where θ is the angle of diffraction. Correspondingly, the asymptotic form of the Hankel function becomes

$$H_0^{(1)}\left[k\sqrt{x^2 + (y-y')^2}\right] \sim \sqrt{\frac{2}{i\pi k\rho}}\, e^{ik\rho}\, e^{-ik\sin\theta\, y'} \quad . \tag{43.39}$$

[We see here again a remnant of the factor

$$e^{-ik\vec{n}\cdot\vec{r}'}$$

characteristic of radiation fields. See, for example, (43.13).] When the wavelength is small compared to the slit, $a \gg \lambda\!\!\!^{-}$, we may again approximate the field in the slit by the incident field,

$$E(y') \simeq E_{inc}(y') = 1 \ .$$

We also recognize that the radiation is predominately forward, implying

$$-\frac{\partial}{\partial x} e^{ik\rho} = -ik\,\frac{x}{\rho}\, e^{ik\rho}$$

$$\simeq -ik\, e^{ik\rho} \ .$$

Note that x/ρ was approximated as 1

Thus, for $x \gg a \gg \lambda\!\!\!^{-}$, the field, (43.37b), may then be approximated by

$$E(x,y) \sim \sqrt{\frac{i}{2\pi k\rho}} \int_{-a/2}^{a/2} dy'(-ik)\, e^{ik\rho}\, e^{-ik\sin\theta\, y'}$$

$$= \sqrt{\frac{i}{2\pi k\rho}}\,(-ik)\, e^{ik\rho}\, \frac{2\sin\left(\frac{ka}{2}\sin\theta\right)}{k\sin\theta} \ , \tag{43.40}$$

ak

representing an outgoing cylindrical wave.

The differential cross section per unit length of the slit is determined by

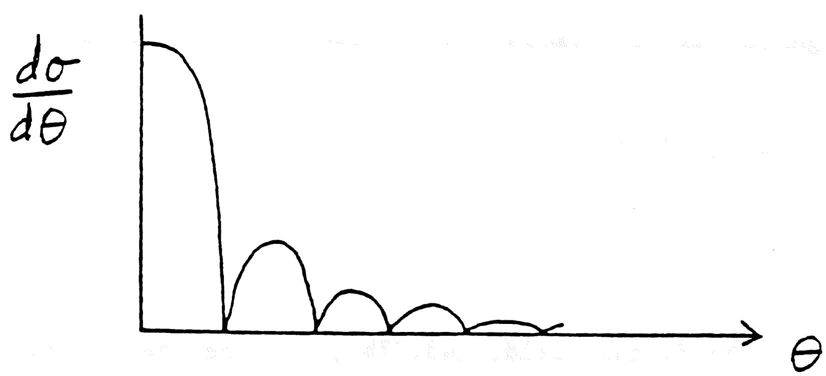
$$d\sigma = \frac{|E|^2}{|E_{inc}|^2}\,\rho d\theta\ ,$$

which, upon use of the asymptotic field, implies

$$\frac{d\sigma}{d\theta} = \frac{2}{\pi k}\left(\frac{\sin\left(\frac{ka}{2}\sin\theta\right)}{\sin\theta}\right)^2$$

$$\simeq \frac{2}{\pi k}\left(\frac{\sin\left(\frac{ka\theta}{2}\right)}{\theta}\right)^2 . \qquad (43.41)$$

The latter holds for $\theta \ll 1$, which is the only region of validity for our result. The resulting diffraction pattern is shown in the figure,



where the zeroes occur at equally spaced points:

$$ka\theta = n\,2\pi\ , \qquad n = \pm 1, \pm 2, \ldots\ . \qquad (43.42)$$

At $\theta = 0$, the differential cross section is

$$\left(\frac{d\sigma}{d\theta}\right)_{\theta=0} = \frac{1}{2\pi k}(ka)^2\ . \qquad (43.43)$$

In the short wavelength limit, we anticipate that the total cross section per unit length is given by the width of the slit. In fact, for $ka >> 1$, most of the contribution arises from values of θ near zero, so that we indeed have

$$\sigma \simeq \int_{-\infty}^{\infty} d\theta \, \frac{2}{\pi k} \left(\frac{\sin\left(\frac{ka}{2}\theta\right)}{\theta} \right)^2$$

$$= \frac{2}{\pi k}\frac{ka}{2} \int_{-\infty}^{\infty} dz \left(\frac{\sin z}{z} \right)^2 = a \,, \tag{43.44}$$

where we have used the integral (31.12).

As a check of consistency, we now turn to the limit $a \to \infty$, for which the slit disappears and the incident plane wave should propagate undisturbed. The field far away from the "slit" is given by (43.37b) and (43.38),

$$E(x,y) = \frac{i}{2} \int dy' \, E(y') \left(-\frac{\partial}{\partial x} \right) \left[\left(\frac{2}{i\pi k} \right)^{1/2} [x^2 + (y-y')^2]^{-1/4} \right.$$

$$\left. \times \, e^{ik\sqrt{x^2 + (y-y')^2}} \right] . \tag{43.45}$$

If we put $E(y') = 1$ for all y', we must recover the incident plane wave, $E(x,y) = e^{ikx}$. We expect that $y' \sim y$ gives the major contribution to the integral, since the field should just advance with constant phase in y, thereby allowing us to use

$$\sqrt{x^2 + (y-y')^2} \simeq x + \frac{(y-y')^2}{2x} \, . \tag{43.46}$$

Inserting this approximation into (43.45), we have the following asymptotic evaluation ($x >> \lambdabar$):

$$E(x,y) \sim \frac{i}{2}\int_{-\infty}^{\infty} dy' \left(-\frac{\partial}{\partial x}\right)\left[\left(\frac{2}{\pi ikx}\right)^{1/2} e^{ikx}\, e^{ik(y-y')^2/2x}\right]$$

$$\sim \frac{-i}{2}\int_{-\infty}^{\infty} dy' \left(\frac{2}{\pi ikx}\right)^{1/2} ik\, e^{ikx}\, e^{ik(y-y')^2/2x}$$

$$= \sqrt{\frac{k}{2\pi ix}}\, e^{ikx}\int_{-\infty}^{\infty} dy'\, e^{ik(y-y')^2/2x} = e^{ikx}\ , \tag{43.47}$$

as is expected. Here we note, as a check of the approximation $|y-y'| \ll x$, that the significant contributions to the Gaussian integral come from the values of y satisfying

$$\frac{|y-y'|}{x} \sim \sqrt{\frac{\lambda\!\!\!^{-}}{x}} \ll 1\ . \tag{43.48}$$

43-4. Diffraction by a Straight Edge

Finally, we consider the diffraction produced by a semi-infinite plane conductor, lying in the region defined by $x = 0$ and $y < 0$. The field produced by such a half plane conductor can be again described by (43.45), where, as a first approximation, we take

$$\begin{aligned} E(y') &\simeq E_{inc} = 1\ , & y' &> 0\ , \\ E(y') &\simeq 0\ , & y' &< 0\ . \end{aligned} \tag{43.49}$$

Using the approximation (43.46), we arrive at the expression

$$E(x,y) \sim \sqrt{\frac{k}{2\pi ix}}\, e^{ikx}\int_{0}^{\infty} dy'\, e^{-\frac{k}{2ix}(y-y')^2}\ , \tag{43.50}$$

which is valid for

$$x \gg \lambda\!\!\!^{-} \quad \text{and} \quad |y-y'| \ll x\ .$$

It is convenient to shift the origin of the y' integration:

$$E(x,y) \sim \sqrt{\frac{k}{2\pi ix}}\, e^{ikx} \int_{-\infty}^{y} dy'\, e^{-\frac{k}{2ix}y'^2}$$

$$= \sqrt{\frac{k}{2\pi ix}}\, e^{ikx} \left(\int_{-\infty}^{\infty} - \int_{y}^{\infty} \right) dy'\, e^{-\frac{k}{2ix}y'^2}$$

$$= e^{ikx} \left\{ 1 - \sqrt{\frac{k}{2\pi ix}} \int_{y}^{\infty} dy'\, e^{-\frac{k}{2ix}y'^2} \right\} . \qquad (43.51)$$

For sufficiently large y,

$$y \gg \sqrt{\lambda\!\!\!^{-} x} , \qquad (43.52)$$

the second term in (43.51) may be neglected, and the wave travels undisturbed:

$$E(x,y) \sim e^{ikx} . \qquad (43.53)$$

On the other hand, for $y = 0$ the integral in (43.51) is

$$\sqrt{\frac{k}{2\pi ix}} \int_{0}^{\infty} dy'\, e^{-\frac{ky'^2}{2ix}} = \frac{1}{2} ,$$

and, in consequence, the amplitude of the wave there is reduced by half:

$$E(x,0) \sim \frac{1}{2} e^{ikx} . \qquad (43.54)$$

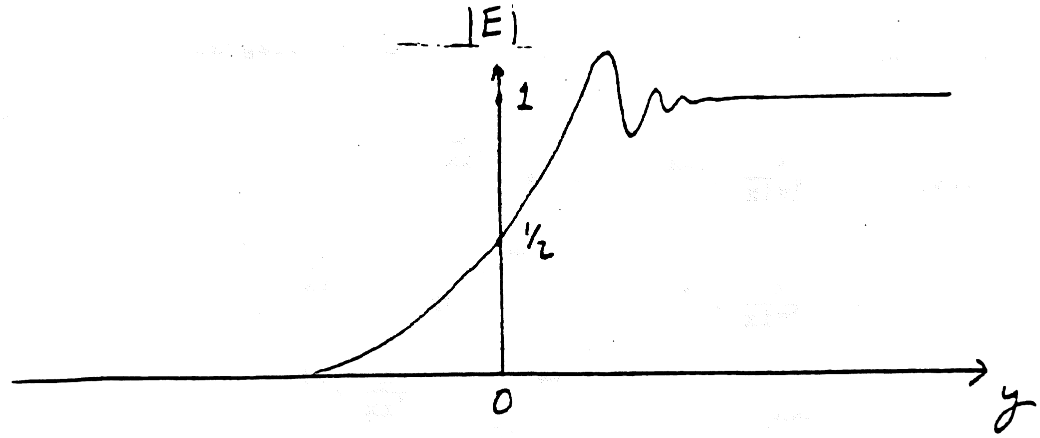
Far below the edge,

$$y < 0 , \quad |y| \gg \sqrt{\lambda\!\!\!^{-} x} , \qquad (43.55)$$

the diffracted field vanishes,

$$E(x,y) \sim 0 . \qquad (43.56)$$

To summarize the above results, we present a rough picture of the magnitude of the field strength as a function of y:



The region over which the intensity varies significantly has width

$$\Delta y \sim \sqrt{\lambda\!\!\!^{-} x} \ , \tag{43.57}$$

a distance small compared to x,

$$\frac{\Delta y}{x} \sim \sqrt{\frac{\lambda\!\!\!^{-}}{x}} \ll 1 \ . \tag{43.58}$$

A quite different limit of diffraction by a straight edge occurs when both x and y are large, while the diffraction angle θ is fixed:

$$\frac{y}{x} = \tan\theta \ . \tag{43.59}$$

We anticipate that the dominant contribution to the scattered field comes from the region near the edge, where y' is small, and in consequence

$$\sqrt{x^2 + (y-y')^2} \simeq \rho - y' \sin\theta \ , \tag{43.60}$$

where

$$\rho = \sqrt{x^2+y^2} \ .$$

Using this approximation in (43.45), we obtain for $\theta \ll 1$,

$$E_{scatt} \sim \sqrt{\frac{i}{2\pi k\rho}}\,(-ik)\, e^{ik\rho} \int_0^{\infty} dy'\, e^{-iky'\sin\theta}$$

$$\sim -\sqrt{\frac{i}{2\pi k\rho}}\,\frac{e^{ik\rho}}{\theta}\,. \tag{43.61}$$

We see the appearance of a cylindrical wave, originating from the edge, $x' = 0$, $y' = 0$. The corresponding differential scattering cross section per unit length is

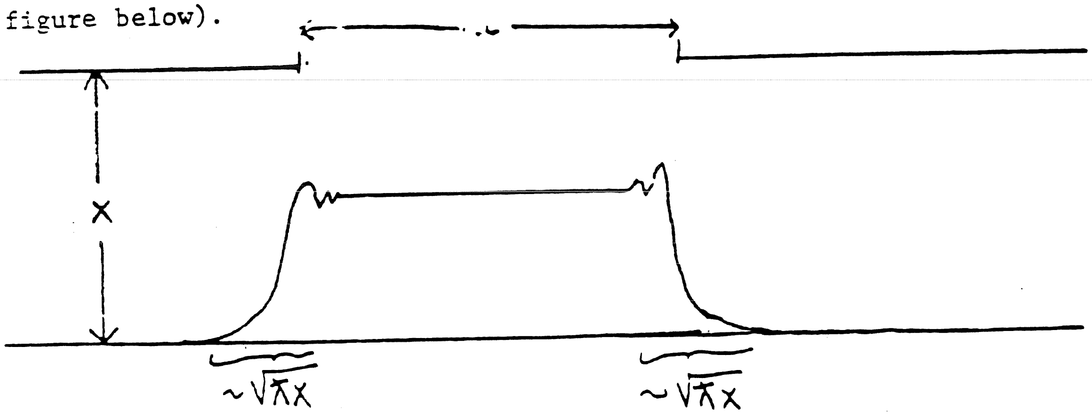
$$\frac{d\sigma}{d\theta} = \frac{|E_{scatt}|^2}{|E_{inc}|^2}\,\rho \simeq \frac{1}{2\pi k}\,\frac{1}{\theta^2}\,. \tag{43.62}$$

[In the next section we will provide an exact treatment of this diffraction problem.]

The above discussion of diffraction by an edge provides a clarification of our earlier consideration of diffraction by a slit. If we make observations by use of a screen sufficiently close to the slit, so that

$$a \gg \sqrt{\lambda x}\,, \tag{43.63}$$

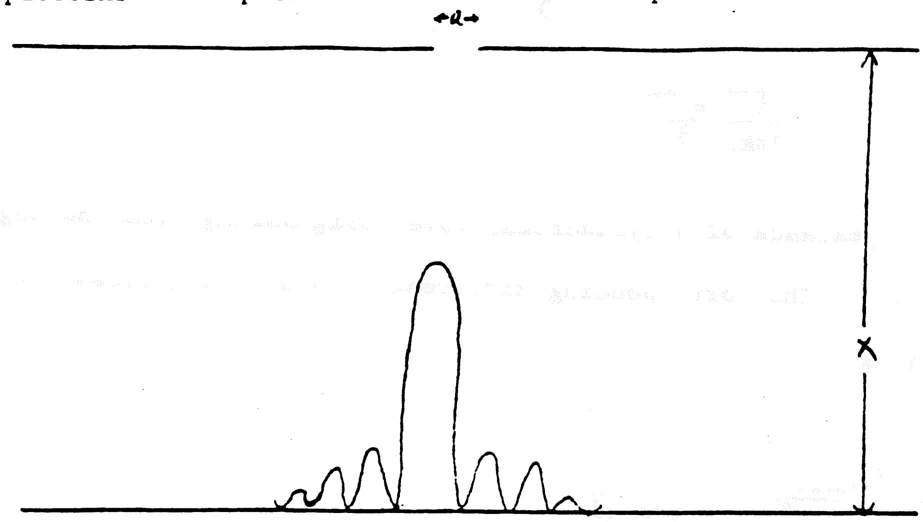
we see a geometrical image of the slit modulated by edge diffraction (see figure below).



However, if we move the screen so far back that

$$\sqrt{\lambda\!\!\!^{-} x} >> a \, , \tag{43.64}$$

the two patterns overlap to form the diffraction pattern derived in (43.41).



Mathematically, the origin of the requirement (43.64) can be traced to the fact that the expansion (43.60) is only valid when terms of order a^2/ρ are negligible, that is

$$\frac{ka^2}{\rho} = \frac{1}{\lambda\!\!\!^{-}} \frac{a^2}{\rho} << 1 \, , \tag{43.65}$$

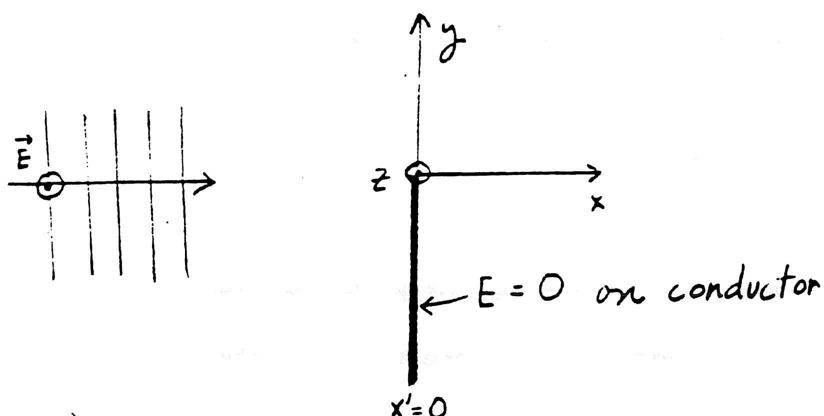
or

$$\lambda\!\!\!^{-}\rho >> a^2 \, . \tag{43.66}$$

Lecture 16

XLIV. DIFFRACTION II

We now adopt a more physical approach to diffraction in which the currents that give rise to the scattered wave are made explicit. We will reconsider diffraction by a semi-infinite metal conductor, for which this method is capable of giving an exact solution. The geometry of the situation is as given in the figure.



We consider $\vec{E}$ to possess only a z-component (z subscript suppressed), and decompose the electric field into incident and scattered parts,

$$E = E_{inc} + E_{scatt} \ . \tag{44.1}$$

We assume the incident field is a normalized plane wave with frequency $\omega = kc$,

$$E_{inc} = e^{ikx} \ . \tag{44.2}$$

The scattered field arises from the induced current that flows on the metal plate, which by symmetry can have only a z-component, $J = J_z$, and has no dependence on z,

$$\frac{\partial}{\partial z} J_z(x,y) = 0 \ , \tag{44.3}$$

in conformity with the properties of the incident electric field. Consequently, there is no charge density and the scattered electric field is expressed by

$$E_{scatt} = ik \int (d\vec{r}\,') \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \frac{1}{c} J(\vec{r}\,') \,, \tag{44.4}$$

which follows from (32.6) and (32.11a). We consider an infinitesimally thin conductor for which we may reduce the three-dimensional integral in (44.4) to a two-dimensional one by introducing the surface current,

$$\int dx'\, J_z = K \,. \tag{44.5}$$

Further, by identifying the z integral with the representation of the Hankel function (41.27), we may express the result as an integration over y' alone:

$$E = E_{inc} + ik \int_{-\infty}^{0} dy'\, \pi i\, H_0^{(1)}[k\sqrt{x^2 + (y-y')^2}]\, \frac{1}{c} K(y') \,. \tag{44.6}$$

For the semi-infinite perfect conductor being considered here, (44.6) is subject to the boundary condition

$$E = 0 \quad \text{for} \quad x = 0,\ y < 0 \,. \tag{44.7}$$

44-1. Approximate Solution

Before finding an exact solution to (44.6), we will first solve this equation by an approximate treatment, based on the fact that, on the conductor and far from the edge, the conducting sheet appears to be infinite. There, the incident wave is totally reflected,

$$E_z = e^{ikx} - e^{-ikx} ,$$

$$B_y = -e^{ikx} - e^{-ikx} , \tag{44.8}$$

which we will assume hold for all $x < 0$, $y < 0$, and that the fields vanish for $x > 0$, $y < 0$. From these forms for the electric and magnetic fields, we can find the induced current through the use of Maxwell's equation

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \dot{\vec{E}} + \frac{4\pi}{c} \vec{J} , \tag{44.9}$$

which becomes here

$$\frac{4\pi}{c} J_z = \partial_x B_y - \frac{1}{c} \dot{E}_z . \tag{44.10}$$

By integrating (44.10) across the conducting surface from just to the left to just to the right,

$$B_y(x = +0) - B_y(x = -0) = \frac{4\pi}{c} K ,$$

and noting that, in our approximation, (44.8),

$$B_y(x = +0) = 0 , \quad B_y(x = -0) = -2 , \tag{44.11}$$

we find the current appropriate to an infinite conducting sheet to be

$$K = \frac{c}{2\pi} . \tag{44.12}$$

Then from (44.6), the electric field everywhere is approximately given by

$$E = e^{ikx} - \frac{k}{2} \int_{-\infty}^{0} dy' \, H_0^{(1)}[k\sqrt{x^2 + (y-y')^2}] . \tag{44.13}$$

If we further use the asymptotic form (43.38) for the Hankel function, together with the approximation that when $|x|$ is large, only small values of $|y-y'|$ are significant,

$$\sqrt{x^2 + (y-y')^2} \sim |x| + \frac{(y-y')^2}{2|x|} ,$$

the expression for the electric field (44.13), becomes

$$E \sim e^{ikx} - \sqrt{\frac{k}{2\pi i|x|}}\, e^{ik|x|} \int_{-\infty}^{0} dy'\, e^{ik(y-y')^2/2|x|} . \qquad (44.14)$$

With the substitution $y'-y \to y'$, the integral in (44.14) has the form

$$\int_{-\infty}^{-y} dy'\, e^{ik\, y'^2/2|x|} . \qquad (44.15)$$

When $y > 0$ and far away from the edge, $y \gg \sqrt{\lambda\!\!\!^{-}|x|}$, the integral (44.15) is negligible, and the wave propagates undisturbed,

$$E \sim e^{ikx} .$$

On the other hand, sufficiently below the edge, $y < 0$, $|y| \gg \sqrt{\lambda\!\!\!^{-}|x|}$, the integral is $\sqrt{\frac{2\pi i}{k}|x|}$, and

$$E \sim e^{ikx} - e^{ik|x|}$$

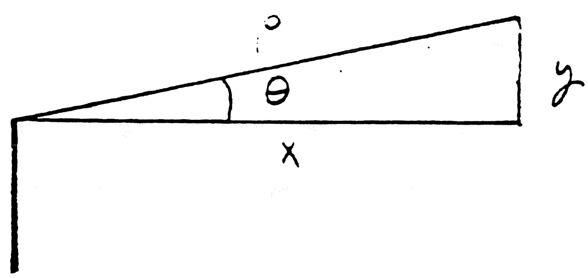
$$= \begin{cases} 0 , & x > 0 , \\ e^{ikx} - e^{-ikx} , & x < 0 , \end{cases}$$

reproducing the boundary condition (44.8). On line with the edge, $y = 0$, the electric field is

$$E \sim e^{ikx} - \frac{1}{2} e^{ik|x|} .$$

These results are identical with those found earlier in Subsection 43-4, but now a physical picture has been provided: The total electric field is the sum of the incoming field plus the field produced by the currents induced in the metal.

We next turn to the limit of large x and y where the angle of diffraction θ is fixed.



In this limit, the dominant contribution at (x,y) is the sum of the incident wave plus the wave scattered by the edge. In terms of the description employing currents, this is given approximately by (44.13), with

$$\sqrt{x^2 + (y-y')^2} \approx \rho - y' \sin\theta .$$

As in (43.61), the scattered field is

$$E_{scatt} \sim -\frac{k}{2} \int_{-\infty}^{0} dy' \sqrt{\frac{2}{\pi k \rho i}}\, e^{ik\rho}\, e^{-ik \sin\theta y'}$$

$$\sim -\sqrt{\frac{i}{2\pi k}}\, \frac{1}{\sqrt{\rho}}\, e^{ik\rho}\, \frac{1}{\sin\theta} , \qquad (44.16)$$

leading to a differential cross section per unit length (valid only for $\theta << 1$)

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k}\frac{1}{\theta^2} , \tag{44.17}$$

in agreement with (43.62).

44-2. Exact Solution for Current

We now seek an exact solution to (44.6) subject to the boundary condition (44.7). This is an integral equation since $K(y')$ and $E(y)$ are interrelated functions. To find the surface current, we consider the electric field on the $x = 0$ plane,

$$E(y) = 1 - \pi k \int_{-\infty}^{\infty} dy' \; H_0^{(1)}[k|y-y'|] \; \frac{1}{c} K(y') . \tag{44.18}$$

In order to solve this equation, we must recognize that an incident plane wave is an over-idealization, one that can be removed by introducing an exponential cutoff in y:

$$E_{inc}(x = 0) = 1 \to e^{-\varepsilon|y|} , \quad \varepsilon \to +0 . \tag{44.19}$$

Note in (44.18) that we have two conditions:

1. $K(y') = 0$ if $y' > 0$ (since there is no conductor there),
2. $E(y) = 0$ if $y < 0$ (since the conductor is perfect).

We have introduced an infinite range of integration in (44.18) in order to employ Fourier transforms:

$$E(\zeta) = \int_{-\infty}^{\infty} dy \; e^{-i\zeta y} \; E(y) ,$$

$$K(\zeta) = \int_{-\infty}^{\infty} dy' \; e^{-i\zeta y'} \; K(y') . \tag{44.20}$$

$-\frac{i}{\zeta - i\varepsilon} + \frac{i}{\zeta + i\varepsilon}$

The Fourier transform of the incident field on the surface is

$2\pi i(-i+i)$

$$\int_{-\infty}^{\infty} dy\, e^{-i\zeta y}\, e^{-\varepsilon|y|} = -\frac{i}{\zeta - i\varepsilon} + \frac{i}{\zeta + i\varepsilon}\,, \tag{44.21}$$

which, as expected, becomes $2\pi\delta(\zeta)$ as $\varepsilon \to 0$. Furthermore, we require the Fourier transform of the Hankel function. Starting from its integral representation, (41.27), using the three-dimensional Fourier transform of Green's function, (41.16), and then integrating over z, we have

$$\begin{aligned}
\pi\, i\, H_0^{(1)}(k|y-y'|) &= \int_{-\infty}^{\infty} dz\, \frac{e^{ik\sqrt{(y-y')^2+z^2}}}{\sqrt{(y-y')^2+z^2}} \\
&= 4\pi \int_{-\infty}^{\infty} dz \int \frac{dk_x\, dk_y\, dk_z}{(2\pi)^3}\, \frac{e^{ik_y(y-y')}\; e^{ik_z z}}{k_x^2 + k_y^2 + k_z^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2} \\
&= 4\pi \int \frac{dk_x\, dk_y}{(2\pi)^2}\, \frac{e^{ik_y(y-y')}}{k_x^2 + k_y^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2}\,.
\end{aligned}$$

Employing the simple contour integral

$$\int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \frac{1}{k_x^2 + k_y^2 - \left(\frac{\omega+i\varepsilon}{c}\right)^2} = \frac{i}{2\sqrt{(k+i\varepsilon)^2 - k_y^2}}\,, \qquad k = \frac{\omega}{c}\,,$$

(where the cuts are chosen not to cross the real k_y axis) we arrive at the following representation for the Hankel function,

$$\pi\, i\, H_0^{(1)}(k|y-y'|) = 2\pi i \int_{-\infty}^{\infty} \frac{dk_y}{2\pi}\, \frac{e^{ik_y(y-y')}}{\sqrt{(k+i\varepsilon)^2 - k_y^2}}\,, \tag{44.22}$$

which supplies the desired Four~~ier~~ ~~transform~~

$$\int_{-\infty}^{\infty} d(y-y')\, e^{-i\zeta(y-y')}\, H_0^{(1)}(k|y-y'|) = \frac{2}{\sqrt{(k+i\varepsilon)^2-\zeta^2}} \quad . \tag{44.23}$$

We now find for the Fourier transform of the integral equation (44.18),

$$E(\zeta) = -\frac{i}{\zeta-i\varepsilon} + \frac{i}{\zeta+i\varepsilon} - \pi k \frac{2}{\sqrt{(k+i\varepsilon)^2-\zeta^2}} \frac{1}{c} K(\zeta) \tag{44.24}$$

where, if we make explicit the regions in which the integrands are non-zero,

$$E(\zeta) = \int_0^{\infty} dy\, e^{-i\zeta y}\, E(y) \quad , \tag{44.25a}$$

$$K(\zeta) = \int_{-\infty}^{0} dy'\, e^{-i\zeta y'}\, K(y') \quad . \tag{44.25b}$$

If these integrals exist for real ζ, they will also exist for complex values of ζ. Anticipating that $E(y)$, $K(y)$ fall off like $e^{-\varepsilon|y|}$ as $|y| \to \infty$, we see that

$$\begin{aligned} &E(\zeta) \text{ exists for } \operatorname{Im}\zeta < \varepsilon \quad , \\ &K(\zeta) \text{ exists for } \operatorname{Im}\zeta > -\varepsilon \quad . \end{aligned} \tag{44.26}$$

We will call the half planes $\operatorname{Im}\zeta < \varepsilon$, $\operatorname{Im}\zeta > -\varepsilon$ the lower half plane (LHP) and upper half plane (UHP), respectively. It is essential to observe that the UHP and the LHP overlap in a strip,

$$-\varepsilon < \operatorname{Im}\zeta < \varepsilon \quad . \tag{44.27}$$

Our physical requirements of boundedness ensure that

$E(\zeta)$ is regular in the LHP ,

$K(\zeta)$ is regular in the UHP .

In order to examine clearly the analytic properties of (44.24), we multiply it by $\sqrt{k+i\varepsilon-\zeta}$:

$$\sqrt{k+i\varepsilon-\zeta}\; E(\zeta) = -i\,\frac{\sqrt{k+i\varepsilon-\zeta}}{\zeta-i\varepsilon} + i\,\frac{\sqrt{k+i\varepsilon-\zeta}}{\zeta+i\varepsilon} - 2\pi k\,\frac{\frac{1}{c}K(\zeta)}{\sqrt{k+i\varepsilon+\zeta}} \;. \tag{44.28}$$

The factors in (44.28) can be chosen to be regular in the following regions,

$\sqrt{k+i\varepsilon-\zeta}$: LHP ,

$\sqrt{k+i\varepsilon+\zeta}$: UHP ,

$\sqrt{k+i\varepsilon-\zeta}/(\zeta-i\varepsilon)$: LHP ,

$\sqrt{k+i\varepsilon-\zeta}/(\zeta+i\varepsilon)$: $-\varepsilon < \mathrm{Im}\,\zeta < \varepsilon$.

The last combination can be written as the sum of terms regular in the LHP and the UHP, respectively,

$$\frac{\sqrt{k+i\varepsilon-\zeta}}{\zeta+i\varepsilon} = \frac{\sqrt{k+i\varepsilon-\zeta} - \sqrt{k+2i\varepsilon}}{\zeta+i\varepsilon} + \frac{\sqrt{k+2i\varepsilon}}{\zeta+i\varepsilon} \;.$$

Thus we can reorganize (44.28) into parts that are regular in the LHP and in the UHP:

$$\sqrt{k+i\varepsilon-\zeta}\; E(\zeta) + i\,\frac{\sqrt{k+i\varepsilon-\zeta}}{\zeta-i\varepsilon} - i\,\frac{\sqrt{k+i\varepsilon-\zeta} - \sqrt{k+2i\varepsilon}}{\zeta+i\varepsilon}$$

$$= i\,\frac{\sqrt{k+2i\varepsilon}}{\zeta+i\varepsilon} - 2\pi k\,\frac{\frac{1}{c}K(\zeta)}{\sqrt{k+i\varepsilon+\zeta}} \;. \tag{44.29}$$

The right hand side of (44.29) is regular in the UHP, the left hand side in the LHP. Since the two functions are regular in a common region [the strip

(44.27)], they may be analytically continued into a function regular for all ζ.

We will now show that this function vanishes at infinity, so that it vanishes everywhere. To this end, we examine (44.25b) in the limit $\zeta \to \infty$, for which only the behavior of $K(y')$ for $y' \to 0$ is significant. Because there is no intrinsic length scale in this limit, the current near the edge must behave as a power of y',

$$K(y') \sim (-y')^{-\alpha} , \qquad y' \to 0 , \tag{44.30}$$

where, in order that the integral (44.25b) exist,

$$\alpha < 1 . \tag{44.31}$$

The behavior of the corresponding Fourier transform of the current for large ζ is therefore

$$\begin{aligned} K(\zeta) &\sim \int_{-\infty}^{0} e^{-i\zeta y'} (-y')^{-\alpha} \, dy' \\ &\sim \frac{1}{\zeta^{1-\alpha}} \to 0 , \qquad \zeta \to \infty . \end{aligned} \tag{44.32}$$

Similarly, since $E(y) = 0$ if $y < 0$, the continuity of E requires the following power law for E near the edge,

$$E(y) \sim y^{\beta} , \quad y \to +0 , \tag{44.33}$$

$$\beta > 0 , \tag{44.34}$$

which is equivalent to the asymptotic statement

$$E(\zeta) = \int_0^\infty e^{-i\zeta y} E(y)\, dy$$

$$\sim \frac{1}{\zeta^{1+\beta}} \to 0 , \quad \zeta \to \infty . \tag{44.35}$$

Hence, the function represented by the analytic continuation of either side of (44.29) vanishes at infinity, and thus, by Cauchy's theorem (mistakenly attributed to Liouville), is zero everywhere. Thus we have solutions for $K(\zeta)$ and $E(\zeta)$:

$$\frac{2\pi}{c} K(\zeta) = \frac{i}{\sqrt{k}} \frac{\sqrt{k+i\varepsilon+\zeta}}{\zeta+i\varepsilon} \tag{44.36a}$$

$$\sim \frac{1}{\sqrt{\zeta}} , \quad |\zeta| >> 1 , \tag{44.36b}$$

$$E(\zeta) = i \left(\frac{1}{\zeta+i\varepsilon} - \frac{1}{\zeta-i\varepsilon} \right)$$

$$- \frac{i\sqrt{k}}{\zeta+i\varepsilon} \frac{1}{\sqrt{k+i\varepsilon-\zeta}} \tag{44.37a}$$

$$\sim \frac{1}{\zeta^{3/2}} , \quad |\zeta| >> 1 . \tag{44.37b}$$

Comparison with (44.32) and (44.35) determines the powers

$$\alpha = \frac{1}{2} , \quad \beta = \frac{1}{2} , \tag{44.38}$$

implying the following spatial dependences,

$$K(y') \sim \frac{1}{\sqrt{|y'|}} , \tag{44.39}$$

$$E(y) \sim \sqrt{y} , \tag{44.40}$$

as $|y'|$, $y \to 0$. We will supply ~~a~~ physical discussion of the meaning of the behavior near the edge in Subsection 44-4.

Lecture 17

44-3. Exact Diffraction Cross Section

The exact current and field in the plane $x = 0$ are given by (44.36a) and (44.37a). By rewriting the expression for the surface current in the form

$$\frac{1}{c} K(\zeta) = \frac{i}{2\pi} \left\{ \frac{1}{\zeta+i\varepsilon} + \frac{1}{\sqrt{k}} \frac{\sqrt{k+\zeta+i\varepsilon} - \sqrt{k}}{\zeta+i\varepsilon} \right\} , \tag{44.41}$$

we can identify the first term as the Fourier transform of

$$\frac{1}{2\pi} \begin{cases} e^{-\varepsilon|y|} , & y < 0 , \\ 0 , & y > 0 , \end{cases} \tag{44.42}$$

which is the current, (44.12), we used in the first approximate solution to this problem, corresponding to the neglect of edge effects. [Here, (44.42) includes the exponential cutoff.] The second term in (44.41) thus gives the correction that must be added in order to obtain the exact current. By considering the behavior of (44.41) for ζ small (which, as we will see below, corresponds to small diffraction angles), we note that the first term is singular as $\zeta \to 0$, while the second is finite. Thus, the first approximation is valid for small angles, as we have previously asserted.

The asymptotic scattered field at fixed angle θ follows by use of (43.39) in (44.6):

$$E_{scatt} \sim -\sqrt{\frac{k}{2\pi i\rho}}\, e^{ik\rho} \int_{-\infty}^{0} dy'\, e^{-ik \sin\theta y'}\, \frac{2\pi}{c} K(y')$$

$$= -\sqrt{\frac{k}{2\pi i\rho}}\, e^{ik\rho}\, \frac{2\pi}{c}\, K(\zeta) \,, \tag{44.43}$$

where we have used the Fourier transform (44.25b) with

$$\zeta = k \sin\theta \,. \tag{44.44}$$

Then from the solution (44.36a) for the surface current, we find for the exact asymptotic scattered field

$$E_{scatt} \sim -\sqrt{\frac{i}{2\pi k\rho}}\, e^{ik\rho}\, \frac{\sqrt{1+\sin\theta}}{\sin\theta} \,. \tag{44.45}$$

As we anticipated, it agrees with the first approximation (44.16) when $\theta \ll 1$. The corresponding exact differential cross section per unit length is

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k}\, \frac{1+\sin\theta}{\sin^2\theta} \,, \tag{44.46}$$

generalizing (44.17). We recognize that the new factor in (44.45), $\sqrt{1+\sin\theta}$, is present in order to enforce the boundary condition that the electric field vanish at $\theta = -\frac{\pi}{2}, \frac{3\pi}{2}$.

Finally, we wish to make contact with the other method, in which the field in the "aperture," not the current on the conducting plate, is employed. That is, we use (43.37b), relating the scattered field to the field in the aperture, and (43.39), the asymptotic form of the Hankel function, to give the scattered electric field for finite diffraction angle θ, $|\theta| < \frac{\pi}{2}$,

$$E_{scatt} \sim \sqrt{\frac{k}{2\pi i}}\, \frac{e^{ik\rho}}{\sqrt{\rho}}\, \cos\theta \int_0^\infty dy\, e^{-ik\sin\theta y}\, E(y) \,, \tag{44.47}$$

where $E(y)$ is the exact electric field in the aperture. The integral in (44.47) is the Fourier transform (44.25a),

$$\int_0^\infty dy\, e^{-ik\,\sin\theta y}\, E(y) = E(\zeta = k\,\sin\theta) = -i\,\frac{1}{k\,\sin\theta}\,\frac{1}{\sqrt{1-\sin\theta}}\,, \tag{44.48}$$

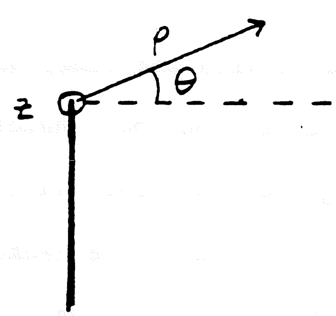
according to (44.37a). The resulting scattered electric field coincides with (44.45).

44-4. Field Near Edge

Here we wish to examine the form of the field near the edge, (44.40), from a different point of view. We may alternatively derive this result by solving the wave equation, (43.3),

$$\left(\nabla^2 + \frac{1}{\lambda\!\!\!^{-2}}\right) E = 0\,, \tag{44.49}$$

near the edge. Since there the field is rapidly varying over a distance small compared to the wavelength, we can omit the $1/\lambda\!\!\!^{-2}$ term. Thus our problem is the electrostatic one of finding the field near the edge of a plane conductor. Using the cylindrical coordinate system shown in the figure,



we write the Laplacian as

$$\nabla^2 = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)\,, \tag{44.50}$$

since there is no z dependence. By separating variables $(\partial^2/\partial\theta^2 \to -m^2)$, we find the characteristic solutions to Laplace's equation to be

$$E \sim \begin{Bmatrix} \sin m\theta \\ \cos m\theta \end{Bmatrix} \rho^m . \tag{44.51}$$

The boundary conditions that the field must vanish on the conducting plane,

$$E = 0 \quad \text{at} \quad \theta = -\frac{\pi}{2} \quad \text{and} \quad \frac{3\pi}{2} , \tag{44.52}$$

imply that the characteristic solutions are

$$E \sim \sin m\left(\theta + \frac{\pi}{2} \right) \rho^m , \tag{44.53}$$

with

$$\sin m \, 2\pi = 0 . \tag{44.54}$$

The smallest value of m consistent with (44.54) is

$$m = \frac{1}{2} , \tag{44.55}$$

giving the electric field

$$E = C\sqrt{\rho} \sin \frac{1}{2} \left(\theta + \frac{\pi}{2} \right) , \tag{44.56}$$

where C is a constant. [Note that the solution (44.45) exhibits this same behavior, since $\sqrt{1+\sin\theta} = \sqrt{2} \sin \frac{1}{2} \left(\theta + \frac{\pi}{2} \right)$.] For $x = 0$, $y > 0$ $(\theta = \pi/2)$, the electric field is

$$E\left(\rho, \frac{\pi}{2} \right) = C \sqrt{\rho} = C \sqrt{y} , \tag{44.57}$$

which is (44.40). Another way of writing (44.56) is satisfies the same boundary conditions, (44.63), so that we have the following asymptotic solution,

$$E \sim -Cx + C|x| = \begin{cases} 0 \ , & x > 0 \ , \\ -2Cx \ , & x < 0 \ . \end{cases} \tag{44.68}$$

Far from the surface, we can no longer neglect $k^2 = 1/\lambda\!\!\!^{-2}$ in (44.49). Consequently, the approximate diffracted wave, the static limit of which is (44.68), is

$$E = -\frac{2C}{k}\sin kx \ , \qquad x < 0 \ . \tag{44.69}$$

This is exact for zero aperture, a = 0, since it represents a standing wave due to total reflection by the conducting plane. The associated magnetic field,

$$ik\, B_y = -\frac{\partial}{\partial x} E_z = 2C\cos kx \ , \tag{44.70}$$

does not vanish on the conducting plane,

$$ik\, B_y(x = 0) = 2C \ ,$$

thereby determining the constant C in terms of the unperturbed magnetic field (that is, in terms of the magnetic field present when the slit is absent). Thus from (44.65), the electric field in the aperture, $|y| < \frac{a}{2}$, is determined by the magnetic field at x = 0,

$$E_z(x = 0) \simeq \frac{ik}{2} B_y(x = 0)\sqrt{\frac{a^2}{4} - y^2} \ , \tag{44.71}$$

for a narrow slit satisfying (44.62). The magnitude of the electric field in the aperture is small compared to the magnetic field there,

$$|E_z| \sim ka|B_y| \ll |B_y| \; , \tag{44.72}$$

and would be zero if (44.69) were exact. Equation (44.71) is the basis for treating the diffraction by a narrow slit.