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Radiation from Plasmas

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RADIATION FROM PLASMAS*

by

John M. Dawson

I. Introduction

This series of five lectures is devoted to radiation from plasmas. In such a short time we cannot begin to cover this subject in detail and hence, we will confine our attention to only a few processes which give rise to radiation. The processes we will consider are those which give rise to bremsstrahlung (charged particle encounters) and to cyclotron and synchrotron radiation. We shall start with a short introduction on black body radiation, its density inside a plasma, and absorption and emission coefficients in terms of collision times.

II. Black Body Radiation

The radiation from a black body is given by the well-known Planck radiation law,

$$\varepsilon(\nu)\, d\nu\, d\Omega\, dA = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}\, d\Omega\, d\nu\, dA \quad , \tag{1}$$

where $\varepsilon(\nu)$ is the energy radiated per unit area, per unit solid angle per unit frequency per second, ν is the frequency, h is Planck's constant, c is the velocity of light, $d\Omega$ is the element of solid angle into which the radiation is emitted, and dA is the element of surface are which is radiating. For frequencies low enough so that $h\nu << kT$, Eq. (1), reduces to the well-known Rayleigh-Jeans law,

*See References

$$\varepsilon(\nu)\, d\nu\, d\Omega\, dA = \frac{2\nu^2}{c^2}\, kT\, d\Omega\, dA\, d\nu \quad . \tag{2}$$

If Eq. (1) is integrated over all solid angles and over all frequencies, one obtains the Stefan Boltzmann law for the total energy emitted per second per unit area:

$$\varepsilon_T\, dA = \sigma T^4\, dA \tag{3}$$

$$\sigma = 4.186 \times 1.355 \times 10^{-12} \text{ joules/cm}^2\text{sec}(^\circ\text{K})^4 \quad .$$

Now most laboratory plasmas have electron temperatures which range from 2500° K (kT = 0.25 ev) for a thermally ionized Cs plasma to perhaps 10^8 $^\circ$K (kT = 10^4 ev) for the plasma in a thermonuclear reactor. If these plasmas were radiating like black bodies they would radiate very large amounts of energy. The thermonuclear plasma would radiate 10^{20} watts/cm^2. The reaction rate for any conceivable device cannot hope to supply this enormous power and even if it could, the container would be instantly vaporized. Even for the Cs plasma the radiation is of the order of 100 watts/cm^2 and even a modest plasma 3 cm in radius would radiate 10^4 watts. Fortunately for us, hot plasmas do not radiate like black bodies unless they are very thick and very dense. The following simple estimate of the bremsstrahlung from a hot plasma illustrates this.

III. Estimate of the Total Bremsstrahlung Emitted by a Fully Ionized Plasma

For a plasma which is hot, fully ionized, and free from magnetic fields the principle source of radiation is bremsstrahlung. Such radiation arises from the acceleration of the free electrons by the ions. We may estimate the total bremsstrahlung emitted by a plasma by the following physical considerations.

Consider an ion of charge z imbedded in a plasma with electron density n_e. The acceleration of an electron by this ion is given by

$$a = \frac{z e^2}{m_e r^2} , \tag{4}$$

where r is the distance between the ion and the electron.

Classically an accelerated electron (acceleration a) radiates energy at the rate

$$P = \frac{2e^2 a^2}{3c^3} . \tag{5}$$

Thus by (4) the power being radiated by the electron under consideration is

$$P = \frac{2z^2 e^6}{3m_e^2 r^4} . \qquad (P \sim |E^2|) \tag{6}$$

Multiplying (6) by the density of electrons and integrating over r gives

$$P = \frac{2z^2 e^6}{3c^3 m_e^2} n_e \int_{r_{min}}^{\infty} \frac{4\pi r^2 dr}{r^4} = \frac{8\pi}{3} \frac{z^2 e^6}{c^3 m_e^2} \frac{n_e}{r_{min}} . \tag{7}$$

Here r_{min} is the distance at which our classical approximation breaks down. In principle, we cannot treat the radiation as coming from a point electron since by the uncertainty * we cannot localize the electron to an arbitrarily small volume. We may expect that for collision distances smaller than the reciprocal De Broglie wave number,

$$\frac{\hbar}{P} = \frac{1}{k} \cong r_{min} , \tag{8}$$

this will be an important effect and that we should use (8) for r_{min}. If we use r_{min} as given by (8) with $\sqrt{m_e kT}$ for P, then (7) becomes

$$P = \frac{16\pi^2 z^2 e^6 n_e}{3c^3 m_e h} \sqrt{\frac{kT_e}{m_e}} . \tag{9}$$

The total radiation due to all ions in a unit volume is

$$P_T = \frac{16\pi^2 z^2 e^6 n_e n_i}{3c^3 m_e h} \sqrt{\frac{kT_e}{m_e}} , \tag{10}$$

or if there is more than one species of ion,

$$P_T = \frac{(4\pi)^2 e^6}{3c^3 m_e h} \sqrt{\frac{kT_e}{m_e}}\, n_e \sum_i z_i^2 n_i . \tag{11}$$

It is of interest to compare Eq. (11) with the more exact result obtained from a quantum treatment and given by Spitzer.[1]

$$P_T = \left(\frac{2\pi kT}{3m_e}\right)^{1/2} \frac{32\pi e^6}{3c^3 m_e h} z^2 n_e n_i \overline{g_{ff}} . \tag{12}$$

Here g_{ff} is a pure number and is called the Gaunt factor. The value one obtains for it depends on the approximation one makes in the quantumechanical theory. Greene[2] gives a discussion of this and gives exact values of it for a wide range of densities and temperatures. For the Born approximation $\overline{g_{ff}}$ is equal to 1.103.

Equations (10) and (12) would agree if $\overline{g_{ff}}$ were equal to 1.08.

From Eq. (10) we can estimate how large the plasma would have to be before it would radiate like a black body. If we had a sphere of plasma of radius R, then according to Eq. (10) it would radiate the amount of power w,

$$w = \frac{4}{3}\pi R^3 P_T = \frac{64}{3} \frac{\pi^3 z^2 e^6 n_e n_i R^3}{c^3 m_e h} \sqrt{\frac{kT_e}{m_e}}$$

$$= 5.95 \times 10^{-27} z^2 n_i n_e T^{1/2} R^3 \text{ ergs/sec} \quad , \qquad (13)$$

where T is in degrees K. Equation (13), of course, neglects reabsorption by the plasma. The larger the plasma the more important is the reabsorption and ultimately for large plasmas a balance is reached where the emission and reabsorption maintain the radiation at the black body level inside the plasma. Such a large plasma would give off black body radiation from its surface. We can obtain an estimate of how large the plasma must be before this happens by equating w to the black body radiation from its surface. From Eq. (3) this latter is given by

$$w_{BB} = 7.12 \times 10^{-4} R^2 T^4 \text{ ergs/sec} \quad . \qquad (14)$$

Equating (13) and (14) we find for R

$$\boxed{R = \frac{1.2 \times 10^{23} \times T^{7/2}}{z^2 n_i n_e} \text{ cm}} \quad . \qquad (15)$$

For a Cs plasma at 3000° K with an ion and electron density of $10^{12}/\text{cm}^3$ and z equal to 1, Eq. (15) gives an R of 1.7×10^{11} cm. For a thermonuclear plasma at a temperature of 10^8 °K and densities of 10^{16} with z =1, R comes out to be 10^{19} cm. Even inside a nuclear explosion where the densities might be as high as 10^{23} and $z \sim 100$ with a T of 10^8, R comes out to be 10 cm. Thus we see that most plasmas of interest to us will not radiate like black bodies, but in reality will emit much less strongly.

While the above argument shows that most laboratory plasmas will be optically thin to the black body radiation as a whole it tells us nothing about the absorption and emission coefficient as functions of frequency. The plasma may be optically thick for some frequencies and optically thin for others.

If we know the absorption coefficient for radiation of frequency ν by a plasma at temperature T then we can obtain the plasma emissivity by requiring that the absorption of black body radiation for frequency ν and temperature T must be made up by the plasma emission.

The absorption coefficient can be obtained from the plasma conductivity or resistivity. While the resistivity is generally a function of frequency[3] we can generally obtain some pretty good approximations to it using very simple models for the collisional processes.

Summing up, we may obtain the emissivity if we know the black body radiation level within the plasma and its rate of absorption. The dielectric properties of the plasma are involved in the first of these while the plasma resistivity enters the second.

IV. Black Body Radiation Inside a Plasma

Let us first look at the black body radiation within a plasma at temperature T. We shall assume that the absorption per wavelength is small so that we can talk about radiation. To find the density of radiation we consider the situation shown in Fig. 1.

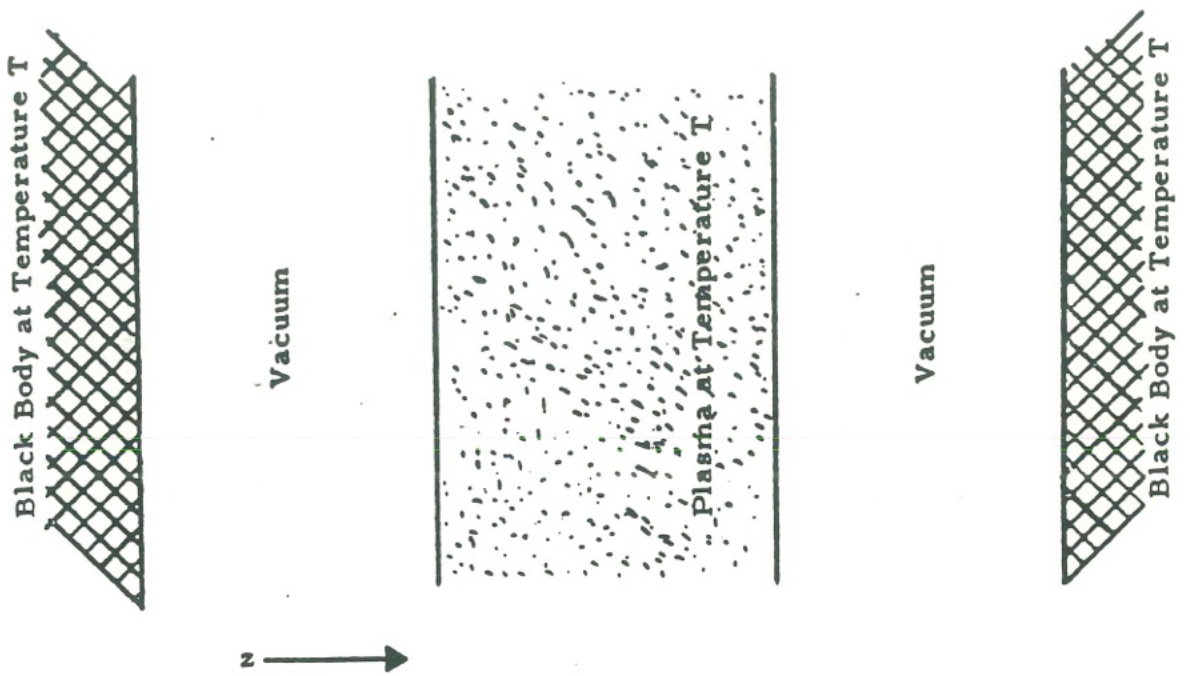

Fig. 1

On the left and right we have black bodies at temperature T. These bodies are slabs which are infinite in the x, y direction and they are normal to the z direction. In between these bodies we have a slab of plasma which is also infinite in the x, y direction and at temperature T. Between the plasma and the black bodies there are vacuum regions. Radiation is emitted by the black bodies and enters the plasma. We shall take the transition from the vacuum to the plasma to be sufficiently gradual so that none of the radiation which can enter the plasma will be reflected (radiation striking the plasma at more than the critical angle to the normal will be totally reflected no matter how slow the transition is). The method of confinement is not specified, but we can imagine that some external forces are applied to the particle in the boundary region which prevent their escape.

Now if the plasma does not absorb any of the radiation then the radiation emitted by one of the black bodies will pass through the plasma and out the other side and be absorbed at the other black body. The density of this radiation inside the plasma must be the equilibrium density for if the plasma absorbed more than it was emitting the black bodies would cool off while the plasma heated up, while if the plasma emitted more than it absorbed the plasma would cool off while the black bodies heated up. In either case the second law of thermodynamics would be violated. This argument applies frequency by frequency since we can impose filters between the black body and the plasma which let only the desired frequency through.

Now consider the radiation with frequency ν in $d\nu$ with direction of propagation lying within $d\theta$ of the normal to the plasma surface (propagating in $d\Omega = \pi d\theta^2$ about the normal) and which strikes a unit area of the surface in time τ.

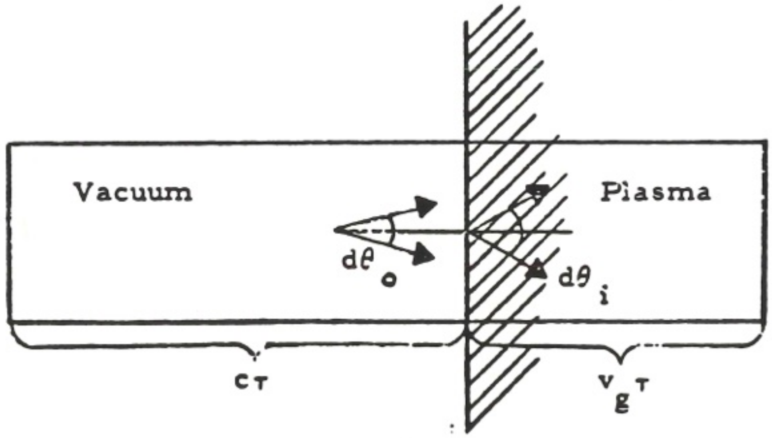

Fig. 2

After entering the plasma the direction of propagation for the radiation will lie within a cone making an angle $d\theta_i$ to the normal. We may obtain $d\theta_i$ by applying Snell's law for the refraction of the waves as they enter the plasma,

$$\frac{c}{v_p} = \frac{\sin\theta_o}{\sin\theta_i} \quad . \tag{16}$$

Here c is the velocity of light in vacuum and v_p is the phase velocity of the waves in the plasma. Now if θ_o is infinitesimal then θ_i will be also, and the sines may be equated to the angles and Eq. (16) may be written as

$$\frac{c}{v_p} = \frac{d\theta_o}{d\theta_i} \quad . \tag{17}$$

The solid angle in which the radiation propagates outside the plasma is equal to $\pi d\theta_o^2$ while inside it is $\pi d\theta_i^2$. Thus by Eq. (17) we have that these solid angles are related by

$$\frac{c^2}{v_p^2} = \frac{d\theta_o^2}{d\theta_i^2} = \frac{d\Omega_o}{d\Omega_i} \quad . \tag{18}$$

Now in the time τ all the radiation in the cylinder of length $c\tau$ enters the plasma. We assume here that $d\theta_o$ is so small that we can neglect the variations in cross-section of the cylinder with distance from the surface and also variations in the distance which the radiation must travel to reach the surface. Now inside the plasma the radiation travels a distance equal to the group velocity ($v_g = \frac{d\omega}{dk}$), times τ, $v_g\tau$, in the time τ. The radiation may be considered a wave pocket and such a packet propagates at the group velocity.

Thus the radiation which was in a volume $c\tau$ and with propagation direction lying in a solid angle $d\Omega_o$ after entering the plasma lies in a volume $v_g\tau$ with propagation direction lying in $d\Omega_i$. If the radiation density per unit frequency interval per unit volume per unit solid angle in the vacuum is $\epsilon_v(\nu)$ then inside the plasma it is given by

$$\epsilon_p(\nu)\, v_g\, d\Omega_i = \epsilon_v(\nu)\, c\, d\Omega_o \tag{19}$$

or by using Eq. (18),

$$\epsilon_p(\nu) = \frac{c^3}{v_p^2\, v_g}\, \epsilon_v(\nu) \quad .^* \tag{20}$$

The density of radiation in the vacuum may be obtained directly from the Planck law (1) by dividing by c, and is

$$\epsilon_v(\nu) = \frac{2h\nu^3}{c^3(e^{h\nu/kT} - 1)} \quad . \tag{21}$$

Substituting this expression into (20) gives

$$\epsilon_p(\nu) = \frac{2h\nu^3}{v_p^2\, v_g(e^{h\nu/kT} - 1)} \quad . \tag{22}$$

For frequencies such that $\frac{h\nu}{kT} << 1$ we expand the exponential in (22) and obtain

$$\epsilon_p(\nu) = \frac{2kT\nu^2}{v_p^2\, v_g} \quad . \tag{23}$$

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*If the phase velocity and group velocities are not in the same direction, Eq. (20) reads

$$\epsilon_p(\nu) = \frac{c^3}{v_p\, \underline{v}_p \cdot \underline{v}_g}\, \epsilon_v(\nu) \quad .$$

Equation (23) gives the energy density in both transverse polarizations. It has been assumed that the phase and group velocities are the same for both of them. If this is not true then we must treat them individually and for each one we get

$$\epsilon_p(\nu) = \frac{kT\,\nu^2}{v_p^2\, v_g} \tag{24}$$

where the phase and group velocities are appropriate for the polarization under consideration. The above arguments apply to any material and not simply to plasmas.

It is of interest to inquire what the equilibrium energy per mode is inside a material. To have the modes well defined we put the material inside a perfectly conducting box of volume V. The electric field must vanish at the walls and only modes which satisfy this criterion can exist, i.e., only modes with an integer number of half wavelengths in the box in each of the x, y, z directions are allowed. The number of modes within the box whose magnitude of k lies between k and k + dk is given by[4]

$$N(k)\,dk = \frac{V}{2\pi^2}\,k^2\,dk \tag{25}$$

or the number per unit volume is

$$n(k)\,dk = \frac{k^2\,dk}{2\pi^2} \quad . \tag{26}$$

The number propagating in the solid angle $d\Omega$ is

$$n(k)\,dk\,\frac{d\Omega}{4\pi} = k^2\,\frac{dk\,d\Omega}{8\pi^3} \quad . \tag{27}$$

If we convert from k to ν we get

$$k = k(\nu)$$

$$dk = \frac{dk}{d\nu} d\nu = \frac{2\pi}{v_g} d\nu$$

$$n(\nu)d\nu d\Omega = \frac{n(k(\nu))}{4\pi} \quad \frac{2\pi \, d\nu d\Omega}{v_g} = \frac{k^2(\nu)}{4\pi^2} \frac{d\nu d\Omega}{v_g} \quad . \tag{28}$$

Dividing the energy density given in (24) by $n(\nu)$ gives the energy per mode,

$$\frac{\epsilon(\nu)}{n(\nu)} = \frac{kT \, \nu^2 \, 4\pi^2 \cancel{v_g}}{v_p^2 \cancel{v_g} \, k^2(\nu)} = kT \tag{29}$$

$$v_p = 2\pi \, \nu/k \quad .$$

Thus each mode of the field has energy kT. Not all of this energy is in the electric and magnetic fields. Some of it is stored in the material, i.e., in polarizing the material and in mass motion of its constituents.

We now apply our results to a uniform plasma which contains no static electric or magnetic fields.

V. The Equilibrium Radiation Energy Density Inside a Field Free Plasma

Let us consider an infinite homogeneous plasma consisting of infinitely heavy ions and mobile electrons and subject to no static electric or magnetic fields. We shall assume that the electromagnetic waves under consideration propagate at speeds much higher than the thermal velocities for the electrons and that hence the electron thermal motion can be neglected. To begin with we shall also neglect collisions between particles. The equation of motion for an electron is then

$$\frac{d\underline{v}}{dt} = -\frac{e\underline{E}}{m} \tag{30}$$

where $-e$, m, $\underline{v}$ are the electron charge mass and velocity and $\underline{E}$ is the electric field. The current is given by

$$\underline{j} = -ne\underline{v} \tag{31}$$

where n is the electron density. In addition to these equations we have Maxwell's equations

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \quad , \tag{32}$$

$$\underline{\nabla} \times \underline{B} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + \frac{4\pi \underline{j}}{c} \quad , \tag{33}$$

$$\underline{\nabla} \cdot \underline{E} = 4\pi\rho \quad , \tag{34}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad . \tag{35}$$

Here the charge and current are in electrostatic units and since we have simply free space with charges and currents imbedded in it no distinction is made between $\underline{B}$ and $\underline{H}$. One also has the equation of continuity for the electrons,

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{j} = 0 \quad . \tag{36}$$

However, this equation may be derived from (33) and (34).

We now look for solutions of these equations which go like $e^{i(k \cdot r - \omega t)}$. Substituting this form in (30), (31), (32), and (33) we obtain

$$i\underline{k} \times \underline{E} = \frac{i\omega}{c}\underline{B} \quad , \tag{37}$$

$$i\underline{k} \times \underline{B} = -\frac{i\omega}{c}\underline{E} + \frac{i4\pi n e^2}{c m \omega}\underline{E} \quad . \tag{38}$$

Crossing (37) with $\underline{k}$ on the left and substituting for $\underline{k} \times \underline{B}$ from (38) gives

$$(\underline{k} \times \underline{k} \times \underline{E}) = -\frac{\omega^2}{c^2}\underline{E} + \frac{\omega_p^2}{c^2}\underline{E} \quad , \tag{39}$$

or

$$\underline{k}(\underline{k} \cdot \underline{E}) - k^2\underline{E} = \frac{\omega_p^2 - \omega^2}{c^2}\underline{E} \quad . \tag{40}$$

$$\omega_p^2 = 4\pi n e^2/m \quad .$$

If one looks for transverse waves, $\underline{k} \cdot \underline{E} = 0$, then Eq. (40) becomes

$$(k^2c^2 + \omega_p^2 - \omega^2)\underline{E} = 0 \quad , \tag{41}$$

while if one looks for longitudinal waves, $\underline{k} \times \underline{E} = 0$, it becomes

$$(\omega_p^2 - \omega^2)\underline{E} = 0 \quad , \tag{42}$$

$$\omega^2 = \omega_p^2 \quad .$$

We are primarily interested in the transverse waves since these couple to the radiation field in the vacuum which is also transverse. From Eq. (41) the dispersion relation for these waves is

$$(k^2c^2 + \omega_p^2 - \omega^2) = 0 \ . \tag{43}$$

The phase velocity is given by

$$v_p = \frac{\omega}{k} = \frac{c}{(1 - \omega_p^2/\omega^2)^{1/2}} \tag{44}$$

while the group velocity

$$v_g = \frac{d\omega}{dk} = c(1 - \omega_p^2/\omega^2)^{1/2} \ . \tag{45}$$

If we substitute (44) and (45) into Eq. (23) for the equilibrium energy density within the plasma we obtain,

$$\epsilon_p(\nu) = \frac{2kT\nu^2}{c^3}\left(1 - \frac{\nu_p^2}{\nu^2}\right)^{1/2} , \tag{46}$$

$$\omega = 2\pi\nu \ .$$

Equation (46) gives the energy density in both polarizations; the energy density of each polarization is just half of this value.

Earlier we saw that the energy per mode in the radiation field was kT. It is interesting to ask what is the fraction of this energy in the electric and magnetic fields and in the electron kinetic energy for this case. The energy density in the electric and magnetic fields is

$$w_E = \frac{E^2}{8\pi} \tag{47}$$

$$w_B = \frac{B^2}{8\pi} \tag{48}$$

while the energy density in the electron motion is

$$w_e = \frac{n m v^2}{2} \quad . \tag{49}$$

By Eq. (37) the magnitude of the magnetic field is related to the magnitude of the electric field by

$$|\underline{B}| = \frac{ck}{\omega} |\underline{E}| = \frac{c}{v_p} |\underline{E}| \quad . \tag{50}$$

Thus the magnetic energy density is related to the electric energy density by

$$w_B = \frac{c^2}{v_p^2} w_E \quad . \tag{51}$$

From Eq. (44) for v_p we find for a plasma

$$w_B = (1 - \frac{\omega_p^2}{\omega^2})\, w_E \quad . \tag{52}$$

By Eqs. (30) and (49) we have for the electron energy density

$$w_e = \frac{n e^2}{2 m \omega^2} E^2 \quad . \tag{53}$$

Thus by (47) and (53) the electric energy density and the electron energy density are related by

$$w_e = \frac{\omega_p^2}{\omega^2} w_E \quad . \tag{54}$$

The fraction of the total energy which is in each of these terms is

$$\frac{w_E}{w_T} = \frac{1}{2} \tag{55}$$

$$\frac{w_B}{w_T} = \frac{1}{2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{56}$$

$$\frac{w_e}{w_T} = \frac{1}{2}\,\frac{\omega_p^2}{\omega^2} \tag{57}$$

$$w_T = w_E + w_B + w_e \quad .$$

The electric field energy is always 1/2 kT per mode. For high frequencies the magnetic energy is also 1/2 kT per mode and there is negligible energy in the particle motion. For frequencies near the plasma frequency there is 1/2 kT per mode in the kinetic energy of the oarticles and negligible energy in the magnetic field. Problem: Show that the energy flux $(E \times B)c/4\pi$ divided by the energy density is the group velocity.

VI. Elementary Estimation of the Absorption Coefficient for a Plasma

We now wish to find the absorption and emission coefficients for a plasma. We may estimate these by simply introducing a collisional damping into Eq. (30). In place of (30) we write for the equation of motion of an electron,

$$\frac{d\underline{v}}{dt} = -\frac{eE}{m} - \frac{\underline{v}}{\tau_c} \quad . \tag{58}$$

Here τ_c is the electron-ion collision time. It depends on density, temperature and on frequency as given by Eq. (59),

$$\tau_c(\nu) = \frac{3}{2} \frac{\pi^{3/2}(kT)^{3/2} m^{1/2}}{z^2 e^4 n_i} \ln\left(\frac{D}{r_{min}} \frac{\nu_p}{\nu}\right) \quad . \tag{59}$$

where D is the Debye length $\left(\sqrt{\frac{kT}{4\pi e^2 n_e}} = D\right)$ and r_{min} is the minimum impact parameter which will be discussed later.

For $\omega = \omega_p$ this collision time is that given by Spitzer.[1] The logarithmatic change in τ_c with frequency is due to the fact that a collision with impact parameter p is not effective in generating or absorbing waves with frequencies greater than v/p. This will be discussed in more detail when we compute bremsstrahlung directly.

If we now use (58) in place of (31) and proceed as before, we find in place of the dispersion relation (43),

$$k^2c^2 - \omega^2 + \frac{\omega_p^2 \omega}{\omega + \frac{i}{\tau_c(\omega)}} = 0 \quad . \tag{60}$$

Since i/τ_c is generally quite small compared to ω, (60) may be approximated by

$$k^2c^2 - \omega^2 + \omega_p^2 \left[1 - \frac{i}{\omega\tau_c}\right] = 0 \quad . \tag{61}$$

Now the rate of absorption of energy for waves of a given wave number k is given in terms of the imaginary part of ω by

$$\frac{dw}{dt} = +2 \mathrm{Im}\, \omega\, w \tag{62}$$

where w is the wave energy. This follows from the fact that the energy is proportional to the square of the amplitude of the wave and the amplitude damps according to the equation.

$$A = A e^{-i\omega t} \quad . \tag{63}$$

From (61) we have for ω,

$$\omega = \pm \sqrt{k^2 c^2 + \omega_p^2 - \frac{i\omega_p^2}{\omega \tau_c}} \quad , \tag{64}$$

$$\omega \cong \pm (k^2c^2 + \omega_p^2)^{1/2}(1 - \frac{i\omega_p^2}{2\omega\tau_c} \frac{1}{k^2c^2 + \omega_p^2}) \quad , \tag{65}$$

$$\omega \cong \pm \omega_o - \frac{i\omega_p^2}{2\omega_o^2 \tau_c} \quad , \tag{66}$$

$$\omega_o = (k^2c^2 + \omega_p^2)^{1/2} \quad .$$

Thus we see that the imaginary part of ω is always negative so that we get damping according to Eq. (62) and that the damping time is given by

$$\tau = \tau_c \frac{\omega_o^2}{\omega_p^2} \quad . \tag{67}$$

We can understand this result physically as follows. The fraction of the wave energy which the electrons have is $\omega_p^2/2\omega^2$, as given by Eq. (57). In one collision time the electrons will dissipate their energy and hence the wave will lose the fraction $\omega_p^2/2\omega^2$ of its total energy. The wave would lose all of its energy is essentially ω^2/ω_p^2 collision times, or in time

$$\tau = \tau_c \frac{\omega^2}{\omega_p^2} \tag{68}$$

as given by (68).

If instead of the absorption in time we had wanted the absorption with distance for a wave of fixed frequency, then we must solve (61) for k in terms of ω and the damping is given by

$$w = w_o e^{2 \mathrm{Im} k \cdot x} \quad . \tag{69}$$

If we solve (61) for k, we get

$$k \cong \pm \frac{(\omega^2 - \omega_p^2)^{1/2}}{c} \left(1 + \frac{i\omega_p^2}{2(\omega^2 - \omega_p^2)\omega\tau_c}\right)$$

$$= \pm \frac{(\omega^2 - \omega_p^2)^{1/2}}{c} \pm \frac{i\omega_p^2}{c\, 2\omega(\omega^2 - \omega_p^2)^{1/2}\tau_c} \quad . \tag{70}$$

If ω is positive and k is positive the wave propagates in the positive direction and the wave damps in that direction, while if k is negative the wave is propagating in the negative direction and again the wave damps in the direction of propagation. Similar results hold for negative ω.

We see from (69) and (70) that the absorption distance 1/2 Im k is given by

$$\frac{c|\omega|\,[\omega^2 - \omega_p^2]^{1/2}\tau_c}{\omega_p^2} = \ell \quad . \tag{71}$$

We see from (67) for the absorption time and (45) for the group velocity that ℓ is given by

$$\ell = \tau v_g \quad ; \tag{72}$$

that is, ℓ is the distance the wave would propagate at the group velocity in an absorption time.

VII. Emission From a Field Free Plasma

We may now estimate the emission from a plasma. The emission is equal to the equilibrium radiation density times the absorption coefficient. Thus from (46) for the equilibrium energy density and (67) for the absorption time, we find for the emissivity

$$\varepsilon(\nu) = \frac{2kT\nu^2}{c^3}\left(1 - \frac{\nu_p^2}{\nu^2}\right)^{1/2} \frac{\nu_p^2}{\nu^2} \frac{1}{\tau_c(\nu)}$$

$$= \frac{2kT\nu_p^2}{c^3\tau_c(\nu)}\left(1 - \frac{\nu_p^2}{\nu^2}\right)^{1/2} \quad . \tag{73}$$

Equation (73) gives the emission per unit volume, per unit time, per unit frequency interval, per unit solid angle in both transverse polarizations. The emission per unit frequency interval, per unit solid angle, per unit cross-sectional area per unit distance of propagation in both polarizations is obtained by dividing (73) by the groups velocity,

$$\varepsilon(\nu) = \frac{2kT\,\nu_p^2}{c^4 \tau_c(\nu)} \quad . \tag{74}$$

For a body to be optically thick for frequency ν the absorption length ℓ given by (71) must be small compared to the size of the body. For high frequencies this size is given by

$$c \, \frac{\omega^2}{\omega_p^2} \, \tau_c = \ell \quad . \tag{75}$$

VIII. Absorption and Emission Due to Electron-Electron Collisions

In the last few sections we have considered only the effects of electron-ion collisions or equivalently, the plasma resistance. Electron-electron collisions do not change the total momentum of the electrons and hence do not change the current. Thus to a first approximation they do not lead to either absorption or emission of radiation. We can also see this by considering individual collisions between two electrons. When two electrons collide their acceleration is equal and opposite and hence the radiation from one is 180 degrees out of phase with that from the other. Thus the two radiation fields cancel each other. Another way to say it is that the two electrons do not have a dipole moment and so they cannot radiate in the dipole order. They do, however, have a quadrupole moment and they do give rise to quadrupole radiation. This, however, is a much weaker process than the dipole radiation from electron-ion encounters. From the macroscopic point of view the corresponding absorption is due to an electron viscosity or to the dissipation due to electrons colliding which have slightly different phases. This is, however, a high frequency viscosity and low frequency results cannot be employed here. These terms are generally smaller than the electron-ion terms by the factor $v_T^2 k^2/\omega^2$.

IX. Direct Classical Calculation of Bremsstrahlung From a Field Free Plasma Neglecting Plasma Shielding Effects

We have just seen how we can estimate the total bremsstrahlung emitted by a plasma and how we can obtain the emission in terms of the collision frequency. We shall now turn to the problem of computing this radiation directly.[5] Here we shall do this classically and use quantum effects only to obtain cut-offs in impact distances as we did earler. We shall begin by neglecting the effect of the plasma on the radiation emitted, that is, we shall treat the emission as if it took place in a vacuum. Later we shall include the plasma effects. We start by finding the dipole radiation due to an electron-ion encounter and then we add up the emission due to all encounters.

The time averaged total power radiated by a dipole with time dependence $\underline{p}(t)$ may be obtained from Eq. (5) for the radiation by an accelerated charge and is given by $\frac{2e^2a^2}{3c^3} = \frac{2e^2\ddot{r}^2}{3c^3}$

$$w = \frac{1}{3c^3 T} \int_{-T}^{T} |\ddot{\underline{p}}(t)|^2 \, dt \quad , \tag{76}$$

where $\underline{p}$ is the dipole moment ($\underline{p} = \sum_i q_i \underline{r}_i(t)$). Here we have imagined that the motion of our dipole occurs only during the time $-T < t < T$ (T may be very large, however). Fourier analyzing $\underline{p}(t)$, Eq. (76) becomes

$$w = \frac{2\pi}{3c^3T}\int_{-\infty}^{\infty}\omega^4\,|\underline{p}(\omega)|^2 d\omega \;, \qquad \frac{1}{3c^3T}\int_{-\infty}^{\infty}\omega^4|\underline{p}(\omega)|^2 d\omega \tag{77}$$

$$\underline{p}(\omega) = \frac{1}{2\pi}\int_{-T}^{T} p(t)\, e^{-i\omega t}\, dt \;. \qquad \tilde{p}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-T}^{T}\underline{p}(t)\, e^{-i\omega t}\, dt \tag{78}$$

Now consider an electron passing an ion as shown in Fig. 3

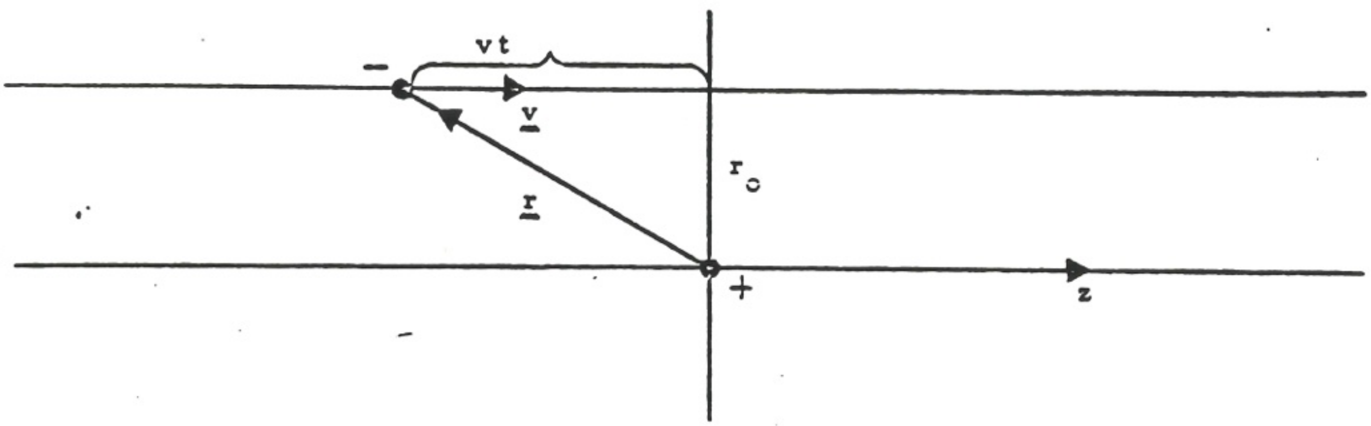

Fig. 3

We shall assume that the deflection is small so that to a first approximation the electric field which the electron sees is the same as it would see if it followed a straight line trajectory. Here we are making use of the fact that except for very high frequencies of the order of v/r_{min}, most of the radiation comes from distant encounters where the deflection is small. We take the trajectory to be in the x, z plane and the velocity before the encounter to be in the z direction. We take the impact parameter to be r_o and we measure time from the instant of closest approach. The electric field which the electron sees is given by

$$E_x = \frac{z e r_o}{(r_o^2 + v^2 t^2)^{3/2}} , \tag{79}$$

$$E_z = \frac{z e v t}{(r_o^2 + v^2 t^2)^{3/2}} . \tag{80}$$

If we Fourier analyze these E fields in time we obtain

$$E_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t} z e r_o \, dt}{(r_o^2 + v^2 t^2)^{3/2}} = -\frac{ze\omega}{\pi v^2} K_1\left(\frac{r_o \omega}{v}\right) \tag{81}$$

$$E_z(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t} z e v t \, dt}{(r_o^2 + v^2 t^2)^{3/2}} = -\frac{ize}{\pi v^2} \omega K_o\left(\frac{r_o \omega}{v}\right) \tag{82}$$

where K_1 and K_o are modified Bessel functions of the second kind.[6]

Now the dipole moment of our system is

$$\underline{P} = - e \underline{r}(t) . \tag{83}$$

Taking the second time derivative of $\underline{P}$ and Fourier analyzing gives

$$\ddot{\underline{P}} = - e \ddot{\underline{r}} , \tag{84}$$

$$- \omega^2 \underline{P}(\omega) = - e \ddot{\underline{r}}_o(\omega) = - \frac{e^2 \underline{E}(\omega)}{m} . \tag{85}$$

$$\underline{P}(\omega) = \frac{e^2 \underline{E}(\omega)}{m\omega^2} \quad . \tag{86}$$

Here $\underline{E}$ is the electric field seen by the electron. If we had simply used the straight line orbit then the only non-zero component of $\underline{P}(\omega)$ is for $\omega = 0$. This gives no radiation as we already know from the fact that unaccelerated charges do not radiate.

If we now substitute $\underline{E}(\omega)$ from (81) and (82) in Eq. (86), then we have for $P(\omega)$

$$P_x(\omega) = \frac{z e^3}{\pi m v^2 \omega} K_1\left(\frac{r_o \omega}{v}\right) \quad , \tag{87}$$

$$P_z(\omega) = \frac{-i z e^3}{\pi m v^2 \omega} K_o\left(\frac{r_o \omega}{v}\right) \quad . \tag{88}$$

Substituting these expressions in Eq. (77) for the dipole radiation gives

$$w = \frac{2}{3c^3 T} \frac{z^2 e^6}{\pi m^2 v^4} \int_{-\infty}^{\infty} \omega^2 \left\{ K_1^2\left(\frac{r_o \omega}{v}\right) + K_c^2\left(\frac{r_o \omega}{v}\right) \right\} d\omega \quad . \tag{89}$$

Equation (89) gives the total power radiated at frequency ω due to a single encounter between an electron and an ion with impact parameter r_o and electron velocity v. To obtain the total power radiated by all the electrons encountering this ion we must multiply (89) by the number of electrons with impact parameters lying between r_o and $r_o + dr_o$ and velocities lying in d^3v centered at v in the time 2T,

$$dn = T\,4\pi\, r_o\, v\, f(v)\, dr_o\, d^3v \quad , \tag{90}$$

and integrate over all r_o and v. Thus we obtain for the total power radiated

$$w = \frac{8z^2e^6}{3c^3m^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{r_{min}}^{\infty} \frac{\omega^2 r_o f(v)}{v^3} \left\{K_1^2\left(\frac{r_o\omega}{v}\right) + K_o^2\left(\frac{r_o\omega}{v}\right)\right\} d\omega\, d^3v\, dr_o \tag{91}$$

where r_{min} is the minimum distance at which we can employ our straight line orbit approximation.

By making use of the identity

$$x[K_1^2(x) + K_o^2(x)] = -\frac{d}{dx}[x\, K_1(x)\, K_o(x)] \tag{92}$$

we may integrate (91) with respect to r_o and we obtain

$$w = \frac{8z^2e^6}{3c^3m^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v) \frac{r_{min}\omega}{v^2} K_1\left(\frac{r_{min}\omega}{v}\right) K_o\left(\frac{r_{min}\omega}{v}\right) d\omega\, d^3v \quad . \tag{93}$$

The energy radiated between ω and $\omega + d\omega$, $w(\omega)d\omega$ is from (93),

$$w(\omega)d\omega = \frac{8z^2e^6 d\omega}{3c^3m^2} \int f(v) \frac{r_{min}\omega}{v^2} K_1\left(\frac{r_{min}\omega}{v}\right) K_o\left(\frac{r_{min}\omega}{v}\right) d^3v \quad . \tag{94}$$

Now r_{min} must be of the order of the larger of the following two numbers:

(1) Reciprocal de Broglie wave number, $\hbar/mv$,

(2) Classical distance of closest approach, ze^2/mv^2.

In general these are very small numbers and unless ω is very large (so large that the finite energy of the photon emitted is important) $r_{min}\ \omega/v$ will be very much less than 1. Thus we may use the asymptotic forms of $K_1(z)$ and $K_o(z)$ for small z. These are

$$\lim_{z\to 0} K_o(z) = -\ln z \ ,$$
$$\lim_{z\to 0} K_1(z) = \frac{1}{z} \ . \tag{95}$$

Equation (94) thus becomes

$$w(\omega)d\omega = \frac{8z^2e^6}{3c^3m}\int f(v)\,\frac{r_{min}\omega}{v^2}\,\frac{v}{r_{min}\omega}\,\ln\frac{v}{r_{min}\omega}\,d^3v$$

$$= n_e\,\frac{16z^2e^6}{3c^3m}\left(\frac{1}{2\pi mkT}\right)^{1/2}\int_o^\infty \frac{mv}{kT}\,\ln\frac{v}{r_{min}\omega}\,e^{-mv^2/2kT}dv \ . \tag{96}$$

If we use for r_{min} the reciprocal de Broglie wave number $\hbar/mv$, and let $x = mv^2/2kT$, then (96) becomes

$$w(\omega)d\omega = n_e\,\frac{16z^2e^6}{3c^3m}\left(\frac{1}{2\pi mkT}\right)^{1/2}\int_o^\infty dx\,e^{-x}\,\ln\frac{2kTx}{\hbar\omega}$$

$$= \frac{16z^2e^6}{3c^3m}\left(\frac{1}{2\pi mkT}\right)^{1/2} n_e\left\{\ln\frac{2kT}{\hbar\omega} - \gamma\right\} \tag{98}$$

where γ is Euler's constant 0.577..... and is given by

$$\gamma = \int e^{-x} \ln x \, dx \quad . \tag{99}$$

If in place of the de Broglie cut-off we had used the classical distance of closest approach, then in place of the term

$$\left\{ \ln \frac{2kT}{\hbar\omega} - \gamma \right\}$$

in (98) we obtain

$$\left\{ \ln \frac{m}{ze^2\omega} \left(\frac{2kT}{m}\right)^{3/2} - \frac{3}{2}\gamma \right\}$$

although we have not treated large angle scatterings or quantum effects correctly here, detailed treatments of these effects[5,7] give essentially the same results. In fact, r_{min} can be adjusted so as to obtain exactly the same results as the detailed calculations. However, since the dependence on r_{min} is logarithmic, even relatively large changes in r_{min} give only a small change in the results. Even the more detailed calculations involve some approximations and the difference between them and the appropriate form obtained above, (98) or (99), is so small that we shall not take them up.

Equation (98) gives the emission for both positive and negative ω. Since the absolute value of the frequency is the only important quantity, we may add the emissions for $\pm\omega$ and restrict ω to values greater than zero. To do this we simply multiply (98) by 2. If we also multiply by the density of ions we obtain the total emission.

$$w_T(\omega)d\omega = \frac{32z^2e^6}{3c^3m}\left(\frac{1}{2\pi mkT}\right)^{1/2} n_e n_i \left\{ \begin{matrix} \ln \frac{2kT}{k\omega} - \gamma \\ \\ \ln \frac{m}{ze^2\omega}\left(\frac{2kT}{m}\right)^{3/2} - \frac{3\gamma}{2} \end{matrix} \right\} . \tag{100}$$

Here the two different cut-offs have been used and the logarithm with smallest argument should be employed.

X. The Influence of the Plasma on the Emission of Bremsstrahlung

There are a number of plasma effects which influence the emission of bremsstrahlung and which have not been taken into account in the above calculations. These are:

(1) The shielding of the ion and electron involved by the electrons of the plasma so that collisions with impact parameters greater than a Debye length do not give rise to radiation.

(2) The radiation is emitted in a plasma and not in a vacuum. The dielectric properties of the plasma modify the emission. No emission takes place for frequencies below the plasma frequency.

(3) An electron is simultaneously colliding with many ions.

We shall take these effects into account by computing the radiation from a dipole embedded in a plasma and then substituting into this formula the time dependent dipole moment due to a shielded electron colliding with one or more ions of the plasma.

XI. Radiation by a Dipole Embedded in a Plasma

We shall now compute the radiation by a dipole embedded in an infinite uniform homogeneous plasma.[8] We shall take the electron to be mobile and the ion to constitute a uniform fixed neutralizing background. We shall take the electron dynamics to be described by the linearized Vlasov equation. This along with Maxwell's equations describe our system.

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} - \frac{e}{m} \underline{E} \cdot \frac{\partial f_o}{\partial \underline{v}} = -\epsilon f \tag{101}$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t} \tag{102}$$

$$\underline{\nabla} \times \underline{B} = \frac{1}{c} \frac{\partial \underline{E}}{\partial t} - \frac{4\pi e}{c} \int \underline{v}\, f\, d^3v + \frac{4\pi}{c} \underline{j}_s \tag{103}$$

$$\underline{\nabla} \cdot \underline{E} = 4\pi \left[-e \int f\, d^3v + \rho_s \right] \tag{104}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad . \tag{105}$$

Here f is the perturbation in the electron distribution function, f_o is the unperturbed distribution function and is taken to be isotropic (this is why the $v \times B \cdot \partial f_o/\partial \underline{v}$ term drops out of (101) since $v \times B$ is perpendicular to $\partial f_o/\partial v$). For the case of equilibrium f_o is Maxwellian but we do not need to make this restriction here. The small damping term, $-\epsilon f$, in Eq. (101) has been introduced for mathematical convenience for determining the contours of integration for certain singular integrals that will appear. In the end we will let ϵ go to zero. The terms $\underline{j}_s$ and $\underline{\rho}_s$ appearing in Eqs. (103) and (104) are source currents and charges. For the case under consideration they are the current and charge densities due to a dipole. For the time being, however, we may leave them arbitrary. We shall assume that they satisfy the continuity equation,

$$\frac{\partial \rho_s}{\partial t} + \underline{\nabla} \cdot \underline{j}_s = 0 \quad . \tag{106}$$

Fourier analyzing Eqs. (101)-(105) in space and time gives

$$(i\omega + i\underline{k} \cdot \underline{v} + \epsilon)f - \frac{e\underline{E}}{m} \cdot \frac{\partial f_o}{\partial \underline{v}} = 0 \quad , \tag{107}$$

$$\underline{k} \times \underline{E} = -\frac{\omega}{c} \underline{B} \quad , \tag{108}$$

$$i\underline{k} \times \underline{B} = \frac{i\omega}{c} \underline{E} - \frac{4\pi e}{c} \int \underline{v} f d^3v + \frac{4\pi}{c} \underline{j}_s \tag{109}$$

$$i\underline{k} \cdot \underline{E} = 4\pi(-e \int f d^3v + \rho_s) \quad , \tag{110}$$

$$i\underline{k} \cdot \underline{B} = 0 \quad . \tag{111}$$

We have omitted the subscripts k, ω from f, E, B, j_s and ρ_s although they are to be understood. Crossing (108) with k on the left and substituting $\underline{k} \times \underline{B}$ from (109) gives

$$(\underline{k} \times \underline{k} \times \underline{E}) = -\frac{\omega^2}{c^2} \underline{E} - \frac{4\pi e^2 \omega}{mc^2} \int \frac{\underline{v}\,\underline{E} \cdot \frac{\partial f_o}{\partial \underline{v}}\, d^3v}{\omega + \underline{k} \cdot \underline{v} - i\epsilon} + \frac{4\pi \omega i}{c^2} \underline{j}_s \tag{112}$$

We may now split $\underline{E}$ and $\underline{j}_s$ into longitudinal and transverse components

$$\underline{E} = E^{\ell} + E^t , \tag{113}$$

where

$$\underline{k} \times \underline{E}^{\ell} = 0 \quad , \tag{114}$$

$$\underline{k} \cdot \underline{E}^{t} = 0 \quad , \tag{115}$$

$$\underline{k} \cdot \underline{E}^{\ell} = kE^{\ell} \quad , \tag{116}$$

$$\underline{k} \times \underline{k} \times \underline{E}^{t} = -k^2 E^{t} \quad . \tag{117}$$

Similar expressions hold for $\underline{j}_s$.

From Eq. (110) we find

$$ikE^{\ell} = -\frac{4\pi e^2}{mi}\,\underline{E} \cdot \int \frac{\frac{\partial f_o}{\partial \underline{v}}\, d^3v}{\omega + k \cdot v - i\epsilon} + 4\pi \rho_s \quad . \tag{118}$$

Now $\omega + \underline{k} \cdot \underline{v} - i\epsilon$ depends only on the component of $\underline{v}$ parallel to $\underline{k}$. Hence the integral on the right-hand side of (118) can be integrated over the two components of v perpendicular to k. The components of $\partial f_o/\partial \underline{v}$ perpendicular to k integrate to zero and only the term parallel to $\underline{k}$ makes a contribution. Thus (118) may be written in the form

$$\left(1 - \frac{4\pi e^2}{mk^2} \int \frac{k \cdot \frac{\partial f_o}{\partial \underline{v}}\, 3^3v}{\omega + k \cdot v - i\epsilon}\right) E^{\ell} = -\frac{i\rho_s}{k} \tag{119}$$

or by (106)

$$E^{\ell} = -\frac{4\pi i\rho_s}{kD^{\ell}(k,\omega)} = +\frac{4\pi i j_s^{\ell}}{\omega D^{\ell}(k,\omega)} \tag{120}$$

where $D^{\ell}(k,\omega)$ is the longitudinal dielectric function,

$$D^{\ell}(k,\omega) = \left(1 - \frac{4\pi e^2}{mk^2} \int \frac{k \cdot \frac{\partial f_o}{\partial v}\, d^3v}{\omega + k \cdot v - i\epsilon}\right). \tag{121}$$

If Eq. (120) is substituted into Eq. (112) for E^{ℓ} and use is made of the continuity equation for ρ_s and $\underline{j}_s$ (Eq. 106), then the E^{ℓ} terma just cancel the j_s^{ℓ} term and Eq. (112) becomes

$$-k^2\underline{E}^t = -\frac{\omega^2}{c^2}\underline{E}^t - \frac{4\pi e^2\omega}{mc^2}\int \frac{\underline{v}\,\underline{E}^t \cdot \frac{\partial f_o}{\partial \underline{v}}\, d^3v}{\omega + k \cdot v - i\epsilon} + \frac{4\pi\omega i}{c^2}\underline{j}_s^t \quad . \tag{122}$$

Now in the integral appearing in (122) only derivatives of f_o with respect to the components of v perpendicular to $\underline{k}$ appear. These terms may be integrated by parts in the perpendicular directions when use is made of the fact that f_o is isotropic. One thus finds that (122) may be written as

$$\left[\omega^2 - k^2c^2 - \frac{4\pi e^2\omega}{m}\int \frac{f_o\, d^3v}{\omega + \underline{k}\cdot\underline{v} - i\epsilon}\right]\underline{E}^t = 4\pi\omega i\,\underline{j}_s^t \ . \tag{123}$$

Solving Eq. (123) for $\underline{E}^t$ gives

$$\underline{E}^t = +\frac{4\pi\omega i}{k^2c^2}\,\frac{\underline{j}_s^t}{D^t(k,\omega)} \tag{124}$$

where $D^t(k,\omega)$ is the transverse dielectric function,

$$D^t(k,\omega) = \left[\frac{\omega^2}{k^2c^2} - 1 - \frac{4\pi e^2\omega}{mk^2c^2}\int \frac{f_o\, d^3v}{\omega + k \cdot v - i\epsilon}\right] \ . \tag{125}$$

XII. Power Radiated by the Sources

We now compute the time average power dissipated by the sources. This is given by

$$\overline{P} = \frac{1}{2T}\int_{-T}^{T}\int \underline{j}_s \cdot \underline{E}\, d^3r\, dt \quad . \tag{126}$$

Substituting the expressions

$$\underline{j}_s = \iint \underline{j}_s(k,\omega)\, e^{i(\underline{k}\cdot\underline{r} + \omega t)} d^3k\, d\omega \quad , \tag{127}$$

$$\underline{E} = \iint \underline{E}(k,\omega)\, e^{i(\underline{k}\cdot\underline{r} + \omega t)} d^3k\, d\omega \quad , \tag{128}$$

for $\underline{j}_s$ and $\underline{E}$ in (126) gives

$$\overline{P} = \frac{1}{2T}\int_{-T}^{T} dt \int d^3r \int d^3k \int d^3k' \int d\omega \int d\omega' \underline{j}_s(\underline{k},\omega)\underline{E}(\underline{k}',\omega')e^{i[(\underline{k}+\underline{k}')\cdot r + (\omega+\omega')t]}$$

$$= \frac{(2\pi)^4}{2T}\iint d^3k\; d\omega\, \underline{j}_s(-\underline{k},-\omega)\; \underline{E}(+\underline{k},+\omega) \quad . \tag{129}$$

Substituting for $\underline{E}$ from (120) and (124) gives

$$\overline{P} = \frac{(2\pi)^4}{2T}\, 4\pi i \iint d^3k\, d\omega \cdot [+\frac{|\underline{j}_s^{\ell}|^2}{\omega D^{\ell}(k,\omega)} + \frac{|\underline{j}_s^{t}|^2\omega}{k^2c^2D^t(k,\omega)}] \quad . \tag{130}$$

XIII. Radiation From a Dipole

Let us now consider the radiation from a dipole oscillating at frequency ω. Our dipole is given by

$$\underline{\rho} = \lim_{\substack{q\to\infty \\ \mathbf{r}_o \to 0 \\ q\mathbf{r}_o \to \mathbf{p}}} \left\{ q\delta[\mathbf{r} - \mathbf{r}_o(t)] - q\delta(\mathbf{r}) \right\} \quad . \tag{131}$$

The current is given by

$$\mathbf{j} = \lim_{\substack{q\to\infty \\ \mathbf{r}_o \to 0 \\ q\mathbf{r}_o \to \mathbf{p}}} q\dot{\mathbf{r}}_o(t)\, \delta[\mathbf{r} - \mathbf{r}_o(t)] \cong \dot{\mathbf{p}}(t)\, \delta(\mathbf{r}) \quad . \tag{132}$$

Fourier analyzing in r and in t gives

$$j(k,\omega) = \frac{i\omega \mathbf{p}(\omega)}{(2\pi)^{\cancel{3}\,4}} \quad . \tag{133}$$

For the longitudinal and transverse components of $\mathbf{j}$ we have

$$j^{\ell} = \frac{\mathbf{k}\cdot\mathbf{j}}{k} = \frac{i\omega \mathbf{k}\cdot\mathbf{p}(\omega)}{k(2\pi)^4} \quad , \tag{134}$$

$$j^{t} = \frac{-\mathbf{k}\times\mathbf{k}\times\mathbf{j}}{k^2} = -\frac{i\omega\,\mathbf{k}\times\mathbf{k}\times\mathbf{p}}{k^2(2\pi)^4} \quad . \tag{135}$$

Substituting (134) and (135) into (130) and integrating over ω gives

$$\overline{P} = \frac{\mathrm{Im}}{(2\pi T}\iint \omega^2 \left\{ + \frac{|\mathbf{k}\cdot\mathbf{p}|^2 D^{\ell}(k,\omega)}{\omega k^2 |D^{\ell}(k,\omega)|^2} + \frac{\omega_o |\mathbf{k}\times\mathbf{p}|^2 D^{t}(k,\omega)}{k^4 c^2 |D^{t}(k,\omega)|^2} \right\} d^3k\, d\omega \tag{136}$$

where use has been made of the fact that $D(-k,-\omega) = D^*(k,\omega)$ for both the longitudinal and transverse dielectric constants.

Now Eq. (136) includes all types of energy loss by the dipole which are contained in the Vlasov treatment. These include energy losses due to the scattering of individual particles by the dipole as well as that radiated away as coherent waves. The latter include both longitudinal and transverse plasma waves.

Since the longitudinal and transverse waves are essentially natural oscillations of the plasma, the k's and ω's associated with them satisfy the condition.

$$D^{\binom{t}{\ell}}(k,\omega) = 0 \quad , \tag{137}$$

that is, such oscillations of the plasma can exist according to (120) and (124) even if there are no sources. Generally speaking these oscillations are damped to some extent so that the only solutions of (137) have either complex k's or ω's. The only damping contained in our treatment is Landau damping while in many cases collisional damping would dominate. We can include collisional damping in a semi-empirical way by setting ϵ equal to the reciprocal of the collision time. However, it turns out that the total radiation emitted is independent of the damping so long as it is small. The damping affects the absorption length for the wave once it is emitted, but it does not influence the emission.

From the above arguments we see that the wave emission is obtained by integrating Eq. (136) over those k's for fixed ω which lie in the vicinity of

$$\mathrm{Re}\, D^{\binom{t}{\ell}}(k,\omega) = 0 \quad . \tag{138}$$

Examining the integrand shows that it has sharp peaks in the vicinity of these values so long as the damping is small or equivalently so long as the imaginary part of D is small.

Turning now to find these resonances we note that the phase velocity of the transverse electromagnetic waves is always large compared to the particle velocity so that we may neglect v compared to ω/k. Equation (125) for $D^t(k,\omega)$ reduces to

$$D^t(k,\omega) \cong \frac{\omega^2 - \omega_p^2}{k^2c^2} - 1 - \frac{i\,\omega_p^2\epsilon}{k^2c^2\omega} \quad . \tag{139}$$

The real part of $D^t(k,\omega)$ vanishes for $k = \pm k_o$

$$k_o = \frac{(\omega^2 - \omega_p^2)^{1/2}}{c} \quad . \tag{140}$$

Thus $D^t(k,\omega)$ is approximately given by

$$k^2c^2D^t(k,\omega) = [k_o^2 - k^2 - \frac{i\epsilon\omega_p^2}{\omega c^2}]c^2 \quad . \tag{141}$$

Substituting this into Eq. (136) gives

$$\overline{P^t} = -\frac{\omega_p^2\epsilon}{2\pi Tc^2}\int\int\int \frac{|P(\omega)|^2\omega^2\sin^3\theta\, k^2 d\theta\, k\, d\omega}{(k_o^2 - k^2)^2 + \epsilon^2\omega_p^4/\omega^2}$$

$$= \frac{2\pi}{3c^3T}\int_{-\infty}^{\infty}\omega^3(\omega^2 - \omega_p^2)^{1/2}\,|p(\omega)|^2\,d\omega \quad . \tag{142}$$

It is interesting to compare this with the classical formula for the radiation emitted in a vacuum (Eq. 77). We see that the difference is that we replace

$$\omega^4 |P(\omega)|^2$$

appearing in the vacuum formula by

$$\omega^3(\omega^2 - \omega_p^2)^{1/2} |P(\omega)|^2 \quad . \tag{143}$$

The vacuum result differs from (143) by the factor $(1 - \omega_p^2/\omega^2)^{1/2}$ in the integrand which is equal to $c^3/V_p^2 V_g$. We note that this factor also appeared in the radiation formula obtained from the resistivity, Eq. (73).

We may find the longitudinal wave emission in a similar fashion. Here we obtain wave emission only if the wavelength is long compared with the Debye length and for ω in the vicinity of the plasma frequency; that is, this is the only region where the real part of D^ℓ vanishes and the imaginary part of D^ℓ is small. For k, ω in this region the phase velocity is large compared with the particle velocity and we may treat ω/k as much larger than v. One then finds that $D^\ell(k, \omega)$ is approximately given by the Bohm and Gross[9] dispersion relation,

$$D^\ell(k, \omega) \cong 1 - \frac{\omega_p^2 - 3k^2 v_T^2}{\omega^2} + \text{damping} \tag{144}$$

where $v_T^2 = kT/m$. Substituting this expression into Eq. (136) we find for the longitudinal radiation

$$\overline{P^\ell} = \frac{2\pi}{T} \int_{-\infty}^{\infty} \frac{\omega^3}{6(\sqrt{3}\, v_T)^3} (\omega^2 - \omega_p^2)^{1/2} |p(\omega)|^2 d\omega \tag{145}$$

$$\omega_p \leq \omega < 1.4\omega_p \quad .$$

This is the same formula as (142) if (142) is divided by 2 and c is replaced by $3\,v_T$. The factor 2 comes from the fact that there is only one polarization for the longitudinal mode while there are two for the transverse wave. Since the dispersion relation for the longitudinal waves is the same as for transverse waves with c replaced by $3v_T$, this substitution follows immediately.

XIV. The Dipole Moment Produced by a Shielded Electron-Ion Encounter

The dipole moment produced by a system of n charges is

$$\underline{p}(t) = \sum_i q_i r_i(t) \tag{146}$$

where q_i is the charge on the $i^{\underline{th}}$ particle and r_i is its position. Now our radiation formula involves the spectrum of $\underline{p}(t)$. If the particles are at rest or move in a straight line, $\underline{p}(\omega)$ is non-zero only for $\omega = 0$ and we get no radiation. We get radiation only because of the acceleration of the charges. Now $\ddot{\underline{p}}$ is given by

$$\ddot{\underline{p}} = \sum_i \sum_j \frac{q_i^2 q_j (\underline{r}_i - \underline{r}_j)}{m_i |\underline{r}_i - \underline{r}_j|^3} \quad , \tag{147}$$

where

$$\ddot{r}_i(t) = \sum_j \frac{q_i q_j (r_i - r_j)}{m_i |r_i - r_j|^3} \quad . \tag{148}$$

From (147) we see that the sums over like particles (electron-electron or ion-ion) make no contributions since these terms cancel out pair by pair (ij against ji). There is no dipole moment due to these interactions. Thus we have for $\ddot{\underline{p}}$,

$$\ddot{\underset{\sim}{p}} = \sum_{\substack{i \\ \text{all} \\ \text{electrons}}} \sum_{\substack{j \\ \text{all} \\ \text{ions}}} \left(+\frac{e^3 z}{m_e} + \frac{z^2 e^3}{m_i}\right) \frac{(\underset{\sim}{r}_i - \underset{\sim}{r}_j)}{|r_i - r_j|^3} \quad . \tag{149}$$

If we take the ions to be infinitely massive then (149) reduces

$$\ddot{\underset{\sim}{p}} = \frac{e^2 z}{m_e} \sum_{\substack{i \\ \text{all} \\ \text{electrons}}} \sum_{\substack{j \\ \text{all} \\ \text{ions}}} + \frac{e(\underset{\sim}{r}_i - \underset{\sim}{r}_j)}{|\underset{\sim}{r}_i - \underset{\sim}{r}_j|^3}$$

$$= \frac{e^2 z}{m_e} \sum_{\substack{j \\ \text{all} \\ \text{ions}}} \underset{\sim}{E}_e(\underset{\sim}{r}_j) \quad , \tag{150}$$

where $\underset{\sim}{E}_e(\underset{\sim}{r}_j)$ is the electric field at the position of the $j^{\underline{th}}$ ion due to all the electrons. Now we may divide $\underset{\sim}{E}_e$ into contributions due to individual electrons where each contribution is the E field produced by an electron shielded by other electrons. We thus write (150) in the form

$$\ddot{\underset{\sim}{p}} = \frac{e^2 z}{m} \sum_{\substack{j \\ \text{all} \\ \text{ions}}} \sum_{\substack{i \\ \text{all} \\ \text{electrons}}} \underset{\sim}{\tilde{E}}_i(\underset{\sim}{r}_j) \quad , \tag{151}$$

where $\tilde{\underline{E}}_i$ is the shielded E field of the $i^{\underline{th}}$ electron.

According to Rostoker[10] the dressed or shielded particles may be treated as statistically independent so that the $\tilde{\underline{E}}_i$ are statistically independent. Hubbard[11] used this approximation to compute the electric field fluctuations in a plasma. Mercer[12] calculated the radiation from a plasma using this formula and the formula for the radiation by a dipole in a vacuum.

If we assume that the ions are uncorrelated then the radiation from each term in the double sum appearing in (150) will be randomly phased with respect to each other and we may simply sum the radiation due to each term to obtain the total radiation. If (151) is Fourier analyzed in t, we find for $\underline{p}(\omega)$

$$\underline{p}(\omega) = -\frac{e^2 z^2}{m\omega^2} \sum_{\substack{j \\ \text{all} \\ \text{ions}}} \sum_{\substack{i \\ \text{all} \\ \text{electrons}}} \tilde{\underline{E}}_i(\underline{r}_j, \omega) \quad . \tag{152}$$

XV. The Shielded Field of an Electron

To obtain the shielded field of an electron we solve the test particle problem of an electron moving through an infinite uniform plasma with fixed ions. The plasma is described by the Vlasov equations (101)-(105) where the source is an electron moving uniformly along a straight line. We wish to keep only the electrostatic or longitudinal E field. This was the only field used in the last section in deriving the fluctuating dipole moment. Further, the transverse or electromagnetic interaction is relativistically small and so it can be neglected if the electron energy is small compared to the electron rest mass. The longitudinal E field is obtained from Eq. (120) and is given by

$$\underline{E}^{\ell}(\underline{k},\omega) = -\frac{4\pi i \underline{k}\rho_s(\underline{k},\omega)}{k^2 D^{\ell}(\underline{k},\omega)} \quad . \tag{153}$$

Now the source charge density is given by

$$\rho_s = -e\,\delta(\underline{r} - \underline{r}_o - \underline{v}_o t) \quad , \tag{154}$$

where $\underline{r}_o$ and $\underline{v}_o$ are the initial position and velocity. Fourier analyzing gives for $\rho_s(k,\omega)$

$$\rho_s(k,\omega) = -\frac{e}{(2\pi)^4}\int_{-T}^{T}\int e^{-i(\omega t + \underline{k}\cdot\underline{r})}\delta(\underline{r} - \underline{r}_o - \underline{v}_o t)d^3r\,dt$$

$$\rho_s(k,\omega) = -\frac{e}{(2\pi)^4}\, e^{-i\underline{k}\cdot\underline{r}_o}\,\frac{2\sin(\omega + \underline{k}\cdot\underline{v}_o)T}{\omega + k\cdot v_o} \quad . \tag{155}$$

where again we have taken the process to be going on from $t = -T$ to $t = T$. Substituting (155) in (153) we obtain

$$\underline{E}^{\ell}(k,\omega) = \frac{i4\pi e}{(2\pi)^4}\,\frac{\underline{k}\,e^{-i\underline{k}\cdot\underline{r}_o}}{k^2 D^{\ell}(k,\omega)}\,\frac{2\sin(\omega + \underline{k}\cdot\underline{v}_o)T}{\omega + \underline{k}\cdot\underline{v}_o} \quad . \tag{156}$$

The dipole moment produced by the passage of this electron by an ion at $\underline{r} = 0$ is given by Eqs. (152) and (156),

$$\underline{p}(\omega) = -\frac{i8\pi e^3 z}{(2\pi)^4 m\omega^2} \int \frac{\underline{k}\, e^{-i\underline{k}\cdot\underline{r}_o} \sin(\omega + \underline{k}\cdot\underline{v}_o)T}{k^2 D^{\ell}(\underline{k},\omega)\ (\omega + \underline{k}\cdot\underline{v}_o)}\, d^3k \quad . \tag{157}$$

As Eq. (157) is written, $\underline{r}_o$ is the vector distance of closest approach. Time is measured from this point in the particle orbit.

Now the quantity which enters into the radiation formula (142) is $p(\omega)^2$. By Eq. (157) this is given by

$$|\underline{p}(\omega)|^2 = \frac{e^6 z^2}{4\pi^6 m^2 \omega^4} \iint \frac{\underline{k}\cdot\underline{k}'\, e^{-i(\underline{k}-\underline{k}')\cdot\underline{r}_o} \sin(\omega + \underline{k}\cdot\underline{v}_o)T \sin(\omega + \underline{k}'\cdot\underline{v}_o)T}{k^2 k'^2 D^{\ell}(\underline{k},\omega) D^{\ell *}(\underline{k}',\omega)(\omega + \underline{k}\cdot\underline{v}_o)(\omega + \underline{k}'\cdot\underline{v}_o)}\, d^3k\, d^3k' . \tag{158}$$

To obtain the total radiation we must sum (158) over all encounters taking place in time $-T \leq t \leq T$. The vector $\underline{r}_o$ is always perpendicular to $\underline{v}_o$ so for given v_o it can be expressed in terms of polar coordinates r_o, θ. The number of encounters with particles whose velocity lies in d^3v centered at $\underline{v}$ with r_o between r_o and $r_o + dr_o$ and θ between θ and $\theta + d\theta$ is

$$2Tv f(\underline{v})\, r_o\, dr_o\, d\theta\, d^3v \quad , \tag{159}$$

where $f(\underline{v})$ is the distribution function. Multiplying Eq. (158) by this factor and integrating over r_o, θ and v gives

$$|\underline{p}(\omega)|^2_T = \frac{e^6 z^2 2T}{4\pi^6 m^2 \omega^4} \iiiint\!\!\iint \Big\{ \frac{v_o f(\underline{v}_o) r_o\, \underline{k}\cdot\underline{k}'\, e^{-i(\underline{k}-\underline{k}')\cdot\underline{r}_o} \sin(\omega + \underline{k}\cdot\underline{v}_o)T \sin(\omega + \underline{k}'\cdot\underline{v}_o)T}{k^2 k'^2 D^{\ell}(\underline{k},\omega) D^{\ell *}(\underline{k}',\omega)(\omega + \underline{k}\cdot\underline{v}_o)(\omega + \underline{k}'\cdot\underline{v}_o)} \cdot d^3k\, d^3k'\, dr_o\, d\theta\, d^3v \Big\} \quad . \tag{160}$$

For large T, $\sin\alpha T/\alpha$ becomes $\pi\delta(\alpha)$ and hence the integral over the components of $\underline{k}$ and $\underline{k}'$ parallel to $\underline{v}$ give π^2 times the integranl evaluated at $\underline{k}\cdot\underline{v}_o = \underline{k}'\cdot\underline{v}_o = \omega$. We achieve the same result by replacing the $(\sin\alpha T)/\alpha$ terms by

$$\frac{1}{v_o}\,\delta\left[(\underline{k}-\underline{k}')\cdot\frac{\underline{v}_o}{|v_o|}\right]\,\delta(\underline{k}\cdot\underline{v}_o+\omega) \quad . \tag{161}$$

Now let us consider the integral over r_o and θ. The terms of interest are

$$\int_0^{2\pi}\int_{r_{min}}^{\infty} e^{-i(\underline{k}-\underline{k}')\cdot\underline{r}_o}\, r_o\,dr_o\,d\theta \quad , \tag{162}$$

where r_{min} is the minimum impact parameter. Since $\underline{r}_o$ is perpendicular to $\underline{v}_o$ only the components of $\underline{k}$ and $\underline{k}'$ perpendicular to $\underline{v}_o$ enter. Let $\underline{K}_\perp$ be equal to $\underline{k}_\perp - \underline{k}'_\perp$, where $\perp$ stands for the components perpendicular to $\underline{v}_o$. We may measure θ from the $\underline{K}_\perp$ direction for fixed $\underline{K}$. Equation (162) may thus be written as

$$\int_{r_{min}}^{\infty}\int_0^{2\pi} e^{-K_\perp r_o\cos\theta}\, r_o\,dr_o\,d\theta = \int_{r_{min}}^{\infty} 2\pi J_o(K_\perp r_o) r_o\,dr_o \tag{163}$$

$$\int_{r_{min}}^{\infty} 2\pi J_o(K_\perp r_o) r_o\,dr = \lim_{R\to\infty}\frac{1}{K_\perp^2}\int_{Kr_{min}}^{KR} 2\pi\frac{d}{dx}[x J_1(x)]\,dx$$

$$= \lim_{R\to\infty}\frac{2\pi}{K_\perp}\left[R J_1(K_\perp R) - r_{min} J_1(K_\perp r_{min})\right] \quad . \tag{164}$$

For the time being let us consider the $RJ_1(KR)$ term. As R becomes larger and larger the contribution of this term to the integral (160) is more and more concentrated to the region $K_\perp = 0$. Thus we may replace $K_\perp$ by 0 in all other terms. But this implies

$$\underset{\sim}{k}_\perp = \underset{\sim}{k}'_\perp \quad . \tag{165}$$

Thus this term's contribution to (160) is

$$|\underset{\sim}{p}(\omega)|^2_{TR} = \frac{Te^6 z^2 \lim_{R\to\infty}}{\pi^2 m^2 \omega^4} \int\iiint \frac{f(\underset{\sim}{v})\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_o+\omega)}{k^2|D^\ell(k,\omega)|^2} \frac{RJ_1(K_\perp R)}{K_\perp} d^3k\, K_\perp dK_\perp d\phi\, d^3v \quad , \tag{166}$$

where use has been made of (161) and $d^2k'_\perp$ has been replaced by $d^2K_\perp = K_\perp dK_\perp d\phi$. Now

$$\int_0^\infty \int_0^{2\pi} R\, J_1(K_\perp R) d\phi\, dK_\perp = 2\pi J_o(0) = 2\pi \quad . \tag{167}$$

Thus Eq. (166) becomes

$$|\underset{\sim}{p}(\omega)|^2_{TR} = \frac{e^6 z^2 2T}{\pi^2 m^2 \omega^2} \iint \frac{f(\underset{\sim}{v})\,\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}+\omega)}{k^2|D^\ell(k,\omega)|^2} d^3k\, d^3v \quad . \tag{168}$$

Equation (168) can be integrated over the component of $\underset{\sim}{v}$ perpendicular to $\underset{\sim}{k}$ for fixed $\underset{\sim}{k}$. For isotropic f's we thus obtain

$$|\underset{\sim}{p}(\omega)|^2_{TR} = \frac{8e^6 z^2 T}{\pi m^2 \omega^4} \int \frac{\tilde{f}(\omega/k)dk}{k|D^\ell(k,\omega)|^2} \tag{169}$$

$$\tilde{f}(v) = \iint f(v) d^2v_\perp \quad ,$$

where $\tilde{i}$ is the one-dimensional velocity distribution function.

Returning now to the term

$$2\pi\, r_{min} J_1(K_\perp r_{min}) \tag{170}$$

in Eq. (164). Since r_{min} is the minimum impact parameter (either $2e^2/kT$ or $\hbar/\sqrt{kTm}$), it is generally a very small distance and this term will not contribute except for very large $K_\perp$. ($J_1(x) \propto x$ for small x.) This implies that $\underline{k}_\perp$ and $\underline{k}'_\perp$ will be large where this term is appreciable. Now for large k, $D(k,\omega)$ goes to 1. For evaluating this term we may replace $D(k,\omega)$ by 1. This is equivalent to using the vacuum field for close impacts. For close impacts the shielding is negligible and so this approximation should be good.

We have already found how the unshielded encounters contribute to $p(\omega)^2$. From Eq. (96) by using (77) we find for $\underline{p}(\omega)^2$ for impact parameters less than r_{min} (still using the straight line approximation)

$$|\underline{p}(\omega)|^2_{T r_{min}} = \frac{4e^6 z^2 T}{2\pi m^2 \omega^4} \int \frac{f(v)}{v}\left(\ln \frac{v}{r_{min}\omega} - \ln \frac{v}{\epsilon\omega}\right) d^3v\ . \tag{171}$$

Here ϵ is a distance much smaller than r_{min} which we have introduced artifically to keep the integrand in (170) finite. We must subtract (171) from (169) to obtain $\underline{p}(\omega)^2_T$. Now (171) diverges logarithmically as ϵ goes to zero and we must let it go to zero. However, (169) diverges logarithmically with k_{max} if we let the limits on k (k_{max}) go to infinity. Since we are limiting ourselves to finite impact parameter and hence finite acceleration of the electrons, the radiation should be finite and no such divergence should arise.

We should choose ϵ and the maximum k in such a way that these divergences cancel out. We may obtain the same result by subtracting the divergent terms to begin with. We thus obtain

$$|\underline{p}(\omega)|_T^2 = \frac{8e^6 z^2 T}{\pi m^2 \omega^4} \left\{ \int_0^{k_{max}} \frac{\tilde{f}(\omega/k)dk}{k|D^{\ell}(k,\omega)|^2} - \tilde{f}(0) \ln k_{max} \right.$$

$$\left. - \frac{1}{4} \int \frac{f(v)}{v} \ln(\frac{v}{r_{min}\omega}) d^3v \right\} \quad . \tag{172}$$

Since the divergences of these two terms are logarithmic and very slow, we can obtain essentially the same result by cutting off the k integral at $1/\bar{r}_{min}$, where $\bar{r}_{min}$ is an appropriate minimum impact parameter.

The total radiation is obtained by substituting (172) in (142) and the power spectrum of the radiation is

$$w(\omega) = \frac{2\pi}{3c^3 T} \omega^3 (\omega^2 - \omega_p^2)^{1/2} |\underline{p}(\omega)|_T^2$$

$$= \frac{16e^6 z^2}{3c^3 m^2 \omega^4} \left\{ \int_0^{k_{max}} \frac{\tilde{f}(\omega/k)dk}{k|D^{\ell}(k,\omega)|^2} - \tilde{f}(0) \ln k_{max} \right.$$

$$\left. - \frac{1}{4} \int \frac{f(v)}{v} \ln(\frac{v}{r_{min}\omega}) d^3v \right\} \quad . \tag{173}$$

XVI. Cyclotron and Synchrotron Radiation

We now turn to the problem of the emission of radiation by electron spiralling about magnetic field lines. Since most laboratory plasmas contain magnetic fields, this process can be of some importance. We shall again start by making a simple estimate of the total radiation emitted. Here we shall treat the radiation as if it is emitted in a vacuum. In principle, one can introduce the plasma dielectric properties as was done in the previous section. However, in practice things are much more complicated here because one gets a complicated tensor dielectric function. Our treatment is good at low plasma densities, but should not be used if the plasma frequency is comparable with the frequency being considered (cyclotron or synchrotron frequency). A complete treatment of this radiation has not been given.

XVII. Elementary Estimate of Cyclotron Radiation by a Plasma

We can compute the radiation from an individual electron moving in a magnetic field from the equation for its acceleration and the classical formula for the radiation from accelerated charge.

$$a = \frac{eBv_\perp}{mc} \quad , \tag{174}$$

($v_\perp$ is the velocity perpendicular to the lines of force),

$$P = \frac{2}{3}\frac{e^2a^2}{c^3} \quad . \tag{175}$$

Substituting (174) in (175) gives for the power radiated,

$$P = \frac{2}{3} \frac{e^4 B^2}{m^2 c^5} v_\perp^2 \quad . \tag{176}$$

If we equate the power radiated to the loss of perpendicular energy $mv_\perp^2/2$ then we find

$$\frac{dmv_\perp^2/2}{dt} = -\frac{2}{3} \frac{e^4 B^2}{m^2 c^5} v_\perp^2 \tag{177}$$

$$w_\perp = w_\perp(o)\, e^{-t/\tau} \quad , \tag{178}$$

$$w_\perp = mv_\perp^2/2 \quad , \tag{179}$$

$$\tau = \frac{3}{4} \frac{m^3 c^5}{e^4 B^2} \quad . \tag{180}$$

For B in gauss and τ in sec, Eq. (180) gives

$$\tau = 1.5 \times 10^8/B^2 \quad .$$

These equations hold only for nonrelativistic energies.

Now if the particle is nonrelativistic all the energy is emitted in a small frequency interval about the cyclotron frequency,

$$\nu = \frac{eB}{2\pi mc} \quad . \tag{181}$$

The width of the emitted line depends on a number of broadening effects. First there is radiation broadening. This is the broadening which results from the fact that a finite length wave train cannot have a pure frequency, but has a frequency width of the order $1/\tau$ where τ is time duration of the wave packet. This minimum line width is given by Eq. (180). Second, the line will be broadened because of collisions. If τ_c is the mean free time between collisions then $1/\tau_c$ is the width of the broadening due to this effect. Thirdly, there is a Doppler broadening of the line due to the random motion of the electrons along the field lines. This broadening is of the order of $v_T/c \cos\theta$ where v_T is the thermal velocity of the electrons along the lines of force and θ is the angle made by the direction of propagation of the radiation with the magnetic field lines. Finally, there will be a broadening due to nonuniformities in the magnetic field. In all of this we have assumed that the plasma density is so low that we can neglect plasma effects on the emission and absorption processes.

If we assume that the primary broadening process is the Doppler broadening and that the emission is isotropic* then we can estimate how large the plasma must be to radiate as a black body at the cyclotron frequency. To do this we proceed as earlier and equate the radiation emitted by all the electrons in a sphere of radius R to the black body radiation from the surface of such a sphere in the cyclotron line. From Eq. (1) the radiation emitted into $d\nu$ about ν into solid angle $d\Omega$ from area dA is

$$dP = \frac{2\nu^2 kT}{c^2} \, d\Omega d\nu dA \; ^{**} \tag{182}$$

*The intensity of the radiation emitted and an angle θ to the magnetic field is proportional to $(2 - \sin^2\theta)$.

**This includes both polarizations for the radiation. Actually the cyclotron radiation from the electron will be polarized so that it would probably be more accurate to divide this by 2.

where we have assumed $h\nu << kT$. Substituting $4\pi R^2$, the surface of the sphere for dA; 2π, the solid angle of a hemisphere dor $d\Omega$ and $\nu v/c$, the width of the cyclotron line for $d\nu$, we get for the black body radiation in the cyclotron line

$$P = \frac{16\pi^2 \nu^3 kT}{c^3} vR^2 \quad . \tag{183}$$

From Eq. (176) we obtain for the total radiation emitted by all the electrons in the sphere

$$P = \frac{4}{3}\pi R^3 \frac{2}{3} \frac{e^4 B^2}{m^2 c^5} v_{\perp}^2 n_o \tag{184}$$

where n_o is the density of electrons. Equating (183) and (184) and substituting

$$\nu = \frac{eB}{2\pi mc}$$

gives the radius at which saturation at the black body level takes place,

$$R = \frac{9}{8\pi^2} \frac{B}{e} \frac{v}{c} \frac{1}{n_o} \quad . \tag{185}$$

For B in gauss and R in cm, (185) reads

$$R = 7.6 \times 10^8 \frac{v}{c} \frac{B}{n_o} \quad . \tag{186}$$

For $\frac{v}{c} = 0.02$ which corresponds to a thermal energy of 100 ev, and for $B = 2 \times 10^4$ gauss and $n_o = 10^{12}$ electrons per cm^3, this gives an R of 0.3 cm. The temperature and density and magnetic field are typical of many laboratory plasmas. However, most such plasmas would be many centimeters in diameter and so they would radiate at the cyclotron frequency like a black body. The total cyclotron radiation will be surface area S times the black body rate or, from (182),

$$P = \frac{4\pi\nu^3 kT}{c^3} \; vS \quad . \tag{187}$$

Another interesting thing to compute is the cross section for the absorption of a cyclotron photon by an electron. We can estimate this from (185) if we equate R to the mean free path of a photon. The mean free path λ is given by

$$n_o \sigma \lambda = 1 \tag{188}$$

where σ is the cross section for absorption. Equating λ with R in (185) gives for σ,

$$\sigma = \frac{8\pi^2}{9} \frac{e}{B} \frac{c}{v} \quad . \tag{189}$$

Equation (189) can be written in the form

$$\sigma = \frac{16\pi^2}{27} \frac{c^2}{\omega_c^2} \frac{c}{\tau \omega_c v} \quad . \tag{190}$$

τ is given by Eq. (180). The quantity c^2/ω_c^2 is essentially the square of the wavelength at the cyclotron frequency and is the natural cross section for the scattering of cyclotron radiation by an electron. The term $c/\tau\omega_c v$ is the ratio of the natural line width $1/\tau$ to the Doppler line width $\omega_c v/c$. This factor is essentially the fraction of the particle moving at the proper speed so as to scatter the Doppler sifted radiation.

XVIII. Synchrotron Radiation

We now turn to the synchrotron radiation emitted by a relativistic electron gyrating in a magnetic field. We shall not give a derivation of the formula for this radiation as it is somewhat tedious and it can be found in a number of places.[13,14,15*] Further, we shall be primarily concerned with mildly relativistic electrons (plasma temperatures in the 10's of keV) and we shall give only a rough treatment which, however, shows the essential features of how and why this radiation is important in thermonuclear devices. More accurate treatments are available in the literature.[16,17,18]

The total power radiated by a single relativistic electron gyrating about a magnetic field with no velocity parallel to B is given by[13]

$$P = \frac{2}{3} \frac{e^4 B^2}{m_o^2 c^5} \frac{v_\perp^2}{(1 - v_\perp^2/c^2)} . \tag{191}$$

Here m_o is the rest mass of the electron and $v_\perp$ is the velocity perpendicular to $\underline{B}$. The emissivity per unit solid angle at an angle θ to $\underline{B}$ in the n^{th} harmonic is

$$\epsilon_n(\theta) = \frac{n^2 e^4 B^2}{2\pi m^2 c^3} (1 - \frac{v^2}{c^2}) \left[\cot^2\theta \, J_n^2(\frac{nv}{c} \sin\theta) + \frac{v^2}{c^2} J_n'^2(\frac{nv}{c} \sin\theta) \right] . \tag{192}$$

Since we are interested in mildly relativistic plasmas we shall treat $v/c << 1$ and approximate $J_n(nv/c \sin\theta)$ by

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*One can obtain the formula for the radiation by substituting the current due to the gyrating electron into Eq. (130) with D_T set equal to $\omega^2/k^2c^2 - 1$ and $D_L = 0$.

$$J_n(\frac{nv}{c}\sin\theta) \approx \frac{1}{n!}(\frac{nv}{2c})^n \sin^n\theta \quad . \tag{193}$$

This approximation breaks down for $n \sim (c/v)^2$. Substituting into Eq. (192) gives

$$\epsilon_n(\theta) \cong \frac{e^4B^2}{2\pi m^2c^3}(1-\frac{v^2}{c^2})(\frac{v^2}{c^2})^n \sin^{2(n-1)}\theta \, n^2(\frac{n}{2})^{2n}\frac{1}{(n!)^2} \quad . \tag{194}$$

We see immediately from this formula that for large n the radiation is strongly peaked around $\theta = \pi/2$. If we make use of Stirling's approximation for n! then

$$n! \cong \sqrt{2\pi}\; n^{1/2}(\frac{n}{\epsilon})^n \tag{195}$$

where ϵ is the base of the natural logarithms. Equation (194) thus becomes

$$\epsilon_n(\theta) = \frac{e^4B^2}{(2\pi)^2m^2c^3}(1-\frac{v^2}{c^2})(\frac{v^2}{c^2})^n \, n(\frac{\epsilon}{2})^{2n}\sin^{2(n-1)}\theta \quad . \tag{196}$$

Multiplying (196) by $2\pi\sin\theta\, d\theta$ and integrating over all θ gives the total power emitted in the $n^{\underline{th}}$ harmonic

$$I_n = \frac{e^4B^2}{(2\pi)m^2c^3}(1-\frac{v^2}{c^2})(\frac{v^2}{c^2}\frac{\epsilon}{2})^n n\sqrt{\pi}\,\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n)}$$

$$\approx \frac{e^4B^2}{2\sqrt{\pi}\,m^2c^3}(1-\frac{v^2}{c^2})(\frac{v^2}{c^2}\frac{\epsilon}{2})^n n^{1/2} \quad . \tag{197}$$

A somewhat more accurate formula is given by Landau and Lifshitz (Ref. 13, p. 281), and is

$$I_n = \frac{e^4 B^2}{2\sqrt{\pi}\, m^2 c^3} \left(1 - \frac{v^2}{c^2}\right)^{5/4} \left[\frac{\frac{v}{c}\epsilon^{\sqrt{1 - v^2/c^2}}}{1 + \sqrt{1 - v^2/c^2}}\right]^{2n} . \tag{198}$$

Equation (198) is accurate so long as $n(1-v^2/c^2)^{3/2} >> 1$ and $v/c << 1$. Comparison with Eq. (197) shows that Eq. (197) is good for much larger n than we had a right to expect.

To estimate the total emission in the $n^{\underline{th}}$ harmonic by a plasma we must multiply (197) by the distribution function for perpendicular velocities.* Since we are dealing with weakly relativistic plasmas we shall use the Maxwell distribution

$$f(v_\perp)\, d^2 v_\perp = \frac{kT}{m}\, e^{-mv_\perp^2/2kT}\, v_\perp d^2 v_\perp \quad . \tag{199}$$

To be more correct we should use the relativistic Maxwellian

$$f(v) d^3 v \alpha\, e^{-\frac{m_o c^2}{\sqrt{1-v^2/c^2}\,kT}} \frac{d^3 v}{(1 - v^2/c^2)^{3/2}} \quad . \tag{200}$$

Multiplying (197) by (199) and integrating gives

$$I_n \cong \frac{e^4 B^2}{2\, \pi\, m^2 c^3}\, n^{1/2} \left(1 - \frac{2}{\epsilon} \frac{kT}{m_o c^2}\right)\left(\frac{nkT}{m_o c^2}\right)^n \quad . \tag{201}$$

Equation (201) applies only if $n^2 kT/m_o c^2 < 1$. The velocity of the particles which contribute most to I_n is $\sqrt{2n}\; v_T$ where v_T is the thermal velocity. Particles whose velocities lie within $\pm\, v_T$ of this velocity contribute primarily to I_n. The central frequency is shifted from $n\,\omega_c$ to

*The parallel velocity alters the power radiated slightly and this effect has been neglected here.

$$\langle \omega_n \rangle = n \frac{eB}{m_o c} (1 - \frac{nkT}{m_o c^2})^{1/2} . \qquad (202)$$

and the width if $v_{\parallel}$ is neglected is

$$\langle \Delta\omega_n^2 \rangle = n^2 \frac{e^2 B^2}{m_o^2 c^2} \frac{kT}{m_o c^2} . \qquad (203)$$

parallel motion gives an spread about [illegible]
This must be as well as [illegible]

The motion of particles along the lines of force gives rise to Doppler shifts which results in an additional spread in ω_n for directions of propagation which are not perpendicular to B. However, since for large n the emission is primarily perpendicular to B this effect is small. There is also some second order Doppler shift due to the motion parallel to B due to the fact that moving clocks run slower. This effect primarily shifts the central frequency by the factor $(1 - kT/m_o c^2)^{1/2}$.

XIX. The Importance of Synchrotron Radiation for Thermonuclear Reactors

One can compute the time it would take an electron to radiate away its energy from Eq. (191). This time is roughly given by

$$\tau \cong \tau_o [\frac{(1 - v^2/c^2)^{1/2} - 1 + v^2/c^2}{v^2/c^2}] \qquad (204)$$

where τ_o is the time it takes a nonrelativistic electron to radiate its energy and is given by Eq. (180). For moderately relativistic electrons τ will not differ greatly from τ_o. As we saw earlier, this time is ~~3~~ 1.5 $\times 10^8/B^2$ sec. For a field of 10^4 gauss the electrons will radiate away their energy in 3 sec.

For a thermonuclear reactor this radiation can be significant. For a D-D reactor operating at 30 kev temperatures (roughly the temperature required to obtain a self-sustaining reaction without synchrotron radiation) and a deuteron density of 6×10^{13} cm^{-3} ($[n_e + n_i]\, kT = B^2/8\pi$, $B = 10^4$ gauss) the mean life-time for reaction of a deuteron is 3000 sec.[19] Thus the electrons would radiate away their energy 1000 times over before the deuterons interacted if they radiated according to Eq. (191). The total energy they would radiate would be

$$E_r = 1000 \times \frac{3}{2} N_e kT = N_e \frac{9}{2} \times 10^7 \text{ ev}$$

$$= N_e \, 0.45 \text{ Mev} . \tag{205}$$

whereas the reactions would deliver to the plasma[19]

$$E_{DD} = \frac{N_D}{2} 0.9 \text{ mev} = 0.45\, N_D \text{ mev} \quad . \tag{206}$$

Thus the electrons would radiate 100 times the energy supplied to keep the plasma hot.

For the D-T reaction the reaction rate is roughly 100 times that for D-D. For a 10 kev D-T plasma (roughly the ignition temperature for D-T with no magnetic field) at a deuteron density of $\times 10^{14}$ cm^{-3} ($[n_e + n_D + n_T]\, kT = B^2/8\pi$, $B = 10^4$ gauss, $n_D = n_T$) the mean reaction time for a deuteron is about 50 sec. The electrons could radiate away their energy 17 times according to (191). The total energy radiated is

$$E_r = N_e \frac{3}{2} \times 10^4 \times 17 = N_e \times 0.26 \text{ mev} \tag{207}$$

while the reaction energy delivered to the plasma is

$$E_{DT} = \frac{N_D}{2} \times 3.5 = N_e \times 1.75 \quad . \tag{208}$$

Thus for this case the reactions deliver to the plasma about 8 times the energy the electrons can radiate.

We see from the above that synchrotron radiation will probably not be serious for a D-T reactor. However, for a D-D reactor it may be quite serious as according to (195) it could be 100 times the plasma heating energy. If the particle density is lower than $B^2/8\pi kT$ the synchrotron radiation would be even more serious. Since both the cyclotron radiation and the particle density scale as B^2, changing the strength of the magnetic field does not alter the above conclusions.

Now although (191) shows that the electrons can radiate 100 times the energy liberated in the D-D reaction, much of this is at the cyclotron frequency and will be reabsorbed. According to our elementary nonrelativistic estimate (Eq. 186) the absorption length for cyclotron radiation for the D-D plasma considered above is of the order of 4×10^{-2} cm. If all the radiation were in the fundamental we should only require that the plasma have dimensions of the order of 100 times this length or 4 cm to reduce the cyclotron radiation to an acceptable level. However, because the cyclotron radiation is so large we need have only 1% of it in higher harmonics before these will carry off more energy than is being delivered to the plasma by the D-D reaction.

As we have seen, the radiation emitted in the n^{th} harmonic by a weakly relativistic plasma is given by

$$\varepsilon_n \propto \left(\frac{nkT}{m_o c^2}\right)^n \quad , \tag{209}$$

$$nkT << m_o c^2 \quad .$$

The ratio of the emission by the n^{th} harmonic to that by the fundamental is

$$\frac{\varepsilon_n}{\varepsilon_1} \propto n^n \left(\frac{kT}{m_o c^2}\right)^{n-1} \quad . \tag{210}$$

Equation (210) is valid only if $n\,kT/m_o c^2 < 1$. We see from this that for $kT \cong 30$ kev (the ignition temperature for D-D with no synchrotron radiation) we must go to n = 4 before this ratio becomes less than 10^{-2}. For higher temperatures the critical n rises very rapidly and our approximate formula can no longer be used.

Now if the radiation is to be reduced to an acceptable level the plasma must be optically thick for all harmonics radiating more than 1% of the fundamental.* Since the high harmonics are emitted much more weakly they are also absorbed much less strongly than the fundamental. Further the black body level for the higher harmonics goes like n^3 if we assume the frequency band width is proportional to the frequency (see Eq. 182). Thus they must build up to a higher level. Again we may find the dimensions of the plasma required for it to be black to the n^{th} harmonic by equating the emission to the black body emission. From (210) and the n^3 dependence of the total radiation we find that the thickness

*Actually we must be sure that the radiation by all harmonic beyond the 1% harmonic do not contribute 1% of the fundamental radiation.

L is

$$L \;=\; L_o \left(\frac{m_o c^2}{kT}\right)^{n-1} \frac{1}{n^{n-3}} {}^{*} \tag{211}$$

where L_o is the thickness at which the plasma is black to the fundamental. For the case of the 30 kev plasma already considered, L_o was 4 cm and n was 4. We thus find for L in this case

$$L \;\cong\; 1200 \text{ cm} \quad . \tag{212}$$

This would be qüite a large size for a reactor. If $B^2/8\pi\,(N_e + N_i)kT$ were less than one we should have to make it even much larger ($L \propto 1/B^2$).

This size can be greatly reduced by the introduction of reflectors. If the reflection coefficient is R then the critical length for which the plasma is essentially black is reduced by the factor (1-R). Reflection coefficients of 90% should be easy to obtain and values even as high as 99.9% might be obtained. [The reflectivity of copper to a microwave of frequency 12×10^{10} ($4\nu_{cyclotron}$ for $B = 10^4$) is about 99.9% at 1000° C.] Thus it would appear that we should have no trouble in reducing L to an acceptable size of the order of 100 cm. There are other factors which also help considerably. The 30 kev ignition temperature for D-D does not take into account the burning of the tritium and He_3 produced. If these are burned it increases the energy delivered to the plasma by about 12 and substantially reduces the ignition temperature. On the other hand, the mirror machines which require much higher operating temperatures

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*This formula assumes that the radiation in the n^{th} harmonic has the same angular dependence which it of course does not. The radiation by the high harmonics is strongly concentrated in the plane of the orbit if there is no motion along B. The motion along B spreads out the angle the radiation makes with the normal to B. Also if the magnetic lines are not straight this will also spread out the radiation. We shall therefore use the crude approximation.

because of the particle losses out the mirror may find the synchrotron radiation a serious factor.

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