$$\vec{n} = \frac{\vec{r}-\vec{r}(t')}{|\vec{r}-\vec{r}(t')|} \quad . \tag{38.31}$$

If we let ϕ be the angle between $\vec{n}$ and $\vec{v}$, the factor

$$\frac{1}{1 - \frac{\vec{n}\cdot\vec{v}}{c}} = \frac{1}{1 - \beta \cos\phi} \quad ,$$

for $\beta \to 1$, is dominated by the small angle region, so we may approximate it by

$$\frac{1}{1 - \beta\left(1 - \frac{\phi^2}{2} + \ldots\right)} \simeq \frac{1}{\frac{1}{2}\,(1-\beta^2+\phi^2)} \quad .$$

As we saw in greater detail for impulsive scattering [see (35.10)] the radiation is therefore concentrated in a narrow angular range near the forward direction

$$\phi \sim \sqrt{1-\beta^2} = \frac{mc^2}{E} \quad , \tag{38.32}$$

which is just the behavior found in the preceding subsection.

Next, we seek a qualitative understanding of the characteristic frequancies radiated by a high energy charged particle moving in a circle with frequency ω_o. Because of the strong directionality of the emitted radiation as expressed by (38.32), the radiation detected at a particular point only arises from a small portion of the orbit, or equivalently, is only emitted during a time interval small compared with the period of revolution, $2\pi/\omega_o$.

This effective emission time interval is of order $\frac{2\pi}{\omega_o}\phi$, so the important frequencies radiated are $\sim \frac{1}{\text{emission time}} \sim \frac{\omega_o}{2\pi}\frac{1}{\phi}$. That is, the smaller the time interval involved in the emission, the higher the frequency emitted. Therefore a typical frequency emitted is,

$$\omega_e \sim \frac{\omega_o}{\sqrt{1-\beta^2}} = \frac{E}{mc^2}\,\omega_o \quad . \tag{38.33}$$

However, this is not what the observer sees since detection time intervals are not equal to emission time intervals. To see the connection between these intervals, the Doppler effect, we recall that these two times are related by (38.30), which implies for the respective time intervals

$$dt' = dt - \frac{1}{c}\,\vec{n}\cdot(-\vec{v}(t')dt')$$

or

$$dt = \left[1 - \frac{1}{c}\,\vec{n}\cdot\vec{v}\right] dt' \sim (1-\beta^2)\; dt', \; \beta \simeq 1 \;, \; \phi \sim 0 \;. \tag{38.34}$$

This is the origin of the denominator factor appearing in (38.29). From (38.34), we see that the time interval of detection is shorter than that of emission by a factor $1-\beta^2$, implying that the detection frequency, ω_d, is highe[r] than the emission frequency by a factor $(1-\beta^2)^{-1}$:

$$\omega_d \sim \frac{1}{(1-\beta^2)}\,\omega_e \sim \frac{1}{(1-\beta^2)^{3/2}}\,\omega_o = \omega_o\left(\frac{E}{mc^2}\right)^3 , \tag{38.35}$$

which is the characteristic frequency found in (38.22).

Lecture 10

XXXIX. PROPAGATION OF RADIATION IN A DIELECTRIC MEDIUM

39-1. Equations for the Normal Modes

We now turn from the mechanisms by which electromagnetic radiation (or light) is produced to how it propagates in a material medium, described at a macroscopic level. Quite quickly, we shall specialize to the case of a dielectric, with inhomogeneity in a single direction only. We begin by restating Maxwell's equations for a macroscopic medium,

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\dot{\vec{D}} + \frac{4\pi}{c}\vec{J} ,$$

$$-\vec{\nabla} \times \vec{E} = \frac{1}{c}\dot{\vec{B}} , \tag{39.1}$$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho ,$$

$$\vec{\nabla}\cdot\vec{B} = 0 . \tag{39.2}$$

Before proceeding, we note that here the scalar equations, (39.2), are not independent of the first set, (39.1), since taking the divergence of the latter, and making use of the current conservation condition, (28.2), we find

$$0 = \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D} - 4\pi\rho) ,$$

$$0 = \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) .$$

Thus, <u>excluding</u> statics, the divergence equations, (39.2), are subsumed within the curl equations, (39.1).

To concentrate our attention on light of a definite frequency ω, we introduce the Fourier transform

$$\int_{-\infty}^{\infty} dt\ e^{i\omega t}\ F(\vec{r},t) = F(\vec{r},\omega) \quad ,$$

which amounts to the replacement in (39.1) of

$$\frac{\partial}{\partial t} \to -i\omega \ ,$$

ϵ were

$D = \epsilon E$

yielding for the vector equations

$$\vec{\nabla} \times \vec{H} = -\frac{i\omega}{c}\vec{D} + \frac{4\pi}{c}\vec{J} \quad ,$$

$+ i k \times H = -\frac{i\omega}{c} D + \frac{4\pi}{c} J$

$- \partial i k \times E = -\frac{i\omega}{c} B$

$$-\vec{\nabla} \times \vec{E} = -\frac{i\omega}{c}\vec{B} \quad . \qquad (39.3)$$

Specifically, we assume a linear, dispersive medium, for which

$$\vec{B} = \mu\vec{H} \ , \qquad \vec{D} = \epsilon\vec{E} \quad , \qquad (39.4)$$

where $\mu(\vec{r},\omega)$ and $\epsilon(\vec{r},\omega)$ are complex functions. We will further assume

$$\mu(\vec{r},\omega) \simeq 1 \quad , \qquad (39.5)$$

since a ferromagnet cannot follow the rapid oscillations of the electromagnetic field in a light wave. It will be sufficient for our purposes to suppose that spatial variation of ϵ occurs in the z direction only,

$$\epsilon(\vec{r},\omega) = \epsilon(z,\omega) \quad .$$

Since the material is then translationally invariant in x and y, we introduce corresponding spatial Fourier transforms:

$$\int dx\ dy\ e^{-i(k_x x + k_y y)}\ F(\vec{r},\omega)$$

$$\equiv \int (d\vec{r}_\perp)\ e^{-i\vec{k}_\perp \cdot \vec{r}_\perp}\ F(\vec{r},\omega)$$

$$\equiv F(z, \vec{k}_\perp, \omega) \quad .$$

Letting $\vec{n}$ be the unit vector pointing in the z direction, we replace the gradient operator by

$$\vec{\nabla} = \vec{\nabla}_\perp + \vec{n}\frac{\partial}{\partial z}$$

$$\rightarrow i\vec{k}_\perp + \vec{n}\frac{\partial}{\partial z} \quad .$$

It is natural to project Maxwell's equations, (39.3), onto spaces parallel to, and perpendicular to, $\vec{n}$. The z components of (39.3) become, using a highly redundant notation,

$$-i\vec{k}_\perp\cdot(\vec{n}\times\vec{B}_\perp) = -\frac{i\omega\varepsilon}{c}E_z + \frac{4\pi}{c}J_z \quad ,$$

$$i\vec{k}_\perp\cdot(\vec{n}\times\vec{E}_\perp) = -\frac{i\omega}{c}B_z \quad , \tag{39.6}$$

while the $\perp$ (x,y) components appear as

$$-\frac{\partial}{\partial z}\vec{B}_\perp + i\vec{k}_\perp B_z = -\frac{i\omega\varepsilon}{c}\vec{n}\times\vec{E}_\perp + \frac{4\pi}{c}\vec{n}\times\vec{J}_\perp \quad ,$$

$$\frac{\partial}{\partial z}\vec{E}_\perp - i\vec{k}_\perp E_z = -\frac{i\omega}{c}\vec{n}\times\vec{B}_\perp \quad . \tag{39.7}$$

For $\omega \neq 0$, (39.6) determines E_z and B_z in terms of $\vec{E}_\perp$ and $\vec{B}_\perp$:

$$B_z = -\frac{c}{\omega}\vec{k}_\perp\cdot(\vec{n}\times\vec{E}_\perp) \quad ,$$

$$E_z = \frac{c}{\omega\varepsilon}\vec{k}_\perp\cdot(\vec{n}\times\vec{B}_\perp) - i\frac{4\pi}{\omega\varepsilon}J_z \quad . \tag{39.8}$$

Inserting these expressions into the $\perp$ set, (39.7), we have the following equations for the $\perp$ components:

$$\frac{\partial}{\partial z} \vec{B}_\perp - \frac{i\omega\varepsilon}{c} \vec{n} \times \vec{E}_\perp + i\vec{k}_\perp \frac{c}{\omega} \vec{k}_\perp \cdot (\vec{n} \times \vec{E}_\perp)$$

$$= -\frac{4\pi}{c} \vec{n} \times \vec{J}_\perp \quad , \tag{39.9a}$$

$$\frac{\partial}{\partial z} \vec{E}_\perp + \frac{i\omega}{c} \vec{n} \times \vec{B}_\perp - i\vec{k}_\perp \frac{c}{\omega\varepsilon} \vec{k}_\perp \cdot (\vec{n} \times \vec{B}_\perp) = \vec{k}_\perp \frac{4\pi}{\omega\varepsilon} J_z \quad , \tag{39.9b}$$

which mix $\vec{B}_\perp = (B_x, B_y)$ with $\vec{n} \times \vec{E}_\perp = (-E_y, E_x)$, and $\vec{E}_\perp = (E_x, E_y)$ with $\vec{n} \times \vec{B}_\perp = (-B_y, B_x)$. Next, as an alternative form of the same equations, we take the cross product of the above equations with $\vec{n}$,

$$\frac{\partial}{\partial z} \vec{n} \times \vec{B}_\perp + \frac{i\omega\varepsilon}{c} \vec{E}_\perp + i\, \vec{n} \times \vec{k}_\perp \frac{c}{\omega} \vec{k}_\perp \cdot (\vec{n} \times \vec{E}_\perp) = \frac{4\pi}{c} \vec{J}_\perp \quad , \tag{39.10a}$$

$$\frac{\partial}{\partial z} \vec{n} \times \vec{E}_\perp - \frac{i\omega}{c} \vec{B}_\perp - i\, \vec{n} \times \vec{k}_\perp \frac{c}{\omega\varepsilon} \vec{k}_\perp \cdot (\vec{n} \times \vec{B}_\perp) = \vec{n} \times \vec{k}_\perp \frac{4\pi}{\omega\varepsilon} J_z \quad , \tag{39.10b}$$

where we have noted that for any vector $\vec{V}_\perp$ in the xy plane, $-\vec{n} \times (\vec{n} \times \vec{V}_\perp) = \vec{V}_\perp$. Now, we project these vector equations on $\vec{k}_\perp$. From (39.9a) and (39.10b) we find a system of equations relating $\vec{k}_\perp \cdot \vec{B}_\perp$ and $\vec{k}_\perp \cdot (\vec{n} \times \vec{E}_\perp)$:

$$\frac{\partial}{\partial z} \vec{k}_\perp \cdot \vec{B}_\perp - \frac{i\omega\varepsilon}{c} \left(1 - \frac{k_\perp^2 c^2}{\omega^2 \varepsilon} \right) \vec{k}_\perp \cdot (\vec{n} \times \vec{E}_\perp) = -\frac{4\pi}{c} \vec{k}_\perp \cdot (\vec{n} \times \vec{J}_\perp) \quad , \tag{39.11a}$$

$$\frac{\partial}{\partial z} \vec{k}_\perp \cdot (\vec{n} \times \vec{E}_\perp) - \frac{i\omega}{c} \vec{k}_\perp \cdot \vec{B}_\perp = 0 \quad , \tag{39.11b}$$

and, from (39.9b) and (39.10a), a system relating $\vec{k}_\perp \cdot \vec{E}_\perp$ and $\vec{k}_\perp \cdot (\vec{n} \times \vec{B}_\perp)$:

$$\frac{\partial}{\partial z} \vec{k}_\perp \cdot \vec{E}_\perp + \frac{i\omega}{c} \left(1 - \frac{k_\perp^2 c^2}{\omega^2 \varepsilon} \right) \vec{k}_\perp \cdot (\vec{n} \times \vec{B}_\perp) = k_\perp^2 \frac{4\pi}{\omega\varepsilon} J_z \quad , \tag{39.12a}$$

$$\frac{\partial}{\partial z} \vec{k}_\perp \cdot (\vec{n} \times \vec{B}_\perp) + \frac{i\omega\varepsilon}{c} \vec{k}_\perp \cdot \vec{E}_\perp = \frac{4\pi}{c} \vec{k}_\perp \cdot \vec{J}_\perp \quad . \tag{39.12b}$$

The vectors $\vec{n}$ and $\vec{k}_\perp$ define a plane, called the plane of incidence, so that the system (39.11) relates the component of $\vec{B}_\perp$ in this plane to the component $E_\perp$ perpendicular to this plane, and vice versa for system (39.12). That is, if we take $\vec{k}_\perp$ in the x direction, we have equations governing B_x and E_y, and B_y and E_x, respectively. These perpendicular components belong together physically (see Subsection 7-2). By combining (39.11a) with (39.11b), and (39.12a) with (39.12b), we convert these systems of first order differential equations into second order ones:

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\omega^2\varepsilon}{c^2} - k_\perp^2\right] \vec{k}_\perp\cdot(\vec{n}\times\vec{E}_\perp) = -\frac{i\omega}{c}\frac{4\pi}{c}\vec{k}_\perp\cdot(\vec{n}\times\vec{J}_\perp) \quad , \tag{39.13a}$$

$$\frac{\partial}{\partial z}\left[\frac{1}{\varepsilon}\frac{\partial}{\partial z}\vec{k}_\perp\cdot(\vec{n}\times\vec{B}_\perp)\right] + \left(\frac{\omega^2}{c^2} - \frac{k_\perp^2}{\varepsilon}\right)\vec{k}_\perp\cdot(\vec{n}\times\vec{B}_\perp)$$

$$= -\frac{i\omega}{c}k_\perp^2\frac{4\pi}{\omega\varepsilon}J_z + \frac{\partial}{\partial z}\left(\frac{4\pi}{\varepsilon c}\vec{k}_\perp\cdot\vec{J}_\perp\right) \quad , \tag{39.13b}$$

where we must remember that ε is a function of z. From these components, perpendicular to the plane of incidence, those in the plane of incidence can be obtained from (39.11b) and (39.12b), and the longitudinal components (the z-components) from (39.8).

When ε is a constant function of z, the differential operators in these last two equations are the same, namely

$$\frac{\partial^2}{\partial z^2} + \frac{\omega^2\varepsilon}{c^2} - k_\perp^2 \quad . \tag{39.14}$$

We further assume that ε is real. Then, depending on the sign of $\frac{\omega^2\varepsilon}{c^2} - k_\perp^2$, the solutions to the corresponding homogeneous differential equations are different:

$$\frac{\omega^2\epsilon}{c^2} - k_\perp^2 < 0 : \quad e^{\pm\sqrt{k_\perp^2 - \frac{\omega^2\epsilon}{c^2}}\, z} , \tag{39.15a}$$

$-\nabla^2 E + \frac{\omega^2\epsilon}{c^2} E + i\frac{4\pi\omega}{c^2} J$

$-\frac{\partial^2}{\partial z^2} E + \left(\frac{\omega^2\epsilon}{c^2} - k_\perp^2\right) E$

$$\frac{\omega^2\epsilon}{c^2} - k_\perp^2 > 0 : \quad e^{\pm i\sqrt{\frac{\omega^2\epsilon}{c^2} - k_\perp^2}\, z} . \tag{39.15b}$$

$\boxed{k_z^2 = \frac{\omega^2\epsilon}{c^2} - k_\perp^2}$

The first possibility is essentially that discussed in electrostatics, in that there is no wave propagation, and so we will not consider it further here. [See Sec. XIII.] For the second possibility, we do have propagating waves. In terms of

$$k_z = \sqrt{\frac{\omega^2\epsilon}{c^2} - k_\perp^2} , \tag{39.16}$$

we construct a plane wave by combining the imaginary exponential structure in (39.15b) with the transverse spatial dependence $e^{i\vec{k}_\perp\cdot\vec{r}_\perp}$:

$$e^{i\vec{k}_\perp\cdot\vec{r}_\perp}\, e^{ik_z z} = e^{i\vec{k}\cdot\vec{r}} , \tag{39.17}$$

which represents a plane wave moving in the direction $\vec{k}$, since the phase is constant on a plane perpendicular to $\vec{k}$. The wavelength, λ, is defined as the distance over which the phase advances by 2π, so

$$|\vec{k}|\,\lambda = 2\pi, \text{ or}$$

$$|\vec{k}| = \frac{2\pi}{\lambda} \equiv \frac{1}{\bar{\lambda}} ; \tag{39.18}$$

$|\vec{k}|$ is called the wavenumber, and $\vec{k}$ the propagation vector. Including the time factor $e^{-i\omega t}$, we find the dependence of a plane wave on space and time to be

$$e^{i(\vec{k}\cdot\vec{r}-\omega t)} .$$

The surface of constant phase advances with time so that

$$\vec{k} \cdot \frac{d\vec{r}}{dt} = \omega \quad ;$$

the phase velocity thus is

$$v = \frac{\omega}{|\vec{k}|} = \omega\lambda\!\!\!^{-} = \nu\lambda \quad . \tag{39.19}$$

From (39.16) we see that the square of the wave number is

$$|\vec{k}|^2 = k_\perp^{\,2} + k_z^{\,2} = \frac{\omega^2\varepsilon}{c^2} \quad ,$$

so that the phase velocity is

$$v = \frac{c}{\sqrt{\varepsilon}} = \frac{c}{n} \quad , \tag{39.20a}$$

$$n = \sqrt{\varepsilon} \quad , \tag{39.20b}$$

as we saw in Subsection 7-2.

39-2. Reflection and Refraction by a Plane Interface: $\perp$ Polarization

We now specialize the above discussion to a situation in which the inhomogeneity in the dielectric constant is due to a plane interface between two dielectric substances:

$$\begin{aligned} z > 0 : \quad & \varepsilon(z) = \varepsilon_2 \quad , \\ z < 0 : \quad & \varepsilon(z) = \varepsilon_1 \quad . \end{aligned} \tag{39.21}$$

Since the relation between field and source is linear, the field produced by a prescribed current can be expressed in terms of an appropriate Green's function. Green's function corresponding to (39.13a) satisfies

$$-\left(\frac{\partial^2}{\partial z^2}+\frac{\omega^2\varepsilon}{c^2}-k_\perp{}^2\right)g(z,z') = \delta(z-z') \quad , \tag{39.22}$$

which as we noted above refers to the component of $\vec{E}_\perp$ perpendicular to the plane of incidence, defined by $\vec{k}_\perp$ and $\vec{n}$, and so we will call this the $\perp$ polarization. We imagine that the source lies in medium 2, that is $z' > 0$. The boundary conditions at the interface are that $\vec{k}_\perp\cdot(\vec{n}\times\vec{E}_\perp)$ and $\vec{k}_\perp\cdot\vec{B}_\perp$ be continuous, which impose the conditions that g and, by (39.11b), $\partial g/\partial z$ be continuous at $z = 0$. For $z \to \pm\infty$ we must have outgoing waves, since the source is localized at z'. The solutions to (39.22) in the three regions have the forms

$$z > z' \quad : \quad g = A\, e^{ik_{z2}z} \quad , \tag{39.23a}$$

$$z' > z > 0 : \quad g = B\, e^{ik_{z2}z} + C\, e^{-ik_{z2}z} \quad , \tag{39.23b}$$

$$z < 0 \quad : \quad g = D\, e^{-ik_{z1}z} \quad , \tag{39.23c}$$

where the subscripts 1 and 2 refer to the value of k_z, as defined by (39.16), in medium 1 and 2, respectively. Note that these equations are very similar to those considered in electrostatics, in fact, when in both dielectrics

$$\frac{\omega^2\varepsilon}{c^2} - k_\perp{}^2 < 0 \quad ,$$

we make the replacement

$$\sqrt{\frac{\omega^2\varepsilon}{c^2} - k_\perp^2} \rightarrow i\sqrt{k_\perp^2 - \frac{\omega^2\varepsilon}{c^2}}$$

in (39.23), and thereby recover the same system in (13.5), except that, here, the wavenumbers are different in the two media.

We now determine the coefficients A through D by satisfying the boundary conditions stated above as well as the discontinuity at $z = z'$. At $z = 0$, continuity of g yields the relation

$$B + C = D \quad , \tag{39.24a}$$

while continuity of $\partial g/\partial z$ requires

$$B - C = -\frac{k_{z1}}{k_{z2}} D \quad . \tag{39.24b}$$

The combination of these supplies

$$B = \frac{1}{2}\left(1 - \frac{k_{z1}}{k_{z2}}\right) D \quad ,$$

$$C = \frac{1}{2}\left(1 + \frac{k_{z1}}{k_{z2}}\right) D \quad . \tag{39.25}$$

(Note that if $\varepsilon_1 = \varepsilon_2$, these coefficients are simply $B = 0$ and $C = D$, which expresses the obvious fact that the entire wave is transmitted, with no reflection, since there is no interface.) The presence of the source term in (39.22) imposes the conditions that, at $z = z'$,

g is continuous, and

$$\left[-\frac{\partial}{\partial z} g\right]_{z=z'-0}^{z=z'+0} = 1 \quad . \tag{39.26}$$

Consequently, there are two additional equations for the constants,

$$A\, e^{ik_{z2}z'} = B\, e^{ik_{z2}z'} + C\, e^{-ik_{z2}z'} , \qquad (39.27a)$$

$$-A\, e^{ik_{z2}z'} + B\, e^{ik_{z2}z'} - C\, e^{-ik_{z2}z'} = \frac{1}{ik_{z2}} . \qquad (39.27b)$$

Equations (39.25) and (39.27) can be solved to yield, successively,

$$C = \frac{i}{2k_2}\, e^{ik_2 z'} , \qquad (39.28a)$$

$$B = \frac{k_2-k_1}{k_2+k_1}\,\frac{i}{2k_2}\, e^{ik_2 z'} , \qquad (39.28b)$$

$$D = \frac{2k_2}{k_2+k_1}\,\frac{i}{2k_2}\, e^{ik_2 z'} , \qquad (39.28c)$$

$$A = \frac{k_2-k_1}{k_2+k_1}\,\frac{i}{2k_2}\, e^{ik_2 z'} + \frac{i}{2k_2}\, e^{-ik_2 z'} , \qquad (39.28d)$$

where we have simplified the notation by dropping all the z subscripts on the k's. Combining these results, we may now write down the Green's function in the two regions,

$$z > 0 : \quad g(z,z') = \frac{i}{2k_2}\, e^{ik_2|z-z'|} + r\,\frac{i}{2k_2}\, e^{ik_2(z+z')} , \qquad (39.29a)$$

$$z < 0 : \quad g(z,z') = t\,\frac{i}{2k_2}\, e^{-ik_1 z}\, e^{ik_2 z'} , \qquad (39.29b)$$

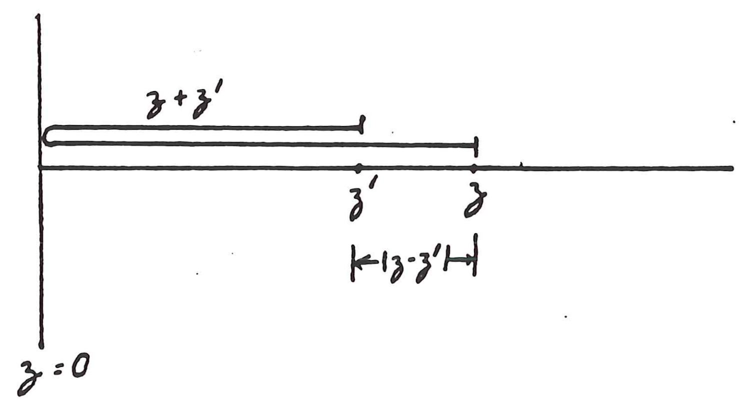
in terms of the reflection and transmission coefficients,

$$r = \frac{k_2-k_1}{k_2+k_1} , \qquad (39.30a)$$

and

$$t = \frac{2k_2}{k_2+k_1} \quad . \tag{39.30b}$$

The two terms in (39.29a) have a simple physical interpretation. The first refers to the direct wave from the source, since $|z-z'|$ is (the z-projection of) the distance from the source, while the second represents the wave reflected from the interface, since $z+z'$ is (the z-projection of) the distance from the source to the interface and back to the observation point z.



The following algebraic identities relating t and r,

$$t = 1 + r \quad , \tag{39.31a}$$

$$k_1 t = k_2(1-r) \quad , \tag{39.31b}$$

restate the continuity of g and g' at $z = 0$, which in turn are consequences of the continuity of the tangential components of $\vec{E}$ and $\vec{B}$ across the interface. Moreover, on combining (39.31a) and (39.31b) to give

$$k_1 t^2 = k_2(1-r^2) \quad , \tag{39.32}$$

we have a statement of conservation of energy, as we will show in Subsection 39-5.

Lecture 11

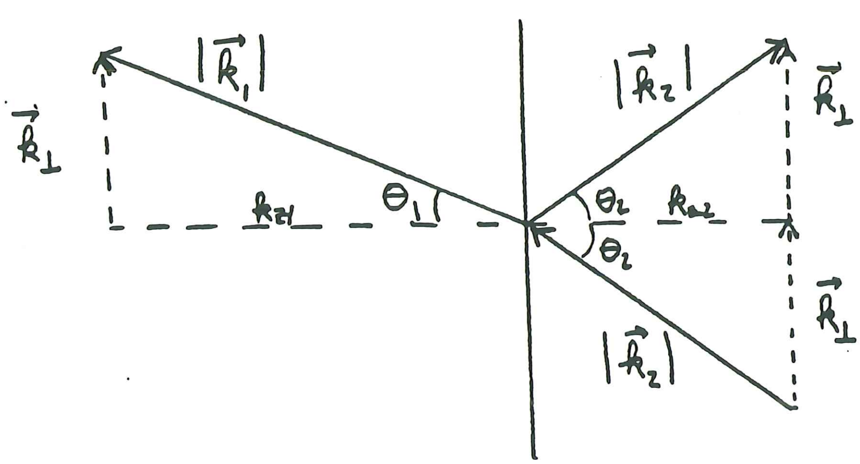
We are here primarily interested in how a plane interface reflects and transmits light, irrespective of how the radiation is produced. Accordingly, for regions to the left of the source, we regroup the factors of Green's function suggestively:

$$0 < z < z' : \quad g(z,z') = \left[e^{-ik_2 z} + re^{ik_2 z}\right]\left(\frac{i}{2k_2} e^{ik_2 z'}\right) , \qquad (39.33a)$$

$$z < 0 : \quad g(z,z') = te^{-ik_1 z}\left(\frac{i}{2k_2} e^{ik_2 z'}\right) . \qquad (39.33b)$$

The common factors in parentheses refer to how the wave is produced, which is not of interest to us at the moment. The other remaining factors describe the reflection and transmission of a plane wave: unit amplitude incident on the interface, amplitude r reflected, amplitude t transmitted.

We recall that Green's function, $g(z,z')$, describes only the z-dependence of the wave propagation; the x and y dependence is contained in the factor $e^{i\vec{k}_\perp \cdot \vec{r}_\perp}$ [recall (39.17)]. Since the transverse momenta are unaltered, while the longitudinal momenta (k_z) change from k_{z2} to k_{z1} as the wave crosses the interface, we have the following geometrical picture for the incident, reflected, and transmitted propagation vectors.



Under reflection k_z merely changes sign, so the angle of incidence equals the angle of reflection. The relation between the wavenumber and the index of refraction is

$$|\vec{k}| = \frac{\omega n}{c} \quad ,$$

with $n = \sqrt{\varepsilon}$. The continuity of $\vec{k}_\perp$ at the interface,

$$|\vec{k}_2|\, \sin\theta_2 = |\vec{k}_1|\, \sin\theta_1 \quad ,$$

then implies Snell's law of refraction (discovered in 1621),

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad , \tag{39.34}$$

relating the angle of incidence θ_2 to the angle of refraction θ_1.

We may re-express the reflection and transmission coefficients given in (39.30) in terms of the angles θ_1 and θ_2 by using

$$\frac{k_z}{|\vec{k}_\perp|} = \cot\theta \quad ,$$

to obtain

$$r = \frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)} \quad , \tag{39.35a}$$

and

$$t = \frac{2\,\sin\theta_1\,\cos\theta_2}{\sin(\theta_1+\theta_2)} \quad , \tag{39.35b}$$

but the wavenumber form is usually preferable. This is particularly evident for normal incidence, where $\theta_1 = \theta_2 = 0$, and (39.35) is indeterminate. On the other hand the wavenumber form, (39.30), is well defined, since when $k_\perp = 0$, $k_z = \frac{\omega}{c} n$, which yields

$$r = -\frac{n_1-n_2}{n_1+n_2} \quad , \tag{39.36a}$$

$$t = \frac{2n_2}{n_1+n_2} \quad . \tag{39.36b}$$

Also from the wavenumber form it is easy to see that if $n_1/n_2 \to \infty$, the coefficients become

$$r = -1 \ , \qquad t = 0 \ ; \tag{39.37}$$

this corresponds to the situation of total reflection by a perfect conductor. We verify this last statement by recalling the plane wave part of the Green's function between the source and the interface, (39.33a):

$$0 < z < z' : \quad g \sim e^{-ik_{z2}z} + re^{ik_{z2}z} \quad .$$

When (39.37) is satisfied, g vanishes at $z = 0$,

$$g \sim (1+r) = 0 \quad ,$$

which expresses the vanishing of the tangential component of $\vec{E}$ on the surface, characteristic of a perfect conductor.

39-3. Reflection and Refraction by a Plane Interface: $\|$ Polarization

We now turn to a consideration of the second polarization state, in which $\vec{E}_\perp$ lies in the plane of incidence and $\vec{B}_\perp$ is perpendicular to that plane. We will call this the $\|$ polarization in contrast to the $\perp$ case discussed above.

According to (39.13b), Green's function in this situation, which relates $\vec{k}_\perp \cdot (\vec{n} \times \vec{B}_\perp)$ to its source, satisfies

$$-\frac{\partial}{\partial z}\left(\frac{1}{\varepsilon}\frac{\partial}{\partial z} g\right) - \frac{k_z^{\,2}}{\varepsilon} g = \delta(z-z') \quad , \tag{39.38}$$

where, again,

$$k_z^{\,2} = \frac{\omega^2}{c^2}\varepsilon - k_\perp^{\,2} \quad .$$

With the plane interface between the two dielectric media located at $z = 0$, the form of the solution in the three regions, when $z' > 0$, is

$$z > z' \quad : \quad g = A\, e^{ik_2 z} \quad , \tag{39.39a}$$

$$0 < z < z' : \quad g = B\, e^{ik_2 z} + C\, e^{-ik_2 z} \quad , \tag{39.39b}$$

$$z < 0 \quad : \quad g = D\, e^{-ik_1 z} \quad , \tag{39.39c}$$

where, as before, the z subscript is suppressed. Here, the boundary conditions at $z = 0$ are the continuity of g (since $\vec{B}_\perp$ is continuous) and the continuity of $\frac{1}{\varepsilon} g'$ [since $\vec{E}_\perp$, which is related to g by (39.12b), is continuous], which furnish the relations

$$B + C = D \quad , \tag{39.40a}$$

$$B - C = -\frac{\varepsilon_2}{\varepsilon_1}\frac{k_1}{k_2} D \quad , \tag{39.40b}$$

implying

$$C = \frac{1}{2}\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\frac{k_1}{k_2}\right) D \quad , \tag{39.41a}$$

$$B = \frac{1}{2}\left(1 - \frac{\varepsilon_2}{\varepsilon_1}\frac{k_1}{k_2}\right) D \quad . \tag{39.41b}$$

At z = z', g is continuous, while from (39.38)

$$-\frac{1}{\varepsilon_2}\frac{\partial}{\partial z} g\Big|_{z=z'-0}^{z=z'+0} = 1 \quad ,$$

so

$$B\,e^{ik_2z'} + C\,e^{-ik_2z'} = A\,e^{ik_2z'} \quad , \tag{39.42a}$$

and

$$ik_2(-A\,e^{ik_2z'} + B\,e^{ik_2z'} - C\,e^{-ik_2z'}] = \varepsilon_2 \quad . \tag{39.42b}$$

The solution to this system of equations is

$$C = \frac{i\varepsilon_2}{2k_2}\,e^{ik_2z'} \quad , \tag{39.43a}$$

$$B = \frac{1 - \frac{\varepsilon_2}{\varepsilon_1}\frac{k_1}{k_2}}{1 + \frac{\varepsilon_2}{\varepsilon_1}\frac{k_1}{k_2}}\,\frac{i\varepsilon_2}{2k_2}\,e^{ik_2z'} \quad , \tag{39.43b}$$

$$D = \frac{2}{1 + \frac{\varepsilon_2}{\varepsilon_1}\frac{k_1}{k_2}}\,\frac{i\varepsilon_2}{2k_2}\,e^{ik_2z'} \quad , \tag{39.43c}$$

$$A = \frac{1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{k_1}{k_2}}{1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{k_1}{k_2}}\,\frac{i\varepsilon_2}{2k_2}\,e^{ik_2z'} + \frac{i\varepsilon_2}{2k_2}\,e^{-ik_2z'} \quad . \tag{39.43d}$$

These are as in the other polarization, as given by (39.28), except that in the amplitudes multiplying the exponentials, where we had k_z, we now have k_z/ε. Green's function is thus

$$z > 0 : \quad g = \frac{i\varepsilon_2}{2k_2} e^{ik_2|z-z'|} + r\,\frac{i\varepsilon_2}{2k_2} e^{ik_2(z+z')} , \qquad (39.44a)$$

$$z < 0 : \quad g = t\,\frac{i\varepsilon_2}{2k_2} e^{-ik_1 z} e^{+ik_2 z'} . \qquad (39.44b)$$

Green's functions for the two polarizations, $\perp$ and $\parallel$, are given by (39.29) and (39.44), respectively, with the corresponding reflection and transmission coefficients given by

$\perp$ mode	$\parallel$ mode
$r = -\dfrac{k_1-k_2}{k_1+k_2}$,	$r = -\dfrac{\frac{k_1}{\varepsilon_1}-\frac{k_2}{\varepsilon_2}}{\frac{k_1}{\varepsilon_1}+\frac{k_2}{\varepsilon_2}}$,
$t = \dfrac{2k_2}{k_1+k_2}$,	$t = \dfrac{2\frac{k_2}{\varepsilon_2}}{\frac{k_1}{\varepsilon_1}+\frac{k_2}{\varepsilon_2}}$.

(39.45)

In both cases the algebraic relation (39.31a) holds true,

$$t = 1 + r ,$$

expressing the continuity of $\vec{E}_\perp$ and of $\vec{B}_\perp$ in the $\perp$ and $\parallel$ polarizations, respectively, but the statement of energy conservation is different in the two situations [recall (39.32)]:

$\perp$ mode $\qquad\qquad$ $\parallel$ mode

$$k_1 t^2 = k_2(1-r^2) \quad , \qquad \frac{k_1}{\varepsilon_1} t^2 = \frac{k_2}{\varepsilon_2}(1-r^2) \quad , \tag{39.46}$$

where, again, the second relation is obtained from the first by the substitution $k_z \to k_z/\varepsilon$.

It is particularly interesting to ask when the reflection coefficient can vanish. For the $\perp$ polarization, $r = 0$ only when $k_{z1} = k_{z2}$, or $\varepsilon_1 = \varepsilon_2$, that is, in the absence of an interface. However, for the $\parallel$ polarization, another solution to $r = 0$ exists. The vanishing of the reflection coefficient requires

$$\frac{k_{z1}}{\varepsilon_1} = \frac{k_{z2}}{\varepsilon_2} \quad ,$$

or, since

$$\frac{k_z^2}{\varepsilon^2} = \frac{\omega^2}{c^2}\frac{1}{\varepsilon} - \frac{k_\perp^2}{\varepsilon^2} \quad ,$$

the condition becomes

$$\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_1\varepsilon_2}\right)\left[\frac{\omega^2}{c^2} - \frac{\varepsilon_2+\varepsilon_1}{\varepsilon_1\varepsilon_2} k_\perp^2\right] = 0 \quad .$$

The new possibility for r vanishing occurs when

$$\frac{\omega}{c} = \sqrt{\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}}\, |\vec{k}_\perp| \quad . \tag{39.47}$$

Geometrically, since

$$|\vec{k}_\perp| = \frac{\omega}{c} n \sin\theta \ , \qquad n = \sqrt{\varepsilon} \ , \tag{39.48}$$

(39.47) is satisfied when the angle of incidence, θ_2, equals θ_B, where

$$\sin\theta_B = \sqrt{\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}} \ , \tag{39.49a}$$

or,

$$\tan\theta_B = \frac{n_1}{n_2} \ . \tag{39.49b}$$

At the incident angle θ_B, the $\|$ mode is completely transmitted, so that the reflected wave is completely polarized perpendicular to the plane of incidence. The phenomenon was discovered by Sir David Brewster about the year 1800, and in consequence θ_B is called Brewster's angle.

To express r and t again in terms of the angle of incidence θ_2 and the angle of refraction θ_1, we use the geometrical relations

$$k_z = |\vec{k}_\perp| \cot\theta \ , \qquad n \sin\theta = \frac{c}{\omega} |\vec{k}_\perp| \ ,$$

to write

$$\frac{k_z}{\varepsilon} = \frac{(\omega/c)^2}{|\vec{k}_\perp|} \frac{1}{2} \sin 2\theta \ .$$

Consequently, for the $\|$ mode, the reflection and transmission coefficients can be given as

$$r = -\frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)} \ , \tag{39.50a}$$

$$t = \frac{2\sin\theta_2 \cos\theta_2}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)} \quad . \tag{39.50b}$$

Here we observe, from (39.50a), that Brewster's angle occurs when the sum of the angle of incidence and the angle of refraction forms a right angle,

$$\theta_1 + \theta_2 = \pi/2 \quad . \tag{39.51}$$

For normal incidence ($\theta_1 = \theta_2 = 0$), as before it is convenient to return to (39.45) and employ

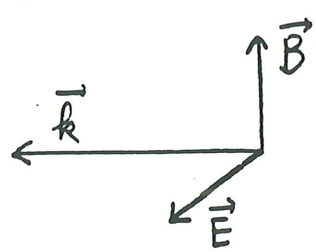
$$\frac{k_z}{\varepsilon} = \frac{\omega}{nc} \quad .$$

For the two polarizations we have [recall (39.36)]

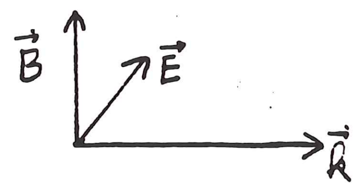
$$\begin{array}{ll} \perp \text{ mode} & \parallel \text{ mode} \\ r = -\dfrac{n_1-n_2}{n_1+n_2} \ , & r = \dfrac{n_1-n_2}{n_1+n_2} \ , \\ t = \dfrac{2n_2}{n_1+n_2} \ , & t = \dfrac{2n_1}{n_1+n_2} \ . \end{array} \tag{39.52}$$

But for normal incidence there can be no physical difference between the two polarizations, since the plane of incidence is not defined. To reconcile the two forms in (39.52), we recognize that the reflection and transmission coefficients for the $\parallel$ mode refer to $\vec{B}$, while those for the $\perp$ mode refer to $\vec{E}$ [see (39.13)]. Under reflection, $\vec{E}$ reverses its sense relative to $\vec{B}$, since the direction of propagation is reversed.

Incident:



Reflected:



So the two forms for the reflection coefficient in (39.52) are physically equivalent. To reach the same conclusion for the transmission coefficients, we recall from Subsection 7-2 that the magnitudes of the electric and magnetic fields in the plane wave are connected by

$$\varepsilon|\vec{E}|^2 = |\vec{B}|^2 \quad ,$$

which relates transmission coefficients referring to electric and magnetic fields by

$$t_E = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\, t_B = \frac{n_2}{n_1}\, t_B \quad , \tag{39.53}$$

which explains the ratio of the two forms for t in (39.52).

39-4. Total Internal Reflection

We now return to Snell's law, (39.34),

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad .$$

Suppose we consider a plane wave going from a region of greater dielectric constant to a region with a smaller dielectric constant, that is

$$n_1 < n_2 \Longrightarrow \sin\theta_1 > \sin\theta_2 \quad . \tag{39.54}$$

For this arrangement, there are incident angles satisfying

$$n_2 \sin\theta_2 > n_1 \quad , \tag{39.55}$$

for which Snell's law can be satisfied only if

$$\sin\theta_1 > 1 \; . \tag{39.56}$$

What does this mean physically? It is helpful to consider the angle of transition into this new regime, where

$$n_2 \sin\theta_2 = n_1 \quad ,$$

or

$$\sin\theta_1 = 1 \quad .$$

At exactly this angle, the transmitted wave travels along the interface since

$$|\vec{k}_1| = |\vec{k}_\perp| \; , \quad k_{z1} = 0 \quad .$$

Now as θ_2 is increased, $|\vec{k}_\perp|^2$ becomes greater than $\frac{\omega^2}{c^2}\varepsilon_1$ so that

$$k_{z1} = \sqrt{\frac{\omega^2}{c^2}\varepsilon_1 - k_\perp^2}$$

becomes imaginary,

$$k_{z1}^2 < 0 \quad \text{or} \quad k_{z1} = i\kappa \quad , \tag{39.57}$$

where κ is real. The propagation of the wave in medium 1 is now described by

$$e^{ik_{z1}|z|} = e^{-\kappa|z|} \quad ,$$

a decreasing exponential: the field penetrates only a short distance into region 1, and no energy is transmitted into that region, it being all reflected. This last fact follows from the form of the reflection coefficients,

$$\perp : \quad r = \frac{k_{z2} - i\kappa_1}{k_{z2} + i\kappa_1} \quad , \tag{39.58a}$$

$$\| : \quad r = \frac{\dfrac{k_{z2}}{\varepsilon_2} - i\,\dfrac{\kappa_1}{\varepsilon_1}}{\dfrac{k_{z2}}{\varepsilon_2} + i\,\dfrac{\kappa_1}{\varepsilon_1}} \quad , \tag{39.58b}$$

both of which have unit magnitude,

$$|r|^2 = 1 \quad .$$

In the next subsection, we will prove that this implies that all the energy is reflected. We have here the situation of total (internal) reflection.

Lecture 12

39-5. Energy Conservation

We now turn to a general consideration of energy flow across the plane interface. Recall Poynting's vector, (7.4), which has as the component normal to the interface

$$S_z = \frac{c}{4\pi} \vec{n}\cdot(\vec{E} \times \vec{H}) \quad , \tag{39.59}$$

where here

$$\vec{H} = \vec{B} \quad .$$

From this the total energy per unit area flowing in the +z direction is computed to be

$$\int dt\, S_z(\vec{r},t) = \frac{c}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\vec{n}\cdot\vec{E}(\vec{r},\omega)^* \times \vec{B}(\vec{r},\omega)$$

$$= \frac{c}{4\pi}\, 2\, \mathrm{Re}\int_{0}^{\infty}\frac{d\omega}{2\pi}\,\vec{n}\cdot\vec{E}(\vec{r},\omega)^* \times \vec{B}(\vec{r},\omega) \quad . \tag{39.60}$$

We next integrate over the x-y plane by introducing the Fourier transform in the $\vec{k}_\perp$ space:

$$\int dxdydt\, S_z = \frac{c}{2\pi}\,\mathrm{Re}\int_{0}^{\infty}\frac{d\omega}{2\pi}\int\frac{(d\vec{k}_\perp)}{(2\pi)^2}\,\vec{n}\cdot\vec{E}(z,\vec{k}_\perp,\omega)^* \times \vec{B}(z,\vec{k}_\perp,\omega) \quad . \tag{39.61}$$

The integrand in (39.61) states that the flow of energy in the +z direction per unit frequency interval and per unit $\vec{k}_\perp$ volume is

$$\frac{c}{2\pi}\,\mathrm{Re}[\vec{n}\cdot\vec{E}(z,\vec{k}_\perp,\omega)^* \times \vec{B}(z,\vec{k}_\perp,\omega)] \quad . \tag{39.62}$$

We adopt a coordinate system such that $\vec{k}_\perp$ is in the x-direction so that the y-axis is perpendicular to the plane of incidence. Then for $\perp$ polarization, the flow of energy in the -z direction is

$$\frac{c}{2\pi}\,\mathrm{Re}[-\vec{n}\cdot\vec{E}^* \times \vec{B}] = \frac{c}{2\pi}\,\mathrm{Re}\, E_y^*\, B_x \quad . \tag{39.63}$$

From the relation between B_x and E_y, (39.11b),

$$B_x = i\,\frac{c}{\omega}\frac{\partial}{\partial z}E_y \quad , \tag{39.64}$$

we see that (39,63) is

$$\frac{c^2}{2\pi\omega}\,\mathrm{Re}\, E_y^*\, i\,\frac{\partial}{\partial z}E_y \quad . \tag{39.65}$$

Now we use (39.33b) for the transmitted field ($z < 0$):

$$E_y = t\, e^{-ik_1 z} ,$$
$$i \frac{\partial}{\partial z} E_y = k_1 E_y , \qquad (39.66)$$

so the energy flowing in medium 1 away from the interface is proportional to

$$\mathrm{Re}\, E_y^* \, i \frac{\partial}{\partial z} E_y \propto \mathrm{Re}(k_1 |t|^2) . \qquad (39.67)$$

For the case of total reflection, k_1 is purely imaginary, so no energy is transmitted into region 1 as stated in the preceding subsection.

In region 2, $z > 0$, (39.33a) expresses the incident and reflected fields as

$$E_y = e^{-ik_2 z} + r\, e^{ik_2 z} ,$$

so that

$$i \frac{\partial}{\partial z} E_y = k_2 (e^{-ik_2 z} - r\, e^{ik_2 z}) .$$

Just outside the interface, at $z = 0+$, where

$$E_y = 1 + r ,$$
$$i \frac{\partial}{\partial z} E_y = k_2 (1-r) , \qquad (39.68)$$

the flow of energy toward the interface is given by

$$\frac{c^2}{2\pi\omega} \mathrm{Re}\, E_y^* \, i \frac{\partial}{\partial z} E_y \propto \mathrm{Re}\, k_2 (1 - |r|^2 + r^* - r) .$$

Since we are considering an incident plane wave, with real k_z, this becomes

$$k_2(1 - |r|^2) \quad . \tag{39.69}$$

The quantities (39.67) (at z = 0-) and (39.69) (at z = 0+) therefore must be identical for two reasons:

(1) The tangential components of $\vec{E}$, $\vec{B}$ must be continuous, since there are no surface currents. [For k_1 real, we derived this equality earlier in (39.32).]

(2) Energy is conserved, since there is no source of energy on the surface.

Thus, if k_1 is real, the statement of energy conservation is

$$k_1|t|^2 = k_2(1 - |r|^2) \quad , \tag{39.70}$$

while if k_1 is imaginary, there is no flow of energy in region 1 and

$$0 = k_2(1 - |r|^2)$$

or

$$|r|^2 = 1 \tag{39.71}$$

which is the situation of total reflection considered in Subsection 39-4.

For || polarization, the flow of energy is given in terms of

$$-\vec{n}\cdot\vec{E}^* \times \vec{B} = -E_x^* B_y$$

where, using (39.12b) in the absence of currents,

$$E_x = \frac{c}{i\omega\varepsilon}\frac{\partial}{\partial z} B_y \quad . \tag{39.72}$$

Thus the flow of energy is proportional to

$$\mathrm{Re}[-B_y^* E_x] \propto \mathrm{Re}\, B_y^* \frac{1}{\varepsilon} i \frac{\partial}{\partial z} B_y \quad , \tag{39.73}$$

with the same constant of proportionality as before. Since now the transmission and reflection coefficients, and the Green's function, refer to B_y, we may obtain the result by simply replacing

$$k_z \to \frac{k_z}{\varepsilon} \tag{39.74}$$

in the previous form. And so, again, the $\|$ form of (39.46) expresses the conservation of energy when k_1 is real; generally, for this polarization, the conservation of energy for a wave reflected and transmitted by an interface between plane dielectrics is stated by

$$\mathrm{Re}\left(\frac{k_1}{\varepsilon_1} |t|^2 \right) = \frac{k_2}{\varepsilon_2} (1 - |r|^2) \quad . \tag{39.75}$$

XL. REFLECTION AND ATTENTUATION BY AN IMPERFECT CONDUCTOR

We here briefly consider the reflection and transmission of electromagnetic radiation by a conductor. A conductor is characterized by Ohm's law, (5.9),

$$\vec{J}_{\text{cond}} = \sigma \vec{E} \quad , \tag{40.1}$$

where $\vec{J}_{\text{cond}}$ is the conduction current and σ is the conductivity. The effect of this current can be incorporated into the previous discussion by noting that Maxwell's equation for the curl of $\vec{H}$ now becomes

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial}{\partial t}\vec{D} + \frac{4\pi}{c}\vec{J}_{\text{cond}} + \frac{4\pi}{c}\vec{J} \quad , \tag{40.2}$$

where $\vec{J}$ is a source, external to the conductor, for $\vec{E}$ and $\vec{H}$. For fields of a definite frequency, that is, whose time dependence is given by $e^{-i\omega t}$, this equation becomes

$$\vec{\nabla} \times \vec{H} = -\frac{i\omega}{c}\left[\varepsilon + i\,\frac{4\pi}{\omega}\,\sigma\right]\vec{E} + \frac{4\pi}{c}\vec{J} \quad . \tag{40.3}$$

This shows that the conductor can be described by an effective dielectric constant,

$$\varepsilon + i\,\frac{4\pi}{\omega}\,\sigma \quad , \tag{40.4}$$

which is complex, expressing the dissipative nature of the conductor. (This was implicit in the considerations in Subsection 5-2.) According to (39.16), the component of the propagation vector normal to the surface of a planar conductor is then also complex:

$$k_z = \left[\frac{\omega^2}{c^2}\left(\varepsilon + i\,\frac{4\pi}{\omega}\,\sigma\right) - k_\perp^{\,2}\right]^{1/2} \quad . \tag{40.5}$$

The simplest situation is that of a good conductor, where the conduction current dominates the displacement current, that is

$$\frac{4\pi\sigma}{\omega} \gg \varepsilon \quad . \tag{40.6}$$

In the following we will assume that σ is real, which can be approximately valid only for sufficiently low frequencies, as (5.10) states. Since there is wave propagation outside the conductor, $k_\perp^{\,2} \lesssim \omega^2/c^2$, and (40.5) is, approximately,

$$k_z \simeq \sqrt{\frac{\omega^2}{c^2}\,i\,\frac{4\pi}{\omega}\,\sigma} = \frac{\omega}{c}\sqrt{\frac{4\pi\sigma}{\omega}}\,\frac{1+i}{\sqrt{2}} \quad ,$$

or (40.7)

$$k_z \simeq \frac{1+i}{\delta} \quad ,$$

where

$$\delta = \frac{c}{\sqrt{2\pi\omega\sigma}} \quad . \tag{40.8}$$

The z dependence of the propagation of a plane wave through the conductor is then given by

$$e^{ik_z|z|} = e^{-|z|/\delta}\, e^{i|z|/\delta} \quad . \tag{40.9}$$

The first factor on the right side of (40.9) represents the exponential attenuation of the wave; the amplitude is reduced by a factor of e^{-1} in the distance δ. This characteristic distance δ is called the skin depth. As indicated by (40.1), the current in a conductor is confined to a region within a distance

$\sim\delta$ of the surface, a distance which becomes smaller as the frequency increases. For a good conductor characterized by (40.6), the ratio of the skin depth to the reduced wavelength of the impinging radiation, $\lambda\!\!\!^{-} \simeq c/\omega$, is very small:

$$\frac{\delta}{\lambda\!\!\!^{-}} \simeq \sqrt{\omega/2\pi\sigma} << 1 \quad . \tag{40.10}$$

Thus, only radiation of long wavelengths (low frequencies) penetrate the conductor appreciably.

The discussion of transmission and reflection given in the previous section remains essentially unchanged except for the replacement of k_z by (40.7) inside the conductor. Therefore, for $\perp$ polarization, the reflection coefficient for a plane wave impinging from a dielectric (labelled 2) onto a flat conductor (labelled 1) is, from (39.30a) (again the z subscript is suppressed),

$$\begin{aligned} r &= \frac{k_2-k_1}{k_2+k_1} = \frac{k_2 - \frac{1+i}{\delta}}{k_2 + \frac{1+i}{\delta}} \\ &\simeq -[1 - k_2\delta(1-i)] \quad , \end{aligned} \tag{40.11}$$

since

$$\frac{1}{\delta} >> k_2 \quad . \tag{40.12}$$

For a perfect conductor, $\sigma \to \infty$, $\delta = 0$, and

$$r = -1 \quad , \tag{40.13}$$

as we saw earlier in (39.37). The fractional amount of power reflected by a good conductor is

$$|r|^2 \simeq 1 - 2k_2\delta \quad , \tag{40.14}$$

so that $2k_2\delta$ is the relative power entering the conductor. For a perfect conductor there is total reflection, with no energy absorbed by the conductor.

Another way of deriving this result, (40.14), is to return to the equation for $\vec{E}_\perp$, (39.9b);

$$\frac{\partial}{\partial z}\vec{E}_\perp + i\frac{\omega}{c}\vec{n}\times\vec{B}_\perp - \frac{ic}{\omega\varepsilon}\vec{k}_\perp[\vec{k}_\perp\cdot(\vec{n}\times\vec{B}_\perp)] = \text{(external currents)} \quad , \tag{40.15}$$

in which the effects of conduction currents are incorporated in the replacement

$$\varepsilon \rightarrow \varepsilon + i\,\frac{4\pi\sigma}{\omega} \quad ,$$

which has a magnitude much greater than that of $\varepsilon \sim 1$ inside a good conductor. Under such circumstances, the third term in (40.15) is negligible, implying, inside the conductor,

$$\frac{\partial}{\partial z}\vec{E}_\perp + i\frac{\omega}{c}\vec{n}\times\vec{B}_\perp \simeq 0 \; . \tag{40.16}$$

The derivative here is proportional to $\vec{E}_\perp$, since the electric field in the conductor attenuates exponentially according to (40.9),

$$\frac{\partial}{\partial z}\vec{E}_\perp = \frac{1-i}{\delta}\vec{E}_\perp \quad , \tag{40.17}$$

which implies that inside a good conductor,

$$\vec{E}_\perp \simeq \frac{1-i}{2}\frac{\omega\delta}{c}\vec{n}\times\vec{B}_\perp \quad , \tag{40.18}$$

relating the tangential components of the electric and magnetic fields. This relation is valid for an arbitrarily shaped surface as long as the radius of curvature of the surface is much larger than the skin depth δ. For a perfect conductor, this supplies the boundary condition that the tangential component of the electric field vanishes on the surface,

$$\vec{E}_{\perp} = 0 \,. \tag{40.19}$$

The energy flow in the $-z$ direction just inside the conductor is now seen to be proportional to [recall (39.62)]

$$\begin{aligned} &-\mathrm{Re}\, \vec{n}\cdot\vec{E}^{*} \times \vec{B} \\ &= \mathrm{Re}\, \frac{1+i}{2}\,\frac{\omega\delta}{c}\,(\vec{n}\times\vec{B}_{\perp}{}^{*})\cdot(\vec{n}\times\vec{B}_{\perp}) \\ &= \frac{\omega\delta}{2c}\,|\vec{B}_{\perp}|^{2} \,. \end{aligned} \tag{40.20}$$

To find the fractional amount of energy absorbed by a good conductor, we approximately evaluate $\vec{B}_{\perp}$ as though $\delta = 0$ (a perfect conductor), and compare (40.20) at $z = 0-$ to the incident flux at $z = 0+$. The latter is constructed in terms of the fields (normalized so that the amplitude of the incident magnetic field is unity), for the two independent polarization states,

$$\perp: \qquad \begin{aligned} B_x &= e^{-ik_2 z} + e^{ik_2 z} \,, \\ E_y &= \frac{\omega}{ck_2}\,(e^{-ik_2 z} - e^{ik_2 z}) \,, \end{aligned} \tag{40.21a}$$

$$\parallel: \qquad \begin{aligned} B_y &= e^{-ik_2 z} + e^{ik_2 z} \,, \\ E_x &= -\frac{ck_2}{\omega\varepsilon_2}\,(e^{-ik_2 z} - e^{ik_2 z}) \,. \end{aligned} \tag{40.21b}$$

Here, we have used (39.64) and (39.72) to relate B_x to E_y, and E_x to B_y, respectively. The incident energy flux arises from the terms proportional to $e^{-ik_2 z}$:

$$\text{Incident flux} \sim \mathrm{Re}\left\{ \frac{\omega}{ck_2} |B_{x\ \mathrm{inc}}|^2,\ \frac{ck_2}{\omega\varepsilon_2} |B_{y\ \mathrm{inc}}|^2 \right\}$$

$$= \frac{1}{4}\left\{ \frac{\omega}{ck_2} |B_x(z=0)|^2,\ \frac{ck_2}{\omega\varepsilon_2} |B_y(z=0)|^2 \right\} . \qquad (40.22)$$

The tangential magnetic field is continuous since, for an imperfect conductor, there is no surface current. Consequently, to determine the fractional amount of power absorbed, we take the ratio of (40.20) to (40.22):

$$\perp : \quad \frac{\frac{\omega\delta}{2c}}{\frac{1}{4}\frac{\omega}{ck_2}} = 2k_2\delta \ , \qquad (40.23a)$$

$$\parallel : \quad \frac{\frac{\omega\delta}{2c}}{\frac{1}{4}\frac{ck_2}{\omega\varepsilon_2}} = 2\left(\frac{\omega}{c}\right)^2 \frac{\varepsilon_2}{k_2}\delta \ . \qquad (40.23b)$$

Of course, (40.23a) agrees with (40.14). [See also Problem 20.]

XLI. GREEN'S FUNCTION IN CYLINDRICAL COORDINATES

41-1. 2 + 1 Dimensional Decomposition of Green's Function

A wave guide, in which conducting walls force a wave to propagate in a single direction (say along the z axis), motivates a consideration of a 2 + 1 dimensional break-up of Green's function for a wave propagating in vacuum with a definite frequency ω. This is analogous to our earlier discussion in electrostatics. (See Sections XII-XVI.) Furthermore, by deriving alternative expressions for Green's function, we are able to obtain additional mathematical properties of Bessel's functions.

The retarded Green's function satisfying (28.22),

$$-\left[\nabla^2 + \left(\frac{\omega}{c} \right)^2 \right] G_\omega(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{41.1}$$

is given by (28.32), that is

$$G_\omega(\vec{r},\vec{r}\,') = \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \quad . \tag{41.2}$$

In deriving this expression we treated all directions on the same footing; if instead we single out the z axis, by introducing a Fourier transform in the $\perp$ (transverse) space,

$$G_\omega(\vec{r},\vec{r}\,') = 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} g(z,z') \quad , \tag{41.3}$$

(41.1) is satisfied providing the reduced Green's function $g(z,z')$ obeys

$$\left[-\frac{\partial^2}{\partial z^2} + k_\perp{}^2 - \left(\frac{\omega}{c} \right)^2 \right] g(z,z') = \delta(z-z') \quad . \tag{41.4}$$

We have seen this differential equation before, (39.22) with $\varepsilon = 1$, and its solution is given by (39.29a) with $r = 0$:

$$g(z,z') = \frac{i\, e^{ik_z|z-z'|}}{2k_z} \tag{41.5}$$

where, for $\omega > 0$,

$$k_z = \begin{cases} \sqrt{\dfrac{\omega^2}{c^2} - k_\perp^2} \ , & \text{if } \dfrac{\omega}{c} > |\vec{k}_\perp| \ , \\ i\sqrt{k_\perp^2 - \dfrac{\omega^2}{c^2}} \ , & \text{if } \dfrac{\omega}{c} < |\vec{k}_\perp| \ . \end{cases} \tag{41.6}$$

In statics ($\omega = 0$) the second possibility in (41.6) occurs. [See (12.16).] Without loss of generality, we set $\vec{r}' = 0$, and define $k = \frac{\omega}{c} > 0$. Then, comparing (41.2) with (41.3) and (41.5), we obtain the identity

$$\frac{e^{ikr}}{r} = 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot\vec{r}_\perp}\, \frac{i\, e^{i\sqrt{k^2-k_\perp^2}|z|}}{2\sqrt{k^2-k_\perp^2}} \ , \tag{41.7}$$

where the square root is defined by (41.6). It is convenient to introduce a cylindrical coordinate system, with

$$\begin{aligned} \vec{r} &= (\rho,0,z), \qquad r = \sqrt{z^2+\rho^2} \ , \\ \vec{k}_\perp &= (\lambda,\phi) \ , \end{aligned} \tag{41.8}$$

from which follow

$$e^{i\vec{k}_\perp\cdot\vec{r}_\perp} = e^{i\lambda\rho\cos\phi}$$

$$(d\vec{k}_\perp) = \lambda\, d\lambda\, d\phi \ .$$

Making use of the integral representation for the Bessel function J_0, (15.2),

$$\int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{i\lambda\rho\,\cos\phi} = J_0(\lambda\rho) \;, \tag{41.9}$$

we find, from (41.7), the identity

$$\frac{e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} = i \int_0^{\infty} \lambda d\lambda\; J_0(\lambda\rho)\, \frac{e^{i\sqrt{k^2-\lambda^2}\,|z|}}{\sqrt{k^2-\lambda^2}} \;. \tag{41.10}$$

In the static limit $[k = 0, \sqrt{k^2-\lambda^2} \to i\lambda]$, we recover the previously derived result (15.4),

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^{\infty} d\lambda\; J_0(\lambda\rho)\; e^{-\lambda|z|} \;. \tag{41.11}$$

41-2. Three Dimensional Fourier Representation for Green's Function

An alternative representation for Green's function can be obtained by taking the three dimensional Fourier transform of (41.1), which immediately leads to the formal solution

$$G_\omega(\vec{r},\vec{r}\,') = 4\pi \int \frac{(d\vec{k})}{(2\pi)^3}\, \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{\vec{k}^2 - \frac{\omega^2}{c^2}} \;. \tag{41.12}$$

The above expression is not well defined since the integrand has a singularity wherever $|\vec{k}| = \omega/c$, corresponding to real wave propagation. To assign meaning to this expression, we must specify the boundary conditions. Recall that the retarded and advanced Green's functions have the explicit forms given in (28.32),

$$G_{\text{ret.}}(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|}\, e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|} \,, \tag{41.13a}$$

$$G_{\text{adv.}}(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|}\, e^{-i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|} \,. \tag{41.13b}$$

Mathematically, both signs of i are possible; a consideration of physical conditions is necessary to select the appropriate solution. If we let the wavenumber be complex, which is the situation occurring in a conductor [for example, see (40.7)]:

$$\frac{\omega}{c} \to \frac{\omega_1}{c} + \frac{i\omega_2}{c} \,, \qquad \omega_2 > 0 \,, \tag{41.14}$$

the retarded and advanced Green's functions become, respectively,

$$G_{\text{ret.}} \to \frac{1}{R}\, e^{i\frac{\omega_1}{c}R}\, e^{-\frac{\omega_2}{c}R} \,, \tag{41.15a}$$

$$G_{\text{adv.}} \to \frac{1}{R}\, e^{-i\frac{\omega_1}{c}R}\, e^{\frac{\omega_2}{c}R} \,, \tag{41.15b}$$

where $R = |\vec{r}-\vec{r}\,'|$. As $R \to \infty$, $G_{\text{adv.}}$ increases without bound, while the physically correct solution, $G_{\text{ret.}}$, goes to zero. Thus the device for defining $G_{\text{ret.}}$ in terms of the representation (41.12), which builds in the boundedness requirement, is to replace ω there by $\omega \to \omega + i\varepsilon$, $\varepsilon \to +0$:

$$G_{\text{ret.}}(\vec{r},\vec{r}\,') = 4\pi \int \frac{(d\vec{k})}{(2\pi)^3} \left. \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{\vec{k}^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2} \right|_{\varepsilon\to+0} \,. \tag{41.16}$$

We can quickly check the validity of this result by explicitly evaluating the integral using spherical coordinates [recall the spherical average of a plane wave seen, for example, in (36.33)]:

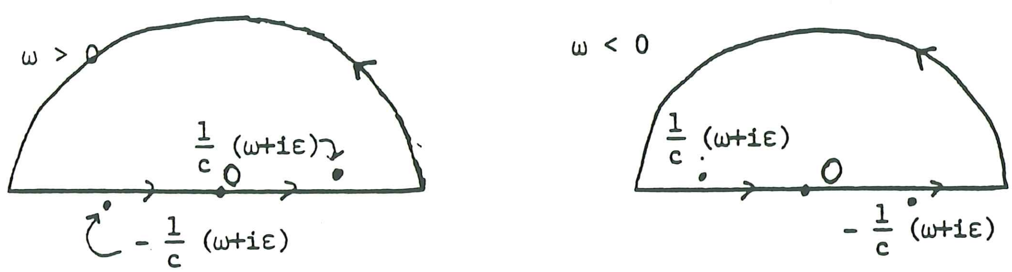
$$G_{ret.}(\vec{r},\vec{r}\,') = 4\pi \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} \frac{\sin kR}{kR} \frac{1}{k^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2}$$

$$= \frac{2}{\pi}\frac{1}{R}\int_0^\infty dk\, k \frac{\sin kR}{k^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2}$$

$$= -\frac{2}{\pi}\frac{1}{R}\frac{d}{dR}\int_0^\infty dk \frac{\cos kR}{k^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2}$$

$$= -\frac{1}{\pi}\frac{1}{R}\frac{d}{dR}\int_{-\infty}^\infty dk \frac{e^{ikR}}{k^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2}\,, \tag{41.17}$$

since the integrand is even in k. Equation (41.17) can be evaluated by expressing it in terms of a contour integral in the complex k plane.



For both $\omega > 0$ and $\omega < 0$, we close the contour in the upper half plane, since the infinite semicircle then makes no contribution (because $R > 0$). Then by the residue theorem, we find

$$G_{ret.}(\vec{r},\vec{r}\,') = -\frac{1}{\pi}\frac{1}{R}\frac{d}{dR}\, 2\pi i \frac{e^{i\frac{\omega}{c}R}}{2\frac{\omega}{c}}$$

$$= \frac{1}{R}e^{i\frac{\omega}{c}R}\,, \tag{41.18}$$

which is the retarded Green's function given in (41.13a).

Lecture 13.

41-3. Hankel Functions

We have thus established the equivalence between two representations for $G_{ret.}$, (41.2) and (41.16),

$$\frac{1}{r} e^{i\frac{\omega}{c}r} = 4\pi \int \frac{(d\vec{k})}{(2\pi)^3} \left. \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2} \right|_{\varepsilon\to+0} . \tag{41.19}$$

Note that if we now perform the integral on k_z, and use the integral representation for J_0, (41.9), we rederive the identity (41.10). [See Problem 21.] If, instead, we rewrite (41.19) in the form of a 1 + 2 dimensional decomposition,

$$\frac{1}{r} e^{i\frac{\omega}{c}r} = 4\pi \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} \frac{e^{i\vec{k}_\perp\cdot\vec{r}_\perp} e^{ik_z z}}{k_\perp^2 + k_z^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2} , \tag{41.20}$$

and first integrate over the angle associated with $\vec{k}_\perp$, we find

$$\frac{1}{r} e^{i\frac{\omega}{c}r} = 4\pi \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{ik_z z} \int_0^{\infty} \frac{\lambda d\lambda}{2\pi} \frac{J_0(\lambda\rho)}{\lambda^2 - \left(\left(\frac{\omega+i\varepsilon}{c}\right)^2 - k_z^2\right)} , \tag{41.21}$$

which, in the static limit, $\omega = 0$, we have already seen in (16.23).

If we substitute the representation (41.21) into the Green's function equation, (41.1), and use the Laplacian in cylindrical coordinates, (15.7), we immediately see that the λ-integral obeys Bessel's equation

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+\kappa^2\right]\int_0^\infty d\lambda\lambda\,\frac{J_0(\lambda\rho)}{\lambda^2-\kappa^2}=0\ , \tag{41.22}$$

for $\rho \neq 0$. We may also obtain this result explicitly from the differential equation satisfied by J_0, (15.8), since

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+\kappa^2\right]\int_0^\infty d\lambda\lambda\,\frac{J_0(\lambda\rho)}{\lambda^2-\kappa^2}$$

$$=\int_0^\infty d\lambda\lambda\,\frac{1}{\lambda^2-\kappa^2}\,(-\lambda^2+\kappa^2)\;J_0(\lambda\rho)=0\ ,\quad \text{for}\quad \rho\neq 0\ ,$$

where we have used a specialization of the orthonormality relation, (15.28a), with $\rho' = 0$,

$$\int_0^\infty d\lambda\lambda\; J_0(\lambda\rho)=\frac{1}{\rho}\,\delta(\rho)\ . \tag{41.23}$$

With the exception of the origin, here is another solution of Bessel's equation, which is called the zeroth order Hankel function of the first kind, defined by

$$\int_0^\infty d\lambda\lambda\,\frac{J_0(\lambda\rho)}{\lambda^2-\left[\left(\frac{\omega+i\varepsilon}{c}\right)^2-k_z^{\,2}\right]}=\frac{\pi i}{2}\,H_0^{(1)}\left(\sqrt{\frac{\omega^2}{c^2}-k_z^{\,2}}\;\rho\right)\ . \tag{41.24}$$

When $\omega/c < |k_z|$, the square root in (41.24) becomes imaginary, $i\sqrt{k_z^{\,2}-\frac{\omega^2}{c^2}}$, and in this case, the Hankel function becomes the modified Bessel function, (16.19),

$$\frac{\pi i}{2}\,H_0^{(1)}\left(\sqrt{\frac{\omega^2}{c^2}-k_z^{\,2}}\;\rho\right)\to K_0\left(\sqrt{k_z^{\,2}-\frac{\omega^2}{c^2}}\;\rho\right)\ . \tag{41.25}$$

In terms of the Hankel function defined by (41.24), the equality (41.21) reads

$$\frac{1}{r} e^{i\frac{\omega}{c}r} = \frac{i}{2}\int_{-\infty}^{\infty} dk_z\, e^{ik_z z}\, H_0^{(1)}\left(\sqrt{\frac{\omega^2}{c^2} - k_z^2}\,\rho\right)$$

$$= i\int_0^{\infty} dk_z\, \cos(k_z z)\, H_0^{(1)}\left(\sqrt{\frac{\omega^2}{c^2} - k_z^2}\,\rho\right) . \qquad (41.26)$$

The representation (41.10) has its ρ dependence expressed in terms of $J_0(\lambda\rho)$, which is a slowly damped oscillatory function, while the associated z dependence is either oscillatory or exponentially damped, depending on whether the square root is real or imaginary. On the other hand, the form (41.26) represents the function in terms of oscillatory functions of z, while the corresponding ρ dependence is either oscillatory or exponentially decreasing. Let us extract the Hankel function in (41.26) by taking the Fourier transform in z:

$$\int_{-\infty}^{\infty} dz\, \frac{e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2 + z^2}}\, e^{-ik_z z} = i\pi\, H_0^{(1)}\left(\sqrt{k^2-k_z^2}\,\rho\right) ,$$

where $k = \omega/c$. Then by setting $k_z = 0$, we obtain another integral representation for the Hankel function,

$$i\pi\, H_0^{(1)}(k\rho) = \int_{-\infty}^{\infty} dz\, \frac{e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} . \qquad (41.27)$$

We will now use this representation to find the asymptotic form of the Hankel function for $k\rho >> 1$. As we will justify a posterori, the main contribution of the integral comes from small values of z/ρ, so that we may expand the square root in the exponential as

$$\sqrt{\rho^2+z^2} \simeq \rho + \frac{z^2}{2\rho} ,$$

and approximate (41.27) by

$$\begin{aligned} i\pi\, H_0^{(1)}(k\rho) &\sim \frac{1}{\rho}\, e^{ik\rho} \int_{-\infty}^{\infty} dz\, e^{-\frac{k}{2i\rho} z^2} \\ &= i\sqrt{\frac{2\pi}{k\rho}}\, e^{i\left(k\rho - \frac{\pi}{4}\right)} . \end{aligned} \tag{41.28}$$

Here the Gaussian integral is evaluated as

$$\int_{-\infty}^{\infty} dz\, e^{-\frac{k}{2i\rho} z^2} = \sqrt{\pi\, \frac{2i\rho}{k}} . \tag{41.29}$$

The consistency of our approximation hinges on the fact that the dominant contribution to the integral, (41.29), comes from small values of z/ρ, for which

$$\frac{z}{\rho} \sim \sqrt{\frac{1}{k\rho}} \ll 1 .$$

As an application of (41.28), we may obtain the asymptotic behavior of J_0 by taking the imaginary part of (41.24):

$$J_0(k\rho) = \mathrm{Re}\, H_0^{(1)}(k\rho) \sim \sqrt{\frac{2}{\pi}\frac{1}{k\rho}}\, \cos\left(k\rho - \frac{\pi}{4} \right) . \tag{41.30}$$

This is a result derived in an alternative way in Problem 22.

XLII. SCATTERING BY SMALL OBSTACLES

When a plane electromagnetic wave interacts with a macroscopic dielectric, we talk of reflection and refraction. For the interaction of the wave with a small object, the appropriate word is scattering. Here, small refers to the size of the object in comparison to the wavelength of the radiation.

42-1. Thomson Scattering

As a first illustration, we consider a single charge e, of mass m, which accelerates due to an applied electric field such as is present in a light wave:

$$m \frac{d\vec{v}}{dt} = e\vec{E} , \tag{42.1}$$

(as long as $|\vec{v}/c| << 1$). An accelerated charge radiates, which gives rise to the "scattered" radiation; the power radiated is given by the Larmor formula, (29.24),

$$P_{scatt} = \frac{2}{3} \frac{e^2}{c^3} (\dot{\vec{v}})^2 = \frac{2}{3} \frac{e^2}{c^3} \left(\frac{e}{m} \vec{E} \right)^2 = \frac{2}{3} \frac{e^4}{m^2 c^3} E^2 . \tag{42.2}$$

This scattered power is to be compared to the incoming power per unit area of the plane wave, which is, in vacuum, given by the magnitude of the Poynting vector,

$$|\vec{S}| = \frac{c}{4\pi} |\vec{E} \times \vec{B}| = \frac{c}{4\pi} |\vec{E}|^2 . \tag{42.3}$$

The ratio of these two quantities defines the <u>cross section</u>,

$$\sigma = \frac{P_{scatt}}{|\vec{S}|} , \tag{42.4}$$

which is the effective area presented by the obstacle to the wave. For the situation considered here, the cross section is

$$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 , \tag{42.5}$$

the so-called Thomson cross section. For an electron, this cross section is conveniently expressed in terms of the length, r_0,

$$r_0 = \frac{e^2}{mc^2} = 2.8 \times 10^{-13} \text{ cm} , \tag{42.6}$$

which, for historical reasons only, goes by the name "classical radius of the electron." Numerically, the Thomson cross section for the electron is about

$$\sigma \approx \frac{2}{3} \times 10^{-24} \text{ cm}^2 . \tag{42.7}$$

What is the angular distribution of the radiation scattered by the charge? We recall the non-relativistic expression (29.22),

$$\frac{dP}{d\Omega} = \frac{1}{4\pi}\frac{e^2}{c^3}(\vec{n} \times \dot{\vec{v}})^2 , \tag{42.8}$$

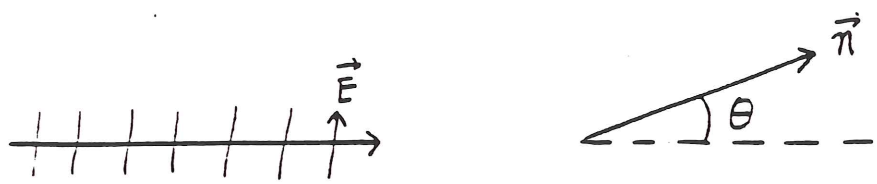
which gives the power radiated in the direction $\vec{n}$ per unit solid angle. Using (42.1), we rewrite this as

$$\frac{dP}{d\Omega} = \frac{1}{4\pi}\frac{e^4}{m^2c^3}(\vec{n} \times \vec{E})^2 ,$$

or, introducing $\vec{e}$ as a unit vector along the direction of $\vec{E}$,

$$\frac{dP}{d\Omega} = \frac{1}{4\pi}\frac{e^4}{m^2c^3}E^2[1 - (\vec{n}\cdot\vec{e})^2] . \tag{42.9}$$

There are two distinct possibilities here: If $\vec{e}$ is perpendicular to the scattering plane defined by $\vec{n}$ and the initial direction of propagation, then $\vec{n}\cdot\vec{e} = 0$. If $\vec{e}$ lies in that plane, then $|\vec{n}\cdot\vec{e}| = \sin\theta$, where, as the figure shows, θ is the scattering angle.



Therefore, for the two choices of $\vec{e}$, the factor in the scattered power is

$$1 - (\vec{n}\cdot\vec{e})^2 = \begin{cases} 1 \quad , & \vec{e} \perp \text{ scattering plane,} \\ \cos^2\theta \quad , & \vec{e} \parallel \text{ scattering plane .} \end{cases} \tag{42.10}$$

Notice that there is no scattering at $\theta = \pi/2$ for the second choice which just re-expresses the fact that there is no radiation emitted in the direction of the acceleration. If the incoming radiation is unpolarized, we have the average of the two possibilities given in (42.10), so that

$$1 - (\vec{n}\cdot\vec{e})^2 \to \frac{1 + \cos^2\theta}{2} \quad .$$

The differential cross section, the effective area for scattering into a given element of solid angle, is defined by

$$\frac{d\sigma}{d\Omega} = \frac{\dfrac{dP_{scatt}}{d\Omega}}{|\vec{S}|} \quad . \tag{42.11}$$

For an unpolarized wave, the differential Thomson cross section is then

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \frac{1 + \cos^2\theta}{2} \quad . \tag{42.12}$$

The total Thomson cross section, (42.5), is recovered when (42.12) is integrated over all solid angles,

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega \quad . \tag{42.13}$$

42-2. Scattering by a Bound Charge

The preceding subsection described scattering by a free particle. We now suppose, instead, that the "electron" is bound by a damped harmonic oscillator force. As we discussed in Subsection 5-2, this interaction is characterized by a natural frequency, ω_o, and a damping constant, γ. The corresponding equation of motion of the electron is given by (5.15):

$$m \frac{d^2\vec{r}}{dt^2} + m\omega_o^2 \vec{r} + m\gamma \frac{d\vec{r}}{dt} = e\vec{E} \quad . \tag{42.14}$$

Suppose the electric field is due to an incoming light wave of a definite frequency ω,

$$\vec{E}(t) = \text{Re}\ \vec{E}e^{-i\omega t} = \frac{1}{2}\left(\vec{E}e^{-i\omega t} + \vec{E}^* e^{i\omega t}\right) \quad , \tag{42.15}$$

so the deviation of the particle from the center of the harmonic force is given by (5.16):

$$\vec{r}(t) = \frac{e}{m}\ \text{Re}\ \frac{\vec{E}e^{-i\omega t}}{\omega_o^2 - \omega^2 - i\gamma\omega} \quad . \tag{42.16}$$

The corresponding radiated power,

$$P_{scatt} = \frac{2}{3}\frac{e^2}{c^3}(\dot{\vec{v}})^2 \tag{42.17}$$

involves

$$\ddot{\vec{r}} = -\frac{e}{m} \text{Re} \frac{\omega^2 \vec{E} e^{-i\omega t}}{\omega_o^2 - \omega^2 - i\gamma\omega} . \tag{42.18}$$

We are interested, not in the instantaneous scattered power, but rather in its time average over one cycle. For a typical time varying quantity,

$$A(t) = \text{Re}\, Ae^{-i\omega t} = \frac{1}{2}(Ae^{-i\omega t} + A^* e^{i\omega t}) ,$$

the time average of its square is

$$\overline{(A(t))^2} = \frac{1}{2}|A|^2 .$$

Thus, the time averaged power scattered is, from (42.17) and (42.18),

$$\overline{P} = \frac{2}{3}\frac{e^2}{c^3}\frac{1}{2}\frac{e^2}{m^2}\omega^4 \frac{|\vec{E}|^2}{(\omega_o^2-\omega^2)^2 + (\gamma\omega)^2} , \tag{42.19}$$

while the average incident energy flux is

$$\overline{|\vec{S}|} = \frac{c}{4\pi}\overline{E^2} = \frac{c}{4\pi}\frac{1}{2}|\vec{E}|^2 , \tag{42.20}$$

both expressed in terms of the complex amplitude $\vec{E}$. The resulting cross section is

$$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_o^2-\omega^2)^2 + (\gamma\omega)^2} . \tag{42.21}$$

This reduces to the Thomson cross section for a free particle when both ω_o and γ are zero. More realistically, the above cross section approaches the Thomson limit,

$$\sigma \to \sigma_{Th} \, , \tag{42.22}$$

when the frequency of light is large compared to the natural frequencies of the bound system,

$$\omega >> \omega_o, \gamma \, .$$

Physically, in this limit, the time scale is set by the period of the incident light; if this is very short, the electron does not have time to be influenced by internal forces so it behaves as though it were free.

In the opposite limit of very low frequencies,

$$\omega << \omega_o \, ,$$

the cross section (42.21) approaches

$$\sigma = \sigma_{Th} \frac{\omega^4}{\omega_o^4} \, , \tag{42.23}$$

which is called the Rayleigh cross section. In this domain, the highest frequencies are scattered the most strongly. This frequency behavior is presumably one of the reasons the sky is blue.

We have the situation of resonant scattering when $\omega = \omega_o$, at which point the cross section becomes

$$\sigma = \sigma_{Th} \frac{\omega_o^2}{\gamma^2} \, . \tag{42.24}$$

For small damping, $\omega_o >> \gamma$, (42.24) is much larger than the Thomson cross section. Of course, γ cannot be zero; energy is necessarily dissipated because, if for no other reason, the particle radiates due to its acceleration (recall Subsection 32-3). We will return to this issue shortly.

42-3. Scattering by a Dielectric Sphere

We do not require a microscopic description in order to calculate the scattering of electromagnetic waves by small objects. The scattering by such an object may be calculated from the dipole radiation formula (29.29),

$$\overline{P_{rad}} = \frac{2}{3}\frac{1}{c^3}\overline{(\ddot{\vec{d}})^2} \rightarrow \frac{2}{3}\frac{\omega^4}{c^3}\overline{d^2} \quad . \tag{42.25}$$

As an example, recall from (21.9) that a dielectric sphere, of radius a, in a static electric field $\vec{E}$ acquires a dipole moment

$$\vec{d} = \frac{\varepsilon-1}{\varepsilon+2}\, a^3\vec{E} \; . \tag{42.26}$$

This formula is applicable to a radiation field whenever $\vec{E}$ varies slowly on the scale set by the radius of the sphere, that is, if

$$\lambda\!\!\!^{-} >> a \; , \tag{42.27}$$

where $\lambda\!\!\!^{-} = c/\omega$. Then the radiated power is

$$\overline{P_{rad}} = \frac{2}{3}\frac{\omega^4}{c^3}\left(\frac{\varepsilon-1}{\varepsilon+2}\, a^3\right)^2 \overline{E^2} \tag{42.28}$$

so, by (42.20), the total cross section for scattering is

$$\sigma = \frac{8\pi}{3}\frac{1}{\lambda\!\!\!^{-4}}\left(\frac{\varepsilon-1}{\varepsilon+2}\, a^3\right)^2 \; . \tag{42.29}$$

Again we see the ω^4 dependence of the cross section characteristic of the scattering of long wavelength radiation.

42-4. Radiation Damping

We now return to the oscillator and consider the damping due to radiation. From (30.7b), the radiation reaction force on the charged particle can be identified from $P = -\vec{F}\cdot\vec{v}$ to be

$$\vec{F} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}} . \tag{42.30}$$

For a definite frequency, this becomes

$$-\frac{2}{3}\frac{e^2}{c^3}\omega^2\vec{v} = -m\gamma_r\vec{v}$$

where here γ_r is the radiative part of the dissipation, which must be present in order that energy be conserved. It coincides with the previously determined damping constant for radiation by a harmonic oscillator, (32.46), as long as the characteristic radiated frequencies are near the resonant frequency ω_o. Recognizing that there may be other forms of dissipation (for example, collisions), we write the total dissipation constant as

$$\gamma = \frac{2}{3}\frac{e^2}{mc^3}\omega^2 + \gamma_d$$

$$= \gamma_r + \gamma_d . \tag{42.31}$$

The energy removed from the incident electromagnetic field feeds both γ_r and γ_d. The rate of energy transfer from this field is, according to (42.16),

$$\overline{(e\vec{E}\cdot\vec{v})} = e\,\overline{\mathrm{Re}(Ee^{-i\omega t})\ \mathrm{Re}\left(\frac{e}{m}\frac{E(-i\omega)\,e^{-i\omega t}}{\omega_o^2 - \omega^2 - i\omega\gamma}\right)} . \tag{42.32}$$

Generically, the time average of such a product is given by

$$\overline{\operatorname{Re} A(t) \operatorname{Re} B(t)} = \overline{\frac{1}{2}\left(Ae^{-i\omega t} + A^* e^{i\omega t}\right) \frac{1}{2}\left(Be^{-i\omega t} + B^* e^{i\omega t}\right)}$$

$$= \frac{1}{4}\left(AB^* + A^* B\right) = \frac{1}{2} \operatorname{Re} A^* B \ ,$$

implying the power transferred to the oscillator,

$$\overline{(e\vec{E}\cdot\vec{v})} = \frac{e^2}{m} \frac{1}{2} |\vec{E}|^2 \operatorname{Re} \frac{-i\omega}{\omega_o^2 - \omega^2 - i\gamma\omega} \ .$$

Consequently, the total power removed from the incident field (not just scattered), is

$$\overline{P}_{tot} = \frac{e^2}{m} \frac{1}{2} |\vec{E}|^2 \frac{\gamma\omega^2}{(\omega_o^2 - \omega^2)^2 + (\gamma\omega)^2} \ , \tag{42.33}$$

which must exceed the scattered power,

$$\overline{P}_{tot} \geq \overline{P}_{scatt} \ . \tag{42.34}$$

The corresponding total cross section is

$$\frac{\overline{P}_{tot}}{\overline{|\vec{S}|}} = \sigma_{tot} = 4\pi \frac{e^2}{mc} \frac{\gamma\omega^2}{(\omega_o^2 - \omega^2)^2 + (\gamma\omega)^2} \ . \tag{42.35}$$

If we use (42.31),

$$\frac{e^2}{mc} \omega^2 = \frac{3}{2} c^2 \gamma_r \ ,$$

we can rewrite (42.35) as

$$\sigma_{tot} = 4\pi \frac{3}{2} c^2 \frac{\gamma_r \gamma}{(\omega_o^2-\omega^2)^2 + (\gamma\omega)^2} \quad . \tag{42.36}$$

At resonance, $\omega = \omega_o$, the total cross section becomes

$$\sigma_{tot}(\omega_o) = 4\pi \frac{3}{2} \left(\frac{c}{\omega_o} \right)^2 \frac{\gamma_r}{\gamma} \tag{42.37a}$$

and therefore if we write $c/\omega_o = \lambda\!\!\!^{-}_o$, and note that

$$\frac{\gamma_r}{\gamma} \leq 1 \quad ,$$

we have the inequality

$$\sigma_{tot}(\omega_o) \leq 6\pi \lambda\!\!\!^{-}_o{}^2 \quad . \tag{42.37b}$$

We now break up σ_{tot} into two pieces corresponding to the two channels for energy loss, given in (42.31)

$$\sigma_{tot} = \sigma_{scatt} + \sigma_{diss} \; , \tag{42.38}$$

where the scattering cross section is identified as

$$\begin{aligned} \sigma_{scatt} &= 4\pi \frac{3}{2} c^2 \frac{(\gamma_r)^2}{(\omega_o^2-\omega^2)^2 + (\gamma\omega)^2} \\ &= \frac{8\pi}{3} \left(\frac{e^2}{mc^2} \right)^2 \frac{\omega^4}{(\omega_o^2-\omega^2)^2 + (\gamma\omega)^2} \end{aligned} \tag{42.39}$$

[which is the same result found in (42.21)], while the dissipation cross section is

$$\sigma_{diss} = 4\pi \frac{3}{2} c^2 \frac{\gamma_r \gamma_d}{(\omega_o^2-\omega^2)^2 + (\gamma\omega)^2} . \tag{42.40}$$

If we ignore γ_d, and consider the high frequency limit $\omega >> \omega_o$, the scattering cross section becomes

$$\sigma_{scatt} = \sigma_{Th} \frac{1}{1 + \left(\frac{\gamma}{\omega}\right)^2} = \frac{\sigma_{Th}}{1 + \left(\frac{2}{3}\frac{e^2}{mc^2}\frac{1}{\lambda\!\!\!^{-}}\right)^2} . \tag{42.41}$$

If $\lambda\!\!\!^{-} >> e^2/mc^2$, the correction to the cross section is very small. But if we were to take literally the limit of very short wavelengths,

$$\lambda\!\!\!^{-} << \frac{e^2}{mc^2} ,$$

the cross section would behave as

$$\sigma_{scatt} \sim \frac{\frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2}{\left(\frac{2}{3}\frac{e^2}{mc^2}\frac{\omega}{c}\right)^2} = 6\pi\lambda\!\!\!^{-2} \tag{42.42}$$

which has the same form as the upper bound (42.37). This limit is completely unbelievable, however, since quantum effects become significant already at the considerably larger distance

$$\frac{\hbar}{mc} = 137 \frac{e^2}{mc^2} . \tag{42.43}$$

Lecture 14

XLIII. DIFFRACTION I

We now turn to the consideration of the scattering of electromagnetic radiation by objects large compared to the wavelength. We begin with a scatterer, consisting of a perfectly conducting screen, with a (macroscopic) hole in it. As we shall see, the predictions of geometrical optics are altered because of the finite wavelength of the radiation. This phenomenon is called diffraction. We first derive a general expression for the diffracted electric field in terms of its boundary values. Although it is an exact expression, we will apply it to three examples of near forward scattering.

43-1. Diffracted Electric Field

Away from the sources, in vacuum ($\varepsilon = \mu = 1$), the pertinent Maxwell equations are

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\dot{\vec{E}}, \quad \vec{\nabla}\cdot\vec{E} = 0,$$
$$-\vec{\nabla} \times \vec{E} = \frac{1}{c}\dot{\vec{B}}. \tag{43.1}$$

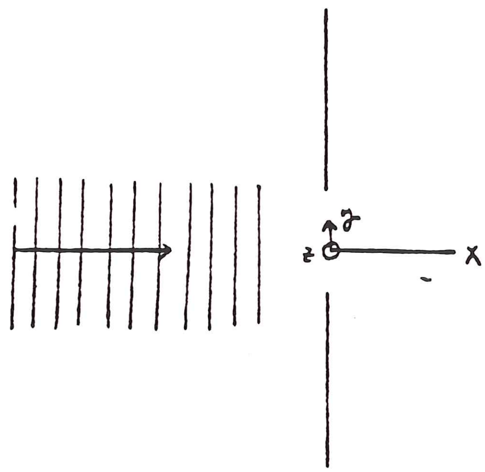
A consequence of these equations is that $\vec{E}$ satisfies the wave equation,

$$-\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) \equiv -\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) + \nabla^2\vec{E} = \frac{1}{c^2}\ddot{\vec{E}},$$

or

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0. \tag{43.2}$$

Specifically, we consider a plane polarized wave normally incident on a perfectly conducting screen, with an aperture. We choose the coordinate system (see figure below) such that x is in the direction of propagation and z is along the direction of polarization of the incident wave.



From our choice of coordinate system, far from the source, the incident electric field has only a z-component. For radiation of a particular frequency ω (with $k = \omega/c$), the wave equation for E_z, for example, becomes

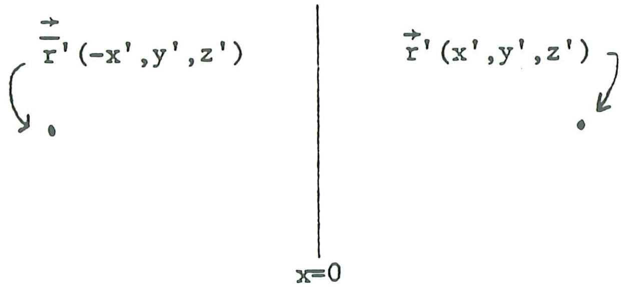
$$(\nabla^2+k^2)\, E_z = 0 \;. \tag{43.3}$$

We are interested in finding the electric field to the right of the screen ($x > 0$), subject to the boundary condition that the tangential electric field vanishes on the screen (since it is a perfect conductor).

The solution to the differential equation (43.3) can be expressed in terms of Green's function, for $x > 0$, which satisfies (28.22),

$$(\nabla^2+k^2)\, G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,') \;, \tag{43.4}$$

subject to the boundary condition that $G = 0$ on the surface $x = 0$. This Green's function refers to the field of a unit point charge in the presence of a perfect conductor lying in the entire $x = 0$ plane, as shown in the figure.



The solution in the region to the right of the conducting plane is the desired Green's function, which may be found by the method of images (cf. Sec. XIV). If the coordinates of the source point $\vec{r}'$ are (x',y',z'), the image point is located at

$$\vec{\bar{r}}' = (-x',y',z') \, , \tag{43.5}$$

in terms of which the solution to (43.4) is

$$G(\vec{r},\vec{r}') = \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} - \frac{e^{ik|\vec{r}-\vec{\bar{r}}'|}}{|\vec{r}-\vec{\bar{r}}'|} \, . \tag{43.6}$$

This is a generalization of the electrostatic Green's function (14.1).

To express the electric field in terms of this Green's function, we first multiply (43.3) by $G(\vec{r}',\vec{r})$ and (43.4) by $E_z(\vec{r}')$. Then subtracting the two resulting expressions, we obtain (dropping the z subscript on E_z for simplicity)

$$\begin{aligned} 4\pi E(\vec{r}')\,\delta(\vec{r}-\vec{r}') &= -E(\vec{r}')(\nabla'^2+k^2)\,G(\vec{r}',\vec{r}) + G(\vec{r}',\vec{r})(\nabla'^2+k^2)\,E(\vec{r}') \\ &\equiv \vec{\nabla}'\cdot[G(\vec{r}',\vec{r})\,\vec{\nabla}'E(\vec{r}') - E(\vec{r}')\,\vec{\nabla}'G(\vec{r}',\vec{r})] \, . \end{aligned} \tag{43.7}$$

The electric field to the right of the screen can now be obtained in terms of its boundary values by integrating (43.7) over the volume to the right of the plane $(x' > 0)$ and using the divergence theorem:

$$4\pi E(\vec{r}) = -\int_{\text{screen + aperture}} dS' \left[G(\vec{r}',\vec{r}) \frac{\partial}{\partial x'} E(\vec{r}') - E(\vec{r}') \frac{\partial}{\partial x'} G(\vec{r}',\vec{r}) \right]$$

$$+ \int_{S_\infty} dS' \left[G(\vec{r}',\vec{r}) \frac{\partial}{\partial r'} E(\vec{r}') - E(\vec{r}') \frac{\partial}{\partial r'} G(\vec{r}',\vec{r}) \right] . \qquad (43.8)$$

The first integral extends over the entire $x' = 0$ plane, the minus sign appearing because the outward normal to the volume is in the $-x'$ direction. The second integral is over a surface which can be taken to be a hemisphere with radius tending toward infinity. Far from the aperture, which acts as the source of the scattered wave, the leading behavior of the electric field, an outgoing spherical wave, is the same as that of the Green's function:

$$G(\vec{r}',\vec{r}) \sim \frac{e^{ikr'}}{r'} , \quad E(\vec{r}') \sim \frac{e^{ikr'}}{r'} . \qquad (43.9)$$

Then, the r' derivative can be effectively replaced by

$$\frac{\partial}{\partial r'} \to ik ,$$

so that the integral in (43.8) over the hemisphere at infinity vanishes. Physically, this follows from the fact that the source of the electric field is not at infinitely remote points, but at the aperture.

Now we incorporate the boundary conditions that $G = 0$ on the entire $x' = 0$ surface, while $E = 0$ on the surface $x' = 0$ except for the aperture (since we are talking about E_z here). Thus the electric field to the right of the screen is

$$4\pi E(\vec{r}) = \int_{\text{aperture}} dS' \, E(\vec{r}') \frac{\partial}{\partial x'} G(\vec{r}',\vec{r}) . \qquad (43.10)$$

The derivative of Green's function can be evaluated from (43.6) by noting that, for the first term,

$$\frac{\partial}{\partial x'} = -\frac{\partial}{\partial x} \quad ,$$

since $|\vec{r}\,'-\vec{r}|$ depends on x'-x, while in the second term,

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} \; ,$$

since $|\vec{r}\,'-\vec{\bar{r}}|$ depends on x'+x. Therefore at x' = 0, the two terms contribute equally. Of particular interest is the electric field far away from the aperture (kr >> 1, r >> r'), for which we may use the expansion (29.1) to write

$$\frac{1}{|\vec{r}-\vec{r}\,'|}\, e^{ik|\vec{r}-\vec{r}\,'|} \sim \frac{1}{r}\, e^{ikr}\, e^{-ik\vec{n}\cdot\vec{r}\,'} \; , \tag{43.11}$$

where $\vec{n}$ is the unit vector in the direction of observation, $\vec{r}$. If we note that

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \; , \tag{43.12}$$

the normal derivative of the Green's function is seen to be

$$\frac{\partial}{\partial x'}\, G(\vec{r}\,',\vec{r})\Big|_{x'=0} \sim \frac{-2ik}{r}\, \cos\theta\, e^{ikr}\, e^{-ik\vec{n}\cdot\vec{r}\,'} \; . \tag{43.13}$$

Therfore, far from the aperture, the electric field is

$$E(\vec{r}) \sim -\frac{ik}{2\pi}\, \cos\theta\, \frac{e^{ikr}}{r} \int_{\text{aperture}} dS'\, e^{-ik\vec{n}\cdot\vec{r}\,'}\, E(\vec{r}\,') \; . \tag{43.14}$$

This expression is valid for any shape of the aperture. This form holds for E_y as well as E_z, but for E_x one must integrate over the whole plane. In the following subsections, we will apply this result to discuss the diffraction by a circular hole, a slit, and a straight edge. We will do this in the approximation that the wavelength of the radiation is small compared to the dimensions of the aperture. Consequently, the wave travels mostly forward, with small angular deviation. We may further reasonably suppose that only E_z is present, and that it has very little z-dependence,

$$E_x \simeq 0\ , \quad E_y \simeq 0 \Longrightarrow \frac{\partial E_z}{\partial z} \simeq 0\ .$$

In the same approximation, we may assume that the field in the hole is just that of the incident wave,

$$E_{hole} \simeq E_{inc}\ , \tag{43.15}$$

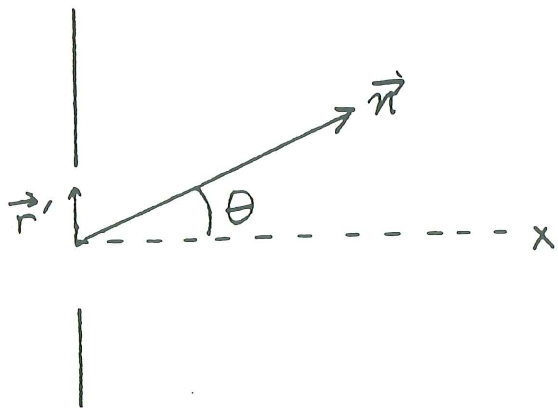
which is valid except near the edges. Since only relative amplitudes enter into our discussion, we take the normally incident wave to have unit amplitude, $E_{inc} = 1$.

43-2. Diffraction by a Circular Aperture

We here specialize the aperture to be of circular shape, having a radius a, with $\lambdabar \ll a$. Using polar coordinates with the origin at the center of the hole, we can then express the electric field to the right of the hole as ($\cos\theta \simeq 1$)

$$E \sim -ik\,\frac{e^{ikr}}{r}\int_0^a \rho\,d\rho \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{-ik\rho\sin\theta\cos\phi}\ , \tag{43.16}$$

where θ is the angle between $\vec{n}$ and the x-axis, and ϕ is the angle between $\vec{r}'$ and the projection of $\vec{n}$ on the plane defined by the hole.



Because the integration over ϕ is identified as J_0, according to (15.2), the electric field is proportional to

$$\int_0^a \rho d\rho\, J_0(k\rho \sin\theta) = \frac{1}{(k \sin\theta)^2} \int_0^{ka \sin\theta} z\, dz\, J_0(z) \ . \tag{43.17}$$

The remaining z-integration can be performed with the aid of Bessel's equation of zeroth order, (15.9),

$$\frac{d}{dz} [z \frac{d}{dz} J_0(z)] + z\, J_0(z) = 0 \ , \tag{43.18}$$

that is,

$$\int_0^{\bar{z}} dz\, z\, J_0(z) = -\bar{z} \frac{d}{d\bar{z}} J_0(\bar{z}) = \bar{z}\, J_1(\bar{z}) \ , \tag{43.19}$$

where we have used the recurrence relation

$$-\frac{d}{dz} J_0(z) = J_1(z) \ , \tag{43.20}$$

which follows from (36.29a) and (15.16). Thus the relative diffracted electric field is

$$E \sim -ik \frac{e^{ikr}}{r} \frac{a}{k \sin\theta} J_1(ka \sin\theta) ,$$

or, consistently making the small angle approximation, $\sin\theta \simeq \theta$,

$$E \sim -ik \frac{e^{ikr}}{r} a^2 \frac{J_1(ka\theta)}{ka\theta} . \tag{43.21}$$

The ratio of diffracted to incident energy fluxes is

$$\frac{|E|^2}{|E_{inc}|^2} = \frac{1}{r^2} (ka^2)^2 \left(\frac{J_1(ka\theta)}{ka\theta} \right)^2 .$$

which, when multiplied by r^2, yields the differential cross section,

$$\frac{d\sigma}{d\Omega} = (ka^2)^2 \left(\frac{J_1(ka\theta)}{ka\theta} \right)^2 . \tag{43.22}$$

We anticipate that the total cross section, in the small wavelength limit,

$$ka >> 1 , \quad \text{or} \quad k^{-1} = \lambda\!\!\!^{-} << a ,$$

will be

$$\sigma \simeq \pi a^2 , \tag{43.23}$$

since this is just the geometrical area of the circular hole, and we are in the domain of "geometrical" optics. This can be shown explicitly by integrating (43.22) over all solid angles. The element of solid angle is approximately

$$d\Omega = 2\pi \sin\theta \, d\theta \simeq 2\pi \, \theta \, d\theta ,$$

since physically the wave is mostly scattered near the forward direction. Mathematically, this last fact is evident because the important values of θ are those for which the argument of the Bessel function is of order one, that is,

$$\theta \sim \frac{1}{ka} = \frac{\lambda\!\!\!^{-}}{a} << 1 \,.$$

The total cross section is, therefore, approximately,

$$\sigma \simeq 2\pi \int_0^{\pi/2} d\theta\, \theta (ka^2)^2 \left(\frac{J_1(ka\theta)}{ka\theta} \right)^2$$

$$\simeq 2\pi\, a^2 \int_0^{ka\frac{\pi}{2} \to \infty} dz\, z \left(\frac{J_1(z)}{z} \right)^2 \,. \tag{43.24}$$

The expectation, (43.23), will be borne out if

$$\int_0^\infty \frac{dz}{z} (J_1(z))^2 = \frac{1}{2} \,. \tag{43.25}$$

is true.

To prove (43.25), we first need an integral representation for J_1^2. This can be achieved by starting from the addition theorem for Bessel's functions, (15.33),

$$J_0(\sqrt{z^2+z'^2-2zz'\cos\phi}) = \sum_{m=-\infty}^{\infty} J_m(z)\, J_m(z')\, e^{im\phi} \,, \tag{43.26}$$

setting $z = z'$,

$$J_0\left(2z \sin\frac{\phi}{2} \right) = \sum_{m=-\infty}^{\infty} [J_m(z)]^2\, e^{im\phi} \,, \tag{43.27}$$

and extracting the $m = 1$ harmonic:

$$\oint \frac{d\phi}{2\pi} e^{-i\phi} J_0\left(2z \sin\frac{\phi}{2} \right) = [J_1(z)]^2 \,. \tag{43.28}$$

We next insert this representation into the integral in (43.25). In order to be able to interchange the order of integration, we introduce a vanishingly small lower limit for z, $z_0 \to 0$, and obtain

$$\int_{z_0}^{\infty} \frac{dz}{z} [J_1(z)]^2 = \oint \frac{d\phi}{2\pi} e^{-i\phi} \int_{x_0}^{\infty} \frac{dx}{x} J_0(x) , \tag{43.29}$$

where z and x are related by

$$2z \left|\sin \frac{\phi}{2}\right| = x . \tag{43.30}$$

Because of the orthonormality properties of the functions $e^{im\phi}$, (15.18), we are only interested in the coefficient of $e^{i\phi}$ in the Fourier series expansion of the x integral. Since J_0 $(x << 1) \simeq 1$, the x integral is

$$\int_{2z_0|\sin \frac{\phi}{2}|}^{\infty} \frac{dx}{x} J_0(x) \simeq \log \frac{1}{2z_0 |\sin \frac{\phi}{2}|} + \text{constant} . \tag{43.31}$$

By means of the following expansion,

$$\log \frac{1}{|\sin \frac{\phi}{2}|} = \frac{1}{2} \log \frac{4}{(1-e^{-i\phi})(1-e^{i\phi})}$$

$$= \log 2 + \frac{1}{2} \left[\left(e^{-i\phi} + \frac{1}{2} e^{-2i\phi} + \dots \right) + \left(e^{i\phi} + \frac{1}{2} e^{2i\phi} + \dots \right)\right] ,$$

we obtain the coefficient of $e^{i\phi}$ to be 1/2, verifying the identity (43.25).

Let us turn to a discussion of the angular distribution given by (43.22). For such small angles that $ka\theta << 1$, we need to know the behavior of $J_1(z)$ for small z, which is easily inferred from the integral representation, (15.19),

$$J_1(z) = \frac{1}{2\pi i} \oint d\phi \, e^{iz \cos\phi} \, e^{-i\phi}$$

$$\simeq \frac{1}{2\pi i} \oint d\phi \; iz \, \frac{e^{i\phi}+e^{-i\phi}}{2} \, e^{-i\phi}$$

$$= \frac{z}{2} , \quad \text{for } z << 1 . \tag{43.32}$$

Therefore, very near to the forward direction, the differential cross section is

$$\frac{d\sigma}{d\Omega} \simeq \frac{1}{4} (ka^2)^2 = \frac{a^2}{4} \left(\frac{a}{\lambda\!\!\!^{-}} \right)^2 . \tag{43.33}$$

Comparing this with the total cross section, (43.23), we see that, since $a/\lambda\!\!\!^{-} >> 1$, the diffraction is mostly forward, consistent with our assumption.

For scattering angles such that $z = ka\theta >> 1$, we make use of the asymptotic behavior of $J_1(z)$, which can be obtained by using the integral representation, (15.19), with $\phi = \phi' - \pi/2$:

$$J_1(z) = \oint \frac{d\phi'}{2\pi} e^{i(z \sin\phi' - \phi')}$$

$$= \int_0^{\pi} \frac{d\phi'}{\pi} \cos(z \sin\phi' - \phi') . \tag{43.34}$$

For $z >> 1$, the main contribution to (43.34) comes from the region near the stationary phase point, $\phi' \simeq \frac{\pi}{2}$; therefore we reintroduce $\phi = \phi' - \pi/2$ to obtain

$$J_1(z) = \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} \cos(z \cos\phi - \phi - \pi/2)$$

$$\cong \mathrm{Re} \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} e^{i(z - \pi/2)} e^{-i \frac{z}{2} \phi^2} ,$$

where we have expanded $z \cos\phi - \phi$ about the stationary phase point, $\phi \simeq -\frac{1}{z}$.
Evaluating the Gaussian integral, we find (for $z \gg 1$)

$$J_1(z) \sim \mathrm{Re}\, e^{i(z - \pi/2)} \frac{1}{\pi} \sqrt{\frac{2\pi}{iz}}$$

$$= \sqrt{\frac{2}{\pi z}} \cos(z - \frac{\pi}{2} - \frac{\pi}{4}) \; . \qquad (43.35)$$

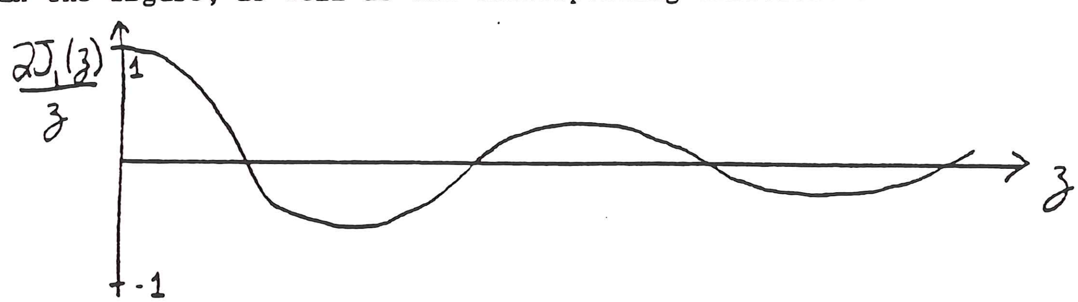
Alternatively, we can obtain this same result by using the recurrence relation (43.20) and the asymptotic form for J_0, (41.30),

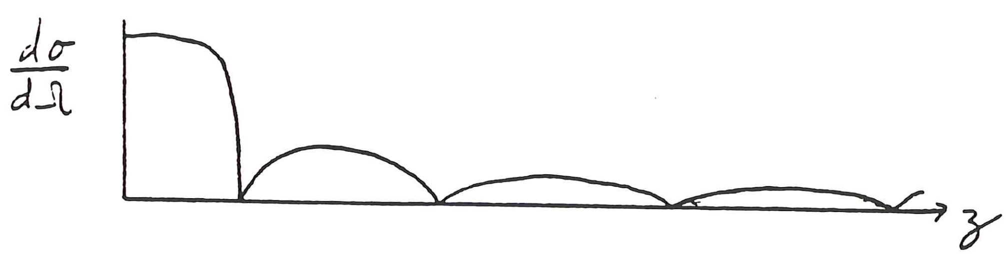
$$J_1(z) = -J_0'(z) \sim -\frac{d}{dz} \sqrt{\frac{2}{\pi z}} \cos(z - \pi/4)$$

$$\sim \sqrt{\frac{2}{\pi z}} \sin(z - \pi/4) \; . \qquad (43.36)$$

This asymptotic formula is already a good approximation for relatively small z.

The behavior of $2J_1(z)/z$, as suggested by (43.32) and (43.35), is sketched in the figure, as well as the corresponding differential cross section.





Here we see quite clearly a diffraction pattern, with minima occurring at the zeroes of $J_1(z)$, denoted by z_1. These zeroes can be found approximately from the asymptotic form (43.35). A comparison of these with the exact values is shown below for the first three ~~zeroes~~.

z_1 (asymptotic)	z_1 (exact)
$\frac{3\pi}{4} + \frac{\pi}{2} = 3.927$	3.832
$\frac{3\pi}{4} + \frac{3\pi}{2} = 7.069$	7.016
$\frac{3\pi}{4} + \frac{5\pi}{2} = 10.210$	10.173

Lecture 15

43-3. Diffraction by a Slit

We now consider another example of diffraction, in which the aperture is a slit running along the entire z-axis. Because of this symmetry in z, when the electric field has no initial z-dependence, it acquires none subsequently. Therefore, we have an essentially two-dimensional problem, for which the preceding analysis applies, leading to (43.10): [Note that we cannot use (43.11) because the slit is infinitely long.]

$$E(x,y) = \frac{1}{4\pi}\int_{x'=0} dy'dz'\ E(y')\ \frac{\partial}{\partial x'}\ G(\vec{r}',\vec{r})\ . \tag{43.37a}$$

The Green's function is given by (43.6), or explicitly,

$$G(\vec{r}',\vec{r}) = \frac{e^{ik[(z'-z)^2 + (\vec{r}_\perp{}'-\vec{r}_\perp)^2]^{1/2}}}{[(z'-z)^2 + (\vec{r}_\perp{}'-\vec{r}_\perp)^2]^{1/2}} - \frac{e^{ik[(z'-z)^2 + (\vec{r}_\perp{}'-\vec{\bar{r}}_\perp)^2]^{1/2}}}{[(z'-z)^2 + (\vec{r}_\perp{}'-\vec{\bar{r}}_\perp)^2]^{1/2}}$$

where

$$\vec{r}_\perp = (x,y)\ ,$$
$$\vec{\bar{r}}_\perp = (-x,y)\ .$$

The z' integral here is identif~~ied as~~ the Hankel function, (41.27),

$$\int_{-\infty}^{\infty} dz' \frac{e^{ik\sqrt{\rho^2 + (z'-z)^2}}}{\sqrt{\rho^2 + (z'-z)^2}} = i\pi H_0^{(1)}(k\rho) .$$

(The Hankel function typically emerges in two-dimensional problems.) Thus the electric field to the right of the slit is

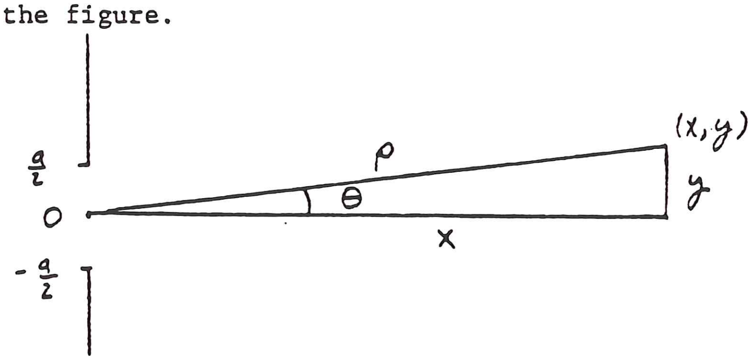
$$E(x,y) = \frac{i}{2}\int_{\text{slit}} dy'\, E(y')\left(-\frac{\partial}{\partial x}\right) H_0^{(1)}\left(k\sqrt{x^2 + (y-y')^2}\right) . \qquad (43.37b)$$

We will be concerned with the diffracted field far away from the slit, for which we can use the asymptotic form (41.28),

$$H_0^{(1)}\left(k\sqrt{x^2 + (y-y')^2}\right) \sim \sqrt{\frac{2}{i\pi k}}\, \frac{1}{[x^2 + (y-y')^2]^{1/4}}\, e^{ik[x^2 + (y-y')^2]^{1/2}} ,$$

$$(kx >> 1) . \qquad (43.38)$$

To represent diffraction by a slit of width a, we adopt the coordinate system shown in the figure.



For a finite slit, since a << x, as well as

$$y' \sim a << \rho = \sqrt{x^2 + y^2} ,$$

we make the following approximation:

$$\sqrt{x^2 + (y-y')^2} \simeq \sqrt{\rho^2 - 2yy'} \sim \rho - \frac{y}{\rho} y' = \rho - \sin\theta\, y' ,$$

where θ is the angle of diffraction. Correspondingly, the asymptotic form of the Hankel function becomes

$$H_0^{(1)}\left(k\sqrt{x^2+(y-y')^2}\right) \sim \sqrt{\frac{2}{i\pi k\rho}}\, e^{ik\rho}\, e^{-ik\sin\theta\, y'} . \tag{43.39}$$

[We see here again a remnant of the factor

$$e^{-ik\vec{n}\cdot\vec{r}'}$$

characteristic of radiation fields. See, for example, (43.13).] When the wavelength is small compared to the slit, $a \gg \lambda\!\!\!^{-}$, we may again approximate the field in the slit by the incident field,

$$E(y') \simeq E_{inc}(y') = 1 .$$

We also recognize that the radiation is predominately forward, implying

$$-\frac{\partial}{\partial x} e^{ik\rho} = -ik\frac{x}{\rho} e^{ik\rho}$$

$$\simeq -ik\, e^{ik\rho} .$$

Note that x/ρ was approximated as 1

Thus, for $x \gg a \gg \lambda\!\!\!^{-}$, the field, (43.37b), may then be approximated by

$$E(x,y) \sim \sqrt{\frac{i}{2\pi k\rho}} \int_{-a/2}^{a/2} dy'(-ik)\, e^{ik\rho}\, e^{-ik\sin\theta\, y'}$$

$ak = 1$

$$= \sqrt{\frac{i}{2\pi k\rho}}\,(-ik)\, e^{ik\rho}\, \frac{2\sin\left(\frac{ka}{2}\sin\theta\right)}{k\sin\theta} , \tag{43.40}$$

representing an outgoing cylindrical wave.

The differential cross section per unit length of the slit is determined by

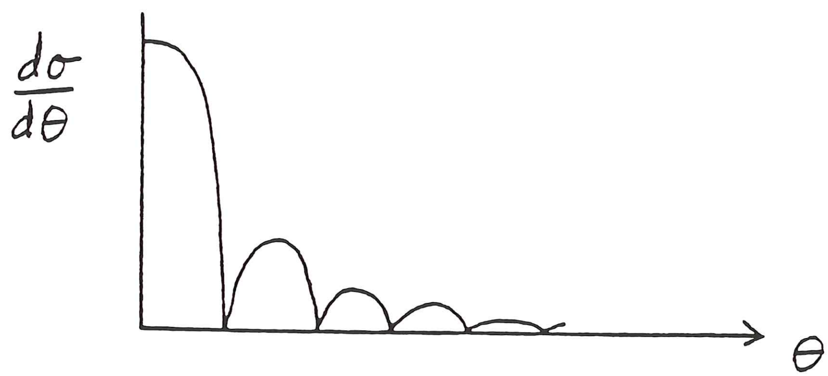
$$d\sigma = \frac{|E|^2}{|E_{inc}|^2} \rho d\theta ,$$

which, upon use of the asymptotic field, implies

$$\frac{d\sigma}{d\theta} = \frac{2}{\pi k} \left(\frac{\sin\left(\frac{ka}{2} \sin\theta \right)}{\sin\theta} \right)^2$$

$$\simeq \frac{2}{\pi k} \left(\frac{\sin\left(\frac{ka\theta}{2} \right)}{\theta} \right)^2 . \tag{43.41}$$

The latter holds for $\theta \ll 1$, which is the only region of validity for our result. The resulting diffraction pattern is shown in the figure,



where the zeroes occur at equally spaced points:

$$ka\theta = n\, 2\pi , \quad n = \pm 1, \pm 2, \ldots . \tag{43.42}$$

At $\theta = 0$, the differential cross section is

$$\left(\frac{d\sigma}{d\theta} \right)_{\theta=0} = \frac{1}{2\pi k} (ka)^2 . \tag{43.43}$$

In the short wavelength limit, we anticipate that the total cross section per unit length is given by the width of the slit. In fact, for ka >> 1, most of the contribution arises from values of θ near zero, so that we indeed have

$$\sigma \simeq \int_{-\infty}^{\infty} d\theta \, \frac{2}{\pi k} \left(\frac{\sin\left(\frac{ka}{2}\theta\right)}{\theta} \right)^2$$

$$= \frac{2}{\pi k} \frac{ka}{2} \int_{-\infty}^{\infty} dz \left(\frac{\sin z}{z} \right)^2 = a , \tag{43.44}$$

where we have used the integral (31.12).

As a check of consistency, we now turn to the limit $a \to \infty$, for which the slit disappears and the incident plane wave should propagate undisturbed. The field far away from the "slit" is given by (43.37b) and (43.38),

$$E(x,y) = \frac{i}{2} \int dy' \, E(y') \left(-\frac{\partial}{\partial x} \right) \left[\left(\frac{2}{i\pi k} \right)^{1/2} [x^2 + (y-y')^2]^{-1/4} \right.$$

$$\left. \times \, e^{ik\sqrt{x^2 + (y-y')^2}} \right] . \tag{43.45}$$

If we put $E(y') = 1$ for all y', we must recover the incident plane wave, $E(x,y) = e^{ikx}$. We expect that $y' \sim y$ gives the major contribution to the integral, since the field should just advance with constant phase in y, thereby allowing us to use

$$\sqrt{x^2 + (y-y')^2} \simeq x + \frac{(y-y')^2}{2x} . \tag{43.46}$$

Inserting this approximation into (43.45), we have the following asymptotic evaluation ($x >> \lambdabar$):

$$E(x,y) \sim \frac{i}{2}\int_{-\infty}^{\infty} dy'\left(-\frac{\partial}{\partial x}\right)\left[\left(\frac{2}{\pi ikx}\right)^{1/2} e^{ikx}\, e^{ik(y-y')^2/2x}\right]$$

$$\sim \frac{-i}{2}\int_{-\infty}^{\infty} dy'\left(\frac{2}{\pi ikx}\right)^{1/2} ik\, e^{ikx}\, e^{ik(y-y')^2/2x}$$

$$= \sqrt{\frac{k}{2\pi ix}}\, e^{ikx}\int_{-\infty}^{\infty} dy'\, e^{ik(y-y')^2/2x} = e^{ikx}\,, \tag{43.47}$$

as is expected. Here we note, as a check of the approximation $|y-y'| << x$, that the significant contributions to the Gaussian integral come from the values of y satisfying

$$\frac{|y-y'|}{x} \sim \sqrt{\frac{\lambda\!\!\!^{-}}{x}} << 1\,. \tag{43.48}$$

43-4. Diffraction by a Straight Edge

Finally, we consider the diffraction produced by a semi-infinite plane conductor, lying in the region defined by $x = 0$ and $y < 0$. The field produced by such a half plane conductor can be again described by (43.45), where, as a first approximation, we take

$$E(y') \simeq E_{inc} = 1\,, \qquad y' > 0\,,$$

$$E(y') \simeq 0\,, \qquad y' < 0\,. \tag{43.49}$$

Using the approximation (43.46), we arrive at the expression

$$E(x,y) \sim \sqrt{\frac{k}{2\pi ix}}\, e^{ikx}\int_{0}^{\infty} dy'\, e^{-\frac{k}{2ix}(y-y')^2}\,, \tag{43.50}$$

which is valid for

$$x >> \lambda\!\!\!^{-} \quad \text{and} \quad |y-y'| << x\,.$$

It is convenient to shift the origin of the y' integration:

$$E(x,y) \sim \sqrt{\frac{k}{2\pi ix}}\, e^{ikx} \int_{-\infty}^{y} dy'\, e^{-\frac{k}{2ix} y'^2}$$

$$= \sqrt{\frac{k}{2\pi ix}}\, e^{ikx} \left(\int_{-\infty}^{\infty} - \int_{y}^{\infty} \right) dy'\, e^{-\frac{k}{2ix} y'^2}$$

$$= e^{ikx} \left\{ 1 - \sqrt{\frac{k}{2\pi ix}} \int_{y}^{\infty} dy'\, e^{-\frac{k}{2ix} y'^2} \right\} . \qquad (43.51)$$

For sufficiently large y,

$$y >> \sqrt{\lambda\!\!\!^{-} x} \ , \qquad (43.52)$$

the second term in (43.51) may be neglected, and the wave travels undisturbed:

$$E(x,y) \sim e^{ikx} \ . \qquad (43.53)$$

On the other hand, for $y = 0$ the integral in (43.51) is

$$\sqrt{\frac{k}{2\pi ix}} \int_{0}^{\infty} dy'\, e^{-\frac{ky'^2}{2ix}} = \frac{1}{2} \ ,$$

and, in consequence, the amplitude of the wave there is reduced by half:

$$E(x,0) \sim \frac{1}{2}\, e^{ikx} \ . \qquad (43.54)$$

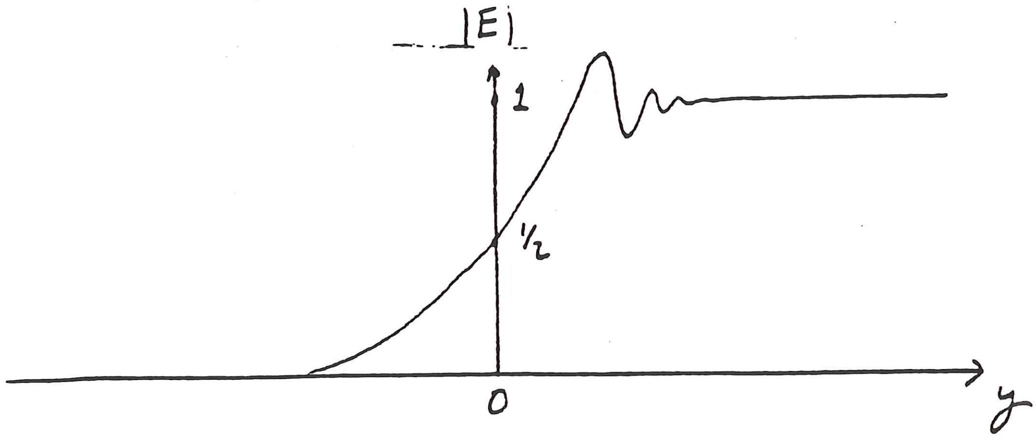
Far below the edge,

$$y < 0 \ , \quad |y| >> \sqrt{\lambda\!\!\!^{-} x} \ , \qquad (43.55)$$

the diffracted field vanishes,

$$E(x,y) \sim 0 \ . \qquad (43.56)$$

To summarize the above results, we present a rough picture of the magnitude of the field strength as a function of y:



The region over which the intensity varies significantly has width

$$\Delta y \sim \sqrt{\lambda\!\!\!^{-} x} \;, \tag{43.57}$$

a distance small compared to x,

$$\frac{\Delta y}{x} \sim \sqrt{\frac{\lambda\!\!\!^{-}}{x}} << 1 \;. \tag{43.58}$$

A quite different limit of diffraction by a straight edge occurs when both x and y are large, while the diffraction angle θ is fixed:

$$\frac{y}{x} = \tan\theta \;. \tag{43.59}$$

We anticipate that the dominant contribution to the scattered field comes from the region near the edge, where y' is small, and in consequence

$$\sqrt{x^2 + (y-y')^2} \simeq \rho - y' \sin\theta \;, \tag{43.60}$$

where

$$\rho = \sqrt{x^2+y^2} \;.$$

Using this approximation in (43.45), we obtain for $\theta << 1$,

$$E_{scatt} \sim \sqrt{\frac{i}{2\pi k\rho}}\,(-ik)\,e^{ik\rho}\int_0^{\infty} dy'\, e^{-iky'\sin\theta}$$

$$\sim -\sqrt{\frac{i}{2\pi k\rho}}\,\frac{e^{ik\rho}}{\theta}\;. \tag{43.61}$$

We see the appearance of a cylindrical wave, originating from the edge, $x' = 0,\ y' = 0$. The corresponding differential scattering cross section per unit length is

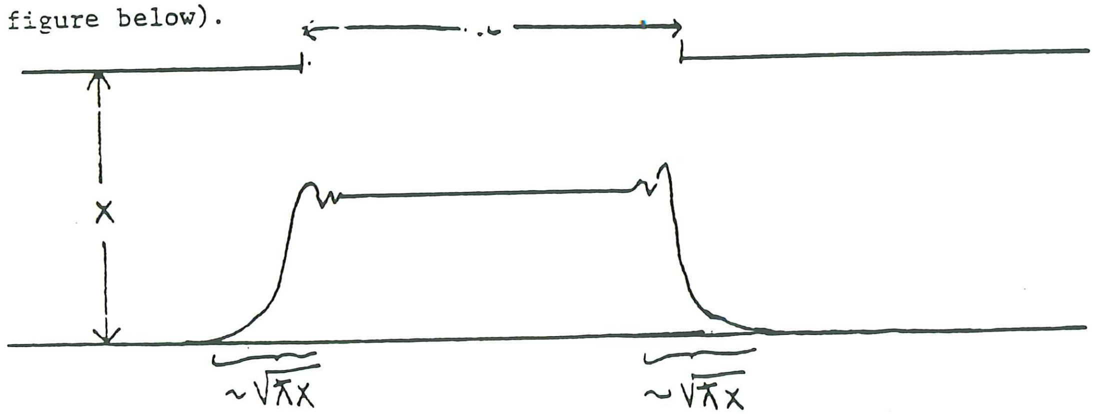
$$\frac{d\sigma}{d\theta} = \frac{|E_{scatt}|^2}{|E_{inc}|^2}\,\rho \simeq \frac{1}{2\pi k}\,\frac{1}{\theta^2}\;. \tag{43.62}$$

[In the next section we will provide an exact treatment of this diffraction problem.]

The above discussion of diffraction by an edge provides a clarification of our earlier consideration of diffraction by a slit. If we make observations by use of a screen sufficiently close to the slit, so that

$$a \gg \sqrt{\bar{\lambda} x}\;, \tag{43.63}$$

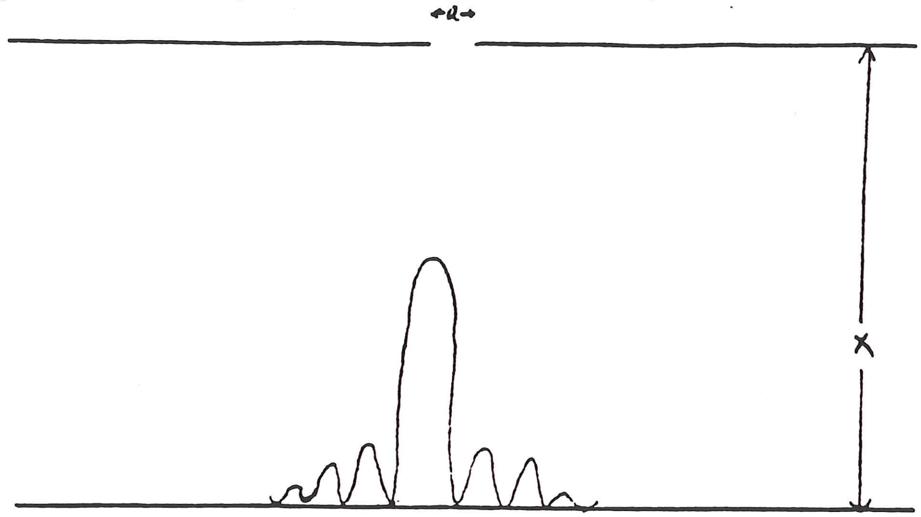
we see a geometrical image of the slit modulated by edge diffraction (see figure below).



However, if we move the screen so far back that

$$\sqrt{\bar{\lambda} x} \gg a \ , \tag{43.64}$$

the two patterns overlap to form the diffraction pattern derived in (43.41).



Mathematically, the origin of the requirement (43.64) can be traced to the fact that the expansion (43.60) is only valid when terms of order a^2/ρ are negligible, that is

$$\frac{ka^2}{\rho} = \frac{1}{\bar{\lambda}}\frac{a^2}{\rho} \ll 1 \ , \tag{43.65}$$

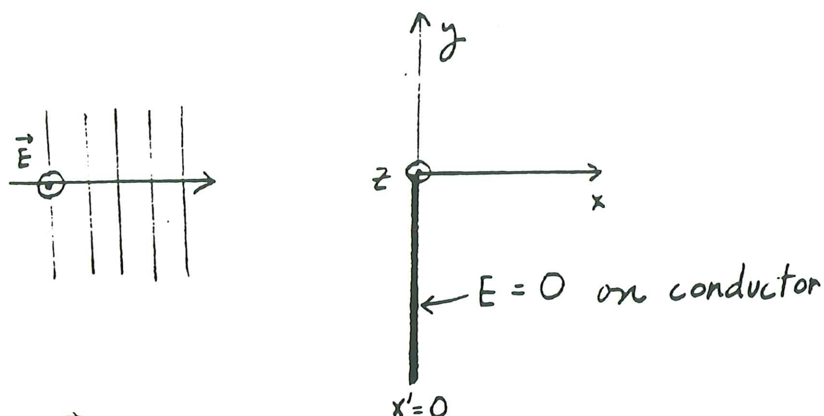
or

$$\bar{\lambda}\rho \gg a^2 \ . \tag{43.66}$$

Lecture 16

XLIV. DIFFRACTION II

We now adopt a more physical approach to diffraction in which the currents that give rise to the scattered wave are made explicit. We will reconsider diffraction by a semi-infinite metal conductor, for which this method is capable of giving an exact solution. The geometry of the situation is as given in the figure.



We consider $\vec{E}$ to possess only a z-component (z subscript suppressed), and decompose the electric field into incident and scattered parts,

$$E = E_{inc} + E_{scatt} \, . \tag{44.1}$$

We assume the incident field is a normalized plane wave with frequency $\omega = kc$,

$$E_{inc} = e^{ikx} \, . \tag{44.2}$$

The scattered field arises from the induced current that flows on the metal plate, which by symmetry can have only a z-component, $J = J_z$, and has no dependence on z,

$$\frac{\partial}{\partial z} J_z(x,y) = 0 \, , \tag{44.3}$$

in conformity with the properties of the incident electric field. Consequently, there is no charge density and the scattered electric field is expressed by

$$E_{\text{scatt}} = ik \int (d\vec{r}\,') \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \frac{1}{c} J(\vec{r}\,') \,, \qquad (44.4)$$

which follows from (32.6) and (32.11a). We consider an infinitesimally thin conductor for which we may reduce the three-dimensional integral in (44.4) to a two-dimensional one by introducing the surface current,

$$\int dx'\, J_z = K \,. \qquad (44.5)$$

Further, by identifying the z integral with the representation of the Hankel function (41.27), we may express the result as an integration over y' alone:

$$E = E_{\text{inc}} + ik \int_{-\infty}^{0} dy'\, \pi i\, H_0^{(1)}[k\sqrt{x^2 + (y-y')^2}]\, \frac{1}{c} K(y') \,. \qquad (44.6)$$

For the semi-infinite perfect conductor being considered here, (44.6) is subject to the boundary condition

$$E = 0 \quad \text{for} \quad x = 0,\ y < 0 \,. \qquad (44.7)$$

44-1. Approximate Solution

Before finding an exact solution to (44.6), we will first solve this equation by an approximate treatment, based on the fact that, on the conductor and far from the edge, the conducting sheet appears to be infinite. There, the incident wave is totally reflected,

$$E_z = e^{ikx} - e^{-ikx} ,$$

$$B_y = -e^{ikx} - e^{-ikx} , \tag{44.8}$$

which we will assume hold for all $x < 0$, $y < 0$, and that the fields vanish for $x > 0$, $y < 0$. From these forms for the electric and magnetic fields, we can find the induced current through the use of Maxwell's equation

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \dot{\vec{E}} + \frac{4\pi}{c} \vec{J} , \tag{44.9}$$

which becomes here

$$\frac{4\pi}{c} J_z = \partial_x B_y - \frac{1}{c} \dot{E}_z . \tag{44.10}$$

By integrating (44.10) across the conducting surface from just to the left to just to the right,

$$B_y(x = +0) - B_y(x = -0) = \frac{4\pi}{c} K ,$$

and noting that, in our approximation, (44.8),

$$B_y(x = +0) = 0 , \quad B_y(x = -0) = -2 , \tag{44.11}$$

we find the current appropriate to an infinite conducting sheet to be

$$K = \frac{c}{2\pi} . \tag{44.12}$$

Then from (44.6), the electric field everywhere is approximately given by

$$E = e^{ikx} - \frac{k}{2} \int_{-\infty}^{0} dy' \, H_0^{(1)}[k\sqrt{x^2 + (y-y')^2}] . \tag{44.13}$$

If we further use the asymptotic form (43.38) for the Hankel function, together with the approximation that when $|x|$ is large, only small values of $|y-y'|$ are significant,

$$\sqrt{x^2 + (y-y')^2} \sim |x| + \frac{(y-y')^2}{2|x|} ,$$

the expression for the electric field (44.13), becomes

$$E \sim e^{ikx} - \sqrt{\frac{k}{2\pi i|x|}}\, e^{ik|x|} \int_{-\infty}^{0} dy'\, e^{ik(y-y')^2/2|x|} . \qquad (44.14)$$

With the substitution $y'-y \rightarrow y'$, the integral in (44.14) has the form

$$\int_{-\infty}^{-y} dy'\, e^{ik\, y'^2/2|x|} . \qquad (44.15)$$

When $y > 0$ and far away from the edge, $y \gg \sqrt{\lambda\!\!\!^{-}|x|}$, the integral (44.15) is negligible, and the wave propagates undisturbed,

$$E \sim e^{ikx} .$$

On the other hand, sufficiently below the edge, $y < 0$, $|y| \gg \sqrt{\lambda\!\!\!^{-}|x|}$, the integral is $\sqrt{\frac{2\pi i}{k}|x|}$, and

$$E \sim e^{ikx} - e^{ik|x|}$$

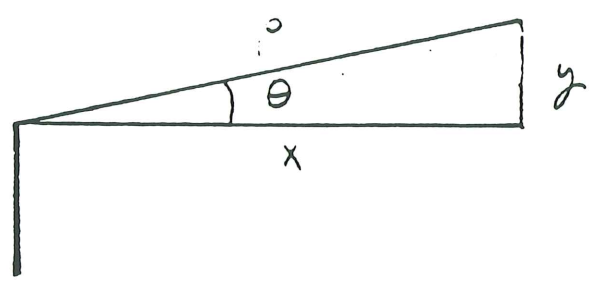
$$= \begin{cases} 0 , & x > 0 , \\ e^{ikx} - e^{-ikx} , & x < 0 , \end{cases}$$

reproducing the boundary condition (44.8). On line with the edge, $y = 0$, the electric field is

$$E \sim e^{ikx} - \frac{1}{2} e^{ik|x|} .$$

These results are identical with those found earlier in Subsection 43-4, but now a physical picture has been provided: The total electric field is the sum of the incoming field plus the field produced by the currents induced in the metal.

We next turn to the limit of large x and y where the angle of diffraction θ is fixed.



In this limit, the dominant contribution at (x,y) is the sum of the incident wave plus the wave scattered by the edge. In terms of the description employing currents, this is given approximately by (44.13), with

$$\sqrt{x^2 + (y-y')^2} \simeq \rho - y' \sin\theta .$$

As in (43.61), the scattered field is

$$E_{scatt} \sim -\frac{k}{2} \int_{-\infty}^{0} dy' \sqrt{\frac{2}{\pi k\rho i}}\, e^{ik\rho}\, e^{-ik \sin\theta y'}$$

$$\sim -\sqrt{\frac{i}{2\pi k}} \frac{1}{\sqrt{\rho}} e^{ik\rho} \frac{1}{\sin\theta} , \qquad (44.16)$$

leading to a differential cross section per unit length (valid only for $\theta \ll 1$)

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k}\frac{1}{\theta^2}, \tag{44.17}$$

in agreement with (43.62).

44-2. Exact Solution for Current

We now seek an exact solution to (44.6) subject to the boundary condition (44.7). This is an integral equation since $K(y')$ and $E(y)$ are interrelated functions. To find the surface current, we consider the electric field on the $x = 0$ plane,

$$E(y) = 1 - \pi k \int_{-\infty}^{\infty} dy'\, H_0^{(1)}[k|y-y'|]\, \frac{1}{c} K(y') . \tag{44.18}$$

In order to solve this equation, we must recognize that an incident plane wave is an over-idealization, one that can be removed by introducing an exponential cutoff in y:

$$E_{inc}(x = 0) = 1 \to e^{-\varepsilon|y|}, \quad \varepsilon \to +0 . \tag{44.19}$$

Note in (44.18) that we have two conditions:

1. $K(y') = 0$ if $y' > 0$ (since there is no conductor there),
2. $E(y) = 0$ if $y < 0$ (since the conductor is perfect).

We have introduced an infinite range of integration in (44.18) in order to employ Fourier transforms:

$$E(\zeta) = \int_{-\infty}^{\infty} dy\, e^{-i\zeta y}\, E(y) ,$$

$$K(\zeta) = \int_{-\infty}^{\infty} dy'\, e^{-i\zeta y'}\, K(y') . \tag{44.20}$$

$$-\frac{i}{\zeta - i\varepsilon} \qquad + \frac{i}{\zeta + i\varepsilon}$$

The Fourier transform of the incident field on the surface is

$$\int_{-\infty}^{\infty} dy\, e^{-i\zeta y}\, e^{-\varepsilon|y|} = -\frac{i}{\zeta - i\varepsilon} + \frac{i}{\zeta + i\varepsilon}\,, \qquad 2\pi i(-i + i)^{-0} \tag{44.21}$$

which, as expected, becomes $2\pi\delta(\zeta)$ as $\varepsilon \to 0$. Furthermore, we require the Fourier transform of the Hankel function. Starting from its integral representation, (41.27), using the three-dimensional Fourier transform of Green's function, (41.16), and then integrating over z, we have

$$\begin{aligned}
\pi i H_0^{(1)}(k|y-y'|) &= \int_{-\infty}^{\infty} dz\, \frac{e^{ik\sqrt{(y-y')^2+z^2}}}{\sqrt{(y-y')^2+z^2}} \\
&= 4\pi \int_{-\infty}^{\infty} dz \int \frac{dk_x\, dk_y\, dk_z}{(2\pi)^3}\, \frac{e^{ik_y(y-y')}\; e^{ik_z z}}{k_x^2 + k_y^2 + k_z^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2} \\
&= 4\pi \int \frac{dk_x\, dk_y}{(2\pi)^2}\, \frac{e^{ik_y(y-y')}}{k_x^2 + k_y^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2}\,.
\end{aligned}$$

Employing the simple contour integral

$$\int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \frac{1}{k_x^2 + k_y^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2} = \frac{i}{2\sqrt{(k+i\varepsilon)^2 - k_y^2}}\,, \qquad k = \frac{\omega}{c}\,,$$

(where the cuts are chosen not to cross the real k_y axis) we arrive at the following representation for the Hankel function,

$$\pi i H_0^{(1)}(k|y-y'|) = 2\pi i \int_{-\infty}^{\infty} \frac{dk_y}{2\pi}\, \frac{e^{ik_y(y-y')}}{\sqrt{(k+i\varepsilon)^2 - k_y^2}}\,, \tag{44.22}$$

which supplies the desired Four~~ier transform~~

$$\int_{-\infty}^{\infty} d(y-y')\, e^{-i\zeta(y-y')}\, H_0^{(1)}(k|y-y'|) = \frac{2}{\sqrt{(k+i\varepsilon)^2-\zeta^2}} \quad . \tag{44.23}$$

We now find for the Fourier transform of the integral equation (44.18),

$$E(\zeta) = -\frac{i}{\zeta-i\varepsilon} + \frac{i}{\zeta+i\varepsilon} - \pi k \frac{2}{\sqrt{(k+i\varepsilon)^2-\zeta^2}} \frac{1}{c} K(\zeta) \tag{44.24}$$

where, if we make explicit the regions in which the integrands are non-zero,

$$E(\zeta) = \int_0^{\infty} dy\, e^{-i\zeta y}\, E(y) \quad , \tag{44.25a}$$

$$K(\zeta) = \int_{-\infty}^{0} dy'\, e^{-i\zeta y'}\, K(y') \quad . \tag{44.25b}$$

If these integrals exist for real ζ, they will also exist for complex values of ζ. Anticipating that $E(y)$, $K(y)$ fall off like $e^{-\varepsilon|y|}$ as $|y| \to \infty$, we see that

$$\begin{aligned} &E(\zeta) \text{ exists for } \operatorname{Im}\zeta < \varepsilon \quad , \\ &K(\zeta) \text{ exists for } \operatorname{Im}\zeta > -\varepsilon \ . \end{aligned} \tag{44.26}$$

We will call the half planes $\operatorname{Im}\zeta < \varepsilon$, $\operatorname{Im}\zeta > -\varepsilon$ the lower half plane (LHP) and upper half plane (UHP), respectively. It is essential to observe that the UHP and the LHP overlap in a strip,

$$-\varepsilon < \operatorname{Im}\zeta < \varepsilon \ . \tag{44.27}$$

Our physical requirements of boundedness ensure that

$E(\zeta)$ is regular in the LHP ,

$K(\zeta)$ is regular in the UHP .

In order to examine clearly the analytic properties of (44.24), we multiply it by $\sqrt{k+i\varepsilon-\zeta}$:

$$\sqrt{k+i\varepsilon-\zeta}\; E(\zeta) = -i\,\frac{\sqrt{k+i\varepsilon-\zeta}}{\zeta-i\varepsilon} + i\,\frac{\sqrt{k+i\varepsilon-\zeta}}{\zeta+i\varepsilon} - 2\pi k\,\frac{\frac{1}{c}\,K(\zeta)}{\sqrt{k+i\varepsilon+\zeta}} \;. \tag{44.28}$$

The factors in (44.28) can be chosen to be regular in the following regions,

$\sqrt{k+i\varepsilon-\zeta}$: LHP ,

$\sqrt{k+i\varepsilon+\zeta}$: UHP ,

$\sqrt{k+i\varepsilon-\zeta}/(\zeta-i\varepsilon)$: LHP ,

$\sqrt{k+i\varepsilon-\zeta}/(\zeta+i\varepsilon)$: $-\varepsilon < \mathrm{Im}\,\zeta < \varepsilon$.

The last combination can be written as the sum of terms regular in the LHP and the UHP, respectively,

$$\frac{\sqrt{k+i\varepsilon-\zeta}}{\zeta+i\varepsilon} = \frac{\sqrt{k+i\varepsilon-\zeta} - \sqrt{k+2i\varepsilon}}{\zeta+i\varepsilon} + \frac{\sqrt{k+2i\varepsilon}}{\zeta+i\varepsilon} \;.$$

Thus we can reorganize (44.28) into parts that are regular in the LHP and in the UHP:

$$\sqrt{k+i\varepsilon-\zeta}\; E(\zeta) + i\,\frac{\sqrt{k+i\varepsilon-\zeta}}{\zeta-i\varepsilon} - i\,\frac{\sqrt{k+i\varepsilon-\zeta} - \sqrt{k+2i\varepsilon}}{\zeta+i\varepsilon}$$

$$= i\,\frac{\sqrt{k+2i\varepsilon}}{\zeta+i\varepsilon} - 2\pi k\,\frac{\frac{1}{c}\,K(\zeta)}{\sqrt{k+i\varepsilon+\zeta}} \;. \tag{44.29}$$

The right hand side of (44.29) is regular in the UHP, the left hand side in the LHP. Since the two functions are regular in a common region [the strip

(44.27)], they may be analytically continued into a function regular for all ζ.

We will now show that this function vanishes at infinity, so that it vanishes everywhere. To this end, we examine (44.25b) in the limit $\zeta \to \infty$, for which only the behavior of $K(y')$ for $y' \to 0$ is significant. Because there is no intrinsic length scale in this limit, the current near the edge must behave as a power of y',

$$K(y') \sim (-y')^{-\alpha} , \qquad y' \to 0 , \tag{44.30}$$

where, in order that the integral (44.25b) exist,

$$\alpha < 1 . \tag{44.31}$$

The behavior of the corresponding Fourier transform of the current for large ζ is therefore

$$\begin{aligned} K(\zeta) &\sim \int_{-\infty}^{0} e^{-i\zeta y'} (-y')^{-\alpha} \, dy' \\ &\sim \frac{1}{\zeta^{1-\alpha}} \to 0 , \qquad \zeta \to \infty . \end{aligned} \tag{44.32}$$

Similarly, since $E(y) = 0$ if $y < 0$, the continuity of E requires the following power law for E near the edge,

$$E(y) \sim y^{\beta} , \qquad y \to +0 , \tag{44.33}$$

$$\beta > 0 , \tag{44.34}$$

which is equivalent to the asymptotic statement

$$E(\zeta) = \int_0^\infty e^{-i\zeta y}\, E(y)\, dy$$

$$\sim \frac{1}{\zeta^{1+\beta}} \to 0 , \quad \zeta \to \infty . \tag{44.35}$$

Hence, the function represented by the analytic continuation of either side of (44.29) vanishes at infinity, and thus, by Cauchy's theorem (mistakenly attributed to Liouville), is zero everywhere. Thus we have solutions for $K(\zeta)$ and $E(\zeta)$:

$$\frac{2\pi}{c} K(\zeta) = \frac{i}{\sqrt{k}} \frac{\sqrt{k+i\varepsilon+\zeta}}{\zeta+i\varepsilon} \tag{44.36a}$$

$$\sim \frac{1}{\sqrt{\zeta}} , \quad |\zeta| >> 1 , \tag{44.36b}$$

$$E(\zeta) = i\left(\frac{1}{\zeta+i\varepsilon} - \frac{1}{\zeta-i\varepsilon}\right)$$

$$- \frac{i\sqrt{k}}{\zeta+i\varepsilon} \frac{1}{\sqrt{k+i\varepsilon-\zeta}} \tag{44.37a}$$

$$\sim \frac{1}{\zeta^{3/2}} , \quad |\zeta| >> 1 . \tag{44.37b}$$

Comparison with (44.32) and (44.35) determines the powers

$$\alpha = \frac{1}{2} , \quad \beta = \frac{1}{2} , \tag{44.38}$$

implying the following spatial dependences,

$$K(y') \sim \frac{1}{\sqrt{|y'|}} , \tag{44.39}$$

$$E(y) \sim \sqrt{y} , \tag{44.40}$$

as $|y'|$, $y \to 0$. We will supply ~~a~~ physical discussion of the meaning of the behavior near the edge in Subsection 44-4.

Lecture 17

44-3. Exact Diffraction Cross Section

The exact current and field in the plane $x = 0$ are given by (44.36a) and (44.37a). By rewriting the expression for the surface current in the form

$$\frac{1}{c} K(\zeta) = \frac{i}{2\pi} \left\{ \frac{1}{\zeta+i\varepsilon} + \frac{1}{\sqrt{k}} \frac{\sqrt{k+\zeta+i\varepsilon} - \sqrt{k}}{\zeta+i\varepsilon} \right\} , \qquad (44.41)$$

we can identify the first term as the Fourier transform of

$$\frac{1}{2\pi} \begin{cases} e^{-\varepsilon|y|} , & y < 0 , \\ 0 , & y > 0 , \end{cases} \qquad (44.42)$$

which is the current, (44.12), we used in the first approximate solution to this problem, corresponding to the neglect of edge effects. [Here, (44.42) includes the exponential cutoff.] The second term in (44.41) thus gives the correction that must be added in order to obtain the exact current. By considering the behavior of (44.41) for ζ small (which, as we will see below, corresponds to small diffraction angles), we note that the first term is singular as $\zeta \to 0$, while the second is finite. Thus, the first approximation is valid for small angles, as we have previously asserted.

The asymptotic scattered field at fixed angle θ follows by use of (43.39) in (44.6):

$$E_{\text{scatt}} \sim -\sqrt{\frac{k}{2\pi i\rho}}\, e^{ik\rho} \int_{-\infty}^{0} dy'\, e^{-ik\sin\theta y'} \frac{2\pi}{c} K(y')$$

$$= -\sqrt{\frac{k}{2\pi i\rho}}\, e^{ik\rho}\, \frac{2\pi}{c}\, K(\zeta) \; , \tag{44.43}$$

where we have used the Fourier transform (44.25b) with

$$\zeta = k \sin\theta \; . \tag{44.44}$$

Then from the solution (44.36a) for the surface current, we find for the exact asymptotic scattered field

$$E_{scatt} \sim -\sqrt{\frac{i}{2\pi k\rho}}\, e^{ik\rho}\, \frac{\sqrt{1+\sin\theta}}{\sin\theta} \; . \tag{44.45}$$

As we anticipated, it agrees with the first approximation (44.16) when $\theta \ll 1$. The corresponding exact differential cross section per unit length is

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k}\, \frac{1+\sin\theta}{\sin^2\theta} \; , \tag{44.46}$$

generalizing (44.17). We recognize that the new factor in (44.45), $\sqrt{1+\sin\theta}$, is present in order to enforce the boundary condition that the electric field vanish at $\theta = -\frac{\pi}{2}, \frac{3\pi}{2}$.

Finally, we wish to make contact with the other method, in which the field in the "aperture," not the current on the conducting plate, is employed. That is, we use (43.37b), relating the scattered field to the field in the aperture, and (43.39), the asymptotic form of the Hankel function, to give the scattered electric field for finite diffraction angle θ, $|\theta| < \frac{\pi}{2}$,

$$E_{scatt} \sim \sqrt{\frac{k}{2\pi i}}\, \frac{e^{ik\rho}}{\sqrt{\rho}} \cos\theta \int_0^\infty dy\, e^{-ik \sin\theta y}\, E(y) \; , \tag{44.47}$$

where $E(y)$ is the exact electric field in the aperture. The integral in (44.47) is the Fourier transform (44.25a),

$$\int_0^\infty dy\ e^{-ik\ \sin\theta y}\ E(y) = E(\zeta = k\ \sin\theta) = -i\ \frac{1}{k\ \sin\theta}\ \frac{1}{\sqrt{1-\sin\theta}}\ , \qquad (44.48)$$

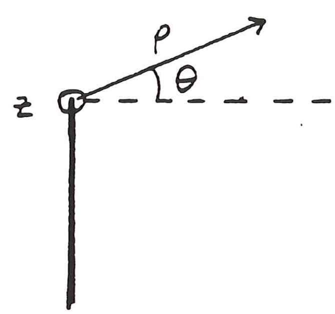
according to (44.37a). The resulting scattered electric field coincides with (44.45).

44-4. Field Near Edge

Here we wish to examine the form of the field near the edge, (44.40), from a different point of view. We may alternatively derive this result by solving the wave equation, (43.3),

$$\left(\nabla^2 + \frac{1}{\lambda\!\!\!^{-2}} \right) E = 0\ , \qquad (44.49)$$

near the edge. Since there the field is rapidly varying over a distance small compared to the wavelength, we can omit the $1/\lambda\!\!\!^{-2}$ term. Thus our problem is the electrostatic one of finding the field near the edge of a plane conductor. Using the cylindrical coordinate system shown in the figure,



we write the Laplacian as

$$\nabla^2 = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} \right)\ , \qquad (44.50)$$

since there is no z dependence. By separating variables ($\partial^2/\partial\theta^2 \to -m^2$), we find the characteristic solutions to Laplace's equation to be

$$E \sim \begin{pmatrix} \sin m\theta \\ \cos m\theta \end{pmatrix} \rho^m . \tag{44.51}$$

The boundary conditions that the field must vanish on the conducting plane,

$$E = 0 \quad \text{at} \quad \theta = -\frac{\pi}{2} \quad \text{and} \quad \frac{3\pi}{2} , \tag{44.52}$$

imply that the characteristic solutions are

$$E \sim \sin m\left(\theta + \frac{\pi}{2} \right) \rho^m , \tag{44.53}$$

with

$$\sin m\, 2\pi = 0 . \tag{44.54}$$

The smallest value of m consistent with (44.54) is

$$m = \frac{1}{2} , \tag{44.55}$$

giving the electric field

$$E = C\sqrt{\rho}\, \sin \frac{1}{2} \left(\theta + \frac{\pi}{2} \right) , \tag{44.56}$$

where C is a constant. [Note that the solution (44.45) exhibits this same behavior, since $\sqrt{1+\sin\theta} = \sqrt{2}\, \sin \frac{1}{2} \left(\theta + \frac{\pi}{2} \right)$.] For $x = 0$, $y > 0$ ($\theta = \pi/2$), the electric field is

$$E\left(\rho, \frac{\pi}{2} \right) = C\sqrt{\rho} = C\sqrt{y} , \tag{44.57}$$

which is (44.40). Another way of writing (44.56) is

$$E = C\ \mathrm{Im}\left[\sqrt{\rho}\ e^{\frac{i}{2}\left(\theta + \frac{\pi}{2}\right)}\right]$$

$$= C\ \mathrm{Im}\left[\sqrt{\rho e^{i\theta}}\ \sqrt{i}\right]$$

$$= C\ \mathrm{Im}\sqrt{i(x+iy)}\ . \tag{44.58}$$

We note that a general solution to the two-dimensional Laplace's equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E = 0\ , \tag{44.59}$$

has the form

$$E = \mathrm{Im}\ f(x+iy) \tag{44.60}$$

where f is a locally regular function. Equation (44.58) has this property away from the origin, and satisfies the appropriate boundary condition,

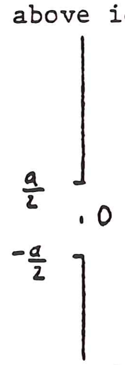
$$E = 0 \quad \text{for} \quad x = 0, \quad y < 0\ , \tag{44.61}$$

since there

$$E = C\ \mathrm{Im}\sqrt{i(x+iy)}$$

$$\rightarrow C\ \mathrm{Im}\sqrt{-y} = 0\ .$$

Next, we apply the above ideas to the situation of a slit.



When the slit is small compared to the wavelength,

$$a \ll \lambda \,, \tag{44.62}$$

the solution to Laplace's equation becomes relevant here. Since the appropriate boundary conditions are

$$E = 0 \quad \text{when} \quad x = 0 \,, \qquad |y| > \frac{a}{2} \,, \tag{44.63}$$

the electric field is found by generalizing our previous result, (44.58),

$$E = C \,\mathrm{Im}\left[i\left(x + i\left(y + \frac{a}{2} \right)\right) i\left(x + i\left(y - \frac{a}{2} \right)\right)\right]^{1/2} . \tag{44.64}$$

Explicitly, we note that on the surface x = 0, this reduces to

$$\begin{aligned} E &= C \,\mathrm{Im} \sqrt{y^2 - \left(\frac{a}{2}\right)^2} \\ &= C \,\mathrm{Im}\, i \sqrt{\frac{a^2}{4} - y^2} \\ &= \begin{cases} 0 \,, & |y| > \frac{a}{2} \,, \\ C \sqrt{\frac{a^2}{4} - y^2} \,, & |y| < \frac{a}{2} \,, \end{cases} \end{aligned} \tag{44.65}$$

To determine the constant C for the diffraction problem, we consider the fields far from the slit,

$$|x| \gg a \,, \quad |y| \,, \tag{44.66}$$

where the static result, (44.64), becomes

$$E \sim C \,\mathrm{Im}\, i|x| = C|x| \,. \tag{44.67}$$

In order to reproduce the static limit in the absence of the slit ($a \to 0$), we superimpose on this field another solution to Laplace's equation, $-Cx$, which satisfies the same boundary conditions, (44.63), so that we have the following asymptotic solution,

$$E \sim -Cx + C|x| = \begin{cases} 0 , & x > 0 , \\ -2Cx , & x < 0 . \end{cases} \tag{44.68}$$

Far from the surface, we can no longer neglect $k^2 = 1/\lambda\!\!\!^{-2}$ in (44.49). Consequently, the approximate diffracted wave, the static limit of which is (44.68), is

$$E = -\frac{2C}{k} \sin kx , \qquad x < 0 . \tag{44.69}$$

This is exact for zero aperture, a = 0, since it represents a standing wave due to total reflection by the conducting plane. The associated magnetic field,

$$ik\, B_y = -\frac{\partial}{\partial x} E_z = 2C \cos kx , \tag{44.70}$$

does not vanish on the conducting plane,

$$ik\, B_y(x = 0) = 2C ,$$

thereby determining the constant C in terms of the unperturbed magnetic field (that is, in terms of the magnetic field present when the slit is absent). Thus from (44.65), the electric field in the aperture, $|y| < \frac{a}{2}$, is determined by the magnetic field at x = 0,

$$E_z(x = 0) \simeq \frac{ik}{2} B_y(x = 0) \sqrt{\frac{a^2}{4} - y^2} , \tag{44.71}$$

for a narrow slit satisfying (44.62). The magnitude of the electric field in the aperture is small compared to the magnetic field there,

$$|E_z| \sim ka|B_y| << |B_y| , \qquad (44.72)$$

and would be zero if (44.69) were exact. Equation (44.71) is the basis for treating the diffraction by a narrow slit.