ADVANCES IN PLASMA PHYSICS

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Radiation from Plasmas*

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Introduction

In this chapter we shall discuss the generation and absorption of radiation both longitudinal and transverse, due to particle encounters. We shall also discuss to some extent the scattering and coupling of the various types of waves by plasma-density fluctuations.

We shall begin by giving some elementary estimates of the absorption and emission of radiation. These calculations are quite useful and informative as they lead quickly to many of the results obtained by the more complete theory. This will be followed by detailed theory for the absorption and emission of radiation by a plasma.

No attempt has been made to include cyclotron or synchrotron radiation.

I. Estimate of the Total Bremsstrahlung Emitted by a Fully Ionized Plasma

For a plasma which is hot, fully ionized, and free from strong static magnetic fields, the principal source of radiation is bremsstrahlung. Such radiation arises from the acceleration of the free electrons by the ions. We may estimate the total bremsstrahlung emitted by such a plasma by the following physical considerations.

Consider an ion of charge z imbedded in a plasma with electron density n_e. The acceleration of an electron by this ion is given by

$$a = ze^2/m_e r^2 \tag{I.1}$$

where r is the distance between the ion and the electron.

Classically, an accelerated electron (acceleration a) radiates energy at the rate

$$P = 2e^2a^2/3c^3 \tag{I.2}$$

Thus, by Eq. (I.1) the power being radiated by the electron under consideration is

$$P = 2z^2e^6/3m_e^2c^3r^4 \tag{I.3}$$

Multiplying Eq. (I.3) by the density of electrons and integrating over r gives

$$P = \frac{2z^2e^6}{3c^3m_e^2} n_e \int_{r_{\min}}^{\infty} \frac{4\pi r^2\,dr}{r^4} = \frac{8\pi}{3}\frac{z^2e^6n_e}{c^3m_e^2r_{\min}} \tag{I.4}$$

Here, $r_{\min}$ is the distance at which our classical approximation breaks down. In principle, we cannot treat the radiation as coming from a point electron since by the uncertainty principle we cannot localize the electron to an arbitrarily small volume. We may expect that for collision distances smaller than the reciprocal de Broglie wave number,

$$1/k = \hbar/p \cong r_{\min} \tag{I.5}$$

this will be an important effect and that we should use Eq. (I.5) for $r_{\min}$. If we do this, setting p equal to $\sqrt{m_ekT}$, then Eq. (I.4) becomes

$$P = (16\pi^2z^2e^6n_e/3c^3m_eh)\sqrt{kT_e/m_e} \tag{I.6}$$

Multiplying Eq. (I.6) by the density of ions gives the total radiation emitted per unit volume,

$$P_T = (16\pi^2z^2e^6n_en_i/3c^3m_eh)\sqrt{kT_e/m_e} \tag{I.7}$$

If there is more than one species of ion present, z^2n_i must be replaced by

$$\sum_i n_iz_i^2 \tag{I.8}$$

where i refers to the species.

It is of interest to compare Eq. (I.7) with the more exact results obtained from a quantum treatment. These may be found in Spitzer (1) and are given by

$$P_T = (2\pi kT/3m_e)^{1/2}(32\pi e^6/3c^3m_eh)z^2n_en_i\bar{g}_{ff} \tag{I.9}$$

Here $\bar{g}_{ff}$ is a pure number and is called the Gaunt factor. The value obtained for it depends on the approximations made in the quantum mechanical theory. Greene (2) discusses this and gives values of it for a wide range of densities and temperatures. For the Born approximation $\bar{g}_{ff}$ is equal to 1.103. Equations (I.7) and (I.9) would agree if $\bar{g}_{ff}$ were equal to 1.08.

One might expect, on the basis of the above arguments, that the acceleration of electrons by one another would also give rise to radiation. However, if we look at two electrons interacting, we see that their accelerations are of equal magnitudes but opposite in direction and so, to a first approximation, their radiation fields cancel. The radiation due to electron–electron interaction appears first in quadrupole order and is smaller than that due to electron–ion interactions by the factor v^2/c^2.

From Eq. (I.7) we can estimate how large the plasma would have to be before it would radiate like a blackbody. If we had a sphere of plasma of radius R, then according to Eq. (I.7) it would radiate an amount of power w,

$$w = \tfrac{4}{3}\pi R^3 P_T = \frac{64}{9}\,\frac{\pi^3 z^2 e^6 n_e n_i R^3}{c^3 m_e h}\sqrt{\frac{kT_e}{m_e}}$$
$$\cong 6 \times 10^{-27} z^2 n_i n_e T^{1/2} R^3 \text{ ergs/sec} \qquad (\text{I.10})$$

where T is in degrees Kelvin. Equation (I.10), of course, neglects reabsorption by the plasma. For large plasmas, reabsorption becomes important and, for sufficient size, a balance is reached where the emission and reabsorption maintain the radiation at the blackbody level. Such a large plasma would give off blackbody radiation from its surface. We estimate the size required for this to happen by equating w to blackbody radiation from the plasma surface. This radiation is given by

$$w_{BB} = \sigma T^4 4\pi R^2 = 7.12 \times 10^{-4}\ R^2 T^4 \text{ ergs/sec} \qquad (\text{I.11})$$

where σ is the Stefan-Boltzmann constant,

$$\sigma = 5.69 \times 10^{-12} \text{ joules/cm}^2 \text{ sec } (^\circ\text{K})^4 \qquad (\text{I.12})$$

Equating Eqs. (I.10) and (I.11) gives for R,

$$R = (1.2 \times 10^{23} \times T^{7/2})/(z^2 n_i n_e) \text{ cm} \qquad (\text{I.13})$$

This is an extremely large distance for most laboratory plasmas. For example, for a 10 eV plasma (10^5 °K) and a density of 10^{12} ions and electrons per cm^3, Eq. (I.13) gives an R of 3×10^{16} cm. Thus we see that most plasmas of interest to us will not radiate like blackbodies but in reality will emit much more weakly.

While the above argument shows that most laboratory plasmas will be optically thin to the blackbody radiation as a whole, it tells us nothing about the absorption and emission coefficients as functions of frequency. The plasma may be optically thick for some frequencies and optically

thin for others. To determine this we must find the absorption coefficient as a function of frequency for the plasma.

If we know the absorption coefficient for radiation at frequency ν, for a plasma at temperature T, then we can obtain the plasma emissivity by requiring that the absorption of blackbody radiation with temperature T at frequency ν must be made up for by the plasma emission.

The absorption coefficient can be obtained from the plasma conductivity or resistivity. While the resistivity may be a function of frequency [indeed, it generally is (3)], one can generally obtain some pretty good approximations to it by using very simple models for the collisional processes.

In addition to the absorption, we require the density of blackbody radiation within the plasma. This we can find in terms of the dielectric properties of the plasma.

II. Blackbody Radiation Inside a Transparent Medium

Let us first look at the blackbody radiation within a plasma at temperature T. We shall assume that the absorption per wavelength is small so that it makes sense to talk about radiation. To find the density of radiation, we consider the Gedanken experiment shown in Figure 1. At the top and bottom we have blackbodies at temperature T. These bodies are slabs which are infinite in the x,y direction and are normal to the z direction. In between these bodies we have a slab of plasma which is also at temperature T. Between the plasma and the blackbodies are vacuum regions. Radiation is emitted by the blackbodies and enters the plasma. We shall take the transition from vacuum to plasma to be sufficiently gradual that none of the radiation that can enter the plasma will be reflected. (We are considering only radiation that can enter the plasma; radiation with frequency below the plasma frequency or radiation striking the surface at more than the critical angle to the normal will be totally reflected no matter how slow the transition.) We do not specify the method of confinement, but we can imagine that some external forces are applied to the particles in the boundary region which prevent their escape.

Now if the plasma does not absorb any of the radiation, then an equilibrium will be set up with the radiation streaming from one blackbody to the other through the plasma. For this case, a certain radiation

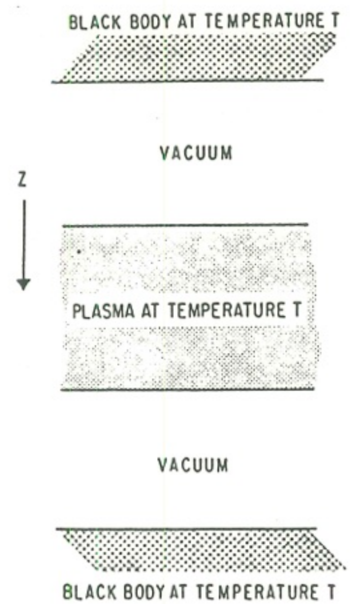

Fig. 1. Plasma in equilibrium with blackbody radiation.

density will be established in the plasma. This radiation density must be the equilibrium level even when there is absorption by the plasma, because if the plasma is absorbing more than it is emitting, it would heat up at the expense of the blackbodies. If it is emitting more than it is absorbing, it will cool down and the blackbodies will be heated. In either case, the second law of thermodynamics will be violated. The argument applies frequency by frequency, since we may place filters between the blackbodies and the plasma which allow exchange of energy in a narrow frequency interval.

Now consider radiation with frequency ν in $d\nu$, with direction of propagation lying within $d\theta$ of the normal to the plasma surface (propagating in $d\Omega = \pi\, d\theta^2$ about the normal), and which strikes a unit area of the surface in time τ. The situation is shown in Figure 2. After entering the plasma, the direction of propagation for the radiation will be within a cone making an angle $d\theta_i$ to the normal. We may obtain $d\theta_i$ by applying Snell's law for the refraction of the waves as they enter the plasma,

$$c/v_p = \sin\theta_0/\sin\theta_i \tag{II.1}$$

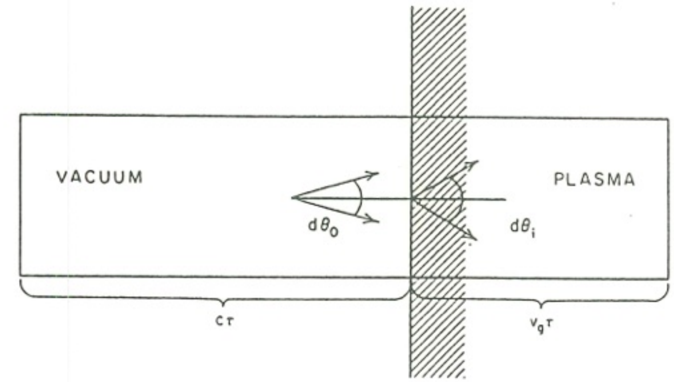

Fig. 2. Radiation entering a plasma.

Here c is the velocity of light in vacuum and v_p is the phase velocity of the waves in the plasma. Now if θ_0 is infinitesimal, then θ_i will be also, and the sines may be equated to the angles. Equation (II.1) may then be written as

$$\frac{c}{v_p} = \frac{d\theta_0}{d\theta_i} \tag{II.2}$$

The solid angle in which the radiation propagates outside the plasma is equal to $\pi\, d\theta_0^2$ while inside it is $\pi\, d\theta_i^2$. Thus by Eq. (II.2), the solid angles are related by

$$\frac{d\Omega_0}{d\Omega_i} = \frac{c^2}{v_p^2} \tag{II.3}$$

Now in the time τ, all the radiation in a cylinder of length $c\tau$ enters the plasma. (We have assumed that $d\theta_0$ is so small that we can neglect the variations in the distance the radiation must travel to reach the surface and can also neglect the variations in the cross section of the cylinder.) Inside the plasma the radiation travels a distance equal to the group velocity ($v_g = d\omega/dk$) times τ in the time τ (the radiation may be considered as a wave packet which propagates at the group velocity).

The radiation which was in the volume $c\tau$ with propagation direction lying in solid angle $d\Omega_0$ after entering the plasma lies in a volume $v_g\tau$ with propagation direction lying in $d\Omega_i$. If the radiation density per unit frequency interval per unit volume per unit solid angle in the vacuum is $\epsilon_v(\nu)$, then inside the plasma it is given by

$$\epsilon_p(\nu) v_g \, d\Omega_i = \epsilon_v(\nu) c \, d\Omega_0 \tag{II.4}$$

or by using Eq. (II.3)

$$\epsilon_p(\nu) = [c^3/(v_p^2 v_g)]\epsilon_v(\nu)^* \tag{II.5}$$

The density of radiation in the vacuum is given by Planck's law

$$\epsilon_v(\nu) = 2h\nu^3/c^3(e^{h\nu/kT} - 1) \tag{II.6}$$

Substituting this expression into Eq. (II.5) gives

$$\epsilon_p(\nu) = 2h\nu^3/v_p^2 v_g(e^{h\nu/kT} - 1) \tag{II.7}$$

For frequencies such that $h\nu/kT \ll 1$ we may expand the exponential in Eq. (II.7) and obtain

$$\epsilon_p(\nu) = 2kT\nu^2/v_p^2 v_g \tag{II.8}$$

Equation (II.8) gives the energy density in both transverse polarizations. It has been assumed that the phase and group velocities are the same for both. If this is not true we must divide Eq. (II.8) by 2 and use the appropriate phase and group velocities for each polarization.

It is of interest to inquire as to what the equilibrium energy per mode is inside a material. To have the modes well defined, we put the material inside a perfectly conducting box of volume V. The electric field must vanish at the walls and only modes which satisfy this criterion can exist, i.e., only modes with an integer number of half wavelengths in the box in each of the x,y,z directions are allowed. The number of modes within the box whose magnitude lies between k and $k + dk$ is given by (4)

$$N(k)\, dk = (V/2\pi^2)k^2 \, dk \tag{II.9}$$

The number per unit volume is

$$n(k)\, dk = 2\frac{k^2 \, dk}{2\pi^2} \tag{II.10}$$

* This formula holds for any isotropic material, not just for field-free plasmas. If the material is nonisotropic as, for example, a plasma containing a static B field, then the phase and group velocities may be in different directions. The arguments may be repeated in this case and we find that Eq. (II.5) should be replaced by

$$\epsilon_p(\nu) = (c^3/v_p \mathbf{v}_p \cdot \mathbf{v}_g)\epsilon_v(\nu)$$

(This includes both polarizations.) The number propagating in the solid angle $d\Omega$ is

$$n(k)\, dk \frac{d\Omega}{4\pi} = 2k^2 \frac{dk \, d\Omega}{8\pi^3} \tag{II.11}$$

Converting from k to ν gives

$$k = k(\nu) \tag{II.12}$$

$$dk = \frac{dk}{d\nu} d\nu = \frac{2\pi}{v_g} d\nu \tag{II.13}$$

$$n(\nu)\, d\nu \, d\Omega = \frac{n[k(\nu)]}{4\pi} \frac{2\pi \, d\nu \, d\Omega}{v_g} = 2 \frac{k^2(\nu)}{4\pi^2} \frac{d\nu \, d\Omega}{v_g} \tag{II.14}$$

Dividing the energy density given in Eq. (II.8) by the density of modes given by (II.14) gives

$$\epsilon(\nu)/n(\nu) = \kappa T \tag{II.15}$$

Thus, each mode of the field has energy κT.

Not all of the energy is in the electric and magnetic fields. Some of it is stored in the material, i.e., in polarizing the material and in mass motion of its constituents. We now apply our result to the case of a uniform plasma which contains no static magnetic fields.

III. Elementary Calculation of Equilibrium Radiation-Energy Density Inside a Field-Free Plasma

Let us consider an infinite homogeneous plasma consisting of infinitely heavy ions and mobile electrons and containing no static electric or magnetic fields. We shall assume that the electromagnetic waves under consideration have phase velocities much higher than the thermal velocity of the electrons and, hence, this thermal motion can be neglected. To begin with, we shall neglect collisions between particles.

The equation of motion for the electrons under these assumptions is

$$\frac{d\mathbf{v}}{dt} = -\frac{e\mathbf{E}}{m} \tag{III.1}$$

where $-e$, m, and $\mathbf{v}$ are the electron charge, mass, and velocity, respectively; $\mathbf{E}$ is the electric field. The current is given by

$$\mathbf{j} = -ne\mathbf{v} \tag{III.2}$$

where n is the electron density. In addition to these equations we have Maxwell's equations,

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{III.3}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi \mathbf{j}}{c} \tag{III.4}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{III.5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{III.6}$$

Here the charge and current are in electrostatic units, and since we have simply free space with charges and currents imbedded in it, no distinction is made between **B** and **H**. We also have the equation of continuity for the electrons,

$$\frac{\partial\rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{III.7}$$

However, this equation may be derived from Eqs. (III.4) and (III.5).

We now look for solutions of these equations of the form

$$\begin{Bmatrix} \mathbf{E}(\mathbf{r},t) \\ \mathbf{B}(\mathbf{r},t) \\ \mathbf{v}(\mathbf{r},t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{E} \\ \mathbf{B} \\ \mathbf{v} \end{Bmatrix} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{III.8}$$

Substituting this form into Eqs. (III.1)–(III.6) gives

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - k^2\mathbf{E} = [(\omega_p^2 - \omega^2)/c^2]\mathbf{E} \tag{III.9}$$

with

$$\omega_p^2 = 4\pi n e^2/m \tag{III.10}$$

If we look only for transverse waves, $\mathbf{k}\cdot\mathbf{E} = 0$, then Eq. (III.9) gives the dispersion relation

$$(k^2c^2 + \omega_p^2 - \omega^2) = 0 \tag{III.11}$$

while if we look for longitudinal waves, $\mathbf{k} \times \mathbf{E} = 0$ and Eq. (III.9) gives

$$\omega^2 = \omega_p^2 \tag{III.12}$$

We are primarily interested in the transverse waves since these couple directly to the radiation field in the vacuum.

From Eq. (III.11), the phase and group velocities are

$$v_p = \omega/k = c/(1 - \omega_p^2/\omega^2)^{1/2} \tag{III.13}$$

$$v_g = \frac{d\omega}{dk} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \tag{III.14}$$

We note that propagation takes place only for $\omega > \omega_p$, and for such frequencies the phase velocity is always greater than c; hence our neglect of thermal motions by comparison was justified. On the other hand, the group velocity is always less than the velocity of light.

If we substitute Eqs. (III.13) and (III.14) into Eq. (III.8), we obtain for the equilibrium energy density within the plasma

$$\epsilon_p(\nu) = (2kT\nu^2/c^3)(1 - \nu_p^2/\nu^2)^{1/2} \tag{III.15}$$

Equation (III.15) gives the energy density in both polarizations.

Earlier, we saw that the energy per mode in the radiation field was kT. It is interesting to ask what fraction of this energy is in the electric and magnetic fields and what fraction is in electron kinetic energy. These energy densities are given by

$$w_E = E^2/8\pi \tag{III.16}$$

$$w_B = B^2/8\pi \tag{III.17}$$

$$w_e = nmv^2/2 \tag{III.18}$$

From Eq. (III.3) the magnitude of the magnetic field is related to the magnitude of the electric field by

$$|\mathbf{B}| = (ck/\omega)|\mathbf{E}| = (c/v_p)|\mathbf{E}| \tag{III.19}$$

Thus w_B is given in terms of w_E by

$$w_B = (c^2/v_p^2)w_E = (1 - \omega_p^2/\omega^2)w_E \tag{III.20}$$

By Eqs. (III.1), (III.18), and (III.16) we have for the electron energy density

$$w_e = (\omega_p^2/\omega^2)w_E \tag{III.21}$$

The fraction of the total energy in each term thus comes out to

$$w_E/w_T = 1/2 \tag{III.22}$$

$$w_B/w_T = 1/2(1 - \omega_p^2/\omega^2) \tag{III.23}$$

$$w_e/w_T = 1/2\,\omega_p^2/\omega^2 \tag{III.24}$$

with

$$w_T = w_E + w_B + w_e \tag{III.25}$$

The electric-field energy is always $\frac{1}{2}kT$ per mode. For high frequencies, the magnetic-field energy is also $\frac{1}{2}kT$ per mode and there is negligible energy in the particle motion. For frequencies near the plasma frequency, there is $\frac{1}{2}kT$ per mode in the kinetic energy of the particles and negligible energy in the magnetic field.

Finally, we may mention that the energy flux, $(E \times B)c/4\pi$, divided by the energy density is equal to the group velocity as it should be if the group velocity is the flow velocity for the energy.

IV. Elementary Estimate of the Absorption and Emission Coefficients for a Plasma

A. *Absorption of Radiation*

We now wish to find the absorption and emission coefficients for a plasma. We may estimate these by simply introducing a collisional damping into Eq. (III.1). Thus we write, in place of Eq. (III.1),

$$\frac{d\mathbf{v}}{dt} = -\frac{e\mathbf{E}}{m} - \frac{\mathbf{v}}{\tau_c} \tag{IV.1}$$

Here, τ_c is the electron–ion collision time. It depends on density, temperature, and frequency, and over a wide range of frequencies it is essentially given by the expression (5)

$$\tau_c(\nu) = \frac{3}{2}\frac{\pi^{3/2}(kT)^{3/2}m^{1/2}}{z^2e^4n_i}\ln\left(\frac{\lambda_D}{r_{\min}}\frac{\nu_p}{(\nu^2+\nu_p^2)^{1/2}}\right) \tag{IV.2}$$

where λ_D is the Debye length ($\lambda_D = \sqrt{kT/4\pi}\, e^2 n_e$) and $r_{\min}$ is the minimum impact parameter. For $r_{\min}$, one usually chooses the larger of the inverse de Broglie wave numbers $(\hbar/\sqrt{m_e kT})$ or the classical distance of closest approach, (kT/ze^2). This will be discussed later. The relatively mild logarithmic dependence of τ_c on ν, for many purposes, can be neglected. This also will be discussed later.

If now we use Eq. (IV.1) in place of Eq. (III.1) and proceed as before, we find in place of the dispersion relation, Eq. (3.11),

$$k^2c^2 - \omega^2 + \omega_p^2\omega/(\omega + i/\tau_c) = 0 \tag{IV.3}$$

Since τ_c is generally quite small compared to ω, Eq. (IV.3) may be approximated by

$$k^2c^2 - \omega^2 + \omega_p^2[1 - i/\omega\tau_c] = 0 \tag{IV.4}$$

Now since the energy of a wave is proportional to its amplitude squared, the rate of energy absorption for a given wave number, k, is given in terms of the imaginary part of ω by

$$\frac{dw}{dt} = -(2\,\mathrm{Im}\,\omega)w \tag{IV.5}$$

Here w is the wave energy.

From Eq. (IV.4) we have for ω,

$$\omega \cong \pm\omega_0 - i\omega_p^2/2\omega_0^2\tau_c \tag{IV.6}$$

$$\omega_0 = (k^2c^2 + \omega_p^2)^{1/2} \tag{IV.7}$$

Thus we see that the imaginary part of ω is always negative so that the waves always damp as they should. The damping time is given by

$$\tau = \tau_c\omega_0^2/\omega_p^2 \tag{IV.8}$$

We can understand this result physically as follows. The fraction of the wave energy which the electrons have is $\omega_p^2/2\omega^2$, as given by Eq. (III.24). The electrons dissipate this energy in essentially a collision time (the dissipation rate is mv_e^2/τ_c) and so the wave will dissipate its energy in ω^2/ω_p^2 collision times as given by Eq. (IV.8).

If instead of absorption in time we want the absorption with distance for a wave of fixed frequency, then we must solve Eq. (IV.4) for k in terms of ω. We then find

$$k \cong \pm\frac{(\omega^2 - \omega_p^2)^{1/2}}{c} \pm \frac{i\omega_p^2}{2c\omega(\omega^2 - \omega_p^2)^{1/2}\tau_c} \tag{IV.9}$$

According to Eq. (IV.9), the waves always damp in the direction of propagation. The absorption distance is

$$l = |2k|^{-1} = c\,\frac{|\omega|(\omega^2 - \omega_p^2)^{1/2}}{\omega_p^2}\,\tau_c \tag{IV.10}$$

From Eq. (IV.8) for the absorption time and Eq. (IV.14) for the group velocity we see that

$$l(\nu) = v_g(\nu)\tau(\nu) \tag{IV.11}$$

For a plasma to be optically thick for frequency ν, $l(\nu)$ must be smaller than the dimensions of the plasma. If this is so and the plasma is in kinetic equilibrium (electrons have a Maxwellian distribution), then the plasma will radiate like a blackbody at frequency ν.

B. *Emission from a Field-Free Plasma*

We may now calculate the emission from the plasma. The emission is equal to the equilibrium radiation density times the absorption coefficient. Thus from Eq. (III.15) for the equilibrium energy density and from the damping time given by Eq. (IV.8) we find for the emissivity

$$\mathscr{E}(\nu) = [2kT\nu_p^2/c^3\tau_c(\nu)](1 - \nu_p^2/\nu^2)^{1/2} \qquad \text{(IV.12)}$$

Equation (IV.12) gives the emission per unit volume, per unit time, per unit solid angle in both transverse polarizations. The emission in both polarizations per unit frequency interval, per unit solid angle, per unit area, per unit distance of propagation is obtained by dividing Eq. (IV.12) by the group velocity and is given by

$$\mathscr{E}(\nu) = 2kT\nu_p^2/c^4\tau_c(\nu) \qquad \text{(IV.13)}$$

V. Calculation of the Absorption Coefficient from a Kinetic Model for the Plasma

A. *Derivation of the Impedance*

We now calculate the resistivity of a plasma (collision time τ) from a simple, but accurate, model for the plasma (3). The model is as follows: We assume that electron dynamics are adequately described by the Vlasov equations, while the ions are regarded as a set of randomly distributed fixed scatterers. We assume that, in addition to the self-consistent electric fields set up around the ions, there is a prevailing spatially uniform field oscillating in time at the frequency ω. We may take this uniform field to result from either a long-wavelength $[k < (\omega/u_0)]$ transverse or longitudinal wave.

This model neglects electron–electron collisions. However, since such collisions do not change the total current, they make no direct contribution to the resistance for long wavelengths. They can have an indirect effect by altering the electron-distribution function and, through it, the collision rate with ions. However, for frequencies large

compared to the collision frequency, such effects are small and can be neglected. This approximation and the model itself can be justified from a more complete kinetic description of the plasma using the B-B-G-K-Y hierarchy (6,7). Electron–electron collisions do contribute to the dissipation of finite-wavelength disturbances. They produce a dissipation which is proportional to k^2. This dissipation is the inverse process to quadrupole emission. We shall discuss quadrupole emission later. Detailed calculations can be found in the literature (8–11).

In the rest frame of the ions, the Vlasov equation for the electrons reads

$$\frac{\partial F}{\partial t} + \mathbf{v}\cdot\frac{\partial F}{\partial \mathbf{r}} - \frac{e}{m}(\mathbf{E}_0 e^{i\omega t} - \nabla\phi)\cdot\frac{\partial F}{\partial \mathbf{v}} = 0 \qquad \text{(V.1)}$$

$$\nabla^2\phi = 4\pi e\left[\int F d^3v - z\sum_j \delta(\mathbf{r} - \mathbf{r}_j)\right] \qquad \text{(V.2)}$$

Here, F is the electron-distribution function, the $\mathbf{E}_0$ term is due to the driving field, ϕ is the self-consistent potential, ze is the charge on the ions, and the $\mathbf{r}_j$ are the positions of the ions.

If we go to the frame oscillating with the electrons by making the transformations

$$\begin{aligned} \boldsymbol{\rho} &= \mathbf{r} - (\epsilon/\omega^2)e^{i\omega t} \\ \mathbf{u} &= \mathbf{v} - (i\epsilon/\omega)e^{i\omega t} \\ t &= t \qquad \epsilon = e\mathbf{E}_0/m \end{aligned} \qquad \text{(V.3)}$$

then Eqs. (V.1) and (V.2) become

$$\frac{\partial F}{\partial t} + \mathbf{u}\cdot\frac{\partial F}{\partial \boldsymbol{\rho}} + \frac{e}{m}\frac{\partial\phi}{\partial\boldsymbol{\rho}}\cdot\frac{\partial F}{\partial \mathbf{u}} = 0 \qquad \text{(V.4)}$$

$$\frac{\partial}{\partial\boldsymbol{\rho}}\left(\frac{\partial\phi}{\partial\boldsymbol{\rho}}\right) = 4\pi e\left[\int F d^3u - z\sum\delta\left(\boldsymbol{\rho} - \frac{\epsilon}{\omega^2}e^{i\omega t} - \mathbf{r}_j\right)\right] \qquad \text{(V.5)}$$

This is simply the Vlasov equation for the electron with a set of oscillating ion sources imbedded in it.

We now linearize these equations about a spatially uniform Maxwellian distribution

$$f_0 = (2\pi)^{-3/2}u_0^3 \exp(-u^2/2u_0^2) \qquad \text{(V.6)}$$

under the assumption that the fine-grained nature of the ions causes a small perturbation in the electron distribution. The resultant equations for the first-order quantities f,ϕ are

$$\frac{\partial f}{\partial t} + \mathbf{u}\cdot\frac{\partial f}{\partial \boldsymbol{\rho}} + \frac{e}{m}\frac{\partial \phi}{\partial \boldsymbol{\rho}}\cdot\frac{\partial f_0}{\partial \mathbf{u}} = 0 \tag{V.7}$$

and

$$\frac{\partial}{\partial \boldsymbol{\rho}}\cdot\left(\frac{\partial \phi}{\partial \boldsymbol{\rho}}\right) = 4\pi e\left[n_0\int f d^3u + n_0 - z\sum_j \delta\left(\boldsymbol{\rho} - \frac{\boldsymbol{\epsilon}}{\omega^2}e^{i\omega t} - \mathbf{r}_j\right)\right] \tag{V.8}$$

The average electron density n_0 is canceled by the average ion density in Eq. (V.8), so that this equation contains only first-order quantities.

Upon Fourier analysis of the spatial variable we have

$$\frac{\partial f_{\mathbf{k}}}{\partial t} - i\mathbf{k}\cdot\mathbf{u}f_{\mathbf{k}} - \frac{e}{m}i\phi_{\mathbf{k}}\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{u}} = 0 \tag{V.9}$$

and

$$\phi_{\mathbf{k}} = \frac{4\pi e}{k^2}\left\{n_0\int f_{\mathbf{k}}d^3u - (2\pi)^{-3}z\sum_j \exp\left[-i\mathbf{k}\cdot\left(\mathbf{r}_j + \frac{\boldsymbol{\epsilon}}{\omega^2}e^{i\omega t}\right)\right]\right\} \tag{V.10}$$

where $k = |\mathbf{k}|$. Since we are interested in the steady-state behavior, we solve these equations under the assumption of vanishing perturbation in the remote past. They can be formally integrated to yield

$$f_{\mathbf{k}} = \frac{ie}{m}\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{u}}\int_{-\infty}^{t}\phi_{\mathbf{k}}(\tau)\exp\left[-i(\mathbf{k}\cdot\mathbf{u})(t-\tau)\right]d\tau \tag{V.11}$$

and

$$\phi_{\mathbf{k}}(t) = \omega_p^2\int_{-\infty}^{t}(\tau - t)\exp\left[-\tfrac{1}{2}k^2u_0^2(\tau - t)^2\right]\phi_{\mathbf{k}}(\tau)\,d\tau + S_{\mathbf{k}}(t) \tag{V.12}$$

where $S_{\mathbf{k}}(t)$ (the source term) is the last term on the right-hand side of Eq. (V.10). In obtaining Eq. (V.12), we have substituted Eq. (V.11) into Eq. (V.10), and made use of Eq. (V.6) to carry out explicitly the velocity integration.

Now we are interested in the solution of these equations in the limit of $\epsilon \to 0$; indeed, it is only in this limit that the usual concept of resistivity is defined. Correspondingly, we expand the source term, writing

$$S_{\mathbf{k}}(t) \cong \frac{ze}{2\pi^2k^2}\left[1 - \frac{i\boldsymbol{\epsilon}}{\omega^2}\cdot\mathbf{k}e^{i\omega t}\right]\sum_j \exp\left[-i\mathbf{k}\cdot\mathbf{r}_j\right] \tag{V.13}$$

and then solve Eq. (V.12). The constant term in $S_{\mathbf{k}}$ gives rise to a static part of $\phi_{\mathbf{k}}$, $\phi_{\mathbf{k}}^s$; $\phi_{\mathbf{k}}^s$ is given by

$$\phi_{\mathbf{k}}^s = \frac{ze}{2\pi^2(k^2 + K^2)}\sum_j \exp\left[-i\mathbf{k}\cdot\mathbf{r}_j\right] \qquad (K = \omega_p/u_0) \tag{V.14}$$

and it is the usual Debye shielded potential. The oscillating term in $S_{\mathbf{k}}$ gives a contribution $\phi_{\mathbf{k}}^{\text{osc}}$,

$$\phi_{\mathbf{k}}^{\text{osc}} = +i\frac{ze}{2\pi^2k^2}\mathbf{k}\cdot\frac{\boldsymbol{\epsilon}}{\omega^2}\frac{e^{i\omega t}}{D_L(\mathbf{k},\omega)}\sum_j \exp\left[-i\mathbf{k}\cdot\mathbf{r}_j\right] \tag{V.15}$$

where $D_L(\mathbf{k},\omega)$ is the dielectric function for the plasma which is given in this case by

$$D_L(\mathbf{k},\omega) = 1 + \omega_p^2\int_0^\infty d\theta\,\theta\exp\left(-i\omega\theta - \tfrac{1}{2}k^2u_0^2\theta^2\right) \tag{V.16}$$

For a more general f_0, D_L may be written as

$$D_L(\mathbf{k},\omega) = \left(1 + \frac{4\pi e^2}{mk^2}\lim_{\epsilon\to 0}\int\frac{\mathbf{k}\cdot\partial f_0/\partial\mathbf{v}\,d^3v}{\omega + \mathbf{k}\cdot\mathbf{v} - i\epsilon}\right) \tag{V.17}$$

The plasma resistivity depends on the momentum transferred from the electrons to the ions. We therefore compute the force on the ions due to the electrons. This is given by

$$\mathbf{F}_{ie} = ze\sum_l \nabla\phi\left[\mathbf{r}_l - \frac{\boldsymbol{\epsilon}}{\omega^2}\exp(i\omega t)\right] \tag{V.18}$$

$$\mathbf{F}_{ie} = ize\sum_l\int d^3k\,\mathbf{k}\phi_{\mathbf{k}}\exp\left\{i\mathbf{k}\cdot\left[\mathbf{r}_l - \frac{\boldsymbol{\epsilon}}{\omega^2}\exp(i\omega t)\right]\right\} \tag{V.19}$$

$$\mathbf{F}_{ie} \cong \frac{4\pi i z^2e^2}{(2\pi)^3}\int\frac{d^3k}{k^2}\mathbf{k}\left\{\left[\frac{1}{D_L(k,0)} + \frac{i\mathbf{k}\cdot\boldsymbol{\epsilon}\exp(i\omega t)}{\omega^2D_L(k,\omega)}\right]\cdot\left[1 - \frac{i\mathbf{k}\cdot\boldsymbol{\epsilon}}{\omega^2}\exp(i\omega t)\right]\right\}\times\sum_{j,l}\exp\left[i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_j)\right] \tag{V.20}$$

We are interested only in the part of this force that varies like $\exp(i\omega t)$, as this is what will be associated with the ac conductivity. (The static part of $\mathbf{F}_{ie}$ vanishes because there can be no net force on the ions for an infinite homogeneous system.) Therefore we have

$$\mathbf{F}_{Ie}(\omega) = \left\{-\frac{4\pi z^2 e^3}{(2\pi)^3 m\omega^2}\int d^3k\,\frac{\mathbf{kk}}{k^2}\left[\frac{1}{D_L(k,0)} - \frac{1}{D_L(k,\omega)}\right] \times \sum_{j,l}\exp\left[i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_j)\right]\right\}\cdot\mathbf{E}_0 \qquad \text{(V.21)}$$

As Eq. (V.21) stands, it applies to a specific set of ion positions. We are not interested in the conductivity for one such configuration (indeed, we could not obtain such detailed knowledge), but are rather interested in its average value. We shall therefore take ensemble averages of Eq. (V.21) over ion configurations and denote the averages by $\langle\,\rangle$. We thus write

$$\langle \mathbf{F}_{Ie}(\omega)\rangle = zeN_I\tilde{\sigma}_1\cdot\mathbf{E}_0$$

$$\tilde{\sigma}_1 = -\frac{4\pi z e^2}{(2\pi)^3 m\omega^2 N_I}\int d^3k\,\frac{\mathbf{kk}}{k^2}\left[\frac{1}{D_L(k,0)} - \frac{1}{D_L(k,\omega)}\right] \times \left\langle\sum_{j,l}\exp\left[i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_j)\right]\right\rangle \qquad \text{(V.22)}$$

where N_I is the total number of ions.

So far we have worked in the oscillating-ion frame, since the ion–electron interaction is most easily computed in this frame. However, the impedance is most straightforwardly calculated in the ion rest frame. From the equations of motion for the electrons, it follows immediately that

$$\frac{d\bar{\mathbf{J}}_e}{dt} = -\frac{e}{m}\overline{\mathbf{G}}_e \qquad \text{(V.23)}$$

where $\bar{\mathbf{J}}_e$ is the average current density $(-n_0e\bar{v})$ and $\overline{\mathbf{G}}_e$ is the average total force per unit volume on the electrons. This force is given by

$$\overline{\mathbf{G}}_e = -n_0e[\mathbf{E}_0 + \overline{\mathbf{E}}_{Ie}(\omega)]\exp(i\omega t) \qquad \text{(V.24)}$$

where $\overline{\mathbf{E}}_{Ie}(\omega)$ is the average electric field produced at an ion by the electrons. This average field is given by

$$\overline{\mathbf{E}}_{Ie}(\omega) = \langle\mathbf{F}_{Ie}\rangle/zeN_I \qquad \text{(V.25)}$$

where N_I is the total number of ions. The $\bar{E}_{Ie}$ term follows from Newton's third law and from the invariance of this quantity under transformation of coordinates to the ion rest frame.

If we now make use of Eqs. (V.22), (V.25), and (V.24), we can convert Eq. (V.23) to

$$\frac{d\bar{\mathbf{J}}(\omega)}{dt} = i\omega\bar{\mathbf{J}} = \frac{n_0e^2}{m}(\mathrm{I} + \tilde{\sigma}_1)\cdot\mathbf{E}_0(\omega) = (\sigma_0 + \sigma_1)\cdot E \qquad \text{(V.26)}$$

Here I is the unit dyadic. If we assume that the electron–ion interaction is weak, as it always is in a high-temperature plasma, then $\tilde{\sigma}_1$ is small compared to I and we may write

$$\mathbf{E}_0 = (4\pi i\omega/\omega_p^2)(\mathrm{I} - \tilde{\sigma}_1)\cdot\bar{\mathbf{J}} = \mathrm{Z}(\omega)\cdot\bar{\mathbf{J}} \qquad \text{(V.27)}$$

where $Z(\omega)$ is the impedance.

B. *Discussion of the Impedance*

In general, the impedance, as given by Eqs. (V.22) and (V.27), is a tensor quantity; the tensorial character arises from anisotropy in the ion distribution. If the ions are isotropically distributed then $\tilde{\sigma}_1$ can be replaced by a scalar times the unit dyadic. In such a case we may treat the conductivity as a scalar. In this case we can find the effective collision time τ_c used in Eq. (IV.1), in terms of $\tilde{\sigma}$. Multiplying Eq. (IV.1) by $-n_0e$ and Fourier analyzing in t gives, for the impedance,

$$Z(\omega) = 4\pi(i\omega/\omega_p^2)(1 - i/\omega\tau_c) \qquad \text{(V.28)}$$

Thus by Eq. (V.27)

$$\tau_c = 1/\omega\,\mathrm{Im}\,\tilde{\sigma} \qquad \text{(V.29)}$$

$$\tilde{\sigma} = \text{scalar magnitude of } \tilde{\sigma}$$

The absorption length for radiation can be found in terms of this τ_c by making use of Eq. (IV.10). If the electrons are in kinetic equilibrium, i.e., they have a Maxwellian velocity distribution, then the emissivities per unit time and per unit length are given in terms of this τ_c by Eqs. (IV.12) and (V.13), respectively.

1. *Random Distribution of Ions*

The simplest case to consider is that for which the ions have a completely random distribution. Detailed calculations of $Z(\omega)$,$(\tilde{\sigma}_1)$, have been made for this case (3). For $\omega/\omega_p \ll 1$ and $\omega/\omega_p \gg 1$, the following analytic expressions can be obtained for $Z(\omega)$:

$\omega/\omega_p \ll 1$,

$$Z(\omega) \cong \frac{4\pi i\omega}{\omega_p^2}\left\{\left[1 - \frac{ze^2\omega_p}{6mu_0^3}\left(1 - \frac{\pi}{8}\right)\right] - i\left(\frac{2}{\pi}\right)^{1/2} \frac{ze^2\omega_p^2}{6mu_0^3\omega}\left[\ln\frac{k_{\max}^2}{K^2} - 1\right]\right\} \quad \text{(V.30)}$$

$\omega/\omega_p \gg 1$,

$$Z(\omega) \cong \frac{4\pi i\omega}{\omega_p^2}\left\{\left[1 - (2\pi)^{1/2}\frac{ze^2\omega_p^2}{6mu_0^3\omega}\right] - i\left(\frac{2}{\pi}\right)^{1/2} \frac{ze^2\omega_p^2}{6mu_0^3\omega}\left[\ln\left(\frac{2k_{\max}^2u_0^2}{\omega^2}\right) - \gamma\right]\right\} \quad \text{(V.31)}$$

Here $k_{\max}$ is the reciprocal of the minimum impact parameter,

$$k_{\max} = 1/r_{\min} \qquad r_{\min} = \text{Max}\,[ze^2/KT;\, \hbar/mu_0] \quad \text{(V.32)}$$

K is the Debye wave number, ω_p/u_0, and γ is Euler's constant, 0.577. . . . The derivation of Eqs. (V.30) and (V.31) can be found in reference 3. These results are in agreement with previous results which apply to either high or low frequencies (12–16).

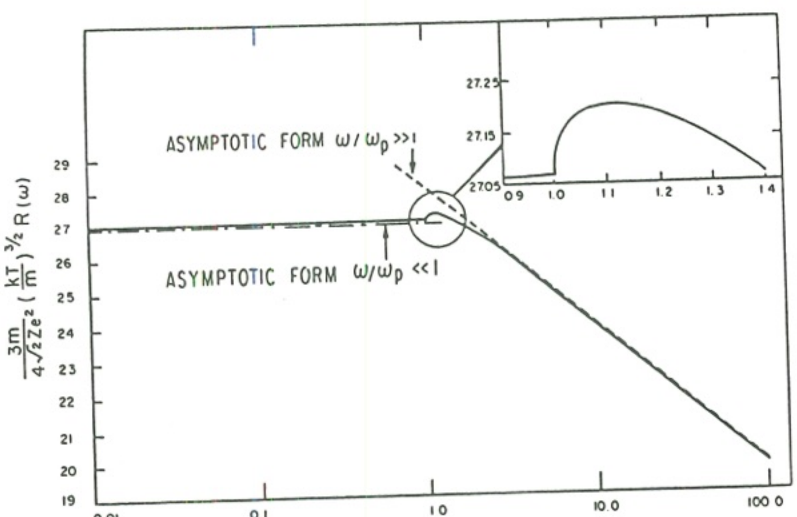

Fig. 3. Resistivity versus frequency.

The quantity $Z(\omega)$ has been found numerically for intermediate values of ω. The results are displayed in Figures 3–6. Figures 3 and 4 show plots of $R(\omega) = \text{Re}\, Z(\omega)$ and $X(\omega) - 4\pi\omega/\omega_p^2$ vs. ω, $X(\omega) = \text{Im}\, Z(\omega)$. For the evaluation of these quantities $k_{\max}$ was chosen to be $2^{-1/2}10^{-7}\omega_p/u_0$. To obtain $R(\omega)$ for a different $k_{\max}$ (different densities and temperatures), one simply adds

$$(2\pi)^{1/2}(2ze^2/3mu_0^3)\ln(2k_{\max}/10^{14}K^2) \quad \text{(V.33)}$$

to the graphed results. The integral giving $\text{Im}\, Z(\omega)$ requires no cutoff and, hence, is insensitive to $k_{\max}$.

The resistivity $R(\omega)$ shows a slight bump just above the plasma frequency. This enhancement of the resistivity is due to the generation of longitudinal plasma oscillations. The imaginary part of the integrand for Eq. (V.22) is plotted in Figure 5 for a number of frequencies. For frequencies slightly in excess of the plasma frequency, an isolated spike appears in the integrand at k_0. The value of k_0 at which the spike occurs corresponds to the excitation of longitudinal waves and is determined

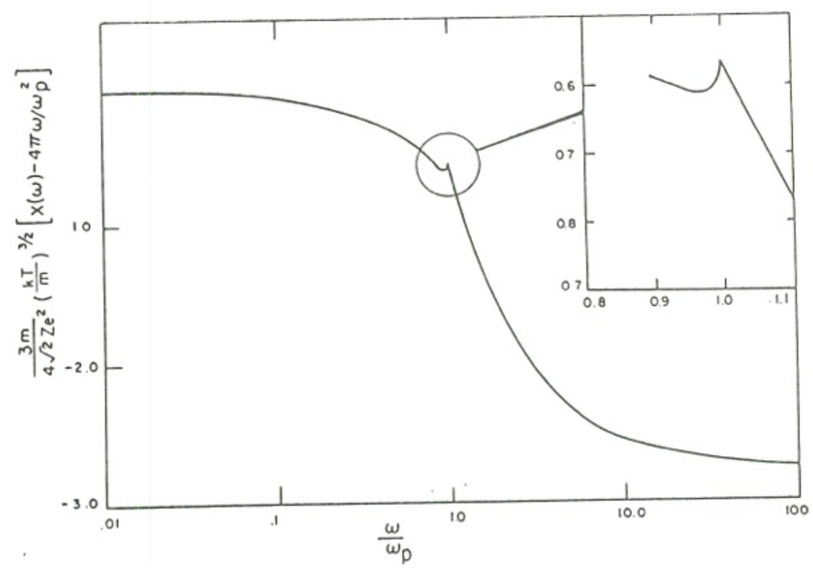

Fig. 4. Reactance versus frequency.

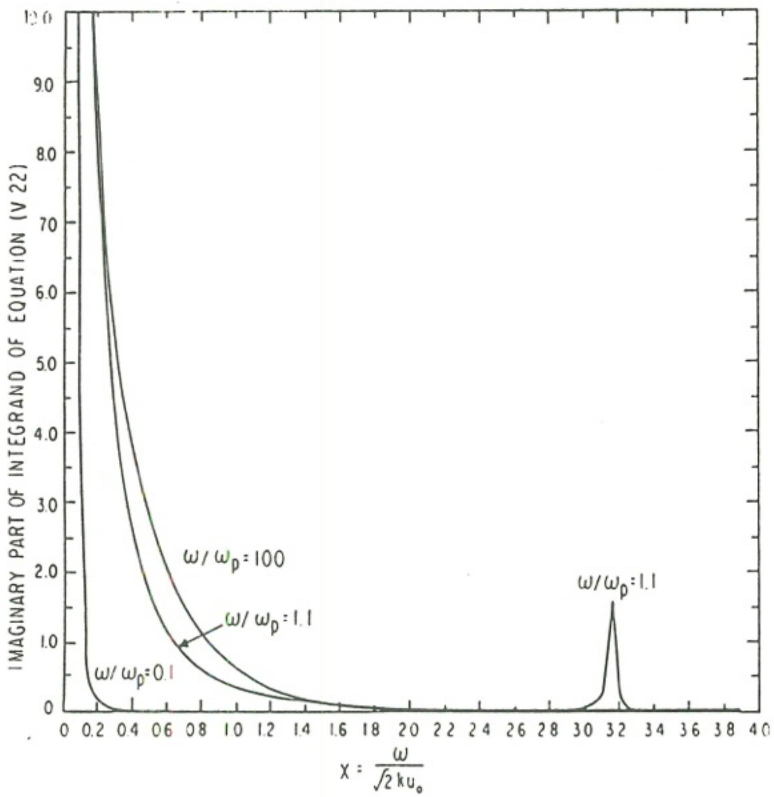

Fig. 5. Imaginary part of the integrand of Eq. (V.22) versus frequency.

by the vanishing of Re $D(k,\omega)$. For ω close to ω_p and $\omega/k \gg u_0$, $D(k,\omega)$ is approximately given by

$$D(k,\omega) \cong \left[1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2u_0^2}{\omega^2}\right)\right] - i\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_p^2\omega}{k^3u_0^3} \exp\left(-\frac{\omega^2}{2k^2u_0^2}\right)$$
$$= \left[1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2u_0^2}{\omega^2}\right)\right] - i\epsilon \qquad \text{(V.34)}$$

We may evaluate the wave contribution to the impedance by integrating Eq. (V.22) over a small range of k's about k_0. Making use of Eqs. (V.27) and (V.34) and integrating Eq. (V.22) only over the vicinity of the spike, we find for the wave impedance

$$Z_w(\omega) = 4\pi z e^2 \omega k_0 / 9 m u_0^2 \omega_p^2 \qquad \text{(V.35)}$$

Equation (V.35) applies only for ω greater than, but in the vicinity of, ω_p

$$\omega_p \leqslant \omega \gtrsim 1.4\omega_p \qquad \text{(V.36)}$$

Although for a purely random distribution of ions this bump is quite small, contributing only one per cent or so to the resistance, for nonrandom distributions it can easily dominate the resistance near the plasma frequency. This will be discussed in more detail shortly.

The drop in the resistance for frequencies large compared to ω_p is due to the contraction of the effective maximum impact parameter from u_0/ω_p to u_0/ω. For these high frequencies, collisions at large distances take place during many oscillations of the field and are thus rendered ineffective.

The plot of $X(\omega)$ in Figure 4 shows the variation of the reactance. This may be thought of as a change in the effective mass of the electrons due to their interaction with the ions. This effect is extremely small for a high-temperature plasma. It also shows the effects of the resonance at the plasma frequency. Figure 6 shows the real part of the integrand of Eq. (V.22) for a number of values of ω. It too shows a singularity due to the excitation of longitudinal oscillations for frequencies slightly in excess of the plasma frequency.

2. *The Effects of Equilibrium Ion Correlations* (17)

Of all nonrandom ion distributions, that which occurs for thermal equilibrium is of particular interest. We must evaluate the ion-density spectrum.

$$(2\pi)^6\langle|n_i(k)|^2\rangle \equiv \sum_{lj} \langle \exp[i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_j)]\rangle \qquad \text{(V.37)}$$

appearing in Eq. (V.22). We have assumed that the ensemble average and sum can be interchanged. The terms for $j = l$ sum to N_i, the total number of ions. There are $N_i(N_i - 1)$ terms for $j \neq l$ and they all have the same ensemble average since the ions are identical. We can evaluate one of these terms by making use of the probability of finding ion j at a distance r from ion l,

$$\langle \exp[i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_j)]\rangle = \int P(\mathbf{r}) \exp(-i\mathbf{k}\cdot\mathbf{r}) d^3r \qquad \text{(V.38)}$$

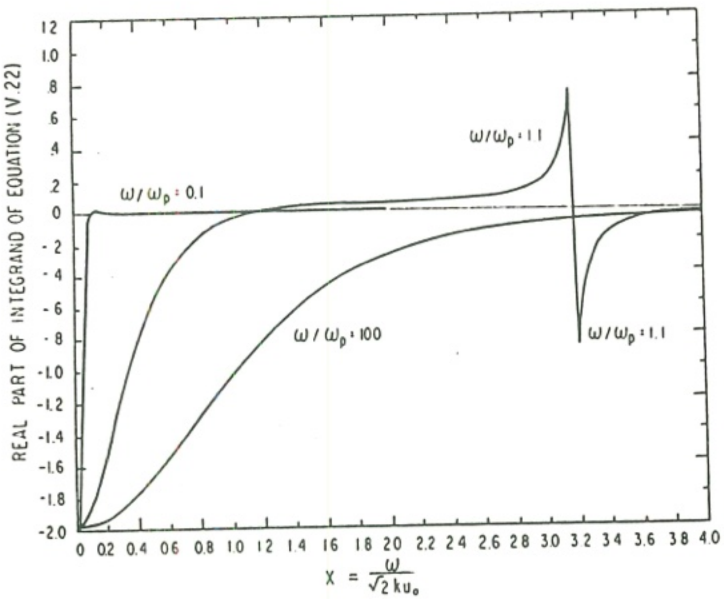

Fig. 6. Real part of the integrand of Eq. (V.22) versus frequency.

Here $\mathbf{r} = \mathbf{r}_j - \mathbf{r}_l$. It is well known that, in equilibrium, the probability of finding a particular ion in a volume element d^3r at a distance r from a given ion is given by

$$P(\mathbf{r})d^3r = \frac{d^3r}{V}\exp\left(-\frac{ze\phi}{\kappa T}\right) \cong \frac{d^3r}{V}\left(1 - \frac{ze\phi}{\kappa T}\right) \tag{V.39}$$

where V is the volume of the plasma (18). Here ϕ is the shielded potential of an ion and is given by

$$\phi = (ze/r)e^{-r/\lambda_T} \tag{V.40}$$

where λ_T is the Debye length for both ions and electrons combined,

$$\lambda_T = \kappa T/4\pi n_e e^2(z+1) \tag{V.41}$$

Substituting Eq. (V.39) into Eq. (V.38) and using Eq. (V.40), we find

$$\langle \exp [i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_j)]\rangle = -\frac{2Tz^2e^2}{V\kappa T}\int \exp(-ikr\cos\theta - r/\lambda_T) r \sin\theta \, d\theta \, dr$$
$$= \frac{4\pi z^2e^2}{V\kappa T}\frac{\lambda_T^2}{1 + k^2\lambda_T^2} \tag{V.42}$$

Multiplying Eq. (V.42) by $N_l(N_l - 1) \cong N_l^2$ and adding the contribution from the $j = l$ terms we obtain for Eq. (V.37)

$$(2\pi)^6\langle|n(k)|^2\rangle = N_l\left[1 - \frac{z}{(1+z) + \lambda^2k^2}\right] \tag{V.43}$$

where λ is the Debye length for the electrons only and $\lambda^2 = \kappa T/4\pi n_e e^2$. Finally, using Eqs. (V.43), (V.22), (V.27), and (V.37) we obtain

$$\tilde{\sigma}_1 = \mathrm{I} - \frac{\omega_p^2}{4\pi i\omega}\mathrm{Z}(\omega)$$
$$= -\frac{4\pi ze^2}{(2\pi)^3 m\omega^2}\int d^3k \frac{\mathbf{kk}}{k^2}\left[\frac{1}{D(k,0)} - \frac{1}{D(k,\omega)}\right]\cdot\left[\frac{1 + k^2\lambda^2}{(1+z) + k^2\lambda^2}\right] \tag{V.44}$$

Again it is possible to evaluate $\tilde{\sigma}_1$ for frequencies both high and low compared with the electron plasma frequency (see appendix B of reference 3 for the method of evaluation). We find for $\mathrm{Z}(\omega)$ in these limits:

For $\omega^2 \ll \omega_p^2$,

$$\mathrm{Z}(\omega) = \frac{4\pi i\omega}{\omega_p^2}\mathrm{I}\left\{1 + \frac{e^2\omega_p^2}{6mu_0^3 z}\left[\pi + z\left(\frac{\pi}{2} + 2\right) - (1+z)^{1/2}(2z + \pi)\right]\right.$$
$$\left. - i\left(\frac{2}{\pi}\right)^{1/2}\frac{ze^2\omega_p^2}{6mu_0^3\omega}\cdot\left[\ln\frac{k_{\max}^2u_0^2}{(1+z)\omega_p^2} - \frac{1}{z}\ln(1+z)\right]\right\} \tag{V.45}$$

while for $\omega^2 \gg \omega_p^2$,

$$\mathrm{Z}(\omega) = \frac{4\pi i\omega}{\omega_p^2}\mathrm{I}\left\{1 - \frac{(2\pi)^{1/2}ze^2\omega_p^2}{6mu_0^3|\omega|} - i\left(\frac{2}{\pi}\right)^{1/2}\frac{ze^2\omega_p^2}{6mu_0^3\omega}\right.$$
$$\left.\cdot\left[\ln(2k_{\max}^2u_0^2/\omega^2) - \gamma\right]\right\} \tag{V.46}$$

The high-frequency result is the same as we obtained for uncorrelated ions. This follows since only k's larger than ω/u_0 contribute in this limit and $\langle|n(k)|^2\rangle$ for these k's is not influenced by the ion correlation. The low-frequency results are influenced by the ion correlations, the primary effect being the replacement of the electron Debye length by the Debye length for both ion and electron, $\lambda/(z+1)$.

In addition to influencing the Debye cutoff for low frequencies, the ion correlations will affect the contribution to the resistance near the plasma frequency made by the generation of longitudinal waves. The k's which contribute to this wave resistance lie close to the value k_0 where $D_L(k,\omega)$ vanishes,

$$k_0^2 \cong (\omega^2 - \omega_p^2)/3u_0^2 \qquad (V.47)$$

For this process to exist, k_0 must be smaller than one. By Eq. (V.44) the integrand is reduced by the factor $(1 + z)^{-1}$ from the uncorrelated result.

3. *Effects of Nonthermal Ion Correlations* (*Ion Waves*)

The reduction of the resistance due to the generation of longitudinal waves found above holds only if the ion-density spectrum is thermal. If, for some reason, nonthermal ion correlations exist, then this wave resistance may be enhanced by many orders of magnitude. For simplicity, we consider an isotropic situation so that $\tilde{\sigma}_1$ is a scalar function of k and ω, times the unit dyadic. From Eq. (V.22) the wave resistance will equal the wave resistance for uncorrelated ions times the factor β,

$$\beta = \frac{(2\pi)^6\langle|n_i(k_0)|^2\rangle}{N_i} = \frac{1}{N_i}\sum_{ji}\langle\exp[ik_0\cdot(\mathbf{r}_i - \mathbf{r}_j)]\rangle \qquad (V.48)$$

where k_0 is given by Eq. (V.47).

Let us estimate how large β might be. Suppose that the ion-density spectrum has a strong maximum at $|\mathbf{k}| = \bar{k}$ and let us assume that the width of the maximum is Δk. Now the mean square density fluctuations due to this group of wave numbers is related to $n(k)$ by

$$\int_V n_{i,\Delta k}^2(r)d^3r = \int_V d^3r\int_{\Delta k} d^3k\int_{\Delta k} d^3k'n_i(k)n_i(k')\exp[i(\mathbf{k}-\mathbf{k}')\cdot r]$$
$$\cong (2\pi)^3\,\overline{|n_i(\bar{k})|^2}\,4\pi\bar{k}^2\,\Delta k = \overline{n_{i,\Delta k}^2(r)}V \qquad (V.49)$$

Here the subscript Δk on $n_i(r)$ means that only the contributions to $n_i(r)$ from the group of wave numbers considered is included; $\overline{|n_i(\bar{k})|^2}$ is the average square value of $n_i(\bar{k})$ for k's lying in the interval considered and $\overline{n_{i,\Delta k}^2(r)}$ is the mean square density fluctuation. Let us take $\overline{n_{i,\Delta k}^2(r)}$ to be the square of some fraction of the mean density, n_i,

$$\overline{n_{i,\Delta k}^2(r)} = \delta^2\bar{n}_i^2 \qquad (V.50)$$

Then Eq. (V.49) gives

$$\overline{|n_i(\bar{k})|^2} = \delta^2\bar{n}_i^2V/(2\pi)^3 4\pi\bar{k}^2\,\Delta k \qquad (V.51)$$

If we substitute $\overline{|n_i(\bar{k})|^2}$ for $\langle|n_i(\bar{k}_0)|^2\rangle$ in Eq. (V.48), assuming that $|k_0|$ lies in Δk, we find

$$\beta \approx \frac{(2\pi)^3}{4\pi}\frac{\delta^2\bar{n}_i}{\bar{k}^2\,\Delta k} = \frac{\delta^2 k}{4\pi\,\Delta k}\bar{n}_i\lambda^3 \qquad (V.52)$$

Now $n_i\lambda^3$ is the number of ions in a cubic wavelength. In a typical laboratory plasma for waves long compared to a Debye length, this can easily exceed 10^6 by many orders of magnitude. Thus, even if δ is relatively small (less than 10^{-2}) and Δk is of the order of $\bar{k}$, β can exceed 10^2. The longitudinal wave resistance can then exceed the normal resistance. [For uncorrelated ions this wave resistance is of the order of one per cent of the normal resistance (see Fig. 3).]

C. *Coupling of Longitudinal and Transverse Waves*

If the driving field is due to the propagation of a transverse wave through the plasma, then the resistance due to the generation of longitudinal waves represents a coupling of transverse to longitudinal waves via the ion-density fluctuations. The inverse process must also exist, i.e., the coupling of longitudinal waves to transverse waves. If the electrons are in kinetic equilibrium (have a Maxwellian velocity distribution), then this latter coupling may be found from detailed balance arguments. These may be found in reference 3. Later we shall calculate this coupling more directly and the calculation applies to any electron-velocity distribution.

If the driving field is produced by a longitudinal wave, then the longitudinal-wave resistance represents a scattering of the wave.

As we have seen above, if the ion-density fluctuations are superthermal, then these coupling and scattering processes can be greatly enhanced. This process can be used to explain solar radio bursts which appear to have their origin in large-amplitude plasma oscillations in the solar atmosphere (19).

VI. Radiation from Sources Embedded in a Plasma

We now proceed to compute the radiation from a plasma directly. To achieve this, we shall proceed in two steps. First, we shall compute the radiation emitted by arbitrary sources embedded in the plasma. Second, we shall determine the appropriate sources to be used in the emission formula.

Proceeding with the first step, we compute the radiation emitted by charges and currents imbedded in an infinite homogeneous Vlasov plasma consisting of mobile electrons plus a smeared-out uniform immobile-ion background. We assume that the electrons satisfy the linearized Vlasov equation. We shall restrict ourselves to isotropic electron-velocity distributions. This restriction simplifies the mathematics because it decouples the longitudinal and transverse fields. For many mildly anisotropic distributions, the coupling will be negligible and the method will apply directly. The method can be extended to arbitrary distributions, but no extensive work has been done in this direction.

Our starting equations are the linearized Vlasov-Maxwell equations, which are written here in their Fourier-analyzed form.

$$-i(\omega - \mathbf{k}\cdot\mathbf{v})f(\mathbf{k},\omega,\mathbf{v}) - \frac{e\mathbf{E}(\mathbf{k},\omega)}{m}\cdot\frac{\partial f_0}{\partial \mathbf{v}} = -\epsilon f(\mathbf{k},\omega,\mathbf{v}) \qquad \text{(VI.1)}$$

$$\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega) = 4\pi i\left[n_0 e\int f(\mathbf{k},\omega,\mathbf{v})d^3v - \rho_s(\mathbf{k},\omega)\right] \qquad \text{(VI.2)}$$

$$\mathbf{k}\cdot\mathbf{B}(\mathbf{k},\omega) = 0 \qquad \text{(VI.3)}$$

$$\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega) = (\omega/c)\mathbf{B}(\mathbf{k},\omega) \qquad \text{(VI.4)}$$

$$\mathbf{k}\times\mathbf{B}(\mathbf{k},\omega) = -(\omega/c)\mathbf{E}(\mathbf{k},\omega) + (4\pi i/c)\left[n_0 e\int f(\mathbf{k},\omega,\mathbf{v})\mathbf{v}d^3v - j_s(\mathbf{k},\omega)\right] \qquad \text{(VI.5)}$$

We have assumed for the Fourier analysis that all quantities vary like $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$.

Here n_0 and f_0 are the unperturbed electron density and distribution function, respectively, with f_0 normalized to unity. The small damping term ϵf has been introduced for mathematical convenience in deciding the proper paths for contour integrations and it will ultimately be set equal to zero. The sources ρ_s and $\mathbf{j}_s$ are assumed to satisfy the continuity equation

$$\frac{\partial \rho_s}{\partial t} + \nabla\cdot\mathbf{j}_s = 0 \qquad \text{(VI.6)}$$

Decomposing E and $\mathbf{j}_s$ into longitudinal and transverse parts, we write, for example,

$$\mathbf{E}(\mathbf{k},\omega) = \mathbf{E}(\mathbf{k},\omega)\cdot\hat{\mathbf{k}}\hat{\mathbf{k}} - \hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{k},\omega)] \qquad \text{(VI.7)}$$

where $\hat{\mathbf{k}}$ is a unit vector in the direction of k. Substituting into Eq. (VI.1)–(VI.6) we find

$$\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega) = -4\pi i\rho_s(\mathbf{k},\omega)/D_L(k,\omega) = -4\pi i\mathbf{k}\cdot\mathbf{j}_s/\omega D_L(k,\omega) \qquad \text{(VI.8)}$$

$$\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega) = [4\pi i\omega\mathbf{k}\times\mathbf{j}_s(\mathbf{k},\omega)]/k^2c^2D_T(k,\omega) \qquad \text{(VI.9)}$$

where k is the magnitude of k and the longitudinal and transverse dielectric functions are given by

$$D_L(k,\omega) = 1 + \frac{\omega_p^2}{k^2}\int\frac{\mathbf{k}\cdot(\partial f_0/\partial\mathbf{v})}{\omega - \mathbf{k}\cdot\mathbf{v} + i\epsilon}d^3v \qquad \text{(VI.10)}$$

$$D_T(k,\omega) = 1 - \frac{\omega^2}{c^2k^2} + \frac{\omega_p^2\omega}{k^2c^2}\int\frac{f_0}{\omega - \mathbf{k}\cdot\mathbf{v} + i\epsilon}d^3v \qquad \text{(VI.11)}$$

We now compute the energy emitted by the sources; that is, we compute

$$w = \lim_{T\to\infty}\int_T^T\int\mathbf{E}(\mathbf{r},t)\cdot\mathbf{j}_s(\mathbf{r},t)d^3r\,dt \qquad \text{(VI.12)}$$

By Parsival's theorem, Eq. (VI.12) can be written as

$$w = \frac{\mathrm{Re}}{2(2\pi)^4}\int\int\mathbf{E}(\mathbf{k},\omega)\cdot\mathbf{j}_s^*(\mathbf{k},\omega)d^3k\,d\omega \qquad \text{(VI.13)}$$

Dropping the integral on ω gives the energy emitted per unit frequency

$$w_\omega = \frac{\mathrm{Re}}{2(2\pi)^4}\int\mathbf{E}(\mathbf{k},\omega)\cdot\mathbf{j}_s^*(\mathbf{k},\omega)d^3k \qquad \text{(VI.14)}$$

We can and shall consider ω as positive. However, the Fourier transform is defined for $-\infty \leqslant \omega \leqslant \infty$. We make this change by multiplying Eq. (VI.14) by a factor of 2 and interpreting ω as $|\omega|$. When this change is made and when the values of the electric field given by Eqs. (VI.10) and (VI.11) are substituted into Eq. (VI.14), we obtain

$$w_\omega = \frac{1}{4\pi^3}\,\mathrm{Im}\int\left[\frac{|\mathbf{k}\cdot\mathbf{j}_s(\mathbf{k},\omega)|^2}{\omega k^2D_L(k,\omega)} - \frac{\omega|\mathbf{k}\times\mathbf{j}_s(\mathbf{k},\omega)|^2}{k^4c^2D_T(k,\omega)}\right]d^3k \qquad \text{(VI.15)}$$

Of particular interest is the case where $\mathbf{j}_s$ is the current density of a test dipole of moment $\mathbf{P}(t)$,

$$\mathbf{j}_s(\mathbf{r},t) = \dot{\mathbf{P}}(t)\,\delta(\mathbf{r}-\mathbf{r}_0) \qquad \text{(VI.16)}$$

$$\mathbf{j}_s(\mathbf{k},\omega) = i\omega\mathbf{P}(\omega)\exp[-i\mathbf{k}\cdot\mathbf{r}_0]$$

When Eq. (VI.16) is Fourier decomposed and the result inserted into Eq. (VI.15), the energy spectrum reduces to

$$w_\omega = \frac{\omega^2}{4\pi^3}\,\mathrm{Im}\int\left\{\frac{|\mathbf{k}\cdot\mathbf{P}(\omega)|^2}{\omega k^2 D_L(k,\omega)} + \frac{\omega[|\mathbf{k}\cdot\mathbf{P}(\omega)|^2 - k^2|\mathbf{P}(\omega)|^2]}{k^4c^2D_T(k,\omega)}\right\}d^3k \qquad \text{(VI.17)}$$

The energy emission at frequency ω and wave number $\mathbf{k}$ is given by the integrand of Eq. (VI.17). We note from Eq. (VI.9) that the transverse wave is always polarized with its electric field in the plane determined by $\mathbf{k}$ and $\mathbf{j}_s$.

Equation (VI.17) gives the total energy expended by the dipole. This includes the energy transferred to individual plasma electrons by encounters with the dipole [witness the logarithmic divergence of the first integral of Eq. (VI.17) for large k^3] as well as the emission of transverse and longitudinal waves. The wave emission is manifested by the contribution to the integral of Eq. (VI.17) from the spikes occurring because of the near-vanishing of D_T and D_L for certain values of k and ω (see Figure 5). This wave emission is obtained only for phase velocities large compared with the electron thermal velocity. Hence, to obtain these contributions, we may utilize the asymptotic forms of D_T and D_L for $ku_0/\omega \ll 1$ (where u_0 is the rms value of v),

$$D_T(k,\omega) \cong 1 - \frac{\omega^2}{c^2k^2} + \frac{\omega_p^2}{c^2k^2} + i\,\mathrm{Im}\,D_T \qquad \text{(VI.18)}$$

$$D_L(k,\omega) \cong 1 - \frac{\omega_p^2}{\omega^2} - \frac{3k^2u_0^2\omega_p^2}{\omega^4} + i\,\mathrm{Im}\,D_L \qquad \text{(VI.19)}$$

Here $\mathrm{Im}\,D_T$ and $\mathrm{Im}\,D_L$ are slowly varying functions of k and ω in the regime considered.* If we set D_T equal to zero, Eq. (VI.18) admits

* The imaginary part of D_L is due to Landau damping. In a real plasma there would be no such damping of the transverse waves since their phase velocity is greater than the velocity of light. Nevertheless, we shall assume that a little damping, say due to collisions, should be included in D_T.

solutions for all $\omega > \omega_p$, while it has been found that the simultaneous near-vanishing of the real and imaginary parts of Eq. (VI.19) occurs only in the approximate frequency range $3\omega_p \leqslant \omega \gtrsim 1.4\omega_p$.

Integration of Eq. (VI.17) over the resonances yields for the wave emission at frequency ω,

$$w_\omega = \frac{1}{2\pi}\begin{Bmatrix}\dfrac{\omega_p^3}{(3^{1/2}u_0)^3}\\[2ex] 2\omega^3/3c^3\end{Bmatrix}(\omega^2-\omega_p^2)^{1/2}|\mathbf{P}(\omega)|^2 \qquad \text{(VI.20)}$$

The upper coefficient represents the energy emission in longitudinal oscillations, while the lower one describes the transverse spectrum.

While the longitudinal emission is restricted to a narrow band of frequencies near the plasma frequency, in this regime it dominates the transverse emission by a factor of the order $(c/u_0)^3$.

We may easily extend these results to the case in which there are a large number of dipoles imbedded throughout the plasma. This case is important because many situations may be decomposed into a large number of dipole components.

In this case the source currents are given by,

$$\mathbf{j}_s(\mathbf{r},t) = \sum_l \dot{\mathbf{P}}_l(t)\,\delta(\mathbf{r}-\mathbf{r}_l)$$

$$\mathbf{j}_s(\mathbf{k},\omega) = \sum_l i\omega\mathbf{P}_l(\omega)\exp[-i\mathbf{k}\cdot r_l] \qquad \text{(VI.21)}$$

Substituting into Eq. (VI.15) and integrating over the resonance region gives for w_ω,

$$w_\omega = \frac{\omega^2}{4\pi^3}\,\mathrm{Im}\int d^3k\left\{\frac{\mathbf{kk}:\sum_{lj}\mathbf{P}_l(\omega)\mathbf{P}_j^*(\omega)\exp[i\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_l)]}{\omega k^2D_L(k,\omega)} + \frac{\omega\left(\mathbf{kk}:\sum_{lj}\mathbf{P}_l(\omega)\mathbf{P}_j^*(\omega)\exp[i\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_l)] - k^2\sum_{lj}\mathbf{P}_l(\omega)\mathbf{P}_j^*(\omega)\exp[i\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_l)]\right)}{k^4c^2D_T(k,\omega)}\right\} \qquad \text{(VI.22)}$$

A third case of interest is that in which the sources constitute a plane wave

$$\mathbf{j}_s = \mathrm{Re}\, \mathbf{j}_s \exp\left[i(k_0 z - \omega_0 t)\right] \tag{VI.23}$$

In line with considering longitudinal and transverse waves, we consider two cases, i.e., $\mathbf{j}_s$ perpendicular and parallel to z. We may use these cases to determine the energy of plane waves in terms of their amplitudes. To find the amplitudes, we shall assume that ω has a small positive imaginary part so that the field builds up from zero at $t = -\infty$ to a finite value at t equals zero. We also assume that the wave has negligible damping so that no energy is dissipated while it is being created.

For this case, we use the equation for the average energy expended per unit volume by the sources,

$$\overline{w} = \frac{1}{V}\int d^3r \int_{-\infty}^{0} \mathbf{j}_s(r,t) \cdot \mathbf{E}(\mathbf{r},t)\, dt \tag{VI.24}$$

Here, V is the volume of the plasma. $\overline{w}$ must also be the average energy density per unit volume carried by the waves.

Making use of Eqs. (VI.8) and (VI.9) and assuming that for $k = k_0$ and $\omega = \mathrm{Re}\, \omega_0$, $D(k,\omega) = 0$, we find for the average energy density in the wave

$$\overline{w} = -\left\{\begin{matrix} \omega\left(\dfrac{\partial D_L}{\partial \omega}\right)_k \\ \dfrac{k^2c^2}{\omega}\left(\dfrac{\partial D_T}{\partial \omega}\right)_k \end{matrix}\right\} \frac{\overline{|E|^2}}{8\pi} \tag{VI.25}$$

Here the bar over $|E|^2$ indicates the spatial average. The upper expression holds for longitudinal waves and the lower for transverse waves. These are familiar expressions (20) and are in agreement with the more specific results obtained earlier [Eq. (III.22)]. If we make use of Eqs. (VI.18) and (VI.19) for $D_{L,T}$ then in the long-wavelength limit, Eq. (VI.25) gives

$$\overline{w} = \overline{|E|^2}/4\pi \tag{VI.26}$$

for both types of waves.

The momentum density associated with the waves is $k\omega/\overline{w}$. This may be obtained from the integral of $E\rho_s$ for longitudinal waves and from $\mathbf{j}_s \times \mathbf{B}/c$ for transverse waves.

VII. The Interaction of First-Order Disturbances in a Vlasov Plasma (21)

A. *The Interactions of First-Order Disturbances*

We now consider the interaction of two small disturbances in a Vlasov plasma. This calculation may be used to determine the scattering and coupling of longitudinal and transverse waves in a plasma and also to justify our procedure for computing bremsstrahlung. We again consider the processes to be going on in an infinite homogeneous plasma. To begin with, we shall allow both ion and electron motion but later we shall restrict ourselves to the case of infinitely massive ions. We assume that there are no static-electric and magnetic fields in the equilibrium.

We start by assuming that some disturbances exist in the plasma which satisfy the linearized Vlasov-Maxwell equations (these may be wave disturbances, $D(k,\omega) = 0$, or they may be of an arbitrary nature). We solve for the interaction of these disturbances by taking the Vlasov-Maxwell equations to second order. The second-order terms in these equations are

$$\frac{\partial f_{\pm 2}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{\pm 2}}{\partial \mathbf{r}} \pm \frac{e}{m_\pm}\left(\mathbf{E}_2 + \frac{\mathbf{v} \times \mathbf{B}_2}{c}\right) \cdot \frac{\partial f_{\pm 0}}{\partial \mathbf{v}} \pm \frac{e}{m_\pm}\left(\mathbf{E}_1 + \frac{\mathbf{v} \times \mathbf{B}_1}{c}\right) \cdot \frac{\partial f_{\pm 1}}{\partial \mathbf{v}} = 0 \tag{VII.1}$$

$$\nabla \times \mathbf{E}_2 = -\frac{1}{c}\left(\frac{\partial \mathbf{B}_2}{\partial t}\right) \tag{VII.2}$$

$$\nabla \times \mathbf{B}_2 = \frac{1}{c}\frac{\partial \mathbf{E}_2}{\partial t} + \frac{4\pi e}{c}\int \mathbf{v}(f_{+2} - f_{-2})\, d^3v \tag{VII.3}$$

$$\nabla \cdot \mathbf{E}_2 = 4\pi e \int (f_{+2} - f_{-2})\, d^3v \tag{VII.4}$$

$$\nabla \cdot \mathbf{B}_2 = 0 \tag{VII.5}$$

Here, + stands for ions and − stands for electrons, respectively; we have taken the ions to be singly charged, and $\mathbf{E}_1$, $\mathbf{B}_1$, and $f_{\pm 1}$ are the first-order quantities associated with the linear disturbance. The initial conditions are that all second-order quantities are zero at $t = 0$.

We now claim that the solution of these equations can be obtained by solving the linearized Vlasov-Maxwell equations with appropriate sources which are given in terms of first-order quantities. To demonstrate this we write

$$f_{\pm 2} = \psi_{\pm 2} + \chi_{\pm 2} \tag{VII.6}$$

The quantities $\psi_{\pm 2}$ will give the source terms while $\chi_{\pm 2}$ give the response of the plasma.

The equation which $\psi_{\pm 2}$ satisfies is

$$\frac{\partial \psi_{\pm 2}}{\partial t} + \mathbf{v} \cdot \frac{\partial \psi_{\pm 2}}{\partial \mathbf{r}} \pm \frac{e}{m_\pm}\left(\mathbf{E}_1 + \frac{\mathbf{v} \times \mathbf{B}_1}{c}\right) \frac{\partial f_{\pm 1}}{\partial \mathbf{v}} = 0 \tag{VII.7}$$

while $\chi_{\pm 2}$ simply satisfies the linearized Vlasov equation

$$\frac{\partial \chi_{\pm 2}}{\partial t} + \mathbf{v} \cdot \frac{\partial \chi_{\pm 2}}{\partial \mathbf{r}} \pm \frac{e}{m_\pm}\left(\mathbf{E}_2 + \frac{\mathbf{v} \times \mathbf{B}_2}{c}\right) \cdot \frac{\partial f_{\pm 0}}{\partial \mathbf{v}} = 0 \tag{VII.8}$$

Initially, $\chi_{\pm 2}$ and $\psi_{\pm 2}$ are both zero.

The proof that this gives a solution is as follows. If Eq. (VII.6) is correct, then Eqs. (VII.2)–(VII.5) give the correct $\mathbf{E}_2$ and $\mathbf{B}_2$, and if $\mathbf{E}_2$ and $\mathbf{B}_2$ are correct, then Eq. (VII.6) satisfies Eq. (VII.1) by virtue of Eqs. (VII.7) and (VII.8).

We see from this solution that $\psi_{\pm 2}$ is determined strictly from $f_{\pm 1}$, $E_{\pm 1}$, and $B_{\pm 1}$, and we may obtain its solution by integrating

$$\left(\mathbf{E}_1 + \frac{\mathbf{v} \times \mathbf{B}_1}{c} \cdot \frac{\partial f_{\pm 1}}{\partial \mathbf{v}}\right)$$

along straight-line particle orbits. When f, as given by Eq. (VII.6), is substituted into Eqs. (VII.3) and (VII.4), the known values of the charges and currents due to $\psi_{\pm 2}$ act as sources for $\mathbf{E}_2$, $\mathbf{B}_2$, and $\chi_{\pm 2}$. [Through Eq. (VII.8), the fields are the shielded fields due to the charge and current densities associated with $\psi_{\pm 2}$.]

We may get a clearer physical picture of what this means if we look more closely at the charges and currents associated with $\psi_{\pm 2}$. We may readily verify, by making use of Eq. (VII.7), that these satisfy the continuity equation

$$\frac{\partial \rho_{2s}}{\partial t} + \nabla \cdot \mathbf{j}_{2s} = 0 \tag{VII.9}$$

with

$$\rho_{2s} = e\int (\psi_{+2} - \psi_{-2})\, d^3v \tag{VII.10}$$

$$\mathbf{j}_{2s} = e\int \mathbf{v}(\psi_{+2} - \psi_{-2})\, d^3v \tag{VII.11}$$

Thus, we need only consider the current $\mathbf{j}_{2s}$ since ρ_{2s} may be found from it. To obtain the current, multiply Eq. (VII.7) by $\pm e\mathbf{v}$ (the $\pm$ sign being appropriate to $\psi_{\pm 2}$) and integrate over $\mathbf{v}$. This gives

$$\begin{aligned} \pm e \int \left[\frac{\partial}{\partial t}(\mathbf{v}\psi_{\pm 2}) + \frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\mathbf{v}\psi_{\pm 2}) \pm \frac{e}{m_\pm}\mathbf{v}\left(\mathbf{E}_1 + \frac{\mathbf{v} \times \mathbf{B}_1}{c}\right)\frac{\partial f_{\pm 1}}{\partial \mathbf{v}}\right] d^3v \\ = \pm e \int \left\{\left[\frac{d(\mathbf{v}\psi_{\pm 2})}{dt}\right]_{\substack{\text{following}\\ \text{noninteracting}\\ \text{orbits}}} \mp \frac{e}{m_\pm}\left(\mathbf{E}_1 + \frac{\mathbf{v} \times \mathbf{B}_1}{c}\right) f_{\pm 1}\right\} d^3v \end{aligned} \tag{VII.12}$$

To obtain the second integral, we have combined the first two terms to give the convective-time derivative along the orbit, and the last term has been integrated by parts. Equation (VII.12) may be rewritten as

$$\begin{aligned} \mathbf{j}_{2s}(\mathbf{r},t) &= e\int \mathbf{v}(\psi_{+2} - \psi_{-2}) d^3v \\ &= \int d^3v \int_0^t d\tau \left[\left(\mathbf{E}_1 + \frac{\mathbf{v} \times \mathbf{B}_1}{c}\right)\left(\frac{e^2 f_{+1}}{m_+} + \frac{e^2 f_1}{m_-}\right)\right]_{\substack{\text{following}\\ \text{noninteracting}\\ \text{orbits}}} \end{aligned} \tag{VII.13}$$

Equation (VII.13) states that the current $\mathbf{j}_{2s}$ is the current produced by the acceleration of the first-order charge densities $\pm e f_{\pm 1}$ by the first-order fields, $\mathbf{E}_1$ and $\mathbf{B}_1$. The fields $\mathbf{E}_2$ and $\mathbf{B}_2$ are those obtained by imbedding these sources in the plasma.

B. *The Energy Associated with Waves*

The energy associated with waves in the plasma is second order in the E field,* as we saw in Section VI, Eq. (VI.25). Now let us suppose that E is expanded in a series,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \cdots \tag{VII.14}$$

where the subscript stands for the order. If this field is associated with a wave, then the wave energy will be given by

$$w \propto \mathbf{E}_1 \cdot \mathbf{E}_1 + \mathbf{E}_1 \cdot \mathbf{E}_2 + \mathbf{E}_1 \cdot \mathbf{E}_3 + \mathbf{E}_2 \cdot \mathbf{E}_2 \tag{VII.15}$$

To second order, we need only keep $\mathbf{E}_1 \cdot \mathbf{E}_1$. To third and fourth order, we should keep $\mathbf{E}_1 \cdot \mathbf{E}_2$ and $\mathbf{E}_1 \cdot \mathbf{E}_3$ terms as well as $\mathbf{E}_2 \cdot \mathbf{E}_2$ terms.

* Provided the amplitude is not too large.

However, if the wave generated in the $\mathbf{E}_2$ order is not present in first order or is only extremely weakly generated in first order, then the wave energy generated can be computed from $\mathbf{E}_2 \cdot \mathbf{E}_2$ or, equivalently, from $\mathbf{E}_2 \cdot \mathbf{j}_{2s}$.

In the general case, there will be interference between the first-order, second-order, etc., processes. However, when there is first-order emission this will generally dominate and second-order processes may be ignored. For waves where there is no first-order emission we must go to second-order processes and interference will not be important (bremsstrahlung is an example). Only in intermediate regions where the first-order emission is very weak will interference play a role. In that case the relative phasing between the first- and third-order fields will determine the importance of this type of effect. If they are randomly phased, then we can again neglect the interference. An important exception arises when one is considering turbulent plasmas in which large-amplitude oscillations are taking place and second- and higher-order terms compete with those of first order (22). In this case, important correlations between the terms will also exist.

VIII. Radiation Due to Electron–Ion Encounters in a Field-Free Plasma

In Section V, we computed the plasma resistivity due to electron–ion collisions. From this we computed the absorption coefficient for waves propagating through the plasma. For the case of kinetic equilibrium for the electrons (Maxwellian distribution), we used the law of detailed balance to find the emissivity. However, this method does not work for any other velocity distribution. A method to compute the emission directly would, therefore, be desirable. Such a method is, in essence, contained in Section VII where it was shown that the second-order disturbances produced by the interaction of first-order disturbances could be written in terms of source currents resulting from the acceleration of the first-order charge density by the first-order fields. For particle encounters, the first-order disturbances are those associated with the random motion of the particles. The fluctuating fields and charge densities are superpositions of the shielded fields of all the particles and their associated charge densities. Detailed discussions of these fluctuations can be found in references 23–25. More detailed justification of the procedure may be found in reference 21.

In this section, we shall confine ourselves to the dipole radiation produced by particle encounters. Such radiation results only from electron–ion encounters; electron–electron and ion–ion encounters do not give rise to it. We shall further assume that the ions are fixed-point particles.

Consider a small volume of plasma in which the encounters are taking place. The dipole moments of the charges contained in this volume are

$$\mathbf{P} = \sum_l q_l \mathbf{r}_l \qquad \text{(VIII.1)}$$

where q_l and $\mathbf{r}_l$ are the charge and position of the lth particle contained in the volume under consideration (27). Differentiating Eq. (VIII.1) with respect to t gives

$$\dot{\mathbf{P}} = \sum_l q_l \dot{\mathbf{r}}_l = \mathbf{j} \qquad \text{(VIII.2)}$$

$$\ddot{\mathbf{P}} = \sum_l q_l \ddot{\mathbf{r}}_l = \dot{\mathbf{j}} \qquad \text{(VIII.3)}$$

We now divide the particles into electrons and ions and write Eq. (VIII.3) as

$$\ddot{\mathbf{P}} = ze \sum_i \ddot{\mathbf{r}}_i - e \sum_e \ddot{\mathbf{r}}_e \qquad \text{(VIII.4)}$$

But from Newton's second law (assuming that all the ions and electrons which are being accelerated are in the same volume)

$$N_e \bar{\ddot{\mathbf{r}}}_e = -(N_i M_i / m_e) \bar{\ddot{\mathbf{r}}}_i \qquad \text{(VIII.5)}$$

where N_e and N_i are the number of electrons and ions in the small volume, and $\bar{\ddot{\mathbf{r}}}_e$ and $\bar{\ddot{\mathbf{r}}}_i$ are the average acceleration of the electrons and ions. (Note that ion–ion and electron–electron terms cancel so that only the acceleration of the electrons by the ions and the ions by the electrons enters here.) Substituting Eq. (VIII.5) into Eq. (VIII.4) gives

$$\ddot{\mathbf{P}} = \left\{ ze - \frac{M_i e}{m_e} \right\} \sum_i \ddot{\mathbf{r}}_i \qquad \text{(VIII.6)}$$

Now for $\ddot{\mathbf{r}}_i$ we have

$$\ddot{\mathbf{r}}_i = ze\mathbf{E}_i / M_i \qquad \text{(VIII.7)}$$

where $\mathbf{E}_i$ is the total electric field seen by the ith ion. Since ion–ion interactions cancel out, we may replace $\mathbf{E}_i$ by $\tilde{\mathbf{E}}_i$, the electric field at the

ith ion due to the electrons. Thus we may write Eq. (VIII.6) as

$$\ddot{\mathbf{P}} = \left\{\frac{z^2e^2}{M_i} - \frac{e^2z}{m_e}\right\} \sum_i \mathbf{E}_i$$

$$\mathbf{P}(\omega) = -\frac{1}{\omega^2}\left\{\frac{z^2e^2}{M_i} - \frac{e^2z}{m_e}\right\} \sum_i \mathbf{E}_i(\omega) \tag{VIII.8}$$

The second time derivative of the dipole moment is given in terms of the electric-field fluctuations seen by the ions.

There is one dipole term caused by each ion and, within the accuracy of the dipole approximation, we may take this dipole to be at the position of the ion. We therefore write for the source current density in accordance with Eq. (VI.21)

$$\mathbf{j}_s(\mathbf{r},\omega) = -\frac{i}{\omega}\left\{\frac{z^2e^2}{M_i} - \frac{e^2z}{m_e}\right\} \sum_i \mathbf{E}_i(\omega)\,\delta(\mathbf{r} - r_i)$$

$$\mathbf{j}_s(k,\omega) = -\frac{i}{\omega}\left\{\frac{z^2e^2}{M_i} - \frac{e^2z}{m_e}\right\} \sum_i \mathbf{E}_i(\omega)\exp\left[-ik\cdot\mathbf{r}_i\right] \tag{VIII.9}$$

Now in accord with the superposition principle of dressed particles, the total fluctuating electric field due to all electrons is the sum of the shielded fields (shielded by electrons only) from all the electrons when the electrons are treated as statistically independent (21,23,25,26). Consequently, we write Eq. (VIII.9) as

$$\mathbf{j}_s(k,\omega) = -\frac{i}{\omega}\left\{\frac{z^2e^2}{M_i} - \frac{e^2z}{m_e}\right\} \sum_{ie} \tilde{\mathbf{E}}_{ie}\exp\left[-ik\cdot\mathbf{r}_i\right] \tag{VIII.10}$$

where $\tilde{\mathbf{E}}_{ie}$ is the shielded electric field produced by the eth electron at the site of the ith ion. There is one term in this expression for each electron–ion encounter; the $\tilde{\mathbf{E}}_{ie}$ are to be treated as uncorrelated.

We now insert Eq. (VIII.10) into Eq. (VI.15), integrate over the resonant spikes, and take the ensemble average (indicated by $\langle\ \rangle$) of the radiation emitted

$$W_\omega = \frac{z^2e^4}{8\pi^2m_e^2}(\omega^2 - \omega_p^2)^{1/2}\int d\Omega_{\mathbf{k}_{L,T}}$$
$$\times\left\{\begin{array}{l}\dfrac{1}{3^{3/2}u_0^3\omega}\displaystyle\sum_{ii'e}\langle\exp\left[i\mathbf{k}_L\cdot(\mathbf{r}_i - \mathbf{r}_{i'})\right]\hat{\mathbf{k}}_L\cdot\tilde{\mathbf{E}}_e(\mathbf{r}_i,\omega)\hat{\mathbf{k}}_L\cdot\tilde{\mathbf{E}}_e^*(\mathbf{r}_{i'},\omega)\rangle\\ \dfrac{1}{\omega c^3}\displaystyle\sum_{ii'e}\langle\exp\left[i\mathbf{k}_T\cdot(\mathbf{r}_i - \mathbf{r}_{i'})\right]\hat{\mathbf{k}}_T\times\tilde{\mathbf{E}}_e(\mathbf{r}_i,\omega)\hat{\mathbf{k}}_T\times\tilde{\mathbf{E}}_e^*(\mathbf{r}_{i'},\omega)\rangle\end{array}\right\} \tag{VIII.11}$$

The upper expression gives the longitudinal emission while the lower one gives the transverse emission. The remaining integration is to be carried out over all directions of emission for $\mathbf{k}_L$ or $\mathbf{k}_T$. The resonant values of $|\mathbf{k}_L|$ and of $|\mathbf{k}_T|$ appearing in Eq. (VIII.11) are determined by the near vanishing of Eqs. (VI.18) and (VI.19), respectively, and are given by

$$|\mathbf{k}_L| = (\omega^2 - \omega_p^2)^{1/2}/3^{1/2}u_0 \tag{VIII.12}$$

$$|\mathbf{k}_T| = (\omega^2 - \omega_p^2)^{1/2}/c \tag{VIII.13}$$

To obtain the shielded field of an electron to insert into Eq. (VIII.11), one solves the test-particle problem for an electron moving through an infinite uniform plasma with smeared-out fixed ions since only electron shielding is to be taken into account. The plasma is described by the Vlasov Eqs. (VI.1)–(VI.5). We retain only the longitudinal electric field; the transverse fields are relativistically small and are negligible if the thermal energy is small compared to the electron rest mass. [The transverse field has already been neglected in the derivation of Eq. (VIII.8).] The longitudinal electric field as given by Eq. (VI.8) is

$$\tilde{E}_e(\mathbf{k},\omega) = -(4\pi i\mathbf{k}/k^2)[\rho_s(\mathbf{k},\omega)/D_L(k,\omega)] \tag{VIII.14}$$

For an electron moving along a straight-line trajectory, the source charge density is given by

$$\rho_s = -e\,\delta(\mathbf{r} - \mathbf{r}_{e0} - \mathbf{v}_{e0}t) \tag{VIII.15}$$

and its Fourier transform is given by

$$\rho_s(\mathbf{k},\omega) = -2\pi e\exp(-i\mathbf{k}\cdot\mathbf{r}_{e0})\,\delta(\omega - \mathbf{k}\cdot\mathbf{v}_{e0}) \tag{VIII.16}$$

Here $\mathbf{r}_{e0}$ and $\mathbf{v}_{e0}$ are the initial position and velocity of the electrons.

Substituting Eq. (VIII.16) into Eq. (VIII.14) and inverting the k transform for the electric field at the ith ion gives

$$\tilde{\mathbf{E}}(\mathbf{r}_i,\mathbf{r}_{e0},\omega) = \frac{ie}{\pi}\int\frac{d^3k\mathbf{k}}{k^2D_L(k,\omega)}\exp\left[i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_{e0})\right]\delta(\omega - \mathbf{k}\cdot\mathbf{v}_{e0}) \tag{VIII.17}$$

To evaluate Eq. (VIII.11) we must compute

$$\sum_e\sum_{ii'}\langle\exp\left[i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_{i'})\right]\tilde{\mathbf{E}}_e(\mathbf{r}_i,\omega)\tilde{\mathbf{E}}_e^*(\mathbf{r}_{i'},\omega)\rangle \tag{VIII.18}$$

This may also be written in terms of the ion-density spectrum by

$$\langle |n_i \tilde{E}(\omega)|^2_{k\ \text{component}} \rangle \tag{VIII.19}$$

In order to perform the average over i and i' in Eq. (VIII.18) we must know the ion correlations. As has been pointed out in Section V, the presence of nonthermal ion-density correlations that might result from large-amplitude ion waves can significantly enhance the absorption coefficient and, consequently, the emission.

Ion correlations can be incorporated into the formalism in a straightforward manner. The procedure is the same as that employed in Section V to compute the resistance when ion correlations exist. Detailed calculations may be found in references 19 and 28. Here, however, we shall restrict ourselves to the case of uncorrelated ions.

With the assumption of uncorrelated ions, the only contribution to Eq. (VIII.11) occurs when $i = i'$. Integrating Eq. (VIII.11) over all directions, we determine the wave contribution to be

$$W\omega = \frac{z^2e^4}{6\pi m_e^2}(\omega^2 - \omega_p^2)^{1/2}\begin{Bmatrix} 1/3^{3/2}u_0^3\omega \\ 2/3c^3\omega \end{Bmatrix} \sum_{ie} |\tilde{\mathbf{E}}_e(\mathbf{r}_i,\omega)|^2 \tag{VIII.20}$$

the summations are to be carried out over all ions and electrons in the plasma. As usual, the upper term refers to longitudinal wave emission and the lower to transverse wave emission.

The wave emission for a single encounter can be obtained by extracting one term from the double summation appearing in Eq. (VIII.20) and inserting the expression for the shielded field derived in Eq. (VIII.17). We do so, choosing the origin of time to be that time corresponding to closest approach in a collision; hence $(\mathbf{r}_i - \mathbf{r}_{e0})$ is the impact parameter, $\mathbf{b}_{ie}$, in the straight-line approximation. The result is

$$W\omega = \frac{z^2e^6(\omega^2 - \omega_p^2)^{1/2}}{6\pi^3 m_e^2}\begin{Bmatrix} 1/3^{3/2}u_0^3\omega \\ 2/3c^3\omega \end{Bmatrix} \iint d^3k\,d^3k'$$
$$\times \frac{\mathbf{k}\cdot\mathbf{k}' \exp\,[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}_{ie}]}{k^2k'^2 D_L(k,\omega)D_L^*(k',\omega)}\,\delta(\omega - \mathbf{k}\cdot\mathbf{v}_{e0})\cdot\delta(\omega - \mathbf{k}'\cdot\mathbf{v}_{e0}) \tag{VIII.21}$$

The total number of electron–ion collisions per unit time per unit volume, characterized by impact parameter $\mathbf{b}_{ie}$ and electron velocity $\mathbf{v}_e$, is

$$dN = n_+n_-|\mathbf{v}_e|\,d\phi\,b_{ie}\,db_{ie}\,f(v_e)d^3v_e \tag{VIII.22}$$

In Eq. (VIII.22), ϕ represents the orientation of $\mathbf{b}_{ie}$ in the plane transverse to $\mathbf{v}_e$; n_+ and n_- are the average number densities of ions and electrons, respectively. The total power radiated per unit volume is obtained by multiplying Eq. (VIII.21) by Eq. (VIII.22) and performing the indicated integrations. This procedure is lengthy but straightforward and we do not give it here. We thus determine the longitudinal (upper coefficient) and transverse (lower) power emission spectra to be

$$P(\omega) = \frac{4z^2e^6n_+n_-}{3\pi m^2}\begin{Bmatrix} 1/3^{3/2}u_0^3\omega \\ 2/3c^3\omega \end{Bmatrix}(\omega^2 - \omega_p^2)^{1/2}$$
$$\iint d^3k\,d^3v_e f(\mathbf{v}_e)\frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v}_e)}{k^2|D_L(k,\omega)|^2} \tag{VIII.23}$$

For the case of thermal equilibrium this expression agrees with that obtained by applying the law of detailed balance to the absorption coefficients obtained in Section V. For frequencies large compared to the plasma frequency, $D_L(k,\omega)$ may be set equal to unity and Eq. (VIII.23) agrees with the result obtained by computing the radiation from classical unshielded electron–ion encounters in a vacuum (5,29).

IX. The Contribution of Electron–Electron Collisions to the Emission of Bremsstrahlung

A. *Source Currents*

We here compute the bremsstrahlung due to electron–electron encounters. We first compute the source currents due to such encounters, realizing that they will contribute first in quadrupole order. The space Fourier transform of the current due to a group of electrons is

$$\mathbf{j}_s(\mathbf{k},t) = -e\sum_e \mathbf{v}_e(t)\exp\,[-i\mathbf{k}\cdot\mathbf{r}_e(t)] \tag{IX.1}$$

where the sum extends over all electrons in the volume of plasma from which the emission is to be computed.

As an aid to isolating the accelerative portion of the current density, which is the desired source current due to encounters, we take two time derivatives of Eq. (IX.1), Fourier transform in time, and obtain for $\mathbf{j}_s(\mathbf{k},\omega)$,

$$\mathbf{j}_s(\mathbf{k},\omega) = -\omega^{-2}\dddot{\mathbf{j}}_s(\mathbf{k},\omega)$$
$$= e\omega^{-2}\sum_e \{[\dddot{\mathbf{v}}_e - i\mathbf{k}\cdot\dot{\mathbf{v}}_e\mathbf{v}_e - 2i\mathbf{k}\cdot\mathbf{v}_e\dot{\mathbf{v}}_e - (\mathbf{k}\cdot\mathbf{v}_e)^2\mathbf{v}_e] \times \exp(-i\mathbf{k}\cdot\mathbf{r}_e)\}_\omega \quad \text{(IX.2)}$$

An order-of-magnitude estimate of the terms in this equation indicates that they are in the ratio

$$1 : ku_0/\omega : ku_0/\omega : k^2u_0^2/\omega^2$$

where the typical value of $\mathbf{v}_e$ has been taken to be the rms velocity, u_0, of the electrons. For small k, we know that the phase velocities of both longitudinal and transverse waves are greater than the electron thermal velocity. Defining, therefore, a smallness parameter $\alpha = ku_0\omega_p^{-1} = k\lambda_D$, we discard terms of $O(\alpha\omega_p/\omega)^2$, retaining $O[\alpha(\omega_p/\omega)]$ terms, since it will be evident shortly that $O(1)$ terms vanish identically for the electron–electron process. [For ω near ω_p, small terms are of $O(\alpha)$; for much higher frequencies they are still smaller.]

As we are interested in bremsstrahlung from electron–electron collisions, the acceleration $\dot{\mathbf{v}}_e$ and its derivative $\ddot{\mathbf{v}}_e$ can be represented as a sum over the contributions $\dot{\mathbf{v}}_{ee'}$ and $\ddot{\mathbf{v}}_{ee'}$ from individual encounters. To $O[\alpha(\omega_p/\omega)]$ Eq. (IX.2) then becomes

$$\mathbf{j}_s(\mathbf{k},\omega) = e\omega^{-2}\sum_{e\neq e'}[(\ddot{\mathbf{v}}_{ee'} - i\mathbf{k}\cdot\dot{\mathbf{v}}_{ee'}\mathbf{v}_e - 2i\mathbf{k}\cdot\mathbf{v}_e\dot{\mathbf{v}}_{ee'})\exp(-i\mathbf{k}\cdot\mathbf{r}_e)]_\omega \quad \text{(IX.3)}$$

and since the sums e and e' extend over the same assemblage of particles, it follows that

$$\mathbf{j}_s(\mathbf{k},\omega) = \frac{e}{2\omega^2}\sum_{e\neq e'}[(\ddot{\mathbf{v}}_{ee'} - i\mathbf{k}\cdot\dot{\mathbf{v}}_{ee'}\mathbf{v}_e - 2i\mathbf{k}\cdot\mathbf{v}_e\dot{\mathbf{v}}_{ee'})\exp(-i\mathbf{k}\cdot\mathbf{r}_e)$$
$$+ (\ddot{\mathbf{v}}_{e'e} - i\mathbf{k}\cdot\dot{\mathbf{v}}_{e'e}\mathbf{v}_{e'} - 2i\mathbf{k}\cdot\mathbf{v}_{e'}\dot{\mathbf{v}}_{e'e})\exp(-i\mathbf{k}\cdot\mathbf{r}_{e'})]_\omega \quad \text{(IX.4)}$$

Now we expand

$$\exp(-i\mathbf{k}\cdot\mathbf{r}_{e'}) = \sum_{n=0}^{\infty}\frac{[i\mathbf{k}\cdot(\mathbf{r}_e - r_{e'})]^n}{n!}\exp(-i\mathbf{k}\cdot\mathbf{r}_e) \quad \text{(IX.5)}$$

Successive terms in the expansion [Eq. (IX.5)] are of $O(\alpha)^n$, since we anticipate that shielding (when properly introduced) will limit the range

of interaction, $\mathbf{r}_e - \mathbf{r}_{e'}$, to a few Debye distances. To order α, $\mathbf{j}_s(\mathbf{k},\omega)$ reduces to

$$\mathbf{j}_s(\mathbf{k},\omega) = \frac{e}{2\omega^2}\sum_{e\neq e'}\{[\ddot{\mathbf{v}}_{ee'} + \ddot{\mathbf{v}}_{e'e}(1 + i\mathbf{k}\cdot\mathbf{r}_{ee'}) - i\mathbf{k}\cdot(\dot{\mathbf{v}}_{ee'}\mathbf{v}_e + \dot{\mathbf{v}}_{e'e}\mathbf{v}_{e'}$$
$$+ 2\mathbf{v}_e\dot{\mathbf{v}}_{ee'} + 2\mathbf{v}_{e'}\dot{\mathbf{v}}_{e'e})]\exp(-i\mathbf{k}\cdot\mathbf{r}_e)\}_\omega \quad \text{(IX.6)}$$

where the relative separation vector $\mathbf{r}_{ee'} = \mathbf{r}_e - \mathbf{r}_{e'}$ has been introduced.

For particles e and e' interacting through a Coulomb potential, we have

$$\dot{\mathbf{v}}_{ee'} = \frac{e^2}{m}\frac{\mathbf{r}_{ee'}}{|\mathbf{r}_{ee'}|^3} = -\dot{\mathbf{v}}_{e'e} \quad \text{(IX.7)}$$

$$\ddot{\mathbf{v}}_{ee'} = \frac{e^2}{m}\left[\frac{(\mathbf{v}_e - \mathbf{v}_{e'})|\mathbf{r}_{ee'}|^2 - \mathbf{r}_{ee'}\mathbf{r}_{ee'}\cdot(\mathbf{v}_e - \mathbf{v}_{e'})}{|\mathbf{r}_{ee'}|^5}\right] = -\ddot{\mathbf{v}}_{e'e} \quad \text{(IX.8)}$$

and the $O(1)$ terms in Eq. (IX.6) are identically vanishing. [For electron–ion collisions $\mathbf{j}_s(\mathbf{k},\omega)$ has a nonzero $O(1)$ component which leads to the dipole bremsstrahlung discussed in Section VIII.] Substituting from the equations of motion and defining the relative velocity vector $\mathbf{v}_{ee'} = \mathbf{v}_e - \mathbf{v}_{e'}$, we obtain for $\mathbf{j}_s(\mathbf{k},\omega)$

$$\mathbf{j}_s(\mathbf{k},\omega) = -\frac{ie^3}{2m\omega^2}\mathbf{k}\cdot\sum_{e\neq e'}\left[\left(2\frac{\mathbf{r}_{ee'}\mathbf{v}_{ee'}}{|\mathbf{r}_{ee'}|^3} + 2\frac{\mathbf{v}_{ee'}\mathbf{r}_{ee'}}{|\mathbf{r}_{ee'}|^3} - 3\frac{\mathbf{r}_{ee'}\mathbf{r}_{ee'}\mathbf{r}_{ee'}\cdot\mathbf{v}_{ee'}}{|\mathbf{r}_{ee'}|^5}\right)\times\exp(-i\mathbf{k}\cdot\mathbf{r}_e)\right]_\omega$$

$$= \frac{i}{6\omega^2}\mathbf{k}\cdot\sum_{e=e'}(\dddot{\mathbf{Q}}_{ee'})_\omega \quad \text{(IX.9)}$$

where, for convenience, we have defined

$$\dddot{\mathbf{Q}}_{ee'} = -\frac{3e^3}{m}\left(2\frac{\mathbf{r}_{ee'}\mathbf{v}_{ee'}}{|\mathbf{r}_{ee'}|^3} + 2\frac{\mathbf{v}_{ee'}\mathbf{r}_{ee'}}{|\mathbf{r}_{ee'}|^3} - 3\frac{\mathbf{r}_{ee'}\mathbf{r}_{ee'}\mathbf{r}_{ee'}\cdot\mathbf{v}_{ee'}}{|\mathbf{r}_{ee'}|^5}\right)\exp(-i\mathbf{k}\cdot\mathbf{r}_e) \quad \text{(IX.10)}$$

(We shall shortly relate $\dddot{\mathbf{Q}}_{ee'}$ to the third time derivate of the $e - e'$ quadrupole tensor.)

We now wish to compute the total quadrupole emission due to an encounter between electrons e and e'. This will contain a part due to the direct interaction of electrons e and e', a part due to the interaction of electron e with the shielding cloud of e' and vice versa, and a part due to the interaction of the shielding cloud of e with that of e'. To compute these we employ the superposition principle of Rostoker (25), and accordingly write,

$$\dddot{\mathbf{Q}}_{ee'} = \dddot{\mathbf{Q}}_{\bar{e}\bar{e}'} + n \int d^3r_{e''} \int d^3v_{e''} [\dddot{\mathbf{Q}}_{e''\bar{e}'} f(e''|\bar{e}) + \dddot{\mathbf{Q}}_{\bar{e}e''} f(e''|\bar{e}')]$$
$$+ n^2 \int d^3r_{e''} \int d^3v_{e''} \int d^3r_{e'''} \dddot{\mathbf{Q}}_{e''e'''} f(e''|\bar{e}) f(e'''|\bar{e}') \quad \text{(IX.11)}$$

In computing the radiation, the superposition principle tells us that we treat $\mathbf{Q}_{ee'}$ as uncorrelated with $\mathbf{Q}_{e''e'''}$ (in the squared form of $\mathbf{j}_s$) provided the pair ee' is different from the pair $e''e'''$.

A bar (-) in Eq. (IX.11) signifies the mean rectilinear motion of the particle(s) is to be used, and thus $\dddot{\mathbf{Q}}_{\bar{e}\bar{e}'}$ is the contribution to $\dddot{\mathbf{Q}}_{ee'}$ due to the direct interaction of e and e' as they follow their rectilinear trajectories. The remainder is the contribution to $\dddot{\mathbf{Q}}_{ee'}$ from shielding, i.e., from all other plasma electrons regarded as field particles interacting with e and e' as the latter move along their rectilinear orbits.

The quantity $f(e''|\bar{e})$ is the perturbation in the one-particle electron distribution (as a function of the phase space coordinates of e'') arising from the rectilinear motion of e. To obtain f, we solve the test-particle problem in which e moves along the trajectory $\bar{\mathbf{r}}_e(t) = \mathbf{r}_{e0} + \bar{\mathbf{v}}_e t$ through an infinite, uniform Vlasov plasma consisting of mobile electrons and a smeared-out neutralizing ion background. Only the longitudinal interaction between electron e and the plasma is retained, the transverse fields being relativistically small if the thermal energy of the electrons is much less than their rest energy. The value of f so derived is

$$f(e''|\bar{e}) = -\frac{e^2}{2\pi^2 m} \int d^3k' \frac{\exp[i\mathbf{k}' \cdot (\mathbf{r}_{e''} - \bar{\mathbf{r}}_e)]}{k'^2 D_L(k', \mathbf{k}' \cdot \bar{\mathbf{v}}_e)} \frac{\mathbf{k}' \cdot \partial f_0 / \partial \mathbf{v}_{e''}}{(\mathbf{k}' \cdot \bar{\mathbf{v}}_e - \mathbf{k}' \cdot \mathbf{v}_{e''} + i\epsilon)} \quad \text{(IX.12)}$$

$D_L(k', \mathbf{k}' \cdot \bar{\mathbf{v}}_e)$ being the usual longitudinal dielectric function, Eq. (VI.10), evaluated for $\omega = \mathbf{k}' \cdot \bar{\mathbf{v}}_e$.

If one uses Eqs. (IX.12), (IX.11), and (IX.10), one finds for $\dddot{\mathbf{Q}}_{ee'}$ after much calculation (see ref. 9)

$$\dddot{\mathbf{Q}}_{ee'} = -\frac{3}{2\pi^2} \frac{ie^3}{m} \exp(-i\mathbf{k} \cdot \bar{\mathbf{r}}_e) \int d^3k' \frac{\exp(i\mathbf{k}' \cdot \bar{\mathbf{r}}_{ee'})}{k'^2}$$
$$\times \left[\frac{\mathbf{k}'\bar{\mathbf{v}}_{e'\perp} + \bar{\mathbf{v}}_{e'\perp}\mathbf{k}'}{D_L(k', -\mathbf{k}' \cdot \bar{\mathbf{v}}_e)} - \frac{\mathbf{k}'\bar{\mathbf{v}}_{e\perp} + \bar{\mathbf{v}}_{e\perp}\mathbf{k}'}{D_L(k', \mathbf{k}' \cdot \bar{\mathbf{v}}_{e'})} \right.$$
$$\left. + \frac{(\mathbf{I}k'^2 - 4\mathbf{k}'\mathbf{k}')(\mathbf{k}' \cdot \bar{\mathbf{v}}_{ee'})}{k'^2 D_L(k', -\mathbf{k}' \cdot \bar{\mathbf{v}}_e) D_L(k', \mathbf{k}' \cdot \bar{\mathbf{v}}_{e'})} \right] \quad \text{(IX.13)}$$

A $\perp$ subscript indicates a vector component perpendicular to $\mathbf{k}'$, and $\mathbf{I}$ is the unit dyadic.

B. *Quadrupole Spectra*

Only the straight-line orbits of the interacting particles e and e' enter Eq. (IX.13). When $\mathbf{j}_s(\mathbf{k},\omega)$ appears quadratically, as it does in Eq. (VI.15), the interacting shielded particles are considered uncorrelated and there occurs only a sum over e and e' rather than the ordinary fourfold sum. Further, the term $\exp(-i\mathbf{k} \cdot \bar{\mathbf{r}}_e)$ in Eq. (IX.13) can be replaced by unity, since in squaring $\mathbf{j}_s$ we obtain a factor, $\exp\{i\mathbf{k} \cdot [\bar{\mathbf{r}}_e(t') - \bar{\mathbf{r}}_e(t)]\}$ (the two different times, t' and t, appear because the two exponentials occur in different Fourier time transforms), which is approximately unity in the long-wavelength approximation.

To shorten the presentation while illustrating the technique, we shall develop only the transverse electron–electron bremsstrahlung spectrum. The longitudinal spectrum is similarly derived, and the result of this calculation will be quoted and discussed.

Substituting the value of $\mathbf{j}_s$ from Eq. (IX.9) into Eq. (VI.15), we obtain

$$\frac{d\bar{P}^T}{d\omega} = -\frac{1}{72\pi^3 T} \frac{1}{\omega^5} \operatorname{Im} \int d^3k \frac{1}{k^2 D_T(k,\omega)} \sum_{e \neq e'} [\mathbf{k} \times \mathbf{k} \cdot (\dddot{\mathbf{Q}}_{ee'})_\omega]$$
$$\cdot [\mathbf{k} \times \mathbf{k} \cdot (\dddot{\mathbf{Q}}_{ee'} + \dddot{\mathbf{Q}}_{e'e})_{-\omega}] \quad \text{(IX.14)}$$

where $\dddot{\mathbf{Q}}_{e'e} = \dddot{\mathbf{Q}}_{ee'}$. The sums still extend over all pairs of electrons. The wave contribution is extracted by integrating locally over the resonance in the $|\mathbf{k}|$ integrand occurring near the $|\mathbf{k}|$ value given by Eq. (VIII.13) and is

$$\frac{d\bar{P}^T}{d\omega} = \frac{1}{72\pi^2 c^5 T} \frac{(\omega^2 - \omega_p^2)^{3/2}}{\omega^3} \int d\Omega \sum_{e \neq e'} [\hat{\mathbf{k}} \times \hat{\mathbf{k}} \cdot (\dddot{\mathbf{Q}}_{ee'})_\omega]$$
$$\cdot [\hat{\mathbf{k}} \times \hat{\mathbf{k}} \cdot (\dddot{\mathbf{Q}}_{ee'})_{-\omega}] \quad \text{(IX.15)}$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ is unit vector in the direction of wave propagation. The integrand of Eq. (IX.15) displays the angular distribution of the emitted transverse radiation.

Indeed, we now recognize that [as used in Eq. (IX.15)] $\dddot{\mathbf{Q}}_{ee'}$ is the third time derivative of the local quadrupole tensor

$$\dddot{\mathbf{Q}}_{ee'} = -\frac{e}{2} [3(\mathbf{r}_e\mathbf{r}_e + \mathbf{r}_{e'}\mathbf{r}_{e'}) - (|\mathbf{r}_e|^2 + |\mathbf{r}_{e'}|^2)\mathbf{I}] \quad \text{(IX.16)}$$

for Coulomb interacting particles e and e'. [The first two terms reduce to Eq. (IX.10) upon differentiation, while the terms involving $\mathbf{I}$ do not contribute to Eq. (IX.15).] The magnetic dipole radiation which ordinarily appears at this level in a multipole expansion (30) vanishes, since the local magnetic dipole moment

$$\mathbf{m}_{ee'} = -\frac{e}{4c}[\mathbf{r}_e \times \mathbf{v}_e + \mathbf{r}_{e'} \times \mathbf{v}_{e'}] \qquad \text{(IX.17)}$$

is time invariant when particles e and e' interact via a Coulomb force.

The total emission is obtained by integrating Eq. (IX.15) over all solid angles

$$\frac{d\bar{P}^T}{d\omega} = \frac{1}{270\pi c^5}\frac{1}{T}\frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3}\sum_{e\neq e'}\{3(\ddot{\mathbf{Q}}_{ee'})_\omega : (\ddot{\mathbf{Q}}_{ee'})_{-\omega} - (\bar{q}_{ee'})_\omega(\bar{q}_{ee'})_{-\omega}\} \qquad \text{(IX.18)}$$

where $\bar{q}_{ee'}$ is a scalar corresponding to Eq. (IX.10) formed by replacing the dyadic product by a scalar product. Its screened form is [cf. Eq. (IX.13)]

$$\bar{q}_{ee'} = \frac{3}{2\pi^2}\frac{ie^3}{m}\exp(-i\mathbf{k}\cdot\bar{\mathbf{r}}_e)\int d^3k'\,\frac{\exp(i\mathbf{k}'\cdot\mathbf{r}_{ee'})}{k'^2} \times \frac{\mathbf{k}'\cdot\bar{\mathbf{v}}_{ee'}}{D_L(k', -\mathbf{k}'\cdot\bar{\mathbf{v}}_e)D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_{e'})} \qquad \text{(IX.19)}$$

Fourier transforming Eqs. (IX.13) and (IX.19) in time, we substitute in Eq. (IX.18) and obtain

$$\begin{aligned}\frac{d\bar{P}^T}{d\omega} = &-\frac{e^6}{30\pi^3 m^2 c^5}\frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3}\frac{1}{T} \times \sum_{e\neq e'}\int d^3k'\int d^3k''\,\frac{\exp[i(\mathbf{k}'+\mathbf{k}'')\cdot(\mathbf{r}_{e0}-\mathbf{r}_{e'0})]}{k'^2k''^2}\\ &\times\Bigg\{3\left[\frac{\mathbf{k}'\bar{\mathbf{v}}_{e'\perp}+\bar{\mathbf{v}}_{e'\perp}\mathbf{k}'}{D_L(k', -\mathbf{k}'\cdot\bar{\mathbf{v}}_e)} - \frac{\mathbf{k}'\bar{\mathbf{v}}_{e\perp}+\bar{\mathbf{v}}_{e\perp}\mathbf{k}'}{D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_{e'})} + \frac{(\mathbf{I}k'^2 - 4\mathbf{k}'\mathbf{k}')(\mathbf{k}'\cdot\bar{\mathbf{v}}_{ee'})}{k'^2 D_L(k', -\mathbf{k}'\cdot\bar{\mathbf{v}}_e)D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_{e'})}\right]\\ &: \left[\frac{\mathbf{k}''\bar{\mathbf{v}}_{e'\perp}+\bar{\mathbf{v}}_{e'\perp}\mathbf{k}''}{D_L(k'', -\mathbf{k}''\cdot\bar{\mathbf{v}}_e)} - \frac{\mathbf{k}''\bar{\mathbf{v}}_{e\perp}+\bar{\mathbf{v}}_{e\perp}\mathbf{k}''}{D_L(k'', \mathbf{k}''\cdot\bar{\mathbf{v}}_{e'})} + \frac{(\mathbf{I}k''^2 - 4\mathbf{k}''\mathbf{k}'')(\mathbf{k}''\cdot\bar{\mathbf{v}}_{ee'})}{k''^2 D_L(k'', -\mathbf{k}''\cdot\bar{\mathbf{v}}_e)D_L(k'', \mathbf{k}''\cdot\bar{\mathbf{v}}_{e'})}\right]\end{aligned}$$

$$-\left.\frac{\mathbf{k}'\cdot\bar{\mathbf{v}}_{ee'}\mathbf{k}''\cdot\bar{\mathbf{v}}_{ee'}}{D_L(k', -\mathbf{k}'\cdot\bar{\mathbf{v}}_e)D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_e)\,D_L(k'', -\mathbf{k}''\cdot\bar{\mathbf{v}}_e)D_L(k'', \mathbf{k}''\cdot\bar{\mathbf{v}}_{e'})}\right\} \times \delta(\omega + \mathbf{k}'\cdot\bar{\mathbf{v}}_{ee'})\,\delta(-\omega + \mathbf{k}''\cdot\bar{\mathbf{v}}_{ee'}) \qquad \text{(IX.20)}$$

To calculate the average power per unit volume emitted over the time interval T, we sum the uncorrelated collisions in the following manner. The number of electrons e' with velocities in the interval $d^3\bar{v}_{e'}$ about $\bar{\mathbf{v}}_{e'}$ which will collide with a given electron e having velocity $\bar{\mathbf{v}}_e$ at an impact distance between $\mathbf{b}_{ee'}$ and $\mathbf{b}_{ee'} + d\mathbf{b}_{ee'}$ in time T is just the number of electrons in the angular element $d\phi$ of a cylindrical shell of radius $|b_{ee'}|$ and length $|\bar{\mathbf{v}}_{ee'}|T$,

$$dN = n|\bar{\mathbf{v}}_{ee'}|T\,d\phi|\mathbf{b}_{ee'}|\,db_{ee'}f_0(\bar{v}_{e'})d^3v_{e'} \qquad \text{(IX.21)}$$

In Eq. (IX.21), ϕ is the angle made by $\mathbf{b}_{ee'}$ with some arbitrary axis in the plane transverse to $\bar{\mathbf{v}}_{ee'}$ ($b_{ee'}$ and $\bar{\mathbf{v}}_{ee'}$ are orthogonal vectors).

For the total average power per unit volume, we multiply a single term in the sum in Eq. (IX.20) by the weighting factor, Eq. (IX.21), and the number density n of electrons e, integrate over all impacts allowable in a small-angle deflection theory, and average over the velocity distributions for particles e and e'. Without loss of generality, the initial displacement $\mathbf{r}_{e0} - \mathbf{r}_{e'0}$ may be considered as the impact vector for each collision. The transverse spectrum reduces upon performing these operations to

$$\frac{d\bar{P}^T}{d\omega} = \frac{8n^2e^6}{15\pi m^2c^5}\frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3}\int d^3\bar{v}_e\int d^3\bar{v}_{e'}f_0(\bar{v}_e)f_0(\bar{v}_{e'})\int d^3k'\,\frac{1}{k'^4} \times\left\{\frac{3k'^2\bar{v}_{e\perp}^2}{|D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_e)|^2} + \frac{8\omega^2}{|D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_e)|^2|D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_{e'})|^2}\right\} \times \delta(\omega + k'\cdot\bar{\mathbf{v}}_{ee'}) \qquad \text{(IX.22)}$$

Before discussing Eq. (IX.22) we shall write down the parallel expression for the emission of quadrupole longitudinal bremsstrahlung. The longitudinal spectrum is obtained by exactly the same process just discussed for transverse waves: Equation (VI.9) becomes the source current for the longitudinal part of Eq. (VI.15), the wave contribution is excerpted by local integration over $|\mathbf{k}|$ values near that given by Eq. (VIII.12), the angular distribution of radiation is integrated over, and finally collisions are summed. The emergent result,

$$\frac{d\bar{P}^L}{d\omega} = \frac{8n^2e^6}{15\pi m^2(3^{1/2}u_0)^5}\frac{(\omega^2-\omega^2)^{3/2}}{\omega^3}\int d^3\bar{v}_e \int d^3\bar{v}_{e'} f_0(\bar{v}_e) f_0(\bar{v}_{e'}) \int d^3k' \frac{1}{k'^4}$$
$$\times \left\{\frac{2k'^2\bar{v}_{e\perp}^2}{D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_e)} + \frac{23\omega^2}{4|D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_e)|^2|D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_{e'})|^2}\right\}$$
$$\times\ \delta(\omega + \mathbf{k}'\cdot\bar{\mathbf{v}}_{ee'}) \quad \text{(IX.23)}$$

is similar in structure to the transverse spectrum, differing by a wavelength ratio $(cu_0{}^{-1}3^{-1/2})^5$ [cf. Eqs. (VIII.12) and (VIII.13)] and numerical factors. These numerical factors arise for two reasons: first, there are two transverse polarizations as opposed to a single longitudinal polarization, and second, for any given collision, the Doppler shift (due to the mean motion of the two electrons through the plasma) for the finite-wavelength radiation differs for longitudinal and transverse waves.

C. *Comments on the Quadrupole Emission*

The first term in both spectra exhibits the customary logarithmic divergence for large values of $|\mathbf{k}'|$ (small impact distances) and must be cut off at a value $1/b_{\min}$ consistent with the small-angle scattering approximation. Because of the slow logarithmic nature of the divergence, it makes little difference whether $b_{\min}$ is taken as the typical 90° deflection impact distance ($b_{\min} \approx e^2/mu^2$) or the de Broglie wavelength ($b_{\min} \approx \hbar/mu$) for an average event. From the nondivergent term, little contribution is obtained for large values of $|\mathbf{k}'|$, so that the integral may here be extended to ∞ with but negligible error.

The $\mathbf{k}'$ integrands of Eqs. (IX.22) and (IX.23) exhibit resonances which correspond to the interaction of electrons through self-generated plasma oscillations. These resonances occur in the small $|\mathbf{k}'|$ region, $|\mathbf{k}'|\lambda_D \ll 1$, where Landau damping of the oscillations (i.e., Im D_L) is small. In the region where resonant effects are important, the principal contribution to the $\mathbf{k}'$ integrals is from the nondivergent terms. The resonance will be most strongly manifest when both $|D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_e)|^2$ and $|D_L(k', \mathbf{k}'\cdot\bar{\mathbf{v}}_e)|^2$ are nearly vanishing or when $\mathbf{k}'\cdot\bar{\mathbf{v}}_e \approx \mathbf{k}'\cdot\bar{\mathbf{v}}_{e'} \approx \pm\omega_p$. Using the δ functions, we then conclude that the resonance appears in the emitted spectra at the harmonic $2\omega_p$, because the possibility of emission at the difference frequency $\omega \approx 0$ is excluded by the shielding factor $(\omega^2 - \omega_p^2)^{3/2}$. Since longitudinal waves of frequency $\omega \cong 2\omega_p$ are strongly Landau damped we expect this nonlinear resonance to

appear only in the transverse bremsstrahlung spectrum. Tidman and Dupree (19) have studied the enhancement of the transverse bremsstrahlung spectrum near $\omega = \omega 2_p$ and have found that certain electron distributions composed of a tenuous energetic component coexisting with a thermal background can exhibit a significant resonance. The observed resonance around $\omega = 2\omega_p$ in the spectra of type II solar radio bursts has thus been explained as arising from enhanced electron–electron bremsstrahlung (31). The accompanying resonance in type II spectra at ω_p is explained as similarly enhanced electron–ion bremsstrahlung and occurs dominantly in dipole order. These Tidman-Dupree results are included in the transverse dipole spectrum of reference 28 and can be extracted by local integration near $\omega = \omega_p$ of Eq. (IV.13) of that paper.

For thermal equilibrium

$$f_0(\bar{v}_e) = (2\pi u_0^2)^{-3/2} \exp{-\bar{v}_e^2/2u_0^2} \quad \text{(IX.24)}$$

and the transverse and longitudinal spectra become

$$\frac{d\bar{P}^T}{d\omega} = \frac{32}{15\pi}\frac{n^2e^6}{m^2c^5}\frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3}u_0\int dk' \frac{1}{k'^3}\int_{-\infty}^{+\infty} dv \exp-\left(\frac{1}{4k'^2}+v^2\right)$$
$$\times\left(\frac{3k'^2}{|D_L^+|^2} + \frac{4}{|D_L^+|^2|D_L^-|^2}\right) \quad \text{(IX.25)}$$

and

$$\frac{d\bar{P}^L}{d\omega} = \frac{32}{15\pi}\frac{n^2e^6}{m^2(3^{1/2}u_0)^5}\frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3}u_0\int dk' \frac{1}{k'^3}\int_{-\infty}^{+\infty} dv \exp-\left(\frac{1}{4k'^2}+v^2\right)$$
$$\times\left(\frac{2k'^2}{|D_L^+|^2} + \frac{23}{8|D_L^+|^2|D_L^-|^2}\right) \quad \text{(IX.26)}$$

where, following the notation of Eq. (VI.19),

$$D_L^{\pm} = D_L\left[k'\frac{\omega}{u}, \omega(1/2 \pm k'v)\right] \quad \text{(IX.27)}$$

The integration variables in Eqs. (IX.25) and (IX.26) have been nondimensionalized, so that the logarithmically divergent term must now be cut off at a value $k'_{\max} \approx u_0/\omega b_{\min}$.

For steady-state thermal equilibrium conditions, the emissions of both transverse and longitudinal electron–electron bremsstrahlung are related to their absorption via collision processes through Kirchhoff's law. Electron–electron collisional damping may be thought of as a viscous effect, since it is only present when the finite wavelength of the spectral components is taken into consideration so that a macroscopic shearing of the electrons (regarded as a fluid) is effected. Alternatively, it may be looked upon as a finite wavelength resistance due to electron–electron collision processes.

This latter approach has been taken by DuBois and Gilinsky (11), who have calculated the dissipative conductivity component which results from thermal equilibrium electron–electron collisions. The conductivity components for transverse and longitudinal waves which we calculate from Eqs. (IX.25) and (IX.26) (the calculation is the same as that carried out in reference 28 for steady-state electron-ion dipole processes) are in exact agreement with those of DuBois and Gilinsky (including the logarithmic term).

Figure 7 is a graph of the integral

$$J = \int_0^\infty dk' \frac{1}{k'^3} \int_{-\infty}^{+\infty} dv \exp - \left(\frac{1}{4k'^2} + v^2\right) \frac{1}{|D_L^+|^2 |D_L^-|^2}$$

common to Eqs. (IX.25) and (IX.26) and normalized to its $D_L \to 1$ value of $2(\Pi)^{1/2}$. Because the logarithmically divergent terms in Eqs.

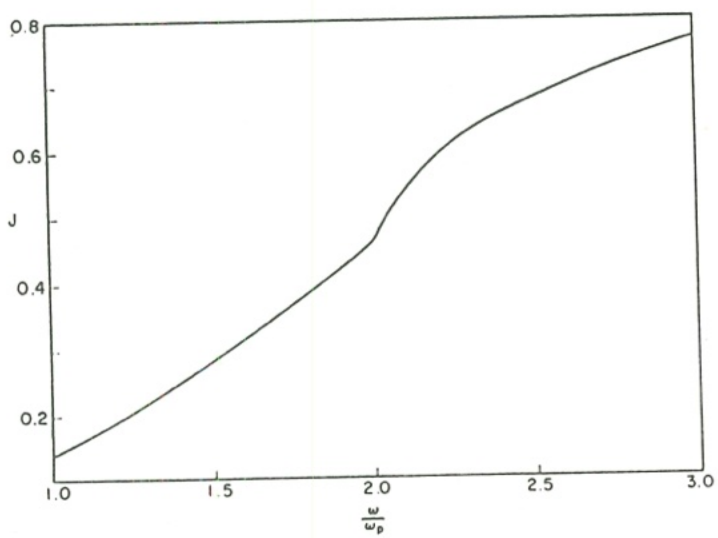

Fig. 7. Plot of the integral J versus frequency.

(IX.25) and (IX.26) are weighted toward large k', where the D_L's are close to unity, this integral displays the primary effect of shielding for thermal equilibrium electron–electron events.

As might be expected, shielding becomes decreasingly important as ω progresses higher above ω_p. This decrease results from the fact that collisions at impact distances comparable with a Debye length (where screening is most effective) take place too slowly to contribute to the radiation at high frequencies. High-frequency spectral components are emitted in closer impact collisions which are essentially two-body in nature. A small but perceptible inflection is noted in Figure 7 near $\omega = 2\omega_p$. This is the aforementioned collective resonance, which for thermal equilibrium conditions adds but a minute contribution to the overall emission because of the low level of longitudinal, Cerenkov-type oscillations in such plasmas (19). A curve similar to Figure 7 has been obtained by DuBois and Gilinsky (11) (cf. Fig. 3 of their paper).

For frequencies $\omega \gtrsim 3\omega_p$, Eq. (IX.17) can be approximately evaluated by ignoring shielding effects, i.e., by replacing $(\omega^2 - \omega_p^2)^{3/2}$ by ω^3 and letting $D_L \to 1$,

$$\frac{d\bar{P}^T}{d\omega} \approx \frac{64n^2e^6}{15m^2c^5} \int d\bar{v}_e \int dk' \frac{1}{k'^3} F(\bar{v}_e) F'\left(\bar{v}_e + \frac{\omega}{k}\right)[k'^2u^2 + 4\omega^2] \quad \text{(IX.28)}$$

In Eq. (IX.28) the F's are one-dimensional electron distributions, and use has been made of the fact that for isotropic distributions

$$\int d^2v_{e\perp} v_{e\perp}^2 f_0 = \tfrac{2}{3}u^2 F \quad \text{(IX.29)}$$

Further evaluation of Eq. (IX.28) depends on the explicit form of F. For purposes of a rough estimate, however, we assume that distributions of interest can be approximated by

$$F(\bar{v}_e) = \frac{1}{2u_0} \qquad |\bar{v}_e| < u_0 \quad \text{(IX.30)}$$
$$= 0 \qquad \text{otherwise}$$

Equation (IX.28) then becomes

$$\frac{dP^T}{d\omega} = \frac{16n^2e^6}{15m^2c^5} u \left\{2 \ln \frac{2u_0}{\omega b_{\min}} + \frac{10}{3}\right\} \quad \text{(IX.31)}$$

For frequencies at which the theory is valid, $\omega \lesssim 0.1\omega_{90°}$, the first term in Eq. (IX.31) dominates. (Toward the high-frequency end, the

10/3 may add a factor of ~ 2 to our estimate.) Comparing this logarithmically dominant portion of Eq. (IX.31) with the high-frequency form of the transverse dipole spectrum as derived from Section VIII, we find that the ratio, quadrupole to dipole,

$$P^{\mathrm{IV}}/P^{\mathrm{II}} \approx (2/5z)u_0^2/c^2$$

(z is the ionic charge), is small even for rather energetic electron distributions. We conclude, therefore, that except for those electron distributions which lead to a considerable enhancement of the bremsstrahlung in the neighborhood of $\omega = 2\omega_p$ via strong collective effects, radiation loss due to electron–electron collisions is negligibly small in the frequency interval considered.

D. *Remarks on Other Effects*

We should at this point discuss briefly two points: k^2 bremsstrahlung corrections from electron–ion collisions, and the role of relativistic kinematics in the higher-order bremsstrahlung emission.

1. *Electron–Ion k^2 Corrections*

A treatment very similar to that presented in the preceding sections has been carried out to assess the effects of electron–ion collisions on higher-order longitudinal and transverse bremsstrahlung spectra. In this problem it is important to carry the electron–ion source current to octupole order, for in Eq. (VI.15) there exists a finite coupling between the dipole and octupole current densities. This dipole–octupole coupling leads to an emission comparable to that obtained from the squared form of the quadrupole $\mathbf{j}_s$. [For isotropic f_0, the dipole–quadrupole interaction vanishes because of the angular symmetry in the integrand of Eq. (VI.15).] The calculation has been carried out assuming uncorrelated ions and for an electron–ion Coulomb interaction force. The superposition principle has also been used to properly account for shielding.

As in the dipole case, a resonance near $\omega = \omega_p$ is found. This resonance arises from a nonlinear coupling between the electron plasma oscillation ($\omega \approx \omega_p$) and the ion wave ($\omega \approx 0$) associated with each of the interacting particles.

In thermal equilibrium, the k^2 electron–ion bremsstrahlung corrections can be compared (via Kirchhoff's law) with the finite-wavelength conductivity corrections calculated by Berk (8) for electron–ion collisions. To logarithmically dominant order, our emission results lead in this case to expressions for the dissipative conductivity components which agree with Berk's calculations for both transverse and longitudinal waves.

For high frequencies the k^2 correction is small compared with the dipole emission in the approximate ratio

$$P^{\mathrm{IV}}/P^{\mathrm{II}} \approx 4u^2/5c^2 \qquad \text{(IX.32)}$$

2. *Relativistic Particle Dynamics*

We have seen in Eqs. (IX.30) and (IX.32) that the transverse spectrum arising from electron–electron collisions and the finite-wavelength correction to the electron–ion emission are both relativistically small in comparison with the electron–ion dipole emission. The question now arises as to whether inclusion of relativistic effects in the particle equations of motion [i.e., replacing Eqs. (IX.7) and (IX.8) and the corresponding electron–ion forms by their relativistic generalizations, including the full electromagnetic interaction between particles] generates corrections in *dipole* order of the same magnitude as those we have calculated.

We argue that relativistic dipole corrections to the electron–electron transverse spectrum will be of $O(u^4/c^4)$ and hence generally smaller than those which we have considered. This follows from the fact that the dipole power emission is proportional to $|\ddot{\mathrm{p}}(\omega)|^2$ (see reference 28), where p is the local dipole moment for each interaction. To lowest nonvanishing order $\ddot{\mathrm{p}} \sim u^2/c^2$ and hence the dipole power is of $O(u^4/c^4)$.

For electron–ion encounters, however, we do expect corrections to the dipole spectrum of $O(u^2/c^2)$. This follows from the fact that $\ddot{\mathrm{p}}(\omega)$ now has an $O(1)$ contribution, so that $|\ddot{\mathrm{p}}(\omega)|^2$ can have an $O(u^2/c^2)$ term for this type of interaction. Relativistic considerations of the problem will lead to dipole electron–ion corrections comparable to those given by Eq. (IX.32). This problem has not yet been attacked.

X. The Scattering and Coupling of Waves of Density Fluctuations

A. *Derivation of Formula*

We now consider the scattering and coupling of waves by density fluctuations. We saw in Section VII that the source currents for this case were given by the acceleration of the first-order density fluctuations by the first-order fields. We shall neglect radiation due to the first-order magnetic field. Since only the electrons are appreciably accelerated, only electron-density fluctuations are important. The electron-density fluctuations can originate in a number of ways. First, the electrons tend to follow the ions maintaining charge neutrality, and so there are electron-density fluctuations associated with the ion-density fluctuations. Second, there are electron-density fluctuations due to the random motion of the electrons. Since an electron surrounds itself with a positive charged Debye cloud which neutralizes its charge (one electron is missing from the cloud) these fluctuations are important only for regions small compared to a Debye length and are effective in scattering waves whose wavelengths are short compared to a Debye length. Third, there are density fluctuations associated with longitudinal electron oscillations.

Here we shall confine ourselves to the scattering and coupling of long-wavelength waves; thus we omit the second source of fluctuations given above. We shall assume that we have a spectrum of waves. If we take the plasma to be contained in a box of volume L^3, then we may write the electric field in the form

$$\mathbf{E}(r,t) = \sum_{\mathbf{k}} \mathbf{E}(\mathbf{k}) \exp\{i[\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t]\} \tag{X.1}$$

The waves can be either longitudinal or transverse depending on whether $\mathbf{E}(\mathbf{k})$ is parallel or perpendicular to $\mathbf{k}$. [$\omega(\mathbf{k})$, of course, depends on the type of wave we have.]

In the long-wavelength limit where thermal motions can be neglected compared to the phase velocities of waves, Eq. (VII.13) for the source current may be written as

$$\frac{\partial \mathbf{j}_{2s}}{\partial t} = \frac{e^2}{m} \mathbf{E}_1(\mathbf{r},t) n_{-1}(\mathbf{r},t) \tag{X.2}$$

Fourier transforming Eq. (X.2) in space and time gives

$$\mathbf{j}_{2s}(\mathbf{k},\omega) = \sum_{k'} \frac{ie^2\mathbf{E}(\mathbf{k}')}{m\omega} n_-[\mathbf{k} - \mathbf{k}', \omega - \omega(\mathbf{k}')] \tag{X.3}$$

We now introduce Eq. (X.3) as the sources in Eq. (VI.15) and extract the wave contribution by integrating over the resonant values

of $|k|$ to obtain

$$W_\omega = \frac{e^4(\omega^2 - \omega_p^2)^{1/2}}{8\pi^2 m_e^2} \int d\Omega_{kL,T} \cdot \sum_{k',k''} \begin{Bmatrix} \dfrac{\omega_p \hat{\mathbf{k}}_L \cdot \mathbf{E}(\mathbf{k}') n_e[\mathbf{k}_L - \mathbf{k}', \omega(\mathbf{k}_L) - \omega(\mathbf{k}')] \hat{\mathbf{k}}_L \cdot \mathbf{E}(-\mathbf{k}'') n_e[-\mathbf{k}_L + \mathbf{k}'', \omega(\mathbf{k}'') - \omega(\mathbf{k}_L)]}{(3)^{3/2} u_0^3 \omega^2} \\ \dfrac{\omega \hat{\mathbf{k}}_T \times \mathbf{E}(\mathbf{k}') n_e[\mathbf{k}_T - \mathbf{k}', \omega(\mathbf{k}_T) - \omega(\mathbf{k}')] \hat{\mathbf{k}}_T \times \mathbf{E}(-\mathbf{k}'') n_e[-\mathbf{k}_T + \mathbf{k}'', \omega(\mathbf{k}'') - \omega(\mathbf{k}_T)]}{c^3 \omega^2} \end{Bmatrix} \tag{X.4}$$

The integration is to be carried out over angles of longitudinal emission, $d\Omega_{kL}$, for the upper multiplying factor and angles of transverse emission, $d\Omega_{kT}$, for the lower one. The values of $|k_T|$ and $|k_L|$ are those which make D_T and D_L given by Eqs. (VI.18) and (VI.19) zero, except for the small imaginary parts.

If we ensemble average Eq. (X.4) and assume that waves of different $\mathbf{k}$ are randomly phased, then the only terms which contribute are those with $\mathbf{k}'$ equal to $\mathbf{k}''$. Furthermore, if the box size is taken to be very large, the sum over $\mathbf{k}'$ may be converted into an integral in the usual way.

Equation (X.4) gives the total energy emitted at frequency ω. If we assume that the process goes on from $t = -T/2$ to $t = T/2$, then we obtain the average power emitted per unit volume by dividing by L^3T,

$$\bar{P}_\omega = \frac{e^4(\omega^2 - \omega_p^2)^{1/2}}{64\pi^5 m_e^2 T} \int d\Omega_{kL,T} \int d^3k' \cdot \begin{Bmatrix} \dfrac{1}{3^{3/2} u_0^3 \omega} |\hat{\mathbf{k}}_L \cdot \mathbf{E}(k')|^2 \cdot |n_-[\mathbf{k}_L - \mathbf{k}', \omega(\mathbf{k}_L) - \omega(\mathbf{k}')]|^2 \\ \dfrac{1}{c^3\omega} |\hat{\mathbf{k}}_T \times \mathbf{E}(\mathbf{k}')|^2 \cdot |n_-[\mathbf{k}_T - \mathbf{k}', \omega(\mathbf{k}_T) - \omega(\mathbf{k}')]|^2 \end{Bmatrix} \tag{X.5}$$

In general, $|n_-(k,\omega)|^2$ will be proportional to T, as we shall see shortly for some special examples, and hence $\bar{P}_\omega$ is independent of T.

For frequencies ω much higher than the plasma frequency where dielectric shielding can be neglected, that part of Eq. (X.5) which represents the scattering of transverse waves reduces to the results obtained by Rosenbluth and Rostoker (32). However, the dielectric corrections are important near the plasma frequency.

We now proceed to consider the particular cases where the electron-density fluctuations appearing in Eq. (X.5) result from (*1*) electron neutralization of ion-density fluctuations and (*2*) longitudinal electron oscillations in the plasma.

B. *Electron Neutralization of Ion-Density Fluctuations*

The electron-density fluctuations can here be related to the ion-density fluctuations. We disregard ion motions, so that the electron distribution perturbed by interactions with discrete ions is independent of time. The perturbation is obtained by solving the linearized Vlasov-Poisson equation with these ion sources,

$$n_e(\mathbf{k},\omega) = n_0 \int f(\mathbf{k},\omega,v)d^3v$$
$$= -\frac{2\pi z \omega_p^2 n_+(\mathbf{k})}{k^2 D_L(k,0)}\,\delta(\omega) P \int \frac{1}{v}\frac{\partial F}{\partial v}\,dv \tag{X.6}$$

where $n_+(\mathbf{k})$ is the Fourier-transformed ion number density, F is the one-dimensional electron distribution normalized to unity, $D_L(\mathbf{k},0)$ is the static longitudinal dielectric function, and P indicates that the principal value of the integral over v is to be taken.

With Eq. (X.6) for the electron-density fluctuations, Eq. (X.5) gives the average power per unit volume scattered and coupled by such density fluctuations. The longitudinal power spectrum becomes

$$\bar{P}_\omega = \frac{z^2e^4}{48(3)^{1/2}\pi^3}\,\frac{1}{m_e^2u_0^3}\,\frac{(\omega^2-\omega_p^2)^{1/2}\omega_p^5}{T\omega^2}\left\{P\int\frac{1}{v}\frac{\partial F}{\partial v}\,dv\right\}^2$$
$$\cdot \int d\Omega_{k_L}\int d^3k'|\hat{\mathbf{k}}_L\cdot\mathbf{E}(\mathbf{k}')|^2$$
$$\cdot \left|\frac{n_+(\mathbf{k}_L-\mathbf{k}')\,\delta[\omega(\mathbf{k}_L)-\omega(\mathbf{k}')]}{(\mathbf{k}_L-\mathbf{k}')^2 D_L(\mathbf{k}_L-\mathbf{k}',0)}\right|^2 \tag{X.7}$$

with an analogous expression for the transverse emission. Here, the limiting procedure which is formally represented by $\delta^2(\omega-\omega_0)$ is to be interpreted as $(T/2\pi)\,\delta(\omega-\omega_0)$ where T is the duration of the emission process. The power emission is thus seen to be independent of T. To apply this to a single wave, the integral $\int d^3k'$ is replaced by $(2\pi)^3/L^3$.

If the ions are uncorrelated, the ensemble averaged value of $|n_+(\mathbf{k}_{L,T}-\mathbf{k}_0)|^2$ is just equal to N, the number of ions in the volume L^3 being considered. However, as mentioned in conjunction with the bremsstrahlung calculation, the presence of strong ion correlations (ion waves) can enhance this factor by many orders of magnitude.

For a single wave with wave vector $\mathbf{k}_0$ and frequency $\omega(\mathbf{k}_0)$ interacting with density fluctuations associated with the interaction of a Maxwellian electron distribution with discrete uncorrelated ions, we can write Eq. (X.7) as

$$P_\omega = \frac{z^2e^4}{12(3)^{1/2}\pi}\,\frac{\kappa^4}{m_e^2u_0^3}\,\frac{(\omega^2(\mathbf{k}_0)-\omega_p^2)^{1/2}}{\omega(\mathbf{k}_0)}\,n_+$$
$$\cdot\int d\Omega_{k_L}\frac{|\hat{\mathbf{k}}_L\cdot\mathbf{E}(\mathbf{k}_0)|^2}{\{|\mathbf{k}_L-\mathbf{k}_0|^2+\kappa^2\}^2}\,\delta[\omega(\mathbf{k}_L)-\omega(\mathbf{k}_0)] \tag{X.8}$$

where $\kappa = \omega_p/u_0$ is the reciprocal Debye distance.

For long-wavelength radiation, $|\mathbf{k}_{L,T}-\mathbf{k}_0| \ll \kappa$ and hence can be neglected in evaluating the integral in Eq. (X.8). Integrating over frequencies, we pick up only the resonance at $\omega(\mathbf{k}_L)=\omega(\mathbf{k}_0)$. We may consequently express Eq. (X.8) (and the corresponding expression for the transverse spectrum in our parallel notation) as

$$\bar{P}_{\omega(k_0)} = \frac{z^2e^4[\omega^2(\mathbf{k}_0)-\omega_p^2]^{1/2}}{m_e^2\omega(\mathbf{k}_0)}\,n_+|\mathbf{E}(\mathbf{k}_0)|^2\begin{Bmatrix}\dfrac{1}{3^{3/2}u_0^3}\\[2ex] \dfrac{2}{3c^3}\end{Bmatrix} \tag{X.9}$$

If $\mathbf{E}(\mathbf{k}_0)$ represents a longitudinal wave, the upper coefficient gives the wave scattering, while the lower one gives the coupling to transverse waves of frequency $\omega(\mathbf{k}_0) \cong \omega_p$. The energy density per frequency interval of a longitudinal wave of frequency $\omega(\mathbf{k}_L)$ and wave vector $\mathbf{k}_L$ can be expressed in terms of its electric-field amplitude by

$$\mathscr{E}_L(\omega)\,d\omega = \frac{1}{4\pi}\,|\mathbf{E}(\mathbf{k}_L)|^2\,\frac{k_L^2}{2\pi}\frac{dk_L}{d\omega}\,d\omega \tag{X.10}$$

Here we have used Eq. (VI.26) for the wave energy density. The factor $(k_L^2/2\pi)\,dk_L/d\omega$ represents the density of k_L states per frequency interval. If we sum Eq. (X.9) over all waves in the frequency interval $d\omega$, we determine the scattered power to be

$$\bar{P}_{\omega,L}\,d\omega = [ze^2/9(3)^{1/2}](\omega_p/m_eu_0^3)(\omega^2-\omega_p^2)^{1/2}\mathscr{E}_L(\omega)\,d\omega \tag{X.11}$$

while the power coupled to transverse waves is

$$\bar{P}_{\omega,T}\, d\omega = \tfrac{1}{3} z e^2 (\omega_p/m_e c^3)(\omega^2 - \omega_p^2)^{1/2} \mathscr{E}_L(\omega)\, d\omega \qquad \text{(X.12)}$$

In deriving Eq. (X.12) we have noted the fact that the longitudinal wave couples to only one of the two transverse polarizations, viz., to that wave polarized in the plane determined by the wave vectors of the incident longitudinal and outgoing transverse waves. Dawson and Oberman (3) have derived these results from absorption calculations and detailed balance arguments. The results given here are in agreement with theirs.

In a very similar manner, we can compute the scattering and coupling when $\mathbf{E}(\mathbf{k}_0)$ is a transverse wave in terms of the energy density $\mathscr{E}^T(\omega)$ for such waves. We find the power coupled to longitudinal waves to be

$$\bar{P}_{\omega L}\, d\omega = [z e^2/9(3)^{1/2}](\omega_p/m_e u_0^3)(\omega^2 - \omega_p^2)^{1/2} \mathscr{E}_T(\omega)\, d\omega \qquad \text{(X.13)}$$

and that scattered as transverse waves (again remarking that the prevailing transverse wave can couple to only one of the two possible polarizations of the scattered transverse wave) to be

$$\bar{P}_{\omega T}\, d\omega = \tfrac{1}{6} z e^2 (\omega_p^2/m_e c^3)[(\omega^2 - \omega_p^2)^{1/2}/\omega] \mathscr{E}_T(\omega)\, d\omega \qquad \text{(X.14)}$$

The latter two results are in agreement with those obtained by Berk (8), who extended the Dawson-Oberman (3) model to include finite wavelength as well as electromagnetic corrections to the absorption coefficient.

We remark in passing that by applying the principle of detailed balance to a thermal equilibrium plasma, we may equate the coupled powers, Eqs. (X.12) and (X.13), to find that the steady-state wave energy densities, $\bar{\mathscr{E}}_L(\omega)$ and $\bar{\mathscr{E}}_T(\omega)$, are in the ratio

$$\bar{\mathscr{E}}_L(\omega) = (c/u_0)^3 [1/6(3)^{1/2}] \bar{\mathscr{E}}_T(\omega) \qquad \text{(X.15)}$$

and that we have equipartition, viz.,

$$|\bar{\mathbf{E}}(\mathbf{k}_L)|^2 = |\bar{\mathbf{E}}(\mathbf{k}_T)|^2 \qquad \text{(X.16)}$$

In Eq. (X.15), $\bar{\mathscr{E}}_T(\omega)$ includes both polarizations of the transverse wave while Eq. (X.16) involves a single transverse polarization. We shall verify Eq. (X.15) directly in Subsection E by balancing emission against absorption.

C. *Cross Sections for Scattering and Coupling of Waves*

We may also use Eqs. (X.11)–(X.14) to compute cross sections for the various processes. To do this we write for the scattered power

$$\bar{P} = \bar{\mathscr{E}} v_g n_+ \sigma \qquad \text{(X.17)}$$

Here $\bar{P}$ is the scattered power, $\bar{\mathscr{E}}$ is the wave energy density, v_g is the group velocity for the waves, n_+ is the ion density, and σ is the scattering cross section. We may note that in the processes being discussed, the ions act as scattering centers. However, it is the electrons that associate themselves with the ions to preserve charge neutrality which really do the scattering. Now the group velocities for the longitudinal and transverse waves are given by

$$v_g = \frac{d\omega}{dk} = \begin{Bmatrix} 3^{1/2} u_0 \\ c \end{Bmatrix} (1 - \omega_p^2/\omega^2)^{1/2} \qquad \text{(X.18)}$$

Making use of Eqs. (X.18), (X.17), and (X.11)–(X.14), we find for the various cross sections

$$\begin{Bmatrix} \sigma_{LL} & \sigma_{LT} \\ \sigma_{TL} & \sigma_{TT} \end{Bmatrix} = \frac{4\pi e^4 z^2}{m_e^2} \begin{Bmatrix} 1/3a^4 & 2/3c^3 a \\ 1/3a^3 c & 2/3c^4 \end{Bmatrix} \qquad \text{(X.19)}$$

where a is $3^{1/2} u_0$. Here the first subscript refers to the incident wave and the second to the scattered wave. The cross section for transverse–transverse scattering is equal to the Thomson cross section. The longitudinal–longitudinal cross section is larger by the factor $c^4/2a^4$. We should again stress that ion correlations can greatly affect the scattering process. The above results for uncorrelated ions give, more or less, the minimum amount of scattering.

D. *Longitudinal Electron Oscillations on the Plasma*

As a second example illustrating the application of Eq. (X.5) we look at the generation of transverse waves by the interaction of two or more longitudinal waves. This problem has been investigated by Sturrock (33), by Aamodt and Drummond (34), and by Boyd (35).

The density fluctuations appearing in Eq. (X.5) are now those electron-density fluctuations associated with the passage of longitudinal waves through the plasma. They can therefore be determined from a solution of Poisson's equation. If we assume that we have the spectrum of longitudinal waves given by Eq. (X.1), the quantity n_e which appears in Eq. (X.5) can be written as

$$n_e[\mathbf{k}_{L,T} - \mathbf{k}', \omega(\mathbf{k}'_{L,T}) - \omega(\mathbf{k}')] = -\frac{i4\pi^3}{e}\sum_{k''} k''|\mathbf{E}(\mathbf{k}'')|\delta(\mathbf{k}_{L,T} - \mathbf{k}' - \mathbf{k}'') \times \delta[\omega(\mathbf{k}_{L,T}) - \omega(\mathbf{k}') - \omega(\mathbf{k}'')] \qquad (X.20)$$

We have already assumed that $\mathbf{k}''$ is the wave vector of a longitudinal wave [and therefore $\mathbf{k}''$ is parallel to $\mathbf{E}(\mathbf{k}'')$ in Eq. (X.20)]. We now further confine ourselves to the situation where $\mathbf{k}'$ also represents a longitudinal wave. The δ functions physically represent the conservation of momentum and energy for the "collision" of these two longitudinal waves.

Since $\mathbf{k}'$ and $\mathbf{k}''$ are both longitudinal waves, $\omega(\mathbf{k}')$ and $\omega(\mathbf{k}'')$ each lie in the range $\omega_p \leq \omega \gtrsim 1.5\omega_p$. The quantity $\omega(\mathbf{k}_{L,T})$ is thus at least as large as $2\omega_p$. Since longitudinal waves cannot propagate at this frequency, we see that it is energetically impossible for two longitudinal waves to couple to a third. We shall thus be concerned only with the transverse part of Eq. (X.5).

We further note that since $\mathbf{k}'$ and $\mathbf{k}''$ are much greater in magnitude than $\mathbf{k}_T$ (except for relativistically energetic plasmas), transverse waves can only be produced by the nearly "head-on collision" of the two longitudinal waves.

If we introduce Eq. (X.20) for the density fluctuations appearing in Eq. (X.5), we obtain for the transverse emission

$$\bar{P}_\omega = \frac{e^2\pi}{4m_e^2T}\frac{[\omega^2(|\mathbf{k}_T|) - \omega_p^2]^{1/2}(\omega|\mathbf{k}_T|)}{c^3} \times \int d\Omega_{k_T}\int d^3k' \frac{|\hat{\mathbf{k}}_T \times \mathbf{E}(\mathbf{k}')|^2}{\omega^2} \times \sum_{k'',k'''} k''k'''|\mathbf{E}(\mathbf{k}'')||\mathbf{E}^*(\mathbf{k}''')| \times \delta(\mathbf{k}_T - \mathbf{k}' - \mathbf{k}'')\,\delta(\mathbf{k}_T - \mathbf{k}' - \mathbf{k}''') \times \delta[\omega(\mathbf{k}_T) - \omega(\mathbf{k}') - \omega(\mathbf{k}'')] \times \delta[\omega(\mathbf{k}_T) - \omega(\mathbf{k}') - \omega(\mathbf{k}''')] \qquad (X.21)$$

Assuming that the $\mathbf{k}''$'s and the $\mathbf{k}'''$'s are dense, we can convert the summations into integrals and use the δ functions on $\mathbf{k}''$ and $\mathbf{k}'''$ to carry out these integrals.

If we, in addition, split the $\mathbf{k}'$ integration so that we consider in pairs, the effects produced by wave $\mathbf{k}'$ scattering off density fluctuations produced by wave $\mathbf{k}_T - \mathbf{k}'$, and wave $\mathbf{k}_T - \mathbf{k}'$ scattering off density fluctuations produced by wave $\mathbf{k}'$, $\bar{P}_\omega$ can be expressed as

$$\bar{P}_\omega = \frac{e^2}{m_e^2c^3}\frac{L^6}{256\pi^5T}(\omega^2(|\mathbf{k}_T|) - \omega_p^2)^{1/2}\omega(|\mathbf{k}_T|) \times \int d\Omega_{k_T}\int d^3k\frac{1}{2}\frac{|\mathbf{k}_T \times \mathbf{k}'|^2}{|\mathbf{k}_T|^2} \times \frac{1}{\omega^2(|\mathbf{k}_T|)}\left[\frac{|\mathbf{k}_T - \mathbf{k}'|^2}{|\mathbf{k}'|^2} + \frac{|\mathbf{k}'|^2}{|\mathbf{k}_T - \mathbf{k}'|^2}\right] \times \{|\mathbf{E}(\mathbf{k}')||\mathbf{E}(\mathbf{k}_T - \mathbf{k}')|\,\delta[\omega(\mathbf{k}_T) - \omega(\mathbf{k}') - \omega(\mathbf{k}_T - \mathbf{k}')]\}^2 \qquad (X.22)$$

where we have used the fact that $\mathbf{E}(\mathbf{k}')$ and $\mathbf{E}(\mathbf{k}_T - \mathbf{k}')$ represent longitudinal waves.

From Eq. (X.22) we note that the two collisional processes are additive. For conditions where the energy spectrum of longitudinal waves is nearly constant, we perform the angular integration

$$\bar{P}_\omega = \frac{e^2}{12\pi}\frac{[\omega^2(\mathbf{k}_T) - \omega_p^2]^{1/2}}{m_e^2c^3}\,\omega(|\mathbf{k}_T|)\frac{L^3}{T}\left(\frac{L}{2\pi}\right)^3 \times \int d^3k'\frac{1}{\omega^2(|\mathbf{k}_T|)}\left\{\frac{\omega^2(|\mathbf{k}_T|) - \omega_p^2}{c^2} + k'^2\right\} \times |\mathbf{E}(\mathbf{k}')|^2|\mathbf{E}(-\mathbf{k}')|^2\,\delta^2[\omega(|\mathbf{k}_T|) - 2\omega(\mathbf{k}')] \qquad (X.23)$$

where again $\delta^2[\omega(\mathbf{k}_T) - 2\omega(\mathbf{k}')]$ is to be interpreted as $(T/2\pi)[\omega(\mathbf{k}')]$. We have substituted for $|\mathbf{k}_T|$ the value given by Eq. (VIII.13).

When we now integrate over frequencies of emission ω, we are left with

$$\bar{P}_\omega = \frac{L^3}{48\pi^2}\frac{e^2}{m_e^2c^3}\left(\frac{L}{2\pi}\right)^3\int d^3k' \times \frac{(4\omega^2(\mathbf{k}') - \omega_p^2)^{1/2}}{\omega(\mathbf{k}')}\left(\frac{4\omega^2(\mathbf{k}') - \omega_p^2}{c^2} + k'^2\right) \times |\mathbf{E}(\mathbf{k}')|^2|\mathbf{E}(-\mathbf{k}')|^2 \qquad (X.24)$$

If we estimate $\bar{P}_\omega$ by assuming that the average value of $\omega(\mathbf{k}')$ for all waves is given by $\bar{\omega}(\mathbf{k}') = \alpha\omega_p$, where $1 \leq \alpha \gtrsim 1.5$, then

$$\bar{P}_\omega = \frac{L^3}{48\pi^2}\frac{(4\alpha^2 - 1)^{1/2}}{\alpha}\frac{e^2}{m_e^2c^3}\left(\frac{L}{2\pi}\right)^3 \int d^3k'$$

$$\times\ [\omega_p^2c^{-2}(4\alpha^2 - 1) + k'^2]|\mathbf{E}(\mathbf{k}')|^2|\mathbf{E}(-\mathbf{k}')|^2 \qquad \text{(X.25)}$$

For long-wavelength longitudinal oscillations, k'^2 may be small by comparison with the first term. Equation (X.25) gives the total scattered power. To obtain the power scattered per unit volume in terms of the mean square of the electric-field strength we must divide by L^6.

E. *Wave Spectra in a Steady-State Plasma*

If we have a plasma maintained in the steady state, we can combine our emission calculations with the absorption calculations to find the steady-state wave spectra in the plasma. Some steady-state plasmas which might be considered are thermal equilibrium plasmas and non-equilibrium but steady discharges. Here we shall confine ourselves to isotropic electron distributions.

The conductivity for the case of uncorrelated ions, as found in Section V, is given by

$$\sigma = \sigma_0(1 + \bar{\sigma}_1) \qquad \text{(X.26)}$$

where

$$\sigma_0 = i\omega_p^2/4\pi\omega \qquad \text{(X.27)}$$

and

$$\bar{\sigma}_1 = \frac{2Ze^2}{3\pi m_e\omega^2}\cdot\int_0^{k_{\max}} dk\, k^3 \left\{\frac{1}{D_L(k,\omega)} - \frac{1}{D_L(k,0)}\right\} \qquad \text{(X.28)}$$

The cutoff in Eq. (X.25) at $k_{\max} \sim mu_0^2/Ze^2$ (u_0 = rms electron velocity) is the usual cutoff introduced in order to prevent the divergence for small impact parameters $(1/k)$.

In terms of σ, the dispersion relation for transverse waves propagating through the plasma is

$$k^2c^2 - \omega^2 - 4\pi i\omega\sigma = 0 \qquad \text{(X.29)}$$

Substituting σ from Eqs. (X.26) and (X.27) into Eq. (X.29) gives

$$k^2c^2 - \omega^2 + \omega_p^2 + \omega_p^2\bar{\sigma}_1 = 0 \qquad \text{(X.30)}$$

If Eq. (X.30) is solved for ω for real k, then the imaginary part of ω gives the time rate of decay of the amplitude of this pure k mode [decay

going as $\exp(-\mathrm{Im}\,\omega t)$]. The e^{-1} time for the energy is $\frac{1}{2}\,\mathrm{Im}\,\omega$. We may solve Eq. (X.30) for the imaginary part of ω by making use of the fact that $|\bar{\sigma}_1|$ is small. We thus find

$$\mathrm{Im}\,\omega = \tfrac{1}{2}[\omega_p^2/\pm(\omega_p^2 + k^2c^2)^{1/2}]\,\mathrm{Im}\,\bar{\sigma}_1 \qquad \text{(X.31)}$$

$$\omega \cong \pm(\omega_p^2 + k^2c^2)^{1/2}$$

Hence, the decay time for transverse waves is

$$\tau_T = \frac{\omega}{\omega_p^2\,\mathrm{Im}\,\bar{\sigma}_1}$$

$$= \frac{3\pi m_e\omega^3}{2Ze^2\omega_p^2}\left(\int_0^{k_{\max}} dk\, k^2\,\mathrm{Im}\frac{1}{D_L(k,\omega)}\right)^{-1} \qquad \text{(X.32)}$$

where we have used the fact that $\mathrm{Im}\,D_L(k,0) = 0$.

In the steady state, the rate of absorption of transverse waves is balanced by their rate of emission, provided we can neglect the escape of radiation from the plasma. The average rate of absorption of transverse-wave energy per unit volume per unit frequency interval is just the steady-state energy density $\bar{\mathscr{E}}_{\omega T}$ divided by the decay time for such transverse waves as those given by Eq. (X.32). If we balance the absorption by the emission of transverse waves as represented by Eq. (VIII.22), we may write the expression for $\bar{\mathscr{E}}_{\omega T}$,

$$\bar{\mathscr{E}}_{\omega T} = \frac{4ze^4n_+n_-\omega^2}{mc^3\omega_p^2}(\omega^2 - \omega_p^2)^{1/2}$$

$$\cdot\frac{\displaystyle\iint d^3k\, d^3v_e f(v_e)\frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v}_e)}{k^2|D_L(k,\omega)|^2}}{\displaystyle\int_0^{k_{\max}} dk\, k^2\frac{\mathrm{Im}\,D_L^*}{|D_L(k,\omega)|^2}} \qquad \text{(X.33)}$$

Equation (X.33) can be somewhat simplified by integrating the numerator over velocity space and introducing the value of $\mathrm{Im}\,D_L^*$ obtained from Eq. (VI.10), viz.,

$$\mathrm{Im}\,D_L^*(k,\omega) = (\pi\omega_p^2/k^2)F'(\omega/k) \qquad \text{(X.34)}$$

In Eq. (X.34) F is once again the one-dimensional distribution function normalized to unity. Equation (X.33) consequently becomes

$$\bar{\mathscr{E}}_{\omega T} = -\frac{16ze^4 n_+ n_-}{mc^3} \frac{\omega^2(\omega^2 - \omega_p^2)^{1/2}}{\omega_p^4} \times \int_0^{k_{\max}} dk \frac{F(\omega/k)}{k|D_L(k,\omega)|^2} \bigg/ \int_0^{k_{\max}} dk \frac{F'(\omega/k)}{|D_L(k,\omega)|^2} \quad \text{(X.35)}$$

The steady-state energy density of longitudinal waves can be similarly calculated. The decay time for such waves is found to be

$$\tau_L = \frac{1}{\omega_p \operatorname{Im} \bar{\sigma}_1} = \frac{3\pi m_e \omega_p}{2ze^2} \cdot \left(\int_0^{k_{\max}} dk \frac{k^2}{D_L(k,\omega)}\right)^{-1} \quad \text{(X.36)}$$

Landau damping has not been included in this calculation. For $k^{-1} > 5\lambda_D$, where λ_D is the Debye length, Landau damping is unimportant for a thermal distribution. One could include it in these calculations. However, if one does this, Cerenkov emission of longitudinal waves (the inverse process to Landau damping) must also be included. Calculations including these effects have been made by Perkins and Salpeter (36). From Eqs. (VIII.22) and (X.36) we thus obtain the steady-state energy level of longitudinal oscillations,

$$\bar{\mathscr{E}}_{\omega L} = \frac{-8ze^4 n_+ n_-}{3(3)^{1/2} m u_0^3} \frac{1}{\omega_p^2} (\omega^2 - \omega_p^2)^{1/2} \times \left[\int_0^{k_{\max}} dk \frac{F(\omega_p/k)}{k|D_L(k,\omega_p)|^2} \bigg/ \int_0^{k_{\max}} dk \frac{F'(\omega_p/k)}{|D_L(k,\omega_p)|^2}\right] \quad \text{(X.37)}$$

In thermal equilibrium the ratio of integrals appearing in Eqs. (X.35) and (X.37) simplified to $-u_0^2/\omega^2$ and the transverse and longitudinal wave energy densities reduce to

$$\bar{\mathscr{E}}_{\omega T} = (1/\pi^2)(m u_0^2/c^3)\omega(\omega^2 - \omega_q^2)^{1/2} \quad \text{(X.38)}$$

$$\bar{\mathscr{E}}_{\omega L} = [1/6(3)^{1/2}\pi^2](m/u_0)\omega_p(\omega^2 - \omega_p^2)^{1/2} \quad \text{(X.39)}$$

From Eqs. (X.38) and (X.39), Eq. (X.15) can be immediately verified.

The density of modes $\rho(\omega)$ is given by

$$\rho(\omega) = \frac{k_0^2(\omega)}{2\pi^2} \frac{dk_0}{d\omega} \quad \text{(X.40)}$$

By dividing $\bar{\mathscr{E}}_{\omega L}$ and $\bar{\mathscr{E}}_{\omega T}$ by $\rho(\omega)$ and using the dispersion relations, Eqs. (VIII.12) and (VIII.13), for longitudinal and transverse waves and taking into account the two possible transverse polarizations, we find

the average energy per mode is

$$\bar{\mathscr{E}}_{\omega L,T}/\rho_{L,T}(\omega) = m u_0^2 = \kappa T \quad \text{(X.41)}$$

where κ is Boltzmann's constant.

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