III. Moments of the Vlasov Equation; Two Fluid MHD

A. Equations for n, $\underline{u}$, $\underset{\sim}{p}$.

We define, for each species (but omitting species indices for the most part), a mean density and velocity,

$$n\ (\underline{x},\ t) = \int d\underline{v}\,\bar{n}f \tag{3.1}$$

$$n\underline{u}\ (\underline{x},\ t) = \int d\underline{v}\,\bar{n}\underline{v}f \quad ; \tag{3.2}$$

a pressure tensor,

$$\underset{\sim}{p} = \int d\underline{v}\,\bar{n}mf\ \Delta\underline{v}\Delta\underline{v} \quad ; \tag{3.3}$$

and a heat flow tensor

$$\underset{\sim}{Q} = Q_{ijk} = \int d\underline{v}\,\bar{n}mf\ \Delta v_i \Delta v_j \Delta v_k \quad , \tag{3.4}$$

where $\Delta\underline{v} = \underline{v} - \underline{u}$.

As in Chapter II, we use $\bar{n}$ to denote the average particle density, $\bar{n} = N/V$, which is constant, while n $(\underline{x},\ t)$ denotes the space and time dependent fluid dynamic density.

Taking the first three moments of the Vlasov equation we find the equation of continuity

$$\partial n/\partial t + \nabla\cdot(n\underline{u}) = 0 \quad ; \tag{3.5}$$

the momentum equation,

$$mn(\partial\underline{u}/\partial t + \underline{u}\cdot\nabla\underline{u}) + \nabla\cdot\underset{\sim}{p} = nq(\underline{E} + \underline{u}\times\underline{B}/c) \quad ; \tag{3.6}$$

and the stress tensor equation

$$(\partial\underset{\sim}{p}/\partial t + \underline{u}\cdot\nabla\underset{\sim}{p}) + \underset{\sim}{p}\nabla\cdot\underline{u} + \underset{\sim}{p}\cdot\nabla\underline{u} + \widetilde{\underset{\sim}{p}\cdot\nabla\underline{u}} - \frac{q}{mc}\,(\underset{\sim}{p}\mathrm{x}\underline{B} + \widetilde{\underset{\sim}{p}\mathrm{x}\underline{B}}) + \nabla\cdot\underset{\sim}{Q} = 0 \tag{3.7}$$

The tilde ∿ denotes transpose of a dyadic or second rank tensor, $\tilde{P}_{ij} = P_{ji}$.

It is clear that, due to the $\underline{v}\cdot\nabla f$ term in the Vlasov equation, the equation for the nth velocity moment of f will involve the (n+1)th moment, so that the set of equations <u>cannot close</u>.

It must be emphasized that the very nature of the fluid dynamic description of the plasma is quite different from that associated with the Vlasov equation or other kinetic equations. The former involves, for each species, a series of functions (n, $\underline{u}$, $\underset{\sim}{p}$, ...) of $\underline{x}$, t, which satisfy an infinite set of coupled partial differential equations; the latter deals with a single function, f (the density in the 6 dimensional phase space) but there are now 7 independent variables ($\underline{v}$, $\underline{x}$, t), not just 4, so the description is much more detailed. Ensemble averaging takes us from Ψ to f; velocity averaging, from f to n, $\underline{u}$, $\underset{\sim}{p}$, etc.

Note that the pressure tensor, $\underset{\sim}{p}$, as defined here is in accord with the usual definition for a continuum fluid, where $\underline{F} = -\underset{\sim}{p}\cdot d\underline{\sigma}$ is the force acting on (i.e., transmitted across) a surface element, $d\underline{\sigma} = \hat{n}d\sigma$, moving with the fluid. (See Fig. 3.1 ; the sign convention is that $\underline{F}$ is the force exerted by region 1 on region 2.) Since force is equivalent to a rate of change of momentum, a microscopic calculation of such a force involves just the momentum transferred through the surface from side 1 to side 2. Each particle of velocity $\Delta\underline{v}$ relative to the surface carries momentum $m\Delta\underline{v}$ and the flux of such particles through the surface from 1 to 2 is $-\bar{n}f\Delta\underline{v}\cdot d\underline{\sigma}d\underline{v}$. Thus

$$\underline{F} = -\underset{\sim}{p}\cdot d\underline{\sigma} = -\int d\underline{v}\bar{n}m\Delta\underline{v}\Delta\underline{v}f\cdot d\underline{\sigma}$$

giving (3.3) for $\underset{\sim}{p}$. Similarly, some of the elements of $\underset{\sim}{Q}$ correspond to a flux of kinetic energy.

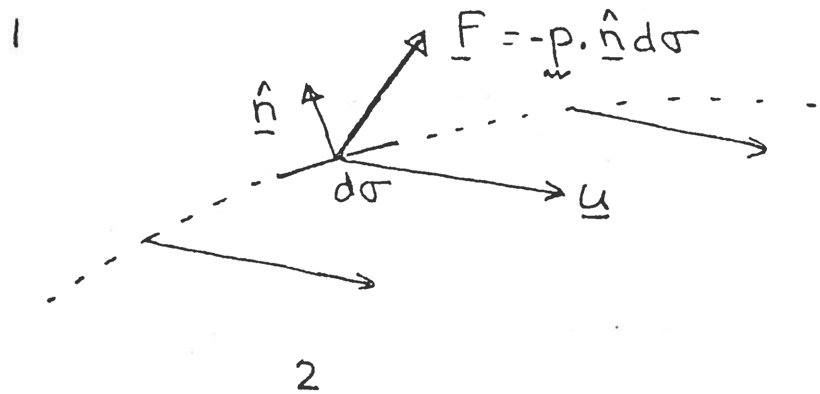

Fig. 3.1 Force transmitted across a surface element $d\underline{\sigma} = \hat{\underline{n}} d\sigma$ moving with the local fluid velocity, $\underline{u}$.

All of the equations (3.5) through (3.7) involve the convective derivative,

$$d/dt \equiv \partial/\partial t + \underline{u}\cdot\nabla$$

so that (3.5), for example, can be written

$$dn/dt + n\nabla\cdot\underline{u} = 0$$

and similarly for (3.6) and (3.7).

As with the Vlasov equation, the fluid equations, (3.5) through (3.7), can be written down on phenomenological grounds. Thus, given any volume V, with closed bounding surface S, conservation of mass gives

$$\frac{\partial}{\partial t}\int_V d\underline{x}\, n + \int_S d\underline{\sigma}\cdot n\underline{u} = 0$$

Using Gauss' theorem gives

$$\int_V d\underline{x}[\partial n/\partial t + \nabla\cdot(n\underline{u})] = 0$$

and since V is arbitrary we have (3.5). Similarly for momentum conservation:

$$(\partial/\partial t)\int_V d\underline{x} nm\underline{u} + \int_S d\underline{\sigma}\cdot(\underset{\approx}{p} + nm\underline{u}\underline{u}) = \int_V d\underline{x}[nq(\underline{E} + \underline{u}\times\underline{B}/c) + \underline{P})$$

whence

$$\partial(nm\underline{u})/\partial t + \nabla\cdot\underset{\approx}{p} + \nabla\cdot(nm\underline{u}\underline{u}) = nq(\underline{E} + \underline{u}\times\underline{B}/c) + \underline{P} \tag{3.8}$$

where $\underline{P}$ denotes the transfer of momentum from one species to the other due to collisions. We can include it in a phenomenological derivation like this without gross inconsistency, but in our deductive treatment such terms appear only when we go to the second order (quasilinear) approximation. If we neglect $\underline{P}$, then this simplifies, on using (3.5), to give just (3.6). In a similar way, conservation of energy gives an equation identical to that obtained by taking the trace of (3.7) when $\underset{\sim}{p}$ is diagonal and $\underset{\approx}{Q}$ is diagonal in two of its three indices. This phenomenological "derivation" is useful in that it makes clear the relation of the fluid equations to conservation of mass, energy and momentum.

B. The Conventional Approximations

For a collision-dominated gas, a similar moment analysis of the Boltzmann equation is carried out. The Boltzmann equation has the form

$$\mathcal{L} f = (\delta f/\delta t)_B$$

where the right hand side represents short range collisions. If these occur often enough to make the mean free path and mean free time (ℓ_c, t_c) short compared to typical macroscopic lengths and times (ℓ_0, t_0), then an expansion in (ℓ_c/ℓ_0), (t_c/t_0) can be carried out, yielding the Hilbert-Enskog-Chapman kinetic theory. In this expansion, it is assumed that collisions keep f nearly Maxwellian and the theory solves for the small corrections to this.

Although this elegant formalism cannot really be justified for hot, dilute plasmas, where collisions are infrequent, it has proved useful to copy some features of this method, namely to simplify the equations in a way which would be correct *if* f were Maxwellian,

$$\bar{n} f_M = n \exp[-(\underline{v}-\underline{u})^2/a^2]/(\pi^{1/2}a)^3 \tag{3.9}$$

with n, $\underline{u}$ and a functions of $\underline{x}$ and t. The n and $\underline{u}$ here are identical with the quantities defined in (3.1), (3.2) and we find that the stress tensor is diagonal,

$$\underset{\sim}{p} = p\,\underset{\sim}{1} = (nma^2/2)\,\underset{\sim}{1} = nT\,\underset{\sim}{1} \tag{3.10}$$

while the heat flow tensor vanishes,

$$\underset{\sim}{Q} = 0 \tag{3.11}$$

We should emphasize that this "Maxwellian closure" approximation, as we shall call it, is useful because it gives a large fraction of the physics correctly in a majority of cases and hence is invaluable as an initial attack on a new problem. In subsequent chapters we shall discuss in detail the differences between results of the fluid approximation and those given by the Vlasov equation.

When we use (3.10) and (3.11), the momentum equation (3.6) is slightly simplified because $\nabla\cdot\underset{\sim}{p}$ becomes simply ∇p. The energy equation (3.7) is enormously simplified, first because, as noted, $\underset{\sim}{Q} = 0$, closing the infinite set of moment equations, and second because we now need only an equation for

$$p = \mathrm{Tr}\underset{\sim}{p}/3$$

and hence can take the trace of (3.7), obtaining

$$\partial p/\partial t + \underline{u}\cdot\nabla p + (5/3)p\nabla\cdot\underline{u} = 0$$

Using (3.5), we can write this as

$$dp/dt - 5/3\ p\,d(\log n)/dt = 0$$

whence

$$d(pn^{-\gamma})/dt = 0 \tag{3.12}$$

with

$$\gamma = 5/3$$

Thus, with the Maxwellian closure approximation, the energy equation gives simply the familiar adiabatic equation of state,

$$pn^{-\gamma} = \text{constant} \tag{3.13}$$

with a γ appropriate to three dimensions [$\gamma = (f+2)/f = 5/3$ when f = number of degrees of freedom is 3]. (We shall see later that in presence of an external magnetic field, a plasma species may, under various circumstances, behave as either a two or a one dimensional gas, with $\gamma = 2$ or 3 respectively.)

Since our "Maxwellian closure" approximation is not rigorously justifiable, we might consider closing the set of moment equations in more cavalier fashion by simply neglecting the energy equation altogether and assuming T = constant. This is just equivalent to assuming $\gamma = 1$ in (3.13) and hence does not usually involve a qualitative change in the results. (We should also remark that in a strict sense (3.12) and (3.13) state that $pn^{-\gamma}$ is constant along a "trajectory", $d\underline{x}/dt = \underline{u}$. This "constant" will be independent of $\underline{x}$ only if $pn^{-\gamma}$ is uniform initially.)

Often we can be content with a still rougher approximation, in which thermal effects are neglected altogether. This corresponds formally to setting p and T equal to 0, and leaves just two equations, (3.5) and (3.6), for the two fluid variables $n, \underline{u}$, plus, of course, Maxwell's equations for $\underline{E}$ and $\underline{B}$. We shall see later that this approximation is useful when all of the phase velocities are large compared to the thermal velocities of the particles. It is often referred to as the "cold" plasma approximation, where, of course, the term "cold" must not be taken too literally.

In summary, the conventional two fluid equations, obtained by averaging the Vlasov equation and invoking Maxwellian closure, are the set

$$\partial n/\partial t + \nabla \cdot n\underline{u} = 0 \tag{3.14}$$

$$nm(\partial \underline{u}/\partial t + \underline{u}\cdot\nabla\underline{u}) + \nabla p = nq(\underline{E} + \underline{u}\times\underline{B}/c) + \underline{P} \tag{3.15}$$

$$pn^{-\gamma} = \text{constant} \tag{3.16}$$

for each species, plus the Maxwell equations

$$\boxed{\begin{aligned} \nabla\cdot\underline{E} &= 4\pi(\sum qn + \rho_e) = 4\pi\rho \\ \nabla\times\underline{B} &= 4\pi c^{-1}(\sum nq\underline{u} + \underline{j}_e) + \dot{\underline{E}}/c = (4\pi\underline{j} + \dot{\underline{E}})/c \end{aligned}}$$ (3.17) (3.18)

$$\boxed{\begin{aligned} \nabla\cdot\underline{B} &= 0 \\ \nabla\times\underline{E} + \dot{\underline{B}}/c &= 0 \end{aligned}}$$ (3.19) (3.20)

In a later chapter, we will discuss the corrections to (3.14) through (3.16) which arise from the deviation of f from a Maxwellian. This leads to "transport theory", i.e. to the phenomena of diffusion, viscosity, thermal conductivity, and electrical resistivity.

C. Linearized Waves and Dispersion Relations

The fluid equations with the Maxwellian closure approximation, together with Maxwell's equations, i.e. the set (3.14) through (3.20), constitute a complete system of equations for 6+5S variables $\underline{E}$, $\underline{B}$, n_α, p_α, $\underline{u}_\alpha$, in a system with S species. (Unless otherwise specified we shall generally consider S = 2, i.e. electrons and one ion species.) There are (8+5S) equations altogether, but (3.17) and (3.19) have the role of initial conditions:

i) From (3.20) it follows that $(\partial/\partial t)\nabla\cdot\underline{B} = 0$

ii) From (3.18), (3.14) and the continuity equation

$$\partial\rho_e/\partial t + \nabla\cdot\underline{j}_e = 0$$

for the external sources it follows that $(\partial/\partial t)(\nabla\cdot\underline{E} - 4\pi\rho) = 0$.

Thus, if (3.17) and (3.19) are satisfied initially, the remaining equations guarantee that they will hold for all time.

Given a system of partial differential equations like these, we can most easily explore their physical content by looking at the linearized theory. We take a simple, even trivial solution of the equations and study small perturbations about it. The simplest such "unperturbed" solution is just

$$n = n_0 = \text{constant} \qquad p = p_0 = \text{constant}$$

We then set

$$n = n_0 + n_1(\underline{x},t) \qquad p = p_0 + p_1(\underline{x},t)$$

and expand in n_1, p_1, $\underline{u}$, $\underline{E}$ and $\underline{B}$, keeping only terms linear in these quantities. This gives the set

$$\left.\begin{aligned}
&\partial n_1/\partial t + n_0\nabla\cdot\underline{u} = 0\\
&\partial\underline{u}/\partial t + \nabla p_1/mn_0 = (q/m)\,\underline{E}\\
&p_1 = \gamma n_1 p_0/n_0 = n_1\gamma T_0\\
&\nabla\cdot\underline{E} = 4\pi(\textstyle\sum qn_1 + \rho_e)\\
&\nabla\times\underline{B} = 4\pi c^{-1}(\textstyle\sum n_0 q\underline{u} + \underline{j}_e) + \dot{\underline{E}}/c\\
&\nabla\cdot\underline{B} = \nabla\times\underline{E} + \dot{\underline{B}}/c = 0
\end{aligned}\right\} \qquad (3.21)$$

which can be solved using three different approaches:

1) The Simple-Minded Plane Wave Substitution
2) The Response to External Fields
3) The Initial Value Formulation .

In this section and the next, we follow method 1), which simply exploits the fact that in the absence of external sources ($\rho_e = \underline{j}_e = 0$) any set of homogeneous linear partial differential equations like (3.21) admits plane wave solutions, with each dependent variable proportional to $\exp[i(\underline{k}\cdot\underline{x}-\omega t)]$. Writing

$$n_1 = \hat{n}_1 \exp[i(\underline{k}\cdot\underline{x}-\omega t)]$$

etc. we immediately obtain from (3.21) a set of 11 independent linear homogeneous equations for the amplitudes $\hat{n}_1$, $\hat{\underline{u}}_1$, etc. These have a nontrivial solution if and only if the determinant of the set

$$D(\underline{k},\omega) = 0$$

The resulting relation between $\underline{k}$ and ω is called a dispersion relation.

The next simplest unperturbed state is one with a constant, uniform magnetic field, $\underline{B}_0$, a case we shall study in great detail. Also interesting is the case of a uniform oscillating electric field, $\underline{E}_0 \sin\omega_0 t$, which leads to parametric instabilities and provides a nice illustration of a time dependent unperturbed state. An important and extensively studied case is that of an unperturbed state with a density gradient, i.e. n_0 a function of $\underline{x}$, which in the presence of a magnetic field, $\underline{B}_0$, gives rise to drift waves and "universal instabilities." We shall return to these more complicated unperturbed states in subsequent chapters.

D. Wave Modes in a Uniform, Unmagnetized Plasma

Instead of forming a $(6+5S)\times(6+5S)$ determinant, we can, when convenient, eliminate variables from the set of linear equations. Using n_1, $\underline{u}_1$ etc. to denote the amplitudes in place of $\hat{n}_1$, $\hat{\underline{u}}_1$, we have the set of equations

$$\omega n_1 = n_0 \underline{k}\cdot\underline{u} \tag{3.22}$$

$$i(\underline{k}c_\alpha^2 n_1/n_0 - \omega\underline{u}) = (q/m)\underline{E} \tag{3.23}$$

$$i\underline{k}\cdot\underline{E} = 4\pi\sum q n_1 \tag{3.24}$$

$$i(\underline{k}\times\underline{B} + \omega\underline{E}/c) = 4\pi c^{-1}\sum n_0 q\underline{u} \tag{3.25}$$

$$\underline{k}\times\underline{E} = \omega\underline{B}/c \tag{3.26}$$

[We have omitted $\underline{k}\cdot\underline{B} = 0$, since it follows from (3.26) but have, for convenience, retained (3.24), even though it follows from (3.22) and (3.25).] We have used the adiabatic equation of state to eliminate p_1, and have introduced the notation

$$c_\alpha^2 = \gamma p_0/n_0 m = \gamma T_0/m = \gamma a^2/2 \quad .$$

We call c_α the "sound speed" for species α because of its formal resemblance to that quantity for a neutral gas.

1. Electrostatic Modes

From the first two equations we easily find

$$n_1/n_0 = \underline{k}\cdot\underline{u}/\omega = i(q/m)\underline{k}\cdot\underline{E}/\omega^2 + k^2 c_\alpha{}^2 n_1/\omega^2 n_0 = [i(q/m)\underline{k}\cdot\underline{E}/\omega^2][1 - (kc_\alpha/\omega)^2]^{-1}$$

Substituting this into (3.24) gives

$$(\underline{k}\cdot\underline{E})[1 - \sum \omega_p{}^2/(\omega^2 - k^2 c_\alpha{}^2)] = 0$$

Thus, longitudinal waves ($\underline{E}$ parallel to $\underline{k}$) are possible only if $\underline{\omega}$ and $\underline{k}$ satisfy the dispersion relation

$$\boxed{1 = \sum \frac{\omega_p{}^2}{\omega^2 - k^2 c_\alpha{}^2}} \qquad (3.27)$$

We call these electrostatic modes since, $\underline{B}$ must vanish if ω satisfies (3.27). (See problem 4). For two species, (3.27) is a biquadratic in ω, and hence trivial to solve explicitly. However, we get a little more physical insight using the following approach.

a) In the limit of heavy ions, $m/M \to 0$, the ion contribution to (3.27) vanishes and we have

$$\omega^2 = \omega_k{}^2 \equiv \omega_{pe}^2 + k^2 c_e{}^2 \qquad (3.28)$$

the Bohm-Gross dispersion equation for electron plasma oscillations or Langmuir waves. Here we see that thermal effects shift the frequency away from the simple (cold plasma) result $\omega = \omega_p$ obtained in Chapter I. There is dispersion (ω depends on $\underline{k}$) and the waves have not only a phase velocity,

$$v_{ph} = \omega/k \qquad (3.29)$$

but also a group velocity

$$v_g = \partial\omega/\partial k = c_e{}^2/v_{ph} \qquad (3.30)$$

Large shifts in frequency occur only if $kc_e \gtrsim \omega_{pe}$, which implies $\gamma^{1/2}k/k_D \gtrsim 1$. We shall see later that the correct Vlasov treatment modifies the results significantly for such short wavelengths, the waves being heavily damped there. As in many problems, the fluid treatment gives partially correct results which are nonetheless useful in describing the principal features of the physics.

If ω is given, then (3.28) can be used to find k. However, if $\omega < \omega_{pe}$, then k is imaginary, implying that there can be no propagating solutions. (We will see later [Chapter V, Section G] that the proper sign of Imk corresponds to spatial damping, not growth.)

b) For $m/M \neq 0$, but still small there are small corrections, of order m/M, to (3.28), which are of no interest. There is also a new mode. If we look for low frequency solutions to (3.27), with

$$\omega \ll kc_e \tag{3.31}$$

then we have

$$1 = -\omega_{pe}^2/k^2c_e^{\;2} + \omega_{pi}^2/(\omega^2 - k^2c_i^{\;2})$$

and hence

$$\omega^2 = k^2c_i^{\;2} + \omega_{pi}^2(1 + \omega_{pe}^2/k^2c_e^{\;2})^{-1} = k^2[c_i^{\;2} + (m/M)c_e^{\;2}/(1+k^2c_e^{\;2}/\omega_{pe}^2)] \tag{3.32}$$

It is clear from (3.32) that (3.31) is satisfied since c_i is of order $c_e(m/M)^{1/2}$. This solution of the dispersion equation, called the ion acoustic mode, differs drastically in character from the Langmuir dispersion equation: for small $k \ll k_D$, $\omega \to \omega_{pe}$ = constant and $v_{ph} \to \infty$ for Langmuir waves, while $\omega \to 0$ and $v_{ph} \to c_s$ = constant for ion acoustic waves.

We shall see later, when studying the Vlasov equation, that like Langmuir waves of short wavelength, ion acoustic waves are heavily damped for $T_e \approx T_i$, so the fluid theory results are not useful in this regime. However, for $T_e >> T_i$, they are valid. In this limit the c_i^2 term in(3.32) is negligible and we have

$$\omega^2 = \Omega_k^2 \equiv k^2 c_s^2 (1 + \gamma k^2/k_D^2)^{-1}$$

where

$$c_s^2 = (m/M)\, c_e^2 = \gamma\, T_e/M \tag{3.33}$$

The restoring forces come from the electron pressure, $n_0 T_e$, the inertia from the ions. While ion motion and hence ion contributions to the charge density are negligible for Langmuir waves, n_e/n_i is of order 1 for ion acoustic waves in the $T_e/T_i >> 1$ limit where they are well defined.

2. Electromagnetic Modes

In general, we can resolve any vector, like $\underline{E}$ or $\underline{u}$, into components parallel and perpendicular to $\underline{k}$, e.g.

$$\underline{E} = E_\ell \hat{\underline{k}} + \underline{E}_t$$

$$\hat{\underline{k}} = \underline{k}/k \qquad E_\ell = \hat{\underline{k}}\cdot\underline{E} \qquad \underline{E}_t = \hat{\underline{k}}\times(\underline{E}\times\hat{\underline{k}})$$

(While the conventional terminology of electromagnetic theory refers to $\underline{E}_\ell$ and $\underline{E}_t$ as "longitudinal" and "transverse" components respectively, they are often referred to in plasma physics as electrostatic and electromagnetic polarizations to avoid confusion with the use of longitudinal and transverse to designate directions relative to an external magnetic field, $\underline{B}_0$, when there is one, rather than to $\hat{\underline{k}}$.)

For a two species plasma, the set (3.22) through (3.26) constitute 13 independent equations for the thirteen variables n_1, u_ℓ and $\underline{u}_t$ (for each species) and E_ℓ, $\underline{E}_t$, $\underline{B}_t$. Fortunately, they decouple into a set of five equations for the longitudinal variables n_1, u_ℓ, E_ℓ, leading to the dispersion relation (3.27), and a set of 8 equations for the transverse variables $\underline{E}_t$, $\underline{B}_t$, $\underline{u}_t$, with a dispersion relation which we now derive.

Substituting (3.26) into (3.25) gives

$$(\omega^2\underline{E}/c^2 - \underline{E}_t k^2) = -4\pi i\omega c^{-2} \sum n_0 q\underline{u} \qquad (3.34)$$

From the longitudiral component of (3.34) we recover just the electrostatic mode discussed in the preceding section. The transverse component, on the other hand, together with the transverse component of (3.23), gives

$$(\omega^2 - k^2c^2)\, \underline{E}_t = - 4\pi i\omega \sum n_0 q\underline{u}_t = \sum \omega_p{}^2 \underline{E}_t$$

or

$$\underline{E}_t (\omega^2 - \omega_p{}^2 - k^2c^2) = 0 \qquad (3.35)$$

Thus, transverse or electromagnetic modes, $\underline{E}_t \neq 0$, are possible only if ω and $\underline{k}$ satisfy the dispersion relation

$$\omega^2 = \omega_p{}^2 + k^2c^2 \qquad (3.36)$$

(For the usual plasma, consisting of electrons and ions, where $m/M \ll 1$, we shall use $\omega_p{}^2$ to denote either $\omega_{pe}^2 + \omega_{pi}^2$ or just ω_{pe}^2, since m/M corrections to the Longmuir frequency are never important.)

As with (3.28) there is dispersion and a cut-off at ω_p. In the limit of vanishing plasma density, these waves go over into the usual transverse solutions, $\omega^2 = k^2c^2$, of Maxwell's equations for vacuum. Note that there are no "thermal" effects, i.e., the dispersion relation (3.36) does not involve the temperature or pressure and hence would be the same for a "cold" plasma. For low frequencies, $\omega << \omega_p$, the dispersion relation (3.36) gives $k = i(\omega_p/c)$ so that an electromagnetic wave incident on an "overdense" plasma (i.e., one with $\omega_p >> \omega$) will penetrate only a distance of order c/ω_{pe}. This is called the "collisionless skin depth" since the phenomenon is similar to the well known skin effect associated with collisional resistivity of a medium.

3. Unified Treatment of Longitudinal and Transverse Modes.

We can write (3.34) in the form

$$(\omega^2/c^2)\, \underline{E} + \underline{k} \times (\underline{k} \times \underline{E}) = -4\pi i\omega \underline{j}_p/c^2,$$

sometimes called the vector wave equation, where

$$\underline{j}_p = \sum n_0\, q\underline{u}$$

is the plasma current. It is clear from (3.22) and (3.23) that $\underline{u}$, for each species must be a linear function of $\underline{E}$, and that hence

$$\underline{j}_p = \underset{\sim}{\sigma} \cdot \underline{E}$$

where the specific form of the conductivity tensor $\underset{\sim}{\sigma}$ can be found from (3.22) and (3.23). Of course, any linearized description of the plasma will give an "Ohm's Law" of this kind, different models leading to different specific forms for $\underset{\sim}{\sigma}$. Substituting this expression for $\underline{j}_p$ into the vector wave equation gives

$$\underset{\sim}{D} \cdot \underline{E} = 0$$

where

$$\underset{\sim}{D} = \underset{\sim}{1} + 4\pi i\underset{\sim}{\sigma}/\omega - (kc/\omega)^2 \, (\underset{\sim}{1} - \underline{k}\,\underline{k}/k^2), \tag{3.37}$$

The condition for this set of homogeneous equations to have a non-zero solution is just the dispersion equation

$$\det \underset{\sim}{D} = 0.$$

whose solutions will simply be (3.36) and the two electrostatic mode solutions, $\omega^2 = \omega_k{}^2$ and $\omega^2 = \Omega_k{}^2$, of (3.27).

E. Response to External Fields

We now return to the set of linearized equations (3.21) but adopt a different viewpoint, one which recognizes the role of the external sources, ρ_{ext} and $\underline{j}_{ext}$, and the associated external fields $\underline{E}_{ext}$, $\underline{B}_{ext}$ generated by them. We assume that ρ_{ext} and $\underline{j}_{ext}$ are plane waves proportional to exp $[i(k\cdot x-\omega t)]$, both $\underline{k}$ and ω now being regarded as given. Then from the longitudinal equations of the set we have

$$n_1 = (n_0 k/\omega)u_\ell = (n_0 k/\omega)[(iq/m\omega)E_\ell + (kc_\alpha{}^2/\omega)n_1/n_0]$$

so that

$$n_1 = \frac{(iqn_0 k/m\omega^2)E_\ell}{1 - k^2 c_\alpha{}^2/\omega^2}$$

and the contribution of species α to the charge density

$$qn_1 = (ik/4\pi)\,[\omega_p^{\,2}(\omega^2 - k^2 c_\alpha^{\,2})^{-1}]E_\ell \tag{3.38}$$

is proportional to E_ℓ. We define the constant of proportionality as the susceptibility, in analogy with the quantity encountered in the classical theory of slowly varying fields in macroscopic media, i.e., we write (3.38) in form

$$qn_1 = (k/4\pi i)\chi E_\ell \tag{3.39}$$

where

$$\chi(\underline{k},\omega) = -\omega_p^{\,2}(\omega^2 - k^2 c_\alpha^{\,2})^{-1} \tag{3.40}$$

is defined as the susceptibility of this species.

Note that E_ℓ here denotes the total longitudinal field, due to both external and plasma sources, and (3.39) shows what charge density this induces in the plasma. The virtue of this approach is that we can calculate χ from the dynamics of the system, treating the field as known and ignoring the self-consistent aspects resulting from Maxwell's equations. Then, once χ is known, these aspects are easily incorporated. Thus, the Poisson equation gives us

$$ikE_\ell = 4\pi(\textstyle\sum qn_1 + \rho_{ext}$$

or

$$E_\ell = -\textstyle\sum \chi E_\ell + (4\pi/ik)\rho_{ext} \tag{3.41}$$

Since, by definition,

$$ikE_{\ell ext} = 4\pi\rho_{ext} \tag{3.42}$$

we have

$$E_\ell(\underline{k},\omega) = E_{\ell ext}(\underline{k},\omega)/\varepsilon_\ell(\underline{k},\omega) \tag{3.43}$$

where

$$\varepsilon_\ell(\underline{k},\omega) = 1 + \sum\chi(\underline{k},\omega) = 1 - \sum\omega_p{}^2(\omega^2-k^2c_\alpha{}^2)^{-1} \tag{3.44}$$

is, again by analogy, called the longitudinal dielectric constant or, reflecting the fact that it is actually a function of $\underline{k}$ and ω, the longitudinal dielectric function. In any case, note that the situation is quite different from the case of slowly varying fields. There we conventionally write $E(\underline{x},t) = E_{ext}(\underline{x},t)/\varepsilon$ where ε is really, to good approximation, a constant, i.e.,independent of $\underline{x}$ and t. The simple proportionality expressed by (3.43) and (3.44), however, holds only for the $(\underline{k},\omega)$ Fourier components; the relation in $(\underline{x},t)$ space is, as we shall see, nonlocal in both $\underline{x}$ and t.

We see from (3.43) that the total response to an external field is greater the smaller ε. For $\varepsilon = 0$ there is an infinite response, but this is just the usual situation of an external source driving a system at one of its natural resonances. (Think of a harmonic oscillator driven at its natural frequency.) We shall see in the next section how to describe this apparently divergent result in sensible physical terms, even when the resonance is not limited by dissipative effects. The fact that this is a natural resonance of the system in absence of external sources (i.e., for $\rho_{ext} = 0$, $\underline{j}_{ext} = 0$) is clear when we note that the dispersion relation (3.27) for longitudinal waves, derived in the previous section, is just

$$\varepsilon_\ell(\underline{k},\omega) = 0 \tag{3.45}$$

In an entirely similar way, we obtain from the transverse components of (3.21) the result

$$\underline{j}_t = \sum n_0 q \underline{u}_t = i\sum(n_0 q^2/m\omega)\underline{E}_t = \sigma\underline{E}_t \tag{3.46}$$

where,
$$\sigma = \sum i\omega_p^{\,2}/4\pi\omega \tag{3.47}$$

Again, we have a simple proportionality of current density to external field, which can then be substituted into Maxwell's equations,

$$\underline{k}\times(\underline{k}\times\underline{E}) + \omega^2\underline{E}/c^2 = -4\pi i\omega c^{-2}\underline{j}$$

The transverse component gives

$$(\omega^2-k^2c^2)\underline{E}_t = -4\pi i\omega\ (\underline{j}_t + \underline{j}_{t,ext}) = -4\pi i\omega\sigma\underline{E}_t - 4\pi i\omega\underline{j}_{t,ext}$$

or

$$\underline{E}_t = -4\pi i\omega\underline{j}_{t,ext}/(\omega^2 - k^2c^2 + 4\pi i\omega\sigma) \tag{3.48}$$

Since $\underline{E}_{t,ext}$ and $\underline{j}_{t,ext}$ must satisfy (3.48) with $\sigma = 0$, we can also write

$$\underline{E}_t = \frac{(\omega^2-k^2c^2)\ \underline{E}_{t,ext}}{(\omega^2-k^2c^2+4\pi i\omega\sigma)} \tag{3.49}$$

Still another common form arises from writing Ampere's equation in a form analogous to that for slowly varying fields in a non-magnetic medium:

$$i\underline{k}\times\underline{B} = 4\pi c^{-1}\underline{j} - i\omega\underline{E}/c = -i\omega\varepsilon_t\underline{E}/c \tag{3.50}$$

with
$$\varepsilon_t = 1 + 4\pi i\sigma/\omega \tag{3.51}$$

Then (3.49) becomes

$$\underline{E}_t = \frac{(\omega^2 - k^2c^2)\underline{E}_{t,ext}}{\omega^2\varepsilon_t - k^2c^2} \tag{3.52}$$

For transverse waves, the dispersion relation (3.36) is thus equivalent to

$$(kc/\omega)^2 = \varepsilon_t \tag{3.53}$$

since (3.47) and (3.51) give

$$\varepsilon_t = 1 + 4\pi i\sigma/\omega = 1 - \sum\omega_p^2/\omega^2 \tag{3.54}$$

Defining an index of refraction,

$$N = kc/\omega \tag{3.55}$$

we have the result familiar from optics,

$$N^2 = \varepsilon_t \tag{3.56}$$

F. Initial Value Problems

We return again to the linearized equations (3.21) but this time we take the viewpoint of an initial value problem: Given the values of n_1, $\underline{u}$, $\underline{E}$ at all $\underline{x}$ at time $t = 0$, we use the equations (3.21) to determine their values at later times. Since any physically sensible problem can be posed as an initial value problem, this approach can settle many otherwise ambiguous or confusing points. Having once understood these, we can use the simpler approaches of the previous two sections, returning to the initial value point of view whenever we encounter difficulty.

Since the unperturbed plasma is assumed uniform over all space, a Fourier transform of the spatial dependence is appropriate. For the time dependence, however, a Laplace or one-sided Fourier transform is most convenient on account of the initial value character of the problem.

We define the Fourier-Laplace transform of n_1 as

$$n_1(\underline{k},\omega) \equiv \int d\underline{x} \int_0^\infty dt \exp[-i(\underline{k}\cdot\underline{x}-\omega t)] n_1(\underline{x},t) \tag{3.57}$$

and similarly for $\underline{u}$, $\underline{E}$, $\underline{B}$. We assume $n_1(\underline{x},t)$ to be sufficiently well behaved at large $\underline{x}$ (e.g., $n_1(\underline{x},t) \to 0$ as $|\underline{x}| \to \infty$) so that the spatial integration converges. For large times, however, n_1 may not decrease; it may even, in the case of an "unstable" plasma, increase exponentially with t. We therefore assume that in (3.57) Im ω is positive and large enough to make the integral convergent. Equivalently, $n_1(\underline{k},\omega)$ is defined by (3.57) for ω sufficiently far up in the complex ω plane. The inverse of (3.57) is

$$n_1(\underline{x},t) = (2\pi)^{-4} \int d\underline{k} \int_C d\omega n_1(\underline{k},\omega) \exp[i(\underline{k}\cdot\underline{x}-\omega t)] \tag{3.58}$$

where $\underline{k}$ is real but the ω integration is along a contour C which lies in the upper half ω plane <u>above</u> any singularities of $n_1(\underline{k},\omega)$.

[Two comments on notation:

i) Laplace transforms are conventionally expressed in terms of $p = -i\omega$, but for plasma physics the notation used here proves more convenient.

ii) Good mathematical notation would use a different symbol, say $\tilde{n}_1(\underline{k},\omega)$ for the transform, but the resulting notational complication is more a nuisance than a help in a subject already carrying a heavy notational burden. For example, sometimes it is convenient to transform only one of the independent variables, i.e., to consider quantities like

$$n_1(\underline{k},t) = \int d\underline{x} \exp(-i\underline{k}\cdot\underline{x}) n_1(\underline{x},t),$$

etc. We shall therefore generally use the same symbol for a physical variable and its transform, adding extra notation, e.g., the $n_1(t=0)$ in (3.59), where needed for clarity.]

In transforming the set (3.21), we note that the Laplace transform of a term like $\partial n_1/\partial t$ is given by

$$\int_0^\infty dt\ \exp(i\omega t)(\partial n_1/\partial t) = -n_1(t=0) - i\omega n_1(\omega) \qquad (3.59)$$

We shall define

$$n_1^{\ 0}(\underline{k}) = n_1(\underline{k},t=0) \equiv n_1(\underline{k},t)\Big|_{t=0} \qquad (3.60)$$

Then the transformed equations (3.21) are

$$\begin{aligned}
& i(n_0\underline{k}\cdot\underline{u} - \omega n_1) = n_1^{\ 0} \\
& i(\underline{k}c_\alpha^{\ 2}n_1/n_0 - \omega\underline{u}) - (q/m)\underline{E} = \underline{u}^0 \\
& i(\underline{k}\times\underline{B} + \omega\underline{E}/c) - 4\pi c^{-1}\textstyle\sum n_0 q\underline{u} = -\underline{E}^0/c \\
& i(\underline{k}\times\underline{E} - \omega\underline{B}/c) = \underline{B}^0/c \\
& i\underline{k}\cdot\underline{B} = i\underline{k}\cdot\underline{E} - 4\pi\textstyle\sum n_1 q = 0
\end{aligned} \qquad (3.61)$$

where, for simplicity, we shall initially consider the case of no external sources ($\rho_{ext} = \underline{j}_{ext} = 0$).

Comparing these with the set (3.22) through (3.26), we see one, essential difference: the earlier equations were homogeneous, whereas the present set, even in the absence of external sources, is inhomogeneous, due to the initial value terms on the right side. To see the effect of this, consider the longitudinal portion. We have

$$\begin{aligned}
n_1/n_0 &= ku_\ell/\omega + in_1^{\ 0}/n_o\omega = (k/\omega)[kc_\alpha^{\ 2}n_1/n_0\omega + (i/\omega)(u_\ell^{\ 0} + qE_\ell/m)] + \\
&+ in_1^{\ 0}/n_0\omega = \frac{(ikq/m\omega^2)E_\ell + i(n_1^{\ 0}/n_0\omega + ku_\ell^{\ 0}/\omega^2)}{1 - k^2c_\alpha^{\ 2}/\omega^2}
\end{aligned} \qquad (3.62)$$

so the charge density is

$$qn_1 = (k/4\pi i)[\chi E_\ell - I] \qquad (3.63)$$

where

$$I \equiv (4\pi q n_0)\ [n_1{}^0\omega/n_0 k + u_\ell{}^0]\ [\omega^2 - k^2 c_\alpha{}^2]^{-1} \tag{3.64}$$

and χ is given by (3.40).
Then Poisson's equation gives

$$E_\ell = (4\pi/ik)\sum q n_1 = -\sum \chi_\alpha E_\ell + \sum I_\alpha \tag{3.65}$$

or

$$E_\ell(\underline{k},\omega) = \sum I_\alpha(\underline{k},\omega)/\varepsilon_\ell(\underline{k},\omega) \tag{3.66}$$

This resembles (3.43), save that the inhomogeneity is here due to the initial values of n_1 and $\underline{u}$ rather than an external field. Had we included the latter, then it is clear that in the numerator of (3.66) we would have $[E_{\ell ext} + \sum I_\alpha]$ in place of $\sum I_\alpha$. The important point, however, is in the physical interpretation of the result (3.66).

Consider the inverse transformation from ω back to t:

$$E_\ell(\underline{k},t) = \int_C (d\omega/2\pi) E_\ell(\underline{k},\omega)\exp(-i\omega t) \tag{3.67}$$

where C is a horizontal contour in the ω plane lying above any singularities of $E_\ell(\underline{k},\omega)$. From (3.44) we have

$$\varepsilon_\ell(\underline{k},\omega) = 1 - \sum \omega_p{}^2(\omega^2 - k^2 c_\alpha{}^2)^{-1} = (\omega^2-\omega_k^2)\ (\omega^2-\Omega_k{}^2)/(\omega^2-k^2c_e{}^2)(\omega^2-k^2c_i{}^2) \tag{3.68}$$

where ω_k and Ω_k indicate the high frequency (Langmuir or Bohm-Gross) and ion acoustic frequencies, respectively,

$$\omega_k{}^2 = \omega_p{}^2 + k^2 c_e{}^2 \qquad\qquad \Omega_k{}^2 = k^2 c_s{}^2 (1 + \gamma k^2/k_D{}^2)^{-1} \tag{3.69}$$

(For the ion acoustic branch we are assuming $T_i/T_e \ll 1$, but this does not affect the general argument.)

From (3.64) and (3.66) we see that the only singularities of $E_\ell(\underline{k},\omega)$ occur at $\omega = \pm\omega_k,\ \pm\Omega_k$. Otherwise the integrand of (3.67) is analytic so we can deform the contour C to a contour C' lying below the real ω (see Fig. 3.2) provided we take proper account of the contribution of singularities lying between C and C':

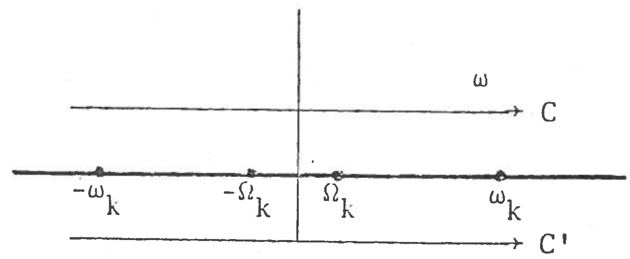

Figure 3.2 Integration contours for evaluating inverse Laplace transforms.

$$E_\ell(\underline{k},t) = \int_{C'} (d\omega/2\pi) E_\ell(\underline{k},\omega)\exp(-i\omega t) + \sum_j (-i)R_j \exp(-i\omega_j t) \qquad (3.70)$$

where the R_j are the residues of $E_\ell(\underline{k},\omega)$ at the singularities ω_j. If the contour C' lies a distance p below the real ω axis, then the first term in (3.70) is proportional to exp(-pt) and becomes as small as we please if C' is far enough below the real ω axis. Thus, we are left with an expression for $E_\ell(\underline{k},t)$ as a superposition of normal modes (i.e., solutions of the homogeneous equations)

$$E_\ell(\underline{k},t) = -i \sum_j \sum_\alpha I_\alpha (\underline{k},\omega_j)(\partial\varepsilon/\partial\omega)^{-1}\Big|_{\omega=\omega_j} \exp(-i\omega_j t) \qquad (3.71)$$

If we want $E_\ell(\underline{x},t)$ we can carry out the integration also over $\underline{k}$, obtaining the longitudinal (or irrotational) portion, $\underline{E}_\ell$, of $\underline{E}$:

$$\underline{E}_\ell(\underline{x},t) = \int \frac{d\underline{k}}{(2\pi)^3} e^{i\underline{k}\cdot\underline{x}} E_\ell(\underline{k},t)\,\hat{\underline{k}}, \qquad \hat{\underline{k}} = \underline{k}/k$$

corresponding to the decomposition

$$\underline{E}(\underline{x},t) = \underline{E}_\ell + \underline{E}_t$$

$$\nabla \times \underline{E}_\ell = 0, \quad \nabla\cdot\underline{E}_t = 0$$

However, the important lesson associated with the initial value formulation is already apparent from (3.71): <u>the roots of the longitudinal dispersion relation simply describe the normal modes which will be excited by an arbitrary choice of initial conditions</u>, n_1^0, u_ℓ^0. (Analysis of the transverse modes can be carried out in a similar way.) <u>This is the significance of</u>

the linear dispersion relation. If we keep it in mind we can often analyze a problem using the simpler approach of either of the two preceeding sections, but interpreting the roots of the dispersion relation according to the initial value problem viewpoint discussed in this section.

It should be clear that had we included external sources as well as the initial value terms in analyzing (3.61), the only effect would be to replace (3.66) by

$$E_\ell(\underline{k},\omega) = [E_{e\ell}(\underline{k},\omega) + \sum I_\alpha]/\varepsilon_\ell(\underline{k},\omega) \tag{3.72}$$

So long as $E_{e\ell}(\underline{k},\omega)$ has no singularities, the only change will be a modification in the values of the R_j. Of course, if our external field is purely periodic,

$$E_{e\ell}(\underline{k},t) = E_0 \exp(-i\omega_0 t) \tag{3.73}$$

we will have

$$E_{e\ell}(\underline{k},\omega) = E_0 i(\omega-\omega_0)^{-1} \tag{3.74}$$

which will add a "steady-state" term, proportional to $\exp(-i\omega_0 t)$, to (3.71). In the special case where ω_0 coincides with one of the natural frequencies ω_j, we will have a second order pole and hence a secular contribution to E_ℓ, of the form $t\exp(-i\omega t)$. Again, this is just the familiar situation of a resonant system driven at one of its natural frequencies.

In general, $\underline{E}(\underline{k},\omega)$ will be a quotient of two terms, the numerator arising from the inhomogeneous terms (external sources or initial values) and the denominator involving $\varepsilon(\underline{k},\omega)$. About the numerator we can make no general statement so far as the location of singularities, if any, is concerned. However, the denominator, ε_ℓ or $(\varepsilon_t\omega^2-k^2c^2)$, is universal in the sense that it is independent of the inhomogeneities which drive the system. Thus, its roots are of general interest for all linearized problems.

G. Two Stream Instabilities

Our discussions of linearized fluid dynamics have been restricted to the case where the unperturbed plasma is uniform, homogeneous and at rest. Under these circumstances, it is not surprising that the natural modes have a bounded, purely oscillatory character. If the plasma is not in equilibrium, we find that some of the waves are unstable, that is, some solutions, ω_j, of the dispersion relation have $\mathrm{Im}\omega_j > 0$ and hence contribute to $\underline{E}(\underline{k},t)$ an exponentially growing term, proportional to $\exp(-i\omega_j t)$. We shall later be discussing a great many examples of this; here we consider one of the simplest cases.

If one charged species in a plasma has a streaming or drift velocity relative to the other component(s), then a perturbation in potential, $\delta\phi$, will cause a change in velocity, $\delta\underline{v}$, which in turn causes a perturbation in density, δn. Since charge density variations give rise to changes in potential, this induces a $\delta\phi$, which can, provided the phase relations are appropriate, amplify the original $\delta\phi$, resulting in an instability.

To study this quantitatively, consider the situation where each species has in the unperturbed state a streaming velocity, V_α. Then we set $\underline{u} = \underline{V} + \underline{u}_1$

and the first two equations of the linearized set (3.21) become

$$(\partial/\partial t + \underline{V}\cdot\nabla)n_1 + n_0\nabla\cdot\underline{u}_1 = 0 \tag{3.75}$$

$$(\partial/\partial t + \underline{V}\cdot\nabla)\underline{u}_1 + c_\alpha^{\ 2}\nabla n_1/n_0 = (q/m)(\underline{E}+\underline{V}\times\underline{B}/c). \tag{3.76}$$

while the Ampere's law equation gives

$$\nabla\times\underline{B} = 4\pi c^{-1}[\sum(n_0 q\underline{u} + n_1 q\underline{V}) + \underline{j}_e] + \dot{\underline{E}}/c \tag{3.77}$$

In general, the $\underline{V}\times\underline{B}$ term will couple the longitudinal and transverse polarizations, but we shall restrict ourselves to the case $\underline{k}$ parallel to $\underline{V}$ where this does not happen. For the longitudinal variables, the only change is then the replacement of $\partial/\partial t$ by $(\partial/\partial t + \underline{V}\cdot\nabla)$ i.e., of ω by $(\omega-\underline{k}\cdot\underline{V}) = (\omega-kV)$ when we take the Fourier-Laplace transform. Carrying through an analysis like that of sections E or F we obtain a result of the form (3.43), (3.66) or, most generally, (3.72) but with ε_ℓ given by

$$\varepsilon_\ell(\underline{k},\omega) = 1 - \sum \omega_p^{\ 2}[(\omega-kV)^2 - k^2c_\alpha^{\ 2}]^{-1} . \tag{3.78}$$

Since the results must be Galilean invariant, we can choose to work in the ion frame; then $V_i = 0$, V_e is the relative electron-ion streaming velocity, and (3.78) becomes

$$1 = \omega_{pe}^2/[(\omega-kV)^2 - k^2c_e^{\ 2}] + \omega_{pi}^2/(\omega^2-k^2c_i^{\ 2}) \tag{3.79}$$

Consider the cold plasma version of this, c_e, $c_i \to 0$, and let

$$s = \omega/\omega_p \qquad \kappa = kV/\omega_p \qquad \delta = m/M \tag{3.80}$$

Then the dispersion relation becomes

$$1 = (s - \kappa)^{-2} + \delta/s^2 \equiv G(s) \tag{3.81}$$

and its general properties can be readily inferred from the graph of G(s) vs s, as shown in Fig.3.3).Clearly, G has a single relative minimum, at $s = s_0$, located on the interval $0 < s_0 < \kappa$. When rationalized, (3.81) is a quartic in s and hence must have four solutions. As we see from Fig. 3, if $1 > G(s_0)$ there are four real solutions, but for $1 < G(s_0)$ there are only two real solutions. The other two must be complex conjugates of one another, since the coefficients of the quartic are all real, and therefore one root must have $\mathrm{Im}\omega = \omega_p \mathrm{Im}s > 0$, corresponding to a growing wave or instability. In general, we want to know three things about any instability:

1) Its threshold, i.e. the conditions which the independently variable parameters of the problem must satisfy in order that $\mathrm{Im}\omega \geq 0$.

2) Its growth rate, i.e., the value of $\mathrm{Im}\omega$ when the threshold has been exceeded.

3) The nonlinear mechanisms which cause saturation and the level at which the instability saturates.

The first two of these can be determined from linear theory, as we shall now illustrate for the two stream instability we are considering. The third is much more difficult, not only because it involves solution of a nonlinear problem but also because there will generally be a number of different physical effects which, a priori, seem like reasonable choices for the saturation mechanism and we must determine, by heuristic arguments or by analysis, which actually leads to the lowest saturation level. We shall defer until much later the discussion of these nonlinear considerations, which have been carried through for a number of specific cases.

From (3.81) we see that $G'(s_0) = 0$ implies

$$(s_0-\kappa)/s_0 = -\delta^{-1/3}$$

or

$$s_0 = \kappa\delta^{1/3}(1 + \delta^{1/3})^{-1} \approx \kappa\delta^{1/3} \tag{3.82}$$

and that

$$G(s_0) = \kappa^{-2}(1 + \delta^{1/3})^3 \approx \kappa^{-2} \tag{3.83}$$

The threshold or condition for instability, $1 < G(s_0)$, thus becomes

$$\kappa = kV/\omega_p < (1 + \delta^{1/3})^{3/2} \approx 1 \tag{3.84}$$

To determine the growth rate, we must actually solve (3.81) for s.

This is a quartic in s, but we can avoid the unpleasant algebra associated with the solution of a quartic by exploiting the smallness of δ. Buneman (PRL 1, 8 (1958)) has pointed out that an expansion in δ reduces (3.81) to a cubic, for which a solution in parametric form can easily be obtained. If we are above threshold, we may still expect that $|s| \sim \kappa\delta^{1/3}$ for the unstable roots, in which case the second term on the right side of (3.81) is of order $\delta^{1/3}$ compared to the first term:

$$(\kappa - s) = (1 - \delta/s^2)^{-1/2} \doteq 1 + \delta/2s^2 \tag{3.85}$$

(We have chosen the sign, in taking the square root, to yield the unstable root of the complex conjugate pair.)

Let

$$s = r\exp(i\theta)$$

Then the imaginary part of (3.85) gives

$$r\sin\theta = (\delta/2r^2)\sin 2\theta$$

or

$$r = (\delta\cos\theta)^{1/3} \tag{3.86}$$

while the real part gives

$$\kappa = 1 + r\cos\theta + (\delta/2r^2)\cos 2\theta = 1 + \delta^{1/3}[\cos^{4/3}\theta + \cos 2\theta/2\cos^{2/3}\theta] \tag{3.87}$$

For each θ, $0 \leq \theta \leq \pi$, (3.86) and (3.87) give r and κ, and hence

and

$$\left.\begin{aligned} \mathrm{Re}\,\omega &= \omega_p\, r\cos\theta; \quad \mathrm{Im}\,\omega = \omega_p r\sin\theta \\ k &= \kappa\omega_p/V \end{aligned}\right\} \tag{3.88}$$

Using θ as a parameter, we can then plot Reω and Imω against k, as shown in Fig. 3.4. That Imω has a maximum follows from setting $d\mathrm{Im}\omega/d\theta = \omega_p\delta^{1/3}(d/d\theta)(\sin\theta\cos^{1/3}\theta) = 0$ which gives

$$\tan\theta = \sqrt{3} \qquad \theta = \pi/3$$

At that point, $\kappa \equiv 1$, $\sin\theta = \sqrt{3}/2$ so the maximum growth rate for this instability is

$$\mathrm{Im}\,\omega_{max} = (3^{1/2}/2)\ (\delta/2)^{1/3}\omega_p$$

and occurs when

$$k = \omega_p/V \quad .$$

Finally, we note that the condition

$$\delta/s^2 = \delta^{1/3}\cos^{-2/3}\theta << 1$$

required for the expansion (3.85) is satisfied for all θ of interest, being violated only for $\theta \approx \pi/2$ where $\mathrm{Im}\,\omega \propto \cos^{1/3}\theta \to 0$.

Although keeping c_e and c_i in (3.79) allows us to consider modification of these results due to thermal effects, we postpone discussion of electron-ion two stream instabilities in a warm plasma to Chapter V, where the effects of temperature are (more conveniently) considered using the linearized Vlasov equation.

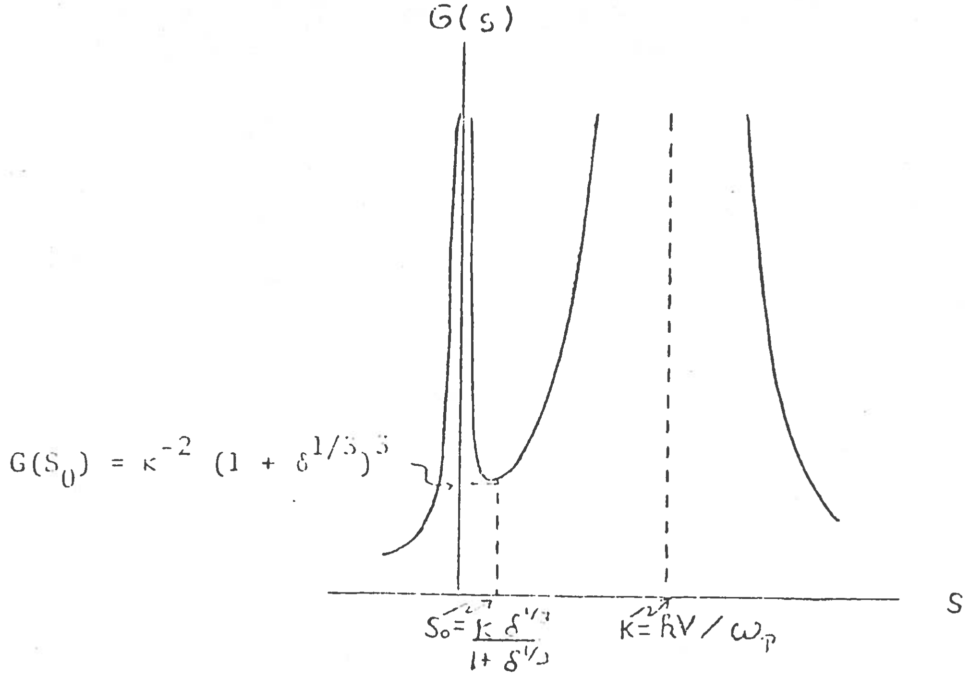

Fig. 3.3 A sketch of the function $G(s) = (s - \kappa)^{-2} + \delta/s^2$ which relates wave number and phase velocity for longitudinal waves in a cold, current-carrying plasma.

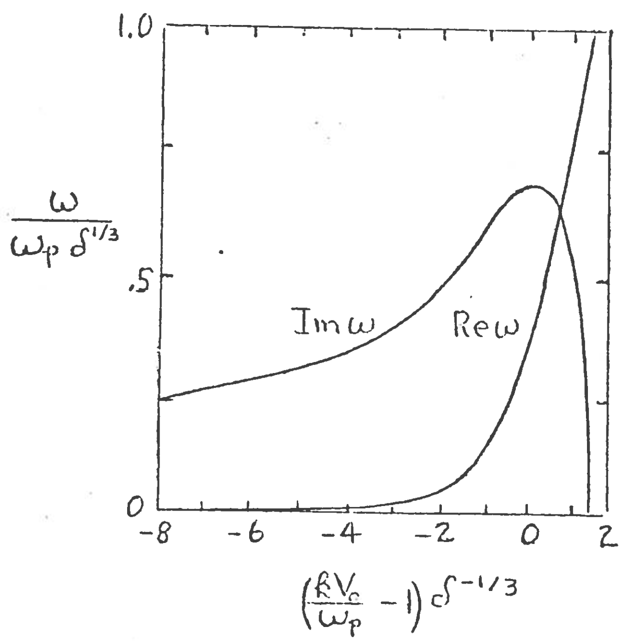

Fig. 3.4 Real and imaginary parts of ω vs. k for the two stream instability in a cold, current carrying plasma. The ions are at rest in the lab frame so the mean electron velocity V_0 is also the drift velocity.