$$\vec{d} = \int (d\vec{r}')\, \vec{r}'\rho(\vec{r}') \quad , \tag{19.4b}$$

$$\overleftrightarrow{q} = \int (d\vec{r}')(3\vec{r}'\vec{r}' - \overleftrightarrow{1}r'^2)\, \rho(\vec{r}') \quad , \tag{19.4c}$$

which are the total charge, the dipole moment vector, and the quadrupole moment dyadic, respectively.

Using this potential, we can now calculate the interaction energy of the charge distribution with an additional point charge e_1 located at a point $\vec{r}$ lying far outside the charge distribution:

$$E = e_1\phi(\vec{r}) = \frac{ee_1}{r} + \vec{d} \cdot \frac{e_1\vec{r}}{r^3} + \frac{1}{2}\, e_1 \frac{1}{r^5}\, \vec{r}\cdot\overleftrightarrow{q}\cdot\vec{r} + \ldots \quad . \tag{19.5}$$

We may alternatively interpret (19.5) as the interaction energy of the various moments of the charge distribution with the field produced by e_1 at the origin, that is

$$E = e\phi - \vec{d}\cdot\vec{E} + \frac{1}{6}\, \vec{\nabla}\cdot\overleftrightarrow{q}\cdot\vec{E} + \ldots \tag{19.6}$$

where

$$\phi = \frac{e_1}{r}$$

$$\vec{E} = \frac{e_1(-\vec{r})}{r^3} \quad .$$

[We have seen this form of the dipole interaction energy before, in Subsection 4-1.] This is a starting point for considering the interaction of one charge distribution with another charge distribution. For example, if one had a dipole $\vec{d}_1$, rather than a charge, e_1, interacting with a charge distribution which had only a dipole moment, $\vec{d}_2$, the interaction energy deduced from (19.6) would be

$$E = -\vec{d}_2 \cdot \left[-\vec{\nabla} \frac{\vec{r}\cdot\vec{d}_1}{r^3} \right] = - \frac{3\vec{r}\cdot\vec{d}_1\ \vec{r}\cdot\vec{d}_2 - \vec{d}_1\cdot\vec{d}_2\ r^2}{r^5} , \tag{19.7}$$

which is the dipole-dipole interaction.

Although this approach could obviously be continued indefinitely, it rapidly becomes unwieldy for higher multipoles. A systematic approach can be based on the use of spherical harmonics. Outside a charge distribution $(r > r')$, the potential (19.1) can be expanded in spherical harmonics according to (18.23), that is

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell m} \frac{r'^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell m}(\theta,\phi) \sqrt{\frac{4\pi}{2\ell+1}}\, Y^{*}_{\ell m}(\theta',\phi') , \tag{19.8}$$

as

$$\phi(\vec{r}) = \sum_{\ell m} \frac{1}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell m}(\theta,\phi)\, \rho_{\ell m} . \tag{19.9}$$

Here, the multipole moments, $\rho_{\ell m}$, are defined by

$$\rho_{\ell m} = \int (d\vec{r}')\, r'^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}\, Y^{*}_{\ell m}(\theta',\phi')\, \rho(\vec{r}') . \tag{19.10}$$

The connection with the previous definition, (19.4), is, for example, given by

$$\ell = 0 : \quad \rho_{00} = e ,$$

$$\ell = 1 : \quad \rho_{11} = -\frac{1}{\sqrt{2}}(d_x - id_y) \quad ,$$

$$\rho_{10} = d_z \quad ,$$

$$\rho_{1-1} = \frac{1}{\sqrt{2}}(d_x + id_y) \quad .$$

Now we return to the consideration of the energy of interaction of a charge distribution, $\rho(\vec{r})$, with an external potential, $\phi(\vec{r})$:

$$E = \int (d\vec{r})\, \rho(\vec{r})\, \phi(\vec{r}) \quad . \tag{19.11}$$

Since the potential is produced by sources outside of the charge distribution, it can be expanded in terms of spherical harmonics,

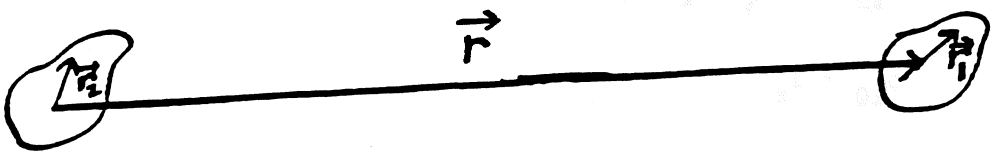
$$\phi(\vec{r}) = \sum_{\ell m} r^{\ell} Y_{\ell m}(\theta,\phi) \sqrt{\frac{4\pi}{2\ell+1}}\, \phi_{\ell m} \quad , \tag{19.12}$$

$\phi_{\ell m}$ being the expansion coefficients. Inserting this multipole expansion for the potential back into (19.11) and using the definition (19.10) for the multipole moments, we obtain the simple expression for the energy of interaction

$$E = \sum_{\ell m} \rho^{*}_{\ell m}\, \phi_{\ell m} \quad , \tag{19.13}$$

generalizing (19.6).

Rather than expressing the interaction energy in the unsymmetrical form (19.13), let us formulate the energy in terms of the interaction of the charge multipole moments of each distribution; that is, we seek a generalization of the dipole-dipole interaction, (19.7). If we let $\vec{r}_1$ and $\vec{r}_2$ be measured from points within ρ_1 and ρ_2 , respectively, while $\vec{r}$ measures the distance between these two origins, as illustrated in the diagram below,



the interaction energy can be written as

$$E = \int (d\vec{r}_1)\,(d\vec{r}_2)\,\frac{\rho_1(\vec{r}_1)\;\rho_2(\vec{r}_2)}{|\vec{r} + \vec{r}_1 - \vec{r}_2|} \quad . \tag{19.14}$$

Since the two charge distributions are non-overlapping, we can expand the denominator occurring here in a double Taylor series:

$$\frac{1}{|\vec{r} + \vec{r}_1 - \vec{r}_2|} = \sum_{\ell_1\ell_2} \frac{(\vec{r}_1\cdot\vec{\nabla})^{\ell_1}}{\ell_1!}\,\frac{(-r_2\cdot\vec{\nabla})^{\ell_2}}{\ell_2!}\,\frac{1}{r} \quad . \tag{19.15}$$

We already know that, for $r > r'$,

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell m} \frac{r'^{\ell}}{r^{\ell+1}}\,\frac{4\pi}{2\ell+1}\,Y_{\ell m}(\theta,\phi)\,Y^*_{\ell m}(\theta',\phi') = \sum_{\ell} \frac{(-\vec{r}\,'\cdot\vec{\nabla})^{\ell}}{\ell!}\,\frac{1}{r} \quad , \tag{19.16}$$

or, equating powers of r' ,

$$\frac{(-\vec{r}\,'\cdot\vec{\nabla})^{\ell}}{\ell!}\,\frac{1}{r} = \frac{4\pi}{2\ell+1}\sum_m r'^{\ell}\,Y_{\ell m}(\theta',\phi')\,r^{-\ell-1}\,Y^*_{\ell m}(\theta,\phi) \quad . \tag{19.17}$$

Further, recall the generating function for the spherical harmonics, (17.14),

$$\frac{(\vec{r}\,'\cdot\vec{a})^{\ell}}{2^{\ell}\ell!} = r'^{\ell}\sum_m \sqrt{\frac{4\pi}{2\ell+1}}\,Y_{\ell m}(\theta',\phi')\,\psi_{\ell m} \quad , \tag{19.18}$$

which is valid for $\vec{a}^2 = 0$. Thus it is permissible to replace $\vec{a}$ by a gradient,

$$\vec{a} \to \vec{\nabla} ,$$

as long as the derivatives act on $1/r$,

$$\vec{a}^2 \to \nabla^2 \frac{1}{r} = 0 , \qquad r > 0 .$$

In this way, a comparison of (19.17) and (19.18) gives the identity

$$\sqrt{\frac{4\pi}{2\ell+1}}\, r^{-\ell-1}\, Y^*_{\ell m}(\theta,\phi) = (-1)^{\ell} 2^{\ell} \psi_{\ell m} \frac{1}{r} , \tag{19.19}$$

where $\psi_{\ell m}$ is now regarded as a differential operator, constructed according to (17.13) from $\vec{\nabla}$. Using (19.18) twice with the above replacement, we obtain

$$\frac{(\vec{r}_1\cdot\vec{\nabla})^{\ell_1}}{\ell_1!}\frac{(-\vec{r}_2\cdot\vec{\nabla})^{\ell_2}}{\ell_2!}\frac{1}{r} = (-1)^{\ell_2}\, 2^{\ell_1} r_1^{\ell_1}\, 2^{\ell_2} r_2^{\ell_2}$$
$$\times \sum_{m_1 m_2} \sqrt{\frac{4\pi}{2\ell_1+1}}\, Y_{\ell_1 m_1}(\theta_1,\phi_1) \sqrt{\frac{4\pi}{2\ell_2+1}}\, Y_{\ell_2 m_2}(\theta_2,\phi_2)$$
$$\times\, \psi_{\ell_1 m_1}\psi_{\ell_2 m_2}\frac{1}{r} . \tag{19.20}$$

According to the definition of $\psi_{\ell m}$, (17.13), the product of two of these functions is

$$\psi_{\ell_1 m_1}\psi_{\ell_2 m_2} = C_{\ell_1\ell_2\, m_1 m_2}\, \psi_{\ell_1+\ell_2,\, m_1+m_2} , \tag{19.21}$$

where

$$C_{\ell_1\ell_2m_1m_2} = \left[\frac{(\ell_1+\ell_2+m_1+m_2)!\;(\ell_1+\ell_2-m_1-m_2)!}{(\ell_1+m_1)!\;(\ell_1-m_1)!\;(\ell_2+m_2)!\;(\ell_2-m_2)!}\right]^{1/2} \quad . \tag{19.22}$$

Then we evaluate the derivative structure in (19.20) by means of (19.19):

$$\psi_{\ell_1+\ell_2,\; m_1+m_2}\frac{1}{r} = (-1)^{\ell_1+\ell_2}\frac{1}{2^{\ell_1+\ell_2}}\sqrt{\frac{4\pi}{2(\ell_1+\ell_2)+1}}\; r^{-\ell_1-\ell_2-1}$$

$$\times\; Y^*_{\ell_1+\ell_2,\; m_1+m_2}(\theta,\phi) \quad . \tag{19.23}$$

Combining (19.14), (19.15), (19.20), (19.21), and (19.23), taking the complex conjugate, and identifying $\rho_{\ell m}$, (19.10), we find for the energy of interaction

$$E = \sum_{\ell_1\ell_2m_1m_2}(-1)^{\ell_1}\left[\frac{4\pi}{2(\ell_1+\ell_2)+1}\right]^{1/2} C_{\ell_1\ell_2m_1m_2}\frac{1}{r^{\ell_1+\ell_2+1}}$$

$$\times\; \rho_{\ell_1m_1}\, Y_{\ell_1+\ell_2,\; m_1+m_2}(\theta,\phi)\, \rho_{\ell_2m_2} \quad . \tag{19.24}$$

If we were to set $\ell_1=\ell_2=1$ we would rederive the dipole-dipole interaction, (19.7). However, this is a completely general result for the interaction between two arbitrary non-overlapping charge distributions, of a remarkably simple and compact form.

XX. Hollow Conducting Sphere

The spherical harmonics are useful in solving problems posessing spherical symmetry. In this and the next section, we will solve two such problems. Here we wish to find Green's function inside a hollow conducting sphere of radius a , which is grounded, that is, the potential is zero on its surface. As usual, the Green's function equation is

$$-\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad . \tag{20.1}$$

The solution must be expressible in terms of spherical harmonics as

$$r < r' : \qquad G = \sum_{\ell m} r^{\ell} Y_{\ell m}(\theta,\phi)\, A_{\ell m} \quad , \tag{20.2a}$$

see 18.15

$$r' < r \leq a : \quad G = \sum_{\ell m} \left(\frac{1}{r^{\ell+1}} - \frac{r^{\ell}}{a^{2\ell+1}} \right) Y_{\ell m}(\theta,\phi)\, B_{\ell m} \quad , \tag{20.2b}$$

see 18.16

where we have imposed the boundary conditions that

G = finite at $r = 0$,

and

$G = 0$ at $r = a$.

To determine the expansion coefficients, $A_{\ell m}$ and $B_{\ell m}$, we use the equations for the continuity of G , (18.17a),

$$r'^{\ell} A_{\ell m} = \left(\frac{1}{r'^{\ell+1}} - \frac{r'^{\ell}}{a^{2\ell+1}} \right) B_{\ell m} \quad , \tag{20.3a}$$

cont. at r=r'

and for the discontinuity of $\frac{\partial}{\partial r} G$, (18.17b) ,

$$\sum_{\ell m}\left[\left(\frac{\ell+1}{r'^{\ell}}+\frac{\ell r'^{\ell+1}}{a^{2\ell+1}}\right)Y_{\ell m}B_{\ell m}+\ell r'^{\ell+1}\,Y_{\ell m}A_{\ell m}\right]$$

$$=4\pi\sum_{\ell m}Y_{\ell m}(\theta,\phi)\,Y^{*}_{\ell m}(\theta',\phi')\quad, \tag{20.3b}$$

at $r = r'$. Solving (20.3a) by introducing $C_{\ell m}$, defined by

$$A_{\ell m}=\left(\frac{1}{r'^{\ell+1}}-\frac{r'^{\ell}}{a^{2\ell+1}}\right)C_{\ell m}\quad, \qquad \left(r'^{-\ell}-\frac{a^{2\ell+1}}{r'^{\ell+1}}\right)C_{\ell m} \tag{20.4a}$$

$$B_{\ell m}=r'^{\ell}\,C_{\ell m}\quad, \qquad \left(\frac{1}{r'^{\ell+1}}\right)C_{\ell m} \tag{20.4b}$$

we find, from (20.3b),

$$C_{\ell m}=\frac{4\pi}{2\ell+1}\,Y^{*}_{\ell m}(\theta',\phi')\quad. \tag{20.5}$$

Therefore, Green's function is

$$G=\sum_{\ell m}\left(\frac{r_<^{\ell}}{r_>^{\ell+1}}-\frac{r^{\ell}r'^{\ell}}{a^{2\ell+1}}\right)\underbrace{\frac{4\pi}{2\ell+1}\,Y_{\ell m}(\theta,\phi)\,Y^{*}_{\ell m}(\theta',\phi')}_{P_\ell(\cos\gamma)}\quad. \tag{20.6}$$

Noticing that

$$\frac{r^{\ell}r'^{\ell}}{a^{2\ell+1}}=\frac{a}{r'}\,\frac{r^{\ell}}{\left(\frac{a^2}{r'}\right)^{\ell+1}}\quad,\qquad \frac{a^2}{r'}>a\geq r\quad,$$

for exterior case replace $\frac{r^{\ell}r'^{\ell}}{a^{2\ell+1}}$ with $\frac{a^{2\ell+1}}{(r\,r')^{\ell+1}}$

we can perform the summation in (20.6) by use of (18.23):

$$G=\frac{1}{|\vec r-\vec r\,'|}-\frac{a}{r'}\,\frac{1}{|\vec r-\vec{\bar r}\,'|} \tag{20.7}$$

for exterior problem get same result

where $\vec{\bar r}\,'$ locates the so-called image point,

$$\vec{r}' = \left(\frac{a^2}{r'}, \theta', \phi' \right) , \tag{20.8}$$

which, of course, lies outside the sphere. Thus we have achieved for the sphere the analog of the image solution given for the conducting plane in (14.1).

Lecture 17.

What is the induced charge density on the inside surface of the sphere? This charge density is proportional to the radial electric field, according to (10.31),

$$4\pi\sigma = -E_r = \frac{\partial}{\partial r} G \bigg|_{r=a} \tag{20.9}$$

since the normal is inward, and so in the negative radial direction. Differentiating (20.6) with respect to $r = r_>$, and using the addition theorem, (18.24), we obtain

$$4\pi\sigma = -\sum_{\ell} (2\ell+1) \frac{1}{a^2} \left(\frac{r'}{a} \right)^{\ell} P_{\ell}(\cos\gamma) , \tag{20.10}$$

where γ is the angle between $\vec{r}$ and $\vec{r}'$, (18.8). Alternatively, we could use the image charge form of Green's function, (20.7), to derive

$$4\pi\sigma = -\frac{1}{a^2} \frac{1-(r'/a)^2}{\left[1 - 2\frac{r'}{a}\cos\gamma + \left(\frac{r'}{a} \right)^2 \right]^{3/2}} , \tag{20.11}$$

which indicates that as $r' \to a$, the only significant charge buildup is near $\gamma = 0$. The total charge can be computed from (20.10) by use of the orthonormality condition, (17.33), which, for $\ell' = 0$, implies

$$\int_0^{\pi} \sin\gamma \, d\gamma \, P_\ell(\cos\gamma) = 2\delta_{\ell 0} \quad . \tag{20.12}$$

Therefore, only the $\ell = 0$ term in (20.10) contributes to the total charge:

$$Q = \int dS\sigma = \int a^2 \sin\gamma \, d\gamma \, d\phi \left[- \frac{1}{4\pi a^2} \right] = -1 \ , \tag{20.13}$$

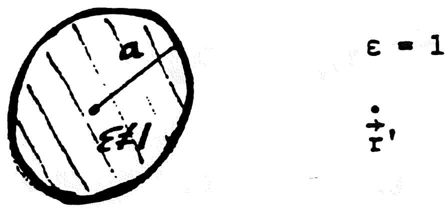
which is the expected result. Of course, it is possible to use (20.11) to compute the total charge, but that is more elaborate.

20.11 ⇒ $4\pi \int \sigma \, dS = 4\pi \int \sigma a^2 \sin\gamma$ [illegible]

$= \int -\sum (2\ell+1) \frac{1}{a^2} \left(\frac{r}{a}\right)^\ell P_\ell(\cos\gamma)$ [illegible]

XXI. Dielectric Sphere

As a second example of the use of spherical harmonics in solving Green's function problems with spherical symmetry, we consider a dielectric sphere of radius a, with a unit point charge outside.



In this case, the Green's function equation is

$$r > a: \quad -\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{21.1a}$$

$$r < a: \quad -\vec{\nabla}\cdot[\varepsilon\vec{\nabla}G(\vec{r},\vec{r}\,')] = 0 \quad , \tag{21.1b}$$

where we will take ε to be a constant. The boundary conditions at $r = a$ are, from (10.24) and (10.26),

$$G \text{ is continuous,} \tag{21.2a}$$

and

$$\left[-\frac{\partial}{\partial r}G\right]_{r=a+0} = \left[-\varepsilon\frac{\partial}{\partial r}G\right]_{r=a-0} \quad . \tag{21.2b}$$

The conditions on G at $r = r'$ are as given in (18.17),

$$G \text{ is continuous,} \tag{21.2c}$$

and

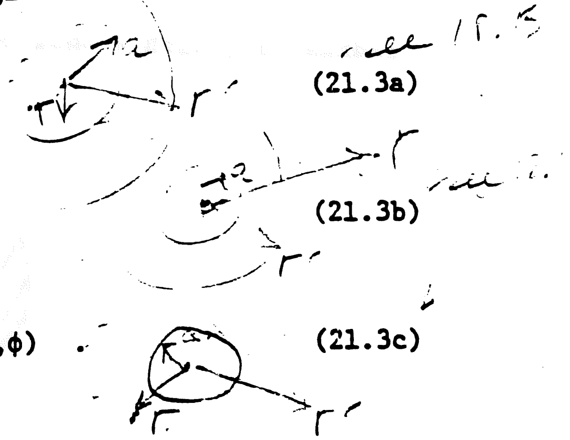
$$\left[-r^2 \frac{\partial}{\partial r} G\right]_{r=r'-0}^{r=r'+0} = 4\pi \underbrace{\frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\phi-\phi')}_{\sum_{\ell m} Y_{\ell m} Y^*_{\ell m}} \quad . \tag{21.2d}$$

As is familiar by now, the solution in the three regions has the form

$$r < a : \qquad G = \sum_{\ell m} r^{\ell} Y_{\ell m}(\theta,\phi)\, A_{\ell m} \quad , \tag{21.3a}$$

$$r > r' : \qquad G = \sum_{\ell m} r^{-\ell-1}\, Y_{\ell m}(\theta,\phi)\, D_{\ell m} \quad , \tag{21.3b}$$

$$a < r < r' : \quad G = \sum_{\ell m} (r^{\ell} B_{\ell m} + r^{-\ell-1}\, C_{\ell m})\, Y_{\ell m}(\theta,\phi) \quad . \tag{21.3c}$$



It is very easy to find the expansion coefficients by use of (21.2):

$$A_{\ell m} = \frac{4\pi\, Y^*_{\ell m}(\theta',\phi')}{\ell(\varepsilon+1)+1}\, r'^{-\ell-1} \quad , \tag{21.4a}$$

$$B_{\ell m} = \frac{4\pi\, Y^*_{\ell m}(\theta',\phi')}{2\ell+1}\, r'^{-\ell-1} \quad , \tag{21.4b}$$

$$C_{\ell m} = -\frac{(\varepsilon-1)\ell}{\ell(\varepsilon+1)+1}\, \frac{4\pi\, Y^*_{\ell m}(\theta',\phi')}{2\ell+1}\, \frac{a^{2\ell+1}}{r'^{\ell+1}} \quad , \tag{21.4c}$$

$$D_{\ell m} = C_{\ell m} + \frac{4\pi\, Y^*_{\ell m}(\theta',\phi')}{2\ell+1}\, r'^{\ell} \quad . \tag{21.4d}$$

Green's function, outside the sphere, is therefore found to be

$$r, r' > a : \quad G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} - \sum_{\ell=1}^{\infty} \frac{(\varepsilon-1)\ell}{\ell(\varepsilon+1)+1}\, \frac{a^{2\ell+1}}{r^{\ell+1}\; r'^{\ell+1}}\, P_{\ell}(\cos\gamma) \quad . \tag{21.5}$$

We now ask what is the leading behavior of this potential when the separation between the point charge and the sphere is large compared to the radius of the sphere, $r' \gg a$. Since the ℓ^{th} term in the sum behaves as

$\left(\frac{a}{r'}\right)^{\ell+1}$, only small values of ℓ contribute. The leading contribution arises from $\ell = 1$,

$$r' \gg a : \quad G(\vec{r},\vec{r}\,') \sim \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{\varepsilon-1}{\varepsilon+2}\frac{a^3}{r^2 r'^2}\cos\gamma \quad . \tag{21.6}$$

Since γ is the angle between $\vec{r}$ and $\vec{r}\,'$,

$$\cos\gamma = \frac{\vec{r}\cdot\vec{r}\,'}{rr'} \quad ,$$

this asymptotic form of Green's function can be rewritten as

$$G(\vec{r},\vec{r}\,') \sim \frac{1}{|\vec{r}-\vec{r}\,'|} + \frac{\vec{r}}{r^3}\cdot\vec{d} \quad , \tag{21.7}$$

see 19.3 here $\gamma = 1$

the two terms of which have simple physical interpretations. The first term is due to the point charge while the second is the potential arising from the induced electric dipole moment of the sphere [cf. (19.3)]. The latter is identified from (21.6) to be

$$\vec{d} = \frac{\varepsilon-1}{\varepsilon+2}\,a^3\left(-\frac{\vec{r}\,'}{r'^3}\right) \quad , \tag{21.8}$$

where $-\vec{r}\,'/r'^3$ is interpreted as the electric field, $\vec{E}(0)$, at the center of the sphere (in the absence of the dielectric) produced by the unit point charge. Since this electric field is essentially constant over the sphere, we recognize that the electric dipole moment induced in a dielectric sphere of radius a by a constant electric field $\vec{E}$ is

$$\vec{d} = \frac{\varepsilon-1}{\varepsilon+2}\,a^3\,\vec{E} \quad . \tag{21.9}$$

We complete this section by writing the expression for Green's function inside the sphere:

$$\begin{matrix} r < a \\ r' > a \end{matrix} \quad : \quad G(\vec{r},\vec{r}\,') = \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{r'^{\,\ell+1}} \frac{2\ell+1}{\ell(\varepsilon+1)+1} P_{\ell}(\cos\gamma) \quad . \tag{21.10}$$

Again, in the situation in which the point charge is located far from the sphere, $r' \gg a$, low values of ℓ predominate:

$$G \sim \frac{1}{r'} + \frac{3}{\varepsilon+2} \frac{\vec{r}\cdot\vec{r}\,'}{r'^{3}} = \frac{1}{r'} - \frac{3}{\varepsilon+2} \vec{r}\cdot\vec{E}(0) = \frac{1}{r'} - \vec{r}\cdot\vec{E} \quad , \tag{21.11}$$

where we identify the electric field in the dielectric as the negative gradient of G , that is

$$\vec{E} = \frac{3}{\varepsilon+2} \vec{E}(0) \quad , \tag{21.12}$$

which is less than $\vec{E}(0)$, if $\varepsilon > 1$.

XXII. Electrostatics: Dielectrics and Conductors

22-1. Variational Principle

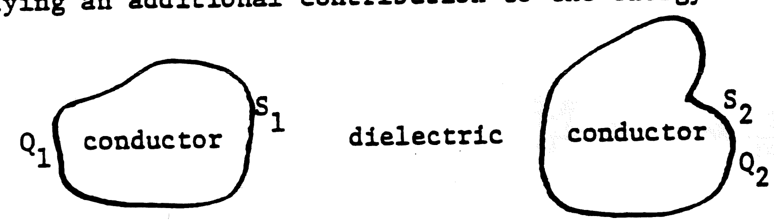
In Section X, we investigated the stationary properties of the electrostatic energy when only dielectrics are present, that is, we had a stationary principle,

$$\delta E = 0 \quad ,$$

where [see (10.4)]

$$E = \int (d\vec{r}) \left[\rho\phi + \frac{\varepsilon}{4\pi} (\vec{E}\cdot\vec{\nabla}\phi + \frac{1}{2} \vec{E}^2) \right] \quad . \tag{22.1}$$

We now wish to generalize this situation to include conductors as well. The new feature here is the existence of surface charges on various conductors implying an additional contribution to the energy:



$$E = \int (d\vec{r}) \left[\rho\phi + \frac{\varepsilon}{4\pi} (\vec{E}\cdot\vec{\nabla}\phi + \frac{1}{2} \vec{E}^2) \right] + \sum_{i=1}^{n} \int dS_i \, \sigma\phi \; , \tag{22.2}$$

where the volume integral extends over all space exterior to the conductors and the surface integral is over all of the conductors, σ being the surface charge density. This energy functional is to be supplemented by the condition that the total charge on each conductor,

$$Q_i = \int dS_i \, \sigma \; , \qquad i = 1,2,\ldots n \quad , \tag{22.3}$$

is fixed. The electrostatic problem is completely specified by the location of the conductors and dielectrics, the free volume charge density, ρ, and the charge on each conductor, Q_i . Note in particular that the surface charge density, σ , is to be determined dynamically. The change of the energy under variations of ϕ, $\vec{E}$, and σ is

$$\delta E = \int (d\vec{r})\left[\rho\delta\phi + \frac{\varepsilon}{4\pi}(\delta\vec{E}\cdot\vec{\nabla}\phi + \vec{E}\cdot\vec{\nabla}\delta\phi + \vec{E}\cdot\delta\vec{E})\right] + \sum_i \int dS_i[\delta\sigma\phi + \sigma\delta\phi] \quad , \tag{22.4}$$

which is subject to the condition that Q_i be constant, that is

$$\int dS_i \; \delta\sigma = 0 \quad . \tag{22.5}$$ ✓

We rewrite the $\vec{\nabla}\delta\phi$ term by means of an integration by parts, which makes use of the identity

$$\frac{\vec{D}}{4\pi} \cdot \vec{\nabla}\delta\phi = \vec{\nabla}\cdot\left(\frac{\vec{D}}{4\pi}\delta\phi\right) - \delta\phi\,\frac{\vec{\nabla}\cdot\vec{D}}{4\pi} \quad . \tag{22.6}$$

The implied surface integral here cannot be discarded since now there are contributions arising from the surfaces of the conductors. If we let $\vec{n}_i$ be the outward normal on the i^{th} conductor, this surface term is

$$\int (d\vec{r})\; \vec{\nabla}\cdot\left(\frac{\vec{D}}{4\pi}\delta\phi\right) = -\sum_i \int dS_i\, \frac{\vec{n}_i\cdot\vec{D}}{4\pi}\,\delta\phi \quad . \tag{22.7}$$ ✓

The variation in the energy, (22.4), now reads

$$\delta E = \int (d\vec{r})\left[\rho\delta\phi + \frac{\varepsilon}{4\pi}\delta\vec{E}\cdot\vec{\nabla}\phi - \delta\phi\,\frac{\vec{\nabla}\cdot\vec{D}}{4\pi} + \frac{1}{4\pi}\vec{D}\cdot\delta\vec{E}\right]$$

$$+ \sum_i \int dS_i \left[- \frac{\vec{n}_i \cdot \vec{D}}{4\pi} \delta\phi + \delta\sigma\phi + \sigma\delta\phi \right] . \tag{22.8}$$

The requirement that the energy be stationary under independent variations in ϕ and $\vec{E}$ then implies, in the interior of the dielectric,

$$\delta\phi : \quad \vec{\nabla}\cdot\vec{D} = 4\pi\rho \quad , \tag{22.9}$$

$$\delta\vec{E} : \quad \vec{E} = -\vec{\nabla}\phi \quad , \tag{22.10}$$

while just outside the surfaces of the conductors,

$$\delta\phi : \quad \vec{n}\cdot\vec{D} = 4\pi\sigma \quad . \quad \text{or} \quad \boxed{-\hat{n}_i \cdot \epsilon\vec{E} = 4\pi\sigma} \quad \hat{n} = -\hat{n}_i \tag{22.11}$$

Finally, the variation in the surface charge density requires

$$\delta E = \sum_i \int dS_i \; \delta\sigma\phi = 0$$

which is subject to the restriction (22.5), implying that each conductor is an equipotential surface,

$$\boxed{\phi = \text{constant on } S_i = \phi_i} \; . \tag{22.12}$$

Thus, the stationary action principle, based on the energy functional (22.2), yields all the physical laws governing electrostatics in the presence of conductors and dielectrics.

22-2. Restricted Forms of the Variational Principle

As in Section X, there are two restricted forms of the variational principle we may discuss. In the first, we take the electric field as being defined by

1. $$\vec{E} = -\vec{\nabla}\phi \quad , \tag{22.13}$$

so that the energy functional becomes

$$E = \int (d\vec{r}) \left[\rho\phi - \frac{\varepsilon}{8\pi} (\vec{\nabla}\phi)^2 \right] + \sum_i \int dS_i \, \sigma\phi \quad . \tag{22.14}$$

The independent variables are ϕ and σ, the latter of which is subject to the condition (22.3). For the second form, $\vec{D}$ is regarded as an independent variable, subject to the condition

2. $$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \, , \qquad \text{inside dielectric} \, , \tag{22.15a}$$

while σ is determined by

$$\vec{n}\cdot\vec{D} = 4\pi\sigma \, , \qquad \text{on surface } S_i \; . \tag{22.15b}$$

To rewrite the energy as a functional of $\vec{D}$ only, we integrate by parts on the $\vec{D}\cdot\vec{\nabla}\phi$ term in (22.2) and use (22.15) to obtain

$$E = \int (d\vec{r}) \, \frac{1}{8\pi} \frac{\vec{D}^2}{\varepsilon} \quad , \tag{22.16}$$

while the subsidiary condition (22.3) becomes

$$Q_i = \int dS_i \, \frac{\vec{n}\cdot\vec{D}}{4\pi} \quad . \tag{22.17}$$

In (22.16), we identify the integrand as the energy density of the field.

Let us now verify that the second restricted form of the variational principle correctly describes the electrostatic situation under consideration. For a finite change in $\vec{D}$, $\vec{D} \to \vec{D} + \delta\vec{D}$, the change in the energy functional (22.16) is

$$\delta E = \int (d\vec{r})\, \frac{1}{4\pi}\, \frac{\vec{D}}{\varepsilon} \cdot \delta\vec{D} + \int (d\vec{r})\, \frac{1}{8\pi}\, \frac{(\delta\vec{D})^2}{\varepsilon} \quad , \tag{22.18}$$

while the constraints read

$$\vec{\nabla}\cdot\delta\vec{D} = 0 \quad , \tag{22.19a}$$

$$\int dS_i\, \frac{\vec{n}\cdot\delta\vec{D}}{4\pi} = 0 \quad . \tag{22.19b}$$

The stationary condition requires that the integral linear in $\delta\vec{D}$ in (22.18) vanishes. To incorporate the constraint (22.19a), we add to (22.18) the volume integral

$\vec{\nabla}\cdot[\delta\vec{D}\,\phi] = \phi\,\vec{\nabla}\cdot\delta\vec{D} + \delta\vec{D}\cdot\vec{\nabla}\phi$

$$0 = \int (d\vec{r})\, \frac{\phi(\vec{r})}{4\pi}\, \vec{\nabla}\cdot\delta\vec{D} = + \sum_i \int dS_i\, \frac{\vec{n}\cdot\delta\vec{D}}{4\pi}\, \phi - \int (d\vec{r})\, \frac{\vec{\nabla}\phi}{4\pi} \cdot \delta\vec{D} \tag{22.20a}$$

where $\phi(\vec{r})$ is an arbitrary function. Likewise, to incorporate the constraint (22.19b), we add to (22.18) a sum of surface integrals,

$$0 = \sum_i \phi_i \int dS_i\, \frac{\vec{n}\cdot\delta\vec{D}}{4\pi} \quad . \tag{22.20b}$$

where ϕ_i is an arbitrary constant. In the resulting form of δE , the variations $\delta\vec{D}$ can be regarded as independent, so that the stationary principle implies, in the volume,

$$\frac{\vec{D}}{\varepsilon} = \vec{E} = -\vec{\nabla}\phi \quad , \tag{22.21}$$

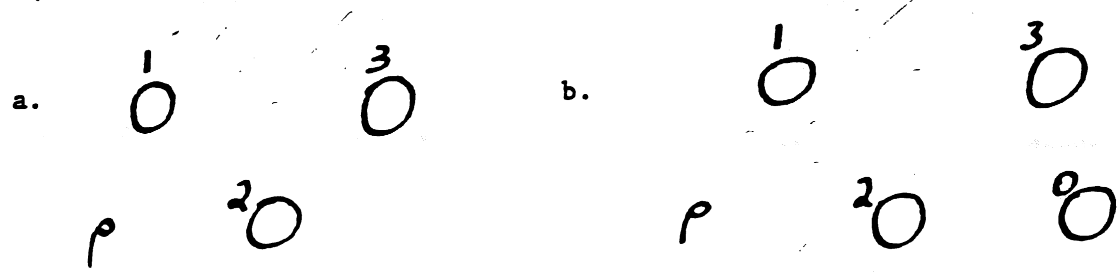
while, on the surfaces,

$$\boxed{\phi = \phi_i} \tag{22.22}$$

In this way, we recover the full set of equations for electrostatics. Moreover, (22.18) also tells us that, for any field other than the correct solution, $\delta E > 0$, that is, the physical field minimizes the energy functional (22.16). This is a statement of Thomson's Theorem: The charges on the surfaces of conductors always readjust themselves in such a way that each conductor becomes as equipotential surface and the total energy of the system is a minimum.

22-3. Introduction of Additional Uncharged Conductor

We now consider a region of space with dielectric constant $\varepsilon(\vec{r})$ bounded by an array of conductors into which we introduce an uncharged conductor at a location where there is no free charge density. We are interested in the change of energy in going from the initial configuration (a) to the final configuration (b). (In the following, the subscript 0 refers to the introduced conductor.)



The energy for (a) is

$$E_a = \int_V (d\vec{r}) \frac{\vec{D}_a^{\,2}}{8\pi\varepsilon} , \tag{22.23}$$

where V is the volume exterior to the conductors and the charge on the i^{th} conductor is

$$\int dS_i \frac{\vec{n}\cdot\vec{D}_a}{4\pi} = Q_i . \tag{22.24}$$

For (b) the energy is

$$E_b = \int_{V-V_0} (d\vec{r}) \, \frac{\vec{D}_b^{\,2}}{8\pi\varepsilon} \quad , \tag{22.25}$$

where now the volume occupied by conductor 0 (V_0) is also excluded, and the charges on the conductors are

$$\int dS_i \, \frac{\vec{n}\cdot\vec{D}_b}{4\pi} = Q_i \quad , \tag{22.26a}$$

$$\int dS_0 \, \frac{\vec{n}\cdot\vec{D}_b}{4\pi} = 0 \quad . \tag{22.26b}$$

The energy, for case (a), satisfies the following inequality,

$$E_a = \left[\int_{V-V_0} + \int_{V_0}\right] (d\vec{r}) \, \frac{\vec{D}_a^{\,2}}{8\pi\varepsilon} > \int_{V-V_0} (d\vec{r}) \, \frac{\vec{D}_a^{\,2}}{8\pi\varepsilon} \quad . \tag{22.27}$$

Although $\vec{D}_a$ is not the correct field for (b), it is an allowable trial function to use in the energy functional, (22.16), because it satisfies all the necessary conditions:

$$\vec{\nabla}\cdot\vec{D}_a = 4\pi\rho \quad ,$$

$$\int dS_i \, \frac{\vec{n}\cdot\vec{D}_a}{4\pi} = Q_i \quad ,$$

$$\int dS_0 \, \frac{\vec{n}\cdot\vec{D}_a}{4\pi} = \int_{V_0} (d\vec{r}) \, \frac{\vec{\nabla}\cdot\vec{D}_a}{4\pi} = 0 \quad ,$$

since, by hypothesis, the region V_0 originally had no charge ($\rho = 0$). According to Thomson's Theorem, the correct field yields a minimum value of the energy functional,

$$E_b = \int_{V-V_0} (d\vec{r})\, \frac{\vec{D}_b^{\,2}}{8\pi\varepsilon} < \int_{V-V_0} (d\vec{r})\, \frac{\vec{D}_a^{\,2}}{8\pi\varepsilon} \quad , \tag{22.28}$$

implying, upon comparison with (22.27),

$$\boxed{E_a > E_b} \quad , \tag{22.29}$$

which states that the introduction of an uncharged conductor lowers the energy of the system.

Lecture 18

22-4. Alternate Variational Principle

In the first restricted version of the variational principle, (22.14), the charges on the conductors, (22.3), are specified. For some purposes, it is more convenient to regard the potentials, ϕ_i , on the surfaces of the conductors as specified, rather than the charges, Q_i . For simplicity we will assume that there is no volume charge density, $\rho = 0$. In order to obtain a new form of the energy functional, we note that the stationary property of (22.14) under the replacement

$$\phi \to \lambda\phi \quad ,$$

for λ infinitesimally different from unity, implies

$$0 = \left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=1} = 2 \int (d\vec{r}) \left(- \frac{\varepsilon}{8\pi} \right) (\vec{\nabla}\phi)^2 + \sum_i \int dS_i\, \sigma\phi \quad . \tag{22.30}$$

Consequently, the energy functional is

$$\boxed{E = \frac{1}{2} \sum_i \int dS_i\, \sigma\phi} \quad , \tag{22.31}$$

which becomes, for the actual field values on the surfaces, $\phi = \phi_i$,

$$E = \frac{1}{2} \sum_i Q_i \, \phi_i \quad . \tag{22.32}$$

Therefore, we obtain another energy functional by combining (22.32) and (22.14),

$$E = \sum_i Q_i \phi_i - \sum_i \int dS_i \, \sigma\phi + \int (d\vec{r}) \, \frac{\varepsilon}{8\pi} (\vec{\nabla}\phi)^2 \quad , \tag{22.33}$$

or, using (22.3),

$$\boxed{E = \sum_i \int dS_i \, \sigma(\phi_i - \phi) + \int (d\vec{r}) \, \frac{\varepsilon}{8\pi} (\vec{\nabla}\phi)^2} \quad . \tag{22.34}$$

Here we regard ϕ and σ to be the variables while ϕ_i is specified [note that here we impose no subsidiary restriction on σ]. Under variations in ϕ and σ , the energy changes by

$$\delta E = \sum_i \int dS_i [\delta\sigma(\phi_i - \phi) - \sigma\delta\phi] + \int (d\vec{r}) \, \frac{\varepsilon}{4\pi} (\vec{\nabla}\phi)\cdot(\vec{\nabla}\delta\phi) \quad , \tag{22.35}$$

which becomes

$$\delta E = \sum_i \int dS_i \left[\delta\sigma(\phi_i - \phi) - \sigma\delta\phi + \frac{\vec{n}\cdot\vec{D}}{4\pi} \delta\phi \right] + \int (d\vec{r}) \, \frac{\vec{\nabla}\cdot\vec{D}}{4\pi} \delta\phi \quad , \tag{22.36}$$

by identifying $\vec{D}$ and integrating by parts. The stationary principle, $\delta E = 0$, implies, from the volume part of (22.36),

$$\delta\phi : \quad \vec{\nabla}\cdot\vec{D} = 0 \quad , \tag{22.37}$$

and from the surface part,

$$\delta\phi : \quad \vec{n}\cdot\vec{D} = 4\pi\sigma \quad , \tag{22.38}$$

$$\delta\sigma : \quad \phi = \phi_i \quad . \tag{22.39}$$

These are the correct equations of electrostatics when there is no volume charge density.

22-5. Green's Function

In our study of electrostatics, we have found Green's functions to be of great use. We will here introduce Green's function in the presence of conductors that are grounded, that is $\phi_i = 0$; the corresponding differential equation is

$$-\vec{\nabla}\cdot[\varepsilon\vec{\nabla}G(\vec{r},\vec{r}')] = 4\pi\delta(\vec{r}-\vec{r}') \quad , \tag{22.40}$$

with the boundary condition

$$G(\vec{r},\vec{r}') = 0 \quad \text{for} \quad \vec{r} \quad \text{on} \quad S_i \quad . \tag{22.41}$$

We will show that this Green's function can be used to solve the electrostatics problem in which the potentials on the conductors are specified. We wish to consider a situation for which the free charge density is zero,

$$\vec{\nabla}\cdot\vec{D} = -\vec{\nabla}\cdot(\varepsilon\vec{\nabla}\phi) = 4\pi\rho = 0 \quad , \tag{22.42}$$

while the potential, ϕ_i , on each conducting surface, S_i , is constant,

$$\phi = \phi_i \quad \text{on} \quad S_i \quad . \tag{22.43}$$

If we multiply (22.40) by $\phi(\vec{r})$ and (22.42) by $G(\vec{r},\vec{r}')$ and subtract, we obtain

$$-\phi(\vec{r})\ \vec{\nabla}\cdot[\epsilon\vec{\nabla}G(\vec{r},\vec{r}')] + G(\vec{r},\vec{r}')\ \vec{\nabla}\cdot(\epsilon\vec{\nabla}\phi(\vec{r})) = 4\pi\delta(\vec{r}-\vec{r}')\ \phi(\vec{r}) \quad . \tag{22.44}$$

Since the left-hand side of (22.44) is a divergence,

$$\vec{\nabla}\cdot[G(\vec{r},\vec{r}')\ \epsilon\vec{\nabla}\phi(\vec{r}) - \phi(\vec{r})\ \epsilon\vec{\nabla}G(\vec{r},\vec{r}')] = 4\pi\delta(\vec{r}-\vec{r}')\ \phi(\vec{r}) \quad , \tag{22.45}$$

when we integrate over the entire volume, V , exterior to all of the conductors, we obtain an integral over a surface S which is made up of all the surfaces of the individual conductors, S_i ,

$$\begin{aligned} 4\pi\phi(\vec{r}') &= \int_S d\vec{S}\cdot[G(\vec{r},\vec{r}')\ \epsilon\vec{\nabla}\phi(\vec{r}) - \phi(\vec{r})\ \epsilon\vec{\nabla}G(\vec{r},\vec{r}')] \\ &= -\sum_i \int dS_i[G(\vec{r},\vec{r}')\ \epsilon\vec{n}_i\cdot\vec{\nabla}\phi(\vec{r}) - \phi(\vec{r})\ \epsilon\vec{n}_i\cdot\vec{\nabla}G(\vec{r},\vec{r}')] \quad . \end{aligned} \tag{22.46}$$

The negative sign occurs since $d\vec{S}$ is directed out of the volume V , and so into the conductors, while $\vec{n}_i$ is the outward normal for the i^{th} conductor. Deleted here is the surface at infinity for which

$$dS \sim R^2 ,$$

$$G \lesssim \frac{1}{R} , \qquad |\vec{\nabla}G| \lesssim \frac{1}{R^2} ,$$

$$\phi \lesssim \frac{1}{R} , \qquad |\vec{\nabla}\phi| \lesssim \frac{1}{R^2} ,$$

so that the corresponding contribution goes to zero as the volume gets arbitrarily large. Now imposing the boundary conditions, (22.41) and (22.43), we obtain the desired expression for the potential, $[\epsilon' = \epsilon(\vec{r}')]$

$$\boxed{\phi(\vec{r}) = \sum_i \phi_i \int dS_i'\ \frac{\epsilon'}{4\pi}\ \vec{n}_i'\cdot\vec{\nabla}'G(\vec{r},\vec{r}')} , \tag{22.47}$$

where we have interchanged the roles of $\vec{r}$ and $\vec{r}'$ and used the symmetry property of G , (11.4). Therefore, if we know G (the potential due to a point charge with zero potential on the conductors), we can calculate the potential, $\phi(\vec{r})$, due to arbitrarily specified potentials on the conductors.

22-6. Capacitance

Once we know the potential, we can compute the surface charge density on the i^{th} conductor by using

$$\sigma_i = \frac{1}{4\pi}\vec{n}_i\cdot(-\epsilon\vec{\nabla}\phi)$$

$$= -\sum_j \phi_j \int dS_j' \frac{\epsilon}{4\pi}\frac{\epsilon'}{4\pi}(\vec{n}_i\cdot\vec{\nabla})(\vec{n}_j'\cdot\vec{\nabla}')\, G(\vec{r},\vec{r}') \quad . \tag{22.48}$$

The total charge on S_i is therefore

$$Q_i = \int dS_i \sigma_i = -\sum_j \phi_j \int dS_i\, dS_j' \frac{\epsilon}{4\pi}\frac{\epsilon'}{4\pi}(\vec{n}_i\cdot\vec{\nabla})(\vec{n}_j'\cdot\vec{\nabla}')\, G(\vec{r},\vec{r}') \quad . \tag{22.49}$$

Occurring here are the coefficients of capacitance, C_{ij} , defined by

$$\boxed{C_{ij} = -\int dS_i\, dS_j' \frac{\epsilon}{4\pi}\frac{\epsilon'}{4\pi}(\vec{n}_i\cdot\vec{\nabla})(\vec{n}_j'\cdot\vec{\nabla}')\, G(\vec{r},\vec{r}')} \quad , \tag{22.50}$$

$Q_i = \sum_j \phi_j C_{ij}$

which are symmetric in i and j ,

$$C_{ij} = C_{ji} \quad . \tag{22.51}$$

The total charge on the i^{th} conductor is thus simply written as

$$\boxed{Q_i = \sum_j C_{ij}\,\phi_j \quad .} \tag{22.52}$$

The energy of a system of charged conductors can be expressed in terms of the coefficients of capacitance by means of (22.32),

$$E = \frac{1}{2} \sum_i Q_i \phi_i$$

$$= \frac{1}{2} \sum_{ij} \phi_i C_{ij} \phi_j \quad . \tag{22.53}$$

There is a consistency check between this expression and the variational principle which employs (22.34). Suppose we vary the potential on conductor i by an amount $\delta\phi_i$. Such a change induces variations in σ and ϕ but the resulting change in the energy from these induced variations is of second order due to the stationary principle. So the first order variation in the energy arises only from the explicit variation of ϕ_i :

$$\delta E = \int dS_i \sigma \delta\phi_i = \delta\phi_i Q_i \quad , \checkmark$$

or,

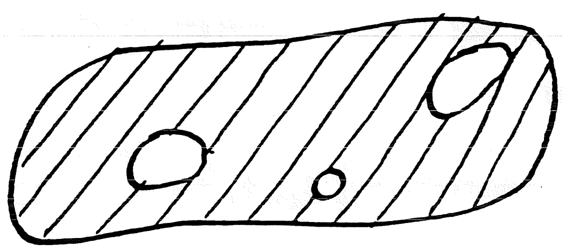
$$\frac{\partial E}{\partial \phi_i} = Q_i \quad \checkmark \tag{22.54}$$

This result is in agreement with that obtained from (22.53),

$$\frac{\partial E}{\partial \phi_i} = \sum_j C_{ij} \phi_j = Q_i \quad ,$$

where we make use of (22.52).

Suppose the system consists of a finite region bounded by conducting surfaces,



that is, there is no surface at infinity. The total charge induced on such a system of grounded conductors by a point charge at any interior point, $\vec{r}'$, is

$$-\frac{1}{4\pi}\sum_i \int dS_i\ \epsilon\vec{n}_i\cdot\vec{\nabla}G(\vec{r},\vec{r}') = -\frac{1}{4\pi}\int (d\vec{r})(-\vec{\nabla})\cdot[\epsilon\vec{\nabla}G(\vec{r},\vec{r}')] = -1 , \quad (22.55)$$

where we have used the first line of (22.48), with ϕ replaced by G, as well as the differential equation satisfied by the Green's function, (22.40). This implies that the coefficients of capacitance, C_{ij}, (22.50), satisfy

$$\sum_i C_{ij} = -\int dS_j'\ \frac{\epsilon'}{4\pi}(\vec{n}_j'\cdot\vec{\nabla}')\sum_i\int dS_i\ \frac{\epsilon}{4\pi}(\vec{n}_i\cdot\vec{\nabla})\ G(\vec{r},\vec{r}')$$

$$= -\int dS_j'\ \frac{\epsilon'}{4\pi}(\vec{n}_j'\cdot\vec{\nabla}')\ (1) = 0 ,$$

that is, the sum of all the coefficients of capacitance referring to a given conductor vanishes,

$$\sum_i C_{ij} = \sum_j C_{ij} = 0 . \quad (22.56)$$

Consequently, the total charge on the conductors is zero when there is no volume charge present:

$$\sum_i Q_i = \sum_{ij} C_{ij}\ \phi_j = 0 . \quad (22.57)$$

Furthermore, for this system, only relative values of the potential are significant. If we were to add a common constant to all potentials, all charges would remain the same:

$$Q_i = \sum_j C_{ij}\ (\phi_j + \text{constant}) = \sum_j C_{ij}\ \phi_j + \text{const}\underbrace{\sum_j C_{ij}}_{=0}$$

As a simple example, consider a closed system bounded by only two conductors.

In this case, in order to satisfy (22.51) and (22.56), we must have

$$C_{11} = -C_{21} = -C_{12} = C_{22} \equiv C \qquad (22.58)$$

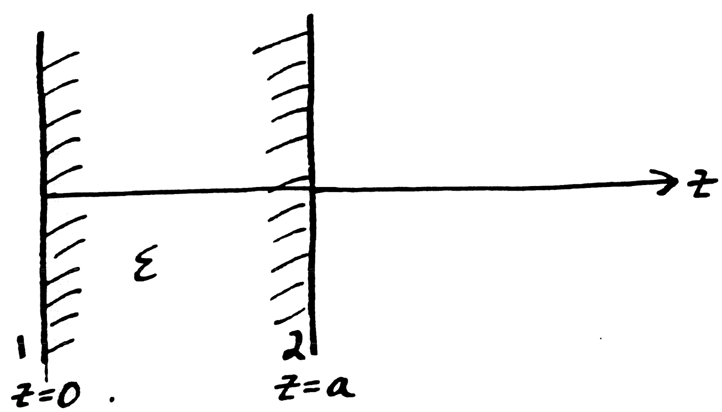
where C is called the capacitance of the system. The charges on the two conductors are

$$Q_1 = -Q_2 = C(\phi_1 - \phi_2) = CV \qquad (22.59)$$

where V is the potential difference between the two conductors, while the energy is

$$E = \frac{1}{2} \sum_{ij} \phi_i \, C_{ij} \, \phi_j = \frac{1}{2} CV^2 \quad . \qquad (22.60)$$

As an application of these ideas, consider a capacitor constructed from two parallel conducting plates of area A. The separation of the plates, a, is assumed to be small compared to the dimensions of the plates, $a << \sqrt{A}$, the approximate Green's function therefore being that of two infinite plates [cf. (16.6)].



The material between the plates is characterized by a dielectric constant, ε. The above discussion applies to this situation so that the system has a capacitance C,

$$C = C_{11} = -\int dS\, dS' \frac{\varepsilon^2}{(4\pi)^2} \left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z'}\right) G\Big|_{z,z'=0} . \tag{22.61}$$

The first surface integral here was previously evaluated in (16.8a) and (16.10a):

$$\int dS\left(-\frac{1}{4\pi}\right)\frac{\partial}{\partial z} G(\vec{r},\vec{r}\,')\Big|_{z=0} = -\frac{1}{\varepsilon}\left(1 - \frac{z'}{a}\right) , \tag{22.62}$$

where we have now included the presence of ε in (22.40). The remaining surface integral is trivial,

$$C = \frac{\varepsilon}{4\pi a}\int dS' = \frac{\varepsilon A}{4\pi a} , \tag{22.63}$$

yielding the well-known result for a parallel plate capacitor.

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XXIII. Magnetostatics

23-1. Variational Principle

We now return to the general action principle of electrodynamics, (9.1), before the specialization to electrostatics. The field part of the Lagrangian in the microscopic description is

$$L = \int (d\vec{r})\Big\{ -\rho\phi + \frac{1}{c}\vec{j}\cdot\vec{A} + \frac{1}{4\pi}\Big[\vec{E}\cdot\Big(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\phi\Big) - \vec{B}\cdot\vec{\nabla}\times\vec{A} + \frac{1}{2}(B^2-E^2)\Big]\Big\} . \qquad (23.1)$$

Again we will consider the static situation where all fields and densities are time independent, in particular

$$\frac{\partial}{\partial t}\vec{A} = 0 . \qquad (23.2)$$

When this condition is imposed, the Lagrangian can be separated into two pieces,

$$L = -\int (d\vec{r})\Big[\rho\phi + \frac{1}{4\pi}\Big(\vec{E}\cdot\vec{\nabla}\phi + \frac{1}{2}E^2\Big)\Big] + \int (d\vec{r})\Big[\frac{1}{c}\vec{j}\cdot\vec{A} + \frac{1}{4\pi}\Big(-\vec{B}\cdot\vec{\nabla}\times\vec{A} + \frac{1}{2}B^2\Big)\Big] , \qquad (23.3)$$

where the first part, as we have seen, corresponds to electrostatics. The second part describes magnetostatics, which is the subject of our investigation here. Notice that this separation is possible only because of the condition (23.2); otherwise, electric and magnetic effects are interrelated. Analogously to our incorporation of dielectrics in electrostatics (see Section X), we here pass to a macroscopic description of fields in permeable media. The energy, for these circumstances, becomes

$$E = -\int (d\vec{r})\left[\frac{1}{c}\vec{J}\cdot\vec{A} + \frac{1}{4\pi\mu}\left(-\vec{B}\cdot\vec{\nabla}\times\vec{A} + \frac{1}{2}B^2\right)\right] , \tag{23.4}$$

where μ is the permeability of the medium. We now have to check that the stationary principle applied to this form of the energy yields the correct equations of magnetostatics. We are to regard $\vec{A}$ and $\vec{B}$ as the independent variables so the variation of the energy is

$$\delta E = -\int (d\vec{r})\left[\frac{1}{c}\vec{J}\cdot\delta\vec{A} - \frac{1}{4\pi}\frac{\vec{B}}{\mu}\cdot\vec{\nabla}\times\delta\vec{A} + \frac{1}{4\pi\mu}\delta\vec{B}\cdot(\vec{B}-\vec{\nabla}\times\vec{A})\right] . \tag{23.5}$$

From the coefficient of $\delta\vec{B}$, we obtain

$$\vec{B} = \vec{\nabla}\times\vec{A} , \tag{23.6a}$$

which is equivalent to

$$\vec{\nabla}\cdot\vec{B} = 0 . \tag{23.6b}$$

By making use of the identity

$$\frac{\vec{B}}{\mu}\cdot\vec{\nabla}\times\delta\vec{A} = \vec{\nabla}\cdot\left(\delta\vec{A}\times\frac{\vec{B}}{\mu}\right) + \delta\vec{A}\cdot\left(\vec{\nabla}\times\frac{\vec{B}}{\mu}\right) , \tag{23.7}$$

and discarding the implied surface integral, we find from the vanishing of the coefficient of $\delta\vec{A}$,

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} , \tag{23.8a}$$

a consequence of which is that only steady currents occur here:

$$\vec{\nabla}\cdot\vec{J} = 0 . \tag{23.8b}$$

As appropriate to macroscopic media, we have introduced the magnetic field,

$$\vec{H} = \frac{\vec{B}}{\mu} \quad . \tag{23.9}$$

Thus we have recovered Maxwell's equations in the static limit, (23.6b) and (23.8a).

As in electrostatics, there is a restricted version of the stationary principle for the energy. If we take

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

as the definition of $\vec{B}$, the expression for the energy becomes

$$E = -\int (d\vec{r}) \left[\frac{1}{c} \vec{J} \cdot \vec{A} - \frac{1}{8\pi\mu} (\vec{\nabla} \times \vec{A})^2 \right] \quad . \tag{23.10}$$

Regarding this as stationary under variations in $\vec{A}$, we derive the equation satisfied by the vector potential,

$$\vec{\nabla} \times \left(\frac{1}{\mu} \vec{\nabla} \times \vec{A} \right) = \frac{4\pi}{c} \vec{J} \quad , \tag{23.11}$$

which coincides with (23.8a).

Proceeding in a manner parallel to the corresponding discussion in electrostatics (see Subsection 10-2), we consider a change in the permeability, $\delta\mu(\vec{r})$. Because of the stationary property, the only first order variation in the energy arises from the explicit appearance of μ in (23.10):

$$\delta E = -\int (d\vec{r}) \frac{\delta\mu}{\mu^2} \frac{1}{8\pi} (\vec{\nabla} \times \vec{A})^2 = -\int (d\vec{r}) \frac{\delta\mu}{8\pi} H^2 \quad , \tag{23.12}$$

which is the analog of (10.13). In particular, by considering $\delta\mu$ to arise from a displacement of the material, we infer the force on the (inhomogeneous) permeable medium to be [cf. (10.17)]

$$\vec{F} = - \int (d\vec{r}) \frac{H^2}{8\pi} \vec{\nabla}\mu \quad ; \tag{23.13}$$

a diamagnetic material, with $\mu < 1$, is repelled from a region of stronger magnetic field.

23-2. Boundary Conditions

The simplest example of an inhomogeneous magnetic material occurs when μ is discontinuous across an interface.

μ_1	μ_2
V_1	V_2

First we consider the boundary conditions that $\vec{B}$ and $\vec{A}$ must satisfy across the interface. The fact that $\vec{B}$ is the curl of $\vec{A}$ implies that the tangential component of $\vec{A}$, $\vec{A}_t$, must be continuous across the boundary, in order that $\vec{B}_t$ be finite:

$$\vec{n}_1 \times \vec{A}_1 + \vec{n}_2 \times \vec{A}_2 = 0 \quad , \tag{23.14a}$$

or,

$$\vec{n}_1 \times (\vec{A}_1 - \vec{A}_2) = 0 \quad , \tag{23.14b}$$

where $\vec{n}_1(\vec{n}_2)$ is the <u>outward</u> normal to $V_1(V_2)$ so that $\vec{n}_2 = -\vec{n}_1$. The relation, (23.14a), is true for all points on the surface. Thus, when we take the divergence of this expression, in which only tangential components of $\vec{\nabla}$ occur, we find

$$\vec{n}_1\cdot\vec{B}_1 + \vec{n}_2\cdot\vec{B}_2 = 0 \quad , \tag{23.15a}$$

or

$$\vec{n}_1\cdot(\vec{B}_1 - \vec{B}_2) = 0 \quad , \tag{23.15b}$$

that is, the normal component of $\vec{B}$ is continuous across the boundary. [We may regard this as a surface version of $\vec{\nabla}\cdot\vec{B} = 0$.]

If we include the possibility that there is a surface current, $\vec{K}$, on the boundary between V_1 and V_2 , we must amend the energy expression, (23.4), to read

$$E = -\int (d\vec{r})\left[\frac{1}{c}\vec{J}\cdot\vec{A} + \frac{1}{4\pi\mu}\left(-\vec{B}\cdot\vec{\nabla}\times\vec{A} + \frac{1}{2}B^2\right)\right] - \int dS\,\frac{1}{c}\vec{K}\cdot\vec{A} \quad . \tag{23.16}$$

In our previous discussion of the variation in the energy, we discarded the surface integral [see (23.7)]; this is no longer permissible because of the presence of the boundary. Consequently, there is a new contribution to the variation of the energy arising from the occurrence of the interface,

$$\begin{aligned}
\delta E &= \int_{V_1}(d\vec{r})\,\vec{\nabla}\cdot\left(\delta\vec{A}_1\times\frac{\vec{H}_1}{4\pi}\right) + \int_{V_2}(d\vec{r})\,\vec{\nabla}\cdot\left(\delta\vec{A}_2\times\frac{\vec{H}_2}{4\pi}\right) - \int dS\,\frac{1}{c}\vec{K}\cdot\delta\vec{A} \\
&= \int dS\left[\vec{n}_1\cdot\frac{\delta\vec{A}_1\times\vec{H}_1}{4\pi} + \vec{n}_2\cdot\frac{\delta\vec{A}_2\times\vec{H}_2}{4\pi} - \frac{1}{c}\vec{K}\cdot\delta\vec{A}\right] \\
&= -\int dS\,\delta\vec{A}\cdot\left[\frac{\vec{n}_1\times\vec{H}_1}{4\pi} + \frac{\vec{n}_2\times\vec{H}_2}{4\pi} + \frac{1}{c}\vec{K}\right] \quad .
\end{aligned} \tag{23.17}$$

x prod is cyclic, so $\vec{H}_1\cdot\hat{n}_1\times\delta\vec{A}_1$ [illegible]
$\delta\vec{A}_1\cdot\vec{H}_1\times\hat{n}_1$ [illegible]
$\therefore\ \delta\vec{A}_1\cdot\hat{n}_1\times\vec{H}_1$ [illegible]

Here we have used the fact that $\vec{A}_t$ must be continuous,

$$\delta\vec{A}_{1t} = \delta\vec{A}_{2t} = \delta\vec{A}_t \quad .$$

We then conclude from the stationary principle on the surface that

$$\vec{n}_1 \times \vec{H}_1 + \vec{n}_2 \times \vec{H}_2 + \frac{4\pi}{c}\vec{K} = 0 \quad . \tag{23.18}$$

When no surface current is present, $\vec{K} = 0$, this states that the tangential component of $\vec{H}$ is continuous. If, in addition, we have $\mu_2 >> \mu_1$ [idealized as $\mu_2 \to \infty$; we might call this a perfect magnetic conductor (see Section XLV)], the magnetic field in medium 2 goes to zero,

$$\vec{H}_2 = \frac{1}{\mu_2}\vec{B}_2 \to 0 \quad ,$$

so that $\vec{H}_1$ is normal to the surface,

$$\vec{n}_1 \times \vec{H}_1 = 0 \quad .$$

This is the same condition satisfied by the electric field at the surface of a conductor.

Lecture 19

23-3. Vector Potential

The fundamental equation of magnetostatics is (23.11),

$$\vec{\nabla} \times \left(\frac{1}{\mu}\vec{\nabla} \times \vec{A}\right) = \frac{4\pi}{c}\vec{J} \quad , \tag{23.19}$$

which reduces for vacuum ($\mu = 1$), to

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} \quad . \tag{23.20}$$

This equation can be simplified by using the fact that it is invariant under a gauge transformation [see (9.27)],

$$\vec{A} \to \vec{A} - \vec{\nabla}\lambda \quad .$$

Because of this gauge freedom, we can usually choose some particular gauge to simplify the problem at hand. In the present situation, the convenient choice of gauge is one for which

$$\vec{\nabla}\cdot\vec{A} = 0 \quad , \tag{23.21}$$

called the radiation, Coulomb, or transverse gauge. To show that it is always possible to choose this gauge, suppose we start with a vector potential, $\vec{A}_o$, which does not satisfy this condition,

$$\vec{\nabla}\cdot\vec{A}_o \neq 0 \quad .$$

It is possible to make a gauge transformation,

$$\vec{A}_o \to \vec{A} = \vec{A}_o - \vec{\nabla}\lambda \quad ,$$

such that (23.21) is satisfied, that is

$$\vec{\nabla}\cdot(\vec{A}_o - \vec{\nabla}\lambda) = 0 \quad ,$$

for then λ is a solution to Poisson's equation,

$$\nabla^2\lambda = \vec{\nabla}\cdot\vec{A}_o \quad .$$

In the radiation gauge, (23.20) becomes

$$-\nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} \quad , \tag{23.22}$$

the solution of which is

$$\vec{A}(\vec{r}) = \frac{1}{c}\int (d\vec{r}\,')\,\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \quad , \tag{23.23}$$

in precise analogy to the solution of the electrostatic problem,

$$-\nabla^2\phi = 4\pi\rho \quad .$$

As a consistency check, we verify explicitly that (23.23) satisfies the radiation gauge condition (23.21),

$$\begin{aligned}
\vec{\nabla}\cdot\vec{A}(\vec{r}) &= \frac{1}{c}\int (d\vec{r}\,')\left(\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\cdot\vec{J}(\vec{r}\,') \\
&= -\frac{1}{c}\int (d\vec{r}\,')\left(\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\cdot\vec{J}(\vec{r}\,') \\
&= \frac{1}{c}\int (d\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|}\vec{\nabla}'\cdot\vec{J}(\vec{r}\,') = 0
\end{aligned}$$

where we have used (23.8b), and the fact that the current distribution is localized.

Once we have the vector potential, we can compute the magnetic field $\vec{B}$:

$$\begin{aligned}
\vec{B} &= \vec{\nabla}\times\vec{A} = \frac{1}{c}\vec{\nabla}\times\int (d\vec{r}\,')\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \\
&= \frac{1}{c}\int (d\vec{r}\,')\,\vec{J}(\vec{r}\,')\times\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \quad .
\end{aligned} \tag{23.24}$$

For a point charge moving with velocity $\vec{v}$, the electric current is

$$\vec{J}(\vec{r}) = e\vec{v}\,\delta(\vec{r}-\vec{R}) \quad , \tag{23.25}$$

where $\vec{R}$ is the position of the particle, which produces the magnetic field

$$\vec{B} = \frac{\vec{v}}{c}\times\frac{e(\vec{r}-\vec{R})}{|\vec{r}-\vec{R}|^3} \quad . \tag{23.26}$$

This has the form

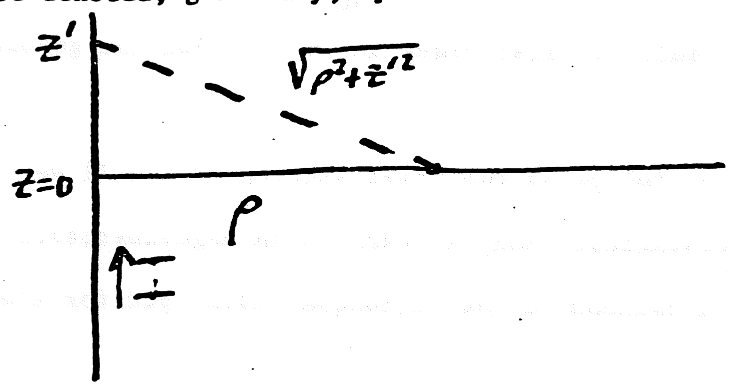
$$\vec{B} = \frac{\vec{v}}{c} \times \vec{E} \quad ,$$

which was our starting point for introducing magnetic fields in Section I. [Of course, we have now transcended the domain of magnetostatics since $\partial\vec{A}/\partial t \neq 0$. However, since $\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ is of order v^2/c^2 , (23.26) is the correct magnetic field to first order in v/c . For the general case, see Section XXVIII.]

In the following two brief sections, we will develop some applications of magnetostatics. Many situations in magnetostatics can be attacked by evident variations on the techniques developed for electrostatics.

XXIV. Macroscopic Current Distributions

The simplest example of a macroscopic current is that which flows in a long straight wire. We will take the wire to lie along the z axis, carrying a current I that flows in the $+z$ direction. We will let the direction of current flow be denoted, generally, by $\vec{n}$.



We wish to find the magnetic induction $\vec{B}$ produced by this current. Since $\vec{B}$ is independent of z , without loss of generality we evaluate it at $z = 0$. For a wire with negligible cross section, any volume integral involving the current density becomes a line integral

$$\int (d\vec{r})\, \vec{J}(\vec{r}\,') \ldots = \int dz'\, dS'\, \vec{n}\, J(\vec{r}\,') \ldots = \int dz'\, \vec{n}\, I \ldots \quad . \tag{24.1}$$

The expression for the magnetic field, (23.24), a distance ρ from the wire, is then reduced to

$$\vec{B} = \frac{I}{c} \int_{-\infty}^{\infty} dz' \left(\vec{\nabla} \frac{1}{\sqrt{\rho^2+z'^2}} \right) \times \vec{n} \quad . \tag{24.2}$$

For a long wire of length $2L$, $L >> \rho$, the basic integral occurring here is

$$\int_{-L}^{L} dz' \frac{1}{\sqrt{\rho^2+z'^2}} = 2 \int_{0}^{L} dz' \frac{1}{\sqrt{z'^2+\rho^2}}$$

$$\simeq 2\left[\ell og \frac{L}{\rho} + \text{constant}\right] , \tag{24.3}$$

the gradient of which is

$$\vec{\nabla}\left(2\,\ell og\frac{L}{\rho} + \text{constant}\right) = -2\,\frac{\vec{\nabla}\rho}{\rho} = -\frac{2}{\rho}\,\hat{\rho} \ . \tag{24.4}$$

The magnetic field produced by this wire is therefore

$$\vec{B} = \frac{2I}{c\rho}\,(\vec{n}\times\hat{\rho}) \ . \tag{24.5}$$

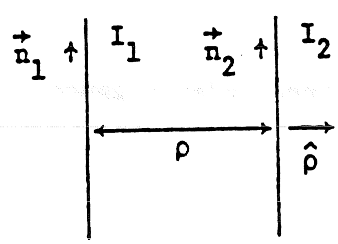
The force exerted on a current distribution by a magnetic field is [see (1.28)]

$$\vec{F} = \frac{1}{c}\int (d\vec{r})\,\vec{J}\times\vec{B} \ , \tag{24.6}$$

which, when specialized to current flowing in a long straight wire, becomes

$$\vec{F} = \int dz\,\frac{I}{c}\,\vec{n}\times\vec{B} \ . \tag{24.7}$$

One possibility is that the magnetic field is produced by a second, parallel, current-carrying wire.



Of course, the total force acting on I_2 due to the magnetic field produced by I_1 is unbounded. The quantity of interest is the force per unit length on I_2 ,

$$\frac{\text{force}}{\text{length}} = \frac{I_2}{c}\vec{n}_2 \times \vec{B} = \frac{I_2}{c}\frac{2I_1}{c\rho}\vec{n}_2 \times (\vec{n}_1 \times \hat{\rho}) \quad . \tag{24.8}$$

For parallel flowing currents, $\vec{n}_1 = \vec{n}_2$, so

$$\frac{\text{force}}{\text{length}} = -\frac{2I_1I_2}{c^2}\frac{\hat{\rho}}{\rho} , \tag{24.9}$$

that is, the force is attractive. If the currents flow in opposite senses, the force is repulsive.

We can also obtain the above result by recalling that the force is the negative gradient of the energy,

$$\vec{F} = -\vec{\nabla}E , \tag{24.10}$$

where the energy is given by (23.10), with $\mu = 1$. Equation (23.10) is quite analogous to the electrostatic energy, (10.7), except for the overall sign, which implies that the sense of attraction or repulsion is reversed when we go from static charge distributions to steady current flows. By integrating by parts on the $(\vec{\nabla} \times \vec{A})^2$ term and then using the differential equation (23.11), with $\mu = 1$, we may rewrite the magnetostatic energy as

$$E = -\frac{1}{2c}\int (d\vec{r})\ \vec{J}\cdot\vec{A} \quad . \tag{24.11}$$

[Notice that (24.11) is gauge invariant, since under a gauge transformation,

$$\vec{A} \rightarrow \vec{A} - \vec{\nabla}\lambda \quad ,$$

the energy does not change,

$$\delta E = \frac{1}{2c}\int (d\vec{r})\ \vec{J}\cdot\vec{\nabla}\lambda = 0 \quad ,$$

since we can integrate by parts and use (23.8b).] Introducing the explicit form for $\vec{A}$, (23.23), we can write the energy in terms of the current density alone,

$$E = -\frac{1}{2}\int (d\vec{r})(d\vec{r}')\,\frac{\frac{1}{c}\vec{J}(\vec{r})\cdot\frac{1}{c}\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\quad, \tag{24.12}$$

which is analogous to the electrostatic result, (11.8), except that its sign is opposite. For the case of two current distributions,

$$\vec{J}(\vec{r}) = \vec{J}_1(\vec{r}) + \vec{J}_2(\vec{r})\quad,$$

the energy expression contains self energies as well as the mutual interaction energy. We are here interested only in the latter, which is

$$E = -\int (d\vec{r})(d\vec{r}')\,\frac{\frac{1}{c}\vec{J}_1(\vec{r})\cdot\frac{1}{c}\vec{J}_2(\vec{r}')}{|\vec{r}-\vec{r}'|}\quad, \tag{24.13}$$

as it is the sole term that contributes to the force, (24.10). For straight wires, this becomes

$$E = -\int dz\,dz'\,\frac{I_1 I_2}{c^2}\,\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{r}-\vec{r}'|}\quad. \tag{24.14}$$

For parallel wires with currents flowing in the same sense,

$$\vec{n}_1\cdot\vec{n}_2 = 1\quad,$$

$$|\vec{r}-\vec{r}'| = \sqrt{\rho^2 + (z-z')^2}\quad,$$

the integration over z' is evaluated as in (24.3), so that

$$E = -\frac{I_1 I_2}{c^2}\, 2\left(\ell og\, \frac{L}{\rho} + \text{constant}\right) \int dz$$

$$= -\frac{I_1 I_2}{c^2}\, 2\left(\ell og\, \frac{L}{\rho} + \text{constant}\right) 2L \quad , \tag{24.15}$$

where we have used the restriction $L/\rho >> 1$. The force can now be calculated from (24.10), or, since E depends only on ρ ,

$$\vec{F} = -\left(\frac{\partial}{\partial\rho} E\right) \hat{\rho} \quad .$$

The force per unit length is therefore

$$\frac{F}{2L} = -\frac{2I_1 I_2}{c^2} \frac{\hat{\rho}}{\rho} \quad , \tag{24.16}$$

which is our previous result, (24.9).

Warren More

XXV. Magnetic Multipoles

25-1. Magnetic Dipole Moment

We now direct our attention to the magnetic field produced by a confined current distribution. If we wish to evaluate the vector potential far outside the current distribution, $|\vec{r}|$ >> dimension of region of current flow (where the origin of the coordinate system is located in the current distribution), we may use the expansion

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \ldots \tag{25.1}$$

$\frac{1}{\sqrt{|r-r'|^2}} = \frac{1}{\sqrt{r^2 - 2\vec{r}\cdot\vec{r}\,' + r'^2}} = \frac{1}{r\sqrt{1 - 2\frac{\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{r'^2}{r^2}}} \simeq \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right)$

in the expression for $\vec{A}$, (23.23). The resulting expansion for the vector potential is then

$$\vec{A}(\vec{r}) = \frac{1}{r}\int (d\vec{r}\,')\,\frac{1}{c}\,\vec{J}(\vec{r}\,') + \frac{\vec{r}}{r^3}\cdot\int (d\vec{r}\,')\,\vec{r}\,'\left(\frac{1}{c}\,\vec{J}(\vec{r}\,')\right) + \ldots \tag{25.2}$$

which is analogous to the expansion for ϕ , (19.3). From current conservation for steady currents,

$$\vec{\nabla}\cdot\vec{J}(\vec{r}) = 0 \quad ,$$

the first term of (25.2) vanishes for a confined current distribution:

$$0 = \int (d\vec{r})\,\vec{r}\,\vec{\nabla}\cdot\vec{J} = \int (d\vec{r})[\vec{\nabla}\cdot(\vec{J}\,\vec{r}) - \vec{J}] = -\int (d\vec{r})\,\vec{J} \quad ,$$

$r_i\partial_k J_k$

$\partial_k(r_i J_k)$

$-J_k\partial_k r_i$

so that there is no $1/r$ term in the expansion of $\vec{A}(\vec{r})$. Physically, there is no Coulomb-like potential for magnetism, when magnetic charge is not present. The leading term in the vector potential expansion is therefore

$$\vec{A}(\vec{r}) = \frac{1}{r^3}\frac{1}{c}\int (d\vec{r}\,')\vec{r}\cdot\vec{r}\,'\;\vec{J}(\vec{r}\,') + \ldots \quad . \tag{25.3}$$

To evaluate this integral, we again use (23.8b) and consider the integral

$$0 = \int (d\vec{r})\, x_i x_j \sum_{k=1}^{3} \nabla_k J_k$$

$x_i x_j \partial_k J_k = \partial_k(x_i x_j J_k) - \int \partial_k(x_i x_j)\, J_k$

$= -x_j J_i - x_i J_j$

$$= -\int (d\vec{r})(x_j J_i + x_i J_j) \quad ,$$

or, in a dyadic notation,

$$\int (d\vec{r})[\vec{r}\,\vec{J} + \vec{J}\,\vec{r}] = 0 \quad . \tag{25.4}$$

Using this fact, we make the following rearrangement:

$$\int (d\vec{r}\,')\, \vec{r}\cdot\vec{r}\,'\, \vec{J} = \frac{1}{2}\int (d\vec{r}\,')\, \vec{r}\cdot[(\vec{r}\,'\vec{J} + \vec{J}\,\vec{r}\,') + (\vec{r}\,'\vec{J} - \vec{J}\,\vec{r}\,')]$$

$$= \frac{1}{2}\int (d\vec{r}\,')\, \vec{r}\cdot(\vec{r}\,'\vec{J} - \vec{J}\,\vec{r}\,')$$

$\epsilon_{ijk} r_j \epsilon_{klm} J_l r_m'$

$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) r_j J_l r_m'$

$$= \frac{1}{2}\int (d\vec{r}\,')(\vec{r}\,' \times \vec{J}) \times \vec{r} \quad . \tag{25.5}$$

sign problems?

$$= \frac{1}{2}\int d\vec{r}\,' (\vec{r} \times \vec{J}) \times \vec{r}\,'$$

$(r\cdot r')J_i - (r\cdot J) r_i$

$= \vec{r}\cdot(\vec{r}\,'\vec{J} - \vec{J}\,\vec{r}\,')$

$(r'\times J)\times r = r\times(J\times r') = (r\times J)\times r'$

The leading term of the vector potential now becomes

$$\vec{A}(\vec{r}) = \frac{\vec{\mu}\times\vec{r}}{r^3} \tag{25.6}$$

$= \frac{1}{r^3}\left[\frac{1}{2c}\int (r\times J)\times r'\; d\vec{r}\,'\right]$

$= \frac{1}{r^3}\;\frac{1}{2c}$

where $\vec{\mu}$ is the magnetic dipole moment, defined by

$$\vec{\mu} = \frac{1}{2c}\int (d\vec{r})\; \vec{r}\times\vec{J}(\vec{r}) \quad . \tag{25.7}$$

[For a point charge, the current density is given by (23.25), so the magnetic dipole moment is

$\vec{v} = \vec{r}\times\vec{\omega}_c \qquad \vec{r} = \frac{\vec{v}\times\omega_c}{\omega_c^2}$

$$\vec{\mu} = \frac{e}{2c}\vec{R}\times\vec{v} \quad ,$$

$\vec{\mu} = \frac{e}{2c}\vec{R}\times\vec{\omega}_c\times\vec{r}$

$= \frac{e}{2c}\frac{v^2}{\omega_c} = \frac{1}{2}v^2\frac{e\,mc}{c\,eB} = \frac{\frac{1}{2}mv^2}{B}$

in agreement with (6.18).]

The leading contribution to the magnetic field can now be calculated from this vector potential,

$\epsilon_{ijk} \partial_j \epsilon_{klm} A_l B_m$

$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j A_l B_m$

$\cancel{A_i \partial_m B_m} - B_m \partial_j A_l$

$\partial_m A_i B_m \quad - \partial_j B_i A_m$

$\vec{A}(\nabla\cdot\vec{B}) - B(\nabla\cdot\vec{A})$

$+ (\vec{B}\cdot\nabla) A - (\vec{A}\cdot\nabla)\vec{B}$

$$\vec{B} = \vec{\nabla} \times \left(\vec{\mu} \times \frac{\vec{r}}{r^3} \right) = \vec{\mu}\, \vec{\nabla} \cdot \frac{\vec{r}}{r^3} - (\vec{\mu}\cdot\vec{\nabla}) \frac{\vec{r}}{r^3}$$

$$= 4\pi\vec{\mu}\delta(\vec{r}) - \vec{\nabla}\left(\frac{\vec{\mu}\cdot\vec{r}}{r^3} \right) , \qquad (25.8)$$

since

$\mu_i \partial_i \frac{r_j}{r^3}$

$$-\vec{\nabla}\frac{1}{r} = \frac{\vec{r}}{r^3} ,$$

$\partial_i \frac{\mu_i r_j}{r^3}$

and consequently

$$0 = \vec{\mu} \times \left(\vec{\nabla} \times \frac{\vec{r}}{r^3} \right) = \vec{\nabla}\left(\frac{\vec{\mu}\cdot\vec{r}}{r^3} \right) - (\vec{\mu}\cdot\vec{\nabla}) \frac{\vec{r}}{r^3} . \quad ? \qquad (25.9)$$

The delta function term in $\vec{B}$ is necessary in order to satisfy $\vec{\nabla}\cdot\vec{B} = 0$:

$\mu_j \partial_j \partial_k f$

$$\vec{\nabla}\cdot\vec{B} = 4\pi(\vec{\mu}\cdot\vec{\nabla})\,\delta(\vec{r}) + \nabla^2(\vec{\mu}\cdot\vec{\nabla})\frac{1}{r} = 0 . \qquad (25.10)$$

$\mu_i \partial_i \quad \partial_k^2 \mu_j \partial_j f$

For $r > 0$, this magnetic field has the same form as that of the electric field produced by an electric dipole moment, which is contained in (19.7), that is

$3\vec{r}\, u\cdot\vec{r} -$

$$\vec{B}(\vec{r}) = \frac{3\vec{r}\,\vec{\mu}\cdot\vec{r} - r^2\vec{\mu}}{r^5} . \qquad (25.11)$$

In general, this is only the leading contribution, since there are higher multipoles. We will not, however, explore these further here.

25-2. Rotating Charged Spherical Shell

An example for which the dipole expression is exact beyond a certain distance will be achieved if a charge e is distributed uniformly over a spherical shell of radius a that is rotating with angular velocity $\vec{\omega}$. If we choose the origin to be at the center of the sphere, the velocity of a point $\vec{r}'$ on the surface is

$$\vec{v} = \vec{\omega} \times \vec{r}' \quad . \tag{25.12}$$

The current density is

$$\vec{J} = \rho\vec{v} \quad ,$$

where here the charge density is entirely concentrated on the surface,

$$\int (d\vec{r}')\, \rho \ldots = \int dS'\, \sigma \ldots \quad ,$$

where the surface charge density is constant,

$$\sigma = \frac{e}{4\pi a^2} \quad .$$

The expression for the vector potential, (23.23), becomes

$$\vec{A}(\vec{r}) = \frac{1}{c}\int dS' \frac{e}{4\pi a^2} \frac{\vec{\omega} \times \vec{r}'}{|\vec{r}-\vec{r}'|} \quad , \tag{25.13}$$

where $\vec{r}$ may be either inside or outside the sphere.

We first calculate $\vec{A}$ inside the sphere, that is, for $|\vec{r}| < a = |\vec{r}'|$. Recalling the expansion, (18.10),

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{a^{\ell+1}} P_\ell(\cos\gamma) \quad , \tag{25.14}$$

and noting that $\vec{r}'$ occurring in (25.13) is related to Y_{1m}, we see that the surface integral selects only $\ell = 1$:

$$\int dS' \, Y_{\ell m} \, Y_{1m'} \sim \delta_{\ell 1} \, \delta_{mm'} \quad .$$

Therefore, only the $\ell = 1$ term in (25.14) will contribute to the integral (25.13),

$$\frac{1}{|\vec{r}-\vec{r}'|} \xrightarrow{\ell=1} \frac{r}{a^2} \frac{\vec{r}\cdot\vec{r}'}{rr'} = \frac{1}{a^3} \vec{r}\cdot\vec{r}' \quad , \tag{25.15}$$

yielding for the vector potential,

$$\vec{A}(\vec{r}) = \frac{e}{4\pi a^2 c} \int dS' (\vec{\omega} \times \vec{r}') \frac{\vec{r}'\cdot\vec{r}}{a^3} \quad . \tag{25.16}$$

Using spherical symmetry, we easily evaluate the integral over the dyadic to be

$$\int dS' \, \vec{r}'\vec{r}' = \int dS' \, \frac{1}{3} \overleftrightarrow{1} r'^2 = \frac{1}{3} a^2 4\pi a^2 \overleftrightarrow{1} \quad . \tag{25.17}$$

Therefore, the vector potential inside the sphere is

$$\vec{A} = \frac{e}{3ac} (\vec{\omega} \times \vec{r}) \equiv \frac{1}{2} \vec{B} \times \vec{r} \quad , \tag{25.18}$$

where, using the result of (6.14), we identify the magnetic field $\vec{B}$ as

$$\vec{B} = \frac{2e}{3ac} \vec{\omega} \quad , \tag{25.19}$$

which is uniform inside the sphere.

We now calculate $\vec{A}$ outside the sphere, where $|\vec{r}| > a = |\vec{r}'|$, and the appropriate expansion is

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{a^{\ell}}{r^{\ell+1}} P_{\ell} \xrightarrow{\ell=1} \frac{a}{r^2} \frac{\vec{r}\cdot\vec{r}'}{rr'} = \frac{1}{r^3} \vec{r}\cdot\vec{r}' \quad , \tag{25.20}$$

since, as before, only $\ell = 1$ contributes to the surface integral. The calculation proceeds as above except for a factor of a^3/r^3 with the result

$$\vec{A} = \frac{ea^2}{3c} \frac{\vec{\omega} \times \vec{r}}{r^3} = \frac{\vec{\mu} \times \vec{r}}{r^3} \quad , \tag{25.21}$$

which, upon comparison with (25.6), allows us to identify the magnetic dipole moment as

$$\vec{\mu} = \frac{ea^2}{3c} \vec{\omega} \quad . \tag{25.22}$$

Notice that $\vec{B}$ is discontinuous across the spherical shell because there is a surface current density. The values of $\vec{B}$ just outside and just inside the surface ($\vec{n}$ = outward normal = $\vec{r}/a$) are

$$r = a+0 : \quad \vec{B}_+ = \frac{3(\vec{\mu}\cdot\vec{n})\vec{n} - \vec{\mu}}{a^3} = \frac{ea^2}{3c} \frac{3(\vec{\omega}\cdot\vec{n})\vec{n} - \vec{\omega}}{a^3} \quad , \tag{25.23}$$

$$r = a-0 : \quad \vec{B}_- = \frac{2e}{3ca} \vec{\omega} \quad , \tag{25.24}$$

so the discontinuity in $\vec{B}$ is

$$\vec{B}_+ - \vec{B}_- = \frac{e}{ca} [(\vec{\omega}\cdot\vec{n})\vec{n} - \vec{\omega}] = \frac{e}{ca} \vec{n} \times (\vec{n} \times \vec{\omega}) \quad . \tag{25.25}$$

Now recall that in vacuum ($\vec{B} = \vec{H}$), the normal component of $\vec{B}$ is continuous while the tangential component is discontinuous if there is a surface current. Written in the notation above, these boundary conditions [(23.15b) and (23.18)] read

$$\vec{n}\cdot(\vec{B}_+-\vec{B}_-) = 0 \quad , \tag{25.26a}$$

$$\vec{n} \times (\vec{B}_+-\vec{B}_-) = \frac{4\pi}{c}\vec{K} \quad . \tag{25.26b}$$

Obviously, (25.26a) is satisfied. From (25.26b) and (25.25), we calculate the surface current density,

$$\vec{K} = \frac{c}{4\pi}\vec{n} \times \left[\frac{e}{ca}\vec{n} \times (\vec{n} \times \vec{\omega})\right] = \frac{e}{4\pi a}\vec{\omega} \times \vec{n} \quad , \tag{25.27}$$

in agreement with the direct result $(\vec{r}\,' = a\vec{n})$,

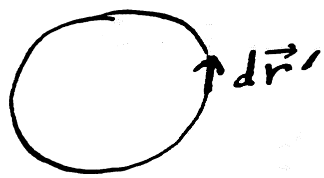
$$\vec{K} = \sigma\vec{v} = \frac{e}{4\pi a^2}(\vec{\omega} \times \vec{n})a = \frac{e}{4\pi a}\vec{\omega} \times \vec{n} \quad .$$

XXVI. Magnetic Scalar Potential

We now return to the macroscopic situation with a steady current flowing in a permeable medium characterized by a constant μ , so that the vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu}{c} \int (d\vec{r}\,') \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} , \tag{26.1}$$

in the Coulomb gauge. In particular, consider a current I flowing in a closed loop,



so that the volume integral is to be replaced by a line integral,

$$\int (d\vec{r}\,') \vec{J} \to \int d\vec{r}\,' \, I ,$$

where $d\vec{r}\,'$ is a directed line element tangential to the wire, in the direction of the current flow. Now the vector potential, (26.1), becomes

$$\vec{A}(\vec{r}) = \frac{\mu}{c} I \oint \frac{d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} , \tag{26.2}$$

which implies for the magnetic induction,

$$\vec{B} = \frac{\mu}{c} I \vec{\nabla} \times \oint \frac{d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} \tag{26.3}$$

or, for the magnetic field,

$$\vec{H} = \frac{I}{c} \oint \left[\vec{\nabla} \frac{1}{|\vec{r}-\vec{r}\,'|} \right] \times d\vec{r}\,' = \frac{I}{c} \oint d\vec{r}\,' \times \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}\,'|} . \tag{26.4}$$

It is convenient to rewrite $\vec{H}$ in terms of a surface integral instead of a line integral. We make use of Stokes' theorem for a vector field, $\vec{V}$, which reads

$$\oint_C d\vec{r}'\cdot\vec{V} = \int_S d\vec{S}'\cdot(\vec{\nabla}' \times \vec{V}) \quad , \tag{26.5}$$

where S is any surface which has the contour C as its boundary. If we replace

$$\vec{V} \rightarrow \vec{V} \times \vec{a}$$

where $\vec{a}$ is an arbitrary constant vector, Stokes' theorem becomes

$$\begin{aligned}\oint_C d\vec{r}' \times \vec{V}\cdot\vec{a} &= \int_S d\vec{S}'\cdot\vec{\nabla}' \times (\vec{V} \times \vec{a}) \\ &= \int_S d\vec{S}'\cdot[(\vec{a}\cdot\vec{\nabla}')\vec{V} - \vec{a}\,\vec{\nabla}'\cdot\vec{V}] \quad .\end{aligned} \tag{26.6}$$

The identity

$$\vec{a} \times (\vec{\nabla}' \times \vec{V}) = \vec{\nabla}'(\vec{a}\cdot\vec{V}) - (\vec{a}\cdot\vec{\nabla}')\,\vec{V} \quad ,$$

allows us to rewrite (26.6) as

$$\oint_C d\vec{r}' \times \vec{V}\cdot\vec{a} = \int_S d\vec{S}'\cdot[\vec{\nabla}'(\vec{V}\cdot\vec{a}) - \vec{a} \times (\vec{\nabla}' \times \vec{V}) - \vec{a}\,\vec{\nabla}'\cdot\vec{V}] \quad . \tag{26.7}$$

Furthermore, if everywhere on the surface S, the vector field satisfies

$$\vec{\nabla}'\cdot\vec{V} = 0 \ , \quad \text{and} \quad \vec{\nabla}' \times \vec{V} = 0 \quad , \tag{26.8}$$

(26.7) reduces to

$$\oint_C d\vec{r}' \times \vec{V} = \int_S (d\vec{S}'\cdot\vec{\nabla}')\,\vec{V} \quad . \tag{26.9}$$

We will apply this result to rewrite (26.4) for which

$$\vec{V} = \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}'|} = -\vec{\nabla} \frac{1}{|\vec{r}-\vec{r}'|} \quad , \tag{26.10}$$

which satisfies the conditions (26.8) as long as $\vec{r} \neq \vec{r}'$, that is, at points outside the wire. We therefore find

$$\vec{H} = -\vec{\nabla} \phi_m \quad , \tag{26.11}$$

where the magnetic scalar potential, ϕ_m , is

$$\phi_m(\vec{r}) = \frac{I}{c} \int d\vec{S}' \cdot \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}'|} \quad . \tag{26.12}$$

According to (1.10), the surface integral,

$$-\int d\vec{S}' \cdot \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}'|} = -\int d\vec{S}' \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \Omega \quad , \tag{26.13}$$

is the solid angle subtended by the current loop at the observation point, so that

$$\phi_m = -\frac{I}{c} \Omega \quad . \tag{26.14}$$

Therefore, a scalar potential for $\vec{H}$ exists for points not on the wire, consistent with the Maxwell equation, (23.8a),

$$\vec{\nabla} \times \vec{H} = 0 \quad . \tag{26.15}$$

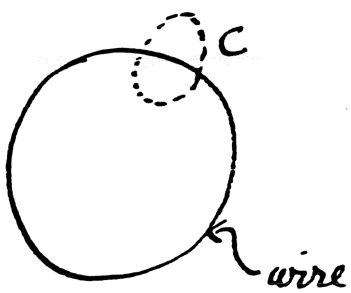
However, this scalar potential is not single-valued. We consider the integral of $\vec{H}$ around a closed path, C , which does not touch the wire,

$$\oint_C d\vec{r} \cdot \vec{H} = -\oint d\vec{r} \cdot \vec{\nabla} \phi_m = -\phi_m \Big]_{0}^{0} \quad . \tag{26.16}$$

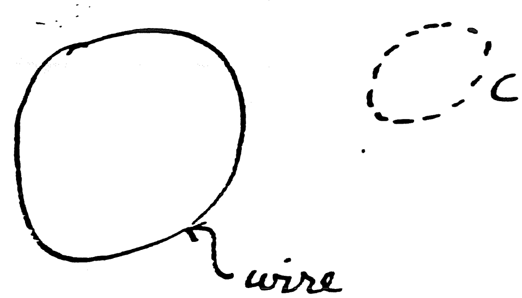
The naive anticipation is that (26.16) would be zero. An alternative calculation of this quantity can be made using Stokes' theorem:

$$\oint_C d\vec{r}\cdot\vec{H} = \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{H}) = \int_S d\vec{S}\cdot\frac{4\pi}{c}\vec{J}$$
$$= \begin{cases} \pm\frac{4\pi}{c}\,I\ , & \text{YES}\ , \\ 0\ , & \text{NO}\ , \end{cases} \tag{26.17}$$

where YES means the wire passes once through the surface S, bounded by the path C, [the $\pm$ sign refers to the relative orientations of $d\vec{S}$ and $\vec{J}$] while NO means the wire does not pass through the surface. Some examples of this are supplied by the following illustrations.



YES



NO

Therefore, contrary to our naive expectation,

$$\left.\phi_m\right]_{\oint} = \begin{cases} \mp\frac{4\pi}{c}\,I\ , & \text{YES}\ , \\ 0\ , & \text{NO}\ , \end{cases} \tag{26.18a}$$

or

$$\left.\Omega\right]_{\oint} = \begin{cases} \pm 4\pi\ , & \text{YES}\ , \\ 0\ , & \text{NO}\ , \end{cases} \tag{26.18b}$$

which means that ϕ_m is a multi-valued function, as required by the fact that $\vec{\nabla}\times\vec{H}$ is not zero everywhere. The discontinuity found in (26.18b) corresponds to the fact that when one crosses the surface S defined by the current loop and used in the evaluation of the solid angle in Ω , there is a change of 4π .

Very far away from the current loop, the solid angle subtended by it is, if we assume that the points of S are localized in the vicinity of the loop,

$$\Omega = -\frac{\vec{r}\cdot\vec{S}}{r^3}, \quad \vec{S} = \int d\vec{S}' , \tag{26.19}$$

so the corresponding magnetic field is

$$\vec{H} = -\vec{\nabla}\left(\frac{\vec{\mu}\cdot\vec{r}}{r^3}\right) , \tag{26.20}$$

which upon comparison with (25.8) identifies the magnetic moment of the current loop to be

$$\vec{\mu} = \frac{I}{c}\vec{S} . \tag{26.21}$$

We obtain the same result if we use the definition of the magnetic moment, (25.7),

$$\vec{\mu} = \frac{1}{2c}\int (d\vec{r}')\, \vec{r}' \times \vec{J}(\vec{r}') = \frac{I}{c}\frac{1}{2}\oint \vec{r}' \times d\vec{r}' = \frac{I}{c}\vec{S} , \tag{26.22}$$

where we have used (26.7) to evaluate the line integral, which is also obvious geometrically.

XXVII. Magnetic Charge II

In the previous sections, we have considered the magnetic fields produced by steady currents with some attention to the attendant vector potential. As we have indicated at various points, an alternative source of a static magnetic field would be static magnetic charge, if such exists. We would here like to consider a few consequences for the vector potential corresponding to such a magnetic field.

Let a magnetic charge, g, be located at the origin so that the magnetic field satisfies

$$\vec{\nabla}\cdot\vec{B} = 4\pi g\,\delta(\vec{r}) \quad , \tag{27.1}$$

which has the solution

$$\vec{B} = g\,\frac{\vec{r}}{r^3} = -\vec{\nabla}\,\frac{g}{r} \quad . \tag{27.2}$$

Away from the origin, $\vec{B}$ is divergenceless,

$$\vec{\nabla}\cdot\vec{B} = 0 \quad ,$$

so we would once again expect $\vec{B}$ to be the curl of a vector potential,

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad . \tag{27.3}$$

However, this cannot be true everywhere since

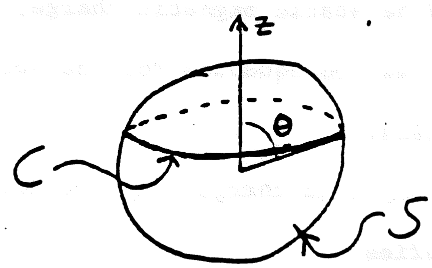
$$\oint d\vec{S}\cdot\vec{\nabla}\times\vec{A} = \int (d\vec{r})\,\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0 \quad , \tag{27.4}$$

while (27.1) implies for a closed surface surrounding the magnetic charge

$$\oint d\vec{S}\cdot\vec{B} = \int (d\vec{r})\ \vec{\nabla}\cdot\vec{B} = 4\pi g\ . \tag{27.5}$$

We now want to find a vector potential that satisfies (27.3) almost everywhere. The simplest possibility is that this equation fails to hold on a line, which we may take to be the +z axis. We apply Stokes' theorem in the form

$$\int_C \vec{A}\cdot d\vec{r} = -\int_S \vec{B}\cdot d\vec{S} \tag{27.6}$$



where C is a circle of constant θ on a sphere of radius r about the origin and S is the lower portion of the spherical surface bounded by C. Equation (27.6) holds since (27.3) is true everywhere on S. [The minus sign appears because we use the outward normal to the surface S.] The surface integral follows trivially from (27.2),

$$\int_S \vec{B}\cdot d\vec{S} = \frac{g}{r^2}\, 2\pi r^2(1+\cos\theta)\ . \tag{27.7}$$

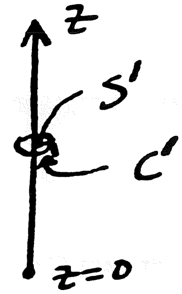
An obvious solution of (27.6) is then

$$\vec{A} = A_\phi\ \hat{\phi} \tag{27.8}$$

where

$$A_\phi = -\frac{g}{r}\,\frac{1+\cos\theta}{\sin\theta}\ . \tag{27.9}$$

The structure of the singularity on the z axis is now isolated by taking the limit $\theta \to 0$, where C' is an infinitesimal circle about the z axis and S' is the enclosed area.



Since (27.10) picks out $(\nabla \times A)_z$, which has the singularity $-4\pi g\ \delta(x)\ \delta(y)$ on the +z axis, we conclude that the magnetic field can be expressed everywhere by

$$\vec{B} = \vec{\nabla} \times \vec{A} + 4\pi g\ \delta(x)\ \delta(y)\ \eta(z)\ \hat{z} \tag{27.11}$$

where $\eta(z)$ is the step function. This result can be confirmed by noting that $\vec{B}$ has the correct divergence, (27.1),

$$\vec{\nabla}\cdot\vec{B} = 0 + 4\pi g\ \delta(x)\ \delta(y)\ \delta(z) \quad . \tag{27.12}$$

The vector potential (27.8) is an example of a whole class of potentials that yield the correct magnetic field except for a one-dimensional set of points, a curve. On this curve, called a string, $\vec{A}$ is singular, whereas the magnetic field is regular, being the curl of $\vec{A}$ plus a compensating singularity on the string.

Warren Mori

Lecture 1

XXVIII. RETARDED GREEN'S FUNCTION AND LIENARD-WIECHERT POTENTIALS

28-1. Potentials and Gauges

In the previous sections, we have primarily confined ourselves to the discussion of electrostatics and magnetostatics. We will now study in general how time-dependent electromagnetic fields are produced by arbitrary charges and currents. In vacuum, we recall that Maxwell's equations are [see (1.27)]

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{j} \quad , \tag{28.1a}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \quad , \tag{28.1b}$$

$$-\vec{\nabla} \times \vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B} \quad , \tag{28.1c}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad , \tag{28.1d}$$

where ρ is the charge, and $\vec{j}$ the current density, and we have assumed that no magnetic charge is present. Notice that the local charge conservation law,

$$\vec{\nabla}\cdot\vec{j} + \frac{\partial}{\partial t}\rho = 0 \quad , \tag{28.2}$$

is not an independent statement, but is derivable from (28.1a) and (28.1b).

To solve Maxwell's equations, we first recognize that the last two equations, (28.1c) and (28.1d), make no reference to charge or current, and they can be identically satisfied by introducing potentials through the definitions

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \tag{28.3}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\phi \quad . \tag{28.4}$$

$A_\mu \to A_\mu + \partial_\mu \lambda$

$F_{i\mu} = \partial_i A_\mu - \partial_\mu A_i = \partial_i \partial_\mu \lambda - \partial_\mu \partial_i \lambda = 0$

As we have observed previously, in Subsection 9-4, the potentials $\vec{A}$ and ϕ are not uniquely defined. Since the magnetic field is the curl of $\vec{A}$, it is unchanged when a gradient is added to $\vec{A}$,

$$\vec{A} \to \vec{A} + \vec{\nabla}\lambda \, , \tag{28.5}$$

where λ is an arbitrary function. In order that this new choice of vector potential does not alter the electric field, (28.4), it is necessary to simultaneously replace the scalar potential by

$$\phi \to \phi - \frac{1}{c}\frac{\partial}{\partial t}\lambda \, . \tag{28.6}$$

This new set of potentials, (28.5) and (28.6), is as acceptable as the original one since only the fields $\vec{B}$ and $\vec{E}$ are physically measurable quantities. This arbitrariness in the choice of potentials is called the gauge freedom of the theory, while the corresponding transformations are called gauge transformations. In the following, we will exploit this freedom in the process of solving the differential equations for the potentials.

Upon substituting the constructions of $\vec{B}$ and $\vec{E}$ in terms of potentials, (28.3) and (28.4), into the first set of Maxwell's equations, we find, from (28.1a),

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{1}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A} \right) + \frac{4\pi}{c}\vec{j} \, ,$$

or

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\vec{A} = -\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi \right) + \frac{4\pi}{c}\vec{j} \, , \tag{28.7}$$

and, from (28.1b),

$$-\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = 4\pi\rho \quad . \tag{28.8}$$

This is a pair of coupled second order differential equations for $\vec{A}$ and ϕ, which may be simplified by utilizing the gauge freedom in defining the potentials. The two most convenient and common choices of gauge are discussed below.

(1) The radiation gauge (or Coulomb gauge) is defined by the condition

$$\vec{\nabla}\cdot\vec{A} = 0 \quad . \tag{28.9}$$

That we can always make this choice was shown in Subsection 23-3. In this gauge, (28.7) and (28.8) reduce to

$$-\nabla^2\phi = 4\pi\rho \quad , \tag{28.10}$$

$$-\Box^2\vec{A} = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\phi) \quad , \tag{28.11}$$

where

$$\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{28.12}$$

is the d'Alembertian. The equation for ϕ, (28.10), is just the same as that in electrostatics (hence the origin of the term "Coulomb gauge") so that ϕ is, in principle, known. The structure on the right hand side of (28.11) is proportional to an effective current, the second term of which is present in order that it be divergenceless:

$$\vec{\nabla}\cdot\left[\vec{j} - \vec{\nabla}\left(\frac{1}{4\pi}\frac{\partial}{\partial t}\phi\right)\right] = \vec{\nabla}\cdot\vec{j} - \frac{1}{4\pi}\frac{\partial}{\partial t}(\nabla^2\phi)$$

$$= \vec{\nabla}\cdot\vec{j} + \frac{\partial}{\partial t}\rho = 0 \quad ,$$

where the last equality follows from charge conservation, (28.2). This relation also entails the consistency of the choice of the radiation gauge in that if we set $\vec{\nabla}\cdot\vec{A}$ equal to zero at one time, it remains zero for all time, since

$$-\Box^2(\vec{\nabla}\cdot\vec{A}) = \frac{4\pi}{c}\vec{\nabla}\cdot\left[\vec{j} - \vec{\nabla}\left(\frac{1}{4\pi}\frac{\partial}{\partial t}\phi\right)\right] = 0 \quad .$$

I'm a little confused about what exactly is being implied here

(2) The Lorentz gauge condition is a relation between vector and scalar potentials,

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi = 0 \quad . \tag{28.13}$$

In this gauge, the equations for $\vec{A}$ and ϕ have the symmetrical form,

$$-\Box^2\vec{A} = \frac{4\pi}{c}\vec{j} \quad , \tag{28.14}$$

$$-\Box^2\phi = 4\pi\rho \quad . \tag{28.15}$$

The consistency of this gauge choice again follows from the fact that charge is conserved,

$$-\Box^2\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi\right) = \frac{4\pi}{c}\left(\vec{\nabla}\cdot\vec{j} + \frac{\partial\rho}{\partial t}\right) = 0 \quad .$$

28-2. Green's Function and Potentials in the Lorentz Gauge

In the following, we will solve the differential equations, (28.14) and (28.15), for the potentials in the Lorentz gauge. Since the potentials are linearly related to their sources, they may be expressed in terms of a Green's function,

$$\phi(\vec{r},t) = \int (d\vec{r}\,')\,dt'\,G(\vec{r}-\vec{r}\,', t-t')\,\rho(\vec{r}\,',t') \quad , \tag{28.16}$$

$$\vec{A}(\vec{r},t) = \int (d\vec{r}')\, dt'\, G(\vec{r}-\vec{r}', t-t') \frac{1}{c} \vec{j}(\vec{r}',t') \quad . \tag{28.17}$$

This Green's function, $G(\vec{r}-\vec{r}', t-t')$, is a function only of relative positions and times because of translational invariance in unbounded space. Since ϕ satisfies (28.15), this Green's function obeys the differential equation

$$-\Box^2 G(\vec{r}-\vec{r}', t-t') = 4\pi\, \delta(\vec{r}-\vec{r}')\, \delta(t-t') \quad , \tag{28.18}$$

which is a four-dimensional generalization of the three-dimensional Green's function equation we studied in electrostatics,

$$-\nabla^2 G(\vec{r}-\vec{r}') = 4\pi\, \delta(\vec{r}-\vec{r}') \quad . \tag{28.19}$$

To solve (28.18), we will analyze its time dependence by making use of the exponential representations

$$\delta(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \quad , \tag{28.20}$$

$$G(\vec{r}-\vec{r}', t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, G_\omega(\vec{r}-\vec{r}') \quad , \tag{28.21}$$

where G_ω satisfies the three-dimensional differential equation,

$$-\left(\nabla^2 + \frac{\omega^2}{c^2} \right) G_\omega(\vec{r}-\vec{r}') = 4\pi\, \delta(\vec{r}-\vec{r}') \quad . \tag{28.22}$$

In the static limit, $\omega \to 0$, (28.22) reduces to (28.19), the solution of which is Coulomb's potential, (12.3):

$$G_{\omega=0}(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} \quad . \tag{28.23}$$

Since G_ω depends only on $\vec{r}-\vec{r}\,'$, we may set $\vec{r}\,' = 0$, without loss of generality in the following discussion. Also, since we are now looking for a spherically symmetrical solution for G_ω, it is natural to use a spherical coordinate system in which the Laplacian here reduces to

$$\nabla^2 \to \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) . \tag{28.24}$$

Therefore, for $r > 0$, we wish to solve the homogeneous equation

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{\omega^2}{c^2}\right] G_\omega(\vec{r}) = 0 , \tag{28.25}$$

subject to the boundary condition that there is a point charge at the origin. The consequence of this requirement is most conveniently extracted by integrating (28.22) over a sphere of vanishing radius r_0 about the origin,

$$\begin{aligned} 4\pi &= -\int (d\vec{r})\ \vec{\nabla}\cdot(\vec{\nabla}G_\omega) \\ &= -\int dS\ \nabla_r\ G_\omega = -4\pi\ r^2\frac{d}{dr}G_\omega\Big|_{r_0\to 0} , \end{aligned}$$

or

$$-r^2\frac{d}{dr}G_\omega(\vec{r})\Big|_{r_0\to 0} = 1 . \tag{28.26}$$

[We have noted that the ω^2/c^2 term in the differential equation does not contribute to the integral since

$$\frac{\omega^2}{c^2}G_\omega \sim \frac{1}{r} , \qquad \text{as} \qquad r \to 0 ,$$

which has vanishing volume integral as $r_0 \to 0$.] To solve (28.25), we introduce g_ω , defined by

$$G_\omega = \frac{1}{r} g_\omega \quad , \tag{28.27}$$

which satisfies the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{\omega^2}{c^2} \right) g_\omega(r) = 0 \ , \quad \text{for} \quad r > 0 \quad , \tag{28.28}$$

where we have used

$$r^2 \frac{d}{dr} G_\omega = r \frac{d}{dr} g_\omega - g_\omega \quad ,$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} G_\omega \right) = \frac{1}{r} \frac{d^2}{dr^2} g_\omega \quad . \tag{28.29}$$

The independent solutions of (28.28) have the form

$$g_\omega \sim e^{\pm i \frac{\omega}{c} r} \quad ,$$

and the corresponding forms for G_ω are

$$G_\omega(r) = \frac{C}{r} e^{\pm i \frac{\omega}{c} r} \quad , \quad \text{for} \quad r > 0 \quad . \tag{28.30}$$

For either choice of + or - sign, the constant C is determined by the boundary condition (28.26) to be

$$C = 1 \quad . \tag{28.31}$$

Therefore, we have two fundamental solutions to (28.22),