$$\bar{E}' = \epsilon^{\frac{1}{2}}\bar{E}, \quad B' = \mu^{\frac{1}{2}}H, \quad \bar{J}' = \epsilon^{-\frac{1}{2}}\bar{J}, \quad \rho' = \epsilon^{-\frac{1}{2}}\rho,$$

$$c' = (\epsilon\mu)^{-\frac{1}{2}}c. \qquad (1\text{-}17.17)$$

This makes it clear that

$$c' = \frac{c}{n}, \quad n = (\epsilon\mu)^{\frac{1}{2}}, \qquad (1\text{-}17.18)$$

is the speed of propagating electromagnetic waves in the medium; the quantity n is called the index of refraction for the medium. Our early results on electromagnetic fields moving in a definite direction can also be applied to propagation in this medium. The perpendicularity and equality of magnitude of $\bar{E}'$ and $\bar{B}'$ imply that

$$\bar{E}\cdot\bar{H} = 0, \quad \epsilon^{\frac{1}{2}}|\bar{E}| = \mu^{\frac{1}{2}}|\bar{H}|. \qquad (1\text{-}17.19)$$

In the usual situation at low frequencies, where $\mu \cong 1$ and $\epsilon > 1$, the speed c' is less than c by the factor $n \cong \epsilon^{\frac{1}{2}}$ and the magnitude of the electric field is less than that of the magnetic field by the same factor.

Dispersive Medium

Suppose we continue to assume that $\mu \cong 1$, but go to higher frequencies, where $\epsilon(\omega)$ has the kind of behavior suggested by the simple model (1-13.22),

$$\epsilon(\omega) = 1 + \frac{4\pi ne^2}{m}\frac{1}{\omega_1^2 - \omega^2 - i\omega\gamma}. \qquad (1\text{-}17.20)$$

When ω is sufficiently below ω_1 that $\omega_1^2 - \omega^2 \gg \gamma\omega$, $\epsilon(\omega)$ is real and greater than unity. But, when ω is above ω_1 to the extent that $\omega^2 - \omega_1^2 \gg \gamma\omega$, $\epsilon(\omega)$ is real and less than unity. Shall we conclude from $c' = c/\epsilon^{\frac{1}{2}}$ that, in a medium where the speed of light depends on frequency—a dispersive medium—energy can flow at a speed greater than c? No!

The identification of the energy density according to (1-17.5) involves a time integration, the building up of the field. But a field with a definite frequency has no such transient behavior; a range of frequencies is required. And, if the properties of the medium are frequency dependent, we have a new situation.

To begin again, we represent the time behavior of the electric field, for example, as a superposition of frequencies—a Fourier integral,

$$\overline{E}(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,\overline{E}(\omega) = \int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\,\overline{E}(\omega)^{*}. \qquad (1\text{-}17.21)$$

This explicit indication that $\overline{E}(t)$ is a real function is also conveyed by

$$\overline{E}(\omega)^{*} = \overline{E}(-\omega). \qquad (1\text{-}17.22)$$

There is a similar representation for the field vector

$$\overline{D}(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,\overline{D}(\omega), \qquad (1\text{-}17.23)$$

with

$$\overline{D}(\omega)^{*} = \overline{D}(-\omega). \qquad (1\text{-}17.24)$$

The frequency dependent dielectric constant definition

$$\overline{D}(\omega) = \epsilon(\omega)\overline{E}(\omega) \qquad (1\text{-}17.25)$$

then also obeys

$$\epsilon(\omega)^{*} = \epsilon(-\omega), \qquad (1\text{-}17.26)$$

as realized in the simple model (1-17.20).

? what is the limitation here

To simplify the following discussion, we retain the restriction $\mu \cong 1$, and also assume that the significant frequencies ω are such that $|\omega^2 - \omega_1{}^2| >> \gamma\omega$. Then $\epsilon(\omega)$ is essentially real although, more generally, it would be the real part of $\epsilon(\omega)$ that we consider. Let us pick out the part of (1-17.5) that refers to electric energy,

$$\frac{\partial}{\partial t} U_E(t) = \frac{1}{4\pi} \bar{E}(t) \cdot \frac{\partial}{\partial t} \bar{D}(t), \qquad (1\text{-}17.27)$$

and insert Fourier integrals for the two fields:

$$\frac{1}{4\pi} \bar{E}(t) \cdot \frac{\partial}{\partial t} \bar{D}(t)$$

$$= \frac{1}{4\pi} \left[\int_{-\infty}^{\infty} d\omega' e^{i\omega' t} \bar{E}(-\omega') \right] \cdot \left[\int_{-\infty}^{\infty} d\omega (-i\omega) e^{-i\omega t} \epsilon(\omega) \bar{E}(\omega) \right]$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega\, d\omega' e^{-i(\omega-\omega')t} [-i\omega\, \epsilon(\omega)]\, \bar{E}(\omega) \cdot \bar{E}(-\omega'). \qquad (1\text{-}17.28)$$

An alternative way of presenting this relabels the integration variables according to the substitutions

$$\omega \rightarrow -\omega', \quad \omega' \rightarrow -\omega, \qquad (1\text{-}17.29)$$

which produce

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega\, d\omega' e^{-i(\omega-\omega')t} [i\omega'\, \epsilon(\omega')]\, \bar{E}(\omega) \cdot \bar{E}(-\omega'). \qquad (1\text{-}17.30)$$

Then, the average of the two equivalent forms gives

$$\frac{\partial}{\partial t} \frac{1}{8\pi} \int d\omega\, d\omega' e^{-i(\omega-\omega')t} \frac{\omega\epsilon(\omega) - \omega'\epsilon(\omega')}{\omega - \omega'} \bar{E}(\omega) \cdot \bar{E}(-\omega'), \qquad (1\text{-}17.31)$$

where we have also taken the step that will lead to the identification of the electric energy density through time integration. Notice at this point that setting $\varepsilon(\omega) = 1$ immediately reproduces the known vacuum energy density, $(1/8\pi)(\overline{E}(t))^2$.

Now suppose that the turning on of the field takes place so slowly that only a very small band of frequencies about a central frequency $\overline{\omega}$ occurs (the precise relation between turning-on time and band width is not required here, but see Problem). Now, with $\omega \cong \omega' \cong \overline{\omega}$ or $-\overline{\omega}$, we have $[\varepsilon(-\omega) = \varepsilon(\omega)]$

$$\frac{\omega\varepsilon(\omega) - \omega'\varepsilon(\omega')}{\omega - \omega'} \to \frac{d}{d\omega}\left[\omega\varepsilon(\omega)\right]\Big|_{\overline{\omega}} \quad . \tag{1-17.32}$$

On the other hand, for $\omega \cong -\omega'$, the above quantity is simply $\varepsilon(\overline{\omega})$. However, this contribution to the energy density oscillates in time with the frequency $2\overline{\omega}$ and will not contribute significantly when averaged over one or two periods. In addition, the spatial dependence will also be rapidly oscillating, suppressing the contribution in spatial averages. We conclude that, effectively, in the sense of such averages,

$$U_E = \frac{1}{8\pi}\left[\frac{d}{d\omega}(\omega\varepsilon)\right](\overline{E}(t))^2 \tag{1-17.33}$$

(now writing ω in place of $\overline{\omega}$). To this is added

$$U_H = \frac{1}{8\pi}(\overline{H}(t))^2. \tag{1-17.34}$$

To find the speed at which energy is propagated in an electromagnetic pulse, we proceed as in Sec. 1-5. The local conservation equation in charge-free space,

$$\frac{\partial}{\partial t} U + \overline{\nabla}\cdot\overline{S} = 0, \tag{1-17.35}$$

$\int r U$

is multiplied by $\bar{r}$ and integrated over the entire pulse:

$$\frac{d}{dt}\int(d\bar{r})\,\bar{r}\,U = \int(d\bar{r})\,\bar{S} = \bar{v}_E \int(d\bar{r})\,U. \qquad (1\text{-}17.36)$$

The equality of magnitudes in the final relation is written out as

$$|\bar{v}_E| \int(d\bar{r})\,\frac{1}{8\pi}\left\{\frac{d}{d\omega}(\omega\epsilon)E^2 + H^2\right\} = \frac{c}{4\pi}\left|\int(d\bar{r})\,\bar{E}\times\bar{H}\right|. \qquad (1\text{-}17.37)$$

Now, what we had to say about such pulses in Eq.(1-17.19) is still true for a nearly monochromatic pulse, with $\epsilon = \epsilon(\omega)$, $\mu = 1$. It is only the interpretation of c' as the energy propagation speed in a dispersive medium that is being challenged. And so, we have

$$|\bar{E}\times\bar{H}| = |\bar{E}|\,|\bar{H}| = \epsilon^{\frac{1}{2}}E^2, \qquad H^2 = \epsilon E^2, \qquad (1\text{-}17.38)$$

and

$$|\bar{v}_E| = \frac{2\epsilon^{\frac{1}{2}}}{\frac{d}{d\omega}(\omega\epsilon) + \epsilon}\,c = \frac{1}{\frac{d}{d\omega}(\omega\epsilon^{\frac{1}{2}})}\,c\,. \qquad (1\text{-}17.39)$$

[For the situation when $\mu \neq 1$, see Problem 4.1.]

In the absence of dispersion, we recover the speed c', here appearing as $c/\epsilon^{\frac{1}{2}}$. Inserting the simple model

$$\epsilon(\omega) = 1 + \frac{4\pi n e^2}{m}\,\frac{1}{\omega_1^2 - \omega^2} \qquad (1\text{-}17.40)$$

we find [using the first form of (1-17.39)] that

$$\frac{1}{c}\,|\bar{v}_E| = \frac{\epsilon^{\frac{1}{2}}}{1 + \frac{4\pi n e^2}{m}\,\frac{\omega_1^2}{(\omega_1^2 - \omega^2)^2}}\,. \qquad (1\text{-}17.41)$$

In the circumstance $\omega < \omega_1$, where

$$\frac{\omega_1^2}{\omega_1^2 - \omega^2} > 1, \tag{I-17.42}$$

we get

$$\frac{1}{c}\,|\vec{v}_E| < \epsilon^{-\frac{1}{2}} < 1, \tag{I-17.43}$$

and for $\omega > \omega_1$, we now find

$$\frac{1}{c}\,|\vec{v}_E| < \epsilon^{\frac{1}{2}} < 1; \tag{I-17.44}$$

the speed of energy flow in a dispersive medium is less than c. (While this conclusion is not altered, the story is more complicated when $|\omega - \omega_1| \sim \gamma$. Treatment of this question by Sommerfeld and Brillouin can be found in L. Brillouin, "Wave Propagation and Group Velocity," Academic Press, New York, 1960.)

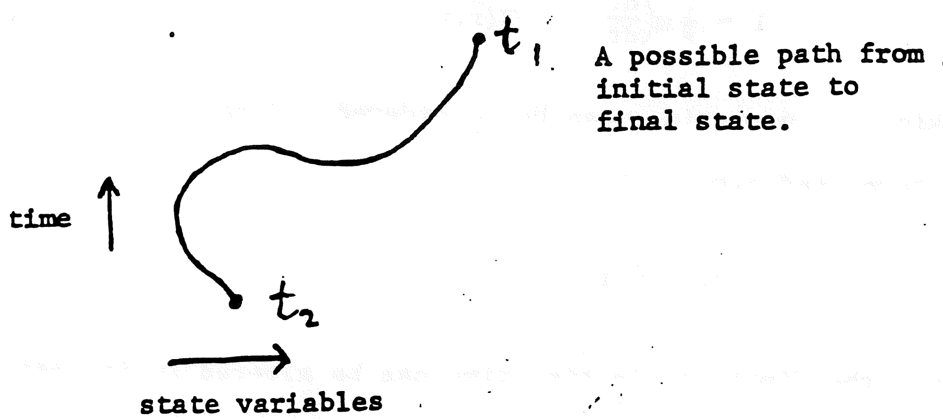
18. Review of Action Principles

The recognition that the electromagnetic field possesses mechanical properties naturally invites a complete description in mechanical terms and we shall supply it. But first it will be helpful to review the Lagrangian-Hamiltonian mechanical principles.

Action, W, is defined as the time integral of the Lagrangian, L, extended from an initial configuration or state at time t_2 up to a final state at time t_1:

$$W_{12} = \int_{t_2}^{t_1} dt\, L. \tag{I-18.1}$$

The integral refers to any path, any line of time development, from the initial to the final state.



The actual time evolution of the system is selected by the principle of stationary action: In response to infinitesimal variations of the integration path, the action W_{12} is stationary—does not have a corresponding infinitesimal change—for variations about the correct path, provided the initial and final configurations are held fixed. This means that, if we allow infinitesimal changes at the initial and final times, including alterations of those times, the only contribution to δW_{12} then comes from the endpoint variations, or

$$\delta W_{12} = G_1 - G_2, \tag{1-18.2}$$

where G_1 and G_2 refer to the state of affairs at the respective terminal times, t_1 and t_2. We now give three different realizations of the action principle, using a single particle for illustration.

Lagrangian Viewpoint

The non-relativistic motion of a particle with mass m in a potential $V(\bar{r},t)$ is described by the Lagrangian

$$L = \tfrac{1}{2}m\left(\frac{d\bar{r}}{dt}\right)^2 - V(\bar{r},t). \tag{I-18.3}$$

Two kinds of variations can be considered. First, a particular motion is altered infinitesimally,

$$\bar{r}(t) \to \bar{r}(t) + \delta\bar{r}(t). \tag{I-18.4}$$

Second, the final and initial time can be altered infinitesimally, by δt_1 and δt_2, respectively. It is more convenient however to think of these time displacements as produced by a continuous variation of the time, $\delta t(t)$, so chosen that, at the endpoints,

$$\delta t(t_1) = \delta t_1, \quad \delta t(t_2) = \delta t_2. \tag{I-18.5}$$

The corresponding change of the time differential in the integral is

$$dt \to d(t+\delta t) = \left(1 + \frac{d\delta t}{dt}\right)dt, \tag{I-18.6}$$

and the related transformation of the time derivative becomes

$$\frac{d}{dt} \to \left(1 - \frac{d\delta t}{dt}\right)\frac{d}{dt}\ . \tag{I-18.7}$$

Now, in carrying out the variation of the action

$$W_{12} = \int_2^1 \left[\tfrac{1}{2}m\frac{(d\bar{r})^2}{dt} - dt\,V\right], \tag{I-18.8}$$

the integration limits are left intact, the time displacement being produced through $\delta t(t)$ subject to (I-18.5). We get

$t \to t + \delta$

$dt \quad d + \frac{d\delta t\, dt}{dt}$

$= dt\left(1 + \frac{d\delta t}{dt}\right)$

$\frac{d}{dt} = \left(1 - \frac{d\delta t}{dt}\right)\frac{d}{dt}$

$$\delta W_{12} = \int_2^1 dt\left[m\frac{d\vec{r}}{dt}\cdot\frac{d}{dt}\delta\vec{r} - \delta\vec{r}\cdot\vec{\nabla}V - \frac{d\delta t}{dt}\tfrac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2 - \frac{d\delta t}{dt}V - \delta t\frac{\partial}{\partial t}V\right], \qquad (1\text{-}18.9)$$

or, with the definition

$$E = \tfrac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2 + V, \qquad (1\text{-}18.10)$$

we have

$$\delta W_{12} = \int_2^1 d\left[m\frac{d\vec{r}}{dt}\cdot\delta\vec{r} - E\,\delta t\right]$$

$$+ \int_2^1 dt\left[-\delta\vec{r}\cdot\left(m\frac{d^2\vec{r}}{dt^2} + \vec{\nabla}V\right) + \delta t\left(\frac{dE}{dt} - \frac{\partial V}{\partial t}\right)\right]. \qquad (1\text{-}18.11)$$

The first term on the right-hand side of (1-18.11) is the integral of a differential and therefore depends only on the endpoints. The principle of stationary action refers to the other term, which must vanish, for arbitrary $\delta\vec{r}(t)$ and $\delta t(t)$, if the unvaried motion is the actual one. Accordingly, that actual motion is governed by

$$m\frac{d^2\vec{r}}{dt^2} = -\vec{\nabla}V, \qquad (1\text{-}18.12)$$

Newton's equation of motion under the force derived from the potential V, and

$$\frac{dE}{dt} = \frac{\partial V}{\partial t}, \qquad (1\text{-}18.13)$$

which, for static potentials, $\partial V/\partial t = 0$, is the equation of energy conservation,

$$\frac{dE}{dt} = 0. \qquad (1\text{-}18.14)$$

What remains in (I-18.11) is a structure of the form (I-18.2), with

$$G = m\frac{d\vec{r}}{dt}\cdot\delta\vec{r} - E\delta t \qquad \text{(I-18.15)}$$

applying at each time boundary; more about this later.

Hamiltonian Viewpoint

The Lagrangian for the same system is now written

$$L = \vec{p}\cdot\frac{d\vec{r}}{dt} - H(\vec{r},\vec{p},t), \qquad \text{(I-18.16)}$$

with

$$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r},t). \qquad \text{(I-18.17)}$$

The result of subjecting $\vec{r}$, $\vec{p}$, and t to infinitesimal variations in

$$W_{12} = \int_2^1 [\vec{p}\cdot d\vec{r} - dtH] \qquad \text{(I-18.18)}$$

is

$$\delta W_{12} = \int_2^1 dt\left[\vec{p}\cdot\frac{d}{dt}\,\delta\vec{r} - \delta\vec{r}\cdot\vec{\nabla}V + \delta\vec{p}\cdot\frac{d\vec{r}}{dt} - \delta\vec{p}\cdot\frac{\vec{p}}{m}\right.$$
$$\left. - \frac{d\delta t}{dt}H - \delta t\,\frac{\partial H}{\partial t}\right], \qquad \text{(I-18.19)}$$

or

$$\delta W_{12} = \int_2^1 d\left[\vec{p}\cdot\delta\vec{r} - \delta tH\right] + \int_2^1 dt\left[-\delta\vec{r}\cdot\left(\frac{d\vec{p}}{dt}+\vec{\nabla}V\right) + \delta\vec{p}\cdot\left(\frac{d\vec{r}}{dt} - \frac{1}{m}\vec{p}\right)\right.$$
$$\left. + \delta t\left(\frac{dH}{dt} - \frac{\partial H}{\partial t}\right)\right]. \qquad \text{(I-18.20)}$$

Now the stationary action principle gives the equations

$$\frac{d\bar{r}}{dt} = \frac{1}{m}\bar{p} = \frac{\partial H}{\partial \bar{p}},$$

$$\frac{d\bar{p}}{dt} = -\nabla V = -\frac{\partial H}{\partial \bar{r}},$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}, \tag{1-18.21}$$

and the endpoint contributions, at each time, are

$$G = \bar{p}\cdot\delta\bar{r} - H\delta t. \tag{1-18.22}$$

In contrast with the Lagrangian differential equations of motion, which involve second derivatives, these Hamiltonian equations contain only first derivatives; they are called first-order equations. Why do we think that the same physical system is being described? Because one of those Hamiltonian equations,

$$\bar{p} = m\frac{d\bar{r}}{dt}, \tag{1-18.23}$$

can be used to convert the other equation of motion into

$$\frac{d}{dt}\left(m\frac{d\bar{r}}{dt}\right) = m\frac{d^2\bar{r}}{dt^2} = -\bar{\nabla}V, \tag{1-18.24}$$

which is the Lagrangian-Newtonian equation of motion. Furthermore, the insertion of (1-18.23) into H gives

$$\frac{1}{2m}\left(m\frac{d\bar{r}}{dt}\right)^2 + V = E, \tag{1-18.25}$$

and the third of the equations in (1-18.21) becomes indentical with (1-18.13). We also note the equivalence of the two versions of G.

But probably the most direct way of seeing that the same physical system is involved comes by writing the Lagrangian of the Hamiltonian viewpoint as

$$L = \frac{m}{2}\left(\frac{d\bar{r}}{dt}\right)^2 - V - \frac{1}{2m}\left(\bar{p} - m\frac{d\bar{r}}{dt}\right)^2. \qquad (1\text{-}18.26)$$

The result of varying $\bar{p}$ in the stationary action principle is to produce

$$\bar{p} = m\frac{d\bar{r}}{dt}. \qquad (1\text{-}18.27)$$

But, if we accept this as the <u>definition</u> of $\bar{p}$, the corresponding term in L disappears and we explicitly regain the Lagrangian description. We are justified in completely omitting the last term on the right side of (1-18.26), despite its dependence on the variables $\bar{r}$ and t, because of its quadratic structure. The explicit contribution to δL is

$$-\frac{1}{m}\left(\bar{p} - m\frac{d\bar{r}}{dt}\right)\cdot\left(\delta\bar{p} - m\frac{d}{dt}\,\delta\bar{r} + m\frac{d\bar{r}}{dt}\frac{d\delta t}{dt}\right), \qquad (1\text{-}18.28)$$

and the equation supplied by the stationary action principle for $\bar{p}$ variations, (1-18.27), also guarantees that there is no contribution here to the results of $\bar{r}$ and t variations.

<u>Third Viewpoint</u>

To the variables of the Hamiltonian description, we add $\bar{v}$:

$$L = \bar{p}\cdot\frac{d\bar{r}}{dt} - H(\bar{r},\bar{p},\bar{v},t),$$

$$H = \bar{p}\cdot\bar{v} - \tfrac{1}{2}mv^2 + V(\bar{r},t). \qquad (1\text{-}18.29)$$

The consequences of $\bar{r}$, $\bar{p}$, and t variations emerge, as before

$$\frac{d\bar{r}}{dt} = \frac{\partial H}{\partial \bar{p}} = \bar{v},$$

$$\frac{d\bar{p}}{dt} = -\frac{\partial H}{\partial \bar{r}} = -\bar{\nabla}V,$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}, \tag{1-18.30}$$

and

$$G = \bar{p}\cdot\delta\bar{r} - H\delta t. \tag{1-18.31}$$

As for the additional variable $\bar{v}$, its time derivative does not occur, and the stationary action principle gives an equation that refers to a single time—an equation of constraint—rather than an equation of motion:

$$0 = \frac{\partial H}{\partial \bar{v}} = \bar{p} - m\bar{v}. \tag{1-18.32}$$

With this description we have the option of returning to either the Hamiltonian or the Lagrangian viewpoint, by adopting suitable equations as definitions. On writing

$$H(\bar{r},\bar{p},\bar{v},t) = \frac{p^2}{2m} + V(\bar{r},t) - \frac{1}{2m}(\bar{p} - m\bar{v})^2, \tag{1-18.33}$$

and adopting

$$\bar{v} = \frac{1}{m}\bar{p} \tag{1-18.34}$$

as the <u>definition</u> of $\bar{v}$, we recover the Hamiltonian description. Alternatively, we can present the Lagrangian as

$$L = \frac{m}{2}\left(\frac{d\bar{r}}{dt}\right)^2 - V + (\bar{p} - m\bar{v})\cdot\left(\frac{d\bar{r}}{dt} - \bar{v}\right) - \frac{m}{2}\left(\frac{d\bar{r}}{dt} - \bar{v}\right)^2 . \qquad (I\text{-}18.35)$$

Then after adopting the following as <u>definitions</u>,

$$\bar{v} = \frac{d\bar{r}}{dt}, \quad \bar{p} = m\bar{v} , \qquad (I\text{-}18.36)$$

the resultant form of L is that of the Lagrangian viewpoint. It might seem that only the definition $\bar{v} = d\bar{r}/dt$, inserted in (I-18.35), suffices to regain the Lagrangian description. But then the next to last term in (I-18.35) would give the following additional contribution to δL, associated with the variation $\delta\bar{r}$:

$$(\bar{p} - m\bar{v})\cdot\frac{d}{dt}\delta\bar{r} . \qquad (I\text{-}18.37)$$

We shall find that it is the third viewpoint that is best adapted to present the mechanical formulation of the electromagnetic field in interaction with charged particles.

19. Invariance and Conservation Laws. Virial Theorem.

There is more content in the principle of stationary action than equations of motion. Suppose one considers a variation such that

$$\delta W_{12} = 0, \tag{1-19.1}$$

independently of the choice of initial and final times. We say that the action, which is left unchanged, is <u>invariant</u> under this alteration of path. Then the stationary action principle (1-18.2) asserts that

$$\delta W_{12} = G_1 - G_2 = 0, \tag{1-19.2}$$

or, there is a quantity $G(t)$ that has the same value for any choice of time t; it is conserved in time. A differential statement of that is

$$\frac{d}{dt} G(t) = 0. \tag{1-19.3}$$

Invariance implies conservation. Here are some examples. Suppose the Hamiltonian of (1-18.16) does not depend explicitly on time, or

$$W_{12} = \int_2^1 [\bar{p}\cdot d\bar{r} - H(\bar{r},\bar{p})dt]. \tag{1-19.4}$$

Then the variation

$$\delta t = \text{constant} \tag{1-19.5}$$

will give $\delta W_{12} = 0$ (see Eq.(1-18.19), with $\delta\bar{r} = 0$, $\delta\bar{p} = 0$, $d\delta t/dt = 0$, $\partial H/\partial t = 0$). The conclusion is that G (Eq.(1-18.22)), which here is just

$$G_t = -H\delta t, \tag{1-19.6}$$

is a conserved quantity, or that

$$\frac{dH}{dt} = 0. \qquad \text{(I-19.7)}$$

This inference, that the Hamiltonian—the energy—is conserved, if there is no explicit time dependence in H, is already present in the third equation of (I-18.21), But now a more general principle is at work.

Next, consider an infinitesimal, rigid rotation, one that maintains the lengths and scalar products of all vectors. Written explicitly for the position vector $\vec{r}$, it is

$$\delta\vec{r} = \delta\vec{\omega} \times \vec{r} \qquad \text{(I-19.8)}$$

where the constant vector $\delta\vec{\omega}$ gives the direction and magnitude of the rotation. (See Fig.)

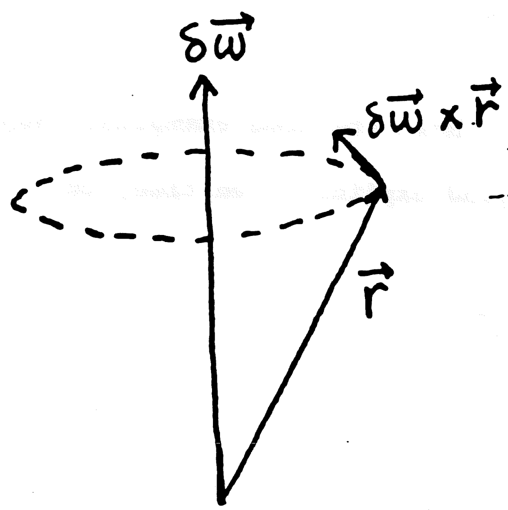


Fig.

$\delta\vec{\omega} \times \vec{r}$ is perpendicular to $\delta\vec{\omega}$ and $\vec{r}$, and represents a rotation of $\vec{r}$ about the $\delta\vec{\omega}$ axis.

Now specialize (I-18.17) to

$$H = \frac{p^2}{2m} + V(r), \qquad \text{(I-19.9)}$$

where $r = |\vec{r}|$, a rotationally invariant structure. Then

$$W_{12} = \int_2^1 [\vec{p}\cdot d\vec{r} - H dt] \qquad \text{(I-19.10)}$$

is also invariant under the rigid rotation, implying the conservation of

$$G_{\delta\vec{\omega}} = \vec{p}\cdot\delta\vec{r} = \delta\vec{\omega}\cdot\vec{r}\times\vec{p}. \qquad (1\text{-}19.11)$$

This is the conservation of angular momentum:

$$\vec{L} = \vec{r}\times\vec{p}, \quad \frac{d}{dt}\vec{L} = 0. \qquad (1\text{-}19.12)$$

Conservation of linear momentum appears analogously when there is, invariance under a rigid translation. For a single particle, the second of Eqs. (1-18.21) tells us immediately that $\vec{p}$ is conserved if V is a constant, say zero. Then, indeed, the action

$$W_{12} = \int_2^1 [\vec{p}\cdot d\vec{r} - \frac{p^2}{2m}\,dt] \qquad (1\text{-}19.13)$$

is invariant under the displacement

$$\delta\vec{r} = \delta\vec{\epsilon} = \text{constant}, \qquad (1\text{-}19.14)$$

and

$$G_{\delta\vec{\epsilon}} = \vec{p}\cdot\delta\vec{\epsilon} \qquad (1\text{-}19.15)$$

is conserved. But the general principle acts just as easily for, say, a system of two particles, a and b, with Hamiltonian

$$H = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} + V(\vec{r}_a - \vec{r}_b). \qquad (1\text{-}19.16)$$

This Hamiltonian and the associated action

$$W_{12} = \int_2^1 [\vec{p}_a\cdot d\vec{r}_a + \vec{p}_b\cdot d\vec{r}_b - H dt] \qquad (1\text{-}19.17)$$

are invariant under the rigid translation

$$\delta\bar{r}_a = \delta\bar{r}_b = \delta\bar{\epsilon}, \tag{1-19.18}$$

with the implication that

$$G_{\delta\bar{\epsilon}} = \bar{p}_a\cdot\delta\bar{r}_a + \bar{p}_b\cdot\delta\bar{r}_b = (\bar{p}_a+\bar{p}_b)\cdot\delta\bar{\epsilon} \tag{1-19.19}$$

is conserved. This is conservation of the total linear momentum

$$\bar{P} = \bar{p}_a + \bar{p}_b, \quad \frac{d}{dt}\bar{P} = 0. \tag{1-19.20}$$

Something that is a bit more general appears when we consider a rigid translation that grows linearly in time:

$$\delta\bar{r}_a = \delta\bar{v}t, \quad \delta\bar{r}_b = \delta\bar{v}t, \tag{1-19.21}$$

using the example of two particles. This gives each particle the common additional velocity $\delta\bar{v}$, and therefore must also change their momenta,

$$\delta\bar{p}_a = m_a\delta\bar{v}, \quad \delta\bar{p}_b = m_b\delta\bar{v}. \tag{1-19.22}$$

The response of the action to this variation is

$$\begin{aligned}\delta W_{12} &= \int_2^1 [(\bar{p}_a+\bar{p}_b)\cdot\delta\bar{v}dt + \delta\bar{v}\cdot(m_a d\bar{r}_a + m_b d\bar{r}_b) - (\bar{p}_a+\bar{p}_b)\cdot\delta\bar{v}dt] \\ &= \int_2^1 d[(m_a\bar{r}_a + m_b\bar{r}_b)\cdot\delta\bar{v}].\end{aligned} \tag{1-19.23}$$

The action is <u>not</u> invariant; its variation has end point contributions. But there is still a conservation law, not of $G = \mathcal{P}\cdot\delta\bar{v}t$, but of

$$\bar{N}\cdot\delta\bar{v},$$

$$\bar{N} = \bar{P}t - (m_a\bar{r}_a + m_b\bar{r}_b). \tag{1-19.24}$$

Written in terms of the center of mass position vector

$$\bar{R} = \frac{m_a\bar{r}_a + m_b\bar{r}_b}{M}, \quad M = m_a + m_b, \tag{1-19.25}$$

the statement of conservation of

$$\bar{N} = \bar{P}t - M\bar{R}, \tag{1-19.26}$$

namely

$$0 = \frac{d\bar{N}}{dt} = \bar{P} - M\frac{d\bar{R}}{dt}, \tag{1-19.27}$$

is the familiar fact that the center of mass of an isolated system moves at the constant velocity given by the ratio of the total momentum to the total mass of that system.

The action principle also supplies useful non-conservation laws. Consider, for constant $\delta\lambda$,

$$\delta\bar{r} = \delta\lambda\bar{r}, \quad \delta\bar{p} = -\delta\lambda\bar{p}, \tag{1-19.28}$$

which leaves $\bar{p}\cdot d\bar{r}$ invariant,

$$\delta(\bar{p}\cdot d\bar{r}) = (-\delta\lambda\bar{p})\cdot d\bar{r} + \bar{p}\cdot(\delta\lambda d\bar{r}) = 0. \tag{1-19.29}$$

But the response of the Hamiltonian

$$H = T(p) + V(\bar{r}), \tag{1-19.30}$$

$$T(p) = \frac{p^2}{2m},$$

is given by the non-invariant form

$$\delta H = \delta\lambda(-2T + \bar{r}\cdot\bar{\nabla}V). \tag{1-19.31}$$

Therefore we have, for an arbitrary time interval,

$$\delta W_{12} = \int_2^1 dt[\delta\lambda(2T - \vec{r}\cdot\vec{\nabla}V)] = G_1 - G_2 = \int_2^1 dt \frac{d}{dt}(\vec{p}\cdot\delta\lambda\vec{r}) \tag{1-19.32}$$

or, the theorem

$$\frac{d}{dt}\vec{r}\cdot\vec{p} - 2T - \vec{r}\cdot(-\vec{\nabla}V) = 0. \tag{1-19.33}$$

This is an example of the mechanical virial theorem referred to in Sec. 1-4.

For the particular situation of the Coulomb potential between charges, $V = \text{const.}/r$, where

$$\vec{r}\cdot\vec{\nabla}V = r\frac{d}{dr}V = -V, \tag{1-19.34}$$

the virial theorem asserts that

$$\frac{d}{dt}(\vec{r}\cdot\vec{p}) = 2T + V. \tag{1-19.35}$$

We apply this to a <u>bound</u> system produced by a force of attraction. On taking the time average of (1-19.35) the time derivative term disappears. That is because, over an arbitrarily long time interval $\tau = t_1 - t_2$, the value of $\vec{r}\cdot\vec{p}(t_1)$ can differ by only a finite amount from $\vec{r}\cdot\vec{p}(t_2)$, and

$$\overline{\frac{d}{dt}(\vec{r}\cdot\vec{p})} = \frac{1}{\tau}\int_{t_2}^{t_1} dt \frac{d}{dt}\vec{r}\cdot\vec{p} = \frac{\vec{r}\cdot\vec{p}(t_1) - \vec{r}\cdot\vec{p}(t_2)}{\tau} \to 0 \tag{1-19.36}$$

as $\tau \to \infty$. The conclusion,

$$2\overline{T} = -\overline{V}, \tag{1-19.37}$$

has been used qualitatively in Sec. 1-10.

Here is one more illustration of a non-conservation law:

$$\delta\vec{r} = \delta\lambda\frac{\vec{r}}{r},$$

$$\delta\vec{p} = -\delta\lambda\left(\frac{\vec{p}}{r} - \frac{\vec{r}\,\vec{p}\cdot\vec{r}}{r^3}\right) = \delta\lambda\frac{\vec{r}\times(\vec{r}\times\vec{p})}{r^3}. \tag{1-19.38}$$

Again $\bar{p}\cdot d\bar{r}$ is invariant

$$\delta(\bar{p}\cdot d\bar{r}) = -\delta\lambda\left(\frac{\bar{p}}{r} - \frac{\bar{r}\,\bar{p}\cdot\bar{r}}{r^3}\right)\cdot d\bar{r} + \bar{p}\cdot\left(\delta\lambda\,\frac{d\bar{r}}{r} - \delta\lambda\bar{r}\,\frac{\bar{r}\cdot d\bar{r}}{r^3}\right) = 0 \tag{1-19.39}$$

and the change of the Hamiltonian (1-19.30) is now

$$\delta H = \delta\lambda\left[-\frac{\bar{L}^2}{mr^3} + \frac{\bar{r}}{r}\cdot\bar{\nabla}V\right]. \tag{1-19.40}$$

The resulting theorem (for $V = V(r)$) is

$$\frac{d}{dt}\left(\frac{\bar{r}}{r}\cdot\bar{p}\right) = \frac{\bar{L}^2}{mr^3} - \frac{dV}{dr} \tag{1-19.41}$$

which, applied to the Coulomb potential, gives the bound state time average relation

$$\frac{L^2}{m}\overline{\left(\frac{1}{r^3}\right)} = -\overline{\left(\frac{V}{r}\right)} . \tag{1-19.42}$$

20. Potentials

The equation of motion for a charged particle acted on by electric and magnetic fields, the Lorentz law of force,

$$\frac{d}{dt} m\vec{v} = e(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}), \tag{1-20.1}$$

is not obviously in the Hamiltonian form

$$\frac{d}{dt} \vec{p} = - \frac{\partial H}{\partial \vec{r}} . \tag{1-20.2}$$

Indeed, we have already noted in Sec. 1-14 where attention focused on a constant magnetic field,

$$\vec{B} = \vec{\nabla} \times (\tfrac{1}{2} \vec{B} \times \vec{r}), \tag{1-20.3}$$

that momentum $\vec{p}$ is not the same as $m\vec{v}$ [Eq. (1-14.22)]:

$$\vec{p} = m\vec{v} + \frac{e}{c} \tfrac{1}{2} \vec{B} \times \vec{r}. \tag{1-20.4}$$

We must now go into this matter generally, for arbitrary fields, if we are to have the proper tools for a full mechanical description.

We see in (1-20.3) an example of the fact that the field equation

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1-20.5}$$

is automatically satisfied if $\vec{B}$ is derived as the curl of another vector,

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{1-20.6}$$

The vector $\vec{A}(\vec{r},t)$ is called the vector potential. And, as in the earlier discussion, the Maxwell field equation

$$\vec{\nabla} \times \vec{E} = - \frac{1}{c} \frac{\partial}{\partial t} \vec{B} = - \frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times A, \tag{1-20.7}$$

presented as

$$\vec{\nabla} \times (\vec{E} + \frac{1}{c} \frac{\partial}{\partial t} \vec{A}) = 0, \tag{1-20.8}$$

shows the existence of an induced electric field, beyond the electrostatic type deduced from a scalar potential, $\phi(\bar{r},t)$,

$$\bar{E} = -\frac{1}{c}\frac{\partial}{\partial t}\bar{A} - \bar{\nabla}\phi. \qquad (1\text{-}20.9)$$

We return to the equation of motion, now written as

$$\frac{d}{dt}m\bar{v} = -e\,\bar{\nabla}\varphi - \frac{e}{c}\left[\frac{\partial}{\partial t}\bar{A} - \bar{v}\times(\bar{\nabla}\times\bar{A})\right]. \qquad (1\text{-}20.10)$$

The identity

$$\bar{v}\times(\bar{\nabla}\times\bar{A}) = \bar{\nabla}(\bar{v}\cdot\bar{A}) - \bar{v}\cdot\bar{\nabla}\bar{A}, \qquad (1\text{-}20.11)$$

and the concept of total time derivative for the function $\bar{A}(\bar{r}(t),t)$,

$$\frac{d}{dt}\bar{A}(\bar{r}(t),t) = \frac{\partial}{\partial t}\bar{A}\ (\bar{r}(t),t) + \bar{v}\cdot\bar{\nabla}\bar{A}(\bar{r}(t),t), \qquad (1\text{-}20.12)$$

with

$$\bar{v} = \frac{d\bar{r}(t)}{dt}, \qquad (1\text{-}20.13)$$

then recasts (1-20.10) as

$$\frac{d}{dt}\bar{p} = -\bar{\nabla}[e(\phi - \frac{1}{c}\bar{v}\cdot\bar{A})]\ . = -\nabla V \qquad (1\text{-}20.14)$$

Here

$$\bar{p} = m\bar{v} + \frac{e}{c}\bar{A}, \qquad (1\text{-}20.15)$$

is the generalization of (1-20.4) to an arbitrary field.

It was stated in our review of mechanical action principles that a viewpoint employing variables $\bar{r}$, $\bar{p}$, and $\bar{v}$ was particularly convenient for describing electromagnetic forces on charged particles. With the explicit, and linear, appearance of $\bar{v}$ in what plays the role of the potential function in (1-20.14), we begin to see the basis for that remark. Indeed, we have only to consult Eq.(1-18.29) to find the appropriate Lagrangian:

$$L = \bar{p}\cdot\left(\frac{d\bar{r}}{dt} - \bar{v}\right) + \frac{1}{2}mv^2 - e\,\varphi + \frac{e}{c}\,\bar{v}\cdot\bar{A}. \qquad (1\text{-}20.16)$$

To recapitulate, the equations resulting from variations of $\overline{p}$, $\overline{r}$, and $\overline{v}$ are, respectively:

$$\frac{d\overline{r}}{dt} = \overline{v},$$

$$\frac{d}{dt}\,\overline{p} = -\overline{\nabla}[e(\varphi - \frac{1}{c}\,\overline{v}\cdot\overline{A})],$$

$$\overline{p} = m\overline{v} + \frac{e}{c}\,\overline{A}. \qquad (1\text{-}20.17)$$

We can now move to either the Lagrangian or the Hamiltonian formulation. For the first, we simply adopt $\overline{v} = d\overline{r}/dt$ as a definition (but see the discussion in Sec. 1-18) and get

$$L = \tfrac{1}{2}m\left(\frac{d\overline{r}}{dt}\right)^2 - e\,\varphi + \frac{e}{c}\frac{d\overline{r}}{dt}\cdot\overline{A}. \qquad (1\text{-}20.18)$$

Alternatively, we use the third equation of (1-20.17) to define

$$\overline{v} = \frac{1}{m}(\overline{p} - \frac{e}{c}\,\overline{A}), \qquad (1\text{-}20.19)$$

and find

$$L = \overline{p}\,\frac{d\overline{r}}{dt} - H,$$

$$H = \frac{1}{2m}(\overline{p} - \frac{e}{c}\,\overline{A})^2 + e\,\varphi. \qquad (1\text{-}20.20)$$

Here we make contact with the energy considerations of Sec. 1-15. For a homogeneous magnetic field, the terms in H that are linear and quadratic in $\overline{A}$ (and therefore in $\overline{B}$), respectively, are the paramagnetic and diamagnetic energy terms of that section.

The obvious generalization of (1-20.16) to a number of charged particles, of possibly different types, is the Lagrangian (the time variable is not made explicit)

$$L_{charges} = \sum_a \left[\bar{p}_a \cdot \left(\frac{d\bar{r}_a}{dt} - \bar{v}_a\right) + \tfrac{1}{2} m_a v_a^2 - e_a \phi(\bar{r}_a) + \frac{e_a}{c} \bar{v}_a \cdot \bar{A}(\bar{r}_a)\right]. \qquad \text{(J-20.21)}$$

21. The Action of the Field

The electromagnetic field is a mechanical system. It contributes its variables to the action, to the Lagrangian of the whole system of charges and fields. In contrast with the point charges, the field is distributed in space. Its Lagrangian should therefore be, not a summation over discrete points, but an integration over all spatial volume elements,

$$L_{field} = \int (d\bar{r}) \mathcal{L}_{field} ; \qquad (1\text{-}21.1)$$

this introduces the Lagrange function, or Lagrangian density, $\mathcal{L}$. The action principle associated with the Lagrangian (1-21.1) must produce the mechanical field equations. What are they? The fundamental importance of potentials in describing charged particle motion indicates some of these, namely [Eqs. (1-20.9) and (1-20.6)]

$$\frac{1}{c}\frac{\partial}{\partial t}\bar{A} = -\bar{E} - \bar{\nabla}\phi,$$

$$\bar{B} = \bar{\nabla}\times\bar{A}, \qquad (1\text{-}21.2)$$

which take the place of

$$\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial}{\partial t}\bar{B},$$

$$\bar{\nabla}\cdot\bar{B} = 0. \qquad (1\text{-}21.3)$$

The other set of equations, stated here for the pure field situation, the absence of charged particles, is

$$\frac{1}{c}\frac{\partial}{\partial t}\bar{E} = \bar{\nabla}\times\bar{B},$$

$$\bar{\nabla}\cdot\bar{E} = 0. \qquad (1\text{-}21.4)$$

We recognize that $\bar{A}(\bar{r},t)$, $\bar{E}(\bar{r},t)$, in analogy with $\bar{r}_a(t)$, $\bar{p}_a(t)$, obey equations of motion while $\phi(\bar{r},t)$, $\bar{B}(\bar{r},t)$, as analogues of $\bar{v}_a(t)$, do not. There are enough clues here to give the structure of $\mathcal{L}$, apart from an overall factor. We anticipate that factor in writing

$$\mathcal{L}_{\text{field}} = \frac{1}{4\pi}\left[\bar{E}\cdot\left(-\frac{1}{c}\frac{\partial}{\partial t}\bar{A} - \bar{\nabla}\varphi\right) - \bar{B}\cdot\bar{\nabla}\times\bar{A} + \tfrac{1}{2}(B^2 - E^2)\right] . \qquad (1\text{-}21.5)$$

To assist in verifying this, we record various contributions to the variation of L_{field}:

$$\delta\bar{E}: \quad \delta L_{\text{field}} = \frac{1}{4\pi}\int(d\bar{r})\,\delta\bar{E}\cdot\left(-\frac{1}{c}\frac{\partial}{\partial t}\bar{A} - \bar{\nabla}\varphi - \bar{E}\right); \qquad (1\text{-}21.6)$$

$$\delta\bar{A}: \quad \delta L_{\text{field}} = -\frac{d}{dt}\left[\frac{1}{4\pi c}\int(d\bar{r})\bar{E}\cdot\delta\bar{A}\right] + \frac{1}{4\pi}\int(d\bar{r})\,\delta\bar{A}\cdot\left[\frac{1}{c}\frac{\partial}{\partial t}\bar{E} - \bar{\nabla}\times\bar{B}\right] , \qquad (1\text{-}21.7)$$

which uses the identity

$$\bar{B}\cdot(\bar{\nabla}\times\delta\bar{A}) = \bar{\nabla}\cdot(\delta\bar{A}\times\bar{B}) + \delta\bar{A}\cdot\bar{\nabla}\times\bar{B} \qquad (1\text{-}21.8)$$

and ignores contributions from infinitely remote surfaces;

$$\delta\bar{B}: \quad \delta L_{\text{field}} = \frac{1}{4\pi}\int(d\bar{r})\,\delta\bar{B}\cdot[\bar{B} - \bar{\nabla}\times\bar{A}]; \qquad (1\text{-}21.9)$$

$$\delta\phi: \quad \delta L_{\text{field}} = \frac{1}{4\pi}\int(d\bar{r})\,\delta\phi\,\bar{\nabla}\cdot\bar{E}, \qquad (1\text{-}21.10)$$

where again a spatial reorganization is involved,

$$\bar{E}\cdot\bar{\nabla}\delta\phi = \bar{\nabla}\cdot(\bar{E}\delta\phi) - \delta\phi\,\bar{\nabla}\cdot\bar{E}. \qquad (1\text{-}21.11)$$

It is now a matter of inspection to conclude that the principle of stationary action, applied to variations of $\bar{E}$ and $\bar{B}$, does produce the field equations (1-21.2), while the stationary requirement on variations of $\bar{A}$ and ϕ yields the pair (1-21.4). The additional appearance of the time derivative term in the $\delta\bar{A}$ variation supplies the terminal contribution:

$$\delta W_{12\text{field}} = \delta \int_2^1 dt\, L_{\text{field}} = G_{1\text{field}} - G_{2\text{field}}, \tag{1-21.12}$$

$$G_{\text{field}} = -\frac{1}{4\pi c}\int (d\bar{r})\bar{E}\cdot\delta\bar{A}, \tag{1-21.13}$$

which refers only to field variations (the analogue of $\sum_a \bar{p}_a\cdot\delta\bar{r}_a$).

That we have correctly chosen the overall factor can now be verified. Let us present L_{field} in the Hamiltonian formulation by eliminating the variables, analogous to the $\bar{v}_a$, that do not obey equations of motion. For that we simply adopt the definition

$$\bar{B} = \bar{\nabla}\times\bar{A}, \tag{1-21.14}$$

and accept the restriction

$$\bar{\nabla}\cdot\bar{E} = 0. \tag{1-21.15}$$

Then the transference of the gradient on ϕ gives

$$L_{\text{field}} = \int (d\bar{r})\left(-\frac{1}{4\pi c}\right)\bar{E}\cdot\frac{\partial}{\partial t}\bar{A} - H_{\text{field}}, \tag{1-21.16}$$

where

$$H_{\text{field}} = \int (d\bar{r})\,\frac{1}{8\pi}\,(E^2+B^2) \tag{1-21.17}$$

will be recognized as the correct total field energy [Eq.(1-4.4)].

22. Action and Reaction

The Lagrangian $L_{charges}$ describes the motions of the charged particles under the action of a given electromagnetic field, as represented by the scalar and vector potentials. The Lagrangian L_{field} describes the development in time of an electromagnetic field without reference to charged particles. Does this mean that we must now hunt for something additional that, added to L_{field}, will describe the effect of the charges on the field? No. We already know it. It is automatically contained in the Lagrangian of the complete system of charges and fields:

$$L = L_{charges} + L_{field}. \tag{1-22.1}$$

There is a piece of $L_{charges}$ that explicitly describes the interaction between the charges and the fields:

$$L_{int.} = \sum_a \left[-e_a\phi(\vec{r}_a) + \frac{e_a}{c}\vec{v}_a\cdot\vec{A}(\vec{r}_a)\right]. \tag{1-22.2}$$

It produces the action of the fields on the charges, and it therefore also produces the (re)action of the charges on the field.

To verify this we must re-examine the implications of the stationary action principle for field variations, specifically, the variations of $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$, which now give additional contributions through $L_{int.}$. For that it is desirable to present $L_{int.}$ in the same form as L_{field}, a three dimensional spatial integration, rather than as a summation over point charges. This is accomplished by introducing the electric charge density and the electric current density,

$$\rho(\vec{r},t) = \sum_a e_a\delta(\vec{r}-\vec{r}_a(t)),$$

$$\vec{j}(\vec{r},t) = \sum_a e_a\,\vec{v}_a(t)\,\delta(\vec{r}-\vec{r}_a(t)), \tag{1-22.3}$$

and indeed

$$L_{int.}(t) = \int(d\bar{r})[-\rho(\bar{r},t)\phi(\bar{r},t) + \frac{1}{c}\bar{J}(\bar{r},t)\cdot\bar{A}(\bar{r},t)], \qquad (I\text{-}22.4)$$

according to the delta function property [see Eq. (I-1.24)] that is illustrated by (t is omitted)

$$\int(d\bar{r})\,\delta(\bar{r}-\bar{r}_a)\phi(\bar{r}) = \int(d\bar{r})\,\delta(\bar{r}-\bar{r}_a)\phi(\bar{r}_a) = \phi(\bar{r}_a). \qquad (I\text{-}22.5)$$

We now restate the consequences of $\bar{A}$ and ϕ variations for the total Lagrangian:

$$\delta A:\quad \delta L = -\frac{d}{dt}\left[\frac{1}{4\pi c}\int(d\bar{r})\bar{E}\cdot\delta\bar{A}\right] + \frac{1}{4\pi}\int(d\bar{r})\,\delta\bar{A}\cdot\left[\frac{4\pi}{c}\bar{J} + \frac{1}{c}\frac{\partial}{\partial t}\bar{E} - \bar{\nabla}\times\bar{B}\right], \qquad (I\text{-}22.6)$$

$$\delta\phi:\quad \delta L = \frac{1}{4\pi}\int(d\bar{r})\,\delta\phi[-4\pi\rho + \bar{\nabla}\cdot\bar{E}]. \qquad (I\text{-}22.7)$$

The anticipated results are here; the principle of stationary action applied to $\bar{A}$ and ϕ variations gives the field equations

$$\bar{\nabla}\times\bar{B} = \frac{1}{c}\frac{\partial}{\partial t}\bar{E} + \frac{4\pi}{c}\bar{J},$$

$$\bar{\nabla}\cdot\bar{E} = 4\pi\rho. \qquad (I\text{-}22.8)$$

These, together with (I-21.2), constitute the full set of Maxwell equations in the presence of moving charges.

23. Dynamics of Charges and Fields

The total Lagrangian (1-22.1) can be presented as

$$L = \sum_a \bar{p}_a \cdot \frac{d\bar{r}_a}{dt} + \int (d\bar{r}) \left(-\frac{1}{4\pi c}\right) \bar{E} \cdot \frac{\partial}{\partial t} \bar{A} - H, \tag{1-23.1}$$

with

$$H = \sum_a \left[\left(\bar{p}_a - \frac{e_a}{c}\bar{A}(\bar{r}_a)\right)\cdot \bar{v}_a - \tfrac{1}{2} m_a v_a^2 + e_a \phi_a(\bar{r}_a)\right]$$

$$+ \int (d\bar{r}) \frac{1}{4\pi} [\bar{E}\cdot\bar{\nabla}\phi + \bar{B}\cdot(\bar{\nabla}\times\bar{A}) + \tfrac{1}{2}(E^2 - B^2)] \tag{1-23.2}$$

appearing as the Hamiltonian of the system. The principle of stationary action, now including variations of the time variable, gives

$$\delta W_{12} = \delta\left[\int_2^1 dt\, L\right] = G_1 - G_2, \tag{1-23.3}$$

where, at any particular time,

$$G = \sum_a \bar{p}_a \cdot \delta\bar{r}_a + \int (d\bar{r}) \left(-\frac{1}{4\pi c}\right) \bar{E}\cdot\delta\bar{A} - H\delta t. \tag{1-23.4}$$

The narrower, Hamiltonian, description is reached by eliminating all variables that do not obey equations of motion, and correspondingly, do not appear in the first variation terms of G. Those superfluous variables are the $\bar{v}_a$, and the fields ϕ, $\bar{B}$. Accordingly, we adopt the following as definitions,

$$\bar{v}_a = \frac{1}{m_a}\left(\bar{p}_a - \frac{e_a}{c}\bar{A}(\bar{r}_a)\right), \tag{1-23.5}$$

$$\bar{B} = \bar{\nabla}\times\bar{A}, \tag{1-23.6}$$

and accept the restriction on $\bar{E}$ given by

$$\bar{\nabla}\cdot\bar{E} = 4\pi\rho. \tag{1-23.7}$$

The latter produces the elimination of ϕ from the Hamiltonian, which now appears as

$$H = \sum_a \frac{\left(\bar{p}_a - \frac{e_a}{c}\bar{A}(\bar{r}_a)\right)^2}{2m_a} + \int(d\bar{r})\,\frac{1}{8\pi}\,(E^2 + B^2). \qquad (1\text{-}23.8)$$

Perhaps, this finally startled you a bit? How could the scalar potential, which is essential electromagnetically, disappear from the dynamical description? Fear not—it's still with us. Think of the aspect of the stationary action principle associated with $\bar{E}$ variations:

$$\delta L = \int(d\bar{r})\left(-\frac{1}{4\pi}\right)\delta\bar{E}\cdot\left(\frac{1}{c}\frac{\partial}{\partial t}\bar{A} + \bar{E}\right) = 0. \qquad (1\text{-}23.9)$$

Do we conclude that $\frac{1}{c}\frac{\partial}{\partial t}\bar{A} + \bar{E} = 0$? That would be true if the $\delta\bar{E}(\bar{r},t)$ were arbitrary. They are not; $\bar{E}$ is subject to the restriction—the constraint—of Eq.(1-23.7) and any change in $\bar{E}$ must obey

$$\bar{\nabla}\cdot\delta\bar{E} = 0. \qquad (1\text{-}23.10)$$

The proper conclusion is that the vector multiplying $\delta\bar{E}$ in (1-23.9) is the gradient of a scalar function,

$$\frac{1}{c}\frac{\partial}{\partial t}\bar{A} + \bar{E} = -\nabla\phi, \qquad (1\text{-}23.11)$$

for that leads to

$$\delta L = \int(d\bar{r})\left(-\frac{1}{4\pi}\right)(\bar{\nabla}\cdot\delta\bar{E})\phi = 0, \qquad (1\text{-}23.12)$$

as required.

24. Mechanical Conservation Laws

Energy. The Hamiltonian (1-23.2), or (1-23.8), is constructed from particle and field variables. It is not an explicit function of the time, and is therefore conserved:

$$\frac{dH}{dt} = 0. \tag{1-24.1}$$

This is conservation of energy, and that energy, written as

$$H = \sum_a \tfrac{1}{2} m_a v_a^2 + \int (d\bar{r}) U \tag{1-24.2}$$

has a simple structure in terms of the sum of particle kinetic energies and the integrated field energy density.

Linear Momentum. Energy is conserved because there is no physically distinguished origin of time; nothing is altered on shifting that origin, displacing all time values by a common constant. Equally well, there is nothing in the Hamiltonian (1-23.2) or (1-23.8) to pick out a particular origin of the spatial coordinates. This will lead to conservation of the momentum vector. An infinitesimal displacement of the whole system by the constant vector $\delta\bar{\epsilon}$ means that

$$\delta\bar{r}_a = \delta\bar{\epsilon}, \tag{1-24.3}$$

but what is the analogous statement for field variables? For any field quantity $F(\bar{r})$, the consequence of a rigid displacement is to change the assigned value at the arbitrary point $\bar{r} + \delta\bar{\epsilon}$ to coincide with the value initially attributed to the point $\bar{r}$

$$(F + \delta F)(\bar{r} + \delta\bar{\epsilon}) = F(\bar{r}). \tag{1-24.4}$$

This being true for all $\bar{r}$, it can equally well be written

$F(r) - \delta F(r) = F(r - \delta\epsilon)$

$$F(\vec{r}) + \delta F(\vec{r}) = F(\vec{r} - \delta\vec{\epsilon}) = F(\vec{r}) - \delta\vec{\epsilon}\cdot\vec{\nabla} F(\vec{r}), \tag{1-24.5}$$

or

$$\delta F(\vec{r}) = -\delta\vec{\epsilon}\cdot\vec{\nabla} F(\vec{r}). \tag{1-24.6}$$

As an example, consider the charge density (omitting t)

$$\rho(\vec{r}) = \sum_a e_a \delta(\vec{r} - \vec{r}_a). \tag{1-24.7}$$

If the positions of all particles, the $\vec{r}_a$, are displaced by $\delta\vec{\epsilon}$, the charge density changes to

$$\rho(\vec{r}) + \delta\rho(\vec{r}) = \sum_a e_a \delta(\vec{r} - \vec{r}_a - \delta\vec{\epsilon}), \tag{1-24.8}$$

where

$$\delta(\vec{r} - \vec{r}_a - \delta\vec{\epsilon}) = \delta(\vec{r} - \vec{r}_a) - \delta\vec{\epsilon}\cdot\vec{\nabla}_{\vec{r}}\, \delta(\vec{r} - \vec{r}_a), \tag{1-24.9}$$

and therefore

$$\delta\rho(\vec{r}) = -\delta\vec{\epsilon}\cdot\vec{\nabla}\rho(\vec{r}). \tag{1-24.10}$$

We now apply this to compute the values of G at a particular time, as produced by a rigid displacement:

$$G_{\delta\vec{\epsilon}} = \sum_a \vec{p}_a\cdot\delta\vec{\epsilon} + \int(d\vec{r})\left(-\frac{1}{4\pi c}\right)\vec{E}\cdot(-\delta\vec{\epsilon}\cdot\vec{\nabla})\vec{A} = \vec{P}\cdot\delta\vec{\epsilon}. \tag{1-24.11}$$

The conserved momentum vector is therefore given by

$$\vec{P} = \sum_a \vec{p}_a + \int(d\vec{r})\,\frac{1}{4\pi c}\,\vec{E}\cdot(\vec{\nabla})\cdot\vec{A}, \tag{1-24.12}$$

which uses the notation

$$\vec{E}\cdot(\vec{\nabla})\cdot\vec{A} = \sum_{k=1}^{3} E_k \vec{\nabla} A_k. \quad = \nabla\times(\vec{E}\times\vec{A}) + \vec{E}\cdot\nabla A \tag{1-24.13}$$

For another way of presenting $\vec{P}$ we return to (1-24.11) and note the identity

$$\delta\vec{\epsilon} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\delta\vec{\epsilon} \cdot \vec{A}) - \delta\vec{\epsilon} \cdot \vec{\nabla}\vec{A} . \tag{1-24.14}$$

Then the field term of $G_{\delta\vec{\epsilon}}$ becomes

$$\int (d\vec{r}) \left(-\frac{1}{4\pi c}\right) \vec{E} \cdot [\delta\vec{\epsilon} \times \vec{B} - \vec{\nabla}(\delta\vec{\epsilon} \cdot \vec{A})] = \int (d\vec{r}) \frac{1}{4\pi c} (\vec{E} \times \vec{B}) \cdot \delta\vec{\epsilon}$$
$$- \int (d\vec{r}) \rho \frac{1}{c} \vec{A} \cdot \delta\vec{\epsilon}, \tag{1-24.15}$$

where the last contribution, produced by partial integration, is also the following sum over charges:

$$-\sum_a \frac{e_a}{c} \vec{A}(\vec{r}_a) \cdot \delta\vec{\epsilon}. \tag{1-24.16}$$

Accordingly, we now have

$$\vec{P} = \sum_a \left(\vec{p}_a - \frac{e_a}{c} \vec{A}(\vec{r}_a)\right) + \int (d\vec{r}) \frac{1}{4\pi c} \vec{E} \times \vec{B} \tag{1-24.17}$$

or

$$\vec{P} = \sum_a m_a \vec{v}_a + \int (d\vec{r}) \vec{G}, \tag{1-24.18}$$

the sum of particle kinetic momenta and the integrated field momentum density.

Angular Momentum. There is nothing in the Hamiltonian to distinguish a particular orientation of the spatial coordinate system, or, equivalently, the whole system can be rigidly rotated with no internally discernable effect. This implies conservation of the angular momentum vector. The infinitesimal rotation $\delta\vec{\omega}$ changes all particle position vectors by (see Fig.)

$$\delta\vec{r}_a = \delta\vec{\omega} \times \vec{r}_a . \tag{1-24.19}$$

As for the effect on a field function, we must now distinguish between scalars and vectors. We illustrate this with examples. The consequence, for the scalar charge density, of the particle displacement (1-24.19) is

$$\delta\rho(\bar{r}) = \sum_a e_a(-\delta\bar{r}_a\cdot\bar{\nabla}_{\bar{r}})\delta(\bar{r}-\bar{r}_a), \tag{1-24.20}$$

which restates the infinitesimal displacement of (1-24.10) without the specialization to constant $\delta\bar{\epsilon}$. The delta function property

$$(\bar{r}-\bar{r}_a)\delta(\bar{r}-\bar{r}_a) = 0, \quad \Rightarrow \vec{r}\,\delta(\vec{r}-\vec{r}_a) = \vec{r}_a\,\delta(\vec{r}-\vec{r}_a) \tag{1-24.21}$$

is now employed to write

$$\delta\bar{r}_a\cdot\bar{\nabla}_{\bar{r}}\delta(\bar{r}-\bar{r}_a) = \delta\bar{\omega}\times\bar{r}_a\cdot\bar{\nabla}_{\bar{r}}\delta(\bar{r}-\bar{r}_a) = \delta\bar{\omega}\times\bar{r}\cdot\bar{\nabla}_{\bar{r}}\delta(\bar{r}-\bar{r}_a) \tag{1-24.22}$$

as justified by the fact that only different, perpendicular, components of $\bar{r}$ and $\bar{\nabla}_{\bar{r}}$ occur here. Hence, we get

$$\delta\rho(\bar{r}) = -\delta\bar{\omega}\times\bar{r}\cdot\bar{\nabla}\rho(\bar{r}), \tag{1-24.23}$$

an example of the effect of a rigid rotation on any scalar function $S(\bar{r})$:

$$(S+\delta S)(\bar{r}+\delta\bar{\omega}\times\bar{r}) = S(\bar{r}), \tag{1-24.24}$$

$$\delta S(\bar{r}) = -\delta\bar{\omega}\times\bar{r}\cdot\bar{\nabla}S(\bar{r}). \tag{1-24.25}$$

For a vector example, we use the electric current density,

$$\bar{J}(\bar{r}) = \sum_a e_a\,\bar{v}_a\,\delta(\bar{r}-\bar{r}_a). \tag{1-24.26}$$

Now the particle velocity vectors are also rotated,

$$\delta\bar{v}_a = \delta\bar{\omega}\times\bar{v}_a, \tag{1-24.27}$$

if $\nabla \times \vec{v} = 0$

then

$\vec{v} = \nabla S(\vec{r})$

now

$\delta S(\vec{r}) \rightarrow -(\delta\vec{\omega} \times \vec{r}) \cdot \nabla S(\vec{r})$

thereby giving an additional contribution

$$\delta\bar{J}(\bar{r}) = -\delta\bar{\omega} \times \bar{r} \cdot \bar{\nabla} \bar{J}(\bar{r}) + \delta\bar{\omega} \times \bar{J}(\bar{r}). \qquad (1\text{-}24.28)$$

$\delta\vec{V} - \nabla \delta S r = -\nabla(\delta\vec{\omega} \times \vec{r}) \cdot \nabla S(\vec{r})$

$= -\delta\vec{\omega} \times$

$\nabla \vec{r}$

This is typical of the effect of the rigid rotation on any vector field, as described in relation to the fixed coordinate system:

$$(\bar{V} + \delta\bar{V})(\bar{r} + \delta\bar{\omega} \times \bar{r}) = \bar{V}(\bar{r}) + \delta\bar{\omega} \times \bar{V}(\bar{r}), \qquad (1\text{-}24.29)$$

$$\delta V(\bar{r}) = -\delta\bar{\omega} \times \bar{r} \cdot \bar{\nabla} \bar{V}(\bar{r}) + \delta\bar{\omega} \times \bar{V}(\bar{r}). \qquad (1\text{-}24.30)$$

In particular, then,

$$\delta\bar{A} = -\delta\bar{\omega} \times \bar{r} \cdot \bar{\nabla} \bar{A} + \delta\bar{\omega} \times \bar{A}, \qquad (1\text{-}24.31)$$

and

$$G_{\delta\bar{\omega}} = \bar{J} \cdot \delta\bar{\omega}\,, \qquad (1\text{-}24.32)$$

where the conserved angular momentum vector $\bar{J}$ is

$$\bar{J} = \sum_a \bar{r}_a \times \bar{p}_a + \int (d\bar{r}) \left(\frac{1}{4\pi c}\right) [\bar{E} \cdot (\bar{r} \times \bar{\nabla}) \cdot \bar{A} + \bar{E} \times \bar{A}]. \qquad (1\text{-}24.33)$$

Again there is an alternative version, now using the identity

$$(\delta\bar{\omega} \times \bar{r}) \times (\bar{\nabla} \times \bar{A}) = \bar{\nabla}(\delta\bar{\omega} \times \bar{r} \cdot \bar{A}) + \delta\bar{\omega} \times \bar{A} - \delta\bar{\omega} \times \bar{r} \cdot \bar{\nabla} \bar{A}, \qquad (1\text{-}24.34)$$

and leading to

$$\bar{J} = \sum_a \bar{r}_a \times m\bar{v}_a + \int (d\bar{r}) \bar{r} \times \bar{G}, \qquad (1\text{-}24.35)$$

the sum of moments of particle kinetic momenta and the integrated moment of the field momentum density.

25. Charge Conservation. Gauge Invariance

An electromagnetic system possesses a conservation law, that of electric charge, which has no place in the usual mechanical framework. How does it fit into our evolving dynamics of charges and fields? First, we must recognize explicitly something about potentials—they are not unique. The vector potential $\bar{A}$ was introduced to produce the magnetic field $\bar{B}$ as $\bar{\nabla}\times\bar{A}$. But the gradient of any scalar function λ could be added to a particular $\bar{A}$,

$$\bar{A} \to \bar{A} + \bar{\nabla}\lambda, \qquad (1\text{-}25.1)$$

without altering its curl, and the magnetic field $\bar{B}$ being represented. There would appear to be an alteration in the electric field:

$$\bar{E} = -\frac{1}{c}\frac{\partial}{\partial t}\bar{A} - \bar{\nabla}\phi \to \bar{E} - \bar{\nabla}\frac{1}{c}\frac{\partial}{\partial t}\lambda, \qquad (1\text{-}25.2)$$

but that can be compensated by changing the scalar potential,

$$\phi \to \phi - \frac{1}{c}\frac{\partial}{\partial t}\lambda. \qquad (1\text{-}25.3)$$

The possibility of modifying the potentials $\bar{A}$ and ϕ in the manner of (1-25.1), (1-25.3) without thereby changing the fields $\bar{E}$ and $\bar{B}$, is called the freedom of gauge transformation. [Reader: Do not transpose gauge into guage! The term had its origin in a now obsolete theory (1918) of Hermann Weyl (1885-1955).]

The Lagrange function of the field, (1-21.5), is not altered by the gauge transformation

$$\bar{A} \to \bar{A} + \bar{\nabla}\lambda, \quad \phi \to \phi - \frac{1}{c}\frac{\partial}{\partial t}\lambda,$$
$$\bar{E} \to \bar{E}, \quad \bar{B} \to \bar{B}; \qquad (1\text{-}25.4)$$

it is a gauge invariant function. How about the Lagrangian of the charges, (1-20.21)? It helps here to anticipate, from the immediate physical significance of the velocities

$$\bar{v}_a = \frac{1}{m_a}\left(\bar{p}_a - \frac{e_a}{c}\bar{A}(\bar{r}_a)\right), \qquad (1\text{-}25.5)$$

that the gauge transformation of $\bar{A}$ must be compensated by a redefinition of the $\bar{p}_a$,

$$\bar{p}_a \rightarrow \bar{p}_a + \frac{e_a}{c}\bar{\nabla}\lambda(\bar{r}_a). \qquad (1\text{-}25.6)$$

Then we find that

$$L_{\text{charge}} \rightarrow L_{\text{charge}} + \sum_a \left[\frac{e_a}{c}\frac{\partial}{\partial t}\lambda(\bar{r}_a,t) + \frac{e_a}{c}\frac{d\bar{r}_a}{dt}\cdot\bar{\nabla}\lambda(\bar{r}_a,t)\right]. \qquad (1\text{-}25.7)$$

There are two ways of looking at this additional term. One is mechanical:

$$\frac{d}{dt}\left[\sum_a \frac{e_a}{c}\lambda(\bar{r}_a,t)\right]; \qquad (1\text{-}25.8)$$

the other is electromagnetic:

$$\int(d\bar{r})\frac{1}{c}\left[\rho(\bar{r},t)\frac{\partial}{\partial t}\lambda(\bar{r},t) + \bar{J}(\bar{r},t)\cdot\bar{\nabla}\lambda(\bar{r},t)\right], \qquad (1\text{-}25.9)$$

where the latter is also equal to

$$\frac{d}{dt}\left[\int(d\bar{r})\,\frac{1}{c}\,\rho(\bar{r},t)\lambda(\bar{r},t)\right] - \int(d\bar{r})\,\frac{1}{c}\,\lambda(\bar{r},t)\left(\frac{\partial}{\partial t}\rho(\bar{r},t) + \bar{\nabla}\cdot\bar{J}(\bar{r},t)\right). \qquad (1\text{-}25.10)$$

In the mechanical viewpoint, the gauge transformation induces a change of the Lagrangian that leads to a change in the action,

$$W_{12} \rightarrow W_{12} + \left[\sum_a \frac{e_a}{c}\lambda(\bar{r}_a,t_1)\right] - \left[\sum_a \frac{e_a}{c}\lambda(\bar{r}_a,t_2)\right], \qquad (1\text{-}25.11)$$

by boundary terms. This means that the stationary action principle will produce exactly the same equations of motion, and constraint equations, despite the gauge transformation—it is the same physical system. In using the electromagnetic viewpoint it helps to think of an arbitrary infinitesimal gauge transformation, $\lambda \rightarrow \delta\lambda$, so that

$$\delta\bar{A} = \bar{\nabla}\,\delta\lambda, \quad \delta\phi = -\frac{1}{c}\frac{\partial}{\partial t}\,\delta\lambda, \tag{1-25.12}$$

induces

$$\delta W_{12} = G_{\delta\lambda 1} - G_{\delta\lambda 2} - \int_{t_2}^{t_1} dt \int (d\bar{r})\,\frac{1}{c}\,\delta\lambda \left(\frac{\partial}{\partial t}\rho + \bar{\nabla}\cdot\bar{J}\right), \tag{1-25.13}$$

with

$$G_{\delta\lambda} = \int (d\bar{r})\,\frac{1}{c}\,\rho\,\delta\lambda\ . \tag{1-25.14}$$

In view of the arbitrary nature of $\delta\lambda(\bar{r},t)$, the stationary action principle now demands that, at every point

$$\frac{\partial}{\partial t}\,\rho(\bar{r},t) + \bar{\nabla}\cdot\bar{J}(\bar{r},t) = 0; \tag{1-25.15}$$

gauge invariance implies local charge conservation. Then the special situation $\delta\lambda = \text{constant}$, where $\delta\bar{A} = \delta\phi = 0$ and W_{12} is certainly invariant, implies a conservation law, that of

$$G_{\delta\lambda} = \frac{1}{c}\,\delta\lambda Q, \tag{1-25.16}$$

in which

$$Q = \int (d\bar{r})\,\rho, \tag{1-25.17}$$

is the conserved total charge.

26. Gauge Invariance and Local Conservation Laws

We have just derived the local conservation law of electric charge. That is a property carried only by the particles, not by the electromagnetic field. In contrast the mechanical properties of energy, linear and angular momentum are attributes of both particles and fields. For these we have conservation laws of total quantities. What about local conservation laws? Early in this development (Sec. I-4) we produced local non-conservation laws; they concentrated on the field and characterized the charged particles as sources (or sinks) of field mechanical properties. It is natural to ask for a more even-handed treatment of both charges and fields. We shall supply it, in the framework of a particular example. The property of gauge invariance will be both a valuable guide, and an aid in simplifying the calculations.

The time displacement of a complete physical system identifies its total energy. This suggests that time displacement of a part of the system provides energetic information about that portion. The ultimate limit of this spatial subdivision, a local description, should appear in response to an (infinitesimal) time displacement that varies arbitrarily in space as well as in time, $\delta t(\bar{r},t)$.

Now we need a clue. How do fields, and potentials, respond to such coordinate-dependent displacements? This is where the freedom of gauge transformations enters: the change of the vector and scalar potentials, by $\bar{\nabla}\lambda(\bar{r},t), -\frac{1}{c}\frac{\partial}{\partial t}\lambda(\bar{r},t)$, respectively, serves as a model for the potentials themselves. The advantage here is that the response of the scalar $\lambda(\bar{r},t)$ to the time displacement can be reasonably taken to be:

$$(\lambda+\delta\lambda)(\bar{r},t+\delta t) = \lambda(\bar{r},t) \qquad (I\text{-}26.1)$$

or

$$\delta\lambda(\bar{r},t) = -\delta t(\bar{r},t)\frac{\partial}{\partial t}\lambda(\bar{r},t). \qquad (I\text{-}26.2)$$

$\tilde{A} = A - \nabla\lambda$

$\nabla\lambda = A$

$\delta A =$

Then we derive

$$\delta(\bar{\nabla}\lambda) = -\delta t\frac{\partial}{\partial t}(\bar{\nabla}\lambda) + \left(-\frac{1}{c}\frac{\partial}{\partial t}\lambda\right)c\,\bar{\nabla}\,\delta t,$$

$$\delta\left(-\frac{1}{c}\frac{\partial}{\partial t}\lambda\right) = -\delta t\left(-\frac{1}{c}\frac{\partial^2}{\partial t^2}\lambda\right) - \left(-\frac{1}{c}\frac{\partial}{\partial t}\lambda\right)\frac{\partial}{\partial t}\,\delta t, \qquad (1\text{-}26.3)$$

$\delta(\nabla\lambda) = -\nabla(\delta t \frac{\partial}{\partial t}\lambda)$

which is immediately generalized to

$$\delta\bar{A} = -\delta t\,\frac{\partial}{\partial t}\,\bar{A} + \phi c\,\bar{\nabla}\,\delta t$$

$$\delta\phi = -\delta t\,\frac{\partial}{\partial t}\,\phi - \phi\,\frac{\partial}{\partial t}\,\delta t, \qquad (1\text{-}26.4)$$

$\vec{A} \to A + \nabla\lambda$

or, equivalently,

$-\delta t \frac{\partial A}{\partial t} + -\phi c \nabla \delta t =$

$$\delta\bar{A} = c\,\delta t\bar{E} + \bar{\nabla}(\phi c\,\delta t),$$

$$\delta\phi = -\frac{1}{c}\frac{\partial}{\partial t}(\phi c\,\delta t). \qquad (1\text{-}26.5)$$

$\delta\vec{A} = \nabla(+\delta t c\phi)$

In the latter form we recognize a gauge transformation, produced by the scalar $\phi c\,\delta t$, which will not contribute to the changes of field strengths. Accordingly, for that calculation we have, effectively, $\delta\bar{A} = c\,\delta t\bar{E}$, $\delta\phi = 0$, leading to

$$\delta\bar{E} = -\frac{1}{c}\frac{\partial}{\partial t}(c\,\delta t\bar{E}) = -\delta t\frac{\partial}{\partial t}\bar{E} - \bar{E}\frac{\partial}{\partial t}\,\delta t,$$

$$\delta\bar{B} = \bar{\nabla}\times(c\,\delta t\bar{E}) = -\delta t\frac{\partial}{\partial t}\bar{B} - \bar{E}\times\bar{\nabla}\,c\,\delta t; \qquad (1\text{-}26.6)$$

the last line employs the field equation $\bar{\nabla}\times\bar{E} = -(1/c)\partial\bar{B}/\partial t$.

In the following we adopt a viewpoint in which such field equations are accepted as consequences of the definition of the fields in terms of potentials. That permits the field Lagrange function (1-21.5) to be simplified:

$$L_{field} = \frac{1}{8\pi}(E^2 - B^2). \qquad (1\text{-}26.7)$$

$A \to A + \nabla\lambda$

$\phi \to \phi + \frac{1}{c}\frac{\partial\lambda}{\partial t}$

$E\cdot\delta E - B\cdot\delta B$

$\frac{E\cdot}{4\pi}$

Then we can apply the field variation (1-26.6) directly, and get

$$\delta\mathcal{L}_{field} = -\delta t\frac{\partial}{\partial t}\mathcal{L}_{field} - \frac{1}{4\pi}E^2\frac{\partial}{\partial t}\delta t - \frac{c}{4\pi}\bar{E}\times\bar{B}\cdot\bar{\nabla}\delta t$$

$$= -\frac{\partial}{\partial t}(\delta t\,\mathcal{L}_{field}) - \frac{1}{8\pi}(E^2+B^2)\frac{\partial}{\partial t}\delta t - \frac{c}{4\pi}\bar{E}\times\bar{B}\cdot\bar{\nabla}\delta t. \qquad (1\text{-}26.8)$$

Before commenting on these last, not unfamiliar, field structures we turn to the charged particles and put them on a somewhat similar footing in terms of a continuous, rather than a discrete, description.

We therefore present the Lagrangian of the charges, (1-20.21), in terms of a corresponding Lagrange function,

$$L_{charges} = \int(d\bar{r})\mathcal{L}_{charges}, \qquad (1\text{-}26.9)$$

where

$$\mathcal{L}_{charges} = \sum_a \mathcal{L}_a \qquad (1\text{-}26.10)$$

and

$$\mathcal{L}_a = \delta(\bar{r}-\bar{r}_a(t))\left[\tfrac{1}{2}m_a v_a(t)^2 - e_a\phi(\bar{r}_a(t),t) + \frac{e_a}{c}\bar{v}_a(t)\cdot\bar{A}(\bar{r}_a(t),t)\right]; \qquad (1\text{-}26.11)$$

the latter adopts the Lagrangian viewpoint, with $\bar{v}_a = d\bar{r}_a/dt$ accepted as a definition. Then the effect of the time displacement on the variables $\bar{r}_a(t)$, taken as

$$(\bar{r}_a+\delta\bar{r}_a)(t+\delta t) = \bar{r}_a(t), \qquad (1\text{-}26.12)$$

$$\delta\bar{r}_a(t) = -\delta t(\bar{r}_a(t),t)\bar{v}_a(t), \qquad (1\text{-}26.13)$$

?

implies the velocity variation

$$\delta\bar{v}_a(t) = -\delta t(\bar{r}_a(t),t)\frac{d}{dt}\bar{v}_a(t) - \bar{v}_a(t)\left[\frac{\partial}{\partial t}\delta t + \bar{v}_a\cdot\bar{\nabla}_a\delta t\right]; \qquad (1\text{-}26.14)$$

the last step exhibits both the explicit and implicit dependences of $\delta t(\bar{r}_a(t),t)$ on t. In computing the variation of $\phi(\bar{r}_a(t),t)$, for example, we combine the potential variation given in (1-26.4), with the effect of $\delta\bar{r}_a$:

$$\delta\phi(\bar{r}_a(t),t) = -\delta t\frac{\partial}{\partial t}\phi - \phi\frac{\partial}{\partial t}\delta t - \delta t\,\bar{v}_a\cdot\bar{\nabla}_a\phi = -\delta t\frac{d}{dt}\phi - \phi\frac{\partial}{\partial t}\delta t, \qquad (1\text{-}26.15\ )$$

and, similarly,

$$\delta\bar{A}(\bar{r}_a(t),t) = -\delta t\frac{\partial}{\partial t}\bar{A} + \phi c\,\bar{\nabla}\delta t - \delta t\,\bar{v}_a\cdot\bar{\nabla}_a\bar{A} = -\delta t\frac{d}{dt}\bar{A} + \phi c\,\bar{\nabla}\delta t. \qquad (1\text{-}26.16\ )$$

The total effect of these variation on $\mathcal{L}_a$ is thus

$$\delta\mathcal{L}_a = -\delta t\frac{d}{dt}\mathcal{L}_a + \delta(\bar{r} - \bar{r}_a(t))\left(-m_a v_a^2 - \frac{e_a}{c}\bar{A}\cdot\bar{v} + e_a\phi\right)\left(\frac{\partial}{\partial t}\delta t + \bar{v}_a\cdot\bar{\nabla}\delta t\right), \qquad (1\text{-}26.17)$$

or

$$\delta\mathcal{L}_a = -\frac{d}{dt}[\delta t\,\mathcal{L}_a] - \delta(\bar{r} - \bar{r}_a(t))E_a\left(\frac{\partial}{\partial t}\delta t + \bar{v}_a\cdot\bar{\nabla}\delta t\right), \qquad (1\text{-}26.18)$$

where

$$E_a = \tfrac{1}{2}m_a v_a^2. \qquad (1\text{-}26.19)$$

We have retained the particle symbol d/dt to the last, but now, being firmly back in the field, space-time, viewpoint, it should be written $\partial/\partial t$, referring to all t dependence, with $\bar{r}$ being held fixed. The union of these various contributions to the variation of the total Lagrange function is

$$\delta\mathcal{L}_{tot} = -\frac{\partial}{\partial t}[\delta t\,\mathcal{L}_{tot}] - U_{tot}\frac{\partial}{\partial t}\delta t - \bar{S}_{tot}\cdot\bar{\nabla}\delta t, \qquad (1\text{-}26.20)$$

where

$$U_{tot} = \frac{1}{8\pi}(E^2 + B^2) + \sum_a \delta(\bar{r} - \bar{r}_a(t))E_a \qquad (1\text{-}26.21)$$

and

$$\bar{S}_{tot} = \frac{c}{4\pi}\bar{E}\times\bar{B} + \sum_a \delta(\bar{r}-\bar{r}_a(t))E_a\bar{v}_a, \qquad (1\text{-}26.22)$$

are physically transparent forms for the total energy density and total energy flux vector.

To focus on what is new in this development we ignore boundary effects in the stationary action principle, by setting the otherwise arbitrary $\delta t(\bar{r},t)$ equal to zero at t_1 and t_2. Then, through partial integrations, we conclude that

$$\delta W_{12} = \int_{t_2}^{t_1} dt \int (d\bar{r})\,\delta t\left(\frac{\partial}{\partial t}U_{tot} + \bar{\nabla}\cdot\bar{S}_{tot}\right) = 0, \qquad (1\text{-}26.23)$$

from which follows the local statement of total energy conservation,

$$\frac{\partial}{\partial t}U_{tot} + \bar{\nabla}\cdot\bar{S}_{tot} = 0. \qquad (1\text{-}26.24)$$

$\nabla\cdot V_a = 0$

$$\frac{\partial \sum_a \delta(r-\vec{r}_a(t))E_a}{\partial t} + \nabla\cdot\sum_a \delta(\vec{r}-\vec{r}_a(t))E_a\vec{V}_a$$

$V_a\cdot\nabla\sum_a \delta(r-r_a(t))E_a$

$$-\sum_a \nabla\delta(\vec{r}-\vec{r}_a(t))\cdot\vec{V}_a(t)\,E_a$$

$$-\sum_a V_a\cdot\nabla\delta(\vec{r}-\vec{r}_a(t))\,E_a$$

$$+\sum_a \delta(r-r_a(t))\frac{\partial E_a}{\partial t}$$

$$+V_a\,\delta(r-r_a)\cdot\nabla E_a$$

$$\frac{d}{dt}\sum_a \delta(\vec{r}-\vec{r}_a(t))E_a = 0$$

$$= \sum_a \delta(r-r_a)\frac{dE_a}{dt}$$

27. Einsteinian Relativity

After that discussion one might well ask whether a unified dynamics of charges and fields has now been attained. No—there is still a major flaw. An electromagnetic pulse is a mechanical system that travels at the speed of light, carrying a mass proportional to the total energy content,

$$m = E/c^2. \tag{1-27.1}$$

In contrast, the masses of the charged particles are fixed quantities that have no reference to the particle's state of motion and its associated energy. Another way of expressing this lack of mechanical unity between fields and particles comes from the physically evident expression for the total momentum density (Problem)

$$\overline{G}_{tot} = \frac{1}{4\pi c}\,\overline{E}\times\overline{B} + \sum_a \delta(\overline{r}-\overline{r}_a(t))\,m_a\overline{v}_a. \tag{1-27.2}$$

The relation

$$\overline{G}_{tot} = \frac{1}{c^2}\,\overline{S}_{tot}, \tag{1-27.3}$$

which is valid for the field terms, does not hold for the particle contributions. Well then, could it be that Newtonian mechanics is mistaken, and that the correct expressions for particle inertia and energy do satisfy $m = (1/c^2)E$? We now follow this unifying suggestion—that the relation between inertia and energy, which the electromagnetic field has disclosed, is, in fact, universally valid.

Consider a single particle in the absence of applied electromagnetic fields. What we are proposing is that the connection between momentum and velocity is actually

$$\overline{p} = \frac{1}{c^2}E\vec{v}. \tag{1-27.4}$$

To this we add the relation of Hamiltonian mechanics,

$$\bar{v} = \frac{\partial E}{\partial \bar{p}} , \qquad (1\text{-}27.5)$$

and deduce that

$$c^2 \bar{p} \cdot d\bar{p} = E dE, \qquad (1\text{-}27.6)$$

which is integrated to

$$E^2 = c^2 p^2 + \text{constant}. \qquad (1\text{-}27.7)$$

We already know (Sec. 1-5) that the added constant is zero for an electromagnetic pulse, moving at the speed c. What is its value for an ordinary particle? The energy (1-27.7) is smallest for $\bar{p} = 0$, when the particle is at rest. Then we write

$$\bar{p} = 0: \quad E = m_o c^2, \qquad (1\text{-}27.8)$$

where m_o is the mass appropriate to zero velocity—the rest mass. Therefore we have, in general,

$$E^2 = c^2 p^2 + (m_o c^2)^2 , \qquad (1\text{-}27.9)$$

or

$$E = (c^2 p^2 + m_o^2 c^4)^{\frac{1}{2}}. \qquad (1\text{-}27.10)$$

Another way of presenting this replaces momentum by velocity in accordance with (1-27.4),

$$E^2 = E^2 \frac{v^2}{c^2} + (m_o c^2)^2, \qquad (1\text{-}27.11)$$

giving

$$E = \frac{m_o c^2}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} , \quad \bar{p} = \frac{m_o \bar{v}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} . \qquad (1\text{-}27.12)$$

$= m_o c^2$

The last momentum construction exhibits the relation to, and the limitation of, the initial Newtonian formulation of particle mechanics. For speeds small in comparison to that of light, $|\bar{v}| << c$, the momentum is $\bar{p} = m_o \bar{v}$ and the particle inertia is a constant. This is the domain of Newtonian mechanics. But even here something is different, as we see from the energy derived from the approximation

$$\frac{|\bar{v}|}{c} << 1: \quad \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \cong 1 + \frac{1}{2}\frac{v^2}{c^2} , \tag{1-27.13}$$

namely,

$$E \cong m_o c^2 + \frac{1}{2} m_o v^2 . \tag{1-27.14}$$

In addition to the Newtonian kinetic energy $\frac{1}{2} m_o v^2$, there is a constant, the rest energy $m_o c^2$, displacing the Newtonian origin of energy. The same thing appears in the momentum form of E, (1-27.10), as

$$|\bar{p}| << m_o c: \quad E \cong m_o c^2 + \frac{p^2}{2m_o} . \tag{1-27.15}$$

For speeds approaching the speed of light we enter a new physical domain, one where the speed of light is an impassable barrier. This we can see from the particle velocity, exhibited as

$$\bar{v} = c \frac{\bar{p}}{\left(p^2 + m_o^2 c^2\right)^{\frac{1}{2}}} ; \tag{1-27.16}$$

it is such that

$$|\bar{v}| \leq c, \tag{1-27.17}$$

with the equality sign occuring only for $m_o = 0$. As for the last conclusion, it is not unreasonable that a system, such as an electromagnetic pulse, which can never be at rest, has no rest mass.

Now we must reconstruct the Lagrangian-Hamiltonian dynamics of particles. The Lagrangian of (1-18.29), omitting the potential V, is

$$L = \bar{p}\cdot\left(\frac{d\bar{r}}{dt} - \bar{v}\right) + \tfrac{1}{2}mv^2. \qquad (1\text{-}27.18)$$

Clearly the Newtonian term $\frac{1}{2}mv^2$ must be replaced by a new function of $\bar{v}$, $L(\bar{v})$:

$$L = \bar{p}\cdot\left(\frac{d\bar{r}}{dt} - \bar{v}\right) + L(\bar{v}), \qquad (1\text{-}27.19)$$

that will reproduce the new forms. We can find $L(\bar{v})$ by using the velocity construction of the energy,

$$E = \bar{p}\cdot\bar{v} - L(\bar{v}), \qquad (1\text{-}27.20)$$

and of the momentum. It is

$$L(\bar{v}) = \frac{m_o\bar{v}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\cdot\bar{v} - \frac{m_o c^2}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} = -m_o c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}. \qquad (1\text{-}27.21)$$

In the Newtonian regime, this $L(\bar{v})$ reproduces the original form to within a constant,

$$\frac{|\bar{v}|}{c} << 1: \quad L(v) = -m_o c^2 + \tfrac{1}{2}m_o v^2. \qquad (1\text{-}27.22)$$

The consistency of the action principle is verified on noting the consequence of a $\bar{v}$ variation:

$$\bar{p} = \frac{\partial L(\bar{v})}{\partial \bar{v}} = \frac{m_o\bar{v}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}. \qquad (1\text{-}27.23)$$

The elimination of $\bar{v}$ produces the Hamiltonian version

$$L = \bar{p}\cdot\frac{d\bar{r}}{dt} - H, \quad H = c(p^2 + m_o^2 c^2)^{\frac{1}{2}} = E. \qquad (1\text{-}27.24)$$

We shall find it especially rewarding to use this new particle dynamics in re-examining a subject previously discussed in the Newtonian framework. The topic is the coordinate translation that grows linearly in time, (1-19.21), or equivalently, the introduction of a new coordinate system with a constant relative velocity. Here we consider only a single particle. The displacement

$$\delta\vec{r}\,(t) = \delta\vec{v}t, \tag{1-27.25}$$

combined with the momentum change

$$\delta\vec{p}(t) = (E/c^2)\,\delta\vec{v}, \tag{1-27.26}$$

induces the following alteration of the action element $dt\,L$:

$$\delta_{\vec{r},\vec{p}}[dt\,L] = \delta_{\vec{r},\vec{p}}[\vec{p}\cdot d\vec{r} - H\,dt] = \vec{p}\cdot\delta\vec{v}\,dt + (H/c^2)\,\delta\vec{v}\cdot d\vec{r} - \vec{p}\cdot\delta\vec{v}\,dt$$

$$= H\,d\left(\frac{1}{c^2}\,\delta\vec{v}\cdot\vec{r}\right). \tag{1-27.27}$$

At the analogous Newtonian stage, (1-19.23), $m(=m_0)$ appeared in place of H/c^2, and we concluded that the action was not invariant, but changed by a differential. Now a totally new situation presents itself. If we also vary the <u>time</u> by

$$\delta t = \frac{1}{c^2}\,\delta\vec{v}\cdot\vec{r}, \tag{1-27.28}$$

the additional contribution, $-H\,d(\delta t)$, will cancel (1-27.27), and the action <u>is</u> invariant under the combined space and time transformations

$$\delta\vec{r} = \delta\vec{v}\,t, \quad \delta t = \frac{1}{c^2}\,\delta\vec{v}\cdot\vec{r}. \tag{1-27.29}$$

In view of the invariance of the action the implied conservation law should now follow directly. Indeed

$$G = \vec{p}\cdot\delta\vec{r} - H\delta t = \delta\vec{v}\cdot\vec{N}, \tag{1-27.30}$$

where $(H = E)$

$$\bar{N} = \bar{p}t - \frac{E}{c^2}\bar{r} \qquad (1\text{-}27.31)$$

is conserved:

$$\frac{d\bar{N}}{dt} = \bar{p} - \frac{E}{c^2}\frac{d\bar{r}}{dt} = 0 \qquad (1\text{-}27.32)$$

and we have recovered our starting point, $m = (1/c^2)E$.

But, from this initial dynamical modification of Newtonian dynamics has now emerged a change in Newtonian kinematics: the absolute distinction between time and space has been removed. That is emphasized by the fact that neither r^2 nor $(ct)^2$ is left unchanged by the transformation (1-27.29), whereas the difference, $r^2 - (ct)^2$, is invariant:

$$\delta[r^2 - (ct)^2] = 2[\bar{r}\cdot\delta\bar{v}\, t - t\delta\bar{v}\cdot\bar{r}] = 0. \qquad (1\text{-}27.33)$$

The physical meaning of this invariance appears on considering an electromagnetic pulse that, at time $t = 0$, is emitted from the origin, $\bar{r} = 0$. Moving at the speed of light, c, the pulse, at time t, will have travelled the distance $r = ct$, so that

$$r^2 - c^2t^2 = 0. \qquad (1\text{-}27.34)$$

The fact that an observer in uniform relative motion will assign different values to the elapsed time, and to the distance traversed, but agree that (1-27.34) is still valid, means that he also measures the speed of light as c. This is Einsteinian relativity.

One might object that it could all be true only for infinitesimal transformations. But, from infinitesimal transformations, finite transformations grow. We make this explicit by letting $\delta\bar{v}$ point along the z-axis, so that

$$\delta x = 0, \quad \delta y = 0,$$

$$\delta z = \frac{\delta v}{c}\, ct, \quad \delta ct = \frac{\delta v}{c}\, z\ . \qquad (1\text{-}27.35)$$

Let us regard this infinitesimal transformation as the result of changing a parameter θ by the infinitesimal amount

$$\delta\theta = \frac{\delta v}{c} . \qquad (1\text{-}27.36)$$

The implied differential equations in the variable θ are

$d\theta = \frac{dv}{c}$

$\frac{dv}{d\theta} = c$

$$\frac{dx(\theta)}{d\theta} = 0, \quad \frac{dy(\theta)}{d\theta} = 0,$$

$$\frac{dz(\theta)}{d\theta} = ct(\theta), \quad \frac{dct(\theta)}{d\theta} = z(\theta). \qquad (1\text{-}27.37)$$

From the latter we derive

$$\frac{d^2z(\theta)}{d\theta^2} = z(\theta), \quad \frac{d^2ct(\theta)}{d\theta^2} = ct(\theta), \qquad (1\text{-}27.38)$$

which are solved by the hyperbolic functions, coshθ and sinhθ. The explicit solutions of these equations that obey the initial conditions

$$z(0) = z, \quad ct(0) = ct, \quad \frac{dz}{d\theta}(0) = ct, \quad \frac{dct}{d\theta}(0) = z \qquad (1\text{-}27.39)$$

$\frac{z(\theta)}{ct(\theta)} = \frac{z + ct\tanh\theta}{z\tanh\theta + ct}$

are

$$z(\theta) = z\cosh\theta + ct\sinh\theta,$$

$\frac{v}{c} =$

$\frac{v}{c} + \tanh\theta / \frac{v}{c}\tanh\theta + 1$

$$ct(\theta) = z\sinh\theta + ct\cosh\theta. \qquad (1\text{-}27.40)$$

Physical interpretation is facilitated by focusing on tanhθ, the ratio of sinhθ and coshθ, which cannot exceed unity in magnitude. We now write

$$\tanh\theta = \frac{v}{c} , \qquad (1\text{-}27.41)$$

which reduces to (1-27.36) for infinitesimal values of these parameters. Then, the constructions

$$\cosh\theta = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} , \quad \sinh\theta = \frac{\frac{v}{c}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \qquad (1\text{-}27.42)$$

satisfy (1-27.41) as well as the hyperbolic relation

$$\cosh^2\theta - \sinh^2\theta = 1. \qquad (1\text{-}27.43)$$

If we distinguish the transformed values of the coordinates by a prime, the transformation equations read

$$z' = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}(z+vt),$$

$$t' = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left(t+\frac{v}{c^2}z\right), \qquad (1\text{-}27.44)$$

along with

$$x' = x, \quad y' = y. \qquad (1\text{-}27.45)$$

We see that the point with coordinates

$$x = 0, \quad y = 0, \quad z = -vt \qquad (1\text{-}27.46)$$

is represented by the transformed coordinates

$$x' = 0, \quad y' = 0, \quad z' = 0. \qquad (1\text{-}27.47)$$

It is the origin of the new reference frame which therefore moves with velocity -v relative to the initial one. (See Fig.)

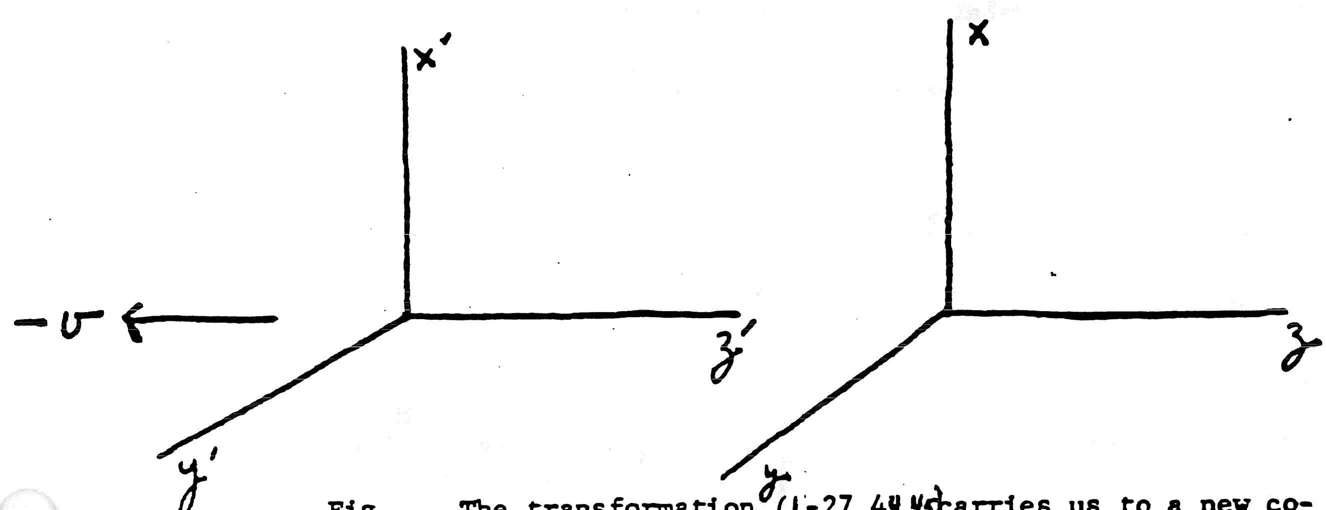


Fig. The transformation (1-27.44,45) carries us to a new coordinate frame moving relative to the original one with a velocity -v along the z-axis.

To see that $r^2-(ct)^2$ is left invariant by these finite transformations, it helps to present (1-27.44) as

$$z'+ct'=\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\frac{1}{2}}(z+ct),$$

$$z'-ct'=\left(\frac{1-\frac{v}{c}}{1+\frac{v}{c}}\right)^{\frac{1}{2}}(z-ct), \qquad (1\text{-}27.48)$$

for it is immediately apparent, on multiplication, that

$$z'^2-(ct')^2=z^2-(ct)^2 \qquad (1\text{-}27.49)$$

and then [Eq.(1.27.45)]

$$r'^2-(ct')^2=r^2-(ct)^2. \qquad (1\text{-}27.50)$$

The space-time coordinate transformations of the new kinematics are called Lorentz transformations, although it was Albert Einstein (1879-1955) who, in 1905, first understood their significance as describing the full physical equivalence of reference frames in uniform relative motion. As an aspect of that equivalence, we note the following. The original reference frame moves with velocity $+v$ relative to the new one (Fig.):

$$x=0,\quad y=0,\quad z=0 \qquad (1\text{-}27.51)$$

implies

$$x'=0,\ y'=0,\ z'=vt', \qquad (1\text{-}27.52)$$

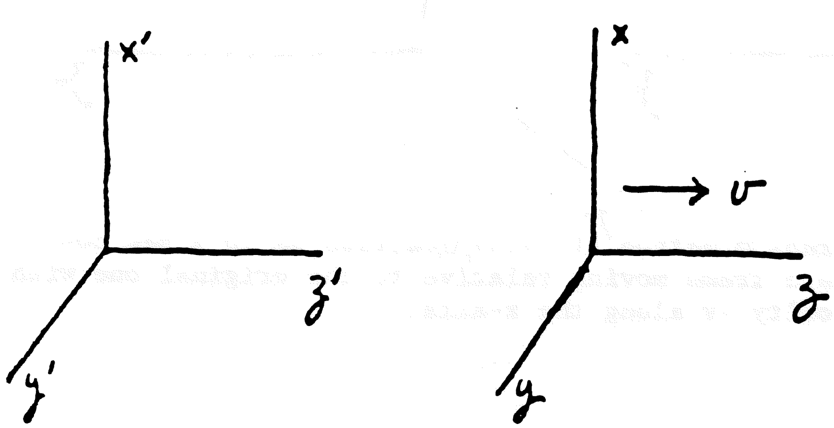


Fig. Motion of original frame relative to new frame

Then, should not the transformation that produces the unprimed coordinates from the primed ones be of precisely the same form as (1-27.44,45) but with the sign of v reversed? Indeed it is, as is evident on rewriting (1-27.48) as

$$z+ct = \left(\frac{1-\frac{v}{c}}{1+\frac{v}{c}}\right)^{\frac{1}{2}} (z'+ct')$$

$$z-ct = \left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\frac{1}{2}} (z'-ct') \,, \tag{1-27.53}$$

together with $x=x'$, $y=y'$. More generally, suppose the coordinate transformation $z, t \rightarrow z', t'$, produced by relative velocity-v_1 along the common $z-z'$ axis, is followed by the transformation $z', t' \rightarrow z'', t''$, produced by relative velocity-v_2 along the common $z'-z''$ axis. Is the net result a transformation $z, t \rightarrow z'', t''$ that is produced by some relative velocity-v? Yes. It suffices to consider just one of the pair of equations analogous to (1-27.48), say

$$z'+ct' = \left(\frac{1+\frac{v_1}{c}}{1-\frac{v_1}{c}}\right)^{\frac{1}{2}} (z+ct) \,, \tag{1-27.54}$$

and, similarly,

$$z''+ct'' = \left(\frac{1+\frac{v_2}{c}}{1-\frac{v_2}{c}}\right)^{\frac{1}{2}} (z'+ct'), \tag{1-27.55}$$

(the other set emerges by the systematic substitution of $-c$ for c) which immediately yields

$$(z''+ct'') = \left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\frac{1}{2}} (z+ct), \tag{1-27.56}$$

with

$$\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\frac{1}{2}} = \left(\frac{1+\frac{v_1}{c}}{1-\frac{v_1}{c}}\right)^{\frac{1}{2}}\left(\frac{1+\frac{v_2}{c}}{1-\frac{v_2}{c}}\right)^{\frac{1}{2}} . \qquad (1\text{-}27.57)$$

For any of the square root factors, the variation of the value of the appropriate v/c from -1 to $+1$ changes the square root from 0 to ∞; it is a positive number, and the product of two positive numbers is again a positive number. In other words, no succession of Lorentz transformations can produce a net transformation with $|v| > c$. The specific value of v in (1-27.57) is identified by writing this relation as

$$\frac{1+\frac{v}{c}}{1-\frac{v}{c}} = \frac{1+\frac{v_1v_2}{c^2}+\frac{v_1+v_2}{c}}{1+\frac{v_1v_2}{c^2}-\frac{v_1+v_2}{c}} , \qquad (1\text{-}27.58)$$

or

$$v = \frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}} . \qquad (1\text{-}27.59)$$

Simple addition of the velocities occurs only in the Newtonian regime, $|v_{1,2}| \ll c$.

We can't end this section without showing that the kinematical space-time transformations of (1-27.44,45) do indeed produce a dynamical unification of charged particles and electromagnetic fields. That requires a study of the behavior of fields and potentials under the infinitesimal Lorentz transformations. This has already been touched on in Problem 3.1, but we prefer to apply our recently developed methods here, beginning with the analogue of (1-26.2) for Lorentz transformations:

$$\delta\lambda(\bar{r},t) = -\left(\delta\bar{v}t\cdot\bar{\nabla}+\frac{1}{c^2}\,\delta\bar{v}\cdot\bar{r}\frac{\partial}{\partial t}\right)\lambda(\bar{r},t) = -\delta_{coor}\lambda(\bar{r},t). \quad (1\text{-}27.60)$$

Differentiation supplies the model for potential variations:

$$\delta\bar{A}(\bar{r},t) = -\delta_{coor}\,\bar{A}(\bar{r},t)+\frac{1}{c}\,\delta\bar{v}\phi(\bar{r},t)\ ,$$

$$\delta\phi(\bar{r},t) = -\delta_{coor}\,\phi(\bar{r},t)+\frac{1}{c}\,\delta\bar{v}\cdot\bar{A}(\bar{r},t), \quad (1\text{-}27.61)$$

which are alternatively presented as

$$\delta\bar{A} = \delta\bar{v}t\times\bar{B}+\frac{1}{c}\,\delta\bar{v}\cdot\bar{r}\bar{E}+\bar{\nabla}(-\delta\bar{v}t\cdot\bar{A}+\frac{1}{c}\,\delta\bar{v}\cdot\bar{r}\phi)\ ,$$

$$\delta\phi = \delta\bar{v}t\cdot\bar{E}-\frac{1}{c}\frac{\partial}{\partial t}(-\delta\bar{v}t\cdot\bar{A}+\frac{1}{c}\,\delta\bar{v}\cdot\bar{r}\phi). \quad (1\text{-}27.62)$$

The use of the latter simplifies the calculation of the field variations; they emerge as

$$\delta\bar{E} = -\delta_{coor}\,\bar{E}-\frac{1}{c}\,\delta\bar{v}\times\bar{B},$$

$$\delta\bar{B} = -\delta_{coor}\,\bar{B}+\frac{1}{c}\,\delta\bar{v}\times\bar{E}. \quad (1\text{-}27.63)$$

Then further differentiation in accordance with the field equations,

$$\rho = \frac{1}{4\pi}\,\bar{\nabla}\cdot\bar{E}$$

$$\bar{J} = \frac{c}{4\pi}\left(\bar{\nabla}\times\bar{B}-\frac{1}{c}\frac{\partial}{\partial t}\,\bar{E}\right), \quad (1\text{-}27.64)$$

yields

$$\delta\rho = -\delta_{coor}\rho+\frac{1}{c^2}\,\delta\bar{v}\cdot\bar{J}$$

$$\delta\bar{J} = -\delta_{coor}\bar{J}+\delta\bar{v}\rho. \quad (1\text{-}27.65)$$

Notice that the Lorentz transformation properties of $\frac{1}{c}\bar{J}$, ρ are of the same form as those for $\bar{A}$, ϕ. We can also recognize from these results that the symbol 'δ' used in Problem , which involved both changes of fields and of coordinates, has just that meaning as applied to any field $F(\bar{r},t)$:

$$'\delta' F(\bar{r},t) = (F+\delta F)(\bar{r}+\delta\bar{r}, t+\delta t) - F(\bar{r},t) = \delta F(\bar{r},t) + \delta_{coor} F(\bar{r},t). \quad (1\text{-}27.66)$$

We must now examine the responses of the various parts of the total Lagrange function. First consider

$$\mathcal{L}_{field} = \frac{1}{8\pi}(E^2 - B^2). \quad (1\text{-}27.67)$$

It is immediately apparent that the contributions of the last terms in the transformation equations (1-27.63) just cancel and

$$\delta\mathcal{L}_{field} = -\delta_{coor}\mathcal{L}_{field}. \quad (1\text{-}27.68)$$

Then we consider the interaction contribution to the Lagrange function [see Eq.(1-22.4)]

$$\mathcal{L}_{int} = -\rho\phi + \frac{1}{c}\bar{J}\cdot\bar{A}. \quad (1\text{-}27.69)$$

Again the last terms in the transformation Eqs.(1-27.61) and (1-27.65) have no net effect, and

$$\delta\mathcal{L}_{int} = -\delta_{coor}\mathcal{L}_{int}. \quad (1\text{-}27.70)$$

Finally, we come to the Lagrange function of the individual particles. For one particle, with the Lagrangian description adopted ($\bar{v} = d\bar{r}/dt$), the Lagrangian (1-27.21) is

$$L_a = -m_{oa}c^2\left(1 - \frac{1}{c^2}v_a^2\right)^{\frac{1}{2}}. \quad (1\text{-}27.71)$$

In contrast with the procedure of (1-26.9), our introduction of the Lagrange function now is dictated by the impossibility of maintaining a common time for particles at different spatial points — that Newtonian concept has disappeared in the world of Einsteinian relativistic kinematics. Accordingly we give each particle its individual time coordinate, and present its contribution to the action as

$$\int dt_a L_a = \int (-m_{oa} c)[(c dt_a)^2 - (d\bar{r}_a)^2]^{\frac{1}{2}} = \int dt (d\bar{r}) \mathcal{L}_a \qquad (1\text{-}27.72)$$

where

$$\mathcal{L}_a(\bar{r},t) = -m_{oa} c \int \delta(\bar{r} - \bar{r}_a)\delta(t - t_a) \quad [(c dt_a)^2 - (d\bar{r}_a)^2]^{\frac{1}{2}} . \qquad (1\text{-}27.73)$$

[Unlike (1-26.11), interaction terms are not included here.] The last integral is extended over the trajectory of the particle, with $\bar{r}_a$ varying as a function of t_a, or, better, with $\bar{r}_a$ and t_a given as functions of some parameter that is not changed by Lorentz transformations. Apart from a sign change and the consideration of infinitesimals, the space-time structure of the square root has just the invariant form in (1-27.33). Then, in response to

$$\delta\bar{r}_a = \delta\bar{v} t_a, \quad \delta t = \frac{1}{c^2}\, \delta\bar{v} \cdot \bar{r}_a, \qquad (1\text{-}27.74)$$

the delta function product becomes

$$\delta(\bar{r} - \bar{r}_a - \delta\bar{v} t_a)\delta(t - t_a - \frac{1}{c^2}\delta\bar{v}\cdot\bar{r}_a) = \delta(\bar{r} - \delta\bar{v}\, t - \bar{r}_a)\delta(t - \frac{1}{c^2}\delta\bar{v}\cdot\bar{r} - t_a), \qquad (1\text{-}27.75)$$

with the last form following from the delta function property, and the resulting change is just

$$-\delta_{coor} \quad [\delta(\bar{r} - \bar{r}_a)\delta(t - t_a)]. \qquad (1\text{-}27.76)$$

We have now verified, for every individual constituent of $\mathcal{L}_{tot}$, that

$$\delta\mathcal{L}_{tot} = -\delta_{coor}\, \mathcal{L}_{tot} \qquad (1\text{-}27.77)$$

More generally expressed: on introducing the transformed particle and field variables associated with the transformed coordinates,

$$\mathcal{L}'(\bar{r}',t') = \mathcal{L}(\bar{r},t); \qquad (1\text{-}27.78)$$

the Lagrange function of the system of interacting charges and fields is invariant under the Lorentz transformations of Einsteinian relativity.

The action

$$W_{12} = \int_2^1 dt(d\bar{r})\mathfrak{L} \qquad (1\text{-}27.79)$$

is also Lorentz invariant because the space-time, four dimensional, element of volume has that property; it suffices to examine the Jacobian determinant of the transformation (1-27.40), for example:

$$\cosh^2\theta - \sinh^2\theta = 1. \qquad (1\text{-}27.80)$$

We shall not repeat the discussion of the various conservation laws in the light of these relativistic modifications. It should be sufficiently evident that such expressions as the total energy density and energy flux vector, (1-26.21) and (1-26.22), will be regained with E_a replaced by the relativistic energy (1-27.12). And, certainly (1-27.3) is now satisfied!

Finally, we supply the finite versions of the Lorentz transformations for the various fields. Proceeding in analogy with (1-27.37), we write (1-27.61) as

$$\frac{dA_x(\theta)}{d\theta} = 0, \quad \frac{dA_y(\theta)}{d\theta} = 0, \quad \frac{dA_z(\theta)}{d\theta} = \phi(\theta), \quad \frac{d\phi(\theta)}{d\theta} = A_z(\theta), \qquad (1\text{-}27.81)$$

where d denotes the total change in the sense of (1-27.66). The finite transformation equations are then

$$A_x'(\bar{r}',t') = A_x(\bar{r},t), \quad A_y'(\bar{r}',t') = A_y(\bar{r},t),$$

$$A_z'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left(A_z(\bar{r},t) + \frac{v}{c}\,\phi(\bar{r},t)\right),$$

$$\phi'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left(\phi(\bar{r},t) + \frac{v}{c}\,A_z(\bar{r},t)\right). \qquad (1\text{-}27.82)$$

The same forms apply with $\bar{A}$, ϕ replaced by $\frac{1}{c}\mathbf{J}$, ρ. The electric and magnetic field equations supplied by (1-27.63) are

$$\frac{dE_z(\theta)}{d\theta} = 0, \quad \frac{dB_z(\theta)}{d\theta} = 0,$$

$$\frac{dE_x}{d\theta}(\theta) = B_y(\theta), \quad \frac{dB_y}{d\theta}(\theta) = E_x(\theta),$$

$$\frac{dE_y}{d\theta}(\theta) = -B_x(\theta), \quad \frac{dB_x}{d\theta}(\theta) = -E_y(\theta) \qquad (1\text{-}27.83)$$

and therefore

$$E_z'(\bar{r}',t') = E_z(\bar{r},t), \quad B_z'(\bar{r}',t') = B_z(\bar{r},t),$$

$$E_x'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left[E_x(\bar{r},t) + \frac{v}{c}B_y(\bar{r},t)\right],$$

$$B_y'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left[B_y(\bar{r},t) + \frac{v}{c}E_x(\bar{r},t)\right], \qquad (1\text{-}27.84)$$

$v_z E_y$

and

$$E_y'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left[E_y(\bar{r},t) - \frac{v}{c}B_x(\bar{r},t)\right],$$

$$B_x'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left[B_x(\bar{r},t) - \frac{v}{c}E_y(\bar{r},t)\right]. \qquad (1\text{-}27.85)$$

The invariance, under finite transformations, of $E^2 - B^2$ is readily apparent, as is the invariance property

$$\bar{E}'(\bar{r}',t')\cdot\bar{B}'(\bar{r}',t') = \bar{E}(\bar{r},t)\cdot\bar{B}(\bar{r},t). \qquad (1\text{-}27.86)$$

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X. Stationary Principles for Electrostatics

10-1. Stationary Principles for the Energy

We will now specialize the general formulation given in the previous section to static circumstances, where there is no time variation. First note that with all time derivatives equal to zero, the Hamiltonian (energy), (9.12), 1.23-2 is stationary with respect to variations in $\vec{p}$, $\vec{r}$, and $\vec{v}$, that is [see also (8.22)-(8.24)] 18-2 see also 1-19.4

$$\frac{\partial H}{\partial \vec{p}} = 0 \ , \quad \frac{\partial H}{\partial \vec{r}} = 0 \ , \quad \frac{\partial H}{\partial \vec{v}} = 0 \ .$$

We now drop all reference to particle velocity in (9.12) to obtain for the static energy

$$E = \int (d\vec{r}) \left[\rho\phi + \frac{1}{4\pi} \vec{E}\cdot\vec{\nabla}\phi + \frac{E^2}{8\pi} + \frac{1}{4\pi} \left(\vec{B}\cdot\vec{\nabla} \times \vec{A} - \frac{1}{2} B^2 \right) \right] \ . \tag{10.1}$$

If there is no motion, $\vec{B} = 0$ as there is no source to produce a magnetic field (we assume, as usual, the absence of magnetic charge); consequently we will temporarily ignore the magnetic terms. We will show that if we require this energy expression be stationary under variations $\delta\phi$ and $\delta\vec{E}$, we recover the equations of electrostatics. The resulting variation in the energy is

$$\delta E = \int (d\vec{r}) \left[\rho\delta\phi + \frac{1}{4\pi} \vec{E}\cdot\vec{\nabla}\delta\phi + \frac{1}{4\pi} \delta\vec{E}\cdot(\vec{\nabla}\phi + \vec{E}) \right] \ ,$$

which may be simplified by use of the identity

$$\vec{E}\cdot\vec{\nabla}\delta\phi = \vec{\nabla}\cdot(\vec{E}\delta\phi) - (\vec{\nabla}\cdot\vec{E})\ \delta\phi \ ,$$

to read

$$\delta E = \int (d\vec{r}) \left[\delta\phi \left(\rho - \frac{\vec{\nabla}\cdot\vec{E}}{4\pi} \right) + \frac{1}{4\pi} \delta\vec{E}\cdot(\vec{\nabla}\phi + \vec{E}) \right] , \tag{10.2}$$

where we have ignored the surface term since we assume the integral extends over the entire region where the fields are non-zero. The stationary requirement on E, $\delta E = 0$, now implies the two basic equations of electrostatics,

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho , \quad \vec{E} = -\vec{\nabla}\phi . \tag{10.3}$$

Of course, the connection between $\vec{E}$ and ϕ implies

$$\vec{\nabla} \times \vec{E} = 0 .$$

Having seen this, we can immediately modify the energy expression, (10.1), to incorporate the effects of a dielectric medium,

$$E = \int (d\vec{r}) \left[\rho\phi + \frac{\epsilon\vec{E}\cdot\vec{\nabla}\phi}{4\pi} + \frac{\epsilon E^2}{8\pi} \right] . \tag{10.4}$$

The validity of this form is indicated by noting that if it is required to be stationary under variations in ϕ and $\vec{D} = \epsilon\vec{E}$, we recover the equations of electrostatics in a dielectric. That is, the variation in the energy,

$$\delta E = \int (d\vec{r}) \left[\delta\phi \left(\rho - \frac{1}{4\pi} \vec{\nabla}\cdot\vec{D} \right) + \frac{1}{4\pi} \delta\vec{D}\cdot(\vec{\nabla}\phi + \vec{E}) \right] = 0 \tag{10.5}$$

implies

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho , \quad \vec{E} = -\vec{\nabla}\phi . \tag{10.6}$$

(As before, in writing the variation in the form of (10.5), we have integrated by parts and omitted the surface term.) We can use two restrictive versions of the above energy functional, (10.4).

1. We adopt as the definition of $\vec{E}$, its construction in terms of the scalar potential,

$$\vec{E} = -\vec{\nabla}\phi \quad ,$$

the curl of which is zero. The energy, as a functional of the potential, is then

ϵE^2

$$E(\phi) = \int (d\vec{r}) \left[\rho\phi - \frac{1}{8\pi}\, \epsilon(\vec{\nabla}\phi)^2 \right] \quad . \tag{10.7}$$

The requirement that E be stationary under the variation of ϕ yields Maxwell's equation

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \quad ,$$

where $\vec{D}$ is defined by

$$\vec{D} = -\epsilon\vec{\nabla}\phi \quad .$$

The energy functional, (10.7), contains yet further information. At the stationary point ϕ , E is an absolute maximum, as is seen by making a finite variation in the potential, $\delta\phi$,

$$E(\phi + \delta\phi) = E(\phi) - \int (d\vec{r})\, \frac{\epsilon}{8\pi}\, (\vec{\nabla}\delta\phi)^2 \le E(\phi) \quad , \tag{10.8}$$

since the linear term in $\delta\phi$ vanishes by the stationary principle. The correct energy, given by the physical ϕ , is a maximum of the functional (10.7). Evaluating the energy functional for an arbitrary potential bounds the energy from below.

Lecture 10

2. If we adopt

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \tag{10.9}$$

as defining $\vec{D}$, replace $\varepsilon\vec{E}$ by $\vec{D}$ in the expression for the energy, (10.4), and integrate by parts on the $\vec{D}\cdot\vec{\nabla}\phi$ term, we obtain

$$E(\vec{D}) = \int (d\vec{r})\, \frac{D^2}{8\pi\varepsilon} \quad , \tag{10.10}$$

which is a functional of $\vec{D}$. How does the stationary principle work here? The variation of (10.10) is

$$\delta E = \int (d\vec{r})\, \frac{1}{4\pi\varepsilon}\, \vec{D}\cdot\delta\vec{D} \quad ,$$

but $\nabla\cdot\delta D = 0$
it is not arbitrary

where the displacement vector is varied subject to the restriction that (10.9) be satisfied:

$$\vec{\nabla}\cdot(\vec{D} + \delta\vec{D}) = 4\pi\rho \quad ,$$

or

$$\vec{\nabla}\cdot\delta\vec{D} = 0 \quad .$$

Therefore, any variation in $\vec{D}$ must be a curl,

$$\delta\vec{D} \equiv \vec{\nabla}\times\vec{A} \quad ,$$

where $\vec{A}$ is an arbitrary, infinitesimal vector, enabling us to write the variation in the energy as

$$\delta E = \int (d\vec{r})\, \frac{\vec{E}}{4\pi}\cdot\vec{\nabla}\times\vec{A} \quad . \tag{10.11}$$