# MODERN PLASMA THEORY

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## B. The Debye Length

The most important phenomenon in a plasma is Debye shielding; the most important parameter is the Debye length associated with this. We therefore begin with the simplest possible description of Debye shielding.

If a point "test charge", Q, is placed in a homogeneous plasma at temperature T, the equilibrium density of field particles having charge q will change from its undisturbed value, n, to

$$\tilde{n} = n \exp(-q\phi/T) \ , \tag{1}$$

where $\phi$ is the self-consistent potential due to the charges (both test and plasma) and T is the temperature. (We shall always use energy units for T in this book, so the Boltzmann constant k will never appear.) Poisson's equation gives (assuming Q is at the origin)

$$\nabla^2\phi = -[4\pi Q\delta(\underline{r}) + 4\pi\Sigma\tilde{n}q] \tag{2}$$

where the summation is over the charge species of the plasma. If we can make the assumption, to be justified a posteriori, that

$$|q\phi/T| << 1 \tag{3}$$

then we have

$$\nabla^2\phi - 4\pi\sum\left(\frac{nq^2}{T}\right)\phi = -4\pi(Q\delta(\underline{r}) + \Sigma nq) \tag{4}$$

For a neutral plasma, the last term on the right side vanishes and we have

$$\nabla^2\phi - K_D{}^2\phi = 4\pi Q\delta(\underline{r}) \tag{5}$$

or

$$\phi = Qe^{-K_D r}/r \tag{6}$$

with $$K_D{}^2 = \sum \frac{4\pi nq^2}{T}$$

In place of the Coulomb potential Q/r which the test charge Q would produce in vacuum, we have a Debye potential which drops off in a distance $K_D{}^{-1}$. The physics involved is clear. The test charge repels (attracts) plasma particles of like (unlike) sign, resulting in a neutralizing charge "cloud," whose dimension increases with T since thermal effects tend to keep the density uniform. Both ions and electrons contribute equally to the effect if $T_e = T_i$. However, we shall define the Debye wavenumber on a single species basis:

$$k_D \equiv (4\pi n_0 q^2/T)^{1/2} \tag{7}$$

Its inverse

$$L_D = k_D{}^{-1} \tag{8}$$

is the Debye length. (Both $r_D$ and $\lambda_D$ are used by some authors in place of $L_D$, but the second of these is ambiguous, since $\lambda$ connotes wavelength and so $\lambda_D$ might reasonably be assumed to denote $2\pi/k_D$ rather than $k_D{}^{-1}$.)

It remains, of course, to justify our approximation (3). We see from (6) that in a literal sense (3) cannot hold, since $\phi \to \infty$ as $r \to 0$. However, the region in which (3) fails can be expected to make a small contribution to the charge density provided that region is small compared to the Debye cloud, i.e. provided $|q\phi/T| << 1$ for $r \sim L_D$. This will be true provided

$$q^2 k_D/T = (k_D{}^3/4\pi n) = (4\pi n L_D{}^3)^{-1} << 1$$

More generally, we can say that the whole Debye shielding picture makes physical sense only if there are many particles in the Debye cloud, i.e. if

$$nL_D{}^3 >> 1 \quad . \tag{9}$$

This dimensionless number is of transcendental importance in plasma physics. As in any many-body problem, development of a coherent theory is possible only if there is some small parameter in terms of which a perturbation expansion can be made. The plasma parameter

$$\varepsilon_p = (nL_D^3)^{-1} \tag{10}$$

plays this role, and all of modern plasma theory is based on the assumption $\varepsilon_p \ll 1$. We note that the ratio of average potential energy to average kinetic energy in a plasma is of order $\varepsilon_p$:

$$\overline{V}/\overline{K} = \varepsilon_p/16\pi \tag{11}$$

It is also significant that $\varepsilon_p$ orders the three basic scale lengths in an unmagnetized plasma, namely $L_D$; $L_T = e^2/T$, the "distance of closest approach"; and $L_n = n^{-1/3}$, the mean interparticle spacing:

$$L_T : L_n : L_D = \varepsilon_p/4\pi : \varepsilon_p^{1/3} : 1 \tag{12}$$

Before leaving the subject of Debye shielding, we note that it is, of course, not restricted to the simplest case -- point test charge, neutral plasma -- discussed here. In fact, so long as the basic condition $\varepsilon_p \ll 1$ which justifies (3) is satisfied, we can generalize (4) to

$$\nabla^2\phi - K_D^2\phi = S(\underline{r})$$

where S is a general source term, i.e. a superposition of external test charges plus the terms due to possible lack of plasma neutrality. The particular solution of this equation,

$$\phi(\underline{r}) = \int dr'[\exp(-K_D R)/4\pi R]S(\underline{r}')$$

$$R = |\underline{r} - \underline{r}'|$$

shows how Debye shielding manifests itself for an arbitrary source term.

## C. Plasma Oscillations

Debye shielding nicely illustrates the collective character of plasma phenomena, i.e. the simultaneous interaction of many particles, but it involves thermal effects in an essential way. Plasma oscillations are the simplest example of a collective phenomenon which can occur even when thermal effects are neglected, although, as we shall see later, their effect can be very important here also. We treat electrons as a simple fluid characterized by density $n(\underline{x},t)$ and velocity $\underline{v}(\underline{x},t)$, and linearize the equations in $\underline{v}$ and $n_1 = n - n_0$, where $n_0$ is the uniform density of background ions, whose motion we may neglect. Then the usual continuity equation,

$$\partial n/\partial t + \nabla \cdot (n\underline{v}) = 0$$

becomes

$$\partial n_1/\partial t + n_0 \nabla \cdot \underline{v} = 0 \tag{13}$$

while the momentum equation is just

$$\partial \underline{v}/\partial t = -(e/m)\underline{E} = +(e/m)\nabla\phi \tag{14}$$

Finally, Poisson's equation gives

$$\nabla^2\phi = 4\pi e n_1 \tag{15}$$

By taking the divergence of (14) and the time derivative of (13) we can eliminate $\underline{v}$, leaving

$$\partial^2 n_1/\partial t^2 = -(n_0 e/m)\nabla^2\phi = -(4\pi n_0 e^2/m)n_1 \tag{16}$$

wo that $n_1$, $\underline{v}$ and $\phi$ all oscillate at the plasma frequency,

$$\omega_p = (4\pi n_0 e^2/m)^{1/2}$$

Again, this is a collective effect, no hint of which can be gleaned from a single particle description. If we define the thermal velocity by a:

$$ma^2/2 = T \tag{17}$$

then we see that $\omega_p$, $k_D$ and a are related by

$$\omega_p = k_D a/2^{1/2} \tag{18}$$

D. The Saha Equation

We shall concentrate on the physics of either fully ionized plasmas or those for which the processes of interest occur on a time scale short compared to ionization and recombination times, but it is useful to know what combination of density and temperature give a substantial degree of ionization. Conventional equilibrium statistical mechanics leads to the Saha euqation, which we quote without proof[1]: The densities of electrons, ions and neutral atoms for a monatomic system in equilibrium are related by

$$n_e n_i/n_0 = g \exp(-I/T)/\lambda\!\!\!^{-}_e{}^3 \tag{19}$$

where I is the first ionization potential and $\lambda\!\!\!^{-}_e$ is the electron deBroglie wavelength,

$$\lambda\!\!\!^{-}_e = (2\pi\hbar^2/mT)^{1/2} \tag{20}$$

(We have assumed T large enough so that the molecule is dissociated if it is diatomic, like $H_2$; and small enough so that multiple ionization can be neglected for heavier atoms, like the rare gases.) The statistical factor, g, is

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[1] A simple but rigorous derivation is given in Chapter 3 of "Plasma Physics in Theory and Application", loc. cit.

defined by

$$g = 2\, Z_i/Z_0$$

where $Z_i$ is the partition function for the ion,

$$Z_i = \sum_j \exp(-E_j/T)$$

where the $E_j$ are energy levels of the ions, and

$$Z_0 = \sum_i \exp(W_i/T)$$

where $W_i$ are the excitation energy levels of the atom measured from its ground state. For order of magnitude arguments, the principal dependence on density and temperature occurs through the $\lambda\!\!\!^{-}_e$ and $\exp(-I/T)$ and we may take $g \approx 1$. While (19) can be written in many forms, one of the most useful is in terms of the degree of ionization,

$$\alpha = n_e(n_e+n_0)^{-1} = n_e/n$$

For a neutral plasma we have

$$\alpha = \frac{(\eta^2+4\eta)^{1/2} - \eta}{2} = \begin{cases} \eta^{1/2} & \eta \ll 1 \\ 1 - \eta^{-1} & \eta \gg 1 \end{cases}$$

where

$$\eta = g \exp(-I/T)/n\lambda\!\!\!^{-}_e{}^3$$

depends only on density and temperature. In particular, we note that the ionization can be substantial at low densities ($n\lambda_e^3 \ll 1$) even if $T < I$.