XXVII. Magnetic Charge II

In the previous sections, we have considered the magnetic fields produced by steady currents with some attention to the attendant vector potential. As we have indicated at various points, an alternative source of a static magnetic field would be static magnetic charge, if such exists. We would here like to consider a few consequences for the vector potential corresponding to such a magnetic field.

Let a magnetic charge, g , be located at the origin so that the magnetic field satisfies

$$\vec{\nabla}\cdot\vec{B} = 4\pi g\,\delta(\vec{r}) \quad , \tag{27.1}$$

which has the solution

$$\vec{B} = g\frac{\vec{r}}{r^3} = -\vec{\nabla}\frac{g}{r} \quad . \tag{27.2}$$

Away from the origin, $\vec{B}$ is divergenceless,

$$\vec{\nabla}\cdot\vec{B} = 0 \quad ,$$

so we would once again expect $\vec{B}$ to be the curl of a vector potential,

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad . \tag{27.3}$$

However, this cannot be true everywhere since

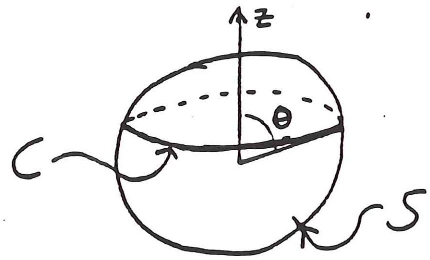
$$\oint d\vec{S}\cdot\vec{\nabla}\times\vec{A} = \int (d\vec{r})\;\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0 \quad , \tag{27.4}$$

while (27.1) implies for a closed surface surrounding the magnetic charge

$$\oint d\vec{S}\cdot\vec{B} = \int (d\vec{r})\ \vec{\nabla}\cdot\vec{B} = 4\pi g\ . \tag{27.5}$$

We now want to find a vector potential that satisfies (27.3) almost everywhere. The simplest possibility is that this equation fails to hold on a line, which we may take to be the +z axis. We apply Stokes' theorem in the form

$$\int_C \vec{A}\cdot d\vec{r} = -\int_S \vec{B}\cdot d\vec{S} \tag{27.6}$$



where C is a circle of constant θ on a sphere of radius r about the origin and S is the lower portion of the spherical surface bounded by C. Equation (27.6) holds since (27.3) is true everywhere on S. [The minus sign appears because we use the outward normal to the surface S.] The surface integral follows trivially from (27.2),

$$\int_S \vec{B}\cdot d\vec{S} = \frac{g}{r^2}\, 2\pi r^2(1 + \cos\theta)\ . \tag{27.7}$$

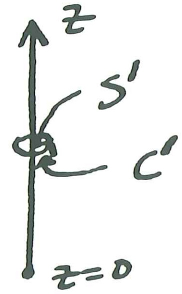
An obvious solution of (27.6) is then

$$\vec{A} = A_\phi\ \hat{\phi} \tag{27.8}$$

where

$$A_\phi = -\frac{g}{r}\,\frac{1 + \cos\theta}{\sin\theta}\ . \tag{27.9}$$

The structure of the singularity on the z axis is now isolated by taking the limit $\theta \to 0$, where C' is an infinitesimal circle about the z axis and S' is the enclosed area.



Since (27.10) picks out $(\nabla \times A)_z$, which has the singularity $-4\pi g\ \delta(x)\ \delta(y)$ on the +z axis, we conclude that the magnetic field can be expressed everywhere by

$$\vec{B} = \vec{\nabla} \times \vec{A} + 4\pi g\ \delta(x)\ \delta(y)\ \eta(z)\ \hat{z} \tag{27.11}$$

where $\eta(z)$ is the step function. This result can be confirmed by noting that $\vec{B}$ has the correct divergence, (27.1),

$$\vec{\nabla}\cdot\vec{B} = 0 + 4\pi g\ \delta(x)\ \delta(y)\ \delta(z) \quad . \tag{27.12}$$

The vector potential (27.8) is an example of a whole class of potentials that yield the correct magnetic field except for a one-dimensional set of points, a curve. On this curve, called a string, $\vec{A}$ is singular, whereas the magnetic field is regular, being the curl of $\vec{A}$ plus a compensating singularity on the string.

Warren Moon

Lecture 1

XXVIII. RETARDED GREEN'S FUNCTION AND LIENARD-WIECHERT POTENTIALS

28-1. Potentials and Gauges

In the previous sections, we have primarily confined ourselves to the discussion of electrostatics and magnetostatics. We will now study in general how time-dependent electromagnetic fields are produced by arbitrary charges and currents. In vacuum, we recall that Maxwell's equations are [see (1.27)]

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{j} \quad , \qquad (28.1a)$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \quad , \qquad (28.1b)$$

$$-\vec{\nabla} \times \vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B} \quad , \qquad (28.1c)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad , \qquad (28.1d)$$

where ρ is the charge, and $\vec{j}$ the current density, and we have assumed that no magnetic charge is present. Notice that the local charge conservation law,

$$\vec{\nabla}\cdot\vec{j} + \frac{\partial}{\partial t}\rho = 0 \quad , \qquad (28.2)$$

is not an independent statement, but is derivable from (28.1a) and (28.1b).

To solve Maxwell's equations, we first recognize that the last two equations, (28.1c) and (28.1d), make no reference to charge or current, and they can be identically satisfied by introducing potentials through the definitions

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \qquad (28.3)$$

$$\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\phi \quad . \qquad (28.4)$$

$A_\mu \to A_\mu + \partial_\mu \lambda$

$F_{i\mu} = \partial_i A_\mu - \partial_\mu A_i = \partial_i \partial_\mu \lambda - \partial_\mu \partial_i \lambda = 0$

As we have observed previously, in Subsection 9-4, the potentials $\vec{A}$ and ϕ are not uniquely defined. Since the magnetic field is the curl of $\vec{A}$, it is unchanged when a gradient is added to $\vec{A}$,

$$\vec{A} \to \vec{A} + \vec{\nabla}\lambda \ , \tag{28.5}$$

where λ is an arbitrary function. In order that this new choice of vector potential does not alter the electric field, (28.4), it is necessary to simultaneously replace the scalar potential by

$$\phi \to \phi - \frac{1}{c}\frac{\partial}{\partial t}\lambda \ . \tag{28.6}$$

This new set of potentials, (28.5) and (28.6), is as acceptable as the original one since only the fields $\vec{B}$ and $\vec{E}$ are physically measurable quantities. This arbitrariness in the choice of potentials is called the gauge freedom of the theory, while the corresponding transformations are called gauge transformations. In the following, we will exploit this freedom in the process of solving the differential equations for the potentials.

Upon substituting the constructions of $\vec{B}$ and $\vec{E}$ in terms of potentials, (28.3) and (28.4), into the first set of Maxwell's equations, we find, from (28.1a),

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{1}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A} \right) + \frac{4\pi}{c}\vec{j} \ ,$$

or

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\vec{A} = -\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi \right) + \frac{4\pi}{c}\vec{j} \ , \tag{28.7}$$

and, from (28.1b),

$$-\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = 4\pi\rho \quad . \tag{28.8}$$

This is a pair of coupled second order differential equations for $\vec{A}$ and ϕ, which may be simplified by utilizing the gauge freedom in defining the potentials. The two most convenient and common choices of gauge are discussed below.

(1) The radiation gauge (or Coulomb gauge) is defined by the condition

$$\vec{\nabla}\cdot\vec{A} = 0 \quad . \tag{28.9}$$

That we can always make this choice was shown in Subsection 23-3. In this gauge, (28.7) and (28.8) reduce to

$$-\nabla^2\phi = 4\pi\rho \quad , \tag{28.10}$$

$$-\Box^2\vec{A} = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\phi) \quad , \tag{28.11}$$

where

$$\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{28.12}$$

is the d'Alembertian. The equation for ϕ, (28.10), is just the same as that in electrostatics (hence the origin of the term "Coulomb gauge") so that ϕ is, in principle, known. The structure on the right hand side of (28.11) is proportional to an effective current, the second term of which is present in order that it be divergenceless:

$$\vec{\nabla}\cdot\left[\vec{j} - \vec{\nabla}\left(\frac{1}{4\pi}\frac{\partial}{\partial t}\phi\right)\right] = \vec{\nabla}\cdot\vec{j} - \frac{1}{4\pi}\frac{\partial}{\partial t}(\nabla^2\phi)$$

$$= \vec{\nabla}\cdot\vec{j} + \frac{\partial}{\partial t}\rho = 0 \quad ,$$

where the last equality follows from charge conservation, (28.2). This relation also entails the consistency of the choice of the radiation gauge in that if we set $\vec{\nabla}\cdot\vec{A}$ equal to zero at one time, it remains zero for all time, since

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$$-\square^2(\vec{\nabla}\cdot\vec{A}) = \frac{4\pi}{c}\vec{\nabla}\cdot\left[\vec{j} - \vec{\nabla}\left(\frac{1}{4\pi}\frac{\partial}{\partial t}\phi\right)\right] = 0 \quad .$$

(2) The Lorentz gauge condition is a relation between vector and scalar potentials,

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi = 0 \quad . \qquad (28.13)$$

In this gauge, the equations for $\vec{A}$ and ϕ have the symmetrical form,

$$-\square^2\vec{A} = \frac{4\pi}{c}\vec{j} \quad , \qquad (28.14)$$

$$-\square^2\phi = 4\pi\rho \quad . \qquad (28.15)$$

The consistency of this gauge choice again follows from the fact that charge is conserved,

$$-\square^2\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi\right) = \frac{4\pi}{c}\left(\vec{\nabla}\cdot\vec{j} + \frac{\partial\rho}{\partial t}\right) = 0 \quad .$$

28-2. Green's Function and Potentials in the Lorentz Gauge

In the following, we will solve the differential equations, (28.14) and (28.15), for the potentials in the Lorentz gauge. Since the potentials are linearly related to their sources, they may be expressed in terms of a Green's function,

$$\phi(\vec{r},t) = \int (d\vec{r}\,')\, dt'\, G(\vec{r}-\vec{r}\,', t-t')\, \rho(\vec{r}\,',t') \quad , \qquad (28.16)$$

$$\vec{A}(\vec{r},t) = \int (d\vec{r}')\, dt'\, G(\vec{r}-\vec{r}', t-t') \frac{1}{c} \vec{j}(\vec{r}',t') \quad . \tag{28.17}$$

This Green's function, $G(\vec{r}-\vec{r}', t-t')$, is a function only of relative positions and times because of translational invariance in unbounded space. Since ϕ satisfies (28.15), this Green's function obeys the differential equation

$$-\Box^2 G(\vec{r}-\vec{r}', t-t') = 4\pi\, \delta(\vec{r}-\vec{r}')\, \delta(t-t') \quad , \tag{28.18}$$

which is a four-dimensional generalization of the three-dimensional Green's function equation we studied in electrostatics,

$$-\nabla^2 G(\vec{r}-\vec{r}') = 4\pi\, \delta(\vec{r}-\vec{r}') \quad . \tag{28.19}$$

To solve (28.18), we will analyze its time dependence by making use of the exponential representations

$$\delta(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \quad , \tag{28.20}$$

$$G(\vec{r}-\vec{r}', t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, G_\omega(\vec{r}-\vec{r}') \quad , \tag{28.21}$$

where G_ω satisfies the three-dimensional differential equation,

$$-\left(\nabla^2 + \frac{\omega^2}{c^2} \right) G_\omega(\vec{r}-\vec{r}') = 4\pi\, \delta(\vec{r}-\vec{r}') \quad . \tag{28.22}$$

In the static limit, $\omega \to 0$, (28.22) reduces to (28.19), the solution of which is Coulomb's potential, (12.3):

$$G_{\omega=0}(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} \quad . \tag{28.23}$$

Since G_ω depends only on $\vec{r}-\vec{r}'$, we may set $\vec{r}' = 0$, without loss of generality in the following discussion. Also, since we are now looking for a spherically symmetrical solution for G_ω, it is natural to use a spherical coordinate system in which the Laplacian here reduces to

$$\nabla^2 \to \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) . \qquad (28.24)$$

Therefore, for $r > 0$, we wish to solve the homogeneous equation

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{\omega^2}{c^2}\right] G_\omega(\vec{r}) = 0 , \qquad (28.25)$$

subject to the boundary condition that there is a point charge at the origin. The consequence of this requirement is most conveniently extracted by integrating (28.22) over a sphere of vanishing radius r_0 about the origin,

$$\begin{aligned} 4\pi &= -\int (d\vec{r})\, \vec{\nabla}\cdot(\vec{\nabla}G_\omega) \\ &= -\int dS\, \nabla_r G_\omega = -4\pi\, r^2 \frac{d}{dr} G_\omega\Big|_{r_0\to 0} , \end{aligned}$$

or

$$-r^2\frac{d}{dr}G_\omega(\vec{r})\Big|_{r_0\to 0} = 1 . \qquad (28.26)$$

[We have noted that the ω^2/c^2 term in the differential equation does not contribute to the integral since

$$\frac{\omega^2}{c^2}G_\omega \sim \frac{1}{r} , \quad \text{as} \quad r\to 0 ,$$

which has vanishing volume integral as $r_0 \to 0$.] To solve (28.25), we introduce g_ω , defined by

$$G_\omega = \frac{1}{r} g_\omega \quad , \tag{28.27}$$

which satisfies the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{\omega^2}{c^2} \right) g_\omega(r) = 0 \; , \quad \text{for} \quad r > 0 \quad , \tag{28.28}$$

where we have used

$$r^2 \frac{d}{dr} G_\omega = r \frac{d}{dr} g_\omega - g_\omega \quad ,$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} G_\omega \right) = \frac{1}{r} \frac{d^2}{dr^2} g_\omega \quad . \tag{28.29}$$

The independent solutions of (28.28) have the form

$$g_\omega \sim e^{\pm i \frac{\omega}{c} r} \quad ,$$

and the corresponding forms for G_ω are

$$G_\omega(r) = \frac{C}{r} e^{\pm i \frac{\omega}{c} r} \quad , \quad \text{for} \quad r > 0 \quad . \tag{28.30}$$

For either choice of $+$ or $-$ sign, the constant C is determined by the boundary condition (28.26) to be

$$C = 1 \quad . \tag{28.31}$$

Therefore, we have two fundamental solutions to (28.22),

$$G_\omega(\vec{r}-\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|}\, e^{\pm i \frac{\omega}{c} |\vec{r}-\vec{r}\,'|} \quad , \tag{28.32}$$

while from (28.21) we now obtain the space-time form of the Green's functions,

$$\begin{aligned} G(\vec{r}-\vec{r}\,', t-t') &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{|\vec{r}-\vec{r}\,'|}\, e^{i\omega\left[\pm \frac{1}{c} |\vec{r}-\vec{r}\,'| - (t-t')\right]} \\ &= \frac{1}{|\vec{r}-\vec{r}\,'|}\, \delta\left(\pm \frac{1}{c} |\vec{r}-\vec{r}\,'| - (t-t')\right) \quad . \end{aligned} \tag{28.33}$$

What is implied by the use of the $+$ or $-$ sign in (28.33)? The choice of the $+$ sign leads to the retarded Green's function,

$$G_{\text{ret.}}(\vec{r}-\vec{r}\,', t-t') = \frac{1}{|\vec{r}-\vec{r}\,'|}\, \delta\left(\frac{1}{c} |\vec{r}-\vec{r}\,'| - (t-t')\right) \quad , \tag{28.34}$$

which is nonvanishing when

$$t = t' + \frac{1}{c} |\vec{r}-\vec{r}\,'| \quad . \tag{28.35}$$

This means that the signal propagates with the speed of light c from the source (at time t') to the observer (at time t); the effect occurs later than the cause. If we pick the $-$ sign, we obtain the advanced Green's function

$$G_{\text{adv.}} = \frac{1}{|\vec{r}-\vec{r}\,'|}\, \delta\left(\frac{1}{c} |\vec{r}-\vec{r}\,'| + t-t'\right) \quad , \tag{28.36}$$

which is non-zero when

$$t = t' - \frac{1}{c} |\vec{r}-\vec{r}\,'| \quad , \tag{28.37}$$

the signal arriving at the observer before it is emitted by the source. Since the latter is not in accordance with elementary ideas of causality, we adopt [illegible] emission and absorption are not [illegible]. So use a mixture

the retarded Green's function as the solution which satisfies the correct time boundary condition. (Actually, both retarded and advanced Green's functions are useful in physics.) We can now obtain explicit expressions for the potentials by substituting (28.34) into (28.16) and (28.17),

$$\phi(\vec{r},t) = \int (d\vec{r}\,')\, dt' \, \frac{\delta\left(\frac{1}{c}\,|\vec{r}-\vec{r}\,'| - (t-t')\right)}{|\vec{r}-\vec{r}\,'|}\, \rho(\vec{r}\,',t') \quad , \tag{28.38}$$

$$\vec{A}(\vec{r},t) = \int (d\vec{r}\,')\, dt' \, \frac{\delta\left(\frac{1}{c}\,|\vec{r}-\vec{r}\,'| - (t-t')\right)}{|\vec{r}-\vec{r}\,'|}\, \frac{1}{c}\,\vec{j}(\vec{r}\,',t') \quad . \tag{28.39}$$

Integrating over t', we obtain the so-called retarded or Lienard-Wiechert potentials,

$$\phi(\vec{r},t) = \int (d\vec{r}\,')\, \frac{1}{|\vec{r}-\vec{r}\,'|}\, \rho\left(\vec{r}\,',\, t - \frac{1}{c}\,|\vec{r}-\vec{r}\,'|\right) \quad , \tag{28.40}$$

$$\vec{A}(\vec{r},t) = \int (d\vec{r}\,')\, \frac{1}{|\vec{r}-\vec{r}\,'|}\, \frac{1}{c}\,\vec{j}\left(\vec{r}\,',\, t - \frac{1}{c}\,|\vec{r}-\vec{r}\,'|\right) \quad . \tag{28.41}$$

These results are elementary generalizations of the potentials for electrostatics and magnetostatics, but now reflecting the finite propagation speed of light.

XXIX. ELECTROMAGNETIC RADIATION—FIELD POINT OF VIEW

29-1. Asymptotic Potentials and Fields

As we have seen, the distinction between static electric and magnetic fields and those produced by time-varying charges and currents is that, in the latter case, we must take into account the finite propagation speed of light. The fact that the time of emission is different from the time of detection is the basis for the existence of electromagnetic radiation, as we will now see. Sufficiently near the source, retardation effects can be neglected. That is, if ρ and $\vec{j}$ do not change appreciably over a time scale of $|\vec{r}-\vec{r}\,'|/c$, the time of emission, t', can be effectively replaced by the time of detection, t, in (28.40) and (28.41), which is to say that the potentials are not very different from those occurring in statics. On the contrary, far away from the source, retardation effects become important. Choosing the origin of coordinates to lie inside the charge distribution, having characteristic dimension a, we may use the expansion, for $r >> a$,

$$\frac{1}{r-\vec{n}\cdot\vec{r}\,'} = \frac{1}{r}\left(1+\frac{\vec{n}\cdot\vec{r}\,'}{r}\right) = \frac{1}{r}+\frac{\vec{n}\cdot\vec{r}\,'}{r^2}$$

$$|\vec{r}-\vec{r}\,'| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}\,' + r'^2} \simeq r - \vec{n}\cdot\vec{r}\,' + O\left(\frac{1}{r}\right) , \qquad (29.1)$$

to derive the asymptotic form of the potentials in a Lorentz gauge [cf. (28.40) and (28.41)],

$$\phi(\vec{r},t) \simeq \frac{1}{r}\int (d\vec{r}\,')\, \rho\left(\vec{r}\,', t - \frac{r}{c} + \frac{1}{c}\vec{n}\cdot\vec{r}\,'\right) , \qquad (29.2)$$

$$\vec{A}(\vec{r},t) \simeq \frac{1}{r}\int (d\vec{r}\,')\, \frac{1}{c}\vec{j}\left(\vec{r}\,', t - \frac{r}{c} + \frac{1}{c}\vec{n}\cdot\vec{r}\,'\right) , \qquad (29.3)$$

$\frac{\vec{n}\cdot\vec{r}}{c}\ \omega$

where

$$\vec{n} = \frac{\vec{r}}{r} \qquad (29.4)$$

$\nabla \vec{n}\cdot\vec{r}' = \vec{r}'\cdot\nabla\vec{n}$

$\frac{1}{r-\vec{n}\cdot\vec{r}'} = \frac{1}{r}\left(1+\frac{\vec{n}\cdot\vec{r}'}{r}\right) = \frac{1}{r} + \frac{\vec{n}\cdot\vec{r}'}{r^2}$

is the unit vector in the direction toward the observation point. In the above equations, the $\vec{n}\cdot\vec{r}'$ term in the expansion of $1/|\vec{r}-\vec{r}'|$ has been deleted since it gives rise to a $1/r^2$ term in the potential, while it has been retained in the expression for the time of emission, t' ,

$$t' \simeq t - \frac{r}{c} + \frac{1}{c}\vec{n}\cdot\vec{r}' \equiv t_r . \tag{29.5}$$

The last term in t_r reflects the finite amount of time it takes radiation to propagate across the source, which can be significant if the source distribution changes rapidly, or, more precisely, when a typical frequency of oscillation of the source distribution is of order c/a.

The fields at large distances can now be calculated by substituting (29.2) and (29.3) into (28.3) and (28.4), and by using the evaluation

$$\begin{aligned}\vec{\nabla}\left[\frac{1}{r} f\left(t - \frac{r}{c} + \frac{1}{c}\vec{n}\cdot\vec{r}' \right)\right] &= -\frac{\vec{n}}{r^2} f(t_r) \\ &\quad - \left[\frac{\vec{n}}{c} + \frac{1}{c}\frac{1}{r}\vec{n}\times(\vec{n}\times\vec{r}')\right]\frac{\partial}{\partial t}\left[\frac{1}{r} f(t_r)\right] \\ &\simeq -\frac{\vec{n}}{c}\frac{\partial}{\partial t}\left[\frac{1}{r} f(t_r)\right] . \end{aligned} \tag{29.6}$$

We see that because of the appearance of r in the time dependences, the fields behave as $1/r$ rather than the behavior $1/r^2$ characteristic of statics, and in particular, for $r >> a$, the field strengths are

$$\vec{B}(\vec{r},t) \simeq -\frac{1}{c}\vec{n}\times\frac{1}{r}\int (d\vec{r}')\,\frac{1}{c}\frac{\partial}{\partial t}\vec{j}(\vec{r}',t_r) , \tag{29.7}$$

$$\vec{E}(\vec{r},t) \simeq \frac{\vec{n}}{c}\frac{1}{r}\int (d\vec{r}')\,\frac{\partial}{\partial t}\rho(\vec{r}',t_r) - \frac{1}{c}\frac{1}{r}\int (d\vec{r}')\,\frac{1}{c}\frac{\partial}{\partial t}\vec{j}(\vec{r}',t_r) . \tag{29.8}$$

$\frac{\partial}{\partial t}\rho$

$t_r = t - |r - r'|$ $\partial_{t_r} \rho(r', t_r) + \nabla' j(r', t_r) = 0$

$\partial_t \rho(r', t_r) - \nabla' j(r', t_r) +$

These two terms in (29.8) can be further combined by using the local charge conservation condition (28.2),

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$$\frac{\partial}{\partial t} \rho(\vec{r}', t_r) = -\vec{\nabla}' \cdot \vec{j}(\vec{r}', t_r) + \frac{\vec{n}}{c} \cdot \frac{\partial}{\partial t_r} \vec{j}(\vec{r}', t_r) \quad , \tag{29.9}$$

where we have used the fact that the divergence operator in (28.2) acts only on the spatial arguments of $\vec{j}$, while $\vec{\nabla}'$ in (29.9) also differentiates the $\vec{r}'$ dependence of t_r . The first integral in (29.8) may be simplified through the use of $(r', t$

$$\int (d\vec{r}')\, \vec{\nabla}' \cdot \vec{j} = 0 \quad , \tag{29.10}$$

since the charge distribution is bounded, and the remaining terms involving the time derivative of $\vec{j}$ can be combined by means of the identity

$$\vec{V} - \vec{n}(\vec{n} \cdot \vec{V}) = -\vec{n} \times (\vec{n} \times \vec{V}) \tag{29.11}$$

to read

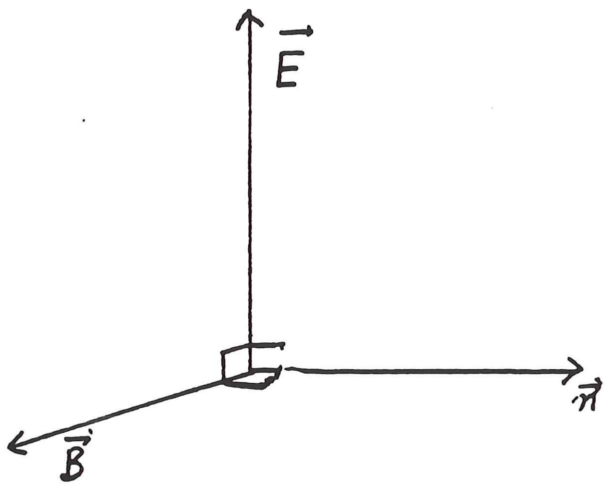
$$\begin{aligned} \vec{E}(\vec{r}, t) &= \vec{n} \times \left[\frac{\vec{n}}{c} \times \frac{1}{r} \int (d\vec{r}')\, \frac{1}{c} \frac{\partial}{\partial t} \vec{j}(\vec{r}', t_r) \right] \\ &= -\vec{n} \times \vec{B}(\vec{r}, t) \quad . \end{aligned} \tag{29.12}$$

We observe that, far from the source distribution, $\vec{E}$ and $\vec{B}$ are perpendicular to each other,

$$\begin{aligned} \vec{E} &= -\vec{n} \times \vec{B} \quad , \\ \vec{B} &= \vec{n} \times \vec{E} \quad , \end{aligned} \tag{29.13}$$

are perpendicular to the direction of propagation $\vec{n}$, and have equal magnitude,

$$E^2 = B^2 \quad .$$



These are the same characteristics seen in Subsection 3-4, where we considered the propagation of electromagnetic waves along a single direction, in terms of the flow of energy and momentum.

29-2. Angular Distribution of Radiated Power

Next we ask at what rate does this time-varying charge and current distribution radiate energy. The amount of energy flowing across a unit area per unit time is expressed by Poynting's vector, (3.5),

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} \quad , \tag{29.14}$$

which points in the direction of propagation $\vec{n}$. Substituting the asymptotic expressions for the fields, (29.7) and (29.12), into (29.14), we may write the rate of energy radiated per unit area in terms of the current distribution:

$$\begin{aligned}\vec{n}\cdot\vec{S} &= \frac{c}{4\pi}(\vec{n} \times \vec{E})\cdot\vec{B} = \frac{c}{4\pi}B^2 \\ &= \frac{1}{4\pi r^2}\frac{1}{c^3}\left[\vec{n} \times \int (d\vec{r}\,')\frac{\partial}{\partial t}\vec{j}(\vec{r}\,',t_r)\right]^2 \quad .\end{aligned} \tag{29.15}$$

no power if J is constant magnetostatics

Rather than the energy crossing an element of area da, we would instead like the energy radiated into a solid angle $d\Omega$,

$$d\Omega = \frac{da}{r^2} \ , \qquad r^2 d\Omega = da \tag{29.16}$$

since the latter measure is independent of how far away the observer is from the source. Therefore, the amount of energy radiated per unit time (the power) per unit solid angle in the direction $\vec{n}$ is

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \left[\vec{n} \times \int (d\vec{r}\,') \frac{\partial}{\partial t} \vec{j}(\vec{r}\,', t_r) \right]^2 , \tag{29.17}$$

while the total power radiated is obtained by integrating this over all solid angles,

$$P = \int d\Omega \left(\frac{dP}{d\Omega} \right) . \tag{29.18}$$

This finite energy flow at large distances is a consequence of the $1/r$ behavior of the fields, which, in turn, arises from the time variation of the current density.

Lecture 2.

29-3. Radiation by an Accelerated Charged Particle

Let us first apply the above general result, (29.17), to the simple example of a particle, with charge e, moving with a velocity $\vec{v}$, small compared with the speed of light, $v/c \ll 1$. If $\vec{R}(t')$ is the position of the charged particle, the corresponding current density is

$$\vec{j}(\vec{r}\,', t') = e\vec{v}(t')\, \delta(\vec{r}\,' - \vec{R}(t')) . \tag{29.19}$$

For this situation, the time of emission, (29.5), of the radiation may be approximated by

$$t_r = t - \frac{r}{c} + \frac{1}{c}\,\vec{n}\cdot\vec{R}(t') \qquad t' = t_r$$

$$\simeq t - \frac{r}{c} \equiv t_e\ , \qquad \text{for } \frac{v}{c} \ll 1\ , \tag{29.20}$$

since $|\vec{R}(t')|$ is bounded by v times a characteristic time. Therefore, the integral in (29.17) is immediately evaluated to be

$$\int (d\vec{r}\,')\,\frac{\partial}{\partial t}\,\vec{j}(\vec{r}\,',t_r) \simeq \frac{d}{dt_e}\int (d\vec{r}\,')\,\vec{j}(\vec{r}\,',t_e) = e\,\frac{d\vec{v}}{dt_e}(t_e)\ , \tag{29.21}$$

is this really true

implying that radiation is produced whenever a charged particle is accelerated. (However, see Sec. XXXIV.) The angular distribution of the radiated power is then given by, from (29.17) and (29.21),

$$\frac{dP}{d\Omega} \simeq \frac{1}{4\pi c^3}\left(\vec{n}\times e\,\frac{d\vec{v}}{dt}\right)^2$$

$$= \frac{e^2}{4\pi c^3}\left[\left(\frac{d\vec{v}}{dt}\right)^2 - \left(\vec{n}\cdot\frac{d\vec{v}}{dt}\right)^2\right]$$

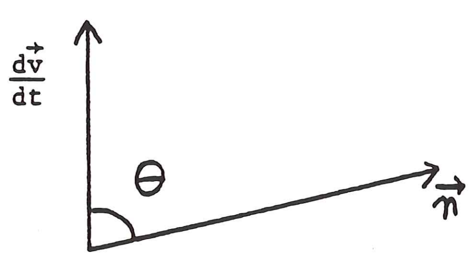
$$= \frac{e^2}{4\pi c^3}\left(\frac{d\vec{v}(t_e)}{dt_e}\right)^2 \sin^2\theta\ , \tag{29.22}$$

$\frac{2e^2v^4}{3c^3r^2} = \frac{2e^2\omega^4r^2}{3c^3}$

$a = \frac{v_{\parallel}^2}{r}$

$v = \omega r$

where θ is the angle between the direction of observation $\vec{n}$ and the direction of the acceleration at the emission time t_e.



Evidently there is no radiation emitted along the direction of the acceleration. By employing the angular integral

$$\int \frac{d\Omega}{4\pi} \sin^2\theta = \frac{2}{3} \quad , \tag{29.23}$$

we obtain the total radiated power:

$$P = \frac{2e^2}{3c^3}\left(\frac{d\vec{v}}{dt}\right)^2 \quad , \quad \text{for } \frac{v}{c} << 1 \quad , \tag{29.24}$$

and this is true only if $\frac{dv}{dt} \neq 0$ for all time

which is called the Larmor formula.

29-4. Dipole Radiation

Next, we generalize the above discussion to a system consisting of many charged particles. For a small system in which particles are all moving with low velocities, the time of emission, t_e , does not vary significantly over the current distribution. Consequently, the integral in (29.17) becomes

$$\int (d\vec{r}\,') \frac{\partial}{\partial t} \vec{j}(\vec{r}\,', t_r) \simeq \frac{d}{dt_e} \int (d\vec{r}\,') \vec{j}(\vec{r}\,', t_e) \quad , \tag{29.25}$$

which is evaluated by means of an identity, derived from current conservation:

$$\begin{aligned} 0 &= \int (d\vec{r})\, \vec{r}\left[\vec{\nabla}\cdot\vec{j} + \frac{\partial}{\partial t}\rho\right] = \int (d\vec{r}) \left[\vec{\nabla}\cdot(\vec{j}\,\vec{r}) - \vec{j} + \vec{r}\frac{\partial}{\partial t}\rho\right] \\ &= -\int (d\vec{r})\, \vec{j}(\vec{r},t) + \frac{d}{dt}\int (d\vec{r})\, \vec{r}\rho(\vec{r},t) \quad . \end{aligned} \tag{29.26}$$

Recalling the definition of the electric dipole moment,

$$\vec{d}(t) = \int (d\vec{r})\, \vec{r}\rho(\vec{r},t) \quad , \tag{29.27}$$

we recognize (29.25) as the second time derivative of $\vec{d}(t_e)$. Therefore, the angular distribution is given by

$$\frac{dP}{d\Omega} \simeq \frac{1}{4\pi c^3} (\vec{n} \times \ddot{\vec{d}})^2 \;, \quad \text{for } \frac{v}{c} \ll 1 \;, \tag{29.28}$$

while the total power is

$$P \simeq \frac{2}{3c^3} (\ddot{\vec{d}})^2 \;, \tag{29.29}$$

where we have used a dot to denote time differentiation. Radiation described by these formulae is called electric dipole radiation. For a single charged particle

$$\vec{d} = e\vec{r} \;, \quad \ddot{\vec{d}} = e \frac{d\vec{v}}{dt} \;,$$

and (29.22) and (29.24) are recovered, as expected. However, for a system of n charged particles, the electric dipole moment is

$$\vec{d} = \sum_{k=1}^{n} e_k \vec{r}_k \;,$$

so the power radiated is not additive, but exhibits interference effects:

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \left(\vec{n} \times \sum_{k=1}^{n} e_k \ddot{\vec{r}}_k \right)^2 \;.$$

Let us now make a better approximation by keeping the v/c correction arising from the $\frac{1}{c} \vec{n}\cdot\vec{r}\,'$ term in the retarded time. The integral over the current in (29.17) now becomes

$$\int (d\vec{r}\,') \, \vec{j}\left(\vec{r}\,', t_e + \frac{1}{c} \vec{n}\cdot\vec{r}\,' \right) \simeq \dot{\vec{d}}(t_e) + \int (d\vec{r}\,') \frac{1}{c} \vec{n}\cdot\vec{r}\,' \frac{\partial}{\partial t_e} \vec{j}(\vec{r}\,', t_e) + \ldots$$

$$= \dot{\vec{d}}(t_e) + \frac{1}{c} \frac{d}{dt} \int (d\vec{r}\,') \, \vec{n}\cdot\left[\frac{1}{2} (\vec{r}\,'\vec{j} + \vec{j}\,\vec{r}\,') + \frac{1}{2} (\vec{r}\,'\vec{j} - \vec{j}\,\vec{r}\,') \right]$$

$$\simeq \dot{\vec{d}}(t_e) - \frac{d}{dt}\int (d\vec{r}\,')\, \vec{n} \times \left[\frac{1}{2c}\vec{r}\,' \times \vec{j}(\vec{r}\,',t_e)\right]$$

$$= \dot{\vec{d}}(t_e) - \vec{n} \times \dot{\vec{\mu}}(t_e) \,, \tag{29.30}$$

where $\vec{\mu}$ is the magnetic dipole moment, (25.7). Here we have neglected the contribution due to

$$\frac{1}{2}\,(\vec{r}\,'\vec{j} + \vec{j}\,\vec{r}\,')$$

since this is an electric quadrupole moment effect,

$$\int (d\vec{r}\,')\,[x_i{}'j_j + x_j{}'j_i] = -\int (d\vec{r}\,')\, x_i{}'x_j{}'\, \vec{\nabla}'\cdot\vec{j}$$

$\vec{C}\cdot \quad - \vec{\partial}\cdot\vec{C}\cdot(x_i x_j)$

$\partial_k \partial_k x_i x_j$

$$= \frac{d}{dt}\int (d\vec{r}\,')\, x_i{}'x_j{}'\rho \rightarrow \frac{1}{3}\dot{q}_{ij} \quad , \tag{29.31}$$

from (19.4c), since the unit dyadic evidently does not contribute to the radiated power. The angular distribution of the radiated power is therefore more accurately given by

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\,[\vec{n} \times (\ddot{\vec{d}} - \vec{n} \times \ddot{\vec{\mu}})]^2$$

$$= \frac{1}{4\pi c^3}\,[(\vec{n} \times \ddot{\vec{d}})^2 + (\vec{n} \times \ddot{\vec{\mu}})^2 + 2\vec{n}\cdot(\ddot{\vec{d}} \times \ddot{\vec{\mu}})] \quad , \tag{29.32}$$

where the last term represents interference between $\vec{d}$ and $\vec{\mu}$, which, since it depends linearly on $\vec{n}$, does not contribute to the total radiated power:

$$P = \frac{2}{3c^3}\,[(\ddot{\vec{d}})^2 + (\ddot{\vec{\mu}})^2] \quad . \tag{29.33}$$

The behavior of the fields at large distances can be obtained by substituting (29.30) into (29.7) and (29.12),

$$\vec{B} \sim -\frac{1}{c^2}\frac{1}{r}\vec{n}\times(\ddot{\vec{d}} - \vec{n}\times\ddot{\vec{\mu}}) \quad , \tag{29.34}$$

$$\vec{E} \sim -\vec{n}\times\vec{B} \sim \frac{1}{c^2}\frac{1}{r}\vec{n}\times(\ddot{\vec{\mu}} + \vec{n}\times\ddot{\vec{d}}) \quad . \tag{29.35}$$

Note that these results are invariant under the replacements $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ together with $\vec{d} \to \vec{\mu}$ and $\vec{\mu} \to -\vec{d}$, which is a manifestation of the symmetry discussed in Sec. II. We here have an indication of the connection between the directions of the electric field for electric and magnetic dipole radiation.

29-5. Potentials in Radiation Gauge

To this point, we have discussed radiation by use of the Lorentz gauge. However, we do have the freedom to choose an arbitrary gauge without affecting the physical results. As an illustration, let us here consider the radiation gauge, which exhibits a certain physical simplicity. In this gauge, the potentials satisfy the differential equations (28.10) and (28.11), that is

$$-\nabla^2\phi = 4\pi\rho \quad , \tag{29.36a}$$

$$-\Box^2\vec{A} = \frac{4\pi}{c}\left(\vec{j} - \frac{1}{4\pi}\vec{\nabla}\frac{\partial}{\partial t}\phi\right) \quad , \tag{29.36b}$$

while the vector potential is subject to the gauge condition (28.9),

$$\vec{\nabla}\cdot\vec{A} = 0 \quad . \tag{29.37}$$

The electric and magnetic fields are obtained from these potentials by the relations (28.3) and (28.4). In order to make contact with what has gone before, we consider the fields at large distances, which are those of interest for radiation. The solution of (29.36a) is the Coulomb potential,

$$\phi(\vec{r},t) = \int (d\vec{r}\,') \frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} ,$$

which, asymptotically, behaves as

$$\phi \sim \frac{e}{r} , \tag{29.38}$$

with e the total charge. Since the gradient of this is inversely proportional to the square of the distance,

$$-\vec{\nabla}\phi \sim \frac{e}{r^3}\vec{r} , \tag{29.39}$$

we can neglect the scalar potential in computing the radiation fields, which decrease only as $1/r$. Therefore, the vector potential alone determines the radiation fields:

$$\vec{E} \simeq -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} , \tag{29.40}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} . \tag{29.41}$$

We also note that the gauge condition (29.37) enforces the transversality of these fields. That is, as a consequence of $\vec{\nabla}\cdot\vec{A} = 0$, we recover the scalar Maxwell equations outside the sources,

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{E} = 0 , \tag{29.42}$$

which, by virtue of (29.6), supplies the relations

$$\vec{n}\cdot\vec{B} = 0 \quad \text{and} \quad \vec{n}\cdot\vec{E} = 0 , \tag{29.43}$$

while $\vec{E} \perp \vec{B}$ follows immediately from (29.40) and (29.41).

To solve (29.36b) for the vector potential, we first write the solution to (29.36a) symbolically as

$$\phi = \frac{1}{-\nabla^2}\, 4\pi\rho \quad , \tag{29.44}$$

the time derivative of which is

$$\begin{aligned} \frac{\partial}{\partial t}\phi &= \frac{1}{-\nabla^2}\, 4\pi \frac{\partial}{\partial t}\rho \\ &= \frac{1}{\nabla^2}\, 4\pi\, \vec{\nabla}\cdot\vec{j} \quad . \end{aligned} \tag{29.45}$$

Consequently, we may rewrite (29.36b) as

$$-\square^2 \vec{A} = \frac{4\pi}{c}\left(\overleftrightarrow{1} - \frac{\vec{\nabla}\vec{\nabla}}{\nabla^2}\right)\cdot\vec{j} \quad , \tag{29.46}$$

which makes the radiation gauge condition (29.37) transparent. The solution to (29.46) may be obtained from that of (28.14) by applying the operator

$$\overleftrightarrow{1} - \frac{\vec{\nabla}\vec{\nabla}}{\nabla^2}$$

to (28.41):

$$\vec{A}(\vec{r},t) = \left(\overleftrightarrow{1} - \frac{\vec{\nabla}\vec{\nabla}}{\nabla^2}\right)\int (d\vec{r}\,')\, \frac{\frac{1}{c}\,\vec{j}(\vec{r}\,', t - \frac{1}{c}|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|} \quad . \tag{29.47}$$

At large distances, by making use of (29.1) and (29.6), we have effectively the replacement

$$\vec{\nabla} \to -\frac{\vec{n}}{c}\frac{\partial}{\partial t} \tag{29.48}$$

so that the operator $\overleftrightarrow{1} - \dfrac{\vec{\nabla}\,\vec{\nabla}}{\nabla^2}$ can be replaced by

$$\overleftrightarrow{1} - \frac{\vec{\nabla}\,\vec{\nabla}}{\nabla^2} \to \overleftrightarrow{1} - \frac{\left(-\dfrac{\vec{n}}{c}\dfrac{\partial}{\partial t}\right)\left(-\dfrac{\vec{n}}{c}\dfrac{\partial}{\partial t}\right)}{\left(-\dfrac{\vec{n}}{c}\dfrac{\partial}{\partial t}\right)^2}$$

$$= \overleftrightarrow{1} - \vec{n}\,\vec{n} \quad . \tag{29.49}$$

Notice that this symbolic notation is convenient when $1/\nabla^2$ can be computed simply. By making use of (29.11) and (29.1), we obtain the asymptotic form of the vector potential, in the radiation gauge, to be

$$\vec{A}(\vec{r},t) \sim (\overleftrightarrow{1} - \vec{n}\,\vec{n}) \cdot \frac{1}{cr}\int (d\vec{r}\,')\, \vec{j}(\vec{r}\,',t_r)$$

$$= -\vec{n} \times \left[\vec{n} \times \frac{1}{cr}\int (d\vec{r}\,')\, \vec{j}(\vec{r}\,',t_r)\right] \quad . \tag{29.50}$$

The resultirg electric and magnetic fields are precisely the same as those found in the Lorentz gauge, (29.7) and (29.12).

XXX. ELECTROMAGNETIC RADIATION—SOURCE POINT OF VIEW

30-1. Conservation of Energy

Having examined the radiation fields, we turn our attention to an examination of the source of the radiated energy. Energy and momentum are transferred from the charges to the electromagnetic field; the rate at which the current does work on the field is

$$-\vec{j}\cdot\vec{E} = \frac{\partial}{\partial t}\left(\frac{E^2+B^2}{8\pi}\right) + \vec{\nabla}\cdot\left(\frac{c}{4\pi}\,\vec{E}\times\vec{B}\right) \quad , \tag{30.1}$$

which is the local statement of energy conservation, (3.7). When (30.1) is integrated over a large volume enclosing the charge and current distributions, the conservation of total energy follows:

$$\int (d\vec{r})(-\vec{j}\cdot\vec{E}) = \frac{d}{dt}\int (d\vec{r})\,\frac{E^2+B^2}{8\pi} + \int d\vec{S}\cdot\frac{c}{4\pi}\,\vec{E}\times\vec{B}$$

$$= \frac{d}{dt}\,E + P \quad , \tag{30.2}$$

or stated in words, the rate at which the charged particles transfer energy to the electromagnetic field is equal to the sum of the rate of increase of the total electromagnetic energy, E , in the volume, and rate of flow of energy, P , out of the surface bounding the volume. Equation (30.2) gives us an alternative way of calculating the radiated power, P , by computing the rate at which energy is transferred to the fields,

$$\int (d\vec{r})(-\vec{j}\cdot\vec{E}) \quad , \tag{30.3}$$

and discarding total time derivative terms, which are not associated with radiation. From this point of view, we need to know the electric field inside the current distribution in contrast to the previous discussion, in which we computed the radiated power by evaluating the fields far from the source.

30-2. Dipole Radiation

To illustrate this method, we again consider dipole radiation produced by a small charge distribution, for which the nonrelativistic approximation is valid, that is, all particle speeds are small compared to the speed of light, $v/c \ll 1$. In this limit, we will require an expression for the field $\vec{E}$ accurate to order $1/c^3$, which means, from the definition (28.4), that the scalar potential ϕ must be expanded in powers of $1/c$ up to order $1/c^3$, while the vector potential $\vec{A}$ need only be expanded up to order $1/c^2$. In the Lorentz gauge, the appropriate expansions of the potentials (28.40) and (28.41) are

$$\phi(\vec{r},t) \simeq \int (d\vec{r}\,')\,\frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} - \int (d\vec{r}\,')\,\frac{\frac{1}{c}\,|\vec{r}-\vec{r}\,'|\,\frac{\partial}{\partial t}\rho}{|\vec{r}-\vec{r}\,'|} \quad \leftarrow 0 \;\; \int\rho = \text{constant}$$

$$+ \int (d\vec{r}\,')\,\frac{\frac{1}{2}\left(\frac{1}{c}\,|\vec{r}-\vec{r}\,'|\right)^2 \frac{\partial^2}{\partial t^2}\rho}{|\vec{r}-\vec{r}\,'|} - \int (d\vec{r}\,')\,\frac{\frac{1}{6}\left(\frac{|\vec{r}-\vec{r}\,'|}{c}\right)^3 \frac{\partial^3}{\partial t^3}\rho}{|\vec{r}-\vec{r}\,'|} + \ldots$$

$$= \int (d\vec{r}\,')\,\frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} + \frac{1}{2c^2}\int (d\vec{r}\,')\,|\vec{r}-\vec{r}\,'|\,\frac{\partial^2}{\partial t^2}\rho(\vec{r}\,',t)$$

$$- \frac{1}{6c^3}\int (d\vec{r}\,')\,(\vec{r}-\vec{r}\,')^2\,\frac{\partial^3}{\partial t^3}\rho(\vec{r}\,',t) + \ldots\;, \tag{30.4}$$

since the total charge e is conserved,

$$\int (d\vec{r}\,')\,\frac{\partial}{\partial t}\rho = \frac{d}{dt}e = 0\;,$$

and

$$\vec{A}(\vec{r},t) \simeq \frac{1}{c}\int (d\vec{r}\,')\,\frac{\vec{j}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} - \frac{1}{c^2}\int (d\vec{r}\,')\,\frac{\partial}{\partial t}\vec{j}(\vec{r}\,',t) + \ldots$$

$$= \frac{1}{c}\int (d\vec{r}\,')\,\frac{\vec{j}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} - \frac{1}{c^2}\ddot{\vec{d}} + \ldots\ , \tag{30.5}$$

where we have used (29.26), and (29.27) for the electric dipole moment $\vec{d}$. The contribution of the $1/c^3$ term in ϕ to $\vec{E}$ can be simplified as follows:

$$-\vec{\nabla}\left[-\frac{1}{6c^3}\int (d\vec{r}\,')(\vec{r}-\vec{r}\,')^2\,\frac{\partial^3}{\partial t^3}\,\rho(\vec{r}\,',t)\right]$$

$$= \left(\frac{\partial}{\partial t}\right)^3 \frac{1}{3c^3}\int (d\vec{r}\,')(\vec{r}-\vec{r}\,')\,\rho(\vec{r}\,',t)$$

$$= -\frac{1}{3c^3}\dddot{\vec{d}}\ . \tag{30.6}$$

The expression for the energy transfer, (30.3), becomes, in this approximation,

$$-\int (d\vec{r})\,\vec{j}\cdot\vec{E} = -\int (d\vec{r})\,\vec{j}\cdot\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}\right)$$

$$= \int (d\vec{r})\left(\frac{\partial\rho}{\partial t}\,\phi + \frac{1}{c}\,\vec{j}\cdot\frac{\partial}{\partial t}\vec{A}\right)$$

$$= \frac{d}{dt}\left\{\frac{1}{2}\int (d\vec{r})(d\vec{r}\,')\,\frac{\rho(\vec{r},t)\,\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\right.$$

≈ $\frac{d}{dt}$ (Electrostatic Energy)

$$\left. + \frac{1}{4c^2}\int (d\vec{r})(d\vec{r}\,')\,\frac{\partial}{\partial t}\rho(\vec{r},t)\,\frac{\partial}{\partial t}\rho(\vec{r}\,',t)\,|\vec{r}-\vec{r}\,'|\right\}$$

$$+ \frac{1}{3c^3}\dot{\vec{d}}\cdot\dddot{\vec{d}} + \frac{d}{dt}\left[\frac{1}{2c^2}\int (d\vec{r})(d\vec{r}\,')\,\frac{\vec{j}(\vec{r},t)\cdot\vec{j}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\right] - \frac{1}{c^3}\dot{\vec{d}}\cdot\dddot{\vec{d}} \tag{30.7a}$$

$$\to -\frac{2}{3c^3}\dot{\vec{d}}\cdot\dddot{\vec{d}} \tag{30.7b}$$

$$= \frac{2}{3c^3}(\ddot{\vec{d}})^2 + \frac{d}{dt}\left(-\frac{2}{3c^3}\dot{\vec{d}}\cdot\ddot{\vec{d}}\right)$$

$$\to \frac{2}{3c^3}(\ddot{\vec{d}})^2 = P\ , \tag{30.7c}$$

where we have again used (29.26) and have set aside total time derivative terms, which do not contribute to the radiation. The power radiated, (30.7c), is the same as that found in the preceding section, given by (29.29).

Lecture 3

30-3. Hamiltonian

As a byproduct of this source approach we may identify the total electromagnetic energy of the system, to order $1/c^2$, by comparing (30.2) with the total time derivative terms in (30.7a),

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int (d\vec{r})(d\vec{r}\,')\,\frac{\rho(\vec{r},t)\,\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} \\
&+ \frac{1}{2c^2}\int (d\vec{r})(d\vec{r}\,')\,\frac{\vec{j}(\vec{r},t)\cdot\vec{j}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} \\
&+ \frac{1}{4c^2}\int (d\vec{r})(d\vec{r}\,')\left(\frac{\partial}{\partial t}\rho(\vec{r},t)\,\frac{\partial}{\partial t}\rho(\vec{r}\,',t)\right)|\vec{r}-\vec{r}\,'| + \dots\,, \qquad (30.8)
\end{aligned}
$$

where the first two terms have the form of the electrostatic and magnetostatic field energies. However, the sign of the current-current interaction is opposite to that of (24.12). The resolution of this apparent discrepancy requires the third term in (30.8), which we rewrite by means of the local charge conservation condition, (28.2):

$$
\begin{aligned}
&\frac{1}{4c^2}\int (d\vec{r})(d\vec{r}\,')\,\vec{\nabla}\cdot\vec{j}(\vec{r},t)\,\vec{\nabla}'\cdot\vec{j}(\vec{r}\,',t)\,|\vec{r}-\vec{r}\,'| \\
&= -\frac{1}{4c^2}\int (d\vec{r})(d\vec{r}\,')\left(\vec{\nabla}'\cdot\vec{j}(\vec{r}\,',t)\,\vec{j}(\vec{r},t)\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\right) \\
&= \frac{1}{4c^2}\int (d\vec{r})(d\vec{r}\,')\,\vec{j}(\vec{r}\,',t)\cdot\vec{\nabla}'\left[\vec{j}(\vec{r},t)\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\right] \\
&= \frac{1}{4c^2}\int (d\vec{r})(d\vec{r}\,')\left\{-\frac{\vec{j}(\vec{r},t)\cdot\vec{j}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} + \frac{[(\vec{r}-\vec{r}\,')\cdot\vec{j}(\vec{r},t)][(\vec{r}-\vec{r}\,')\cdot\vec{j}(\vec{r}\,',t)]}{|\vec{r}-\vec{r}\,'|^3}\right\}\,,
\end{aligned}
$$

(30.9)

which, when combined with the first two terms in (30.8) yields

$$E(t) = \frac{1}{2}\int (d\vec{r})(d\vec{r}')\,\frac{\rho(\vec{r},t)\,\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|} + \frac{1}{4c^2}\int (d\vec{r})(d\vec{r}')\left\{\frac{\vec{j}(\vec{r},t)\cdot\vec{j}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\right.$$

$$\left. + \frac{[(\vec{r}-\vec{r}')\cdot\vec{j}(\vec{r},t)][(\vec{r}-\vec{r}')\cdot\vec{j}(\vec{r}',t)]}{|\vec{r}-\vec{r}'|^3}\right\} . \tag{30.10}$$

For point charges, the charge and current densities are

$$\rho(\vec{r},t) = \sum_k e_k\,\delta(\vec{r}-\vec{r}_k(t)) \quad , \tag{30.11a}$$

$$\vec{j}(\vec{r},t) = \sum_k e_k\vec{v}_k\,\delta(\vec{r}-\vec{r}_k(t)) \quad , \tag{30.11b}$$

so the terms in (30.10) referring to the mutual interaction of the particles are

$$E_{field} = \frac{1}{2}\sum_{k\neq\ell}\frac{e_k e_\ell}{r_{k\ell}} + \frac{1}{4c^2}\sum_{k\neq\ell} e_k e_\ell\left[\frac{\vec{v}_k\cdot\vec{v}_\ell}{r_{k\ell}} + \frac{(\vec{r}_{k\ell}\cdot\vec{v}_k)(\vec{r}_{k\ell}\cdot\vec{v}_\ell)}{r_{k\ell}^3}\right] , \tag{30.12}$$

where

$$\vec{r}_{k\ell} \equiv \vec{r}_k - \vec{r}_\ell \quad .$$

To this we must add the particle kinetic energy, from (9.36),

$$E_{particle} = \sum_k m_k c^2\left(\frac{1}{\sqrt{1-\frac{v_k^2}{c^2}}} - 1\right)$$

$$\simeq \sum_k \left(\frac{1}{2}m_k v_k^2 + \frac{3}{8}m_k\frac{v_k^4}{c^2} + \ldots\right) , \quad \frac{v_k^2}{c^2} \ll 1 \quad , \tag{30.13}$$

in order to obtain the total energy of the system:

$$E_{total} = E_{field} + E_{particle} \quad . \tag{30.14}$$

We now wish to describe this mechanical system in Hamiltonian language. We recall [from (8.13)] that the Hamiltonian is related to the Lagrangian by

$$H = \sum_k \vec{p}_k \cdot \vec{v}_k - L(\vec{r}_i, \vec{v}_i) \quad , \tag{30.15}$$

where the canonical momentum $\vec{p}_k$ is defined by

$$\vec{p}_k = \frac{\partial L(\vec{r}_i, \vec{v}_i)}{\partial \vec{v}_k} \quad . \tag{30.16}$$

To determine the Lagrangian, we substitute (30.16) into (30.15),

$$H = \sum_k \frac{\partial L(\vec{r}_i, \vec{v}_i)}{\partial \vec{v}_k} \cdot \vec{v}_k - L(\vec{r}_i, \vec{v}_i) \quad , \tag{30.17}$$

where H is given in terms of $\vec{v}$ by (30.14). Equating terms on either side of (30.17) that are independent of $\vec{v}$, quadratic in $\vec{v}$, and quartic in $\vec{v}$, we find for the Lagrangian

$$L = \sum_k \left(\frac{1}{2} m_k v_k^2 + \frac{1}{8} m_k \frac{v_k^4}{c^2} \right) + \frac{1}{4c^2} \sum_{k \neq \ell} e_k e_\ell \left(\frac{\vec{v}_k \cdot \vec{v}_\ell}{r_{k\ell}} + \frac{(\vec{r}_{k\ell} \cdot \vec{v}_k)(\vec{r}_{k\ell} \cdot \vec{v}_\ell)}{r_{k\ell}^3} \right)$$

$$- \frac{1}{2} \sum_{k \neq \ell} \frac{e_k e_\ell}{r_{k\ell}} \quad . \tag{30.18}$$

Note that the particle term in (30.18) agrees with the expansion of (9.37). The canonical momenta are given by (30.16),

$$\vec{p}_k = m_k \vec{v}_k + \frac{1}{2} m_k \frac{v_k^2 \vec{v}_k}{c^2} + \frac{1}{2c^2} \sum_{\ell \neq k} e_k e_\ell \left[\frac{\vec{v}_\ell}{r_{k\ell}} + \frac{\vec{r}_{k\ell}(\vec{r}_{k\ell} \cdot \vec{v}_\ell)}{r_{k\ell}^3} \right] \quad , \tag{30.19}$$

in terms of which the Hamiltonian of the system is

$$H = \frac{1}{2} \sum_{k \neq \ell} \frac{e_k e_\ell}{r_{k\ell}} + \sum_k \left(\frac{p_k^2}{2m_k} - \frac{1}{8} \frac{p_k^4}{m_k^3 c^2} \right)$$

$$- \frac{1}{4c^2} \sum_{k \neq \ell} \frac{e_k e_\ell}{m_k m_\ell} \left[\frac{\vec{p}_k \cdot \vec{p}_\ell}{r_{k\ell}} + \frac{(\vec{r}_{k\ell} \cdot \vec{p}_k)(\vec{r}_{k\ell} \cdot \vec{p}_\ell)}{r_{k\ell}^3} \right] , \qquad (30.20)$$

where we have consistently kept terms up to order $1/c^2$. This form of the Hamiltonian is appropriate for small systems and has application in both atomic and nuclear physics. It is called the Darwin Hamiltonian when it is applied classically, and the Breit Hamiltonian when it is applied quantum mechanically (with an accompanying re-expression in terms of Dirac matrices).

From the general discussion given in Sec. IX, the canonical momentum can be expressed in terms of the vector potential

$$\vec{p}_k = \frac{m_k \vec{v}_k}{\sqrt{1 - \frac{v_k^2}{c^2}}} + \frac{e_k}{c} \vec{A}(\vec{r}_k, t) , \qquad (30.21)$$

where we have used (9.36) in generalizing (9.5b). Upon comparison with (30.19), we obtain an explicit form of the vector potential, $\vec{A}(\vec{r}_k, t)$, which, when generalized to an arbitrary position, reads

$$\vec{A}(\vec{r}, t) = \sum_\ell \frac{e_\ell}{2c} \left[\frac{\vec{v}_\ell}{|\vec{r} - \vec{r}_\ell|} + \frac{[(\vec{r} - \vec{r}_\ell) \cdot \vec{v}_\ell](\vec{r} - \vec{r}_\ell)}{|\vec{r} - \vec{r}_\ell|^3} \right]$$

$$= \sum_\ell \frac{e_\ell}{c} \left[\frac{\vec{v}_\ell}{|\vec{r} - \vec{r}_\ell|} - \frac{1}{2} \vec{\nabla}(\vec{v}_\ell \cdot \vec{\nabla} |\vec{r} - \vec{r}_\ell|) \right] . \qquad (30.22)$$

We notice that this vector potential satisfies the radiation gauge condition (28.9):

$$\vec{\nabla}\cdot\vec{A} = \sum_{\ell} \frac{e_\ell}{c}\left[\vec{v}_\ell\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}_\ell|} - \frac{1}{2}\vec{v}_\ell\cdot\vec{\nabla}(\nabla^2|\vec{r}-\vec{r}_\ell|)\right] = 0 \quad ,$$

since

$$\nabla^2|\vec{r}-\vec{r}_\ell| = \frac{2}{|\vec{r}-\vec{r}_\ell|} \quad ,$$

and, as required, it also satisfies the differential equation (28.11) to order $1/c^2$:

$$\begin{aligned} -\Box^2\vec{A} &\simeq -\nabla^2\vec{A} \\ &= \sum_{\ell}\frac{e_\ell}{c}\left[4\pi\vec{v}_\ell\,\delta(\vec{r}-\vec{r}_\ell) + \frac{1}{2}\vec{\nabla}\left(\vec{v}_\ell\cdot\vec{\nabla}\frac{2}{|\vec{r}-\vec{r}_\ell|}\right)\right] \\ &\simeq \frac{4\pi}{c}\vec{j}(\vec{r},t) - \vec{\nabla}\frac{1}{c}\frac{\partial}{\partial t}\phi(\vec{r},t) \quad . \end{aligned}$$

Furthermore, if we rewrite the last term of (30.20) in terms of $\vec{A}$, (30.22), we obtain

$$-\frac{1}{2c}\sum_k e_k\vec{v}_k\cdot\vec{A}_k(\vec{r},t) = -\frac{1}{2c}\int(d\vec{r})\,\vec{j}(\vec{r},t)\cdot\vec{A}(\vec{r},t) \quad , \qquad (30.23)$$

which has the form of the magnetostatic field energy, (24.11), which is therefore correctly given by (30.8). As we commented in Section XXIV, because of the negative sign in (30.23), "like" currents attract each other.

Warren Mori

XXXI. SIMPLE MODEL OF AN ANTENNA

We have been discussing a small system, in which the time delay effects are not great. For an example of the opposite situation, let us consider an oversimplified model of an antenna. In this model we have a wire of length ℓ , and of negligible cross section, carrying a current density flowing in the z direction:

$$J_z = I\delta(x)\,\delta(y)\,\sin\omega t\ , \quad -\frac{\ell}{2} \le z \le \frac{\ell}{2}\ , \qquad (31.1)$$

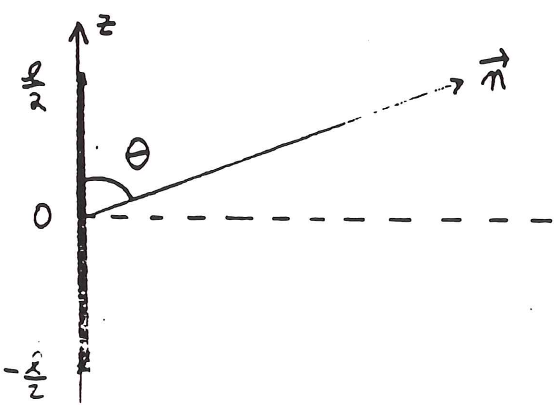
which has the property

$$\vec{\nabla}\cdot\vec{J} = \frac{\partial}{\partial z} J_z = \begin{cases} 0 & \text{for} \quad -\frac{\ell}{2} < z < \frac{\ell}{2}\ , \\ \neq 0 & \text{for} \quad z = \pm\frac{\ell}{2}\ . \end{cases}$$

From the local charge conservation condition (28.2),

$$0 = \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial z} J_z\ ,$$

we see that (31.1) implies that charge will build up at both ends of the antenna. In any realistic model, J_z will depend on z . We lack this dependence since our model assumes that the antenna is fed at every point along its length. Even though this model is oversimplified, it possesses many of the significant characteristics of a real antenna.



To compute the power radiated, (29.17), we evaluate the integral

$$\int (d\vec{r}\,') \frac{1}{c}\frac{\partial}{\partial t} J_z\left(\vec{r}\,', t - \frac{r}{c} + \frac{1}{c}\vec{n}\cdot\vec{r}\,'\right)$$

$$= \frac{\omega}{c} I \int_{-\ell/2}^{\ell/2} dz' \cos\omega\left(t - \frac{r}{c} + \frac{1}{c} z' \cos\theta\right)$$

$$= \frac{\omega}{c} I \int_{-\ell/2}^{\ell/2} dz' \cos\omega\left(t - \frac{r}{c}\right) \cos\left(\frac{\omega z'}{c}\cos\theta\right)$$

$$= \frac{2\frac{\omega}{c} I \cos\omega\left(t - \frac{r}{c}\right) \sin\left(\frac{\omega\ell}{2c}\cos\theta\right)}{\frac{\omega}{c}\cos\theta} , \tag{31.2}$$

where, as indicated in the figure, θ denotes the angle between the direction of observation and the antenna. The angular distribution of the radiated power, at the observation time t, is then

$$\frac{dP(t)}{d\Omega} = \frac{1}{4\pi c}\sin^2\theta \frac{I^2 \cos^2\omega\left(t - \frac{r}{c}\right) 4 \sin^2\left(\frac{\omega\ell}{2c}\cos\theta\right)}{\cos^2\theta} , \tag{31.3}$$

which when averaged over one cycle of oscillation, becomes

$$\frac{dP}{d\Omega} = \frac{I^2}{2\pi c} \frac{\sin^2\theta \sin^2\left(\frac{\omega\ell}{2c}\cos\theta\right)}{\cos^2\theta} . \tag{31.4}$$

To rewrite (31.4) in terms of more convenient parameters, we recognize that, far from the antenna, the fields oscillate periodically both in space and in time:

$$E,H \sim \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left(\omega t - \frac{\omega}{c} r\right) = \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left(2\pi\nu t - \frac{2\pi\nu}{c} r\right) ,$$

so we identify the relation between the frequency ν (cycles/sec) and wavelength λ to be

$$\lambda\nu = c \quad , \tag{31.5a}$$

or

$$\frac{\omega}{c} = \frac{2\pi}{\lambda} \quad . \tag{31.5b}$$

Therefore the radiation of the system is characterized by two parameters: the length of the antenna, ℓ , and the wavelength of the radiation, λ . The combination that appears in the expression for the power, (31.4), is

$$\frac{\omega}{c}\frac{\ell}{2} = \frac{\pi\ell}{\lambda} \quad , \tag{31.6}$$

in terms of which the angular distribution is

$$\frac{dP}{d\Omega} = \frac{I^2}{2\pi c}\,\frac{\sin^2\theta\,\sin^2\left(\frac{\pi\ell}{\lambda}\cos\theta\right)}{\cos^2\theta} \quad . \tag{31.7}$$

In particular, in the direction perpendicular to the antenna, $\theta = \frac{\pi}{2}$, the radiated power is proportional to the square of the length of the antenna ,

$$\left.\frac{dP}{d\Omega}\right|_{\theta=\pi/2} = \frac{I^2}{2\pi c}\left(\frac{\pi\ell}{\lambda}\right)^2 \quad . \tag{31.8}$$

To appreciate the characteristic features of this radiation, we will consider the application of this general formula, (31.7), to three special circumstances:

1. $\lambda \gg \ell$.

For a short antenna, $\ell \ll \lambda$, the approximation

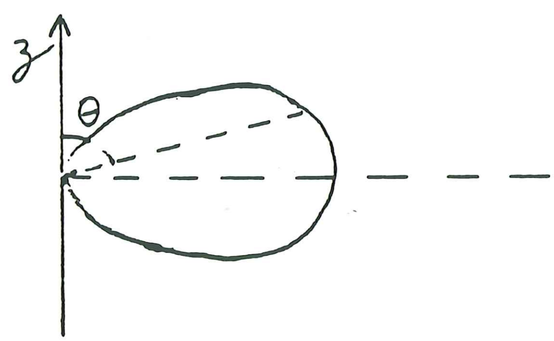
$$\frac{\sin^2\left(\frac{\pi \ell}{\lambda}\cos\theta\right)}{\cos^2\theta} \simeq \left(\frac{\pi \ell}{\lambda}\right)^2$$

holds true for all angles. The resulting radiation pattern may be alternatively derived from the dipole radiation formula, (29.22), which is appropriate to a small system:

$$\frac{dP}{d\Omega} \simeq \frac{I^2}{2\pi c}\left(\frac{\pi \ell}{\lambda}\right)^2 \sin^2\theta \quad .$$

2. $\lambda > \ell$.

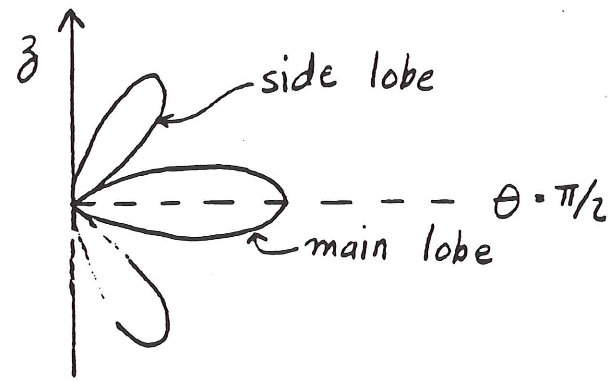
When $\frac{\pi \ell}{\lambda} < \pi$, the argument of the factor $\sin^2\left(\frac{\pi \ell}{\lambda}\cos\theta\right)$ goes from 0 to something less than π when the angle θ varies from $\frac{\pi}{2}$ to 0 . Therefore, the only angles at which the power radiated vanishes are 0 and π , so the radiation pattern has a single lobe.



(In this and the following figure, axial symmetry about the z-axis is to be understood.)

3. $\lambda < \ell < 2\lambda$.

When $\pi < \frac{\pi \ell}{\lambda} < 2\pi$ there is an additional zero in the radiated power at the angle $\theta = \cos^{-1}\left(\frac{\lambda}{\ell}\right)$. Consequently the radiation pattern exhibits both a main lobe and two side lobes.



Evidently, as ℓ/λ increases, more and more side lobes appear.

The total power radiated by this antenna may be obtained by integrating (31.7) over all angles:

$$
\begin{aligned}
P &= \int_0^{\pi} 2\pi \sin\theta d\theta \, \frac{I^2}{2\pi c} \, \frac{\sin^2\theta \sin^2\left(\frac{\pi\ell}{\lambda}\cos\theta\right)}{\cos^2\theta} \\
&= \frac{I^2}{c}\int_{-\pi/2}^{\pi/2} \cos\chi d\chi \, \cos^2\chi \, \frac{\sin^2\left(\frac{\pi\ell}{\lambda}\sin\chi\right)}{\sin^2\chi} \\
&= \frac{2I^2}{c}\int_0^{\pi\ell/\lambda} \left(\frac{\pi\ell}{\lambda}\right) dz \, \frac{\sin^2 z}{z^2}\left[1 - \frac{z^2}{\left(\frac{\pi\ell}{\lambda}\right)^2}\right] ,
\end{aligned} \tag{31.9}
$$

where we have made the successive changes of variables,

$$
\begin{aligned}
\chi &= \frac{\pi}{2} - \theta , \\
z &= \frac{\pi\ell}{\lambda}\sin\chi .
\end{aligned} \tag{31.10}
$$

If $\frac{\pi\ell}{\lambda} >> 1$, the second term in (31.9) is negligible compared with the first, so we find

$$P \simeq \frac{2I^2}{c}\frac{\pi\ell}{\lambda}\int_0^\infty dz\,\frac{\sin^2 z}{z^2}$$

$$= \frac{\pi^2 I^2}{c}\frac{\ell}{\lambda}\ , \tag{31.11}$$

where we have used the integral

$$\int_0^\infty dz\,\frac{\sin^2 z}{z^2} = \int_0^\infty dz\,\frac{\sin 2z}{z} = \int_0^\infty dt\,\frac{\sin t}{t} = \frac{\pi}{2}\ . \tag{31.12}$$

Here we observe that the total radiated power, (31.11), increases linearly with ℓ , while the power radiated in the direction perpendicular to the antenna, (31.8), is proportional to ℓ^2 ; that is, as the length of the antenna increases, a larger and larger fraction of the radiated power is concentrated near $\theta = \pi/2$.

To see how much energy is radiated into a very small angular range near $\chi = 0$ (or $\theta = \pi/2$), we consider the power radiated into the main lobe by a long antenna

$$0 < \chi < \frac{\lambda}{\ell} << 1\ .$$

Following the same procedure used to obtain the total power radiated, (31.9), we find for the total power radiated into the main lobe

$$P_{\text{main lobe}} \simeq \frac{2I^2}{c}\int_0^{\lambda/\ell} d\chi\,\frac{\sin^2\left(\frac{\pi\ell}{\lambda}\chi\right)}{\chi^2}$$

$$= \frac{2I^2}{c}\frac{\pi\ell}{\lambda}\int_0^\pi dz\,\frac{\sin^2 z}{z^2}\ . \tag{31.13}$$

The fraction of the energy radiated into the main lobe is obtained by taking the ratio of (31.13) to (31.11):

$$\frac{2}{\pi}\int_0^{\pi} dz\, \frac{\sin^2 z}{z^2} = \frac{2}{\pi}\int_0^{2\pi} dt\, \frac{\sin t}{t}$$

$$= \frac{2}{\pi}\int_0^{\infty} dt\, \frac{\sin t}{t} - \frac{2}{\pi}\int_{2\pi}^{\infty} dt\, \frac{\sin t}{t}$$

$$= 1 - \frac{1}{\pi^2} + \frac{1}{2\pi^4} - \ldots \simeq 0.90 \quad , \tag{31.14}$$

where the infinite series is derived by integrating by parts repeatedly. Over 90% of the power is radiated into the main lobe, which has angular width λ/ℓ , implying that the radiation from the antenna is highly directional, a feature characteristic of large systems. In contrast, small systems, for which the dipole approximation is valid, typically have angular distributions proportional to $\sin^2\theta$.

Lecture 4

XXXII. SPECTRAL DISTRIBUTION OF RADIATION

32-1. Spectral and Angular Distribution

In the previous sections, we discussed the angular distribution of the radiation produced by a time varying charge and current distribution. Here we will turn our attention to the spectral characteristics of this radiation, that is, its dependence on frequency, or wavelength. To investigate this dependence, we return to our starting point, the potentials in the Lorentz gauge, given by (28.38) and (28.39),

$$\phi(\vec{r},t) = \int (d\vec{r}')\, dt' \, \frac{\delta\left(\frac{1}{c}\,|\vec{r}-\vec{r}'| - (t-t')\right)}{|\vec{r}-\vec{r}'|}\, \rho(\vec{r}',t') \;, \tag{32.1}$$

$$\vec{A}(\vec{r},t) = \int (d\vec{r}')\, dt' \, \frac{\delta\left(\frac{1}{c}\,|\vec{r}-\vec{r}'| - (t-t')\right)}{|\vec{r}-\vec{r}'|}\, \frac{1}{c}\,\vec{j}(\vec{r}',t') \;. \tag{32.2}$$

In deriving these results, we had used the spectral representation [cf. (28.33)]

$$\delta\left(\frac{1}{c}\,|\vec{r}-\vec{r}'| - (t-t')\right) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega\left[\frac{1}{c}\,|\vec{r}-\vec{r}'| - (t-t')\right]} \;. \tag{32.3}$$

If we now reinsert (32.3) into (32.1) and (32.2), and carry out the t' integration by introducing the Fourier transform

$$\int_{-\infty}^{\infty} dt'\, e^{i\omega t'}\, f(t') = f(\omega) \;, \tag{32.4a}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t'}\, f(\omega) = f(t') \;, \tag{32.4b}$$

we obtain the Fourier transformed versions of (32.1) and (32.2):

Exact

$$\phi(\vec{r},\omega) = \int (d\vec{r}\,') \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \rho(\vec{r}\,',\omega) \quad , \tag{32.5}$$

$$\vec{A}(\vec{r},\omega) = \int (d\vec{r}\,') \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \frac{1}{c}\vec{j}(\vec{r}\,',\omega) \quad . \tag{32.6}$$

We observe that if $f(t')$ is a real function of t', its Fourier transform, $f(\omega)$, satisfies the condition

$$f(\omega)^* = f(-\omega) \; ; \tag{32.7}$$

consequently,

$$f(\omega)^* \, f(\omega) = |f(\omega)|^2 = f(-\omega)\, f(\omega) \tag{32.8}$$

is a real positive number, symmetric under the interchange $\omega \to -\omega$. This implies that the algebraic sign of ω is not significant, since only its magnitude enters into physical quantities.

Let us focus our attention on the radiation fields, far from the sources. Following the procedure given in Subsection 29-1, in particular, using the expansion (29.1) for $|\vec{r}-\vec{r}\,'|$ in (32.5) and (32.6), we obtain the asymptotic expression for the potentials, in terms of spatial Fourier transforms,

$$\phi(\vec{r},\omega) \simeq \frac{e^{i\frac{\omega}{c}r}}{r} \int (d\vec{r}\,')\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}\,'} \rho(\vec{r}\,',\omega) \quad , \tag{32.9}$$

$$A(\vec{r},\omega) \simeq \frac{e^{i\frac{\omega}{c}r}}{r} \int (d\vec{r}\,')\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}\,'} \frac{1}{c}\vec{j}(\vec{r}\,',\omega) \quad . \tag{32.10}$$

Evidently, the effectiveness of radiation with a given wavelength and direction of propagation depends upon the Fourier analysis of the time and spatial dependences of the charges and currents. In the exponential, the term $\frac{\omega}{c}\vec{n}\cdot\vec{r}'$, which is of the order of the ratio of the size of the system to the wavelength of the radiation, is significant for all but small systems. The corresponding field strengths can now be computed from the time Fourier transforms of (28.3) and (28.4):

$$\vec{E}(\vec{r},\omega) = i\,\frac{\omega}{c}\,\vec{A}(\vec{r},\omega) - \vec{\nabla}\phi(\vec{r},\omega) \quad , \tag{32.11a}$$

$$\vec{B}(\vec{r},\omega) = \vec{\nabla}\times\vec{A}(\vec{r},\omega) \quad , \tag{32.11b}$$

where we have used the effective replacement

$$\frac{\partial}{\partial t} \to -i\omega \tag{32.12}$$

since

$$\int_{-\infty}^{\infty} dt\; e^{i\omega t}\,\frac{\partial}{\partial t}\,F(\vec{r},t) = -i\omega\,F(\vec{r},\omega) \quad , \tag{32.13}$$

provided that $F(\vec{r},t) \to 0$ as $t \to \pm\infty$. Physically, the time boundary conditions state that in the infinite past and the infinite future, nothing is happening: what is significant for our observation takes place in a finite time interval only. The measure of significant variation in r is the wavelength, $\lambda = \frac{2\pi c}{\omega}$, as indicated by the effective replacement for the gradients in (32.11),

$$\vec{\nabla} \to i\,\frac{\omega}{c}\,\vec{n} \tag{32.14}$$

[recall (29.6)]. Consequently, the asymptotic forms of the electric and magnetic fields are

$$\vec{E}(\vec{r},\omega) \sim i\frac{\omega}{c}\frac{e^{i\frac{\omega}{c}r}}{r}\int(d\vec{r}')\,e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}'}\frac{1}{c}\vec{j}(\vec{r}',\omega)$$

$$- i\frac{\omega}{c}\frac{e^{i\frac{\omega}{c}r}}{r}\vec{n}\int(d\vec{r}')\,e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}'}\rho(\vec{r}',\omega) \quad , \qquad (32.15a)$$

$$\vec{B}(\vec{r},\omega) \sim i\frac{\omega}{c}\frac{e^{i\frac{\omega}{c}r}}{r}\vec{n}\times\int(d\vec{r}')\,e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}'}\frac{1}{c}\vec{j}(\vec{r}',\omega) \quad . \qquad (32.15b)$$

By using the Fourier transformed version of the local charge conservation condition, (28.2),

$$-i\omega\rho(\vec{r},\omega) + \vec{\nabla}\cdot\vec{j}(\vec{r},\omega) = 0 \quad , \qquad (32.16)$$

we may rewrite the second term of (32.15a) as

$$-\frac{1}{c}\frac{e^{i\frac{\omega}{c}r}}{r}\vec{n}\int(d\vec{r}')\,e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}'}\vec{\nabla}'\cdot\vec{j}(\vec{r}',\omega)$$

$$= -\frac{i\omega}{c}\frac{e^{i\frac{\omega}{c}r}}{r}\vec{n}\,\vec{n}\cdot\int(d\vec{r}')\,e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}'}\frac{1}{c}\vec{j}(\vec{r}',\omega) \quad . \qquad (32.17)$$

The electric field now becomes

$$\vec{E}(\vec{r},\omega) \sim \frac{e^{i\frac{\omega}{c}r}}{r}\,i\frac{\omega}{c}(\overleftrightarrow{1} - \vec{n}\,\vec{n})\cdot\int(d\vec{r}')\,e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}'}\frac{1}{c}\vec{j}(\vec{r}',\omega)$$

$$= -\frac{e^{i\frac{\omega}{c}r}}{r}\,i\frac{\omega}{c}\vec{n}\times\left[\vec{n}\times\int(d\vec{r}')\,e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}'}\frac{1}{c}\vec{j}(\vec{r}',\omega)\right]$$

$$= -\vec{n}\times\vec{B}(\vec{r},\omega) \quad , \qquad (32.18)$$

which reconfirms (29.13).

Before proceeding, we remark that the relation between the two terms in (32.15a) can also be obtained by using the Lorentz gauge condition; this is not surprising since the consistency of the Lorentz gauge depends upon current conservation. The Fourier transform of (28.13) reads

$$\vec{\nabla}\cdot\vec{A}(\vec{r},\omega) - i\frac{\omega}{c}\phi(\vec{r},\omega) = 0 \quad , \tag{32.19}$$

which becomes, upon using the asymptotic replacement (32.14),

$$i\frac{\omega}{c}[\vec{n}\cdot\vec{A}(\vec{r},\omega) - \phi(\vec{r},\omega)] = 0 \quad ,$$

or

$$\phi(\vec{r},\omega) \sim \vec{n}\cdot\vec{A}(\vec{r},\omega) \quad , \tag{32.20}$$

from which the reduction (32.18) follows.

The instantaneous flux of energy, at a particular time t, is given by Poynting's vector

$$\vec{S}(\vec{r},t) = \frac{c}{4\pi}\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t) \quad , \tag{32.21}$$

so the total radiated energy crossing a unit area of surface normal to $\vec{S}$ is

$$\int_{-\infty}^{\infty} dt\, \vec{S}(\vec{r},t) = \frac{c}{4\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\vec{E}(\vec{r},\omega)^{*} e^{i\omega t} \times \vec{B}(\vec{r},t)$$

$$= \frac{c}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\vec{E}(\vec{r},\omega)^{*} \times \vec{B}(\vec{r},\omega)$$

$$= \vec{n}\,\frac{c}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}|\vec{B}(\vec{r},\omega)|^{2} \quad , \tag{32.22}$$

using (32.18), and the fact that $\vec{n}\cdot\vec{B} = 0$. The energy flows in the direction of $\vec{n}$ and the energy radiated per unit area perpendicular to this direction is

$$\int_{-\infty}^{\infty} dt\ \vec{n}\cdot\vec{S}(\vec{r},t) = \frac{c}{4\pi^2}\int_0^{\infty} d\omega\,|\vec{B}(\vec{r},\omega)|^2 \quad , \tag{32.23}$$

where we have used the symmetry property (32.8). As before, it is more useful to consider the total energy radiated into the solid angle $d\Omega$ [see (29.16)],

$$\int_{-\infty}^{\infty} dt(\vec{n}\cdot\vec{S})\ r^2 d\Omega = d\Omega r^2 \frac{c}{4\pi^2}\int_0^{\infty} d\omega\,|\vec{B}(\vec{r},\omega)|^2$$

$$\equiv d\Omega \int_0^{\infty} d\omega\ \frac{dE(\omega)}{d\Omega} \quad , \tag{32.24}$$

where [cf. Problems 4 and 5]

$$\frac{dE(\omega)}{d\Omega} = \frac{\omega^2}{4\pi^2 c^3}\left|\vec{n}\times\int (d\vec{r})\ e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\ \vec{j}(\vec{r},\omega)\right|^2$$

$$= \frac{\omega^2}{4\pi^2 c}\left\{\left|\int (d\vec{r})\ e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\ \frac{1}{c}\vec{j}(\vec{r},\omega)\right|^2\right.$$

$$\left. - \left|\int (d\vec{r})\ e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\ \rho(\vec{r},\omega)\right|^2\right\} \quad , \tag{32.25}$$

is the general expression for the spectral distribution, the energy radiated per unit frequency per unit solid angle in the direction of observation $\vec{n}$. This equation is the analog of Eq. (29.17) for $dP/d\Omega$.

Lecture 5

32-2. Spectral Distribution for Dipole Radiation

As an application of the general result, (32.25), we consider a small system with typical dimension a much smaller than the reduced wavelength of the radiation, that is, $\lambda\!\!\!^{-} \equiv \frac{\lambda}{2\pi} >> a$. Then, the exponential factor in (32.25) can be approximated by unity, since

$$\frac{\omega}{c}\vec{n}\cdot\vec{r} = \frac{1}{\lambda\!\!\!^{-}}\vec{n}\cdot\vec{r} << 1 \quad , \tag{32.26}$$

whence the spectral distribution becomes

$$\begin{aligned}\frac{dE(\omega)}{d\Omega} &\simeq \frac{\omega^2}{4\pi^2 c^3}\left|\vec{n} \times \int (d\vec{r})\ \vec{j}(\vec{r},\omega)\right|^2 \\ &= \frac{\omega^2}{4\pi^2 c^3}\sin^2\theta \left|\int (d\vec{r})\ \vec{j}(\vec{r},\omega)\right|^2\end{aligned} \tag{32.27}$$

where θ is the angle between the observation direction and the direction of the average current flow. For the small system discussed here, the only reference to the direction of observation, $\vec{n}$, occurs as a multiplicative factor, implying the $\sin^2\theta$ behavior exhibited above, characteristic of dipole radiation. For larger systems, $\vec{n}$ also enters in the exponential so that the angular distribution could be completely different. The total energy radiated per unit frequency range can be obtained by integrating (32.27) over all angles,

$$E(\omega) = \frac{2}{3}\frac{\omega^2}{\pi c^3}\left|\int (d\vec{r})\ \vec{j}(\vec{r},\omega)\right|^2 \quad , \tag{32.28}$$

where we have used (29.23).

Suppose we further specialize to a single point charge in non-relativistic motion, corresponding to the current density

$$\vec{j}(\vec{r},t) = e\vec{v}(t)\ \delta(\vec{r}-\vec{r}(t)) \ , \tag{32.29}$$

implying for the volume integral of the current density

$$\int (d\vec{r})\ \vec{j}(\vec{r},t) = e\vec{v}(t) \quad , \tag{32.30a}$$

which is the time derivative of the dipole moment; the Fourier transform of this is

$$\int (d\vec{r})\ \vec{j}(\vec{r},\omega) = e\vec{v}(\omega) \quad . \tag{32.30b}$$

The energy radiated per unit frequency interval, $E(\omega)$, is

$$\begin{aligned} E(\omega) &= \frac{2}{3}\frac{e^2}{\pi c^3}\ \omega^2 |\vec{v}(\omega)|^2 \\ &= \frac{2}{3}\frac{e^2}{\pi c^3}\ |\dot{\vec{v}}(\omega)|^2 \quad , \end{aligned} \tag{32.31}$$

where $\dot{\vec{v}}(\omega)$ is the Fourier transform of $\dot{\vec{v}}(t)$:

$$\dot{\vec{v}}(\omega) = \int_{-\infty}^{\infty} dt\ e^{i\omega t}\ \dot{\vec{v}}(t) = -i\omega\vec{v}(\omega) \quad . \tag{32.32}$$

The total energy radiated,

$$E_{rad} = \int_0^{\infty} E(\omega)\ d\omega = \frac{2}{3}\frac{e^2}{\pi c^3}\int_0^{\infty} d\omega\ \omega^2 |\vec{v}(\omega)|^2 \quad , \tag{32.33}$$

can also be obtained from the Larmor formula, (29.24):

$$E_{rad} = \int_{-\infty}^{\infty} dt\ P(t)$$

$$= \frac{2}{3}\frac{e^2}{c^3}\int_{-\infty}^{\infty} dt\ |\dot{\vec{v}}(t)|^2$$

$$= \frac{2}{3}\frac{e^2}{\pi c^3}\int_0^{\infty} d\omega\ \omega^2 |\vec{v}(\omega)|^2 \quad ,$$

as expected. In the above, we have used (32.32), and the theorem

$$\int_{-\infty}^{\infty} dt |f(t)|^2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |f(\omega)|^2 = \int_0^{\infty} \frac{d\omega}{\pi} |f(\omega)|^2 \quad . \tag{32.34}$$

Thus we see that we can calculate the total energy radiated by a small system through the use either of the spectral distribution, (32.28), or of the Larmor formula for the power, (29.29). The equivalence of these two descriptions is demonstrated generally in Problem 5, where the spectral distribution $dE(\omega)/d\Omega$, (32.25), is derived directly from the power spectrum, $dP(t)/d\Omega$, (29.17).

32-3. Damped Harmonic Motion

As a further simple application of the spectral distribution, consider a model in which a charged particle undergoes damped motion in a Hooke's law potential,

$$\ddot{\vec{r}} = -\omega_0^2 \vec{r} - \gamma\dot{\vec{r}} \quad , \tag{32.35}$$

where, due to the radiation produced by the accelerating charged particle, there is a damping force represented by $-\gamma\dot{\vec{r}}$, which we will assume to be small:

$$\frac{\gamma}{\omega_0} << 1 \ . \tag{32.36}$$

This model is often taken as an oversimplified description of a bound electron inside an atom. (Recall Subsection 5-2.) Given the initial conditions that at $t = 0$, the particle has a displacement $\vec{a}$ from the force center and has zero velocity, the solution to (32.35) when (32.36) holds is approximately

$$\vec{r}(t) \simeq \vec{a} \cos\omega_o t \; e^{-\frac{1}{2}\gamma t} , \quad \text{for} \quad t > 0 , \tag{32.37}$$

which exhibits the fact that many oscillations are necessary before significant damping occurs. The velocity of the particle,

$$\vec{v}(t) \simeq -\vec{a}\,\omega_o \sin\omega_o t \; e^{-\frac{1}{2}\gamma t} , \quad \text{for} \quad t > 0 , \tag{32.38}$$

has the Fourier transform

$$\begin{aligned}
\vec{v}(\omega) &= \int_0^\infty dt(-\vec{a}\omega_o)\, e^{i\omega t} \sin\omega_o t \; e^{-\frac{1}{2}\gamma t} \\
&= \frac{i}{2}\vec{a}\omega_o \int_0^\infty dt \left[-e^{i\left(\omega - \omega_o + i\frac{\gamma}{2}\right)t} + e^{i\left(\omega + \omega_o + i\frac{\gamma}{2}\right)t} \right] \\
&= \frac{\omega_o \vec{a}}{2} \left(\frac{1}{\omega - \omega_o + \frac{i}{2}\gamma} - \frac{1}{\omega + \omega_o + \frac{i}{2}\gamma} \right) .
\end{aligned} \tag{32.39}$$

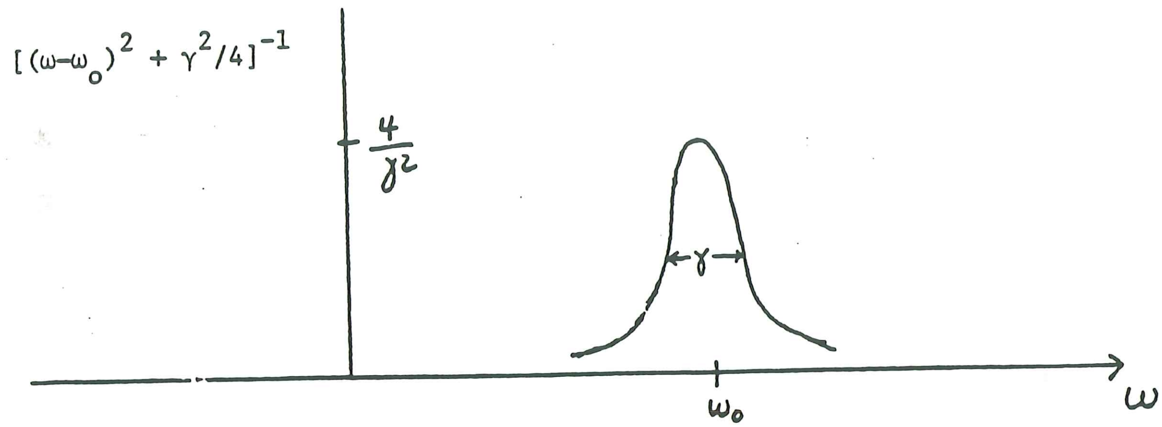
Without loss of generality, we may assume $\omega > 0$ here, in which case the two terms are very different. Only the first denominator can be small, implying that radiation is predominantly emitted with frequencies $\omega \simeq \omega_o$. Therefore, we approximate the square of the magnitude of (32.39) by

$$|\vec{v}(\omega)|^2 \simeq \frac{\omega_o^2 a^2}{4} \frac{1}{(\omega-\omega_o)^2 + \gamma^2/4} , \tag{32.40}$$

implying for the energy radiated per unit frequency range, (32.31),

$$E(\omega) \approx \frac{2}{3}\frac{e^2}{\pi c^3}\omega_o^2\frac{\omega_o^2 a^2}{4}\frac{1}{(\omega-\omega_o)^2+\gamma^2/4}\,, \quad \text{for} \quad \omega \sim \omega_o\,. \tag{32.41}$$

The behavior of $[(\omega-\omega_o)^2+\gamma^2/4]^{-1}$ is plotted in the figure below:



which exhibits what is called the Lorentzian line shape. Since when $|\omega-\omega_o| = \gamma/2$, the intensity is half that at ω_o, where the intensity is maximum, γ is the width of the spectrum at half-maximum intensity. In the limit of negligible damping, $\gamma \to 0$, only one frequency, ω_o, is emitted, with infinite intensity. In general, a range of frequencies is emitted, with γ being a measure of the sharpness of the spectrum. That is, if the system decays slowly (γ small), the emission line is very narrow, while if it decays rapidly (γ large), it is very broad.

To demonstrate that no significant amount of energy is radiated outside of this sharp peak, we evaluate the total energy radiated in the peak by integrating (32.41) over all frequencies:

$$E_{rad} = \int_0^\infty d\omega\, E(\omega)$$

$$\simeq \frac{2}{3}\frac{e^2}{\pi c^3}\omega_o^2\left(\frac{\omega_o^2a^2}{4}\right)\int_0^\infty \frac{d\omega}{(\omega-\omega_o)^2+\gamma^2/4}$$

$$\simeq \frac{2}{3}\frac{e^2}{\pi c^3}\omega_o^2\,\frac{\omega_o^2a^2}{4}\,\frac{2\pi}{\gamma}\ , \tag{32.42}$$

where we have used the integral

$$\int_0^\infty \frac{d\omega}{(\omega-\omega_o)^2+\gamma^2/4} = \int_{-2\omega_o/\gamma}^\infty \frac{(\gamma/2)\ dx}{\frac{\gamma^2}{4}(1+x^2)} \qquad x = \frac{2\omega}{\gamma} - \frac{2\omega_o}{\gamma}$$

$$\simeq \frac{2}{\gamma}\int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{2\pi}{\gamma}\ . \tag{32.43}$$

We wish to compare this radiated energy to the original energy of the oscillator. Since the latter is given by

$$E_{initial} = \frac{1}{2}m\omega_o^2a^2\ , \tag{32.44}$$

the radiated energy, (32.42), can be rewritten as

$$E_{rad} = E_{initial}\left(\frac{2}{3}\frac{e^2}{mc^3}\frac{\omega_o^2}{\gamma}\right)\ . \tag{32.45}$$

Assuming that the Lorentzian peak adequately accounts for the energy radiated and that there are no other forms of energy dissipation, we learn, from the conservation of energy,

$$\gamma = \frac{2}{3}\frac{e^2}{mc^3}\omega_o^2\ . \tag{32.46}$$

Since this is the same result found in Problem 2 where we calculate the power radiated by the oscillator and identify γ from

$$P = -\frac{dE}{dt} = \gamma E\ , \qquad E = \text{energy of oscillator,} \tag{32.47}$$

we conclude that (32.41) is an adequate representation of the energy spectrum. Besides thus demonstrating that most of the radiated energy is contained within the peak, we have also established self-consistency in that the damping of the oscillator is shown to arise from the reaction of the radiation back on the radiating system.

XXXIII. MACROSCOPIC SPECTRAL DISTRIBUTION AND CERENKOV RADIATION

33-1. Spectral Distribution at a Particular Macroscopic Time

In the previous section, we derived a general expression for the spectral distribution, (32.25), which accounts for the radiation from all times, $t \to -\infty$ to $t \to +\infty$. Here we wish to obtain a generalization applicable to a limited epoch. In so doing, we must note that time and frequency are complementary; that is, a time interval of many periods is required in order to identify a corresponding frequency. We first rewrite (32.25) in the space-time form:

For field so approximate in that sense

$$\frac{dE(\omega)}{d\Omega} = \frac{\omega^2}{4\pi^2 c^3}\left[\vec{n}\times\int(d\vec{r})\,e^{i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\int dt\,e^{-i\omega t}\,\vec{j}(\vec{r},t)\right]$$

$$\cdot\left[\vec{n}\times\int(d\vec{r}\,')\,e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}\,'}\int dt'\,e^{i\omega t'}\,\vec{j}(\vec{r}\,',t')\right], \qquad (33.1)$$

and focus our attention on the part involving time integrations:

$$\int dt\,dt'\,e^{-i\omega(t-t')}\,\vec{j}(\vec{r},t)\,\vec{j}(\vec{r}\,',t)$$

$$= \int dT\,d\tau\,e^{-i\omega\tau}\,\vec{j}\left(\vec{r},\,T+\frac{\tau}{2}\right)\vec{j}\left(\vec{r}\,',\,T-\frac{\tau}{2}\right), \qquad (33.2)$$

where we have introduced the average time and the time difference,

$$T = \frac{1}{2}(t+t'), \quad \tau = t-t', \quad dtdt' = dTd\tau. \qquad (33.3)$$

$\frac{1}{2}(dt-dt')(dt+dt') = \frac{dt^2-dt'^2}{2}$

Jacobian $\begin{vmatrix}\frac{\partial T}{\partial t} & \frac{\partial T}{\partial t'}\\ \frac{\partial \tau}{\partial t} & \frac{\partial \tau}{\partial t'}\end{vmatrix} = \begin{vmatrix}1/2 & 1/2\\ 1 & -1\end{vmatrix} = \frac{1}{2}+\frac{1}{2}=1$ ✓

From the exponential structure of (33.2), we infer that the important range of τ that contributes to the integral is of order $1/\omega$, thus setting the time scale for the emission of radiation. This microscopic time scale may be much smaller than macroscopic time intervals; for example, for visible light, $\tau \sim 10^{-15}$ sec. The time T is then interpreted as the average (macroscopic) time of emission, which can be specified only to within a time of order τ. Substituting (33.2) into (33.1), we infer the power spectrum at time T,

$$\frac{dE(\omega)}{d\Omega} = \int dT \, \frac{dP(\omega,T)}{d\Omega} , \tag{33.4a}$$

where

$$\frac{dP(\omega,T)}{d\Omega} = \frac{\omega^2}{4\pi^2 c}\int_{-\infty}^{\infty} d\tau \, e^{-i\omega\tau}\left[\vec{n}\times\int (d\vec{r})\, e^{i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\,\frac{1}{c}\vec{j}\left(\vec{r}, T+\frac{\tau}{2}\right)\right]$$
$$\cdot\left[\vec{n}\times\int (d\vec{r}\,')\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}\,'}\,\frac{1}{c}\vec{j}\left(\vec{r}\,', T-\frac{\tau}{2}\right)\right] , \tag{33.4b}$$

or, alternatively, using the second form of (32.25),

$$\frac{dP(\omega,T)}{d\Omega} = \frac{\omega^2}{4\pi^2 c}\int_{-\infty}^{\infty} d\tau \, e^{-i\omega\tau}\left\{\left[\int (d\vec{r})\, e^{i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\,\frac{1}{c}\vec{j}\left(\vec{r}, T+\frac{\tau}{2}\right)\right]\cdot\right.$$
$$\cdot\left[\int (d\vec{r}\,')\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}\,'}\,\frac{1}{c}\vec{j}\left(\vec{r}\,', T-\frac{\tau}{2}\right)\right]$$
$$\left.-\left[\int (d\vec{r})\, e^{i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\,\rho\left(\vec{r}, T+\frac{\tau}{2}\right)\right]\left[\int (d\vec{r}\,')\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}\,'}\,\rho\left(\vec{r}\,', T-\frac{\tau}{2}\right)\right]\right\} . \tag{33.4c}$$

33-2. Cerenkov Radiation

As an application of (33.4), we consider the radiation produced by a charged particle moving with constant velocity $\vec{v}$, for which the charge and current densities are

$$\rho(\vec{r},t) = e\,\delta(\vec{r}-\vec{v}t) ,$$
$$\vec{j}(\vec{r},t) = e\vec{v}\,\delta(\vec{r}-\vec{v}t) . \tag{33.5}$$

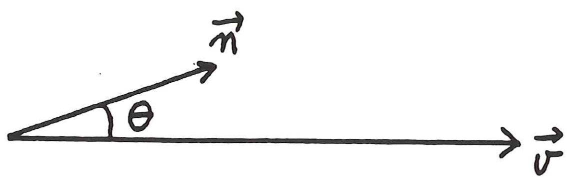
The spatial integrals in (33.4c) are trivially evaluated:

$$\int (d\vec{r})\, e^{\pm i \frac{\omega}{c} \vec{n}\cdot\vec{r}} \begin{Bmatrix} \rho\left(\vec{r},\, T \pm \frac{\tau}{2}\right) \\ \frac{1}{c}\vec{j}\left(\vec{r},\, T \pm \frac{\tau}{2}\right) \end{Bmatrix} = e^{\pm i \frac{\omega}{c} \vec{n}\cdot\vec{v}\left(T \pm \frac{\tau}{2}\right)} \begin{Bmatrix} e \\ \frac{e}{c}\vec{v} \end{Bmatrix} . \tag{33.6}$$

When these are substituted into (33.4c), the T dependence disappears, as expected, and we obtain

$$\begin{aligned} \frac{dP(\omega)}{d\Omega} &= \frac{\omega^2}{4\pi^2 c} \int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\, e^2\left(\frac{v^2}{c^2} - 1\right) e^{i \frac{\omega}{c} \vec{n}\cdot\vec{v}\tau} \\ &= \frac{\omega^2}{4\pi^2 c}\, e^2\left(\frac{v^2}{c^2} - 1\right) \int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau\left(1 - \frac{\vec{n}\cdot\vec{v}}{c}\right)} \\ &= \frac{\omega^2 e^2}{4\pi^2 c}\left(\frac{v^2}{c^2} - 1\right) 2\pi\delta\left(\omega\left(1 - \frac{v}{c}\cos\theta\right)\right) \end{aligned} \tag{33.7}$$

where θ is the angle between the direction of observation, $\vec{n}$, and the velocity of the particle, $\vec{v}$.



The δ-function implies that there is no radiation, since

$$\frac{v}{c}\cos\theta < 1 .$$

This is the familiar result that a charged particle moving with a constant velocity in vacuum does not radiate.

However, if it were possible that

$$\frac{v}{c} > 1 , \tag{33.8}$$

the argument of the delta function could vanish, and radiation would be emitted by the charged particle. Is there any way of effectively satisfying (33.8)? In a medium, light can move with a speed, c', less than c, and correspondingly the speed of a particle can be greater than c'. Now does the particle radiate? Recall that the macroscopic Maxwell's equations, (4.31), for a medium with dielectric constant ε and magnetic permeability μ, can be put into vacuum form (1.27), by the redefinitions (recall Subsection 7-2)

$$\vec{E}' = \sqrt{\varepsilon}\,\vec{E} , \quad \vec{H}' = \sqrt{\mu}\,\vec{H} , \quad c' = \frac{c}{\sqrt{\varepsilon\mu}} , \quad \rho' = \frac{1}{\sqrt{\varepsilon}}\rho , \quad \vec{j}' = \frac{1}{\sqrt{\varepsilon}}\vec{j} . \tag{33.9}$$

Therefore, the power radiated when a charged particle is moving with constant velocity in a non-magnetic medium ($\mu = 1$) of index of refraction $n = \sqrt{\varepsilon}$ can be obtained immediately from (33.7) by the substitutions $e \to e/n$ and $c \to c/n$:

$$\frac{dP(\omega)}{d\Omega} = \frac{\omega^2}{4\pi^2\left(\frac{c}{n}\right)}\left(\frac{e}{n}\right)^2\left[\frac{v^2}{\left(\frac{c}{n}\right)^2} - 1\right] 2\pi\delta\left(\omega\left(1 - \frac{v}{\left(\frac{c}{n}\right)}\cos\theta\right)\right) . \tag{33.10}$$

Thus, indeed there is radiation if the condition

$$\frac{nv}{c}\cos\theta = 1 , \tag{33.11a}$$

or

$$\cos\theta = \frac{c}{nv} < 1 , \tag{33.11b}$$

is satisfied. Here we see that, for a charged particle moving with a constant velocity inside a medium characterized by an index of refraction $n > 1$, electromagnetic radiation can be emitted by the charged particle if the criterion

$$v > \frac{c}{n}$$

is satisfied. Such a medium can be easily found for fast particles. The radiation is emitted on a cone described by (33.11b), and because of its unique characteristics, is especially suited for determining the velocities of relativistic charged particles. This phenomenon is called Cerenkov radiation. We emphasize that the condition (33.11b) can only be satisfied when $n > 1$ which means that, because the index of refraction depends on frequency, that is, media are dispersive, it can only be satisfied for a finite range of frequencies (typically, in the optical region). Moreover, this dispersion implies that different frequencies are emitted at different angles. This radiation is commonly seen in water moderated nuclear reactors as blue light surrounding the core.

The frequency spectrum of the radiated power can be obtained by integrating (33.10) over all angles,

$$\begin{aligned} P(\omega) &= \int d\Omega \, \frac{dP(\omega)}{d\Omega} \\ &= \omega^2 \frac{e^2}{nc} \frac{n^2 v^2}{c^2} \left[1 - \frac{c^2}{n^2 v^2} \right] \int_{-1}^{1} d(\cos\theta) \; \delta\left[\omega\left(1 - \frac{nv}{c} \cos\theta \right)\right] \\ &= \omega \frac{e^2 v}{c^2} \left[1 - \frac{c^2}{n^2(\omega)\, v^2} \right], \quad \text{if} \quad n(\omega) > \frac{c}{v}, \end{aligned} \qquad (33.12)$$

and the total radiated power is

$$-\frac{dE}{dt} = P = \int_0^\infty d\omega\, P(\omega)$$

$$= \int d\omega\, \omega \frac{e^2 v}{c^2} \left[1 - \frac{c^2}{n^2(\omega)\, v^2} \right] , \qquad (33.13a)$$

where E is the energy of the particle. In practice, it is more convenient to consider the energy lost per unit distance traveled by the particle, since this is what can be directly measured by a Cerenkov counter:

$$-\frac{dE}{dz} = \int d\omega\, \omega \frac{e^2}{c^2} \left[1 - \frac{c^2}{n^2(\omega)\, v^2} \right] . \qquad (33.13b)$$

In (33.13) it is understood that the ω integration extends only over the range where $n(\omega) > c/v$. Finally we note that detection is a quantum process, involving photons. The energy of a photon of frequency $\nu = \omega/2\pi$ is $h\nu = \hbar\omega$, where $h = 2\pi\hbar$ is Planck's constant. Therefore, the number of photons emitted per unit length is

$$\frac{dN}{dz} = \alpha \int \frac{d\omega}{c} \left[1 - \frac{c^2}{n^2(\omega)\, v^2} \right] , \qquad (33.14)$$

where the fine structure constant α is defined by

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137} . \qquad (33.15)$$

To obtain an order of magnitude estimate, we suppose that over a certain range of ω, $\Delta\omega$, the $1 - \frac{c^2}{n^2 v^2}$ factor is of order 1:

in practice

$$\frac{dN}{dz} \sim \alpha \frac{\Delta\omega}{c} \sim \frac{\alpha}{\lambda\!\!\!^-} = \frac{1}{137\lambda\!\!\!^-} , \qquad (33.16)$$

where we have noted that, typically, the range of wavelengths is of the same order of magnitude as the wavelengths themselves. Equation (33.16) implies that, roughly speaking, in a distance of 137 wavelengths, one photon is emitted. In the visible spectrum, where $\lambda \sim 10^{-5}$ cm, about 10^3 photons/cm are emitted. (A more accurate estimate is 10^2 photons/cm.)

In reality in a distance of $\frac{137}{1-\eta}\lambda$ one photon is emitted

in other words $\eta \sim .9$

This seems much more logical anyways

XXXIV. Constant Acceleration

Lecture 6

Next, let us examine the characteristics of the radiation emitted by a uniformly accelerated charged particle. For simplicitly we will assume that the particle is non-relativistic (which remains valid only for a finite length of time), but the conclusions we will draw are independent of that simplification. The equations of motion of such a particle are

$$\vec{r}(t) = \frac{1}{2}\vec{a}\,t^2 \quad , \tag{34.1a}$$

$$\vec{v}(t) = \vec{a}\,t \quad , \tag{34.1b}$$

in terms of which we construct the current density

$$\vec{j}(\vec{r},t) = e\vec{v}(t)\,\delta[\vec{r} - \vec{r}(t)] \quad . \tag{34.1c}$$

The power spectrum, at time T, (33.4b), becomes $(T+\frac{\tau}{2})(T-\frac{\tau}{2})$

$$\begin{aligned}
\frac{dP(\omega,T)}{d\Omega} &= \frac{\omega^2}{4\pi^2 c^3}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\,(\vec{n}\times\vec{a})^2\, e^2\,[T^2 - (\tau^2/4)] \\
&\qquad \times \exp\left[i\,\frac{\omega}{c}\,\vec{n}\cdot\frac{\vec{a}}{2}\left([T + (\tau/2)]^2 - [T - (\tau/2)]^2\right)\right] \\
&= \frac{\omega^2 e^2}{4\pi^2 c^3}(\vec{n}\times\vec{a})^2\int_{-\infty}^{\infty} d\tau\,\exp\left[-i\omega\tau[1 - \frac{1}{c}\,\vec{n}\cdot\vec{v}(T)]\right]\,[T^2 - (\tau^2/4)] \\
&= \frac{\omega^2 e^2}{2\pi c^3}(\vec{n}\times\vec{a})^2\left\{T^2\,\delta\left[\omega\left(1 - \frac{\vec{n}\cdot\vec{v}}{c}\right)\right] + \frac{1}{4}\,\delta''\left[\omega\left(1 - \frac{\vec{n}\cdot\vec{v}}{c}\right)\right]\right\} ,
\end{aligned} \tag{34.2}$$

where we have identified the second derivative of the δ-function according to

$$\int_{-\infty}^{\infty} d\tau\tau^2\, e^{-i\tau x} = -\,2\pi\delta''(x) \quad . \tag{34.3}$$

For $\omega > 0$, the argument of the delta functions in (34.2) never vanishes,

$$\frac{dP(\omega,T)}{d\Omega} = 0 \quad , \qquad \text{for } \omega > 0 \quad . \tag{34.4}$$

The power, (34.2), is non-zero only for $\omega = 0$, which corresponds to static fields. Such fields do not correspond to radiation, and a uniformly accelerated charged particle does not radiate. On the other hand, the total radiated power can be computed from the Larmor formula, (29.24):

$$P = \frac{2}{3}\frac{e^2}{c^3}(\dot{\vec{v}})^2 = \frac{2}{3}\frac{e^2}{c^3}a^2 \neq 0 \quad , \tag{34.5}$$

which seems to indicate that a uniformly accelerated charge does radiate. How can we reconcile the above two seemingly contradictory results, (34.4) and (34.5)? Actually there is no logical contradiction, for we recognize that the power spectrum is obtained by a Fourier decomposition of the total radiated power. Since the latter is a constant here, the spectrum consists only of zero frequency, which, to reiterate, does not represent radiation. The Larmor formula thus is not applicable to this situation. (But, see later.) what does this mean

It is instructive to consider this radiation process from the source point of view, discussed in Subsection 30-2. Remember that, there, we obtained the following expression for the rate at which charges in a small, non-relativistic system do work on the electromagnetic field [cf. (30.7)],

$$-\int (d\vec{r})\vec{E}\cdot\vec{j} = \frac{d}{dt}E - \frac{2}{3c^3}\dot{\vec{d}}\cdot\dddot{\vec{d}} \quad , \tag{34.6}$$

where we identified E as the electromagnetic field energy while the remaining term is the power radiated. To obtain the Larmor formula, we neglected a further total time derivative. If we now use the above expression for the power radiated,

$$P = -\frac{2}{3c^3}\dot{\vec{d}}\cdot\dddot{\vec{d}} = -\frac{2}{3}\frac{e^2}{c^3}\vec{v}\cdot\ddot{\vec{v}} \quad , \tag{34.7}$$

since the dipole moment of a point charge is

$$\vec{d} = e\vec{r} \quad , \tag{34.8}$$

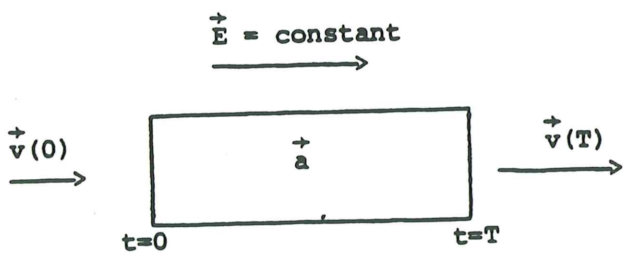
we immediately see that there is no radiation produced by a uniformly accelerated charged particle.

Having persuaded you that there is no radiation produced by uniform acceleration, we now enlarge the picture and recognize that radiation is associated with the whole history of a process, not just a particular period of time. What we have assumed so far is a situation, in which, for all time, the particle undergoes uniform acceleration, which, if nothing else, violates our non-relativistic treatment since eventually the velocity of the particle will become comparable with the speed of light. The essential point to recognize is that uniform acceleration for all time is an idealization of the realistic situation in which uniform acceleration is only experienced for a finite time interval.

As an example, we consider the acceleration of a charged particle by a linear accelerator. Suppose originally the particle moves with a small constant speed $\vec{v}(0)$. At a particular time $t = 0$, it enters a region where a uniform electric field is applied, causing the particle to undergo a constant acceleration $\vec{a}$. After a period of time T, the particle is ejected from the accelerator with a velocity

$$\vec{v}(T) = \vec{v}(0) + \vec{a}T \quad , \tag{34.9}$$

which is still assumed to be small compared to the speed of light. This acceleration process is represented in the figure below.



The spectral distribution of the energy radiated by this accelerated particle is given by (32.31), which can be rewritten in the form

$$E(\omega) = \frac{2}{3}\frac{e^2}{c^3}\frac{1}{\pi}\frac{1}{\omega^2}|\ddot{\vec{v}}(\omega)|^2 \quad , \tag{34.10}$$

since, from (32.32),

$$\dot{\vec{v}}(\omega) = -i\omega\vec{v}(\omega) \quad ,$$

and

$$\ddot{\vec{v}}(\omega) = -i\omega\dot{\vec{v}}(\omega) \quad .$$

The quantity $\ddot{\vec{v}}(\omega)$ can be calculated by noting that since the acceleration is discontinuous at $t = 0$ and $t = T$, the derivative of $\vec{a}$ has an impulse at these two times:

$$\ddot{\vec{v}}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \dddot{\vec{v}}(t)$$

$$= \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, [\vec{a}\delta(t) - \vec{a}\delta(t-T)]$$

$$= \vec{a}(1 - e^{i\omega T})$$

$$= -2i\vec{a}\, e^{i\omega T/2} \sin\left(\frac{1}{2}\,\omega T\right) \quad . \qquad (34.11)$$

The square of the magnitude of (34.11) is

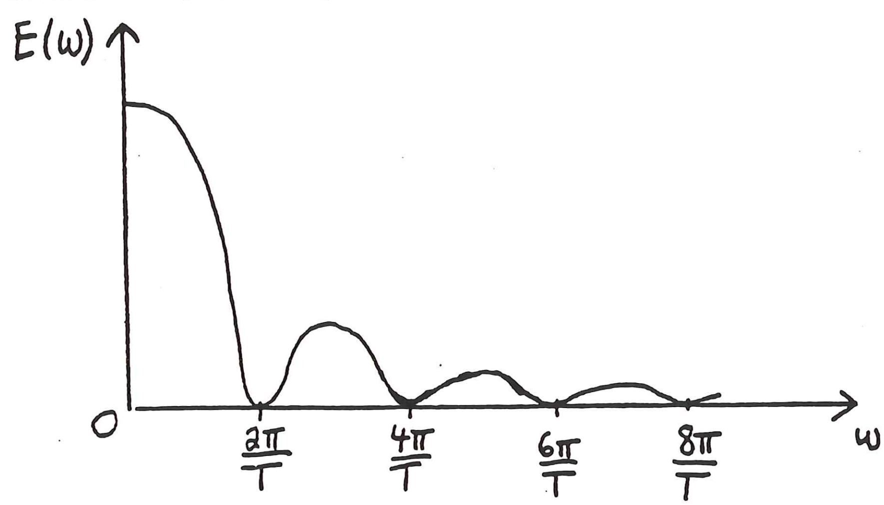
$$|\ddot{\vec{v}}(\omega)|^2 = 4a^2 \sin^2 \frac{1}{2}\,\omega T \quad , \qquad (34.12)$$

so that the spectral distribution of radiated energy is

$$E(\omega) = \frac{2}{3}\,\frac{e^2}{c^3}\,\frac{1}{\pi}\,\frac{4a^2}{\omega^2}\,\sin^2 \frac{1}{2}\,\omega T \quad , \qquad (34.13)$$

which is nonzero in general. Of course, there is no contradiction between this result and (34.4), since we are no longer talking about uniform acceleration for all time. We also observe that the contributions from the changes of acceleration at $t = 0$ and $t = T$ interfere with each other. This is incompatible with locality in time, that is, we cannot specify at which time the radiation is emitted.

To see what are the important frequencies being radiated, we plot the frequency distribution, (34.13), as a function of ω.



The significant range of frequencies is set by the time of acceleration, T,

$$\omega \sim 1/T \quad , \tag{34.14}$$

that is, as T becomes longer, $E(\omega)$ is more and more concentrated at low values of ω. In the limit when $T \to \infty$, we recover the situation of uniform acceleration discussed above. Moreover, we see again that the emission spectrum is not time analyzable, since the relation (34.14) implies that it takes a time of the order of the whole process to determine the frequency. The total energy radiated is obtained from (34.13) by integrating over all frequencies,

$$\begin{aligned}
E_{rad} &= \int_0^\infty d\omega\, E(\omega) = \frac{2}{3}\frac{e^2}{c^3}\frac{4a^2}{\pi}\int_0^\infty \frac{d\omega}{\omega^2}\sin^2\frac{\omega T}{2} \\
&= \frac{2}{3}\frac{e^2}{c^3}a^2T\left(\frac{2}{\pi}\int_0^\infty \frac{dx}{x^2}\sin^2 x\right) \\
&= \frac{2}{3}\frac{e^2}{c^3}a^2T \quad ,
\end{aligned} \tag{34.15}$$

where we have used the integral (31.12). From (34.15), we observe that the total energy radiated per unit time is

$$\frac{E_{rad}}{T} = \frac{2}{3}\frac{e^2}{c^3}a^2 \quad ,$$

which is precisely that obtained from the Larmor formula, (34.5), which refers only to uniform acceleration.

Finally let us calculate, from the source point of view, the radiated power using (34.7),

$$\begin{aligned}
P &= -\frac{2}{3}\frac{e^2}{c^3}\vec{v}\cdot\ddot{\vec{v}} \\
&= -\frac{2}{3}\frac{e^2}{c^3}\vec{a}\cdot[\vec{v}(0)\,\delta(t) - \vec{v}(T)\,\delta(t-T)] \quad ,
\end{aligned} \tag{34.16}$$

which incorrectly attributes the radiation entirely to the beginning and the end of the acceleration process. However, the total radiated energy

$$E_{rad} = \int_{-\infty}^{\infty} dt\, P$$

$$= \frac{2}{3}\frac{e^2}{c^3}\vec{a}\cdot[\vec{v}(T) - \vec{v}(0)]$$

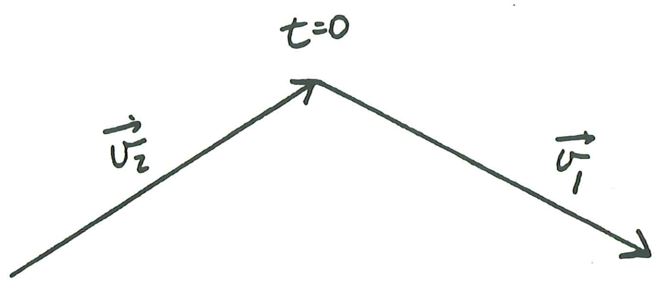
$$= \frac{2}{3}\frac{e^2}{c^3}a^2 T \quad ,$$

is the same as (34.15).

Thus we have seen that the question of whether there is radiation produced by a uniformly accelerated charge is only properly answered by taking into account the beginning and ending of the acceleration. Facetiously, we may say that a uniformly accelerated charge radiates because it is not uniformly accelerated.

XXXV. RADIATION BY IMPULSIVE SCATTERING

As an idealization of a scattering process, consider one in which a charged particle abruptly changes its velocity from a constant value $\vec{v}_2$ to another constant value $\vec{v}_1$. We will calculate the radiation produced by this accelerated particle, bearing in mind that this description can be realistic only for radiation of sufficiently low frequencies since these cannot probe the detailed character of the particle motion.



The charge and current densities before and after the deflection act, which is assumed to take place at $t = 0$, are

$$\begin{aligned} \rho &= e\delta(\vec{r}-\vec{v}_2 t) , \\ \vec{j} &= e\vec{v}_2\delta(\vec{r}-\vec{v}_2 t) , \end{aligned} \qquad \text{for} \quad t < 0 , \tag{35.1a}$$

and

$$\begin{aligned} \rho &= e\delta(\vec{r}-\vec{v}_1 t) , \\ \vec{j} &= e\vec{v}_1\delta(\vec{r}-\vec{v}_1 t) . \end{aligned} \qquad \text{for} \quad t > 0 , \tag{35.1b}$$

The spectral distribution for the radiated energy can be computed by substituting (35.1) into (32.25), where we encounter the integral

$$\int (d\vec{r})\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\, \vec{j}(\vec{r},\omega) = \int (d\vec{r})\, dt\, e^{i\omega t}\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\, \vec{j}(\vec{r},t)$$

$$= e\vec{v}_2 \int_{-\infty}^{0} dt\, e^{i\omega t\left(1 - \frac{1}{c}\vec{n}\cdot\vec{v}_2\right)} + e\vec{v}_1 \int_{0}^{\infty} dt\, e^{i\omega t\left(1 - \frac{1}{c}\vec{n}\cdot\vec{v}_1\right)}$$

$$= i\,\frac{e}{\omega}\left[\frac{\vec{v}_1}{1 - \frac{1}{c}\vec{n}\cdot\vec{v}_1} - \frac{\vec{v}_2}{1 - \frac{1}{c}\vec{n}\cdot\vec{v}_2}\right] . \qquad (35.2)$$

Here we have used the effective evaluations

$$\int_0^{\infty} dt\, e^{i\lambda t} = \frac{i}{\lambda} , \qquad (35.3a)$$

$$\int_{-\infty}^{0} dt\, e^{i\lambda t} = -\frac{i}{\lambda} , \qquad (35.3b)$$

since physical quantities cannot depend on what transpired at infinitely remote times. As a consistency check of (35.3), we can calculate the corresponding integral on ρ directly,

$$\int (d\vec{r})\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}} \rho(\vec{r},\omega) = i\,\frac{e}{\omega}\left[\frac{1}{1 - \frac{1}{c}\vec{n}\cdot\vec{v}_1} - \frac{1}{1 - \frac{1}{c}\vec{n}\cdot\vec{v}_2}\right] , \qquad (35.4)$$

or by means of the charge conservation condition, (32.16),

$$\int (d\vec{r})\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}} \rho(\vec{r},\omega) = \frac{1}{i\omega}\int (d\vec{r})\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\, \vec{\nabla}\cdot\vec{j}(\vec{r},\omega)$$

$$= \frac{\vec{n}}{c}\cdot\left(\int (d\vec{r})\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\, \vec{j}(\vec{r},\omega)\right) .$$

Indeed, if we take the scalar product of (35.2) with $\vec{n}/c$, we have (35.4):

$$i\,\frac{e}{\omega}\left(\frac{\frac{1}{c}\,\vec{n}\cdot\vec{v}_1}{1-\frac{1}{c}\,\vec{n}\cdot\vec{v}_1}-\frac{\frac{1}{c}\,\vec{n}\cdot\vec{v}_2}{1-\frac{1}{c}\,\vec{n}\cdot\vec{v}_2}\right)=i\,\frac{e}{\omega}\left(\frac{1}{1-\frac{1}{c}\,\vec{n}\cdot\vec{v}_1}-\frac{1}{1-\frac{1}{c}\,\vec{n}\cdot\vec{v}_2}\right).$$

We now immediately obtain the spectral distribution

$$\frac{dE(\omega)}{d\Omega}=\frac{e^2}{4\pi^2c^3}\left|\vec{n}\times\left(\frac{\vec{v}_1}{1-\frac{1}{c}\,\vec{n}\cdot\vec{v}_1}-\frac{\vec{v}_2}{1-\frac{1}{c}\,\vec{n}\cdot\vec{v}_2}\right)\right|^2. \tag{35.5}$$

Since (35.5) is independent of the frequency, the implied total radiated energy is unbounded. This unphysical result is due to the idealization that the scattering occurs instantaneously, that is, our assumption that the particle changes its velocity abruptly. Realistically, this change occurs over some finite period of time, T, so that our result, (35.5) holds only when $\omega \ll 1/T$. In the non-relativistic limit, where $|\vec{v}_{1,2}/c| \ll 1$, (35.5) reduces to

$$\begin{aligned}\frac{dE(\omega)}{d\Omega}&\simeq\frac{e^2}{4\pi^2c^3}\,|\vec{n}\times(\vec{v}_1-\vec{v}_2)|^2\\&=\frac{e^2}{4\pi^2c^3}\,(\vec{v}_1-\vec{v}_2)^2\,\sin^2\theta\,,\end{aligned} \tag{35.6}$$

$\omega \sim \frac{1}{T}$

where θ is the angle between $\vec{v}_1-\vec{v}_2$ and $\vec{n}$. Integrating this over all angles, we obtain the energy radiated at the frequency ω

$$E(\omega)=\frac{2}{3}\,\frac{e^2}{\pi c^3}\,(\vec{v}_1-\vec{v}_2)^2\,, \tag{35.7}$$

which can also be easily obtained from (32.31), since the Fourier transform of the derivative here is simply

$$\dot{\vec{v}}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \dot{\vec{v}}(t) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, (\vec{v}_1-\vec{v}_2)\,\delta(t) = \vec{v}_1-\vec{v}_2 \quad . \quad (35.8)$$

Because either denominator in (35.5) has the structure

$$1 - \frac{1}{c}\vec{n}\cdot\vec{v}_i = 1 - \frac{v_i}{c}\cos\theta_i \quad ,$$

where $\vec{v}_i$ is either $\vec{v}_1$ or $\vec{v}_2$, and θ_i is the angle between $\vec{n}$ and $\vec{v}_i$ we see that the radiation in the ultrarelativistic limit $(|\vec{v}_i| \simeq c)$ is preferentially emitted near the direction of the velocity of the particle (the forward direction), either before or after the scattering act. On the other hand, this behavior is softened by the numerator factor $|\vec{n}\times\vec{v}_i| = v_i \sin\theta_i$, which forbids radiation in the exactly forward direction, $\theta_i = 0$. In either region of significant radiation, characterized by $\theta_i \ll 1$, only one term in (35.5) makes a major contribution (in the following we drop the subscript i):

$$\begin{aligned}
\left(\frac{\vec{n}\times\vec{v}/c}{1-\frac{v}{c}\cos\theta}\right)^2 &= \left(\frac{v}{c}\right)^2 \sin^2\theta\, \frac{1}{\left(1-\frac{v}{c}\cos\theta\right)^2} \\
&\simeq \frac{\theta^2}{\left(1-\frac{v}{c}+\frac{\theta^2}{2}\right)^2} \simeq \frac{4\theta^2}{\left(1-\frac{v^2}{c^2}+\theta^2\right)^2} \\
&\simeq \begin{cases} \dfrac{4}{\theta^2}\,, & \text{if } \sqrt{1-\dfrac{v^2}{c^2}} \ll \theta \ll 1\,, \\ 0\,, & \text{if } \theta = 0\,. \end{cases}
\end{aligned} \quad (35.9)$$

Therefore, the maximum intensity occurs at

$$\theta = \sqrt{1-\frac{v^2}{c^2}} = \frac{mc^2}{E} \quad (35.10)$$

(which we have implicitly assumed to be small compared to the scattering angle), and most of the radiation is emitted in a small angular range near this angle. As the particle moves faster, the peaking of the radiation is more pronounced. This is a fundamental difference between radiation produced by relativistic ($v \sim c$) and non-relativistic ($v \ll c$) particles. We will see this again as we now turn to a discussion of synchrotron radiation.

XXXVI. SYNCHROTRON RADIATION I

Lecture 7

A charged particle, moving in a circular orbit, undergoes centripetal acceleration and therefore radiates electromagnetic energy. The physical circumstances under which this process occurs arise in betatrons and synchrotrons, in which charged particles are guided in a circle by external magnetic fields. In this and the following two sections, we will explore the characteristics of this synchrotron radiation. We begin by considering some kinematics.

36-1. Motion of a Charged Particle in a Homogeneous Magnetic Field

If E is the energy of the charged particle and $\vec{p}$ is its momentum, then the (relativistic) equations of motion of the particle in a magnetic field $\vec{B}$ are

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\,\vec{v} \times \vec{B} \;, \tag{36.1}$$

$$\frac{dE}{dt} = 0 \;, \tag{36.2}$$

where the momentum is related to the velocity, $\vec{v}$, by

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\beta^2}} = \frac{E}{c^2}\,\vec{v} \;, \qquad \beta \equiv \frac{v}{c} \;, \tag{36.3}$$

and the energy is

$$E = \frac{mc^2}{\sqrt{1-\beta^2}} \;. \tag{36.4}$$

Since the magnetic force is always perpendicular to the direction of motion, no work is done on the particle, and consequently the energy of the particle is conserved, as stated by (36.2). This fact, together with (36.3), enables us to rewrite (36.1) in the form

$$\frac{E}{c^2}\frac{d\vec{v}}{dt} = \frac{e}{c}\,\vec{v}\times\vec{B} \,, \tag{36.5a}$$

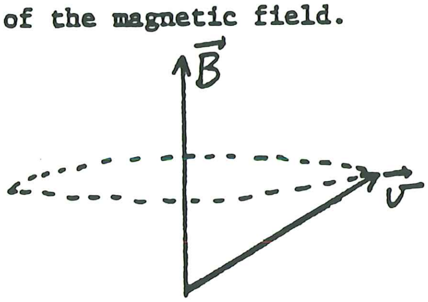
or

$$\frac{d\vec{v}}{dt} = \left(-\frac{ec}{E}\,\vec{B}\right)\times\vec{v} \;. \tag{36.5b}$$

This implies a constant deflection of the velocity vector, that is, $\vec{v}$ precesses with angular velocity $\vec{\omega}_o$,

$$\vec{\omega}_o = -\frac{ec}{E}\,\vec{B} \,, \tag{36.6a}$$

about the direction of the magnetic field.



The angular speed of this precession is the Larmor frequency,

$$\omega_o = \frac{ec}{E}\,B \,, \tag{36.6b}$$

which reduces, in the non-relativistic limit, to the cyclotron frequency,

$$\omega_o \approx \frac{eB}{mc} \,, \quad \frac{v}{c} \ll 1 \;. \tag{36.6c}$$

The component of the velocity parallel to the magnetic field, according to (36.5), is constant. In practice, we constrain the motion such that $\vec{v}$ is perpendicular to $\vec{B}$. Then the particle moves with angular speed ω_o in a circle of radius R,

$$R = \frac{v}{\omega_o} = \frac{\beta E}{eB} \quad . \tag{36.7}$$

The relation between the momentum of the particle and the radius of the circular orbit,

$$p = \frac{E}{c^2} v = \frac{e}{c} BR \quad , \tag{36.8}$$

supplies us with a practical method of measuring the momentum of a relativistic particle. When the information thus obtained is coupled with that derived from a Cerenkov counter, which measures the speed of the particle according to (33.11), we can determine the mass of the particle:

$$m = \frac{\sqrt{1-\beta^2}}{v} p \quad . \tag{36.9}$$

36-2. Spectrum of Synchrotron Radiation

We now proceed to calculate the radiation emitted by a charged particle moving in a circle. For a point particle, the charge and current densities are given by

$$\rho(\vec{r},t) = e\delta(\vec{r}-\vec{r}(t)) \quad , \tag{36.10a}$$

$$\vec{j}(\vec{r},t) = e\vec{v}(t)\ \delta(\vec{r}-\vec{r}(t)) \quad , \tag{36.10b}$$

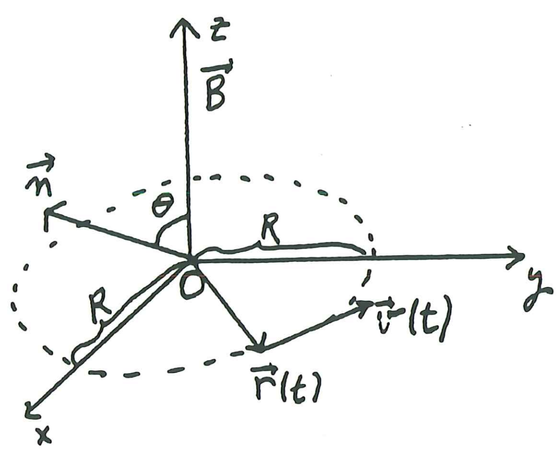
where $\vec{r}(t)$ is the position vector of the particle at a time t, and $\vec{v}(t)$ is its velocity. Substituting (36.10) into (33.4c), we obtain the spectrum of the power, emitted at time T, into the element of a solid angle in the direction $\vec{n}$,

$$\frac{dP(\omega,T)}{d\Omega} = \frac{\omega^2 e^2}{4\pi^2 c}\int_{-\infty}^{\infty} d\tau e^{-i\omega\tau}\left[\frac{1}{c^2}\vec{v}\left(T+\frac{\tau}{2}\right)\cdot\vec{v}\left(T-\frac{\tau}{2}\right)-1\right]$$

$$\times \exp\left\{i\,\frac{\omega}{c}\,\vec{n}\cdot\left[\vec{r}\left(T+\frac{\tau}{2}\right)-\vec{r}\left(T-\frac{\tau}{2}\right)\right]\right\} . \tag{36.11}$$

This description in terms of an average macroscopic time can be important, for then it is possible to consider the effect of a slow alteration in the parameters describing the motion.

It is convenient to choose the coordinate system such that the particle is moving in a circle of radius R about the origin in the xy plane, and the magnetic field $\vec{B}$ is directed along the +z direction. Also, without loss of generality, we choose the observation direction, $\vec{n}$, to lie in the xz plane, making an angle θ with the +z axis.



Then we have, for a convenient choice of initial conditions,

$$\vec{n} = (\sin\theta,\ 0,\ \cos\theta) \quad , \tag{36.12}$$

$$x(t) = R\cos\omega_o t \quad , \tag{36.13a}$$

$$y(t) = R\sin\omega_o t \quad , \tag{36.13b}$$

$$z(t) = 0 \quad , \tag{36.13c}$$

where ω_o and R are given by (36.6b) and (36.7), respectively. The corresponding velocity is given by

$$v_x(t) = -v \sin\omega_o t \quad , \tag{36.14a}$$

$$v_y(t) = v \cos\omega_o t \quad , \tag{36.14b}$$

$$v_z(t) = 0 \quad , \tag{36.14c}$$

where the speed v satisfies

$$v = R\omega_o \quad . \tag{36.15}$$

Using these explicit representations for $\vec{r}(t)$ and $\vec{v}(t)$, we may simplify the integrand of (36.11) by means of the following:

$$\frac{1}{c^2}\,\vec{v}\left(T + \frac{\tau}{2}\right)\cdot\vec{v}\left(T - \frac{\tau}{2}\right) = \beta^2 \cos\omega_o\tau \tag{36.16}$$

(a fact which is apparent geometrically), and

$$\vec{n}\cdot\vec{r}\left(T \pm \frac{\tau}{2}\right) = \sin\theta \; x\left(T \pm \frac{\tau}{2}\right) = R \sin\theta \cos\omega_o\left(T \pm \frac{\tau}{2}\right) \quad , \tag{36.17}$$

$$\begin{aligned} e^{\pm i \frac{\omega}{c} \vec{n}\cdot\vec{r}\left(T \pm \frac{\tau}{2}\right)} &= e^{\pm i \frac{\omega}{c} R \sin\theta \cos\omega_o\left(T \pm \frac{\tau}{2}\right)} \\ &= \sum_{m=-\infty}^{\infty} (\pm i)^m \, e^{\pm im\omega_o\left(T \pm \frac{\tau}{2}\right)} \, J_m\left(\frac{\omega}{c} R \sin\theta\right) . \end{aligned} \tag{36.18}$$

In the last equation, we have used the generating function for the Bessel functions of integer order, (15.17),

$$e^{iz \cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(z) \quad . \tag{36.19}$$

With these evaluations, we obtain the power spectrum, (36.11), in the form

$$\frac{dP(\omega,T)}{d\Omega} = \frac{\omega^2 e^2}{4\pi^2 c}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}[\beta^2 \cos\omega_o\tau - 1]$$

$$\times \sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} i^m e^{im\omega_o\left(T+\frac{\tau}{2}\right)} J_m\left(\frac{\omega}{c} R \sin\theta\right)$$

$$\times (-i)^{m'} e^{-im'\omega_o\left(T-\frac{\tau}{2}\right)} J_{m'}\left(\frac{\omega}{c} R \sin\theta\right) . \tag{36.20}$$

Now we recall that the emission time T represents an average over many periods of the motion, since many oscillations are required to identify a frequency. Here, it is sufficient to consider the average of (36.20) over one period, by using the relation

$$\langle e^{i(m-m')\omega_o T}\rangle_{\text{one period}} = \delta_{mm'} . \tag{36.21}$$

After this averaging procedure, (36.20) becomes

$$\frac{dP(\omega)}{d\Omega} = \frac{\omega^2 e^2}{4\pi^2 c}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}[\beta^2 \cos\omega_o\tau - 1]$$

$$\times \sum_{m=-\infty}^{\infty} e^{im\omega_o\tau}\left[J_m\left(\frac{\omega}{c} R \sin\theta\right)\right]^2 . \tag{36.22}$$

Incidentally, we remark that this result, (36.22), can also be obtained by combining the two exponentials directly as follows:

$$\langle e^{i\frac{\omega}{c}\vec{n}\cdot\left[\vec{r}\left(T+\frac{\tau}{2}\right)-\vec{r}\left(T-\frac{\tau}{2}\right)\right]}\rangle_{\text{one period}}$$

$$= \langle e^{i\frac{\omega}{c}R\sin\theta\left(-2\sin\omega_o T\sin\frac{1}{2}\omega_o\tau\right)}\rangle_{\text{one period}}$$

$$= J_o\left(\frac{2\omega R}{c}\sin\theta\,\sin\frac{\omega_o\tau}{2}\right)$$

$$= \sum_{m=-\infty}^{\infty}\left[J_m\left(\frac{\omega R}{c}\sin\theta\right)\right]^2 e^{im\omega_o\tau} \quad . \tag{36.23}$$

Here we have used the integral representation of J_o, (15.2),

$$J_o(x) = \int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{ix\cos\phi} \quad , \tag{36.24}$$

and the addition theorem for the Bessel functions, (15.33),

$$J_o\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right) = \sum_{m=-\infty}^{\infty} J_m(k\rho)\,e^{im\phi}J_m(k\rho')\,e^{-im\phi'} \quad , \tag{36.25}$$

with $\rho = \rho' = R$ and $\phi-\phi' = \omega_o\tau$.

Exploiting the exponential representation for $\cos\omega_o\tau$,

$$\cos\omega_o\tau = \frac{1}{2}\,[e^{i\omega_o\tau} + e^{-i\omega_o\tau}] \quad , \tag{36.26}$$

we can easily perform the τ integration in (36.22) by means of the integral

$$\int_{-\infty}^{\infty} d\tau e^{-i\omega\tau}\,e^{im\omega_o\tau} = 2\pi\delta(\omega-m\omega_o) \quad . \tag{36.27}$$

Consequently, the spectrum is discrete in that only harmonics of the Larmor frequency, (36.6b), are radiated. That is, the emitted frequencies are multiples of ω_o, $\omega = m\omega_o$, where the integer m is positive since $\omega > 0$. Thus the power spectrum can be written as a sum of the contributions of each harmonic,

$$\frac{dP(\omega,T)}{d\Omega} = \sum_{m=1}^{\infty} \delta(\omega-m\omega_o) \frac{dP_m(T)}{d\Omega} , \tag{36.28a}$$

where the power radiated into the mth harmonic is

$$\frac{dP_m}{d\Omega} = \frac{m^2\omega_o^2 e^2}{2\pi c} \left[\frac{\beta^2}{2} (J^2_{m+1} + J^2_{m-1}) - J_m^{\ 2} \right] , \tag{36.28b}$$

the argument of the Bessel functions being

$$\frac{\omega R}{c} \sin\theta = m\beta \sin\theta \ . \tag{36.28c}$$

In (36.28a) the T dependence could arise from any slow variation in the parameters ω_o and β. We may further simplify (36.28b) by using the following recurrence relations for Bessel's functions, found in Problem I-15,

$$J_{m-1}(z) - J_{m+1}(z) = 2J'_m(z) , \tag{36.29a}$$

$$J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z) , \tag{36.29b}$$

which imply that

$$\frac{1}{2} [(J_{m+1})^2 + (J_{m-1})^2] - \frac{1}{\beta^2} (J_m)^2 = (J'_m)^2 + \left(\frac{J_m}{\beta \tan\theta} \right)^2 . \tag{36.30}$$

Therefore, the angular distribution for the power radiated into the mth harmonic is

$$\frac{dP_m}{d\Omega} = \frac{\omega_o}{2\pi} \frac{e^2}{R} \beta^3 m^2 \left\{ [J'_m(m\beta \sin\theta)]^2 + \left[\frac{J_m(m\beta \sin\theta)}{\beta \tan\theta} \right]^2 \right\} . \tag{36.31}$$

36-3. Total Power Emitted into the mth Harmonic

To obtain the total power radiated into the mth harmonic, we could integrate (36.31) over all angles. However, we find it simpler to return to the general expression (36.11) and perform the integration over the angles first, since there all the angular dependence is in the exponential factor. If we define $\vec{s}$ to be

$$\vec{s} = \vec{r}\left(T + \frac{\tau}{2}\right) - \vec{r}\left(T - \frac{\tau}{2}\right) \quad , \tag{36.32a}$$

$$|\vec{s}| = \sqrt{R^2 + R^2 - 2R^2 \cos\omega_0\tau} = 2R\left|\sin\frac{1}{2}\omega_0\tau\right| \quad , \tag{36.32b}$$

the angular integration we encounter in this way is

$$\begin{aligned}\int d\Omega\, e^{i\frac{\omega}{c}\vec{n}\cdot\vec{s}} &= 2\pi\int_0^\pi \sin\chi\, d\chi\, e^{i\frac{\omega}{c}|\vec{s}|\cos\chi} \\ &= 4\pi\frac{\sin\frac{\omega}{c}|\vec{s}|}{\frac{\omega}{c}|\vec{s}|} \\ &= \frac{2\pi c}{\omega R}\frac{\sin\left(\frac{2\omega}{c}R\sin\frac{\omega_0\tau}{2}\right)}{\sin\frac{\omega_0\tau}{2}} \quad .\end{aligned} \tag{36.33}$$

The resulting frequency distribution of the radiated power is, from (36.11)

$$\begin{aligned}P(\omega) &= \frac{\omega}{2\pi}\frac{e^2}{R}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}(\beta^2\cos\omega_0\tau - 1)\frac{\sin\left(\frac{2\omega}{c}R\sin\frac{\omega_0\tau}{2}\right)}{\sin\frac{\omega_0\tau}{2}} \\ &\equiv \frac{\omega}{2\pi}\frac{e^2}{R}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau} f(\omega_0\tau) \quad .\end{aligned} \tag{36.34}$$

The challenge now is to carry out the τ integration. We observe that $f(\omega_0\tau)$ is a periodic function,

$$f(\omega_0\tau) = f(\omega_0\tau + 2n\pi) \quad , \qquad n = \text{integer} \; , \tag{36.35}$$

so that it may be represented by a Fourier series,

$$f(\omega_0\tau) = \sum_{m=-\infty}^{\infty} e^{im\omega_0\tau} f_m \; , \tag{36.36a}$$

where the Fourier coefficient f_m is given by

$$f_m = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-im\phi} f(\phi) \; . \tag{36.36b}$$

Inserting the representation (36.36) into (36.34), and carrying out the then trivial τ integration, (36.27), we obtain the spectrum of the radiated power:

$$P(\omega) = \sum_{m=1}^{\infty} \delta(\omega - m\omega_0) P_m \; , \tag{36.37a}$$

where the total power radiated in the mth harmonic is expressed as a Fourier coefficient,

$$\begin{aligned} P_m &= \frac{e^2}{R} m\omega_0 \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-im\phi} (\beta^2 \cos\phi - 1) \frac{\sin\left(2m\beta \sin \frac{\phi}{2} \right)}{\sin \frac{\phi}{2}} \\ &= \frac{e^2}{R} m\omega_0 \int_0^{\pi} \frac{d\phi}{\pi} \cos m\phi \left(\beta^2 - 1 - 2\beta^2 \sin^2 \frac{\phi}{2} \right) \frac{\sin\left(2m\beta \sin \frac{\phi}{2} \right)}{\sin \frac{\phi}{2}} \; . \end{aligned}$$

(37.37b)

Not surprisingly, this can be rewritten in terms of Bessel functions. Starting from the integral representation (15.19),

$$i^m J_m(z) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-im\phi} e^{iz\cos\phi} , \tag{36.38a}$$

we make the substitution $\phi \to \phi - \pi/2$, leading to

$$J_m(z) = \int_0^{\pi} \frac{d\phi}{\pi} \cos(z \sin\phi - m\phi) . \tag{36.38b}$$

Breaking this integral at $\pi/2$ and substituting $\phi \to \pi - \phi$ for the range $\pi/2$ to π, we finally find

$$J_m(z) = \int_0^{\pi/2} \frac{d\phi}{\pi} [\cos(z \sin\phi - m\phi) + \cos(z \sin\phi + m\phi - m\pi)] . \tag{36.38c}$$

This supplies the following integral representation for the Bessel functions of even order:

$$\begin{aligned} J_{2m}(z) &= \int_0^{\pi/2} \frac{d\phi}{\pi} [\cos(z \sin\phi - 2m\phi) + \cos(z \sin\phi + 2m\phi)] \\ &= \int_0^{\pi} \frac{d\phi}{\pi} \cos\left(z \sin\frac{\phi}{2} \right) \cos m\phi , \end{aligned} \tag{36.39}$$

where $\phi \to \frac{1}{2}\phi$ in the second line. The integral in (36.37b) can now be expressed in terms of the derivative and the integral of J_{2m}, which are represented by

$$J'_{2m}(z) = -\int_0^{\pi} \frac{d\phi}{\pi} \sin\frac{\phi}{2} \cos m\phi \sin\left(z \sin\frac{\phi}{2} \right) , \tag{36.40}$$

and

$$\int_0^x dz\, J_{2m}(z) = \int_0^\pi \frac{d\phi}{\pi} \cos m\phi \frac{\sin\left(x \sin\frac{\phi}{2}\right)}{\sin\frac{\phi}{2}} . \tag{36.41}$$

Therefore, the total power radiated into the mth harmonic is

$$P_m = \frac{e^2}{R} m\omega_o \left[2\beta^2 J'_{2m}(2m\beta) - (1-\beta^2)\int_0^{2m\beta} dz\, J_{2m}(z) \right] . \tag{36.42}$$

36-4. Total Radiated Power

To find the total radiated power, we could sum (36.42) over all m. However, as before, it is much simpler to return to an earlier stage, this time to (36.34), and first perform the frequency integration, before doing the τ integration. An intermediate form for the radiated power is

$$P = \int_0^\infty d\omega\, P(\omega) = \frac{1}{2}\int_{-\infty}^\infty d\omega\, P(\omega)$$

$$= \frac{e^2}{R}\frac{1}{2\pi}\int_{-\infty}^\infty d\tau \frac{\beta^2 \cos\omega_o\tau - 1}{\sin\frac{1}{2}\omega_o\tau}\frac{1}{2}\int_{-\infty}^\infty d\omega\, \omega e^{-i\omega\tau}\frac{1}{2i}\left[e^{i\frac{2\omega}{c}R\sin\frac{\omega_o\tau}{2}} - e^{-i\frac{2\omega}{c}R\sin\frac{\omega_o\tau}{2}}\right]$$

$$= \frac{e^2}{R}\frac{\omega_o}{4}\int_{-\infty}^\infty d\phi \frac{\beta^2\cos\phi - 1}{\sin\frac{\phi}{2}}\left[\delta'\left(\phi - 2\beta\sin\frac{\phi}{2}\right) - \delta'\left(\phi + 2\beta\sin\frac{\phi}{2}\right)\right] , \tag{36.43}$$

where we have made use of the representation

$$\int_{-\infty}^\infty d\omega\, \omega e^{-i\omega\lambda} = i\frac{d}{d\lambda}\int_{-\infty}^\infty d\omega e^{-i\omega\lambda} = 2\pi i\, \delta'(\lambda) . \tag{36.44}$$

Equation (36.43) can be evaluated in a straightforward manner. However, it is simpler to combine the delta functions there by means of

$$\frac{1}{2\beta\sin\frac{\phi}{2}}\left[\delta'\left(\phi+2\beta\sin\frac{\phi}{2}\right)-\delta'\left(\phi-2\beta\sin\frac{\phi}{2}\right)\right]=\int_{-1}^{1}d\lambda\,\delta''(y)\quad, \tag{36.45}$$

where

$$y\equiv\phi+2\lambda\beta\sin\frac{\phi}{2}\quad, \tag{36.46}$$

so that we may rewrite the total power as

$$P=-\frac{e^2}{2R}\omega_o\beta\int_{-1}^{1}d\lambda\int_{-\infty}^{\infty}dy\,\frac{\beta^2\cos\phi-1}{1+\lambda\beta\cos\frac{\phi}{2}}\,\delta''(y)\quad. \tag{36.47}$$

Here the y integration can be performed by integrating by parts twice and noting that the support of the delta function occurs at $\phi=0$:

$$\begin{aligned}\frac{d^2}{dy^2}\left(\frac{\beta^2\cos\phi-1}{1+\lambda\beta\cos\frac{\phi}{2}}\right)\Bigg|_{y=0}&=\frac{1}{1+\lambda\beta\cos\frac{\phi}{2}}\frac{d}{d\phi}\left(\frac{1}{1+\lambda\beta\cos\frac{\phi}{2}}\frac{d}{d\phi}\frac{\beta^2\cos\phi-1}{1+\lambda\beta\cos\frac{\phi}{2}}\right)\Bigg|_{\phi=}\\&=\frac{1}{(1+\lambda\beta)^2}\frac{d^2}{d\phi^2}\left(\frac{\beta^2\cos\phi-1}{1+\lambda\beta\cos\frac{\phi}{2}}\right)\Bigg|_{\phi=0}\\&=\frac{2}{(1+\lambda\beta)^3}\left(-\frac{\beta^2}{2}+\frac{\lambda\beta}{8}\frac{\beta^2-1}{1+\lambda\beta}\right)\quad.\end{aligned} \tag{36.48}$$

The remaining λ integral can now be easily evaluated, yielding [see problem 12]

$$P=-\frac{e^2}{R}\omega_o\beta\int_{-1}^{1}d\lambda\,\frac{1}{(1+\lambda\beta)^3}\left(-\frac{\beta^2}{2}+\frac{\lambda\beta}{8}\frac{\beta^2-1}{1+\lambda\beta}\right)$$

$$= \frac{2}{3}\frac{e^2}{R}\omega_o \frac{\beta^3}{(1-\beta^2)^2}$$

$$= \frac{2}{3}\frac{e^2}{R}\omega_o\beta^3\left(\frac{E}{mc^2}\right)^4 , \qquad (36.49)$$

which is the exact result for the total radiated power.

In the non-relativistic limit, $\beta << 1$, (36.49) reduces to the Larmor formula, (29.24),

$$P_{N.R.} = \frac{2}{3}\frac{e^2}{R}\omega_o\beta^3$$

$$= \frac{2}{3}\frac{e^2}{c^3}\left(\frac{d\vec{v}}{dt}\right)^2 , \qquad (36.50)$$

since, for circular motion, the centripetal acceleration is

$$\left|\frac{d\vec{v}}{dt}\right| = \frac{v^2}{R} = v\omega_o . \qquad (36.51)$$

For a relativistic particle, the power radiated is larger by the factor $(E/mc^2)^4$, signifying that high energy charged particles moving in circular orbits emit substantial synchrotron radiation. In one period, $T = 2\pi/\omega_o$, the total energy radiated is

$$\Delta E = PT = \frac{4\pi}{3}\frac{e^2}{R}\beta^3\left(\frac{E}{mc^2}\right)^4 . \qquad (36.52)$$

In practical units, if we express energy in electron-volts (eV), and the radius of the accelerator in meters, then for $\beta \simeq 1$, the energy loss per cycle by an electron is

$$\Delta E(\text{keV}) = 88.5\ \frac{E^4(\text{GeV})}{R(\text{m})} , \qquad (36.53)$$

where

$$1 \text{ GeV} = 10^6 \text{ keV} = 10^9 \text{ eV} \quad . \tag{36.54}$$

Inserting typical numbers for an electron synchrotron, R = 10 meters and E = 10 GeV, we find for the energy loss per cycle

$$\Delta E = 88.5 \text{ MeV} \quad ,$$

which is quite substantial. For this reason electron synchrotrons are impractical for energies greater than ~10 GeV. On the other hand, the radiation from any existing or projected proton synchrotron is quite negligible, being smaller, for the same energy, than electron synchrotron radiation by the factor $(m_e/m_p)^4 \sim 10^{-13}$.

Lecture 8

XXXVII. SYNCHROTRON RADIATION II - POLARIZATION

The polarization state of an electromagnetic wave is determined by the direction of its electric field of which there are two independent possibilities, each normal to the direction of propagation, $\vec{n}$. The results obtained in the previous section referred to the sum of powers radiated into both polarization states. We now investigate the polarization characteristics of synchrotron radiation. One way in which this information is practically useful lies in studying astrophysical objects; for example, the Crab Nebula is inferred to emit synchrotron radiation, because of the unique polarization characteristics of the latter. In order to calculate the power radiated into each polarization state, we return to the basic formula (33.4b). However, we recall that the latter equation was obtained from (32.24) in which energy flow was expressed in terms of $|\vec{B}|$. It is desirable to shift the emphasis from $|\vec{B}|^2$ to $|\vec{E}|^2$, since it is the electric field that is measured when polarization is determined. We do this by noting that since $\vec{E}$ and $\vec{B}$ are related by (32.18), (33.4b) can be equivalently written as

$$\frac{dP(\omega,T)}{d\Omega} = \frac{\omega^2}{4\pi^2 c}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\left[-\vec{n}\times\left(\vec{n}\times\int (d\vec{r})\, e^{i\frac{\omega}{c}\vec{n}\cdot\vec{r}}\,\frac{1}{c}\vec{j}\left(\vec{r}, T+\frac{\tau}{2}\right)\right)\right]$$

$$\cdot\left[-\vec{n}\times\left(\vec{n}\times\int (d\vec{r}\,')\, e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{r}\,'}\,\frac{1}{c}\vec{j}\left(\vec{r}\,', T-\frac{\tau}{2}\right)\right)\right] , \quad (37.1)$$

where the terms in the square brackets are proportional to the electric field strength.

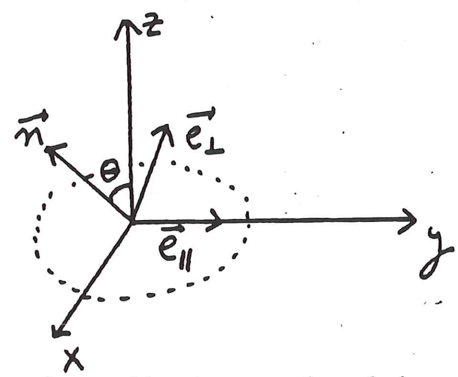
To isolate the effect of each polarization state, we look at the two components of $\vec{E}$, in the plane perpendicular to $\vec{n}$, separately. We choose the coordinate system (which is naturally set by the synchrotron) as before, with

$\vec{n}$ given by (36.12). We define two polarization orientations as follows: For "parallel polarization," the electric field is in the direction of $\vec{e}_{\parallel}$, which lies in the orbital plane and is perpendicular to $\vec{n}$, that is, $\vec{e}_{\parallel}$ points in the +y direction,

$$\vec{e}_{\parallel} = (0,\ 1,\ 0)\ . \tag{37.2}$$

The "perpendicular polarization" vector, $\vec{e}_{\perp}$, is perpendicular to both $\vec{e}_{\parallel}$ and $\vec{n}$, with $\vec{e}_{\parallel}$, $\vec{e}_{\perp}$, and $\vec{n}$ forming a right-handed system of unit vectors:

$$\vec{e}_{\perp} = (-\cos\theta,\ 0,\ \sin\theta)\ . \tag{37.3}$$



It is sufficient to compute the radiation produced in a single polarization state since the result of the previous section then supplies the power radiated in the other state. It is simpler to consider the $\vec{e}_{\parallel}$ polarization. The partial intensity with this polarization arises from the product of y components of the vectors in square brackets in (37.1). Since $\vec{n}$ lies in the xz plane, this component is

$$[-\vec{n} \times (\vec{n} \times \vec{j})]_y = [\vec{j} - \vec{n}(\vec{n}\cdot\vec{j})]_y = j_y\ , \tag{37.4}$$

and the resulting contribution to the power spectrum radiated in the parallel polarization state is

$$\left(\frac{dP(\omega,T)}{d\Omega}\right)_{\parallel} = \frac{\omega^2}{4\pi^2 c}\int_{-\infty}^{\infty} d\tau\ e^{-i\omega\tau}\int (d\vec{r})(d\vec{r}\,')\ e^{i\frac{\omega}{c}\vec{n}\cdot(\vec{r}-\vec{r}\,')}$$

$$\times \frac{1}{c}\, j_y\!\left(\vec{r},\ T+\frac{\tau}{2}\right)\frac{1}{c}\, j_y\!\left(\vec{r}\,',\ T-\frac{\tau}{2}\right)\ . \tag{37.5}$$

For synchrotron radiation, the current density is described by (36.10b), with position vector and velocity given by (36.13) and (36.14), respectively, so that (37.5) reduces to

$$\left(\frac{dP(\omega,T)}{d\Omega}\right)_{\parallel} = \frac{\omega^2 e^2}{4\pi^2 c}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\, e^{i\frac{\omega}{c}\vec{n}\cdot\left[\vec{r}\left(T+\frac{\tau}{2}\right)-\vec{r}\left(T-\frac{\tau}{2}\right)\right]}$$

$$\times \frac{1}{c^2} v_y\left(T+\frac{\tau}{2}\right) v_y\left(T-\frac{\tau}{2}\right) . \tag{37.6}$$

We now can evaluate (37.6) by following closely the procedure given in the previous section. Therefore, instead of the factor

$$\frac{1}{c^2}\vec{v}\left(T+\frac{\tau}{2}\right)\cdot\vec{v}\left(T-\frac{\tau}{2}\right) - 1 = \beta^2 \cos\omega_o\tau - 1 \tag{37.7a}$$

appearing in (36.11), we now encounter

$$\frac{1}{c^2} v_y\left(T+\frac{\tau}{2}\right) v_y\left(T-\frac{\tau}{2}\right) = \beta^2 \cos\omega_o\left(T+\frac{\tau}{2}\right)\cos\omega_o\left(T-\frac{\tau}{2}\right)$$

$$= \frac{\beta^2}{2}(\cos 2\omega_o T + \cos\omega_o\tau) . \tag{37.7b}$$

The time average of (37.6) involves not only (36.23) but also the new evaluation [most easily done starting with (36.18)],

$$\left\langle \cos 2\omega_o T\, e^{i\frac{\omega}{c}\vec{n}\cdot\left[\vec{r}\left(T+\frac{\tau}{2}\right)-\vec{r}\left(T-\frac{\tau}{2}\right)\right]}\right\rangle_{\text{one period}}$$

$$= \sum_{m=-\infty}^{\infty} (-1)\, e^{im\omega_o\tau} J_{m+1}\left(\frac{\omega R}{c}\sin\theta\right) J_{m-1}\left(\frac{\omega R}{c}\sin\theta\right) . \tag{37.8}$$

Therefore, instead of the combination

$$(\beta^2 \cos\omega_o\tau - 1)\sum_{m=-\infty}^{\infty} e^{im\omega_o\tau}\left[J_m\left(\frac{\omega}{c}R\sin\theta\right)\right]^2 , \tag{37.9a}$$

appearing in (36.22), we have here, for parallel polarization, the term

$$\frac{1}{2}\beta^2 \sum_{m=-\infty}^{\infty} e^{im\omega_o\tau}[-J_{m+1}J_{m-1} + \cos\omega_o\tau\, J_m^{\,2}] \quad . \tag{37.9b}$$

The τ integration now leads to the power radiated into the mth harmonic being proportional to

$$\frac{\beta^2}{2}\left(\frac{J_{m+1}^2 + J_{m-1}^2}{2} - J_{m+1}J_{m-1}\right) = \frac{1}{4}\beta^2(-J_{m+1} + J_{m-1})^2 = \beta^2(J_m')^2 \quad , \tag{37.10}$$

where we have used the recurrence relation (36.29a). Thus, the two terms in (36.31) represent, respectively, the radiation in the states characterized by $\vec{e}_{\parallel}$ and $\vec{e}_{\perp}$:

$$\left(\frac{dP_m}{d\Omega}\right)_{\parallel} = \frac{\omega_o}{2\pi}\frac{e^2}{R}\beta^3 m^2[J_m'(m\beta\sin\theta)]^2 \quad , \tag{37.11a}$$

$$\left(\frac{dP_m}{d\Omega}\right)_{\perp} = \frac{\omega_o}{2\pi}\frac{e^2}{R}\beta^3 m^2\left[\frac{J_m(m\beta\sin\theta)}{\beta\tan\theta}\right]^2 \quad . \tag{37.11b}$$

what this means is that for radiation in the plane of rotation where $\tan\theta = \infty$ all radiation is $\parallel$

on the other hand for radiation $\perp$ to the plane of motion $\sin\theta = \tan\theta = 0$ then all the q then

Think about this some more

$$\left[\frac{J_m(m\beta\sin\theta)}{\beta\tan\theta}\right]^2 \to \frac{\left(\frac{m\beta\sin\theta}{2}\right)^{m2}}{(\beta\tan\theta)^m} = \frac{\frac{m\beta\theta^{2m}}{2}}{\beta\theta^{2}}$$

so again all $m \neq 1 \to 0$ radiation is in

Lecture 9

XXXVIII. SYNCHROTRON RADIATION III - HIGH ENERGY LIMIT

38-1. Range of Important Harmonics

For most practical applications, we are interested in the frequency spectrum of the power radiated by high energy electrons. In the limit $\beta \to 1$ the power radiated into the mth harmonic, (36.42), is approximately

$$P_m \simeq \omega_o \frac{e^2}{R} 2m\, J'_{2m}(2m) \quad , \tag{38.1a}$$

which becomes for large harmonic number, $m >> 1$,

$$P_m \simeq \frac{3^{1/6}}{\pi} \Gamma(2/3)\ \omega_o \frac{e^2}{R} m^{1/3} \quad , \tag{38.1b}$$

where we have used the asymptotic form of $J'_{2m}(2m)$,

$$J'_{2m}(2m) \simeq \frac{3^{1/6}}{2\pi} \Gamma(2/3)\ m^{-2/3} \quad , \qquad m >> 1 \quad , \tag{38.2}$$

which is already good to within 15% for $m = 1$. We will derive (38.2) in the following subsection. The total power radiated is thus roughly estimated by summing (38.1b) over all harmonics:

$$P = \sum_{m=1}^{\infty} P_m \sim \sum_{m=1}^{m_c} m^{1/3} \quad , \tag{38.3}$$

where we have noted that the continuing increase in the power radiated into higher and higher harmonics must break down for sufficiently large m, since the total power radiated, (36.49), is finite. Consequently, we have cut off the sum at m_c, the critical harmonic number. In terms of this cutoff, the total power radiated is

$$P \sim m_c^{4/3} \quad , \tag{38.4a}$$

which, on the other hand, is, by (36.49), proportional to $(E/mc^2)^4$:

$$P \sim \left(\frac{E}{mc^2}\right)^4 \quad . \tag{38.4b}$$

Therefore, we conclude that the order of magnitude of m_c is

$$m_c \sim \left(\frac{E}{mc^2}\right)^3 \quad . \tag{38.5}$$

Roughly speaking, then, the maximum frequency of radiation is of the order

$$\omega_c \sim \omega_o\left(\frac{E}{mc^2}\right)^3 \quad , \tag{38.6a}$$

which implies the shortest wavelength radiated is about

$$\lambda\!\!\!^{-}_c \sim \left(\frac{mc^2}{E}\right)^3 R \quad , \tag{38.6b}$$

where we have used the relations

$$\omega = \frac{c}{\lambda\!\!\!^{-}} \, , \quad \text{and} \quad \omega_o \approx \frac{c}{R} \quad . \tag{38.7}$$

For an electron, the shortest wavelength radiated is approximately

$$\lambda\!\!\!^{-}_c(\text{Å}) \sim \frac{R(\text{m})}{E^3(\text{GeV})} \quad , \tag{38.8}$$

where one Angstrom, Å = 10^{-8} cm, is a characteristic x-ray wavelength. Thus, the synchrotron radiation produced by a high energy electron is characterized by very large harmonic numbers; and consequently one gets visible, ultraviolet, and x-ray radiation from a typical accelerator.

38-2. Asymptotic Form for $J'_{2m}(2m)$.

We now furnish a derivation of the asymptotic formula for $J'_{2m}(2m)$, (38.2). Starting from the integral representation for the Bessel function [see (36.38b)]

$$J_{2m}(z) = \int_0^\pi \frac{d\phi}{\pi} \cos(z \sin\phi - 2m\phi) \quad , \tag{38.9}$$

we obtain

$$J'_{2m}(z) = -\int_0^\pi \frac{d\phi}{\pi} \sin\phi \, \sin(z \sin\phi - 2m\phi) \quad , \tag{38.10a}$$

which, when we set $z = 2m$, becomes

$$J'_{2m}(2m) = -\int_0^\pi \frac{d\phi}{\pi} \sin\phi \, \sin(2m(\sin\phi - \phi)) \quad . \tag{38.10b}$$

If 2m is large, the integrand of (38.10b) oscillates rapidly as ϕ varies, leading to destructive interference except near $\phi = 0$. (This is the basis of stationary phase or saddle point evaluations.) The main contribution comes from that value of ϕ satisfying the stationary condition

$$\frac{d}{d\phi}(\sin\phi - \phi) = 0 \quad , \tag{38.11a}$$

which implies, as expected,

$$\cos\phi = 1 \;, \quad \text{or} \quad \phi = 0 \quad . \tag{38.11b}$$

Therefore, making use of the approximations, for $\phi \ll 1$,

$$\sin\phi - \phi \simeq -\frac{\phi^3}{6} \quad , \tag{38.12a}$$

$$\sin\phi \simeq \phi \quad , \tag{38.12b}$$

and noting the range of integration can be extended to infinity with negligible error, we obtain

$$J'_{2m}(2m) \simeq -\mathrm{Im}\int_0^\infty \frac{d\phi}{\pi}\,\phi\, e^{-i\frac{m}{3}\phi^3}$$

$$= -\mathrm{Im}\left(\frac{3}{m}\right)^{2/3} \frac{e^{-i\frac{\pi}{3}}}{\pi}\int_0^\infty dt\left(\frac{1}{3}\,t^{-2/3}\right) t^{1/3} e^{-t}$$

?

$$= -\mathrm{Im}\left(\frac{3}{m}\right)^{2/3} \frac{\Gamma(2/3)}{3\pi}\, e^{-i\frac{\pi}{3}}$$

$$= \frac{3^{1/6}}{2\pi}\,\frac{\Gamma(2/3)}{m^{2/3}} \quad , \quad \text{for } m >> 1 \quad , \tag{38.13}$$

let $t = i\frac{m}{3}\varphi^3$

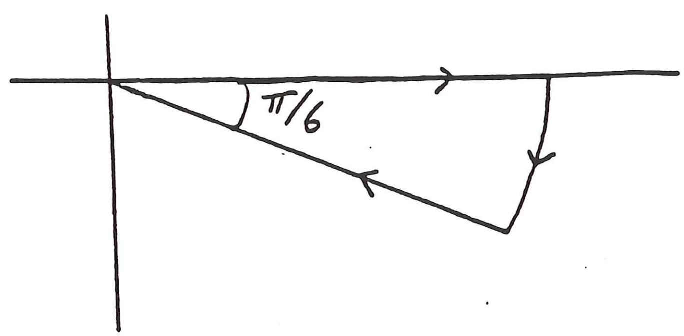
so $dt = i m \varphi^2 d\varphi$

$d\varphi = \frac{dt}{i m \varphi^2} = \frac{dt}{i m \left(\frac{t\,3}{m i}\right)^{2/3}}$

so $\int_0^\infty \frac{dt}{\pi}\left(\frac{1}{3}t^{-2/3}\right)t^{1/3}e^{-t} \overset{?}{=} \frac{\Gamma(2/3)}{3\pi}$

which is just (38.2). In the above evaluation, we have used Cauchy's theorem to perform a change of contour,

ϕ plane:



and have used the definition of the gamma function

$$\Gamma(z) = \int_0^\infty dt\; t^{z-1}\, e^{-t} \quad . \tag{38.14}$$

Notice that (38.13) is valid for m either integer or half-integer.

38-3. Spectral Distribution

We now want to improve upon the qualitative discussion of the high energy power spectrum given in subsection 38-1. As we have previously remarked, the approximation (38.1) breaks down for sufficiently large m. This can be traced to the fact that, in (36.42), β is not exactly equal to 1. Consequently, both terms there contribute but we will concentrate on the first one as it contains all the essential characteristics of the radiation. [See problem 13.] Thus we seek an asymptotic expression for $J'_{2m}(2m\beta)$, starting from the integral representation (38.10a):

$$J'_{2m}(2m\beta) = -\int_0^{\pi} \frac{d\phi}{\pi} \sin\phi \, \sin 2m(\beta \sin\phi - \phi) \quad . \tag{38.15}$$

As before, in the limit when $\beta \to 1$, $m \gg 1$, the main contribution comes from the region near $\phi = 0$. Therefore, we may expand the integrand in (38.15) as follows:

$$\begin{aligned} \sin\phi \, \sin 2m(\beta \sin\phi - \phi) &\simeq \phi \, \sin 2m\left[\beta\left(\phi - \frac{\phi^3}{3!} \right) - \phi \right] \qquad 2 \sim 1+\beta \\ &= \phi \, \sin\left(2m\left[-\phi(1-\beta) - \frac{1}{6}\beta\phi^3 \right]\right) \\ &\simeq -\phi \, \sin\left[m\left((1-\beta^2)\phi + \frac{1}{3}\phi^3 \right)\right] \\ &= -\sqrt{1-\beta^2}\, x \, \sin\left[m(1-\beta^2)^{3/2} \cdot \left(x + \frac{1}{3}x^3 \right)\right] \quad , \end{aligned} \tag{38.16}$$

where we have introduced the change of scale

$$\phi = \sqrt{1-\beta^2}\, x \quad . \tag{38.17}$$

As a result, in this limit, (38.15) can be approximated by

$$J'_{2m}(2m\beta) \simeq (1-\beta^2)\int_0^\infty \frac{dx}{\pi}\, x \sin\left[m(1-\beta^2)^{3/2}\left(x + \frac{1}{3}x^3 \right)\right]$$

$$= \frac{(1-\beta^2)}{\pi}\, \mathrm{Im} \int_0^\infty dx\, x\, e^{im(1-\beta^2)^{3/2}\left[x + \frac{1}{3}x^3 \right]} \quad . \qquad (38.18)$$

For m fixed and β approaching unity in such a way that $m(1-\beta^2)^{3/2} \ll 1$, the significant contribution to (38.18) comes from the region where x is large, and (38.18) reduces to (38.13):

$$J'_{2m}(2m\beta) \simeq (1-\beta^2)\int_0^\infty \frac{dx}{\pi}\, x \sin\left[\frac{m}{3}(1-\beta^2)^{3/2}\, x^3 \right]$$

$$= \int_0^\infty \frac{d\phi}{\pi}\, \phi \sin\left[\frac{m}{3}\phi^3 \right] \quad ,$$

where all reference to the speed of the particle has disappeared. However, for sufficiently large m, the parameter $m(1-\beta^2)^{3/2}$ becomes large, and the integrand undergoes rapid oscillations in x except near the stationary points, which satisfy

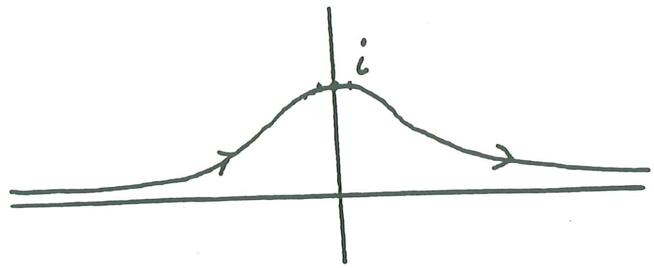
$$\frac{d}{dx}\left[x + \frac{1}{3}x^3 \right] = 1 + x^2 = 0 \quad ; \qquad (38.19a)$$

that is, the stationary phase points are located at

$$x = \pm i \quad . \qquad (38.19b)$$

By extending the region of integration from $-\infty$ to $+\infty$, we evaluate (38.18) asymptotically by following the standard procedure of the saddle point method (or the method of steepest descents). We deform the contour of integration so that it passes through the stationary point $x = i$, because then the dominant contribution comes from the vicinity of that point.

LX



path of steepest decent is where

In the neighborhood of $x = i$, we let

$$x = i + \xi \quad ,$$

where ξ is real, to take advantage of the saddle point character. For small ξ,

$\frac{\partial \psi}{\partial \mathrm{Re}\, x} = i\left(\frac{2}{3}+\xi^2\right)$ so it is stationary to first order in ξ as well

$$\begin{aligned} x + \frac{1}{3} x^3 &= (i+\xi) + \frac{1}{3} (i+\xi)^3 \\ &\simeq i\left(\frac{2}{3} + \xi^2\right) \quad , \end{aligned} \tag{38.20}$$

the exponential factor in (38.18) becomes

$$e^{-\frac{2}{3} m(1-\beta^2)^{3/2}} \; e^{-m(1-\beta^2)^{3/2}\,\xi^2} \quad , \tag{38.21}$$

which falls off exponentially on both sides of $x = i$. The resulting Gaussian integral in (38.18) leads to the following asymptotic form:

$$J'_{2m}(2m\beta) \sim \frac{1}{2} \frac{(1-\beta^2)^{1/4}}{\sqrt{\pi m}} e^{-\frac{2}{3} m(1-\beta^2)^{3/2}} \quad , \quad m(1-\beta^2)^{3/2} \gg 1 \quad .$$

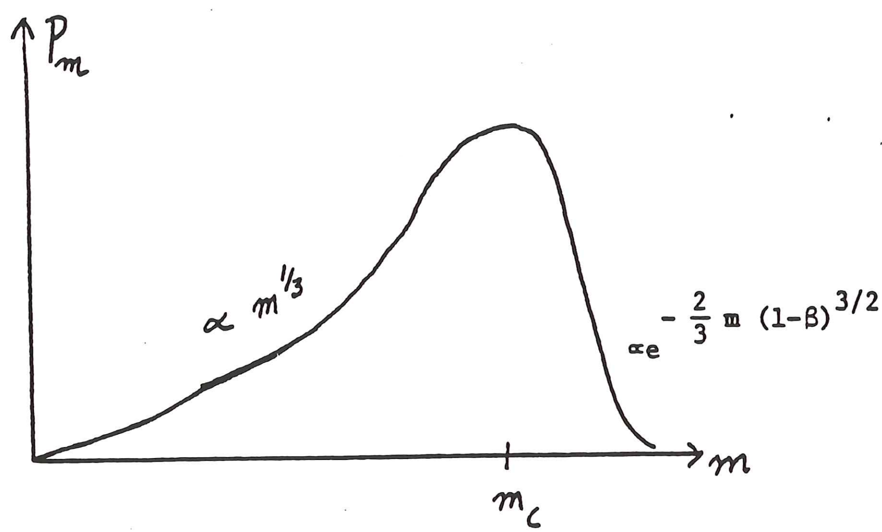
Thus, for very large harmonic numbers, the power spectrum decreases exponentially in contrast to the behavior for smaller values of m where it increases like $m^{1/3}$. The transition between these two regimes occurs near the critical harmonic number, m_c, for which

$$m_c(1-\beta^2)^{3/2} \sim 1 \quad ,$$

or

$$m_c \sim (1-\beta^2)^{-3/2} = \left(\frac{E}{mc^2}\right)^3 \quad , \tag{38.22}$$

supporting our previous estimate, (38.5). The bulk of the radiation is emitted with harmonic numbers near m_c. The qualitative shape of the spectrum is shown below.



38-4. Angular Distribution

We now go back one more stage and examine the angular distribution in the high energy limit. For the radiation in the plane of the orbit $(\theta = \pi/2)$, (36.31) becomes

$$\frac{dP_m}{d\Omega} = \frac{\omega_o}{2\pi} \frac{e^2}{R} \beta^3 m^2 [J_m'(m\beta)]^2 \quad . \tag{38.23}$$

If we further let $\beta \to 1$ and $m_c >> m >> 1$, we find from (38.1) and (38.2),

$$\frac{dP_m}{2\pi d\theta} \simeq \frac{\omega_o}{2\pi}\frac{e^2}{R}m^2\left[\frac{3^{1/6}}{2\pi}\Gamma(2/3)\ (2/m)^{2/3}\right]^2$$

$$\sim \omega_o \frac{e^2}{R} m^{2/3} \sim \frac{P_m}{m^{-1/3}} . \qquad (38.24)$$

Here we compare P_m, which increases as $m^{1/3}$, with $dP_m/d\theta$, which behaves as $m^{2/3}$, from which it is evident that the radiation, for large harmonic numbers, is confined to a small angular range around $\theta = \pi/2$ of width

$$\Delta\theta \sim m^{-1/3} . \qquad (38.25)$$

Since most of the radiation is emitted with harmonic numbers in the neighborhood of

$$m_c \sim \left(\frac{E}{mc^2}\right)^3 ,$$

the radiation is concentrated in an angular range about the plane of the electron orbit of the order

$$\Delta\theta \sim \frac{mc^2}{E} = \sqrt{1-\beta^2} . \qquad (38.26)$$

This radiation, concentrated around $\theta = \pi/2$, is predominantly polarized in the plane of the orbit, according to (37.11). More precisely, it can be easily shown (see Problems 14 and 15) that in the ultrarelativistic limit the ratio of the power radiated with parallel polarization to that with perpendicular polarization is

$$\frac{P_{\parallel}}{P_{\perp}} = 7 , \quad \text{for} \quad \beta \simeq 1 , \qquad (38.27)$$

while in the non-relativistic limit,

$$\frac{P_{\parallel}}{P_{\perp}} = 3 \ , \quad \text{for} \quad \beta << 1 \ . \tag{38.28}$$

That is, for any value of β, the radiation is strongly polarized, the degree of polarization increasing with the speed of the charged particle. This characteristic distinguishes synchrotron radiation from thermal radiation, and was, for example, the clue to understanding the origin of the nonthermal radiation emitted by the Crab Nebula.

38-5. Qualitative Description

One might ask whether there is a more direct way of seeing that, for synchrotron radiation produced by a high energy charged particle, the characteristic frequency emitted is so much larger than the Larmor frequency, as stated by (38.6a), and that the angular distribution is so strongly peaked in the forward direction, as suggested by (38.26). We here wish to emphasize the simple physics behind these striking features. For a point charge, the current density, $\vec{j}(\vec{r},t)$, is given by (36.10b), so that the vector potential, in the Lorentz gauge, can be computed from (28.41) to be

$$\vec{A}(\vec{r},t) = \frac{\frac{e}{c}\,\vec{v}(t')}{|\vec{r}-\vec{r}(t')|\left[\,1 - \frac{\vec{n}}{c}\cdot\vec{v}(t')\,\right]} \ , \tag{38.29}$$

where $\vec{r}(t')$ is the position vector of the particle at the emission time t', while t is the detection time, which are related by (28.35),

$$t = t' + \frac{1}{c}\,|\vec{r}-\vec{r}(t')| \ , \tag{38.30}$$

and $\vec{n}$ is the direction of observation