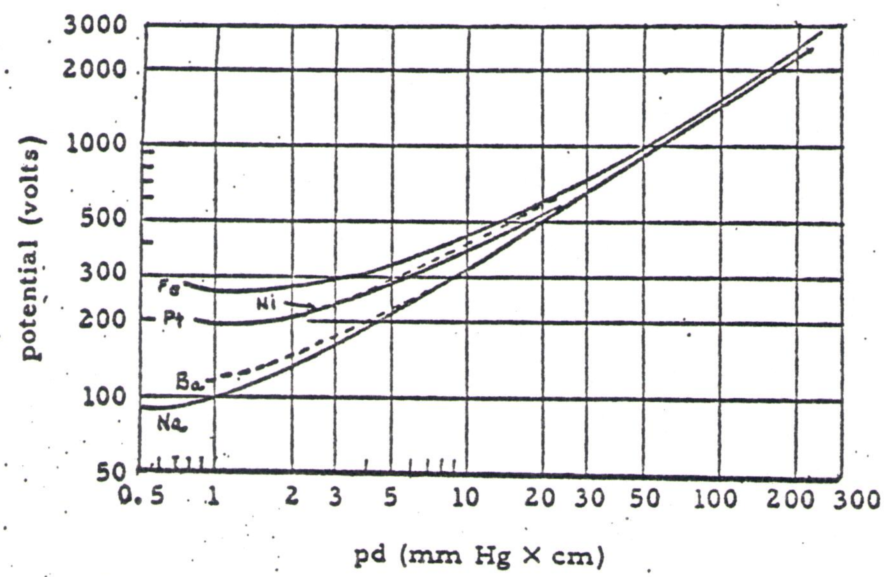

Fig. 20. Breakdown potential in argon between plates for various cathode materials.

[As shown in Brown.]

$$A^{+} + e \rightarrow A + h\nu \tag{59}$$

or may be given to a third particle (three-body recombination). The case where the third body is another electron is of particular interest.

$$A^{+} + e + e \rightarrow A + e \tag{60}$$

There are many cases of low-temperature plasmas where molecular ions are formed which may recombine and dissociate (dissociative recombination).

$$AB^{+} + e \rightarrow A + B \tag{61}$$

the excess energy being carried off as kinetic energy of the atoms.

The process by which an ion and free electron end up in the ground state may be quite complex. The electron may be captured in the ground or in an excited state, from whence it may cascade down, emitting additional radiation, or it may be raised to a higher level or re-ionized by a photon; it may engage in additional collisions which result in transitions up or down. Instead of using cross sections, it is customary to make use of a recombination coefficient α defined by

$$\frac{d n_i}{dt} = \alpha n_i n_e \tag{62}$$

Thus α is essentially $\overline{\sigma v}$ for the process.

At low electron densities, three-body and electron collisions with excited atoms are negligible; then radiative recombination would be expected to dominate.

Calculations for radiative recombination in hydrogen yield approximately

$$\alpha \sim 6 \times 10^{-11}\ T^{-\frac{1}{2}} \qquad (T \text{ in } {}^\circ K) \tag{63}$$

at temperatures below 1 eV, falling somewhat more rapidly as the temperature rises. Thus at 1 eV the cross section for radiative recombination is about

$$\frac{\alpha}{v} = \sigma = \frac{6 \times 10^{-11}}{5 \times 10^{7}\ 10^{2}} \sim 10^{-20} \tag{64}$$

At higher pressures other processes become dominant. Measured dissociative recombination coefficients, at about 400°K are given in Table II. Little information is available on temperature dependence.

In the case of three-body recombination, α must depend on density. For $n_e > 10^{10}$, an approximate equation for α (three-body) is

$$\alpha \sim 3 \times 10^{-22}\, n \left(\frac{T}{1000}\right)^{4/5} \tag{65}$$

More exact calculated values are given in Table III.

IV. Particle Orbits

We have seen that interactions between individual pairs of charged particles are to some good approximation negligible. On the other hand, when many particles interact the interactions may be important. The fields due to large numbers of charged particles will be smooth on the average. There will be small fluctuations due to the presence of individual particles, but these are more or less unimportant. Thus we may approximate what goes on by treating the system as a collection of particles, each one moving in the smoothed-out fields of all the other particles plus, of

Table I. Relative probabilities of different types of collision of electrons in atomic hydrogen (Massey and Burhop)

Type of Collision	Energy of incident electrons (eV) 100	200	400	1,000	10,000
	Percentage of all collisions				
Elastic	12.2	10.2	9.8	8.7	6.5
Excitation of 2-quantum levels	33.5	33.6	39.0	42.8	45.3
Excitation of 3-quantum levels	5.9	5.8	6.8	6.3	7.0
Excitation of 4-quantum levels	2.2	2.0	2.2	2.4	2.6
Excitation of 5-quantum levels	1.0	0.9	1.0	1.2	1.2
Excitation of higher quantum levels	1.7	1.7	2.0	2.2	2.3
All discrete levels	44.3	44.0	51.0	54.8	58.4
Ionization	43.5	45.8	39.2	36.5	35.1
Total cross section (units πa_o^2)	2.45	1.50	0.79	0.37	0.049

Table II. Dissociative Recombination Coefficients (Measured)

He	10^{-8}	H	3×10^{-8}
Ne	2×10^{-7}	N_2^+	6×10^{-7}
A	7×10^{-7}	N_3^+	$\sim 2 \times 10^{-6}$
Kr	3×10^{-7}	N_4^+	$\sim 2 \times 10^{-6}$
Xe	2×10^{-6}		

Table III.

n	250°K	1,000°K	4,000°K	16,000°K	64,000°K
10^8	8.8×10^{-11}	4.1×10^{-12}	9.2×10^{-13}	3×10^{-13}	10^{-13}
10^{10}	2.8×10^{-9}	1.9×10^{-11}	1.4×10^{-12}	3.2×10^{-13}	10^{-13}
10^{12}	2.6×10^{-7}	3.9×10^{-10}	4.4×10^{-12}	4.3×10^{-13}	10^{-13}
10^{14}	2.6×10^{-5}	2.9×10^{-8}	5.1×10^{-11}	1×10^{-12}	1.2×10^{-13}
10^{16}	2.6×10^{-3}	2.9×10^{-6}	2.3×10^{-9}	5×10^{-12}	1.9×10^{-13}

(at low temperature, $\alpha \propto n$)

course, any externally-applied fields. One can gain much insight into the behavior of plasmas by investigating the motion of single charged particles in artibrary electric and magnetic fields. One can go further and make the fields consistent with the motion of all the particles. That is, we must determine the fields from Maxwell's equations and the particle motion from the Lorentz force acting on a particle. Maxwell's equations are

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}}{c} \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \tag{4}$$

From Eqs. (2) and (4) we get the equation of continuity

$$\dot{\rho} + \vec{\nabla} \cdot \vec{j} = 0 \tag{5}$$

where

$$\rho = \sum_i q_i \overline{n_i} \tag{6}$$

$$\vec{j} = \sum_i q_i \overline{n_i \vec{v}_i} \tag{7}$$

while the Lorentz force law reads

$$m_i \frac{d\vec{v}_i}{dt} = q_i(\vec{E}_i + \vec{v}_i \times \vec{B}) \tag{8}$$

Here ρ and j are the charge and current densities due to the plasma particles, i refers to the i^{th} species, $\overline{n_i}$ is the number density of the i^{th} species at a point r . These are the so-called Vlasov equations for a plasma. We will take these up later. Here we shall start by investigating single particle motions.

A. Cyclotron Motion

The equation of motion for a charged particle in an electric and magnetic field is

$$m \frac{d\vec{v}}{dt} = q\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) \tag{9}$$

We can immediately obtain the energy conservation law by dotting both sides with $\vec{v}$ and integrating with respect to time

$$m\vec{v} \cdot \frac{d\vec{v}}{dt} = q\,\vec{v} \cdot \vec{E} \tag{10}$$

$$\frac{d}{dt}\,\frac{m\vec{v} \cdot \vec{v}}{2} = q\,\vec{v} \cdot \vec{E} \tag{11}$$

or

$$\Delta\left(\frac{m\,\vec{v} \cdot \vec{v}}{2}\right) = \int q\,\vec{v} \cdot \vec{E}\,dt = \int_{\text{orbit}} q\,\vec{E} \cdot d\vec{s} \tag{12}$$

where $d\vec{s}$ is a vector element of the orbit. If E is electrostatic and hence derivable from a potential, this Eq. (12) may be written in the form

$$\frac{mv^2}{2} + q\phi = \text{constant} \tag{13}$$

$$\vec{E} = -\vec{\nabla}\phi$$

Eq. (13) says that the change in the kinetic energy is equal to the work done by the electric field. The magnetic field does no work on the particle, since the force it exerts on the particle is always perpendicular to the velocity.

Returning now to Eq. (9), the solution of this equation for arbitrary E and B would in general be very complicated. We shall therefore look at some simple situations out of which we could build up more complicated situations.

The simplest possible situation is, of course, that of spatially uniform E and B fields. Let us first consider the case of a uniform B. Since the magnetic force is perpendicular to both v and B, there is no force on the particle in the direction of B and the velocity in this direction is constant. We need not consider this velocity any more. The velocity perpendicular to the magnetic field is constant in magnitude. One can obtain the radius of this circle by balancing the centrifugal force against the magnetic force.

$$\frac{m v_\perp^2}{r} = \frac{q v_\perp B}{c} \tag{14}$$

or

$$r = \frac{v_\perp mc}{qB} \tag{15}$$

The quantity qB/mc is the angular frequency of the particle and is called the cyclotron frequency

$$\omega_c = \frac{qB}{mc} \tag{16}$$

The radius r is called the Larmor radius.

If one wishes to be more formal in solving for the motion, one can proceed as follows. Let the direction of B be the z direction. Then $v_\perp$ has x and y components and Eq. (9) can be written in the form

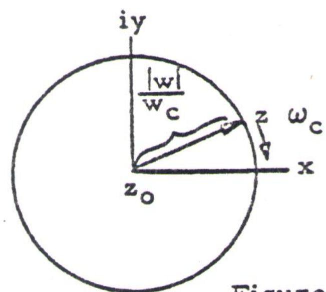

Figure 21

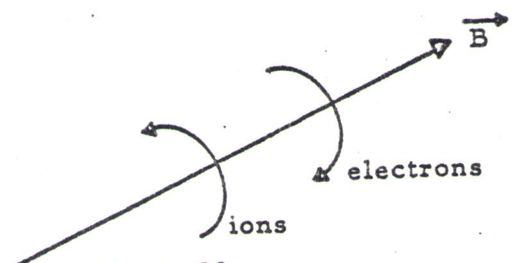

Figure 22

Problem: Find the cyclotron frequency for electrons and protons in a field of 10,000 gauss; in a field of .1 gauss. The latter is roughly the strength of the earth's field at 4,000 miles. Compute the radius of gyration of a 1 MeV proton in a field of .1 gauss.

B. Magnetic Moment

A current loop has a dipole moment associated with it of magnitude

$$\mu = \frac{IA}{c} \tag{24}$$

where I is the current (in esu units) and A is the area of the loop. The circular orbit of a charge in a magnetic field is the area of the loop. The circular orbit of a charge in a magnetic field on the average constitutes a current loop. The average current is the average charge per unit time which passes a point on the orbit. This is $\frac{1}{\tau} q/c$, where τ is the period. The magnetic moment is thus

$$\mu = \frac{q}{c\tau}\pi r^2 = \frac{q\omega_c}{c\,2\pi}\pi\frac{v_\perp^2}{\omega_c^2} = \frac{\pi q}{2\pi c}\frac{v_\perp^2}{qB/mc} \tag{25}$$

$$= mv^2/2B = W_\perp/B$$

Here $W_\perp$ is the energy of the particle due to its perpendicular motion.

In addition to the magnitude $W_\perp/B$, the magnetic moment has a direction associated with it — the direction of the magnetic dipole with the equivalent magnetic moment. From Fig. 22 and the right-hand rule, we see that the current loop is such as to reduce the field inside of itself, and hence the magnetic moment has a direction opposite to the direction of the field.

$$\vec{\mu} = -\frac{W_\perp}{B^2}\vec{B} \tag{26}$$

C. Magnetization

In a plasma containing many particles the magnetic field produced by all the magnetic moments can be appreciable. To compute this effect we must make use of the Maxwell equation

$$\vec{\nabla} \times \vec{B} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}}{c} \tag{27}$$

First we shall assume that all quantities are time-independent, or at least vary so slowly that we can neglect the $\partial E/\partial t$ term. Second, for the time being we imagine that j is produced only by plasma particles. Equation (27) may be written in the integral form

$$\oint_c \vec{B} \cdot d\vec{\ell} = \int_s \vec{\nabla} \times \vec{B} \cdot d\vec{A} = \frac{4\pi}{c} \int_s \vec{j} \cdot d\vec{A} \tag{28}$$

where c is a curve bounding an area s, $d\vec{\ell}$ is an element of the curve c, and $d\vec{A}$ is a vector element of the surface s and has the direction of the normal to s [Eq. (28) follows from the Stokes theorem]. To compute the integral on j in Eq. (28) we must find the current normal to s or the total charge crossing s per unit time. Consider the situation shown in Fig. 23.

Let us compute the total charge crossing s per unit time. Orbits like 2, which intersect s twice, give no transfer of charge across s. On the other hand, orbits like 1 and 3, which intersect s only once, transfer an amount of charge q (the charge on the particle) every time they cross s. Thus only those orbits which loop the curve c contribute to the current through s. Now they transfer the charge q (the charge on a particle) for each revolution, or they transfer charge at the rate q/τ_c.

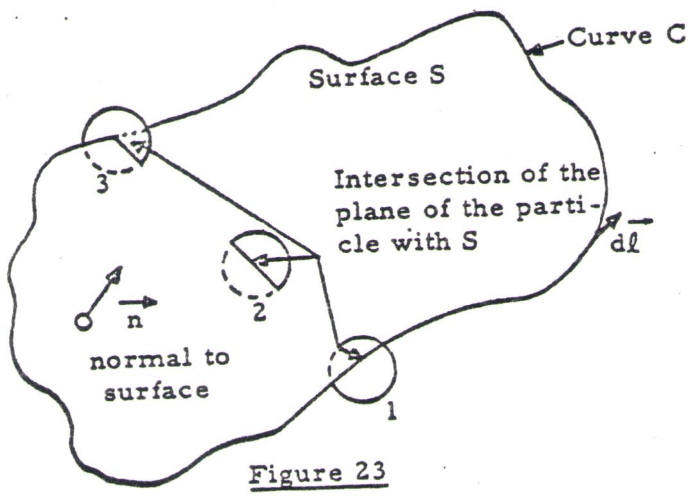
Curve C
Surface S
Intersection of the
plane of the parti-
cle with S
$\vec{dl}$
3
2
1
$\vec{n}$
normal to
surface

Figure 23

The number of orbits which loop c in a length $|d\vec{\ell}|$ is the density of such orbits times the areas of such orbits normal to $d\vec{\ell}$, times $|d\vec{\ell}|$ $(q/\tau_c)a_n |d\vec{\ell}| = Ia_n |d\vec{\ell}|$.

Now we must take account of whether charge is transferred across the surface in the positive or negative direction. We must compute the net charge crossing in the positive direction. If the magnetic moment $\vec{\mu} = \frac{I\vec{A}}{c}$ is in the direction of $d\vec{\ell}$ then the current crosses the surface in the positive direction, while if $\vec{\mu}$ is opposite to $d\vec{\ell}$ the current crosses the surface in the negative direction (a little consideration using the right-hand rule shows this). Thus the total current crossing s is given by

$$\int_s \vec{j} \cdot d\vec{A} = \int_c NI\vec{a} \cdot d\vec{\ell} = c\int_c N\vec{\mu} \cdot d\vec{\ell} \tag{29}$$

Eq. (28) may thus be written in the form

$$\int_c \vec{B} \cdot d\vec{\ell} = 4\pi \int_c N\vec{\mu} \cdot d\vec{\ell} = \int_s \vec{\nabla} \times \vec{B} \cdot d\vec{A} = 4\pi \int_s \vec{\nabla} \times N\vec{\mu} \cdot d\vec{A} \tag{30}$$

Writing $\vec{M} = N\vec{\mu}$, we may write

$$\int_s \vec{\nabla} \times \vec{B} \cdot d\vec{A} = 4\pi \int_s \vec{\nabla} \times \vec{M} \cdot d\vec{A} \tag{31}$$

or

$$\vec{\nabla} \times (\vec{B} - 4\pi \vec{M}) = 0$$

If we do not neglect $\partial\vec{E}/\partial t$, and if there are currents other than those contributed by the plasma particles (let us denote such currents by j_e), then we can proceed in a similar manner and we obtain Eq. (32) in place of Eq. (31).

$$\vec{\nabla} \times (\vec{B} - 4\pi \vec{M}) = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}_e}{c} \tag{32}$$

In the classical treatment of magnetic materials, $\vec{B} - 4\pi\vec{M}$ would have been called $\vec{H}$, and Eq. (32) would then read

$$\vec{\nabla} \times \vec{H} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}_e}{c} \tag{33}$$

For normal materials and small magnetic fields, the magnetization is proportional to B

$$\vec{M} = \alpha \vec{B} \tag{34}$$

so that H is also proportional to B . Here, however,

$$\vec{M} = -\frac{N W_\perp \vec{B}}{B^2} \tag{35}$$

and is proportional to $1/B$. Thus α, and also the magnetic permeability $[1/(1-4\pi\alpha)]$ are not constant. H is not useful; we will use only B .

We may substitute Eq. (35) in Eq. (32) and obtain

$$\vec{\nabla} \times \left[\vec{B}\left(1 - 4\pi\frac{N W_\perp}{B^2}\right)\right] = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}_e}{c} \tag{36}$$

From this we see that the particles begin to make an important contribution to the field when their energy density $N W_\perp$ becomes comparable to the energy density of the magnetic field $B^2/8\pi$.

D. The Electric Field Drift

Next in simplicity to the constant magnetic field case is the situation in which we have constant electric and magnetic fields. From Eq. (9) we have for the component of the motion along the magnetic field

$$m\frac{d v_\parallel}{dt} = q E_\parallel \tag{37}$$

We can integrate this equation to obtain

$$v_{\parallel} = \frac{qE_{\parallel}}{m} t + v_{\parallel o} \tag{38}$$

$$x_{\parallel} = \frac{qE_{\parallel}}{m}\frac{t^2}{2} + v_{\parallel o}t + x_{\parallel o} \tag{39}$$

Thus the particle freely accelerates along the field.

The equation of motion for the components of $\vec{v}$ perpendicular to $\vec{B}$ is

$$m\frac{d\vec{v}_{\perp}}{dt} = q\left(\vec{E}_{\perp} + \frac{\vec{v}_{\perp}\times\vec{B}}{c}\right) \tag{40}$$

Now in this equation both the electric force and the magnetic force are perpendicular to $\vec{B}$ and it is possible to balance them. If they are balanced, $d\vec{v}_{\perp}/dt$ is 0 , and the particle moves with a constant perpendicular velocity. Let us equate the electric and magnetic forces

$$\vec{E}_{\perp} + \frac{\vec{v}_{\perp}\times\vec{B}}{c} = 0 \tag{41}$$

Crossing this with $\vec{B}$ gives

$$\vec{B}\times\vec{E}_{\perp} + \frac{\vec{B}\times(\vec{v}_{\perp}\times\vec{B})}{c} = \vec{B}\times\vec{E}_{\perp} + \frac{\vec{v}_{\perp}B^2}{c} = 0 \tag{42}$$

or solving for $\vec{v}_{\perp}$

$$\vec{v}_{\perp} = \frac{(\vec{E}_{\perp}\times\vec{B})c}{B^2} \tag{43}$$

Let us denote this velocity by $\vec{v}_E$ and write for $\vec{v}_{\perp}$

$$\vec{v}_{\perp} = \vec{v}_1 + \vec{v}_E \tag{44}$$

Substituting in Eq. (9) the $\vec{v}_E \times \vec{B}$ term cancels the E term and we get

$$m \frac{d\vec{v}_1}{dt} = q \frac{\vec{v}_1 \times \vec{B}}{c} \qquad (45)$$

This is the same equation that one gets without an E field, and hence $\vec{v}_1$ rotates at a constant rate. The motion of the particle is thus a drift with a uniform velocity v_E plus a rotation about the magnetic field lines. We should note that the drift velocity is independent of the charge.

Motion across a magnetic field gives rise to an E field. The transformation law for E in going from one frame to another, moving with a velocity v relative to it, is (for velocities small compared to light)

$$\vec{E}' = \vec{E} + \frac{\vec{v} \times \vec{B}}{c} \qquad (46)$$

For $v = v_E$ the right side vanishes: in the frame moving with the drift velocity there is no electric field and hence the particle sees only a magnetic field and moves accordingly.

We may also view the drift in another way, which is illustrated in Fig. 24.

As a positive charge spirals about the magnetic field its energy changes due to the E field. It moves faster on the upper part of its orbit and the curvature is smaller here than on the lower part of its orbit; hence the drift. For a negative charge, on the other hand, the velocity is larger on the lower part of the orbit, but since the direction of rotation is opposite to that for a positive charge, the resultant drift is in the same direction.

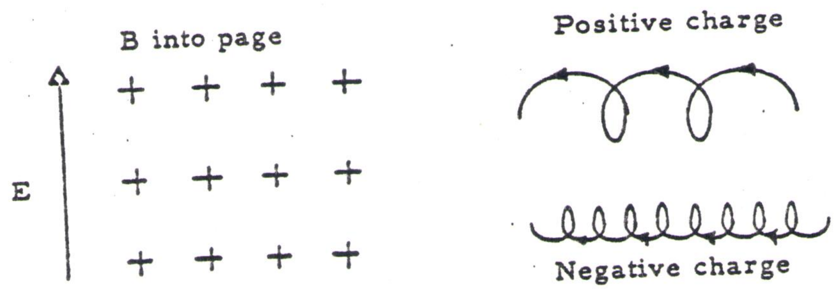

Figure 24

For a neutral plasma, since the two types of charges drift in the same direction at the same rate, there is no current. No net work is done on either type of particle, since they drift perpendicular to E .

E. The Effect of Gravity or Other External Force

If a uniform gravitational force or other external force acts on the particle in addition to the magnetic force, then the equation of motion for the particle is given by

$$m \frac{d\vec{v}}{dt} = \vec{F} + \frac{q}{c} \vec{v} \times \vec{B} \tag{47}$$

where $\vec{F}$ is the external force. We may replace $\vec{F}$ by an equivalent electric field force.

$$\vec{F} = q\vec{E}$$

or

$$\vec{E} = \vec{F}/q \tag{48}$$

and take over all our results from the previous section. The particle drifts across the magnetic field with the equivalent of an $\vec{E} \times \vec{B}$ velocity, which is now

$$\vec{v}_F = \frac{c}{q} \frac{\vec{F} \times \vec{B}}{B^2} \tag{49}$$

Superimposed upon this drift is the usual Larmor motion about the magnetic field. If $\vec{F}$ has a component parallel to $\vec{B}$, the particle accelerates in this direction at a constant rate. We see from Eq. (49) that the direction of drift (unlike that found for the E field case) depends on the sign of the charge on the particle. Thus an external force acting on a neutral cloud will cause charges of different sign to drift in opposite directions, giving rise to a current. The current gives rise to a $j \times B$ force which balances the external force.

Problem: Prove the statement in the last sentence of part E.

F. The Effect of a Time-Varying Electric Field

Let us imagine that our particle is subject to a spatially uniform B field and a spatially uniform E which has a constant direction in space, but whose magnitude varies in time. We will take the E field to be perpendicular to B since the magnetic field plays no role in the motion of a particle parallel to it. We shall further assume that the electric field is changing at a rate which is slow by comparison with the Larmor motion. That is, we assume

$$\frac{1}{E}\,\frac{1}{\omega_c}\,\frac{dE}{dt} << 1 \tag{50}$$

We will first derive what happens on simple physical grounds. Since the E field is changing slowly with time, to a first approximation the particle will move with an instantaneous $E \times B$ drift velocity plus Larmor motion. Thus $\vec{v}_\perp$ is given by

$$\vec{v}_\perp = \frac{c\,\vec{E}(t) \times \vec{B}}{B^2} + \vec{v}_{Larmor} \tag{51}$$

Now, since the drift velocity changes with time, so does the kinetic energy of the particle. Let us average this change over one Larmor period so as to remove the periodic changes already discussed. We then get

$$\frac{d}{dt}\frac{m}{2}v_{\perp}^{2} = m\vec{v}_E \cdot \frac{d\vec{v}_E}{dt} = m\frac{c^2}{B^2}\frac{dE^2/2}{dt} \tag{52}$$

The last equality follows from the fact that we assumed that E was constant in direction and perpendicular to B . Now this energy must be supplied by the electric field or

$$q v_{\parallel E} E = \frac{c^2 m}{B^2} E \frac{dE}{dt} \tag{53}$$

$$v_{\parallel E} = \frac{c^2 m}{q B^2}\frac{dE}{dt} \tag{54}$$

where $v_{\parallel E}$ denotes the component of the velocity parallel to E .

We may derive this result from still another physical argument. The sum of all the forces on a particle (including inertial forces) must be zero. Now we may treat the inertial force due to the changing $E \times B$ velocity as an external force; a gravitational force, if one wishes. Then according to our previous analysis of the drift of a particle subject to an external force, the particle will acquire a drift velocity given by

$$\vec{v}_F = c\frac{\vec{F}\times\vec{B}}{qB^2} \tag{55}$$

Substituting for $\vec{F}$ $- m\,dv/dt$ (the inertial force or effective gravitational force is in the opposite direction to the acceleration):

$$\vec{F} = -\frac{mc}{B^2}\frac{d\vec{E}}{dt}\times\vec{B} \tag{56}$$

and we find for $\vec{v}_{\frac{dE}{dt}} \equiv v_p$

$$\vec{v}_p = -\frac{mc^2}{qB^2}\left(\frac{d\vec{E}}{dt} \times \vec{B}\right) \times \vec{B}/B^2 \tag{57}$$

or

$$\vec{v}_p = +\frac{mc^2}{qB^2}\frac{d\vec{E}}{dt} \tag{58}$$

since by assumption E is perpendicular to B . This method has the advantage that it applies even if the direction of E is changing with time, but E must still remain perpendicular to B . This approach also applies only when E varies slowly on the scale of the Larmor frequency, for only then can we neglect the Larmor motion and the forces associated with it.

Finally, we may derive these results formally from Eq. (9). Again we write

$$\vec{v}_\perp = \vec{v}_E + \vec{v}_1 \tag{59}$$

$$\vec{v}_E = c\,\frac{\vec{E} \times \vec{B}}{B^2} \tag{60}$$

Substituting in Eq. (9) gives

$$m\left(\frac{d\vec{v}_E}{dt} + \frac{d\vec{v}_1}{dt}\right) = \frac{q}{c}\,\vec{v}_1 \times \vec{B} \tag{61}$$

We set $v_1 = v_2 + v_p$, so that Eq. (61) becomes

$$m\frac{dv_E}{dt} + \frac{m\,dv_2}{dt} + \frac{m\,dv_p}{dt} = \frac{q}{c}v_2 \times B + \frac{q}{c}v_p \times B \tag{62}$$

and in a manner similar to the previous case we define v_p so as to cancel the $m\frac{dv_E}{dt}$ term, i.e.,

$$v_p \equiv \frac{mc^2}{qB^2}\frac{dE}{dt} \tag{63}$$

so that

$$\frac{q}{c}\vec{V}_p \times \vec{B} = \frac{q\,mc^2}{c\,q\,B^2}\frac{d\vec{E}}{dt}\times\vec{B} = mc\,\frac{\frac{d\vec{E}}{dt}\times\vec{B}}{B^2} = m\,\frac{d\vec{V}_E}{dt} \tag{64}$$

Then Eq. (62) becomes

$$m\frac{d\vec{V}_p}{dt} + m\frac{d\vec{V}_2}{dt} = \frac{q}{c}\vec{V}_2\times\vec{B} \tag{65}$$

If the first term is negligible, then as before $\vec{v}_2$ describes the Larmor motion in a frame moving with velocity $\vec{v}_p + \vec{v}_E$. Essentially we are utilizing a Taylor expansion (in time) of the electric field, having thus far found drifts corresponding to $\vec{E}$ and $d\vec{E}/dt$; the remaining term, $d\vec{v}_p/dt$ is a $d^2\vec{E}/dt^2$ term.

The $d\vec{v}_p/dt$ may be dropped provided that

$$\left| m\frac{d\vec{V}_p}{dt}\right| \ll \left|\frac{q}{c}\vec{V}_2\times\vec{B}\right| \tag{66}$$

or

$$\left|\frac{mc^2}{qB^2}\frac{d^2\vec{E}}{dt^2}\right| \ll \left|\frac{q}{c}\vec{V}_2\times\vec{B}\right| \quad \text{and,} \tag{67}$$

assuming that $\vec{E} \sim \vec{E}e^{i\omega t}$,

$$1 \ll \frac{q^2B^2}{m^2c^2}\left(\frac{B}{cE}\right)\frac{V_2}{\omega^2} \tag{68}$$

or

$$1 \ll \frac{\omega_c^2}{\omega^2}\frac{V_2}{V_E}. \tag{69}$$

Thus the next higher term may be dropped, unless $v_E \gg v_2$, or ω is comparable with ω_c .

Problem: By setting $\vec{v}_2 = \vec{v}_3 + \vec{v}_p$, find the drift corresponding to d^2E/dt^2 .

G. The Effective Dielectric Constant of a Plasma in a Magnetic Field

We have just seen that when a time-varying electric field is applied to a plasma in a magnetic field ($\vec{E} \perp \vec{B}$), particle drifts parallel to E arise, given by Eq. (58). The drifts are opposite for oppositely charged particles, so that a current arises in the plasma. The work that the $\vec{E}$ field does on this current is just what is required to get the plasma moving with the $c\,\dfrac{\vec{E} \times \vec{B}}{B^2}$ velocity.

The currents in the direction of $\vec{E}$ may be thought of as polarization currents; the plasma becomes polarized in this direction (Fig. 25).

To treat the plasma like a dielectric we divide the current into a plasma current and into currents due to external sources, in a manner analogous to the case of magnetization.

$$\vec{J} = \vec{J}_p + \vec{J}_e. \tag{70}$$

We have

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}_e + \frac{4\pi}{c}\vec{J}_p + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}. \tag{71}$$

Now if the plasma current is proportional to $\dfrac{\partial \vec{E}}{\partial t}$, as we have just found it to be when $\vec{E}$ is perpendicular to $\vec{B}$, then we can combine the last two terms on the right-hand side of Eq. (71)

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}_e + \left[\frac{4\pi c}{B^2}\sum_i n_i m_i + \frac{1}{c}\right]\frac{\partial \vec{E}}{\partial t} \tag{72}$$

(This only applies if $\vec{E}$ is perpendicular to $\vec{B}$, $\vec{J}_p = \frac{c^2}{B^2}\sum_i n_i m_i \frac{\partial \vec{E}}{\partial t}$).

We may set

$$\vec{D} = \left[\frac{4\pi\rho c^2}{B^2} + 1\right]\vec{E} = \epsilon \vec{E} \tag{73}$$

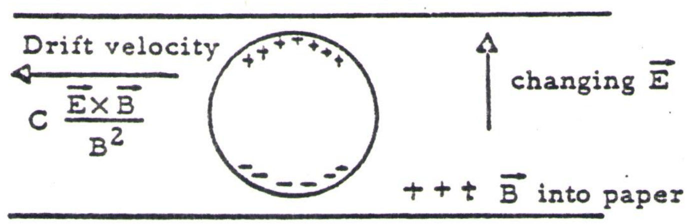
Drift velocity
$c\,\frac{\vec{E}\times\vec{B}}{B^2}$
changing $\vec{E}$
+ + + $\vec{B}$ into paper

Figure 25

and then Eq. (72) reads

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}_c + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}. \tag{74}$$

Further, we may divide the charge density into an internal and an external part.

$$\sigma = \sigma_p + \sigma_e \tag{75}$$

where σ_p has its sources in the charges within the plasma and is related to $\vec{j}_p$ by the continuity equation

$$\frac{\partial \sigma_p}{\partial t} + \vec{\nabla} \cdot \vec{j}_p = 0. \tag{76}$$

We then have

$$\vec{\nabla} \cdot \vec{E} = 4\pi(\sigma_p + \sigma_e) \tag{77}$$

and

$$\frac{\partial \sigma_p}{\partial t} + \vec{\nabla} \cdot \left[\frac{\rho c^2}{B^2} \frac{\partial \vec{E}}{\partial t}\right] = 0 \tag{78}$$

or

$$\frac{\partial}{\partial t}\left[\sigma_p + \frac{\rho c^2}{B^2} \vec{\nabla} \cdot \vec{E}\right] = 0 \tag{79}$$

or

$$\sigma_p = -\frac{\rho c^2}{B^2} \vec{\nabla} \cdot \vec{E} \tag{80}$$

If σ_p is 0 when E is 0, then from Eq. (77)

$$\nabla \cdot \left[1 + \frac{4\pi \rho c^2}{B^2}\right] \vec{E} = 4\pi \sigma_e \tag{81}$$

or

$$\vec{\nabla} \cdot \vec{D} = 4\pi \sigma_e. \tag{82}$$

<u>Problem:</u> Find the capacitance of a parallel plate condenser with plasma between its plates and with a magnetic field parallel to the surface of the plates. Assume that there is an insulating layer of infinitesimal thickness isolating the plates from the plasmas. Show that the energy per unit area stored in the capacitor, $\frac{1}{2} cv^2$, is stored as kinetic energy.

H. Time-Varying $\vec{B}$

Let us consider the case in which $\vec{B}$ is spatially uniform, at least in the region visited by our particles, but in which its magnitude varies with time. Because of the time variation of $\vec{B}$ there will be $\vec{E}$ fields set up. These give rise to the $\vec{E} \times \vec{B}$ and $\dot{\vec{E}} \times \vec{B}$ drifts just discussed. However, here we are not so interested in these effects as we are in the fact that $\vec{E}$ has a curl and hence will do work on a circulating charge. We will imagine that we have subtracted out the mean $\vec{E} \times \vec{B}$ drift.

Now the change in the perpendicular energy of the particles is given by

$$\delta W_\perp = -q \int \vec{E} \cdot d\vec{l}. \qquad (83)$$

But now if we go around a closed orbit, then

$$\oint \vec{E} \cdot d\vec{l} = \int \vec{\nabla} \times \vec{E} \cdot d\vec{A} = -\int \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}. \qquad (84)$$

If $\partial \vec{B}/\partial t$ is essentially constant over an orbit, then we may replace the last integral by

$$(\pm) \frac{\pi a^2}{c} \frac{\partial \vec{B}}{\partial t}$$

where a is the Larmor radius $v_1/\omega_c = a$.

The $(\pm)$ is determined by whether $\partial \vec{B}/\partial t$ is parallel or antiparallel to $d\vec{A}$. Now the direction of $d\vec{A}$ is determined by the direction of $d\vec{l}$ since the direction $d\vec{l}$ must be the direction in which the particle moves. If $\vec{B}$ is taken in the z direction, then $d\vec{A}$ is antiparallel to $\vec{B}$ for ions and parallel to $\vec{B}$ for electrons. Thus the sign is opposite to that of the charge. We thus find for $\delta W_\perp$

$$\delta W_\perp = \frac{|q|}{c} \pi a^2 \frac{\partial B}{\partial t} \quad \text{(in scalar form)} \tag{85}$$

or

$$\delta W_\perp = \frac{|q|}{c} \pi \frac{v_\perp^2 m^2 c^2}{q^2 B^2} \frac{\partial B}{\partial t} \tag{86}$$

or

$$\delta W_\perp = W_\perp \frac{2\pi mc}{|q| B} \frac{1}{B} \frac{\partial B}{\partial t} = W_\perp \frac{2\pi}{\omega_c B} \frac{\partial B}{\partial t} \tag{87}$$

thus

$$\frac{\delta W_\perp}{W_\perp} = \frac{\delta B}{B} \quad \text{for one orbit.} \tag{88}$$

This equation may also be written in the form

$$\delta\left(\frac{W_\perp}{B}\right) = 0 \quad \text{per orbit.} \tag{89}$$

Hence

$$\frac{W_\perp}{B} = \mu \cong \text{constant.} \tag{90}$$

The magnetic moment is thus approximately constant. It is not a strict constant since the above treatment requires that $\partial B/\partial t$ be essentially constant throughout an orbit. If B were changed instantaneously (very quickly, in the time it takes light to cross an orbit) from one value to another, then $W_\perp$ will not change since the particle will not have moved during the time B is changing. Thus $W_\perp/B$ will have changed. The magnetic moment is called an adiabatic invariant, since it is constant to a high degree of accuracy for slow variations of B . (It has been shown that the change in μ is exponentially small in the $B/\dot{B}$.)

There is a simple physical way to look at the constancy of μ . Now μ is equal to

$$\mu = \frac{W_\perp}{B} = \frac{m}{2}\frac{(a\omega_c)^2}{B}$$
$$= \frac{m}{2}\frac{a^2 e^2 B^2}{m^2 c^2 B} = \frac{e^2}{2mc^2} a^2 B \qquad (91)$$

where a is the Larmor radius.

Thus if μ is constant, a^2B or the flux through the orbit is constant. The orbit thus looks like a little superconducting current loop and no flux can cross it. This should not be surprising, since we have put in no mechanism for dissipating the current.

I. Spatially-Varying Magnetic Field

So far we have considered only magnetic fields which are spatially uniform over the regions visited by the particle. We wish now to consider magnetic fields which are not spatially uniform but which vary slowly with position. Here slowly means that variations of the magnetic field over a Larmor orbit are small, or

$$\frac{|\vec{\nabla}\vec{B}|\cdot|a|}{B} \ll 1. \qquad (92)$$

We can then find the particle's motion as a perturbation (locally) from what it would have in a spatially-uniform field. To this end we Taylor expand $\vec{B}$ about some point $\vec{r}$; $\vec{r}$ may be a function of t .

$$\vec{B}(\vec{r}+\vec{\rho}) = \vec{B}(\vec{r}) + \vec{\rho}\cdot\vec{\nabla}\vec{B}(\vec{r}). \qquad (93)$$

We will in general choose $\vec{r}$ to be the position of the guiding center for the particle — i.e., the instantaneous center of gyration. We shall consider the various elements of the tensor $\vec{\nabla}\vec{B}$ in turn. In the case of each set of terms, we will see first the influence of the terms on the shape of the field lines, and then find the effect on the particle orbits.

$$\vec{\nabla}\vec{B} = \begin{pmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial B_x}{\partial y} & \frac{\partial B_x}{\partial z} \\ \frac{\partial B_y}{\partial x} & \frac{\partial B_y}{\partial y} & \frac{\partial B_y}{\partial z} \\ \frac{\partial B_z}{\partial x} & \frac{\partial B_z}{\partial y} & \frac{\partial B_z}{\partial z} \end{pmatrix}. \tag{94}$$

(1) <u>The Effect of Diagonal Terms (Converging and Diverging Lines of Force)</u>

First consider the effects of the diagonal terms. Since $\vec{\nabla}\cdot\vec{B} = 0$, these terms are not all independent but their sum must vanish. We will choose a coordinate system such that r is 0 and such that the magnetic field at the origin points in the z direction. We have for the local magnetic field (neglecting off-diagonal terms for the moment, since their effect will be found shortly)

$$B_z = B_0 + \left[\frac{\partial B_z}{\partial z}\right]_0 z, \tag{95}$$

$$B_y = \left[\frac{\partial B_y}{\partial y}\right]_0 y, \tag{96}$$

and

$$B_x = \left[\frac{\partial B_x}{\partial x}\right]_0 x. \tag{97}$$

First let us see what these terms imply about the lines of force. The equations for a line of force are

$$\frac{dx}{dz} = \frac{B_x}{B_z} \tag{98}$$

and

$$\frac{dy}{dz} = \frac{B_y}{B_z}. \tag{99}$$

To 0 order (x and y or derivative terms neglected) we have

$$\frac{dx}{dz} = \frac{dy}{dz} = 0. \tag{100}$$

Thus $x = x_o$, $y = y_o$, and the lines are straight and parallel to the z axis as expected, since we chose the z direction as the direction in which the major part of B points. To first order in x and y we have

$$\frac{dx}{dz} = \left(\frac{\partial B_x}{\partial x}\right)_o \frac{x}{B_o} \tag{101}$$

and

$$\frac{dy}{dz} = \left(\frac{\partial B_y}{\partial y}\right)_o \frac{y}{B_o} \tag{102}$$

or

$$x = \left(\frac{\partial B_x}{\partial x}\right)_o \frac{xz}{B_o} + x_o \tag{103}$$

and

$$y = \left(\frac{\partial B_y}{\partial y}\right)_o \frac{yz}{B_o} + y_o. \tag{104}$$

Thus the lines of force are tilted as shown in Fig. 26. The lines of force are diverging or converging. Further, since $\vec{\nabla} \cdot \vec{B} = 0$, we have, to lowest order for B_z

$$\frac{\partial B_z}{\partial z} = -\left\{\left(\frac{\partial B_x}{\partial x}\right)_o + \left(\frac{\partial B_y}{\partial y}\right)_o\right\} \tag{105}$$

or

$$B_z = B_o - \left\{\left(\frac{\partial B_x}{\partial x}\right)_o + \left(\frac{\partial B_y}{\partial y}\right)_o\right\} z. \tag{106}$$

Thus the strength of the main magnetic field (the B_z component) varies along the z direction or varies as one moves along the magnetic

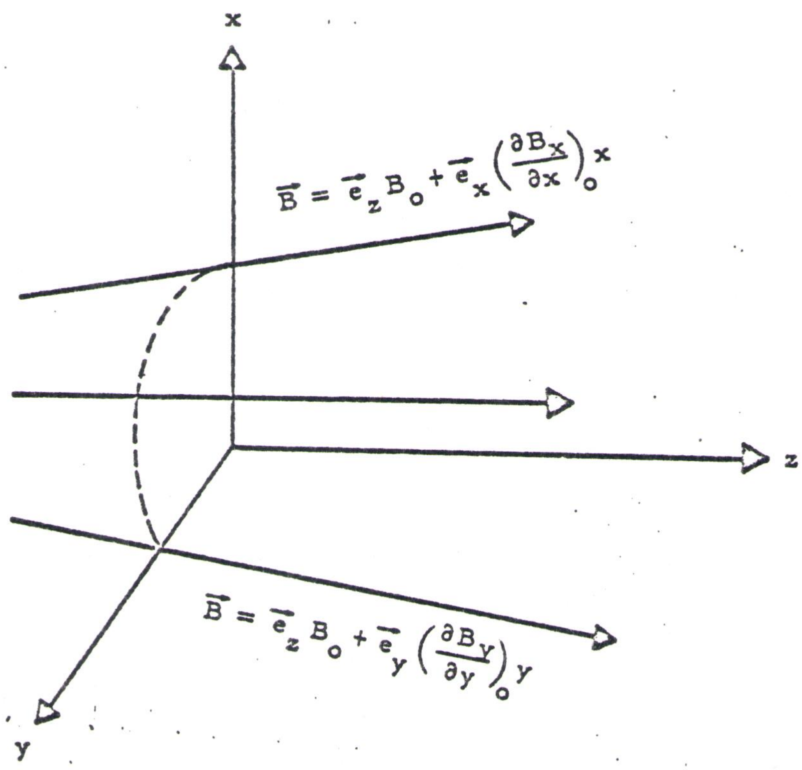
x
$\vec{B} = \vec{e}_z B_o + \vec{e}_x \left(\frac{\partial B_x}{\partial x}\right)_o x$
z
$\vec{B} = \vec{e}_z B_o + \vec{e}_y \left(\frac{\partial B_y}{\partial y}\right)_o y$
y

Figure 26