LECTURE NOTES

Selected Topics on Asymptotic Methods

by

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Mathematical methods dealing with asymptotic expansions are numerous and varied and have received the attention of many writers during the last 60 years. Many such methods, however, are highly specialized and have been devised to solve specific problems. For this reason we propose to concentrate in these lectures on the methods of the widest applicability and greatest usefulness, namely: (1) the saddle-point method of integration; (2) the method of stationary phase, and (3) Langer's method, the WKB approximation, and the question of turning points. All of which should prove of value to researchers in space physics and in theoretical plasma physics.

1. SADDLE-POINT METHOD OF INTEGRATION

The saddle-point method of integration, frequently referred to as the method of steepest descents, was first applied by Debye (1909) to obtain integral representations of Bessel functions of large order from which ensue readily useful asymptotic expansions. Historically, however, the method of steepest descents is to be traced to a posthumous paper by Riemann, Werke, 405; see Watson (1944), footnote, p. 235.

Since then the method has had numerous applications and its theory has been described by many authors among which we might cite Brillouin (1916), Courant and Hilbert (1931), Ott (1943), Watson (1922, 1944), Jeffreys and Jeffreys (1950), van der Waerden (1951), Morse and Feschbach (1953), Erdélyi (1956), Brekhovskikh (1960), Jeffreys (1962), Copson (1965), and Baños (1966). However, all of these authors, with the single exception of Baños (1966), inject ab initio the presence of a large parameter in the exponential behaviour of the integrand. We show below that this notion is not at all necessary, and that the correct interpretation of Watson's lemma (Section 1.5) indicates that what has to be large is the radius of convergence of the power series expansion of the factor in the integrand which multiplies the principal exponential behaviour. The two statements are essentially equivalent, but the latter interpretation, as we shall see, fixes unambiguously the size of the large parameter and its precise role in the estimate of the error in the asymptotic expansion.

In addition to the correct and more enlightened interpretation of Watson's lemma, we present here a major improvement on the saddle-point method of integration which can be applied when there exists a first order pole in the close vicinity of the saddle-point. We describe in Section 1.10 a general method for the subtraction of the pole from the integrand and the evaluation of the contribution from the pole in closed form, in terms of the error function of complex argument or else in terms of the plasma dispersion function. We conclude this Chapter by discussing the multiple saddle-point method of integration as well as the method of stationary phase. We show in what respect the latter is completely equivalent to the method of steepest descents.

1.1 BASIC INTEGRAL FORM

In applying the method of steepest descents to the asymptotic evaluation of an infinite integral*, we find that, through a suitable conformal transformation of the variable of integration, it is always possible to reduce our primitive integral to the basic form.

$$I = \int_C F(w)e^{\phi(w)}dw, \qquad (1.1)$$

in which w is the (complex) variable of integration and

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* The case of definite integrals between finite limits will be discussed in due course.

in which $w = 0$ is a saddle-point defined by the property

$$\phi'(w) = 0 \text{ when } w = 0; \quad \text{i.e., } w = 0 \text{ is S.P.,} \tag{1.2}$$

and in which C denotes the path of steepest descents in the w plane which originates at infinity, passes through the S.P. $w = 0$, and terminates at infinity, altogether satisfying the requirement

$$\operatorname{Im}\{\phi(w)\} = \operatorname{Im}\{\phi(0)\} \text{ on } C, \tag{1.3}$$

in accordance with the precepts explained, for example, in Watson (1944, p. 235). Henceforth the path of integration C will be referred to as P.S.D., i.e., path of steepest descents.

The function $F(w)$ in the basic integral form (1.1) frequently is identically unity, but it need not be. In fact, it must be emphasized at this juncture that the exponent $\phi(w)$ need not incorporate the totality of the exponential behaviour of the integrand in (1.1), a fact which some authors fail to recognize. Consequently, the functions $F(w)$ in our basic integral (1.1) may well exhibit exponential behaviour (essential singularities at infinity), but only so long as the convergence of the integral is guaranteed beforehand.

1.2 TRANSFORMATION TO THE x PLANE

First we rewrite our basic integral (1.1) in the equivalent form

$$I = \int_C F(w) e^{\phi(w)} dw = e^{\phi(0)} \int_C F(w) e^{\overbrace{\phi(w) - \phi(0)}^{-x^2}} dw \tag{1.4}$$

by extracting from the integrand the constant factor $e^{\phi(0)}$. Next, we observe that, by virtue of (1.3), the new exponent in the second form of (1.4), $\phi(w) - \phi(0)$, is zero when $w = 0$, is real and negative on the entire path C and tends to $-\infty$ at the extremities of the P.S.D. Hence, it is suggested that we introduce the new variable of integration x defined by the conformal transformation

$$x^2 = \phi(0) - \phi(w) \tag{1.5}$$

which maps the P.S.D. C in the w plane onto the axis of reals in the x plane. Thus, we rewrite (1.4) in the form

$$I = e^{\phi(0)}\int_{-\infty}^{\infty} F(w(x))\,\frac{dw(x)}{dx}\,e^{-x^2}\,dx \tag{1.6}$$

in which $w = w(x)$ in accordance with (1.5). Finally, by bisecting the path of integration in (1.6), we obtain the compact form

$$I = e^{\phi(0)}\int_{0}^{\infty} \Phi(x)e^{-x^2}\,dx, \tag{1.7}$$

wherein

$$\Phi(x) = F(w(x))\,\frac{dw(x)}{dx} + F(w(-x))\frac{dw(-x)}{dx} \tag{1.8}$$

is an even function of x, $\Phi(-x) = \Phi(x)$, by construction.

Thus, the evaluation of the basic form (1.1) has now been reduced to the evaluation of the integral (1.7), which, as shown below, proceeds quite readily by first expanding the function $\Phi(x)$ into a power series in x^2 and then integrating term by term. The result is a divergent or asymptotic series which we discuss in section 1.4.

1.3 ILLUSTRATIVE EXAMPLE FOR $\phi(w)$

Before undertaking the asymptotic evaluation of (1.7), it is desirable to illustrate the above concepts by considering a specific example of the exponent $\phi(w)$. In particular, we are interested in (a) determining the location of the saddle points (S.P.); (b) sketching the paths of steepest descents (P.S.D.), by ascertaining the asymptotes at infinity and the passage through the saddle points; (c) similarly sketching the paths of steepest ascents (P.S.A.); (d) similarly sketching the so-called contours of stationary phase (C.S.P.). It is always possible (in principle) to obtain mathematical equations for the above paths: P.S.D., P.S.A., and C.S.P., but it is seldom worth the trouble. A simple sketch of the paths in question will usually give us all the information we need.

1.3.1 Paths of Steepest Descents - Let us choose, by way of illustration, the exponent

$$\phi(w) = ia \cos w, \tag{1.9}$$

where a is a positive definite parameter, $a > 0$. Putting $\phi'(w) = 0 = -ia \sin w$ and solving for w, we ascertain that there exists an infinity (a denumerable set of measure zero) of saddle points located at

$$w_s = n\pi, \; n = 0, \pm 1, \ldots \tag{1.10}$$

on the real axis of the w plane. We shall choose $n = 0$ to designate our operating S.P. in accordance with (1.2).

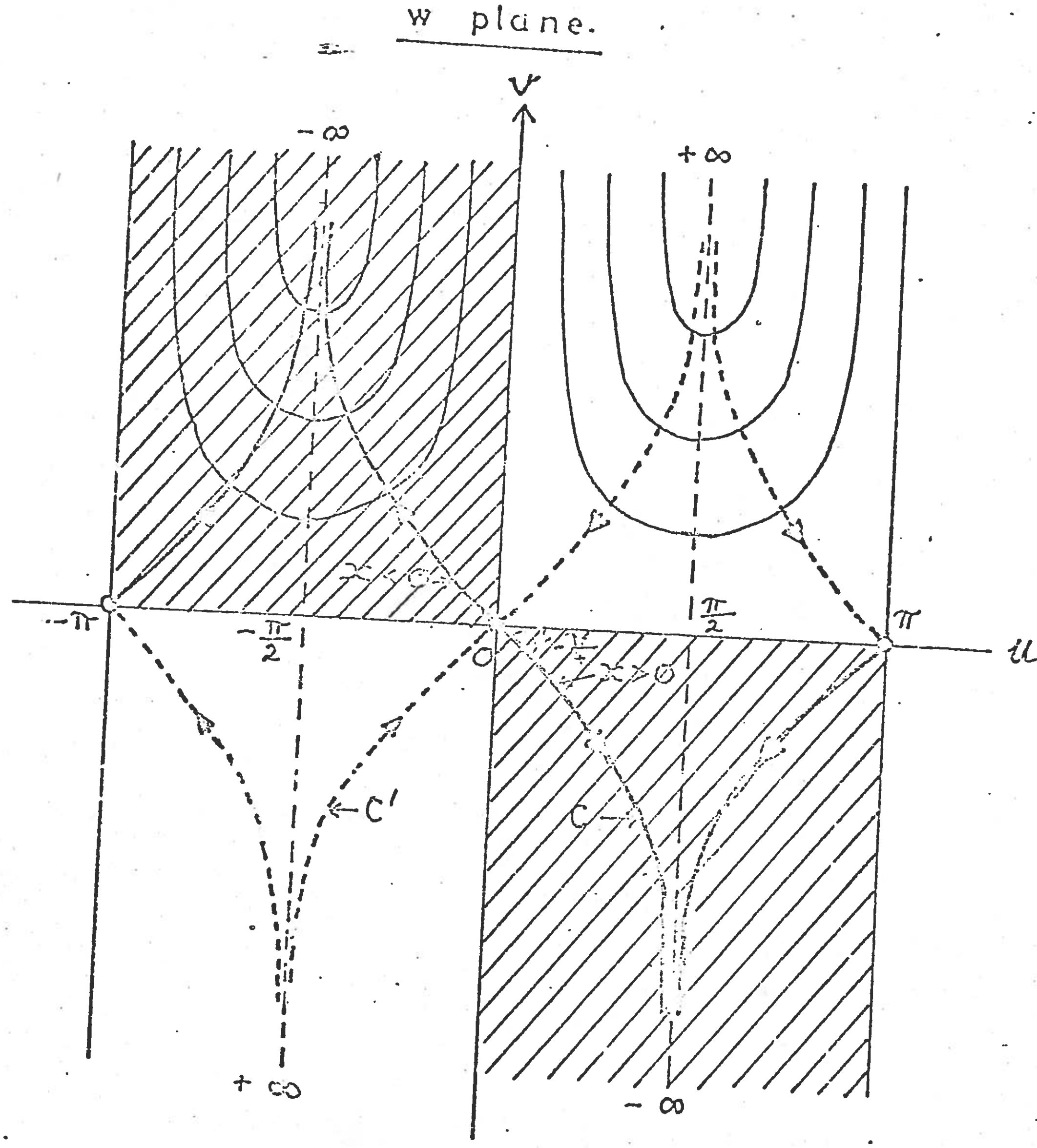

FIG. 1 - The w plane illustrating the P.S.D. (full lines) and the P.S.A. (dotted lines) associated with the exponent $\phi(w) = ia \cos w$, $a > 0$, and the hills and valleys of the transformation.

The conformal transformation (1.5) now becomes, with $n = 0$,

$$x^2 = \phi(0) - \phi(w) = ia(1-\cos w) = 2ia \sin^2 \frac{w}{2}. \qquad (1.11)$$

Putting $w = u + iv$ and confining our attention to the principal full-period strip, $-\pi \leqslant u \leqslant \pi$, we can readily ascertain from the requirement (1.3) that the path of steepest descents C, associated with the exponent (1.9) and the chosen S.P. at $w = 0$, is a symmetric curve (see Fig. 1) with vertical asymptotes at $u = \pm\frac{1}{2}\pi$ and crossing the S.P. at the origin $w = 0$ at an angle of $-\frac{1}{4}\pi$ with the axis of reals.

To verify the above statements we observe first that, asymptotically, $\cos w \to \frac{1}{2}e^{v-iu}$ as $v \to \infty$ in the upper half-plane, whence making use of (1.11) we note that

$$x^2 \xrightarrow[v\to\infty]{} -ia\ \tfrac{1}{2}e^{v-iu} = \tfrac{1}{2}ae^{v} \cdot e^{-i(\frac{\pi}{2}+u)} \longrightarrow +\infty \text{ if } u=-\frac{\pi}{2}$$

and with x rigorously real on the P.S.D., we must have $u \to -\frac{1}{2}\pi$ for the left-hand asymptote. Similar considerations in the lower half-plane show that $u \to \frac{1}{2}\pi$ for the right-hand asymptote, as shown in Fig. 1.

Next, we investigate the passage of the P.S.D. C of Fig. 1 through the chosen S.P. at $w = 0$. From the second form of (1.11) we have

$$x^2 \xrightarrow[w\to 0]{} 2ia\ \frac{w^2}{4} = \tfrac{1}{2}iaw^2$$

and, upon extracting the square root we obtain

$$x \xrightarrow[w\to 0]{} \pm\sqrt{a/2}\ we^{i\pi/4}$$

the double sign indicating the <u>sense</u> in which we choose to traverse the path C with x going from $-\infty$ to $+\infty$. Thus, if we discard the minus sign we find that $x > 0$ when $\arg\{w\} = -\frac{\pi}{4}$, which says that we have chosen to traverse the S.P. from the upper half-plane (u.h.p.) onto the lower half-plane (l.h.p.), with $x < 0$ in the u.h.p. and $x > 0$ in the l.h.p. as shown in Fig. 1. In Fig. 1 we still have two other S.P., at $w = \pm\pi$. We leave it as an exercise to the reader to show that the passage through these two S.P. is as indicated in Fig. 1.

1.3.2 Paths of Steepest Ascents - As shown above, the exponent $-x^2$ in (1.7) tends to $-\infty$ as $v \to \pm\infty$ along the P.S.D. C. Associated with this path C there is a symmetric orthogonal trajectory, labelled C' in Fig. 1, along which the exponent $-x^2$ tends to $+\infty$ as $v \to \pm\infty$, which we have chosen to call the path of steepest ascents (P.S.A.) for reasons that are more clearly expounded in (1.3.3) below. Making use of the procedure outlined above, the reader should have no trouble in determining the vertical asymptotes of the path C' and its passage through the S.P. at $w = 0$, as well as the half portions of P.S.A. associated with the S.P. at $w = \pm\pi$.

1.3.3 Contours of Stationary Phase - We have seen that, on the P.S.D. C associated with the S.P. $w = 0$, the exponent $-x^2 = \phi(w) - \phi(0)$ in (1.4), is real and negative and tends to $-\infty$ at the extremities of the path. Likewise, we have seen that, on the P.S.A. C' associated with the S.P. $w = 0$, the exponent $-x^2 = \phi(w) - \phi(0)$ is real and positive and tends to $+\infty$ at the extremities of the path. It is clear that in between these two orthogonal trajectories, C and C', there exist the so-called contours of stationary phase (C.S.P.) along which the exponent $-x^2 = \phi(w) - \phi(0)$ is rigorously pure imaginary. In the present instance, Eq. (1.11), it is clear that the contours of stationary phase are given, for the principal full period strip of Fig. 1, by the segment $-\pi \leq u \leq \pi$, $v = 0$, and by the three vertical lines $u = 0$, $u = \pm\pi$, $-\infty < v < \infty$. Hence, the cross-hatched area in Fig. 1 corresponds to the condition Re $\{-x^2\} < 0$, and conversely the remaining half-strips correspond to Re $\{-x^2\} > 0$. Therefore, if we plot the contours Re $\{-x^2\}$ = Re $\{\phi(w) - \phi(0)\}$ = const., as sketched in the upper half of Fig. 1, we would obtain a picture of a relief map in which the cross-hatched region depicts the valleys and the remaining portions the mountain ridges of a topographic map. It is now clear that a path through the pass at the origin descends most rapidly on either side of the S.P. along the contour C, which is the reason for calling it the path of steepest descents. Conversely, the orthogonal path through the pass at the origin ascends most rapidly on either side of the S.P. along the contour C', which is the reason for calling it the path of steepest ascents. In Fig. 1 the arrows drawn along the above paths indicate the direction of most rapid descent away from the S.P. or towards it, as the case might be.

1.3.4 <u>General Formulas for the Passage through a S.P.</u> - Returning to Fig. 1 and to the S.P. at $w = 0$, we see that the paths C and C', of steepest descents and ascents respectively, intersect each other at right angles, and that there are two contours of stationary phase, likewise orthogonal to each other, which bisect the angles between C and C'. This result is quite general for a S.P. of the 1st order and is not a special property of the particular exponent $\phi(w)$ chosen in (1.9) as illustrative example. Thus, quite generally, we write in the vicinity of a S.P.

$$-x^2 = \phi(w) - \phi(w_s) = \frac{1}{2}\,\phi''(w_s)(w - w_s)^2 + \ldots \qquad (1.12)$$

in which $w = w_s$ is a S.P., i.e. $\phi'(w_s) = 0$, and in which we assume that $\phi''(w_s) \neq 0$. When this is the case, we call $w = w_s$ a S.P. of the 1st order and, as we have seen, there are <u>two</u> P.S.D. leading away from the S.P. When $\phi''(w_s) = 0$, but $\phi'''(w_s) \neq 0$, then $w = w_s$ is a S.P. of the 2nd order and, this time, there are <u>three</u> paths of steepest descents leading away from w_s, and so on for higher order saddle points. In these lectures we confine ourselves exclusively to S.P. of the 1st order, which are the ones of most common occurrence.

<u>P.S.D. & P.S.A.</u>

We write

$$\mp x^2 = \phi(w) - \phi(w_s) = \frac{1}{2}\,\phi''(w_s)(w - w_s)^2 + \ldots \qquad (1.13)$$

where the <u>minus</u> sign is associated with the P.S.D. and the <u>plus</u> sign with the P.S.A. Upon extracting the square root we obtain

$$\text{either} \quad x = \pm\sqrt{-\tfrac{1}{2}\phi''(w_s)}\,(w - w_s), \quad \text{P.S.D.}$$

$$\text{or} \quad x = \pm\sqrt{+\tfrac{1}{2}\phi''(w_s)}\,(w - w_s), \quad \text{P.S.A.}$$

from which we determine $\arg\{w - w_s\}$ in either case by imposing the requirement that x be real on the respective path. It is recalled that the double sign outside the radicals above has only to do with the sense of traversal through the S.P. It is clear that the two paths governed by (1.13) intersect each other at right angles.

C.S.P.

This time we write, instead of (1.13),

$$\pm ix^2 = \phi(w) - \phi(w_s) = \tfrac{1}{2}\phi''(w_s)(w - w_s)^2 + \ldots, \qquad (1.14)$$

the double sign indicating the existence of two orthogonal contours of stationary phase passing through the S.P. Upon solving for x we obtain

$$x = \pm\, e^{\pm i\pi/4} \sqrt{\tfrac{1}{2}\phi''(w_s)}\,(w - w_s) \qquad (1.15)$$

from which we determine $\arg\{w - w_s\}$ from the requirement that x be real on the respective paths. Again the double sign in front has only to do with the sense of traversal through the S.P. Comparing all the expressions for x given above show clearly that the C.S.P. bisect the angles between the orthogonal P.S.D. and P.S.A., which was to be shown.

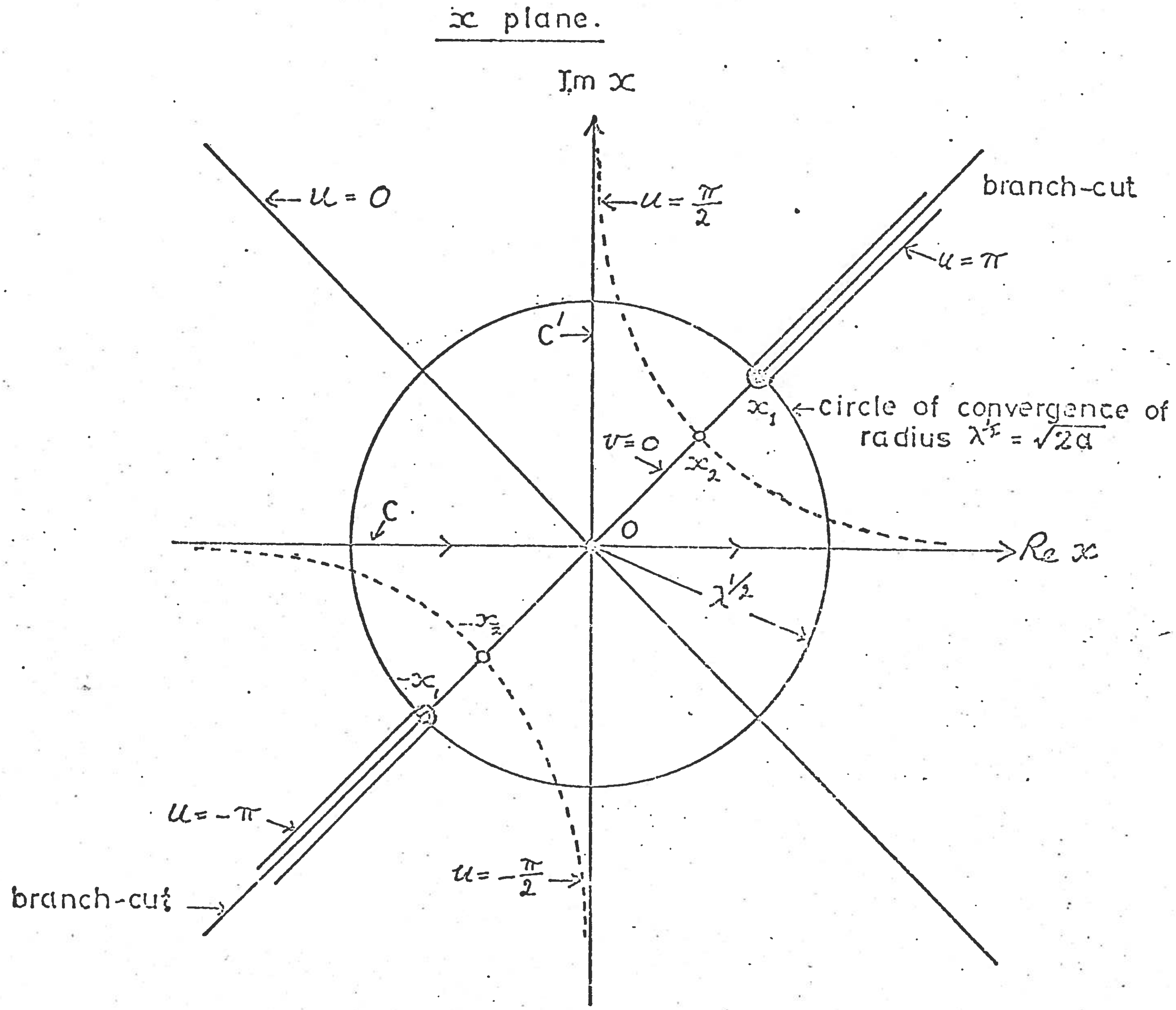

FIG. 2 - The x plane illustrating the conformal transformation $x^2 = ia(1 - \cos w)$, the full-period $(-\pi \leq u \leq \pi)$ strip of the w plane (Fig. 1), and the mapping of the P.S.D. C onto the real axis.

1.3.5 Mapping onto the x plane - We return once more to the conformal transformation (1.11) and we proceed to illustrate the mapping of the principal full-period strip of Fig. 1 onto the x plane, which is given in Fig. 2.

We know, by construction, that the P.S.D. C maps onto the axis of reals (the arrows in Fig. 2 indicate the sense of traversal along the path, from $-\infty$ to $+\infty$), and that the path of steepest ascents (P.S.A.) C' maps onto the axis of imaginaries. The chosen S.P. at $w = 0$ maps into the origin $x = 0$. The neighbouring S.P. at $w = \pm\pi$ map onto the points $x = \pm x_1$, thus

$$x^2 = ia(1 - \cos w)$$

$$w = \pm\pi,\ x = \pm x_1;\quad x_1^2 = 2ia,\ x_1 = e^{i\pi/4}\sqrt{2a} \tag{1.16}$$

The segment between $-x_1$ and $+x_1$ represents the mapping of the segment $v = 0$, $-\pi \leqslant u \leqslant \pi$. We must recall that all other S.P. (other than the chosen S.P.) become branch points of the transformation. Hence, as indicated in Fig. 2, the half-lines at 45° joining the branch points x_1 and $-x_1$ to the point at infinity are branch cuts and their borders represent respectively the mapping of the vertical lines (of Fig. 1) at $u = \pm\pi$. The mapping of the vertical lines at $u = \pm\frac{\pi}{2}$ are given by the dotted equilateral hyperbolas shown in Fig. 2. The intercepts with the mapping of the segment $v = 0$ occur at $x = \pm x_2$, thus

$$w = \pm\frac{\pi}{2},\ x = \pm x_2;\quad x_2^2 = ia,\ x_2 = e^{i\pi/4}\sqrt{a} \tag{1.17}$$

We conclude this section with a few remarks concerning the circle of convergence shown in Fig. 2

Returning for the moment to the integral (1.4), let us assume for the present that F(w), when expressed as a function of x through the conformal transformation (1.11), exhibits no singularities within the circle of convergence of Fig. 2; F(w) for example might be identically unity, or a polynomial in w, or else an entire function. Then, returning to (1.6), we observe that the factor dw(x)/dx that appears in the integrand exhibits a power series in x valid within the radius of convergence $\lambda^{\frac{1}{2}} = |x_1| = \sqrt{2a}$, which follows immediately from the inverse of the conformal transformation (1.11), namely

$$w = w(x) = 2 \sin^{-1} \frac{x}{\sqrt{2ia}} \tag{1.18}$$

The importance of the radius of convergence of the power series expansion in x^2 of the function $\phi(x)$ defined by Eq. (1.8), in determining the usefulness and applicability of the ensuing asymptotic expansion, will shortly become apparent. Here we merely wish to state that anything that happens, for example in the way of singularities (poles or branch points), outside the circle of convergence will not affect the asymptotic results we are about to obtain.

1.4 DIVERGENT SERIES EXPANSION

Having disposed of our illustrative example for the exponent $\phi(w)$ in (1.4), we now return to the evaluation of the integral (1.7), which we now rewrite in the form

$$e^{-\phi(0)} I = \int_0^\infty \Phi(x) e^{-x^2} dx, \tag{1.19}$$

where the function $\Phi(x)$, defined by (1.8), exhibits the power series expansion

$$\Phi(x) = \sum_{m=0}^\infty A_{2m} x^{2m}, \quad |x| < \lambda^{\frac{1}{2}}, \tag{1.20}$$

where $\lambda^{\frac{1}{2}}$ denotes the radius of convergence; i.e., the function $\Phi(x)$ exhibits its nearest singularity to the origin at $x = x_0$ where $|x_0| = \lambda^{\frac{1}{2}}$ by definition. It is important to emphasize at this juncture that the function $\Phi(x)$, because of the way it was constructed, can never be an entire function and, therefore, the radius of convergence $\lambda^{\frac{1}{2}}$ is always finite. The statement is true even when the function $F(w)$ might be identically unity, for the factor dw/dx in (1.8), which must be computed by inverting the conformal transformation (1.5), always exhibits branch points in the finite plane which arise from all other saddle points, as illustrated in the last paragraph of the preceding Section.

Returning now to the integral (1.19), we replace the function $\Phi(x)$ by its power series expansion (1.20) to obtain

$$e^{-\phi(0)} I = \int_0^\infty \sum_{m=0}^\infty A_{2m} x^{2m} e^{-x^2} dx, \tag{1.21}$$

where, it is important to recall, the power series expansion within the integral sign, has a <u>finite</u> radius of convergence. Notwithstanding, and with full knowledge of the fact that we are about to commit serious violence against mathematical rigour, we interchange the order of summation and integration, to obtain the divergent series

$$e^{-\phi(0)}I = \sum_{m=0}^{\infty} A_{2m} \int_0^{\infty} x^{2m}\, e^{-x^2}\, dx$$

$$= \frac{\sqrt{\pi}}{2} \sum_{m=0}^{\infty} (\tfrac{1}{2})_m\, A_{2m}$$

$$= \frac{\sqrt{\pi}}{2} \{A_0 + \tfrac{1}{2}A_2 + \tfrac{3}{4}A_4 + \ldots\}, \qquad (1.22)$$

where use has been made of the well-known integrals

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} x^{2m}\, e^{-x^2}\, dx = (\tfrac{1}{2})_m, \qquad (1.23)$$

and which, as shown below, is asymptotic in the sense of Poincaré. It bears emphasizing that the process of interchanging the order of summation and integration, followed by term by term integration from zero to infinity, that is, beyond the domain of applicability of the power series expansion (1.21), is mathematically untenable. Nonetheless, the resulting series (1.22), which is divergent, actually proves very useful in the approximate evaluation of the integral by judiciously limiting the number of terms employed.

The integrals (1.23) have been expressed in terms of the generalized factorial notation which, with n = 0, 1, 2, ..., is defined by

$$(a)_0 = 1,\ (a)_1 = a,\ (a)_2 = a(a+1), \ldots, \qquad (1.24)$$

$$(a)_n = a(a+1) \ldots (a+n-1)$$

whence $(1)_n = n!$ Accordingly we have, in (1.23),

$$(\tfrac{1}{2})_n = 1 \cdot \tfrac{1}{2} \cdot \tfrac{3}{2} \cdot \ldots \cdot \frac{2n-1}{2} = \frac{(2n-1)!!}{2^n} \tag{1.25}$$

where the double factorial is given by

$$(2n-1)!! = 1 \cdot 1.3 \ldots (2n-1) = \frac{(2n)!}{2^n\, n!}, \tag{1.26}$$

with the aid of which we may write the alternative forms

$$(\tfrac{1}{2})_n = \frac{(2n-1)!!}{2^n} = \frac{(2n)!}{2^{2n} n!} = \frac{\Gamma(2n+1)}{2^{2n}\Gamma(n+1)} = \frac{1}{2^{2n-1}} \frac{\Gamma(2n)}{\Gamma(n)}$$

$$\xrightarrow[n \gg 1]{} \sqrt{2}\, n^n\, e^{-n}, \tag{1.27}$$

where the last form ensues upon replacing the gamma functions by their Stirling's approximation for large n. The important fact that emerges is that the coefficients $(\frac{1}{2})_n$ grow without limit, as $n \to \infty$, which is the reason why the series (1.22) is divergent no matter how well behaved the expansion coefficients A_{2m} in (1.20) might be.

1.5 WATSON'S LEMMA

To establish that the expansion (1.22) is asymptotic in the sense of Poincaré we need to invoke (our own modification of) Watson's lemma (1944, p. 236). To this end let us make a change of scale in the variable of integration by putting

$$x = \lambda^{\frac{1}{2}} y \qquad \text{and} \qquad \Phi(x) = \Psi(y) \tag{1.28}$$

Our typical integral (1.19) now becomes

$$e^{-\phi(0)} I = \lambda^{\frac{1}{2}} \int_0^\infty \Psi(y) e^{-\lambda y^2} dy, \tag{1.29}$$

in which, by virtue of (1.20) and (1.28), the function $\Psi(y)$ exhibits the power series expansion

$$\Psi(y) = \sum_{m=0}^{\infty} A_{2m} \lambda^m y^{2m}, \quad |y|<1, \tag{1.30}$$

with unity radius of convergence. In terms of the new variable of integration Watson's lemma, as applied to the problem at hand, may be phrased as follows:

Lemma: let $\Psi(y)$ be analytic within the unit circle $|y|<1$, that is, let $\Psi(y)$ possess the power series expansion (1.30). Assume further that

$$|\Psi(y)| < Ay^{2p} e^{\eta\lambda y^2}, \tag{1.31}$$

where A is a positive number independent of y, p is a positive integer or zero, and η is a positive number, $0 \leqslant \eta < 1$, whenever $y \leqslant 1$ or $y \geqslant 1$, $0 \leqslant y < \infty$. Then the asymptotic expansion (1.22) is valid in the sense of Poincaré.

Proof: to establish the above result we note from (1.30) and (1.31) that, if $M \geqslant p$ is a fixed integer, a constant B can be found such that

$$\left|\Psi(y) - \sum_{m=0}^{M-1} A_{2m}\lambda^m y^{2m}\right| \leqslant By^{2M} e^{\eta\lambda y^2}, \tag{1.32}$$

whenever $y \geq 0$, whether $y \leq 1$ or $y \geq 1$, and therefore we can rewrite (1.22) in the form

$$e^{-\phi(0)} I = \frac{\sqrt{\pi}}{2}\left\{ \sum_{m=0}^{M-1} (\tfrac{1}{2})_m A_{2m} + R_M \right\}, \qquad (1.33)$$

where R_M, the remainder after M terms, is bounded as follows:

$$|R_M| < \lambda^{\frac{1}{2}} B \frac{2}{\sqrt{\pi}} \int_0^\infty y^{2M} e^{-(1-\eta)\lambda y^2} dy$$

$$= \frac{B}{(1-\eta)^{\frac{1}{2}}} (\tfrac{1}{2})_M \frac{1}{\left[(1-\eta)\lambda\right]^M}$$

$$= O\{(1-\eta)\lambda\}^{-M}, \qquad (1.34)$$

and therefore we have established the expansion*

$$e^{-\phi(0)} I = \frac{\sqrt{\pi}}{2}\{ \sum_{m=0}^{M-1} (\tfrac{1}{2})_m A_{2m} + O\left[(1-\eta)\lambda\right]^{-M} \}, \qquad (1.35)$$

which is useful and of practical value only when $(1-\eta)\lambda >> 1$, for only then does the error or remainder become tolerably small. It is recalled that $\lambda^{\frac{1}{2}}$ is the radius of convergence of the power series expansion (1.20), and that η is a parameter, $0 \leq \eta < 1$, which gives a measure of the (adverse) exponential behaviour that the function $\phi(x)$ might have. It is clear that the integral (1.19) converges only if $\eta < 1$, and that the expansion (1.35) is useful only if $\eta << 1$ (preferably $\eta = 0$) and $\lambda >> 1$.

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* Asymptotic a la Poincaré means that $|R_M| \to 0$ as $\lambda \to \infty$ with M fixed, whereas $|R_M| \to \infty$ as $M \to \infty$, with λ fixed.

Two remarks are very much in order concerning our own modification of Watson's lemma. The first modification has to do with the presence of the factor y^{2p} in (1.31) and the subsequent statement that $M \geqslant p$ in (1.32). This is necessary to provide for the contingency, which often occurs in practice, whenever the first coefficient A_0, or the first few coefficients A_{2m} in (1.20) actually vanish. The second modification, of more profound significance, has to do with the change of scale introduced in Eq. (1.28), from which ensues the true role played in the asymptotic expansion (1.35) by the radius of convergence of the power series expansion (1.20). It is now clear that we had no need to introduce explicitly a large parameter in the exponent of our basic integral (1.1), as has been invariably done by all previous authors, and that the large parameter emerges in a natural way when we show that $|R_M| = O\left[(1-\eta)\lambda\right]^{-M}$, where λ is the square of the radius of convergence of the power series expansion (1.20). In the practical application of the theory it turns out, of course, that λ is given in terms of some physical parameter, which itself will have to be large in order to render the asymptotic expansion (1.35) of practical value. Thus, in the illustrative example discussed in Section 1.3.5, where we dealt with the exponent $\phi(w) = ia \cos w$, we showed that $\lambda = 2a$. Hence, the resulting asymptotic expansion would turn out in reciprocal powers of 2a and, if a is some physical parameter, it will better be large, $a >> 1$, in order that the resulting expansion be useful.

1.6 INVERSION OF THE POWER SERIES EXPANSION FOR x^2

We have seen that the asymptotic evaluation of our basic integral form (1.1), through the conformal transformation (1.5) that maps the P.S.D. onto the real axis in the x plane, had been reduced to the evaluation of the integral (1.7) in which x is the variable of integration. To evaluate (1.7) asymptotically, as given by (1.35), we must first expand $\Phi(x)$ into the power series (1.20). To this end we must invert the conformal transformation (1.5) to express $w = w(x)$ as a power series in x, and then we must proceed to the evaluation of the expansion coefficients A_{2m} in (1.20). As we shall see this procedure proves to be quite onerous algebraically in all but the simplest cases. In fact, even the inversion of the conformal transformation (1.5) is not in general a simple matter, except when we are fortunate to encounter an exponent $\phi(w)$ like the one we used in (1.9). Here, as we have seen, the conformal transformation (1.11) admits at once the inversion (1.18), which can be readily expanded into a power series in x.

When we are not thus fortunate, we can proceed quite generally as follows. Starting from (1.5), making use of a Taylor's series expansion, we can write

$$
\begin{aligned}
x^2 &= \phi(0) - \phi(w) \\
&= w^2\{c_0 + c_1 w + c_2 w^2 + \dots\} \\
&= c_0 w^2\{1 + \frac{c_1}{c_0} w + \frac{c_2}{c_0} w^2 + \dots\}, \quad c_0 \neq 0, \qquad (1.36)
\end{aligned}
$$

which is certainly valid for w sufficiently small. Next, making use of standard formulas that one finds for example in Abramowitz (1964), we extract the square root of both sides of (1.36) to obtain

$$x = \sqrt{c_0}\, w\{1 + \tfrac{1}{2}\frac{c_1}{c_0} w + \left[\tfrac{1}{2}\frac{c_2}{c_0} - \tfrac{1}{8}\frac{c_1^2}{c_0^2}\right] w^2 + \cdots\}$$

$$= \sqrt{c_0}\, w + \frac{\sqrt{c_0}}{2}\frac{c_1}{c_0} w^2 + \sqrt{c_0}\left[\tfrac{1}{2}\frac{c_2}{c_0} - \tfrac{1}{8}\frac{c_1^2}{c_0^2}\right] w^3 + \cdots \qquad (1.37)$$

Finally, again making use of standard formulas, we invert the series (1.37) to write

$$w = w(x) = a_0 x + \tfrac{1}{2} a_1 x^2 + \tfrac{1}{3} a_2 x^3 + \cdots, \qquad (1.38)$$

whence

$$\frac{dw}{dx} = a_0 + a_1 x + a_2 x^2 + \cdots, \qquad (1.39)$$

where the expansion coefficients a_0, a_1, $\cdots$ are given by the formulas

$$a_0 = c_0^{-\frac{1}{2}}$$

$$a_1 = c_0^{-1}\left[-\frac{c_1}{c_0}\right] \qquad (1.40)$$

$$a_2 = c_0^{-3/2}\left[\frac{15}{8}\frac{c_1^2}{c_0^2} - \frac{3}{2}\frac{c_2}{c_0}\right]$$

etc.

in terms of the expansion coefficients c_0, c_1, c_2, $\cdots$. It is abundantly clear that higher order coefficients in (1.40) rapidly become considerably more complicated and thus we omit them here.

1.7 TWO-TERM EXPANSION FORMULA

According to Eq. (1.35), a two-term expansion formula is obtained by putting $M = 2$. Thus we have

$$I = \tfrac{1}{2}\sqrt{\pi}\, e^{\phi(0)}\{A_0 + \tfrac{1}{2}A_2 + O(\lambda)^{-2}\}, \tag{1.41}$$

where $\lambda^{\frac{1}{2}}$ is the radius of convergence of the power series expansion (1.20), and where we have put $\eta = 0$, which implies that $F(w)$ in (1.1) exhibits no exponential behaviour. Thus, our problem has been reduced to the computation of the expansion coefficients A_0 and A_2. To this end we first expand the Function $F(w)$ into a Taylor's series about $w = 0$,

$$F(w) = F(0) + wF'(0) + \tfrac{1}{2}w^2F''(0) + \cdots. \tag{1.42}$$

Next, making use of (1.38) we express successive powers of w in terms of x to obtain, after some algebraic rearrangements,

$$F(w(x)) = F(0)\{1 + a_0\frac{F'(0)}{F(0)}x + \tfrac{1}{2}\left[a_1\frac{F'(0)}{F(0)} + a_0^2\frac{F''(0)}{F(0)}\right]x^2 + \cdots\}; \tag{1.43}$$

and then we rewrite (1.39) in a similar form

$$\frac{dw(x)}{dx} = a_0\{1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \cdots\}. \tag{1.44}$$

Whence, multiplying the series (1.43) and (1.44) we obtain finally for $\Phi(x)$ in (1.8) the expansion

$$\phi(x) = F(w(x))\,\frac{dw(x)}{dx} + F(w(-x))\,\frac{dw(-x)}{dx}$$

$$= 2a_0F(0)\left\{1 + \left[\frac{3a_1}{2}\,\frac{F'(0)}{F(0)} + \frac{a_0^2}{2}\,\frac{F''(0)}{F(0)} + \frac{a_2}{a_0}\right]x^2 + \cdots\right\}$$

$$= A_0 + A_2x^2 + \cdots, \qquad (1.45)$$

where

$$A_0 = 2a_0F(0)$$

$$A_2 = 2a_0F(0)\left[\frac{3a_1}{2}\,\frac{F'(0)}{F(0)} + \frac{a_0^2}{2}\,\frac{F''(0)}{F(0)} + \frac{a_2}{a_0}\right] \qquad (1.46)$$

which completes our computation. Thus, substituting into (1.41), our desired two-term expansion formula becomes

$$I = \sqrt{\pi}\, a_0F(0)e^{\phi(0)}\left\{1+\tfrac{1}{2}\left[\frac{3a_1}{2}\,\frac{F'(0)}{F(0)}+\frac{a_0^2}{2}\,\frac{F''(0)}{F(0)}+\frac{a_2}{a_0}\right]+O(\lambda^{-2})\right\}, \qquad (1.47)$$

where $\lambda^{\frac{1}{2}}$ is the smaller of the radii of convergence of the expansions (1.43) and (1.44).

When, as it often happens, $F(w) = 1$, then $F(0) = 1$ and all higher derivatives vanish in (1.47) leading to the much simpler result

$$I = \sqrt{\pi}\, a_0e^{\phi(0)}\left\{1 + \tfrac{1}{2}\,\frac{a_2}{a_0} + O(\lambda^{-2})\right\}, \qquad (1.48)$$

where, this time, $\lambda^{\frac{1}{2}}$ is the radius of convergence of the expansion~~s~~ (1.44).

1.8 LEADING TERM ONLY

When dealing with asymptotic expansions it happens very often that we can be content with the leading term of the expansion. Thus, from (1.47) we have at once the leading term

$$I = \sqrt{\pi}\; a_0\, F(0) e^{\phi(0)} \{1 + O(\tfrac{1}{\lambda})\}, \tag{1.49}$$

where, according to (1.40), $a_0 = c_0^{-\frac{1}{2}}$. But now, from (1.36) we have $x^2 = \phi(0) - \phi(w) = -\frac{1}{2}\, \phi''(0)w^2 + \cdots$ $= c_0 w^2 + \cdots$, whence $c_0 = -\frac{1}{2}\, \phi''(0)$ and $a_0 = 1/\sqrt{-\frac{1}{2}\phi''(0)}$. Substituting into (1.49), the leading term formula becomes

$$I = \sqrt{2\pi}\, \frac{F(0) e^{\phi(0)}}{\sqrt{-\phi''(0)}}\, \{1 + O(\tfrac{1}{\lambda})\}, \tag{1.50}$$

wherein, we must recall, $\lambda = |\phi(0) - \phi(w_i)|$, in which $w = w_1$ is the singularity nearest to the S.P. at $w = 0$.

However, and this is a very important point that we cannot emphasize too strongly, if we only want the leading term of the asymptotic expansion, then we do not need at all the heavy machinary of the preceding pages. In fact, starting from (1.1), we can proceed as follows:

$$\begin{aligned} I &= \int_C F(w) e^{\phi(w)} dw = e^{\phi(0)} \int_C F(w) e^{\phi(w) - \phi(0)} dw \\ &= e^{\phi(0)} \int_C \{F(0) + \cdots\} e^{\frac{1}{2}\, \phi''(0) w^2}\, dw. \end{aligned} \tag{1.51}$$

Now, to lowest order, we put $-x^2 = \frac{1}{2}\phi''(0)w^2$, or $x = \sqrt{-\frac{1}{2}\phi''(0)}\,w$, whence $w = x/\sqrt{-\frac{1}{2}\phi''(0)}$, and integrating with respect to x from $-\infty$ to $+\infty$, which is <u>not</u> the same thing as integrating along the path of steepest descents C, we obtain from (1.51) approximately

$$I \sim \frac{F(0)e^{\phi(0)}}{\sqrt{-\frac{1}{2}\phi''(0)}} \underbrace{\int_{-\infty}^{\infty} e^{-x^2}\,dx}_{\sqrt{\pi}} = \sqrt{2\pi}\,\frac{F(0)e^{\phi(0)}}{\sqrt{-\phi''(0)}}\,, \tag{1.52}$$

exactly in agreement with (1.50). If the S.P. occurs at $w = w_s$, instead of $w = 0$, then we have

$$I \sim \sqrt{2\pi}\,\frac{F(w_s)e^{\phi(w_s)}}{\sqrt{-\phi''(w_s)}}\,.$$

This heuristic approach can be extremely useful in practice because of its simplicity. It is often justified by saying that our formula (1.52) is like a mean value theorem which picks out the contribution from the vicinity of the S.P. where supposedly the integrand has its largest value. Since this is not always the case as we shall see below, the argument is hardly valid. The full justification of the formula is the heavy machinery that we have so laboriously set up, for only then do we have a proper estimate of the remainder. One last remark is in order: we note that integrating with respect to x in (1.52), from $-\infty$ to $+\infty$, is tantamount to integrating in the w plane along a straight line tangent to the P.S.D. at the chosen S.P. Referring to Fig. 1, for example, this means that the straight line path of integration would take us <u>outside</u> of the principal full-period strip in which we had been working, which is all right in principle, for we recognize that whatever happens outside the circle of convergence of Fig. 2 is of little consequence in the final result.

1.9 LEADING TERM WHEN F(0) = 0

It happens sometimes that the function $F(w)$ in (1.1) vanishes at the S.P. $w = 0$, $F(0) \equiv 0$. We can no longer say, as many writers state, that the integrand is largest in the vicinity of the S.P., for in fact the integrand is zero at the S.P. Nonetheless, our two-term expansion formula (1.47), can immediately be put to good use by outright putting $F(0) = 0$. Thus, we obtain, with $F(0) = 0$,

$$I \sim \sqrt{\pi}\, a_0 e^{\phi(0)} \tfrac{1}{4}\left[3a_1 F'(0) + a_0^2 F''(0)\right]$$

$$= \sqrt{\pi} e^{\phi(0)} \frac{a_0^3}{4}\left[F''(0) + \frac{3a_1}{a_0^2} F'(0)\right], \tag{1.53}$$

which we will now endeavor to rewrite in the manner of the leading term formula (1.51). To this end we note from the expansion

$$x^2 = \phi(0) - \phi(w) = -\tfrac{1}{2}\phi''(0)w^2 - \tfrac{1}{6}\phi'''(0)w^3 - \cdots$$

$$= w^2\left[-\tfrac{1}{2}\phi''(0) - \tfrac{1}{6}\phi'''(0)w - \cdots\right],$$

upon comparison with (1.36), that

$$c_0 = -\tfrac{1}{2}\phi''(0); \quad c_1 = -\tfrac{1}{6}\phi'''(0); \cdots .$$

Making use of (1.40) we have

$$a_0 = c_0^{-\frac{1}{2}} = \frac{1}{\sqrt{-\frac{1}{2}\phi''(0)}}; \quad a_0^2 = \frac{1}{-\frac{1}{2}\phi''(0)}$$

$$a_1 = -\frac{c_1}{c_0^2} = \frac{2}{3}\frac{\phi'''(0)}{\left[\phi''(0)\right]^2} .$$

whence, finally, the coefficient $3a_1/a_0^2$ in (1.53) becomes

$$-\frac{3a_1}{a_0^2} = \frac{\phi'''(0)}{\phi''(0)} .$$

Substituting the pertinent values into (1.53) yields the desired form,

$$I \sim \tfrac{1}{2}\sqrt{2\pi}\,\frac{F''(0) - \frac{\phi'''(0)}{\phi''(0)}F'(0)}{\left[-\phi''(0)\right]^{3/2}}\,e^{\phi(0)}, \tag{1.54}$$

which is the leading term of the asymptotic expansion when $F(0) = 0$.

It often happens that the P.S.D. is symmetric about the S.P. at $w = 0$, in which case $\phi'''(0) = 0$; or else $F(w)$ is an even function of w, in which case $F'(0) = 0$. In either case, then, our leading term formula assumes the much simpler form

$$I \sim \sqrt{2\pi}\,\frac{\tfrac{1}{2}F''(0)e^{\phi(0)}}{\left[-\phi''(0)\right]^{3/2}}, \tag{1.55}$$

which should be contrasted with (1.52). It so happens that (1.55), but <u>not</u> (1.54) please note, can be deduced with the simple heuristic approach that led us to (1.52). To this end we proceed as follows: noting that, by hypothesis, $F(0) = 0$ and $F'(0) = 0$, we write in the manner of (1.51)

$$I = e^{\phi(0)}\int_c \{\tfrac{1}{2}F''(0)w^2 + \dots\}e^{\frac{1}{2}\phi''(0)w^2}\,dw$$

$$= \tfrac{1}{2}F''(0)e^{\phi(0)}\int_c w^2 e^{\frac{1}{2}\phi''(0)w^2}\,dw,$$

and upon writing $-x^2 = \tfrac{1}{2}\phi''(0)w^2$, we have integrating with respect to x from $-\infty$ to $+\infty$,

$$I \sim \frac{\frac{1}{2}F''(0)e^{\phi(0)}}{\left[-\frac{1}{2}\phi''(0)\right]^{3/2}} \underbrace{\int_{-\infty}^{\infty} x^2 e^{-x^2} dx}_{\frac{1}{2}\sqrt{\pi}} = \sqrt{2\pi}\, \frac{\frac{1}{2}F''(0)e^{\phi(0)}}{\left[-\phi''(0)\right]^{3/2}}, \qquad (1.56)$$

precisely in accord with (1.55), which was to be shown.

1.10 SUBTRACTION OF A 1st ORDER POLE FROM VICINITY OF S.P.

It is clear from the foregoing analysis (Section 1.5) and in particular from the expansion (1.35) that the magnitude of the remainder, as given by (1.34), limits the applicability of our asymptotic expansions to fairly large values of λ, where $\lambda^{\frac{1}{2}}$ represents the radius of convergence of the power series expansion of the function $\Phi(x)$, defined by (1.8), about the origin in the x plane. Expressed otherwise, it is the singularity of $\Phi(x)$ occurring nearest to the origin in the x plane which governs the behaviour of the asymptotic expansion. If it happens, however, that the singularity nearest to the origin is a pole of the 1st order (or of higher order), then it turns out that the radius of convergence can be enlarged to the <u>next</u> nearest singularity by the subtraction of the pole, thus enhancing the range of applicability of the resulting asymptotic expansion which now exhibits a remainder R_M with a smaller upper bound.

Thus, we assume that the function $\Phi(x)$, defined by (1.8), exhibits a pair of simple poles at $x = \pm x_0$, where $\lambda^{\frac{1}{2}} = |x_0|$ denotes the radius of convergence of the power series expansion (1.20), and that the next nearest singularity of $\Phi(x)$, presumably an algebraic singularity and not a pole, occurs at $x = x_1$ with $|x_1| = \mu^{\frac{1}{2}}$ and $\mu > \lambda$ by hypothesis. Then, the function

$$\Psi(x) = \Phi(x) - \frac{2x_0 C}{x^2 - x_0^2}, \qquad (1.57)$$

where C is defined by

$$C = \lim_{x \to x_0} \{\frac{x^2 - x_0^2}{2x_0} \Phi(x)\}, \qquad (1.58)$$

is analytic for $|x| < \mu^{\frac{1}{2}}$. Hence, in accordance with Eq. (1.20), $\Psi(x)$ admits the power series expansion

$$\Psi(x) = \sum_{m=0}^{\infty} B_{2m}\, x^{2m}, \quad |x| < \mu^{\frac{1}{2}}, \qquad (1.59)$$

with expansion coefficients

$$B_{2m} = A_{2m} + 2C\, x_0^{-1-2m}. \qquad (1.60)$$

Solving for $\Phi(x)$ in Eq. (1.57) and substituting into Eq. (1.19) we obtain

$$e^{-\phi(0)} I = \int_0^{\infty} \{\Psi(x) + \frac{2x_0 C}{x^2 - x_0^2}\} e^{-x^2} dx = w_s + w_p, \qquad (1.61)$$

where w_p, the contribution arising from the pole, is given by the integral

$$w_p = 2x_0 C \int_0^{\infty} \frac{e^{-x^2} dx}{x^2 - x_0^2}, \qquad (1.62)$$

which, as shown below, can be expressed in closed form in terms of the complex error function or, alternatively, in terms of the plasma dispersion function, and where w_s, the contribution over the path through the saddle-point, according to Eqs. (1.35) and (1.59), is obtained from

$$w_s = \int_0^\infty \Psi(x)e^{-x^2}dx = \frac{\sqrt{\pi}}{2}\{\sum_{m=0}^{M-1} (\tfrac{1}{2})_m B_{2m} + R_M\}, \qquad (1.63)$$

in which the order of magnitude of the remainder after M terms is given by

$$|R_M| = O\left[(1-\eta)\mu\right]^{-M}, \qquad (1.64)$$

in accordance with the analysis leading to Eq. (1.34). It is clear that, with $\mu > \lambda$, the asymptotic expansion (1.63) has a smaller remainder after M terms than the original expansion (1.35), and hence a greater range of applicability. Thus, we have shown that the subtraction of the pair of poles from the integrand effectively improves, from the point of view of applications, the asymptotic behaviour of the resulting expansion. There remains only the evaluation of the integral (1.62).

Thus, we recall that, according to Fried and Conte (1961), the plasma dispersion function $Z(\zeta)$ may be defined from the equivalent integrals

$$Z(\zeta) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{e^{-t^2}dt}{t-\zeta} = \frac{2\zeta}{\sqrt{\pi}}\int_0^\infty \frac{e^{-t^2}dt}{t^2-\zeta^2}, \; Im\{\zeta\}>0, \qquad (1.65)$$

and, for $Im\{\zeta\}<0$, one uses the analytic continuation of the above integrals in the customary fashion. Alternatively, according to Baños and Johnston (1969), the plasma dispersion function may be expressed in terms of the complex error function,

$$Z(\zeta) = i \sqrt{\pi}\, e^{-\zeta^2}\left[1 - \mathrm{erf}(-i\zeta)\right], \qquad (1.66)$$

which is valid quite generally as written for all values of $\mathrm{Im}\{\zeta\}$, and in which the error function is defined by the integral

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \qquad (1.67)$$

Thus, making use of (1.65) and (1.66), we can at once evaluate w_p in (1.62) in terms of the plasma dispersion function or in terms of the complex error function as follows:

$$w_p = \sqrt{\pi}\; C\; Z(x_0) = i\; \pi C e^{-x_0^2}\left[1 - \mathrm{erf}(-ix_0)\right], \qquad (1.68)$$

and, since the plasma dispersion function and the complex error function have by now been widely tabulated, we may regard the evaluation of (1.62) as accomplished.

The foregoing analysis is quite general, but also purely formal. In the practical application of the theory the most burdensome part of the analysis is the computation of the expansion coefficients B_{2m} defined by (1.60), which become progressively more complicated in higher orders as we have seen (Section 1.7 et seq.). However, if we are content with results in lowest order, we can once again proceed heuristically as we did in the derivation of Eqs. (1.52) and (1.56). To this end we return to our basic integral (1.1), wherein we assume that $F(w) = G(w)/(w-w_0)$

exhibits a simple pole at $w = w_o$, which <u>does not</u> lie on the path of steepest descents C (for reasons that will be made clear later), and where G(w) is otherwise analytic in a sufficiently large domain containing the saddle point. We shall assume that the chosen S.P. occurs this time not at w = 0, but at $w = w_s$, a fixed point, and that w_o, the location of the pole is close to w_s and may in fact approach w_s as a limit. Accordingly we have

$$I = \int_C F(w)e^{\phi(w)}dw = \int_C \frac{G(w)}{w-w_o}\, e^{\phi(w)}dw$$

$$\sim \sqrt{2\pi}\, \frac{G(w_s)e^{\phi(w_s)}}{(w_s-w_o)\sqrt{-\phi''(w_s)}} \xrightarrow[w_o \to w_s]{} \infty, \qquad (1.69)$$

where the latter leading term asymptotic form is obtained from the preceding integral by the direct application of our leading term formula (1.52), with due regard to the fact that the S.P. now occurs at $w = w_s$. The fact that our leading term result in (1.69) diverges to infinity as $w_o \to w_s$ merely dramatizes how catastrophic it is to have a pole in the close vicinity of the S.P. and how important, therefore, it is to remove such a pole, which we now proceed to do making use of the simple heuristic approach.

To this end we expand G(w) in the second integral in (1.69) into a Taylor's series about $w = w_o$, the location of the pole, and we subtract and add (within the integral sign) the leading term of the expansion, to obtain in lowest order:

$$I = \int_c \frac{G(w_o)+(w-w_o)G'(w_o)+\cdots-G(w_o)}{w-w_o} e^{\phi(w)} dw + G(w_o)\int_c \frac{e^{\phi(w)}}{w-w_o} dw$$

$$= G'(w_o)\int_c e^{\phi(w)} dw + \cdots + G(w_o)\int_c \frac{e^{\phi(w)}}{w-w_o} dw$$

$$= \sqrt{2\pi}\,\frac{G'(w_o)e^{\phi(w_s)}}{\sqrt{-\phi''(w_s)}} + \cdots + G(w_o)e^{\phi(w_s)}\int_c \frac{e^{\phi(w)-\phi(w_s)}}{w-w_o} dw, \qquad (1.70)$$

To deal with the 2nd integral, which represents the contribution from the pole, we proceed as many times before by writing

$$-x^2 = \phi(w)-\phi(w_s) = \tfrac{1}{2}\phi''(w_s)(w-w_s)^2+\cdots$$

whence we have

$$x = \sqrt{-\tfrac{1}{2}\phi''(w_s)}\;(w-w_s)$$

$$x_o = \sqrt{-\tfrac{1}{2}\phi''(w_s)}\;(w_o-w_s)$$

and

$$w-w_o = (w-w_s)-(w_o-w_s) = \frac{x-x_o}{\sqrt{-\tfrac{1}{2}\phi''(w_s)}}$$

with

$$dw = \frac{dx}{\sqrt{-\tfrac{1}{2}\phi''(w_s)}}.$$

Thus, changing the variable of integration in the 2nd integral of (1.70) from w to x in accordance with the above, the integral in question becomes

$$G(w_o)e^{\phi(w_s)}\int_C \frac{e^{\frac{1}{2}\phi''(w_s)(w-w_s)^2+\cdots}}{w-w_o}dw = G(w_o)e^{\phi(w_s)}\int_{-\infty}^{\infty}\frac{e^{-x^2}dx}{x-x_o}$$

$$= \sqrt{\pi}\, Z(x_o)G(w_o)e^{\phi(w_s)}, \qquad (1.71)$$

in which the latter form ensues from (1.65) upon integrating with respect to x from $-\infty$ to $+\infty$ and not along the P.S.D. C, which in itself is part of the philosophy of the lowest order computation. Combining the results embodied in (1.70) and (1.71), we have finally in lowest order the asymptotic expansion for the integral in (1.69), namely

$$I = \int_C \frac{G(w)}{w-w_o}e^{\phi(w)}dw \sim \sqrt{2\pi}\frac{G'(w_o)e^{\phi(w_s)}}{\sqrt{-\phi''(w_s)}} + \sqrt{\pi}\, Z(x_o)G(w_o)e^{\phi(w_s)}, \qquad (1.72)$$

in which nothing catastrophic happens as $w_o \to w_s$.

In the above we assumed initially that the pole at $w = w_o$ did not lie on the P.S.D. C. We did this to avoid talking about principal value integrals and about the (ambiguous) contribution from the infinitesimal semi-circle about such a pole. Such matters are taken care of automatically by the plasma dispersion function $Z(x_o)$ which appears in (1.72) and, hence, we are now permitted to lift the initial restriction on the location of the pole.

1.11 MULTIPLE SADDLE POINT METHOD OF INTEGRATION

In Section (1.8) we obtained an important formula, Eq. (1.52), for the leading term of an asymptotic expansion of our basic integral form (1.1). As we have seen, when the S.P. occurs at $w = w_s$, the formula assumes the form

$$I \sim \sqrt{2\pi}\,\frac{F(w_s)e^{\phi(w_s)}}{\sqrt{-\phi''(w_s)}}\,. \tag{1.73}$$

Now consider, in complete analogy with our basic integral form (1.1), the n-fold integral

$$I = \int\cdots\int F(w_1,\cdots,w_n)e^{\Phi(w_1,\cdots,\,w_n)}dw_1\cdots dw_n, \tag{1.74}$$

which we shall assume exists (converges) when the integrations are carried out over <u>specified</u> paths of integration in each of the (complex) variables $w_1,\cdots,w_n$, and the only restriction that we impose is that the limits of integration, whether finite or infinite, be constants independent of the variables of integration.

Next, assume that there exists a (multiple) saddle point in complex n-space (there may be several),

$$w_i^s = (w_1^s,\ \cdots,\ w_n^s), \tag{1.75}$$

whose coordinates are determined by solving simultaneously the n equations

$$\frac{\partial\Phi}{\partial w_i} = 0, \ (i = 1,\ \cdots,\ n). \tag{1.76}$$

Then, expand the exponent in (1.74) about the multiple S.P. (1.75), thus

$$\Phi(w_1,\cdots,\ w_n) = \Phi(w_1^s,\cdots,\ w_n^s) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(w_i-w_i^s)(w_j-w_j^s)\Phi_{ij} + \cdots$$

where

$$\Phi_{ij} \equiv \left.\frac{\partial^2\Phi}{\partial w_i \partial w_j}\right|_{\substack{w_i = w_i^s \\ w_j = w_j^s}} , \quad (i,\ j = 1, \cdots, n) \tag{1.77}$$

Accordingly, the <u>leading term</u> of the asymptotic expansion of (1.74) about the S.P. (1.75) is given by

$$I = F(w_i^s)e^{\Phi(w_i^s)}\int\cdots\int \exp\{\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(w_i - w_i^s)(w_j - w_j^s)\Phi_{ij}\} \times dw_1\cdots dw_n, \tag{1.78}$$

where the 2nd order differential coefficients Φ_{ij}, $i,j = 1,\cdots, n$, have been evaluated at the chosen S.P. in accordance with (1.77). To evaluate the n-fold integral in (1.78) asymptotically, we institute an orthogonal transformation of coordinates

$$x_i = a_{i\alpha}(w_\alpha - w_\alpha^s), \quad w_i - w_i^s = a_{\alpha i}x_\alpha, \quad i = 1, \cdots, n, \tag{1.79}$$

where we have used the summation convention for repeated indices, and which satisfies the orthogonality and normalization conditions, for $i,\ j = 1,\ \cdots,\ n$,

$$\begin{aligned} a_{i\alpha}\ a_{j\alpha} &= \delta_{ij} \quad \text{(rows)} \\ a_{\alpha i}\ a_{\alpha j} &= \delta_{ij} \quad \text{(columns)} \\ |a_{ij}| &= 1 \quad \text{(determinant)} \end{aligned}$$

and such that the homogeneous quadratic form in the exponent of (1.78) becomes the sum of squares

$$(w_\alpha - w_\alpha^s)(w_\beta - w_\beta^s)\Phi_{\alpha\beta} = x_\alpha^2\ \Phi'_{\alpha\alpha}, \tag{1.80}$$

where (by choice of orthogonal transformation)

$$\Phi'_{ij} = a_{i\alpha} a_{j\beta} \Phi_{\alpha\beta} = 0 \quad \text{when } i \neq j; \tag{1.81}$$

that is, we have diagonalized the symmetric tensor Φ_{ij} by rotation in n-space onto the principal axes.

Thus, making use of (1.80), the n-fold integral in (1.78) becomes

$$\int \cdots \int \exp\{\tfrac{1}{2} \sum_{i=1}^{n} x_i^2 \Phi'_{ii}\} dx_1 \cdots dx_n, \tag{1.82}$$

since the element of volume in (1.78) is given by

$$dw_1 \cdots dw_n = \frac{\partial[(w_1 - w_1^s), \cdots, (w_n - w_n^s)]}{\partial(x_1, \cdots, x_n)} dx_1 \cdots dx_n$$

and the Jacobian of the transformation in the present case is $J = |a_{ij}| = 1$. Proceeding in the spirit of our earlier heuristic method we now put

$$-\xi_i^2 = \tfrac{1}{2} x_i^2 \Phi'_{ii}, \quad i=1, \cdots, n,$$

whence

$$dx_i = \frac{d\xi_i}{\sqrt{-\frac{1}{2}\Phi'_{ii}}}, \quad i=1, \cdots, n.$$

Our integral (1.82) can now be evaluated asymptotically as

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\xi_1^2 - \cdots - \xi_n^2} \frac{d\xi_1}{\sqrt{-\frac{1}{2}\Phi'_{11}}} \cdots \frac{d\xi_n}{\sqrt{-\frac{1}{2}\Phi'_{nn}}} = \frac{(\sqrt{2\pi})^n}{\{(-)^n \prod_{i=1}^{n} \Phi'_{ii}\}^{\frac{1}{2}}}$$

But $\prod_{i=1}^{n} \Phi'_{ii}$ = determinant = $|\Phi_{ij}|$, sometimes referred to as Hesse's determinant, whence finally substituting back into

Eq. (1.78) we find that the leading term of the asymptotic expansion of the primitive n-fold integral (1.74) is given by the formula

$$I \sim \frac{(\sqrt{2\pi})^n \, F(w_i{}^s)}{\{(-)^n |\Phi_{ij}|\}^{\frac{1}{2}}} \, e^{\Phi(w_i{}^s)}, \tag{1.83}$$

which is the (demonstrated) generalization of the leading term (1.73) for a single integration. In (1.83) we understand that

$$F(w_i^s) = F(w_1^s, \ldots, w_n^s); \quad \Phi(w_i^s) = \Phi(w_1^s, \ldots, w_n^s).$$

It is to be understood further that the method developed above yields only the leading term (1.83), and that it can not be extended to obtain higher order terms in the manner of the expansion (1.35), nor does it appear possible to make any estimates of the remainder associated with (1.83). Nonetheless, the leading term formula for a multiple integral, which to our knowledge was first employed by Bremmer (1949), proves to be extremely useful when physical considerations alone suffice to support its validity.

1.12 THE METHOD OF STATIONARY PHASE

It has been our contention all along that the method of stationary phase is no different from the method of steepest descents, for they are completely equivalent and both give exactly the same result, at least insofar as the leading term of the expansion is concerned. The principle of stationary phase has been applied to numerous mathematical problems as well as physical situations, but it appears to be difficult to formulate in a precise manner. Perhaps the best available theorem was given by Watson (1920). Poincaré discussed the method of stationary phase applied to integrals involving analytic functions, and the connection of his work with the method of steepest descents was indicated by Copson (1946). More recently Erdélyi (1956) made a valiant effort to put the whole theory on a sound mathematical basis and went further than all his predecessors by providing a scheme for the computation of higher order terms, even though this entailed an inordinate amount of extremely heavy analysis at the most mathematically sophisticated level and even though his entire analysis must be restricted exclusively to a real variable of integration. When all is said and done, Erdélyi (loc. cit.) in his illustrative examples obtains the very same asymptotic expansions that he had obtained by the method of steepest descents, which proves the complete equivalence of both methods.

It is my opinion, however, that since both methods are completely equivalent, little purpose is served by a thoroughgoing study of the principle of stationary phase with all its cumbersomeness and limitations. Below we present the derivation of the leading term making use of the principle of stationary phase. Following Erdélyi (1956) let us consider the integral

$$I(x) = \int_a^b F(w)e^{ixh(w)}dw, \qquad (1.84)$$

in which x is a <u>large</u> positive variable, and h(w) is a real function of the <u>real</u> variable w. According to Stokes and Kelvin, the major contribution to the value of the integral arises from the immediate vicinity of the end points of the interval of integration and from the vicinity of those points at which h(w) is stationary, i.e., $h'(w)=0$; and in lowest order of approximation the contribution of stationary points, if there are any, is more important than the contribution from the end points.

Suppose that F(w) is continuous and h(w) is twice continuously differentiable, and let $w = w_s$ be the only stationary point of h(w), $a < w_s < b$, $h'(w_s)=0$ and $h''(w_s)>0$ (Please note last restriction). In the assumption that the neighbourhood of w_s will give the principal contribution to the integral (which assumes that $F(w_s) \neq 0$, although Erdélyi does not say so), we introduce a new variable of integration u by the substitution

$$u^2 = h(w) - h(w_s)$$

to obtain

$$I(x) \sim \int_{w_s-\varepsilon}^{w_s+\varepsilon} F(w)e^{ixh(w)}dw = \int_{-u_1}^{u_2} F(w)e^{ix\left[h(w_s)+u^2\right]}\frac{2u}{h'(w)}\,du$$

where $u_1 = \left[h(w_s-\varepsilon) - h(w_s)\right]^{\frac{1}{2}}$ and $u_2 = \left[h(w_s+\varepsilon) - h(w_s)\right]^{\frac{1}{2}}$. Since only the neighbourhood of u=0 is all that matters, we may replace $F(w)$ by $F(w_s)$ and $2u/h'(w)$ by $\left[2/h''(w_s)\right]^{\frac{1}{2}}$ which is the limit of $2u/h'(w)$ as $u \to 0$ or $w \to w_s$, so that

$$I(x) \sim \frac{\sqrt{2}}{\sqrt{h''(w_s)}} F(w_s) \int_{-u_1}^{u_2} e^{ix\left[u^2 + h(w_s)\right]} du.$$

By the same argument we may extend the integration from $-e^{i\pi/4}\infty$ to $+e^{i\pi/4}\infty$ (although Erdélyi puts it from $-\infty$ to $+\infty$), and we finally obtain

$$I(x) \sim \sqrt{2\pi} \frac{F(w_s) e^{ixh(w_s) + i\pi/4}}{\sqrt{xh''(w_s)}}$$

which we may rewrite as follows:

$$I(x) \sim \sqrt{2\pi} \frac{F(w_s) e^{ixh(w_s)}}{\sqrt{-ixh''(w_s)}}. \qquad (1.85)$$

We will now show that the above formula, Eq. (1.85), is completely equivalent to our leading term formula, Eq. (1.50) or Eq. (1.52), that we had earlier derived by more rigorous methods and without all the attending restrictions specified above. In fact, all we need to do is to identify the exponent, in (1.84), by putting

$$\phi(w) = ixh(w), \qquad (1.86)$$

with which Erdélyi's formula (1.85) becomes

$$I \sim \sqrt{2\pi} \frac{F(w_s) e^{\phi(w_s)}}{\sqrt{-\phi''(w_s)}}, \qquad (1.87)$$

completely in accord with (1.52). But please note that, in our own derivation, we did not have to impose any of the restrictions specified above. We have shown that the method of steepest descents and the principle of stationary phase are completely equivalent. Hence, from now on, we shall stay with the method of steepest descents, which is more rigorous and which we understand better. For one thing we know that we do not need to invoke a "large parameter", and we do understand what it is that has to be large.

By way of illustration of the preceding concepts let us consider the integral

$$I(a) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ia \cos w} dw, \tag{1.88}$$

in which the path of integration is strictly along the contour of stationary phase and in which the exponent $\phi(w) = ia \cos w$ has been fully treated in Section 1.3. We note that $w=0$ is a stationary point precisely half-way between the limits of integration. We will now show how the main contribution arises from the passage through the S.P. (stationary point or saddle point) at $w=0$, and to what extent the contribution from the end-points at $w = \pm\frac{\pi}{2}$ can be regarded as negligible. For this purpose we find it convenient to proceed quite generally by rewriting (1.88) in the form

$$\begin{aligned} I &= \int_{-\alpha}^{\alpha} e^{\phi(w)} dw \\ &= e^{\phi(0)} \int_{-\alpha}^{\alpha} e^{\phi(w) - \phi(0)} dw \\ &\sim e^{\phi(0)} \int_{-\alpha}^{\alpha} e^{\frac{1}{2}\phi''(0) w^2} dw \\ &= \frac{e^{\phi(0)}}{\sqrt{-\frac{1}{2}\phi''(0)}} \int_{-x_1}^{x_1} e^{-x^2} dx, \end{aligned} \tag{1.89}$$

where $x_1 = \alpha\sqrt{-\frac{1}{2}\phi''(0)}$. Recalling the definition (1.67) of the complex error function, we have from (1.89)

$$I \sim \sqrt{2\pi}\,\frac{e^{\phi(0)}}{\sqrt{-\phi''(0)}}\,\mathrm{erf}(x_1). \tag{1.90}$$

Next, making use of (1.66), we observe that

$$\mathrm{erf}(-i\zeta) = 1 - \frac{Z(\zeta)e^{\zeta^2}}{i\sqrt{\pi}} \sim 1 + \frac{e^{\zeta^2}}{i\sqrt{\pi}\zeta} \tag{1.91}$$

for $|\zeta|>>1$, and $\mathrm{Im}\{\zeta\}>0$. See Baños and Johnston (1969).

Thus, returning to our primitive integral (1.88), we put $\phi(w) = ia\cos w$, $\phi(0) = ia$, $\phi''(0) = -ia$, $\alpha = \frac{\pi}{2}$, whence

$$x_1 = \alpha\sqrt{-\tfrac{1}{2}\phi''(0)} = \tfrac{1}{2}\pi e^{i\pi/4}\sqrt{a/2} = -i\cdot\tfrac{1}{2}\pi e^{3\pi i/4}\sqrt{a/2}$$

$$\text{or}\quad \zeta = \tfrac{1}{2}\pi e^{3\pi i/4}\sqrt{a/2}$$

$$\text{and}\quad \zeta^2 = -i\pi^2 a/8$$

Hence

$$\mathrm{erf}(x_1) = \mathrm{erf}(-i\zeta) \sim 1 + \frac{e^{-i\pi^2 a/8}}{\frac{1}{2}\pi e^{5\pi i/4}\sqrt{\pi a/2}}$$

$$= 1 - \frac{2e^{-i\pi^2 a/8-\pi i/4}}{\pi\sqrt{\pi a/2}}$$

and, substituting into (1.90), we have finally

$$I(a) \sim \sqrt{2\pi}\,\frac{e^{ia}}{\sqrt{ia}}\{1 - \frac{2e^{-i\pi^2 a/8-\pi i/4}}{\pi\sqrt{\pi a/2}} + O(\frac{1}{2a})\}, \tag{1.92}$$

from which we deduce that, if $\frac{1}{2}\pi\sqrt{\pi a/2}>>1$, then we can neglect the contribution from the end points, and the integral becomes simply the leading term

$$I(a) \sim \sqrt{2\pi}\, \frac{e^{ia}}{\sqrt{ia}}, \tag{1.93}$$

in accordance with the precepts such as expounded by Erdélyi at the beginning of the present Section. One last remark concerning (1.92) is in order here. With reference to Fig. 1 we can interpret the result given by noting that the leading term itself corresponds to the integral along the path of steepest descents C, and that the contribution from the end points represents the sum of the integrals in the w plane, from $w = -\frac{\pi}{2}$ to $w = -\frac{\pi}{2}+i\infty$, and from $w = \frac{\pi}{2}-i\infty$ to $w = \frac{\pi}{2}$, which all together represent the deformation of the primitive path of integration, from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ along the contour of stationary phase onto the path of steepest descents C.

2. THE ASYMPTOTIC PROPERTIES OF AIRY FUNCTIONS

The Airy functions have a venerable history of more than 100 years, and are well known in mathematical physics because of their frequent occurrence in numerous problems in classical as well as in modern physics. In addition, the study of their asymptotic properties gives us the simplest non-trivial example of the application of the method of steepest descents, of the principle of analytic continuation, and of the Stokes phenomenon which is always associated with the asymptotic expansions of analytic functions. In these lectures we first introduce three integral representations which satisfy the canonical form of Airy's differential equation, in terms of which we then define the Airy functions Ai(z) and Bi(z), which are real when z is real, which are entire functions, and which constitute the most convenient fundamental set of solutions of Airy's equation. We then compute their leading term asymptotic expansions for $z > 0$, and then, by the process of analytic continuation, we extend these leading term formulas to $z < 0$. In so doing we examine in detail the attendant Stokes phenomenon. As an example of the methods presented in the preceding Chapter, we present here a two-term expansion formula for Ai(z) when $z > 0$. Finally, we deduce the so-called anti-canonical form of Airy's equation and we show how to obtain a fundamental set of solutions in terms of the Airy functions Ai(z) and Bi(z) previously defined. The latter effort is undertaken in

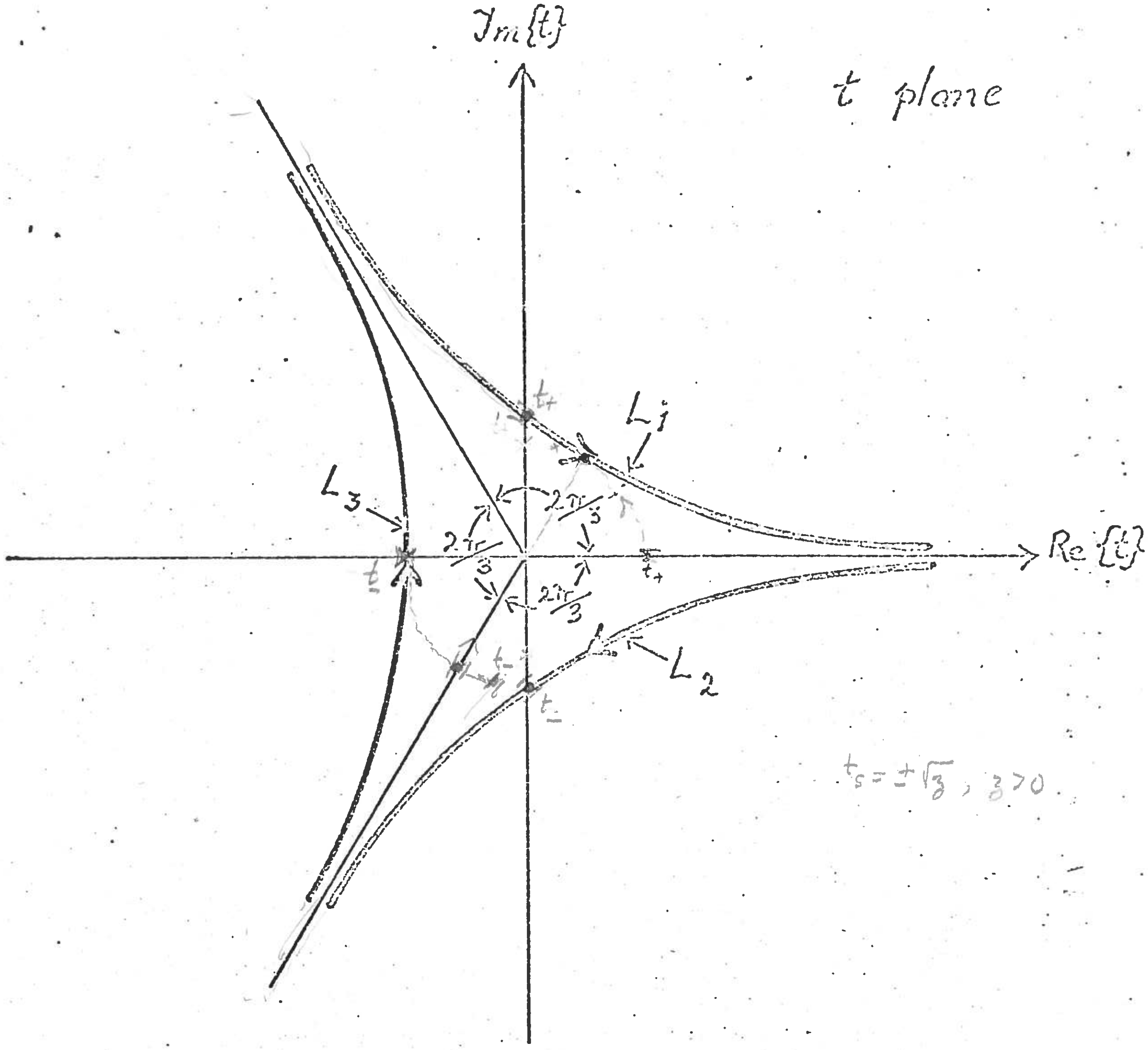


FIG. 3 - The t plane and the paths of integration L_1, L_2, and L_3 associated with the integral representations $I_i(z) = \int_{L_i} e^{zt - \frac{1}{3}t^3} dt$, $i = 1, 2, 3$.

preparation for the study of Langer's method, the WKB approximation, and the question of turning points, which we take up in Chapter 3.

2.1 INTEGRAL REPRESENTATIONS

Consider the functions $I_i(z)$, $i = 1, 2, 3$, defined by the integral representations

$$I_i(z) = \int_{L_i} e^{zt - \frac{1}{3}t^3}\, dt; \quad i = 1, 2, 3, \tag{2.1}$$

in which the paths of integration $L_i (i = 1, 2, 3)$, as shown in Fig. 3, are asymptotic to the rays at $2\pi/3$ with each other (to guarantee the convergence of the integrals), but are otherwise arbitrary. The chosen sense of traversal along the paths L_1, L_2, and L_3 is the one indicated by arrows in Fig. 3.

2.2 AIRY'S DIFFERENTIAL EQUATION

To ascertain the differential equation which the functions defined by (2.1) satisfy, we compute for $i = 1, 2, 3$,

$$\frac{d^2 I_i}{dz^2} = \int_{L_i} t^2 e^{zt - \frac{1}{3}t^3}\, dt$$

by differentiating under the sign of integration, and then we form the combination

$$I_i'' - zI_i = -\int_{L_i} (z - t^2) e^{zt - \frac{1}{3}t^3}\, dt = -\int_{L_i} \frac{d}{dt}\left(e^{zt - \frac{1}{3}t^3}\right) dt = 0$$

by the choice of paths of integration L_1, L_2, and L_3 as illustrated in Fig. 3. The above result shows that the functions $I_i(z)$, $i = 1, 2, 3$ satisfy the canonical forms of Airy's equation

$$\frac{d^2u}{dz^2} - zu = 0, \tag{2.2}$$

which is a 2nd order, homogeneous, linear differential equation. Consequently, any two of the three functions $I_i(z)$, $i = 1, 2, 3$, defined above constitute a pair of fundamental solutions of (2.2), and all three together satisfy the condition

$$I_1(z) + I_2(z) + I_3(z) = 0, \tag{2.3}$$

which is immediately obvious from Fig. 3 upon invoking Cauchy's theorem.

2.3 DEFINITION OF THE AIRY FUNCTIONS Ai(z) AND Bi(z)

In terms of our basic integral representations (2.1), we now define the classical fundamental set of solutions of Airy's equation (2.2) by means of the formulas

$$\mathrm{Ai}(z) = \frac{I_3(z)}{2\pi i} = \frac{1}{2\pi i}\int_{L_3} e^{zt-\frac{1}{3}t^3}\,dt; \tag{2.4}$$

$$\mathrm{Bi}(z) = \frac{I_1(z)-I_2(z)}{2\pi} = \frac{1}{2\pi}\left\{\int_{L_1} e^{zt-\frac{1}{3}t^3}\,dt - \int_{L_2} e^{zt-\frac{1}{3}t^3}\,dt\right\}, \tag{2.5}$$

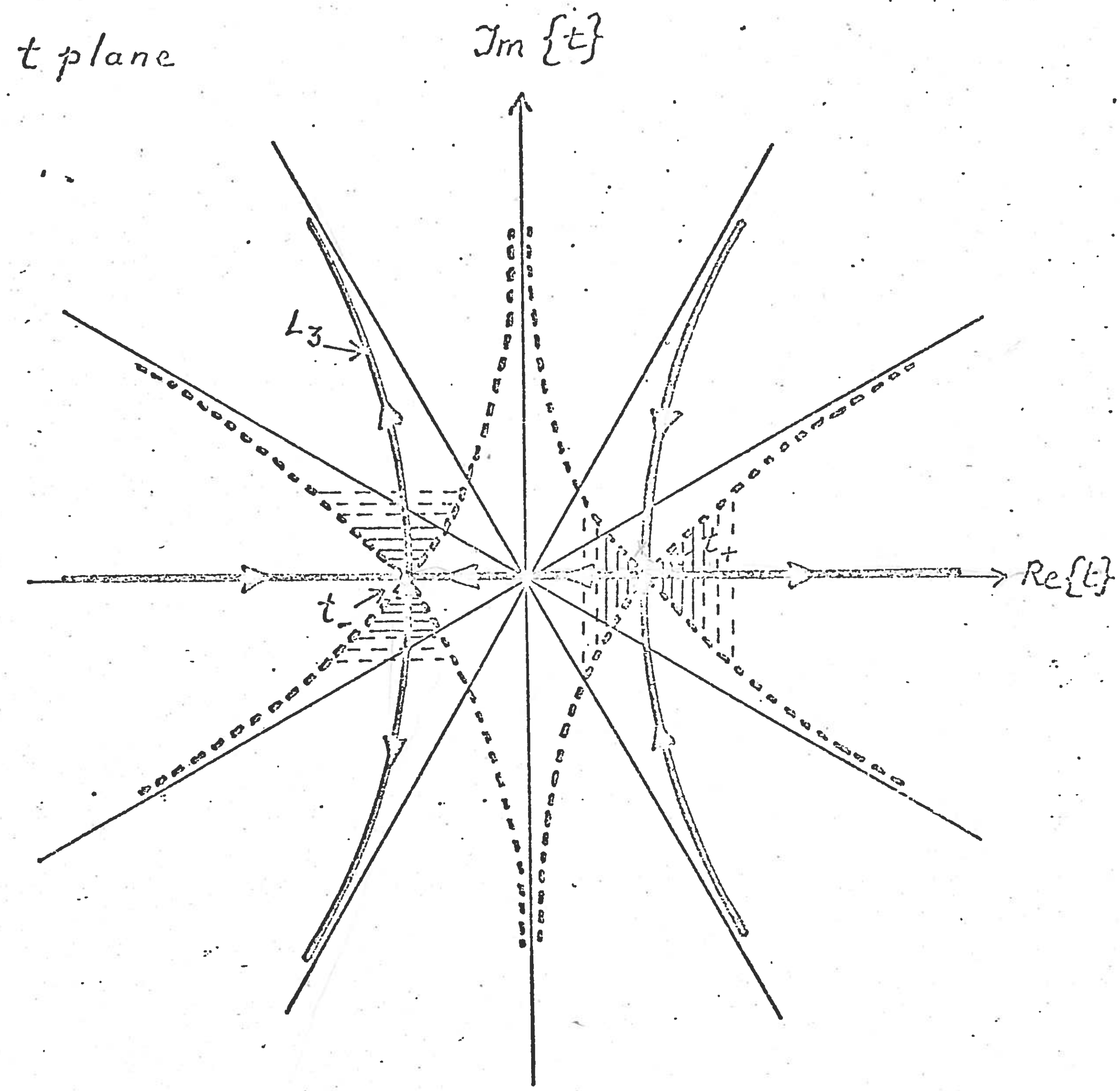
which have the property that both are entire functions which are real when z is real. That they are linearly independent is obvious by construction, but a direct computation shows that their Wronskian determinant is given by

$$\mathrm{Ai}(z)\mathrm{Bi}'(z) - \mathrm{Ai}'(z)\mathrm{Bi}(z) = \frac{1}{\pi} \qquad (2.6)$$

which confirms the fact that Ai(z) and Bi(z) constitute a fundamental set of solutions of Airy's equation. We should mention in passing that Watson (1922, Sec. 6.4) showed that Airy's equation can be transformed to the differential equation satisfied by Bessel functions of order $^1/_3$, an interesting and useful fact which, however, we shall not make any further use of in these lectures. The interested reader is referred to Erdélyi (1956) for a useful short tabulation of the properties of Airy functions in terms of Bessel functions of order $^1/_3$.

2.4 LEADING TERM ASYMPTOTIC EXPANSIONS FOR $z > 0$

In the present Section we shall endeavor to deduce leading term asymptotic expansions for Ai(z) and Bi(z) in sectors for arg{z} which include the positive half of the z axis and, therefore, are valid for $z > 0$. For the purpose we first undertake an analysis of the exponent $f(t) = zt - \frac{1}{3}t^3$ in the manner of Section 1.3, including the study of the passage through the saddle points and the evaluation of the leading term asymptotic expansions arising



<u>FIG. 4</u> - The t plane illustrating the paths of steepest descents and steepest ascents (full lines), and the contours of stationary phase (dotted lines) associated with the exponent $f(t) = zt - \frac{1}{3}t^3$, $z > 0$, and the two saddle points at $t = t_\pm = \pm z^{\frac{1}{2}}$. The arrows indicate the direction of most rapid descent along the respective paths. In the case of the P.S.D. the arrows point <u>away</u> from the corresponding S.P., whereas the contrary is true in the case of the P.S.A. The shaded regions indicate the so-called "valleys" of the transformation for each S.P.

from the corresponding paths of steepest descents. Similarly, we undertake the computation of the leading term asymptotic expansions which are valid in sectors of arg{z} comprising the negative half of the z axis and, therefore, are valid for $z < 0$. In this process we are led to the study of the process of analytic continuation with its attendant Stokes phenomenon.

2.4.1 <u>Analysis of the Characteristic Exponent</u> - The asymptotic behaviour of the Airy functions (2.4) and (2.5) is governed by the properties of the exponent

$$
\begin{aligned}
&f(t) = zt - \tfrac{1}{3}t^3 \\
&f'(t) = 0 = z - t^2; \quad \text{S.P. at } t = t_\pm = \pm z^{\frac{1}{2}} \qquad (2.7) \\
&f''(t) = -2t
\end{aligned}
$$

which says, as indicated, that there exist two S.P. located at $t = t_\pm = \pm z^{\frac{1}{2}}$. When $z > 0$, the S.P. lie symmetrically on the real axis of the t plane as indicated in Fig. 4. Further, from (2.7) we have

$$f(t_\pm) = \pm\tfrac{2}{3}z^{3/2} = \pm\zeta; \quad \zeta \equiv \tfrac{2}{3}z^{3/2}, \qquad (2.8)$$

and

$$-\tfrac{1}{2}f''(t_\pm) = t_\pm = \pm z^{\frac{1}{2}} = \pm|z|^{\frac{1}{2}}e^{i\theta/2}; \quad z = |z|e^{i\theta} \qquad (2.9)$$

On Fig. 4 we exhibit the paths of steepest descents and ascents, as well as the contours of stationary phase corresponding to the exponent (2.7) when $z > 0$, which have been sketched in the manner of Fig. 1. We leave it to the interested reader to work out the details such as were presented in Section 1.3. Here, we merely wish to point out that, as shown below, the chosen passage through the S.P. at $t = t-$ is from the lower half-plane onto the upper half-plane and, therefore, the path of integration L_3 of Fig. 3 can readily be deformed to coincide exactly with the P.S.D. through $t-$. Similarly, as shown below, the chosen passage through the S.P. at $t = t+$ is from right to left along the real axis, but this time the P.S.D. through $t+$ goes directly to join the S.P. at $t-$. This is only true for z rigorously real and positive. Generally speaking, access from one S.P. to another, by way of P.S.D., can only be had via the point at infinity as clearly illustrated in Fig. 1. The situation depicted in Fig. 4, where two S.P. are joined by a P.S.D. is a singular case of infrequent occurrence. In fact, we show below, that this can only occur when $\theta = \arg\{z\} = 0, \pm 2\pi/3$, whereas in the case of Fig. 1, it can never occur at all.

2.4.2 Passage through the S.P. at $t = t\pm$ - Proceeding quite generally, as in Section 1.3.4, we now write

$$-x^2 = f(t) - f(t\pm) = \tfrac{1}{2}f''(t\pm)(t-t\pm)^2 + \cdots, \qquad (2.10)$$

and, upon extracting the square root, we obtain

$$x = \pm\sqrt{-\tfrac{1}{2}f''(t\pm)}\,(t-t\pm), \qquad (2.11)$$

where, we recall, the double sign in front of the radical determines for us the sense of traversal through the S.P. as x goes from negative to positive values. In this

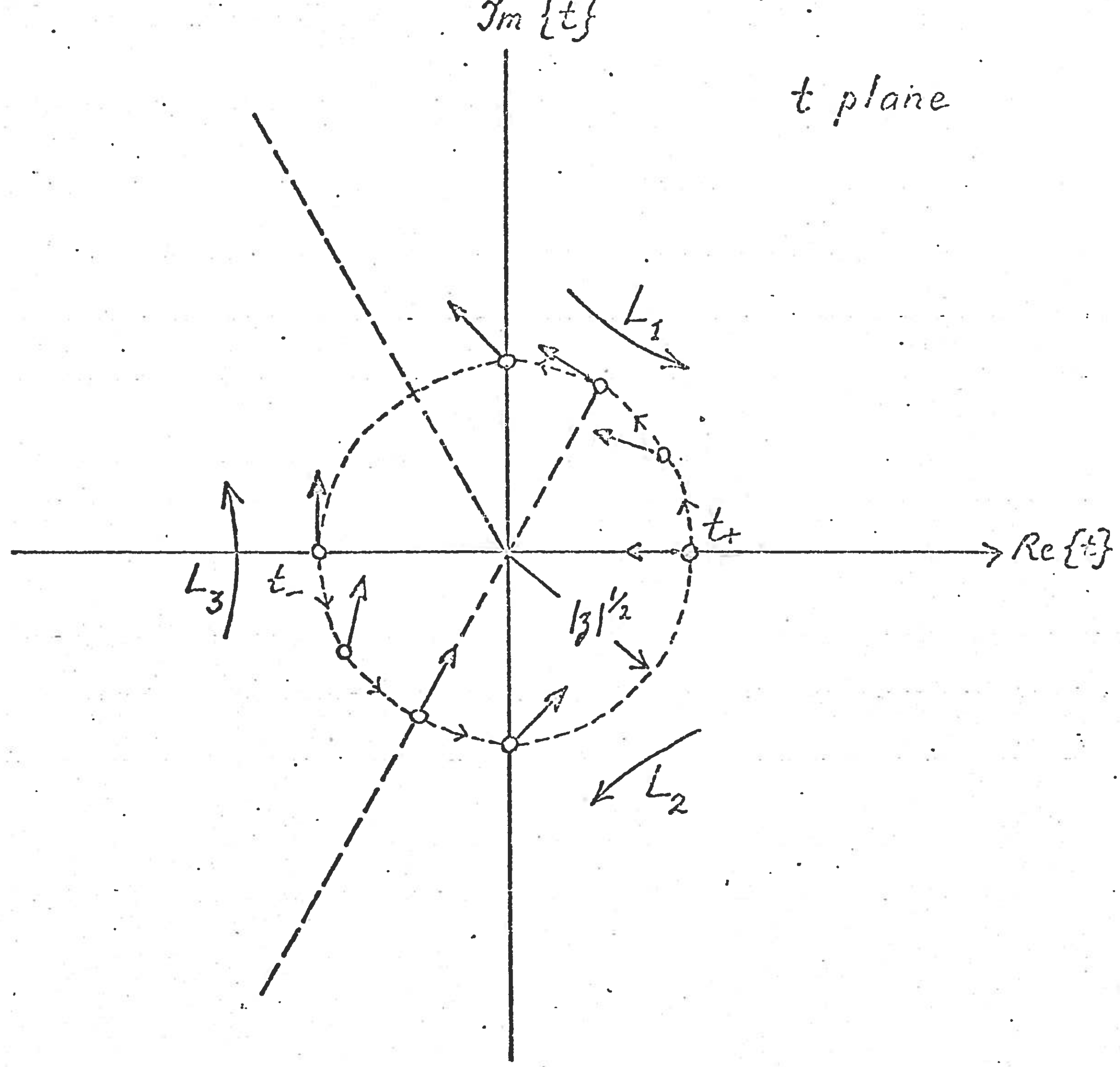


FIG. 5 - The t plane showing the location of the saddle points t_+ and t_-, $t_\pm = \pm z^{\frac{1}{2}} = \pm|z|^{\frac{1}{2}}e^{i\theta/2}$, for $\theta = 0$, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, π, the arrows indicating the tangents to the respective paths of steepest descents and the chosen sense of traversal through the corresponding S.P. The figure also shows the three sectors corresponding to the paths of integration L_1, L_2, and L_3 of Fig. 3.

analysis we discard by choice the upper sign, so from now on the radical, wherever it appears, will have a minus sign in front. Thus, making use of (2.9) we now have in the vicinity of the S.P.

$$x = -\sqrt{\pm}\ |z|^{\frac{1}{4}} e^{i\theta/4} w; \quad w = t - t_{\pm}, \tag{2.12}$$

from which we construct the following table:

t_+:- $x = -|z|^{\frac{1}{4}} e^{i\theta/4} w$

$\arg\{x\} = 0 = -\pi + \frac{\theta}{4} + \arg\{w\}$

$\arg\{w\} = \pi - \frac{\theta}{4}$

θ	$\arg\{w\}$
0	π
$\frac{\pi}{3}$	$11\pi/12$
$\frac{2\pi}{3}$	$5\pi/6$
π	$3\pi/4$

t_-:- $x = -i|z|^{\frac{1}{4}} e^{i\theta/4} w$

$\arg\{x\} = 0 = -\frac{\pi}{2} + \frac{\theta}{4} + \arg\{w\}$

$\arg\{w\} = \frac{\pi}{2} - \frac{\theta}{4}$

θ	$\arg\{w\}$
0	$\pi/2$
$\pi/3$	$5\pi/12$
$2\pi/3$	$\pi/3$
π	$\pi/4$

which is illustrated in Fig. 5. We assume that $z = |z|e^{i\theta}$ and that the modulus $|z|$ remains fixed while $\theta = \arg\{z\}$, varies from 0 to π. Negative values of θ are not considered, because the situation is perfectly symmetric. In Fig. 5 we show the location of the S.P. $t_{\pm} = \pm z^{\frac{1}{2}} = \pm|z|^{\frac{1}{2}} e^{i\theta/2}$, as $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$, the arrows indicating by their inclination the sense of traversal through the corresponding S.P. as given by the values of $\arg\{w\}$ tabulated above.

2.4.3 Leading Terms Corresponding to the S.P. $t_{\pm}$ — As shown above and as illustrated in Fig. 5 we have, for every value of $z = |z|e^{i\theta}$, two S.P. at $t = t_\pm$. It now becomes of interest to compute the leading terms of the asymptotic expansions of the integrals

$$I_\pm = \int_{C_\pm} e^{zt-\frac{1}{3}t^3}\,dt, \tag{2.13}$$

where $C_\pm$ denote the P.S.D. through the S.P. $t_\pm$, respectively. Making use of our familiar formula (1.50), we have at once

$$I_+ \sim \sqrt{\pi}\,\frac{e^{f(t_+)}}{\sqrt{-\frac{1}{2}f''(t_+)}} = \sqrt{\pi}\,\frac{e^{\zeta}}{-z^{\frac{1}{4}}}\{1 + O(\tfrac{1}{\zeta})\}; \tag{2.14}$$

$$I_- \sim \sqrt{\pi}\,\frac{e^{f(t_-)}}{\sqrt{-\frac{1}{2}f''(t_-)}} = \sqrt{\pi}\,\frac{e^{-\zeta}}{-iz^{\frac{1}{4}}}\{1 + O(\tfrac{1}{\zeta})\}, \tag{2.15}$$

where we have made use of (2.9) to compute $\sqrt{-\frac{1}{2}f''(t_\pm)}$ as $-\sqrt{t_\pm}$, in accordance with our chosen determination of the square root. The order of magnitude of the remainder in the above formulas is given by the square of the radius of convergence

$$\lambda = |f(t_+) - f(t_-)| = \tfrac{4}{3}\,z^{3/2} = 2\zeta, \tag{2.16}$$

in accordance with (2.8). We note that, when z is real and positive, $\zeta = \frac{2}{3}\,z^{3/2} > 0$ and I_+ grows exponentially as $\zeta\to\infty$, whereas I_- is exponentially attenuated. More generally we find, since

$$\zeta = \tfrac{2}{3}\,z^{3/2} = \tfrac{2}{3}|z|^{3/2}(\cos\tfrac{3\theta}{2} + i\sin\tfrac{3\theta}{2}) \tag{2.17}$$

that whenever $\mathrm{Re}\{\zeta\}>0$, I_+ is dominant and I_- is evanescent, and the converse is true when $\mathrm{Re}\{\zeta\}<0$. Thus, we have

$$-\tfrac{\pi}{3}<\theta<\tfrac{\pi}{3},\ \mathrm{Re}\{\zeta\}>0;\quad \text{and } -\pi<\theta<-\tfrac{\pi}{3} \text{ or } \tfrac{\pi}{3}<\theta<\pi,\ \mathrm{Re}\{\zeta\}<0. \tag{2.18}$$

2.4.4 Leading Term for Ai(z) where $z>0$.- We observe from Fig. 5 that, as long as $\theta = \arg\{z\}$ lies in the range $-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$, which is bisected by $\theta = 0$ or $z > 0$, then the S.P. t_- lies within the sector corresponding to the path of integration L_3. Since the asymptotes of the P.S.D. through t_- are independent of z and coincide with the asymptotes of the path L_3, we can write at once from (2.4) and (2.14)

$$Ai(z) = \frac{I_3(z)}{2\pi i} = \frac{I_-(z)}{2\pi i}$$

$$\sim \tfrac{1}{2}\, \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\zeta}\{1+O(\tfrac{1}{\zeta})\}, \quad -\frac{2\pi}{3}<\theta<\frac{2\pi}{3}, \qquad (2.19)$$

where $\zeta = \frac{2}{3} z^{3/2}$, and which is real and positive when $z > 0$.

2.4.5 Leading Term for Bi(z) when $z > 0$. - When z is rigorously real and positive we have, from (2.5) and Fig. 4, that the paths L_1 and L_2 can be deformed to pass through the S.P. t_- and t_+ to give us

$$2\pi\, Bi(z) = I_1 - I_2 = (-\tfrac{1}{2}I_- - I_+)+(\tfrac{1}{2}I_- - I_+) = -2I_+; \qquad (2.20)$$

when $\theta > 0$ slightly, the S.P. t_+ and t_- move off the real axis in the t plane as shown in Fig. 5, and according to (2.5) we now have

$$2\pi\, Bi(z) = I_1 - I_2 = -I_+ +(I_- - I_+) = I_- - 2I_+; \qquad (2.21)$$

and when $\theta < 0$ slightly, the S.P. t_+ and t_- move off the real axis in the opposite direction to that shown in Fig. 5, with the result that we now obtain

$$2\pi\ Bi(z) = I_1 - I_2 = (-I_- - I_+) + (-I_+) = -I_- - 2I_+ . \qquad (2.22)$$

We note that the average of (2.21) and (2.22) yields precisely (2.20), which is certainly valid in the limit $\theta \to 0$. In any case, the most important fact is that, for θ in the range $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$, we know from (2.18) that I_+ is dominant and I_- is evanescent and can be neglected when we are looking for the dominant term of an asymptotic expansion. Thus we have, regardless of which formula above we use,

$$Bi(z) = \frac{I_1(z) - I_2(z)}{2\pi} = -\frac{I_+(z)}{\pi}$$

$$\sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{\zeta}\{1 + O(\tfrac{1}{\zeta})\}, \quad -\frac{\pi}{3} < \theta < \frac{\pi}{3}, \qquad (2.23)$$

in which the latter form was obtained making use of the leading term (2.14) and wherein $\zeta = \frac{2}{3} z^{3/2}$ as previously defined. It is clear that Bi(z), like Ai(z) in (2.19), is real and positive when $z > 0$.

It is important to point out at this juncture that the apparent differences which stem from (2.20), (2.21) and (2.22), when examined asymptotically, constitute a classical example of the Stokes phenomenon, in which the various expressions differ depending on arg{z}. But it is essential to remember that, asymptotically, these differences are negligible in comparison with the dominant term.

2.5 LEADING TERM ASYMPTOTIC EXPANSION FOR $z < 0$

In the present section we shall endeavour to deduce leading term asymptotic expansions for Ai(z) and Bi(z) in sectors of arg{z} which include the negative half of the z axis and, therefore, are valid for $z < 0$. For the purpose we shall make use of the principle of analytic continuation by tracing the locus of the S.P. $t_{\pm}$ as arg{z} varies from 0 to π, much as we have already done in section 2.4.2 and Fig. 5, and then expressing Ai(z) and Bi(z) from (2.4) and (2.5) in terms of the leading terms I_+ and I_-, already derived in (2.14) and (2.15), but which we must now re-write in terms of $-z$, by putting

$$z = e^{i\pi}(ze^{-i\pi}) = e^{i\pi}(-z) \tag{2.24}$$

in which $-z > 0$ when z is real and negative. We shall see below that this process of analytic continuation brings to the fore the so-called Stokes phenomenon quite generally associated with the asymptotic expansions of analytic functions.

2.5.1 <u>Leading term for Ai(z) when $z < 0$</u>. - We have seen that, when $z > 0$ or more precisely when $-\frac{2\pi}{3}<\theta<\frac{2\pi}{3}$, the Airy function Ai(z) is given by (2.19) exclusively in terms of $I_-(z)$ as

$$Ai(z) = \frac{I_-(z)}{2\pi i}, \quad -\frac{2\pi}{3}<\theta<\frac{2\pi}{3}, \tag{2.25}$$

but now we find that, when $z < 0$ or more precisely when $\frac{2\pi}{3}<\theta<\frac{4\pi}{3}$, which includes the negative half of the real axis in the z plane, Eq. (2.4) and the analysis of Fig. 5 lead us to the expression

$$Ai(z) = \frac{I_3(z)}{2\pi i} = \frac{I_-(z)+I_+(z)}{2\pi i}, \quad \frac{2\pi}{3}<\theta<\frac{4\pi}{3}, \tag{2.26}$$

which is readily established from Fig. 5 for z rigorously real and negative. When we expand (2.25) and (2.26) asymptotically we of course obtain different results since we are talking about different values of $\arg\{z\}$, but the apparent discontinuity in terms of I_+ and I_-, which accrues when θ crosses the Stokes line at $2\pi/3$ is simply the manifestation of the Stokes phenomenon. To proceed with the leading term calculation of (2.26), we first compute from (2.24), for $z < 0$, the following quantities:

$$
\begin{aligned}
z^{\frac{1}{4}} &= e^{i\pi/4}(-z)^{\frac{1}{4}} \\
z^{\frac{1}{2}} &= e^{i\pi/2}(-z)^{\frac{1}{2}} = i(-z)^{\frac{1}{2}} \qquad \begin{cases} t_+ = e^{i\pi/2}(-z)^{1/2} \\ t_- = e^{3\pi i/2}(-z)^{1/2} \end{cases} \\
z^{3/2} &= e^{3\pi i/2}(-z)^{3/2} = -i(-z)^{3/2}; \quad \eta \equiv \tfrac{2}{3}(-z)^{3/2}
\end{aligned}
\tag{2.27}
$$

whence, recalling our choice of sign for the square root in (2.11), we now write making use of (2.9),

$$
\begin{aligned}
\sqrt{-\tfrac{1}{2}f''(t_+)} &= -\sqrt{t_+} = -e^{i\pi/4}(-z)^{\frac{1}{4}}; \\
\sqrt{-\tfrac{1}{2}f''(t_-)} &= -\sqrt{t_-} = -e^{3\pi i/4}(-z)^{\frac{1}{4}};
\end{aligned}
\tag{2.28}
$$

which follow from the 2nd of (2.27) upon recalling that $t_\pm = \pm z^{\frac{1}{2}}$. Thus, making use of (2.26), we have

$$
\begin{aligned}
2\pi i \,\mathrm{Ai}(z) = I_3 &= I_- + I_+ \\
&\sim \sqrt{\pi}\,\frac{e^{f(t_-)}}{\sqrt{-\frac{1}{2}f''(t_-)}} + \frac{\sqrt{\pi}\,e^{f(t_+)}}{\sqrt{-\frac{1}{2}f''(t_+)}} \\
&= \frac{-\sqrt{\pi}}{(-z)^{\frac{1}{4}}}\left\{\frac{e^{i\eta}}{e^{3\pi i/4}} + \frac{e^{-i\eta}}{e^{\pi i/4}}\right\},
\end{aligned}
\tag{2.29}
$$

where $\eta = \frac{2}{3}(-z)^{3/2}$ in accordance with the last of (2.27). Multiplying both sides of (2.29) by $-i$, and solving for

$\cos(\eta-\frac{\pi}{4}) = \cos(\eta+\frac{\pi}{4}-\frac{\pi}{2})$
$= \sin(\eta+\frac{\pi}{4})$

Ai(z) we obtain finally

$$Ai(z) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}}(-z)^{-\frac{1}{4}}e^{-\pi i/4}\{e^{i\eta}+ie^{-i\eta}\}$$

$$= \pi^{-\frac{1}{2}}(-z)^{-\frac{1}{4}}\cos(\eta-\tfrac{\pi}{4})\{1+O(\tfrac{1}{\eta})\},\ \tfrac{2\pi}{3}<\theta<\tfrac{4\pi}{3}\ , \qquad (2.30)$$

$\sin(\eta+\frac{\pi}{4})$

which is seen to be real when z is real and $z < 0$. Although (2.30) was derived assuming that $z < 0$, we know, as pointed out in (2.26) that the expression given is valid within the indicated range in arg{z}.

2.5.2 <u>Leading Term for Bi(z) when $z < 0$</u>. – Proceeding as in the preceding Section we have at once, from (2.5) and Fig. 5,

$$Bi(z) = \frac{I_1(z)-I_2(z)}{2\pi} = \frac{I_-(z)-I_+(z)}{2\pi}\ ,\ \frac{2\pi}{3}<\theta<\frac{4\pi}{3}\ , \qquad (2.31)$$

which should be contrasted with (2.20), (2.21) and (2.22) which abide when $z > 0$. The contrast in this case is once again an example of the Stokes phenomenon. Making use of the results tabulated in (2.27) and (2.28) we have, from (2.31),

$$2\pi\, Bi(z) = I_1 - I_2 = I_- - I_+$$

$$\sim \frac{-\sqrt{\pi}}{(-z)^{\frac{1}{4}}}\{\frac{e^{i\eta}}{e^{3\pi i/4}} - \frac{e^{-i\eta}}{e^{\pi i/4}}\}\ , \qquad (2.32)$$

where, it is recalled, $\eta = \frac{2}{3}(-z)^{3/2}$. Solving for Bi(z) we finally obtain

$$Bi(z) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}}(-z)^{-\frac{1}{4}}e^{-\pi i/4}\{ie^{i\eta}+e^{-i\eta}\}$$

$$= \pi^{-\frac{1}{2}}(-z)^{-\frac{1}{4}}\cos(\eta+\tfrac{\pi}{4})\{1+O(\tfrac{1}{\eta})\},\ \tfrac{2\pi}{3}<\theta<\tfrac{4\pi}{3}\ , \qquad (2.33)$$

which again is seen to be real when z is real and $z < 0$.

It is interesting at this juncture to contrast the asymptotic behaviour of $Ai(z)$ and $Bi(z)$ as deduced from the above and preceding Sections. We have seen that, as $z\to\infty$, $Bi(z)$ and $Ai(z)$ are monotonic, with $Bi(z)$ growing exponentially while $Ai(z)$ being exponentially attenuated. On the other hand, as $z\to-\infty$, both functions are highly oscillatory and, in fact, behave like $(-z)^{-\frac{1}{4}}\cos\{\frac{2}{3}(-z)^{3/2}\pm\frac{\pi}{4}\}$, which shows that the amplitudes decrease slowly as $(-z)^{-\frac{1}{4}}$, while the wavelength of their oscillations, defined from

$$\frac{2\pi(-z)}{\lambda} = \frac{2}{3}(-z)^{3/2} \quad \text{or} \quad \lambda = \frac{3\pi}{(-z)^{\frac{1}{2}}} \xrightarrow[z\to-\infty]{} 0$$

becomes gradually smaller as $z\to-\infty$ as indicated in the table below:

$-z$	$(-z)^{\frac{1}{2}}$	λ
1	1	9.42
4	2	4.71
9	3	3.14
16	4	2.35
25	5	1.88

2.6 TWO-TERM EXPANSION FORMULA FOR $Ai(z)$ and $z > 0$

We have seen that, when $F(w)=1$ in our basic integral (1.1), our two-term expansion formula reduces to the simple form (1.48), which we now propose to apply to the function

$Ai(z)$. To this end we must compute the necessary coefficients a_o and a_2 after reducing the integral (2.4) to the basic form (1.1) and deforming the path of integration L_3 to coincide with the path S.D. passing through $t = t_- = -z^{\frac{1}{2}}$, $z > 0$. For the purpose we write $t = -z^{\frac{1}{2}} + w$, whence we have

$$\begin{aligned} x^2 &= f(t_-)-f(t) = -\tfrac{2}{3}z^{3/2}-zt+\tfrac{1}{3}t^3 \\ &= -z^{\frac{1}{2}}w+\tfrac{1}{3}w^3 = w^2\{-z^{\frac{1}{2}}+\tfrac{1}{3}w\}, \end{aligned} \tag{2.34}$$

which, upon comparison with (1.36), yields the coefficients

$$c_o = -z^{\frac{1}{2}}; \quad c_1 = \tfrac{1}{3}; \quad c_2 = 0, \text{ etc.} \tag{2.35}$$

Then, making use of (1.40), we obtain

$$a_o = \frac{1}{-iz^{\frac{1}{4}}} \quad \text{and} \quad a_2 = \frac{-i}{z^{7/4}} \cdot \frac{5}{24}, \tag{2.36}$$

whence

$$\frac{a_2}{a_o} = -\frac{i}{z^{7/4}} \cdot \frac{5}{24} \cdot -iz^{\frac{1}{4}} = -\frac{5}{24}\,\frac{1}{z^{3/2}}. \tag{2.37}$$

Finally, substituting (2.36) and (2.37) into (1.48), making use of the definition (2.4), we obtain the desired two-term expansion formula,

$$Ai(z) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}e^{-\zeta}\{1-\frac{5}{48}\,z^{-3/2}+O(\zeta^{-2})\}, \tag{2.38}$$

where, we recall, $\zeta = \frac{2}{3}z^{3/2}$. We also recall that the above result was derived for $z > 0$, but that it is actually valid, like (2.19), in the wider range $-\frac{2\pi}{3}<\theta<\frac{2\pi}{3}$. We leave it to the interested reader to derive other two-term expansion formulas for $Ai(z)$ and $Bi(z)$ for all the cases not treated here.

2.7 SOLUTIONS OF THE ANTI-CANONICAL FORM OF AIRY'S EQUATION

As pointed out in the preamble to Section 2.2, the canonical form of Airy's equation, written in terms of the real variable x, may now be written as

$$u'' - xu = 0, \tag{2.39}$$

where $u = u(x)$ is the dependent variable. As shown in the preceding Sections, the form of the solutions of (2.39), when examined asymptotically, exhibit the following leading term expansions:

$$\begin{matrix} x>0 \\ x\to\infty \end{matrix} \quad \left\{ \begin{aligned} Ai(x) &= \tfrac{1}{2}\pi^{-\frac{1}{2}}x^{-\frac{1}{4}}e^{-\zeta}\{1+O(\tfrac{1}{\zeta})\} \\ Bi(x) &= \pi^{-\frac{1}{2}}x^{-\frac{1}{4}}e^{\zeta}\{1+O(\tfrac{1}{\zeta})\} \end{aligned} \right. \tag{2.40}$$

$$\begin{matrix} x<0 \\ x\to-\infty \end{matrix} \quad \left\{ \begin{aligned} Ai(x) &= \pi^{-\frac{1}{2}}(-x)^{-\frac{1}{4}}\cos(\eta-\tfrac{\pi}{4})\{1+O(\tfrac{1}{\eta})\} \\ Bi(x) &= \pi^{-\frac{1}{2}}(-x)^{-\frac{1}{4}}\cos(\eta+\tfrac{\pi}{4})\{1+O(\tfrac{1}{\eta})\} \end{aligned} \right. \tag{2.41}$$

where $\zeta = \frac{2}{3}x^{3/2}$ and $\eta = \frac{2}{3}(-x)^{3/2}$, with $\zeta > 0$ when $x > 0$ and $\eta > 0$ when $x < 0$ or $-x > 0$.

But now let us change x into -x in Airy's equation (2.39), and to emphasize that we have made a change in the independent variable let us put $v(x) = u(-x)$, thus introducing a new dependent variable. Thus we obtain what we have chosen to label the anti-canonical form of Airy's equation, namely

$$v'' + xv = 0, \tag{2.42}$$

where $v = v(x)$. A moment's reflection tells us at once that the solutions of (2.42) are trivially related to the solutions of (2.39). In fact, without doing further work we can assert that a fundamental set of solutions of (2.42) is given by Ai(-x) and Bi(-x), where Ai(x) and Bi(x) are the familiar Airy functions first defined by (2.4) and (2.5), and whose leading term asymptotic expansions, for z = x real, are given in (2.40) and (2.41). Thus, finally, the leading term asymptotic forms which are solutions of (2.42), become

$$\begin{array}{l} x>0 \\ x\to\infty \end{array} \left\{ \begin{array}{l} Ai(-x) = \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \cos(\zeta - \frac{\pi}{4})\{1+0(\frac{1}{\zeta})\} \\ Bi(-x) = \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \cos(\zeta + \frac{\pi}{4})\{1+0(\frac{1}{\zeta})\} \end{array} \right. \qquad (2.43)$$

$$\begin{array}{l} x<0 \\ x\to-\infty \end{array} \left\{ \begin{array}{l} Ai(-x) = \frac{1}{2}\pi^{-\frac{1}{2}} (-x)^{-\frac{1}{4}} e^{-\eta}\{1+0(\frac{1}{\eta})\} \\ Bi(-x) = \pi^{-\frac{1}{2}} (-x)^{-\frac{1}{4}} e^{\eta}\{1+0(\frac{1}{\eta})\} \end{array} \right. \qquad (2.44)$$

where, as before, $\zeta = \frac{2}{3}x^{3/2}$ and $\eta = \frac{2}{3}(-x)^{3/2}$. We shall presently make use of the above solutions in a study of Langer's method that we take up in the next Chapter.

3. LANGER'S METHOD, THE WKB APPROXIMATION, AND THE QUESTION OF TURNING POINTS

We now undertake, as an example and important application of the methods developed in the two preceding Chapters, a study of Langer's method which was devised many years ago, in conjunction with quantum mechanical problems, to deal with the question of turning points. More precisely, we recall that the WKB approximation provides asymptotic solutions of 2nd order linear differential equations which are perfectly adequate for values of the independent variable sufficiently removed from a turning point, that is, from such point (or points) at which the <u>coefficient</u> of the dependent variable either vanishes or becomes infinite. In these lectures, however, we shall confine ourselves exclusively to turning points of the vanishing class. In the vicinity of such points the WKB approximation utterly fails, whereas Langer's method provides an ingenious scheme which gives us at once a solution uniformly valid in the vicinity of a turning point and which blends smoothly with the asymptotic solutions valid far away from the turning point, such as might have been obtained, e.g., by the WKB approximation. Thus, we first present here an elementary introduction to the WKB method, making use of the asymptotic properties of the Airy functions discussed in Chapter 2 to illustrate the familiar application of the method as well as its failure in the vicinity of the turning point at the origin $x = 0$. Next, we give a fairly detailed exposition of the essence of Langer's method and, finally, we illustrate its application by considering a simple one-dimensional propagation problem with a single turning point.

3.1 PREAMBLE ON THE WKB METHOD

Consider first the scalar, one-dimensional Helmholtz equation

$$u'' + k^2 u(x) = 0 \tag{3.1}$$

in which x is a real variable and k is a real propagation constant, $k > 0$. As is well known, the solutions of (3.1) are given by the progressive and regressive waves

$$u(x) = A\, e^{\pm ikx} \tag{3.2}$$

where A is an arbitrary constant. The problem we wish to solve, however, arises when the square of the propagation constant in (3.1) is no longer a constant, but becomes a function of x. Thus, instead of (3.1), we now want to solve the equation

$$u'' + k_0^2\, g(x) u(x) = 0, \tag{3.3}$$

where k_0 is a constant, $k_0 > 0$, usually referred to as a "large parameter", and where $g(x)$ is a single-valued, continuous and differentiable real function of the real variable x. It can be shown that (3.3) admits a solution of the form

$$u(x) = e^{ik_0 f(x)} \{A_0(x) + \frac{A_1(x)}{k_0} + \frac{A_2(x)}{k_0{}^2} + \cdots\}, \tag{3.4}$$

in which we assume that $k_0 >> 1$, and in which $f(x)$, the function in the exponent, and the expansion amplitudes $A_0(x)$, $A_1(x)$, ..., are to be determined by substituting the expansion (3.4) into the left-hand side of (3.3), employing termwise differentiation and rearranging the terms by successive decreasing powers of k_0. Equating to zero the coefficients of successive powers of k_0, whose reciprocal is now being employed as an ordering parameter, yields a series of recursion differential equations with the aid of which we ought to be able, in principle, to determine the unknown functions $f(x)$, $A_0(x)$, $A_1(x)$, etc. This complicated procedure gives us the series (3.4) as formal solution of (3.3), but by now the interested reader will recognize that (3.4) is nothing more than an asymptotic expansion in the sense of Poincaré, such as we discussed in Chapter 1. In fact, the only difference is that the saddle point method of integration yields an asymptotic expansion for an integral, whereas the WKB method provides an asymptotic expansion for the solution of a linear 2nd order differential equation. But, it is well known that the solutions of such differential equations can sometimes be represented by means of suitable integral representations, which in turn can be treated asymptotically by the saddle point method of integration. Thus, when this is the case, both methods are completely equivalent: the saddle point method deals with an integral representation which may be the solution of a linear 2nd order differential equation, whereas the WKB method deals directly with the differential equation itself, but both methods yield equivalent asymptotic expansions. However, those of us who have dealt with both methods recognize that the integral representation method, when applicable, followed by saddle point integration, possesses distinct advantages over the WKB method, particularly as regards the matters of analytic continuation, the attendant Stokes phenomenon, and the computation of connection formulas among the various asymptotic forms. However, in all fairness to the WKB method, we must admit that very often we cannot readily obtain integral representations and, therefore, our only recourse is the WKB method.

These matters, however, fall outside the scope of the present lectures. We have already learned (Chapter 1), that leading term asymptotic formulas can be extremely useful. Hence, let us take another good look at the WKB method, but this time let us be content with the leading term. That is, instead of (3.4), let us now postulate a solution of the form

$$u(x) = A(x)e^{ik_o f(x)}, \tag{3.5}$$

where we still have to determine the function f(x) in the exponent and the amplitude function A(x). Substituting (3.5) into (3.3), and rearranging terms by decreasing powers of k_o, we obtain

$$k_o^2(g-f'^2)A + ik_o(2f'A'+f''A) + A'' = 0, \tag{3.6}$$

which can only be satisfied by equating to zero the coefficients of k_o^2, k_o, and $k_o^0 = 1$. But the last says that $A'' = 0$, which of course is not true in general. However, at this juncture, most elementary text-books provide us with the following "hand-waving" argument: if $k_o >> 1$ and if $g(x)$ in (3.3) varies very "gently" in one wavelength, then we may assume that $A''(x)$ in (3.6) can be neglected in comparison with the other two terms and, therefore, we may put $A'' = 0$ outright. This is what we call the leading term approximation of the WKB method, the validity of which can in any case be verified a posteriori.

Thus we proceed by first neglecting A'' in (3.6), and then equating to zero the coefficients of k_o^2 and k_o. The first yields

$$f'^2 = g; \quad f' = \pm\sqrt{g}; \quad f(x) = \pm\int^x \sqrt{g(s)}\,ds, \tag{3.7}$$

and the latter can be written as

$$\frac{2A'}{A} + \frac{f''}{f'} = 0 \tag{3.8}$$

which upon integration gives us

$$A^2 f' = \text{const}; \quad A = \frac{C}{\sqrt{f'}} = \frac{C}{\sqrt[4]{g}} . \tag{3.9}$$

Thus, finally, substituting (3.7) and (3.9) into the postulated solution (3.5), we obtain the familiar lowest order WKB approximation

$$u(x) = \frac{C}{\{g(x)\}^{\frac{1}{4}}}\, e^{\pm i k_o \int^x \sqrt{g(s)}\,ds} . \tag{3.10}$$

We note, first of all, that if $g(x) \equiv 1$ we merely recover, as it should be, the <u>exact</u> solution (3.2). We observe next that, if $g(x)\to 0$ as $x\to x_0$, say, then $u(x)\to\infty$ as $x\to x_0$, which quite dramatically points out to the failure of the WKB approximation in the vicinity of the turning point $x = x_0$.

Now a word as to the validity of the leading term approximation (3.10). Despite the frequent text-book pronouncements that k_0 must be a large parameter, we have so far made no use of this assumption in deriving (3.10). We show below that the lowest order WKB approximation (3.10) is completely equivalent to our leading term asymptotic formula (1.50) derived by the saddle point method of integration, but the important point to emphasize is that the proper interpretation of Watson's lemma has given us exact information as to <u>what</u> parameter must be large, and also has given us a fair estimate of the error associated with the leading term. Neither one of these two important facts can be deduced from the manner in which (3.10) was derived. The only thing we can say is that A'' in (3.6) ought to be negligible (in some fashion) for the validity of the lowest order approximation (3.10), in which x must be sufficiently far removed (in some fashion) from the nearest turning point (if any). All of this, of course, is most unsatisfactory from a mathematical point of view. However, we see from (3.9), that $A(x)\sim\{g(x)\}^{-\frac{1}{4}}$, whence

$$A'' \sim -\tfrac{1}{4}\, g^{-5/4} g'' + \frac{5}{16}\, g^{-9/4} g'^{2} . \tag{3.11}$$

Therefore, if $g(x)\equiv 1$ then $A''\equiv 0$ and, as we have seen, the WKB approximation becomes exact. If, on the other hand,

$g(x)$ is a linear function of x, say $g(x)=x$, then $A''\sim x^{-9/4}\to 0$ as $x\to\infty$, which would say that, in this case, the WKB method yields a "good" asymptotic approximation (as $x\to\infty$). Finally, and this is a case of common occurrence, assume that $g(x)\to$const as $x\to\infty$, in which case $A''\to 0$ and (3.10) again becomes a good approximation. Below we illustrate both cases by means of specific examples.

3.2 THE AIRY EQUATION AS AN IMPORTANT EXAMPLE

Let us now return to the anti-canonical form of Airy's equation

$$v'' + xv = 0, \tag{3.12}$$

which we introduced in Section 2.7, and whose asymptotic solutions are given by (2.43) and (2.44). Comparing this equation with (3.3) we see that $k_0\equiv 1$ (no large parameter of course), and $g(x)=x$. Thus, making use of (3.7), we have for $x > 0$:

$$f' = \pm\sqrt{g} = \pm\sqrt{x}$$

$$f = \pm\int_0^x \sqrt{s}\, ds = \pm\frac{2}{3}\, x^{3/2} = \pm\zeta$$

whence, from (3.10), we have at once

$$v(x) = \frac{\text{const}}{x^{\frac{1}{4}}}\, e^{\pm i\zeta},\quad x > 0, \tag{3.13}$$

which we see contains the essential features of the solutions given in (2.43). For example, we see that $Ai(-x)$ in (2.43) is a linear combination of the two linearly independent solutions given by (3.13). Proceeding similarly for $x < 0$, we now have from (3.7), after writing $s = -(-s)$,

$$f = \pm i \int_0^x \sqrt{-s}\, ds$$

$$= \pm i \int_0^{-x} \sqrt{s}\, ds = \pm i \tfrac{2}{3}(-x)^{3/2} = \pm i\eta$$

whence, from (3.10), we now have

$$v(x) = \frac{\text{const}}{(-x)^{\frac{1}{4}}} e^{\pm\eta}, \quad x < 0, \tag{3.14}$$

which we see at once exhibits the essential features of the solutions given in (2.44).

We have seen that the lowest order WKB approximation (3.10) gives us, except for constant factors, precisely the same asymptotic forms that we had earlier obtained for Airy's equation by the more sophisticated and more powerful method of steepest descents, and we can say that we have established the equivalence of both methods. Certainly we understand that, in the case of Airy's equation, the validity of the WKB approximation, as pointed out in the preceding Section, stems from the fact that A'' in (3.6) tends to vanish as $|x|\to\infty$, and this has nothing to do with the fact that $k_o=1$, and therefore is not a "large" parameter.

3.3 ON LANGER'S METHOD

Suppose we wish to solve the one-dimensional Helmholtz equation

$$\frac{d^2w}{dz^2} + k_o^2\, g(z)w(z) = 0, \quad k_o = \text{const}>0, \tag{3.15}$$

which is identical in form to (3.3), for the whole interval $-\infty<z<\infty$. We shall assume that $g(z)$ is a single-valued, continuous and differentiable real function of the real variable z subject to the following provisos:

$$\begin{aligned} &g(0) = 0 \\ &g(z) > 0 \text{ when } z > 0 \\ &g(z) < 0 \text{ when } z < 0 \\ &g(z) = g_0 z + \cdots; \quad g_0 = (dg/dz)_{z=0} \end{aligned} \tag{3.16}$$

which says that $z = 0$ is the only turning point. In the vicinity of $z = 0$, Eq. (3.15) becomes approximately

$$w'' + k_o^2\, g_0\, z\, w(z) = 0, \tag{3.17}$$

which is reminiscent of Airy's equation (3.12) and, therefore, near $z = 0$ we expect solutions of (3.15) to behave like solutions of (3.17). On the other hand, for $|z|\to\infty$ or, say, for $|z|>>1$, we can always find asymptotic solutions of (3.15) by the WKB approximation. The problem at hand, however, is to obtain a single solution uniformly valid in the whole interval, $-\infty<z<\infty$, which blends smoothly the solution about $z = 0$ with the asymptotic solutions. This problem has received considerable attention in the past from a number of investigators among which we note Gans (1915), Jeffreys (1924), Kramers (1926), Langer (1931, 1932, 1934, 1937, 1939), and Jones (1964). Below we present the essence of Langer's method as contained in his 1937 paper.

Langer introduces a transformation of both the dependent variable w and of the independent variable z in (3.15) to make it look like (3.17), or nearly so as we shall see. Langer's transformation is essentially a "stretching" operation which insures that solutions about z = 0 blend smoothly with solutions for $|z|\to\infty$. Following Langer, we introduce a new independent variable $\xi = \xi(z)$ and we factorize the dependent variable in the form

$$w(z) = u(z)v(\xi), \tag{3.18}$$

thus introducing a new dependent variable, $v(\xi)$. We now choose $u(z)$ to insure that the term $dv/d\xi$ is absent from the new equation. Substituting (3.18) into (3.15) we obtain

$$u\xi'^2 \frac{d^2v}{d\xi^2} + (u\xi'' + 2u'\xi')\frac{dv}{d\xi} + (u'' + k_0^2 gu)v(\xi) = 0. \tag{3.19}$$

The requirement that the term $dv/d\xi$ be absent from the final equation gives us at once the requirement

$$u\xi'' + 2u'\xi' = 0$$

or

$$\frac{\xi''}{\xi'} + 2\frac{u'}{u} = 0$$

which, upon integration, yields

$$u^2\xi' = \text{const}; \quad u = \frac{A}{\sqrt{\xi'}}, \tag{3.20}$$

which establishes a connection between the factor $u(z)$ in (3.18) and the new dependent variable $\xi = \xi(z)$ as yet undetermined. This latter functional dependence is obtained by putting

$$\xi\xi'^2 = g(z); \quad \xi^{\frac{1}{2}}\xi' = \pm\sqrt{g(z)}, \tag{3.21}$$

with the aid of which we have, from (3.20),

$$u = \frac{A}{\sqrt{\xi'}} = A\left(\frac{g}{\xi}\right)^{-\frac{1}{4}}. \tag{3.22}$$

Finally, introducing (3.20) and (3.21) into (3.19), we obtain the desired (simpler) equation

$$\frac{d^2v}{d\xi^2} + \left(\frac{u''}{u\xi'^2} + k_o^2\,\xi\right)v(\xi) = 0, \tag{3.23}$$

which looks nearly like (3.17) but for the extra term $u''/u\xi'^2$ that we shall presently dispose of.

The solution of our primitive equation (3.15) may now be written in the form

$$w(z) = A\left[\frac{g(z)}{\xi(z)}\right]^{-\frac{1}{4}} v(\xi), \tag{3.24}$$

where $\xi = \xi(z)$ in accordance with (3.21) and where $v(\xi)$ is a solution of (3.23). Now, recalling the provisos (3.16) and invoking the 2nd of (3.21), we choose $\xi(0) = 0$ and we choose $\xi(z)$ to have the same sign as $g(z)$ or z. Thus, discarding the minus sign in the 2nd of (3.21) and integrating we obtain, for $z \geq 0$,

$$\frac{2}{3}\xi^{3/2} = \int_0^z \sqrt{g(s)}\,ds \xrightarrow[z\to 0+]{} \int_0^z \sqrt{g_o\,s+\cdots}\,ds$$

$$= g_o^{\frac{1}{2}}\,\frac{2}{3}z^{3/2}$$

$$\xi = g_o^{1/3}\,z+\cdots, \quad z \geq 0, \tag{3.25}$$

and proceeding similarly for $z \leqslant 0$ (after multiplying both sides of the equation by $e^{3\pi i/2}$), we have

$$\tfrac{2}{3}(-\xi)^{3/2} = -\int_{0}^{z}\sqrt{-g(s)}\,ds$$

$$= \int_{0}^{-z}\sqrt{-g(-s)}\,ds \xrightarrow[z\to 0-]{} \int_{0}^{-z}\sqrt{g_0 s+\cdots}\;ds$$

$$= g_0^{\frac{1}{2}}\,\tfrac{2}{3}(-z)^{3/2}$$

$$-\xi = g_0^{1/3}(-z)+\cdots,\quad z \leqslant 0. \qquad (3.26)$$

The only task remaining for us is to obtain a solution of the reduced equation (3.23). For the purpose, suppose we merely neglect the obnoxious term $u''/u\xi'^2$, which we can do in lowest approximation as shown by Langer (1937), who also gives the necessary recursion equations to compute the higher order terms. Thus, to this approximation, Eq. (3.23) reduces to the simpler equation

$$\frac{d^2v}{d\xi^2} + k_0^2\,\xi\, v(\xi) = 0, \qquad (3.27)$$

which we propose to reduce further by introducing the change in scale

$$\xi = k_0^{-2/3}x;\quad k_0\xi^{3/2} = x^{3/2};\quad k_0\tfrac{2}{3}\xi^{3/2} = \tfrac{2}{3}x^{3/2}, \qquad (3.28)$$

with the aid of which (3.27) becomes

$$v'' + x\,v(x) = 0, \qquad (3.29)$$

which we recognize is our old friend (2.42), the anti-canonical form of Airy's equation, whose asymptotic solutions are tabulated in (2.43) and (2.44). Thus our task is now formally accomplished, for all that is required

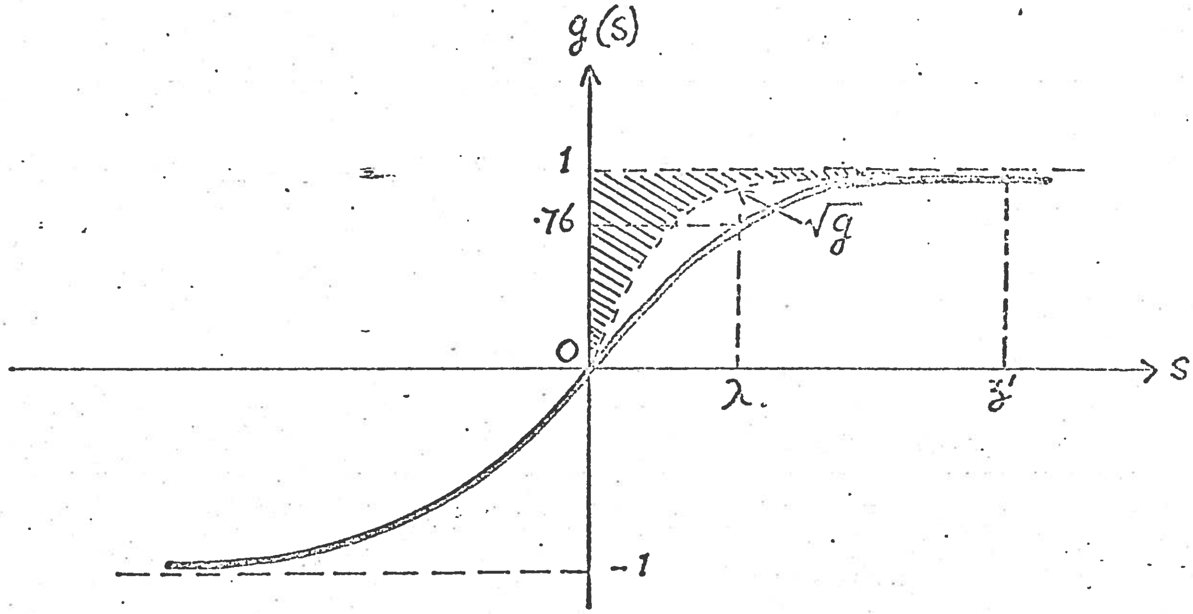


FIG. 6

is to substitute in (3.24) the appropriate Airy function (or linear combination thereof) which satisfies the asymptotic conditions of (3.15) as $|z|\to\infty$, conditions which we can always ascertain with the aid of the WKB approximation. However, all of these matters are best illustrated by considering a specific example.

3.4 ILLUSTRATIVE EXAMPLE

We now propose to construct a solution of the one-dimensional Helmholtz equation (3.15), subject to appropriate asymptotic forms as $|z|\to\infty$, by making use of the (lowest order) Langer's formula (3.24). For the purpose we select a specific example by putting

$$g(z)=\tanh\frac{z}{\lambda}=\frac{z}{\lambda}-\frac{1}{3}\left(\frac{z}{\lambda}\right)^3+\cdots;\quad g_0=\frac{1}{\lambda} \tag{3.30}$$

which we illustrate as $g = g(s)$ in Fig. 6. We notice that the selected function satisfies all the provisos of (3.16) with $g_o = 1/\lambda$, where λ is the scale length.

Our first problem is to determine the functional relation $\xi = \xi(z)$ as $|z|\to\infty$. Thus, from (3.25) and Fig. 6 with $z' >> \lambda$, we have for $z > 0$:

$$\begin{aligned}\tfrac{2}{3}\xi^{3/2} &= \int_0^z\sqrt{g(s)}\,ds \xrightarrow[z\to\infty]{} \int_0^{z'}\sqrt{g(s)}\,ds+\int_{z'}^{z}1\cdot ds\\ &= z-z'+\int_0^{z'}\sqrt{g(s)}\,ds = z-z_0,\end{aligned} \tag{3.31}$$

where z_0 is a constant which, making use of (3.30), becomes

$$z_0 = z' - \int_0^{z'} \sqrt{g(s)}\, ds = \int_0^{z'} \{1-\sqrt{\tanh \tfrac{s}{\lambda}}\}ds$$

$$= \lambda \int_0^{z'/\lambda} \{1-\sqrt{\tanh t}\}dt \simeq \lambda \int_0^{\infty}\{1-\sqrt{\tanh t}\}dt,\quad z' >> \lambda, \tag{3.32}$$

and which, according to Fig. 6, is represented by the shaded area. We notice that z_0 is proportional to λ and that $z_0 \to 0$ as $\lambda \to 0$. Thus, recalling (3.28), and making use of (3.31), we have the limit

$$\tfrac{2}{3}\, x^{3/2} = k_0 \tfrac{2}{3} \xi^{3/2} \xrightarrow[z\to\infty]{} k_0(z-z_0). \tag{3.33}$$

For $z < 0$, we proceed similarly making use of (3.26), to obtain

$$\tfrac{2}{3}(-\xi)^{3/2} = -\int_0^{z} \sqrt{-g(s)}\, ds = \int_0^{-z} \sqrt{-g(-s)}ds = \int_0^{-z} \sqrt{g(s)}ds, \tag{3.34}$$

where the latter form arises from the fact that our choice (3.30) satisfies the condition $g(s) = -g(-s)$. Thus, we have at once

$$\tfrac{2}{3}(-\xi)^{3/2} \xrightarrow[z\to-\infty]{} -z-z_0 = -(z+z_0), \tag{3.35}$$

and from (3.28)

$$\tfrac{2}{3}(-x)^{3/2} = k_0 \tfrac{2}{3}(-\xi)^{3/2} \xrightarrow[z\to-\infty]{} -k_0(z+z_0), \tag{3.36}$$

where z_0 is given by (3.32).

Next, we return to our primitive equation (3.15) and examine its asymptotic behaviour in the light of the chosen $g(z) = \tanh \frac{z}{\lambda}$. Thus, we have at once

$$z\to\infty,\quad g\to 1,\quad w'' + k_0^2 w = 0,\quad w \sim e^{\pm i k_0 z}$$

$$z\to-\infty,\quad g\to -1,\quad w^2 - k_0^2 w = 0,\quad w \sim e^{\pm k_0 z}$$

in which the asymptotic forms were obtained by mere inspection without recourse to the WKB approximation because of the simple asymptotic behaviour of $g(z)$ in (3.30). We would now like to obtain a solution of (3.15) which asymptotically behaves as follows

$$\begin{aligned} &w(z) \sim e^{-ik_0 z} + Re^{ik_0 z} \text{ as } z \to \infty; \\ &\text{and} \\ &w(z) \sim Se^{k_0 z} \text{ as } z \to -\infty, \end{aligned} \tag{3.37}$$

that is, we are sending an incident (regressive) wave of unit amplitude, $w_i = e^{-ik_0 z}$, and we are bouncing off a reflected (progressive) wave, $w_r = Re^{ik_0 z}$, which we detect for $z \to \infty$. When $z \to -\infty$, we are aware of an exponentially attenuated (evanescent) or scattered wave, $w_s = Se^{k_0 z}$. Our problem is to compute the reflection coefficient R and the "scattering" coefficient S, making use of Langer's method and, specifically, of his lowest order formula (3.24).

We recall that $v(\xi)$ in (3.24) is a solution of Airy's equation (2.42) and, examining the asymptotic forms (2.43) and (2.44), in the light of (3.33) and (3.35), we conclude that the only Airy function satisfying the asymptotic conditions (3.37) is $Ai(-k_0^{2/3}\xi)$, in which $x = k_0^{2/3}\xi$ in accordance with (3.28). Thus, we assert that, putting $A = 2\pi^{\frac{1}{2}}k_0^{1/6}$ in (3.24), gives us the desired solution, namely

$$w(z) = 2\pi^{\frac{1}{2}}k_0^{1/6}\left(\frac{\xi}{g}\right)^{\frac{1}{4}}Ai(-k_0^{2/3}\xi), \tag{3.38}$$

and all we have to do now is to show that (3.38) does indeed satisfy the asymptotic conditions (3.37), except of course for constant coefficients which in turn should allow us to compute the coefficients R and S in (3.37). First, let us examine the behaviour of (3.38) as $z\to-\infty$,

$$w(z)\xrightarrow[z\to-\infty]{} 2\pi^{\frac{1}{2}}k_o^{1/6}\left(\frac{-\xi}{-g}\right)^{\frac{1}{4}}\cdot\tfrac{1}{2}\pi^{-\frac{1}{2}}(-k_o^{2/3}\xi)^{-\frac{1}{4}}e^{k_o(z+z_o)}$$

$$\sim e^{k_o(z+z_o)}, \tag{3.39}$$

in complete accord with the 2nd of (3.37). To arrive at the above result we took into account the fact that $-g\to1$ as $z\to-\infty$ (see Fig. 6), and we made use of the 1st of (2.44) and of (3.36) to arrive at the asymptotic form of $Ai(-k_o^{2/3}\xi)$, $\xi=\xi(z)$, indicated above. Finally, making use of the 1st of (2.43) and of (3.33), and proceeding as above we have, as $z\to\infty$,

$$w(z)\xrightarrow[z\to\infty]{} 2\pi^{\frac{1}{2}}k_o^{1/6}\left(\frac{\xi}{g}\right)^{\frac{1}{4}}\cdot\pi^{-\frac{1}{2}}(k_o^{2/3}\xi)^{-\frac{1}{4}}\cos\{k_o(z-z_o)-\tfrac{\pi}{4}\}$$

$$\sim e^{ik_o(z-z_o)-i\pi/4}+e^{-ik_o(z-z_o)+i\pi/4}, \tag{3.40}$$

which exhibits the superposition of a progressive wave and a regressive wave, in complete accord with (3.37).

Comparing (3.40) with the 1st of (3.37) gives us at once the reflection coefficient R,

$$R=\frac{e^{-ik_oz_o-i\pi/4}}{e^{ik_oz_o+i\pi/4}}=-i\ e^{-2ik_oz_o}, \tag{3.41}$$

and, similarly, comparing (3.39) with the 2nd of (3.37) yields the scattering coefficient S,

$$S=\frac{e^{k_oz_o}}{e^{ik_oz_o+i\pi/4}}=e^{k_oz_o-ik_oz_o-i\pi/4}, \tag{3.42}$$

in both of which z_0 is given by (3.32). Thus, we have shown that our proposed solution (3.38), based on Langer's lowest order formula, gives us a solution of (3.15) which is uniformly valid for $-\infty<z<\infty$ and which satisfies the imposed asymptotic conditions as $|z|\to\infty$.

But we must not forget that we have arrived at the preceding results by altogether neglecting the term $u''/u\xi'^2$ in Eq. (3.23) to obtain the simpler equation (3.27), which we know how to solve in terms of Airy functions. We show below that the neglect of this term is completely equivalent, asymptotically, to neglecting the term A'' in (3.6) when dealing with the WKB approximation, and so we expect our asymptotic results (3.39) and (3.40), as well as the reflection and scattering coefficients (3.41) and (3.42), to be excellent approximations as $|z|\to\infty$ <u>regardless</u> of what value we assign to the scale length λ in our specific example (3.30). On the other hand, as shown below, we expect our lowest order solution (3.38) to represent a good approximation to the solution of our primitive equation (3.15), in the vicinity of the turning point $z = 0$, only if λ is bounded away from zero, more precisely if $\lambda = O(\lambda_0)$ or larger, where λ_0 is the wavelength associated with $k_0 = 2\pi/\lambda_0$.

To proceed with the analysis we note, from (3.20), that $u = A\xi'^{-\frac{1}{2}}$, whence

$$u'' = \tfrac{3}{4}\, A\, \xi'^{-5/2}\, \xi''^2 - \tfrac{1}{2}\, A\, \xi'^{-3/2}\, \xi'''$$

and, dividing by $u\xi'^2 = A\xi'^{3/2}$, we obtain for the neglected term the expression

$$\frac{u''}{u\xi'^2} = \tfrac{3}{4}\,\xi'^{-4}\,\xi''^2 - \tfrac{1}{2}\,\xi'^{-3}\,\xi''' ; \qquad (3.43)$$

which is quite general so far, but which we shall now examine in the light of our specific example. First, let us consider the asymptotic behaviour. From (3.31) we have

$$\tfrac{2}{3}\,\xi^{3/2} \xrightarrow[z\to\infty]{} z - z_0 \simeq z, \qquad (3.44)$$

since z_0, as given by (3.32), is a finite constant proportional to our scale length λ. Then, from (3.44), casting aside numerical factors which prove irrelevant in the present analysis we have, as $z\to\infty$,

$$\begin{aligned} \xi &\sim z^{2/3} \\ \xi' &\sim z^{-1/3} \\ \xi'' &\sim z^{-4/3} \\ \xi''' &\sim z^{-7/3} \end{aligned}$$

and, substituting these values in (3.43), we find for the neglected term the behaviour

$$\frac{u''}{u\xi'^2} \sim z^{-4/3} \xrightarrow[z\to\infty]{} 0, \qquad (3.45)$$

with a completely analogous result as $z\to-\infty$. Thus we conclude that, asymptotically, the neglected term can have no effect on our lowest order results. The above conclusions are valid for any finite $\lambda \geq 0$. Putting $\lambda = 0$ converts our function $g(s)$ in Fig. 6 into a step-function and, from (3.32), we find $z_0 = 0$ when $\lambda = 0$. In this case we obtain, from (3.41) and (3.42), the exact reflection and scattering coefficients

$$R = -i \text{ and } S = e^{-i\pi/4}, \tag{3.46}$$

which can be verified by elementary methods.

The situation in the vicinity of the turning point $z = 0$ is, however, something else. To proceed with the analysis, we make use of (3.30) to obtain the power series expansion

$$\sqrt{g(s)} = \lambda^{-\frac{1}{2}} s^{\frac{1}{2}} - \frac{1}{6}\lambda^{-5/2} s^{5/2} + \cdots, \quad s \geqslant 0,$$

and returning to (3.25) we compute, for $z > 0$,

$$\tfrac{2}{3}\xi^{3/2} = \int_0^z \sqrt{g(s)}\,ds = \tfrac{2}{3}\lambda^{-\frac{1}{2}} z^{3/2} - \frac{1}{21}\lambda^{-5/2} z^{7/2} + \cdots$$

from which we deduce in succession

$$\begin{aligned}
\xi &= \lambda^{-1/3}\{z - \frac{\lambda^{-2}}{21} z^3 + \cdots\} \\
\xi' &= \lambda^{-1/3}\{1 - \frac{\lambda^{-2}}{7} z^2 + \cdots\} \\
\xi'' &= -\tfrac{2}{7}\lambda^{-7/3} z + \cdots \\
\xi''' &= -\tfrac{2}{7}\lambda^{-7/3}
\end{aligned}$$

Substituting the above values into (3.43) we find for the neglected term the expansion

$$\frac{u''}{u\xi'^2} = \frac{1}{7}\lambda^{-4/3}\{1 + \frac{6}{7}\lambda^{-2} z^2 + \cdots\}, \tag{3.47}$$

which is certainly valid as $z \to 0$ for $z \geqslant 0$. To lowest order, then, we find that the coefficient of $v(\xi)$ in (3.23), becomes

$$\begin{aligned}
\frac{u''}{u\xi'^2} + k_0^2\xi \xrightarrow[z\to 0+]{} &\ \frac{1}{7}\lambda^{-4/3} + k_0^2\lambda^{-1/3} z + O(z^2) \\
&\cong \lambda^{-1/3} k_0^2\{z + \frac{1}{7k_0^2\lambda}\},
\end{aligned} \tag{3.48}$$

which says two things: (1) that λ, our scale length in (3.30), must be bounded away from zero, $\lambda > 0$; and (2) that, in lowest order approximation, the turning point no longer occurs at $z = 0$ as in (3.15), but at

$$z = -\frac{1}{7k_o^2\lambda} = -\frac{\lambda_o}{14\pi(k_o\lambda)} , \qquad (3.49)$$

where λ_o is the wavelength associated with the wave phenomenon as $z\to\infty$, that is, $k_o = 2\pi/\lambda_o$. We conclude that, if $14\pi(k_o\lambda)>>1$, and if λ_o itself is small (which means that k_o is large), then the shift (3.49) in the location of the turning point becomes negligible and our proposed solution (3.38) becomes a good approximation uniformly valid over the entire interval $-\infty<z<\infty$. Notice that this is the first time, in the present set of lectures, when we had to invoke <u>explicitly</u> a large parameter, namely k_o^2.

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