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Introduction to Plasma Physics

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I. Introduction

A plasma may be roughly defined as a system containing mobile charges, either positive, negative, or both, in which the electric and magnetic interactions between particles play a dominant role in the dynamics of the systems. Some examples of plasmas are:

(1) Gases which are heated to such high temperatures that some electrons are detached from the constituent atoms and molecules.

(2) The gas in any discharge. Here the electrons gain sufficient energy from the applied electric field that they can ionize other atoms and molecules.

(3) Interstellar gas which is ionized by the radiation from stars.

(4) The ionosphere, where the outer layers of the earth's atmosphere are ionized by solar radiation.

(5) The mobile electrons in metals and semiconductors. Here the perturbations caused by neighboring atoms weaken the binding of the electrons to the atoms to such an extent that some of the electrons become free to move about through the material.

In addition to these systems there are systems which contain mobile charges but which are not generally classified as plasmas. Examples of systems in this category are salt solutions, or gases at temperatures where there are very few free electrons. Systems of this type contain so many neutrals that collisions between the neutrals and charged particles mask many of the properties which are usually associated with a plasma (for example, plasma oscillations

of all types are suppressed). One can go continuously from a system which is not a plasma to one which is, as the example of raising the temperature of a gas shows. We shall not say much about the electron plasmas in solids or about conducting solutions, though some of our consideration will be applicable to these systems.

Plasma has been called by some people the fourth state of matter. In spite of the fact that there is no sharp transition from the gas to the plasma state, there is much justification for this point of view. One essential feature of a plasma, as compared with most other fluids, is the ability to carry current and so to interact strongly with electric and magnetic fields. This property leads to a great wealth of phenomena not exhibited by nonplasmas.

To produce a plasma by heating a gas we must heat it to such a temperature that collisions between particles are able to detach electrons. Since typical binding energies for electrons in atoms (first ionization potential) range from about 4 eV for Cs to 24 eV for He, the colliding particles must have comparable energies. A temperature of 11,600°K corresponds to an average energy of 1 eV.[1]

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1 The Maxwell-Boltzmann velocity distribution function is described in Appendix I.

$$kT = 1 \text{ eV} \quad \text{for} \quad T = 11{,}600°\text{K} \tag{1}$$

Thus we see that temperatures of thousands of degrees are required.

Problem: Compare the average kinetic energy per particle with the energy in eV/particle involved in the following changes of state:

(a) melting of water
(b) boiling of water at STP

We do not strictly require kT to equal the ionization energy to obtain appreciable ionization. In a gas in thermal equilibrium there are always particles with considerably more energy than the average, and collisions between these particles can produce ionization. The charged particle density is determined by a balance between such ionization and recombination of the ions and electrons. In a low-density gas the ions and electrons relatively rarely approach one another, and recombination rates may be quite low. Thus a large percentage ionization can exist even when $kT << E_{ionization}$.

For a gas in thermal equilibrium, the degree of ionization depends only on the temperature. Every process that can produce ionization can take place in reverse to give recombination. At thermodynamic equilibrium, these rates are the same. In fact, every process is exactly balanced by the inverse process (law of detailed balance).

For thermal equilibrium the degree of ionization is determined by the Saha equation. This equation has essentially the form

$$\frac{n_e n_i}{n_o} \cong \frac{1}{\lambda^3} \frac{e^{-E_i/kT}}{g_i} \tag{2}$$

$\lambda^2 = h^2/2m_e kT$ (Debroglie wavelength for electron of average energy)

Here n_e, n_i, and n_o are the electron, ion and neutral densities, E_i is the ionization energy, g_i is the number of ground state levels, m_e is the electron mass, and h is Planck's constant. A derivation of this formula lies outside the scope of these notes (see Kittel, Statistical Mechanics), but we point out the physical significance of the terms.

For a singly ionized substance with overall charge neutrality, $n_i = n_e$ and Eq. (2) can be written in the form

$$\frac{n_e}{n_o} \cong \frac{V_e}{\lambda^3} \frac{e^{-E_i/kT}}{g_i} \tag{3}$$

where $V_e = 1/n_e$ is the volume per electron. Now the number of free (continuum) states for an electron in a volume V is essentially V/λ^3. Thus V_e/λ^3 is the number of free states per electron (number of free states for a volume containing 1 electron). If there is one free electron in the volume V_e, then there is one ion and this ion has g_i ground states. Thus in the volume V_e there are V_e/λ^3 unbound states g_i ground states available to the electron. The ground states must be weighted by the factor $e^{E_i/kT}$ (the probability of occurrence of a state of energy E in thermal equilibrium is $e^{-E/kT}$; the binding energy is negative, $-E_i$). Thus the ratio of the number of unbound states to ground states is $\frac{V_e}{\lambda^3} \frac{e^{-E_i kT}}{g_i}$, which is also the ratio of the number of electrons to neutrals, n_e/n_o, and this gives the above Saha equation. In this illustration we have neglected bound states other than the ground state.

<u>Problem</u>: Find the percent ionization for H at densities of n = 1 (interstellar space), $n = 10^{14}$ (solar surface), 10^{19} (atmospheric density), 10^{22} (liquid density), and temperatures of .1 eV, 1,160°K, 1 eV, 10 eV, 100 eV, $E_i = 13.6$ eV, $g_i = 1$.

Besides the thermal equilibrium plasma there are many types of plasmas which are not in thermal equilibrium. Examples of these are: most discharge plasmas, and the interstellar gas which is ionized by the radiation from stars.

For discharge-type plasmas, the electrons gain enough energy from the electric field so that they can knock other electrons from the constituent atoms of the gas. Thus, typically the electron energies in a discharge are of the order of a few eV. To heat them to a higher temperature than this is difficult, because the electrons lose so much energy in ionizing and exciting the gas atoms. These losses can only be overcome with large currents and high powers.

The plasma in a gas discharge is rarely in thermal equilibrium. The mean free paths of the particles are usually greater than the size of the container. Thus the ions and neutral atoms, though heated slightly by electron collisions, remain at temperatures close to that of the walls; i.e., kT_{wall} of about 1/40 eV. As we shall see later, electric fields occur naturally near the walls which reflect most of the electrons, thus isolating them from the wall. Also, the radiation level is substantially below the level corresponding to thermodynamic equilibrium; the plasma is optically thin.

<u>Problem:</u> Calculate the power that would be radiated per cm^2 by a 10 eV plasma if it were to radiate as a blackbody.

Even though the radiation levels are low compared with blackbody levels, a lot of energy is carried off by radiation. The electrons, in colliding with the atoms of the gas, raise them to excited states. Many of the excited states give up their energy immediately as radiation. Since energetic electrons are generally more efficient at exciting and ionizing, the energetic electrons tend to be brought down to lower energies and their number is depleted. Thus such discharges contain fewer energetic electrons than would an equilibrium plasma with the same average energy.

Rather than having a Maxwellian distribution,

$$f(v) \propto e^{-mv^2/2kT} \tag{4}$$

such plasmas have distributions like that shown in Fig. 1, which are more or

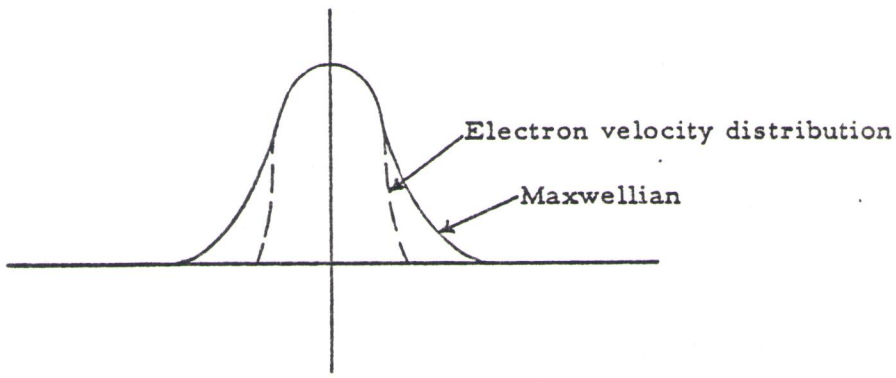


Figure 1

less sharply cut off. A distribution which is often used is the Druyvesteyn distribution

$$f(v) \propto e^{-v^4/v_o^4} \tag{5}$$

In contrast to the plasma in thermodynamic equilibrium, the properties of nonequilibrium plasmas are strongly dependent on the cross sections for the various processes. Of course when such a plasma is in a steady state, the rate of ionization must be balanced by the rate of recombination. However, here each process does not have to be balanced by the inverse process. For example, we could have ionization taking place in the discharge and all the recombination taking place at the wall of the tube.

Because of the many processes and their various cross sections, the analysis of gas discharges or other nonequilibrium plasmas can become

extremely complicated. In the next section we summarize the most important collision processes.

II. Particle Interactions, Neutrality, and Collective Behavior

As has already been mentioned, in order for us to produce a plasma, particle energies (electron, ion, atomic, or photon) must be at least comparable to the ionization energy of the constituent atoms. The electrons which are detached will have about the same energy. Particle energies of the order of 10% or even perhaps a few percent might be considered comparable. Now the magnitude of the binding energy of an electron to an atom is about equal to its potential energy (perhaps one half of it) when bound to the atom. When it becomes detached, its mean distance from an ion is much greater than atomic dimensions, and consequently the mean potential energy of the electrons is much less than the mean kinetic energy; a basic property of all plasmas. For example, consider a Cs plasma at a density of 10^{12} electrons/cm^3 and a temperature of 2500°K (use of the Saha equation shows that Cs under these conditions will have a high percent ionization). Now the mean distance between an electron and an ion is 10^{-4} cm and the corresponding potential energy is

$$\frac{e^2}{r} = \frac{e^2}{a_o}\frac{a_o}{r} = 27 \times \frac{5 \times 10^{-9}}{10^{-4}}$$
$$= 135 \times 10^{-5}\ \text{eV} \simeq 10^{-3}\ \text{eV}. \tag{6}$$

(The Cs is singly ionized.)

Thus the free electron as it moves through a plasma is, most of the time, in regions of small potential; so that the effect on the trajectory is small except during the relatively rare close approaches — i.e., during collisions. The

cumulative effect of many small collisions will, however, turn out to be important. On the other hand, because of the long-range nature of the electric interactions, even slight excesses of electric charge over extended regions can produce potential energies and hence interactions which are not negligible compared to the particle kinetic energies. To illustrate this, consider an infinite homogeneous plasma. Let us ask how much energy it would take to move all the electrons in a spherical region of radius r to its surface.

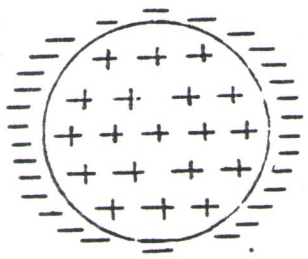


Figure 2

We hold the ions fixed at a uniform density. The potential energy stored in the electric field is

$$W_E = \int \frac{E^2}{8\pi}\, dv \tag{7}$$

The electric field is radially outward and is given by

$$E(\rho) = \begin{cases} \dfrac{1}{\rho^2}\,\dfrac{4}{3}\,\pi\, n_i\, Ze\,\rho^3 & \rho < r \\ 0 & \rho > r \end{cases} \tag{8}$$

where ρ is the radius, n_i is the density of ions, Ze is the charge on an ion, $\frac{4}{3}\pi n_i Ze\rho^3$ is the charge Q within the sphere of radius ρ, and Q/ρ^2 is the field it produces. Substituting Eq. (8) into Eq. (7) gives

$$W_E = \int \frac{2\pi}{9} n_i^2 z^2 e^2 \rho^2 4\pi \rho^2 d\rho$$

$$= \frac{8\pi^2}{9 \times 5} n_i^2 z^2 e^2 r^5 = \frac{8\pi^2}{9 \times 5} n_e^2 e^2 r^5 \tag{9}$$

where n_e is the undisturbed electron density.

<u>Problem:</u> Consider a plasma with an electron temperature of 10 eV and a density of 10^{14} electrons and ions per cubic centimeter. Compare the average kinetic energy of the electrons with the work per electron required to remove all the electrons from a sphere of 1 cm radius. (The ions are held at the initial density.)

Let us equate W_E to the kinetic energy of all the electrons within the sphere

$$\frac{3}{2} kT \frac{4}{3} \pi n_e r^3 = \frac{8\pi^2}{9 \times 5} n_e^2 e^2 r^5 \tag{10}$$

$$r^2 = 45 \frac{kT}{4\pi n_e e^2} \tag{11}$$

This is the radius of a sphere where the electrons just have enough kinetic energy to remove themselves from the sphere; i.e., such a motion is energetically possible though extremely unlikely. For larger spheres there is not enough energy, while for smaller radii there is sufficient energy. The quantity λ_D given by

$$\lambda_D^2 = \frac{kT}{4\pi n_e e^2} = \frac{v_T^2}{\omega_p^2} \qquad \omega_p^2 \equiv \frac{4\pi n_e e^2}{m}$$

$$v_T^2 = kT/m \tag{12}$$

is called the Debye length. (It was first introduced by Debye in the study of electrolytes.) It is a measure of the size of a region in which appreciable

charge density fluctuations can occur. The quantity ω_p is called the plasma frequency (as originally defined by Tonks and Langmuir). Regions with dimensions much larger than a Debye length must be neutral to a very high degree. This is another basic characteristic of a plasma.

It is illuminating to write Eq. (12) in the following form

$$\frac{\lambda_D^2}{d^2} = \frac{kT\,d^3}{d^2\,4\pi e^2} = \frac{kT}{e^2/d\,4\pi} = \frac{1}{4\pi}\,\frac{kT}{\phi(d)} \tag{13}$$

where we have replaced $1/n_e$ by d^3, where d is the interparticle spacing and $\phi(d)$ is the potential of an electron (or ion) evaluated at the distance d. We have already shown that in a plasma kT must be large compared to $\phi(d)$, and hence λ_D is large compared to d. Thus in a plasma there are many particles in a Debye sphere. For the Cs example given ($T = 2500°K$, $n_e = 10^{12}$), $kT/\phi(d) = 250$ and $\lambda_D = 4.5\,d$: a Debye sphere contains $\frac{4}{3}\pi\lambda_D^3 n_e \cong 360$ particles. For a typical plasma in the stellarator, $kT = 50\text{ eV} = 5 \times 10^5 °K$, $n_e = 10^{12}$ and $\lambda_D = 70\,d$: the number of particles in a Debye sphere is 10^6. We shall have more to say about the Debye length later in the course.

The quantity ω_p in Eq. (12) arises whenever collective electron motion takes place in a plasma. As an example, consider the case where all the electrons in an infinite sheet of thickness L are shifted by a distance δ. Then there are two regions, one charged positive and the other charged negative, separated by the distance L. The charge σ per unit area is $en\delta$ and the electric field in the region L is

$$E = 4\pi\sigma \tag{14}$$

The force/area on the electrons in L is

$$(Lne)\,E = 4\pi n^2 L e^2 \delta \tag{15}$$

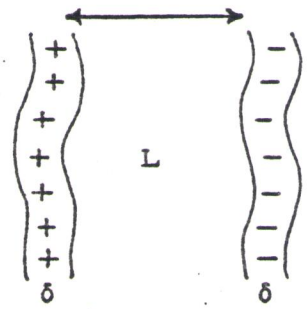


Figure 3

This force acts on a mass of Lnm (where m is the electron mass) so the equation of motion is

$$-4\pi n^2 L e^2 \delta = Lnm \frac{d^2\delta}{dt^2} \tag{16}$$

which has as a solution

$$\delta = A e^{-i\omega_p t} \tag{17}$$

The electrons thus oscillate at angular frequency

$$\omega_p = \left(\frac{4\pi ne^2}{m}\right)^{\frac{1}{2}} \quad \text{or}$$
$$f_p \sim 9 \times 10^3 (n)^{\frac{1}{2}} \tag{18}$$

The force on the ions is the same, but the much greater ion mass results in a relatively small motion which is neglected here. The plasma frequency is essentially a measure of the minimum response time of a plasma to an applied electric field. In a time of the order of $1/f_p$ the electrons will move in such a fashion as to reduce an applied electric field. Thus a plasma is transparent to electromagnetic waves above the plasma frequency. Below this frequency the plasma acts like a good conductor; the waves are reflected.

Of course, if the collision frequency is higher than the plasma frequency, such motion is quickly damped; the plasma becomes an absorbing medium.

Plasma physics deals with an enormous range of physical parameters. Fig. 4 summarizes the regions of physical interest; a temperature range of 10^0 to 10^9; a density range of 10^{-2} to 10^{18}. The plasma frequency — which is a measure of density — is shown at the right. The Debye length (a function of n and T) is shown by the diagonal lines.

We have described some of the basic ideas of plasma physics. Collisions complicate matters; therefore, proceeding further we briefly survey various collisional processes.

III. Collisions and Atomic Processes

It is not possible to take collisions completely into account in describing a plasma. First of all there is a very large variety of types of collisions. Secondly, relatively few cross sections are well known, either on the basis of theory or experiment, although recent research has enlarged our knowledge greatly. Nevertheless, it does prove possible in many cases of practical interest to get reasonable agreement between experiment and theory on the basis of what is known.

In the following sections we attempt to provide brief descriptions of the

principal types of collisions involved in plasma problems and to give typical values of the cross sections. It is hopeless to attempt to provide complete data. The intent is to provide sufficient information to enable order-of-magnitude estimates of collisional effects. If order-of-magnitude estimates indicate collisions to be important, then the detailed cross sections may be sought out in the literature.

Data is given in several different units in the literature. Most recent data is presented in units of cm^2/atom. Also in common use is the dimension $\pi a_o^2 = .88 \times 10^{-16}$ cm^2, where a_o is h^2/me^2, the Bohr radius of the hydrogen atom; also, since atomic cross sections are frequently in the region 10^{-16} cm^2, angstroms squared — $Å^2$ — are used. In the older literature, particularly with respect to elastic collisions, the cross sections are given in terms of collision probability per cm per unit pressure (mm Hg at 0°). To change them into cm^2/atom one must divide by the number of atoms per cm^3 at a pressure of 1 mm Hg. Thus a collision probability of one in the above units corresponds to a cross section of 0.283×10^{-16} cm^2/atom.

If we watch a given "test particle" proceed through a region where the target or field particle density is n, then the probable number of a given type of collisions per cm path length is simply σn, where σ is the cross section for the particular type of interaction. The average distance traveled between collisions, or mean free path, is then $1/n\sigma$. The number of collisions per second, collision frequency, is $n\sigma v$, where v is the relative velocity of field and test particles. If the field particles are in motion or if we are concerned with the average behavior of a large number of test particles moving at different velocities,

then since σ for most processes is a function of v, we must average the product σv over all velocities. Thus if $f(v)\,dv$ is the number of particles having velocities between v and $v+dv$, we may write

$$\int f(v)\,\sigma(v)\,v\,dv \equiv n\overline{\sigma v} \tag{1}$$

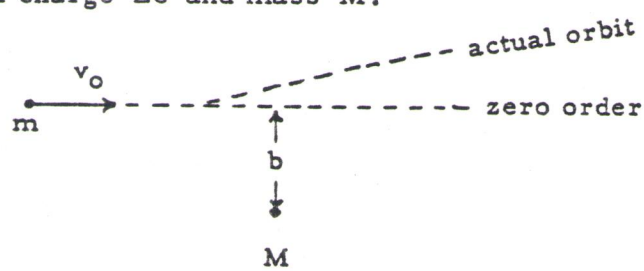
If instead of one there are n_t test particles per cubic centimeter, then the total number of collisions per second per cubic centimeter of type described by a given σ is

$$\text{Col/sec} = n_t\, n\overline{\sigma v} \tag{2}$$

The product $\overline{\sigma v}$ is usually called the rate coefficient.

Coulomb collisions

This is the process of greatest importance to us. We consider the problem of a light particle of charge ze and mass m, approaching with velocity v_o a stationary heavy particle of charge Ze and mass M.



Let us assume that the deflection angle is small so that in zero order approximation the particle is undeflected, with a distance of minimum approach the same as the impact parameter, b. Most of the deflection is due to the force exerted during the time the particles are close together — a time approximately b/v_o, and a force of about zZe^2/b^2. Then the momentum acquired perpendicular to v is

$$p_\perp = F_\perp \Delta t \sim \frac{zZe^2}{b^2}\,\frac{b}{v_o} = \frac{zZe^2}{v_o b} \tag{3}$$

$$I\,d(\theta)\,d\Omega = I(\theta)\,2\pi \sin\theta\, d\theta = 2\pi\, bdb$$

$$I(\theta) = \frac{bdb}{\sin\theta d\theta} \tag{9}$$

Eliminating b from Eq. (9) by Eq. (7) yields the Rutherford scattering formula

$$I(\theta) = \left(\frac{zZe^2}{2m\,v_o^2}\right)^2 \cos e^4 \frac{\theta}{2} \tag{10}$$

Multiple Coulomb Scattering

We consider a group of electrons initially moving in the z-direction. It is assumed that there are very many ions and due to the long range of the Coulomb forces there are far more small-angle than large-angle collisions. On the average, of course, there are as many deflections in any particular direction as in the opposite direction. However, there will be scattering due to statistical fluctuations.

For a given electron the total velocity in the x-direction, acquired after n individual scatterings is

$$\Delta v_x = (\Delta v_x)_1 + (\Delta v_x)_2 + \ldots + (\Delta v_x)_N \ . \tag{11}$$

For an ensemble of many electrons, we consider the average of Δv_x over all the collection. Then due to the equal probability of any given scattering and its opposite,

$$\overline{\Delta v_x} = \overline{\Delta v_y} = 0 \ . \tag{12}$$

Let us instead look at the average of Δv_x^2, designated

$$\left\langle \Delta v_x^2 \right\rangle = \overline{[(\Delta v_x)_1 + (\Delta v_x)_2 + \ldots\ldots]^2} \tag{13}$$

The sum will then consist of two types of terms, $\overline{(\Delta v_x)_n^2}$ and $\overline{(\Delta v_x)_n (\Delta v_x)_m}$.

The second term is zero due to the fact that collisions are completely uncorrelated.

If successive collisions $(\Delta v_x)_i$ are the same in magnitude, then

$$\left\langle (\Delta v_x)^2 \right\rangle = N \overline{(\Delta v_x)^2} \tag{14}$$

Because of the axial symmetry

$$\left\langle \Delta v_x^2 \right\rangle = \left\langle \overline{\Delta v_y^2} \right\rangle \equiv \left\langle \Delta v_\perp^2 \right\rangle \tag{15}$$

$$\Delta v_\perp^2 = v_o^2 \sin^2 \theta \tag{16}$$

$$\tan \frac{\theta}{2} = \frac{b_o}{b} \tag{17}$$

$$\sin \theta = \frac{2 b_o/b}{1 + (b_o/b)^2} \tag{18}$$

$$(\Delta v_\perp)^2 = \frac{4 v_o^2 (b/b_o)^2}{[1 + (b/b_o)^2]^2} \quad . \tag{19}$$

The cross section for collision within db of b is $2\pi b\, db$. The number of collisions per unit path length is $nI d\Omega = 2\pi n b\, db$. At each collision the change in velocity squared is given by Eq. (19). Then the average change in $v_\perp^2$ suffered by an electron per cm path length is

$$\left\langle (\Delta v_\perp)^2 \right\rangle = \int_o^{b_m} \frac{4 v_o^2 (b/b_o)^2}{[1 + (b/b_o^2)]^2} 2\pi n b\, db \quad , \tag{20}$$

where b_m is the largest distance at which the Coulomb potential is applicable, in view of the shielding of the ion charge due to the surrounding electrons. The point is discussed further in Spitzer[2].

2 L. Spitzer, Jr., Physics of Fully Ionized Gases (Interscience Publishers, New York, N. Y., 1962), 2nd ed.

$$\left\langle (\Delta v_\perp)^2 \right\rangle = 8\pi n v_o^2 b_o^2 \int_o^{b_m/b_o} \frac{x^3\,dx}{(1+x^2)^2} \quad . \tag{21}$$

$$\left\langle (\Delta v_\perp)^2 \right\rangle = 4\pi n v_o^2 b_o^2 \left\{ \ln\left[1+\left(\frac{b_m}{b_o}\right)^2\right] + \frac{1}{1+(b_m/b_o)^2} - 1 \right\}. \tag{22}$$

Assuming b_m/b_o and $\ln(b_m/b_o) >> 1$,

$$\left\langle (\Delta v_\perp)^2 \right\rangle = 8\pi n v_o^2 b_o^2 \ln(b_m/b_o) \quad . \tag{23}$$

It is customary to take as b_m, the Debye length λ_D. The $\ln(b_m/b_o)$ term for almost all conditions of interest is between 10 and 20.

We may roughly say that when $(\Delta v_x)^2 = v^2$ the particle has been scattered by 90°.

$$\frac{\text{multiple}}{\text{single}} \sim \frac{8\pi n b_o^2 \ln(b_m/b_o)}{n\pi b_o^2} = 8\ln(b_m/b_o) \equiv 8\ln\Lambda \tag{24}$$

Thus large angle deflections due to multiple scattering are about two orders of magnitude more probable than those due to single scattering.

For force fields which fall off as r^{-m} with $m \geq 3$, it is no longer true that multiple scattering dominates.

The Coulomb scattering described here is elastic (we have neglected radiation effects): kinetic energy is conserved. In the center of mass

coordinate system the individual particle energies are unchanged by a collision. This is no longer true in the laboratory frame. For example, the light particle engaging in a large angle collision with a stationary heavy particle loses roughly a fraction $2m/M$ of its energy at each collision.

Relaxation Times

When a plasma is heated by electric forces the velocity distribution may differ markedly from kinetic equilibrium (Maxwell-Boltzmann distribution). It is important to be able to estimate the time required for the distribution to relax by means of collisions toward the M-B distribution (self-collision time). Furthermore, the heating effects on ions and electrons are usually quite different. We need also an idea of the time required for ions and electrons to approach the same temperature (equipartition time).

Exact solutions require extensive numerical computations, but we can get approximate results based on Eq. 23. This equation was derived for scattering of a light particle by a heavy one. If we had used center-of-mass coordinates, the problem would have been formally identical. The velocity v would then represent the relative motion of the two particles, and m the reduced mass $m_1 m_2 / m_1 + m_2$. It would only be necessary to transform the scattering angle from center-of-mass coordinates to the laboratory frame. If the two particles have the same mass and we consider the problem as one of the slowing down of fast particles by collisions with slow ones (assumed stationary), then a 90° scattering angle in the c of m frame becomes a 45° scattering in the laboratory frame. We assume that by the time a particle has been scattered by a like particle through an angle of 90°, energy exchange has been effected. (This is, of course, not a reasonable assumption if the masses are very different.)

We found the expression (19) for the scattering per unit path length. The scattering per unit time is then Eq. (19) multiplied by v. Eliminating b_o by Eq. (6) we get

$$\frac{d\left\langle v_x^2\right\rangle}{dt} = \frac{8\pi z^2 e^4 \ln \Lambda n}{m^2 v_o} \tag{25}$$

and $\left\langle v_x^2\right\rangle$ increases linearly with time. If we now set $\left\langle v_x^2\right\rangle = 4v_o^2$ (i.e., multiply by dt and integrate from 0 to τ_c) corresponding to large angle scattering in the laboratory frame, we can solve for the time τ

$$\tau_c = \frac{4v_o^3 m^2}{8\pi z^2 e^4 n \ln \Lambda} \tag{26}$$

Let us view this in terms of a group of particles at high temperature T being slowed by cold particles of the same type then using the relationship

$$\frac{3}{2} kT = \frac{1}{2} m v_o^2 \tag{27}$$

we get

$$\tau_c = \frac{m^{\frac{1}{2}} (3kT)^{3/2}}{2\pi z^2 e^4 n \ln \Lambda} \tag{28}$$

This compares with the relation given by Spitzer

$$\tau = \frac{m^{\frac{1}{2}} (3kT)^{3/2}}{8 \times 0.714 \pi z^2 e^4 n \ln \Lambda} = \frac{11.4 A^{\frac{1}{2}} T^{3/2}}{n z^2 \ln \Lambda} \tag{29}$$

Thus if the temperature is the same, hydrogen ions require $\sqrt{M/m}$ or 43 times as long as electrons to approach a M-B distribution. For any other ions the time is of course longer. What about equipartition times between ions and electrons? Each collision between a light and heavy particle transfers

only a fraction of order m/M of the energy. The relative velocity of two electrons is roughly the same as the relative velocity of electron and ion, thus roughly speaking electron-ion equipartition takes $\left(\frac{M}{m}\right)$ times longer than electron-electron self-collision time. Thus the ratio of electron-electron to ion-ion to electron-ion times are roughly

$$\tau_{ee} : \tau_{ii} : \tau_{ie} = 1 : \sqrt{M/m} : M/m$$

provided the temperature is the same in all cases.

Conductivity

We can get an approximate expression for the conductivity of an ionized gas using the collision cross sections that we have calculated. The model adopted is the so-called Lorentz model, where the conductivity is assumed to be entirely due to electrons moving in a stationary ion background. Our simplified calculation of the conductivity of a Lorentz gas will assume that the electrons are completely stopped at each large angle collision and that between collisions the electrons are uniformly accelerated by the applied electric field. The drift velocity is

$$v_d = \frac{at}{2} = \frac{Ee\,t_c}{2} \tag{1}$$

It is reasonable to assume that the collisions most effective in stopping the drift motion are those with ions — i.e., we use the ion-electron large angle scattering time which depends on ion density n_i

$$t_c = \frac{v_o^3 m^2}{8\pi z^2 e^4 n_i \ln \Lambda} \tag{2}$$

The resistivity (in emu) is

$$\eta = \frac{cE}{j} = \frac{cE}{n_e e v_d / c} \tag{3}$$

where j and cE are in emu.

Using Eq. (1) and (2) in (3), and substituting $z n_i = n_e$, we get

$$\eta = \frac{8\pi}{3^{3/2}} \frac{c^2 e^2 z \ln \Lambda}{m^{\frac{1}{2}} (kT)^{3/2}} \sim 5 \frac{c^2 e^2 z \ln \Lambda}{m^{\frac{1}{2}} (kT)^{3/2}} \tag{4}$$

This estimate should be high for the following reason. Since the collision time goes as $1/v^3$, the electron of higher than average energy should be less affected by collisions and would therefore be the dominant current carrier, whereas we have assumed T_e appropriate for the average. When the calculation is done properly, the conductivity drops by a factor of about 5 (as though we had all electrons of about 3 × average energy). When electron-electron collisions are also included, this raises the resistivity by about a factor of 2 (for hydrogen), giving numerically

$$\begin{aligned} \eta &= 6.53 \times 10^{12} \frac{\ln \Lambda}{T^{3/2}} \text{ emu} \\ &= 6.53 \times 10^{3} \frac{\ln \Lambda}{T^{3/2}} \text{ ohm cm} \end{aligned} \tag{5}$$

Electron-Atom Elastic Collisions

At electron energies below that of the lowest excited state, only elastic collisions with neutral atoms in the ground state can take place. According to the correspondence principle, a description in classical terms is valid only for large impact parameters. In this region, the main contribution to the interaction potential is due to the polarization of the neutral atom by the electric field of the electron. At a separation of r, the electric field produced by the

electron is

$$E = -\frac{e}{r^2} \tag{6}$$

This field causes a polarization of the atom proportional to E, or

$$P = \alpha E = -\alpha \frac{e}{r^2} \tag{7}$$

where α is the atomic polarizability and is of the order of 10^{-24} cm^3. The force on an electric dipole is the product of the dipole moment and the electric field gradient. Thus the force

$$F = \left(-\alpha \frac{e}{r^2}\right) \frac{d}{dr}\left(-\frac{e}{r^2}\right) = -2\alpha \frac{e}{r^5} . \tag{8}$$

The potential is thus an inverse 4th power one. There is not the same problem of divergent cross sections as in the Coulomb case. Notice that at $r = 10^{-8}$ cm (where the impact parameter is too small for validity of this description) the force is equal to the Coulomb force, while at $r = 10^{-7}$ cm the force has fallen to 10^{-3} of the Coulomb force and is now negligible.

In the case of ion-atom collisions, however, the wavelength of the ion is very small compared with the impact parameter: the classical description is valid. Now it turns out that for a central force which varies as r^{-n}, the cross section varies as $r^{-n/4}$. Thus the ion-atom cross section varies as v^{-1}; the collision frequency, $n\sigma v$, is independent of velocity, thus greatly simplifying the calculation of transport phenomena. At small separations, as the ion begins to overlap the atom, thus there arises a strong repulsive force. The net overall result is sketched in Fig. 5.

For the important range of impact parameters in the case of electron-atom scattering, one must apply quantum mechanical techniques.

Returning to electron scattering, it may be described as the interaction of an incoming train of Debroglie waves representing the electrons, passing through the force field of the atom. The impact parameter is replaced by the quantized angular momentum. As the particles enter the field of force of the atom, the wavelengths change as a function of the potential U, as determined by the Schrodinger equations

$$\nabla^2 \psi + \frac{8\pi m}{h^2} (E - U) \psi = 0 \qquad (9)$$

The potential distribution for the hydrogen atom may be calculated by a number of methods. For more complex atoms there exist several ways of approximating this potential function. The matching of the waves of incoming and outgoing electrons with the waves in the region where the $V \neq 0$ then determines the scattering angle. The form of the potential function depends on the distribution of the orbiting electrons about the nucleus. Thus it is not surprising that chemically similar atoms have similar scattering properties.

The theory is in fair agreement with observations, both in terms of total and differential cross sections, except at very low energies below 1 eV where the measurements become more difficult. We shall provide here only a summary of the more important cross sections in Figs. 6 and 7.

Of particular interest is the dip especially apparent in the case of the heavier rare gases (Fig. 6) at an electron energy of about 1 eV — the Ramsauer

effect. This is essentially a resonance effect. The phase shift of the electron wave in crossing the atom is close to an integral multiple of 2π.

Momentum Transfer Cross Section

The fractional energy loss of an electron of mass m scattered by an atom of mass M is [neglecting terms of order $(m/M)^2$]

$$2(1 - \cos\theta)\frac{m}{M} . \tag{10}$$

Problem: Derive Eq. (10).

The total fractional energy loss per unit length is then

$$\epsilon = \int n(1 - \cos\theta)\frac{2m}{M} I\, 2\pi \sin\theta\, d\theta \tag{11}$$

$$\epsilon \equiv \frac{2m}{M}\sigma_m n \tag{12}$$

where

$$\sigma_m = 2\pi \int_0^\pi I(1 - \cos\theta)\sin\theta\, d\theta \tag{13}$$

and is called the momentum transfer cross section. The average fractional energy loss per collision is then Eq. (12) divided by the total number of collisions per unit length, σn

$$\frac{\Delta E}{E} = \frac{2m}{M}\frac{\sigma_m}{\sigma_c} . \tag{14}$$

In most cases the momentum transfer cross section is within 10% of the total collision cross section, σ_c, as may be seen in Fig. 8.

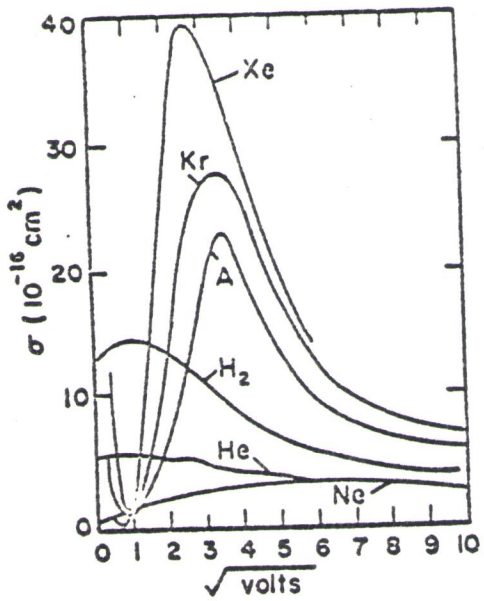


Fig. 6. Total collision cross sections in Ne, A, Kr, and Xe; R. B. Brode, Rev. Mod. Phys. 5, 257 (1933).

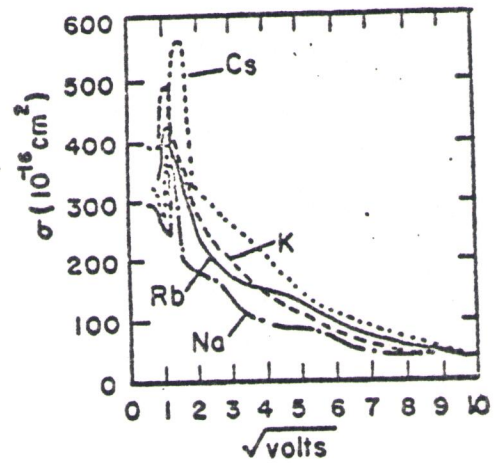


Fig. 7. Total collision cross sections in the alkali metals; R. B. Brode, Rev. Mod. Phys. 5, 257 (1933).

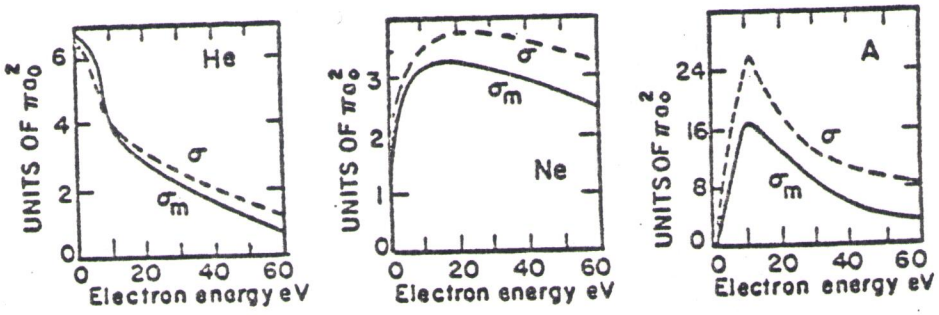


Fig. 8. Comparison of total collision cross section σ with momentum transfer cross section σ_m for He, Ne, and A. (Massey and Burhop)

Inelastic Electron-Atom Collisions

If the electron energy is sufficiently high, excitation and ionization become possible. These are, of course, quantum mechanical problems. For energies well above threshold energies (by a factor of 5 or more), the Born approximation works well and calculations are simplified.

For a complete description of the processes going on in a partially ionized plasma we would need to know the cross sections for excitation to all levels and to the continuum. The information is meager. Fairly complete calculations exist for helium and they are in fair agreement (except near threshold) with experimental observations. Experimental measurements of excitation cross sections are very difficult. Calculations for atomic hydrogen, of course, can also be carried out, but most experiments involve collisions with molecular hydrogen. For allowed transitions the calculated cross section rises linearly from zero above the threshold energy to a maximum at about twice threshold (experimentally observed peaks tend to be at somewhat higher energies) and falls off as $(\ln v)/v^2$ at high energies.[1] For forbidden transitions, the cross section falls a little more rapidly, as $1/v^2$. If electron exchange is involved as, for example, in transitions between triplet and singlet levels in helium, the cross section falls off very rapidly beyond the peak (see Fig. 10).

[1] Fig. 9 shows calculated and observed cross sections for two allowed transitions.

The ionization cross section can be measured with considerable accuracy. Here again the cross sections rise rapidly to a maximum at several times the threshold energy, falling asymptotically as $\ln v/v^2$.

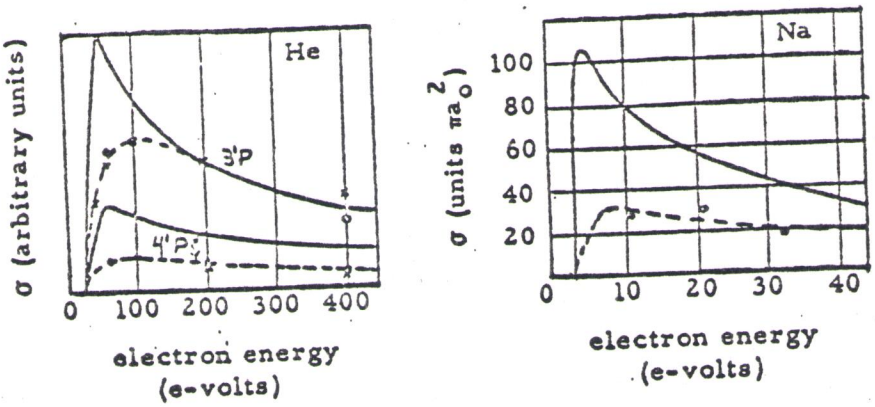


Fig. 9. Cross-sections for optically allowed transitions in He and Na.

——— calculated
------observed

[Massey and Burhop.]

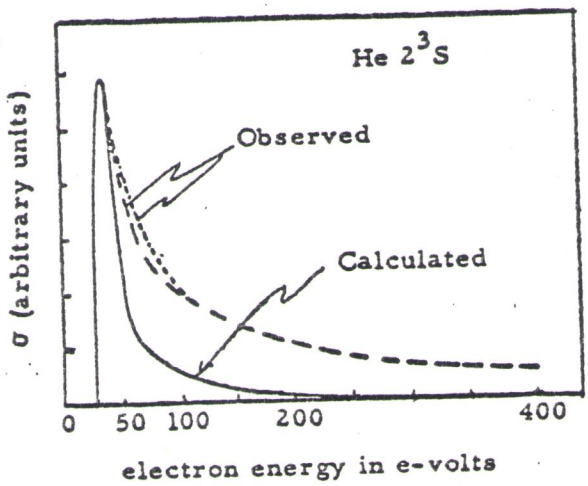


Fig. 10. Excitation cross-sections for the He 2^3S metastable level.

[Massey and Burhop.]

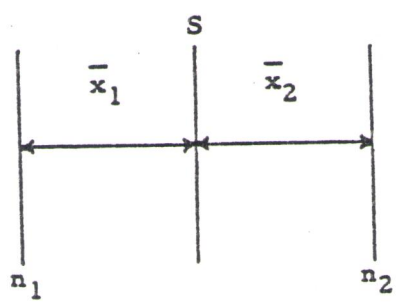
(p. 39a)

Table I compares calculated elastic ionization and excitation cross sections for atomic hydrogen at energies high enough to allow the use of the Born approximation.

Diffusion and Mobility in Partially Ionized Gases

We derive here an elementary description of electron diffusion and mobility processes in slightly ionized gases where elastic collisions predominate.

We consider first the diffusion of electrons (or ions) through a neutral gas. The concentration of charged particles is so low that electron-neutral collisions dominate over Coulomb collisions, and the temperature is low enough that elastic collisions dominate. The temperature is assumed uniform and the density of electrons is a function of x only. There will then be a net flow of electrons from the region of higher to the region of lower concentration.



The flux Γ_2 crossing surface S from the right consists of electrons which on the average engaged in a collision at a distance $\bar{x}_2$ from S where the electron density is n_2. The flux from the left came from a distance $\bar{x}_1$ where the electron density was n_1. Since the background neutral gas density

is constant, $\bar{x}_2 = \bar{x}_1$ and

$$n_1 = n_o - \frac{dn}{dx}\bar{x} \tag{15}$$

$$n_2 = n_o + \frac{dn}{dx}\bar{x} \tag{16}$$

This of course may not be legitimate if the density varies too rapidly — e.g., in the case of shocks.

From kinetic theory we recall that the total number of particles crossing a unit plane for a Maxwellian distribution is $\frac{1}{4}n\bar{v}$. The net flux crossing S is then

$$\Gamma = \frac{1}{4}n_1\bar{v}_1 - \frac{1}{4}n_2\bar{v}_2 \tag{17}$$

$$\Gamma = -\frac{1}{2}\frac{dn}{dx}\bar{x}\bar{v} \tag{18}$$

But from kinetic theory the mean free path is related to the mfp across a plane by

$$\bar{x} = \frac{2}{3}\lambda \tag{19}$$

$$\Gamma = -\frac{1}{3}\lambda\bar{v}\frac{\partial n}{\partial x} \tag{20}$$

$$\Gamma = -\frac{1}{3}\frac{v^2}{\nu_c}\frac{\partial n}{\partial x} \equiv -D\frac{\partial n}{\partial x} \tag{21}$$

where the diffusion coefficient is

$$D = +\frac{1}{3}\frac{v^2}{\nu_c} \tag{22}$$

where ν_c is an effective collision frequency.

If we consider the three-dimensional case,

$$\Gamma = - D \nabla n \tag{23}$$

$$\nabla \cdot \Gamma = - \frac{\partial n}{\partial t} \tag{24}$$

$$\frac{\partial n}{\partial t} = D \nabla^2 n \quad \text{(the diffusion equation)} \tag{25}$$

If D varies with position, then we must write

$$\Gamma = - \nabla D n \ . \tag{26}$$

We can also construct a simple model of mobility. Again we consider a case where collisions with neutrals are dominant. We consider an electron in a uniform electric field E . Then the electron acquires, in addition to its thermal motion, an accelerated motion along $-E$. On the average, a fraction α of the drift momentum is lost at each collision. If the collision frequency is ν_c' , the equation of motion for the average electron is

$$m \frac{dv}{dt} - m v \alpha \nu_c' = - e E \ . \tag{27}$$

We now consider a steady state so that the collisional drag of the neutrals compensates for the electric field, and define the effective collision frequency ν_c as $\alpha \nu_c'$. We can now solve for the average drift motion

$$v = \left(\frac{e}{m \nu_c}\right) E = \mu E \tag{28}$$

where μ is called the mobility. If the drift velocity is much smaller than the thermal velocity, ν_c , and thus μ , will be independent of E . If the diffusion coefficient is divided by the mobility, we obtain the Einstein

relation

$$\frac{D}{\mu} = \frac{1}{3}\frac{v^2}{\nu_c}\left(\frac{m\nu_c}{e}\right) = \frac{1}{3}\frac{mv^2}{e} \tag{29}$$

which depends on kinetic energy only. If the particles are in thermal equilibrium, then

$$\frac{3}{2}KT = \frac{1}{2}mv^2 \tag{30}$$

and Eq. (29) becomes

$$\frac{D}{\mu} = \frac{kT}{e} \tag{31}$$

a result valid for Maxwell-Boltzmann distribution.

Since ν_c is directly proportional to p, we expect the drift velocity to be proportional to E/p so long as the drift velocity remains small compared with the thermal velocity. At high E/p this inequality is no longer satisfied and the drift velocity is no longer proportional to E/p.

This theory is obviously oversimplified. We have seen, for example, that the elastic collision frequency is a strong function of velocity. The electron drift velocity as a function of E/p is shown in Figs. 11 to 14.

Ion Mobility

Similar arguments hold for ion mobility. Once again for low E/p one expects the drift velocity to be proportional to E/p. Because of the greater mass, ion mobilities are of course far smaller than electron mobilities. The case of ions drifting through neutrals of the same kind — e.g., helium ions in helium atoms — is subject to charge exchange collision and the mobilities are smaller. (See Fig. 15.)

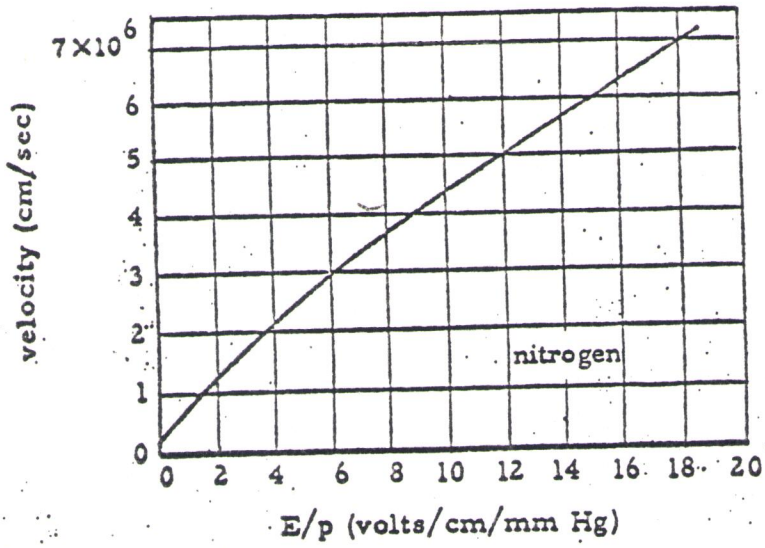


Fig. 11. Drift velocity of electrons in nitrogen as a function of E/p.

[R. A. Nielsen, Phys. Rev. 50, 950 (1936).
L. Colli and U. Facchini, Rev. Sci. Instr. 23, 39 (1952).]

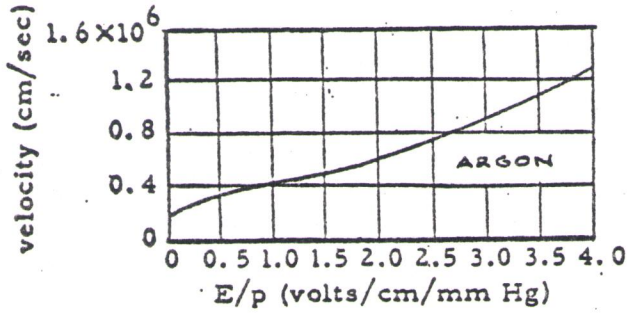


Fig. 12. Drift velocity of electrons in argon as a function of E/p.

[R. A. Nielsen, Phys. Rev. 50, 950 (1936).
L. Colli and U. Facchini, Rev. Sci. Instr. 23, 39 (1952).
J. M. Kirshner and D. S. Toffolo, J. Appl. Phys. 23, 594 (1952)]

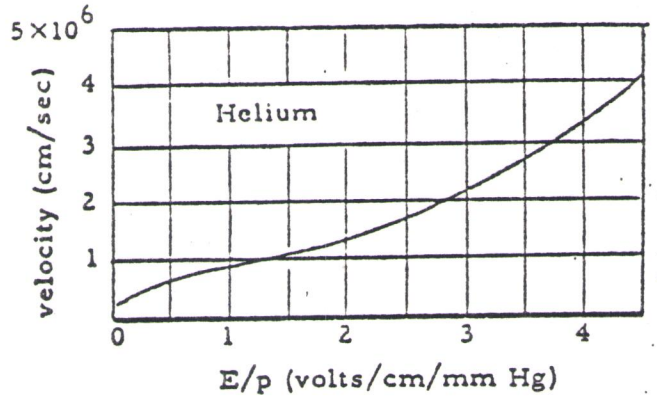


Fig. 13. Drift velocity of electrons in helium as a function of E/p.

[R. A. Nielsen, Phys. Rev. 50, 950 (1936).
J. A. Hornbeck, Phys. Rev. 83, 374 (1951).]

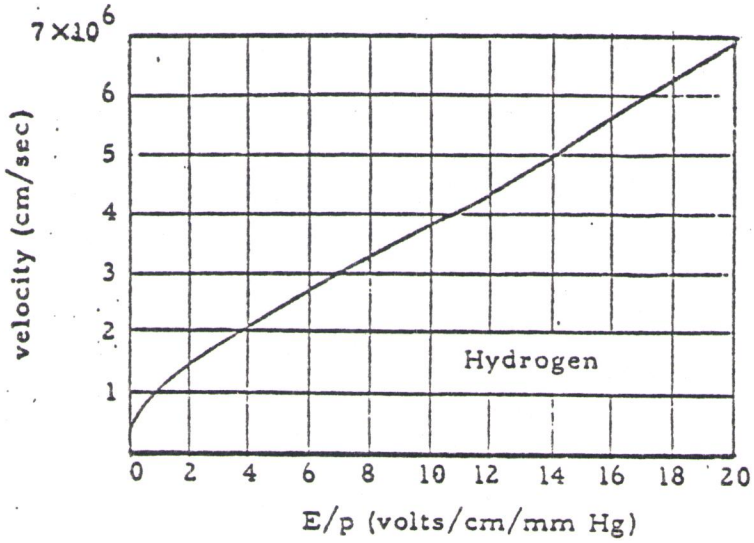


Fig. 14. Drift velocity of electrons in hydrogen as a function of E/p.

[N. E. Bradbury and R. A. Nielsen, Phys. Rev. 49, 388 (1936).]

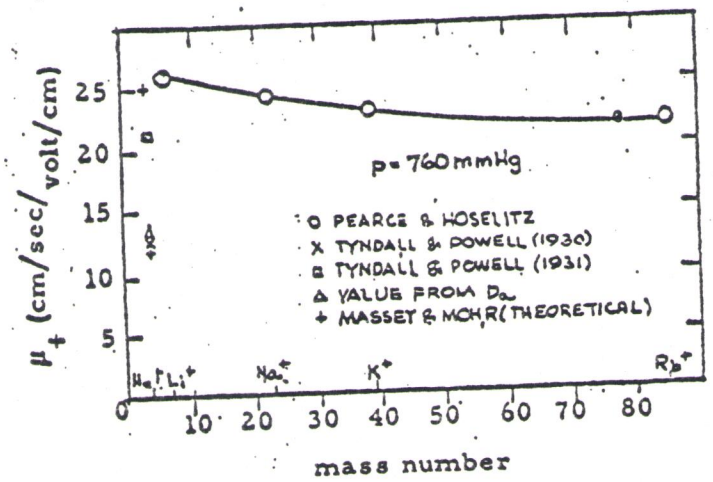


Fig. 15. Mobility of various positive ions in helium.

Charge Exchange

An ion passing through a gas may pick up an electron from a neutral atom (or from another ion). This is called charge exchange. In a partially ionized gas this produces a marked effect on both diffusion and mobility. For example, an ion drifting in an electric field may pick up an electron from a neutral. The neutralized particle continues with almost the same velocity and is replaced by an ion moving at the random velocity it had as a neutral. Thus the drifting ion is on the average replaced by a nondrifting ion. This process represents an additional collision mechanism which further reduces the average drift velocity.

An electron which transfers between like atoms requires no energy. A transfer between unlike atoms will involve a difference in energy difference designated ΔE. The electron orbital motion is very rapid compared with the time of impact between two heavy particles. The interaction is then adiabatic; the energy levels have a chance to readjust slowly.

The classical analogue is that of transfer to an oscillator of frequency ν by a pulse of length τ. The maximum excitation of the classical oscillator will take place if the pulse has a maximum Fourier amplitude at frequency ν. This will occur if τ is $1/\nu$. But τ is roughly a/v, where a is the range of the interaction; i.e.,

$$\tau\nu = 1 = \frac{a\nu}{v} . \tag{32}$$

To now transfer this to a quantum mechanical picture we replace ν by $\Delta E/h$. Then the charge exchange cross section will be a maximum if

$$a\,\Delta E/hv \sim 1 . \tag{33}$$

Thus, for like atoms the charge exchange cross section increases as the relative velocity decreases. This is usually termed resonant charge-exchange. For unlike atoms (nonresonant charge-exchange) the cross sections exhibit a maximum as shown in Figs. 16 and 17.

Ambipolar Diffusion

We consider a steady state plasma in a container. The electron and ion losses are assumed to be compensated by additional ionization processes taking place in the volume. First let us consider the loss processes before steady state is reached. If the initial electron density n_e is equal to the initial ion density n_i and $T_e \geq T_i$, the diffusion coefficient for electrons D_e being much greater than that for ions D_i, the initial loss of electrons to the wall exceeds the ion losses. This sets up an electric field toward the wall which decreases the electron loss rate and increases the ion loss rate. This process continues until the electric field has increased to a value which makes the ion and electron loss rates equal. This is the steady state referred to above. For the electrons and ions, we write for the fluxes

$$\Gamma_i = - D_i \nabla n_i + \mu_i E n_i \tag{34}$$

$$\Gamma_e = - D_e \nabla n_e - \mu_e E n_e \ . \tag{35}$$

The electric field here is the sheath field caused by the excess electron loss. There is no applied field. At steady state the total losses are equal. We assume that the fluxes of both particles are the same in all regions.

$$\Gamma_i = \Gamma_e = \Gamma_a \tag{36}$$

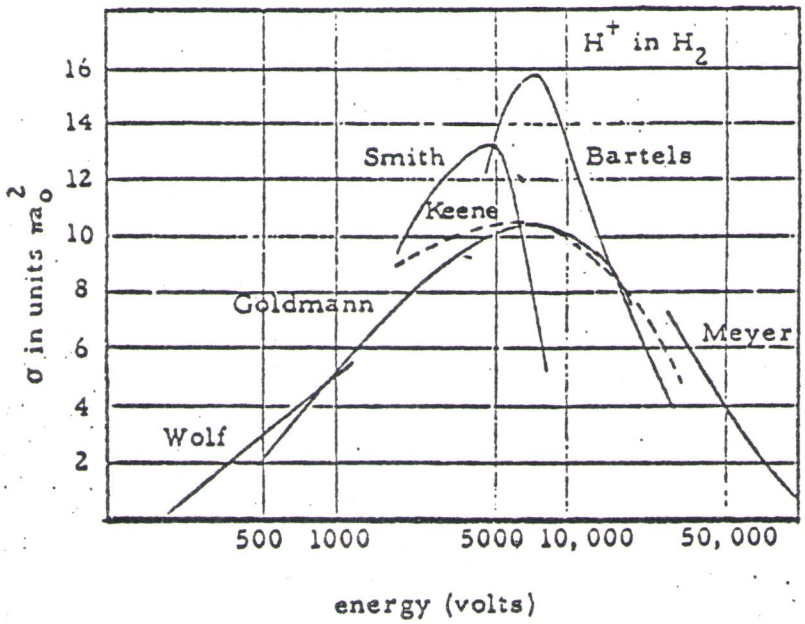


Fig. 16. Charge transfer cross-section of H^+ in H_2. [Massey and Burhop]

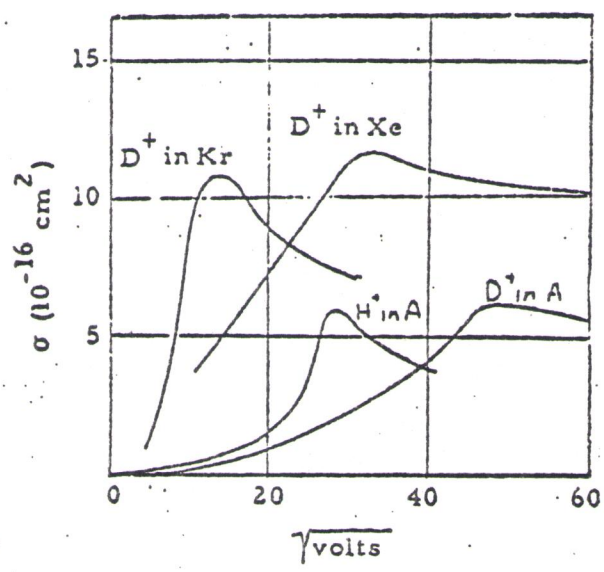


Fig. 17. Charge transfer cross-sections of H^+ in argon; D^+ in argon, krypton, and xenon.
[J. B. Hasted, Proc. Roy. Soc. (London) A212, 235 (1952), as show in Brown.]

using the subscript a to refer to ambipolar. We assume that $n_i \approx n_e = n$; that is, the ion and electron density are initially sufficiently large that the requisite electric field may be created by a slight inequality in n_i and n_e. Eqs. (34) and (35) become

$$\Gamma_a = - D_i \nabla n + \mu_i E n \tag{37}$$

$$\Gamma_a = - D_e \nabla n + \mu_e E n \ . \tag{38}$$

Eliminating E between Eqs. (37) and (38), we get

$$\Gamma_a = - D_a \nabla n \tag{39}$$

where

$$D_a = \left(\frac{D_i \mu_e + D_e \mu_i}{\mu_i + \mu_e} \right) \ . \tag{40}$$

From the Einstein relation we have

$$\frac{D_i}{\mu_i} = \frac{kT_i}{e} \ ; \qquad \frac{D_e}{\mu_e} = \frac{kT_e}{e} \ . \tag{41}$$

If we now assume that

$$\alpha T_i = T_e \tag{42}$$

where $\alpha \geq 1$,

$$\frac{D_i}{\mu_i} = \frac{1}{\alpha} \frac{D_e}{\mu_e} \ . \tag{43}$$

Using Eq. (43) to eliminate D_e from Eq. (40), and assuming $\mu_i << \mu_e$, we find

$$D_a = D_i (1 + \alpha) \ . \tag{44}$$

This is another way of saying that the loss rate of both types of charged particle is essentially determined by the loss rate of the slower.

Note that while we have given an effective diffusion coefficient, mobility is definitely involved. There is an electric field. From Eqs. (37) and (38) we can eliminate Γ_a, giving

$$E = \frac{\nabla n}{n}\left(\frac{D_i - D_e}{\mu_i + \mu_e}\right) \tag{45}$$

which under the approximations $D_i < D_e$, $\mu_i < \mu_e$ becomes

$$E = -\frac{\nabla n}{n}\frac{D_e}{\mu_e} = \frac{\nabla n}{n}\frac{kT_e}{e} \quad . \tag{46}$$

We assume that $(n_e - n_i) << n$. Let us find the conditions under which this is valid. From the Maxwell equation for the divergence of E, we have

$$\nabla \cdot E = 4\pi\, e(n_i - n_e) \sim \frac{E}{L} \, , \tag{47}$$

where L is a length characteristic of the region where $n_i \neq n_e$. From Eqs. (46) and (47),

$$E = L\, 4\pi e(n_i - n_e) = -\frac{\nabla n}{n}\frac{kT_e}{e} \quad . \tag{48}$$

We approximate ∇n as n/L. In steady state, any region where there is a gradient of the density ambipolar diffusion must be taking place; therefore the electric field must exist over the same region as the density gradient. We therefore use the same L here. Thus we have

$$\frac{n_i - n_e}{n} = -\frac{kT}{4\pi n e^2 L^2} = -\frac{\lambda_D^2}{L^2} \quad . \tag{49}$$

Thus the condition that $n_i - n_e << n$ becomes one that the Debye length be small compared with the region over which the density changes.

Breakdown and the First Townsend Coefficient

We consider a neutral gas of density n_g to which is applied a uniform electric field E, say in the $-x$ direction. At some point $x = x_o$ electrons are injected at density n_o. The electric field is sufficiently high that ionizing collisions take place, so that as the electron drifts in the x direction, n is a function of x.

Since each electron produces ν_i secondary electrons per second, and there are at any time n electrons per cubic centimeter, the rate of increase of n due to ionization is just $n\nu_i$, where $\nu_i = n_g \overline{\sigma v}$. Under the influence of the electric field, the electrons drift across the boundaries. The difference between the number entering and leaving unit volume per second is $\bar{v}\nabla n$, where $\bar{v}$ is a constant since E and p, and thus the mobility are constant. Thus, for a steady state

$$\nu_i n = \bar{v} \frac{dn}{dx} \tag{50}$$

$$n = n_o e^{\alpha x} \tag{51}$$

where

$$\alpha = \frac{\nu_i}{v} \tag{52}$$

and is called the first Townsend coefficient. $\bar{v}$ is a function of E/p. ν_i is proportional to pressure (at fixed $\bar{v}$). Thus, α/p is a function of E/p. Typical values are shown in Figs. 18 and 19.

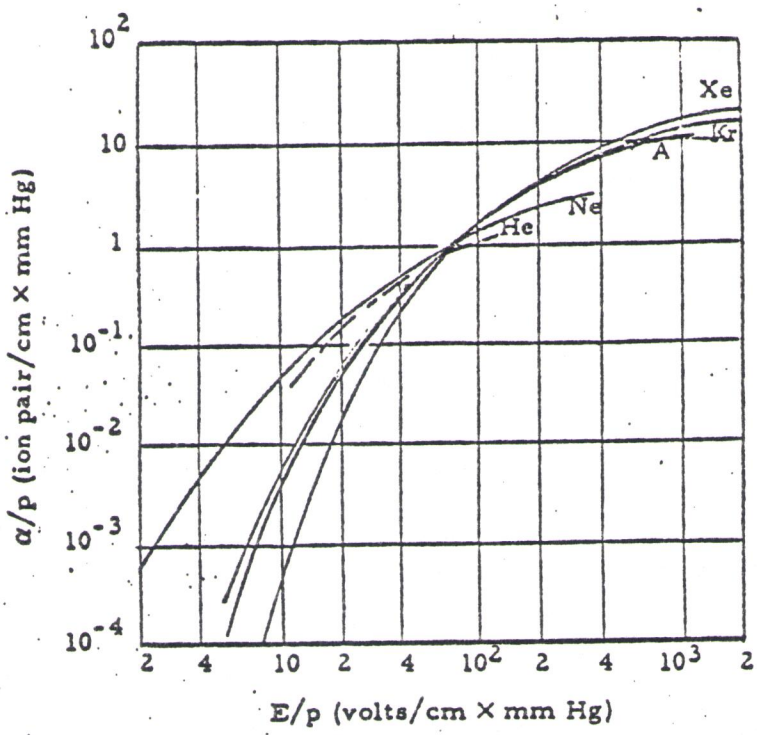


Fig. 18. First Townsend ionization coefficients in noble gases.
[A. von Engel, Handbuch der Physik, Springer Verlag, Berlin (1956) Vol. 21, p. 504, as shown in Brown.]

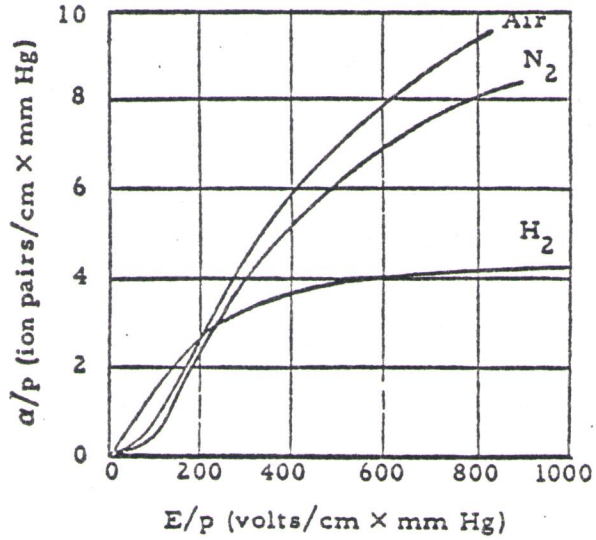


Fig. 19. First Townsend ionization coefficients of air, N_2 and H_2.
[A. von Engel, Handbuch der Physik, Springer Verlag, Berlin (1956) Vol. 21, p. 504, as shown in Brown.]

This, of course, applies to dc discharges. When ac fields are applied the dominant losses may be quite different. In an ac field the particle position oscillates with a limited amplitude (in the absence of collisions). If the dimension (along E) is large compared with this amplitude, mobility losses no longer dominate. Diffusion becomes dominant. See, for example, Brown.

We have shown how an initial density of electrons n_o builds up exponentially in space. In order to effect breakdown, a feedback mechanism is needed. Consider, for example, a discharge between electrodes separated by a distance d . Ions striking the cathode may produce secondary emission as one feedback mechanism. Assuming a cascading density as in Eq. (51), the ion production per unit time in the whole length is

$$\int_o^d n(x)\,\nu_i\,dx = \nu_i \frac{n_o}{\alpha}(e^{\alpha d} - 1) \tag{53}$$

where n_o is the total number at the cathode. Then if γ_i is the ratio of secondary electron production rate to the column ion production rate, the rate of secondary electron production is

$$\Gamma_e = \gamma_i \nu_i \frac{n_o}{\alpha}(e^{\alpha d} - 1) \ . \tag{54}$$

If n_s is the number of secondaries per second, then since $\Gamma_e = n_s \bar{v}_e$

$$n_s = \gamma_i n_o (e^{\alpha d} - 1) \ . \tag{55}$$

Then is n_o' is the initiating flux (perhaps one electron),

$$n_o = \gamma_i n_o (e^{\alpha d} - 1) + n_o' \tag{56}$$

$$n(d) = \frac{n_o' e^{\alpha d}}{1 - \gamma_i(e^{\alpha d} - 1)} \tag{57}$$

If E/p is increased, α increases and the condition may be achieved that the denominator of Eq. (57) approaches zero,

$$1 - \gamma_i(e^{\alpha d} - 1) = 0 \tag{58}$$

thus effecting breakdown. An additional feedback mechanism is photo emission from the cathode. The sum of γ_i for secondary emission and photo emission is called the second Townsend coefficient. The value of the quantity γ_i depends on the nature of the electrodes and on the geometry.

If we plot the voltage required for breakdown between two electrodes against the product of pressure p times electrode separation d, the so-called Paschen curve results. Fig. 20 shows the typical form of this curve. For small pd there is very little cascading. More secondary emission is needed so a larger potential is required.

This is an approximate description of one type of breakdown. It obviously does not apply to discharges which are dominated by other factors such diffusion or negative ion formation.

Electron-Ion Recombination

Electron-ion recombination is an important process in low-temperature plasmas such as gas discharges or the ionosphere. Recombination of an electron and an atomic ion may be quite a complex process.. An ion may capture an electron in the ground or excited state. The excess energy involved in the process may be radiated (radiative recombination).