222 A. 10/30

Today

Linearized Equation.

1. From last time.

$$m_s n_s \left[\frac{\partial}{\partial t} \underline{v}_s + \underline{v}_s \cdot \nabla \underline{v}_s \right] = q_s n_s \underline{E} - \nabla \cdot \overleftrightarrow{P}_s$$

- high frequency. $\rightarrow \frac{\partial}{\partial t}$ is large.
- $| q_s n_s \vec{E}_s | >> | \nabla P_s | \longrightarrow$ neglect plasma pressure.
- breakdown for low freq. because $\frac{\partial}{\partial t}$ is small $\rightarrow$ # cannot neglect pressure.

Linearized High freq. response. $\longrightarrow$ Plasma wave equation.

(I) $\frac{\partial}{\partial t} \underline{v}_s = \frac{q_s}{m_s} \underline{E} - \nu_c \underline{V}$

(II) $\frac{\partial}{\partial t} n_s + \nabla (n_{os} \underline{v}_s) = 0$

(III) $\nabla \cdot \vec{E} = 4\pi \sum_s \tilde{n}_s q_s$

(IV) $\nabla \times \vec{B} = \frac{4\pi}{c} \sum_s q_s n_{os} \underline{v}_s + \frac{1}{c} \frac{\partial}{\partial t} \underline{E}$

(V) $\boxed{\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \underline{B}}$ $\longrightarrow$ Faraday's Eq. is true whether or not there is a plasma.

- $\underline{V}_{os} = 0$
- $n_s = n_{os} + \tilde{n}_s$

  ↑ Time independent.

- Take $\frac{\partial}{\partial t}$ of (II)

$$\frac{\partial^2}{\partial t^2} \tilde{n}_s + \nabla \cdot \left(n_s \frac{q_s}{m_s} \underline{E} \right) = 0$$

$$\boxed{\frac{\partial^2}{\partial t^2} \tilde{n}_s + \frac{n_{os} q_s}{m_s} \nabla \cdot \left(4\pi \sum_j q_j \tilde{n}_j \right) = 0} \longrightarrow **$$

For electrons. $q_s = -e$, $m_s = m$

For ions. $q_s = Ze$, $m_s = M$

$\longrightarrow$ zeroth-order density. $n_{oe} = Z n_{oi}$

for electrons.

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\tilde{n}_e + \dfrac{4\pi n_{0e} e^2}{m}\tilde{n}_e = \dfrac{4\pi n_{0e} e}{m} Ze\,\tilde{n}_i \\[2ex] \dfrac{\partial^2}{\partial t^2}\tilde{n}_e + \dfrac{4\pi n_{0i} (Ze)^2}{M}\tilde{n}_i = \dfrac{4\pi n_{0i} (Ze)}{M} e\,\tilde{n}_e \end{cases}$$

• Define

$$\omega_{pe}^2 = \frac{4\pi n_{0e} e^2}{m}$$

$$\omega_{pi}^2 = \frac{4\pi n_{0i} (Ze)^2}{M} = \frac{4\pi n_{0e} Z e^2}{M}$$

∴ The coupled oscillator equation becomes

$$\boxed{\begin{aligned} \frac{\partial^2}{\partial t^2}\tilde{n}_e + \omega_{pe}^2 \tilde{n}_e &= \frac{M}{m}\omega_{pi}^2 \tilde{n}_i \\ \frac{\partial^2}{\partial t^2}\tilde{n}_i + \omega_{pi}^2 \tilde{n}_i &= \frac{m}{M}\omega_{pe}^2 \tilde{n}_e \end{aligned}}$$

→ Find Normal Modes

→ Spatial Derivatives are not present.

Assume $\tilde{n}_j = \tilde{n}_{0j}\, e^{-i\omega t}$

solve for eigenmodes

So, our goal is,

- solve for ω's
- solve for motion of the eigenmodes.

$$(-\omega^2 - \omega_{pe}^2)\,\tilde{n}_e = \frac{M}{m}\omega_{pi}^2\,\tilde{n}_i$$

$$(-\omega^2 - \omega_{pi}^2)\,\tilde{n}_i = \frac{m}{M}\omega_{pe}\,\tilde{n}_e$$

$$\longrightarrow (-\omega^2 + \omega_{pe}^2)\,\cancel{\tilde{n}_e} = \cancel{\frac{M}{m}}\,\omega_{pi}^2 \cdot \cancel{\frac{m}{M}}\,\frac{\omega_{pe}^2}{(-\omega^2 + \omega_{pi}^2)}\,\cancel{\tilde{n}_e}$$

$$(-\omega^2 + \omega_{pe}^2)(+\omega^2 - \omega_{pi}^2) = \omega_{pe}^2\,\omega_{pi}^2$$

$$\omega^4 - \omega^2(\omega_{pe}^2 + \omega_{pi}^2) + \cancel{\omega_{pe}^2\omega_{pi}^2} = \cancel{\omega_{pe}^2\omega_{pi}^2}$$

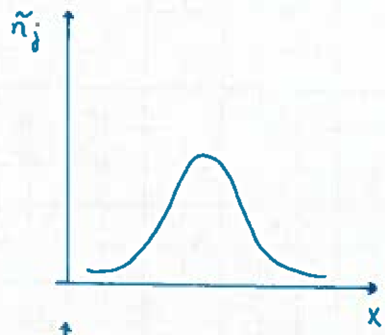
2 roots.

- $\omega = \pm\sqrt{\omega_{pe}^2 + \omega_{pi}^2}$ → OK root because high frequency.
- $\omega = 0$ → Bad root! (NOT CORRECT)

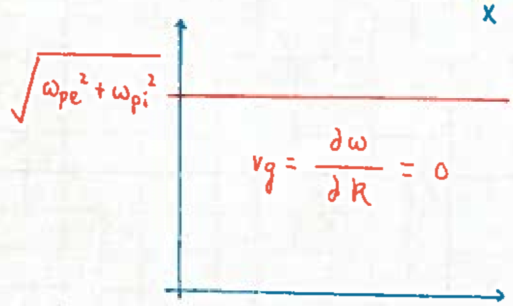
Why is the $\omega = 0$ term incorrect?

$$\boxed{\frac{\partial}{\partial t} \tilde{v}_s = \frac{q_s}{m_s} \underline{E}}$$

Because there is no other force to balance the $\vec{E}$ field. hence, $\omega = 0$ root is not valid.

- 

Because $\exists!$ spatial dependence, therefore, a disturbance will not spread in space, which is counterintuitive.

- 

$\longrightarrow$ ω is independent of k in this approximation ("cold" plasma oscillators)

- Where is the plasma dielectric?

$$\frac{\partial}{\partial t} n_s = -\nabla \cdot (n_s \underline{v_s})$$

$$\frac{\partial}{\partial t} n_s = \nabla \cdot \left(\frac{n_s \underline{v_s}}{i\omega} \right)$$

also. $$\frac{\partial}{\partial t} \underline{v_s} = \frac{q_s}{m_s} \underline{E}$$

$$\boxed{v_s = \frac{q_s \underline{E}}{-i\omega m}}$$

Go to Poisson's Eq.

$$\boxed{\begin{aligned} \nabla \cdot \underline{E} &= \sum_s 4\pi \tilde{n}_s q_s \\ &= \nabla \cdot \left(\sum_s \frac{\cancel{4\pi} n_s q_s^2}{\cancel{4\pi}\, \omega^2 m} \vec{E} \right) \end{aligned}}$$

$$\nabla \cdot \left[\left(1 - \sum_s \left(\frac{\omega_{ps}^2}{\omega^2} \right) \right) \vec{E} \right] = 0.$$

Recall.

$$\nabla \cdot \vec{D} = 0.$$

$$\longrightarrow \boxed{\epsilon(\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}} \longrightarrow$$ Dielectric of plasma for high-frequency response.

NOTE:

$$\epsilon(\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

is valid even if plasma is non-uniform....

- Further on $\longrightarrow$ temperature corrections.
- Electrostatic waves.

2. E + M waves $\longrightarrow$ w/ 2-moments High freq. description.

$$\underbrace{\nabla \times \underline{B}}_{\text{vacuum feature}} = \underbrace{\frac{4\pi}{c} \sum_s q_s n_{os} \underline{v}_s}_{\underline{j} \text{ is the actual plasma current.}} + \underbrace{\frac{1}{c} \frac{\partial}{\partial t} \vec{E}}_{\text{vacuum feature.}}$$

apply $\left(\frac{\partial}{\partial t}\right)$

$$\nabla \times \left| -c \nabla \times \underline{E} \right| = \frac{4\pi}{c} \sum_s q_s n_{os} \frac{\partial}{\partial t} \underline{v}_s + \frac{1}{c} \frac{\partial^2}{\partial t^2} \vec{E}$$

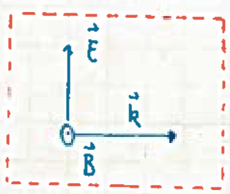
$$\boxed{-\nabla \times \nabla \times \underline{E} = \frac{4\pi}{c^2} \sum_s \frac{n_{os} q_s^2}{m_s} \underline{E} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \underline{E}}$$

$$\nabla \times \nabla \times \underline{E} = \nabla(\nabla \cdot \underline{E}) - \nabla^2 \underline{E}$$

Consider EM mode, i.e., $\boxed{\nabla \cdot \underline{E} = 0}$

$$\nabla^2 \underline{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \underline{E} - \frac{1}{c^2} \sum_s \omega_{ps}^2 \underline{E} = 0 \quad \longrightarrow$$

NO Fourier analysis yet...



For harmonic time-dependence

$$\underline{E}(\underline{r}, t) = \underline{E}(\underline{r}) e^{-i\omega t}.$$

$$\nabla^2 \underline{E}(\underline{r}) + \frac{\omega^2}{c^2} \underline{E}(\underline{r}) - \frac{1}{c^2} \sum_s \omega_{ps}^2 \underline{E}(\underline{r}) = 0$$

$$\longrightarrow \nabla^2 \underline{E}(\underline{r}) + \frac{\omega^2}{c^2} \underbrace{\left[1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \right]}_{\epsilon(\omega)} \underline{E}(\underline{r}) = 0.$$

Also, Fourier analyze in space.

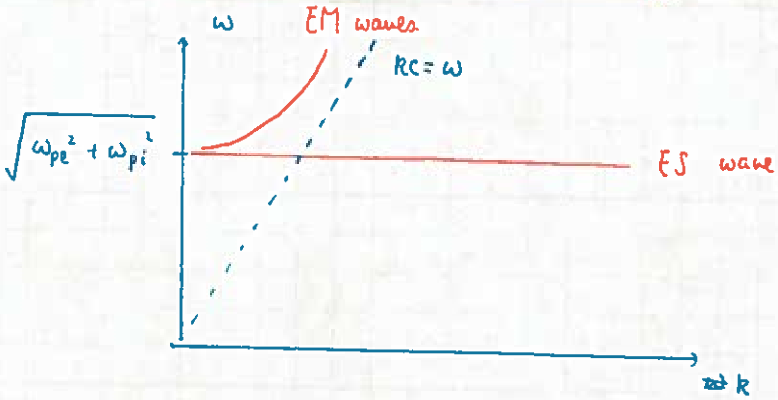
$$\vec{E}(\underline{r};t) = \underline{E}_0 \, e^{i\vec{k}\cdot\vec{x} - i\omega t}.$$

$$\nabla^2 \underline{E}(\underline{r}) + k_0^2 (\epsilon(\omega)) = 0$$

$$\rightarrow \boxed{(-k^2 + k_0^2 \epsilon(\omega)) \, \underline{E}(\underline{r}) = 0.}$$

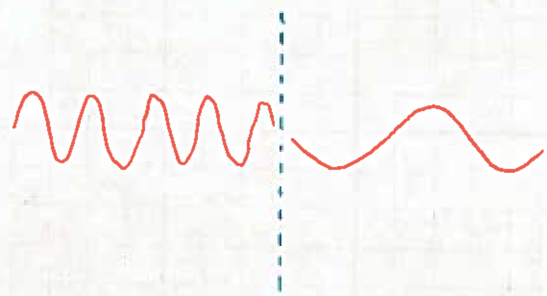
$$k^2 = k_0^2 \, \epsilon(\omega)$$

$$\boxed{\omega^2 = k^2 c^2 + (\omega_{pe}^2 + \omega_{pi}^2)}$$



2 properties

- $$k^2 = k_0^2 \, \epsilon(\omega)$$
  $$\quad \llcorner \; \epsilon(\omega) < 1$$



i.e.

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\right)$$

$\rightarrow$ k in plasma is less than k in vacuum

- Group velocity.

$$2k c^2 = 2\omega \frac{\partial \omega}{\partial k}. \longrightarrow c^2 = \frac{\omega}{k} \frac{\partial \omega}{\partial k}.$$

$$\boxed{v_{group} \, v_{ph} = c^2}$$