- L. Landau
  - part (i) → temporal problem. → easier.
  - part (ii) → spatial problem

Today.

Collisionless Pla.

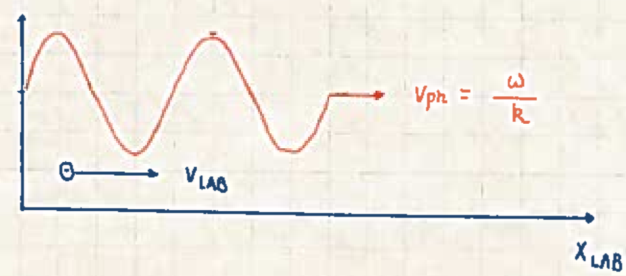
Moment desc.

- $$\boxed{\frac{\partial}{\partial t} f_\alpha + \underline{v} \cdot \nabla_x f + \underline{a} \cdot \nabla_v f = 0}$$

- The equation
$$\frac{\partial}{\partial t} f_\alpha + \underline{v} \cdot \nabla_x f_\alpha + \underline{a} \cdot \nabla_v f = 0$$
  - Dominantes by collective field.
  - neglect near-neighbor interaction.
  - key phenomenon → Landau Damping

→ $\boxed{\text{Wave - Particle interaction is dominant}}$

- (In lab frame)
$$\frac{d}{dt} \underline{v}_{Lab} = \frac{q}{m} E(t) \sin(k x_{LAB} - \omega t)$$



- In the wave frame
  - $x_{LAB} = x + \frac{\omega}{k} t.$
  - $\boxed{E(x,t) = E(t) \sin(kx)}$ → wave - frame → no more t - dependence

- In wave frame for e^-.
$$\frac{d}{dt} v_x(t) = -\frac{e}{m} E(t) \sin(kx)$$

∴ for $\boxed{\frac{d}{dt} E = 0}$ → constant amplitude approximation.

$$\frac{d}{dt} v(t) = -\frac{e}{m} E_0 \sin kx$$

$$= \frac{\partial}{\partial x}\left[\frac{eE_0}{mk} \cancel{\sin} \cos kx\right]$$

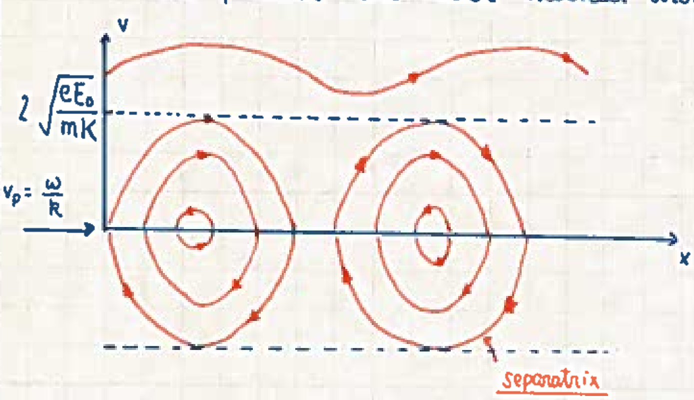
i.e., a potential

$$\boxed{mv\frac{dv}{dt} = \frac{dx}{dt}\cdot\frac{\partial}{\partial x}\left[\frac{eE_0}{k}\cos(kx)\right]} \longrightarrow \text{CONSERVATION LAW.}$$

- $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \frac{eE_0}{k}\cdot\left\{\cos(kx) - \cos(kx_0)\right\}$

- $\boxed{v = \left\{v_0^2 + \frac{2eE_0}{mk}\left[\cos(kx) - \cos(kx_0)\right]\right\}^{1/2}}$

  - if $|v_0| > 2\sqrt{\frac{eE_0}{mk}} \longrightarrow v$ is never 0. i.e., the particles are never turned around. !!



- Trapped Particles

$$\frac{1}{2}|v_0| < \sqrt{\frac{eE_0}{mk}}$$

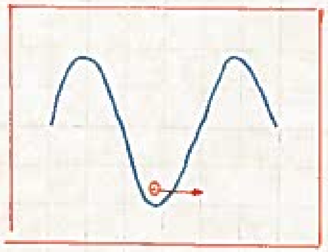
- Passing Particles

$$\frac{1}{2}|v_e| > \sqrt{\frac{eE_0}{mk}}$$

- Note that $v_e \sim \sqrt{E_0}$, i.e., trapping is a non-linear effect. However, Landau Damping is a linear effect, i.e., trapping is not a physical picture for Landau Damping.

- Bounce frequency calculation

(Bounce freq. cont)



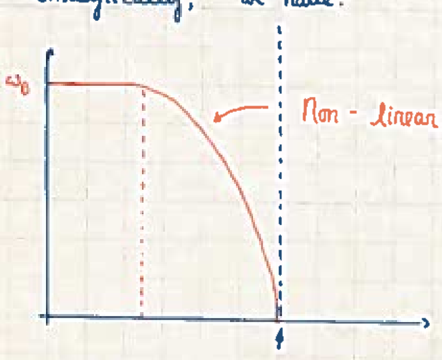
$$\frac{d^2}{dt^2} x = -\frac{e}{m} E_0 \sin(kx)$$

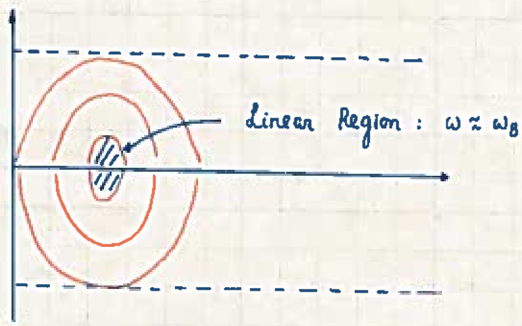
- (Small Oscillation)

$$\frac{d^2}{dt^2} \delta x + \frac{e}{m} E_0 k \, \delta x = 0$$

$$\longrightarrow \boxed{\omega_B = \sqrt{\frac{e E_0 k}{m_e}}} \longrightarrow \text{DEF. of bounce frequency.}$$

- $v_{TRAP} = 2 \dfrac{\omega_B}{k}$
- Analytically, we have.





- If $\dfrac{d}{dt} E = 0$
  - ~~No exchange in energy~~
  - No net exchange of momentum and energy between particle and wave $\longrightarrow$ $\boxed{\text{no damping}}$
  - Trapped particles are a non-linear feature.
    - # of particles trapped

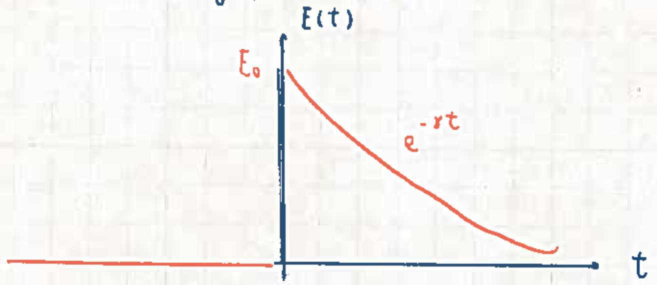
$$\boxed{n_T \sim f\left(v = \frac{\omega}{k}\right) \cdot \underbrace{\left[4 \frac{\omega_B}{k}\right]}_{2 v_{TRAPPED}}}$$

$$\boxed{n_T \sim \sqrt{E_0}}$$

Landau Damping Environment

- If Linear phenomenon $\longrightarrow$ no trapped particles (???)
- Exchange of energy + momentum $\longrightarrow$ $\boxed{\dfrac{d}{dt} E(t) \neq 0}$

- Real damping proc. is :



→ $E(t) \sim E_0\, e^{-\gamma t}$

→ Wave has to disappear before trapping occurs

$$\boxed{\gamma >> \omega_B}$$

- Momentum Transfer between electrons and a damped wave.

  - Eq. of motion

$$\frac{d}{dt} v = \frac{q}{m} E_0\, e^{-\gamma t} \cos(kx)$$

$$\boxed{\begin{aligned} v(t) &= v_0 + \frac{q}{m}\int_0^t E_0\, e^{-\gamma t'} \cos(k\,x(t'))\, dt' \\ x(t) &= x_0 + \int_0^t v(t')\, dt' \end{aligned}}$$

  - The above equation is not solvable exactly, because $v(t)$ is a function of $x(t)$, and vice versa...

- Perturbation Theory
  i.e., small E_0 → $\boxed{\omega_B << \gamma}$

$$\boxed{\begin{aligned} &\bullet\ v(t) = v_0 + v^{(1)}(t) + v^{(2)}(t) + \dots \\ &\bullet\ x(t) = (x_0 + v_0(t) \times t) + x^{(1)}(t) + x^{(2)}(t) + \dots \end{aligned}}$$

- What we want is the momentum transfer.

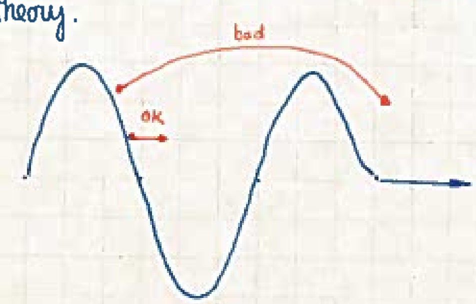
$$\boxed{\Delta p(t) = m\left[v(t) - v_0\right]} \rightarrow \text{for 1 particle}$$

Also interested in momentum transfer in a collection of particles

- look at.

$$v^{(1)} = \frac{qE_0}{m}\int_0^t dt'\; e^{-\gamma t} \cos\left[k(x_0 + v_0 t) + \cancelto{0}{x^{(1)}(t) + x^{2}(t) + \dots} \right]$$

- i.e., small spatial displacement is necessary for pert. theory.



$$\boxed{v^{(1)}(t) = \frac{qE_0}{2m}\int_0^t dt'\; e^{-\gamma t}\, e^{ikx_0}\, e^{ikv_0 t} + c.c.}$$

- No momentum exchange in the first order...

Today.

Wave - Particle Interaction

- Perturbation Expansion.
  - Wave frame.

$$v = v^{(0)} + v^{(1)} + v^{(2)} + \dots$$

$$x = (x_0 + v_0 t) + x^{(1)} + x^{(2)} + \dots$$

- Our assumption:

$$\boxed{kx \ll 1}$$

- $$v^{(1)} = -\frac{eE_0}{m}\int_0^t dt' \left[\frac{e^{ikx_0} e^{ikv_0 t}}{2} + c.c. \right]$$

$$\boxed{\mathcal{E} = \frac{eE_0}{2m}} \qquad \boxed{e^{-\gamma t}}$$

$$v^{(1)} = -\mathcal{E}\int_0^t dt' \left[e^{ikx_0} e^{ikv_0 t} + c.c. \right]$$

$$= -\mathcal{E}\frac{e^{ikx_0}}{ikv_0 - \gamma}\left[e^{-\gamma t} e^{ikv_0 t} - 1 \right] + c.c.$$

$$\boxed{\Delta p^{(1)} = m v^{(1)} = -m\mathcal{E}\frac{e^{ikx_0}}{ikv_0 - \gamma}\left[e^{-\gamma t} e^{ikv_0 t} - 1 \right] + c.c.}$$

- $\langle \Delta p^{(1)} \rangle_{x_0} = 0$ if x_0 is uniformly distributed.

  $\longrightarrow$ no global modification.

(uniform in space)

- Landau damping breaks down in a non-uniform plasma because of effects in $\Delta p^{(1)}$

- Early time behavior.

  - $\Delta p^{(1)}(t \to 0^+) \simeq -m\mathcal{E}t\, e^{ikx_0} + c.c.$

  - $$\boxed{\Delta p^{(1)}(t \to 0^+) = -2m\mathcal{E}t \cos kx_0} \leftarrow$$ particle sees a constant E field @ early time.

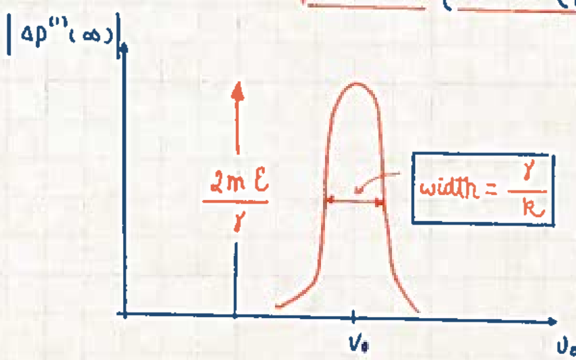
limit as $t \to \infty$

$$\Delta p^{(1)} = - m \mathcal{E} e^{ikx_0} \frac{(-1)}{ikv_0 - \gamma} + c.c.$$

$$\Delta p^{(1)} = m \mathcal{E} \left\{ \frac{-ikv_0 - \gamma \; (e^{ikx_0})}{(kv_0)^2 + \gamma^2} + \frac{\left[ikv_0 - \gamma \right] e^{-ikx_0}}{(kv_0)^2 + \gamma^2} \right\}$$

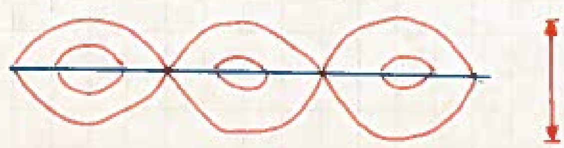
$$\Delta p^{(1)}(t \to \infty) = \frac{m \mathcal{E}}{(kv_0)^2 + \gamma^2} \left\{ \cos kx_0 \cdot \left[-2\gamma \right] + 2kv_0 \sin kx_0 \right\}$$

$$= \boxed{2m \mathcal{E} \left\{ \frac{kv_0 \sin kx_0 - \gamma \cos kx_0}{(kv_0)^2 + \gamma^2} \right\}}$$



- $\frac{2m\mathcal{E}}{\gamma}$ = maximum momentum transfer
- in NL analysis, the trapping width is

$$\boxed{\Delta v = \frac{\omega_B}{k}}$$



width $\sim \frac{\omega_B}{k}$

- In Landau damping, width is set by γ/k, not ω_B/k !!

• 2nd order analysis.

$$\boxed{\langle \Delta p^{(2)} \rangle_{x_0}}$$

$$x^{(1)}(t) = \int_0^t dt' \; v^{(1)}(t')$$

$$= \int_0^t dt' \; - \mathcal{E} e^{ikx_0} \left[\frac{e^{-\gamma t} e^{ikv_0 t} - 1}{(ikv_0 - \gamma)} \right] + c.c.$$

$$x^{(1)}(t) = -\mathcal{E}\, e^{ikx_0}\left\{ \frac{e^{-\gamma t} e^{ikv_0 t} - 1}{(ikv_0 - \gamma)^2} - \frac{t}{(ikv_0 - \gamma)} \right\} + \text{c.c.}$$

$$v(t) = -\underbrace{\frac{eE_0}{m}}_{2\mathcal{E}} \int_0^t dt'\, e^{-\gamma t'} \cos\left[kx_0 + kv_0 t' + kx^{(1)} \right]$$

$$\cos(kx_0 + kv_0 t)\cos(kx^{(1)}) \longrightarrow$$ do not contribute in the 2nd order perturbation

$$- \sin(kx_0 + kv_0 t)\sin(kx^{(1)})$$

- $$\langle v^{(2)}(t)\rangle_{x_0} = \left\langle + \frac{eE_0}{m} \int_0^t dt'\, e^{-\gamma t} \sin(k_0 x + kv_0 t)\left[kx^{(1)}(t')\right] \cancel{+\text{ c.c.}} \right\rangle_{x_0}$$

$$= \frac{k\mathcal{E}}{i} \left\langle \int_0^t dt'\, e^{-\gamma t} e^{ikv_0 t'} e^{ikx_0} x^{(1)}(t') \right\rangle_{x_0} + \text{ c.c.}$$

$$= \frac{k\mathcal{E}}{i} \int_0^t dt'\, e^{-\gamma t} e^{ikv_0 t} \left\{ -\mathcal{E}\left[\frac{e^{-\gamma t} e^{-ikv_0 t} - 1}{(ikv_0 + \gamma)^2} + \frac{t}{ikv_0 + \gamma} \right] \right\}$$ This sign is corred.

$$+ \text{ c.c.}$$

- Landau does everything in Laplace space, which hides the secular term in $x^{(1)}(t)$

$$\langle v^{(2)}(\infty)\rangle_{x_0} = ik\mathcal{E}^2 \left\{ \frac{1}{(ikv_0+\gamma)^2} \cdot \left[\frac{1}{2\gamma} - \frac{(-1)}{ikv_0 - \gamma} \right] + \left(\frac{1}{ikv_0 + \gamma}\right)\left(\frac{1}{(ikv_0 - \gamma)^2}\right) \right\}$$

$$+ \text{ c.c.}$$

$$\boxed{\int_0^\infty dt'\, e^{-\gamma t} e^{ikv_0 t}\, t = -\frac{\partial}{\partial \gamma} \cdot \left[\frac{-1}{(ikv_0 - \gamma)} \right] = \frac{1}{(ikv_0 - \gamma)^2}}$$

- $$\langle v^{(2)}(\infty)\rangle_{x_0} = \frac{ik\mathcal{E}^2}{(ikv_0 + \gamma)} \left\{ \frac{1}{ikv_0 + \gamma_0}\left[\frac{1}{2\gamma} + \frac{1}{ikv_0 - \gamma} \right] + \frac{1}{(ikv_0 - \gamma)^2} \right\}$$

$$+ \text{ c.c.}$$

$$= \frac{k\mathcal{E}^2}{\left[(kv_0)^2 + \gamma^2\right]} \left[i\gamma + kv_0 \right] \cdot \left\{ \text{same} \right\} + \text{ c.c.}$$

$$= \frac{2k\mathcal{E}^2}{\left[(kv_0)^2 + \gamma^2\right]} \text{Re}\left[(i\gamma + kv_0)\left\{ \text{same} \right\} \right]$$

- $$\mathrm{Re}\left[(i\gamma + kv_0) \cdot \left[-\frac{1}{(kv_0)^2+\gamma^2} + \frac{1}{2\gamma}\,\frac{(-ikv_0+\gamma)}{(kv_0)^2+\gamma^2} + \frac{\left[\gamma^2-(kv_0)^2\right] + 2i\,kv_0\gamma}{\left[\gamma^2-(kv_0)^2\right]^2 + (2kv_0\gamma)^2} \right] \right]$$

$$= \cancel{\frac{\gamma k v_0}{2\gamma\left[(kv_0)^2+\gamma^2\right]}} - \cancel{\frac{1}{2}\,\frac{kv_0}{(kv_0)^2+\gamma^2}} + \frac{kv_0\left[\gamma^2-(kv_0)^2\right] - 2kv_0\gamma^2}{\left[\gamma^2-(kv_0)^2\right]^2 + (2kv_0\gamma)^2}$$

- $$\langle v^{(2)}(\infty)\rangle_{x_0} = \frac{2k\varepsilon^2\cdot(-1)}{\left[(kv_0)^2+\gamma^2\right]} \cdot \left\{ \frac{(kv_0)\left[(kv_0)^2+\gamma^2\right]}{\left[\gamma^2+(kv_0)^2\right]^2} \right\}$$

$$\boxed{\langle v^{(2)}(\infty)\rangle_{x_0} = (-1)\,2k\,\varepsilon^2\,\frac{kv_0}{\left[\gamma^2+(kv_0)^2\right]^2}}$$