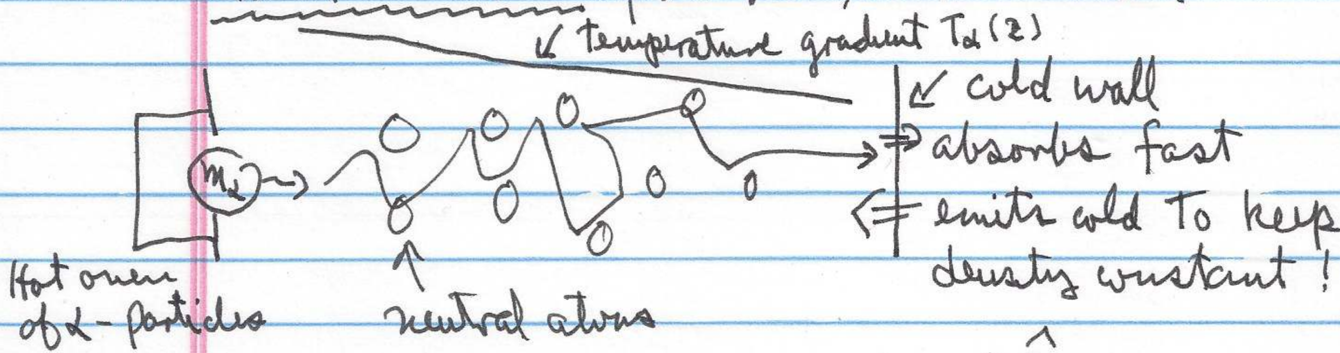
Lect #1 $\boxed{\text{Heat conduction}}$ from Relaxation Model



assume $L_T^{-1} \equiv \left| \frac{d}{dz} \ln T_\alpha \right|$ is such $L_T >> l_{mfp}$

$\Rightarrow$ a lot of collisions before collected by wall

$\Rightarrow$ no ballistic Transport!

Krook Relaxation Model with Maxwellian Reference function

$$f_{\alpha 0} = \frac{n_{\alpha 0}}{\left[2\pi \frac{T_\alpha(z)}{m_\alpha}\right]^{3/2}} \exp\left[-\frac{v^2}{2\frac{T_\alpha(z)}{m_\alpha}}\right]$$ || symmetric has no heat flux!

with $\frac{\partial}{\partial z} n_{\alpha 0} = 0 \Rightarrow$ no density gradient -- pure Thermal Transport

$$\underbrace{\cancel{\frac{\partial}{\partial t}}}_{0 \text{ by S.S.}} f_\alpha + \underline{v} \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} \underbrace{\cancel{\underline{E}}}_{\text{by assumption}}{}^{0} \cdot \frac{\partial}{\partial \underline{v}} f_\alpha = -\nu_\alpha (f_\alpha - f_{\alpha 0})$$

Small deviation from Thermal equilibrium:

$$f_\alpha = f_{\alpha 0} + \delta f_\alpha \quad \text{with} \quad |\delta f_\alpha| << f_{\alpha 0}$$

$$\Rightarrow \quad \delta f_\alpha = -\frac{1}{\nu_\alpha} \underline{v} \cdot \nabla f_{\alpha 0}$$

Need Heat Flux (a 222B problem) $\quad \underline{Q}_\alpha \equiv \int d^3v \, \frac{m v^2}{2} \underline{v} \, \delta f_\alpha$

Lect #1 (cont.) but ∇ is only along $\hat{z}$

$$\Rightarrow \quad \underline{Q}_\alpha = -\frac{m_\alpha}{2}\int \frac{d^3v}{\nu_\alpha}\, \underline{v}\, v^2 v_z \frac{\partial}{\partial z} f_{\alpha 0}$$

$$= -\frac{m_\alpha}{2}\frac{\partial}{\partial z}\left[\int \frac{d^3v}{\nu_\alpha} \frac{(v_x\hat{x}+v_y\hat{y}+v_z\hat{z})\, v^2 v_z \left[\exp\left(-\frac{v^2}{2\bar{v}_\alpha^2(z)}\right)\right]}{(2\pi \bar{v}_\alpha^2(z))^{3/2}}\right] n_{\alpha 0}$$

by symmetry only z-survives

$$\underline{Q}_\alpha = -\frac{m_\alpha}{2} n_{\alpha 0}\frac{\partial}{\partial z}\left\{\int \frac{d^3v}{\nu_\alpha} \frac{(v_x^2+v_y^2+v_z^2)\, v_z^2\, \hat{z}\, \exp\left[-\frac{v^2}{2\bar{v}_\alpha^2}\right]}{(2\pi \bar{v}_\alpha^2)^{3/2}}\right\}$$

scale $u_j = \frac{v_j}{\bar{v}_\alpha}$, $\bar{v}_\alpha^4 = \frac{T_\alpha^2}{m_\alpha}$

$$\underline{Q}_\alpha = -\frac{m_\alpha}{2} n_{\alpha 0}\, \hat{z}\frac{\partial}{\partial z}\left\{\left(\frac{T_\alpha(z)}{m_\alpha}\right)^2 \int \frac{d^3u\, (u_x^2+u_y^2+u_z^2)\, u_z^2\, e^{-u^2/2}}{(2\pi)^{3/2}\,\nu_\alpha}\right\}$$

or,
$$\underline{Q}_\alpha = -\frac{n_{\alpha 0}}{2m_\alpha}\, \hat{z}\frac{\partial}{\partial z}\left\{T_\alpha^2(z)\left\langle \frac{1}{\nu_\alpha}\right\rangle\right\}$$

with
$$\left\langle \frac{1}{\nu_\alpha}\right\rangle \equiv \int \frac{d^3u}{(2\pi)^{3/2}} \frac{(u_x^2+u_y^2+u_z^2)\, u_z}{\nu_\alpha} e^{-u^2/2}$$

For intuition - say ν_α is a "constant"

$$\left\langle \frac{1}{\nu_\alpha}\right\rangle \to \frac{1}{\nu_\alpha}\left[\frac{1}{(2\pi)^{3/2}}\int d^3u\, u_z^4\, e^{-u^2/2} + 2\right]$$

From $\int d^2u\, (u_x^2+u_y^2)\, e^{-(u_x^2+u_y^2)/2} \int du_z\, u_z^2\, e^{-u_z^2/2}$

but by parts,
$$\int_{-\infty}^{\infty} \frac{du_z\, u_z^4\, e^{-u_z^2/2}}{(2\pi)^{1/2}} = 3$$

Lect #1 (cont--) $\langle \frac{1}{\nu_\alpha} \rangle \to \frac{1}{\nu_\alpha} [\underbrace{3+2}_{}]$

from 3D geometry is 5

$$\underline{Q}_\alpha \to -\nabla \left\{ \frac{n_{\alpha 0}}{2 m_\alpha} T_\alpha^2 \frac{5}{\nu_\alpha} \right\}$$

Careful how Thermal conductivity K_α is defined! inside or outside ??

If one says $\underline{Q}_\alpha = -K_\alpha \nabla T_\alpha$ [Fick's Law]

for constant $\nu_\alpha \Rightarrow K_\alpha = \frac{5 n_{\alpha 0} T_\alpha}{m_\alpha \nu_\alpha}$

but the single species Diffusion coefficient is $D_\alpha = \frac{\bar{v}_\alpha^2}{\nu_\alpha}$

$$\Rightarrow \boxed{K_\alpha = 5 n_{\alpha 0} D_\alpha}$$

from Higher degrees of freedom!

ExTrapolate The scaling to different collision processes.

For neutral collisions $\frac{1}{\nu_\alpha} \sim \frac{1}{\bar{v}_\alpha} \sim \frac{1}{T_\alpha^{1/2}}$

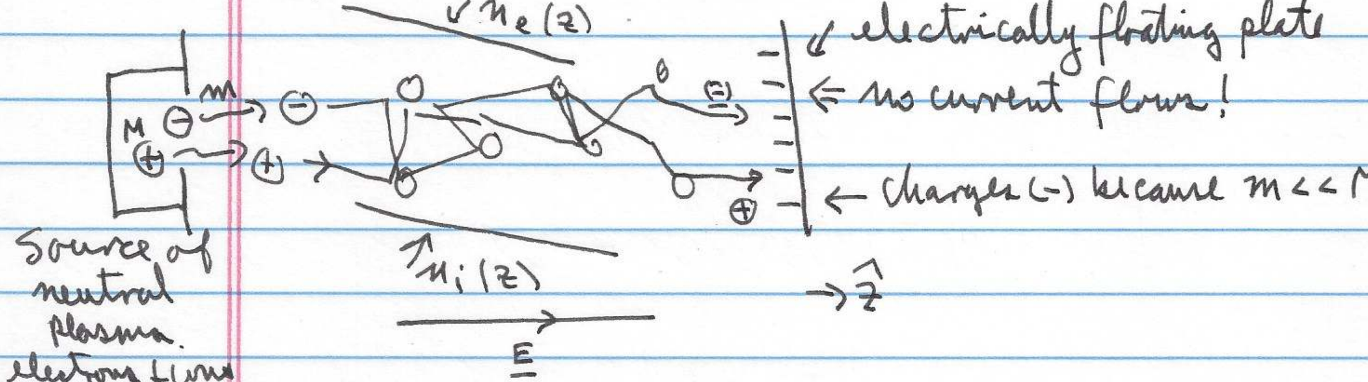
$$\Rightarrow [Q_\alpha]_{neutral} \sim \nabla \left(\frac{T_\alpha^2}{T_\alpha^{1/2}} \right) = \nabla \left(T_\alpha^{3/2} \right) = \frac{3}{2} T_\alpha^{1/2} \nabla T_\alpha$$

$$\Rightarrow [K_\alpha]_{neutral} \sim T_\alpha^{1/2}$$

For Coulomb collisions $\frac{1}{\nu_\alpha} \sim T_\alpha^{3/2} \Rightarrow [Q_\alpha]_{Coul} \sim \nabla (T_\alpha^{7/2})$

$$\Rightarrow [Q_\alpha]_{Coul} \sim \frac{7}{2} T_\alpha^{5/2} \nabla T_\alpha \Rightarrow [K_\alpha]_{Coul} \sim \frac{7}{2} T_\alpha^{5/2}$$

Lect #1 (cont.) **Ambipolar Diffusion** - The real process in a plasma -
- Assume constant Temperature - but now allow for self-consistent $\underline{E}$ - field



Now there is charge flow + mass flow induced by $\underline{E}$ field

$$(\underline{J}_\alpha)_{Total} = \underbrace{-D_\alpha \nabla n_\alpha}_{\text{single species diffusion}} + \underbrace{\mu_\alpha \underline{E}}_{\text{Mobility}} \quad ; \quad \alpha = \begin{cases} e & \text{for electrons} \\ i & \text{for ions} \end{cases}$$

Mass current density of species α

single species diffusion - we know! $D_\alpha = \frac{\bar{v}_\alpha^2}{\nu_\alpha}$

Algebra is simpler for singly charged ions $q = +e \to$ do that

Total current density: $\underline{j}_{Total} = q\underline{J}_i - e\underline{J}_e$

Steady-State $\Rightarrow$ $\underline{j}_{Total} = 0 \Rightarrow q\underline{J}_i = e\underline{J}_e$

+ for $q = e \Rightarrow \boxed{\underline{J}_i = \underline{J}_e}$ $\Rightarrow n_e \simeq n_i \equiv n$

Global charge neutrality: $L_N >> \lambda_D$ with $L_N^{-1} \equiv \left|\frac{d}{dz}\ln n\right|$

In 1-D

$$\underline{J}_e = -D_e \nabla n + \mu_e \underline{E}$$

$$\underline{J}_i = -D_i \nabla n + \mu_i \underline{E}$$

in Figure above

Lect #1 (cont.) Subtract: $\underline{J}_e - \underline{J}_i = 0 = -(D_e - D_i)\nabla n + (\mu_e - \mu_i)\underline{E}$

Solve for the unknown self-consistent electric field $\boxed{\underline{E} = \dfrac{D_e - D_i}{\mu_e - \mu_i}\nabla n}$

+ now determine the fluxes (say electrons)

$$\underline{J} = \underline{J}_e = -D_e \nabla n + \mu_e \frac{(D_e - D_i)}{\mu_e - \mu_i}\nabla n = \frac{-D_e(\mu_e - \mu_i) + \mu_e(D_e - D_i)}{\mu_e - \mu_i}\nabla n$$

$$\underline{J} = \frac{-\cancel{D_e\mu_e} + D_e\mu_i + \cancel{\mu_e D_e} - \mu_e D_i}{\mu_e - \mu_i}\nabla n = -\left[\frac{\mu_i D_e - \mu_e D_i}{\mu_i - \mu_e}\right]\nabla n$$

Define the ambipolar Diffusion coefficient $\underline{J} = -D_A \nabla n$

from which $\boxed{D_A = \dfrac{\mu_i D_e - \mu_e D_i}{\mu_i - \mu_e}}$ $\Leftarrow$ is a collective response both e + i determine it!

Find connection to parameters: $D_e = \dfrac{\bar{v}_e^2}{\nu_e}$, $D_i = \dfrac{\bar{v}_i^2}{\nu_i}$] single species

$$\mu_e = -\frac{\omega_{pe}^2}{e 4\pi \nu_e}, \quad \mu_i = +\frac{\omega_{pi}^2}{e 4\pi \nu_i}$$

$$\Rightarrow \mu_i - \mu_e = \frac{1}{4\pi e}\left[\frac{\omega_{pi}^2}{\nu_i} + \frac{\omega_{pe}^2}{\nu_e}\right] = ne\left[\frac{1}{M\nu_i} + \frac{1}{m\nu_e}\right]$$

$$+ \quad \mu_i D_e = \frac{\omega_{pi}^2}{4\pi e \nu_i}\frac{\bar{v}_e^2}{\nu_e}; \quad \mu_e D_i = \frac{\omega_{pe}^2}{4\pi e \nu_e}\frac{\bar{v}_i^2}{\nu_i}$$

$$\Rightarrow \mu_i D_e - \mu_e D_i = \frac{ne}{\nu_e \nu_i}\left[\frac{T_e}{Mm} + \frac{T_i}{Mm}\right] = \frac{ne}{\nu_e \nu_i Mm}\left[T_e + T_i\right]$$

Finally, $D_A = \dfrac{\dfrac{ne}{\nu_e\nu_i}\dfrac{1}{Mm}\left[T_e + T_i\right]}{\dfrac{ne}{Mm\nu_e\nu_i}\left[m\nu_e + M\nu_i\right]} = \boxed{\dfrac{T_e + T_i}{m\nu_e + M\nu_i}}$

Lect#1 (cont--) For a plasma in Thermal equilibrium $T_e = T_i$

$$\Rightarrow \quad D_A = 2\,\frac{T}{m\nu_e + M\nu_i} \qquad + \text{ since } M >> m$$

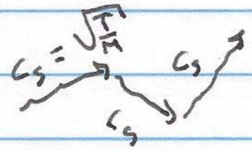
$$D_A \to 2\,\frac{T}{M\nu_i} = 2\,\frac{\bar{v}_i^2}{\nu_i} = 2D_i$$ ← The single species diffus.

↑ The role of self-consistent electric field!

-- Physics behind process

- Collective Diffusion is controlled by heavy species -- but it is faster!
  - at Thermal equilibrium by a factor of 2 -

The process can be viewed as a random scattering of IAW



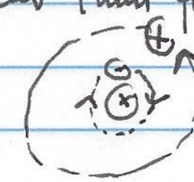
$\Delta r = c_s \Delta t \quad , \quad \Delta t = \frac{1}{\nu_i}$

$$\frac{(\Delta r)^2}{\Delta t} = \frac{c_s^2}{\nu_i}$$

(*) Across a magnetic field The roles are reversed!

Since Larmor radius of electrons is smaller than for ions

- The electrons are The "massive species"



(**) Start Discussion of magnetic field effects!

First topic is a Transition from unmagnetized to ⇒ magnetized

A Bizarre effect: "The Weibel-Fried Instability"

-- papers were sent earlier -

Lect #1 (cont.) Very unusual + surprising effects – presently is considered in many different areas -- because it generates spontaneous magnetic fields out of non-thermal equilibrium!

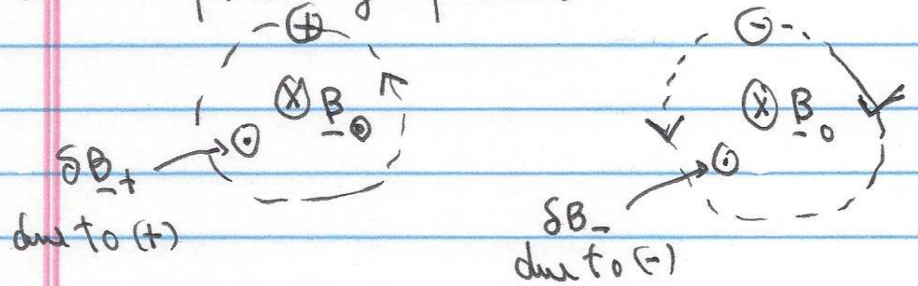
Some applications are: • Origin of B-fields in universe

- Asymmetries in scattering in Quark-gluon plasmas
- Formation of jet structures in ↗

The surprise arises from theorems derived from Statistical Mechanics known as Ms Van Leeuwen's Theorem + also from Niels Bohr Ph Thesis.

– There are no permanent magnetic fields generated by matter in thermal equilibrium -- i.e, matter is intrinsically Diamagnetic!

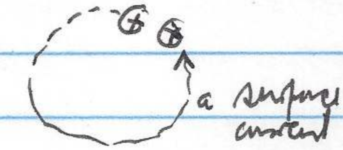
Can be seen from single particle orbits:



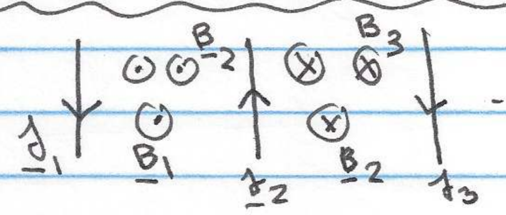
Magnetized particles cancel the field that makes them rotate

⇒ Can not progressively increase or sustain a magnetic field by particle orbits!

But could generate a field from Boundary effects



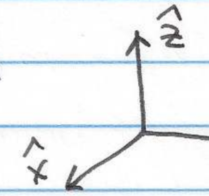
What is needed to generate a net magnetic field?



A system of organized currents that add coherently

Lect #1 (cont.) In thermal equilibrium the currents cannot organize because the entropy is maximum – the organization of the current system lowers the entropy -- Thus we need to consider a "non-thermal equilibrium plasma" + to generate B-fields need to consider E+M waves $\Rightarrow$ Return to a HW problem in 222B – E+M wave in a Kinetic plasma!

Lect #2. Geometry



$\underline{B}_0 = 0 \Leftarrow$ no zero order but $\delta\underline{B}(\underline{r},t)$ emerges spontaneously!

Collisionless – Hot plasma – keep only electrons – dominate current!

Vlasov Eq: $\frac{\partial}{\partial t} f + \underline{v}\cdot\nabla f - \frac{e}{m}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)\cdot\frac{\partial}{\partial \underline{v}} f_0 = 0$

↑ Linearize at first because $\underline{B}=0$

Pure E+M process $\Rightarrow$ no electrostatic phenomena $\nabla\cdot(\delta\underline{E})=0$

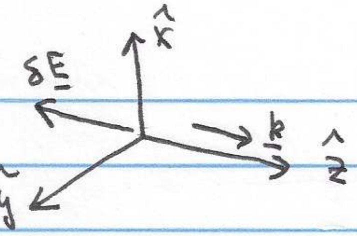
Assume phenomena is all transverse $\Rightarrow$ $\nabla\cdot(\delta\underline{J})=0$

In Fourier space this means that $\underline{k}\cdot\hat{\underline{E}}(\underline{k},\omega)=0$ + $\underline{k}\cdot\hat{J}(\underline{k},\omega)=0$

As in the previous cartoon $\downarrow \tilde{J}$ $\longrightarrow$ direction of variation

Since plasma has no zero-order gradients can conveniently rotate the coordinate system to make $\underline{k} \parallel \hat{z}$

Lect #2 (cont.)



$\delta \underline{E}$ and $\delta \underline{B}$ are in (x,y) plane

Do a Fourier T in both $\underline{k}$ and ω - Being sloppy + not using a proper LT - If we encounter a singularity from wave-particle interaction ⇒ an analytic continuation is to be implemented - we know how to do that!

$$\delta f = \frac{\tilde{f}(\underline{k}, \underline{v}, \omega)\, e^{i(\underline{k}\cdot\underline{r} - \omega t)}}{2} + c.c.$$

$$\delta \underline{E} = \frac{\tilde{\underline{E}}(\underline{k},\omega)}{2} e^{i(\underline{k}\cdot\underline{r} - \omega t)} + c.c. \ ; \quad \delta \underline{B} = \frac{\tilde{\underline{B}}(\underline{k},\omega)}{2} e^{i(\underline{k}\cdot\underline{r} - \omega t)} + c.c.$$

For this geometry $\underline{k}\cdot\underline{r} = kz$ + $\tilde{\underline{E}} = \tilde{E}_x \hat{x} + \tilde{E}_y \hat{y}$

+ $\tilde{\underline{E}} \cdot \frac{\partial}{\partial \underline{v}} f_0 = \tilde{E}_x \frac{\partial}{\partial v_x} f_0 + \tilde{E}_y \frac{\partial}{\partial v_y} f_0$ + $\frac{\partial}{\partial t} \to -i\omega$, $\nabla \to i\underline{k}$

⇒ $\underline{v}\cdot\nabla = v_z \frac{\partial}{\partial z} \Rightarrow ikv_z$ now apply to Vlasov Eq:

$$i(kv_z - \omega)\tilde{f} - \frac{e}{m}\left(\tilde{E}_x \frac{\partial}{\partial v_x} f_0 + \tilde{E}_y \frac{\partial}{\partial v_y} f_0\right) - \frac{e}{mc}(\underline{v} \times \tilde{\underline{B}}) \cdot \frac{\partial}{\partial \underline{v}} f_0 = 0$$

Faraday's Law says: $\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial}{\partial t}\underline{B} \Rightarrow \underline{k} \times \tilde{\underline{E}} = \frac{\omega}{c}\tilde{\underline{B}}$

We need: $\underline{v} \times \tilde{\underline{B}} = \underline{v} \times \frac{c}{\omega} \underline{k} \times \tilde{\underline{E}} = \frac{c}{\omega}\left[\underline{k}(\underline{v}\cdot\tilde{\underline{E}}) - \tilde{\underline{E}}(\underline{v}\cdot\underline{k})\right]$

$$= \frac{c}{\omega}\Big[\underbrace{k\hat{z}(v_x \tilde{E}_x + v_y \tilde{E}_y)}_{\text{along } \underline{k}} - k v_z \underbrace{(\tilde{E}_x \hat{x} + \tilde{E}_y \hat{y})}_{\text{along } \tilde{\underline{E}}}\Big]$$

i.e., magnetic pushes particles along the direction of propagation + across!

Lect #2 (cont.) What is needed to generate a δf is a contribution from

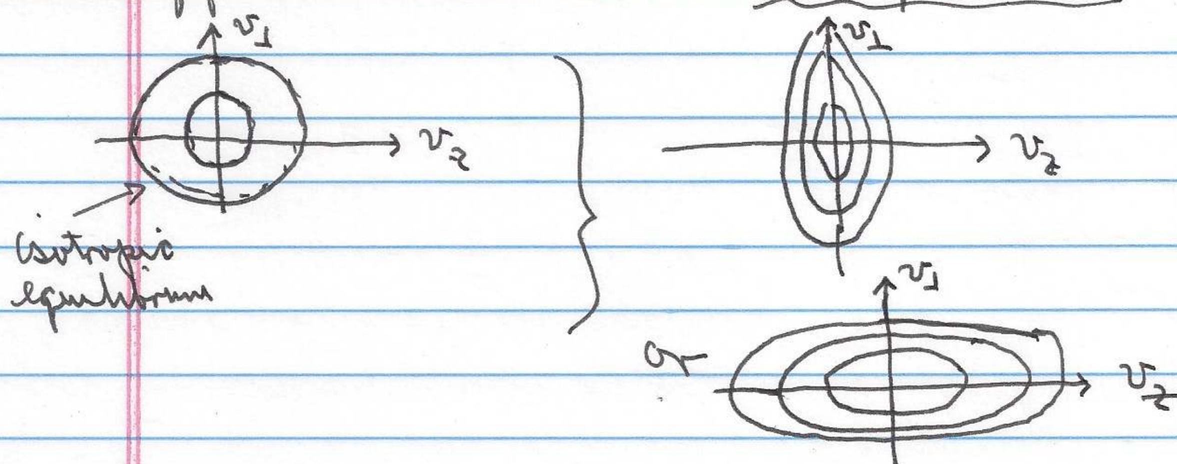
$$\frac{\underline{v}\times\tilde{\underline{B}}}{c}\cdot\frac{\partial}{\partial \underline{v}} f_0 =$$

$$\frac{1}{\omega}\left[k(v_x\tilde{E}_x + v_y \tilde{E}_y)\frac{\partial}{\partial v_z} f_0 - k v_z \tilde{E}_x \frac{\partial}{\partial v_x} f_0 - k v_z \tilde{E}_y \frac{\partial}{\partial v_y} f_0\right]$$

The Non-Thermal equilibrium feature appears Through f_0

i.e., $f_0(v_\perp, v_z) \neq f_0(v)$ where $v_\perp \equiv \sqrt{v_x^2+v_y^2}$

Cartoon of possible contours that are not in Equilibrium



a typical example might be a Two-Temperature Maxwellian

$$f_0(v_\perp, v_z) = n_0\, g_\perp(v_\perp)\, g_z(v_z)$$

$$= n_0 \frac{\exp\left[-\frac{v_\perp^2}{2\bar{v}_\perp^2}\right]}{(2\pi \bar{v}_\perp^2)} \frac{\exp\left[-\frac{v_z^2}{2\bar{v}_\parallel^2}\right]}{(2\pi \bar{v}_\parallel^2)^{1/2}}$$

with $\bar{v}_\perp^2 \equiv \frac{T_\perp}{m}$, $\bar{v}_\parallel^2 \equiv \frac{T_\parallel}{m}$ } 2 different Temperatures

How to make This? External Heating or Explosions!

waves That move preferentially in 1 direction, or boundaries That cool axially or $\perp$

Lect #2 (cont..) to the partial velocity derivatives

$$\frac{\partial}{\partial v_x} = \frac{\partial}{\partial v_\perp}\frac{\partial v_\perp}{\partial v_x} \;,\quad \frac{\partial}{\partial v_y} = \frac{\partial}{\partial v_\perp}\frac{\partial v_\perp}{\partial v_y}$$

$$\frac{\partial v_\perp}{\partial v_x} = \frac{v_x}{v_\perp} \quad \& \quad \frac{\partial v_\perp}{\partial v_y} = \frac{v_y}{v_\perp}$$

recast

$$\left(\frac{\underline{v}\times\tilde{B}}{c}\right)\cdot\frac{\partial}{\partial \underline{v}} f_0 = \frac{k}{\omega}\left[(v_x\tilde{E}_x + v_y\tilde{E}_y)\frac{\partial}{\partial v_z}f_0 - v_z\left(\frac{v_x}{v_\perp}\tilde{E}_x + \frac{v_y}{v_\perp}\tilde{E}_y\right)\frac{\partial}{\partial v_\perp}f_0\right]$$

$$= \frac{k}{\omega}(v_x\tilde{E}_x + v_y\tilde{E}_y)\left[\frac{\partial}{\partial v_z}f_0 - \frac{v_z}{v_\perp}\frac{\partial}{\partial v_\perp}f_0\right]$$

Now include the effect of the electrical force

$$\frac{e}{m}\left[\tilde{\underline{E}} + \frac{\underline{v}\times\tilde{\underline{B}}}{c}\right]\cdot\frac{\partial}{\partial \underline{v}}f_0 = \frac{e}{m}\left\{\tilde{E}_x\frac{v_x}{v_\perp}\frac{\partial}{\partial v_\perp}f_0 + \tilde{E}_y\frac{v_y}{v_\perp}\frac{\partial}{\partial v_\perp}f_0 + \frac{k}{\omega}(v_x\tilde{E}_x + v_y\tilde{E}_y)\left[\frac{\partial}{\partial v_z}f_0 - \frac{v_z}{v_\perp}\frac{\partial}{\partial v_\perp}f_0\right]\right.$$

$$= \frac{e}{m}(v_x\tilde{E}_x + v_y\tilde{E}_y)\left\{\underbrace{\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}f_0}_{\text{electric force}} + \underbrace{\frac{k}{\omega}\left(\frac{\partial}{\partial v_z}f_0 - \frac{v_z}{v_\perp}\frac{\partial}{\partial v_\perp}f_0\right)}_{\text{magnetic force}}\right]$$

$$= \frac{e}{m}(v_x\tilde{E}_x + v_y\tilde{E}_y)\left\{\left(1 - \frac{kv_z}{\omega}\right)\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}f_0 + \frac{k}{\omega}\frac{\partial}{\partial v_z}f_0\right\}$$

now obtain the perturbed distribution from

$$\tilde{f} = \frac{\frac{e}{m}\left(\tilde{\underline{E}} + \frac{\underline{v}\times\underline{B}}{c}\right)\cdot\frac{\partial}{\partial \underline{v}}f_0}{i(kv_z - \omega)}$$

cancels the wave-particle resonance

$$\tilde{f} = \frac{e}{im\omega}(v_x\tilde{E}_x + v_y\tilde{E}_y)\left\{\frac{1}{v_z - \frac{\omega}{k}}\frac{\partial}{\partial v_z}f_0 - \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}f_0\right\}$$

Lect#2 (cont..) There is not wave-particle resonance from the $\tilde{E}_\perp$ it cancels exactly, but discover a new effect a wave-particle resonance due to the magnetic force along the "$\underline{k}$ direction"
This not an electrostatic resonance -- it is a magnetic resonance & arises only in a kinetic description - not in fluid approximation

Compare the new magnetic wave-particle interaction to what we learned in 222B about electrostatic modes & Landau damping

For electrostatic resonance: $\tilde{f} \rightarrow \dfrac{e}{im} \dfrac{\tilde{E}_z \frac{\partial}{\partial v} f_0}{kv_z - \omega} = \dfrac{e}{imk} \dfrac{\tilde{E}_z \frac{\partial}{\partial v_z} f_0}{v_z - \omega/k}$

For magnetic resonance: $\tilde{f} \rightarrow \dfrac{e}{im\omega} \dfrac{\left[v_x \tilde{E}_x + v_y \tilde{E}_y\right] \frac{\partial}{\partial v_z} f_0}{v_z - \omega/k}$

$\Rightarrow$ The dynamics of electrostatic wave-particle interaction are controlled by the bounce frequency $\omega_B = \sqrt{\dfrac{ek\tilde{E}_z}{m}}$ "The Bounce Frequency"

Nonlinear effect!
$\Rightarrow$ By analogy the equivalent "Trapping frequency" for the Magnetic wave-particle resonance depends on the effective electric field

$$E_z \rightarrow \frac{1}{\omega/k} \left[v_x \tilde{E}_x + v_y \tilde{E}_y\right]$$

& the equivalent Magnetic Trapping frequency is

$$\left[\omega_B\right]_{Magnetic} \rightarrow \sqrt{\frac{ek^2}{m\omega} (v_x \tilde{E}_x + v_y \tilde{E}_y)}$$

which explicitly depends on the particle velocity (v_x, v_y) so particles of different (v_x, v_y) bounce at different rates $\Rightarrow$ There is an intrinsic "phase-mixing".

Lect # 2 (cont...) Since we know $\tilde{f}$ – the current can be found + from that the magnetic field!

$$\tilde{\underline{j}} = -e \int d^3v\, \underline{v}\, \tilde{f}(\underline{k}, \underline{v}, \omega)$$

↑ $\tilde{j}(\underline{k}, \omega)$

Ampere's Law: $\nabla \times \underline{B} = \frac{4\pi}{c}\underline{j} + \frac{1}{c}\frac{\partial}{\partial t}\underline{E} \Rightarrow i\underline{k} \times \tilde{\underline{B}} = \frac{4\pi}{c}\tilde{\underline{j}} - \frac{i\omega}{c}\tilde{\underline{E}}$

Faraday's Law: $\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial}{\partial t}\underline{B} \Rightarrow \underline{k} \times \tilde{\underline{E}} = \frac{\omega}{c}\tilde{\underline{B}}$

but $\underline{k} \times \underline{k} \times \tilde{\underline{E}} = \underline{k}(\cancel{\underline{k} \cdot \tilde{\underline{E}}})^{0} - k^2 \tilde{\underline{E}}$

$\Rightarrow \quad -i\frac{k^2 c}{\omega}\tilde{\underline{E}} = \frac{4\pi}{c}\tilde{\underline{j}} - i\frac{\omega}{c}\tilde{\underline{E}}$

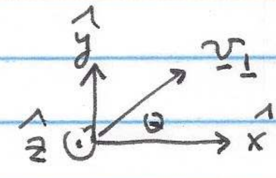
$\Rightarrow \quad i\frac{\omega}{c}\left[1 - \frac{k^2c^2}{\omega^2}\right]\tilde{\underline{E}} = \frac{4\pi}{c}\tilde{\underline{j}}$ For a pure E+M wave

or $\boxed{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]\tilde{\underline{E}} = \frac{4\pi\omega}{ic^2}\tilde{\underline{j}}}$ $\Rightarrow$ if $\tilde{\underline{j}} = 0 \Rightarrow \frac{\omega}{c} = k$ vacuum E+M

Write out component by component – say along $\hat{x}$ -- answer is the same! for $\hat{j}$

$$\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]\tilde{E}_x = \frac{4\omega}{ic^2}\tilde{j}_x$$

with $\tilde{j}_x = -e\int d^3v\, v_x \tilde{f}$ but



cylindrical coord. RHS

$\Rightarrow \quad v_x = v_\perp \cos\theta \quad ; \quad v_y = v_\perp \sin\theta$

and $d^3v = dv_z\, v_\perp\, d\theta\, dv_\perp$

we need $(v_x \tilde{E}_x + v_y \tilde{E}_y) = v_\perp(\cos\theta\,\tilde{E}_x + \sin\theta\,\tilde{E}_y)$

↑ only survives integration over θ

↑ vanishes upon θ integration!

Lect #2 (cont.) and since $\int_0^{2\pi} d\theta \cos^2\theta = \pi$

results in $\tilde{j}_x = -e \int_{-\infty}^{\infty} dv_z \int_0^{\infty} v_\perp dv_\perp (v_\perp \pi) \frac{e}{im\omega} v_\perp \left\{ \frac{\frac{\partial}{\partial v_z} f_0}{v_z - \omega/k} - \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \right\} \tilde{E}_x$

The $\tilde{E}_x$ cancels & results in a dispersion relation connecting (ω, k)

$$\boxed{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right] = \frac{4\pi\omega}{ic^2} \frac{(-e^2)}{im\omega} \pi \int_{-\infty}^{\infty} dv_z \int_0^{\infty} dv_\perp v_\perp^3 \left[\frac{\frac{\partial}{\partial v_z} f_0}{v_z - \omega/k} - \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \right]}$$

Behavior is determined by a competition between the response along $\underline{E}$ and $\underline{k}$ - The response along $\underline{E}$ is fluid-like

- The response along $\underline{k}$ has wave-particle resonance due to magnetic push!

→ clean-up with $\omega_p^2 = \frac{4\pi e^2 n_0}{m}$

$$\left[\left(\frac{\omega}{c}\right)^2 - k^2\right] = \left(\frac{\omega_p}{c}\right)^2 \pi \left[I_\parallel + I_\perp \right]$$

with $I_\parallel \equiv \frac{1}{n_0} \int_{-\infty}^{\infty} dv_z \int_0^{\infty} dv_\perp v_\perp^3 \frac{\frac{\partial}{\partial v_z} f_0}{v_z - \omega/k}$

$$I_\perp \equiv \frac{-1}{n_0} \int_{-\infty}^{\infty} dv_z \int_0^{\infty} dv_\perp v_\perp^3 \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} f_0$$

Taking a bi-Maxwellian dist.

$$\frac{1}{n_0} f_0(v_\perp, v_z) = \frac{e^{-v_z^2/2\bar{v}_\parallel^2}}{(2\pi \bar{v}_\parallel^2)^{1/2}} \frac{e^{-v_\perp^2/2\bar{v}_\perp^2}}{(2\pi \bar{v}_\perp^2)} \quad \text{with} \quad \bar{v}_\parallel^2 = \frac{T_\parallel}{m}, \quad \bar{v}_\perp^2 = \frac{T_\perp}{m}$$

Lect #2 (cont.) work each integral

$$I_\perp = \frac{1}{2\pi \bar{v}_\perp^2} \underbrace{\int_{-\infty}^{\infty} \frac{dv_z\, e^{-v_z^2/2\bar{v}_\parallel^2}}{(2\pi \bar{v}_\parallel^2)^{1/2}}}_{1} \int_0^\infty dv_\perp\, v_\perp^2 \underbrace{\frac{\partial}{\partial v_\perp}\left[e^{-v_\perp^2/2\bar{v}_\perp^2}\right]}_{\text{Do by parts}}$$

$$I_\perp = -2 \int_0^\infty \frac{dv_\perp\, v_\perp}{(2\pi \bar{v}_\perp^2)} e^{-v_\perp^2/2\bar{v}_\perp^2} = +2\left(\frac{1}{2\pi}\right)\int_0^\infty dv_\perp \underbrace{\frac{\partial}{\partial v_\perp}\left[e^{-v_\perp^2/2\bar{v}_\perp^2}\right]}_{\text{perfect differential}}$$

$I_\perp = -\frac{1}{\pi}$, i.e., independent of $\bar{v}_\perp$!

$\Rightarrow$ This is not a kinetic contribution is a bulk-fluid response!

The parallel interaction has more structure

$$I_\parallel = \int_{-\infty}^{\infty} dv_z \frac{\frac{\partial}{\partial v_z}\left[e^{-v_z^2/2\bar{v}_\parallel^2}\right]}{(v_z - \omega/k)(2\pi \bar{v}_\parallel^2)^{1/2}} \underbrace{\int_0^\infty dv_\perp \frac{v_\perp^3\, e^{-v_\perp^2/2\bar{v}_\perp^2}}{(2\pi \bar{v}_\perp^2)}}_{\text{Do This first}}$$

here $\frac{v_\perp}{\bar{v}_\perp^2} e^{-v_\perp^2/2\bar{v}_\perp^2} = -\frac{\partial}{\partial v_\perp}\left[e^{-v_\perp^2/2\bar{v}_\perp^2}\right]$

This part is: $$-\int_0^\infty \frac{dv_\perp}{2\pi} v_\perp^2 \frac{\partial}{\partial v_\perp}\left[e^{-v_\perp^2/2\bar{v}_\perp^2}\right] = \frac{1}{\pi}\underbrace{\int_0^\infty dv_\perp\, v_\perp\, e^{-v_\perp^2/2\bar{v}_\perp^2}}$$

$$-\frac{\bar{v}_\perp^2}{\pi}\int_0^\infty dv_\perp \frac{\partial}{\partial v_\perp}\left[e^{-v_\perp^2/2\bar{v}_\perp^2}\right] = +\frac{\bar{v}_\perp^2}{\pi}$$

After $\perp$ contribution is done.

$$I_\parallel = \frac{\bar{v}_\perp^2}{\pi}\int_{-\infty}^{\infty} dv_z \frac{\frac{\partial}{\partial z}\left[e^{-v_z^2/2\bar{v}_\parallel^2}\right]}{v_z - \omega/k}$$ which is something we explored in detail in 222B!

Lect #2 (cont.)

$$I_{\parallel} = \left(\frac{\bar{v}_\perp^2}{\pi}\right)(-1)\frac{1}{(2\pi \bar{v}_\parallel^2)^{1/2}}\frac{1}{\bar{v}_\parallel^2}\int_{-\infty}^{\infty} dv_z \, \frac{v_z \, e^{-v_z^2/2\bar{v}_\parallel^2}}{v_z - \omega/k}$$

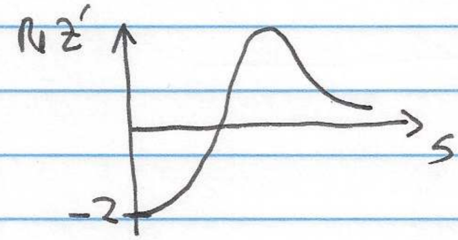
scale to: $t \equiv \dfrac{v_z}{\sqrt{2}\,\bar{v}_\parallel}$

results in $$I_\parallel = -\frac{\bar{v}_\perp^2}{\pi \bar{v}_\parallel^2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dt \, \frac{t\, e^{-t^2}}{t-s}$$

with $s \equiv \dfrac{\omega}{\sqrt{2}\,k\bar{v}_\parallel}$ the usual kinetic parameter

but from 222B $$Z'(s) = -\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} dt\,\frac{t\,e^{-t^2}}{t-s}$$ is the derivative of the plasma dispersion function!

with



⇒ The contribution from parallel dynamics is:

$$I_\parallel = \frac{\bar{v}_\perp^2}{\pi \bar{v}_\parallel^2}\left(\frac{1}{2}\right) Z'\left(\frac{\omega}{\sqrt{2}\,k\,\bar{v}_\parallel}\right)$$

Results in the full Kinetic Dispersion relation

$$\boxed{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right] = \frac{\omega_p^2}{c^2}\left[1 + \frac{T_\perp}{2T_\parallel} Z'\left(\frac{\omega}{\sqrt{2}\,k\bar{v}_\parallel}\right)\right]}$$

since $\omega_p^2 \propto \frac{1}{m}$ This result can be interpreted as a renormalization of the electron mass

Lect #2 (cont.) -- The collective response creates an effective electron mass

$$m^* \to \frac{m}{1 + \frac{T_\perp}{2T_\parallel} Z'\left(\frac{\omega}{\sqrt{2}\, k \bar{v}_\parallel}\right)}$$

That varies with the speed of the wave! and depending on the parameters it can become negative

- As usual - our mantra - there are adiabatic, inertial + resonant regimes

Consider the inertial regime: $s \gg 1 \Rightarrow Z'(s) \to \frac{1}{s^2} = \frac{2k^2 \bar{v}_\parallel^2}{\omega^2}$

$\Rightarrow$ Results in

$$\omega^2 - k^2c^2 = \omega_p^2 \left[1 + \frac{\bar{v}_\perp^2}{2\bar{v}_\parallel^2}\left(\frac{2k^2\bar{v}_\parallel^2}{\omega^2}\right)\right]$$

$$= \omega_p^2 \Big[\underbrace{1 + \frac{k^2 \bar{v}_\perp^2}{\omega^2}}_{\text{The electrons become lighter!}}\Big]$$

multiply by $\omega^2 \Rightarrow$ $\omega^4 - k^2c^2\omega^2 = \omega_p^2\omega^2 + k^2 v_\perp^2 \omega_p^2$

which is a quadric

$$\omega^4 - (\omega_p^2 + k^2c^2)\,\omega^2 - \omega_p^2 k^2 v_\perp^2 = 0 \quad \Leftarrow \text{a quadratic in } \omega^2$$

new term from magnetic force!

solve for ω^2

$$\omega^2 = \frac{(\omega_p^2 + k^2c^2) \pm \left[(\omega_p^2 + k^2c^2)^2 + 4\omega_p^2 k^2 v_\perp^2\right]^{1/2}}{2}$$

There are 2 roots: The (+) is the usual E+M with a thermal correction that you did as HW in 222B

Lect #2 (cont.) say for small thermal effect $k^2\bar{v}_\perp^2 \ll \omega_p^2$

$\Rightarrow$ expand the square root

$$\omega_+^2 = \underbrace{\omega_p^2 + k^2c^2}_{\text{usual}} + \frac{4}{4}\underbrace{\frac{\omega_p^2 k^2 \bar{v}_\perp^2}{\omega_p^2+k^2c^2}}_{\text{thermal correction}}$$

but there is a new root! The (−)

$$\omega_-^2 = \left\{\omega_p^2 + k^2c^2 - (\omega_p^2+k^2c^2)\left[1 + \frac{4}{2}\frac{\omega_p^2 k^2 \bar{v}_\perp^2}{(\omega_p^2+k^2c^2)^2}\right]\right\}\frac{1}{2}$$

cancels

The usual effect of plasma oscillations cancels out!

$$\omega_-^2 = -\frac{\omega_p^2 k^2 \bar{v}_\perp^2}{\omega_p^2+k^2c^2}$$

or $\omega_- = \pm i \dfrac{\omega_p k \bar{v}_\perp}{\sqrt{\omega_p^2+k^2c^2}}$ not an oscillator because plasma oscillations have been cancelled out!

it is a purely growing mode! in which short wavelengths grow faster!

Check for consistency of approximation of "inertial regime"

$\Rightarrow$ $s \equiv \dfrac{\omega}{\sqrt{2}\,k\bar{v}_\parallel} \gg 1$ for ω use $|\omega|$ to estimate

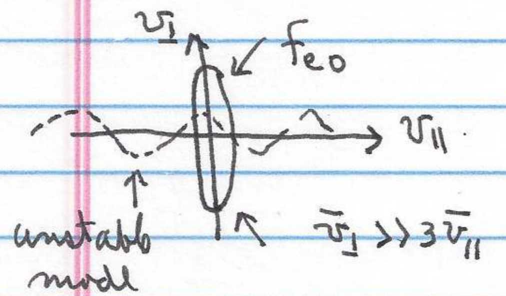
$$\frac{\omega_p \bar{v}_\perp}{\bar{v}_\parallel \sqrt{2}\,(\omega_p^2+k^2c^2)^{1/2}} \gg 1 \Rightarrow \sim 3 \text{ in practice}$$

or $\dfrac{\bar{v}_\perp}{\bar{v}_\parallel} > 3\sqrt{2}\left(1+\dfrac{k^2c^2}{\omega_p^2}\right)^{1/2}$ or $\boxed{\dfrac{\bar{v}_\perp}{\bar{v}_\parallel} > 3\sqrt{2}}$

for this limit to apply!

Lect #2 (cont.) In this inertial regime $\omega_- \simeq i k \bar{v}_\perp$

increases with k for small k



What happens as $\underline{\underline{k}}\uparrow$? The $\frac{\omega}{\sqrt{2}\, k \bar{v}_\parallel}$ enters the extreme adiabatic regime

back to exact disp relation:

$$\omega^2 - k^2c^2 = \omega_p^2\left[1 + \frac{1}{2}\frac{T_\perp}{T_\parallel} Z'\left(\frac{\omega}{\sqrt{2}\, k \bar{v}_\parallel}\right)\right]$$

for $k\uparrow$ $Z'(s) \to -2$

$$\Rightarrow \quad \omega^2 - k^2c^2 \to \omega_p^2\left[1 - \frac{T_\perp}{T_\parallel}\right]$$

or $$\omega^2 = \omega_p^2\left[1 - \frac{T_\perp}{T_\parallel}\right] + k^2c^2$$

+ now $m^* \to \dfrac{m}{1 - \frac{T_\perp}{T_\parallel}}$

+ for $T_\perp > T_\parallel$ $m^* < 0$

In this case the Higgs mechanism yields negative mass!

In this limit as $k \to 0$ mode can be unstable for $T_\perp > T_\parallel$

From this expression can estimate the max k when mode goes stable by setting

$$0 = \omega_p^2\left[1 - \frac{T_\perp}{T_\parallel}\right] + k_{MAX}^2 c^2$$

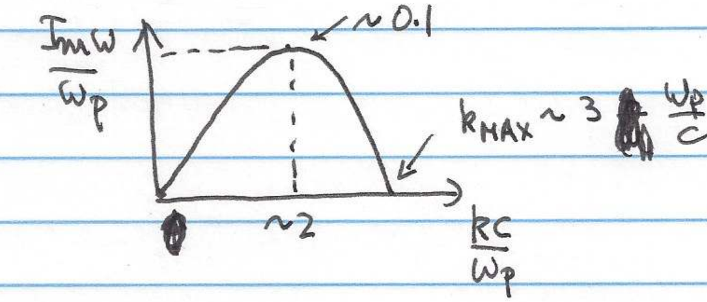
Lect #2 (cont--)

$$k_{MAX} = \left(\frac{T_\perp}{T_\parallel} - 1\right)^{1/2} \underbrace{\frac{\omega_p}{c}}_{\text{Natural scaling}}$$

The modes excited are "short scale" ~ electron skin depth

⇒ This Mechanism does not create "global magnetic fields"

What the instability tries to do is accelerate particles along k to make the effective "$T_\parallel$" ↑ -- but it does not succeed in practice -- process saturates by particle trapping + results in coherent trapping oscillations not - heating. The numerical solution of the dispersion relation looks like



The saturation level is estimated when the growth rate equals the bounce frequency

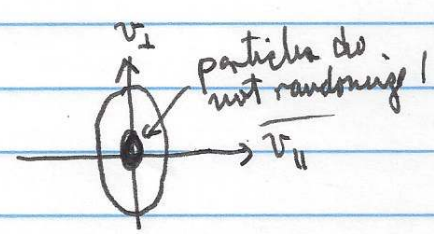
$$\langle\omega_B\rangle \sim \sqrt{\frac{e k \bar{v}_\perp \tilde{B}_y}{mc}} \simeq Im\,\omega$$

or

$$\frac{e\tilde{B}_y}{mc} \simeq (0.1)^2 \frac{\omega_p^2}{k\bar{v}_\perp}$$

The fluctuation gyro frequency

$$\Rightarrow \frac{\tilde{\Omega}}{\omega_p} \sim 10^{-2}\left(\frac{c}{\bar{v}_\perp}\right) \Rightarrow$$



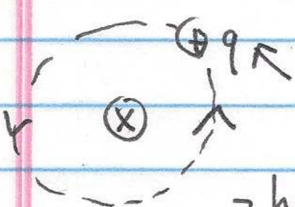
Lect #3 . Particle orbits in a magnetic field.

You have seen it many times - This is the last for you to cement it in your brain cells so that in the future when you see a complex magnetic topology you can plunge into the discussions.

"Nonrelativistic" Magnetic force: $\underline{F}_B = q\,\frac{\underline{v}\times\underline{B}}{c}$ in cgs

[B] is in Gauss and 1 Tesla $= 10^4$ G

- This by itself is a purely relativistic effect! even without "γ"
- Because there is no magnetic charge



in rest frame of charge $\underline{v} = 0 \Rightarrow$ There is no magnetic force!

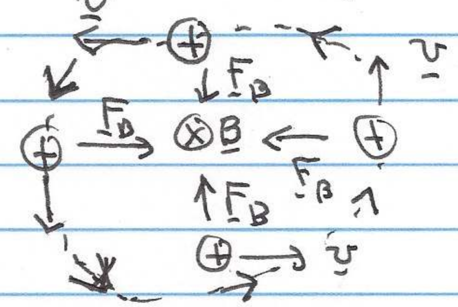
- but the charge says - I do not follow a straight line $\Rightarrow$ some force must act on me! But what force acts on a charge?

- It must be an electric field $\underline{E}' = \frac{\underline{v}\times\underline{B}}{c}$ - So actually, the magnetic force does not exist - what truly exists is the transformation of a magnetic field into $\rightarrow \underline{E}'$ in the rest frame of the charge! - It is odd that relativity was always present but its effect was masked

Lect #3 (cont.) Important to hard-wire in your brain the direction of rotation

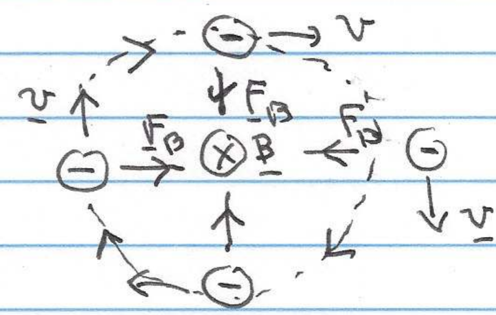
For (+) charge

The Right-Hand direction is



(+) Rotates anti-clockwise when looking into $\underline{B}$

For (−) charge



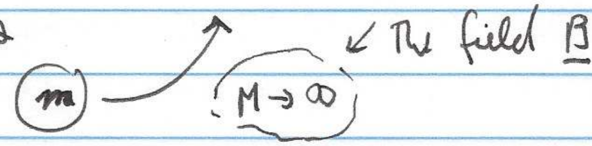
(−) Rotates clockwise

Energy:
$$\underline{v} \cdot m \frac{d}{dt} \underline{v} = \underline{v} \cdot \left[\frac{q}{c} \underline{v} \times \underline{B} \right] = 0$$

$$\frac{d}{dt}\left[\frac{1}{2} m \underline{v}^2 \right] = 0 \Rightarrow KE = \frac{1}{2} m |\underline{v}|^2 = \text{constant}$$

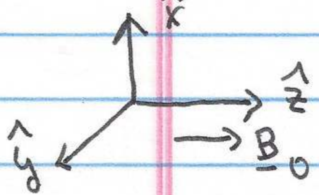
$\Rightarrow$ $\underline{B}$-fields can never transfer energy to a charge $\Rightarrow$ any change in energy in a system must come from $\underline{E}$-fields!

But $\underline{B}$ changes momentum $\Rightarrow$ it is like colliding with an invisible object of ∞ mass



$\Rightarrow$ There must be something hidden in the vacuum that absorbs the momentum change - - we will see later what this quantity is .. it is the magnetic vector potential $\underline{A}$ -

Lect #3 (cont..) Let's do the orbits carefully - will need for other studies



in $\hat{x}$: $m\frac{d}{dt}v_x = \frac{q}{c}v_y B_0$

in $\hat{y}$: $m\frac{d}{dt}v_y = -\frac{q}{c}v_x B_0$

apply $\frac{d}{dt}$ to $\hat{x}$ $\Rightarrow$ $$m\frac{d^2}{dt^2}v_x = \frac{qB_0}{c}\left(-\frac{q v_x B_0}{mc}\right)$$

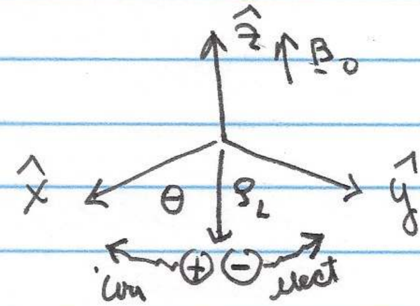
$\Rightarrow$ $\frac{d^2}{dt^2}v_x + \left(\frac{qB_0}{mc}\right)^2 v_x = 0 \Rightarrow$ SHO with freq. $\Omega \equiv \frac{|q|B_0}{mc}$

my convention is for Ω to be always > 0 and to the signs by inspection - other authors like to put a sign on Ω - I find it dangerous!

Ω : cyclotron frequency, gyrofrequency, Larmor frequency, etc..

since $v_x^2 + v_y^2 = \text{const}$ & SHO is $\sin\Omega t$ or $\cos\Omega t$ $\Rightarrow$

motion is a circle



$\Rightarrow$ $\theta(t) = -\frac{q}{|q|}\Omega t + \theta_0$

keeps signs straight

initial angle

The position in (x, y) is $\underline{r}_L = x_L\hat{x} + y_L\hat{y}$

with $x_L(t) = \rho_L\cos\theta$; $y_L(t) = \rho_L\sin\theta$

Lect #3 (cont..). $\dot{x}_L(t) = -\rho_L \dot{\theta} \sin\theta$, $\dot{y}_L(t) = \rho_L \dot{\theta} \cos\theta$

& $(\dot{x}_L)^2 + (\dot{y}_L)^2 = \rho_L^2 (\dot{\theta})^2 = v_\perp^2(0)$

$\Rightarrow \rho_L = \dfrac{v_\perp(0)}{\Omega}$ is the size of the circle -- "the Larmor radius"

Some #'s worth remembering: $\Omega_e = 1.76 \times 10^7 B \ \frac{\text{rad}}{\text{sec}}$

$\Omega_i = \frac{z}{\mu} 9.58 \times 10^3 B \ \frac{\text{rad}}{\text{sec}}$

$\frac{q}{e} \equiv z$; $\mu \equiv \frac{M}{M_{proton}}$

In lab work one uses "f" -- the frequency "f" $= \dfrac{\Omega}{2\pi}$

useful values: $\dfrac{\Omega_e}{2\pi} \simeq 2.8 \ \dfrac{\text{GHz}}{\text{kG}}$ say LAPD $B \simeq 1\,\text{kG} \Rightarrow \boxed{2.8 \text{ GHz}}$

for Earth $B \doteq \frac{1}{2}\,G \Rightarrow \dfrac{\Omega_e}{2\pi} \simeq 2.8 \times 10^9 \times \frac{1}{2} \times 10^{-3} \Rightarrow \boxed{1.4 \text{ MHz}}$

in ionosphere O^+ is dominant species $\Rightarrow \boxed{f_{O^+} \sim 50 \text{ Hz}}$

but important to have intuition for associated E&M wavelengths because it determines the hardware required to sample

for electrons in LAPD: $\lambda = \dfrac{c}{f} = \dfrac{3 \times 10^{10}}{2.8 \times 10^9} \simeq 10$ cm

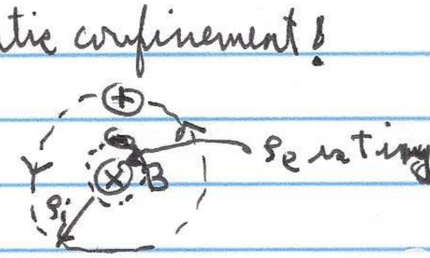
for electrons in ITER $(B \doteq 50\,\text{kG}) \Rightarrow \lambda \simeq 2$ mm

for electrons in ionosphere: $\lambda = \dfrac{3 \times 10^{10}}{1.4 \times 10^6} \simeq 200$ meters

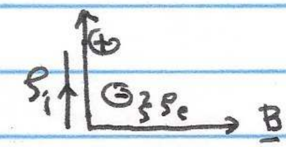
Scaling of $\langle \rho_L \rangle \sim \dfrac{\langle v_\perp \rangle}{\Omega} \sim \dfrac{\sqrt{T}\sqrt{m}}{B}$ } important to have hard wired in your brain!

Lect #3 (cont.) Fundamental ~~constraint~~ obstacle for magnetic confinement!

$$\frac{\langle \rho_i \rangle}{\langle \rho_e \rangle} = \sqrt{\frac{M}{m}} > 40 \quad \Rightarrow$$



or better view

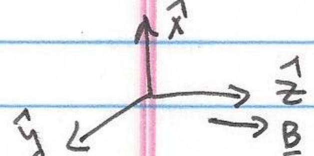


$\Rightarrow$ Hard to achieve neutrality because ⊕ + ⊖ + separated by large distance $\Rightarrow$ charge neutrality requires that e - move along $\underline{B}$

$\Rightarrow$ parallel currents can be unstable and noisy - anything that stops parallel currents cause $\tilde{E}_\perp$ to grow!

Mysterious entity that causes collisions with vacuum

-- What is inside the vacuum that scatters charges?



$$m \frac{d}{dt} v_x = \frac{q}{c} v_y B \quad \text{with} \quad v_y = \frac{dy}{dt}$$

$$\Rightarrow \frac{d}{dt}\left[m v_x - \frac{q}{c} y B \right] = 0$$

$\Rightarrow$ There is a quantity $\quad p_x = \underbrace{m v_x}_{\text{Mechanical momentum}} - \underbrace{\frac{q}{c} y B}_{\text{field Momentum}} \quad$ That is constant

What does $\frac{q}{c} yB$ represent? - Look at magnetic vector potential

$$\underline{B} = \nabla \times \underline{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = B \hat{z} \Rightarrow \partial_x A_y - \partial_y A_x = B \text{, can set } A_y = A_z = 0$$

$$\Rightarrow A_x = -yB$$

Lect #3 (cont.) $$\frac{d}{dt}\left(m v_x + \frac{q}{c} A_x\right) = 0$$

can interpret : $m v_x \Rightarrow$ material momentum ; $\frac{q A_x}{c} \Rightarrow$ vacuum momentum

but $P_{Total} = P_{Matter} + P_{Vacuum} = \text{constant}$

$\Rightarrow$ repeat for each component $\Rightarrow$ $\underline{P} = m\underline{v} + \frac{q}{c}\underline{A}$ is what is conserved.

This is known as the "Canonical Momentum" -- it shows that "$\underline{A}$" is very real - not just a mathematical aid to do algebra

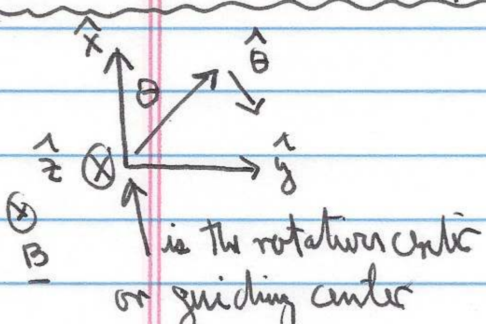
- Hamiltonian in a magnetic field - B does not enter, $\underline{E}$ does not enter either!

$$H = KE + PE$$

$$= \frac{(\underline{P}_{Matter})^2}{2m} + q\phi = \boxed{\frac{(\underline{P} - \frac{q}{c}\underline{A})^2}{2m} + q\phi}$$

here do not confuse $\underline{P}$ with the particle momentum!

Orbits relative to Lab frame -- issue of confusion - will need later

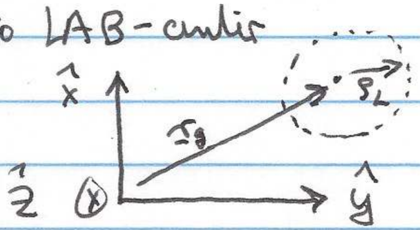


$$\theta(t) = -\frac{q}{|q|}\Omega t + \theta_0$$

$$x(t) = \rho_L \cos\theta = \rho_L \cos\left[-\frac{q}{|q|}\Omega t + \theta_0\right]$$

$$y(t) = \rho_L \sin\theta = \rho_L \sin\left[-\frac{q}{|q|}\Omega t + \theta_0\right]$$

but relative to LAB-center

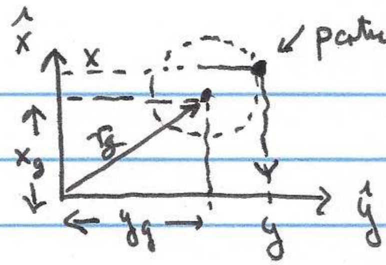


$$\underline{r}_g = x_g\hat{x} + y_g\hat{y}$$

Lect #3 (cont...)

$$\begin{cases} x(t) = x_g + \rho_L \cos\left[\frac{q}{|q|}\Omega t + \theta_0\right] \\ y(t) = y_g + \rho_L \sin\left[\frac{-q}{|q|}\Omega t + \theta_0\right] \end{cases}$$

are LAB positions



but what are x_g and y_g? -- get from initial conditions

$$x(0) = x_g + \rho_L \cos\theta_0 \equiv x_0 ; \quad y(0) = y_g + \rho_L \sin\theta_0 \equiv y_0$$

solve for them $\quad x_g = x(0) - \rho_L \cos\theta_0 ; \quad y_g = y(0) - \rho_L \sin\theta_0$

plug back:

$$x(t) = x_0 + \rho_L \left\{ \cos\left[-\frac{q}{|q|}\Omega t + \theta_0\right] - \cos\theta_0 \right\}$$

$$y(t) = y_0 + \rho_L \left\{ \sin\left[-\frac{q}{|q|}\Omega t + \theta_0\right] - \cos\theta_0 \right\}$$

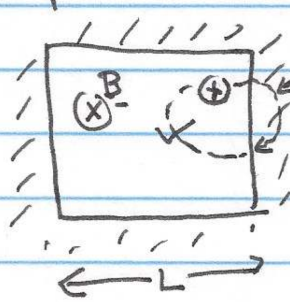
will use many times keep handy!

Check at $t = 0 \Rightarrow \quad x(t=0) = x_0$ and $y(t=0) = y_0$

Now, (x_0, y_0) are relative to LAB and θ_0 is angle relative to gyration center

Remember always: "Particle rotates about its guiding center not the center of the LAB frame"

A constraint for confinement inside a device



at $t = 0$ ⊕ is born by ionization if its orbit intersects wall it is lost during its first gyration

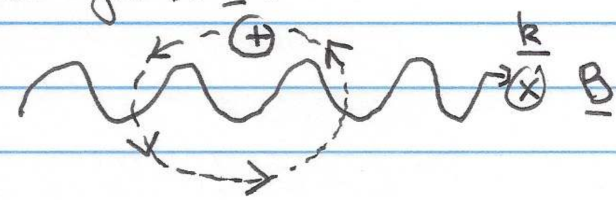
$\Rightarrow$ For magnetic confinement need $L \gg \rho_L$

Lect#3 (cont.) Two important consequences of magnetized orbits:

Ω is a very sharp oscillator $\Rightarrow$ it causes a new resonance

For a wave at fixed frequency ω $\Rightarrow$ $\underbrace{\omega - k_z v_z}_{\text{Doppler shifted frequency felt by particle}} = \Omega$ ← oscillator

and for waves travelling $\perp$ to $\underline{B}$:



during one rotation the ⊕ charge sees many different values of the electric field

$\Rightarrow$ net effect is an average over the orbit $\Rightarrow$ the result depends on

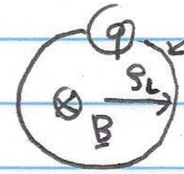
$$\langle E_\perp \rangle \sim \frac{J_n(k_\perp \rho_L)}{\omega - n\Omega} \quad \text{for } n = 1, 2, 3, \ldots$$

Next we introduce a related averaging effect that is very useful for time scales far from these resonances, i.e.,

for slow times $\tau \gg \frac{2\pi}{\Omega}$ + smooth spatial variations

compared to ρ_L, i.e., $L \gg \rho_L$ - The essence is that one

takes a photograph of the moving charge with a camera that has a <u>long exposure time</u>



see a continuous charge distribution or "ring" that carries a net current

The magnitude of the current flowing in the ring $|I| = \frac{|q|}{\frac{2\pi}{\Omega}} = |q|\frac{\Omega}{2\pi}$

But every current ring has <u>a magnetic moment</u>

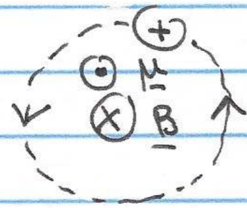
$|\underline{\mu}| = \text{Area}\,\frac{|I|}{c}$ in cgs where Area is bounded by ring

Lect #3 (cont.) For the rotating charge the magnetic moment is

$$|\underline{\mu}| = \frac{|q|\Omega}{2\pi c}(\pi \rho_L^2) = \frac{|q|}{2c}\,\Omega\,\frac{v_\perp^2}{\Omega^2} = \frac{|q|}{2c}\frac{v_\perp}{\Omega} = \frac{|q|\,v_\perp}{2c\,\frac{|q|B}{mc}}$$

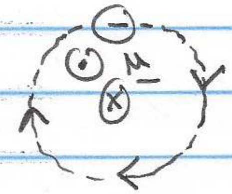
$$\boxed{|\underline{\mu}| = \frac{\frac{1}{2}m v_\perp^2}{B}} = \frac{U_\perp}{B} \leftarrow \perp \text{ KE}$$

The direction of $\underline{\mu}$ is by RH rule



for (+) charge

and for electrons



but current is opposite to $\underline{v}$

$\Rightarrow$ $\underline{\mu}$ also opposes $\underline{B}$ $\Rightarrow$ independent of charge sign

$\Rightarrow$ In general $\quad \underline{\mu} = -\dfrac{\frac{1}{2}m v_\perp^2}{B}\,\hat{B}$

How much energy does the magnetic loop have?

$$U_B = -\underline{\mu}\cdot\underline{B} = -(-\underline{\mu}\cdot\hat{B}B) = \frac{1}{2}m v_\perp^2 = (KE)_\perp$$

i.e., The loop magnetic energy is the same as the $(KE)_\perp$

$\Rightarrow$ Careful not to double count!

If you use loop concept : $-\underline{\mu}\cdot\underline{B}$ is the energy

If you use the particle concept : $\frac{1}{2}m v_\perp^2$ is the energy

<u>but</u> do <u>not add them</u>!

The total kinetic energy of a magnetized charge is

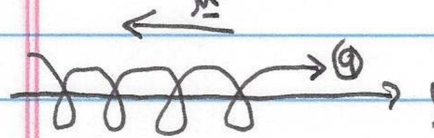
$$(KE)_{Total} = (KE)_{\parallel} + (KE)_{\perp} = \frac{1}{2}m v_{\parallel}^2 \underbrace{- \underline{\mu}\cdot\underline{B}}_{\text{removes particle nature from } \perp \text{ direction}}$$

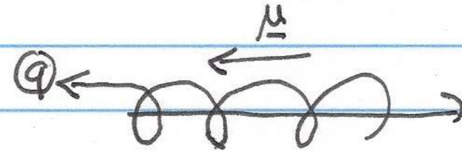
Lect#3 (cont.) The exact kinetic description requires $f = f(v_\parallel, v_\perp)$

but there is an "averaged" reduced description that is very useful for slow effects -- but with less information - you do not know where the particle is in the $\perp$ direction - in this description

$$f \to \langle f \rangle = \langle f \rangle (v_\parallel, \mu) \quad \text{i.e, reduced phase-space } (v_\parallel, \mu)$$

useful to remember that direction of $\underline{\mu}$ does not depend on $v_\parallel$:



and


The current in the loop generates its own magnetic field $\delta \underline{B}$ whose direction is parallel to $\underline{\mu} \Rightarrow$ always opposes $\underline{B}$ - If the plasma is in thermal equilibrium & has no gradients - then there is no change in $\underline{B}$ due to Van Leeuwen's Theorem -- but gradients can generate edge currents that cause a net reduction of the ambient $\underline{B}$ - The ability to change $\underline{B}$ is measured by quantity known as "beta" $\beta \equiv \dfrac{n U_\perp}{\frac{B^2}{8\pi}} = \dfrac{n |\underline{\mu} \cdot \underline{B}|}{B^2/8\pi}$

in LAPD $\sim 10^{-4}$ is negligible, in Tokamaks $\sim 3\%$ has small effects but in the solar wind $\beta \gtrsim 1$ is a dominant feature

Lect #4. <u>Adiabatic invariants</u> -- Powerful concept in general

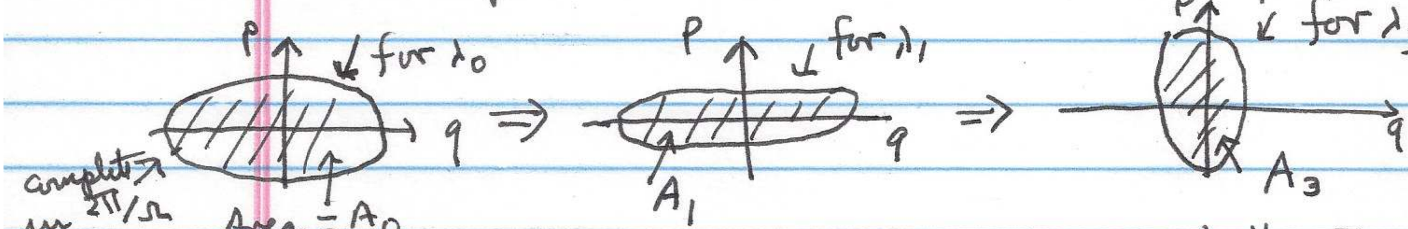
Applies to "adiabatic regime" of wave-particle interaction $\frac{\omega}{k} \ll \bar{v}$
- For magnetized charges $\underline{\mu}$ is an adiabatic invariant - we will not show it here -- left as HW problem -

The general Dynamical definition for a system governed by a Hamiltonian $H(p, q, \lambda)$
↑ parameter to be varied!

If this system has periodic solutions (could be fully nonlinear) then the integral $\oint p\, dq$ over 1 cycle of oscillations is essentially invariant, i.e., <u>constant</u> as λ is varied slowly - by slowly it means $\omega \equiv \left| \frac{1}{\lambda}\frac{d\lambda}{dt} \right| \ll \Omega$ ↑ the frequency of oscillation & also

That $\omega - kv \neq \Omega$ for a spatial variation with "k"

Geometrical interpretation is the area in phase-space



if $\lambda_0 \to \lambda_3$ "slowly" then $\boxed{A_0 = A_1 = A_3}$ for "τ" $\gg \frac{2\pi}{\Omega}$

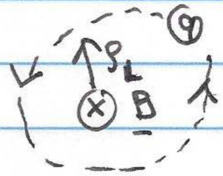
⇒ The phase-space "looks different" ~~for~~ but the area bounded by the orbit <u>is the same</u>!

Lect#4 (cont..) For magnetic moment this has a deep physical meaning

if $\underline{B} = \underline{B}(\underline{r}, t)$ & $\nu = \frac{1}{|\underline{B}|}\left|\frac{d}{dt}\underline{B}\right| << \Omega$, $\underline{\mu}$ is invariant

provided $\omega << \Omega$ for $\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v}\cdot\nabla$

The E+M reason is:



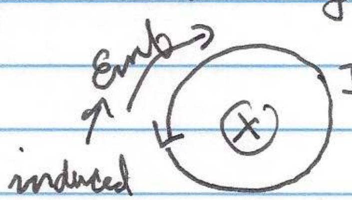
Flux through orbit is $\Phi_B = B \cdot \text{Area} = B\pi\rho_L^2$

$$\Phi_B = \pi \frac{B v_\perp^2}{\frac{|q|^2 B^2}{m^2c^2}} = \underbrace{\left(\pi\frac{mc}{|q|^2}\right)}_{\text{constants}}\underbrace{\left(\frac{m v_\perp^2}{B}\right)}_{2\mu} = \text{const } |\underline{\mu}|$$

$\Rightarrow$ if Φ_B remains constant $\Rightarrow$ $|\underline{\mu}|$ remains constant!

but $\frac{d}{dt}\Phi_B = -\text{Emf}$ $\Rightarrow$ if one tries to change $\underline{\mu}$ by changing $\underline{B}$

then Emf is induced!

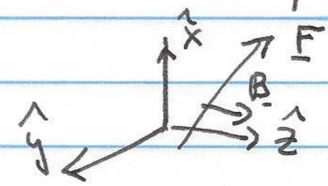


i.e., it behaves as a superconducting wire!

but it fails if $\omega = \Omega$ because energy can be take out or put-in

- We will make use of the invariance of $\underline{\mu}$ later.

Effect of constant force.



$$m\frac{d}{dt}\underline{v} = \underline{F} + \frac{q}{c}\underline{v}\times\underline{B}$$

$\hat{z}$: $m\frac{dv_z}{dt} = F_z \Rightarrow v_z(t) = v_z(0) + \frac{F_z}{m}t \Rightarrow z(t) = z(0) + v_z(0)t + \frac{F_z}{2m}t^2$

$\Rightarrow$ Particle free falls along the magnetic field independently of what happens to $\underline{v}_\perp$

Lect # 4 (cont..) Examine $\perp$ direction: $m \frac{d}{dt} \underline{v}_\perp = \underline{F}_\perp + \frac{q}{c} \underline{v}_\perp \times \underline{B}$

$\Uparrow$ constant needs to be balanced by a constant velocity piece i.e., $\langle \underline{v}_\perp \rangle$

allow $\underline{v}_\perp = \langle \underline{v}_\perp \rangle + \tilde{\underline{v}}_\perp$

with $\frac{d}{dt} \langle \underline{v}_\perp \rangle = 0$

plug-in $\Rightarrow$ $\underbrace{m \frac{d}{dt} \tilde{\underline{v}}_\perp}_{\text{Time dependent}} = \underbrace{\underline{F}_\perp + \frac{q}{c} \langle \underline{v}_\perp \rangle \times \underline{B}}_{\text{constant}} + \underbrace{\frac{q}{c} \tilde{\underline{v}}_\perp \times \underline{B}}_{\text{Time dependent}}$

To balance Eq need $0 = \underline{F}_\perp + \frac{q}{c} \langle \underline{v}_\perp \rangle \times \underline{B}$ and $m \frac{d}{dt} \tilde{\underline{v}}_\perp = \frac{q}{c} \tilde{\underline{v}}_\perp \times \underline{B}$

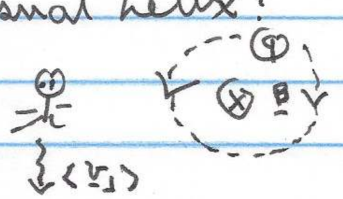
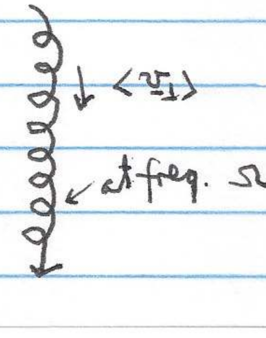
apply $\underline{B} \times \rightarrow \Rightarrow$ $0 = \underline{B} \times \underline{F}_\perp + \frac{q}{c} \underbrace{\underline{B} \times \langle \underline{v}_\perp \rangle \times \underline{B}}_{\langle \underline{v}_\perp \rangle B^2 - \underline{B}(\cancel{\underline{B} \cdot \langle \underline{v}_\perp \rangle})^{\,0}}$

or $\langle \underline{v}_\perp \rangle = - \frac{\underline{B} \times \underline{F}_\perp}{B^2} \frac{c}{q} = \boxed{\frac{c}{q} \frac{\underline{F}_\perp \times \hat{\underline{B}}}{B}}$ called The $\boxed{\text{F cross B drift!}}$

Two important features: 1) depends on sign of charge
2) strength $\downarrow$ as $B \uparrow$

Obviously result is not correct as $B \to 0 \Rightarrow \langle \underline{v}_\perp \rangle \to \infty$ -- so in this limit a relativistic analysis is required $\Rightarrow$ $\boxed{\text{a HW problem}}$

In a frame moving with velocity $\langle \underline{v}_\perp \rangle$ relative To the LAB The orbit is The usual helix!

in Frame  but in LAB frame orbit is open 

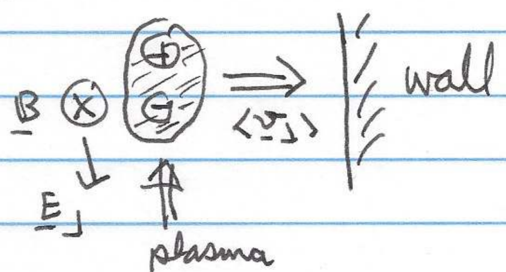
Lect#4 (cont--) Important case: Electrical Force $\underline{F}_\perp = q\,\underline{E}_\perp$

Then $\langle \underline{v}_\perp \rangle = \frac{c}{q}\,\frac{q\,\underline{E}_\perp \times \hat{\underline{B}}}{B} = c\,\frac{\underline{E}_\perp \times \hat{\underline{B}}}{B}$ ⇒ Charge cancels! Called $E \otimes B$ Drift

ions and electrons drift in the same direction!

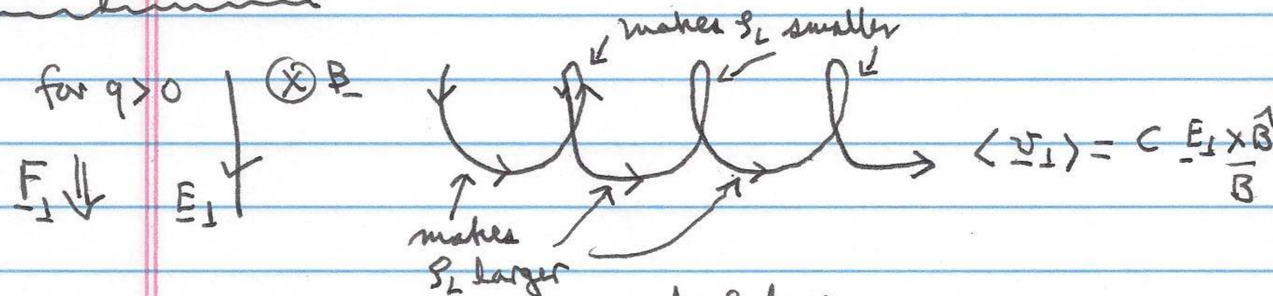
Immediate consequence: 1) $\underline{E}_\perp$ does not induce a charge separation
2) $\underline{E}_\perp$ fields cannot drive currents!
3) Net plasma can move across $\underline{B}$

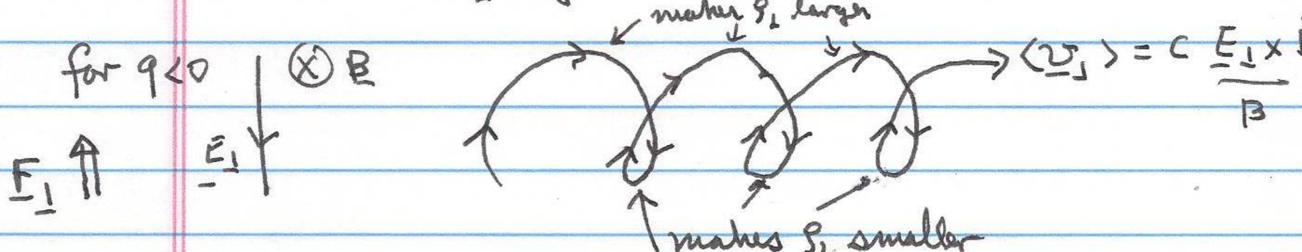
very dangerous!



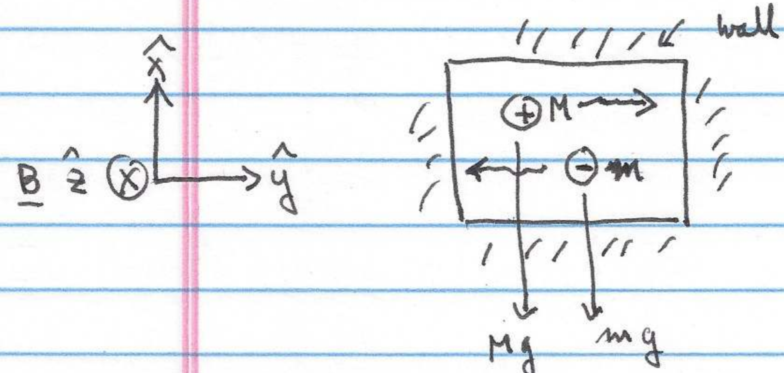
To generate $\underline{j}_\perp$ need neutral collisions! or Time dependence

Sense of rotation due to $\underline{E} \times \underline{B}$ -- can conclude from $\rho_L = \frac{v_\perp}{\Omega}$





Lect #4 (cont.) Effect of non-electrical force, e.g., gravity or neutral collisions



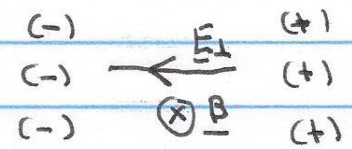
$$\langle \underline{v}_\perp \rangle = \frac{c}{qB}(-m_j g \hat{x}) \times \hat{z}$$

$$\langle \underline{v}_\perp \rangle_e = \frac{c}{eB} mg(-\hat{y})$$

$$\langle \underline{v}_\perp \rangle_i = -\frac{c}{ZeB} Mg(-\hat{y})$$

In this picture the electrons go left and the ions go right!

-- There is now a charge separation ⇒ $\underline{E}_\perp$ is generated



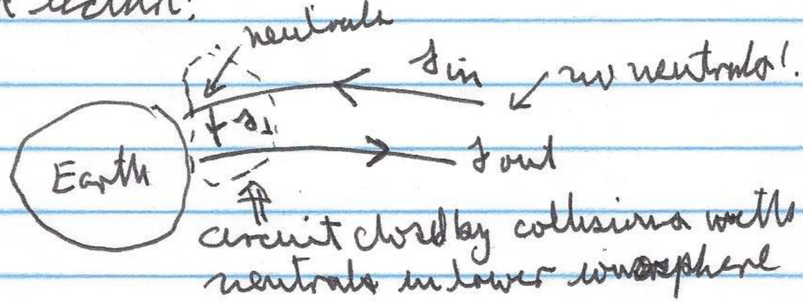
+ This $\langle \underline{v}_\perp \rangle = \frac{c}{B} \underline{E}_\perp \times \hat{z}$ is downward!

i.e., the whole plasma as a "blob" falls along the direction of gravity! plasma is dumped onto the bottom of the container!

(✱✱) This is the problem with toroidal magnetic confinement devices -- the solution is to provide an additional rotation that mixes the top + bottom of the surrounding walls -- success of Tokamak using an internal magnetic field generated by a current! -- will examine in future lecture!

Example of ⊥ currents:

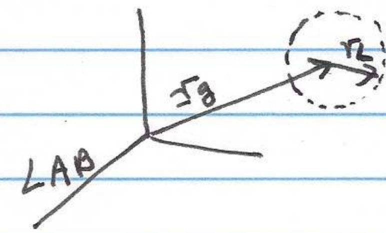
(Earth's global current system)



Lect #4 (cont..) In reality $\underline{B} = B(\underline{r}, t)$ because it is generated by discrete currents that are turned on + off -- How to handle?

Guiding Center Model

(separates scales, spatial + Temporal)



local Larmor orbit about a uniform $\underline{B}$ evaluated at $\underline{r}_g$

$$\underline{r}(t) \simeq \underline{r}_g(t) + \underline{r}_L(t)$$

↑ Lab position -- associate a guiding center frequency "ω_g" $\equiv \frac{1}{|\underline{r}_g|}\left|\frac{d}{dt}\underline{r}_g\right|$

That satisfies: $\omega_g \ll \frac{1}{|\underline{r}_L|}\left|\frac{d}{dt}\underline{r}_L\right| \simeq \Omega(\underline{r}_g)$

+ Further slow variation $\quad L_B \gg |\underline{r}_L|$ with $L_B^{-1} \equiv \frac{1}{|\underline{B}|}|\nabla \underline{B}|$

and also $\quad \frac{1}{|\underline{B}|}\left|\frac{d}{dt}\underline{B}\right| \ll \Omega$

with these orderings - make a Taylor expansion in $\underline{r}_L$ - The small quantity

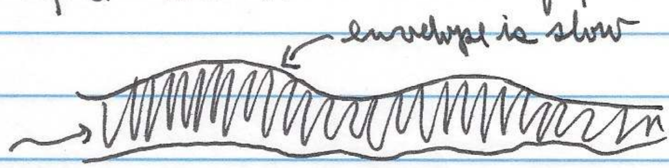
$$\underline{B}(\underline{r}(t)) = \underline{B}(\underline{r}_g + \underline{r}_L) \simeq \underline{B}(\underline{r}_g) + \underline{r}_L \cdot \nabla \underline{B}(\underline{r}_g) + \cancel{\mathcal{O}\left(\left(\frac{r_L}{L_B}\right)^2\right)}$$

i.e., a 1st order expansion in $\frac{|\underline{r}_L|}{L_B}$

Examine Eq. of motion to separate time scales (without external forces)

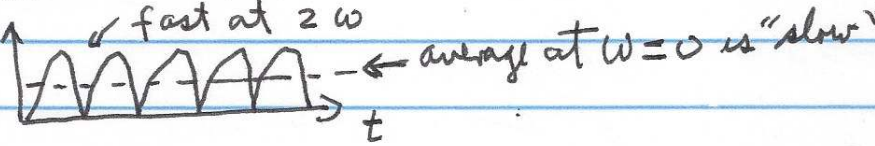
$$m\frac{d^2}{dt^2}(\underline{r}_g + \underline{r}_L) = \frac{q}{c}(\dot{\underline{r}}_g + \dot{\underline{r}}_L) \otimes \left[\underline{B}(\underline{r}_g) + \underline{r}_L \cdot \nabla \underline{B}(\underline{r}_g)\right]$$

This contains The full behavior - a mix of fast and slow Time scales



} idea is to separate The Two behaviors by inspection

Lect #4 (cont..) Remember that oscillatory quantities can generate through a nonlinearity - fast + slow terms -

like $(\sin\omega t)^2 \rightarrow$ 

Fast Terms: $m \frac{d^2}{dt^2} \underline{r}_L = \frac{q}{c} \dot{\underline{r}}_L \times \underline{B}(\underline{r}_g)$

$\uparrow$ fast $\quad \uparrow$ fast $\quad \uparrow$ slow

Slow terms: $m \frac{d^2}{dt^2} \underline{r}_g = \frac{q}{c} \dot{\underline{r}}_g \times \underline{B}(\underline{r}_g) + \frac{q}{c} \langle \dot{\underline{r}}_L \otimes \underline{r}_L \cdot \nabla \underline{B}(\underline{r}_g) \rangle$

$\uparrow$ slow $\quad \uparrow$ slow $\quad \uparrow$ slow $\quad$ fast ~~like~~ $\quad \uparrow$ slow

average over one cycle $\frac{2\pi}{\Omega(\underline{r}_g)}$

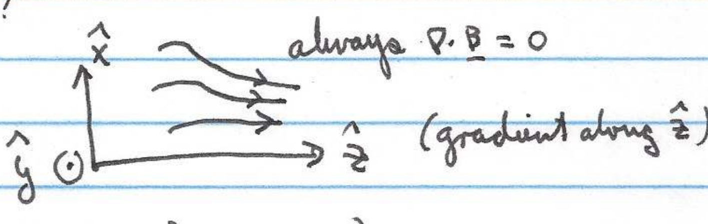
The fast Eq. is identical to the result for a uniform field $\Rightarrow$ we know well!

i.e., sinusoidal solution with frequency $\Omega(\underline{r}_g)$

What is tricky in this formulation is that the term $\underline{r}_L \cdot \nabla$ generates $\parallel$ and $\perp$ terms - thus the additional constraint is required

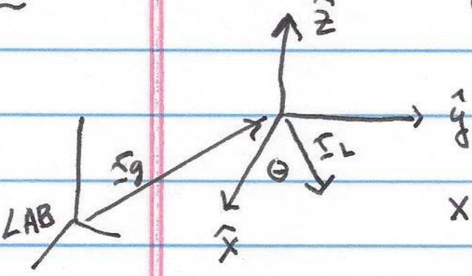
$$\underbrace{\frac{|\underline{v}_\parallel \cdot \nabla \underline{B}|}{|\underline{B}|}}_{\text{is the local Doppler shift!}} \ll \Omega$$

Consider $\underline{B}$ with parallel gradient



$\underline{B} = B_z \hat{z} + B_x \hat{x}$ with $\nabla \cdot \underline{B} = 0 \Rightarrow \frac{\partial}{\partial x} B_x = -\frac{\partial}{\partial z} B_z$

Lect #4 (cont..) Assume weak gradient $\Rightarrow |B_x| \ll |B_z|$ small bend in field line



$\hat{y}$ ⇐ center of coordinate is now at $\underline{r}_g$ location

$$\underbrace{x_L = |\underline{r}_L| \cos\theta, \quad y_L = |\underline{r}_L| \sin\theta}_{\text{fast terms of orbit}}, \quad \theta(t) = -\frac{q}{|q|}\Omega t + \theta_0$$

need $\underline{r}_L \cdot \nabla = (|\underline{r}| \cos\theta\, \hat{x} + |\underline{r}_L| \sin\theta\, \hat{y}) \odot (\partial_x \hat{x} + \cancel{\partial_y \hat{y}}^{\,0} + \partial_z \hat{z})$

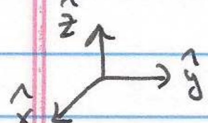
$\Rightarrow \quad \underline{r}_L \cdot \nabla = |\underline{r}_L| \cos\theta \frac{\partial}{\partial x} \quad \Rightarrow \quad \underline{r}_L \cdot \nabla \underline{B} = |\underline{r}_L| \cos\theta \frac{\partial}{\partial x}\left[B_x \hat{x} + B_z \hat{z}\right]$

but $\frac{\partial}{\partial x} B_z = 0$ because $B_z = B_z(z) \Rightarrow \quad \underline{r}_L \cdot \nabla \underline{B} = |\underline{r}_L| \cos\theta \frac{\partial}{\partial x} B_x \hat{x}$

and $\dot{\underline{r}}_L = \frac{d}{dt} x_L \hat{x} + \frac{d}{dt} y_L \hat{y}$ with $\frac{d}{dt} x_L = -|\underline{r}_L| \sin\theta\, \dot{\theta}$; $\frac{dy_L}{dt} = |\underline{r}_L| \cos\theta\, \dot{\theta}$

and $\dot{\theta} = -\frac{q}{|q|}\Omega$

Construct: $\dot{\underline{r}}_L \otimes (\underline{r}_L \cdot \nabla \underline{B}) = -\frac{q}{|q|}\Omega |\underline{r}_L| \left[-\sin\theta\, \hat{x} + \cos\theta\, \hat{y}\right] \otimes |\underline{r}_L| \cos\theta \frac{\partial}{\partial x} B_x \hat{x}$

but  $\Rightarrow \quad \hat{y} \otimes \hat{x} = -\hat{z}$

$$\Rightarrow \quad \dot{\underline{r}}_L \otimes (\underline{r}_L \cdot \nabla \underline{B}) = \frac{q}{|q|}\Omega |\underline{r}_L|^2 \cos^2\theta \frac{\partial}{\partial x} B_x \hat{z}$$

need to average $\langle \dot{\underline{r}}_L \otimes (\underline{r}_L \cdot \nabla \underline{B}) \rangle = \frac{q}{|q| 2}\Omega |\underline{r}_L|^2 \frac{\partial}{\partial x} B_x \hat{z}$

$$= -\frac{1}{2}\frac{q}{|q|}\Omega |\underline{r}_L|^2 \frac{\partial}{\partial z} B_z \hat{z}$$

& The slow Eq. becomes

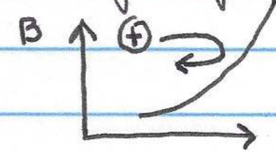
$$m \frac{d^2}{dt^2}\underline{r}_g = \frac{q}{c}\dot{\underline{r}}_g \times \underline{B}(\underline{r}_g) \underbrace{- \frac{q^2}{c\,2}\Omega \frac{v_\perp^2}{\Omega^2}\frac{\partial}{\partial z} B_z \hat{z}}_{\text{The Mirror force}}$$

$$\boxed{\langle \underline{F} \rangle = -|\underline{\mu}| \frac{\partial}{\partial z} B_z \hat{z}}$$

Lect #4 (cont.) The term $F_z = -|\underline{\mu}| \frac{\partial}{\partial z} B_z$ is simply $-\nabla U_B$ with $U_B = -\underline{\mu} \cdot \underline{B}$

Check - since $\underline{\mu} = -\frac{m v_\perp^2}{2B}\hat{B} \Rightarrow U_B = -\left(-\frac{m v_\perp^2}{2B}\right)\hat{B}\cdot\underline{B} = \frac{1}{2} m v_\perp^2$

The effect of this force is for increasing magnetic field



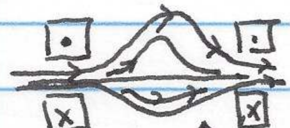
is a magnetic wall!

$\Rightarrow$ it prevents charges from entering a region of strong magnetic field

- Note it is independent of charge!

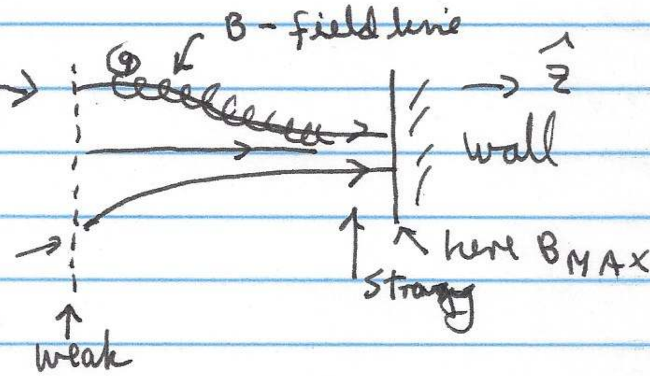
$\Rightarrow$ The entire plasma can be reflected

Can exploit as an attractive confinement device -- A Mirror Machine



$\swarrow$ external coils can provide local increase in B } generic geometry

Mirror loss to an end wall



Consider a charge q launched at the midpoint where field is B_{Min}

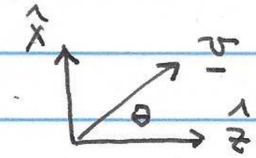
There are 2 conserved quantities : The total energy, The magnetic moment

A barely trapped (or confined) particle stops at the wall, i.e., $\boxed{v_{\parallel} = 0 \text{ at } B_{MAX}}$

- The energy of such particle is $U = |\underline{\mu}| B_{MAX}$

Lect #4 (cont..) – But $|\underline{\mu}|$ is the same as its value at B_{Min}

$$\Rightarrow \quad |\underline{\mu}| = \frac{\frac{1}{2} m v_\perp^2}{B_{Min}} \quad \text{where}$$



↑ at midpoint where $B = B_{Min}$

and $v_\perp = v \sin\theta$

$$\Rightarrow \quad |\underline{\mu}| = \frac{\left(\frac{1}{2} m v^2\right) \sin^2\theta}{B_{Min}} \quad \& \quad U = \frac{1}{2} m v^2 = \text{constant}$$

or $\quad |\underline{\mu}| = U \dfrac{\sin^2\theta}{B_{Min}}$ $\quad$ but at the wall $|\underline{\mu}| = \dfrac{U}{B_{MAX}}$

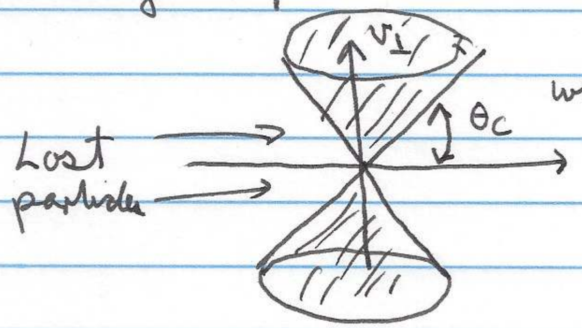
~~Mirror~~ Ratio

$$\Rightarrow \quad \sin^2\theta = \boxed{\frac{B_{Min}}{B_{MAX}}} \quad \text{or} \quad \sin\theta = \pm\sqrt{\frac{B_{Min}}{B_{MA}}}$$

This is the special or critical angle for $v_{\parallel} = 0$ at wall

if θ is ~~less~~ smaller the particle is lost, if larger it reflects

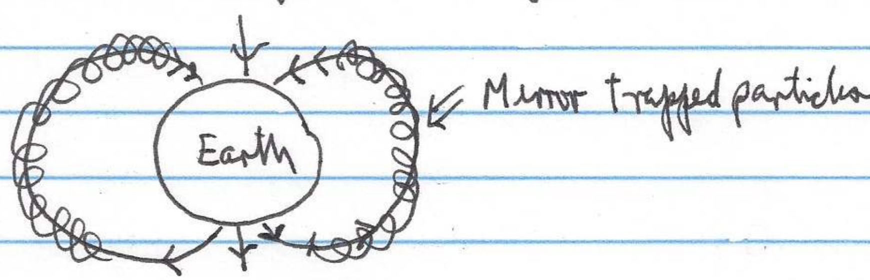
This critical angle defines a "loss cone" in $(v_\perp, v_\parallel)$ space



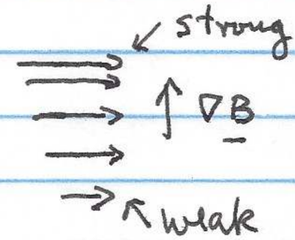
with $\sin\theta_C = \sqrt{\dfrac{B_{Min}}{B_{MAX}}}$

The quantity $\dfrac{B_{MAX}}{B_{Min}}$ is "The Mirror Ratio" – The figure of merit for a simple mirror Machine

Lect #4 (cont..) The earth's magnetic field provides a natural mirror machine That confines energetic particles for extended periods of time -- These Energetic trapped particles are currently of great concern because they can destroy communication satellites --



Lecture #5. Force due to a transverse gradient. In this case The typical magnetic field configuration looks like



The associated force can be calculated from The average slow term

$$\frac{q}{c} \langle \dot{\underline{r}}_L \otimes \underline{r}_L \cdot \nabla \underline{B}(\underline{r}_g) \rangle$$ as was done for The parallel gradient

- I will not do it - but it is a problem in HW # 1 - for you to do. Here I will use a simple derivation motivated by The result for The mirror force $\underline{F} = -\nabla U_B$ in general for a magnetic moment $\underline{\mu}$

where $U_B = -\underline{\mu} \cdot \underline{B}$.

Apply This concept to $\underline{B} = B(x)\hat{z} \Rightarrow \underline{F}_\nabla = \nabla(\underline{\mu} \cdot \underline{B})$

but $\underline{\mu} = -|\underline{\mu}|\hat{B} \Rightarrow \underline{\mu} \cdot \underline{B} = -|\underline{\mu}| B$

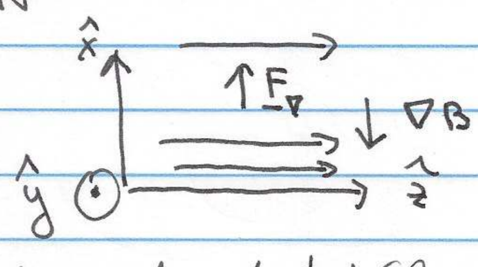
Lect #5 (cont...) for this case $\quad \underline{F}_{\nabla} = \hat{x}\,\frac{\partial}{\partial x}\left[\,|\underline{\mu}|\,B(x)\right]$

is invariant!

$\Rightarrow \underline{F}_{\nabla} = -|\underline{\mu}|\,\hat{x}\,\frac{\partial}{\partial x}B(x)$

it opposes the increase in B-field!

For example:



just as in ∥ gradient case a charge is pushed away from regions of strong B - into weak regions

again the effect is independent of (q)

What is now different from the mirror force is that this push is across $\underline{B}$ $\Rightarrow$ it causes a drift! called "the gradient drift"

In general for any slow force there is a drift, now

$$\boxed{\underline{v}_{\nabla} = \frac{c}{q}\,\frac{\underline{F}_{\nabla}\times\hat{B}}{B}}$$

Apply to this case: $\quad \underline{v}_{\nabla} = \frac{c}{q}\,\frac{-|\underline{\mu}|\,\hat{x}\,\frac{\partial}{\partial x}B \otimes \hat{B}}{B} = -\frac{c}{q}\,\frac{|\underline{\mu}|}{B}\,\frac{\partial B}{\partial x}\,\underbrace{\hat{x}\otimes\hat{z}}_{-\hat{y}}$

or $\quad \underline{v}_{\nabla} = \frac{c}{q}\,\frac{|\underline{\mu}|}{B}\,\frac{\partial}{\partial x}B\,\hat{y}$

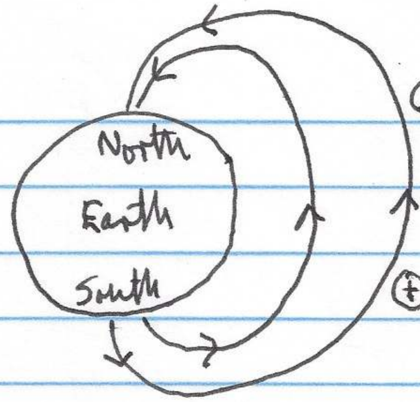
depends on charge $\Rightarrow$ generates a current + charge separation

In the previous picture the ions (+) flow into page + electrons (−) out!

This has an important consequence for currents around Earth

Lect #5 (cont..)

Side view



⊖ ⊗ flow in

$\leftarrow \nabla B$

$\longrightarrow \underline{F}_\nabla$ is away from Earth

⊕ ⊙ flow out

$\Rightarrow$ There is a net current that circulates around the Earth

Top View above North pole

Net current is called "The Ring Current"

How large are gradient drifts? $|\underline{v}_\nabla| = \frac{c}{|q|} |\underline{\mu}| \frac{1}{L_B}$

where $\frac{1}{L_B} = \frac{1}{|B|} \left| \frac{\partial}{\partial x} B \right|$ is the gradient scale length

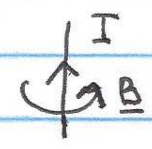
$$\Rightarrow \quad |\underline{v}_\nabla| = \frac{c}{|q|} \frac{\frac{1}{2} m v_\perp^2}{B} \frac{1}{L_B} = v_\perp \left(\frac{\rho_L}{2 L_B} \right)$$

↑ has to be small for guiding center to apply

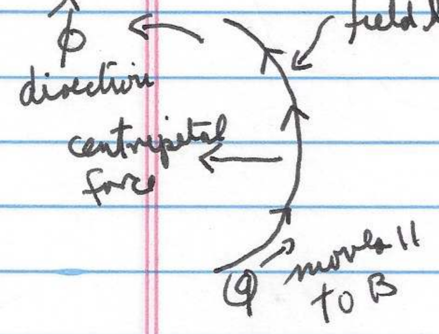
$$\Rightarrow \quad \boxed{\frac{|\underline{v}_\nabla|}{v_\perp} < 10^{-1} \quad \text{Typically}}$$ but can have big effects!

Curvature drift - Result of another slow force due to curved field line

- In general field lines are curved because B-fields are made by wires carrying a current!



Lect#5 (cont--) Consider the motion of charge whose velocity lies along the direction of a $\underline{B}$ field that has a weak curvature



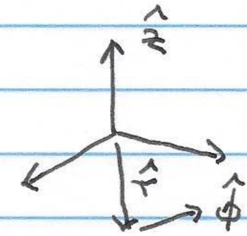
Centripetal force $= m a_r \hat{r}$ with $a_r = -\dfrac{v_\phi^2}{r}$ } is inward towards center of rotation

Put in $\underline{F} = m\underline{a}$ The $-\dfrac{m v_\phi^2}{r}$ term can be moved to the other side an appears with a (+) sign $\Rightarrow$ $+\dfrac{m v_\phi^2}{r}\hat{r} \equiv$ Centrifugal force

- It is not a "True force" but a consequence of being in a non-inertial frame -- Nevertheless - it appears very real if you drive an F1 race car through a tight curve in the Suzuka circuit - the acceleration exceeds 5g $\Rightarrow$ means your head weighs 5 times as much! -- It is the same for an ion going around a tokamak -- it feels an outward push - But now this push is $\perp$ to the direction of $\underline{B}$ $\Rightarrow$ it results in another drift -- "The curvature drift"

$$\underline{v}_{curvature} = \frac{c}{q}\,\frac{\underline{F}_{centrifugal}}{B} \otimes \hat{B}$$

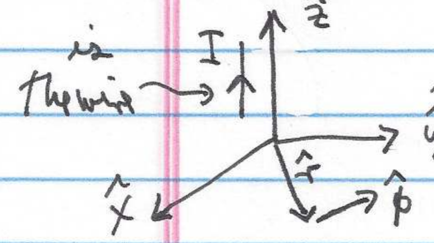
using a cylindrical coordinate system $\hat{B} = \hat{\phi}$



$$\Rightarrow \quad \underline{v}_{curvature} = \frac{c}{q}\,\frac{m v_\phi^2}{r}\,\underbrace{\hat{r} \otimes \hat{\phi}}_{\hat{z}}$$

Lect #5 (cont.) In general the curvature and gradient drifts appear together

Let's assess the situation for a generic $\underline{B}$ generated by a wire (infinite)



by Ampere's law (right-hand rule)

$\underline{B} = B(r)\,\hat{\phi}$

in cgs $B(r) = \dfrac{2I}{cr} \Rightarrow \underline{B} = \dfrac{2I}{cr}\,\hat{\phi}$

$\Rightarrow$ from previous calculation $\underline{v}_c = \dfrac{c}{qB}\,\dfrac{m v_\parallel^2}{r}\,\hat{z}$

because $B \sim \frac{1}{r} \Rightarrow$ it has a gradient that points inward!

$$\underline{F}_\nabla = \nabla(\underline{\mu}\cdot\underline{B}) = -\frac{m v_\perp^2}{2B}\nabla B \quad \text{with } \nabla B = \hat{r}\frac{\partial}{\partial r}B = -\frac{B}{r}\hat{r}$$

$$\Rightarrow \underline{F}_\nabla = \frac{m v_\perp^2}{2B}\,\frac{B}{r}\,\hat{r} = \frac{m v_\perp^2}{2r}\,\hat{r}$$ in the same direction as the centrifugal force

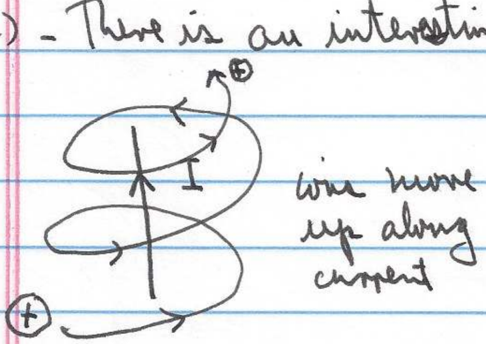
$\Rightarrow$ The curvature and gradient drifts are additive!

$$\underline{v}_{total} = \underline{v}_\nabla + \underline{v}_c = \boxed{\frac{c}{qB}\, m\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\frac{1}{r}\,\hat{z}}$$

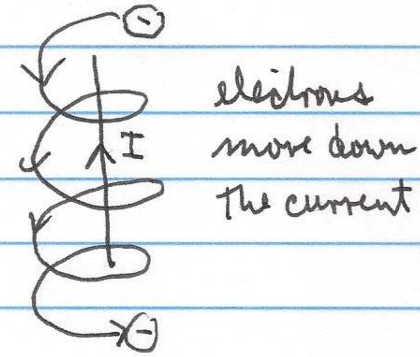
Some important properties:

1) This is a true single particle kinetic effect -- not a fluid effect
2) Direction depends on sign of the charge $\Rightarrow$ leads to currents and charge separation
3) It is proportional to the mass because it is an inertial effect
4) In thermal equilibrium $\langle v_\parallel^2\rangle + \langle v_\perp^2\rangle \propto \frac{T}{m}$
$\Rightarrow$ effect becomes proportional to temperature! T

Lect #5 (cont..) - There is an interesting hidden coupling in the final result



and

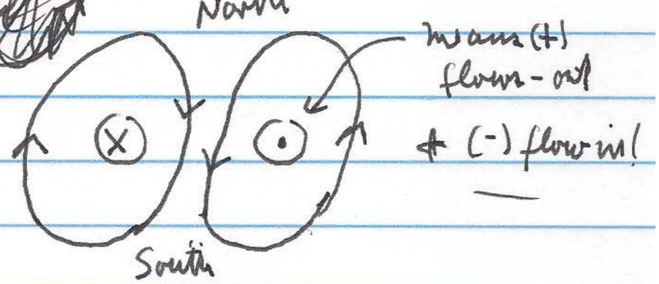


If one views the current as being due to positive carriers - then these carriers induce the surrounding - like charges to follow them - But if one views the current as being due to moving electrons - then the wire electrons induce the surrounding electrons to follow them!

(*) Consequence is -- if you can determine the direction of the I that makes $\underline{B}$ $\Rightarrow$ you can immediately tell which way the different plasma species will drift - just by inspection -- no need for any formulas!

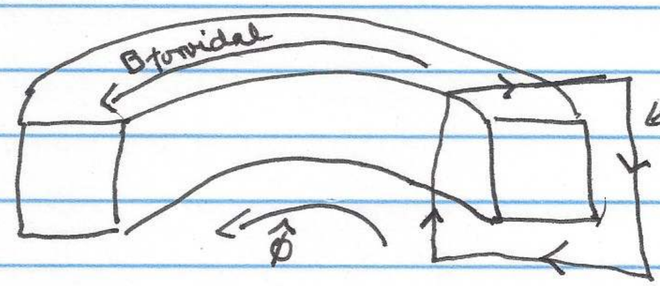
~~[illegible]~~ :

For Earth - The current coil is like

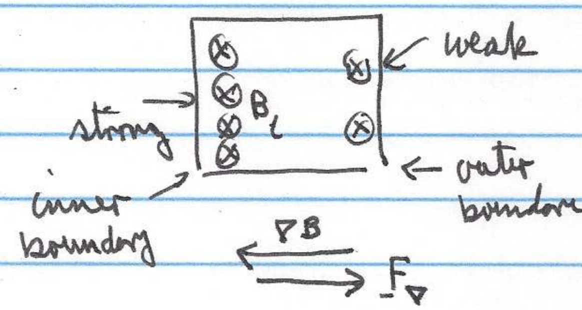


Lect #5 (cont.) The curvature + gradient drifts render a pure torroidal field not suitable for confining a plasma in a finite container

The picture of a torroidal solenoid that makes a torroidal field is

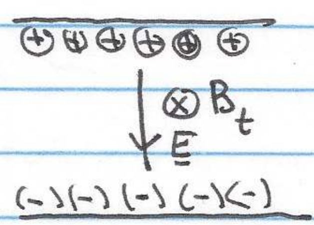


$B_\phi \propto \frac{1}{R}\hat{\phi}$ $\Rightarrow$ cross section



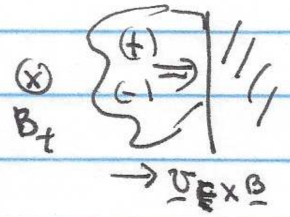
$\Rightarrow$ ⊕ ions move to top boundary

⊖ ions move to bottom boundary



Charge separation generates downward E-field

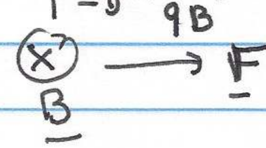
$\Rightarrow$ Results in $E \times B$ drifts towards the outer wall -- and is independent of charge $\Rightarrow$ The whole plasma is lost



Lect #5 (cont.) Time dependent Force $\underline{F}(t) \Rightarrow$ "Polarization Drift"

Found that for $\frac{d}{dt}\underline{F} = 0$ there is a drift

$$\uparrow \underline{v}_D = \frac{c}{qB}\,\underline{F}\times\hat{\underline{B}}$$



- Now consider a "slowly" changing case: $\underline{\omega} = \frac{1}{|\underline{F}|}\left|\frac{d\underline{F}}{dt}\right| << \Omega$

$\Rightarrow$ Develop an expansion in $\frac{\omega}{\Omega}$ that in zero-order is $\frac{c}{qB}\,\underline{F}\times\hat{\underline{B}}$

Eq. of motion: $\quad m\frac{d}{dt}\underline{v} = \underline{F}(t) + \frac{q}{c}\,\underline{v}\times\underline{B}$

let $\quad \underline{v} = \underline{v}^{(0)} + \underline{v}^{(1)} + \underline{v}^{(2)} + \cdots \quad$ where $\quad \underline{v}^{(j)} \propto \left(\frac{\omega}{\Omega}\right)^j$

$$\Rightarrow \quad m\frac{d}{dt}\left[\underline{v}^{(0)} + \underline{v}^{(1)} + \cdots\right] = \underline{F}(t) + \frac{q}{c}\left[\underline{v}^{(0)} + \underline{v}^{(1)} + \cdots\right]\times\underline{B}$$

solve by iteration -- use: $\underline{v}^{(j-1)}$ to obtain $\underline{v}^{(j)}$

To 1st order:

$$m\frac{d}{dt}\underline{v}^{(1)} = -m\frac{d}{dt}\underline{v}^{(0)} + \underbrace{\left[\underline{F}(t) + \frac{q}{c}\,\underline{v}^{(0)}\times\underline{B}\right]}_{0 \text{ by zero-order}} + \frac{q}{c}\,\underline{v}^{(1)}\times\underline{B}$$

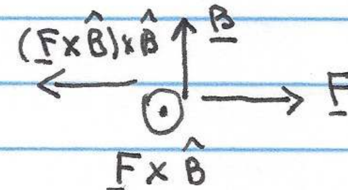
$$\Rightarrow \quad m\frac{d}{dt}\underline{v}^{(1)} = \boxed{-m\frac{d}{dt}\underline{v}^{(0)}} + \frac{q}{c}\,\underline{v}^{(1)}\times\underline{B}$$

a new effective force like $\underline{F}$ + is slow!

This leads to $\quad \underline{v}^{(1)} \to \frac{c}{qB}\,\underbrace{\left[-m\frac{d}{dt}\underline{v}^{(0)}\right]}_{\text{The new}}\times\hat{\underline{B}}$ "F $\times$ B" drift!

Lect #5 (cont.) $\frac{d}{dt}\underline{v}^{(0)} = \frac{c}{qB}\frac{d}{dt}\underline{F}\times\hat{B}$

$\Rightarrow \quad \underline{v}^{(1)} = -\left(\frac{c}{qB}\right)^2 m \left[\frac{d}{dt}\underline{F}\times\hat{B}\right]\times\hat{B}$



$\boxed{\underline{v}^{(1)} = \frac{mc^2}{q^2B^2}\frac{d}{dt}\underline{F}}$ called "The Polarization Drift": $\underline{v}_p$

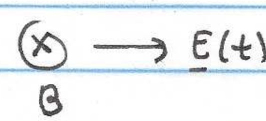
1) is in the direction of $\frac{d}{dt}\underline{F}$ $\Rightarrow$ can push matter across B-field

2) is proportional to $m \Rightarrow$ is dominated by the ions

3) it is independent of sign of charge $\Rightarrow$ electrons + ions go together in same direction

Example: Time dependent electric field $\underline{E}(t) \perp$ to $\underline{B}$

$$\underline{v}_p = \frac{mc^2}{q^2B^2}q\frac{d}{dt}\underline{E}$$



Plasma current: $\underline{j} = \sum_s' q_s n_s (\underline{v}_p)_s = \sum_s' \frac{m_s c^2}{B^2} n_s \frac{d}{dt}\underline{E}$

Ampere's law: $\nabla\times\tilde{\underline{B}} = \frac{4\pi}{c}\underline{j} + \frac{1}{c}\frac{\partial}{\partial t}\tilde{\underline{E}} = \left[\sum_s' \frac{4\pi n_s m_s}{B^2}c + \frac{1}{c}\right]\frac{\partial}{\partial t}\underline{E}$

$$= \frac{1}{c}\left[\sum_s'\left(\frac{c}{B^2/4\pi n_s m_s}\right)^2 + 1\right]\frac{\partial}{\partial t}\underline{E}$$

but $\sum_s'$ dominated by ions

$$\Rightarrow \quad \nabla\times\tilde{\underline{B}} = \frac{1}{c}\left[\underbrace{\left(\frac{c}{V_A}\right)^2}_{\text{plasma}} + \underbrace{1}_{\text{vacuum}}\right]\frac{\partial}{\partial t}\underline{E}$$

$V_A = \frac{B^2}{4\pi n M}$ is the "Alfvén speed"

$\rightsquigarrow$ Typically $\sim 10^8$ cm/sec.

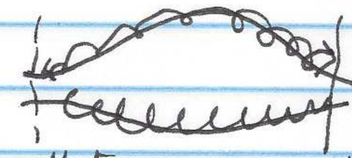
Lect #5 (cont--) $\Rightarrow \left(\frac{c}{V_A}\right)^2 \gg 1$ vacuum is not important $\perp$ to $\underline{B}$-field

$$\Rightarrow \quad \nabla \times \underline{\tilde{B}} = \frac{1}{c}\left(\frac{c}{V_A}\right)^2 \frac{\partial}{\partial t} \underline{\tilde{E}}$$

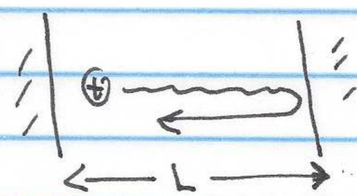
$\Rightarrow$ Leads to waves with $\underline{\tilde{E}}_\perp$ propagating with $\frac{\omega}{k_\parallel} = v_A = \frac{B}{\sqrt{4\pi n M}}$

- These are the Alfvén waves -- to be discussed later

Second Adiabatic Invariant - For systems in which reflections occur rapidly -- as in a mirror B-field



Do it simply:



$$\oint p_\parallel \, dq_\parallel = \text{const or invariant} = m v_\parallel (2L) \quad \text{if } L \text{ changes slowly}$$

$$= 4 \, \tfrac{1}{2} m v_\parallel^2 \, \frac{L}{v_\parallel} = \text{invariant} \quad \underline{\text{but}} \quad \tfrac{1}{2} m v_\parallel^2 = U \text{ the energy is a constant!}$$

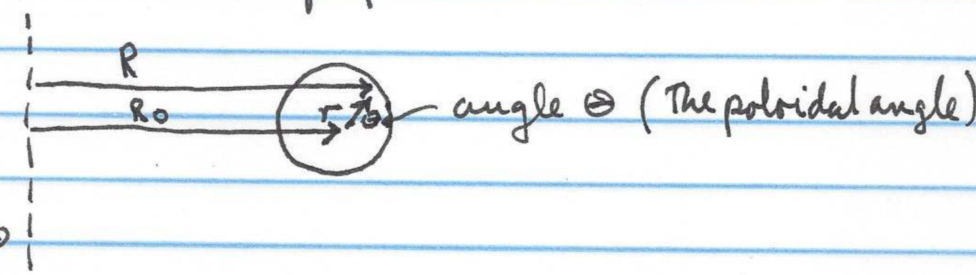
$\Rightarrow \frac{L}{v_\parallel}$ is an invariant & equal to the Bounce-Time!

$\Rightarrow$ for systems in which Bounce motion is fast compared to the change, then $\frac{L}{v_\parallel}$ is an invariant -- for mirror reflections

$$J_\parallel \equiv \oint \frac{ds}{v_\parallel} = \oint \frac{ds}{\sqrt{U - |\underline{\mu}| B(s)}} \quad \text{is an invariant}$$

This implies in a reduced description: $\boxed{f \rightarrow f(U, \underline{\mu}, J_\parallel)}$ is Vlasov Equilibrium!

Lect#6 (cont.) A more clear view of plasma X-section



Distance from center of torus to a pt inside plasma is: R
Distance from center of plasma to a pt is: r
Angle relative to mid-plane: θ
Radial dimension of plasma is: a

$$R = R_0 + r\cos\theta$$

Over The past 50 years about ~100 tokamaks have been built around The world - 5 at UCLA -

The construction constraints make The parameter $\epsilon \equiv \frac{a}{R_0} \simeq \frac{1}{3}$ for most of Them - with rare exceptions

"The standard Theory of Tokamaks" assumes ϵ is a "small parameter" and results are given as expansions in ϵ

$\epsilon \to 0$ is a cylindrical plasma

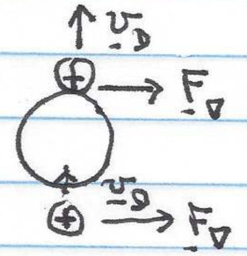
The aspect ratio is $\frac{R_0}{a}$ and ϵ is The inverse aspect ratio

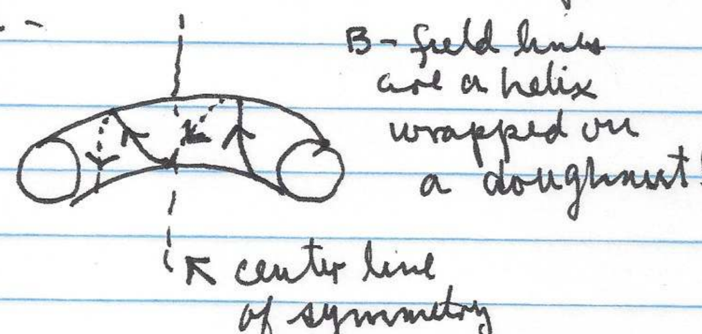
"There are some compact Tokamaks" with $\epsilon \simeq 1$ -- They are plagued with construction difficulties -- The NSTX at Princeton -
But -- There is group of scientists who like Them -

Lect #6   Tokamak orbits - Important to know the concepts even if you presently do not follow the subject -- in the future you may have job in this area -- has been the case with many students who took 222 series -

Understand $0^{\underline{th}}$ order geometry:

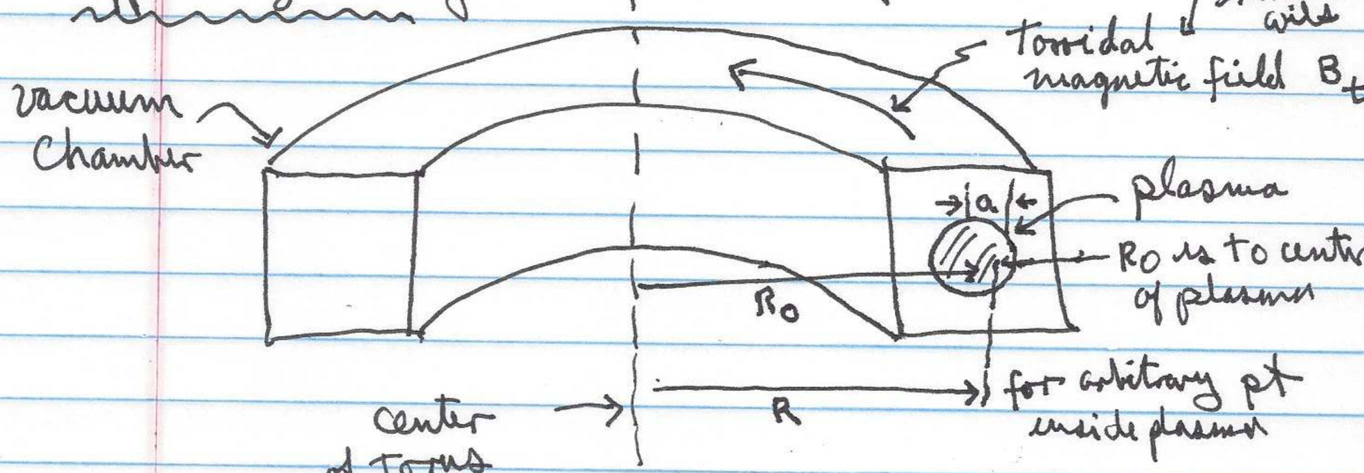
in X-section view





$\Rightarrow$ The key idea is that a particle that follows a helical field has a self-correcting orbit that cancels the uni-directional upward drift (for +) - that exists in a purely Toroidal field line -- This means that on average - a charge remains confined within the surface that is spanned by the helical field line $\equiv$ "magnetic surface"

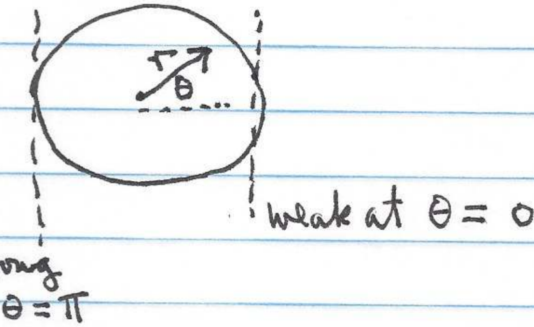
The typical geometry is: (show only half)



Lect #6 (cont.) Spatial dependence of Toroidal field -- generated by toroidal solenoid

$$B_t = \frac{B_0 R_0}{R} \quad \text{at a pt } R \text{ from centerline}$$

$$B_t = \frac{B_0 R_0}{R_0 + r\cos\theta}$$



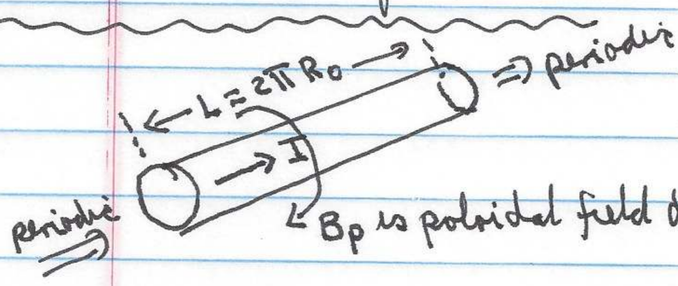
Since $r < a$ $\Rightarrow$ $\frac{r}{R_0} < \epsilon$ which is assumed to be small

$\Rightarrow$ Conventional Tokamak approximation: $B_t \simeq B_0\left(1 - \frac{r}{R_0}\cos\theta\right)$

with B_0 The value of B_t at plasma center

$\Rightarrow$ There is a mirror ratio $= \frac{B_{MAX}}{B_{Min}} \simeq 1 + 2a$ That plays a key role once we include The helical field line

Zero-order model of a tokamak - a periodic cylinder as $\epsilon \to 0$



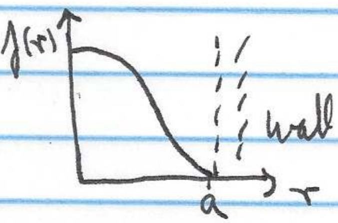
Length of cylinder is circumference of the actual torus

$$L = 2\pi R_0$$

Lect #6 (cont) Easy to calculate B_p due to cylindrical current

$$B_p(r) = \frac{2I(r)}{cr} \quad \text{in cgs}$$

with $$I(r) = \int_0^r dr' \, 2\pi r' j(r')$$
↑ current density

For Typical Ohmic operation 

The peak current is $j(0)$

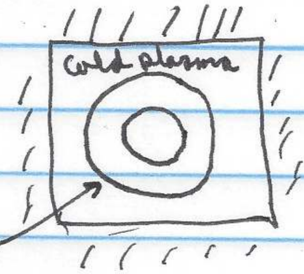
but in some advanced scenarios it is possible to have 

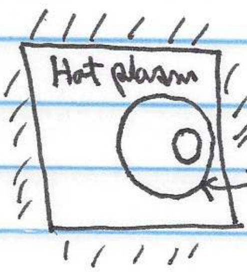
Near plasma center: $$I(r) \simeq j(0)\pi r^2$$

Consequences: $$B_p(r\to 0) = \frac{2\pi}{c} j(0)\, r \text{ , i.e., it goes to zero}$$

but importantly: $$\lim_{r\to 0} \frac{B_p}{r} = \frac{2\pi}{c} j(0) \text{ is finite!}$$

As plasmas get hot -- The pressure increases - and this pushes against the toroidal field magnetic pressure - i.e., $\beta \equiv \frac{nT}{B_t^2/8\pi}$ is finite

Consequences: 

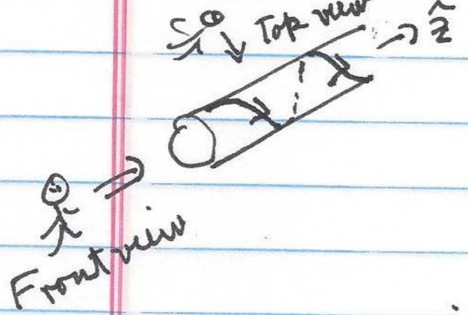


constant p surfaces are concentric circles

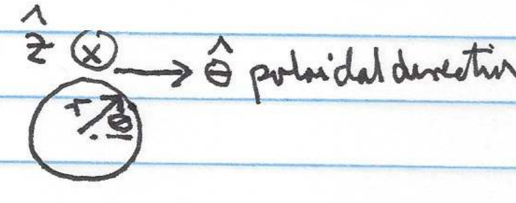
constant p surfaces are shifted outward & are not concentric.

Lect #6 (cont.) Examine single particle orbit in combined $\underline{B}_{Total} = \underline{B}_p + \underline{B}_t$

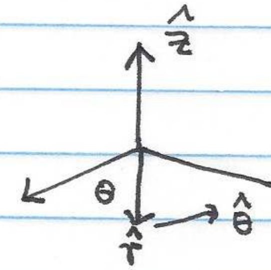
The total field is a helix that lies on a periodic cylinder



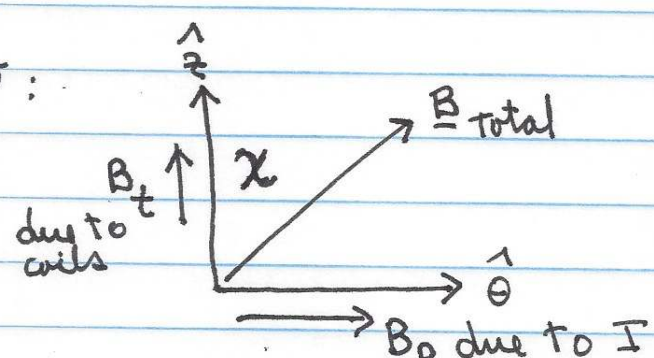
In front view



in 3 perspective:



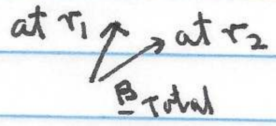
Now go to Top view:



$$\tan\chi = \frac{B_p}{B_t} \sim \frac{1}{10}$$ ↖ in typical machine ⇒ ↑ picture not to real scale!

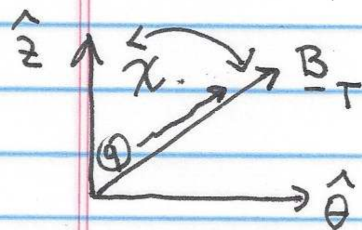
In general $\chi = \chi(r)$ ⇒ magnetic field has "shear" i.e., field lines are twisted - χ is the "pitch angle" of a field line that lies on ~~cylinder~~ cylinder of radius r

The pitch is determined by the radial variation of $f(r)$



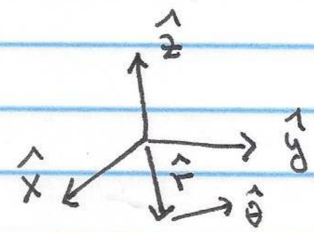
⇒ good for confinement!

Lect #6 (cont.) Properties of passing particle – it follows direction of local $\underline{B}_{Total}$ – it does not mirror – ideally $\underline{v} = v_{\parallel} \hat{B}$



$v_z = v_{\parallel} \cos \chi \; ; \; v_\theta = v_{\parallel} \sin \chi$

but in 3D coordinate system with origin at plasma center



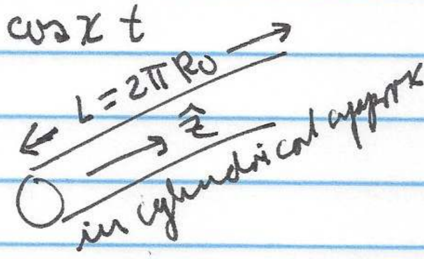
$v_\theta = r\dot{\theta} = v_{\parallel} \sin \chi$

$$\Rightarrow \quad \dot{\theta} = \frac{v_{\parallel}}{r} \sin \chi \quad \Rightarrow \quad \theta = \frac{v_{\parallel}}{r} \sin \chi \, t$$

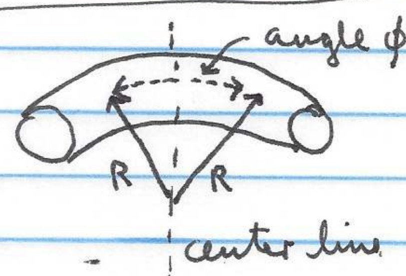
$$\dot{z} = v_{\parallel} \cos \chi \quad \Rightarrow \quad z = v_{\parallel} \cos \chi \, t$$

$$\Rightarrow \quad \boxed{\theta = \frac{z}{r} \tan \chi}$$

+ remember



but in a real Torus



For large R: $R\phi \simeq z$ ← The position along cylinder

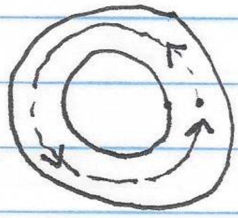
But $\underline{B}_T$ is a helix ⇒ ϕ motion is simultaneous with θ-motion

⇒ In terms of (θ, ϕ) ⇒ $\boxed{\theta = \frac{z}{r} \tan \chi = \frac{R\phi}{r} \tan \chi}$

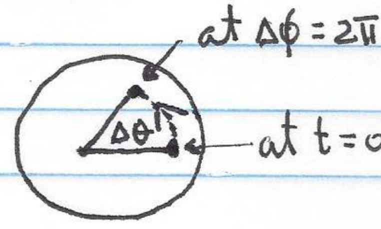
Lect #6 (cont) How much does θ (poloidal angle) change in 1 torroidal transit?

For this $\Delta\phi = 2\pi$ -- is change in torroidal angle

Top view of torus



$\Rightarrow$ Front view of plasma



The amount of $\Delta\theta \equiv \iota$ is "iota" $\iota = \frac{2\pi R}{r} \tan\chi$

Is called the "Rotational Transform" & is central to the theory of Tokamak confinement

How long does it take to move ι? now $\Delta z = v_{\parallel} \cos\chi \, \Delta t$

$$2\pi R = v_{\parallel} \cos\chi \, \Delta t \Rightarrow \Delta t \equiv t_{transit} = \frac{2\pi R}{v_{\parallel} \cos\chi}$$

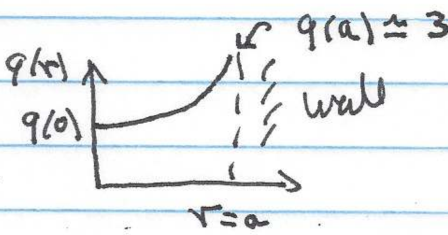
This Δt is the "Transit time"

In tokamak work -- the jargon is $q \equiv \frac{2\pi}{\iota} = \frac{r}{R} \frac{1}{\tan\chi}$ ← not the charge!

but $\tan\chi = \frac{B_p}{B_t} \Rightarrow$ $\boxed{q = q(r) = \frac{r}{R} \frac{B_t}{B_p}}$

q is called the "safety factor"

For a typical tokamak



Strong constraint from MHD is $q(0) > 1$ for a stable current

If not problems arise!

Lect #6 (cont.) What is $q(0)$ in terms of parameters?

$$q(r) = \frac{r}{R}\frac{B_t}{B_p} = \frac{B_t}{R}\frac{1}{B_p/r}$$

essentially R_0 $\Rightarrow \lim_{r\to 0} q(r) \to \frac{B_t}{R_0}\frac{1}{\lim_{r\to 0}\frac{B_p}{r}}$

but from the B_p calculation: $\lim_{r\to 0}\frac{B_p}{r} \to \frac{2\pi j(0)}{c}$

$\Rightarrow$ For stable operation $\frac{B_t}{R_0}\frac{1}{\frac{2\pi j(0)}{c}} > 1$

$\Rightarrow$ $\boxed{j(0) < \frac{B_t c}{2\pi R_0}}$ $\Rightarrow$ There is a limit on the peak current!

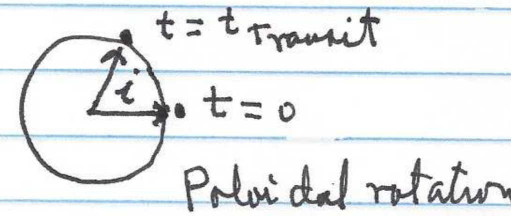
Once you select the size & strength of Field - the plasma must be run below a certain current $\Rightarrow$ the available heating from current is limited $\Rightarrow$ in general a "Hot Tokamak" must have auxilliary heating - not from the current

What Happens if $j(0) > \frac{B_t c}{2\pi R_0}$?

Instability grows & induces radial transport that makes $j(0)\downarrow$
- system develops relaxation oscillations --
jargon is "Sawtooth" oscillations

Lect #6 (cont.) Physical meaning of "q" with $q = \frac{2\pi}{\iota}$

or $\boxed{q\iota = 2\pi}$



q is how many transits along the toroidal direction are required for the particle to make a $\Delta\theta = 2\pi$ change poloidally, i.e., return to $\theta(t=0)$.

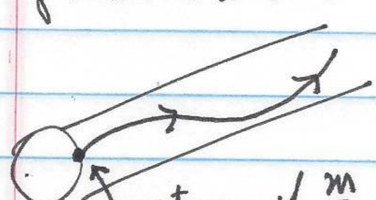
If $q = \frac{m}{n}$ with $m = 1, 2, \ldots$; $n = 1, 2 \ldots$

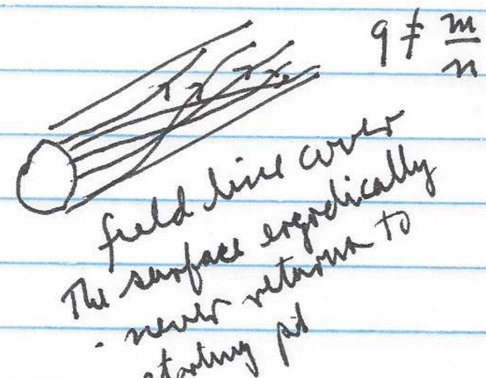
i.e., rational then $m\iota = 2\pi n$

⇒ The particle coherently returns to its starting pt

⇒ This is bad for stability because an initial perturbation does not disappear

but if q is irrational ⇒ $\neq \frac{m}{n}$ then the particle never returns to its initial point + a fluctuation simply smears out!





⇒ There are bad surfaces embedded within good surfaces in a Tokamak plasma

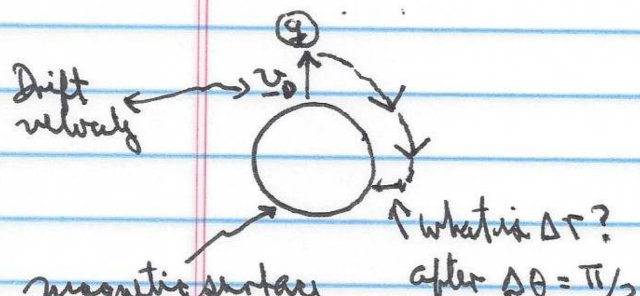
Lect #6 (cont-) How far does a particle travel to return to the same θ?

It takes "q" transits + each transit is $2\pi R_0 \Rightarrow \boxed{2\pi q R_0}$

This can be interpreted as the "effective length" or "connection length" of a tokamak -- it boosts the cylinder by a factor "q"

Deviation from magnetic surface -- for a passing particle

here call electrical charge $\equiv \mathfrak{q}$



$$\Delta r = v_D \, \Delta t$$

time to go $\pi/2$

Here v_D is due to curvature drift $\Rightarrow$ $v_D = \dfrac{m v_{\parallel}^2}{R_0} \dfrac{c}{\mathfrak{q} B_t}$

$$\Delta t = \frac{1}{4} \frac{\text{"}L\text{"}}{v_{\parallel}} = \frac{1}{4} \frac{2\pi q R_0}{v_{\parallel}} = \frac{\pi q R_0}{2 v_{\parallel}}$$

$$\Rightarrow \Delta r = \frac{m v_{\parallel}^2}{R_0} \frac{c}{\mathfrak{q} B_t} \frac{\pi q R_0}{2 v_{\parallel}} = \frac{\pi}{2} \left(\frac{mc}{\mathfrak{q} B_t}\right) q \, v_{\parallel} \quad \text{but } q = \frac{r}{R} \frac{B_t}{B_p}$$

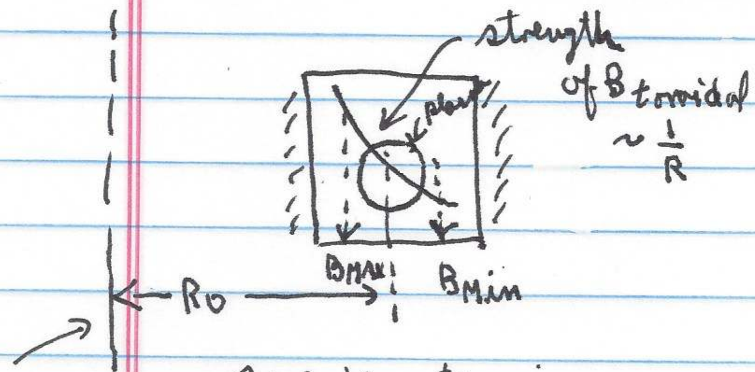
$$\Delta r = \frac{\pi}{2} \frac{r}{R} \frac{v_{\parallel}}{\Omega_p} \quad \text{with } \Omega_p \equiv \frac{\mathfrak{q} B_p}{mc} \text{ is poloidal gyrofrequency}$$

and $\rho_p \equiv \dfrac{v_{\parallel}}{\Omega_p}$ is "poloidal Larmor radius"

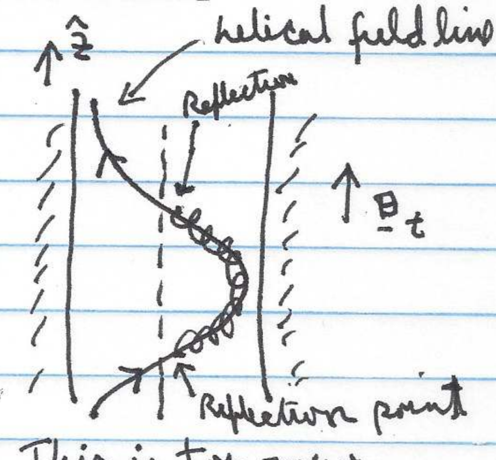
$\Rightarrow$ $\boxed{\Delta r = \dfrac{\pi}{2} \dfrac{r}{R} \rho_p}$ shows that what is really doing the confinement is B_p not B_t!

and $\dfrac{r}{R} < \epsilon$

Lect #6 (cont.) Tokamak is a continuous Mirror Machine!



This is X-section view



This is top view

There is another class of orbits: "Trapped particles" That reflect between conjugate points located toroidally where B_{MAX} is encountered by particle

Now there is a $\underline{F} = \nabla(\underline{\mu} \cdot \underline{B}) \perp \underline{B}$ => There is a drift That can be large if $v_\perp$ is large => These are The dangerous orbits -- They can strike The wall.

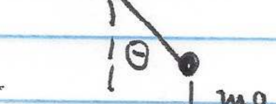
Energy: $\frac{1}{2} m v_\parallel^2 + \mu B = U = U(r, \theta)$ because $\begin{cases} v_\theta = r\dot{\theta} = v_\parallel \sin\chi \\ v_\parallel = \dfrac{r\dot{\theta}}{\sin\chi} \end{cases}$

$$\Rightarrow \quad U = \frac{1}{2} \frac{m r^2 \dot{\theta}^2}{\sin^2\chi} + \mu B_0 \left(1 - \frac{r}{R} \cos\theta\right)$$

or a scaled energy: $U' = \sin^2\chi\, U = \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{\mu B_0 r}{R} \sin^2\chi \cos\theta + \underset{\text{a constant}}{\mu B_0 \sin^2\chi}$

$\Updownarrow$ identical

by Hamiltonian for a pendulum is



$$H = \frac{1}{2} m r^2 \dot{\theta}^2 - m g r \cos\theta$$

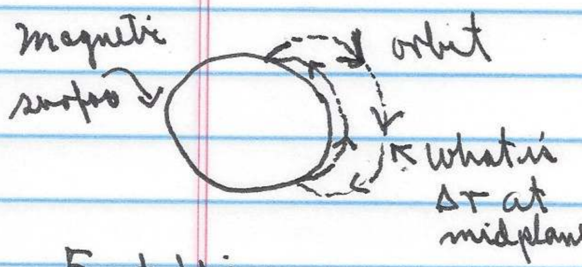
Lect #6 (cont.) For a simple pendulum the oscillation frequency

is $\omega = \sqrt{\frac{g}{r}}$ so what is "g" for a trapped particle?

by inspection "g" $\rightarrow \frac{\mu B_0}{m R_0} \sin^2 \chi$

$\Rightarrow$ For bottom of trapping well $\boxed{\omega_{\text{Trapped}} = \sqrt{\frac{\mu B_0 \sin^2\chi}{m R_0 r}}}$

How far does a trapped particle deviate from magnetic surface?



Front View

$$\Delta r = v_{\nabla B} \left(\frac{1}{4} \frac{2\pi}{\omega_T} \right)$$

$$\Delta r = \frac{m v_\perp^2}{2R} \frac{c}{q B_0} \left(\frac{\pi}{2} \frac{\sqrt{2 R_0 r}}{v_\perp^2 \sin\chi} \right)$$

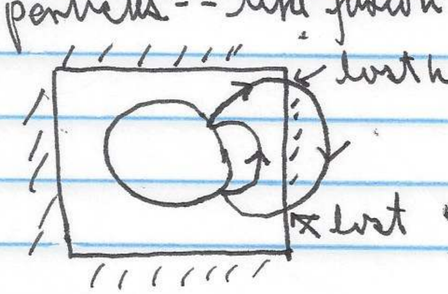
which can be put in the form: $\boxed{\Delta r = \frac{\pi}{4} \rho_p}$

where here $\rho_p = \frac{v_\perp}{\Omega_p}$

$\Rightarrow$ Excursion of trapped particles is larger than $\frac{1}{\epsilon}$ compared to excursion of passing particles

- The orbits of trapped particles are called "Banana orbits"

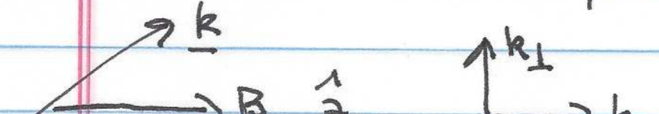
+ for energetic particles -- like fusion α- particles can strike wall



Lect #7 . AC-Linear response of Cold, Magnetized plasma (thousands of papers!)

It is central to understanding behavior of a magnetized plasma

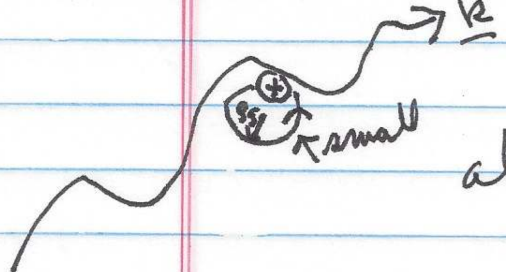
Recall: "Cold" means inertial response $\frac{\omega}{k_\parallel} \gg \bar{v}_e, \bar{v}_i$



we use both $k_\parallel = k_z$ to mean the same

also, $\omega \neq n\Omega_s$ with $n = 1, 2, \ldots$ and $\Omega_s = \frac{|q_s| B_0}{m_s c}$

+ $k_\perp \rho_s \ll 1$ with $\rho_s \equiv \frac{\bar{v}_s}{\Omega_s}$ $\Rightarrow$ Small Larmor radius!



all quantities $\Rightarrow$ $\begin{pmatrix} \underline{E} \\ \underline{B} \\ \underline{v} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \tilde{\underline{E}} \\ \tilde{\underline{B}} \\ \tilde{\underline{v}} \end{pmatrix} e^{i(\underline{k}\cdot\underline{r} - \omega t)} + \text{c.c.}$

Cold fluid Eq. of motion: $m \frac{d}{dt} \underline{v} = q\underline{E} + \frac{q}{c} \underline{v} \times \underline{B} - \underbrace{m\nu \underline{v}}_{\text{model effective drag}}$

Note here $\underline{v}$ is "fluid velocity" not particle velocity

so, in general here $\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla$

Consider "Linear Response" -- for "small" $|\underline{E}|$

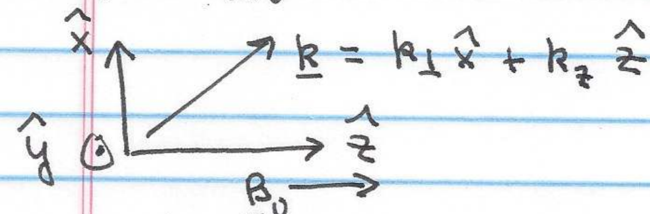
There is a linear connection -- $\tilde{\underline{v}}(\underline{k}, \omega) = \overleftrightarrow{\mu} \cdot \tilde{\underline{E}}$

Note here the symbol $\overleftrightarrow{\mu}$ is Not the magnetic moment!

$\overleftrightarrow{\mu}$ is the "Mobility Tensor"

Lect #7 (cont) In general $\overleftrightarrow{\mu} = \overleftrightarrow{\mu}(\underline{k}, \omega)$ but since $\underline{k}$ enters due to Doppler shifts $\Rightarrow$ For cold plasmas without flows
$\overleftrightarrow{\mu} = \overleftrightarrow{\mu}(\omega)$ it only depends on frequency!
But for a kinetic situation -- The $\underline{k}$-dependence is important

For uniform $\underline{B}_0$ The coordinate system can be rotated to make



i.e., $\underline{k} \cdot \hat{y} = 0$

In absence of zero-order flows The linear contribution from $\underline{v} \times \underline{B}$ is only $\underline{v} \times \underline{B}_0$ and $\underline{v} \cdot \cancel{\nabla} \underline{v}$ is 2nd order

Linearized Eq. of motion is: $m \frac{\partial}{\partial t} \underline{v} = q\underline{E} + \frac{q}{c} \underline{v} \times \underline{B}_0 - m\nu \underline{v}$

For harmonic fields $\frac{\partial}{\partial t} \to -i\omega$

in $\hat{z}$ direction: $-i\omega m \tilde{v}_z = q\tilde{E}_z - m\nu \tilde{v}_z$

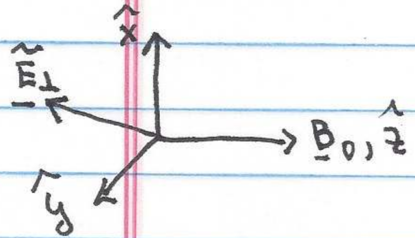
or $(-i\omega + \nu)\tilde{v}_z = \frac{q}{m}\tilde{E}_z \Rightarrow$ Define $W \equiv \omega + i\nu$

$\Rightarrow \tilde{v}_z = \frac{iq}{mW} \tilde{E}_z$

by inspection $\mu_{zz} = \frac{iq}{mW}$ and $\mu_{zx} = 0$, $\mu_{zy} = 0$

where $\overleftrightarrow{\mu} = \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix}$

Lect#7 (cont.) In transverse directions:



$$\hat{x}: \; -i\omega m \tilde{v}_x = q\tilde{E}_x + \frac{q}{c}\tilde{v}_y B_0 - m\nu \tilde{v}_x$$

$$\hat{y}: \; -i\omega m \tilde{v}_y = q\tilde{E}_y - \frac{q}{c}\tilde{v}_x B_0 - m\nu \tilde{v}_x$$

or equivalently:

$$\hat{x}: \; -iW\tilde{v}_x = \frac{q}{m}\tilde{E}_x + \frac{qB_0}{mc}\tilde{v}_y$$

$$\hat{y}: \; -iW\tilde{v}_y = \frac{q}{m}\tilde{E}_y - \frac{qB_0}{mc}\tilde{v}_x$$

Re-arrange terms to form a matrix structure + the sign operator $\sigma \equiv \frac{q}{|q|}$

$$\hat{x}: \; (1)\tilde{v}_x + \frac{\sigma\Omega}{iW}\tilde{v}_y = -\frac{q}{imW}\tilde{E}_x$$

$$\hat{y}: \; -\frac{\sigma\Omega}{iW}\tilde{v}_x + (1)\tilde{v}_y = -\frac{q}{imW}\tilde{E}_y$$

Solve by Cramer's rule with determinant D

$$D \equiv \begin{vmatrix} 1, & \frac{\sigma\Omega}{iW} \\ -\frac{\sigma\Omega}{iW}, & 1 \end{vmatrix} = 1 - \left(\frac{\sigma\Omega}{W}\right)^2 = 1 - \frac{\Omega^2}{W^2} = \frac{W^2 - \Omega^2}{W^2}$$

we will need $\frac{1}{D} = \frac{W^2}{W^2 - \Omega^2}$ which suggest a resonance at $W = \Omega$

recall that in all this $\Omega = \frac{|q|B_0}{mc} > 0$

$$\tilde{v}_x = \frac{1}{D}\begin{vmatrix} -\frac{q}{imW}\tilde{E}_x, & \frac{\sigma\Omega}{iW} \\ -\frac{q}{imW}\tilde{E}_y, & 1 \end{vmatrix} = \left[-\frac{q}{imW}\tilde{E}_x - \frac{q\sigma\Omega}{mW^2}\tilde{E}_y\right]\frac{1}{D}$$

$\Rightarrow$ by inspection $\boxed{\mu_{xz} = 0}$

Lect #7 (cont.) now for $\tilde{v}_y$

$$\tilde{v}_y = \frac{1}{D}\begin{vmatrix} 1 & , -\frac{q}{im W}\tilde{E}_x \\ -\frac{\sigma\Omega}{iW} & , -\frac{q}{imW}\tilde{E}_y \end{vmatrix} = \left[-\frac{q}{imW}\tilde{E}_y + \frac{q\sigma W}{mW^2}\tilde{E}_x\right]\frac{1}{D}$$

+ by inspection again $\boxed{\mu_{yz} = 0}$

collecting all the terms shows $\mu_{yy} = \frac{-\frac{q}{imW}}{D}$; $\mu_{yx} = \frac{q\sigma\Omega}{DW^2 m}$

collecting all the components + factoring out $\frac{iq}{mW}$ results in

$$\overleftrightarrow{\mu} = \frac{iq}{mW}\begin{pmatrix} \frac{W^2}{W^2-\Omega^2}, & \frac{i\sigma\Omega W}{W^2-\Omega^2}, & 0 \\ -\frac{i\sigma\Omega W}{W^2-\Omega^2}, & \frac{W^2}{W^2-\Omega^2}, & 0 \\ 0, & 0, & 1 \end{pmatrix}$$

This contains all the info about cold, magnetized plasmas!
MHD is automatically recovered in the limit $\omega \to 0$. The $E\times B$ drifts and polarization drifts are obtained for $\omega \ll \Omega$.

Let's become familiar with different frequency domains

1. High frequency $\omega \gg \Omega, \nu$ + recall $W = \omega + i\nu \to \omega$

$$\overleftrightarrow{\mu} \to \frac{iq}{m\omega}\begin{pmatrix} 1, & 0; & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix}$$ i.e., the unmagnetized result!

Lect #7 (cont.) Which means $\tilde{\underline{v}} = \overleftrightarrow{\mu} \cdot \tilde{\underline{E}} = \frac{iq}{m\omega} \overleftrightarrow{I} \cdot \tilde{\underline{E}} = \frac{iq}{m\omega} \tilde{\underline{E}}$ ✓

But in a neutral plasma there are typically 2 species - ions + electrons

$\Rightarrow$ There is a $\overleftrightarrow{\mu}_s$ for each and with ion mass $M \gtrsim 2000\, m$ ↑ electron mass

This means it is possible to have situations in which

$\omega \ll \Omega_e$ but $\omega \gg \Omega_i$, i.e., electrons are strongly magnetized

but the ions are unmagnetized.

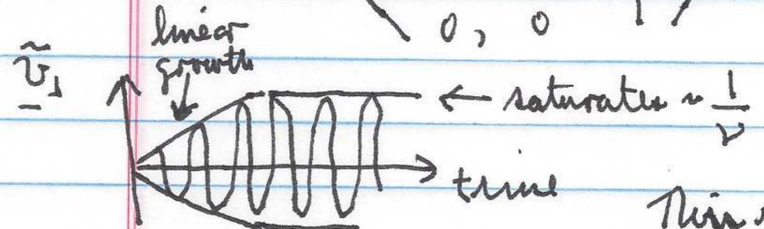
2. Near Resonant behavior : $\omega \approx \Omega$ and $\Omega \gg \nu \to$ weak collisions

Then $W^2 - \Omega^2 \to (\omega + i\nu)^2 - \Omega^2 \simeq 2i\nu\Omega$

$$\overleftrightarrow{\mu} \to \frac{iq}{m\Omega} \begin{pmatrix} \frac{\Omega}{2i\nu}, & \frac{\sigma\Omega}{2\nu}, & 0 \\ -\frac{\sigma\Omega}{2\nu}, & \frac{\Omega}{2i\nu}, & 0 \\ 0, & 0 & 1 \end{pmatrix}$$

The $\tilde{v}_\perp \to \infty$ as $\omega \to \Omega$

The response is a driven harmonic oscillator



This is an attractive heating channel

- proper description needs kinetic formulation -

3. Low frequency behavior: $\omega \ll \Omega$ but $\Omega \gg \nu$ -- weak collisions

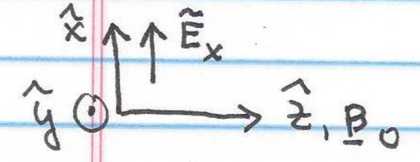
$$\overleftrightarrow{\mu} \to \frac{iq}{mW} \begin{pmatrix} -\frac{W^2}{\Omega^2}, & -i\frac{\sigma W}{\Omega}, & 0 \\ +i\frac{\sigma W}{\Omega}, & -\frac{W^2}{\Omega^2}, & 0 \\ 0, & 0, & 1 \end{pmatrix}$$

This still retains

$W = \omega + i\nu$ ↑ small

Lect #7 (cont.) What is the physical meaning of these terms?

To understand consider the simplest polarization $\tilde{\underline{E}} = \tilde{E}_x \hat{x}$



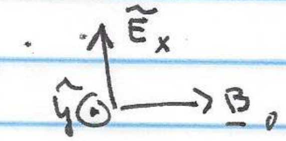
From the matrix $\Rightarrow$ $\tilde{v}_x = -\frac{iq\omega}{m\Omega^2}\tilde{E}_x = \mu_{xx}\tilde{E}_x$

and $\tilde{v}_y = -\frac{q\sigma}{m\Omega}\tilde{E}_x = \mu_{yx}\tilde{E}_x$

+ $\tilde{v}_z = 0$

Analyze the $\tilde{v}_y$ term + $q\sigma = |q|$ for both σ values

$\Rightarrow$ $\tilde{v}_y = -\frac{|q|}{m\frac{|q|B_0}{mc}} = \boxed{-\frac{c}{B_0}\tilde{E}_x}$ $\Rightarrow$ look at



$\Rightarrow$ $\tilde{\underline{E}} \times \hat{\underline{B}} = -\hat{y}$

or $\tilde{v}_y$ is simply the $\underline{E}\times\underline{B}$ velocity!

$\Rightarrow$ it comes from the off-diagonal

$\Rightarrow$ The off-diagonal terms μ_{xy} + μ_{yx} represent the $\underline{E}\times\underline{B}$ drift generalized to all orders of $\frac{\omega}{\Omega}$

But what about the diagonal term? $\tilde{v}_x = -\frac{iq}{m\Omega^2}(\omega + i\nu)\tilde{E}_x$

first consider $\omega >> \nu$ - collisions not important $\Rightarrow$ $\tilde{v}_x = \frac{-iq\omega}{m\Omega^2}\tilde{E}_x$

but $-i\omega\tilde{E}_x \Rightarrow \frac{d}{dt}E_x$ $\Rightarrow$ $v_x = \frac{q}{m\Omega^2}\frac{d}{dt}E_x$

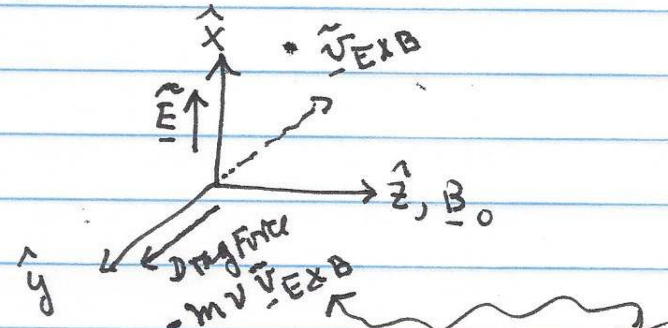
or $\boxed{v_x = \frac{mc^2}{qB_0^2}\frac{d}{dt}E_x}$ is the polarization drift!

Lect #7 (cont.) Now consider strong collisions $\omega << \nu << \Omega$ but Magnetized!

$$\tilde{v}_x \to \frac{q\nu}{m\Omega^2}\tilde{E}_x$$ What is This?

It is the result of a double-drift!

Now need 3D to see it



The drag force associated with the $E\times B$ drift is $(-m\nu)\frac{c}{B_0}\underline{E}\times\hat{B}$

$$\underline{F}_{Drag} = m\nu\frac{c}{B_0}\tilde{E}_x\,\hat{y}$$

For a force also exerts a drift $\quad \underline{v}_F = \frac{c}{qB_0}\underline{F}\times\hat{B}$

apply to this case:

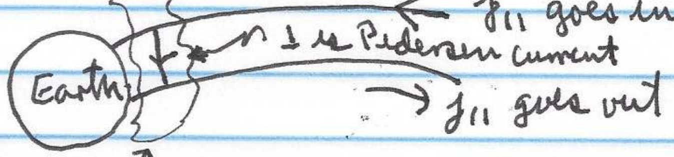
$$\underline{v}_{Drag} = \frac{c}{qB_0}\frac{m\nu c}{B_0}\tilde{E}_x\,\underbrace{\hat{y}\times\hat{z}}_{\hat{x}}$$ || Proportional To mass!

$$\underline{v}_{Drag} = \frac{q}{m}\frac{\nu}{\Omega^2}\tilde{E}_x\,\hat{x}$$ ✓ yes, is identical

$\Rightarrow$ This term due to the diagonal μ_{xx} can generate a current due to ~~electron~~ ion-neutral collision called the

"Pedersen Current" - It is an inertial effect + is thus dominated by the ions due to $M >> m$

Lect #7 (cont.) This drag drift plays an important role because it allows currents to flow $\perp$ to $\underline{B}$. -- Coulomb collisions cancel currents - out because E×B is the same for ions & electrons => There is no relative drag force. The Pedersen current is responsible for current closure in the lower ionosphere of currents injected from Magnetosphere along field lines



(Here $\nu = 0$)

neutrals in ionosphere provide ν

Now that we know the fluid velocities => can find the currents!

$$\underline{\tilde{J}} = \sum_s q_s n_s \underline{\tilde{v}}_s = \sum_s q_s n_s \overleftrightarrow{\mu}_s \cdot \underline{\tilde{E}}$$

with n_s the zero-order density of species 's'

but there is a tensor conductivity $\quad \underline{\tilde{J}} = \overleftrightarrow{\sigma}(\omega) \cdot \underline{\tilde{E}}$

that depends on ω -- for a hot plasma $\quad \overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}(\underline{k}, \omega)$ ↑also

which means $\quad \overleftrightarrow{\sigma}(\omega) = \sum_s q_s n_s \overleftrightarrow{\mu}_s(\omega)$

explicitly: $$\overleftrightarrow{\sigma}(\omega) = \sum_s \frac{i\omega_{ps}^2}{4\pi W} \begin{pmatrix} \frac{W^2}{W^2 - \Omega_s^2}, & \frac{i\lambda W \Omega_s}{W^2 - \Omega_s^2}, & 0 \\ -\frac{i\lambda W \Omega_s}{W^2 - \Omega_s^2}, & \frac{W^2}{W^2 - \Omega_s^2}, & 0 \\ 0, & 0, & 1 \end{pmatrix}$$

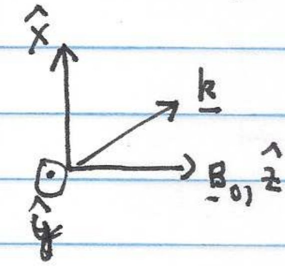
here we use $\lambda = \begin{cases} + & \text{for ions} \\ - & \text{for electrons} \end{cases}$

with $\quad W = \omega + i\nu_s$

Lect #8. (cont.) Now make $(\tilde{\underline{J}}, \tilde{\underline{E}})$ self-consistent using Maxwell's Eqs.

$$\begin{pmatrix} \underline{E}\\ \underline{B}\\ \underline{J}\end{pmatrix} = \frac{1}{2} \begin{pmatrix}\tilde{\underline{E}}\\ \tilde{\underline{B}}\\ \tilde{\underline{J}}\end{pmatrix} e^{i(\underline{k}\cdot\underline{r}-\omega t)} + CC \Rightarrow \text{plane waves}$$

with $\underline{k} = k_\perp \hat{x} + k_z \hat{z}$



Faraday's Law: $\nabla\times \underline{E} = -\frac{1}{c} \frac{\partial}{\partial t} \underline{B} \Rightarrow \boxed{\underline{k}\times\tilde{\underline{E}} = \frac{\omega}{c}\tilde{\underline{B}}} \Rightarrow$ let $k_0 \equiv \frac{\omega}{c}$

Ampere's Law: $\nabla\times \underline{B} = \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial}{\partial t}\underline{E} \Rightarrow \boxed{i\underline{k}\times\tilde{\underline{B}} = \frac{4\pi}{c}\tilde{\underline{J}} - i\frac{\omega}{c}\tilde{\underline{E}}}$

but $\tilde{\underline{J}} = \overleftrightarrow{\sigma}\cdot\tilde{\underline{E}}$

$$\Rightarrow \quad i\,\underline{k}\times \tilde{\underline{B}} = i\frac{\omega}{c}\left[\frac{4\pi}{i\omega}\overleftrightarrow{\sigma} - \overleftrightarrow{I}\right]\cdot\tilde{\underline{E}}$$

Define The Dielectric Tensor: $\overleftrightarrow{\epsilon}(\omega) \equiv \overleftrightarrow{I} - \frac{4\pi}{i\omega}\overleftrightarrow{\sigma}$

Note This is ω + not W a common mistake!

$$\Rightarrow \quad \boxed{\underline{k}\times\tilde{\underline{B}} = -k_0\overleftrightarrow{\epsilon}\cdot\tilde{\underline{E}}}$$ This is Ampere's Law in $(\underline{k},\omega)$ form

insert from Faraday's law $\Rightarrow \quad \underline{k}\times \underline{k}\times\tilde{\underline{E}} = -k_0^2\overleftrightarrow{\epsilon}\cdot\tilde{\underline{E}}$

or $\quad \left[\underline{k}(\underline{k}\cdot\tilde{\underline{E}}) - k^2\tilde{\underline{E}}\right] = -k_0^2 \overleftrightarrow{\epsilon}\cdot\tilde{\underline{E}}$

$$\Rightarrow \quad (\underline{k}\underline{k} - k^2 \overleftrightarrow{I} + k_0^2\overleftrightarrow{\epsilon})\cdot\tilde{\underline{E}} = \boxed{0}$$ $\Leftarrow$ This is in absence of external sources or antennas

in general $\quad \tilde{\underline{J}} = \tilde{\underline{J}}_{plasma} + \tilde{\underline{J}}_{external}$

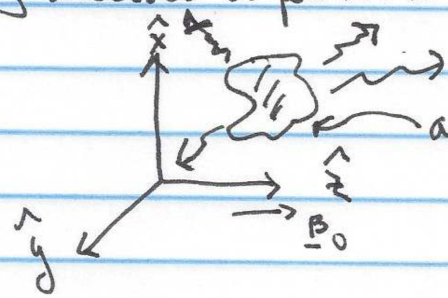
in The presence of external currents: $\quad \overleftrightarrow{M}\cdot\tilde{\underline{E}} = \frac{4\pi k_0}{ic}\tilde{\underline{J}}_{ext}$

Lect #7 (cont.) The formal solution to the problem of exciting waves in a plasma is given by

$$\underline{E}(\underline{r},t) = \int \frac{d^3k}{(2\pi)^3} \, \frac{4\pi}{ic} k_0 \, e^{i(\underline{k}\cdot\underline{r}-\omega t)} \, \overleftrightarrow{M}^{-1} \cdot \underline{\tilde{J}}_{ext}$$

with $\overleftrightarrow{M} \equiv \underline{k}\,\underline{k} - k^2 \overleftrightarrow{I} + k_0^2 \overleftrightarrow{\epsilon}$

This contains all the problems associated with antennas radiating inside a plasma that is magnetized



antenna given by $\underline{\tilde{J}}_{ext}(\underline{k},\omega)$

The normal modes of a magnetized plasma, i.e., not driven by an antenna are obtained from:

$$\mathrm{Det}\,|\overleftrightarrow{M}| = 0 \quad \text{or} \quad \mathrm{Det}\,|\underline{k}\,\underline{k} - k^2 \overleftrightarrow{I} + k_0^2 \overleftrightarrow{\epsilon}| = 0$$

where $\overleftrightarrow{\epsilon} = \begin{pmatrix} \epsilon_{xx}, & \epsilon_{xy}, & 0 \\ \epsilon_{yx}, & \epsilon_{yy}, & 0 \\ 0, & 0, & \epsilon_{zz} \end{pmatrix}$ for a 'cold' plasma without flow

in which from $\overleftrightarrow{\mu}$ one obtains -- here for $\nu = 0$, i.e., $W \to \omega$

$$\epsilon_{xx} = \epsilon_{yy} \equiv \epsilon_\perp = 1 - \sum_s{}' \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}$$

$$\epsilon_{xy} = -i \sum_s{}' \frac{\lambda_s \omega_{ps}^2 \Omega_s}{\omega(\omega^2 - \Omega_s^2)} \, , \quad \epsilon_{yx} = -\epsilon_{xy}$$

$$\epsilon_{zz} \equiv \epsilon_\parallel = 1 - \sum_s{}' \frac{\omega_{ps}^2}{\omega^2}$$

Lect # 8 . Investigate the Normal or "collective" modes of a cold, magnetized plasma

i.e., solutions to $\text{Det}\,|\overleftrightarrow{M}| = 0$

For $\underline{k} = k_\perp \hat{x} + k_z \hat{z} \Rightarrow \underline{k}\,\underline{k} = k_\perp^2 \hat{x}\hat{x} + k_\perp k_z \hat{x}\hat{z} + k_\perp k_z \hat{z}\hat{x} + k_z^2 \hat{z}\hat{z}$

Recall $\overleftrightarrow{M} = \underline{k}\,\underline{k} - k^2 \overleftrightarrow{I} + k_0^2 \overleftrightarrow{\epsilon}$ and $k^2 = k_\perp^2 + k_z^2$

$$\Rightarrow \text{Det}\,\overleftrightarrow{M} = \begin{vmatrix} -k_z^2 + k_0^2 \epsilon_\perp, & k_0^2 \epsilon_{xy}, & k_\perp k_z \\ k_0^2 \epsilon_{yx}, & -k^2 + k_0^2 \epsilon_\perp, & 0 \\ k_\perp k_z, & 0, & -k_\perp^2 + k_0^2 \epsilon_{\parallel} \end{vmatrix} = 0$$

Useful to introduce the "Index of Refraction" $n_j \equiv \frac{k_j}{k_0}$

$$\Rightarrow \begin{vmatrix} \epsilon_\perp - n_z^2, & \epsilon_{xy}, & n_\perp n_z \\ \epsilon_{yx}, & \epsilon_\perp - n^2, & 0 \\ n_\perp n_z, & 0, & \epsilon_{\parallel} - n_\perp^2 \end{vmatrix} = 0 \quad \text{with } n^2 = n_\perp^2 + n_z^2$$

There are clumsy + easy ways to expand -- The easy is to take advantage of the 2-zeros - Start from bottom left-corner + go right!

$$0 = (n_\perp n_z)(-n_\perp n_z)(\epsilon_\perp - n^2) - 0 + (\epsilon_{\parallel} - n_\perp^2)\left[(\epsilon_\perp - n_z^2)(\epsilon_\perp - n^2) - \epsilon_{xy}\epsilon_{yx}\right]$$

$$\boxed{0 = -(n_\perp n_z)^2 (n^2 - \epsilon_\perp) + (n_\perp^2 - \epsilon_{\parallel})\left[(n_z^2 - \epsilon_\perp)(n^2 - \epsilon_\perp) - \epsilon_{xy}\epsilon_{yx}\right]}$$

where the order of $\epsilon_\perp - n^2 \to (-1)(n^2 - \epsilon_\perp)$ is reversed to symmetrize

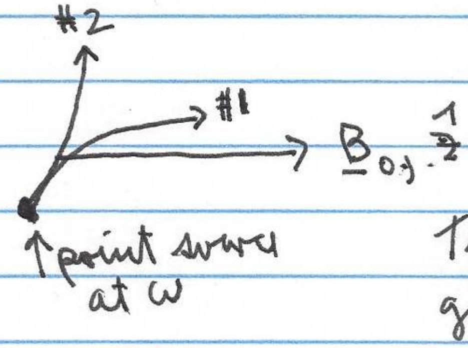
Lect #8 (cont.) Counting the powers of n_j it shows that this is a polynomial of n^4 order - i.e., a quartic - known as the "Booker Quartic".

Since all the n_j appear as n_j^2 => 2 solutions are trivial because $-n_j$ also solves the Eq. => right-left + up-down symmetry is preserved because there are no gradients of plasma parameters.

=> For a given frequency "ω" there are 2 different modes in a cold, magnetized plasma. If kinetic effects are retained then ϵ_{ij} depends on k and the order increases which imply more modes exist - - These are the "Bernstein modes" - But we now only consider the cold case - The consequence of the 2 modes is

An antenna at fixed ω radiates 2 different waves #1 and #2 with up-down and right-left symmetry

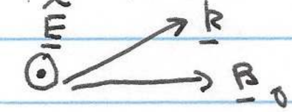
The system is "birefringent" - Energy from a point source follows different paths



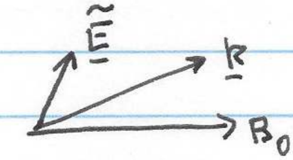
Typically one mode has a group velocity more along $\underline{B}_0$ and the other more $\perp$ to $\underline{B}_0$

Lect #8 (cont.) In general The 2 different modes have different polarization

One is purely E&M :



The other is mixed E&M + electrostatic :

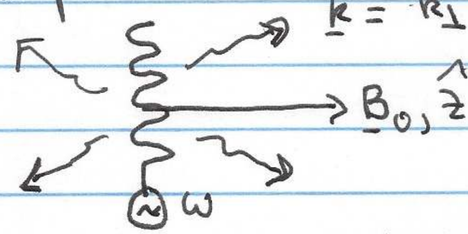


The complicated-looking dispersion relation can be solved exactly - analytically by defining auxialliary variables -- This is not so widely known -- but it is extremely useful in solving complicated problems because it is just a plug-in, There is no need for root finding -- I will send you the expressions separately -

Proceed to examine various limits of Booker Quartic.

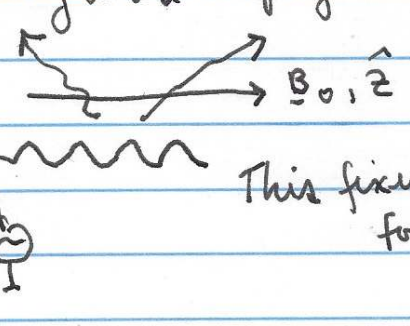
There are 2 important physical situations defined by how the external currents or antennae are arranged - here is how it is done in practice

$\underline{k} = k_\perp \hat{x} + k_z(k_\perp, \omega)\,\hat{z}$



This fixes $k_\perp$ + The plasma selects k_z for a given real ω

another commonly used configuration is:



$\underline{k} = k_\perp(k_\parallel, \omega)\,\hat{x} + k_\parallel \hat{z}$

This fixes $k_\parallel$ + The plasma selects $k_\perp$ for a given real ω

note we use $k_z = k_\parallel$ interchangeably

Lect #8 (cont.) General Disp. relation is:

$$0 = -(n_\perp^2 n_\parallel^2)\left[n^2 - \epsilon_\perp\right] + (n_\perp^2 - \epsilon_\parallel)\left[(n_\parallel^2 - \epsilon_\perp)(n^2 - \epsilon_\perp) - \epsilon_{xy}\epsilon_{yx}\right]$$

Examine Parallel Propagation: $n_\perp = 0$ $\quad \xrightarrow{\underline{k}} \underline{B}_0, \hat{z}$

$$\Rightarrow n^2 = n_\parallel^2$$

or $$0 = (-\epsilon_\parallel)\left[(n_\parallel^2 - \epsilon_\perp)^2 - \epsilon_{xy}\epsilon_{yx}\right]$$

There are two possibilities : $\epsilon_\parallel = 0$ and $(n_\parallel^2 - \epsilon_\perp)^2 - \epsilon_{xy}\epsilon_{yx} = 0$

The mode corresponding to $\epsilon_\parallel = 0$ is the cold plasma oscillations of 222A

$$0 = 1 - \sum_s{}' \frac{\omega_{ps}^2}{\omega^2} \quad \text{or} \quad \omega = \pm(\omega_{pe}^2 + \omega_{pi}^2)^{1/2} \simeq \pm\omega_{pe}$$

If kinetic effects are included then $\epsilon_\parallel = 0$ contains all the Physics of 222 B, i.e., Langmuir waves, IAW, Landau Damping streaming instabilities, etc... In other words, the electrostatic phenomena of an unmagnetized plasma exists along $\underline{B}_0$!

The new magnetized effects appear in the other possible zero

$$(n_\parallel^2 - \epsilon_\perp)^2 - \epsilon_{xy}\epsilon_{yx} = 0$$

but recall $\epsilon_{xy} = -i\sum_s{}' \frac{\lambda_s \Omega_s \omega_{ps}^2}{\omega(\omega^2 - \Omega_s^2)}$ and $\epsilon_{yx} = \overset{\text{note}}{-}\epsilon_{xy}$

$$\Rightarrow (n_\parallel^2 - \epsilon_\perp^2) - \epsilon_{xy}(-\epsilon_{xy}) = (n_\parallel^2 - \epsilon_\perp^2)^2 + (\epsilon_{xy})^2 = 0$$

or $$\left[(n_\parallel^2 - \epsilon_\perp^2) - i\epsilon_{xy}\right]\left[(n_\parallel^2 - \epsilon_\perp^2) + i\epsilon_{xy}\right] = 0$$

Lect #8 (cont.) As expected, there are 2 different roots for a given ω

$$n_\parallel^2 = \epsilon_\perp + i\,\epsilon_{xy} \qquad \text{call the \#1 root (or the + root)}$$

$$n_\parallel^2 = \underbrace{\epsilon_\perp}_{\text{is polarization current}} - i\,\underbrace{\epsilon_{xy}}_{\text{is the } E\times B \text{ current}} \qquad \text{call the \#2 root (or the − root)}$$

where $\epsilon_\perp = 1 - \sum_s{}' \dfrac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}$ and $\epsilon_{xy} = -i \sum_s{}' \dfrac{\lambda_s \Omega_s \omega_{ps}^2}{\omega(\omega^2 - \Omega_s^2)}$

For root #1 (+ root): $\epsilon_\perp + i\epsilon_{xy} = 1 - \sum_s{}' \dfrac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}\left[1 - \dfrac{\lambda_s \Omega_s}{\omega}\right]$

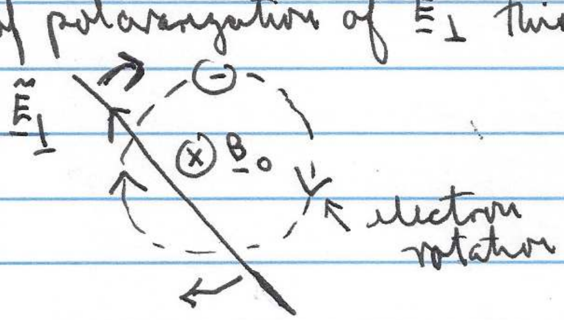
For root #2 (− root): $\epsilon_\perp - i\epsilon_{xy} = 1 - \sum_s{}' \dfrac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}\left[1 + \dfrac{\lambda_s \Omega_s}{\omega}\right]$

Write out #1 explicitly for ions + electrons

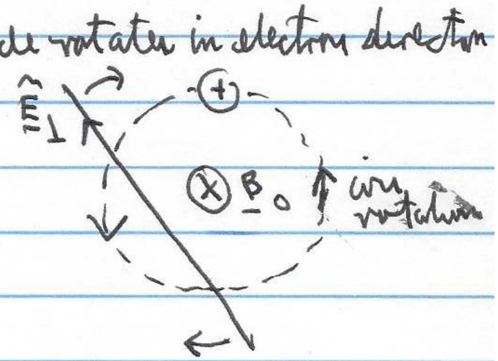
$$\epsilon_\perp + i\epsilon_{xy} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2}\left[1 + \frac{\Omega_e}{\omega}\right] - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}\left[1 - \frac{\Omega_i}{\omega}\right]$$

$$n_\parallel^2 = 1 - \underbrace{\frac{\omega_{pe}^2}{\omega(\omega - \Omega_e)}}_{\text{singular for electrons}} - \underbrace{\frac{\omega_{pi}^2}{\omega(\omega + \Omega_i)}}_{\text{non-resonant for ions}}$$

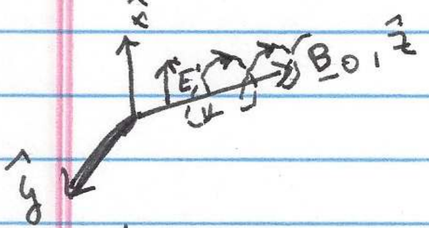
In terms of polarization of $\tilde{\underline{E}}_\perp$ this means mode rotates in electron direction



and for ions



Lect #8 (cont.) This root #1 represents a Right-Handed polarized mode that propagates along the magnetic field.



Note that rotation direction is relative to $\underset{\sim}{B}_0$, not to $\underset{\sim}{k}$

if $\underset{\sim}{k}$ is reversed, i.e, propagation is against $\underset{\sim}{B}_0$

The polarization remains the same - i.e., in the direction of electron rotation! - This gets confused at times -

Now examine root #2 (- root) for ions and electrons

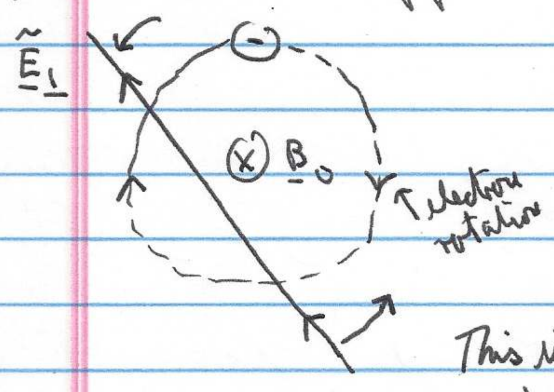
$$\epsilon_\perp - i\epsilon_{xy} = 1 - \frac{\omega_{pe}^2}{\omega^2-\Omega_e^2}\left[1-\frac{\Omega_e}{\omega}\right] - \frac{\omega_{pi}^2}{\omega^2-\Omega_i^2}\left[1+\frac{\Omega_i}{\omega}\right]$$

$$n_\parallel^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega+\Omega_e)} - \frac{\omega_{pi}^2}{\omega(\omega-\Omega_i)}$$

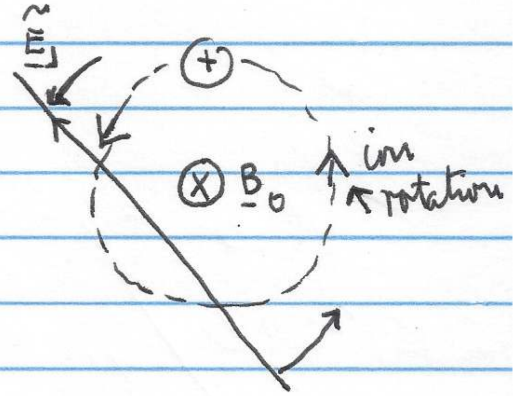
non-resonant for electrons

singular for ions

This mode now has opposite polarization



and for ions



This is a Left-Handed polarized mode relative to $\underset{\sim}{B}_0$ direction

Lecture #8 (cont.) The jargon that is used to refer to these modes is

(+ root) → Resonates with electrons ⇒ "R-mode"

(− root) → Resonates with ions ⇒ "L-mode"

Before getting into further details about wave propagation it is important to develop a quantitative "feel" for the magnitude of the parameters involved

$$\omega_{pe} = 5.6 \times 10^4 \sqrt{n}\ \frac{\text{rad}}{\text{sec}} \ ; \ \Omega_e = 1.8 \times 10^7\, B\ \frac{\text{rad}}{\text{sec}} \quad \left\{ \begin{array}{l} n \text{ in cm}^{-3} \\ B \text{ in Gauss} \end{array} \right.$$

The degree of magnetization in a plasma is determined by the ratio

$$\frac{\omega_{pe}}{\Omega_e} \left\{ \begin{array}{l} \to \infty \Rightarrow \text{unmagnetized} \\ \to 0 \Rightarrow \text{strongly magnetized} \end{array} \right.$$

in # This is:
$$\frac{\omega_{pe}}{\Omega_e} = \frac{5.6 \times 10^4 \sqrt{n}}{1.8 \times 10^7\, B} \simeq 3 \times 10^{-3} \frac{\sqrt{n}}{B}$$

Look at some important cases of current interest.

LAPD: $n \simeq 10^{12}\ \text{cm}^{-3}$, $B \simeq 1\ \text{kg}$ $\Rightarrow \frac{\omega_{pe}}{\Omega_e} = \frac{3 \times 10^{-3}}{10^3}\sqrt{10^{12}} \simeq 3 > 1$

ITER: $n \simeq 10^{14}\ \text{cm}^{-3}$, $B \simeq 50\ \text{kg}$ $\Rightarrow \frac{\omega_{pe}}{\Omega_e} = \frac{3 \times 10^{-3}}{5 \times 10^4}\sqrt{10^{14}} \simeq 0.6 < 1$

Ionosphere: $n \simeq 10^5\ \text{cm}^{-3}$, $B \simeq \frac{1}{2}\ \text{G}$ $\Rightarrow \frac{\omega_{pe}}{\Omega_e} = \frac{3 \times 10^{-3}}{1/2}\sqrt{10^5} \simeq 2 > 1$

The lesson learned from this exercise is that over a wide range of parameter values in actual plasma situations

$$\frac{\omega_{pe}}{\Omega_e} \sim \mathcal{O}(1) \quad \text{i.e., a parameter of order 1}$$

Lect #8 (cont). - A side remark on practical numerology that may be of use to you in the future - How to estimate construction cost?

How can you guess the cost of ITER?

From the magnetization Table one sees that ITER has a factor (5) larger magnetization than LAPD - but the energy of magnetic fields goes as $B^2 d$ to achieve that ITER has to increase the B field by a factor (50) from LAPD

So, we can guess the cost of ITER by comparing to LAPD cost

$$\text{Cost of ITER} \sim (50)^2 (\text{cost of LAPD}) \sim 25 \times 10^2 \times 10^7 \$$$

$$\sim \$25 \text{ Billion} - \text{which is not far-off.}$$

And for the construction of RF equipment needed to explore the AC properties that we are currently studying - a useful metric is $1/watt => the RF system needed to heat ITER is planned to operate at 40 MW => This will require ~ $\boxed{\$40 \text{ Million}}$ - But what if you apply this guessing to your μW oven at home? - Such an oven operates at ~ 1 kW => you should pay $1,000 but you know that you can get one at Costco or Walmart for less than $100 - i.e., off by factor of (10) - - Why? - Is mass production!

The 1st generation μW were ~ $1,000 but look at it now.
- The same rule applies to comparing a Ferrari vs a Ford today!

Lect #8 (cont.) The other ion-relevant parameters are

$$\Omega_i = \frac{|q|B_0}{MC} = \frac{m}{M}\Omega_e \quad \text{and} \quad \omega_{pi} = \sqrt{\frac{4\pi e^2 n}{M}} = \sqrt{\frac{m}{M}}\,\omega_{pe}$$

Their ratio is: $\frac{\omega_{pi}}{\Omega_i} = \sqrt{\frac{M}{m}}\,\frac{\omega_{pe}}{\Omega_e} > 40\,\frac{\omega_{pe}}{\Omega_e}$

but since in general $\frac{\omega_{pe}}{\Omega_e}$ is $\mathcal{O}(1) \Rightarrow \frac{\omega_{pi}}{\Omega_i} \gtrsim 40$

and in $\epsilon_\perp$ and ϵ_{xy} the quantity enters as $\left(\frac{\omega_{pi}}{\Omega_i}\right)^2 > 10^3$

So, in making practical approximations one should keep in mind these values

Examine R and L modes at high-frequency: $\omega \gg \Omega_i, \omega_{pi}$

ions are unmagnetized!

$$R \Rightarrow \quad n_\parallel^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega-\Omega_e)} - \frac{\omega_{pi}^2}{\omega(\omega+\Omega_i)}$$

negligible

$$n_\parallel^2 \to 1 - \frac{\omega_{pe}^2}{\omega(\omega-\Omega_e)} - \frac{\omega_{pi}^2}{\omega^2}$$

and further for $\omega \gg \omega_{pi}$ can be dropped

$\Rightarrow$ response at high-frequency is dominated by electrons

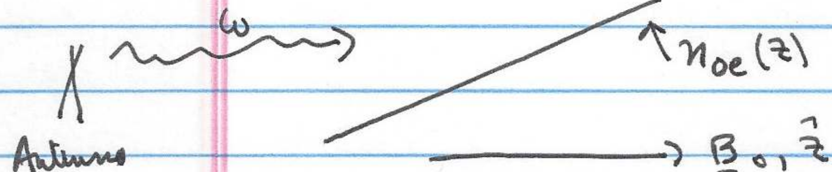
$$n_\parallel^2 \to \cancel{1-\frac{\omega_{pe}^2}{\omega(\omega-\Omega_e)}}\; 1 - \frac{\omega_{pe}^2}{\omega(\omega-\Omega_e)} \quad \text{for R-root}$$

and likewise for the L-root

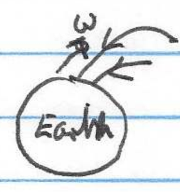
$$n_\parallel^2 \to 1 - \frac{\omega_{pe}^2}{\omega(\omega+\Omega_e)}$$

What is the consequence for propagation?

Lect #8 (cont.) Say there is a density gradient along the B-field

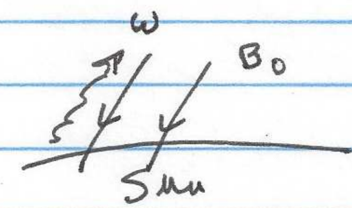


Typical scenario could be:



a radio transmitter in Alaska

or a wave radiated from the surface of the sun:



for R-mode: There is a cut-off where $n_{\parallel} = 0$

for $n_{\parallel}^2 > 0$ the mode propagates + for $n_{\parallel}^2 < 0$ it is evanescent

Cut-off is at $\quad 1 - \dfrac{\omega_p^2}{\omega(\omega - \Omega_e)} = 0 \Rightarrow \quad \omega_p^2 = \omega(\omega - \Omega_e)$

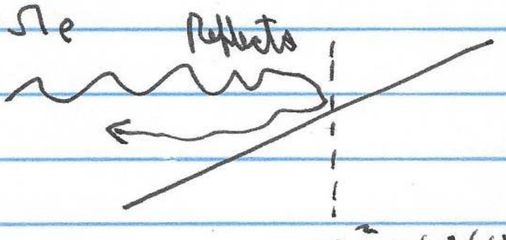
Since $\omega_p^2 > 0 \Rightarrow$ This exists only if $\omega > \Omega_e$

for $\omega < \Omega_e \Rightarrow n_{\parallel}^2 > 0$ for all ω_{pe} !

$\Rightarrow$

propagate all along into plasma

but for $\omega > \Omega_e$



a value less than the unmagnetized case $\omega_{pe}^2 = \omega^2$

$$\omega_{pe}^2 = \omega(\omega - \Omega_e) = \omega^2\left(1 - \frac{\Omega_e}{\omega}\right)$$

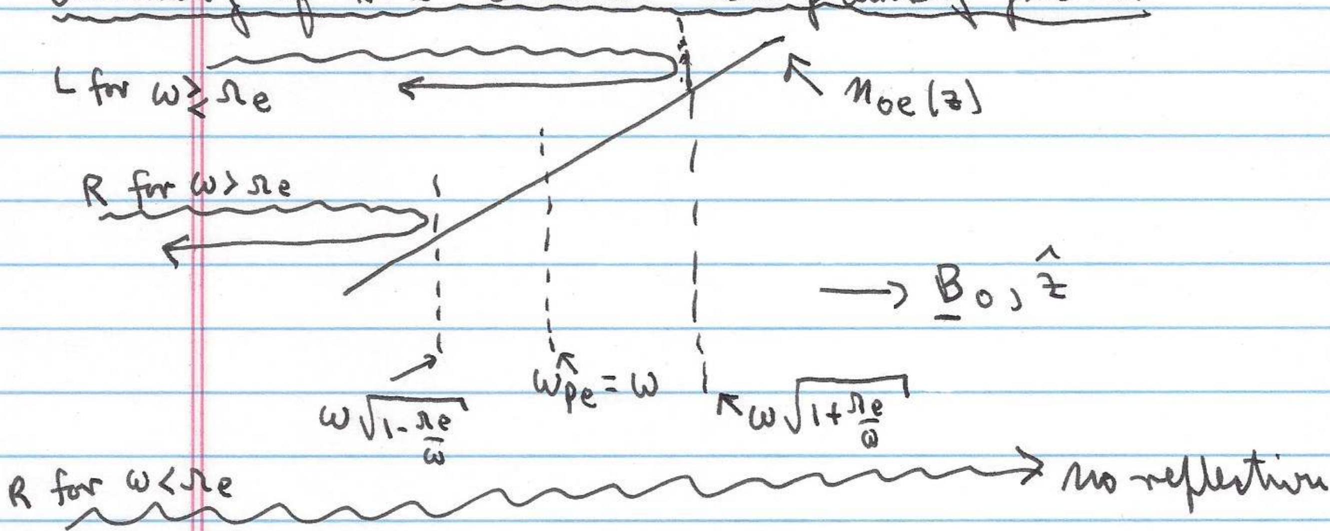
Lect #9 _ Reflection conditions for L-mode

now $\quad n_{\parallel}^2 = 1 - \dfrac{\omega_{pe}^2}{\omega(\omega+\Omega_e)}$

cut-off at $\quad \omega_{pe}^2 = \omega(\omega+\Omega_e) = \omega^2\left(1+\dfrac{\Omega_e}{\omega}\right) > \omega^2$

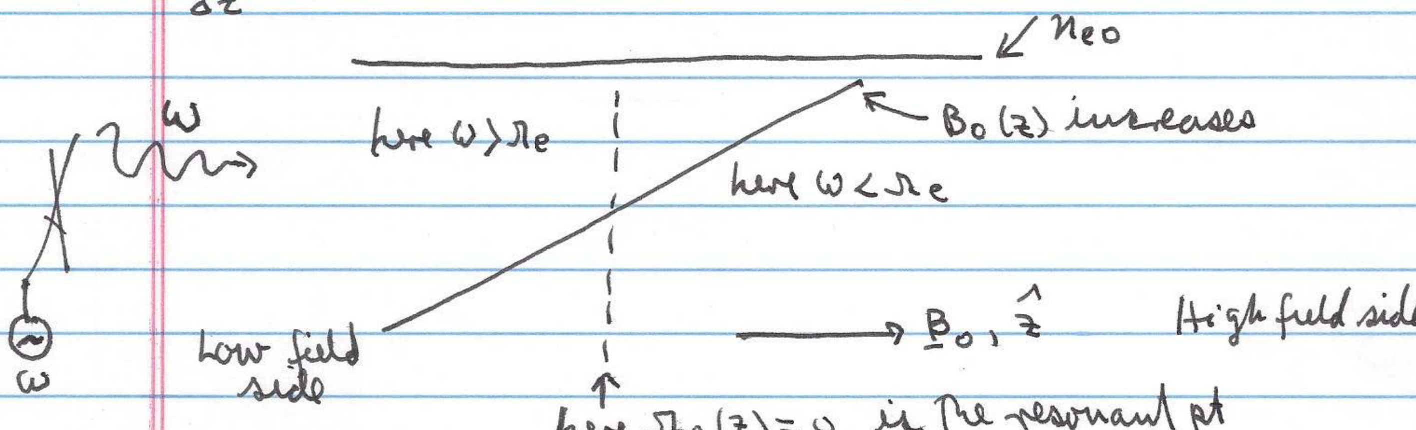
$\Rightarrow$ it penetrates deeper than in the $B_0 = 0$ case.

Summary of R-L behavior along density gradient



R-L behavior along a magnetic field gradient $\Rightarrow B_0(z)$

Assume $\dfrac{d}{dz} n_{eo} = 0 \Rightarrow$ constant density



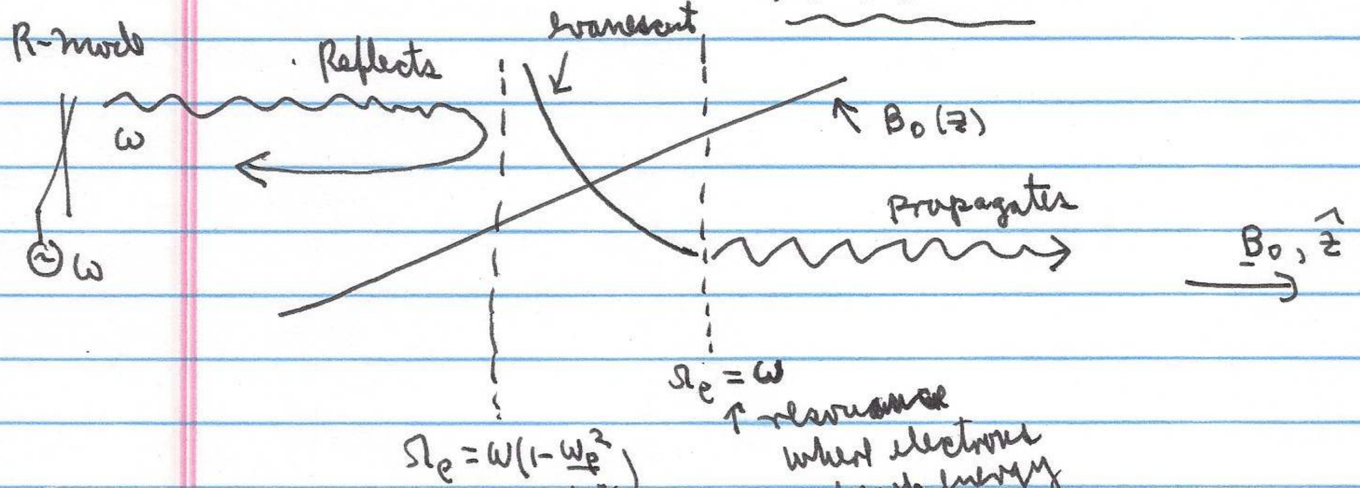
Relevant to Magnetic Mirror geometry.

Lect #9 (cont) For The R-mode - The resonant mode for electrons -

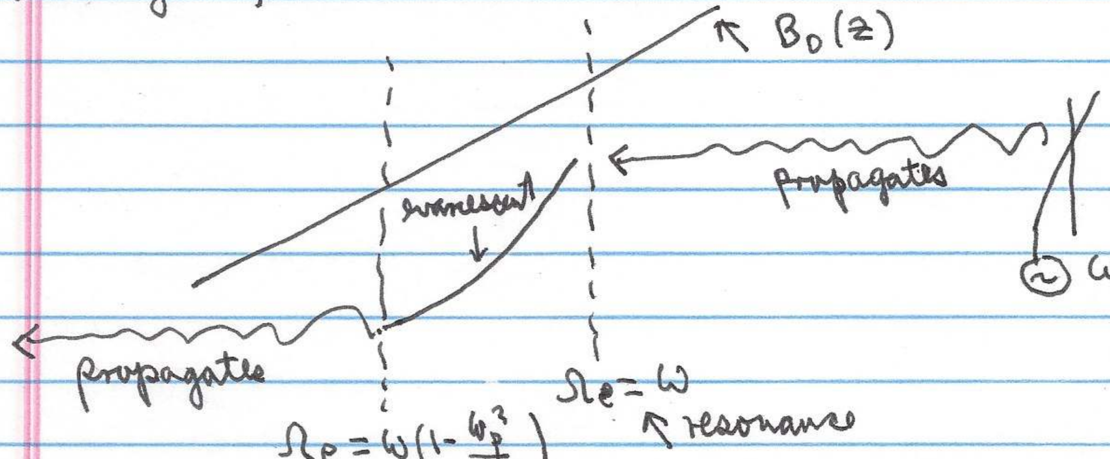
$$n_z^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega - \Omega_e)} \Rightarrow \text{cut-off at } \omega(\omega-\Omega_e) = \omega_{pe}^2$$

or at a value $\frac{\omega^2 - \omega_{pe}^2}{\omega} = \Omega_e(z)$

or $\boxed{\Omega_e(z) = \omega\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)}$ exists if $\omega > \omega_{pe}$ and occurs before resonance is achieved



But The situation can be reversed with an antenna located on The high-field side



These Two situations are called Budden - Tunneling described by a singular differential Eq. $\Rightarrow$ Transmission + Reflection coefficients have analytic formulas!

Lect #9 (cont) The ~~[illegible]~~ behavior of the L-mode in this geometry is boring $n_{\parallel}^2 = 1 - \dfrac{\omega_{pe}^2}{\omega(\omega + \Omega_e)}$

It propagates if $\omega(\omega + \Omega_e) > \omega_{pe}^2$

Is evanescent if $\omega(\omega + \Omega_e) < \omega_{pe}^2$

Recall that all these properties are for $\omega \gg \Omega_i, \omega_{pi}$ where the electron response dominates.

Now consider the low frequency behavior where ions participate

$\omega \ll \Omega_e$ but can be arbitrary relative to Ω_i

Consider the R-mode : $n_{\parallel}^2 = 1 - \dfrac{\omega_{pe}^2}{\omega(\underset{\uparrow \text{ negligible}}{\omega} - \Omega_e)} - \dfrac{\omega_{pi}^2}{\omega(\omega + \Omega_i)}$

$$n_{\parallel}^2 \to 1 + \frac{\omega_{pe}^2}{\omega \Omega_e} - \frac{\omega_{pi}^2}{\omega(\omega + \Omega_i)}$$

This is a peculiar term because it is mass independent!

$$\frac{\omega_{pe}^2}{\Omega_e} = \frac{4\pi e^2 n_0}{m \frac{eB_0}{mc}} = \frac{4\pi e^2 n_0}{M \frac{eB_0}{Mc}} = \frac{\omega_{pi}^2}{\Omega_i}$$

$\Rightarrow$ it looks like an ion contribution but in fact it is due to "electrons"

$$\Rightarrow \quad n_{\parallel}^2 = 1 + \frac{\omega_{pi}^2}{\omega}\left[\frac{1}{\Omega_i} - \frac{1}{\Omega_i(\omega + \Omega_i)}\right] = 1 + \frac{\omega_{pi}^2}{\omega}\left[\frac{\omega + \overbrace{\Omega_i - \Omega_i}^{\text{cancel}}}{\Omega_i(\omega + \Omega_i)}\right]$$

$$\boxed{n_{\parallel}^2 = 1 + \frac{\omega_{pi}^2}{\Omega_i(\omega + \Omega_i)}}$$

$\Rightarrow$ This R mode has no resonances or cut-offs it propagates without unusual features.

Lect #9 (cont.) Examine L-mode at low frequency $\omega \ll \Omega_e$

$$n_{\parallel}^2 = 1 - \underbrace{\frac{\omega_{pe}^2}{\omega(\omega+\Omega_e)}}_{\text{neglect}} - \frac{\omega_{pi}^2}{\omega(\omega-\Omega_i)} \rightarrow 1 - \underbrace{\frac{\omega_{pe}^2}{\omega\Omega_e}}_{\text{as before is } \frac{\omega_{pi}^2}{\omega\Omega_i}} - \frac{\omega_{pi}^2}{\omega(\omega-\Omega_i)}$$

$$\Rightarrow \quad n_{\parallel}^2 = 1 - \frac{\omega_{pi}^2}{\omega}\left[\frac{1}{\Omega_i} + \frac{1}{\omega-\Omega_i}\right] = \boxed{1 - \frac{\omega_{pi}^2}{\Omega_i(\omega-\Omega_i)}}$$

This mode is more interesting: 1) for $\omega < \Omega_i$ it propagates for all density values

2) it has a resonance at $\omega = \Omega_i$

3) it is evanescent for $\omega > \Omega_i$

Extremely low frequency (ELF): There is a degeneracy when $\omega \ll \Omega_i$

$$\text{R-mode}: (n_{\parallel}^2)_R = 1 + \frac{\omega_{pi}^2}{\Omega_i(\omega+\Omega_i)} \rightarrow 1 + \frac{\omega_{pi}^2}{\Omega_i^2}$$

$$\text{L-mode}: (n_{\parallel}^2)_L = 1 - \frac{\omega_{pi}^2}{\Omega_i(\omega-\Omega_i)} \rightarrow 1 + \frac{\omega_{pi}^2}{\Omega_i^2}$$

but for all plasmas of interest $\frac{\omega_{pi}^2}{\Omega_i^2} \gtrsim 10^3 \Rightarrow$ The 1 is negligible

This means that at ELF conditions the vacuum (displacement current) is not important.

$$\Rightarrow \quad (n_{\parallel}^2)_R = (n_{\parallel}^2)_L = \frac{\omega_{pi}^2}{\Omega_i^2} = \frac{k_{\parallel}^2 c^2}{\omega^2}$$

or $\boxed{\frac{\omega}{k_{\parallel}} = \frac{\Omega_i}{\omega_{pi}} c}$ $\Rightarrow$ a renormalized speed of light!
vacuum current is replaced by plasma current

Lect #9 (cont.) This "renormalized" speed of light is called the "Alfvén speed"

$$v_A \equiv \frac{\Omega_i}{\omega_{pi}} c = \frac{eB_0}{Mc\sqrt{\frac{4\pi e^2 n_0}{M}}} c = \boxed{\frac{B_0}{\sqrt{4\pi n_0 M}}}$$

$$v_A = \frac{2.18 \times 10^{11}\ B}{\sqrt{\mu}\sqrt{n}} \ \frac{cm}{sec} \quad \text{with} \quad \mu = \frac{M}{M_{proton}}; \quad B \text{ in Gauss}, \ n \text{ in } cm^{-3}$$

Let's check some typical values of the Alfvén speed:

LAPD: $n \simeq 10^{12}\ cm^{-3}$, $B \simeq 1\ kG$, $\mu = 4 \Rightarrow He$

$$v_A \rightarrow \frac{2 \times 10^{11} \times 10^{3}}{2 \times 10^{6}} = 10^{8}\ cm/sec$$

ITER: $n \simeq 10^{14}\ cm^{-3}$, $B \simeq 50\ kG$, $\mu = 2 \Rightarrow D$

$$v_A \rightarrow \frac{2 \times 10^{11} \times 5 \times 10^{4}}{\sqrt{2} \times 10^{7}} = \frac{10}{\sqrt{2}} \times 10^{8}\ \frac{cm}{sec} \simeq 8 \times 10^{8}\ \frac{cm}{sec}$$

Ionosphere: $n \simeq 10^{5}\ cm^{-3}$, $B \simeq \frac{1}{2}\ G$, $\mu = 16 \Rightarrow O$

$$v_A \rightarrow \frac{2 \times 10^{11} \times \frac{1}{2}}{4 \times \sqrt{10^5}} \simeq \frac{10^{11}}{12 \times 10^{2}} \simeq \frac{10^{8}}{1.2} \simeq 10^{8}\ \frac{cm}{sec}$$

$\Rightarrow$ In general over a very wide range of conditions

$$10^{8}\ \frac{cm}{sec} \leq v_A \lesssim 10^{9}\ \frac{cm}{sec}$$

$\Rightarrow$ a good guess is always in the mid $10^{8}\ \frac{cm}{sec}$ $\Rightarrow \left(\frac{c}{v_A}\right)^2 > 10^3$

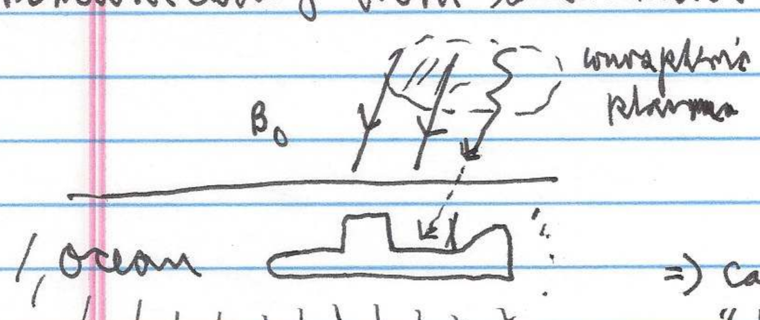
Lect #9 (cont.) The low-frequency L-mode is part of a larger class of modes called "Shear Alfvén Waves".

and the low-frequency R-mode is part of the "Compressional Alfvén Wave" family.

Shear Alfvén waves are slower because $\boxed{(n_\parallel^2)_L > (n_\parallel^2)_R}$

⇒ A more universal nomenclature -- following from the Booker Quartic is "slow" and "fast mode" rather than the more instructive "R-L" labels -- but in practice all of these names are used and there is much confusion in the literature because of it - Sometimes people working on the same problem do not realize it because they use different names.

A feature of practical importance is that a magnetized plasma shortens the wavelength of a low-frequency E+M wave in vacuum ⇒ Becomes an attractive feature for communicating with submarines

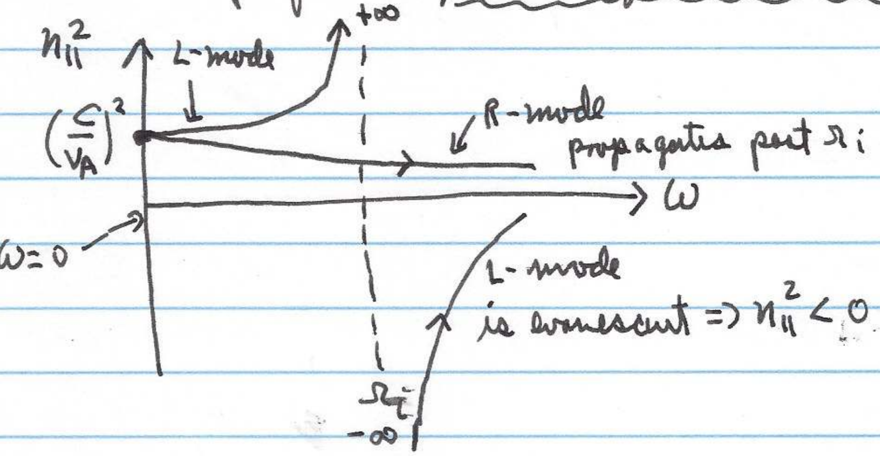


$$\lambda_{Alfvén} \simeq \lambda_0 \left(\frac{v_A}{c}\right) \sim \frac{\lambda_0}{300}$$

(λ_0: vacuum)

⇒ can use plasma-based "effective antennas" to launch waves in range of $\frac{\omega}{2\pi} = 50$ Hz

Lect #9 (cont.) Summary of the features of the R-L modes at low frequency



Examine intermediate frequency regime: $\Omega_i << \omega_{pi} << \omega << \Omega_e, \omega_{pe}$

Already know from above plot that the R-mode propagates in this band - The question is how this "Fast Alfvén Wave" behaves for frequencies beyond Ω_i -

$$(n_{\parallel}^2)_R = 1 - \underbrace{\frac{\omega_{pe}^2}{\omega(\omega - \Omega_e)}}_{} - \underbrace{\frac{\omega_{pi}^2}{\omega(\omega+\Omega_i)}}_{} \quad \text{is exact}$$

negligible — now is $\frac{\omega_{pi}^2}{\omega^2} << 1$

$$\Rightarrow \quad (n_{\parallel}^2)_R \to 1 + \frac{\omega_{pe}^2}{\omega \Omega_e}$$

but $\frac{\omega_{pe}^2}{\omega\Omega_e} = \left(\frac{\Omega_e}{\omega}\right)\left(\frac{\omega_{pe}}{\Omega_e}\right)^2$ but we know $\frac{\omega_{pe}}{\Omega_e}$ is $\mathcal{O}(1)$

This is large $\Rightarrow \frac{\omega_{pe}^2}{\omega\Omega_e} >> 1 \Rightarrow$ neglect vacuum current!

$$n_{\parallel}^2 = \frac{\omega_{pe}^2}{\omega\Omega_e} = \frac{k_{\parallel}^2 c^2}{\omega^2} \quad \text{or} \quad \boxed{k_{\parallel} = \frac{\omega_{pe}}{c}\sqrt{\frac{\omega}{\Omega_e}}}$$

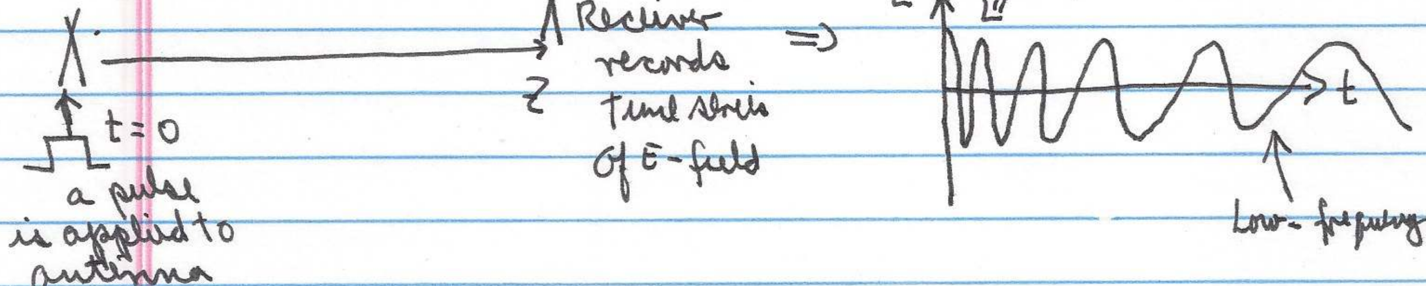
This has a peculiar "square root" dependence very different from the familiar sound waves or light waves

Lect #9 (cont.) Consider the group velocity $(v_g)_{\parallel} = \frac{\partial \omega}{\partial k_{\parallel}}$

apply $\frac{\partial}{\partial k_{\parallel}}$ to disp relation $\Rightarrow$ $1 = \frac{\omega_p}{c\sqrt{\Omega_e}} \frac{1}{2} \frac{1}{\omega^{1/2}} \left(\frac{\partial \omega}{\partial k_{\parallel}}\right)$

$\Rightarrow (v_g)_{\parallel} = 2c \frac{\sqrt{\Omega_e}}{\omega_{pe}} \omega^{1/2}$ ↑ depends on frequency

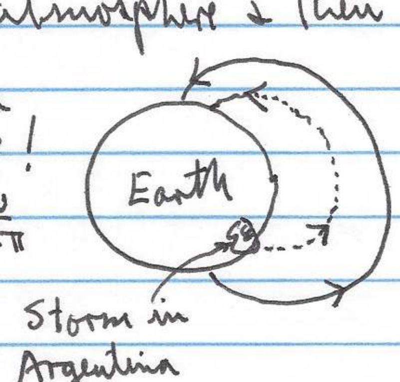
$\Rightarrow$ low frequencies are slower than large frequencies



This tone-stretching motivates the name "whistler waves"

Typically heard by Ham-radio operators as background noise because they are naturally generated by lightning pulses in the lower atmosphere & then propagate into the ionosphere

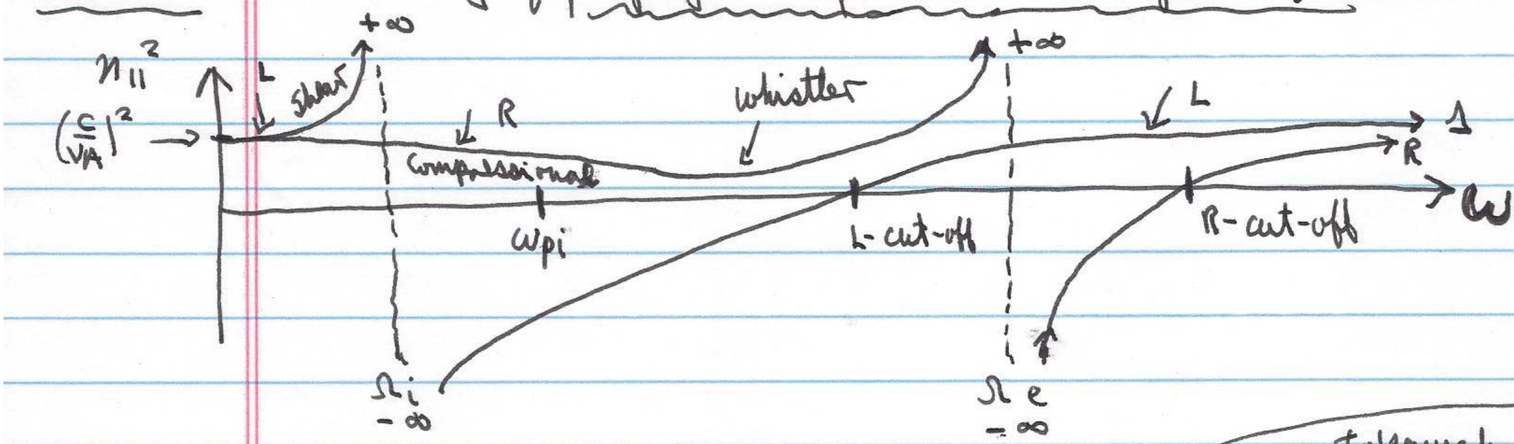
Frequency is in audio range!
$5 - 10 \text{ kHz} = \frac{\omega}{2\pi}$
& $\frac{\Omega_e}{2\pi} = 1.4 \text{ MHz}$



This is a cartoon of how a storm in Argentina can excite a whistler received in Canada

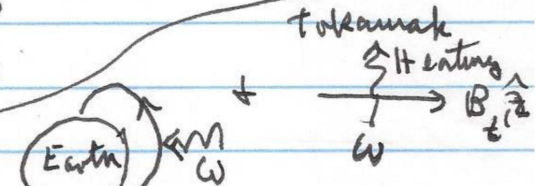
- Such global propagations were studied for many years from a dedicated station in Antartica called "Sipple Station"
- Of current central interest to magnetosphere research -

Lect #9 (cont.) Summary of propagation parallel to magnetic field



Note This is not to scale because $\frac{\Omega_e}{\Omega_i} > 2000$

of great significance to: Earth &



Now proceed to examine propagation $\perp$ to $\underline{B}_0$ $\Rightarrow$ $n_{\parallel}^2 = 0$

$$\uparrow \underline{k} = k_{\perp}\hat{x} \qquad \longrightarrow \underline{B}_{0},\ \hat{z}$$

Return to general Booker Quartic

$$0 = -(n_{\parallel} n_{\perp})^2 (n^2 - \epsilon_{\perp}) + (n_{\perp}^2 - \epsilon_{\parallel})\left[(n_{\parallel}^2 - \epsilon_{\perp})(n^2 - \epsilon_{\perp}) - \epsilon_{xy}\epsilon_{yx}\right]$$

now $n^2 = n_{\perp}^2$ $\Rightarrow$ $0 = (n_{\perp}^2 - \epsilon_{\parallel})\left[-\epsilon_{\perp}(n_{\perp}^2 - \epsilon_{\perp}) - \epsilon_{xy}\epsilon_{yx}\right]$

Again there are 2 non-trivial roots

Root #1: $\quad 0 = n_{\perp}^2 - \epsilon_{\parallel} \Rightarrow \boxed{n_{\perp}^2 = \epsilon_{\parallel}}$

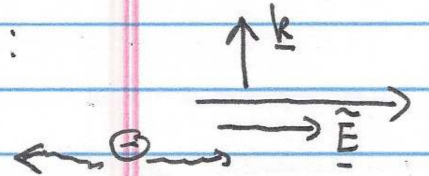
Root #2: $\quad 0 = -\epsilon_{\perp}(n_{\perp}^2 - \epsilon_{\perp}) - \epsilon_{xy}\epsilon_{yx} \Rightarrow \boxed{n_{\perp}^2 = \dfrac{\epsilon_{\perp}^2 - \epsilon_{xy}\epsilon_{yx}}{\epsilon_{\perp}}}$

Examine #1:

$$\frac{k_{\perp}^2 c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \Rightarrow \boxed{\omega^2 = \omega_{pe}^2 + k_{\perp}^2 c^2}$$

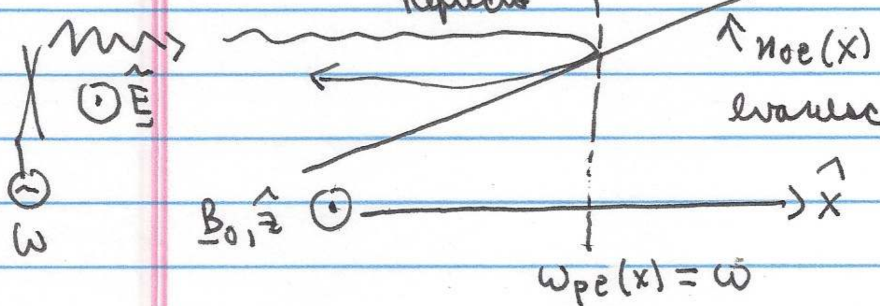
Identical to unmagnetized plasma!

Lect #9 (cont.) The reason why the magnetic field does not appear for this mode is:



The electric field is $\parallel$ to $\underline{B}_0$ + the electrons can oscillate freely along $\underline{B}_0$ without a magnetic force.

The properties of the mode are identical to what we saw in <u>222A</u>



evanescent for $\omega < \omega_{pe}$ for all $\underline{B}_0$

Because nothing is changed by $\underline{B}_0$ this is called the "O-mode" or ordinary mode.

<u>Lect #10</u>. Examine Root #2 - it has a lot more structure because the electric field is $\perp$ to $\underline{B}_0$ $\Rightarrow$ $n_\perp^2 = \dfrac{\epsilon_\perp^2 - \epsilon_{xy}\epsilon_{yx}}{\epsilon_\perp}$

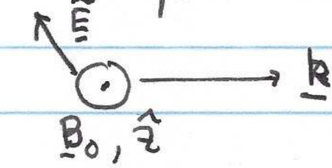
but $\epsilon_{yx} = -\epsilon_{xy}$ $\Rightarrow$ $\epsilon_\perp^2 - \epsilon_{xy}\epsilon_{yx} = \epsilon_\perp^2 + (\epsilon_{xy})^2 = (\epsilon_\perp + i\epsilon_{xy})(\epsilon_\perp - i\epsilon_{xy})$

$$\Rightarrow \quad n_\perp^2 = \frac{(\epsilon_\perp + i\epsilon_{xy})(\epsilon_\perp - i\epsilon_{xy})}{\epsilon_\perp}$$

<u>but</u> we have already done the related algebra when we studied $\underline{k} \parallel \underline{B}_0$ case. We found earlier that $(n_\parallel^2)_R = \epsilon_\perp - i\epsilon_{xy}$ and $(n_\parallel^2)_L = \epsilon_\perp + i\epsilon_{xy}$

$$\Rightarrow \quad n_\perp^2 = \frac{(n_\parallel^2)_R (n_\parallel^2)_L}{\epsilon_\perp}$$

Lect#10 (cont.) The reason for the mixture of these two effects is that the polarization of the electric field is



$\Rightarrow$ if one decomposes $\tilde{\underline{E}}$ it can be viewed as a mixture of R and L responses that manifests as the product of the pure $(n_\parallel^2)_R$ and $(n_\parallel^2)_L$ but it is not the sum

What is now fundamentally new is that a collective resonance (not a particle resonance) can occur for conditions that make $\epsilon_\perp = 0$.

Note This is not a resonance like Ω_e or Ω_i it is like ω_{pe} for an unmagnetized plasma. The fact that it is at $\epsilon_\perp = 0$ brings out our earlier knowledge from 222B that electrostatic modes correspond to the condition $\epsilon = 0$ but now ϵ is a Tensor. For $\epsilon_\parallel = 0$ one has the ion acoustic and Langmuir waves, now for $\epsilon_\perp = 0$ there are new electrostatic modes.

In terms of parameters,
$$n_\perp^2 = \frac{\left[1 - \frac{\omega_{pe}^2}{\omega(\omega - \Omega_e)} - \frac{\omega_{pi}^2}{\omega(\omega + \Omega_i)}\right]\left[1 - \frac{\omega_{pe}^2}{\omega(\omega + \Omega_e)} - \frac{\omega_{pi}^2}{\omega(\omega - \Omega_i)}\right]}{1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}}$$

Do these modes have a cyclotron resonance, i.e., $n_\perp^2 \to \infty$ at $\omega \to \Omega_i$ or $\omega \to \Omega_e$?

Intuitively one may expect such an effect + the expression for $n_\perp^2$ is full of singular denominators -- but --

Take the limit as $\omega \to \Omega_e$ in $n_\perp^2$

$$n_\perp^2 \to \frac{\left[-\frac{\omega_{pe}^2}{\omega(\omega - \Omega_e)}\right]\left[1 - \frac{\omega_{pe}^2}{2\Omega_e^2}\right]}{\left(-\frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2}\right)} \to \left[1 - \frac{\omega_{pe}^2}{2\Omega_e^2}\right]\frac{\omega^2 - \Omega_e^2}{(\omega - \Omega_e)\,\omega}$$

$$\to 2\left[1 - \frac{\omega_{pe}^2}{2\Omega_e^2}\right]$$

Finite!

Lect #10 (cont.) The behavior for $\omega \to \Omega_i$ is similar

$$n_\perp^2 \to \left[+\frac{\omega_{pe}^2}{\Omega_i\Omega_e} - \frac{\omega_{pi}^2}{2\Omega_i^2}\right]\left[-\frac{\omega_{pi}^2}{\omega(\omega-\Omega_i)}\right]\left[\frac{-(\omega^2-\Omega_i^2)}{\omega_{pi}^2}\right] = 2\left[\frac{\omega_{pe}^2}{\Omega_i\Omega_e} - \frac{\omega_{pi}^2}{2\Omega_i^2}\right]$$

which is also finite

An important consequence is that these modes cannot be used to heat a plasma by cyclotron resonance absorption – although they can transport E+M energy across B_0 –

High-Frequency behavior: $\omega \gg \Omega_i, \omega_{pi}$; $\omega \sim \Omega_e, \omega_{pe}$ ⇒ pure electron response

$$n_\perp^2 \to \frac{\left[1-\frac{\omega_{pe}^2}{\omega(\omega-\Omega_e)}\right]\left[1-\frac{\omega_{pe}^2}{\omega(\omega+\Omega_e)}\right]}{1-\frac{\omega_{pe}^2}{\omega^2-\Omega_e^2}}$$

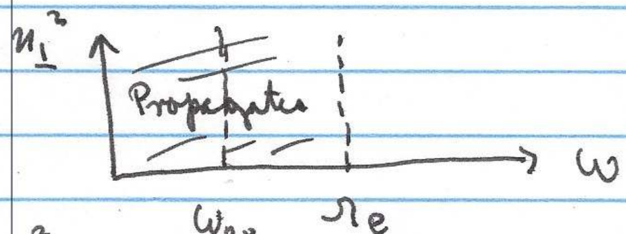
if $\omega < \Omega_e$ ⇒ $\text{sign}\, n_\perp^2 = \text{sign}\left[1-\frac{\omega_{pe}^2}{\omega(\omega+\Omega_e)}\right]$

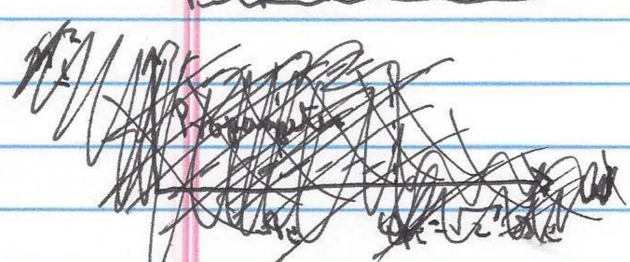
⇒ Mode propagates for $n_\perp^2 > 0$ ⇒ $\omega(\omega+\Omega_e) > \omega_{pe}^2$

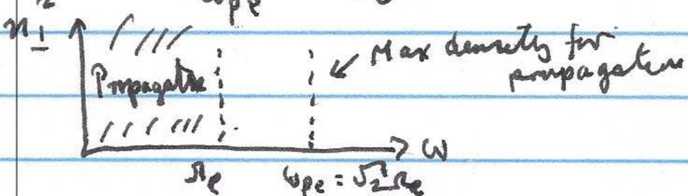
What does this imply? say $\omega = \alpha\Omega_e$ with $\alpha < 1$

$$\Rightarrow \underbrace{\alpha(1+\alpha)}_{\text{Max is 2}} > \left(\frac{\omega_{pe}}{\Omega_e}\right)^2 \Rightarrow \frac{\omega_{pe}}{\Omega_e} < \sqrt{2}$$

There are propagation bands







Lect #10 (cont..) Examine propagation for $\omega > \Omega_e$

$$n_\perp^2 = \frac{\left[1 - \frac{\omega_{pe}^2}{\omega(\omega - \Omega_e)}\right]\left[1 - \frac{\omega_{pe}^2}{\omega(\omega + \Omega_e)}\right]}{1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2}}$$

There are 2 cut-offs - The "R" and "L" at

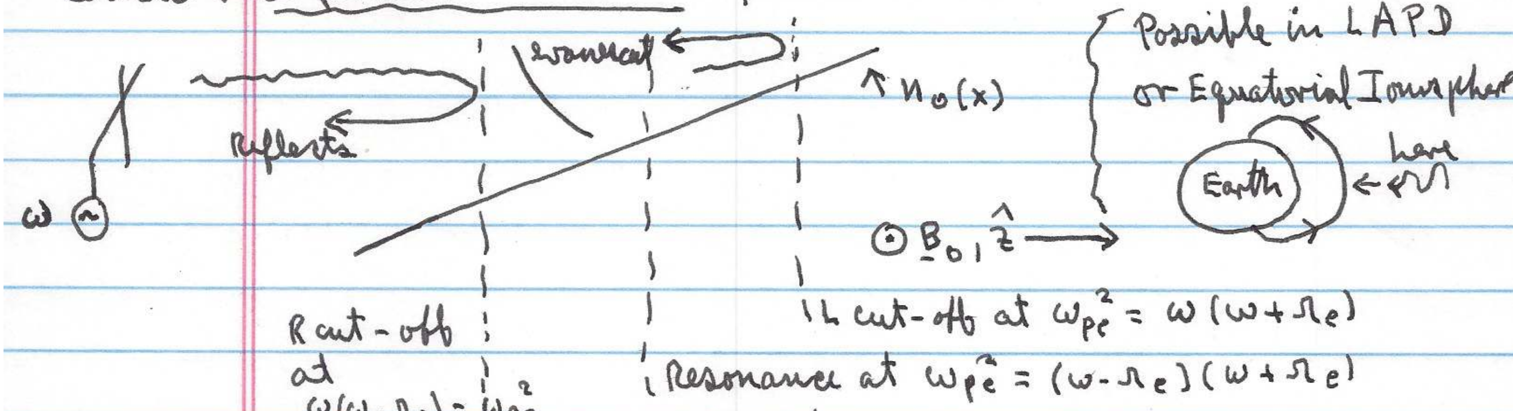
R- cut-off : $\omega(\omega - \Omega_e) = \omega_{pe}^2$

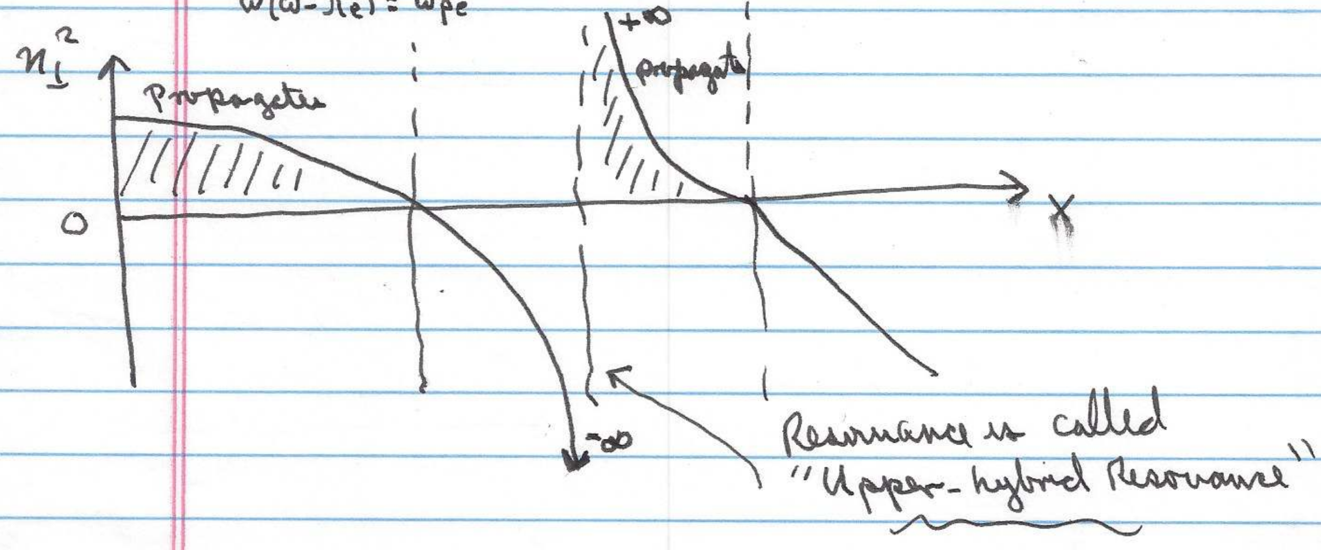
L- cut-off : $\omega(\omega + \Omega_e) = \omega_{pe}^2$ -- note this also exists for $\omega < \Omega_e$

And there is a resonance - where $n_\perp^2 \to \infty$.

Resonance : $\omega^2 = \omega_{pe}^2 + \Omega_e^2$ or $(\omega - \Omega_e)(\omega + \Omega_e) = \omega_{pe}^2$

Consider a possible scenario for concreteness - B_0 is uniform!





Resonance is called "Upper-hybrid Resonance"

Lect#10 (cont.) Important consequences of $\omega > \Omega_e$ propagation in $\nabla n_0 \perp \underline{B}_0$

1) Behavior consists of resonance - tunneling - propagation features also described by Budden Eq. - Reflection, Transmission and absorption coefficients are known analytically

2) Efficient Transfer of energy occurs at the upper-hybrid point - a collective resonance - not a particle resonance - proper description requires kinetic effects -- waves known as "Electron Bernstein waves" are excited

3) The second reflection at the L-cut-off introduces a delay in a second signal received at the ground in ionospheric experiments - The second ~~sens~~ signal received on the ground is called "Z-trace".

Intermediate frequency behavior $\perp$ to $\underline{B}_0$: $\omega << \Omega_e$ but $\omega \sim \Omega_i$ or ω_{pi}

$$n_\perp^2 \Rightarrow \frac{\left[1+\frac{\omega_{pe}^2}{\omega\Omega_e}-\frac{\omega_{pi}^2}{\omega(\omega+\Omega_i)}\right]\left[1-\frac{\omega_{pe}^2}{\omega\Omega_e}-\frac{\omega_{pi}^2}{\omega(\omega-\Omega_i)}\right]}{1+\frac{\omega_{pe}^2}{\Omega_e^2}-\frac{\omega_{pi}^2}{\omega^2-\Omega_i^2}}$$

+ we already know that $\frac{\omega_{pe}^2}{\Omega_e} = \frac{\omega_{pi}^2}{\Omega_i}$ & the features of Shear + Compressional Modes $\parallel$ to $\underline{B}_0$

$$n_\perp^2 \rightarrow \frac{\overbrace{\left[1+\frac{\omega_{pi}^2}{\Omega_i(\omega+\Omega_i)}\right]}^{\text{"R"}}\overbrace{\left[1-\frac{\omega_{pi}^2}{\Omega_i(\omega-\Omega_i)}\right]}^{\text{"L"}}}{1+\frac{\omega_{pe}^2}{\Omega_e^2}-\frac{\omega_{pi}^2}{\omega^2-\Omega_i^2}}$$

Important: for $\omega < \Omega_i$ there is propagation $\perp$ to $\underline{B}_0$ for all densities

but for $\omega > \Omega_i$ there is a cut-off at $\omega_{pi}^2 = \Omega_i(\omega-\Omega_i)$

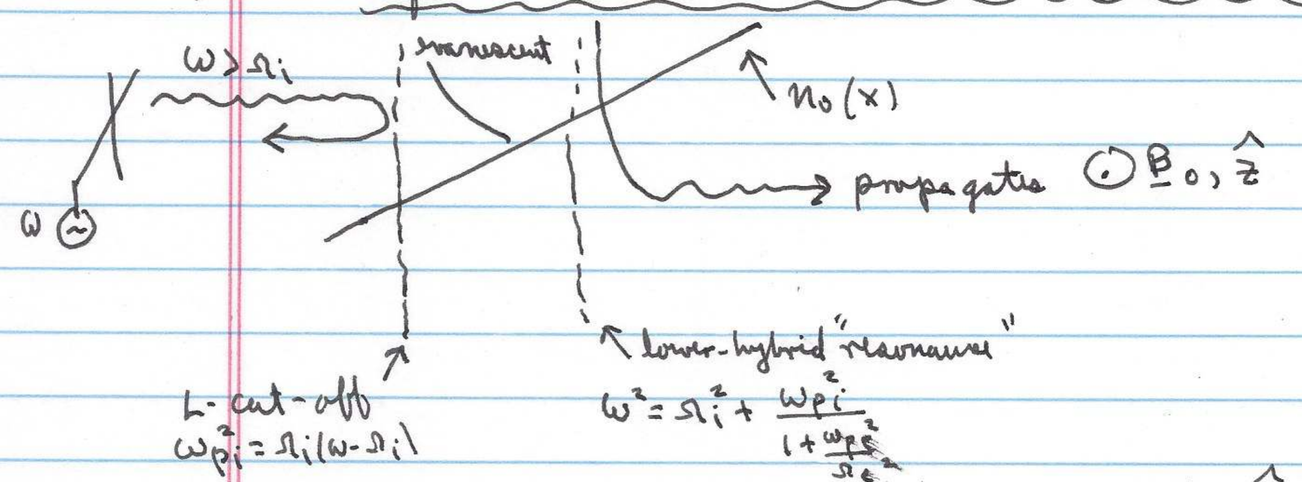
The "L cut-off"

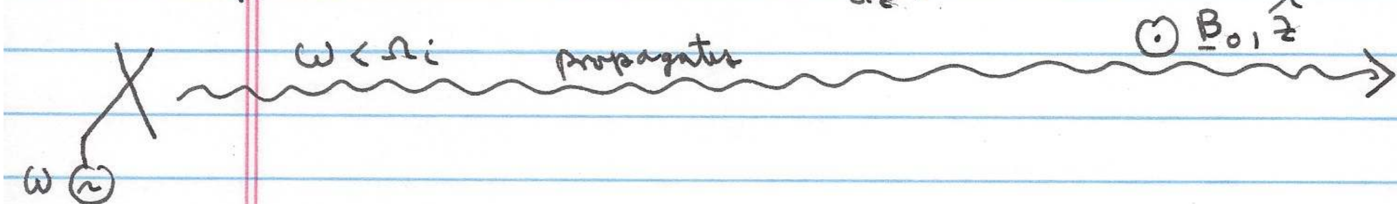
Lect #10 (cont.) Now the collective resonance at $\epsilon_\perp = 0$ is given by

$$(\omega - \Omega_i)(\omega + \Omega_i) = \frac{\omega_{pi}^2}{1 + \frac{\omega_{pe}^2}{\Omega_e^2}} \quad \text{or} \quad \omega^2 = \Omega_i^2 + \frac{\omega_{pi}^2}{1 + \frac{\omega_{pe}^2}{\Omega_e^2}}$$

which is known as the "lower-hybrid resonance"

Again, it is useful to consider a condition where $\nabla n_0 \perp \underline{B}_0$





The compressional mode with $\omega > \Omega_i$ is the main candidate for heating ions in a tokamak fusion device -- A topic of much present study is the possible nonlinear effects that can occur at the edge of the plasma

Examine the parameter dependence of lower-hybrid-resonance (LHR)

$\epsilon_\perp = 0 \implies \omega_{LH}^2 = \Omega_i^2 + \dfrac{\omega_{pi}^2}{1 + \frac{\omega_{pe}^2}{\Omega_e^2}}$ ← electron shielding because it is at low frequency

for $\omega \ll \Omega_e$

compare to upper-hybrid: $\omega_{UH}^2 = \Omega_e^2 + \omega_{pe}^2$ ← no shielding

Lect #10 (cont.) Physically The UH & LH resonances are The same — The mixing of Two oscillators — at Ω_e with ω_{pe}

— at Ω_i with ω_{pi}

but The ω_{pi} oscillation is shielded by the electrons!

One can view the electron shielding as causing an effective ion mass $M^* \to M\underbrace{\left(1 + \frac{\omega_{pe}^2}{\Omega_e^2}\right)}_{\text{Typically is } O(1)}$

In The LH expression $\omega_{LH}^2 = \left[\left(\frac{1}{\frac{\omega_{pi}^2}{\Omega_i^2}}\right) + \frac{1}{1 + \frac{\omega_{pe}^2}{\Omega_e^2}}\right] \omega_{pi}^2$

we know That $\frac{\omega_{pi}^2}{\Omega_i^2}$ is $O(10^3)$ and $\frac{\omega_{pe}^2}{\Omega_e^2}$ is $O(1)$

$\Rightarrow$ The Ω_i^2 term is negligible $\Rightarrow$ LH is simply a shielded ω_{pi} oscillation That does not exist in an unmagnetized plasma (shown in 222A)

$$\Rightarrow \quad \omega_{LH} \to \frac{\omega_{pi}}{\sqrt{1 + \frac{\omega_{pe}^2}{\Omega_i^2}}}$$

slightly smaller Than ω_{pi} for most plasma conditions

Consider extreme limits: Very large $B_0 \Rightarrow \frac{\omega_{pe}^2}{\Omega_e^2} \ll 1$

$$\Rightarrow \quad \boxed{\omega_{LH} \to \omega_{pi}}$$

Weak magnetic fields: $\frac{\omega_{pe}^2}{\Omega_e^2} \gg 1 \Rightarrow \omega_{LH} \to \frac{\omega_{pi}}{\omega_{pe}}\Omega_e$

$$\text{or } \omega_{LH} = \sqrt{\Omega_e^2 \frac{m}{M}} = \boxed{\sqrt{\Omega_e \Omega_i}}$$

peculiar geometric mean

independent of density!

attractive possibility for ion-heating but plagued by nonlinearities at plasma edge

Lecture # 10(cont) Extreme low frequency $\perp$ to B_0 : $\omega << \Omega_i$ (ELF)

$$n_\perp^2 = \frac{\left[1 + \frac{\omega_{pi}^2}{\Omega_i(\omega+\Omega_i)}\right]\left[1 - \frac{\omega_{pi}^2}{\Omega_i(\omega-\Omega_i)}\right]}{1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2-\Omega_i^2}}$$

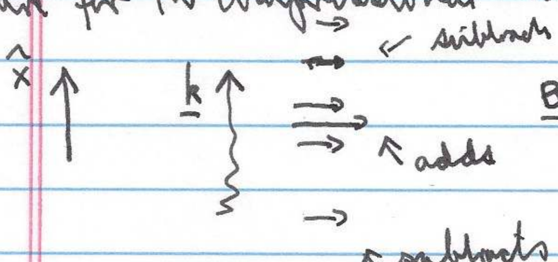
as we found before -- There is no cut-off & vacuum & electron terms are negligible

$$n_\perp^2 \rightarrow \frac{\left(\frac{\omega_{pi}^2}{\Omega_i^2}\right)\left(\frac{\omega_{pi}^2}{\Omega_i^2}\right)}{\frac{\omega_{pi}^2}{\Omega_i^2}} = \left(\frac{\omega_{pi}}{\Omega_i}\right)^2 = \left(\frac{c}{V_A}\right)^2$$

or $$\frac{k_\perp^2 c^2}{\omega^2} = \frac{c^2}{V_A^2} \Rightarrow \boxed{\frac{\omega}{k_\perp} = V_A = \frac{B_0}{\sqrt{4\pi M n_0}}}$$ propagates at the Alfvén speed

but now There is only 1 Alfvén wave - before, for $k \parallel \underline{B}_0$ we found 2 modes -- In $\perp$ direction only the "Compressional" Mode with $\tilde{E} \perp$ to B_0 exists. The equivalent of the shear mode ~~[illegible]~~ becomes the "O-mode" which we know does not propagate for $\omega < \omega_{pe}$ $\Rightarrow$ The "shear" Alfvén mode does not exist for $k_\parallel = 0$.

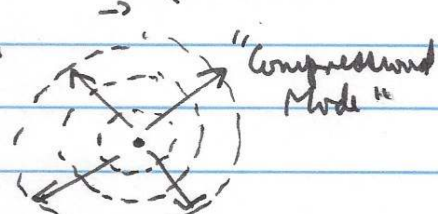
The picture for the compressional mode is ($\hat{\underline{B}}$ is $\parallel$ to $\underline{B}_0$)



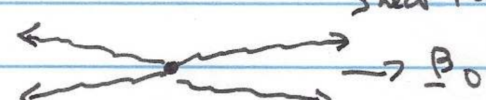
$\underline{B}_{Total} = \underline{B}_0 + \hat{\underline{B}}$ { Total effect is to compress the Total magnetic field lines

For 3-D propagation

point source radiates spherical wave fronts like light in vacuum but $c \rightarrow V_A$



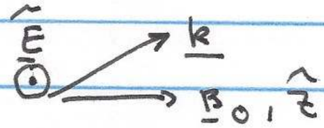
"Shear Mode"



propagates along cones

Lect #11 Oblique propagation is a mixture of ∥ and ⊥ features but important to realize a new feature appears

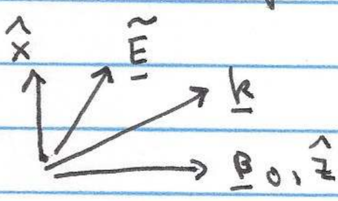
For $\tilde{\underline{E}}$ ⊥ to ($\underline{k}$ + $\underline{B}_0$)



The mode is purely electromagnetic

There is no charge separation - Kinetically there is no wave-particle resonance due to $\tilde{\underline{E}}$ - The kinetic effects are due to the "Magnetic force ∥ B_0" that we encountered in the Weibel instability - This not present in a "cold" plasma

For $\tilde{\underline{E}}$ on the ($\underline{k}$, $\underline{B}_0$) plane



The mode is a mixture of E&M and electrostatics

Now $\tilde{\underline{E}}$ has a direct wave-particle interaction due to $\tilde{E}_{\parallel}$ + also a "Magnetic force term"

These mixed modes become important near $\epsilon_\perp \to 0$ - The hybrid resonances - Let's examine this world of mixed E&M and electrostatics for a cold plasma

as usual $\underline{E}(\underline{r},t) = \dfrac{\tilde{\underline{E}}(\underline{r},t)\, e^{-i\omega t}}{2} + c.c.$

Poisson's Eq. is always exact: $\nabla \cdot \left[\overleftrightarrow{\epsilon}(\omega) \cdot \tilde{\underline{E}}(\underline{r},\omega) \right] = 0$ for a cold plasma

- In a cold plasma there is no $\underline{k}$ dependence -

What does electrostatic mean? $\underline{E} = -\nabla\phi - \dfrac{1}{c}\dfrac{\partial}{\partial t}\underline{A}$ ← magnetic vector potential

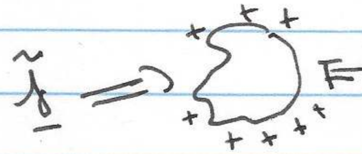
electrostatic potential

with $\begin{pmatrix} \phi \\ \underline{A} \end{pmatrix} = \begin{pmatrix} \tilde{\phi} \\ \tilde{\underline{A}} \end{pmatrix} \dfrac{e^{-i\omega t}}{2} + c.c.$

$$\Rightarrow \quad \nabla\cdot\left[\overleftrightarrow{\epsilon}\cdot\tilde{\underline{E}}\right] = -\nabla\cdot\left[\overleftrightarrow{\epsilon}\cdot\nabla\tilde{\phi}\right] + i\frac{\omega}{c}\nabla\cdot\left[\overleftrightarrow{\epsilon}\cdot\underline{A}\right]$$

$$\Rightarrow \quad \nabla\cdot\left(\overleftrightarrow{\epsilon}\cdot\nabla\tilde{\phi}\right) = i k_0 \nabla\cdot\left(\overleftrightarrow{\epsilon}\cdot\tilde{\underline{A}}\right)$$

Lect #11 (cont.) This means that the term $i k_0 \nabla\cdot(\overleftrightarrow{\epsilon}\,\tilde{\underline{A}})$ acts like an effective charge separation that generates a temporally modulated "electrostatic potential" -- but a gradient in $\overleftrightarrow{\epsilon}$ and/or $\tilde{\underline{A}}$ is required



non-uniform patch in density or $\underline{B}$

The structure of the potential generated is of the form of solutions to

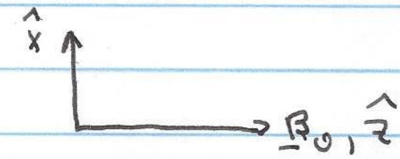
$$\boxed{\nabla\cdot(\overleftrightarrow{\epsilon}\cdot\nabla\tilde{\phi}) = S}$$

where S is the E+M generated charge source

The extreme limit of pure electrostatics requires that

$|k_0 \nabla\cdot(\overleftrightarrow{\epsilon}\,\tilde{\underline{A}})| \ll 1$ for neglect of E+M features in oblique propagation

Pure electrostatic modes - Total neglect of $\tilde{\underline{A}}$



$$\nabla \to \hat{x}\,\partial_x + \hat{z}\,\partial_z$$

$$(\hat{x}\,\partial_x + \hat{z}\,\partial_z)\cdot(\hat{x}\,\epsilon_\perp \partial_x \tilde{\phi} + \hat{z}\,\epsilon_\parallel \partial_z \tilde{\phi}) = 0$$

$$\Rightarrow \quad \partial_x(\epsilon_\perp \partial_x \tilde{\phi}) + \partial_z(\epsilon_\parallel \partial_z \tilde{\phi}) = 0$$

If plasma parameters are uniform $\Rightarrow$ $\boxed{\partial_z^2 \tilde{\phi} + \frac{\epsilon_\perp}{\epsilon_\parallel}\partial_x^2 \tilde{\phi} = 0}$

Compare with typical Eq in space-time for a wave: $\frac{\partial^2}{\partial z^2}\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0$
for a generic scalar ψ

The solutions are $\psi = \underbrace{\psi_+(z - ct)}_{\text{Right-going wave at speed } c} + \underbrace{\psi_-(z + ct)}_{\text{Left-going wave at speed } c}$

Now make a comparison with the electrostatic Equation

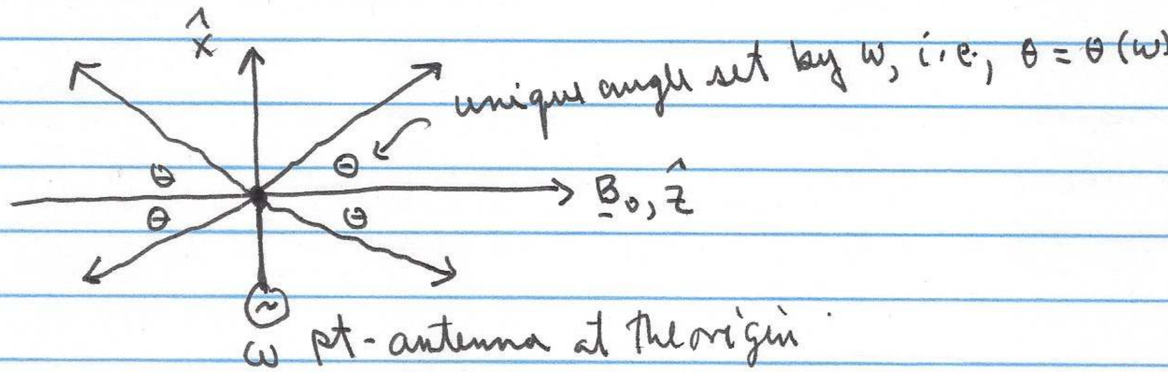
Lect#11 (cont-) Electrostatic solution is $\tilde{\phi}(z,x) = \tilde{\phi}_+\left(z - \sqrt{-\frac{\epsilon_\parallel}{\epsilon_\perp}}\,x\right) + \tilde{\phi}_-\left(z + \sqrt{-\frac{\epsilon_\parallel}{\epsilon_\perp}}\,x\right)$

where the identification is made $\frac{1}{c^2} \to -\frac{\epsilon_\perp}{\epsilon_\parallel}$ and $x \to t$

For these signals to propagate in the (x,z) plane $\Rightarrow$ $\boxed{\frac{\epsilon_\parallel}{\epsilon_\perp} < 0}$

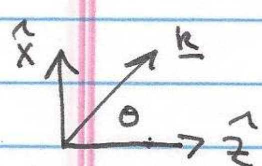
The overall space-time dependence of the signals is:

$$\phi(\underline{x},t) = \left[\frac{\phi_+\left(z - \sqrt{-\frac{\epsilon_\parallel}{\epsilon_\perp}}\,x\right) + \phi_-\left(z + \sqrt{-\frac{\epsilon_\parallel}{\epsilon_\perp}}\,x\right)}{2}\right] e^{-i\omega t} + \text{c.c.}$$



In terms of plane waves: if $\tilde{\phi} \sim e^{i(k_\perp x + k_\parallel z)}$

$$\Rightarrow \quad -k_\perp^2 \epsilon_\perp - k_\parallel^2 \epsilon_\parallel = 0 \quad \text{or} \quad \frac{k_\perp^2}{k_\parallel^2} = -\frac{\epsilon_\parallel}{\epsilon_\perp}$$



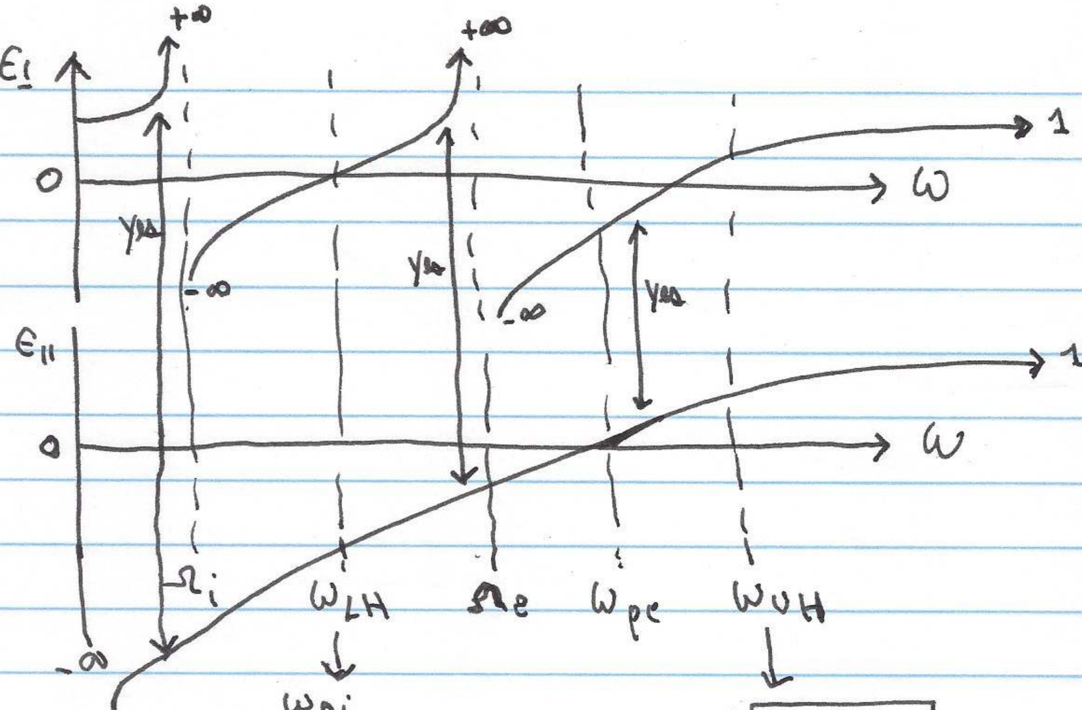
$$\tan\theta = \frac{k_\perp}{k_\parallel} = \pm\sqrt{-\frac{\epsilon_\parallel}{\epsilon_\perp}}$$

Need to explore in what frequency range are these modes possible?

$$\epsilon_\perp = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}, \quad \epsilon_\parallel = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}$$

requirements are: $-\frac{\epsilon_\parallel}{\epsilon_\perp} < 0$ and $\epsilon_\perp \simeq 0$ $\Rightarrow$ plot these vs ω

Lect #11 (cont.)



$$\omega_{LH} \to \frac{\omega_{pi}}{\sqrt{1+\frac{\omega_{pe}^2}{\Omega_e^2}}} \qquad \omega_{UH} \to \sqrt{\omega_{pe}^2+\Omega_e^2}$$

The "yes" are possibilities but only bands that simultaneous have $\epsilon_\perp \simeq 0$ are slightly above ω_{LH} and perhaps below ω_{UH}

$\Rightarrow$ Examine around ω_{LH} $\Rightarrow$ $\Omega_i \ll \omega \sim \omega_{pi} \ll \Omega_e, \omega_{pe}$

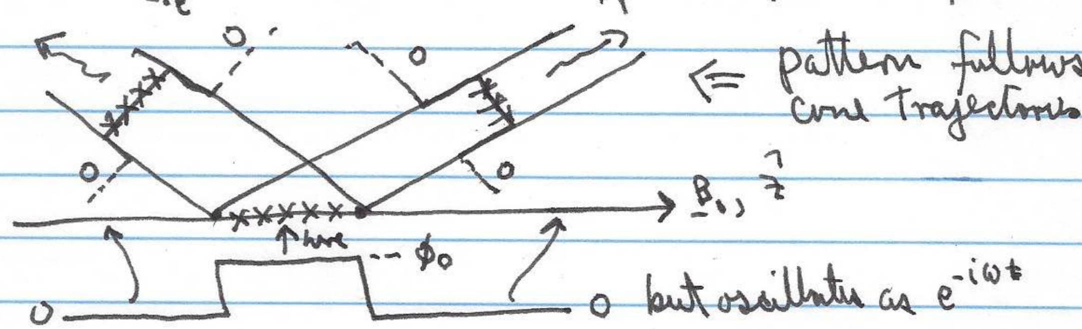
$$\frac{k_\perp^2}{k_\parallel^2} = -\frac{\epsilon_\parallel}{\epsilon_\perp} \to \frac{-\left(-\frac{\omega_{pe}^2}{\omega^2}\right)}{1+\frac{\omega_{pe}^2}{\Omega_e^2}-\frac{\omega_{pi}^2}{\omega^2}} = +\frac{\frac{\omega_{pe}^2}{\omega^2}}{1+\frac{\omega_{pe}^2}{\Omega_e^2}}\frac{1}{\left(1-\frac{\omega_{LH}^2}{\omega^2}\right)}$$

or

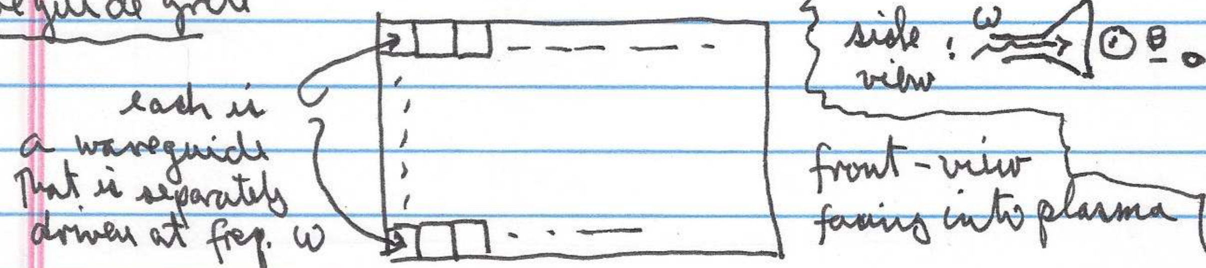
$$\frac{k_\perp^2}{k_\parallel^2} = \frac{\omega_{pe}^2}{1+\frac{\omega_{pe}^2}{\Omega_e^2}}\frac{1}{(\omega^2-\omega_{LH}^2)}$$

$\Rightarrow$ propagates for $\omega > \omega_{LH}$ as expected from the plot

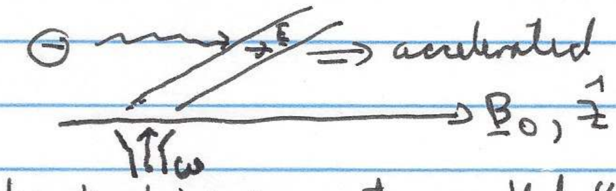
Radiation from a finite parallel source $\Rightarrow$



Lecture #11 (cont.) The step-function type of oscillation at a boundary can be realized with metallic plates or at high-frequency with a "waveguide grill"



each waveguide radiates a pattern of The type sketched in previous page – The interaction is with electrons as They Traverse the patches



This is done in Tokamaks to drive currents – called "LH current-drive"

These electrostatic modes are called "Lower-hybrid waves" and display a dispersion relation analogous to a Langmuir wave

$$\omega^2 = \omega_{LH}^2 + \frac{\omega_{pe}^2}{1+\frac{\omega_{pe}^2}{\Omega_e^2}} \frac{k_{\parallel}^2}{k_{\perp}^2}$$

& for Langmuir waves: $\omega^2 = \omega_{pe}^2 + 3k^2\bar{v}_e^2 \Rightarrow 3\bar{v}_e^2 \Longleftrightarrow \dfrac{\omega_{pe}^2/k_{\perp}^2}{1+\frac{\omega_{pe}^2}{\Omega_e^2}}$

So, some nonlinearities That are studied for Langmuir waves are extrapolated to The behavior of LH-waves.

Consider The other possible band of electrostatic modes -- "Upper-Hybrid Modes"

Now There are 2 possibilities : $\omega_{pe} > \Omega_e$ or $\omega_{pe} < \Omega_e$

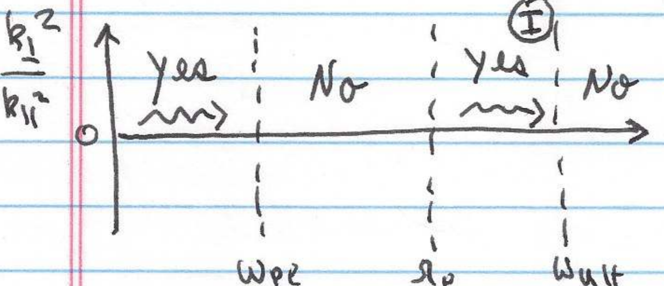
Let's see what the two imply

Lect #11 (cont.)

$$\frac{k_\perp^2}{k_\parallel^2} = -\frac{\epsilon_\parallel}{\epsilon_\perp} \rightarrow -\frac{1-\omega_{pe}^2/\omega^2}{1-\frac{\omega_{pe}^2}{\omega^2-\Omega_e^2}} = -\frac{\left(1-\frac{\omega_{pe}^2}{\omega^2}\right)(\omega^2-\Omega_e^2)}{\omega^2-\omega_{uH}^2}$$

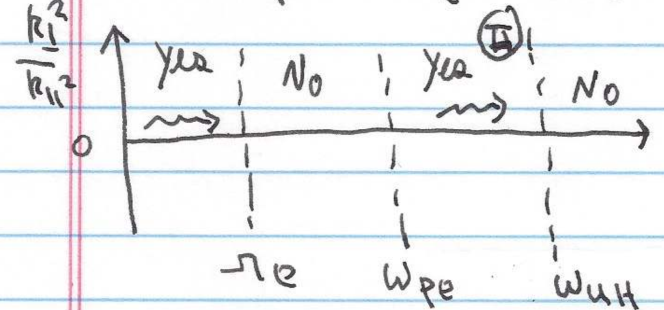
with $\omega_{uH} = \sqrt{\omega_{pe}^2+\Omega_e^2}$

Strong B_0



$\Rightarrow$ There is no electrostatic mode above ω_{uH}

weak B_0



Consider the strongly magnetized limit where $\Omega_e \gg \omega_{pe}$

and $\omega \sim \omega_{uH}$ $\Rightarrow$ $\frac{k_\perp^2}{k_\parallel^2} \simeq -\frac{\omega^2-\Omega_e^2}{\omega^2-\omega_{uH}^2}$

Corresponds to region I

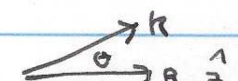
or $\boxed{\omega^2 = \Omega_e^2 + \frac{\omega_{pe}^2}{1+\frac{k_\parallel^2}{k_\perp^2}}}$ which is slightly below ω_{uH}

$= \Omega_e^2 + \omega_{pe}^2 \sin^2\theta$

These modes would be present in the Malmberg experimental set-up that you reviewed in 222B - But there the limit $B_0 \rightarrow \infty$ was taken and these were thrown away because $\omega \rightarrow \infty$ & is the high-frequency branch.

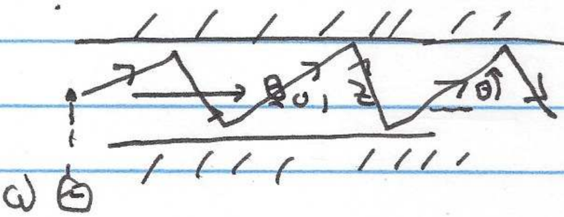
Now consider still strongly magnetized but low frequency: $\omega \ll \Omega_e$

$$\frac{k_\perp^2}{k_\parallel^2} \simeq -\left(1-\frac{\omega_{pe}^2}{\omega^2}\right) \Rightarrow 1+\frac{k_\perp^2}{k_\parallel^2} = \frac{\omega_{pe}^2}{\omega^2} \Rightarrow \omega^2 = \omega_{pe}^2\frac{k_\parallel^2}{k_\perp^2+k_\parallel^2}$$



$k_\parallel = k\cos\theta$

Lect #11 (cont.) – These are the $\boxed{\omega = \omega_{pe}\cos\theta}$ modes ~~in~~ Malmberg's experiment of 222B. The modes bounce from the walls and correspond to the band $\omega < \omega_{pe}$ in the previous page.



Consider the weakly magnetized limit where $\omega \gg \Omega_e$

$$\frac{k_\perp^2}{k_\parallel^2} \to -\frac{(\omega^2-\omega_{pe}^2)}{(\omega^2-\omega_{UH}^2)}\frac{(\omega^2-\overset{\downarrow \text{neglect}}{\Omega_e^2})}{\omega^2} = -\frac{\omega^2-\omega_{pe}^2}{\omega^2-\omega_{UH}^2}$$

or $\omega^2 = \omega_{UH}^2 - \frac{k_\parallel^2}{k_\perp^2}\omega^2 + \omega_{pe}^2\frac{k_\parallel^2}{k_\perp^2} \Rightarrow \omega^2\left(1+\frac{k_\parallel^2}{k_\perp^2}\right) = \omega_{UH}^2 + \omega_{pe}^2\frac{k_\parallel^2}{k_\perp^2}$

since $\omega_{UH}^2 = \omega_{pe}^2+\Omega_e^2 \Rightarrow \omega^2\left(1+\frac{k_\parallel^2}{k_\perp^2}\right) = \omega_{pe}^2\left(1+\frac{k_\parallel^2}{k_\perp^2}\right)+\Omega_e^2$

Corresponds to region II $\Rightarrow$ $\boxed{\omega^2 = \omega_{pe}^2+\Omega_e^2\sin^2\theta} < \omega_{UH}^2$

These are the cold plasma oscillations in a weak magnetic field!

All these modes are obtained by neglecting the induction electric field that appears through the magnetic vector potential $\underline{A}$ in Poisson's Eq.

(*) But we found earlier an exact description that does not make such an approximation, i.e., the Booker Quartic

Question: Where are these electrostatic modes buried inside the structure of the Booker Quartic ??

$$\boxed{0 = -(n_\parallel n_\perp)^2(n^2-\epsilon_\perp) + (n_\perp^2-\epsilon_\parallel)\left[(n^2-\epsilon_\perp)(n^2-\epsilon_\perp)-\epsilon_{xy}\epsilon_{yx}\right]}$$

Lect #11 (cont..) The hint is that Poisson's Eq. without the induction field is: $0 = k_\perp^2 \epsilon_\perp + k_\parallel^2 \epsilon_\parallel$ - This is pure electrostatics and there is no contribution from the ϵ_{xy} Term $\Rightarrow$ These modes neglect the $\underline{E} \otimes \underline{B}$ currents! They are all due to polarization currents alone ..

$\Rightarrow$ Explore from Booker Quartic what is required to drop ϵ_{xy} contribution

$$\Rightarrow \quad |(n^2 - \epsilon_\perp)(n_\parallel^2 - \epsilon_\perp)| \gg |\epsilon_{xy}|^2$$

Near the hybrid resonances where $\epsilon_\perp \to 0 \Rightarrow |n n_\parallel| \gg |\epsilon_{xy}|$

But also, if $\epsilon_\perp \to \infty$ this could be satisfied. Let's assume that it can be done + explore the consequences - The Booker Quartic becomes

same

$$0 \simeq -(n_\parallel n_\perp)^2 (n^2 - \epsilon_\perp) + (n_\perp^2 - \epsilon_\parallel)(n^2 - \epsilon_\perp)(n_\parallel^2 - \epsilon_\perp)$$

$$0 \simeq +(n^2 - \epsilon_\perp)\left[\underbrace{-(n_\parallel n_\perp)^2 + (n_\perp^2 - \epsilon_\parallel)(n_\parallel^2 - \epsilon_\perp)}_{-n_\parallel^2 n_\perp^2 + n_\perp^2 n_\parallel^2 - n_\perp^2 \epsilon_\perp - n_\parallel^2 \epsilon_\parallel + \epsilon_\parallel \epsilon_\perp}\right]$$

cancel

factors into $\underline{\underline{2}}$ different modes!

$$0 = n^2 - \epsilon_\perp \Rightarrow \boxed{k^2 = k_0^2 \epsilon_\perp} \quad \text{with } k_0 = \frac{\omega}{c}$$

and, $$0 = -n_\perp^2 \epsilon_\perp - n_\parallel^2 \epsilon_\parallel + \epsilon_\parallel \epsilon_\perp \Rightarrow \boxed{k_\perp^2 \epsilon_\perp + k_\parallel^2 \epsilon_\parallel = k_0^2 \epsilon_\parallel \epsilon_\perp}$$

This last mode is the structure that reduces to the "pure electrostatic" and now one can obtain a more precise condition for the neglect of inductive fields:

Lect #11 (cont.) rewrite as: $k_\perp^2 \epsilon_\perp \left(1 - \frac{k_0^2}{k_\perp^2}\frac{\epsilon_{\parallel}}{2}\right) + k_{\parallel}^2 \epsilon_{\parallel}\left(1 - \frac{k_0^2}{k_{\parallel}^2}\frac{\epsilon_\perp}{2}\right) = 0$

There is a dual condition: $\boxed{n_\perp \gg \sqrt{\frac{\epsilon_{\parallel}}{2}} \text{ and } n_{\parallel} \gg \sqrt{\frac{\epsilon_\perp}{2}}}$ That property should always be checked - a posteriori - when using electrostatic descriptions -- Obviously the modes need to be obliquely propagating for both conditions to apply.

But when the conditions do not meet these requirements the induction field must be retained ⇒ results in a mixed electrostatic / electromagnetic mode given by

$$\boxed{k_\perp^2 \epsilon_\perp + k_{\parallel}^2 \epsilon_{\parallel} = k_0^2 \epsilon_\perp \epsilon_{\parallel}}$$

Apply to the regime of ELF in which $\omega \ll \Omega_i$ where

$\epsilon_{\parallel} \to -\frac{\omega_{pe}^2}{\omega^2}$ and $\epsilon_\perp \to \frac{\omega_{pi}^2}{\Omega_i^2} = \left(\frac{c}{V_A}\right)^2$ Then the conditions imply that

$$k_\perp > k_0 \frac{\omega_{pe}}{\omega\sqrt{2}} = \frac{\omega_{pe}}{c\sqrt{2}} \quad + \quad k_{\parallel} > \frac{\omega}{V_A\sqrt{2}}$$

⇒ For $\lambda_\perp < \frac{c}{\omega_p}$ there are electrostatic modes

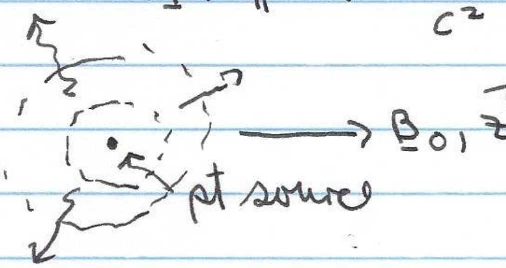
+ simultaneously $\lambda_{\parallel} < \lambda_{Alfven}$ is the domain of pure electrostatics at low-frequency

Before we focus on the details of the mixed ES/E+M for shear Alfvén waves - let's examine what is the other root obtained when ϵ_{xy} is neglected

Lect #11 (cont.) The mode given by $n^2 = \epsilon_\perp$ is obtained by neglecting ϵ_{xy} but it is not electrostatic. It is actually a "renormalized" electromagnetic wave, but now inside a magnetized plasma. The dispersion relation becomes: $k_\perp^2 + k_\parallel^2 = k^2 = \frac{\omega^2}{c^2}\epsilon_\perp$

$\Rightarrow$ it is isotropic

spherical phase-fronts →



in spite of medium being anisotropic - The speed of these modes becomes: $\frac{\omega}{k} = v_p = v_{group} = \frac{c}{\sqrt{\epsilon_\perp}}$

and in the low frequency regime: $\omega << \Omega_i$ $\epsilon_\perp \Rightarrow \left(\frac{c}{v_A}\right)^2$

$\Rightarrow$ These new light waves propagate at the Alfvén speed with $k^2 = \frac{\omega^2}{V_A^2}$ with $\tilde{\underline{E}} \perp \underline{k}$

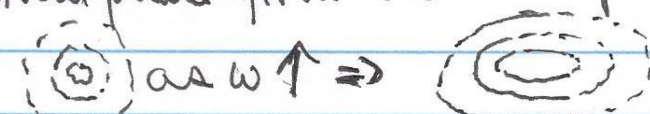
These modes are called "Compressional Alfvén Waves"

In this case the neglect of ϵ_{xy} is because: $|\epsilon_{xy}| = \left(\frac{\omega}{\Omega_i}\right)|\epsilon_\perp|$

$\Rightarrow$ for small $\frac{\omega}{\Omega_i}$ the polarization current is dominant.

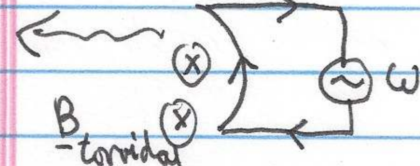
-- But compressional modes can propagate for $\omega > \Omega_i$ --

as $\omega\uparrow$ the spherical phase-fronts become elliptical



due to ϵ_{xy}

Lect#11 (cont.) The compressional Alfvén mode is the preferred wave to heat plasmas to fusion conditions + in ITER a 40 MW system is presently being built. These modes require antennas in which the current flows in the "poloidal direction"

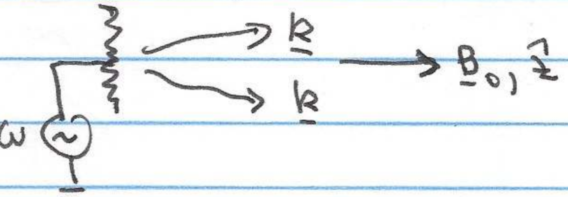


because they propagate $\perp \underline{B}_0$ -- they can deliver E+M energy to the center of the plasma. The absorption is anticipated to occur where $B_{toroidal}(R)$ results in $\omega = 2\,\Omega_i(R)$ - i.e., second harmonic cyclotron absorption on Deuterium ions

Lecture#12 _ A modern perspective on shear Alfvén waves.

Now examine the mixed es/EM relation : $k_\perp^2 \epsilon_\perp + k_\parallel^2 \epsilon_\parallel = k_0^2 \epsilon_\perp \epsilon_\parallel$

- For a "cold plasma" $\Rightarrow$ inertial regime where $\frac{\omega}{k_\parallel} >> \bar{v}_e$ - These modes are called "inertial Alfvén waves". The excitation scheme uses a structure $\perp$ to $\underline{B}_0$



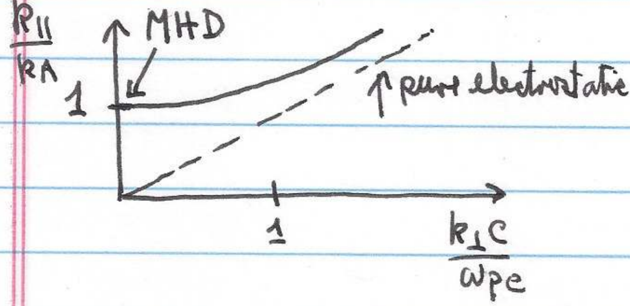
The source sets $k_\perp$ + the plasma responds with a $k_\parallel$ at the set frequency ω

$$k_\parallel^2 = k_0^2 \epsilon_\perp \left(1 - \frac{k_\perp^2}{k_0^2} \frac{1}{\epsilon_\parallel} \right)$$

for low frequency $\omega << \Omega_i \Rightarrow \epsilon_\perp \to \left(\frac{c}{V_A}\right)^2 + \epsilon_\parallel \to - \frac{\omega_{pe}^2}{\omega^2}$

$$\Rightarrow \quad k_\parallel^2 = \frac{\omega^2}{V_A^2} \left(1 + \frac{k_\perp^2 c^2}{\omega_{pe}^2} \right)$$

Lect #12 (cont.) or $\frac{k_\parallel}{k_A} = \sqrt{1+\left(\frac{k_\perp c}{\omega_{pe}}\right)^2}$ with $k_A = \frac{\omega}{V_A}$

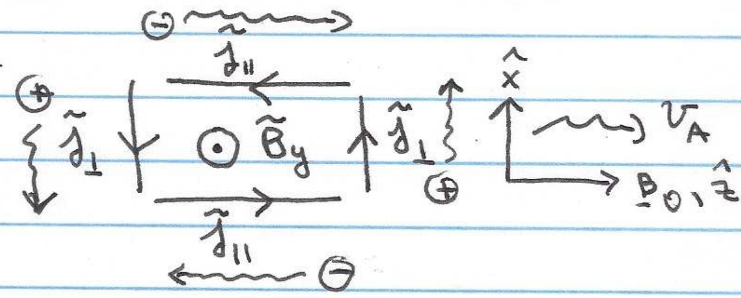


The ideal MHD limit is recovered for $\frac{k_\perp c}{\omega_{pe}} \ll 1$

+ as $k_\perp \uparrow \Rightarrow$ the mode becomes a cone like the LH waves!

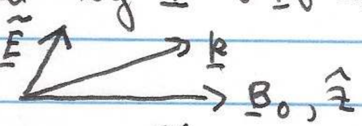
in ideal MHD - there are no electric fields $\parallel$ to $\underline{B}_0$, but now since induction effects are regulated by free electron motion along $\underline{B}_0$, finite $\tilde{E}_\parallel$ develops due to the 'skin-effect' i.e., an oscillating current generates finite $\tilde{B}$ within $\frac{c}{\omega_{pe}}$

- The associated physical picture with these modes are propagating current loops



Along $\underline{B}_0$ the current is carried by electrons due to $\tilde{E}_\parallel$ push

Perpendicular to $\underline{B}_0$ the current is carried by ions due to the $\frac{\partial}{\partial t}\underline{E}$ polarization drift - the closed loops generate a fluctuating magnetic field $\tilde{B}_y \perp$ to $\underline{B}_0 \Rightarrow$ shear polarization and there is an oblique $\tilde{\underline{E}}$



$\tilde{E}_\parallel$ is due to induction and $\tilde{E}_\perp$ is electrostatic

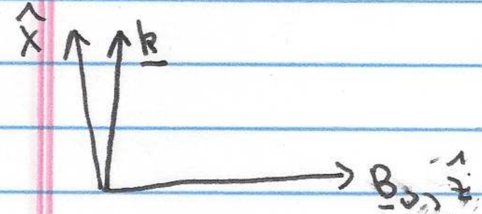
Lect #12 (cont.) The "cone" regime corresponds to $\frac{k_\perp c}{\omega_{pe}} \gg 1$

$$\Rightarrow k_\parallel^2 = k_A^2\left(1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}\right) \Rightarrow \frac{k_\parallel}{k_\perp} = \frac{k_A c}{\omega_{pe}} = \left(\frac{\omega}{\omega_{pe}}\right)\frac{c}{V_A} = \frac{\omega}{\omega_{pe}}\frac{\omega_{pi}}{\Omega_i}$$

but $\frac{\omega_{pi}}{\omega_{pe}} = \sqrt{\frac{m}{M}} \Rightarrow \frac{k_\perp}{k_\parallel} = \left(\frac{\Omega_i}{\omega}\right)\sqrt{\frac{M}{m}}$

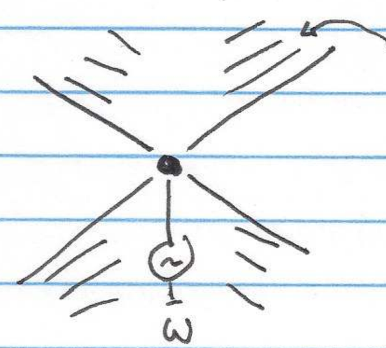
both are large

$\Rightarrow$



$\Rightarrow$ The phase fronts are nearly $\perp$ to $\underline{B}_0$

$\Rightarrow$



phase-fronts

But what is the direction of energy flow? -- Determined by group velocity!

with $(v_g)_\parallel = \frac{\partial\omega}{\partial k_\parallel}$ but $k_\parallel^2 = \frac{\omega^2}{V_A^2}\left(1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}\right)$

$$\Rightarrow \quad 2k_\parallel = 2\omega\frac{\partial\omega}{\partial k_\parallel}\frac{(1+k_\perp^2 c^2/\omega_{pe}^2)}{V_A^2}$$

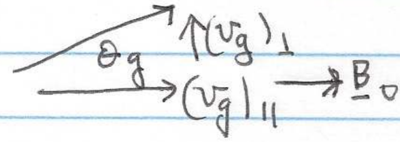
$$\text{or} \quad (v_g)_\parallel = \frac{V_A^2}{1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}}\frac{k_\parallel}{\omega} = \frac{V_A^2}{1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}}\left(1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}\right)^{1/2}\frac{1}{V_A}$$

$$\text{or} \quad (v_g)_\parallel = \frac{V_A}{\sqrt{1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}}} \Rightarrow \text{as } k_\perp \uparrow \Rightarrow (v_g)_\parallel \downarrow$$

Now need $(v_g)_\perp = \frac{\partial\omega}{\partial k_\perp} \Rightarrow \quad 0 = \frac{2\omega\frac{\partial\omega}{\partial k_\perp}}{V_A^2}\sqrt{1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}} + \frac{\omega^3}{V_A^2}\frac{\frac{k_\perp c^2}{\omega_{pe}^2}}{\sqrt{1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}}}$

Lect #12 (cont.) solve for $\frac{\partial \omega}{\partial k_\perp} \Rightarrow (v_g)_\perp = -\frac{\omega}{2} \frac{k_\perp c^2/\omega_{pe}^2}{1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}}$

Now find the "group angle": $\tan\theta_g \equiv \frac{(v_g)_\perp}{(v_g)_\parallel}$



$$\tan\theta_g = -\frac{\frac{\omega}{2}\frac{k_\perp c^2/\omega_{pe}^2}{1+k_\perp^2 c^2/\omega_{pe}^2}}{\frac{V_A}{\sqrt{1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}}}} = -\frac{\omega}{2V_A}\frac{k_\perp c^2/\omega_{pe}^2}{\sqrt{1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}}}$$

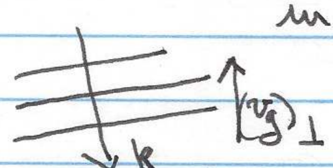
but $v_A = c\frac{\Omega_i}{\omega_{pi}} \Rightarrow \tan\theta_g = -\frac{\omega}{2\Omega_i}\frac{\omega_{pi}}{\omega_{pe}}\frac{(k_\perp c/\omega_{pe})}{\sqrt{1+\frac{k_\perp^2 c^2}{\omega_{pe}^2}}}$

& $\frac{\omega_{pi}}{\omega_{pe}} = \sqrt{\frac{m}{M}} \Rightarrow$ $\boxed{\tan\theta_g = -\frac{\omega}{2\Omega_i}\sqrt{\frac{m}{M}}\frac{k_\perp c/\omega_{pe}}{\sqrt{1+k_\perp^2 c^2/\omega_{pe}^2}}}$

Properties: 1) θ_g is very small $\simeq$ 3–5% $\Rightarrow$ energy propagates nearly $\parallel$ to $\underline{B}_0$ – But there is a finite spread

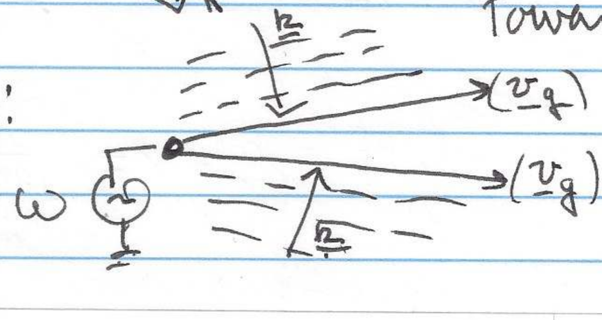
2) Spread increases as $k_\perp$ & saturates at $\theta_g \simeq -\frac{\omega}{2\Omega_i}\sqrt{\frac{m}{M}}$

3) The wave is a "backward wave" $(v_g)_\perp \propto -k_\perp$ in $\perp$ direction
yes



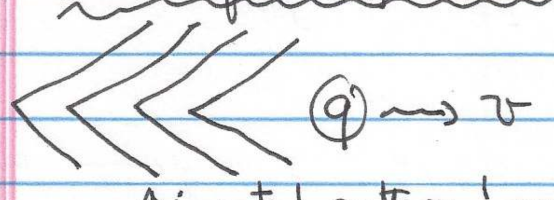
The phase fronts converge towards the source!

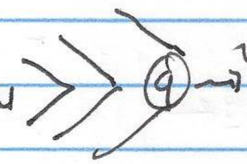
The overall picture is:



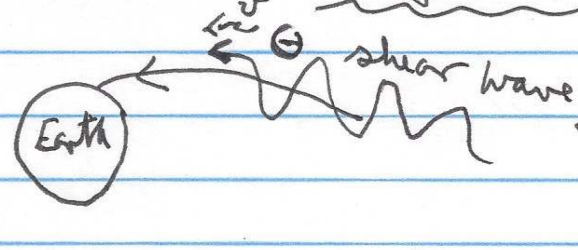
"is a spreading narrow cone"

Lect#12 (cont.) The consequence for Cerenkov radiation of inertial Shear waves is



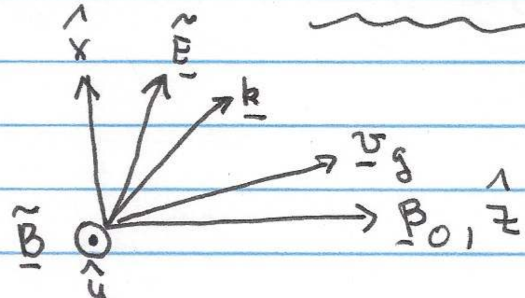
↑ inverted pattern! -- The usual wakes are like 

Why pay so much attention to these modes? -- They regulate current changes & can result in ∥ electron acceleration ⇒ play an important role in the generation of auroras



⇐ electron acceleration requires a finite $E_\parallel$
- in ideal MHD $E_\parallel = 0$!

Here is the wave topology:



Let's estimate what is the possible velocity gain

Ampere's Law: $\nabla \times \tilde{\underline{B}} = -ik_0 \overleftrightarrow{\epsilon} \cdot \tilde{\underline{E}}$

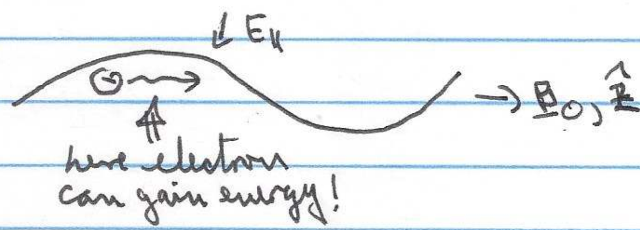
proportional to $k_\perp$

For a plane wave: $ik_\perp \tilde{B}_y = -ik_0 \epsilon_\parallel \tilde{E}_\parallel \Rightarrow \tilde{E}_\parallel = -\frac{k_\perp}{k_0}\frac{1}{\epsilon_\parallel}\tilde{B}_y$

but at low frequency: $\epsilon_\parallel \to -\frac{\omega_{pe}^2}{\omega^2} \Rightarrow |\tilde{E}_\parallel| = \frac{k_\perp c}{\omega_{pe}^2}\omega|\tilde{B}_y| = \left(\frac{k_\perp c}{\omega_p}\right)\left(\frac{\omega}{\omega_{pe}}\right)|\tilde{B}_y|$

a very small #

but along $\underline{B}_0$



Lect#12 (cont.) The interval over which electron gains energy is: $\Delta t \sim \frac{\pi}{\omega}$

$$\Rightarrow \quad |\Delta v_{\parallel}| = \frac{e}{m}|\tilde{E}_{\parallel}|\Delta t = \frac{e}{m}\frac{\pi}{\omega}|\tilde{E}_{\parallel}| = \frac{e}{m}\frac{\pi}{\cancel{\omega}}\left(\frac{k_\perp c}{\omega_{pe}}\right)\left(\frac{\cancel{\omega}}{\omega_{pe}}\right)|\tilde{B}_y|$$

The frequency cancels out!

$$\Rightarrow \quad \frac{|\Delta v_{\parallel}|}{c} = \pi \frac{e}{mc}\frac{1}{\omega_{pe}}\left(\frac{k_\perp c}{\omega_{pe}}\right)|\tilde{B}_y|$$

$$\Rightarrow \quad \frac{|\Delta v_{\parallel}|}{c} = \pi\left(\frac{\Omega_e}{\omega_{pe}}\right)\left(\frac{k_\perp c}{\omega_{pe}}\right)\frac{|\tilde{B}_y|}{B_0} \quad \text{with } \Omega_e = \frac{eB_0}{mc}$$

since $\frac{\omega_{pe}}{\Omega_e} \sim \mathcal{O}(1) \Rightarrow$ for $\frac{|\tilde{B}_y|}{B_0} \sim 10^{-1} \Rightarrow \boxed{\frac{|\Delta v_{\parallel}|}{c} \sim 10^{-1}}$

$\Rightarrow$ electron energy gain in one cycle $\sim$ 3 keV

-- The small $\tilde{E}_{\parallel}$ is compensated by the long interaction time!

The properties previously explored are for the inertial regime where $\frac{\omega}{k_{\parallel}} \gg \bar{v}_e$ but for Alfvén waves $\frac{\omega}{k_{\parallel}} \sim V_A$ - Let's see what this implies

$$\left.\begin{array}{l} \bar{v}_e = 4\times 10^7 \sqrt{T_e}\ \frac{cm}{sec} \\ \text{\& Typically } v_A \simeq 5\times 10^8 \frac{cm}{sec} \end{array}\right\} \Rightarrow T_e < 100\ eV \text{ for inertial regime}$$

But as $T_e \uparrow \Rightarrow \bar{v}_e \uparrow$ &/or $B\downarrow \Rightarrow v_A \downarrow$ -- There is the opposite regime - the adiabatic regime - where $\frac{\omega}{k_{\parallel}} \ll \bar{v}_e$

- This is typical of weakly magnetized/hot plasmas such as the "Solar Wind" - These waves are called "Kinetic Alfvén Waves" - let's explore their properties

Lect #12 (cont.) - The general dispersion relation for shear Alfvén waves is $k_\perp^2 \epsilon_\perp + k_\parallel^2 \epsilon_\parallel = k_0^2 \epsilon_\perp \epsilon_\parallel$ with $\epsilon_\parallel \to 1 - \frac{1}{2}\frac{k_D^2}{k_\parallel^2} Z'\left(\frac{\omega}{\sqrt{2}\, k_\parallel \bar{v}_e}\right)$

with Z' the plasma dispersion function from 222B.

in adiabatic regime $Z' \to -2 \Rightarrow \quad \epsilon_\parallel \to 1 + \frac{k_D^2}{k_\parallel^2} \Rightarrow k_\parallel^2 \epsilon_\parallel \to k_D^2$

$$\Rightarrow \quad k_\perp^2 \epsilon_\perp + k_D^2 = k_0^2 \frac{k_D^2}{k_\parallel^2} \epsilon_\perp \qquad \text{now solve for } k_\parallel = k_\parallel(\omega, k_\perp)$$

$$\Rightarrow \quad k_\parallel^2 = \frac{k_0^2 k_D^2 \epsilon_\perp}{k_D^2 + k_\perp^2 \epsilon_\perp} = \frac{k_0^2 \epsilon_\perp}{1 + \frac{k_\perp^2 \epsilon_\perp}{k_D^2}} = \frac{\omega^2/v_A^2}{1 + \frac{k_\perp^2 \epsilon_\perp}{k_D^2}}$$

What is $\frac{\epsilon_\perp}{k_D^2}$? $= \frac{\omega_{pi}^2}{\Omega_i^2} \frac{1}{\omega_{pe}^2/\bar{v}_e^2} = \frac{m}{M}\frac{1}{\Omega_i^2}\frac{T_e}{m} = \frac{T_e/M}{\Omega_i^2} = \boxed{\left(\frac{c_s}{\Omega_i}\right)^2}$

with $c_s = \sqrt{\frac{T_e}{M}}$ the ion-sound speed!

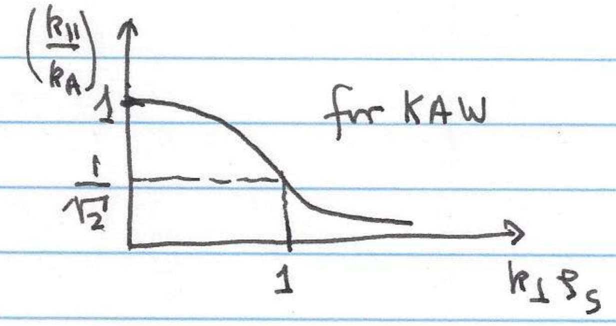
$\Rightarrow$ Discover that there is another characteristic scale length associated with "electron adiabatic response in a magnetized plasma"

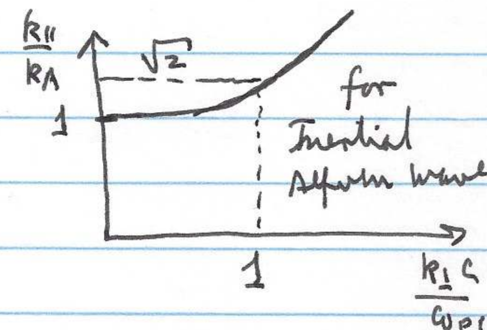
It is called $\rho_s \equiv \frac{c_s}{\Omega_i}$ "The ion-sound gyroradius"

or the the Larmor radius of an ion with velocity equal to the sound-speed -- Now the dispersion relation for Kinetic Alfvén waves becomes: $k_\parallel = \frac{k_A}{\sqrt{1 + (k_\perp \rho_s)^2}}$ $\quad k_A \leftrightarrow \omega/v_A$

or $\frac{\omega}{k_\parallel} = V_A \sqrt{1 + (k_\perp \rho_s)^2} \quad \Rightarrow v_p \uparrow \text{ as } k_\perp \uparrow$

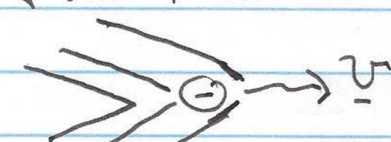
Lect #12 (cont.)





In both cases the role of $k_\perp \uparrow$ is to bring mode into the resonant regime where $\frac{\omega}{k_\parallel} \sim \bar{v}_e$

- We will not do the algebra here -- but indicate that the KAW is a "forward wave" - i.e., the usual case & it exhibits a normal V-shape for its Cerenkov cone :



There is another important electrostatic resonance for a magnetized plasma that arises from $\epsilon_\perp \to 0$ -- Already learned about the upper-hybrid $\Rightarrow$ $\omega_{UH} = \sqrt{\omega_{pe}^2 + \Omega_e^2}$

& the lower-hybrid $\Rightarrow$ $\omega_{LH} = \dfrac{\omega_{pi}}{\sqrt{1 + \left(\frac{\omega_{pe}}{\Omega_e}\right)^2}}$

- These are for plasmas with a "single ion species"

But there are several interesting situations in which more than one ion species is present - in particular -- for fusion plasmas one starts out with D, T and hopes to generate fast He (or alpha) particles - the presence of multiple ions allows for new "hybrid resonances"

Lect #12 (cont.) Another example of multiple-ion plasmas is the ionosphere with O + H ions!

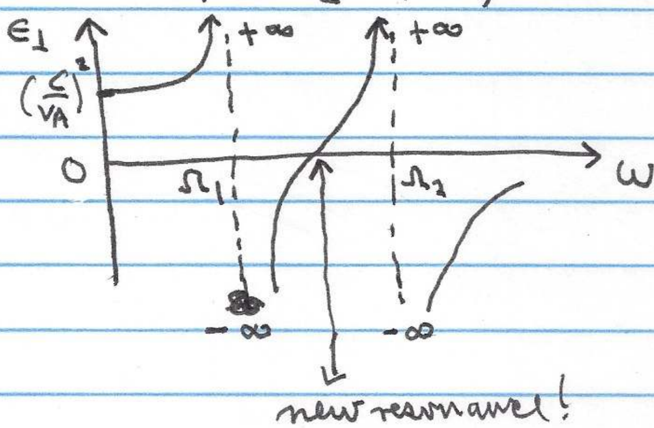
Consider $\epsilon_\perp = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \sum_s \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2}$; $\omega_{pj}^2 = \frac{4\pi q_j^2 n_j}{M_j}$

(first term: electrons; sum: multiple species ions)

for the relevant low frequencies $\epsilon_\perp \to - \sum_j \frac{\omega_{pi}^2}{\omega^2 - \Omega_j^2}$ because $\frac{\omega_{pe}}{\Omega_e} \sim \mathcal{O}(1)$

To understand issues consider 2 ions where $\Omega_1 < \Omega_2$ (i.e., #1 + #2)

$$\epsilon_\perp \to - \left[\frac{\omega_{p1}^2}{\omega^2 - \Omega_1^2} + \frac{\omega_{p2}^2}{\omega^2 - \Omega_2^2} \right] \Longrightarrow$$



where the new Alfvén speed is:

$$\boxed{V_A = \frac{\Omega_1 \Omega_2}{\sqrt{\omega_{p1}^2 \Omega_2^2 + \omega_{p2}^2 \Omega_1^2}}}$$

+ $\epsilon_\perp = 0$ at a frequency $\omega_{ii} = \sqrt{\frac{\omega_{p1}^2 \Omega_2^2 + \omega_{p2}^2 \Omega_1^2}{\omega_{p1}^2 + \omega_{p2}^2}}$

Called - $\boxed{\text{The Ion-Ion Hybrid Frequency}}$ which the analog of ω_{UH} + ω_{LH}

- This resonance provides another possible option to heat plasmas + has proven very effective in Tokamaks

- The singularities at $\epsilon_\perp = 0$ need to be resolved by the inclusion of kinetic effects that lead to a new class of modes -- The electron and ion Bernstein waves [EBW, IBW]

Lect #12 (cont.) Before ending the survey of Alfvén waves I would like to emphasize an important constraint that experimentalists + computer simulation researchers need to be aware of – a size constraint – This pertains to the Low-frequency Compressional Alfvén wave – Whose dispersion relation is: $k^2 = k_A^2 = \frac{\omega^2}{V_A^2} = k_\perp^2 + k_\parallel^2$

since $\omega \sim \Omega_i$ $\Rightarrow$ The region of interest is $k_\perp^2 = \frac{\Omega_i^2}{V_A^2} - k_\parallel^2$

and $V_A = c\,\frac{\Omega_i}{\omega_{pi}}$ $\Rightarrow$ ~~the smallest~~ The smallest $\perp$ wavelength

is $\boxed{(\lambda_\perp)_{min} = 2\pi\,\frac{c}{\omega_{pi}}}$ $\Rightarrow$ In The LAB in order to probe features related to these modes

need machines whose Transverse dimension is $> \frac{c}{\omega_{pi}}$ + likewise for computer simulations -- The systems must be large --

In actual #'s: $\frac{c}{\omega_{pi}} \simeq 2.28\times10^7\,\frac{1}{Z}\sqrt{\frac{\mu}{n_i}}$ cm

which for ITER at $n_i \sim 10^{14}\ cm^{-3}$ $\Rightarrow$ $\frac{c}{\omega_{pi}} \sim 2$ cm || Yes plenty of activity

for LAPD at $n_i \sim 10^{12}\ cm^{-3}$, He $\Rightarrow$ $\frac{c}{\omega_{pi}} \sim 40$ cm || not possible

Space Plasmas at $n_i \sim 1\ cm^{-3}$, H $\Rightarrow$ $\frac{c}{\omega_{pi}} \sim 200$ km

Lect #13 . Kinetic description of magnetized plasmas (collisionless)

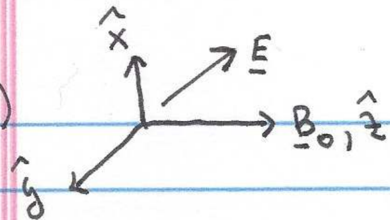
Major new features: 1) Wave particle resonance $\Rightarrow$ $\omega - k_\parallel v_\parallel = n\Omega$, $n = 1, 2, 3 \ldots$

2) $\tilde{E}_\perp$ is averaged over a Larmor radius $\rho_s = \frac{v_\perp}{\Omega_s}$ $\Rightarrow$ $\langle \tilde{E}_\perp \rangle \to E_0\, J_n^2(k_\perp \rho_s)$

3) There are new modes between harmonics $n\Omega_s < \omega < (n+1)\Omega_s$

4) There is now $\epsilon_{xz} \neq 0$, $\epsilon_{zx} \neq 0$ due to parallel magnetic force

Lect #13 (cont..)



Zero-order plasma: infinite, uniform & collisionless.

Each charged species has a zero-order dist. function $f_0 = f_0(v_\perp^2, v_\parallel)$ ← or "v_z"
- Do not keep track of species label "s" - do that at the end. No net drifts - symmetric in v_z.

Consider usual linear response: $f = f_0 + \delta f$ with $|\delta f| \ll f_0$

The electromagnetic fields are: $\underline{E}$, $\underline{B} = \underline{B}_0 + \delta\underline{B}$, no zero-order E_0!

$\uparrow$ uniform, time independent (refers to $\underline{B}_0$)

Linearized Vlasov Eq:
$$\frac{\partial}{\partial t}\delta f + \underline{v}\cdot\nabla\delta f + \left[\frac{q\underline{E}}{m} + \frac{q}{mc}\underline{v}\times\delta\underline{B}\right]\cdot\nabla_v f_0 + \frac{q}{mc}(\underline{v}\times\underline{B}_0)\cdot\frac{\partial}{\partial\underline{v}}\delta f = \boxed{0}$$

Separate into free streaming + driven effect of fluctuations:
$$\frac{\partial}{\partial t}\delta f + \underline{v}\cdot\nabla\delta f + \frac{q}{m}(\underline{v}\times\underline{B}_0)\cdot\frac{\partial}{\partial\underline{v}}\delta f = -\frac{q}{m}\left[\underline{E} + \frac{\underline{v}\times\delta\underline{B}}{c}\right]\cdot\nabla_v f_0$$

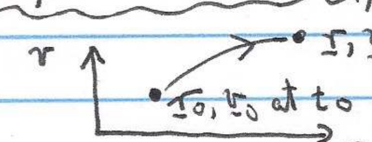
where $\underline{E} = \underline{E}(\underline{r}, t)$ and $\delta\underline{B} = \delta\underline{B}(\underline{r}, t)$ - The Eq. has the form:
$$\boxed{\frac{D}{Dt}\delta f = S(\underline{r}, \underline{v}, t)}$$

$\frac{D}{Dt}$: Total time derivative along spiraling path due to $\underline{B}_0$

S: a source of modifications produced by field fluctuations

We are going to solve using the "orbit functional technique" of 222A

recall that in absence of collisions the past + future can be mapped by time-reversal invariance



Lect #13 (cont.) The orbit functional Technique is The same mathematical method as: "integration along characteristics" or "Green's function method"

The key idea is That if The mapping: $\underline{r}_0(\underline{r}, \underline{v}, t)$ and $\underline{v}_0(\underline{r}, \underline{v}, t)$ is known Then The answer to the driven Eq. with source $S'(\underline{r}, \underline{v}, t)$ - The answer is obtained by integrating The source along The orbits -- This is an approach also used in 222B to tackle Landau's 2nd half -- the spatial problem. The result is:

$$\delta f(\underline{r}, \underline{v}, t) = \delta f(t=-\infty) + \int_{-\infty}^{t} dt' \, S'(\underbrace{\underline{r}_0(\underline{r}, \underline{v}, t'); \underline{v}_0(\underline{r}, \underline{v}, t')}_{\text{assume known Trajectories}}; t')$$

The variable t' used as a dummy variable corresponds physically To the initial time t_0 in The cartoon used to illustrate The mapping!

The mapping conditions are: $\frac{\partial}{\partial t'} \underline{r}_0 = \underline{v}_0$

$$\frac{\partial}{\partial t'} \underline{v}_0 = \frac{q}{mc} \underline{v}_0 \times \underline{B}_0$$

with The constraint That $\underline{v}_0(\text{at } t'=t) = \underline{v}$ and $\underline{r}_0(\text{at } t'=t) = \underline{r}$

$$\Rightarrow \delta f(\underline{r}, \underline{v}, t) = -\frac{q}{m} \int_{-\infty}^{t} dt' \left[\underline{E}(\underline{r}_0, t') + \frac{\underline{v}_0 \times \delta \underline{B}(\underline{r}_0, t')}{c} \right] \cdot \frac{\partial}{\partial \underline{v}_0} f_0(v_{\perp 0}^2, v_{z0})$$

$\boxed{\text{where it is assumed That } \delta f(t=-\infty) = 0}$

note That in the integral wherever There is $\underline{r} \to \underline{r}_0$ and where There is a $\underline{v} \to \underline{v}_0$

(*) Now we get sloppy based on our mastery of 222B!

- We are going To use a FT in Time + we know That we can get in Trouble like Vlasov -- but have no fear because we know That we should do a LT To properly deal with singularities

Lect#13 (cont.) If we run into a singular denominator $\Rightarrow$ use analytic continuation + that will lead to the equivalent of the plasma dispersive function Z or Z'. -- Proceed with simpler FT algebra -

Harmonic Representation of fluctuations.

$$\begin{pmatrix} \underline{E} \\ \delta \underline{B} \\ \delta f \\ \underline{J} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \tilde{\underline{E}} \\ \tilde{\underline{B}} \\ \tilde{f} \\ \tilde{\underline{J}} \end{pmatrix} e^{i(\underline{k}\cdot\underline{r} - \omega t)} + c.c.$$

& now plug in these forms into previous integral

$$e^{i(\underline{k}\cdot\underline{r}-\omega t)}\,\tilde{f}(\underline{k},\omega,\underline{v}) = -\frac{q}{m}\int_{-\infty}^{t} dt' \left[\tilde{\underline{E}}\, e^{i(\underline{k}\cdot\underline{r}_0 - \omega t')} + \frac{\underline{v}_0 \times \tilde{\underline{B}}}{c}\, e^{i(\underline{k}\cdot\underline{r}_0-\omega t')}\right]\cdot \frac{\partial}{\partial \underline{v}_0} f_0(v_{\perp 0}^2, v_{z0})$$

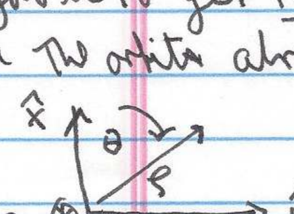
move inside integral & define $\tau \equiv t' - t$

$$\tilde{f}(\underline{k},\omega,\underline{v}) = -\frac{q}{m}\int_{-\infty}^{t} dt' \left[\tilde{\underline{E}}\, e^{i[\underline{k}\cdot(\underline{r}_0-\underline{r}) - \omega\tau]} + \frac{\underline{v}_0 \times \tilde{\underline{B}}}{c}\, e^{i[\underline{k}\cdot(\underline{r}_0-\underline{r})-\omega\tau]}\right]\cdot \frac{\partial}{\partial \underline{v}_0} f_0(v_{\perp 0}^2, v_{z0})$$

Define the position difference $\underline{R} \equiv \underline{r}_0 - \underline{r}$ and integrate over $d\tau = dt'$

$$\Rightarrow \tilde{f}(\underline{k},\omega,\underline{v}) = -\frac{q}{m}\int_{-\infty}^{0} d\tau \left[\tilde{\underline{E}}\, e^{i[\underline{k}\cdot\underline{R} - \omega\tau]} + \frac{\underline{v}_0 \times \tilde{\underline{B}}}{c}\, e^{i[\underline{k}\cdot\underline{R}-\omega\tau]}\right]\cdot \frac{\partial}{\partial \underline{v}_0} f_0(v_{\perp 0}^2, v_{z0})$$

The job is to get $\underline{R}$ + plug into integral + do the integral - We have done the orbits already - but for better feel - repeat here again.



$\rho = \frac{v_\perp}{\Omega}$; $\theta = C_\theta - \lambda\Omega t$ with $\lambda = \begin{cases} +1 & \text{for electrons} \\ -1 & \text{for ions}\end{cases}$

and C_θ is a constant determined from initial conditions, at $t = t'$, $\theta = \theta_0 \Rightarrow \theta_0 = C_\theta - \lambda\Omega t'$

or $C_\theta = \theta_0 + \lambda\Omega t' \Rightarrow \theta = \theta_0 - \lambda\Omega(t - t') \Rightarrow \boxed{\theta_0 = \theta - \lambda\Omega\tau}$ mapping of θ

Lect #13 (cont.) Now follow the same procedure for (x,y) coordinates

$$x = c_x + \rho \cos\theta \ , \quad y = c_y + \rho \sin\theta$$

at $t = t' \Rightarrow x = x_0 \Rightarrow$ $x_0 = c_x + \rho\cos\theta_0 \Rightarrow c_x = x_0 - \rho\cos\theta_0$

& $\quad y = y_0 \Rightarrow c_y = y_0 - \rho \sin\theta_0$

$$\Rightarrow \quad x = x_0 - \rho\cos\theta_0 + \rho\cos\theta \Rightarrow \boxed{x_0 = x + \rho\left[\cos\theta_0 - \cos(\theta_0 + \lambda\Omega\tau)\right]}$$

& likewise, $\quad y_0 = y + \rho\left[\sin\theta_0 - \sin(\theta_0 + \lambda\Omega\tau)\right]$

& the integral needs $\quad \underline{R} \equiv \underline{r}_0 - \underline{r}$

$$\Rightarrow \boxed{\underline{R}_\perp = \rho\left[\cos\theta_0 - \cos(\theta_0+\lambda\Omega\tau)\right]\hat{x} + \rho\left[\sin\theta_0 - \sin(\theta_0 + \lambda\Omega\tau)\right]\hat{y}}$$

and in the $\hat{z}$ direction $\boxed{R_z = z_0 - z = v_z\tau}$

But we also need the velocity functionals

$$\frac{\partial}{\partial t'}\underline{r}_0 = \underline{v}_0 \quad \text{and since } \tau = t' - t \Rightarrow \frac{\partial}{\partial t'} = \frac{\partial}{\partial\tau}$$

$$\Rightarrow \quad \frac{\partial}{\partial\tau}\underline{r}_0 = \underline{v}_0 \Rightarrow \boxed{v_{z0} = \frac{\partial}{\partial\tau} z_0 = v_z} \text{ it is a constant}$$

$$\boxed{\begin{aligned} v_{x0} &= \frac{\partial x_0}{\partial\tau} = +\rho\lambda\Omega\sin(\theta_0 + \lambda\Omega\tau) \\ v_{y0} &= \frac{\partial y_0}{\partial\tau} = -\rho\lambda\Omega\cos(\theta_0 + \lambda\Omega\tau) \end{aligned}}$$

From which $\quad v_{x0}^2 + v_{y0}^2 = \rho^2\Omega^2 = v_\perp^2$ - as we know, it is a constant

Now examine how these terms enter into the integral

- Because zero-order is uniform and symmetric $\Rightarrow \underline{k} = k_\perp\hat{x} + k_z\hat{z}$

$$\Rightarrow \boxed{\underline{k}\cdot\underline{R} = k_\perp\rho\left[\cos\theta_0 - \cos(\theta_0 + \lambda\Omega\tau)\right] + k_z v_z\tau}$$

Lect #13 (cont.) But $\frac{\partial}{\partial \underline{v}_0}$ derivatives are needed

$$\Rightarrow \frac{\partial}{\partial v_{x0}} = \frac{\partial}{\partial v_\perp}\frac{\partial v_\perp}{\partial v_{x0}} \quad \text{but } v_\perp = \sqrt{v_{x0}^2 + v_{y0}^2} \Rightarrow \frac{\partial v_\perp}{\partial v_{x0}} = \frac{v_{x0}}{v_\perp} \Rightarrow \boxed{\frac{\partial}{\partial v_{x0}} = \frac{v_{x0}}{v_\perp}\frac{\partial}{\partial v_\perp}}$$

and likewise $\boxed{\frac{\partial}{\partial v_{y0}} = \frac{v_{y0}}{v_\perp}\frac{\partial}{\partial v_\perp}}$ and since $f_0(v_{\perp 0}^2, v_{z0}) = f_0(v_\perp, v_z)$

Then, $$\boxed{\tilde{\underline{E}} \cdot \frac{\partial}{\partial \underline{v}_0} f_0 = \hat{E}_z \frac{\partial}{\partial v_z} f_0 + \hat{E}_x \frac{v_{x0}}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 + \hat{E}_y \frac{v_{y0}}{v_\perp}\frac{\partial}{\partial v_\perp} f_0}$$

In the integral there are effects arising from the electrical force and from the magnetic force, i.e., from

$$\underbrace{\frac{q}{m}\tilde{\underline{E}} \cdot \frac{\partial}{\partial \underline{v}_0} f_0}_{\text{due to this } \tilde{f}_E} + \underbrace{\frac{q}{mc}(\underline{v}_0 \times \tilde{\underline{B}}) \cdot \frac{\partial}{\partial \underline{v}_0} f_0}_{\text{due to this } \tilde{f}_B}$$

$\Rightarrow$ separate as: $\tilde{f} = \tilde{f}_E(\underline{k}, \omega, \underline{v}) + \tilde{f}_B(\underline{k}, \omega, \underline{v})$

Examine the $\tilde{f}_E$ and $\tilde{f}_B$ separately.

$$\tilde{f}_E \rightarrow \begin{pmatrix} (\tilde{f}_E)_x \\ (\tilde{f}_E)_y \\ (\tilde{f}_E)_z \end{pmatrix} = -\frac{q}{m}\int_{-\infty}^{0} d\tau\, e^{i[\underline{k}\cdot\underline{R} - \omega\tau]} \begin{pmatrix} \hat{E}_z \frac{\partial}{\partial v_z} f_0 \\ \hat{E}_x \frac{v_{x0}}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \\ \hat{E}_y \frac{v_{y0}}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \end{pmatrix}$$

consists of 3 terms from $(\hat{E}_x, \hat{E}_y, \hat{E}_z)$

Take out the trivial parts

in $\hat{x}$: $(\tilde{f}_E)_x = -\frac{q}{m}\hat{E}_x \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \; \mathcal{J}_x$

in $\hat{y}$: $(\tilde{f}_E)_y = -\frac{q}{m}\hat{E}_y \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \; \mathcal{J}_y$

in $\hat{z}$: $(\tilde{f}_E)_z = -\frac{q}{m}\hat{E}_z \frac{\partial}{\partial v_z} f_0 \; \mathcal{J}_z$

These are the actual integrals where the orbits appear

Lect # 13 (cont.) The orbit integrals due to Electrical force are:

$$\begin{pmatrix} \mathcal{J}_z \\ \mathcal{J}_x \\ \mathcal{J}_y \end{pmatrix} = \int_{-\infty}^{0} d\tau\, e^{i[k_z v_z \tau - \omega\tau]}\, e^{i k_\perp \rho[\cos\theta_0 - \cos(\theta_0 + \lambda\Omega\tau)]} \circledast \begin{pmatrix} 1 \\ v_\perp \lambda \sin[\theta_0 + \lambda\Omega\tau] \\ -v_\perp\lambda\cos[\theta_0+\lambda\Omega\tau] \end{pmatrix}$$

Let's work out 2 of these integrals to appreciate where the new magnetized results come from.

The key identity in dealing with kinetic effects in magnetized plasma relates to the $e^{i\underline{k}_\perp \cdot \underline{r}_\perp}$ phase-factor - seen above -

$$e^{ia\sin\alpha} = \sum_{n=-\infty}^{\infty} J_n(a)\, e^{in\alpha} \quad \text{called the "Bessel Identity"}$$

but in the integral what appears is the cos not the sin

so we need, $\sin(\alpha + \pi/2) = \sin\alpha \cancelto{0}{\cos\pi/2} + \cancelto{1}{\sin\pi/2}\cos\alpha = \cos\alpha$

$$\Rightarrow \quad e^{ia\cos\alpha} = \sum_{n=-\infty}^{\infty} J_n(a)\, e^{in\alpha} e^{in\pi/2} \quad \text{but } e^{i\pi/2} = i$$

$$\Rightarrow \quad e^{ia\cos\alpha} = \sum_{n=-\infty}^{\infty} (i)^n J_n(a)\, e^{in\alpha} \quad \text{and} \quad e^{-ia\cos\alpha} = \sum_{n=-\infty}^{\infty} (-i)^n J_n(a)\, e^{-in\alpha}$$

now apply to $\mathcal{J}_z$ above - now a product of two $\Rightarrow$ <u>different dummy indices</u>

$$\mathcal{J}_z = \int_{-\infty}^{0} d\tau\, e^{i(k_z v_z - \omega\tau)} \sum_n \sum_l J_n(k_\perp\rho) J_l(k_\perp\rho) (i)^n (-i)^l e^{in\theta_0} e^{-il(\theta_0 + \lambda\Omega\tau)}$$

(*) At this point our ignorance shows-up - what do we do with the <u>θ_0 terms</u>? -- The usual approach is to say - they are uniformly distributed, so average over them, i.e., consider

$$\langle \mathcal{J}_z \rangle_{\theta_0} = \frac{1}{2\pi}\int_0^{2\pi} d\theta_0\, \mathcal{J}_z$$

Lect #13 (cont.) The θ_0 enters as $\int_0^{2\pi} \frac{d\theta_0}{2\pi} e^{i(n-l)\theta_0} = \delta_{n,l}$ Kronecker Delta

$\Rightarrow$ in the sum over l only the $l = n$ contributes and gives $1 + (i)^n(-i)^l = 1$

$$\Rightarrow \langle \mathcal{I}_z \rangle_{\theta_0} = \sum_n J_n^2(k_\perp \rho) \int_{-\infty}^{0} d\tau \, e^{i(k_z v_z - \omega)\tau} e^{-in\lambda\Omega\tau}$$

(*) Another point of ignorance is what to do with oscillation at $\tau \to -\infty$

$\Rightarrow$ assume the fields as "adiabatically turned-on" $\Rightarrow$ They vanish at $\tau \to -\infty$

$$\text{Then} \quad \langle \mathcal{I}_z \rangle_{\theta_0} \to \sum_{n=-\infty}^{\infty} \frac{J_n^2(k_\perp \rho)}{i[k_z v_z - \omega - n\lambda\Omega]}$$

and this implies that the effect produced by $\tilde{E}_z$ results in the modification

$$\boxed{(\tilde{f}_E)_z \to -\frac{q}{im} \sum_{n=-\infty}^{\infty} \frac{J_n^2(k_\perp \rho)}{[k_z v_z - \omega - n\lambda\Omega]} \tilde{E}_z \frac{\partial}{\partial v_z} f_0}$$

which clearly shows the 2 key effects of $\underline{B}_0$

1) Electric field $\tilde{E}_z \to J_n^3(k_\perp \rho) \tilde{E}_z$

2) Generalized wave-particle resonance

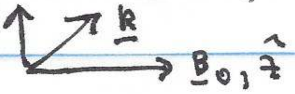
$$\boxed{\frac{\omega}{k_z} = v_z - n\lambda\frac{\Omega}{k_z}} \quad \text{for } n = 0, \pm 1, \pm 2, ---$$

~~Check Plates~~ if $k_\perp \to 0$ or $\rho \to 0$ $\Rightarrow$ $J_n^2(k_\perp \rho) = \delta_{n,0}$

$+$ $\boxed{(\tilde{f}_E)_z \to \frac{q}{im} \frac{\tilde{E}_z \frac{\partial}{\partial v_z} f_0}{(\omega - k_z v_z)}}$ The unmagnetized result of 222B

$\Rightarrow$ Langmuir waves, IAW + Beam instabilities $\parallel$ to $\underline{B}_0$ are the same!

Lect #13 (cont.) Let's explore one of the $\perp$ to $\underline{B}_0$ terms due to the $\underline{\tilde{E}}$-field in the $\hat{x}$ direction $\quad (\tilde{f}_E)_x \propto \mathcal{J}_x \quad$ recall



with
$$\mathcal{J}_x = \int_{-\infty}^{0} d\tau\, \underbrace{e^{i(k_z v_z - \omega)\tau}\, e^{i k_\perp \rho[\cos\theta_0 - \cos(\theta_0 + \lambda\Omega\tau)]}}_{\text{know how to do from } \mathcal{J}_z \text{ calculation}}\, \underbrace{\sin(\theta_0 + \lambda\Omega\tau)}$$

This is the new part $\quad = \dfrac{e^{i(\theta_0 + \lambda\Omega\tau)} - e^{-i(\theta_0 + \lambda\Omega\tau)}}{2i}$

using the Bessel Identity again,

$$\mathcal{J}_x = \frac{1}{2i}\int_{-\infty}^{0} d\tau\, e^{i(k_z v_z - \omega)\tau} \sum_n{}' \sum_l{}' J_n(k_\perp\rho) J_l(k_\perp\rho)\, i^n \left(\frac{1}{i}\right)^l \begin{cases} e^{in\theta_0} e^{-il\theta_0} e^{i\theta_0} e^{-il\lambda\Omega\tau} \otimes e^{i\lambda\Omega\tau} \\ - e^{in\theta_0} e^{-il\theta_0} e^{-i\theta_0} e^{-il\lambda\Omega\tau} \otimes e^{-i\lambda\Omega\tau} \end{cases}$$

Implement the $\langle\,\rangle_{\theta_0} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} d\theta_0$ average

but now there are two terms from the $\sin(\,)$ $\Rightarrow$ for one: $(l-1) = n$ survives

for the other: $(l+1) = n$ survives

$$\langle \mathcal{J}_x \rangle_{\theta_0} = \frac{1}{2i}\int_{-\infty}^{0} d\tau\, e^{i(k_z v_z - \omega)\tau} \sum_n{}' J_n \left[\begin{array}{l} J_{n+1}\, e^{i\lambda\Omega\tau}\, e^{-i\lambda(n+1)\Omega\tau}\, \overbrace{i^n \left(\frac{1}{i}\right)^{n+1}}^{1/i} \\ - J_{n-1}\, e^{-i\lambda\Omega\tau}\, e^{-i\lambda\Omega\tau(n-1)}\, \underbrace{i^n \left(\frac{1}{i}\right)^{n-1}}_{i} \end{array}\right]$$

$$\langle \mathcal{J}_x \rangle_{\theta_0} = -\frac{1}{2}\int_{-\infty}^{0} d\tau\, e^{i(k_z v_z - \omega - n\lambda\Omega)\tau} \sum_n{}' J_n (J_{n+1} + J_{n-1})$$

-- assuming adiabatic turn-on at $t = -\infty$

$$\Rightarrow \langle \mathcal{J}_x \rangle_{\theta_0} = -\frac{1}{2} \sum_n{}' \frac{J_n (J_{n+1} + J_{n-1})}{i(k_z v_z - \omega - n\lambda\Omega)} \quad + \text{ all } J_l = J_l(k_\perp \rho)$$

Lect #13 (cont.) Recall the definition $(\tilde{f}_E)_x = -\frac{q}{m}\tilde{E}_x \frac{\lambda v_\perp}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \, \mathcal{I}_x$

results in: $\langle(\tilde{f}_E)_x\rangle_{\theta_0} = -\frac{q}{m}\tilde{E}_x \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \left[-\frac{1}{2}\sum_n{}' \frac{J_n(J_{n+1}+J_{n-1})}{i(k_z v_z - \omega - \lambda n\Omega)}\right.$

or,
$$\boxed{\langle(\tilde{f}_E)_x\rangle_{\theta_0} = +\frac{q}{2mi}\sum_n{}' \frac{J_n(J_{n+1}+J_{n-1})}{(k_z v_z - \omega - \lambda n\Omega)}\frac{\lambda v_\perp}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \, \tilde{E}_x}$$

This expression will be used later to extract the electron Bernstein modes propagating with $\underline{k} \perp$ to $\underline{B}_0$.

The algebra behind $\langle \mathcal{I}_y\rangle_{\theta_0}$ is similar to that for $\langle \mathcal{I}_x\rangle_{\theta_0}$ and is not repeated here. The analogous result yields

$$\boxed{\langle(\tilde{f}_E)_y\rangle_{\theta_0} = -\frac{q}{2m}\sum_n{}' \frac{J_n(J_{n+1}-J_{n-1})}{(k_z v_z - \omega - \lambda n\Omega)}\frac{\lambda v_\perp}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \, \tilde{E}_y}$$

Now examine the algebra associated with the magnetic force

$$\tilde{f}_B = -\frac{q}{m}\int_{-\infty}^{0} d\tau \, e^{i[\underline{k}\cdot\underline{R}-\omega\tau]} \left(\frac{\underline{v}_0}{c}\times\tilde{\underline{B}}\right)\cdot\frac{\partial}{\partial \underline{v}_0} f_0$$

in which $\tilde{B}$ is eliminated using Faraday's law: $\underline{k}\times\tilde{\underline{E}} = k_0 \tilde{\underline{B}}$

$$\Rightarrow \underline{v}_0 \times \underline{k} \times \tilde{\underline{E}} = \underline{k}\,(\underline{v}_0\cdot\tilde{\underline{E}}) - (\underline{k}\cdot\underline{v}_0)\tilde{\underline{E}}$$

The following vector algebra is identical to that done for the Weibel instability & is not repeated here. The result is:

$$\boxed{(\underline{v}_0\times\underline{k}\times\tilde{\underline{E}})\cdot\frac{\partial}{\partial \underline{v}_0} f_0 = v_z\left[(k_z\tilde{E}_x - k_x\tilde{E}_z)v_{0x} + k_z\tilde{E}_y v_{0y}\right]\left\{\frac{1}{v_z}\frac{\partial}{\partial v_z} f_0 - \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} f_0\right\}}$$

Lect #13 (cont.) - They important feature is that $\tilde{f}_B \neq 0$ if the distribution function $f_0 = f_0(v_\perp^2, v_\parallel)$ is anisotropic, i.e., of the type that leads to the Weibel instability -

It is useful to compare the structure of the magnetic and electric modifications

$$\tilde{f}_B = -\frac{q v_z}{m} \int_{-\infty}^{0} d\tau \, e^{i[\underline{k}\cdot\underline{R} - \omega\tau]} \left\{ (k_z \tilde{E}_x - k_\perp \tilde{E}_z) v_{0x} + k_z \tilde{E}_y v_{0y} \right\} \otimes \left\{ \frac{1}{v_z}\frac{\partial}{\partial v_z} f_0 - \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \right\}$$

and

$$\tilde{f}_E = -\frac{q}{m} \int_{-\infty}^{0} d\tau \, e^{i[\underline{k}\cdot\underline{R} - \omega\tau]} \left\{ \tilde{E}_z \frac{\partial}{\partial v_z} f_0 + \frac{v_{x0}}{v_\perp}\frac{\partial}{\partial v_\perp} f_0 \tilde{E}_x + \frac{v_{y0}}{v_\perp}\frac{\partial f_0}{\partial v_\perp} \tilde{E}_y \right\}$$

from which it can ~~been~~ seen that the integrals $\langle \partial_x \rangle_{00}$ and $\langle \partial_y \rangle_{00}$ are common to both & have already been done => the full answer for $\tilde{f} = \tilde{f}_E + \tilde{f}_B$ is known

The magnetized, kinetic dielectric tensor is obtained from the conductivity tensor $\overleftrightarrow{\sigma}(\underline{k}, \omega)$ through the relation

$$\overleftrightarrow{\epsilon}(\underline{k}, \omega) = \overleftrightarrow{I} - \frac{4\pi}{i\omega} \overleftrightarrow{\sigma}(\underline{k}, \omega)$$

with

$$\tilde{J}_i = \sum_j \sigma_{ij} \tilde{E}_j \quad \text{for } i = x, y, z$$

and $\tilde{J}_i$ is the i^{th} component of the fluctuating current $\tilde{J}(\underline{k}, \omega)$ obtained by performing the integral

$$\tilde{J}_i = q \int d^3v \, v_i (\tilde{f}_E + \tilde{f}_B)$$

Lect #14 Let's do one of the components to illustrate some of the common features -- Consider a Maxwellian distribution that is isotropic

$$f_0 = \frac{n_0 \exp\left(-\frac{v_z^2}{2\bar{v}^2}\right)\exp\left(-\frac{v_\perp^2}{2\bar{v}^2}\right)}{(2\pi\bar{v}^2)^{1/2}\;(2\pi\bar{v}^2)}$$

in this case $\tilde{f}_B = 0$. Focus on ϵ_{zz} associated with $\tilde{E}_z$

Then $\tilde{j}_z = q\int d^3v\, v_z (\tilde{f}_E)_z$ with $\tilde{f}_z = -\frac{q}{mi}\sum_n \frac{J_n^2(k_\perp\rho)\frac{\partial}{\partial v_z}f_0}{(k_z v_z - \omega - \lambda n\Omega)}\tilde{E}_z$

and by inspection, $\rho = v_\perp/\Omega$

$$\sigma_{zz} = -\frac{q^2}{mi}\int d^3v\, v_z \sum_n \frac{J_n^2(k_\perp\rho)}{(k_z v_z - \omega - \lambda n\Omega)}\frac{\partial}{\partial v_z} f_0$$

now $d^3v = 2\pi v_\perp dv_\perp dv_z$ $\Rightarrow$ have 2 integrals to do

$$\sigma_{zz} = -\frac{q^2}{mi}\frac{(2\pi n_0)}{(2\pi\bar{v}^2)^{3/2}}\int_{-\infty}^{\infty} dv_z \int_0^{\infty} dv_\perp v_\perp \sum_n \frac{v_z \frac{\partial}{\partial v_z} e^{-v_z^2/2\bar{v}^2} J_n^2\left(\frac{k_\perp v_\perp}{\Omega}\right) e^{-v_\perp^2/2\bar{v}^2}}{k_z v_z - \omega - \lambda n\Omega}$$

+ realize that

$$\frac{v_z}{k_z v_z - \omega - \lambda n\Omega} = \frac{1}{k_z}\left[1 + \frac{U_n}{v_z - U_n}\right]$$

where $U_n \equiv \frac{\omega + \lambda n\Omega}{k_z}$ $\Rightarrow$ Does not contribute to v_z integral

$$\Rightarrow \sigma_{zz} = -\frac{q^2}{mi}\frac{(2\pi n_0)}{(2\pi\bar{v}^2)^{3/2}}\sum_n U_n \int_{-\infty}^{\infty} dv_z \frac{\left(-\frac{v_z}{\bar{v}^2}\right)e^{-v_z^2/2\bar{v}^2}}{v_z - U_n}\int_0^{\infty} dv_\perp v_\perp J_n^2\left(\frac{k_\perp v_\perp}{\Omega}\right)e^{-v_\perp^2/2\bar{v}^2}$$

as usual let $t = \frac{v_z}{\sqrt{2}\,\bar{v}}$ $\Rightarrow$ is $-\frac{2}{\sqrt{2}\,\bar{v}}\int_{-\infty}^{\infty} dt \frac{t\, e^{-t^2}}{t - \xi_n}$ with $\xi_n \equiv \frac{\omega + \lambda n\Omega}{\sqrt{2}\, k_z \bar{v}}$

Lect #14 (cont.) But from 222B - you are experts on the Z' function

$$Z'(\xi_n) = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \, \frac{t\, e^{-t^2}}{t-\xi_n}$$

$$\Rightarrow \sigma_{zz} = -\frac{q^2}{mi}\frac{(2\pi n_0)}{(2\pi\bar{v}^2)^{3/2}}\frac{1}{k_z}\sqrt{\frac{\pi}{2}}\frac{1}{\bar{v}}\sum_n{}' U_n Z'(\xi_n)\int_0^\infty dv_\perp \, v_\perp J_n^2\!\left(\frac{k_\perp v_\perp}{\Omega}\right) e^{-v_\perp^2/2\bar{v}^2}$$

Examine the $\perp$ integral, again in it let $t = \frac{v_\perp}{\sqrt{2}\,\bar{v}}$

to become $\quad 2\bar{v}^2 \int_0^\infty dt\, t\, J_n^2(\not{b}\, t)\, e^{-t^2} \quad$ with $\not{b} \equiv \frac{\sqrt{2}\, k_\perp \bar{v}}{\Omega}$

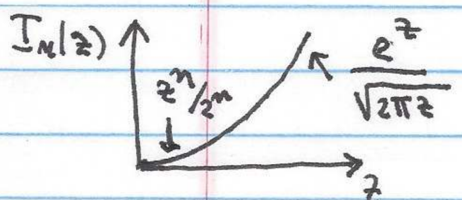
This is the typical integral that appears in all the kinetic magnetized plasmas and which has an exact analytic answer

from the identity $\quad \int_0^\infty dt\, t\, J_n^2(\not{b}\, t)\, e^{-at^2} = \frac{1}{2a^2}\, e^{-\not{b}^2/2a^2}\, I_n\!\left(\frac{\not{b}^2}{2a}\right)$

in our case $a = 1$

$$\Rightarrow \int_0^\infty dv_\perp\, v_\perp J_n^2\!\left(\frac{k_\perp v_\perp}{\Omega}\right) e^{-v_\perp^2/2\bar{v}^2} = (2\bar{v}^2)\left(\frac{1}{2}\right) e^{-\not{b}^2/2}\, I_n\!\left(\frac{\not{b}^2}{2}\right)$$

with I_n The modified Bessel function of order "n"



and $\quad \frac{\not{b}^2}{2} \equiv b = \left(\frac{k_\perp \bar{v}}{\Omega}\right)^2 = (k_\perp \bar{\rho})^2$

with $\bar{\rho} = \frac{\bar{v}}{\Omega}$ The thermal Larmor radius

$$\Rightarrow \boxed{\sigma_{zz} = -\frac{q^2}{mi}\frac{(2\pi n_0)}{(2\pi\bar{v}^2)^{3/2}}\frac{1}{k_z}\sqrt{\frac{\pi}{2}}\frac{1}{\bar{v}}\,\bar{v}^2\sum_n{}' U_n Z'(\xi_n)\, e^{-b^2} I_n(b^2)}$$

Lect #14 (cont.) The connection to the dielectric is $\epsilon_{zz} = 1 - \frac{4\pi\sigma_{zz}}{i\omega}$

$$\Rightarrow \epsilon_{zz} = 1 - \frac{4\pi\sigma_{zz}}{i\omega} = 1 - \frac{\omega_p^2}{\omega}\frac{(2\pi)}{(2\pi\bar v^2)^{3/2}}\frac{1}{k_z}\sqrt{\frac{\pi}{2}}\,\bar v \sum_n{}' Z'(\xi_n)\, U_n\, e^{-b^2} I_n(b^2)$$

useful to define $\Lambda_n(b^2) \equiv e^{-b^2} I_n(b^2)$

or
$$\boxed{\epsilon_{zz} = 1 - \frac{1}{2}\frac{k_D^2}{k_z^2}\sum_{n=-\infty}^{\infty}{}'\left(1+\frac{\lambda n\Omega}{\omega}\right) Z'\left(\frac{\omega+\lambda n\Omega}{\sqrt{2}\,k_z\bar v}\right)\Lambda_n\left((k_\perp\bar\rho)^2\right)}$$

with $k_D = \frac{\omega_p}{\bar v}$ the corresponding Debye wave number

for $k_\perp \to 0$ only $n=0$ survives and $I_0(0) = 1$

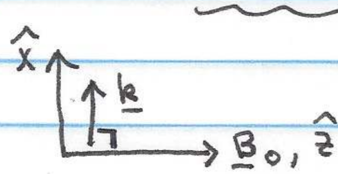
$$\Rightarrow \boxed{\epsilon_{zz} \to 1 - \frac{1}{2}\frac{k_D^2}{k_z^2} Z'\left(\frac{\omega}{\sqrt{2}\,k_z\bar v}\right)}$$ The unmagnetized result of 222B

$\Rightarrow$ Langmuir & IAW & Beam instabilities $\parallel$ to $\underline{B}_0$ behave as we learned in 222B. But now the oblique modes such as LH waves can experience a resonance at $\omega + \lambda n\Omega = 0$, i.e., cyclotron absorption at Harmonics!

For LH $\Rightarrow$ $\omega = n\Omega_i$, i.e., ion cyclotron heating by a LH wave!

Lect #14 – Examine the new modes that arise in hot, magnetized plasma

- Pure electrostatic modes $\perp$ to $\underline{B}_0$



Using FT in $(\underline{k}, \omega)$

Poisson's Eq is: $i k_\perp \tilde{E}_x = -4\pi e \int d^3v\, (\tilde{f}_E)_x$

Lect #14 (cont.) with $\left(\tilde{f}_E\right)_x = \frac{q\lambda}{2m} \sum_{n=-\infty}^{\infty}{}' \frac{J_n(J_{n+1}+J_{n-1})}{i(k_z v_z - \omega - \lambda n \Omega)} \frac{\partial}{\partial v_\perp} f_0 \, \tilde{E}_x$

where, $J_\ell = J_\ell\left(\frac{k_\perp v_\perp}{\Omega}\right)$; Now for pure EBW $\Rightarrow k_z = 0, q = -e, \lambda = -1$

and $\Omega = \frac{eB_0}{mc}$ is the electron cyclotron frequency - since

$\omega \sim \Omega_e \Rightarrow$ ion motion is $\underline{\underline{\text{not}}}$ important -- is all electron physics

$$\Rightarrow \left(\tilde{f}_E\right)_x \to \frac{e}{2m} \sum_{n=-\infty}^{\infty}{}' \frac{J_n(J_{n+1}+J_{n-1})}{i(-\omega + n\Omega)} \frac{\partial}{\partial v_\perp} f_0 \, \tilde{E}_x$$

This can be made simpler using the recursion formula for Bessel functions

$$J_{n+1} + J_{n-1} = \frac{2n}{k_\perp \rho} J_n \qquad \text{for } J_\ell = J_\ell(k_\perp \rho) \; ; \; \rho = \frac{v_\perp}{\Omega}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}{}' \frac{J_n(J_{n+1}+J_{n-1})}{(n\Omega - \omega)} = \frac{2}{k_\perp \rho} \sum_{n=-\infty}^{\infty}{}' \frac{n J_n^2}{(n\Omega - \omega)} \quad \text{and since } J_{-n}^2 = J_{+n}^2$$

The sum can be performed entirely over $n \geq 0$

$$\Rightarrow \sum_{n=-\infty}^{\infty}{}' \frac{J_n(J_{n+1}+J_{n-1})}{n\Omega - \omega} = \frac{2}{k_\perp \rho} \sum_{n=0}^{\infty}{}' \left[\frac{n J_n^2}{n\Omega - \omega} - \frac{n J_{-n}^2}{(-n\Omega - \omega)}\right]$$

$$= \frac{2\,(2\Omega)}{k_\perp \rho} \sum_{n=0}^{\infty}{}' \frac{n^2 J_n^2}{(n\Omega)^2 - \omega^2}$$

Back to Poisson's Eq: (cancelling $\tilde{E}_x$)

$$i k_\perp = -4\pi e \left(\frac{e}{2mi}\right) \frac{4\Omega^2}{k_\perp} \frac{n_0}{(2\pi \bar{v}^2)^{3/2}} \underbrace{\int_{-\infty}^{\infty} dv_z \int_0^{\infty} dv_\perp v_\perp 2\pi \frac{1}{v_\perp} \sum_{n=0}^{\infty}{}' \frac{n^2 J_n^2\left(\frac{k_\perp v_\perp}{\Omega}\right)}{(n\Omega)^2 - \omega^2}}_{\left(e^{-v_z^2/2\bar{v}^2}\right)\left(e^{-v_\perp^2/2\bar{v}^2}\right)} \quad (*)$$

Lect #14 (cont.) - The v_z integral gives 1 using the normalization factor

$$\Rightarrow \quad i k_\perp = -\frac{\omega_p^2}{2i}\left(\frac{1}{\bar{v}^2}\right)\left(\frac{4\Omega^2}{k_\perp}\right)\int_0^\infty dv_\perp \frac{v_\perp}{v_\perp} \sum_{n=0}^{\infty}{}' \frac{n^2 J_n^2\left(\frac{k_\perp v_\perp}{\Omega}\right)\frac{\partial}{\partial v_\perp}\left(e^{-v_\perp^2/2\bar{v}^2}\right)}{(n\Omega)^2-\omega^2}$$

The behavior is determined by the properties of the generic integral

$$G_n(k_\perp) \equiv -2k_D^2\Omega^2\int_0^\infty dv_\perp\, J_n^2\left(\frac{k_\perp v_\perp}{\Omega}\right)\frac{\partial}{\partial v_\perp} e^{-v_\perp^2/2\bar{v}^2}$$

where $k_D^2 = \frac{\omega_p^2}{\bar{v}}$, the usual Debye wave number.

In this expression since $\frac{\partial}{\partial v_\perp} e^{-v_\perp^2/2\bar{v}^2} < 0 \Rightarrow G_n > 0$ for all $k_\perp$

With this definition Poisson's Eq. becomes

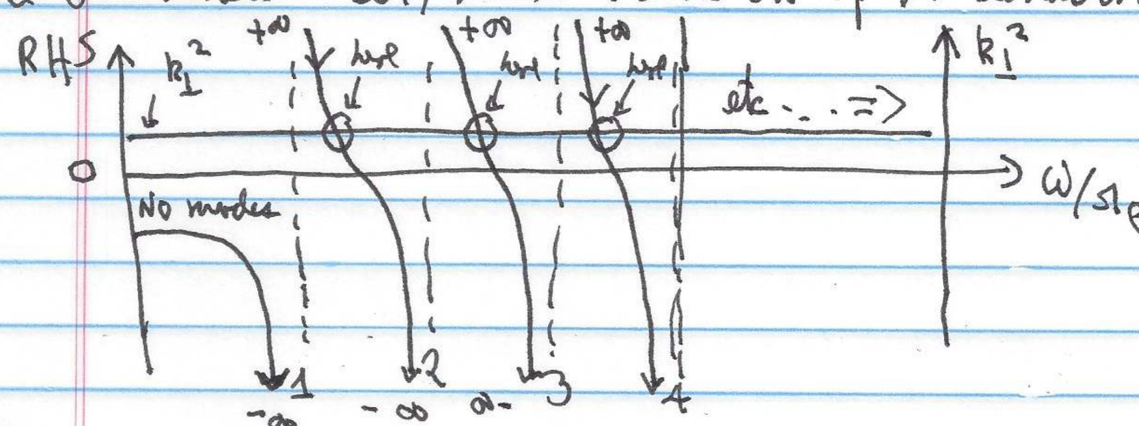
$$\boxed{k_\perp^2 = -\sum_{n=0}^{\infty}{}' \frac{n^2 G_n(k_\perp)}{(n\Omega)^2-\omega^2}}$$

where here $\Omega = \Omega_e$ since High-frequency is considered

It is clear that for $\omega < \Omega_e$ the RHS is < 0 & as $\omega \uparrow$

There are ∞ as one of the integer resonances is encounter, i.e., $\omega = n\Omega_e$

As ω crosses each of those resonances the RHS flips from $+\infty \to -\infty$, and viceversa - Let, make a sketch of the behavior of the RHS



Lect 14 (cont.) The self-consistent solutions to Poisson's Eq., i.e., The collective modes of The system, corresponds to the situations when The RHS is equal, numerically, to The value of $k_\perp^2$. Thus, by superposing a constant line of $k_\perp^2$ in The same graph, The intersects yield The values of ω for which There is a mode with That value of $k_\perp$

That is indicated by The circles in The previous graph. It is found from The graph That no modes exist for $\omega < \Omega_e$, and between the harmonics : $n\Omega_e < \omega < (n+1)\Omega_e$ There is one mode having That value of $k_\perp$, i.e., There is an ∞ # modes with The same $k_\perp$. But, for a given ω, fixed by an antenna, There is a unique $k_\perp$ for every ω between The harmonics of Ω_e.

These modes are the electron Bernstein modes (EBW). Their role is to limit The ~~resonance~~ encounter in The cold wave propagation at The upper-hybrid resonances, i.e., in parameter regimes where $k_\perp \uparrow \infty$ for The X-mode, a mode-conversion process arises in which The E+M energy is ~~Trans~~ Transferred to The EBW, This is analogous to The Transfer from E+M $\Rightarrow$ Langmuir waves in The unmagnetized case.

(*) There are modes similar TO EBW in The low frequency domain, i.e., between harmonics of Ω_i -- but These modes are affected by The electron response $\parallel$ to $\underline{B}_0$ -- so there are two possibilities $\frac{\omega}{k_\parallel} \gg \bar{v}_e$ or $\frac{\omega}{k_\parallel} \ll \bar{v}_e$ for These modes!

$\Rightarrow$ Go over references on Bernstein Modes!

Lect #15. Overview of Magnetohydrodynamics (MHD)

Old + very large subject -- probably the most used by plasma researchers. Very popular among space scientists + fusion experts. Does not consider any particle feature - Plasma is a blob - Intended to study - very slow + very large spatial scales $\Rightarrow \omega \ll \Omega_i$. We derived the underlying Eqs in 222A by making questionable approximations.

$$\underset{\text{mass density}}{\rho} \frac{d}{dt} \underline{v} = - \nabla \cdot \underset{\text{stress tensor}}{\overleftrightarrow{P}} + \frac{1}{c} \underset{\text{total electrical current}}{\underline{J}} \times \underline{B} + \underbrace{\underline{F}_{external}}_{\text{acts on C.M.}}$$

Since no particles exist $\Rightarrow$ Need to invent models for $\overleftrightarrow{P}$ and $\underline{J}$

$\rho = \underset{\text{average ion mass}}{M} \, \underset{\text{particle density}}{n}$ with a continuity of mass Eq.

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \underline{v}) = 0$$

Maxwell's Eqs:

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J} + \frac{1}{c} \frac{\partial}{\partial t} \underline{E}$$ neglect because $\omega \ll \Omega_i$

$$\nabla \times \underline{E} = - \frac{1}{c} \frac{\partial}{\partial t} \underline{B}$$ // Note that it is exact and no contribution of matter ever enters into it!

There is no Poisson's Eq. because $\boxed{n_e = n_i}$ $\Rightarrow$ All the $\underline{E}$ - fields are inductive - which according to Faraday's law $\Rightarrow \frac{\partial}{\partial t} \underline{B} \neq 0$

Most common approach: $\nabla \cdot \overleftrightarrow{P} = \nabla P \Rightarrow$ scalar pressure

Lect #15 (cont.) Modeling Transport (A key feature at slow-time & large scales)

$$\frac{d}{dt}\left(p\rho^{-\gamma}\right) = 0$$ => adiabatic compression of the "plasma blob"

$$\nabla\cdot\underline{v} = 0$$ => incompressible blob

Modeling electrical conduction:

Ideal MHD: $\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = 0$ [perfect conductor]

or

Resistive MHD: $\underline{J} = \sigma\left[\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right]$ [a "generalized Ohm's Law"]

$\uparrow$ a scalar whose value is unknown

Is just an attempt to acknowledge that electrons exist!

In the electron fluid Eq. of motion:

$$\underbrace{m\frac{d}{dt}\underline{v}_e}_{\text{neglect}} = -e\underline{E} - \underbrace{\frac{\nabla\cdot\overleftrightarrow{P}_e}{n_e}}_{\text{neglect}} - \frac{e}{c}\underline{v}_e\times\underline{B} - m\nu_{ei}(\underline{v}_e - \underline{v}_i)$$

This gives

$$\left[\underline{E} + \frac{\underline{v}_e\times\underline{B}}{c}\right] = -\frac{m\nu_{ei}}{e}(\underline{v}_e - \underline{v}_i)$$

$$= \underbrace{\frac{m\nu_{ei}}{e^2 n_e}}_{\text{say is } \frac{1}{\omega_{pe}^2/4\pi\nu_{ei}} = \frac{1}{\sigma}}\ \underbrace{(-en_e)(\underline{v}_e - \underline{v}_i)}_{\text{say is } \underline{J}}$$

and say here $\underline{v}_e$ is "$\underline{v}$"

Then $\left[\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right] = \frac{\underline{J}}{\sigma}$ but.....

Lect #15 (cont.) The bulk magnetic force admits an interesting physical picture: $\underline{f}_B = \frac{\underline{J} \times \underline{B}}{c}$ is the magnetic force density

but Ampere's law says. $\underline{B}$ is determined by $\underline{J}$, where now it is assumed that there are no external currents within the region sampled $\Rightarrow$ $\nabla \times \underline{B} = \frac{4\pi}{c}\underline{J} \Rightarrow \underline{f}_B = \frac{1}{4\pi}(\nabla \times \underline{B}) \times \underline{B}$

but there is an identity in your NRL formulary:

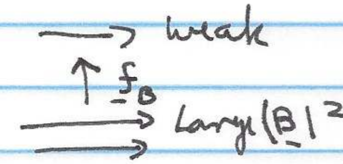
$$(\nabla \times \underline{B}) \times \underline{B} = (\underline{B}\cdot\nabla)\underline{B} - \frac{1}{2}\nabla(|\underline{B}|^2)$$ That applies to

any vector field, not just $\underline{B}$ fields $\Rightarrow$ the magnetic force density

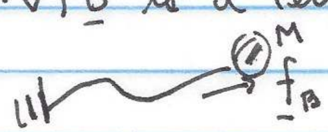
has two features $\Rightarrow$ $$\boxed{\underline{f}_B = -\nabla\left(\frac{|\underline{B}|^2}{8\pi}\right) + \frac{1}{4\pi}(\underline{B}\cdot\nabla)\underline{B}}$$

But always remember - that what goes on are the orbits that we learned in lectures 3-6 earlier!

The term $-\nabla\left[\frac{|\underline{B}|^2}{8\pi}\right]$ (U_B the field energy density) is a compressional force



while the term $\frac{1}{4\pi}(\underline{B}\cdot\nabla)\underline{B}$ is a tension force like pulling on a string



$\Rightarrow$ if magnetic field line is bent it will pull on the "blob"

The compressional force on Bulk-Matter can be very real

at $|\underline{B}| = 5\,kG \Rightarrow \frac{|\underline{B}|^2}{8\pi} \sim 1$ atm $\Rightarrow$ for ITER is ~ 100 atm

+ need to be very careful with every bolt!

Lect #15 (cont.) The goal of MHD is to make the plasma properties disappear and just think as the medium being the "object" to which a blob is attached => find an evolution Eq. for $\underline{B}$ + forget the plasma!

Use the resistive MHD - Ohm's Law to eliminate $\underline{j}$, i.e.,

apply $\nabla \times$ to $\left[\underline{j} = \sigma \left(\underline{E} + \frac{\underline{v} \times \underline{B}}{c} \right) \right]$

+ use Ampere's Law with $\nabla \times \underline{j} = \frac{c}{4\pi} \nabla \times (\nabla \times \underline{B}) = -\frac{c}{4\pi} \nabla^2 \underline{B}$

because $\nabla \cdot \underline{B} = 0$

$$\Rightarrow \quad -\frac{c}{4\pi} \nabla^2 \underline{B} = \sigma \underbrace{(\nabla \times \underline{E})}_{-\frac{1}{c}\frac{\partial}{\partial t}\underline{B} \text{ by Faraday's law}} + \frac{\sigma}{c} \nabla \times (\underline{v} \times \underline{B})$$

$$\Rightarrow \quad -\frac{c^2}{4\pi\sigma} \nabla^2 \underline{B} + \frac{\partial}{\partial t} \underline{B} = \nabla \times (\underline{v} \times \underline{B})$$

+ we need help again from the NRL formulary for the identity

$$\nabla \times (\underline{v} \times \underline{B}) = \underline{v}(\nabla \cdot \underline{B}) - \underline{B}(\nabla \cdot \underline{v}) + (\underline{B} \cdot \nabla)\underline{v} - (\underline{v} \cdot \nabla)\underline{B}$$

collect in the form of a "Fluid evolution Eq.": i.e., $\frac{\partial}{\partial t} + \underline{v} \cdot \nabla$

$$\frac{\partial}{\partial t} \underline{B} + \underline{v} \cdot \nabla \underline{B} = \frac{c^2}{4\pi\sigma} \nabla^2 \underline{B} - \underline{B}(\nabla \cdot \underline{v}) + \underline{B} \cdot \nabla \underline{v}$$

or,

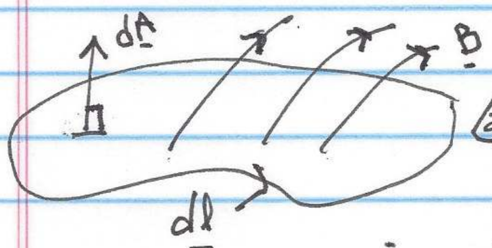
$$\boxed{\frac{d}{dt} \underline{B} = \frac{c^2}{4\pi\sigma} \nabla^2 \underline{B} - \underline{B}(\nabla \cdot \underline{v}) + \underline{B} \cdot \nabla \underline{v}}$$

Total change in $\underline{B}(\underline{r}, t)$ | Diffusion by resistivity | Flux compression or expansion | parallel velocity gradient

Lect #15 (cont.) -- The velocity terms can be put on the left-side to give an interpretation:

$$\underbrace{\frac{d}{dt}}_{\text{Total Derivative}}\underline{B} + \underbrace{\underline{B}(\nabla\cdot\underline{v}) - \underline{B}\cdot\nabla\underline{v}}_{\substack{\text{Topological}\\ \text{changes on}\\ \text{hypothetical area}}} = \underbrace{\frac{c^2}{4\pi\sigma}\nabla^2\underline{B}}_{\text{Resistive Diffusion}}$$

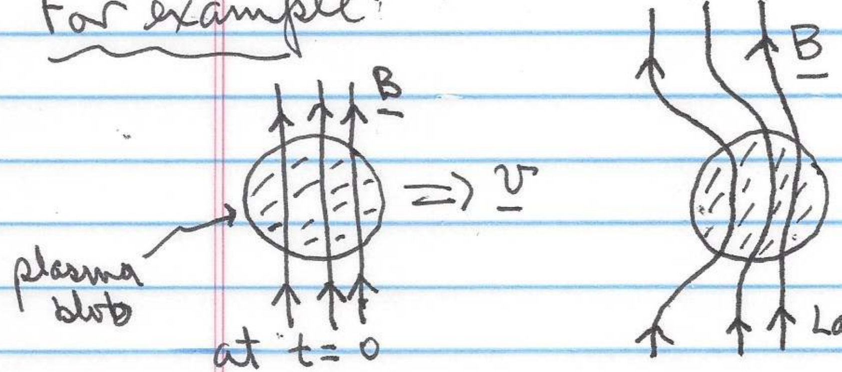
if $\sigma \to \infty \Rightarrow$ blob cannot pass through a magnetic barrier and/or $\underline{B}$ field cannot exit or enter the blob



if $\sigma \to \infty \Rightarrow$ Magnetic flux Φ_B remains constant

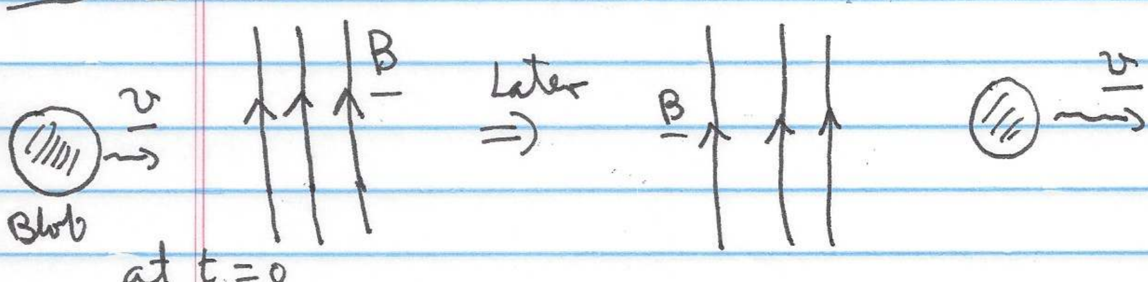
~~where~~ direction of $d\underline{A}$ + $d\underline{l}$ are determined by velocity field $\underline{v} = \underline{v}(\underline{r}, t)$

For example:



meaning of frozen flux

For small σ



Lect #15 (cont.) Validity of frozen flux picture (ideal MHD)

⇒ time scale for diffusion >> Time scale for $\underline{B}$ field lines to change.

From our knowledge of Alfven waves ⇒ know $\underline{B}$ can move at speed

$v_A = \sqrt{\frac{|\underline{B}|^2}{4\pi \rho}}$ (ρ: Mn) but for $\underline{B}$ to follow blobs ⇒ $|\underline{v}| \sim v_A$

⇒ The $|\underline{v} \cdot \nabla \underline{B}| \sim \frac{v_A B}{L}$ with L the typical spatial scale

but the diffusion term is $\frac{c^2}{4\pi\sigma} \nabla^2 B$ ⇒ scales as $\frac{c^2}{4\pi\sigma} \frac{1}{L^2} B$

⇒ The characteristic B-field diffusion time is $\frac{1}{\tau_D} \sim \frac{c^2}{4\pi\sigma} \frac{1}{L^2}$

or $\tau_d = \frac{4\pi\sigma L^2}{c^2}$ with the convective time scale: $\tau_c = \frac{L}{v_A}$

The condition for frozen flux is $\tau_d >> \tau_c$ ⇒ $\frac{\tau_d}{\tau_c} >> 1$

The ratio $\frac{\tau_d}{\tau_c} = \frac{4\pi\sigma v_A L}{c^2}$ is known as the Magnetic Reynolds # $\equiv R_M$

Some typical values are:

System	R_M
Earth's core	10^2
Ionosphere	10
Solar Corona	10^{11}
Fusion Plasma	10^4

But problem is that systems with large R_M means collisionless ⇒ not in thermal equilibrium ⇒ behavior is highly kinetic/nonlinear!

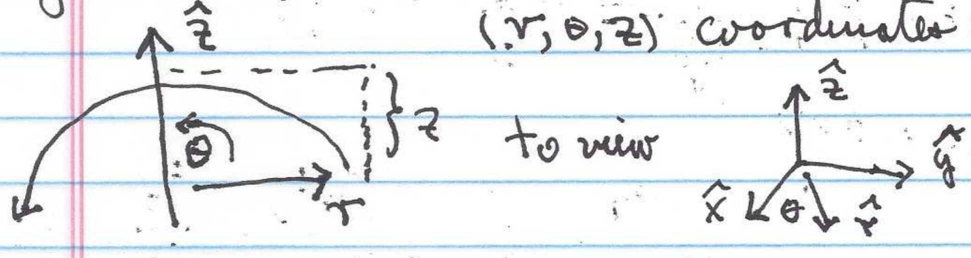
-- Bottom line is that it is always questionable!

Lect #15 (cont.) Axially symmetric MHD equilibria -- applicable to

Toroidal devices, pinches or symmetric mirrors

- Use cylindrical coordinates

(r, θ, z) coordinates



but now assume $\frac{\partial}{\partial\theta} = 0$ but Must make sure that $\nabla\cdot\underline{B} = 0$

But $\nabla\cdot\underline{B} = 0$ is analogous to incompressible hydrodynamics ($\nabla\cdot\underline{v} = 0$)

$\Rightarrow$ use hydro concepts

$$\nabla\cdot\underline{B} = \frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \cancelto{0}{\frac{1}{r}\frac{\partial}{\partial\theta}B_\theta} + \frac{\partial}{\partial z}B_z$$

$$\Rightarrow \quad 0 = \frac{\partial}{\partial r}(r B_r) + r\frac{\partial}{\partial z}B_z \quad \text{or} \quad 0 = \frac{\partial}{\partial r}(r B_r) + \frac{\partial}{\partial z}(r B_z)$$

Introduce a stream function: $\psi = \psi(r, z)$ "The Flux Function" in MHD

such that $\quad r B_r = -\frac{\partial\psi}{\partial z}$ and $r B_z = \frac{\partial\psi}{\partial r}$

Note it says nothing about B_θ! but satisfies $\nabla\cdot\underline{B} = 0$ automatically

because $\quad \frac{\partial}{\partial r}\left(-\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial r}\right) = 0 \quad \checkmark$

The idea is to obtain and Eq. for ψ + then from that determine the equilibrium conditions - Force balance requires

$$0 = -\nabla p + \frac{\underline{J}\times\underline{B}}{c} = -\nabla p + \frac{1}{4\pi}\underbrace{(\nabla\times B)\times\underline{B}}_{}$$ is nonlinear in B!

Lect #15 (cont.) The next steps are boring algebra more fun in a blackboard but needed for completeness

We need $(\nabla \times \underline{B}) \times \underline{B} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ (\nabla\times\underline{B})_r, & (\nabla\times\underline{B})_\theta, & (\nabla\times\underline{B})_z \\ B_r, & B_\theta & B_z \end{vmatrix}$

$$= \hat{r}\left[B_z (\nabla\times\underline{B})_\theta - (\nabla\times\underline{B})_z B_\theta\right] + \hat{\theta}\left[B_r(\nabla\times\underline{B})_z - B_z(\nabla\times\underline{B})_r\right]$$

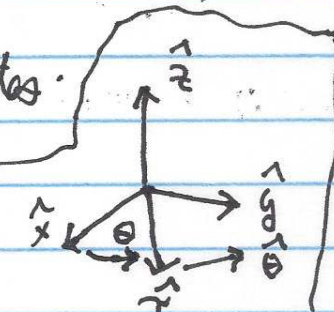
$$+ \hat{z}\left[B_\theta (\nabla\times\underline{B})_r - B_r(\nabla\times\underline{B})_\theta\right]$$

but for this need the Curl in cylindrical coordinates

$\hat{r}\times\hat{\theta} = \hat{z}$

$\hat{z}\times\hat{r} = \hat{\theta}$

$\hat{\theta}\times\hat{z} = \hat{r}$



$$(\nabla\times\underline{B})_r = -\frac{\partial B_\theta}{\partial z} + \frac{1}{r}\cancelto{0}{\frac{\partial}{\partial\theta}} B_z$$

$$(\nabla\times\underline{B})_\theta = \frac{\partial}{\partial z} B_r - \frac{\partial B_z}{\partial r}$$

$$(\nabla\times\underline{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(r B_\theta) - \frac{1}{r}\cancelto{0}{\frac{\partial}{\partial\theta}} B_r$$

Find the $\hat{r}$ component of the Equilibrium condition:

$$0 = -\frac{\partial}{\partial r} P + \frac{1}{4\pi}\left\{ B_z \underbrace{\left(\frac{\partial}{\partial z}B_r - \frac{\partial}{\partial r}B_z\right)}_{(\nabla\times\underline{B})_\theta} - B_\theta \underbrace{\left\{\frac{1}{r}\frac{\partial}{\partial r}(r B_\theta) - 0\right\}}_{(\nabla\times\underline{B})_z}\right.$$

use the ψ variable for (B_r, B_z)

$$0 = -4\pi\frac{\partial}{\partial r}P - \frac{B_\theta}{r}\frac{\partial}{\partial r}(rB_\theta) + \frac{1}{r}\frac{\partial\psi}{\partial r}\left[\frac{\partial}{\partial z}\overbrace{\left(-\frac{1}{r}\frac{\partial}{\partial z}\psi\right)}^{B_r} - \frac{\partial}{\partial r}\overbrace{\left(\frac{1}{r}\frac{\partial}{\partial r}\psi\right)}^{B_z}\right]$$

or

$$0 = +4\pi\frac{\partial}{\partial r}P + \frac{B_\theta}{r}\frac{\partial}{\partial r}(rB_\theta) + \frac{1}{r^2}\frac{\partial\psi}{\partial r}\left[\frac{\partial^2}{\partial z^2}\psi + \frac{\partial^2}{\partial r^2}\psi - \frac{1}{r}\frac{\partial}{\partial r}\psi\right]$$

Lect #15 (cont.) Defining a new differential operator

$$\Delta^2 \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}$$ Note: it is not the Laplacian!

$$\Rightarrow \boxed{0 = 4\pi \frac{\partial}{\partial r} p + \frac{B_\theta}{r}\frac{\partial}{\partial r}(r B_\theta) + \frac{1}{r^2}\frac{\partial \psi}{\partial r}\Delta^2 \psi} \quad \hat{r}-\text{Eq.}$$

The same procedure is now applied to the $\hat{\theta}$ pressure balance

$$0 = -\frac{1}{r}\cancelto{0}{\frac{\partial}{\partial \theta}} p + \frac{1}{4\pi}\left\{ B_r \underbrace{\left[\frac{1}{r}\frac{\partial}{\partial r}(r B_\theta) - 0\right]}_{(\nabla \times \underline{B})_z} - B_z \underbrace{\left[-\frac{\partial B_\theta}{\partial z} + 0\right]}_{(\nabla \times \underline{B})_r} \right\}$$

multiply by $4\pi r$

$$0 = -\frac{\partial \psi}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(r B_\theta)\right] + \frac{\partial \psi}{\partial r}\left[\frac{\partial}{\partial z} B_\theta\right]$$

multiply by r

$$\Rightarrow 0 = \frac{\partial \psi}{\partial r}\frac{\partial}{\partial z}(r B_\theta) - \frac{\partial \psi}{\partial z}\frac{\partial}{\partial r}(r B_\theta)$$

but this combination of derivative products is nothing but

$$\begin{vmatrix} \frac{\partial \psi}{\partial r} & \frac{\partial \psi}{\partial z} \\ \frac{\partial}{\partial r}(r B_\theta) & \frac{\partial}{\partial z}(r B_\theta) \end{vmatrix} = \frac{J(\psi, r B_\theta)}{(r, z)}$$

The Jacobian of $(\psi, r B_\theta)$ with respect to the variables (r, z)

but if $J(\psi, r B_\theta) = 0 \Rightarrow \boxed{r B_\theta = f(\psi)}$

i.e., ψ & $r B_\theta$ are not independent $\Rightarrow r B_\theta$ is a function of ψ

but at this stage it is an unknown function!

Lect 15 (cont.) Now examine the $\hat{z}$ component of Force balance

$$0 = -\frac{\partial}{\partial z} P + \frac{1}{4\pi}\left\{ -B_r \underbrace{\left[\frac{\partial}{\partial z} B_r - \frac{\partial}{\partial r} B_z\right]}_{(\nabla \times \underline{B})_\theta} + B_\theta \underbrace{\left[-\frac{\partial}{\partial z} B_\theta + 0\right]}_{(\nabla \times \underline{B})_r} \right\}$$

introduce ψ

$$0 = -\frac{\partial}{\partial z} P + \frac{1}{4\pi}\left\{ -B_\theta \frac{\partial}{\partial z} B_\theta - \left(-\frac{1}{r}\frac{\partial \psi}{\partial z}\right)\left[-\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right]\right\}$$

$$0 = -\frac{\partial}{\partial z} P + \frac{1}{4\pi}\left\{ -B_\theta \frac{\partial}{\partial z} B_\theta - \frac{1}{r}\frac{\partial \psi}{\partial z}\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right]\right\}$$

all terms are (-)

$$\Rightarrow 0 = +\frac{\partial}{\partial z} P (4\pi) + B_\theta \frac{\partial}{\partial z} B_\theta + \frac{1}{r^2}\frac{\partial \psi}{\partial z} \underbrace{\left[\frac{\partial^2}{\partial z^2}\psi + \frac{\partial^2}{\partial r^2}\psi - \frac{1}{r}\frac{\partial \psi}{\partial r}\right]}_{\Delta^2 \psi}$$

$$\Rightarrow \boxed{0 = 4\pi \frac{\partial}{\partial z} P + B_\theta \frac{\partial}{\partial z} B_\theta + \frac{1}{r^2}\frac{\partial \psi}{\partial z}\Delta^2 \psi} \quad \hat{z} - Eq.$$

Compare now $\hat{r}$ + $\hat{z}$ Eqs. :

$$\hat{z} \Rightarrow -4\pi \frac{\partial P}{\partial z} = \frac{1}{r^2}(rB_\theta)\frac{\partial}{\partial r}(rB_\theta) + \frac{1}{r^2}\frac{\partial \psi}{\partial z}\Delta^2 \psi$$

+ since $rB_\theta = f(\psi)$ $\Rightarrow$ $\frac{\partial}{\partial r}(rB_\theta) = \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial r}$ + define $f' \equiv \frac{\partial f}{\partial \psi}$

$$\Rightarrow \boxed{-4\pi \frac{\partial P}{\partial z} = \frac{1}{r^2}\frac{\partial \psi}{\partial z}\left[f f' + \Delta^2 \psi\right]}$$ is $\hat{z}$ Eq in terms of ψ

$$\hat{r} = -4\pi \frac{\partial}{\partial r} P = \frac{1}{r^2}(rB_\theta)\frac{\partial}{\partial r}(rB_\theta) + \frac{1}{r^2}\frac{\partial \psi}{\partial r}\Delta^2 \psi$$

or $$\boxed{-4\pi \frac{\partial}{\partial r} P = \frac{1}{r^2}\frac{\partial \psi}{\partial r}\left[f f' + \Delta^2 \psi\right]}$$ is $\hat{r}$ Eq. in terms of ψ

Lect#15 (cont.) Multiply $\hat{z}$ Eq by $\frac{1}{4\pi}\frac{\partial}{\partial r}\psi$ and $\hat{r}$ Eq. by $\frac{1}{4\pi}\frac{\partial}{\partial z}\psi$

and subtract $\Rightarrow$ $\frac{\partial p}{\partial r}\frac{\partial \psi}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial \psi}{\partial r} = 0$

The Jacobian again

which is identical to: $\begin{vmatrix} \frac{\partial}{\partial r}p & \frac{\partial}{\partial z}p \\ \frac{\partial \psi}{\partial r} & \frac{\partial \psi}{\partial z} \end{vmatrix} = \frac{J(p,\psi)}{(r,z)} = 0$

$\Rightarrow$ $p = g(\psi)$ where g is some unknown function of ψ

Now one can return to either the $\hat{r}$ or $\hat{z}$ Eq. to obtain a nonlinear Eq. that determines the equilibrium flux function, say the $\hat{r}$ Eq. with $\frac{\partial}{\partial r}p = \frac{\partial}{\partial \psi}g\,\frac{\partial \psi}{\partial r}$

$\Rightarrow$ $-4\pi\frac{\partial \psi}{\partial r}g' = \frac{1}{r^2}\frac{\partial \psi}{\partial r}\left[ff' + \Delta^2\psi\right]$ + with $g' \equiv \frac{\partial g}{\partial \psi}$

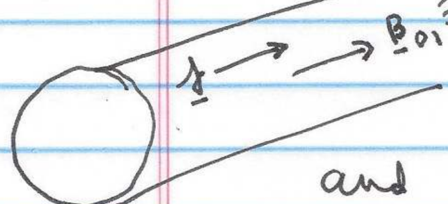
which yields:
$$\boxed{\Delta^2\psi + ff' = -4\pi r^2 g'}$$

Known as the Grad-Shafranov Eq + is the starting pt for studying MHD equilibrium of axisymmetric systems. To solve this Eq - subject to boundary conditions - two functions must be specified: $f(\psi) = rB_\theta$ set by external coils

and $p = g(\psi)$ set by heating and transport:

Lect #16 (cont) — <u>Constant current cylinder</u>: A useful example to obtain a feeling for the function Ψ that satisfies the <u>Grad-Shafranov Eq</u>: $\boxed{\Delta^2\Psi + f f' = -4\pi r^2 g'}$ where $g(\Psi) = p$ the pressure and the quantity $rB_\theta = f(\Psi)$ needs to be specified for the chosen configuration with the operator $\Delta^2 \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}$

and $f' \equiv \frac{\partial f}{\partial \Psi}$, $g' \equiv \frac{\partial g}{\partial \Psi}$



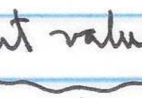
here $B_z = B_0$

and $B_z = \frac{1}{r}\frac{\partial \Psi}{\partial r}$; $B_r = -\frac{1}{r}\frac{\partial \Psi}{\partial z}$

A <u>B_r would destroy the desired confinement</u> $\Rightarrow B_r = 0 \Rightarrow \frac{\partial \Psi}{\partial z} = 0$ & $\Psi = \Psi(r)$ only!

& since $\underset{\substack{\uparrow \\ \text{constant}}}{B_0} = B_z = \frac{1}{r}\frac{\partial \Psi}{\partial r} \Rightarrow \Psi = \underset{\substack{\uparrow \\ \text{a constant}}}{\Psi_0} + \frac{r^2}{2} B_0$

but $B_0 \pi r^2 = \oint \underline{B} \cdot d\underline{s} = \Phi_B$ the magnetic flux through a loop of radius r



$\Rightarrow \Psi \propto$ magnetic flux !

$\Rightarrow$ "<u>A given constant value of Ψ implies a certain radius r</u>"

$\Rightarrow$ In Tokamak work is common to by-pass plots vs r in terms of <u>plots vs. Ψ</u> for a given measurable quantity

For this simple case one can apply Ampere's Law directly

$$\oint \underline{B} \cdot d\underline{l} = \frac{4\pi}{c} I(r) = \frac{4\pi}{c}(J\pi r^2) = 2\pi r B_\theta \Rightarrow \boxed{f = rB_\theta = \left(\frac{2\pi J}{c}\right) r^2}$$

Lect #16 (cont.) But we found that $\psi = \psi_0 + \frac{r^2 B_0}{2}$ & setting $\psi = 0$ at $r = 0$ yields $\psi_0 = 0$ and the connection is $r = \sqrt{\frac{2\psi}{B_0}}$

with the intuition now being that in regions of uniform current $r \sim \sqrt{\psi}$ for interpreting Tokamak plots -- but in reality the relationship is more complicated because $I(r) = \int_0^r dr' r' j(r')$

Now replace r in terms of ψ to extract $\boxed{f = \left(\frac{4\pi j}{c B_0}\right)\psi}$

Which means a simpler situation where f is linear in ψ

But, to obtain the equilibrium state of this system one needs to find $p = p(\psi)$ -- which in reality is a result of the heating scheme and heat transport -- which are issues outside MHD!

But MHD equilibrium demands a specific $p(\psi) = g$ in order to satisfy the Grad-Shafranov Eq. -- let's see what it must be

We need $\Delta^2\psi = \frac{\partial^2}{\partial r^2}\psi - \frac{1}{r}\frac{\partial}{\partial r}\psi = \frac{\partial^2}{\partial r^2}\left[\frac{r^2 B_0}{2}\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r^2 B_0}{2}\right]$

$\Delta^2\psi = B_0 - B_0 = \boxed{0}$ $\Rightarrow$ $\cancel{\Delta^2\psi} + f f' = -4\pi r^2 g'$

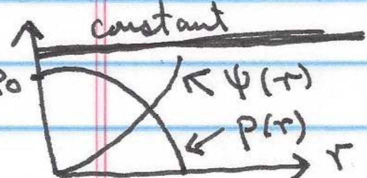
but we know $f \Rightarrow$ can solve for unknown $g = g(\psi) = p(\psi)$

$$\frac{\partial g}{\partial \psi} = \frac{\partial p}{\partial \psi} = -\frac{f f'}{4\pi r^2} = -\left(\frac{4\pi j}{c B_0}\right)\psi\left(\frac{4\pi j}{c B_0}\right)\frac{1}{4\pi r^2}$$

or $$\frac{\partial p}{\partial \psi} = -\left(\frac{4\pi j}{c B_0}\right)^2 \frac{\psi}{4\pi\left(\frac{2\psi}{B_0}\right)} = -\frac{2\pi j^2}{c^2 B_0} \leftarrow \text{a constant!}$$

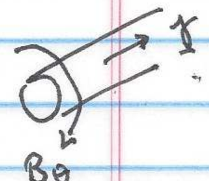
Lect # 16 cont. integrate to yield $\boxed{p(\psi) = p_0 - \frac{2\pi j^2}{c^2 B_0}\psi}$

Now we have all the ingredients of the MHD equilibrium

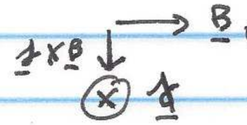


MHD demands that this pressure profile must develop in order for the pinching effect of the B_θ field does not collapse the plasma - but MHD does not say how this can be achieved - If it is possible to be achieved by some means, there is next issue as to the stability of the equilibrium - that requires a separate consideration

The result of this simple Grad-Shafranov calculation can be also easily obtained from "pressure balance".



$$\boxed{0 = -\nabla p + \frac{\underline{j} \times \underline{B}}{c}}$$



or $\underline{j} \times \underline{B} = j B_\theta (-\hat{r})$

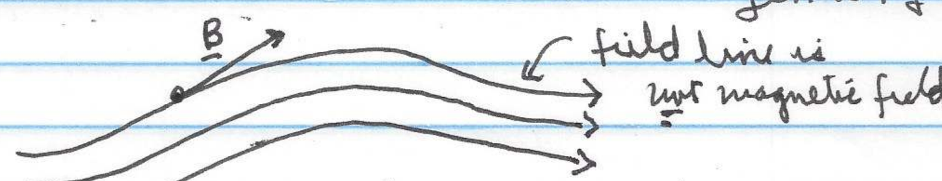
and $\nabla p = \frac{\partial}{\partial r} p \, \hat{r}$

$$\Rightarrow \quad \frac{\partial p}{\partial r} = -\frac{j B_\theta}{c} \qquad \text{but} \quad B_\theta = \frac{2\pi r j}{c} \quad \text{from Ampere's Law}$$

$$\text{or} \quad \frac{\partial p}{\partial r} = -\frac{2\pi j^2}{c^2} r \Rightarrow \quad p(r) = p_0 - \frac{\pi j^2}{c^2} r^2 = \boxed{p_0 - \frac{2\pi j^2}{c^2 B_0}\psi}$$

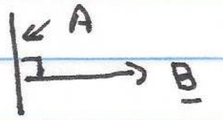
Elementary Concepts of MHD Stability - useful for complicated geometry

Magnetic flux tube:



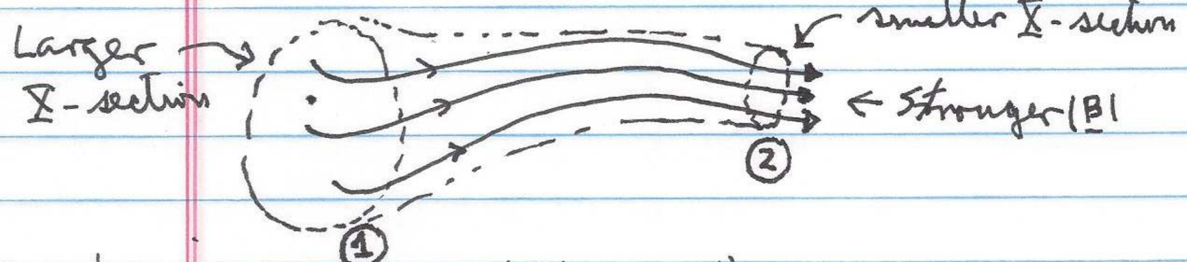
$\underline{B}$ field is tangent vector at a given point on a field lines!

Lect #16 (cont.) Construct a mathematical area element $A \perp$ to the magnetic field vector



$\Rightarrow \Phi = |\underline{B}| A$ is magnetic flux through A

Now think of Φ as an independent parameter that will be used to label a bundle of field lines – i.e., a geometrical object that encloses a certain volume of magnetized plasma



The dashed object is the "flux tube"

Spatial Volume Contained within a flux tube: as above between ① + ②

$$V = \int_{①}^{②} A\, dl = \int_{①}^{②} \Phi \frac{dl}{|\underline{B}|}$$ if we use Φ = constant to label the given tube

$$\Rightarrow \quad V = \Phi \int_{①}^{②} \frac{dl}{|\underline{B}|}$$ with the integral being a line integral along a "field line" on which $|\underline{B}|$ varies

Magnetic Energy Stored in a flux tube: Energy density $u_B = \frac{|\underline{B}|^2}{8\pi}$

$$U_B = \int_{①}^{②} \underbrace{A\, dl}_{dV}\, u_B = \int_{①}^{②} \frac{\Phi}{|\underline{B}|}\, dl \frac{|\underline{B}|^2}{8\pi} = \frac{\Phi}{8\pi} \int_{①}^{②} |\underline{B}|\, dl = \frac{\Phi^2}{8\pi} \int_{①}^{②} \frac{dl}{A}$$

Thermal Energy within a flux tube: $u_T = \frac{3}{2} nT$ not "flow energy"

For ideal gas law $p = nT \Rightarrow u_T = \frac{3}{2} p$

Lect #16 (cont.) Using the ideal gas ratio of specific heats $\gamma = 5/3$; $\frac{3}{2} = \frac{1}{\gamma - 1}$

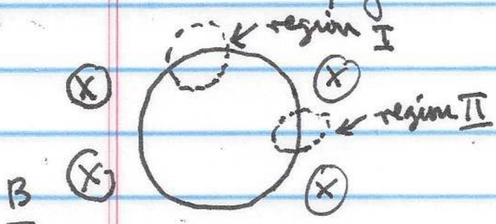
$$u_T = \frac{p}{\gamma - 1} \Rightarrow U_T = \int_{①}^{②} \underbrace{A\,dl}_{dV}\, u_T = \int_{①}^{②} \frac{\Phi}{|\underline{B}|}\, dl\, \frac{p}{\gamma-1} = \frac{\Phi}{\gamma-1}\int_{①}^{②} p\,\frac{dl}{|\underline{B}|}$$

but if there is no flow $\parallel$ to $\underline{B}$ $\Rightarrow$ p is constant along each dl

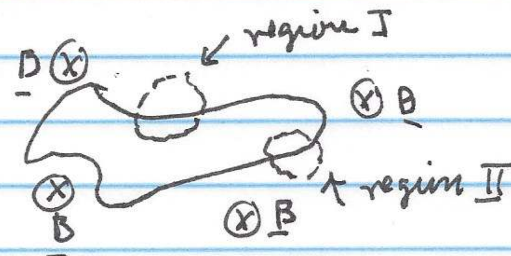
$$\Rightarrow \quad U_T = \frac{p}{\gamma-1}\left[\Phi \int_{①}^{②} \frac{dl}{|\underline{B}|}\right] = \frac{pV}{\gamma - 1}$$ ← volume of flux tube!

MHD Stability Test -- Assumes an equilibrium is known --

Examine the total energy change $\delta(U_B + U_T)$ under an adiabatic topological deformation of a given equilibrium configuration



configuration α



configuration β

Q: Is there an increase in energy in going from configuration α to β?

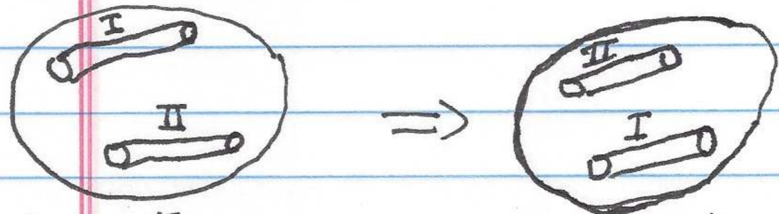
if $\delta(U_B + U_T) > 0$ system is stable!

if $\delta(U_B + U_T) = 0$ is marginally stable

if $\delta(U_B + U_T) < 0$ is unstable!

Note the distinction made between regions (I) and (II) and α, β in both α + β regions (I) + (II) exist but they may contain different total energy in α and β

Lect #16 (cont.) Interchange Instability -- A common situation --



configuration α $\Rightarrow$ configuration β

i.e., $\text{I} \rightleftarrows \text{II}$

Mathematically what is meant: $\int_{\text{I}} \frac{dl}{A} \rightleftarrows \int_{\text{II}} \frac{dl}{A}$ } a topological exchange

Initial Magnetic Energy (in state α): $(U_B)_\alpha = \frac{\Phi_{\text{I}}^2}{8\pi} \int_{\text{I}} \frac{dl}{A} + \frac{\Phi_{\text{II}}^2}{8\pi} \int_{\text{II}} \frac{dl}{A}$

Final Magnetic Energy (in state β): $(U_B)_\beta = \frac{\Phi_{\text{I}}^2}{8\pi} \int_{\text{II}} \frac{dl}{A} + \frac{\Phi_{\text{II}}^2}{8\pi} \int_{\text{I}} \frac{dl}{A}$

Calculate the change in Magnetic Energy:

$$\delta U_B = (U_B)_\beta - (U_B)_\alpha = \frac{(\Phi_{\text{I}}^2 - \Phi_{\text{II}}^2)}{8\pi} \left[\int_{\text{II}} \frac{dl}{A} - \int_{\text{I}} \frac{dl}{A} \right]$$

The most dangerous (unstable) interchanges are those in which the magnetic energy is not changed -- but the rearrangement cause a change in the thermal energy due to compression or expansion -- worst case when $\delta U_B = 0 \Rightarrow \Phi_{\text{I}} = \Phi_{\text{II}}$

The rearrangement causes the interchange of tubes having equal magnetic flux.

Lect #16 (cont.) For this case of $\Phi_I = \Phi_{II}$ the plasma is stable if $\delta U_T > 0$

Let's evaluate that:

$$(U_T)_\alpha = \frac{(P_I)_\alpha V_I}{\gamma - 1} + \frac{(P_{II})_\alpha V_{II}}{\gamma - 1}$$

interchange now means P_I goes into volume V_{II} and becomes $(P_I)_\beta$

and P_{II} in volume V_{II} goes into volume V_I and becomes $(P_{II})_\beta$

$$(U_T)_\beta = \frac{(P_{II})_\beta V_I}{\gamma - 1} + \frac{(P_I)_\beta V_{II}}{\gamma - 1}$$

We need a rule for how the pressures are changed! — This requires kinetic Theory & Transport! -- which is outside MHD

Assume adiabatic compression of an ideal gas! $\boxed{PV^\gamma = \text{constant}}$

$$\Rightarrow \quad (P_I)_\beta = (P_I)_\alpha \frac{V_I^\gamma}{V_{II}^\gamma} \quad \text{and} \quad (P_{II})_\beta = (P_{II})_\alpha \frac{V_{II}^\gamma}{V_I^\gamma}$$

$$\delta U_T = \frac{1}{\gamma - 1}\left\{ (P_{II})_\alpha \frac{V_{II}^\gamma}{V_I^\gamma} V_I + (P_I)_\alpha \frac{V_I^\gamma}{V_{II}^\gamma} V_{II} - (P_I)_\alpha V_I - (P_{II})_\alpha V_{II} \right\}$$

Now everything is about configuration α -- and two tubes within $\alpha \Rightarrow$ drop α notation and consider two adjacent flux tubes

$$(P_I)_\alpha \to P \quad ; \quad (P_{II})_\alpha \to P + \delta P$$

$$V_I \to V \quad ; \quad V_{II} \to V + \delta W$$

$$\Rightarrow \delta U_T = \frac{1}{\gamma - 1}\left\{ (P + \delta P)\frac{(V + \delta V)^\gamma}{V^{\gamma - 1}} + P\frac{V^\gamma}{(V + \delta V)^{\gamma - 1}} - PV - (P + \delta P)(V + \delta V) \right\}$$

Lect #16 (cont.) Neglecting third order ~~or~~ higher terms like $(\delta V)^3$ or $(\delta V)^2 \delta p$ – results in quadratic order expression

$$\boxed{\delta U_T = \delta p\, \delta V + \gamma p \frac{(\delta V)^2}{V}}$$

The linear terms cancel out exactly

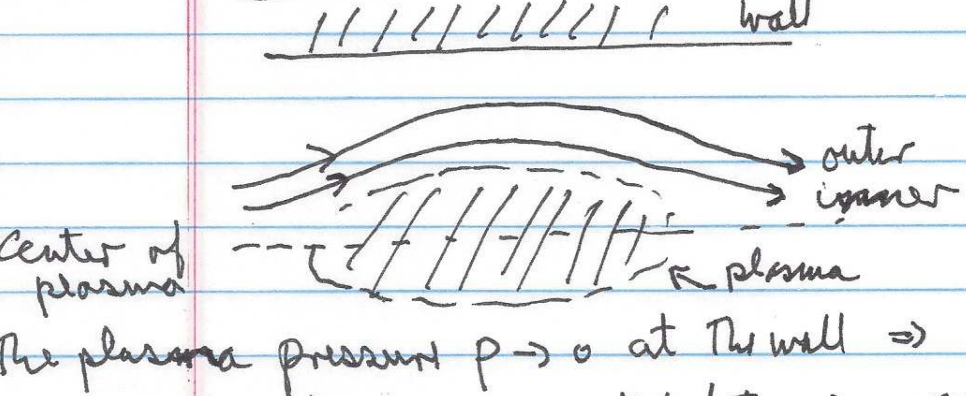
which can also be put in the form: $\boxed{\delta U_T = p\, \delta V \left\{ \frac{\delta p}{p} + \gamma \frac{\delta V}{V} \right\}}$

or equivalently $\boxed{\delta U_T = \frac{\delta V}{V^\gamma}\, \delta(p V^\gamma)}$

Since the contribution from $\gamma p \frac{(\delta V)^2}{V} > 0 \Rightarrow$ This is a stabilizing effect which implies $\boxed{\delta p\, \delta V < 0}$ is a necessary condition for instability

and whenever $\delta p\, \delta V > 0$ the configuration is stable!

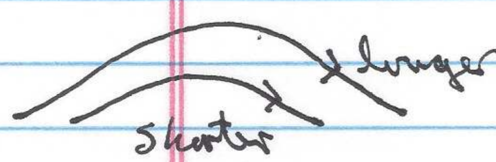
Examine the behavior near the edge of a confined plasma



This looks like a mirror configuration

The plasma pressure $p \to 0$ at the wall $\Rightarrow$ $\delta p < 0$ going from the center outwards $\Rightarrow$ stability is achieved if $\boxed{\delta V < 0}$ as one moves out towards the wall

Lect #16 (cont.) Since the volume of a flux tube is strongly dependent on the length of a field line the condition $\delta V < 0$ implies that field lines should be shorter on the outside than the inside, e.g.,



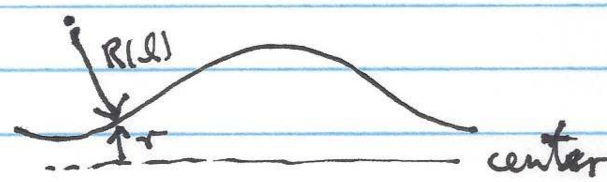
This is bad!

This is good!

Since geometrically the length is a consequence of the curvature - this result can be interpreted in terms of of "good curvature" vs "bad curvature":

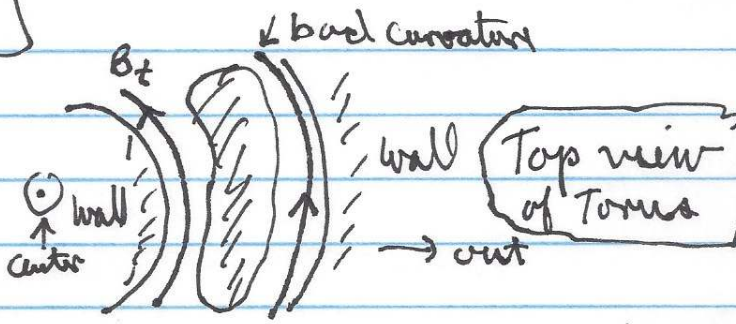
A simple mirror like  has bad curvature and is unstable to interchange of flux tubes.

A more detailed mathematical statement can be obtained in terms of the curvature function $R(l)$



$$\boxed{\int \frac{dl}{r B^2 R(l)} > 0 \quad \text{is stable}}$$

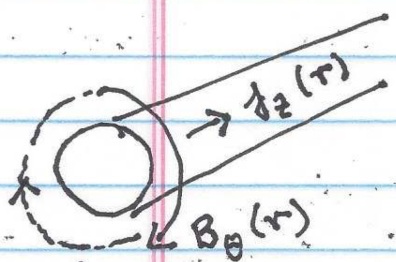
For a tokamak plasma



The inner side is stable but outer side is unstable $\Rightarrow$ need poloidal field to average out the effect

Lect #16 (cont.) Stability of a Theta pinch

Now there is no axial field, i.e., $B_z = 0$



field has bad curvature

$$\underline{B}_{Total} = B_\theta(r)\,\hat{\theta}$$

by Ampere's law: $B_\theta(r) = \dfrac{2I(r)}{cr}$

Volume of a flux tube at fixed $r = V(r)$

with $V = \Phi \oint \dfrac{d\ell}{|\underline{B}|} = \Phi \displaystyle\int_0^{2\pi} \dfrac{r\,d\theta}{B_\theta(r)} = \dfrac{2\pi r \Phi}{2I(r)}\,cr$

$\Rightarrow V = \dfrac{\pi r^2}{I(r)} c\,\Phi$ $\Rightarrow$ For constant current density $I(r) = j\pi r^2$

$\Rightarrow \quad V = \dfrac{c\,\Phi}{j} \quad \Rightarrow \quad \delta V = 0$ and $\delta U_T = \delta p\,\delta V + \gamma p \dfrac{(\delta V)^2}{V}$

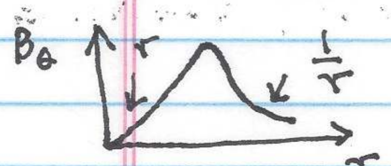
$\Rightarrow$ The interchange of a flux tube causes $\delta U_T = 0$ $\Rightarrow$ such a plasma is marginally stable!

Now consider a nonuniform equilibrium called the Bennet Pinch

In This model $j(r) \propto n \propto p(r)$

Specifically, now $B_\theta(r)$ is legislated + the other quantities need to be determined

$$B_\theta(r) = B_0 \frac{r/r_0}{1 + (r/r_0)^2}$$



now $(\nabla \times \underline{B})_z = \dfrac{4\pi}{c} j_z(r)$

$\dfrac{1}{r}\dfrac{\partial}{\partial r}(r B_\theta) = \dfrac{4\pi}{c} j_z(r)$ ← can solve for

Lect #16 (cont.) Use Force Balance $0 = -\nabla p + \frac{\underline{j} \times \underline{B}}{c}$ for equilibrium

$$\underline{j} \times \underline{B} = j_z B_\theta (-\hat{r}) \quad ; \quad \nabla p = \frac{\partial}{\partial r} p(r)\,\hat{r}$$

$$\Rightarrow \frac{\partial p}{\partial r} = -\frac{j_z B_\theta}{c} = -\frac{1}{4\pi} \underbrace{\left[\frac{1}{r}\frac{\partial}{\partial r}(r B_\theta)\right]}_{(\nabla \times \underline{B})_z \left(\frac{c}{4\pi}\right)} B_\theta$$

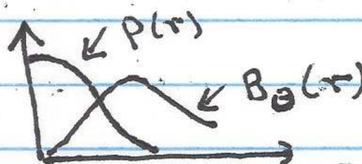
$$\frac{\partial p}{\partial r} = -\frac{B_0^2}{4\pi} \frac{(1/r_0)}{1 + \left(\frac{r}{r_0}\right)^2} \frac{\partial}{\partial r}\left[\underbrace{\frac{r^2/r_0}{1+(r/r_0)^2}}_{\to\ 1 - \frac{1}{1+(r/r_0)^2}}\right]$$

or $$\frac{\partial p}{\partial r} = -\frac{B_0^2}{4\pi} \frac{2r/r_0^2}{[1+(r/r_0)^2]^3}$$

integrate to find $$p(r) = \frac{B_0^2/8\pi}{[1+(r/r_0)^2]^2} + \text{constant}$$

say $p(r=0) = p_0 \Rightarrow$ $$p(r) = \frac{p_0}{[1+(r/r_0)^2]^2} \quad \text{with } p_0 = \frac{B_0^2}{8\pi}$$

Summary of Equilibrium



+ also $$j_z(r) = \frac{c}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}[r B_\theta] = \frac{c}{4\pi}\frac{B_0}{r}\frac{\partial}{\partial r}\left[1 - \frac{1}{1+(r/r_0)^2}\right]$$

$$j_z = \frac{c}{4\pi} B_0 \frac{2/r_0^2}{[1+(r/r_0)^2]^2} = \frac{j_0}{[1+(r/r_0)^2]^2} \quad \text{with } j_0 = \frac{cB_0}{2\pi r_0^2}$$

This is a special case where $\boxed{j \propto p}$

But is this equilibrium stable?

Lect #16 (cont.) Check the interchange instability for this system

$$\delta U_T = \delta p\, \delta V + \gamma p \frac{(\delta V)^2}{V} = p\, \delta V \left[\frac{\delta p}{p} + \gamma \frac{\delta V}{V} \right]$$

now $$V = \Phi \int \frac{dl}{|\underline{B}|} = \Phi \frac{2\pi r}{B_\theta(r)} = \Phi \frac{2\pi r_0}{B_0}\left[1 + (r/r_0)^2\right]$$

for a flux tube at radial position r

The change in volume for the tube is δV

$$\delta V = \Phi \frac{2\pi r_0}{B_0}\, \delta\left[(r/r_0)^2\right]$$

and the change in pressure is $$\delta p = \delta\left[\frac{B_0^2/8\pi}{[1+(r/r_0)^2]^2}\right]$$ ← squared

or $$\delta p = \frac{B_0^2}{8\pi} \frac{(-2)}{[1+(r/r_0)^2]^3}\, \delta\left[(r/r_0)^2\right] = -2 \frac{p}{1+(r/r_0)^2}$$

$$\Rightarrow \quad \frac{\delta p}{p} = \frac{-2}{1+(r/r_0)^2}\, \delta\left[(r/r_0)^2\right]$$

and $$\frac{\delta V}{V} = \frac{\delta\left[(r/r_0)^2\right]}{1+(r/r_0)^2}$$

$$\Rightarrow \quad \delta U_T = \frac{B_0^2/8\pi}{[1+(r/r_0)^2]^2}\, \Phi \frac{2\pi r_0}{B_0}\, \delta\left[(r/r_0)^2\right] \left\{ -\frac{2\,\delta\left[(r/r_0)^2\right]}{1+(r/r_0)^2} + \gamma \frac{\delta\left[(r/r_0)^2\right]}{1+(r/r_0)^2} \right\}$$

$$= \underbrace{\frac{\Phi\, (B_0/4)\, r_0}{[1+(r/r_0)^2]^3}}_{>0} \underbrace{\left(\delta\left[(r/r_0)^2\right]\right)^2}_{>0} \underbrace{\left\{-2 + \gamma\right\}}_{<0}$$

↑ 5/3 for ideal gas

$\Rightarrow \delta U_T < 0 \Rightarrow$ unstable to interchange $\Rightarrow$ pinch develops flutes