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INTRODUCTION TO PLASMA PHYSICS

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Introduction to Plasma Physics

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I. Introduction

A plasma may be roughly defined as a material containing mobile charges, either negative or positive, or both and in which the electric and magnetic interactions between particles play a dominant role in the dynamics of the system. In most cases there are equal numbers of positive and negative charges so that the system as a whole is neutral. However, there are cases of interest in which the plasma is not neutral and even cases where it consists of a single type of charged particle; we will consider mainly neutral plasmas but will include a short discussion of non-neutral plasma. Because of the long range nature of electric and magnetic forces, plasmas exhibit collective motions; motions in which many particles move coherently. Because of this the physics of plasmas is extremely rich as we will see during this course.

Some examples of plasmas are:

(1) Gases which are heated to such a high temperature that some or all electrons are detached from the constituent atoms and molecules. Examples are fusion reactor plasmas, plasmas generated by strong shock waves, plasmas created by laser heated materials, etc.

(2) The gas in any discharge. Here the electrons gain sufficient energy from the applied electric field that they can ionize other atoms and

molecules. The discharge can be DC or AC. Examples are lightning, fluorescent lights, neon lights, all types of laboratory discharges, and many others.

(3) Interstellar gas which is ionized by the ultraviolet light from stars. A special case is the ionization of interstellar gas by ultraviolet and x-rays from novas and supernovas. These plasmas have quite low temperatures.

(4) Plasma blowing off of stars (the solar wind, stellar winds). Plasma produced by the ionization of interstellar gas by energetic particles generated by pulsars (the Crab Nebula, for example).

(5) The ionosphere of the Earth and other planets produced by ultraviolet radiation from the Sun.

(6) The mobile electrons in metals and semiconductors. Here the perturbations caused by neighboring atoms weaken the binding of the electrons to the atoms to such an extent that some of the electrons are free to move through the material.

(7) The hot dense material in the interior of stars. This can range from the material at the center of the Sun ($T = 10^7$K, $\rho = 100$ gm/cm^3) to that in white dwarfs ($T = 10^7$K, $\rho = 10^6$ gm/cm^3) to that at the core of a supernova ($T = 2 \times 10^{10}$K, $\rho = 10^{15}$ gm/cm^3).

In addition, there are systems which contain mobile charges but which are not generally classified as plasmas. Examples of systems of this type are salt solutions, gases heated to temperatures where there are very few free electrons, and the free electrons in semiconductors where collisions are important. Systems of this type contain so many neutral molecules and collisions between the charged particles and neutrals are so dominant that many of the properties that we generally associate with plasmas (namely collective motions) are masked. However, such systems can exhibit some of the properties of plasmas and some plasma concepts are useful for such systems. A system can pass continuously from a non-plasma state to a plasma state. (For example, by gradually raising the temperature of a gas.)

From the above examples we see that the plasma state encompasses an enormous range of physical parameters ranging from temperatures of 100's of degrees Kelvin to more than 10^{10}K, and densities from a few particles per m^3 to 10^{38} per cm^3.

Plasma has been called the fourth state of matter. In spite of the fact that there is no sharp transition from a gas to the plasma state, there is much justification for this point of view. This is because plasmas behave so differently from the non-conducting gases and fluids that we are much more familiar with. The mobile charges in a plasma allow it to carry electric current. It thus interacts strongly with electric and magnetic fields. This leads to a great wealth of phenomena not exhibited by non-plasmas. The physics of plasmas is much richer than that of air and water which we are much more familiar with. Until quite recently man had little direct experience with plasma. He was of course familiar with lightning and he could see the sun and stars but he could not handle it or manipulate it; he could not touch it or feel it or control it in any way. Only in the last 100 years with the rise of the electrical industry and the electronics industry has man been able to experiment with plasma. In fact only in the last 40 or 50 years or so with the development of the controlled fusion program and the exploration of space has man started to investigate plasmas in detail and discover just how rich and complex their physics is.

To produce a plasma by heating a gas, we must heat it to such a temperature that collisions between particles are able to knock off electrons. Since first ionization energies for atoms range from roughly 4 eV for Cs to 24 eV for He, the colliding energies must be comparable. A temperature of 11,600° K corresponds to an average thermal energy of 1 eV (kT = 1 eV), therefore we see that temperatures of thousands of degrees are required.

We do not strictly require kT to equal the ionization temperature to obtain appreciable ionization. It is well known that the thermal energy distribution for molecules in a gas (taken to be classical) is a Maxwell Boltzmann distribution,

$$P(E) = A \exp(-E/kT) \quad , \tag{1}$$

where P(E) is the probability of having an energy E and A is a normalizing constant. In a gas at thermal equilibrium, there are always particles with considerably more energy than the average and collisions between these particles can produce ionization. The electron density is determined by a balance between such ionization and recombination of the ions and electrons. In a low-density gas, the electrons and ions rarely approach each other and recombination can be quite slow (later we will give some estimates of what these rates are). Thus, a substantial percentage of ionization can exist even when kT << E(ionization).

For a gas in thermal equilibrium, the degree of ionization depends only on the temperature. Every process that can produce ionization can take place in reverse to give recombination. In thermal equilibrium, each process and its inverse takes place at exactly the same rate (the law of detailed balance). For most laboratory plasmas, one does not have thermal equilibrium; if nothing else, thermal equilibrium would require enormous levels of radiation. A plasma at a temperature of 10 eV (by the Stephan Boltzmann Law) would radiate 6 x 10^9 watts per square cm. of its surface. Needless to say, to maintain a plasma radiating at such a rate would require a huge input power; fortunately, low density laboratory plasmas are optically thin and radiate much more feebly than this. They are not in radiation equilibrium and often are not in equilibrium in other ways; we will examine these shortly.

For thermal equilibrium the degree of ionization is determined by the Saha equation. This equation has the form

$$(n_e\ n_i\ /n_0) \simeq (1/\lambda^3 g_i)\ \exp(-E_i/kT\ )\ , \qquad (2)$$

$$\lambda^2 = h^2/2m_e kT$$

where λ^2 is the square of the DeBroglie wave length for the average electron energy.

Here n_e, n_i, n_o are the electron, ion and neutral densities, E_i, is the ionization energy, g_i , is the the number of ground state levels of the atom, m_e is the electron mass, and h is Planck's constant. A derivation of this formula is outside the scope of this course (see Kittel, Statistical Mechanics), but we will point out the physical significance of the terms.

For a singly ionized substance with overall charge neutrality, $n_e = n_i$ and Eq. (2) can be written in the form

$$(n_e / n_0) = (V_e / \lambda^3 g_i) \exp(-E_i / kT) \ , \tag{3}$$

where $V_e = 1/n_e$ is the volume per electron. Now the number of free (continuum) states for an electron in a volume V is proportional to V/λ^3; thus, V_e/λ^3 is the number of free states per electron (number of free states for a volume containing one electron). If there is one free electron in the volume V_e, then there is one ion and this ion has g_i ground states. The probability that the electron is in one of the g_i ground states is $\exp(E_i/kT)$ times larger than the probability that it is in a free state so the ratio n_e/n_o is given by Eq. (3). In this discussion we have neglected bound states other than the ground state; these are less populated than the ground state by the appropriate Boltzmann factor. Since they are less numerous than the free states, they generally can be neglected.

Besides thermal equilibrium plasmas there are many types of plasmas which are not in thermal equilibrium. Examples of these are:

(1) most electrical discharges (fluorescent lights, neon signs, Aurora, sparks, etc.),

(2) interstellar gas which is ionized by radiation from stars,

(3) the ionospheres of planets,

(4) the solar and stellar winds and many others.

For discharge-type plasmas, the electrons gain enough energy from electric fields so that they can knock other electrons from the constituent atoms of the gas. This requires the electron energies in the discharge to be roughly a few eV as ionization energies of atoms and molecules range from 3.9 eV for Cs to 24.5 eV for He (it is, of course, the energetic tails of the electron distributions that do the ionizing). Thus, the electron temperature in a florescent light is about 30,000° K; the ions on the other hand are only slightly hotter than room temperature because they are in good thermal contact with the walls of the tube and the electrons transfer energy to them slowly. It is difficult to heat the electrons to a much higher temperature because they lose so much energy in ionizing and exciting the atoms inside the discharge tube. These losses can only be overcome with very high powers and currents; any mechanism that insulates the discharge from its cold surroundings, such as magnetic plasma confinement, can greatly help in reducing power requirements.

The plasmas in gas discharges are rarely in thermal equilibrium. The mean free paths for particle encounters are often comparable to the size of the container. Thus, ions and neutral atoms, though slightly heated by collisions with hot electrons, remain pretty much at the temperature of the walls; wall temperatures are typically of the order of (1/30) eV. As we shall see later, electric fields develop naturally near the walls that tend to confine the electrons to the discharge and isolate them from the walls.

Virtually all laboratory plasmas are not in thermal equilibrium with the electromagnetic radiation field. The self-absorption lengths are so long and the rates of emission are so low that most radiation freely leaves the plasma; there is not enough room for the plasma to reach a balance between emission and absorption (the plasma is optically thin.) If this were not true the plasma in a florescent light would be 600 times brighter than the surface of the sun.

Even though the radiation levels are low compared with black body levels, a lot of energy is carried off by radiation. The electrons, in colliding with the atoms and molecules of the gas, raise their bound electrons to excited states. Many excited states give up their energy immediately as radiation. Since energetic electrons are generally more efficient at exciting and ionizing, the energetic electrons tend to be brought down to lower energies by these processes and their number is depleted. Thus, such discharges contain fewer energetic electrons than would a plasma in thermal equilibrium. The velocity distribution for electrons, f(v), in a thermal equilibrium plasma is a Maxwellian

$$f(v) \propto \exp(-m_e v^2/2kT) \quad , \tag{4}$$

where v is the electron velocity, k is Boltzmann's constant, and T is the temperature. Such a distribution function is shown in Fig. 1.

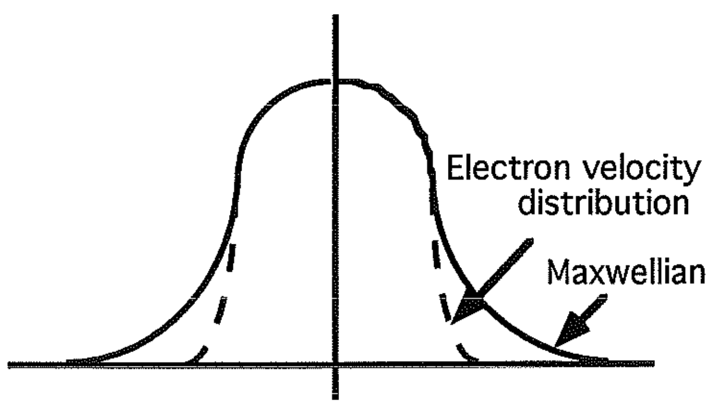


Figure 1

For a gas discharge plasma the tail dies out more rapidly; a distribution that is often used is the Druyvesteyn distribution

$$f(v) \propto \exp(-v^4/v_0{}^4) \qquad (5)$$

where v_0 is determined by the average electron energy.

As we have already said, for a plasma in thermal equilibrium, all processes satisfy the condition of detailed balance; that is every process and its inverse are exactly in balance. For a non-equlibrium plasma, this no longer holds. For example, we can have ionization taking place in the plasma and recombination taking place at the walls. If the plasma is in steady-state, then the rate of ionization must equal the rate of recombination but different processes can be involved in ionization and recombination. The distributions, the excited levels of atoms, the ionization levels, the chemical species and the radiation levels are all strongly dependent on cross-sections. It is the non-equlibrium properties of such plasmas which make them so rich in phenomenon and useful in practice (for example, lasers, plasma etching, light sources, etc). Because of the large number of processes that are involved in gas discharges and their various cross-sections, the analysis of non-equlibrium plasmas can become very complex. Most of these notes will be devoted to the case of fully ionized plasmas where the multitude of atomic processes need not be considered. However, atomic processes do play a central role in nearly all laboratory plasmas and in many natural plasmas. Therefore, we will give some attention to these processes using a semi-classical approach which gives reasonable estimates of their size and which can be used to estimate the role and importance of such processes in situations of interest.

II. Particle Interactions, Coulomb Collisions, Ionization, Excitation, Recombination, Radiation

(a) Debye Shielding

As already mentioned, in order for us to produce a plasma, particle energies (electron, ion, atomic, and photon) must be at least comparable to the ionization energy of the constituent atoms. The electrons which are detached will have similar energies (energies of 10% or even 1% might be considered similar). Furthermore, when the electron is removed from an atom, it wanders around in the space between atoms and in general is quite far away (compared to atomic dimensions) from the positive ions. The electron ion potential energies are quite small compared to the ionization energies throughout most of space so that the electron kinetic energy is much greater than the potential energy almost everywhere. For example, consider a laboratory Q machine plasma of Cs with an electron density of $n_e = 10^{12}/cm^3$ and at a temperature of 2500° K (0.22 eV). From the Saha equation, Eq. 2, we find the degree of ionization is very high. Now the mean distance between an electron and an ion is 10^{-4} cm. This corresponds to a potential of

$$e^2/r = (e^2/a_0)(a_0/r) = 27\text{eV}\ (5 \times 10^{-9}/10^{-4}) = 1.35 \times 10^{-3}\ \text{eV}\ . \qquad (6)$$

Here a_0 is the Bohr radius for an electron in a hydrogen atom and 27 eV is its potential energy at that radius. The electron kinetic energy is more than two orders of magnitude larger than this potential energy.

As they move through a plasma, the free electrons are usually in regions of small potential. The effect of the electric fields on their trajectories is small except during the relatively rare close approaches, i.e., during a collision, of an electron to an ion or another electron; the same is true for ions if they have even a modest temperature (in the example above, even if the ions are at room temperature [.03 eV] their mean kinetic energy is more than 20 times the mean potential energy between two ions).

While the interactions between pairs of charged particles is generally quite small, the sum of the fields of many ions or electrons can be important. This is because of the long range nature of the Coulomb interaction which falls off as r^{-2}. If we consider the number of electrons in a spherical shell of radius r and thickness dr centered on a given electron, then this number is equal to $4\pi n_e r^2 dr$. We see that this number times r^{-2} is independent of r. Of course, if the electrons are distributed evenly over the sphere, the electric field at the center will be zero. However, if we move only a small fraction of the electrons, say 1%, from the left-hand side to the right-hand side of all the spheres, then each spherical shell will contribute as much to the electric field as every other shell. If this continues out to some large distance, a very large electric field will be produced. Of course, since the electrons (and ions) are free to move, they quickly move from regions where there are excess electrons towards regions where there are excess ions and thus attempt to neutralize regions of charge imbalance. Actually the electrons do not stop once charge balance is established; because of their motion they overshoot the charge balance state. Regions that start with an excess of electrons end up with a deficit while regions that initially had a deficit end up with an excess. The resulting electric fields stop the electron motions; the process then repeats in the reverse direction; the electron density oscillates about the neutral state. Such charge density oscillations are called plasma oscillations; the characteristic frequency for these oscillations is the so-called plasma frequency, ω_p

$$\omega_p{}^2 = 4\pi n_e e^2/m_e \ , \tag{7}$$

$$\omega_p \simeq 6 \times 10^4 (n_e)^{1/2} \quad ,$$

where ω_p is in sec^{-1} and n_e is in cm^{-3}.

To gain an idea of how strong space charge effects are in a plasma, let us consider an infinite homogeneous plasma and let us ask how much energy it would take to move all the electrons in a spherical region of radius r to its surface. The situation is illustrated in Fig. 2.

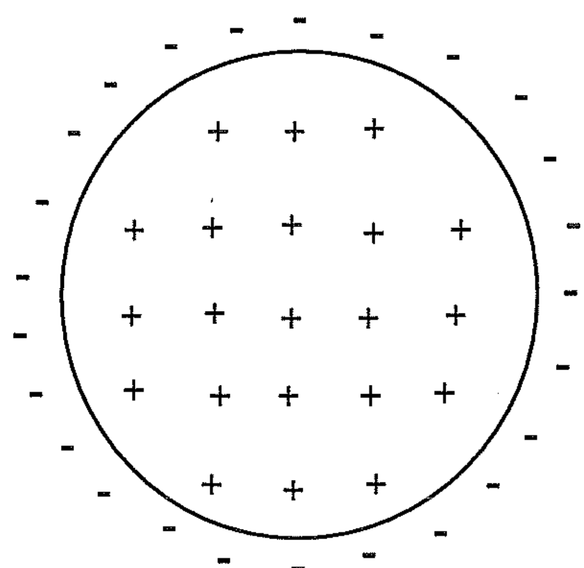


Figure2

We will hold the ions fixed at their initial uniform density. The potential energy is stored in the electric field; energy density equals $E^2/4\pi$; the total potential is given by

$$W = \int \frac{E^2}{8\pi} dV$$

The electric field is radial outward from the center of the sphere and is given by

$$E(\rho) = (4/3)\ \pi n_i\ Ze\rho, \qquad \rho < r \tag{8}$$

$$0, \qquad \rho > r.$$

Carrying out the integration gives

$$W = (8/45)\pi^2 n_e{}^2 e^2 r^5 \tag{9}$$

Let us equate W to the electron kinetic energy in the sphere.

$$K = (4/3)\pi n_e r^3 (3/2)kT = W$$

$$r^2 = 45(kT/4\pi n_e e^2). \tag{10}$$

This is the radius of a sphere where the electron thermal energy is just enough to remove themselves from the sphere; such a motion is energetically possible but extremely unlikely. For larger spheres there is not enough energy to separate the electrons from the ions while for smaller spheres there is sufficient energy. If the sphere is one tenth this big, only one percent of the kinetic energy is required for the electrons removal. For smaller spheres we can expect that large fluctuations in electron density will take place because of the random motions of the electrons; the fluctuations will be essentially the same as those that would occur in a neutral gas

($\delta N^2 \propto N$). For larger spheres the deviations from neutrality will be greatly suppressed due to the large amount of energy that is required to create them.

The quantity λ_D given by the expression

$$\lambda_D{}^2 = kT/4\pi n_e e^2 = v_T{}^2/\omega_p{}^2 , \tag{11}$$

$$\omega_p{}^2 \equiv 4\pi n_e e^2/m ,$$

and

$$v_T{}^2 = kT/m$$

is called the Debye length (it was first introduced in the 1930's by Debye in the study of electrolytes). It is a measure of the size of a region in which appreciable deviations from charge neutrality occur in a thermal plasma. The quantity ω_p is called the plasma frequency (originally defined by Tonks and Langmuir) and it gives the basic response frequency of the electrons to applied electric fields as we shall see presently. The Debye length is the electron thermal velocity divided by the plasma frequency. A thermal electron moving through a plasma pushes other electrons away; at a distance of a Debye length or greater, the plasma electrons have time to adjust to the passing electron so as to neutralize its electric field and shield the rest of the plasma from its presence. We have made this argument for an electron. Later on in the course we will see that the Debye length is also the distance at which an ions field is shielded out, even though the ion is moving much slower. In this case the electrons move past the ion before they can respond for distances less than the Debye length.

It is illuminating to write Eq. (11) in the following form,

$$\lambda_D{}^2/d^2 = kTd^3/d^2 4\pi e^2 = kT/(4\pi e^2/d) = kT/4\pi\phi[d] \ , \tag{12}$$

where $d^3 = 1/n_e$; d is the inter-particle spacing and $\phi[d]$ is the potential of an electron (or ion) evaluated at a separation d. We have already shown that in a plasma kT is much larger than $\phi[d]$. Hence, λ_D is large compared to d. Thus, in a plasma there are many particles in a Debye sphere (sphere of radius λ_D). For the Cs plasma example given, T = 2500° K, and $n_c = 10^{12}$, $\lambda_D = 5d$ and the number of particles per Debye sphere is, $N_D = 600$; for a typical Fusion plasma, T = 10^8 K, $n_e = 10^{14}$, $\lambda_D \sim 400d$ and $N_D = 2.5 \times 10^8$; at the center of the sun, T = 10^7 K, $n_e = 6 \times 10^{25}$, $\lambda_D \sim 16d$.

The quantity ω_p in Eq. (11) is the frequency at which electrons can respond to a disturbance or it is the natural frequency of vibration of the electron density fluctuations. We can see this from the following simple calculation. Consider an infinite uniform plasma. Let us imagine that we displace all the electrons in a slab of plasma of thickness L (the slab is perpendicular to x and infinite in the y, z directions) by a distance δ. See Fig. 3. Then we will create two regions, one will be charged positive since the electrons have been moved out of that region and the other will be charged negative since we have moved extra electrons into that region. The total charge per unit area σ contained in these areas is $\pm\, en_e\delta$ and the electric field in the region of the slab is (the situation is like that of a parallel plate capacitor)

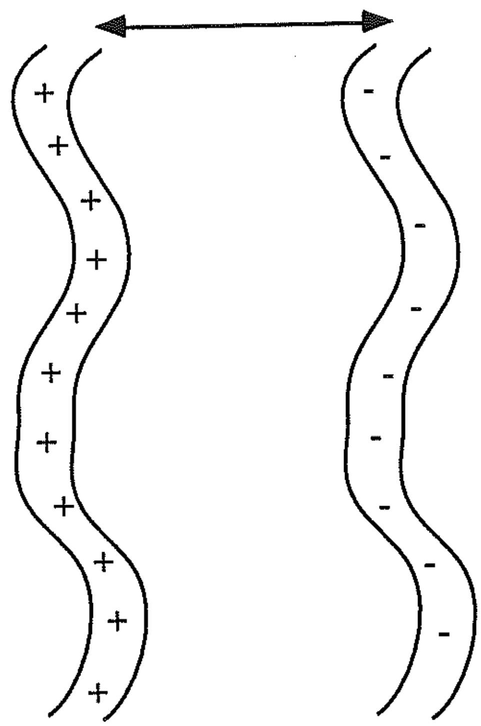


Figure 3

$$E = 4\pi\sigma = 4\pi e n_e \delta \ . \tag{13}$$

The force on the electrons in the slab is

$$F = -eE = -4\pi e^2 n_e \delta \ . \tag{14}$$

From Newton's laws of motion, we have

$$m_e d^2x/dt_2 = m_e d^2\delta/dt^2 = - 4\pi e^2 n_e \delta \ , \tag{15}$$

and

$$d^2\delta/dt^2 = -(4\pi e^2 n_e/m_e)\delta = -\omega_p{}^2\delta \ , \tag{16}$$

This is the equation of motion of a simple harmonic oscillator which oscillates at the plasma frequency. In numerical terms we have

$$\omega_p = 5.64 \times 10^4 \ n_e{}^{1/2} \ , \tag{17}$$

and

$$f_p = \omega_p/2\pi = 8.97 \times 10^3 \ n_e{}^{1/2} \ , \tag{18}$$

The force on the ions is of equal magnitude but oppositely directed; however, due to their large mass, their motion is relatively small and has been neglected here. We have also neglected effects associated with the random thermal motions of the electrons; these will be taken up later in the course.

The plasma frequency is a measure of the minimum response time of the electrons to an electric field, $\tau_p \sim 1/\omega_p$. For times longer than τ_p, the electrons will move in such a way as to reduce an applied electric field. Thus, a plasma is transparent to electromagnetic waves with frequencies higher than the plasma frequency; below the plasma frequency the plasma acts like a good conductor and electromagnetic waves in this region are reflected; we will study this in much more detail when we get to plasma waves.

Of course if the collision frequency is higher than the plasma frequency, such motions are impeded and electron density oscillations are quickly damped. The plasma also becomes an absorbing medium for electromagnetic waves. For hot plasmas, as we have already emphasized, the particle kinetic energy is much larger than the potential energy of interaction between pairs of particles so their deflections, as they pass one another, are small (collisions are weak). Thus, most plasmas of interest are only weakly collisional and plasma oscillations are an important aspect of their behavior.

III. Collisions and Atomic Processes

It is not possible to take collisions completely into account in describing a plasma. First of all there are a very large variety of collisions. Secondly, for plasmas containing atoms and molecules that have not been fully striped of their electrons, relatively few cross-sections are well known, either on the basis of theory or from experimental measurements. Recent research has greatly increased our knowledge in this area and ongoing research continually adds to it. However, the number of possibilities is so enormous that this will be a field of research for a long time to come. Despite this it is possible to make reasonable estimates of many types of collisions and these prove very useful when planing experiments or when trying to interpret experimental data.

In this section we will provide a brief treatment of the most important types of collisions involved in laboratory plasmas and we will show how to make reasonable estimates of the sizes of the various cross-sections. The intent is to provide sufficient information to enable order-of-magnitude estimates of collisional effects to be made. If such estimates indicate that particular collisional processes are important for situations of interest, then detailed cross-sections can be sought in the literature.

Data on cross-sections is given in several different units in the literature. Much recent data is given in cm^2/atom. Also in common use is the dimension $\pi a_0^2 = 0.88 \times 10^{-16}$ cm^2, where a_0 is $h^2/m_e e^2$, the Bohr radius of the hydrogen atom. Since atomic cross-sections are frequently in the region of 10^{-16} cm^2, Angstroms squared, Å^2, are used. In the older literature, particularly with respect to elastic collisions, cross-sections are given in terms of collision probability per cm per unit of pressure (mm Hg at 0° C). To change them into cm^2/molecule, one must divide by the number of molecules per cm^3 at a pressure of 1 mm Hg; this is 3.53×10^{16}.

If we watch a given "test particle" proceed through a region where the density of target or field particles is n, then the probable number of a given type of collision per cm of path length is simply σn, where σ is the cross-section for the particular type of interaction. The average distance traveled between collisions, or mean free path, is then $1/n\sigma$. The number of collisions per second, collision frequency, is $n\sigma v$, where v is the relative velocity of the test particle through the field particles. If the field particles are in motion or if we are concerned with the average behavior of a large number of test particles moving at different velocities, then since σ (for most processes) is a function of velocity we must average σv over all velocities. Thus, if $f(v)d^3v$ is the number of particles having velocities in a small volume of velocity space d^3v about a velocity $\mathbf{v}$ and our test particle has a velocity $\mathbf{v}_0$, then we may write

$$\int f(\mathbf{v})\sigma(\mathbf{v}-\mathbf{v}_0)|\mathbf{v}-\mathbf{v}_0|\, d^3v = n\langle\sigma v\rangle \quad . \tag{19}$$

The quantity $n\langle\sigma v\rangle$ is called the rate coefficient. If instead of one particle there are n_t test particles per cubic centimeter, we must multiply Eq. (19) by n_t to get the total number of reactions of the type described by σ. If there is more than one type of reaction, then we must make similar calculations for every process involved.

Coulomb Collisions

This is a process of great importance to us; not only is it a basic process that goes on in fully ionized plasmas but it will also provide us with the basis for estimating a wide variety of other collisional processes.

We consider the problem of a light particle (test particle) of charge ze and mass m approaching at a velocity v_0 a stationary heavy (immobile) particle (field particle) of charge Ze. The situation is shown in Fig. 4.

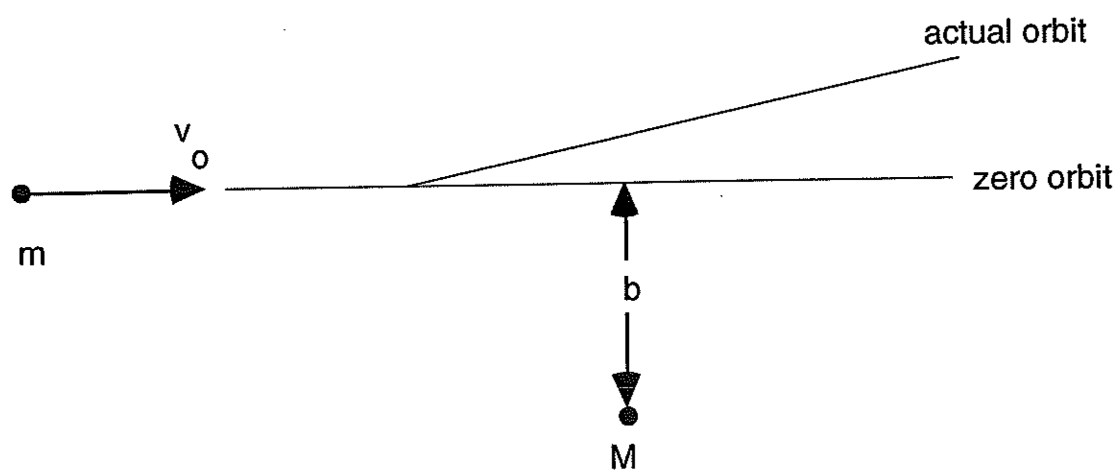


Figure 4

Coulomb Collisions

Let us assume that the deflection angle is small so that in the zero order approximation the particle follows a straight line; denote the minimum distance from this straight line orbit and the field particle by b; this is called the impact parameter.

Most of the deflection is due to the force exerted during the time the particles are close together, this occurs for a time interval $\tau = 2b/v_0$ centered at the time of closest approach and its magnitude is roughly zZe^2/b^2. The momentum acquired by the test particle perpendicular to v_0 is

$$\Delta p_\perp = F_\perp \tau \sim (zZe^2/b^2)2b/v_0 = 2zZe^2/bv_0 \ . \tag{20}$$

The scattering angle is

$$\Theta = \Delta p_\perp/p_0 = 2zZe^2/bmv_0^2 \ . \tag{21}$$

An accurate calculation yields

$$\tan(\Theta/2) = zZe^2/bmv_0^2 \tag{22}$$

(see for example, Goldstein's "Classical Mechanics"). For small angles of deflection Eq. (21) and (22) are equal.

Note that the magnitude of the scattering angle depends on the impact parameter and the initial velocity and is the same magnitude, but of opposite sign, for like and unlike charges. The impact parameter for a 90° scattering (from Eq. 22) is

$$b_0 = zZe^2/mv_0^2 \ , \tag{23}$$

i.e., the distance at which the potential energy is twice the original kinetic energy. The scattering angle for arbitrary b may be written as

$$\tan(\Theta/2) = b_0/b \tag{24}$$

For small angle deflections, $\Theta = 2b_0/b$; since b_0 is usually quite small, there are very few large angle deflections and we can use this approximation so long as we cut it off at approximately b_0.

All particles that pass within db of b will be scattered within a corresponding $d\Theta$ of $\Theta(b)$. We can define a differential cross-section $I(\Theta,\phi,v)$, where I is the cross-section for scattering into a small solid angle $d\Omega$ about the angles Θ and ϕ; spherical polar coordinates are used, Θ is the angle of deflection with respect to the initial direction of motion of the test particle (we take this to be the z direction) and ϕ is the angle the orbit plane makes with the x, z plane (x is perpendicular to the z axis and its direction can be chosen to be the most convenient for making calculations). The total cross-section is obtained by integrating $I(\Theta,\phi,v)$ over all solid angles,

$$\sigma(v) = \int_{4\pi} I(\Theta,\phi,v)\ d\Omega\ . \tag{25}$$

By cylindrical symmetry $I(\Theta,\phi,v)$ is independent of ϕ and we can write

$$I(\Theta,v)2\pi\sin\Theta d\Theta = 2\pi b db\ . \tag{26}$$

Eliminating b from Eqs. (24) and (26) yields the Rutherford scattering formula

$$I(\Theta) = (zZe^2/2mv_0^2)^2\operatorname{cosec}^4(\Theta/2)\ . \tag{27}$$

For small values of Θ this is approximately

$$I(\Theta) = 16(zZe^2/mv_0{}^2)^2/\Theta^4 \,. \tag{27a}$$

The total cross-section is obviously infinite; it is the contributions that come from small Θ which give this divergence. Small Θ come from large values of b; for distances greater than the Debye length the plasma shields out the E field of the field particle and we should not include impact parameters greater than this. We will investigate this in much more detail shortly.

<u>Multiple Coulomb Scattering</u>

Because of the long range of the Coulomb force, there are many more small angle scatterings than large angle scatterings ($\Theta \sim 1$). It turns out that not only are there many more small angle scatterings than large angle scatterings, but their overall effect is much more important. We will show this shortly; however, first we will compute the effect of many small angle scatterings.

We consider a group of electrons which are initially moving in the z direction through a plasma that we will consider infinite and homogeneous. To begin with we will consider only their scattering by ions that we will treat as infinitely massive. On the average there are as many deflections up as down, and as many to the right as to the left. On the average there is no deflection. This does not mean that if we look at a particular particle that it will not be deflected. Any one electron will by chance encounter more ions on one side than on another; there will be a symmetrical spread of deflections about the z axis.

For a given electron the total velocity in the x-direction, acquired after N individual scatterings, will be

$$\Delta v_x = (\Delta v_x)_1 + (\Delta v_x)_2 + (\Delta v_x)_3 + \bullet\bullet\bullet + (\Delta v_x)_N \, . \tag{28}$$

For the ensemble (collection) of all the electrons in the group, we consider the average of Δv_x. Then due to the equality of a scattering in one direction and in the opposite direction, we have

$$<\Delta v_x> = <\Delta v_y> = 0 \, . \tag{29}$$

From our comments above, however, it is clear that this does not mean that there is no deflection; to get a measure of this let us look at the average of $\Delta v_x{}^2$; this is

$$<\Delta v_x{}^2> = <[(\Delta v_x)_1 + (\Delta v_x)_2 + (\Delta v_x)_3 + \bullet\bullet\bullet + (\Delta v_x)_N]^2> \tag{30}$$

The sum will consists of two types of terms, $<(\Delta v_x)_n{}^2>$ and $<(\Delta v_x)_n(\Delta v_x)_m{}^2>$. The second type of these terms averages to zero if we assume that the ions are randomly distributed and uncorrelated correlated. Thus, we are left with only the first type of term; since the plasma is uniform and homogeneous, $<(\Delta v_x)_n{}^2>$ is the same for all n. We therefore get

$$<\Delta v_x{}^2> = N<(\Delta v_x)_1{}^2> \, . \tag{31}$$

Furthermore, by symmetry we have

$$<\Delta v_x{}^2> = <\Delta v_y{}^2> \equiv <\Delta v_\perp{}^2>/2 \, . \tag{32}$$

We also have

$$\Delta v_\perp{}^2 = v_0{}^2 \sin^2 \Theta \sim v_0{}^2 \Theta^2 . \tag{33}$$

Using Eq. (24) we have

$$\sin\Theta = 2(b_0/b)/(1 + [b_0/b]^2) , \tag{34}$$

and

$$\Delta v_\perp{}^2 = [4v_0{}^2(b/b_0)^2]/(1 + [b/b_0]^2)^2 . \tag{35}$$

The cross-section for a collision within db of b is $2\pi bdb$. The number of collisions of this type per unit path length is $nI(\Theta,v)d\Omega = 2\pi nbdb$. At each collision, the change in the $\Delta v_\perp{}^2$ is given by Eq. (35). Thus, the average total change in $\Delta v_\perp{}^2$ suffered by a test electron per cm of path length traveled in the plasma is

$$d\langle\Delta v_\perp{}^2\rangle/dz = \int_0^{b_m} \{[4v_0{}^2(b/b_0)^2]/(1 + [b/b_0]^2)^2\} 2\pi nbdb , \tag{36}$$

where b_m is the maximum distance at which the Coulomb potential can be applied in view of the shielding of the ion charge by the plasma ($b_m \sim \lambda_D$). Spitzer discusses this point (see L. Spitzer, Jr., Physics of Fully Ionized Gases [Interscience Publishers, New York, N. Y., 1962]; 2nd ed.) and we will look at it more deeply later on in the course.

$$d\langle\Delta v_\perp{}^2\rangle/dz = 8\pi n_f v_0{}^2 b_0{}^2 \textbf{ Error } [x^3/(1 + x^2)^2]dx \qquad (37)$$

$$d\langle\Delta v_\perp{}^2\rangle/dz = 4\pi n_f v_0{}^2 b_0{}^2 \{\ln[1 + (b_m/b_0)^2 + 1/[1 + (b_m/b_0)^2] -1\}. \qquad (38)$$

It is customary (later on we will show that it is correct) to take for b_m the Debye length, λ_D. The quantity b_0 is the impact parameter for a 90° collision, the separation distance at which the potential energy is equal to twice the initial kinetic energy; it is generally of the order of or smaller than atomic dimensions. On the other hand, b_m is much larger than the inter-particle spacing. Assuming b_m/b_0 and $\ln(b_m/b_0) >> 1$, Eq. (38) becomes

$$d\langle\Delta v_\perp{}^2\rangle/dz = 8\pi n_f v_0{}^2 b_0{}^2 \ln(b_m/b_0). \qquad (39a)$$

Inserting the value of $b_0(b_0 = zZe^2/mv_0{}^2)$ from Eq. (23) gives

$$d\langle\Delta v_\perp{}^2\rangle/dz = 8\pi n_f(zZe^2/m_t v_0)^2 \ln(\Lambda) \qquad (39b)$$

where Λ is b_m/b_0. Typically the term $\ln(\Lambda)$ is between 10 and 20 for conditions of interest.

We may roughly say that when $\langle\Delta v_\perp{}^2\rangle = v^2$ the particle has been scattered through 90°. If we use this criterion we may compare the importance of small angle scattering to that of large angle scattering. We find from this

$$(\text{multiple scattering})/(\text{single } 90° \text{ scattering}) \sim [8\pi v_0{}^2 b_0{}^2 \ln(b_m/b_0)]/\pi b_0{}^2. \qquad (40)$$

Thus, it is about two orders of magnitude more probable that a particle is scattered through 90° by multiple small angle scatterings than it is for it to happen by one large angle scattering.

If the force between particles went as r^{-m}, then for $m \geq 3$ this is no longer true and large angle scatterings dominate.

The Coulomb scattering described here is elastic (we have neglected radiation, multiple ionization processes, and excitation of partially striped atoms): kinetic energy is conserved. In the center of mass coordinate system the individual particle energies are unchanged by a collision. This is no longer true in the laboratory frame. For example, if a light particle scatters from a stationary heavy particle by 180°, the light particle looses 2m/M of its energy. We shall now look at this more completely.

Energy Transfer and Changes in Parallel Velocity for Collisions of Particles with Finite m/M Ratios

Let us now look at encounters between two particles with finite mass ratios. To start with we will take one of the particles moving (the test particle) and one to be at rest (the field particle); this of course depends on the frame of reference we use. We saw in the last section that for a hot plasma, a good approximation is to assume to lowest order that the particles move along their undisturbed orbits. We then calculate their deflection by integrating the acceleration they would feel as they follow these lowest order orbits. For the case under consideration the zero order orbit has the test particle moving along a straight line orbit and the field particle at rest. The situation is the same as shown in Fig. 4 and our small angle deflection calculation for the test particle of the last section applies.

Now from conservation of momentum we have

$$m_t \Delta v_{t\perp} + m_f \Delta v_{f\perp} = 0 . \tag{41}$$

We also have from conservation of energy

$$\Delta E_f = m_f \Delta v_{f\perp}{}^2/2 = (m_t{}^2/m_f 2)\Delta v_{t\perp}{}^2 . \tag{42}$$

The change in the energy of the field particle must come from the energy of the test particle.

$$\Delta E_t = -m_f \Delta v_{f\perp}{}^2/2 = -(m_t{}^2/m_f 2)\Delta v_{t\perp}{}^2 , \tag{43}$$

where the last equality follows by making use of Eq. (41). We may sum up the loss of energy given by Eq. (43) for encounters with many field particles; we thus get

$$<\Delta E_t> = -(m_t{}^2/m_f 2) <\Delta v_{t\perp}{}^2> . \tag{44}$$

Now using Eq. (39), we find for the energy lost per unit distance in the plasma

$$d<\Delta E_t>/dz = -(m_t{}^2/2m_f) 8\pi n_f v_0{}^2 b_0{}^2 \ln(b_m/b_0) . \tag{45}$$

Substituting in $b_0 = zZe^2/mv_0{}^2$ from Eq. (23) gives

$$d\langle \Delta E_t \rangle/dz = -\,(m_t^2/2m_f)8\pi n_f(zZe^2/m_t v_0)^2 \ln(\Lambda)\ , \qquad (46)$$

where Λ is given by

$$\Lambda = b_m/(zZe^2/mv_0^2)\ . \qquad (47)$$

Since $\ln(\Lambda)$, varies only slowly with Λ, $\ln(\Lambda)$ is typically between 10 and 20. We can also find the time rate of change of $\langle \Delta E_t \rangle$ by multipling equation (46) by $dz/dt = v_0$. This gives

$$d\langle \Delta E_t \rangle/dt = -(m_t^2/2m_f)8\pi n_f v_0(zZe^2/m_t v_0)^2 \ln(\Lambda)\ . \qquad (48)$$

Next let us consider the change in the velocity parallel to z; this can also be obtained from conservation of energy. From conservation of energy in an encounter we have

$$m_t\ v_0^2/2 = m_t\ ([v_0 + \Delta v_{tz}]^2 + \Delta v_{t\perp}^2)/2 + m_f\ (\Delta v_{f\perp}^2 + \Delta v_{fz}^2)/2\ . \qquad (49)$$

Assuming that the Δv_z^2 terms are negligible, as can be shown at the end of the calculation, Eq. (49) leads to

$$m_t\ (2[v_0\ \Delta v_{tz}] + \Delta v_{t\perp}^2) + m_f\ \Delta v_{f\perp}^2 = 0\ , \qquad (50)$$

or

$$\Delta v_{tz} = -(m_t\ \Delta v_{t\perp}^2 + m_f\ \Delta v_{f\perp}^2\ )/2m_t\ v_0\ , \qquad (51)$$

$$\langle \Delta v_{tz} \rangle = -(1 + m_t/m_f)(\langle \Delta v_{t\perp}^2 \rangle/2v_0)\ , \qquad (52a)$$

$$d\langle \Delta v_{tz}\rangle/dz = -(1 + m_t/m_f)(1/2v_0)d\langle \Delta v_{t\perp}{}^2\rangle/dz \ . \tag{52}$$

When $\langle \Delta v_{t\perp}{}^2\rangle \simeq v_0{}^2$, a large angle deflection will have occurred. We can estimate the distance (mean free path) required for this to happen by multipling Eq. (39) by l_{MFP} and equating the result to $v_0{}^2$. The result is

$$l_{MFP,\Theta} = (m_t v_0{}^2)^2/\{8\pi n_f(zZe^2)^2 \ln(\Lambda)\}. \tag{53}$$

We may also calculate the change in energy of a test particle in passing a distance l_{MFP} through a plasma by multipling Eq. (46) by l_{MFP}; this gives

$$\Delta E_t = -(m_t{}^2/2m_f)v_0{}^2 = -E_t m_t/m_f. \tag{54}$$

If $m_f \gg m_t$, as in the case of the scattering of electrons by ions, then the loss in energy per large angle deflection is m_t/m_f of the initial energy. To loose all its energy, the test particle must undergo m_f/m_t large angle deflections.

If $m_t \gg m_f$, as in the case of the scattering of ions by electrons, then the loss in energy per large angle deflection is m_t/m_f of the initial energy which is much larger than the initial energy. This means the particle (ion) is brought to rest (by the light electrons) long before it is significantly scattered. We may define a mean free path for energy loss in the same way we did for large angle scattering; i.e., we multiply Eq. (46) by $l_{MFP,E}$ and equate it to the negative of the initial energy. This gives

$$l_{MFP,E} = (m_f/m_t)(m_t v_0{}^2)^2/\{8\pi n_f(zZe^2)^2 \ln(\Lambda)\} \tag{55}$$

$$l_{MFP,E} = (m_f/m_t)l_{MFP,\Theta}$$

We may also compute a mean free path for stopping of the velocity in the initial direction of motion; to do this we multiply Eq. (51) by $l_{MFP,v}$ and equate it to the negative of the initial velocity. This gives

$$l_{MFP,v} = (m_t v_0{}^2)^2/(\{1 + m_t/m_f\}\{4\pi n_f(zZe^2)^2\ln(\Lambda)\})$$

$$l_{MFP,v} = 2l_{MFP,\Theta}/(1 + m_t/m_f). \tag{56}$$

If $m_t \gg m_f$, the particle is brought to rest in $m_f/2m_t$ of a large angle scattering distance.

Other quantities of interest are scattering times, stopping times and energy loss times. We can roughly get these from the mean free paths by dividing by the initial velocity, $t = l_{MFP}/v_0$. Using this relationship we get the following:

$$\tau(\Theta) = m_t{}^2 v_0{}^3/\{8\pi n_f(zZe^2)^2\ln(\Lambda)\}. \tag{57}$$

$$\tau(E) = (m_f/m_t)m_t{}^2 v_0{}^3/\{8\pi n_f(zZe^2)^2\ln(\Lambda)\}. \tag{58}$$

$$\tau(v) = m_t{}^2 v_0{}^3/(\{1 + m_t/m_f\}\{4\pi n_f(zZe^2)^2\ln(\Lambda)\}). \tag{59}$$

From these we have the following relations:

$$\tau(E) = (m_f/m_t)\tau(\Theta) \tag{60}$$

$$\tau(v) = 2\tau(\Theta)/\{1 + m_t/m_f\}\ . \tag{61}$$

The formulas (53) to (61) were derived assuming the field particle was at rest. It is clear that they can usually be applied to the interaction of electrons with ions since ions generally have relatively slow motions. We can also apply them to the interaction of very energetic ions (say fusion reaction products) with ions and if the electrons are not too hot with the electrons. Of course these formulas can be applied accurately to the interactions of the energetic tails of the electron distribution with the bulk of the electrons and likewise for the energetic ion tail with the bulk of the ions. These formulas can also be roughly applied to electron-electron and ion-ion interactions in a thermal plasma by using the thermal velocity for v_0. In this case, while the field particles are moving their motion is roughly of the same order as that of the test particle and large errors will not be made.

One case where they cannot be applied is to the slowing down of most ions by electrons or to the exchange of energy between most ions and the electrons. However, we can find the rate of energy loss by the ions to the electrons by making use of the property of detailed balance; for a thermal plasma, the ions must be losing energy to the electrons at the same rate they are receiving energy from the electrons. Equation (58) gives the time it takes for a set of test particles to transfer their energy to the field particles. If we let the test particles be electrons and the field particles be ions and if we use the electron thermal velocity for v_0, then this equation gives us the time for the electrons to give their energy to the ions. By detailed balance this must also be the time the ions take to transfer their energy to the electrons.

We can also find the slowing down or stopping time for an ion by the electrons. The ion energy is proportional to their velocity squared $E_i = M_i v_i^2/2$; thus,

$$dE_i/dt = M_i\, v_i dv_i/dt$$

or

$$E_i^{-1} dE_i/dt = 2v_i^{-1} dv_i/dt\ , \qquad (62)$$

The slowing down time for the ions is twice the energy loss time or twice the number given by Eq. (58). As the ions move through the electrons, they gradually slow down. For slow moving ions, we may expect the drag force to be proportional to their velocity; this implies that the stopping time is the same for all velocities (so long as the velocity is small compared to the thermal velocity of the electrons). A more rigorous treatment verifies this.

The following table summarizes the results just obtained for collision times:

SUMMARY OF COULOMB COLLISION TIMES

$$\tau_{ei}(\Theta) = m_e^2 v_T^3/\{8\pi n_i (Ze^2)^2 \ln(\Lambda)\}. \qquad (I.1)$$

$$\tau_{ei}(E) = (m_i/m_e) m_e^2 v_T^3/\{8\pi n_i (Ze^2)^2 \ln(\Lambda)\}. \qquad (I.2)$$

$$\tau_{ei}(v_T) = m_e^2 v_T^3/(\{1 + m_e/m_i\}\{4\pi n_i (Ze^2)^2 \ln(\Lambda)\}). \qquad (I.3)$$

$$\tau_{ee}(\Theta) = m_e^2 v_T^3/\{8\pi n_e e^4 \ln(\Lambda)\}. \qquad (I.4)$$

$$\tau_{ee}(E) = m_e^2 v_T^3/\{8\pi n_e e^4 \ln(\Lambda)\}. \qquad (I.5)$$

$$\tau_{ee}(v_T) = m_e^2 v_T^3/\{8\pi n_e e^4 \ln(\Lambda)\}. \qquad (I.6)$$

$$\tau_{ii}(\Theta) = m_i^2 v_T^3/\{8\pi n_i(Ze^2)^2 \ln(\Lambda)\}. \qquad (I.7)$$

$$\tau_{ii}(E) = m_i^2 v_T^3/\{8\pi n_i(Ze^2)^2 \ln(\Lambda)\}. \qquad (I.8)$$

$$\tau_{ii}(v_T) = m_i^2 v_T^3/\{8\pi n_i(Ze^2)^2 \ln(\Lambda)\}. \qquad (I.9)$$

$$\tau_{ie}(E) = (m_i/m_e) m_e^2 v_{T,e}^3/\{8\pi n_e Z e^4 \ln(\Lambda)\}. \qquad (I.10)$$

$$\tau_{ie}(v_{0,i}) = (m_i/m_e) m_e^2 v_{T,e}^3/\{8\pi n_e Z e^4 \ln(\Lambda)\}. \qquad (I.11)$$

The corresponding mean free paths are obtained from these times by multiplying them by the appropriate thermal velocities. In cases where the formula can be applied to non-thermal situations, one simply uses the appropriate velocity to obtain the corresponding mean free paths.

Electrical Conductivity of a Plasma

We can use the results we have just obtained to compute the electrical conductivity of a plasma to a good approximation. For this we use a simple fluid model for the electrons; we write Newton's equation of motion for the electrons as follows

$$m_e\, d\langle v_e\rangle/dt = -eE - m_e \langle v_e\rangle/\tau_{ei}(v_{Te}) . \qquad (63)$$

Here $\langle v_e\rangle$ is the average velocity of the electrons through the ions, E is the electric field, and $\tau_{ei}(v_{Te})$ is the collision time associated with a thermal electron. We look at a steady state so we set $d\langle v_e\rangle/dt$ equal to zero. Then solving for $\langle v_e\rangle$, we find

$$\langle v_e \rangle = (-eE/m_e)\tau_{ei}(v_{Te}) . \tag{64}$$

Multiplying this equation by $-en_e$ gives us the current density

$$-en_e\langle v_e \rangle = j = (n_e e^2/m_e)\tau_{ei}(v_e)E . \tag{65}$$

From this we get

$$j = \sigma E = (\omega_{pe}{}^2/4\pi)\tau_{ei}(v_{Te})E ,$$

and

$$\sigma = (\omega_{pe}{}^2/4\pi)\tau_{ei}(v_{Te}) . \tag{66}$$

The resistivity is the reciprocal of σ so we have

$$\eta = 4\pi/\omega_{pe}{}^2\tau_{ei}(v_{Te}) . \tag{67}$$

If we use the $\tau_{ei}(v_T)$ obtained from Eq. (I.3) and neglect the ratio m_e/m_i, then we find for η

$$\eta = \{16\pi^2 n_e Z e^4 \ln(\Lambda)\}/[\omega_{pe}{}^2 m_e{}^2 v_{Te}{}^3] .$$

Using $v_{Te}{}^2 = 3kT_e/m_e$ we get

$$\eta = \{4\pi Z e^2 m_e{}^{1/2} \ln(\Lambda)\}/[3kT_e]^{3/2} . \tag{68}$$

We can compare this with the result given by Spitzer (see L. Spitzer, Jr., Physics of Fully Ionized Gases [Interscience Publishers, New York, N.Y., 1962]; 2nd ed.) which was obtained by solving the Boltzmann equation for the electrons with only electron ion collisions included; he gives

$$\eta = \{\pi^{3/2}Ze^2m_e{}^{1/2}\ln(\Lambda)\}/2[2kT_e]^{3/2}. \qquad (69)$$

Numerically Spitzer's value of η is given by

$$\eta = 3x10^{-3}Z\ln(\Lambda)\}/T_e{}^{3/2}\text{ohm-cm} \qquad (70)$$

where T_e is in eV. The value of η that we find is 1.6 times larger than that found by Spitzer; however all the dependences are correct. We can see that we should find a larger value than Spitzer since the electron collision rate is proportional to their velocity cubed; thus, higher velocity electrons will contribute more to the conductivity (and less to the resistivity) than lower velocity electrons. To get a more correct value we should replace $v_{Te} = (3kT_e/m_e)^{1/2}$ by $v_{Te} = (<v^3>)^{1/3}$. This gives $(<v^3>)^{1/3} = 2.08(kT_e/m_e)^{1/2}$; this reduces the resistivity by a factor of 1.74 which brings it into pretty good agreement with that given by Spitzer.

In the above calculation we have neglected electron-electron collisions; only electron-ion collisions are kept. This is known as a Lorentz model for the conductivity or resistivity. This model is appropriate because electron-electron collisions do not change the current directly; electron momentum is conserved for such collisions and so is the total electron current. Although they do not directly change the electron current, electron-electron collisions do affect the current indirectly. We have seen that the current is preferentially carried by the energetic electrons. These electrons collide with the lower energy bulk electrons and transfer their current to them. The lower energy electrons collide more rapidly with the ions and so dissipate the current more rapidly. Spitzer has carried out detailed calculations of the resistivity for both the Lorentz model and from the kinetic equations including electron-electron collisions. If the ionic charge is one, electron-electron collisions increase the resistivity by a factor of 1.7 (approximately 2) over that for the Lorentz model. For large ionic charges, the factor by which the resistivity is increased decreases (roughly as [1+ 1/Z], for more precise values see the Spitzer reference). Numerically Spitzer gives the following formula for the plasma resistivity in the important case of Z equal one.

$$\eta = 5.2\text{x}10^{-3}\ln(\Lambda)\}/T_e^{3/2} \text{ ohm-cm} \qquad (71)$$

An interesting point about the resistivity is that it is independent of the density. This is because as the density goes up, the collision time decreases in proportion to the density and so the drift velocity of the electrons decreases in proportion to the density. However, the number of carriers increases in proportion to the density so that the current density, $j \propto n\langle v_e\rangle$, is independent of n. Of course for a given E, the drift velocity becomes larger and larger as the density gets lower and at some point the drift will approach the thermal velocity and the resistivity (conductivity) is modified by the drift itself; the resistivity becomes nonlinear at this point.

It is informative to compute the conductivity of some typical plasmas. For a laboratory plasma of hydrogen at 10 eV, Eq. (72) gives (setting $\ln(\Lambda)$ equal to 10) $\eta = 1.7 \times 10^{-3}$ ohm-cm. For a fusion plasma at 10^4 eV we get $\eta = 5.2 \times 10^{-8}$ ohm-cm which is much lower than that for room temperature copper ($\eta = 1.7 \times 10^{-6}$ ohm-cm). We might also try to apply this formula to metallic copper. In this case we must use the Fermi energy of the electrons instead of the temperature; this is roughly 10 eV. Since the resistivity is independent of density, this would give the same value for the resistivity that we found for the 10 eV laboratory plasma. This value is about 1000 times too large. It is interesting that for mercury the resistivity is 10^{-4} ohm-cm., which is only one order of magnitude better than our laboratory plasma. Actually for Hg the $\ln(\Lambda)$ term is only about 1 - 2 so the conductivity is about an order or magnitude larger, or the resistivity an order of magnitude smaller. We get roughly the right value of the resistivity from our plasma formula. The reason for the discrepancy for Cu is due to the high degree of order in the arrangement of the Cu atoms in the metal so that they do not act like a lot of random scatterers; for Hg this is not such a large effect. Of course one should make these calculation quantum mechanically to be correct.

Runaway Electrons

From the above calculations we have seen that the collision time and the mean free paths depend on the particles energy. The collision time of an electron goes essentially as its energy to the three halves power [see Eq. (57)]. If an electric field is applied to a plasma to drive current, the very energetic electrons will very weakly collide and will freely accelerate. They will gain energy faster than they are scattered. As they gain energy, collisions become weaker and weaker and so they become more and more collisionless; they will "run away". We can explore this phenomenon by a calculation that parallels our conductivity calculation. Consider the electrons that start with a velocity v_0. We write a fluid equation for the acceleration of these electrons

$$m_e\, d\Delta v(v_0)/dt = -eE - m_e\, \Delta v(v_0)/\tau_{ei}(v_0). \tag{72}$$

Let us now compute the steady state $\Delta v(v_0)$ that is achieved; this is

$$\Delta v(v_0) = -\,[(eE/m_e)\tau_{ei}(v_T)][v_0{}^3/v_T{}^3] \tag{73}$$

For $[(eE/m_e)\tau_{ei}(v_T)][v_0{}^2/v_T{}^3] \geq 1$, $\Delta v(v_0) \geq v_0$ and the collision rate is dropping faster than the electrons can be scattered; this is known as the Dreicer runaway condition. When the Dreicer condition holds, it implies that there will be a significant change in velocity and hence a large reduction in the collision rate before a collision has taken place. The collision rate will drop fast and the particle will freely accelerate, i.e., "run away".

<u>Low Energy Electron Atom and Ion Atom Collisions</u>

Let us consider a low energy charged particle (electron or ion) neutral atom collision. The situation is illustrated in Fig. 5 below.

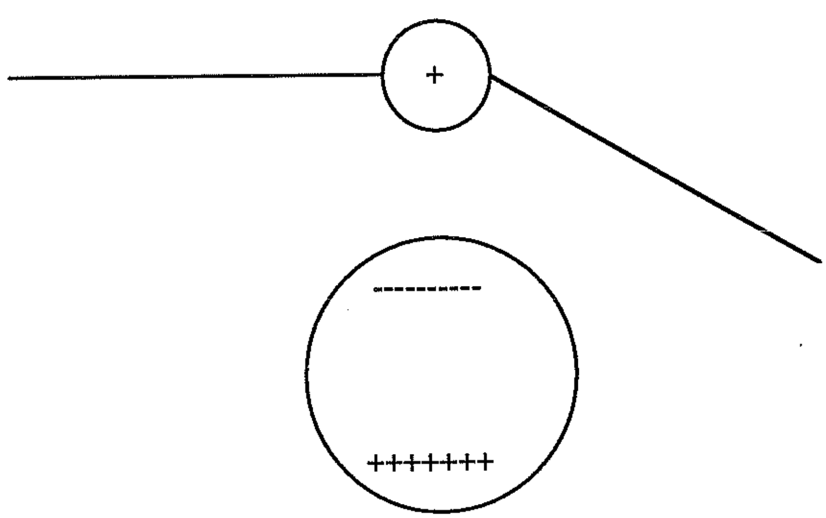


Figure 5

The electric field of the charged particle polarizes the atom as it passes. For an ion as is shown, the electrons are pulled towards it and the nucleus is pushed away. The polarization of the atom produces a dipole moment **P**.

$$\mathbf{P} = \alpha\mathbf{E} = \alpha Q\mathbf{r}/r^3 \,. \tag{74}$$

Here α is the atomic polarizability, Q is the charge on the particle, **r** is the vector separation distance, and r is the magnitude of the separation. The electric field produced by the atomic dipole at the charged particle is

$$E = 2P/r^3. \tag{75}$$

This field is always in a direction so as to give attraction between the charge and the atom and is directed along the line joining them. The attractive force is

$$F = Q\,E = 2Q^2\,\alpha/r^5. \tag{76}$$

We can obtain an approximation to α by regarding the atomic electrons as being harmonically bound to the nuclei. Let Δr be the displacement of the electron cloud relative to the nucleus. Then our harmonic approximation gives us the following relation

$$d^2\Delta r/dt^2 = -\omega_a{}^2\Delta r\; -eE/m_e. \tag{77}$$

Assuming that the time variation of E is slow compared to the atomic response time, $\omega_a{}^{-1}$, we find for Δr

$$\Delta r = -eE/m_e\omega_a{}^2 . \tag{78}$$

The dipole moment is given by

$$P = -e\Delta r = e^2E/m_e\omega_a{}^2 . \tag{79}$$

This gives for the polarizability

$$\alpha = e^2/m_e\omega_a{}^2. \tag{80}$$

Now ω_a is related to the ionization energy I roughly by $h\omega_a/2\pi = I$ (this approximation can be justified from quantum mechanics). This gives

$$\alpha = (e^2/m_e)(h/2\pi I)^2 \tag{81}$$

If we now use the relations

$$p = h/\lambda \ , \ p^2/2m_e = K = (h/\lambda)^2/2m_e \ ,$$

from quantum mechanics and $e\Phi = 2K$ ($-e\Phi$ is the potential energy) from the Virial Theorem for a system interacting by Coulomb forces, we find

$$e\Phi - K = I \simeq K \sim (h/\lambda)^2/2m_e \ . \tag{82}$$

Using Eqs. (82) and (83) we get

$$\alpha = 2(e^2/I)\lambda^2)/2\pi \simeq a_A{}^3 \quad (83)$$

where $a_A{}^3$ is the volume of the atom, a_A is the atomic radius. The following table lists I, α, and αI^2 for the noble gases. By Eq. (82) αI^2 would be constant; we see from the table that it varies by roughly a factor of 5 in going from He to Xe. For the heavier more complex atoms, our simple one electron model is clearly too simple but it does remarkably well.

Table 1

Table I - Diameters of Some Atomic Ions, In Å

		O^{--}	2.6	S^{--}	3.5	Se^{--}	3.5
H^-	2.5	F^-	2.7	Cl^-	3.6	Br^-	3.9
He	1.9	Ne	2.3	A	2.8	Kr	3.2
Li^+	1.6	Na^+	2.0	K^+	2.7	Rb^+	3.0
Be^{++}	0.7	Mg^{++}	1.6	Ca^{++}	2.1	Sr^{++}	2.5
B^{+++}	—	Al^{+++}	1.2	Sc^{+++}	1.7	Y^{+++}	2.1
C^{++++}	0.4	Si^{++++}	0.8	Ti^{++++}	1.3	Zr^{++++}	1.7
N^{+++++}	0.3	P^{+++++}	0.7				

Mean Polarizabilities of Inert Gases and Atomic Ions.
(The numbers denote $\overline{\alpha} \cdot 10^{24}$ cm.^{3})

		F^-	0.99	Cl^-	3.05	Br^-	4.17	I^-	2.8
He	0.202	Ne	0.392	A	1.629	Kr	2.46	Xe	4.00
Li^+	0.075	Na^+	0.21	K^+	0.87	Rb^+	1.81	Cs^+	2.79
		Mg^{++}	0.12			Sr^{++}	1.42		
		Al^{+++}	0.065						
		Si^{++++}	0.043						

Ionization and Excitation

The calculations of ionization and excitation cross-sections require quantum mechanics (see Mott, N.F. and H.S.W. Massey, "The Theory of Atomic Collisions", 2nd edition [Clarendon Press, Oxford, 1949], and Massey, H.S.W. and E.H.S. Burhop, "Electronic and Ionic Impact Phenomena", [Clarendon Press, Oxford, 1952]). However, it is possible to make reasonable estimates of these from a semi-classical calculations; we shall do this now.

We use the results we developed in our treatment of Coulomb collisions. We consider an energetic electron impinging on an atom and ask what the cross-section is for imparting the ionization energy or more to an atomic electron. For simplicity we will consider the atomic electrons at rest; we could improve the model by allowing them to have a distribution of energies but we will see that we can get reasonably good results without doing this. From our Coulomb scattering calculations, Eqs. (33), (34), and (42), we get for the energy transfer

$$\Delta W = 2e^4/(m_e v_0^2\, b^2) = [\Phi(b)]^2/K \; . \tag{84}$$

The cross-section for transferring energy ΔW or more is

$$\sigma = \pi b^2 = \pi e^4/(K\,\Delta W) \; . \tag{85}$$

To get ionization, ΔW must be the ionization energy I. Thus, we get

$$\sigma_i = \pi e^4/(K\,I) = 729 x \pi a_0^2/IK \; . \tag{86}$$

This formula predicts that the cross-section decreases as K^{-1}; it is clear that it must fail when K approaches I; we cannot get ionization when K is below I. Furthermore, when K approaches I the motion of the atomic electron becomes more and more important and should be taken into account. We have seen that to treat Coulomb collisions in a plasma we had to introduce maximum and minimum distance cut offs; these gave the $\ln\Lambda$ factor. If we include a distribution of energies for the atomic electrons and averaged it in computing energy transfers or if we were more precise and made a quantum mechanical calculation, then such a ln factor would enter the result. A ln factor that makes the cross-section go to zero at K equal I is lnK/I. In fact the following is found to be a good approximation to ionization cross-sections,

$$\sigma_i = [\pi e^4 \ln(K/I)]/KI = [729 x \pi a_0{}^2 \ln(K/I)]/IK$$
$$\sigma_i = [5.7 x 10^{-14} \ln(K/I)]/IK \quad . \tag{87}$$

This expression has a maximum for K equal to eI (e is the base of natural logarithms) or roughly 2.7xI. This gives a peak cross-section of $1.8x10^{-14}/I^2$. For atomic hydrogen this gives a peak ionization cross-section of $2x10^{-16}$ for

Figure 6

Fig.6 - Ionization cross sections σ_i for atomic and molecular hydrogen by electron impact. After W.L. Fite and R.T. Brackman, *Phys. Rev.*, **112**, 1141 (1958).

37eV electrons compared to a quantum calculation of 0.7×10^{-16} for about 50 eV electrons (see D.J. Rose and M. Clark, Jr., "Plasmas and Controlled Fusion", The M.I.T. Press, Third Printing, 1973). Figure 6 is a plot of σ_i = vs. K/I.

We can also estimate the total excitation cross-section this way. For this we need to transfer an amount of energy between the first excited level and the ionization energy. Replacing I in Eq. (87) by the first excitation energy and subtracting the cross-section for ionization (since transfer of this much energy gives ionization and not excitation) gives

$$\sigma_E = \pi e^4\{[\ln(K/E_1)]/KE_1 - [\ln(K/I)]/KI\}. \tag{88}$$

This cannot tell us which excited states are generated, but it allows us to estimate the energy transferred to the atoms and radiated away.

Radiative Recombination

For plasma formation and maintenance, we must consider not only the ionization processes (and radiative energy loss processes) but we should also look at recombination which removes charged particles from the free state. Conceptually the simplest of these, though often not a very important process from the numerical point of view, is radiative recombination. In this case we have an electron passing an ion (shown in Fig. 7); the acceleration of the electron causes it to radiate and if it radiates enough energy it will go into a bound orbit. Strictly speaking, this problem must be treated by using quantum mechanics. However, we will find that a semi-classical semi-quantum treatment will get us in the right range.

For an electron to be captured, it must radiate away at least its initial kinetic energy. The rate of radiation by a classical electron is given by

$$P = (2e^2a^2)/(3c^3). \tag{89}$$

The acceleration of an electron as it passes a charge Z at a distance ρ is

$$a = (Ze^2)/(m_e\rho^2). \tag{90}$$

Thus, the power radiated is

$$P = (2Z^2e^6)/(3m_e{}^2c^3\rho^4). \tag{91}$$

The total energy radiated is roughly

$$\Delta W = Pt \sim 2\rho P/v\ . \tag{92}$$

Substituting in Eq. (91) for P gives

$$\Delta W = (4Z^2e^6)/(3m_ec^3\rho^3m_ev) = m_ev^2/2. \tag{93}$$

One might be tempted to use the classical electron velocity in this formula; however if one does this one finds that the electron must pass so close to the nucleus that the uncertainty principal comes into play. From the uncertainty principal there is a relation between the the size of a region, ρ, into which we squeeze an electron and its momentum; this is $p\rho = h/2\pi$. This gives a velocity $v = h/2\pi m_e\rho$; we use this velocity in the denominator of the left-hand side of Eq. (93). Substituting these expressions into our expression for ΔW gives the expression for capture

$$\Delta W = (8\pi Z^2 e^6)/(3m_e c^3 \rho^2 h) = m_e v^2/2. \tag{94}$$

The cross-section for capture is

$$\pi\rho^2 = (16\pi^2 Z^2 e^6)/(3m_e^2 v_e^2 c^3 h). \tag{95}$$

In numerical values this is

$$\sigma = \pi\rho^2 = 1.2\text{x}10^{-21} Z^2/K \text{ cm}^2 . \tag{96}$$

To find the rate of recombination in a plasma, we must average this cross-section over the velocity distribution of the electrons; for a thermal plasma we should average it over a Maxwellian. The result for a thermal plasma is

$$\langle\sigma v\rangle = \alpha Z^2 [2m/\pi kT]^{1/2}, \tag{97}$$
$$\alpha = 1.2\text{x}10^{-21} ,$$

or in numerical values

$$\langle\sigma v\rangle = 4.0 \text{ x } 10^{-14} Z^2/[kT]^{1/2}. \tag{98}$$

Spitzer gives the result of a much more detailed Quantum Mechanical calculation; he gives

$$\langle\sigma v\rangle_{Spitzer} = 1.94 \times 10^{-13} Z^2/[kT]^{1/2}\Phi(\beta), \tag{99}$$

where $\Phi(\beta)$ is a numerical factor that is a function of temperature and the ionic charge, $\beta = 13.5Z^2/T$. Spitzer gives the following table for $\Phi(\beta)$.

β	1	2	5	10	100	1000
$\Phi(\beta)$	0.96	1.26	1.69	2.02	3.2	4.3

The expression that we get is about four times smaller than that given by Spitzer which is not surprising given the crudeness of our calculation.

The rate of recombination is obtained from

$$dn_e/dt = -\langle\sigma v\rangle n_e n_i \sim -n_e n_i\, 4.9 \times 10^{-14} Z^2/[kT]^{1/2}. \tag{100}$$

Three-Body Recombination

One recombination process that is important in low temperature and high density plasmas is that of three body recombination. In this case two electrons encounter a single ion at the same time; one of the electrons gives up enough energy to the other that it cannot escape the ion. This is of course a complex three-body problem and cannot be solved exactly. However, by using the work we have done on Coulomb collisions and ionization of atoms, we can get a pretty good answer that agrees favorably with the more detailed calculations that have been carried out.

Consider a low temperature plasma for which $kT << I$. We calculate the recombination rate using the law of detailed balance for thermal equilibrium and the rate of ionization of highly excited atoms.

We first compute the number of nuclei with an electron bound with energy of kT. These are the important atoms. If the binding is much larger than kT, then there are very few electrons in the plasma that can kick the electron out (ionize the atom) and ionization will be very slow. The orbiting electron is constantly loosing energy by radiation and is becoming more and more tightly bound. It therefore has effectively recombined. On the other hand if the electron is less tightly bound than kT, nearly all the electrons in the plasma are capable of ionizing the atom and it will be ionized in very short order.

It is possible to treat these highly excited atoms classically by the correspondence principal. The density of such atoms is roughly

$$n_a = n_e n_i (4\pi/3)\ r_0{}^3\ \underline{(\exp[ef/kT])},\ r_0 = eZ/kT. \qquad (101)$$

The exponential factor that is underlined has exponent 1 and to the accuracy we are working, we will treat the whole quantity as one. The factor, $n_e(4p/3)\ r_0{}^3$ is the number of electrons that we expect to find in a sphere of radius r_0; a very large fraction of them will be bound to the ion.

The rate of ionization of these highly excited atoms is given by

$$dn_a/dt = -\ n_e n_a <\sigma v> = -n_e{}^2 n_i (4\pi/3)\ r_0{}^3 <\sigma v>. \qquad (102)$$

From the section on the ionization of atoms we found

$$\sigma_i = [\pi e^4 \ln(K/I)]/KI \quad ,$$

since virtually every electron has an energy around kT and can ionize the atom, we can approximate this by

$$\sigma_i = \pi e^4/T^2 \, . \tag{103}$$

We approximate the velocity by the thermal velocity, i.e., $v_T = 6 \times 10^7 T^{1/2}$ and we approximate $<\sigma v>$ by

$$<\sigma v> = (\pi e^4/T^2) v_T = 6 \times 10^7 (\pi e^4/T^{3/2}). \tag{104}$$

Using this in Eq. (101) gives

$$dn_a/dt = -\, n_e n_a <\sigma v> = -6 \times 10^7 n_e n_a (\pi e^4/T^{3/2}). \tag{105}$$

Now using Eq. (100) for n_a with the underlined exp set equal to one, we get

$$dn_a/dt = -6 \times 10^7 n_e{}^2 n_i (4\pi/3)\; r_0{}^3\; (\pi e^4/T^{3/2}). \tag{106}$$

Replacing r_0 by Ze^2/T, we get

$$dn_a/dt = -6 \times 10^7 n_e{}^2 n_i (4\pi/3)\; (Ze^2)^3\; (\pi e^4/T^{9/2}). \tag{107}$$

By plugging in numbers we get

$$dn_a/dt = -7 \times 10^{-26} n_e^2 n_i Z^3/T^{9/2}. \tag{108}$$

Since highly excited atoms with binding energy about T will be rapidly formed and destroyed in the plasma, we expect them to be in thermal equilibrium. Thus, their rate of destruction by electron collisions should equal their rate of formation by the inverse process of three-body recombination in accordance with the law of detailed balance. We thus have for the rate of three-body recombination into these highly excited atoms

$$dn_e/dt = - 7 \times 10^{-26} n_e^2 n_i Z^3/T^{9/2}. \tag{109}$$

As we have already commented, for electrons that are just a little more tightly bound than this, the rate of detachment will become so slow that they have little chance of re-ionizing before the electron's radiation carries them so far down into the potential well that they cannot ionize again.

We do not expect that the rate of formation of these slightly more tightly bound states will be significantly different from that which we have just calculated. We take Eq. (108) to be roughly the three-body recombination coefficient for a plasma.

Detailed calculations, based on quantum mechanics, of the three-body recombination rate have been made by Hinnov and Herschberg (Phys. Rev. 125, 795, [1962], of course some approximations are made in this derivation also). They give the following expression for the three-body recombination rate

$$dn_e/dt = -\alpha n_e^2 n_i Z^3/T^{9/2} = - 5.6 \times 10^{-27} n_e^2 n_i Z^3/T^{9/2}. \tag{110}$$

This expression is about a factor of eight smaller than our result and to some degree has been checked experimentally; however, all the functional dependences are the same.

Other Recombination Effects

There are a great many other effects that lead to recombination. To treat them in any detail would be a course in itself. We will, however, mention some of them and the physics behind them so that the student will be aware of them and can look them up if he has need of them.

(1) Molecular Ion Dissociative Recombination: here an electron encounters a molecular ion and recombines with it; the molecule breaks apart, the fragments carrying off the recombination energy. An example is

$$e + H_2^+ \longleftrightarrow 2H .$$

(2) Dielectronic Recombination: here an electron encounters a partially stripped atom (an atom which has lost only a fraction of its electrons). The incoming electron encounters one of the bound electrons and transfers a sufficient amount of its energy to it so that it cannot immediately escape; an atom with two excited electrons is produced. Such an atom has sufficient energy to eject an electron and return to its initial state or possibly an excited state. However, the excited electrons are radiating and if they have time to radiate away their energy before such auto detachment takes place, recombination will have occurred. The rate of radiation is roughly proportional to Z^6; the rate of re-

encounter is inversely proportional to the volume of the electron cloud which goes roughly as Z^3 and iis nversely proportional to the relative energies of the two electrons to the three halves power, this goes roughly as Z^{-3}. Thus, the radiative loss takes over at high Z and is important for the capture of electrons by heavy impurities in fusion machines and by heavy atoms in the sun's corona.

Charge Exchange

Another very important process that plays a large role in many plasmas is that of charge exchange. In this process, an ion passes a neutral atom and an electron is transfered from the neutral to the ion. The ion is thus neutralized and an ion is left behind. This process is particularly important for a collision between an ion and an atom of the same type; in this case the electron transfer involves no change in energy; the transfer is therefore a resonant process. Some processes where charge exchange is important are the following:

(1) Charge exchange has a large effect on the flow of ion current across a magnetic field. A moving ion exchanges an electron with a stationary neutral. A moving neutral is produced which freely crosses the magnetic field; also a stationary ion is produced which has to be accelerated from rest by existing electric fields (we will look at this in much more detail shortly).

(2) The above is an important process in star formation: two important effects are involved here. First the collapsing stellar gas must slip through the interstellar magnetic field. If it cannot do this, then the magnetic field will be compressed as the cloud contracts. Without this slip, the product BA (B is the

magnetic field and A is a typical cross-sectional area) remains constant as the cloud carries the magnetic flux along with it. Since very large radial contractions are involved ($\sim 10^8$) and hence even larger area contractions are involved ($\sim 10^{16}$), even starting with very small interstellar magnetic fields, very large magnetic fields will be produced before the star forms. The resulting magnetic forces (pressure) will prevent further collapse.

The second effect is that the gas cloud must be able to get rid of its angular momentum in order to collapse. The interstellar magnetic field is anchored in the vast amount of plasma that surrounds the collapsing cloud. The ions are tied to this magnetic field (as we shall see later) and so are rotating very slowly. The spinning collapsing neutral gas cloud transfers momentum to the ions through charge exchange and the ions in turn transfer the momentum to the magnetic field (by twisting it up) which carries it out to the surrounding interstellar gas.

(3) Charge exchange is an important process for fusion reactors for a number of reasons:

(a) It is a serious source of cooling of the fusion plasma. A neutral atom can enter a fusion plasma from the wall; if it charge exchanges with a fuel atom, it will produce a fast neutral that can freely cross the confining magnetic field and escape; the initial atom from the wall remains as a cold ion in the plasma.

(b) The escaping fast neutrals do have a use; they can be collected by a suitable detector and their energy distribution can be measured. This allows one to measure the ion temperature, or even the ion velocity distribution function of the plasma. For this reason, sometimes beams of modest energy neutrals are injected into the plasma so that the escaping energetic neutrals can be analyzed.

(c) Energetic neutral beams can be used to heat plasmas to fusion energies. One starts by accelerating protons, deuterons, or tritons to energies in the range of 100 to 200 KeV; these are then passed through a neutral gas cell where the energetic ions are converted to energetic neutrals through charge exchange. The energetic neutrals cross the confining magnetic field and penetrate the fusion plasma. Some of the neutrals are ionized by encounters with the deuterons, tritons, and electrons of the plasma. Others charge exchange with the plasma ions but even in this case a relatively low energy neutral leaves the plasma and a highly energetic ion remains in the plasma. In all cases the plasma is heated.

Semi-Quantitative Calculation of Charge Exchange

Let us consider a proton approaching a neutral hydrogen atom as shown:

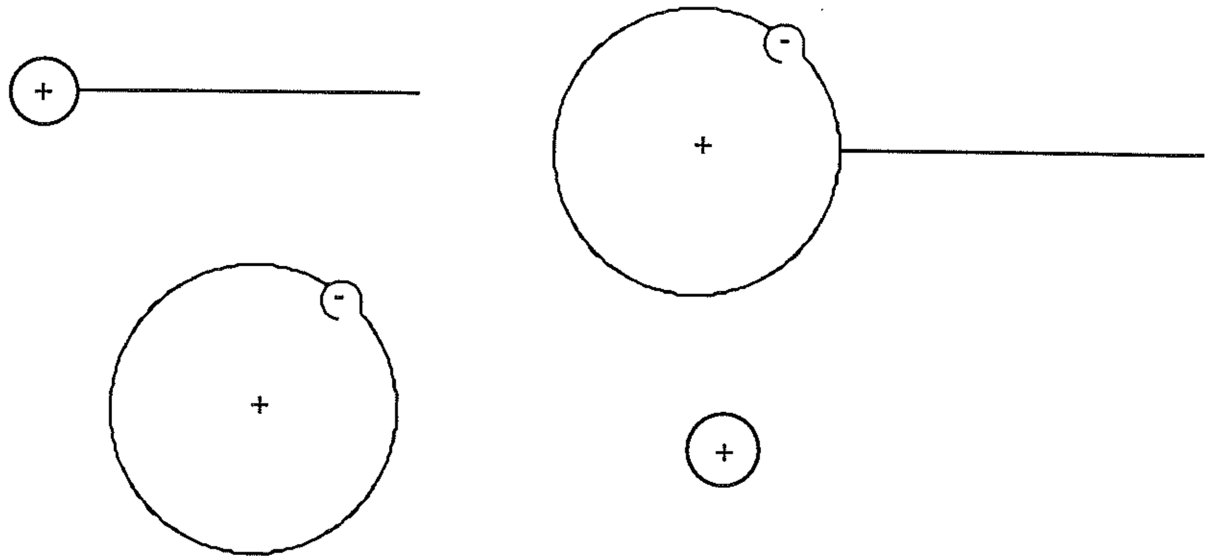


Fig. 7

There are quantum orbits for the electron orbiting each of the nuclei. To transfer from one atom to the other, the electron must pass through a region of negative kinetic energy and must tunnel through this region. This has to be treated quantum mechanically. In order to do this we will expand the wave functions of the combined system in terms of the wave functions of two isolated atoms.

During charge exchange, the ions are generally moving past each other slowly when compared to the electron orbital velocity. We can find the electronic wave function as if the protons were at rest at whatever their instantaneous separation is (the Born Oppenheimer approximation). We may write the wave function as a combination of the wave functions of the two protons. By the symmetry of the system we can write the combined wave functions as symmetric and antisymmetric combinations of the two possible atomic wave functions. Thus, we have

$$\Psi_I = \alpha_I (\Psi_1 + \Psi_2), \ \ \Psi_{II} = \alpha_{II} (\Psi_1 - \Psi_2). \qquad (111)$$

Since charge transfer generally takes place for proton separations large compared to the size of the atoms, Ψ_1 is very small in the region where Ψ_2 is appreciable and vise versa. Now α_I and α_{II} are determined by the normalization conditions

$$\int \Psi_I x \Psi_I^* d\tau = 1, \int \Psi_{II} x \Psi_{II}^* d\tau = 1. \tag{112}$$

By Eq. (110) this is equal to

$$\alpha_I\alpha_I^* \int (\Psi_1 + \Psi_2) x(\Psi_1 + \Psi_2)^* d\tau = 1, \tag{113}$$

and

$$\alpha_{II}\alpha_{II}^* \int (\Psi_1 - \Psi_2) x(\Psi_1 - \Psi_2)^* d\tau = 1. \tag{114}$$

By our argument that Ψ_1 is small where Ψ_2 is appreciable and vise versa, all the cross products involving Ψ_1's and Ψ_2's make negligible contributions. The squares of the Ψ_1's and Ψ_2's integrate to 1 so that Eqs. (112) and (113) give

$$2\alpha_I\alpha_I^* = 1, \tag{115}$$

$$2\alpha_{II}\alpha_{II}^* = 1, \tag{116}$$

and by symmetry

$$\alpha_I = \alpha_{II} = 2^{-1/2}. \tag{117}$$

We can now compute the energies associated with these two states from

$$E = \int \Psi H \Psi^* d\tau, \tag{118}$$

where H is the Hamiltonian for the system,

$$(h/2\pi)^2 \nabla^2 + e^2/|\mathbf{r}-\mathbf{R}_1| + e^2/|\mathbf{r}-\mathbf{R}_2| = H. \tag{119}$$

We get

$$E_I = \int \Psi_I H \Psi_I^* d\tau = (1/2)\int (\Psi_1 + \Psi_2)\, H(\Psi_1^* + \Psi_2^*) d\tau\ , \tag{120}$$

and

$$E_{II} = \int \Psi_{II} H \Psi_{II}^* d\tau = (1/2)\int (\Psi_1 - \Psi_2)\, H(\Psi_1^* - \Psi_2^*) d\tau. \tag{121}$$

There are two important integrals that contribute to the difference in energies; these are

$$I_1 = \int \Psi_1 (e^2/|\mathbf{r}-\mathbf{R}_2|) \Psi_1^*\, d\tau, \tag{122}$$

and

$$I_2 = \int \Psi_2 (e^2/|\mathbf{r}-\mathbf{R}_2|) \Psi_1^*\, d\tau. \tag{123}$$

Integrals with the subscripts 1 and 2 interchanged also enter but by symmetry are identical to the corresponding integrals in Eq. (121) and (122). The energies are

$$E_I \cong E_0 - (1/2)(I_1 - I_2), \tag{124}$$

and

$$E_{II} \cong E_0 - (1/2)(I_1 + I_2). \tag{125}$$

The difference in energy is

$$\Delta E = E_I - E_{II} = - I_2. \tag{126}$$

The major contribution to I_2 comes from the region around the mid-point between the two protons. To a good approximation I_2 is given by

$$I_2 = (8e^2/a_0)(\exp[-R/a_0]) , \tag{127}$$

where a_0 is the Bohr radius. If we denote the initial neutral atom by the subscript 1 and the initial proton by the subscript 2, then at t equal zero the wave function for the system is $\Psi_1 = 2^{1/2}(\Psi_I + \Psi_{II})$.

The two wave functions Ψ_I and Ψ_{II} have different frequencies, $\omega_I = 2\pi E_I/h$ and $\omega_{II} = 2\pi E_{II}/h$. Thus, as time goes on the two wave functions will get out of phase with each other; when Ψ_{II} is π out of phase with Ψ_I, then the wave function of the system will be Ψ_2 and the electron will have transfered from the initial neutral atom to the initial proton.

To estimate when this will happen we note that the difference in frequencies of the two states is largest when the separation of the nuclei is the smallest, i.e., at the distance of closest approach. Let us call this distance R_0; the difference in frequencies of the two states at that separation is

$$\Delta\omega = (16\pi e^2/ha_0)(\exp[-R_0/a_0]) = (8\omega_0)(\exp[-R_0/a_0]) , \tag{128}$$

where ω_0 is the orbital frequency associated with the ground state of hydrogen. The time the particles remain close together is roughly

$$\tau = 2R_0/V_0 . \tag{129}$$

Multipling $\Delta\omega$ by t and equating it to π gives the distance R_0 at which we can expect charge exchange to take place

$$(\Delta\omega)\tau = (16\omega_0)(R_0/V_0)(\exp[-R_0/a_0]) = \pi. \tag{130}$$

The strongest dependence on R_0 is in the exponential; we may expect R_0/a_0 to be about 10; if we assume this to be true then (128) gives

$$(16a_0\omega_0)(R_0/(a_0V_0) = (16v_e/V_0\ )(R_0/a_0) \sim 160v_e/V_0 , \tag{131}$$

where v_e is the orbital velocity of an electron in a Bohr hydrogen atom (velocity corresponding to 13.5eV). For protons in the range of 10eV, the right hand side of (129) is ~7200 which comes pretty close to making $R_0/a_0 = 10$. Even for very energetic ions with $V_0 = v_e$, (proton energy equal to 27,000 eV) $R_0/a_0 \sim 4$. If we take $R_0/a_0 = 10$, the charge exchange cross-section is $\sigma = 100\pi a_0^2 \sim 0.8 \times 10^{-14}$

For creating energetic neutral beams for heating fusion reactor plasmas, positively charged deuterons are accelerated to energies between 100 and 200 KeV. These high energy ions are then passed through a neutral gas cell where some of the ions are converted to neutrals through charge exchange; these neutrals can freely cross the magnetic field confining the plasma. They enter the hot plasma where they are ionized and deposit their energy in the plasma. These energetic ions also have large fusion cross-sections and significantly contribute to the fusion reaction rate. The fraction of the energetic ion beam that ends up as neutral atoms is determined by a balance between charge exchange and reionization by energetic neutral gas neutral ionizing collisions. At 100 KeV the charge exchange cross-section and the re-ionization cross-sections are about equal and about half the ion beam gets neutralized; at higher energies the charge exchange cross-section falls off while the ionization cross-section remains rather flat so that the fraction of the beam that ends up as neutral falls off.

Particle Orbits

We have seen that interactions between individual pairs of charged particles are to some approximation negligible. On the other hand, when many particles are interacting the fields can become quite large and the interactions can become quite important (this lead to the plasma oscillations we saw earlier). The fields due to a large number of charged particles are rather smooth with small fluctuations superimposed on them due to the discrete nature of the particles. The small fluctuations are responsible for scattering and collisions, while the macroscopic smoothed out fields give the collective motions of the plasma (coherent motion of many particles). Thus, to the extent that we can neglect collisions, we can approximate the plasma motion by treating the system as a collection of charged particles, each one moving in the smoothed-out field of all the other particles plus, of course, any externally-applied fields. We will see that we can gain much insight into the behavior of plasmas by investigating the motions of single charged particles in arbitrary electric and magnetic fields. We can go further and make the fields consistent with the motion of all the particles. We determine the fields from Maxwell's equations and the charges and currents associated with the particle motions. The particle motions are determined from the Lorentz force acting on the particles. Maxwell's equations are:

$$\nabla \times \mathbf{E} = -(1/c)\partial \mathbf{B}/\partial t \,, \tag{132}$$

$$\nabla \times \mathbf{B} = (1/c)\partial \mathbf{E}/\partial t + 4\pi \mathbf{j}/c \,, \tag{133}$$

$$\nabla \bullet \mathbf{B} = 0 \,, \tag{134}$$

and

$$\nabla \bullet \mathbf{E} = 4\pi\rho \,, \tag{135}$$

From Eqs. (133) and (136) we get the equation of continuity

$$\partial\rho/\partial t + \nabla \bullet \mathbf{j} = 0 \,, \tag{136}$$

where

$$\rho = \Sigma_i q_i \langle n_i \rangle \,, \tag{137}$$

and

$$\mathbf{j} = \Sigma_i q_i \langle v_i n_i \rangle \,. \tag{138}$$

Here ρ and $\mathbf{j}$ are the charge and current densities due to the plasma particles, i refers to the ith species of particle. The particles move according to their equation of motion:

Non-relativistic case

$$m_i d\mathbf{v}_i/dt = q_i(\mathbf{E}_i + [\mathbf{v}_i X \mathbf{B}_i]/c), \tag{139}$$

or in the relativistic case by

$$d\mathbf{p}_i/dt = q_i(\mathbf{E}_i + [\mathbf{v}_i X \mathbf{B}_i]/c), \tag{140}$$

with

$$\mathbf{p}_i = \gamma m_i \mathbf{v}_i, \ \gamma = (1- [v/c]^2)^{-1/2}.$$

Here the subscript i on **E** and **B** means that these quantities are to be evaluated at the position of particle i. These are the so-called Vlasov equations for a plasma. We will be studying them in some detail throughout this course.

A. <u>Cyclotron Motion</u>

The equation of motion of a particle is given by Eqs. (139) or (140). If we dot Eq. (139) with $\mathbf{v}_i$ then we get the energy equation

$$d(m_i v_i^2/2)/dt = q_i \mathbf{v}_i \cdot \mathbf{E}_i \,. \tag{141}$$

The magnetic term drops out of this equation since the magnetic force is always perpendicular to the velocity and hence does no work. Integrating gives

$$\Delta(m_i v_i^2/2) = \int q_i \mathbf{v}_i \cdot \mathbf{E}_i \, dt = \int_{\text{orbit}} q_i \, \mathbf{E}_i \cdot d\mathbf{s} \,, \tag{142}$$

where $d\mathbf{s}$ is a vector element of the orbit. If $\mathbf{E}$ is an electrostatic field (i.e., derivable from a potential, $\mathbf{E} = -\nabla\Phi$), then Eq. (142) may be written in conservation of energy form.

$$(m_i v_i^2/2) + q_i \Phi_i = \text{constant} \tag{143}$$

Returning to Eq. (139), the solution of this equation for arbitrary $\mathbf{E}$ and $\mathbf{B}$ can be very complicated. As we often do in physics, we will build up the more complicated motions from results obtained by looking at a number of simple cases.

The simplest possible situations of course involve spatially uniform, time independent $\mathbf{E}$ and $\mathbf{B}$ fields. Let us first consider the case of a uniform $\mathbf{B}$. Since the magnetic force is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$, there is no force on the particle in the $\mathbf{B}$ direction and the velocity in that direction is constant. We need not consider this velocity any more. The velocity perpendicular to the magnetic field has a constant magnitude by Eq. (142), the particle moves in a circular orbit about the magnetic field. One can obtain the radius of this circle by balancing the centrifugal force against the the magnetic force.

$$m v_\perp{}^2/r = q v_\perp B/c \ , \tag{144}$$

or

$$r = (mcv_\perp/qB).$$

The quantity qB/mc is the angular frequency of the particle and is called the cyclotron frequency, ω_c = qB/mc. The radius is called the Larmor radius. Numerically these are given by:

$$\omega_{ce} \cong 2x10^7 B \ ,$$

$$\omega_{cp} \cong 10^4 B \ ,$$

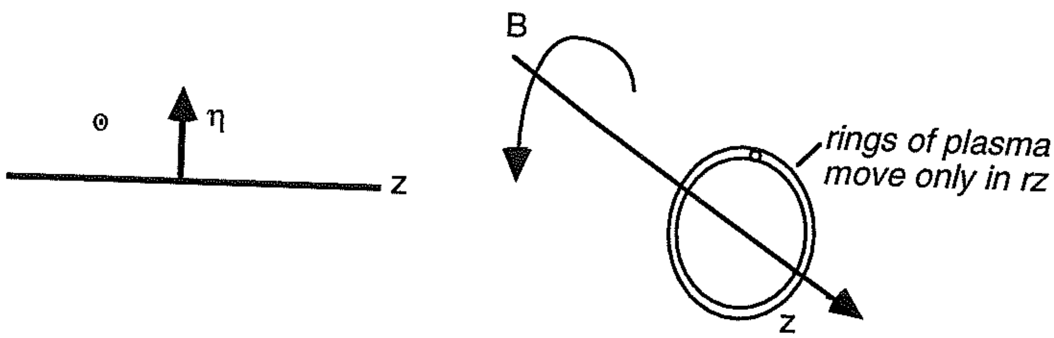
and

$$\omega_{c,Z,A} \cong (Z/A)10^4 B \ ,$$

where Z is the ionic charge and A is the atomic weight.

The General Two Dimensional Cylindrical Time Dependent Problem

Let us consider a situation in which the **B** lines are circles centered on the z axis. We consider only motion in the r,z directions (there is no variation in the Q direction) so that the circular symmetry is preserved. We use cylindrical coordinates with only r,z variations. We are still working in the large q/m limit so that the only important velocity is the $(\mathbf{E} \times \mathbf{B})c/B^2$ velocity; the other drifts, however, contribute to the current. The situation is as shown in the Figure.



The equations for **B**, **V**, and **E**:

$$\mathbf{B} = \mathbf{e}_\Theta B(r,z) \tag{IS.1}$$

$$\mathbf{V} = (\mathbf{E} \times \mathbf{B})c/B^2, \tag{IS.2}$$

$$\mathbf{E} = -(\mathbf{V} \times \mathbf{B})/c, \tag{IS.3}$$

$\mathbf{E}$ has only r,z components because of the assumed perfect conductivity of the plasma. We take the plasma number density to be N (equal numbers of ions and electrons with the ions having charge e). The density satisfies the continuity equation

$$\partial N/\partial t + \nabla \bullet N\mathbf{V} = 0 \qquad \text{IS.4}$$

or

$$[(dN(r,z)/dt)_{FTM}]/N(r,z) + \nabla \bullet \mathbf{V} = 0. \qquad \text{IS.5}$$

The subscript FTM means following the motion of an element of the plasma. We also have Maxwells equations

$$\nabla \times \mathbf{E} = -(\partial \mathbf{B}/\partial t)/c \qquad \text{IS.6}$$

$$\nabla \times \mathbf{B} = (\partial \mathbf{E}/\partial t)/c + 4\pi \mathbf{j}/c\ . \qquad \text{IS.7}$$

Again we use the vector identity

$$\nabla \times (\mathbf{U} \times \mathbf{V}) = \mathbf{V}\bullet\nabla\mathbf{U} - \mathbf{U}\bullet\nabla\mathbf{V} + \mathbf{U}(\nabla\bullet\mathbf{V}) - \mathbf{V}(\nabla\bullet\mathbf{U}) \qquad \text{IS.8}$$

and replacing $\mathbf{U}$ by $\mathbf{B}$ and $\mathbf{V}$ by $\mathbf{V}$, we get

$$\nabla \times (\mathbf{V} \times \mathbf{B}) = -\nabla \times (\mathbf{B} \times \mathbf{V}) = \mathbf{B}\bullet\nabla\mathbf{V} - \mathbf{V}\bullet\nabla\mathbf{B} + \mathbf{V}(\nabla\bullet\mathbf{B}) - \mathbf{B}(\nabla\bullet\mathbf{V}). \qquad \text{IS.9}$$

The $\mathbf{V}(\nabla\bullet\mathbf{B})$ is zero because $\nabla\bullet\mathbf{B}$ is zero. If we write $\mathbf{B}\bullet\nabla\mathbf{V}$ in component form we get

$$\mathbf{e}_\Theta B(r,z)\bullet\{[\mathbf{e}_r(\partial/\partial r) + (\mathbf{e}_\Theta/r)(\partial/\partial\Theta) + \mathbf{e}_z(\partial/\partial z)][\mathbf{e}_r V_r + \mathbf{e}_z V_z]\} =$$

$$(B(r,z)/r)(\partial \mathbf{e}_r V_r/\partial\Theta) = [\mathbf{e}_\Theta B(r,z)][V_r/r] \qquad \text{IS.10}$$

since dotting with $\mathbf{e}_\Theta$ keeps only the Θ derivitive and $\mathbf{e}_r$ is the only quantity with a Θ dependence. Now substituting E from IS.3 in IS.6 and making use of IS.9 and IS.10 gives

$$(\partial \mathbf{e}_\Theta B(r,z)/\partial t) = [\mathbf{e}_\Theta B(r,z)][V_r/r] - \mathbf{V}\bullet\nabla[\mathbf{e}_\Theta B(r,z)] - \mathbf{e}_\Theta B(r,z)\nabla\bullet\mathbf{V}. \qquad \text{IS.11}$$

Now

$$\mathbf{V}\bullet\nabla[\mathbf{e}_\Theta B(r,z)] = \mathbf{e}_\Theta(\mathbf{V}\bullet\nabla B) + B(r,z)\mathbf{V}\bullet\nabla\, \mathbf{e}_\Theta \qquad \text{IS.12}$$

$$\nabla\, \mathbf{e}_\Theta = \mathbf{e}_\Theta \mathbf{e}_r,\ \mathbf{V}\bullet\nabla\, \mathbf{e}_\Theta = 0, \text{ since } \mathbf{V} \text{ has no } \Theta \text{ component.} \qquad \text{IS.13}$$

Therefore, IS.11 gives

$$(\partial \mathbf{e}_\Theta B(r,z)/\partial t) = - \mathbf{e}_\Theta B(r,z)\nabla\bullet\mathbf{V} - \mathbf{e}_\Theta \mathbf{V}\bullet\nabla[B(r,z)] + [\mathbf{e}_\Theta B(r,z)][V_r/r] \qquad \text{IS.14}$$

or

$$B(r,z)][V_r/r] - \mathbf{V}\bullet\nabla[B(r,z)] - B(r,z)\nabla\bullet\mathbf{V} = \partial B(r,z)/\partial t \qquad \text{IS.15}$$

$$\partial B(r,z)/\partial t + \nabla\bullet B(r,z)\mathbf{V} - B(r,z)][V_r/r] = 0 \qquad \text{IS.16}$$

This equation can be rewritten as

$$[(dB(r,z)/dt)_{FTM}]/B(r,z) + \nabla \bullet V - V_r/r] = 0. \qquad \text{IS.17}$$

Equation IS.17 is similar to but not exactly the same as the continuity equation for N; in fact using the continuity equation, IS.5, we can write IS.17 as

$$[(dB(r,z)/dt)_{FTM}]/B(r,z) - [(dN(r,z)/dt)_{FTM}]/N(r,z) - V_r/r = 0. \qquad \text{IS.18}$$

Now V_r/r is $(dr/dt)_{FTM}]/r$, so that IS.17 becomes

$$(dB(r,z)/dt)_{FTM}]/B(r,z)-[(dN(r,z)/dt)_{FTM}]/N(r,z)-(dr/dt)_{FTM}]/r=0 \qquad \text{IS.19}$$

Equation IS.19 can be integrated directly giving

$$\ln B(r,z) - \ln N(r,z) - \ln r = \text{constant} \qquad \text{IS.20}$$

or

$$B(r,z)/[N(r,z)r] = B(r_0,z_0)/[N(r_0,z_0)r_0] \qquad \text{IS.21}$$

Here r_0, z_0 is the r,z position that a particle starts out at $t = 0$. There is a simple physical interpretation of this result. Consider a ring of plasma of dimensions dr_0, dz_0 starting at r_0, z_0. Let it move to position r,z where its dimensions are drdz. The magnetic flux through drdz must be the same as that through dr_0dz_0; or

$$B_0dr_0dz_0 = Bdrdz = \Delta\Phi_0. \qquad \text{IS.22}$$

On the other hand, the total number of particles in the ring is conserved; this gives

$$N_0r_0dr_0dz_0 = Nrdrdz. \qquad \text{IS.23}$$

Dividing IS.22 by IS.23 gives IS.21; the magnetic field changes with the cross section area of the ring so as to keep the flux constant while the density also changes because of changes in the circumference of the plasma ring.