V. THE LINEARIZED VLASOV EQUATION FOR A UNIFORM UNMAGNETIZED PLASMA

Although it constitutes the lowest order approximation in the expansion in fluctuations discussed in Chapter II, the Vlasov equation describes a very large number of important physical phenomena. In fact, we have seen that even the moment approximations to the Vlasov equation, i.e. two fluid and one fluid MHD, discussed in the last two chapters, lead to quite a variety of linearized wave modes. In the present chapter, we begin to study the character of the full Vlasov equation. As we have seen in the fluid examples, linearization provides a good starting point for exploring a complex system of equations.

For each species, the full Vlasov equation is

$$[\partial/\partial t + \underline{v}\cdot\nabla+(q/m)(\underline{E} + \underline{v}\times\underline{B}/c) \cdot \nabla_v]f = 0 \tag{5.1}$$

to which we must adjoin the Maxwell equations

$$\begin{aligned} &\nabla\cdot\underline{E} = 4\pi[\textstyle\sum\int d\underline{v}\bar{n}qf + \rho_e] && \nabla\cdot\underline{B} = 0 \\ &\nabla\times\underline{E} + \dot{\underline{B}}/c = 0 && \nabla\times\underline{B} = 4\pi c^{-1}[\textstyle\sum\int d\underline{v}\bar{n}q\underline{v}f + \underline{j}_e] + \dot{\underline{E}}/c \end{aligned} \tag{5.2}$$

In choosing an "unperturbed" state, i.e., a known solution of these equations, we shall begin with the simplest choice: a uniform plasma, time-independent, with no external uniform, time-independent fields.

A. Response to External Fields and Initial Values

1. Dielectric Functions for Isotropic f_0

For the uniform, time-independent unmagnetized plasma we set

$$f(\underline{x},\underline{v},t) = f_0(\underline{v}) + f_1(\underline{x},\underline{v},t)$$

where f_1, $\underline{E}$ and $\underline{B}$ are first order quantities and we neglect terms of second or higher order. Note that (5.1) is identically satisfied to zeroth order, regardless of the velocity dependence of f_0.

2. A Simple Example: Lorentzian f_0

A class of distribution functions which greatly simplify the calculation of the dielectric functions is

$$f_0(\underline{v}) = N(v^2+a^2)^{-p}$$

where p is an integer, N is a suitable normalizing factor, and a has the significance of a "thermal" velocity. The simplest choice is $p = 2$,

$$f_0(\underline{v}) = (a/\pi^2)(v^2+a^2)^{-2}$$

which leads to

$$f_\ell(v_\ell) = \int d\underline{v}_t f_0(\underline{v}) = (a/\pi)(v_\ell^2+a^2)^{-1} \tag{5.18}$$

(Since this f_ℓ leads to an infinite kinetic energy density, $\int dv f_\ell mv^2/2$, some caution is necessary, but generally it gives qualitatively useful results. The kinetic energy is finite if $p > 2$, but the root finding algebra is not quite so simple.)

Consider the limit $m/M \to 0$ (neglect of ion contributions). Then (5.14) gives

$$\varepsilon_\ell = 1 - (\omega_p{}^2/k^2)\int dv f_\ell{}'(v)/(v-u) = 1 - (\omega_p{}^2/k^2(d/du)\int dv f_\ell(v)(v-u)^{-1} \tag{5.19}$$

where the phase velocity

$$u = \omega/k$$

<u>must not</u> be confused with the mean velocity, $\underline{u}$, used in the fluid treatment of Chapter III. With f_ℓ given by (5.18), we thus need to compute

$$(a/\pi)\int dv(a^2+v^2)^{-1}(v-u)^{-1} = -(u+ia)^{-1} \tag{5.20}$$

which is most easily done by the method of residues, closing the contour in the lower half plane and <u>remembering that Imu > 0</u>. Then where the $\underline{V}$ for each species may be assumed parallel (in absence of $\underline{B}_0$, a common streaming of both species is irrelevant) and we consider waves with $\underline{k}$ parallel to $\underline{V}$. The Lorentz force term in the Vlasov equation is then

$$\underline{v}\times\underline{B}\cdot\nabla v f_0 = -2\underline{v}\times\underline{B}\cdot\underline{V}\ F'[(\underline{v}-\underline{V})^2]$$

which vanishes on integration over $\underline{v}_t$.

Moreover, in calculating the linearized density, n_1, the terms involving $\underline{E}_t$ will still vanish,

$$\int d\underline{v}\ \underline{E}_t\cdot\nabla_v f_0(\underline{k}\cdot\underline{v}-\omega)^{-1} = 0$$

as in (5.8). Thus, we can correctly use the expression (5.14) for ε_ℓ even though f_0 is not isotropic.

B. Techniques for Solving the Longitudinal Dispersion Relation

1. The Hilbert Transform of f_ℓ

Typical f_ℓ functions are characterized by a thermal spread or velocity width, a , so that af_ℓ is a dimensionless function of (v/a). Thus, it is convenient to write the susceptibility (5.15) as

$$\chi = -\int dv(\omega_p{}^2/k^2)f_\ell{}'(v)(v-u)^{-1} = -(\omega_p{}^2/k^2a^2)(d/ds)\int dw a f_\ell(w)(w-s)^{-1}$$

where w = v/a is a dimensionless velocity variable and

$$s = \omega/ka = u/a \tag{5.23}$$

is a dimensionless phase velocity. For real s, the function

$$Z(s) = a\int dw f_\ell(w)(w-s)^{-1} \quad , \tag{5.24}$$

with the integral defined as a principal value, is called the Hilbert transform. We shall use the same term for the integral (5.24) with Ims > 0.

For the Lorentzian, discussed in section A, we have

$$Z = Z_L(s) \equiv -(s+i)^{-1} \tag{5.25}$$

Later we shall discuss the Z which corresponds to a Maxwellian f_0. In any case, setting $\omega_p{}^2/a^2 = k_D{}^2/2$ we can write (5.14) as

$$\varepsilon_\ell(\underline{k},\omega) = 1 - \sum(k_D{}^2/2k^2)\, Z'(\omega/ka) \tag{5.26}$$

where the sum is over species. Thus, the properties of the appropriate Z function determine those of ε_ℓ.

2. General Expression for Nearly Real Roots; Landau Damping and Growth

We now return to the full longitudinal dielectric function (5.14) and consider the solution of the dispersion equation which we write as

$$\varepsilon_\ell(\underline{k},\omega) = 1 - (\Omega_p{}^2/k^2)\int dv\; F'(v)/(v-u) = 0 \tag{5.27}$$

where $u = \omega/k$ and we have defined an effective distribution function

$$F(v) = \sum \omega_p{}^2 f_\ell(v)/\Omega_p{}^2$$

$$\Omega_p{}^2 = \sum \omega_p{}^2$$

the sums being over species.

For weakly damped roots, i.e. those with

$$\omega = \omega_R + i\gamma \qquad |\gamma| \ll \omega_R \tag{5.28}$$

an approximate analytical solution is possible and often very useful. Writing the phase velocity $u = \omega/k$ as

$$u = u_1 + iu_2$$

and expanding in u_2 we have

$$\int dv\, \frac{F'(v)}{v-u} = \int dv F'(v) \left[\frac{1}{v-u_1-i\varepsilon} + \frac{iu_2}{(v-u_1-i\varepsilon)^2} + \cdots\right] \tag{5.29}$$

where the $i\varepsilon$ is necessary as a reminder of how to evaluate the integral as $u_2 \to 0$. Integrating the last term by parts, we have

$$\int dv\, \frac{F'(v)}{(v-u)} = P\int dv\, \frac{F'(v)}{(v-u_1)} + i\pi F'(u_1) + iu_2\left[P\int \frac{dvF''(v)}{v-u_1} + i\pi F''(u_1)\right] + O(u_2{}^2) \tag{5.30}$$

When we use this expansion in (5.27), we must retain the imaginary term proportional to u_2, since otherwise we could not satisfy the imaginary part of the equation, but we can drop the $-\pi u_2 F''(u_1)$ term since, for small u_2, it gives only a small correction to the real part of (5.27). Thus from the real part we have an equation

$$k^2/\Omega_p^{\ 2} = P\int dv\ F'(v)/(v-u_1) \tag{5.31}$$

to determine u_1. Once it is known, we calculate u_2 from the imaginary part of (5.27):

$$u_2 = -\pi F'(u_1)/P\int dv F''(v)(v-u_1)^{-1} \tag{5.32}$$

We can write (5.32) in a different form by differentiating (5.31) with respect to k to get

$$2k/\Omega_p^{\ 2} = \left[P\int dv\ \frac{F''(v)}{(v-u_1)} \right] \frac{du_1}{dk} \tag{5.33}$$

and substituting this in (5.32) to give

$$\boxed{\gamma = ku_2 = -(\pi\Omega_p^{\ 2}/2)F'(u_1)(du_1/dk)} \tag{5.34}$$

which is a generally valid expression for weak Landau damping. For typical dispersion relations [e.g. Langmuir waves, where $u_1 \approx \omega_p/k$, or ion acoustic waves, where $u_1^2 \approx c_s^{\ 2}(1+k^2/k_D^{\ 2})^{-1}$], $du_1/dk < 0$, and so γ has the sign of $F'(u_1)$: a negative slope gives damped waves, while a positive slope leads to growing waves or instabilities.

So far we have obtained Landau damping as a formal, mathematical consequence of the properties of ε. As is clear from the analysis, it is the "resonant particles", i.e., those with $v \approx u_1$ which give rise to Imε and hence to γ = Imω. In fact, if there are no such particles, i.e., if f and f' vanish in the neighborhood of u_1, then (5.27) can be satisfied by a purely real u and there will be no growth or damping. In a frame moving with velocity u_1, the resonant particles see an almost constant electric field, and hence have a much stronger interaction with the wave than the other, non-resonant particles, which experience only a rapidly oscillating field. It is the exchange of energy between resonant particles and the wave electric field which gives rise to the Landau damping or growth (sometimes called inverse Landau damping. Since particles with v slightly greater than u_1 give up energy to the wave while those with v slightly less than u_1 absorb energy, we can understand why γ is proportional to F'(v). In section F we consider the energy exchange in more detail and give an alternate derivation of (5.34) which is more "physical", in that we examine the particles directly, but also, unfortunately, involves much more algebra than the argument given here, using the method of Landau's original paper.

3. The Large Phase Velocity Approximation

A further approximation is possible when u_1 is large compared to the typical velocities where $F(v) \neq 0$. Since this cannot hold for ion acoustic waves, where $u_1 << a_e$, we shall illustrate this approximation using Langmuir waves. There, the ionic contributions give only m/M corrections, so we may ignore them, i.e., replace F by f_ℓ for electrons and let $\Omega_p^2 \to \omega_p^2$. An asymptotic expansion of (5.31) then gives

$$k^2/\omega_p{}^2 = -\frac{d}{du_1}\int dv\,\frac{f_\ell(v)}{u_1}\left[1+\frac{v}{u_1}+\frac{v^2}{u_1{}^2}+\ldots\right] = u_1{}^{-2}[1+3a^2/2u_1{}^2+\ldots] \tag{5.35}$$

where $a^2 \equiv 2\int dvf_\ell(v)v^2$ is, by assumption, small compared to (ω_p/k). Solving (5.35) by iteration gives

$$u_1{}^2 = (\omega_p/k)^2[1 + 3k^2a^2/2\omega_p{}^2] \tag{5.36}$$

or

$$\omega_R = \omega_p\,[1 + 3k^2a^2/2\omega p^2]^{1/2} \tag{5.37}$$

Since (5.36) gives $du_1/dk = -u_1/k$ we have from (5.34)

$$u_2 \doteq (\pi/2)u_1{}^3f_\ell'(u_1) \tag{5.38}$$

These results can also be written

$$\boxed{\begin{aligned}\omega_R/\omega_p &= (1 + 3k^2a^2/2\omega_p{}^2)^{1/2}\\ \gamma/\omega_p &= (\pi/2)u_1{}^2f_\ell'(u_1)\end{aligned}} \tag{5.39}$$

If f_ℓ is Maxwellian, these become

$$\omega_R/\omega_p = (1 + 3k^2/k_D{}^2)^{1/2} \tag{5.40}$$

i.e. the Bohm-Gross dispersion relation (3.28) for a one-dimensional $(\gamma = 3)$ plasma, and

$$\gamma/\omega_p = \sqrt{\pi}\,(k_D{}^2/2k^2)^{3/2}e^{-k_D{}^2/2k^2}\,e^{-3/2} \tag{5.41}$$

the expression first obtained by Landau (save for the factor $e^{-3/2}$ which is due to J.D. Jackson). Note that $k \doteq k_D$ separates the cases of weak and strong Landau damping; the latter case, obtained when $k \gtrsim k_D$, of course violates the assumption of small damping which underlies the approximate evaluations of this section and the preceeding one, so (5.41) cannot be trusted there. For small k, however, it is correct, and shows that γ goes strongly to zero.

As a final remark, we stress the fact that (5.34) is correct for small damping, whether or not $u_1 >> 1$, and can be useful even if (5.31) must be solved by some other approximation than the asymptotic expansion (5.35).

4. The Plasma Dispersion Function

If the damping is not small, numerical methods of solving (5.27) cannot be avoided. Because f_ℓ is frequently chosen to be Maxwellian, it is useful to know the general properties of its Hilbert transform, called the plasma dispersion function, Z_M,

$$Z_M(s) \equiv \pi^{-1/2} \int_{-\infty}^{\infty} dt e^{-t^2}/(t-s) \qquad \text{Im}s > 0 \qquad (5.42)$$

with Z_M defined by analytic continuation in the lower half s plane. Henceforth, we shall simply denote this function as Z, i.e., drop the subscript M for Maxwellian. As s moves below the real axis, one way of analytically continuing Z is to deform the contour of integration from the real t axis to a curve, C, which always lies below s. Integration along this contour is equivalent to integration along the real axis plus integration around the pole at t = s (see Fig. 5.1)

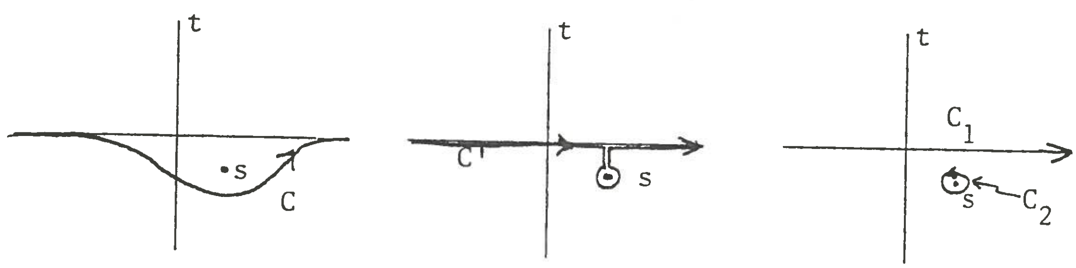


Fig.5.1. Deformation of contour for computation of Z(s).
Integration of $\epsilon^{-t^2}/(t-s)$ along C,C' or C_1+C_2 gives the same result.

Thus, (5.42) is supplemented by

$$Z(s) = \pi^{-1/2} \int_{-\infty}^{\infty} dt e^{-t^2}/(t-s) + 2i\pi^{1/2} e^{-s^2} \qquad \text{Im}s < 0 \qquad (5.43)$$

For some purposes, it is convenient to have an expression for Z valid in the whole plane. Differentiating (5.42) gives a differential equation

$$Z'(s) = \pi^{-1/2} \int_{-\infty}^{\infty} dt e^{-t^2} (t-s)^{-2} = \pi^{-1/2} \int_{-\infty}^{\infty} (-2t) e^{-t^2} (t-s)^{-1}$$

or

$$Z' = -2(1 + sZ) \tag{5.44}$$

which is useful in itself. Writing it as

$$Z' + 2sZ = -2$$

and using the integrating factor e^{s^2} gives

$$Z(s)e^{s^2} = Z(s_0)e^{s_0^2} - 2\int_{s_0}^{s} dt' e^{t'^2}$$

Choosing $s_0 \to i\infty$, we have

$$Z(s) = -2e^{-s^2}\int_{i\infty}^{s} dt' e^{t'^2}$$

or with $t = it'$

$$\boxed{Z(s) = 2ie^{-s^2}\int_{-\infty}^{is} dt e^{-t^2}} \tag{5.45}$$

an expression valid for all s. In terms of the error function of complex argument

$$\mathrm{erf}(x) \equiv (2/\pi^{1/2})\int_0^x dt e^{-t^2}$$

we can write

$$Z(s) = 2ie^{-s^2}[\pi^{1/2}/2 + \int_0^{is} dt e^{-t^2}] = i\sqrt{\pi}\, e^{-s^2}[1 + \mathrm{erf}(is)]$$

Although this reduces Z to the "known" and, in fact, tabulated error function, a direct tabulation of Z itself is useful, to avoid the nuisance of adding 1 and multiplying by e^{-s^2}. This is given in "The Plasma Dispersion Function" by B.D. Fried and S. Conte, Academic Press (New York) 1961.

From (5.42), it is easy to obtain the asymptotic expression for large s in the upper half plane,

$$Z(s) = -\pi^{-1/2} \int_{-\infty}^{\infty} dt (e^{-t^2}/s)(1 + t/s + t^2/s^2 + \ldots) =$$

$$= -s^{-1}(1 + 1/2s^2 + \ldots) \qquad \begin{array}{l} |s| >> 1 \\ \text{Im} s > 0 \end{array} \tag{5.46}$$

The coefficients in the power series (small s) expansion are most easily obtained from (5.44). Since (5.45) gives

$$Z(0) = i\sqrt{\pi} \tag{5.47}$$

we have from (5.44)

$$Z'(s) = -2(1 + sZ) = -2$$

$$Z''(s) = -2(sZ' + Z) = -2i\sqrt{\pi}$$

$$Z(s) = i\sqrt{\pi} - 2s - i\sqrt{\pi}s^2 + \ldots \qquad |s| << 1 \tag{5.48}$$

Plots of real and imaginary parts of Z and Z' are shown in Figs. 5.2 and 5.3.

5. Langmuir and Ion Acoustic Waves; Comparison with Two Fluid Approximation

We now consider the solutions of (5.27) for an electron-ion plasma without using the small γ approximation discussed in sections 2 and 3. For definitions, we assume both species to have Maxwellian distributions, so the dispersion relation involves the plasma dispersion function, Z, of section

$$\varepsilon = 1 - (k_D{}^2/2k^2)[Z'(\omega/ka_e) + (T_e/T_i)Z'(\omega/ka_i)] = 0 \tag{5.49}$$

or

$$2k^2/k_D{}^2 = Z'(s_e) + \theta Z'(s_i) \tag{5.50}$$

$$s_e = \omega/ka_e \qquad s_i = \omega/ka_i = s_e\sqrt{\theta/\delta}$$

$$\delta = m/M \qquad \theta = T_e/T_i$$

The imaginary part of (5.50)

$$0 = \left[\mathrm{Im}\, Z'(s_e) + \theta Z'(s_i)\right] \tag{5.51}$$

defines a locus in the s_e plane. At each point on that locus, we can compute

$$2k^2/k_D{}^2 = \mathrm{Re}Z'(s_e) + \theta \mathrm{Re}Z'(s_i) \tag{5.52}$$

If, and only if, the right side of (5.52) is positive, this point is a valid solution of (5.50).

In Fig. 5.4 we show the results of this procedure in the $s_e = x+iy$ plane for the case $m/M \to 0$, i.e. neglecting the $Z'(s_i)$ contribution. Only the highest curve corresponds to observable oscillations (namely, Langmuir waves) the others representing modes which are strongly damped in one cycle.
The small damping approximation

derived in sections 2 and 3 is also shown; note how poor it is for $Re(s) < 3$. While we show only $x > 0$, the figure is symmetric about the y-axis. The multiplicity of roots as we move down in the plane is a consequence of the highly oscillatory term e^{-s^2} in (5.43) for $Im(s)$ negative and large but no physical manifestation of these roots has been seen, due no doubt to their strong damping.

If we now include the ion dynamics (i.e., the $Z(s_i)$ term in ε), then it is clear that on or below the line $y = -|x|$, the ion term $e^{-s_i^2}$ will completely dominate the equation, since $|s_i| >> |s_e|$. Thus, the curves below $y = -|x|$ will be the same as in Fig. 5.4 save for a reduction in scale by $(\theta/\delta)^{1/2}$. Above this, the ion and electron terms will be comparable; the result, for $\theta = 1$, $M/m = 1836$(H^+ ions) is shown in Fig.5.5 . The Langmuir branch, which does not appear on this magnified scale, is virtually unchanged, but there is now a new branch, the ion acoustic waves. Unlike the Langmuir waves, these are strongly damped ($Ims/Res \sim 1$) even for $k = 0$. (Since ω_R/k is of order a_i, strong ion Landau damping is to be expected.) However, as $\theta = T_e/T_i$ is raised, they become less damped; with increasing phase velocity (larger ω_R/ka_i) we would expect the ion Landau damping to decrease. For example, with $\theta = 25$, the top two ion acoustic branches are as shown in Fig.5.6 , and the damping per cycle is clearly very small. It is interesting to observe that $k = 0$ no longer corresponds to the minimum damping. In fact, at $k = 0$ the phase velocity is large enough so that Landau damping on the ions is negligible and there is only the weak electron Landau damping (weak because $\omega_R/ka_e << 1$, so f_e is nearly flat at ω_R/k). As k increases, the phase velocity drops; ω_R/k falls at an even flatter part of f_e; and so the damping drops. With further increase of k (and decrease of ω_R/k) the phase velocity comes into the tail of the ion distribution function, and then ion Landau damping becomes important, resulting in an increase in damping. Although an analytic approximation is possible in the case $\theta >> 1$, we defer it to section C, where we will obtain it as a special case of the streaming instability discussion.

C. Longitudinal Instabilities

So far, we have considered in this chapter only situations where the roots of the dispersion equation lie in the lower half ω plane (stable waves). Just as in the fluid theory, growing waves (roots in the upper half plane) can occur for suitable f_ℓ, in particular those which involve streaming of one species of particles. For the longitudinal wave case represented by (5.27), a simple theorem gives the necessary condition on f_ℓ for instabilities.

1. Jackson's Theorem

Theorem: If $F(v)$ in (5.27) is a single-humped curve, meaning that $G(v) = F'(v)$ has one and only one zero, $v = v_0$, with

$$G(v)(v-v_0) < 0, \quad v \neq 0 \quad , \tag{5.53}$$

then (5.27) has no roots in the upper half plane.

Proof: Assume

$$u = u_1 + iu_2 \qquad u_2 > 0$$

is such a root. Then

$$1 = \int dv \, \frac{G(v)[v-u_1+iu_2]}{(v-u_1)^2+u_2^2}$$

From the imaginary part of this we have

$$\int dv \, \frac{G(v)}{(v-u_1)^2+u_2^2} = 0 \tag{5.54}$$

From the real part, we have

$$1 = \int dv \, \frac{G(v)(v-u_1)}{(v-u_1)^2+u_2^2} \tag{5.55}$$

Multiply (5.54) by (u_1-v_0) and add it to (5.55) giving

$$1 = \int dv \, \frac{G(v)(v-v_0)}{(v-u_1)^2+u_2^2} \tag{5.56}$$

Since (5.53) shows that the right side of (5.56) is negative definite, we have a reductio ad absurdum, which establishes the theorem.

Thus, we see that a double humped character of F is necessary (but not, as we shall see, sufficient) to have unstable waves. Clearly, relative streaming (electrons vs. ions, or two classes of electrons) can result in a double humped F, so we consider that case now.

2. Current-Driven Ion Acoustic Instability

In the fluid theory we have seen that a streaming of electrons relative to ions can give rise to an instability. Although this is one of the simplest of the microscopic instabilities, it has a ubiquitous character, since its possible occurance must be considered whenever currents flow in the plasma.

Suppose each species has a Maxwellian distribution in its own rest frame:

$$\begin{aligned} f_{0e}(\underline{v}) &= (\pi^{1/2}a_e)^{-3}\exp[-(\underline{v}-\underline{V}_e)^2/a_e^{\,2}] \\ f_{0i}(\underline{v}) &= (\pi^{1/2}a_i)^{-3}\exp[-(\underline{v}-\underline{V}_i)^2/a_i^{\,2}] \end{aligned} \qquad (5.57)$$

Then

$$\varepsilon_\ell = 1 - (k_D^{\,2}/2k^2)\{Z'[(\omega-kV_e)/ka_e] + \theta Z'[(\omega-kV_i)/ka_i]\} \qquad (5.58)$$

where

$$\theta = T_e/T_i$$

We shall work in the ion rest frame, setting

$$V_i = 0 \qquad\qquad V_e = V \quad .$$

Two cases lend themselves to simple analysis:

a) $\theta = T_e/T_i = 1$

The only easy result here is the threshold. There ω is real and since

$$\mathrm{Im}Z(s) = \sqrt{\pi}\, e^{-s^2} \qquad\qquad s \text{ real}$$

it is easy to decompose ε_ℓ into real and imaginary parts. Setting the imaginary part equal to zero gives

$$s_e e^{-s_e^2} = -s_i e^{-s_i^2}$$

where

$$s_e = (\omega - kV)/ka_e \quad , \qquad s_i = \omega/ka_i \quad ,$$

One solution of this (and, as it turns out, the only one which gives an unstable mode) is

$$s_i = -s_e = s$$

From the real part of (5.58) we then have

$$k^2/k_D^2 = \mathrm{Re}Z'(s)$$

since ReZ'(s) is an even function of s for real s. Also

$$V = sa_e(1 + \delta^{1/2})$$

From the graph of ReZ'(s), Fig.5.3, we see that the minimum drift which gives non-negative k^2/k_D^2 will be associated with the root of ReZ', which is approximately .925 and the corresponding k value will be 0. Thus, in this case long wavelength waves ($k \to 0$) are the most unstable and the threshold is at

$$V = .925\ a_e(1 + \delta^{1/2})$$

i.e. at a drift velocity nearly equal to the electron thermal speed. We also see from Fig.5.3 that there will be no marginally stable solution if k^2/k_D^2 exceeds the maximum value of ReZ'(s), namely .569, which occurs at s = 1.5, or $V = 1.5a_e(1+\delta^{1/2})$. Stabilization at large k/k_D occured also in our two-fluid analysis of this problem, and in both cases, for given $k > 0$ (and, in the Vlasov case, $k^2 < .569k_D^2$) there is an upper bound on V for instability.

b) $\theta = T_e/T_i >> 1$

Here we make the "ansatz" (provisional assumption) that

$$|s_e| << 1 \quad , \quad |s_i| >> 1 \tag{5.59}$$

which we then verify a posteriori. Using the power series approximation for the electron Z and the asymptotic expression for the ion Z we have

$$\varepsilon_\ell = 1 - (k_D{}^2/2k^2)[-2(1 + s_e i\sqrt{\pi}) + \theta s_i{}^{-2}] = 0 \tag{5.60}$$

which we can "solve" for s_i:

$$s_i{}^2 = \frac{\theta}{2(\hat{k}^2+1)}\left(1 + \frac{s_e i\sqrt{\pi}}{\hat{k}^2+1}\right)^{-1} \qquad \hat{k} \equiv k/k_D \tag{5.61}$$

This is not really a solution, since both s_i and s_e involve the unknown phase velocity, ω/k, which we are trying to find, but assuming $s_e \ll 1$, we can solve (5.61) iteratively. Neglecting the s_e term gives

$$s_i = s_i{}^0 = \left[\frac{\theta}{2(\hat{k}^2+1)}\right]^{1/2}$$

or

$$\omega = kc_s/(1 + k^2/k_D{}^2)^{1/2} \qquad c_s = (T_e/M)^{1/2} \tag{5.62}$$

which is just the two-fluid dispersion equation, derived earlier (Chap. III), in the limit $T_i \ll T_e$. Iterating, we have

$$s_e = (a_i s_i{}^0 - V)/a_e$$

and

$$s_i = s_i{}^0\,[1 - i\sqrt{\pi}\,(s_i{}^0 a_i - V)/a_e(1 + \hat{k}^2)] \tag{5.63}$$

or

$$\omega = \Omega + i\Gamma = \frac{kc_s}{(1+k^2/k_D{}^2)^{1/2}}\left\{1 - \frac{i\sqrt{\pi}}{2}\,\frac{\left[\dfrac{c_s}{(1+k^2/k_D{}^2)^{1/2}} - V\right]}{a_e(1+k^2/k_D{}^2)}\right\} \tag{5.64}$$

From this result we see that:

a) Our ansatz (5.59) is satisfied, provided k and V are not too large. Since we have assumed $\theta \gg 1$, we will have

$$|s_i|^2 \approx \theta/2(\hat{k}^2 + 1) \gg 1$$

unless k exceeds $k_D\sqrt{\theta} = k_{Di}$, the ion Debye wave number, and certainly $s_i >> 1$ for $k < k_D$. Since $s_i^0 a_i/a_e < \sqrt{m/M}$, s_e will be small so long as V/a_e is small.

b) In absence of drift (V = 0), ion acoustic waves are weakly damped in the $T_e/T_i >> 1$ limit, with

$$|\Gamma/\Omega| < c_s/a_e = \sqrt{m/2M}$$

This contrasts markedly with the equal temperature result, $|\Gamma/\Omega| \gtrsim 0.4$.

c) The threshold for instability is just c_s, which is smaller, by about $(m/2M)^{1/2}$, than in the equal temperature case. Of course, this is not surprising, since the Landau damping in absence of drift is reduced by a similar factor.

The sign of Γ is determined by a competition between the two stream instability, which tends to make $\Gamma > 0$, and the Landau damping. Using the extreme asymptotic limit $Z(s_i) \doteq s_i^{-2}$ is equivalent to neglecting the ion Landau damping, but for $\theta >> 1$ the phase velocity is much greater than a_i so this is a good approximation: the number of resonant ions ($v \doteq \Omega/k$) is vanishingly small. We have kept the electron Landau damping, which is small not because there are few resonant particles (in fact, ω/k falls almost at the maximum of f_{e0}) but because the <u>slope</u> of the distribution function is small there. These considerations are illustrated schematically in Fig. 5.7. Of course, as k increases, the phase velocity drops (because of the $(1 + k^2/k_D^{\ 2})^{-1/2}$ factor) and eventually comes down into the ion distribution function. Then ion Landau damping becomes important; Γ/Ω is not small; and our approximations are invalid. A similar effect occurs for decreasing T_e/T_i, as illustrated in section B.5.

It is important to note the differences between the instabilities of the inverse Landau damping type, discussed here, and the two stream instabilities studied in Chapter II.

i) In the fluid theory, $\varepsilon(k,\omega)$ is real for k, ω real, but $\varepsilon = 0$ may still have imaginary or complex roots in the ω plane for real k. However, no resonant wave-particle interaction is involved. For example, two counter-streaming, cold electron fluids (with a static neutralizing ion background) have an unstable mode with purely imaginary ω. Hence the wave is at rest in the lab frame, whereas all of the electrons are streaming rapidly ($v = \pm V$) to left or right in that frame. The instability in such cases arises from a mechanism very much like the "bunching" phenomenon in a klystron.

ii) For instabilities of the Landau type, ε is complex for real and we can distinguish two parts: a "real" or Hermitian portion which is real for real ω and k and an "imaginary" or anti-Hermitian portion which is purely imaginary.

$$\varepsilon(\omega) = \varepsilon_R(\omega) + i\,\varepsilon_I(\omega) \qquad \omega \text{ real}$$

(We suppress the k dependance for simplicity.) If $|\varepsilon_I| << |\varepsilon_R|$, then an approximate solution of $\varepsilon = 0$ is $\omega = \omega_R$, where

$$\varepsilon_R\,(\omega_R) = 0$$

A better solution, including an imaginary part of ω, can be obtained by expanding around ω_R:

$$\omega = \omega_R + i\gamma$$

$$\varepsilon = \varepsilon_R(\omega_R + i\gamma) + i\,\varepsilon_I(\omega_R + i\gamma) =$$

$$= i\gamma \frac{\partial \varepsilon_R}{\partial \omega} + i\,\varepsilon_I(\omega_R) + \text{-----} = 0$$

or

$$\gamma = -\left(\frac{\varepsilon_I}{\partial \varepsilon_R/\partial \omega}\right)_{\omega = \omega_R}$$

This procedure leads to the same results as the small γ approximation (5.34).

iii) The terms "reactive" or "non-resonant" instability are sometimes used to describe cases of type i while those of type ii are called "resistive" or "resonant" instabilities. A significant difference between the two types is illustrated by the case of a relative drift, V, of electrons relative to ions. If $V \gg a_e$, we have the case studied by Buneman and presented in Chapter III. Leaving aside numerical factors, we note from 3.86 and (3.88) that $(\gamma/\omega_p) \approx (m/M)^{1/3}$ for $V \approx \omega_p/k$. On the other hand, for $T_e \gg T_i$ the ion acoustic waves have

$$(\gamma/\omega_p) \approx (k/k_D)\ (V/a_e)\ (m/M)^{\frac{1}{2}} \approx (m/M)^{\frac{1}{2}}$$

An account of the transition from one case to the other is given by O'Neil .

D. Initial Value Effects: Free Streaming and Phase Mixing

As we have seen in section A, with an isotropic f_0 and suitable conditions on the linearized variables (no external transverse currents, no initial $\underline{E}_t$ and $\underline{B}$, and an isotropic initial distribution function, g) the electric field is purely longitudinal and the problem becomes essentially one dimensional. Then our solutions (5.5) and (5.13) for f_1 and E_ℓ become

$$f_1(k,v,\omega) = \frac{g_\ell(k,v) - \frac{q}{m} E_\ell f_\ell'(v)}{i(kv-\omega)} \tag{5.65}$$

$$E_\ell(k,\omega) = \frac{E_e(k,\omega) - (4\pi/k)\int dv n_0 q g_\ell/(kv-\omega)}{\varepsilon_\ell(k,\omega)} = N(k,\omega)/\varepsilon_\ell(k,\omega) \tag{5.66}$$

Consider the simple case where N is an entire function of ω (no singularities in the finite plane) and f_ℓ is stable so the roots, ω_j, of ε_ℓ are all in the lower half ω plane. Then $E_\ell(k,t)$ is a superposition of damped waves, $e^{-i\omega_j t}$. However, $f_1(k,v,\omega)$ has, besides the roots at ω_j, due to E_ℓ, also a root at $\omega = kv$, so that

$$f_1(k,v,t) = +\sum_j \left[\frac{qN(k,\omega_j)f_\ell'(v)}{m(kv-\omega_j)(\partial\varepsilon_\ell/\partial\omega)_{\omega_j}} \right] e^{-i\omega_j t} + [g_\ell(k,v) - \tag{5.67}$$

$$- (q/m)E_\ell(k,kv)f_\ell'(v)]e^{-ikvt}$$

is a superposition of damped waves <u>plus</u> a term which is purely oscillatory, with frequency kv. This last term is sometimes referred to as the <u>free-streaming</u> contribution.

Its physical origin can be understood in a simple way. If there is initially a density modulation, of wave number k, in those particles which have velocity v, and if we neglect electric forces altogether (i.e., consider neutral particles) then as this modulation travels, at velocity v, it will appear in the laboratory as a density modulation of wave number k and frequency kv (Doppler shift). Expressing this mathematically, we have, for q = 0,

$$\partial f/\partial t + v\partial f/\partial x = 0$$

whose general solution (treating v as a parameter) is

$$f(x,v,t) = F(x-vt)$$

where F is an arbitrary function, determined by the initial distribution function $f(x,v,t = 0)$. If $g(k,v)$ is the Fourier transform of this, ~~the initial condition~~

$$f(x,v,t=0) = \int g(k,v)e^{ikx}dk/2\pi$$

then

$$F(x) = \int ge^{ikx}dk/2\pi$$

and so

$$f(x,v,t) = F(x - vt) = \int ge^{ik(x-vt)}dk/2$$

It follows that

$$f(k,v,t) = g(k,v)e^{-ikvt}$$

and

$$f(k,v,\omega) = \frac{g(k,v)}{i(kv-\omega)}$$

[cf. the first term in (5.65)] If the particles are charged, then the density perturbation causes an electric field and the free streaming contribution takes the form of the second term in (5.67).

If f has a term which is purely oscillating and hence undamped, why is there no corresponding term in E_ℓ or n_1? The reason is, of course, that n_1 (from which we compute E_ℓ, by Poisson's equation) involves a velocity integration, so we get not e^{-ikvt} but rather something proportional to

$$H(t) = \int dvh(v)e^{-ikvt}$$

where $h(v)$ is either $f_\ell'(v)\,E_\ell(k, kv)$ or $g(k,v)$. This integral decreases rapidly with increasing t, a phenomenon referred to as <u>phase mixing</u>. The Riemann-Lebesque lemma (Whittaker-Watson, Modern Analysis §9.41) guarantees that if $h(v)$ is of limited total fluctuation, then $H(t)$ vanishes as $1/kt$. A less rigorous statement, of greater practical value, is that $H(t)$ dies off in a time of order $1/ka$, where a characterizes the width of $h(v)$. A simple example is the Lorentzian, $h(v) = (a/\pi)(v^2+a^2)^{-1}$, for which $H(t) = e^{-kat}$.

If f_ℓ is unstable, so that one or more of the ω_j is in the upper half plane, then at large times the growing exponential will dominate the free streaming term, which can therefore be safely neglected, unless we are interested in the initial transient behavior. If f_ℓ is stable, then at long times the free streaming term is more important, in f_1, than the Landau damped terms. Many of the quantities of physical interest in a plasma involve integrations over v, so that it is generally correct to invoke "phase mixing" as a justification for dropping the free streaming terms. However, it can lead to mistakes in some instances. Moreover, just these terms play an essential role in the very pretty phenomena known as plasma wave echos, which we will discuss when considering nonlinear problems.

As a final remark, we emphasize that <u>this</u> phase mixing is <u>not</u> to be confused with Landau damping. In a neutral gas, an initial perturbation in f_1 will cause a density perturbation which decays in a time of order 1/ka, due to phase mixing. In a plasma, such a perturbation will cause a density perturbation which has two components: one which decays due to phase mixing (the velocity integral of the second term in (5.67)) and another (coming from the first term of (5.67)) which oscillates at a characteristic frequency (e.g., ω_p) and which, for long wavelengths (small k) decays very slowly, in a time much greater than 1/ka. For example, with Langmuir waves we have for Maxwellian f_ℓ ,

$$\tau_{\text{phase mixing}}/\tau_{\text{Landau}} = |\gamma_L/\gamma_{pm}| \doteq \sqrt{\pi}\,(u/a)^4\, e^{-(u/a)^2} < .96$$

where $u = \omega_p/k$ is the Langmuir wave phase velocity and we set $\gamma_{pm} \doteq ka$. For small k (large u) this ratio becomes as small as we please. Moreover, Landau damping occurs in f_1 itself, as well as in n_1 and E, while phase mixing affects only the latter.

The notion that Landau damping is associated with phase mixing arises in a different context, namely the relation of the Landau initial value problem, discussed here, to the steady state "normal mode" solutions of N. Van Kampen, which we take up in ~~the next section~~ a later chapter.

E. Dynamical Shielding of a Test Particle

Suppose that we have no initial disturbance but do have an external source consisting of a "test charge" Q, moving with velocity V through the plasma, passing the point $\underline{x}_0$ at time 0. The associated external charge and current densities are

$$\rho_e(\underline{x},t) = Q\delta[\underline{x} - \underline{x}_0 - \underline{V}t] \; ; \quad \underline{j}_e = \underline{V}\rho_e \tag{5.68}$$

with Fourier-Laplace transforms

$$\rho_e(\underline{k},\omega) = \frac{Qe^{-i\underline{k}\cdot\underline{x}_0}}{i(\underline{k}\cdot\underline{V}-\omega)} \quad ; \; \underline{j}_e(k,\omega) = \underline{V}\rho_e(\underline{k},\omega) \; . \tag{5.69}$$

How does the plasma respond to the passage of this test charge, i.e., what are the total fields? Assuming $f_0(\underline{v})$ to be isotropic, we can, as before, divide the problem into longitudinal and transverse parts. Since we shall eventually want to transform back from $\underline{k}$ to $\underline{x}$ in order to see the behavior in real space, it is convenient to write (5.7) in the equivalent coordinate space form,

$$E(\underline{x},t) = \underline{E}_\ell + \underline{E}_t \tag{5.70}$$

with

$$\nabla\times E_\ell = \nabla\cdot\underline{E}_t = 0 \tag{5.71}$$

It follows from (5.71) that we can write $\underline{E}_\ell$ in terms of a scalar potential,

$$\underline{E}_\ell = -\nabla\varphi \; .$$

Then Poisson's equation gives, neglecting initial value contributions,

$$\varphi(\underline{k},\omega) = 4\pi\rho_e(\underline{k},\omega)/k^2\varepsilon_\ell/\underline{k},\omega) = \frac{(4\pi Q/k^2)e^{-i\underline{k}\cdot\underline{x}_0}}{i(\underline{k}\cdot\underline{V}-\omega)\varepsilon_\ell(\underline{k},\omega)} \tag{5.72}$$

Transforming back to $(\underline{x},t)$ we have

$$\varphi(\underline{x},t) = 4\pi Q \int \frac{d\underline{k}d\omega}{(2\pi)^4} \frac{e^{i[\underline{k}\cdot(\underline{x}-\underline{x}_0)-\omega t]}}{k^2 i(\underline{k}\cdot\underline{V}-\omega)\varepsilon_\ell(\underline{k},\omega)} \tag{5.73}$$

If we carry out the ω integration first we find a pole at $\omega = \underline{k}\cdot\underline{V}$ in addition to those at the roots, ω_j, of ε_ℓ. We shall now show that the former represents a dynamic version of Debye shielding, reducing to the familiar form in the V = 0 limit, while the latter corresponds to Cerenkov radiation of those Langmuir waves, if any, whose phase velocity is less than V.

Let $X = \underline{x} - \underline{x}_0 - \underline{V}_t$, and write

$$\varphi(\underline{x},t) = \varphi_D(\underline{X}) + \varphi_C(\underline{x},t)$$

where the Debye shielded potential is

$$\varphi_D(\underline{X}) \equiv 4\pi Q \int \frac{d\underline{k}}{(2\pi)^3} \frac{e^{i\underline{k}\cdot\underline{X}}}{k^2 \varepsilon_\ell(\underline{k},\underline{k}\cdot\underline{V})} \tag{5.74}$$

while the Cerenkov radiation term is

$$\varphi_C(\underline{x},t) \equiv -4\pi Q \sum_j \int \frac{d\underline{k}}{(2\pi)^3} \frac{e^{i[\underline{k}\cdot(\underline{x}-\underline{x}_0)-\omega_j t]}}{k^2(\omega_j - \underline{k}\cdot\underline{V})(\partial\varepsilon_\ell/\partial\omega)_{\omega_j}} \tag{5.75}$$

(Of course, the roots ω_j are functions of $\underline{k}$) In the limit $\underline{V} \to 0$, the denominator of φ_D involves

$$\varepsilon_\ell(\underline{k},0) = 1 - \fint d\underline{v}\ (\omega_p{}^2/k^2)\ \frac{\underline{k}\cdot d f_\ell/d\underline{v}}{\underline{k}\cdot\underline{v}} \tag{5.76}$$

For a Maxwellian,

$$\int d\underline{v}(\omega_p{}^2/k^2)\ \frac{\underline{k}\cdot df_\ell/d\underline{v}}{\underline{k}\cdot\underline{v}} = -2\omega_p{}^2/k^2a^2 = -k_D{}^2/k^2$$

so, setting $K_D{}^2 = k_{De}{}^2 + k_{De}{}^2$ as in Chapter I, we have

$$\varepsilon_e(\underline{k},0) = 1 + K_D{}^2/k^2 \tag{5.77}$$

Thus, for V = 0

$$\varphi_D(\underline{X}) = 4\pi Q \int \frac{d\underline{k}}{(2\pi)^3} \frac{e^{i\underline{k}\cdot\underline{X}}}{k^2 + k_D{}^2} = \frac{4\pi Q}{X}\, e^{-K_D X}\ , \tag{5.78}$$

the same Debye shielded potential derived in Chapter I.

When $V \neq 0$, the integration over $\underline{k}$ in (5.74) can only be performed approximately (e.g. by an expansion in V/a) or numerically, but the result, for any stable f_ℓ, is simply a distortion of the Debye cloud.

The "Cerenkov" term, φ_C, is a superposition of damped plane waves, weighted with resonant denominators, $(\omega_j - \underline{k}\cdot\underline{V})$. For a stable plasma, $\mathrm{Im}\,\omega_j < 0$ so none of these denominators can vanish. However, we know that when $k \ll k_D$, Langmuir waves are weakly damped; if in addition $T_i/T_e \ll 1$, so are ion acoustic waves. Let θ denote the angle between $\underline{k}$ and the line of travel of the test particle and consider the conditions under which

$$\omega_j - \underline{k} \cdot \underline{V} = k(\omega_j/k - V\cos\theta) \tag{5.79}$$

can be small. For Langmuir waves, the Landau damping will be large unless $\omega_j/k > a_e$, so the minimum V which can give a sharply resonant denominator for Langmuir waves is $V \doteq a_e$. If $V > a_e$, then a "wake" with half angle $\theta \lesssim \theta_c$, where

$$\cos\theta_c = V/a_e \quad ,$$

will be stimulated by the test particle (see Fig. 8); this wake consists of Langmuir waves propagating along $\underline{k}$, damping in the course of time. In a plasma with $T_i/T_e \ll 1$, a test particle with $V > c_s$ will, similarly, excite an ion acoustic wake, with half angle given by

$$\cos\theta_c = V/c_s$$

So far we have considered only the longitudinal field response to the test charge. In a similar way, we can find the transverse field response to $\underline{j}_{et}$ using ε_t. We also note that the test particle will experience a retarding force due to the charges and currents induced in the plasma. It its mass is finite, then its velocity, heretofore assumed constant, will decrease. Thus, from the longitudinal part of the field we have for the rate of energy loss by the test particle

$$\frac{dW}{dt} = Q\underline{\hat{E}}_\ell \cdot \underline{V} = -Q\nabla\hat{\varphi} \cdot \underline{V}$$

where $\hat{\varphi}$ is the potential due to plasma particles, i.e. φ with the self-field effects subtracted,

$$\hat{\varphi}(\underline{k},\omega) = \varphi(\underline{k},\omega) - \varphi_e(\underline{k},\omega) = \varphi_e(\underline{k},\omega)[1/\varepsilon_\ell - 1] = (4\pi\rho_e/k^2)(1/\varepsilon_\ell - 1). \tag{5.80}$$

Then

$$\frac{dW}{dt} = -4\pi Q^2 \int \frac{d\underline{k}d\omega}{(2\pi)^4} \frac{e^{i(\underline{k}\cdot\underline{V}-\omega)t}\underline{k}\cdot\underline{V}}{k^2(\underline{k}\cdot\underline{V}-\omega)}\left(\frac{1}{\varepsilon_\ell(\underline{k},\omega)} - 1\right) \tag{5.81}$$

an expression which, like that for φ, can be evaluated in various limiting cases.

F. Linearization About an Oscillating Distribution; Parametric Instabilities

So far we have considered only the case of an unperturbed distribution, f_0, which is independent of $\underline{x}$ and t. For the sake of variety (and because it leads to interesting applications) we consider briefly the case of a plasma subjected to a uniform, time varying electric field, $\underline{E}_0(t)$, i.e. we linearize about an f_0 which is the solution to the Vlasov equation in presence of $\underline{E}_0(t)$. The field $\underline{E}_0$ could be produced by external conductors -- e.g. a set of condenser plates outside the plasma -- or it could be the field of a high intensity radio frequency or laser beam, whose wavelength is large enough to justify treating $\underline{E}_0$ as spatially uniform. We shall assume $\underline{E}_0$ to be a periodic function of time, e.g. a sinusoidal function or a superposition of such functions.

If we look for an f_0 which is isotropic in $\underline{v}$ and independent of $\underline{x}$, then there will be no induced fields in the plasma and f_0 simply satisfies

$$\partial f_0/\partial t = (q/m)\underline{E}_0(t) \cdot \nabla_v f_0 = 0 \tag{5.82}$$

If $\underline{V}_0(t)$ obeys

$$m d\underline{V}_0/dt = q\underline{E}_0 \tag{5.83}$$

then

$$f_0(\underline{x},\underline{v},t) = F[\underline{v}-\underline{V}_0(t)] \tag{5.84}$$

will be a solution of (5.82) for arbitrary F. For simplicity, we shall consider the case where F is isotropic.

We now study perturbations about this oscillating solution, i.e., set

$$f(\underline{x},\underline{v},t) = f_0(\underline{v},t) + f_1(\underline{x},\underline{v},t) \ .$$

The linearized Vlasov equation for f_1 and $\underline{E}_1$ is then

$$\partial f_1/\partial t + \underline{v} \cdot \nabla f_1 + (q/m)[\underline{E}_0 \cdot \nabla_v f_1 + \underline{E}_1 \cdot \nabla_v f_0] = 0 \tag{5.85}$$

$$\nabla \cdot \underline{E}_1 = 4\pi \int d\underline{v} n q f_1$$

if we neglect the $\underline{v} \times \underline{B}_1$ term (which now does not vanish even for isotropic f_0), thus restricting our discussion to electrostatic perturbations. (Henceforth, $\underline{E}$ will mean the longitudinal component, $\underline{E}_\ell$, in this section.) Although this equation appears to be more complicated than previous cases because of the $\underline{E}_0$ term, a simple transformation reduces (5.85) to familiar form. Define $\underline{V}_0$ by (5.83) and let $\underline{X}_0(t)$ be determined by

$$d\underline{X}_0/dt = \underline{V}_0 \tag{5.86}$$

Of course, $\underline{X}_0(t)$ is just the oscillating trajectory of a particle in the applied field, $\underline{E}_0$, and we would expect things to be particularly simple in a frame moving with X_0. The coordinates in this frame are

$$\begin{aligned} \underline{\tilde{x}} &= \underline{x} - \underline{X}_0(t) \\ \underline{\tilde{v}} &= \underline{v} - \underline{V}_0(t) \end{aligned} \tag{5.87}$$

and the distribution function is

$$\tilde{f}(\underline{\tilde{x}},\underline{\tilde{v}},t) = f(\underline{x},\underline{v},t) = f(\tilde{x}+\underline{X}_0,\underline{\tilde{v}}+\underline{V}_0,t) \tag{5.88}$$

Then

$$\partial f_1/\partial t + \underline{v}\cdot\nabla f_1 + (q/m)\underline{E}_0\cdot\nabla_v f_1 = (\partial\tilde{f}_1/\partial t - \underline{V}_0\cdot\nabla\tilde{f}_1 - (q/m)\underline{E}_0\cdot\nabla_v\tilde{f}_1) +$$

$$+ \underline{v}\cdot\nabla\tilde{f}_1 + (q/m)\underline{E}_0\cdot\nabla_v\tilde{f}_1 = (\partial/\partial t + \underline{\tilde{v}}\cdot\nabla)\tilde{f}_1$$

i.e., in the oscillating frame the effects of $\underline{E}_0$ are transformed away and we have just the usual linearized Vlasov system,

$$(\partial/\partial t + \underline{\tilde{v}}\cdot\nabla)\tilde{f}_1 + (q/m)\underline{\tilde{E}}_1\cdot\nabla_v F(\underline{\tilde{v}}) = 0 \qquad (5.89)$$

$$\nabla\cdot\underline{\tilde{E}}_1 = 4\pi\int d\underline{v}nq\tilde{f}_1$$

[Our solution of the unperturbed Vlasov equation (5.82) can be considered as a simple example of this transformation to the oscillating frame. In fact, the transformation (5.88) brings the Vlasov equation to the form

$$[\partial/\partial t + \underline{v}\cdot\nabla + (q/m)(\underline{E}-\underline{E}_0)\cdot\nabla_v]\tilde{f} = 0$$

Writing

$$\tilde{f} = \tilde{f}_0 + \tilde{f}$$

$$\underline{\tilde{E}} = \underline{\tilde{E}}_0 + \underline{\tilde{E}}_1$$

we have to zero order

$$(\partial/\partial t + \underline{v}\cdot\nabla)\tilde{f}_0 = 0$$

of which a particular solution is just

$$\tilde{f}_0(\underline{\tilde{x}},\underline{\tilde{v}},t) = F(\underline{\tilde{v}})$$

for arbitrary F, i.e. (5.84) to first order we get just (5.89).]

It follows immediately from (5.89) and the work in section A that in the oscillating frame we have for the density perturbation

$$q\tilde{n}_1 = -(ik/4\pi)\chi\tilde{E}_1 + q\tilde{N}_1 \qquad (5.90)$$

where χ is the usual linear susceptibility, computed with $F(\underline{v})$ as unperturbed distribution function:

$$\chi = \int dv \frac{\omega_p^2}{k^2} \frac{\underline{k}\cdot dF/d\underline{v}}{\omega-\underline{k}\cdot\underline{v}} \tag{5.91}$$

The last term in (5.90) arises from the initial values of $\tilde{f}_1$,

$$\tilde{N}_1 = in_0\int d\underline{v}\ \tilde{f}_1(\underline{k},\underline{v},t=0)/(\omega-\underline{k}\cdot\underline{v}) \tag{5.92}$$

Of course, the oscillating frame will be different for each species. However, in the case of the ions, we introduce only errors of order m/M if we neglect the difference between the lab frame and the oscillating frame for ions. Poisson's equation is valid in both the oscillating electron frame and in the lab frame:

$$\boxed{\begin{aligned} ikE_1 &= 4\pi e(n_{1i}-n_{1e}) \\ ik\tilde{E}_1 &= 4\pi e(\tilde{n}_{1i}-\tilde{n}_{1e}) \end{aligned}} \tag{5.93}$$

Using the second form, we obtain from (5.90)

$$e\tilde{n}_{1e} = e\chi_e(\tilde{n}_{1i} - \tilde{n}_{1e}) + e\tilde{N}_{1e}$$

or

$$\boxed{\tilde{n}_{1e} = (\chi_e/\varepsilon_e)\tilde{n}_{1i} + \tilde{N}_{1e}/\varepsilon_e} \tag{5.94}$$

where

$$\varepsilon_e = 1 + \chi_e$$

would be the dielectric function in absence of ions. Similarly, from the corresponding equation for ions we have

$$\boxed{n_{1i} = (\chi_i/\varepsilon_i)n_{1e} + N_{1e}/\varepsilon_i} \qquad (5.95$$

$$\varepsilon_i = 1 + \chi_i$$

The set (5.94) and (5.95) may be reduced to a single (integral) equation for one scalar function in six different ways: we may work in either the lab or oscillating frame; and we may eliminate the electron density, eliminate the ion density, or, using Poisson's equation (5.93), eliminate both densities in favor of the electrostatic potential or electric field. All of these manipulations are facilitated by introducing a convolution operator, $\Delta_{\underline{k}}$, which relates the Fourier-Laplace transform $A(\underline{k},\omega)$ of any function $A(\underline{x},t)$ to the transform $\tilde{A}(\underline{k},\omega)$ of its oscillating frame counterpart, $\tilde{A}(\tilde{\underline{x}},t)$, since the density functions in these equations are all Fourier-Laplace transforms, $n_1(\underline{k},\omega)$ and $\tilde{n}_1(\underline{k},\omega)$. We have

$$\begin{aligned}\tilde{A}(\underline{k},\omega) &= \int_0^\infty dt\, d\tilde{\underline{x}}\, \tilde{A}(\tilde{x},t)e^{-i(\underline{k}\cdot\tilde{\underline{x}}-\omega t)} = \\ &= \int_0^\infty dt\, d\underline{x}\, A(\underline{x},t)e^{-i[\underline{k}\cdot(\underline{x}-X_0)-\omega t]} = \\ &= \int_0^\infty dt\, e^{i\underline{k}\cdot X_0(t)+i\omega t} A(\underline{k},t)\end{aligned} \tag{5.96}$$

Define

$$\boxed{\Delta_{\underline{k}}(\omega) = \int_0^\infty dt\, e^{i\omega t} e^{ik\cdot X_0(t)}} \tag{5.97}$$

so that

$$e^{i\underline{k}\cdot X_0(t)} = \int \frac{d\omega'}{2\pi}\, \Delta_k(\omega')e^{-i\omega' t}$$

then

$$\begin{aligned}\tilde{A}(\underline{k},\omega) &= \int_0^\infty dt \int \frac{d\omega'}{2\pi}\, \Delta_k(\omega')e^{i(\omega-\omega')t}A(\underline{k},t) \\ &= \int \frac{d\omega'}{2\pi}\, \Delta_k(\omega')A(\underline{k},\omega-\omega')\end{aligned}$$

For notational convenience, we can define a convolution operator, Δ_k:

$$\boxed{\tilde{A}(\underline{k},\omega) = \Delta_{\underline{k}}\cdot A \equiv \int \frac{d\omega'}{2\pi}\, \Delta_{\underline{k}}(\omega')A(k,\omega-\omega')} \tag{5.98}$$

The convolution operation involves only ω ; $\underline{k}$ is simply a parameter. It is easy to see that the inverse operator Δ_k^{-1} which takes us from $\tilde{A}$ to A,

$$A(k,\omega) = \Delta_k^{-1}\cdot\tilde{A}$$

is just

$$\Delta_{\underline{k}}^{-1} = \Delta_{-\underline{k}}$$

Note that the integration contour, ω', in (5.98) must lie in the upper half plane, i.e. above any singularities of $\Delta_{\underline{k}}(\omega')$. Of course, $\omega-\omega'$ must also lie above any singularities of A.

We now return to the set (5.94) and (5.95). Of the six possibilities, for eliminating all but one dependent variable, the simplest is to work with the lab frame ion density, $n_{1i}(\underline{k},\omega)$. We have, dropping the 1 subscript,

$$(\varepsilon_i/\chi_i)n_i(\underline{k},\omega) = \Delta_{\underline{k}}^{-1}\cdot\tilde{n}_e + N_i/\chi_i = \Delta_{\underline{k}}^{-1}\,[(\chi_e/\varepsilon_e)\Delta_{\underline{k}}\cdot n_i + \tilde{N}_e/\varepsilon_e] + N_i/\chi_i$$

or

$$(1+\chi_i^{-1})n_i = \Delta_{\underline{k}}^{-1}\cdot[(1-\varepsilon_e^{-1})\Delta_{\underline{k}}\cdot n_i] + \Delta_{\underline{k}}^{-1}\cdot\tilde{N}_e/\varepsilon_e + N_i/\chi_i$$

Subtracting n_i from both sides leaves

$$\boxed{n_i(\underline{k},\omega) = -\chi_i\Delta_k^{-1}\cdot(\varepsilon_e^{-1}\Delta_{\underline{k}}\cdot n_i) + N} \tag{5.99}$$

with

$$N = N_i + \chi_i\Delta_k^{-1}\,(\tilde{N}_e/\varepsilon_e) \tag{5.100}$$

This is an integral equation for $n_i(k,\omega)$, valid for arbitrary, oscillating $\underline{E}_0(t)$, which cannot be further simplified without some specific assumption about $\underline{E}_0(t)$.

We shall consider only the simplest case,

$$\underline{E}_0(t) = 2\underline{E}_0\cos\omega_0 t$$

(The factor of 2 is included to make our final results agree with those in the standard literature on parametric instabilities.) Then

$$\underline{k}\cdot\underline{X}_0(t) = \underline{k}\cdot(2e/m\omega_0^{\,2})\underline{E}_0\cos\omega_0 t = \rho\cos\omega_0 t \tag{5.101}$$

where

$$\rho = (2e/m\omega_0^{\,2})\underline{k}\cdot\underline{E}_0$$

is 2π times the ratio of the electron excursion distance in the field $\underline{E}_0$ to the wavelength, $\lambda = 2\pi/k$, of the perturbation we are studying. We shall presently assume $\rho << 1$, as is the case for most experimental realizations.

We can now evaluate $\Delta_{\underline{k}}(\omega)$, using the Bessel function identity,

$$e^{i\rho\sin\theta} = \sum_{-\infty}^{\infty} J_n(\rho)e^{in\theta} \quad : \tag{5.102}$$

$$\left.\begin{aligned} \Delta_{\underline{k}}(\omega) &= \int_0^\infty e^{i\omega t}e^{i\rho\sin\omega_0 t} = i\sum_{-\infty}^{\infty}\frac{J_n(\rho)}{\omega+n\omega_0} \\ \Delta_{\underline{k}}^{-1}(\omega) &= \Delta_{-\underline{k}}(\omega) = i\sum_{-\infty}^{\infty}\frac{J_n(-\rho)}{\omega+n\omega_0} = i\sum\frac{J_n(\rho)}{\omega-n\omega_0} \end{aligned}\right\} \tag{5.103}$$

It follows that

$$\tilde{A}(\omega) = \Delta_{\underline{k}}\cdot A = \int\frac{d\omega'}{2\pi}\sum\frac{J_n(\rho)}{\omega'+n\omega_0}A(\omega-\omega') \tag{5.104}$$

If $A(\omega)$ has no singularities in the upper half plane, then we can close the ω' integration in the lower half plane, obtaining

$$\boxed{\tilde{A}(\omega) = \sum J_n(\rho)A(\omega+n\omega_0)} \tag{5.105}$$

Similarly, using $\Delta_{\underline{k}}^{-1}$ gives

$$A(\omega) = \sum J_n(\rho)A(\omega-n\omega_0)$$

The quantities needed for the integral equation (5.99) are then

$$\tilde{n}_i(\underline{k},\omega) = \Delta_{\underline{k}}\cdot n_i = \sum J_n n_i(\underline{k},\omega+n\omega_0) \tag{5.106}$$

and

$$\Delta_{\underline{k}}^{-1}\cdot(\varepsilon_e^{-1}\tilde{n}_i) = \sum J_{n'}\varepsilon_e^{-1}(\underline{k},\omega-n'\omega_0)J_n n_i[\underline{k},\omega+(n-n')\omega_0]$$

so the integral equation (5.99) takes the form

$$n_i(\underline{k},\omega) = -\chi_i(\underline{k},\omega) \sum_{n,n'=-\infty}^{\infty} \frac{J_n(\rho)J_{n'}(\rho)}{\varepsilon_e(\underline{k},\omega-n'\omega_0)} n_i[\underline{k},\omega+(n-n')\omega_0] + N(\underline{k},\omega) \tag{5.107}$$

with

$$N(\underline{k},\omega) = N_i(\underline{k},\omega) + \chi_i \sum \frac{J_n J_{n'} N_e}{\varepsilon_e(\underline{k},\omega-n'\omega_0)} [\underline{k},\omega+(n-n')\omega_0] \tag{5.108}$$

Note that both $N_i(\underline{k},\omega)$ and $\chi_i(\underline{k},\omega)$ will be small unless ω is in the ionic range ($\omega \lesssim \omega_{pi}$) so we can assume N is negligible outside that range.

In general, the density at ω is linked to that at $\omega+n\omega_0$ for all n, so we have an infinite set of coupled equations. However, we know that ions will not respond to high frequency fields, i.e. that $n_i(\underline{k},\omega)$ is negligible unless ω is in the ionic range, say $\omega \lesssim \omega_{pi}$. Henceforth, in analyzing (5.107) we shall assume ω to be small, in this sense. (Formally, this is a consequence of the similar character of the inhomogeneous term, N, noted above.)

We now invoke the assumption that $\rho << 1$. Then the leading terms of the sum in (5.107) are $n=n'=0$, giving

$$\left(1 + \frac{\chi_i}{\varepsilon_e}\right) n_i = N$$

or

$$\varepsilon n_i = \varepsilon_e N$$

where

$$\varepsilon = 1 + \chi_e + \chi_i$$

is the usual dielectric function. The ion response to the initial perturbations,

$$n_i(\underline{k},\omega) = \frac{\varepsilon_e N(\underline{k},\omega)}{\varepsilon(\underline{k},\omega)}$$

will have poles at the zeros of ε, and for stable distribution functions, f_0, these will all lie in the lower half plane. In general, the remaining terms in (5.107), involving $J_1(\rho)$, etc., will give only small corrections to this result. However, noting the ε_e in the denominator of these terms, we see that if

$(-n'\omega_0)$ is near the Bohm-Gross frequency ω_k, then $\varepsilon_e(\omega-n'\omega_0)$ will be small and the corresponding terms in the sum may become comparable to the n=n'=0 term. Of course, with ω_0 in the electronic range, $n_i[\omega+(n-n')\omega_0]$ will, as noted, be negligible unless n=n'. Thus, the double sum in (5.107) reduces to a single sum. This is an enormous simplification, since (5.107) then becomes a single, scalar equation for $n_i(\underline{k},\omega)$, rather than a set of coupled equations, i.e. we have

$$n_i(\underline{k},\omega) = N(\underline{k},\omega)/D(\underline{k},\omega) \tag{5.109}$$

where

$$D(\underline{k},\omega) = 1 + \chi_i \sum \frac{J_n^{\,2}(\rho)}{\varepsilon_e(\omega-n\omega_0)} \tag{5.110}$$

and the usual dispersion relation, $\varepsilon = 0$, is replaced by $D = 0$. As we shall now show, this can have roots in the upper half plane, even for stable distribution functions.

Consider the case where $\omega_0 \gtrsim \omega_p$, so the dominant terms in D are $n = 0$ and $n = 1$, giving the dispersion equation

$$D = 1 + \frac{\chi_i(\omega)}{\varepsilon_e(\omega)} + \chi_i(\omega)\,\frac{\rho^2}{4}\left[\frac{1}{\varepsilon_e(\omega-\omega_0)} + \frac{1}{\varepsilon_e(\omega+\omega_0)}\right] = 0 \tag{5.111}$$

or

$$\varepsilon(\omega)/\chi_i(\omega) = -\varepsilon_e(\omega)(\rho^2/4)[\varepsilon_e^{-1}(\omega-\omega_0) + \varepsilon_e^{-1}(\omega+\omega_0)] \tag{5.112}$$

where we have omitted the ubiquitous $\underline{k}$ dependence for notational simplicity. The simplest form of this dispersion relation results from the use of fluid approximations for χ_i and χ_e. We can view this as the limiting case where

$$|\omega/ka_i| \gg 1 \qquad |\omega/ka_e| \ll 1 \qquad |\omega_0/ka_e| \gg 1 \tag{5.113}$$

which is reasonable for ω in the ionic range, $(\omega \lesssim \omega_{pi})$; $k \lesssim k_e$; and $T_i \ll T_e$. Then

$$\chi_i(\omega) = -(\omega_{pi}^2/k^2a_i^2)Z'(\omega/ka_i) \rightarrow -\omega_{pi}^2/\omega^2$$

$$\chi_e(\omega) = -(\omega_{pe}^2/k^2a_e^2)Z'(\omega/ka_e) \rightarrow k_e^2/k^2 \tag{5.114}$$

$$\chi_e(\omega\pm\omega_0) \rightarrow -\frac{\omega_k^2}{(\omega_0\pm\omega)^2}$$

ω_k being the Bohm-Gross frequency,

$$\omega_k^2 = \omega_p^2(1 + 3k^2/k_e^2) \tag{5.115}$$

and so

$$\varepsilon_e(\omega\pm\omega) = 1 + \chi_e \doteq \frac{\omega_0^2-\omega_k^2\pm 2\omega\omega_0}{\omega_0^2} \doteq \pm\frac{2}{\omega_0}(\omega\pm\delta) \tag{5.116}$$

where

$$\delta = \omega_0-\omega_k \tag{5.117}$$

is the "mismatch" between the pump frequency and ω_k. This gives

$$D = 1 - \Omega_k^2/\omega^2 + (\omega_{pi}^2\omega_0\rho^2/4)\delta\omega^{-2}(\omega^2-\delta^2)^{-1} \tag{5.118}$$

where

$$\Omega_k = kc_s(1 + k^2/k_e^2)^{-1/2} \tag{5.119}$$

is the ion acoustic frequency. The dispersion equation $D = 0$ is then a quadratic in $s = \omega^2$:

$$s^2-s(\delta^2+\Omega_k^2) + \mu\delta + (\Omega_k\delta)^2 = 0 \tag{5.120}$$

where

$$\mu \equiv (\omega_{pi}^2\omega_0\rho^2/4) \tag{5.121}$$

is proportional to the pump energy density, E_0^2. The solutions,

$$s = [\delta^2+\Omega_k^2 \pm \sqrt{(\delta^2+\Omega_k^2)^2 - 4\mu\delta - 4(\delta\Omega_k)^2}]/2 \tag{5.122}$$

$$= [\delta^2+\Omega_k^2 \pm \sqrt{(\delta^2-\Omega_k^2)^2 - 4\mu\delta}]/2 \tag{5.123}$$

can give instabilities ($Im\omega > 0$) if s is either complex, or real and negative. We now examine each of these possibilities.

i) s complex.

From (5.123) we see that this occurs when

$$\mu > (\delta^2 - \Omega_k^{\ 2})^2/4\delta \tag{5.124}$$

Since $\mu > 0$, this can happen only for $\delta > 0$, i.e. for pump frequencies above ω_k. For exact frequency matching,

$$\delta = \omega_0 - \omega_k = \Omega_k \quad , \tag{5.125}$$

there is no threshold for this instability; this is a consequence of our neglect of all dissipation or damping effects, an omission we shall remedy shortly. Later, in our study of nonlinear effects, we shall encounter this same phenomenon under the guise of mode coupling, i.e. the decay of a pump photon (ω_0) into a Langmuir wave or "plasmon" (ω_k) and an ion acoustic wave or "phonon" (ω_k). Conservation of action (corresponding to quantum mechanical conservation of energy, $\hbar\omega$) then makes the decay condition (5.125) intuitively obvious and accounts for the name "decay instability" given to this $\delta > 0$ branch. If (5.125) is not satisfied, then the pump power threshold for the instability is, for fixed k and ω_0, given by (5.124). We note that for given ω_0 and ω_p, the k corresponding to frequency matching is obtained by solving (5.125) for k. When $k << k_e$, so that $\Omega_k \doteq kc_s$, we have

$$\omega_0 - \omega_p(1 + 3k^2/2k_e^{\ 2}) = kc_s \tag{5.126}$$

or

$$(k/k_D) = \frac{(m/M+6\Delta)^{1/2} - (m/M)^{1/2}}{3} \tag{5.127}$$

where

$$\Delta \equiv (\omega_0 - \omega_p)/\omega_p \quad .$$

A graphical construction for finding k is shown in Fig. 9.

From (5.123) it is easy to get explicit expressions for the real and imaginary parts of ω. With $\omega = (\omega_1 + i\omega_2)$ we have

$$\omega^2 = \omega_1^{\ 2}-\omega_2^{\ 2} + 2i\omega_1\omega_2 = s = [\delta^2+\Omega_k^{\ 2} + i\sqrt{4\mu\delta-(\delta^2-\Omega_k^{\ 2})^2}\,]/2$$

or

$$\omega_1^{\ 2} - \omega_2^{\ 2} = (\delta^2 + \Omega_k^{\ 2})/2 \equiv \alpha$$

$$\omega_1\omega_2 = \sqrt{4\mu\delta-(\delta^2-\Omega_k^{\ 2})^2}/4 \equiv \beta$$

whose solution is

$$\omega_1^{\ 2} = [2\sqrt{\delta^2\Omega_k^{\ 2}+\mu\delta} + \delta^2 + \Omega_k^{\ 2}]/4$$

$$\omega_2^{\ 2} = [2\sqrt{\delta^2\Omega_k^{\ 2}+\mu\delta} - \delta^2-\Omega_k^{\ 2}]/4 \tag{5.128}$$

Thus, when (5.125) is satisfied, i.e. for $\delta = \Omega_k$ and $\mu << \Omega_k$ we have

$$\omega_r \approx \Omega_k$$

$$\omega_I \approx (\mu/\Omega_k)^{1/2} \tag{5.129}$$

and we can look at this as an ion acoustic wave driven unstable by the pump.

Actually, there will be an associated instability in n_e at a high frequency, $\omega' \approx \omega_0$, as can be seen directly from (5.94):

$$n_e(\omega') = \Delta^{-1}\tilde{n}_e = \Delta^{-1}\cdot[(\chi_e/\varepsilon_e)\Delta\cdot n_i] \doteq -\Delta^{-1}\left[\frac{1}{\varepsilon_e}\,\Delta\cdot n_i\right] =$$

$$= -\sum \frac{J_{n'}J_n n_i[\omega'+(n-n')\omega_0]}{\varepsilon_e(\omega'-n'\omega_0)} \tag{5.130}$$

if we neglect the inhomogeneous term, $\tilde{N}_{1e}$, as well as $n_i(\omega')$. Using (5.108), we then have

$$n_e(\omega') = -\sum \frac{J_{n'}J_n N[\omega'+(n-n')\omega_0]}{\varepsilon_e(\omega'-n'\omega_0)D[\omega'+(n-n')\omega_0]} \tag{5.131}$$

so to each root, $\omega = \Omega$, of D corresponds a pole of $n_e(\omega')$ at

$$\omega' = \Omega + (n'-n)\omega_0$$

The ε_e factor will be resonant for $n' = 0, \pm 2$ but the lowest order contribution will, of course, be from $n' = 0$ since $J_2 \propto \rho^2$. Since $N(\omega) \neq 0$ only for small ω, n is restricted to ± 1 and we have

$$n_e(\omega') = -\frac{\rho}{2\varepsilon_e(\omega')}\left[\frac{N(\omega'+\omega_0)}{D(\omega'+\omega_0)} - \frac{N(\omega'-\omega_0)}{D(\omega'-\omega_0)}\right] \tag{5.132}$$

Thus, to each zero, $\omega = \Omega$, of $D(\omega)$ in the upper half plane there correspond unstable poles at $n_e(\omega')$ at $\omega' = \Omega \pm \omega_0$.

Consider the situation near threshold, $\mu << \Omega_k$, where one root of D is at $\Omega \doteq \Omega_k$, and we have the frequency matching condition $\omega_0 - \omega_k \doteq \omega_k$. The corresponding poles in $n_e(\omega')$ will be at $\Omega_k + \omega_0 \doteq \omega_k + 2\Omega_k$ and $\Omega_k - \omega_0 \doteq \omega_k$ so the respective weights will be proportional to $\varepsilon_e^{-1}(\omega_k+2\Omega_k)$ and $\varepsilon_e^{-1}(\omega_k)$, respectively. Since $\varepsilon_e(\omega_k) \approx 0$, the largest response will be at $\Omega_k-\omega_0 \doteq -\omega_k$. Similarly, corresponding to the root $\Omega \doteq -\Omega_k$ of D there will be poles in $n_e(\omega')$ at $\omega' = -\Omega_k\pm\omega_0$ and, again, the dominant one will be at $\omega_0-\Omega_k \doteq \omega_k$. Consequently, near threshold we have (considering only positive frequencies) an ion instability near Ω_k and a dominant electron instability near $\omega_0-\Omega_k \doteq \omega_k$, with a weaker instability near $\omega_0+\Omega_k$. While we cannot consider here the nonlinear effects which ultimately stabilize this instability, we would expect a steady state spectrum with peaks near the frequencies corresponding to these instabilities, i.e. an ionic line at Ω_k and an electronic sideband at $\omega_0-\Omega_k$, plus a weaker sideband at $\omega_0+\Omega_k$. For higher powers, the disparity between the upper and lower electronic sidebands should diminish.

We now return to the other possibility for unstable ω in the solution of (5.118), namely

<u>ii) $s < 0$</u> .

From (5.122) we see that this can occur only if $\delta < 0$ and

$$\mu > |\delta|\Omega_k^{\,2} \tag{5.133}$$

Again, there is zero threshold, this time for $\delta = 0$. Since ω^2 is negative, ω is purely imaginary and we have a non-oscillatory, purely growing instability. In this respect, it is similar to the instability associated with two symmetric, counterstreaming electron beams. In fact, averaged over an ion period, the electron distribution function has just such a character, since the electrons stream with velocity $V_0 = eE_0/m\omega_0$ in one direction, say at $t = 0$, and with equal velocity in the opposite direction at $t = \pi/\omega_0$. In view of these considerations, this branch is called the oscillating two stream instability. Since it corresponds to a root of D with zero real part, we see from (5.132) that the associated electron spectral line will be at ω_0.

Since we have so far used the simplest approximations for χ_i, χ_e, with no damping or dissipation effects included, we have found that the decay and the oscillating two stream instabilities both have zero thresholds if the respective frequency matching condition ($\delta = \Omega_k$ or $\delta = 0$) is satisfied. If we use the correct kinetic expressions for χ_i, χ_e, including Landau damping, the minimum thresholds will be non-zero, but the analysis is onerous and necessarily numerical. A first approximation to the effects of damping can be obtained if we simply modify χ_i and χ_e in the fashion which corresponds to the inclusion of dissipation in a fluid dynamic derivation of the χ's. If we include a phenomenological momentum transfer term

$$P = -nm\nu\underline{u}$$

in the momentum equation (cf. the discussion following equation (3.8),) the linearized fluid equations (3.21) are modified only by having a term $-\nu\underline{u}$ on the right side of the second equation of the set. If we then compute χ as in the derivation of (3.20), we obtain in place of (3.20) the result

$$\chi(\underline{k},\omega) = -\frac{\omega_p^2}{\omega(\omega+i\nu)-k^2c_\alpha^2} \doteq -\frac{\omega_p^2}{(\omega+i\gamma)^2-k^2c_\alpha^2} \tag{5.134}$$

where we have assumed $\nu \ll \omega$ and defined $\gamma = \nu/2$. This resonant expression for χ is a convenient approximation, and its consequences for the parametric instability problem were explored by Nishikawa (J. Phys. Soc. Jap. 24, 916, 1152 (1968)). We shall simply summarize the results here.

As before, we assume $kc_i/|\omega+i\gamma| \ll 1$, $|\omega+i\gamma|/kc_e \ll 1$ and $\omega_0/kc_e \gg 1$, so we can write

$$\left.\begin{aligned}
\chi_i(\omega) &= -\frac{\omega_{pi}^{\ 2}}{(\omega+i\Gamma)^2} \\
\chi_e(\omega) &= k_D^{\ 2}/k^2 \\
\chi_e(\omega\pm\omega_0) &= -\frac{\omega_p^{\ 2}}{(\omega+i\gamma\pm\omega_0)^2-3k^2/2a_e^{\ 2}}
\end{aligned}\right\} \tag{5.135}$$

$$\varepsilon_e(\omega\pm\omega_0) \doteq \pm\frac{2}{\omega_0}(\omega\pm\delta_s+i\gamma)$$

where γ and Γ denote the electron and ion damping terms. (To be precise, (5.135) follows from (5.134) if we choose $c_e^{\ 2} = a_e^{\ 2}/2$, corresponding to $\gamma = 1$, in evaluating $\chi_i(\omega)$, but choose $c_e^{\ 2} = 3a_e^{\ 2}/2$, corresponding to $\gamma = 3$, in computing $\chi_i(\omega\pm\omega_0)$. This is physically reasonable inasmuch as electronic processes can be expected to be isothermal for low frequencies, $\omega << \omega_p$, but adiabatic for $\omega \gtrsim \omega_p$. However, these numerical factors should not be taken too seriously; (5.135) is best regarded as a simple ad hoc method of approximately including small dissipative effects.)

Substituting (5.135) into (5.112), we obtain

$$\boxed{\omega^2+2i\Gamma - \Omega_k^{\ 2} + \frac{(K\delta/\omega_p)}{(\omega+i\gamma)^2-\delta^2} = 0} \tag{5.136}$$

where

$$\Lambda^2 = \varepsilon_e(\omega)\rho^2/4 = (E_0^{\ 2}\cos^2\theta/4\pi nT)(1 + k^2/k_e^{\ 2}) \tag{5.137}$$

is a measure of the pump power, θ being the angle between $\underline{k}$ and $\underline{E}_0$, and

$$K = \Lambda^2\omega_p^{\ 2}\Omega_k^{\ 2} = \rho^2\omega_{pi}^{\ 2}\omega_{pe}^{\ 2}/4$$

Of course, for $\gamma,\Gamma \to 0$, (5.136) reduces to (5.118).

Guided by our experience with the dissipationless case, we first look for an instability with imaginary ω, $\omega = iy$, corresponding to the O.T.S. case. Then (5.136) gives

$$2\Gamma y = -y^2 - \omega_1{}^2 - \frac{(K\delta/\omega_p)}{(y+\gamma)^2+\delta^2} \tag{5.138}$$

and $y > 0$ is possible only if $\delta < 0$, just as in the undamped case. If we consider marginal stability, $y = 0$, we have

$$K = -\omega_p \Omega_k{}^2 (\delta^2+\gamma^2)/\delta \tag{5.139}$$

which, for fixed k, takes its minimum value,

$$K_{min} = 2\gamma\omega_p\Omega_k{}^2 \tag{5.140}$$

at $\delta = -\gamma$. Thus, there is now a non-zero minimum threshold for given k, which we can write as

$$\boxed{\Lambda_{th}^2 = 2\gamma/\omega_p} \quad , \tag{5.141}$$

a form which is nearly independent of k so long as $k/k_D << 1$. Just at threshold, we have instability only at $\delta = -\gamma$ (not $\delta = 0$, as in the undamped case). Above threshold, δ need only lie on the range

$$-\gamma[p + \sqrt{p^2-1}] < \delta < -\gamma[p - \sqrt{p^2-1}]$$

where

$$p = K/K_{min} = \Lambda^2\omega_p/2\gamma \quad .$$

For powers far above threshold, the optimum δ and assoicated growth rate are independent of γ:

$$-\delta_{opt} = (\Lambda^2\omega_p\Omega_k{}^2/2)^{1/3} = y_{max} \tag{5.142}$$

a result which is valid provided $y > \Gamma, \gamma, \Omega_k$, i.e. provided

$$\Lambda^2 > \Gamma^3/\omega_p\Omega_k{}^2 \quad , \gamma^3/\omega_p\Omega_k{}^2 \quad , \Omega_k/\omega_p \quad .$$

To analyze the decay instability, we must in general allow ω to be complex. For the marginal stability case, however, i.e., just at the threshold, ω is real and equating the real and imaginary parts of (5.136) to zero gives, after some elementary algebra,

$$\omega^2 = \frac{\gamma\Omega_k^{\ 2} + \Gamma(\delta^2+\gamma^2)}{\Gamma + \gamma} \tag{5.143}$$

and

$$K = \frac{\omega_p\gamma\Gamma}{\delta}\left\{4\delta^2 + \frac{(\Omega_k^{\ 2}+\gamma^2+2\gamma\Gamma-\delta^2)^2}{(\gamma+\Gamma)^2}\right\} \tag{5.144}$$

We now vary δ to minimize K, considering two regimes for computational convenience:

a) $\Omega_k^{\ 2} >> \gamma^2,\ \gamma\Gamma$.

Then (5.144) can be written

$$K = (\omega_p\gamma\Gamma/\delta)\left[4\delta^2 + \frac{(\Omega_k^{\ 2}-\delta^2)^2}{(\gamma+\Gamma)^2}\right]$$

whose minimum value,

$$K_{min} = 4\omega_p\gamma\Gamma\Omega_k(1 - \Gamma^2/4\Omega_k^{\ 2}) \quad , \tag{5.145}$$

occurs at

$$\delta^2 = \Omega_k^{\ 2} - \Gamma^2 \tag{5.146}$$

In this case, the threshold for the decay instability is

$$\boxed{\Lambda_D^{\ 2} = \frac{4\gamma\Gamma}{\omega_p\Omega_k}(1 - \Gamma^2/4\Omega_k^{\ 2})} \tag{5.147}$$

b) $\gamma >> \Omega_k >> \Gamma$.

Then (5.144) can be written as

$$K = \frac{\omega_p\Gamma}{\delta\gamma}(\gamma^2 + \delta^2) \tag{5.148}$$

whose minimum value

$$K_{min} = (16\sqrt{3}/9)\omega_p\gamma^2\Gamma$$

occurs at $\delta = \gamma/\sqrt{3}$. The threshold is then

$$\boxed{\Lambda_D{}^2 = (16\sqrt{3}/9)\,\frac{\gamma^2\Gamma}{\omega_p\Omega_k{}^2}} \tag{5.149}$$

which is larger by $(4\sqrt{3}/9)(\gamma/\Omega_k)$ than $\Lambda_D{}^2$. There is now considerable experimental work showing the existence of these parametric instabilities, and, some, albeit much less certain, indicating their occurance in high power r.f. irradiation of the ionosphere and high energy laser interactions with solid matter.

G. Steady State ("Boundary Value") Problems; Real ω, Complex k.

So far, we have discussed initial value problems, in which the dynamical variables (n, $\underline{u}$, $\underset{\sim}{p}$, etc. in fluid theory; $f(\underline{x},\underline{v})$ for the Vlasov equation) are specified at $t = 0$ and we study their time evolution. At early times there will be "transients," depending in an essential way on the initial conditions, but at long times the behavior is dominated by the least damped modes, for a stable plasma, or the fastest growing modes, for an unstable plasma. While the amplitudes of these depend upon the initial conditions, the frequency and decay (or growth) rate do not.

Although this represents the simplest kind of physically well-posed problem within the domain of linear theory, it does not, unfortunately, correspond to most laboratory experiments with coherent, small amplitude waves. These are rarely carried out in the spirit of an initial value problem, i.e. by exciting initially, say, a wave of definite $\underline{k}$ and observing its time development. It is far more convenient, experimentally, to impose a steady-state excitation, with constant frequency, ω_0, at some point in the plasma and observe the spatial development of the wave. In simple media, there is a close connection between the two kinds of experiment: if a wave of wavenumber k_0 is found to have a real frequency ω_0 and a weak growth rate γ (positive or negative) then we expect that a steady-state wave, with frequency ω_0, will have wavenumber $k_0 + i\gamma/v_g$, i.e. a wave length $2\pi/k_0$ and a spatial growth rate γ/v_g, where $v_g = \partial\omega/\partial k$ is the group velocity. This expectation arises from the fact that if $v_g \neq 0$, then

$$\varepsilon(k_0,\omega_0 + i\gamma) = 0$$

implies

$$\varepsilon(k_0 + i\kappa,\omega_0) = 0$$

with

$$\kappa \doteq \gamma/v_g$$

if $|\gamma/\omega_0|$ and $|\kappa/k_0|$ are small. However, whereas the long time behavior of the waves in an initial value problem is determined simply by the roots of the dispersion equation, $\varepsilon = 0$, a similar statement does not apply to the spatially asymptotic (large x) behavior for steady-state excitation. Since a detailed analysis of the problem can involve considerable mathematical complexity, we shall only consider here the essential aspects of steady-state excitation.

Although these problems are sometimes designated as "boundary value" problems, the experimental data rarely, if ever, include sufficient information to actually determine the boundary conditions for the particles, as is necessary for the Vlasov equation. Idealized boundary conditions, such as specular reflection -- $f(v_x,v_y,v_z) = f(-v_x,v_y,v_z)$ at a boundary plane normal to the x axis, for instance -- are sometimes assumed in theoretical analyses, but the resulting calculations are still not very simple. An easier approach is to retain the idealization of a homogeneous, unbounded plasma used in the initial value problem and to assume that a grid or probe, idealized in the sense of being transparent to particles, is inserted into the plasma. Starting at $t = 0$, currents or voltages applied to this probe excite external fields of frequency ω_0. If the sheath effects associated with the plasma perturbations produced by the grid or probe can be made negligible (by suitable biasing and/or by using small probes or very fine grid wires), then this model gives a reasonably good description of typical experimental situations.

(In a sense, we have not left the realm of initial value problems, since we assume the external signals begin at a definite time, $t = 0$. This corresponds to physical reality: all experiments are actually initial value problems and what is phenomenologically considered "steady-state excitation at frequency ω_0" is really just the time asymptotic behavior of a system excited in the manner described in the preceeding paragraph.)

While any of the linearized modes can be excited by a suitable probe configuration, we shall discuss here only the case of one-dimensional longitudinal waves; the basic techniques and ideas can most easily be understood for this case and can be readily generalized to more complicated modes.

1) Grid Excitation of Longitudinal Waves

From a theoretical viewpoint, the simplest excitation structure is a plane dipole, consisting of a pair of plane, parallel grids, at $x = \pm d$, inserted into the plasma and driven push-pull, i.e. with a potential $\pm\phi_0 e^{-i\omega_0 t}$ applied to $x = \pm d$. The external electric field is then

$$E_0(x,t) = (\phi_0/2d)e^{-i\omega_0 t}\,\theta(d^2-x^2) \qquad (5.150)$$

where θ is the unit step function,

$$\theta(u) = \begin{cases} u & u > 0 \\ 0 & u < 0 \end{cases} \qquad (5.151)$$

The Fourier-Laplace transform of $E_0(x,t)$ is

$$E_0(k,\omega) = \int_0^\infty dt \int_{-\infty}^\infty dx\ e^{-i(kx-\omega t)} E_0(x,t) =$$

$$= \left(\frac{i\phi_0}{\omega-\omega_0} \right) \left(\frac{\sin kd}{kd} \right) = \frac{i\phi_0}{\omega-\omega_0} Q(k) \tag{5.152}$$

where

$$Q(k) = [\sin(kd)]/kd$$

Of course, the pair of grids correspond to a dipole only in the limit $d \to 0$, in which case the form factor Q approaches 1, independent of k. This limiting case is convenient for many calculations but whenever we encounter nonphysical results or convergence problems associated with large k we must recall that Q falls off, as k^{-1}, for $k >> d^{-1}$. We also note that the more complex grid structures sometimes used in experiments simply give rise to a different form factor, Q(k).

Given $E_0(k,\omega)$, we know that the total field is simply

$$E(k,\omega) = E_0(k,\omega)/\varepsilon(k,\omega)$$

and the computation of E(x,t) requires only the inversion of the Fourier-Laplace transform,

$$E(x,t) = \int_C \frac{d\omega}{2\pi} \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{E_0(k,\omega)}{\varepsilon(k,\omega)} e^{i(kx-\omega t)} \tag{5.153}$$

where, as usual C is a horizontal contour in the ω plane lying above any singularities of $E(k,\omega)$. In discussing the properties of this inversion, we consider separately the stable and unstable cases.

2) <u>Propagation in a Stable Plasma</u>

Here "stable" means that $\varepsilon(k,\omega)$ has no roots in the upper half ω plane for real k. Of course, for given ω there will, in general, be one or more roots of $\varepsilon(k,\omega)$ in the complex k plane, and these may lie above or below the real k axis. Suppose that we now deform the contour C by bringing it down to the real ω axis, leaving the k integration along the real k axis, as originally specified. The assumption of stability guarantees that we will encounter, thereby, no singuilarity of the integrand, $E(k,\omega)$, until we reach the pole of $E_0(k,\omega)$ at $\omega = \omega_0$ on the real axis. All other singularities lie in the lower half ω plane (for real k) so the behavior at large times is dominated by the pole at $\omega = \omega_0$:

$$E(x,t) \underset{t\to\infty}{\longrightarrow} \phi_0 e^{-i\omega_0 t} \int_{-\infty}^{\infty} \frac{dkQ(k)e^{ikx}}{2\pi\varepsilon(k,\omega_0)} = \hat{E}(x)\phi_0 e^{-i\omega_0 t} \tag{5.154}$$

Thus, we obtain the expected steady-state time dependence, with frequency ω_0, the x dependence being determined by the k integration,

$$\hat{E}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{Q(k)e^{ikx}}{\varepsilon(k,\omega_0)} \tag{5.155}$$

If $x > d$, then for reasonable ε we can close the k contour in the upper half plane [$Q(k)e^{ikx} \to 0$ on the semi-circle at ∞]. Any roots of $\varepsilon(k,\omega_0)$ in the upper half k plane, say $k = k_R + ik_I$, $k_I > 0$, will give decaying contributions, proportional to $e^{-k_I x}$. Similarly, for $x < 0$ we can close the k contour below, again obtaining terms which decrease with increasing $|x|$. <u>Thus, even though $\varepsilon(k,\omega_0) = 0$ has roots with Imk < 0, we will not obtain spatially growing waves.</u> A term proportional to $e^{-k_I x}$ with $k_I < 0$ arises only for $x < 0$, so it corresponds <u>not</u> to a growing wave, as it would for $x > 0$, but to a <u>decaying</u> or <u>evanescent</u> wave. [This is why we noted, in the discussion of the Bohm-Gross dispersion relation (3.28), that even though solving for k with $\omega = \omega_0 < \omega_p$ gives

$$kc_e = \pm i(\omega_p^{\,2} - \omega_0^{\,2})^{1/2}$$

only the sign corresponding to spatial damping is to be used.]

For algebraic ε, of the sort obtained in fluid theory [cf (3.44)], the k integration (5.155), can often be carried through explicitly, in closed form, yielding simple exponential spatial damping. For the transcendental ε encountered in the Vlasov analysis, however, some further consideration is required. When $k > 0$, we have defined the susceptibility $\chi(k,\omega)$, for each species as

$$\chi(k,\omega) = -\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} dv\,\frac{f_\ell'(v)}{v-\omega/k} \tag{5.156}$$

for $\mathrm{Im}\omega > 0$ and as the analytic continuation of this for $\mathrm{Im}\omega \le 0$. It is sometimes convenient to introduce the function which is given by (5.156) for $\mathrm{Im}\omega < 0$ and by the analytic continuation thereof for $\mathrm{Im}\omega \ge 0$. We introduce the notation

$$-(k^2/\omega_p^2)\chi_+(k,\omega) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} dv\,\frac{f_\ell'(v)}{v-\omega/k} & \mathrm{Im}\omega > 0 \\ \text{analytic continuation of this} & \mathrm{Im}\omega \le 0 \end{cases} \tag{5.157}$$

and

$$-(k^2/\omega_p^2)\chi_-(k,\omega) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{dv f_\ell'(v)}{v-\omega/k} & \mathrm{Im}\omega < 0 \\ \text{analytic continuation of this} & \mathrm{Im}\omega \ge 0 \end{cases} \tag{5.158}$$

A similar notation is used for $\varepsilon_\pm = 1 + \sum_{\text{species}} (\chi_\alpha)_\pm$. From these definitions we have, for all ω,

$$\chi_+(k,\omega) - \chi_-(k,\omega) = -2\pi i(\omega_p^2/k^2)f_\ell'(\omega/k) \tag{5.159}$$

In our earlier discussions of ε in this chapter, we have taken k to be inherently positive. Since the Laplace transform theory dictates that we analytically continue functions of ω from the upper half plane downward, the appropriate χ function for $k > 0$ is always χ_+. However, for $k < 0$ we must use χ_-! Thus, in the integration (5.155), the $\varepsilon(k,\omega_0)$ in the denominator must be taken as ε_+ for $k > 0$ and as ε_- for $k < 0$, that is, we should write (5.155) as

$$\hat{E}(x) = \int_{-\infty}^{0} dk\; Q(k) e^{ikx}/\varepsilon_{-}(k,\omega_0) + \int_{0}^{\infty} dk\; Q(k) e^{ikx}/\varepsilon_{+}(k,\omega_0) \quad . \tag{5.160}$$

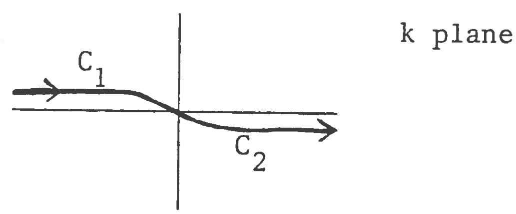
Since k and ω_0 are <u>both real</u>, we can write the $\varepsilon_\pm$ in (5.160) in terms of

$$\chi_\pm(k,\omega_0) = -\frac{\omega_p^{\;2}}{k^2} \int_{-\infty}^{\infty} dv \frac{f_\ell'(v)}{v-\omega_0/(k\mp i\eta)} \tag{5.161}$$

[from which we see, incidentally, that <u>for real k,ω</u> $\chi_+ = (\chi_-)^*$.] Equivalently, we can revert to the original form, (5.155); define

$$\varepsilon(k,\omega) = 1 - (\omega_p^{\;2}/k^2) \int dv \frac{f_\ell'(v)}{v-\omega/k} \tag{5.162}$$

provided ω/k is not real; and use for the k integration in (5.155) a contour like that shown in Fig. 5.10.



<u>Fig. 5.10</u>: Contour of integration in the k plane for Eq. (5.155) using (5.162) for ε.

To carry out the integral (5.155), it is convenient to deform the left half of the contour, C_1, in Fig. 5.10 so that it lies on (more accurately, just above) the positive k axis, as shown in Fig. 5.11.

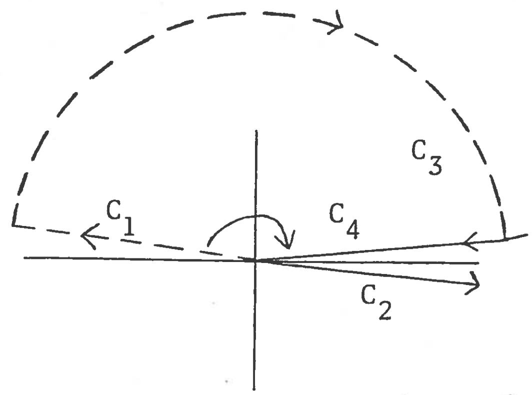


Fig. 5.11: Deformation of the contour C_1 in Fig. 5.10 to the new contour C_3+C_4 or, effectively C_4, since $\int_{C_3} \to 0$.

That is, we write (5.160) as

$$E = \left(\int_{C_1} + \int_{C_2} \right) \frac{dk}{2\pi} \frac{Q(k)e^{ikx}}{\varepsilon(k,\omega_0)} \tag{5.163}$$

and set

$$\int_{C_1} = \int_{C_3} + \int_{C_4} + S$$

where S is a sum of residues at the poles, if any, of $[Q(k)e^{ikx}/\varepsilon_-(k,\omega_0)]$ in the upper half k plane. We need not worry about singularities other than those associated with zeros of ε_-: the function $\varepsilon_+(k,\omega)$ is generally well-behaved in the upper half of the ω plane for real k, and correspondingly, in the lower half of the k plane for real ω; the same is true of ε_- with the substitutions "lower" $\leftrightarrow$ "upper". Thus the contour $(C_1 + C_3 - C_4)$ encloses no singularities of ε_- or of Qe^{ikx} (provided $x > d$).

[Whether or not $\varepsilon_-(k,\omega_0)$ has roots in the upper half plane depends on ω_0 and, of course, our choice of $f_\ell(v)$. For example, with the Lorentzian, (5.18), we have in the limit $m/M \to 0$

$$\varepsilon_-(k,\omega_0) = 1 - \frac{\omega_p^2}{(\omega_0 - ika)^2} \tag{5.164}$$

The roots of $\varepsilon_- = 0$ are then

$$k = -i(\omega_0 \pm \omega_p)/a$$

so there are no roots in the upper half plane for $\omega_0 > \omega_p$. There is one for $\omega_0 < \omega_p$, corresponding to Debye shielding of the excitation probe; for $\omega_0 << \omega_p$ it approaches $i\omega_p/a \approx ik_D$. In the case where ω_0 is near ω_p the Lorentzian can be misleading (since the phase velocity of the waves is large and we know the Lorentzian falls off too slowly at large v). For Maxwellian f_ℓ we have

$$\varepsilon_-(k,\omega_0) = 1 - (\omega_p^2/\omega^2)s^2 Z_-'(s) \qquad s \equiv \omega/ka \tag{5.165}$$

(where $Z_-(s)$ is defined by (5.42) for Ims < 0 and by its analytic continuation for Ims $\geq$ 0)

If we look for roots of ε_- with s large and nearly real we can use the asymptotic expansion of Z_-:

$$\tilde{\omega}^2 \equiv \omega^2/\omega_p^2 Z_-'(s) \doteq s^2\left[\frac{1}{s^2}\left(1+\frac{3}{2s^2}\right) + 2i\pi^{1/2} s e^{-s^2}\right] \tag{5.166}$$

from which follows

$$s^2 = \frac{3/2}{\tilde{\omega}^2 - 1 - 2i\pi^{1/2}s^3 e^{-s^2}} \doteq s_0^2\left[1 + \frac{4i\pi^{1/2}}{3} s^5 e^{-s^2}\right] \tag{5.167}$$

provided

$$s_0^2 \equiv \frac{3}{2(\tilde{\omega}^2-1)} >> 1 \quad . \tag{5.168}$$

In the s plane, the two roots given by (5.167) both lie in the upper half s plane

$$s = \pm s_0\left[1 \pm \frac{2i\pi^{1/2}}{3} s_0^5 e^{-s_0^2}\right] \tag{5.169}$$

and hence in the lower half k plane.]

Returning to the calculation of E(x), we see that the contribution from the contour C_3 vanishes [$e^{ikx} \to 0$ there more strongly than $\varepsilon(k,\omega_0)$] leaving

$$\hat{E} = S + \int_{C_4} + \int_{C_1} = S + \int_0^\infty \frac{dk}{2\pi} Q(k)e^{ikx}[1/\varepsilon_+(k,\omega_0) - 1/\varepsilon_-(k,\omega_0)] =$$

$$= S + \int_0^\infty \frac{dk}{2\pi} \frac{Q(k)e^{ikx} i(\omega_p^2/k^2) f_\ell'(\omega_0/k)}{|\varepsilon(k,\omega_0)|^2} \tag{5.170}$$

where we have used (5.159) and the fact that $\varepsilon_+ = (\varepsilon_-)^*$ for real arguments. If the upper half k plane roots of $\varepsilon_-(k,\omega_0)$ are located at $k = k_*$, then we can write

$$S = i\sum_\alpha \frac{Q(k_\alpha)e^{ik_\alpha x}}{(\partial\varepsilon/\partial k)_\alpha} \tag{5.171}$$

i.e. a sum of damped terms. There remains only the evaluation of the integral in (5.170), which we can write as

$$I(x) = i\int_0^\infty \frac{dk}{2\pi} \frac{Q(k)e^{ikx}\omega_p{}^2 f_\ell{}'(\omega_0/k)}{k^2|\varepsilon(k,\omega_0)|^2} = \frac{\omega_p{}^2}{\omega_0}\int_0^\infty \frac{dv}{2\pi} \frac{Q(\omega_0/v)e^{i\omega_0 x/v} f_\ell{}'(v)}{|\varepsilon(\omega_0/v,\omega_0)|^2} \qquad (5.172)$$

where we have, for convenience, introduced $v \equiv \omega/k$. If x_0 is large, the integral can conveniently be evaluated by the saddle-point method.

For example, with

$$f_\ell(v) = e^{-v^2/a^2}/a\pi^{1/2}$$

the integrand contains the rapidly varying factor $e^{\psi(v)}$, where

$$\psi(v) = i\omega_0 x/v - v^2/a^2 = iz/u - u^2 \qquad (5.173)$$

and we have introduced dimensionless velocity and spatial variables,

$$u = v/a = \omega_0/ka \qquad\qquad z = \omega_0 x/a$$

We deform the contour of integration from the positive real v axis to a new contour, $\tilde{C}$, called the path of steepest descent (p.s.d.). This is a contour which a) passes through the saddle point $v = v_s$, defined by

$$\psi'(v_s) = 0 \ , \qquad (5.174)$$

and b) satisfies

$$\mathrm{Im}[\psi(v)/\psi(v_s)] = 0$$

The p.s.d. has the virtue that along it $e^{\psi(v)}$ is a non-oscillating function, whereas on the real v axis e^{ψ} is a rapidly oscillating function for large z. From (5.173) we have

$$d\psi/du = -\frac{iz}{u^2} - 2u = 0$$

or

$$u_s = (-iz/2)^{1/3} = (z/2)^{1/3}e^{-\pi i/6} \qquad (5.175)$$

and

$$\left.\frac{d^2\psi}{du^2} = -2 + \frac{2iz}{u^3}\right|_{u=u_s} = -6$$

[Clearly, (5.174) has three solutions, the other two differing from (5.175) by $e^{\pm 2\pi i/3}$. We have selected the one appropriate to positive z and consider henceforth only the case $z > 0$. Of course, a similar analysis can be carried through for $z < 0$.] The p.s.d. contour $\tilde{C}$ determined from (5.173) is shown in Fig. 5.12

(see page 53a)

Since the higher order derivatives approach zero for large z, the approximation

$$\psi(u) = \psi(u_s) \doteq -3(u-u_s)^2 \tag{5.176}$$

is adequate for the asymptotic (large x) region. Indeed, for $|z| >> 1$, $e^{\psi(u)-\psi(u_s)}$ will be sharply peaked about the point $u = u_s$ $(\delta u/u_s \sim z^{-1/3})$, so we can evaluate all other factors in the integrand of $u = u_s$, leaving only the integration of e^{ψ} which we approximate as

$$\int_{\tilde{C}} e^{\psi(v)} dv = ae^{\psi_s} \int_{\tilde{C}} du\, e^{\psi-\psi_s} \doteq$$

$$\doteq ae^{\psi_s} \int_{\tilde{C}} du\, e^{-3(u-u_s)^2} =$$

$$= ae^{\psi_s} (\pi/e)^{1/2}$$

where

$$\varphi_s \equiv \varphi(v_s) = 3(z/2)^{2/3} e^{2\pi i/3} \tag{5.177}$$

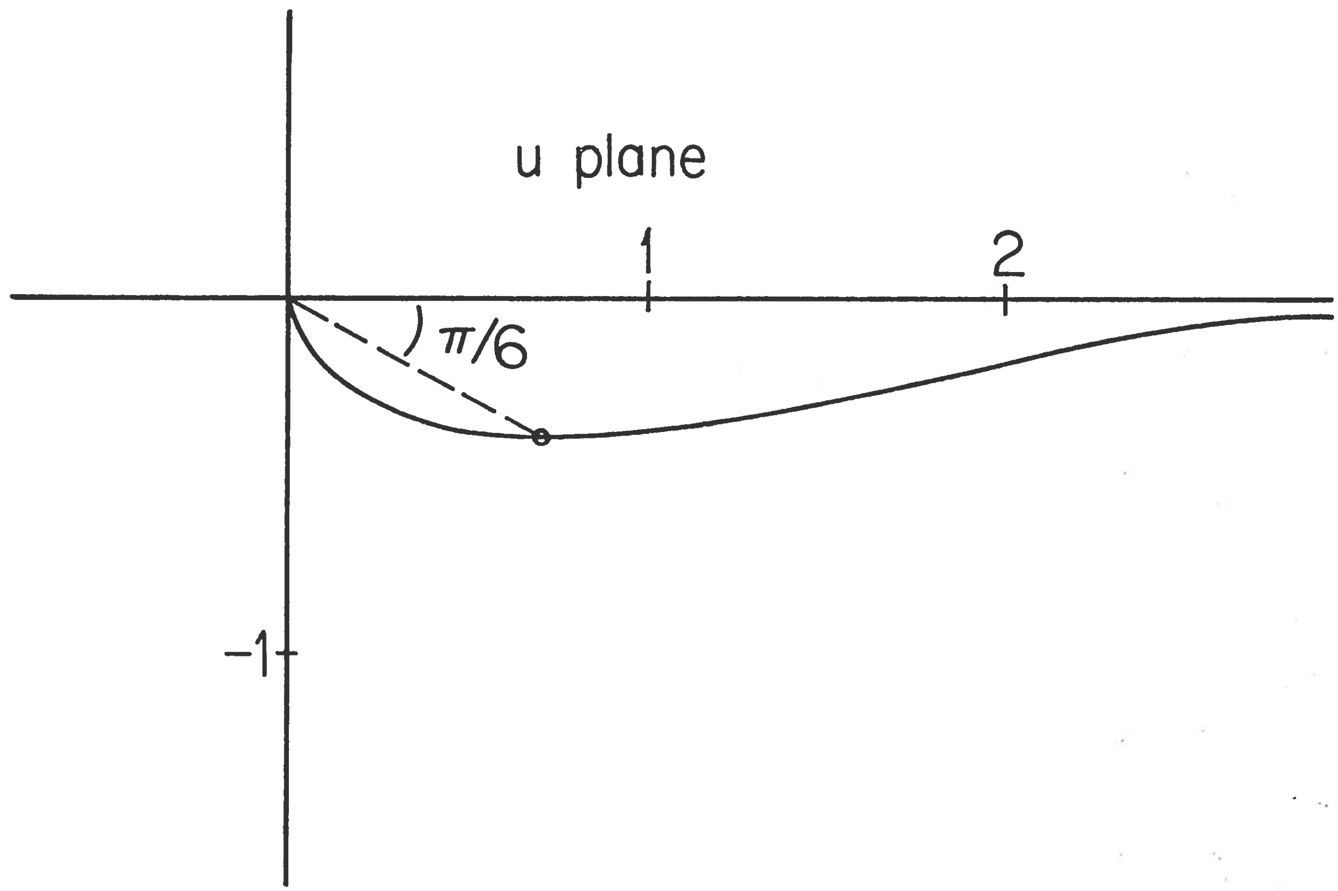


Fig. 5.12: Path of steepest descent for a Maxwellian f_ℓ. The saddle point is shown for $z = 1$.

Finally, then

$$I = - \left[\frac{(\omega_p{}^2/\omega_0 a)}{\pi\sqrt{3}} \frac{u_s Q_s}{|\varepsilon_s|^2} \right] \exp\left[\frac{3}{2} (z/2)^{2/3}(-1 + i\sqrt{3}) \right] \tag{5.178}$$

where the subscript s denotes evaluation of Q and ε at

$$k_s = (\omega_0/au_s) = (\omega_0/a)(2/z)^{1/3}e^{\pi i/6} \tag{5.179}$$

While the factor in square brackets has, through k_s, a weak dependence on z, the principal variation is through the exponential term. As expected, there is damping for large z, but instead of the usual exponential damping, $\log E \propto -x$ we have the unexpected result $\log E \propto -x^{2/3}$. Moreover, the oscillations arising from Imlog(I) have, because of the $z^{2/3}$, a wavelength which decreases as z increases.

In deforming the contour of integration in (5.172) from the real axis to $\tilde{C}$ we must, of course, account for any roots of $|\varepsilon|^2$ which may be encountered. If there are roots of ε_- ,they would already have been included in S, and hence need only be subtracted from the sum (5.171). If there are roots, k_β, of ε_+, they will contribute a term

$$T = -i\sum_i \frac{Q(k_\beta)e^{ik_\beta x}}{(\partial\varepsilon_+/\partial k)_\beta} \tag{5.180}$$

Thus, we can finally express (5.170) as

$$E = S + I + T \tag{5.181}$$

with S, T, I given by (5.171), (5.180) and (5.178). Depending upon the value of ω_0 and the size of z, we may find that one of the exponential terms in S or T dominates (5.181). In such a region, we expect to see waves of constant wavelength with simple exponential damping. In other regions a combination of two or more such terms may dominate, giving interference effects. (NB. Any attempt to observe this experimentally requires careful