Substituting into (370) gives

$$\overline{W_J} = \frac{1}{4L}\int_{-\infty}^{+\infty} \frac{\omega^2 G_q(\omega)\,d\omega}{\left[\omega-\omega_0-\frac{(X-iR)}{2L}\right]\left[\omega+\omega_0-\frac{(X-iR)}{2L}\right]}$$

$$\cdot \quad \frac{1}{\left[\omega-\omega_0-\frac{X+iR}{2L}\right]\left[\omega+\omega_0+\frac{X+iR}{2L}\right]} \tag{374}$$

Now for large L the primary contribution to the integral comes from the vicinity $\omega = \pm\omega_0$, i.e., the poles. Assuming that $G_q(\omega)$ is well behaved for large frequencies, we may evaluate (374) by contour integration. We thus find

$$W_J = \frac{2\pi i}{4L}\ \omega_0^2\, G_q(\omega_0)\left\{\frac{2}{4\omega_0^2\ \frac{iR}{L}}\right\} \tag{375}$$

or

$$W_J = \frac{\pi}{4}\ \frac{G_q(\omega_0)}{R} = \frac{kT}{2} \tag{376}$$

$$G_q(\omega_0) = \frac{2kTR}{\pi} \tag{377}$$

In terms of the frequency f we have

$$G(f) = 2\pi\, G(\omega) \tag{378}$$

so

$$G_q(f) = 4kTR = 4kT\ \mathrm{Re}\, Z \tag{379}$$

For very high frequencies or low temperatures where quantum effects must be taken into account, we must use the average energy of a quantum mechanical oscillator in place of KT . Thus we replace KT by

$$\frac{1}{2}hf + \frac{hf}{(e^{hf/kT}-1)} \tag{380}$$

Equation (379) then becomes

$$G(f) = 4hf\left\{\frac{1}{2} + \frac{1}{e^{hf/kT}-1}\right\} \mathrm{Re}\, Z(f) \tag{381}$$

<u>Nyquist's Theorem for the Voltage Fluctuations in a Continuous Medium</u>

We may obtain the voltage fluctuations in a continuous conductor by means of similar arguments to those used to obtain the fluctuations across a resistor. Here, however, rather than hooking the system into one external LC ciruit, we must hook the system into an LC circuit at each point; thus, we must use a continuous lossless medium. For this purpose we imagine that our conductor is imbedded in a uniform medium of weakly charged particles which are bound elastically to an immobile background, i.e., it is a large number of harmonic oscillators which are fixed in space. We assume also that the background carries a neutralizing charge so that there is no net charge in the medium. We assume that the only interaction with the conductor is through the electric field. The conductor and the medium freely premeate each other.

We take for the equations of our medium,

$$\ddot{\underline{r}} = -\omega_0^2 \underline{r} + \frac{\epsilon \underline{E}}{m} \tag{385}$$

$$\underline{J}_M = \epsilon n_0 \dot{\underline{r}} \tag{386}$$

$$\frac{\partial \rho_M}{\partial t} = -\nabla \cdot \underline{J}_M = -\epsilon n_0 \nabla \cdot \dot{\underline{r}} \tag{387}$$

$$\nabla \cdot \underline{E}_M = 4\pi \rho_M \tag{388}$$

Here r is the displacement of an oscillator particle from its equilibrium, and it is assumed to be small. ω_0 is the natural frequency of the oscillators, ϵ and M are the charge and mass on one of the oscillator particles, E is the total electric field, J_M is the current in the medium, ρ_M is the net charge in the medium, and E_M is the electric field due to the medium.

There are two possibilities for $\underline{E}$. It can be either transverse or longitudinal. We shall start by investigating the longitudinal electric fluctuations. We shall assume that the conductor is isotropic so that the current is parallel to the $\underline{E}$ field.

We assume that the current in the conductor is proportional to the driving field or

$$\underline{E}_M(k,\omega) + \underline{E}_J(k,\omega) = Z(k,\omega)\, \underline{J}_C(k,\omega) \tag{389}$$

Here E_q is the generator field due to the random motion of the electrons. The field produced by the medium also drives current in the conductor. Z is the impedance, and J_e is the total current in the conductor. In addition to the generator field, there is an electric field due to the systematic motion of electrons in the conductor. Let us call that field $\underset{\sim}{E_c}$. Then from Poisson's equation

$$\nabla \cdot (\underset{\sim}{E_s} + \underset{\sim}{E_q}) = 4\pi \rho_c \tag{390}$$

where ρ_c is the net charge density in the conductor. Both $\underset{\sim}{E_c}$ and $\underset{\sim}{E_q}$ have their origin in the conductor's electrons. The charge ρ_c must satisfy the continuity equation

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \underset{\sim}{J_c} = 0 \tag{391}$$

Taking the time derivative of (390) and substituting in $\frac{\partial \rho_c}{\partial t}$ from (391) gives

$$\nabla \cdot \left(\frac{\partial [\underset{\sim}{E_s} + \underset{\sim}{E_q}]}{\partial t} + 4\pi \underset{\sim}{J_c} \right) = 0 \tag{392}$$

For longitudinal currents and fields J_c and E are parallel to k($\nabla \times \underset{\sim}{E}$ and $\nabla \times \underset{\sim}{J}$ are zero) and (392) gives

$$\frac{\partial}{\partial t} (\underset{\sim}{E_s} + \underset{\sim}{E_q}) + 4\pi J_c = 0 \tag{393}$$

Now the total electric field is the sum of that due to the conductor and that due to the medium. Thus

$$\underset{\sim}{E} = \underset{\sim}{E_s} + \underset{\sim}{E_q} + \underset{\sim}{E_M} \tag{394}$$

Fourier analyzing we write

$$A(\vec{r},t) = \frac{1}{(2\pi)^{1/2}} \sum_k \int_{-\infty}^{+\infty} A(k,\omega)\, e^{i(\vec{k}\cdot\vec{r}+\omega t)}\, d\omega$$

where A is any one of the quantities $\underset{\sim}{E}_c$, $\underset{\sim}{E}_g$, $\underset{\sim}{E}_M$, $\underset{\sim}{J}_M$, ρ_M, ρ_c, $\underset{\sim}{r}$.

, Substituting into equations (385) and (388) gives

$$\underset{\sim}{r}(k,\omega) = \frac{\epsilon E}{M(\omega_0^2-\omega^2)}, \quad \dot{\underset{\sim}{r}}(k,\omega) = \frac{i\omega\epsilon E}{M(\omega_0^2-\omega^2)} \tag{395}$$

$$\underset{\sim}{J}_M(k,\omega) = \frac{n_0\epsilon^2}{M}\,\frac{i\omega \underset{\sim}{E}}{(\omega_0^2-\omega^2)} \tag{396}$$

$$\rho_M(k,\omega) = -\frac{n_0\epsilon^2}{M}\,\frac{\underset{\sim}{k}\cdot\underset{\sim}{E}}{(\omega_0^2-\omega^2)} \tag{397}$$

$$\underset{\sim}{E}_M(k,\omega) = -\frac{4\pi n_0\epsilon^2}{M}\,\frac{\underset{\sim}{E}}{(\omega_0^2-\omega^2)} = \frac{4\pi i\epsilon n_0\dot{\underset{\sim}{r}}}{\omega} \tag{398}$$

And from (393)

$$\underset{\sim}{E}_s + \underset{\sim}{E}_g = \frac{4\pi i J_c}{\omega} = \frac{4\pi i}{\omega}\,\frac{(\underset{\sim}{E}_M+\underset{\sim}{E}_g)}{Z(k,\omega)} \tag{399}$$

Substituting $\underset{\sim}{E} = \underset{\sim}{E}_c + \underset{\sim}{E}_g + \underset{\sim}{E}_M$ into (398) gives

$$\underset{\sim}{E}_M = -\frac{\omega_{pM}^2\,(\underset{\sim}{E}_s + \underset{\sim}{E}_g)}{[(\omega_0^2-\omega^2)+\omega_{pM}^2]} \tag{400}$$

Substituting (400) into (399) gives

$$(\underset{\sim}{E}_s+\underset{\sim}{E}_g)\left[\frac{Z\omega}{4\pi i} + \frac{\omega_{pM}^2}{(\omega_0^2-\omega^2)+\omega_{pM}^2}\right] = \underset{\sim}{E}_g \tag{401}$$

We are now interested in computing the average kinetic energy of the oscillators. We have from (395) and (398)

$$\dot{\underset{\sim}{r}}(k,\omega) = \frac{-i\omega}{4\pi n_0 e} \underset{\sim}{E}_M(k,\omega) \tag{402}$$

Now from (398) we have

$$\underset{\sim}{E}_M = \frac{-\omega_{pM}^2}{(\omega_0^2-\omega^2)} (\underset{\sim}{E}_M + \underset{\sim}{E}_s + \underset{\sim}{E}_g) \tag{403}$$

or

$$[(\omega_0^2-\omega^2)+\omega_{pM}^2]\underset{\sim}{E}_M = -\omega_{pM}^2(\underset{\sim}{E}_s+\underset{\sim}{E}_g)$$

$$\underset{\sim}{E}_M = \frac{-\omega_{pM}^2(\underset{\sim}{E}_s+\underset{\sim}{E}_g)}{[(\omega_0^2-\omega^2)+\omega_{pM}^2]} \tag{404}$$

From (399) and (404), we have

$$\underset{\sim}{E}_s = \frac{4\pi i}{\omega}\frac{\underset{\sim}{E}_M}{Z} + \left[\frac{4\pi i}{\omega}\frac{1}{Z} - 1\right]\underset{\sim}{E}_g \tag{405}$$

$$[(\omega_0^2-\omega^2)+\omega_{pM}^2]\underset{\sim}{E}_M + \omega_{pM}^2(\underset{\sim}{E}_s+\underset{\sim}{E}_g) = 0 \tag{406}$$

Substituting E_s from (405) into (406) gives

$$\left[\omega_0^2-\omega^2+\omega_{pM}^2+\omega_{pM}^2\frac{4\pi i}{\omega Z}\right]\underset{\sim}{E}_M + \frac{\omega_{pM}^2 4\pi i}{\omega Z}\underset{\sim}{E}_g = 0$$

or

$$\underset{\sim}{E}_M = \frac{-4\pi i\,\omega_{pM}^2 \underset{\sim}{E}_g}{\omega Z\left[(\omega_0^2-\omega^2)+\omega_{pM}^2+\omega_{pM}^2\frac{4\pi i}{\omega Z}\right]} \tag{407}$$

Now making use of equation (398) for $\dot{\underset{\sim}{r}}$ in terms of E_M, we find

$$\dot{\underset{\sim}{r}}(k,\omega) = -\frac{4\pi\epsilon}{MZ}\,\frac{\underset{\sim}{E}_g}{\left[\omega_0^2-\omega^2+\omega_{pM}^2\left(1+\frac{4\pi i}{\omega Z}\right)\right]} \tag{408}$$

We now calculate the average kinetic energy of the imposed medium for wave number k and equate it to .

$$\frac{n_0 M}{4\pi T^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\, d\omega' \sum_{k,k'} \dot{r}(k,\omega)\,\dot{r}(k',\omega')\,e^{i(\omega+\omega')t+i(k+k')\cdot r}\,dt\,d^3r$$

$$W_k = \frac{L^3 n_0 M}{2T}\int |\dot{r}(k,\omega)|^2 d\omega$$

$$= \frac{n_0 L^3}{2T}\,\frac{(4\pi\epsilon)^2}{M}\int \frac{|E_g(k,\omega)|^2/|Z|^2\, d\omega}{\left[\omega^2-\omega_0^2+\omega_{pM}^2\left(1+\frac{4\pi i}{\omega Z}\right)\right]\left[\omega^2-\omega_0^2+\omega_{pM}^2\left(1-\frac{4\pi i}{\omega Z^*}\right)\right]} \tag{409}$$

Again let $Z = R+iX$; $4\pi i/\omega Z$ is equal to

$$\frac{4\pi(iR+X)}{\omega(R^2+X^2)}.$$

We will now let the mass and the spring constant for the imposed oscillators go to ∞ in such a way as to keep ω_0 fixed; however, we keep their charge ϵ and number density finite so ω_{pM}^2 goes to zero. We evaluate

(409) in this limit. We have

$$\frac{n_0 M L^3}{2T} \int |\dot{r}(k,\omega)|^2 d\omega \approx \frac{L^3 n_0 \epsilon^2 (4\pi)^2}{2MT|Z(k,\omega_0)|^2} |E_\xi(k,\omega_0)|^2 \times$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{\left[\omega-\omega_0+\frac{2\pi(iR+X)\omega_{PM}^2}{\omega_0^2|Z(k,\omega_0)|^2}\right]\left[\omega+\omega_0+\frac{2\pi(iR+X)\omega_{PM}^2}{\omega_0^2|Z(k,\omega_0)|^2}\right] \times \left[\omega-\omega_0+\frac{2\pi(-iR+X)\omega_{PM}^2}{\omega_0^2|Z(k,\omega_0)|^2}\right]\left[\omega+\omega_0+\frac{2\pi(-iR+X)\omega_{PM}^2}{\omega_0^2|Z(k,\omega_0)|^2}\right]}$$

$$= \frac{L^3 4\pi}{T^2} \frac{2|E_\xi(k,\omega_0)|^2}{4R} = \frac{\kappa T}{2} \tag{410}$$

Thus

$$\frac{\pi L^3 |E_\xi(k,\omega_0)|^2}{2RT} = \frac{\kappa T}{2}. \tag{411}$$

Now the noise per unit frequency interval for the given k is

$$\tilde{G}(k,\omega) = \frac{2|E_\xi(k,\omega_0)|^2}{T}. \tag{412}$$

Thus the noise in the interval $d\omega$ is

$$\tilde{G}(k,\omega)d\omega = \frac{2\kappa T R}{\pi L^3} d\omega. \tag{413}$$

The noise in the frequency interval f is

$$\tilde{G}(k,f)df = \tilde{G}(k,\omega) 2\pi df = \frac{4\kappa T R}{L^3} df. \tag{414}$$

Now this gives the noise for one single $\underline{k}$. If we want the noise is a bank of k's, then we must multiply this by the number of k's in the interval. This number is given by

$$dN = \left(\frac{L}{2\pi}\right)^3 k^2 d\Omega\, dk \tag{415}$$

where k is the magnitude of the k vector and $d\Omega$ is the element of solid angle which the k vector lies in. Multiplying (414) by (415) gives

$$\tilde{G}(k, f)\, df\, dN = \frac{4kTR}{(2\pi)^3}\, df\, k^2 d\Omega\, dk. \tag{416}$$

If instead of using k we had used $\mathcal{K} = \frac{k}{2\pi}$ (similar to f and ω), we would have obtained

$$\breve{G}(\mathcal{K}, f)\, df\, \mathcal{K}^2 d\Omega\, d\mathcal{K} = 4kTR\, df\, \mathcal{K}^2 d\Omega\, d\mathcal{K}, \tag{417}$$

Relation to the Dielectric Constant

The impedence is related to the dielectric constant and hence we can relate $\overline{|E_g(k,\omega)|^2}$ to the dielectric constant. To find this relation, we set E_q equal to 0 (we are only looking for the coherent responce to an external field) and solve for the electric field which results from an imposed field due to charges outside the conductor, i.e., from the fictitious fluid. From (389), we have

$$J_c = \frac{E_f}{Z}, \tag{418}$$

From (383), we find the E_f produced by the current in the conductor

$$E_c = \frac{4\pi i}{\omega} \frac{E_f}{Z}. \tag{419}$$

The total E field is

$$E = E_f + E_c = E_f\left(1 + \frac{4\pi i}{\omega Z}\right). \tag{420}$$

The dielectric constant ϵ is defined by the equation

$$\epsilon E = E_f. \tag{421}$$

Thus we have

$$\epsilon = \frac{1}{1 + \frac{4\pi i}{\omega Z}}. \tag{422}$$

We could use this equation to find a relation between E_g and ϵ. However, this relation is somewhat complicated, and we shall proceed differently so as to relate ϵ to the actual fluctuating fields appearing in the conductor rather than the generator voltages.

Now E_g is the voltage put out by the generators in the conductor. It would appear only if no currents were allowed to flow. The actual E fields which will appear in the conductor are those which remain when the fictitious fluid is removed. Setting $E_f = 0$, we have from (394),

$$E = E_s + E_g \tag{423}$$

and from (389)

$$J_c = \frac{E_g}{Z} \tag{424}$$

and finally from (393)

$$i\omega\,(E_s + E_g) = i\omega E = -4\pi J_c \tag{425}$$

Combining (424) and (425) gives

$$E = \frac{4\pi i}{\omega Z} E_g \tag{426}$$

or

$$\overline{|E|^2} = \left(\frac{4\pi}{\omega |Z|}\right)^2 |E_g|^2. \tag{427}$$

or making use of (412) and (416) gives

$$G_E(k,\omega)\, d\omega k^2 d\Omega\, dk = \left(\frac{2}{\omega\pi}\right)^2 \frac{\Theta R\, d\omega\, dk\, k^2 d\Omega}{Z Z^*}. \tag{428}$$

From (422) for ϵ we have

$$I_m\left(\frac{1}{\epsilon}\right) = \frac{4\pi R}{\omega Z Z^*}, \tag{429}$$

Combining (429) and (428) gives

$$G_E(k,\omega) = \left[\frac{4\pi}{\omega} I_m\left(\frac{1}{\epsilon}\right)\right] \frac{4\,\Theta}{(2\pi)^4}. \tag{430}$$

Relation to Fluctuating Currents

While we have treated the fluctuations in a conductor as if they were produced by a generator of voltage E_g, we may also think of them as arising from random currents in the conductor. It is worthwhile to look at the problem from this point of view. To find out what the basic fluctuating currents are, we should short out all E fields so that the fields resulting from the currents do not modify the currents. To do this we make the mass of a particle of the fictitious fluid go to zero while their number density and charge go to ∞ so that the conductivity of the fictitious fluid is ∞. Then the total E field must be zero for otherwise an infinite current would arise in the fictitious fluid to cancel it out by equations (385)-(388). Thus, we set $E = 0$ or

$$E_F = -E_s - E_g \tag{431}$$

From (389) we have

$$J_r = J_c = -\frac{E_s}{Z} \tag{432}$$

, here J_r is the shorted generator current, r stands for random

and from (393), we get

$$i\omega(E_s + E_g) = \frac{4\pi}{Z} E_s \tag{433}$$

or

$$E_s = \frac{-E_g}{1 + \frac{4\pi i}{\omega Z}} . \tag{434}$$

The current in the conductor is thus given by

$$J_r = -\frac{E_c}{Z} = -\frac{E_g}{Z + \frac{4\pi i}{\omega}} \tag{435}$$

$$\text{or} \quad |J_r|^2 = \frac{\omega^2}{(4\pi)^2} \frac{|E|^2}{\left(1+\frac{4\pi i}{Z\omega}\right)\left(1-\frac{4\pi i}{Z^*\omega}\right)}$$

$$|J_r|^2 = \frac{\omega^2}{(4\pi)^2} \epsilon\epsilon^* |E|^2 = \frac{|\epsilon|^2}{|Z|^2} |E_g|^2. \tag{436}$$

If J_r existed alone, then from the continuity equation and Poisson's equation it would produce an electric field E_r which satisfies the equation

$$i\omega E_r + 4\pi J_r = 0. \tag{437}$$

Thus from 436

$$|E_r|^2 = |\epsilon E|^2 \tag{438}$$

or

$$|E|^2 = \left|\frac{E_r}{\epsilon}\right|^2. \tag{439}$$

The actual fluctuating currents flowing in our material when we are not driving it are given by the continuity equation and Poisson's equation

$$J = -\frac{i\omega E}{4\pi} \tag{440}$$

or

$$|J|^2 = \left(\frac{\omega}{4\pi}\right)^2 |E|^2 = \frac{|J_r|^2}{|\epsilon|^2}. \tag{441}$$

SECTION F

Fluctuations in a Plasma

Let us now consider the case of an infinite uniform homogeneous isotropic plasma of mobile electrons and fixed smeared out ions. We assume there is no static magnetic field in the plasma, and we assume the plasma is in thermal equilibrium. We wish to find the fluctuating fields and currents within the plasma. We shall assume the plasma is at a high temperature so that collisions can be neglected. Then the plasma is described by the Vlasov equation plus Maxwell's equations. We shall first consider the longitudinal fields and currents.

The equations describing the plasma are the Vlasov equation,

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{e}{m} \nabla \phi \cdot \frac{\partial f}{\partial \underline{v}} = 0 , \qquad \text{(F1)}$$

and Poisson's equation

$$\nabla^2 \phi = 4\pi\rho = -4\pi e \left[\int f d^3 v - n_o \right] . \qquad \text{(F2)}$$

Here n_o is the background number density for the ions or the mean electron density.

Now linearize these equations about a spatially uniform Maxwellian distribution

$$f_o(v) = \frac{n_o}{(2\pi)^{3/2} u_T^3} \exp \left\{ - \frac{v^2}{2 u_T^2} \right\} . \qquad \text{(F3)}$$

We assume that all disturbances we are interested in are small. Writing

$$f(v) = f_0(v) + f_1(v) \tag{F4}$$

$$E = E_1 = -\nabla\phi \quad . \tag{F5}$$

(There is no zero order E field) and substituting into (F1) and (F2) gives

$$\frac{\partial f_1}{\partial t} + \underline{v}\cdot\frac{\partial f_1}{\partial \underline{r}} + \frac{e}{m}\nabla\phi\cdot\frac{\partial f_0}{\partial \underline{v}} = -\nu f_1 \tag{F6}$$

$$\nabla^2\phi = -4\pi e\int f_1\, d^3v \quad . \tag{F7}$$

Here we have added a small collisional damping term $-\nu f_1$ to the right-hand side of (F6) so as to make certain singular integrals we shall encounter shortly well defined. We shall take the limit as v goes to zero in the end.

Now to apply Nyquist's theorem, we must compute the impedance of the plasma or its dielectric function. To do this, we add the source charge

$$\rho_s = \rho_s e^{i(\underline{k}\cdot\underline{r} + \omega t)} \tag{F8}$$

to the right hand side of (F8). We then look for the response of the plasma which is also of the form of (F8). Equations (F6) and (F7) thus become,

$$(i\omega + i\underline{k}\cdot\underline{v} + \nu) f_1 = -\frac{e}{m} i\phi\,\underline{k}\cdot\frac{\partial f_0}{\partial \underline{v}} \tag{F9}$$

$$-k^2\phi = 4\pi e\int f_1\, d^3v + 4\pi\rho_s \quad . \tag{F10}$$

Solving (F9) for f_1 and substituting into (F10) gives

$$-k^2\phi = -\frac{4\pi e^2}{m}\phi\int\frac{\underline{k}\cdot\frac{\partial f_0}{\partial \underline{v}}\, d^3v}{(\omega + \underline{k}\cdot\underline{v} - i\nu)} + 4\pi\rho_s \tag{F11}$$

or

$$k^2 \phi \left(1 - \frac{4\pi e^2}{m k} \int \frac{\left(\frac{\partial F_0}{\partial \underline{v}}\right)_{\parallel \text{ to } \underline{k}} d^3 v}{(\omega + \underline{k} \cdot \underline{v} - i\nu)} \right) = -4\pi \rho_s \quad \text{(F12)}$$

$$\phi = \frac{-4\pi \rho_s}{k^2 \epsilon(k, \omega)} \quad , \quad \text{(F13)}$$

where ϵ is the dielectric function

$$\epsilon = 1 - \frac{4\pi e^2}{m k} \int \frac{\tilde{F}_0' \, d v_k}{(\omega + k v_k - i\nu)} \quad . \quad \text{(F14)}$$

In writing down (F14), we have integrated over the velocities perpendicular to $\underline{k}$.

$$\tilde{F}_0 = n_0 \exp\{-v_k^2/2u_T^2\} / (2\pi)^{1/2} u_T \quad . \quad \text{(F15)}$$

We may substitute expression (F14) into Eq. (430) to obtain the power spectrum for the fluctuating field in the plasma

$$G_E(k, \omega) = \frac{4\pi}{\omega} \operatorname{Im} \left[\frac{1}{1 - \frac{4\pi e^2}{m k} \int \frac{\tilde{F}_0' \, d v_k}{(\omega + k v_k - i\nu)}} \right] \frac{4 \Theta}{(2\pi)^4} . \quad \text{(F16)}$$

To see more clearly what this means we compute the random currents which give rise to this field using equation (436). We thus find from (436) and (F14)

$$G_{J_r}(k, \omega) = \frac{\omega}{\pi} \operatorname{Im} \left[1 - \frac{4\pi e^2}{m k} \int \frac{\tilde{F}_0' \, d v_k}{(\omega + k v_k - i\nu)} \right] \frac{4 \Theta}{(2\pi)^4} \quad , \quad \text{(F17)}$$

Now we are interested in this expression as v goes to zero. In this limit

$$\operatorname*{Im}_{v \to 0} \epsilon \to \frac{\omega_p^2 \pi \tilde{F}_o'(\omega/k)}{k^2} = \frac{\omega_p^2 \pi \omega}{k^3 \sqrt{2\pi}\, u_T^3} e^{-\frac{\omega^2}{2k^2 u_T^2}} \qquad \text{(F18)}$$

$$\omega_p^2 = \frac{4\pi n_o e^2}{m} \quad .$$

Substituting into (F17) gives

$$G_{J_r}(k,\omega) = \frac{\omega_p^2 \omega^2 \; m}{\sqrt{2\pi}\, k^3 u_T (2\pi)^4} \exp\left\{ -\frac{\omega^2}{2k^2 u_T^2} \right\} \quad . \qquad \text{(F19)}$$

To interpret this result, we imagine that the particles did not interact. The current would then be given by

$$J_r = -\sum_{\ell} e\, v_\ell\, \delta(\underline{r} - \underline{r}_\ell - \underline{v}_\ell t) \ , \qquad \text{(F20)}$$

where r_ℓ and v_ℓ are the initial position, and velocity of the particle. Fourier analyzing in space gives

$$J_r(k,t) = -\frac{e}{(2\pi)^{3/2}} \sum_{\ell} v_\ell\, e^{i\underline{k}\cdot(\underline{r}_\ell + \underline{v}_\ell t)} \cdot \qquad \text{(F21)}$$

Fourier analyzing in time and assuming the process goes on from $-T/2$ to $T/2$ gives

$$J_r(k,\omega) = -\frac{2e}{(2\pi)^2} \sum_{\ell} v_\ell\, e^{-i\underline{k}\cdot\underline{r}_\ell} \frac{\sin\left[\underline{k}\cdot\underline{v}_\ell + \omega\right]\frac{T}{2}}{\omega + \underline{k}\cdot\underline{v}_\ell} \ . \qquad \text{(F22)}$$

We now square J and take its ensemble average assuming the particles are uncorrelated

$$|J_r(k,\omega)|^2 = \frac{4e^2}{(2\pi)^4}\sum v_l^2 \frac{\sin^2[k\cdot v_l + \omega]\frac{T}{2}}{(\omega + k\cdot v_l)^2}$$

$$= \frac{4Ne^2}{(2\pi)^4}\int_{-\infty}^{\infty} \frac{v^2 e^{-v^2/2u_T^2}\sin^2[k\cdot v + \omega]\frac{T}{2}}{(2\pi)^{3/2}u_T^3(\omega + k\cdot v)^2}\, d^3v , \tag{F23}$$

where N is the total number of electrons. Integrating over the two components of v perpendicular to k gives

$$|J_r(k,\omega)|^2 = \frac{4Ne^2}{(2\pi)^4}\int \frac{[2u_T^2 + v_k^2]\, e^{-v_k^2/2u_T^2}}{(2\pi)^{1/2}u_T} \frac{\sin^2[kv_k + \omega]\frac{T}{2}}{(\omega + kv_k)^2}\, dv_k . \tag{F24}$$

This may be integrated to give

$$\overline{|J_r(k,\omega)|^2} = \frac{2\pi Ne^2 T}{(2\pi)^4(2\pi)^{1/2}u_T k}\left[2u_T^2 + \frac{\omega^2}{k^2}\right] e^{-\omega^2/2k^2u_T^2} . \tag{F25}$$

The Power spectrum is defined as

$$G_{J_r}(k,\omega) = \frac{2\,\overline{|J(k,\omega)|^2}}{L^3 T} = \frac{\omega_p^2 m\, e^{-\omega^2/2k^2u_T^2}}{(2\pi)^4(2\pi)^{1/2}u_T k}\left[2u_T^2 + \frac{\omega^2}{k^2}\right] . \tag{F26}$$

If we had considered only the component of J_r parallel to k, then only ω^2/k^2 appears, and we have

$$G_{J_{r_\parallel}}(k,\omega) = \frac{\omega_p^2 m\,\omega^2 \exp\{-\omega^2/2k^2u_T^2\}}{(2\pi)^4(2\pi)^{1/2}u_T k^3} . \tag{F27}$$

Equation (F27) is identical with (F19) and shows that the random currents which appear when the E field is shorted out is juat that due to the random motion of the particles. By Eq. (436), the random fields which actually appear in the plasma are obtained by shielding these currents

$$|E|^2 = \frac{(4\pi)^2}{\omega^2}\frac{|J_r|^2}{(\epsilon\,\epsilon^*)} \,. \tag{F28}$$

This is Rostoker's superposition principal.

If instead of the component of J_r parallel to k we had kept one of the components of J_r perpendicular to k, then (F26) is replaced by

$$G_{J_{r_\perp}}(k,\omega) = \frac{\omega_p^2\, m\, U_r\, e^{-\omega^2/2k^2U_T^2}}{(2\pi)^4(2\pi)^{1/2}\,k} \,. \tag{F29}$$

This is for two perpendicular directions. For just one, G_J should be divided by 2. We shall use this result shortly when we consider transverse fluctuations.

Transverse Field Fluctuations

To find the transverse fluctuations, we will employ the generalized Nyquist theorem. For the transverse waves, we will work with the vector potential $\underline{A}$. The equations that $\underline{A}$ satisfies are

$$\nabla^2\underline{A} - \frac{1}{c^2}\frac{\partial^2\underline{A}}{\partial t^2} = \frac{4\pi}{c}(\underline{j} + \underline{j}_s)$$

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t}\,, \quad B = \nabla\times\underline{A}\,, \quad \nabla\cdot A = 0 \tag{F30}$$

Here j_s is a source current and j is the current due to the conductor.

This interaction Hamiltonian is

$$H_1 = \frac{\underset{\sim}{J_s} \cdot \underset{\sim}{A_c}}{c} + \frac{\underset{\sim}{J_s} \cdot \underset{\sim}{A_s}}{c} - = \frac{2 \underset{\sim}{J_s} \cdot \underset{\sim}{A_s}}{c} \tag{F31}$$

where A_c is the part of the vector potential due to the current in the conductor. Fourier analyzing (F30) gives

$$(k^2 c^2 - \omega^2)\underset{\sim}{A} = -4\pi c\ (\underset{\sim}{J_c} + \underset{\sim}{J_s}) \quad . \tag{F32}$$

We assume that the current in the conductor is related to the electric field by

$$\underset{\sim}{J_c}(k,\omega) = K(k,\omega) \cdot \underset{\sim}{E}(k,\omega) \tag{F33}$$

or from (F30)

$$\underset{\sim}{J_c}(k,\omega) = -\frac{i\omega}{c} K(k,\omega)\ \underset{\sim}{A}(k,\omega) \quad . \tag{F34}$$

Substituting into (F32) gives

$$\left[k^2 c^2 - \omega^2 - 4\pi i \omega K(k,\omega)\right] \underset{\sim}{A} = -4\pi c\, \underset{\sim}{J_s} \tag{F35}$$

or

$$\underset{\sim}{A} = -4\pi c \frac{\underset{\sim}{J_s}}{k^2 c^2 - \omega^2 - 4\pi i \omega K(k,\omega)} \quad . \tag{F36}$$

Now the A due to the source currents is

$$\underset{\sim}{A_s} = -\frac{4\pi c\ \underset{\sim}{J_s}}{(k^2 c_2 - \omega^2)} \quad . \tag{F37}$$

Thus the A due to the current in the conductors is

$$\underset{\sim}{A}_c = -4\pi c \underset{\sim}{J}_s \left[\frac{1}{k^2c^2 - \omega^2 - 4\pi i\omega K(k,\omega)} - \frac{1}{k^2c^2 - \omega^2} \right] . \tag{F38}$$

Identifying $\lambda \underset{\sim}{J}_s$ with $\underset{\sim}{r}$ and A with $\frac{\underset{\sim}{A}_c}{c}$ in the generalized Nyquist theorem, we see from equations (2) and (7) of that section that $x(k,\omega)$ is

$$X(k,\omega) = -2\pi \left[\frac{1}{k^2c^2 - \omega^2 - 4\pi i\omega K(k,\omega)} - \frac{1}{k^2c^2 - \omega^2} \right] . \tag{F39}$$

We thus find from equation (15) of that section

$$\begin{aligned} G_{A_c}(k,\omega) &= \frac{c^2 4\Theta}{\omega(2\pi)^4} \, I_m X(k,\omega) \\ &= \frac{8\Theta}{(2\pi)^5} \, c^2 \frac{Re\, K(k,\omega)}{\left[(k^2c^2-\omega^2)^2 + 8\pi\omega K_i + (4\pi)^2\omega^2 K K^*\right]} . \end{aligned} \tag{F40}$$

We may ask what is the power spectrum of the source currents which give rise to these fluctuations in A_c . The source currents are given in terms of $\underset{\sim}{A}$ by

$$\underset{\sim}{J}_s = -\frac{\Pi}{4\pi c} \underset{\sim}{A} \tag{F41}$$

where

$$\Pi = \left[k^2c^2 - \omega^2 - 4\pi i\omega K(k,\omega) \right] . \tag{F42}$$

Thus

$$G_{J_s}(k,\omega) = \frac{\mathbb{I}\,\mathbb{I}^*}{(4\pi)^2 c^2}\, G_{A_c} \qquad \text{(F43)}$$

or by (F40)

$$G_{J_s}(k,\omega) = \frac{2\Theta}{(2\pi)^4}\, \mathrm{Re}\, K(k,\omega) \quad . \qquad \text{(F44)}$$

<u>$K(k,\omega)$ for a Plasma</u>

We now derive the conductivity for a plasma. We assume the plasma satisfies the Vlasov equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial \underset{\sim}{r}} - \frac{\epsilon \underset{\sim}{E}}{m} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} = -\nu f \quad . \qquad \text{(F45}$$

Fourier analyzing

$$(i\omega + k \cdot v + \nu) f = \frac{\epsilon \underset{\sim}{E}}{m} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} \qquad \text{(F46)}$$

$$f = -\frac{i\epsilon \underset{\sim}{E} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}}}{(\omega + k \cdot v - i\nu)} \quad . \qquad \text{(F47)}$$

Computing the current

$$J = i \int \frac{e^2 \underset{\sim}{E} \cdot \frac{\partial f_0}{\partial v}\, \underset{\sim}{v}}{m\,(\omega + k \cdot v - i\nu)}\, d^3 v \quad . \qquad \text{(F48)}$$

Now we are considering a transverse fields so $\underset{\sim}{E} \perp$ to $\underset{\sim}{k}$. Chose a coordinate system with one axis along $\underset{\sim}{E}$, one along $\underset{\sim}{k}$, and the third perpendicular to

E and k . Then we have

$$\underset{\sim}{E} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} = \frac{E\, v_E\, n_0\, e^{-v^2/2 u_T^2}}{(2\pi)^{3/2}\, u_T^5} \tag{F49}$$

and

$$\underset{\sim}{J} = i\, \frac{e^2 n_0 E}{(2\pi)^{3/2}} \int \frac{v_E\, (\hat{e}_E \cdot v_E + \hat{e}_k \cdot v_k + \hat{e}_\perp \cdot v_\perp)\, e^{-v^2/2 u_T^2}}{m\, u_T^5\, (\omega + k v_k - i\nu)}\, dv_E\, dv_k\, dv_\perp \, . \tag{F50}$$

The $v_\perp$ term gives zero when integrated over $v_\perp$; the v_k term gives zero when integrated over v_E ; the only term which remains is the v_E

$$\underset{\sim}{j} = -i\, \frac{e^2 n_0 E}{m (2\pi)^{1/2} u_T} \int \frac{e^{-v_k^2/2 u_T^2}}{(\omega + k v_k - i\nu)}\, dv_k \, . \tag{F51}$$

Thus K is equal to

$$K = \frac{i e^2 n_0}{m (2\pi)^{1/2} u_T} \int_{-\infty}^{\infty} \frac{e^{-v_k^2/2 u_T^2}}{(\omega + k v_k - i\nu)}\, dv_k \, . \tag{F52}$$

The power spectrum of the source currents is given by (F44)

$$G_{JS} = \frac{2\Theta}{(2\pi)^4}\, \mathrm{Re}\, K \, , \tag{F53}$$

From (F52) taking the limit as

$$\mathrm{Re}\, K = \frac{\pi e^2 n_0}{m (2\pi)^{1/2} u_T k}\, e^{-\omega^2/2k^2 u_T^2} = \frac{\omega_p^2\, e^{-\omega^2/2k^2 u_T^2}}{4 (2\pi)^{1/2} u_T k} \, . \tag{F54}$$

Substituting into (F53) gives

$$G_{J_\perp} = \frac{\omega_p^2 \, m \, v_T^2}{2(2\pi)^4 (2\pi)^{1/2} v_T k} \, e^{-\omega^2/2k^2 v_T^2} , \qquad \text{(F55)}$$

(taking into account that this is for just one perpendicular component of j). This agrees with equation (F29) which we found for the power spectrum of the current due to the particles when they were treated moving randomly (no correlations).