$$\vec{J}_m = -c\left[-\hat{e}_r \frac{\partial}{\partial z}\left(\frac{P_\perp}{B}\right) + \hat{e}_z \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r P_\perp}{B}\right)\right]. \qquad (25)$$

From the gradients we have, from Eq. (9), Section V,

$$\vec{J}_G = \frac{c P_\perp}{B^2}\, \vec{n} \times \vec{\nabla}(\vec{B}\cdot\vec{n}) \qquad (26)$$

or

$$\vec{J}_G = \frac{c P_\perp}{B^2}\, \hat{e}_\theta \times \left[\hat{e}_r \frac{\partial B}{\partial r} + \hat{e}_z \frac{\partial B}{\partial z}\right] \qquad (27)$$

or

$$\vec{J}_G = \frac{c P_\perp}{B^2}\left[-\hat{e}_z \frac{\partial B}{\partial r} + \hat{e}_r \frac{\partial B}{\partial z}\right]. \qquad (28)$$

Finally, from Eq. (8), Section V, we get the curvature currents.

$$\vec{J}_R = 2c\frac{N W_\parallel}{B^2}\, \vec{B} \times (\vec{n}\cdot\vec{\nabla})\vec{n}$$

or

$$= \frac{c P_\parallel}{B^2}\, B\, \hat{e}_\theta \times (\vec{e}_\theta \cdot \vec{\nabla})\hat{e}_\theta$$

or

$$= \frac{c P_\parallel}{B r}\, \hat{e}_z \qquad (29)$$

where $P_\parallel$ is the gas kinetic pressure parallel to the B lines.

Now from $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$ we get the following equations for B

$$\frac{1}{r}\frac{\partial}{\partial r}(r B) = \frac{4\pi}{c} j_z \qquad (30)$$

and

$$\frac{\partial B}{\partial z} = -\frac{4\pi}{c} j_r. \qquad (31)$$

Adding up the currents given by Eqs. (25), (28), and (29), and substituting in Eqs. (30) and (31) gives

$$\frac{1}{r}\frac{\partial}{\partial r}(rB) = -4\pi\,\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rP_\perp}{B}\right) - \frac{4\pi P_\perp}{B^2}\frac{\partial B}{\partial r} + \frac{4\pi}{r}\frac{P_{\parallel}}{B} \tag{32}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}(rB) = -\frac{4\pi}{Br}\frac{\partial}{\partial r}(rP_\perp) + \frac{4\pi P_{\parallel}}{rB} \tag{33}$$

and

$$\frac{\partial B}{\partial z} = -4\pi\left[\frac{\partial}{\partial z}\left(\frac{P_\perp}{B}\right) + \frac{P_\perp}{B^2}\frac{\partial B}{\partial z}\right]. \tag{34}$$

Eq. (34) can be immediately reduced to

$$\frac{\partial B}{\partial z} = -\frac{4\pi}{B}\frac{\partial P_\perp}{\partial z} \tag{35}$$

or

$$\frac{B^2}{8\pi} + P_\perp = f(r) \tag{36}$$

where $f(r)$ is an arbitrary function of r.

This equation says that we have pressure balance in the z direction for each value of r. Again, $B^2/8\pi$ is the magnetic pressure and $P_\perp$ is the particle pressure perpendicular to the lines of force. Since the parallel pressure exerts no force in the z direction, the perpendicular pressure is the only force which must be balanced by the magnetic field in this direction.

Turning now to Eq. (33), we may write this equation in the form

$$\frac{B}{4\pi}\frac{\partial}{\partial r}(rB) + \frac{\partial}{\partial r}(rP_\perp) + P_{\parallel} = 0 \tag{37}$$

or

$$\frac{\partial}{\partial r}\left[r\left[\frac{B^2}{8\pi} + P_\perp\right]\right] + \frac{B^2}{8\pi} - P_{\parallel} = 0. \tag{38}$$

This equation is again the equation for the balance of forces on a little element.

$B^2/8\pi$ is the magnetic pressure and $P_\perp$ is particle pressure perpendicular to the lines. The term $B^2/8\pi$ is a term we should get if the lines of force were under a tension of magnitude $B^2/8\pi$, while the $P_\parallel$ term is equivalent to there being a compressional stress of magnitude $P_\parallel$ in the column. See Fig. 38 for the details of the forces.

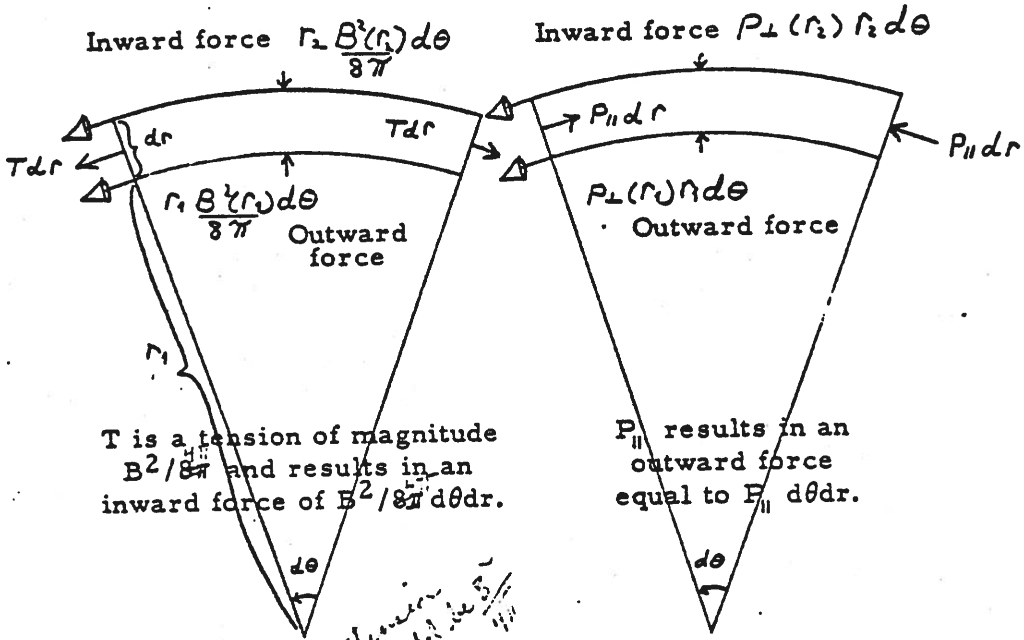

Figure 38

Later we will show that the idea of the lines exerting a pressure perpendicular to themselves, of magnitude $B^2/8\pi$, and being under tension of magnitude $B^2/4\pi$, can be formalized.

Substituting for $B^2/8\pi$ from Eq. (36) in Eq. (38) gives

$$\frac{\partial}{\partial r}\left[r\, f(r)\right] + f(r) = P_\perp + P_\parallel. \qquad (39)$$

. Now if the plasma is to be confined in the z direction, then $P_\perp$ and $P_\parallel$ must go to zero for large z and hence they must be functions of z . However, the left-hand side of Eq. (39) is independent of z ; it is only a function of r . This means we cannot have a stationary confined plasma with this type of field. The reason for this is easy to find. If we look at the particle drifts we see that electrons and ions drift in opposite directions because of both their parallel and perpendicular motions. This will lead to a charge separation and hence to an $\vec{E} \times \vec{B}/B^2$ drift. We may examine this in detail for the case in which the plasma pressure is negligible compared to the magnetic pressure. In this case we can find B from Eq. (33) by putting $P_\parallel = P_\perp = 0$.

One finds

$$rB = \text{constant} \tag{40}$$

or

$$B = B_0 r_0 / r \tag{41}$$

or

$$\vec{B} = \hat{e}_\theta \, B_0 r_0 / r. \tag{42}$$

The field falls off as $1/r$; this is the field produced by an infinite straight wire carrying current $j = c B_0 r_0 / 2$, or the field which exists in a uniform circular toroidal solenoid. From Eqs. (25), (28), (29) and (42),

$$\vec{j}_m + \vec{j}_G = \hat{e}_r \frac{c}{B_0 r_0} \frac{\partial}{\partial z}(r P_\perp) - \hat{e}_z \frac{c}{B_0 r_0} \frac{\partial}{\partial r}(r P_\perp) \tag{43}$$

and

$$\vec{j}_R = \hat{e}_z \frac{c P_\parallel}{B_0 r_0}. \tag{44}$$

In addition to these currents, there will now also be a current due to $\vec{E}$.

$$\vec{J}_{\dot{E}} = \frac{c^2}{B^2}\rho\dot{\vec{E}} \quad \text{where } \rho = \sum_i N_i m_i. \tag{45}$$

From Maxwell's equations — see Eq. (14) — we have

$$\vec{\nabla}\cdot\left(\dot{\vec{E}} + 4\pi\vec{J}\right) = 0. \tag{46}$$

We get from Eqs. (46), (45), (44), and (43)

$$\vec{\nabla}\cdot\dot{\vec{E}}\left[1 + \frac{4\pi\rho c^2}{B^2}\right] + 4\pi\vec{\nabla}\cdot\left[\vec{J}_H + \vec{J}_G + \vec{J}_R\right] = 0 \tag{47}$$

or

$$\vec{\nabla}\cdot\dot{\vec{E}}\left[1 + \frac{4\pi\rho c^2}{B^2}\right] + \frac{4\pi c}{B_0 r_0}\frac{\partial}{\partial z}\left[P_{\parallel} + P_{\perp}\right] = 0. \tag{48}$$

If we assume that $\vec{E}$ has only a z component (strictly speaking, one should solve the above equation along with $\nabla\times E = 0$, assuming B is negligible; we could find a configuration of pressures so that our assumption was true), this equation gives

$$\dot{E}_z\left[1 + \frac{4\pi\rho c^2}{B^2}\right] + \frac{4\pi c}{B_0 r_0}\left[P_{\parallel} + P_{\perp}\right] = 0, \tag{49}$$

[Again strictly speaking, the right-hand side of Eq. (49) could be an arbitrary function of t, which must be determined from boundary conditions. This correction comes about because of fringing fields outside the plasma, which will change the effective mass. We neglect them here.]

We obtain from Eq. (49)

$$\dot{E}_z = -\frac{4\pi c}{B_0 r_0}\left[\frac{P_{\parallel} + P_{\perp}}{\left(1 + \frac{4\pi\rho c^2}{B^2}\right)}\right]. \tag{50}$$

Problem: Find the correct E if $P_\perp$ and $P_\parallel$ are uniform inside a small torus of minor radius r. In finding E, assume you can treat the torus as a cylinder. Also, find the correct rate of drop in a gravitational field of a cylindrical plasma of radius r_o and with uniform density, when similar boundary corrections are included in that calculation.

$$\dot{\vec{E}} = -\hat{e}_z \frac{4\pi c\,(P_\parallel + P_\perp)}{B_o r_o} \frac{1}{(1+4\pi\rho^2 c^2 r^2/B_o^2 r_o^2)}. \tag{51}$$

This E gives rise to a drift velocity

$$\vec{V}_E = \frac{c\vec{E}\times\vec{B}}{B^2}. \tag{52}$$

Taking the time derivative of this equation gives

$$\dot{\vec{V}}_E = c\frac{\dot{\vec{E}}\times\vec{B}}{B^2} = \hat{e}_r \frac{4\pi c^2 r\,(P_\perp + P_\parallel)}{B_o^2 r_o^2\left(1+\dfrac{4\pi\rho c^2 r^2}{B_o^2 r_o^2}\right)}. \tag{53}$$

---

* Here we have neglected the change in B due to the motion. This is equivalent to setting $\vec{v}_* = 0$ or neglecting this term compared to B. This type of correction will be discussed in the next section; see Eq. (79).

If $\rho c^2 >> B_o^2$,

$$\dot{\vec{V}}_E = \hat{e}_r \frac{P_\parallel + P_\perp}{\rho r}. \tag{54}$$

The column accelerates in the r direction: it is not confined by the magnetic field. The reason is, of course, similar to that involved in the plasma column falling in a gravitational field.

D. The Cylindrical Pinch

If we allow the plasma to be uniform in the z direction (no z dependence), then Eqs. (38) or (39) allow a solution where $P_{\parallel}$ and $P_{\perp}$ are functions of r alone. This solution approximates the so-called linear pinch, where we have a long column of plasma through which a large current passes. The current produces a magnetic field which confines the gas.

Problem: (a) If one adds a uniform B_z to the toroidal B_θ field, can one obtain a confined plasma in a torus? not within [illegible]

(b) Verify that for the vacuum magnetic field around an infinite straight wire carrying a current that the tension of the magnetic lines balances the magnetic pressure.

IX. The General Two-Dimensional Problem (Time-Dependent)

Let us again take the B lines to be parallel, straight, and pointing in the z direction. We assume that there is no variation of any of the quantities in the z direction. We further assume that $\vec{v}_E$ is the largest velocity in the problem and we identify it with the mass velocity of the plasma. Finally, we assume that the plasma is approximately electrically neutral and denote the common number density by N . From conservation of particles we have the continuity equation

$$\frac{\partial N}{\partial t} + \vec{\nabla} \cdot (N \vec{V}_E) = 0. \tag{55}$$

or

$$\frac{1}{N} \frac{dN}{dt} = \vec{\nabla} \cdot \vec{V}_E. \tag{56}$$

where dN/dt is the time rate of change of density moving with the particles.

Second, we have Maxwell's induction equation

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}. \tag{57}$$

By making use of the relation of $\vec{v}_E$ to $\vec{E}$

$$\vec{E} = -\frac{\vec{V}_E \times \vec{B}}{c} \tag{58}$$

we can write Eq. (57) in the form

$$\vec{\nabla} \times (\vec{V}_E \times \vec{B}) = \frac{\partial \vec{B}}{\partial t}. \tag{59}$$

Recalling

$$\vec{\nabla} \times (\vec{u} \times \vec{v}) = (\vec{v}\cdot\vec{\nabla})\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{v} + \vec{u}(\vec{\nabla}\cdot\vec{v}) - \vec{v}(\vec{\nabla}\cdot\vec{u}) \tag{60}$$

we get

$$\cancelto{0}{(\vec{B}\cdot\vec{\nabla})\vec{V}_E} - (\vec{V}_E\cdot\vec{\nabla})\vec{B} + \cancelto{0}{\vec{V}_E(\vec{\nabla}\cdot\vec{B})} - \vec{B}(\vec{\nabla}\cdot\vec{V}_E) = \frac{\partial \vec{B}}{\partial t} \tag{61}$$

The first term is zero since $\vec{B}$ has only a z component and all quantities vary only in x and y direction, so that ∇_z is zero. The third term is zero from the Maxwell equation.

Now we may write $\vec{B}$ as $\hat{n}B$, where $\hat{n}$ is a unit vector in the z direction. Then Eq. (61) becomes

$$-\vec{V}_E\cdot\vec{\nabla}(B\hat{n}) - \hat{n}B(\vec{\nabla}\cdot\vec{V}_E) = \hat{n}\frac{\partial B}{\partial t} \tag{62}$$

or

$$\hat{n}\left[\frac{\partial B}{\partial t} + (\vec{V}_E\cdot\vec{\nabla})B + B\vec{\nabla}\cdot\vec{V}_E\right] = 0 \tag{63}$$

or

$$\frac{\partial B}{\partial t} + (\vec{\nabla}\cdot\vec{V}_E)B = 0 \tag{64}$$

thus

$$\frac{1}{B}\frac{dB}{dt} = \vec{\nabla}\cdot\vec{V}_E. \tag{65}$$

Again the derivative follows the motion. From Eqs. (56) and (65)

$$\frac{1}{B}\frac{dB}{dt} = \frac{1}{N}\frac{dN}{dt}. \tag{66}$$

Therefore,

$$-\frac{N}{B^2}\frac{dB}{dt} + \frac{1}{B}\frac{dN}{dt} = 0 \tag{67}$$

or

$$\frac{d}{dt}\left(\frac{N}{B}\right)_{\substack{\text{FOLLOWING}\\ \text{MOTION}}} = 0. \tag{68}$$

Eq. (68) is simply a consequence of the field being carried with the particles. If we draw a curve (in reality a cylinder) in the plasma and move the curve with the plasma, then both the number of particles inside the curve and the flux through it do not change with time. We already have shown this in the section on the motion of magnetic lines of force, and hence we could have written Eq. (68) down immediately. The above calculation gives another demonstration of this fact for the special case considered here.

We now compute the currents. The magnetization current is given by

$$\vec{j}_M = -c\,\vec{\nabla}\times\left(\frac{p_\perp \hat{n}}{B}\right) = c\,\hat{n}\times\vec{\nabla}\left(\frac{p_\perp}{B}\right) \tag{69}$$

while the gradient current is

$$j_G = c\frac{p_\perp}{B^2}\,\hat{n}\times\vec{\nabla}B. \tag{70}$$

If we also allow an external force per unit volume of the form

$$N\vec{F} = \vec{F} \tag{71}$$

then this gives rise to a current

$$\vec{j}_F = c\,\frac{\vec{E}\times\hat{n}}{B}. \tag{72}$$

Before summing the currents, we need to modify the polarization current. We do not use Eq. (7), Section V, because that was derived on the assumption that $\vec{B}$ was constant, while here we want to allow $\vec{B}$ to be time-dependent. We have, from the equation of motion,

$$m\,\frac{d\vec{V}}{dt} = q\left(\vec{E} + \frac{\vec{V}}{c}\times\vec{B}\right). \tag{73}$$

Writing

$$\vec{V} = \vec{V}_E + \vec{V}_1 \tag{74}$$

where

$$\vec{V}_E = c\,\frac{\vec{E}\times\vec{B}}{B^2} \qquad \text{following the particle,} \tag{75}$$

we have

$$m\left[\frac{d\vec{V}_E}{dt} + \frac{d\vec{V}_1}{dt}\right] = \frac{q}{c}(\vec{V}_1\times\vec{B}). \tag{76}$$

Again, let $\vec{v}_1 = \vec{v}_2 + \vec{v}_p$, where $\vec{v}_p$ is chosen to satisfy the equation

$$m\,\frac{d\vec{V}_E}{dt} = mc\,\frac{d}{dt}\,\frac{\vec{E}\times\vec{B}}{B^2}\Bigg|_{\substack{\text{following}\\ \text{the particle}}} = \frac{q}{c}\,\vec{V}_P\times\vec{B} \tag{77}$$

or (keeping in mind that d/dt means following the particle)

$$\vec{V}_P = \frac{mc^2}{q}\,\frac{\vec{B}}{B^2}\times\frac{d}{dt}\left(\frac{\vec{E}\times\vec{B}}{B^2}\right)$$

$$= \frac{mc^2}{q}\,\frac{\hat{n}}{B}\times\frac{d}{dt}\left(\frac{\vec{E}\times\hat{n}}{B}\right) \tag{78}$$

or, since $\vec{E}\perp B$,

$$\vec{V}_P = \frac{mc^2}{q}\,\frac{1}{B}\,\frac{d}{dt}\left(\frac{\vec{E}}{B}\right). \tag{79}$$

Eq. (76) then becomes

$$m \frac{d\vec{V}_2}{dt} + m \frac{d\vec{V}_P}{dt} = \frac{q}{c} \vec{V}_2 \times \vec{B}. \tag{80}$$

As in our previous arguments for a time-varying E , the term $d\vec{v}_p/dt$ can be neglected if $\vec{E}$ and $\vec{B}$ vary slowly on the time scale associated with the cyclotron frequency. Then Eq. (80) just describes the Larmor motion about the lines of force. The polarization current is obtained from Eq. (79) by adding subscript i to m and q , multiplying by $q_i N_i$ and summing over all species. We thus find for $\vec{j}_p$

$$\vec{j}_P = \frac{\rho c^2}{B} \frac{d}{dt}\left(\frac{\vec{E}}{B}\right) \quad \text{where } \rho = \sum_i N_i m_i. \tag{81}$$

Summing up the currents $\vec{j}_m$, $\vec{j}_g$, $\vec{j}_c$, and $\vec{j}_p$, and substituting in the Curl-B Maxwell equation gives

$$\vec{\nabla} \times \hat{n} B = -\hat{n} \times \vec{\nabla} B + B \vec{\nabla} \times \hat{n} =$$

$$\frac{4\pi}{c}\left[\frac{c\,\hat{n}}{B} \times \vec{\nabla} P_\perp + \frac{c\,\vec{F} \times \hat{n}}{B} + \frac{\rho c^2}{B} \frac{d}{dt}\left(\frac{\vec{E}}{B}\right)\right]. \tag{82}$$

Dotting both sides with $\hat{n}$ gives zero, as we should expect, since there is no motion or force parallel to $\vec{B}$. Crossing with $\hat{n}$ on the left gives

$$\vec{\nabla} B = (\hat{n} \times \vec{\nabla} B) \times \hat{n} =$$

$$= -\frac{4\pi}{c}\left[\frac{c}{B} \vec{\nabla} P_\perp - c\frac{\vec{F}}{B} - \frac{\rho c^2}{B} \frac{d}{dt}\left(\frac{\hat{n} \times \vec{E}}{B}\right)\right] \tag{83}$$

or

$$-\vec{\nabla}\left[\frac{B^2}{8\pi} + P_\perp\right] + \vec{F} = \rho \frac{d\vec{V}_E}{dt}. \tag{84}$$

Dropping the E from the v gives

$$\rho \frac{d\vec{V}}{dt} = \vec{F} - \vec{\nabla}\left[P_\perp + \frac{B^2}{8\pi}\right]. \tag{85}$$

Here again we see that the magnetic field acts as if it exerts a pressure of magnitude $B^2/8\pi$.

Now we see that we have three equations — (85), (64), and (55) — for the four unknowns, v, N, B, and $P_\perp$. We need one more equation: an equation to relate $P_\perp$ to N. We can obtain this from the adiabatic invariant μ.

We have

$$P_\perp = N W_\perp = N^2 \left(\frac{W_\perp}{B}\right)\left(\frac{B}{N}\right) \tag{86}$$

Now both B/N and $W_\perp/B$ are constant following the motion, so this tells us that $P_\perp$ is proportional to N^2 or

$$P_\perp = P_{\perp 0} \left(\frac{N}{N_0}\right)^2 \tag{87}$$

or

$$\frac{P_\perp}{N^2} \propto \frac{P_\perp}{\rho^2} = \text{CONSTANT} \tag{88}$$

and thus

$$\frac{d}{dt}\left(\frac{P_\perp}{\rho^2}\right) = 0 \tag{89}$$

all following the motion.

In the section on the longitudinal invariant it was shown that under a slow compression the energy in $v_\parallel$ was related to the pressure in the same way as would be the case for a one-dimensional gas (one degree of freedom). We may show that the above law, Eq. (89), is the same law we would obtain for a two-dimensional gas (two degrees of freedom). The relation between the pressure and the density for an ideal gas undergoing an adiabatic change in volume is

$$\frac{p}{\rho^{\gamma}} = \frac{p_0}{\rho_0^{\gamma}}. \tag{90}$$

Again, γ is given by

$$\gamma = \frac{n+2}{2} \tag{91}$$

where n is the number of degrees of freedom involved in the process. We see that if we set γ equal to 2 in Eq. (90) we get Eq. (89). Thus the gas behaves like an ideal two-dimensional gas, as we might expect for this two-dimensional problem.

Here we might also note that the magnetic field also behaves like a gas with a γ of 2. We note that following the motion N/B or B/N is constant. Hence

$$\frac{B^2}{N^2} \propto \frac{B^2}{8\pi\rho^2} = \text{CONSTANT} \tag{92}$$

and thus the magnetic field also behaves like an ideal gas with $\gamma = 2$.

Summarizing, we have derived two-dimensional hydromagnetics from orbit theory, and the following set of equations describes the motion of the system

$$\rho \frac{d\vec{V}}{dt} = \vec{F} - \vec{\nabla}\left[p_\perp + \frac{B^2}{8\pi}\right], \tag{93}$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\rho\vec{V}, \tag{94}$$

$$\frac{\partial B}{\partial t} = -\vec{\nabla}\cdot B\vec{V}, \tag{95}$$

and

$$\frac{d}{dt}\left(\frac{p_\perp}{\rho^2}\right) = 0. \tag{96}$$

Problem: Consider a two-dimensional plasma, for which $\vec{F}$ is obtainable from a potential $\vec{F} = -\vec{\nabla}\phi$ and which is confined by rigid walls that are perfectly conducting so that no magnetic flux can enter or leave.

Show that the energy W is conserved

$$W = \int \left[\frac{\rho v^2}{2} + \rho\phi + P_\perp + \frac{B^2}{8\pi} \right] d\tau$$

X. Magnetoacoustic Waves

We may now apply our results on the general two-dimensional problem to a number of special cases. The first such problem we will consider is that of magnetoacoustic wave propagation in a spatially-uniform homogeneous infinite plasma.

Our Eqs. (93), (94), (95), and (96) are nonlinear in the variables n or ρ, $\vec{B}$, $P_\perp$ and $\vec{v}$. In order to obtain linear equations we assume that the wave amplitude is small. In equilibrium the plasma density ρ_o, and the pressure P_o are constant throughout the plasma, the plasma velocity $\vec{v}_o$ is zero, and the magnetic field $\vec{B}_o$ is unidirectional (along z) and has uniform strength. The amplitude of the components associated with the wave will be designated by the subscript 1 — i.e., $\vec{B}_1$, $\vec{v}_1$, n_1 or ρ_1, $P_{\perp 1}$. To demonstrate the procedure, consider Eq. (96) as applied to the equilibrium, which, carrying out the differentiation is

$$\frac{\partial P_{\perp o}}{\partial t} = 2\,\frac{P_{\perp o}}{n_o}\,\frac{\partial n_o}{\partial t}. \tag{97}$$

For the perturbed state the pressure and density become $P_{\perp o} + P_{\perp 1}$ and $n_o + n_1$, so we have

$$\frac{\partial(P_{\perp 0}+P_{\perp 1})}{\partial t} = 2\,\frac{P_{\perp 0}+P_{\perp 1}}{n_0+n_1}\,\frac{\partial(n_0+n_1)}{\partial t} \tag{98}$$

which can be written

$$\left[\cancelto{0}{\frac{\partial P_{\perp 0}}{\partial t} - 2\frac{P_{\perp 0}}{n_0+n_1}\frac{\partial n_0}{\partial t}}\right]$$

$$-2\left[\cancelto{0}{\frac{P_{\perp 1}}{n_0+n_1}\frac{\partial n_0}{\partial t}} + \frac{P_{\perp 1}}{n_0+n_1}\frac{\partial n_1}{\partial t}\right] + \left[\frac{\partial P_{\perp 1}}{\partial t} - 2\frac{P_{\perp 0}}{n_0+n_1}\frac{\partial n_1}{\partial t}\right] = 0. \tag{99}$$

The first bracket is zero from Eq. (97) (or since P_o and n_o are not functions of time). The next term is zero, since n_o is not a function of time, while the following term is neglected since it involves the product of two small quantities $P_{\perp 1}$ and n_1. So there remains, dropping n_1 as compared with n_o in the denominator,

$$\frac{\partial P_{\perp 1}}{\partial t} - 2\,\frac{P_{\perp 0}}{n_0}\,\frac{\partial n_1}{\partial t} = 0\,. \tag{100}$$

A similar process carried out on Eqs. (93), (94), and (95) yields

$$\frac{\partial n_1}{\partial t} = n_0\,\vec{\nabla}\cdot\vec{V_1} \tag{101}$$

$$\frac{\partial \vec{B_1}}{\partial t} = \vec{B_0}\,\vec{\nabla}\cdot\vec{V_1} \tag{102}$$

$$\rho_0\,\frac{\partial \vec{V_1}}{\partial t} = -\vec{\nabla}\left[P_\perp + B^2/8\pi\right]_1 \tag{103}$$

Since the magnetic field behaves like a gas with $\gamma = 2$ as well as $P_\perp$, we can also write

$$\frac{\partial}{\partial t}\left[P_\perp + \frac{B^2}{8\pi}\right]_1 = \frac{2}{n_0}\left[P_{\perp 0} + \frac{B_0^2}{8\pi}\right]\frac{\partial n_1}{\partial t}. \tag{104}$$

Eq. (104) can be immediately integrated to give

$$\left[p_\perp + \frac{B^2}{8\pi}\right]_1 = \frac{2}{n_0}\left[p_{\perp 0} + \frac{B_0^2}{8\pi}\right] n_1 + \tilde{n}(x, y). \tag{105}$$

Here $\tilde{n}$ is an arbitrary function of x and y . However, since the perturbed pressure and magnetic field must be zero when n_1 is zero, $\tilde{n}$ must be zero.

Substituting Eq. (105) in Eq. (103) gives

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\frac{2}{n_0}\left[p_{\perp 0} + \frac{B_0^2}{8\pi}\right] \vec{\nabla} n_1. \tag{106}$$

Taking the time derivative of Eq. (101) and the divergence of Eq. (106) gives

$$\frac{\partial^2 n_1}{\partial t^2} = \frac{2}{\rho_0}\left[p_{\perp 0} + \frac{B_0^2}{8\pi}\right] \nabla^2 n_1. \tag{107}$$

Eq. (107) is a wave equation for waves propagating with velocity

$$V^2 = \frac{2}{\rho_0}\left[p_{\perp 0} + \frac{B_0^2}{8\pi}\right]. \tag{108}$$

These waves are called magnetoacoustic waves. For small magnetic fields they reduce to ordinary acoustic waves propagating at the acoustic velocity

$$V^2 = 2\,\frac{p_{\perp 0}}{\rho_0}. \tag{109}$$

The magnetic field increases the effective pressure in the gas. These waves show no dispersion and propagate at a constant velocity. The waves are longitudinal because $\vec{v}_1$ is parallel to $\vec{\nabla} n_1$ by Eq. (106), and hence will be in the direction of $\vec{k}$ or the direction of wave propagation if we Fourier-analyze Eq. (107).

Problem: (1) Do the wave solutions of Eq. (107) include all possible motions of the plasma?

(2) How is it that we have lost the inertia of the magnetic field (or equivalently the energy which goes into the electric field) in this calculation? Can you include it? What effect does this have on the energy conservation calculation given on page 103?

XI. The Rayleigh-Taylor Instability

As a second example of the use of our two-dimensional hydromagnetic equations we will consider the problem of the Rayleigh-Taylor instability. The situation here is shown in Fig. 39.

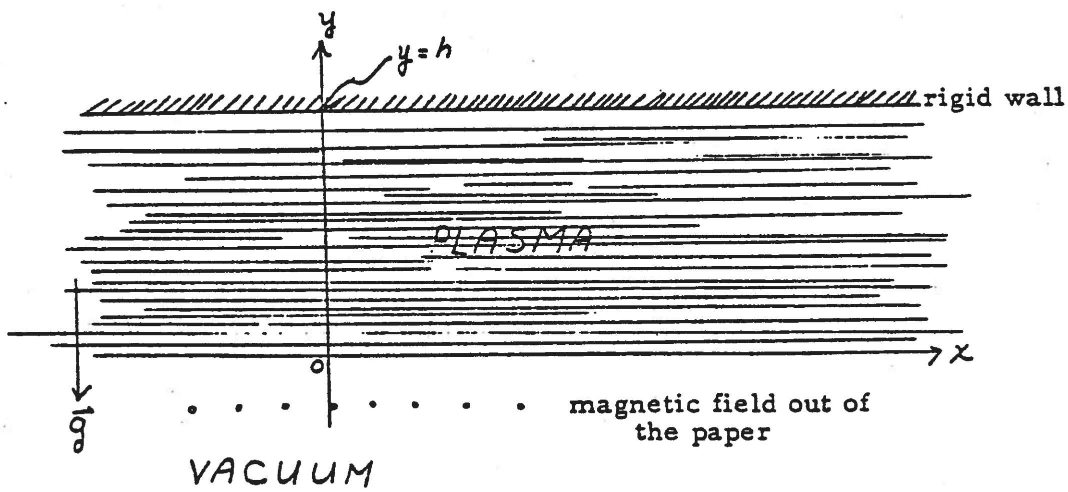

Figure 39

We have a slab of plasma of thickness h supported against a gravitational field $\vec{g}$ by a magnetic field. We take the plasma to lie entirely above the x axis and to be separated from a vacuum region below the x axis by a sharp boundary. The equilibrium conditions are obtained by setting all time derivatives and velocities equal to zero in Eqs. (93) and (96).

$$0 = -\rho\vec{g} - \vec{\nabla}\left[p_\perp + \frac{B^2}{8\pi}\right]. \tag{110}$$

Rather than treat the most general problem, we will treat the specific situation in which the equilibrium plasma density and pressure are constant within the slab. Eq. (110) then reduces to

$$\rho_0 g - \frac{1}{8\pi}\frac{\partial B^2(y)}{\partial y} = 0 \quad (y>0) \tag{111}$$

or

$$\frac{B_\rho^2(y)}{8\pi} = \frac{B_\rho^2(0)}{8\pi} - \rho_0 g y \quad (y>0). \tag{112}$$

At the boundary (y = 0) the magnetic field strength must jump so as to balance the plasma pressure. The vacuum field is given by

$$B_v^2 = B_\rho^2(0) + 8\pi p_{\perp 0}. \tag{113}$$

The general procedure now would be to linearize Eqs. (93) to (96) for small departures from this equilibrium and to look for wave solutions to the resultant equations. Before doing this we will, however, make one further approximation. Here we are interested in the gravitational instability. For this mode the velocities and speeds of propagation are in general small compared to the velocity of a magnetoacoustic wave. This suggests that the plasma's compressibility can play only a small role in the motion and so we look for a solution assuming an incompressible plasma. We may justify this assumption a posteriori. Our set of equations then reduces to

$$\rho_0 \frac{d\vec{V}}{dt} = -\hat{y}\rho_0 g - \vec{\nabla}\left[P_{\perp 0} + \frac{B_0^2}{8\pi}\right] \tag{114}$$

and

$$\vec{\nabla}\cdot\vec{V} = 0. \tag{115}$$

The pressure equations (96) and (92) cannot be used here because ρ does not change. The incompressibility assumption assumes that the pressure is a very strong function of ρ, and hence we can have any pressure for the same density. The pressure must be determined by making it self-consistent with the motion.

In addition to Eqs. (114) and (115) we have the boundary condition that at the wall $y = h$, the normal velocity to the wall is zero

$$v_y = 0 \qquad y = h$$

The other boundary condition at the plasma vacuum interface is given in Eq. (113).

We now linearize Eqs. (114) and (115) to obtain

$$\rho_0 \frac{\partial \vec{V}}{\partial t} = -\vec{\nabla}\,\Pi_1 \tag{116}$$

and

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \tag{117}$$

where

$$\Pi = \left[P_{\perp 0} + \frac{B_0^2}{8\pi}\right], \tag{118}$$

We now look for solutions which go like

$$e^{i\omega t}. \tag{119}$$

From Eq. (116) we get

$$\vec{V} = \frac{i}{\rho_0 \omega}\vec{\nabla}\,\Pi_1 \tag{120}$$

and from Eqs. (117) and (120) we have

$$\nabla^2 \Pi_1 = 0. \tag{121}$$

The solutions to this equation of interest here are

$$\Pi_1 = \Pi_{1\pm} \; e^{ikx \pm ky} \tag{122}$$

or

$$\Pi_1 = (\Pi_{1+} \, e^{ky} + \Pi_{1-} e^{-ky}) e^{ikx}. \tag{123}$$

From Eqs. (120) and (123) we obtain for $\vec{v}$

$$\vec{v} = \frac{i}{\rho_0 \omega} \{ \hat{y} k e^{ikx} (\Pi_{1+} e^{+ky} - \Pi_{1-} e^{-ky})$$

$$+ \hat{x} \, ik \, e^{ikx} (\Pi_{1+} e^{+ky} + \Pi_{1-} e^{-ky}) \} \tag{124}$$

The boundary condition that $v_y = 0$ at $y = h$ gives

$$\Pi_{1+} e^{kh} - \Pi_{1-} e^{-kh} = 0 \tag{125}$$

or

$$\Pi_{1-} = \Pi_{1+} e^{2kh}. \tag{126}$$

Hence

$$\Pi_1 = \Pi_{1+} (e^{ky} + e^{k(2h-y)}) e^{ikx}. \tag{127}$$

Now at the plasma vacuum interface the boundary condition requires that

$$\Pi_p = \Pi_v = \frac{B_v^2}{8\pi} \tag{128}$$

since the magnetic pressure must be constant throughout this region. Before we can satisfy this condition we must find where the new boundary is.

This is obtained from the equation of motion for the boundary

$$\dot{y} = v_y \tag{129}$$

From this equation and Eqs. (124) and (126), substituting $y = 0$ to obtain the lowest order change in y, we get

$$\delta y = \frac{1}{\rho_0 \omega^2} k e^{ikx} \Pi_{1+}[1 - e^{2kh}]. \tag{130}$$

Now to first order $\pi = \pi_0 + \pi_1$ at the new boundary is given by

$$\Pi_v = \Pi_p = \Pi_0(y=0) + \Pi_1(y=0) + \left.\frac{\partial \Pi_0}{\partial y}\right|_{y=0} \delta y. \tag{131}$$

From Eqs. (128) and (131), and since $\pi_0 = \pi_p$, we have

$$\Pi_1(y=0) + \left.\frac{\partial \Pi_0}{\partial y}\right|_{y=0} \delta y = 0, \tag{132}$$

but

$$\left.\frac{\partial \Pi_0}{\partial y}\right|_{y=0} \delta y = -\rho_0 g \tag{133}$$

hence

$$\Pi_1(y=0) - \rho_0 g \, \delta y = 0. \tag{134}$$

Substituting the solutions we have found for π_1 and δy [Eqs. (127) and (130)] in Eq. (134) gives

$$\Pi_{1+}(1 + e^{2kh}) - \frac{g}{\omega^2} k \Pi_{1+}(1 - e^{2kh}) 0 \tag{135}$$

or

$$\omega^2 = g \frac{k(1 - e^{2kh})}{(1 + e^{2kh})} < 0 \quad \text{for all } k. \tag{136}$$

For large h this reduces to

$$\omega^2 = -gk. \tag{137}$$

If g is positive ($\vec{g}$ acting in the negative y direction), this gives an instability with growth rate $1/\tau = \omega$

or
$$\tau = \frac{1}{\sqrt{gk}}. \tag{138}$$

If g is negative ($\vec{g}$ acting in the positive y direction), then one gets only stable oscillations.

We may form the following physical picture of how this instability comes about. Consider the rippled plasma surface as is shown in Fig. 40.

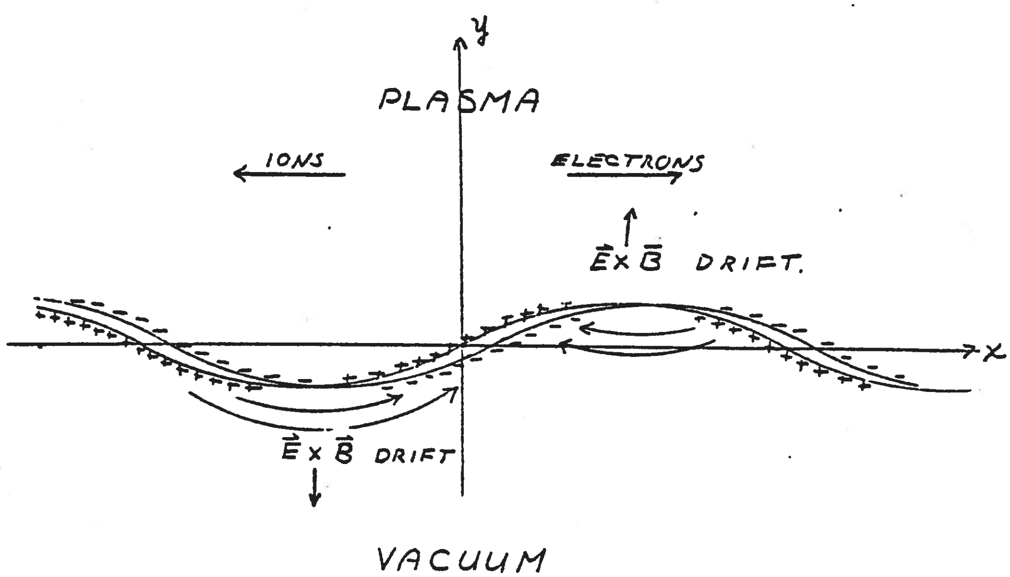

Figure 40

The gravitational field acts in the negative y direction; the magnetic field is out of the paper. Because of the gravitational field the ions drift to the left and the electrons drift to the right. If the surface is flat, no electric field develops, but if the surface is rippled the two charges are separated along the boundary by these drifts, as illustrated by the solid and dashed curves. The resultant $E \times B$ drift is in the negative y direction in the regions where the plasma has been displaced downward and is in the positive y direction where it has been displaced upward. The electric field causes the perturbation to grow.

Problem: Show that the incompressibility approximation was a good one by using π_1 to compute ρ_1 and $\dot{\rho}_1$. What is the phase relationship of all the quantities associated with the wave? Is there an $\vec{E}$ field?

XII. Alfvén Waves

As another example of the use of orbit theory we now calculate the propagation of waves parallel to $\vec{B}$ (Alfvén waves). The equilibrium situation is the same as for the case of magnetoacoustic waves; however, the propagation vector is parallel to $\vec{B}$. In this case the wave magnetic field is perpendicular to $\vec{B}$, so that we cannot use our two-dimensional hydromagnetic equation but must calculate the currents and fields directly. We assume that all quantities go like

$$e^{i(kz-\omega t)} \tag{139}$$

and that there is no variation in the x, y directions. We shall linearize about the equilibrium state, writing all quantities as $A_o + A_1$, where A_o is the equilibrium value and A_1 is the perturbation.

From $\vec{\nabla} \cdot \vec{B} = 0$ we immediately get

$$\vec{k} \cdot \vec{B}_1 = 0 \tag{140}$$

or

$$\hat{z} \cdot \vec{B}_1 = 0 \tag{141}$$

(this also implies that the magnitude of $\vec{B}$ is unchanged to first order $|B| = \sqrt{B_o^2 + B_1 \cdot B_1}$). Thus $\vec{B}_1$ has only x and y components. From Eq. (6) in section V we have, for the perturbed magnetization current,

$$\vec{j}_{M_1} = ci\,\vec{k}\times\vec{M}_1 = -ci\,\vec{k}\times\left[\left(\frac{p_\perp}{B^2}\right)_1 \vec{B}_o + \frac{p_\perp}{B^2}\vec{B}_1\right] \quad (142)$$

or

$$j_{M_1} = -ic\,\frac{p_{\perp o}}{B_o^2}\,\vec{k}\times\vec{B}_1. \quad (143)$$

The lines will be bent in the motion, so there will arise a first order curvature current. This is obtained from Eq. (8) in section V and is given by

$$\vec{j}_R = \frac{c\,p_{\parallel o}}{B_o^2}\,\hat{z}\times\left[(\hat{z}\cdot i\vec{k})\frac{\vec{B}_1}{B_o}\right] \quad (144)$$

or

$$\vec{j}_R = \frac{c\,p_{\parallel o}}{B_o^2}\,i\vec{k}\times\vec{B}_1. \quad (145)$$

Next, the first order gradient current is zero since the magnitude of $\vec{B}$ is not changed to first order

$$\vec{j}_{G_1} = 0. \quad (146)$$

And finally, we have the polarization current which is obtained from Eq. (7) in section V and is given by

$$\vec{j}_{P_1} = \frac{\rho_o c^2}{B_o^2}(-i\omega\vec{E}_{\perp 1}). \quad (147)$$

There is no $\vec{E}_\parallel$ since none of the currents above is along $\vec{B}$, and further, any such $\vec{E}$ would be canceled out by motion of charges along $\vec{B}$ if it were to develop.

We now substitute in Maxwell's equations to obtain

$$i\,\vec{k}\times\vec{B}_1 = -\frac{i\omega}{c}\vec{E}_1 + \frac{4\pi}{c}\left[\frac{-ic\,P_{\perp 0}}{B_0^2}\vec{k}\times\vec{B}_1\right]$$

$$+\ \frac{4\pi}{c}\left[\frac{ic\,P_{\parallel 0}}{B_0^2}\vec{k}\times\vec{B}_1 - \frac{i\,\omega\rho_0 c^2}{B_0^2}\vec{E}_1\right], \tag{148}$$

$$i\,\vec{k}\times\vec{E}_1 = i\frac{\omega}{c}\vec{B}_1, \tag{149}$$

or

$$\left[1+\frac{4\pi}{B_0^2}(P_{\perp 0}-P_{\parallel 0})\right]\vec{k}\times\vec{B}_1 = -\frac{\omega}{c}\left[1+\frac{4\pi\rho_0 c^2}{B_0^2}\right]\vec{E}_1. \tag{150}$$

Crossing with $\vec{k}$ and substituting $\vec{k}\times\vec{E}_1$ gives

$$\left[1+\frac{4\pi}{B_0^2}(P_{\perp 0}-P_{\parallel 0})\right]k^2\vec{B}_1 = \frac{\omega^2}{c^2}\left[1+\frac{4\pi\rho_0 c^2}{B_0^2}\right]\vec{B}_1. \tag{151}$$

This gives the dispersion relation

$$\omega^2 = c^2k^2\frac{\left[B_0^2/4\pi + (P_{\perp 0}-P_{\parallel 0})\right]}{\left[B_0^2/4\pi + \rho_0 c^2\right]}. \tag{152}$$

For negligible ρ_0 and P this equation simply gives

$$\omega^2 = k^2c^2. \tag{153}$$

which is a light wave traveling along the magnetic lines of force.

For isotropic pressure, $P_{\parallel 0} = P_{\perp 0}$ and $\rho_0 c^2 >> B^2/4\pi$, this equation reduces to

$$\omega^2 = k^2\frac{B_0^2}{4\pi\rho_0 c^2}. \tag{154}$$

These waves are known as Alfvén waves. They are transverse waves propagating along $\vec{B}$.

Finally, we note that if

$$\frac{B_o^2}{4\pi} + P_{\perp o} - P_{\parallel o} < 0 \qquad (155)$$

then

$$\omega^2 < 0 \qquad (156)$$

and we have an instability. The instability arises because of the motion of the particles along the curved lines of force.

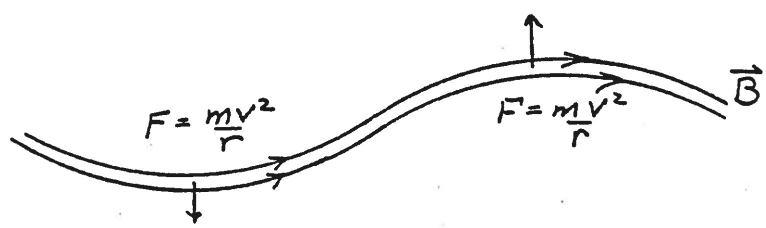

Figure 41

The centrifugal force which the particle exerts on the lines of force tends to distort them further. If this force can overcome the tension in the lines of force ($B_o^2/4\pi$) the system is unstable.

The perpendicular pressure also supplies a restoring force. According to Eq. (143), $\vec{J}_m$, which is the source of this $P_\perp$ term, is in the direction $\vec{B}_1 \times \vec{k}$. This produces a $\vec{J} \times \vec{B}_o$ force which is in the direction $\vec{B}_1 \times \vec{k} \times \vec{B}_o$ which is opposite to $\vec{B}_1$.

Another argument demonstrating this effect is that work must be done against the perpendicular pressure to set up $\vec{B}_1$. We have from the constancy of μ that

$$\frac{W_\perp}{B} = \frac{W_{\perp o}}{B_o} . \qquad (157)$$

Thus

$$\Delta W_\perp = \frac{W_{\perp 0}}{B_0} \Delta B. \tag{158}$$

But B is given by

$$B = \sqrt{B_0^2 + B_1^2} \cong B_0 \left(1 + \frac{B_1^2}{2B_0^2}\right) \tag{159}$$

(there can be no second order change to $B_z = B_o$ because $\vec{\nabla} \cdot \vec{B} = \frac{\partial B}{\partial z} = 0$).

Thus ΔB and $\Delta W_\perp$ are given by

$$\Delta B = B_1^2 / 2 B_0 \tag{160}$$

and

$$\Delta W_\perp = \frac{W_{\perp 0}}{2} \left(\frac{B_1}{B_0}\right)^2. \tag{161}$$

XIII. The Boltzmann Equation Approach

Up to this point we have been working with individual particle orbits. We have been able to combine the orbit calculations with Maxwell's equations for some simple cases to obtain gross plasma motions. However, we have not looked into the problem associated with the particles having a distribution of velocities or into the effects of collisions. We should like to investigate these effects. Our starting point will be the Boltzmann equation. Both limits of small and large collision rates can be treated from this equation (see appendix of Spitzer's book).

We start by defining a phase space for the particles. This space is a six-dimensional space; three of the coordinates are the position coordinates for a particle and the other three are the velocity coordinates. Given the position and velocity of a particle, we know its position in phase space, and vice versa, if we know the position of a particle in phase space we know its position and velocity. There is a point in phase space associated with every particle in the system. This set of points forms a dust or gas in phase space. As the particles move around and change their velocities, the dust moves around in the phase space. If the system contains a great many particles, then the dust will be very dense and we may treat it as a fluid. We may then define a density of points in phase space by the relation

$$f(\vec{r},\vec{v})\,\Delta^3 r\,\Delta^3 v = \text{Number of particles in } \Delta^3 r\,\Delta^3 v \text{ centered at } r \text{ and } v \tag{162}$$

where $\Delta^3 r$ is one element of volume in ordinary space and $\Delta^3 v$ an element of volume in velocity space. In order for this to be meaningful we must be able to choose $\Delta^3 r$ and $\Delta^3 v$ sufficiently large so that there are many particles in $\Delta^3 r\ \Delta^3 v$ and yet sufficiently small so that f does not change appreciably from one cell to the next. We assume that this is so and that f can be treated as a continuous function. Now as the particles move around, f may change with time. Since the number of particles is conserved, the changes in f must be such as to give this conservation. Let us first look at how f changes if there are no collisions between particles.

If particles flow out of a volume in phase space the density must decrease — that is, we may treat the flow of phase points like the flow of a fluid, and may write a continuity equation in this six-dimensional space. The continuity equation is

$$\frac{\partial f}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \tag{163}$$

where $\vec{J}$ is the current of phase points and $\vec{\nabla}$ is the six-dimensional gradient operator in phase space. $\vec{\nabla}$ is given by

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} + \hat{u}\frac{\partial}{\partial u} + \hat{v}\frac{\partial}{\partial v} + \hat{w}\frac{\partial}{\partial w} \qquad (164)$$

where u, v, and w are the velocities in the x, y, and z directions, respectively. The current $\vec{J}$ is given by

$$\begin{aligned} \vec{J} &= \vec{V}f \\ &= \hat{x}u + \hat{y}v + \hat{z}w + \hat{u}a_x + \hat{v}a_y + \hat{w}a_z \end{aligned} \qquad (165)$$

where $\vec{V}$ is the six-dimensional velocity of the phase points and a_x, a_y, and a_z are the accelerations (velocity in the u, v, w directions) in the x, y, and z directions.

If we have velocity-independent forces, then substituting these relations in Eq. (163) gives

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + a_x\frac{\partial f}{\partial u} + a_y\frac{\partial f}{\partial v} + a_z\frac{\partial f}{\partial w} = 0 \qquad (166)$$

or, vectorially,

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}_r f + \vec{a}\cdot\vec{\nabla}_v f = 0 \qquad (167)$$

where

$$\nabla_r = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \qquad (168)$$

and

$$\nabla_v = \hat{u}\frac{\partial}{\partial u} + \hat{v}\frac{\partial}{\partial v} + \hat{w}\frac{\partial}{\partial w}. \qquad (169)$$

In general, if the forces are velocity-dependent, Eq. (163) cannot be written in the form of Eq. (166), but the accelerations must be left inside the differentiation. However, for the important case of magnetic forces one again obtains Eq. (166), even though the a's are functions of the velocity. To see this, consider

$$\vec{\nabla}_v \cdot \left[(\vec{V} \times \vec{B}) f \right] = \vec{\nabla}_v \cdot \left[\hat{u} (v B_z - w B_y) f \right.$$

$$\left. + \hat{v} (w B_x - u B_z) f + \hat{w} (u B_y - v B_x) f \right]. \tag{170}$$

For the $\hat{u}$ term the acceleration $[v B_z - w B_y]$ is independent of u and hence it can be carried to the other side of $\vec{\nabla}_v \cdot$. Thus

$$\vec{\nabla}_v \cdot \left[(\vec{V} \times \vec{B}) f \right] = (\vec{V} \times \vec{B}) \cdot \vec{\nabla}_v f. \tag{171}$$

Hence for electromagnetic forces, Eq. (163) becomes

$$\frac{\partial f}{\partial t} + \vec{V} \cdot \vec{\nabla}_r f + \frac{q}{m} (\vec{E} + \vec{V} \times \vec{B}) \cdot \vec{\nabla}_v f = 0. \tag{172}$$

When the particles collide with each other we must add to Eq. (172) a term $\left[\frac{\partial f}{\partial t} \right]_{coll}$ which gives the change of f due to collisions. If we also add external forces $\vec{F}$ which are not electromagnetic, then Eq. (172) becomes

$$\frac{\partial f}{\partial t} + \vec{V} \cdot \vec{\nabla}_r f + \left[\frac{q}{m} (\vec{E} + \vec{V} \times \vec{B}) + \frac{\vec{F}}{m} \right] \cdot \vec{\nabla}_v f = \left[\frac{\partial f}{\partial t} \right]_{coll}. \tag{173}$$

To find the change in f due to collisions, $\left[\frac{\partial f}{\partial t} \right]_{coll}$, we must look at the details of the collisional processes. We will do this later and for the time being simply carry these terms along symbolically as $\left[\frac{\partial f}{\partial t} \right]_{coll}$.

If we have more than one species of particle then there is one equation of the form of Eq. (173) for every species of particle. Also there will be a collision term for each species of particle with which a collision takes place.

XIV. Transfer Equations

A. General Equations

Let Q be some function of the velocities of the particles. For example, it might be one of the components of the momentum, or energy. The average value of Q at any spatial point is given by

$$\overline{Q(\vec{r},t)} = \frac{1}{n(\vec{r},t)} \iiint\limits_{\substack{\text{ALL} \\ \text{VELOCITIES}}} Q\, f(x,y,z,u,v,w,t)\, du\, dv\, dw. \tag{174}$$

Although Q is not a function of $\vec{r}$ and t, and is only a function of $\vec{v}$, $\overline{Q}$ is only a function of $\vec{r}$ and t because f is a function of $\vec{r}$ and t.

We are now interested in the time rate of change of $\overline{Q}$. We have, from Eqs. (174) and (173),

$$\frac{\partial(n\overline{Q})}{\partial t} = \iiint Q \frac{\partial f}{\partial t}\, du\, dv\, dw =$$

$$-\iiint Q\left[(\vec{v}\cdot\vec{\nabla}_r) f + \left(\left[\frac{q}{m}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) + \frac{\vec{F}}{m}\right]\cdot\vec{\nabla}_v\right) f - \left(\frac{\partial f}{\partial t}\right)_{COLL}\right] du\, dv\, dw. \tag{175}$$

The first term on the right is

$$\vec{\nabla}_r \cdot \iiint \vec{v}\, Q\, f\, d^3v = \vec{\nabla}_r \cdot n\overline{\vec{v}Q} \tag{176}$$

since $\vec{\nabla}_r$ does not operate on $\vec{v}$. The second term on the right-hand side,

$$-\iiint Q\left(\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]\cdot\vec{\nabla}_v\right)f\,d^3v, \tag{177}$$

can be integrated by parts. Consider the contribution to the integral which comes from the acceleration in the x direction, i.e.,

$$-\iiint Q\left(\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]_x\frac{\partial f}{\partial u}\right)d^3v. \tag{178}$$

Integrating by parts with respect to u, keeping v and w fixed, gives

$$-\iint Q\left[\frac{q}{m}\left[\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right]+\frac{\vec{F}}{m}\right]_x f\Big|_{u=-\infty}^{u=\infty}dv\,dw$$

$$+\iiint\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]_x\frac{\partial Q}{\partial u}f\,du\,dv\,dw$$

$$=\left(\frac{q}{m}\vec{E}+\frac{\vec{F}}{m}\right)_x\cdot n\,\overline{\frac{\partial Q}{\partial u}}+\frac{q}{m}n\overline{\left[\left[\frac{\vec{V}\times\vec{B}}{c}\right]_x\frac{\partial Q}{\partial u}\right]} \tag{179}$$

since, as we have seen, $\frac{\partial}{\partial u}\left(\frac{\vec{V}\times\vec{B}}{c}\right)_x=0.$

Thus we can write Eq. (177) in the form

$$-\iiint Q\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]\cdot\vec{\nabla}_v\,f\,d^3v$$

$$=\left[\frac{q\vec{E}}{m}+\frac{\vec{F}}{m}\right]\cdot n\,\overline{\vec{\nabla}_v Q}+\frac{qn}{m}\overline{\left[\left(\frac{\vec{V}\times\vec{B}}{c}\right)\cdot\vec{\nabla}_v Q\right]} \tag{180}$$

the dot product $\left(\frac{\vec{v}\times\vec{B}}{c}\right)\cdot\vec{\nabla}_v Q$ being defined by comparing Eqs. (179) and (180). Then Eq. (175) takes the form

$$\frac{\partial(n\overline{Q})}{\partial t} = - \vec{\nabla}_r \cdot n\overline{(\vec{V} Q)} + \left[\frac{g}{m}\vec{E} + \frac{\vec{F}}{m}\right] \cdot n\overline{\vec{\nabla}_v Q}$$

$$+ \frac{g n}{m}\overline{\left[\left(\frac{\vec{V}\times\vec{B}}{c}\right)\cdot\vec{\nabla}_v Q\right]} + \left[\frac{\partial(n\overline{Q})}{\partial t}\right]_{\text{COLL.}} \tag{181}$$

B. Specific Examples of Q

(1) Q = 1, The Continuity Equation

As the simplest nontrivial example of Q we consider Q = 1. For this case, Eq. (181) becomes

$$\frac{\partial n}{\partial t} = - \vec{\nabla}_r \cdot (n\overline{\vec{V}}) \tag{182}$$

if collisions conserve particles. If particles are not conserved in a collision, then Eq. (181) takes the form

$$\frac{\partial n}{\partial t} = - \vec{\nabla}_r \cdot (n\overline{\vec{V}}) + \left(\frac{\partial n}{\partial t}\right)_{\text{COLL.}} \tag{183}$$

This last case arises when we have ionization or recombination. Eq. (182) is the familiar continuity equation, while Eq. (183) is the form this equation takes when we have sources or sinks for particles.

From now on, unless otherwise stated, we shall assume that no ionization or recombination takes place.

(2) $Q = \vec{v}$, Conservation of Momentum

The next simplest function which Q can be is $\vec{v}$. For this case Eq. (181) becomes

$$\frac{\partial(n\overline{\vec{V}})}{\partial t} = - \vec{\nabla}_r \cdot (n\overline{\vec{V}\vec{V}}) - \left[\frac{g}{m}\vec{E} + \frac{\vec{F}}{m}\right] n$$

$$- \frac{g}{m} n \frac{\overline{\vec{V}}\times\vec{B}}{c} + \left[\frac{\partial}{\partial t}(n\overline{\vec{V}})\right]_{\text{COLL.}} \tag{184}$$

The second term involves the gradient of the momentum transfer dyadic. The physical meaning of this term is described in the next section, where we have made use of the fact that $\vec{A}\,\vec{\nabla}_v \vec{v} = \vec{A}$, or, in terms of dyadics

$$\vec{\nabla}_v \vec{v} = \underset{\approx}{I} \tag{185}$$

where $\underline{I}$ is the unit dyadic. It should be emphasized that there is one equation of the form (184) for each species of particle. By writing

$$\sigma = q n \tag{186}$$

and

$$\vec{J} = q n \overline{\vec{v}} \tag{187}$$

and by noting that from conservation of momentum collisions with particles of the same species can make no contribution to

$$\left[\frac{\partial (n\overline{\vec{v}})}{\partial t}\right]_{COLL}$$

we can write Eq. (184) in the form

$$m\left[\frac{\partial (n\overline{\vec{v}})}{\partial t} + \nabla_r \cdot n\overline{\vec{v}\vec{v}} - \sigma\vec{E} - n\overline{\vec{F}} - \frac{\vec{J}\times\vec{B}}{c}\right] = m\left[\frac{\partial (n\overline{\vec{v}})}{\partial t}\right]_{\substack{COLL\ WITH \\ OTHER\ SPECIES}} \tag{188}$$

If we further write

$$\vec{v} = \vec{c} + \overline{\vec{v}} \tag{189}$$

where $\vec{c}$ is the deviation of $\vec{v}$ from its average value, then we have for $\overline{\vec{v}\,\vec{v}}$

$$\overline{\vec{v}\vec{v}} = \overline{(\vec{c}+\overline{\vec{v}})(\vec{c}+\overline{\vec{v}})} = \overline{\vec{c}\,\vec{c}} + \overline{\vec{v}}\,\overline{\vec{v}} \tag{190}$$

since $\overline{\vec{v}\,\vec{c}}$ and $\overline{\vec{c}\,\vec{v}}$ are zero.

The first term of Eq. (188) may be written

$$\frac{\partial (n\overline{\vec{v}})}{\partial t} = n\frac{\partial \overline{\vec{v}}}{\partial t} + \overline{\vec{v}}\frac{\partial n}{\partial t}. \tag{191}$$

Using the continuity equation, this becomes

$$\frac{\partial(n\overline{\vec{V}})}{\partial t} = n\frac{\partial\overline{\vec{V}}}{\partial t} + \overline{\vec{V}}(-\overline{\vec{V}}\cdot\vec{\nabla}_r n - n\,\vec{\nabla}_r\cdot\overline{\vec{V}}). \tag{192}$$

Furthermore, the second term in Eq. (188), when Eq. (190) is applied, has the term

$$\vec{\nabla}_r\cdot n\overline{\vec{V}}\,\overline{\vec{V}} = \overline{\vec{V}}(\overline{\vec{V}}\cdot\vec{\nabla}_r n) + \overline{\vec{V}}\,n(\vec{\nabla}_r\cdot\overline{\vec{V}}) + n(\overline{\vec{V}}\cdot\vec{\nabla}_r)\,\overline{\vec{V}} \tag{193}$$

Using Eqs. (190), (192), and (193), Eq. (188) becomes

$$mn\left[\frac{\partial}{\partial t} + \overline{\vec{V}}\cdot\vec{\nabla}_r\right]\overline{\vec{V}} + \vec{\nabla}_r\cdot(nm\,\overline{\vec{C}\vec{C}}) - \sigma\vec{E}$$

$$-n\vec{F} - \frac{\vec{J}\times\vec{B}}{c} = \left[\frac{\partial\vec{P}}{\partial t}\right]_{\text{COLL. WITH OTHER SPECIES}}. \tag{194}$$

Eq. (194) is the force equation or the equation for conservation of momentum. The quantity $\underline{\underline{\Pi}}$

$$nm\,\overline{\vec{C}\vec{C}} = \underline{\underline{\Pi}} \tag{195}$$

is called the stress tensor. The right-hand side of Eq. (194) is the momentum transferred to the species under consideration, by collisions with other species of particles. If the mean velocities of the particles are small compared to their thermal velocities, then this term will be proportional to the difference in mean velocities. It can be written as a resistivity to the relative flow times the relative velocity.

The stress tensor contains not only the pressure but also the viscous stresses as well. For a general case, the stress tensor is hard to compute. However, for the case of a very high collision rate the velocity distribution will be isotropic and $\underset{\sim}{\Pi}$ will have only diagonal terms. All the diagonal terms will be equal and $\underset{\sim}{\Pi}$ can be replaced by

$$\underset{\sim}{\Pi} = p \underset{\sim}{I} \tag{196}$$

where $\underset{\sim}{I}$ is the unit dyadic and p is the scalar pressure.

When the collision rate is large, but not so large that it can be considered infinite, then the off-diagonal terms give the viscous stresses and can be written down in terms of the shear viscosity. In addition, the diagonal terms are no longer all equal, the difference arising from the bulk viscosity.

If the collision rate is small, then in general we must compute the stress tensor from the Boltzmann equation. However, there are a few cases here when $\underset{\sim}{\Pi}$ takes on a particularly simple form. If time variations are slow compared to the cyclotron frequency and spatial variations are much larger than a Larmor radius, then the velocity distribution must be symmetric in the two directions perpendicular to the magnetic field. In this case the stress tensor is again diagonal in a coordinate system with one axis along the direction of the magnetic field, and the two terms which come from the velocities perpendicular to the magnetic field are equal. That is, if we take $\bar{B}$ to be in the z direction, then $\underset{\sim}{\Pi}$ has the form

$$\underset{\sim}{\Pi} = \hat{x}\hat{x}\,\Pi_\perp + \hat{y}\hat{y}\,\Pi_\perp + \hat{z}\hat{z}\,\Pi_{\parallel}. \tag{197}$$

Here again one must, in general, solve the Boltzmann equation to find $\Pi_\perp$ and $\Pi_{\parallel}$. However, there are certain simple cases when they can be obtained simply. In particular, when the gradients along the field direction are small, so that variations in this direction can be neglected, then $\Pi_\perp$ can be obtained from the adiabatic law for a two-dimensional gas

$$\Pi_\perp = \Pi_{\perp 0}\, \rho^2/\rho_0^2 \tag{198}$$

as we have already seen in our treatment of the two-dimensional motions of a plasma. In certain cases $\Pi_{\parallel}$ is also given by an adiabatic law, that for a one-dimensional gas

$$\Pi_{\parallel} = \Pi_{\parallel 0}\, \rho^3/\rho_0^3. \tag{199}$$

We saw an example of this earlier when we treated the longitudinal invatiant. The adiabatic conditions are usually obtained when mixing of particles or equivalently energy flow along the lines can be neglected. This happens when the system is closed in the z direction, or for the case of wave propagation when the phase velocity along the field is much faster than the root mean square particle velocity in that direction.

When both these adiabatic laws can be employed, we say that the gas obeys a double adiabatic equation of state.

(3) The Momentum Transfer Dyadic

We consider a small cube with edges $\vec{e}_x dx$, $\vec{e}_y dy$, and $\vec{e}_z dz$ (Fig. 42). In one second all the particles within a distance v_x from face 1 will cross this area, each carrying a momentum $m\vec{v}$. The

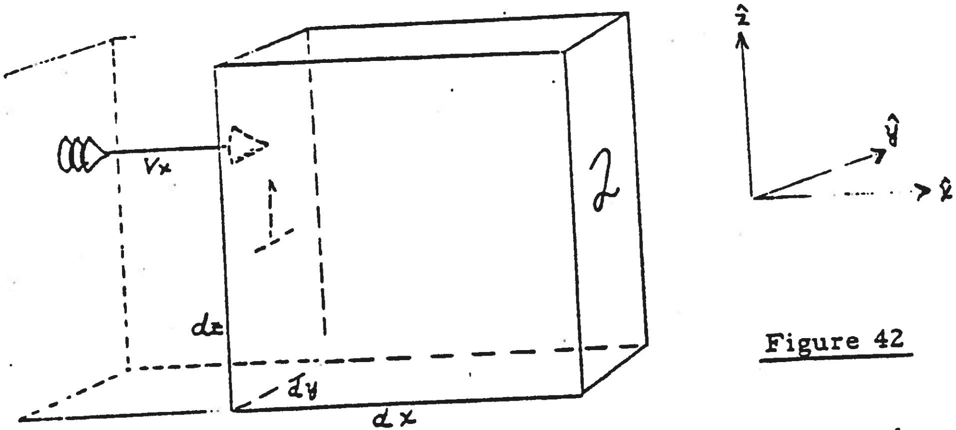

Figure 42

number of particles in this volume is $n v_x dy\, dz$. Thus the momentum entering face 1 per second is

$$m\vec{v}\,(n v_x\, dy\, dz) = mn\, v_x \vec{v}\, dy\, dz. \tag{200}$$

The momentum leaving face 2 per second is

$$mn v_x \vec{v}\, dy\, dz + \frac{\partial}{\partial x}(mn v_x \vec{v}\, dy\, dz)\, dx. \tag{201}$$

Thus the net momentum gain per second through these two surfaces is

$$-\frac{\partial}{\partial x}(mn\, v_x \vec{v}\, dx\, dy\, dz). \tag{202}$$

The other surfaces similarly contribute

$$-\frac{\partial}{\partial y}(mn\, v_y \vec{v}\, dx\, dy\, dz) - \frac{\partial}{\partial z}(mn\, v_z \vec{v}\, dx\, dy\, dz). \tag{203}$$

The increase of momentum per unit time per unit volume is just the sum of Eqs. (202) and (203), which, converting to dyadic notation, is simply

$$\vec{\nabla}_r \cdot \underline{\underline{\Pi}} = \vec{\nabla}_r \cdot (mn\, \vec{v}\vec{v}). \tag{204}$$

(4) $Q = \vec{v} \cdot \vec{v}$, Conservation of Energy

Let us now look at $Q = \vec{v} \cdot \vec{v}$. This will give us an equation for the time development of the kinetic energy density. Substituting in Eq. (181) gives

$$\frac{\partial(n\overline{\vec{v}\cdot\vec{v}})}{\partial t} = -\vec{\nabla}_r\cdot(n\overline{\vec{v}(\vec{v}\cdot\vec{v})})$$

$$+\left[\frac{q}{m}\vec{E}+\frac{\vec{F}}{m}\right]\cdot 2n\overline{\vec{v}}+\left[\frac{\partial}{\partial t}(n\overline{\vec{v}\cdot\vec{v}})\right]_{coll}. \qquad (205)$$

Here we have made use of the fact that

$$\vec{\nabla}_v(\vec{v}\cdot\vec{v}) = 2\vec{v} \qquad (206)$$

and that

$$(\vec{v}\times\vec{B})\cdot\vec{v} = 0. \qquad (207)$$

Again, writing

$$\vec{v} = \vec{c}+\overline{\vec{v}} \qquad (208)$$

gives

$$\overline{\vec{v}\cdot\vec{v}} = \overline{\vec{c}\cdot\vec{c}}+\overline{\vec{v}}\cdot\overline{\vec{v}} \qquad (209)$$

and

$$\nabla_r\cdot(n\,\overline{v(v\cdot v)}) =$$

$$\vec{\nabla}_r\cdot\left\{n\overline{\left[(\vec{c}+\overline{\vec{v}})[(\vec{c}+\overline{\vec{v}})\cdot(\vec{c}+\overline{\vec{v}})]\right]}\right\} =$$

$$\nabla_r\cdot\left\{n\,\overline{\vec{c}[\vec{c}\cdot\vec{c}]}+2n\overline{\vec{c}\vec{c}}\cdot\overline{\vec{v}}+n\,\overline{\vec{v}}\;\overline{\vec{c}\cdot\vec{c}}\right.$$

$$\left.+n\,\overline{\vec{v}}\,(\overline{\vec{v}}\cdot\overline{\vec{v}})+2\,\overline{\overline{\vec{v}}(\vec{c}\cdot\overline{\vec{v}})}+\overline{\vec{c}}\;\overline{\vec{v}}\cdot\overline{\vec{v}}\right\} \qquad (210)$$

where the last two terms are zero, since $\overline{\vec{v}}$ is a constant which may be taken outside the average.

Substituting Eq. (210) in Eq. (205) and making use of the continuity equation (182) gives

$$n\left[\frac{\partial}{\partial t}\left(\frac{m}{2}\overline{V}\cdot\overline{V}\right) + (\overline{V}\cdot\vec{\nabla}_r)\left(\frac{m}{2}\overline{V}\cdot\overline{V}\right)\right]$$

$$+n\left[\frac{\partial}{\partial t}\left(\frac{m}{2}\overline{\vec{c}\cdot\vec{c}}\right) + (\overline{V}\cdot\nabla_r)\left(\frac{m}{2}\overline{\vec{c}\cdot\vec{c}}\right)\right]$$

$$+\nabla_r\cdot\left[nm\,\overline{\vec{c}\vec{c}}\cdot\overline{V} + \frac{nm}{2}\overline{V}\,\overline{\vec{c}\cdot\vec{c}}\right]$$

$$+\nabla_r\cdot\left[\frac{nm}{2}\overline{\vec{c}(\vec{c}\cdot\vec{c})}\right] =$$

$$= \vec{E}\cdot\vec{J} + \vec{F}\cdot n\overline{V} + \left[\frac{\partial(nK)}{\partial t}\right]_{\substack{\text{COLL WITH} \\ \text{OTHER SPECIES}}} \tag{211}$$

where we have written

$$\vec{J} = q\,n\,\overline{V} \tag{212}$$

and K is the total mean kinetic energy per particle, and from conservation of energy K for the species of particles under consideration does not change for collisions with itself.

Problem: Verify Eq. (211).

The terms appearing in Eq. (211) have the following meanings: The first two terms on the left are the convective time derivative of the energy of mean motion; the second two terms on the left form the convective time derivative of the energy of the random motion about the mean — i.e., the heat energy. The fifth term is the divergence of the stress tensor dotted with the mean velocity. The stress tensor dotted with the mean velocity gives the rate at which the fluid stresses are doing work, so this is the divergence of this rate of doing work. The sixth term is the divergence of the flux of random energy due to the mean motion and the seventh term is the divergence of the flux of random motion due to the random motions. This seventh term is called the heat flow. The first term on the right-hand side is the rate at which the electric field does work on the current; the second term on the right-hand side is the rate at which the external force does work on the fluid. Finally, the last term on the right-hand side is the rate at which the species of particles under consideration lose energy to other types of particles.

Again, if we have large collision rates the stress tensor can be written in terms of the viscous stresses which are proportional to the gradient of $\overline{\overline{v}}$, while the heat flow in this case is proportional to the temperature gradient. In general, the thermal conductivity may be a tensor, particularly for a plasma in a magnetic field, and the heat flow will be given by

$$\vec{H} = \underset{\sim}{k} \cdot \vec{\nabla}_r T. \tag{213}$$

The energy exchange between different species of particles will have two terms — one proportional to the mean relative drifts squared and the other proportional to the temperature differences (i.e., differences in mean random energy.

Here, as in the case of the momentum equation when collisions are infrequent, one cannot write simple expressions for these terms. If collisions are negligible, then we are in general forced to solve the full collisionless Boltzmann equation along with Maxwell's equations to find these quantities.

XV. The Basic Fluid Equations for a Two-Component Plasma

Let us look at the problem of a plasma composed of electrons and one species of ion with charge ze. We have just found the conservation equations for particles, momentum, and energy for a single species. We may apply these equations to our present problem. First, consider the momentum equation (194). There is one such equation for the electrons and one for the ions. We will drop the bar from these equations, since everything will be understood to stand for averages from now on.

$$m_e n_e \left(\frac{\partial \vec{v}_e}{\partial t} + \vec{v}_e \cdot \vec{\nabla}_r \vec{v}_e\right) + \vec{\nabla}_r \cdot \underset{\sim}{\Pi_e} + n_e e \vec{E} - n_e \vec{F}_e$$

$$+ n_e e \frac{\vec{v}_e \times \vec{B}}{c} = \left[\frac{\partial \vec{P}_e}{\partial t}\right]_{i.e.} \tag{214}$$

$$m_i n_i \left(\frac{\partial \vec{v}_i}{\partial t} + (\vec{v}_i \cdot \vec{\nabla}_r)\vec{v}_i\right) + \vec{\nabla}_r \cdot \underset{\sim}{\Pi_i} - n_i z e \vec{E} - n_i \vec{F}_i$$

$$- n_i z e \frac{\vec{v}_i \times \vec{B}}{c} = \left[\frac{\partial \vec{P}_i}{\partial t}\right]_{e.i.} = - \left[\frac{\partial \vec{P}_e}{\partial t}\right]_{i.e} \tag{215}$$

It is possible to work with these equations directly in investigating the behavior of a plasma. This is particularly useful if only one species of particle (for example, electrons) plays a significant role in the motion. However, often the two species move together as a fluid, and it is then useful to work with two new variables — the mean velocity and the current. We therefore define the mass velocity $\vec{v}$ by

$$\vec{V} = \frac{n_i m_i \vec{V}_i + n_e m_e \vec{V}_e}{n_i m_i + n_e m_e}, \tag{216}$$

the current $\vec{J}$ by

$$\vec{J} = e\,(z n_i \vec{V}_i - n_e \vec{V}_e), \tag{217}$$

the density ρ by

$$\rho = n_i m_i + n_e m_e, \tag{218}$$

and the charge density by

$$\sigma = e\,(n_i z - n_e). \tag{219}$$

In addition to these relations, we will need the equations of continuity for both types of particles

$$\frac{\partial n_e}{\partial t} = -\vec{\nabla}_r \cdot (n_e \vec{V}_e) \tag{220}$$

and

$$\frac{\partial n_i}{\partial t} = -\vec{\nabla}_r \cdot (n_i \vec{V}_i). \tag{221}$$

By multiplying Eq. (220) by m_e and Eq. (221) by m_i and adding, we immediately obtain the general continuity equation for mass

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_r \cdot (\rho \vec{V}) = 0. \tag{222}$$

By multiplying Eq. (220) by $-e$ and Eq. (221) by $z e$ and adding, we obtain the equation for conservation of charge

$$\frac{\partial \sigma}{\partial t} + \vec{\nabla}_r \cdot \vec{J} = 0. \tag{223}$$

These equations replace the two continuity equations (220) and (221).

Before proceeding we will make two approximations. First, we will assume that $1 >> z\, m_e/m_i$. Since the ratio is always less than 2000^{-1}, this is a very good approximation. Second, we shall assume that the ion and electron densities are equal. At the very beginning of the course we saw that the total kinetic energy of the electrons in a Debye sphere was just enough to remove them from the sphere against the resulting attractive $\vec{E}$ field. For regions much larger than this, only very small deviations from equal numbers can occur. Thus we set

$$n_e = z n_i. \tag{224}$$

This does not mean that σ must be zero; however, it must simply be much smaller than $n_e e$. To find σ we must use Eq. (223). We can use this to check the approximation (224). With these approximations, Eqs. (216) and (217) for $\vec{v}$ and $\vec{j}$ become

$$\vec{V} = \vec{V}_i + \frac{z m_e}{m_i} \vec{V}_e \tag{225}$$

and

$$\vec{j} = n_e e (z\vec{V}_i - \vec{V}_e). \tag{226}$$

We can find $\vec{v}_i$ and $\vec{v}_e$ in terms of $\vec{v}$ and $\vec{j}$ from Eqs. (225) and (226).

$$\vec{V}_i = \vec{V} + \frac{i m_e z \vec{J}}{m_i n e e} \tag{227}$$

and

$$\vec{V}_e = \vec{V} - \frac{\vec{J}}{n_e e}. \tag{228}$$

We shall use the form of Eq. (188) for Eqs. (214) and (215); however, splitting off the stress tensor. They have been written again below for convenience. Letting $\vec{\nabla}_r \cdot (n \vec{V} \vec{V}) = \vec{V}(\vec{V} \cdot \vec{\nabla}_r n) + \vec{V} n (\vec{\nabla}_r \cdot \vec{V}) + n(\vec{V} \cdot \vec{\nabla}_r \vec{V})$,

$$m_e \left[\frac{\partial}{\partial t}(n_e \vec{V}_e) + \underbrace{\vec{V}_e (\vec{V}_e \cdot \vec{\nabla}_r n_e) + n_e \vec{V}_e (\vec{\nabla}_r \cdot \vec{V}_e) + n_e (\vec{V}_e \cdot \vec{\nabla}_r \vec{V}_e)}_{\vec{\nabla}_r \cdot (n_e \vec{V}_e \vec{V}_e)} \right]$$

$$+ \vec{\nabla}_r \cdot \underline{\underline{\Pi}}_e + n_e e \vec{E} - n_e \vec{F}_e + n_e e \left(\frac{\vec{V}_e \times \vec{B}}{c} \right) = \left[\frac{\partial \vec{P}_e}{\partial t} \right]_{ie} \tag{229}$$

$$m_i\Big[\frac{\partial}{\partial t}(n_i\vec{v}_i) + \underbrace{\vec{v}_i(\vec{v}_i\cdot\vec{\nabla}_r n_i) + n_i\vec{v}_i(\vec{\nabla}_r\cdot\vec{v}_i) + n_i(\vec{v}_i\cdot\vec{\nabla}_r\vec{v}_i)}_{\vec{\nabla}_r\cdot(n_i\vec{v}_i\vec{v}_i)}\Big]$$

$$+ \vec{\nabla}_r\cdot\underset{\sim}{\Pi_i} - n_i z e\vec{E} - n_i\vec{F}_i - n_i z e\left(\frac{\vec{v}_i\times\vec{B}}{c}\right) = \left[\frac{\partial\vec{P}_i}{\partial t}\right]_{ei}. \qquad (230)$$

We add Eqs. (229) and (230), term by term, below. With the approximations $z n_i = n_e$ and $z m_e << m_i$, adding the first term of the ion and electron equations gives

$$\frac{\partial}{\partial t}[m_e n_e\vec{v}_e + m_i n_i\vec{v}_i] = \frac{\partial}{\partial t}[\vec{v}(m_e n_e + m_i n_i)]$$

$$+ \frac{\partial}{\partial t}\left[-\frac{n_e m_e}{n_e e}\vec{j} + \frac{n_i m_i m_e z}{m_i n_e e}\vec{j}\right]$$

$$= \frac{\partial(\rho\vec{v})}{\partial t}. \qquad (231)$$

Addition of the second term of the ion and of the electron equations, when $\vec{v}_i$ and $\vec{v}_e$ are expressed in terms of $\vec{v}$ and $\vec{j}$, will involve $\vec{v}\vec{v}$, $\vec{j}\vec{j}$, and mixed ($\vec{v}\vec{j}$ and $\vec{j}\vec{v}$) terms. The $\vec{v}\vec{v}$ term is simply

$$\vec{\nabla}_r\cdot[(n_i m_i + n_e m_e)\vec{v}\vec{v}] = \vec{\nabla}_r\cdot[\rho\vec{v}\vec{v}]. \qquad (232)$$

The $\vec{j}\vec{j}$ term is

$$\vec{\nabla}_r\cdot\left[n_i m_i\left(\frac{m_e z}{m_i n_e e}\right)^2\vec{j}\vec{j} + n_e m_e\left(\frac{1}{n_e e}\right)^2\vec{j}\vec{j}\right]$$

$$\vec{\nabla}_r\cdot\left[\vec{j}\vec{j}\frac{m_e}{n_e e^2}\left(\frac{m_e z}{m_i} + 1\right)\right] \approx \vec{\nabla}_r\cdot\frac{m_e}{n_e e^2}\vec{j}\vec{j}. \qquad (233)$$

The mixed term is

$$\vec{\nabla}_r \cdot n_e m_e\left[-\frac{1}{n_e e}(\vec{j}\vec{V}+\vec{V}\vec{j})\right] + \vec{\nabla}_r \cdot n_i m_i\left[\frac{m_e z}{m_i n_e e}(\vec{j}\vec{V}+\vec{V}\vec{j})\right]$$

$$= \nabla_r \cdot\left[-\frac{m_e}{e}(\vec{j}\vec{V}+\vec{V}\vec{j}) + \frac{m_e}{e}(\vec{j}\vec{V}+\vec{V}\vec{j})\right] = 0. \qquad (234)$$

Summing all the terms gives

$$\frac{\partial}{\partial t}(\rho\vec{V}) + \vec{\nabla}_r \cdot\left[\rho\vec{V}\vec{V} + \frac{m_e}{n_e e^2}\vec{j}\vec{j}\left[1+\frac{m_e z}{m_i}\right] + \underset{\sim}{\Pi}_e + \underset{\sim}{\Pi}_i\right]$$

$$-\sigma\vec{E} - (n_e\vec{F}_e + n_i\vec{F}_i) - \frac{\vec{j}\times\vec{B}}{c} = 0 \qquad (235)$$

or using the continuity equation and writing

$$\underset{\sim}{\Pi}_T = \underset{\sim}{\Pi}_e + \underset{\sim}{\Pi}_i + \frac{m_e}{n_e e^2}\left[1+\frac{m_e z}{m_i}\right]\vec{j}\vec{j} \qquad (236)$$

gives

$$\rho\underbrace{\left(\frac{\partial}{\partial t}+\vec{V}\cdot\vec{\nabla}_r\right)}_{=d/dt}\vec{V} + \vec{\nabla}_r\cdot\underset{\sim}{\Pi}_T - (n_e\vec{F}_e + n_i\vec{F}_i)$$

$$-\sigma\vec{E} - \frac{\vec{j}\times\vec{B}}{c} = 0. \qquad (237)$$

The $\vec{j}\,\vec{j}$ term in the total stress tensor Π_T arises because the mean velocity for both ions and electrons is different from the mean velocity for either species and these are what were split out when we computed the stress tensors. Further, the ion-electron collision terms have canceled out because of conservation of momentum.

To obtain the current equation we multiply Eq. (229) by $-e/m_e$ and Eq. (230) by ez/m_i and add, term by term. The first term is

$$m_i\frac{ez}{m_i}\frac{\partial}{\partial t}(n_i\vec{V}_i) - \frac{e}{m_e}m_e\frac{\partial}{\partial t}(n_e\vec{V}_e)$$

$$= \frac{\partial}{\partial t}(n_i\vec{V}_i ez - n_e\vec{V}_e e) = \frac{\partial\vec{j}}{\partial t}. \qquad (238)$$

The second term is

$$e z \; \vec{\nabla}_r \cdot (n_i \vec{v}_i \vec{v}_i) - e \; \vec{\nabla}_r \cdot (n_e \vec{v}_e \vec{v}_e) \tag{239}$$

This will again involve $\vec{v}\vec{v}$, $\vec{j}\vec{j}$ and mixed terms. The $\vec{v}\vec{v}$ term is

$$\vec{\nabla}_r \cdot (e z n_i \; \vec{V}\vec{V}) - \vec{\nabla}_r \cdot (e n_e \vec{V}\vec{V}) = 0 . \tag{240}$$

The $\vec{j}\vec{j}$ term is

$$\vec{\nabla}_r \cdot \left[e z \; n_i \left(\frac{m_e z}{m_i n_e} e \right)^2 \vec{j}\vec{j} \right] - \vec{\nabla}_r \cdot \left[e n_e \left(\frac{1}{n_e e} \right)^2 \vec{j}\vec{j} \right]$$

$$= \vec{\nabla}_r \cdot \vec{j}\vec{j} \left(\frac{m_e^2 z^2}{m_i^2 e n_e} - \frac{1}{n_e e} \right)$$

$$= \vec{\nabla}_r \cdot \left[\vec{j}\vec{j} \left(\frac{m_e^2 z^2}{m_i^2} - 1 \right) \frac{1}{n_e e} \right] = - \vec{\nabla}_r \cdot \left(\frac{1}{n_e e} \vec{j}\vec{j} \right). \tag{241}$$

The $\vec{v}\vec{j}$ term is

$$\vec{\nabla}_r \cdot \left[e z n_i \left(\vec{V}\vec{j} + \vec{j}\vec{V} \right) \frac{m_e z}{m_i n_e e} \right] + \vec{\nabla}_r \cdot \left[e n_e \left(\vec{V}\vec{j} + \vec{j}\vec{V} \right) \frac{1}{n_e e} \right]$$

$$= \vec{\nabla}_r \cdot \left[\left(\frac{m_e z}{m_i} + 1 \right) \left(\vec{V}\vec{j} + \vec{j}\vec{V} \right) \right] \approx \vec{\nabla}_r \cdot \frac{1}{n_e e} \vec{j}\vec{j} \tag{242}$$

and writing $\left[\frac{\partial \vec{P}_e}{\partial t} \right]_{ie} = \alpha \, (\vec{V}_i - \vec{V}_e)$ we get

$$\frac{\partial \vec{j}}{\partial t} - \vec{\nabla}_r \cdot \left[\frac{\vec{j}\vec{j}}{n_e e} \right] + \vec{\nabla}_r \cdot \left[\vec{V}\vec{j} + \vec{j}\vec{V} \right]$$

$$+ \vec{\nabla}_r \cdot \left[\frac{z e \Pi_i}{m_i} - \frac{e \Pi_e}{m_e} \right] - \frac{n_e e^2}{m_e} \left[1 + \frac{z m_e}{m_e} \right] \vec{E}$$

$$+ n_e e \left[\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i} \right] - \frac{n_e e^2}{m_e} \frac{\vec{V} \times \vec{B}}{c} + \frac{e}{m_e c} \vec{j} \times \vec{B} = - \frac{\alpha \vec{j}}{n_e m_e} \tag{243}$$

or, multiplying by $+m_e/n_e e^2$, gives

$$\frac{m_e}{n_e e^2}\frac{\partial \vec{j}}{\partial t} + \frac{m_e}{n_e e^2}\vec{\nabla}_r \cdot \left\{ -\frac{\vec{j}\vec{j}}{n_e e} + (\vec{v}\vec{j} + \vec{j}\vec{v}) + \frac{e}{m_e}\left[\frac{Z m_e}{m_i} \underset{\sim}{\Pi}_i - \underset{\sim}{\Pi}_e \right] \right\}$$

$$-\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) + \frac{1}{n_e e c}\,\vec{j}\times\vec{B} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) = -\underset{\sim}{\eta}\cdot\vec{j}. \tag{244}$$

Here $\underset{\sim}{\eta}$ is the conductivity, and in principle it should be a tensor for a plasma in a magnetic field. If we linearize Eqs. (237) and (244) in $\vec{j}$ and $\vec{v}$ — that is, we assume j and v are small and drop all second order terms (vv, jv, jj) — then these equations become

$$\rho\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}_r \cdot \underset{\sim}{\Pi}_T - (n_e \vec{F}_e + n_i \vec{F}_i) - \sigma\vec{E} - \frac{\vec{j}\times\vec{B}}{c} = 0 \tag{245}$$

and

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{j}}{\partial t}\right] + \frac{1}{e n_e}\vec{\nabla}_r \cdot \left[\frac{Z m_e}{m_i}\underset{\sim}{\Pi}_i - \underset{\sim}{\Pi}_e\right] - \left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)$$

$$+ \frac{\vec{j}\times\vec{B}}{n_e e c} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) = -\underset{\sim}{\eta}\cdot\vec{j}. \tag{246}$$

If $\underset{\sim}{\Pi}_i$ is of the order of $\underset{\sim}{\Pi}_e$, then the ion stress tensor can be neglected in Eq. (246). If $\partial\vec{j}/\partial t$, $\vec{B}$, $\vec{F}_e$, and $\vec{F}_i$ are negligible, then Eq. (246) reduces to

$$\vec{E} = \underset{\sim}{\eta}\cdot\vec{j} \tag{247}$$

which is Ohm's law with η the resistivity. We may therefore think of Eq. (246), or more generally Eq. (244), as a generalized Ohm's law.

The terms in Eq. (245) are fairly clear — $\rho\,\partial\vec{v}/\partial t$ is the inertial term, $\vec{\nabla}_r \cdot \underset{\sim}{\Pi}_T$ is the force due to the material stresses, the $\vec{F}$ terms are the external forces, and $\vec{j}\times\vec{B}$ is the magnetic force. The terms in

Eq. (246) have the following meaning. The $\partial \vec{j}/\partial t$ term is due to the inertia of the current. In most cases where the current is carried primarily by the electron, it comes from the electron inertia. The term involving the stress tensors arises because the pressure or stresses due to one species tends to accelerate that species relative to the other, creating a current. The $\vec{E} + \frac{\vec{v} \times \vec{B}}{c}$ term is the electric field as seen by an observer moving with the fluid. The $\vec{j} \times \vec{B}$ term arises because the ions and electrons carry different fractions of the current and have different masses so that they are accelerated differently, and this tends to give rise to a current or to a balancing $\vec{E}$ field. This gives rise to the Hall effect. The $\vec{F}$ terms arise from differential accelerations of the two species due to the external forces. If $\vec{F}$ arises from a gravitational field, this term cancels out. The $\eta \cdot \vec{j}$ term is, of course, the resistivity.

XVI. Summary of the Macroscopic Equations

Summing up, the macroscopic equations for the fluid are

$$\rho \frac{d\vec{V}}{dt} + \vec{\nabla}\cdot \underline{\Pi} - (n_e \vec{F}_e + n_i \vec{F}_i) - \frac{\vec{J}\times\vec{B}}{c} = 0, \qquad (248)$$

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{J}}{\partial t} + \vec{\nabla}\cdot\left[-\frac{\vec{J}\vec{J}}{n_e e} + (\vec{V}\vec{J} + \vec{J}\vec{V}) + e\left(\frac{z\,\Pi_i}{m_i} - \frac{\Pi_e}{m_e}\right)\right]\right]$$
$$-\left(\vec{E} + \frac{\vec{V}\times\vec{B}}{c}\right) + \frac{\vec{J}\times\vec{B}}{n_e e c} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) = -\eta\cdot\vec{J}, \qquad (249)$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0, \qquad (250)$$

and

$$\frac{\partial\sigma}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0. \qquad (251)$$

In addition we have Maxwell's equations

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \qquad (252)$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi\vec{J}}{c}, \qquad (253)$$

$$\vec{\nabla}\cdot\vec{B} = 0, \qquad (254)$$

and

$$\vec{\nabla}\cdot\vec{E} = 4\pi\sigma. \qquad (255)$$

In addition to these equations we need equations to determine the π's . We can proceed a long way, however, by making some simplifying assumptions about π . Examples of choices for π are:

(1) We assume the particle stresses are negligible and neglect π altogether.

(2) We may assume that we need use only a scalar pressure p

$$\underset{\sim}{\Pi} = p\,\underline{I}. \qquad (256)$$

and that p satisfies a $\gamma = 5/3$ law.

(3) We may assume a double adiabatic law with the pressure perpendicular to the lines of force determined from a $\gamma = 2$ law and with the pressure parallel to the lines of force determined by a $\gamma = 3$ law.

(4) We may use fluid equations, including viscosity, heat conduction, exchange of energy between parallel and perpendicular degrees of freedom and between electrons and ions.

One must examine the physics of the situation under consideration to determine which, if any, of the above approximations is pertinent.

XVII. Approximations to the Equation for the Current

The first and most usual approximation we will make to Eq. (249) is that of linearization. That is, we will neglect the $\vec{j}\,\vec{j}$ and $\vec{v}\,\vec{j}$ terms so that Eq. (249) becomes

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{J}}{\partial t} + \vec{\nabla}\cdot e\left(\frac{z\,\Pi_i}{m_i} - \frac{\Pi_e}{m_e}\right)\right] - \left(\vec{E} + \frac{\vec{V}\times\vec{B}}{c}\right)$$

$$+ \frac{1}{n_e e c}\,\vec{J}\times\vec{B} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) = -\eta\cdot\vec{J}. \qquad (257)$$

Second, in most applications the external forces are either nonexistent or negligible and can be dropped (for the case of gravitational forces they cancel). Third, if the electron and ion thermal ~~velocities~~ energies are comparable, the π_i and π_e are roughly equal and we may neglect π_i/m_i compared to π_e/m_e. With these approximations, Eq. (257) reduces to

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{J}}{\partial t} - \vec{\nabla}\cdot\frac{e\,\Pi_e}{m_e}\right] - \left[\vec{E} + \frac{\vec{V}\times\vec{B}}{c}\right]$$

$$+ \frac{1}{n_e e c}\,\vec{J}\times\vec{B} = -\eta\cdot\vec{J}. \qquad (258)$$

If we solve the linearized version of Eq. (248) for $\vec{j} \times \vec{B}/c$ (also neglecting $\vec{F}$), we obtain

$$\frac{\vec{j} \times \vec{B}}{c} = \rho \frac{\partial \vec{v}}{\partial t} + \vec{\nabla} \cdot \underset{\sim}{\Pi}_T \tag{259}$$

and substituting this in Eq. (258) gives

$$\frac{4\pi}{\omega_p^2}\left[\frac{\partial \vec{j}}{\partial t} + \vec{\nabla} \cdot \frac{e}{m_e} \underset{\sim}{\Pi}_i\right] - \frac{\rho}{n_e e} \frac{\partial \vec{v}}{\partial t}$$

$$- \vec{E} + \frac{\vec{v} \times \vec{B}}{c} = - \eta \cdot \vec{j}. \tag{260}$$

If we now look for a steady state solution and further assume that the pressure gradient is negligible, then Eq. (260) reduces to

$$\vec{E} + \vec{v} \times \vec{B}/c = \eta \cdot \vec{j}. \tag{261}$$

This says that the current is driven by the electric field seen in a frame moving with the fluid. Finally, if the conductivity of the fluid is very high, then η is negligible and Eq. (261) becomes

$$\vec{E} + \frac{\vec{v} \times \vec{B}}{c} = 0 \tag{262}$$

For a high-temperature plasma η becomes very small, as we shall see later, and this may become a good approximation provided the other assumptions made are valid. A fluid for which Eq. (262) is satisfied is said to be a perfect conductor. You will note that it satisfies our criterion that the lines of force move with the fluid (equation 19 in section VII).

XVIII. Discussion of the Relation Between Macroscopic and Microscopic Velocities

Let us consider the steady state case with isotropic pressure, comparable ion and electron pressures, zero resistivity, and small $\vec{v}$ and $\vec{j}$ (linearized equations). Eqs. (248) and (249) give

$$\vec{\nabla} p - (n_e \vec{F}_e + n_i \vec{F}_i) - \frac{\vec{j} \times \vec{B}}{c} = 0, \tag{263}$$

$$0 = -\frac{1}{n_e e} \vec{\nabla} P_e - \left(\vec{E} + \frac{\vec{V} \times \vec{B}}{c}\right) + \frac{1}{n_e e c} \vec{j} \times \vec{B} + \frac{m_e}{e}\left(\frac{\vec{F}_e}{m_e} - \frac{\vec{F}_i}{m_i}\right) \tag{264}$$

where

$$p = P_e + P_i \tag{265}$$

or, substituting $\vec{j} \times \vec{B}/c$ from Eq. (263) in Eq. (264), Eq. (264) becomes

$$\frac{1}{n_e e} \vec{\nabla} P_i - \left(\vec{E} + \frac{\vec{V} \times \vec{B}}{c}\right) - \frac{\vec{F}_i}{Z e} = 0. \tag{266}$$

We should like to compare the motions given by these equations with those for individual particles. We shall consider various situations as we did for the case of particle drifts.

A. Uniform Plasma Density and $\vec{B}$ Field, with No External Force, $\vec{E}$ Perpendicular to $\vec{B}$

For this case, as we have already seen,

$$\vec{E}_\perp \doteq - \frac{\vec{V} \times \vec{B}}{c}, \tag{267}$$

and

$$\vec{V}_\perp = \frac{c\, \vec{E}_\perp \times \vec{B}}{B^2} \tag{268}$$

This is the same drift velocity we found for individual particle motions.

B. Uniform $\vec{B}$, no $\vec{E}$ or $\vec{F}$, with a Gradient of $\vec{P}$

For this case Eq. (266) becomes

$$\vec{\nabla} P_i = n_e e \frac{\vec{V} \times \vec{B}}{c}. \tag{269}$$

Thus, $\vec{\nabla} P_i$ is perpendicular to $\vec{B}$. Also, by Eq. (263)

$$\vec{\nabla} P = \frac{\vec{j} \times \vec{B}}{c}. \tag{270}$$

$\vec{\nabla} P$ is perpendicular to $\vec{B}$. (If the ion and electron densities and temperatures are the same, then $P_i = P_e$.) Now we may solve Eq. (269) for $\vec{v}$ just as we did in the case of a uniform $\vec{E}$ field. Crossing Eq. (269) with $\vec{B}$ on the right gives

$$\vec{V} = -\frac{c}{n_e e}\left[\frac{\vec{\nabla} P_i \times \vec{B}}{B^2}\right]. \tag{271}$$

We see that there is a macroscopic velocity for the plasma in this case. However, from the particle orbit point of view the particles do not drift in a uniform B field. How does this drift arise?

The reason for the macroscopic motion is the following. The velocity $\vec{v}$ of the fluid in a little element of volume is the mean velocity of all particles within that volume. Those particles whose orbits lie entirely within the volume will not contribute to the mean velocity since they are moving up as much as down (see Fig. 43). However, near the edge of the volume there are some particles whose orbits lie only partially within the volume, as shown in Fig. 43. All those particles with their centers of gyration lying on the right side of the volume element $d\tau = dx\,dy\,dz$ contribute to a net downward velocity, while all those particles with centers lying on the

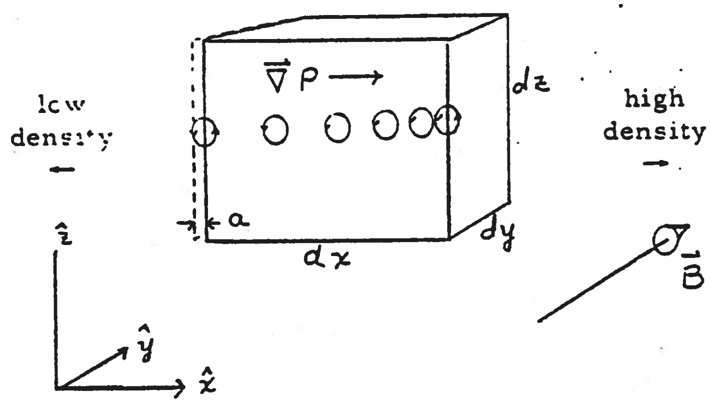

Figure 43

left-hand side of the volume contribute to a net upward velocity. If there are more particles on the right than on the left, the result will be a net downward motion of the fluid in $d\tau$. A quick estimation of the size of this effect shows that it can give Eq. (271). The downward momentum contributed by the particles on the right is

$$-m n_2 a v_\perp dy\,dz = -n_2 \frac{m v_\perp^2}{eB} mc\,dy\,dz \tag{272}$$

while the upward momentum contributed by the particles on the left is

$$m n_1 a v_\perp dy\,dz = n_1 \frac{m v_\perp^2}{eB} mc\,dy\,dz \tag{273}$$

Here a is the larmor radius and na is the number of particles per unit area within a larmor radius of the surface $d\tau$. The net downward momentum is

$$-\frac{m^2 c v_\perp^2}{eB}[n_2 - n_1]\,dy\,dz = -\frac{m^2 c v_\perp^2}{eB}\vec{\nabla} n\,dx\,dy\,dz \tag{274}$$

Dividing by the total mass of material in $d\tau$ gives

$$V_z = -\frac{m^2 c v_\perp^2}{eB}\frac{\vec{\nabla} n\,dx\,dy\,dz}{mn\,dx\,dy\,dz} = -\frac{mc v_\perp^2}{eBn}\nabla n = -\frac{c\,\nabla_x P_i}{n_e e B} \tag{275}$$

where it has been assumed that the temperature is independent of position and hence also v^2, so that it can be taken inside $\vec{\nabla}$ ($P_i = m_i v^2$, to order m_e/m_i only the ions contribute). We see that Eq. (275) is identical to Eq. (271) if the x direction is the same as $\vec{\nabla} P_i$.

C. Uniform Pressure, Nonuniform $\vec{B}$, no $\vec{E}$ or $\vec{F}$

For this case Eq. (266) gives

$$\frac{\vec{\nabla} \times \vec{B}}{c} = 0$$

or

$$\vec{V}_\perp = 0. \tag{276}$$

where $\perp$ means perpendicular to $\vec{B}$.

Here we have no macroscopic velocity. On the other hand, from the particle orbit point of view the particles are drifting. How do we explain this?

Consider a small element of volume $d\tau = dx\,dy\,dz$ with $\vec{B}$ out of the paper and $\vec{\nabla}B$ in the x direction, as shown in Fig. 44.

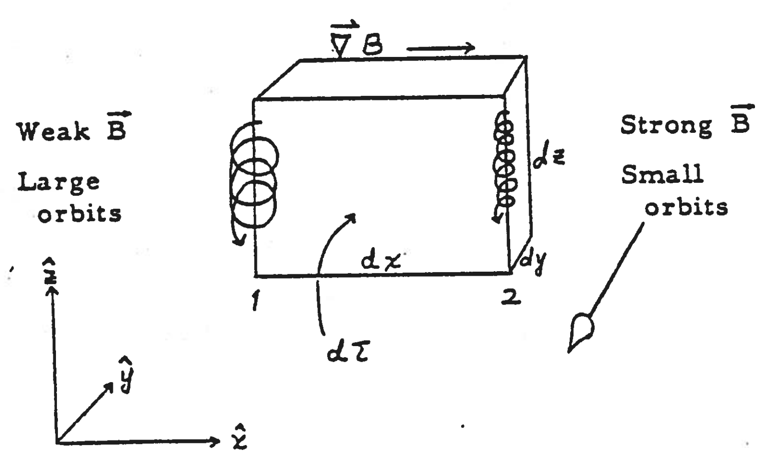

Figure 44

Now the particle orbits within the volume contribute a net momentum given by Eq. (147) section IV, which for this case is downward of magnitude

$$\frac{c m v_\perp^2 \nabla B}{e B^2} m n \, dx\, dy\, dz \tag{277}$$

The net velocities due to particles whose centers of gyration lie outside $d\tau$ but whose guiding centers intersect $d\tau$, however, cancel this. The upward momentum contribution due to particles at face 1 is

$$+\frac{n v_\perp^2}{e B_1} m^2 c \, dy\, dz \tag{278}$$

while the downward contribution due to particles at face 2 is

$$\frac{n v_\perp^2}{e B_2} m^2 c \, dy\, dz = \left[\frac{n v_\perp^2}{e B_1} m^2 c + \frac{\partial}{\partial x}\left(\frac{n v_\perp^2}{e B} m^2 c\right) dx\right] dy\, dz \, . \tag{279}$$

The net momentum upward due to both faces 1 and 2 is

$$\frac{n v_\perp^2 m^2 c}{e} \times \frac{\nabla B \, d\tau}{B^2} \tag{280}$$

which is just what is required to cancel the drift given by Eq. (277).

XIX. Diffusion of Magnetic Fields Through Matter and of Plasma Across Magnetic Fields

A. Diffusion of a Magnetic Field through a Solid Conductor

As the simplest example of the diffusion of a magnetic field through a conducting material we will consider the diffusion of a magnetic field through a solid conductor. We shall adopt the simple ohm's law

$$\vec{E} = \eta \vec{j} \tag{281}$$

as the equation which determines the current in terms of $\vec{E}$. This can be obtained from our general equation (249) by neglecting the inertial terms, the nonlinear terms, the pressure terms, the Hall term, and the external forces. We have also set the velocity equal to zero. In addition to this equation we have the Maxwell equations

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \tag{282}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j}, \tag{283}$$

and

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{284}$$

We have dropped the displacement current term ($\dot{\vec{E}}$) since here we are primarily interested in low-frequency phenomena where it is negligible. We must, however, keep the $\partial \vec{B}/\partial t$ term because it is needed to determine $\vec{E}$ which drives $\vec{j}$. In addition to these we should add

$$\vec{\nabla} \cdot \vec{j} = 0 \tag{285}$$

for if this is violated charges build up rapidly, producing large $\vec{E}$

fields which alter the current so as to prevent further buildup of charge. However, Eq. (285) is automatically satisfied because of Eq. (283). We now substitute Eq. (281) in Eq. (283), obtaining

$$\vec{\nabla} \times \vec{B} = \frac{4\pi \vec{E}}{c\eta}. \tag{286}$$

Taking the curl of Eq. (286) and making use of Eq. (282) and Eq. (284) gives

$$-\nabla^2 \vec{B} = -\frac{4\pi}{c^2\eta} \frac{\partial \vec{B}}{\partial t}. \tag{287}$$

Only $\vec{B}$'s which satisfy Eq. (284) as well as Eq. (287) are acceptable.

Now Eq. (287) is a diffusion equation with a diffusion coefficient

$$D = \frac{c^2\eta}{4\pi}. \tag{288}$$

Thus the larger the resistivity the larger is the rate of diffusion of the field through the matter.

In order to obtain a physical understanding of the meaning of this equation, consider the following simple situation. Imagine that we have a conducting slab of material which is infinite in the xy plane and has thickness 2d in the z direction (see Fig. 45).

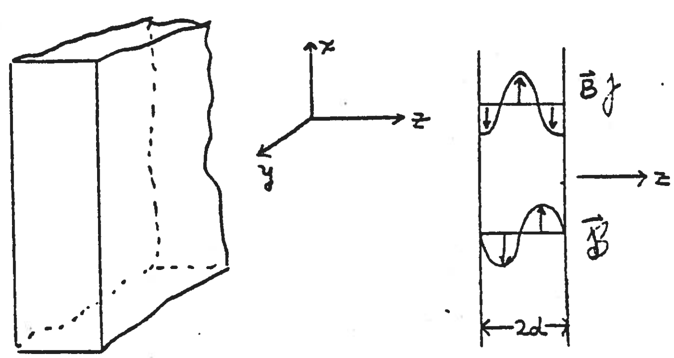

Figure 45

We take the initial $\vec{j}$ to be of the form

$$\vec{j} = \hat{x}\, j_0 \cancel{\sin} \frac{\pi z}{d} \tag{289}$$

and hence the initial $\vec{B}$ is given by

$$\frac{\partial B_y}{\partial z} = -\frac{4\pi}{c} j_0 \cancel{\cos} \frac{\pi z}{d} \tag{290}$$

or, on integration

$$B_y = -\frac{4 j_0 d}{c} \cancel{\cos} \frac{\pi z}{d} + B_0. \tag{291}$$

Since we want $\vec{B}$ to vanish at $z = \pm\infty$, we choose $B_o = 0$.

Now since things are uniform in the x,y direction, there can be no variations in these directions. Further, if there are no x and z components to B initially, none will arise according to Eq. (287). Thus, in this case Eq. (287) becomes

$$\frac{c^2\eta}{4\pi} \frac{\partial^2 B_y}{\partial z^2} = \frac{\partial B_y}{\partial t}. \tag{292}$$

Writing

$$B_y = A(t) \overset{\sin}{\cancel{\cos}} \frac{\pi z}{d} \tag{293}$$

and substituting in Eq. (292) gives

$$-\frac{c^2\eta}{4\pi} \frac{\pi^2}{d^2} A(t) = \frac{\partial A}{\partial t} \tag{294}$$

or

$$A = A_0 e^{-t/\tau} \tag{295}$$

where

$$\tau = \frac{4d^2}{c^2\eta\pi} = \frac{d^2}{\pi^2 D^2} \tag{296}$$

and from Eq. (291)

$$A_0 = -\frac{4 j_0 d}{c}. \tag{297}$$

Thus the magnetic field decays exponentially with time.

To find the physical significance of τ we compute the magnetic energy per unit area in the x y direction and the rate of dissipation of this energy by the opposition to the current offered by the resistance. The magnetic energy is given by

$$\int_{-d}^{d} \frac{B^2}{8\pi}\, dz = \int_{-d}^{d} \frac{A^2}{8\pi} \cancel{\cos^2} \sin^2 \frac{\pi z}{d}\, dz = \frac{A^2 d}{8\pi}. \tag{298}$$

The current density in terms of A is given by

$$j = \frac{cA}{4d} \tag{299}$$

and the rate of dissipation of energy is

$$\int_{-d}^{d} \eta j^2\, dz = \eta \frac{c^2 A^2}{16 d}. \tag{300}$$

If we divide the magnetic energy by its rate of dissipation we obtain a time t

$$t = \frac{A^2 d}{8\pi} \Big/ \eta \frac{c^2 A^2}{16 d} = \frac{2 d^2}{\pi \eta c^2}. \tag{301}$$

Thus, according to Eq. (301), B^2 should decay on the time scale $t = 2d^2/\pi \eta c^2$. We can obtain the same result qualitatively from Eq. (287) if we assume the characteristic distances over which $\vec{B}$ varies is L and its decay time is τ. We then have approximately, from Eq. (287),

$$\frac{c^2 \eta}{4\pi} \frac{B}{L^2} \cong \frac{B}{\tau} \tag{302}$$

or,

$$\frac{c^2 \eta}{4\pi} \frac{B^2}{L^2} \cong B^2/\tau. \tag{303}$$

But from Eq. (283)

$$B/L \cong 4\pi j/c \tag{304}$$

or

$$\eta j^2 4\pi \simeq B^2/\tau \text{ or } \eta j^2 \cong \frac{B^2}{4\pi\tau}. \tag{305}$$

The left-hand side of Eq. (305) is the rate of dissipation of energy by the current and the right-hand side is the rate of decay of magnetic energy.

(In the above example the magnetic field diffuses out of the slab into the vacuum region.)

If we had kept the $\dot{\vec{E}}$ term we would see the $\vec{B}$ field propagate away from the slab at the speed of light upon emerging from the slab. If we had kept this term, Eq. (287) would become

$$\nabla^2 \vec{B} = \frac{4\pi}{c^2\eta}\frac{\partial \vec{B}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}. \tag{306}$$

The flux between planes $z = \pm\infty$, or more exactly, between $z = \pm ct + d$ is conserved, provided there is no magnetic field outside the slab initially. However, the magnetic energy density which is proportional to B^2, decays in time.

Problem: Prove the above statement.

B. Diffusion of a Plasma Across a Magnetic Field

As the simplest example of the diffusion of a plasma across a magnetic field we will consider of a uniform circular plasma cylinder in the case of a very strong field so that the gas pressure is negligible compared to the magnetic pressure. We can thus neglect the effect of the plasma on the magnetic field. We shall take the field to be uniform and to point in the z direction. We shall also assume that the diffusion is slow so that the inertial forces can be neglected. The nonlinear terms and external forces will be dropped. We shall assume that we can use a scalar pressure. Eqs. (248) and (249) thus become

$$\vec{\nabla} p - \frac{\vec{j} \times \vec{B}}{c} = 0 \tag{307}$$

and

$$\vec{E} + \frac{\vec{v} \times \vec{B}}{c} - \frac{\vec{j} \times \vec{B}}{n_e e c} = \eta \vec{j}. \tag{308}$$

From Eq. (307), $\vec{\nabla} P$ is perpendicular to $\vec{B}$. There can be no pressure variations along $\vec{B}$. Solving Eq. (307) for the part of $\vec{j}$ which is perpendicular to $\vec{B}$ gives

$$\vec{j}_\perp = -c \frac{\vec{\nabla} p \times \vec{B}}{B^2}. \tag{309}$$

One may readily verify that $\vec{\nabla} \cdot \vec{j}_\perp$ is zero, so no accumulation of charge takes place. Thus no other currents are required — i.e., $\vec{j}_\parallel$ to $\vec{B}$ — and we will assume that none exist. We may also drop $\vec{E}$ because $\vec{\nabla} \times \vec{E}$ is zero since $\dot{\vec{B}}$ is, and the only $\vec{E}$ fields that can exist are electrostatic fields arising from charges. These need not exist, and we shall assume there are none.*

---

* Such charges may exist and in principle they should be determined from Eqs. (307) and (308) by requiring $\vec{\nabla} \cdot \vec{j} = 0$; i.e., no accumulation of charge must take place. This in general requires a $\vec{j}_\parallel$, and hence an $\vec{E}_\parallel$ by Eq. (308). This $\vec{E}$, along with $\vec{\nabla} \times \vec{E} = 0$, suffice to determine $\vec{E}$. If one cannot find a $\vec{j}_\parallel$ which makes $\vec{\nabla} \cdot \vec{j} = 0$, then one must take account of the fact that charges are accumulating and that a time-dependent $\vec{E}$ exists, hence an acceleration of the plasma and also a polarization current. This was the case of the plasma column dropping in a magnetic field. Since no $\vec{j}_\parallel$ is needed here, we ignore it.

One other source of $\vec{E}$ field exists here. That is, an electrostatic field with E_x and E_y components which is the same in all z planes. Such a field would exist if the plasma were rotating. Such a field would have to be put in by introducing charges initially, for it will not arise by itself because $\vec{\nabla} \cdot \vec{j} = 0$. Again we ignore it.

Substituting Eqs. (307) and (309) in (308) then gives

$$\frac{\vec{v}\times\vec{B}}{c} - \frac{\vec{\nabla}P}{n_e e} = -\eta c \frac{\vec{\nabla}P\times\vec{B}}{B^2} \tag{310}$$

Here we are treating η as a scalar. Crossing Eq. (310) on the right with $\vec{B}$ and solving for $\vec{v}_\perp$ gives

$$\vec{v}_\perp = -c\frac{\vec{\nabla}P\times\vec{B}}{n_e e B^2} - \eta\frac{c^2\vec{\nabla}P}{B^2} \tag{311}$$

The first term on the right is a velocity perpendicular to both $\vec{\nabla}P$ and $\vec{B}$, and hence perpendicular to the density gradients if we assume a uniform temperature. This motion lies in the surfaces of constant pressure and leads to no loss of plasma.

The second term on the right-hand side is antiparallel to the pressure gradient and hence to the density gradient if the temperature is uniform. The motion carries the plasma from regions of higher density to regions of lower density.

The following argument shows the physical cause of this diffusion. The rate of dissipation of energy by the current per unit volume is just ηj^2.

By Eq. (309) this is

$$\eta j^2 = c^2\eta\,|\vec{\nabla}P|^2/B^2. \tag{312}$$

This energy must be supplied by the work done by the pressure of the gas as it expands. For an element of convecting fluid which is being pushed outward by gas pressure inside, from following a fluid element we get

$$\frac{dP}{dt} = 0 = \frac{\partial P}{\partial t} + \vec{v}\cdot\vec{\nabla}P. \tag{313}$$

At a given position

$$\frac{\partial P}{\partial t} = - \vec{V} \cdot \vec{\nabla} P. \tag{314}$$

But since pressure is thermal energy per unit volume, then the work done on the convecting fluid is $\vec{V} \cdot \vec{\nabla} P$. Equating this with Eq. (312) gives

$$\vec{V}_{\nabla P} = c^2 \eta \, \frac{\vec{\nabla} P}{B^2} \tag{315}$$

which is the same as Eq. (311).

A second way to view this diffusion is the following. The ions and electrons must drift through each other to provide the current required by pressure balance. Because of collisions there is a drag of one species on the other, the drags being equal and opposite in direction if we consider singly-charged ions, or proportional to z if they are multiply-charged. The force per electron is (by equation 308) ([illegible]

$$- e \eta \vec{j} = \vec{F}_e \tag{316}$$

and per ion

$$z e \eta \vec{j} = \vec{F}_i \tag{317}$$

If we view these as external forces to each species, then these forces give rise to drifts. The drifts are in the same direction and have the same magnitude for each species. The drift velocity is

$$\vec{V}_D = \frac{c}{q} \frac{\vec{F} \times \vec{B}}{B^2} = - c \eta \, \frac{\vec{j} \times \vec{B}}{B^2} \tag{318}$$

or

$$= c^2 \eta \, \vec{\nabla} P / B^2 \tag{319}$$

The diffusion of a plasma given in Eq. (319), across the field, is obviously related to the diffusion of a magnetic field across a plasma, as described by Eqs. (287) and (302). Let us write Eq. (302) in a somewhat different way

$$\frac{L}{T} = \frac{c^2 \eta}{4\pi L} \sim V_B, \tag{320}$$

We compare this with that part of Eq. (319) that describes the outward diffusion of the plasma, which we will approximate

$$V_\perp = -\frac{\eta c^2}{B^2}\frac{P}{L}. \tag{321}$$

Then

$$\frac{V_\perp}{V_B} \sim \frac{P}{B^2/4\pi}. \tag{322}$$

XX. Waves in a Plasma

To investigate the propagation of waves in a plasma we shall use the macroscopic equations for ions and electrons, Eqs. (214) and (215), along with the appropriate continuity equations and Maxwell's equations, Eqs. (252) to (255). We do this rather than use the fluid equations just derived because it will give us an insight into the role played by each species of particle. Also, the procedure used is readily generalized to include more than two species of particles. We shall neglect external forces, and we shall use the linearized versions of these equations assuming no velocity or electric field in the equilibrium. We also take the plasma to be infinite and homogeneous in equilibrium. The equations to be used are the following:

$$N_e m_e \frac{\partial \vec{V}_e}{\partial t} + \vec{\nabla}\cdot \underset{\sim}{\Pi_e} + N_e e \vec{E} + \frac{N_e e \vec{V} \times \vec{B}_0}{c} = \left(\frac{\partial \vec{P}_e}{\partial t}\right)_{i.e.}, \tag{323}$$

$$N_i m_i \frac{\partial \vec{V}_i}{\partial t} + \vec{\nabla} \cdot \underset{\sim}{\pi}_i - N_i z e \vec{E} - \frac{N_i z e \vec{V}_i \times \vec{B}_0}{c} = -\left(\frac{\partial \vec{P}_e}{\partial t}\right)_{ie}, \tag{324}$$

$$\frac{\partial n_e}{\partial t} + N_e \vec{\nabla} \cdot \vec{V}_e = 0, \tag{325}$$

$$\frac{\partial n_i}{\partial t} + N_i \vec{\nabla} \cdot \vec{V}_i = 0, \tag{326}$$

$$\sigma = n_i z e - n_e e, \tag{327}$$

$$\vec{j} = N_i z e \vec{V}_i - N_e e \vec{V}_e, \tag{328}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \tag{329}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{j}, \tag{330}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\sigma, \tag{331}$$

and

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{332}$$

Here N_e, N_i are the equilibrium number densities for ions and electrons and to preserve charge neutrality $N_e = z N_i$; B_o is the equilibrium magnetic field. The perturbed densities, velocities, and stress tensors are n_e, n_i, $\vec{v}_e$, $\vec{v}_i$, $\underset{\sim}{\pi}_e$, $\underset{\sim}{\pi}_i$, while $\left(\frac{\partial \vec{P}_e}{\partial t}\right)_{ei}$ is the rate of change of the electron momentum due to collisions with ions (a first order quantity). The perturbed electric and magnetic fields are denoted by $\vec{E}$ and $\vec{B}$. To proceed we also need equations to determine the $\underset{\sim}{\pi}_e$'s. To start out, we shall assume they are negligible.

A. Waves in a Cold Plasma

1. No Magnetic Field in the Equilibrium and No Collisions

For this case Eqs. (323) and (324) reduce to

$$N_e m_e \frac{\partial V_e}{\partial t} + N_e e E = 0 \tag{333}$$

and

$$N_i m_i \frac{\partial V_i}{\partial t} - z N_i e E = 0, \tag{334}$$

Combining Eqs. (333) and (334) so as to obtain the time derivative of the current gives

$$\frac{\partial \vec{J}}{\partial t} - \left[\frac{N_e e^2}{m_e} + \frac{N_i z^2 e^2}{m_i}\right]\vec{E} = 0. \tag{335}$$

Taking the time derivative of Eq. (330) and substituting it into the curl of Eq. (329) gives

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi}{c^2}\frac{\partial \vec{J}}{\partial t} \tag{336}$$

or by making use of vector identities and Eq. (335)

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{4\pi}{c^2}\left[\frac{N_e e^2}{m_e} + \frac{N_i z^2 e^2}{m_e}\right]\vec{E} = 0. \tag{337}$$

If we set

$$4\pi\left[\frac{N_e e^2}{m_e} + \frac{N_i z^2 e^2}{m_i}\right] = \omega_p^2, \tag{338}$$

the electron plasma frequency to order m_e/m_i, then Eq. (337) becomes

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{\omega_p^2}{c^2}\vec{E} = 0. \tag{339}$$

a. Transverse Waves

Let us first consider the case $\vec{\nabla}\cdot\vec{E} = 0$. We see from Eq. (335) that if $\vec{\nabla}\cdot\vec{E}$ is zero then the time derivative of $\vec{\nabla}\cdot\vec{j}$ is zero and hence the second time derivative of σ is zero by $\frac{\partial \sigma}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$. Since $\vec{\nabla}\cdot\vec{E} = 4\pi\sigma$, the second time derivative of $\vec{\nabla}\cdot\vec{E}$ is zero. Thus, if $\vec{\nabla}\cdot\vec{E}$ and $\vec{\nabla}\cdot\vec{j}$ are zero initially, they will remain zero forever, so this is a consistent approximation.

Eq. (339) now reduces to

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \frac{\omega_p^2}{c^2}\vec{E} = 0. \tag{340}$$

If we Fourier-analyze $\vec{E}$ in time and space — i.e., write

$$\vec{E} = \vec{E}_1 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}, \tag{341}$$

Eq. (340) gives

$$\left[-k^2 + \frac{\omega^2 - \omega_p^2}{c^2}\right]\vec{E}_1 = 0. \tag{342}$$

The condition $\vec{\nabla}\cdot\vec{E} = 0$ gives

$$\vec{k}\cdot\vec{E}_1 = 0. \tag{343}$$

Thus these waves are transverse. Eq. (342) gives the dispersion relation

$$k^2c^2 = \omega^2 - \omega_p^2. \tag{344}$$

We see from Eq. (344) that if ω^2 is less than ω_p^2, k^2 is negative. This means that k will be imaginary and hence E will become exponentially large or exponentially small as one goes to ∞ in either the $+\vec{k}$ or $-\vec{k}$ direction. The first is not allowed by energy consideration; the second corresponds to a ~~damped~~ wave going as $e^{-\vec{k}\cdot\vec{r}}$, where

$$k = \frac{\sqrt{\omega_p^2 - \omega^2}}{c}. \tag{345}$$

We may compute the phase velocity of the waves given by Eq. (344). It is

$$V_p^2 = \frac{\omega^2}{k^2} = c^2 \Big/ \left(1 - \frac{\omega_p^2}{\omega^2}\right). \tag{346}$$

The index of refraction $n = c/v_p$ is

$$n = \left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2}. \tag{347}$$

We see from Eqs. (346) or (347) that for $\omega > \omega_p$ the phase velocity is always greater than c .

Another quantity which is of interest is the group velocity V_g which is given by

$$V_g = \frac{d\omega}{dk}. \tag{348}$$

From Eq. (344) we have

$$2 k c^2 dk = 2\omega d\omega \tag{349}$$

or

$$\frac{d\omega}{dk} = \frac{k}{\omega} c^2 = \frac{c^2}{V_p} \tag{350}$$

thus

$$V_g = \frac{d\omega}{dk} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}. \tag{351}$$

Thus the group velocity is always less than the velocity of light. The group velocity is usually the velocity of propagation of energy, unless one is in a region of strong anomalous dispersion which usually is associated with strong absorption, or for optically active media, strong stimulated emission.

One other quantity of interest for these waves is the energy density

$$W = \frac{E^2}{8\pi} + \frac{B^2}{8\pi} + \frac{\rho V^2}{2}. \tag{352}$$

From Eq. (329) we have

$$\frac{i\omega \vec{B}}{c} = i\vec{k} \times \vec{E}. \tag{353}$$

The magnetic field is ~~90° out of~~ phase and perpendicular to the electric field. We get

$$\frac{B^2}{8\pi} = \frac{c^2 k^2}{\omega^2} \frac{E^2}{8\pi} \tag{354}$$

while from Eq. (333) we have

$$i\omega N_e m_e \vec{V}_e = + N_e e \vec{E}. \tag{355}$$

The electron velocity is also 90° out of phase with the electric field, therefore

$$\vec{V}_e = \mp \frac{ie}{m_e \omega} \vec{E} \tag{356}$$

while from Eq. (334) we have, for the ions,

$$\vec{V}_i = + \frac{iZe\vec{E}}{m_i \omega}. \tag{357}$$

Hence the peak particle kinetic energy

$$\frac{N_e m_e V_e^2 + N_i m_i V_i^2}{2} = \left[\frac{N_e e^2}{m_e} + \frac{N_i Z^2 e^2}{m_i}\right] \frac{E^2}{2\omega^2}. \tag{358}$$

The peak energy density of the magnetic field and particles ~~(which are in phase)~~ is then

$$W_t = \frac{E^2}{8\pi}\left[\frac{c^2 k^2 + \omega_p^2}{\omega^2}\right] = \frac{E^2}{8\pi}. \tag{359}$$

~~and the energy at one time is entirely in the electric field; 90° later it is in the magnetic field and particle kinetic energy.~~

In summary, we have found that waves similar to electromagnetic waves in a vacuum are propagated in a plasma for $\omega > \omega_p$. In the limit as the plasma density goes to zero, the waves go over to the usual electromagnetic waves. For a wave incident on a plasma such that $\omega < \omega_p$, the wave is attenuated.

Problem: Show that the energy flux given by Poynting's Theorem $c(\vec{E} \times \vec{B})$ divided by the energy density is equal to the group velocity. This can be thought of as the velocity of propagation of energy.

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b. Longitudinal Oscillations

Let us now consider the case where $\vec{\nabla} \times \vec{E} = \vec{k} \times \vec{E} = 0$; i.e., $\vec{k}$ is parallel to $\vec{E}$. Then Eq. (336) gives, for this case,

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial j}{\partial t} = 0. \tag{360}$$

Fourier-analyzing Eq. (360) and Eq. (335) gives

$$-\omega^2 \vec{E} - 4\pi i \omega \vec{j} = 0 \tag{361}$$

and

$$-i\omega\vec{j} - \frac{\omega_p^2}{4\pi}\vec{E} = 0. \tag{362}$$

Eliminating $i\omega\vec{j}$ between Eqs. (361) and (362) gives

$$\left(-\omega^2 + 4\pi\frac{\omega_p^2}{4\pi}\right)\vec{E} = 0 \tag{363}$$

so that the dispersion relation is simply

$$\omega^2 = \omega_p^2. \tag{364}$$

Since $\vec{\nabla} \cdot \vec{E} \neq 0$, there is also an oscillation of charge density that goes as $e^{\pm i\omega_p t}$. From this we see that the component of $\vec{E}_k$ parallel to $\vec{k}$ is determined by σ_k or the longitudinal part of $\vec{E}$ is determined by σ_k. All k's oscillate at this frequency. Thus there is no dependence of ω on k.

The phase velocity of the waves is thus proportional to $1/k$ and is given by

$$V_p = \frac{\omega}{k} = \frac{\omega_p}{k} \tag{365}$$

and the group velocity is zero.

$$V_g = \frac{d\omega}{dk} = 0, \tag{366}$$

These longitudinal waves do not propagate.

c. <u>The Effect of Collisions</u>

(1) <u>Transverse Waves</u>

We may easily include the effects of electron-ion collisions on the propagation of transverse and longitudinal waves through an infinite homogeneous plasma containing no zero order magnetic field and for which the thermal motions are unimportant. We do this by simply adding a term, $-R(v_e - v_i)$, to the right-hand side of Eq. (333), and $R(v_e - v_i)$ to the right-hand side of Eq. (334). Equivalently, we may substitute

$$\vec{E} - \eta \vec{J} \quad \text{for} \quad \vec{E} \tag{367}$$

in the right-hand side of Eq. (335). Here η is the usual resistivity. Eq. (335) thus becomes

$$\frac{\partial \vec{J}}{\partial t} - \frac{\omega_p^2}{4\pi} \vec{E} = -\frac{\omega_p^2}{4\pi} \eta \vec{J}. \tag{368}$$

Here, as with η equal to zero, we may divide the modes into transverse and longitudinal oscillations. First let us look at transverse oscillations with $\vec{\nabla} \cdot \vec{E} = 0$. Fourier-analyzing Eqs. (336) and (368) gives

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\vec{E} - \frac{4\pi i\omega}{c^2}\vec{j} = 0, \tag{369}$$

$$\vec{k}\cdot\vec{E} = 0, \tag{370}$$

$$-i\omega\vec{j} - \frac{\omega_p^2}{4\pi}\vec{E} = -\frac{\omega_p^2}{4\pi}\eta\vec{j} \tag{371}$$

where everything goes like $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$. Solving Eq. (371) for $\vec{j}$ in terms of $\vec{E}$ and substituting in Eq. (369) gives

$$\left[k^2 - \frac{\omega^2}{c^2} + \frac{4\pi i\omega\omega_p^2}{c^2(4\pi i\omega - \omega_p^2\eta)}\right]\vec{E} = 0 \tag{372}$$

from which we obtain the dispersion relation

$$c^2k^2 = \omega^2 - \frac{4\pi i\omega\omega_p^2}{(4\pi i\omega - \omega_p^2\eta)} \tag{373}$$

or

$$c^2k^2 = \omega^2 - \omega_p^2\left[\frac{(4\pi\omega)^2 - 4\pi\omega i\eta\omega_p^2}{(4\pi\omega)^2 + \omega_p^4\eta^2}\right]. \tag{374}$$

From Eq. (374) we see that k^2 has a positive imaginary part, as shown in Fig. 46. The root k with the positive real part has a positive imaginary part, while the root with the negative real part has a negative imaginary part, as shown.

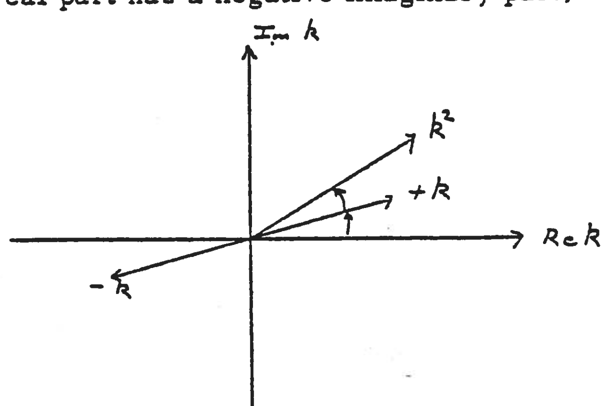


Figure 46

In either case the waves die out as one goes in the $\vec{r}$ direction parallel to k .

Problems: (1) Simplify equation for large and small ω.

(2) If Eq. (374) is solved for ω for given k, k real, show that the imaginary part of ω is always negative.

(2) Longitudinal Waves

Here, as in the case of transverse waves, we may proceed as we did in the case of no resistance. We noted there that $\vec{\nabla} \times \vec{E}$ is zero for this type of oscillation, so that Eq. (336) takes the form

$$\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial \vec{j}}{\partial t} = 0. \tag{375}$$

Eq. (375), along with Eq. (368) for the current, determines these oscillations. Fourier-analyzing Eqs. (375) and (368) gives

$$-\omega^2 \vec{E} - 4\pi i \omega \vec{j} = 0 \tag{376}$$

and

$$\vec{j} = \frac{\omega_p^2 \vec{E}}{(\omega_p^2 \eta - 4\pi i \omega)}. \tag{377}$$

Substitution of Eq. (377) into Eq. (376) gives

$$\left[\omega^2 + \frac{\omega_p^4\, 4\pi i \omega}{(\omega_p^2 \eta - 4\pi i \omega)}\right]\vec{E} = 0 \tag{378}$$

or

$$4\pi i \omega(\omega^2 - \omega_p^2) - \omega^2 \omega_p^2 \eta = 0, \tag{379}$$

$$4\pi i \omega\left[\omega^2 - \omega_p^2 + \frac{i}{4\pi}\,\omega\,\omega_p^2 \eta\right] = 0, \tag{380}$$

gives

$$\omega = \frac{1}{2}\left[\frac{i\,\omega_p^2 \eta}{4\pi} \pm \sqrt{\frac{-\omega_p^4 \eta^2}{(4\pi)^2} + 4\omega_p^2}\right] \tag{381}$$

or

$$\omega = \pm \omega_p \left(1 - \frac{\omega_p^2 \eta^2}{4^3 \pi^2}\right)^{1/2} - \frac{i \omega_p^2 \eta}{8\pi}. \qquad (382)$$

Thus for $\omega_p \eta << 1$, $\omega \sim \omega_p$ and the imaginary part is small compared with the real part. In addition, we must impose the condition that

$$\vec{\nabla} \times \vec{E} = 0 \qquad (383)$$

or

$$\vec{k} \times \vec{E} = 0. \qquad (384)$$

2. Waves in an Infinite Homogeneous Cold Plasma Containing a Uniform Magnetic Field

a. Basic Equations

We now turn to the problems of the propagation of waves through a cold plasma containing a uniform magnetic field. We shall start by neglecting collisions. We obtain our basic equations from Eqs. (323) to (332) by dropping the stress tensor and the collision term. These basic equations are

$$N_e m_e \frac{\partial \vec{V}_e}{\partial t} + N_e e\left[\vec{E} + \frac{\vec{V}_e \times \vec{B}_o}{c}\right] = 0 \qquad (385)$$

and

$$N_i m_i \frac{\partial \vec{V}_i}{\partial t} - N_i Z e\left[\vec{E} + \frac{\vec{V}_i \times \vec{B}}{c}\right] = 0, \qquad (386)$$

plus the equations of continuity and Maxwell's equations.

We shall take $\vec{B}_o$ to be in the z direction. Again, taking the curl of Eq. (252) and substituting into it the time derivative of Eq. (253) gives

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi}{c^2} \frac{\partial \vec{J}}{\partial t}. \qquad (387)$$

Again we look for solutions which go like

$$e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad (388)$$

Substituting this form into Eqs. (385) and (386) we obtain

$$-i\omega N_e m_e \vec{V}_e + N_e e \frac{\vec{V}_e \times \vec{B}_0}{c} = -N_e e \vec{E} \qquad (389)$$

$$-i\omega N_i m_i \vec{V}_i - N_i Z e \frac{\vec{V}_i \times \vec{B}_0}{c} = N_i Z e \vec{E}. \qquad (390)$$

Writing Eqs. (389) and (390) in component form gives

$$w_e = -\frac{ie}{m_e \omega} E_z, \qquad (391)$$

$$w_i = \frac{iZe}{m_i \omega} E_z, \qquad (392)$$

$$-i\omega u_e + \omega_{ce} v_e = -\frac{e E_x}{m_e}, \qquad (393)$$

$$-i\omega v_e - \omega_{ce} u_e = \frac{-e E_y}{m_e}, \qquad (394)$$

$$-i\omega u_i - \omega_{ci} v_i = \frac{Ze E_x}{m_i}, \qquad (395)$$

$$-i\omega v_i + \omega_{ci} u_i = \frac{Ze E_y}{m_i} \qquad (396)$$

where u, v, and w are the x, y, z components of the velocity and

$$\omega_c = |q| \frac{B_0}{mc}. \qquad (397)$$

Solving for u_e, v_e, u_i, and v_i gives

$$u_e = \frac{-e}{m_e} \frac{(i\omega E_x + \omega_{ce} E_y)}{\omega^2 - \omega_{ce}^2}, \qquad (398)$$

$$v_e = \frac{-e}{m_e} \frac{(-\omega_{ce} E_x + i\omega E_y)}{\omega^2 - \omega_{ce}^2}, \qquad (399)$$

$$u_i = \frac{Ze}{m_i} \frac{(i\omega E_x - \omega_{ci} E_y)}{\omega^2 - \omega_{ci}^2}, \qquad (400)$$

and

$$v_i = \frac{Ze}{m_i}\,\frac{(\omega_{ci} E_x + i\omega E_y)}{\omega^2 - \omega_{ci}^2} \tag{401}$$

Fourier-analyzing Eq. (387) and writing it in component form gives

$$\left(k^2 - k_x^2 - \frac{\omega^2}{c^2}\right)E_x - k_x k_y E_y - k_x k_z E_z = \frac{4\pi i\omega}{c^2} j_x, \tag{402}$$

$$-k_y k_x E_x + \left(k^2 - k_y^2 - \frac{\omega^2}{c^2}\right)E_y - k_y k_z E_z = \frac{4\pi i\omega}{c^2} j_y, \tag{403}$$

and

$$-k_z k_x E_x - k_z k_y E_y + \left(k^2 - k_z^2 - \frac{\omega^2}{c^2}\right)E_z = \frac{4\pi i\omega}{c^2} j_z. \tag{404}$$

Summing electron and ion contributions to j, using Eqs. (391), (392), and (398) to (401) for the v's, and inserting these in Eqs. (402) to (404) gives

$$\left[k^2 - k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2\omega^2}{c^2(\omega^2 - \omega_{ce}^2)} + \frac{\omega_{pi}^2\omega^2}{c^2(\omega^2 - \omega_{ci}^2)}\right] E_x$$

$$-\left[k_x k_y + \frac{i\omega\,\omega_{ce}\,\omega_{pe}^2}{c^2(\omega^2 - \omega_{ce}^2)} - \frac{i\omega\,\omega_{ci}\,\omega_{pi}^2}{c^2(\omega^2 - \omega_{ci}^2)}\right] E_y$$

$$-k_x k_z E_z = 0, \tag{405}$$

$$-\left[k_x k_y - \frac{i\omega}{c^2}\frac{\omega_{pe}^2\omega_{ce}}{(\omega^2 - \omega_{ce}^2)} + \frac{i\omega\,\omega_{pi}^2\omega_{ci}}{c^2(\omega^2 - \omega_{ci}^2)}\right] E_x$$

$$+\left[k^2 - k_y^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2\omega^2}{c^2(\omega^2 - \omega_{ce}^2)} + \frac{\omega_{pi}^2\omega^2}{c^2(\omega^2 - \omega_{ci}^2)}\right] E_y$$

$$-k_y k_z E_z = 0, \tag{406}$$

and

$$-k_z k_x E_x - k_z k_y E_y + \left[k^2 - k_z^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2 + \omega_{pi}^2}{c^2}\right] E_z = 0. \tag{407}$$

We can write these three equations in tensor notation as

$$\overline{\overline{K}} \cdot \vec{E} = 0 = \underset{\sim}{K} \cdot \vec{E} = 0 \tag{408}$$

where K is the tensor

$$\begin{pmatrix} \left[k^2 - k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2\omega^2}{c^2(\omega^2-\omega_{ce}^2)} + \frac{\omega_{pi}^2\omega^2}{c^2(\omega^2-\omega_{ci}^2)}\right] & \left[\frac{i\omega\omega_{ci}\omega_{pi}^2}{c^2(\omega^2-\omega_{ci}^2)} - \frac{i\omega\omega_{ce}\omega_{pe}^2}{c^2(\omega^2-\omega_{ce}^2)} - k_x k_y\right] & -k_x k_z \\ \left[-\frac{i\omega\omega_{ci}\omega_{pi}^2}{c^2(\omega^2-\omega_{ci}^2)} + \frac{i\omega\omega_{ce}\omega_{pe}^2}{c^2(\omega^2-\omega_{ce}^2)} - k_x k_y\right] & \left[k^2 - k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2\omega^2}{c^2(\omega^2-\omega_{ce}^2)} + \frac{\omega_{pi}^2\omega^2}{c^2(\omega^2-\omega_{ci}^2)}\right] & -k_y k_z \\ -k_x k_z & -k_z k_y & \left[k^2 - k_z^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2+\omega_{pi}^2}{c^2}\right] \end{pmatrix}$$

and where

$$\omega_{pe}^2 = \frac{4\pi N_e e^2}{m_e} \quad \text{and} \quad \omega_{pi}^2 = \frac{4\pi N_i Z^2 e^2}{m_i}. \tag{409}$$

In order to have a nontrivial solution to Eq. (408) ($\vec{E} \neq 0$) the determinant of K must be zero. Before proceeding we may note that because of cylindrical symmetry about the magnetic field direction or z direction we can choose the xz plane to be the plane determined by $\vec{k}$ and $\vec{B}$. Hence, without loss of generality we may choose $k_y = 0$.

b. Propagation Parallel to B_0

We shall begin our investigation of the roots of

$$|\underset{\sim}{K}| = 0 \tag{410}$$

by looking at waves propagating in the z direction, $k_x = k_y = 0$. Our dispersion relation is then

$$\begin{vmatrix} A & iB & 0 \\ -iB & A & 0 \\ 0 & 0 & C \end{vmatrix} = 0 = (A+B)(A-B)C \qquad (411)$$

where

$$A = k^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2\omega^2}{c^2(\omega^2-\omega_{ce}^2)} + \frac{\omega_{pi}^2\omega^2}{c^2(\omega^2-\omega_{ci}^2)} \qquad (412)$$

$$B = -\frac{\omega}{c^2}\left[\frac{\omega_{ce}\omega_{pe}^2}{(\omega^2-\omega_{ce}^2)} - \frac{\omega_{ci}\omega_{pi}^2}{(\omega^2-\omega_{ci}^2)}\right] \qquad (413)$$

$$C = \left[-\frac{\omega^2}{c^2} + \frac{\omega_{pe}^2+\omega_{pi}^2}{c^2}\right]. \qquad (414)$$

The three solutions of Eq. (411) are obviously

$$A-B=0, \quad A+B=0, \quad C=0. \qquad (415)$$

The root $C = 0$ gives

$$\omega^2 = \omega_{pe}^2 + \omega_{pi}^2 \equiv \omega_p^2 \qquad (416)$$

and when this condition ($C = 0$) is inserted in Eq. (408) we find that

$$E_x = E_y = 0, \; E_z \quad \text{is arbitrary.} \qquad (417)$$

This oscillation has only an E_z associated with it. Since there is no motion across the field lines, the magnetic field does not influence the motion and the oscillation is the same as we obtained in the absence of a magnetic field. The root $A + B = 0$ applied to Eq. (408) gives

$$A E_x - iAE_y = 0, \; E_x = iE_y \text{ and } E_z = 0, \qquad (418)$$

while $A - B = 0$ similarly gives

$$E_x = -iE_y, \; E_z = 0. \qquad (419)$$

Thus there is no E_z and E_x and E_y are 90° out of phase with each other. These are transverse waves which propagate along $\vec{B}$ and are right and left circularly polarized. Handedness is here defined with respect to $\vec{k}$. The positive sign always corresponds to rotation about B_o in the negative θ direction. The plus sign corresponds to a left-hand polarized wave for propagation in the $+z$ direction, $k > 0$, and a right-hand polarized wave for propagation in the $-z$ direction, $k < 0$. The reverse is true for the minus sign.

OK

Let us write out the two roots $A \pm B = 0$.

$$\omega^2 - k^2c^2 = \frac{\omega_{pe}^2\omega^2}{\omega^2-\omega_{ce}^2} + \frac{\omega_{pi}^2\omega^2}{\omega^2-\omega_{pi}^2} \pm \omega\left[\frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2} - \frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2}\right]$$

$$= \frac{\omega_{pe}^2\omega(\omega \mp \omega_{ce})}{\omega^2-\omega_{ce}^2} + \frac{\omega_{pi}^2\omega(\omega \pm \omega_{ci})}{\omega^2-\omega_{ci}^2}$$

$$= \frac{\omega_{pe}^2\omega}{\omega \pm \omega_{ce}} + \frac{\omega_{pi}^2\omega}{\omega \mp \omega_{ci}}. \tag{420}$$

Now note that formal substitution of $-\omega$ for $+\omega$ leaves the left side unchanged and merely interchanges $\pm$ with $\mp$. That is, we may drop out the lower sign and merely look for solutions of Eq. (420) for $+$ and $-\omega$. The solutions for $+\omega$ corresponding to the upper sign are for $A + B = 0$, i.e., $E_x = iE_y$, and for $-\omega$ to $A - B = 0$, i.e., $E_x = -iE_y$.

So Eq. (420) is now simply

$$k^2c^2 - \omega^2 = \frac{\omega_{pi}^2\omega}{\omega_{ci}-\omega} - \frac{\omega_{pe}^2\omega}{\omega_{ce}+\omega}. \tag{421}$$

Let us look at the dispersion relation, Eq. (421), in various limits. First let us assume that ω is much smaller than ω_{pi} and ω_{ci}.

Then we can write Eq. (421) in the form

$$\omega^2 - k^2c^2 = \frac{\omega_{pe}^2\,\omega(\omega-\omega_{ce})}{(\omega^2-\omega_{ce}^2)} + \frac{\omega_{pi}^2\,\omega(\omega+\omega_{ci})}{(\omega^2-\omega_{ci}^2)}. \tag{422}$$

Dropping the ω^2 in the denominator and rewriting

$$k^2c^2 - \omega^2 = \frac{\omega_{pe}^2\omega^2}{\omega_{ce}^2} + \frac{\omega_{pi}^2\omega^2}{\omega_{ci}^2} - \left[\frac{\omega_{pe}^2\omega_{ce}}{\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega_{ci}^2}\right]\omega. \tag{423}$$

The last two terms drop out

$$\omega\left[\frac{\omega_{pi}^2}{\omega_{ci}} - \frac{\omega_{pe}^2}{\omega_{ce}}\right] = \omega\left[\frac{4\pi N_i Z^2 e^2/m_i}{ZeB/m_ic} - \frac{4\pi N_e e^2/m_e}{eB/m_ec}\right] = 0 \tag{424}$$

so we have for Eq. (423)

$$k^2c^2 - \omega^2 = \omega^2\left[\frac{4\pi N_e e^2/m_e}{e^2B^2/m_e^2c^2} + \frac{4\pi N_i Z^2e^2/m_i}{e^2B^2Z^2/m_i^2c^2}\right] \tag{425}$$

or

$$k^2c^2 - \omega^2 = \frac{4\pi\omega^2c^2}{B^2}\left[N_em_e + N_iM_i\right] \tag{426}$$

or

$$k^2 = \frac{\omega^2}{c^2}\left[1 + \frac{4\pi\rho c^2}{B^2}\right]. \tag{427}$$

The quantity in the brackets is the familiar low-frequency dielectric constant we saw back at Eq. (100) and the wave is just the Alfvén wave found earlier (equation (370)) for zero pressure. Since these waves are nondispersive, the group velocity is equal to the phase velocity. Further, both polarizations propagate at the same velocity. This nondispersive property holds only in the limit of $\omega^2 << \omega_{pi}^2$ and $\omega^2 << \omega_{ci}^2$.

Let us return now to Eq. (421). We plot the right and left-hand sides vs ω for a fixed value of k^2. Such a plot is shown in Fig. 47.

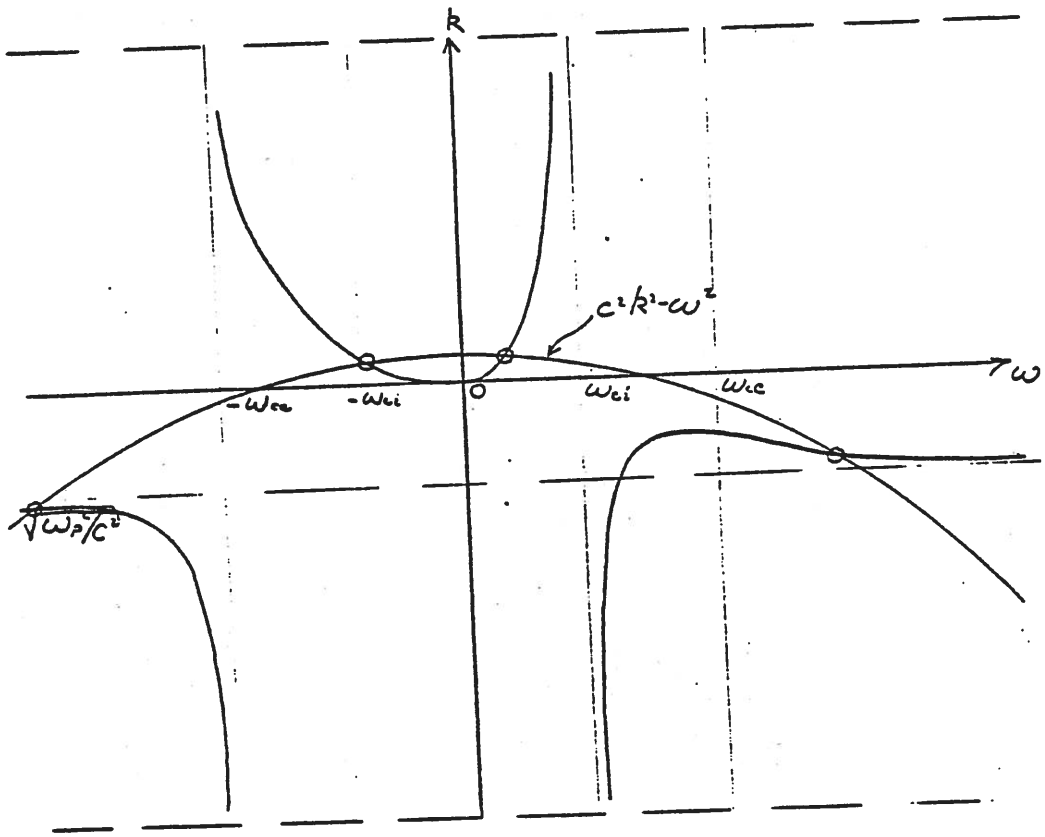


Figure 47

The first thing we note is that Eq. (420) has a singularity at $\omega = \omega_{ci}$ and $\omega = -\omega_{ce}$. Since positive ω corresponds to rotation of $\vec{E}$ about $\vec{B}$ in the negative θ direction, the direction in which ions rotate, it is natural that there should be a singular behavior at $\omega = +\omega_{ci}$. Likewise, since negative ω always corresponds to rotation of $\vec{E}$ about $\vec{B}$ in the positive θ direction, which is the direction in which electrons rotate, it is natural that there should be a singularity at $\omega = -\omega_{ce}$.

Now every place the $c^2k^2 - \omega^2$ curve crosses the curve for the right-hand side of Eq. (420) we obtain a root of the dispersion relation. We note that there are four such crossings for every value of k^2. Thus for a given value of k there are four values of ω which satisfy the dispersion relation. We note that Eq. (420) would give a 4th degree polynomial if it were rationalized, and so this is exactly the number of roots we should find. We may also note that this is the proper number from the point of view of degrees of freedom. The plasma consists of electrons and ions. There is one degree of freedom of the plasma for a given k and polarization per species, which in this case is 2. For each degree of freedom we require two constants to specify its state exactly — a displacement and a velocity. Thus the amplitudes of the four modes found here give four constants which the displacement and velocity of the electrons and ions could be written in terms of.

Now we may look at what happens as we vary k^2 from 0 to ∞. We note that for small k or long wavelength the curve for the right-hand side is parabolic near $\omega = 0$ and that the intersections with $k^2c^2 - \omega^2$ in this region will give $k^2 \alpha \, \omega^2$. These are just the Alfvén modes we already have found. The other two roots for small k^2c^2 are at

$$\omega = \frac{-(\omega_{ce} - \omega_{ci}) \pm \sqrt{(\omega_{ce} + \omega_{ci})^2 + 4(\omega_{pe}^2 + \omega_{pi}^2)}}{2} \tag{428}$$

If the magnetic field is weak so that ω_{ci} and ω_{ce} are negligible, then this gives $\omega = \omega_p$. These are, however, transverse waves at the plasma frequency.

As k^2 is increased now, one of the Alfvén modes goes to a wave whose frequencies are near to the ion cyclotron frequency, while the other goes into a wave whose frequencies are near the electron cyclotron frequency. Near the cyclotron frequencies there are a great many k's or waves for which the frequency of the waves are nearly the same — the appropriate cyclotron frequency. Situations like this, where ω stays finite and k goes to ∞, are called resonances. We see that for these waves the group velocity is very small

$$V_g = \frac{d\omega}{dk} \tag{429}$$

since ω changes only very slightly while k changes by a very large amount. This is not surprising because these waves correspond to the motion of little elements of the electron or ion fluid circulating at their cyclotron frequency. Since these particles are not moving there is no propagation of these waves. What propagation does take place comes about because a gyrating particle induces motion in its neighbors, but this is a relatively weak effect.

Finally we note that as k^2 goes to ∞, ω is large compared with ω_{ce} and ω_{ci}; the other two roots move out to

$$\omega^2 = c^2k^2 + \omega_p^2. \tag{430}$$

This is the same mode we would have found if the magnetic field were not there.

We have just looked at the roots for ω when k^2 is given. It is also of interest to ask, given an ω, what are the roots for k. Again we make a plot, Fig. 48, which is similar to Fig. 47, but in this case we include curves for several values of k^2.

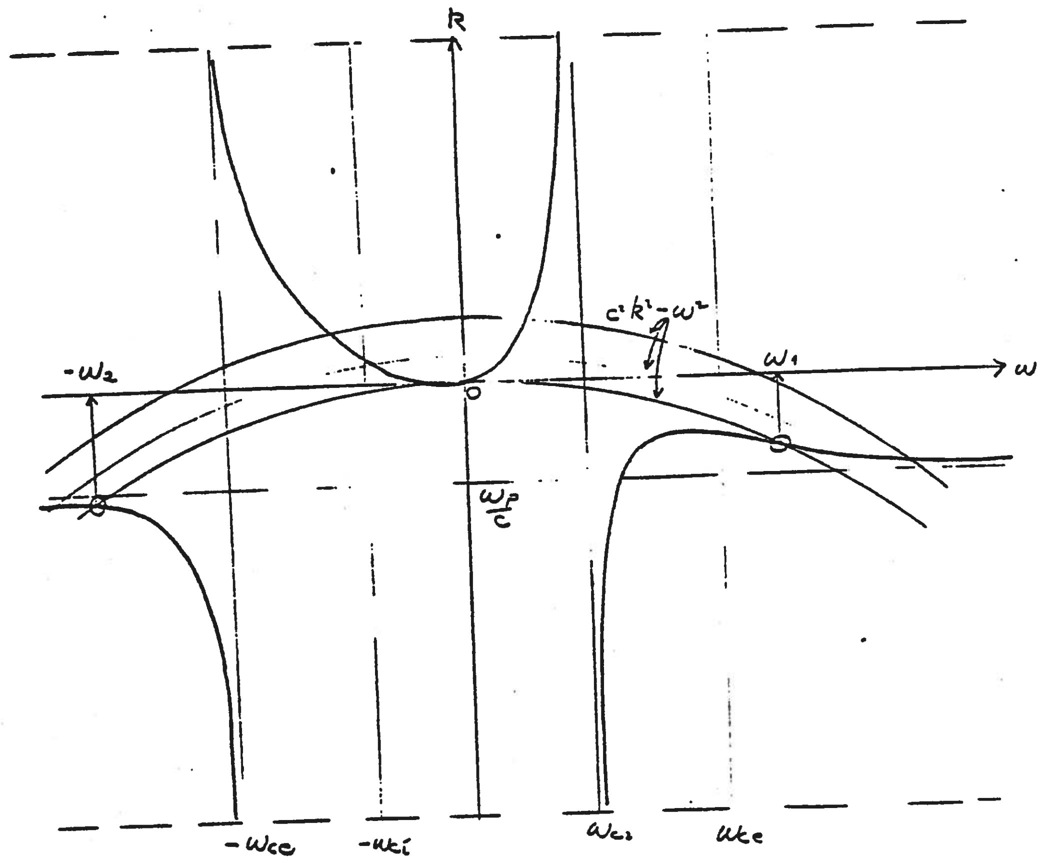


Figure 48

As we change k^2 from plus zero to plus infinity, the point at which the parabola intersects the axis moves from zero to ∞ and all parabolas lie above the one $-\omega^2$. Thus we see that there are no intersections of these curves with the right-hand side for ω between ω_{ci} and ω_1 and $-\omega_{ce}$ and $-\omega_2$. Thus if the frequency lies in either of the regions

$$\omega_{ci} \le \omega \le \omega_1 \tag{431}$$

and

$$-\omega_2 \le \omega \le -\omega_{ce} \tag{432}$$

there is no ~~real~~ positive root for k^2. In fact, we see that we get roots in these regions only if k^2 is negative since these parabolas then lie below $-\omega^2$ and go to $-\infty$ as k^2 goes to $-\infty$. This means that k is purely imaginary. Hence the waves do not propagate for ω lying in either of these regions. Such waves die out in either the plus z or minus z direction, and become exponentially large in the other direction. We can have these waves only if the plasma is bounded in the z direction. Then a wave incident upon the plasma surface, with frequency lying in one of these regions, will give rise to one of these waves which will die out exponentially as one goes further and further into the plasma. Since there is no dissipation here, the wave is reflected at the surface and is simply nonpropagating in the plasma.

c. Propagation Perpendicular to $\vec{B}_o$

We shall now look at propagation perpendicular to $\vec{B}_o$.

We therefore set $k_z = 0$ and $k = k_y$, $k_x = 0$. From Eq. (409) our dispersion relation is now

$$0 = \left\| \begin{matrix} \left[-\frac{\omega^2}{c^2} + \frac{\omega_{pe}^2\omega^2}{c^2(\omega^2-\omega_{ce}^2)} + \frac{\omega_{pi}^2\omega^2}{c^2(\omega^2-\omega_{ci}^2)}\right] & -\frac{i\omega}{c^2}\left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right] & 0 \\ \frac{i\omega}{c^2}\left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right] & \left[k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2\omega^2}{c^2(\omega^2-\omega_{ce}^2)} + \frac{\omega_{pi}^2\omega^2}{c^2(\omega^2-\omega_{ci}^2)}\right] & 0 \\ 0 & 0 & \left[k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2+\omega_{pi}^2}{c^2}\right] \end{matrix} \right\| . \tag{433}$$

Again we see that the determinant splits and we get roots for

$$k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2+\omega_{pi}^2}{c^2} = 0 \tag{434}$$

and for

$$0 = \left[c^2k_x^2 - \omega^2\left[1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right]\right]\left[1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right] + \left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right]^2 . \tag{435}$$

The modes obtained from Eq. (434) have only an E_z. The E_x and E_y components of the electric field are zero, as can be shown by substituting the solution for ω in $\underset{\sim}{K} \cdot \vec{E} = 0$. Since the $\vec{E}$ field is along z and hence the motion is along z, the magnetic field plays no role in this mode. It is the same transverse mode that we obtain for a uniform plasma without a magnetic field. It propagates only if

$$\omega^2 > \omega_{pe}^2 + \omega_{pi}^2 . \tag{436}$$

If Eq. (436) is not satisfied, k_x^2 is negative and the wave does not propagate.

Returning now to Eq. (435), let us first look at the case of small ω. To order ω^2 this equation reduces to

$$c^2 k_x^2 - \omega^2\left[1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} + \frac{\omega_{pi}^2}{\omega_{ci}}\right] = 0 \tag{437}$$

or

$$c^2 k^2 = \omega^2\left[1 + \frac{4\pi\rho c^2}{B^2}\right]. \tag{438}$$

These are transverse hydromagnetic waves propagating perpendicular to the magnetic field. They are what we called magnetoacoustic waves earlier. The dispersion relation agrees with Eq. (108) if we put $P_{\perp o}$ equal to zero and if we neglect 1 compared to $\frac{4\pi\rho c^2}{B^2}$.

To investigate the general case we again resort to a graphical technique. First we solve Eq. (435) for $c^2 k_x^2 - \omega^2$.

$$c^2 k_x^2 - \omega^2 = -\frac{\left(\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right)^2}{\left[1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right]} + \omega^2\left[\frac{\omega_{pe}^2}{\omega_{ce}^2-\omega^2} + \frac{\omega_{pi}^2}{\omega_{ci}^2-\omega^2}\right] \tag{439}$$

Since all terms in ω are quadratic, the plot will be symmetric. A plot of the left- and right-hand sides of Eq. (439) are shown in Fig. 49 (page 179). First we observe that the right-hand side of Eq. (439) has no singularities at $\omega = \pm\omega_{ci}$ or $\omega = \pm\omega_{ce}$. At these values of ω the singularities cancel out. There are, however, singularities where

$$1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2} = 0 \tag{440}$$

or

$$\omega^4 - \omega^2\left[\omega_{ce}^2 + \omega_{ci}^2 + \omega_{pe}^2 + \omega_{pi}^2\right] + \left[\omega_{ce}^2\omega_{ci}^2 + \omega_{pe}^2\omega_{ci}^2 + \omega_{pi}^2\omega_{ce}^2\right] = 0 \tag{441}$$

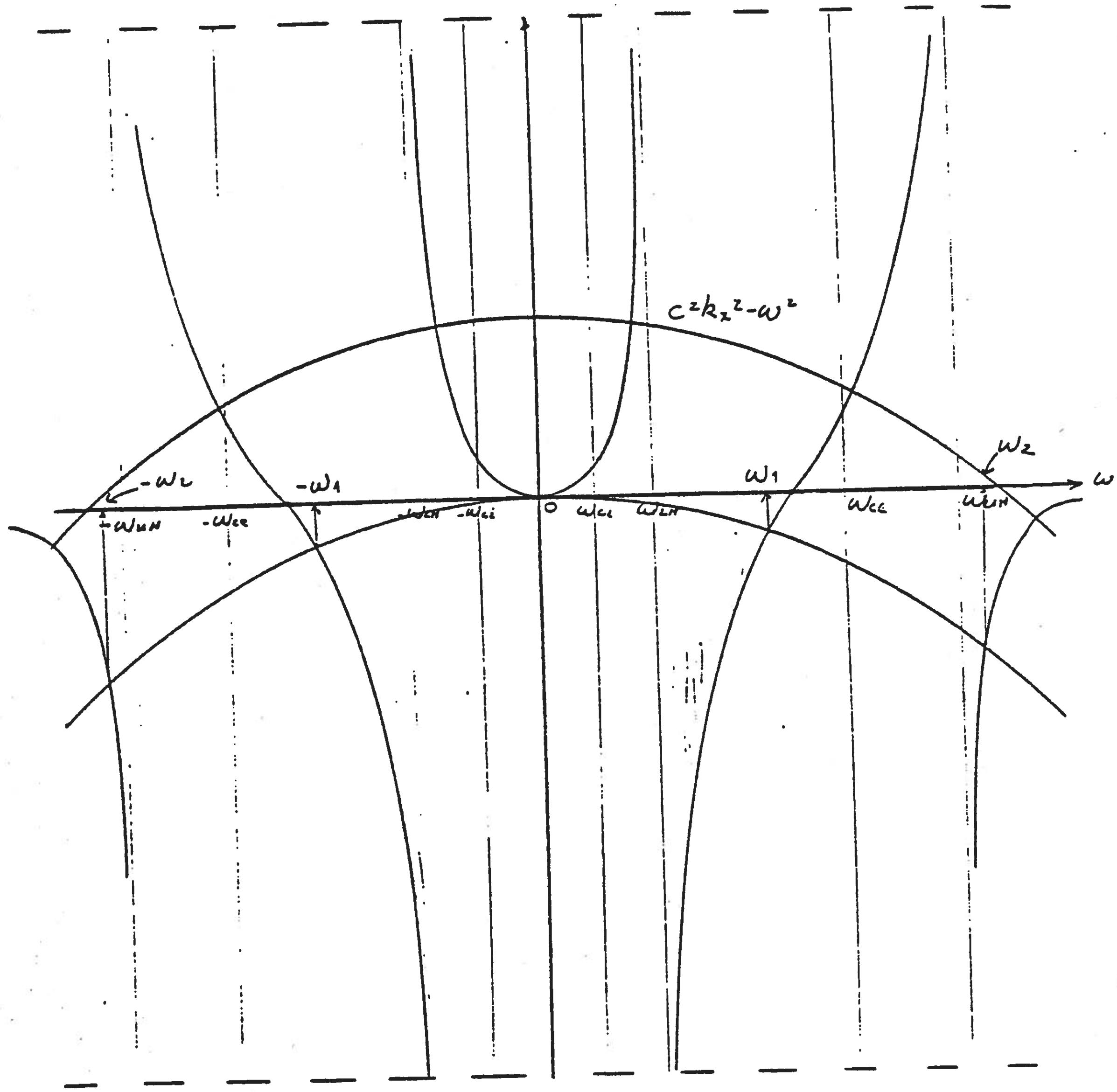


Figure 49

Eq. (441) has, ~~for~~ four roots, one in the vicinity of $\omega^2 = \omega_{ci}^2$ and the other in the vicinity of $\omega^2 = \omega_{ce}^2$. Since $\omega_{ce}^2 >> \omega_{ci}^2$, the lower mode can be found approximately by neglecting ω^2 in $\omega_{pe}^2/\omega^2 - \omega_{ce}^2$. We thus have

$$(\omega^2-\omega_{ci}^2)\left(1+\frac{\omega_{pe}^2}{\omega_{ce}^2}\right) - \omega_{pi}^2 = 0 \qquad (442)$$

or

$$\omega^2 = \frac{\omega_{pi}^2}{1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}} + \omega_{ci}^2 . \qquad (443)$$

For $\omega_{pe} > \omega_{ce}$ this becomes $\omega^2 \sim \omega_{ci}\,\omega_{ce} + \omega_{ci}^2 \sim \omega_{ci}\,\omega_{ce}$.

If we assume that $\omega_{ce}^2 >> \omega_{pi}^2$, then we can approximately find the high-frequency root by neglecting $\omega_{pi}^2/\omega^2 - \omega_{ci}^2$. The upper root is thus given approximately by

$$\omega^2 = \omega_{pe}^2 + \omega_{ce}^2 . \qquad (444)$$

For this upper mode the ions cannot respond and the electrons are primarily responsible for the oscillation. Both the magnetic field and the electric field produced by charge separation contribute to the restoring force. Thus the natural frequency is not the cyclotron frequency or the plasma frequency, but the combination of them given by Eq. (444).

For the lower frequency mode the natural frequency is no longer the ion cyclotron frequency but depends also on the ion plasma frequency and the electron plasma and cyclotron frequencies. The ion plasma frequency enters because of charge separation due to the ion motion. The electrons are prevented from following because of the magnetic field. However, because of the $\dot{\vec{E}}$ drift they execute in the x direction they do tend to move with the ions in this direction and this reduces the charge separation. This may also be thought of as a dielectric effect. At these low frequencies the electrons behave like a dielectric medium just as the total plasma did for frequencies low compared to ω_{ci}. The effective dielectric constant is

$$\epsilon = 1 + \frac{4\pi N_e m_e c^2}{B_0^2}. \tag{445}$$

The two resonant frequencies found above are known as the upper and lower hybrid frequencies.

For ω lying in the regions

$$|\omega_{LH}| < |\omega| < |\omega_1|, \quad |\omega_{UH}| < |\omega| < |\omega_2| \tag{446}$$

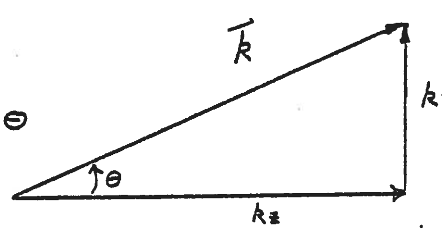
(ω_1 and ω_2 are shown in Fig. 49), there is no positive value of k_x^2 which satisfies the dispersion relation. However, there are negative k^2's which will satisfy it and these represent non-propagating waves.

The modes just found are elliptically polarized in the yx plane. They contain both longitudinal and transverse components of $\vec{E}$.

d. Propagation at an Arbitrary Angle

Here we will use the magnitude of $\vec{k}$ and the angle that $\vec{k}$ makes with the z direction,

$k_x = k \sin\theta$



$k_z = k\cos\theta$.

Figure 50

$$0 = \begin{vmatrix} \left[k^2\cos^2\theta - \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right]\right] & -i\frac{\omega}{c^2}\left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right] & -k^2\sin\theta\times\cos\theta \\ \left[\frac{i\omega}{c^2}\left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right]\right] & \left[k^2 - \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right]\right] & 0 \\ -k^2\sin\theta\cos\theta & 0 & k^2\sin^2\theta - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2+\omega_{pi}^2}{c^2} \end{vmatrix} \tag{447}$$

or

$$\left[k^2\sin^2\theta - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2+\omega_{pi}^2}{c^2}\right]\left\{\left[k^2\cos^2\theta - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right)\right]\times\right.$$

$$\left.\left[k^2 - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right)\right] - \frac{\omega^2}{c^4}\left[\frac{\omega_{pe}^2\omega_{ce}}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2\omega_{ci}}{\omega^2-\omega_{ci}^2}\right]^2\right\}$$

$$- k^4\sin^2\theta\cos^2\theta\left[k^2 - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\right)\right] = 0. \tag{448}$$

First let us look at low-frequency propagation. Our dispersion relation, Eq. (448), reduces to

$$0 = \begin{vmatrix} k^2\cos^2\theta - \frac{\omega^2}{c^2}\left(1 + \frac{4\pi\rho c^2}{B_0^2}\right) & 0 & -k^2\sin\theta\cos\theta \\ 0 & k^2 - \frac{\omega^2}{c^2}\left(1 + \frac{4\pi\rho c^2}{B_0^2}\right) & 0 \\ -k^2\sin\theta\cos\theta & 0 & k^2\sin^2\theta - \frac{\omega^2}{c^2} \; \frac{\omega_{pe}^2+\omega_{pi}^2}{c^2} \end{vmatrix} \tag{449}$$

Again the determinant splits into two factors which we set separately equal to zero.

$$k^2 - \frac{\omega^2}{c^2}\left(1 + \frac{4\pi\rho c^2}{B_0^2}\right) \tag{450}$$

and

$$\left[k^2\cos^2\theta - \frac{\omega^2}{c^2}\left(1 + \frac{4\pi\rho c^2}{B_0^2}\right)\right]\left[k^2\sin^2\theta - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2 + \omega_{pi}^2}{c^2}\right]$$

$$- k^2\sin^2\theta\cos^2\theta. \tag{451}$$

Eq. (450) simply gives a hydromagnetic wave propagating at an arbitrary angle to the magnetic field. It has only an E_y component. Expanding out Eq. (451) gives

$$k^2\left(\cos^2\theta\left[\frac{\omega^2}{c^2} - \frac{\omega_{pe}^2 + \omega_{pi}^2}{c^2}\right] + \sin^2\theta\,\frac{\omega^2}{c^2}\left[1 + \frac{4\pi\rho c^2}{B_0^2}\right]\right)$$

$$- \frac{\omega^2}{c^2}\left[1 + \frac{4\pi\rho c^2}{B_0^2}\right]\left[\frac{\omega^2}{c^2} - \frac{\omega_{pe}^2 + \omega_{pi}^2}{c^2}\right] = 0. \tag{452}$$

$$k^2 = \frac{\dfrac{\omega^2}{V_A^2}\left[\dfrac{\omega^2}{c^2} - \dfrac{\omega_{pe}^2 + \omega_{pi}^2}{c^2}\right]}{\cos^2\theta\left[\dfrac{\omega^2}{c^2} - \dfrac{\omega_{pe}^2 + \omega_{pi}^2}{c^2}\right] + \sin^2\theta\,\dfrac{\omega^2}{V_A^2}} \tag{453}$$

where

$$V_A^2 = \frac{c^2}{1 + \dfrac{4\pi\rho c^2}{B_0^2}}.$$

Since we are looking at small ω,

$$\omega^2 << \omega_{pe}^2 + \omega_{pi}^2 \tag{454}$$

and we can approximate Eq. (453) by

$$k^2 V_A^2 \left[\cos^2\theta - \frac{c^2\omega^2}{V_A^2(\omega_{pe}^2 + \omega_{pi}^2)} \sin^2\theta \right] = \omega^2. \tag{455}$$

You get positive k's only if

$$\cos^2\theta - \frac{c^2\omega^2}{V_A^2(\omega_{pe}^2 + \omega_{pi}^2)} \sin^2\theta \geq 0. \tag{456}$$

When the equality holds in Eq. (456), k goes to infinity and the wavelength goes to zero. For small ω the inequality, Eq. (456), will only be violated for θ near $\pi/2$. The modification of the Alfvén wave is due to the inertial effects of the electrons moving back and forth along the field lines.

B. Thermal Effects and Landau Damping

1. Use of a Pressure - No Magnetic Field

The problem of the effects of the thermal motions of particles on the propagation of waves through a plasma is one of vast scope and we cannot hope to treat it adequately here. However, we can get some idea of how they affect wave propagation by looking at their effects on the simplest case — wave propagation in an infinite homogeneous plasma containing no magnetic field.

Let us first adopt a fluid approach to this problem and use an adiabatic law to relate the pressure to the density,

$$\frac{P}{n^{\gamma}} = \frac{P_0}{n_0^{\gamma}} . \qquad (695)^{*}$$

The linearized equations of motion for the electrons are

$$N_e m_e \frac{\partial \vec{v}_e}{\partial t} = -N_e e \vec{E} - \vec{\nabla} P_e \qquad (696)$$

and

$$\frac{\partial n_e}{\partial t} + N_e \vec{\nabla} \cdot \vec{v}_e = 0 \qquad (697)$$

while those for the ions are

$$N_i m_i \frac{\partial \vec{v}_i}{\partial t} = - N_i Z e \vec{E} - \vec{\nabla} P_i \qquad (698)$$

and

$$\frac{\partial n_i}{\partial t} + N_i \vec{\nabla} \cdot \vec{v}_i = 0. \qquad (699)$$

Using our pressure law we have

$$P = P_0 \left(\frac{n}{n_0}\right)^{\gamma}, \qquad (700)$$

$$\delta P = \frac{\gamma P_0}{n_0} \delta n , \qquad (701)$$

$$\vec{\nabla} p = \frac{\gamma P_0}{n_0} \vec{\nabla} n , \quad \text{or} \qquad (702)$$

---

* Note: Equation numbers (457) through (694) have been omitted.

$$\vec{\nabla} p_e = \frac{\gamma p_{0e}}{N_e} \vec{\nabla} n_e \tag{703}$$

and

$$\vec{\nabla} p_i = \frac{\gamma p_{0i}}{N_i} \vec{\nabla} n_i \tag{704}$$

[Small p is the perturbation in pressure.]

Now from Eqs. (336), (327), and (328) we have

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2 \vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial \vec{J}}{\partial t} = 0 \tag{705}$$

where also

$$\sigma = n_i Ze - n_e e, \tag{706}$$

$$\vec{J} = N_i Ze \vec{v}_i - N_e e \vec{v}_e, \tag{707}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\sigma, \tag{708}$$

and

$$\frac{\partial \sigma}{\partial t} = -\vec{\nabla}\cdot\vec{J}. \tag{709}$$

First let us look at transverse waves. Since $\vec{J}$ is perpendicular to $\vec{k}$, we assume that $\vec{\nabla}\cdot\vec{E} = 0$. By taking the divergence of Eq. (696) we obtain

$$N_e m_e \frac{\partial}{\partial t} \vec{\nabla}\cdot\vec{v}_e = -\nabla^2 p_e. \tag{710}$$

Similiarly from (698)

$$N_i m_i \frac{\partial}{\partial t} \vec{\nabla}\cdot\vec{v}_i = -\nabla^2 p_i. \tag{711}$$

Now if p_e and p_i are zero, then the time derivatives of $\vec{\nabla}\cdot\vec{v}_e$ and $\vec{\nabla}\cdot\vec{v}_i$ are zero and hence by the continuity equations for n_e, n_i, and σ, the second derivatives of these quantities are zero. Hence if they and their first derivatives are zero, they will remain zero for a short interval of time. But, if these are zero the second time derivatives of P_e, P_i and σ are also zero, and hence they do not develop in time. Thus we get a consistent solution by setting

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{712}$$

$$n_e = n_i = 0, \tag{713}$$

and

$$p_e = p_i = 0. \tag{714}$$

Thus the pressure plays no role in the propagation of these modes and we get the same solution that we obtained earlier in Eq. (344).

2. Phase Mixing and Landau Damping

a. A Model For Landau Damping

We just investigated the effects of the thermal motions of the particles of a plasma by using a pressure. If collisions are very frequent then this approach is justified. However, in a hot plasma collisional effects are often weak, particularly when dealing with the propagation of high-frequency waves. To find the thermal effects here we must return to the Boltzmann equation with the collision term set equal to zero. However, we must keep the self-consistent fields. Before doing this we shall investigate a simplified model which, however, makes much of the physics of what we shall find clear.

Our model is the following. We consider a number of streams of electrons flowing through each other and through a fixed neutralizing background. We take the streams to be infinite in extent and we take them all to be flowing in the x direction. Each stream has a different x velocity and in the undisturbed state all the electrons within a stream move with a uniform velocity (no random motion within a stream). We assume that the only force acting on the electrons is that due to the self-consistent electric field. We look for longitudinal waves propagating in the x direction (E parallel to x). The linearized equations of motion for the σ th beam are

$$\frac{\partial v_\sigma}{\partial t} + V_\sigma \frac{\partial v_\sigma}{\partial x} = - \frac{eE}{m_\sigma}. \tag{715}$$

$$\frac{\partial n_\sigma}{\partial t} + V_\sigma \frac{\partial n_\sigma}{\partial t} + N_0 \frac{\partial v_\sigma}{\partial x} = 0 \tag{716}$$

Here v_σ and n_σ are the perturbations in the velocity and number density of the particles in the σ th beam, while V_σ and N_σ are the corresponding unperturbed quantities. The electric field $\vec{E}$ is determined from Poisson's equation,

$$\frac{\partial E}{\partial x} = - 4\pi e \sum_\sigma n_\sigma. \tag{717}$$

We now look for wave solutions where all quantitied vary like

$$e^{i(\omega t - kx)}. \tag{718}$$

Substituting into Eqs. (715) - (717) gives

$$i(\omega - kV_\sigma)v_\sigma = -\frac{eE}{m_\sigma} \tag{719}$$

$$i(\omega - kV_\sigma)n_\sigma + ikN_\sigma v_\sigma = 0, \tag{720}$$

and

$$ikE = -4\pi e \sum_\sigma n_\sigma. \tag{721}$$

Solving Eq. (719) for v_σ in terms of E gives

$$v_\sigma = \frac{ieE}{m_\sigma(\omega - kV_\sigma)}. \tag{722}$$

Substituting Eq. (722) into Eq. (720) and solving for n_σ gives

$$n_\sigma = \frac{-ikeEN_\sigma}{m_\sigma(\omega - kV_\sigma)^2} \tag{723}$$

Finally, substituting Eq. (723) into Eq. (721) gives

$$E = \frac{4\pi e^2}{m} E \sum_\sigma \frac{N_\sigma}{(\omega - kV_\sigma)^2} \tag{724}$$

or

$$1 = \frac{4\pi e^2}{m} \sum_\sigma \frac{N_\sigma}{(\omega - kV_\sigma)^2}. \tag{725}$$

This is the dispersion relation which ω and k must satisfy. If we plot the left and right-hand sides of Eq. (725) against ω for fixed k, we get a diagram like that shown in Fig. 51.

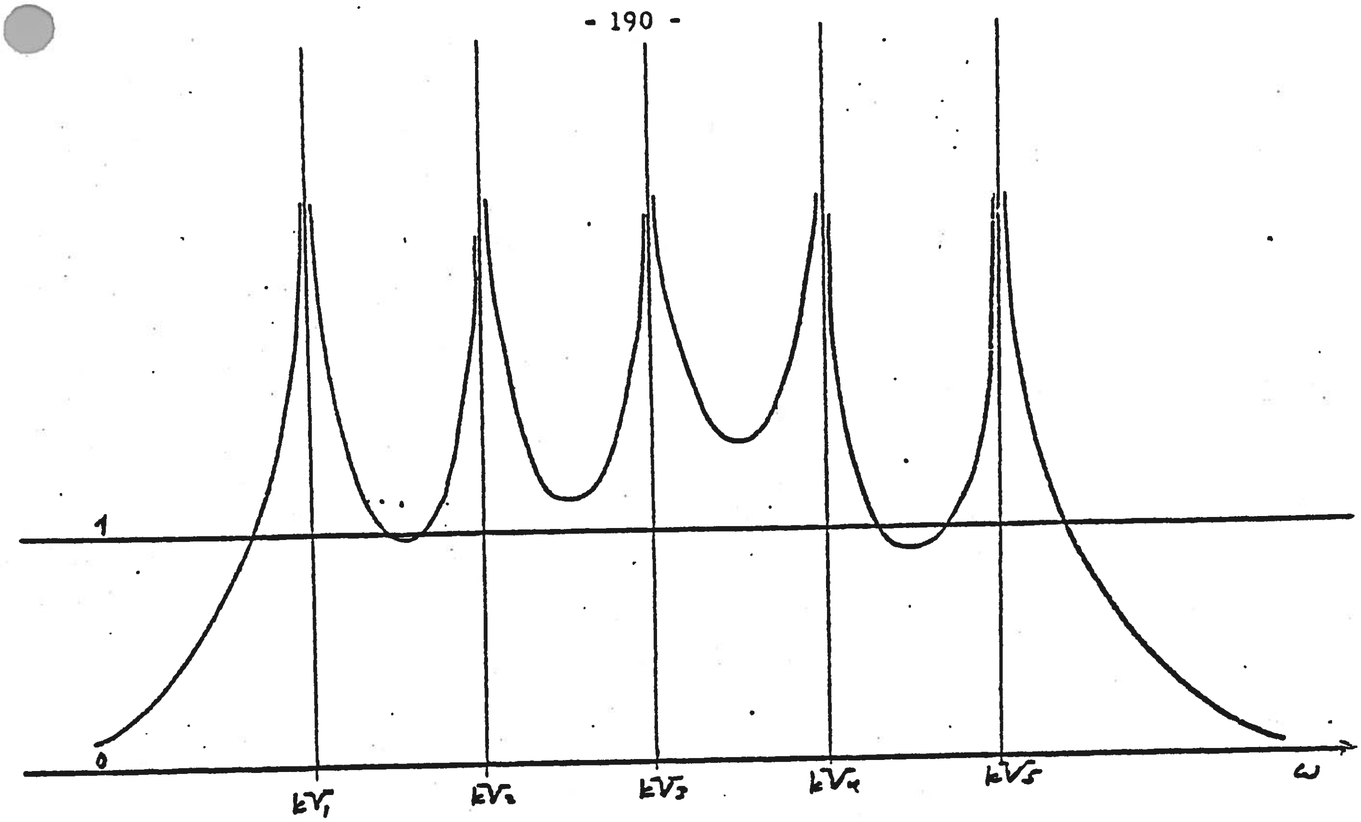


Figure 51

The sum on the right-hand side becomes infinite every time ω takes on the value of one of the kV_σ. Each of the points at which the curve of the right-hand side of Eq. (725) crosses 1 is a real root of Eq. (725). These crossings give all the real roots of Eq. (725). However, in general there are also complex roots or ω's which solve Eq. (725). There are, in fact, twice as many roots to Eq. (725) as there are streams. This is most quickly seen by writing Eq. (725) in polynomial form. One gets a polynomial of degree 2 M (M = number of beams).

At this point the amplitude of the modes is arbitrary. Since the equations are linear, any multiple of a solution is also a solution. We shall specify the solution we are considering by setting

$$E = \frac{4\pi e i}{k}. \tag{726}$$

Then any solution can be obtained by multiplying by an arbitrary constant.

Each of the ω's which satisfies the dispersion relation gives a possible mode of oscillation of the system for the given k. The system has one longitudinal degree of freedom per beam for fixed k. It takes two constants to specify the state of a beam for fixed k, the amplitudes of n_k and v_k. Thus to specify the state of M beams requires 2M constants. The 2M amplitudes of the normal modes we have found supply just this number of constants, so we expect this to be a complete set of normal modes.

The modes we have just found satisfy an orthogonality relation. Since orthogonality between the various k's is standard, we shall concern ourselves only with the orthogonality between modes with various ω's for fixed k. Let ω and ω' be two solutions of the dispersion relation, Eq. (725), for fixed k. Let $n_\sigma, n_\sigma', \nu_\sigma,$ and ν_σ' be the corresponding values of the n_σ's and ν_σ's. Now return to Eqs. (719) - (721) and eliminate ν_σ from Eq. (719) by using Eq. (720) and E from Eq. (719) by using Eq. (721).

Thus we obtain

$$(\omega - kV_\sigma)^2 n_\sigma = \frac{4\pi e^2}{m} N_\sigma \sum_\mu n_\mu . \tag{727}$$

Now multiply Eq. (727) by the normalized perturbed number density

$$n_\sigma'/N_\sigma \tag{728}$$

and Eq. (727) with prime quantities by

$$n_\sigma/N_\sigma . \tag{729}$$

Subtracting the second of these results from the first gives

$$\left[(\omega - kV_\sigma)^2 - (\omega' - kV_\sigma)^2\right] \frac{n_\sigma n_\sigma'}{N_\sigma} = \frac{4\pi e^2}{m} \sum_\mu (n_\sigma' n_\mu - n_\mu' n_\sigma) . \tag{730}$$

If we now sum this over σ the right-hand side gives zero, as can be seen by interchanging the dummy variables σ and μ. We thus obtain

$$0 = (\omega - \omega') \sum_\sigma \left[\omega + \omega' - 2kV_\sigma\right] \frac{n_\sigma n_\sigma'}{N_\sigma} . \tag{731}$$

If $\omega \neq \omega'$, then

$$\sum_\sigma \left[\omega + \omega' - 2kV_\sigma\right] \frac{n_\sigma n_\sigma'}{N_\sigma} = 0 \tag{732}$$

while if $\omega = \omega'$ then Eq. (732) need not be true and we may set

$$\sum_\sigma (\omega - kV_\sigma) \frac{n_\sigma^2}{N_\sigma} = H(\omega, k), \tag{733}$$

or from Eqs. (726) and (723) we have

$$n_\sigma(\omega, k) = \frac{4\pi e^2}{m} \frac{N_\sigma}{(\omega - kV_\sigma)^2} . \tag{734}$$

Substituting this in Eq. (733) gives

$$H(\omega, k) = \left[\frac{4\pi e^2}{m}\right]^2 \sum_\sigma \frac{N_\sigma}{(\omega - kV_\sigma)^3}. \tag{735}$$

We shall also find the expression for $v_\sigma(\omega, k)$ useful. This is obtained by substituting Eq. (734) into Eq. (720)

$$v_\sigma(\omega, k) = \frac{4\pi e^2}{mk} \frac{1}{\omega - kV_\sigma}. \tag{736}$$

We may use Eqs. (732), (733), (735), and (736) to solve the general initial value problem. Again, since the Fourier analysis in k is straightforward, we restrict ourselves to a single k. Let the amplitudes of the k^{th} Fourier components of n_σ and v_σ at t = 0 be $n_\sigma(k)$ and $v_\sigma(k)$. The n_σ's and v_σ's may be expanded in terms of the normal modes so that we may write

$$n_\sigma(k, t) = \sum_\omega C(\omega, k)\, n_\sigma(\omega, k)\, e^{i\omega t}, \tag{737}$$

$$v_\sigma(k, t) = \sum_\omega C(\omega, k)\, v_\sigma(\omega, k)\, e^{i\omega t}, \tag{738}$$

and

$$E(k, t) = \sum_\omega \frac{i4\pi e}{k} C(\omega, k)\, e^{i\omega t}, \tag{739}$$

or using Eqs. (734) and (736), Eqs. (737) and (738) become

$$n_\sigma(k, x, t) = \frac{4\pi e^2}{m} \sum_\omega \frac{C(\omega, k)}{(\omega - kV_\sigma)^2} N_\sigma\, e^{i(\omega t - kx)} \tag{740}$$

$$v_\sigma(k, x, t) = \frac{4\pi e^2}{mk} \sum_\omega \frac{C(\omega, k)}{(\omega - kV_\sigma)^2}\, e^{i(\omega t - kx)}. \tag{741}$$

Here the sum is over all ω's which satisfy the dispersion relation, Eq. (725), for the given k. For time t = 0, Eqs. (740) and (741) become

$$n_\sigma(k) = \frac{4\pi e^2}{m} \sum_\omega \frac{C(\omega,k)\, N_\sigma}{(\omega - kV_\sigma)^2}, \tag{742}$$

$$v_\sigma(k) = \frac{4\pi e^2}{mk} \sum_\omega \frac{C(\omega,k)}{\omega - kV_\sigma}. \tag{743}$$

If Eq. (742) is multiplied by $(\omega' - kV_\sigma)\dfrac{n_\sigma(\omega',k)}{N_\sigma}$ or

$$\frac{4\pi e^2}{m} \frac{(\omega' - kV_\sigma)}{(\omega' - kV_\sigma)^2} = (\omega' - kV_\sigma)\frac{n_\sigma(\omega',k)}{N_\sigma} \tag{744}$$

and if Eq. (743) is multiplied by $k\, n_\sigma(\omega',k)$ or

$$\frac{4\pi e^2}{m} \frac{k N_\sigma}{(\omega' - kV_\sigma)^2} = k\, n_\sigma(\omega',k). \tag{745}$$

and if the two expressions are added and summed over σ, we obtain

$$\frac{4\pi e^2}{m} \sum_\sigma \left\{ \frac{N_\sigma(k)}{(\omega' - kV_\sigma)} + \frac{k V_\sigma(k)}{(\omega' - kV_\sigma)^2} \right\}$$

$$= \left(\frac{4\pi e^2}{m}\right)^2 \sum_\sigma \sum_\omega \frac{C(\omega,k)\,(\omega + \omega' - 2kV_\sigma)\, N_\sigma(\omega,k)}{(\omega - kV_\sigma)^2\,(\omega' - kV_\sigma)^2}$$

$$= 2H(\omega',k)\, C(\omega',k) \tag{746}$$

or

$$C(\omega',k) = \frac{\dfrac{4\pi e^2}{m} \displaystyle\sum_\sigma \left\{ \frac{n_\sigma(k)}{\omega' - kV_\sigma} + \frac{k V_\sigma(k)}{(\omega' - kV_\sigma)^2} \right\}}{2H(\omega',k)} \tag{747}$$

Thus we have found the C's in terms of the initial state of the beams.

Phase Mixing

On the basis of what we have just done we may form the following physical picture of how an initial disturbance will develop in time. In general an initial perturbation will contain all possible modes. The amplitude of each mode will depend on the details of the initial perturbation. These modes will start out more or less in phase, so as to add up to the initial disturbance. However, all the modes have different frequencies, so that as time goes on they will get out of phase with each other. Thus as time goes on they will no longer add coherently, and so all coherent effects will die out. The electric field is the sum total of the electric fields due to all the waves, and hence is a coherent effect. Thus as time goes on it will die out or appear damped. This effect is called phase mixing and the damping is often called Landau damping. If there are instabilities with appreciable growth rates, then we must qualify the above statement because the growing $\overset{\circ}{E}$ field due to the unstable modes will ultimately dominate the picture.

These arguments can be developed quantitatively by treating the initial value problem in the limit of a great many beams (strictly speaking, an infinite number of beams which approximates a continuous distribution). Rather than carry out this rather tedious limiting process we will return to the beginning and proceed in the more conventional way by using Laplace transform methods on the Vlasov equation. The conventional method has the advantage that the mathematics is more mechanical, while the above gives us a better insight into the physics of what is going on.

b. Conventional Treatment of Landau Damping

We now take up the more conventional treatment of Landau damping. We shall still restrict ourselves to an infinite homogeneous plasma with a fixed uniform background of ions. We shall look at longitudinal waves propagating in the x direction. We shall assume that magnetic effects are negligible and that the electric field can be determined from Poisson's equation. Since the motions perpendicular to the x direction play no role in the oscillations, we may neglect them and reduce the problem to a one-dimensional one. Our basic equations are the linearized collisionless Boltzmann equation for the electron and Poisson's equation.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f_0}{\partial v} = 0, \tag{748}$$

$$\frac{\partial E}{\partial x} = -4\pi e \int f\, dv. \tag{749}$$

Again we Fourier-analyze in x space, and Eqs. (748) and (749) become

$$\frac{\partial f_k}{\partial t} + ikvf_k - \frac{eE_k}{m}\frac{\partial f_0}{\partial v} = 0, \tag{750}$$

$$ikE_k = -4\pi e\int f_k\,dv. \tag{751}$$

We shall drop the subscript k from now on and understand that we are talking about a single k component.

We now Laplace transform these equations. The Laplace transform of f(t) is defined by

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt. \tag{752}$$

$$s = \sigma + iw \qquad \sigma \geq 0 \tag{753}$$

Re σ must be sufficiently large so that Eq. (752) converges. The function f(t) is given in terms of its Laplace transform by

$$f(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\,e^{st}\,ds \tag{754}$$

The Laplace transform of $\partial f(t)/\partial t$ is given by

$$\begin{aligned} \mathcal{L}[f'(t)] &= \int_0^\infty f'(t)\,e^{-st}\,dt \\ &= f(t)\,e^{-st}\Big|_0^\infty + \int_0^\infty sf(t)\,e^{-st}\,dt \\ &= -f(0) + s\mathcal{L}[f(t)]. \end{aligned} \tag{755}$$

Laplace transforming Eq. (750) and Eq. (751) gives

$$(s+ikv)\,F(s,v) - \frac{eE(s)}{m}\frac{\partial f_0}{\partial v} = f(0,v), \tag{756}$$

$$ikE(s) = -4\pi e\int_{-\infty}^{\infty} F(s,v)\,dv, \tag{757}$$

Solving Eq. (756) for F(s,v) gives

$$F(s,v) = \frac{f(0,v)}{(s+ikv)} + \frac{eE(s)}{m}\,\frac{\partial f_0(v)/\partial v}{s+ikv}. \tag{758}$$

Substituting into Eq. (757) gives

$$E(s)\left[1+\frac{4\pi e^2}{imk}\int_{-\infty}^{\infty}\frac{(\partial f_0(v)/\partial v)\,dv}{s+ikv}\right] = -\frac{4\pi e}{ik}\int_{-\infty}^{\infty}\frac{f(0,v)\,dv}{s+ikv} \tag{759}$$

or

$$E(s) = \frac{\dfrac{i4\pi e}{k}\displaystyle\int_{-\infty}^{\infty}\frac{f(0,v)\,dv}{s+ikv}}{\left[1-\dfrac{i4\pi e^2}{mk}\displaystyle\int_{-\infty}^{\infty}\frac{(\partial f_0/\partial v)\,dv}{s+ikv}\right]}. \tag{760}$$

The perturbed number density will also be a quantity of interest. This is given by

$$n(s) = -\frac{ikE(s)}{4\pi e} = \frac{\displaystyle\int_{-\infty}^{\infty}\frac{f(0,v)\,dv}{s+ikv}}{\left[1-\dfrac{i4\pi e^2}{mk}\displaystyle\int_{-\infty}^{\infty}\frac{(\partial f_0/\partial v)\,dv}{s+ikv}\right]}. \tag{761}$$

To find E(t) ~~of~~ or n(t) we must invert these Laplace transforms. We have

$$E(t) = \frac{2e}{k}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{-\infty}^{\infty}\frac{f(0,v)\,dv\,e^{st}\,ds}{(s+ikv)\left[1-\dfrac{4\pi e^2 i}{mk}\displaystyle\int_{-\infty}^{\infty}\frac{(\partial f_0/\partial v')\,dv'}{s+ikv'}\right]}. \tag{762}$$

If we assume that we may invert the order of the v and s integration, this expression becomes

$$E(t) = \frac{2e}{k}\int_{-\infty}^{\infty} dv\, f(0,v) \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{st}\, ds}{(s+ikv)\left\{1 - \frac{4\pi e^2 L}{mK}\int_{-\infty}^{\infty}\frac{(\partial f_0/\partial v')\,dv'}{s+ikv'}\right\}}. \tag{763}$$

Now the contour of integration for s is that shown in Fig. 52.

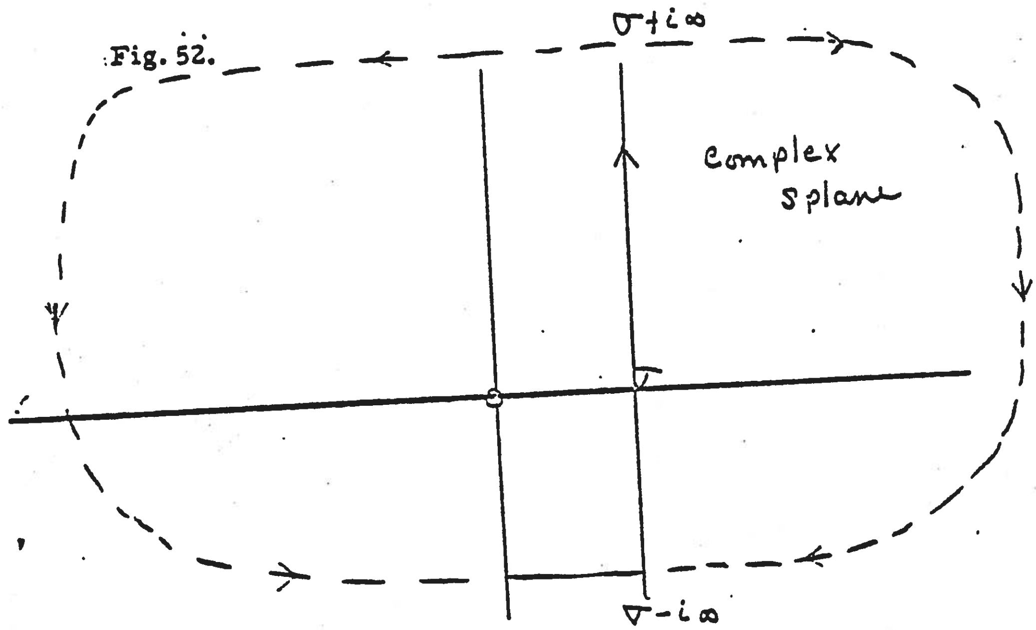


Figure 52

For t less than zero, we may close the contour by the large semicircle to the right. The integral along the semicircle gives nothing because e^{st} vanishes along it. We thus get $2\pi i$ times the sum of the residues of the poles of the integrand inside this contour. However, s + ikv has no zeros inside this contour, and

$$\left\{ 1 - \frac{4\pi e^2 i}{mk} \int_{-\infty}^{\infty} \frac{(\partial f_0/\partial v')\, dv'}{s + ikv'} \right\} \tag{764}$$

also must have no zeros in this region for the following reason. If it vanished for a value of s in this region, say for

$$S = \sigma_0 + i\omega_0$$

then there would be a motion for the plasma for which the E field had the time development

$$E(t) = E\, e^{(\sigma_0 + i\omega_0)t}. \tag{765}$$

But if σ_0 is greater than σ, then the Laplace transform of E, obtained from Eq. (752), would not exist. Hence σ must be larger than $\mathrm{Re}\, S$ for any root of Eq. (764).

Now for $t > 0$ we should like to close the contour on the large semicircle to the left, for e^{st} vanishes on this semicircle. For

$$\frac{1}{s + ikv} \tag{766}$$

this is all right. However, for

$$D(s,k) = \left\{ 1 - \frac{4\pi e^2 i}{mk} \int_{-\infty}^{\infty} \frac{(\partial f_0/\partial v')\, dv'}{s + ikv'} \right\}. \tag{767}$$

This is not all right, because we have defined Eq. (767) only for Re S > maximum value of Re S for S, a root of Eq. (767).

To close the contour as desired we must define this function for all values of s (in particular, $Re\, S<0$) and in such a way that $D(s,k)$ is analytic if we are to apply the residue theorem. Thus we must analytically continue $D(s,k)$ into half plane $Re\, S<0$. Now for $Re\, S$ greater than zero, $D(s,k)$ is perfectly analytic and no problem arises (the integral is a sum of analytic function). However, when S takes on a real value the integrand becomes singular and we must define how the integrand is to be taken. If $D(s,k)$ is to be analytic, then $D(s,k)$ must be continuous. This will be true provided the poles of the integrand always stay on the same side of the contour of integration for v' . If they were to jump across the contour, then the value of the integral would jump by $2\pi i$ times the residue at the pole. Thus as S approaches the real axis we must distort the v' contour to go under $v' = i S/k$ (taking k positive for convenience).

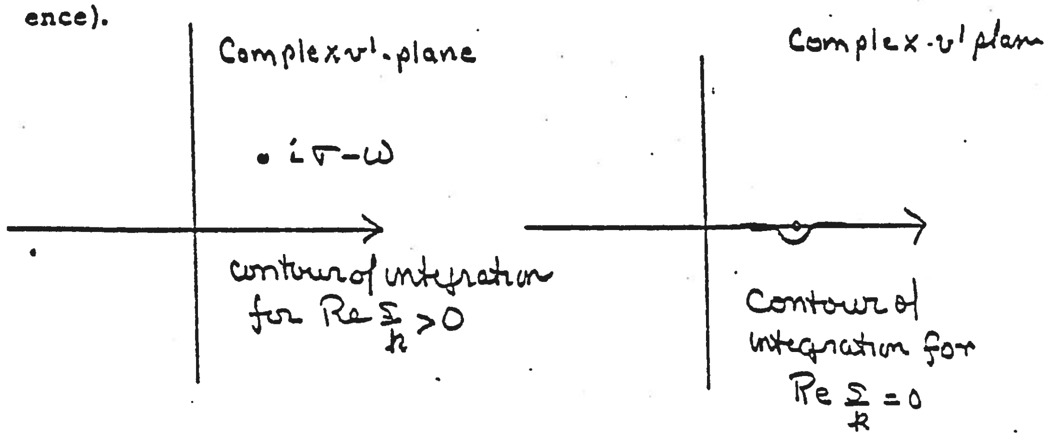


Figure 53

Now as we let $Re\, s < 0$ we must always loop the v' contour under is/k to maintain the continuity of $D(s,k)$.

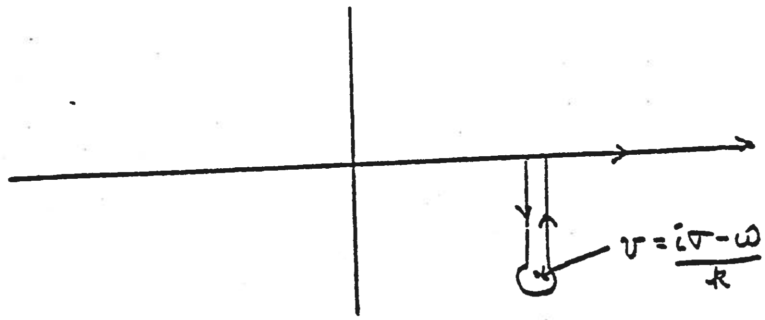


Figure 54

Thus for $Re\, s < 0$, $D(s,k)$ is defined by

$$D(s,k) = 1 - \frac{4\pi e^2 i}{mk} \int_{-\infty}^{\infty} \frac{(\partial f_0/\partial v')\, dv'}{s + ikv'} + \frac{8\pi^2 e^2 i}{mk^2} f_0'(is/k)$$

Along real axis; $\qquad Re\, s < 0. \qquad (768)$

With this definition we may close the s contour on the large semicircle with $Re\, s < 0$ and Eq. (763) becomes

$$E_k(t) = \frac{4\pi i e}{k} \int_{-\infty}^{\infty} \frac{f_k(0,v)\, dv\, e^{-ikvt}}{D(-ikv,k)}$$

$$+ \frac{4\pi i e}{k} \sum_l e^{s_l t} \, \mathrm{Res} \frac{1}{D(s_l,k)} \int_{-\infty}^{\infty} \frac{f_k(0,v)\, dv}{s_l + ikv} \qquad (769)$$

where the fact that E and f refer to a single Fourier component has been restored explicitly, where the s_l are the zeros of $D(s,k)$ and $\mathrm{Res}\, \frac{1}{D(s_l,k)}$ is the residue of D^{-1} at these roots.

We may obtain n(t) from E(t) by multiplying by $-ik/4\pi e$. We thus have for n(t)

$$n_k(t) = \int_{-\infty}^{\infty} \frac{f_k(0,v)\,dv\,e^{-ikvt}}{D(-ikv, v)}$$

$$+ \sum_{\ell} \int_{-\infty}^{\infty} \frac{f_k(0,v)\,dv\,e^{is_\ell t}}{s_\ell + ikv} \operatorname{Res} \frac{1}{D(s_\ell, k)}. \tag{770}$$

Let us first look at Eq. (770) in the limit of e going to zero. In this limit D goes to one by Eq. (767), hence D has no zeros and the sum in Eq. (770) goes out. Eq. (770) then becomes

$$n_k(t) = \int_{-\infty}^{\infty} f_k(0,v)\,e^{-ikvt}\,dv. \tag{771}$$

This is exactly the formula we would get for a gas of noninteracting particles. To see this more clearly we include the x dependence in Eq. (771).

$$n_k(x,t) = \int_{-\infty}^{\infty} f_k(0,v)\,e^{ik(x-vt)}\,dv. \tag{772}$$

This equation says that a disturbance in the particles traveling with velocity v, one which is at x_0 at t = 0 will be found at position x such that

$$x - vt = x_0 \tag{773}$$

or

$$x = x_0 + vt.$$

Thus it is simply carried along by the particles. We see from this that the term

$$\frac{4\pi i e}{k} \int_{-\infty}^{\infty} \frac{f_k(o,v)\, e^{-ikvt}\, dv}{D(-ikv, v)} \tag{774}$$

appearing in Eq. (769) for E gives the E field produced by the free streaming of the particles. The appearance of $D(-ikv, v)$ in the denominator gives the modification of the individual particle field due to all the other particles or plasma. D is the dielectric constant for the plasma. The sum appearing in Eq. (769) gives the electric field due to collective motions of the particles or to plasma oscillation. The time dependence of these modes is

$$e^{i\Omega t}$$

and thus is determined by the zeros of the dielectric constant D. The integral over $f_k(o,v)$ simply determines the amplitude of each mode. Thus the important thing here is the zero of D which we shall now investigate.

We have for D

$$D(s,k) = \left\{ 1 - \frac{4\pi e^2 i}{mk} \int_{-\infty}^{\infty} \frac{(\partial f_0/\partial v)\, dv}{s + ikv} \right\} \tag{775}$$

(below all zeros of $s+ikv$)

Let us restrict ourselves to a Maxwellian distribution

$$f_0 = \frac{n_0}{\sqrt{2\pi}} \frac{e^{-\frac{v^2}{2v_0^2}}}{v_0} \tag{776}$$

We shall also write $s = i\omega + \nu$

Then we have for D

$$D = 1 - \frac{\omega_p^2}{k} \int_{-\infty}^{\infty} \frac{-v\, e^{-\frac{v^2}{2U_0^2}}\, dv}{\sqrt{2\pi}\, U_0^3\, (\omega - i\sigma + kv)} \tag{777}$$

(below all zeros of $\omega - i\sigma + kv$).

Let us first look for zeros of D for k small. We expect the oscillations of the plasma to be near the plasma frequency so we will assume that ω remains finite as k goes to zero. Then ω/k will be much larger than the thermal velocity U_0 as k goes to zero.

$$D = 1 + \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{v\, e^{-\frac{v^2}{2U_0^2}}\, dv}{\sqrt{2\pi}\, U_0^3 \left(v + \frac{\omega}{k} - \frac{i\sigma}{k}\right)} \tag{778}$$

(below all zeros of $v + \frac{\omega}{k} - i\frac{\sigma}{k}$).

Now since ω/k is large we may expand the denominator to obtain (writing $-iS = \omega - i\sigma$)

$$D \simeq 1 + \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{v\, e^{-v^2/2U_0^2}\, dv}{\sqrt{2\pi}\, U_0^3} \frac{1}{\left(\frac{-iS}{k}\right)} \left(1 + \frac{kv}{iS} - \frac{k^2v^2}{S^2} + \frac{k^3v^3}{iS^3} \ldots\right) \tag{779}$$

or

$$D \simeq 1 + \frac{\omega_p^2}{k^2 U_0^2} \left[\frac{k^2 U_0^2}{S^2} + \frac{3k^4 U_0^4}{S^5} \ldots\right]. \tag{780}$$

For small k the solution to this equation is roughly

$$S = i\omega_p \tag{781}$$

or the oscillations are approximately at the plasma frequency. For k small but not quite negligible, we can substitute the solution, Eq. (781), into the S^4 term of Eq. (780) and find a corrected value of

$$S^2 = -(\omega_p^2 + 3k^2v_0^2) \tag{782}$$

or

$$\omega^2 = \omega_p^2 + 3k^2v_0^2. \tag{783}$$

This is the dispersion relation first obtained by Bohm and Gross and is sometimes called the Bohm and Gross dispersion relation.

The roots we have found for ω are real. However, if ω is real then the integrand of Eq. (778) has a singularity on the real axis and we have not treated this properly. The approximation we make in expanding the denominator of Eq. (778) breaks down when v becomes comparable to ω/k. However, the integrand is very small for such values of v, because of the $e^{-v^2/2v_0^2}$ dependence. The only place where we make a serious error is right near the singularity. We may add the contribution from this region to D to obtain a corrected dispersion relation. We shall assume that σ is very small. Then we may ignore the variation of $v e^{-v^2/2v_0^2}$ over the region where the singularity makes a contribution.

$$\frac{\omega}{k} - |\sigma| < v < \frac{\omega}{k} + |\sigma|. \tag{784}$$

Now we may evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dv}{v + \frac{\omega}{k} - i\frac{\nabla}{k}} \qquad (785)$$

and we find it is equal to πi for $\frac{\nabla}{k} > 0$ and $-\pi i$ for $\nabla/k < 0$. However, if $\nabla/k < 0$ we must loop the contour under the pole, as already discussed, so that for $\nabla/k < 0$ we must add $2\pi i$ to the value $-\pi i$, and hence we get πi for both ∇/k greater than or less than 0. Thus the continuity of D is preserved, as it must be. We then find for D

$$D \simeq 1 - \frac{\omega_p^2}{\omega^2}\left[1 + \frac{3k^2 v_0^2}{\omega_p^2 \; \omega^2}\right] + \frac{\pi i \omega_p^2}{k^2} \frac{\frac{\omega}{k} e^{-\frac{\omega^2}{2k^2 v_0^2}}}{\sqrt{2\pi}\, v_0^3} = 0. \qquad (786)$$

If we substitute the solution for ω obtained from Eq. (783) into the imaginary part of D we can again solve approximately for ω and we obtain

$$\omega^2 = \omega_p^2 + 3k^2 v_0^2 + i\sqrt{\frac{\pi}{2}} \frac{\omega_p^2}{k^3} \frac{\omega^3}{v_0^3} e^{-\frac{\omega^2}{2k^2 v_0^2}} \qquad (787)$$

or

$$\omega = \sqrt{\omega_p^2 + 3k^2 v_0^2}\left(1 + i\sqrt{\frac{\pi}{2}} \; \frac{\omega_p^2}{2} \frac{(\omega_p^2 + 3k^2 v_0^2)}{k^2 v_0^3} e^{-\frac{\omega_p^2 + 3k^2 v_0^2}{2k^2 v_0^2}} \cdots \right. \qquad (788)$$

The imaginary part of ω is such as to give damping. One finds no growing solutions. We should expect this since a Maxwell distribution is a thermal equilibrium distribution and should be stable.

One might think that the zeros of D give normal modes of the plasma, and this is in a sense true. They give time dependences for E and n which are pure exponentials. However, if one returns to the Vlasov equations and looks for solutions which have this time dependence, one will find no such pure solutions. To find such solutions we would require

$$D = 1 - \frac{4\pi e^2}{mk} \int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{s + ikv}\, dv = 0. \qquad (789)$$
real axis

This is the expression for D only for the real part of $S > 0$. However, the roots we found had $\mathrm{Re}\, S < 0$, so that we cannot excite them by themselves. The change in the definition of D for $\mathrm{Re}\, S < 0$ was required by analyticity and was forced on us by the Laplace transform.

If one looks at f one finds that it contains terms which go like e^{ikvt} as well as terms of the form e^{st}, so a pure e^{st} dependence for all quantities does not exist.

Finally, these modes are not the modes found in the beam analysis. The beam modes contain all the individual particle motions as well as any collective motions. The beam modes are true normal modes, whereas the Landau-damped modes are not.

c. <u>An Energy Treatment of Landau Damping</u>

We may also derive expression (786) for Landau damping from the law of conservation of energy. Again we use the one-dimensional Vlasov equation and include only the longitudinal electric forces.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f}{\partial v} = 0 \tag{790}$$

and

$$\frac{\partial E}{\partial x} = -4\pi e\int f\,dv. \tag{791}$$

If we take the time derivative of Eq. (791) and make use of Eq. (790) we have

$$\frac{\partial^2 E}{\partial t\,\partial x} = 4\pi e\int\left[v\frac{\partial f}{\partial x}dv - \frac{eE}{m}\frac{\partial f}{\partial v}\right]dv \tag{792}$$

from which we obtain

$$\frac{\partial}{\partial x}\left[\frac{\partial E}{\partial t} - 4\pi e\int v f\,dv\right] = 0 \tag{793}$$

or

$$\frac{\partial}{\partial x}\left[\frac{\partial E}{\partial t} + 4\pi j\right] = 0. \tag{794}$$

We may integrate Eq. (794) with respect to x to obtain

$$\frac{\partial E}{\partial t} + 4\pi j = c(t) \tag{795}$$

where c(t) may be a function of time. Now if there is a place, say $-\infty$, where E and j vanish for all times, then Eq. (795) becomes

$$\frac{\partial E}{\partial t} + 4\pi j = 0. \tag{796}$$

Multiplying by E and integrating over all x gives

$$\frac{\partial}{\partial t}\Phi = \frac{\partial}{\partial t}\int \frac{E^2}{8\pi}\,dx = -\int Ej\,dx. \qquad (797)$$

The left-hand side of Eq. (797) is the energy in the electric field, while the right-hand side is the rate at which the electric field does work on the current, thus the rate of loss of energy by the electric field.

Next, let us compute the rate of change of the kinetic energy of the particles. To do this we multiply Eq. (790) by $mv^2/2$ and integrate over all v and x.

$$\frac{\partial}{\partial t}\iint \frac{mv^2 f}{2}\,dv\,dx + \iint \frac{mv^3}{2}\frac{\partial f}{\partial x}\,dv\,dx$$

$$- \iint \frac{e}{m} E \frac{mv^2}{2}\frac{\partial f}{\partial v}\,dv\,dx = 0. \qquad (798)$$

Now if f is undisturbed at the ends of the interval for the x interval of integration $(-\infty,\infty)$, then the middle integral goes out (infinite homogeneous plasma) and we have

$$\frac{\partial}{\partial t}\iint \frac{mv^2}{2} f\,dv\,dx - \iint \frac{e}{m} E \frac{mv^2}{2}\frac{\partial f}{\partial v}\,dv\,dx = 0 \qquad (799)$$

We may integrate the last integral in Eq. (799) by parts with respect to v to obtain

$$\iint \frac{eE}{2} v^2 \frac{\partial f}{\partial v}\,dv\,dx = -\iint eEvf\,dv\,dx \qquad (800)$$

$$= \int Ej\,dx.$$

Thus Eq. (799) becomes

$$\frac{\partial}{\partial t}K = \frac{\partial}{\partial t}\iint \frac{mv^2}{2} f\,dv\,dx = \int Ej\,dx. \qquad (801)$$

We see from Eqs. (797) and (801) that the total energy

$$\int \frac{E^2}{8\pi}dx + K = W \qquad (802)$$

is conserved ($\partial w/\partial t = 0$).

We shall now apply Eqs. (797) and (801) to the problem of Landau damping. To do this we observe that the particles moving at nearly the phase velocity of the wave were the ones responsible for the damping in our earlier treatment of it. These particles are very strongly perturbed by the wave because they see almost a constant E field. The frequency which they see is the doppler-shifted frequency $\omega' = \omega - kv$, which is very small. The E field accelerates them for times of the order of $1/\omega'$ (neglecting second order effects) which is a very long time and hence their perturbed velocity is very large. We shall thus divide the electrons into two groups — those with velocities considerably different from the phase velocity (main plasma) and those with velocities approximately equal to the phase velocity (resonant electrons). This division is illustrated in Fig. 55.

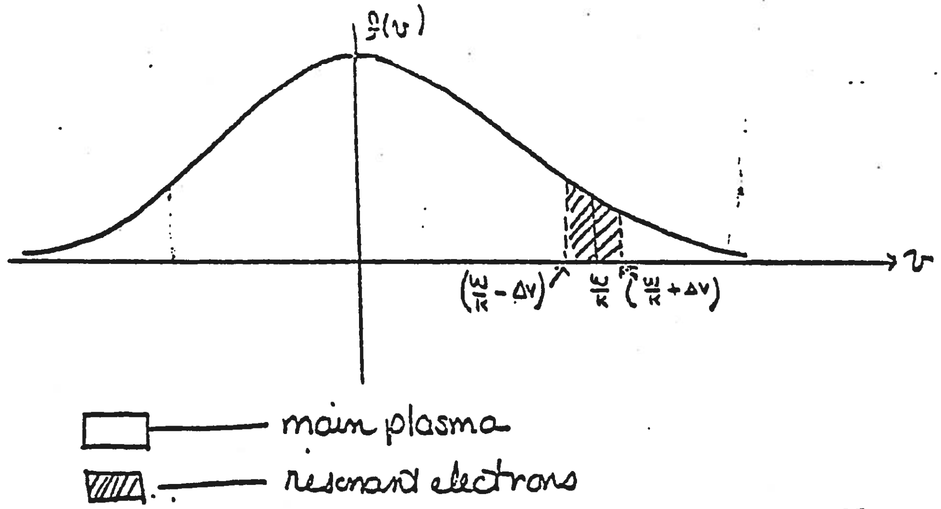


Figure 55

The size of the region around the resonance is not uniquely determined. However, all that is necessary is that we be able to choose a Δv small compared to ω/k.

Now we realize that the wave is primarily carried by the particles in the main plasma, while the resonant electrons are primarily responsible for the damping. We therefore first look for waves carried by the main plasma. The Vlasov Eqs. (748) and (749) govern their motion, so if we look for solution of the form $E = E \sin(kx - \omega t)$, we obtain from these equations, $f = f \cos(kx - \omega t)$

$$(\omega - kv) f = \frac{eE}{m} \frac{\partial f_0}{\partial v} \tag{803}$$

and

$$kE = -4\pi e \int_{\text{Main Plasma (M.P.)}} f\, dv \tag{804}$$

Here the label MP on the integral appearing in Eq. (804) implies that this integral is to be taken only over the main plasma electrons. We will neglect the electric field produced by the resonant electrons because there are so few of them.

Solving Eq. (803) for f and substituting in Eq. (804) for E gives

$$f = \frac{eE}{m} \frac{\partial f_0/\partial v}{\omega - kv} \cos(kx - \omega t) \tag{805}$$

$$E = \frac{-4\pi e^2}{mk} E \int_{MP} \frac{(\partial f_0/\partial v)\, dv}{\omega - kv} \tag{806}$$

Eq. (806) gives us the dispersion relation when E is cancelled

$$1 = -\frac{4\pi e^2}{mk} \int_{MP} \frac{(\partial f_0/\partial v)\, dv}{\omega - kv}. \tag{807}$$

We note that no difficulty arises in evaluating the integral appearing in Eq. (807) because f_o and f_o' are zero in the vicinity of $\omega = -kv$. Eq. (807) can be converted to Eq. (808) by integrating by parts and by observing that $f_o(\pm\infty) = 0$ and $f_o(+\omega/k \pm \Delta V) = 0$.

$$1 = \frac{4\pi e^2}{m} \int_{MP} \frac{f_0\, dv}{(\omega \pm kv)^2}. \tag{808}$$

We may use Eqs. (801) and (797) to evaluate the wave energy. First there is the electric field energy, $E^2/8\pi$. The average of this per unit length is

$$\phi = \frac{E^2}{16\pi} \tag{809}$$

To obtain the kinetic energy (the change in the kinetic energy of the plasma due to the oscillation) we use Eqs. (801) and (748). Solving Eq. (748) for E in terms of f gives

$$E = \frac{m}{e} \frac{1}{\frac{\partial f_0}{\partial v}} \left[\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} \right]. \tag{810}$$

Now $\int E j \, dx$ is

$$\iint -e v f \, E \, dv \, dx = -m \iint \frac{v}{\frac{\partial f_0}{\partial v}} f \left[\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} \right] dv \, dx \tag{811}$$

$$= -\frac{\partial}{\partial t} \int \frac{mv}{2} \frac{f^2}{\frac{\partial f_0}{\partial v}} dv \, dx. \tag{812}$$

From Eqs. (812) and (801) we see that we can identify the time derivative of K with the time derivation of

$$-\iint \frac{mv}{2} \frac{f^2}{\frac{\partial f_0}{\partial v}} dv \, dx. \tag{813}$$

Thus K and Eq. (813) can at most differ by a constant. We can, in fact, show that if K is properly defined this quantity is in fact K. (Since K is second order in the amplitude, a second order change in f can also give a second order change in K. Since the second order f is arbitrary — we are working with f_1 — this leads to an ambiguity in K). Taking ~~this as~~ (813) to be K and substituting in f from Eq. (805) gives

$$K = -\frac{E^2e^2}{2m}\iint_{MP}\frac{v\,\partial f_0/\partial v}{(\omega-kv)^2}\cos^2(kv-\omega t)\,dx\,dv \tag{814}$$

or per unit length

$$K = -\frac{E^2e^2}{4m}\iint_{MP}\frac{v\,\partial f_0/\partial v}{(\omega-kv)^2}\,dv. \tag{815}$$

Combining Eqs. (809) and (815) we get the total wave

energy per unit length

$$W = K + \Phi = \frac{E^2}{16\pi}\left\{1 - \frac{4\pi e^2}{m}\int_{MP}\frac{v\,\partial f_0/\partial v\,dv}{(\omega-kv)^2}\right\}. \tag{816}$$

From the dispersion relation, Eq. (807), we may write

for 1

$$1 = -\frac{4\pi e^2}{mk}\int_{MP}\frac{(\partial f_0/\partial v)\,dv}{\omega-kv}. \tag{817}$$

Substituting into Eq. (816) gives for the energy density

$$W = -\frac{E^2}{16\pi}\,\frac{4\pi e^2}{mk}\,\omega\int_{MP}\frac{\partial f_0/\partial v\,dv}{(\omega-kv)^2}. \tag{818}$$

Now we have from the dispersion relation

$$\frac{4\pi e^2}{m}\left[-\frac{1}{k^2}\int_{MP}\frac{\partial f_0/\partial v}{\omega-kv}\,dv + \frac{1}{k}\int_{MP}\frac{v\,\partial f_0/\partial v}{(\omega-kv)^2}\,dv\right]dk$$

$$-\frac{4\pi e^2}{mk}\,d\omega\int_{MP}\frac{\partial f_0/\partial v\,dv}{(\omega-kv)^2} = 0 \tag{819}$$

or

$$\frac{4\pi e^2}{mk^2}\left[-2\int_{MP}\frac{\partial f_0/\partial v\,dv}{\omega-kv} + \omega\int\frac{\partial f_0/\partial v}{(\omega-kv)^2}\,dv\right]$$

$$= \frac{4\pi e^2}{mk}\,\frac{d\omega}{dk}\int\frac{\partial f_0/\partial v}{(\omega-kv)^2}\,dv.$$

(820)