VIII. Plasmas in Slowly Varying Fields

Many applications of plasma physics, such as the confinement of fusion plasmas and the study of ionospheric, magnetospheric and astrophysical plasmas, involve magnetic fields which are non-uniform, and, sometimes, time-varying; applied electric fields may also be present. For such cases, the analysis given in Chapter VII is of limited value. Even when linearization of the Vlasov equation is valid, the unperturbed orbits are more complicated than the helical trajectories given by (7.4) and (7.5). An understanding of the trajectories in these more complex fields is valuable, since, as we have seen, the integration over orbits provides a powerful means of obtaining solutions or approximate solution of the Vlasov equation. In this chapter we first study these trajectories using the "guiding center" approximation, and then show how this approximation leads to useful appropriate versions of the Vlasov or fluid equations. In section A, where we examine the single particle trajectories, the $\underline{E}$ and $\underline{B}$ fields are regarded as given, so we are simply solving Newton's equations of motion for a single charged particle. We have already seen how the integration over orbits gives f, from which follow $\underline{j}$ and, in linearized theory, $\underset{\sim}{\sigma}$, $\underset{\sim}{\varepsilon}$ and hence the self-consistent and collective aspects. This is demonstrated in section B, where we obtain the Drift Kinetic Equation, which is a kinetic equation for the motion of guiding centers. Taking velocity moments gives a guiding center version of magneto-hydrodynamics, known as the Chew-Goldberger-Low approximation, which is discussed in section C.

A. Guiding Center Theory

Electronic digital computers make it easy to solve the equations of motion in principle but it is hard to develop physical intuition from specific numerical solutions. Moreover, if the fields are slowly varying (on the scale of r_c or Ω^{-1}) most of the computing time is spent simply following the particle gyration about the magnetic field, whereas the quantity of interest is really the slow change in this cyclotron motion. Just for this case, where numerical computation is inefficient and, in the absence of suitable precautions, inaccurate, analytic techniques are available, namely the <u>guiding center</u> theory of H. Alfven; its principle features are as follows.

In a uniform magnetic field, we know that a charged particle follows a circular helix, i.e., rotates at frequency $\Omega = qB/mc$ about a "guiding center" which moves with constant velocity along the electric field. We shall see that:

i) If a uniform electric field is also present, the gyration is unchanged but the guiding center (g.c.) drifts, transverse to $\underline{B}$, with a velocity $\underline{u}_E = c\underline{E} \times \underline{B}/B^2$. Slow time variation of $\underline{E}$ gives rise to an additional drift, $\underline{u}_{\dot{E}} = c\dot{\underline{E}}/B\Omega$.

ii) Non-uniformity or time variation of $\underline{B}$ also cause additional drifts, normal to $\underline{B}$.

iii) The partition of kinetic energy between the motion of the g.c. itself and the energy, $W_\perp = mv_\perp^2/2$, of gyration about the g.c. is determined by conservation of both the total kinetic energy and the magnetic moment $\mu = W_\perp/B$.

iv) In situations of special symmetry, additional invariants (action, canonical monenta, etc.) also exist.

We shall begin our discussion with elementary derivations of i) through iii) and then give a more formal demonstration.

1. Elementary Derivations

a) Uniform $\underline{B}$ field.

Although this case has already been examined in Chapter VII, we review it briefly here to establish the notation used in more complex examples. We choose an orthogonal set of unit vectors, $\underline{e}$, $\underline{e}_2$, $\underline{e}_3$ with $\underline{e}$ along $\underline{B}$, and set

$$\underline{v} = u\underline{e} + w\underline{\tau} \qquad \underline{\tau}^2 = \underline{\tau}\cdot\underline{\tau} = 1 \tag{1}$$

which implicitly defines u, w, and $\underline{\tau}$, u being the parallel component and w the gyration velocity. It we take the scalar product of the equation of motion,

$$\dot{\underline{v}} = \dot{u}\underline{e} + \dot{w}\underline{\tau} + w\dot{\underline{\tau}} = \underline{v} \times \underline{\Omega} = w\Omega\,\underline{\tau} \times \underline{e} \tag{2}$$

with $\underline{e}$ and with $\underline{\tau}$ it follows that

$$\dot{u} = 0, \quad u = \text{const}; \quad \dot{w} = 0, \ w = \text{constant}$$

and

$$\dot{\underline{\tau}} = \Omega\underline{\tau} \times \underline{e} \tag{3}$$

where we have used the fact that

$$\underline{e}\cdot\underline{\tau} = 0 \quad \text{implies} \quad \underline{e}\cdot\dot{\underline{\tau}} = 0$$

and (4)

$$\underline{\tau}^2 = 1 \quad \text{implies} \quad \underline{\tau}\cdot\dot{\underline{\tau}} = 0$$

If we add to (3) the initial condition $\underline{\tau}(0) = \underline{e}_2$, then the solution of (3) is

$$\underline{\tau} = \underline{e}_2 \cos\Omega t - \underline{e}_3 \sin\Omega t \tag{5}$$

From (1) and (5) we have

$$\dot{\underline{r}} = d\underline{r}/dt = \underline{v} = u\underline{e} + w(\underline{e}_2 \cos\Omega t - \underline{e}_3 \sin\Omega t)$$

which integrates to

$$\underline{r} = ut\underline{e} + (w/\Omega)\,\underline{\rho} + \underline{r}_0 \tag{6}$$

where

$$\underline{\rho} = \underline{e}_2 \sin\Omega t + \underline{e}_3 \cos\Omega t = \underline{e} \times \underline{\tau} \tag{7}$$

and $[\underline{r}_0 + (w/\Omega)\underline{e}_3]$ is the particle position at $\tau = 0$. Clearly, (6) describes gyration, with radius $r_c = (w/\Omega)$ about a g.c. moving with velocity u along $\underline{B}_o$.

b) Uniform $\underline{E}$ and $\underline{B}$ fields

We solve the equation of motion

$$\dot{\underline{v}} = d\underline{v}/d\tau = (q/m)(\underline{E} + \underline{v} \times \underline{B}/c) \tag{8}$$

by setting

$$\underline{v} = \underline{u}_E + \underline{v}_1 \qquad \underline{u}_E = c\,\underline{E} \times \underline{B}/B^2 \tag{9}$$

(We assume here that $E/B << 1$ so that $\underline{u}_E$ is non-relativistic; a relativistic version of the procedure followed here also exits.) Then (8) gives

$$\dot{\underline{v}}_1 = (q/m)\,(E_{\parallel}\,\underline{e} + \underline{v}_1 \times \underline{B}/c) \tag{10}$$

whose solution, following the same approach as in a), is clearly

$$\underline{v}_1 = u\underline{e} + w\underline{\tau} \tag{11}$$

with w = const and $\underline{\tau}$ given by (5), the only difference being that u is no longer constant,

$$\dot{u} = qE_{\parallel}/m \tag{12}$$

Thus, in a frame moving with velocity $\underline{u}_E$, we have gyration, at frequency Ω, radius r_c, about a g.c. accelerating along $\underline{B}$ as given by (12).

A physical picture of the origin of this $\underline{E} \times \underline{B}$ drift is easily obtained. As we can see in Fig. 8.1, the cyclotron motion of a positively charged particle in crossed electric (E_x) and magnetic (B_z) fields is altered because the particle is accelerated as it moves up (toward larger x) and decelerated as it moves down. Thus, its velocity, and hence its cyclotron radius, is larger at the top of the orbit and smaller at the bottom, leading to a cyloidal motion, as shown, and a drift in the $\underline{E} \times \underline{B}$ direction.

c) Uniform B and Non-electromagnetic Force, $\underline{F}$

Since the equation of motion is just (8) with $q\underline{E}$ replaced by $\underline{F}$, we again have gyration in a frame moving with drift velocity

$$\underline{u}_F = c\, \underline{F} \times \underline{B}/qB^2 \tag{13}$$

about a g.c., whose motion along $\underline{B}$ is accelerated by $F_{\parallel}$. Since $\underline{u}_F$, unlike $\underline{u}_E$, is different for different species, this drift will generally give rise to a current, whereas an $\underline{E}$ field does not.

d) Uniform $\underline{B}$ and Uniform, Time-Dependent $\underline{E}$.

Again using (9) we have

$$\dot{\underline{v}}_1 = (q/m)\, E_{\parallel}\, \underline{e} + \underline{v}_1 \times \underline{\Omega} - \dot{\underline{u}}_E =$$

$$= (q/m)\, E_{\parallel}\, \underline{e} + \underline{v}_2 \times \underline{\Omega}$$

where

$$\underline{v}_2 \times \underline{\Omega} \equiv \underline{v}_1 \times \underline{\Omega} - \underline{\dot{u}}_E = (\underline{v}_1 - c\ \underline{\dot{E}}/B\Omega) \times \underline{\Omega}$$

i.e., $$\underline{v}_1 - \underline{v}_2 = \underline{u}_{\dot{E}} \equiv c\ \underline{\dot{E}}/B\Omega \tag{14}$$

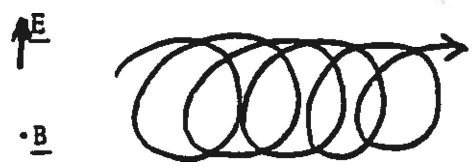

Fig. 8.1 Cycloidal trajectory of a postively charged particle in crossed $\underline{E}$ and $\underline{B}$ fields. The particle is accelerated as it moves up, decelerated as it moves down, so that w and $r_c = w/\Omega$ are larger at the top of the orbit than at the bottom, leading to the drift in the $\underline{E} \times \underline{B}$ direction. The character of the trajectory varies with the value of $U = u_E/w$.

Thus

$$\underline{\dot{v}}_2 = (q/m)\ E_{\parallel}\ \underline{e} + \underline{v}_2 \times \underline{\Omega} - \underline{\dot{u}}_{\dot{E}} \tag{15}$$

If the last term is small compared to the others, then we have the usual cyclotron motion about a g.c. which drifts with velocity $\underline{u}_E + \underline{u}_{\dot{E}}$. Let ω denote a typical frequency for $\underline{E}$, i.e., $|\underline{\dot{E}}| \sim \omega\ |\underline{E}|$. Then

$$|\underline{\dot{u}}_{\dot{E}}|/|\underline{\dot{v}}_2| \sim (c\omega^2 E/B\Omega)/\Omega v_2 = (\omega/\Omega)^2 u_E/v_2$$

which will indeed be small if $\omega << \Omega$ and $u_E \lesssim v_2$.

e. Time Dependent $\underline{\dot{B}}$.

Time varying $\underline{B}$ implies an electric field,

$$\nabla \times \underline{E} = -\ \underline{\dot{B}}/c \tag{16}$$

If $\underline{B}$ varies little during one cyclotron period ($\omega << \Omega$, where ω is a typical frequency for $\underline{B}$), then the orbit will be almost a circle. Due to $\underline{E}$, the particle energy will change, during one cycle, by

$$\Delta W_\perp = \Delta(mw^2/2) = q \oint_C \underline{E} \cdot d\underline{s}$$

(the direction of integration along C being in the direction of gyration for positively charged particles, i.e., in a left-hand sense relative to $\underline{B}$.) From (16) and Stoke's law, we have

$$\Delta W_\perp = q \int\!\!\int_S \underline{\dot{B}} \cdot d\underline{\sigma} \doteq \pi r_c^{\ 2} \underline{q}(\partial B/\partial t) \tag{17}$$

where S is an element of area enclosed by C. The average rate of increase of $W_\perp$ is then

$$\dot{W}_\perp = \Delta W_\perp \Omega/2\pi = (q\Omega r_c^{\ 2}/2)\dot{B} =$$
$$= (qw^2/2\Omega)\ \dot{B} = W_\perp \dot{B}/B$$

This is gives

$$\partial(\ell n W_\perp - \ell n B)/\partial t = 0$$

i.e.,

$$\partial\mu/\partial t = 0 \qquad \mu \equiv W_\perp/B = \text{constant} \tag{18}$$

Since (17) holds only in the limit of slowly varying fields, $|\omega/\Omega| << 1$, μ is called an adiabatic invariant. Actually, μ is only the first term of an asymptotic series in ω/Ω; it is possible to construct a quantity invariant to any desired order in ω/Ω.

We note that μ can also be written as IA/c, where $I = 2\pi q/\Omega$ is the current associated with the gyrating particle and $A = \pi r_c{}^2$ is the enclosed area. This is the usual definition of magnetic moment in electromagnetic theory. Since $\mu = (q^2/2\pi mc^2)\ \Phi$ where $\phi = \pi r_c{}^2 B$ is the flux through the cyclotron orbit we see that ϕ is also an adiabatic invariant. Finally, we can write $\mu = (q/2mc)L$, where $\underline{L} = [\underline{r} \times m\underline{v}]$ is the orbital angular momentum of the gyrating particle; it follows that L is likewise an adiabatic invariant.

As this, admittedly approximate, derivation shows, the invariance of μ rests solely on the time dependence of B at the particle and should therefore be equally valid whether this time dependence arises from motion of the particle through a spatially varying magnetic field or from local variations of B. Since the average particle velocity is $u\underline{e} + \underline{u}_E$, we can generalize (18) to

$$D\mu/Dt \equiv \partial\mu/\partial t + (u\underline{e} + \underline{u}_E) \cdot \Delta\mu = 0$$

f. Guiding Center Drifts Due to Gradients of B

If B varies principally in the direction of $\underline{B}$ itself, as in a gently converging solenoidal field, the principal effect is an acceleration of the g.c. as the particle moves into weaker field regions (and a deceleration as it moves towards stronger fields). This follows immediately from the conservation of total kinetic energy in a static magnetic field:

$$K = m(w^2 + u^2)/2 = \mu B + mu^2/2 = \text{const.}$$

or

$$u^2 = 2(K - \mu B)/m \tag{19}$$

If s is a coordinate along $\underline{B}$, then for motion of the g.c. along s we see that μB plays the role of a potential energy. Moreover, $u = \dot{s}$ and

$$\dot{u} = u\partial u/\partial s = -\partial(\mu B/m)\partial s \tag{20}$$

so the g.c. experiences an effective force equal to $-\mu\partial B/\partial s$. An increasing B therefore acts as a "magnetic mirror", which will reflect particles whose parallel kinetic energy at $s = s_o$ is less than the height, $\mu[B_{max} - B(s_o)]$, of the effective potential barrier.

A quite different effect arises from variation of B in a direction perpendicular to $\underline{B}$. As we see from Fig. 8.2., a weak variation of B_z in the x direction is similar to an E_x field in making r_c larger at the top of the orbit and smaller at the bottom (assuming $\partial B_z/\partial x < 0$). It is clear from the figure that the direction of the resulting drift will be along $q\underline{B} \times \nabla B$. To estimate its magnitude, we compare the change in r_c across the orbit due to an $\underline{E}$ field (where we know the drift velocity) and that due to ∇B. Since constancy of μ implies

$$\Delta\mu/\mu = 2\Delta w/w - \Delta B/B = 0$$

where Δ denotes the difference between the top and bottom of the orbit in Figure 8.2, we must take into account the change in both w and B in computing the variation in r_c associated with ∇B:

$$(\Delta r_c)_{\nabla B} = |\Delta(w/\Omega)| = |\Delta w/\Omega - w\Delta\Omega/\Omega^2| =$$
$$= |w\Delta\Omega/2\Omega^2 - w\Delta\Omega/\Omega^2| = w\Delta\Omega/2\Omega^2 = r_c w\ |\nabla B|/B\Omega$$

The change in r_c due to an electric field arises only from the variation in w:

$$(\Delta r_c)_E = \Delta w/\Omega = \Delta W_\perp/mw\Omega = 2r_c qE/mw\Omega$$

The magnitude of the ∇B drift is then

$$u_{\nabla B} = u_E\ (\Delta r_c)_{\nabla B}/(\Delta r_c)_E = (w^2/2\Omega^2)|\nabla B|B$$

and since we see from Figure 8.2 that it must be in the direction of $q\underline{B} \times \nabla B$ we have finally

$$\underline{u}_{\nabla B} = (w^2/2\Omega)\ \underline{B} \times \nabla B/B^2 = (w^2/2\Omega^2)\ \underline{e} \times \nabla\Omega \qquad (21)$$

(As we will show in section 2, this expression not only gives the correct order of magntiude, as we might expect, but also the correct numerical factor. This must be regarded as somewhat fortuitous since we have used the invariance of μ between one part of the orbit and another, whereas our derivation of the constancy of μ involved an averaging over a complete cyclotron orbit.)

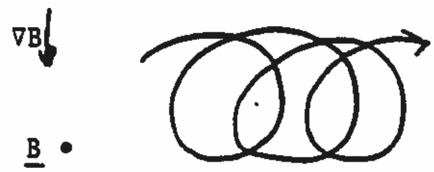

Fig. 8.2 Trajectory of a positively charged particle in a field with ∇B normal to B. The cyclotron frequency Ω is smaller and hence the cyclotron radius $r_c = w/\Omega$ is larger at the top than at the bottom of the orbit (w being constant), leading to the drift in the B x ∇B direction. For negatively charged particles, the direction of the cyclotron motion is reversed and the drift is therefore in the -B x ∇B direction.

Finally, curvature of the magnetic field also leads to a drift across $\underline{B}$. A particle whose g.c. follows a curved or moving magnetic field experiences a pseudo-force (due to centripetal acceleration plus acceleration associated with mirroring)

$$\underline{F} = -m\, D(u\underline{e} + \underline{u}_E)/Dt = -m[\partial/\partial t + (u\underline{e} + \underline{u}_E)\cdot\nabla](u\underline{e} + \underline{u}_E)$$

and hence a drift, $\underline{u}_F = c\underline{F} \times \underline{B}/qB^2$, as given by (13). For static, curved magnetic field lines, with B constant along a line and $\underline{E} = 0$, $\underline{F}$ is just the centrifugal force

$$\underline{F} = -mu^2\underline{e}\cdot\nabla\underline{e} = -mu^2 d\underline{e}/ds = (mu^2/R)\hat{\underline{R}}$$

directed along the line from the local center of curvature to the particle, R being the radius of curvature. The drift $\underline{u}_F$ is the called the "curvature drift", and is given by

$$\underline{u}_R = -(u^2/R\Omega)\underline{e} \times \hat{\underline{R}} \tag{22}$$

As a simple example of both ∇B and curvature drifts, consider the circular field lines surrounding a long wire carrying current $\underline{I} = I\hat{\underline{z}}$. The Biot-Savart law gives $\underline{B} = (2I/cr)(\hat{\underline{z}} \times \hat{\underline{r}})$, where r is the radial distance from the wire; since the lines are circles, we also have r = R. Then

$$\nabla B = (B/r)\hat{\underline{r}}$$

and (21) gives

$$\underline{u}_{\nabla B} = (w^2/2\Omega R)\hat{\underline{z}} \tag{23}$$

while (22) yields

$$\underline{u}_R = (u^2/R\Omega)\hat{\underline{z}} = 2(u^2/w^2)\underline{u}_{\nabla B} \tag{24}$$

Both the curvature and ∇B effects cause a drift along the current axis (in the direction of $\underline{I}$, for a positive particle), the net drift being

$$\underline{u}_T = \hat{\underline{z}}(u^2 + w^2/2)/\Omega R \tag{25}$$

2. Formal Derivation of Guiding Center Theory

We follow here the approach of A. Banos (J. Plasma Phys. 1, 305 (1967)) which provides a systematic and reasonable treatment without excessive formalism. The basic idea of g.c. theory is that the cyclotron gyration of the particle is rapid compared to all other time scales

$$\Omega >> \omega,\ v/L$$

where ω and L typify the time rate of change and the spatial scale length of $\underline{B}(\underline{r},t)$. We therefore seek an asymptotic expansion in Ω^{-1}, averaging over the rapid cyclotron motion to obtain equations of motion on the slow time scale $(\omega^{-1}, L/v)$. (Since $w \leq v$, we are, of course, assuming $r_c L = w/\Omega L << 1$).

We begin with the particle equation of motion

$$\ddot{\underline{r}} = \dot{\underline{v}} = q\,\underline{E}/m + \underline{v} \times \underline{\Omega} \tag{26}$$

and, just as in section 1, set

$$\underline{v} = u\,\underline{e} + \underline{u}_E + w\underline{\tau} \qquad \underline{e} = \underline{B}/B \tag{27}$$

where $\underline{u}_E = c\underline{E} \times \underline{B}/B^2$ is the usual electric field drift. (We shall assume $\underline{E}$ to be constant in space and time; the extension to include varying $\underline{E}$ is straightforward, but not particularly instructive). Clearly u is the parallel component of $\underline{v}$ and $w\underline{\tau}$ is the perpendicular component not included in $\underline{u}_E$, i.e., the gryation velocity, $\underline{e}$ and $\underline{\tau}$ being orthogonal unit vectors,

$$\left.\begin{aligned} &e^2 = \underline{e} \cdot \underline{e} = \tau^2 = \underline{\tau} \cdot \underline{\tau} = 1 \qquad \underline{e} \cdot \underline{\tau} = 0 \\ &\text{Note that} \\ &\underline{e} \cdot \dot{\underline{e}} = \underline{\tau} \cdot \dot{\underline{\tau}} = 0 \qquad \underline{e} \cdot \dot{\underline{\tau}} = -\dot{\underline{e}} \cdot \underline{\tau} \end{aligned}\right\} \tag{28}$$

In a formal sense, u, w and $\underline{\tau}$ are defined in terms of the instantaneous $\underline{v}$ and the local $\underline{E}$ and $\underline{B}$ by (27), taking account of (28) and the definition of $\underline{u}_E$.

If we now introduce a third, orthogonal unit vector,

$$\underline{\rho} = \underline{e} \times \underline{\tau} \tag{29}$$

then we can define the position, R, of the g.c. by

$$\underline{R} = \underline{r} - \underline{\rho} w/\Omega \tag{30}$$

In addition to the orthonormal set ($\underline{e}$, $\underline{\tau}$, $\underline{\rho}$), the latter two of which will be spinning about $\underline{e}$ at frequency Ω, we also introduce another set ($\underline{e}$,$\underline{e}_2$, $\underline{e}_3$) where $\underline{e}_2$ and $\underline{e}_3$ are orthogonal to each other and to $\underline{e}$. While $\underline{e}_2$ and $\underline{e}_3$, like $\underline{e}$, will be functions of $\underline{r}$ and t, we assume that this dependence is slow, compared to L^{-1} and Ω, i.e., that $\underline{e}_2$ and $\underline{e}_3$ change only as required by the slow variations of $\underline{e}$. The relations of these various vectors are summarized in Fig. 8.3, where we show also the angle θ (measured positive in a clockwise sense) between $\underline{\tau}$ and $\underline{e}_2$. Substituting (27) into (26) we obtain, as our fundamental equation of motion,

$$\dot{\underline{v}} = (q/m)\, E_{\parallel}\underline{e} - w\Omega\underline{\rho} \tag{31}$$

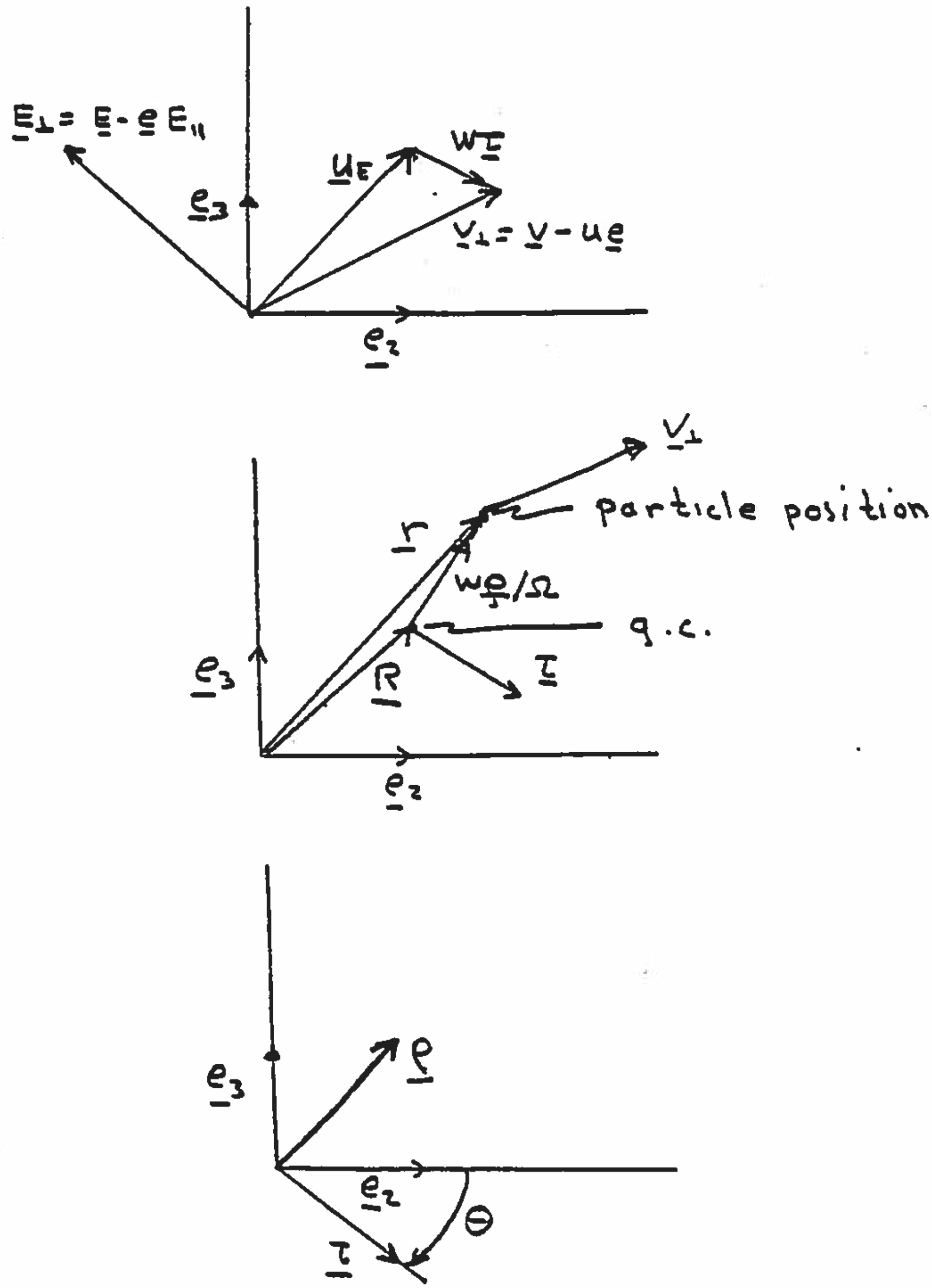

Fig. 8.3 Relation of position and velocity vectors involved in g.c. theory.

We note in passing that one particular choice for $\underline{e}_2$ and $\underline{e}_3$ is to use the principal normal and binormal vectors associated with the field line. To review briefly the pertinent ideas of differential geometry, we note that, as illustrated in Fig. 8.4, the vector $\partial\underline{e}/\partial s$, where s is arc length along the field line, points toward the local center of curvature and has a magnitude of R^{-1}, where R is the local radius of curvature. If we set

$$\partial\underline{e}/\partial s = R^{-1}\underline{n}$$

then $\underline{n}$ is defined as the principal normal and

$$\underline{b} = \underline{e} \times \underline{n}$$

is called the binormal. (NB. Do not mistake $\underline{b}$ for the unit vector tangent to the field line, $\underline{e} \equiv \underline{B}/B$.) If the field line lies in a plane, then $\underline{B}$ will be constant along the field line. If the field line is a space curve, then

$$\frac{\partial\underline{b}}{\partial s} = \underline{e} \times (\partial\underline{n}/\partial s)$$

It follows that

$$\frac{\partial\underline{b}}{\partial s} \cdot \underline{b} = \frac{\partial\underline{b}}{\partial s} \cdot \underline{e} = 0$$

and we can write

$$\frac{\partial\underline{b}}{\partial s} = \tau\underline{n}$$

The quantity τ, which is called the torsion, measures the twisting of the field line in space. If we chose $e_2 = \underline{n}$, $\underline{e}_3 = \underline{b}$, then these vectors will only change with $\underline{r}$ or t as required by the field line geometry. In the following discussion, we do not assume $\underline{e}_2$ and $\underline{e}_3$ to be chosen in this way; it is only necessary that they rotate slowly relative to $\underline{b}$ and $\underline{n}$.

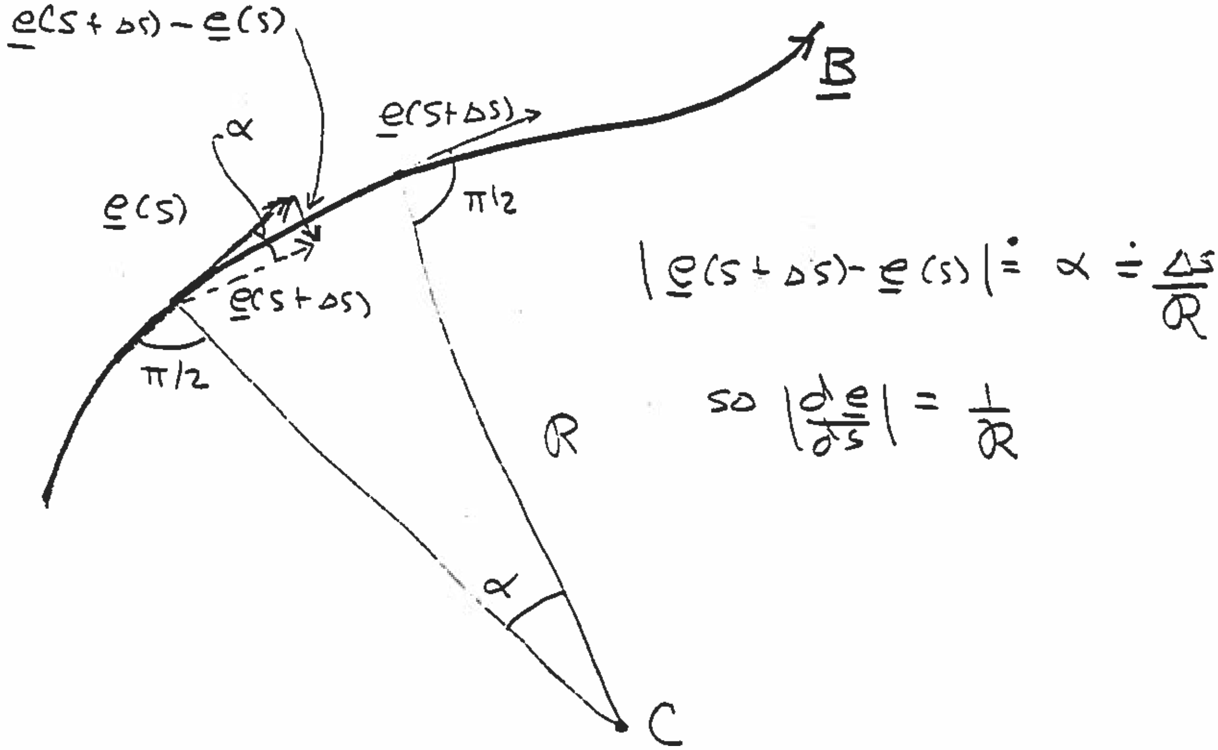

Fig. 8.4 Differential geometry for a field line. The normals to the field line at s and s + Δs interect (in the limit Δs → 0) at the center of curvature C, so d$\underline{e}$/ds also points to C.

a) <u>Equations of motion for u and w</u>

The slow (g.c.) motion is given by u and w so we derive equations for u and w and then take a time average. We have from (27)

$$\begin{aligned} \dot{u} &= du/dt = d(\underline{v} \cdot \underline{e})/dt = \dot{\underline{e}} \cdot \underline{v} + \dot{\underline{v}} \cdot \underline{e} = \\ &= \dot{\underline{e}} \cdot (\underline{u}_E + w\underline{\tau}) + (q/m)E_{\parallel} \end{aligned} \tag{32}$$

where we have used (28). Since u and w are functions of time not only through the intrinsic time dependence of $\underline{E}$ and $\underline{B}$ but also through the dependence of these fields upon $\underline{r}(t)$, we can write

$$\dot{u} = du/dt = (D/Dt + w\underline{\tau} \cdot \nabla)u \tag{33}$$

where

$$D/Dt = \partial/\partial t + (u\underline{e} + \underline{u}_E) \cdot \nabla \tag{34}$$

Similarly, from (27) and (34)

$$\begin{aligned} Dw/Dt + w\underline{\tau} \cdot \nabla\, w = \dot{w} &= (d/dt)[\underline{\tau} \cdot (\underline{v} - \underline{u}_E)] = \\ &= \dot{\underline{\tau}} \cdot (\underline{v} - \underline{u}_E) - \underline{\tau} \cdot \dot{\underline{u}}_E = u\underline{e} \cdot \dot{\underline{\tau}} - \underline{\tau} \cdot \dot{\underline{u}}_E = \\ &= -\underline{\tau} \cdot (u\dot{\underline{e}} + \dot{\underline{u}}_E) = -\underline{\tau} \cdot D(u\underline{e} + \underline{u}_E)/Dt - w\,\underline{\tau}\,\underline{\tau} : \nabla(u\underline{e} + \underline{u}_E) \end{aligned} \tag{35}$$

where we have again used (28). Finally, differentiating

$$\underline{\rho} \cdot \underline{v} = \underline{\rho} \cdot \underline{u}_E$$

gives

$$\dot{\underline{\rho}} \cdot (u\underline{e} + \underline{u}_E + w\underline{\tau}) - w\Omega = \dot{\underline{\rho}} \cdot \underline{u}_E + \underline{\rho} \cdot \dot{\underline{u}}_E$$

or

$$\dot{\underline{\rho}} \cdot \underline{\tau} = \Omega + \underline{\rho} \cdot (\dot{\underline{u}}_E + u\dot{\underline{e}})/w \tag{36}$$

We see from Fig. 8.3 that

$$\begin{aligned} \underline{\tau} &= \underline{e}_2 \cos\theta - \underline{e}\ \ \sin\theta \\ \underline{\rho} &= \underline{e} \times \underline{\tau} = \underline{e}_2 \sin\theta + \underline{e}_3 \cos\theta \end{aligned} \tag{37}$$

so

$$\dot{\underline{\rho}} \cdot \underline{\tau} = \dot{\theta} + \underline{e}_2 \cdot \dot{\underline{e}}_3$$

Substituting this in (37) gives

$$\dot{\theta} = \Omega + \dot{\underline{e}}_2 \cdot \underline{e}_3/w + \underline{\rho} \cdot (\dot{\underline{u}}_E + u\dot{\underline{e}}) \tag{38}$$

So far we have made __no__ approximations; (32), (35) and (39) are __completely__ equivalent to (26). Now we make the basic assumption of g.c. theory, namely that Ω is large compared to ω or v/L. We introduce a dimensionless parameter, η, and assume that both ω/Ω and $v/L\Omega$ are of order η. Then the second and third terms of (39) are of order η relative to the first and to the lowest order in η in we can set

$$\dot{\theta} = \Omega \qquad\qquad \theta = \Omega t + \theta_o$$

In computing time averages we will actually average over θ. Terms linear in $\sin\theta$ or $\cos\theta$ then average to zero (e.g., $\langle\underline{\tau}\rangle = \langle\underline{\rho}\rangle = 0$) and terms like $\sin^2\theta$ and $\cos^2\theta$ average to 1/2, where <> denotes an average over θ.

We now make use of this in calculating $\langle\dot{u}\rangle$ and $\langle\dot{w}\rangle$. From (32) and (33) we have

$$\langle\dot{u}\rangle = Du/Dt = \langle(D\underline{e}/Dt + w\underline{\tau} \cdot \nabla\underline{e}) \cdot (u\underline{e} + w\underline{\tau})\rangle + (q/m)E_{\shortparallel} \tag{39}$$

while (35) gives

$$<\dot{w}> = Dw/Dt = <-\underline{\tau} \cdot D(u_{\underline{e}} + \underline{u}_E)/Dt> - w\, \underline{\tau}\, \underline{\tau} \cdot \nabla(\underline{u}_E + \underline{u}_E)$$

For both calculations we need

$$<\underline{\tau} \cdot \dot{\underline{e}}> = w < \underline{\tau}\, \underline{\tau}>: \nabla\underline{e} \tag{41}$$

From (36) we have

$$<\underline{\tau}\, \underline{\tau}> = (\underline{e}_2\, \underline{e}_2 + \underline{e}_3\, \underline{e}_3)/2 = (\underline{1} - \underline{e}\, \underline{e})/2 \tag{42}$$

so

$$<\underline{\tau}\, \underline{\tau}> : \nabla\underline{e} = (\nabla \cdot \underline{e} - \underline{e}\, \underline{e} : \nabla\underline{e})/2$$

We note that

$$\underline{e} \cdot \nabla = \partial/\partial s$$

where s denotes arc length along a magnetic field line and

$$\underline{e} \cdot \partial\underline{e}/\partial s = \partial(\underline{e} \cdot \underline{e}/2)/\partial s = 0$$

Also,

$$\nabla \cdot \underline{e} = \nabla \cdot (\underline{B}/B) = \underline{B} \cdot \nabla B^{-1} = -e \cdot \nabla B/B = -(\partial B/\partial s)/B$$

It follows that

$$<\underline{\tau}\, \underline{\tau}> : \nabla\underline{e} = \nabla \cdot \underline{e}/2 = -(\partial B/\partial s)/2B \tag{43}$$

Then, (39) gives

$$<<\dot{u}> = Du/Dt = (q/m)E_{\parallel} + \underline{u}\underline{e} \cdot D\underline{e}/Dt - (w^2/2B)\partial B/\partial s \tag{44}$$

To compute $\dot{w}$ we need, in addition to (44), also

$$\langle \underline{\tau} \cdot \dot{\underline{u}}_E \rangle = w \langle \underline{\tau}\, \underline{\tau} \rangle : \nabla \underline{u}_E =$$

$$= (wc/2)\ (\nabla \cdot (\underline{E} \times \underline{e}/b) - \underline{e} \cdot \partial(\underline{E} \times \underline{e}/B)/\partial s] \qquad (45)$$

We can simplify (45) by using Maxwell's equations and the identity

$$0 = \nabla(\underline{e} \cdot \underline{e}/2) = \underline{e} \cdot \nabla\, \underline{e} + \underline{e} \times (\nabla \times \underline{e})$$

from which follows

$$\partial \underline{e}/\partial s = -\underline{e} \times (\nabla \times \underline{e}) \qquad (46)$$

For the first term on the right hand side of (45), we have

$$\nabla \cdot (\underline{E} \times \underline{e}/B) = \underline{e} \cdot (\nabla \times \underline{E})/B + \underline{E} \times \underline{e} \cdot \nabla B^{-1} - \underline{E} \cdot (\nabla \times \underline{e})/B =$$

$$-(\partial B/\partial t)/cB\ -\ \underline{u}_E \cdot (\nabla B)/Bc - \underline{E} \cdot (\nabla \times \underline{e})/B$$

while for the second term, using (46),

$$\underline{e} \cdot \underline{E} \times (\partial \underline{e}/\partial s)/B = \underline{e} \cdot (\underline{e} \cdot \underline{E}\, \nabla \times \underline{e} - \underline{e}\, \underline{E} \cdot \nabla \times \underline{e})/B =$$

$$= [E_{\parallel}\, \underline{e} \cdot \nabla \times \underline{e} - \underline{E} \cdot (\nabla \times \underline{e})]/B$$

Since the last terms in these two expressions cancel, we have

$$\langle \underline{\tau} \cdot \dot{\underline{u}}_E \rangle = w \langle \underline{\tau}\, \underline{\tau} \rangle :\ \nabla \underline{u}_E = (c/2)[\nabla \cdot (\underline{E} \times \underline{e}/B) - \underline{e} \cdot \partial(\underline{E} \times \underline{e}/B)/\partial s]$$

$$= -\ (w/2B)\ [\partial B/\partial t + \underline{u}_E \cdot \nabla B + cE_{\parallel}\, \underline{e} \cdot \nabla \times \underline{e}] \qquad (47)$$

Combining (43) and (47) gives for the quantity which appears in the second term of (39)

$$\langle \underline{\tau}\, \underline{\tau} \rangle :\ \nabla\ (u\underline{e} + \underline{u}_E) = u\ \langle \underline{\tau}\, \underline{\tau} \rangle :\ \nabla \underline{e} + \langle \underline{\tau}\, \underline{\tau} \rangle :\ \nabla \underline{u}_E =$$

$$= -\ [\partial B/\partial t + (u_{\underline{e}} + \underline{u}_E) \cdot \nabla B + cE_{\parallel}\, \underline{e} \cdot (\nabla \times \underline{e})]/2B \qquad (48)$$

Substituting this into (39) we have

$$\langle\dot{w}\rangle = Dw/Dt = (w/2B)\ [DB/Dt + cE_{\parallel}\ \underline{e} \cdot \nabla \times \underline{e}] \tag{49}$$

The last term of (49) is usually omitted from discussions of lowest order g.c. theory because it is of order η relative to the first term. The size of $E_{\parallel}$ is set by (32),

$$(q/m)E_{\parallel} \sim \omega u$$

so the ratio of the two terms on the right side of (49) is

$$|cE_{\parallel}\ \underline{e} \cdot \nabla \times \underline{e}|/|DB/Dt| \sim (cm\omega u/qL)/\omega B =$$
$$= u/L\Omega = \theta(\eta)$$

Moreover, if $\underline{B}$ is a vacuum field, $\nabla \times \underline{B} = 0$, then

$$\underline{e} \cdot \nabla \times \underline{e} = \underline{e} \cdot (\nabla \times \underline{B}/B - \underline{e} \times \nabla B/B^2) = 0$$

so the last term vanishes identically. In any case, we have to lowest order

$$2D\ell nw/Dt - D\ell nB/Dt = 0$$

or

$$D\mu/Dt = 0 \tag{50}$$

Note that this justifies the results of the heuristic proof given in section 1, where we showed that $\partial\mu/\partial t = 0$ and, on physical grounds, extended this to the total time variation, including a convective term. We see that the appropriate derivative is indeed D/Dt, i.e., the local time variation plus that due to the parallel motion, $u\underline{e}$, and the $\underline{E} \times \underline{B}$ drift, $\underline{u}_E$. Having shown that μ is constant to lowest order to η, we can write (44) as

$$Du/Dt = (q/m)E_{\parallel} + \underline{u}_E \cdot D\underline{e}/Dt - \partial(\mu B)\partial \tag{51}$$

showing that μB plays the role of a potential for the parallel motion.

In summary, we have now shown that, starting from the exact equation of motion, (26) or (31), and averging over the cyclotron motion we obtain, in lowest order, the invariance of μ, as stated in (50), and the phenomenon of magnetic mirroring, as expressed in (51). (The $\underline{u}_E \cdot [D\underline{e}/D\tau]$ term simply accounts for the apparent change of μ arising from the change in $\underline{e}$, i.e., the change in what is meant by "parallel" component of $\underline{v}$; it is kinematic in origin, rather than dynamic.) A particle gyrates about a guiding center, which drifts across $\underline{B}$ with velocity $\underline{u}_E$ and moves along $\underline{B}$ with velocity u, the gyration energy, $W_\perp$, varying with B so that μ remains constant and the parallel velocity, u, varying according to (51). We have not obtained the curvature and ∇B drifts discussed in Section 1 because they are of order η, but we do find then when we examine the motion of the g.c. itself.

b. The Guiding Center Drifts

Since the position of the g.c. is defined by $R = r - \underline{\rho}\, w/\Omega$ we have

$$\underline{R} = \underline{u}_E + u\underline{e} + \underline{\rho} w\dot{\Omega}/\dot{\Omega}^2 + [w\underline{\tau} - \dot{\underline{\rho}}w/\Omega - \dot{w}\underline{\rho}/\Omega]$$

The quantity of interest is, as usual, the time or phase average $\langle\dot{\underline{R}}\rangle = D\underline{R}/Dt$. To leading order (η^0), (52) gives $\langle\underline{R}\rangle = u\underline{e} + \underline{u}_E$, as expected. The terms of order ν result from the terms in $\dot{\underline{\rho}}$, $\dot{\Omega}\underline{\rho}$ and $w\underline{\rho}$ which all have non-vanishing time averages, thanks to the $w\underline{\tau}\cdot\nabla$ part of the d/dt operator, (33). The calculation is simplified by using the relation obtained by taking the cross product of $\underline{e}$ with $\dot{\underline{v}}$:

$$\underline{e} \times \dot{\underline{v}} = u\underline{e} \times \dot{\underline{e}} + \underline{e} \times \underline{u}_E + \underline{\rho}\dot{w} + w\, \underline{e} \times \dot{\underline{\tau}} = w\Omega\underline{\tau} \qquad (53)$$

where the last term follows directly from (31). Since $\underline{\tau} = \underline{\rho} \times \underline{e}$, we have

$$\underline{e} \times \dot{\underline{\tau}} = \underline{e} \times (\dot{\underline{\rho}} \times \underline{e} + \underline{\rho} \times \underline{e}) = \dot{\underline{\rho}} - \underline{e}\,\underline{e} \cdot \dot{\underline{\rho}} \qquad (54)$$

From (53) and (54) we have

$$u\underline{e} \times \dot{\underline{e}} + \underline{e} \times \dot{\underline{u}}_E = \Omega(w\underline{\tau} - \underline{\rho}\,\dot{w}/\Omega - w\dot{\underline{\rho}}/\Omega) + w\,\underline{e}\,\underline{e} \cdot \dot{\underline{\rho}} \qquad (55)$$

so (52) can be written

$$\dot{\underline{R}} - (\underline{u}_E + u\underline{e}) = \Omega^{-1}\,[\underline{e} \times (u\dot{\underline{e}} + \dot{\underline{u}}_E) + w\,\underline{e}\,\underline{\rho} \cdot \dot{\underline{e}} + w\,\underline{\rho}\,\dot{\Omega}/\Omega] \qquad (56)$$

It is now clear that the right hand side is of order η. To compute the average of the last two terms on the right side of (56) we simply need

$$<\underline{\rho}\,\underline{\tau}> = <\underline{e} \times \underline{\tau}\,\underline{\tau}> = \underline{e} \times \ (\underset{\sim}{1} - \underline{e}\,\underline{e})/2 = \underline{e} \times \underset{\sim}{1}/2$$

or, in index notation,

$$<\rho_i\tau_j> = e_m\epsilon_{imj}/2$$

Thus,

$$w\ <\underline{\rho}\cdot\dot{\Omega}> = w\ <\underline{\rho}\,\underline{\tau}> \cdot \nabla\Omega = (w/2)\ \underline{e} \times \nabla\Omega$$

$$<\underline{\rho} \cdot \dot{\underline{e}}> = w<\underline{\rho} \cdot (\underline{\tau} \cdot \nabla)\underline{e}> = w(\underline{e} \times \nabla) \cdot \underline{e}/2 = (w/2)\ \underline{e} \cdot \nabla \times \underline{e}$$

Then

$$<\dot{\underline{R}}> = D\underline{R}/Dt = \underline{u}_E + u\underline{e} - [D(u\underline{e} + \underline{u}_E)/Dt] \times \underline{e}/\Omega + (w^2/2\Omega^2)\ \underline{e} \times \nabla\Omega$$

$$+ (w^2/2\Omega)\ \underline{e}\,\underline{e} \cdot \nabla \times \underline{e} =$$

$$= \underline{u}_E + [u + (w^2/2\Omega)\ \underline{e} \cdot \nabla \times \underline{e}]\ \underline{e} + (w^2/2\Omega)\ \underline{B} \times \nabla B/B^2 + c(\underline{F}/q) \times \underline{B}/B^2 \qquad (57)$$

where

$$\underline{F} = -m\, D(u\underline{e} + \underline{u}_E)/Dt \tag{58}$$

We thus obtain a correction (of order η) to the parallel velocity u plus the drifts discussed in section 1, that is the ∇B drift

$$\underline{u}_{\nabla B} = (w^2/2\Omega)\ \underline{B} \times \nabla B/B^2$$

and the so-called curvature drift

$$\underline{u}_F = (c\ \underline{F}/q) \times B/B^2$$

The pseudo-force $\underline{F}$ is due to the particle's curvilinear motion typically arising from a curved (or moving) magnetic field line. Of course, a variable $\underline{E} \times \underline{B}$ drift would, as we see from (58), also contribute to $\underline{F}$ and hence to $\underline{u}_{\nabla B}$.

c. Energy Conservation

The time-averaged kinetic energy is $K = \langle mv^2/2\rangle = m(u_E{}^2 + u^2 + w^2)/2$ and we would expect its time rate of change to be given by the average rate at which the $\underline{E}$ and $\underline{B}$ fields do work on the particle. To demonstrate this we take the scalar product of (8) with $\underline{v}$:

$$(m/2)d(v^2)/dt = m\ \underline{v} \cdot \dot{\underline{v}} = q\ \underline{v} \cdot \underline{E}$$

Since $mv^2/2 = K + m\ w\underline{\tau} \cdot \underline{u}_E$ we have

$$(d/dt)\ (K + mw\underline{\tau} \cdot \underline{u}_E) = q\ E \cdot (u\underline{e} + w\tau) \tag{59}$$

On taking the usual time average we have

$$DK/Dt = -m \langle \dot{w}\underline{\tau} + w\dot{\underline{\tau}}\rangle \cdot \underline{u}_E - mw \langle \underline{\tau} \cdot \dot{\underline{u}}_e\rangle + q\ E_{\shortparallel} u \tag{60}$$

From (27) and (31),

$$\dot{\underline{v}} = u\dot{\underline{e}} + u\dot{\underline{e}} + \dot{\underline{u}}_E + \dot{w}\underline{\tau} + w\dot{\underline{\tau}} = (q/m)\ E_{\shortparallel}\ \underline{e} - w\Omega\underline{\rho} \tag{61}$$

On taking the time average of the scalar product of this with $\underline{u}_E$ we obtain

$$\underline{u}_E \cdot \langle\dot{w}\underline{\tau} + w\dot{\underline{\tau}}\rangle = -\underline{u}_E \cdot \langle u\dot{\underline{e}} + \dot{\underline{u}}_E\rangle = -(c/B)\ \underline{E} \times \underline{e} \cdot D(u\underline{e} + \underline{u}_E)/Dt \tag{62}$$

Using this with (47) we find

$$DK/Dt = \mu[\partial B/\partial t + \underline{u}_E\cdot\nabla B + cE_{\shortparallel}\ \underline{e} \cdot \nabla \times \underline{e}] + $$
$$+ mc(\underline{E} \times \underline{e}/B) \cdot D(u\underline{e} + \underline{u}_E)/Dt + q\ E_{\shortparallel}u \tag{63}$$

or

$$DK/Dt = \mu\partial B/\partial t + c\mu\ \underline{E} \cdot (e \times \nabla B/B + \underline{e}\ \underline{e} \cdot \nabla \times \underline{e}) + $$
$$+ (q/\Omega)\ \underline{E} \cdot \underline{e} \times D\ (u\underline{e} + \underline{u}_E)/Dt + qE_{\shortparallel}u \tag{64}$$

Since (57) gives

$$q\underline{E} \cdot (D\underline{R}/Dt) = q\ E_{\shortparallel}u + \mu c\underline{E} \cdot [\underline{e} \times \nabla B/B + \underline{e}\ \underline{e} \cdot \nabla \times e]$$
$$+ (q/\Omega)\ \underline{E} \cdot \underline{e} \times D(u\underline{e} + \underline{u}_E)/Dt \tag{65}$$

we have

$$\frac{DK}{Dt} = \mu\partial B/\partial t + q\ \underline{E} \cdot DR/Dt \tag{66}$$

i.e., the increase in average kinetic energy results from work done by $\underline{E}$ on the guiding center motion plus work done by the inductive field, associated with $\partial B/\partial t$, in increasing w^2. (If $(\partial B/\partial t) = 0$, the convective change $(\underline{u}_E + u\underline{e}) \cdot \nabla B$, causes only an exchange between the parallel and perpendicular parts of K, not a net change.)

To summarize our discussion of guiding center theory for a single charged particle in given $\underline{E}$ and $\underline{B}$ fields, we have shown that, to lowest order in a small parameter η, which is of order ω/Ω and r_c/L, the particle motion can be described as a gyration, at frequency Ω, about a guiding center (g.c.) $\underline{R}$,

$$\begin{aligned} \underline{r} &= \underline{R} + (w/\Omega)\underline{\rho} \\ \underline{\rho} &= \underline{e}_2 \sin\Omega t + \underline{e}_3 \cos\Omega t \end{aligned} \qquad (67)$$

where $\underline{e}_2$ and $\underline{e}_3$ are perpendicular to $\underline{e} = \underline{B}/B$ and equal to (or rotating slowly with respect to) the principal normal and binormal, $\underline{n}$ and $\underline{b}$. The equation of motion for the guiding center (averaged over the cyclotron gyration) is

$$D\underline{R}/Dt = \underline{u}_E + (u + u_1)\underline{e} + \underline{u}_{\nabla B} + \underline{u}_F \qquad (68)$$

where

$$\underline{u}_E = c\underline{E} \times \underline{B}/B^2 = c\underline{E} \times \underline{e}/B$$

is the $\underline{E} \times \underline{B}$ drift velocity;

$$\underline{u}_{\nabla B} = (w^2/2\Omega)(\underline{B} \times \nabla B)/B^2 \qquad (69)$$

is the gradient B drift which arises whenever $\underline{B}$ varies in a direction perpendicular to $\underline{B}$, and

$$\underline{u}_F = c(\underline{F}/q) \times B/B^2 \qquad (70)$$

is the drift arising from the pseudo force

$$\underline{F} = -m\, D(u\underline{e} + \underline{u}_E)/Dt \qquad (71)$$

experienced as the g.c. follows curved (or time varying) magnetic field lines and, simultaneously, drifts across the field lines with a variable $\underline{E} \times \underline{B}$ drift velocity. The velocity parallel to $\underline{B}$ consists of a component u which is accelerated by $E_{\parallel}$ and by variations of $\underline{B}$ along $\underline{B}$ (as in a magnetic mirror)

$$Du/Dt = (q/m)E_{\parallel} - \mu\underline{e} \cdot \nabla B + \underline{u}_E \cdot (D\underline{e}/Dt) \tag{72}$$

plus a parallel drift velocity

$$u_1 = (w^2/2\Omega)\underline{e} \cdot \nabla \times \underline{e} \tag{73}$$

which vanishes for a vacuum field, $\nabla \times \underline{B} = 0$. The perpendicular drifts $\underline{u}_{\nabla B}$ and $\underline{u}_F$ and the parallel drift $u_1\underline{e}$ are of order η compared to $\underline{u}_e + \underline{u}_E$ so that to lowest order the g.c. moves with velocity u along $\underline{B}$ and drifts with velocity $\underline{u}_E$ across it, this motion then being modified by the first order perpendicular and parallel drifts $\underline{u}_{\nabla B}$, $\underline{u}_F$ and $u_1\underline{e}$. The instantaneous particle velocity is

$$\underline{v} = d\underline{r}/dt = u\underline{e} + w\underline{\tau} + \underline{u}_E$$

where

$$\underline{\tau} = \underline{e}_2 \cos\Omega t - \underline{e}_3 \sin\Omega t$$

and w, the gyration velocity, satisfies

$$\mu \equiv mw^2/2B = \text{const}$$

or, equivalently

$$Dw/Dt = (w/2B)(DB/Dt) \tag{74}$$

if we drop the last term (of order η) in (49).

B. Collective Effects in the Guiding Center Approximation: The Drift Kinetic Equation and the Chew-Goldberger-Low Approximation

Having examined the guiding center (g.c.) approximation for the motion of single particles in given $\underline{E}$ and $\underline{B}$ fields, we now consider how the real problem -- the self-consistent determination of both the fields and the plasma response -- can be solved when the basic conditions which justify the g.c. approximation ($\Omega >> \omega$, v/L) are satisfied. At the level of kinetic theory we shall derive an approximate version of the Vlasov equation known as the Drift Kinetic Equation (DKE), which is very useful for studying plasma behavior in complex magnetic geometries such as fusion confinement configurations or planetary ionospheres and magnetospheres. At the level of fluid theory we shall derive a version of one fluid MHD known as the Chew-Goldberger-Low (CGL) approximation which differs from that presented in Chapter IV in allowing a diagonal but non-scalar pressure tensor with different $p_{\parallel}$ and $p_{\perp}$.

1. The Drift Kinetic Equation (DKE)

In general, the Vlasov equation simply states that f is constant along the particle trajectories. When the g.c. approximation is valid ($\Omega >> \omega$, v/L) we can replace the exact particle equations of motion with the equations derived in section A, in which ($\underline{x}$, $\underline{v}$) are replaced by the g.c. variables ($\underline{R}$, u, w) and the velocity azimuth angle has been eliminated. The g.c. equations of motion are given by (67) through (74). If we introduce a distribution function F($\underline{R}$, u, w, t) which depends only on the g.c. variables, we would expect it to satisfy

$$\partial F/\partial t + (D\underline{R}/Dt) \cdot (\partial F/\partial \underline{R}) + (Du/Dt)(\partial F/\partial u) + (Dw/Dt)\partial F/\partial w = 0 \qquad (75)$$

with $D\underline{R}/Dt$, Du/Dt, Dw/Dt given by (68) through (74). In fact, as we shall now show, if we start with the usual Vlasov equation and expand in Ω^{-1} (i.e., in ω/Ω and $v/L\Omega$) we obtain just (75) and (67) through (69). Not surprisingly, the demonstration involves much of the algebra used in Section A, although the point of view here is different, $\underline{x}$ and t $\underline{v}$ being now the usual independent variables in the distribution function; thus, $\underline{x}$ and $\underline{v}$ are not functions of t, as in the single particle description.

We begin with the usual Vlasov equation

$$\partial f/\partial t + \underline{v} \cdot \nabla f + (q/m)(\underline{E} + \underline{v} \times \underline{B}/c) \cdot \partial f/\partial \underline{v} = 0 \tag{76}$$

and make a formal change of variables from $\underline{x},\underline{v}$ to $\underline{R}$, u, w, ϕ as prescribed by the following equations (which are partly explicit, partly implicit, but completely specify the transformation):

$$\underline{R} \equiv \underline{x} - (w/\Omega)\underline{\rho} \tag{77}$$

$$\underline{v} = u\underline{e} + w\underline{\tau} + \underline{u}_E \tag{78}$$

$$\underline{\tau} = \underline{e}_2 \cos\theta - \underline{e}_3 \sin\theta \tag{79}$$

$$\underline{\rho} = \underline{e} \times \underline{\tau} = \underline{e}_2 \sin\theta + \underline{e}_3 \cos\theta \tag{80}$$

with $\underline{e}_2$, $\underline{e}_3$ and $e = \underline{B}/B$ forming an orthonormal set, $\underline{e}_2 \times \underline{e}_3 = \underline{e}$. It then follows that $\underline{\tau}$ and $\underline{\rho}$ are unit vectors. We consider $\underline{B}(\underline{x},t)$ and $\underline{e}(\underline{x},t)$ to be given functions, as usually done in considering the Vlasov equation, and chose $\underline{e}_2$, $\underline{e}_3$ as in section A. Of course, we can see use (78) and (79) to write (u, w, ϕ) as explicit functions of ($\underline{x}$, $\underline{v}$, t):

$$u = \underline{e} \cdot \underline{v}$$

$$w = |\underline{w}|; \quad \underline{w} = \underline{v} - \underline{v} \cdot \underline{e}\,\underline{e} - \underline{u}_E = \underline{e} \times (\underline{v} \times \underline{e}) - \underline{u}_E$$

$$\phi = -\tan^{-1} [(\underline{e}_3 \cdot w)\ (\underline{e}_2 \cdot \underline{w})^{-1}]$$

and these, with (80), then give $\underline{R}$ as an explicit function of $\underline{x}$ and $\underline{v}$. However, the set (77) through (80) is usually both sufficient and convenient.

We write the distribution function as

$$f(\underline{x}, \underline{v}, t) = f\ (\underline{R} - (w/\Omega)\underline{\rho}, u, w, \phi, t) \equiv F\ (\underline{R}, u, w, \phi, t) \tag{81}$$

and proceed to derive from (76) the corresponding equation for F. This requires that we express terms containing derivatives of f, such as $\partial f/\partial t$, $\underline{v} \cdot \nabla f$ and

$$(q/m)\ (\underline{E} + \underline{v} \times B/c) \cdot \partial f/\partial \underline{v} = (qE_{\parallel}/m)\ \underline{e} \cdot \frac{\partial f}{\partial \underline{v}} - w\Omega\underline{\rho} \cdot \partial f/\partial \underline{v},$$

in terms of derivatives of F. Straighforward application of the chain rule of differentiation yields

$$\partial f/\partial t = \partial F/\partial t + (\partial F/\partial \underline{R}) \cdot (\partial \underline{R}/\partial t) + (\partial F/\partial u)(\partial u/\partial t) +$$
$$+ (\partial F/\partial w)(\partial w/\partial t) + (\partial F/\partial \phi)(\partial \phi/\partial t) \tag{82}$$

Of course, in calculating $(\partial \underline{R}/\partial t)$, $\partial u/\partial t$, etc., $\underline{x}$ and $\underline{v}$ are held constant. Thus,

$$\partial u/\partial t = \partial(\underline{e} \cdot \underline{v}) / \partial t = (\underline{u}_E + w\underline{\tau}) \cdot \partial e/\partial t \tag{83}$$

$$\partial w/\partial t = \underline{\tau} \cdot \partial(w\underline{\tau})/\partial t = \underline{\tau} \cdot \partial(\underline{v} - u\underline{e} - \underline{u}_E)\ /\partial t =$$
$$= -\underline{\tau} \cdot \partial(u\underline{e} + \underline{u}_E)/\partial t \qquad (84)$$

$$\partial\underline{R}/\partial t = -\partial(w\underline{\rho}/\Omega)/\partial t \qquad (85)$$

N.B. Although our notation here is similar to that used in Section A -- u for the parallel component of $\underline{v}$, w for the magnitude of the perpendicular component of ($\underline{v}$ - $\underline{u}_E$), etc. -- the point of view is quite different. Here u, w, etc. are functions of $\underline{x}$ and t only because the dependence of $\underline{B}$, and hence $\underline{e}$, on $\underline{x}$ and t causes a similar variation of the coordinate system. Since $\underline{v}$ is not a function of t, u changes with t only because the variation in $\underline{B}$ and $\underline{e}$ changes what is meant by the "parallel" component. The change in u does not arise from particle dynamics, as was the case in the single particle treatment of section A.

For $\partial f/\partial x_i$ we have a set of equations exactly analogous to (82) through (85) with x_i in place of t. Finally, the velocity derivatives needed for (76) are

$$\underline{e} \cdot (\partial f/\partial\underline{v}) = \partial F/\partial u \qquad (86)$$

and $\underline{\rho} \cdot \partial f/\partial\underline{v}$. In evaluating the last of these, it is convenient to consider f temporarily as a function of ($\underline{x}$, u, $\underline{w}$, t) with $\underline{w} = w\underline{\tau} = \underline{v} - u\underline{e} - \underline{u}_E$. For fixed $\underline{x}$ and t, and hence fixed $\underline{e}$, we can choose the yz axes along $\underline{e}_2$ and $\underline{e}_3$. Then,

$$(\partial/\partial\phi)f(\underline{x}, u, \underline{w}, \tau) = (\partial f/\partial\underline{w})_{\underline{x},\, u,\, t} \cdot (\partial\underline{w}/\partial\phi) = (\partial f/\partial\underline{v})_{\underline{x}t} \cdot (\partial\underline{w}/\partial\phi) =$$
$$= -[\partial f/\partial v_y)\ w \sin\phi + (\partial f/\partial v_z)\ w \cos\phi] = -w\underline{\rho} \cdot (\partial f/\partial\underline{v}) \qquad (87)$$

Finally, then,

$$w\underline{\rho}\cdot(\partial f/\partial \underline{v}) = (\partial f/\partial\phi)_{\underline{x},u,w,t} = \partial F/\partial\phi - (w/\Omega)(\partial F/\partial\underline{R})\cdot(\partial\underline{\rho}/\partial\phi) =$$
$$= \partial F/\partial\phi - (w/\Omega)\,\underline{\tau}\cdot(\partial F/\partial R)$$

and

$$(q/m)(\underline{E} + \underline{v}\times\underline{B}/c) = (qE_{\shortparallel}/m)\;\partial F/\partial u + \Omega\partial F/\partial\phi - w\underline{\tau}\cdot\partial F/\partial\underline{R} \tag{88}$$

Substituting $\partial f/\partial t$, $\partial f/\partial x_i$, etc. into (76) we find

$$\partial F/\partial t + (\partial F/\partial\underline{R})\cdot[\partial\underline{R}/\partial t + v_i\;\partial\underline{R}/\partial x_i - w\underline{\tau}] + (\partial F/\partial u)[qE_{\shortparallel}/m + \partial u/\partial t +$$
$$+ \underline{v}\cdot\nabla u] + (\partial F/\partial w)\;[\partial w/\partial t + \underline{v}\cdot\nabla w] + \partial F/\partial\phi\;[\partial\phi/\partial t + \underline{v}\cdot\nabla\phi + \Omega] = 0 \tag{89}$$

This equation is completely equivalent to the Vlasov equation (76); we have simply made a change of variables from ($\underline{x}$, $\underline{v}$, t) to ($\underline{R}$, u, w, ϕ, t).

We now introduce the g.c. approximation. We assume ω/Ω and $v/L\;\Omega$ to be or order η, where $\eta << 1$, and we expand F in powers of η(or Ω^{-1})

$$F = F^{(0)} + F^{(1)} + ---$$

where $F^{(n)}$ is of order η^n. This expansion is, of course, quite different from the expansions in the field amplitudes used in our previous studies of the linearized Vlasov equation. The DKE which we shall presently derive is approximate only in the sense that $\eta << 1$. The DKE and the Maxwell equations still constitute a nonlinear system, quite analogous to the Vlasov-Maxwell system, which we may, but need not, linearize.

If we divide (89) by Ω, there are only two terms which are not clearly of order η so to lowest order we have

$$\partial F^{(0)}/\partial\phi \;-\; (E_{\shortparallel}/B)c\partial F^{(0)}/\partial u = 0 \tag{90}$$

The general solution to this is

$$F^{(0)} = G(\phi + uB/cE_{\parallel}) \tag{91}$$

where G is an arbitrary function. However, $F^{(0)}$ must be periodic in ϕ with period 2π and must vanish for $|u| \to \infty$. Since no function G exists which satisfies both of these conditions, the expansion in Ω^{-1} can be carried through consistently only if $(E_{\parallel}c/B)\partial F^{(0)}/\partial u \sim (E_{\parallel}c/uB)$ is, in fact, of order η or higher, so that the second term in (90) can be dropped. Since

$$E_{\parallel}c/uB = qE_{\parallel}/m\Omega u$$

this simply means that the change in parallel velocity produced by $E_{\parallel}$ in one cyclotron period must be small (of order η) compared to u.

Our lowest order equation then gives simply

$$\partial F^{(0)}/\partial\phi = 0, \tag{92}$$

that is, $F^{(0)}$ must be independent of the velocity azimuth angle ϕ. Thus, to lowest order

$$F(\underline{R}, u, w, \phi, t) = F^{(0)} (\underline{R}, u, w, t) \tag{93}$$

and the corresponding Vlasov function f is

$$f (\underline{x}, \underline{v}, t) = F^{(0)} (\underline{x} - w\underline{\rho}/\Omega, u, w, t)$$

Note that, through $\underline{\rho}$, f is a function of ϕ, even though F is independent of ϕ.

To first order (89) gives

$$\partial F^{(0)}/\partial t + (\partial F^{(0)}/\partial\underline{R}) \cdot [\partial\underline{R}/\partial t + v_i\ \partial\underline{R}/\partial x_i + w\tau] +$$
$$(\partial F^{(0)}/\partial u)(qE_{\parallel}/m + \partial u/\partial t + \underline{v} \cdot \nabla\underline{u}) + (\partial F^{(0)}/\partial w)\ [\partial w/\partial t + \underline{v} \cdot \nabla w] = \Omega\ \partial F^{(1)}/\partial\phi \tag{94}$$

If $F^{(0)}$ were known, we would consider this as an equation for $F^{(1)}$. However, just the requirement that $F^{(1)}$ be periodic in ϕ and that therefore

$$\int_0^{2\pi} d\phi(\partial F^{(1)}/\partial\phi) = 0 \tag{95}$$

already imposes a requirement of $F^{(0)}$, namely that the left side of (94) averaged over ϕ should vanish. Since the derivatives of $F^{(0)}$ with respect to $\underline{R}$, u, w and t are all like $F^{(0)}$ itself, independent of ϕ, we need only compute the averages of their coefficients. Thus, taking the ϕ average of (94), we have

$$\partial F^{(0)}/\partial t + (\partial F^{(0)}/\partial \underline{R}) \cdot \langle\delta\underline{R}/\partial t\rangle + (\partial F^{(0)}/\partial u)(qE_{\shortparallel}/m + \langle\delta u/\delta t\rangle) + \\ + (\partial F^{(0)}/\partial w)\langle\partial w/\partial t\rangle = 0 \tag{96}$$

where

$$\langle\ \rangle = (2\pi)^{-1}\int_0^{2\pi} d\phi$$

and

$$\delta/\delta t \equiv \partial/\partial t + (u\underline{e} + \underline{u}_E + w\underline{\tau}) \cdot \nabla \tag{97}$$

From (83) and the analogous equation for ∇u, we have

$$\langle\delta u/\delta t\rangle = \underline{u}_E \cdot [\frac{\partial \underline{e}}{\partial t} + (u\underline{e} + \underline{u}_E) \cdot \nabla\underline{e}] + w^2 \langle\underline{\tau}\ \underline{\tau}\rangle\colon \nabla\underline{e} = \\ = \underline{u}_E \cdot D\underline{e}/Dt - (\mu/m)\partial B/\partial s \tag{98}$$

where we have used the D/Dt notation defined in (34) and also made use of (43), which involves only averages over ϕ. Similarly, from (84) and the analogous equation for ∇w we have, since $\langle\partial w/\partial t\rangle = 0$ and $\langle(u\underline{e} + \underline{u}_E)\rangle\cdot\nabla w\rangle = 0$,

$$<\delta w/\delta t> = <w\underline{\tau} \cdot \nabla w> = -w <\underline{\tau}\ \underline{\tau}>: \nabla\ (u\underline{e} + \underline{u}_E) =$$
$$= (w/2B)\,[DB/Dt + cE_{\shortparallel}\ (\underline{e} \cdot \nabla \times \underline{e})] \qquad (99)$$

where we have used (48).

Finally, we need to compute

$$<\delta\underline{R}/\delta t> =<\delta\underline{x}/\delta t - \delta(w\underline{\rho}/\Omega)/\delta t> =$$
$$= (u\underline{e} + \underline{u}_E) - <\delta(w\underline{\rho})/\delta t>/\Omega + <w^2\underline{\rho}\ \underline{\tau} \cdot \nabla\Omega)\Omega^2 \qquad (100)$$

To evaluate the second term in (100) we use the identity $\underline{\rho} = \underline{e} \times \underline{\tau}$ to write

$$<\delta(w\underline{\rho})/\delta\ > = <\delta(w\underline{e} \times \underline{\tau})/\delta t> = \underline{e} \times <\delta w\underline{\tau}/\delta t> + <\delta\underline{e}/\delta t) \times w\underline{\tau}> \qquad (101)$$

For the second term in (101) we have

$$<(\delta\underline{e}/\delta t) \times w\underline{\tau}>_i = <w^2\ \underline{\tau} \cdot \nabla\underline{e} \times \underline{\tau}>_i = w^2 <\tau_j\ (\partial e_k/\partial x_j)\ \tau_\ell\ \varepsilon_{k\ell i} =$$
$$= (w^2/2)\ (\nabla \times \underline{e} + \underline{e} \cdot \nabla\ \underline{e} \times \underline{e}) = (-w^2/2)\ [\nabla \times \underline{e} - (\underline{e} \times \nabla \times \underline{e}) \times \underline{e}]$$
$$= -(w^2/2)\ \underline{e}\ \underline{e} \cdot \nabla \times \underline{e}$$

To evaluate the first term in (101) we use the fact that $\delta\underline{v}/\delta t = 0$ so

$$\underline{e} \times <\delta w\underline{\tau}/\delta t> = -\underline{e} \times <\delta(u\underline{e} + \underline{u}_E)/\delta t> = -u\underline{e} \times <\delta\underline{e}/\delta t> - \underline{e} \times <\delta\underline{u}_E/\delta t> =$$
$$= -\underline{e} \times D(u\underline{e} + \underline{u}_E)/Dt$$

Thus,

$$<\delta(w\underline{\rho})/\delta t = -(w^2/2)\ \underline{e}\ \underline{e} \cdot \nabla \times \underline{e} - \underline{e} \times D(u\underline{e} + \underline{u}_E)/Dt \qquad (103)$$

Finally, we have

$$<\underline{\rho}\ \underline{\tau} \cdot \nabla\Omega> = <\underline{e} \times \underline{\tau}\ \underline{\tau} \cdot \nabla\Omega> = \underline{e} \times (\underset{\sim}{1} - \underline{e}\ \underline{e}) \cdot \nabla\Omega/2 = \underline{e} \times \nabla\Omega/2 \qquad (104)$$

and substituting this and (103) into (100) gives

$$\delta\underline{R}/\delta t = u\underline{e} + \underline{u}_E + (w^2/2\Omega^2)\ \underline{e} \times \nabla\Omega + (w^2/2\Omega)\underline{e} \cdot \nabla \times \underline{e} + $$
$$+ \underline{e} \times [D(u\underline{e} + \underline{u}_E)/Dt]/\Omega \qquad (105)$$

In summary, to lowest order in an expansion in Ω^{-1} the distribution function satisfying the Vlasov equation has the form

$$f\ (\underline{x}, \underline{v}, t) = F^{(0)}(\underline{R}, u, w, t) \qquad (106)$$

with

$$\left.\begin{array}{lll} & u \equiv \underline{e} \cdot \underline{v} \qquad\quad \underline{e} \equiv \underline{B}/B & \\ w \equiv |\underline{w}| & \underline{w} \equiv \underline{v} - u\underline{e} - \underline{u}_E & \underline{u}_E = c\underline{E} \times \underline{B}/B \\ & \underline{R} \equiv \underline{x} - \underline{\rho}\ w/\Omega \qquad \underline{\rho} \equiv \underline{e} \times \underline{\tau} & \end{array}\right\} \qquad (107)$$

where $F^{(0)}$ satisfies the <u>drift kinetic equation</u>

$$\partial F^{(0)}/\partial t + \langle\delta\underline{R}/\delta t\rangle \cdot \partial F^{(0)}/\partial\underline{R} + [qE_{\parallel}/m + \langle\delta u/\delta t\rangle]\ \partial F^{(0)}/\partial w +$$
$$+ \langle\delta w/\delta t\rangle\ \partial F^{(0)}/\partial w = 0 \qquad (108)$$

with

$$\left.\begin{array}{l} \langle\delta\underline{R}/\delta t\rangle = (u + u_1)\underline{e} + \underline{u}_E + \underline{u}_{\nabla B} + \underline{u}_F \\ \underline{u}_{\nabla B} = (w^2/2\Omega)\ \underline{B} \times \nabla B/B^2 \\ \underline{u}_F = -[D(u\underline{e} + \underline{u}_E)/Dt] \times \underline{B}/\Omega \\ u_1 = (w^2/2\Omega)\underline{e} \cdot \nabla \times \underline{e} \\ \langle\delta u/\delta t\rangle \equiv \underline{u}_E \cdot D\underline{e}/Dt - (\mu/m)\partial B/\partial s \\ \langle\delta w/\delta t\rangle \equiv (w/2B)\ [DB/Dt + cE_{\parallel}\ \underline{e} \cdot \nabla \times \underline{e}] \end{array}\right\} \qquad (109)$$

Although $F^{(0)}$ is independent of the velocity azimuth angle ϕ,

$$f(\underline{x}, \underline{v}, t) = F\ (\underline{x} - w\underline{\rho}/\Omega, u, w, t)$$

does depend on ϕ, albiet the dependence will be weak: by assumption $F^{(0)}$ has a scale length $L >> r_c = w/\Omega$, so that

$$f(\underline{x}, \underline{v}, t) \doteq F^{(0)} (\underline{x}, u, w, t) - (\underline{w\rho}/\Omega) \cdot \nabla F^{(0)}(\underline{x}, u, w, t) \quad (110)$$

The ϕ dependence of f is contained in the second term and since the second term is of order η compared to the first, it is comparable to the next term $F^{(1)}$ in the Ω^{-1} expansion. If we neglect $F^{(1)}$, i.e., work only to lowest order in Ω^{-1}, we should also drop the $\nabla F^{(0)}$ term in (110).

2. The Chew-Goldberger-Low Equation

Although we could, in principle, continue the expansion of F to higher orders in Ω^{-1}, it becomes quite onerous to go beyond $F^{(0)}$. If, however, we are content with a fluid description, i.e., one valid for low frequencies and long wavelengths, it is possible to obtain tractable equations correct to first order in Ω^{-1} for the velocity-averaged quantities. As we shall see, it is not necessary to actually solve for $F^{(1)}$; we need only the first order charge and current density, and it is possible to eliminate them by using Maxwell's equations.

The change of variables (81) is not necessary in this case. We simply expand $f(\underline{x}, \underline{v}, t)$ in Ω^{-1}:

$$f = f^{(0)} + f^{(1)} + \cdots$$

To zeroth order we have, as before,

$$\partial f^{(0)}/\partial\phi = 0$$

and to first order

$$\left.\begin{array}{l} \partial f^{(0)}/\partial t + \underline{v} \cdot \nabla f^{(0)} = -(q/mc)\ (\partial/\partial \underline{v}) \cdot \underline{V} \times \underline{B} f^{(1)} \\ \text{where} \quad \underline{V} = \underline{v} - \underline{u}_E; \qquad \underline{u}_E = c\underline{E} \times B/B^2 \end{array}\right\} \tag{111}$$

and we have, for simplicity, assumed $E_{\parallel} = 0$. As before, ϕ denotes the azimuth angle of $\underline{V}_{\perp} = w\underline{\tau}$. Taking the first two velocity moments (i.e., $\int d\underline{v}$ and $\int d\underline{v}\ \underline{v}$) of (111) we obtain first the continuity equation

$$\partial n/\partial t + \nabla \cdot n\underline{u} = 0 \tag{112}$$

where

$$(n,\ n\ \underline{u}) \equiv \int d\underline{v}\ (1,\underline{v})\ \overline{n}\ f^{(0)} \tag{113}$$

and then a momentum equation

$$\partial(\rho\ \underline{u})/\partial t + \nabla \cdot (\underset{\sim}{p} + \rho\ \underline{u}\ \underline{u}) = (q/c)\ \overline{n} \int d\underline{v}\ \underline{V} \times \underline{B} f^{(1)} \tag{114}$$

The average density of both species is denoted by $\overline{n}$, as in Chap. III, and

$$\underset{\sim}{p} = \int d\underline{v}\ \overline{n}\ m\ (\underline{v} - \underline{u})\ (\underline{v} - \underline{u})\ f^{(1)} \tag{115}$$

$$\rho = nm$$

N.B. Do not confuse $\underline{u}(\underline{x},t)$, defined by (113) with the parallel component $u = \underline{v} \cdot \underline{e}\ (\underline{x},t)$ used in the discussion of the DKE. The confusion could be avoided by using the symbols $n^{(0)}$, $\underline{u}^{(0)}$ for the quantities defined by (113), since the averaging is done with the lowest order distribution function $f^{(0)}$, but this would lead to unwarranted notational clutter.

As usual, we can use the continuity equation to obtain an alternative form of the momentum equation

$$\rho d\underline{u}/dt + \nabla \cdot \underset{\sim}{p} = (q/c)\ \overline{n} \int d\underline{v}\ w\underline{\tau} \times \underline{B} f^{(1)} \tag{116}$$

with d/dt defined as the usual fluid dynamic convective derivative

$$d/dt = \partial/\partial t + \underline{u} \cdot \nabla$$

(not to be confused with the d/dt used in the single particle analysis of section A).

The perpendicular component of $\underline{u}$ is (for both species) simply $\underline{u}_E$ since

$$n\,\underline{u}_{\perp} = \int d\underline{v}\,\overline{n}\,(\underline{V}_{\perp} + \underline{u}_E)f^{(0)} = n\underline{u}_E \tag{117}$$

The pressure tensor $\underset{\sim}{p}$ is also simplified by the fact that $f^{(0)}$ is independent of ϕ:

$$\begin{aligned}\underset{\sim}{p} &= m\int d\underline{v}\,\overline{n}\,[(v_{\parallel} - u_{\parallel})\underline{e} + w\underline{\tau}][(v_{\parallel} - u_{\parallel})\underline{e} + w\underline{\tau}]f^{(0)} = \\ &= p_{\parallel}\,\underline{e}\,\underline{e} + p_{\perp}(\underset{\sim}{1} - \underline{e}\,\underline{e})\end{aligned} \tag{118}$$

with

$$\begin{aligned}p_{\parallel} &= \overline{n}m \int d\underline{v}\,(v_{\parallel} - u_{\parallel})^2 f^{(0)} \\ p_{\perp} &= \overline{n}m \int dv\, w^2 f^{(0)}/2\end{aligned} \tag{119}$$

Although we have treated the two species alike so far, it is consistent to treat electrons to lowest order and ions to first order. Since $\omega_{ci}/\omega_{ce} = m/M$ and $r_{ce}/r_{ci} = (m/M)\,(T_e/T_i)^{1/2}$, effects of finite ω/Ω or r_c/L can be important for ions even though they are negligible for electrons. Thus, we set

$$f_i = f^{(0)} + f^{(1)}$$

and simply use f_e to denote the electron distribution function to lowest order. The electron equations are then

$$\begin{aligned}&\partial\rho_e/\partial t + \nabla\cdot\rho_e\,\underline{u}_e = 0\\ &\rho_e\,d\underline{u}_e/dt + \nabla\cdot\underset{\sim}{p}_e = 0\end{aligned}$$

with $\underset{\sim}{p}_e$ still to be determined. The ion momentum equation, however involves not only $\underset{\sim}{\rho}_i$ but also the term

$$\underline{P} \equiv (e/c)\,\bar{n} \int d\underline{v}\; w\underline{\tau} \times \underline{B} f^{(1)}$$

on the right side of (116), in which the unknown function $f^{(1)}$ appears. As we now show, this term can be expressed entirely in terms of $\underline{u}$ and $\underline{B}$ by making use of Maxwell's equations.

The current density

$$\underline{j} = \bar{n}e \int dv\; \underline{v}\, (f^{(0)} + f^{(1)} - f_e) \tag{120}$$

can be split into two parts

$$\underline{j} = \underline{j}_o + \underline{j}_1 \tag{121}$$

where

$$\underline{j}_o \equiv \bar{n}e \int d\underline{v}\; \underline{u}_i\, (f^{(0)} + f^{(1)} - f_e) = \underline{u}_i\, \nabla \cdot \underline{E}/4\pi \tag{122}$$

is a convective current and

$$\begin{aligned} \underline{j}_1 \equiv e\bar{n} \int d\underline{v}(\underline{v} - \underline{u}_i)(f^{(0)} + f^{(1)} - f_e) = \\ = e\bar{n} \int d\underline{v}(\underline{v} - \underline{u}_i)(f^{(1)} - f_e) \end{aligned} \tag{123}$$

the last form being a consequence of the definition (113) of $\underline{u}_i$. Then

$$\begin{aligned} \underline{j}_1 \times \underline{B} = e\bar{n} \int d\underline{v}\; (f^{(1)} - f_e)\; (\underline{u}_E + w\underline{\tau} - \underline{u}_i) \times \underline{B} \\ - e\bar{n} \int d\underline{v}\; f^{(1)}\; w\underline{\tau} \times \underline{B} = c\underline{P} \end{aligned} \tag{124}$$

since $(\underline{u}_E - \underline{u}_i) \times \underline{B} = 0$, as we see from (117), and f_e is independent of ϕ. We note that (124) expresses $\underline{P}$ in terms of $\underline{j}_1$ and that, in turn,

$$\begin{aligned} \underline{j}_1 \times \underline{B} = (\underline{j} - \underline{j}_o) \times \underline{B} = (c/4\pi)(\nabla \times \underline{B} - c^{-1}\partial\, \underline{E}/\partial t) \times B - \\ - \nabla \cdot \underline{E}\, \underline{u}_i/4\pi) \times \underline{B} \end{aligned} \tag{125}$$

can be expressed entirely in terms of $\underline{u}_i$ if we use (111) and (117) to write

$$\underline{E} = -\underline{u}_E \times \underline{B}/c = -\underline{u}_i \times \underline{B}/c \tag{126}$$

Thus, we obtain a set of equations for the time evolution of ρ_i, $\underline{u}_i$, and $\underline{B}$, (using (126) and Faraday's Law, for the latter):

$$\partial\rho_i/\partial t + \nabla \cdot \rho_i \underline{u}_i = 0 \tag{127}$$

$$\rho_i d\underline{u}_i/dt + \nabla \cdot \underset{\sim}{p}_i = \underline{P} = (4\pi)^{-1} [\nabla \times \underline{B} + \partial(\underline{u}_i \times \underline{B})/\partial t + \underline{u}_i \nabla \cdot (\underline{u}_i \times \underline{B})] \times \underline{B} \tag{128}$$

$$\partial\underline{B}/\partial t = \nabla \times (\underline{u}_i \times \underline{B}) \tag{129}$$

with $\underset{\sim}{p}_i$ still to be determined.

To determine $\underset{\sim}{p}_e$ and $\underset{\sim}{p}_i$ we take the second velocity moments of (111),

$$\begin{aligned} &(\partial/\partial t) \int d\underline{v}\, \overline{n}\, \underline{m}\, \underline{v}\, \underline{v}\, f^{(0)} + \nabla \cdot \int dv\, \overline{n}\, \underline{m}\, \underline{v}\, \underline{v}\, \underline{v}\, f^{(0)} = \\ &= -(e/c) \int d\underline{v}\, \overline{n}\, \underline{v}\, \underline{v}\, (\partial/\partial \underline{v}) \times (\underline{V} \times \underline{B})\, f^{(1)} = \\ &= (e/c) \int d\underline{v}\, \overline{n}\, (\underline{v}\, \underline{V} \times \underline{B} + \underline{V} \times \underline{B}\, \underline{v})\, f^{(1)} \equiv \underset{\sim}{R} \end{aligned} \tag{130}$$

Again (in the case of ions), the unknown function $f^{(1)}$ appears, but we shall find that, like the $f^{(1)}$ term in the momentum equation, it can be eliminated. For the first term in (130) we have

$$\int d\underline{v}\, \overline{n}\, m\, \underline{v}\, \underline{v}\, f^{(0)} = \rho\, \underline{u}\, \underline{u} + \underset{\sim}{p} \tag{131}$$

For the second term we have (using index notation since we must deal with a tensor of third rank)

$$\int d\, \underline{v}\, \overline{n}m\, v_i\, v_j\, v_k\, f^{(0)} = \rho\, u_i\, u_j\, u_k + (u_i\, p_{jk} + u_j\, p_{ki} + u_k\, p_{ij}) + q_{ijk} \tag{132}$$

where

$$q_{ijk} \equiv \int d\underline{v}\, \tilde{n}\, m\, (v-u)_i\, (v-u)_j\, (v-u)_k \tag{133}$$

is the heat flow tensor. Using (131) and (132) we can write (130) as

$$(\partial/\partial t)\rho\underline{u}\,\underline{u} + \nabla \cdot (\rho\, \underline{u}\,\underline{u}\,\underline{u}) + \partial\underset{\sim}{p}/\partial t + \nabla \cdot (\underline{u}\,\underset{\sim}{p} + \underset{\sim}{p}\,\underline{u}) + \underline{u}\,\nabla \cdot \underset{\sim}{p} + {} $$
$$+ \widetilde{\nabla\underline{u}} \cdot p = \underset{\sim}{R} - \nabla \cdot \underset{\sim}{q} \tag{134}$$

(Note that $\underset{\sim}{p}$ is a symmetric tensor, $\tilde{\underset{\sim}{p}} = \underset{\sim}{p}$.) Using the continuity equation (112) and the momentum equation (114) we can simplify the first two terms in (134).

$$(\partial/\partial t)\,(\rho\, \underline{u}\,\underline{u}) + \nabla \cdot (\rho\underline{u}\,\underline{u}\,\underline{u}) = \rho(\partial/\partial t)\,\underline{u}\,\underline{u} + \rho\, \underline{u} \cdot \nabla\, \underline{u}\,\underline{u} =$$
$$= (\rho\, d\underline{u}/dt)\,\underline{u} + \underline{u}\,\rho\, d\underline{u}/dt = (\underline{P} - \nabla \cdot \underset{\sim}{p})\underline{u} + \underline{u}\,(\underline{P} - \nabla \cdot \underset{\sim}{p}) \tag{135}$$

Then (134) gives

$$d\underset{\sim}{p}/dt + \nabla \cdot \underline{u}\,\underset{\sim}{p} + \underset{\sim}{p} \cdot \nabla\,\underline{u} + \widetilde{\nabla u} \cdot \underset{\sim}{p} = (\underset{\sim}{R} - \nabla \cdot \underset{\sim}{q} - \underline{u}\,\underline{P} - \underline{P}\,\underline{u}) \equiv$$
$$\equiv \underset{\sim}{S} - \nabla \cdot \underset{\sim}{q} \tag{136}$$

In (112), (114) and (136) we have a set of fluid equations (for each species) giving the evolution of ρ, $\underline{u}$ and $\underset{\sim}{p}$. The equations fail to close for two reasons: the occurance of higher order moments, namely $\underset{\sim}{q}$ in (136), and, in the case of ions, the appearance of $f^{(1)}$ in $\underline{P}$ and $\underset{\sim}{R}$. (There are no terms analogous to $\underline{P}$ and $\underset{\sim}{R}$ in the electron equations since we have dropped $f_e^{(1)}$.) As we have seen, $\underline{P}$ can be eliminated and we shall presently see that the same is true of $\underset{\sim}{R}$. Moreover, $\underset{\sim}{q}$ can be simplified, just as $\underline{P}$ was, as a consequence of the azimuthal symmetry of $f^{(0)}$. Since

$$\underline{v} - \underline{u} = \underline{v} - \underline{u}_E - u\underline{e} = w\underline{\tau} + (v_{\shortparallel} - u)\underline{e} \tag{137}$$

we have

$$q_{ijk} = \int d\underline{v}\, \bar{n}\, m\, f^{(0)}\, (v-u)_i\, (v-u)_j\, (v-u)_k =$$

$$= \int d\underline{v}\, \bar{n}\, m\, f^{(0)}\, [w^2\, (\tau_i \tau_j\, e_k + \tau_k\, \tau_i\, e_j + \tau_j\, \tau_k\, e_i) + (v_{\shortparallel}-u)^3\, e_i\, e_j\, e_k].$$

Thus

$$\underset{\sim}{q} = q_{\shortparallel}\, e_i\, e_j\, e_k + q_{\perp}\, (\delta_{ij}\, e_k + \delta_{jk}\, e_i + \delta_{ki}\, e_j) \tag{138}$$

with

$$\begin{aligned} q_{\shortparallel} &= \int d\underline{v}\, \bar{n}\, m\, f^{(0)'}\, (v_{\shortparallel} - u)^3 \\ q_{\perp} &= \int d\underline{v}\, \bar{n}\, m\, f^{(0)}\, w^2\, (v_{\shortparallel} - u)/2 \end{aligned} \tag{139}$$

Taking the third velocity moments of (111) would give equations for $q_{\shortparallel}$ and $q_{\perp}$, but also introduce new moments. The essence of what is called the "double adiabatic" or Chew-Goldberger-Law approximation is to drop $\underset{\sim}{q}$, thus closing the equations.

Even if we neglect $\underset{\sim}{q}$, we must still eliminate $\underset{\sim}{S}$ from (136). This occurs in a natural way when we operate on (136) by contracting it with $\underline{ee}$ and $\underset{\sim}{1}$ to obtain equations for the evolution of $p_{\shortparallel}$ and $p_{\perp}$. Since

$$\underset{\sim}{p} = (p_{\shortparallel} - p_{\perp})\, \underline{ee} + p_{\perp}\, \underset{\sim}{1}$$

and $\underline{e}$ is a unit vector, $\underline{e} \cdot (d\underline{e}/dt) = 0$, we have

$$\underline{e} \cdot (d\underset{\sim}{p}/dt) \cdot \underline{e} = dp_{\parallel}/dt \tag{140}$$

$$\mathrm{Tr}\ (d\underset{\sim}{p}/dt) = d\ (p_{\parallel} + 2p_{\perp})\ dt \tag{141}$$

It is interesting that $\underset{\sim}{S}$ can be expressed in terms of the first order correction to the pressure tensor $\underset{\sim}{p}^{(1)}$. We have

$$\underset{\sim}{S} = (\underset{\sim}{R} - \underline{Pu} - \underline{uP}) = (e/c \int d\underline{v}\ \bar{n}\ f^{(1)}\ [(\underline{v}-\underline{u})\ \underline{V} \times \underline{B} + \underline{V} \times \underline{B}\ (\underline{v}-\underline{u})]$$

Since it follows from (117) that

$$\underline{V} \times \underline{B} = (\underline{v}-\underline{u}_E) \times \underline{B} = (\underline{v}-\underline{u}) \times \underline{B}$$

we can write

$$\underset{\sim}{S} = (e/c) \int d\underline{v}\ \bar{n}\ f^{(1)}\ [(\underline{v}-\underline{u})\ (\underline{v}-\underline{u}) \times \underline{B} - \underline{B} \times (\underline{v}-\underline{u})\ (\underline{v}-\underline{u})] \tag{142}$$

Thus

$$\underset{\sim}{S} = \Omega\ (\underset{\sim}{p}^{(1)} \times \underline{e} - \underline{e} \times \underset{\sim}{p}^{(1)}) \tag{143}$$

where $\underset{\sim}{p}^{(1)}$ is the correction to the pressure tensor arising from $f^{(1)}$. However, for our purposes (142) is sufficient since from that equation it follows that

$$\underline{e} \cdot \underset{\sim}{S} \cdot \underline{e} = \mathrm{Tr}\ \underset{\sim}{S} = 0 \tag{144}$$

Neglecting $\underline{q}$ and taking the trace and $\underline{ee}$ projection of (136) we obtain

$$dp_{\parallel}/dt + \nabla \cdot \underline{u}\ p_{\parallel} + 2p_{\parallel}\ \underline{ee} : \underline{u} = 0 \tag{145}$$

and

$$(d/dt)\ (p_{\parallel} + 2p_{\perp}) + \nabla \cdot \underline{u}\ (p_{\parallel} + 2p_{\perp}) +$$
$$2\ (p_{\parallel} - p_{\perp})\ \underline{ee}: \nabla u + 2p_{\perp}\ \nabla \cdot \underline{u} = 0 \tag{146}$$

Subtracting (146) from (145) we have

$$dp_{\perp}/dt + 2p_{\perp}\ \nabla \cdot \underline{u} - p_{\perp}\ \underline{ee}: \nabla u = 0 \tag{147}$$

The equations (145) and (147) hold for each species and, together with the continuity and momentum equations, give a closed set of equations for ρ, $\underline{u}$ and $\underset{\sim}{p}$. They can be put in more convenient form by expressing $\nabla \cdot \underline{u}$ and $\underline{ee}:\nabla\underline{u}$ in terms of B and ρ, as follows. From the continuity equation we have

$$\nabla \cdot \underline{u} = -\rho^{-1}\ (d\rho/dt) = -d\ (\ln\rho)/dt \tag{148}$$

From Faraday's law together with (117) we obtain

$$\partial\underline{B}/\partial t = \nabla \times (\underline{u} \times \underline{B}) = \underline{B} \cdot \nabla\underline{u} - \underline{B}\ \nabla \cdot \underline{u} - \underline{u} \cdot \nabla B$$

which gives

$$dB/dt = \underline{e} \cdot d\underline{B}/dt = B(\underline{ee}:\nabla\underline{u} - \nabla \cdot \underline{u})$$
$$= B(\underline{ee} \cdot \nabla\underline{u} + d\ \ln\rho/dt)$$

or finally

$$\underline{ee}:\nabla\underline{u} = (d/dt)\ \ln\ (B/\rho) \tag{149}$$

Dividing (145) by $p_{\parallel}$, (147) by $p_{\perp}$, and using (148) and (149) we find

$$(d/dt)\ (p_{\parallel}\ B^2/\rho^3) = 0 \tag{150}$$

and

$$(d/dt)\ (p_\perp/\rho B) = 0 \tag{151}$$

In other words, along a fluid trajectory we have, for each species, two constants, $p_\parallel B^2/\rho^3$ and $p_\perp/\rho B$. This is to be compared with the result for a neutral monatomic gas, where the constancy of $p\rho^{-\gamma}$, with $\gamma = 5/3$, is the familiar adiabatic gas law. Since we have here two constants (for each species) the CGL formulation is, as we have remarked, often called the double adiabatic approximation.

We note that the constancy of $p_\perp/\rho B$ is essentially equivalent to the constancy of the magnetic moment μ for each particle, since $p_\perp/\rho$ is the average perpendicular energy at a point for a given species and hence equal to $\overline{\mu}B$. The other adiabatic law (150) could then be derived from this and the relation

$$(d/dt)\ (p_\parallel\, p_\perp^{\ 2}/\rho^5) = 0 \tag{152}$$

which is a generalization of the familiar adiabatic law to the case of a gas with uncoupled parallel and perpendicular pressure components.

If the heat flow tensor is retained, then it is a matter of straightforward algebra, using (138), to show that in place of (150) and (151) we obtain

$$(d/dt)\ \ln\ (p_\parallel B^2/\rho^3) = -\{\nabla\cdot(q_\parallel + q_\perp)\ \underline{e}] + 2\underline{e}\cdot\nabla\, q_\perp\}/p_\parallel \tag{153}$$

$$(d/dt)\ (p_\perp/\rho B) = -\ [\nabla\cdot(q_\perp\ \underline{e}) + q_\perp\ \nabla\cdot e]/p_\perp \tag{154}$$

The relation of the DKE and CGL equations to the Vlasov equation and the single particle g.c. theory are summarized in Fig. 8. The exact orbit equations are the characteristics of the Vlasov equation

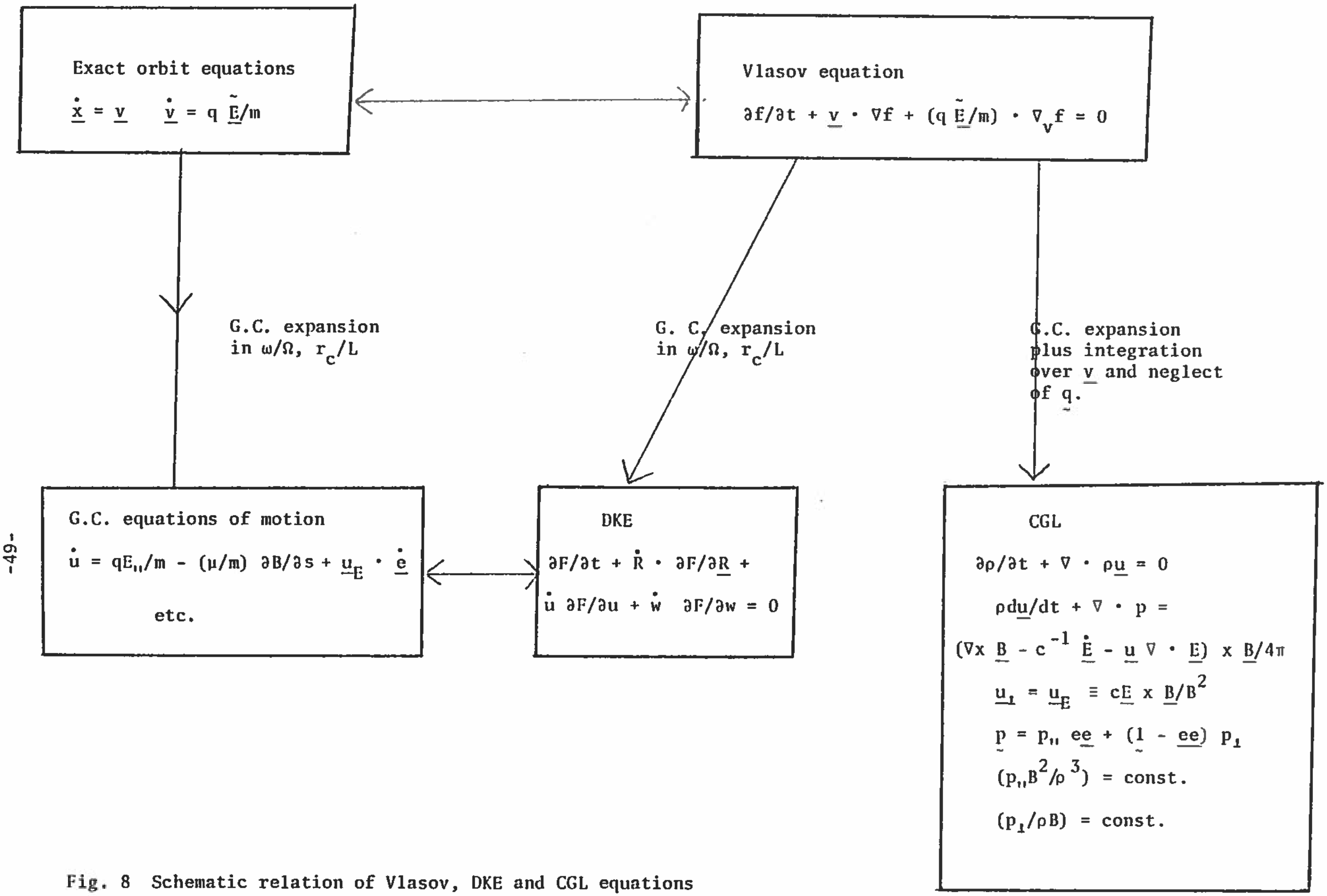

Fig. 8 Schematic relation of Vlasov, DKE and CGL equations

and, conversely, the Vlasov equation states that f is constant along an exact particle trajectory in phase space. Precisely the same relations exist between the g.c. equations of motion and the DKE, these being obtained from the exact equations of motion and from the Vlasov equation, respectively via an expansion in ω/Ω and r_c/L. The CGL equations are a fluid dynamic formulation obtained from the Vlasov equation by this same expansion plus an integration over velocity to yield moment equations.