$$D = |(\vec{r}-\vec{r}\,')_{\perp}| = [\rho^2 + \rho'^2 -$$

A complete eigenfunction decompo

addition theorem (15.33). In (1

function, K_0 , the modified Bes

integral representation

$$K_0(\kappa\rho) = \int_0^\infty kdk \frac{J_0(k\rho)}{k^2+\kappa^2} \quad .$$

In terms of this new function,

parallel plates is

$$G(\vec{r},\vec{r}\,') = \frac{4}{a}\sum_{n=1}^{\infty} \sin\frac{n\pi}{a} z \; s$$

This form becomes particularly

modified Bessel function, K_0 ,

pression [see problem 24]

$$K_0(t) \sim \sqrt{\frac{\pi}{2t}}\, e^{-t} , \qquad t \to \infty$$

Due to this exponentially decre

verse distance between the poin

plates, the first term of (16.2

$$D >> a , \qquad G \sim \sqrt{\frac{8}{Da}} \sin\frac{\pi}{a}$$

Notice that we can do the

(12.12), which applies to free

$$G(\vec{r},\vec{r}\,') = 4\pi \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{e^{ik_z(z-z')}}{k_\perp^2 + k_z^2}$$

$$= \frac{2}{\pi} \int kdk\ J_0(kD) \int_0^{\infty} dk_z \frac{\cos k_z(z-z')}{k^2 + k_z^2}$$

$$= \frac{2}{\pi} \int_0^{\infty} dk_z\ \cos k_z(z-z')\ K_0(k_z D) \quad . \tag{16.23}$$

In particular, if in (16.23) we take $\vec{r}\,' = 0$ and use ρ to denote $|\vec{r}_\perp|$, we learn that [cf. (15.4)]

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^{\infty} dk\ J_0(k\rho)\ e^{-k|z|} = \frac{2}{\pi} \int_0^{\infty} dk\ \cos kz\ K_0(k\rho) \quad . \tag{16.24}$$

Observe that the J_0 form has exponential damping in the z direction but oscillations in the ρ direction, while the K_0 form exhibits the reverse behavior, damping in ρ and oscillations in z. Since $1/r$ is a solution to Laplace's equation, for $r \neq 0$, it is impossible to get damping in all directions (since then the second derivatives would be of the same sign and could not sum to zero).

XVII. Spherical Harmonics

17-1. Solutions to Laplace's Equation

The fundamental solution to Laplace's equation in unbounded space is $1/r$,

$$\nabla^2 \frac{1}{r} = 0 \quad , \qquad r > 0 \quad . \tag{17.1}$$

In terms of this solution, we can generate a large number of others. For example, taking $\vec{a}$ to be a constant vector,

$$\nabla^2 \, (\vec{a}\cdot\vec{\nabla}) \, \frac{1}{r} = 0 \quad , \tag{17.2}$$

$\nabla^2 \, (\vec{a}\cdot\vec{\nabla}) \, f(r) =$

$\partial_i A_m B_m$

$B_m \partial_i A_m +$

$\nabla(\nabla\cdot$

$\nabla\cdot(\nabla(\vec{A}\cdot\vec{B})) =$

$\nabla\cdot$

we find

$$\vec{a}\cdot\vec{\nabla} \, \frac{1}{r} = - \frac{\vec{a}\cdot\vec{r}}{r^3} \tag{17.3}$$

$\partial_i^2 \, (\vec{A}\cdot\vec{B})$

$= \partial_i^2 A_m B_m$

$= A_m \partial_i^2 B_m$

is also a solution for $r \neq 0$. Continuing this operation, we see that

$$(\vec{a}_1\cdot\vec{\nabla})(\vec{a}_2\cdot\vec{\nabla}) \, \frac{1}{r} = \frac{3(\vec{a}_1\cdot\vec{r})(\vec{a}_2\cdot\vec{r}) - (\vec{a}_1\cdot\vec{a}_2)r^2}{r^5} \tag{17.4}$$

is yet a third solution. This process can be repeated an indefinite number of times, to yield the following solution to Laplace's equation,

$$[\vec{a}_1\cdot\vec{\nabla} \; \vec{a}_2\cdot\vec{\nabla} \; \ldots \; \vec{a}_\ell\cdot\vec{\nabla}] \, \frac{1}{r} = \frac{1}{r^{2\ell+1}} \, f_\ell(\vec{r}) \quad , \tag{17.5}$$

where $f_\ell(\vec{r})$ is a homogeneous function of $\vec{r}$ of degree ℓ. We also observe that $f_\ell(\vec{r})$ itself is a solution to Laplace's equation. [This follows from the inversion theorem, that if $\phi(\vec{r})$ is a solution to Laplace's equation, so is $\frac{1}{r}\phi\left(\frac{\vec{r}}{r^2}\right)$. See problem 22.]

Thus, our attention is directed to solutions that are homogeneous polynomials of degree ℓ . The above construction provides examples for $\ell = 0,1,2$. (Here we denote (x,y,z) by (x_1,x_2,x_3).)

ℓ	f_ℓ	number of independent solutions
0	1	1
1	x_1,x_2,x_3	3
2	$3x_mx_n - \delta_{mn}r^2$	5

Why are there only five independent solutions for $\ell = 2$? A symmetrical tensor has six independent components but because of the constraint that the tensor satisfies Laplace's equation, it must be traceless, leaving but five independent components.

The general polynomial of degree ℓ can be constructed from the monomials

$$x_1^{k_1}\, x_2^{k_2}\, x_3^{k_3} , \qquad k_1 + k_2 + k_3 = \ell .$$

How many of these monomials are there? To answer this, we first ask the analogous question in two dimensions: how many monomials of the form

$$x_1^{k_1}\, x_2^{k_2}$$

are there with $k_1 + k_2 = n$? The answer to this question is simple since if k_1 goes from 0 to n , k_2 must go from n to 0 , giving $n+1$ possibilities

Thus to answer our three-dimensional question, we first assign a definite value to k_3 ,

$$k_1 + k_2 = \ell - k_3 \quad .$$

The number of monomials with this value of k_3 is

$$\ell - k_3 + 1 \quad ,$$

so the number of homogeneous polynomials of degree ℓ is

$$\sum_{k_3=0}^{\ell} (\ell - k_3 + 1) = \frac{1}{2}(\ell+1)(\ell+2) \quad .$$

From this set of polynomials, we wish to find those combinations which are solutions to Laplace's equation. Since ∇^2 acting on a homogeneous polynomial of degree ℓ produces a homogeneous polynomial of degree $\ell-2$, of which there are

$$\frac{1}{2}(\ell-2+1)(\ell-2+2) = \frac{1}{2}\ell(\ell-1)$$

independent ones, there are $\frac{1}{2}\ell(\ell-1)$ restrictions on the polynomials, that is, the number of independent solutions to Laplace's equation of degree ℓ is

$$\frac{1}{2}(\ell+1)(\ell+2) - \frac{1}{2}\ell(\ell-1) = 2\ell+1 \quad .$$

For the cases $\ell = 0,1,2$, this agrees with what we found above. The solutions we find in this way are called <u>solid harmonics</u>, $Y_\ell(\vec{r})$. To emphasize the fact that they are homogeneous polynomials of degree ℓ , the solid harmonic may be written in terms of a surface (or spherical) harmonic, $Y_\ell\left(\frac{\vec{r}}{r}\right)$:

$$Y_\ell(\vec{r}) = r^\ell Y_\ell\left(\frac{\vec{r}}{r}\right) \tag{17.6a}$$

$$\rightarrow \frac{1}{r^{\ell+1}} Y_\ell\left(\frac{\vec{r}}{r}\right) \tag{17.6b}$$

where the latter form, also a solid harmonic, results from inversion and is the solution constructed in (17.5).

17-2. Spherical Harmonics

Our next task is to devise a way to systematically and conveniently generate the spherical harmonics as functions of the spherical angles θ and ϕ. We first ask under what condition is the polynomial,

$$(\vec{a}\cdot\vec{r})^\ell \quad , \tag{17.7}$$

a solution to Laplace's equation? Since

$$\vec{\nabla}(\vec{a}\cdot\vec{r})^\ell = \ell(\vec{a}\cdot\vec{r})^{\ell-1}\,\vec{a} \quad ,$$

we see that the Laplacian acting on this polynomial is

$$\nabla^2(\vec{a}\cdot\vec{r})^\ell = \ell(\ell-1)(\vec{a}\cdot\vec{r})^{\ell-2}\,\vec{a}^2 \quad ,$$

which is Laplace's equation if $\vec{a}^2$ is zero (necessarily, $\vec{a}$ must then be complex). A convenient way to write $\vec{a}^2$ is

$$\vec{a}^2 = (a_1 - ia_2)(a_1 + ia_2) + a_3^2 \quad , \tag{17.8}$$

suggesting that the condition that $\vec{a}^2$ be zero can be automatically satisfied if we write

$$a_1 + ia_2 = 2\xi_-^{\;2} \quad ,$$

$$a_1 - ia_2 = -2\xi_+^{\;2} \quad ,$$

$$a_3 = 2\xi_+\xi_- \quad , \qquad (17.9)$$

where $\xi_\pm$ are two arbitrary complex numbers. Then we have

$$\vec{a}\cdot\frac{\vec{r}}{r} = \frac{1}{2}\,(a_1-ia_2)\,\frac{x+iy}{r} + \frac{1}{2}\,(a_1+ia_2)\,\frac{x-iy}{r} + a_3\,\frac{z}{r}$$

$$= -\xi_+^{\;2}\,\sin\theta\; e^{i\phi} + \xi_-^{\;2}\,\sin\theta\; e^{-i\phi} + 2\xi_+\xi_-\,\cos\theta \quad , \qquad (17.10)$$

and the polynomial (17.7),

$$(\vec{a}\cdot\vec{r})^\ell = r^\ell\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^\ell \quad ,$$

can be rewritten in terms of $\xi_\pm$ as

$$\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^\ell = \left(\frac{\xi_+^{\;2}\,e^{i\phi}}{\sin\theta}\right)^\ell \left[\left(\frac{\xi_-}{\xi_+}\,\sin\theta\; e^{-i\phi}\right)^2 + 2\,\frac{\xi_-}{\xi_+}\,\sin\theta\; e^{-i\phi}\,\cos\theta - \sin^2\theta\right]^\ell$$

$$= \left(\frac{\xi_+^{\;2}\,e^{i\phi}}{\sin\theta}\right)^\ell \left[\left(\frac{\xi_-}{\xi_+}\,\sin\theta\; e^{-i\phi} + \cos\theta\right)^2 - 1\right]^\ell$$

$$= \frac{\xi_+^{\;2\ell}\,e^{i\phi\ell}}{\sin^\ell\theta}\sum_{m=-\ell}^{\ell}\frac{\left(\frac{\xi_-}{\xi_+}\,\sin\theta\; e^{-i\phi}\right)^{\ell-m}}{(\ell-m)!}\left[\frac{d}{d(\cos\theta)}\right]^{\ell-m}(\cos^2\theta-1)^\ell \quad ,$$

where, in the last line, we have employed a convenient form of a Taylor expansion. In this way, we have constructed, from polynomials of degree ℓ, $2\ell+1$ independent functions which are solutions to Laplace's equation, the coefficients of the powers of $\xi_\pm$ in the expansion

$$\left[\vec{a} \cdot \frac{\vec{r}}{r}\right]^{\ell} = 2^{\ell}\ell! \sum_{m=-\ell}^{\ell} \frac{\xi_+^{\ell+m}\,\xi_-^{\ell-m}}{\sqrt{(\ell+m)!\,(\ell-m)!}} \sqrt{\frac{(\ell+m)!}{(\ell-m)!}}\,(\sin\theta)^{-m}\,e^{im\phi}$$

$$\times \left[\frac{d}{d(\cos\theta)}\right]^{\ell-m} \frac{(\cos^2\theta-1)^{\ell}}{2^{\ell}\ell!} \quad . \tag{17.12}$$

Lecture 15

All we need is a normalization factor in order to define the spherical harmonics. Employing the notation for the monomials for $\theta \to \pi \to (-1)^{\ell}\frac{\xi_+^{\ell}\xi_-^{\ell}}{\ell!}$

$$\frac{\xi_+^{\ell+m}\,\xi_-^{\ell-m}}{\sqrt{(\ell+m)!\,(\ell-m)!}} = \psi_{\ell m} \quad , \qquad \underset{\theta \to 0}{\psi_{\ell m} Y_{\ell m}} \to \frac{1}{2^{\ell}\ell!}\left[2\xi_+\xi_-\right]^{\ell} \tag{17.13}$$

$= \xi_+^{\ell}\xi_-^{\ell}/\ell!$

we obtain the generating function for the spherical harmonics, $Y_{\ell m}(\theta,\phi)$,

$$\frac{(\vec{a}\cdot\vec{r})^{\ell}}{2^{\ell}} = r^{\ell} \sum_{m=-\ell}^{\ell} \psi_{\ell m}\sqrt{\frac{4\pi}{2\ell+1}}\,Y_{\ell m}(\theta,\phi) \quad , \tag{17.14}$$

where, according to (17.12),

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell+m)!}{(\ell-m)!}}\,e^{im\phi}\,(\sin\theta)^{-m}\left[\frac{d}{d(\cos\theta)}\right]^{\ell-m}\frac{(\cos^2\theta-1)^{\ell}}{2^{\ell}\ell!} \quad , \tag{17.15}$$

in which $-\ell \le m \le \ell$, with $\ell = 0,1,2,\ldots$. An alternative form can be derived by noting that the left hand side of (17.14) is unaltered by the transformation [see (17.10)]

$$\xi_+ \leftrightarrow \xi_- \ , \quad \theta \to -\theta \ , \quad \phi \to -\phi \quad ,$$

which implies that the spherical harmonics must remain unchanged under the substitutions

$$m \to -m \ , \quad \theta \to -\theta \ , \quad \phi \to -\phi \ .$$

In this way, we learn that

$$Y_{\ell m}(\theta,\phi) = Y_{\ell,-m}(-\theta,-\phi) \quad , \tag{17.16}$$

or, using the explicit form (17.15), we obtain the alternate version

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, e^{im\phi}\, (-\sin\theta)^m \left[\frac{d}{d(\cos\theta)}\right]^{\ell+m} \frac{(\cos^2\theta-1)^\ell}{2^\ell \ell!} \ . \tag{17.17}$$

Sometimes it is convenient to separate $Y_{\ell m}$ into its θ and ϕ dependences,

$$Y_{\ell m}(\theta,\phi) = \frac{e^{im\phi}}{\sqrt{2\pi}}\, \Theta_{\ell m}(\theta) \quad , \tag{17.18}$$

where

$$\Theta_{\ell m}(\theta) = \sqrt{\frac{2\ell+1}{2}}\sqrt{\frac{(\ell\pm m)!}{(\ell\mp m)!}}\, (\pm\sin\theta)^{\mp m} \left[\frac{d}{d(\cos\theta)}\right]^{\ell\mp m} \frac{(\cos^2\theta-1)^\ell}{2^\ell \ell!} \ . \tag{17.19}$$

17-3. Orthonormality Condition

The particular factors that occur in the definition of $Y_{\ell m}$ are such as to make the spherical harmonics an orthonormal set of functions. To see this, consider the product of two generating functions, with parameters $\vec{a}$ and $\vec{a}^*$, integrated over all angles:

$$\int d\Omega \left(\vec{a}^* \cdot \frac{\vec{r}}{r}\right)^\ell \left(\vec{a} \cdot \frac{\vec{r}}{r}\right)^{\ell'} \tag{17.20}$$

with

$$d\Omega = \sin\theta \, d\theta \, d\phi \, . \tag{17.21}$$

This integral can only contain rotationally invariant combinations, that is, it has to be a function of scalars constructed from $\vec{a}$ and $\vec{a}^*$. Since $\vec{a}^2 = \vec{a}^{*2} = 0$, the only such scalar is $\vec{a}^* \cdot \vec{a}$. Therefore, there must be an equal number of factors of $\vec{a}$ and $\vec{a}^*$, which means that the integral (17.20) is zero unless $\ell = \ell'$; we have

$$\int d\Omega \left(\vec{a}^* \cdot \frac{\vec{r}}{r} \right)^{\ell} \left(\vec{a} \cdot \frac{\vec{r}}{r} \right)^{\ell'} = \delta_{\ell\ell'} \, C_\ell (\vec{a}^* \cdot \vec{a})^{\ell} \, . \tag{17.22}$$

To calculate C_ℓ we consider a particular form of $\vec{a}$:

$$\vec{a} = (1, i, 0) \, , \quad \vec{a}^* = (1, -i, 0) \, .$$

The quantities appearing in (17.22) are then

$$\vec{a}^* \cdot \frac{\vec{r}}{r} = \sin\theta \, e^{-i\phi} \, , \quad \vec{a} \cdot \frac{\vec{r}}{r} = \sin\theta \, e^{i\phi} \, ,$$

$$\vec{a}^* \cdot \vec{a} = 2 \, ,$$

implying that the integral, (17.22), for $\ell = \ell'$, is

$$\begin{aligned} C_\ell 2^\ell &= \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, (\sin\theta \, e^{-i\phi})^\ell \, (\sin\theta \, e^{i\phi})^\ell \\ &= 2\pi \int_{-1}^{1} d(\cos\theta) \, (1-\cos^2\theta)^\ell \\ &= 4\pi \int_0^1 dx (1-x^2)^\ell \\ &= 4\pi \frac{[2^\ell \ell!]^2}{(2\ell+1)!} \, . \end{aligned} \tag{17.23}$$

The final integral in (17.23) is evaluated as follows. Defining

$$I_\ell = \int_0^1 dx(1-x^2)^\ell \ , \quad I_0 = 1 \ ,$$

we integrate by parts, for $\ell > 0$, to derive the recursion formula

$$I_\ell = 2\ell \, I_{\ell-1} - 2\ell \, I_\ell \ ,$$

which implies that

$$\begin{aligned} I_\ell &= \frac{2\ell}{2\ell+1} I_{\ell-1} \\ &= \frac{2\ell}{2\ell+1} \frac{2\ell-2}{2\ell-1} \frac{2\ell-4}{2\ell-3} \cdots I_0 \\ &= \frac{[2^\ell \ell!]^2}{(2\ell+1)!} \ , \end{aligned}$$

the last form being valid for $\ell \geq 0$. Alternatively, we could evaluate I_ℓ in terms of the beta function, $B(m,n)$,

$$\begin{aligned} I_\ell &= \frac{1}{2} \int_{-1}^{1} dx(1-x)^\ell \, (1+x)^\ell \\ &= 2^{2\ell} \int_0^1 dt \, t^\ell (1-t)^\ell \qquad \left[t = \frac{1-x}{2} \right] \\ &= 2^{2\ell} B(\ell+1, \ell+1) \\ &= 2^{2\ell} \frac{\ell! \, \ell!}{(2\ell+1)!} \end{aligned}$$

where we have noted that for integer m and n ,

$$B(m,n) = \frac{(m-1)! \, (n-1)!}{(m+n-1)!} \ .$$

We have therefore learned that

$$\int d\Omega \, \frac{(\vec{a}^* \cdot \vec{r}/r)^\ell}{2^\ell \ell!} \frac{(\vec{a} \cdot \vec{r}/r)^{\ell'}}{2^{\ell'} \ell'!} = \delta_{\ell\ell'} \, 4\pi \frac{1}{2^\ell (2\ell+1)!} \, (\vec{a}^* \cdot \vec{a})^\ell$$

$$= \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell'}^{\ell'} \psi_{\ell m}^* \psi_{\ell' m'} \sqrt{\frac{4\pi}{2\ell+1}} \sqrt{\frac{4\pi}{2\ell'+1}} \int d\Omega \, Y_{\ell m}^*(\theta,\phi) \, Y_{\ell' m'}(\theta,\phi) \quad , \tag{17.24}$$

where we have used the generating function (17.14). What we now must do is extract the coefficient of $\psi_{\ell m}^* \psi_{\ell' m'}$ from $(\vec{a}^* \cdot \vec{a})^\ell$, which is achieved as follows:

$$\frac{(\vec{a}^* \cdot \vec{a})^\ell}{2^\ell (2\ell+1)!} = \frac{(\xi_+^* \xi_+ + \xi_-^* \xi_-)^{2\ell}}{(2\ell+1)!}$$

$$= \frac{1}{(2\ell+1)!} \sum_{m=-\ell}^{\ell} \frac{(2\ell)!}{(\ell+m)! \, (\ell-m)!} (\xi_+^* \xi_+)^{\ell+m} (\xi_-^* \xi_-)^{\ell-m}$$

$$= \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \psi_{\ell m}^* \psi_{\ell m} \tag{17.25}$$

where we have used (17.9), (17.13), and the binomial expansion. Comparing this with (17.24), we obtain the orthonormality condition for the spherical harmonics:

$$\int d\Omega \, Y_{\ell m}^*(\theta,\phi) \, Y_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'} \delta_{mm'} \quad . \tag{17.26}$$

When $Y_{\ell m}$ is separated as in (17.18), the orthonormality condition reads

$$\int_0^\pi \sin\theta \, d\theta \, \Theta_{\ell m}(\theta) \, \Theta_{\ell' m}(\theta) = \delta_{\ell\ell'} \quad . \tag{17.27}$$

17-4. Legendre Polynomials

A few special cases of $\Theta_{\ell m}$ can be easily extracted from (17.19):

$$\Theta_{\ell\ell}(\theta) = \sqrt{\frac{(2\ell+1)!}{2}}\, \frac{(-1)^{\ell}\,(\sin\theta)^{\ell}}{2^{\ell}\ell!} \quad , \tag{17.28}$$

$$\Theta_{\ell,-\ell}(\theta) = \sqrt{\frac{(2\ell+1)!}{2}}\, \frac{(\sin\theta)^{\ell}}{2^{\ell}\ell!} \quad , \tag{17.29}$$

$$\begin{aligned}\Theta_{\ell,0}(\theta) &= \sqrt{\frac{2\ell+1}{2}} \left[\frac{d}{d(\cos\theta)}\right]^{\ell} \frac{(\cos^2\theta-1)^{\ell}}{2^{\ell}\ell!} \\ &= \sqrt{\frac{2\ell+1}{2}}\, P_{\ell}(\cos\theta) \quad .\end{aligned} \tag{17.30}$$

Occurring in (17.30) is the Legendre polynomial of order ℓ ,

$$P_{\ell}(\cos\theta) = \left[\frac{d}{d(\cos\theta)}\right]^{\ell} \frac{(\cos^2\theta-1)^{\ell}}{2^{\ell}\ell!} \qquad \pm\sin\theta^{\mp m} \frac{d^{\mp m}}{d(\cos\theta)} P_{\ell}(\cos\theta) = \Theta_{\ell m} \tag{17.31}$$

which is so normalized that

$$P_{\ell}(1) = 1 \quad . \tag{17.32}$$

According to (17.27), the Legendre polynomials satisfy the following orthonormality condition: see 17.30

$$\int_{-1}^{1} d(\cos\theta)\, P_{\ell}(\cos\theta)\, P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'} \quad . \tag{17.33}$$

XVIII. Coulomb Potential and Spherical Harmonics

The motivation for constructing the solid harmonics was that they formed, in terms of homogeneous functions, a particular set of solutions to Laplace's equation. Since these harmonics are functions of the spherical angles θ and ϕ, Laplace's equation should be expressed in spherical coordinates, where the Laplacian has the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] . \tag{18.1}$$

Thus, since the solid harmonics, (17.6),

$$Y_{\ell m}(\vec{r}) = \left\{\begin{matrix} r^{\ell} \\ r^{-\ell-1}\end{matrix}\right\} Y_{\ell m}(\theta,\phi) , \tag{18.2}$$

are solutions to

$$\nabla^2 Y_{\ell m}(\vec{r}) = 0 , \qquad r \neq 0 , \tag{18.3}$$

and since

$$\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\left\{\begin{matrix} r^{\ell} \\ r^{-\ell-1}\end{matrix}\right\} = \ell(\ell+1)\left\{\begin{matrix} r^{\ell} \\ r^{-\ell-1}\end{matrix}\right\} , \tag{18.4}$$

the differential equation satisfied by the spherical harmonics, $Y_{\ell m}$, is

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \ell(\ell+1)\right] Y_{\ell m}(\theta,\phi) = 0 . \tag{18.5}$$

When the θ and ϕ dependence of $Y_{\ell m}$ is separated as in (17.18), the differential equation for $\Theta_{\ell m}$ is

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]\Theta_{\ell m}(\theta)=0 \quad . \tag{18.6}$$

The fundamental solution of Laplace's equation is the Coulomb potential, (12.3), for $\vec{r} \neq \vec{r}\,'$, which, written in spherical coordinates, is

$$\frac{1}{|\vec{r}-\vec{r}\,'|}=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} \quad , \tag{18.7}$$

where γ is the angle between $\vec{r}$ and $\vec{r}\,'$, explicitly,

$$\cos\gamma=\cos\theta\,\cos\theta'+\sin\theta\,\sin\theta'\,\cos(\phi-\phi') \quad . \tag{18.8}$$

We now expand (18.7) as

$$\frac{1}{\sqrt{r_>^2+r_<^2-2r_>r_<\cos\gamma}}=\sum_{\ell=0}^{\infty}\left(\frac{r_<^{\ell}}{r_>^{\ell+1}}\right)(\text{Polynomial of degree } \ell \text{ in } \cos\gamma) \quad , \tag{18.9}$$

where $r_>(r_<)$ is the greater (lesser) of r and r' . The polynomial of degree ℓ appearing here is a solution to (18.5), and so must be a linear combination of $Y_{\ell m}(\theta,\phi)$'s , $-\ell \leq m \leq \ell$. On the other hand, as we will show below, this is just the Legendre polynomial in $\cos\gamma$, (17.31), that is

$$\frac{1}{|\vec{r}-\vec{r}\,'|}=\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_{\ell}(\cos\gamma) \quad . \tag{18.10}$$

For $\gamma = 0$, this expansion is trivially

$$\frac{1}{r_>-r_<} = \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}}$$

which supplies the normalization condition

$$P_\ell(1) = 1 \quad , \tag{18.11}$$

as is required for the Legendre polynomials [see (17.32)].

We now wish to expand the Coulomb potential

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} \quad , \tag{18.12}$$

which satisfies the inhomogeneous Green's function equation

$$-\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{18.13}$$

in terms of the solutions to the homogeneous Laplace's equation, (18.3). In spherical coordinates, the delta function is

$$\delta(\vec{r}-\vec{r}\,') = \frac{1}{r^2}\,\delta(r-r')\,\frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\phi-\phi') \quad , \tag{18.14}$$

while the Laplacian is given by (18.1). For $r < r'$, (18.10) shows that the solution to (18.13) can be expanded in powers of r ,

$$r < r' : \qquad G = \sum_{\ell m} r^\ell Y_{\ell m}(\theta,\phi)\, A_{\ell m} \quad , \tag{18.15}$$

while for $r > r'$, the expansion is in terms of powers of $1/r$,

$$r > r' : \qquad G = \sum_{\ell m} r^{-\ell-1}\, Y_{\ell m}(\theta,\phi)\, B_{\ell m} \quad . \tag{18.16}$$

The expansion coefficients, $A_{\ell m}$ and $B_{\ell m}$, depending on r', θ' and ϕ', are to be determined by the conditions on Green's function near the source:

$$G \text{ is continuous at } r = r' ; \tag{18.17a}$$

and

$$\left[-r^2 \frac{\partial}{\partial r} G \right]_{r'-0}^{r'+0} = 4\pi \frac{1}{\sin\theta} \delta(\theta-\theta') \delta(\phi-\phi') . \tag{18.17b}$$

?

These two conditions imply, respectively,

$$r'^{\ell} A_{\ell m} = \frac{1}{r'^{\ell+1}} B_{\ell m} , \tag{18.18a}$$

$$\sum_{\ell m} \left[(\ell+1) \frac{1}{r'^{\ell}} B_{\ell m} Y_{\ell m} + \ell r'^{\ell+1} A_{\ell m} Y_{\ell m} \right] = 4\pi \frac{1}{\sin\theta} \delta(\theta-\theta') \delta(\phi-\phi') . \tag{18.18b}$$

If we write

$$A_{\ell m} = r'^{-\ell-1} C_{\ell m} , \quad B_{\ell m} = r'^{\ell} C_{\ell m} , \tag{18.19}$$

(18.18a) is satisfied automatically, while (18.18b) reads

$$\sum_{\ell m} (2\ell+1) C_{\ell m} Y_{\ell m}(\theta,\phi) = 4\pi \frac{1}{\sin\theta} \delta(\theta-\theta') \delta(\phi-\phi') . \tag{18.20}$$

The use of the orthonormality condition, (17.26), now yields

$$C_{\ell m} = \frac{4\pi}{2\ell+1} Y^{*}_{\ell m}(\theta',\phi') . \tag{18.21}$$

By substituting this into (18.20) we obtain the completeness statement for the spherical harmonics,

$$\sum_{\ell m} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi') = \frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\phi-\phi') \quad , \tag{18.22}$$

which allows us to expand any function of θ and ϕ in terms of spherical harmonics. We therefore have obtained such an expansion for the Green's function

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell m} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi') \quad . \tag{18.23}$$

Comparing this with the alternative representation, (18.10), we obtain the relation

Addition theorem

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi') \quad . \tag{18.24}$$

We must finally show that this function of $\cos\gamma$ actually is the Legendre polynomial, (17.31). This is easily done by considering a particular coordinate system, where

$$\theta' = 0 \implies \gamma = \theta \quad .$$

From (17.19), we learn that

$$Y_{\ell m}(\theta',\phi') \propto (\sin\theta')^{|m|} \quad ,$$

implying [see (17.30)]

$$Y_{\ell m}(0,\phi') = \delta_{m0}\sqrt{\frac{2\ell+1}{4\pi}} \quad ,$$

so that only the $m = 0$ term contributes to the right-hand side of (18.24), which is therefore, by (17.30),

$$\frac{4\pi}{2\ell+1}\sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell(\cos\theta)\sqrt{\frac{2\ell+1}{4\pi}} = P_\ell(\cos\theta) \quad .$$

Thus we have proved that the function of $\cos\gamma$ occurring in (18.10) is indeed Legendre's polynomial. The relation (18.24) is called the addition theorem for spherical harmonics.

Lecture 16

XIX. Multipoles

In terms of the above discussion of spherical harmonics, we now make a general analysis of the potential, due to a given charge distribution, $\rho(\vec{r}\,')$, outside of that distribution.

The potential is given by (1.2), or

$$\phi(\vec{r}) = \int (d\vec{r}\,') \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} , \tag{19.1}$$

where, for convenience, we will choose the origin to lie within the charge distribution. If r is large compared to the characteristic dimensions of the charge distribution, we may expand the Coulomb potential as follows:

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (-x\cdot\nabla)^n$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r} - \vec{r}\,'\cdot\vec{\nabla}\frac{1}{r} + \frac{1}{2}(\vec{r}\,'\cdot\vec{\nabla})^2\frac{1}{r} + \ldots$$

$$= \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \frac{1}{2}\frac{1}{r^5}\vec{r}\cdot[3\vec{r}\,'\vec{r}\,' - \overleftrightarrow{1}r'^2]\cdot\vec{r} + \ldots , \tag{19.2}$$

so that the potential, in its leading behavior for large distances, has the form

$$\phi(\vec{r}) = \frac{e}{r} + \frac{\vec{r}\cdot\vec{d}}{r^3} + \frac{1}{2}\frac{1}{r^5}\vec{r}\cdot\overleftrightarrow{q}\cdot\vec{r} + \ldots . \tag{19.3}$$

Occurring here are the first three moments of the charge distribution,

$$e = \int (d\vec{r}\,') \rho(\vec{r}\,') , \tag{19.4a}$$

$$\vec{d} = \int (d\vec{r}')\, \vec{r}'\rho(\vec{r}') \quad , \tag{19.4b}$$

$$\overleftrightarrow{q} = \int (d\vec{r}')(3\vec{r}'\vec{r}' - \overleftrightarrow{1}r'^2)\, \rho(\vec{r}') \quad , \tag{19.4c}$$

which are the total charge, the dipole moment vector, and the quadrupole moment dyadic, respectively.

Using this potential, we can now calculate the interaction energy of the charge distribution with an additional point charge e_1 located at a point $\vec{r}$ lying far outside the charge distribution:

$$E = e_1\phi(\vec{r}) = \frac{ee_1}{r} + \vec{d}\cdot\frac{e_1\vec{r}}{r^3} + \frac{1}{2}\, e_1 \frac{1}{r^5}\, \vec{r}\cdot\overleftrightarrow{q}\cdot\vec{r} + \dots \quad . \tag{19.5}$$

We may alternatively interpret (19.5) as the interaction energy of the various moments of the charge distribution with the field produced by e_1 at the origin, that is

$$E = e\phi - \vec{d}\cdot\vec{E} + \frac{1}{6}\, \vec{\nabla}\cdot\overleftrightarrow{q}\cdot\vec{E} + \dots \tag{19.6}$$

where

$$\phi = \frac{e_1}{r}$$

$$\vec{E} = \frac{e_1(-\vec{r})}{r^3} \quad .$$

[We have seen this form of the dipole interaction energy before, in Subsection 4-1.] This is a starting point for considering the interaction of one charge distribution with another charge distribution. For example, if one had a dipole $\vec{d}_1$, rather than a charge, e_1 , interacting with a charge distribution which had only a dipole moment, $\vec{d}_2$, the interaction energy deduced from (19.6) would be

$$E = -\vec{d}_2 \cdot \left[-\vec{\nabla} \frac{\vec{r}\cdot\vec{d}_1}{r^3} \right] = - \frac{3\vec{r}\cdot\vec{d}_1 \; \vec{r}\cdot\vec{d}_2 - \vec{d}_1\cdot\vec{d}_2 \; r^2}{r^5} \quad , \tag{19.7}$$

which is the dipole-dipole interaction.

Although this approach could obviously be continued indefinitely, it rapidly becomes unwieldy for higher multipoles. A systematic approach can be based on the use of spherical harmonics. Outside a charge distribution $(r > r')$, the potential (19.1) can be expanded in spherical harmonics according to (18.23), that is

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell m} \frac{r'^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell m}(\theta,\phi) \sqrt{\frac{4\pi}{2\ell+1}}\, Y^{*}_{\ell m}(\theta',\phi') \quad , \tag{19.8}$$

as

$$\phi(\vec{r}) = \sum_{\ell m} \frac{1}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell m}(\theta,\phi)\, \rho_{\ell m} \quad . \tag{19.9}$$

Here, the multipole moments, $\rho_{\ell m}$, are defined by

$$\rho_{\ell m} = \int (d\vec{r}')\, r'^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}\, Y^{*}_{\ell m}(\theta',\phi')\, \rho(\vec{r}') \quad . \tag{19.10}$$

The connection with the previous definition, (19.4), is, for example, given by

$$\ell = 0 : \quad \rho_{00} = e \quad ,$$

$$\ell = 1 : \quad \rho_{11} = -\frac{1}{\sqrt{2}}(d_x - i d_y) \ ,$$

$$\rho_{10} = d_z \ ,$$

$$\rho_{1-1} = \frac{1}{\sqrt{2}}(d_x + i d_y) \ .$$

Now we return to the consideration of the energy of interaction of a charge distribution, $\rho(\vec{r})$, with an external potential, $\phi(\vec{r})$:

$$E = \int (d\vec{r})\, \rho(\vec{r})\, \phi(\vec{r}) \ . \tag{19.11}$$

Since the potential is produced by sources outside of the charge distribution, it can be expanded in terms of spherical harmonics,

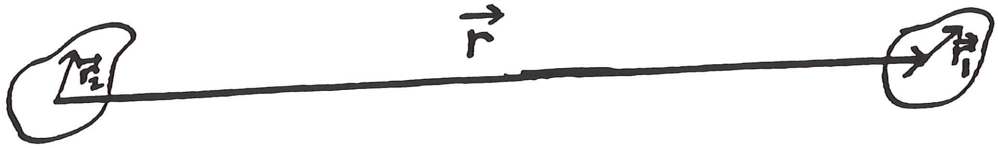
$$\phi(\vec{r}) = \sum_{\ell m} r^{\ell} Y_{\ell m}(\theta,\phi) \sqrt{\frac{4\pi}{2\ell+1}}\, \phi_{\ell m} \ , \tag{19.12}$$

$\phi_{\ell m}$ being the expansion coefficients. Inserting this multipole expansion for the potential back into (19.11) and using the definition (19.10) for the multipole moments, we obtain the simple expression for the energy of interaction

$$E = \sum_{\ell m} \rho^{*}_{\ell m}\, \phi_{\ell m} \ , \tag{19.13}$$

generalizing (19.6).

Rather than expressing the interaction energy in the unsymmetrical form (19.13), let us formulate the energy in terms of the interaction of the charge multipole moments of each distribution; that is, we seek a generalization of the dipole-dipole interaction, (19.7). If we let $\vec{r}_1$ and $\vec{r}_2$ be measured from points within ρ_1 and ρ_2 , respectively, while $\vec{r}$ measures the distance between these two origins, as illustrated in the diagram below,



the interaction energy can be written as

$$E = \int (d\vec{r}_1)\,(d\vec{r}_2)\,\frac{\rho_1(\vec{r}_1)\;\rho_2(\vec{r}_2)}{|\vec{r} + \vec{r}_1 - \vec{r}_2|} \quad . \tag{19.14}$$

Since the two charge distributions are non-overlapping, we can expand the denominator occurring here in a double Taylor series:

$$\frac{1}{|\vec{r} + \vec{r}_1 - \vec{r}_2|} = \sum_{\ell_1 \ell_2} \frac{(\vec{r}_1 \cdot \vec{\nabla})^{\ell_1}}{\ell_1!}\,\frac{(-\vec{r}_2 \cdot \vec{\nabla})^{\ell_2}}{\ell_2!}\,\frac{1}{r} \quad . \tag{19.15}$$

We already know that, for $r > r'$,

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell m} \frac{r'^{\ell}}{r^{\ell+1}}\,\frac{4\pi}{2\ell+1}\,Y_{\ell m}(\theta,\phi)\,Y^*_{\ell m}(\theta',\phi') = \sum_{\ell} \frac{(-\vec{r}\,'\cdot\vec{\nabla})^{\ell}}{\ell!}\,\frac{1}{r} \quad , \tag{19.16}$$

or, equating powers of r' ,

$$\frac{(-\vec{r}\,'\cdot\vec{\nabla})^{\ell}}{\ell!}\,\frac{1}{r} = \frac{4\pi}{2\ell+1} \sum_{m} r'^{\ell}\,Y_{\ell m}(\theta',\phi')\,r^{-\ell-1}\,Y^*_{\ell m}(\theta,\phi) \quad . \tag{19.17}$$

Further, recall the generating function for the spherical harmonics, (17.14),

$$\frac{(\vec{r}\,'\cdot\vec{a})^{\ell}}{2^{\ell}\ell!} = r'^{\ell} \sum_{m} \sqrt{\frac{4\pi}{2\ell+1}}\,Y_{\ell m}(\theta',\phi')\,\psi_{\ell m} \quad , \tag{19.18}$$

which is valid for $\vec{a}^2 = 0$. Thus it is permissible to replace $\vec{a}$ by a gradient,

$$\vec{a} \to \vec{\nabla} \ ,$$

as long as the derivatives act on $1/r$,

$$\vec{a}^2 \to \nabla^2 \frac{1}{r} = 0 \ , \qquad r > 0 \ .$$

In this way, a comparison of (19.17) and (19.18) gives the identity

$$\sqrt{\frac{4\pi}{2\ell+1}}\, r^{-\ell-1}\, Y^*_{\ell m}(\theta,\phi) = (-1)^{\ell} 2^{\ell} \psi_{\ell m} \frac{1}{r} \ , \tag{19.19}$$

where $\psi_{\ell m}$ is now regarded as a differential operator, constructed according to (17.13) from $\vec{\nabla}$. Using (19.18) twice with the above replacement, we obtain

$$\begin{aligned} \frac{(\vec{r}_1\cdot\vec{\nabla})^{\ell_1}}{\ell_1!} \frac{(-\vec{r}_2\cdot\vec{\nabla})^{\ell_2}}{\ell_2!} \frac{1}{r} &= (-1)^{\ell_2} 2^{\ell_1} r_1^{\ell_1} 2^{\ell_2} r_2^{\ell_2} \\ &\times \sum_{m_1 m_2} \sqrt{\frac{4\pi}{2\ell_1+1}}\, Y_{\ell_1 m_1}(\theta_1,\phi_1) \sqrt{\frac{4\pi}{2\ell_2+1}}\, Y_{\ell_2 m_2}(\theta_2,\phi_2) \\ &\times \psi_{\ell_1 m_1} \psi_{\ell_2 m_2} \frac{1}{r} \ . \end{aligned} \tag{19.20}$$

According to the definition of $\psi_{\ell m}$, (17.13), the product of two of these functions is

$$\psi_{\ell_1 m_1} \psi_{\ell_2 m_2} = C_{\ell_1 \ell_2\, m_1 m_2}\, \psi_{\ell_1+\ell_2,\, m_1+m_2} \ , \tag{19.21}$$

where

$$C_{\ell_1\ell_2m_1m_2} = \left[\frac{(\ell_1+\ell_2+m_1+m_2)!\ (\ell_1+\ell_2-m_1-m_2)!}{(\ell_1+m_1)!\ (\ell_1-m_1)!\ (\ell_2+m_2)!\ (\ell_2-m_2)!}\right]^{1/2} . \tag{19.22}$$

Then we evaluate the derivative structure in (19.20) by means of (19.19):

$$\psi_{\ell_1+\ell_2,\ m_1+m_2}\frac{1}{r} = (-1)^{\ell_1+\ell_2}\frac{1}{2^{\ell_1+\ell_2}}\sqrt{\frac{4\pi}{2(\ell_1+\ell_2)+1}}\ r^{-\ell_1-\ell_2-1}$$

$$\times\ Y^*_{\ell_1+\ell_2,\ m_1+m_2}(\theta,\phi) . \tag{19.23}$$

Combining (19.14), (19.15), (19.20), (19.21), and (19.23), taking the complex conjugate, and identifying $\rho_{\ell m}$, (19.10), we find for the energy of interaction

$$E = \sum_{\ell_1\ell_2m_1m_2} (-1)^{\ell_1}\left[\frac{4\pi}{2(\ell_1+\ell_2)+1}\right]^{1/2} C_{\ell_1\ell_2m_1m_2}\frac{1}{r^{\ell_1+\ell_2+1}} .$$

$$\times\ \rho_{\ell_1m_1}\ Y_{\ell_1+\ell_2,\ m_1+m_2}(\theta,\phi)\ \rho_{\ell_2m_2} . \tag{19.24}$$

If we were to set $\ell_1=\ell_2=1$ we would rederive the dipole-dipole interaction, (19.7). However, this is a completely general result for the interaction between two arbitrary non-overlapping charge distributions, of a remarkably simple and compact form.

XX. Hollow Conducting Sphere

The spherical harmonics are useful in solving problems posessing spherical symmetry. In this and the next section, we will solve two such problems. Here we wish to find Green's function inside a hollow conducting sphere of radius a , which is grounded, that is, the potential is zero on its surface. As usual, the Green's function equation is

$$-\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad . \tag{20.1}$$

The solution must be expressible in terms of spherical harmonics as

$$r < r' : \qquad G = \sum_{\ell m} r^{\ell} Y_{\ell m}(\theta,\phi)\, A_{\ell m} \quad , \tag{20.2a}$$

see 18.15

$$r' < r \leq a : \quad G = \sum_{\ell m} \left(\frac{1}{r^{\ell+1}} - \frac{r^{\ell}}{a^{2\ell+1}} \right) Y_{\ell m}(\theta,\phi)\, B_{\ell m} \quad , \tag{20.2b}$$

see 18.16

where we have imposed the boundary conditions that

$$G = \text{finite at } r = 0 \ ,$$

and

$$G = 0 \text{ at } r = a \ .$$

To determine the expansion coefficients, $A_{\ell m}$ and $B_{\ell m}$, we use the equations for the continuity of G , (18.17a),

$$r'^{\ell} A_{\ell m} = \left(\frac{1}{r'^{\ell+1}} - \frac{r'^{\ell}}{a^{2\ell+1}} \right) B_{\ell m} \quad , \tag{20.3a}$$

and for the discontinuity of $\frac{\partial}{\partial r} G$, (18.17b) ,

$$\sum_{\ell m}\left[\left(\frac{\ell+1}{r'^{\ell}}+\frac{\ell r'^{\ell+1}}{a^{2\ell+1}}\right)Y_{\ell m}B_{\ell m}+\ell r'^{\ell+1}\,Y_{\ell m}A_{\ell m}\right]$$

$$=4\pi\sum_{\ell m}Y_{\ell m}(\theta,\phi)\,Y^{*}_{\ell m}(\theta',\phi')\quad, \tag{20.3b}$$

at $r = r'$. Solving (20.3a) by introducing $C_{\ell m}$, defined by

$$A_{\ell m}=\left(\frac{1}{r'^{\ell+1}}-\frac{r'^{\ell}}{a^{2\ell+1}}\right)C_{\ell m}\quad, \qquad \left(r^{-\ell}-\frac{a^{2\ell+1}}{r^{\ell+1}}\right)C_{\ell m} \tag{20.4a}$$

$$B_{\ell m}=r'^{\ell}\,C_{\ell m}\quad, \qquad \left(\frac{1}{r^{\ell+1}}\right)C_{\ell m} \tag{20.4b}$$

we find, from (20.3b),

$$C_{\ell m}=\frac{4\pi}{2\ell+1}\,Y^{*}_{\ell m}(\theta',\phi')\quad. \tag{20.5}$$

Therefore, Green's function is

$$G=\sum_{\ell m}\left(\frac{r_<^{\ell}}{r_>^{\ell+1}}-\frac{r^{\ell}r'^{\ell}}{a^{2\ell+1}}\right)\underbrace{\frac{4\pi}{2\ell+1}\,Y_{\ell m}(\theta,\phi)\,Y^{*}_{\ell m}(\theta',\phi')}_{P_\ell(\cos\gamma)}\quad. \tag{20.6}$$

Noticing that

$$\frac{r^{\ell}r'^{\ell}}{a^{2\ell+1}}=\frac{a}{r'}\,\frac{r^{\ell}}{\left(\frac{a^{2}}{r'}\right)^{\ell+1}}\quad,\qquad \frac{a^{2}}{r'}>a\geq r\quad,$$

for exterior can replace $\frac{r^\ell r'^\ell}{a^{2\ell+1}}$ with $\frac{a^{2\ell+1}}{(r r')^{\ell+1}}$

we can perform the summation in (20.6) by use of (18.23):

$$G=\frac{1}{|\vec{r}-\vec{r}\,'|}-\frac{a}{r'}\,\frac{1}{|\vec{r}-\bar{\vec{r}}\,'|} \tag{20.7}$$

for exterior problem get same result

where $\bar{\vec{r}}\,'$ locates the so-called image point,

$$\vec{r}' = \left(\frac{a^2}{r'} , \theta', \phi' \right) , \tag{20.8}$$

which, of course, lies outside the sphere. Thus we have achieved for the sphere the analog of the image solution given for the conducting plane in (14.1).

Lecture 17.

What is the induced charge density on the inside surface of the sphere? This charge density is proportional to the radial electric field, according to (10.31),

$$4\pi\sigma = -E_r = \frac{\partial}{\partial r} G \bigg|_{r=a} \tag{20.9}$$

since the normal is inward, and so in the negative radial direction. Differentiating (20.6) with respect to $r = r_>$, and using the addition theorem, (18.24), we obtain

$$4\pi\sigma = -\sum_\ell (2\ell+1) \frac{1}{a^2} \left(\frac{r'}{a} \right)^\ell P_\ell(\cos\gamma) , \tag{20.10}$$

where γ is the angle between $\vec{r}$ and $\vec{r}'$, (18.8). Alternatively, we could use the image charge form of Green's function, (20.7), to derive

$$4\pi\sigma = -\frac{1}{a^2} \frac{1-(r'/a)^2}{\left[1 - 2\frac{r'}{a}\cos\gamma + \left(\frac{r'}{a} \right)^2 \right]^{3/2}} , \tag{20.11}$$

which indicates that as $r' \to a$, the only significant charge buildup is near $\gamma = 0$. The total charge can be computed from (20.10) by use of the orthonormality condition, (17.33), which, for $\ell' = 0$, implies

$$\int_0^{\pi} \sin\gamma \, d\gamma \, P_\ell(\cos\gamma) = 2\delta_{\ell 0} \quad . \tag{20.12}$$

Therefore, only the $\ell = 0$ term in (20.10) contributes to the total charge:

$$Q = \int dS\sigma = \int a^2 \sin\gamma \, d\gamma \, d\phi \left[- \frac{1}{4\pi a^2} \right] = -1 \; , \tag{20.13}$$

which is the expected result. Of course, it is possible to use (20.11) to compute the total charge, but that is more elaborate.

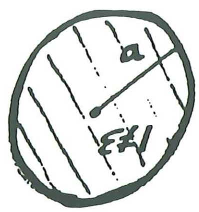
20.11 ⇒ $\int \sigma \, dS = \int \sigma a^2 \sin\gamma \, d\gamma \, d\phi$

$= \int -\sum (2\ell+1) \frac{1}{a^2} \left(\frac{r}{a}\right)^{\ell} P_\ell(\cos\gamma) \ldots$

$\ell = 1$

XXI. Dielectric Sphere

As a second example of the use of spherical harmonics in solving Green's function problems with spherical symmetry, we consider a dielectric sphere of radius a, with a unit point charge outside.



$\varepsilon = 1$

$\vec{r}'$

In this case, the Green's function equation is

$$r > a: \quad -\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{21.1a}$$

$$r < a: \quad -\vec{\nabla}\cdot[\varepsilon\vec{\nabla}G(\vec{r},\vec{r}\,')] = 0 \quad , \tag{21.1b}$$

where we will take ε to be a constant. The boundary conditions at $r = a$ are, from (10.24) and (10.26),

$$G \text{ is continuous,} \tag{21.2a}$$

and

$$\left[-\frac{\partial}{\partial r}G\right]_{r=a+0} = \left[-\varepsilon\frac{\partial}{\partial r}G\right]_{r=a-0} \quad . \tag{21.2b}$$

The conditions on G at $r = r'$ are as given in (18.17),

$$G \text{ is continuous,} \tag{21.2c}$$

and

$$\left[-r^2 \frac{\partial}{\partial r} G\right]_{r=r'-0}^{r=r'+0} = 4\pi \frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\phi-\phi') \quad . \tag{21.2d}$$

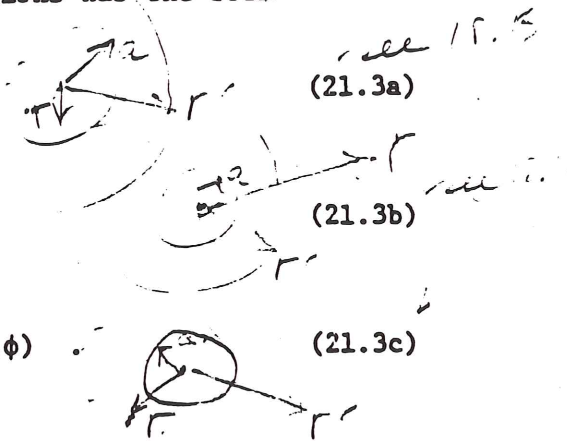
$= \sum_{lm} Y_{lm} Y^*_{lm}$

As is familiar by now, the solution in the three regions has the form

$$r < a : \qquad G = \sum_{\ell m} r^{\ell} Y_{\ell m}(\theta,\phi)\, A_{\ell m} \quad , \tag{21.3a}$$

$$r > r' : \qquad G = \sum_{\ell m} r^{-\ell-1}\, Y_{\ell m}(\theta,\phi)\, D_{\ell m} \quad , \tag{21.3b}$$

$$a < r < r' : \quad G = \sum_{\ell m} (r^{\ell} B_{\ell m} + r^{-\ell-1} C_{\ell m})\, Y_{\ell m}(\theta,\phi) \quad . \tag{21.3c}$$



It is very easy to find the expansion coefficients by use of (21.2):

$$A_{\ell m} = \frac{4\pi\, Y^*_{\ell m}(\theta',\phi')}{\ell(\varepsilon+1)+1}\, r'^{-\ell-1} \quad , \tag{21.4a}$$

$$B_{\ell m} = \frac{4\pi\, Y^*_{\ell m}(\theta',\phi')}{2\ell+1}\, r'^{-\ell-1} \quad , \tag{21.4b}$$

$$C_{\ell m} = -\frac{(\varepsilon-1)\ell}{\ell(\varepsilon+1)+1}\, \frac{4\pi\, Y^*_{\ell m}(\theta',\phi')}{2\ell+1}\, \frac{a^{2\ell+1}}{r'^{\ell+1}} \quad , \tag{21.4c}$$

$$D_{\ell m} = C_{\ell m} + \frac{4\pi\, Y^*_{\ell m}(\theta',\phi')}{2\ell+1}\, r'^{\ell} \quad . \tag{21.4d}$$

Green's function, outside the sphere, is therefore found to be

$$r,r' > a : \quad G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \sum_{\ell=1}^{\infty} \frac{(\varepsilon-1)\ell}{\ell(\varepsilon+1)+1}\, \frac{a^{2\ell+1}}{r^{\ell+1}\, r'^{\ell+1}}\, P_{\ell}(\cos\gamma) \quad . \tag{21.5}$$

We now ask what is the leading behavior of this potential when the separation between the point charge and the sphere is large compared to the radius of the sphere, $r' \gg a$. Since the ℓ^{th} term in the sum behaves as

$\left(\frac{a}{r'}\right)^{\ell+1}$, only small values of ℓ contribute. The leading contribution arises from $\ell = 1$,

$$r' \gg a : \quad G(\vec{r},\vec{r}\,') \sim \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{\varepsilon-1}{\varepsilon+2}\frac{a^3}{r^2 r'^2}\cos\gamma \quad . \tag{21.6}$$

Since γ is the angle between $\vec{r}$ and $\vec{r}\,'$,

$$\cos\gamma = \frac{\vec{r}\cdot\vec{r}\,'}{rr'} \quad ,$$

this asymptotic form of Green's function can be rewritten as

$$G(\vec{r},\vec{r}\,') \sim \frac{1}{|\vec{r}-\vec{r}\,'|} + \frac{\vec{r}}{r^3}\cdot\vec{d} \quad , \tag{21.7}$$

see 19.3 where $q = 1$

the two terms of which have simple physical interpretations. The first term is due to the point charge while the second is the potential arising from the induced electric dipole moment of the sphere [cf. (19.3)]. The latter is identified from (21.6) to be

$$\vec{d} = \frac{\varepsilon-1}{\varepsilon+2}a^3\left(-\frac{\vec{r}\,'}{r'^3}\right) \quad , \tag{21.8}$$

where $-\vec{r}\,'/r'^3$ is interpreted as the electric field, $\vec{E}(0)$, at the center of the sphere (in the absence of the dielectric) produced by the unit point charge. Since this electric field is essentially constant over the sphere, we recognize that the electric dipole moment induced in a dielectric sphere of radius a by a constant electric field $\vec{E}$ is

$$\vec{d} = \frac{\varepsilon-1}{\varepsilon+2}a^3\,\vec{E} \quad . \tag{21.9}$$

We complete this section by writing the expression for Green's function inside the sphere:

$$\begin{matrix} r < a \\ r' > a \end{matrix} \quad : \quad G(\vec{r},\vec{r}\,') = \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{r'^{\ell+1}} \frac{2\ell+1}{\ell(\varepsilon+1)+1} P_{\ell}(\cos\gamma) \quad . \tag{21.10}$$

Again, in the situation in which the point charge is located far from the sphere, $r' \gg a$, low values of ℓ predominate:

$$G \sim \frac{1}{r'} + \frac{3}{\varepsilon+2} \frac{\vec{r}\cdot\vec{r}\,'}{r'^3} = \frac{1}{r'} - \frac{3}{\varepsilon+2} \vec{r}\cdot\vec{E}(0) = \frac{1}{r'} - \vec{r}\cdot\vec{E} \quad , \tag{21.11}$$

where we identify the electric field in the dielectric as the negative gradient of G , that is

$$\vec{E} = \frac{3}{\varepsilon+2} \vec{E}(0) \quad , \tag{21.12}$$

which is less than $\vec{E}(0)$, if $\varepsilon > 1$.

XXII. Electrostatics: Dielectrics and Conductors

22-1. Variational Principle

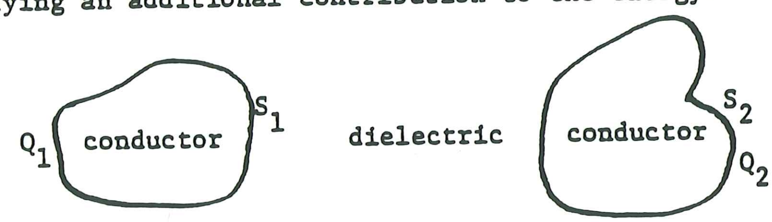
In Section X, we investigated the stationary properties of the electrostatic energy when only dielectrics are present, that is, we had a stationary principle,

$$\delta E = 0 \quad ,$$

where [see (10.4)]

$$E = \int (d\vec{r}) \left[\rho\phi + \frac{\varepsilon}{4\pi} (\vec{E}\cdot\vec{\nabla}\phi + \frac{1}{2}\vec{E}^2) \right] \quad . \tag{22.1}$$

We now wish to generalize this situation to include conductors as well. The new feature here is the existence of surface charges on various conductors implying an additional contribution to the energy:



$$E = \int (d\vec{r}) \left[\rho\phi + \frac{\varepsilon}{4\pi} (\vec{E}\cdot\vec{\nabla}\phi + \frac{1}{2}\vec{E}^2) \right] + \sum_{i=1}^{n} \int dS_i \, \sigma\phi \quad , \tag{22.2}$$

where the volume integral extends over all space exterior to the conductors and the surface integral is over all of the conductors, σ being the surface charge density. This energy functional is to be supplemented by the condition that the total charge on each conductor,

$$Q_i = \int dS_i \, \sigma \quad , \qquad i = 1,2,\ldots n \quad , \tag{22.3}$$

is fixed. The electrostatic problem is completely specified by the location of the conductors and dielectrics, the free volume charge density, ρ, and the charge on each conductor, Q_i. Note in particular that the surface charge density, σ, is to be determined dynamically. The change of the energy under variations of ϕ, $\vec{E}$, and σ is

$$\delta E = \int (d\vec{r}) \left[\rho\delta\phi + \frac{\varepsilon}{4\pi} (\delta\vec{E}\cdot\vec{\nabla}\phi + \vec{E}\cdot\vec{\nabla}\delta\phi + \vec{E}\cdot\delta\vec{E}) \right]$$

$$+ \sum_i \int dS_i [\delta\sigma\phi + \sigma\delta\phi] \quad , \tag{22.4}$$

which is subject to the condition that Q_i be constant, that is

$$\int dS_i \; \delta\sigma = 0 \quad . \tag{22.5}$$ ✓

We rewrite the $\vec{\nabla}\delta\phi$ term by means of an integration by parts, which makes use of the identity

$$\frac{\vec{D}}{4\pi} \cdot \vec{\nabla}\delta\phi = \vec{\nabla}\cdot\left(\frac{\vec{D}}{4\pi} \delta\phi \right) - \delta\phi \frac{\vec{\nabla}\cdot\vec{D}}{4\pi} \quad . \tag{22.6}$$

The implied surface integral here cannot be discarded since now there are contributions arising from the surfaces of the conductors. If we let $\vec{n}_i$ be the outward normal on the i^{th} conductor, this surface term is

$$\int (d\vec{r})\; \vec{\nabla}\cdot\left(\frac{\vec{D}}{4\pi} \delta\phi \right) = - \sum_i \int dS_i \frac{\vec{n}_i\cdot\vec{D}}{4\pi} \delta\phi \quad . \tag{22.7}$$ ✓

The variation in the energy, (22.4), now reads

$$\delta E = \int (d\vec{r}) \left[\rho\delta\phi + \frac{\varepsilon}{4\pi} \delta\vec{E}\cdot\vec{\nabla}\phi - \delta\phi \frac{\vec{\nabla}\cdot\vec{D}}{4\pi} + \frac{1}{4\pi} \vec{D}\cdot\delta\vec{E} \right]$$

$$+\sum_i \int dS_i \left[-\frac{\vec{n}_i \cdot \vec{D}}{4\pi}\, \delta\phi + \delta\sigma\phi + \sigma\delta\phi \right] \quad . \tag{22.8}$$

The requirement that the energy be stationary under independent variations in ϕ and $\vec{E}$ then implies, in the interior of the dielectric,

$$\delta\phi : \quad \vec{\nabla}\cdot\vec{D} = 4\pi\rho \quad , \tag{22.9}$$

$$\delta\vec{E} : \quad \vec{E} = -\vec{\nabla}\phi \quad , \tag{22.10}$$

while just outside the surfaces of the conductors,

$$\delta\phi : \quad \vec{n}\cdot\vec{D} = 4\pi\sigma \quad . \qquad \text{or} \quad \boxed{-\hat{n}_i \cdot \epsilon\vec{E} = 4\pi\sigma} \qquad \hat{n} = -\hat{n}_i \tag{22.11}$$

Finally, the variation in the surface charge density requires

$$\delta E = \sum_i \int dS_i \; \delta\sigma\phi = 0$$

which is subject to the restriction (22.5), implying that each conductor is an equipotential surface,

$$\boxed{\phi = \text{constant on } S_i = \phi_i} \quad . \tag{22.12}$$

Thus, the stationary action principle, based on the energy functional (22.2), yields all the physical laws governing electrostatics in the presence of conductors and dielectrics.

22-2. Restricted Forms of the Variational Principle

As in Section X, there are two restricted forms of the variational principle we may discuss. In the first, we take the electric field as being defined by

1, $$\vec{E} = -\vec{\nabla}\phi \quad , \tag{22.13}$$

so that the energy functional becomes

$$E = \int (d\vec{r}) \left[\rho\phi - \frac{\varepsilon}{8\pi} (\vec{\nabla}\phi)^2 \right] + \sum_i \int dS_i \, \sigma\phi \quad . \tag{22.14}$$

The independent variables are ϕ and σ, the latter of which is subject to the condition (22.3). For the second form, $\vec{D}$ is regarded as an independent variable, subject to the condition

2. $$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \ , \qquad \text{inside dielectric} \ , \tag{22.15a}$$

while σ is determined by

$$\vec{n}\cdot\vec{D} = 4\pi\sigma \ , \qquad \text{on surface } S_i \ . \tag{22.15b}$$

To rewrite the energy as a functional of $\vec{D}$ only, we integrate by parts on the $\vec{D}\cdot\vec{\nabla}\phi$ term in (22.2) and use (22.15) to obtain

$$E = \int (d\vec{r}) \, \frac{1}{8\pi} \frac{\vec{D}^2}{\varepsilon} \quad , \tag{22.16}$$

while the subsidiary condition (22.3) becomes

$$Q_i = \int dS_i \, \frac{\vec{n}\cdot\vec{D}}{4\pi} \quad . \tag{22.17}$$

In (22.16), we identify the integrand as the energy density of the field.

Let us now verify that the second restricted form of the variational principle correctly describes the electrostatic situation under consideration. For a finite change in $\vec{D}$, $\vec{D} \to \vec{D} + \delta\vec{D}$, the change in the energy functional (22.16) is

$$\delta E = \int (d\vec{r})\, \frac{1}{4\pi} \frac{\vec{D}}{\varepsilon} \cdot \delta\vec{D} + \int (d\vec{r})\, \frac{1}{8\pi} \frac{(\delta\vec{D})^2}{\varepsilon} \quad , \tag{22.18}$$

while the constraints read

$$\vec{\nabla}\cdot\delta\vec{D} = 0 \quad , \tag{22.19a}$$

$$\int dS_i\, \frac{\vec{n}\cdot\delta\vec{D}}{4\pi} = 0 \quad . \tag{22.19b}$$

The stationary condition requires that the integral linear in $\delta\vec{D}$ in (22.18) vanishes. To incorporate the constraint (22.19a), we add to (22.18) the volume integral

$\vec{\nabla}\cdot[\delta\vec{D}\,\phi] = \phi\,\vec{\nabla}\cdot\delta\vec{D} + \delta\vec{D}\cdot\vec{\nabla}\phi$

$$0 = \int (d\vec{r})\, \frac{\phi(\vec{r})}{4\pi}\, \vec{\nabla}\cdot\delta\vec{D} = -\sum_i \int dS_i\, \frac{\vec{n}\cdot\delta\vec{D}}{4\pi}\, \phi - \int (d\vec{r})\, \frac{\vec{\nabla}\phi}{4\pi} \cdot \delta\vec{D} \tag{22.20a}$$

where $\phi(\vec{r})$ is an arbitrary function. Likewise, to incorporate the constraint (22.19b), we add to (22.18) a sum of surface integrals,

$$0 = \sum_i \phi_i \int dS_i\, \frac{\vec{n}\cdot\delta\vec{D}}{4\pi} \quad . \tag{22.20b}$$

where ϕ_i is an arbitrary constant. In the resulting form of δE, the variations $\delta\vec{D}$ can be regarded as independent, so that the stationary principle implies, in the volume,

$$\frac{\vec{D}}{\varepsilon} = \vec{E} = -\vec{\nabla}\phi \quad , \tag{22.21}$$

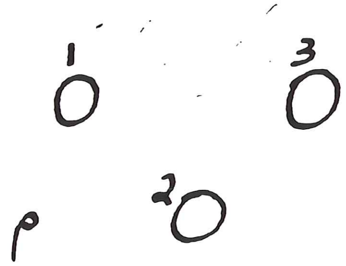
while, on the surfaces,

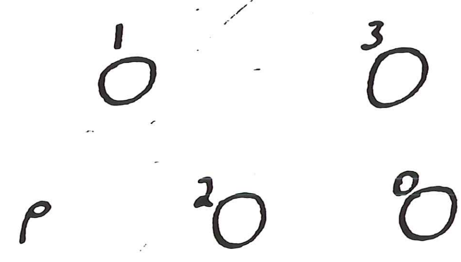
$$\boxed{\phi = \phi_i} \tag{22.22}$$

In this way, we recover the full set of equations for electrostatics. Moreover, (22.18) also tells us that, for any field other than the correct solution, $\delta E > 0$, that is, the physical field minimizes the energy functional (22.16). This is a statement of Thomson's Theorem: The charges on the surfaces of conductors always readjust themselves in such a way that each conductor becomes as equipotential surface and the total energy of the system is a minimum.

22-3. Introduction of Additional Uncharged Conductor

We now consider a region of space with dielectric constant $\varepsilon(\vec{r})$ bounded by an array of conductors into which we introduce an uncharged conductor at a location where there is no free charge density. We are interested in the change of energy in going from the initial configuration (a) to the final configuration (b). (In the following, the subscript 0 refers to the introduced conductor.)

a. 

b. 

The energy for (a) is

$$E_a = \int_V (d\vec{r})\, \frac{\vec{D}_a^{\,2}}{8\pi\varepsilon} , \tag{22.23}$$

where V is the volume exterior to the conductors and the charge on the i^{th} conductor is

$$\int dS_i\, \frac{\vec{n}\cdot\vec{D}_a}{4\pi} = Q_i . \tag{22.24}$$

For (b) the energy is

$$E_b = \int_{V-V_0} (d\vec{r})\, \frac{\vec{D}_b^{\,2}}{8\pi\varepsilon} \quad , \tag{22.25}$$

where now the volume occupied by conductor 0 (V_0) is also excluded, and the charges on the conductors are

$$\int dS_i\, \frac{\vec{n}\cdot\vec{D}_b}{4\pi} = Q_i \quad , \tag{22.26a}$$

$$\int dS_0\, \frac{\vec{n}\cdot\vec{D}_b}{4\pi} = 0 \quad . \tag{22.26b}$$

The energy, for case (a), satisfies the following inequality,

$$E_a = \left[\int_{V-V_0} + \int_{V_0}\right] (d\vec{r})\, \frac{\vec{D}_a^{\,2}}{8\pi\varepsilon} > \int_{V-V_0} (d\vec{r})\, \frac{\vec{D}_a^{\,2}}{8\pi\varepsilon} \quad . \tag{22.27}$$

Although $\vec{D}_a$ is not the correct field $V-V_0$ for (b), it is an allowable trial function to use in the energy functional, (22.16), because it satisfies all the necessary conditions:

$$\vec{\nabla}\cdot\vec{D}_a = 4\pi\rho \quad ,$$

$$\int dS_i\, \frac{\vec{n}\cdot\vec{D}_a}{4\pi} = Q_i \quad ,$$

$$\int dS_0\, \frac{\vec{n}\cdot\vec{D}_a}{4\pi} = \int_{V_0} (d\vec{r})\, \frac{\vec{\nabla}\cdot\vec{D}_a}{4\pi} = 0 \quad ,$$

since, by hypothesis, the region V_0 originally had no charge $(\rho = 0)$. According to Thomson's Theorem, the correct field yields a minimum value of the energy functional,

$$E_b = \int_{V-V_0} (d\vec{r})\, \frac{\vec{D}_b^{\,2}}{8\pi\varepsilon} < \int_{V-V_0} (d\vec{r})\, \frac{\vec{D}_a^{\,2}}{8\pi\varepsilon} \,, \tag{22.28}$$

implying, upon comparison with (22.27),

$$\boxed{E_a > E_b} \,, \tag{22.29}$$

which states that the introduction of an uncharged conductor lowers the energy of the system.

Lecture 18

22-4. Alternate Variational Principle

In the first restricted version of the variational principle, (22.14), the charges on the conductors, (22.3), are specified. For some purposes, it is more convenient to regard the potentials, ϕ_i , on the surfaces of the conductors as specified, rather than the charges, Q_i . For simplicity we will assume that there is no volume charge density, $\rho = 0$. In order to obtain a new form of the energy functional, we note that the stationary property of (22.14) under the replacement

$$\phi \to \lambda\phi \,,$$

for λ infinitesimally different from unity, implies

$$0 = \left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=1} = 2 \int (d\vec{r}) \left(- \frac{\varepsilon}{8\pi} \right) (\vec{\nabla}\phi)^2 + \sum_i \int dS_i\, \sigma\phi \,. \tag{22.30}$$

Consequently, the energy functional is

$$\boxed{E = \frac{1}{2} \sum_i \int dS_i\, \sigma\phi} \,, \tag{22.31}$$

which becomes, for the actual field values on the surfaces, $\phi = \phi_i$,

$$E = \frac{1}{2} \sum_i Q_i \phi_i \quad . \tag{22.32}$$

Therefore, we obtain another energy functional by combining (22.32) and (22.14),

$$E = \sum_i Q_i \phi_i - \sum_i \int dS_i \, \sigma\phi + \int (d\vec{r}) \frac{\varepsilon}{8\pi} (\vec{\nabla}\phi)^2 \quad , \tag{22.33}$$

or, using (22.3),

$$\boxed{E = \sum_i \int dS_i \, \sigma(\phi_i - \phi) + \int (d\vec{r}) \frac{\varepsilon}{8\pi} (\vec{\nabla}\phi)^2} \quad . \tag{22.34}$$

Here we regard ϕ and σ to be the variables while ϕ_i is specified [note that here we impose no subsidiary restriction on σ]. Under variations in ϕ and σ , the energy changes by

$$\delta E = \sum_i \int dS_i [\delta\sigma(\phi_i - \phi) - \sigma\delta\phi] + \int (d\vec{r}) \frac{\varepsilon}{4\pi} (\vec{\nabla}\phi)\cdot(\vec{\nabla}\delta\phi) \quad , \tag{22.35}$$

which becomes

$$\delta E = \sum_i \int dS_i \left[\delta\sigma(\phi_i - \phi) - \sigma\delta\phi + \frac{\vec{n}\cdot\vec{D}}{4\pi} \delta\phi \right] + \int (d\vec{r}) \frac{\vec{\nabla}\cdot\vec{D}}{4\pi} \delta\phi \quad , \tag{22.36}$$

by identifying $\vec{D}$ and integrating by parts. The stationary principle, $\delta E = 0$, implies, from the volume part of (22.36),

$$\delta\phi : \quad \vec{\nabla}\cdot\vec{D} = 0 \quad , \tag{22.37}$$

and from the surface part,

$$\delta\phi : \quad \vec{n}\cdot\vec{D} = 4\pi\sigma \quad , \tag{22.38}$$

$$\delta\sigma : \quad \phi = \phi_i \quad . \tag{22.39}$$

These are the correct equations of electrostatics when there is no volume charge density.

22-5. Green's Function

In our study of electrostatics, we have found Green's functions to be of great use. We will here introduce Green's function in the presence of conductors that are grounded, that is $\phi_i = 0$; the corresponding differential equation is

$$-\vec{\nabla}\cdot[\epsilon\vec{\nabla}G(\vec{r},\vec{r}\,')] = 4\pi\delta(\vec{r}-\vec{r}\,') \quad , \tag{22.40}$$

with the boundary condition

$$G(\vec{r},\vec{r}\,') = 0 \quad \text{for} \quad \vec{r} \quad \text{on} \quad S_i \quad . \tag{22.41}$$

We will show that this Green's function can be used to solve the electrostatics problem in which the potentials on the conductors are specified. We wish to consider a situation for which the free charge density is zero,

$$\vec{\nabla}\cdot\vec{D} = -\vec{\nabla}\cdot(\epsilon\vec{\nabla}\phi) = 4\pi\rho = 0 \quad , \tag{22.42}$$

while the potential, ϕ_i , on each conducting surface, S_i , is constant,

$$\phi = \phi_i \quad \text{on} \quad S_i \quad . \tag{22.43}$$

If we multiply (22.40) by $\phi(\vec{r})$ and (22.42) by $G(\vec{r},\vec{r}\,')$ and subtract, we obtain

$$-\phi(\vec{r})\ \vec{\nabla}\cdot[\epsilon\vec{\nabla}G(\vec{r},\vec{r}')] + G(\vec{r},\vec{r}')\ \vec{\nabla}\cdot(\epsilon\vec{\nabla}\phi(\vec{r})) = 4\pi\delta(\vec{r}-\vec{r}')\ \phi(\vec{r}) \quad . \tag{22.44}$$

Since the left-hand side of (22.44) is a divergence,

$$\vec{\nabla}\cdot[G(\vec{r},\vec{r}')\ \epsilon\vec{\nabla}\phi(\vec{r}) - \phi(\vec{r})\ \epsilon\vec{\nabla}G(\vec{r},\vec{r}')] = 4\pi\delta(\vec{r}-\vec{r}')\ \phi(\vec{r}) \quad , \tag{22.45}$$

when we integrate over the entire volume, V, exterior to all of the conductors, we obtain an integral over a surface S which is made up of all the surfaces of the individual conductors, S_i,

$$\begin{aligned} 4\pi\phi(\vec{r}') &= \int_S d\vec{S}\cdot[G(\vec{r},\vec{r}')\ \epsilon\vec{\nabla}\phi(\vec{r}) - \phi(\vec{r})\ \epsilon\vec{\nabla}G(\vec{r},\vec{r}')] \\ &= -\sum_i \int dS_i[G(\vec{r},\vec{r}')\ \epsilon\vec{n}_i\cdot\vec{\nabla}\phi(\vec{r}) - \phi(\vec{r})\ \epsilon\vec{n}_i\cdot\vec{\nabla}G(\vec{r},\vec{r}')] \quad . \end{aligned} \tag{22.46}$$

The negative sign occurs since $d\vec{S}$ is directed out of the volume V, and so into the conductors, while $\vec{n}_i$ is the outward normal for the i^{th} conductor. Deleted here is the surface at infinity for which

$$dS \sim R^2 ,$$

$$G \lesssim \frac{1}{R} , \qquad |\vec{\nabla}G| \lesssim \frac{1}{R^2} ,$$

$$\phi \lesssim \frac{1}{R} , \qquad |\vec{\nabla}\phi| \lesssim \frac{1}{R^2} ,$$

so that the corresponding contribution goes to zero as the volume gets arbitrarily large. Now imposing the boundary conditions, (22.41) and (22.43), we obtain the desired expression for the potential, $[\epsilon' = \epsilon(\vec{r}')]$

$$\boxed{\phi(\vec{r}) = \sum_i \phi_i \int dS_i'\ \frac{\epsilon'}{4\pi}\ \vec{n}_i'\cdot\vec{\nabla}'G(\vec{r},\vec{r}')} \ , \tag{22.47}$$

where we have interchanged the roles of $\vec{r}$ and $\vec{r}'$ and used the symmetry property of G, (11.4). Therefore, if we know G (the potential due to a point charge with zero potential on the conductors), we can calculate the potential, $\phi(\vec{r})$, due to arbitrarily specified potentials on the conductors.

22-6. Capacitance

Once we know the potential, we can compute the surface charge density on the i^{th} conductor by using

$$\sigma_i = \frac{1}{4\pi}\vec{n}_i\cdot(-\epsilon\vec{\nabla}\phi)$$

$$= -\sum_j \phi_j \int dS_j'\, \frac{\epsilon}{4\pi}\frac{\epsilon'}{4\pi}(\vec{n}_i\cdot\vec{\nabla})(\vec{n}_j{}'\cdot\vec{\nabla}')\, G(\vec{r},\vec{r}') \quad . \tag{22.48}$$

The total charge on S_i is therefore

$$Q_i = \int dS_i\sigma_i = -\sum_j \phi_j \int dS_i\, dS_j'\, \frac{\epsilon}{4\pi}\frac{\epsilon'}{4\pi}(\vec{n}_i\cdot\vec{\nabla})(\vec{n}_j{}'\cdot\vec{\nabla}')\, G(\vec{r},\vec{r}') \quad . \tag{22.49}$$

Occurring here are the coefficients of capacitance, C_{ij}, defined by

$$\boxed{C_{ij} = -\int dS_i\, dS_j'\, \frac{\epsilon}{4\pi}\frac{\epsilon'}{4\pi}(\vec{n}_i\cdot\vec{\nabla})(\vec{n}_j{}'\cdot\vec{\nabla}')\, G(\vec{r},\vec{r}')} \quad , \tag{22.50}$$

$Q_i = \sum_j \phi_j C_{ij}$

which are symmetric in i and j,

$$C_{ij} = C_{ji} \quad . \tag{22.51}$$

The total charge on the i^{th} conductor is thus simply written as

$$\boxed{Q_i = \sum_j C_{ij}\,\phi_j \quad .} \tag{22.52}$$

The energy of a system of charged conductors can be expressed in terms of the coefficients of capacitance by means of (22.32),

$$E = \frac{1}{2} \sum_i Q_i \phi_i$$

$$= \frac{1}{2} \sum_{ij} \phi_i C_{ij} \phi_j \quad . \tag{22.53}$$

There is a consistency check between this expression and the variational principle which employs (22.34). Suppose we vary the potential on conductor i by an amount $\delta\phi_i$. Such a change induces variations in σ and ϕ but the resulting change in the energy from these induced variations is of second order due to the stationary principle. So the first order variation in the energy arises only from the explicit variation of ϕ_i:

$$\delta E = \int dS_i \sigma \delta\phi_i = \delta\phi_i Q_i \quad , ✓$$

or,

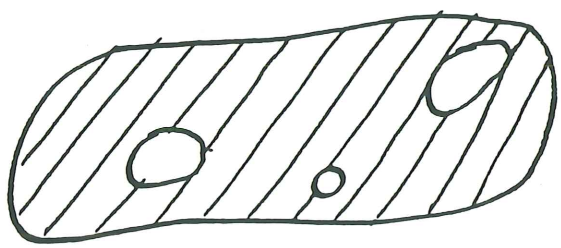
$$\frac{\partial E}{\partial \phi_i} = Q_i \quad ✓ \tag{22.54}$$

This result is in agreement with that obtained from (22.53),

$$\frac{\partial E}{\partial \phi_i} = \sum_j C_{ij} \phi_j = Q_i \quad ,$$

where we make use of (22.52).✓

Suppose the system consists of a finite region bounded by conducting surfaces,



that is, there is no surface at infinity. The total charge induced on such a system of grounded conductors by a point charge at any interior point, $\vec{r}'$, is

$$-\frac{1}{4\pi}\sum_i \int dS_i\, \epsilon\vec{n}_i\cdot\vec{\nabla}G(\vec{r},\vec{r}') = -\frac{1}{4\pi}\int (d\vec{r})(-\vec{\nabla})\cdot[\epsilon\vec{\nabla}G(\vec{r},\vec{r}')] = -1 , \qquad (22.55)$$

where we have used the first line of (22.48), with ϕ replaced by G, as well as the differential equation satisfied by the Green's function, (22.40). This implies that the coefficients of capacitance, C_{ij}, (22.50), satisfy

$$\sum_i C_{ij} = -\int dS_j'\, \frac{\epsilon'}{4\pi}(\vec{n}_j'\cdot\vec{\nabla}') \sum_i \int dS_i\, \frac{\epsilon}{4\pi}(\vec{n}_i\cdot\vec{\nabla})\, G(\vec{r},\vec{r}')$$
$$= -\int dS_j'\, \frac{\epsilon'}{4\pi}(\vec{n}_j'\cdot\vec{\nabla}')\,(1) = 0 ,$$

that is, the sum of all the coefficients of capacitance referring to a given conductor vanishes,

$$\sum_i C_{ij} = \sum_j C_{ij} = 0 . \qquad (22.56)$$

Consequently, the total charge on the conductors is zero when there is no volume charge present:

$$\sum_i Q_i = \sum_{ij} C_{ij}\,\phi_j = 0 . \qquad (22.57)$$

Furthermore, for this system, only relative values of the potential are significant. If we were to add a common constant to all potentials, all charges would remain the same:

$$Q_i = \sum_j C_{ij}\,(\phi_j + \text{constant}) = \sum_j C_{ij}\,\phi_j + \text{const}\,\underbrace{\sum_j C_{ij}}_{=0}$$

As a simple example, consider a closed system bounded by only two conductors.

In this case, in order to satisfy (22.51) and (22.56), we must have

$$C_{11} = -C_{21} = -C_{12} = C_{22} \equiv C \tag{22.58}$$

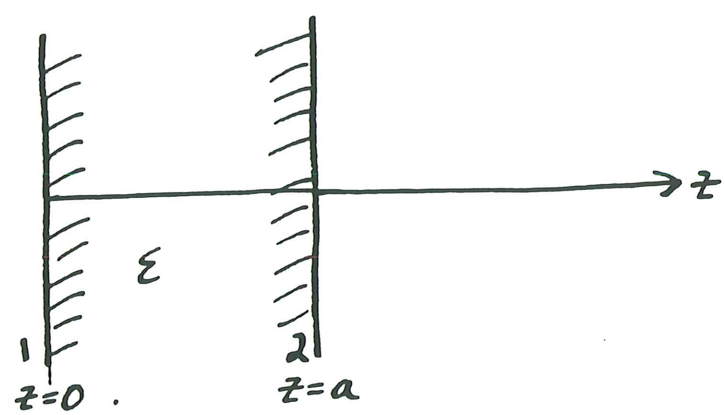
where C is called the capacitance of the system. The charges on the two conductors are

$$Q_1 = -Q_2 = C(\phi_1-\phi_2) = CV \tag{22.59}$$

where V is the potential difference between the two conductors, while the energy is

$$E = \frac{1}{2}\sum_{ij} \phi_i C_{ij} \phi_j = \frac{1}{2} CV^2 \quad . \tag{22.60}$$

As an application of these ideas, consider a capacitor constructed from two parallel conducting plates of area A. The separation of the plates, a, is assumed to be small compared to the dimensions of the plates, $a \ll \sqrt{A}$, the approximate Green's function therefore being that of two infinite plates [cf. (16.6)].



The material between the plates is characterized by a dielectric constant, ε. The above discussion applies to this situation so that the system has a capacitance C,

$$C = C_{11} = -\int dS\, dS' \frac{\varepsilon^2}{(4\pi)^2} \left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z'}\right) G\Big|_{z,z'=0} . \tag{22.61}$$

The first surface integral here was previously evaluated in (16.8a) and (16.10a):

$$\int dS\left(-\frac{1}{4\pi}\right)\frac{\partial}{\partial z} G(\vec{r},\vec{r}\,')\Big|_{z=0} = -\frac{1}{\varepsilon}\left(1 - \frac{z'}{a}\right) , \tag{22.62}$$

where we have now included the presence of ε in (22.40). The remaining surface integral is trivial,

$$C = \frac{\varepsilon}{4\pi a}\int dS' = \frac{\varepsilon A}{4\pi a} , \tag{22.63}$$

yielding the well-known result for a parallel plate capacitor.

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XXIII. Magnetostatics

23-1. Variational Principle

We now return to the general action principle of electrodynamics, (9.1), before the specialization to electrostatics. The field part of the Lagrangian in the microscopic description is

$$L = \int (d\vec{r}) \Big\{ -\rho\phi + \frac{1}{c}\vec{j}\cdot\vec{A} + \frac{1}{4\pi}\Big[\vec{E}\cdot\Big(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\phi\Big) - \vec{B}\cdot\vec{\nabla}\times\vec{A} + \frac{1}{2}(B^2-E^2)\Big]\Big\} \quad . \tag{23.1}$$

Again we will consider the static situation where all fields and densities are time independent, in particular

$$\frac{\partial}{\partial t}\vec{A} = 0 \quad . \tag{23.2}$$

When this condition is imposed, the Lagrangian can be separated into two pieces,

$$L = -\int (d\vec{r})\Big[\rho\phi + \frac{1}{4\pi}\Big(\vec{E}\cdot\vec{\nabla}\phi + \frac{1}{2}E^2\Big)\Big] + \int (d\vec{r})\Big[\frac{1}{c}\vec{j}\cdot\vec{A} + \frac{1}{4\pi}\Big(-\vec{B}\cdot\vec{\nabla}\times\vec{A} + \frac{1}{2}B^2\Big)\Big] \quad , \tag{23.3}$$

where the first part, as we have seen, corresponds to electrostatics. The second part describes magnetostatics, which is the subject of our investigation here. Notice that this separation is possible only because of the condition (23.2); otherwise, electric and magnetic effects are interrelated. Analogously to our incorporation of dielectrics in electrostatics (see Section X), we here pass to a macroscopic description of fields in permeable media. The energy, for these circumstances, becomes

$$E = -\int (d\vec{r})\left[\frac{1}{c}\vec{J}\cdot\vec{A} + \frac{1}{4\pi\mu}\left(-\vec{B}\cdot\vec{\nabla}\times\vec{A} + \frac{1}{2}B^2\right)\right] , \tag{23.4}$$

where μ is the permeability of the medium. We now have to check that the stationary principle applied to this form of the energy yields the correct equations of magnetostatics. We are to regard $\vec{A}$ and $\vec{B}$ as the independent variables so the variation of the energy is

$$\delta E = -\int (d\vec{r})\left[\frac{1}{c}\vec{J}\cdot\delta\vec{A} - \frac{1}{4\pi}\frac{\vec{B}}{\mu}\cdot\vec{\nabla}\times\delta\vec{A} + \frac{1}{4\pi\mu}\delta\vec{B}\cdot(\vec{B}-\vec{\nabla}\times\vec{A})\right] . \tag{23.5}$$

From the coefficient of $\delta\vec{B}$, we obtain

$$\vec{B} = \vec{\nabla}\times\vec{A} , \tag{23.6a}$$

which is equivalent to

$$\vec{\nabla}\cdot\vec{B} = 0 . \tag{23.6b}$$

By making use of the identity

$$\frac{\vec{B}}{\mu}\cdot\vec{\nabla}\times\delta\vec{A} = \vec{\nabla}\cdot\left(\delta\vec{A}\times\frac{\vec{B}}{\mu}\right) + \delta\vec{A}\cdot\left(\vec{\nabla}\times\frac{\vec{B}}{\mu}\right) , \tag{23.7}$$

and discarding the implied surface integral, we find from the vanishing of the coefficient of $\delta\vec{A}$,

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} , \tag{23.8a}$$

a consequence of which is that only steady currents occur here:

$$\vec{\nabla}\cdot\vec{J} = 0 . \tag{23.8b}$$

As appropriate to macroscopic media, we have introduced the magnetic field,

$$\vec{H} = \frac{\vec{B}}{\mu} \quad . \tag{23.9}$$

Thus we have recovered Maxwell's equations in the static limit, (23.6b) and (23.8a).

As in electrostatics, there is a restricted version of the stationary principle for the energy. If we take

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

as the definition of $\vec{B}$, the expression for the energy becomes

$$E = -\int (d\vec{r}) \left[\frac{1}{c} \vec{J} \cdot \vec{A} - \frac{1}{8\pi\mu} (\vec{\nabla} \times \vec{A})^2 \right] \quad . \tag{23.10}$$

Regarding this as stationary under variations in $\vec{A}$, we derive the equation satisfied by the vector potential,

$$\vec{\nabla} \times \left(\frac{1}{\mu} \vec{\nabla} \times \vec{A} \right) = \frac{4\pi}{c} \vec{J} \quad , \tag{23.11}$$

which coincides with (23.8a).

Proceeding in a manner parallel to the corresponding discussion in electrostatics (see Subsection 10-2), we consider a change in the permeability, $\delta\mu(\vec{r})$. Because of the stationary property, the only first order variation in the energy arises from the explicit appearance of μ in (23.10):

$$\delta E = -\int (d\vec{r}) \frac{\delta\mu}{\mu^2} \frac{1}{8\pi} (\vec{\nabla} \times \vec{A})^2 = -\int (d\vec{r}) \frac{\delta\mu}{8\pi} H^2 \quad , \tag{23.12}$$

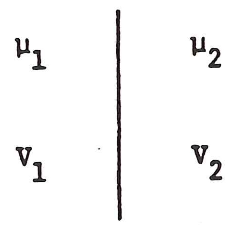
which is the analog of (10.13). In particular, by considering $\delta\mu$ to arise from a displacement of the material, we infer the force on the (inhomogeneous) permeable medium to be [cf. (10.17)]

$$\vec{F} = -\int (d\vec{r})\, \frac{H^2}{8\pi}\, \vec{\nabla}\mu \quad ; \tag{23.13}$$

a diamagnetic material, with $\mu < 1$, is repelled from a region of stronger magnetic field.

23-2. Boundary Conditions

The simplest example of an inhomogeneous magnetic material occurs when μ is discontinuous across an interface.



First we consider the boundary conditions that $\vec{B}$ and $\vec{A}$ must satisfy across the interface. The fact that $\vec{B}$ is the curl of $\vec{A}$ implies that the tangential component of $\vec{A}$, $\vec{A}_t$, must be continuous across the boundary, in order that $\vec{B}_t$ be finite:

$$\vec{n}_1 \times \vec{A}_1 + \vec{n}_2 \times \vec{A}_2 = 0 \quad , \tag{23.14a}$$

or,

$$\vec{n}_1 \times (\vec{A}_1 - \vec{A}_2) = 0 \quad , \tag{23.14b}$$

where $\vec{n}_1(\vec{n}_2)$ is the <u>outward</u> normal to $V_1(V_2)$ so that $\vec{n}_2 = -\vec{n}_1$. The relation, (23.14a), is true for all points on the surface. Thus, when we take the divergence of this expression, in which only tangential components of $\vec{\nabla}$ occur, we find

$$\vec{n}_1\cdot\vec{B}_1 + \vec{n}_2\cdot\vec{B}_2 = 0 \quad , \tag{23.15a}$$

or

$$\vec{n}_1\cdot(\vec{B}_1 - \vec{B}_2) = 0 \quad , \tag{23.15b}$$

that is, the normal component of $\vec{B}$ is continuous across the boundary. [We may regard this as a surface version of $\vec{\nabla}\cdot\vec{B} = 0$.]

If we include the possibility that there is a surface current, $\vec{K}$, on the boundary between V_1 and V_2 , we must amend the energy expression, (23.4), to read

$$E = -\int (d\vec{r})\left[\frac{1}{c}\vec{J}\cdot\vec{A} + \frac{1}{4\pi\mu}\left(-\vec{B}\cdot\vec{\nabla}\times\vec{A} + \frac{1}{2}B^2\right)\right] - \int dS\,\frac{1}{c}\vec{K}\cdot\vec{A} \quad . \tag{23.16}$$

In our previous discussion of the variation in the energy, we discarded the surface integral [see (23.7)]; this is no longer permissible because of the presence of the boundary. Consequently, there is a new contribution to the variation of the energy arising from the occurrence of the interface,

$$\begin{aligned}
\delta E &= \int_{V_1}(d\vec{r})\,\vec{\nabla}\cdot\left(\delta\vec{A}_1\times\frac{\vec{H}_1}{4\pi}\right) + \int_{V_2}(d\vec{r})\,\vec{\nabla}\cdot\left(\delta\vec{A}_2\times\frac{\vec{H}_2}{4\pi}\right) - \int dS\,\frac{1}{c}\vec{K}\cdot\delta\vec{A} \\
&= \int dS\left[\vec{n}_1\cdot\frac{\delta\vec{A}_1\times\vec{H}_1}{4\pi} + \vec{n}_2\cdot\frac{\delta\vec{A}_2\times\vec{H}_2}{4\pi} - \frac{1}{c}\vec{K}\cdot\delta\vec{A}\right] \\
&= -\int dS\,\delta\vec{A}\cdot\left[\frac{\vec{n}_1\times\vec{H}_1}{4\pi} + \frac{\vec{n}_2\times\vec{H}_2}{4\pi} + \frac{1}{c}\vec{K}\right] \quad .
\end{aligned} \tag{23.17}$$

x prod is cyclic, so $\vec{H}_1\cdot\hat{n}_1\times\delta\vec{A}_1 =$ $\delta\vec{A}_1\cdot\vec{H}_1\times\hat{n}_1$ ...
$\therefore \delta\vec{A}_1\cdot\hat{n}_1\times\vec{H}_1$ ...

Here we have used the fact that $\vec{A}_t$ must be continuous,

$$\delta\vec{A}_{1t} = \delta\vec{A}_{2t} = \delta\vec{A}_t \quad .$$

We then conclude from the stationary principle on the surface that

$$\vec{n}_1 \times \vec{H}_1 + \vec{n}_2 \times \vec{H}_2 + \frac{4\pi}{c}\vec{K} = 0 \quad . \tag{23.18}$$

When no surface current is present, $\vec{K} = 0$, this states that the tangential component of $\vec{H}$ is continuous. If, in addition, we have $\mu_2 >> \mu_1$ [idealized as $\mu_2 \to \infty$; we might call this a perfect magnetic conductor (see Section XLV)], the magnetic field in medium 2 goes to zero,

$$\vec{H}_2 = \frac{1}{\mu_2}\vec{B}_2 \to 0 \quad ,$$

so that $\vec{H}_1$ is normal to the surface,

$$\vec{n}_1 \times \vec{H}_1 = 0 \quad .$$

This is the same condition satisfied by the electric field at the surface of a conductor.

Lecture 19

23-3. Vector Potential

The fundamental equation of magnetostatics is (23.11),

$$\vec{\nabla} \times \left(\frac{1}{\mu}\vec{\nabla} \times \vec{A}\right) = \frac{4\pi}{c}\vec{J} \quad , \tag{23.19}$$

which reduces for vacuum ($\mu = 1$), to

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} \quad . \tag{23.20}$$

This equation can be simplified by using the fact that it is invariant under a gauge transformation [see (9.27)],

$$\vec{A} \to \vec{A} - \vec{\nabla}\lambda \quad .$$

Because of this gauge freedom, we can usually choose some particular gauge to simplify the problem at hand. In the present situation, the convenient choice of gauge is one for which

$$\vec{\nabla}\cdot\vec{A} = 0 \quad , \tag{23.21}$$

called the radiation, Coulomb, or transverse gauge. To show that it is always possible to choose this gauge, suppose we start with a vector potential, $\vec{A}_o$, which does not satisfy this condition,

$$\vec{\nabla}\cdot\vec{A}_o \neq 0 \quad .$$

It is possible to make a gauge transformation,

$$\vec{A}_o \to \vec{A} = \vec{A}_o - \vec{\nabla}\lambda \quad ,$$

such that (23.21) is satisfied, that is

$$\vec{\nabla}\cdot(\vec{A}_o - \vec{\nabla}\lambda) = 0 \quad ,$$

for then λ is a solution to Poisson's equation,

$$\nabla^2\lambda = \vec{\nabla}\cdot\vec{A}_o \quad .$$

In the radiation gauge, (23.20) becomes

$$-\nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} \quad , \tag{23.22}$$

the solution of which is

$$\vec{A}(\vec{r}) = \frac{1}{c}\int (d\vec{r}\,')\,\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \quad , \tag{23.23}$$

in precise analogy to the solution of the electrostatic problem,

$$-\nabla^2\phi = 4\pi\rho \quad .$$

As a consistency check, we verify explicitly that (23.23) satisfies the radiation gauge condition (23.21),

$$\begin{aligned}
\vec{\nabla}\cdot\vec{A}(\vec{r}) &= \frac{1}{c}\int (d\vec{r}\,')\left(\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\cdot\vec{J}(\vec{r}\,') \\
&= -\frac{1}{c}\int (d\vec{r}\,')\left(\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\cdot\vec{J}(\vec{r}\,') \\
&= \frac{1}{c}\int (d\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|}\vec{\nabla}'\cdot\vec{J}(\vec{r}\,') = 0
\end{aligned}$$

where we have used (23.8b), and the fact that the current distribution is localized.

Once we have the vector potential, we can compute the magnetic field $\vec{B}$:

$$\begin{aligned}
\vec{B} = \vec{\nabla}\times\vec{A} &= \frac{1}{c}\vec{\nabla}\times\int (d\vec{r}\,')\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \\
&= \frac{1}{c}\int (d\vec{r}\,')\,\vec{J}(\vec{r}\,')\times\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \quad .
\end{aligned} \tag{23.24}$$

For a point charge moving with velocity $\vec{v}$, the electric current is

$$\vec{J}(\vec{r}) = e\vec{v}\,\delta(\vec{r}-\vec{R}) \quad , \tag{23.25}$$

where $\vec{R}$ is the position of the particle, which produces the magnetic field

$$\vec{B} = \frac{\vec{v}}{c}\times\frac{e(\vec{r}-\vec{R})}{|\vec{r}-\vec{R}|^3} \quad . \tag{23.26}$$

This has the form

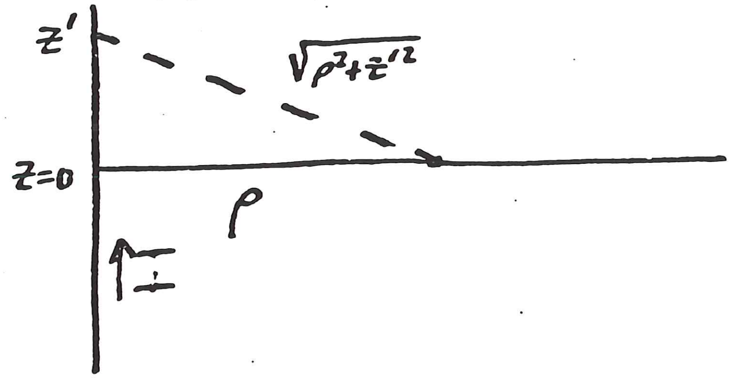
$$\vec{B} = \frac{\vec{v}}{c} \times \vec{E} ,$$

which was our starting point for introducing magnetic fields in Section I. [Of course, we have now transcended the domain of magnetostatics since $\partial\vec{A}/\partial t \neq 0$. However, since $\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ is of order v^2/c^2 , (23.26) is the correct magnetic field to first order in v/c . For the general case, see Section XXVIII.]

In the following two brief sections, we will develop some applications of magnetostatics. Many situations in magnetostatics can be attacked by evident variations on the techniques developed for electrostatics.

XXIV. Macroscopic Current Distributions

The simplest example of a macroscopic current is that which flows in a long straight wire. We will take the wire to lie along the z axis, carrying a current I that flows in the $+z$ direction. We will let the direction of current flow be denoted, generally, by $\vec{n}$.



We wish to find the magnetic induction $\vec{B}$ produced by this current. Since $\vec{B}$ is independent of z , without loss of generality we evaluate it at $z = 0$. For a wire with negligible cross section, any volume integral involving the current density becomes a line integral

$$\int (d\vec{r})\, \vec{J}(\vec{r}\,') \ldots = \int dz'\, dS'\, \vec{n}\, J(\vec{r}\,') \ldots = \int dz'\, \vec{n}\, I \ldots \quad . \tag{24.1}$$

The expression for the magnetic field, (23.24), a distance ρ from the wire, is then reduced to

$$\vec{B} = \frac{I}{c} \int_{-\infty}^{\infty} dz' \left(\vec{\nabla} \frac{1}{\sqrt{\rho^2+z'^2}} \right) \times \vec{n} \quad . \tag{24.2}$$

For a long wire of length $2L$, $L >> \rho$, the basic integral occurring here is

$$\int_{-L}^{L} dz' \frac{1}{\sqrt{\rho^2+z'^2}} = 2 \int_{0}^{L} dz' \frac{1}{\sqrt{z'^2+\rho^2}}$$

$$\simeq 2\left[\ell og \frac{L}{\rho} + \text{constant}\right] , \tag{24.3}$$

the gradient of which is

$$\vec{\nabla}\left(2\,\ell og \frac{L}{\rho} + \text{constant}\right) = -2\,\frac{\vec{\nabla}\rho}{\rho} = -\frac{2}{\rho}\,\hat{\rho} . \tag{24.4}$$

The magnetic field produced by this wire is therefore

$$\vec{B} = \frac{2I}{c\rho}\,(\vec{n} \times \hat{\rho}) . \tag{24.5}$$

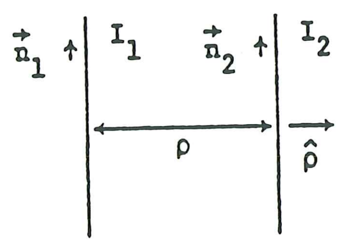
The force exerted on a current distribution by a magnetic field is [see (1.28)]

$$\vec{F} = \frac{1}{c}\int (d\vec{r})\ \vec{J} \times \vec{B} , \tag{24.6}$$

which, when specialized to current flowing in a long straight wire, becomes

$$\vec{F} = \int dz\,\frac{I}{c}\,\vec{n} \times \vec{B} . \tag{24.7}$$

One possibility is that the magnetic field is produced by a second, parallel, current-carrying wire.



Of course, the total force acting on I_2 due to the magnetic field produced by I_1 is unbounded. The quantity of interest is the force per unit length on I_2 ,

$$\frac{\text{force}}{\text{length}} = \frac{I_2}{c}\vec{n}_2 \times \vec{B} = \frac{I_2}{c}\frac{2I_1}{c\rho}\vec{n}_2 \times (\vec{n}_1 \times \hat{\rho}) \quad . \tag{24.8}$$

For parallel flowing currents, $\vec{n}_1 = \vec{n}_2$, so

$$\frac{\text{force}}{\text{length}} = -\frac{2I_1I_2}{c^2}\frac{\hat{\rho}}{\rho} , \tag{24.9}$$

that is, the force is attractive. If the currents flow in opposite senses, the force is repulsive.

We can also obtain the above result by recalling that the force is the negative gradient of the energy,

$$\vec{F} = -\vec{\nabla}E , \tag{24.10}$$

where the energy is given by (23.10), with $\mu = 1$. Equation (23.10) is quite analogous to the electrostatic energy, (10.7), except for the overall sign, which implies that the sense of attraction or repulsion is reversed when we go from static charge distributions to steady current flows. By integrating by parts on the $(\vec{\nabla} \times \vec{A})^2$ term and then using the differential equation (23.11), with $\mu = 1$, we may rewrite the magnetostatic energy as

$$E = -\frac{1}{2c}\int (d\vec{r})\ \vec{J}\cdot\vec{A} \quad . \tag{24.11}$$

[Notice that (24.11) is gauge invariant, since under a gauge transformation,

$$\vec{A} \rightarrow \vec{A} - \vec{\nabla}\lambda ,$$

the energy does not change,

$$\delta E = \frac{1}{2c}\int (d\vec{r})\ \vec{J}\cdot\vec{\nabla}\lambda = 0 ,$$

since we can integrate by parts and use (23.8b).] Introducing the explicit form for $\vec{A}$, (23.23), we can write the energy in terms of the current density alone,

$$E = -\frac{1}{2}\int (d\vec{r})(d\vec{r}\,')\,\frac{\frac{1}{c}\vec{J}(\vec{r})\cdot\frac{1}{c}\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} , \tag{24.12}$$

which is analogous to the electrostatic result, (11.8), except that its sign is opposite. For the case of two current distributions,

$$\vec{J}(\vec{r}) = \vec{J}_1(\vec{r}) + \vec{J}_2(\vec{r}) ,$$

the energy expression contains self energies as well as the mutual interaction energy. We are here interested only in the latter, which is

$$E = -\int (d\vec{r})(d\vec{r}\,')\,\frac{\frac{1}{c}\vec{J}_1(\vec{r})\cdot\frac{1}{c}\vec{J}_2(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} , \tag{24.13}$$

as it is the sole term that contributes to the force, (24.10). For straight wires, this becomes

$$E = -\int dz\,dz'\,\frac{I_1 I_2}{c^2}\,\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{r}-\vec{r}\,'|} . \tag{24.14}$$

For parallel wires with currents flowing in the same sense,

$$\vec{n}_1\cdot\vec{n}_2 = 1 ,$$

$$|\vec{r}-\vec{r}\,'| = \sqrt{\rho^2 + (z-z')^2} ,$$

the integration over z' is evaluated as in (24.3), so that

$$E = -\frac{I_1 I_2}{c^2}\, 2\left[\ell og \frac{L}{\rho} + \text{constant} \right] \int dz$$

$$= -\frac{I_1 I_2}{c^2}\, 2\left[\ell og \frac{L}{\rho} + \text{constant} \right] 2L \quad , \qquad (24.15)$$

where we have used the restriction $L/\rho >> 1$. The force can now be calculated from (24.10), or, since E depends only on ρ ,

$$\vec{F} = -\left(\frac{\partial}{\partial\rho} E \right) \hat{\rho} \quad .$$

The force per unit length is therefore

$$\frac{F}{2L} = -\frac{2I_1 I_2}{c^2} \frac{\hat{\rho}}{\rho} \quad , \qquad (24.16)$$

which is our previous result, (24.9).

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XXV. Magnetic Multipoles

25-1. Magnetic Dipole Moment

We now direct our attention to the magnetic field produced by a confined current distribution. If we wish to evaluate the vector potential far outside the current distribution, $|\vec{r}| >>$ dimension of region of current flow (where the origin of the coordinate system is located in the current distribution), we may use the expansion

$$\frac{1}{\sqrt{|\vec{r}-\vec{r}'|^2}} = \frac{1}{\sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2}} = \frac{1}{r\sqrt{1 - \frac{2\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}}} \simeq \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}'}{r^2}\right)$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \ldots \tag{25.1}$$

in the expression for $\vec{A}$, (23.23). The resulting expansion for the vector potential is then

$$\vec{A}(\vec{r}) = \frac{1}{r}\int (d\vec{r}')\,\frac{1}{c}\,\vec{J}(\vec{r}') + \frac{\vec{r}}{r^3}\cdot\int (d\vec{r}')\,\vec{r}'\left[\frac{1}{c}\,\vec{J}(\vec{r}')\right] + \ldots \tag{25.2}$$

which is analogous to the expansion for ϕ, (19.3). From current conservation for steady currents,

$$\vec{\nabla}\cdot\vec{J}(\vec{r}) = 0 \quad ,$$

the first term of (25.2) vanishes for a confined current distribution:

$$0 = \int (d\vec{r})\,\vec{r}\,\vec{\nabla}\cdot\vec{J} = \int (d\vec{r})\,[\vec{\nabla}\cdot(\vec{J}\,\vec{r}) - \vec{J}] = -\int (d\vec{r})\,\vec{J} \quad ,$$

$r_i \partial_k J_k$

$\partial_k(r_i J_k)$

$- J_k \partial_k r_i$

so that there is no $1/r$ term in the expansion of $\vec{A}(\vec{r})$. Physically, there is no Coulomb-like potential for magnetism, when magnetic charge is not present. The leading term in the vector potential expansion is therefore

$$\vec{A}(\vec{r}) = \frac{1}{r^3}\frac{1}{c}\int (d\vec{r}')\,\vec{r}\cdot\vec{r}'\,\vec{J}(\vec{r}') + \ldots \quad . \tag{25.3}$$

To evaluate this integral, we again use (23.8b) and consider the integral

$$0 = \int (d\vec{r})\, x_i x_j \sum_{k=1}^{3} \nabla_k J_k$$

$$= -\int (d\vec{r})\,(x_j J_i + x_i J_j) \quad ,$$

$x_i x_j \partial_k J_k = \partial_k(x_i x_j J_k) - \int \partial_k(x_i x_j)\, J_k$

$= -x_j J_i - x_i J_j$

or, in a dyadic notation,

$$\int (d\vec{r})[\vec{r}\,\vec{J} + \vec{J}\,\vec{r}] = 0 \quad . \tag{25.4}$$

Using this fact, we make the following rearrangement:

$$\int (d\vec{r}\,')\, \vec{r}\cdot\vec{r}\,'\,\vec{J} = \frac{1}{2}\int (d\vec{r}\,')\, \vec{r}\cdot[(\vec{r}\,'\vec{J} + \vec{J}\,\vec{r}\,') + (\vec{r}\,'\vec{J} - \vec{J}\,\vec{r}\,')]$$

$$= \frac{1}{2}\int (d\vec{r}\,')\, \vec{r}\cdot(\vec{r}\,'\vec{J} - \vec{J}\,\vec{r}\,')$$

$$= \frac{1}{2}\int (d\vec{r}\,')(\vec{r}\,'\times\vec{J})\times\vec{r} \quad . \tag{25.5}$$

$= \frac{1}{2}\int d\vec{r}\,'(\vec{r}\times\vec{J})\times\vec{r}\,'$

$\epsilon_{ijk}\, r_j\, \epsilon_{klm} J_l r'_m$

$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) r_j J_l r'_m$

Sign problems?

$(r\cdot r')J_i - (r\cdot J) r'_i$

$= \vec{r}\cdot(\vec{r}\,'\vec{J} - \vec{J}\,\vec{r}\,')$

$(r'\times J)\times r$

$= r\times(J\times r')$

$= (r\times J)\times r'$

The leading term of the vector potential now becomes

$$\vec{A}(\vec{r}) = \frac{\vec{\mu}\times\vec{r}}{r^3} \tag{25.6}$$

$-\frac{1}{r^3}\frac{1}{2c}\int (r\times J)\times r'\; d\vec{r}\,'$

$= \frac{1}{r^3}\;\frac{1}{2c}$

where $\vec{\mu}$ is the magnetic dipole moment, defined by

$$\vec{\mu} = \frac{1}{2c}\int (d\vec{r})\; \vec{r}\times\vec{J}(\vec{r}) \quad . \tag{25.7}$$

[For a point charge, the current density is given by (23.25), so the magnetic dipole moment is

$$\vec{\mu} = \frac{e}{2c}\vec{R}\times\vec{v} \quad ,$$

$\vec{v} = \vec{r}_c\times\vec{\omega}_c \qquad \vec{r} = \frac{\vec{v}\times\omega_c}{\omega_c^2}$

$\vec{\mu} = \frac{e}{2c}\vec{R}\times\vec{\omega}_c\times\vec{r}$

$= \frac{e}{2c}\frac{v^2}{\omega_c} = \frac{1}{2}\frac{v^2 e}{c}\frac{mc}{eB} = \frac{\frac{1}{2}mv^2}{B}$

in agreement with (6.18).]

The leading contribution to the magnetic field can now be calculated from this vector potential,

$$\vec{B} = \vec{\nabla} \times \left(\vec{\mu} \times \frac{\vec{r}}{r^3} \right) = \vec{\mu}\, \vec{\nabla} \cdot \frac{\vec{r}}{r^3} - (\vec{\mu}\cdot\vec{\nabla}) \frac{\vec{r}}{r^3}$$

$$= 4\pi\vec{\mu}\delta(\vec{r}) - \vec{\nabla}\left(\frac{\vec{\mu}\cdot\vec{r}}{r^3} \right) , \tag{25.8}$$

$\epsilon_{ijk}\partial_j \epsilon_{klm} A_l B_m$

$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j A_l B_m$

$\cancel{A_l \partial_m} B_m - B_m \partial_j A_l$

$\partial_m A_i B_m - \partial_j B_i A_m$

$\vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$

since

$$-\vec{\nabla}\frac{1}{r} = \frac{\vec{r}}{r^3} ,$$

$\mu_i \partial_i \frac{r_j}{r^3}$

$\partial_i \mu_i \frac{r_j}{r^3}$

and consequently

$$0 = \vec{\mu} \times \left(\vec{\nabla} \times \frac{\vec{r}}{r^3} \right) = \vec{\nabla}\left(\frac{\vec{\mu}\cdot\vec{r}}{r^3} \right) - (\vec{\mu}\cdot\vec{\nabla}) \frac{\vec{r}}{r^3} . \tag{25.9}$$

?

The delta function term in $\vec{B}$ is necessary in order to satisfy $\vec{\nabla}\cdot\vec{B} = 0$:

$$\vec{\nabla}\cdot\vec{B} = 4\pi(\vec{\mu}\cdot\vec{\nabla})\, \delta(\vec{r}) + \nabla^2(\vec{\mu}\cdot\vec{\nabla}) \frac{1}{r} = 0 . \tag{25.10}$$

$\mu_i \partial_i$

$u_j \partial_j \partial_k f$

$\partial_k^2 \mu_j \partial_j f$

For $r > 0$, this magnetic field has the same form as that of the electric field produced by an electric dipole moment, which is contained in (19.7), that is

$$\vec{B}(\vec{r}) = \frac{3\vec{r}\,\vec{\mu}\cdot\vec{r} - r^2\vec{\mu}}{r^5} . \tag{25.11}$$

$3\vec{r}\, u\cdot\vec{r}$ -

In general, this is only the leading contribution, since there are higher multipoles. We will not, however, explore these further here.

25-2. Rotating Charged Spherical Shell

An example for which the dipole expression is exact beyond a certain distance will be achieved if a charge e is distributed uniformly over a spherical shell of radius a that is rotating with angular velocity $\vec{\omega}$. If we choose the origin to be at the center of the sphere, the velocity of a point $\vec{r}'$ on the surface is

$$\vec{v} = \vec{\omega} \times \vec{r}' \quad . \tag{25.12}$$

The current density is

$$\vec{J} = \rho\vec{v} \quad ,$$

where here the charge density is entirely concentrated on the surface,

$$\int (d\vec{r}')\, \rho \ldots = \int dS'\, \sigma \ldots \quad ,$$

where the surface charge density is constant,

$$\sigma = \frac{e}{4\pi a^2} \quad .$$

The expression for the vector potential, (23.23), becomes

$$\vec{A}(\vec{r}) = \frac{1}{c} \int dS' \frac{e}{4\pi a^2} \frac{\vec{\omega} \times \vec{r}'}{|\vec{r}-\vec{r}'|} \quad , \tag{25.13}$$

where $\vec{r}$ may be either inside or outside the sphere.

We first calculate $\vec{A}$ inside the sphere, that is, for $|\vec{r}| < a = |\vec{r}'|$. Recalling the expansion, (18.10),

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{a^{\ell+1}} P_\ell(\cos\gamma) \quad , \tag{25.14}$$

and noting that $\vec{r}'$ occurring in (25.13) is related to Y_{1m}, we see that the surface integral selects only $\ell = 1$:

$$\int dS' \, Y_{\ell m} \, Y_{1m'} \sim \delta_{\ell 1} \, \delta_{mm'} \quad .$$

Therefore, only the $\ell = 1$ term in (25.14) will contribute to the integral (25.13),

$$\frac{1}{|\vec{r}-\vec{r}'|} \xrightarrow{\ell=1} \frac{r}{a^2} \frac{\vec{r}\cdot\vec{r}'}{rr'} = \frac{1}{a^3} \vec{r}\cdot\vec{r}' \quad , \tag{25.15}$$

yielding for the vector potential,

$$\vec{A}(\vec{r}) = \frac{e}{4\pi a^2 c} \int dS' (\vec{\omega} \times \vec{r}') \frac{\vec{r}'\cdot\vec{r}}{a^3} \quad . \tag{25.16}$$

Using spherical symmetry, we easily evaluate the integral over the dyadic to be

$$\int dS' \, \vec{r}'\vec{r}' = \int dS' \, \frac{1}{3} \overleftrightarrow{1} \, r'^2 = \frac{1}{3} \, a^2 4\pi a^2 \, \overleftrightarrow{1} \quad . \tag{25.17}$$

Therefore, the vector potential inside the sphere is

$$\vec{A} = \frac{e}{3ac} (\vec{\omega} \times \vec{r}) \equiv \frac{1}{2} \vec{B} \times \vec{r} \quad , \tag{25.18}$$

where, using the result of (6.14), we identify the magnetic field $\vec{B}$ as

$$\vec{B} = \frac{2e}{3ac} \vec{\omega} \quad , \tag{25.19}$$

which is uniform inside the sphere.

We now calculate $\vec{A}$ outside the sphere, where $|\vec{r}| > a = |\vec{r}'|$, and the appropriate expansion is

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{a^{\ell}}{r^{\ell+1}} P_{\ell} \xrightarrow{\ell=1} \frac{a}{r^2} \frac{\vec{r}\cdot\vec{r}\,'}{rr'} = \frac{1}{r^3} \vec{r}\cdot\vec{r}\,' \quad , \tag{25.20}$$

since, as before, only $\ell = 1$ contributes to the surface integral. The calculation proceeds as above except for a factor of a^3/r^3 with the result

$$\vec{A} = \frac{ea^2}{3c} \frac{\vec{\omega} \times \vec{r}}{r^3} = \frac{\vec{\mu} \times \vec{r}}{r^3} \quad , \tag{25.21}$$

which, upon comparison with (25.6), allows us to identify the magnetic dipole moment as

$$\vec{\mu} = \frac{ea^2}{3c} \vec{\omega} \quad . \tag{25.22}$$

Notice that $\vec{B}$ is discontinuous across the spherical shell because there is a surface current density. The values of $\vec{B}$ just outside and just inside the surface ($\vec{n}$ = outward normal = $\vec{r}/a$) are

$$r = a+0 \;:\; \vec{B}_+ = \frac{3(\vec{\mu}\cdot\vec{n})\vec{n} - \vec{\mu}}{a^3} = \frac{ea^2}{3c} \frac{3(\vec{\omega}\cdot\vec{n})\vec{n} - \vec{\omega}}{a^3} \quad , \tag{25.23}$$

$$r = a-0 \;:\; \vec{B}_- = \frac{2e}{3ca} \vec{\omega} \quad , \tag{25.24}$$

so the discontinuity in $\vec{B}$ is

$$\vec{B}_+ - \vec{B}_- = \frac{e}{ca} [(\vec{\omega}\cdot\vec{n})\vec{n} - \vec{\omega}] = \frac{e}{ca} \vec{n} \times (\vec{n} \times \vec{\omega}) \quad . \tag{25.25}$$

Now recall that in vacuum ($\vec{B} = \vec{H}$), the normal component of $\vec{B}$ is continuous while the tangential component is discontinuous if there is a surface current. Written in the notation above, these boundary conditions [(23.15b) and (23.18)] read

$$\vec{n}\cdot(\vec{B}_+ - \vec{B}_-) = 0 \quad , \tag{25.26a}$$

$$\vec{n} \times (\vec{B}_+ - \vec{B}_-) = \frac{4\pi}{c}\vec{K} \quad . \tag{25.26b}$$

Obviously, (25.26a) is satisfied. From (25.26b) and (25.25), we calculate the surface current density,

$$\vec{K} = \frac{c}{4\pi}\vec{n} \times \left[\frac{e}{ca}\vec{n} \times (\vec{n} \times \vec{\omega})\right] = \frac{e}{4\pi a}\vec{\omega} \times \vec{n} \quad , \tag{25.27}$$

in agreement with the direct result ($\vec{r}' = a\vec{n}$) ,

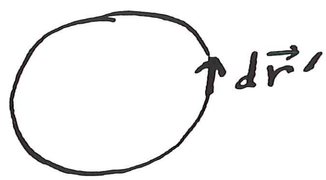
$$\vec{K} = \sigma\vec{v} = \frac{e}{4\pi a^2}(\vec{\omega} \times \vec{n})a = \frac{e}{4\pi a}\vec{\omega} \times \vec{n} \quad .$$

XXVI. Magnetic Scalar Potential

We now return to the macroscopic situation with a steady current flowing in a permeable medium characterized by a constant μ, so that the vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu}{c}\int (d\vec{r}\,')\,\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} , \tag{26.1}$$

in the Coulomb gauge. In particular, consider a current I flowing in a closed loop,



so that the volume integral is to be replaced by a line integral,

$$\int (d\vec{r}\,')\,\vec{J} \rightarrow \int d\vec{r}\,'\,I ,$$

where $d\vec{r}\,'$ is a directed line element tangential to the wire, in the direction of the current flow. Now the vector potential, (26.1), becomes

$$\vec{A}(\vec{r}) = \frac{\mu}{c}\,I\oint \frac{d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} , \tag{26.2}$$

which implies for the magnetic induction,

$$\vec{B} = \frac{\mu}{c}\,I\,\vec{\nabla}\times\oint \frac{d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} \tag{26.3}$$

or, for the magnetic field,

$$\vec{H} = \frac{I}{c}\oint\left[\vec{\nabla}\,\frac{1}{|\vec{r}-\vec{r}\,'|}\right]\times d\vec{r}\,' = \frac{I}{c}\oint d\vec{r}\,'\times\vec{\nabla}'\,\frac{1}{|\vec{r}-\vec{r}\,'|} . \tag{26.4}$$

It is convenient to rewrite $\vec{H}$ in terms of a surface integral instead of a line integral. We make use of Stokes' theorem for a vector field, $\vec{V}$, which reads

$$\oint_C d\vec{r}'\cdot\vec{V} = \int_S d\vec{S}'\cdot(\vec{\nabla}' \times \vec{V}) \quad , \tag{26.5}$$

where S is any surface which has the contour C as its boundary. If we replace

$$\vec{V} \to \vec{V} \times \vec{a}$$

where $\vec{a}$ is an arbitrary constant vector, Stokes' theorem becomes

$$\begin{aligned}\oint_C d\vec{r}' \times \vec{V}\cdot\vec{a} &= \int_S d\vec{S}'\cdot\vec{\nabla}' \times (\vec{V} \times \vec{a}) \\ &= \int_S d\vec{S}'\cdot[(\vec{a}\cdot\vec{\nabla}')\vec{V} - \vec{a}\,\vec{\nabla}'\cdot\vec{V}] \quad .\end{aligned} \tag{26.6}$$

The identity

$$\vec{a} \times (\vec{\nabla}' \times \vec{V}) = \vec{\nabla}'(\vec{a}\cdot\vec{V}) - (\vec{a}\cdot\vec{\nabla}')\,\vec{V} \quad ,$$

allows us to rewrite (26.6) as

$$\oint_C d\vec{r}' \times \vec{V}\cdot\vec{a} = \int_S d\vec{S}'\cdot[\vec{\nabla}'(\vec{V}\cdot\vec{a}) - \vec{a} \times (\vec{\nabla}' \times \vec{V}) - \vec{a}\,\vec{\nabla}'\cdot\vec{V}] \quad . \tag{26.7}$$

Furthermore, if everywhere on the surface S , the vector field satisfies

$$\vec{\nabla}'\cdot\vec{V} = 0 \ , \quad \text{and} \quad \vec{\nabla}' \times \vec{V} = 0 \quad , \tag{26.8}$$

(26.7) reduces to

$$\oint_C d\vec{r}' \times \vec{V} = \int_S (d\vec{S}'\cdot\vec{\nabla}')\,\vec{V} \quad . \tag{26.9}$$

We will apply this result to rewrite (26.4) for which

$$\vec{V} = \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}'|} = -\vec{\nabla} \frac{1}{|\vec{r}-\vec{r}'|} , \tag{26.10}$$

which satisfies the conditions (26.8) as long as $\vec{r} \neq \vec{r}'$, that is, at points outside the wire. We therefore find

$$\vec{H} = -\vec{\nabla} \phi_m , \tag{26.11}$$

where the magnetic scalar potential, ϕ_m , is

$$\phi_m(\vec{r}) = \frac{I}{c} \int d\vec{S}' \cdot \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}'|} . \tag{26.12}$$

According to (1.10), the surface integral,

$$-\int d\vec{S}' \cdot \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}'|} = -\int d\vec{S}' \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \Omega , \tag{26.13}$$

is the solid angle subtended by the current loop at the observation point, so that

$$\phi_m = -\frac{I}{c} \Omega . \tag{26.14}$$

Therefore, a scalar potential for $\vec{H}$ exists for points not on the wire, consistent with the Maxwell equation, (23.8a),

$$\vec{\nabla} \times \vec{H} = 0 . \tag{26.15}$$

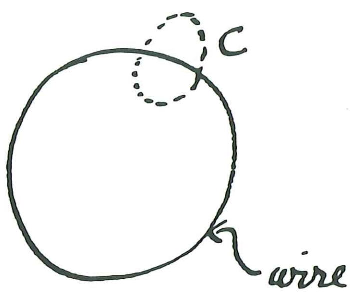
However, this scalar potential is not single-valued. We consider the integral of $\vec{H}$ around a closed path, C , which does not touch the wire,

$$\oint_C d\vec{r} \cdot \vec{H} = -\oint d\vec{r} \cdot \vec{\nabla} \phi_m = -\phi_m \Big]_0^0 . \tag{26.16}$$

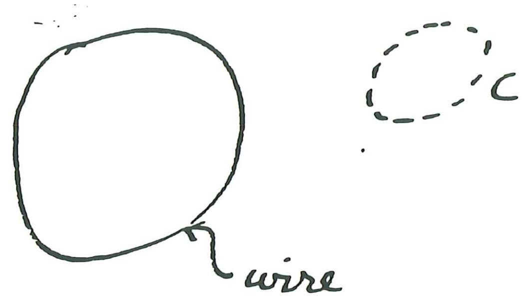
The naive anticipation is that (26.16) would be zero. An alternative calculation of this quantity can be made using Stokes' theorem:

$$\oint_C d\vec{r}\cdot\vec{H} = \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{H}) = \int_S d\vec{S}\cdot\frac{4\pi}{c}\vec{J}$$
$$= \begin{cases} \pm\dfrac{4\pi}{c}\,I \;, & \text{YES} \;, \\ 0 \;, & \text{NO} \;, \end{cases} \tag{26.17}$$

where YES means the wire passes once through the surface S , bounded by the path C , [the $\pm$ sign refers to the relative orientations of $d\vec{S}$ and $\vec{J}$] while NO means the wire does not pass through the surface. Some examples of this are supplied by the following illustrations.



YES



NO

Therefore, contrary to our naive expectation,

$$\phi_m \Big]_{\oint} = \begin{cases} \mp\dfrac{4\pi}{c}\,I \;, & \text{YES} \;, \\ 0 \;, & \text{NO} \;, \end{cases} \tag{26.18a}$$

or

$$\Omega \Big]_{\oint} = \begin{cases} \pm 4\pi \;, & \text{YES} \;, \\ 0 \;, & \text{NO} \;, \end{cases} \tag{26.18b}$$

which means that ϕ_m is a multi-valued function, as required by the fact that $\vec{\nabla}\times\vec{H}$ is not zero everywhere. The discontinuity found in (26.18b) corresponds to the fact that when one crosses the surface S defined by the current loop and used in the evaluation of the solid angle in Ω, there is a change of 4π.

Very far away from the current loop, the solid angle subtended by it is, if we assume that the points of S are localized in the vicinity of the loop,

$$\Omega = -\frac{\vec{r}\cdot\vec{S}}{r^3}, \quad \vec{S} = \int d\vec{S}', \tag{26.19}$$

so the corresponding magnetic field is

$$\vec{H} = -\vec{\nabla}\left(\frac{\vec{\mu}\cdot\vec{r}}{r^3}\right), \tag{26.20}$$

which upon comparison with (25.8) identifies the magnetic moment of the current loop to be

$$\vec{\mu} = \frac{I}{c}\vec{S}. \tag{26.21}$$

We obtain the same result if we use the definition of the magnetic moment, (25.7),

$$\vec{\mu} = \frac{1}{2c}\int (d\vec{r}')\, \vec{r}' \times \vec{J}(\vec{r}') = \frac{I}{c}\frac{1}{2}\oint \vec{r}' \times d\vec{r}' = \frac{I}{c}\vec{S}, \tag{26.22}$$

where we have used (26.7) to evaluate the line integral, which is also obvious geometrically.