Solving this equation for $\frac{4\pi e^2}{mk}\int\frac{\partial f_0/\partial v\, dv}{(\omega-kv)^2}$ gives

$$\frac{2}{k} = -\frac{\omega}{k}\left[1-\frac{k}{\omega}\frac{d\omega}{dk}\right]\frac{4\pi e^2}{mk}\int_{mp}\frac{\partial f_0/\partial v}{(\omega-kv)^2}\,dv \tag{821}$$

or

$$\frac{4\pi e^2}{mk}\,\omega\int\frac{\partial f_0/\partial v}{(\omega-kv)^2}\,dv = -\frac{2}{1-\frac{k}{\omega}\frac{d\omega}{dk}}, \tag{822}$$

Substituting Eq. (822) into Eq. (818) gives

$$w = \frac{\varepsilon^2}{8\pi}\,\frac{1}{1-\frac{k}{\omega}\frac{d\omega}{dk}}. \tag{823}$$

We must now find the rate at which the resonant particles absorb energy. The Vlasov equation, (790), also applies to them and Eq. (805) also gives solutions for them. However, solutions (805) diverge for $v = \frac{\omega}{k}$. If the initial perturbation is smooth in the vicinity of ω/k, then no such singular f will appear. For simplicity we will assume that f is initially zero for the resonant particles. To satisfy these initial conditions we must add to the solution (805) a solution of the homogeneous collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0 \tag{824}$$

The general solution to this equation is

$$f = f_i\,(x - vt, v) \tag{825}$$

where i refers to the initial f, $f_i(x)$ is the initial value of f. We see from Eq. (805) that if we choose f_i to cancel f we get

$$f_i = -\frac{eE}{m}\frac{\partial f_0/\partial v}{\omega - kv}\cos k(x-vt) \tag{826}$$

The full f is given by

$$f = \frac{eE}{m}\frac{\partial f_0/\partial v}{\omega - kv}\left\{\cos(kx-\omega t) - \cos k(x-vt)\right\} \tag{827}$$

The f given by Eq. (827) has no singularities for any finite length of time.

We now compute the current due to the resonant particles.

$$j_r = -e\int v\, f_r = -\frac{e^2E}{m}\int dv\, v\,\frac{\partial f_0/\partial v}{\omega - kv}\left\{\cos(kx-\omega t) - \cos k(x-vt)\right\}. \tag{828}$$

We assume that we may replace v and $\partial f_0/\partial v$ by their values at ω/k

$$j_r \approx -\frac{e^2E}{m}\frac{\omega}{k}\frac{\partial f_0(\omega/k)}{\partial v}\int dv\,\frac{\cos(kx-\omega t) - \cos k(x-vt)}{\omega - kv}. \tag{829}$$

Writing

$$\cos(kx-\omega t) - \cos k(x-vt) = -2\sin\tfrac{1}{2}(2kx-\omega t-kvt)\sin\tfrac{1}{2}(kvt-\omega t), \tag{830}$$

and substituting in Eq. (829) gives

$$j_r \approx \frac{2e^2E}{m}\frac{\omega}{k}\frac{\partial f_0(\omega/k)}{\partial v}\int\frac{\sin\frac{1}{2}(2kx-\omega t-kvt)\sin\frac{1}{2}(kvt-\omega t)\,dv}{\omega - kv}. \tag{831}$$

Multiplying j_r by $E\sin(kx-\omega t)$ and integrating over x and v gives the rate at which work is done on the resonant particles.

$$\frac{dw_r}{dt} = \frac{2e^2E^2}{m}\,\frac{\omega}{k}\,\frac{\partial f_0(\omega/k)}{\partial v}\iint \sin(kx-\omega t) \tag{832}$$

$$\times\,\frac{\sin\frac{1}{2}(2kx-\omega t-kvt)\,\sin\frac{1}{2}(kvt-\omega t)}{\omega-kv}\,dx\,dv.$$

From this we find for the average energy absorbed per unit length

$$\frac{dw_r}{dt} = \pi\,\frac{e^2E^2}{2m}\,\frac{\omega}{k}\,\frac{\partial f_0(\omega/k)}{\partial v}. \tag{833}$$

Equating the energy gained by the resonant particles to that lost by the wave gives

$$\frac{dw}{dt} = \frac{1}{8\pi\left(1-\frac{k}{\omega}\frac{d\omega}{dk}\right)}\,\frac{dE^2}{dt} = -\frac{dw_r}{dt} = -\frac{\pi e^2\omega}{2mk}\,\frac{\partial f_0(\omega/k)}{\partial v}\,E^2. \tag{834}$$

The damping rate for the wave is

$$\gamma = 8\pi^2\left(1-\frac{k}{\omega}\frac{d\omega}{dk}\right)\frac{e^2\omega}{2mk}\,\frac{\partial f_0}{\partial v}(\omega/k) \tag{835}$$

or

$$\gamma = \pi\left(1-\frac{k}{\omega}\frac{d\omega}{dk}\right)\frac{\omega_p^2\,\omega}{k\,n_0}\,\frac{\partial f_0(\omega/k)}{\partial v}. \tag{836}$$

n_0 is the unperturbed number density. If f_0 had been normalized to 1, then $\frac{1}{n_0}\frac{\partial f_0(\omega/k)}{\partial v}$ would be replaced by simply $\frac{\partial f_0(\omega/k)}{\partial v}$. This is in agreement with our previous result (equation 788).

Solution of the Vlasov Equation

We will now look at another method of solving the Vlasov equation which is fundamental because it can be extended to a plasma made of discrete particle rather than a continuous phase fluid.

Consider the field free Vlasov equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial \underset{\sim}{r}} - \frac{e\underset{\sim}{E}}{m} \cdot \frac{\partial f}{\partial v} = 0$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = -4\pi c \, [\int f dv - n_i]$$

Since we are interested in small amplitude disturbances we linearize these equations

$$f = f_o + f_1$$

$$\underset{\sim}{E} = \underset{\sim}{E}_1$$

$$\int f_o d^3v = n_i$$

$$\frac{\partial f_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f_1}{\partial r} - \frac{eE}{m} \cdot \frac{\partial f_o}{\partial v} = 0$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E}_1 = -4\pi e \int f_1 \, d^3v$$

To solve these equations we observe the following, we may divide f_1 into two parts

$$f_1 = \psi_1 + \chi_1$$

where ψ_1 satisfies the equation

$$\frac{\partial \psi_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \psi_1}{\partial \underset{\sim}{r}} = 0$$

and $\psi_1 = f_1$ at $t = 0$ $\quad$ $\chi = 0$ at $t - 0$

χ satisfies the equation

$$\frac{\partial \chi_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \chi_1}{\partial} - \frac{eE}{n} \cdot \frac{\partial f_o}{\partial \underset{\sim}{v}} = 0$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E}_1 = 4\pi 2 \int f_1 d^3v = -4\pi e \int (\psi_1 + \chi_1) d^3v$$

It is clear that $\psi_1 + \chi_1$ satisfies the Vlasov equation; if $\underset{\sim}{E}$ is correct $\psi_1 +$ χ_1 is the correct f_1 and if $\psi_1 + \chi_1$ is the correct f_1 then $\underset{\sim}{E}$ is correct.

Now ψ_1 develops according to the free streaming of the initial f_1

$$\psi_1 (\underset{\sim}{r}, \underset{\sim}{v}, 0) = f_1(\underset{\sim}{r}, \underset{\sim}{v}, 0)$$

$$\psi_1 (\underset{\sim}{r}, \underset{\sim}{v}, t) = \psi_1 (\underset{\sim}{r} - \underset{\sim}{v}t, \underset{\sim}{v}, 0)$$

Inserting in the equation for ψ_1

$$\frac{\partial \psi_1}{\partial t} = - \underset{\sim}{v} \cdot \nabla \psi_1$$

$$\therefore \frac{\partial \psi_1}{\partial t} + \underset{\sim}{v} \cdot \nabla \psi_1 = - \underset{\sim}{v} \cdot \nabla \psi_1 + \underset{\sim}{v} \cdot \nabla \psi_1 = 0$$

A specific solution for ψ_1 is

$$\delta(\underset{\sim}{r} - \underset{\sim}{r}_o)\, \delta(\underset{\sim}{v} - \underset{\sim}{v}_o) \rightarrow \delta(\underset{\sim}{r} - \underset{\sim}{v}t - \underset{\sim}{r}_o)\, \delta(\underset{\sim}{v} - \underset{\sim}{v}_o) = \delta(\underset{\sim}{r} - \underset{\sim}{v}_o t - \underset{\sim}{r}_o)\delta(\underset{\sim}{v} - \underset{\sim}{v}_o)$$

ψ_1 can be written as a sum of δ functions

$$\psi_1(\underset{\sim}{r}, \underset{\sim}{v}, t) = \int \psi_1(\underset{\sim}{r}_o, \underset{\sim}{v}_o, 0)\, d^3r_o d^3v_o\, \delta(\underset{\sim}{r} - [\underset{\sim}{r}_o + \underset{\sim}{v}_o t])\, \delta(\underset{\sim}{v} - \underset{\sim}{v}_o)$$

$\psi_1(\underset{\sim}{r}_o, \underset{\sim}{r}_o, 0)\, d^3\, {}_o d^3v_o$ is the number of particles starting at $\underset{\sim}{r}_o$, v_o in $d^3 r_o d^3 v_o$.

Now ψ_1 is a known function of space and time and can be regarded as

Since the equation for χ_1 is linear, the solution obtained when there are many driving sources is the sum of the solutions obtained for the sources one at a time. Since we have seen that ψ_1 can be broken up into a sum of δ function, we can obtain the general solution if we can solve the equation for a single δ function source.

$$\frac{\partial \chi_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \chi_1}{\partial \underset{\sim}{r}} - \frac{e\underset{\sim}{E}_1}{m} \cdot \frac{\partial f_o}{\partial \underset{\sim}{v}} = 0$$

$$\frac{\partial \cdot \underset{\sim}{E}}{\partial \underset{\sim}{r}} = - 4\pi e \int \chi_1 d^3v - 4\pi e\, \delta(\underset{\sim}{r} - \underset{\sim}{v}t)$$

$$f_1 = \int d^3r_o d^3v_o\, \psi_1(r_o, v_o, 0)\, \{\delta(\underset{\sim}{r} - [\underset{\sim}{r}_o + \underset{\sim}{v}_o t]\, \delta(\underset{\sim}{v} - \underset{\sim}{v}_o)$$

$$+ \chi_1 (\underset{\sim}{r}, \underset{\sim}{v}, t; \underset{\sim}{\ }_o, v_o)$$

Where $\psi_1(\underset{\sim}{r}, \underset{\sim}{v}, t; \underset{\sim}{r}_o, \underset{\sim}{v}_o)$ is the solution for ψ_1 when a unit charge starts at r_o, v_o at time 0.

Problem. Generalize the above for the full set of Maxwell's Equation assuming that the undistrubed plasma contains no static electric or magnetic fields.

Field due to sources embedded in a plasma. We generalize to the full Maxwell Field.

$$\frac{\partial f_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f_1}{\partial \underset{\sim}{r}} - \frac{e}{m} (\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}) \cdot \frac{\partial f_o}{\partial \underset{\sim}{r}} = - \nu f_1$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = - \frac{1}{c} \frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{1}{c} \frac{\partial \underset{\sim}{E}}{\partial t} - \frac{4\pi e}{m} \int \underset{\sim}{v}\, f_1 d^3v + 4\pi \underset{\sim}{j}_s$$

$$\nabla \cdot E = -4\pi e \int f_1 d^3v + 4\pi\rho_s$$

$$\nabla \cdot B = 0$$

Choose f_o to be isotropic $f_o(v^2)$. Assume that the source charges and currents satisfy the continuity equation

$$\frac{\partial\rho_s}{\partial t} + \nabla \cdot j_s = 0$$

Fourier analyze in r and t

$$f(r, v, t) = \frac{1}{(2\pi)^2} \int f(k, \omega, v)\, e^{i(k\cdot r - \omega t)} d^3k d\omega$$

$$f(k, \omega, v) = \frac{1}{(2\pi)^2} \int f(r, v, t)\, e^{-i(k\cdot r - \omega t)} d^3r dt$$

$$-i(\omega - k\cdot v)\, f_1 - \frac{eE}{m} \cdot \frac{\partial f_o}{\partial v} = - \nu f_1$$

$$ik \times E = \frac{i\omega B}{c}$$

$$ik \times B = \frac{i\omega}{c} E - \frac{4\pi e}{m} \int v\, f_1 d^3v + 4\pi j_s$$

$$ik \cdot B = 0$$

$$ik \cdot E = -4\pi e \int f_1 d^3v + 4\pi\rho_s$$

$$-i\omega\rho_s + ik \cdot j_s = 0$$

$$\rho_s = \frac{\underset{\sim}{k} \cdot \underset{\sim}{j}_s}{\omega}$$

Decompose $\underset{\sim}{E}$, $\underset{\sim}{B}$ and $\underset{\sim}{j}_s$ into longitudinal and transverse components. For example

$$\underset{\sim}{E}_L \times \underset{\sim}{k} = 0 \qquad \underset{\sim}{E}_L \cdot \underset{\sim}{k} = \underbrace{E_L k}_{\text{magnitudes}} \qquad \underset{\sim}{E}_L = \hat{\underset{\sim}{k}}\,\hat{\underset{\sim}{k}} \cdot \underset{\sim}{E}$$

$$\underset{\sim}{E}_T \cdot \underset{\sim}{k} = 0 \qquad \underset{\sim}{E}_T = -\hat{\underset{\sim}{k}} \times [\hat{\underset{\sim}{k}} \times \underset{\sim}{E}] = \underset{\sim}{E} - \hat{\underset{\sim}{k}}\,\hat{\underset{\sim}{k}} \cdot \underset{\sim}{E}$$

$$\hat{\underset{\sim}{k}} = \frac{\underset{\sim}{k}}{|k|}$$

$$i\underset{\sim}{k} \cdot \underset{\sim}{E} = -4\pi e \int f_1 d^3v + 4\pi\rho_s$$

$$\underset{\sim}{k} \cdot \underset{\sim}{E} = 4\pi e i \int f_1 d^3v - 4\pi i\rho_s$$

$$f_1 = \frac{i\,\frac{eE}{m} \cdot \frac{\partial f_o}{\partial \underset{\sim}{v}}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu}$$

$$\underset{\sim}{k} \cdot \underset{\sim}{E} = -\frac{4\pi e^2}{m} \int \frac{\underset{\sim}{E} \cdot \frac{\partial f_o}{\partial \underset{\sim}{v}}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu}\, d^3v = 4\pi i\rho_s$$

$$\underset{\sim}{k} \cdot \underset{\sim}{E} + \frac{4\pi e^2}{m} \int \frac{\underset{\sim}{E} \cdot \partial f_o/\partial \underset{\sim}{v}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu} = -4\pi i\rho_s$$

Now for $\displaystyle\int \frac{\underset{\sim}{E} \cdot \partial f_o/\partial \underset{\sim}{v}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu)} d^3v$

the denominator depends only on the component of v || to k.

We can write the expression as

$$\int \frac{E_{\parallel}\, \partial f_o/\partial v_{\parallel} + \underset{\sim}{E}_{\perp} \cdot \partial f_o/\partial v_{\perp}}{(\omega - k\, v_{\parallel} + i\nu)} d^3v = \int \frac{E_{\parallel}\, \partial f_o/\partial v_{\parallel}\, d^3v}{(\omega - kv_{\parallel} + i\nu)}$$

Integration of the $E_{\perp}$ term over $v_{\perp}$ given 0

$$\int \frac{E_{\parallel}\, \partial f_o/\partial v_{\parallel}\, d^3v}{(\omega - kv_{\parallel} + i\nu)} = \frac{\underset{\sim}{k} \cdot \underset{\sim}{E}}{k^2} \int \frac{k \cdot \partial f_o/\partial \underset{\sim}{v}\, d^3v}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu)}$$

$$\underset{\sim}{k} \cdot \underset{\sim}{E}\, [1 + \frac{4\pi e^2}{mk^2} \int \frac{\underset{\sim}{k} \cdot \partial f_o/\partial \underset{\sim}{v}\, d^3v}{\omega - \underset{\sim}{k} \cdot v + i\nu}] = -\, 4\pi i\rho_s$$

$$\underset{\sim}{k} \cdot \underset{\sim}{E} = \frac{-4\pi i\rho_s}{D_L(\underset{\sim}{k}, \omega)} = \frac{-4\pi i\underset{\sim}{k} \cdot \underset{\sim}{j}_s}{\omega\, D_L(\underset{\sim}{k}, \omega)}$$

$$D_L\, (\underset{\sim}{k}, \omega) = 1 + \frac{4\pi e^2}{mk^2} \int \frac{\underset{\sim}{k} \cdot \partial f_o/\partial \underset{\sim}{v}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu)}\, d^3v$$

This gives the longitudinal E field; now proceeding to the transverse field.

$$\underset{\sim}{k} \times \underset{\sim}{k} \times \underset{\sim}{E} = \frac{\omega}{c}\, \underset{\sim}{k} \times \underset{\sim}{B} = -\, \frac{\omega^2}{c^2}\, \underset{\sim}{E} + \frac{4\pi e}{m} \frac{i\omega}{c^2} \int \underset{\sim}{v}\, f_1 d^3v - \frac{-4\pi j_s(k, \omega)\omega}{c^2}$$

$$\underset{\sim}{k} \times \underset{\sim}{k} \times \underset{\sim}{E} = -\frac{\omega^2}{c^2}\underset{\sim}{E} - \frac{4\pi e^2}{mc^2}\,\omega \int \frac{\underset{\sim}{v}\underset{\sim}{E} \cdot \partial f_o/\partial \underset{\sim}{v} d^3v}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\nu)}$$

$$- \frac{-4\pi i\, \underset{\sim}{j}_s(k,\underset{\sim}{\omega})}{c^2}$$

Problem. Show that if E_L satisfies the equation above that E_L drops out of this equation for $\underset{\sim}{k} \times \underset{\sim}{k} \times \underset{\sim}{E}$.

Let us take k to be in the z direction. Consider the components of $E_\perp$ to k

$$\int \frac{(\underset{\sim}{e}_x v_x + \underset{\sim}{e}_y v_y + \underset{\sim}{e}_z v_z)E_x \partial f_o/\partial v_x d^3v}{\omega - kv_z + i\nu}$$

for the v_y and v_z terms $\int$ over v_x and get 0

$$\int \frac{\underset{\sim}{e}_x E_x v_x\ \partial f_o/\ \underset{\sim}{v}_x\ dv_x dv_z dv_z}{(\omega - kv_z + i\nu)}$$

$$= -\underset{\sim}{e}_x E_x \int \frac{f_o d^3v}{(\ -kv_z + i\nu)}$$

Likewise for the y component

$$\underset{\sim}{k} \times \underset{\sim}{k} \times E_\perp = [-\frac{\omega^2}{c^2} + \frac{4\pi e^2}{mc^2}\,\omega \int \frac{f_o d^3v}{\omega - kv_z + i\nu}]\ \underset{\sim}{E}_\perp - \frac{4i\pi \underset{\sim}{j}_\perp(k,\omega)}{c^2}$$

$$[-k^2c^2 + \omega^2 - \frac{4\pi e^2\omega}{m} \int \frac{f_o d^3v}{\omega - kv_z + i\nu}]\ k \times E_\perp = -4\pi i\omega k \times \underset{\sim}{j}_s$$

$$\underset{\sim}{k} \times \underset{\sim}{E} = \frac{4\pi i\omega \underset{\sim}{k} \times \underset{\sim}{j}_s}{k^2c^2 D_T(k,\underset{\sim}{\omega})}$$

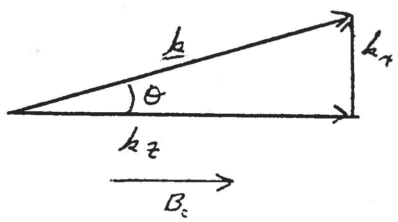
$$D_T = 1 - \frac{\omega^2}{k^2c^2} + \frac{4\pi e^2\omega}{k^2mc^2} \quad \frac{f_o dv^3}{(\omega - \underset{\sim}{k} \cdot v + i\nu)}$$

Problem. Find the o's of D_T for $\omega/k >> V_T$. Are there roots of D_T for $|\omega/k| \lesssim v_T$, if so find then

✱ Pages 226-233 form an insert after page 180

Propagation at an Arbitrary Angle

Use the magnitude of $\underset{\sim}{k}$ and the angle with respect to B



$$k_x = k \sin \theta, \; k_z = \cos \theta, \; k_y = 0$$

The equations are

$$\left(k^2 \cos\theta - \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ci}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right]\right)E_x - \frac{i\omega}{c^2}\left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right]E_y - k^2\sin\theta\cos\theta E_z$$

$$\frac{i\omega}{c^2}\left[-\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right] E_x + \left[k^2 - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right)\right] E_y = 0 -$$

$$k^2\sin\theta\cos\theta E_x + \left[k^2\sin^2\theta - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2 + \omega_{pi}^2}{c^2}\right] E_z = 0$$

Let us first look at the low frequency limit

$$\begin{vmatrix} k^2c^2\cos^2\theta - \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right) & 0 & -c^2k^2\sin\theta\cos\theta \\ 0 & c^2k^2 - \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right) & 0 \\ -c^2k^2\sin\theta\cos\theta & 0 & c^2k^2\sin^2\theta - \omega^2 + \omega_{pe}^2 + \omega_{pi}^2 \end{vmatrix} = 0$$

Again the determinant splits into two factors

$$k^2c^2 - \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right) = 0$$

$$\left[k^2c^2\cos^2\theta - \omega^2\left(1 + \frac{4\pi\rho c^2}{B^2}\right)\right]\left[k^2c^2\sin^2\theta - \omega^2 + \omega_{pe}^2 + \omega_{pi}^2\right]$$

$$-k^2c^2\sin^2\theta\cos^2\theta = 0$$

$$k^2c^2\sin^2\theta\cos^2\theta = 0$$

The first wave has only an E_y component and propagates at an arbitrary angle to B

$$\frac{k^2c^2}{(1 + \frac{4\pi\rho c^2}{B^2})} = \omega^2 \qquad k^2V_A{}^2 = \omega^2$$

The motion is $\underset{\sim}{E} \times \underset{\sim}{B}$ and therefore has only an x component. It is a mixture of a transverse and a longitudinal mode. If pressure times had been included, the dispersion would have been modified from the alfven waves. For the second case

$$k^2c^2\ [(\omega^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2)\cos^2\theta + \omega^2(1 + \frac{4\pi\rho c^3}{B^2})\sin^2\theta] -$$

$$- \omega^2\ (1 + \frac{4\pi\rho c^2}{B^2})\ (\omega^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2) = 0$$

Since $\omega_{pe}{}^2$ is generally quite large, we get approximately

$$k^2c^2\cos^2\theta - \omega^2(1 + \frac{4\pi\rho c^2}{B^2}) = 0$$

$$k^2V_k{}^2\cos^2\theta = \omega^2$$

$$k_z{}^2V_A{}^2 = \omega^2$$

$$V_g = \nabla_{\underset{\sim}{k}}\omega = \underset{\sim}{e}_x\frac{\partial\omega}{\partial k_x} + \underset{\sim}{e}_z\frac{\partial\omega}{\partial k_x} = \underset{\sim}{e}_zV_A$$

The waves propagate only along z

$$\omega_{pe} \to \infty \quad \text{requires} \quad E_z = 0$$

E_x arbitrary $E_y = 0$

motion which is in the $\underset{\sim}{E} \times \underset{\sim}{B}$ direction is only in the y direction. Each xz plane oscillates independently of every other xz plane.

If we do not consider $\omega_{pe}{}^2$ to be infinitely large then

$$k^2c^2 = \frac{\omega^2(1 + \frac{4\pi\rho c^2}{B^2})\ (\omega^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2)}{(\omega^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2)\cos^2\theta + \omega^2(1 + \frac{4\pi\rho c^2}{B^2})\sin^2\theta}$$

Get a resonance when, iek $\rightarrow \infty$

$$(\omega^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2)\cos^2\theta + \omega^2(1 + \frac{4\pi\rho c^2}{B^2})\sin^2\theta = 0$$

$$\text{or} \quad \cot z\theta = \frac{\omega^2(1 + \frac{4\pi\rho c^2}{B^2})}{\omega_{pe}{}^2 + \omega_{pi}{}^2 - \omega^2} \simeq \frac{\omega^2}{\omega_{pe}{}^2} \quad (1 + \frac{4\pi\rho e^2}{B^2}) << 1$$

$$\theta = \frac{\pi}{2} - \delta\theta$$

$$\delta\theta = \frac{\omega^2}{\omega_{pe}{}^2}(1 + \frac{4\pi\rho c^2}{B^2})$$

$\omega_{ce} \rightarrow \infty$, ions infinitely heavy

$$(k^2\cos^2\theta - \frac{\omega^2}{c^2})E_x \quad 0E_y \quad -k\sin\theta\cos\theta E_z = 0$$

$$0\ E_x \quad (k^2 - \frac{\omega^2}{c^2})E_y \quad 0E_z = 0$$

$$-k^2\sin\theta\cos\theta E_x \qquad 0 \qquad (k^2\sin^2\theta - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2}{c^2})E_z = 0$$

$$(k^2 - \frac{\omega^2}{c^2})E_y = 0$$

$$k^2c^2 = \omega^2 \qquad E_z \text{ arbitrary} \qquad E_y = 0$$

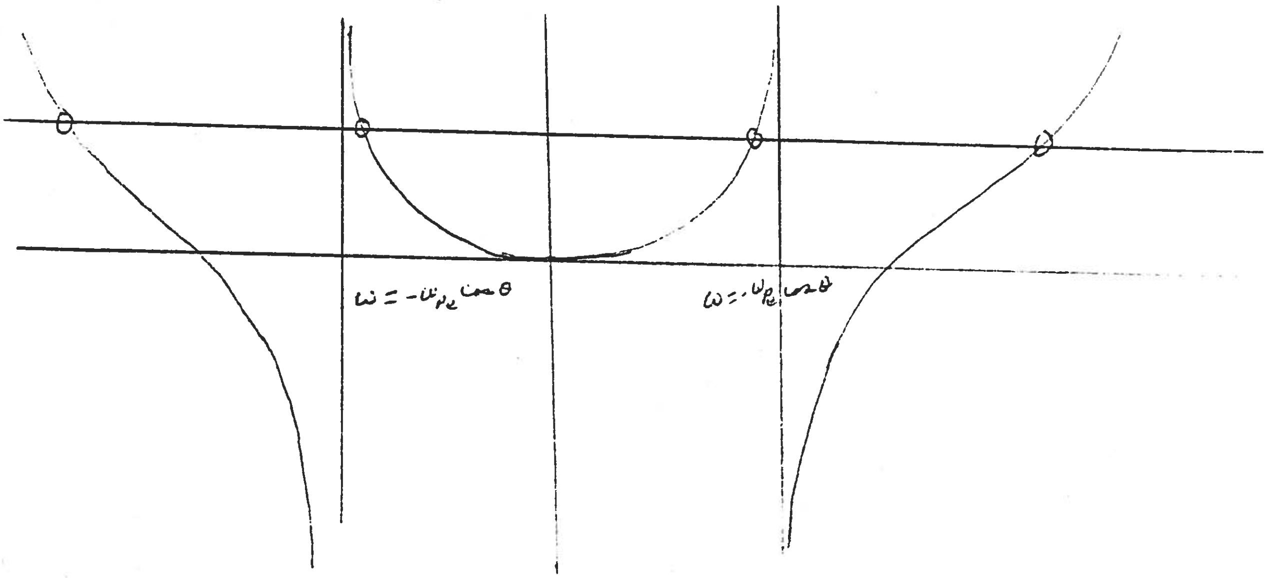
$$(c^2k^2\cos^2\theta - \omega^2)\ (c^2k^2\sin^2\theta - \omega^2 + \omega_{pe}^2) - k^4c^4\sin^2\theta\cos^2\theta = 0$$

$$-k^2c^2\ \omega^2 + k^2c^2(\omega_{pe}^2\cos^2\theta) + \omega^2(\omega^2 - \omega_{pe}^2) = 0$$

$$k^2c^2 = \frac{\omega^2(\omega^2 - \omega_{pe}^2)}{\omega^2 - \omega_{pe}^2\cos^2\theta} = \frac{\omega^2(1 - \frac{\omega_{pe}^2}{\omega^2})}{1 - \frac{\omega_{pe}^2}{\omega^2}\cos^2\theta}$$

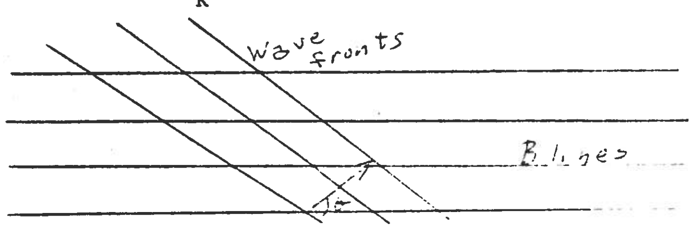
$$k^2c^2 = \frac{\omega^2(1 - \frac{\omega_{pe}^2}{\omega^2})}{1 - \frac{\omega_{pe}^2}{\omega^2}\cos^2\theta}$$

Plot the left and right hand sides



Resonance at $\omega = \omega_{pe}\cos\theta$

$\cos\theta = \frac{k_z}{k}$



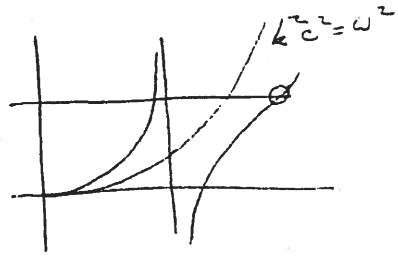
This mode is electrostatic but only the component of E || to B is effective so the effective restoring force is $E\,\frac{k_z}{k}$ and the effective inertia is

$$\frac{m}{\cos\theta} = \frac{mk}{k_z} \quad \frac{mk}{k_z}\,Z = \frac{-eEk_z}{k}$$

$$\ddot{Z} = -\omega_p^2 \frac{k_z^2}{k^2} Z$$

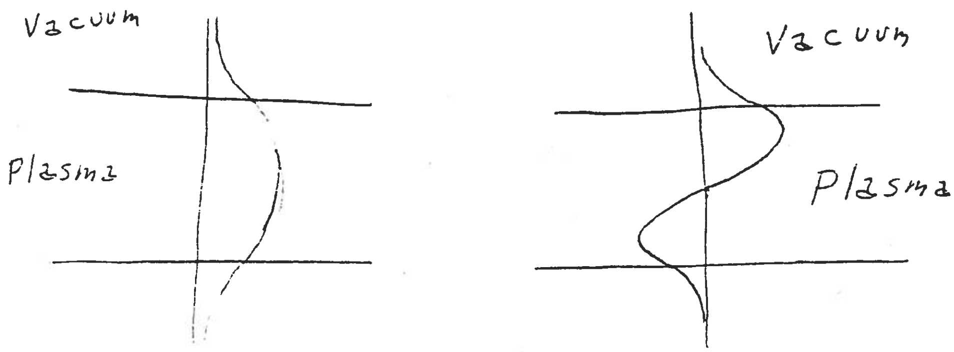
For the other root

$$k^2c^2 > \omega^2$$



$$\frac{\omega^2}{k^2} = V_p < c^2$$

Therefore, the phase velocity is less than the velocity of light. This wave will remain in the plasma slab



Fundemental Mode

Ion cyclotron waves, zero mass electron, zero mass electrons implies $E_z = 0$. Also, we can neglect $\frac{\omega_{pe}^2 \omega_{ce}}{-\omega_{ce}^2}$

compared to $\frac{\omega_{pi}^2 \omega_{ci}}{\omega^2 - \omega_{ci}}$ since the last term is near a resonance

$$\begin{vmatrix} k^2\cos^2\theta - \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{pi}^{\ 2}}{\omega^2 - \omega_{ci}^{\ 2}}\right] & - \frac{i\omega\ \omega_{pi}^{\ 2}\ \omega_{ci}}{\omega^2 - \omega_{ci}^{\ 2}} \\ \frac{i\omega\omega_{pi}^{\ 2}\omega_{ci}}{c(\omega^2 - \omega_{ci}^{\ 2}} & k^2 - \frac{\omega^2}{c^2}(1 - \frac{\omega_{pi}^{\ 2}}{\omega^2 - \omega_{ci}^{\ 2}} \end{vmatrix} = 0$$

$$[c^2k^2\cos^2\theta - \omega_{ci}^{\ 2}\ (1 - \frac{\omega_{pi}^{\ 2}}{\omega^2 - \omega_{ci}^{\ 2}})]\ [k^2 - \omega_{ci}^{\ 2}\ (1 - \frac{\omega_{pi}^{\ 2}}{\omega^2 - \omega_{ci}^{\ 2}})] -$$

$$- \frac{\omega_{ci}^{\ 4}\ \omega_{pi}^{\ 4}}{(\omega^2 - \omega_{ci}^{\ 2})^2} = 0$$

$$k^4c^4\cos^2\theta - k^2c^2\omega_{ci}^{\ 2}\ (1 - \frac{\omega_{pi}^{\ 2}}{\omega^2 - \omega_{ci}^{\ 2}})\ (1 + \cos^2\theta) + \omega_{ci}^{\ 4}$$

$$+ \frac{4\omega_{ci}^{\ 4}\ \omega_{pi}^{\ 4}}{\omega^2 - \omega_{ci}^{\ 2}} = 0$$

$$\frac{\omega_{ci}^{\ 4}\ \omega_{pi}^{\ 4}}{\omega^2 - \omega_{ci}^{\ 2}}\ [4 + \frac{k^2c^2}{\omega_{ci}^{\ 2}\ \omega_{pi}^{\ 2}}\ (1 + \cos^2\theta)] = k^2c^2\ \omega_{ci}^{\ 2}\ (1 + \cos^2\theta)$$

$$- k^4c^4\cos^2\theta - \omega_{ci}^{\ 4}$$

$$\omega - \omega_{ci} = \frac{\dfrac{\omega_{ci}^{3}}{2}\,\omega_{pi}^{4}\left[4 + \dfrac{k^2c^2}{\omega_{ci}^{2}\,\omega_{pi}^{2}}(1 + \cos^2\theta)\right]}{k^2c^2\,\omega_{ci}^{2}(1 + \cos^2\theta) - k^4c^4\cos^2\theta - \omega_{ci}^{2}}$$

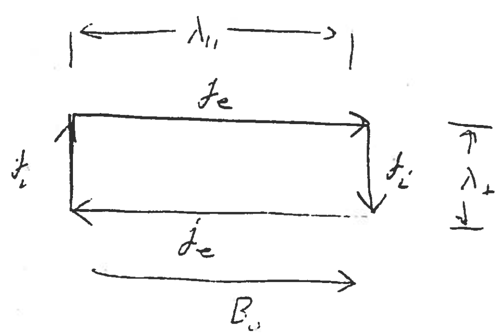
In general, $k^2c^2 >> \omega_{ci}^{2}$

$$\omega = \omega_{ci} - \frac{\dfrac{\omega_{ci}\,\omega_{pi}^{2}}{2}(1 + \cos^2\theta)}{k^2c^2\cos^2\theta}$$

$$\omega = \omega_{ci} - \frac{\omega_{ci}\,\omega_{pi}^{2}}{2k^2c^2}\,\frac{(1 + \cos^2\theta)}{\cos^2\theta} = \omega_{ci}\left(1 - \frac{\omega_{pi}^{2}}{2k^2c^2\cos^2\theta} - \frac{\omega_{pi}^{2}}{2k^2c^2}\right)$$

agrees with Stix.

Can get this answer from a simple model in which ion current flows across B and electron current flows along B to close the circuit. E_z must be zero because of zero mass electrons. The ion motions are produced by a combined space charge field and an inductive field which add up to give $E_z = 0$.



The Energy Principal

We should now like to derive the energy principal for the stability of an ideal MHD fluid. This is a very useful principal since one of the principal things we should like to know about a plasma configuration is whether or not it is stable.

First we might consider the stability of simple mechanical systems. To be specific, consider a pendulum made of a weight and a rigid stick

Consider the two situations shown. Both are in equilibrium, i.e., no force is acting on the weight. For case (1) if the weight is displaced it oscillates stably about the equilibrium while in the second case a slight displacement in leads to a force which tends to move it further away from the equilibrium and the system is unstable. For both cases

$$mr\ddot{\theta} = -Kr\theta \qquad \text{where } K = \pm mg.$$

Multipling by $r\dot{\theta}$ gives

$$mr^2\ddot{\theta}\dot{\theta} = -Kr^2\dot{\theta}\theta$$

Integrating with respect to time

$$mr^2\frac{\dot{\theta}^2}{2} = -Kr^2\frac{\theta^2}{2} + W$$

$$\Rightarrow \quad mr^2\frac{\dot{\theta}^2}{2} + Kr^2\frac{\theta^2}{2} = W$$

Since the kinetic energy term in intrinsically positive if K is positive the displacement θ is limited. If K is negative however then $\dot{\theta}$' and θ can increase continually. The term $Kr^2\theta^2/2$ is the potential energy due to the displacement from equilibrium. If it is positive then we must do work to displace the system and it cannot of its own move away from the equilibrium. On the other hand if the change in potential energy is negative the system of itself can move away from the equilibrium.

You can Fourie analyze in time

$$\theta = e^{i\omega t}\theta \quad \Rightarrow \quad -\omega^2 m r\theta = -K r\theta$$

$$-\omega^2 = \frac{-K\cancel{r}\cancel{\theta}}{m\cancel{r}\cancel{\theta}}$$

If K is positive ω in real and the motion is oscillitory while if K is negative ω is imaginary with the system is unstable.

We wish to apply these ideas to a plasma. A plasma is however, more complex because it has an ∞ number of degrees of freedom. The plasma at any given point can be moved arbitrarily relative to any other point.

$$m_i \ddot{\xi}(i) = -\sum_j a_{ij}\,\xi(j)$$

$$-\omega_k^2\,\xi_k(i) = -\sum_j \frac{a_{ij}}{m_i}\,\xi_k(j)$$

$$\sum_j \left(\frac{a_{ij}}{m_i} - \omega^2\delta_{ij}\right)\xi(j) = 0$$

ω's are the solutions of

$$\text{Determinant}\ \left| a_{ij} - \omega^2\delta_{ij} \right| = 0$$

For N degrees of freedom this is an N by N determinant and gives a polynonid of order N in ω^2, there are N solutions of ω^2, denote them by ω_k. For each ω_k there is a set of $\xi_k(j)$, only the ratios of the $\xi_k(j)$'s are specified.

$$-\omega_k^2\,\xi_k(i) = -\sum_j a_{ij}\,\xi_k(j)$$

Multiply by $\xi_k(i)$

$$-\omega_k^2 \sum_i m_i\,\xi_k^2(i) = -\sum_i a_{ij}\,\xi_k(i)\,\xi_k(j)$$

Normalize so that

$$\sum_i m_i\,\xi_k^2(i) = 1.$$

Let ω_k^2 and ω_1^2 and $\xi_k(i)$ and $\xi_1(i)$ be two solutions

$$-\omega_k^2 m_i \xi_k(i) = -\sum_j a_{ij} \xi_k(j)$$

$$-\omega_l^2 m_i \xi_l(i) = -\sum_j a_{ij} \xi_l(j)$$

multiply the first by $\xi_1(i)$ and the second by $\xi_k(i)$ Σ over and subtract

$$(\omega_k^2 - \omega_l^2) \sum_i m_i \xi_k(i) \xi_l(i) = -\sum_{ij} \left(a_{ij} \xi_l(i) \xi_k(j) - a_{ji} \xi_l(j) \xi_k(i) \right) = 0$$

since $a_{ij} = a_{ji}$

$$\therefore \quad \sum_i m_i \xi_k(i) \xi_l(i) = 0$$

Choose

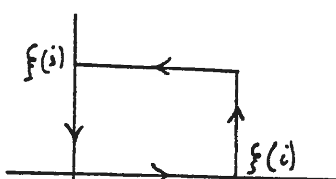
$$\sum_i m_i \xi_k^2(i) = 1$$

$$\sum_i m_i \xi_k^2(i) \xi_l(i) = \delta_{kl}$$

Aside : $\sum_j a_{ij} \xi(j) =$ force on i^{th} particle

$\sum_i a_{ji} \xi(i) =$ force on j^{th} particle

Consider that only particles i and j are moved around the diagram shown



The work done is

$$\int a_{ii} \xi_i \, d\xi_i + \int a_{ij} \xi_j \, d\xi_i + \int a_{ji} \xi_i \, d\xi_j + \int a_{jj} \xi_j \, d\xi_j + a_{jj} \xi_j^2/2$$

$$= a_{ii} \frac{\xi_i^2}{2} + a_{ji} \xi_i \xi_j - a_{ij} \xi_j \xi_i - a_{ii} \xi_i^2/2 - a_{jj} \xi_j^2/2$$

$a_{ij} = a_{ji}$ ~~is~~ if the work done in going around this circuit is to be 0 or if the system is to be conservative.

$$\xi(i) = \sum_k e^{i\omega_k t} A_k \xi_k(i)$$

Stable if all ω_k^2 are positive.

Unstable if one ω_k^2 is negative.

$$-\omega_k^2 \sum_i m_i \xi_k^2(i) = -\sum_{ij} a_{ij}\, \xi_k(i)\, \xi_k(j)$$

The sum on the left is intrinsically positive so that you get stability if the sum on the right is positive, get instability if it is negative.

Let $\xi(i)$ be a displacement of the system

$$\xi(i) = \sum_k \alpha_k \xi_k(i)$$

$$\sum_j a_{ij}\, \xi(j) = \sum_{kj} a_{ij}\, \alpha_k\, \xi_k(j) = \sum_k m_i\, \omega_k^2\, \alpha_k\, \xi_k(i)$$

Multiply by ξ_i and sum over i

$$\sum_{ik\ell} \omega_k^2\, \alpha_k \alpha_i\, m_i\, \xi_k(i)\, \xi_\ell(i) = \sum_k \omega_k^2 \alpha_k^2 = \delta W$$

by the normalization

δW can only be negative if one of the ω_k^2's is negative. If one of the ω_k^2 is negative then we can find a displacement which makes the energy negative, namely the eigen function for the negative ω^2.

$$m_i\, \ddot{\xi}(i) = -\sum_j a_{ij}\, \xi(j)$$

$$m_i\, \ddot{\xi}(i)\, \dot{\xi}(i) = -\sum_j a_{ij}\, \xi(j)\, \dot{\xi}(i)$$

Sum over j ;

$$\sum_i m_i\, \ddot{\xi}(i)\, \dot{\xi}(i) = -\sum_{ij} a_{ij}\, \xi(j)\, \dot{\xi}(i) = -\frac{1}{2} \sum_{ij} a_{ij} \left(\xi(i)\, \xi(j)\right)^{\bullet}$$

$$\sum_i m_i\, \dot{\xi}_i^2/2 + \frac{1}{2} \sum_{ij} a_{ij}\, \xi(i)\, \xi(j) = W_T$$

The ideal MHD equations are

$$\rho \frac{d\underline{v}}{dt} = -\nabla p + \frac{\underline{J}\times\underline{B}}{c}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\underline{v} = 0$$

$$\underline{\nabla}\times\underline{B} = \frac{4\pi\underline{J}}{c}$$

$$\underline{\nabla}\times\underline{E} = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = 0 \qquad E_{\parallel} = 0$$

$$\frac{p}{\rho^{\gamma}} = \text{const} = \frac{p_0}{\rho_0^{\gamma}}$$

The equilibrium is given by

$$\nabla P_0 = \frac{\underline{J}\times\underline{B}}{c} = \frac{1}{4\pi}(\underline{\nabla}\times\underline{B}_0)\times\underline{B}_0$$

The first order equation of motion (assuming $v_o \equiv 0$)

$$\rho_0\dot{\underline{v}} = -\nabla p + \frac{1}{4\pi}\left\{(\nabla\times B)\times B_0 + (\nabla\times B_0)\times B_1\right\}$$

Let $\xi(\Omega_o,t)$ be the displacement of an element of the fluid $\perp$ to Bs. Displacements $\parallel$ to B need not lead to a restoring force, $\underline{v} = \dot{\xi}$ (However when pressure is included $\parallel$ displacements also generally lead to restoring forces and our treatment includes those.) $\underline{E} + \dfrac{\dot{\underline{\xi}}\times\underline{B}_0}{c} = 0$

$$\underline{\nabla}\times\underline{E} = -\underline{\nabla}\times\left(\frac{\dot{\underline{\xi}}\times\underline{B}_0}{c}\right) = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\underline{B} = \underline{\nabla}\times(\underline{\xi}\times B_0) = Q.$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho_0 \dot{\underset{\sim}{\xi}} = 0$$

$$\frac{d\rho}{dt} + \rho_0 \nabla \cdot \underset{\sim}{V} = 0$$

following the motion.

$$\rho + \nabla \cdot \rho_0 \underset{\sim}{\xi} = 0$$

$$\rho = -\rho_0 \nabla \cdot \underset{\sim}{\xi}$$

$$\rho = -\nabla \cdot \rho_0 \underset{\sim}{\xi} = -(\rho_0 \nabla \cdot \xi + \underset{\sim}{\xi} \cdot \nabla \rho_0)$$

$$\frac{d}{dt} \frac{P}{\rho^\gamma} = 0$$

$$\left(\frac{\partial P}{\partial t} + \underset{\sim}{V} \cdot \underset{\sim}{\nabla} P\right] - \frac{\gamma P}{\rho}\left(\frac{\partial \rho}{\partial t} + \underset{\sim}{V} \cdot \underset{\sim}{\nabla} \rho\right) = 0 \qquad ?$$

$$\frac{\partial P}{\partial t} + \dot{\xi} \cdot \underset{\sim}{\nabla} P_0 + \gamma P_0 \nabla \cdot \dot{\xi} = 0$$

$$P = -\{\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_0 + \gamma P_0 \underset{\sim}{\nabla} \cdot \xi\}$$

$$\rho_0 \ddot{\underset{\sim}{\xi}} = \underset{\sim}{\nabla}\{\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_0 + \gamma P_0 \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}\} +$$

$$+ \frac{1}{4\pi}\{\underset{\sim}{\nabla} \times [(\underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B_0})) \times \underset{\sim}{B_0}] + (\underset{\sim}{\nabla} \times \underset{\sim}{B_0}) + \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B_0})\}$$

$$\rho_0 \ddot{\underset{\sim}{\xi}} = F(\underset{\sim}{\xi})$$

can seperate ξ in time and space.

$$\underset{\sim}{\xi} = T(t)\, \underset{\sim}{\xi}(\underset{\sim}{r_0})$$

$$\rho_0 \ddot{T}\, \underset{\sim}{\xi}(r_0) = T F(\underset{\sim}{\xi}(r_0))$$

$$\ddot{T} = -\omega_k^2 T \qquad \omega_k^2 = -\frac{F(\xi(r_0)) \cdot \xi}{\rho_0(r_0)\, \xi^2(r_0)}$$

$$T = T_1 e^{i\omega_k t} + T_2 e^{-i\omega_k t}$$

$$-\rho_0 \omega_k^2 \underset{\sim}{\xi_k}(r_0) = F(\xi_k(r_0))$$

$$\underset{\sim}{\xi}(\underset{\sim}{r_0}, t) = \sum_k a_k \, \mathrm{EXP}(i\omega_k t)\, \xi_k(r_0) \qquad (252)$$

The ξ_k form a set of normal modes. In the normal way there is an orthogonality relation

$$\frac{1}{2}\int \rho_0\, \underset{\sim}{\xi}_k(r_0)\cdot\underset{\sim}{\xi}_l(r_0)\, d\tau_0 = \delta_{kl}$$

By analogy with a set of harmonic oscillators

$$\delta W = -\tfrac{1}{2}\int \underset{\sim}{\xi}\cdot \underset{\sim}{F}(\xi)\, d\tau$$

The system is stable provided this is positive

$$-\frac{1}{2}\int \sum_{kl} \underset{\sim}{a_k}\, Re\, e^{i\omega_k t}\cdot \sum_l \underset{\sim}{F}(\xi_l)\, Re\, e^{i\omega_l t}\, d\tau$$

$$\delta W = \frac{1}{2}\sum_{kl} a_k a_l \omega_l^2 \int \rho_0\, \xi_k\cdot \xi_l\, d\tau\, Re\, e^{i\omega_k t}\, Re\, e^{-i\omega_l t}$$

$$= +\frac{1}{2}\sum_k a_k^2\, \omega_k^2 \cos^2\omega_k t$$

Stable or unstable according to whether or not

$$\int \Big[\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}\left\{\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla} p_0 + \gamma p_0 \underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}\right\} + \frac{\xi\cdot}{4\pi}\Big\{ \big(\nabla\times[(\nabla\times\xi\times B_0)]\times B_0\big)$$
$$+ (\nabla\times B_0)\times[\nabla\times(\xi\times B_0)]\Big]\Big\}\, d\tau$$

is negative or positive. Negative ω_k^2 leads to imaginary ω_k's and to exponentially growing solutions. Positive ω_k^2 to stable oscillations.

$$\rho_o \ddot{\xi} = F(\xi)$$

$$\underset{\sim}{F}(\underset{\sim}{\xi}) = \underset{\sim}{\nabla} \{\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \gamma P_o \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}\} + \frac{1}{4\pi} \; [\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_o))] \times \underset{\sim}{B}_o +$$

$$+ (\underset{\sim}{\nabla} \times \underset{\sim}{B}_o) \times \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_o)\}$$

$$- \omega_k^2 \rho_o \underset{\sim}{\xi}(\underset{\sim}{r}_o) = \underset{\sim}{F}(\underset{\sim}{\xi}_k(\underset{\sim}{r}_o))$$

The ξ_k are a complete set of normal modes. Normalize so that

$$\int \rho_o(\underset{\sim}{r}_o) \; \xi_k^2 d\tau = 1$$

$$- \omega_k^2 \int \rho_o \xi_k^2 d\tau = \int \underset{\sim}{\xi}_k \cdot \underset{\sim}{F}(\underset{\sim}{\xi}_k) d\tau$$

$$\omega_k^2 = - \int \underset{\sim}{\xi}_k \cdot \underset{\sim}{F}(\xi_k) d\tau.$$

If $- \int \underset{\sim}{\xi}_k \cdot \underset{\sim}{F}(\xi_k) d\tau$ is negative.

The system is unstable.

If the system is unstable there is some disturbance which makes the potential energy negative; if we can find a disturbance which makes the energy negative the system is unstable. For many problems we are interested in the plasma not filling the whole space, there will be a boundary between plasma and vacuum and we must find the contributions from displacing the surface and distorting the vacuum fields. To this end we manipulate the expression we get.

We use the relation

$$\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} \{(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \gamma P_o \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})\} = \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} \; (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \gamma P_o \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})$$

$$- (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \gamma P_o \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}) \; \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}$$

Writing

$$\nabla \times (\xi \times B_o) = Q = \delta B$$

$$\nabla \cdot [(\xi \times B_o) \times Q] = Q \cdot [\nabla \times (\xi \times B_o)] - (\xi \times B_o) \cdot (\nabla \times Q)$$

$$= Q^2 - (\xi \times B_o) \cdot \nabla \times Q = Q^2 - \xi \cdot (B_o \times \nabla \times Q)$$

Also,

$$\xi \cdot [(\nabla \times B_o) \times Q] = - [\nabla \times B_o] \cdot (\xi \times Q)$$

$$\delta W = - \frac{1}{2} \int \xi \cdot \{\nabla(\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi) + \frac{1}{4\pi} [(\nabla \times Q) \times B_o + (\nabla \times B_o) \times Q]\} d\tau$$

$$= \frac{1}{2} \int \{(\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi) \nabla \cdot \xi - \nabla \cdot \xi (\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi)$$

$$+ \frac{1}{4\pi} [Q^2 - \nabla \cdot [(\xi \times B_o) \times Q]] + \frac{1}{4\pi} \nabla \times B_o \cdot (\xi \times Q)\} d\tau$$

$$= \frac{1}{2} \int_{\text{plasma}} \{(\nabla \cdot \xi) \xi \cdot \nabla P_o + \gamma P_o (\nabla \cdot \xi)^2 + \frac{Q^2}{4\pi} + \frac{1}{4\pi} (\nabla \times B_o) \cdot (\xi \times Q)\} d\tau$$

$$- \frac{1}{2} \int_{\text{surface}} \{\xi (\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi) + \frac{1}{4\pi} (\xi \times B_o) \times Q\} \cdot ds.$$

Let us manipulate the surface term, the quantities in it must be evaluated at the position of the surface. We have a point in space

$$P_1 = - (\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi)$$

and

$$B_1 = \nabla \times (\xi \times B_o) = Q$$

Now the value of P_1 at the displaced surface point that started at r_s is

$$P_1(R_s, t) = P_1(\underset{\sim}{r}_s, t) + \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o = -\gamma P_o \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}.$$

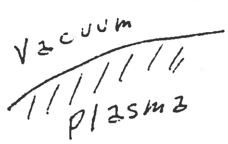
Can also get that from

$$\frac{dp}{dt} = - \gamma P \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}$$

following the motion, for B_1 we have

$$B_1(R_s) = \underset{\sim}{Q} + \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} \underset{\sim}{B}_o$$

Now along the surface we must have pressure balance



$$\left(P + \frac{B^2}{8\pi}\right)_{\text{fluid}} = \frac{B^2 \text{vacuum}}{8\pi}$$

To first order

$$\left(P_1 + \frac{B_o \cdot B_1}{4\pi}\right)_{\text{fluid}} = \left(\frac{B_o\, B_{iv}}{4\pi}\right)_{\text{vacuum}}$$

$$- \gamma P_o \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} + \frac{\underset{\sim}{B}_o}{4\pi} \cdot [\underset{\sim}{Q} + \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} B_o] = \frac{\underset{\sim}{B}_{ov}}{4\pi} \cdot [\underset{\sim}{B}_{iv} + \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} B_{ov}]$$

Let us now substitute this relation into the surface integral

$$- \frac{1}{2} \int_s \{\underset{\sim}{\xi} \, (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \gamma P_o \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}) - \frac{\underset{\sim}{Q} \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_o)}{4\pi}\} \cdot \underset{\sim}{ds}$$

$$= - \frac{1}{2} \int_s \{\underset{\sim}{\xi} \, (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \gamma P_o \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} + \frac{\underset{\sim}{B}_o - \underset{\sim}{\xi}(\underset{\sim}{Q} \cdot \underset{\sim}{B}_o)}{4\pi}\} \cdot \underset{\sim}{ds}$$

Since the plasma surface must be parallel to $\underset{\sim}{B}$, $\underset{\sim}{B} \cdot \underset{\sim}{\nabla} P = 0$ therefore B lies in surfaces of constant P, $[- \nabla P + \frac{j \times B}{c} = 0]$ and the $\underset{\sim}{B}(\underset{\sim}{Q} \cdot \underset{\sim}{\xi}) \cdot ds$ term gives 0, proceeding the surface integral becomes

$$= - \frac{1}{2} \int_s \underset{\sim}{\xi}\{\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \gamma P_o \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} - \frac{\underset{\sim}{Q} \cdot \underset{\sim}{B}_o}{4\pi}\} \cdot \underset{\sim}{ds}$$

$$= - \frac{1}{2} \int_s (\underset{\sim}{\xi} \, \{\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \frac{\underset{\sim}{B}_o \cdot (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} \underset{\sim}{B}_o)}{4\pi}$$

$$- \frac{\underset{\sim}{B}_{ov}}{4\pi} \cdot [\underset{\sim}{B}_{iv} + \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} B_{ov}]\}) \cdot \underset{\sim}{ds}$$

We also have

$$\underset{\sim}{B}_o \cdot (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} \underset{\sim}{B}_o) = B_{ok} \, \xi_i \frac{\partial B_{ok}}{\partial X_i} = \xi_i \, \frac{\partial B_{ok}{}^2/2}{\partial X_i} = \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} B_o{}^2/2$$

The integral becomes

$$- \frac{1}{2} \int_s \{\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_o + \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} \, (\frac{B_{op}{}^2}{\delta\pi} - \frac{B_{ov}{}^2}{\gamma\pi}) - \frac{\underset{\sim}{B}_{ov} \cdot \underset{\sim}{B}_{iv}}{4\pi}\} \underset{\sim}{\xi} \cdot \underset{\sim}{ds}.$$

Since $P + \frac{B_p^2}{8\pi} = \frac{B_v^2}{8\pi}$ holds everywhere on the boundary

$$\frac{\partial}{\partial x_{\parallel}} \left(P_o + \frac{B_{op}^2}{\delta\pi} - \frac{B_{ov}^2}{\delta\pi}\right) = 0$$

$$- \frac{1}{2} \int_s (\underset{\sim}{\xi} \cdot \underset{\sim}{m})^2 \left[\underset{\sim}{\nabla} \left(P_o + \frac{B_o^2}{8\pi}\right)\right] \cdot \underset{\sim}{ds} + \frac{1}{8\pi} \int_s \underset{\sim}{B}_{ov} \cdot \underset{\sim}{B}_{iv} \underset{\sim}{\xi} \cdot \underset{\sim}{ds}$$

We can get the last integral in a slightly different form. Write in the vacuum

$$\underset{\sim}{B}_1 = \underset{\sim}{\nabla} \times \underset{\sim}{A}_1$$

$$\underset{\sim}{E}_1 = - \frac{1}{c} \underset{\sim}{\dot{A}}_1$$

The Coulomb gauge has been adopted so that the scalar potential does not appear. For an observer riding on the surface of the plasma, the tangential component of E is continuous. In the plasma E is zero, therefore, in the vacuum E must be 0. Going back to the lab frame

$$\underset{\sim}{E}_v = - \frac{\underset{\sim}{\dot{\xi}} \times \underset{\sim}{B}_o}{c}$$

$$\underset{\sim}{n} \times \underset{\sim}{E}_v = - \frac{\underset{\sim}{n} \times (\underset{\sim}{\dot{\xi}} \times B_i)}{c} = \frac{-\underset{\sim}{\dot{\xi}} \overbrace{(\underset{\sim}{n} \cdot \underset{\sim}{B}_v)}^{0} + \underset{\sim}{B}_v (\underset{\sim}{n} \cdot \underset{\sim}{\dot{\xi}})}{c}$$

$$\underset{\sim}{n} \cdot \underset{\sim}{E}_v = \frac{(\underset{\sim}{n} \cdot \underset{\sim}{\dot{\xi}})\, \underset{\sim}{B}_v}{c} = - \frac{1}{c} \underset{\sim}{n} \times \underset{\sim}{\dot{A}}_1$$

$$\underset{\sim}{n} \times \underset{\sim}{\dot{A}}_1 = - (\underset{\sim}{n} \cdot \underset{\sim}{\dot{\xi}})\, \underset{\sim}{B}_v$$

$$\frac{1}{8\pi} \int \underset{\sim}{B}_{ov} \cdot \underset{\sim}{B}_{iv}\, \xi \cdot \underset{\sim}{ds} = >$$

$$= \frac{1}{8\pi}\int (\underset{\sim}{n} \cdot \underset{\sim}{\xi}\, ds\, \underset{\sim}{B}_{ov}) \cdot \underset{\sim}{B}_{iv} = \frac{1}{\pi}\int - (d\underset{\sim}{s} \times \underset{\sim}{A}_1) \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{A}_1)$$

$$= \frac{1}{8\pi}\int d\underset{\sim}{s} \cdot [(\underset{\sim}{\nabla} \times \underset{\sim}{A}_1) \times \underset{\sim}{A}_1]$$

because the normal points out of the plasma and into the vacuum.

$$= \frac{1}{8\pi}\int d\tau\, \underset{\sim}{\nabla} \cdot [(\underset{\sim}{\nabla} \times \underset{\sim}{A}_1) \times \underset{\sim}{A}_1]$$

$$= \frac{1}{8\pi}\int \{\underset{\sim}{A}_1 \cdot [\underset{\sim}{\nabla} \times \overset{\overset{0}{||}}{\underset{\sim}{\nabla}} \times \underset{\sim}{A}_1] - (\underset{\sim}{\nabla} \times \underset{\sim}{A}_1) \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{A}_1)\}\, d\tau$$

$$= \frac{1}{8\pi}\int (\underset{\sim}{\nabla} \times \underset{\sim}{A}_1)^2\, d\tau = \frac{1}{8\pi}\int \underset{\sim}{B}_{iv}^{\,2}\, d\tau.$$

The surface integral becomes

$$- \frac{1}{2}\int_s (\underset{\sim}{\xi} \cdot \underset{\sim}{n})^2\, [\underset{\sim}{\nabla}\,(P_o + \frac{B^2}{\delta\pi})] \cdot d\underset{\sim}{s} + \frac{1}{8\pi}\int_{\text{vacuum}} \underset{\sim}{B}_{iv}^{\,2}\, d\tau$$

$$\delta W = \frac{1}{2}\int \{(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P + \gamma P_o(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 + \frac{\underset{\sim}{Q}^2}{4\pi} + \frac{(\underset{\sim}{\nabla} \times \underset{\sim}{B}) \cdot (\underset{\sim}{\xi} \times \underset{\sim}{Q})}{4\pi}\quad d\tau$$

$$- \frac{1}{2}\int \{(\underset{\sim}{n} \cdot \underset{\sim}{\xi})^2 \cdot \underbrace{[\underset{\sim}{\nabla} P_o + \frac{B^2}{\delta\pi}]}_{\text{jump}} \cdot d\underset{\sim}{s} + \frac{1}{8\pi}\int_{\text{vacuum}} B_{iv}^{\,2}\, d\tau.$$

Application of the Energy Principle

Seek the ξ which minimizes δW.

Must normalize ξ, one choice is $\left(\int \xi^2 d\tau\right) = 1$ [If a normalization if not used, δW can be made arbitrarily large by choosing ξ arbitrarily large] but we may use any other convenient normalizing condition.

Consider a force free magnetic field enclosed by a rigid conducting boundary parallel to B. Assume the system is filled with fluid so that the surface and vacuum terms do not enter.

$$\nabla \times B_0 = \frac{4\pi j}{c}$$

$$\frac{j \times B_0}{c} = 0 \qquad (j \parallel B)$$

$$\nabla \times B_0 = \alpha B_0, \qquad \nabla \cdot j = 0 \qquad \nabla \cdot B_0 = 0$$

We find the solution to the equilibrium equations which are independent of Z and θ inside a circular cylinder with α = constant.

Could take α constant or a variable.

$$\delta W = \frac{1}{8\pi}\int [Q^2 + \alpha B \cdot (\xi \cdot Q)]d\tau, \qquad Q = \vec{\nabla} \times (\xi \times B_0) = B_1$$

Let $R = \xi \times B_0$ (R is like the vector potential) $\nabla \times R = B_1$

$$W = \frac{1}{8\pi}\int [(\nabla \times R)^2 + \alpha B_0 \cdot (\xi \times [\nabla \times R])]d\tau$$

$$= \frac{1}{8\pi}\int [(\nabla \times R)^2 - \alpha[\nabla \times R] \cdot [\xi \times B_0]]d\tau$$

$$= \frac{1}{8\pi}\int [(\nabla \times R)^2 - \alpha R \cdot (\nabla \times R)]d\tau$$

For the normalization condition use

condition: $\frac{1}{8\pi} \int \alpha \underset{\sim}{R} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{R}) d\tau$ = constant

Introduce this condition by means of Lagrange multiplier λ

$$I = \frac{1}{8\pi} \int [(\underset{\sim}{\nabla} \times \underset{\sim}{R})^2 - (\lambda + 1)\, \alpha\, \underset{\sim}{R} \cdot \underset{\sim}{\nabla} \times \underset{\sim}{R}] d\tau = \int L\, d\tau$$

This is of the form

$$I = \int L\,(x, q, q')\, d\tau,$$

q' is the derivative of q. We minimize I, the minimum is obtained when the Euler Lagrange equation holds

$$\frac{\partial L}{\partial q} = \frac{d}{dx}\frac{\partial L}{\partial q'}, \quad \frac{L}{R_i} = \frac{d}{dx}\frac{\partial L}{\partial R_{ij}}, \quad R_{ij} = \frac{\partial R_i}{\partial x_k}$$

$$\frac{\partial L}{\partial R_x} = \frac{d}{dx}\frac{\partial L}{\partial R_{x,x}} + \frac{d}{dy}\frac{\partial L}{\partial R_{x,y}} + \frac{d}{dz}\frac{\partial L}{\partial R_{x,z}}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{R} = \begin{vmatrix} \underset{\sim}{e}_x & \underset{\sim}{e}_y & \underset{\sim}{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ R_x & R_y & R_z \end{vmatrix} = \begin{array}{l} \underset{\sim}{e}_x\,(R_{z,y} - R_{y,z}) \\ +\, \underset{\sim}{e}_y\,(R_{x,z} - R_{z,x}) \\ +\, \underset{\sim}{e}_z\,(R_{y,x} - R_{x,y}) \end{array}$$

$$L = R_{z,y}{}^2 + R_{y,z}{}^2 + R_{x,z}{}^2 + R_{z,x}{}^2 + R_{y,x}{}^2 + R_{x,y}{}^2 - 2\, R_{z,y}R_{y,z}$$

$$- 2\, R_{x,z}R_{z,x} - 2\, E_{y,x}R_{x,y}$$

$$- (\lambda + 1)\alpha\, \{R_x(R_{z,y} - R_{y,z}) + R_y(R_{x,z} - R_{z,x}) + R_z(R_{y,x} - R_{x,y})\}$$

$$\frac{\partial L}{\partial R_x} = -(\lambda + 1)\alpha\, (R_{z,y} - R_{y,z}) = -(\lambda + 1)\, \alpha\, (\underset{\sim}{\nabla} \times \underset{\sim}{R})_x$$

$$\frac{\partial L}{\partial R_{xx}} = 0$$

$$\frac{\partial L}{\partial R_{xy}} = 2(R_{x,y} - R_{y,x}) + (\lambda + 1)\alpha\, R_z = -2(\nabla \times R)_z + (\lambda + 1)\alpha R_z$$

$$\frac{\partial L}{\partial R_{xz}} = 2(\nabla \times R)_y - (\lambda + 1)\alpha\, R_y$$

$$\frac{\partial}{\partial y}\frac{\partial L}{\partial R_{x,y}} + \frac{\partial}{\partial z}\frac{\partial L}{\partial R_{x,z}} = -2\frac{\partial}{\partial y}(\nabla \times R)_z + 2\frac{\partial}{\partial z}(\nabla \times R)_y +$$

$$+ (\lambda + 1)\ \alpha\ \left(\frac{\partial R_z}{\partial y} - \frac{\partial R_y}{\partial z}\right)$$

$$= -2(\nabla \times (\nabla \times R))_x + (\lambda + 1)\ \alpha\ (\nabla \times R)_x$$

(take α = constant)

The Euler Lagrange equations are

$$-2(\nabla \times (\nabla \times R)) + (\lambda + 1)\ \alpha\ (\nabla \times R) = -(\lambda + 1)\ \alpha\ (\nabla \times R)$$

$$\nabla \times (\nabla \times R) = (\lambda + 1)\ \alpha\ \nabla \times R$$

The solution of this gives the minimum (or maximum) δW. Use the vector identity

$$b \cdot (\nabla \times a) = \nabla \cdot (a \times b) + a \cdot (\nabla \times b)$$

or, $(\nabla \times R) \cdot (\nabla \times R) = \nabla \cdot (R \times [\nabla \cdot R]] + R \cdot (\nabla \times [\nabla \times R])$

$$\alpha R \cdot (\nabla \times R) = \frac{1}{\lambda+1} R \cdot (\nabla \times \nabla \times R) = \frac{1}{\lambda+1}[(\nabla \times R)^2 + \nabla \cdot ([\nabla \times R] \cdot R)]$$

Substituting in the energy equation

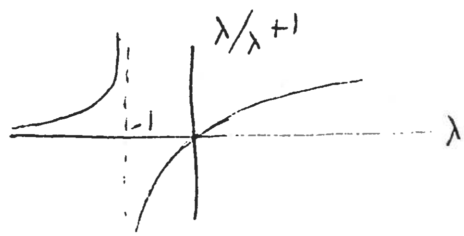
$$\delta W = \frac{1}{8\pi} \int [(\nabla \times R)^2 - \frac{1}{\lambda+1} \{(\nabla \times R)^2 + \nabla \cdot ([\nabla \times R] \times R)\}]d\tau$$

(recall $R = \xi \times B$)

at the boundary $\underset{\sim}{B}$ and $\underset{\sim}{\xi}$ are ll to the surface. Therefore R $\perp$ to the surface and $[\underset{\sim}{\nabla} \times \underset{\sim}{R}] \times \underset{\sim}{R}$ lies in the surface. The divergence term therefore vanishes.

$$\delta W = \frac{1}{8\pi} \frac{\lambda}{\lambda + 1} \int (\underset{\sim}{\nabla} \times \underset{\sim}{R})^2 d\tau$$

This is only negative if $-1 < \lambda < 0$



To solve for λ we must solve for R

$$\underset{\sim}{\nabla} \times [\underset{\sim}{\nabla} \times \underset{\sim}{R}] = (\lambda + 1)\, \alpha\, \underset{\sim}{\nabla} \times \underset{\sim}{R}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{Q} = \alpha(\lambda + 1)\, \underset{\sim}{Q} = \beta \underset{\sim}{Q}$$

$$\frac{1}{r}\frac{\partial Q_z}{\partial \phi} - \frac{\partial Q_z}{\partial z} = \beta Q_r$$

$$\frac{\partial Q_r}{\partial z} - \frac{\partial Q_z}{\partial r} = \beta Q_\phi$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(rQ_\phi) - \frac{\partial Q_r}{\partial \phi}\right] = \beta Q_z$$

Look for solutions that go like $e^{i(kZ+m\phi)}$ in cylindrical coordinates

$$\frac{imQ_z}{r} - ikQ_\phi = \beta Q_r$$

$$ikQ_r - \frac{\partial}{\partial r} Q_z = \beta Q_\phi$$

$$\frac{1}{r} [\frac{\partial}{\partial r} (rQ_\phi) - imQ_r] = \beta Q_z$$

$$\beta Q_r + ikQ_\phi = \frac{imQ_z}{r}$$

$$ikQ_r - \beta Q = \frac{\partial Q_z}{\partial r}$$

$$Q_\phi = \frac{\beta \frac{\partial Q_z}{\partial r} + \frac{kmQ_z}{r}}{k^2 - \beta^2}$$

$$Q_r = \frac{\frac{im\beta Q_z}{r} + ik \frac{\partial Q_z}{\partial r}}{k^2 - \beta^2}$$

$$\frac{1}{r} [\frac{\partial}{\partial r} \{\beta r \frac{\frac{\partial Q_z}{\partial r} + km Q_z}{k^2 - \beta^2}\} - \frac{\frac{m^2 \beta Q_z}{r} - mk \frac{\partial Q_z}{\partial r}}{k^2 - \beta^2}] = \beta Q_z$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial Q_z}{\partial r} - \frac{m^2 Q_z}{r^2} - (k^2 - \beta^2) Q_z = 0$$

$$\frac{\partial^2 Q_z}{r^2} + \frac{1}{r} \frac{\partial Q_z}{\partial r} - [\frac{m^2}{r^2} + \gamma^2] Q_z = 0 \quad , \quad \gamma^2 = k^2 - \beta^2$$

Let $\rho = \gamma r$, $\frac{\partial}{\partial r} = \frac{\partial}{\partial \rho} \frac{\partial \rho}{\partial r} = \gamma \frac{\partial}{\partial \rho}$

$$\gamma^2 \frac{\partial^2 Q_z}{\partial \rho^2} + \frac{r^2}{\rho} \frac{\partial Q_z}{\partial \rho} - \gamma^2 [\frac{m^2}{\rho^2} + 1] Q_z = 0$$

$$Q_z = I_m[r\ (k^2 - \beta^2)^{1/2}]$$

If $\gamma^2 < 0$, we get

$$Q_z = J_m[r\ (k^2 - \beta^2)^{1/2}] = J_m\ [r\ (\alpha^2(\lambda \pm 1)^2 - k^2)^{1/2}]$$

$$\alpha^2(\lambda + 1)^2 - k^2 = \frac{Z_m^{\ 2}}{r_o^{\ 2}}$$

$$(\lambda + 1)^2 - \frac{1}{\alpha^2}\ [k^2 + \frac{Z_m^{\ 2}}{r_o^{\ 2}}]$$

$$\lambda = -1 \pm \frac{1}{\alpha}\ (k^2 + \frac{Z_m^{\ 2}}{r_o^{\ 2}})^{1/2}$$

unstable if $\frac{1}{\alpha}\ (k^2 + \frac{Z_m^{\ 2}}{r_o^{\ 2}})^{1/2} < 1$

or since we can make κ as small as we like,

unstable if $\frac{Z_m}{\alpha r} < 1$ or $r < \frac{Z_m}{\alpha}$

α is the scale length for B_o so it is unstable if the column extends beyond the first 0.

Interchange Instability

1. Stability of incompressible plasma
2. Contribution of compressibility

We have shown that the potential energy of a plasma magentic field system changes according to

$$\delta W = \frac{1}{2} \int_{\text{plasma}} \{(\nabla \cdot \xi)\xi \cdot \nabla P_o + \gamma P_o(\nabla \cdot \xi)^2 + \frac{Q^2}{4\pi} + \frac{1}{4\pi} (\nabla \times B_o) \cdot (\xi \times Q)\} \, d\tau$$

$$- \frac{1}{2} \int_{\text{surface}} \{\xi(\xi \cdot \nabla P_o + \gamma P_o \nabla \cdot \xi) + \frac{(\xi \times B_o) \times Q}{4\pi}\} \cdot ds$$

Where the surface term can be converted to

$$+ \frac{1}{2} \int_S (\xi \cdot n)^2 \, [\nabla (P_o + \frac{B_o^2}{8\pi})] \cdot ds + \frac{1}{8\pi} \int_{\text{vacuum}} B_{iv}^2 \, d\tau$$

jumps across the boundary outside value - inside value

Putting $\gamma = \infty$ (incompressible) can only increase the stability of the system. For an incompressible plasma all perturbations must have $\nabla \cdot \xi = 0$ or the plasma energy would increase by an ∞ amount.

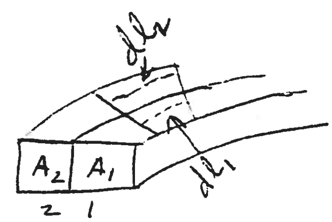
Replacing the vacuum region by a pressureless fluid increases the stability.

Introducing rigid perfectly conducting walls can only increase the stability.

Any constraint on the system inhibits the type of distortions the system is free to make and hence there are fewer ways it can move to decrease its energy.

Interchange Instability

Consider the plasma as incompressible since this only increases the stability. Interchange two flux tubes of equal-volume.



$$\delta W_M = \delta \int \frac{B_1^{\ 2}}{8\pi} A_1 d\ell_1 + \delta \int \frac{B_2^{\ 2}}{8\pi} A_2 d\ell_2$$

$$= \delta \left[\frac{\phi^2}{8\pi} \int \frac{d\ell}{A}\right]_1 + \delta \left[\frac{\phi^2}{8\pi} \int \frac{d\ell}{A}\right]_2$$

$$\delta W_M = \frac{1}{8\pi} \left[\phi_1^{\ 2} \left(\int \frac{d\ell_2}{A_2} - \int \frac{d\ell_1}{A_1}\right) + \phi_2^{\ 2} \left(\int \frac{d\ell_1}{A_1} - \int \frac{d\ell_2}{A_2}\right)\right]$$

$$= \frac{1}{8\pi} (\phi_2^{\ 2} - \phi_1^{\ 2}) \left(\int \frac{d\ell_1}{A_1} - \int \frac{d\ell_2}{A_2}\right)$$

$$= - \frac{\delta\phi^2}{8\pi} \delta \int \frac{d\ell}{A}$$

For incompressibility $A_1 \, d\ell_1 = A_2 d\ell_2$ [The element of length $d\ell$, gets interchanged with element $d\ell_2$].

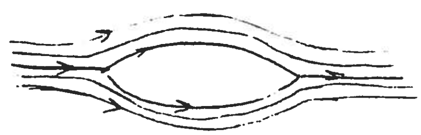
$$\delta W_M = \frac{1}{8\pi} (\phi_2^{\ 2} - \phi_1^{\ 2}) \left\{\int \left(\frac{d\ell_1}{A_1} - \frac{d\ell_2}{A_2} \frac{d\ell_r}{A_1} \frac{A_1}{d\ell_1}\right)\right\}$$

$$\delta W_M = \frac{1}{8\pi} (\phi_2^{\ 2} - \phi_1^{\ 2}) \left\{\int \frac{d\ell_1}{A_1} \left(1 - \frac{d\ell_2^{\ 2}}{d\ell_2 \, A_2} \frac{A_1}{d\ell_1^{\ 2}} \, d\ell_1\right)\right\}$$

$$= \frac{1}{8\pi} (\phi_2^{\ 2} - \phi_1^{\ 2}) \left\{\int \frac{d\ell_1}{A_1} \left(1 - \frac{d\ell_2^{\ 2}}{d\ell_1^{\ 2}}\right)\right\}$$

I. Mirror stability

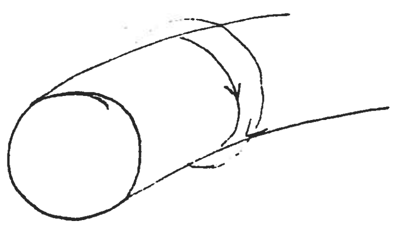
Imagine the field interpenetrates a thin region of the plasma, inside the plasma B is 0

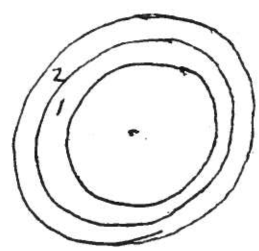


B increases rapidly outward, i.e. $\phi_2 > \phi_1$

$d\ell_2 > d\ell_1$ therefore $\delta W_M < 0$ and the system is MHD unstable.

II. Pinch, thin transition layer





$d\ell_2 = r_2 d\theta \qquad d\ell_1 = r_1 d\theta$

B increases outward, i.e. $\phi_2 > \phi_1$ and

$$\delta W_M = \frac{1}{8\pi}(\phi_2{}^2 - \phi_1{}^2)\left\{\int \frac{d\ell\, r_1 d\theta}{A_1}\left(1 - \frac{r_2^{\,2}}{r_1^{\,2}}\right)\right\} < 0$$

III. Uniform Current Pinch

$$B = \frac{\pi j r^2}{2\pi\, c} = \frac{j}{2c} r, \quad A_1 B_1 = \phi_1 \quad , \qquad A_2 B_2 = \phi_2$$

$$A_1 \frac{j\Lambda_1}{2c} = \phi_1 \quad , \quad A_2 \frac{j\Lambda_2}{2c} = \phi_2$$

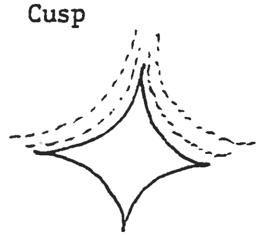
$$A_1 \Lambda_1 2\pi = V_1 \quad , \quad A_2 \Lambda_2 2\pi = V_2$$

$$V_1 = V_2$$

Therefore $\phi_2 = \phi_1$ and $\delta W_M = 0$.

Incompressible motions do not change the energy. It might still be unstable if compressibility were included.

IV. Cusp



$$\phi_2 > \phi_1, \; d\ell_2 < d\ell_1, \; \delta W > 0$$

stable to incompressible motion.

V. Change in energy due to pressure

$$PV^{\gamma} = P_o V_o^{\gamma}$$

$$\delta W = - \int_{V_o}^{V} PdV = - \int_{V_o}^{V} \frac{P_o V_o^{\gamma}}{V^{\gamma}} dV = \frac{P_o V_o^{\gamma}}{\gamma - 1} \left[\frac{1}{V^{\gamma - 1}}\right]_{V_o}^{V}$$

$$= \frac{P_o V_o^{\gamma}}{(\gamma - 1)} \left[\frac{1}{V^{\gamma-1}} - \frac{1}{V_o^{\ \gamma-1}}\right]$$

$$\delta W_P = \frac{P_1V_1{}^\gamma}{\gamma - 1}\left[\frac{1}{V_2{}^{\gamma-1}} - \frac{1}{V_1{}^{\gamma-1}}\right] + \frac{P_2V_2{}^\gamma}{\gamma - 1}\left[\frac{1}{V_1{}^{\gamma-1}} - \frac{1}{V_2{}^{\gamma-1}}\right]$$

$$= \frac{1}{\gamma - 1}[P_2V_2{}^\gamma - P_1V_1{}^\gamma]\left[\frac{1}{V_1{}^{\gamma-1}} - \frac{1}{V_2{}^{\gamma-1}}\right]$$

$$\frac{1}{V_1{}^{\gamma-1}} - \frac{1}{V_2{}^{\gamma-1}} = \frac{1}{V_1{}^{\gamma-1}} - \frac{1}{(V_1 + \delta V)^{\gamma-1}} = (1 - \gamma)\frac{\delta V}{V_1{}^\gamma}$$

$$\delta W_P = \delta(PV^\gamma)\delta V/V^\gamma$$

If we interchange two flux tubes of equal flux $\phi_1 = \phi_2$ and the magnetic energy chance is 0 so the only energy change is due to the pressure.

$$\delta W_P = \frac{\delta(PV^\gamma)\delta V}{V^\gamma} = \frac{\delta V(\ PV^\gamma + \gamma PV^{\gamma-1}\delta V)}{V^\gamma}$$

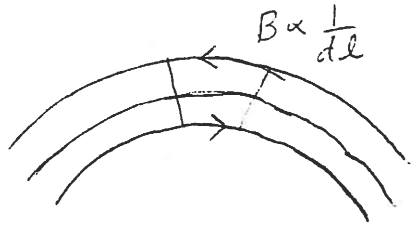
$$= \delta V(\delta P + \frac{\gamma P\delta V}{V}) \quad \begin{array}{ll} > 0 & \text{stability} \\ < 0 & \text{instability} \end{array}$$

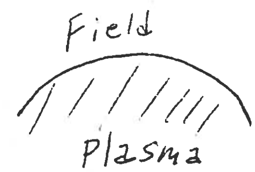
As one goes near the walls $P \to 0$. The P term can be ignored, $\delta P < 0$; therefore, we have instability if $\delta V > 0$, that is if the volume of a flux tube increases outward or if

$$\delta \int A d\ell = \delta \int AB\frac{d\ell}{B} = \phi\ \delta \int \frac{d\ell}{B}$$

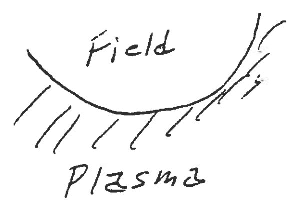
Get instability if $\int \frac{d\ell}{B}$ increases outward

If $B \alpha \frac{1}{d\ell}$ then the contribution from this goes like dl^2.





Unstable



Stable

Let δV increase outward

Then get stability if

$$\delta P + \frac{\gamma P}{V}\,\delta V \geq 0$$

$$\frac{\delta P}{P} = -\gamma\,\frac{\delta V}{V}$$

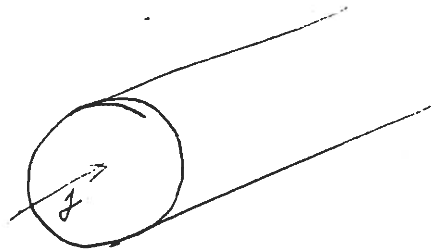
$$\ell n \frac{P}{P_o} = -\ell n\,\frac{V^{\gamma}}{V_o^{\ \gamma}}$$

$$P = P_o\left(\frac{V_o}{V}\right)^{\gamma}$$

Thus, even though to achieve absolute stability with the plasma isolated from the walls δV must be negative if V increases outward one can have a rapid drop in P as one goes outward.

How are the above stability conditions affected if one uses the double adiabatic law rather than a simple adiabatic law?

Consider a cylindrical column of plasma carrying current, J(r)



Consider the interchange of two neighboring flux tubes. Find the current distributions and pressure distribution which gives stability.

VI. Application of the Equations of Motion

$$\rho_o \ddot{\boldsymbol{\xi}} = \mathbf{F}(\boldsymbol{\xi})$$

$$\mathbf{F}(\boldsymbol{\xi}) = \boldsymbol{\nabla}\{\boldsymbol{\xi} \cdot \boldsymbol{\nabla} P_o + \gamma P_o \boldsymbol{\nabla} \cdot \boldsymbol{\xi}\} + \frac{1}{4\pi}\{[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}_o))] \times \mathbf{B}_o$$

$$+ (\boldsymbol{\nabla} \times \mathbf{B}_o) \times \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}_o)\}$$

Consider the simple case of an infinite uniform plasma with straight field lines

$$\boldsymbol{\nabla} P_o = 0, \qquad \boldsymbol{\nabla} \times B_o = 0$$

$$\mathbf{F}(\boldsymbol{\xi}) = \gamma P_o \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{\xi} + \frac{1}{4\pi}(\boldsymbol{\nabla} \times \mathbf{Q}) \times \mathbf{B}_o$$

$$\mathbf{Q} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}_o) = \boldsymbol{\xi}(\boldsymbol{\nabla} \cdot \mathbf{B}_o) + (\mathbf{B}_o \cdot \boldsymbol{\nabla})\boldsymbol{\xi} - \mathbf{B}_o(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\mathbf{B}_o = 0$$

$$\mathbf{Q} = (\mathbf{B}_o \cdot \boldsymbol{\nabla})\boldsymbol{\xi} - \mathbf{B}_o(\boldsymbol{\nabla} \cdot \boldsymbol{\xi})$$

First look for solutions for which $\nabla \cdot \xi = 0$. Take B_o to be in the Z direction

$$\rho_o \ddot{\xi} = \frac{1}{4\pi} [\nabla \times (B_o \cdot \nabla \xi)] \times B_o$$

$\perp z$, therefore no z part to the acceleration, $\xi_z = 0$

$$\rho_o \ddot{\xi} = \frac{B_o^2}{4\pi} [\nabla \times \frac{\partial \xi}{\partial z}] \times \hat{\ell}_z = - \frac{B_o^2}{4\pi} \frac{\partial^2 \xi}{\partial_z 2}$$

$$\omega^2 = \frac{k_z^2 B_o^2}{4\pi \rho^2} = k_z^2 V_A^2$$

For compressible motions look for solutions that go like $e^{ik \cdot r}$

$$-\rho_o \omega^2 \xi = -\gamma \rho_o k\, k \cdot \xi - \frac{B_o^2}{4\pi} \{[k \times (k \times [\xi \times \hat{\ell}_z])] \times \hat{\ell}_z\} (k \times [(-k \cdot \xi) \hat{\ell}_z + k_z \xi]) \times \hat{\ell}_z$$

$$-k \cdot \xi \ (k \times \ell_z) \times \ell_z + k_z (k \times \xi) \times \hat{\ell}_z \ (\hat{\ell}_z k_z - k) + k_z \ (k_z \xi - \xi_z k)$$

$$-\rho_o \omega^2 \xi = -\gamma P_o kk \cdot \xi - \frac{B_o^2}{4\pi} \{-(\hat{\ell}_z k_z - k) \ k \cdot \xi + k_z^2 \xi - kk_z \xi_z\}$$

$$= -\gamma \rho_o kk \cdot \xi - \frac{B_o^2}{4\pi} \{-\hat{\ell}_z k_z k \cdot \xi + k(k \cdot \xi - k_z \xi_z)\}$$

$$\rho_o \omega^2 \xi = \gamma \rho_o k (k \cdot \xi) + \frac{1}{4\pi} [k \times (B \cdot k) \xi_o] \times B_o - \frac{1}{4\pi} (k \times B [k \cdot \xi_o]) \times B$$

$k || B$

$$\rho_o \omega^2 \xi = \gamma \rho_o k^2 \xi_{\parallel} \hat{\ell}_{\parallel} + \frac{B_o^2}{4\pi} k^2 \xi_{\perp} \hat{\ell}_{\perp}$$

two root

$$\omega^2 = \frac{\gamma \rho_o k^2}{P} \qquad \xi || B$$

$$\omega^2 = \frac{B^2 k^2}{4\pi\rho_o} \qquad \xi \perp B$$

$k_\perp B$

$$P_o \omega^2 \underset{\sim}{\xi} = \rho_o \underset{\sim}{k}(\underset{\sim}{k} \cdot \underset{\sim}{\xi}) + \frac{\underset{\sim}{k}}{4\pi} B_o^2 (\underset{\sim}{k} \cdot \underset{\sim}{\xi})$$

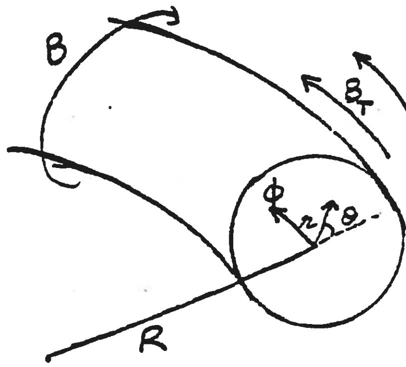
The right hand side is $\parallel$ to $\underset{\sim}{k}$ so only motion $\parallel$ $\underset{\sim}{k}$ enters

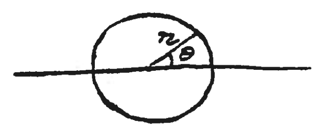
$$\omega^2 = \frac{\gamma P_o k^2 + \frac{B_o^2 k^2}{4\pi}}{\rho_o} = k^2 \left[\frac{\gamma P_o}{\rho_o} + \frac{B_o^2}{4\pi\rho_o}\right]$$

Problem: Find the dispersion relation for arbitrary direction of propagation of the waves. Compute the phase and group velocities.

Neoclassical Diffusion in a Torus (Tokamak)

We looked at diffusion of plasma across a B-field of straight lines. Let us now look at diffusion in a more complex field, that of a Tokamak. We consider it to be a circular torus carrying a uniform current across its cross-section.





The field lines are helical about the minor axis at least in the limit of the radius being small compared to R.

If we neglect particle drifts then the particles will follow the field lines. Because the toroidal field goes as

$$B = \frac{B_0 R}{R + r\cos\theta} \approx B_0\left(1 - \frac{r}{R}\cos\theta\right)$$

Neglecting the contribution of the poloidal field to the strength of B, we can take the strength of B to go as given by this formula. Because the strength of the field increases at the inside, particles following the field lines will see the field increase as they go, towards the center and some of them will be mirror reflected.

$$m\frac{dv_{\parallel}}{dt} = -\mu\frac{dB}{ds}$$

$$v_{\parallel}^2 + v_{\perp}^2 = W$$

$$\frac{v_{\perp}^2}{B} = \text{constant}$$

Let our reference point be on the outside of the torus.

$$v_{\perp in}^2 = v_{\perp out}^2 \frac{B_{MAX}}{B_{Min}}$$

For particles which just reflect

$$v_{\perp out}^2 \frac{B_{Max}}{B_{Min}} = v_{\perp out}^2 + v_{\parallel out}^2$$

$$v_{\parallel out}^2 = v_{\perp out}^2 \left(\frac{B_{Max}}{B_{Min}} - 1 \right)$$

$$v_{\parallel out} = v_{\perp out} \sqrt{\frac{B_{Max}}{B_{Min}} - 1}$$

$$B_{Max} = B_0 \left(1 + \frac{r}{R} \right)$$

$$B_{Min} = B_0 \left(1 - \frac{r}{R} \right)$$

$$\frac{B_{Max}}{B_{Min}} = 1 + \frac{2r}{R}$$

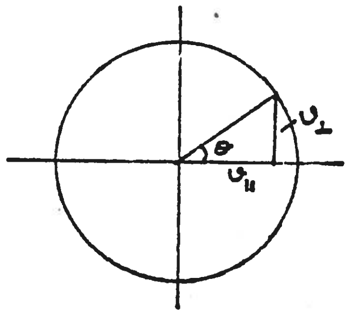
$$v_{\parallel out} = v_{\perp out} \sqrt{\frac{2r}{R}}$$

Even for rather small values of r, $v_\parallel$ is rather large. For r = .1R

$$v_{\parallel out} = v_{\perp out} \sqrt{.2} = .45\, v_{\perp out}$$

For Taylor's tokamak R = 2.5r and $v_\parallel \approx v_\perp$.

This implies a large fraction of the particles are reflected. The fraction that is reflected is equal to the solid angle



The angle dividing passing from Reflecting

$$\cos\theta = \frac{v_\parallel}{\sqrt{v_\parallel^2 + v_\perp^2}} = \frac{\cancel{v_\perp}\sqrt{\frac{2r}{R}}}{\cancel{v_\perp}\sqrt{1+\frac{2r}{R}}}$$

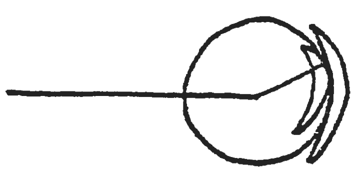
The fraction that is reflected is

$$F = \frac{4\pi - 2\int_0^\theta 2\pi \sin\theta \, d\theta}{4\pi} = 1 - 1 + \cos\theta = \cos\theta$$

$$F = \frac{\sqrt{\frac{2r}{R}}}{\sqrt{1+\frac{2r}{R}}}$$

For $r = .1R$ $F = .4$

As for the trapped particles, they will oscillate back and forth along the field lines if we neglect the drifts. When toroidal drifts are included the particles will move off the field lines. Their orbits in the r, θ plane are in the shape of bananas.



$$\omega_B^2 = \frac{\mu B_0 i^2 r}{mR} \quad , \quad \mu = \frac{m v_\perp^2}{B_0}$$

$$\omega_B^2 = v_\perp^2 i^2 \frac{r}{R} \quad , \quad \omega_B = v_\perp i \sqrt{\frac{r}{R}}$$

$$\Delta r = \frac{v_T^{\cancel{2}}}{\omega_c R \cancel{v_T} i} \sqrt{\frac{R}{r}}$$

Diffusion

ν_{eff} in the small r/R lim

$$\Delta r^2 \nu_{eff} = D$$

$$\Delta v_\parallel^2 = v_T^2 \nu t$$

$$\nu_{eff} = \nu \frac{R}{r}$$

$$\nu_{eff} = \frac{1}{t} = \frac{v_T^2}{\Delta v_\parallel^2} \nu = \frac{R}{2r} \nu$$

$$D = \frac{v_T^2}{\omega_c^2 \cancel{R^2} i^2} \frac{\cancel{R^2}}{r^2} \nu$$

We must multiply this by the fraction of trapped particles

$$F = \sqrt{\frac{2r}{R}}$$

$$D = \frac{\rho_e^2}{i^2 r^2} \nu \sqrt{\frac{2r}{R}}$$

The above derivation has assumed that particles make a complete transit around a banana before making a collision

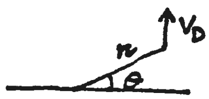
$$\tau_B \nu \frac{R}{r} < 1$$

If this is not true then the result must be modified. Suppose the effective collision frequency is much larger than the bounce frequency, then particles will execute a small fraction of a banana before they jump to another banana. The distance they will go is roughly

$$\Delta r = \Delta r_{banana} \frac{\tau_{coll.}}{\tau_b} = \frac{\Delta r_{banana}}{\nu \tau_b}$$

$$D = \frac{\Delta r^2_{banana}}{\nu^{\cancel{2}} \tau_b^2} \cancel{\nu} = \frac{\Delta r^2_{banana}}{\nu \tau_b^2}$$

We can estimate the thickness of the bananas as follows. If the particle stays on a magnetic surface, r does not change and θ oscillates back and forth. It is the displacement in r we are interested in, $\dot{r} = (v_D)_r$



$$V_D = \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{\omega_c R}$$

$$(v_D)_r = v_D \sin\theta$$

$\theta = \theta_M \sin \omega_B t$, ω_B is the bounce frequency (see discussion below)
θ_M is the maximum excursion in θ

$$\dot{r} = v_D \sin\left[\theta_M \sin \omega_B t\right] \approx v_D \theta_M \sin\omega_B t$$

$$\Delta r = \frac{v_D \theta_M}{\omega_B} \cos\omega_B t$$

The most important particles for the diffusion are those with θ_M large of the order of 1

Bouncing

$$\Delta r \approx \frac{v_D}{\omega_B} \approx \frac{v_T^2}{\omega_c R \omega_B}$$

$$m\frac{dv_\parallel}{dt} = -\mu \frac{dB}{ds}$$

$$B = B_0\left(1 + \frac{r}{R}\cos\theta\right)$$

$\theta = iz$ $\qquad$ i is the rotational transform, z is the distance traveled around the torus

$$B = B_0\left(1 + \frac{r}{R}\cos iz\right)$$

$\underset{\sim}{s} \parallel \underset{\sim}{z}$ approx.

$$\frac{dB}{dz} = B_0 \frac{ir}{R}\sin iz \approx B_0 \frac{i^2 r}{R} z$$

$$m\ddot{z} = -\mu B_0 \frac{i^2 r}{R} z$$

$$D = \cancel{\frac{v_T^2}{\omega_c^2 i^2 R r}} \quad \cancel{\frac{v_T i}{2\nu}\sqrt{\frac{r}{R}}} \quad \frac{v_T^2}{\omega_c^2 i^2 R \eta} \quad \frac{v_T^2 i^2 \eta}{R \nu}$$

$$\cancel{=} \ \cancel{\frac{v_T^3}{\omega_c^2 i\, r^{1/2} R^{3/2}\, 2\nu}} = \frac{v_T^4}{\omega_c^2 R^2 \nu}$$

A more precise description of Bananas

By circular symmetry the angular momentum is conserved

$$P_\phi = \left(m v_\phi + \frac{q}{c} A_\phi\right)(R + r\cos\theta)$$

A_ϕ is determined by

$$\nabla^2 A_\phi = -\frac{4\pi}{c} j_\phi$$

We also have conservation of energy

$$\frac{m}{2}\left(v_\perp^2 + v_\parallel^2\right) = W$$

and conservation of magnet moment

$$\frac{m v_\perp^2}{2B} \cong \frac{m v_\perp^2 (R + r\cos\theta)}{2 B_0 R}$$

$v_\parallel \approx v_\phi$ more precisely $v_\phi + irv_\theta$

Take a uniform current density

$$A_\phi = A_0 r$$

$$\left(m v_\phi + \frac{q}{c}\frac{A_0 r^2}{2}\right)(R + r\cos\theta) = P_\phi$$

$$\frac{m}{2}\left(v_\perp^2 + v_\phi^2\right) = W$$

$$\frac{m v_\perp^2}{2B_0}\left(1 + \frac{r}{R}\cos\theta\right) = \mu$$

$$v_\perp^2 = \frac{\tilde{\mu}}{1 + \frac{r}{R}\cos\theta}, \quad v_\phi = \sqrt{\frac{2W}{m} - \frac{\tilde{\mu}}{1 + \frac{r}{R}\cos\theta}}$$

$$\left\{ m\sqrt{\frac{2W}{m} - \frac{\tilde{\mu}}{1 + \frac{r}{R}\cos\theta}} + \frac{q}{c}\frac{A_0 r^2}{2} \right\}\left\{ R + r\cos\theta \right\} = P_\phi$$

gives r as a function of θ.

To a good approximation if the banana size is small

$$mv_\phi + \frac{q}{c}\frac{A_0 r_1^2}{2} = \tilde{P} = \frac{P_\phi}{R} = -mv_\phi + \frac{q}{c}\frac{A_0 r_2^2}{2}\ ,$$ for the reverse trip

$$\frac{q}{c} A_0 \frac{(r_1^2 - r_2^2)}{2} = 2mv_\phi$$

$$\frac{q}{c} A_0 r \Delta r = 2mv_\phi$$

$$\Delta r = \frac{2mv_\phi}{\frac{qA_0 r}{c}} = \frac{2mv_\phi c}{qB}$$

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CLASS NOTES

1981

Physics 222 A, B, C

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