Brownian Motion as a Natural Limit to Measuring Processes

Reference: Bowling Barnes and Silverman, Reviews of Modern Physics **6**, 162 (1934).

History: After Einstein's remarkable analysis of the phenomenon of Brownian movement in 1905, several people discussed som further consequences of his theory. Notable among these were Smoluchowski (1906), Langevin (1908) and Frau G. L. de Haas - Lorentz (1913). Frau de Haas - Lorentz in particular, gave a very complete and systematic account of the theory of Brownian motion at that time; in addition, she investigated for the first time several phenomena connected with Brownian motion which can significantly influence some measurement processes. Her work was however, somehow largely overlooked in the literature. In 1925, Moll and Burger observed experimentally the deflections of the zero of an ordinary moving coil galvanometer. In their experiment, they magnified the deflection from a galvanometer several hundred times (using a so called 'thermorelay') and recorded them photographically; their main discovery was that the zero point was far from being stationary and showed small fluctuations. Their galvanometer was of course well shielded from thermal, electrical, magnetic and mechanical disturbances and so they attributed this residual unsteadiness to disturbances of a microseismic nature. It was left to Ising (1926) to give a correct description of these fluctuations in the position of zero of a galvanometer. He made a careful study of the nature of these small zero deflections and from them calculated the mean square deviation; he then conclusively showed that they were for most part, purely of Brownian motion origin. Ising's paper can be considered as a classic contribution to the subject, since it is the first paper giving a correct interpretation of such zero point deviations in measuring instruments (these deviations were indeed observed by various experimenters even before Moll and Burger, but were never correctly identified).

General: On the basis of the equipartition theorem we know that associated with each degree of freedom is a mean kinetic energy of $\frac{1}{2}kT$. One can then use the laws of statistical mechanics to calculate in perfect generality, the Brownian movement fluctuations in a system, due to this $\frac{1}{2}kT$ energy. This is possible since the average energy of these random motions will be exactly the same for all systems at the same temperature (so long as they are each in thermodynamic equilibrium with their surroundings) entirely independent of the nature of the systems and the mechanism which produces them. However, just because of this independence from the detailed mechanism of energy fluctuations, it is very unsatisfactory to attempt in a particular case to say that the source of the existing variations is this or that effect. Furthermore, we may not hope to decrease the magnitude of these fluctuations by removing one of these supposed causes (let us say by evacuating a galvanometer and thus preventing bombardment by air molecules). There must be, regardless of what the system and the cause may be, a mean kinetic energy of exactly $\frac{1}{2}kT$ associated with each degree of freedom of the system.

Now in all measurements, the magnitude of the measured quantity is determined by the co-ordinate of an indicator (let us say x) and this co-ordinate is subject to slight deflections of amount $\delta x = (\overline{x^2})^{1/2}$; these fluctuations in x are responsible for the theoretical limit to the sensitivity of a measuring instrument. We can predict this limit quite easily as follows.

Let the sensitivity of any instrument be defined by

$$\sigma = (\frac{\delta x}{\delta q})$$

where δq is the change in the quantity to be measured and δx the corresponding deflection produced by this change. For any measuring instrument, these also exists a relation of the form

$$A\,\delta x = B\,\delta q$$

where A is a 'generalized' directional force depending on the moving system and B may be regarded as a deviation factor. A is to be regarded as a generalized force in the sense that $\frac{1}{2}A(\delta x)^2$ gives the work necessary for a small deviation δx ; as an illustrative example of what B is, if δx is a steady deflection then $B\,\delta q$ is the turning moment produced by δq. If we now replace δx by $(\overline{x^2})^{1/2}$, we get

$$\overline{\delta q} = (A/B)(\overline{x^2})^{1/2} \quad .$$

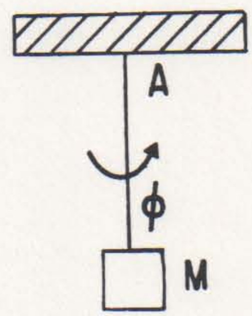
But $\quad \frac{1}{2}A\overline{x^2} = \frac{1}{2}kT = \epsilon$. $\quad$ Therefore $\quad \overline{\delta q} = (2\epsilon A/B)^{1/2}$.

This gives us the average error in a single measurement of the quantity q. In order to estimate its magnitude for any particular instrument, we must find the relation between A and B which is characteristic of that particular instrument.

It should be emphasized here that the limitation of measuring processes we are going to discuss is just in this sense of <u>single measurements</u>. One could for example, make many measurements of the zero of any instrument and thus accurately determine the zero and the most probable deviations from the zero; one could next apply a quantity smaller than δq to the instrument and again make a great many readings of the new position. The difference between the two positions would give us a measure of the quantity much smaller than δq, the individual deflection for which would have been hardly discernible. So, to emphasize once more, <u>we are discussing limits on single measurements</u>.

Let us consider various cases of interest now.

1. <u>Suspended Mirror</u>:



Let M be a very light mirror suspended upon a fine quartz fiber of torsion constant A. The motion of this system may be characterized by one co-ordinate ϕ, where ϕ is the angle by which the system has rotated from its position of equilibrium. Since it is a motion of one degree of freedom, we may expect that the system will oscillate back and forth with a Brownian motion of such magnitude that

$$\tfrac{1}{2} A \overline{\phi^2} = \epsilon = \tfrac{1}{2} kT \quad . \tag{1.1}$$

To get an idea of the order of magnitude of these oscillations consider $T = 18^{\circ}c$; then $\epsilon = 2 \times 10^{-14}$ ergs. For $A = 10^{-6}$ dynes cm/radian^2 (a typical thin quartz fiber), we find

$$\overline{\delta\phi} = (\overline{\phi^2})^{1/2} = 2 \times 10^{-4} \text{ radians} \quad .$$

Gerlach carried out some experiments to verify the relation (1.1). He allowed a beam of light to be reflected from such a mirror onto a distant scale and then made a study of the inherent zero unsteadiness of the system. With a quartz fiber a few tenths of a micron in diameter and a few cms long, a mirror $0.8 \times 1.6 \text{ mm}^2$ and a scale at 1.5 meters, Brownian movements of several cms were observed. The mean square value of $\overline{\phi^2}$ agreed very closely with the predicted value.

Kappler investigated experimentally the effect of background gas pressure on the observed Brownian movement of the mirror. He recorded the zero deviations of the mirror photographically and concluded that although the detailed nature of the individual Brownian movement pictures were very different for different pressures, yet the mean $\overline{\phi^2}$ was remarkably the same for all of them and agreed closely with that predicted by the equation given above. Theoretically, the dependence of the form of Brownian movement on background pressure was studied by Uhlenbeck and Goudsmit (1929). They developed the displacement of the small mirror for a time interval long compared with the free period of the system into a Fourier series and found that whereas the individual Fourier components $\overline{\phi_k^2}$ were explicit functions of pressure and the molecular weight of the surrounding gas molecules, the net deflection

$$\overline{\phi^2} = \sum_{k=0}^{\infty} \overline{\phi_k^2}$$

obtained after summing the Fourier components only depends on the temperature of the gas. This conclusion agrees beautifully with the results of Kappler's experiment.

We may finally conclude that for any suspended system with a fixed torsion constant A, there is a definite limit set by the Brownian motion. However, as may readily be seen, the sensitivity of the instrument may be increased by diminishing A. This is so because for Brownian motion

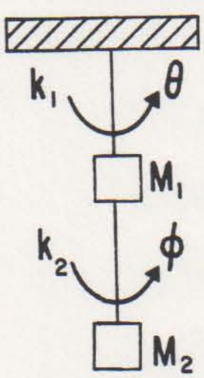
$$\tfrac{1}{2} A \overline{\phi^2} = \epsilon \quad \text{or} \quad \overline{\delta\phi} = (\overline{\phi^2})^{1/2} = \left(\frac{2\epsilon}{A}\right)^{1/2} \propto \frac{1}{A^{1/2}}$$

whereas for a constant turning moment M produced by the measured quantity, the deflection

$$\phi = \frac{M}{A} \propto \frac{1}{A} \quad .$$

Reduction of A therefore leads to a gain of the true deflection over Brownian motion deflection by a factor $A^{-1/2}$. It should be noted however that a reduction of A also leads to an increase in the period of vibration. Therefore the true limiting factors for the sensitivity of an ideal instrument are mechanical stability and the patience necessary to read long period instruments.

2. <u>Compound Torsion Pendulum:</u>



As an illustration of the theory of Brownian movement for a system with two degrees of freedom (in general, on degrees of freedom) we consider the problem of a double torsion pendulum worked out by Condon. Suppose, by attaching a second mirror some distance up the fiber, we cause the system to have a motion characterized by two degrees of freedom. We now require the two angles θ and ϕ to describe the complete motion. According to our knowledge of the equipartition theory, the system should now have a mean kinetic and potential energy equal to kT. Let us proceed to investigate the motion of such a system. We have T = Kinetic energy = $\frac{1}{2}(I_1 \dot{\theta}^2 + I_2 \dot{\phi}^2)$

$$V = \text{Potential energy } \tfrac{1}{2}[k_1\theta^2 + k_2(\phi - \theta)^2]$$

I's denote the relevant moments of inertia and the k's the relevant torsion constants. Defining

$$\Theta = \theta \sqrt{I_1} \quad \text{and} \quad \Phi = \phi \sqrt{I_2}$$

we have

$$T = \tfrac{1}{2}(\dot{\Theta}^2 + \dot{\Phi}^2), \; V = \tfrac{1}{2}(\omega_1^2 \,\Theta^2 + \omega_2^2 \,\Phi^2 - 2\lambda^2 \Theta\Phi)$$

where

$$\omega_1^2 = (k_1 + k_2)/I_1 \;, \quad \omega_2^2 = k_2/I_2 \;, \quad \lambda^2 = k_2/(I_1 I_2)^{1/2} \;.$$

ω_1 and ω_2 are respectively the natural frequencies of oscillation of the upper mirror with the lower one fixed and of the lower one with the upper one fixed. If we write down the equations of motion (define a Lagrangian and use the Lagrange's equations) we get

$$\ddot{\Theta} + \omega_1^2 \Theta - \lambda^2 \Phi = 0$$

$$\ddot{\Phi} + \omega_2^2 \Phi - \lambda^2 \Theta = 0 \quad .$$

Let $\Theta = A\, e^{i\omega t}$ and $\Phi = B\, e^{i\omega t}$; then we obtain for ω,

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) - \lambda^4 = 0$$

which has two roots

$$\omega_\pm^2 = \frac{\omega_1^2 + \omega_2^2}{2} \pm [\lambda^4 + (\frac{\omega_1^2 - \omega_2^2}{2})^2]^{1/2} \quad .$$

These are the frequencies of normal modes of vibration. Let x and y be the normal co-ordinates so that

$$\Theta = x \cos\alpha - y \sin\alpha$$

$$\Phi = x \sin\alpha + y \cos\alpha \ .$$

α can be determined by demanding that the potential energy V should contain no xy terms (since x and y are normal co-ordinates). This gives

$$2 \tan 2\alpha = \lambda^2 / (\omega_2^2 - \omega_1^2)$$

and
$$T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) \ , \quad V = \tfrac{1}{2}(\omega_-^2 x^2 + \omega_+^2 y^2)$$

where $\omega_\pm$ are the normal mode frequencies defined above. Since the Hamiltonian of the system is now composed additively of a function of x and of y, the two co-ordinates will be independent statistically and we shall have $\overline{xy} = 0$. Also, by equipartition theorem

$$\tfrac{1}{2}\omega_-^2 \overline{x^2} = \tfrac{1}{2} kT \ , \quad \tfrac{1}{2}\omega_+^2 \overline{y^2} = \tfrac{1}{2} kT$$

so that

$$\overline{x^2} = (kT/\omega_-^2) \quad ; \qquad \overline{y^2} = (kT/\omega_+^2).$$

The motion of Θ is given by

$$\begin{aligned} \overline{\Theta^2} &= \overline{x^2} \cos^2\alpha + \overline{y^2} \sin^2\alpha \\ &= \tfrac{1}{2}(\overline{x^2} + \overline{y^2}) + \tfrac{1}{2}(\overline{x^2} - \overline{y^2}) \cos 2\alpha \\ &= kT \left(\frac{\omega_2^2}{\omega_1^2 \omega_2^2 - \lambda^4} \right) \end{aligned}$$

where we have substituted for $\overline{x^2}$, $\overline{y^2}$ and $\cos 2\alpha$. Similarly,

$$\overline{\Phi^2} = kT \left(\frac{\omega_1^2}{\omega_1^2 \omega_2^2 - \lambda^4} \right) \quad .$$

Substituting for ω_1, ω_2, λ, Φ and Θ we get

$$\tfrac{1}{2} k \overline{\theta^2} = \tfrac{1}{2} kT$$

and $$\overline{\phi^2} = (\frac{kT}{k_2}) + (\frac{kT}{k_1}) .$$

The first equation shows that the upper mirror fluctuates as if the lower mirror were not present. The second equation shows that the lower mirror fluctuates due to Brownian movement with its own fluctuations, as it would if the upper mirror were fixed ($\frac{1}{2} k_2 \overline{\phi^2} = \frac{1}{2} kT$), plus those of the upper body ($\frac{1}{2} k_1 \overline{\theta^2} = \frac{1}{2} kT$). This motion of the lower mirror is also exactly the same as if the upper mirror were not present [since in that case the torsion constant K would be given by $(1/K) = (1/k_1) + (1/k_2)$].

We can obtain the degree of correlation between the two mirrors $\overline{\theta\phi}$ by evaluating

$$\overline{\Theta\Phi} = \overline{x^2} \sin\alpha \cos\alpha - \overline{y^2} \sin\alpha \cos\alpha.$$

This gives

$$\overline{\theta\phi} = (kT/k_1) ,$$

or the mean value of $\overline{\theta\phi}$ is the same as that of $\overline{\theta^2}$. Similarly

$$\overline{(\phi - \theta)^2} = \overline{\phi^2} + \overline{\theta^2} - \overline{2\theta\phi} = (kT/k_2) ,$$

i.e., the fluctuations in the twist of the second mirror relative to the first are the same as if the upper mirror were fixed.

3. Elastic Rods and Strings:

Smoluchowski (1906) investigated the Brownian movement of the various points of a string fastened at one end. He proved that for a string with radius a and density ρ, the position of the lower end fluctuates by Λ where

$$\overline{\Lambda^2} = (2kT/\pi a^2 \rho g) .$$

The accuracy of this equation was tested experimentally by Prizbram in a very interesting manner. He observed the Brownian movements of long chains of bacteria when one of the ends of the various chains was fastened to the cover glass. Houdijk did a less exotic experiment to verify the above relation. Together with Zeeman, he suspended quartz and platinum fibers vertically with their upper ends fastened and observed the movements of the lower ends of these fibers.

The problem of a string fastened at both ends was discussed by Van Lear and Uhlenbeck. They also were the first to consider the Brownian movement of thin elastic rods and derived expressions for some measurable quantities; most of this work has been verified experimentally.

From the above general discussion of strings and rods it is clear that any instrument in which they form part of the system, will be subject also to an ultimate limit of sensitivity as a result of Brownian movements.

4. Vibrating Membranes:

Let us assume that in a given instrument a membrane is to be used, for example for the measurement of a pressure p or a change of pressure δp (as a typical example, consider an instrument like the aneroid barometer). This membrane with its individual boundary conditions and characterizing constraints will possess a position of equilibrium and its motion can be described by one co-ordinate. Again, by applying the equipartition theorem, we know that the membrane will vibrate constantly and that its average potential and kinetic energies will each be equal to $\frac{1}{2}kT$. The sensitivity of the instrument will be given by $\sigma = (\delta x/\delta p)$ where δx is the deflection produced by the pressure δp. The fluctuations about the equilibrium position x_o are therefore equivalent to slight changes of the pressure, and so every measurement will be in error by an amount δp. One can evaluate the magnitude of δp in terms of the constants of the instrument by setting up a potential energy equation for the deformed membrane and an equation for the free period etc. There is thus a minimum pressure, which can be measured with an instrument such as the aneroid barometer.

Brownian movement of vibrating membranes raises an interesting question regarding the human ear. Is the intensity threshold, measured in terms of minimum audible pressure, determined by the above mentioned δp? The ear drum is vibrating constantly as if the incident pressure were fluctuating by δp. This should produce a faint background noise beneath which no sounds of smaller intensity may be distinguished. If the ear were infinitely sensitive, in other words, there would be due to Brownian movement of the ear drum, an ever present "noise."

5. Chemical Balance:

Consider a chemical balance with arms of length a. Let the loads at the ends of the two arms be m and $m + \delta m$. Let δx be the angle through which the balance beam is deflected and $\frac{1}{2}A(\delta x)^2$ be the amount of work necessary for this deflection. A is then a sort of restoring force and we have

$$A\,\delta x = a\,g\,\delta m \quad .$$

The sensitivity

$$\sigma = \frac{\delta x}{\delta m} = (\frac{ag}{A}) \; ; \text{ thus } \overline{\delta m} = (\frac{A}{ag}) \; \overline{\delta x} \; .$$

Now, the undamped period of oscillation of the balance may be written as

$$T_o = 2\pi a \left(\frac{m_o + 2m}{A}\right)^{1/2}$$

where m_o is the mass of the balance beam.

$$\therefore \frac{A^{1/2}}{ag} = \frac{2\pi}{gT_o} (m_o + 2m)^{1/2} \; .$$

We have further for Brownian movement deflections δx,

$$\overline{\delta x} = (\overline{x^2})^{1/2} = (\frac{2\epsilon}{A})^{1/2} \; .$$

Thus

$$\overline{\delta m} = \frac{2\pi}{gT_o} [2\epsilon (m_o + 2m)]^{1/2} \; .$$

This is the average error in the measurement of mass m which arises because of the uncertainty of the exact zero deflection position of the beam arising due to Brownian motion.

For $T = 18^o$ c ($\epsilon = 2 \times 10^{-14}$ ergs), $g = 981$ cm sec^{-2}, $m_o \ll m$, $m = 10^3$ gm and $T_o = 10$ secs., we get

$$\overline{\delta m} = 5.72 \times 10^{-9} \text{ gms.}$$

Needless to say, this particular limit $\overline{\delta m} = 10^{-9}$ gms has not yet been approached experimentally.

6. <u>Spring Balance</u>:

One can show similarly that the sensitivity of a spring balance is limited by a zero unsteadiness $\overline{\delta x} = (\overline{x^2})^{1/2}$, such that

$$\tfrac{1}{2} A \overline{x^2} = \epsilon = \tfrac{1}{2} kT \quad .$$

Here we have

$$g \, \delta m = A \, \delta x$$

so that $$\sigma = (\delta x/\delta m) = (g/A).$$

The free period is obviously given by

$$T_o = 2\pi \, [(M_o + m)/A]^{1/2}$$

where M_o is the mass of the spring and pan. We finally arrive at

$$\overline{\delta m} = (\overline{\delta x}) \frac{A}{g} = \frac{2\pi}{gT_o} [\, 2\epsilon(M_o + m)]^{1/2} \quad .$$

Again for $M_o << m$ and the same parameters as above, we get a limit in the sensitivity as 10^{-9} gms.

The result then of Brownian movement, in the case of either type of balance, is an unsteady zero and therefore a limited sensitivity. This unsteadiness is the same as would be produced if the mass m on the balance were fluctuating by an amount $\overline{\delta m}$.

7. Other Mechanical Cases:

Let us mention a few other cases briefly. The atoms which make a steel meter stick are constantly in motion and so the length of the stick is in the final analysis, a quantity which fluctuates statistically. Accepting this, it is clear that all measurements of lengths with such an instrument are also subject to uncertainities.

If it is desired to determine the length by some form of an interferometer, one can see at once that here too a limit must be reached as the mirror supports will fluctuate with respect to their relative positions.