22. Action and Reaction

The Lagrangian $L_{charges}$ describes the motions of the charged particles under the action of a given electromagnetic field, as represented by the scalar and vector potentials. The Lagrangian L_{field} describes the development in time of an electromagnetic field without reference to charged particles. Does this mean that we must now hunt for something additional that, added to L_{field}, will describe the effect of the charges on the field? No. We already know it. It is automatically contained in the Lagrangian of the complete system of charges and fields:

$$L = L_{charges} + L_{field}. \tag{1-22.1}$$

There is a piece of $L_{charges}$ that explicitly describes the interaction between the charges and the fields:

$$L_{int.} = \sum_a \left[-e_a \phi(\bar{r}_a) + \frac{e_a}{c} \bar{v}_a \cdot \bar{A}(\bar{r}_a) \right]. \tag{1-22.2}$$

It produces the action of the fields on the charges, and it therefore also produces the (re)action of the charges on the field.

To verify this we must re-examine the implications of the stationary action principle for field variations, specifically, the variations of $\phi(\bar{r},t)$ and $\bar{A}(\bar{r},t)$, which now give additional contributions through $L_{int.}$. For that it is desirable to present $L_{int.}$ in the same form as L_{field}, a three dimensional spatial integration, rather than as a summation over point charges. This is accomplished by introducing the electric charge density and the electric current density,

$$\rho(\bar{r},t) = \sum_a e_a \delta(\bar{r} - \bar{r}_a(t)),$$

$$\bar{j}(\bar{r},t) = \sum_a e_a \bar{v}_a(t) \delta(\bar{r} - \bar{r}_a(t)), \tag{1-22.3}$$

and indeed

$$L_{\text{int.}}(t) = \int (d\bar{r})[-\rho(\bar{r},t)\phi(\bar{r},t) + \frac{1}{c}\bar{J}(\bar{r},t)\cdot\bar{A}(\bar{r},t)], \tag{I-22.4}$$

according to the delta function property [see Eq. (I-1.24)] that is illustrated by (t is omitted)

$$\int (d\bar{r})\,\delta(\bar{r}-\bar{r}_a)\phi(\bar{r}) = \int (d\bar{r})\,\delta(\bar{r}-\bar{r}_a)\phi(\bar{r}_a) = \phi(\bar{r}_a). \tag{I-22.5}$$

We now restate the consequences of $\bar{A}$ and ϕ variations for the total Lagrangian:

$$\delta A: \quad \delta L = -\frac{d}{dt}\left[\frac{1}{4\pi c}\int (d\bar{r})\bar{E}\cdot\delta\bar{A}\right] + \frac{1}{4\pi}\int (d\bar{r})\,\delta\bar{A}\cdot\left[\frac{4\pi}{c}\bar{J} + \frac{1}{c}\frac{\partial}{\partial t}\bar{E} - \bar{\nabla}\times\bar{B}\right], \tag{I-22.6}$$

$$\delta\phi: \quad \delta L = \frac{1}{4\pi}\int (d\bar{r})\,\delta\phi[-4\pi\rho + \bar{\nabla}\cdot\bar{E}]. \tag{I-22.7}$$

The anticipated results are here; the principle of stationary action applied to $\bar{A}$ and ϕ variations gives the field equations

$$\bar{\nabla}\times\bar{B} = \frac{1}{c}\frac{\partial}{\partial t}\bar{E} + \frac{4\pi}{c}\bar{J},$$

$$\bar{\nabla}\cdot\bar{E} = 4\pi\rho. \tag{I-22.8}$$

These, together with (I-21.2), constitute the full set of Maxwell equations in the presence of moving charges.

23. Dynamics of Charges and Fields

The total Lagrangian (1-22.1) can be presented as

$$L = \sum_a \bar{p}_a \cdot \frac{d\bar{r}_a}{dt} + \int (d\bar{r}) \left(-\frac{1}{4\pi c}\right) \bar{E} \cdot \frac{\partial}{\partial t} \bar{A} - H, \qquad (1\text{-}23.1)$$

with

$$H = \sum_a \left[\left(\bar{p}_a - \frac{e_a}{c} \bar{A}(\bar{r}_a)\right) \cdot \bar{v}_a - \tfrac{1}{2} m_a v_a^2 + e_a \phi_a(\bar{r}_a)\right]$$

$$+ \int (d\bar{r}) \frac{1}{4\pi} [\bar{E} \cdot \bar{\nabla}\phi + \bar{B} \cdot (\bar{\nabla} \times \bar{A}) + \tfrac{1}{2}(E^2 - B^2)] \qquad (1\text{-}23.2)$$

appearing as the Hamiltonian of the system. The principle of stationary action, now including variations of the time variable, gives

$$\delta W_{12} = \delta\left[\int_2^1 dt\, L\right] = G_1 - G_2, \qquad (1\text{-}23.3)$$

where, at any particular time,

$$G = \sum_a \bar{p}_a \cdot \delta\bar{r}_a + \int (d\bar{r}) \left(-\frac{1}{4\pi c}\right) \bar{E} \cdot \delta\bar{A} - H\delta t. \qquad (1\text{-}23.4)$$

The narrower, Hamiltonian, description is reached by eliminating all variables that do not obey equations of motion, and correspondingly, do not appear in the first variation terms of G. Those superfluous variables are the $\bar{v}_a$, and the fields ϕ, $\bar{B}$. Accordingly, we adopt the following as definitions,

$$\bar{v}_a = \frac{1}{m_a}\left(\bar{p}_a - \frac{e_a}{c} \bar{A}(\bar{r}_a)\right), \qquad (1\text{-}23.5)$$

$$\bar{B} = \bar{\nabla} \times \bar{A}, \qquad (1\text{-}23.6)$$

and accept the restriction on $\bar{E}$ given by

$$\bar{\nabla} \cdot \bar{E} = 4\pi\rho. \qquad (1\text{-}23.7)$$

The latter produces the elimination of ϕ from the Hamiltonian, which now appears as

$$H = \sum_a \frac{\left(\bar{p}_a - \frac{e_a}{c}\bar{A}(\bar{r}_a)\right)^2}{2m_a} + \int(d\bar{r})\,\frac{1}{8\pi}\,(E^2+B^2). \qquad (1\text{-}23.8)$$

Perhaps, this finally startled you a bit? How could the scalar potential, which is essential, electromagnetically, disappear from the dynamical description? Fear not—it's still with us. Think of the aspect of the stationary action principle associated with $\bar{E}$ variations:

$$\delta L = \int(d\bar{r})\left(-\frac{1}{4\pi}\right)\delta\bar{E}\cdot\left(\frac{1}{c}\frac{\partial}{\partial t}\bar{A}+\bar{E}\right) = 0. \qquad (1\text{-}23.9)$$

Do we conclude that $\frac{1}{c}\frac{\partial}{\partial t}\bar{A}+\bar{E}=0$? That would be true if the $\delta\bar{E}(\bar{r},t)$ were arbitrary. They are not; $\bar{E}$ is subject to the restriction—the constraint—of Eq.(1-23.7) and any change in $\bar{E}$ must obey

$$\bar{\nabla}\cdot\delta\bar{E} = 0. \qquad (1\text{-}23.10)$$

The proper conclusion is that the vector multiplying $\delta\bar{E}$ in (1-23.9) is the gradient of a scalar function,

$$\frac{1}{c}\frac{\partial}{\partial t}\bar{A}+\bar{E} = -\nabla\phi, \qquad (1\text{-}23.11)$$

for that leads to

$$\delta L = \int(d\bar{r})\left(-\frac{1}{4\pi}\right)(\bar{\nabla}\cdot\delta\bar{E})\phi = 0, \qquad (1\text{-}23.12)$$

as required.

24. Mechanical Conservation Laws

Energy. The Hamiltonian (1-23.2), or (1-23.8), is constructed from particle and field variables. It is not an explicit function of the time, and is therefore conserved:

$$\frac{dH}{dt} = 0. \qquad (1\text{-}24.1)$$

This is conservation of energy, and that energy, written as

$F(r + \delta r) = F(r)$

$$H = \sum_a \tfrac{1}{2} m_a v_a^2 + \int (d\bar{r}) U \qquad (1\text{-}24.2)$$

has a simple structure in terms of the sum of particle kinetic energies and the integrated field energy density.

Linear Momentum. Energy is conserved because there is no physically distinguished origin of time; nothing is altered on shifting that origin, displacing all time values by a common constant. Equally well, there is nothing in the Hamiltonian (1-23.2) or (1-23.8) to pick out a particular origin of the spatial coordinates. This will lead to conservation of the momentum vector. An infinitesimal displacement of the whole system by the constant vector $\delta\bar{\epsilon}$ means that

$$\delta\bar{r}_a = \delta\bar{\epsilon}, \qquad (1\text{-}24.3)$$

but what is the analogous statement for field variables? For any field quantity $F(\bar{r})$, the consequence of a rigid displacement is to change the assigned value at the arbitrary point $\bar{r} + \delta\bar{\epsilon}$ to coincide with the value initially attributed to the point $\bar{r}$

$$(F + \delta F)(\bar{r} + \delta\bar{\epsilon}) = F(\bar{r}). \qquad (1\text{-}24.4)$$

This being true for all $\bar{r}$, it can equally well be written

$F(r) - \delta F(r) = F(r - \delta\epsilon)$

$$F(\bar{r}) + \delta F(\bar{r}) = F(\bar{r} - \delta\bar{\epsilon}) = F(\bar{r}) - \delta\bar{\epsilon}\cdot\bar{\nabla} F(\bar{r}), \tag{1-24.5}$$

or

$$\delta F(\bar{r}) = -\delta\bar{\epsilon}\cdot\bar{\nabla} F(\bar{r}). \tag{1-24.6}$$

As an example, consider the charge density (omitting t)

$$\rho(\bar{r}) = \sum_a e_a \delta(\bar{r} - \bar{r}_a). \tag{1-24.7}$$

If the positions of all particles, the $\bar{r}_a$, are displaced by $\delta\bar{\epsilon}$, the charge density changes to

$$\rho(\bar{r}) + \delta\rho(\bar{r}) = \sum_a e_a \delta(\bar{r} - \bar{r}_a - \delta\bar{\epsilon}), \tag{1-24.8}$$

where

$$\delta(\bar{r} - \bar{r}_a - \delta\bar{\epsilon}) = \delta(\bar{r} - \bar{r}_a) - \delta\bar{\epsilon}\cdot\bar{\nabla}_{\bar{r}}\, \delta(\bar{r} - \bar{r}_a), \tag{1-24.9}$$

and therefore

$$\delta\rho(\bar{r}) = -\delta\bar{\epsilon}\cdot\bar{\nabla}\rho(\bar{r}). \tag{1-24.10}$$

We now apply this to compute the values of G at a particular time, as produced by a rigid displacement:

$$G_{\delta\bar{\epsilon}} = \sum_a \bar{p}_a\cdot\delta\bar{\epsilon} + \int (d\bar{r})\left(-\frac{1}{4\pi c}\right)\bar{E}\cdot(-\delta\bar{\epsilon}\cdot\bar{\nabla})\bar{A} = \bar{P}\cdot\delta\bar{\epsilon}. \tag{1-24.11}$$

The conserved momentum vector is therefore given by

$$\bar{P} = \sum_a \bar{p}_a + \int (d\bar{r})\, \frac{1}{4\pi c}\, \bar{E}\cdot(\bar{\nabla})\cdot\bar{A}, \tag{1-24.12}$$

which uses the notation

$$\bar{E}\cdot(\bar{\nabla})\cdot\bar{A} = \sum_{k=1}^{3} E_k \bar{\nabla} A_k. \quad = \nabla\times(\vec{E}\times\vec{A}) + \vec{E}\cdot\nabla A \tag{1-24.13}$$

For another way of presenting $\overline{P}$ we return to (1-24.11) and note the identity

$$\delta\overline{\epsilon} \times (\overline{\nabla} \times \overline{A}) = \overline{\nabla}(\delta\overline{\epsilon} \cdot \overline{A}) - \delta\overline{\epsilon} \cdot \overline{\nabla} \overline{A} . \tag{1-24.14}$$

Then the field term of $G_{\delta\overline{\epsilon}}$ becomes

$$\int (d\overline{r}) \left(-\frac{1}{4\pi c}\right) \overline{E} \cdot [\delta\overline{\epsilon} \times \overline{B} - \overline{\nabla}(\delta\overline{\epsilon} \cdot \overline{A})] = \int (d\overline{r}) \frac{1}{4\pi c} (\overline{E} \times \overline{B}) \cdot \delta\overline{\epsilon}$$
$$- \int (d\overline{r}) \rho \frac{1}{c} \overline{A} \cdot \delta\overline{\epsilon}, \tag{1-24.15}$$

where the last contribution, produced by partial integration, is also the following sum over charges:

$$-\sum_a \frac{e_a}{c} \overline{A}(\overline{r}_a) \cdot \delta\overline{\epsilon}. \tag{1-24.16}$$

Accordingly, we now have

$$\overline{P} = \sum_a \left(\overline{p}_a - \frac{e_a}{c} \overline{A}(\overline{r}_a)\right) + \int (d\overline{r}) \frac{1}{4\pi c} \overline{E} \times \overline{B} \tag{1-24.17}$$

or

$$\overline{P} = \sum_a m_a \overline{v}_a + \int (d\overline{r}) \overline{G}, \tag{1-24.18}$$

the sum of particle kinetic momenta and the integrated field momentum density.

Angular Momentum. There is nothing in the Hamiltonian to distinguish a particular orientation of the spatial coordinate system, or, equivalently, the whole system can be rigidly rotated with no internally discernable effect. This implies conservation of the angular momentum vector. The infinitesimal rotation $\delta\overline{\omega}$ changes all particle position vectors by (see Fig.)

$$\delta\overline{r}_a = \delta\overline{\omega} \times \overline{r}_a . \tag{1-24.19}$$

As for the effect on a field function, we must now distinguish between scalars and vectors. We illustrate this with examples. The consequence, for the scalar charge density, of the particle displacement (1-24.19) is

$$\delta\rho(\bar{r}) = \sum_a e_a(-\delta\bar{r}_a\cdot\bar{\nabla}_{\bar{r}})\delta(\bar{r}-\bar{r}_a) , \tag{1-24.20}$$

which restates the infinitesimal displacement of (1-24.10) without the specialization to constant $\delta\bar{\epsilon}$. The delta function property

$$(\bar{r}-\bar{r}_a)\delta(\bar{r}-\bar{r}_a) = 0, \quad \Rightarrow \vec{r}\,\delta(\vec{r}-\vec{r}_a) = \vec{r}_a\,\delta(\vec{r}-\vec{r}_a) \tag{1-24.21}$$

is now employed to write

$$\delta\bar{r}_a\cdot\bar{\nabla}_{\bar{r}}\,\delta(\bar{r}-\bar{r}_a) = \delta\bar{\omega}\times\bar{r}_a\cdot\bar{\nabla}_{\bar{r}}\,\delta(\bar{r}-\bar{r}_a) = \delta\bar{\omega}\times\bar{r}\cdot\bar{\nabla}_{\bar{r}}\,\delta(\bar{r}-\bar{r}_a) , \tag{1-24.22}$$

as justified by the fact that only different, perpendicular, components of $\bar{r}$ and $\bar{\nabla}_{\bar{r}}$ occur here. Hence, we get

$$\delta\rho(\bar{r}) = -\delta\bar{\omega}\times\bar{r}\cdot\bar{\nabla}\rho(\bar{r}), \tag{1-24.23}$$

an example of the effect of a rigid rotation on any scalar function $S(\bar{r})$:

$$(S+\delta S)(\bar{r}+\delta\bar{\omega}\times\bar{r}) = S(\bar{r}), \tag{1-24.24}$$

$$\delta S(\bar{r}) = -\delta\bar{\omega}\times\bar{r}\cdot\bar{\nabla}S(\bar{r}). \tag{1-24.25}$$

For a vector example, we use the electric current density,

$$\bar{j}(\bar{r}) = \sum_a e_a\,\bar{v}_a\,\delta(\bar{r}-\bar{r}_a). \tag{1-24.26}$$

Now the particle velocity vectors are also rotated,

$$\delta\bar{v}_a = \delta\bar{\omega}\times\bar{v}_a, \tag{1-24.27}$$

If $\nabla \times \vec{v} = 0$ then $\vec{v} = \nabla S(\vec{r})$

now $\delta S(\vec{r}) \to -(\delta\vec{\omega} \times \vec{r}) \cdot \nabla S(\vec{r})$

$\delta\vec{v} - \nabla \delta S r = -\nabla(\delta\vec{\omega} \times \vec{r}) . \nabla S(\vec{r}) = -\delta\vec{\omega} \times \underline{\nabla \vec{r}}$

thereby giving an additional contribution

$$\delta\bar{J}(\bar{r}) = -\delta\bar{\omega} \times \bar{r} \cdot \bar{\nabla} \bar{J}(\bar{r}) + \delta\bar{\omega} \times \bar{J}(\bar{r}). \qquad (1\text{-}24.28)$$

This is typical of the effect of the rigid rotation on any vector field, as described in relation to the fixed coordinate system:

$$(\bar{V} + \delta\bar{V})(\bar{r} + \delta\bar{\omega} \times \bar{r}) = \bar{V}(\bar{r}) + \delta\bar{\omega} \times \bar{V}(\bar{r}), \qquad (1\text{-}24.29)$$

$$\delta V(\bar{r}) = -\delta\bar{\omega} \times \bar{r} \cdot \bar{\nabla} \bar{V}(\bar{r}) + \delta\bar{\omega} \times \bar{V}(\bar{r}). \qquad (1\text{-}24.30)$$

In particular, then,

$$\delta\bar{A} = -\delta\bar{\omega} \times \bar{r} \cdot \bar{\nabla} \bar{A} + \delta\bar{\omega} \times \bar{A}, \qquad (1\text{-}24.31)$$

and

$$G_{\delta\bar{\omega}} = \bar{J} \cdot \delta\bar{\omega} \,, \qquad (1\text{-}24.32)$$

where the conserved angular momentum vector $\bar{J}$ is

$$\bar{J} = \sum_a \bar{r}_a \times \bar{p}_a + \int (d\bar{r}) \left(\frac{1}{4\pi c}\right) [\bar{E} \cdot (\bar{r} \times \bar{\nabla}) \cdot \bar{A} + \bar{E} \times \bar{A}]. \qquad (1\text{-}24.33)$$

Again there is an alternative version, now using the identity

$$(\delta\bar{\omega} \times \bar{r}) \times (\bar{\nabla} \times \bar{A}) = \bar{\nabla}(\delta\bar{\omega} \times \bar{r} \cdot \bar{A}) + \delta\bar{\omega} \times \bar{A} - \delta\bar{\omega} \times \bar{r} \cdot \bar{\nabla} \bar{A}, \qquad (1\text{-}24.34)$$

and leading to

$$\bar{J} = \sum_a \bar{r}_a \times m\bar{v}_a + \int (d\bar{r}) \bar{r} \times \bar{G}, \qquad (1\text{-}24.35)$$

the sum of moments of particle kinetic momenta and the integrated moment of the field momentum density.

25. Charge Conservation. Gauge Invariance

An electromagnetic system possesses a conservation law, that of electric charge, which has no place in the usual mechanical framework. How does it fit into our evolving dynamics of charges and fields? First, we must recognize explicitly something about potentials—they are not unique. The vector potential $\bar{A}$ was introduced to produce the magnetic field $\bar{B}$ as $\bar{\nabla}\times\bar{A}$. But the gradient of any scalar function λ could be added to a particular $\bar{A}$,

$$\bar{A} \to \bar{A} + \bar{\nabla}\lambda\,, \qquad \text{(I-25.1)}$$

without altering its curl, and the magnetic field $\bar{B}$ being represented. There would appear to be an alteration in the electric field:

$$\bar{E} = -\frac{1}{c}\frac{\partial}{\partial t}\bar{A} - \bar{\nabla}\phi \to \bar{E} - \bar{\nabla}\frac{1}{c}\frac{\partial}{\partial t}\lambda, \qquad \text{(I-25.2)}$$

but that can be compensated by changing the scalar potential,

$$\phi \to \phi - \frac{1}{c}\frac{\partial}{\partial t}\lambda. \qquad \text{(I-25.3)}$$

The possibility of modifying the potentials $\bar{A}$ and ϕ in the manner of (I-25.1), (I-25.3) without thereby changing the fields $\bar{E}$ and $\bar{B}$, is called the freedom of gauge transformation. [Reader: Do not transpose gauge into guage! The term had its origin in a now obsolete theory (1918) of Hermann Weyl (1885-1955).]

The Lagrange function of the field, (I-21.5), is not altered by the gauge transformation

$$\bar{A} \to \bar{A} + \bar{\nabla}\lambda, \quad \phi \to \phi - \frac{1}{c}\frac{\partial}{\partial t}\lambda\,,$$

$$\bar{E} \to \bar{E}, \quad \bar{B} \to \bar{B}; \qquad \text{(I-25.4)}$$

it is a gauge invariant function. How about the Lagrangian of the charges, (I-20.21)? It helps here to anticipate, from the immediate physical significance of the velocities

$$\bar{v}_a = \frac{1}{m_a}\left(\bar{p}_a - \frac{e_a}{c}\bar{A}(\bar{r}_a)\right), \qquad (1\text{-}25.5)$$

that the gauge transformation of $\bar{A}$ must be compensated by a redefinition of the $\bar{p}_a$,

$$\bar{p}_a \rightarrow \bar{p}_a + \frac{e_a}{c}\bar{\nabla}\lambda(\bar{r}_a). \qquad (1\text{-}25.6)$$

Then we find that

$$L_{\text{charge}} \rightarrow L_{\text{charge}} + \sum_a \left[\frac{e_a}{c}\frac{\partial}{\partial t}\lambda(\bar{r}_a,t) + \frac{e_a}{c}\frac{d\bar{r}_a}{dt}\cdot\bar{\nabla}\lambda(\bar{r}_a,t)\right]. \qquad (1\text{-}25.7)$$

There are two ways of looking at this additional term. One is mechanical:

$$\frac{d}{dt}\left[\sum_a \frac{e_a}{c}\lambda(\bar{r}_a,t)\right]; \qquad (1\text{-}25.8)$$

the other is electromagnetic:

$$\int(d\bar{r})\frac{1}{c}\left[\rho(\bar{r},t)\frac{\partial}{\partial t}\lambda(\bar{r},t) + \bar{J}(\bar{r},t)\cdot\bar{\nabla}\lambda(\bar{r},t)\right], \qquad (1\text{-}25.9)$$

where the latter is also equal to

$$\frac{d}{dt}\left[\int(d\bar{r})\,\frac{1}{c}\,\rho(\bar{r},t)\lambda(\bar{r},t)\right] - \int(d\bar{r})\,\frac{1}{c}\,\lambda(\bar{r},t)\left(\frac{\partial}{\partial t}\rho(\bar{r},t) + \bar{\nabla}\cdot\bar{J}(\bar{r},t)\right). \qquad (1\text{-}25.10)$$

In the mechanical viewpoint, the gauge transformation induces a change of the Lagrangian that leads to a change in the action,

$$W_{12} \rightarrow W_{12} + \left[\sum_a \frac{e_a}{c}\lambda(\bar{r}_a,t_1)\right] - \left[\sum_a \frac{e_a}{c}\lambda(\bar{r}_a,t_2)\right], \qquad (1\text{-}25.11)$$

by boundary terms. This means that the stationary action principle will produce exactly the same equations of motion, and constraint equations, despite the gauge transformation--it is the same physical system. In using the electromagnetic viewpoint it helps to think of an arbitrary infinitesimal gauge transformation, $\lambda \rightarrow \delta\lambda$, so that

$$\delta\bar{A} = \bar{\nabla}\delta\lambda, \quad \delta\phi = -\frac{1}{c}\frac{\partial}{\partial t}\delta\lambda, \tag{1-25.12}$$

induces

$$\delta W_{12} = G_{\delta\lambda 1} - G_{\delta\lambda 2} - \int_{t_2}^{t_1} dt \int (d\bar{r}) \frac{1}{c}\delta\lambda \left(\frac{\partial}{\partial t}\rho + \bar{\nabla}\cdot\bar{j}\right), \tag{1-25.13}$$

with

$$G_{\delta\lambda} = \int (d\bar{r}) \frac{1}{c}\rho\delta\lambda . \tag{1-25.14}$$

In view of the arbitrary nature of $\delta\lambda(\bar{r},t)$, the stationary action principle now demands that, at every point

$$\frac{\partial}{\partial t}\rho(\bar{r},t) + \bar{\nabla}\cdot\bar{j}(\bar{r},t) = 0; \tag{1-25.15}$$

gauge invariance implies local charge conservation. Then the special situation $\delta\lambda = \text{constant}$, where $\delta\bar{A} = \delta\phi = 0$ and W_{12} is certainly invariant, implies a conservation law, that of

$$G_{\delta\lambda} = \frac{1}{c}\delta\lambda Q, \tag{1-25.16}$$

in which

$$Q = \int (d\bar{r})\rho, \tag{1-25.17}$$

is the conserved total charge.

26. Gauge Invariance and Local Conservation Laws

We have just derived the local conservation law of electric charge. That is a property carried only by the particles, not by the electromagnetic field. In contrast the mechanical properties of energy, linear and angular momentum are attributes of both particles and fields. For these we have conservation laws of total quantities. What about local conservation laws? Early in this development (Sec. I-4) we produced local <u>non</u>-conservation laws; they concentrated on the field and characterized the charged particles as sources (or sinks) of field mechanical properties. It is natural to ask for a more even-handed treatment of both charges and fields. We shall supply it, in the framework of a particular example. The property of gauge invariance will be both a valuable guide, and an aid in simplifying the calculations.

The time displacement of a complete physical system identifies its total energy. This suggests that time displacement of a part of the system provides energetic information about that portion. The ultimate limit of this spatial subdivision, a local description, should appear in response to an (infinitesimal) time displacement that varies arbitrarily in space as well as in time, $\delta t(\bar{r},t)$.

Now we need a clue. How do fields, and potentials, respond to such coordinate-dependent displacements? This is where the freedom of gauge transformations enters: the change of the vector and scalar potentials, by $\bar{\nabla}\lambda(\bar{r},t), -\frac{1}{c}\frac{\partial}{\partial t}\lambda(\bar{r},t)$, respectively, serves as a model for the potentials themselves. The advantage here is that the response of the scalar $\lambda(\bar{r},t)$ to the time displacement can be reasonably taken to be:

$$(\lambda + \delta\lambda)(\bar{r}, t + \delta t) = \lambda(\bar{r},t) \qquad \text{(I-26.1)}$$

or

$$\delta\lambda(\bar{r},t) = -\delta t(\bar{r},t)\frac{\partial}{\partial t}\lambda(\bar{r},t). \qquad \text{(I-26.2)}$$

Then we derive

$$\delta(\bar{\nabla}\lambda) = -\delta t\frac{\partial}{\partial t}(\bar{\nabla}\lambda) + \left(-\frac{1}{c}\frac{\partial}{\partial t}\lambda\right)c\,\bar{\nabla}\,\delta t,$$

$$\delta\left(-\frac{1}{c}\frac{\partial}{\partial t}\lambda\right) = -\delta t\left(-\frac{1}{c}\frac{\partial^2}{\partial t^2}\lambda\right) - \left(-\frac{1}{c}\frac{\partial}{\partial t}\lambda\right)\frac{\partial}{\partial t}\,\delta t, \qquad (1\text{-}26.3)$$

which is immediately generalized to

$$\delta\bar{A} = -\delta t\,\frac{\partial}{\partial t}\,\bar{A} + \phi c\,\bar{\nabla}\,\delta t$$

$$\delta\phi = -\delta t\,\frac{\partial}{\partial t}\,\phi - \phi\,\frac{\partial}{\partial t}\,\delta t, \qquad (1\text{-}26.4)$$

or, equivalently,

$$\delta\bar{A} = c\,\delta t\bar{E} + \bar{\nabla}(\phi c\,\delta t),$$

$$\delta\phi = -\frac{1}{c}\frac{\partial}{\partial t}(\phi c\,\delta t). \qquad (1\text{-}26.5)$$

In the latter form we recognize a gauge transformation, produced by the scalar $\phi c\,\delta t$, which will not contribute to the changes of field strengths. Accordingly, for that calculation we have, effectively, $\delta\bar{A} = c\,\delta t\bar{E}$, $\delta\phi = 0$, leading to

$$\delta\bar{E} = -\frac{1}{c}\frac{\partial}{\partial t}(c\,\delta t\bar{E}) = -\delta t\frac{\partial}{\partial t}\bar{E} - \bar{E}\frac{\partial}{\partial t}\,\delta t,$$

$$\delta\bar{B} = \bar{\nabla}\times(c\,\delta t\bar{E}) = -\delta t\frac{\partial}{\partial t}\bar{B} - \bar{E}\times\bar{\nabla}c\,\delta t; \qquad (1\text{-}26.6)$$

the last line employs the field equation $\bar{\nabla}\times\bar{E} = -(1/c)\partial\bar{B}/\partial t$.

In the following we adopt a viewpoint in which such field equations are accepted as consequences of the definition of the fields in terms of potentials. That permits the field Lagrange function (1-21.5) to be simplified:

$$L_{\text{field}} = \frac{1}{8\pi}(E^2 - B^2). \qquad (1\text{-}26.7)$$

$E\cdot\delta E - B\cdot\delta B$

$= \frac{E\cdot}{4\pi}$

Then we can apply the field variation (1-26.6) directly, and get

$$\delta\mathcal{L}_{field} = -\delta t\frac{\partial}{\partial t}\mathcal{L}_{field} - \frac{1}{4\pi}E^2\frac{\partial}{\partial t}\delta t - \frac{c}{4\pi}\bar{E}\times\bar{B}\cdot\bar{\nabla}\delta t$$

$$= -\frac{\partial}{\partial t}(\delta t\,\mathcal{L}_{field}) - \frac{1}{8\pi}(E^2+B^2)\frac{\partial}{\partial t}\delta t - \frac{c}{4\pi}\bar{E}\times\bar{B}\cdot\bar{\nabla}\delta t. \qquad (1\text{-}26.8)$$

Before commenting on these last, not unfamiliar, field structures we turn to the charged particles and put them on a somewhat similar footing in terms of a continuous, rather than a discrete, description.

We therefore present the Lagrangian of the charges, (1-20.21), in terms of a corresponding Lagrange function,

$$L_{charges} = \int(d\bar{r})\mathcal{L}_{charges}, \qquad (1\text{-}26.9)$$

where

$$\mathcal{L}_{charges} = \sum_a \mathcal{L}_a \qquad (1\text{-}26.10)$$

and

$$\mathcal{L}_a = \delta(\bar{r}-\bar{r}_a(t))\left[\tfrac{1}{2}m_a v_a(t)^2 - e_a\phi(\bar{r}_a(t),t) + \frac{e_a}{c}\bar{v}_a(t)\cdot\bar{A}(\bar{r}_a(t),t)\right]; \qquad (1\text{-}26.11)$$

the latter adopts the Lagrangian viewpoint, with $\bar{v}_a = d\bar{r}_a/dt$ accepted as a definition. Then the effect of the time displacement on the variables $\bar{r}_a(t)$, taken as

$$(\bar{r}_a + \delta\bar{r}_a)(t+\delta t) = \bar{r}_a(t), \qquad (1\text{-}26.12)$$

$$\delta\bar{r}_a(t) = -\delta t(\bar{r}_a(t),t)\bar{v}_a(t), \qquad (1\text{-}26.13)$$

?

implies the velocity variation

$$\delta\bar{v}_a(t) = -\delta t(\bar{r}_a(t),t)\frac{d}{dt}\bar{v}_a(t) - \bar{v}_a(t)\left[\frac{\partial}{\partial t}\delta t + \bar{v}_a\cdot\bar{\nabla}_a\delta t\right]; \qquad (1\text{-}26.14)$$

the last step exhibits both the explicit and implicit dependences of $\delta t(\bar{r}_a(t),t)$ on t. In computing the variation of $\phi(\bar{r}_a(t),t)$, for example, we combine the potential variation given in (1-26.4), with the effect of $\delta\bar{r}_a$:

$$\delta\phi(\bar{r}_a(t),t) = -\delta t\frac{\partial}{\partial t}\phi - \phi\frac{\partial}{\partial t}\delta t - \delta t\,\bar{v}_a\cdot\bar{\nabla}_a\phi = -\delta t\frac{d}{dt}\phi - \phi\frac{\partial}{\partial t}\delta t, \qquad (1\text{-}26.15)$$

and, similarly,

$$\delta\bar{A}(\bar{r}_a(t),t) = -\delta t\frac{\partial}{\partial t}\bar{A} + \phi c\,\bar{\nabla}\delta t - \delta t\,\bar{v}_a\cdot\bar{\nabla}_a\bar{A} = -\delta t\frac{d}{dt}\bar{A} + \phi c\,\bar{\nabla}\delta t. \qquad (1\text{-}26.16)$$

The total effect of these variation on $\mathcal{L}_a$ is thus

$$\delta\mathcal{L}_a = -\delta t\frac{d}{dt}\mathcal{L}_a + \delta(\bar{r}-\bar{r}_a(t))\left(-m_a v_a^2 - \frac{e_a}{c}\bar{A}\cdot\bar{v} + e_a\phi\right)\left(\frac{\partial}{\partial t}\delta t + \bar{v}_a\cdot\bar{\nabla}\delta t\right), \qquad (1\text{-}26.17)$$

or

$$\delta\mathcal{L}_a = -\frac{d}{dt}[\delta t\,\mathcal{L}_a] - \delta(\bar{r}-\bar{r}_a(t))E_a\left(\frac{\partial}{\partial t}\delta t + \bar{v}_a\cdot\bar{\nabla}\delta t\right), \qquad (1\text{-}26.18)$$

where

$$E_a = \tfrac{1}{2}m_a v_a^2. \qquad (1\text{-}26.19)$$

We have retained the particle symbol d/dt to the last, but now, being firmly back in the field, space-time, viewpoint, it should be written $\partial/\partial t$, referring to all t dependence, with $\bar{r}$ being held fixed. The union of these various contributions to the variation of the total Lagrange function is

$$\delta\mathcal{L}_{tot} = -\frac{\partial}{\partial t}[\delta t\,\mathcal{L}_{tot}] - U_{tot}\frac{\partial}{\partial t}\delta t - \bar{S}_{tot}\cdot\bar{\nabla}\delta t, \qquad (1\text{-}26.20)$$

where

$$U_{tot} = \frac{1}{8\pi}(E^2+B^2) + \sum_a \delta(\bar{r}-\bar{r}_a(t))E_a \qquad (1\text{-}26.21)$$

and

$$\bar{S}_{tot} = \frac{c}{4\pi}\,\bar{E}\times\bar{B} + \sum_a \delta(\bar{r} - \bar{r}_a(t))E_a\bar{v}_a, \tag{1-26.22}$$

are physically transparent forms for the total energy density and total energy flux vector.

To focus on what is new in this development we ignore boundary effects in the stationary action principle, by setting the otherwise arbitrary $\delta t(\bar{r},t)$ equal to zero at t_1 and t_2. Then, through partial integrations, we conclude that

$$\delta W_{12} = \int_{t_2}^{t_1} dt \int (d\bar{r})\,\delta t\left(\frac{\partial}{\partial t}U_{tot} + \bar{\nabla}\cdot\bar{S}_{tot}\right) = 0, \tag{1-26.23}$$

from which follows the local statement of total energy conservation,

$$\frac{\partial}{\partial t}U_{tot} + \bar{\nabla}\cdot\bar{S}_{tot} = 0. \tag{1-26.24}$$

$\nabla\cdot V_a = 0$

$$\frac{\partial \sum_a \delta(r - \vec{r}_a(t))\,E_a}{\partial t} + \nabla\cdot\sum_a \delta(\vec{r} - \vec{r}_a(t))\,E_a\vec{V}_a$$

$V_a\cdot\nabla\sum_a \delta(r - r_a(t))\,E_a$

$$-\sum_a \nabla\delta(\vec{r} - \vec{r}_a(t))\cdot\vec{V}_a(t)\,E_a$$

$$-\sum_a V_a\cdot\nabla\delta(\vec{r} - \vec{r}_a(t))\,E_a \qquad + \sum_a \delta(r - r_a(t))\frac{\partial E_a}{\partial t}$$

$$+ V_a\,\delta(r - r_a)\cdot\nabla E_a$$

$$\frac{d}{dt}\sum_a \delta(\vec{r} - \vec{r}_a(t))\,E_a = 0 \qquad = \sum_a \delta(r - r_a)\frac{dE_a}{dt}$$

27. Einsteinian Relativity

After that discussion one might well ask whether a unified dynamics of charges and fields has now be attained. No—there is still a major flaw. An electromagnetic pulse is a mechanical system that travels at the speed of light, carrying a mass proportional to the total energy content,

$$m = E/c^2. \qquad (1\text{-}27.1)$$

In contrast, the masses of the charged particles are fixed quantities that have no reference to the particle's state of motion and its associated energy. Another way of expressing this lack of mechanical unity between fields and particles comes from the physically evident expression for the total momentum density (Problem)

$$\bar{G}_{tot} = \frac{1}{4\pi c}\,\bar{E}\times\bar{B} + \sum_a \delta(\bar{r}-\bar{r}_a(t))m_a\bar{v}_a. \qquad (1\text{-}27.2)$$

The relation

$$\bar{G}_{tot} = \frac{1}{c^2}\,\bar{S}_{tot}, \qquad (1\text{-}27.3)$$

which is valid for the field terms, does not hold for the particle contributions. Well then, could it be that Newtonian mechanics is mistaken, and that the correct expressions for particle inertia and energy do satisfy $m = (1/c^2)E$? We now follow this unifying suggestion—that the relation between inertia and energy, which the electromagnetic field has disclosed, is, in fact, universally valid.

Consider a single particle in the absence of applied electromagnetic fields. What we are proposing is that the connection between momentum and velocity is actually

$$\bar{p} = \frac{1}{c^2}E\vec{v}. \qquad (1\text{-}27.4)$$

To this we add the relation of Hamiltonian mechanics,

$$\bar{v} = \frac{\partial E}{\partial \bar{p}} \quad ; \qquad (1\text{-}27.5)$$

and deduce that

$$c^2 \bar{p} \cdot d\bar{p} = E dE, \qquad (1\text{-}27.6)$$

which is integrated to

$$E^2 = c^2 p^2 + \text{constant}. \qquad (1\text{-}27.7)$$

We already know (Sec. 1-5) that the added constant is zero for an electromagnetic pulse, moving at the speed c. What is its value for an ordinary particle? The energy (1-27.7) is smallest for $\bar{p} = 0$, when the particle is at rest. Then we write

$$\bar{p} = 0: \quad E = m_o c^2, \qquad (1\text{-}27.8)$$

where m_o is the mass appropriate to zero velocity—the rest mass. Therefore we have, in general,

$$E^2 = c^2 p^2 + (m_o c^2)^2 , \qquad (1\text{-}27.9)$$

or

$$E = (c^2 p^2 + m_o^2 c^4)^{\frac{1}{2}}. \qquad (1\text{-}27.10)$$

Another way of presenting this replaces momentum by velocity in accordance with (1-27.4),

$$E^2 = E^2 \frac{v^2}{c^2} + (m_o c^2)^2, \qquad (1\text{-}27.11)$$

giving

$$E = \frac{m_o c^2}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}, \quad \bar{p} = \frac{m_o \bar{v}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \quad . \qquad (1\text{-}27.12)$$

The last momentum construction exhibits the relation to, and the limitation of, the initial Newtonian formulation of particle mechanics. For speeds small in comparison to that of light, $|\bar{v}| << c$, the momentum is $\bar{p} = m_o\bar{v}$ and the particle inertia is a constant. This is the domain of Newtonian mechanics. But even here something is different, as we see from the energy derived from the approximation

$$\frac{|\bar{v}|}{c} << 1: \quad \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \cong 1 + \frac{1}{2}\frac{v^2}{c^2}\,, \qquad (1\text{-}27.13)$$

namely,

$$E \cong m_o c^2 + \tfrac{1}{2} m_o v^2 . \qquad (1\text{-}27.14)$$

In addition to the Newtonian kinetic energy $\frac{1}{2}m_o v^2$, there is a constant, the rest energy $m_o c^2$, displacing the Newtonian origin of energy. The same thing appears in the momentum form of E, (1-27.10), as

$$|\bar{p}| << m_o c: \quad E \cong m_o c^2 + \frac{p^2}{2m_o} . \qquad (1\text{-}27.15)$$

For speeds approaching the speed of light we enter a new physical domain, one where the speed of light is an impassable barrier. This we can see from the particle velocity, exhibited as

$$\bar{v} = c\,\frac{\bar{p}}{\left(p^2 + m_o^2 c^2\right)^{\frac{1}{2}}}\,; \qquad (1\text{-}27.16)$$

it is such that

$$|\bar{v}| \leq c, \qquad (1\text{-}27.17)$$

with the equality sign occuring only for $m_o = 0$. As for the last conclusion, it is not unreasonable that a system, such as an electromagnetic pulse, which can never be at rest, has no rest mass.

Now we must reconstruct the Lagrangian-Hamiltonian dynamics of particles. The Lagrangian of (1-18.29), omitting the potential V, is

$$L = \bar{p}\cdot\left(\frac{d\bar{r}}{dt} - \bar{v}\right) + \tfrac{1}{2}mv^2. \qquad (1\text{-}27.18)$$

Clearly the Newtonian term $\frac{1}{2}mv^2$ must be replaced by a new function of $\bar{v}$, $L(\bar{v})$:

$$L = \bar{p}\cdot\left(\frac{d\bar{r}}{dt} - \bar{v}\right) + L(\bar{v}), \qquad (1\text{-}27.19)$$

that will reproduce the new forms. We can find $L(\bar{v})$ by using the velocity construction of the energy,

$$E = \bar{p}\cdot\bar{v} - L(\bar{v}), \qquad (1\text{-}27.20)$$

and of the momentum. It is

$$L(\bar{v}) = \frac{m_o\bar{v}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}\cdot\bar{v} - \frac{m_o c^2}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = -m_o c^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}. \qquad (1\text{-}27.21)$$

In the Newtonian regime, this $L(\bar{v})$ reproduces the original form to within a constant,

$$\frac{|\bar{v}|}{c} << 1: \quad L(v) = -m_o c^2 + \tfrac{1}{2}m_o v^2. \qquad (1\text{-}27.22)$$

The consistency of the action principle is verified on noting the consequence of a $\bar{v}$ variation:

$$\bar{p} = \frac{\partial L(\bar{v})}{\partial \bar{v}} = \frac{m_o\bar{v}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}. \qquad (1\text{-}27.23)$$

The elimination of $\bar{v}$ produces the Hamiltonian version

$$L = \bar{p}\cdot\frac{d\bar{r}}{dt} - H, \quad H = c(p^2 + m_o^2c^2)^{\frac{1}{2}} = E. \qquad (1\text{-}27.24)$$

We shall find it especially rewarding to use this new particle dynamics in re-examining a subject previously discussed in the Newtonian framework. The topic is the coordinate translation that grows linearly in time, (1-19.21), or equivalently, the introduction of a new coordinate system with a constant relative velocity. Here we consider only a single particle. The displacement

$$\delta\vec{r}\,(t) = \delta\bar{v}t, \tag{1-27.25}$$

combined with the momentum change

$$\delta\bar{p}(t) = (E/c^2)\,\delta\bar{v}, \tag{1-27.26}$$

induces the following alteration of the action element $dt\,L$:

$$\delta_{\bar{r},\bar{p}}[dt\,L] = \delta_{\bar{r},\bar{p}}[\bar{p}\cdot d\bar{r} - H\,dt] = \bar{p}\cdot\delta\bar{v}\,dt + (H/c^2)\,\delta\bar{v}\cdot d\bar{r} - \bar{p}\cdot\delta\bar{v}\,dt$$

$$= H\,d\left(\frac{1}{c^2}\,\delta\bar{v}\cdot\bar{r}\right). \tag{1-27.27}$$

At the analogous Newtonian stage, (1-19.23), $m(=m_0)$ appeared in place of H/c^2, and we concluded that the action was not invariant, but changed by a differential. Now a totally new situation presents itself. If we also vary the time by

$$\delta t = \frac{1}{c^2}\,\delta\bar{v}\cdot\bar{r}, \tag{1-27.28}$$

the additional contribution, $-H\,d(\delta t)$, will cancel (1-27.27), and the action is invariant under the combined space and time transformations

$$\delta\bar{r} = \delta\bar{v}\,t, \quad \delta t = \frac{1}{c^2}\,\delta\bar{v}\cdot\bar{r}. \tag{1-27.29}$$

In view of the invariance of the action the implied conservation law should now follow directly. Indeed

$$G = \bar{p}\cdot\delta\bar{r} - H\delta t = \delta\bar{v}\cdot\bar{N}, \tag{1-27.30}$$

where $(H = E)$

$$\bar{N} = \bar{p}t - \frac{E}{c^2}\bar{r} \quad (1\text{-}27.31)$$

is conserved:

$$\frac{d\bar{N}}{dt} = \bar{p} - \frac{E}{c^2}\frac{d\bar{r}}{dt} = 0 \quad (1\text{-}27.32)$$

and we have recovered our starting point, $m = (1/c^2)E$.

But, from this initial dynamical modification of Newtonian dynamics has now emerged a change in Newtonian kinematics: the absolute distinction between time and space has been removed. That is emphasized by the fact that neither r^2 nor $(ct)^2$ is left unchanged by the transformation (1-27.29), whereas the difference, $r^2 - (ct)^2$, is invariant:

$$\delta[r^2 - (ct)^2] = 2[\bar{r}\cdot\delta\bar{v}\,t - t\delta\bar{v}\cdot\bar{r}] = 0. \quad (1\text{-}27.33)$$

The physical meaning of this invariance appears on considering an electromagnetic pulse that, at time $t = 0$, is emitted from the origin, $\bar{r} = 0$. Moving at the speed of light, c, the pulse, at time t, will have travelled the distance $r = ct$, so that

$$r^2 - c^2t^2 = 0. \quad (1\text{-}27.34)$$

The fact that an observer in uniform relative motion will assign different values to the elapsed time, and to the distance traversed, but agree that (1-27.34) is still valid, means that he also measures the speed of light as c. This is Einsteinian relativity.

One might object that it could all be true only for infinitesimal transformations. But, from infinitesimal transformations, finite transformations grow. We make this explicit by letting $\delta\bar{v}$ point along the z-axis, so that

$$\delta x = 0, \quad \delta y = 0,$$

$$\delta z = \frac{\delta v}{c}ct, \quad \delta ct = \frac{\delta v}{c}z\,. \quad (1\text{-}27.35)$$

Let us regard this infinitesimal transformation as the result of changing a parameter θ by the infinitesimal amount

$$\delta\theta = \frac{\delta v}{c} . \qquad (1\text{-}27.36)$$

The implied differential equations in the variable θ are

$d\theta = \frac{dv}{c}$

$\frac{dv}{d\theta} = c$

$$\frac{dx(\theta)}{d\theta} = 0, \quad \frac{dy(\theta)}{d\theta} = 0,$$

$$\frac{dz(\theta)}{d\theta} = ct(\theta), \quad \frac{dct(\theta)}{d\theta} = z(\theta). \qquad (1\text{-}27.37)$$

From the latter we derive

$$\frac{d^2z(\theta)}{d\theta^2} = z(\theta), \quad \frac{d^2ct(\theta)}{d\theta^2} = ct(\theta), \qquad (1\text{-}27.38)$$

which are solved by the hyperbolic functions, cosh θ and sinh θ. The explicit solutions of these equations that obey the initial conditions

$$z(0) = z, \quad ct(0) = ct, \quad \frac{dz}{d\theta}(0) = ct, \quad \frac{dct}{d\theta}(0) = z \qquad (1\text{-}27.39)$$

$\frac{z(\theta)}{ct(\theta)} = \frac{z + ct \tanh\theta}{z \tanh\theta + ct}$

are

$$z(\theta) = z \cosh\theta + ct \sinh\theta,$$

$\frac{v}{c} =$

$\frac{v}{c} + \tanh\theta \,/\, \frac{z}{ct}\tanh\theta + 1$

$$ct(\theta) = z \sinh\theta + ct \cosh\theta. \qquad (1\text{-}27.40)$$

Physical interpretation is facilitated by focusing on tanh θ, the ratio of sinh θ and cosh θ, which cannot exceed unity in magnitude. We now write

$$\tanh\theta = \frac{v}{c} , \qquad (1\text{-}27.41)$$

which reduces to (1-27.36) for infinitesimal values of these parameters. Then, the constructions

$$\cosh\theta = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}, \quad \sinh\theta = \frac{\frac{v}{c}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \qquad (1\text{-}27.42)$$

satisfy (1-27.41) as well as the hyperbolic relation

$$\cosh^2\theta - \sinh^2\theta = 1. \qquad (1\text{-}27.43)$$

If we distinguish the transformed values of the coordinates by a prime, the transformation equations read

$$z' = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}(z + vt),$$

$$t' = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left(t + \frac{v}{c^2}z\right), \qquad (1\text{-}27.44)$$

along with

$$x' = x, \quad y' = y. \qquad (1\text{-}27.45)$$

We see that the point with coordinates

$$x = 0, \quad y = 0, \quad z = -vt \qquad (1\text{-}27.46)$$

is represented by the transformed coordinates

$$x' = 0, \quad y' = 0, \quad z' = 0. \qquad (1\text{-}27.47)$$

It is the origin of the new reference frame which therefore moves with velocity -v relative to the initial one. (See Fig.   )

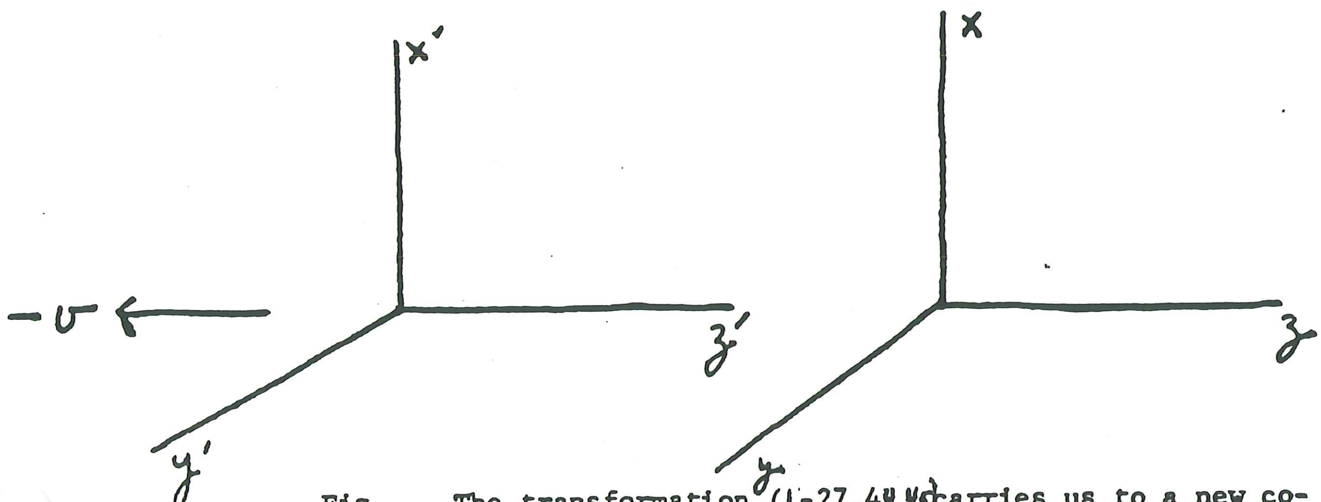

Fig. The transformation (1-27.44,45) carries us to a new coordinate frame moving relative to the original one with a velocity -v along the z-axis.

To see that $r^2-(ct)^2$ is left invariant by these finite transformations, it helps to present (1-27.44) as

$$z'+ct'=\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\frac{1}{2}}(z+ct),$$

$$z'-ct'=\left(\frac{1-\frac{v}{c}}{1+\frac{v}{c}}\right)^{\frac{1}{2}}(z-ct), \qquad (1\text{-}27.48)$$

for it is immediately apparent, on multiplication, that

$$z'^2-(ct')^2=z^2-(ct)^2 \qquad (1\text{-}27.49)$$

and then [Eq.(1.27.45)]

$$r'^2-(ct')^2=r^2-(ct)^2. \qquad (1\text{-}27.50)$$

The space-time coordinate transformations of the new kinematics are called Lorentz transformations, although it was Albert Einstein (1879-1955) who, in 1905, first understood their significance as describing the full physical equivalence of reference frames in uniform relative motion. As an aspect of that equivalence, we note the following. The original reference frame moves with velocity $+v$ relative to the new one (Fig.):

$$x=0,\quad y=0,\quad z=0 \qquad (1\text{-}27.51)$$

implies

$$x'=0,\ y'=0,\ z'=vt', \qquad (1\text{-}27.52)$$

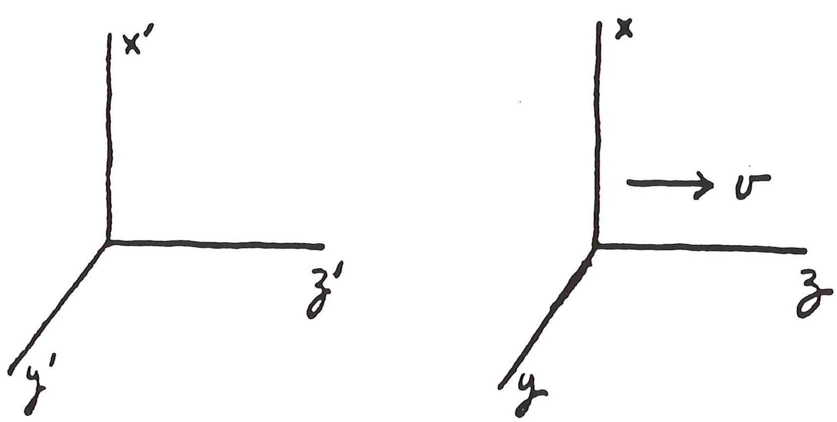

Fig. Motion of original frame relative to new frame

Then, should not the transformation that produces the unprimed coordinates from the primed ones be of precisely the same form as (1-27.44,45) but with the sign of v reversed? Indeed it is, as is evident on rewriting (1-27.48) as

$$z+ct = \left(\frac{1-\frac{v}{c}}{1+\frac{v}{c}}\right)^{\frac{1}{2}} (z'+ct')$$

$$z-ct = \left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\frac{1}{2}} (z'-ct') , \qquad (1\text{-}27.53)$$

together with $x=x'$, $y=y'$. More generally, suppose the coordinate transformation $z, t \to z', t'$, produced by relative velocity v_1 along the common $z-z'$ axis, is followed by the transformation $z', t' \to z'', t''$, produced by relative velocity v_2 along the common $z'-z''$ axis. Is the net result a transformation $z, t \to z'', t''$ that is produced by some relative velocity v? Yes. It suffices to consider just one of the pair of equations analogous to (1-27.48), say

$$z'+ct' = \left(\frac{1+\frac{v_1}{c}}{1-\frac{v_1}{c}}\right)^{\frac{1}{2}} (z+ct) , \qquad (1\text{-}27.54)$$

and, similarly,

$$z''+ct'' = \left(\frac{1+\frac{v_2}{c}}{1-\frac{v_2}{c}}\right)^{\frac{1}{2}} (z'+ct'), \qquad (1\text{-}27.55)$$

(the other set emerges by the systematic substitution of $-c$ for c) which immediately yields

$$(z''+ct'') = \left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\frac{1}{2}} (z+ct), \qquad (1\text{-}27.56)$$

with

$$\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\frac{1}{2}} = \left(\frac{1+\frac{v_1}{c}}{1-\frac{v_1}{c}}\right)^{\frac{1}{2}}\left(\frac{1+\frac{v_2}{c}}{1-\frac{v_2}{c}}\right)^{\frac{1}{2}} . \tag{1-27.57}$$

For any of the square root factors, the variation of the value of the appropriate v/c from -1 to $+1$ changes the square root from 0 to ∞; it is a positive number, and the product of two positive numbers is again a positive number. In other words, no succession of Lorentz transformations can produce a net transformation with $|v| > c$. The specific value of v in (1-27.57) is identified by writing this relation as

$$\frac{1+\frac{v}{c}}{1-\frac{v}{c}} = \frac{1+\frac{v_1v_2}{c^2}+\frac{v_1+v_2}{c}}{1+\frac{v_1v_2}{c^2}-\frac{v_1+v_2}{c}} , \tag{1-27.58}$$

or

$$v = \frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}} . \tag{1-27.59}$$

Simple addition of the velocities occurs only in the Newtonian regime, $|v_{1,2}| \ll c$.

We can't end this section without showing that the kinematical space-time transformations of (1-27.44,45) do indeed produce a dynamical unification of charged particles and electromagnetic fields. That requires a study of the behavior of fields and potentials under the infinitesimal Lorentz transformations. This has already been touched on in Problem 3.1, but we prefer to apply our recently developed methods here, beginning with the analogue of (1-26.2) for Lorentz transformations:

$$\delta\lambda(\bar{r},t) = -\left(\delta\bar{v}t\cdot\bar{\nabla}+\frac{1}{c^2}\,\delta\bar{v}\cdot\bar{r}\frac{\partial}{\partial t}\right)\lambda(\bar{r},t) = -\delta_{coor}\lambda(\bar{r},t). \qquad (1\text{-}27.60)$$

Differentiation supplies the model for potential variations:

$$\delta\bar{A}(\bar{r},t) = -\delta_{coor}\,\bar{A}(\bar{r},t)+\frac{1}{c}\,\delta\bar{v}\phi(\bar{r},t)\ ,$$

$$\delta\phi(\bar{r},t) = -\delta_{coor}\,\phi(\bar{r},t)+\frac{1}{c}\,\delta\bar{v}\cdot\bar{A}(\bar{r},t), \qquad (1\text{-}27.61)$$

which are alternatively presented as

$$\delta\bar{A} = \delta\bar{v}t\times\bar{B}+\frac{1}{c}\,\delta\bar{v}\cdot\bar{r}\bar{E}+\bar{\nabla}(-\delta\bar{v}t\cdot\bar{A}+\frac{1}{c}\,\delta\bar{v}\cdot\bar{r}\phi)\ ,$$

$$\delta\phi = \delta\bar{v}t\cdot\bar{E}-\frac{1}{c}\frac{\partial}{\partial t}(-\delta\bar{v}t\cdot\bar{A}+\frac{1}{c}\,\delta\bar{v}\cdot\bar{r}\phi). \qquad (1\text{-}27.62)$$

The use of the latter simplifies the calculation of the field variations; they emerge as

$$\delta\bar{E} = -\delta_{coor}\,\bar{E}-\frac{1}{c}\,\delta\bar{v}\times\bar{B},$$

$$\delta\bar{B} = -\delta_{coor}\,\bar{B}+\frac{1}{c}\,\delta\bar{v}\times\bar{E}. \qquad (1\text{-}27.63)$$

Then further differentiation in accordance with the field equations,

$$\rho = \frac{1}{4\pi}\,\bar{\nabla}\cdot\bar{E}$$

$$\bar{J} = \frac{c}{4\pi}\left(\bar{\nabla}\times\bar{B}-\frac{1}{c}\frac{\partial}{\partial t}\,\bar{E}\right), \qquad (1\text{-}27.64)$$

yields

$$\delta\rho = -\delta_{coor}\rho+\frac{1}{c^2}\,\delta\bar{v}\cdot\bar{J}$$

$$\delta\bar{J} = -\delta_{coor}\bar{J}+\delta\bar{v}\rho. \qquad (1\text{-}27.65)$$

Notice that the Lorentz transformation properties of $\frac{1}{c}\bar{J}$, ρ are of the same form as those for $\bar{A}$, ϕ. We can also recognize from these results that the symbol 'δ' used in Problem , which involved both changes of fields and of coordinates, has just that meaning as applied to any field $F(\bar{r},t)$:

$$'\delta' F(\bar{r},t) = (F+\delta F)(\bar{r}+\delta\bar{r}, t+\delta t) - F(\bar{r},t) = \delta F(\bar{r},t) + \delta_{coor} F(\bar{r},t). \quad (1\text{-}27.66)$$

We must now examine the responses of the various parts of the total Lagrange function. First consider

$$\mathcal{L}_{field} = \frac{1}{8\pi}(E^2 - B^2). \quad (1\text{-}27.67)$$

It is immediately apparent that the contributions of the last terms in the transformation equations (1-27.63) just cancel and

$$\delta\mathcal{L}_{field} = -\delta_{coor}\mathcal{L}_{field}. \quad (1\text{-}27.68)$$

Then we consider the interaction contribution to the Lagrange function [see Eq.(1-22.4)]

$$\mathcal{L}_{int} = -\rho\phi + \frac{1}{c}\bar{J}\cdot\bar{A}. \quad (1\text{-}27.69)$$

Again the last terms in the transformation Eqs.(1-27.61) and (1-27.65) have no net effect, and

$$\delta\mathcal{L}_{int} = -\delta_{coor}\mathcal{L}_{int}. \quad (1\text{-}27.70)$$

Finally, we come to the Lagrange function of the individual particles. For one particle, with the Lagrangian description adopted ($\bar{v} = d\bar{r}/dt$), the Lagrangian (1-27.21) is

$$L_a = -m_{oa}c^2\left(1 - \frac{1}{c^2}v_a^2\right)^{\frac{1}{2}}. \quad (1\text{-}27.71)$$

In contrast with the procedure of (1-26.9), our introduction of the Lagrange function now is dictated by the impossibility of maintaining a common time for particles at different spatial points — that Newtonian concept has disappeared in the world of Einsteinian relativistic kinematics. Accordingly we give each particle its individual time coordinate, and present its contribution to the action as

$$\int dt_a L_a = \int(-m_{oa}c)[(c\,dt_a)^2 - (d\bar{r}_a)^2]^{\frac{1}{2}} = \int dt(d\bar{r})\mathcal{L}_a \qquad (1\text{-}27.72)$$

where

$$\mathcal{L}_a(\bar{r},t) = -m_{oa}c\int \delta(\bar{r}-\bar{r}_a)\delta(t-t_a)\;[(c\,dt_a)^2 - (d\bar{r}_a)^2]^{\frac{1}{2}}\,. \qquad (1\text{-}27.73)$$

[Unlike (1-26.11), interaction terms are not included here.] The last integral is extended over the trajectory of the particle, with $\bar{r}_a$ varying as a function of t_a, or, better, with $\bar{r}_a$ and t_a given as functions of some parameter that is not changed by Lorentz transformations. Apart from a sign change and the consideration of infinitesimals, the space-time structure of the square root has just the invariant form in (1-27.33). Then, in response to

$$\delta\bar{r}_a = \delta\bar{v}t_a, \quad \delta t = \frac{1}{c^2}\,\delta\bar{v}\cdot\bar{r}_a, \qquad (1\text{-}27.74)$$

the delta function product becomes

$$\delta(\bar{r}-\bar{r}_a-\delta\bar{v}t_a)\delta(t-t_a-\frac{1}{c^2}\delta\bar{v}\cdot\bar{r}_a) = \delta(\bar{r}-\delta\bar{v}\,t-\bar{r}_a)\delta(t-\frac{1}{c^2}\delta\bar{v}\cdot\bar{r}-t_a), \qquad (1\text{-}27.75)$$

with the last form following from the delta function property, and the resulting change is just

$$-\delta_{coor}\;[\delta(\bar{r}-\bar{r}_a)\delta(t-t_a)]. \qquad (1\text{-}27.76)$$

We have now verified, for every individual constituent of $\mathcal{L}_{tot}$, that

$$\delta\mathcal{L}_{tot} = -\delta_{coor}\mathcal{L}_{tot}\;. \qquad (1\text{-}27.77)$$

More generally expressed, on introducing the transformed particle and field variables associated with the transformed coordinates,

$$\mathcal{L}'(\bar{r}',t') = \mathcal{L}(\bar{r},t); \qquad (1\text{-}27.78)$$

the Lagrange function of the system of interacting charges and fields is invariant under the Lorentz transformations of Einsteinian relativity.

The action

$$W_{12} = \int_2^1 dt(d\bar{r})\mathcal{L} \qquad (1\text{-}27.79)$$

is also Lorentz invariant because the space-time, four dimensional, element of volume has that property; it suffices to examine the Jacobian determinant of the transformation (1-27.40), for example:

$$\cosh^2\theta - \sinh^2\theta = 1. \qquad (1\text{-}27.80)$$

We shall not repeat the discussion of the various conservation laws in the light of these relativistic modifications. It should be sufficiently evident that such expressions as the total energy density and energy flux vector, (1-26.21) and (1-26.22), will be regained with E_a replaced by the relativistic energy (1-27.12). And, certainly (1-27.3) is now satisfied!

Finally, we supply the finite versions of the Lorentz transformations for the various fields. Proceeding in analogy with (1-27.37), we write (1-27.61) as

$$\frac{dA_x(\theta)}{d\theta} = 0, \quad \frac{dA_y(\theta)}{d\theta} = 0, \quad \frac{dA_z(\theta)}{d\theta} = \phi(\theta), \quad \frac{d\phi(\theta)}{d\theta} = A_z(\theta), \qquad (1\text{-}27.81)$$

where d denotes the total change in the sense of (1-27.66). The finite transformation equations are then

$$A_x'(\bar{r}',t') = A_x(\bar{r},t), \quad A_y'(\bar{r}',t') = A_y(\bar{r},t),$$

$$A_z'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left(A_z(\bar{r},t) + \frac{v}{c}\,\phi(\bar{r},t)\right),$$

$$\phi'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left(\phi(\bar{r},t) + \frac{v}{c}\,A_z(\bar{r},t)\right). \qquad (1\text{-}27.82)$$

The same forms apply with $\bar{A}$, ϕ replaced by $\frac{1}{c}\bar{J}$, ρ. The electric and magnetic field equations supplied by (1-27.63) are

$$\frac{dE_z(\theta)}{d\theta} = 0, \quad \frac{dB_z(\theta)}{d\theta} = 0,$$

$$\frac{dE_x}{d\theta}(\theta) = B_y(\theta), \quad \frac{dB_y}{d\theta}(\theta) = E_x(\theta),$$

$$\frac{dE_y}{d\theta}(\theta) = -B_x(\theta), \quad \frac{dB_x}{d\theta}(\theta) = -E_y(\theta) \qquad (1\text{-}27.83)$$

and therefore

$$E_z'(\bar{r}',t') = E_z(\bar{r},t), \quad B_z'(\bar{r}',t') = B_z(\bar{r},t),$$

$$E_x'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left[E_x(\bar{r},t) + \frac{v}{c}B_y(\bar{r},t)\right],$$

$$B_y'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left[B_y(\bar{r},t) + \frac{v}{c}E_x(\bar{r},t)\right], \qquad (1\text{-}27.84)$$

$v_z E_y$

and

$$E_y'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left[E_y(\bar{r},t) - \frac{v}{c}B_x(\bar{r},t)\right],$$

$$B_x'(\bar{r}',t') = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\left[B_x(\bar{r},t) - \frac{v}{c}E_y(\bar{r},t)\right]. \qquad (1\text{-}27.85)$$

The invariance, under finite transformations, of $E^2 - B^2$ is readily apparent, as is the invariance property

$$\bar{E}'(\bar{r}',t')\cdot\bar{B}'(\bar{r}',t') = \bar{E}(\bar{r},t)\cdot\bar{B}(\bar{r},t). \qquad (1\text{-}27.86)$$

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(7th.SET)

X. Stationary Principles for Electrostatics

10-1. Stationary Principles for the Energy

We will now specialize the general formulation given in the previous section to static circumstances, where there is no time variation. First note that with all time derivatives equal to zero, the Hamiltonian (energy), (9.12), 1.23-2 is stationary with respect to variations in $\vec{p}$, $\vec{r}$, and $\vec{v}$, that is [see also (8.22)-(8.24)] 18-2 see also 1-19.4

$$\frac{\partial H}{\partial \vec{p}} = 0 , \quad \frac{\partial H}{\partial \vec{r}} = 0 , \quad \frac{\partial H}{\partial \vec{v}} = 0 .$$

We now drop all reference to particle velocity in (9.12) to obtain for the static energy

$$E = \int (d\vec{r}) \left[\rho\phi + \frac{1}{4\pi} \vec{E}\cdot\vec{\nabla}\phi + \frac{E^2}{8\pi} + \frac{1}{4\pi} \left(\vec{B}\cdot\vec{\nabla}\times\vec{A} - \frac{1}{2} B^2 \right) \right] . \tag{10.1}$$

If there is no motion, $\vec{B} = 0$ as there is no source to produce a magnetic field (we assume, as usual, the absence of magnetic charge); consequently we will temporarily ignore the magnetic terms. We will show that if we require this energy expression be stationary under variations $\delta\phi$ and $\delta\vec{E}$, we recover the equations of electrostatics. The resulting variation in the energy is

$$\delta E = \int (d\vec{r}) \left[\rho\delta\phi + \frac{1}{4\pi} \vec{E}\cdot\vec{\nabla}\delta\phi + \frac{1}{4\pi} \delta\vec{E}\cdot(\vec{\nabla}\phi + \vec{E}) \right] ,$$

which may be simplified by use of the identity

$$\vec{E}\cdot\vec{\nabla}\delta\phi = \vec{\nabla}\cdot(\vec{E}\delta\phi) - (\vec{\nabla}\cdot\vec{E})\,\delta\phi ,$$

to read

$$\delta E = \int (d\vec{r}) \left[\delta\phi \left(\rho - \frac{\vec{\nabla}\cdot\vec{E}}{4\pi} \right) + \frac{1}{4\pi} \delta\vec{E}\cdot(\vec{\nabla}\phi + \vec{E}) \right] , \tag{10.2}$$

where we have ignored the surface term since we assume the integral extends over the entire region where the fields are non-zero. The stationary requirement on E , $\delta E = 0$, now implies the two basic equations of electrostatics,

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho , \quad \vec{E} = -\vec{\nabla}\phi . \tag{10.3}$$

Of course, the connection between $\vec{E}$ and ϕ implies

$$\vec{\nabla} \times \vec{E} = 0 .$$

Having seen this, we can immediately modify the energy expression, (10.1), to incorporate the effects of a dielectric medium,

$$E = \int (d\vec{r}) \left[\rho\phi + \frac{\varepsilon\vec{E}\cdot\vec{\nabla}\phi}{4\pi} + \frac{\varepsilon E^2}{8\pi} \right] . \tag{10.4}$$

The validity of this form is indicated by noting that if it is required to be stationary under variations in ϕ and $\vec{D} = \varepsilon\vec{E}$, we recover the equations of electrostatics in a dielectric. That is, the variation in the energy,

$$\delta E = \int (d\vec{r}) \left[\delta\phi \left(\rho - \frac{1}{4\pi} \vec{\nabla}\cdot\vec{D} \right) + \frac{1}{4\pi} \delta\vec{D}\cdot(\vec{\nabla}\phi + \vec{E}) \right] = 0 \tag{10.5}$$

implies

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho , \quad \vec{E} = -\vec{\nabla}\phi . \tag{10.6}$$

(As before, in writing the variation in the form of (10.5), we have integrated by parts and omitted the surface term.) We can use two restrictive versions of the above energy functional, (10.4).

1. We adopt as the definition of $\vec{E}$, its construction in terms of the scalar potential,

$$\vec{E} = -\vec{\nabla}\phi \quad ,$$

the curl of which is zero. The energy, as a functional of the potential, is then

ϵE^2

$$E(\phi) = \int (d\vec{r}) \left[\rho\phi - \frac{1}{8\pi} \underbrace{\epsilon(\vec{\nabla}\phi)^2} \right] \quad . \tag{10.7}$$

The requirement that E be stationary under the variation of ϕ yields Maxwell's equation

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \quad ,$$

where $\vec{D}$ is defined by

$$\vec{D} = -\epsilon\vec{\nabla}\phi \quad .$$

The energy functional, (10.7), contains yet further information. At the stationary point ϕ, E is an absolute maximum, as is seen by making a finite variation in the potential, $\delta\phi$,

$$E(\phi + \delta\phi) = E(\phi) - \int (d\vec{r}) \frac{\epsilon}{8\pi} (\vec{\nabla}\delta\phi)^2 \leq E(\phi) \quad , \tag{10.8}$$

since the linear term in $\delta\phi$ vanishes by the stationary principle. The correct energy, given by the physical ϕ, is a maximum of the functional (10.7). Evaluating the energy functional for an arbitrary potential bounds the energy from below.

Lecture 10

2. If we adopt

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \tag{10.9}$$

as defining $\vec{D}$, replace $\varepsilon\vec{E}$ by $\vec{D}$ in the expression for the energy, (10.4), and integrate by parts on the $\vec{D}\cdot\vec{\nabla}\phi$ term, we obtain

$$E(\vec{D}) = \int (d\vec{r})\, \frac{D^2}{8\pi\varepsilon} \quad , \tag{10.10}$$

which is a functional of $\vec{D}$. How does the stationary principle work here? The variation of (10.10) is

$$\delta E = \int (d\vec{r})\, \frac{1}{4\pi\varepsilon}\, \vec{D}\cdot\delta\vec{D} \quad ,$$

but $\nabla\cdot\delta D = 0$
it is not arbitrary

where the displacement vector is varied subject to the restriction that (10.9) be satisfied:

$$\vec{\nabla}\cdot(\vec{D} + \delta\vec{D}) = 4\pi\rho \quad ,$$

or

$$\vec{\nabla}\cdot\delta\vec{D} = 0 \quad .$$

Therefore, any variation in $\vec{D}$ must be a curl,

$$\delta\vec{D} \equiv \vec{\nabla}\times\vec{A} \quad ,$$

where $\vec{A}$ is an arbitrary, infinitesimal vector, enabling us to write the variation in the energy as

$$\delta E = \int (d\vec{r})\, \frac{\vec{E}}{4\pi}\cdot\vec{\nabla}\times\vec{A} \quad . \tag{10.11}$$

An integration by parts then implies the irrotational property of $\vec{E}$,

$$\vec{\nabla} \times \vec{E} = 0 \quad .$$

The advantage to this functional form of the energy, (10.10), is seen by again considering finite variations (called $\delta\vec{D}$). If $\vec{D}$ is the correct physical solution, then $E(\vec{D})$ is an absolute minimum,

$$E(\vec{D} + \delta\vec{D}) = E(\vec{D}) + \int (d\vec{r}) \frac{(\delta\vec{D})^2}{8\pi\varepsilon} \geq E(\vec{D}) \quad , \tag{10.12}$$

(since the linear term in $\delta\vec{D}$ is zero due to the stationary principle). Therefore, the correct energy is the minimum value of (10.10) while an arbitrary $\vec{D}$ [compatible with (10.9)] will give an upper bound to E . These bounds, (10.8) and (10.12), are useful for finding approximate solutions when exact solutions are difficult or impossible to obtain.

10-2. Force on Dielectrics

As a simple application, consider the effect of a small change in the dielectric constant, $\delta\varepsilon$,

$$\varepsilon(\vec{r}) \rightarrow \varepsilon(\vec{r}) + \delta\varepsilon(\vec{r}) \quad .$$

The first order change in the energy, from the form (10.7), is

$$\delta E(\phi) = -\int (d\vec{r}) \frac{\delta\varepsilon(\vec{\nabla}\phi)^2}{8\pi} = -\int (d\vec{r}) \frac{\delta\varepsilon E^2}{8\pi} \quad . \tag{10.13}$$

The potential ϕ is an implicit function of ε but the first order variation so induced vanishes by the stationary principle. Equivalently, from the second form, (10.10), the first order variation in the energy is

$$\delta E(\vec{D}) = -\int (d\vec{r}) \frac{1}{8\pi} \left(\frac{\vec{D}}{\varepsilon}\right)^2 \delta\varepsilon = -\int (d\vec{r}) \frac{\delta\varepsilon\, E^2}{8\pi} \quad , \tag{10.14}$$

in agreement with (10.13).

An example is provided by the infinitesimal displacement of an uncharged inhomogeneous dielectric. If the material is displaced by $\delta\vec{r}$, the new dielectric constant at $\vec{r}$ is the old dielectric constant at $\vec{r} - \delta\vec{r}$:

$$\varepsilon(\vec{r}) \rightarrow \varepsilon(\vec{r}) + \delta\varepsilon(\vec{r}) = \varepsilon(\vec{r}-\delta\vec{r}) = \varepsilon(\vec{r}) - \delta\vec{r}\cdot\vec{\nabla}\varepsilon(\vec{r}) \quad ,$$

or

$$\delta\varepsilon(\vec{r}) = -\delta\vec{r}\cdot\vec{\nabla}\varepsilon(\vec{r}) \quad . \tag{10.15}$$

Then, the change in energy is

$$\delta E = \delta\vec{r}\cdot\int (d\vec{r})(\vec{\nabla}\varepsilon) \frac{E^2}{8\pi} = -\vec{F}\cdot\delta\vec{r} \quad . \tag{10.16}$$

We therefore identify the force, $\vec{F}$, on the dielectric due to the inhomogeneity of the medium to be

$$\vec{F} = -\int (d\vec{r}) \frac{E^2}{8\pi} \vec{\nabla}\varepsilon \quad . \tag{10.17}$$

This same result, (10.17), can also be derived from the stress tensor, (7.13), with $\vec{H} = 0$,

$$\overleftrightarrow{T} = \overleftrightarrow{1} \frac{\varepsilon E^2}{8\pi} - \frac{\varepsilon\vec{E}\,\vec{E}}{4\pi} \quad . \tag{10.18}$$

Since the stress tensor describes the outward flow of momentum per unit area, the total force <u>on</u> a body bounded by a closed surface S is

$$\vec{F} = -\int_S d\vec{S}\cdot\overleftrightarrow{T} = -\int (d\vec{r})\, \vec{\nabla}\cdot\overleftrightarrow{T} \quad . \tag{10.19}$$

The divergence of the stress tensor is

$$\vec{\nabla}\cdot\overleftrightarrow{T} = (\vec{\nabla}\varepsilon)\,\frac{E^2}{8\pi} + \frac{1}{4\pi}\,D_i\,\vec{\nabla}E_i - \frac{\vec{\nabla}\cdot\vec{D}}{4\pi}\,\vec{E} - \frac{(\vec{D}\cdot\vec{\nabla})\vec{E}}{4\pi} \quad . \tag{10.20}$$

Since for electrostatics,

$$D_i\vec{\nabla}E_i - (\vec{D}\cdot\vec{\nabla})\vec{E} = \vec{D}\times(\vec{\nabla}\times\vec{E}) = 0 \quad ,$$

$\frac{\partial\varepsilon}{\partial r} = F$

$\delta\varepsilon = \delta r \cdot F$

and when no free charge is present,

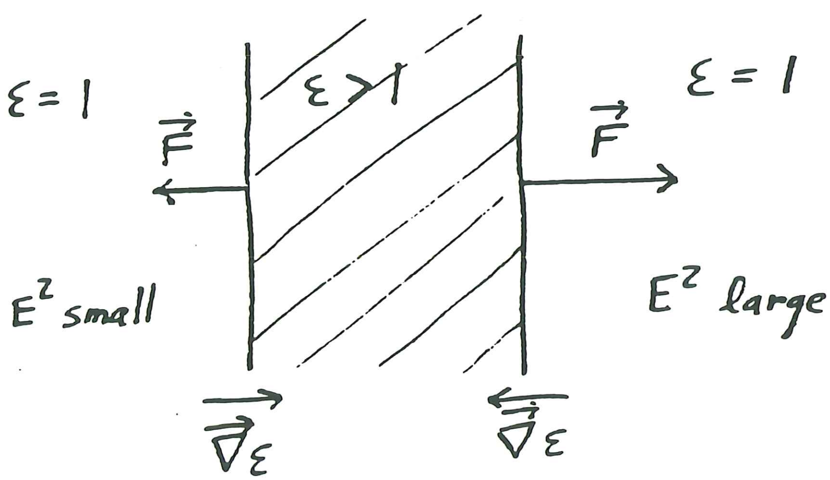
$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho = 0 \quad ,$$

the divergence reduces to

$$\vec{\nabla}\cdot\overleftrightarrow{T} = (\vec{\nabla}\varepsilon)\,\frac{E^2}{8\pi}$$

so that the force calculated from (10.19) is identical with (10.17).

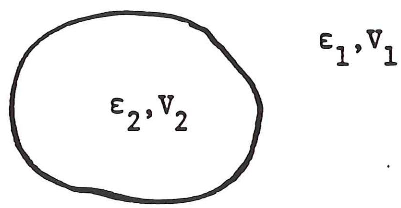
As an application of the above result, consider a slab of dielectric material, with ε = constant, immersed in an inhomogeneous electric field. The gradient of ε arises from the discontinuity of the dielectric constant between vacuum and medium. The situation might be described by the diagram below.



If E^2 is small on the left and large on the right, the dielectric material will be pulled to the right, that is, into the region of strong field. [Note that this depends on $\varepsilon > 1$ so that the directions of $\vec{\nabla}\varepsilon$ are as shown in the diagram.]

10-3. Boundary Conditions

An arrangement of objects with different values of ε that are in contact with each other provides the simplest example of an inhomogeneous dielectric constant. Imagine we have two volumes, V_1 and V_2, with dielectric constants ε_1 and ε_2, respectively, sharing a common surface, as shown in the figure.



Because of this discontinuity of ε on the surface, when the energy expression (10.4) is varied and expressed in the form (10.5), there is an additional contribution from the surface term previously omitted. In the interior of both V_1 and V_2, the same arguments as those given in Subsection 10-1 still apply, so that we have the same equations of motion,

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \; ; \quad \vec{\nabla}\times\vec{E} = 0 \; .$$

The additional surface term, which is now the total variation in the energy, is

$$\delta E = \int_{V_1} (d\vec{r})\, \vec{\nabla}\cdot\left(\frac{\vec{D}}{4\pi}\,\delta\phi\right) + \int_{V_2} (d\vec{r})\, \vec{\nabla}\cdot\left(\frac{\vec{D}}{4\pi}\,\delta\phi\right)$$

$$= \int_S dS \left[\vec{n}_1 \cdot \frac{\vec{D}_1}{4\pi} \delta\phi_1 + \vec{n}_2 \cdot \frac{\vec{D}_2}{4\pi} \delta\phi_2 \right] , \tag{10.22}$$

where S is the common surface and $\vec{n}_1$ and $\vec{n}_2$ are the oppositely directed outward normals for V_1 and V_2, respectively:

$$\vec{n}_1 = -\vec{n}_2 \quad . \tag{10.23}$$

What must be the connection between ϕ_1 and ϕ_2 and between $\vec{D}_1$ and $\vec{D}_2$? We insist that ϕ must be continuous across the boundary, in order that $\vec{E}(= -\vec{\nabla}\phi)$ exist there, so that the boundary condition that ϕ satisfies is

$$\phi_1 = \phi_2 , \qquad \delta\phi_1 = \delta\phi_2 \quad . \tag{10.24}$$

The variation of the energy, (10.22), at the stationary point is therefore

$$\delta E = \int_S dS \, \frac{1}{4\pi} \, [\vec{n}_2 \cdot (\vec{D}_2 - \vec{D}_1)] \, \delta\phi = 0 \tag{10.25}$$

or, since $\delta\phi$ is arbitrary,

$$\vec{n}_2 \cdot (\vec{D}_2 - \vec{D}_1) = 0 \quad . \tag{10.26}$$

The normal component of $\vec{D}$ is continuous across the boundary between the two media, because of our implicit assumption that there is no free surface charge density ($\sigma = 0$). If $\sigma \neq 0$, there is one further term in the variation of the energy (10.4),

$$\int_S dS \, \sigma \, \delta\phi \quad .$$

The resulting generalization of (10.25) is

$$\delta E = \int_S dS \, \frac{1}{4\pi} \, [\vec{n}_2 \cdot (\vec{D}_2 - \vec{D}_1) + 4\pi\sigma] \, \delta\phi = 0 \quad , \qquad (10.27)$$

so correspondingly

$$\vec{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = 4\pi\sigma \qquad (10.28)$$

is the general boundary condition on the normal component of $\vec{D}$. We may regard (10.28) as the surface version of the volume statement

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho \quad .$$

Likewise, there is a surface statement corresponding to $\vec{\nabla} \times \vec{E} = 0$. This follows from the continuity of ϕ , (10.24), which implies that of the tangential derivative (that is, the component of the gradient parallel to the surface):

$$(\vec{n}_2 \times \vec{\nabla})(\phi_1 - \phi_2) = 0$$

or

$$\vec{n}_2 \times (\vec{E}_1 - \vec{E}_2) = 0 \quad . \qquad (10.29)$$

Equation (10.29) states that the tangential component of the electric field is continuous.

As a special application of the above discussion of dielectric interfaces, we consider the surface of a conductor. Inside the conductor the current density is linearly related to the electric field [see (5.9)]. Since in the static situation there is no flow of charge, $\vec{E} = 0$ everywhere inside the conductor. The continuity of the tangential component of $\vec{E}$, (10.29), then implies

$$\vec{n} \times \vec{E} = 0 \tag{10.30}$$

on the surface of the conductor. Moreover, since $\vec{E}$ and $\vec{D}$ vanish inside, the other boundary condition, (10.28), implies

$$\vec{n} \cdot \vec{D} = 4\pi\sigma \tag{10.31}$$

just outside the conductor, where $\vec{n}$ is the outward normal to the surface, and again σ is the surface charge density.

XI. Introduction to Green's Functions

In a dielectric medium the differential equation satisfied by the scalar potential ϕ in electrostatics is

$$-\vec{\nabla}\cdot[\varepsilon\vec{\nabla}\phi] = 4\pi\rho \quad . \tag{11.1}$$

Our task is to find the solution for ϕ , for a given charge distribution. Since (11.1) is a linear differential equation relating ρ and ϕ , the solution is given as a linear integral expression:

$$\phi(\vec{r}) = \int (d\vec{r}\,')\, G(\vec{r},\vec{r}\,')\, \rho(\vec{r}\,') \quad , \tag{11.2}$$

where we have introduced Green's function, $G(\vec{r},\vec{r}\,')$. It is evident that this Green's function is the potential at $\vec{r}$ arising from a unit point charge at $\vec{r}\,'$, so that it satisfies the differential equation

$$-\vec{\nabla}\cdot[\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,')] = 4\pi\delta(\vec{r}-\vec{r}\,') \quad . \tag{11.3}$$

Equation (11.2) expresses the fact that the potential due to a charge distribution is simply the sum of the contributions of each of the charges. Once we have Green's function, the solution for any charge distribution is a matter of integration.

An important property of Green's function is that it satisfies the so-called reciprocity condition,

$$G(\vec{r},\vec{r}\,') = G(\vec{r}\,',\vec{r}) \quad . \tag{11.4}$$

A first derivation of this follows from a consideration of the energy of the system. A convenient expression for the energy can be derived from (10.7) by making the replacement

$\lambda \to \lambda + \delta\lambda$

$\delta E =$

$$\phi \to \lambda\phi$$

where ϕ is the physical solution (which can be recovered by letting $\lambda = 1$) and λ is a constant. The energy functional, (10.7), evaluated at $\lambda\phi$ is

$$E(\lambda\phi) = \lambda \int (d\vec{r})\ \rho\phi - \lambda^2 \int (d\vec{r})\ \frac{\varepsilon(\vec{\nabla}\phi)^2}{8\pi} , \tag{11.5}$$

which is required to be stationary under variations in λ about $\lambda = 1$. That is, the derivative of the energy with respect to λ at $\lambda = 1$ is zero:

$$\frac{\partial}{\partial\lambda} E(\lambda\phi) = 0 \quad \text{at} \quad \lambda = 1 ,$$

or

$$0 = \int (d\vec{r})\ \rho\phi - 2 \int (d\vec{r})\ \frac{\varepsilon(\vec{\nabla}\phi)^2}{8\pi} . \tag{11.6}$$

Therefore, the energy of the physical configuration can be written in two alternative forms:

$$E = \int (d\vec{r})\ \varepsilon \frac{\vec{E}^2}{8\pi} = \frac{1}{2} \int (d\vec{r})\ \rho\phi . \tag{11.7}$$

If we use the second of these expressions [as well as (11.2)]

$$\begin{aligned} E &= \frac{1}{2} \int (d\vec{r})\ \rho(\vec{r})\ \phi(\vec{r}) \\ &= \frac{1}{2} \int (d\vec{r})(d\vec{r}\,')\ \rho(\vec{r})\ G(\vec{r},\vec{r}\,')\ \rho(\vec{r}\,') \\ &= \frac{1}{2} \int (d\vec{r})(d\vec{r}\,')\ \rho(\vec{r})\ G(\vec{r}\,',\vec{r})\ \rho(\vec{r}\,') \end{aligned} \tag{11.8}$$

(where the last form has used $\vec{r} \leftrightarrow \vec{r}\,'$), we see that only the symmetrical part of $G(\vec{r},\vec{r}\,')$ contributes to the energy. This proves (11.4) because the energy completely determines the potential function of an arbitrary charge distribution, and thereby, Green's function. The latter remark follows from the stationary property of the energy functional, as applied to a variation of the charge distribution:

$$\delta E = \int (d\vec{r})\ \delta\rho(\vec{r})\ \phi(\vec{r}) \quad .$$

Another proof of (11.4) can be achieved through the use of the differential equation directly. The equations satisfied by the Green's function due to point charges at $\vec{r}\,'$ and $\vec{r}\,''$, respectively, are

$$-\vec{\nabla}\cdot[\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,')] = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{11.9a}$$

$$-\vec{\nabla}\cdot[\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,'')] = 4\pi\delta(\vec{r}-\vec{r}\,'') \quad . \tag{11.9b}$$

We now multiply (11.9a) by $G(\vec{r},\vec{r}\,'')$ and (11.9b) by $G(\vec{r},\vec{r}\,')$, subtract the resulting equations, and then integrate over all space. These manipulations lead to

$$\begin{aligned} &4\pi[G(\vec{r}\,',\vec{r}\,'') - G(\vec{r}\,'',\vec{r}\,')] \\ &\quad = \int (d\vec{r})\ \{G(\vec{r},\vec{r}\,')\ \vec{\nabla}\cdot[\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,'')] - G(\vec{r},\vec{r}\,'')\ \vec{\nabla}\cdot[\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,')]\} \\ &\quad = \int (d\vec{r})\ \{\vec{\nabla}\cdot[G(\vec{r},\vec{r}\,')\ \varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,'') - G(\vec{r},\vec{r}\,'')\ \varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,')] \\ &\qquad - [\vec{\nabla}G(\vec{r},\vec{r}\,')]\cdot\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,'') + [\vec{\nabla}G(\vec{r},\vec{r}\,'')]\cdot\varepsilon(\vec{r})\ \vec{\nabla}G(\vec{r},\vec{r}\,')\} = 0 \quad , \end{aligned} \tag{11.10}$$

where the surface term at infinity vanishes for a localized charge distribution. Thus we have again proved the reciprocity relation, (11.4).

Lecture 11

XII. Electrostatics in Free Space

The simplest electrostatic situation is for the vacuum, $\varepsilon = 1$. The differential equation for the potential is then Poisson's equation,

$$-\nabla^2\phi(\vec{r}) = 4\pi\rho(\vec{r}) \quad , \tag{12.1}$$

so that the corresponding Green's function equation is

ind. $-\varepsilon\nabla^2$ [illegible]

$$-\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad . \tag{12.2}$$

The solution to (12.2) is the well-known Coulomb potential, since it is the potential at $\vec{r}$ produced by a unit point source at $\vec{r}\,'$:

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} \quad . \tag{12.3}$$

We will now derive (12.3) from the differential equation (12.2). In order to do this, we require an integral representation for the delta function. We recall that the latter is defined by the properties

$$\delta(x-x') = 0 \quad \text{if} \quad x \neq x' \quad , \tag{12.4a}$$

while it is so singular at $x = x'$ that

$$\int_{-\infty}^{\infty} dx\ \delta(x-x') = 1 \quad . \tag{12.4b}$$

Now note that the elementary integral $(\varepsilon > 0)$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}\, e^{-\epsilon|k|}$$

$$= \int_{0}^{\infty} \frac{dk}{\pi} \cos k(x-x')\, e^{-\epsilon k}$$

$$= \frac{1}{\pi} \frac{\epsilon}{(x-x')^2 + \epsilon^2} = Re\, \frac{i}{\pi} \frac{1}{(x-x') \mp i\epsilon}$$

becomes, in the limit $\epsilon \to 0$,

$$\frac{1}{\pi} \frac{\epsilon}{(x-x')^2 + \epsilon^2} \to \begin{cases} 0 \,, & x \neq x' \,, \\ \infty \,, & x = x' \,, \end{cases} \qquad (12.5)$$

while

$$\int_{-\infty}^{\infty} d(x-x')\, \frac{1}{\pi} \frac{\epsilon}{(x-x')^2 + \epsilon^2} = 1 \,, \qquad (12.6)$$

independent of the non-zero value of ϵ . Therefore, an integral representation for the delta function is

$$\delta(x-x') = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}\, e^{-\epsilon|k|} \,, \qquad (12.7a)$$

or, formally,

$$\delta(x-x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')} \,, \qquad (12.7b)$$

with the reference to ϵ and its limiting procedure being left implicit. The corresponding representation for the three-dimensional delta function is

$$\delta(\vec{r}-\vec{r}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$= \int \frac{(d\vec{k})}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} . \tag{12.8}$$

We now employ the above representation of the delta function to solve (12.2) for the Green's function. This can be very simply accomplished if we note that

$$\vec{\nabla} e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} = i\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} , \tag{12.9}$$

or, effectively,

$$\vec{\nabla} \to i\vec{k} ,$$

so that we can read off the solution to (12.2),

$$G(\vec{r},\vec{r}\,') = 4\pi \int \frac{(d\vec{k})}{(2\pi)^3}\, \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{k^2} . \tag{12.10}$$

We can verify that this is in fact the known result, (12.3), by using spherical coordinates for $\vec{k}$, with $\vec{r}-\vec{r}\,'$ pointing along the z-axis,

$$\vec{k}\cdot(\vec{r}-\vec{r}\,') = kR\cos\theta ,$$

$$R = |\vec{r}-\vec{r}\,'| ,$$

$$(d\vec{k}) = k^2 dk\; 2\pi d(\cos\theta) ,$$

so that

$$G(\vec{r},\vec{r}\,') = 4\pi \int \frac{k^2 dk\; 2\pi d(\cos\theta)}{8\pi^3}\, \frac{e^{ikR\cos\theta}}{k^2}$$

$$= \frac{1}{\pi} \int_0^\infty dk\, \frac{1}{ikR}\,(e^{ikR} - e^{-ikR})$$

$$= \frac{2}{\pi}\frac{1}{R}\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{R} \;. \tag{12.11}$$

In the above, we have treated all three directions of space on an equal footing. However, we need not do this. We can separate out one direction, say that of z, and treat it differently. The reason this is useful is because there are geometries in which physically interesting quantities vary in only a single direction. Singling out the z direction, we can write the Green's function, (12.10), as

$$G(\vec{r},\vec{r}\,') = 4\pi \int \frac{dk_x\,dk_y}{(2\pi)^2}\, e^{i[k_x(x-x') + k_y(y-y')]} \times \int \frac{dk_z}{2\pi} \frac{e^{ik_z(z-z')}}{k_x^2 + k_y^2 + k_z^2} \;. \tag{12.12}$$

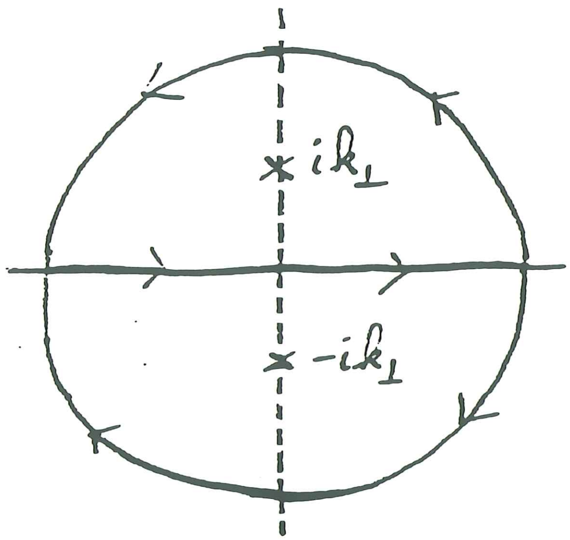
Adopting the nomenclature that the two-dimensional space of x and y is transverse ($\perp$) to the selected direction, we write the first part of (12.12) as

$$\int \frac{(d\vec{k}_\perp)}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \;. \tag{12.13}$$

The remaining integration over k_z ($k_\perp^2 = k_x^2 + k_y^2$ and $k_\perp \geq 0$)

$$\int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{e^{ik_z(z-z')}}{k_\perp^2 + k_z^2} = \frac{1}{2k_\perp} e^{-k_\perp|z-z'|} \;, \tag{12.14}$$

is performed by doing a contour integration as indicated:



$$z-z' > 0: \quad 2\pi i \frac{1}{2\pi}\frac{1}{2ik_\perp} e^{-k_\perp(z-z')} ,$$

$$z-z' < 0: \quad -2\pi i \frac{1}{2\pi}\frac{1}{-2ik_\perp} e^{k_\perp(z-z')} .$$

We have therefore recast Green's function into the form

$$G(\vec{r},\vec{r}\,') = 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} g(z,z',k_\perp) , \qquad (12.15)$$

where, for free space, the "reduced" Green's function is

$$g(z,z',k_\perp) = \frac{1}{2k_\perp} e^{-k_\perp|z-z'|} . \qquad (12.16)$$

The form (12.15) applies to any problem which is translationally invariant in x and y but not necessarily in z. The representation is particularly adapted to the situation in which the dielectric constant is only a function of z. For the case at hand, where the Green's function satisfies (12.2), we can easily derive the differential equation for $g(z,z',k_\perp)$, as follows:

$$-\nabla^2 G = 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \left[k_\perp^2 - \frac{\partial^2}{\partial z^2}\right] g(z,z',k_\perp)$$

$$= 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \delta(z-z') , \quad \leftarrow \text{from 12.10}$$

which implies

$$\left[k_\perp^2 - \frac{\partial^2}{\partial z^2}\right] g(z,z',k_\perp) = \delta(z-z') \quad . \tag{12.17}$$

This equation is solved by noting that for $z \neq z'$,

$$\left[k_\perp^2 - \frac{\partial^2}{\partial z^2}\right] g(z,z',k_\perp) = 0 \quad , \qquad g(z,z',k_\perp) = \frac{1}{2k_\perp} e^{-k_\perp |z-z'|} \tag{12.18}$$

while the discontinuity in the derivative of g at $z = z'$ is

$$- \frac{\partial}{\partial z} g(z,z',k_\perp)\Bigg|_{z'-0}^{z'+0} = 1 \quad . \tag{12.19}$$

Imposing boundedness at infinity yields the solution (12.16), of course.

XIII. Half-Infinite Dielectric Slab

13-1. Green's Function

We now apply the above representation, (12.15), to find the Green's function for the simplest situation involving a non-uniform dielectric constant, that of a half infinite dielectric slab, which possesses the required translational invariance in x and y :

ε = constant | $\varepsilon = 1$

$\longrightarrow z$

$z=0$

The Green's function equation, (11.3),

$$-\vec{\nabla}\cdot[\varepsilon(z)\,\vec{\nabla}G(\vec{r},\vec{r}\,')] = 4\pi\delta(\vec{r}-\vec{r}\,') \quad ,$$

becomes, in the two regions,

$$z > 0 : \quad -\nabla^2 G = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{13.1a}$$

$$z < 0 : \quad -\varepsilon\nabla^2 G = 4\pi\delta(r-r') \quad . \tag{13.1b}$$

To solve these equations, we must impose appropriate boundary conditions. Across the interface between the vacuum and the medium, the normal component of the displacement vector must be continuous [see (10.26)],

$$\left(-\frac{\partial}{\partial z}\right) G\Big|_{z=+0} = \varepsilon\left(-\frac{\partial}{\partial z}\right) G\Big|_{z=-0} \quad . \tag{13.2}$$

Of course, G must be continuous across the interface because it represents the potential of a point charge. The reduced Green's function, $g(z,z',k_\perp)$, introduced in (12.15), satisfies the differential equations [see (12.17)]

$$z > 0 : \quad \left(- \frac{\partial^2}{\partial z^2} + k_\perp^2 \right) g = \delta(z-z') \quad , \tag{13.3a}$$

$$z < 0 : \quad \varepsilon \left(- \frac{\partial^2}{\partial z^2} + k_\perp^2 \right) g = \delta(z-z') \quad , \tag{13.3b}$$

subject to the boundary conditions

$$g\Big|_{-0} = g\Big|_{+0} \quad , \tag{13.4a}$$

$$\varepsilon \frac{\partial}{\partial z} g\Big|_{-0} = \frac{\partial}{\partial z} g\Big|_{+0} \quad . \tag{13.4b}$$

In the following we will solve this problem by assuming that $z' > 0$ (that is, the unit charge lies in the vacuum and not in the dielectric). (For the converse situation see Problem ~~14.~~ 7.2) The solutions to (13.3) and (13.4) can be expressed in terms of the solutions, $e^{k_\perp z}$ and $e^{-k_\perp z}$, of the corresponding homogeneous equation. The forms of the solution in the three regions are as follows:

\# of boundary conditions

3 4

2 each at $z=z'$ and $z=0$

$$z < 0 : \qquad g = A\, e^{k_\perp z} \quad , \tag{13.5a}$$

$$z > z' : \qquad g = B\, e^{-k_\perp z} \quad , \tag{13.5b}$$

$$0 < z < z' : \quad g = C\, e^{k_\perp z} + D\, e^{-k_\perp z} \quad , \tag{13.5c}$$

where the single exponentials in (13.5a) and (13.5b) are required by the boundary condition that g remain finite for $|z| \to \infty$. The boundary conditions at $z = 0$, (13.4), require

$$A = C + D \quad , \tag{13.6a}$$

$$\varepsilon k_\perp A = k_\perp (C-D) \quad . \tag{13.6b}$$

from which we infer

$$C = \frac{\varepsilon+1}{2} A , \qquad D = - \frac{\varepsilon-1}{2} A . \tag{13.6c}$$

Just as in the situation mentioned at the end of the last section [see (12.19)], the singularity in the differential equation requires, at $z = z'$,

$$g \Big|_{z=z'-0}^{z=z'+0} = 0 , \tag{13.7a}$$

$$- \frac{\partial}{\partial z} g \Big|_{z=z'-0}^{z=z'+0} = 1 , \tag{13.8a}$$

which imply, explicitly,

$$B e^{-k_\perp z'} = C e^{k_\perp z'} + D e^{-k_\perp z'} , \tag{13.7b}$$

$$k_\perp B e^{-k_\perp z'} + k_\perp (C e^{k_\perp z'} - D e^{-k_\perp z'}) = 1 . \tag{13.8b}$$

The solution to this system of equations, (13.6c), (13.7b), and (13.8b), is

$$A = \frac{2}{\varepsilon+1} \frac{1}{2k_\perp} e^{-k_\perp z'} , \qquad B = - \frac{\varepsilon-1}{\varepsilon+1} \frac{1}{2k_\perp} e^{-k_\perp z'} + \frac{1}{2k_\perp} e^{k_\perp z'} ,$$

$$C = \frac{1}{2k_\perp} e^{-k_\perp z'} , \qquad D = - \frac{\varepsilon-1}{\varepsilon+1} \frac{1}{2k_\perp} e^{-k_\perp z'} . \tag{13.9}$$

Inserting these coefficients back into (13.5), we find for the reduced Green's function, g , in the two media,

$$z > 0 : \quad g = \frac{1}{2k_\perp} \left[e^{-k_\perp |z-z'|} - \frac{\varepsilon-1}{\varepsilon+1} e^{-k_\perp (z+z')} \right] , \tag{13.10a}$$

$$z < 0 : \quad g = \frac{1}{2k_\perp}\frac{2}{\varepsilon+1} e^{-k_\perp(z'-z)}$$

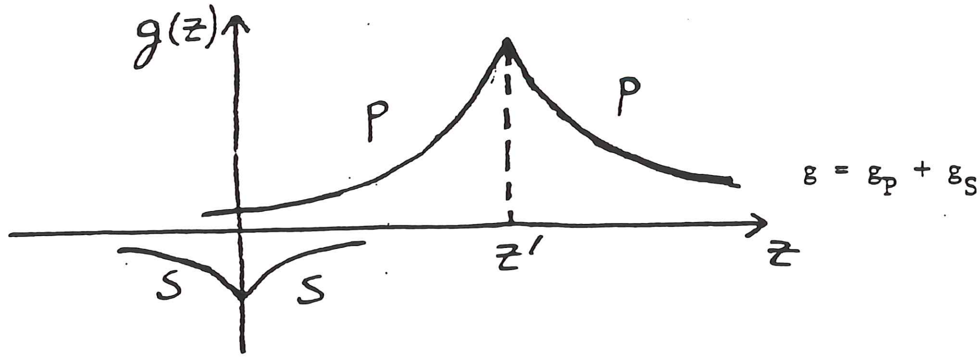
$$= \frac{1}{2k_\perp}\left[e^{-k_\perp|z-z'|} - \frac{\varepsilon-1}{\varepsilon+1} e^{-k_\perp(z'-z)} \right] . \qquad (13.10b)$$

Of course, if we set $\varepsilon = 1$, we recover the vacuum result, (12.16), as we must.

It is helpful to analyze the Green's function we have found in terms of primary and secondary fields. The primary field results from the point charge at z' and so is represented by the Green's function (12.16). The secondary field is due to the bound charge built up on the interface and is given by

$$g_S = -\frac{\varepsilon-1}{\varepsilon+1}\frac{1}{2k_\perp} e^{-k_\perp z'} e^{-k_\perp|z|} . \qquad (13.11)$$

The situation is illustrated by the following picture for $\varepsilon > 1$ (P for primary and S for secondary).



In order to easily identify the full Green's function, we note that we can write (13.10) as a sum of exponentials of the form appearing in (12.16). That is, we write

$$z > 0 : \quad g = \frac{1}{2k_\perp}\left[e^{-k_\perp|z-z'|} - \frac{\varepsilon-1}{\varepsilon+1} e^{-k_\perp|z-\bar{z}'|} \right] , \qquad (13.12a)$$

$$z < 0 : \quad g = \frac{1}{2k_\perp} \frac{2}{\varepsilon+1} e^{-k_\perp |z-z'|} \quad , \tag{13.12b}$$

where we have introduced $\bar{z}'$, the image point of z' , defined by

$$\bar{z}' = -z' \quad . \tag{13.13}$$

In the two regions, we can interpret (13.12) as follows. For $z > 0$, the Green's function appears to describe the potential due to two point charges, one, of magnitude unity, at $\vec{r}' = (x',y',z')$, and another, the image charge, of magnitude $-\frac{\varepsilon-1}{\varepsilon+1}$, at the image point $\bar{\vec{r}}' = (x',y',\bar{z}')$. For $z < 0$, only one point charge appears, of magnitude $\frac{2}{\varepsilon+1}$, located at $\vec{r}'$. In either medium, the total effective charge is the same. With this interpretation, the full Green's function may be written down immediately,

$$z > 0 : \quad G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{\varepsilon-1}{\varepsilon+1} \frac{1}{|\vec{r}-\bar{\vec{r}}'|} \quad , \tag{13.14a}$$

$$z < 0 : \quad G(\vec{r},\vec{r}') = \frac{2}{\varepsilon+1} \frac{1}{|\vec{r}-\vec{r}'|} \quad . \tag{13.14b}$$

Lecture 12

13-2. Force between Charge and Dielectric

Having calculated Green's function, we can now consider the energy of interaction between charges and the dielectric. The total energy, (11.8),

$$E = \frac{1}{2} \int (d\vec{r})(d\vec{r}') \rho(\vec{r}) G(\vec{r},\vec{r}') \rho(\vec{r}') \quad , \tag{13.15}$$

includes the mutual interactions of the charges. We are not interested in this but rather in the <u>change</u> of the energy due to the introduction of the dielectric. To calculate this change, we let G_0 be Green's function in vacuum, while G is Green's function in the presence of the dielectric, as found above. Therefore, the interaction energy between the charge distribution and the dielectric is

$$E \text{ (due to dielectric)} = \frac{1}{2}\int (d\vec{r})(d\vec{r}')\, \rho(\vec{r})\, [G(\vec{r},\vec{r}') - G_0(\vec{r},\vec{r}')]\, \rho(\vec{r}') \quad . \tag{13.16}$$

Evaluating this for a point charge at position $\vec{r}_0$, with $z_0 > 0$,

$$\rho(\vec{r}) = e\delta(\vec{r}-\vec{r}_0) \quad ,$$

and making use of (13.14a), we find the energy of interaction to be

$$E = \frac{e^2}{2}\, [G(\vec{r},\vec{r}') - G_0(\vec{r},\vec{r}')]_{\vec{r},\vec{r}'=\vec{r}_0} = -\frac{e^2}{2}\,\frac{\varepsilon-1}{\varepsilon+1}\,\frac{1}{2z_0} \quad . \tag{13.17}$$

Is this a physically meaningful result? If $\varepsilon > 1$, $E < 0$ so that there is a force of attraction pulling the dielectric toward the charge and into the region of higher fields, in agreement with the earlier discussion of Subsection 10-2. The magnitude of this force is

$$F = -\frac{\partial E}{\partial(-z_0)} = \frac{\varepsilon-1}{\varepsilon+1}\,\frac{e^2}{-z_0^2} = \frac{\left|-\frac{\varepsilon-1}{\varepsilon+1}\, e\right| |e|}{(2z_0)^2} \quad , \tag{13.18}$$

which can be interpreted as the force between the charge and the image charge.

The field point of view, as opposed to that of action-at-a-distance, provides an alternate derivation of this result. To calculate the force on the dielectric, we calculate the normal component of the flow of momentum into the dielectric. In terms of the stress tensor, this force is

$$F = -\int dxdy\, T_{zz} \tag{13.19}$$

where the integration is over a surface just outside the dielectric, at $z = +0$. Correspondingly, we use the vacuum form of T_{zz}, (3.11),

$$T_{zz} = -\frac{1}{8\pi}\left(E_z^2 - E_x^2 - E_y^2\right) . \tag{13.20}$$

Since Green's function is the potential of a unit point charge, the electric field is

$$\begin{aligned} \vec{E}(\vec{r}) &= -\vec{\nabla} e\, G(\vec{r},\vec{r}_0) \\ &= e\left[\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} - \frac{\varepsilon-1}{\varepsilon+1}\frac{\vec{r}-\bar{\vec{r}}_0}{|\vec{r}-\bar{\vec{r}}_0|^3}\right] . \end{aligned} \tag{13.21}$$

If we adopt a coordinate system such that

$$\vec{r}_0 = (0,0,z_0) , \quad \vec{r} = (x,y,0) = (\vec{r}_\perp,0) ,$$

we find

$$\vec{E}_\perp = \frac{2e}{\varepsilon+1}\frac{\vec{r}_\perp}{(x^2+y^2+z_0^2)^{3/2}}$$

$$E_z = -\frac{2\varepsilon e}{\varepsilon+1}\frac{z_0}{(x^2+y^2+z_0^2)^{3/2}} .$$

Using polar coordinates on the surface,

$$r_\perp^2 = x^2 + y^2 = \rho^2 ,$$

$$dxdy = 2\pi\rho d\rho ,$$

we easily evaluate the force to be

$$F = \frac{e^2}{8\pi} \int_0^\infty 2\pi\, \rho d\rho \left[\frac{4\varepsilon^2}{(\varepsilon+1)^2} \frac{z_o^2}{(\rho^2+z_o^2)^3} - \frac{4}{(\varepsilon+1)^2} \frac{\rho^2}{(\rho^2+z_o^2)^3} \right]$$

$$= \frac{\varepsilon-1}{\varepsilon+1} \frac{e^2}{4z_o^2} , \tag{13.22}$$

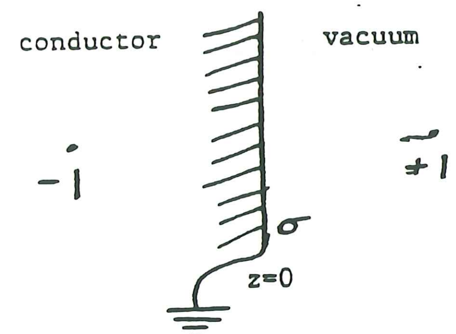
in agreement with (13.18).

XIV. Infinite Conducting Plate

In the limit that $\varepsilon \to \infty$, (13.14) becomes

$$G(\vec{r},\vec{r}\,') = \begin{cases} \dfrac{1}{|\vec{r}-\vec{r}\,'|} - \dfrac{1}{|\vec{r}-\bar{\vec{r}}\,'|} \quad , & z > 0 \; , \\[2ex] 0 \quad , & z < 0 \; , \end{cases} \tag{14.1}$$

which is obviously Green's function for a grounded conductor. For $z > 0$, Green's function can be interpreted as the potential of a unit point charge at $\vec{r}\,' = (0,0,z')$ and an image charge of strength -1 at $\bar{\vec{r}}\,' = (0,0,-z')$.



For such a unit point charge, we now calculate the surface charge density, σ , induced on the conductor. We know, from (10.31), that

$$4\pi\sigma = E_z\Big|_{z=+0} = -\frac{\partial}{\partial z} G\Big|_{z=+0} \quad . \tag{14.2}$$

We have two forms for Green's function:

$$\begin{aligned} G(\vec{r},\vec{r}\,') &= \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{1}{|\vec{r}-\bar{\vec{r}}\,'|} \\ &= 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \frac{1}{2k_\perp}\left[e^{-k_\perp|z-z'|} - e^{-k_\perp(z+z')} \right] . \end{aligned} \tag{14.3}$$

From the first form, we learn that $(\rho^2 = x^2 + y^2)$

$$4\pi\sigma = -\frac{2z'}{(\rho^2+z'^2)^{3/2}} \quad , \tag{14.4}$$

while from the second form

$$4\pi\sigma = -4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp}\, e^{-k_\perp z'} \quad . \tag{14.5}$$

The equivalence of (14.4) and (14.5) will be exploited in the next Section. For now, let us check that in both cases the total induced charge on the surface of the conductor is -1 , the strength of the image charge. The first form, (14.4), yields for the total charge

$$Q = \int_0^\infty 2\pi\,\rho d\rho \left(-\frac{1}{2\pi} \right) \frac{z'}{(\rho^2+z'^2)^{3/2}} = -1 \quad , \tag{14.6}$$

while the second form, (14.5), implies

$$\begin{aligned} Q &= -\int dxdy\, \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\, e^{ik_x(x-x')}\, e^{ik_y(y-y')}\, e^{-\sqrt{k_x^2+k_y^2}\,z'} \\ &= -\int \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\, 2\pi\,\delta(k_x)\, 2\pi\,\delta(k_y)\, e^{-\sqrt{k_x^2+k_y^2}\,z'} \\ &= -1 \quad . \end{aligned} \tag{14.7}$$

In deriving (14.7), we have used the fact that [cf. (12.7b)], for example,

$$\int_{-\infty}^{\infty} d(x-x')\, e^{ik_x(x-x')} = 2\pi\,\delta(k_x) \quad . \tag{14.8}$$

XV. Bessel Functions

Useful mathematical identities can be obtained if we solve a physical problem by using different representations for the Green's function. In particular, through the consideration of situations where physical quantities vary only in a single direction, we learn of the properties of the important class of functions called Bessel functions. An illustration of this was encountered in the last section, where we obtained two forms for the surface charge density, (14.4) and (14.5), the identity of which may be written as (for $z > 0$)

$$-\frac{2z}{(\rho^2+z^2)^{3/2}} = -4\pi \int \frac{k dk\, d\phi}{(2\pi)^2} e^{ik\rho \cos\phi} e^{-kz} \quad , \tag{15.1}$$

where $\rho = |(\vec{r}-\vec{r}\,')_\perp|$ and we have introduced polar coordinates for $\vec{k}_\perp$. The ϕ integral here is defined as the Bessel function of zeroth order, J_0,

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{ik\rho \cos\phi} = J_0(k\rho) \quad , \tag{15.2}$$

in terms of which (15.1) may be expressed as the following relation satisfied by J_0.

$$\frac{z}{(\rho^2+z^2)^{3/2}} = \int_0^\infty k dk\, J_0(k\rho)\, e^{-kz} \quad . \tag{15.3}$$

Since this result was obtained by equating z derivatives [cf. (14.2)], it may be immediately integrated to yield

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty dk\, J_0(k\rho)\, e^{-k|z|} \quad . \tag{15.4}$$

(The integration constant vanishes since both sides go to zero as $|z| \to \infty$.) Note that (15.4) may be directly derived by equating the two Green's functions (12.15,16) and (12.3). Here we recognize two different forms for the potential of a unit point charge at the origin, which satisfies Laplace's equation,

$$\nabla^2 \phi = 0 \quad , \tag{15.5}$$

except at the origin. Actually, the integrand of the right hand side of (15.4) is also a solution of Laplace's equation, that is

$$\nabla^2 [J_0(k\rho)\, e^{-k|z|}] = 0 \quad , \tag{15.6}$$

for $\rho \neq 0$, $z \neq 0$. To prove this, it is convenient to express the Laplacian in cylindrical coordinates,

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \quad , \tag{15.7}$$

leading to the reduced form for (15.6) ,

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + k^2\right] J_0(k\rho) = 0 \quad , \tag{15.8}$$

or, with $t = k\rho$,

$$\left[\frac{1}{t}\frac{d}{dt}\left(t\frac{d}{dt}\right) + 1\right] J_0(t) = 0 \quad . \tag{15.9}$$

Inserting here the integral representation of the Bessel function, (15.2), and multiplying by t , we have

$$\int_0^{2\pi} \frac{d\phi}{2\pi} [-t \cos^2\phi + i \cos\phi + t]\, e^{it \cos\phi}$$

$$= \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{d}{d\phi} [i \sin\phi \, e^{it \cos\phi}] = 0 \quad , \tag{15.10}$$

thereby proving (15.6), and establishing the equation satisfied by the Bessel function of order zero, (15.9).

Looking back at (12.15) and (12.16), we see that the basic solution to Laplace's equation that we have here encountered is

$$e^{i\vec{k}_\perp \cdot \vec{r}_\perp} e^{-k_\perp |z|} \quad . \tag{15.11}$$

We integrated this over ϕ to obtain $J_0(k\rho)\, e^{-k|z|}$, but even without such integration this satisfies Laplace's equation for $z \neq 0$. This is obvious since we have the effective replacements

$$\nabla_\perp^{\,2} \to -k_\perp^{\,2} \ , \quad \left(\frac{\partial}{\partial z}\right)^2 \to k_\perp^{\,2} \quad .$$

These basic solutions, unlike those involving J_0, are not cylindrically symmetric. Using polar coordinates,

$$\vec{k}_\perp \cdot \vec{r}_\perp = k\rho \cos\phi \quad , \tag{15.12}$$

and defining

$$k\rho = t \quad , \tag{15.13}$$

$$e^{i\phi} = \frac{1}{i} u \quad , \tag{15.14}$$

we employ the first exponential in (15.11) as a generating function for Bessel functions of integer order, J_m,

$$e^{ik\rho\cos\phi} = e^{\frac{1}{2}t\left(u-\frac{1}{u}\right)} = \sum_{m=-\infty}^{\infty} u^m J_m(t) \quad . \tag{15.15}$$

Since the generating function is invariant under the substitution

$$u \to -\frac{1}{u} \quad ,$$

that is,

$$\sum_{-\infty}^{\infty} u^m J_m(t) = \sum_{-\infty}^{\infty} (-1)^m u^{-m} J_m(t) = \sum_{-\infty}^{\infty} (-1)^m u^m J_{-m}(t) \quad ,$$

we learn that Bessel functions of positive and negative orders are related by

$$J_{-m}(t) = (-1)^m J_m(t) \quad . \tag{15.16}$$

By writing (15.15) in terms of ϕ ,

$$e^{ik\rho\cos\phi} = \sum_{-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho) \quad , \tag{15.17}$$

and using the orthogonality condition

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{-im'\phi} e^{im\phi} = \delta_{mm'} \quad , \tag{15.18}$$

we obtain an integral representation for J_m ,

$$i^m J_m(k\rho) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(k\rho\cos\phi - m\phi)} \quad , \tag{15.19}$$

$m \to -m$

$i^{-m} J_{-m}(k\rho) = -(\quad) = -i^m J_m$

which contains the result for J_0 , (15.2). Since this is the Fourier coefficient of $e^{ik\rho\cos\phi}$, we see that another set of solutions to Laplace's equation is

$$e^{im\phi}\, J_m(k\rho)\, e^{-k|z|} \quad . \tag{15.20}$$

The Laplacian, (15.7), acting on this solution yields the differential equation satisfied by J_m,

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) - \frac{m^2}{\rho^2} + k^2\right] J_m(k\rho) = 0 \quad , \tag{15.21}$$

or

$$\left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{m^2}{t^2} + 1\right] J_m(t) = 0 \quad . \tag{15.22}$$

Lecture 13

With an eye towards developing further properties of Bessel functions, we make some additional remarks about delta functions. We first recall the integral representation for the three-dimensional delta function, (12.8),

$$\int \frac{(d\vec{k})}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} = \delta(\vec{r}-\vec{r}\,') \quad ,$$

which has evident analogs in two and one dimensions. The defining property of the delta function,

$$f(\vec{r}) = \int (d\vec{r}\,')\, \delta(\vec{r}-\vec{r}\,')\, f(\vec{r}\,') \quad ,$$

becomes, in this representation,

$$f(\vec{r}) = \int \frac{(d\vec{k})}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \int (d\vec{r}\,') \, e^{-i\vec{k}\cdot\vec{r}\,'} f(\vec{r}\,') \quad . \tag{15.23}$$

This equation has the form of a completeness statement, that is, any function can be constructed from a linear combination of basic functions. Here these functions are $e^{i\vec{k}\cdot\vec{r}}$, with the expansion coefficients as indicated in (15.23).

The two-dimensional analog of (12.8) is

$$\int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} = \delta((\vec{r}-\vec{r}\,')_\perp) = \frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\phi-\phi') \quad , \tag{15.24}$$

where we have introduced polar coordinates $\vec{r}_\perp(\rho,\phi)$ and $\vec{r}_\perp{}'(\rho',\phi')$. Correspondingly, if we use polar coordinates for $\vec{k}_\perp(k,\alpha)$, (15.24) becomes

$$\int \frac{k\,dk\,d\alpha}{(2\pi)^2} e^{ik\rho\cos(\phi-\alpha)} e^{-ik\rho'\cos(\phi'-\alpha)} = \frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\phi-\phi') \quad . \tag{15.25}$$

We next expand the exponentials here by use of the generating function expression, (15.17) [with $\phi \to \phi-\alpha$] together with its complex conjugate [with $\phi \to \phi'-\alpha$], and perform the α integration by means of (15.18). The result of these operations is the identity

$$\int_0^\infty \frac{k\,dk}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho)\, e^{-im\phi'} J_m(k\rho') = \frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\phi-\phi') \quad . \tag{15.26}$$

Again this is a completeness statement, this time for the functions

$e^{im\phi}\, J_m(k\rho)$:

$$f(\rho,\phi) = \int_0^\infty \frac{k\,dk}{2\pi} \sum_{-\infty}^{\infty} e^{im\phi} J_m(k\rho) \int \rho' d\rho'\, d\phi'\, e^{-im\phi'} J_m(k\rho')\, f(\rho',\phi') \quad . \tag{15.27}$$

We may now easily isolate the individual ρ and ϕ dependencies of (15.26).

If we multiply (15.26) by $e^{-im(\phi-\phi')}$ and integrate over ϕ, we select a particular value of m, according to (15.18), so that we obtain

$$\int_0^\infty kdk\, J_m(k\rho)\, J_m(k\rho') = \frac{1}{\rho}\,\delta(\rho-\rho') \quad . \tag{15.28a}$$

This states that $J_m(k\rho)$ is a complete set of functions of ρ for <u>any</u> m :

$$f(\rho) = \int_0^\infty kdk\, J_m(k\rho) \int_0^\infty \rho' d\rho'\, J_m(k\rho')\, f(\rho') \quad . \tag{15.28b}$$

Putting this information back into (15.26), we determine the completeness relation for the functions $e^{im\phi}$:

$$\sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\, e^{im(\phi-\phi')} = \delta(\phi-\phi') \quad , \tag{15.29a}$$

and correspondingly

$$f(\phi) = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^{2\pi} \frac{d\phi'}{2\pi}\, e^{-im\phi'}\, f(\phi') \quad , \tag{15.29b}$$

which is the Fourier series expansion for $f(\phi)$.

A further property of the Bessel functions may be obtained by expanding $e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}$ using (15.17), and integrating over α [as we did in going from (15.25) to (15.26)]:

$$\int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp} = \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho)\, e^{-im\phi'} J_m(k\rho') \quad . \tag{15.30}$$

On the other hand, from

$$\vec{k}_\perp \cdot (\vec{r}-\vec{r}\,')_\perp = k|(\vec{r}-\vec{r}\,')_\perp| \cos(\alpha-\Phi)$$

where Φ is the polar angle for the vector $(\vec{r}-\vec{r}\,')_\perp$, the integral on the left-hand side of (15.30) [with the shift of variable $\alpha \to \alpha + \Phi$] is immediately identified as the Bessel function of zeroth order, (15.2),

$$\int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{ik|(\vec{r}-\vec{r}\,')_\perp| \cos\alpha} = J_0(k|(\vec{r}-\vec{r}\,')_\perp|) \quad , \qquad (15.31)$$

where

$$|(\vec{r}-\vec{r}\,')_\perp| = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi-\phi')} \quad . \qquad (15.32)$$

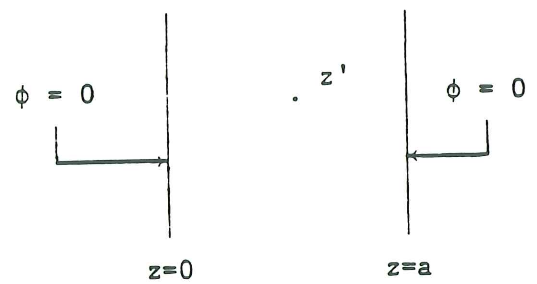
Therefore we have derived the <u>addition theorem</u> for the Bessel functions of integer order:

$$J_0[k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi-\phi')}]$$
$$= \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho)\, e^{-im\phi'} J_m(k\rho') \quad . \qquad (15.33)$$

XVI. Parallel Conducting Plates

16-1. Reduced Green's Function

Having developed some mathematical machinery, let us now turn to another essentially one-dimensional problem, that of the potential due to a point charge between two parallel grounded conducting plates, as illustrated in the figure.



The Green's function is defined by the differential equation

$$-\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{16.1a}$$

together with the boundary conditions

$$G = 0 \quad \text{at} \quad z = 0, a \quad . \tag{16.1b}$$

Since the geometry depends only on the z coordinate, the Green's function can be written in the form (12.15), where the reduced Green's function $g(z,z',k_\perp)$ satisfies (12.17). As in (13.5) the solutions are still of the form $e^{\pm kz}$ (with $k = |\vec{k}_\perp|$); the linear combinations that satisfy the boundary conditions are

$$z < z' : \quad g = A \sinh kz \quad , \tag{16.2a}$$

$$z > z' : \quad g = B \sinh k(a-z) \quad . \tag{16.2b}$$

The constants A and B are to be determined by the conditions on g in the neighborhood of $z = z'$ [recall (13.7a) and (13.8a)],

$$g \text{ continuous},$$

$$-\frac{\partial}{\partial z} g \bigg|_{z=z'-0}^{z=z'+0} = 1 \quad ,$$

which leads to the equations

$$A \sinh kz' = B \sinh k(a-z') \quad , \tag{16.3a}$$

$$kB \cosh k(a-z') + kA \cosh kz' = 1 \quad . \tag{16.3b}$$

It is convenient to satisfy (16.3a) by letting

$$A = C \sinh k(a-z') \quad ,$$
$$B = C \sinh kz' \quad , \tag{16.4}$$

which, when substituted into (16.3b) yields

$$kC \sinh ka = 1 \quad . \tag{16.5}$$

The reduced Green's function is thus found to be

$$g(z,z',k) = \frac{\sinh kz_< \sinh k(a-z_>)}{k \sinh ka} \quad , \tag{16.6}$$

where $z_>(z_<)$ is the greater (lesser) of z and z' . Note that the reciprocity condition, (11.4), is satisfied because $g(z,z',k)$ is symmetrical.

16-2. Induced Charge

One application of this result lies in calculating the charge densities on the conducting plates induced by a unit point charge at z' . According to (10.31), these charge densities are

$$z = 0 : \quad 4\pi\sigma = E_z = -\frac{\partial}{\partial z} G \ , \tag{16.7a}$$

$$z = a : \quad 4\pi\sigma = -E_z = \frac{\partial}{\partial z} G \ . \tag{16.7b}$$

As in Section XIV, it is simplest to calculate the total charge induced on each plate:

$$Q(z=0) = -\frac{1}{4\pi} 4\pi \int dxdy \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp} \frac{\partial}{\partial z} g(z,z',k)\bigg|_{z=0} \ , \tag{16.8a}$$

$$Q(z=a) = \frac{1}{4\pi} 4\pi \int dxdy \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp} \frac{\partial}{\partial z} g(z,z',k)\bigg|_{z=a} \ . \tag{16.8b}$$

We have seen such integrals previously [see (14.~~7~~) and (14.8)]. The spatial integrations over x and y yield $(2\pi)^2\,\delta(\vec{k}_\perp)$ while the subsequent $\vec{k}_\perp$ integration sets $k = 0$ so that

$$g(z,z',0) = \frac{1}{a} z_<(a-z_>) \ . \tag{16.9}$$

The total induced charges are therefore

$$Q(z=0) = -\left(1 - \frac{z'}{a}\right) \ , \tag{16.10a}$$

$$Q(z=a) = -\frac{z'}{a} \ , \tag{16.10b}$$

with the total induced charge on both plates being -1, of course.

Lecture 14

16-3. Eigenfunction Expansion

We now wish to investigate the properties of $g(z,z',k)$, (16.6), considered as a function of a complex variable k. First notice that g is even in k and is finite at $k = 0$. The behavior of g when the real part of k is large and positive is

$$g \sim \frac{\frac{1}{2}e^{kz_<}\frac{1}{2}e^{k(a-z_>)}}{k\frac{1}{2}e^{ka}} = \frac{1}{2k}e^{-k(z_>-z_<)} = \frac{1}{2k}e^{-k|z-z'|} .$$

This limiting form is the reduced Green's function for empty space, (12.16), which is evidently bounded (in fact, goes to zero) as $\operatorname{Re} k \to \infty$. The singularities of $g(z,z',k)$ occur on the imaginary axis where

$$\sinh ka = 0 , \tag{16.11}$$

that is, at the points where

$$ka = in\pi , \qquad n = \pm 1, \pm 2, \ldots . \tag{16.12}$$

The behavior of g in the neighborhood of this singularity is

$$g \sim \frac{i \sin \frac{n\pi}{a} z_< \, i \sin \frac{n\pi}{a}(a-z_>)}{i\frac{n\pi}{a}\left(k - i\frac{n\pi}{a}\right) a \cos n\pi}$$

$$= \frac{1}{n\pi}\frac{\sin\frac{n\pi}{a}z \sin\frac{n\pi}{a}z'}{ik + \frac{n\pi}{a}} , \tag{16.13}$$

that is, g possesses a simple pole there. Therefore, since apart from these poles, g is a bounded, analytic function of k, it is expressible entirely as a sum over the pole contributions given in (16.13),

$$g(z,z',k) = \sum_{n\neq 0} \frac{1}{n\pi} \frac{\sin \frac{n\pi}{a} z \sin \frac{n\pi}{a} z'}{\frac{n\pi}{a} + ik} \quad .$$

Combining together the contributions from n and $-n$, we obtain an alternative representation for the reduced Green's function,

$$g(z,z',k) = \sum_{n=1}^{\infty} \frac{2}{a} \frac{\sin \frac{n\pi}{a} z \sin \frac{n\pi}{a} z'}{k^2 + \left(\frac{n\pi}{a}\right)^2} \quad . \tag{16.14}$$

Lecture 14

To verify explicitly that (16.14) is the Green's function, we must check that it is a solution of the differential equation,

$$\left(-\frac{\partial^2}{\partial z^2} + k^2\right) g(z,z',k) = \delta(z-z') \quad , \tag{16.15}$$

satisfying the boundary conditions,

$$g = 0 \quad \text{at} \quad z = 0,a \quad .$$

The latter conditions are trivially satisfied by (16.14), while the differential equation (16.15) will be confirmed if

$$\frac{2}{a} \sum_{n=1}^{\infty} \sin \frac{n\pi}{a} z \sin \frac{n\pi}{a} z' = \delta(z-z') \quad . \tag{16.16}$$

The correctness of this follows from the completeness relation (15.29):

$$\frac{2}{a} \sum_{n=1}^{\infty} \sin \frac{n\pi}{a} z \sin \frac{n\pi}{a} z' = \frac{1}{a} \sum_{n=-\infty}^{\infty} \sin \frac{n\pi}{a} z \sin \frac{n\pi}{a} z'$$

$$= \frac{1}{2a} \sum_{-\infty}^{\infty} \left[\cos \frac{n\pi}{a} (z-z') - \cos \frac{n\pi}{a} (z+z')\right]$$

$$= \frac{1}{2a} \sum_{-\infty}^{\infty} \left[e^{i \frac{n\pi}{a} (z-z')} - e^{i \frac{n\pi}{a} (z+z')} \right]$$

$$= \frac{\pi}{a} \left[\delta\left(\frac{\pi}{a} (z-z') \right) - \delta\left(\frac{\pi}{a} (z+z') \right) \right] = \delta(z-z') , \qquad 0 < z,z' < a ,$$

where we have used the elementary fact that

$$\lambda\delta(\lambda x) = \delta(x) , \quad \lambda > 0 . \tag{16.17}$$

Notice that (16.16) is a discrete analogue of the continuum representation of the delta function, (12.7b),

$$\int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} = \delta(z-z') .$$

We have here another completeness statement, that any function, $f(z)$, defined on the interval 0 to a , and vanishing on the boundaries, can be constructed in terms of $\sin \frac{n\pi}{a} z$.

16-4. Green's Function

An eigenfunction expansion for Green's function, $G(\vec{r},\vec{r}')$, can now be obtained by substituting the corresponding form for the reduced Green's function, (16.14), into (12.15). The Bessel function of zeroth order, J_0 , (15.31), is introduced upon performing the angular integration associated with $\vec{k}_\perp$, so that Green's function becomes

$$G(\vec{r},\vec{r}') = 2 \int_0^{\infty} kdk\, J_0(kD)\, \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{a} z \sin \frac{n\pi}{a} z'}{k^2 + \left(\frac{n\pi}{a} \right)^2} , \tag{16.18}$$

where