IV. ONE FLUID MHD

A. Heuristic Considerations

In the preceeding two chapters we have derived the Vlasov equation as the lowest (first order) approximation to the microscopic kinetic equation and, in turn, obtained the two-fluid equations by taking velocity moments of the Vlasov equation and invoking the Maxwellian closure approximation. We also showed how the Vlasov and fluid equations can be heuristically "derived" directly from conservation of particles in the six dimensional phase space and from mass, momentum, and energy conservation, respectively. Here, we reverse the order, using a heuristic approach to indicate what form should be expected for one fluid MHD and then showing that equations of this character can, indeed, be derived from the two fluid MHD formulation of Chapter III with suitable approximations.

For a neutral fluid, with mass density ρ_m, velocity $\underline{u}$ and scalar pressure p, conservation of mass, momentum and energy give equations similar to those obtained for a single species in Chapter III, *i.e.* (3.5), (3.6) and (3.14) with one difference: for a neutral fluid there is, of course, no force corresponding to the $nq\underline{E}$ term of (3.6). However, if the fluid is electrically conducting, then its motion may give rise to an electrical current density, $\underline{j}$, and a consequent force, $\underline{j}\times\underline{B}/c$, in presence of a magnetic field. Thus, we expect equations of the form

$$\partial\rho_m/\partial t + \nabla\cdot\rho_m\underline{u} = 0 \tag{4.1}$$

$$\rho_m(\partial\underline{u}/\partial t + \underline{u}\cdot\nabla\underline{u}) + \nabla p = \underline{j}\times\underline{B}/c \tag{4.2}$$

$$p\rho^{-\gamma} = \text{constant} \tag{4.3}$$

As usual, we adjoin to these dynamic equations the Maxwell equations (for a neutral fluid)

$$\left.\begin{aligned} \nabla\cdot\underline{E} &= \nabla\cdot\underline{B} = 0 \\ \nabla\times\underline{E} &+ \dot{\underline{B}}/c = 0 \\ \nabla\times\underline{B} &= 4\pi c^{-1}\underline{j} + \dot{\underline{E}}/c \end{aligned}\right\} \qquad (4.4)$$

However, we need some additional, "constitutive" equation expressing $\underline{j}$ in terms of $\underline{E}$ and $\underline{B}$, introduced on an ad hoc or phenomenological basis similar to that used for the dielectric constant ε in the electrodynamics of uniform continuous media. The simplest assumption is that of scalar conductivity: in the local rest frame of the fluid we assume $\underline{j}_{RF} = \sigma\underline{E}_{RF}$, and hence in the Lab frame

$$\underline{j} = \underline{j}_{RF} = \sigma(\underline{E} + \underline{u} \times \underline{B}/c) \qquad (4.5)$$

Equations (4.1) through (4.5) constitute the simplest form of conventional, one fluid MHD. In the next section, we show how to derive, from two fluid MHD, a set of equations similar to those obtained here by macroscopic and phenomenological considerations.

B. Derivation from Two Fluid MHD

We start from the equations in the form (3.14), (3.15) and define

$$\rho_m = n_e m + n_i M \qquad (4.6)$$

$$\rho_m \underline{u} = n_e m \underline{u}_e + n_i M \underline{u}_i \qquad (4.7)$$

Our first simplifying approximation is to assume charge neutrality

$$n_e = n_i = n \qquad (4.8)$$

so that

$$\underline{u} = (\underline{u}_i + m\underline{u}_e/M) \qquad (4.9)$$

and $\rho_m = nM$

As usual, we neglect corrections of order m/M. In absence of external sources, we have

$$\underline{j} = en(\underline{u}_i - \underline{u}_e) \tag{4.10}$$

Then (4.9) and (4.10) can be solved for

$$\underline{u}_i = \underline{u} + m\underline{j}/neM \tag{4.11}$$

and

$$\underline{u}_e = \underline{u} - \underline{j}/ne \tag{4.12}$$

We now take linear combinations of the equations (3.14) for the two species. We shall retain the collisional momentum transfer terms, $\underline{P}$, even though we have not derived them in a rigorous way, since only then do we obtain the conventional form of one fluid MHD. We shall assume that only electron-ion collisions are involved and we shall assume a simple "frictional" form for these,

$$P_e = -P_i = nm_e\nu(\underline{u}_e - \underline{u}_i)$$

where ν is called the electron-ion collision frequency. (In a subsequent chapter we shall derive the collisional terms in a rigorous way, discuss the circumstances under which this simple form is valid and show how ν can be computed.) Adding the continuity equations, (3.14), for each species, weighted by the respective masses, gives the expected continuity equation (4.1)

$$\partial\rho_m/\partial t + \nabla\cdot\rho_m\underline{u} = 0 \tag{4.13}$$

or

$$\partial n/\partial t + \nabla\cdot n\underline{u} = 0 \tag{4.14}$$

Similarly, it follows from the energy equations in (3.14) that

$$pn^{-\gamma} = \text{constant} \tag{4.15}$$

where

$$p = p_e + p_i$$

Note that (4.14) and (4.15) are formally the same as the continuity and energy equations in (3.14). However, in the latter case each variable refers to a particular species (the species label having been suppressed for notational convenience), whereas here p denotes total pressure; $\underline{u}$ is defined by (4.7); and $n = n_e = n_i$. There remain only the momentum equations. Taking their sum gives

$$nM\partial\underline{u}/\partial t + n(m_e\underline{u}_e\cdot\nabla u_e + M\underline{u}_i\cdot\nabla\underline{u}_i) + \nabla(p_e + p_e) = \underline{j}\times\underline{B}/c$$

or, using (4.11) and (4.12) to simplify the second term,

$$nM(\partial\underline{u}/\partial t + \underline{u}\cdot\nabla\underline{u}) + \nabla p - \underline{j}\times\underline{B}/c = -(m/e^2)\underline{j}\cdot\nabla(\underline{j}/n) \qquad (4.16)$$

The right side of (4.16) can be neglected, and we recover the conventional form (4.2) if we make the very reasonable assumption that $Mu^2 >> m(j/ne)^2$ i.e., that the kinetic energy of fluid flow is much greater than that associated with current flow. Then (4.15) takes the anticipated form, (4.2).

Finally, taking the difference of the momentum equations for each species, divided by the respective masses, gives

$$\partial\underline{j}/\partial t - e\nabla p_e/m + (e/mc)\underline{j}\times\underline{B} - (ne^2/m)(\underline{E}+\underline{u}\times\underline{B}/c) + \nu\underline{j} = \underline{j}(\nabla\cdot\underline{u} + \underline{u}\cdot\nabla n/n) -$$
$$- \underline{j}\cdot\nabla\underline{u} - ne\underline{u}\cdot\nabla(\underline{j}/ne) + \underline{j}\cdot\nabla(\underline{j}/ne) \qquad (4.17)$$

The approximation conventionally made in deriving the MHD equations is that

$$(j/ne)\tau/L \quad \text{and} \quad u\tau/L << 1 \qquad (4.18)$$

where L is a typical scale length for n and $\underline{u}$, $L \sim u/|\nabla u| \sim n/|\nabla n| \sim j/|\nabla j|$, and τ is the time scale on which $\underline{j}$ varies, $\tau \sim j|\partial\underline{j}/\partial t|^{-1}$. Then the right hand side of (4.17) can be neglected compared to $\partial\underline{j}/\partial t$. [If $\nu\tau >> 1$, we justify the neglect of these terms by replacing τ by ν^{-1} in (4.18).] This gives the "Generalized Ohm's Law",

$$(\underline{E}+\underline{u}\times\underline{B}/c) = (m\nu/ne^2)\underline{j} + (m/ne^2)\partial\underline{j}/\partial t + (M/e)d\underline{u}/dt + \nabla p_i/ne \qquad (4.19)$$

where we have used the momentum equation to eliminate $\underline{j}\times\underline{B}$.

In many cases of physical interest, we can neglect some of the terms on the right side of (4.19). Keeping only the first term gives the Simple Ohm's Law (4.5) with electrical conductivity, σ, and resistivity, $\eta \equiv \sigma^{-1}$, where

$$\sigma = \eta^{-1} \equiv ne^2/m\nu$$

Thus the first term on the right side of (4.19) is associated with ohmic resistivity. The second arises from electron inertia; the third, from ion inertia; and the last from non-zero ion pressure. Neglecting all terms on the right gives what is called the "infinite conductivity" approximation,

$$\underline{E} + \underline{u} \times \underline{B}/c = 0 \tag{4.20}$$

In this particularly simple limit, the one fluid MHD equations become reasonably tractable and their consequences have been explored in considerable detail.

At present, a large body of MHD literature, including many developments of considerable mathematical elegance, exists. As can be seen from the block diagram in Fig. 2.1, one fluid MHD is quite far "down the line" of approximations, but it forms an indispensable guide to the physics of complex magnetic geometries, like those found in toroidal controlled fusion experiments (e.g. the Tokamak) or in astrophysical and geophysical problems (e.g., the solar wind and the earth's magnetosphere). In section D, we mention a few simple consequences of the general MHD equations, (4.1) through (4.4) plus the generalized Ohm's law (4.19). However, we first discuss some of the physics they contain by the same device used in the previous chapter, the linearization about simple equilibrium solutions.

C. Linearized Waves

The one-fluid MHD equations, as derived above, are

$$\left.\begin{aligned} &\partial n/\partial t + \nabla\cdot n\underline{u} = 0 \\ &nM(\partial\underline{u}/\partial t+\underline{u}\cdot\nabla\underline{u}) + \nabla p = \underline{j}\times\underline{B}/c \\ &\quad pn^{-\gamma} = \text{constant} \end{aligned}\right\}$$

$$\left.\begin{aligned} \nabla\cdot\underline{B} &= \nabla\cdot\underline{E} = 0 \\ \nabla\times\underline{E} &+ \dot{\underline{B}}/c = 0 \\ \nabla\times\underline{B} &= 4\pi c^{-1}\underline{j}+\dot{\underline{E}}/c \\ \underline{E}+\underline{u}\times\underline{B}/c &= \eta\underline{j} + (m/ne^2)\dot{\underline{j}} + (M/e)\dot{\underline{u}} + \nabla p_i/ne \end{aligned}\right\} \tag{4.21}$$

The simplest non-trivial solution about which we can linearize is the uniform time-independent solution

$$n = n_0 \qquad \underline{B} = \underline{B}_0$$

We shall use the technique of simple-minded plane wave substitution (cf. Chapter III, section D) to find the dispersion equation for this system; from our discussion in Chapter III, we know how to extend the results to physically posed problems with external sources or specified initial values. Thus, we set

$$\begin{aligned} n &= n_0 + n_1 \exp[i(\underline{k}\cdot\underline{x}-\omega t)] \\ \underline{u} &= \underline{u} \exp[i(\underline{k}\cdot\underline{x}-\omega t)] \\ &\text{etc.} \end{aligned} \tag{4.22}$$

obtaining

$$-\omega n_1 + n_0\underline{k}\cdot\underline{u} = 0 \tag{4.23}$$

$$-n_0 M\omega\underline{u} + M\underline{k}c_s^2 n_1 = -i\underline{j} \times \underline{B}_0/c \tag{4.24}$$

$$\underline{k}\cdot\underline{B}_1 = \underline{k}\cdot\underline{E} = 0 \tag{4.25}$$

$$\underline{k}\times\underline{E} = \omega\underline{B}_1/c \tag{4.26}$$

$$\underline{k}\times\underline{B}_1 = -i4\pi c^{-1}\underline{j} - \omega\underline{E}/c \tag{4.27}$$

$$\underline{E} + \underline{u}\times\underline{B}_0/c = \eta\underline{j} + i[\underline{k}p_i/n_0 e - \omega(M/e)\underline{u} - \omega(m/ne^2)\underline{j}] \tag{4.28}$$

Before examining the general solution of this set, we consider the simplest possible case: waves propagating along the magnetic field $\underline{B}_0$, in the "infinite conductivity" approximation, with the displacement current neglected. For the transverse component of $\underline{u}$, i.e. the one perpendicular to $\underline{k}$ and $\underline{B}_0$, we have from the momentum equation, (4.24)

$$\underline{u}_t = i\underline{j} \times \underline{B}_0/n_0 M\omega c$$

Using Ampere's law, (4.27) and Faraday's law, (4.26) we have

$$\underline{u}_t = -(\underline{k}\times\underline{B}_1) \times \underline{B}_0/4\pi n_0 M\omega = -kB_0\underline{B}_1/4\pi n_0 M\omega = -kB_0 c(\underline{k}\times\underline{E})/4\pi n_0 M\omega^2$$

Finally, the infinite conductivity form of Ohm's law gives

$$\underline{u}_t = kB_0\underline{k} \times (\underline{u}\times\underline{B}_0)/4\pi n_0 M\omega^2 = (kc_A/\omega)^2\underline{u}_t$$

where

$$c_A = (B_0^{\,2}/4\pi n_0 M)^{1/2}$$

is the Alfven speed. The existence of transverse waves ($\underline{E}$ and $\underline{B}_1$, as well as $\underline{u}$ being normal to $\underline{B}_0$) was first pointed out by Hannes Alfven and these Alfven waves constitute one of the principal new features of MHD. In Section D we shall discuss one of the common physical pictures associated with these waves.

We now return to the full set of linearized MHD equations, (4.23) through (4.28). It is easy to see that when $\omega/kc << 1$, we can neglect the displacement current in (4.27): from (4.25), (4.26) and (4.27) we have

$$-i4\pi\underline{j} = \underline{k}\times(\underline{k}\times\underline{E})c^2/\omega + \omega\underline{E} = (\omega^2-k^2c^2)\underline{E}/\omega \approx -k^2c^2\underline{E}/\omega$$

We can then substitute (4.23) and (4.27) into (4.24), obtaining an equation for $\underline{u}$ in terms of $\underline{B}_1$:

$$\omega^2\underline{u} - \underline{k}c_s^{\,2}\underline{k}\cdot\underline{u} = i\omega\underline{j} \times \underline{B}_0/n_0 Mc = -\omega(\underline{k}\times\underline{B}_1) \times \underline{B}_0/n_0 M4\pi \qquad (4.29)$$

Substituting (4.27) and (4.28) into (4.26) gives an equation for $\underline{B}_1$ in terms of $\underline{u}$:

$$\underline{B}_1 = -\underline{k}\times[(\underline{u}\times\underline{b})/\omega + i\underline{u}/\Omega_i]B_0/Q \tag{4.30}$$

where we have introduced the ion cyclotron frequency

$$\Omega_i = eB_0/Mc \quad ;$$

the quantity

$$Q \equiv 1 + (kc/\omega_p)^2(1 + i\nu/\omega) \quad ;$$

and a unit vector, $\underline{b} = \underline{B}_0/B_0$. Of the various terms in the generalized Ohm's law, the ion inertia is represented in (4.30) by the $\underline{u}/\Omega_i$ term whose contribution will be small if $\omega/\Omega_i << 1$. The electron inertia effects appear in the $(kc/\omega_p)^2$ term of Q and they will become significant only for wavelengths comparable to the collisionless skin depth, c/ω_{pe}. Finally, the finite conductivity effects are contained in the $i\nu/\omega$ term in Q. It is convenient to introduce a modified Alfven velocity,

$$\tilde{c}_A = c_A Q^{-1/2}$$

which reduces to c_A in the limit of small k. Then substituting (4.30) into (4.29) gives an equation involving $\underline{u}$ alone of the form

$$\boxed{\begin{aligned}&[\omega^2-(\underline{k}\cdot\underline{\tilde{c}}_A)^2]\underline{u}= \underline{k}[(\underline{k}\cdot\underline{u})(c_s{}^2+\tilde{c}_A{}^2) - \underline{k}\cdot\underline{\tilde{c}}_A\underline{u}\cdot\underline{\tilde{c}}_A - \\ &-i(\omega/\Omega_i)\tilde{c}_A\underline{k}\times\underline{u}\cdot\underline{\tilde{c}}_A] - \underline{k}\cdot\underline{\tilde{c}}_A[\underline{k}\cdot\underline{u}\underline{\tilde{c}}_A - (i\omega/\Omega_i)\underline{k}\times\underline{u}\underline{\tilde{c}}_A]\end{aligned}} \tag{4.31}$$

This is a very useful dispersion equation, which has been derived by several authors. It's properties have been most thoroughly explored in an article by Formisano and Kennel (J. of Plasma Physics $\underline{3}$, 55 (1969)). We shall restrict ourselves here to the "classical" case, (infinite conductivity, long wavelength, low frequency), where the ω/Ω_i term can be dropped and $Q \to 1$, $\tilde{c}_A \to c_A$. (In a subsequent chapter, we consider the corrections associated with larger k,ω.) For this case, we set

$$V = \omega/k \qquad \hat{\underline{k}} = \underline{k}/k$$

$$x_1 = \hat{\underline{k}}\cdot\underline{u} \qquad x_2 = \underline{b}\cdot\underline{u}$$

Then (4.31) reduces to

$$(V^2 - c_A^2\cos^2\theta)\underline{u} = \hat{\underline{k}}[x_1(c_s^2 + c_A^2) - x_2 c_A^2\cos\theta] - c_A^2\cos\theta\, x_1\underline{b} \tag{4.32}$$

where θ is the angle between $\underline{k}$ and $\underline{B}_0$. The dot product of (4.32) with $\hat{\underline{k}}$ and with $\underline{b}$ gives

$$\begin{aligned}(V^2-c_s^2-c_A^2)x_1 + c_A^2\cos\theta\, x_2 &= 0\\ c_s^2\cos\theta\, x_1 - V^2 x_2 &= 0\end{aligned} \tag{4.33}$$

and hence a dispersion equation

$$F(V) = V^4 - V^2(c_s^2+c_A^2) + c_s^2 c_A^2\cos^2\theta = 0 \tag{4.34}$$

as the condition that the components of $\underline{u}$ along $\underline{k}$ and $\underline{B}_0$ cannot both be zero. If they are, (4.32) can still be satisfied with a $\underline{u}$ orthogonal to both $\underline{k}$ and $\underline{B}_0$ provided

$$V^2 = c_A^2\cos^2\theta \tag{4.35}$$

The solution of (4.32) gives

$$V^2 = (1/2)[c_s^2+c_A^2 \pm \{(c_s^2+c_A^2)^2 - 4c_s^2 c_A^2\cos^2\theta\}^{1/2}] \tag{4.36}$$

the corresponding modes being called <u>fast waves</u> (+ sign) or <u>slow waves</u> (- sign). Discussion of their properties is conveniently divided into two cases:

1) <u>$c_s \leq c_A$; low β</u>

We note that

$$c_s^2/c_A^2 = \frac{4\pi n(\gamma_i T_i + \gamma_e T_e)}{B^2} = (\gamma_i\beta_i + \gamma_e\beta_e)/2$$

where the quantity

$$\beta = nT/(B^2/8\pi) \quad ,$$

the ratio of kinetic to magnetic pressure, is one of the basic MHD parameters. Thus, this case is referred to as "low β". For $\theta = 0$, (4.34) gives

$$V^2 = (1/2)[c_s^2+c_A^2 \pm (c_A^2-c_s^2)]$$

so the fast wave has velocity c_A, the slow wave velocity c_s. For $\theta = 90°$, the fast wave has

$$V^2 = (c_s^2+c_A^2) \quad ;$$

this speed is called the magneto-sonic speed, since the restoring forces are partially magnetic pressure $(B^2/8\pi)$ and partially acoustic pressure (nT). The slow wave has $V = 0$. For other values of θ, the solutions (4.34) are best represented graphically, by a polar plot of V vs. θ as shown in Fig. 4.1a. Since

$$(c_s^2+c_A^2 - 4c_s^2c_A^2\cos^2\theta) > (c_s^2-c_A^2)^2 \tag{4.37}$$

if follows from (4.36) that, for $c_s < c_A$,

$$V_s < c_s < c_A < V_f \tag{4.38}$$

a property obvious from Fig. 4.1.We have also shown in Fig. 4.1a the solution (4.33), which is called the <u>intermediate wave</u>; denoting the speeds of the three solutions by V_s, V_i and V_f we find that the relation

$$V_s \le V_i \le V_f \tag{4.39}$$

holds at all θ. To prove this, we substitute $V = V_i = c_A \cos\theta$ into the function $F(V)$ defined by (4.34). Since V_s and V_f are roots of $F(V) = 0$, we have

$$F(V_i) = (V_i - V_s)\ (V_i - V_f) =$$

$$= c_A{}^4 \cos^4\theta - c_A{}^2 \cos^2\theta\ (c_s{}^2 + c_A{}^2) + c_s{}^2\ c_A{}^2 \cos^2\theta$$

$$= -c_A{}^4 \cos^2\theta \sin^2\theta < 0 \qquad (4.40)$$

It follows that either

$$V_s < V_i < V_f$$

or (4.41)

$$V_f < V_i < V_s$$

since the latter contradicts the property $V_s < V_f$, as required by (4.38) the former relation, (4.39) must hold.

2) $\underline{c_s \geq c_A}$; high β

The polar plot for this case is shown in Fig.4.1b. Since (4.36) is symmetric under the interchange $c_s \to c_A$, the arguments of the preceeding paragraph show, mutatis mutandis, that in this case

$$V_s < c_A < c_s < V_f \tag{4.42}$$

and that (4.39) is still valid.

D. Elementary General Properties of the MHD Equations

In the approximation where only the scalar ohmic contribution to (4.19) is retained, we have from (4.4) and (4.5)

$$\partial \underline{B}/\partial t = -c\nabla \times (\underline{j}/\sigma - \underline{u}\times\underline{B}/c) = (c^2/4\pi\sigma)\nabla^2\underline{B} + \nabla\times(\underline{u}\times\underline{B}) \tag{4.43}$$

if we neglect the displacement current. In absence of flow, we have a classical diffusion equation with diffusion coefficient

$$D = (c^2/4\pi\sigma) \tag{4.44}$$

An initial magnetic field, and the currents which support it, will decay due to Ohmic dissipation, on a time scale

$$\tau = L^2/D$$

where L is a typical scale length.

The relative magnitude of the two terms in (4.43) will be of order

$$R_m \equiv 4\pi\sigma u L/c^2 \tag{4.45}$$

which, by analogy with the Reynolds number encountered in viscous flow of a neutral gas, is called the magnetic Reynolds number. When $R_m \ll 1$, diffusion dominates. When $R_m \gg 1$, then (4.43) can be approximated by

$$\partial \underline{B}/\partial t = \nabla x(\underline{u}\times\underline{B}) \tag{4.46}$$

the equation of "frozen-in magnetic flux". Consider an arbitrary closed curve, C, moving with the fluid. Then the rate of change of magnetic flux, ϕ, through any surface, S, bounded by C is

$$d\phi/dt = (d/dt)\iint_S \underline{B}\cdot d\underline{\sigma} = \iint_S d\underline{\sigma} \cdot (\partial\underline{B}/\partial t) + \int_C \underline{B} \cdot \underline{u}\times d\underline{s}$$

where the last term takes into account the motion of C. Stoke's law gives then

$$d\phi/dt = \iint_S d\underline{\sigma} \cdot [\partial\underline{B}/\partial t + \nabla x(\underline{B}\times\underline{u})]$$

which, according to (4.46), vanishes. The physical reason is clear: motion of the fluid induces electric fields, and the resultant currents, in the infinite conductivity limit, generate a magnetic field just sufficient to keep ϕ constant as C moves. If we represent the magnetic field by lines of magnetic flux, then the picture of magnetic field lines moving with ("frozen into") the fluid is certainly consistent with $d\phi/dt = 0$, and can be useful in heuristic discussions of complex MHD phenomena.

As a simple example of the use of the frozen-in concept, we give a heuristic derivation of the Alfven speed for waves propagating along $\underline{B}_0$. The Lorentz force term, $\underline{j}\times\underline{B}/c$, can, with neglect of displacement current, be written as

$$\underline{j}\times\underline{B}/c = (\nabla\times\underline{B})x\underline{B}/4\pi = \nabla\cdot(\underline{B}\ \underline{B}/4\pi - B^2\underset{\sim}{1}/8\pi) \quad ,$$

i.e., the magnetic portion of the Maxwell stress tensor is

$$\underset{\sim}{S} = \underline{B}\ \underline{B}/4\pi - (B^2/8\pi)\ \underset{\sim}{1},$$

Corresponding to an isotropic magnetic pressure, $B^2/8\pi$, and a tension along the lines, of magnitude $B^2/4\pi$ per unit area. If we imagine the field lines frozen to the fluid, and vice versa, then we have field lines with tension $T = B^2/4\pi$ and linear density $\rho_m = nM$, both per unit area. The elementary formula for the velocity of transverse wave propagation along a stretched string, $v_s = (T/\rho_m)^{1/2}$, then gives just $v_s = c_A$ for disturbances which ripple the magnetic field lines, i.e. have $\underline{B}_1$ and $\underline{u}$ normal to $\underline{B}_0$. Likewise, for the magneto-sonic waves, propagating across B, the elementary formula for acoustic waves in a gas, $v_s = \sqrt{\gamma p/\rho} = c_s$ goes over to the magnetosonic velocity, $(c_s{}^2+c_A{}^2)^{1/2}$ when we add to γp the magnetic contribution, $\gamma_m p_m = 2(B^2/8\pi)$, the set of field lines behaving like a two dimensional (hence $\gamma_m = 2$) gas.