$

from which we obtain

$$\frac{\partial}{\partial x}\left[\frac{\partial E}{\partial t} - 4\pi e\int v f\,dv\right] = 0 \tag{793}$$

or

$$\frac{\partial}{\partial x}\left[\frac{\partial E}{\partial t} + 4\pi j\right] = 0. \tag{794}$$

We may integrate Eq. (794) with respect to x to obtain

$$\frac{\partial E}{\partial t} + 4\pi j = c(t) \tag{795}$$

where c(t) may be a function of time. Now if there is a place, say $-\infty$, where E and j vanish for all times, then Eq. (795) becomes

$$\frac{\partial E}{\partial t} + 4\pi j = 0. \tag{796}$$

Multiplying by E and integrating over all x gives

$$\frac{\partial}{\partial t}\bar{\Phi} = \frac{\partial}{\partial t}\int \frac{E^2}{8\pi}\,dx = -\int E j\,dx. \tag{797}$$

The left-hand side of Eq. (797) is the energy in the electric field, while the right-hand side is the rate at which the electric field does work on the current, thus the rate of loss of energy by the electric field.

Next, let us compute the rate of change of the kinetic energy of the particles. To do this we multiply Eq. (790) by $mv^2/2$ and integrate over all v and x .

$$\frac{\partial}{\partial t}\iint \frac{mv^2 f}{2}\,dv\,dx + \iint \frac{mv^3}{2}\frac{\partial f}{\partial x}\,dv\,dx$$

$$- \iint \frac{e}{m} E\, \frac{mv^2}{2}\,\frac{\partial f}{\partial v}\,dv\,dx = 0. \tag{798}$$

Now if f is undisturbed at the ends of the interval for the x interval of integration $(-\infty, \infty)$, then the middle integral goes out (infinite homogeneous plasma) and we have

$$\frac{\partial}{\partial t}\iint \frac{mv^2}{2} f\,dv\,dx - \iint \frac{e}{m} E\,\frac{mv^2}{2}\frac{\partial f}{\partial v}\,dv\,dx = 0 \tag{799}$$

We may integrate the last integral in Eq. (799) by parts with respect to v to obtain

$$\begin{aligned}\iint \frac{eE}{2} v^2 \frac{\partial f}{\partial v}\,dv\,dx &= -\iint e E v f\,dv\,dx \\ &= \int E j\,dx.\end{aligned} \tag{800}$$

Thus Eq. (799) becomes

$$\frac{\partial}{\partial t} K = \frac{\partial}{\partial t} \iint \frac{mv^2}{2} f \, dv \, dx = \int E j \, dx . \qquad (801)$$

We see from Eqs. (797) and (801) that the total energy

$$\int \frac{E^2}{8\pi} dx + K = W \qquad (802)$$

is conserved ($\partial w / \partial t = 0$).

We shall now apply Eqs. (797) and (801) to the problem of Landau damping. To do this we observe that the particles moving at nearly the phase velocity of the wave were the ones responsible for the damping in our earlier treatment of it. These particles are very strongly perturbed by the wave because they see almost a constant E field. The frequency which they see is the doppler-shifted frequency $\omega' = \omega - kv$, which is very small. The E field accelerates them for times of the order of $1/\omega'$ (neglecting second order effects) which is a very long time and hence their perturbed velocity is very large. We shall thus divide the electrons into two groups — those with velocities considerably different from the phase velocity (main plasma) and those with velocities approximately equal to the phase velocity (resonant electrons). This division is illustrated in Fig. 55.

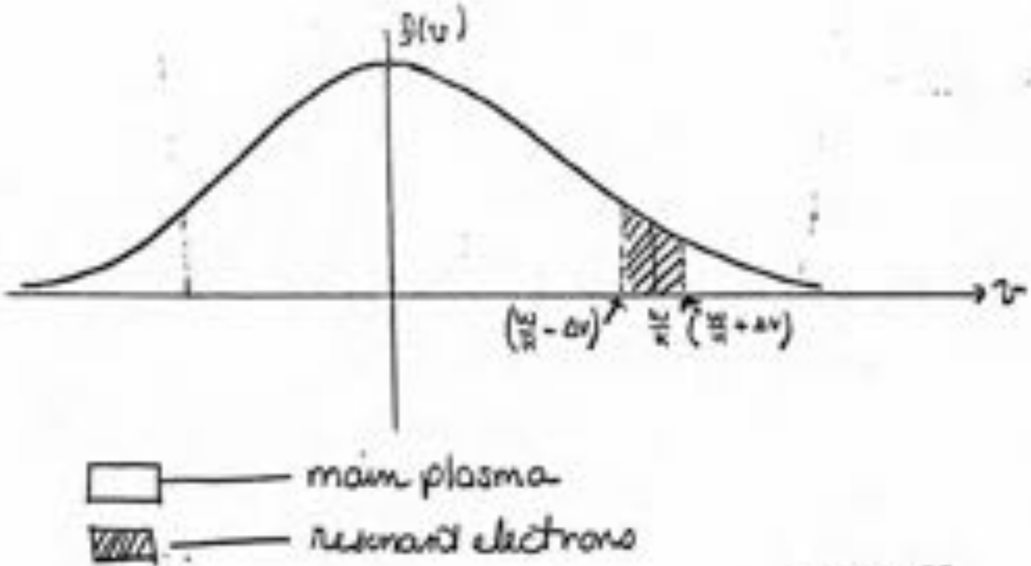


Figure 55

The size of the region around the resonance is not uniquely determined. However, all that is necessary is that we be able to choose a Δv small compared to ω/k.

Now we realize that the wave is primarily carried by the particles in the main plasma, while the resonant electrons are primarily responsible for the damping. We therefore first look for waves carried by the main plasma. The Vlasov Eqs. (748) and (749) govern their motion, so if we look for solution of the form

$E = E \sin (kx - \omega t)$, we obtain from these equations,

$f = f \cos (kx - \omega t)$

$$(\omega - kv)\, f = \frac{eE}{m} \frac{\partial f_0}{\partial v} \qquad (803)$$

and

$$ikE = -4\pi e \int_{\text{Main Plasma (M.P.)}} f\, dv \qquad (804)$$

Here the label MP on the integral appearing in Eq. (804) implies that this integral is to be taken only over the main plasma electrons. We will neglect the electric field produced by the resonant electrons because there are so few of them.

Solving Eq. (803) for f and substituting in Eq. (804) for E gives

$$f = \frac{eE}{m} \frac{\partial f_0/\partial v}{\omega - kv} \cos(kx - \omega t) \tag{805}$$

$$E = \frac{-4\pi e^2}{mk} E \int_{MP} \frac{(\partial f_0/\partial v)\, dv}{\omega - kv} \tag{806}$$

Eq. (806) gives us the dispersion relation when E is cancelled

$$1 = -\frac{4\pi e^2}{mk} \int_{MP} \frac{(\partial f_0/\partial v)\, dv}{\omega - kv}. \tag{807}$$

We note that no difficulty arises in evaluating the integral appearing in Eq. (807) because f_0 and f_0' are zero in the vicinity of $\omega = -kv$. Eq. (807) can be converted to Eq. (808) by integrating by parts and by observing that $f_0(\pm\infty) = 0$ and $f_0(+\omega/k \pm \Delta V) = 0$.

$$1 = \frac{4\pi e^2}{m} \int_{MP} \frac{f_0\, dv}{(\omega \pm kv)^2}. \tag{808}$$

We may use Eqs. (801) and (797) to evaluate the wave energy. First there is the electric field energy, $E^2/8\pi$. The average of this per unit length is

$$\phi = \frac{E^2}{16\pi} \tag{809}$$

To obtain the kinetic energy (the change in the kinetic energy of the plasma due to the oscillation) we use Eqs. (801) and (748). Solving Eq. (748) for E in terms of f gives

$$E = \frac{m}{e} \frac{1}{\frac{\partial f_0}{\partial v}} \left[\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} \right]. \tag{810}$$

Now $\int E j \, dx$ is

$$\iint -evf\, E\, dv\, dx = -m \iint \frac{v}{\frac{\partial f_0}{\partial v}} f \left[\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} \right] dv\, dx \tag{811}$$

$$= -\frac{\partial}{\partial t} \int \frac{mv}{2} \frac{f^2}{\frac{\partial f_0}{\partial v}}\, dv\, dx. \tag{812}$$

From Eqs. (812) and (801) we see that we can identify the time derivative of K with the time derivation of

$$-\iint \frac{mv}{2} \frac{f^2}{\frac{\partial f_0}{\partial v}}\, dv\, dx. \tag{813}$$

Thus K and Eq. (813) can at most differ by a constant. We can, in fact, show that if K is properly defined this quantity is in fact K. (Since K is second order in the amplitude, a second order change in f can also give a second order change in K. Since the second order f is arbitrary — we are working with f_1 — this leads to an ambiguity in K). Taking (813) for ~~this as~~ K and substituting in f from Eq. (805) gives

$$K = -\frac{E^2 e^2}{2m} \iint_{mp} \frac{v\,\partial f_0/\partial v}{(\omega - kv)^2} \cos^2(kv - \omega t)\,dx\,dv \tag{814}$$

or per unit length

$$K = -\frac{E^2 e^2}{4m} \iint_{mp} \frac{v\,\partial f_0/\partial v}{(\omega - kv)^2}\,dv. \tag{815}$$

Combining Eqs. (809) and (815) we get the total wave energy per unit length

$$W = K + \mathcal{E} = \frac{E^2}{16\pi} \left\{ 1 - \frac{4\pi e^2}{m} \int_{mp} \frac{v\,\partial f_0/\partial v\,dv}{(\omega - kv)^2} \right\}. \tag{816}$$

From the dispersion relation, Eq. (807), we may write for 1

$$1 = -\frac{4\pi e^2}{mk} \int_{mp} \frac{(\partial f_0/\partial v)\,dv}{\omega - kv}. \tag{817}$$

Substituting into Eq. (816) gives for the energy density

$$W = -\frac{E^2}{16\pi}\,\frac{4\pi e^2}{mk}\,\omega \int_{mp} \frac{\partial f_0/\partial v\,dv}{(\omega - kv)^2}. \tag{818}$$

Now we have from the dispersion relation

$$\frac{4\pi e^2}{m}\left[-\frac{1}{k^2}\int_{mp} \frac{\partial f_0/\partial v}{\omega - kv}\,dv + \frac{1}{k}\int_{mp} \frac{v\,\partial f_0/\partial v}{(\omega - kv)^2}\,dv\right] dk$$

$$-\frac{4\pi e^2}{mk}\,d\omega \int_{mp} \frac{\partial f_0/\partial v\,dv}{(\omega - kv)^2} = 0 \tag{819}$$

or

$$\frac{4\pi e^2}{mk^2}\left[-2\int_{mp} \frac{\partial f_0/\partial v\,dv}{\omega - kv} + \omega \int \frac{\partial f_0/\partial v}{(\omega - kv)^2}\,dv\right]$$

$$= \frac{4\pi e^2}{mk}\,\frac{d\omega}{dk} \int \frac{\partial f_0/\partial v}{(\omega - kv)^2}\,dv. \tag{820}$$

Solving this equation for $\frac{4\pi e^2}{mk}\int\frac{\partial f_0/\partial v\, dv}{(\omega-kv)^2}$, gives

$$\frac{2}{k} = -\frac{\omega}{k}\left[1-\frac{k}{\omega}\frac{d\omega}{dk}\right]\frac{4\pi e^2}{mk}\int_{nr}\frac{\partial f_0/\partial v}{(\omega-kv)^2}\,dv \tag{821}$$

or

$$\frac{4\pi e^2}{mk}\,\omega\int\frac{\partial f_0/\partial v\, dv}{(\omega-kv)^2} = -\frac{2}{1-\frac{k}{\omega}\frac{d\omega}{dk}}, \tag{822}$$

Substituting Eq. (822) into Eq. (818) gives

$$w = \frac{E^2}{8\pi}\,\frac{1}{1-\frac{k}{\omega}\frac{d\omega}{dk}}. \tag{823}$$

We must now find the rate at which the resonant particles absorb energy. The Vlasov equation, (790), also applies to them and Eq. (805) also gives solutions for them. However, solutions (805) diverge for $v = \frac{\omega}{k}$. If the initial perturbation is smooth in the vicinity of ω/k, then no such singular f will appear. For simplicity we will assume that f is initially zero for the resonant particles. To satisfy these initial conditions we must add to the solution (805) a solution of the homogeneous collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0 \tag{824}$$

The general solution to this equation is

$$f = f_i\,(x - vt, v) \tag{825}$$

where i refers to the initial f, $f_i(x)$ is the initial value of f. We see from Eq. (805) that if we choose f_i to cancel f we get

$$f_r = -\frac{eE}{m}\,\frac{\partial f_0/\partial v}{\omega - kv}\cos k(x - vt) \tag{826}$$

The full f is given by

$$f = \frac{eE}{m}\,\frac{\partial f_0/\partial v}{\omega - kv}\left\{\cos(kx - \omega t) - \cos k(x - vt)\right\} \tag{827}$$

<u>The f given by Eq. (827) has no singularities for any finite length of time.</u> ← statement with relation to many bodies

We now compute the current due to the resonant particles.

$$j_r = -e\int v\, f_r = -\frac{e^2E}{m}\int dv\,\frac{v\,\partial f_0/\partial v}{\omega - kv}\left\{\cos(kx - \omega t) - \cos k(x - vt)\right\}. \tag{828}$$

We assume that we may replace v and $\partial f_0/\partial v$ by their values at ω/k

$$j_r \approx -\frac{e^2E}{m}\,\frac{\omega}{k}\,\frac{\partial f_0}{\partial v}(\omega/k)\int dv\,\frac{\cos(kx - \omega t) - \cos k(x - vt)}{\omega - kv}. \tag{829}$$

Writing

$$\cos(kx - \omega t) - \cos k(x - vt) = -2\sin\tfrac{1}{2}(2kx - \omega t - kvt)\sin\tfrac{1}{2}(kvt - \omega t), \tag{830}$$

and substituting in Eq. (829) gives

$$j_r \approx \frac{2e^2E}{m}\,\frac{\omega}{k}\,\frac{\partial f_0}{\partial v}\left(\frac{\omega}{k}\right)\int \frac{\sin\frac{1}{2}(2kx - \omega t - kvt)\sin\frac{1}{2}(kvt - \omega t)}{\omega - kv}\,dv. \tag{831}$$

Multiplying j_r by $E\sin(kx - \omega t)$ and integrating over x and v gives the rate at which work is done on the resonant particles.

$$\frac{dw_r}{dt} = \frac{2e^2E^2}{m}\frac{\omega}{k}\frac{\partial f_0(\omega/k)}{\partial v}\iint \sin(kx-\omega t)$$
$$\times\frac{\sin\frac{1}{2}(2kx-\omega t-kvt)\sin\frac{1}{2}(kvt-\omega t)}{\omega-kv}\,dx\,dv. \tag{832}$$

From this we find for the average energy absorbed per unit length

$$\frac{dw_r}{dt} = \pi\frac{e^2E^2}{2m}\frac{\omega}{k}\frac{\partial f_0}{\partial v}(\omega/k). \tag{833}$$

Equating the energy gained by the resonant particles to that lost by the wave gives

$$\frac{dw}{dt} = \frac{1}{8\pi\left(1-\frac{k}{\omega}\frac{d\omega}{dk}\right)}\frac{dE^2}{dt} = -\frac{dw_r}{dt} = -\frac{\pi e^2\omega}{2mk}\frac{\partial f_0(\omega/k)}{\partial v}E^2. \tag{834}$$

The damping rate for the wave is

$$\gamma = 8\pi^2\left(1-\frac{k}{\omega}\frac{d\omega}{dk}\right)\frac{e^2\omega}{2mk}\frac{\partial f_0}{\partial v}(\omega/k) \tag{835}$$

$$\gamma = \pi\left(1-\frac{k}{\omega}\frac{d\omega}{dk}\right)\frac{\omega_p^2\omega}{k\,n_0}\frac{\partial f_0}{\partial v}(\omega/k). \tag{836}$$

It goes from 835 to 836

n_0 is the unperturbed number density. If f_0 had been normalized to 1, ~~then~~ and $\frac{1}{n_0}\frac{\partial f_0}{\partial v}(\omega/k)$ replaces ~~would be replaced by simply~~ $\frac{\partial f_0}{\partial v}(\omega/k)$. This is in agreement with our previous result (equation 788).

Solution of the Vlasov Equation

We will now look at another method of solving the Vlasov equation which is fundamental because it can be extended to a plasma made of discrete particle rather than a continuous phase fluid.

Consider the field free Vlasov equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial \underset{\sim}{r}} - \frac{e\underset{\sim}{E}}{m} \cdot \frac{\partial f}{\partial v} = 0$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = -4\pi e \, [\int f dv - n_i]$$

Since we are interested in small amplitude disturbances we linearize these equations

$$f = f_o + f_1$$

$$\underset{\sim}{E} = \underset{\sim}{E}_1$$

$$\int f_o d^3v = n_i$$

$$\frac{\partial f_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f_1}{\partial \underset{\sim}{r}} - \frac{e\underset{\sim}{E}}{m} \cdot \frac{\partial f_o}{\partial v} = 0$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E}_1 = -4\pi e \int f_1 \, d^3v$$

To solve these equations we observe the following, we may divide f_1 into two parts

$$f_1 = \psi_1 + \chi_1$$

where ψ_1 satisfies the equation

$$\frac{\partial \psi_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \psi_1}{\partial \underset{\sim}{r}} = 0$$

and $\phi_1 = f_1$ at $t = 0$ $\quad \chi = 0$ at $t = 0$

χ satisfies the equation

$$\frac{\partial \chi_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \chi_1}{\partial} - \frac{eE}{m} \cdot \frac{\partial f_o}{\partial \underset{\sim}{v}} = 0$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E}_1 = 4\pi 2 \int f_1 d^3v = -4\pi e \int (\phi_1 + \chi_1) d^3v$$

It is clear that $\phi_1 + \chi_1$ satisfies the Vlasov equation; if $\underset{\sim}{E}$ is correct $\phi_1 + \chi_1$ is the correct f_1 and if $\phi_1 + \chi_1$ is the correct f_1 then $\underset{\sim}{E}$ is correct.

Now ϕ_1 develops according to the free streaming of the initial f_1

$$\phi_1 (\underset{\sim}{x}, \underset{\sim}{v}, 0) = f_1(\underset{\sim}{x}, \underset{\sim}{v}, 0)$$

$$\phi_1 (\underset{\sim}{x}, \underset{\sim}{v}, t) = \phi_1 (\underset{\sim}{x} - \underset{\sim}{v}t, \underset{\sim}{v}, 0)$$

Inserting in the equation for ϕ_1

$$\frac{\partial \phi_1}{\partial t} = - \underset{\sim}{v} \cdot \nabla \phi_1$$

$$\therefore \frac{\partial \phi_1}{\partial t} + \underset{\sim}{v} \cdot \nabla \phi_1 = - \underset{\sim}{v} \cdot \nabla \phi_1 + \underset{\sim}{v} \cdot \nabla \phi_1 = 0$$

A specific solution for ϕ_1 is

$$\delta(\underset{\sim}{x} - \underset{\sim}{x}_o)\, \delta(\underset{\sim}{v} - \underset{\sim}{v}_o) + \delta(\underset{\sim}{x} - \underset{\sim}{v}t - \underset{\sim}{x}_o)\, \delta(\underset{\sim}{v} - \underset{\sim}{v}_o) = \delta(\underset{\sim}{x} - \underset{\sim}{v}_o t - \underset{\sim}{x}_o)\delta(\underset{\sim}{v} - \underset{\sim}{v}_o)$$

ϕ_1 can be written as a sum of δ functions

$$\phi_1(\underset{\sim}{x}, \underset{\sim}{v}, t) = \int \phi_1(\underset{\sim}{x}_o, \underset{\sim}{v}_o, 0)\, d^3r_o d^3v_o\, \delta(\underset{\sim}{x} - [\underset{\sim}{x}_o + \underset{\sim}{v}_o t])\, \delta(\underset{\sim}{v} - \underset{\sim}{v}_o)$$

$\phi_1(\underset{\sim}{x}_o, \underset{\sim}{x}_o, 0)\, d^3\ {}_o d^3v_o$ is the number of particles starting at $\underset{\sim}{x}_o$, v_o in $d^3x_o d^3v_o$.

Now ϕ_1 is a known function of space and time and can be regarded ...

Since the equation for χ_1 is linear, the solution obtained when there are many driving sources is the sum of the solutions obtained for the sources one at a time. Since we have seen that ϕ_1 can be broken up into a sum of δ function, we can obtain the general solution if we can solve the equation for a single δ function source.

$$\frac{\partial \chi_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \chi_1}{\partial \underset{\sim}{x}} - \frac{e\underset{\sim}{E}_1}{m} \cdot \frac{\partial f_o}{\partial \underset{\sim}{v}} = 0$$

$$\frac{\partial \cdot \underset{\sim}{E}}{\partial \underset{\sim}{x}} = - 4\pi e \int \chi_1 d^3v - 4\pi e\, \delta(\underset{\sim}{x} - \underset{\sim}{v}t)$$

$$f_1 = \int d^3x_o d^3v_o\, \phi_1(x_o, v_o, 0)\, \{\delta(\underset{\sim}{x} - [\underset{\sim}{x}_o + \underset{\sim}{v}_o t]\, \delta(\underset{\sim}{v} - \underset{\sim}{v}_o)$$

$$+ \chi_1 (\underset{\sim}{x}, \underset{\sim}{v}, t; \underset{\sim}{x}_o, v_o)$$

Where $\phi_1(\underset{\sim}{x}, \underset{\sim}{v}, t; \underset{\sim}{x}_o, \underset{\sim}{v}_o)$ is the solution for ϕ_1 when a unit charge starts at x_o, v_o at time 0.

Problem. Generalize the above for the full set of Maxwell's Equation assuming that the undistrubed plasma contains no static electric or magnetic fields.

Field due to sources embedded in a plasma. We generalize to the full Maxwell Field.

$$\frac{\partial f_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f_1}{\partial \underset{\sim}{x}} - \frac{e}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right) \cdot \frac{\partial f_o}{\partial \underset{\sim}{x}} = - \nu f_1$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = - \frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t} - \frac{4\pi e}{m}\int \underset{\sim}{v}\, f_1 d^3v + 4\pi \underset{\sim}{j}_s$$

$$\nabla \cdot E = -4\pi e \int f_1 d^3v + 4\pi\rho_s$$

$$\nabla \cdot B = 0$$

Choose f_o to be isotropic $f_o(v^2)$. Assume that the source charges and currents satisfy the continuity equation

$$\frac{\partial\rho_s}{\partial t} + \nabla \cdot j_s = 0$$

Fourier analyze in ρ and t

$$f(\rho, v, t) = \frac{1}{(2\pi)^2} \int f(k, \omega, v)\, e^{i(k\cdot\rho - \omega t)} d^3k d\omega$$

$$f(k, \omega, v) = \frac{1}{(2\pi)^2} \int f(\rho, v, t)\, e^{-i(k\cdot\rho - \omega t)} d^3\rho dt$$

$$-i(\omega - k\cdot v)\, f_1 - \frac{eE}{m} \cdot \frac{\partial f_o}{\partial v} = -\nu f_1$$

$$ik \times E = \frac{i\omega B}{c}$$

$$ik \times B = \frac{i\omega}{c} E - \frac{4\pi e}{m} \int v\, f_1 d^3v + 4\pi j_s$$

$$ik \cdot B = 0$$

$$ik \cdot E = -4\pi e \int f_1 d^3v + 4\pi\rho_s$$

$$-i\omega\rho_s + ik \cdot j_s = 0$$

$$\rho_s = \frac{\underline{k} \cdot \underline{j}_s}{\omega}$$

Decompose $\underline{E}$, $\underline{B}$ and $\underline{j}_s$ into longitudinal and transverse components. For example

$$\underline{E}_L \times \underline{k} = 0 \qquad \underline{E}_L \cdot \underline{k} = \underset{\text{magnitudes}}{E_L k} \qquad \underline{E}_L = \hat{\underline{k}}\,\hat{\underline{k}} \cdot \underline{E}$$

$$\underline{E}_T \cdot \underline{k} = 0 \qquad \underline{E}_T = -\hat{\underline{k}} \times [\hat{\underline{k}} \times \underline{E}] = \underline{E} - \hat{\underline{k}}\,\hat{\underline{k}} \cdot \underline{E}$$

$$\hat{\underline{k}} = \frac{\underline{k}}{|k|}$$

$$i\underline{k} \cdot \underline{E} = -4\pi e \int f_1 d^3v + 4\pi\rho_s$$

$$\underline{k} \cdot \underline{E} = 4\pi e i \int f_1 d^3v - 4\pi i \rho_s$$

$$f_1 = \frac{i\,\frac{e\underline{E}}{m} \cdot \frac{\partial f_o}{\partial \underline{v}}}{\omega - \underline{k} \cdot \underline{v} + i\nu}$$

$$\underline{k} \cdot \underline{E} = -\frac{4\pi e^2}{m} \int \frac{\underline{E} \cdot \frac{\partial f_o}{\partial \underline{v}}}{\omega - \underline{k} \cdot \underline{v} + i\nu}\, d^3v - 4\pi i \rho_s$$

$$\underline{k} \cdot \underline{E} + \frac{4\pi e^2}{m} \int \frac{\underline{E} \cdot \partial f_o/\partial \underline{v}}{\omega - \underline{k} \cdot \underline{v} + i\nu} = -4\pi i \rho_s$$

Now for $\displaystyle\int \frac{\underset{\sim}{E} \cdot \partial f_o/\partial \underset{\sim}{v}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu)} d^3v$

the denominator depends only on the component of v || to k.

We can write the expression as

$$\int \frac{E_{\parallel}\ \partial f_o/\partial v_{\parallel} + \underset{\sim}{E}_{\perp} \cdot \partial f_o/\partial v_{\perp}}{(\omega - k\ v_{\parallel} + i\nu)} d^3v = \int \frac{E_{\parallel}\ \partial f_o/\partial v_{\parallel}\ d^3v}{(\omega - kv_{\parallel} + i\nu)}$$

Integration of the $E_{\perp}$ term over $v_{\perp}$ given 0

$$\int \frac{E_{\parallel}\ \partial f_o/\partial v_{\parallel}\ d^3v}{(\omega - kv_{\parallel} + i\nu)} = \frac{\underset{\sim}{k} \cdot \underset{\sim}{E}}{k^2} \int \frac{k \cdot \partial f_o/\partial v\ d^3v}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu)}$$

$$\underset{\sim}{k} \cdot \underset{\sim}{E}\ [1 + \frac{4\pi e^2}{mk^2} \int \frac{\underset{\sim}{k} \cdot \partial f_o/\partial \underset{\sim}{v}\ d^3v}{\omega - \underset{\sim}{k} \cdot v + i\nu}] = - 4\pi i\rho_s$$

$$\underset{\sim}{k} \cdot \underset{\sim}{E} = \frac{-4\pi i\rho_s}{D_L(\underset{\sim}{k}, \omega)} = \frac{-4\pi i\underset{\sim}{k} \cdot \underset{\sim}{j}_s}{\omega\, D_L(\underset{\sim}{k}, \omega)}$$

$$D_L\ (\underset{\sim}{k}, \omega) = 1 + \frac{4\pi e^2}{mk^2} \int \frac{\underset{\sim}{k} \cdot \partial f_o/\partial \underset{\sim}{v}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu)}\ d^3v$$

This gives the longitudinal E field; now proceeding to the transverse field.

$$\underset{\sim}{k} \times \underset{\sim}{k} \times \underset{\sim}{E} = \frac{\omega}{c}\, \underset{\sim}{k} \times \underset{\sim}{B} = - \frac{\omega^2}{c^2} \underset{\sim}{E} + \frac{4\pi e}{m} \frac{i\omega}{c^2} \int \underset{\sim}{v}\ f_1 d^3v - \frac{-4\pi j_s(k, \omega)\omega}{c^2}$$

$$\underset{\sim}{k} \times \underset{\sim}{k} \times \underset{\sim}{E} = -\frac{\omega^2}{c^2}\underset{\sim}{E} - \frac{4\pi e^2}{mc^2}\omega \int \frac{\underset{\sim}{v}\underset{\sim}{E} \cdot \partial f_o/\partial \underset{\sim}{v} d^3v}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu)}$$

$$- \frac{-4\pi i\, \underset{\sim}{j}_s(\underset{\sim}{k},\omega)}{c^2}$$

Problem. Show that if E_L satisfies the equation above that E_L drops out of this equation for $\underset{\sim}{k} \times \underset{\sim}{k} \times \underset{\sim}{E}$.

Let us take k to be in the z direction. Consider the components of $E_\perp$ to k

$$\int \frac{(\underset{\sim}{e}_x v_x + \underset{\sim}{e}_y v_y + \underset{\sim}{e}_z v_z)E_x \partial f_o/\partial v_x d^3v}{\omega - kv_z + i\nu}$$

for the v_y and v_z terms $\int$ over v_x and get 0

$$\int \frac{\underset{\sim}{e}_x E_x v_x\ \partial f_o/\ v_x\ dv_x dv_z dv_z}{(\omega - kv_z + i\nu)}$$

$$= -\underset{\sim}{e}_x E_x \int \frac{f_o d^3v}{(\ -kv_z+i\nu)}$$

Likewise for the y component

$$\underset{\sim}{k} \times \underset{\sim}{k} \times E_\perp = [-\frac{\omega^2}{c^2} + \frac{4\pi e^2}{mc^2}\omega \int \frac{f_o d^3v}{\omega - kv_z + i\nu}]\ \underset{\sim}{E}_\perp - \frac{4i\pi \underset{\sim}{j}_\perp(k,\omega)}{c^2}$$

$$[-k^2c^2 + \omega^2 - \frac{4\pi e^2\omega}{m} \int \frac{f_o d^3v}{\omega - kv_z + i\nu}]\ k \times E_\perp = -4\pi i\omega k \times \underset{\sim}{j}_s$$

$$\underset{\sim}{k} \times \underset{\sim}{E} = \frac{4\pi i\omega \underset{\sim}{k} \times \underset{\sim}{j}_s}{k^2c^2 D_T(\underset{\sim}{k},\omega)}$$

$$D_T = 1 - \frac{\omega^2}{k^2c^2} + \frac{4\pi e^2\omega}{k^2mc^2} \int \frac{f_o dv^3}{(\omega - \underset{\sim}{k} \cdot v + i\nu)}$$

Problem. Find the o's of D_T for $\omega/k \gg V_T$. Are there roots of D_T for $|\omega/k| \le v_T$, if so find them

The Energy Principal

We should now like to derive the energy principal for the stability of an ideal MHD fluid. This is a very useful principal since one of the principal things we should like to know about a plasma configuration is whether or not it is stable.

First we might consider the stability of simple mechanical systems. To be specific, consider a pendulum made of a weight and a rigid stick

Consider the two situations shown. Both are in equilibrium, i.e., no force is acting on the weight. For case (1) if the weight is displaced it oscillates stably about the equilibrium while in the second case a slight displacement in leads to a force which tends to move it further away from the equilibrium and the system is unstable. For both cases

$$Mr\ddot{\theta} = -Kr\theta \qquad \text{where } K = \pm mg.$$

Multipling by $r\dot{\theta}$ gives

$$Mr^2\ddot{\theta}\dot{\theta} = -Kr^2\dot{\theta}\theta$$

Integrating with respect to time

$$mr^2\frac{\dot{\theta}^2}{2} = -Kr^2\frac{\theta^2}{2} + W$$

$$\Rightarrow mr^2\frac{\dot{\theta}^2}{2} + Kr^2\frac{\theta^2}{2} = W$$

Since the kinetic energy term in intrinsically positive if K is positive the displacement θ is limited. If K is negative however then $\dot{\theta}$ and θ can increase continually. The term $Kr^2\theta^2/2$ is the potential energy due to the displacement from equilibrium. If it is positive then we must do work to displace the system and it cannot of its own move away from the equilibrium. On the other hand if the change in potential energy is negative the system of itself can move away from the equilibrium.

You can Fourie analyze in time

$$\theta = e^{i\omega t}\theta \quad \Rightarrow \quad -\omega^2 m r \theta = -K r \theta$$

$$-\omega^2 = \frac{-K \cancel{r} \cancel{\theta}}{m \cancel{r} \cancel{\theta}}$$

If K is positive ω in real and the motion is oscillitory while if K is negative ω is imaginary with the system is unstable.

We wish to apply these ideas to a plasma. A plasma is however, more complex because it has an ∞ number of degrees of freedom. The plasma at any given point can be moved arbitrarily relative to any other point.

$$m_i \ddot{\xi}(i) = -\sum_j a_{ij}\, \xi(j)$$

$$-\omega_k^2\, \xi_k(i) = -\sum_j \frac{a_{ij}}{m_i}\, \xi_k(j)$$

$$\sum_j \left(\frac{a_{ij}}{m_i} - \omega^2 \delta_{ij}\right) \xi(j) = 0$$

ω's are the solutions of

$$\text{Determinant } \left| a_{ij} - \omega^2 \delta_{ij} \right| = 0$$

For N degrees of freedom this is an N by N determinant and gives a polynonid of order N in ω^2, there are N solutions of ω^2, denote them by ω_k. For each ω_k there is a set of $\xi_k(j)$, only the ratios of the $\xi_k(j)$'s are specified.

$$-\omega_k^2\, \xi_k(i) = -\sum_j a_{ij}\, \xi_k(j)$$

Multiply by $\xi_k(i)$

$$-\omega_k^2 \sum_i m_i\, \xi_k^2(i) = -\sum_i a_{ij}\, \xi_k(i)\, \xi_k(j)$$

Normalize so that

$$\sum_i m_i\, \xi_k^2(i) = 1$$

Let ω_k^2 and ω_l^2 and $\xi_k(i)$ and $\xi_l(i)$ be two solutions

$$-\omega_k^2 m_i \xi_k(i) = -\sum_j a_{ij}\xi_k(j)$$

$$-\omega_l^2 m_i \xi_l(i) = -\sum_j a_{ij}\xi_l(j)$$

multiply the first by $\xi_l(i)$ and the second by $\xi_k(i)$ Σ over and subtract

$$(\omega_k^2-\omega_l^2)\sum_i m_i \xi_k(i)\xi_l(i) = -\sum_{ij}\left(a_{ij}\xi_l(i)\xi_k(j) - a_{ji}\xi_l(j)\xi_k(i)\right) = 0$$

since $a_{ij} = a_{ji}$

$$\therefore \quad \sum_i m_i \xi_k(i)\xi_l(i) = 0$$

Choose

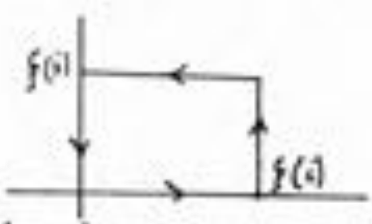
$$\sum_i m_i \xi_k^2(i) = 1$$

$$\sum_i m_i \xi_k^*(i)\xi_l(i) = \delta_{kl}$$

Aside: $\sum_j a_{ij}\xi(j) =$ force on i^{th} particle

$\sum_i a_{ji}\xi(i) =$ force on j^{th} particle

Consider that only particles i and j are moved around the diagram shown



The work done is

$$\int a_{ii}\xi_i\,d\xi_i + \int a_{ij}\xi_j\,d\xi_i + \int a_{ji}\xi_i\,d\xi_j + \int a_{jj}\xi_j\,d\xi_j + a_{jj}\xi_j^2/2$$

$$= a_{ii}\frac{\xi_i^2}{2} + a_{ji}\xi_i\xi_j - a_{ij}\xi_j\xi_i - a_{ii}\xi_i^2/2 - a_{jj}\xi_j^2/2$$

$a_{ij} = a_{ji}$ ~~if~~ if the work done in going around this circuit is to be 0 or if the system is to be conservative.

$$\xi(i) = \sum e^{i\omega_k t} A_k \xi_k(i)$$

Stable if all ω_k^2 are positive.

Unstable if one ω_k^2 is negative.

$$-\omega_k^2 \sum_i m_i \xi_k^2(i) = -\sum_{ij} a_{ij}\, \xi_k(i)\, \xi_k(j)$$

The sum on the left is intrinsically positive so that you get stability if the sum on the right is positive, get instability if it is negative.

Let $\xi(i)$ be a displacement of the system

$$\xi(i) = \sum_k \alpha_k \xi_k(i)$$

$$\sum_j a_{ij}\, \xi(j) = \sum_{kj} a_{ij} \alpha_k \xi_k(j) = \sum_k m_i \omega_k^2 \alpha_k \xi_k(i)$$

Multiply by ξ_i and sum over i

$$\sum_{ik\ell} \omega_k^2 \alpha_k \alpha_\ell\, m_i\, \xi_k(i)\, \xi_\ell(i) = \sum_k \omega_k^2 \alpha_k^2 = \delta W$$

by the normalization

δW can only be negative if one of the ω_k^2's is negative. If one of the ω_k^2 is negative then we can find a displacement which makes the energy negative, namely the eigen function for the negative ω^2.

$$m_i\, \ddot{\xi}(i) = -\sum_j a_{ij}\, \xi(j)$$

$$m_i\, \ddot{\xi}(i)\, \dot{\xi}(i) = -\sum_j a_{ij}\, \xi(j)\, \dot{\xi}(i)$$

Sum over j ;

$$\sum_i m_i\, \ddot{\xi}(i)\, \dot{\xi}(i) = -\sum_{ij} a_{ij}\, \xi(j)\, \dot{\xi}(i) = -\frac{1}{2} \sum_{ij} a_{ij} \left(\xi(i)\, \xi(j)\right)^{\bullet}$$

$$\sum_i m_i\, \dot{\xi}_i^2/2 + \frac{1}{2} \sum_{ij} a_{ij}\, \xi(i)\, \xi(j) = W_T$$

The ideal MHD equations are

$$\rho \frac{d\underline{v}}{dt} = -\nabla p + \frac{\underline{j}\times\underline{B}}{c}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\underline{v} = 0$$

$$\nabla\times\underline{B} = \frac{4\pi\underline{j}}{c}$$

$$\nabla\times\underline{E} = -\frac{1}{c}\frac{\partial\underline{B}}{\partial t}$$

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = 0 \qquad E_{\parallel} = 0$$

$$\frac{p}{\rho^{\gamma}} = \text{const} = \frac{p_0}{\rho_0^{\gamma}}$$

The equilibrium is given by

$$\nabla p_0 = \frac{\underline{j}\times\underline{B}}{c} = \frac{1}{4\pi}(\nabla\times\underline{B}_0)\times\underline{B}_0$$

The first order equation of motion (assuming $v_0 \equiv 0$)

$$\rho_0\dot{\underline{v}} = -\nabla p + \frac{1}{4\pi}\left\{(\nabla\times B)\times B_0 + (\nabla\times B_0)\times B_1\right\}$$

Let $\xi(\Omega_0,t)$ be the displacement of an element of the fluid $\perp$ to Bs. Displacements $\parallel$ to B need not lead to a restoring force, $\underline{v} = \dot{\xi}$ (However when pressure is included $\parallel$ displacements also generally lead to restoring forces and our treatment includes those.) $\underline{E} + \dfrac{\dot{\underline{\xi}}\times\underline{B}_0}{c} = 0$

$$\nabla\times\underline{E} = -\nabla\times\left(\frac{\dot{\underline{\xi}}\times\underline{B}_0}{c}\right) = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\underline{B} = \nabla\times(\underline{\xi}\times B_0) = Q.$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho_0 \dot{\underline{\xi}} = 0 \qquad \frac{d\rho}{dt} + \rho_0 \nabla \cdot \underline{v} = 0$$

$$\rho + \nabla \cdot \rho_0 \underline{\xi} = 0 \qquad \rho = -\rho_0 \nabla \cdot \underline{\xi}$$ following the motion.

$$\rho = -\nabla \cdot \rho_0 \underline{\xi} = -(\rho_0 \nabla \cdot \xi + \underline{\xi} \cdot \nabla \rho_0)$$

$$\frac{d}{dt} \frac{p}{\rho^\gamma} = 0$$

$$\left(\frac{\partial p}{\partial t} + \underline{v} \cdot \underline{\nabla} p\right] - \frac{\gamma p}{\rho}\left(\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla} \rho\right) = 0$$

$$\frac{\partial p}{\partial t} + \dot{\xi} \cdot \underline{\nabla} p_0 + \gamma p_0 \nabla \cdot \dot{\xi} = 0$$

$$p = -\{\underline{\xi} \cdot \underline{\nabla} p_0 + \gamma p_0 \underline{\nabla} \cdot \xi\}$$

$$\rho_0 \ddot{\underline{\xi}} = \underline{\nabla}\{\underline{\xi} \cdot \underline{\nabla} p_0 + \gamma p_0 \underline{\nabla} \cdot \underline{\xi}\} +$$

$$+ \frac{1}{4\pi}\{\underline{\nabla} \times [(\underline{\nabla} \times (\underline{\xi} \times \underline{B}_0)) \times \underline{B}_0] + (\underline{\nabla} \times \underline{B}_0) + \underline{\nabla} \times (\underline{\xi} \times \underline{B}_0$$

$$\rho_0 \ddot{\underline{\xi}} = F(\underline{\xi})$$

can separate ξ in time and space.

$$\underline{\xi} = T(t)\, \underline{\xi}(\underline{r_0})$$

$$\rho_0 \ddot{T}\, \underline{\xi}(r_0) = T F(\underline{\xi}(r_0))$$

$$\ddot{T} = -\omega_k^2 T \qquad \omega_k^2 = -\frac{F(\xi(r_0)) \cdot \xi}{\rho_0(r_0)\, \xi^*(r_0)}$$

$$T = T_1 e^{i\omega_k t} + T_2 e^{-i\omega_k t}$$

$$-\rho_0 \omega_k^2\, \underline{\xi}_k(r_0) = F(\xi_k(r_0))$$

The ξ_k form a set of normal modes. In the normal way there is an orthogonality relation

$$\frac{1}{2}\int \rho_0\, \underline{\xi}_k(r_0)\cdot\underline{\xi}_\ell(r_0)\, d\tau_0 = \delta_{k\ell}$$

By analogy with a set of harmonic oscillators

$$\delta W = -\tfrac{1}{2}\int \underline{\xi}\cdot\underline{F}(\underline{\xi})\, d\tau$$

The system is stable provided this is positive

$$-\frac{1}{2}\int \sum_{k\ell} \underline{a}_k\, Re\, e^{i\omega_k t}\cdot \sum_{\ell} \underline{F}(\xi_\ell)\, Re\, e^{i\omega_\ell t}\, d\tau$$

$$\delta W = \frac{1}{2}\sum_{k\ell} a_k a_\ell \omega_\ell^2 \int \rho_0\, \xi_k\cdot\xi_\ell\, d\tau\, Re\, e^{i\omega_k t}\, Re\, e^{-i\omega_\ell t}$$

$$= +\frac{1}{2}\sum_k a_k^2\, \omega_k^2 \cos^2\omega_k t$$

Stable or unstable according to whether or not

$$\int \Big[\underline{\xi}\cdot\underline{\nabla}\left\{\underline{\xi}\cdot\underline{\nabla}P_0 + \gamma P_0 \underline{\nabla}\cdot\underline{\xi}\right\} + \frac{\underline{\xi}\cdot}{4\pi}\Big\{ \left(\nabla\times[(\nabla\times\xi\times B_0)]\times B_0\right) + (\nabla\times B_0)\times\left[\nabla\times(\xi\times B_0)\right]\Big\}\Big]\, d\tau$$

is negative or positive. Negative ω_k^2 leads to imaginary ω_k's and to exponentially growing solutions. Positive ω_k^2 to stable oscillations.

$$\rho_o \ddot{\xi} = F(\xi)$$

$$\underline{F}(\underline{\xi}) = \underline{\nabla}\,[\underline{\xi} \cdot \underline{\nabla P}_o + \gamma P_o \underline{\nabla} \cdot \underline{\xi}) + \frac{1}{4\pi}\,[\underline{\nabla} \times (\underline{\nabla} \times (\underline{\xi} \times \underline{B}_o))] \times \underline{B}_o +$$

$$+ (\underline{\nabla} \times \underline{B}_o) \times \underline{\nabla} \times (\underline{\xi} \times \underline{B}_o))$$

$$-\omega_k^2 \rho_o \xi(\underline{\mathbf{x}}_o) = \underline{F}(\underline{\xi}_k(\underline{\mathbf{x}}_o))$$

The ξ_k are a complete set of normal modes. Normalize so that

$$\int \rho_o(\underline{\mathbf{x}}_o)\, \xi_k^2 d\tau = 1$$

$$-\omega_k^2 \int \rho_o \xi_k^2 d\tau = \int \underline{\xi}_k \cdot \underline{F}(\underline{\xi}_k) d\tau$$

$$\omega_k^2 = -\int \underline{\xi}_k \cdot \underline{F}(\xi_k) d\tau.$$

If $-\int \underline{\xi}_k \cdot \underline{F}(\xi_k) d\tau$ is negative.

The system is unstable.

If the system is unstable there is some disturbance which makes the potential energy negative; if we can find a disturbance which makes the energy negative the system is unstable. For many problems we are interested in the plasma not filling the whole space, there will be a boundary between plasma and vacuum and we must find the contributions from displacing the surface and distorting the vacuum fields. To this end we manipulate the expression we get.

We use the relation

$$\underline{\xi} \cdot \underline{\nabla}\,\{(\underline{\xi} \cdot \underline{\nabla P}_o + \gamma P_o \underline{\nabla} \cdot \underline{\xi})\} = \underline{\nabla} \cdot \underline{\xi}\,(\underline{\xi} \cdot \underline{\nabla P}_o + \gamma P_o \underline{\nabla} \cdot \underline{\xi})$$

$$- (\underline{\xi} \cdot \underline{\nabla P}_o + \gamma P_o \underline{\nabla} \cdot \underline{\xi})\, \underline{\nabla} \cdot \underline{\xi}$$

Writing

$$\nabla \times (\xi \times B_o) = Q = \delta B$$

$$\nabla \cdot [(\xi \times B_o) \times Q] = Q \cdot [\nabla \times (\xi \times B_o)] - (\xi \times B_o) \cdot (\nabla \times Q)$$

$$= Q^2 - (\xi \times B_o) \cdot \nabla \times Q = Q^2 - \xi \cdot (B_o \times \nabla \times Q)$$

Also,

$$\xi \cdot [(\nabla \times B_o) \times Q] = - [\nabla \times B_o] \cdot (\xi \times Q)$$

$$\delta W = - \frac{1}{2} \int \xi \cdot \{\nabla(\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi) + \frac{1}{4\pi} [(\nabla \times Q) \times B_o + (\nabla \times B_o) \times Q]\} d\tau$$

$$= \frac{1}{2} \int \{(\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi) \nabla \cdot \xi - \nabla \cdot \xi (\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi)$$

$$+ \frac{1}{4\pi} [Q^2 - \nabla \cdot [(\xi \times B_o) \times Q]] + \frac{1}{4\pi} \nabla \times B_o \cdot (\xi \times Q)\} d\tau$$

$$= \frac{1}{2} \int_{\text{plasma}} \{(\nabla \cdot \xi) \xi \cdot \nabla P_o + \gamma P_o (\nabla \cdot \xi)^2 + \frac{Q^2}{4\pi} + \frac{1}{4\pi} (\nabla \times B_o) \cdot (\xi \times Q)\} d\tau$$

$$- \frac{1}{2} \int_{\text{surface}} \{\xi (\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi) + \frac{1}{4\pi} (\xi \times B_o) \times Q\} \cdot ds.$$

Let us manipulate the surface term, the quantities in it must be evaluated at the position of the surface. We have a point in space

$$P_1 = - (\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi)$$

and

$$B_1 = \nabla \times (\xi \times B_o) = Q$$

Now the value of P_1 at the displaced surface point that started at $\underline{A}_s$ is

$$P_1(R_s, t) = P_1(\underline{A}_s, t) + \underline{\xi} \cdot \underline{\nabla} P_o = -\gamma P_o \underline{\nabla} \cdot \underline{\xi}.$$

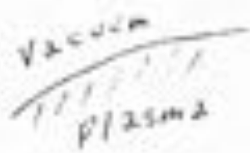
Can also get that from

$$\frac{dp}{dt} = -\gamma P \underline{\nabla} \cdot \underline{\xi}$$

following the motion, for B_1 we have

$$B_1(R_s) = \underline{Q} + \underline{\xi} \cdot \underline{\nabla B}_o$$

Now along the surface we must have pressure balance



$$\left(P + \frac{B^2}{8\pi}\right)_{\text{fluid}} = \frac{B^2 \text{vacuum}}{8\pi}$$

To first order

$$\left(P_1 + \frac{B_o \cdot B_1}{4\pi}\right)_{\text{fluid}} = \left(\frac{B_o \, B_{1v}}{4\pi}\right)_{\text{vacuum}}$$

$$-\gamma P_o \underline{\nabla} \cdot \underline{\xi} + \frac{B_o}{4\pi} \cdot [\underline{Q} + \underline{\xi} \cdot \underline{\nabla} B_o] = \frac{\underline{B}_{ov}}{4\pi} \cdot [\underline{B}_{1v} + \underline{\xi} \cdot \underline{\nabla} B_{ov}]$$

Let us now substitute this relation into the surface integral

$$- \frac{1}{2} \int_s [\xi (\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi) - \frac{Q \times (\xi \times B_o)}{4\pi}] \cdot ds$$

$$= - \frac{1}{2} \int_s [\xi (\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi + \frac{B_o - \xi (Q \cdot B_o)}{4\pi}] \cdot ds$$

Since the plasma surface must be parallel to B, $B \cdot \nabla P = 0$ therefore B lies in surfaces of constant P, $[- \nabla P + \frac{j \times B}{c} = 0]$ and the $B(Q \cdot \xi) \cdot ds$ term gives 0, proceeding the surface integral becomes

$$= - \frac{1}{2} \int_s \xi (\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi - \frac{Q \cdot B_o}{4\pi}) \cdot ds$$

$$= - \frac{1}{2} \int_s (\xi (\xi \cdot \nabla P_o + \frac{B_o \cdot (\xi \cdot \nabla B_o)}{4\pi}$$

$$- \frac{B_{ov}}{4\pi} \cdot [B_{iv} + \xi \cdot \nabla B_{ov}]]) \cdot ds$$

We also have

$$B_o \cdot (\xi \cdot \nabla B_o) = B_{ok} \xi_i \frac{\partial B_{ok}}{\partial X_i} = \xi_i \frac{\partial B_{ok}{}^2/2}{\partial X_i} = \xi \cdot \nabla B_o{}^2/2$$

The integral becomes

$$- \frac{1}{2} \int_s (\xi \cdot \nabla P_o + \xi \cdot \nabla (\frac{B_{op}{}^2}{8\pi} - \frac{B_{ov}{}^2}{\gamma\pi}) - \frac{B_{ov} \cdot B_{iv}}{4\pi}) \xi \cdot ds.$$

Since $P + \frac{B_p^2}{8\pi} = \frac{B_v^2}{8\pi}$ holds everywhere on the boundary

$$\frac{\partial}{\partial x_\parallel}\left(P_o + \frac{B_{op}^2}{8\pi} - \frac{B_{ov}^2}{8\pi}\right) = 0$$

$$-\frac{1}{2}\int_s (\underline{\xi}\cdot\underline{n})^2\left[\underline{\nabla}\left(P_o + \frac{B_o^2}{8\pi}\right)\right]\cdot d\underline{s} + \frac{1}{8\pi}\int_s \underline{B}_{ov}\cdot\underline{B}_{iv}\,\underline{\xi}\cdot d\underline{s}$$

We can get the last integral in a slightly different form. Write in the vacuum

$$\underline{B}_1 = \underline{\nabla}\times\underline{A}_1$$

$$\underline{E}_1 = -\frac{1}{c}\dot{\underline{A}}_1$$

The Coulomb gauge has been adopted so that the scalar potential does not appear. For an observer riding on the surface of the plasma, the tangential component of E is continuous. In the plasma E is zero, therefore, in the vacuum E must be 0. Going back to the lab frame

$$\underline{E}_v = -\frac{\dot{\underline{\xi}}\times\underline{B}_o}{c}$$

$$\underline{n}\times\underline{E}_v = -\frac{\underline{n}\times(\dot{\underline{\xi}}\times\underline{B}_i)}{c} = \frac{-\dot{\underline{\xi}}\overbrace{(\underline{n}\cdot\underline{B}_v)}^{0} + \underline{B}_v(\underline{n}\cdot\dot{\underline{\xi}})}{c}$$

$$\underline{n}\cdot\underline{E}_v = \frac{(\underline{n}\cdot\dot{\underline{\xi}})\,\underline{B}_v}{c} = -\frac{1}{c}\,\underline{n}\times\dot{\underline{A}}_1$$

$$\underline{n}\times\dot{\underline{A}}_1 = -(\underline{n}\cdot\dot{\underline{\xi}})\,\underline{B}_v$$

$$\frac{1}{8\pi}\int \underline{B}_{ov}\cdot\underline{B}_{iv}\,\xi\cdot d\underline{s} = >$$

$$= \frac{1}{8\pi}\int (\underset{\sim}{n} \cdot \underset{\sim}{\xi}\, ds\, \underset{\sim}{B}_{ov}) \cdot \underset{\sim}{B}_{iv} = \frac{1}{\pi}\int - (d\underset{\sim}{s} \times \underset{\sim}{A}_1) \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{A}_1)$$

$$= \frac{1}{8\pi}\int d\underset{\sim}{s} \cdot [(\underset{\sim}{\nabla} \times \underset{\sim}{A}_1) \times \underset{\sim}{A}_1]$$

because the normal points out of the plasma and into the vacuum.

$$= \frac{1}{8\pi}\int d\tau\, \underset{\sim}{\nabla} \cdot [(\underset{\sim}{\nabla} \times \underset{\sim}{A}_1) \times \underset{\sim}{A}_1]$$

$$= \frac{1}{8\pi}\int \{\underset{\sim}{A}_1 \cdot [\underset{\sim}{\nabla} \times \overset{0}{\overset{||}{\underset{\sim}{\nabla}}} \times \underset{\sim}{A}_1] - (\underset{\sim}{\nabla} \times \underset{\sim}{A}_1) \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{A}_1)\}\, d\tau$$

$$= \frac{1}{8\pi}\int (\underset{\sim}{\nabla} \times \underset{\sim}{A}_1)^2\, d\tau = \frac{1}{8\pi}\int \underset{\sim}{B}_{iv}{}^2\, d\tau.$$

The surface integral becomes

$$- \frac{1}{2}\int_s (\underset{\sim}{\xi} \cdot \underset{\sim}{n})^2\, [\underset{\sim}{\nabla}\, (P_o + \frac{B^2}{8\pi})] \cdot d\underset{\sim}{s} + \frac{1}{8\pi}\int_{\text{vacuum}} \underset{\sim}{B}_{iv}{}^2\, d\tau$$

$$\delta W = \frac{1}{2}\int \Big((\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}P + \gamma P_o(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 + \frac{Q^2}{4\pi} + \frac{(\underset{\sim}{\nabla} \times \underset{\sim}{B}) \cdot (\underset{\sim}{\xi} \times \underset{\sim}{Q})}{4\pi}\Big)\, d\tau$$

$$- \frac{1}{2}\int \Big((\underset{\sim}{n} \cdot \underset{\sim}{\xi})^2 \cdot \underbrace{[\underset{\sim}{\nabla}P_o + \frac{B^2}{8\pi}]}_{\text{jump}} \cdot d\underset{\sim}{s} + \frac{1}{8\pi}\int_{\text{vacuum}} B_{iv}{}^2\, d\tau.$$

Application of the Energy Principle

Seek the ξ which minimizes δW. $(\int \xi^2 d\tau) = 1$

Must normalize ξ, one choice is [If a normalization if not used, δW can be made arbitrarily large by choosing ξ arbitrarily large] but we may use any other convenient normalizing condition.

Consider a force free magnetic field enclosed by a rigid conducting boundary parallel to $\underset{\sim}{B}$. Assume the system is filled with fluid so that the surface and vacuum terms do not enter.

$$\nabla \times \underset{\sim}{B}_0 = \frac{4\pi \underset{\sim}{j}}{c}$$

$$\frac{\underset{\sim}{j} \times \underset{\sim}{B}_0}{c} = 0 \qquad (\underset{\sim}{j} \parallel \underset{\sim}{B})$$

$$\nabla \times \underset{\sim}{B}_0 = \alpha \underset{\sim}{B}_0, \quad \nabla \cdot \underset{\sim}{j} = 0 \qquad \nabla \cdot \underset{\sim}{B}_0 = 0$$

We find the solution to the equilibrium equations which are independent of Z and θ inside a circular cylinder with α = constant.

Could take α constant or a variable.

$$\delta W = \frac{1}{8\pi}\int [\underset{\sim}{Q}^2 + \alpha \underset{\sim}{B} \cdot (\underset{\sim}{\xi} \cdot \underset{\sim}{Q})]d\tau, \qquad \underset{\sim}{Q} = \nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0) = \underset{\sim}{B}_1$$

Let $\underset{\sim}{R} = \underset{\sim}{\xi} \times \underset{\sim}{B}_0$ ($\underset{\sim}{R}$ is like the vector potential) $\nabla \times \underset{\sim}{R} = \underset{\sim}{B}_1$

$$W = \frac{1}{8\pi}\int [(\nabla \times \underset{\sim}{R})^2 + \alpha\, \underset{\sim}{B}_0 \cdot (\underset{\sim}{\xi} \times [\nabla \times \underset{\sim}{R}])]d\tau$$

$$= \frac{1}{8\pi}\int [(\nabla \times \underset{\sim}{R})^2 - \alpha[\nabla \times \underset{\sim}{R}] \cdot [\underset{\sim}{\xi} \times \underset{\sim}{B}_0]]d\tau$$

$$= \frac{1}{8\pi}\int [(\nabla \times \underset{\sim}{R})^2 - \alpha \underset{\sim}{R} \cdot (\nabla \times \underset{\sim}{R})]d\tau$$

For the normalization condition use

condition: $\frac{1}{8\pi} \int \alpha \underset{\sim}{R} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{R}) d\tau = \text{constant}$

Introduce this condition by means of Lagrange multiplier λ

$$I = \frac{1}{8\pi} \int [(\underset{\sim}{\nabla} \times \underset{\sim}{R})^2 - (\lambda + 1) \alpha \underset{\sim}{R} \cdot \underset{\sim}{\nabla} \times \underset{\sim}{R}] d\tau = \int L \, d\tau$$

This is of the form

$$I = \int L \, (x, q, q') \, d\tau ,$$

q' is the derivative of q. We minimize I, the minimum is obtained when the Euler Lagrange equation holds

$$\frac{\partial L}{\partial q} - \frac{d}{dx} \frac{\partial L}{\partial q'} , \quad \frac{L}{R_i} = \frac{d}{dx} \frac{\partial L}{\partial R_{ij}} , \quad R_{ij} = \frac{\partial R_i}{\partial x_k}$$

$$\frac{\partial L}{\partial R_x} = \frac{d}{dx} \frac{\partial L}{\partial R_{x,x}} + \frac{d}{dy} \frac{\partial L}{\partial R_{x,y}} + \frac{d}{dz} \frac{\partial L}{\partial R_{x,z}}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{R} = \begin{vmatrix} \underset{\sim}{e}_x & \underset{\sim}{e}_y & \underset{\sim}{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ R_x & R_y & R_z \end{vmatrix} = \begin{matrix} \underset{\sim}{e}_x (R_{z,y} - R_{y,z}) \\ + \underset{\sim}{e}_y (R_{x,z} - R_{z,x}) \\ + \underset{\sim}{e}_z (R_{y,x} - R_{x,y}) \end{matrix}$$

$$L = R_{z,y}{}^2 + R_{y,z}{}^2 + R_{x,z}{}^2 + R_{z,x}{}^2 + R_{y,x}{}^2 + R_{x,y}{}^2 - 2 R_{z,y} R_{y,z}$$

$$- 2 R_{x,z} R_{z,x} - 2 E_{y,x} R_{x,y}$$

$$- (\lambda + 1) \alpha \{ R_x (R_{z,y} - R_{y,z}) + R_y (R_{x,z} - R_{z,x}) + R_z (R_{y,x} - R_{x,y}) \}$$

$$\frac{\partial L}{\partial R_x} = -(\lambda + 1) \alpha \, (R_{z,y} - R_{y,z}) = -(\lambda + 1) \, \alpha \, (\underset{\sim}{\nabla} \times \underset{\sim}{R})_x$$

$$\frac{\partial L}{\partial R_{xx}} = 0$$

$$\frac{\partial L}{\partial R_{xy}} = 2(R_{x,y} - R_{y,x}) + (\lambda + 1)\alpha\, R_z = -2(\nabla \times R)_z + (\lambda + 1)\alpha R_z$$

$$\frac{\partial L}{\partial R_{xz}} = 2(\nabla \times R)_y - (\lambda + 1)\alpha\, R_y$$

$$\frac{\partial}{\partial y}\frac{\partial L}{\partial R_{x,y}} + \frac{\partial}{\partial z}\frac{\partial L}{\partial R_{x,z}} = -2\frac{\partial}{\partial y}(\nabla \times R)_z + 2\frac{\partial}{\partial z}(\nabla \times R)_y +$$

$$+ (\lambda + 1)\,\alpha\,\left(\frac{\partial R_z}{\partial y} - \frac{\partial R_y}{\partial z}\right)$$

$$= -2(\nabla \times (\nabla \times R))_x + (\lambda + 1)\,\alpha\,(\nabla \times R)_x$$

(take α = constant)

The Euler Lagrange equations are

$$-2(\nabla \times (\nabla \times R)) + (\lambda + 1)\,\alpha\,(\nabla \times R) = -(\lambda + 1)\,\alpha\,(\nabla \times R)$$

$$\nabla \times (\nabla \times R) = (\lambda + 1)\,\alpha\,\nabla \times R$$

The solution of this gives the minimum (or maximum) δW. Use the vector identity

$$b \cdot (\nabla \times a) = \nabla \cdot (a \times b) + a \cdot (\nabla \times b)$$

or, $(\nabla \times R) \cdot (\nabla \times R) = \nabla \cdot (R \times [\nabla \cdot R]] + R \cdot (\nabla \times [\nabla \times R])$

$$\alpha R \cdot (\nabla \times R) = \frac{1}{\lambda+1} R \cdot (\nabla \times \nabla \times R) = \frac{1}{\lambda+1}\left[(\nabla \times R)^2 + \nabla \cdot ([\nabla \times R] \cdot R)\right]$$

Substituting in the energy equation

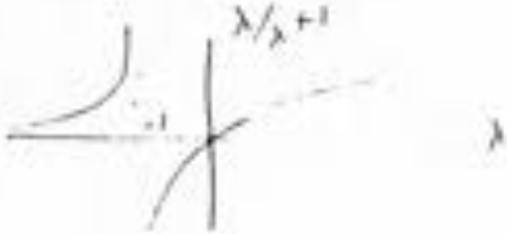
$$\delta W = \frac{1}{8\pi}\int[(\nabla \times R)^2 - \frac{1}{\lambda+1}\{(\nabla \times R)^2 + \nabla \cdot ([\nabla \times R] \times R)\}]d\tau$$

(recall $R = \xi \times B$)

at the boundary $\underline{B}$ and $\underline{\xi}$ are $\parallel$ to the surface. Therefore R $\perp$ to the surface and $[\nabla \times R] \times R$ lies in the surface. The divergence term therefore vanishes.

$$\delta W = \frac{1}{8\pi} \frac{\lambda}{\lambda + 1} \int (\nabla \times R)^2 d\tau$$

This is only negative if $-1 < \lambda < 0$



To solve for λ we must solve for R

$$\nabla \times [\nabla \times R] = (\lambda + 1)\, \alpha\, \nabla \times R$$

$$\nabla \times Q = \alpha(\lambda + 1)\, Q = \beta Q$$

$$\frac{1}{r} \frac{\partial Q_z}{\partial \phi} - \frac{\partial Q_z}{\partial z} = \beta Q_r$$

$$\frac{\partial Q_r}{\partial z} - \frac{\partial Q_z}{\partial r} = \beta Q_\phi$$

$$\frac{1}{r} [\frac{\partial}{\partial r} (r Q_\phi) - \frac{\partial Q_r}{\partial \phi}] = \beta Q_z$$

Look for solutions that go like $e^{i(kZ+m\phi)}$ in cylindrical coordinates

$$\frac{imQ_z}{r} - ikQ_\phi = \beta Q_r$$

$$ikQ_r - \frac{\partial}{\partial r} Q_z = \beta Q_\phi$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(rQ_\phi) - imQ_r\right] = \beta Q_z$$

$$\beta Q_r + ikQ_\phi = \frac{imQ_z}{r}$$

$$ikQ_r - \beta Q = \frac{\partial Q_z}{\partial r}$$

$$Q_\phi = \frac{\beta \frac{\partial Q_z}{\partial r} + \frac{kmQ_z}{r}}{k^2 - \beta^2}$$

$$Q_r = \frac{\frac{im\beta Q_z}{r} + ik \frac{\partial Q_z}{\partial r}}{k^2 - \beta^2}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(\beta r \frac{\frac{\partial Q_z}{\partial r} + km\, Q_z}{k^2 - \beta^2}\right) - \frac{\frac{m^2 \beta Q_z}{r} - mk \frac{\partial Q_z}{\partial r}}{k^2 - \beta^2}\right] = \beta Q_z$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial Q_z}{\partial r} - \frac{m^2 Q_z}{r^2} - (k^2 - \beta^2)\, Q_z = 0$$

$$\frac{\partial^2 Q_z}{\partial r^2} + \frac{1}{r}\frac{\partial Q_z}{\partial r} - \left[\frac{m^2}{r^2} + \gamma^2\right] Q_z = 0 \quad , \quad \gamma^2 = k^2 - \beta^2$$

Let $\rho = \gamma r$, $\frac{\partial}{\partial r} = \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial r} = \gamma \frac{\partial}{\partial \rho}$

$$\gamma^2 \frac{\partial^2 Q_z}{\partial \rho^2} + \frac{\gamma^2}{\rho}\frac{\partial Q_z}{\partial \rho} - \gamma^2 \left[\frac{m^2}{\rho^2} + 1\right] Q_z = 0$$

$$Q_z = I_m[r\,(k^2 - \beta^2)^{1/2}]$$

If $\gamma^2 < 0$, we get

$$Q_z = J_m[r\,(k^2 - \beta^2)^{1/2}] = J_m\,[r\,(\alpha^2(\lambda + 1)^2 - k^2)^{1/2}]$$

$$\alpha^2(\lambda + 1)^2 - k^2 = \frac{Z_m^{\;2}}{r_o^{\;2}}$$

$$(\lambda + 1)^2 = \frac{1}{\alpha^2}\,[k^2 + \frac{Z_m^{\;2}}{r_o^{\;2}}]$$

$$\lambda = -1 \pm \frac{1}{\alpha}\,(k^2 + \frac{Z_m^{\;2}}{r_o^{\;2}})^{1/2}$$

unstable if $\frac{1}{\alpha}\;(k^2 + \frac{Z_m^{\;2}}{r_o^{\;2}})^{1/2} < 1$

or since we can make κ as small as we like,

unstable if $\frac{Z_m}{\alpha r} < 1$ or $r < \frac{Z_m}{\alpha}$

α is the scale length for B_o so it is unstable if the column extends beyond the first 0.

Interchange Instability

1. Stability of incompressible plasma
2. Contribution of compressibility

We have shown that the potential energy of a plasma magentic field system changes according to

$$\delta W = \frac{1}{2} \int_{\text{plasma}} \left((\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \gamma P_o(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 + \frac{Q^2}{4\pi} + \frac{1}{4\pi} (\underset{\sim}{\nabla} \times \underset{\sim}{B}_o) \cdot (\underset{\sim}{\xi} \times \underset{\sim}{Q})\right) d\tau$$

$$- \frac{1}{2} \int_{\text{surface}} \left\{\underset{\sim}{\xi}(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \gamma P_o \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}) + \frac{(\underset{\sim}{\xi} \times \underset{\sim}{B}_o) \times \underset{\sim}{Q}}{4\pi}\right\} \cdot d\underset{\sim}{s}$$

Where the surface term can be converted to

$$+ \frac{1}{2} \int_{S} (\underset{\sim}{\xi} \cdot \underset{\sim}{n})^2 \left[\underset{\sim}{\nabla} \left(P_o + \frac{B_o^{\,2}}{8\pi}\right)\right] \cdot d\underset{\sim}{s} + \frac{1}{8\pi} \int_{\text{vacuum}} B_{1v}^2 \, d\tau$$

jumps across the boundary outside value - inside value

Putting $\gamma = \infty$ (incompressible) can only increase the stability of the system. For an incompressible plasma all perturbations must have $\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} = 0$ or the plasma energy would increase by an ∞ amount.

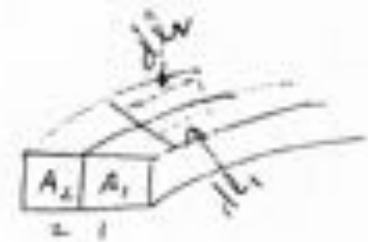
Replacing the vacuum region by a pressureless fluid increases the stability.

Introducing rigid perfectly conducting walls can only increase the stability.

Any constraint on the system inhibits the type of distortions the system is free to make and hence there are fewer ways it can move to decrease its energy.

Interchange Instability

Consider the plasma as incompressible since this only increases the stability. Interchange two flux tubes of equal volume.



$$\delta W_M = \delta \int \frac{B_1^2}{8\pi} A_1 d\ell_1 + \delta \int \frac{B_2^2}{8\pi} A_2 d\ell_2$$

$$= \delta \left[\frac{\phi^2}{8\pi} \int \frac{d\ell}{A}\right]_1 + \delta \left[\frac{\phi^2}{8\pi} \int \frac{d\ell}{A}\right]_2$$

$$\delta W_M = \frac{1}{8\pi} \left[\phi_1^2 \left(\int \frac{d\ell_2}{A_2} - \int \frac{d\ell_1}{A_1}\right) + \phi_2^2 \left(\int \frac{d\ell_1}{A_1} - \int \frac{d\ell_2}{A_2}\right)\right]$$

$$= \frac{1}{8\pi} (\phi_2^2 - \phi_1^2) \left(\int \frac{d\ell_1}{A_1} - \int \frac{d\ell_2}{A_2}\right)$$

$$= -\frac{\delta\phi^2}{8\pi} \delta \int \frac{d\ell}{A}$$

For incompressibility $A_1\, d\ell_1 = A_2 d\ell_2$ [The element of length $d\ell_1$ gets interchanged with element $d\ell_2$].

$$\delta W_M = \frac{1}{8\pi} (\phi_2^2 - \phi_1^2) \left(\int \left(\frac{d\ell_1}{A_1} - \frac{d\ell_2}{A_2} \frac{d\ell_2}{A_1} \frac{A_1}{d\ell_1}\right)\right)$$

$$\delta W_M = \frac{1}{8\pi} (\phi_2^2 - \phi_1^2) \left(\int \frac{d\ell_1}{A_1} \left(1 - \frac{d\ell_2^2}{d\ell_2\, A_2} \frac{A_1}{d\ell_1^2}\; d\ell_1\right)\right)$$

$$= \frac{1}{8\pi} (\phi_2^2 - \phi_1^2) \left(\int \frac{d\ell_1}{A_1} \left(1 - \frac{d\ell_2^2}{d\ell_1^2}\right)\right)$$

I. Mirror stability

Imagine the field interpenetrates a thin region of the plasma, inside the plasma B is 0

B increases rapidly outward, i.e. $\phi_2 > \phi_1$

$d\ell_2 > d\ell_1$ therefore $\delta W_M < 0$ and the system is MHD unstable.

II. Pinch, thin transition layer

$d\ell_2 = r_2 d\theta \qquad d\ell_1 = r_1 d\theta$

B increases outward, i.e. $\phi_2 > \phi_1$ and

$$\delta W_M = \frac{1}{8\pi}\left(\phi_2{}^2 - \phi_1{}^2\right)\left\{\int \frac{d\ell\, r_1 d\theta}{A_1}\left(1 - \frac{r_2{}^2}{r_1{}^2}\right)\right\} < 0$$

III. Uniform Current Pinch

$$B = \frac{\pi j r^2}{2\pi c} = \frac{1}{2c} j A_1 B_1 = \phi_1 \quad , \qquad A_2 B_2 = \phi_2$$

$$A_1 \frac{J\beta_1}{2c} = \phi_1 \quad , \quad A_2 \frac{J\beta_2}{2c} = \phi_2$$

$$A_1\beta_1 2\pi = V_1 \quad , \quad A_2\beta_2 2\pi = V_2$$

$$V_1 = V_2$$

Therefore $\phi_2 = \phi_1$ and $\delta W_M = 0$.

Incompressible motions do not change the energy. It might still be unstable if compressibility were included.

IV. Cusp

$\phi_2 > \phi_1$, $d\ell_2 < d\ell_1$, $\delta W > 0$

stable to incompressible motion.

V. Change in energy due to pressure

$$PV^{\gamma} = P_o V_o^{\gamma}$$

$$\delta W = -\int_{V_o}^{V} PdV = -\int_{V_o}^{V} \frac{P_o V_o^{\gamma}}{V^{\gamma}} dV = \frac{P_o V_o^{\gamma}}{\gamma - 1} \left[\frac{1}{V^{\gamma-1}}\right]_{V_o}^{V}$$

$$= \frac{P_o V_o^{\gamma}}{(\gamma-1)} \left[\frac{1}{V^{\gamma-1}} - \frac{1}{V_o^{\gamma-1}}\right]$$

$$\delta W_P = \frac{P_1 V_1^{\gamma}}{\gamma - 1}\left[\frac{1}{V_2^{\gamma-1}} - \frac{1}{V_1^{\gamma-1}}\right] + \frac{P_2 V_2^{\gamma}}{\gamma - 1}\left[\frac{1}{V_1^{\gamma-1}} - \frac{1}{V_2^{\gamma-1}}\right]$$

$$= \frac{1}{\gamma - 1}\left[P_2 V_2^{\gamma} - P_1 V_1^{\gamma}\right]\left[\frac{1}{V_1^{\gamma-1}} - \frac{1}{V_2^{\gamma-1}}\right]$$

$$\frac{1}{V_1^{\gamma-1}} - \frac{1}{V_2^{\gamma-1}} = \frac{1}{V_1^{\gamma-1}} - \frac{1}{(V_1 + \delta V)^{\gamma-1}} = (1 - \gamma)\,\frac{\delta V}{V_1^{\gamma}}$$

$$\delta W_P = \delta(PV^{\gamma})\delta V/V^{\gamma}$$

If we interchange two flux tubes of equal flux $\phi_1 = \phi_2$ and the magnetic energy chance is 0 so the only energy change is due to the pressure.

$$\delta W_P = \frac{\delta(PV^{\gamma})\delta V}{V^{\gamma}} = \frac{\delta V(\ PV^{\gamma} + \gamma PV^{\gamma-1}\delta V)}{V^{\gamma}}$$

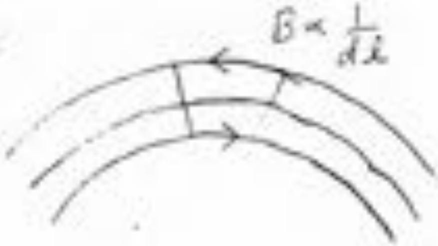
$$= \delta V\left(\delta P + \frac{\gamma P \delta V}{V}\right) \quad \begin{matrix} > 0 & \text{stability} \\ < 0 & \text{instability} \end{matrix}$$

As one goes near the walls $P \to 0$. The P term can be ignored, $\delta P < 0$; therefore, we have instability if $\delta V > 0$, that is if the volume of a flux tube increases outward or if

$$\delta \int A d\ell = \delta \int AB \frac{d\ell}{B} = \phi\, \delta \int \frac{d\ell}{B}$$

Get instability if $\int \frac{d\ell}{B}$ increases outward

If $B \alpha \frac{1}{d\ell}$ then the contribution from this goes like dl^2.



Let δV increase outward

Then get stability if

$$\delta P + \frac{\gamma P}{V}\,\delta V \geq 0$$

$$\frac{\delta P}{P} = -\gamma\,\frac{\delta V}{V}$$

$$\ln\frac{P}{P_o} = -\ln\frac{V^{\gamma}}{V_o^{\,\gamma}}$$

$$P = P_o\left(\frac{V_o}{V}\right)^{\gamma}$$

Thus, even though to achieve absolute stability with the plasma isolated from the walls δV must be negative if V increases outward one can have a rapid drop in P as one goes outward.

How are the above stability conditions affected if one uses the double adiabatic law rather than a simple adiabatic law?

Consider a cylindrical column of plasma carrying current, J(r)

Consider the interchange of two neighboring flux tubes. Find the current distributions and pressure distribution which gives stability.

VI. Application of the Equations of Motion

$$\rho_0 \ddot{\underline{\xi}} = \underline{F}(\underline{\xi})$$

$$\underline{F}(\underline{\xi}) = \underline{\nabla}(\underline{\xi} \cdot \underline{\nabla} P_0 + \gamma P_0 \underline{\nabla} \cdot \underline{\xi}) + \frac{1}{4\pi} \{[\underline{\nabla} \times (\underline{\nabla} \times (\underline{\xi} \times \underline{B}_0))] \times \underline{B}_0$$

$$+ (\underline{\nabla} \times \underline{B}_0) \times \underline{\nabla} \times (\underline{\xi} \times \underline{B}_0)\}$$

Consider the simple case of an infinite uniform plasma with straight field lines

$$\underline{\nabla} P_0 = 0, \qquad \underline{\nabla} \times \underline{B}_0 = 0$$

$$\underline{F}(\underline{\xi}) = \gamma P_0 \underline{\nabla}\underline{\nabla} \cdot \underline{\xi} + \frac{1}{4\pi} (\underline{\nabla} \times \underline{Q}) \times \underline{B}_0$$

$$\underline{Q} = \underline{\nabla} \times (\underline{\xi} \times \underline{B}_0) = \underline{\xi}(\underline{\nabla} \cdot \underline{B}_0) + (\underline{B}_0 \cdot \underline{\nabla})\underline{\xi} - \underline{B}_0(\underline{\nabla} \cdot \underline{\xi}) - (\underline{\xi} \cdot \underline{\nabla})\underline{B}_0 = 0$$

$$\underline{Q} = (\underline{B}_0 \cdot \underline{\nabla})\underline{\xi} - \underline{B}_0(\underline{\nabla} \cdot \underline{\xi})$$

First look for solutions for which $\nabla \cdot \xi = 0$. Take B_o to be in the z direction

$$\rho_o \ddot{\xi} = \frac{1}{4\pi} [\nabla \times (B_o \cdot \nabla \xi)] \times B_o$$

$\hat{z}$, therefore no z part to the acceleration, $\xi_z = 0$

$$\rho_o \ddot{\xi} = \frac{B_o^2}{4\pi} [\nabla \times \frac{\partial \xi}{\partial z}] \times \hat{z}_z = -\frac{B_o^2}{4\pi} \frac{\partial^2 \xi}{\partial_z^2}$$

$$\omega^2 = \frac{k_z^2 B_o^2}{4\pi \rho^2} = k_z^2 V_A^2$$

For compressible motions look for solutions that go like $e^{ik\cdot r}$

$$-\rho_o \omega^2 \xi = -\gamma p_o k\, k \cdot \xi - \frac{B_o^2}{4\pi}([k\times(k\times[\xi\times\hat{z}_z])]\times\hat{z}_z)(k\times[(-k\cdot\xi)\hat{z}_z + k_z \xi])\times\hat{z}_z$$

$$-k\cdot\xi\ (k\times\hat{z}_z)\times\hat{z}_z + k_z(k\times\xi)\times\hat{z}_z\ (\hat{z}_z k_z - k) + k_z\ (k_z \xi - \xi_z k)$$

$$-\rho_o \omega^2 \xi = -\gamma P_o kk \cdot \xi - \frac{B_o^2}{4\pi} \{-(\hat{z}_z k_z - k)\ k\cdot\xi + k_z^2 \xi - kk_z \xi_z\}$$

$$= -\gamma p_o kk \cdot \xi - \frac{B_o^2}{4\pi} \{-\hat{z}_z k_z k \cdot \xi + k(k\cdot\xi - k_z \xi_z)\}$$

$$\rho_o \omega^2 \xi = \gamma p_o k(k \cdot \xi) + \frac{1}{4\pi} [k\times(B\cdot k)\xi_o] \times B_o - \frac{1}{4\pi} (k\times B[k\cdot\xi_o]) \times B$$

$k||B$

$$\rho_o \omega^2 \xi = \gamma p_o k^2 \xi_{\parallel} \hat{\xi}_{\parallel} + \frac{B_o^2}{4\pi} k^2 \xi_\perp \hat{\xi}_\perp$$

two root

$$\omega^2 = \frac{\gamma p_o k^2}{P} \qquad \xi||B$$

$$\omega^2 = \frac{B^2 k^2}{4\pi\rho_o} \qquad \zeta \perp B$$

$k \perp B$

$$P_o \omega^2 \underset{\sim}{\zeta} = \rho_o \underset{\sim}{k}(\underset{\sim}{k} \cdot \underset{\sim}{\zeta}) + \frac{\underset{\sim}{k}}{4\pi} B_o^{\,2} (\underset{\sim}{k} \cdot \underset{\sim}{\zeta})$$

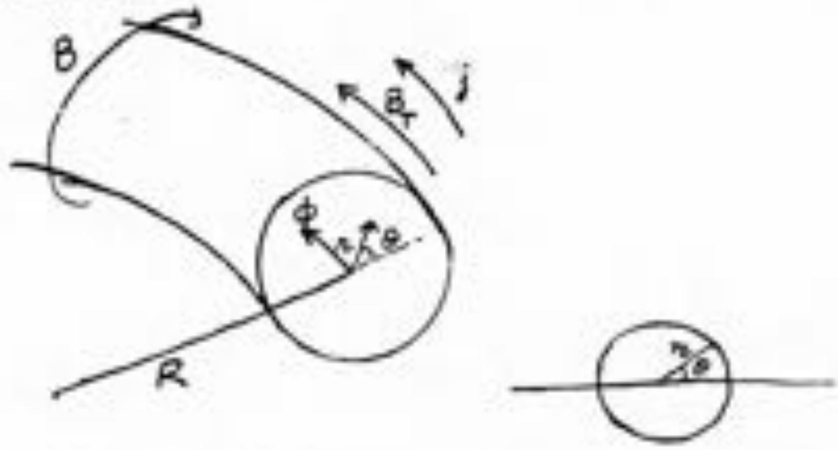
The right hand side is $\parallel$ to $\underset{\sim}{k}$ so only motion $\parallel$ $\underset{\sim}{k}$ enters

$$\omega^2 = \frac{\gamma P_o k^2 + \dfrac{B_o^{\,2} k^2}{4\pi}}{\rho_o} = k^2 \left[\frac{\gamma P_o}{\rho_o} + \frac{B_o^{\,2}}{4\pi\rho_o}\right]$$

Problem: Find the dispersion relation for arbitrary direction of propagation of the waves. Compute the phase and group velocities.

Neoclassical Diffusion in a Torus (Tokamak)

We looked at diffusion of plasma across a B-field of straight lines. Let us now look at diffusion in a more complex field, that of a Tokamak. We consider it to be a circular torus carrying a uniform current across its cross-section.



The field lines are helical about the minor axis at least in the limit of the radius being small compared to R.

If we neglect particle drifts then the particles will follow the field lines. Because the toroidal field goes as

$$B = \frac{B_0 R}{R + r\cos\theta} \approx B_0\left(1 - \frac{r}{R}\cos\theta\right)$$

Neglecting the contribution of the poloidal field to the strength of B, we can take the strength of B to go as given by this formula. Because the strength of the field increases at the inside, particles following the field lines will see the field increase as they go, towards the center and some of them will be mirror reflected.

$$m\frac{dv_{\parallel}}{dt} = -\mu\frac{dB}{ds}$$

$$v_{\parallel}^2 + v_{\perp}^2 = W$$

$$\frac{v_{\perp}^2}{B} = \text{constant}$$

Let our reference point be on the outside of the torus.

$$v_{\perp in}^2 = v_{\perp out}^2 \frac{B_{MAX}}{B_{Min}}$$

For particles which just reflect

$$v_{\perp out}^2 \frac{B_{Max}}{B_{Min}} = v_{\perp out}^2 + v_{\parallel out}^2$$

$$v_{\parallel out}^2 = v_{\perp out}^2 \left(\frac{B_{Max}}{B_{Min}} - 1 \right)$$

$$v_{\parallel out} = v_{\perp out} \sqrt{\frac{B_{Max}}{B_{Min}} - 1}$$

$$B_{Max} = B_0 \left(1 + \frac{r}{R} \right)$$

$$B_{min} = B_0 \left(1 - \frac{r}{R} \right)$$

$$\frac{B_{Max}}{B_{Min}} = 1 + \frac{2r}{R}$$

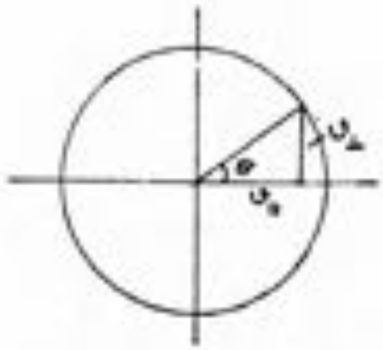
$$v_{\parallel out} = v_{\perp out} \sqrt{\frac{2r}{R}}$$

Even for rather small values of r, $v_{\parallel}$ is rather large. For r = .1R

$$v_{\parallel out} = v_{\perp out} \sqrt{.2} = .45\, v_{\perp out}$$

For Taylor's tokamak R = 2.5r and $v_{\parallel} \gtrsim v_{\perp}$.

This implies a large fraction of the particles are reflected. The fraction that is reflected is equal to the solid angle



The angle dividing passing from Reflecting

$$\cos\theta = \frac{v_\parallel}{\sqrt{v_\parallel^2+v_\perp^2}} = \frac{\cancel{v_\perp}\sqrt{\frac{2r}{R}}}{\cancel{v_\perp}\sqrt{1+\frac{2r}{R}}}$$

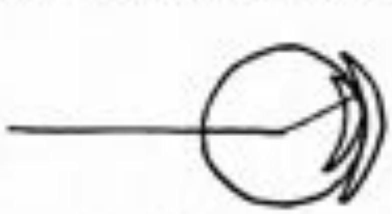
The fraction that is reflected is

$$F = \frac{4\pi - 2\int_0^\theta 2\pi \sin\theta \, d\theta}{4\pi} = 1 - 1 + \cos\theta = \cos\theta$$

$$F = \frac{\sqrt{\frac{2r}{R}}}{\sqrt{1+\frac{2r}{R}}}$$

For r = .1R F = .4

As for the trapped particles, they will oscillate back and forth along the field lines if we neglect the drifts. When toroidal drifts are included the particles will move off the field lines. Their orbits in the r, θ, plane are in the shape of bananas.



$$\omega_B^2 = \cancel{\frac{\mu B_0 i^2 r}{m R^2}} \quad \frac{\mu\, \Pi\, i^2 B_0}{(2\pi)^2 R^3\, m}, \quad \mu = \frac{m v_\perp^2}{2 B_0}$$

$$\omega_B^2 = \frac{v_\perp^2 i^2 r}{2(2\pi)^2 R^3}, \quad \omega_B = \frac{v_\perp i}{2\pi}\sqrt{\frac{r}{2R^3}}$$

$$\Delta r = \frac{v_T^{\cancel{2}}}{\omega_c R \cancel{v_T} i}\sqrt{\frac{R}{r}}$$

Diffusion

ν_{eff} in the small r/R lim

$$\Delta r^2 \nu_{eff} = D$$

$$\Delta v_\parallel^2 = v_T^2 \nu t$$

$$\nu_{eff} = \nu \frac{R}{r}$$

$$\nu_{eff} = \frac{1}{t} = \frac{v_T^2}{\Delta v_\parallel^2}\nu = \frac{R}{2r}\nu$$

$$D = \frac{v_T^2}{\omega_c^2 \cancel{R^2} i^2}\,\frac{\cancel{R^2}}{r^2}\nu$$

We must multiply this by the fraction of trapped particles

$$F = \sqrt{\frac{2r}{R}}$$

$$D = \frac{\rho_c^2}{i^2 r^2}\nu\sqrt{\frac{2r}{R}}$$

The above derivation has assumed that particles make a complete transit around a banana before making a collision

$$\tau_B \nu \frac{R}{r} < 1$$

If this is not true then the result must be modified. Suppose the effective collision frequency is much larger than the bounce frequency, then particles will execute a small fraction of a banana before they jump to another banana. The distance they will go is roughly

$$\Delta r = \Delta r_{banana}\frac{\tau_{coll}}{\tau_b} = \frac{\Delta r_{banana}}{\nu \tau_b}$$

$$D = \frac{\Delta r^2_{banana}}{\nu^{\cancel{2}} \tau_b^2}\cancel{\nu} = \frac{\Delta r^2_{banana}}{\nu \tau_b^2}$$

We can estimate the thickness of the bananas as follows. If the particle stays on a magnetic surface, r does not change and θ oscillates back and forth. It is the displacement in r we are interested in, $\dot{r} = (v_D)_r$

$$V_D = \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{\omega_c R}$$

$(v_D)_r = v_D \sin\theta$

$\theta = \theta_m \sin\omega_B t$, ω_B is the bounce frequency (see discussion below)
θ_m is the maximum excursion in θ

$\dot{r} = v_D \sin\left[\theta_m \sin\omega_B t\right] \sim v_D\theta_m \sin\omega_B t$

$$\Delta r = \frac{v_D\theta_m}{\omega_B}\cos\omega_B t$$

The most important particles for the diffusion are those with θ_m large of the order of 1

Bouncing

$$\Delta r \approx \frac{v_D}{\omega_B} \approx \frac{v_\perp^2}{\omega_c R \omega_B}$$

$$m\frac{dv_\parallel}{dt} = -\mu\frac{dB}{ds}$$

$$B = B_0\left(1 + \frac{r}{R}\cos\theta\right)$$

$\theta = iz$ — i is the rotational transform, z is the distance traveled around the torus

$$B = B_0\left(1 + \frac{r}{R}\cos iz\right)$$

$s \parallel z$ approx.

$$\frac{dB}{dz} = B_0\frac{ir}{R}\sin iz \simeq B_0\frac{i^2 r}{R}z$$

$$m\ddot{z} = -\mu B_0\frac{i^2 r}{R}z$$

$$D = \cancel{\frac{v_T^2}{\omega_c^2 \iota^2 R r}} \; \cancel{\frac{v_T \iota}{2\iota}\sqrt{\frac{r}{R}}} \quad \frac{v_T^2}{\omega_c^2 \iota^2 R r} \; \frac{v_T^2 \iota^2 r}{R \nu} \leftarrow$$

$$\cancel{\neq \frac{v_T^3}{\omega_c^2 \iota\, r^{1/2} R^{3/2}\, 2\iota}} = \frac{v_T^4}{\omega_c^2 R^2 \nu}$$

<u>A more precise description of Bananas</u>

By circular symmetry the angular momentum is conserved

$$P_\phi = \left(m v_\phi + \frac{q}{c} A_\phi\right)\left(R + r\cos\theta\right)$$

A_ϕ is determined by

$$\nabla^2 A_\phi = -\frac{4\pi}{c} j_\phi$$

We also have conservation of energy

$$\frac{m}{2}\left(v_\perp^2 + v_\parallel^2\right) = W$$

and conservation of magnet moment

$$\frac{m v_\perp^2}{2B} \cong \frac{m v_\perp^2 \left(R + r\cos\theta\right)}{2 B_0 R}$$

$v_\parallel \gtrsim v_\phi$ more precisely $v_\phi + i r v_\theta$

Take a uniform current density

$$A_\phi = A_0 r$$

$$\left(m v_\phi + \frac{q}{c}\frac{A_0 r^2}{2}\right)\left(R + r\cos\theta\right) = P_\phi$$

$$\frac{m}{2}\left(v_\perp^2 + v_\phi^2\right) = W$$

$$\frac{m v_\perp^2}{2B_0}\left(1 + \frac{r}{R}\cos\theta\right) = \mu$$

$$v_\perp^2 = \frac{\tilde{\mu}}{1 + \frac{r}{R}\cos\theta}, \quad v_\phi = \sqrt{\frac{2W}{m} - \frac{\tilde{\mu}}{1 + \frac{r}{R}\cos\theta}}$$

$$\left\{ m\sqrt{\frac{2W}{m} - \frac{\tilde{\mu}}{1 + \frac{r}{R}\cos\theta}} + \frac{q}{c}\frac{A_0 r^2}{2} \right\}\left\{ R + r\cos\theta \right\} = P_\phi$$

gives r as a function of θ.

To a good approximation if the banana . size is small

$$m v_\phi + \frac{q}{c}\frac{A_0 r_1^2}{2} = \check{p} = \frac{P_\phi}{R} = -m v_\phi + \frac{q}{c}\frac{A_0 r_2^2}{2}$$, for the reverse trip

$$\frac{q}{c} A_0 \frac{(r_1^2 - r_2^2)}{2} = 2 m v_\phi$$

$$\frac{q}{c} A_0 r \Delta r = 2 m v_\phi$$

$$\Delta r = \frac{2 m v_\phi}{\frac{q A_0 r}{c}} = \frac{2 m v_\phi c}{q B}$$