the current $\vec{J}$ by

$$\vec{J} = e\,(z n_i \vec{V_i} - n_e \vec{V_e}), \tag{217}$$

the density ρ by

$$\rho = n_i m_i + n_e m_e, \tag{218}$$

and the charge density by

$$\sigma = e\,(n_i z - n_e). \tag{219}$$

In addition to these relations, we will need the equations of continuity for both types of particles

$$\frac{\partial n_e}{\partial t} = - \vec{\nabla}_r \cdot (n_e \vec{V_e}) \tag{220}$$

and

$$\frac{\partial n_i}{\partial t} = - \vec{\nabla}_r \cdot (n_i \vec{V_i}). \tag{221}$$

By multiplying Eq. (220) by m_e and Eq. (221) by m_i and adding, we immediately obtain the general continuity equation for mass

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_r \cdot (\rho \vec{V}) = 0. \tag{222}$$

By multiplying Eq. (220) by $-e$ and Eq. (221) by $z e$ and adding, we obtain the equation for conservation of charge

$$\frac{\partial \sigma}{\partial t} + \vec{\nabla}_r \cdot \vec{J} = 0. \tag{223}$$

These equations replace the two continuity equations (220) and (221).

Before proceeding we will make two approximations. First, we will assume that $1 >> z\, m_e/m_i$. Since the ratio is always less than 2000^{-1}, this is a very good approximation. Second, we shall assume that the ion and electron densities are equal. At the very beginning of the course we saw that the total kinetic energy of the electrons in a Debye sphere was just enough to remove them from the sphere against the resulting attractive $\vec{E}$ field. For regions much larger than this, only very small deviations from equal numbers can occur. Thus we set

$$n_e = z\, n_i . \tag{224}$$

This does not mean that σ must be zero; however, it must simply be much smaller than $n_e e$. To find σ we must use Eq. (223). We can use this to check the approximation (224). With these approximations, Eqs. (216) and (217) for $\vec{v}$ and $\vec{j}$ become

$$\vec{v} = \vec{v}_i + \frac{z\, m_e}{m_i} \vec{v}_e \tag{225}$$

and

$$\vec{j} = n_e e \left(z \vec{v}_i - \vec{v}_e \right). \tag{226}$$

We can find $\vec{v}_i$ and $\vec{v}_e$ in terms of $\vec{v}$ and $\vec{j}$ from Eqs. (225) and (226).

$$\vec{v}_i = \vec{v} + \frac{m_e z\, \vec{j}}{m_i n_e e} \tag{227}$$

and

$$\vec{v}_e = \vec{v} - \frac{\vec{j}}{n_e e} . \tag{228}$$

We shall use the form of Eq. (188) for Eqs. (214) and (215); however, splitting off the stress tensor. They have been written again below for convenience. Letting $\vec{\nabla}_r \cdot (n \vec{v} \vec{v}) = \vec{v}(\vec{v} \cdot \vec{\nabla}_r n) + \vec{v} n (\vec{\nabla}_r \cdot \vec{v}) + n(\vec{v} \cdot \vec{\nabla}_r \vec{v})$,

$$m_e \Big[\frac{\partial}{\partial t}(n_e \vec{v}_e) + \underbrace{\vec{v}_e(\vec{v}_e \cdot \vec{\nabla}_r n_e) + n_e \vec{v}_e (\vec{\nabla}_r \cdot \vec{v}_e) + n_e (\vec{v}_e \cdot \vec{\nabla}_r \vec{v}_e)}_{\vec{\nabla}_r \cdot (n_e \vec{v}_e \vec{v}_e)} \Big]$$

$$+ \vec{\nabla}_r \cdot \underset{\sim}{\Pi_e} + n_e e \vec{E} - n_e \vec{F}_e + n_e e \left(\frac{\vec{v}_e \times \vec{B}}{c} \right) = \left[\frac{\partial \vec{P}_e}{\partial t} \right]_{ie} \tag{229}$$

$$m_i\left[\frac{\partial}{\partial t}(n_i\vec{v}_i) + \underbrace{\vec{v}_i(\vec{v}_i\cdot\vec{\nabla}_r n_i) + n_i\vec{v}_i(\vec{\nabla}_r\cdot\vec{v}_i) + n_i(\vec{v}_i\cdot\vec{\nabla}_r\vec{v}_i)}_{\vec{\nabla}_r\cdot(n_i\vec{v}_i\vec{v}_i)}\right]$$

$$+ \vec{\nabla}_r\cdot\underset{\sim}{\Pi_i} - n_i z e\,\vec{E} - n_i\,\vec{F}_i - n_i z e\left(\frac{\vec{v}_i\times\vec{B}}{c}\right) = \left[\frac{\partial\vec{P}_i}{\partial t}\right]_{ei}. \qquad (230)$$

We add Eqs. (229) and (230), term by term, below. With the approximations $z n_i = n_e$ and $z m_e << m_i$, adding the first term of the ion and electron equations gives

$$\frac{\partial}{\partial t}\left[m_e n_e\vec{v}_e + m_i n_i\vec{v}_i\right] = \frac{\partial}{\partial t}\left[\vec{v}(m_e n_e + m_i n_i)\right]$$

$$+ \frac{\partial}{\partial t}\left[-\frac{n_e m_e}{n_e e}\vec{j} + \frac{n_i m_i m_e z}{m_i n_e e}\vec{j}\right]$$

$$= \frac{\partial(\rho\vec{v})}{\partial t}. \qquad (231)$$

Addition of the second term of the ion and of the electron equations, when $\vec{v}_i$ and $\vec{v}_e$ are expressed in terms of $\vec{v}$ and $\vec{j}$, will involve $\vec{v}\,\vec{v}$, $\vec{j}\,\vec{j}$, and mixed ($\vec{v}\,\vec{j}$ and $\vec{j}\,\vec{v}$) terms. The $\vec{v}\,\vec{v}$ term is simply

$$\vec{\nabla}_r\cdot\left[(n_i m_i + n_e m_e)\,\vec{v}\,\vec{v}\right] = \vec{\nabla}_r\cdot\left[\rho\,\vec{v}\,\vec{v}\right]. \qquad (232)$$

The $\vec{j}\,\vec{j}$ term is

$$\vec{\nabla}_r\cdot\left[n_i m_i\left(\frac{m_e z}{m_i n_e e}\right)^2\vec{j}\,\vec{j} + n_e m_e\left(\frac{1}{n_e e}\right)^2\vec{j}\,\vec{j}\right]$$

$$\vec{\nabla}_r\cdot\left[\vec{j}\,\vec{j}\,\frac{m_e}{n_e e^2}\left(\frac{m_e z}{m_i} + 1\right)\right] \approx \vec{\nabla}_r\cdot\frac{m_e}{n_e e^2}\,\vec{j}\,\vec{j}. \qquad (233)$$

The mixed term is

$$\vec{\nabla}_r \cdot n_e m_e\left[-\frac{1}{n_e e}(\vec{j}\vec{V}+\vec{V}\vec{j})\right] + \vec{\nabla}_r \cdot n_i m_i\left[\frac{m_e z}{m_i n_e e}(\vec{j}\vec{V}+\vec{V}\vec{j})\right]$$

$$= \nabla_r \cdot \left[-\frac{m_e}{e}(\vec{j}\vec{V}+\vec{V}\vec{j}) + \frac{m_e}{e}(\vec{j}\vec{V}+\vec{V}\vec{j})\right] = 0. \tag{234}$$

Summing all the terms gives

$$\frac{\partial}{\partial t}(\rho\vec{V}) + \vec{\nabla}_r \cdot \left[\rho\vec{V}\vec{V} + \frac{m_e}{n_e e^2}\vec{j}\vec{j}\left[1+\frac{m_e z}{m_i}\right] + \underset{\sim}{\Pi_e} + \underset{\sim}{\Pi_i}\right]$$

$$-\sigma\vec{E} - (n_e\vec{F}_e + n_i\vec{F}_i) - \frac{\vec{j}\times\vec{B}}{c} = 0 \tag{235}$$

or using the continuity equation and writing

$$\underset{\sim}{\Pi_T} = \underset{\sim}{\Pi_e} + \underset{\sim}{\Pi_i} + \frac{m_e}{n_e e^2}\left[1+\frac{m_e z}{m_i}\right]\vec{j}\vec{j} \tag{236}$$

gives

$$\rho\underbrace{\left(\frac{\partial}{\partial t}+\vec{V}\cdot\vec{\nabla}_r\right)}_{=d/dt}\vec{V} + \vec{\nabla}_r\cdot\underset{\sim}{\Pi_T} - (n_e\vec{F}_e + n_i\vec{F}_i)$$

$$-\sigma\vec{E} - \frac{\vec{j}\times\vec{B}}{c} = 0. \tag{237}$$

The $\vec{j}\,\vec{j}$ term in the total stress tensor Π_T arises because the mean velocity for both ions and electrons is different from the mean velocity for either species and these are what were split out when we computed the stress tensors. Further, the ion-electron collision terms have canceled out because of conservation of momentum.

To obtain the current equation we multiply Eq. (229) by $-e/m_e$ and Eq. (230) by ez/m_i and add, term by term. The first term is

$$m_i\frac{ez}{m_i}\frac{\partial}{\partial t}(n_i\vec{V}_i) - \frac{e}{m_e}m_e\frac{\partial}{\partial t}(n_e\vec{V}_e)$$

$$= \frac{\partial}{\partial t}(n_i\vec{V}_i ez - n_e\vec{V}_e e) = \frac{\partial\vec{j}}{\partial t}. \tag{238}$$

The second term is

$$e z \; \vec{\nabla}_r \cdot (n_i \vec{v}_i \vec{v}_i) - e \; \vec{\nabla}_r \cdot (n_e \vec{v}_e \vec{v}_e) \tag{239}$$

This will again involve $\vec{v}\,\vec{v}$, $\vec{j}\,\vec{j}$ and mixed terms. The $\vec{v}\,\vec{v}$ term is

$$\vec{\nabla}_r \cdot (e z n_i \; \vec{V}\vec{V}) - \vec{\nabla}_r \cdot (e n_e \vec{V}\vec{V}) = 0 . \tag{240}$$

The $\vec{j}\,\vec{j}$ term is

$$\vec{\nabla}_r \cdot \left[e z \; n_i \left(\frac{m_e z}{m_i n_e} e \right)^2 \vec{j}\vec{j} \right] - \vec{\nabla}_r \cdot \left[e n_e \left(\frac{1}{n_e e} \right)^2 \vec{j}\vec{j} \right]$$

$$= \vec{\nabla}_r \cdot \vec{j}\vec{j} \left(\frac{m_e^2 z^2}{m_i^2 e n_e} - \frac{1}{n_e e} \right)$$

$$= \vec{\nabla}_r \cdot \left[\vec{j}\vec{j} \left(\frac{m_e^2 z^2}{m_i^2} - 1 \right) \frac{1}{n_e e} \right] = - \vec{\nabla}_r \cdot \left(\frac{1}{n_e e} \vec{j}\vec{j} \right). \tag{241}$$

The $\vec{v}\,\vec{j}$ term is

$$\vec{\nabla}_r \cdot \left[e z n_i \left(\vec{V}\vec{j} + \vec{j}\vec{V} \right) \frac{m_e z}{m_i n_e e} \right] + \vec{\nabla}_r \cdot \left[e n_e \left(\vec{V}\vec{j} + \vec{j}\vec{V} \right) \frac{1}{n_e e} \right]$$

$$= \vec{\nabla}_r \cdot \left[\left(\frac{m_e z}{m_i} + 1 \right) \left(\vec{V}\vec{j} + \vec{j}\vec{V} \right) \right] \simeq \vec{\nabla}_r \cdot \frac{1}{n_e e} \vec{j}\vec{j} \tag{242}$$

and writing $\left[\dfrac{\partial \vec{P}_e}{\partial t} \right]_{ie} = \alpha \; (\vec{V}_i - \vec{V}_e)$ we get

$$\frac{\partial \vec{j}}{\partial t} - \vec{\nabla}_r \cdot \left[\frac{\vec{j}\vec{j}}{n_e e} \right] + \vec{\nabla}_r \cdot \left[\vec{V}\vec{j} + \vec{j}\vec{V} \right]$$

$$+ \vec{\nabla}_r \cdot \left[\frac{z e \, \underline{\Pi}_i}{m_i} - \frac{e \, \underline{\Pi}_e}{m_e} \right] - \frac{n_e e^2}{m_e} \left[1 + \frac{z m_e}{m_e} \right] \vec{E}$$

$$+ n_e e \left[\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i} \right] - \frac{n_e e^2}{m_e} \frac{\vec{V} \times \vec{B}}{c} + \frac{e}{m_e c} \vec{j} \times \vec{B} = - \frac{\alpha}{n_e m_e} \vec{j} \tag{243}$$

or, multiplying by $+m_e/n_e e^2$, gives

$$\frac{m_e}{n_e e^2}\frac{\partial \vec{j}}{\partial t} + \frac{m_e}{n_e e^2}\vec{\nabla}_r \cdot \left\{ -\frac{\overrightarrow{jj}}{n_e e} + (\vec{\nabla}\vec{j} + \vec{j}\vec{\nabla}) + \frac{e}{m_e}\left[\frac{z\,m_e}{m_i}\underset{\sim}{\Pi}_i - \underset{\sim}{\Pi}_e\right]\right\}$$

$$-\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) + \frac{1}{n_e e c}\,\vec{j}\times\vec{B} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) = -\underset{\sim}{\eta}\cdot\vec{j}. \qquad (244)$$

Here $\underset{\sim}{\eta}$ is the conductivity, and in principle it should be a tensor for a plasma in a magnetic field. If we linearize Eqs. (237) and (244) in $\vec{j}$ and $\vec{v}$ — that is, we assume j and v are small and drop all second order terms (vv, jv, jj) — then these equations become

$$\rho\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}_r \cdot \underset{\sim}{\Pi}_T - (n_e\vec{F}_e + n_i\vec{F}_i) - \sigma\vec{E} - \frac{\vec{j}\times\vec{B}}{c} = 0 \qquad (245)$$

and

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{j}}{\partial t}\right] + \frac{1}{e\,n_e}\vec{\nabla}_r\cdot\left[\frac{z\,m_e}{m_i}\underset{\sim}{\Pi}_i - \underset{\sim}{\Pi}_e\right] - \left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)$$

$$+\ \frac{\vec{j}\times\vec{B}}{n_e e c} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) = -\underset{\sim}{\eta}\cdot\vec{j}. \qquad (246)$$

If $\underset{\sim}{\Pi}_i$ is of the order of $\underset{\sim}{\Pi}_e$, then the ion stress tensor can be neglected in Eq. (246). If $\partial\vec{j}/\partial t$, $\vec{B}$, $\vec{F}_e$, and $\vec{F}_i$ are negligible, then Eq. (246) reduces to

$$\vec{E} = \underset{\sim}{\eta}\cdot\vec{j} \qquad (247)$$

which is Ohm's law with η the resistivity. We may therefore think of Eq. (246), or more generally Eq. (244), as a generalized Ohm's law.

The terms in Eq. (245) are fairly clear — $\rho\,\partial\vec{v}/\partial t$ is the inertial term, $\vec{\nabla}_r\cdot\underset{\sim}{\Pi}_T$ is the force due to the material stresses, the $\vec{F}$ terms are the external forces, and $\vec{j}\times\vec{B}$ is the magnetic force. The terms in

Eq. (246) have the following meaning. The $\partial\vec{j}/\partial t$ term is due to the inertia of the current. In most cases where the current is carried primarily by the electron, it comes from the electron inertia. The term involving the stress tensors arises because the pressure or stresses due to one species tends to accelerate that species relative to the other, creating a current. The $\vec{E} + \frac{\vec{v} \times \vec{B}}{c}$ term is the electric field as seen by an observer moving with the fluid. The $\vec{j} \times \vec{B}$ term arises because the ions and electrons carry different fractions of the current and have different masses so that they are accelerated differently, and this tends to give rise to a current or to a balancing $\vec{E}$ field. This gives rise to the Hall effect. The $\vec{F}$ terms arise from differential accelerations of the two species due to the external forces. If $\vec{F}$ arises from a gravitational field, this term cancels out. The $\eta \cdot \vec{j}$ term is, of course, the resistivity.

XVI. Summary of the Macroscopic Equations

Summing up, the macroscopic equations for the fluid are

$$\rho \frac{d\vec{v}}{dt} + \vec{\nabla}\cdot \underset{\sim}{\Pi} - (n_e \vec{F}_e + n_i \vec{F}_i) - \frac{\vec{j}\times\vec{B}}{c} = 0, \tag{248}$$

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{j}}{\partial t} + \vec{\nabla}\cdot\left[-\frac{\vec{j}\vec{j}}{n_e e} + (\vec{v}\vec{j} + \vec{j}\vec{v}) + e\left(\frac{z\,\Pi_i}{m_i} - \frac{\Pi_e}{m_e}\right)\right]\right]$$
$$-\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) + \frac{\vec{j}\times\vec{B}}{n_e e c} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) = -\eta\cdot\vec{j}, \tag{249}$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0, \tag{250}$$

and

$$\frac{\partial\sigma}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0. \tag{251}$$

In addition we have Maxwell's equations

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \tag{252}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi\vec{j}}{c}, \tag{253}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{254}$$

and

$$\vec{\nabla}\cdot\vec{E} = 4\pi\sigma. \tag{255}$$

In addition to these equations we need equations to determine the π's . We can proceed a long way, however, by making some simplifying assumptions about π . Examples of choices for π are:

(1) We assume the particle stresses are negligible and neglect π altogether.

(2) We may assume that we need use only a scalar pressure p

$$\underset{\sim}{\Pi} = p\,\underset{\sim}{I}. \tag{256}$$

and that p satisfies a $\gamma = 5/3$ law.

(3) We may assume a double adiabatic law with the pressure perpendicular to the lines of force determined from a $\gamma = 2$ law and with the pressure parallel to the lines of force determined by a $\gamma = 3$ law.

(4) We may use fluid equations, including viscosity, heat conduction, exchange of energy between parallel and perpendicular degrees of freedom and between electrons and ions.

One must examine the physics of the situation under consideration to determine which, if any, of the above approximations is pertinent.

XVII. Approximations to the Equation for the Current

The first and most usual approximation we will make to Eq. (249) is that of linearization. That is, we will neglect the $\vec{j}\,\vec{j}$ and $\vec{v}\,\vec{j}$ terms so that Eq. (249) becomes

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{j}}{\partial t} + \vec{\nabla}\cdot e\left(\frac{z\Pi_i}{m_i} - \frac{\Pi_e}{m_e}\right)\right] - \left(\vec{E} + \frac{\vec{V}\times\vec{B}}{c}\right)$$

$$+ \frac{1}{n_e e c}\,\vec{j}\times\vec{B} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) = -\eta\cdot\vec{j}. \qquad (257)$$

Second, in most applications the external forces are either nonexistent or negligible and can be dropped (for the case of gravitational forces they cancel). Third, if the electron and ion thermal ~~velocities~~ energies are comparable, the π_i and π_e are roughly equal and we may neglect π_i/m_i compared to π_e/m_e. With these approximations, Eq. (257) reduces to

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{j}}{\partial t} - \vec{\nabla}\cdot\frac{e\,\Pi_e}{m_e}\right] - \left[\vec{E} + \frac{\vec{V}\times\vec{B}}{c}\right]$$

$$+ \frac{1}{n_e e c}\,\vec{j}\times\vec{B} = -\eta\cdot\vec{j}. \qquad (258)$$

If we solve the linearized version of Eq. (248) for $\vec{j} \times \vec{B}/c$ (also neglecting $\vec{F}$), we obtain

$$\frac{\vec{j} \times \vec{B}}{c} = \rho \frac{\partial \vec{v}}{\partial t} + \vec{\nabla} \cdot \underset{\sim}{\Pi}_T \tag{259}$$

and substituting this in Eq. (258) gives

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{j}}{\partial t} + \vec{\nabla} \cdot \frac{e}{m_e} \underset{\sim}{\Pi}_i\right] - \frac{\rho}{n_e e} \frac{\partial \vec{v}}{\partial t}$$

$$- \vec{E} + \frac{\vec{v} \times \vec{B}}{c} = - \eta \cdot \vec{j}. \tag{260}$$

If we now look for a steady state solution and further assume that the pressure gradient is negligible, then Eq. (260) reduces to

$$\vec{E} + \vec{v} \times \vec{B}/c = \eta \cdot \vec{j}. \tag{261}$$

This says that the current is driven by the electric field seen in a frame moving with the fluid. Finally, if the conductivity of the fluid is very high, then η is negligible and Eq. (261) becomes

$$\vec{E} + \frac{\vec{v} \times \vec{B}}{c} = 0 \tag{262}$$

For a high-temperature plasma η becomes very small, as we shall see later, and this may become a good approximation provided the other assumptions made are valid. A fluid for which Eq. (262) is satisfied is said to be a perfect conductor. You will note that it satisfies our criterion that the lines of force move with the fluid (equation 19 in section VII).

XVIII. Discussion of the Relation Between Macroscopic and Microscopic Velocities

Let us consider the steady state case with isotropic pressure, comparable ion and electron pressures, zero resistivity, and small $\vec{v}$ and $\vec{j}$ (linearized equations). Eqs. (248) and (249) give

$$\vec{\nabla} p - (n_e \vec{F}_e + n_i \vec{F}_i) - \frac{\vec{j} \times \vec{B}}{c} = 0, \tag{263}$$

$$0 = -\frac{1}{n_e e} \vec{\nabla} p_e - \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) + \frac{1}{n_e e c} \vec{j} \times \vec{B} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) \tag{264}$$

where

$$p = p_e + p_i \tag{265}$$

or, substituting $\vec{j} \times \vec{B}/c$ from Eq. (263) in Eq. (264), Eq. (264) becomes

$$\frac{1}{n_e e} \vec{\nabla} p_i - \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) - \frac{\vec{F}_i}{Ze} = 0. \tag{266}$$

We should like to compare the motions given by these equations with those for individual particles. We shall consider various situations as we did for the case of particle drifts.

A. Uniform Plasma Density and $\vec{B}$ Field, with No External Force, $\vec{E}$ Perpendicular to $\vec{B}$

For this case, as we have already seen,

$$\vec{E}_\perp \doteq -\frac{\vec{v} \times \vec{B}}{c}, \tag{267}$$

and

$$\vec{v}_\perp = \frac{c\, \vec{E}_\perp \times \vec{B}}{B^2} \tag{268}$$

This is the same drift velocity we found for individual particle motions.

B. Uniform $\vec{B}$, no $\vec{E}$ or $\vec{F}$, with a Gradient of $\vec{P}$

For this case Eq. (266) becomes

$$\vec{\nabla} P_i = n_e e \frac{\vec{V} \times \vec{B}}{c}. \tag{269}$$

Thus, $\vec{\nabla} P_i$ is perpendicular to $\vec{B}$. Also, by Eq. (263)

$$\vec{\nabla} P = \frac{\vec{j} \times \vec{B}}{c}. \tag{270}$$

$\vec{\nabla} P$ is perpendicular to $\vec{B}$. (If the ion and electron densities and temperatures are the same, then $P_i = P_e$.) Now we may solve Eq. (269) for $\vec{v}$ just as we did in the case of a uniform $\vec{E}$ field. Crossing Eq. (269) with $\vec{B}$ on the right gives

$$\vec{V} = -\frac{c}{n_e e} \left[\frac{\vec{\nabla} P_i \times \vec{B}}{B^2} \right]. \tag{271}$$

We see that there is a macroscopic velocity for the plasma in this case. However, from the particle orbit point of view the particles do not drift in a uniform B field. How does this drift arise?

The reason for the macroscopic motion is the following. The velocity $\vec{v}$ of the fluid in a little element of volume is the mean velocity of all particles within that volume. Those particles whose orbits lie entirely within the volume will not contribute to the mean velocity since they are moving up as much as down (see Fig. 43). However, near the edge of the volume there are some particles whose orbits lie only partially within the volume, as shown in Fig. 43. All those particles with their centers of gyration lying on the right side of the volume element $d\tau = dx\,dy\,dz$ contribute to a net downward velocity, while all those particles with centers lying on the

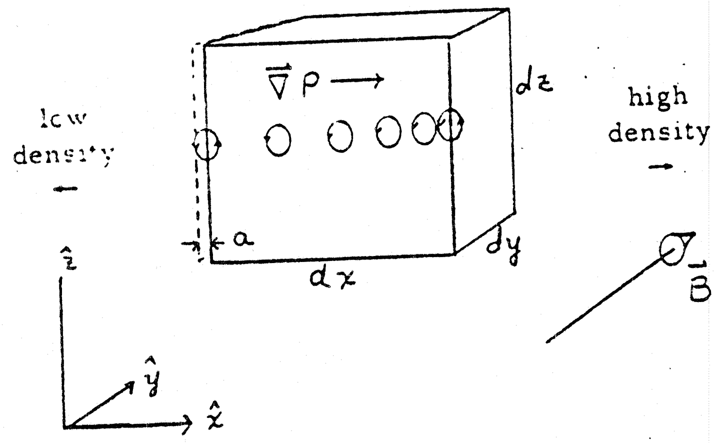

Figure 43

left-hand side of the volume contribute to a net upward velocity. If there are more particles on the right than on the left, the result will be a net downward motion of the fluid in $d\tau$. A quick estimation of the size of this effect shows that it can give Eq. (271). The downward momentum contributed by the particles on the right is

$$-mn_2\, a V_\perp\, dy\, dz = -n_2 \frac{mV_\perp^2}{eB} mc\, dy\, dz \tag{272}$$

while the upward momentum contributed by the particles on the left is

$$mn_1\, a V_\perp\, dy\, dz = n_1 \frac{mV_\perp^2}{eB} mc\, dy\, dz \tag{273}$$

Here a is the larmor radius and na is the number of particles per unit area within a larmor radius of the surface $d\tau$. The net downward momentum is

$$-\frac{m^2 c V_\perp^2}{eB}[n_2 - n_1]\, dy\, dz = -\frac{m^2 c V_\perp^2}{eB} \vec{\nabla} n\, dx\, dy\, dz \tag{274}$$

Dividing by the total mass of material in $d\tau$ gives

$$V_z = -\frac{m^2 c V_\perp^2}{eB} \frac{\vec{\nabla} n\, dx\, dy\, dz}{mn\, dx\, dy\, dz} = -\frac{mcV_\perp^2}{eBn} \nabla n = -\frac{c\, \nabla_x P_i}{n_e e B} \tag{275}$$

where it has been assumed that the temperature is independent of position and hence also v^2, so that it can be taken inside $\vec{\nabla}$ ($P_i = m_i v^2$, to order m_e/m_i only the ions contribute). We see that Eq. (275) is identical to Eq. (271) if the x direction is the same as $\vec{\nabla} P_i$.

C. Uniform Pressure, Nonuniform $\vec{B}$, no $\vec{E}$ or $\vec{F}$

For this case Eq. (266) gives

$$\frac{\vec{\nabla} \times \vec{B}}{c} = 0$$

or

$$\vec{V}_\perp = 0. \qquad (276)$$

where $\perp$ means perpendicular to $\vec{B}$.

Here we have no macroscopic velocity. On the other hand, from the particle orbit point of view the particles are drifting. How do we explain this?

Consider a small element of volume $d\tau = dx\,dy\,dz$ with $\vec{B}$ out of the paper and $\vec{\nabla} B$ in the x direction, as shown in Fig. 44.

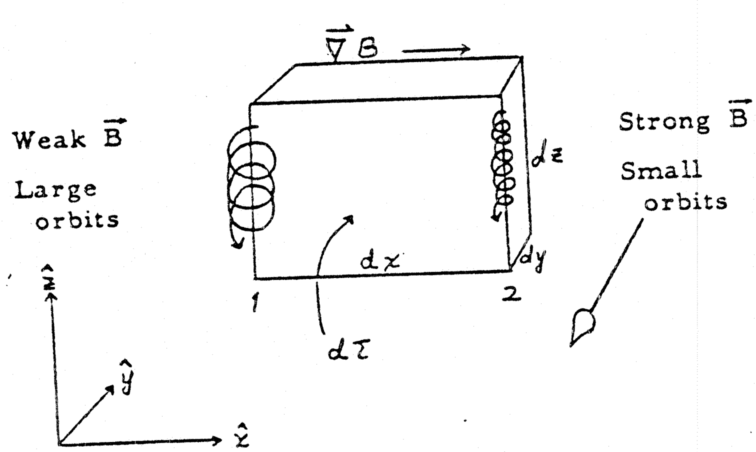

Figure 44

Now the particle orbits within the volume contribute a net momentum given by Eq. (147) section IV, which for this case is downward of magnitude

$$\frac{cm V_\perp^2 \nabla B}{e B^2} m n \, dx\, dy\, dz \tag{277}$$

The net velocities due to particles whose centers of gyration lie outside $d\tau$ but whose guiding centers intersect $d\tau$, however, cancel this. The upward momentum contribution due to particles at face 1 is

$$+ \frac{n V_\perp^2}{e B_1} m^2 c \; dy\, dz \tag{278}$$

while the downward contribution due to particles at face 2 is

$$\frac{n V_\perp^2}{e B_2} m^2 c \; dy\, dz = \left[\frac{n V_\perp^2}{e B_1} m^2 c + \frac{\partial}{\partial x}\left(\frac{n V_\perp^2}{e B} m^2 c\right) dx\right] dy\, dz . \tag{279}$$

The net momentum upward due to both faces 1 and 2 is

$$\frac{n V_\perp^2 m^2 c}{e} \times \frac{\nabla B \, d\tau}{B^2} \tag{280}$$

which is just what is required to cancel the drift given by Eq. (277).

XIX. Diffusion of Magnetic Fields Through Matter and of Plasma Across Magnetic Fields

A. Diffusion of a Magnetic Field through a Solid Conductor

As the simplest example of the diffusion of a magnetic field through a conducting material we will consider the diffusion of a magnetic field through a solid conductor. We shall adopt the simple ohm's law

$$\vec{E} = \eta \vec{j} \tag{281}$$

as the equation which determines the current in terms of $\vec{E}$. This can be obtained from our general equation (249) by neglecting the inertial terms, the nonlinear terms, the pressure terms, the Hall term, and the external forces. We have also set the velocity equal to zero. In addition to this equation we have the Maxwell equations

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \tag{282}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}, \tag{283}$$

and

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{284}$$

We have dropped the displacement current term ($\dot{\vec{E}}$) since here we are primarily interested in low-frequency phenomena where it is negligible. We must, however, keep the $\partial\vec{B}/\partial t$ term because it is needed to determine $\vec{E}$ which drives $\vec{j}$. In addition to these we should add

$$\vec{\nabla} \cdot \vec{j} = 0 \tag{285}$$

for if this is violated charges build up rapidly, producing large $\vec{E}$

fields which alter the current so as to prevent further buildup of charge. However, Eq. (285) is automatically satisfied because of Eq. (283). We now substitute Eq. (281) in Eq. (283), obtaining

$$\vec{\nabla} \times \vec{B} = \frac{4\pi \vec{E}}{c\eta}. \tag{286}$$

Taking the curl of Eq. (286) and making use of Eq. (282) and Eq. (284) gives

$$-\nabla^2 \vec{B} = -\frac{4\pi}{c^2\eta} \frac{\partial \vec{B}}{\partial t}. \tag{287}$$

Only $\vec{B}$'s which satisfy Eq. (284) as well as Eq. (287) are acceptable.

Now Eq. (287) is a diffusion equation with a diffusion coefficient

$$D = \frac{c^2\eta}{4\pi}. \tag{288}$$

Thus the larger the resistivity the larger is the rate of diffusion of the field through the matter.

In order to obtain a physical understanding of the meaning of this equation, consider the following simple situation. Imagine that we have a conducting slab of material which is infinite in the xy plane and has thickness 2d in the z direction (see Fig. 45).

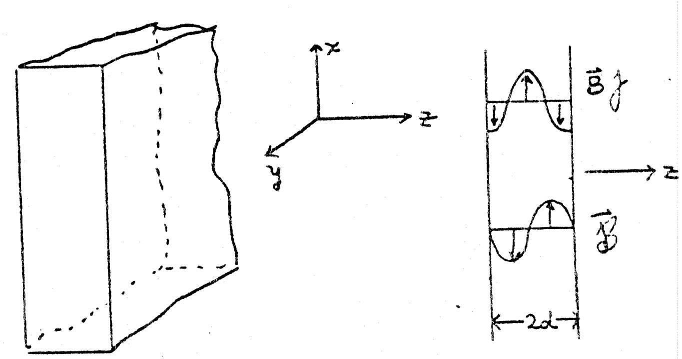

Figure 45