Due to such fluctuations in the length, a simple pendulum must have a fluctuating time period; so the time measurements will be uncertain. Even neglecting the fluctuations in the length, the pendulum still does not give absolutely accurate values of time intervals, since as a result of having one degree of freedom, it must have a random mean kinetic energy equal to $\frac{1}{2}kT$.

8. Galvanometer:

Probably one of the most interesting and important cases where the Brownian movement limits the sensitivity of a measuring instrument is that of the galvanometer.

Let us assume a circuit of resistance R and self inductance L kept for simplicity at a temperature T (these parameters could refer to the coil of a galvanometer). If i denotes the spontaneous current fluctuations, we can use the equipartition theory to write

$$\frac{1}{2} L \overline{i^2} = \frac{1}{2} kT \quad .$$

It is now useful to imagine these currents to be set up by a haphazard emf E. We have thus effectively replaced the galvanometer coil by a generator with an emf E and an internal impedance $(R + j\omega L)$. We shall show in our later discussion of the Nyquist theorem that it is perfectly reasonable to associate a fluctuating emf with any dissipative element in the circuit; as a matter of fact this is another form of fluctuation—dissipation theorem which we have already discussed at several times. It is then possible to write down a differential equation for the circuit viz.

$$L \frac{di}{dt} + R i = E$$

which yields by integration for the value i_t of current at instant t

$$i_t = i_o e^{-(R/L)t} + \frac{e^{-(R/L)t}}{L} \int_o^t E(t') e^{(R/L)t'} dt' \quad .$$

To obtain the effect of one disturbance upon a later one, we multiply this equation by i_o and average over time to get

$$\overline{i_o i_t} = i_o^2 \; e^{-(R/L)t} \quad .$$

The second term vanishes since i_o is independent of any later emf's. We thus find that the two kicks i_o and i_t, separated by a time interval t are correlated, the factor of correlation being $e^{-(Rt/L)}$. This correlation becomes vanishingly small where $t >> (L/R)$.

It is now possible to compute the average current strength j over a time τ

$$j = \frac{1}{\tau} \int_o^\tau i \, dt$$

$$\therefore \overline{j^2} = \frac{1}{\tau^2} \int_o^\tau dt \int_o^\tau dt' \; \overline{i_t i_{t'}}$$

$$= \frac{2}{\tau^2} \int_o^\tau dt \int_t^\tau dt' \; \overline{i_t i_{t'}}$$

$$= \frac{2 \overline{i^2}}{\tau^2} \int_o^\tau dt \int_o^{\tau - t} e^{-(R/L)s} \, ds \quad .$$

If $\tau >> (L/R)$ we can replace $(\tau - t)$ by ∞ without introducing much error so that

$$\overline{j^2} = \frac{2 \overline{i^2} L}{\tau^2 R} \int_o^\tau dt = \frac{2 \overline{i^2} L}{R \tau} \quad .$$

Using $\frac{1}{2}L\overline{i^2} = \frac{1}{2}kT$, we get

$$\overline{\delta i} = (\overline{j^2})^{1/2} = (\frac{2kT}{R\tau})^{1/2} \rightarrow \text{current fluctuations; similarly,}$$

$$\overline{\delta v} = (2kTR/\tau)^{1/2} \rightarrow \text{voltage fluctuations.}$$

If the galvanometer is critically damped, the above analysis is to be modified; only the numerical coefficient changes however and we get (Ising and Zernike)

$$\overline{\delta^2} = (\frac{\pi kT}{R\tau})^{1/2} \quad , \qquad \overline{\delta v} = (\frac{\pi kTR}{\tau})^{1/2} \quad .$$

Ising compared his theory with the experiments of Moll and Burger; reducing their zero deflections to a voltage equivalent form, Ising found that observed $\overline{\delta v} = 9.22 \times 10^{-10}$ volts compared with a value of 6.3×10^{-10} volts computed using the above relation (with τ replaced by T_o, the period of the undamped galvanometer). Thus Brownian movement fluctuations lead to uncertainty in any single measurement of current or voltage.

A direct proof that the galvanometer fluctuations are truly due to Brownian motion was offered by Ornstein and his co-workers. The classical theory shows the dependence of the motion on temperature. It is necessary then, only to carry the general theory over to the specific case of the galvanometer and to compare the calculations with the experimental results obtained by observing the change of the inherent fluctuations of a galvanometer with the change of temperature. Practically, it is hard to change the temperature of the galvanometer itself; it is easier to insert a resistance coil whose temperature may be varied at will. This requires the development of a formula to cover the case of a system with two components which are not at the same temperature.

Consider a system with two inductances L_1, L_2 and two resistances R_1, R_2 at temperatures T_1, T_2 (thus L_1, R_1, T_1, may refer to the galvanometer coil and L_2, R_2, T_2 to the external resistance). It is well known from Brownian motion theory that if f(z) is the spontaneous Brownian force at time z, then

$$\overline{ff} = \int \overline{f(z)\, f(z+x)}\ dx = I = 2kT\ R$$

where R is a measure of the drag force on the Brownian particle. It can be shown that for the electrical case $\overline{EE} = 2kTR$ (where E is the electromotive force and R the electrical resistance). For a two coil system with the coils at different temperatures we may then write that $\overline{E_1E_1} = 2kT_1R_1$, $\overline{E_2E_2} = 2kT_2R_2$ where E_1 and E_2 are the emf associated with generators replacing coils 1 and 2. We shall further assume that $\overline{E_1E_2} = 0$, as the Brownian fluctuations in one coil will be completely independent of the other. The equation for our circuit is $(L_1 + L_2)\ di/dt + (R_1 + R_2)\ i = E_1 + E_2$. Integrating for i , we get

$$i = e^{-[(R_1+R_2)/(L_1+L_2)]t} \{i_o + \int_o^t [E_1(z) + E_2(z)]\ e^{[(R_1+R_2)/(L_1+L_2)]z}\ dz\ \} \quad .$$

Neglecting the i_o term since it decays exponentially, squaring and taking the average, we get

$$\tfrac{1}{2}(L_1 + L_2)\ \overline{i^2} = \tfrac{1}{2}k\ (\frac{R_1T_1 + R_2T_2}{R_1 + R_2}) \quad .$$

For $T_1 = T_2$ we get the usual equipartition law; otherwise we obtain a weighted value in which the coil of higher resistance—temperature product plays the dominant role.

We are now in a position to estimate the Brownian motion of the galvanometer system. Neglecting the inducatances L, the equation of motion and the electrical equation for the galvanometer—resistance system may be written as

$$K \frac{d^2\theta}{dt^2} + \beta \frac{d\theta}{dt} + \alpha\theta + qi = f$$

and

$$(R_1 + R_2)\, i - q \frac{d\theta}{dt} = E_1 + E_2$$

where K is the moment of inertia of the galvanometer coil, β is a damping constant taking care of air damping etc., q is the galvanometer coil effective flux and α is the couple per unit radian twist. Eliminating i between the two equations we get

$$K \frac{d^2\theta}{dt^2} + \left(\beta + \frac{q^2}{R_1 + R_2}\right) \frac{d\theta}{dt} + \alpha\theta = f - \frac{q}{R_1 + R_2}(E_1 + E_2) \ .$$

If we proceed as usual to solve for θ and evaluate $\overline{\theta^2}$ we get (neglecting the air damping β in comparison to the electromagnetic damping term $q^2/(R_1 + R_2)$)

$$\tfrac{1}{2}\alpha\overline{\theta^2} = \tfrac{1}{2} k \left(\frac{R_1T_1 + R_2T_2}{R_1 + R_2}\right) \ .$$

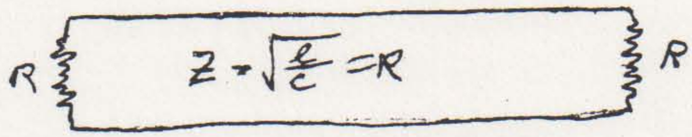
Thus the Brownian fluctuations as measured in the potential energy of rotation are exactly equivalent to the energy of current fluctuations. These results can be checked experimentally by varying T_2' and observing the corresponding change in the disturbance of zero. This is what Ornstein and his co-workers did; they obtained a very good fit of experiment with theory.

Conclusion:

We may conclude by saying that practically every process which might be used to make physical measurements is in some way limited by Brownian motion. We may also state simply that since matter and energy are ultimately divided into small units (atoms, molecules, electrons, photons, etc.), therefore the natural limit of sensitivity is reached by any method of measuring as soon as that method begins to detect the individual effects of these small units.

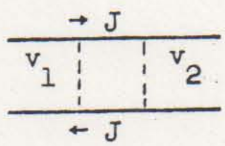
INSERT V.1

Aside on Transmission Line



ℓ = inductance per unit length of line

c = capacitance per unit length



$$C\,\frac{\partial v}{\partial t} = -\,\frac{\partial J}{\partial x} = \frac{\partial Q}{\partial t} \qquad \text{From } cv = Q \qquad (1)$$

$$\frac{\partial v}{\partial x} = -\,\frac{\ell \partial J}{\partial t} \qquad \text{From } L\,\frac{\partial J}{\partial t} = v$$

V is the voltage around the circuit.

(2)

V is the voltage difference between the two wires of the line, J is the current in the line, it is equal and opposide in the two wires. There can be no E field parallel to a wire since we take them to be perfectly conducting.

Taking the x derivative (1) and the time derivative of (2) and eliminating V gives

$$\ell c\,\frac{\partial^2 J}{\partial t^2} - \frac{\partial^2 J}{\partial x^2} = 0 \qquad (3)$$

INSERT V.2

This is the wave equation and the velocity of propagation in

$$v = \frac{1}{\sqrt{\ell c}} \qquad (4)$$

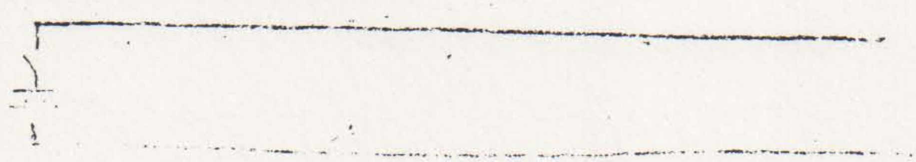
By taking the time derivative of (1) and the x derivative of (2) and eliminating J we can also get a wave equation for V

$$\ell c \frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = 0 \qquad (5)$$

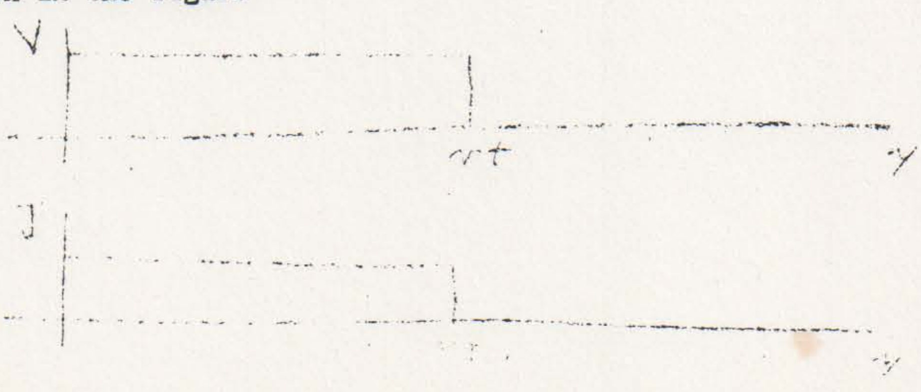
The solution to these equations are

$$J = J(x \pm vt) \;,\; V = V(x \pm vt) \qquad (6)$$

where J and V are arbitrary functions of $x \pm t$. Now suppose we have a semi-infinite transmission



line with a battery and switch at the end. If the switch is suddenly closed, a voltage suddenly appears across the line at the end. As time goes on the voltage propagates along the line. Likewise a current starts to flow in the line to charge it up. Thus V and J have the form shown in the figure



INSERT V.3

Integrating equation 1 from 0 to ∞ gives

$$\int \frac{\partial J}{\partial x}\, dx = - c \int \frac{\partial V}{\partial t}\, dx = - c.\int \frac{\partial V}{\partial x} \frac{dx}{dt}\, dx \qquad (7)$$

$$- J_o = - c V_o v \qquad (8)$$

or

$$J_o = c V_o v . \qquad (9)$$

The transmission line looks like it has a resistance R ,

$$R = \frac{1}{c} v = \sqrt{\frac{\ell}{c}} , \qquad (10)$$

to the battery.

INSERT III.1

Generalization of the Central Limit Theorem

Let $\underset{\sim}{r}$ be a random vector. It may be a vector in an n_o dimensional space. The sum of N $\underset{\sim}{r}$'s is also a random vector.

$$\underset{\sim}{s}_N = \sum_{n=1}^{N} \underset{\sim}{r}_n \qquad \text{III. 1}$$

We wish to compute the probability that $\underset{\sim}{s}_n$ takes on the value $\underset{\sim}{s}$ for N large.

$$P_{N+1}(\underset{\sim}{s}) = \sum_{\underset{\sim}{r}} P_N(\underset{\sim}{s} - \underset{\sim}{r}) P_1(\underset{\sim}{r}) \qquad \text{III. 2}$$

$$P_{N+1}(\underset{\sim}{s}) = \sum_{\gamma} \left[P_N(\underset{\sim}{s}) - \underset{\sim}{r} \cdot \frac{\partial P_N(\underset{\sim}{s})}{\partial \underset{\sim}{s}} + \frac{\underset{\sim}{r}\,\underset{\sim}{r}}{2} : \frac{\partial^2 P_N(\underset{\sim}{s})}{\partial \underset{\sim}{s}\, \partial \underset{\sim}{s}} \; P_1(\underset{\sim}{r}) \right] \qquad \text{III. 3}$$

$$P_{N+1}(\underset{\sim}{s}) - P_N(\underset{\sim}{s}) = \frac{\overline{\underset{\sim}{r}\,\underset{\sim}{r}}}{2} : \frac{\partial^2 P_N(\underset{\sim}{s})}{\partial \underset{\sim}{s}\; \partial \underset{\sim}{s}} \qquad \text{III. 4}$$

$$P = \eta(N)\, S(\underset{\sim}{s}) \qquad \text{III. 5}$$

$$\frac{\eta(N+1) - \eta(N)}{\eta(N)} = \frac{\overline{\underset{\sim}{r}\,\underset{\sim}{r}}}{2} : \frac{\dfrac{\partial^2 S(\underset{\sim}{s})}{\partial \underset{\sim}{s}\; \partial \underset{\sim}{s}}}{S(\underset{\sim}{s})} \qquad \text{III. 6}$$

$$\frac{\eta(N+1) - \eta(N)}{\eta(N)} = -\frac{1}{\tilde{N}} \qquad \text{III. 7}$$

INSERT III. 2

$$\eta(N) = e^{-N/\tilde{N}} \qquad \text{III. 8}$$

$$e^{-\alpha} - 1 = -\tilde{N}^{-1} \qquad \text{III. 9}$$

$\alpha \approx \tilde{N}^{-1}$ if $\tilde{N}$ is large and hence α is small

$$S = S_o(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{s}} \qquad \text{III. 10}$$

$$\frac{\overline{\underset{\sim}{r}\,\underset{\sim}{r}}}{2} : \underset{\sim}{k}\,\underset{\sim}{k} = -\tilde{N}^{-1} \qquad \text{III. 11}$$

$\overline{r\ r}$ is symmetric .

We write

$$\frac{\overline{\underset{\sim}{r}\,\underset{\sim}{r}}}{2} : \underset{\sim}{k}\,\underset{\sim}{k} \text{ as } \underset{\sim}{k}\cdot\frac{\overline{\underset{\sim}{r}\,\underset{\sim}{r}}}{2}\cdot\underset{\sim}{k} \qquad \text{III. 12}$$

$$P(N, \underset{\sim}{s}) = S(\underset{\sim}{k})\, e^{-\underset{\sim}{k}\cdot\frac{\overline{\underset{\sim}{r}\,\underset{\sim}{r}}}{2}\, kN + i\underset{\sim}{k}\cdot\underset{\sim}{s}}\, d\underset{\sim}{k} \qquad \text{III. 13}$$

$$\underset{N\to\infty}{} S(0)\int e^{-k\cdot\frac{\overline{\underset{\sim}{r}\,\underset{\sim}{r}}}{2}\cdot kN + i\underset{\sim}{k}\cdot\underset{\sim}{s}}\, d\underset{\sim}{k} \qquad \text{III. 14}$$

$$\underset{\sim}{k}\cdot\overline{\underset{\sim}{r}\,\underset{\sim}{r}}\cdot\underset{\sim}{k} = k_i\,\overline{r_i r_j}\,k_j \ . \qquad \text{III. 15}$$

The equation

$$\sum_{ij} k_i\,\overline{r_i r_j}\,k_j = \text{constant} \qquad \text{III. 16}$$

is the equation of an ellipsoid in the k space. The principal axes of the ellipsoid are the directions for which the distance to the ellipsoid is a maximum. The distance is

$$\ell^2 = \sum_i k_i^2 \qquad \text{III. 17}$$

$$2\ell\, d\ell = \sum_i 2k_i\, dk_i = 2\sum_{ij} \delta_{ij}\, k_j\, dk_i \quad . \qquad \text{III. 18}$$

We have the constraints

$$2\sum_{ij} \overline{r_i\, r_j}\, k_j\, dk_i = 0 \quad . \qquad \text{III. 19}$$

Multiplying III. 19 by the Lagrange multiplier $1/\lambda$ and subtracting from III. 18 gives

$$\sum_{ij} (\overline{r_i\, r_j} - \lambda\, \delta_{ij})\, k_j\, dk_i = 0 \quad . \qquad \text{III. 20}$$

Since the dk_i are arbitrary, the only way to satisfy this is to have all the coefficients of the dk_i's equal to zero.

Thus,

$$\sum_j (\overline{r_i\, r_j} - \lambda\, \delta_{ij})\, k_j = 0 \quad . \qquad \text{III. 21}$$

There are in general n_o eigen values of λ for this equation; n_o is the dimensionality of the space. Associated with each λ_n is an eigen vector