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Introduction to Plasma Physics

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I. Introduction

A plasma may be roughly defined as a system containing mobile charges, either positive, negative, or both, in which the electric and magnetic interactions between particles play a dominant role in the dynamics of the systems. Some examples of plasmas are:

(1) Gases which are heated to such high temperatures that some electrons are detached from the constituent atoms and molecules.

(2) The gas in any discharge. Here the electrons gain sufficient energy from the applied electric field that they can ionize other atoms and molecules.

(3) Interstellar gas which is ionized by the radiation from stars.

(4) The ionosphere, where the outer layers of the earth's atmosphere are ionized by solar radiation.

(5) The mobile electrons in metals and semiconductors. Here the perturbations caused by neighboring atoms weaken the binding of the electrons to the atoms to such an extent that some of the electrons become free to move about through the material.

In addition to these systems there are systems which contain mobile charges but which are not generally classified as plasmas. Examples of systems in this category are salt solutions, or gases at temperatures where there are very few free electrons. Systems of this type contain so many neutrals that collisions between the neutrals and charged particles mask many of the properties which are usually associated with a plasma (for example, plasma oscillations of all types are suppressed). One can go continuously from a system which is not a plasma to one which is, as the example of raising the temperature of a gas shows. We shall not say much about the electron plasmas in solids or about conducting solutions, though some of our consideration will be applicable to these systems.

Plasma has been called by some people the fourth state of matter. In spite of the fact that there is no sharp transition from the gas to the plasma state, there is much justification for this point of view. One essential feature of a plasma, as compared with most other fluids, is the ability to carry current and so to interact strongly with electric and magnetic fields. This property leads to a great wealth of phenomena not exhibited by nonplasmas.

To produce a plasma by heating a gas we must heat it to such a temperature that collisions between particles are able to detach electrons. Since typical binding energies for electrons in atoms (first ionization potential) range from about 4 eV for Cs to 24 eV for He, the colliding particles must have comparable energies. A temperature of 11,600°K corresponds to an average energy of 1 eV.[1]

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[1] The Maxwell-Boltzmann velocity distribution function is described in Appendix I.

$$kT = 1 \text{ eV} \quad \text{for} \quad T = 11{,}600^\circ\text{K} \tag{1}$$

Thus we see that temperatures of thousands of degrees are required.

<u>Problem:</u> Compare the average kinetic energy per particle with the energy in eV/particle involved in the following changes of state:

(a) melting of water
(b) boiling of water at STP

We do not strictly require kT to equal the ionization energy to obtain appreciable ionization. In a gas in thermal equilibrium there are always particles with considerably more energy than the average, and collisions between these particles can produce ionization. The charged particle density is determined by a balance between such ionization and recombination of the ions and electrons. In a low-density gas the ions and electrons relatively rarely approach one another, and recombination rates may be quite low. Thus a large percentage ionization can exist even when $kT << E_{ionization}$.

For a gas in thermal equilibrium, the degree of ionization depends only on the temperature. Every process that can produce ionization can take place in reverse to give recombination. At thermodynamic equilibrium, these rates are the same. In fact, every process is exactly balanced by the inverse process (law of detailed balance).

For thermal equilibrium the degree of ionization is determined by the Saha equation. This equation has essentially the form

$$\frac{n_e n_i}{n_o} \cong \frac{1}{\lambda^3} \frac{e^{-E_i/kT}}{g_i} \tag{2}$$

$\lambda^2 = h^2/2m_e kT$ (Debroglie wavelength for electron of average energy)

Here n_e, n_i, and n_o are the electron, ion and neutral densities, E_i is the ionization energy, g_i is the number of ground state levels, m_e is the electron mass, and h is Planck's constant. A derivation of this formula lies outside the scope of these notes (see Kittel, Statistical Mechanics), but we point out the physical significance of the terms.

For a singly ionized substance with overall charge neutrality, $n_i = n_e$ and Eq. (2) can be written in the form

$$\frac{n_e}{n_o} \cong \frac{V_e}{\lambda^3} \frac{e^{-E_i/kT}}{g_i} \tag{3}$$

where $V_e = 1/n_e$ is the volume per electron. Now the number of free (continuum) states for an electron in a volume V is essentially V/λ^3. Thus V_e/λ^3 is the number of free states per electron (number of free states for a volume containing 1 electron). If there is one free electron in the volume V_e, then there is one ion and this ion has g_i ground states. Thus in the volume V_e there are V_e/λ^3 unbound states g_i ground states available to the electron. The ground states must be weighted by the factor $e^{E_i/kT}$ (the probability of occurrence of a state of energy E in thermal equilibrium is $e^{-E/kT}$; the binding energy is negative, $-E_i$). Thus the ratio of the number of unbound states to ground states is $\frac{V_e}{\lambda^3} \frac{e^{-E_i kT}}{g_i}$, which is also the ratio of the number of electrons to neutrals, n_e/n_o, and this gives the above Saha equation. In this illustration we have neglected bound states other than the ground state.

Problem: Find the percent ionization for H at densities of $n = 1$ (interstellar space), $n = 10^{14}$ (solar surface), 10^{19} (atmospheric density), 10^{22} (liquid density), and temperatures of .1 eV, 1,160°K, 1 eV, 10 eV, 100 eV, $E_i = 13.6$ eV, $g_i = 1$.

Besides the thermal equilibrium plasma there are many types of plasmas which are not in thermal equilibrium. Examples of these are: most discharge plasmas, and the interstellar gas which is ionized by the radiation from stars.

For discharge-type plasmas, the electrons gain enough energy from the electric field so that they can knock other electrons from the constituent atoms of the gas. Thus, typically the electron energies in a discharge are of the order of a few eV. To heat them to a higher temperature than this is difficult, because the electrons lose so much energy in ionizing and exciting the gas atoms. These losses can only be overcome with large currents and high powers.

The plasma in a gas discharge is rarely in thermal equilibrium. The mean free paths of the particles are usually greater than the size of the container. Thus the ions and neutral atoms, though heated slightly by electron collisions, remain at temperatures close to that of the walls; i.e., kT_{wall} of about 1/40 eV. As we shall see later, electric fields occur naturally near the walls which reflect most of the electrons, thus isolating them from the wall. Also, the radiation level is substantially below the level corresponding to thermodynamic equilibrium; the plasma is optically thin.

Problem: Calculate the power that would be radiated per cm^2 by a 10 eV plasma if it were to radiate as a blackbody.

Even though the radiation levels are low compared with blackbody levels, a lot of energy is carried off by radiation. The electrons, in colliding with the atoms of the gas, raise them to excited states. Many of the excited states give up their energy immediately as radiation. Since energetic electrons are generally more efficient at exciting and ionizing, the energetic electrons tend to be brought down to lower energies and their number is depleted. Thus such discharges contain fewer energetic electrons than would an equilibrium plasma with the same average energy.

Rather than having a Maxwellian distribution,

$$f(v) \propto e^{-mv^2/2kT} \tag{4}$$

such plasmas have distributions like that shown in Fig. 1, which are more or

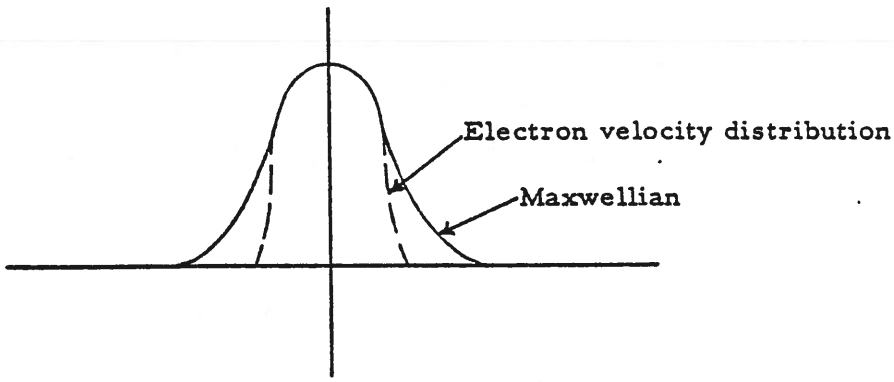


Figure 1

less sharply cut off. A distribution which is often used is the Druyvesteyn distribution

$$f(v) \propto e^{-v^4/v_o^4} \tag{5}$$

In contrast to the plasma in thermodynamic equilibrium, the properties of nonequilibrium plasmas are strongly dependent on the cross sections for the various processes. Of course when such a plasma is in a steady state, the rate of ionization must be balanced by the rate of recombination. However, here each process does not have to be balanced by the inverse process. For example, we could have ionization taking place in the discharge and all the recombination taking place at the wall of the tube.

Because of the many processes and their various cross sections, the analysis of gas discharges or other nonequilibrium plasmas can become extremely complicated. In the next section we summarize the most important collision processes.

II. Particle Interactions, Neutrality, and Collective Behavior

As has already been mentioned, in order for us to produce a plasma, particle energies (electron, ion, atomic, or photon) must be at least comparable to the ionization energy of the constituent atoms. The electrons which are detached will have about the same energy. Particle energies of the order of 10% or even perhaps a few percent might be considered comparable. Now the magnitude of the binding energy of an electron to an atom is about equal to its potential energy (perhaps one half of it) when bound to the atom. When it becomes detached, its mean distance from an ion is much greater than atomic dimensions, and consequently the mean potential energy of the electrons is much less than the mean kinetic energy; a basic property of all plasmas. For example, consider a Cs plasma at a density of 10^{12} electrons/cm^3 and a temperature of 2500°K (use of the Saha equation shows that Cs under these conditions will have a high percent ionization). Now the mean distance between an electron and an ion is 10^{-4} cm and the corresponding potential energy is

$$\begin{aligned} \frac{e^2}{r} &= \frac{e^2}{a_o}\,\frac{a_o}{r} = 27 \times \frac{5 \times 10^{-9}}{10^{-4}} \\ &= 135 \times 10^{-5}\ \text{eV} \simeq 10^{-3}\ \text{eV}. \end{aligned} \tag{6}$$

(The Cs is singly ionized.)

Thus the free electron as it moves through a plasma is, most of the time, in regions of small potential; so that the effect on the trajectory is small except during the relatively rare close approaches — i.e., during collisions. The cumulative effect of many small collisions will, however, turn out to be important. On the other hand, because of the long-range nature of the electric interactions, even slight excesses of electric charge over extended regions can produce potential energies and hence interactions which are not negligible compared to the particle kinetic energies. To illustrate this, consider an infinite homogeneous plasma. Let us ask how much energy it would take to move all the electrons in a spherical region of radius r to its surface.

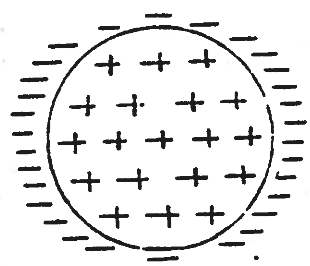


Figure 2

We hold the ions fixed at a uniform density. The potential energy stored in the electric field is

$$W_E = \int \frac{E^2}{8\pi}\, dv \tag{7}$$

The electric field is radially outward and is given by

$$E(\rho) = \begin{cases} \dfrac{1}{\rho^2}\,\dfrac{4}{3}\,\pi\, n_i\, Ze\,\rho^3 & \rho < r \\ 0 & \rho > r \end{cases} \tag{8}$$

where ρ is the radius, n_i is the density of ions, Ze is the charge on an ion, $\frac{4}{3}\pi n_i Ze\rho^3$ is the charge Q within the sphere of radius ρ, and Q/ρ^2 is the field it produces. Substituting Eq. (8) into Eq. (7) gives

$$W_E = \int \frac{2\pi}{9} n_i^2 z^2 e^2 \rho^2 4\pi \rho^2 d\rho$$

$$= \frac{8\pi^2}{9 \times 5} n_i^2 z^2 e^2 r^5 = \frac{8\pi^2}{9 \times 5} n_e^2 e^2 r^5 \tag{9}$$

where n_e is the undisturbed electron density.

Problem: Consider a plasma with an electron temperature of 10 eV and a density of 10^{14} electrons and ions per cubic centimeter. Compare the average kinetic energy of the electrons with the work per electron required to remove all the electrons from a sphere of 1 cm radius. (The ions are held at the initial density.)

Let us equate W_E to the kinetic energy of all the electrons within the sphere

$$\frac{3}{2} kT \frac{4}{3} \pi n_e r^3 = \frac{8\pi^2}{9 \times 5} n_e^2 e^2 r^5 \tag{10}$$

$$r^2 = 45 \frac{kT}{4\pi n_e e^2} \tag{11}$$

This is the radius of a sphere where the electrons just have enough kinetic energy to remove themselves from the sphere; i.e., such a motion is energetically possible though extremely unlikely. For larger spheres there is not enough energy, while for smaller radii there is sufficient energy. The quantity λ_D given by

$$\lambda_D^2 = \frac{kT}{4\pi n_e e^2} = \frac{v_T^2}{\omega_p^2} \qquad \omega_p^2 \equiv \frac{4\pi n_e e^2}{m}$$

$$v_T^2 = kT/m \tag{12}$$

is called the Debye length. (It was first introduced by Debye in the study of electrolytes.) It is a measure of the size of a region in which appreciable charge density fluctuations can occur. The quantity ω_p is called the plasma frequency (as originally defined by Tonks and Langmuir). Regions with dimensions much larger than a Debye length must be neutral to a very high degree. This is another basic characteristic of a plasma.

It is illuminating to write Eq. (12) in the following form

$$\frac{\lambda_D^2}{d^2} = \frac{kT\,d^3}{d^2\,4\pi e^2} = \frac{kT}{e^2/d\,4\pi} = \frac{1}{4\pi}\,\frac{kT}{\phi(d)} \tag{13}$$

where we have replaced $1/n_e$ by d^3, where d is the interparticle spacing and $\phi(d)$ is the potential of an electron (or ion) evaluated at the distance d. We have already shown that in a plasma kT must be large compared to $\phi(d)$, and hence λ_D is large compared to d. Thus in a plasma there are many particles in a Debye sphere. For the Cs example given ($T = 2500°K$, $n_e = 10^{12}$), $kT/\phi(d) = 250$ and $\lambda_D = 4.5\,d$: a Debye sphere contains $\frac{4}{3}\pi\lambda_D^3 n_e \cong 360$ particles. For a typical plasma in the stellarator, $kT = 50\text{ eV} = 5 \times 10^5 °K$, $n_e = 10^{12}$ and $\lambda_D = 70\,d$: the number of particles in a Debye sphere is 10^6. We shall have more to say about the Debye length later in the course.

The quantity ω_p in Eq. (12) arises whenever collective electron motion takes place in a plasma. As an example, consider the case where all the electrons in an infinite sheet of thickness L are shifted by a distance δ. Then there are two regions, one charged positive and the other charged negative, separated by the distance L. The charge σ per unit area is $en\delta$ and the electric field in the region L is

$$E = 4\pi\sigma \tag{14}$$

The force/area on the electrons in L is

$$(Lne)\,E = 4\pi n^2 L e^2 \delta \tag{15}$$

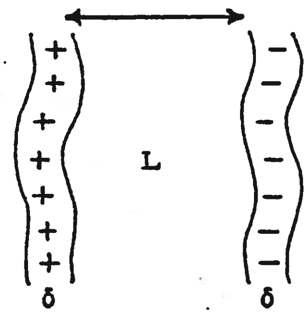


Figure 3

This force acts on a mass of Lnm (where m is the electron mass) so the equation of motion is

$$-4\pi n^2 L e^2 \delta = Lnm \frac{d^2\delta}{dt^2} \tag{16}$$

which has as a solution

$$\delta = A e^{-i\omega_p t} \tag{17}$$

The electrons thus oscillate at angular frequency

$$\omega_p = \left(\frac{4\pi n e^2}{m}\right)^{\frac{1}{2}} \quad \text{or} \quad f_p \sim 9 \times 10^3 (n)^{\frac{1}{2}} \tag{18}$$

The force on the ions is the same, but the much greater ion mass results in a relatively small motion which is neglected here. The plasma frequency is essentially a measure of the minimum response time of a plasma to an applied electric field. In a time of the order of $1/f_p$ the electrons will move in such a fashion as to reduce an applied electric field. Thus a plasma is transparent to electromagnetic waves above the plasma frequency. Below this frequency the plasma acts like a good conductor; the waves are reflected.

Of course, if the collision frequency is higher than the plasma frequency, such motion is quickly damped; the plasma becomes an absorbing medium.

Plasma physics deals with an enormous range of physical parameters. Fig. 4 summarizes the regions of physical interest; a temperature range of 10^0 to 10^9; a density range of 10^{-2} to 10^{18}. The plasma frequency — which is a measure of density — is shown at the right. The Debye length (a function of n and T) is shown by the diagonal lines.

We have described some of the basic ideas of plasma physics. Collisions complicate matters; therefore, proceeding further we briefly survey various collisional processes.

III. Collisions and Atomic Processes

It is not possible to take collisions completely into account in describing a plasma. First of all there is a very large variety of types of collisions. Secondly, relatively few cross sections are well known, either on the basis of theory or experiment, although recent research has enlarged our knowledge greatly. Nevertheless, it does prove possible in many cases of practical interest to get reasonable agreement between experiment and theory on the basis of what is known.

In the following sections we attempt to provide brief descriptions of the principal types of collisions involved in plasma problems and to give typical values of the cross sections. It is hopeless to attempt to provide complete data. The intent is to provide sufficient information to enable order-of-magnitude estimates of collisional effects. If order-of-magnitude estimates indicate collisions to be important, then the detailed cross sections may be sought out in the literature.

Data is given in several different units in the literature. Most recent data is presented in units of cm^2/atom. Also in common use is the dimension $\pi a_o^2 = .88 \times 10^{-16}$ cm^2, where a_o is h^2/me^2, the Bohr radius of the hydrogen atom; also, since atomic cross sections are frequently in the region 10^{-16} cm^2, angstroms squared — $Å^2$ — are used. In the older literature, particularly with respect to elastic collisions, the cross sections are given in terms of collision probability per cm per unit pressure (mm Hg at 0°). To change them into cm^2/atom one must divide by the number of atoms per cm^3 at a pressure of 1 mm Hg. Thus a collision probability of one in the above units corresponds to a cross section of 0.283×10^{-16} cm^2/atom.

If we watch a given "test particle" proceed through a region where the target or field particle density is n, then the probable number of a given type of collisions per cm path length is simply σn, where σ is the cross section for the particular type of interaction. The average distance traveled between collisions, or mean free path, is then $1/n\sigma$. The number of collisions per second, collision frequency, is $n\sigma v$, where v is the relative velocity of field and test particles. If the field particles are in motion or if we are concerned with the average behavior of a large number of test particles moving at different velocities, then since σ for most processes is a function of v, we must average the product σv over all velocities. Thus if $f(v)\,dv$ is the number of particles having velocities between v and $v + dv$, we may write

$$\int f(v)\,\sigma(v)\,v\,dv \equiv n\overline{\sigma v} \tag{1}$$

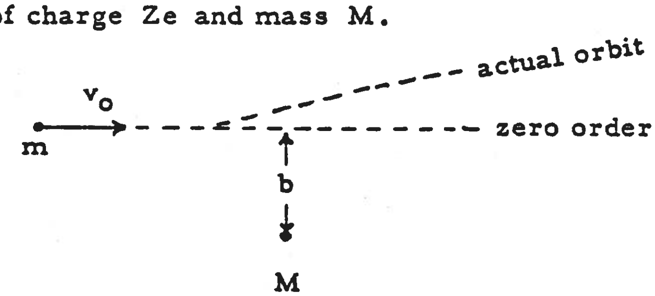
If instead of one there are n_t test particles per cubic centimeter, then the total number of collisions per second per cubic centimeter of type described by a given σ is

$$\text{Col/sec} = n_t n \overline{\sigma v} \tag{2}$$

The product $\overline{\sigma v}$ is usually called the rate coefficient.

Coulomb collisions

This is the process of greatest importance to us. We consider the problem of a light particle of charge ze and mass m, approaching with velocity v_o a stationary heavy particle of charge Ze and mass M.



Let us assume that the deflection angle is small so that in zero order approximation the particle is undeflected, with a distance of minimum approach the same as the impact parameter, b. Most of the deflection is due to the force exerted during the time the particles are close together — a time approximately b/v_o, and a force of about zZe^2/b^2. Then the momentum acquired perpendicular to v is

$$p_\perp = F_\perp \Delta t \sim \frac{zZe^2}{b^2}\,\frac{b}{v_o} = \frac{zZe^2}{v_o b} \tag{3}$$

The scattering angle is then

$$\theta \sim \frac{p_\perp}{p} = \frac{zZe^2}{bv}\,\frac{1}{mv} = \frac{zZe^2}{mbv^2} \tag{4}$$

A more careful calculation[1] yields

[1] See, for example, Goldstein's "Classical Mechanics."

$$\tan\frac{\theta}{2} = \frac{zZe^2}{mv^2 b} \tag{5}$$

Note that the magnitude of the scattering angle depends on the impact parameter and the initial relative velocity and is the same in magnitude for like and unlike charges. The impact parameter for 90° scattering is thus

$$b_o = \frac{zZe^2}{mv_o^2} \tag{6}$$

i.e., a distance where the potential energy is twice the original kinetic energy. The scattering angle for arbitrary b may then be expressed

$$\tan\frac{\theta}{2} = \frac{b_o}{b} \tag{7}$$

The total scattering cross section is obviously infinite. All the particles which pass within θb of b will then be scattered within a corresponding dθ of θ(b). We define a differential cross section I(θ,ϕ), where

$$\int_{4\pi} I(\theta,\phi,v)\, d\Omega = \sigma(v) \tag{8}$$

Then because of the cylindrical symmetry

$$I d(\theta)\, d\Omega = I(\theta)\, 2\pi \sin\theta\, d\theta = 2\pi\, b db$$

$$I(\theta) = \frac{b db}{\sin\theta d\theta} \tag{9}$$

Eliminating b from Eq. (9) by Eq. (7) yields the Rutherford scattering formula

$$I(\theta) = \left(\frac{zZe^2}{2m v_o^2}\right)^2 \cos e^4 \frac{\theta}{2} \tag{10}$$

Multiple Coulomb Scattering

We consider a group of electrons initially moving in the z-direction. It is assumed that there are very many ions and due to the long range of the Coulomb forces there are far more small-angle than large-angle collisions. On the average, of course, there are as many deflections in any particular direction as in the opposite direction. However, there will be scattering due to statistical fluctuations.

For a given electron the total velocity in the x-direction, acquired after n individual scatterings is

$$\Delta v_x = (\Delta v_x)_1 + (\Delta v_x)_2 + \ldots + (\Delta v_x)_N \ . \tag{11}$$

For an ensemble of many electrons, we consider the average of Δv_x over all the collection. Then due to the equal probability of any given scattering and its opposite,

$$\overline{\Delta v_x} = \overline{\Delta v_y} = 0 \ . \tag{12}$$

Let us instead look at the average of Δv_x^2, designated

$$\left\langle \Delta v_x^2 \right\rangle = \overline{[(\Delta v_x)_1 + (\Delta v_x)_2 + \ldots\ldots]^2} \tag{13}$$

The sum will then consist of two types of terms, $\overline{(\Delta v_x)_n^2}$ and $\overline{(\Delta v_x)_n (\Delta v_x)_m}$.

The second term is zero due to the fact that collisions are completely uncorrelated.

If successive collisions $(\Delta v_x)_i$ are the same in magnitude, then

$$\left\langle (\Delta v_x)^2 \right\rangle = N\overline{(\Delta v_x)^2} \tag{14}$$

Because of the axial symmetry

$$\left\langle \Delta v_x^2 \right\rangle = \left\langle \overline{\Delta v_y^2} \right\rangle \equiv \left\langle \Delta v_\perp^2 \right\rangle \tag{15}$$

$$\Delta v_\perp^2 = v_o^2 \sin^2 \theta \tag{16}$$

$$\tan \frac{\theta}{2} = \frac{b_o}{b} \tag{17}$$

$$\sin \theta = \frac{2 b_o / b}{1 + (b_o/b)^2} \tag{18}$$

$$(\Delta v_\perp)^2 = \frac{4 v_o^2 (b/b_o)^2}{[1 + (b/b_o)^2]^2} \quad . \tag{19}$$

The cross section for collision within db of b is $2\pi b\,db$. The number of collisions per unit path length is $nId\Omega = 2\pi nb\,db$. At each collision the change in velocity squared is given by Eq. (19). Then the average change in $v_\perp^2$ suffered by an electron per cm path length is

$$\left\langle (\Delta v_\perp)^2 \right\rangle = \int_o^{b_m} \frac{4v_o^2 (b/b_o)^2}{[1 + (b/b_o^2)]^2} \, 2\pi nb\,db \quad , \tag{20}$$

where b_m is the largest distance at which the Coulomb potential is applicable, in view of the shielding of the ion charge due to the surrounding electrons. The point is discussed further in Spitzer[2].

2 L. Spitzer, Jr., Physics of Fully Ionized Gases (Interscience Publishers, New York, N. Y., 1962), 2nd ed.

$$\left\langle (\Delta v_\perp)^2 \right\rangle = 8\pi n v_o^2 b_o^2 \int_0^{b_m/b_o} \frac{x^3\,dx}{(1+x^2)^2} \quad . \tag{21}$$

$$\left\langle (\Delta v_\perp)^2 \right\rangle = 4\pi n v_o^2 b_o^2 \left\{ \ln\left[1+\left(\frac{b_m}{b_o}\right)^2\right] + \frac{1}{1+(b_m/b_o)^2} - 1 \right\}. \tag{22}$$

Assuming b_m/b_o and $\ln(b_m/b_o) >> 1$,

$$\left\langle (\Delta v_\perp)^2 \right\rangle = 8\pi n v_o^2 b_o^2 \ln(b_m/b_o) \quad . \tag{23}$$

It is customary to take as b_m, the Debye length λ_D. The $\ln(b_m/b_o)$ term for almost all conditions of interest is between 10 and 20.

We may roughly say that when $(\Delta v_x)^2 = v^2$ the particle has been scattered by 90°.

$$\frac{\text{multiple}}{\text{single}} \sim \frac{8\pi n b_o^2 \ln(b_m/b_o)}{n\pi b_o^2} = 8\ln(b_m/b_o) \equiv 8\ln\Lambda \tag{24}$$

Thus large angle deflections due to multiple scattering are about two orders of magnitude more probable than those due to single scattering.

For force fields which fall off as r^{-m} with $m \geq 3$, it is no longer true that multiple scattering dominates.

The Coulomb scattering described here is elastic (we have neglected radiation effects): kinetic energy is conserved. In the center of mass coordinate system the individual particle energies are unchanged by a collision. This is no longer true in the laboratory frame. For example, the light particle engaging in a large angle collision with a stationary heavy particle loses roughly a fraction $2m/M$ of its energy at each collision.

Relaxation Times

When a plasma is heated by electric forces the velocity distribution may differ markedly from kinetic equilibrium (Maxwell-Boltzmann distribution). It is important to be able to estimate the time required for the distribution to relax by means of collisions toward the M-B distribution (self-collision time). Furthermore, the heating effects on ions and electrons are usually quite different. We need also an idea of the time required for ions and electrons to approach the same temperature (equipartition time).

Exact solutions require extensive numerical computations, but we can get approximate results based on Eq. 23. This equation was derived for scattering of a light particle by a heavy one. If we had used center-of-mass coordinates, the problem would have been formally identical. The velocity v would then represent the relative motion of the two particles, and m the reduced mass m_1m_2/m_1+m_2. It would only be necessary to transform the scattering angle from center-of-mass coordinates to the laboratory frame. If the two particles have the same mass and we consider the problem as one of the slowing down of fast particles by collisions with slow ones (assumed stationary), then a 90° scattering angle in the c of m frame becomes a 45° scattering in the laboratory frame. We assume that by the time a particle has been scattered by a like particle through an angle of 90°, energy exchange has been effected. (This is, of course, not a reasonable assumption if the masses are very different.)

We found the expression (19) for the scattering per unit path length. The scattering per unit time is then Eq. (19) multiplied by v. Eliminating b_o by Eq. (6) we get

$$\frac{d\langle v_x^2\rangle}{dt} = \frac{8\pi z^2 e^4 \ln\Lambda n}{m^2 v_o} \tag{25}$$

and $\langle v_x^2\rangle$ increases linearly with time. If we now set $\langle v_x^2\rangle = 4v_o^2$ (i.e., multiply by dt and integrate from 0 to τ_c) corresponding to large angle scattering in the laboratory frame, we can solve for the time τ

$$\tau_c = \frac{4v_o^3 m^2}{8\pi z^2 e^4 n \ln\Lambda} \tag{26}$$

Let us view this in terms of a group of particles at high temperature T being slowed by cold particles of the same type then using the relationship

$$\frac{3}{2}kT = \frac{1}{2}m v_o^2 \tag{27}$$

we get

$$\tau_c = \frac{m^{\frac{1}{2}}(3kT)^{3/2}}{2\pi z^2 e^4 n \ln\Lambda} \tag{28}$$

This compares with the relation given by Spitzer

$$\tau = \frac{m^{\frac{1}{2}}(3kT)^{3/2}}{8\times 0.714\pi z^2 e^4 n \ln\Lambda} = \frac{11.4 A^{\frac{1}{2}} T^{3/2}}{n z^2 \ln\Lambda} \tag{29}$$

Thus if the temperature is the same, hydrogen ions require $\sqrt{M/m}$ or 43 times as long as electrons to approach a M-B distribution. For any other ions the time is of course longer. What about equipartition times between ions and electrons? Each collision between a light and heavy particle transfers only a fraction of order m/M of the energy. The relative velocity of two electrons is roughly the same as the relative velocity of electron and ion, thus roughly speaking electron-ion equipartition takes $\left(\frac{M}{m}\right)$ times longer than electron-electron self-collision time. Thus the ratio of electron-electron to ion-ion to electron-ion times are roughly

$$\tau_{ee} : \tau_{ii} : \tau_{ie} = 1 : \sqrt{M/m} : M/m$$

provided the temperature is the same in all cases.

Conductivity

We can get an approximate expression for the conductivity of an ionized gas using the collision cross sections that we have calculated. The model adopted is the so-called Lorentz model, where the conductivity is assumed to be entirely due to electrons moving in a stationary ion background. Our simplified calculation of the conductivity of a Lorentz gas will assume that the electrons are completely stopped at each large angle collision and that between collisions the electrons are uniformly accelerated by the applied electric field. The drift velocity is

$$v_d = \frac{at}{2} = \frac{Eet_c}{2} \tag{1}$$

It is reasonable to assume that the collisions most effective in stopping the drift motion are those with ions — i.e., we use the ion-electron large angle scattering time which depends on ion density n_i

$$t_c = \frac{v_o^3 m^2}{8\pi z^2 e^4 n_i \ln \Lambda} \tag{2}$$

The resistivity (in emu) is

$$\eta = \frac{cE}{j} = \frac{cE}{n_e e v_d / c} \tag{3}$$

where j and cE are in emu.

Using Eq. (1) and (2) in (3), and substituting $z n_i = n_e$, we get

$$\eta = \frac{8\pi}{3^{3/2}} \frac{c^2 e^2 z \ln \Lambda}{m^{\frac{1}{2}} (kT)^{3/2}} \sim 5 \frac{c^2 e^2 z \ln \Lambda}{m^{\frac{1}{2}} (kT)^{3/2}} \tag{4}$$

This estimate should be high for the following reason. Since the collision time goes as $1/v^3$, the electron of higher than average energy should be less affected by collisions and would therefore be the dominant current carrier, whereas we have assumed T_e appropriate for the average. When the calculation is done properly, the conductivity drops by a factor of about 5 (as though we had all electrons of about 3 × average energy). When electron-electron collisions are also included, this raises the resistivity by about a factor of 2 (for hydrogen), giving numerically

$$\begin{aligned} \eta &= 6.53 \times 10^{12} \frac{\ln \Lambda}{T^{3/2}} \text{ emu} \\ &= 6.53 \times 10^{3} \frac{\ln \Lambda}{T^{3/2}} \text{ ohm cm} \end{aligned} \tag{5}$$

<u>Electron-Atom Elastic Collisions</u>

At electron energies below that of the lowest excited state, only elastic collisions with neutral atoms in the ground state can take place. According to the correspondence principle, a description in classical terms is valid only for large impact parameters. In this region, the main contribution to the interaction potential is due to the polarization of the neutral atom by the electric field of the electron. At a separation of r, the electric field produced by the electron is

$$E = -\frac{e}{r^2} \tag{6}$$

This field causes a polarization of the atom proportional to E, or

$$P = \alpha E = -\alpha \frac{e}{r^2} \tag{7}$$

where α is the atomic polarizability and is of the order of 10^{-24} cm^3. The force on an electric dipole is the product of the dipole moment and the electric field gradient. Thus the force

$$F = \left(-\alpha \frac{e}{r^2}\right) \frac{d}{dr} \left(-\frac{e}{r^2}\right) = -2\alpha \frac{e}{r^5} . \tag{8}$$

The potential is thus an inverse 4th power one. There is not the same problem of divergent cross sections as in the Coulomb case. Notice that at $r = 10^{-8}$ cm (where the impact parameter is too small for validity of this description) the force is equal to the Coulomb force, while at $r = 10^{-7}$ cm the force has fallen to 10^{-3} of the Coulomb force and is now negligible.

In the case of ion-atom collisions, however, the wavelength of the ion is very small compared with the impact parameter: the classical description is valid. Now it turns out that for a central force which varies as r^{-n}, the cross section varies as $r^{-n/4}$. Thus the ion-atom cross section varies as v^{-1}; the collision frequency, $n\sigma v$, is independent of velocity, thus greatly simplifying the calculation of transport phenomena. At small separations, as the ion begins to overlap the atom, thus there arises a strong repulsive force. The net overall result is sketched in Fig. 5.

For the important range of impact parameters in the case of electron-atom scattering, one must apply quantum mechanical techniques.

Returning to electron scattering, it may be described as the interaction of an incoming train of Debroglie waves representing the electrons, passing through the force field of the atom. The impact parameter is replaced by the quantized angular momentum. As the particles enter the field of force of the atom, the wavelengths change as a function of the potential U, as determined by the Schrodinger equations

$$\nabla^2 \psi + \frac{8\pi m}{h^2} (E - U) \psi = 0 \tag{9}$$

The potential distribution for the hydrogen atom may be calculated by a number of methods. For more complex atoms there exist several ways of approximating this potential function. The matching of the waves of incoming and outgoing electrons with the waves in the region where the $V \neq 0$ then determines the scattering angle. The form of the potential function depends on the distribution of the orbiting electrons about the nucleus. Thus it is not surprising that chemically similar atoms have similar scattering properties.

The theory is in fair agreement with observations, both in terms of total and differential cross sections, except at very low energies below 1 eV where the measurements become more difficult. We shall provide here only a summary of the more important cross sections in Figs. 6 and 7.

Of particular interest is the dip especially apparent in the case of the heavier rare gases (Fig. 6) at an electron energy of about 1 eV — the Ramsauer effect. This is essentially a resonance effect. The phase shift of the electron wave in crossing the atom is close to an integral multiple of 2π.

Momentum Transfer Cross Section

The fractional energy loss of an electron of mass m scattered by an atom of mass M is [neglecting terms of order $(m/M)^2$]

$$2(1 - \cos\theta)\frac{m}{M} . \tag{10}$$

Problem: Derive Eq. (10).

The total fractional energy loss per unit length is then

$$\epsilon = \int n(1 - \cos\theta)\frac{2m}{M} I\, 2\pi \sin\theta\, d\theta \tag{11}$$

$$\epsilon \equiv \frac{2m}{M}\sigma_m n \tag{12}$$

where

$$\sigma_m = 2\pi \int_0^\pi I(1 - \cos\theta)\sin\theta\, d\theta \tag{13}$$

and is called the momentum transfer cross section. The average fractional energy loss per collision is then Eq. (12) divided by the total number of collisions per unit length, σn

$$\frac{\Delta E}{E} = \frac{2m}{M}\frac{\sigma_m}{\sigma_c} . \tag{14}$$

In most cases the momentum transfer cross section is within 10% of the total collision cross section, σ_c , as may be seen in Fig. 8.

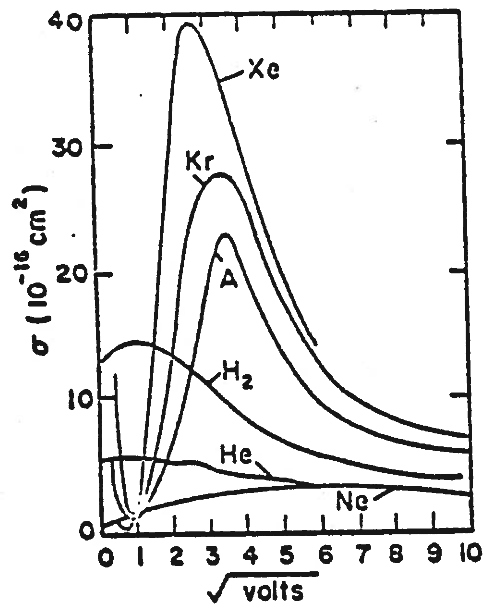


Fig. 6. Total collision cross sections in Ne, A, Kr, and Xe; R. B. Brode, Rev. Mod. Phys. 5, 257 (1933).

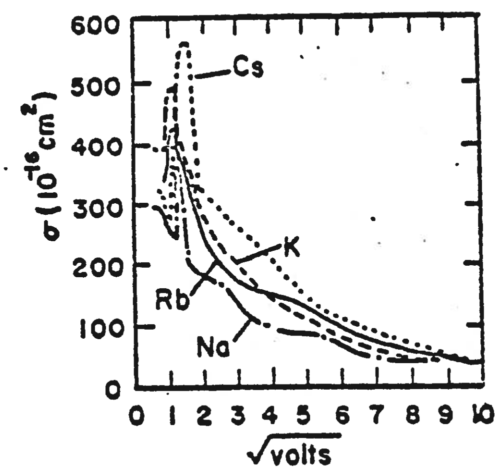


Fig. 7. Total collision cross sections in the alkali metals; R. B. Brode, Rev. Mod. Phys. 5, 257 (1933).

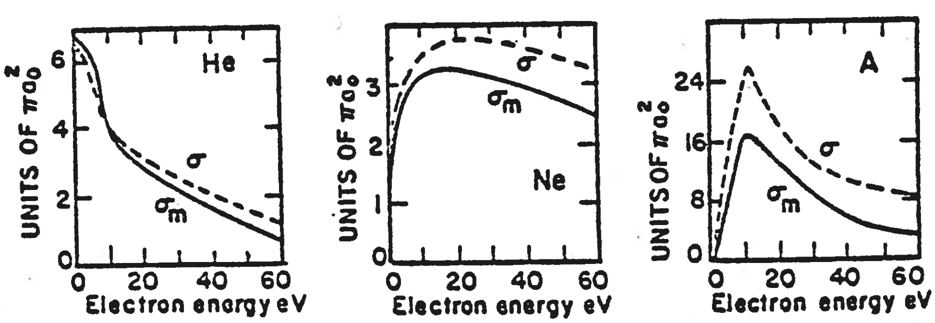


Fig. 8. Comparison of total collision cross section σ with momentum transfer cross section σ_m for He, Ne, and A. (Massey and Burhop)

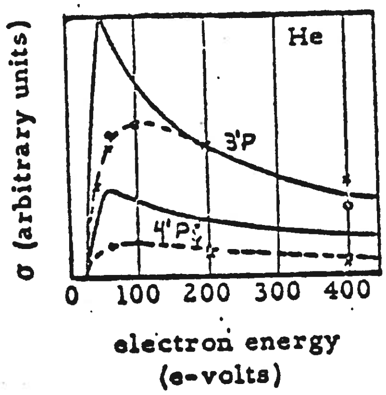
Inelastic Electron-Atom Collisions

If the electron energy is sufficiently high, excitation and ionization become possible. These are, of course, quantum mechanical problems. For energies well above threshold energies (by a factor of 5 or more), the Born approximation works well and calculations are simplified.

For a complete description of the processes going on in a partially ionized plasma we would need to know the cross sections for excitation to all levels and to the continuum. The information is meager. Fairly complete calculations exist for helium and they are in fair agreement (except near threshold) with experimental observations. Experimental measurements of excitation cross sections are very difficult. Calculations for atomic hydrogen, of course, can also be carried out, but most experiments involve collisions with molecular hydrogen. For allowed transitions the calculated cross section rises linearly from zero above the threshold energy to a maximum at about twice threshold (experimentally observed peaks tend to be at somewhat higher energies) and falls off as $(\ell n\, v)/v^2$ at high energies.[1] For forbidden transitions, the cross section falls a little more rapidly, as $1/v^2$. If electron exchange is involved as, for example, in transitions between triplet and singlet levels in helium, the cross section falls off very rapidly beyond the peak (see Fig. 10).

The ionization cross section can be measured with considerable accuracy. Here again the cross sections rise rapidly to a maximum at several times the threshold energy, falling asymptotically as $\ell n\, v/v^2$.

[1] Fig. 9 shows calculated and observed cross sections for two allowed transitions.



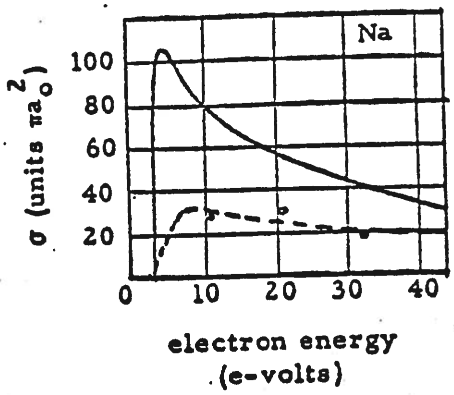


Fig. 9. Cross-sections for optically allowed transitions in He and Na.

——— calculated
------ observed

[Massey and Burhop.]

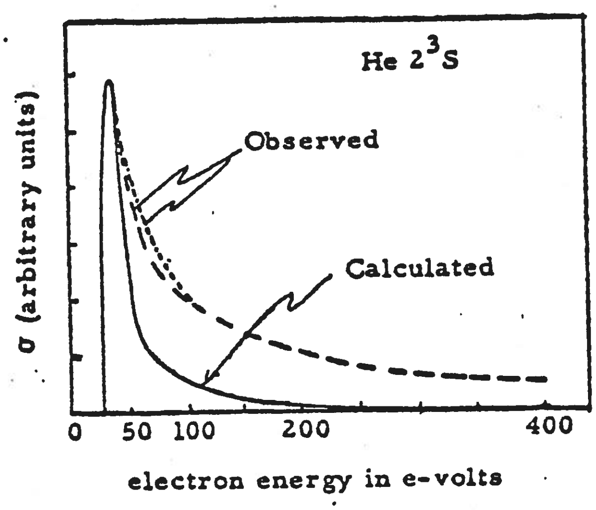


Fig. 10. Excitation cross-sections for the He 2^3S metastable level.

[Massey and Burhop.]

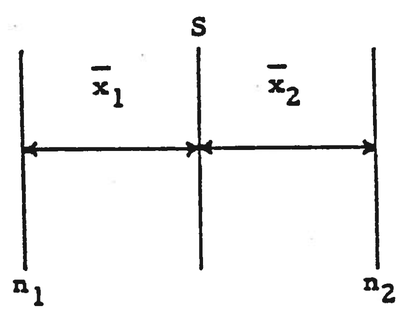
(p.39a)

Table I compares calculated elastic ionization and excitation cross sections for atomic hydrogen at energies high enough to allow the use of the Born approximation.

Diffusion and Mobility in Partially Ionized Gases

We derive here an elementary description of electron diffusion and mobility processes in slightly ionized gases where elastic collisions predominate.

We consider first the diffusion of electrons (or ions) through a neutral gas. The concentration of charged particles is so low that electron-neutral collisions dominate over Coulomb collisions, and the temperature is low enough that elastic collisions dominate. The temperature is assumed uniform and the density of electrons is a function of x only. There will then be a net flow of electrons from the region of higher to the region of lower concentration.



The flux Γ_2 crossing surface S from the right consists of electrons which on the average engaged in a collision at a distance $\bar{x}_2$ from S where the electron density is n_2. The flux from the left came from a distance $\bar{x}_1$ where the electron density was n_1. Since the background neutral gas density is constant, $\bar{x}_2 = \bar{x}_1$ and

$$n_1 = n_o - \frac{dn}{dx}\bar{x} \tag{15}$$

$$n_2 = n_o + \frac{dn}{dx}\bar{x} \tag{16}$$

This of course may not be legitimate if the density varies too rapidly — e.g., in the case of shocks.

From kinetic theory we recall that the total number of particles crossing a unit plane for a Maxwellian distribution is $\frac{1}{4}n\bar{v}$. The net flux crossing S is then

$$\Gamma = \frac{1}{4}n_1\bar{v}_1 - \frac{1}{4}n_2\bar{v}_2 \tag{17}$$

$$\Gamma = -\frac{1}{2}\frac{dn}{dx}\bar{x}\,\bar{v} \tag{18}$$

But from kinetic theory the mean free path is related to the mfp across a plane by

$$\bar{x} = \frac{2}{3}\lambda \tag{19}$$

$$\Gamma = -\frac{1}{3}\lambda\bar{v}\frac{\partial n}{\partial x} \tag{20}$$

$$\Gamma = -\frac{1}{3}\frac{v^2}{\nu_c}\frac{\partial n}{\partial x} \equiv -D\frac{\partial n}{\partial x} \tag{21}$$

where the diffusion coefficient is

$$D = +\frac{1}{3}\frac{v^2}{\nu_c} \tag{22}$$

where ν_c is an effective collision frequency.

If we consider the three-dimensional case,

$$\Gamma = -D\,\nabla n \tag{23}$$

$$\nabla\cdot\Gamma = -\frac{\partial n}{\partial t} \tag{24}$$

$$\frac{\partial n}{\partial t} = D\,\nabla^2 n \quad \text{(the diffusion equation)} \tag{25}$$

If D varies with position, then we must write

$$\Gamma = -\nabla Dn\ . \tag{26}$$

We can also construct a simple model of mobility. Again we consider a case where collisions with neutrals are dominant. We consider an electron in a uniform electric field E. Then the electron acquires, in addition to its thermal motion, an accelerated motion along $-E$. On the average, a fraction α of the drift momentum is lost at each collision. If the collision frequency is ν_c', the equation of motion for the average electron is

$$m\frac{dv}{dt} - mv\alpha\nu_c' = -eE\ . \tag{27}$$

We now consider a steady state so that the collisional drag of the neutrals compensates for the electric field, and define the effective collision frequency ν_c as $\alpha\nu_c'$. We can now solve for the average drift motion

$$v = \left(\frac{e}{m\nu_c}\right)E = \mu E \tag{28}$$

where μ is called the mobility. If the drift velocity is much smaller than the thermal velocity, ν_c, and thus μ, will be independent of E. If the diffusion coefficient is divided by the mobility, we obtain the Einstein relation

$$\frac{D}{\mu} = \frac{1}{3}\frac{v^2}{\nu_c}\left(\frac{m\nu_c}{e}\right) = \frac{1}{3}\frac{mv^2}{e} \tag{29}$$

which depends on kinetic energy only. If the particles are in thermal equilibrium, then

$$\frac{3}{2}KT = \frac{1}{2}mv^2 \tag{30}$$

and Eq. (29) becomes

$$\frac{D}{\mu} = \frac{kT}{e} \tag{31}$$

a result valid for Maxwell-Boltzmann distribution.

Since ν_c is directly proportional to p, we expect the drift velocity to be proportional to E/p so long as the drift velocity remains small compared with the thermal velocity. At high E/p this inequality is no longer satisfied and the drift velocity is no longer proportional to E/p.

This theory is obviously oversimplified. We have seen, for example, that the elastic collision frequency is a strong function of velocity. The electron drift velocity as a function of E/p is shown in Figs. 11 to 14.

Ion Mobility

Similar arguments hold for ion mobility. Once again for low E/p one expects the drift velocity to be proportional to E/p. Because of the greater mass, ion mobilities are of course far smaller than electron mobilities. The case of ions drifting through neutrals of the same kind — e.g., helium ions in helium atoms — is subject to charge exchange collision and the mobilities are smaller. (See Fig. 15.)

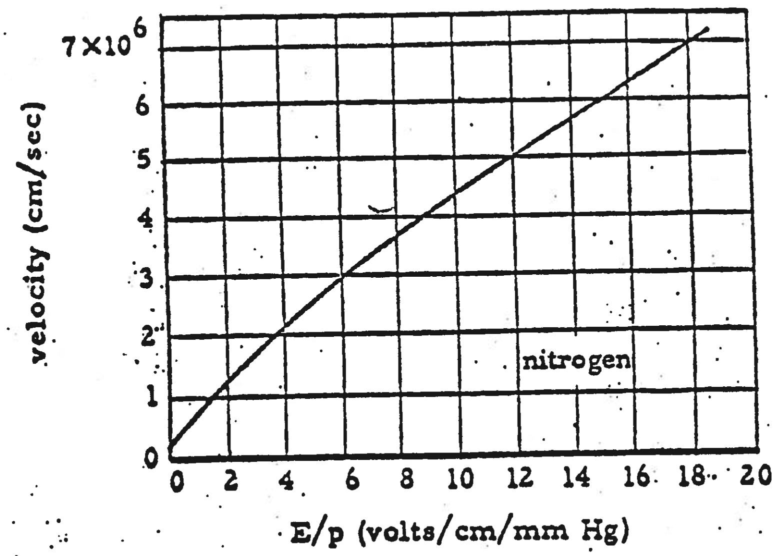


Fig. 11. Drift velocity of electrons in nitrogen as a function of E/p.

[R. A. Nielsen, Phys. Rev. 50, 950 (1936).
L. Colli and U. Facchini, Rev. Sci. Instr. 23, 39 (1952).]

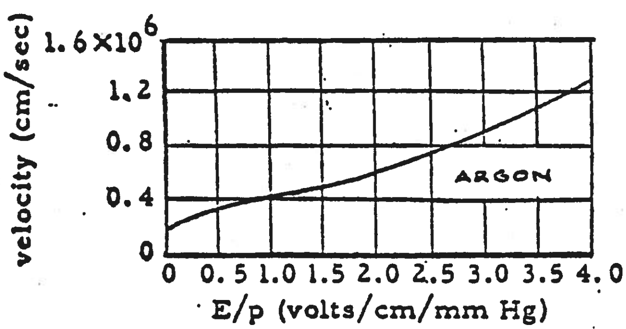


Fig. 12. Drift velocity of electrons in argon as a function of E/p.

[R. A. Nielsen, Phys. Rev. 50, 950 (1936).
L. Colli and U. Facchini, Rev. Sci. Instr. 23, 39 (1952).
J. M. Kirshner and D. S. Toffolo, J. Appl. Phys. 23, 594 (1952)]

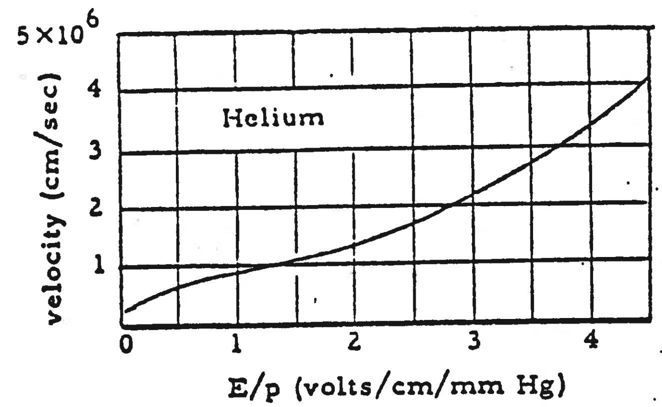


Fig. 13. Drift velocity of electrons in helium as a function of E/p.

[R. A. Nielsen, Phys. Rev. 50, 950 (1936).
J. A. Hornbeck, Phys. Rev. 83, 374 (1951).]

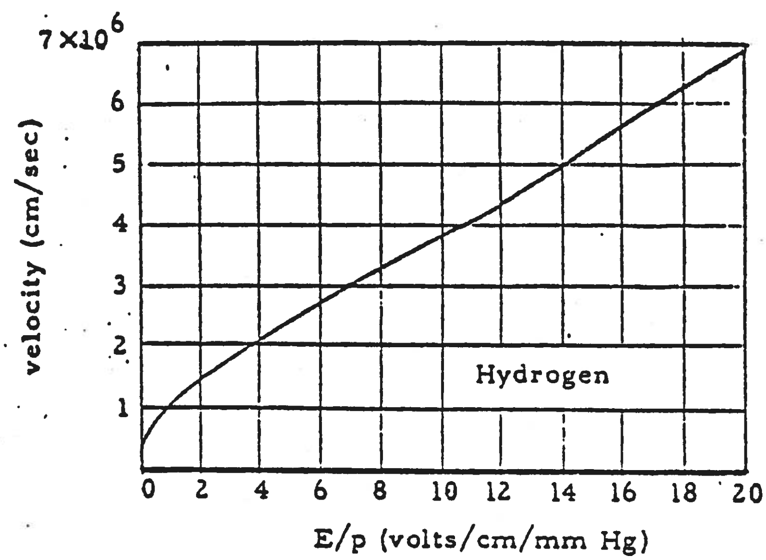


Fig. 14. Drift velocity of electrons in hydrogen as a function of E/p.

[N. E. Bradbury and R. A. Nielsen, Phys. Rev. 49, 388 (1936).]

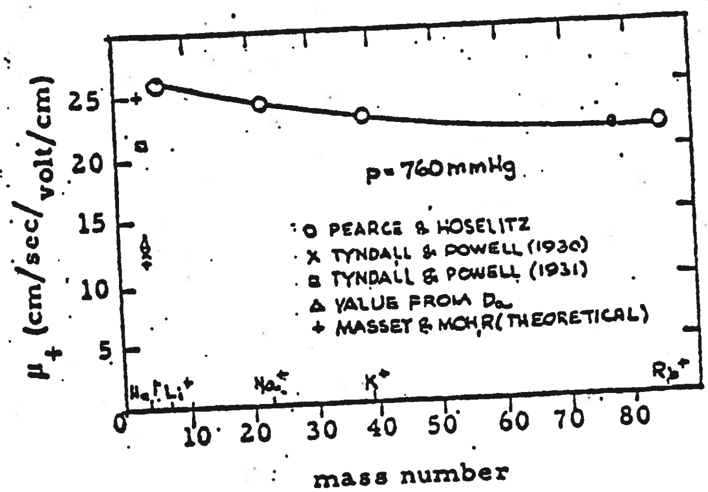


Fig. 15. Mobility of various positive ions in helium.

Charge Exchange

An ion passing through a gas may pick up an electron from a neutral atom (or from another ion). This is called charge exchange. In a partially ionized gas this produces a marked effect on both diffusion and mobility. For example, an ion drifting in an electric field may pick up an electron from a neutral. The neutralized particle continues with almost the same velocity and is replaced by an ion moving at the random velocity it had as a neutral. Thus the drifting ion is on the average replaced by a nondrifting ion. This process represents an additional collision mechanism which further reduces the average drift velocity.

An electron which transfers between like atoms requires no energy. A transfer between unlike atoms will involve a difference in energy difference designated ΔE. The electron orbital motion is very rapid compared with the time of impact between two heavy particles. The interaction is then adiabatic; the energy levels have a chance to readjust slowly.

The classical analogue is that of transfer to an oscillator of frequency ν by a pulse of length τ. The maximum excitation of the classical oscillator will take place if the pulse has a maximum Fourier amplitude at frequency ν. This will occur if τ is $1/\nu$. But τ is roughly a/v, where a is the range of the interaction; i.e.,

$$\tau \nu = 1 = \frac{a\nu}{v} . \tag{32}$$

To now transfer this to a quantum mechanical picture we replace ν by $\Delta E/h$. Then the charge exchange cross section will be a maximum if

$$a \Delta E/hv \sim 1 . \tag{33}$$

Thus, for like atoms the charge exchange cross section increases as the relative velocity decreases. This is usually termed resonant charge-exchange. For unlike atoms (nonresonant charge-exchange) the cross sections exhibit a maximum as shown in Figs. 16 and 17.

Ambipolar Diffusion

We consider a steady state plasma in a container. The electron and ion losses are assumed to be compensated by additional ionization processes taking place in the volume. First let us consider the loss processes before steady state is reached. If the initial electron density n_e is equal to the initial ion density n_i and $T_e \geq T_i$, the diffusion coefficient for electrons D_e being much greater than that for ions D_i, the initial loss of electrons to the wall exceeds the ion losses. This sets up an electric field toward the wall which decreases the electron loss rate and increases the ion loss rate. This process continues until the electric field has increased to a value which makes the ion and electron loss rates equal. This is the steady state referred to above. For the electrons and ions, we write for the fluxes

$$\Gamma_i = - D_i \nabla n_i + \mu_i E n_i \tag{34}$$

$$\Gamma_e = - D_e \nabla n_e - \mu_e E n_e \ . \tag{35}$$

The electric field here is the sheath field caused by the excess electron loss. There is no applied field. At steady state the total losses are equal. We assume that the fluxes of both particles are the same in all regions.

$$\Gamma_i = \Gamma_e = \Gamma_a \tag{36}$$

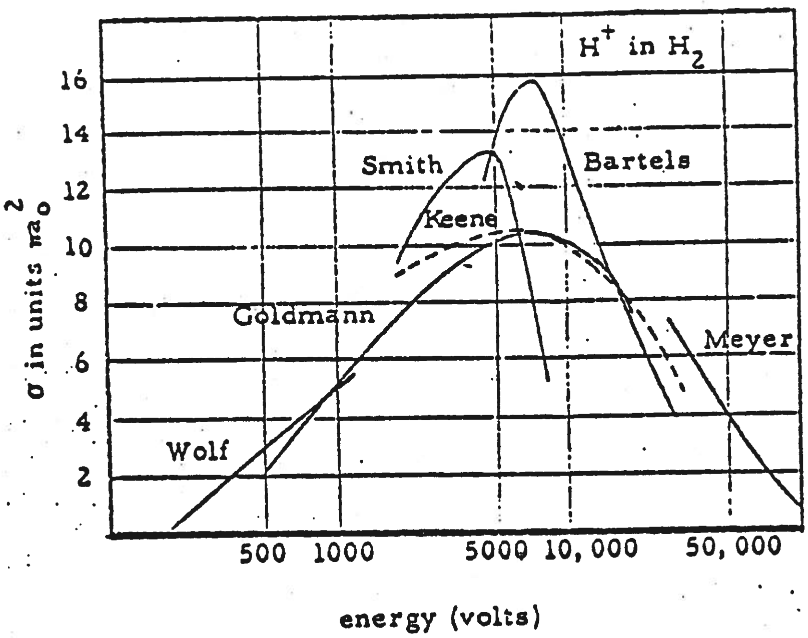


Fig. 16. Charge transfer cross-section of H^+ in H_2. [Massey and Burhop]

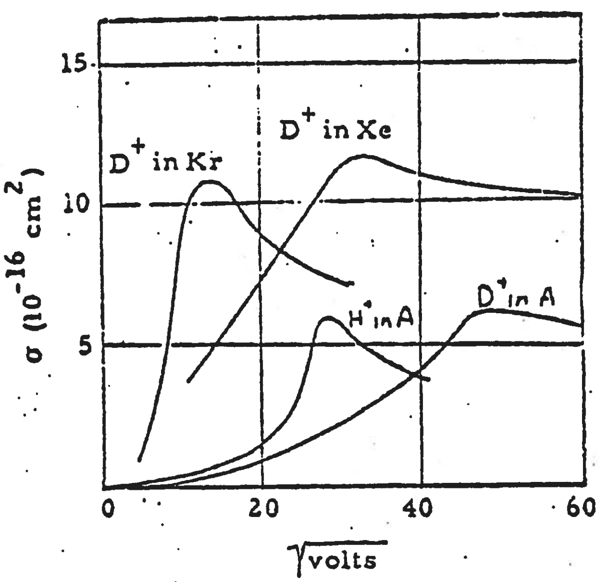


Fig. 17. Charge transfer cross-sections of H^+ in argon; D^+ in argon, krypton, and xenon.
[J. B. Hasted, Proc. Roy. Soc. (London) A212, 235 (1952), as show in Brown.]

using the subscript a to refer to ambipolar. We assume that $n_i \approx n_e = n$; that is, the ion and electron density are initially sufficiently large that the requisite electric field may be created by a slight inequality in n_i and n_e . Eqs. (34) and (35) become

$$\Gamma_a = - D_i \nabla n + \mu_i E n \tag{37}$$

$$\Gamma_a = - D_e \nabla n + \mu_e E n \ . \tag{38}$$

Eliminating E between Eqs. (37) and (38), we get

$$\Gamma_a = - D_a \nabla n \tag{39}$$

where

$$D_a = \left(\frac{D_i \mu_e + D_e \mu_i}{\mu_i + \mu_e} \right) \ . \tag{40}$$

From the Einstein relation we have

$$\frac{D_i}{\mu_i} = \frac{kT_i}{e} \ ; \qquad \frac{D_e}{\mu_e} = \frac{kT_e}{e} \ . \tag{41}$$

If we now assume that

$$\alpha T_i = T_e \tag{42}$$

where $\alpha \geq 1$,

$$\frac{D_i}{\mu_i} = \frac{1}{\alpha} \frac{D_e}{\mu_e} \ . \tag{43}$$

Using Eq. (43) to eliminate D_e from Eq. (40), and assuming $\mu_i << \mu_e$, we find

$$D_a = D_i (1 + \alpha) \ . \tag{44}$$

This is another way of saying that the loss rate of both types of charged particle is essentially determined by the loss rate of the slower.

Note that while we have given an effective diffusion coefficient, mobility is definitely involved. There is an electric field. From Eqs. (37) and (38) we can eliminate Γ_a, giving

$$E = \frac{\nabla n}{n}\left(\frac{D_i - D_e}{\mu_i + \mu_e}\right) \tag{45}$$

which under the approximations $D_i < D_e$, $\mu_i < \mu_e$ becomes

$$E = -\frac{\nabla n}{n}\frac{D_e}{\mu_e} = \frac{\nabla n}{n}\frac{kT_e}{e} \quad . \tag{46}$$

We assume that $(n_e - n_i) << n$. Let us find the conditions under which this is valid. From the Maxwell equation for the divergence of E, we have

$$\nabla \cdot E = 4\pi\, e(n_i - n_e) \sim \frac{E}{L} , \tag{47}$$

where L is a length characteristic of the region where $n_i \neq n_e$. From Eqs. (46) and (47),

$$E = L\, 4\pi e(n_i - n_e) = -\frac{\nabla n}{n}\frac{kT_e}{e} \quad . \tag{48}$$

We approximate ∇n as n/L. In steady state, any region where there is a gradient of the density ambipolar diffusion must be taking place; therefore the electric field must exist over the same region as the density gradient. We therefore use the same L here. Thus we have

$$\frac{n_i - n_e}{n} = -\frac{kT}{4\pi n e^2 L^2} = -\frac{\lambda_D^2}{L^2} \quad . \tag{49}$$

Thus the condition that $n_i - n_e << n$ becomes one that the Debye length be small compared with the region over which the density changes.

Breakdown and the First Townsend Coefficient

We consider a neutral gas of density n_g to which is applied a uniform electric field E, say in the $-x$ direction. At some point $x = x_o$ electrons are injected at density n_o. The electric field is sufficiently high that ionizing collisions take place, so that as the electron drifts in the x direction, n is a function of x.

Since each electron produces ν_i secondary electrons per second, and there are at any time n electrons per cubic centimeter, the rate of increase of n due to ionization is just $n\nu_i$, where $\nu_i = n_g \overline{\sigma v}$. Under the influence of the electric field, the electrons drift across the boundaries. The difference between the number entering and leaving unit volume per second is $\bar{v}\nabla n$, where $\bar{v}$ is a constant since E and p, and thus the mobility are constant. Thus, for a steady state

$$\nu_i n = \bar{v}\frac{dn}{dx} \tag{50}$$

$$n = n_o e^{\alpha x} \tag{51}$$

where

$$\alpha = \frac{\nu_i}{v} \tag{52}$$

and is called the first Townsend coefficient. $\bar{v}$ is a function of E/p. ν_i is proportional to pressure (at fixed $\bar{v}$). Thus, α/p is a function of E/p. Typical values are shown in Figs. 18 and 19.

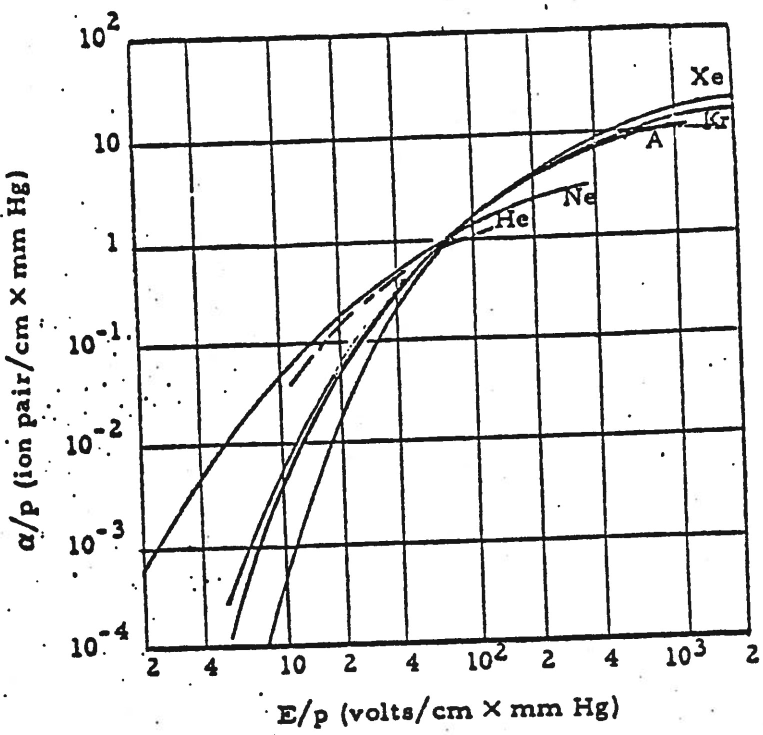


Fig. 18. First Townsend ionization coefficients in noble gases. [A. von Engel, Handbuch der Physik, Springer Verlag, Berlin (1956) Vol. 21, p. 504, as shown in Brown.]

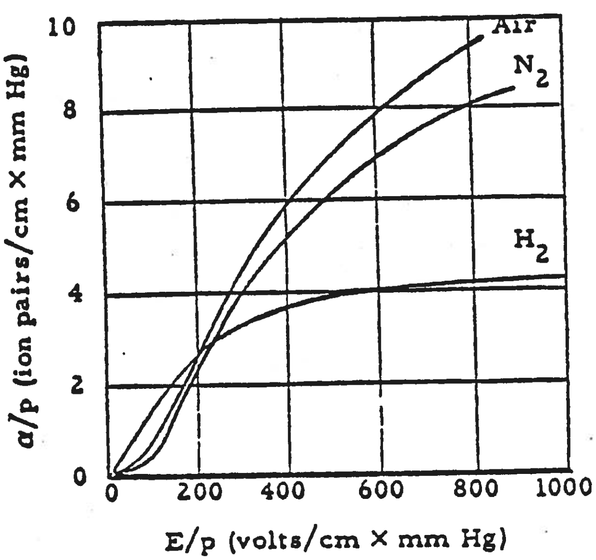


Fig. 19. First Townsend ionization coefficients of air, N_2 and H_2. [A. von Engel, Handbuch der Physik, Springer Verlag, Berlin (1956) Vol. 21, p. 504, as shown in Brown.]

This, of course, applies to dc discharges. When ac fields are applied the dominant losses may be quite different. In an ac field the particle position oscillates with a limited amplitude (in the absence of collisions). If the dimension (along E) is large compared with this amplitude, mobility losses no longer dominate. Diffusion becomes dominant. See, for example, Brown.

We have shown how an initial density of electrons n_o builds up exponentially in space. In order to effect breakdown, a feedback mechanism is needed. Consider, for example, a discharge between electrodes separated by a distance d. Ions striking the cathode may produce secondary emission as one feedback mechanism. Assuming a cascading density as in Eq. (51), the ion production per unit time in the whole length is

$$\int_o^d n(x)\,\nu_i\,dx = \nu_i \frac{n_o}{\alpha}(e^{\alpha d} - 1) \tag{53}$$

where n_o is the total number at the cathode. Then if γ_i is the ratio of secondary electron production rate to the column ion production rate, the rate of secondary electron production is

$$\Gamma_e = \gamma_i \nu_i \frac{n_o}{\alpha}(e^{\alpha d} - 1) \ . \tag{54}$$

If n_s is the number of secondaries per second, then since $\Gamma_e = n_s \bar{v}_e$

$$n_s = \gamma_i n_o (e^{\alpha d} - 1) \ . \tag{55}$$

Then is n_o' is the initiating flux (perhaps one electron),

$$n_o = \gamma_i n_o (e^{\alpha d} - 1) + n_o' \tag{56}$$

$$n(d) = \frac{n_o' e^{\alpha d}}{1 - \gamma_i(e^{\alpha d} - 1)} \tag{57}$$

If E/p is increased, α increases and the condition may be achieved that the denominator of Eq. (57) approaches zero,

$$1 - \gamma_i(e^{\alpha d} - 1) = 0 \tag{58}$$

thus effecting breakdown. An additional feedback mechanism is photo emission from the cathode. The sum of γ_i for secondary emission and photo emission is called the second Townsend coefficient. The value of the quantity γ_i depends on the nature of the electrodes and on the geometry.

If we plot the voltage required for breakdown between two electrodes against the product of pressure p times electrode separation d , the so-called Paschen curve results. Fig. 20 shows the typical form of this curve. For small pd there is very little cascading. More secondary emission is needed so a larger potential is required.

This is an approximate description of one type of breakdown. It obviously does not apply to discharges which are dominated by other factors such diffusion or negative ion formation.

Electron-Ion Recombination

Electron-ion recombination is an important process in low-temperature plasmas such as gas discharges or the ionosphere. Recombination of an electron and an atomic ion may be quite a complex process.. An ion may capture an electron in the ground or excited state. The excess energy involved in the process may be radiated (radiative recombination).

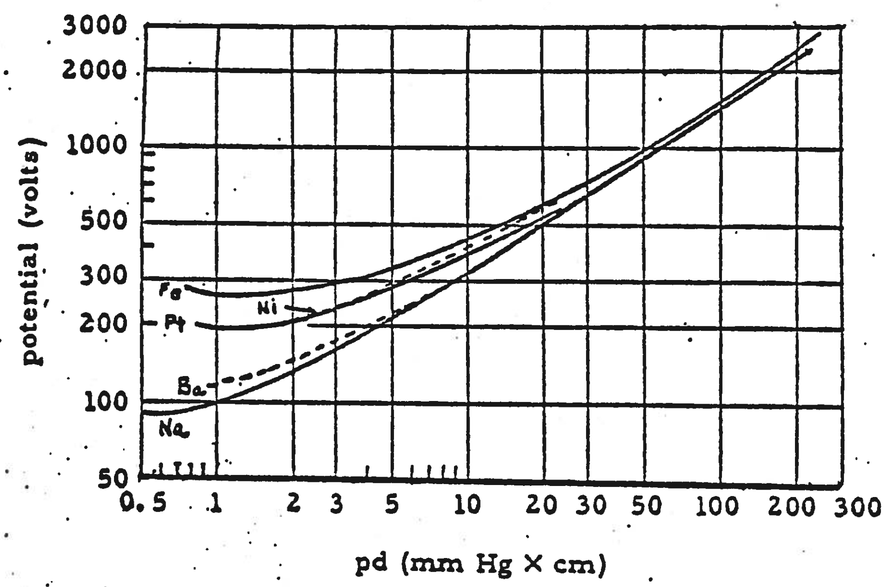


Fig. 20. Breakdown potential in argon between plates for various cathode materials.

[As shown in Brown.]

$$A^+ + e \rightarrow A + h\nu \tag{59}$$

or may be given to a third particle (three-body recombination). The case where the third body is another electron is of particular interest.

$$A^+ + e + e \rightarrow A + e \tag{60}$$

There are many cases of low-temperature plasmas where molecular ions are formed which may recombine and dissociate (dissociative recombination).

$$AB^+ + e \rightarrow A + B \tag{61}$$

the excess energy being carried off as kinetic energy of the atoms.

The process by which an ion and free electron end up in the ground state may be quite complex. The electron may be captured in the ground or in an excited state, from whence it may cascade down, emitting additional radiation, or it may be raised to a higher level or re-ionized by a photon; it may engage in additional collisions which result in transitions up or down. Instead of using cross sections, it is customary to make use of a recombination coefficient α defined by

$$\frac{dn_i}{dt} = \alpha n_i n_e \tag{62}$$

Thus α is essentially $\overline{\sigma v}$ for the process.

At low electron densities, three-body and electron collisions with excited atoms are negligible; then radiative recombination would be expected to dominate.

Calculations for radiative recombination in hydrogen yield approximately

$$\alpha \sim 6 \times 10^{-11}\, T^{-\frac{1}{2}} \qquad (T \text{ in } {}^\circ K) \tag{63}$$

at temperatures below 1 eV, falling somewhat more rapidly as the temperature rises. Thus at 1 eV the cross section for radiative recombination is about

$$\frac{\alpha}{v} = \sigma = \frac{6 \times 10^{-11}}{5 \times 10^{7}\ 10^{2}} \sim 10^{-20} \tag{64}$$

At higher pressures other processes become dominant. Measured dissociative recombination coefficients, at about 400°K are given in Table II. Little information is available on temperature dependence.

In the case of three-body recombination, α must depend on density. For $n_e > 10^{10}$, an approximate equation for α (three-body) is

$$\alpha \sim 3 \times 10^{-22}\, n \left(\frac{T}{1000}\right)^{4/5} \tag{65}$$

More exact calculated values are given in Table III.

IV. Particle Orbits

We have seen that interactions between individual pairs of charged particles are to some good approximation negligible. On the other hand, when many particles interact the interactions may be important. The fields due to large numbers of charged particles will be smooth on the average. There will be small fluctuations due to the presence of individual particles, but these are more or less unimportant. Thus we may approximate what goes on by treating the system as a collection of particles, each one moving in the smoothed-out fields of all the other particles plus, of

Table I. Relative probabilities of different types of collision of electrons in atomic hydrogen (Massey and Burhop)

Type of Collision	Energy of incident electrons (eV)				
	100	200	400	1,000	10,000
	Percentage of all collisions				
Elastic	12.2	10.2	9.8	8.7	6.5
Excitation of 2-quantum levels	33.5	33.6	39.0	42.8	45.3
Excitation of 3-quantum levels	5.9	5.8	6.8	6.3	7.0
Excitation of 4-quantum levels	2.2	2.0	2.2	2.4	2.6
Excitation of 5-quantum levels	1.0	0.9	1.0	1.2	1.2
Excitation of higher quantum levels	1.7	1.7	2.0	2.2	2.3
All discrete levels	44.3	44.0	51.0	54.8	58.4
Ionization	43.5	45.8	39.2	36.5	35.1
Total cross section (units πa_o^2)	2.45	1.50	0.79	0.37	0.049

Table II. Dissociative Recombination Coefficients (Measured)

He	10^{-8}	H	3×10^{-8}
Ne	2×10^{-7}	N_2^+	6×10^{-7}
A	7×10^{-7}	N_3^+	$\sim 2 \times 10^{-6}$
Kr	3×10^{-7}	N_4^+	$\sim 2 \times 10^{-6}$
Xe	2×10^{-6}		

Table III.

n	250°K	1,000°K	4,000°K	16,000°K	64,000°K
10^8	8.8×10^{-11}	4.1×10^{-12}	9.2×10^{-13}	3×10^{-13}	10^{-13}
10^{10}	2.8×10^{-9}	1.9×10^{-11}	1.4×10^{-12}	3.2×10^{-13}	10^{-13}
10^{12}	2.6×10^{-7}	3.9×10^{-10}	4.4×10^{-12}	4.3×10^{-13}	10^{-13}
10^{14}	2.6×10^{-5}	2.9×10^{-8}	5.1×10^{-11}	1×10^{-12}	1.2×10^{-13}
10^{16}	2.6×10^{-3}	2.9×10^{-6}	2.3×10^{-9}	5×10^{-12}	1.9×10^{-13}

(at low temperature, $\alpha \propto n$)

course, any externally-applied fields. One can gain much insight into the behavior of plasmas by investigating the motion of single charged particles in artibrary electric and magnetic fields. One can go further and make the fields consistent with the motion of all the particles. That is, we must determine the fields from Maxwell's equations and the particle motion from the Lorentz force acting on a particle. Maxwell's equations are

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}}{c} \tag{2}$$

$$\vec{\nabla} \bullet \vec{B} = 0 \tag{3}$$

$$\vec{\nabla} \bullet \vec{E} = 4\pi\rho \tag{4}$$

From Eqs. (2) and (4) we get the equation of continuity

$$\dot{\rho} + \vec{\nabla} \bullet \vec{j} = 0 \tag{5}$$

where

$$\rho = \sum_i q_i \overline{n_i} \tag{6}$$

$$\vec{j} = \sum_i q_i \overline{n_i \vec{v}_i} \tag{7}$$

while the Lorentz force law reads

$$m_i \frac{d\vec{v}_i}{dt} = q_i(\vec{E}_i + \vec{v}_i \times \vec{B}) \tag{8}$$

Here ρ and j are the charge and current densities due to the plasma particles, i refers to the i^{th} species, $\overline{n_i}$ is the number density of the i^{th} species at a point r . These are the so-called Vlasov equations for a plasma. We will take these up later. Here we shall start by investigating single particle motions.

A. Cyclotron Motion

The equation of motion for a charged particle in an electric and magnetic field is

$$m\,\frac{d\vec{v}}{dt} = q\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) \tag{9}$$

We can immediately obtain the energy conservation law by dotting both sides with $\vec{v}$ and integrating with respect to time

$$m\,\vec{v}\cdot\frac{d\vec{v}}{dt} = q\,\vec{v}\cdot\vec{E} \tag{10}$$

$$\frac{d}{dt}\,\frac{m\,\vec{v}\cdot\vec{v}}{2} = q\,\vec{v}\cdot\vec{E} \tag{11}$$

or

$$\Delta\left(\frac{m\,\vec{v}\cdot\vec{v}}{2}\right) = \int q\,\vec{v}\cdot\vec{E}\,dt = \int_{\text{orbit}} q\,\vec{E}\cdot d\,\vec{s} \tag{12}$$

where $d\,\vec{s}$ is a vector element of the orbit. If E is electrostatic and hence derivable from a potential, this Eq. (12) may be written in the form

$$\frac{mv^2}{2} + q\phi = \text{constant} \tag{13}$$

$$\vec{E} = -\vec{\nabla}\phi$$

Eq. (13) says that the change in the kinetic energy is equal to the work done by the electric field. The magnetic field does no work on the particle, since the force it exerts on the particle is always perpendicular to the velocity.

Returning now to Eq. (9), the solution of this equation for arbitrary E and B would in general be very complicated. We shall therefore look at some simple situations out of which we could build up more complicated situations.

The simplest possible situation is, of course, that of spatially uniform E and B fields. Let us first consider the case of a uniform B . Since the magnetic force is perpendicular to both v and B , there is no force on the particle in the direction of B and the velocity in this direction is constant. We need not consider this velocity any more. The velocity perpendicular to the magnetic field is constant in magnitude. One can obtain the radius of this circle by balancing the centrifugal force against the magnetic force.

$$\frac{m v_{\perp}^2}{r} = \frac{q v_{\perp} B}{c} \tag{14}$$

or

$$r = \frac{v_{\perp} mc}{qB} \tag{15}$$

The quantity qB/mc is the angular frequency of the particle and is called the cyclotron frequency

$$\omega_c = \frac{qB}{mc} \tag{16}$$

The radius r is called the Larmor radius.

If one wishes to be more formal in solving for the motion, one can proceed as follows. Let the direction of B be the z direction. Then $v_{\perp}$ has x and y components and Eq. (9) can be written in the form

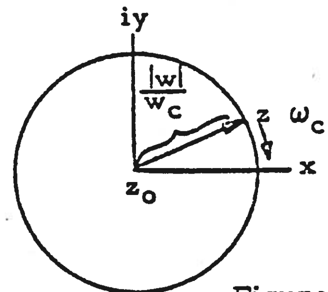


Figure 21

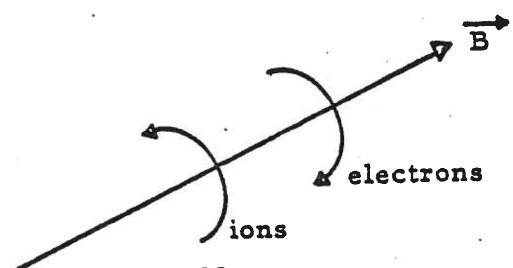


Figure 22

Problem: Find the cyclotron frequency for electrons and protons in a field of 10,000 gauss; in a field of .1 gauss. The latter is roughly the strength of the earth's field at 4,000 miles. Compute the radius of gyration of a 1 MeV proton in a field of .1 gauss.

B. Magnetic Moment

A current loop has a dipole moment associated with it of magnitude

$$\mu = \frac{IA}{c} \tag{24}$$

where I is the current (in esu units) and A is the area of the loop. The circular orbit of a charge in a magnetic field is the area of the loop. The circular orbit of a charge in a magnetic field on the average constitutes a current loop. The average current is the average charge per unit time which passes a point on the orbit. This is $\frac{1}{\tau} q/c$, where τ is the period. The magnetic moment is thus

$$\mu = \frac{q}{c\tau} \pi r^2 = \frac{q\omega_c}{c\,2\pi} \pi \frac{v_\perp^2}{\omega_c^2} = \frac{\pi q}{2\pi c} \frac{v_\perp^2}{qB/mc} \tag{25}$$

$$= mv^2/2B = W_\perp/B$$

Here $W_\perp$ is the energy of the particle due to its perpendicular motion.

In addition to the magnitude $W_\perp/B$, the magnetic moment has a direction associated with it — the direction of the magnetic dipole with the equivalent magnetic moment. From Fig. 22 and the right-hand rule, we see that the current loop is such as to reduce the field inside of itself, and hence the magnetic moment has a direction opposite to the direction of the field.

$$\vec{\mu} = -\frac{W_\perp}{B^2}\vec{B} \tag{26}$$

C. Magnetization

In a plasma containing many particles the magnetic field produced by all the magnetic moments can be appreciable. To compute this effect we must make use of the Maxwell equation

$$\vec{\nabla} \times \vec{B} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}}{c} \tag{27}$$

First we shall assume that all quantities are time-independent, or at least vary so slowly that we can neglect the $\partial E/\partial t$ term. Second, for the time being we imagine that j is produced only by plasma particles. Equation (27) may be written in the integral form

$$\oint_c \vec{B} \cdot d\vec{\ell} = \int_s \vec{\nabla} \times \vec{B} \cdot d\vec{A} = \frac{4\pi}{c} \int_s \vec{j} \cdot d\vec{A} \tag{28}$$

where c is a curve bounding an area s , $d\vec{\ell}$ is an element of the curve c , and $d\vec{A}$ is a vector element of the surface s and has the direction of the normal to s [Eq. (28) follows from the Stokes theorem]. To compute the integral on j in Eq. (28) we must find the current normal to s or the total charge crossing s per unit time. Consider the situation shown in Fig. 23.

Let us compute the total charge crossing s per unit time. Orbits like 2 , which intersect s twice, give no transfer of charge across s . On the other hand, orbits like 1 and 3 , which intersect s only once, transfer an amount of charge q (the charge on the particle) every time they cross s . Thus only those orbits which loop the curve c contribute to the current through s . Now they transfer the charge q (the charge on a particle) for each revolution, or they transfer charge at the rate q/τ_c .

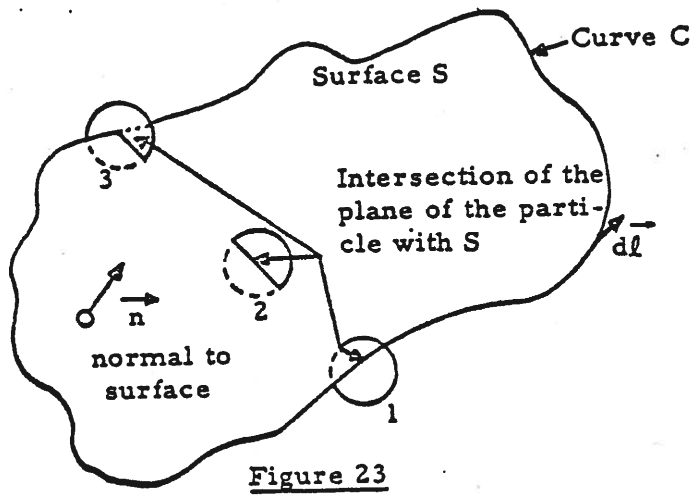

Curve C
Surface S
Intersection of the plane of the particle with S
$\vec{d\ell}$
3
2
1
$\vec{n}$
normal to surface

Figure 23

The number of orbits which loop c in a length $|d\vec{\ell}|$ is the density of such orbits times the areas of such orbits normal to $d\vec{\ell}$, times $|d\vec{\ell}|$ $(q/\tau_c) a_n |d\vec{\ell}| = I a_n |d\vec{\ell}|$.

Now we must take account of whether charge is transferred across the surface in the positive or negative direction. We must compute the net charge crossing in the positive direction. If the magnetic moment $\vec{\mu} = \frac{I\vec{A}}{c}$ is in the direction of $d\vec{\ell}$ then the current crosses the surface in the positive direction, while if $\vec{\mu}$ is opposite to $d\vec{\ell}$ the current crosses the surface in the negative direction (a little consideration using the right-hand rule shows this). Thus the total current crossing s is given by

$$\int_s \vec{j} \cdot d\vec{A} = \int_c N I \vec{a} \cdot d\vec{\ell} = c \int_c N \vec{\mu} \cdot d\vec{\ell} \tag{29}$$

Eq. (28) may thus be written in the form

$$\int_c \vec{B} \cdot d\vec{\ell} = 4\pi \int_c N\vec{\mu} \cdot d\vec{\ell} = \int_s \vec{\nabla} \times \vec{B} \cdot d\vec{A} = 4\pi \int_s \vec{\nabla} \times N\vec{\mu} \cdot d\vec{A} \tag{30}$$

Writing $\vec{M} = N\vec{\mu}$, we may write

$$\int_s \vec{\nabla} \times \vec{B} \cdot d\vec{A} = 4\pi \int_s \vec{\nabla} \times \vec{M} \cdot d\vec{A} \tag{31}$$

or

$$\vec{\nabla} \times (\vec{B} - 4\pi \vec{M}) = 0$$

If we do not neglect $\partial\vec{E}/\partial t$, and if there are currents other than those contributed by the plasma particles (let us denote such currents by j_e), then we can proceed in a similar manner and we obtain Eq. (32) in place of Eq. (31).

$$\vec{\nabla} \times (\vec{B} - 4\pi \vec{M}) = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}_e}{c} \tag{32}$$

In the classical treatment of magnetic materials, $\vec{B} - 4\pi\vec{M}$ would have been called $\vec{H}$, and Eq. (32) would then read

$$\vec{\nabla} \times \vec{H} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}_e}{c} \qquad (33)$$

For normal materials and small magnetic fields, the magnetization is proportional to B

$$\vec{M} = \alpha \vec{B} \qquad (34)$$

so that H is also proportional to B. Here, however,

$$\vec{M} = -\frac{N W_\perp \vec{B}}{B^2} \qquad (35)$$

and is proportional to $1/B$. Thus α, and also the magnetic permeability $[1/(1 - 4\pi\alpha)]$ are not constant. H is not useful; we will use only B.

We may substitute Eq. (35) in Eq. (32) and obtain

$$\vec{\nabla} \times \left[\vec{B}\left(1 - 4\pi \frac{N W_\perp}{B^2}\right)\right] = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{j}_e}{c} \qquad (36)$$

From this we see that the particles begin to make an important contribution to the field when their energy density $N W_\perp$ becomes comparable to the energy density of the magnetic field $B^2/8\pi$.

D. The Electric Field Drift

Next in simplicity to the constant magnetic field case is the situation in which we have constant electric and magnetic fields. From Eq. (9) we have for the component of the motion along the magnetic field

$$m \frac{d v_\parallel}{dt} = q E_\parallel \qquad (37)$$

We can integrate this equation to obtain

$$v_{\parallel} = \frac{qE_{\parallel}}{m} t + v_{\parallel o} \tag{38}$$

$$x_{\parallel} = \frac{qE_{\parallel}}{m} \frac{t^2}{2} + v_{\parallel o} t + x_{\parallel o} \tag{39}$$

Thus the particle freely accelerates along the field.

The equation of motion for the components of $\vec{v}$ perpendicular to $\vec{B}$ is

$$m \frac{d\vec{v}_{\perp}}{dt} = q \left(\vec{E}_{\perp} + \frac{v_{\perp} \times \vec{B}}{c} \right) \tag{40}$$

Now in this equation both the electric force and the magnetic force are perpendicular to $\vec{B}$ and it is possible to balance them. If they are balanced, $d\vec{v}_{\perp}/dt$ is 0, and the particle moves with a constant perpendicular velocity. Let us equate the electric and magnetic forces

$$\vec{E}_{\perp} + \frac{\vec{v}_{\perp} \times \vec{B}}{c} = 0 \tag{41}$$

Crossing this with $\vec{B}$ gives

$$\vec{B} \times \vec{E}_{\perp} + \frac{\vec{B} \times (\vec{v}_{\perp} \times \vec{B})}{c} = \vec{B} \times \vec{E}_{\perp} + \frac{\vec{v}_{\perp} B^2}{c} = 0 \tag{42}$$

or solving for $\vec{v}_{\perp}$

$$\vec{v}_{\perp} = \frac{(\vec{E}_{\perp} \times \vec{B})c}{B^2} \tag{43}$$

Let us denote this velocity by $\vec{v}_E$ and write for $\vec{v}_{\perp}$

$$\vec{v}_{\perp} = \vec{v}_1 + \vec{v}_E \tag{44}$$

Substituting in Eq. (9) the $\vec{v}_E \times \vec{B}$ term cancels the E term and we get

$$m \frac{d\vec{v}_1}{dt} = q \frac{\vec{v}_1 \times \vec{B}}{c} \tag{45}$$

This is the same equation that one gets without an E field, and hence $\vec{v}_1$ rotates at a constant rate. The motion of the particle is thus a drift with a uniform velocity v_E plus a rotation about the magnetic field lines. We should note that the drift velocity is independent of the charge.

Motion across a magnetic field gives rise to an E field. The transformation law for E in going from one frame to another, moving with a velocity v relative to it, is (for velocities small compared to light)

$$\vec{E}' = \vec{E} + \frac{\vec{v} \times \vec{B}}{c} \tag{46}$$

For $v = v_E$ the right side vanishes: in the frame moving with the drift velocity there is no electric field and hence the particle sees only a magnetic field and moves accordingly.

We may also view the drift in another way, which is illustrated in Fig. 24.

As a positive charge spirals about the magnetic field its energy changes due to the E field. It moves faster on the upper part of its orbit and the curvature is smaller here than on the lower part of its orbit; hence the drift. For a negative charge, on the other hand, the velocity is larger on the lower part of the orbit, but since the direction of rotation is opposite to that for a positive charge, the resultant drift is in the same direction.

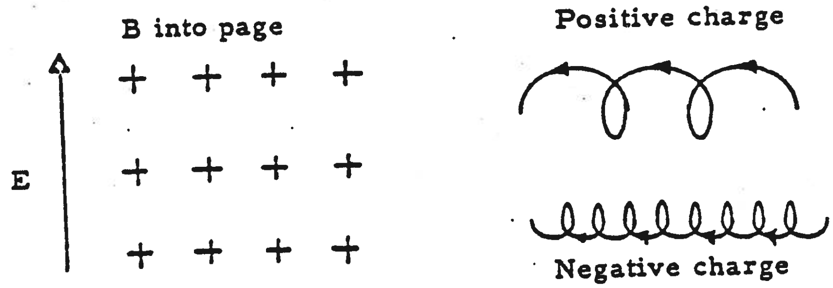


Figure 24

For a neutral plasma, since the two types of charges drift in the same direction at the same rate, there is no current. No net work is done on either type of particle, since they drift perpendicular to E .

E. The Effect of Gravity or Other External Force

If a uniform gravitational force or other external force acts on the particle in addition to the magnetic force, then the equation of motion for the particle is given by

$$m \frac{d\vec{v}}{dt} = \vec{F} + \frac{q}{c} \vec{v} \times \vec{B} \tag{47}$$

where $\vec{F}$ is the external force. We may replace $\vec{F}$ by an equivalent electric field force.

$$\vec{F} = q\vec{E}$$

$$\text{or} \tag{48}$$

$$\vec{E} = \vec{F}/q$$

and take over all our results from the previous section. The particle drifts across the magnetic field with the equivalent of an $\vec{E} \times \vec{B}$ velocity, which is now

$$\vec{v}_F = \frac{c}{q} \frac{\vec{F} \times \vec{B}}{B^2} \tag{49}$$

Superimposed upon this drift is the usual Larmor motion about the magnetic field. If $\vec{F}$ has a component parallel to $\vec{B}$, the particle accelerates in this direction at a constant rate. We see from Eq. (49) that the direction of drift (unlike that found for the E field case) depends on the sign of the charge on the particle. Thus an external force acting on a neutral cloud will cause charges of different sign to drift in opposite directions, giving rise to a current. The current gives rise to a $j \times B$ force which balances the external force.

Problem: Prove the statement in the last sentence of part E.

F. The Effect of a Time-Varying Electric Field

Let us imagine that our particle is subject to a spatially uniform B field and a spatially uniform E which has a constant direction in space, but whose magnitude varies in time. We will take the E field to be perpendicular to B since the magnetic field plays no role in the motion of a particle parallel to it. We shall further assume that the electric field is changing at a rate which is slow by comparison with the Larmor motion. That is, we assume

$$\frac{1}{E}\frac{1}{\omega_c}\frac{dE}{dt} << 1 \tag{50}$$

We will first derive what happens on simple physical grounds. Since the E field is changing slowly with time, to a first approximation the particle will move with an instantaneous $E \times B$ drift velocity plus Larmor motion. Thus $\vec{v}_\perp$ is given by

$$\vec{v}_\perp = \frac{c\vec{E}(t) \times \vec{B}}{B^2} + \vec{v}_{Larmor} \tag{51}$$

Now, since the drift velocity changes with time, so does the kinetic energy of the particle. Let us average this change over one Larmor period so as to remove the periodic changes already discussed. We then get

$$\frac{d}{dt}\frac{m}{2}v_\perp^2 = m\vec{v}_E \cdot \frac{d\vec{v}_E}{dt} = m\frac{c^2}{B^2}\frac{dE^2/2}{dt} \tag{52}$$

The last equality follows from the fact that we assumed that E was constant in direction and perpendicular to B . Now this energy must be supplied by the electric field or

$$q v_{\parallel E} E = \frac{c^2 m}{B^2} E \frac{dE}{dt} \tag{53}$$

$$v_{\parallel E} = \frac{c^2 m}{qB^2}\frac{dE}{dt} \tag{54}$$

where $v_{\parallel E}$ denotes the component of the velocity parallel to E .

We may derive this result from still another physical argument. The sum of all the forces on a particle (including inertial forces) must be zero. Now we may treat the inertial force due to the changing $E \times B$ velocity as an external force; a gravitational force, if one wishes. Then according to our previous analysis of the drift of a particle subject to an external force, the particle will acquire a drift velocity given by

$$\vec{v}_F = c\frac{\vec{F}\times\vec{B}}{qB^2} \tag{55}$$

Substituting for $\vec{F}$ $-m\,dv/dt$ (the inertial force or effective gravitational force is in the opposite direction to the acceleration):

$$\vec{F} = -\frac{mc}{B^2}\frac{d\vec{E}}{dt}\times\vec{B} \tag{56}$$

and we find for $\vec{v}_{\frac{dE}{dt}} \equiv v_p$

$$\vec{v}_p = -\frac{mc^2}{qB^2}\left(\frac{d\vec{E}}{dt} \times \vec{B}\right) \times \vec{B}/B^2 \tag{57}$$

or

$$\vec{v}_p = +\frac{mc^2}{qB^2}\frac{d\vec{E}}{dt} \tag{58}$$

since by assumption E is perpendicular to B. This method has the advantage that it applies even if the direction of E is changing with time, but E must still remain perpendicular to B. This approach also applies only when E varies slowly on the scale of the Larmor frequency, for only then can we neglect the Larmor motion and the forces associated with it.

Finally, we may derive these results formally from Eq. (9). Again we write

$$\vec{v}_\perp = \vec{v}_E + \vec{v}_1 \tag{59}$$

$$\vec{v}_E = c\,\frac{\vec{E} \times \vec{B}}{B^2} \tag{60}$$

Substituting in Eq. (9) gives

$$m\left(\frac{d\vec{v}_E}{dt} + \frac{d\vec{v}_1}{dt}\right) = \frac{q}{c}\,\vec{v}_1 \times \vec{B} \tag{61}$$

We set $v_1 = v_2 + v_p$, so that Eq. (61) becomes

$$m\frac{dv_E}{dt} + \frac{m\,dv_2}{dt} + \frac{m\,dv_p}{dt} = \frac{q}{c}\,v_2 \times B + \frac{q}{c}\,v_p \times B \tag{62}$$

and in a manner similar to the previous case we define v_p so as to cancel the $m\,\frac{dv_E}{dt}$ term, i.e.,

$$v_p \equiv \frac{mc^2}{qB^2}\frac{dE}{dt} \tag{63}$$

so that

$$\frac{q}{c}\vec{V}_P \times \vec{B} = \frac{q\,mc^2}{c\,q\,B^2}\frac{d\vec{E}}{dt} \times \vec{B} = mc\,\frac{\frac{d\vec{E}}{dt} \times \vec{B}}{B^2} = m\,\frac{d\vec{V}_E}{dt} \tag{64}$$

Then Eq. (62) becomes

$$m\frac{d\vec{V}_P}{dt} + m\frac{d\vec{V}_2}{dt} = \frac{q}{c}\vec{V}_2 \times \vec{B} \tag{65}$$

If the first term is negligible, then as before $\vec{v}_2$ describes the Larmor motion in a frame moving with velocity $\vec{v}_p + \vec{v}_E$. Essentially we are utilizing a Taylor expansion (in time) of the electric field, having thus far found drifts corresponding to $\vec{E}$ and $d\vec{E}/dt$; the remaining term, $d\vec{v}_p/dt$ is a $d^2\vec{E}/dt^2$ term.

The $d\vec{v}_p/dt$ may be dropped provided that

$$\left| m\,\frac{d\vec{V}_P}{dt}\right| \ll \left|\frac{q}{c}\vec{V}_2 \times \vec{B}\right| \tag{66}$$

or

$$\left|\frac{mc^2}{q\,B^2}\frac{d^2\vec{E}}{dt^2}\right| \ll \left|\frac{q}{c}\vec{V}_2 \times \vec{B}\right| \quad \text{and,} \tag{67}$$

assuming that $\vec{E} \sim \vec{E}e^{i\omega t}$,

$$1 \ll \frac{q^2B^2}{m^2c^2}\left(\frac{B}{cE}\right)\frac{V_2}{\omega^2} \tag{68}$$

or

$$1 \ll \frac{\omega_c^2}{\omega^2}\frac{V_2}{V_E}. \tag{69}$$

Thus the next higher term may be dropped, unless $v_E \gg v_2$, or ω is comparable with ω_c.

Problem: By setting $\vec{v}_2 = \vec{v}_3 + \vec{v}_p^{\,\circ}$, find the drift corresponding to d^2E/dt^2.

G. The Effective Dielectric Constant of a Plasma in a Magnetic Field

We have just seen that when a time-varying electric field is applied to a plasma in a magnetic field ($\vec{E} \perp \vec{B}$), particle drifts parallel to E arise, given by Eq. (58). The drifts are opposite for oppositely charged particles, so that a current arises in the plasma. The work that the $\vec{E}$ field does on this current is just what is required to get the plasma moving with the $c\dfrac{\vec{E} \times \vec{B}}{B^2}$ velocity.

The currents in the direction of $\vec{E}$ may be thought of as polarization currents; the plasma becomes polarized in this direction (Fig. 25).

To treat the plasma like a dielectric we divide the current into a plasma current and into currents due to external sources, in a manner analogous to the case of magnetization.

$$\vec{J} = \vec{J}_p + \vec{J}_e. \tag{70}$$

We have

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}_e + \frac{4\pi}{c} \vec{J}_p + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}. \tag{71}$$

Now if the plasma current is proportional to $\dfrac{\partial \vec{E}}{\partial t}$, as we have just found it to be when $\vec{E}$ is perpendicular to $\vec{B}$, then we can combine the last two terms on the right-hand side of Eq. (71)

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}_e + \left[\frac{4\pi c}{B^2} \sum_i n_i m_i + \frac{1}{c}\right] \frac{\partial \vec{E}}{\partial t} \tag{72}$$

(This only applies if $\vec{E}$ is perpendicular to $\vec{B}$, $\vec{J}_p = \frac{c^2}{B^2} \sum_i n_i m_i \frac{\partial \vec{E}}{\partial t}$).

We may set

$$\vec{D} = \left[\frac{4\pi\rho c^2}{B^2} + 1\right] \vec{E} = \epsilon \vec{E} \tag{73}$$

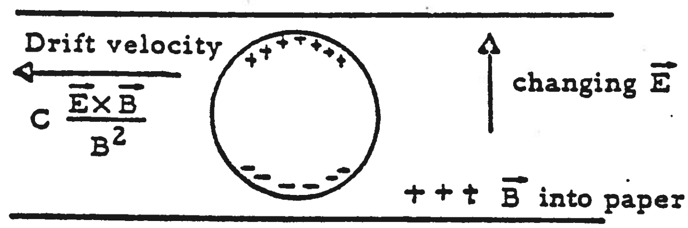

Drift velocity
$c\,\frac{\vec{E}\times\vec{B}}{B^2}$
changing $\vec{E}$
+ + + $\vec{B}$ into paper

Figure 25

and then Eq. (72) reads

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}_e + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}. \tag{74}$$

Further, we may divide the charge density into an internal and an external part.

$$\sigma = \sigma_p + \sigma_e \tag{75}$$

where σ_p has its sources in the charges within the plasma and is related to $\vec{j}_p$ by the continuity equation

$$\frac{\partial \sigma_p}{\partial t} + \vec{\nabla} \cdot \vec{j}_p = 0. \tag{76}$$

We then have

$$\vec{\nabla} \cdot \vec{E} = 4\pi(\sigma_p + \sigma_e) \tag{77}$$

and

$$\frac{\partial \sigma_p}{\partial t} + \vec{\nabla} \cdot \left[\frac{\rho c^2}{B^2} \frac{\partial \vec{E}}{\partial t} \right] = 0 \tag{78}$$

or

$$\frac{\partial}{\partial t} \left[\sigma_p + \frac{\rho c^2}{B^2} \vec{\nabla} \cdot \vec{E} \right] = 0 \tag{79}$$

or

$$\sigma_p = - \frac{\rho c^2}{B^2} \vec{\nabla} \cdot \vec{E} \tag{80}$$

If σ_p is 0 when E is 0, then from Eq. (77)

$$\vec{\nabla} \cdot \left[1 + \frac{4\pi \rho c^2}{B^2} \right] \vec{E} = 4\pi \sigma_e \tag{81}$$

or

$$\vec{\nabla} \cdot \vec{D} = 4\pi \sigma_e. \tag{82}$$

Problem: Find the capacitance of a parallel plate condenser with plasma between its plates and with a magnetic field parallel to the surface of the plates. Assume that there is an insulating layer of infinitesimal thickness isolating the plates from the plasmas. Show that the energy per unit area stored in the capacitor, $\frac{1}{2} cv^2$, is stored as kinetic energy.

H. Time-Varying $\vec{B}$

Let us consider the case in which $\vec{B}$ is spatially uniform, at least in the region visited by our particles, but in which its magnitude varies with time. Because of the time variation of $\vec{B}$ there will be $\vec{E}$ fields set up. These give rise to the $\vec{E} \times \vec{B}$ and $\dot{\vec{E}} \times \vec{B}$ drifts just discussed. However, here we are not so interested in these effects as we are in the fact that $\vec{E}$ has a curl and hence will do work on a circulating charge. We will imagine that we have subtracted out the mean $\vec{E} \times \vec{B}$ drift.

Now the change in the perpendicular energy of the particles is given by

$$\delta W_\perp = -q \int \vec{E} \cdot d\vec{l}. \tag{83}$$

But now if we go around a closed orbit, then

$$\oint \vec{E} \cdot d\vec{l} = \int \vec{\nabla} \times \vec{E} \cdot d\vec{A} = -\int \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}. \tag{84}$$

If $\partial \vec{B}/\partial t$ is essentially constant over an orbit, then we may replace the last integral by

$$(\pm) \frac{\pi a^2}{c} \frac{\partial \vec{B}}{\partial t}$$

where a is the Larmor radius $v_\perp/\omega_c = a$.

The ($\pm$) is determined by whether $\partial \vec{B}/\partial t$ is parallel or antiparallel to $d\vec{A}$. Now the direction of $d\vec{A}$ is determined by the direction of $d\vec{l}$ since the direction $d\vec{l}$ must be the direction in which the particle moves. If $\vec{B}$ is taken in the z direction, then $d\vec{A}$ is antiparallel to $\vec{B}$ for ions and parallel to $\vec{B}$ for electrons. Thus the sign is opposite to that of the charge. We thus find for $\delta W_\perp$

$$\delta W_\perp = \frac{|q|}{c} \pi a^2 \frac{\partial B}{\partial t} \quad \text{(in scalar form)} \tag{85}$$

or

$$\delta W_\perp = \frac{|q|}{c} \pi \frac{v_\perp^2 m^2 c^2}{q^2 B^2} \frac{\partial B}{\partial t} \tag{86}$$

or

$$\delta W_\perp = W_\perp \frac{2\pi mc}{|q| B} \frac{1}{B} \frac{\partial B}{\partial t} = W_\perp \frac{2\pi}{\omega_c B} \frac{\partial B}{\partial t} \tag{87}$$

thus

$$\frac{\delta W_\perp}{W_\perp} = \frac{\delta B}{B} \quad \text{for one orbit.} \tag{88}$$

This equation may also be written in the form

$$\delta\left(\frac{W_\perp}{B}\right) = 0 \quad \text{per orbit.} \tag{89}$$

Hence

$$\frac{W_\perp}{B} = \mu \cong \text{constant.} \tag{90}$$

The magnetic moment is thus approximately constant. It is not a strict constant since the above treatment requires that $\partial B/\partial t$ be essentially constant throughout an orbit. If B were changed instantaneously (very quickly, in the time it takes light to cross an orbit) from one value to another, then $W_\perp$ will not change since the particle will not have moved during the time B is changing. Thus $W_\perp/B$ will have changed. The magnetic moment is called an adiabatic invariant, since it is constant to a high degree of accuracy for slow variations of B . (It has been shown that the change in μ is exponentially small in the $B/\dot{B}$.)

There is a simple physical way to look at the constancy of μ . Now μ is equal to

$$\mu = \frac{W_\perp}{B} = \frac{m}{2}\frac{(a\omega_c)^2}{B}$$
$$= \frac{m}{2}\frac{a^2 e^2 B^2}{m^2 C^2 B} = \frac{e^2}{2mC^2} a^2 B \qquad (91)$$

where a is the Larmor radius.

Thus if μ is constant, a^2B or the flux through the orbit is constant. The orbit thus looks like a little superconducting current loop and no flux can cross it. This should not be surprising, since we have put in no mechanism for dissipating the current.

I. Spatially-Varying Magnetic Field

So far we have considered only magnetic fields which are spatially uniform over the regions visited by the particle. We wish now to consider magnetic fields which are not spatially uniform but which vary slowly with position. Here slowly means that variations of the magnetic field over a Larmor orbit are small, or

$$\frac{|\vec{\nabla}\vec{B}|\cdot|a|}{B} \ll 1. \qquad (92)$$

We can then find the particle's motion as a perturbation (locally) from what it would have in a spatially-uniform field. To this end we Taylor expand $\vec{B}$ about some point $\vec{r}$; $\vec{r}$ may be a function of t .

$$\vec{B}(\vec{r}+\vec{\rho}) = \vec{B}(\vec{r}) + \vec{\rho}\cdot\vec{\nabla}\vec{B}(\vec{r}). \qquad (93)$$

We will in general choose $\vec{r}$ to be the position of the guiding center for the particle — i.e., the instantaneous center of gyration. We shall consider the various elements of the tensor $\vec{\nabla}\vec{B}$ in turn. In the case of each set of terms, we will see first the influence of the terms on the shape of the field lines, and then find the effect on the particle orbits.

$$\vec{\nabla}\vec{B} = \begin{pmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial B_x}{\partial y} & \frac{\partial B_x}{\partial z} \\ \frac{\partial B_y}{\partial x} & \frac{\partial B_y}{\partial y} & \frac{\partial B_y}{\partial z} \\ \frac{\partial B_z}{\partial x} & \frac{\partial B_z}{\partial y} & \frac{\partial B_z}{\partial z} \end{pmatrix}. \quad (94)$$

(1) <u>The Effect of Diagonal Terms (Converging and Diverging Lines of Force)</u>

First consider the effects of the diagonal terms. Since $\vec{\nabla} \cdot \vec{B} = 0$, these terms are not all independent but their sum must vanish. We will choose a coordinate system such that r is 0 and such that the magnetic field at the origin points in the z direction. We have for the local magnetic field (neglecting off-diagonal terms for the moment, since their effect will be found shortly)

$$B_z = B_0 + \left[\frac{\partial B_z}{\partial z}\right]_0 z. \quad (95)$$

$$B_y = \left[\frac{\partial B_y}{\partial y}\right]_0 y, \quad (96)$$

and

$$B_x = \left[\frac{\partial B_x}{\partial x}\right]_0 x. \quad (97)$$

First let us see what these terms imply about the lines of force. The equations for a line of force are

$$\frac{dx}{dz} = \frac{B_x}{B_z} \quad (98)$$

and

$$\frac{dy}{dz} = \frac{B_y}{B_z}. \quad (99)$$

To 0 order (x and y or derivative terms neglected) we have

$$\frac{dx}{dz} = \frac{dy}{dz} = 0. \tag{100}$$

Thus $x = x_o$, $y = y_o$, and the lines are straight and parallel to the z axis as expected, since we chose the z direction as the direction in which the major part of B points. To first order in x and y we have

$$\frac{dx}{dz} = \left(\frac{\partial B_x}{\partial x}\right)_o \frac{x}{B_o} \tag{101}$$

and

$$\frac{dy}{dz} = \left(\frac{\partial B_y}{\partial y}\right)_o \frac{y}{B_o} \tag{102}$$

or

$$x = \left(\frac{\partial B_x}{\partial x}\right)_o \frac{xz}{B_o} + x_o \tag{103}$$

and

$$y = \left(\frac{\partial B_y}{\partial y}\right)_o \frac{yz}{B_o} + y_o. \tag{104}$$

Thus the lines of force are tilted as shown in Fig. 26 . The lines of force are diverging or converging. Further, since $\vec{\nabla} \cdot \vec{B} = 0$, we have, to lowest order for B_z

$$\frac{\partial B_z}{\partial z} = -\left\{\left(\frac{\partial B_x}{\partial x}\right)_o + \left(\frac{\partial B_y}{\partial y}\right)_o\right\} \tag{105}$$

or

$$B_z = B_o - \left\{\left(\frac{\partial B_x}{\partial x}\right)_o + \left(\frac{\partial B_y}{\partial y}\right)_o\right\} z. \tag{106}$$

Thus the strength of the main magnetic field (the B_z component) varies along the z direction or varies as one moves along the magnetic

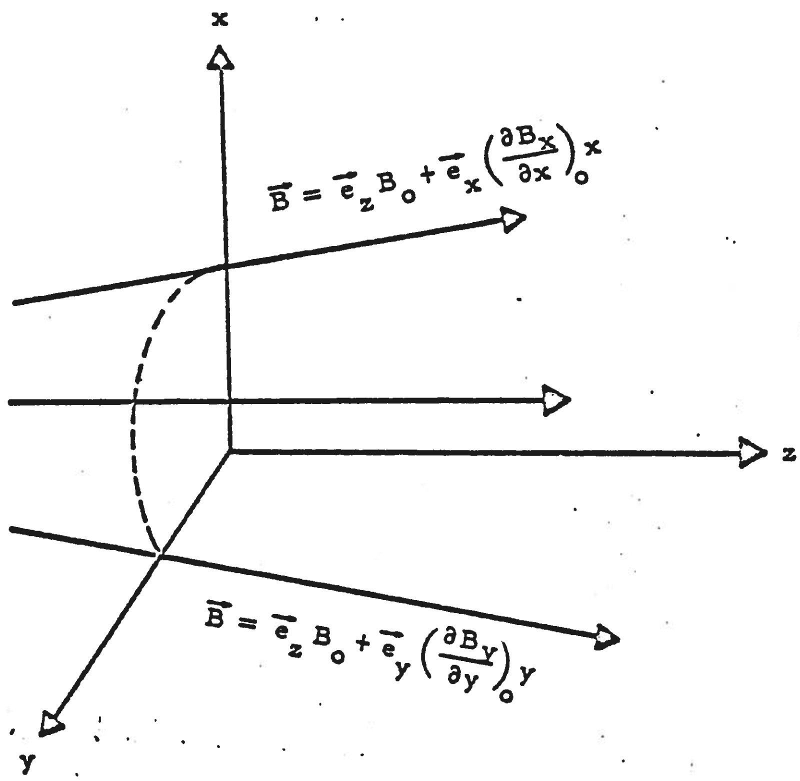

x
$\vec{B} = \vec{e}_z B_o + \vec{e}_x \left(\frac{\partial B_x}{\partial x}\right)_o x$
z
$\vec{B} = \vec{e}_z B_o + \vec{e}_y \left(\frac{\partial B_y}{\partial y}\right)_o y$
y

Figure 26

field lines, since to zero order they are in the z direction (unless the $\partial B_x/\partial x$ and $\partial B_y/\partial y$ terms cancel).

Before computing in detail what will happen to particle orbits, we may look at this problem in view of what we have already learned. Suppose the particle is gyrating about B_z and at the same time moving along the B lines — i.e., in the z direction. Then it will see a magnetic field whose strength is changing in time. By our assumption that $\vec{B}$ is slowly varying in space, this will be a slow time variation provided the motion of the particle along $\vec{B}$ is not extremely fast. From our treatment of the time-varying $\vec{B}$ field we should expect that the perpendicular energy of the particle would vary in such a way as to keep the magnetic moment constant

$$W_\perp = |B|\mu = |B| \frac{W_{\perp 0}}{B_0}. \qquad (107)$$

Now the particle's energy must be constant since the magnetic field does no work on it and hence there must be an equal and opposite change in the parallel energy of the particle.

$$W_\perp + W_\parallel = W = \text{constant}, \qquad (108)$$

$$W_\parallel = W - |B|\mu = W_{\parallel 0} + W_{\perp 0} - |B|\mu. \qquad (109)$$

Thus $|\vec{B}|\mu$ acts like a potential for the motion along the lines of force. Eq. (109) may be written in differential form. For the time interval dt we have

$$dW_\parallel = mv_\parallel dv_\parallel = -\mu \frac{d|B|}{dz} v_\parallel \, dt \qquad (110)$$

or

$$m\frac{dv_{\parallel}}{dt} = -\mu\frac{d|B|}{dz}. \tag{111}$$

This should be a familiar form. The force on a magnetic dipole is the product of the dipole moment and the field gradient. The negative sign results from the fact that the dipole is diamagnetic. Equivalently we can see that this force comes about because of the interaction of the particle's perpendicular motion with the radial $\vec{B}$ field, as shown in Fig. 27. These conclusions are actually borne out by the more detailed calculations which we shall now give. The equations of motion for our particle are

$$m\frac{d\vec{v}}{dt} = \frac{q}{c}\vec{v}\times\vec{B}. \tag{112}$$

Substituting in $\vec{B}$ from Eqs. (95), (96), and (97), and writing in component form gives

$$m\frac{dv_z}{dt} = \frac{q}{c}\left[v_x\left(\frac{\partial B_y}{\partial y}\right)_0 y - v_y\left(\frac{\partial B_x}{\partial x}\right)_0 x\right], \tag{113}$$

$$m\frac{dv_x}{dt} = \frac{q}{c}\left[v_y B_0 - v_z\left(\frac{\partial B_y}{\partial y}\right)_0 y\right], \tag{114}$$

and

$$m\frac{dv_y}{dt} = -\frac{q}{c}\left[v_x B_0 - v_z\left(\frac{\partial B_x}{\partial x}\right)_0 x\right]. \tag{115}$$

We consider a particle whose center of gyration (guiding center) is instantaneously at the origin — i.e., z = 0 . Now the zero order solutions of Eqs. (114) and (115) are

$$w = w_0 e^{-i\omega_c t} = v_x + i v_y$$

and

$$\xi = -\frac{w_0}{i\omega_c} e^{-i\omega_c t} = x + i y$$

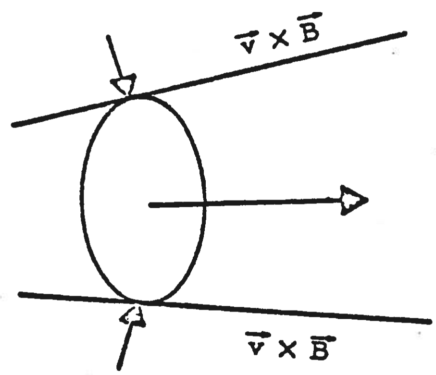


Figure 27

where $$\omega_c = \frac{qB}{mc}.$$

We may choose $\mathcal{z}_0$ to be 0 by proper choice of phase or coordinates; also, we may choose w_0 to be real. Substituting in Eq. (113) gives

$$m\frac{dV_z}{dt} = \frac{q w_0^2}{c\omega_c}\left(-\sin^2\omega_c t\left(\frac{\partial B_y}{\partial y}\right)_0 - \cos^2\omega_c t\left(\frac{\partial B_x}{\partial x}\right)_0\right). \qquad (116)$$

Averaging over one orbit gives

$$m\overline{\frac{dV_z}{dt}} = \frac{q w_0^2}{2c\omega_c}\left[\left(\frac{\partial B_x}{\partial x}\right)_0 + \left(\frac{\partial B_y}{\partial y}\right)_0\right] =$$

$$= \frac{-q w_0^2}{2c\omega_c}\left(\frac{\partial B_z}{\partial z}\right)_0. \qquad (117)$$

Now w_0^2 is $2W_\perp/m$, so that we may rewrite Eq. (117) in the form

$$m\overline{\frac{dV_z}{dt}} = -\frac{W_\perp}{B}\left(\frac{\partial B_z}{\partial z}\right) = -\mu\left(\frac{\partial B_z}{\partial z}\right) \qquad (118)$$

Now multiplying both sides of Eq. (118) by v_z gives

$$\frac{m}{2}\overline{\frac{d}{dt}(V_z)^2} = -\mu\frac{dB_z}{dt}. \qquad (119)$$

Multiplying Eq. (114) by v_x and Eq. (115) by v_y and adding gives

$$\frac{m}{2}\frac{d}{dt}(V_x^2+V_y^2) = \frac{qV_z}{c}\left[x V_y\left(\frac{\partial B_x}{\partial x}\right)_0 - y V_x\left(\frac{\partial B_y}{\partial y}\right)_0\right] \qquad (120)$$

This is just $-v_z$ times Eq. (113), so that we find

$$\frac{m}{2}\frac{d(V_z^2)}{dt} = -\frac{m}{2}\frac{d(V_\perp^2)}{dt} \quad \text{where} \quad V_\perp^2 = V_x^2+V_y^2 \qquad (121)$$

which is just the equation for conservation of energy. Thus we may write Eq. (119) in the form

$$\frac{dW_\perp}{dt} = \frac{m}{2}\frac{d(v_\perp^2)}{dt} = \mu \frac{dB_z}{dt} = \frac{W_\perp}{B}\frac{dB_z}{dt} \tag{122}$$

or

$$\frac{d}{dt}\left(\frac{W_\perp}{B}\right) = \frac{d\mu}{dt} = 0. \tag{123}$$

Thus the magnetic moment is constant in this spatially-varying B field as well as in a time-varying field (provided the variations are not too rapid) and our previous analysis is justified.

(2) Effects of $(\partial B_x/\partial z)$ and $(\partial B_y/\partial z)$ (Curvature of the Lines of Force)

Let us now look at terms of the form $(\partial B_x/\partial z)_o$ and $(\partial B_y/\partial z)_o$. We need consider only one of these, since by appropriate orientation of the xy plane the other can be eliminated. That is, if we choose our x axis to point in the direction $(\partial B_\perp/\partial z)$ ($B_\perp$ is the component of $\vec{B}$ perpendicular to z), then we get only a $(\partial B_x/\partial z)$ term.

Let us again look at what this implies about the shape of the lines of force. We return to Eq. (98) for the lines of force, and again to zero order in derivatives we find straight lines of force. To first order we find

$$\frac{dx_1}{dz} = \frac{\left(\frac{\partial B_x}{\partial z}\right)_o z}{B_o} \tag{124}$$

or

$$x_1 = x_o + \frac{z^2}{2B_o}\left(\frac{\partial B_x}{\partial z}\right)_o \tag{125}$$

The lines of force are curved as shown in Fig. 28. For small z we find that the curve may be regarded as a segment of a circle.

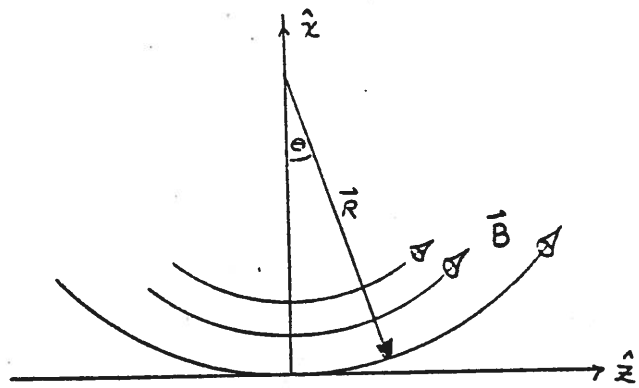


Figure 28

From this figure we have

$$\frac{B_x}{B_o} = \tan\theta \cong \theta \cong \frac{z}{R}. \tag{126}$$

Thus

$$z/R = B_x/B_o = \frac{z}{B_o}\left(\frac{\partial B_x}{\partial z}\right)_o \tag{127}$$

or

$$R = \frac{B_o}{\left(\frac{\partial B_x}{\partial z}\right)_o} \tag{128}$$

In general the vector radius of curvature $\vec{R}$ of a curve is given in terms of the unit tangent to the curve $\vec{n}_t$ (in this case $\vec{n}_t = \frac{\vec{B}}{|B|}$). The relation is

$$\frac{\vec{R}}{|R^2|} = -(\vec{n}\cdot\vec{\nabla})\vec{n} \tag{129}$$

To solve this problem we introduce the local cylindrical coordinates such that the axis of the cylinder is perpendicular to the local plane of the field lines and so that it passes through their center of curvature, as shown in Fig. 29(a).

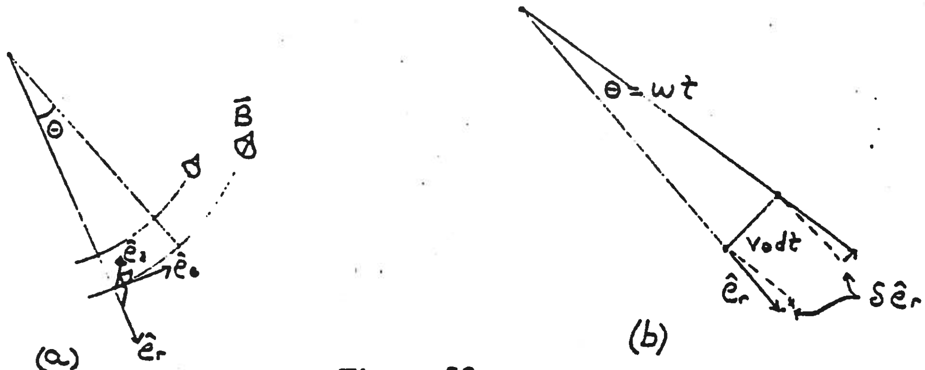


Figure 29

We choose axes so that the unit vector e_θ lies along B, the e_r unit vector is perpendicular to B and in the plane of B and points away from the center of curvature and e_z is chosen normal to θ and r so that e_r, e_θ, e_z forms a right-handed coordinate system. Locally the magnetic field has only a θ component. We have for $d\vec{v}/dt$

$$\frac{d\vec{V}}{dt} = \hat{e}_r \frac{dV_r}{dt} + V_r \frac{d\hat{e}_r}{dt} + \hat{e}_\theta \frac{dV_\theta}{dt} + V_\theta \frac{d\hat{e}_\theta}{dt} + \hat{e}_z \frac{dV_z}{dt} \tag{130}$$

and from Fig. 29(b)

$$\frac{d\hat{e}_r}{dt} = \frac{V_\theta}{r} \hat{e}_\theta, \qquad \frac{d\hat{e}_\theta}{dt} = -\frac{V_\theta}{r} \hat{e}_r. \tag{131}$$

Our equation of motion becomes

$$\begin{aligned} m\frac{d\vec{V}}{dt} &= m \left\{ \hat{e}_r \left[\frac{dV_r}{dt} - \frac{V_\theta^2}{r}\right] + \hat{e}_\theta \left[\frac{dV_\theta}{dt} + \frac{V_\theta V_r}{r}\right] + \hat{e}_z \frac{dV_z}{dt} \right\} \\ &= \frac{q}{c} \vec{V} \times \vec{B} = \frac{q B_0}{c} \left(-\hat{e}_r V_z + \hat{e}_z V_r\right). \end{aligned} \tag{132}$$

The scalar equation representing e_θ terms gives

$$\frac{dV_\theta}{dt} = -\frac{V_\theta V_r}{r} \tag{133}$$

or

$$\frac{dv_\theta}{v_\theta} = -\frac{dr}{r} \quad \left(\text{where } v_r = \frac{dr}{dt}\right). \tag{134}$$

$$v_\theta = \frac{v_0 r_0}{r}. \tag{135}$$

This just says that the angular momentum $\propto v_\theta r$ about the center of curvature of the lines is conserved and this leads to slight fluctuations of the v_θ as the particle gyrates about the B lines. The other two scalar equations are

$$m\left(\frac{dv_r}{dt} - \frac{v_\theta^2}{r}\right) = -\frac{q}{c} v_z B_0 \tag{136}$$

and

$$m\left(\frac{dv_z}{dt}\right) = \frac{q}{c} v_r B_0. \tag{137}$$

Now if we neglect the slight fluctuations in v_θ just found (these are of the order of the ratio of the Larmor radius to the radius of curvature of the lines of force and are hence small by the assumption that variations of B over regions of the size of a Larmor orbit are small), then these are the equations for the gyration of a particle about a uniform magnetic field when subjected to an external force of magnitude $+\dfrac{m v_\theta^2}{r}$. This is the centrifugal force which acts on the particle when it follows the curved field lines. It gives rise to a drift in the z direction equal to

$$v_{zD} = \frac{c m v_\theta^2}{r q B_0} = \frac{2 W_{\parallel} c}{R q B}. \tag{138}$$

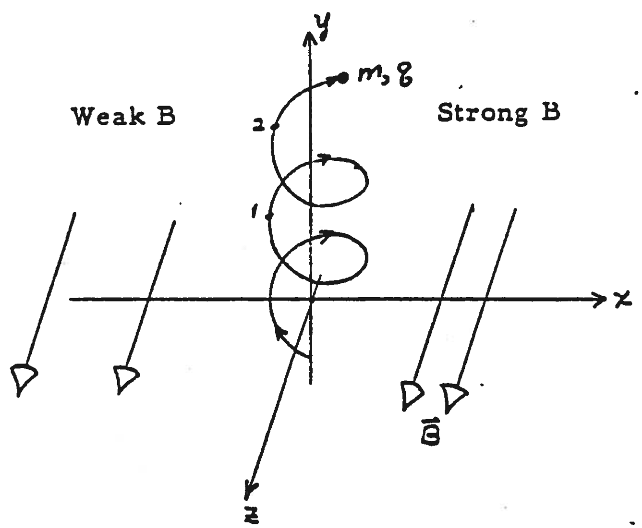
The v_{zD} drift results in a current since it depends on q. This current, in turn, produces the centripetal force required for the circular motion.

We can write Eq. (138) in vector form

$$\vec{V}_r = \frac{2cW_{\parallel}}{q\,B^2}\,\vec{B}\times(\vec{n}\cdot\vec{\nabla})\vec{n} \qquad (139)$$

(3) <u>Effect of $\partial B_z/\partial x$ and $\partial B_z/\partial y$</u>

These terms do not give rise to any slope or curvature of the B lines, but simply state that the strength of the magnetic field varies in the xy plane. Again we need only consider one of these terms, since we can choose a coordinate system in which the other is zero. That is, we can, say, choose the x axis so that it lies along $\vec{\nabla}_\perp B_z$ and then $\partial B_z/\partial y = 0$, $\vec{\nabla}_\perp$ means $(\vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y})$. The situation is shown in Fig. 30.



<u>Figure 30</u>

If the charged particle were to execute an undisplaced circular orbit, the force in the negative x direction, while the particle is in the right half orbit in the stronger magnetic field, would exceed the opposite force during the time the particle is in the left half orbit. The drift along y produces a $\vec{j}\times\vec{B}$ force along x which just compensates this.

A simple calculation of this effect starts out by noting that the average force along x must be zero. We average over one cycle of the periodic motion

$$\int_{t_1}^{t_2} F_x \, dt = 0 \tag{140}$$

where

$$F_x = \frac{e}{c} V_y B_z(x) = \frac{e}{c} V_y \left(B_0 + x\left(\frac{\partial B_z}{\partial x}\right)_0\right). \tag{141}$$

Substituting Eq. (141) in Eq. (140),

$$\int_{t_1}^{t_2} B_0 V_y \, dt + \int_{t_1}^{t_2} x \left(\frac{\partial B_z}{\partial x}\right)_0 V_y \, dt = 0 \tag{142}$$

thus

$$\delta y = y_2 - y_1 = -\frac{1}{B_0}\left(\frac{\partial B_z}{\partial x}\right)_0 \int_{t_1}^{t_2} x V_y \, dt \tag{143}$$

since B_0 and $\left(\frac{\partial B_z}{\partial x}\right)_0$ are constants. Since the field changes are small by assumption, the orbits are only slightly disturbed from circular ones: we use for the integral of $x v_y$ dt over one period simply πa^2, where a is the Larmor radius, giving

$$\delta y = -\frac{1}{B_0}\left(\frac{\partial B_z}{\partial x}\right)_0 \pi a^2 \tag{144}$$

which can be written

$$\delta y = -\frac{1}{B_0} \frac{\partial B_z}{\partial x} \left(\frac{2\pi}{\omega_c}\right)\left(\frac{m}{2} V_\perp^2\right) \frac{c}{e B_0} \tag{145}$$

where δy is the displacement of the orbit in a time of one cycle, $2\pi/\omega_c$.

Then the drift velocity is δy divided by $2\pi/\omega_c$

$$V_y = -\frac{c W_\perp}{q B_0^2}\frac{\partial B_z}{\partial x} \tag{146}$$

or, in general, since the drift is in the direction $\vec{n} \times \vec{\nabla} B$,

$$V_G = \frac{c W_\perp}{q B^2}\, \vec{n} \times \vec{\nabla}(\vec{B}\cdot\vec{n}). \tag{147}$$

(4) Effects of $\partial B_x/\partial y$ and $\partial B_y/\partial x$

These components of $\vec{\nabla}\vec{B}$ represent shear or twisting of the magnetic lines of force, as shown in Fig. 31.

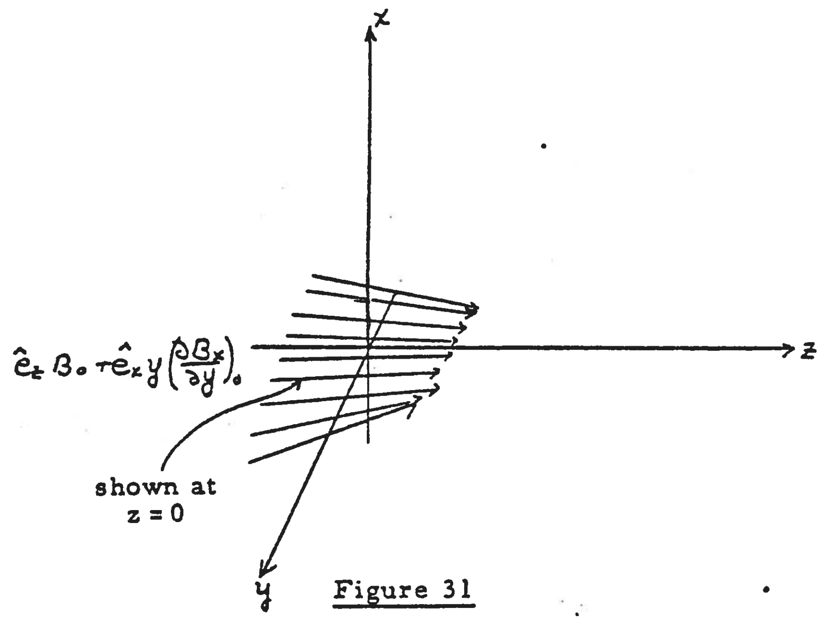


Figure 31

We can solve for their effects in the same manner as we did for $(\partial B_z/\partial x)$. They give rise only to driving terms at $2\omega_c$ in the w_1 equation, and hence result in distortions of the orbit, but give rise to no net drift of the particles.

V. Summary of Drifts and Currents

A. Drifts

Electric $$\vec{V}_E = c\,\frac{\vec{E}\times\vec{B}}{B^2} \qquad (1)$$

Dielectric, $\dot{E}$ $$\vec{V}_{\dot{E}} = \frac{mc^2}{qB^2}\,\underbrace{\vec{n}\times(\dot{\vec{E}}\times\vec{n})}_{\dot{\vec{E}}_\perp} \qquad (2)$$

Curvature $$\vec{V}_R = \frac{2cW_\parallel}{qB^2}\left[\vec{B}\times(\vec{n}\cdot\vec{\nabla})\vec{n}\right] \qquad (3)$$

$$\vec{n} = \vec{B}/|B|$$

Gradient $$\vec{V}_G = \frac{cW_\perp}{qB^2}\left[\vec{n}\times\vec{\nabla}(\vec{B}\cdot\vec{n})\right] \qquad (4)$$

External Force $$\vec{V}_F = \frac{c}{q}\,\frac{\vec{F}\times\vec{B}}{B^2} \qquad (5)$$

B. Currents

Magnetization $$\vec{J}_m = c\,\vec{\nabla}\times\vec{M} \qquad (6)$$

$$M = -NW_\perp\,\frac{\vec{B}}{B^2}$$

Polarization $$\vec{J}_P = \frac{\epsilon}{4\pi}\vec{n}\times(\dot{\vec{E}}\times\vec{n}) = \frac{\epsilon}{4\pi}\dot{\vec{E}} \qquad (7)$$

$$\epsilon = \frac{4\pi Nmc^2}{B^2}$$

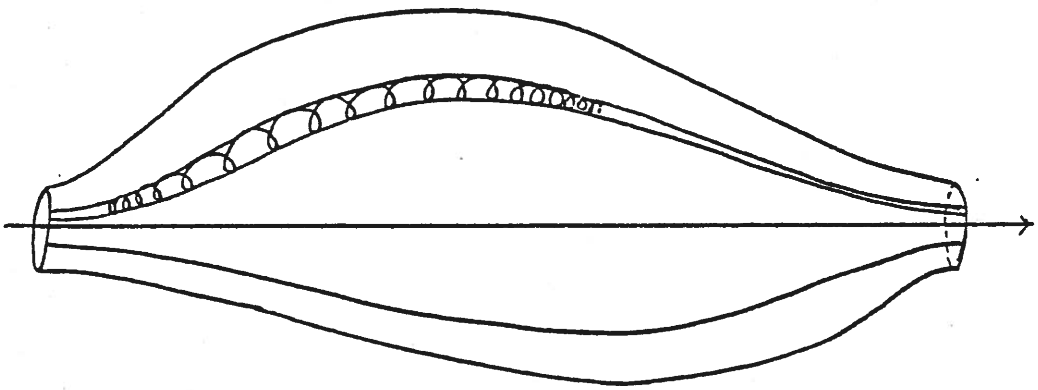
Curvature $$\vec{J}_R = \frac{2NcW_\parallel}{B^2}\,\underbrace{\vec{B}\times\left((\vec{n}\cdot\vec{\nabla})\vec{n}\right)}_{R\times\vec{B}/R^2} \qquad (8)$$

Gradient $$\vec{J}_G = \frac{NcW_\perp}{B^2}\,\vec{n}\times\vec{\nabla}(\vec{B}\cdot\vec{n}) \qquad (9)$$

External Force $$\vec{J}_F = Nc\,\frac{\vec{F}\times\vec{B}}{B^2} \qquad (10)$$

VI. The Longitudinal Invariant

We have already shown that the magnetic moment μ of a particle is an adiabatic invariant. Because of this property a particle may be trapped between regions of high magnetic field, as shown in Fig. 32.



The particle executes a periodic motion back and forth between the regions of large B. There is a second adiabatic invariant associated with this motion which we will now investigate.

There is an adiabatic invariant associated with any periodic motion. This invariant is the action associated with the motion

$$J = \oint p\, dq \tag{1}$$

where p is the momentum, q is the position coordinate, and the integral is to be taken over a closed orbit. For the motion considered here, the appropriate invariant is

$$J = m \int \vec{V}_{\parallel} \cdot d\vec{s} \tag{2}$$

where $\vec{v}_{\parallel}$ is the velocity parallel to the magnetic lines of force, and $\vec{ds}$ is an element of arc length along the magnetic field lines.

In our earlier work we found

$$m\frac{dv_{\parallel}}{dt} = -\mu\frac{\partial B}{\partial s} \tag{3}$$

or

$$m\frac{ds}{dt}\frac{dv_{\parallel}}{ds} = mv_{\parallel}\frac{dv_{\parallel}}{ds} = -\mu\frac{\partial B}{\partial s}. \tag{4}$$

Eq. (4) can be integrated directly to yield an energy relation for the motion

$$W = \frac{mv_{\parallel}^2}{2} + \mu B. \tag{5}$$

Thus if B is time independent, W will be conserved. However, for time-varying B, W will also vary with time.

We may solve Eq. (5) for $v_{\parallel}$ and thus have

$$v_{\parallel} = \pm\sqrt{\frac{2}{m}(W-\mu B)}. \tag{6}$$

Now let B vary slowly with time (slowly means slow compared to the period of oscillation between regions of strong magnetic field — i.e., between magnetic mirrors). We may compute the change in J during one period.

$$\Delta J = \sqrt{\frac{m}{2}}\left\{\oint\frac{\Delta W-\mu\Delta B}{\pm\sqrt{W-\mu B}}ds\right\}. \tag{7}$$

Here ΔW and ΔB are the changes in W and B during one period. The integral is understood to be evaluated at one instant of time (fixed W and B). Now ΔB is given by

$$\Delta B(s) = \frac{\partial B}{\partial s}T \tag{8}$$

where T is to be the period of one oscillation. We have for T

$$T = \oint \frac{ds}{v_{\parallel}} = \sqrt{\frac{m}{2}} \oint \frac{ds}{\pm\sqrt{W - \mu B}} \tag{9}$$

and hence $\Delta B(s)$ is given by

$$\Delta B = \frac{\partial B}{\partial t} \sqrt{\frac{m}{2}} \oint \frac{ds}{\pm\sqrt{W - \mu B}}. \tag{10}$$

The change in W is given by

$$\Delta W = \int_0^T \frac{dW}{dt}\, dt; \tag{11}$$

the integration is to be carried out along an orbit.

From Eq. (5) we have

$$W = \frac{m v_{\parallel}^2}{2} + \mu B = \frac{m \dot{s}^2}{2} + \mu B \tag{12}$$

and

$$\frac{dW}{dt} = m \dot{s} \ddot{s} + \mu \frac{dB}{dt} = m \dot{s} \ddot{s} + \mu \frac{\partial B}{\partial t} + \mu \frac{\partial B}{\partial s} \dot{s} \tag{13}$$

implies d/dt. But $m \ddot{s} = -\mu \frac{\partial B}{\partial s}$ and hence

$$\frac{dW}{dt} = \mu \frac{\partial B}{\partial t}. \tag{14}$$

This is quite a reasonable result, for μB is the effective potential energy and Eq. (13) states that the rate of change of the particle's energy is equal to the rate of change of the potential energy at the point where the particle is located; or, to express this result in another way, the particle does not change energy in moving through an arbitrary magnetic field, unless the field changes in time. From Eq. (14) we find for the change in ΔW .

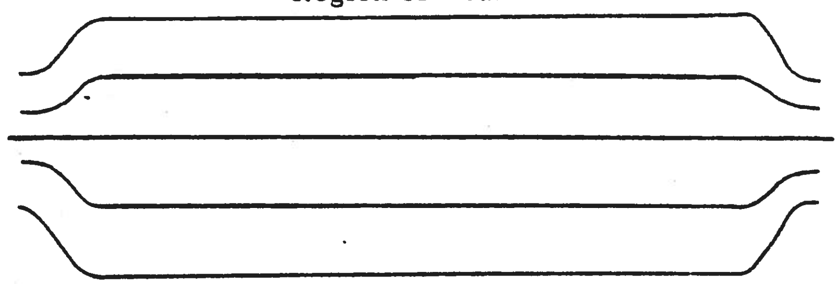


Figure 33

becomes strong is much shorter than L and so we can neglect these regions in evaluating J (we have sharp reflecting boundaries). For this situation J is given by

$$J = m\oint V_{\parallel}\, dS = 2m\,|V_{\parallel}|L \cong \sqrt{2mW}\; 2L. \tag{17}$$

Here we have assumed that μB is negligible between mirrors. Now if the distance between the mirrors changes, then W must change in order to keep J constant.

$$\frac{dJ}{dt} = 2\sqrt{2mW}\,\frac{dL}{dt} + \frac{\sqrt{2m}}{\sqrt{W}}\,L\,\frac{dW}{dt} = 0 \tag{18}$$

or

$$\frac{dW}{W} = -2\,\frac{dL}{L} \tag{19}$$

or

$$W = W_0\left(\frac{L_0}{L}\right)^2 \tag{20}$$

thus

$$V = \frac{V_0 L_0}{L}. \tag{21}$$

Thus if the mirrors move towards each other and L decreases v increases, while if they move apart v decreases. According to Eq. (20) the longitudinal energy or temperature of a gas being compressed between approaching mirrors is proportional to $1/L^2$. Now if we have an ideal gas and compress it adiabatically, then the temperature and volume are related by

$$TV^{\gamma-1} = T_0 V_0^{\gamma-1} \tag{22}$$

where T is the temperature or mean energy per particle, V is the volume and γ is $(n + 2)/n$, where n is the number of degrees of freedom involved in the compression. Here the volume is proportional to L, and only one degree of freedom is involved, the degree associated with the motion back and forth between the mirrors. Thus our adiabatic formula would lead to

$$T = T_0 \frac{L_0^2}{L^2} \tag{23}$$

in agreement with Eq. (20).

This offers one method for heating a gas. However, this means is limited because as the parallel velocity increases it becomes more and more difficult for the mirror fields to trap the particles, and ultimately they escape. Fermi proposed that such a mechanism may be responsible for the acceleration of particles up to cosmic ray energies. Particles would be trapped between magnetic fields associated with large gas clouds. If the clouds are moving towards each other the particle would gain energy until it had sufficient energy to escape. By repeated trappings and compressions, particles could gain energy. Of course if the particle were trapped between the clouds which were separating, it would lose energy. However, in such processes, on the average, particles gain energy if for no other reason than the fact that they can gain an unlimited amount of energy, but they can never lose more than they have.

VII. The Motion of Magnetic Lines of Force

It is sometimes stated that in a plasma in which particle collisions can be neglected the lines of force move with the particles. We will now look at this concept in some detail.

First, this statement is outside the original framework of Maxwell's equations, for in these equations it is not necessary to assign a persistent identity to the field lines. Second, the statement will clearly be true only in the limit of large q/m, when the excursion of the particle involved in the Larmor motion can be neglected. In this limit all drifts are negligible except the $\vec{E} \times \vec{B}$ drift provided $\vec{E} \cdot \vec{B}$ is zero. If $\vec{E} \cdot \vec{B} \neq 0$, then particles are strongly accelerated along field lines, $W_{\parallel}$ is proportional to q and curvature drifts are also important. In this limit the particle moves with the $\vec{E} \times \vec{B}$ drift velocity, and hence we wish to assign this velocity to the field lines. We shall show that we can do this when the component of $\vec{E}$ parallel to $\vec{B}$ is 0, and that the mapping of the B field, which results from this motion, (1) preserves lines of force and (2) preserves the flux through any closed curve.

Consider two particles on the same line of force at time $t = 0$ and which are separated by a small distance $\Delta \ell$.

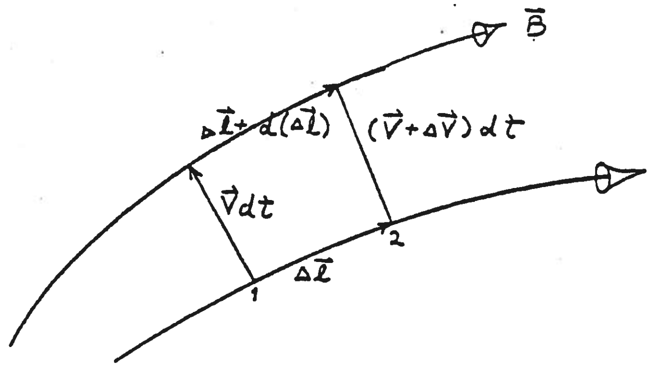


Figure 34

To show that a line of force remains a line of force we must show that $\vec{\Delta l}$ remains parallel to $\vec{B}$ under the transformation, or that

$$\frac{d}{dt}(\vec{\Delta l} \times \vec{B}) = \frac{d(\vec{\Delta l})}{dt} \times \vec{B} + \vec{\Delta l} \times \frac{d\vec{B}}{dt} = 0. \qquad (1)$$

Now that we have for $d\,\vec{\Delta l}$

$$d\,\vec{\Delta l} = \underbrace{\left[\vec{\Delta l} + (\vec{V} + (\vec{\Delta l}\cdot\vec{\nabla})\vec{V})dt\right]}_{\text{Position of point 2 at dt.}} - \underbrace{\left[\vec{\Delta l} + \vec{V}dt\right]}_{\text{Position of point 1 at dt.}} \qquad (2)$$

or

$$\frac{d\,\vec{\Delta l}}{dt} = (\vec{\Delta l}\cdot\vec{\nabla})\vec{V}. \qquad (3)$$

We must now compute B at the displaced point

$$d\vec{B} = \left[\frac{\partial\vec{B}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\,\vec{B}\right]dt. \qquad (4)$$

Now

$$\vec{V} = \frac{c\,\vec{E}\times\vec{B}}{B^2}, \qquad \text{(assuming } \vec{E}\perp\vec{B}) \qquad (5)$$

and hence

$$\vec{V} \times \vec{B} = \frac{c(\vec{E} \times \vec{B})}{B^2} \times \vec{B} = -c\vec{E} \tag{6}$$

since $\vec{E} \cdot \vec{B}$ is taken to be zero.

From Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{7}$$

and hence from Eq. (6)

$$\vec{\nabla} \times (\vec{V} \times \vec{B}) = \frac{\partial \vec{B}}{\partial t} \tag{8}$$

or

$$(\vec{B} \cdot \vec{\nabla})\vec{V} - (\vec{V} \cdot \vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla} \cdot \vec{V}) + \vec{V}\overset{0}{\cancel{(\vec{\nabla} \cdot \vec{B})}} = \frac{\partial \vec{B}}{\partial t}. \tag{9}$$

Thus we find for dB/dt

$$\frac{d\vec{B}}{dt} = (\vec{B} \cdot \vec{\nabla})\vec{V} - \vec{B}(\vec{\nabla} \cdot \vec{V}). \tag{10}$$

Substituting Eqs. (10) and (3) in Eq. (1) gives

$$\frac{d}{dt}(\vec{\Delta l} \times \vec{B}) = ((\vec{\Delta l} \cdot \vec{\nabla})\vec{V}) \times \vec{B} + \vec{\Delta l} \times \left[(\vec{B} \cdot \vec{\nabla})\vec{V} - \vec{B}(\vec{\nabla} \cdot \vec{V})\right]. \tag{11}$$

Now $\vec{\Delta \ell}$ is a vector along the direction of $\vec{B}$. Hence we may replace $\vec{\Delta \ell}$ by $\epsilon\vec{B}$ in the above expression and we immediately see that the right-hand side of Eq. (11) is zero.

$$\frac{d}{dt}(\vec{\Delta l} \times \vec{B}) = 0. \tag{12}$$

Thus the transformation takes lines into lines.

Now to prove that V_E is flux-preserving, we must show that the flux through an area Δs which follows the motion remains constant. If $\Delta\phi$ is the flux through the area, we must show that

$$\frac{d\phi}{dt} = \frac{d}{dt}\int \vec{B}\cdot d\vec{S} = 0. \tag{13}$$

Now ϕ changes for two reasons: first because $\vec{B}$ changes, and secondly because the area changes. The change due to the changing $\vec{B}$ is given by

$$\left(\frac{\partial\phi}{\partial t}\right)_S = \int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} = -c\int (\vec{\nabla}\times\vec{E})\cdot d\vec{S}. \tag{14}$$

The change in ϕ due to the distortion in s (keeping B constant) is given by

$$\left(\frac{\partial\phi}{\partial t}\right)_B = \int_c \vec{B}\cdot(\vec{V}\times d\vec{l}) \tag{15}$$

where c is the bounding curve (see Fig. 35).

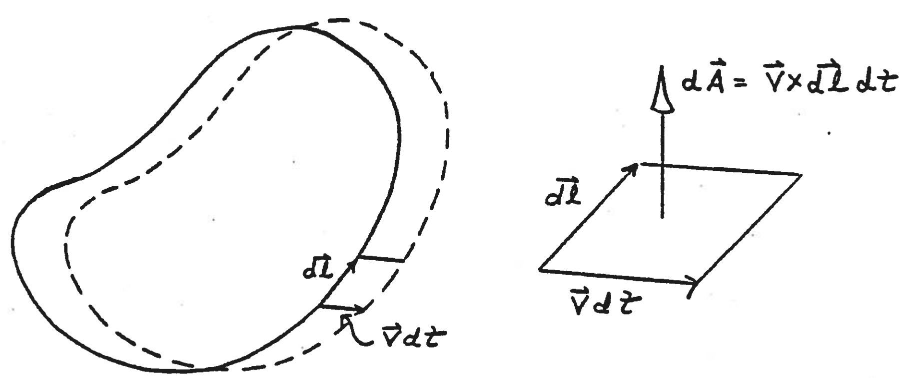


Figure 35

We may interchange the dot and cross products in Eq. (15) to obtain

$$\left(\frac{\partial\phi}{\partial t}\right)_B = -\int_c (\vec{V}\times\vec{B})\cdot d\vec{l}. \tag{16}$$

Converting Eq. (16) into a surface integral gives

$$\left(\frac{\partial\phi}{\partial t}\right)_B = -\int \vec{\nabla}\times(\vec{V}\times\vec{B})\cdot d\vec{S}. \tag{17}$$

Combining Eqs. (17) and (14) gives

$$\frac{d\phi}{dt} = \left(\frac{\partial\phi}{\partial t}\right)_s + \left(\frac{\partial\phi}{\partial t}\right)_B = -\int \vec{\nabla} \times \left(c\vec{E} + \vec{V} \times \vec{B}\right) \cdot d\vec{s}. \tag{18}$$

In order for this to hold true for every possible surface implies that the integrand must be 0 everywhere, or

$$\vec{\nabla} \times \left(c\vec{E} \times \vec{V} \times \vec{B}\right) = 0. \tag{19}$$

Now if $\vec{v}$ is given by

$$\vec{V} = \frac{c\vec{E} \times \vec{B}}{B^2},$$

then Eq. (19) becomes

$$c\,\vec{\nabla} \times \left(\cancelto{0}{\vec{E}} - \cancelto{0}{\vec{E}} + \frac{\vec{B}(\vec{B} \cdot \vec{E})}{B^2}\right) = 0 \tag{20}$$

or

$$\vec{\nabla} \times \left[\frac{\vec{B}(\vec{B} \cdot \vec{E})}{B^2}\right] = 0. \tag{21}$$

Eq. (21) is automatically satisfied if $\vec{B} \cdot \vec{E}$ is zero. We also needed this condition to show that lines went into lines. Thus we see that if $\vec{E} \cdot \vec{B}$ is zero we can assign the velocity $c\vec{E} \times \vec{B}/B^2$ to the lines and this takes lines into lines and preserves the flux through any surface. In a perfect conductor, where inertia can be neglected, $\vec{E} \cdot \vec{B}$ must be zero, for if it were not so the charges would immediately move so as to eliminate $\vec{E}$ parallel to B. To the extent to which this is true for a plasma, the plasma particles are stuck to lines of force.

VIII. Applications of Orbit Theory

A. Static, Straight B Lines, No External Force

We take $\vec{B}$ to be in the z direction. From $\vec{\nabla} \cdot \vec{B} = 0$ we have

$$\frac{\partial B_z}{\partial z} = 0. \tag{1}$$

Since $\vec{B}$ only has a z component, we also have

$$(\vec{B} \cdot \vec{\nabla}) \vec{B} = 0. \tag{2}$$

We further assume that $\vec{E}$ is zero, that the plasma is neutral, and that the particle number and energy densities are independent of z. We now sum up the currents. First the magnetization current is obtained from Eq. (6), Section V.

$$\begin{aligned} \vec{J}_m &= -c\, \vec{\nabla} \times \left(\overline{NW_\perp} \frac{\vec{B}}{B^2} \right) = c\, \vec{n} \times \vec{\nabla} \left(\frac{\overline{NW_\perp}}{|B|} \right) \\ &= \frac{c}{|B|} \vec{n} \times \vec{\nabla}(\overline{NW_\perp}) - c \frac{\overline{NW_\perp}}{B^2} \vec{n} \times \vec{\nabla} |B|. \end{aligned} \tag{3}$$

Secondly we have the current due to a gradient in $\vec{B}$. This we obtain from Eq. (9), Section V.

$$\vec{J}_G = c \frac{\overline{NW_\perp}}{B^2} \vec{n} \times \vec{\nabla} (\vec{B} \cdot \vec{n}). \tag{4}$$

Adding these two currents gives

$$\vec{J}_m + \vec{J}_G = c \frac{\vec{n}}{|B|} \times \vec{\nabla}(\overline{NW_\perp}) = c\, \frac{\vec{B}}{B^2} \times \vec{\nabla}(\overline{NW_\perp}). \tag{5}$$

Finally, we must use Maxwell's equations to get a self-consistent solution. From

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} \tag{6}$$

and Eq. (5) we have

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{n}|B|) = -\vec{n} \times \vec{\nabla}|B| = 4\pi \cancel{\left(\frac{c}{c}\right)}^{1} \frac{\vec{n}}{|B|} \times \vec{\nabla}(\overline{NW_\perp}) \tag{7}$$

or

$$\vec{n} \times \left\{ \vec{\nabla}|B| + \frac{4\pi}{|B|} \vec{\nabla}(\overline{NW_\perp}) \right\} = 0. \tag{8}$$

Since the term in the brackets is perpendicular to $\vec{n}$,

$$\vec{\nabla} \left(\frac{B^2}{8\pi} + \overline{NW_\perp} \right) = 0 \tag{9}$$

or

$$\left[\frac{B^2}{8\pi} + (\overline{NW_\perp}) \right] = \text{CONSTANT.} \tag{10}$$

$B^2/8\pi$ is the pressure associated with the magnetic field lines, while $\overline{NW_\perp}$ is the pressure of the plasma perpendicular to B . Eq. (10) says that the sum of these pressures is constant, or that we have pressure balance.

B. Plasma in a Gravitational Field which is Perpendicular to a Magnetic Field Whose Lines are Straight

Again we take the direction of the magnetic field to be in the z direction, and we take the gravitational field to be in the negative y direction. We assume all quantities are independent of z .

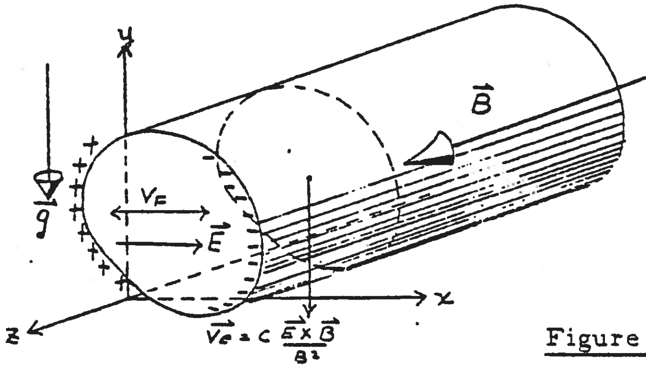


Figure 36

We shall further assume that $\overline{NW_\perp}$, the perpendicular pressure, is negligibly small compared to $B^2/8\pi$. From Eq. (10) we have

$$\frac{B^2}{8\pi} + \overline{NW_\perp} = \text{constant}$$

Hence

$$\frac{B^2}{8\pi}\left[1 + \frac{\overline{NW_\perp}}{B^2/8\pi}\right] = \frac{B^2}{8\pi}(1+\beta) = \text{constant} \tag{11}$$

or

$$B \cong \text{constant} \tag{12}$$

under our assumptions. The quantity β is the ratio of the perpendicular gas pressure to the magnetic field pressure. Under this approximation we can neglect the variations in B due to the gas pressure. We will therefore take B to be constant.

We now ask what will happen if we suddenly release such a plasma. Here an E field will develop and we must include its effects.

First, the particles have a drift due to the gravitational field which is given by Eq. (5), Section V.

$$\vec{V}_F = \frac{c\,m}{q}\,\frac{\vec{g}\times\vec{B}}{B^2}. \tag{13}$$

Ions and electrons move in opposite directions, so that a current is set up, given by Eq. (10), Section V; however, the resulting charge separation tends to oppose this current. The resulting $\vec{E}$ field enters in two ways: first, because $\vec{E}$ is time-dependent it gives rise to a polarization current; and second, because of $\vec{E}$ there is an $\vec{E}\times\vec{B}$ drift of the whole plasma in the $-y$ direction. To compute the time-dependence of E we must use Maxwell's equations (making use of the result which we already found, that the plasma behaves like a dielectric). From Eqs. (4) and (5), Section IV, we have

$$\nabla \cdot \left[\frac{\dot{\vec{E}}}{4\pi} + \dot{\vec{j}}\right] = 0. \tag{14}$$

If we assume that $\vec{\nabla} \times \vec{E}$ is zero ($\vec{B}$ negligible), and also that $\vec{\nabla} \times \vec{j} = 0$ (this is reasonable because there is nothing to make currents circulate in the problem), then

$$\dot{\vec{E}} = 4\pi \dot{\vec{j}}. \tag{15}$$

Now $\vec{j}$ has two parts, one coming from $\vec{g}$ and the other from $\dot{\vec{E}}$. These are obtained from Eqs. (7) and (10), Section V, and are given by

$$\vec{j}_{\dot{E}} = \frac{c^2}{B^2} Nm \dot{\vec{E}} \tag{16}$$

and

$$\vec{j}_g = \frac{c\,Nm}{B^2} \vec{g} \times \vec{B} = -\hat{e}_x \frac{c\,Nm\,g}{|B|} \tag{17}$$

where use has been made of the geometry in writing down $\vec{j}_g$.

Thus from Eq. (15) we have

$$\dot{\vec{E}}\left(1 + \frac{4\pi c^2 Nm}{B^2}\right) - \hat{e}_x \frac{4\pi c\,Nm\,g}{|B|} = 0 \tag{18}$$

or

$$\dot{\vec{E}} = \frac{\hat{e}_x \frac{4\pi c\,Nm\,g}{|B|}}{1 + \frac{4\pi c^2 Nm}{B^2}}. \tag{19}$$

Now the E_x which results from Eq. (19) gives rise to an $\vec{E} \times \vec{B}$ drift of the whole plasma in the $-y$ direction. We have

$$V_y = -\frac{c E_x}{B}. \tag{20}$$

Taking the derivative of Eq. (20) and substituting $\dot{\vec{E}}$ from Eq. (19) gives

$$\dot{V}_y = -\frac{4\pi c^2 Nmg}{B^2 + 4\pi c^2 Nm}. \tag{21}$$

If Nmc^2 is much larger than $B^2/4\pi$, then Eq. (21) reduces to

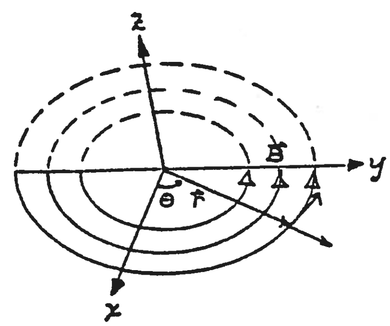
$$\dot{V}_y = -g. \tag{22}$$

Thus the plasma falls freely in the gravitational field just as if $\vec{B}$ were not there. The modification by the factor $\frac{1}{1 + \frac{B^2}{4\pi N m c^2}}$ is due to the fact that not all the gravitational energy goes into kinetic energy of the plasma, but some of it must go into the $\vec{E}$ field needed to give the $\vec{E} \times \vec{B}$ drift. It is readily verified that the ratio of the electric field energy per unit volume to particle kinetic energy is $\frac{B^2}{4\pi N m c^2}$. This results in a slight change in the effective mass of the plasma. It is not surprising that the magnetic field cannot hinder the falling of the plasma. Because of the boundaries, no sustained current can flow in the plasma, and since the only force that the magnetic field can exert on the plasma is a $j \times B$ force, there can be no magnetic force except the small one which results from the polarization current and which changes the effective mass of the plasma. (The inductance and inertia effects on the polarization currents have been neglected, and these will also give rise to slight changes in the effective mass.) Finally, it should be stated that the approximations of $\vec{\nabla} \times \vec{E}$ and $\vec{\nabla} \times \vec{j}$ break down near the surface of the plasma. The particles at the surface do not feel the full $\vec{E}$ field and some are scraped off. This is a difficult problem and has not been solved yet.

<u>Problem:</u> Show that the ratio of the electric field energy per unit volume to the particle kinetic energy (associated with the drift) per unit volume is $B^2/4\pi Nmc^2$ for the falling plasma.

C. <u>Curved Field Lines</u>

We now consider the case in which the field lines are circles centered on the z axis. This case applies either to the plasma in a torus or to the pinch effect. We shall use the cylindrical coordinates appropriate to the problem. Again from $\vec{\nabla} \cdot \vec{B} = 0$ we have that B does not vary in the θ direction. We also take all other quantities to be independent of z .



<u>Figure 37</u>

For the magnetization currents we have

$$\vec{J}_m = c\, \vec{\nabla} \times \vec{M} \tag{23}$$

or

$$\vec{J}_m = -c\, \vec{\nabla} \times \frac{p_\perp \vec{B}}{B^2} = -c\, \vec{\nabla} \times \frac{\hat{e}_\theta\, p_\perp}{B}. \tag{24}$$