

## Chapter 1

### Charges and Fields

The teaching of electromagnetic theory is something like that of American History in school; you get it again and again. Well, this is the end of the line. Here is where we put it all together, and yet, not quite, since it is still classical electrodynamics and the final goal is quantum electrodynamics. This preoccupation reflects the all-pervasive nature of electromagnetism, with implications ranging from the farthest galaxies to the interiors of the fundamental particles. In particular, the properties of ordinary matter, including those properties classified as chemical and biological, depend only on electromagnetic forces, in conjunction with the microscopic laws of quantum mechanics.

## MAXWELL'S EQUATIONS

### 1. Electrostatics

Our intention is to move toward the general picture as quickly as possible, starting with a review of electrostatics. We take for granted the phenomenology of electric charge, including the Coulomb law of force between charges of dimensions small in comparison with their separation. This is expressed by the interaction energy,  $E$ , of a system of such charges in otherwise empty space ~~a vacuum~~:  
exp

$$E = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{e_i e_j}{r_{ij}} \quad (1-1.1)$$

Here  $e_i$  is the charge of the  $i^{\text{th}}$  particle while

$$r_{ij} = |\vec{r}_i - \vec{r}_j| \quad (1-1.2)$$

is the separation between the  $i^{\text{th}}$  and  $j^{\text{th}}$  particles. (Throughout this book we use the Gaussian system of units.) As we shall see, this starting point, (1-1.1), summarizes all the experimental facts of electrostatics. The energy of interaction of an individual charge with the rest of the system can be emphasized by rewriting (1-1.1) as

$$E = \frac{1}{2} \sum_i e_i \sum_{j \neq i} \frac{e_j}{r_{ij}} = \frac{1}{2} \sum_i e_i \phi_i$$

(1-1.3)

electrostatic

where we have introduced the potential at the location of the  $i^{\text{th}}$  charge that is attributable to all the other charges,

$$\phi_i = \sum_{j \neq i} \frac{e_j}{r_{ij}}.$$

(1-1.4)

This is an action-at-a-distance point of view, in which the charge at a given point interacts with charges at other, distant points. Another approach, which generalizes and transcends action at a distance, employs the field concept, a field being a local quantity, defined at every point of space. We take a first step in this direction by considering the potential as a field, which is defined everywhere, not just where the point charges are located. This generalized potential function, or simply the potential,  $\phi(\mathbf{r})$ , is

$$\phi(\mathbf{r}) = \sum_j \frac{e_j}{|\mathbf{r} - \mathbf{r}_j|}$$

(1-1.5)

Here we now treat every charge on an equal footing, which means that in (1-1.5) we sum over all charges  $e_j$ . In terms of this potential, which is different from  $\phi_i$ , the energy can be written as

$$E = \frac{1}{2} \sum_i e_i \phi(\vec{r}_i) - \sum_i E_i$$

(1-1.6)

The last part of (1-1.6) is not to be understood numerically, but rather, as an instruction to remove terms in the first sum that refer to a single particle. In other words, we remove "self-action," leaving the mutual interactions between particles. The field concept naturally leads to self-action.

The notion of force is derived from that of energy, as we can see by considering the work done as a result of a spatial displacement. If we displace the  $i^{\text{th}}$  charge by the amount  $\delta \vec{r}_i$ , the energy changes by an amount

$$\delta E = \delta \vec{r}_i \cdot (\vec{\nabla}_i E) = - \delta \vec{r}_i \cdot \vec{F}_i ,$$

(1-1.7)

where  $\vec{F}_i$  is the force acting on the  $i^{\text{th}}$  point charge. Comparing this with the energy expression, (1-1.1), we find the force on the  $i^{\text{th}}$  particle to be

$$\vec{F}_i = -\vec{\nabla}_i \sum_{j \neq i} \frac{e_i e_j}{r_{ij}} = -\vec{\nabla}_i e_i \sum_{j \neq i} \frac{e_j}{r_{ij}} = -\vec{\nabla}_i e_i \phi(\vec{r}_i) \quad (1-1.8)$$

In the last form, we have substituted  $\phi(\vec{r}_i)$  for  $\phi_i$ , and it would appear that an extra self-action contribution has been introduced. To see that this is not true, we first argue physically that the difference between  $\phi(\vec{r}_i)$  and  $\phi_i$  is independent of position, and so self-action does not contribute to the force. Mathematically, what is this additional, unwanted term? It is the negative gradient of the self-energy:

$$-\vec{\nabla}_{\vec{r}_i} \frac{e_i^2}{|\vec{r}-\vec{r}_i|} = e_i^2 \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} \quad (1-1.9)$$

Can we make sense of this? We could define the limit by arbitrarily picking a vector  $\vec{\epsilon}$  of fixed direction and letting its length approach zero:

$$\vec{r} = \vec{r}_i + \vec{\epsilon}, \quad \vec{\epsilon} \rightarrow 0, \quad (1-1.10)$$

but we do better to average this with the result of approaching  $\vec{r}_i$  from the opposite direction.

$$\vec{r} = \vec{r}_i - \vec{\epsilon}, \quad \vec{\epsilon} \rightarrow 0. \quad (1-1.11)$$

Then the additional term, (1-1.9), for any particular  $\vec{\epsilon}$ , is

$$\frac{1}{2} e_i^2 \left[ \frac{\vec{\epsilon}}{|\vec{\epsilon}|^3} - \frac{\vec{\epsilon}}{|\vec{\epsilon}|^3} \right] = 0. \quad (1-1.12)$$

More elaborate limits, such as an average over all directions, can be used but the simple procedure of (1-1.12) suffices. Therefore we can employ  $\phi(\vec{r})$  in (1-1.8), with the implicit use of the two-sided limit (1-1.12), to calculate the force.

With the force given in terms of the gradient of a field (the potential), the electric field,  $\vec{E}$ , can now be defined as

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}), \quad (1-1.13)$$

so that the force on a point charge  $e_i$  located at  $\vec{r}_i$  is

$$\vec{F}_i = e_i \vec{E}(\vec{r}_i). \quad (1-1.14)$$

The electric field  $\vec{E}(\vec{r})$  enters as a function calculable at  $\vec{r}$  in terms of the (distant) point charges located at  $\vec{r}_j$ ,

$$\vec{E}(\vec{r}) = \sum_j e_j \frac{\vec{r} - \vec{r}_j}{|\vec{r} - \vec{r}_j|^3}. \quad (1-1.15)$$

We now proceed to remove all reference to the initial action-at-a-distance description by characterizing  $\vec{E}(\vec{r})$  in terms of its properties at, or in the infinitesimal neighborhood of the point  $\vec{r}$ .

From its definition as the negative gradient of the potential it follows that  $\vec{E}(\vec{r})$  obeys

$$\vec{\nabla} \times \vec{E}(\vec{r}) = -\vec{\nabla} \times \vec{\nabla}\phi(\vec{r}) = 0; \quad (1-1.16)$$

the curl of the electric field vector is zero. The other elementary differential operation that can be applied to a vector field is the divergence. To find  $\vec{\nabla} \cdot \vec{E}$ , we consider a related integral statement. The integral of the normal component of  $\vec{E}$  over a closed surface  $S$  bounding a volume  $V$ , the <sup>electric</sup> flux, is:



Fig. 1.

$$\oint_S d\vec{s} \cdot \vec{E}(\vec{r}) = \sum_j e_j \oint_S d\vec{s} \cdot \frac{\vec{r} - \vec{r}_j}{|\vec{r} - \vec{r}_j|} \frac{1}{|\vec{r} - \vec{r}_j|^2} = \sum_j e_j \oint_S d\Omega_j \quad (1-1.17)$$

Here,  $d\Omega_j$  is an element of solid angle, which is defined in the following manner. The element of area perpendicular to the line from the  $j^{\text{th}}$  charge is

$$d\vec{s} \cdot \frac{\vec{r} - \vec{r}_j}{|\vec{r} - \vec{r}_j|} \quad (1-1.18)$$

which, when divided by the square of the distance from the  $j^{\text{th}}$  charge gives the solid angle  $d\Omega_j$  subtended by  $dS$ , as seen from the  $j^{\text{th}}$  charge. There are now two possible situations: either  $e_j$  is inside, or it is outside the closed surface  $S$ .

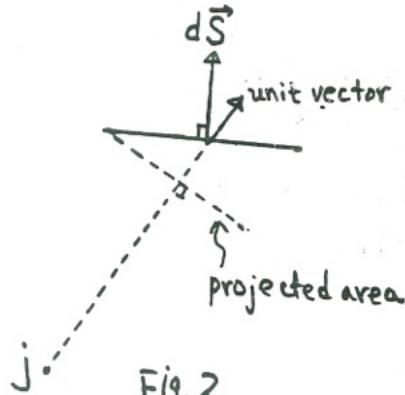


Fig. 2

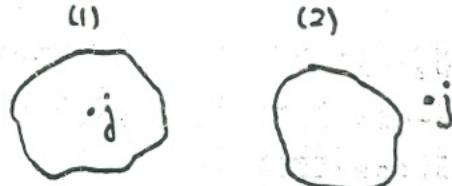


Fig. 3

Correspondingly, the integral over all solid angles in the two situations is

$$(1) \oint_S d\Omega_j = 4\pi \quad , \quad \begin{matrix} e_j \\ \nearrow \\ \text{(inside the closed volume)} \end{matrix}, \quad (1-1.19)$$

$$(2) \oint_S d\Omega_j = 0 \quad , \quad \begin{matrix} e_j \\ \nearrow \\ \text{(outside the closed volume)} \end{matrix}.$$

Hence, the electric flux through a closed surface is proportional to the enclosed charge:

$$\oint_S d\vec{s} \cdot \vec{E}(\vec{r}) = \sum_{j \text{ in } V} 4\pi e_j \quad (1-1.20)$$

This is the theorem of Carl <sup>A</sup>Friedrich Gauss (1777-1855).

With our aim of deriving local statements in mind, we generalize the idea of point charges to that of a <sup>continuous</sup> distribution of charge, as measured by  $\rho(\vec{r})$ , the volume density of charge at the point  $\vec{r}$ . Then, the total charge in a volume  $V$  is obtained by integrating the charge density over that region:

$$\sum_{j \text{ in } V} e_j = \int_V (\vec{dr}) \rho(\vec{r}) . \quad (1-1.21)$$

*following*

(Throughout this book we use the notation for the element of volume:

$$(\vec{dr}) = dx dy dz . . . \quad (1-1.22)$$

For point charges, the charge density is zero except at the location of the charges,

$$\rho(\vec{r}) = \sum_j e_j \delta(\vec{r}-\vec{r}_j) , \quad (1-1.23)$$

where the (three-dimensional)  $\delta$ -function is defined by

$$\int_V (\vec{dr}) \delta(\vec{r}-\vec{r}_j) = \begin{cases} 0 & \text{if } \vec{r}_j \text{ outside } V \\ 1 & \text{if } \vec{r}_j \text{ inside } V \end{cases} . \quad (1-1.24)$$

Now, the flux statement, (1-1.20) becomes

$$4\pi \int_V (\vec{dr}) \rho(\vec{r}) = \oint_S \vec{dS} \cdot \vec{E}(\vec{r}) = \int_V (\vec{dr}) \vec{\nabla} \cdot \vec{E}(\vec{r}) , \quad (1-1.25)$$

by use of the divergence theorem relating surface and volume integrals.

Since (1-1.25) is true for an arbitrary volume  $V$ , the integrands must be equal, so we obtain the equation satisfied by the divergence of  $\vec{E}$ ,

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = 4\pi \rho(\vec{r}) . \quad (1-1.26)$$

These differential equations for the curl and divergence of  $\vec{E}$ , (1-1.16) and (1-1.26), respectively, completely characterize  $\vec{E}$  when appropriate boundary conditions are imposed. It is evident from (1-1.15) that the magnitude of the electric field becomes vanishingly small with increasing distance from a collection of charges:

$$|\vec{E}(\vec{r})| \rightarrow 0, \text{ as } r \rightarrow \infty. \quad (1-1.27)$$

One can also specify how rapidly this occurs. But it is interesting that the weak boundary condition (1-1.27) already suffices for a unique solution to the differential equations, (1-1.16) and (1-1.26). To show this we suppose that  $\vec{E}_1$  and  $\vec{E}_2$  are two such solutions. The difference,  $\vec{\xi} = \vec{E}_1 - \vec{E}_2$ , satisfies

$$\vec{\nabla} \cdot \vec{\xi} = 0, \quad \vec{\nabla} \times \vec{\xi} = 0, \quad (1-1.28)$$

$$|\vec{\xi}| \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

from which we must prove that  $\vec{\xi} = 0$ . The identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\xi}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{\xi}) - \nabla^2 \vec{\xi}, \quad (1-1.29)$$

combined with the vanishing of  $\vec{\nabla} \times \vec{\xi}$  and  $\vec{\nabla} \cdot \vec{\xi}$ , implies that

$$-\nabla^2 \vec{\xi} = 0. \quad (1-1.30)$$

Let the single function  $\xi(\vec{r})$  be any component of the vector field  $\vec{\xi}(\vec{r})$ ; it obeys

$$-\nabla^2 \xi(\vec{r}) = 0. \quad (1-1.31)$$

We present this as the everywhere valid statement

$$0 = -\xi \nabla^2 \xi = -\vec{\nabla} \cdot (\underbrace{\xi \vec{\nabla} \xi}_{\text{term 1}}) + (\vec{\nabla} \xi)^2, \quad (1-1.32)$$

or

$$(\vec{\nabla} \xi)^2 - \nabla^2 \frac{1}{2} \xi^2 = 0. \quad (1-1.33)$$

Now we integrate over  $V(R)$ , the volume of a sphere of radius  $R$  centered at the origin - an arbitrary point. The divergence theorem can be applied to the second term on the left of (1-1.33),

$$-\int_{V(R)} (\vec{dr}) \cdot (\vec{\nabla} \frac{1}{2} \epsilon^2) = -\int_{S(R)} dS \cdot \vec{\nabla} \frac{1}{2} \epsilon^2 \quad (1-1.34)$$

$$= -\int_{S(R)} dS \frac{\partial}{\partial r} \frac{1}{2} \epsilon^2 ;$$

where  $S(R)$  is the surface of the sphere. Using the relation between an element of area and an element of solid angle,  $dS = R^2 d\Omega$ , we can present the surface integral as

$$-4\pi R^2 \frac{d}{dR} \langle \frac{1}{2} \epsilon^2 \rangle_R \quad (1-1.35)$$

where

$$\langle \frac{1}{2} \epsilon^2 \rangle_R = \frac{1}{4\pi} \int d\Omega \frac{1}{2} \epsilon^2 \quad (1-1.36)$$

is the average value of  $\frac{1}{2} \epsilon^2$  over the surface of the sphere.

And so we have learned that

$$\int_{V(R)} (\vec{dr}) (\vec{\nabla} \epsilon)^2 - 4\pi R^2 \frac{d}{dk} \langle \frac{1}{2} \epsilon^2 \rangle_R = 0. \quad (1-1.37)$$

(See Prob. ).

The decisive step now is to divide by the factor  $4\pi R^2$ , and then integrate over  $R$  from zero to infinity:

$$\int_0^\infty dR \frac{1}{4\pi R^2} \int_{V(R)} (\vec{dr}) (\vec{\nabla} \epsilon)^2 + \langle \frac{1}{2} \epsilon^2 \rangle_0 = 0, \quad (1-1.38)$$

which finally incorporates the boundary condition that  $\epsilon$  vanishes at all infinitely remote points. Everything on the left side of (1-1.38) is non-negative, and it all adds up to zero. Accordingly, every individual contribution must be zero. This tells us quite explicitly that  $\epsilon = 0$  at the origin, which is anywhere, and, consistently, that  $\vec{\nabla} \epsilon = 0$  everywhere, or, that  $\epsilon$  is a constant, which is required to be zero by the boundary condition. This being true of any component, we conclude that the vector  $\vec{\epsilon} = 0$ .

We have proved the "uniqueness theorem" of electrostatics, that the differential equations, (1-1.16) and (1-1.26), have a unique solution when the boundary condition, (1-1.27) is imposed.

Thus, from the Coulomb energy we have derived the equations of Electrostatics:

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho, \quad \frac{\partial \phi}{\partial t} = 0, \quad (1-1.39)$$
$$\vec{\nabla} \times \vec{E} = 0, \quad \frac{\partial}{\partial t} \vec{E} = 0,$$

where the time independence has been made explicit. We are now going to remove the restriction to static conditions by letting the charges move in a particularly simple way. The equations that emerge from this discussion will then be accepted as applicable to more general motions, as justified by various tests of internal consistency.

## 2. Inference of Maxwell's Equations

We introduce time dependence by letting all charges move with a common velocity  $\vec{v}$ . (This situation is produced by viewing a

static arrangement of charges in a coordinate system that moves with relative velocity  $-\vec{v}$ .) At first we take  $|\vec{v}|$  to be very small in comparison with a critical speed  $c$  that will be identified with the speed of light. To catch up with the moving charges one would have to move with their velocity,  $\vec{v}$ . Accordingly, the time derivative in that co-moving coordinate system, in which the charges are at rest, is the sum of explicit time dependent,  $\frac{\partial}{\partial t}$ , and coordinate dependent,  $\vec{v} \cdot \vec{\nabla}$ , contributions,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}. \quad (1-2.1)$$

Thus, we now write the differential equation for the charge density in (1-1.39) as

$$\frac{d}{dt} \rho(\vec{r}, t) = \frac{\partial}{\partial t} \rho(\vec{r}, t) + \vec{v} \cdot \vec{\nabla} \rho(\vec{r}, t) = 0, \quad (1-2.2)$$

or

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0. \quad (\vec{\nabla} \cdot \vec{v} \equiv 0 \text{ here}) \quad (1-2.3)$$

We recognize here a particular example of the charge flux vector or current density,

$$\vec{j} = \rho \vec{v}. \quad (1-2.4)$$

The relation

$$\frac{\partial}{\partial t} \rho(\vec{r}, t) + \vec{\nabla} \cdot \vec{j}(\vec{r}, t) = 0 \quad (1-2.5)$$

is the general expression of the local conservation of charge. Conservation demands that the charge within an arbitrary volume  $V$  decreases only in consequence of the outward flux over the bounding surface  $S$ ,

$$-\frac{d}{dt} \int_V (\vec{d}\vec{r}) \rho(\vec{r}, t) = \oint_S d\vec{S} \cdot \vec{j} = \int_V (\vec{d}\vec{r}) \vec{\nabla} \cdot \vec{j}(\vec{r}, t), \quad (1-2.6)$$

from which (1-2.5) follows. We also note that (1-2.5) continues to apply even when  $\vec{v}$  ceases to be constant,

$$\vec{v} \rightarrow \vec{v}(\vec{r}, t). \quad (1-2.7)$$

One can proceed similarly with the time derivative equation for the electric field,

$$0 = \frac{d}{dt} \vec{E}(\vec{r}, t) = \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) + (\vec{v} \cdot \vec{\nabla}) \vec{E}(\vec{r}, t). \quad (1-2.8)$$

The vector identity ( $\vec{v}$  is constant)

$$\vec{\nabla} \times (\vec{v} \times \vec{E}) = \vec{v} (\vec{\nabla} \cdot \vec{E}) - (\vec{v} \cdot \vec{\nabla}) \vec{E}, \quad (1-2.9)$$

together with (1-1.26) and (1-2.4) then gives

$$\begin{aligned} \vec{\nabla} \times (\vec{v} \times \vec{E}) &= 4\pi \rho \vec{v} + \frac{\partial}{\partial t} \vec{E} \\ &= 4\pi \vec{j} + \frac{\partial}{\partial t} \vec{E}. \end{aligned} \quad (1-2.10)$$

Notice that the divergence of this equation recovers (1-2.5), the local charge conservation condition. In  $\vec{v} \times \vec{E}$ , we recognize something new, associated with motion, that is outside the framework of electrostatics. We define a vector  $\vec{B}(\vec{r}, t)$ , the magnetic induction vector, as

$$\vec{v} \times \vec{E} = c \vec{B}, \quad (1-2.11)$$

where  $c$  is the constant with the dimensions of velocity that will be identified as the speed of light. (We shall also speak of  $\vec{B}$  as the magnetic field; a precise distinction is made in Sec. 1-8.) We have learned that the curl of  $\vec{B}$  is

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial}{\partial t} \vec{E} + \frac{4\pi}{c} \vec{j}. \quad (1-2.12)$$

What, then, is the divergence of  $\vec{B}$ ? According to its definition, (1-2.11),

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\frac{\vec{v}}{c} \times \vec{E}) = -\frac{\vec{v}}{c} \cdot \vec{\nabla} \times \vec{E} = 0. \quad (1-2.13)$$

Furthermore, in the co-moving coordinate system where the charges are at rest —static—the new vector  $\vec{B}$  should also not change in time:

$$\frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{B} = 0. \quad (1-2.14)$$

Again we use an identity, analogous to (1-2.9),

$$\vec{\nabla} \times (\vec{v} \times \vec{B}) = \vec{v} (\vec{\nabla} \cdot \vec{B}) - (\vec{v} \cdot \vec{\nabla}) \vec{B} = -(\vec{v} \cdot \vec{\nabla}) \vec{B}, \quad (1-2.15)$$

to arrive at

$$\frac{\partial}{\partial t} \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}), \quad (1-2.16)$$

which is consistent with  $\vec{\nabla} \cdot \vec{B} = 0$ .

What do we do now? We need one experimental fact: Light is an electromagnetic oscillation. The evidence for this is overwhelming. As examples, we note that electric and magnetic fields are known to influence the emission, propagation, and absorption of light; and that radio and infrared waves, which differ only in wavelength from visible light, are emitted by electric charge oscillations. What must be done so that this fact is built into the equations we are inferring? The existence of electromagnetic waves means that the equations determining the electric field have solutions of the form

$$E \sim f(z-ct) \quad (1-2.17)$$

where  $c$  is the speed of the waves. Such waves, propagating in the  $z$ -direction, satisfy the second order differential equation

$$\frac{\partial^2}{\partial z^2} \vec{E} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} \quad (1-2.18)$$

for an arbitrary direction of propagation, the corresponding wave equation is

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} . \quad (1-2.19)$$

More precisely, ~~this~~<sup>should</sup> equation hold far from the charges that produce the field. The left side of the equation can be written as (cf. (1-1.29))

$$\nabla^2 \vec{E} = -\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) , \quad (1-2.20)$$

since  $\vec{\nabla} \cdot \vec{E} = 0$  outside the charge distribution, while, with the use of (1-2.12) and (1-2.16), the right side becomes ( $j = 0$  outside the charge distribution)

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = \frac{1}{c} \vec{\nabla} \times [\vec{\nabla} \times (\vec{v} \times \vec{B})] . \quad (1-2.21)$$

This shows that the desired differential equation will hold if

$$\vec{E} = -\frac{\vec{v}}{c} \times \vec{B} . \quad (1-2.22)$$

But this cannot be a completely correct statement, for then  $\vec{v} \rightarrow 0$  would require  $\vec{E} \rightarrow 0$ . However, all that is needed is that the curl of the vectors on the two sides of this equation be equal,

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \left( \frac{\vec{v}}{c} \times \vec{B} \right) , \quad (1-2.23)$$

or, using (1-2.16),

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B} . \quad (1-2.24)$$

This is a generalization to the time dependent situation of the electrostatic law  $\vec{\nabla} \times \vec{E} = 0$ . Incidentally, there is no real inconsistency with our earlier use of  $\vec{\nabla} \times \vec{E} = 0$  to derive  $\vec{\nabla} \cdot \vec{B} = 0$  in (1-2.13). What now appears is

$$\vec{\nabla} \cdot \vec{B} = -\frac{\vec{v}}{c} \cdot \vec{\nabla} \times \vec{E} = \frac{\vec{v}}{c} \cdot \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 , \quad (1-2.25)$$

first, because

$$\frac{\vec{v}}{c} \cdot \vec{B} = \frac{\vec{v}}{c} \cdot \left( \frac{\vec{v}}{c} \times \vec{E} \right) = 0, \quad (1-2.26)$$

but more significantly, because  $\frac{\vec{v}}{c} \cdot \vec{B}$  contains two factors of  $\vec{v}/c$ ; in this discussion, that is considered to be completely negligible ( $|\vec{v}/c| \ll 1$ ).

We now collect these equations of Electromagnetism:

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial}{\partial t} \vec{E} + \frac{4\pi}{c} \vec{j}, \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho; \quad (1-2.27)$$

$$-\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial}{\partial t} \vec{B}, \quad \vec{\nabla} \cdot \vec{B} = 0;$$

\*  
they are Maxwell's equations, which, as local, differential, field equations, are no longer restricted to the initial assumption of a common velocity for all charges. To complete the dynamical picture we ask: What replaces (1-1.14) to describe the force on an electric charge, when that charge moves with some velocity  $\vec{v}$  in given electric and magnetic fields  $\vec{E}$  and  $\vec{B}$ ? There is no problem if the charge is at rest, as observed in a co-moving coordinate system. But, in moving with velocity  $\vec{v}$  to catch up with this charge, all other charges -those responsible for  $\vec{E}$  and  $\vec{B}$ - are given an additional counter velocity  $-\vec{v}$ . Then (1-2.22) indicates the existence of an additional electric field,

$$- \left( -\frac{\vec{v}}{c} \times \vec{B} \right) = \frac{\vec{v}}{c} \times \vec{B}, \quad (1-2.28)$$

suggesting that the required force is

$$\vec{F} = e \left[ \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right]. \quad (1-2.29)$$

In the following pages, we shall demonstrate the internal consistency of the Maxwell's equations, which supply the fields

produced by prescribed motions of the charges, and the (Lorentz) force law (1-2.29), describing the motion of the charges in the presence of prescribed fields. That demonstration of consistency will hold without reference to the initial restriction that all particle speeds are small compared to  $c$ .

We have arrived at the Maxwell-Lorentz electrodynamics by combining three ingredients: the laws of electrostatics; the Galileo-Newton principle of relativity (charges at rest, and charges with a common velocity viewed by a co-moving observer, are physically indistinguishable); the existence of electromagnetic waves that travel in a vacuum at the speed  $c$ . The historical line of development was otherwise. Until the beginning of the nineteenth century, electricity and magnetism were unrelated phenomena. The discovery in 1820 by Hans Christian Oersted (1777-1851) that an electric current influences a magnet — creates a magnetic field — is formulated, for stationary currents, in the field equation

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}. \quad (1-2.30)$$

The symbol  $c$  that appears in this equation is the ratio of electromagnetic and electrostatic units of electricity. Then, in 1831, Michael Faraday (1791-1867) discovered that relative motion of a wire and a magnet induces a voltage in the wire — creates an electric field. Such is the content of

$$-\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial}{\partial t} \vec{B}, \quad (1-2.31)$$

which extends the magnetostatic relation

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (1-2.32)$$

that expresses the empirical absence of single magnetic poles. Finally, in 1864, James Clark Maxwell (1831-1879) recognized that

the restriction to stationary currents in (1-2.30), as expressed by  $\vec{\nabla} \cdot \vec{j} = 0$ , was removed in

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial}{\partial t} \vec{E}, \quad (1-2.33)$$

when joined to the electrostatic equation

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho. \quad (1-2.34)$$

The deduction of the existence of electromagnetic waves that travel at the speed  $c$ , in remarkable numerical agreement with the speed of light, was confirmed in 1887 by Heinrich Rudolf Hertz (1857-1894). It was the conflict between the existence of this absolute speed  $c$  and the relativity concept of Newtonian mechanics that set the stage for Einsteinian relativity. Already at the age of 16, Albert Einstein (1879-1955) had recognized this paradox: To a co-moving Newtonian observer, light waves should oscillate in ~~space~~, but not move; however, Maxwell's equations admit no such solutions. Einsteinian relativity is an outgrowth of Maxwellian electrodynamics, not the other way about. That is the spirit in which electrodynamics is developed as a self-contained subject in this book.

### 3. Magnetic Charge

The development of Maxwell's equations in Sec. 1-2 contained an implicit assumption that must be dragged into the open. When it came to (1-2.22),

$$\vec{E} = -\frac{\vec{v}}{c} \times \vec{B}, \quad (1-3.1)$$

with its implication that static electric charges produce no electric field, we knew better than to accept this and altered it to

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \left( \frac{\vec{v}}{c} \times \vec{B} \right), \quad (1-3.2)$$

thereby admitting, for  $\vec{v} = 0$ , a static electric field, one obeying  $\vec{\nabla} \times \vec{E} = 0$ . Why then did we earlier accept without question the relation ((1-2.11))

$$\vec{B} = \frac{\vec{v}}{c} \times \vec{E}, \quad (1-3.3)$$

with its implication that all magnetic fields are attributable to the motion of electric charges? That is the (1820) hypothesis of André Marie Ampère (1775-1836). But is it true? An affirmative response is conventional, but let's keep an open mind about this. Again, all that is required is the curl relation

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left( \frac{\vec{v}}{c} \times \vec{E} \right), \quad (1-3.4)$$

admitting the possibility, for  $\vec{v} = 0$ , of a static magnetic field obeying  $\vec{\nabla} \times \vec{B} = 0$ , that has its origin in magnetic charge. If  $\rho_m$  is the density of such magnetic charge, the analogy with electrostatics indicates that

$$\vec{\nabla} \cdot \vec{B} = 4\pi \rho_m. \quad (1-3.5)$$

den

Now the time independence of  $\vec{B}$  when one moves with all the charges asserts, analogously to (1-2.89) that

$$0 = \frac{d}{dt} \vec{B} = \frac{\partial}{\partial t} \vec{B} + (\vec{v} \cdot \vec{\nabla}) \vec{B}$$

$$= \frac{\partial}{\partial t} \vec{B} + \vec{v}(\vec{\nabla} \cdot \vec{B}) - \vec{\nabla} \times (\vec{v} \times \vec{B}), \quad (1-3.6)$$

or, using (1-3.2) and (1-3.5), that

$$-\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial}{\partial t} \vec{B} + \frac{4\pi}{c} \vec{j}_m, \quad (1-3.7)$$

where

$$\vec{j}_m = \rho_m \vec{v} \quad (1-3.8)$$

is the magnetic current density. Notice that (1-3.7) implies the conservation of magnetic charge:

$$\frac{\partial}{\partial t} \rho_m + \vec{\nabla} \cdot \vec{j}_m = 0. \quad (1-3.9)$$

The complete set of Maxwell's equations, when magnetic charge is present, now reads

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial}{\partial t} \vec{E} + \frac{4\pi}{c} \vec{j}_e, \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho_e,$$

$$-\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial}{\partial t} \vec{B} + \frac{4\pi}{c} \vec{j}_m, \quad \vec{\nabla} \cdot \vec{B} = 4\pi \rho_m, \quad (1-3.10)$$

where we have used the subscript e to denote densities for electric charge. Observe that these equations are invariant in form under the replacements

$$\rho_e \rightarrow \rho_m, \quad \vec{E} \rightarrow \vec{B}, \quad \vec{j}_e \rightarrow \vec{j}_m, \quad (1-3.11)$$

$$\rho_m \rightarrow -\rho_e, \quad \vec{B} \rightarrow -\vec{E}, \quad \vec{j}_m \rightarrow -\vec{j}_e.$$

The generalized force law of Hendrick Antoon Lorentz (1835-1928),

suggested by this symmetry, is

$$\vec{F} = e(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) + g(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}), \quad (1-3.12)$$

for a particle carrying both electric and magnetic charge,  $e$  and  $g$ , respectively.

Until recently the search for magnetic charge had been fruitless. In 1975, an atomic track was observed suggestive of a magnetically charged particle. That interpretation was criticized and the claim subsequently withdrawn. Nevertheless, there remain cogent reasons for thinking that magnetic charge is securely locked within nuclear constituents and will someday come to light. [See, for example, "A Magnetic Model of Matter", <sup>(1962)</sup> Science, Volume 165, pages 757-761]. In any event, this provocative possibility emphasises that electrodynamics is far from being a closed subject.

#### 4. Conservation Laws

To check the physical consistency of the Maxwell-Lorentz electrodynamics, we examine the action of , and the reaction on, the sources of the electromagnetic fields. To be precise, we ask whether there is a proper balance in the exchange of energy, momentum, and angular momentum between the charged particles and the fields. As we shall see, the Maxwell-Lorentz system, as it stands, implies the conservation of these mechanical properties without reference to any limitation on the speeds of the particles.

##### Conservation of Energy

We start with a consideration of the rate at which work is done on the particles, that is, the rate of energy transfer — the power absorbed by the particles. For one particle, we know that the rate at which work is done on it is

$$\vec{F} \cdot \vec{v} = \vec{e}v \cdot \vec{E} + \vec{g}v \cdot \vec{B}$$

$$= \int (\vec{dr}) [\vec{j}_e \cdot \vec{E} + \vec{j}_m \cdot \vec{B}] \quad (1-4.1)$$

where we have used the Lorentz force law, (1-3.12), and the expression for the currents, (1-2.4) and (1-3.8), for a point particle. We interpret this equation as meaning, even for general current distributions, that  $\vec{j}_e \cdot \vec{E} + \vec{j}_m \cdot \vec{B}$  is the rate of energy transfer from the field to the particles, per unit volume. Then, through elimination of the currents by use of Maxwell's equations, this rate can be rewritten as

$$\vec{j}_e \cdot \vec{E} + \vec{j}_m \cdot \vec{B} = \frac{c}{4\pi} \vec{E} \cdot \left( -\frac{1}{c} \frac{\partial}{\partial t} \vec{E} + \vec{\nabla} \times \vec{B} \right) + \frac{c}{4\pi} \left( -\frac{1}{c} \frac{\partial}{\partial t} \vec{B} - \vec{\nabla} \times \vec{E} \right) \cdot \vec{B}$$

$$= -\frac{\partial}{\partial t} \left( \frac{\vec{E}^2 + \vec{B}^2}{8\pi} \right) - \vec{\nabla} \cdot \left( \frac{c}{4\pi} \vec{E} \times \vec{B} \right). \quad (1-4.2)$$

The general form of any local conservation law, as in (1-3.9), suggests the following interpretations:

(1) In the absence of charges ( $\vec{j}_e = \vec{j}_m = 0$ ), this is the local conservation law

$$\frac{\partial}{\partial t} \frac{\vec{E}^2 + \vec{B}^2}{8\pi} + \vec{\nabla} \cdot \frac{c}{4\pi} \vec{E} \times \vec{B} = 0 \quad (1-4.3)$$

We label the two objects appearing here as

$$\text{energy density: } U = \frac{\vec{E}^2 + \vec{B}^2}{8\pi}, \quad (1-4.4)$$

$$\text{energy flux vector: } \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}. \quad (1-4.5)$$

(John Henry Poynting (1852-1914))

(2) In the presence of charges, the relation (1-4.2) is

$$\frac{\partial}{\partial t} U + \vec{\nabla} \cdot \vec{S} + \vec{j}_e \cdot \vec{E} + \vec{j}_m \cdot \vec{B} = 0, \quad (1-4.6)$$

which, after integrating over an  $\underset{\Delta}{\text{volume}} V$ , becomes

$$\frac{d}{dt} \int_V (\vec{dr}) \cdot \vec{U} + \int_S d\vec{s} \cdot \vec{s} + \int_V (\vec{dr}) \cdot [\vec{j}_e \cdot \vec{E} + \vec{j}_m \cdot \vec{B}] = 0 \quad (1-4.7)$$

The three terms here are identified, respectively, as the rate of change of electromagnetic energy within the volume; the rate of flow of electromagnetic energy out of the volume, across the bounding surface S; and, the rate of transfer of electromagnetic energy to the charged particles. Thus (1-4.6) gives a complete account of energy conservation.

### Conservation of Momentum

Next we consider the force on a particle, the rate of change of its momentum,

$$\begin{aligned} \vec{F} &= e(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) + q(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}) \\ &= \int (\vec{dr}) [\rho_e \vec{E} + \frac{1}{c} \vec{j}_e \times \vec{B} + \rho_m \vec{B} - \frac{1}{c} \vec{j}_m \times \vec{E}] \\ &\equiv \int (\vec{dr}) \vec{f} \end{aligned} \quad (1-4.8)$$

where  $\vec{f}$  is the force density vector. Removing reference to the generalized charge and current densities by use of Maxwell's equations, we rewrite the force density vector,  $\vec{f}$ , as

$$\begin{aligned} \vec{f} &= \frac{1}{4\pi} [(\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{\nabla} \cdot \vec{B}) \vec{B}] \\ &+ \frac{1}{4\pi} [(\frac{1}{c} \frac{\partial}{\partial t} \vec{E} + \vec{\nabla} \times \vec{B}) \times \vec{B} + \vec{E} \times (-\frac{1}{c} \frac{\partial}{\partial t} \vec{B} - \vec{\nabla} \times \vec{E})] \\ &= -\frac{\partial}{\partial t} \left( \frac{1}{4\pi c} \vec{E} \times \vec{B} \right) + \frac{1}{4\pi} \left[ -\vec{E} \times (\vec{\nabla} \times \vec{E}) + (\vec{\nabla} \cdot \vec{E}) \vec{E} \right. \\ &\quad \left. - \vec{B} \times (\vec{\nabla} \times \vec{B}) + (\vec{\nabla} \cdot \vec{B}) \vec{B} \right]. \end{aligned} \quad (1-4.9)$$

The quadratic structure in  $\vec{E}$  occurring here is

$$\begin{aligned} -\vec{E} \times (\vec{\nabla} \times \vec{E}) + (\vec{\nabla} \cdot \vec{E}) \vec{E} &= -\vec{\nabla} \left( \frac{1}{2} \vec{E}^2 \right) + (\vec{E} \cdot \vec{\nabla}) \vec{E} + (\vec{\nabla} \cdot \vec{E}) \vec{E} \\ &= \vec{\nabla} \cdot \left( \vec{1} \frac{1}{2} \vec{E}^2 + \vec{E} \vec{E} \right), \end{aligned} \quad (1-4.10)$$

which introduces dyadic notation including the unit dyadic  $\vec{1}$ , with components

$$\vec{1}_{kl} = \delta_{kl} = \begin{cases} 1, & k=l, \\ 0, & k \neq l. \end{cases} \quad (1-4.11)$$

Accordingly, the force density vector is

$$\vec{f} = -\frac{\partial}{\partial t} \left( \frac{1}{4\pi c} \vec{E} \times \vec{B} \right) - \vec{\nabla} \cdot \left( \vec{1} \frac{\vec{E}^2 + \vec{B}^2}{8\pi} - \frac{\vec{E}\vec{E} + \vec{B}\vec{B}}{4\pi} \right). \quad (1-4.12)$$

We interpret this equation physically by identifying

$$\text{momentum density vector: } \vec{G} = \frac{1}{4\pi c} \vec{E} \times \vec{B}, \quad (1-4.13)$$

momentum flux dyadic (stress tensor):

$$\vec{T} = \vec{1} \frac{\vec{E}^2 + \vec{B}^2}{8\pi} - \frac{\vec{E}\vec{E} + \vec{B}\vec{B}}{4\pi}. \quad (1-4.14)$$

When  $\vec{f} = 0$ , we get the local statement of the conservation of momentum of the electromagnetic field. A full account of momentum balance is contained in

$$\frac{\partial}{\partial t} \vec{G} + \vec{\nabla} \cdot \vec{T} + \vec{f} = 0. \quad (1-4.15)$$

The volume integral of this equation for electromagnetic momentum is interpreted analogously to the energy result, (1-4.7).

The components of the stress tensor are given by

$$T_{kl} = \delta_{kl} U - \frac{E_k E_l + B_k B_l}{4\pi} \quad (1-4.16)$$

*al*  
Notice that it is a symmetric tensor,  $T_{lk} = T_{kl}$ , which, as we shall see in the next subsection, is required in order to obtain a local conservation law for angular momentum. The trace of  $\mathbf{T}$ , the sum of the diagonal elements  $T_{kk}$ , is just the energy density,

$$\sum_k T_{kk} = U. \quad (1-4.17)$$

We also note that the Poynting vector, (1-4.5), is proportional to the momentum density,

$$\vec{S} = c^2 \vec{G}. \quad (1-4.18)$$

On displaying the meaning of the respective sides of this relation:

$$\text{energy density} \times \text{velocity} = c^2 (\text{mass density} \times \text{velocity}) \quad (1-4.19)$$

we meet the first example of the relativistic connection between energy and mass,  $E = mc^2$ .

### Conservation of Angular Momentum, Virial Theorem

After linear momentum comes angular momentum. Let us write out (1-4.15) in component form as

$$\frac{\partial}{\partial t} G_l + \sum_j \nabla_j T_{jl} + f_l = 0, \quad (1-4.20)$$

and then multiply by the  $k^{\text{th}}$  component of the vector  $\vec{r}$ ,

$$\vec{r}_k = x_k. \quad (1-4.21)$$

$\vec{T} \times \vec{f}$

We rearrange this, using

$$\begin{aligned}\sum_j x_k \nabla_j T_{jl} &= \sum_j \nabla_j (x_k T_{jl}) - \sum_j T_{jl} \nabla_j x_k \\ &= \sum_j \nabla_j (x_k T_{jl}) - T_{kl},\end{aligned}\quad (1-4.22)$$

since

$$\nabla_j x_k = \delta_{jk}, \quad (1-4.23)$$

to get

$$\frac{\partial}{\partial t} (x_k G_l) + \sum_j \nabla_j (x_k T_{jl}) - T_{kl} + x_k f_l = 0. \quad (1-4.24)$$

Now, for a single particle, the rate of change of angular momentum is the torque

$$\vec{\tau} = \vec{r} \times \vec{f} = \int (d\vec{r}) \vec{r} \times \vec{f}, \quad (1-4.25)$$

which, in volume integrated form is no longer restricted to a single particle. The relation between the vector torque density  $\vec{r} \times \vec{f}$  and the component, tensor, notation is illustrated by

$$(\vec{r} \times \vec{f})_3 = x_1 f_2 - x_2 f_1. \quad (1-4.26)$$

Accordingly, we write the analogue of (1-4.24) with  $k$  and  $l$  interchanged and subtract the two equations to produce

$$\frac{\partial}{\partial t} (x_k G_l - x_l G_k) + \sum_j \nabla_j (x_k T_{jl} - x_l T_{jk}) + (x_k f_l - x_l f_k) = 0. \quad (1-4.27)$$

Notice the very important cancellation of  $T_{kl}$  and  $T_{lk}$ , which expresses the symmetry of the tensor (1-4.16). That is what gives (1-4.27) the form of a local conservation law in the absence of charges, leading to the identification of the electromagnetic angular momentum quantities

$$\text{angular momentum density vector: } \vec{r} \times \vec{G}, \quad (1-4.28)$$

$$\text{angular momentum flux tensor: } x_k^T_{jl} - x_l^T_{jk}. \quad (1-4.29)$$

The interpretation of (1-4.27) as a local angular momentum conservation for fields and particles proceeds as before.

In another application of (1-4.24), we place  $k = l$  and sum. With the aid of the relation (1-4.17)

this gives

$$\frac{\partial}{\partial t}(\vec{r} \cdot \vec{G}) + \vec{\nabla} \cdot (\vec{T} \cdot \vec{r}) - U + \vec{r} \cdot \vec{f} = 0, \quad (1-4.30)$$

which we call the electromagnetic virial theorem, in analogy with the mechanical virial theorem of Rudolf Clausius (1822-1888). (See Sec. 1-19).

## 5. Electromagnetic Pulses, Another Conservation Law

In this section, we restrict our attention to electromagnetic fields in domains free of charged particles, specifically, moving, finite regions occupied by electromagnetic fields — electromagnetic pulses. The total electromagnetic energy of such a pulse is constant in time:

$$\frac{d}{dt}E = \int_{\text{pulse}} (dr) \frac{\partial}{\partial t} U = - \int_{\text{pulse}} (dr) \vec{\nabla} \cdot \vec{S} = 0, \quad (1-5.1)$$

inasmuch as the resulting surface integral, conducted over an enclosing surface on which all fields vanish, equals zero. Similar considerations apply to the total electromagnetic linear and angular momenta,

$$\frac{d}{dt} \vec{P} = \int_{\text{pulse}} (d\vec{r}) \frac{\partial}{\partial t} \vec{G} = - \int_{\text{pulse}} (d\vec{r}) \vec{\nabla} \cdot \vec{T} = 0, \quad (1-5.2)$$

$$\frac{d}{dt} \vec{J} = \int_{\text{pulse}} (d\vec{r}) \frac{\partial}{\partial t} (\vec{r} \times \vec{G}) = - \int_{\text{pulse}} (d\vec{r}) \vec{\nabla} \cdot (-\vec{T} \times \vec{r}) = 0. \quad (1-5.3)$$

We now derive another conserved quantity, using the equality of  $\vec{S}$  and  $c^2 \vec{G}$  much as the equality of  $T_{kl}$  and  $T_{lk}$  led to the conservation of angular momentum:

$$0 = \vec{x}_l \left( \frac{\partial}{\partial t} U + \vec{\nabla} \cdot \vec{S} \right) - c^2 t \left( \frac{\partial}{\partial t} G_l + \sum_k \nabla_k T_{kl} \right) \\ = \frac{\partial}{\partial t} (\vec{x}_l U - c^2 t G_l) + \sum_k \nabla_k (S_k \vec{x}_l - c^2 t T_{kl}) + \underbrace{c^2 G_l - S_l}_{=0}, \quad (1-5.4)$$

and

$$\frac{\partial}{\partial t} (\vec{r} U - c^2 t \vec{G}) + \vec{\nabla} \cdot (\vec{S} \vec{r} - c^2 t \vec{T}) = 0. \quad (1-5.5)$$

The resulting conservation law is

$$\frac{d}{dt} \int_{\text{pulse}} (d\vec{r}) (\vec{r} U - c^2 t \vec{G}) = 0. \quad (1-5.6)$$

We recognize one volume integral here,

$$\int_{\text{pulse}} (d\vec{r}) \vec{G} = \vec{P}, \quad (1-5.7)$$

which vector is constant in time. Accordingly, we can rewrite (1-5.6) as

$$\frac{d}{dt} \int (d\vec{r}) \vec{r} \cdot U(\vec{r}, t) = c^2 \vec{P}. \quad (1-5.8)$$

The energy-weighted position vector integral defines, at every time, a certain point  $\langle \vec{r} \rangle_E(t)$ , such that

$$\int_{\text{pulse}} (d\vec{r}) (\vec{r} - \langle \vec{r} \rangle_E(t)) U(\vec{r}, t) = 0, \quad (1-5.9)$$

or

$$\int_{\text{pulse}} (d\vec{r}) \vec{U}(\vec{r}, t) = E \langle \vec{r} \rangle_E(t). \quad (1-5.10)$$

Thus the motion of this energy-centroid vector is such that

$$\frac{E}{c^2} \frac{d}{dt} \langle \vec{r} \rangle_E(t) = \vec{P}, \quad (1-5.11)$$

*to say* that the energy centroid moves with constant velocity,

$$\frac{d}{dt} \langle \vec{r} \rangle_E = \vec{v}_E, \quad (1-5.12)$$

*the momentum being that velocity multiplied by a mass,*

$$m = E/c^2. \quad (1-5.13)$$

The application of the virial theorem, (1-4.3C), to <sup>an</sup> electro-magnetic pulse supplies another velocity. We infer that

$$\frac{d}{dt} \int_{\text{pulse}} (d\vec{r}) \vec{r} \cdot \vec{G} = \int_{\text{pulse}} (d\vec{r}) \vec{U} = E, \quad (1-5.14)$$

which introduces another position vector, with momentum weighting, such that

$$\int_{\text{pulse}} (d\vec{r}) \left( \vec{r} - \langle \vec{r} \rangle_P(t) \right) \cdot \vec{G}(\vec{r}, t) = 0, \quad (1-5.15)$$

or

$$\int_{\text{pulse}} (d\vec{r}) \vec{r} \cdot \vec{G}(\vec{r}, t) = \langle \vec{r} \rangle_P(t) \cdot \vec{P}. \quad (1-5.16)$$

The resulting velocity relation for

$$\frac{d}{dt} \langle \vec{r} \rangle_P(t) = \vec{v}_P, \quad (1-5.17)$$

is

$$c^2 \frac{\vec{P}}{E} = \frac{\partial \vec{M}_E}{\partial \vec{r}} = \vec{V}_E \quad V_p \cdot V_E = c^2$$
$$\vec{V}_p \cdot \vec{P} = E, \quad V_p \cdot P = E \quad (1-5.18)$$

which combines with (1-5.11) to yield

$$\vec{V}_p \cdot \vec{V}_E = c^2. \quad (1-5.19)$$

If the flow of energy and momentum takes place in a single direction, it would be reasonable to expect that these mechanical properties are being transported with a single velocity:

$$\vec{V}_E = \vec{V}_p = \vec{V}, \quad (1-5.20)$$

which then has a definite magnitude

$$\vec{V} \cdot \vec{V} = c^2, \quad |\vec{V}| = c. \quad (1-5.21)$$

That electromagnetic waves move at the speed  $c$  was an input to our inference of the Maxwell equations. Here it is recovered in a consideration about energy and momentum that is consistent with mechanical properties. Indeed from

$$E = \vec{V} \cdot \vec{P}, \quad \vec{P} = \frac{E}{c^2} \vec{V}, \quad (1-5.22)$$

we learn that

$$E^2 = P^2 c^2, \quad E = |\vec{P}| c, \quad (1-5.23)$$

while Hamiltonian mechanics (see Sec. I-18) asserts that the velocity of such a system is

$$\vec{v} = \frac{\partial E}{\partial \vec{P}} = c \frac{\partial |\vec{P}|}{\partial \vec{P}} = c \frac{\vec{P}}{|\vec{P}|}, \quad (1-5.24)$$

which restates the nature of the motion — parallel to the momentum, with the speed  $c$ .

There is another, somewhat more direct, mechanical proof that electromagnetic pulses travel — propagate — at the speed  $c$ . We multiply the local conservation of energy statement, (1-4.3), applicable outside the charge distribution, by  $r^2 - (ct)^2$ :

$$(r^2 - (ct)^2) \left( \frac{\partial}{\partial t} U(\vec{r}, t) + \vec{\nabla} \cdot \vec{S}(\vec{r}, t) \right) = 0, \quad (1-5.25)$$

and then rearrange this as

$$\frac{\partial}{\partial t} \left[ (r^2 - (ct)^2) U(\vec{r}, t) \right] + \vec{\nabla} \cdot \left[ (r^2 - (ct)^2) \vec{S}(\vec{r}, t) \right] = 2c^2 \left[ \vec{r} \cdot \vec{G}(\vec{r}, t) - t U(\vec{r}, t) \right]. \quad (1-5.26)$$

The result of spatial integration over the pulse, expressed in the language of energy and momentum weighted averages, is

$$\frac{d}{dt} E \left( \langle r^2 \rangle_E(t) - c^2 t^2 \right) = 2c^2 \left[ \langle \vec{r} \rangle_P(t) \cdot \vec{P} - t \cdot \vec{E} \right], \quad (1-5.27)$$

where we already know, from (1-5.14) and (1-5.16), that

$$\frac{d}{dt} \left[ \langle \vec{r} \rangle_P(t) \cdot \vec{P} - t \cdot \vec{E} \right] = 0. \quad (1-5.28)$$

With the integration constant for the latter differential equation expressed by the value at  $t = 0$ , we have

$$\frac{d}{dt} \left( \langle r^2 \rangle_E(t) - c^2 t^2 \right) = 2c^2 \langle \vec{r} \rangle_P(0) \cdot \frac{\vec{P}}{E}, \quad (1-5.29)$$

and then

$$\langle r^2 \rangle_E(t) = \langle r^2 \rangle_E(0) + 2ct \langle \vec{r} \rangle_P(0) \cdot \frac{\vec{P} \cdot \vec{c}}{E} + c^2 t^2. \quad (1-5.30)$$

After a sufficient lapse of time, the last term on the right hand side dominates, and

$$[\langle r^2 \rangle_E(t)]^{\frac{1}{2}} \approx ct; \quad (1-5.31)$$

the average distance to the energy centrum of the pulse grows linearly in time — the centroid moves at the speed  $c$ . The whole of (1-5.30) can be interpreted by an analogy with a group of particles that, at time  $t = 0$ , are set off with various positions and velocities, thereafter to move with those constant velocities,

$$\vec{r}(t) = \vec{r}(0) + \vec{v}t . \quad (1-5.32)$$

The square of this vector, averaged over the various particles, is

$$\langle \vec{r}(t)^2 \rangle = \langle \vec{r}(0)^2 \rangle + 2t \langle \vec{r}(0) \cdot \vec{v} \rangle + \langle \vec{v}^2 \rangle t^2 , \quad (1-5.33)$$

and the two kinds of squared distance averages can be identified, along with

$$\langle \vec{r}(0) \cdot \vec{v} \rangle = \langle \vec{r} \rangle_{\text{P}}(0) \cdot \frac{\vec{P}c^2}{E} , \quad (1-5.34)$$

if

$$v^2 = c^2 . \quad (1-5.35)$$

Returning to the simple situation of energy and momentum flow in a single direction, we ask: what are the electromagnetic fields doing to maintain the mechanical relations? The meaning, in terms of fields, of  $E = c |\vec{P}|$  is

$$\int(d\vec{r}) \frac{1}{8\pi} (E^2 + B^2) = \left| \int(d\vec{r}) \frac{1}{4\pi} \vec{E} \times \vec{B} \right| , \quad (1-5.36)$$

where the volume integrations are extended over the pulse. Now, a sum of vectors of given magnitudes is of maximum <sup>magnitude</sup> when all those vectors are parallel, which is to say that

$$\int(d\vec{r}) \frac{1}{2} (E^2 + B^2) \leq \int(d\vec{r}) |\vec{E} \times \vec{B}| , \quad (1-5.37)$$

with the equality sign applying only when  $\vec{E} \times \vec{B}$  everywhere points in the same direction, that of the pulse's total momentum or velocity. On the other hand, here is a different inequality, applicable at any point in space, *It is based on the identity*

$$|\vec{E} \times \vec{B}|^2 = E^2 B^2 - (\vec{E} \cdot \vec{B})^2$$

$$= \left[ \frac{1}{2}(E^2 + B^2) \right]^2 - \left[ \frac{1}{2}(E^2 - B^2) \right]^2 - (\vec{E} \cdot \vec{B})^2 \quad (1-5.38)$$

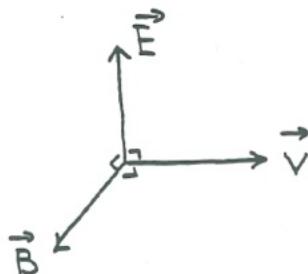
from which follows

$$\frac{1}{2}(E^2 + B^2) \geq |\vec{E} \times \vec{B}|, \quad \begin{array}{l} E^2 - B^2 + 2EB \geq 0 \\ EB \geq 0 \\ (E+B)^2 \geq 0 \\ |B|^2 = \frac{E^2 + B^2}{2} \end{array} \quad (1-5.39)$$

with the equality sign holding only for

$$E^2 = B^2, \quad \vec{E} \cdot \vec{B} = 0. \quad (1-5.40)$$

The two inequalities, (1-5.37) and (1-5.39), are incompatible unless the equality signs do apply. Accordingly, the electric and magnetic fields in a uni-directional pulse are, everywhere within the pulse, of equal magnitude, mutually perpendicular, and perpendicular to the direction of motion of the pulse.



These results, with additional details, will also emerge from explicit solutions of the Maxwell equations ( ). And with these added details we shall be able to discuss the angular momentum carried by an electromagnetic pulse.

## MACROSCOPIC ELECTRODYNAMICS

### 6. Force on an Atom

The Maxwell-Lorentz description gives a microscopic account of electromagnetic phenomena at the classical level, one that ranges from the simplest two-particle system to the detailed behavior of all particles in a macroscopic system. But we have no use for such a detailed description of a  $\Sigma$  many-particle system when our measurements refer to macroscopic properties of that system, such measurements being insensitive to the behavior of an individual atom. Instead of the needlessly detailed atomic theory, we should employ a macroscopic electrodynamics, one expressed in the language of the macroscopic measurements that are made on the system. This is a phenomenological theory, by which <sup>is</sup> meant a theory that operates at the level of the phenomena being correlated and predicted, while maintaining the possibility of contact with a more fundamental theory—here microscopic electrodynamics—that operates at a deeper level. That contact exists to the extent that the macroscopic measurements can be considered to be averages, over very many atoms, of the results of hypothetical microscopic measurements.

To begin, we consider an atom, an electrically neutral collection of point electric charges,

$$\sum_k e_k = 0, \quad (1-6.1)$$

that are bound together in a small (on macroscopic standards) region. We want to consider the effect on such a system of electric and magnetic fields that vary very little over the spatial dimensions of that system. The total force on the atom at a given time, the sum of the forces on its constituents, is

$$\vec{F} = \sum_k \left[ e_k \vec{E}(\vec{r}_k) + e_k \frac{\vec{v}_k}{c} \times \vec{B}(\vec{r}_k) \right]. \quad (1-6.2)$$

It is convenient to refer the position of all charges,  $\vec{r}_k$ , to

a point within the atom,  $\vec{R}$ . For what follows,  $\vec{R}$  is quite arbitrary, but a natural choice, mechanically, is the center of mass of the system. The statement of slow variation of the fields over the atom is expressed by

$$\vec{E}(\vec{r}_k) = \vec{E}(\vec{R}) + [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_{\vec{R}}] \vec{E}(\vec{R}) + \dots, \quad (1-6.3)$$

and similarly for  $\vec{B}(\vec{r}_k)$ , in which the subsequent terms are considered negligible. Now the total force on the atom becomes

$$\begin{aligned} \vec{F} = & \left( \sum_k e_k \right) \vec{E}(\vec{R}) + \sum_k e_k [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_{\vec{R}}] \vec{E}(\vec{R}) + \left( \sum_k e_k \frac{\vec{v}_k}{c} \right) \times \vec{B}(\vec{R}) \\ & + \sum_k e_k \frac{\vec{v}_k}{c} \times [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_{\vec{R}}] \vec{B}(\vec{R}) + \dots. \end{aligned} \quad (1-6.4)$$

The first term on the right hand side, involving the total charge of the system, is zero. The next, leading, term introduces the electric dipole moment of the system, at the given time,

$$\vec{d} = \sum_k e_k (\vec{r}_k - \vec{R}) = \sum_k e_k \vec{r}_k, \quad (1-6.5)$$

and we can recognize that the third term on the right hand side of (1-6.4) contains the time derivative of the dipole moment,

$$\frac{d}{dt} \vec{d} = \sum_k e_k \vec{v}_k. \quad (1-6.6)$$

**¶** The combination of the second and third terms in (1-6.4) is (the time variable is understood)

$$\begin{aligned} & (\vec{d} \cdot \vec{\nabla}_{\vec{R}}) \vec{E}(\vec{R}) + \frac{1}{c} \left( \frac{d}{dt} \vec{d} \right) \times \vec{B}(\vec{R}) \\ & = \frac{d}{dt} \left[ \frac{1}{c} \vec{d} \times \vec{B}(\vec{R}) \right] - \frac{1}{c} \vec{d} \times \left( \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}_{\vec{R}} \right) \vec{B}(\vec{R}) + (\vec{d} \cdot \vec{\nabla}_{\vec{R}}) \vec{E}(\vec{R}), \end{aligned} \quad (1-6.7)$$

where

$$\vec{V} = \frac{d \vec{R}}{dt}, \quad (1-6.8)$$

or, since

$$\vec{d} \times (\vec{\nabla}_R \times \vec{E}(\vec{R})) + (\vec{d} \cdot \vec{\nabla}_R) \vec{E}(\vec{R}) = \vec{\nabla}_R (\vec{d} \cdot \vec{E}(\vec{R})) , \quad (1-6.9)$$

we get for this combination :

$$\vec{\nabla}_R (\vec{d} \cdot \vec{E}(\vec{R})) - \frac{1}{c} (\vec{V} \cdot \vec{\nabla}_R) \vec{d} \times \vec{B}(\vec{R}) + \frac{d}{dt} \left[ \frac{1}{c} \vec{d} \times \vec{B}(\vec{R}) \right] . \quad (1-6.10)$$

Now to the last term of (1-6.4),

$$\begin{aligned} & \sum_k \frac{e_k}{c} \vec{v}_k \times [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_R] \vec{B}(\vec{R}) \\ &= \sum_k \frac{e_k}{c} \vec{V} \times [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_R] \vec{B}(\vec{R}) + \sum_k \frac{e_k}{c} (\vec{v}_k - \vec{V}) \times [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_R] \vec{B}(\vec{R}) \end{aligned} \quad (1-6.11)$$

where the initial term of the right side is

$$\frac{1}{c} \vec{V} \times (\vec{d} \cdot \vec{\nabla}_R) \vec{B}(\vec{R}) . \quad (1-6.12)$$

This contribution combines with the second of the three terms in (1-6.10) to give

$$\begin{aligned} & \frac{1}{c} \vec{V} \times (\vec{d} \cdot \vec{\nabla}_R) \vec{B}(\vec{R}) - \frac{1}{c} (\vec{V} \cdot \vec{\nabla}) \vec{d} \times \vec{B}(\vec{R}) \\ &= \frac{1}{c} [(\vec{d} \cdot \vec{\nabla}_R) \vec{V} - (\vec{V} \cdot \vec{\nabla}_R) \vec{d}] \times \vec{B}(\vec{R}) \\ &= \frac{1}{c} [(\vec{d} \times \vec{V}) \times \vec{\nabla}_R] \times \vec{B}(\vec{R}) , \end{aligned} \quad (1-6.13)$$

or ( $\vec{\nabla} \cdot \vec{B} = 0$  : no magnetic charge!)

$$\vec{\nabla}_R \left[ \frac{1}{c} (\vec{d} \times \vec{V}) \cdot \vec{B}(\vec{R}) \right] . \quad (1-6.14)$$

Collecting the various results to this point, we have

$$\vec{F} = \vec{\nabla}_R [\vec{d} \cdot \vec{E}(\vec{R})] + \vec{\nabla}_R \left[ \frac{1}{c} (\vec{d} \times \vec{V}) \cdot \vec{B}(\vec{R}) \right] + \frac{d}{dt} \left[ \frac{1}{c} \vec{d} \times \vec{B}(\vec{R}) \right] + \vec{F}_B , \quad (1-6.15)$$

in which

$$\vec{F}_B = \sum_k \frac{e_k}{c} (\vec{v}_k - \vec{V}) \times [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_R] \vec{B}(\vec{R}) . \quad (1-6.16)$$

This last part of the force is rearranged as follows,

$$\begin{aligned}\vec{F}_B &= \frac{d}{dt} \sum_k \frac{e_k}{c} (\vec{r}_k - \vec{R}) \times [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_R] \vec{B}(\vec{R}) \\ &\quad - \sum_k \frac{e_k}{c} (\vec{r}_k - \vec{R}) \times [(\vec{v}_k - \vec{V}) \cdot \vec{\nabla}_R] \vec{B}(\vec{R}) \\ &\quad - \sum_k \frac{e_k}{c} (\vec{r}_k - \vec{R}) \times [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_R] \frac{d}{dt} \vec{B}(\vec{R}).\end{aligned}\quad (1-6.17)$$

We must now recall the restricted nature of this description: the electric and magnetic fields change only slightly over the dimensions of the system. And so we recognize that the first of the three right hand terms of (1-6.17) is a small correction to what is already present in (1-6.15) as

$$\frac{d}{dt} \left[ \frac{1}{c} \vec{d} \times \vec{B}(\vec{R}) \right], \quad (1-6.18)$$

and is therefore to be neglected. Furthermore the last term of (1-6.17), well approximated as

$$\sum_k e_k (\vec{r}_k - \vec{R}) \times [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_R] \vec{\nabla}_R \times \vec{E}(\vec{R}), \quad (1-6.19)$$

is of the order of magnitude of the omitted terms in the expansion (1-6.4), and is therefore also to be neglected. An average of the initial form of  $F_B$  with the single remaining contribution of (1-6.17) now gives

$$\begin{aligned}\vec{F}_B &= \frac{1}{2} \sum_k \frac{e_k}{c} (\vec{v}_k - \vec{V}) \times [(\vec{r}_k - \vec{R}) \cdot \vec{\nabla}_R] \vec{B}(\vec{R}) \\ &\quad - \frac{1}{2} \sum_k \frac{e_k}{c} (\vec{r}_k - \vec{R}) \times [(\vec{v}_k - \vec{V}) \cdot \vec{\nabla}_R] \vec{B}(\vec{R}) \\ &= \frac{1}{2} \sum_k \frac{e_k}{c} \left\{ [(\vec{r}_k - \vec{R}) \times (\vec{v}_k - \vec{V})] \times \vec{\nabla}_R \right\} \times \vec{B}(\vec{R}).\end{aligned}\quad (1-6.20)$$

What has finally emerged here is the magnetic dipole moment of the system,

$$\vec{\mu} = \frac{1}{2} \sum_k \frac{e_k}{c} (\vec{r}_k - \vec{R}) \times (\vec{v}_k - \vec{V}), \quad (1-6.21)$$

in terms of which

$$\vec{F}_B = (\vec{\mu} \times \vec{\nabla}_R) \times \vec{B}(R) = \vec{\nabla}_R (\vec{\mu} \cdot \vec{B}(R)). \quad (1-6.22)$$

(It is, of course, the similarity of this structure to  $\vec{\nabla}_R (\vec{d} \cdot \vec{E}(R))$  that justifies the identification of  $\vec{\mu}$  as the magnetic analogue of  $\vec{d}$ .) We also recognize that a contribution of this form already appears in (1-6.15), bearing the information that a moving electric dipole also acts as a magnetic dipole. The comparison of the two effects, characterized by  $\frac{1}{c} \vec{d} \times \vec{V}$  and  $\vec{\mu}$ , is that between the typical speeds of the relatively heavy atoms, and of the light electrons in the interior of the atoms. Accordingly, we neglect the motion <sup>and</sup> effects of the atoms and finally write

$$\vec{F} = \vec{\nabla}_R (\vec{d} \cdot \vec{E}(R) + \vec{\mu} \cdot \vec{B}(R)) + \frac{d}{dt} \left( \frac{1}{c} \vec{d} \times \vec{B}(R) \right). \quad (1-6.23)$$

In the absence of time variation, what remains is a force associated with the respective potential energies of given electric dipole in an electric field,

$$- \vec{d} \cdot \vec{E}, \quad (1-6.24)$$

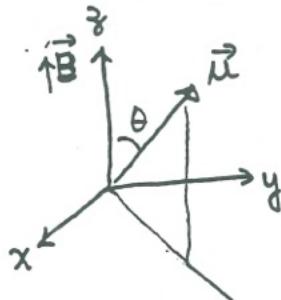
and of a given magnetic dipole in a magnetic field,

$$- \vec{\mu} \cdot \vec{B}. \quad (1-6.25)$$

The energy interpretation does more than supply the rectangular components of force as negative gradients with respect to position coordinates. It also produces torques as negative gradients with respect to angles. Take the example of a magnetic dipole in a magnetic field that, at a particular point in space, is directed along the z-axis. Then the magnetic potential energy is

$$-|\vec{\mu}| |\vec{B}| \cos \theta , \quad (1-6.26)$$

where  $\theta$  is the angle between  $\vec{\mu}$  and the magnetic field.



The implied internal torque (the reference point of this torque is at the position of the dipole) is then

$$\frac{\partial}{\partial \theta} (|\vec{\mu}| |\vec{B}| \cos \theta) = -|\vec{\mu}| |\vec{B}| \sin \theta , \quad (1-6.27)$$

which can be represented by a vector perpendicular to the plane formed by  $\vec{\mu}$  and  $\vec{B}$ ,

$$\vec{\tau} = \vec{\mu} \times \vec{B} . \quad (1-6.28)$$

We shall now derive this vectorial result directly, along with its electric counterpart; for simplicity, additional time derivative terms are omitted. The torque, the moment of the force about the center of the charge distribution at  $\vec{R}$ , is

$$\vec{\tau} = \sum_k (\vec{r}_k - \vec{R}) \times [e_k \vec{E}(\vec{r}_k) + \frac{1}{c} e_k \vec{v}_k \times \vec{B}(\vec{r}_k)] . \quad (1-6.29)$$

To get the leading term here it suffices to ignore the changes of  $\vec{E}$  and  $\vec{B}$  over the atom and replacing these fields by their values at  $\vec{R}$ . Thus the electric torque is just

$$\vec{\tau}_E = \left[ \sum_k e_k (\vec{r}_k - \vec{R}) \right] \times \vec{E}(\vec{R}) = \vec{d} \times \vec{E}(\vec{R}) , \quad (1-6.30)$$

as expected in analogy with (1-6.28). In deriving the magnetic torque, we first make the unimportant change

$$\vec{v}_k \rightarrow \vec{v}_k - \vec{V}, \quad (1-6.31)$$

and then transfer the time derivative to get

$$\begin{aligned} \vec{\tau}_B &= \sum_k (\vec{r}_k - \vec{R}) \times \left[ \frac{1}{c} e_k (\vec{v}_k - \vec{V}) \times \vec{B}(\vec{R}) \right] \\ &\rightarrow - \sum_k (\vec{v}_k - \vec{V}) \times \left[ \frac{1}{c} e_k (\vec{r}_k - \vec{R}) \times \vec{B}(\vec{R}) \right], \end{aligned} \quad (1-6.32)$$

which omits the  $-\frac{1}{c} \frac{\partial}{\partial t} \vec{B} = \vec{\nabla} \times \vec{E}$  contribution as negligible in comparison with  $\vec{\tau}_E$ . The average of the two forms can be simplified, using the vector identity (see Problem )

$$\vec{X} \times (\vec{Y} \times \vec{Z}) + (\vec{X} \times \vec{Z}) \times \vec{Y} = (\vec{X} \times \vec{Y}) \times \vec{Z}, \quad (1-6.33)$$

to produce

$$\begin{aligned} \vec{\tau}_B &= \left[ \frac{1}{2c} \sum_k \frac{e_k}{c} (\vec{r}_k - \vec{R}) \times (\vec{v}_k - \vec{V}) \right] \times \vec{B}(\vec{R}) \\ &= \vec{\mu} \times \vec{B}(\vec{R}), \end{aligned} \quad (1-6.34)$$

as anticipated.

## 7. Force on a Macroscopic Body

Now that we know the force on an atom in a slowly varying (on the atomic scale) electromagnetic field, what is the force on the very large number of atoms constituting a macroscopic body? It is the sum of the forces on the individual atoms. To the extent that the atomic force varies slowly from one atom to another, the summation can be replaced by a volume integration weighted by the atomic density  $n(r)$ , the number of atoms per unit volume at the macroscopic point  $\vec{r}$ :

$$\vec{F} = \int (d\vec{r}) n(\vec{r}) \left[ \vec{d} \times (\vec{\nabla} \times \vec{E}) + (\vec{d} \cdot \vec{\nabla}) \vec{E} + \vec{\mu} \times (\vec{\nabla} \times \vec{B}) + (\vec{\mu} \cdot \vec{\nabla}) \vec{B} + \frac{d}{dt} \left( \frac{1}{c} \vec{d} \times \vec{B} \right) \right]. \quad (1-7.1)$$

Notice that we have rewritten (1-6.23) with the aid of the identities

$$\vec{\nabla} (\vec{d} \cdot \vec{E}) = \vec{d} \times (\vec{\nabla} \times \vec{E}) + (\vec{d} \cdot \vec{\nabla}) \vec{E}, \quad (1-7.2)$$

$$\vec{\nabla} (\vec{\mu} \cdot \vec{B}) = \vec{\mu} \times (\vec{\nabla} \times \vec{B}) + (\vec{\mu} \cdot \vec{\nabla}) \vec{B}. \quad (1-7.3)$$

At this point, we should recognize that the atomic dipole moments at a given time could well vary from one macroscopic point in the body to another, in response to varying electromagnetic conditions. We make this explicit by introducing two macroscopically significant vectors,

$$\vec{P}(\vec{r}, t) = n(\vec{r}) \vec{d}(\vec{r}, t), \quad (1-7.4)$$

$$\vec{M}(\vec{r}, t) = n(\vec{r}) \vec{\mu}(\vec{r}, t), \quad (1-7.5)$$

the electric and the magnetic polarization vector, respectively. The resulting macroscopic form of the total force at time  $t$  is

$$F(t) = \int (d\vec{r}) \left[ \vec{P}(\vec{r}, t) \times (\vec{\nabla} \times \vec{E}(\vec{r}, t)) + (\vec{P}(\vec{r}, t) \cdot \vec{\nabla}) \vec{E}(\vec{r}, t) + \vec{M}(\vec{r}, t) \times (\vec{\nabla} \times \vec{B}(\vec{r}, t)) + (\vec{M}(\vec{r}, t) \cdot \vec{\nabla}) \vec{B}(\vec{r}, t) + \frac{\partial}{\partial t} \left( \frac{1}{c} \vec{P}(\vec{r}, t) \times \vec{B}(\vec{r}, t) \right) \right]. \quad (1-7.6)$$

We proceed to simplify this in various ways such as the use of one Maxwell equation to get

$$\vec{P} \times (\vec{\nabla} \times \vec{E}) + \frac{\partial}{\partial t} \left( \frac{1}{c} \vec{P} \times \vec{B} \right) = \left( \frac{1}{c} \frac{\partial}{\partial t} \vec{P} \right) \times \vec{B}, \quad (1-7.7)$$

and through the application of the identity

$$\vec{\nabla}(\vec{M} \cdot \vec{B}) = \vec{M} \times (\vec{\nabla} \times \vec{B}) + (\vec{M} \cdot \vec{\nabla}) \vec{B} + \vec{B} \times (\vec{\nabla} \times \vec{M}) + (\vec{B} \cdot \vec{\nabla}) \vec{M}, \quad (1-7.8)$$

which is a generalization of (1-7.3). All subsequent steps involve the statement that the integral is extended over a volume that includes the whole body, so that, on the bounding surface of that volume,  $n(\vec{r}) = 0$ . This means that in performing partial integrations through the use of the divergence theorem, as in

$$(\vec{P} \cdot \vec{\nabla}) \vec{E} = \vec{\nabla} \cdot (\vec{P} \vec{E}) - (\vec{\nabla} \cdot \vec{P}) \vec{E}, \quad (1-7.9)$$

the surface integral terms vanish. In effect, then, for this example,

$$(\vec{P} \cdot \vec{\nabla}) \vec{E} \rightarrow -(\vec{\nabla} \cdot \vec{P}) \vec{E}, \quad (1-7.10)$$

and similarly, using  $\vec{\nabla} \cdot \vec{B} = 0$ , (1-7.8) gives

$$\vec{M} \times (\vec{\nabla} \times \vec{B}) + (\vec{M} \cdot \vec{\nabla}) \vec{B} \rightarrow (\vec{\nabla} \times \vec{M}) \times \vec{B}. \quad (1-7.11)$$

The immediate result is

$$\vec{F} = \int(d\vec{r}) \left[ -(\vec{\nabla} \cdot \vec{P}) \vec{E} + (\vec{\nabla} \times \vec{M} + \frac{1}{c} \frac{\partial}{\partial t} \vec{P}) \times \vec{B} \right], \quad (1-7.12)$$

which is precisely as though the body contained (effective) charge and current densities:

$$\rho_{\text{eff}}(\vec{r}, t) = -\vec{\nabla} \cdot \vec{P}(\vec{r}, t), \quad (1-7.13)$$

$$\vec{J}_{\text{eff}}(\vec{r}, t) = c \vec{\nabla} \times \vec{M}(\vec{r}, t) + \frac{\partial}{\partial t} \vec{P}(\vec{r}, t), \quad (1-7.14)$$

which correctly obey the conservation equation

$$\vec{\nabla} \cdot \vec{J}_{\text{eff}} + \frac{\partial}{\partial t} \rho_{\text{eff}} = 0. \quad (1-7.15)$$

It is left to the reader to verify (Problem ) that the total torque on the body, the sum over all atoms of the external torques:

$$\int(d\vec{r}) n \vec{r} \times [\vec{d} \times (\vec{v} \times \vec{E}) + (\vec{d} \cdot \vec{v}) \vec{E} + \vec{\mu} \times (\vec{v} \times \vec{B}) + (\vec{\mu} \cdot \vec{v}) \vec{B} + \frac{d}{dt} (\frac{1}{c} \vec{d} \times \vec{B})], \quad (1-7.16)$$

and of the integral torques:

$$\int(d\vec{r}) n [\vec{d} \times \vec{E} + \vec{\mu} \times \vec{B}], \quad \text{Torque about atom center} \quad (1-7.17)$$

is properly reproduced as the integrated moment of the effective force density

$$\vec{\tau} = \int(d\vec{r}) \vec{r} \times [\rho_{\text{eff}} \vec{E} + \frac{1}{c} \vec{j}_{\text{eff}} \times \vec{B}]. \quad (1-7.18)$$

## 8. Macroscopic Electrodynamics

Now we construct a phenomenological macroscopic electrodynamics. And what is that? Nothing more than the form in which electrodynamics first arose, in the pre-atomic period, when only the properties of bulk matter were involved. But the challenge here is to derive the phenomenological theory from the microscopic Maxwell-Lorentz description. Both theories will employ concepts that are abstracted from the kinds of measurements that are appropriate to their level of description. Being insensitive to the contribution of individual atoms and unresponsive to space and time displacements on the atomic scale, macroscopic measurement will be related to microscopic measurement by averages over space and time intervals that are macroscopically small but microscopically large. We express this relation between any macroscopic field  $F(\vec{r}, t)$  and its microscopic counterpart,  $f(\vec{r}, t)$  as

$$\overline{f(\vec{r}, t)} = \overline{\overline{f(\vec{r}, t)}} = \frac{1}{T} \int_T dt' \frac{1}{V} \int_V (d\vec{r}') f(\vec{r} + \vec{r}', t + t') , \quad (1-8.1)$$

where  $T$  and  $V$  are time and space measures of the averaging intervals. This is a linear relation, in the sense that

$$\overline{f_1 + f_2} = \overline{f}_1 + \overline{f}_2 , \quad \overline{\lambda f} = \lambda \overline{f} , \quad (1-8.2)$$

where  $\lambda$  is a constant. From that it follows that the averaged derivative of a function is the derivative of the average:

$$\overline{\frac{\partial}{\partial t} f(\vec{r}, t)} = \frac{\partial}{\partial t} \overline{f(\vec{r}, t)} , \quad \overline{\vec{\nabla} f(\vec{r}, t)} = \vec{\nabla} \overline{f(\vec{r}, t)} . \quad (1-8.3)$$

The effective charge and current densities derived in the preceding section are an example of macroscopic measured quantities, disclosed by slowly varying electric and magnetic fields. Accordingly, they constitute the macroscopic counterpart of the microscopic charge and current densities of electric charge bound in atoms,

$$\overline{\rho_{\text{bound}}(\vec{r}, t)} = - \vec{\nabla} \cdot \vec{P}(\vec{r}, t) , \quad (1-8.4)$$

$$\overline{j_{\text{bound}}(\vec{r}, t)} = c \vec{\nabla} \times \vec{M}(\vec{r}, t) + \frac{\partial}{\partial t} \vec{P}(\vec{r}, t) . \quad (1-8.5)$$

Experiment informs us that there are also charges free to move within conducting bodies. For these we use the notation

$$\overline{\rho_{\text{free}}(\vec{r}, t)} = \rho(\vec{r}, t) , \quad \overline{\vec{j}_{\text{free}}(\vec{r}, t)} = \vec{j}(\vec{r}, t) , \quad (1-8.6)$$

and the conservation equation applies,

$$\frac{\partial}{\partial t} \overline{\rho(\vec{r}, t)} + \vec{\nabla} \cdot \vec{j}(\vec{r}, t) = 0 , \quad (1-8.7)$$

inasmuch as the bound charge is independently conserved, (1-7.15).

And, now, from the microscopic field equations:

$$\vec{\nabla} \times \vec{b} = \frac{1}{c} \frac{\partial}{\partial t} \vec{e} + \frac{4\pi}{c} (\vec{j}_{\text{bound}} + \vec{j}_{\text{free}}), \quad \vec{\nabla} \cdot \vec{e} = 4\pi (\rho_{\text{bound}} + \rho_{\text{free}}), \\ -\vec{\nabla} \times \vec{e} = \frac{1}{c} \frac{\partial}{\partial t} \vec{b}, \quad \vec{\nabla} \cdot \vec{b} = 0, \quad (1-8.8)$$

we derive the macroscopic Maxwell's equations:

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial}{\partial t} \vec{E} + \frac{4\pi}{c} \left( c \vec{\nabla} \times \vec{M} + \frac{\partial}{\partial t} \vec{P} + \vec{J} \right), \quad \vec{\nabla} \cdot \vec{E} = 4\pi (-\vec{\nabla} \cdot \vec{P} + \rho) \\ -\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial}{\partial t} \vec{B}, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad (1-8.9)$$

Then, with the introduction of new macroscopic field vectors — the displacement vector

$$\vec{D} = \vec{E} + 4\pi \vec{P}, \quad (1-8.10)$$

and the magnetic field vector

$$\vec{H} = \vec{B} - 4\pi \vec{M} \quad (1-8.11)$$

(recall that  $\vec{B}$  is called the magnetic induction) — we get the historical form of the Maxwell equations

$$\vec{\nabla} \times \vec{H} = \frac{1}{c} \frac{\partial}{\partial t} \vec{D} + \frac{4\pi}{c} \vec{J}, \quad \vec{\nabla} \cdot \vec{D} = 4\pi \rho, \\ -\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial}{\partial t} \vec{B}, \quad \vec{\nabla} \cdot \vec{B} = 0. \quad (1-8.12)$$

The charge conservation equation, (1-8.7), is a consequence of the first pair of field equations. As microscopically smooth distributions, the density and flux of free charge will serve to measure the macroscopic fields  $\vec{E}$  and  $\vec{B}$ . That is exhibited in the force expression

$$\vec{F} = \int (d\vec{r}) \left( \rho \vec{E} + \frac{1}{c} \vec{J} \times \vec{B} \right). \quad \begin{matrix} \text{Is this a} \\ \text{solution} \end{matrix} \quad (1-8.13)$$

To completely fix the Maxwell equations, the specific electromagnetic nature of the matter must be introduced by giving

the current  $\vec{J}$ , and the polarization vectors  $\vec{P}$  and  $\vec{M}$  (or equivalent  $\vec{D}$  and  $\vec{H}$ ), in terms of the fields  $\vec{E}$  and  $\vec{B}$ , as well as various material parameters. We limit this discussion to the more usual forms of matter and, since a full quantum mechanical treatment would lead us too far afield, simple classical models are used — they are not qualitatively misleading.

## SIMPLE MICROSCOPIC MODELS, ELECTRIC PROPERTIES

### 9. Conductivity

First consider a free electric charge, an electron, moving under the influence of an external electric field, and subject to collisions with the atoms of the substance. The electric field accelerates the charge, the collisions slow it down. Our model represents the effect of the collisions by a frictional force that is proportional — and opposed — to the velocity. The equation of motion for the particle of mass  $m$  is

$$m \frac{d}{dt} \vec{v}(t) = -m\gamma \vec{v}(t) + e \vec{E}(t), \quad \gamma > 0. \quad (1-9.1)$$

(The variation of the electric field with position is ignored here — the velocities of interest are <sup>of</sup> very small magnitude compared with  $c$ ). The friction constant  $\gamma$  is given a physical interpretation by considering the situation for  $\vec{E} = 0$ :

$$\frac{d}{dt} \vec{v}(t) = -\gamma \vec{v}(t), \quad \vec{v}(t) = e^{-\gamma t} \vec{v}(0); \quad (1-9.2)$$

any initial velocity decreases exponentially in time, with  $\gamma$  supplying the characteristic decay time. The general solution of (1-9.1) is found by rewriting it as

$$\frac{d}{dt} (e^{\gamma t} \vec{v}(t)) = \frac{e}{m} e^{\gamma t} \vec{E}(t), \quad (1-9.3)$$

then  
and integrating from  $t' = -\infty$  (a time before any field has been applied),

to  $t$  (the time of observation),

$$\vec{v}(t) = \frac{e}{m} \int_{-\infty}^t dt' e^{-\gamma(t-t')} \vec{E}(t') . \quad (1-9.4)$$

Notice that the response of the system under the action of an external electric field is non-local in time. (That is, the velocity at a given time depends on the electric field at earlier times.) The main contribution to the integral comes from the region of time differences that are of the order of  $1/\gamma$ .

The current density for a single charge is proportional to its velocity. If  $n_f$  is the (constant) density of <sup>free</sup> conduction electrons, then  $\vec{j}$  is

$$\vec{j} = n_f e \vec{v} = \frac{n_f e^2}{m} \int_{-\infty}^t dt' e^{-\gamma(t-t')} \vec{E}(t') . \quad (1-9.5)$$

For the particular example of a constant electric field, this reduces to

$$\vec{j} = \frac{n_f e^2}{m \gamma} \vec{E} = \sigma \vec{E} , \quad (1-9.6)$$

which is a statement of Ohm's law (Georg Simon Ohm, 1787-1854),  $\sigma$  being the static conductivity. (It can be obtained directly as the static solution of (1-9.1).) A more general situation arises when the electric field exhibits harmonic variation (i.e., has a definite frequency),

$$E \sim \cos(\omega t + \phi) = \operatorname{Re}(\vec{E} e^{-i(\omega t + \phi)}) , \quad \operatorname{Re} : \text{real part}, \quad (1-9.7)$$

or in terms of the complex amplitude  $\vec{E}(\omega)$ ,

$$\vec{E}(t) = \operatorname{Re}(\vec{E}(\omega) e^{-i\omega t}) . \quad (1-9.8)$$

Now, the current density, (1-9.5), becomes

$$\begin{aligned}\vec{J}(t) &= \operatorname{Re} \frac{n_f e^2}{m} \vec{E}(\omega) \int_{-\infty}^t dt' e^{-\gamma(t-t')} e^{-i\omega t'} \\ &= \operatorname{Re} \left[ \frac{n_f e^2}{m} \frac{1}{\gamma-i\omega} \vec{E}(\omega) e^{-i\omega t} \right],\end{aligned}\quad (1-9.9)$$

there displayed is a complex amplitude for  $\vec{J}(t)$ ,  $\vec{J}(\omega)$ , and the complex conductivity  $\sigma(\omega)$ , defined by

$$\vec{J}(\omega) = \sigma(\omega) \vec{E}(\omega), \quad (1-9.10)$$

is

$$\sigma(\omega) = \frac{n_f e^2}{m} \frac{1}{\gamma-i\omega}. \quad (1-9.11)$$

For  $\omega = 0$ , we regain the static conductivity given in (1-9.6).

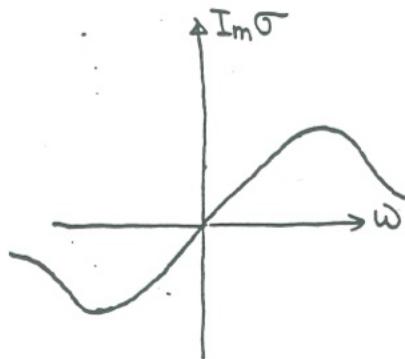
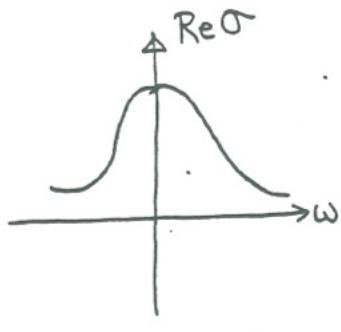
The conductivity is a function of  $i\omega$ , which means that complex conjugation (symbol: \*) is equivalent to changing the sign of  $\omega$ :

$$\sigma(\omega)^* = \sigma(-\omega). \quad (1-9.12)$$

This says that the real and imaginary parts of  $\sigma(\omega)$ ,

$$\sigma(\omega) = \frac{n_f e^2}{m} \frac{\gamma+i\omega}{\gamma^2+\omega^2} = \operatorname{Re} \sigma_A^{(w)} + i \operatorname{Im} \sigma_A^{(w)}, \quad (1-9.13)$$

are, respectively, even and odd functions of  $\omega$ .



Finally, we note that the integral of the conductivity over all frequencies is

$$\int_{-\infty}^{\infty} d\omega \sigma(\omega) = \frac{n_f e^2}{m} \cdot 2 \int_0^{\infty} d\omega \frac{\gamma}{\gamma^2 + \omega^2} = \frac{n_f e^2}{m} \cdot 2 \int_0^{\infty} \frac{dx}{1+x^2}$$

$$= \pi \frac{n_f e^2}{m}; \quad (1-9.14)$$

the integral is independent of the frictional constant  $\gamma$ . As such it provides a model independent way of determining  $n_f$  experimentally. What underlies this is the simplicity of the response to an electric field pulse that is localized at time  $t = 0$ . Without time in which to act, the frictional forces are effectively absent (see Problem 3.4).

#### 10. Dielectric Constant

The simplest model of the force that binds an electron to an atom is one that increases linearly with the displacement from an equilibrium position. To this we add a frictional force, proportional to the velocity, which now represents the dissipation of the particle energy into radiation (More about that later). This equation of motion is

$$m \frac{d}{dt} \vec{v} = -m\omega_0^2 \vec{r} - m\gamma \vec{v} + e\vec{E}, \quad \vec{v} = \frac{d\vec{r}}{dt}, \quad (1-10.1)$$

where the equilibrium point is the origin. The parameter  $\omega_0$  is the natural (angular) frequency for vibration when the dissipative force is omitted and no driving electric field is acting:

$$\frac{d^2}{dt^2} \vec{r} + \omega_0^2 \vec{r} = 0;$$

$$\frac{\partial^2}{\partial t^2} \vec{r} + \vec{F}$$

(1-10.2)

the solution of this equation is a combination of  $\cos \omega_0 t$  and  $\sin \omega_0 t$  time behavior. When a harmonic electric field does act,

$$\vec{E}(t) = \operatorname{Re} [\vec{E}(\omega) e^{-i\omega t}], \quad (1-10.3)$$

the induced displacement of the system is

$$\vec{r}(t) = \operatorname{Re} \left[ \frac{\frac{e}{m} \vec{E}(\omega) e^{-i\omega t}}{-\omega^2 - i\omega\gamma + \omega_0^2} \right], \quad (1-10.4)$$

as one verifies by applying the differential operator

$$\frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2. \quad (1-10.5)$$

Under the usual circumstance of  $\gamma \ll \omega_0$ , the amplitude of the induced oscillation becomes very large for  $\omega = \omega_0$ , the condition of resonance.

The electric polarization vector, the density of atoms multiplied by the induced dipole moment, is

$$\begin{aligned} \vec{P}(t) &= n_b e \vec{r}(t) \\ &= \operatorname{Re} [\chi_e(\omega) E(\omega) e^{-i\omega t}], \end{aligned} \quad (1-10.6)$$

which defines the frequency dependent electric susceptibility

$$\chi_e(\omega) = \chi_e(-\omega)^*. \quad (1-10.7)$$

It is given by

$$\chi_e(\omega) = \frac{n_b e^2}{m} \frac{1}{-\omega^2 - i\omega\gamma + \omega_0^2}, \quad (1-10.8)$$

and, in particular, the real value for  $\omega = 0$ , the static susceptibility, is

$$\chi_e = \frac{n_b e^2}{m \omega_0^2} > 0. \quad (1-10.9)$$

It is interesting to estimate the order of magnitude of this pure number for a dense substance, where

$$n_b l^3 \sim 1, \quad (1-10.10)$$

and  $l$  is a measure both of the distance between atoms, and of the dimension of atoms. Then

$$\chi_e \sim \frac{e^2/l}{m \omega_0^2 l^2}. \quad (1-10.11)$$

The parameter  $\omega_0$  of the model, an inverse atomic time, should be of order  $v/l$ , where  $v$  is a characteristic atomic speed. Thus  $m \omega_0^2 l^2$  is of the order of the electron kinetic energy, while  $e^2/l$  measures the magnitude of the potential energy in the realistic system, bound by electrostatic forces. But these two energies are essentially equal (Virial theorem, see Sec. 1-19). We conclude that

$$\chi_e \sim 1 \quad (1-10.12)$$

in a dense medium.

The displacement vector, (1-8.10), is

$$\begin{aligned} \vec{D}(t) &= \vec{E}(t) + 4\pi \vec{P}(t) \\ &= \text{Re} \left[ (1 + 4\pi \chi_e(\omega)) E(\omega) e^{-i\omega t} \right], \end{aligned} \quad (1-10.13)$$

which defines the frequency dependent dielectric constant

-50-

[This restriction is actually unnecessary - see Problem.]

$$\epsilon(\omega) = 1 + 4\pi \chi_e(\omega) = \epsilon(-\omega)^*, \quad (1-10.14)$$

$$\vec{D}(\omega) = \epsilon(\omega) \vec{E}(\omega). \quad (1-10.15)$$

In particular, the real, static dielectric constant  $\epsilon$  is

$$\epsilon = 1 + 4\pi \chi_e > 1, \quad (1-10.16)$$

and the excess over unity can be significant in dense substances.

Again, there is an integral over all frequencies that is independent of the frictional parameter  $\gamma$ . It is

$$\int_{-\infty}^{\infty} d\omega (-i\omega) \chi_e(\omega) = \pi \frac{n_b e^2}{m}. \quad (1-10.17)$$

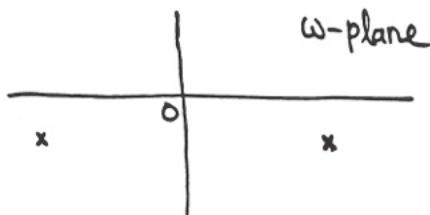
With a rearrangement of the denominator in (1-10.8), this integral reads

$$-\frac{n_b e^2}{m} \int_{-\infty}^{\infty} d\omega \frac{-i\omega}{(\omega + \frac{i}{2}\gamma)^2 - \omega_0'^2}, \quad \omega_0'^2 = \omega_0^2 - (\frac{1}{2}\gamma)^2. \quad (1-10.18)$$

For the following discussion, we assume that  $\gamma < 2\omega_0$ ,  $\omega^2 > 0$ . Now think of  $\omega$  as a complex variable and note that the integrand has singularities at (see Fig. )

$$\omega = -\frac{i}{2}\gamma \pm \omega_0', \quad (1-10.19)$$

which lie in the lower half plane.



Accordingly the integrand is everywhere regular in the upper half plane, and the integral over a path that is closed by a large semicircle in the upper half plane equals

zero. The integral of interest, (1-10.18), is therefore the negative of that over the semicircle of arbitrarily large radius, which is

$$\frac{n_b e^2}{m} \int dw \frac{-i}{\omega} = \pi n_b \frac{e^2}{m} . \quad (1-10.20)$$

In retrospect, the same method could have been applied to the conductivity integral, (1-9.14), which is obtained by putting  $\omega_0 = 0$ . The two phenomena being discussed are just the free electron and bound electron contributions to the total current:

$$\vec{J} + \frac{\partial}{\partial t} \vec{P} = \text{Re} \left[ (\sigma(\omega) - i\omega \chi_e(\omega)) \vec{E}(\omega) e^{-i\omega t} \right], \quad (1-10.21)$$

and the integral relations, which are united in

$$\int_{-\infty}^{\infty} dw [\sigma(\omega) - i\omega \chi_e(\omega)] = (n_f + n_b) \pi \frac{e^2}{m}, \quad (1-10.22)$$

express the fact that, in the response to an electric pulse localized at time zero, only the inertia of the electrons matter — frictional and binding forces have no time in which to act.

## 11. Plasma

Consider a situation in which there is both conduction of free charge and a dielectric medium, as conveyed by the conduction current

$$\vec{J} = \sigma \vec{E} = \frac{\sigma}{\epsilon} \vec{D}. \quad (1-11.1)$$

At sufficiently low frequencies,  $\sigma$  and  $\epsilon$  are effectively frequency independent constants. Then the local charge conservation equation,

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \vec{J} = 0 , \quad (1-11.2)$$

becomes

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \left( \frac{\sigma}{\epsilon} \vec{D} \right) = 0 . \quad (1-11.3)$$

In the interior of a homogeneous substance, the use of  $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$  produces the differential equation

$$\frac{\partial}{\partial t} \rho + \frac{\sigma}{\epsilon} 4\pi\rho = 0 . \quad (1-11.4)$$

Starting at time  $t = 0$ , the solution at any later time,

$$\rho(\vec{r},t) = e^{-\frac{4\pi\sigma}{\epsilon}t} \rho(\vec{r},0) , \quad (1-11.5)$$

shows that charge disappears from the interior of the conducting body at a rate measured by

$$\gamma' = \frac{4\pi\sigma}{\epsilon} . \quad (1-11.6)$$

Where does the charge go? Why, to the surface of the conductor (see Problem ).

The use of the static conductivity here assumes that the measure of the rate of change,  $\gamma'$ , is small compared to the frictional constant  $\gamma$ ,

$$\gamma' = \frac{4\pi\sigma}{\epsilon} \ll \gamma , \quad (1-11.7)$$

or, using (1-9.6), that

$$\frac{4\pi}{\epsilon} \frac{n_f e^2}{m} \ll \gamma^2 . \quad (1-11.8)$$

This must be satisfied by some combination of a small density of free charge and a high coefficient of friction. But what of

the opposite situation, where there is a high density of free charge and little friction, as encountered in a plasma? For that, it may be clearer to return to the equation of motion (1-9.1), presented as the differential equation

$$(\frac{\partial}{\partial t} + \gamma) \vec{J} = \frac{n_f e^2}{m} \vec{E} = \frac{n_f e^2}{m \epsilon} \vec{D} \quad (1-11.9)$$

for the current

$$\vec{J} = n_f e \vec{v} \quad (1-11.10)$$

Now we can use the charge conservation equation, in the form

$$(\frac{\partial}{\partial t} + \gamma) \frac{\partial \phi}{\partial E} + \nabla \cdot (\frac{\partial}{\partial t} + \gamma) \vec{J} = 0, \quad (1-11.11)$$

to get

$$(\frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t} + \omega_p^2) \phi = 0, \quad (1-11.12)$$

where

$$\omega_p^2 = \frac{4\pi}{\epsilon} \frac{n_f e^2}{m} \quad (1-11.13)$$

Here is the same kind of differential operator encountered in (1-10.5), and the solutions of such an equation are generally damped oscillations. But, first, if the rate of change is small on the scale set by  $\gamma$ , the equation simplifies to

$$(\frac{\partial}{\partial t} + \gamma') \phi = 0, \quad \gamma' = \frac{\omega_p^2}{\gamma} \ll \gamma, \quad (1-11.14)$$

which will be recognized as the previous result, including the restrictive condition (1-11.8). But at the other limit where change occurs rapidly relative to  $\gamma$ , it is the  $\gamma$  term

that can be approximately neglected,

$$\left( \frac{\partial^2}{\partial t^2} + \omega_p^2 \right) \rho = 0, \quad (1-11.15)$$

and the charge density oscillates at the angular frequency  $\omega_p$ , the plasma frequency.

The appearance of  $\epsilon$  in this plasma frequency is a reminder of a restriction still in force, that  $\omega_p$  be small compared with  $\omega_0$ , the characteristic atomic frequency, near which the frequency dependence of  $\epsilon(\omega)$  can no longer be ignored. There is an extreme plasma circumstance in which  $\omega_0$  is not relevant. Let the physical conditions be such that all atoms are completely ionized, removing the distinction between free and bound charge. Then the entire charge density can be viewed as the result of polarization,

$$\rho = -\vec{\nabla} \cdot \vec{P}, \quad (1-11.16)$$

arising from the displacement of the electrons ( $\vec{r}$ ) relative to the oppositely charged heavy ions,

$$\vec{P} = n e \vec{r}. \quad (1-11.17)$$

That displacement changes in time, responding to the electric field as

$$m \frac{d^2 \vec{r}}{dt^2} = e \vec{E}, \quad (1-11.18)$$

or.

$$\frac{\partial^2}{\partial t^2} \vec{P} = \frac{n e^2}{m} \vec{E}. \quad (1-11.19)$$

From this, and the field equation

$$\vec{\nabla} \cdot \vec{E} = 4\pi (-\vec{\nabla} \cdot \vec{P}), \quad (1-11.20)$$

we derive

$$\left( \frac{\partial^2}{\partial t^2} + \omega_p^2 \right) (\vec{\nabla} \cdot \vec{E}) = 0, \quad (1-11.21)$$

$$\omega_p^2 = 4\pi \frac{n e^2}{m}, \quad (1-11.22)$$

as the description of the plasma oscillations. Not unexpectedly, the plasma frequency now is the previous one, (1-11.13), with  $\epsilon = 1$ .

## 12. Polar Molecules

In these discussions only induced dipole moments, those created by an applied electric field, have been considered. But what about permanent electric dipole moments? Do individual atoms possess such static properties? Well, with the exception of atomic hydrogen where the orbital motion respects a preferred direction in space (as in the classical elliptic orbits), atomic electric dipole moments change direction in space so rapidly in response to the fast electronic motion, that no average effect survives. But things are different with atomic compounds - molecules - specifically, those of a polar nature. In the example of  $H^+Cl^-$ , the hydrogenic electron is transferred to form the chlorine ion, and a dipole moment is associated with the relative motion of the heavy ions. Other examples of polar molecules associated with familiar substances are  $H_2O$ ,  $SO_2$ ,  $NH_3$  and  $CH_3Cl$ . For such molecules, in isolation, it is not misleading to think of a permanent electric dipole moment that changes its spatial orientation only in response to the slow rotation of the molecule.

But molecules are not ordinarily isolated; they exist in an environment in which other molecules collide with them at a rate determined by the temperature of the substance. The effect of these collisions is to remove any particular spatial orientation of the dipole moments; it still requires an electric field to provide a preferred direction. But now there is a competition between the organizing effect of the electric field, with its preference for lower values of the energy

$$E = - \vec{d} \cdot \vec{E} = - | \vec{d} | | \vec{E} | \cos \theta, \quad (1-12.1)$$

and the disorganizing effect of the ambient temperature  $T$ . For a static

field, the net balance of that competition is expressed by the Boltzmann factor

$$e^{-E/kT} \quad (1-12.2)$$

where  $k$ , the constant of Ludwig Boltzmann (1844-1906) has the value

$$k = 1.381 \times 10^{-16} \text{ erg/degree Kelvin} . \quad (1-12.3)$$

In thermal equilibrium, the fraction of dipole moments that are directed within the solid angle



$$d\Omega = \sin \theta \, d\theta \, d\phi \quad (1-12.4)$$

is proportional to the product of  $d\Omega$  with the Boltzmann factor:

$$\frac{1}{2} \frac{d\Omega}{4\pi} e^{\vec{d} \cdot \vec{E}/kT}, \quad (1-12.5)$$

where the choice of

$$\bar{z} = \int \frac{d\Omega}{4\pi} e^{\vec{d} \cdot \vec{E}/kT} \quad (1-12.6)$$

ensures that the totality of such fractions equals unity. Now, the average dipole moment is computed as

$$\langle \vec{d} \rangle_T = \frac{1}{2} \int \frac{d\Omega}{4\pi} \vec{d} e^{\vec{d} \cdot \vec{E}/kT} = \frac{1}{2} kT \frac{1}{\bar{E}} \bar{z}. \quad (1-12.7)$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{x + \frac{1}{6}x^3}{x - \frac{1}{6}x^3} = \frac{2 + x^2}{2 - x^2}$$

For sufficiently weak fields (see later), the exponential Boltzmann factor can be expanded,

$$\begin{aligned} Z &\equiv \int \frac{d\Omega}{4\pi} \left[ 1 + \frac{\vec{d} \cdot \vec{E}}{kT} + \frac{1}{2} \left( \frac{\vec{d} \cdot \vec{E}}{kT} \right)^2 + \dots \right] \\ &= 1 + \frac{1}{6} \left( \frac{|\vec{d}| |\vec{E}|}{kT} \right)^2 + \dots \end{aligned}$$

$$\begin{aligned} &\approx \frac{2+x^2}{2x(1+\frac{x^2}{6})} \\ &= \frac{(2+x^2)}{2x} \left( 1 - \frac{x^2}{6} \right) \\ &\stackrel{(1-12.8)}{=} \frac{1}{x} + \frac{k}{2} - \frac{x}{6} \end{aligned}$$

In this uniformly weighted average over all directions where  $\vec{d}$  and  $-\vec{d}$  appear with equal weight, the average of  $\vec{d}$  is zero, and that of the square of some component, say  $d_z^2$ , is the same as any other component, so that

$$\langle d_x^2 \rangle = \langle d_y^2 \rangle = \langle d_z^2 \rangle = \frac{1}{3} (\vec{d})^2. \quad (1-12.9)$$

The leading contribution to (1-12.7) is produced by

$$\frac{\partial Z}{\partial \vec{E}} = \frac{1}{3} \frac{d^2}{(kT)^2} \vec{E}, \quad (1-12.10)$$

and  $Z$  in the denominator is replaced by unity in arriving at the linear approximation,

$$\langle \vec{d} \rangle_T \approx \frac{1}{3} \frac{d^2}{kT} \vec{E}. \quad (1-12.11)$$

To suggest the circumstances for which this is applicable, let  $|\vec{d}|$  equal the electron charge,

$$e = 4.803 \times 10^{-10} \text{ esu}, \quad (1-12.12)$$

multiplied by the typical atomic dimension

$$1 \text{ \AA} = 10^{-8} \text{ cm.} \quad (1-12.13)$$

Then, at the nominal room temperature  $T = 300^\circ\text{K}$ , the weak field condition  $|d||\vec{E}| \ll kT$  requires that

$$|\vec{E}| \ll 10^4 \text{ esu} = 3 \times 10^6 \text{ volts/cm}; \quad (1-12.14)$$

ordinary electric field strengths are comfortably below this limit.

With a density of  $n_{\text{mol}}$  polar molecules per unit volume, the (weak field) contribution to the polarization vector is

$$\vec{P}_{\text{mol}} = n_{\text{mol}} \frac{d^2}{3kT} \vec{E}, \quad (1-12.15)$$

and the complete static susceptibility becomes

$$\chi_e = \chi_{e,\text{atom}} + n_{\text{mol}} \frac{d^2}{3kT}. \quad (1-12.16)$$

Through the dependence on temperature of the latter part, one can separate the two effects experimentally and determine the magnitude of the dipole moment. As an example, one of the largest observed moments, ~~that of~~ that of potassium chloride, is slightly more ~~than~~ twice than the value,  $4.8 \times 10^{-18}$ , that was used in the estimate of (1-12.14).

The Boltzmann factor describes the situation of a static

electric field. But suppose the electric field is vibrating, or rotating, with a definite frequency. Owing to the large inertia of molecules, it takes a significant time for the dipole moment to readjust or relax into the configuration demanded by the new direction of the electric field. With increasing frequency the ability to readjust decreases and eventually the dipole moment ceases to follow the variations of the electric field. Experiments indicate that this occurs already for radio frequencies; at infra-red and visible light frequencies, the dipole moments of polar molecules are effectively inert.

A simple quantitative version of this picture combines two ideas. First, in the absence of the electric field, any net dipole moment relaxes to zero with a characteristic decay rate, or inverse relaxation time,  $\gamma$ , and, in the presence of a static field, the average moment is given by (1-12.11). These are united in the differential equation

$$\frac{d}{dt} \langle \vec{d}(t) \rangle = -\gamma \left[ \langle \vec{d}(t) \rangle - \frac{d^2}{3kT} \vec{E}(t) \right]. \quad (1-12.17)$$

The steady state solution for

$$\vec{E}(t) = Re(\vec{E} e^{-i\omega t}) \quad (1-12.18)$$

is directly verified to be

$$\langle \vec{d}(t) \rangle = Re \left( \frac{d^2}{3kT} \frac{\gamma}{\gamma - i\omega} \vec{E} e^{-i\omega t} \right). \quad (1-12.19)$$

The implications for the static situation,  $\omega = 0$ , and for the high frequency limit,  $\omega \gg \gamma$ , are as described. We see in (1-12.19) that, for a field of frequency  $\omega$ , the static susceptibility contribution of the polar molecules is multiplied by

$$\frac{1}{1-i\omega\tau} \quad (1-12.20)$$

where

$$\tau = \frac{1}{\gamma} \quad (1-12.21)$$

is the relaxation time of the electric dipole moments.

### 13. The Driving Field

We must not hurry on without exposing a simplifying assumption that is not generally correct. In computing the dipole moment induced in atoms and the aligning of the permanent dipoles of polar molecules, we have taken the electric field that drives the microscopic system to be

$$\vec{E}(\vec{r}, t) = \overline{\vec{e}(\vec{r}, t)}. \quad (1-13.1)$$

However, the microscopic field  $\vec{e}$  includes not only the field of all other atoms or molecules, but also the field of the particular system being considered. That self-action, of an electrostatic nature, does not belong here; it is only forces of external origin that count. The microscopic field that drives the given system is thus

$$\vec{e}_{\text{drive}} = \vec{e} - \vec{e}_{\text{self}}, \quad (1-13.2)$$

but this acquires a useful macroscopic form only when the sources of  $\vec{e}_{\text{drive}}$  are sufficiently far away. What we are about to do should not be expected to apply for a strongly polar liquid or solid where the forces produced by neighboring molecules could be the dominant effect.

Think of each molecule, say, as surrounded by a sphere of radius  $a$ , the volume of which,  $V = \frac{4\pi}{3}a^3$ , is the average volume per molecule, the inverse of the density,

$$n = \frac{1}{V}. \quad (1-13.3)$$

Then  $\vec{e}_{\text{drive}}$  at any point within the microscopic system will not differ appreciably from its average over the volume  $V$ ,  $\vec{e}_{\text{gain}}$  provided no significant contribution is produced by neighboring molecules. That gives the driving field

$$\vec{E}_{\text{drive}} = \overline{\vec{e}}_{\text{drive}} = \overline{\vec{e}} - \overline{\vec{e}}_{\text{self}} = \overline{\vec{E}} - \overline{\vec{e}}_{\text{self}} \quad (1-13.4)$$

which also embodies the assumption that the average over  $V$  is already sufficiently representative of a macroscopic average that the field  $\overline{\vec{E}}$  can be introduced. Now to the calculation of  $\overline{\vec{e}}_{\text{self}}$ , where [(1-1.13)]

$$\overline{\vec{e}}_{\text{self}}(\vec{r}) = -\nabla \sum_k \frac{e_k}{|\vec{r} - \vec{r}_k|} = \sum_k \overline{\nabla}_k \frac{e_k}{|\vec{r} - \vec{r}_k|}, \quad (1-13.5)$$

refers to all the charges in the particular system. That gives, for the negative of  $\overline{\vec{e}}_{\text{self}}$ ,

$$-\overline{\vec{e}}_{\text{self}} = \sum_k (-\overline{\nabla}_k) \int_V (d\vec{r}) \frac{e_k/V}{|\vec{r}_k - \vec{r}|}. \quad (1-13.6)$$

An individual term here,

$$\overline{\vec{e}}_k(\vec{r}_k) = -\overline{\nabla}_k \int_V (d\vec{r}) \frac{e_k/V}{|\vec{r}_k - \vec{r}|}, \quad (1-13.7)$$

is the electric field, produced at the point  $\vec{r}_k$  within the sphere, arising from a uniform charge distribution of density  $e_k/V$ . As such, it obeys the differential equation [(1-1.26)]

$$\overline{\nabla}_k \cdot \overline{\vec{e}}_k = 4\pi \frac{e_k}{V} = \overline{\nabla}_k \cdot \frac{4\pi}{3} \frac{e_k}{V} (\vec{r}_k - \vec{R}) \quad (1-13.8)$$

where  $\vec{R}$  is the position vector of the center of the sphere. The last form merely uses (with subscript  $k$ ) the fact that

$$\vec{\nabla} \cdot (\vec{r} - \vec{R}) = \vec{\nabla} \cdot \vec{r} = 3. \quad (1-13.9)$$

And now the spherical symmetry of the situation, telling us that the electric field at a point must be directed along the line from the center, immediately yields (see problem 3.3)

$$\vec{e}_k = \frac{4\pi}{3} n e_k (\vec{r}_k - \vec{R}), \quad (1-13.10)$$

and then

$$\begin{aligned} -\vec{E}_{\text{pert}} &= \frac{4\pi}{3} n \sum_k e_k (\vec{r}_k - \vec{R}) = \frac{4\pi}{3} n \vec{d} \\ &= \frac{4\pi}{3} \vec{P}. \end{aligned} \quad (1-13.11)$$

We conclude that the correct driving field (subject to the caveats mentioned) is

$$\vec{E}_{\text{drive}} = \vec{E} + \frac{4\pi}{3} \vec{P}. \quad (1-13.12)$$

To appreciate what effect this has on our earlier results, let us denote by  $\chi'_e$  what was previously, and incorrectly, called the susceptibility. For static fields, then,

$$\chi'_e = n_b \frac{e^2}{m \omega_0^2} + n_{\text{mol}} \frac{d^2}{3kT} \quad (1-13.13)$$

and now we have

$$\vec{P} = \chi'_e \vec{E}_{\text{drive}} = \chi'_e (\vec{E} + \frac{4\pi}{3} \vec{P}) \quad (1-13.14)$$

or

$$\chi_e = \chi'_e (1 + \frac{4\pi}{3} \chi_e). \quad (1-13.15)$$

From this form, or the equivalent versions

$$\frac{\chi_e}{1 + \frac{4\pi}{3} \chi_e} = \chi'_e, \quad \chi_e = \frac{\chi'_e}{1 - \frac{4\pi}{3} \chi'_e}, \quad (1-13.16)$$

it is clear that the earlier identification of  $\chi_e$  with  $\chi'_e$  is valid only when  $\chi_e \ll 1$ , as in substances of low density. Yet another way of presenting matters, one that introduces the dielectric constant according to

$$4\pi \chi_e = \epsilon - 1, \quad (1-13.17)$$

namely,

$$\frac{\epsilon - 1}{\epsilon + 2} = \frac{4\pi}{3} \chi'_e, \quad (1-13.18)$$

is known, from its historical origins, as the Clausius-Mossotti formula [O. F. Mossotti (1850), R. Clausius (1879); the years cited indicate the dates of significant publication].

When one turns to fields of definite frequency, the names change, the relation

$$\frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 2} = \frac{4\pi}{3} \chi_e'(\omega) \quad (1-13.19)$$

being called that of Lorentz and Lorenz [L. Lorenz (1869), H. A. Lorentz (1880)]. For the particular example of a non-polar substance, where [(1-10.8)]

$$\chi_e'(\omega) = \frac{n_b e^2}{m} \frac{1}{-\omega^2 - i\omega\gamma + \omega_0^2}, \quad (1-13.20)$$

the formula

$$4\pi \chi_e(\omega) = \frac{4\pi \chi_e'(\omega)}{1 - \frac{4\pi}{3} \chi_e'(\omega)} = \frac{1}{(4\pi \chi_e'(\omega))^{-1} - \frac{1}{3}} \quad (1-13.21)$$

immediately gives

$$\chi_e(\omega) = \frac{n_b e^2}{m} \frac{1}{-\omega^2 - i\omega\gamma + \omega_1^2}, \quad (1-13.22)$$

in which

$$\omega_1^2 = \omega_0^2 - \frac{4\pi}{3} \frac{n_b e^2}{m}. \quad (1-13.23)$$

Thus the only effect is to lower the characteristic frequency of the atomic oscillators. That is expressed in terms of the static dielectric constant as

$$\omega_1^2 = \frac{3}{\epsilon + 2} \omega_0^2 < \omega_0^2. \quad (1-13.24)$$

Accordingly, the integral relation (1-10.17), which does not refer to this frequency, remains intact (see problem ).

Here is an experimental example of a situation in which the modification of the driving field is both significant and accurate. The static dielectric constant of non-polar nitrogen gas has been measured at low and high densities. At the density  $0.06604 \text{ gm/cm}^3$ , the observed value of  $\epsilon - 1$  is 0.03109. If  $\epsilon - 1$  were equal to  $4\pi N e$ , which is proportional to the density, the value predicted for  $0.5780 \text{ gm/cm}^3$  - slightly more than half the density of water - would be  $\epsilon - 1 = 0.2721$ . The measured value is 9% larger: 0.29633. If, however, one uses the Clausius - Mossotti relation (1-13.18), it is  $(\epsilon - 1)/(\epsilon + 2)$  that is proportional to the density. Now the value of  $\epsilon - 1$  predicted for the larger density is 0.2959, which falls short of the measurement only by 0.1%.

## SIMPLE MICROSCOPIC MODELS. MAGNETIC PROPERTIES

### 14. Induced Electric Field

An imposed electric field does work on individual charges at the rate

$$e \vec{E} \cdot \vec{v}, \quad (1-14.1)$$

and thereby changes the energy of the system. In contrast, the rate at which an imposed magnetic field does work on a moving charge is

$$\frac{e}{c} (\vec{v} \times \vec{B}) \cdot \vec{v} = 0. \quad (1-14.2)$$

But we know quite well experimentally that magnetic fields do influence material properties. What's going on? The subtlety here is that one can not ignore the act of turning on the magnetic field. A changing magnetic field induces an electric field, as described by the Faraday-Maxwell equation

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B}; \quad (1-14.3)$$

that electric field does work on the system and changes its energy.

For the purpose of exhibiting the induced electric field, the following vector identity is useful ( $\vec{V}$  is a constant vector)

$$\vec{\nabla} \times \frac{1}{2} (\vec{V} \times \vec{r}) = \frac{1}{2} (\vec{V} \vec{\nabla} \cdot \vec{r} - \vec{V} \cdot \vec{\nabla} \vec{r}) = \vec{V}. \quad (1-14.4)$$

This is applied to a magnetic field that is spatially constant microscopically,

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times \left( \frac{1}{2} \vec{B} \times \vec{r} \right), \quad (1-14.5)$$

with the inference that the induced electric field is

$$\vec{E}_{ind} = -\frac{1}{2c} \frac{\partial}{\partial t} \vec{B} \times \vec{r}. \quad \text{refers to position in space} \quad (1-14.6)$$

Although we are interested in atomic systems that may have a number of charges, the following study explicitly considers only a single charge moving about the origin; its extension will be clear. The induced field, (1-14.6), changes the energy at the rate

$$\begin{aligned} \frac{dE}{dt} &= e \vec{E}_{ind} \cdot \vec{v} = -\frac{e}{2c} \vec{r} \times \vec{v} \cdot \frac{\partial}{\partial t} \vec{B} \\ &= -\vec{\mu} \cdot \frac{\partial}{\partial t} \vec{B}, \end{aligned} \quad (1-14.7)$$

where we have recognized, in the simplified model, the magnetic moment  $\vec{\mu}$  that is more generally given by (1-6.21),

$$\vec{\mu} = \frac{1}{2} \sum_k \frac{e_k}{c} (\vec{r}_k - \vec{R}) \times (\vec{v}_k - \vec{V}). \quad (1-14.8)$$

In a small time interval,  $\delta t$ , the energy changes by the amount  $\delta E = \delta t \left( \frac{dE}{dt} \right)$ , or

$$\delta E = -\vec{\mu} \cdot \delta \vec{B} \quad (1-14.9)$$

where  $\delta \vec{B}$  is the change of  $\vec{B}$  in that time interval. We have ~~meet~~ this relation before in connection with spatial variation. Under the action of ~~the~~ force 1-6.22, the spatial displacement  $\delta \vec{R}$  alters the energy by

$$\delta E = -\delta \vec{R} \cdot \vec{F} = -\vec{\mu} \cdot \delta \vec{B} \quad (1-14.10)$$

where

$$\delta \vec{B} = (\delta \vec{R} \cdot \vec{\nabla}_R) \vec{B}(\vec{R}). \quad (1-14.11)$$

This indicates the generality of the relation (1-14.9), which is also conveyed by the differential statement

$$-\frac{dE}{d\vec{B}} = \vec{\mu}. \quad (1-14.12)$$

The anticipated electric analogue of this emerges from

$$\frac{dE_0}{dt} = e \vec{E} \cdot \vec{v} \quad (1-14.13)$$

where  $E_0$  is the energy expression in the absence of the applied spatially constant field  $\vec{E}(t)$ . We now write ( $\vec{v} = \frac{d}{dt} \vec{r}$ )

$$\frac{dE_0}{dt} = \frac{d}{dt} (e \vec{E} \cdot \vec{r}) - e \vec{r} \cdot \frac{d}{dt} \vec{E} \quad (1-14.14)$$

which, with the definition

$$E = E_0 - \vec{E} \cdot \vec{r}, \quad (1-14.15)$$

becomes

$$\frac{dE}{dt} = - \vec{d} \cdot \frac{\partial}{\partial t} \vec{E}. \quad (1-14.16)$$

Here, involving the electric dipole moment  $\vec{d}$ , is the required form. It also can be expressed more generally as

$$\delta E = - \vec{d} \cdot \delta \vec{E} \quad (1-14.17)$$

or

$$-\frac{\partial E}{\partial \vec{E}} = \vec{d}. \quad (1-14.18)$$

In this situation, the definition of the energy has been altered, by the addition of the potential energy of the system in the field  $\vec{E}$ :  $-\vec{E} \cdot \vec{d}$ . The analogue of that for a magnetic field appears on considering momentum. The equation of motion for the single charge is

$$\frac{d}{dt} m \vec{v} = e \vec{E}_{atom} + e \left( -\frac{1}{2c} \frac{\partial}{\partial t} \vec{B} \times \vec{r} \right) + \frac{e}{c} \vec{v} \times \vec{B} \quad (1-14.19)$$

where  $\vec{E}_{atom}$  is the atomic electrostatic field, directed radially out from the origin. Now observe that

$$-\frac{e}{2c} \frac{\partial}{\partial t} \vec{B} \times \vec{r} + \frac{e}{c} \vec{v} \times \vec{B} = \frac{d}{dt} \left( -\frac{e}{2c} \vec{B} \times \vec{r} \right) + \frac{1}{2} \frac{e}{c} \vec{v} \times \vec{B} \quad (1-14.20)$$

which gives

$$\frac{d}{dt} \vec{p} = e \vec{E}_{\text{atom}} + \frac{1}{2} \frac{e}{c} \vec{v} \times \vec{B}, \quad (1-14.21)$$

where

*emf flance*

$$\vec{p} = m \vec{v} + \frac{e}{2c} \vec{B} \times \vec{r}. \quad (1-14.22)$$

In this redefinition of momentum, we find the kinetic momentum  $m\vec{v}$  supplemented by the potential momentum  $(\frac{e}{2c})\vec{B} \times \vec{r}$ .

It is interesting to consider the moment of this force equation,

$$\vec{r} \times \frac{d}{dt} \vec{p} = \frac{d}{dt} (\vec{r} \times \vec{p}) - \vec{v} \times \vec{p} = \frac{e}{2c} \vec{r} \times (\vec{v} \times \vec{B}), \quad (1-14.23)$$

which uses the vanishing moment of the radial atomic field,

$$\vec{r} \times \vec{E}_{\text{atom}} = 0. \quad (1-14.24)$$

Now, according to (1-14.22), we have

$$\vec{v} \times \vec{p} = \vec{v} \times \left( m \vec{v} + \frac{e}{2c} \vec{B} \times \vec{r} \right) = \frac{e}{2c} \vec{v} \times (\vec{B} \times \vec{r}), \quad (1-14.25)$$

leading to

$$\begin{aligned} \frac{d}{dt} (\vec{r} \times \vec{p}) &= \frac{e}{2c} \left[ \vec{r} \times (\vec{v} \times \vec{B}) + \vec{v} \times (\vec{B} \times \vec{r}) \right] \\ &= \frac{e}{2c} (\vec{r} \times \vec{v}) \times \vec{B} \end{aligned} \quad (1-14.26)$$

*for*

by an application of the vector identity (1-6.33). We shall

write this as

$$\frac{d}{dt} \vec{L} = \vec{\mu} \times \vec{B}, \quad (1-14.27)$$

$$\vec{L} = \vec{r} \times \vec{p}, \quad (1-14.28)$$

where the recognition [(1-6.28)] that the right hand side of (1-14.27) is the torque exerted on the system supplies the dynamical identification of  $\vec{L}$ , the moment of the momentum, as the angular momentum of the system.

The possibility of replacing  $\vec{v}$  by  $\vec{p}$ , in accordance with [(1-14.22)]

$$\vec{v} = \frac{1}{m} \vec{p} - \frac{e}{2mc} \vec{B} \times \vec{r}, \quad (1-14.29)$$

then gives the magnetic moment as the sum of two parts,

$$\vec{\mu} = \vec{\mu}_0 - \frac{e^2}{4mc^2} \vec{r} \times (\vec{B} \times \vec{r}), \quad (1-14.30)$$

where

$$\vec{\mu}_0 = \frac{e}{2mc} \vec{L}. \quad (1-14.31)$$

Note that the torque equation, (1-14.27), can also be presented as

$$\frac{d}{dt} \vec{\mu}_0 = \frac{e}{2mc} \vec{\mu} \times \vec{B}. \quad (1-14.32)$$

### 15. Magnetic Energy. Diamagnetism

Now to find out how the energy of the system differs from  $E_0$ , its value in the absence of the applied magnetic field. Let the magnetic field grow from zero to its final value  $\vec{B}$  as

$$\vec{B}(t) = \lambda(t) \vec{B} \quad (1-15.1)$$

where  $\lambda(t)$  increases slowly from 0 to 1. According to (1-14.7), the rate of change of the energy at time  $t$  is

$$\frac{dE}{dt} = -\vec{\mu}(t) \cdot \frac{\partial}{\partial t} \vec{B}(t) = -\vec{\mu}(t) \cdot \vec{B} \frac{d\lambda(t)}{dt}. \quad (1-15.2)$$

Then, using the construction of  $\vec{\mu}(t)$  given in (1-14.30), we have

$$\vec{\mu}(t) \cdot \vec{B} = \vec{\mu}_0(t) \cdot \vec{B} - \frac{e^2}{4mc^2} (\vec{B} \times \vec{r})^2 \lambda(t), \quad (1-15.3)$$

since it is  $\vec{B}(t)$  that occurs in the second term of  $\vec{\mu}(t)$ .

Next, observe that the torque equation, (1-14.32), gives, at time  $t$ ,

$$\frac{d\vec{\mu}_0(t)}{dt} \cdot \vec{B} = \frac{e}{2mc} \vec{\mu}(t) \times \vec{B} \lambda(t) \cdot \vec{B} = 0, \quad (1-15.4)$$

which is to say that  $\vec{\mu}_0(t) \cdot \vec{B}$  does not change in time.

We have now reached this stage:

$$\frac{dE}{dt} = -\vec{\mu}_0 \cdot \vec{B} \frac{d\lambda(t)}{dt} + \frac{e^2}{4mc^2} (\vec{B} \times \vec{r})^2 \lambda(t) \frac{d\lambda(t)}{dt}, \quad (1-15.5)$$

where  $\vec{\mu}_0 \cdot \vec{B}$  is a constant. In contrast,  $(\vec{B} \times \vec{F}(t))^2$  changes rapidly in time owing to the atomic motion of the charge. All that matters here, however, is the time average of  $(\vec{B} \times \vec{F}(t))^2$  over one period of the atomic motion, a time interval that is small on the scale of the change in  $\lambda(t)$ . In the following, we understand that time average without an explicit indication. The weakness of magnetic forces in atoms ( $\frac{e}{c} \ll 1$ ), compared with electric forces, justifies computing  $(\vec{B} \times \vec{F})^2$  without reference to the applied magnetic field. We can now carry out the time integration that establishes the field  $\vec{B}$  at its final value:

$$\int dt \frac{d\lambda(t)}{dt} = \int_0' d\lambda = 1, \quad (1-15.6)$$

$$\int dt \lambda(t) \frac{d\lambda(t)}{dt} = \int_0' \frac{1}{2} d\lambda^2 = \frac{1}{2}, \quad (1-15.7)$$

resulting in

$$E = E_0 - \vec{\mu}_0 \cdot \vec{B} + \frac{1}{2} \frac{e^2}{4m c^2} (\vec{B} \times \vec{F})^2. \quad (1-15.8)$$

One correctly suspects that, for a system of several charges centered about  $\vec{R}$ , the last term is replaced by

$$\frac{1}{2} \sum_k \frac{e_k^2}{4m_k c^2} [\vec{B} \times (\vec{r}_k - \vec{R})]^2. \quad (1-15.9)$$

In the first magnetic energy term,  $-\vec{\mu}_0 \cdot \vec{B}$ , we recognize the contribution of a permanent dipole. We defer consideration of this effect and initially suppose that  $\vec{\mu}_0 = 0$ ,

as would be realized in a system of zero angular momentum. It is a quantum rather than classical assertion that a system with  $\vec{L} = 0$  has no preferred direction associated with its orbital motion; it is spherically symmetrical. But, in any event, the summation over many atomic systems with random orientation will bring about an effective averaging over all directions, as expressed by

$$(\vec{B} \times \vec{r})^2 = B^2 r^2 - (\vec{B} \cdot \vec{r})^2 \rightarrow \frac{2}{3} B^2 r^2. \quad (1-15.10)$$

From this average energy,

$$E = E_0 + \frac{1}{3} \frac{e^2}{4m c^2} r^2 B^2, \quad (1-15.11)$$

we compute the average induced magnetic moment, (1-14.12),

$$\vec{\mu} = - \frac{\partial E}{\partial \vec{B}} = - \frac{e^2}{6m c^2} r^2 \vec{B}, \quad (1-15.12)$$

and the macroscopic magnetic polarization,

$$\vec{M} = n \vec{\mu} = -n \frac{e^2}{6m c^2} r^2 \vec{B}. \quad (1-15.13)$$

That the induced magnetization in atoms is opposite in sign to the inducing field is a microscopic example of what is called Lenz's law [Heinrich Lenz (1804-1865)]. Bodies for which this universal effect is the dominant one are called diamagnetic. They are repelled by regions of strong magnetic field, where the energy is increased.

It is traditional to relate the magnetic polarization to the field  $\vec{H}$  in defining the magnetic susceptibility,

$$\vec{M} = \chi_m \vec{H}. \quad (1-15.13)$$

Then the magnetic induction vector  $\vec{B}$  is

$$\begin{aligned} \vec{B} &= \vec{H} + 4\pi \vec{M} = (1 + 4\pi \chi_m) \vec{H} \\ &= \mu \vec{H} \end{aligned} \quad (1-15.14)$$

which defines the permeability

$$\mu = 1 + 4\pi \chi_m. \quad (1-15.15)$$

We see that the linear relation between  $\vec{M}$  and  $\vec{B}$ ,

$$\vec{M} = \chi_m \vec{H} = \frac{\chi_m}{\mu} \vec{B}, \quad (1-15.16)$$

actually yields

$$\frac{\chi_m}{1 + 4\pi \chi_m} = -n \frac{e^2}{6mc^2} r^2, \quad (1-15.17)$$

although only a small error is made in writing

$$\chi_m = -n \frac{e^2}{6mc^2} r^2, \quad (1-15.18)$$

inasmuch as  $-\chi_m \ll 1$ . That follows on approximate comparison with  $\chi_e$  [(1-10.9)],

$$-\frac{\chi_m}{\chi_e} = \frac{1}{6} \left( \frac{\omega_0 r}{c} \right)^2 = \frac{1}{6} \left( \frac{v}{c} \right)^2 \sim 10^{-5} - 10^{-7}, \quad (1-15.19)$$

which, through the intermediaries of characteristic atomic times ( $1/\omega_0$ ) and distances ( $r$ ) compares typical atomic speeds ( $v$ ) to the speed of light ( $c$ ):  $v/c \sim 10^{-2} - 10^{-3}$ . The simple formula (1-15.17) suggests that the ratio of susceptibility to density for a given substance is independent of temperature. This is almost universally valid experimentally. In the example of water, where the mass density is practically constant, the measured susceptibility is close to  $-0.7 \times 10^{-6}$ .

The major exception is ~~the metal~~ with the largest diamagnetic susceptibility  $\rightarrow$  ~~bismuth~~. Its susceptibility at room temperature is about  $-1.3 \times 10^{-6}$ , but decreases significantly with rising temperature. Here the quantum insights of the modern theory of metals are indispensable. In summary, a diamagnetic substance is one with negative susceptibility and permeability  $\mu < 1$ .

Now suppose there is a permanent magnetic dipole moment that can be partly aligned by a magnetic field, against the disorganizing effect of temperature. This is entirely analogous to permanent electric dipoles in an electric field, and we merely transcribe (1-12.15) into magnetic language:

$$\vec{M} = n \frac{\mu_0^2}{3kT} \vec{B}, \quad (1-15.20)$$

which is appropriate to the weak field circumstance

$$\mu_0 B \ll kT.$$

(1-15.21)

Inasmuch as the typical magnitudes of  $\mu_0/d$  are of order  $v/c \sim 10^{-2} - 10^{-3}$ , the upper limit to  $B$ , at room temperature, is in the range of millions of gauss. Substances with positive magnetic susceptibilities are called paramagnetic. For this class of materials, the permeability  $\mu > 1$ . The simple models indicate the ratio of paramagnetic to diamagnetic susceptibility to be of the order

$$\frac{\chi_m \text{ para}}{\chi_m \text{ dia}} \sim \frac{mv^2}{kT}$$

(1-15.22)

where  $mv^2$  is related to the magnitude of atomic energies. Now, atoms are perfectly stable at room temperature but can be dissociated at temperatures  $T \sim 10^4 - 10^5 \text{ }^\circ\text{K}$ , which indicates that the ratio  $mv^2/kT$  is in the hundreds range, at room temperature. This is in general agreement with the observation that paramagnetic gaseous oxygen at standard pressure and room temperature has a (positive) susceptibility about one fifth the susceptibility of water, although the molecular density of the oxygen is less than a thousandth that of water. The susceptibilities of paramagnetic substances are still so small compared with unity (for liquid oxygen,  $\chi_m = 3 \times 10^{-4}$ ) that, as in the diamagnetic discussion, the distinction between  $B$  and  $H$  can be ignored, and (1-15.20) presented as

$$\chi_m = n \frac{\mu_0^2}{3kT} . \quad (1-15.23)$$

The inverse dependence on temperature was discovered experimentally by Pierre Curie (1859-1906).

Again we have persisted in an error. Just as the microscopic driving electric field is not  $\vec{E}$ , so its magnetic counterpart is not  $\vec{B}$ . We must remove the average field produced by the system itself:

$$\vec{B}_{\text{drive}} = \vec{B} - \overline{\vec{b}_{\text{self}}} . \quad (1-15.24)$$

For simplicity, we again use the model with a single charge. We recall from (1-2.11) that the magnetic field associated with a moving charge is

$$\vec{b} = \frac{\vec{v}}{c} \times \vec{e} . \quad (1-15.25)$$

Accordingly, the result of the electric discussion, (1-13.11), gives immediately

$$\overline{\vec{b}_{\text{self}}} = \frac{\vec{v}}{c} \times \overline{\vec{e}_{\text{self}}} = \frac{\vec{v}}{c} \times \left( -\frac{4\pi e}{3} \vec{r} \right) n , \quad (1-15.26)$$

or

$$\overline{\vec{b}_{\text{self}}} = \frac{8\pi}{3} n \frac{e}{2c} \vec{r} \times \vec{v} = \frac{8\pi}{3} n \vec{\mu} = \frac{8\pi}{3} \vec{M} . \quad (1-15.27)$$

The outcome,

$$\vec{B}_{\text{drive}} = \vec{B} - \frac{8\pi}{3} \vec{M} = \vec{H} + \frac{4\pi}{3} \vec{M}, \quad (1-15.28)$$

is analogous to the electrical result if  $\vec{H}$  is considered to be the counterpart of  $\vec{E}$  (in accord with tradition).

To the extent that we can ignore the distinction between  $\vec{H}$  and

$$\vec{B} = (1 + 4\pi\chi_m) \vec{H}, \quad (1-15.29)$$

owing to  $|\chi_m| \ll 1$ , we also need not concern ourselves with the difference between  $\vec{H}$  and

$$\vec{B}_{\text{drive}} = (1 + \frac{4\pi}{3}\chi_m) \vec{H}. \quad (1-15.30)$$

## 16. Ferromagnetism

The history of magnetism did not begin with the phenomena of paramagnetism and diamagnetism, which were first recognized by Faraday in 1845. The ancients were familiar with the remarkable properties of Magnesian stone, the iron oxide  $\text{Fe}_3\text{O}_4$ . The term ferromagnetism refers to the property of such substances, primarily members of the iron group, of exhibiting permanent magnetization. A simple model of this effect was introduced by Pierre Weiss (1865-1940), who effectively postulated that the driving magnetic field within ferromagnets is, not (l-15.28), but

$$\vec{B}_{\text{drive}} = \vec{H} + \lambda \vec{M}, \quad \lambda \gg 1. \quad (\text{l-16.1})$$

The computation of the average atomic moment is both simpler (compare problem ) and more in the quantum spirit if we suppose that the moment is only aligned parallel or antiparallel to the driving field, which points along the z-axis. That gives

$$\langle \mu_z \rangle_T = \frac{1}{z} \frac{1}{2} \left[ \mu e^{\frac{\mu B_{\text{drive}}}{kT}} - \mu e^{-\frac{\mu B_{\text{drive}}}{kT}} \right]$$

$$= \frac{\mu}{z} \sinh \frac{\mu B_{\text{drive}}}{kT}, \quad (\text{l-16.2})$$

where

$$z = \frac{1}{2} \left[ e^{\frac{\mu B_{\text{drive}}}{kT}} + e^{-\frac{\mu B_{\text{drive}}}{kT}} \right] = \cosh \frac{\mu B_{\text{drive}}}{kT}, \quad (\text{l-16.3})$$

so that

$$\langle \mu_z \rangle_T = \mu \tanh \left[ \frac{\mu}{kT} B_{\text{drive}} \right]. \quad (1-16.4)$$

The magnitude of the polarization vector is then

$$M = n\mu \tanh \left[ \frac{\mu}{kT} (H + \lambda M) \right]. \quad (1-16.5)$$

Another way of presenting this is

$$\frac{M}{n\mu} = \tanh \left[ \frac{\mu H}{kT} + \frac{T_c}{T} \frac{M}{n\mu} \right], \quad (1-16.6)$$

in which

$$kT_c = \lambda n\mu^2. \quad (1-16.7)$$

The possible existence of a magnetization in the absence of the field  $H$  is implied by the equation

$$\frac{M}{n\mu} = \tanh \left[ \frac{T_c}{T} \frac{M}{n\mu} \right]. \quad (1-16.8)$$

In Fig. there is plotted the left side of the equation, and also the right side with examples of the two situations  $T > T_c$  and  $T < T_c$ . For  $T > T_c$ , the curves for the respective sides intersect only at  $M = 0$ ; there is no magnetization. But, for  $T < T_c$ , there is also an intersection at a positive value of  $M/n\mu$  that is less than unity; a permanent magnetization can exist. The critical

temperature  $T_c$  above which no permanent magnetization is possible is called the Curie temperature. Its value is of the order  $T_c \sim 1000^\circ K$ . For greater temperatures, a magnetic field  $H$  must be present to produce magnetization. On considering weak fields ( $\frac{\mu H}{kT} \ll 1$ ) and values of  $T$  not too close to  $T_c$  (see below), the argument of the function on the right side of (1-16.6) is small, permitting the approximation

$$\frac{M}{n\mu} \approx \frac{\mu H}{kT} + \frac{T_c}{T} \frac{M}{n\mu} , \quad (1-16.9)$$

or ( $M = \chi_m H$ )

$$\chi_m = \frac{n\mu^2}{k} \frac{1}{T-T_c} = \frac{1}{d} \frac{T_c}{T-T_c} , \quad (1-16.10)$$

which is valid provided

$$\frac{T-T_c}{T_c} \gg \frac{\mu H}{kT_c} (\ll 1). \quad (1-16.11)$$

The system is now paramagnetic.

To estimate the parameter  $\lambda$ , we insert  $\mu \sim er(v/c)$  into (1-16.7) :

$$kT_c \sim \lambda \left(\frac{v}{c}\right)^2 \frac{e^2}{r} (nr^3). \quad (1-16.12)$$

In a solid at ordinary densities, the product of  $n$  ( $\sim 10^{23}/\text{cm}^3$ ) with  $r^3 (10^{-8} \text{ cm})^3$ , the fraction of the solid occupied by atoms, is of order  $\frac{1}{10}$ . And a tenth of the atomic energy magnitude  $e^2/r$  is not far from the value  $kT_c$ .

We conclude that

$$\lambda \sim \left(\frac{e}{v}\right)^2, \quad (1-16.13)$$

which is the inverse of the typical factor that relates magnetic energy to electric energy. The clear suggestion is that the underlying mechanism of ferromagnetism is not magnetic, but electrical, in origin. The quantum theory of ferromagnetism initiated by Werner Heisenberg (1901-1976) vindicates this conclusion.

## 17. Energy and Momentum in Macroscopic Electrodynamics

We now turn to the consideration of the distribution of energy and momentum of electromagnetic fields within material media, following closely the development of Sec. I-4 (however, here we assume no magnetic charge is present). We will base our discussion on the macroscopic form of Maxwell's equations, (I-8, I-12). Accordingly, the rate at which the field does work on the free charges is

$$\bar{J} \cdot \bar{E} = \bar{E} \cdot \left[ \frac{c}{4\pi} \bar{\nabla} \times \bar{H} - \frac{1}{4\pi} \frac{\partial}{\partial t} \bar{D} \right]. \quad (\text{I-17.1})$$

If we add to this the parallel equation, appropriate to the absence of magnetic charge,

$$0 = \bar{H} \cdot \left[ -\frac{c}{4\pi} \bar{\nabla} \times \bar{E} - \frac{1}{4\pi} \frac{\partial}{\partial t} \bar{B} \right], \quad (\text{I-17.2})$$

we obtain the suggestive form

$$\bar{J} \cdot \bar{E} = -\bar{\nabla} \cdot \left[ \frac{c}{4\pi} \bar{E} \times \bar{H} \right] - \frac{1}{4\pi} \left[ \bar{E} \cdot \frac{\partial}{\partial t} \bar{D} + \bar{H} \cdot \frac{\partial}{\partial t} \bar{B} \right]. \quad (\text{I-17.3})$$

Our aim is to write this result as a local energy conservation law. The divergence term immediately identifies the Poynting (energy flux) vector:

$$\bar{S} = \frac{c}{4\pi} \bar{E} \times \bar{H}. \quad (\text{I-17.4})$$

It has the same form as that of the microscopic flux, (I-4.5), except that here  $\bar{B}$  is replaced by  $\bar{H}$ . Somewhat less immediate is the identification of the last term in (I-17.3). Is it the time derivative of an energy density  $U$ ? If there does exist some quantity  $U$  such that

$$\frac{\partial}{\partial t} U = \frac{1}{4\pi} \left[ \bar{E} \cdot \frac{\partial}{\partial t} \bar{D} + \bar{H} \cdot \frac{\partial}{\partial t} \bar{B} \right], \quad (\text{I-17.5})$$

we would have a local statement of energy conservation,

$$\frac{\partial}{\partial t} U + \vec{\nabla} \cdot \vec{S} + \vec{J} \cdot \vec{E} = 0. \quad (\text{I-17.6})$$

Similarly, we consider the rate at which momentum is transferred to the free charges. The relevant pair of equations here is

$$\rho \vec{E} + \frac{1}{c} \vec{J} \times \vec{B} = \frac{1}{4\pi} (\vec{\nabla} \cdot \vec{D}) \vec{E} + \frac{1}{4\pi} \left[ \vec{\nabla} \times \vec{H} - \frac{1}{c} \frac{\partial}{\partial t} \vec{D} \right] \times \vec{B}, \quad (\text{I-17.7})$$

$$0 = \frac{1}{4\pi} (\vec{\nabla} \cdot \vec{B}) \vec{H} + \frac{1}{4\pi} \left[ \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial}{\partial t} \vec{B} \right] \times \vec{D}. \quad (\text{I-17.8})$$

They are added to yield

$$\begin{aligned} \rho \vec{E} + \frac{1}{c} \vec{J} \times \vec{B} &= - \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c} \vec{D} \times \vec{B} \right] - \frac{1}{4\pi} \left[ \vec{D} \times (\vec{\nabla} \times \vec{E}) - (\vec{\nabla} \cdot \vec{D}) \vec{E} \right] \\ &\quad - \frac{1}{4\pi} \left[ \vec{B} \times (\vec{\nabla} \times \vec{H}) - (\vec{\nabla} \cdot \vec{B}) \vec{H} \right]. \end{aligned} \quad (\text{I-17.9})$$

The time derivative term identifies the momentum density,

$$\vec{G} = \frac{1}{4\pi c} \vec{D} \times \vec{B}, \quad (\text{I-17.10})$$

which is akin to (I-17.13), but with  $\vec{D}$  replacing  $\vec{E}$ . Are the last two terms of

(I-17.9) the divergence of a tensor:

$$\vec{\nabla} \cdot \vec{T} \stackrel{?}{=} \frac{1}{4\pi} [\vec{D} \times (\vec{\nabla} \times \vec{E}) - (\vec{\nabla} \cdot \vec{D}) \vec{E} + \vec{B} \times (\vec{\nabla} \times \vec{H}) - (\vec{\nabla} \cdot \vec{B}) \vec{H}]. \quad (\text{I-17.11})$$

Then, a local law of momentum conservation holds,

$$\frac{\partial}{\partial t} \vec{G} + \vec{\nabla} \cdot \vec{T} + \rho \vec{E} + \frac{1}{c} \vec{J} \times \vec{B} = 0. \quad (\text{I-17.12})$$

Non-dispersive Medium

We cannot proceed further without specific assumptions about the properties of the material medium. The simplest hypothesis is that of a homogeneous, isotropic, non-dispersive medium,

$$\bar{D} = \epsilon \bar{E},$$

$$\bar{B} = \mu \bar{H}, \quad (\text{I-17.13})$$

where  $\epsilon$  and  $\mu$  are constants. This is not unrealistic for many substances, with appropriate restrictions on the frequency range. Now  $U$  and  $\tilde{T}$  do exist, as illustrated by

$$\frac{1}{4\pi} \bar{E} \frac{\partial}{\partial t} \epsilon \bar{E} = \frac{\partial}{\partial t} \left[ \frac{\epsilon}{8\pi} \bar{E}^2 \right], \quad (\text{I-17.14})$$

$$\frac{1}{4\pi} [\bar{D} \times (\bar{\nabla} \times \bar{E}) - (\bar{\nabla} \cdot \bar{D}) \bar{E}] = \bar{\nabla} \cdot \left[ \frac{\epsilon}{8\pi} \tilde{T} \bar{E}^2 - \frac{\epsilon}{4\pi} \bar{E} \bar{E} \right].$$

The explicit forms are

$$U = \frac{1}{8\pi} (\epsilon \bar{E}^2 + \mu \bar{H}^2), \quad (\text{I-17.15})$$

$$\tilde{T} = \tilde{T} U - \frac{1}{4\pi} (\epsilon \bar{E} \bar{E} + \mu \bar{H} \bar{H}),$$

and we recall that

$$\bar{S} = \frac{c}{4\pi} \bar{E} \times \bar{H}, \quad \bar{G} = \frac{1}{4\pi c} \bar{D} \times \bar{B} = \frac{\epsilon \mu}{c^2} \bar{S} = \frac{S'}{c'^2} \quad (\text{I-17.16})$$

It is interesting that these expressions, along with the Maxwell equations, can be transformed into the appearance of the vacuum field equations. That is done by redefining the charges and fields, together with the speed of light:

$$\bar{E}' = \epsilon^{\frac{1}{2}} \bar{E}, \quad B' = \mu^{\frac{1}{2}} H, \quad J' = e^{-\frac{1}{2}} J, \quad \rho' = e^{-\frac{1}{2}} \rho,$$

$$c' = (\epsilon\mu)^{-\frac{1}{2}} c.$$

(17.17)

This makes it clear that

$$c' = \frac{c}{n}, \quad n = (\epsilon\mu)^{\frac{1}{2}}, \quad (17.18)$$

is the speed of propagating electromagnetic waves in the medium; the quantity  $n$  is called the index of refraction for the medium. Our early results on electromagnetic fields moving in a definite direction can also be applied to propagation in this medium. The perpendicularity and equality of magnitude of  $\bar{E}'$  and  $\bar{B}'$  imply that

$$\bar{E} \cdot \bar{H} = 0, \quad \epsilon^{\frac{1}{2}} |\bar{E}| = \mu^{\frac{1}{2}} |\bar{H}|. \quad (17.19)$$

In the usual situation at low frequencies, where  $\mu=1$  and  $\epsilon>1$ , the speed  $c'$  is less than  $c$  by the factor  $n=\epsilon^{\frac{1}{2}}$  and the magnitude of the electric field is less than that of the magnetic field by the same factor.

#### Dispersive Medium

Suppose we continue to assume that  $\mu=1$ , but go to higher frequencies, where  $\epsilon(\omega)$  has the kind of behavior suggested by the simple model (13.22),

$$\epsilon(\omega) = 1 + \frac{4\pi n e^2}{m} \frac{1}{\omega_1^2 - \omega^2 - i\omega\gamma}. \quad (17.20)$$

When  $\omega$  is sufficiently below  $\omega_1$  that  $\omega_1^2 - \omega^2 \gg \gamma\omega$ ,  $\epsilon(\omega)$  is real and greater than unity. But, when  $\omega$  is above  $\omega_1$  to the extent that  $\omega^2 - \omega_1^2 \gg \gamma\omega$ ,  $\epsilon(\omega)$  is real and less than unity. Shall we conclude from  $c' = c/\epsilon^{\frac{1}{2}}$  that, in a medium where the speed of light depends on frequency—a dispersive medium—energy can flow at a speed greater than  $c$ ? No!

The identification of the energy density according to (I-17.5) involves a time integration, the building up of the field. But a field with a definite frequency has no such transient behavior; a range of frequencies is required. And, if the properties of the medium are frequency dependent, we have a new situation.

To begin again, we represent the time behavior of the electric field, for example, as a superposition of frequencies—a Fourier integral,

$$\bar{E}(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \bar{E}(\omega) = \int_{-\infty}^{\infty} d\omega e^{i\omega t} \bar{E}(\omega)^*. \quad (\text{I-17.21})$$

This explicit indication that  $\bar{E}(t)$  is a real function is also conveyed by

$$\bar{E}(\omega)^* = \bar{E}(-\omega). \quad (\text{I-17.22})$$

There is a similar representation for the field vector

$$\bar{D}(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \bar{D}(\omega), \quad (\text{I-17.23})$$

with

$$\bar{D}(\omega)^* = \bar{D}(-\omega). \quad (\text{I-17.24})$$

The frequency dependent dielectric constant definition

$$\bar{D}(\omega) = \epsilon(\omega) \bar{E}(\omega) \quad (\text{I-17.25})$$

then also obeys

$$\epsilon(\omega)^* = \epsilon(-\omega), \quad (\text{I-17.26})$$

as realized in the simple model (I-17.20).

? what is the  
limitation here

91

To simplify the following discussion, we retain the restriction  $\mu = 1$ ,

and also assume that the significant frequencies  $\omega$  are such that

$|\omega^2 - \omega_1^2| \gg \gamma\omega$ . Then  $\epsilon(\omega)$  is essentially real although, more generally, it would be the real part of  $\epsilon(\omega)$  that we consider. Let us pick out the part of (I-17.5) that refers to electric energy,

$$\frac{\partial}{\partial t} U_E(t) = \frac{1}{4\pi} \bar{E}(t) \cdot \frac{\partial}{\partial t} \bar{D}(t), \quad (\text{I-17.27})$$

and insert Fourier integrals for the two fields:

$$\begin{aligned} & \frac{1}{4\pi} \bar{E}(t) \cdot \frac{\partial}{\partial t} \bar{D}(t) \\ &= \frac{1}{4\pi} \left[ \int_{-\infty}^{\infty} d\omega' e^{i\omega' t} \bar{E}(-\omega') \right] \cdot \left[ \int_{-\infty}^{\infty} d\omega (-i\omega) e^{-i\omega t} \epsilon(\omega) \bar{E}(\omega) \right] \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega d\omega' e^{-i(\omega-\omega') t} [-i\omega \epsilon(\omega)] \bar{E}(\omega) \cdot \bar{E}(-\omega'). \end{aligned} \quad (\text{I-17.28})$$

An alternative way of presenting this relabels the integration variables according to the substitutions

$$\omega \rightarrow -\omega', \quad \omega' \rightarrow -\omega, \quad (\text{I-17.29})$$

which produce

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega d\omega' e^{-i(\omega-\omega') t} [i\omega' \epsilon(\omega')] \bar{E}(\omega) \cdot \bar{E}(-\omega'). \quad (\text{I-17.30})$$

Then, the average of the two equivalent forms gives

$$\frac{\partial}{\partial t} \frac{1}{8\pi} \int d\omega d\omega' e^{-i(\omega-\omega') t} \frac{\omega \epsilon(\omega) - \omega' \epsilon(\omega')}{\omega - \omega'} \bar{E}(\omega) \cdot \bar{E}(-\omega'), \quad (\text{I-17.31})$$

where we have also taken the step that will lead to the identification of the electric energy density through time integration. Notice at this point that setting  $\epsilon(\omega) = 1$  immediately reproduces the known vacuum energy density,  $(1/8\pi)(\bar{E}(t))^2$ .

Now suppose that the turning on of the field takes place so slowly that only a very small band of frequencies about a central frequency  $\bar{\omega}$  occurs (the precise relation between turning-on time and band width is not required here, but see Problem ). Now, with  $\omega \approx \omega' \approx \bar{\omega}$  or  $-\bar{\omega}$ , we have

$[\epsilon(-\omega) = \epsilon(\omega)]$

$$\frac{\omega\epsilon(\omega) - \omega'\epsilon(\omega')}{\omega - \omega'} - \frac{d}{d\omega} [\omega\epsilon(\omega)] \Big|_{\bar{\omega}} . \quad (I-17.32)$$

On the other hand, for  $\omega \approx -\omega'$ , the above quantity is simply  $\epsilon(\bar{\omega})$ . However, this contribution to the energy density oscillates in time with the frequency  $2\bar{\omega}$  and will not contribute significantly when averaged over one or two periods. In addition, the spatial dependence will also be rapidly oscillating, suppressing the contribution in spatial averages. We conclude that, effectively, in the sense of such averages,

$$U_E = \frac{1}{8\pi} \left[ \frac{d}{d\omega} (\omega\epsilon) \right] (\bar{E}(t))^2 \quad (I-17.33)$$

(now writing  $\omega$  in place of  $\bar{\omega}$ ). To this is added

$$U_H = \frac{1}{8\pi} (\bar{H}(t))^2. \quad (I-17.34)$$

To find the speed at which energy is propagated in an electromagnetic pulse, we proceed as in Sec. I-5. The local conservation equation in charge-free space,

$$\frac{\partial}{\partial t} U + \vec{\nabla} \cdot \vec{S} = 0, \quad (I-17.35)$$

is multiplied by  $\bar{r}$  and integrated over the entire pulse:

$$\frac{d}{dt} \int (d\bar{r}) \bar{r} U = \int (d\bar{r}) \bar{S} = \bar{v}_E \int (d\bar{r}) U. \quad (I-17.36)$$

The equality of magnitudes in the final relation is written out as

$$|\bar{v}_E| \int (d\bar{r}) \frac{1}{8\pi} \left\{ \frac{d}{d\omega} (\omega\epsilon) E^2 + H^2 \right\} = \frac{c}{4\pi} \left| \int (d\bar{r}) \bar{E} \times \bar{H} \right|. \quad (I-17.37)$$

Now, what we had to say about such pulses in Eq.(I-17.19) is still true for a nearly monochromatic pulse, with  $\epsilon = \epsilon(\omega)$ ,  $\mu = 1$ . It is only the interpretation of  $c'$  as the energy propagation speed in a dispersive medium that is being challenged. And so, we have

$$|\bar{E} \times \bar{H}| = |\bar{E}| |\bar{H}| = \epsilon^{\frac{1}{2}} E^2, \quad H^2 = \epsilon E^2, \quad (I-17.38)$$

and

$$|\bar{v}_E| = \frac{2\epsilon^{\frac{1}{2}}}{\frac{d}{d\omega} (\omega\epsilon) + \epsilon} c = \frac{1}{\frac{d}{d\omega} (\omega\epsilon^{\frac{1}{2}})} c. \quad (I-17.39)$$

[For the situation when  $\mu \neq 1$ , see Problem 4.1.]

In the absence of dispersion, we recover the speed  $c'$ , here appearing as  $c/\epsilon^{\frac{1}{2}}$ . Inserting the simple model

$$\epsilon(\omega) = 1 + \frac{4\pi n e^2}{m} \frac{1}{\omega_1^2 - \omega^2} \quad (I-17.40)$$

we find [using the first form of (I-17.39)] that

$$\frac{1}{c} |\bar{v}_E| = \frac{\epsilon^{\frac{1}{2}}}{1 + \frac{4\pi n e^2}{m} \frac{\omega_1^2}{(\omega_1^2 - \omega^2)^2}}. \quad (I-17.41)$$

In the circumstance  $\omega < \omega_1$ , where

$$\frac{\omega_1^2}{\omega^2 - \omega^2} > 1, \quad (I-17.42)$$

we get

$$\frac{1}{c} |\vec{v}_E| < \epsilon^{-\frac{1}{2}} < 1, \quad (I-17.43)$$

and for  $\omega > \omega_1$ , we now find

$$\frac{1}{c} |\vec{v}_E| < \epsilon^{\frac{1}{2}} < 1; \quad (I-17.44)$$

the speed of energy flow in a dispersive medium is less than c. (While this conclusion is not altered, the story is more complicated when  $|\omega - \omega_1| \sim \gamma$ . Treatment of this question by Sommerfeld and Brillouin can be found in L. Brillouin, "Wave Propagation and Group Velocity," Academic Press, New York, 1960.)

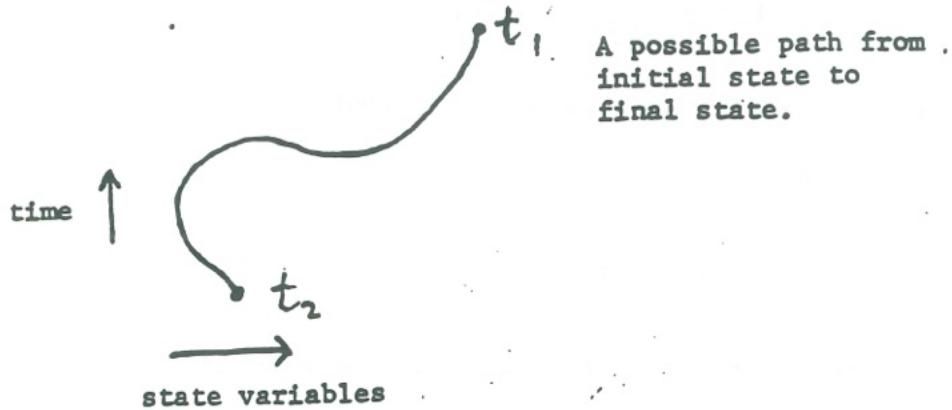
### 18. Review of Action Principles

The recognition that the electromagnetic field possesses mechanical properties naturally invites a complete description in mechanical terms and we shall supply it. But first it will be helpful to review the Lagrangian-Hamiltonian mechanical principles.

Action,  $W$ , is defined as the time integral of the Lagrangian,  $L$ , extended from an initial configuration or state at time  $t_2$  up to a final state at time  $t_1$ :

$$W_{12} = \int_{t_2}^{t_1} dt L. \quad (I-18.1)$$

The integral refers to any path, any line of time development, from the initial to the final state.



The actual time evolution of the system is selected by the principle of stationary action: In response to infinitesimal variations of the integration path, the action  $W_{12}$  is stationary—does not have a corresponding infinitesimal change—for variations about the correct path, provided the initial and final configurations are held fixed. This means that, if we allow infinitesimal changes at the initial and final times, including alterations of those times, the only contribution to  $\delta W_{12}$  then comes from the endpoint variations, or

$$\delta W_{12} = G_1 - G_2, \quad (I-18.2)$$

where  $G_1$  and  $G_2$  refer to the state of affairs at the respective terminal times,  $t_1$  and  $t_2$ . We now give three different realizations of the action principle, using a single particle for illustration.

#### Lagrangian Viewpoint

The non-relativistic motion of a particle with mass  $m$  in a potential  $V(\vec{r}, t)$  is described by the Lagrangian

96

$$L = \frac{1}{2} m \left( \frac{d\bar{r}}{dt} \right)^2 - v(\bar{r}, t). \quad (\text{I-18.3})$$

Two kinds of variations can be considered. First, a particular motion is altered infinitesimally,

$$\bar{r}(t) = \bar{r}(t) + \delta\bar{r}(t). \quad (\text{I-18.4})$$

Second, the final and initial time can be altered infinitesimally, by  $\delta t_1$  and  $\delta t_2$ , respectively. It is more convenient however to think of these time displacements as produced by a continuous variation of the time,  $\delta t(t)$ , so chosen that, at the endpoints,

$$\delta t(t_1) = \delta t_1, \quad \delta t(t_2) = \delta t_2. \quad (\text{I-18.5})$$

The corresponding change of the time differential in the integral is

$$dt = d(t + \delta t) = \left( 1 + \frac{d\delta t}{dt} \right) dt, \quad (\text{I-18.6})$$

and the related transformation of the time derivative becomes

$$\frac{d}{dt} = \left( 1 - \frac{d\delta t}{dt} \right) \frac{d}{dt}. \quad (\text{I-18.7})$$

Now, in carrying out the variation of the action

$$W_{12} = \int_2^1 \left[ \frac{1}{2} m \frac{(d\bar{r})^2}{dt} - dt v \right], \quad (\text{I-18.8})$$

the integration limits are left intact, the time displacement being produced through  $\delta t(t)$  subject to (I-18.5). We get

$$\begin{aligned} dt & \rightarrow t+dt \\ dt & = dt + \frac{d\delta t}{dt} dt \\ & = dt(1 + \frac{d\delta t}{dt}) \\ \frac{d}{dt} & = \cancel{\frac{d}{dt}} \quad (1 - \frac{d\delta t}{dt}) \frac{d}{dt} \end{aligned}$$

$$\delta W_{12} = \int_2^1 dt \left[ m \frac{d\bar{r}}{dt} \frac{d}{dt} \delta\bar{r} - \delta\bar{r} \cdot \bar{\nabla} V - \frac{d\delta t}{dt} \frac{1}{2} m \left( \frac{d\bar{r}}{dt} \right)^2 - \frac{d\delta t}{dt} V - \delta t \frac{\partial}{\partial t} V \right], \quad (1-18.9)$$

or, with the definition

$$E = \frac{1}{2} m \left( \frac{d\bar{r}}{dt} \right)^2 + V, \quad (1-18.10)$$

we have

$$\begin{aligned} \delta W_{12} &= \int_2^1 d \left[ m \frac{d\bar{r}}{dt} \cdot \delta\bar{r} - E \delta t \right] \\ &+ \int_2^1 dt \left[ -\delta\bar{r} \cdot \left( m \frac{d^2\bar{r}}{dt^2} + \bar{\nabla} V \right) + \delta t \left( \frac{dE}{dt} - \frac{\partial V}{\partial t} \right) \right]. \end{aligned} \quad (1-18.11)$$

of (1-18.11)

The first term on the right-hand side is the integral of a differential and therefore depends only on the endpoints. The principle of stationary action refers to the other term, which must vanish, for arbitrary  $\delta\bar{r}(t)$  and  $\delta t(t)$ , if the unvaried motion is the actual one. Accordingly, that actual motion is governed by

$$m \frac{d^2\bar{r}}{dt^2} = -\bar{\nabla} V, \quad (1-18.12)$$

Newton's equation of motion under the force derived from the potential  $V$ ,

and

$$\frac{dE}{dt} = \frac{\partial V}{\partial t}, \quad (1-18.13)$$

which, for static potentials,  $\partial V / \partial t = 0$ , is the equation of energy conservation,

$$\frac{dE}{dt} = 0. \quad (1-18.14)$$

What remains in (I-18.11) is a structure of the form (I-18.2), with

$$G = m \frac{d\bar{r}}{dt} \cdot \delta\bar{r} - E\delta t \quad (\text{I-18.15})$$

applying at each time boundary; more about this later.

### Hamiltonian Viewpoint

The Lagrangian for the same system is now written

$$L = \bar{p} \frac{d\bar{r}}{dt} - H(\bar{r}, \bar{p}, t), \quad (\text{I-18.16})$$

with

$$H = \frac{\bar{p}^2}{2m} + V(\bar{r}, t). \quad (\text{I-18.17})$$

The result of subjecting  $\bar{r}$ ,  $\bar{p}$ , and  $t$  to infinitesimal variations in

$$w_{12} = \int_2^1 [\bar{p} \cdot d\bar{r} - dtH]. \quad (\text{I-18.18})$$

is

$$\begin{aligned} \delta w_{12} = & \int_2^1 dt \left[ \bar{p} \frac{d}{dt} \delta\bar{r} - \delta\bar{r} \cdot \bar{V} + \bar{p} \frac{d\bar{r}}{dt} - \bar{p} \frac{\bar{p}}{m} \right. \\ & \left. - \frac{d\delta t}{dt} H - \delta t \frac{\partial H}{\partial t} \right], \end{aligned} \quad (\text{I-18.19})$$

or

$$\begin{aligned} \delta w_{12} = & \int_2^1 d \left[ \bar{p} \cdot \delta\bar{r} - \delta t H \right] + \int_2^1 dt \left[ -\delta\bar{r} \cdot \left( \frac{d\bar{p}}{dt} + \bar{V} \right) + \bar{p} \cdot \left( \frac{d\bar{r}}{dt} - \frac{1}{m} \bar{p} \right) \right. \\ & \left. + \delta t \left( \frac{dH}{dt} - \frac{\partial H}{\partial t} \right) \right]. \end{aligned} \quad (\text{I-18.20})$$

Now the stationary action principle gives the equations

$$\frac{d\bar{r}}{dt} = \frac{1}{m} \bar{p} = \frac{\partial H}{\partial p},$$

$$\frac{d\bar{p}}{dt} = -\nabla V = -\frac{\partial H}{\partial r},$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t},$$

(I-18.21)

and the endpoint contributions, at each time, are

$$G = \bar{p} \cdot \delta\bar{r} - H\delta t.$$

(I-18.22)

In contrast with the Lagrangian differential equations of motion, which involve second derivatives, these Hamiltonian equations contain only first derivatives; they are called first-order equations. Why do we think that the same physical system is being described? Because one of those Hamiltonian equations,

$$\bar{p} = m \frac{d\bar{r}}{dt},$$

(I-18.23)

can be used to convert the other equation of motion into

$$\frac{d}{dt} \left( m \frac{d\bar{r}}{dt} \right) = m \frac{d^2\bar{r}}{dt^2} = -\bar{\nabla}V,$$

(I-18.24)

which is the Lagrangian-Newtonian equation of motion. Furthermore, the insertion of (I-18.23) into H gives

$$\frac{1}{2m} \left( m \frac{d\bar{r}}{dt} \right)^2 + V = E,$$

(I-18.25)

and the third of the equations in (I-18.21) becomes identical with (I-18.13). We also note the equivalence of the two versions of G.

But probably the most direct way of seeing that the same physical system is involved comes by writing the Lagrangian of the Hamiltonian viewpoint as

$$L = \frac{m}{2} \left( \frac{d\bar{r}}{dt} \right)^2 - v - \frac{1}{2m} \left( \bar{p} - m \frac{d\bar{r}}{dt} \right)^2. \quad (I-18.26)$$

The result of varying  $\bar{p}$  in the stationary action principle is to produce

$$\bar{p} = m \frac{d\bar{r}}{dt}. \quad (I-18.27)$$

But, if we accept this as the definition of  $\bar{p}$ , the corresponding term in  $L$  disappears and we explicitly regain the Lagrangian description. We are justified in completely omitting the last term on the right side of (I-18.26), despite its dependence on the variables  $\bar{r}$  and  $t$ , because of its quadratic structure. The explicit contribution to  $\delta L$  is

$$- \frac{1}{m} \left( \bar{p} - m \frac{d\bar{r}}{dt} \right) \cdot \left( \delta \bar{p} - m \frac{d}{dt} \delta \bar{r} + m \frac{d\bar{r}}{dt} \frac{d\delta t}{dt} \right), \quad (I-18.28)$$

and the equation supplied by the stationary action principle for  $\bar{p}$  variations, (I-18.27), also guarantees that there is no contribution here to the results of  $\bar{r}$  and  $t$  variations.

### Third Viewpoint

To the variables of the Hamiltonian description, we add  $\bar{v}$ :

$$L = \bar{p} \frac{d\bar{r}}{dt} - H(\bar{r}, \bar{p}, \bar{v}, t),$$

$$H = \bar{p} \cdot \bar{v} - \frac{1}{2} mv^2 + v(\bar{r}, t). \quad (I-18.29)$$

The consequences of  $\bar{r}$ ,  $\bar{p}$ , and  $t$  variations emerge as before

$$\frac{d\bar{r}}{dt} = \frac{\partial H}{\partial \bar{p}} = \bar{v},$$

$$\frac{d\bar{p}}{dt} = - \frac{\partial H}{\partial \bar{r}} = -\bar{v}\bar{v},$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}, \quad (|-18.30)$$

and

$$G = \bar{p} \cdot \delta \bar{r} - H \delta t. \quad (|-18.31)$$

As for the additional variable  $\bar{v}$ , its time derivative does not occur, and the stationary action principle gives an equation that refers to a single time—an equation of constraint, rather than an equation of motion:

$$0 = \frac{\partial H}{\partial \bar{v}} = \bar{p} - m\bar{v}. \quad (|-18.32)$$

With this description we have the option of returning to either the Hamiltonian or the Lagrangian viewpoint, by adopting suitable equations as definitions. On writing

$$H(\bar{r}, \bar{p}, \bar{v}, t) = \frac{p^2}{2m} + V(\bar{r}, t) - \frac{1}{2m} (\bar{p} - m\bar{v})^2, \quad (|-18.33)$$

and adopting

$$\bar{v} = \frac{1}{m} \bar{p} \quad (|-18.34)$$

as the definition of  $\bar{v}$ , we recover the Hamiltonian description. Alternatively, we can present the Lagrangian as

$$L = \frac{m}{2} \left( \frac{d\vec{r}}{dt} \right)^2 - V + (\vec{p} - m\vec{v}) \cdot \left( \frac{d\vec{r}}{dt} - \vec{v} \right) - \frac{m}{2} \left( \frac{d\vec{r}}{dt} - \vec{v} \right)^2 . \quad (1-18.35)$$

Then after adopting the following as definitions,

$$\vec{v} = \frac{d\vec{r}}{dt}, \quad \vec{p} = m\vec{v} \quad , \quad (1-18.36)$$

the resultant form of  $L$  is that of the Lagrangian viewpoint. It might seem that only the definition  $\vec{v} = d\vec{r}/dt$ , inserted in (1-18.35), suffices to regain the Lagrangian description. But then the next to last term ← in (1-18.35) would give the following additional contribution to  $\delta L$ , associated with the variation  $\delta\vec{r}$ :

$$(\vec{p} - m\vec{v}) \cdot \frac{d}{dt} \delta\vec{r} . \quad (1-18.37)$$

We shall find that it is the third viewpoint that is best adapted to present the mechanical formulation of the electromagnetic field in interaction with charged particles.

## 19. Invariance and Conservation Laws. Virial Theorem.

There is more content in the principle of stationary action than equations of motion. Suppose one considers a variation such that

$$\delta W_{12} = 0, \quad (\text{I-19.1})$$

independently of the choice of initial and final times. We say that the action, which is left unchanged, is invariant under this alteration of path. Then the stationary action principle (I-18.2) asserts that

$$\delta W_{12} = G_1 - G_2 = 0, \quad (\text{I-19.2})$$

or, there is a quantity  $G(t)$  that has the same value for any choice of time  $t$ ; it is conserved in time. A differential statement of that is

$$\frac{d}{dt} G(t) = 0. \quad (\text{I-19.3})$$

Invariance implies conservation. Here are some examples. Suppose the Hamiltonian of (I-18.16) does not depend explicitly on time, or

$$W_{12} = \int_2^1 [\bar{p} \cdot d\bar{r} - H(\bar{r}, \bar{p}) dt]. \quad (\text{I-19.4})$$

Then the variation

$$\delta t = \text{constant} \quad (\text{I-19.5})$$

will give  $\delta W_{12} = 0$  (see Eq. (I-18.19), with  $\delta \bar{r} = 0$ ,  $\delta \bar{p} = 0$ ,  $d\delta t/dt = 0$ ,  $\partial H/\partial t = 0$ ).

The conclusion is that  $G$  (Eq. (I-18.22)), which here is just

$$G_t = -H\delta t, \quad (\text{I-19.6})$$

is a conserved quantity, or that

$$\frac{dH}{dt} = 0. \quad (\text{I-19.7})$$

This inference, that the Hamiltonian—the energy—is conserved, if there is no explicit time dependence in  $H$ , is already present in the third equation of (I-18.21), But now a more general principle is at work.

Next, consider an infinitesimal, rigid rotation, one that maintains the lengths and scalar products of all vectors. Written explicitly for the position vector  $\vec{r}$ , it is

$$\delta\vec{r} = \delta\vec{\omega} \times \vec{r} \quad (\text{I-19.8})$$

where the constant vector  $\delta\vec{\omega}$  gives the direction and magnitude of the rotation.

(See Fig. )

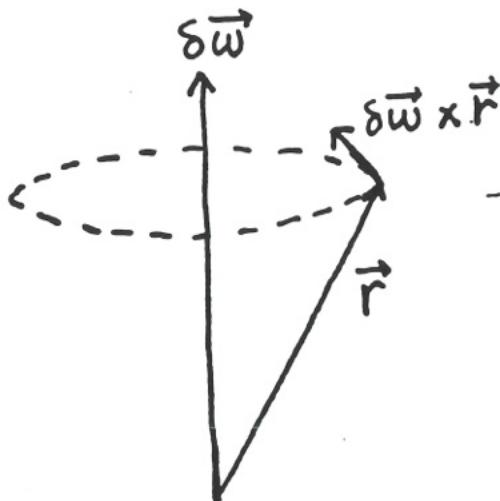


Fig.

$\delta\vec{\omega} \times \vec{r}$  is perpendicular to  $\delta\vec{\omega}$  and  $\vec{r}$ , and represents a rotation of  $\vec{r}$  about the  $\delta\vec{\omega}$  axis.

Now specialize (I-18.17) to

$$H = \frac{p^2}{2m} + V(r), \quad (\text{I-19.9})$$

where  $r = |\vec{r}|$ , a rotationally invariant structure. Then

$$W_{12} = \int_2^1 [\vec{p} \cdot d\vec{r} - H dt] \quad (\text{I-19.10})$$

is also invariant under the rigid rotation, implying the conservation of

$$G_{\frac{1}{\delta\omega}} = \bar{p} \cdot \delta\bar{r} = \delta\bar{\omega} \cdot \bar{r} \times \bar{p}. \quad (1-19.11)$$

This is the conservation of angular momentum:

$$\bar{L} = \bar{r} \times \bar{p}, \quad \frac{d}{dt} \bar{L} = 0. \quad (1-19.12)$$

Conservation of linear momentum appears analogously when there is invariance under a rigid translation. For a single particle, the second of Eqs. (1-18.21) tells us immediately that  $\bar{p}$  is conserved if  $v$  is a constant, say zero. Then, indeed, the action

$$W_{12} = \int_2^1 [\bar{p} \cdot d\bar{r} - \frac{p^2}{2m} dt] \quad (1-19.13)$$

is invariant under the displacement

$$\vec{\delta r} = \vec{\delta e} = \text{constant}, \quad (1-19.14)$$

and

$$G_{\frac{1}{\delta e}} = \bar{p} \cdot \delta\bar{e} \quad (1-19.15)$$

is conserved. But the general principle acts just as easily for, say, a system of two particles, a and b, with Hamiltonian

$$H = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} + v(\bar{r}_a - \bar{r}_b). \quad (1-19.16)$$

This Hamiltonian and the associated action

$$W_{12} = \int_2^1 [\bar{p}_a \cdot d\bar{r}_a + \bar{p}_b \cdot d\bar{r}_b - Hdt] \quad (1-19.17)$$

are invariant under the rigid translation

$$\delta\bar{r}_a = \delta\bar{r}_b = \delta\bar{\epsilon}, \quad (1-19.18)$$

with the implication that

$$G_{\delta\bar{\epsilon}} = \bar{p}_a \cdot \delta\bar{r}_a + \bar{p}_b \cdot \delta\bar{r}_b = (\bar{p}_a + \bar{p}_b) \cdot \delta\bar{\epsilon} \quad (1-19.19)$$

is conserved. This is conservation of the total linear momentum

$$\bar{P} = \bar{p}_a + \bar{p}_b, \quad \frac{d}{dt} \bar{P} = 0. \quad (1-19.20)$$

Something that is a bit more general appears when we consider a rigid translation that grows linearly in time:

$$\delta\bar{r}_a = \delta\bar{v}t, \quad \delta\bar{r}_b = \delta\bar{v}t, \quad (1-19.21)$$

using the example of two particles. This gives each particle the common additional velocity  $\delta\bar{v}$ , and therefore must also change their momenta,

$$\delta\bar{p}_a = m_a \delta\bar{v}, \quad \delta\bar{p}_b = m_b \delta\bar{v}. \quad (1-19.22)$$

The response of the action to this variation is

$$\begin{aligned} \delta W_{12} &= \int_2^1 [(\bar{p}_a + \bar{p}_b) \cdot \delta\bar{v}dt + \delta\bar{v} \cdot (m_a d\bar{r}_a + m_b d\bar{r}_b) - (\bar{p}_a + \bar{p}_b) \cdot \delta\bar{v}dt] \\ &= \int_2^1 d[(m_a \bar{r}_a + m_b \bar{r}_b) \cdot \delta\bar{v}]. \end{aligned} \quad (1-19.23)$$

The action is not invariant; its variation has end point contributions. But there is still a conservation law, not of  $G = \bar{P} \cdot \delta\bar{v}$ , but of

$$\bar{N} \cdot \delta\bar{v},$$

$$\bar{N} = \bar{P}t - (m_a \bar{r}_a + m_b \bar{r}_b). \quad (1-19.24)$$

Written in terms of the center of mass position vector

$$\bar{R} = \frac{m_a \bar{r}_a + m_b \bar{r}_b}{M}, \quad M = m_a + m_b, \quad (I-19.25)$$

the statement of conservation of

$$\bar{N} = \bar{P}t - M\bar{R}, \quad (I-19.26)$$

namely

$$0 = \frac{d\bar{N}}{dt} = \bar{P} - M \frac{d\bar{R}}{dt}, \quad (I-19.27)$$

is the familiar fact that the center of mass of an isolated system moves at the constant velocity given by the ratio of the total momentum to the total mass of that system.

The action principle also supplies useful non-conservation laws. Consider, for constant  $\delta\lambda$ ,

$$\delta\bar{r} = \delta\lambda\bar{r}, \quad \delta\bar{p} = -\delta\lambda\bar{p}, \quad (I-19.28)$$

which leaves  $\bar{p} \cdot d\bar{r}$  invariant,

$$\delta(\bar{p} \cdot d\bar{r}) = (-\delta\lambda\bar{p}) \cdot d\bar{r} + \bar{p} \cdot (\delta\lambda d\bar{r}) = 0. \quad (I-19.29)$$

But the response of the Hamiltonian

$$H = T(p) + V(r), \quad (I-19.30)$$

$$T(p) = \frac{p^2}{2m},$$

is given by the non-invariant form

$$\delta H = \delta\lambda(-2T + \bar{r} \cdot \bar{\nabla}V). \quad (I-19.31)$$

Therefore we have, for an arbitrary time interval,

$$\delta W_{12} = \frac{1}{2} \int dt [\delta \lambda (2T - \bar{r} \cdot \bar{\nabla} V)] = G_1 - G_2 = \frac{1}{2} \int dt \frac{d}{dt} (\bar{p} \cdot \delta \lambda \bar{r}) \quad (I-19.32)$$

or, the theorem

$$\frac{d}{dt} \bar{r} \cdot \bar{p} - 2T - \bar{r} \cdot (-\bar{\nabla} V) = 0. \quad (I-19.33)$$

This is an example of the mechanical virial theorem referred to in Sec. I-4.

For the particular situation of the Coulomb potential between charges,

$V = \text{const.}/r$ , where

$$\bar{r} \cdot \bar{\nabla} V = r \frac{d}{dr} V = -\bar{V}, \quad (I-19.34)$$

<sup>i</sup>  
the viral theorem asserts that  
 $\bar{r} \cdot \bar{p}$

$$\frac{d}{dt} (\bar{r} \cdot \bar{p}) = 2T + V. \quad (I-19.35)$$

We apply this to a bound system produced by a force of attraction. On taking the time average of (I-19.35) the time derivative term disappears. That is because, over an arbitrarily long time interval  $\tau = t_1 - t_2$ , the value of  $\bar{r} \cdot \bar{p}(t_1)$  can differ by only a finite amount from  $\bar{r} \cdot \bar{p}(t_2)$ , and

$$\overline{\frac{d}{dt}(\bar{r} \cdot \bar{p})} = \frac{1}{\tau} \int_{t_2}^{t_1} dt \frac{d}{dt} \bar{r} \cdot \bar{p} = \frac{\bar{r} \cdot \bar{p}(t_1) - \bar{r} \cdot \bar{p}(t_2)}{\tau} = 0 \quad (I-19.36)$$

as  $\tau \rightarrow \infty$ . The conclusion,

$$2\bar{T} = -\bar{V}, \quad (I-19.37)$$

has been used qualitatively in Sec. I-10.

Here is one more illustration of a non-conservation law:

$$\delta \bar{r} = \delta \lambda \frac{\bar{r}}{r},$$

$$\delta \bar{p} = -\delta \lambda \left( \frac{\bar{p}}{r} - \frac{\bar{r} \bar{p} \cdot \bar{r}}{r^3} \right) = \delta \lambda \frac{\bar{r} \times (\bar{r} \times \bar{p})}{r^3}. \quad (I-19.38)$$

Again  $\vec{p} \cdot d\vec{r}$  is invariant

$$\delta(\vec{p} \cdot d\vec{r}) = -\delta\lambda\left(\frac{\vec{p}}{r} - \frac{\vec{r}\vec{p} \cdot \vec{r}}{r^3}\right) \cdot d\vec{r} + \vec{p} \cdot \left(\delta\lambda \frac{d\vec{r}}{r} - \delta\lambda \vec{r} \frac{\vec{r} \cdot d\vec{r}}{r^3}\right) = 0 \quad (I-19.39)$$

and the change of the Hamiltonian (I-19.30) is now

$$\delta H = \delta\lambda \left[ -\frac{\vec{L}^2}{mr^3} + \frac{\vec{r} \cdot \vec{\nabla} V}{r} \right]. \quad (I-19.40)$$

The resulting theorem (for  $V = V(r)$ ) is

$$\frac{d}{dt} \left( \frac{\vec{r} \cdot \vec{p}}{r} \right) = \frac{\vec{L}^2}{mr^3} - \frac{dV}{dr} \quad (I-19.41)$$

which, applied to the Coulomb potential, gives the bound state time average relation

$$\frac{L^2}{m} \left( \overline{\frac{1}{r^3}} \right) = - \left( \overline{\frac{V}{r}} \right). \quad (I-19.42)$$

## 20. Potentials

The equation of motion for a charged particle acted on by electric and magnetic fields, the Lorentz law of force,

$$\frac{d}{dt} m\vec{v} = e(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}), \quad (I-20.1)$$

is not obviously in the Hamiltonian form

$$\frac{d}{dt} \vec{p} = - \frac{\partial H}{\partial \vec{r}}. \quad (I-20.2)$$

Indeed, we have already noted in Sec. I-14 where attention focused on a constant magnetic field,

$$\vec{B} = \vec{\nabla} \times (\frac{1}{c} \vec{B} \times \vec{r}), \quad (I-20.3)$$

that momentum  $\vec{p}$  is not the same as  $m\vec{v}$  [Eq.(I-14.22)]:

$$\vec{p} = m\vec{v} + \frac{e}{c} \frac{1}{c} \vec{B} \times \vec{r}. \quad (I-20.4)$$

We must now go into this matter generally, for arbitrary fields, if we are to have the proper tools for a full mechanical description.

We see in (I-20.3) an example of the fact that the field equation

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (I-20.5)$$

is automatically satisfied if  $\vec{B}$  is derived as the curl of another vector,

$$\vec{B} = \vec{\nabla} \times \vec{A}. \quad (I-20.6)$$

The vector  $\vec{A}(\vec{r}, t)$  is called the vector potential. And, as in the earlier discussion, Maxwell the field equation

$$\vec{\nabla} \times \vec{E} = - \frac{1}{c} \frac{\partial}{\partial t} \vec{B} = - \frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A}, \quad (I-20.7)$$

presented as

$$\vec{\nabla} \times (\vec{E} + \frac{1}{c} \frac{\partial}{\partial t} \vec{A}) = 0, \quad (I-20.8)$$

11

shows the existence of an induced electric field, beyond the electrostatic type deduced from a scalar potential,  $\phi(\vec{r}, t)$ ,

$$\vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \phi. \quad (I-20.9)$$

We return to the equation of motion, now written as

$$\frac{d}{dt} m\vec{v} = -e \vec{\nabla} \phi - \frac{e}{c} \left[ \frac{\partial}{\partial t} \vec{A} - \vec{v} \times (\vec{\nabla} \times \vec{A}) \right]. \quad (I-20.10)$$

The identity

$$\vec{v} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{v} \cdot \vec{A}) - \vec{v} \cdot \vec{\nabla} \vec{A}, \quad (I-20.11)$$

and the concept of total time derivative for the function  $\vec{A}(\vec{r}(t), t)$ ,

$$\frac{d}{dt} \vec{A}(\vec{r}(t), t) = \frac{\partial}{\partial t} \vec{A}(\vec{r}(t), t) + \vec{v} \cdot \vec{\nabla} \vec{A}(\vec{r}(t), t), \quad (I-20.12)$$

with

$$\vec{v} = \frac{d\vec{r}(t)}{dt}, \quad (I-20.13)$$

then recasts (I-20.10) as

$$\frac{d}{dt} \vec{p} = -\vec{\nabla} [e(\phi - \frac{1}{c} \vec{v} \cdot \vec{A})] = -\vec{\nabla} V \quad (I-20.14)$$

Here

$$\vec{p} = m\vec{v} + \frac{e}{c} \vec{A}, \quad (I-20.15)$$

C is the generalization of (I-20.4) to an arbitrary field.

It was stated in our review of mechanical action principles that a view-point employing variables  $\vec{r}$ ,  $\vec{p}$ , and  $\vec{v}$  was particularly convenient for describing electromagnetic forces on charged particles. With the explicit, and linear, appearance of  $\vec{v}$  in what plays the role of the potential function in (I-20.14), we begin to see the basis for that remark. Indeed, we have only to consult Eq. (I-18.29) to find the appropriate Lagrangian:

$$L = \vec{p} \cdot \left( \frac{d\vec{r}}{dt} - \vec{v} \right) + \frac{1}{2} m\vec{v}^2 - e\phi + \frac{e}{c} \vec{v} \cdot \vec{A}. \quad (I-20.16)$$

To recapitulate, the equations resulting from variations of  $\bar{p}$ ,  $\bar{r}$ , and  $\bar{v}$  are, respectively:

$$\frac{d\bar{r}}{dt} = \bar{v},$$

$$\frac{d}{dt} \bar{p} = -\bar{\nabla} [e(\varphi - \frac{1}{c} \bar{v} \cdot \bar{A})],$$

$$\bar{p} = m\bar{v} + \frac{e}{c} \bar{A}. \quad (I-20.17)$$

We can now move to either the Lagrangian or the Hamiltonian formulation. For the first, we simply adopt  $\bar{v} = d\bar{r}/dt$  as a definition (but see the discussion in Sec. I-18) and get

$$L = \frac{1}{2}m\left(\frac{d\bar{r}}{dt}\right)^2 - e\varphi + \frac{e}{c} \frac{d\bar{r}}{dt} \cdot \bar{A}. \quad (I-20.18)$$

Alternatively, we use the third equation of (I-20.17) to define

$$\bar{v} = \frac{1}{m}(\bar{p} - \frac{e}{c} \bar{A}), \quad (I-20.19)$$

and find

$$L = \bar{p} \cdot \frac{d\bar{r}}{dt} - H,$$

$$H = \frac{1}{2m}(\bar{p} - \frac{e}{c} \bar{A})^2 + e\varphi. \quad (I-20.20)$$

Here we make contact with the energy considerations of Sec. I-15. For a homogeneous magnetic field, the terms in  $H$  that are linear and quadratic in  $\bar{A}$  (and therefore in  $\bar{B}$ , respectively, are the paramagnetic and diamagnetic energy terms of that section.

The obvious generalization of (I-20.16) to a number of charged particles, of possibly different types, is the Lagrangian (the time variable is not made explicit)

$$L_{\text{charges}} = \sum_a \left[ \bar{p}_a \cdot \left( \frac{d\bar{r}_a}{dt} - \bar{v}_a \right) + \frac{1}{2} m_a v_a^2 - e_a \phi(\bar{r}_a) + \frac{e_a}{c} \bar{v}_a \cdot \bar{A}(\bar{r}_a) \right]. \quad (J-20.21)$$

## 21. The Action of the Field

The electromagnetic field is a mechanical system. It contributes its variables to the action, to the Lagrangian of the whole system of charges and fields. In contrast with the <sup>point</sup> charges, the field is distributed in space. Its Lagrangian should therefore be, not a summation over discrete points, but an integration over all spatial volume elements,

$$L_{\text{field}} = \int (d\vec{r}) \mathcal{L}_{\text{field}} ; \quad (I-21.1)$$

this introduces the Lagrange function, or Lagrangian density,  $\mathcal{L}$ . The action principle associated with the Lagrangian (I-21.1) must produce the <sup>mechanical</sup> field equations. What are they? The fundamental importance of potentials in describing charged particle motion indicates some of these, namely [Eqs.(I-20.9) and (I-20.6)]

$$\frac{1}{c} \frac{\partial}{\partial t} \vec{A} = -\vec{E} - \vec{\nabla} \phi,$$

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad (I-21.2)$$

which take the place of

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B},$$

$$\vec{\nabla} \cdot \vec{B} = 0. \quad (I-21.3.)$$

The other set of equations, stated here for the pure field situation, the absence of charged particles, is

$$\frac{1}{c} \frac{\partial}{\partial t} \vec{E} = \vec{\nabla} \times \vec{B},$$

$$\vec{\nabla} \cdot \vec{E} = 0. \quad (I-21.4)$$

We recognize that  $\vec{A}(\vec{r}, t)$ ,  $\vec{E}(\vec{r}, t)$ , in analogy with  $\vec{r}_a(t)$ ,  $\vec{p}_a(t)$ , obey equations of motion while  $\phi(\vec{r}, t)$ ,  $\vec{B}(\vec{r}, t)$ , as analogues of  $\vec{v}_a(t)$ , do not. There are enough clues here to give the structure of  $L$ , apart from an overall factor. We anticipate that factor in writing

$$L_{\text{field}} = \frac{1}{4\pi} \left[ \vec{E} \cdot \left( -\frac{1}{c} \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \phi \right) - \vec{B} \cdot \vec{\nabla} \times \vec{A} + \frac{1}{2} (B^2 - E^2) \right]. \quad (I-21.5)$$

To assist in verifying this, we record various contributions to the variation of  $L_{\text{field}}$ :

$$\delta \vec{E}: \quad \delta L_{\text{field}} = \frac{1}{4\pi} \int (d\vec{r}) \delta \vec{E} \cdot \left( -\frac{1}{c} \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \phi - \vec{E} \right); \quad (I-21.6)$$

$$\delta \vec{A}: \quad \delta L_{\text{field}} = - \frac{d}{dt} \left[ \frac{1}{4\pi c} \int (d\vec{r}) \vec{E} \cdot \delta \vec{A} \right] + \frac{1}{4\pi} \int (d\vec{r}) \delta \vec{A} \cdot \left[ \frac{1}{c} \frac{\partial}{\partial t} \vec{E} - \vec{\nabla} \times \vec{B} \right], \quad (I-21.7)$$

which uses the identity

$$\vec{B} \cdot (\vec{\nabla} \times \delta \vec{A}) = \vec{\nabla} \cdot (\delta \vec{A} \times \vec{B}) + \delta \vec{A} \cdot \vec{\nabla} \times \vec{B} \quad (I-21.8)$$

and ignores contributions from infinitely remote surfaces;

$$\delta \vec{B}: \quad \delta L_{\text{field}} = \frac{1}{4\pi} \int (d\vec{r}) \delta \vec{B} \cdot [\vec{B} - \vec{\nabla} \times \vec{A}]; \quad (I-21.9)$$

$$\delta \phi: \quad \delta L_{\text{field}} = \frac{1}{4\pi} \int (d\vec{r}) \delta \phi \vec{\nabla} \cdot \vec{E}, \quad (I-21.10)$$

where again a spatial reorganization is involved,

$$\vec{E} \cdot \vec{\nabla} \delta \phi = \vec{\nabla} \cdot (\vec{E} \delta \phi) - \delta \phi \vec{\nabla} \cdot \vec{E}. \quad (I-21.11)$$

It is now a matter of inspection to conclude that the principle of stationary action, applied to variations of  $\vec{E}$  and  $\vec{B}$ , does produce the field equations (I-21.2), while the stationary requirement on variations of  $\vec{A}$  and  $\phi$  yields the pair (I-21.4). The additional appearance of the time derivative term

116

in the  $\delta\vec{A}$  variation supplies the terminal contribution:

$$\delta W_{12\text{field}} = \delta \int_2^t dt L_{\text{field}} = G_{1\text{field}} - G_{2\text{field}}, \quad (I-21.12)$$

$$G_{\text{field}} = -\frac{1}{4\pi c} \int (d\vec{r}) \vec{E} \cdot \delta\vec{A}, \quad (I-21.13)$$

which refers only to field variations (the analogue of  $\sum_a \vec{p}_a \cdot \delta\vec{r}_a$ ).

That we have correctly chosen the overall factor can now be verified. Let us present  $L_{\text{field}}$  in the Hamiltonian formulation by eliminating the variables, analogous to the  $\vec{v}_a$ , that do not obey equations of motion. For that we simply adopt the definition

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad (I-21.14)$$

and accept the restriction

$$\vec{\nabla} \cdot \vec{E} = 0. \quad (I-21.15)$$

Then the transference of the gradient on  $\phi$  gives

$$L_{\text{field}} = \int (d\vec{r}) \left( -\frac{1}{4\pi c} \vec{E} \cdot \frac{\partial}{\partial t} \vec{A} - H_{\text{field}} \right), \quad (I-21.16)$$

where

$$H_{\text{field}} = \int (d\vec{r}) \frac{1}{8\pi} (E^2 + B^2) \quad (I-21.17)$$

will be recognized as the correct total field energy [Eq. (I-4.4)].