VI. General Properties of the Full Vlasov Equation

In the previous five chapters, we have examined the properties of the "first order" or "Vlasov" approximation to the Klimontovich equation (2.6) by 1) looking at the fluid limits, in Chaps. III and IV, and 2) studying the "linearized" Vlasov equation, i.e., the behavior in the neighborhood of the very simple solution, $f(\underline{x},\underline{v},t) = f_0(\underline{v})$, in Chap. V. Even within the fluid limit, our emphasis has been on the linear theory -- wave properties, dielectric functions, etc. In the following chapter, we shall extend our study of the linearized Vlasov equation to the case of a magnetized plasma. Before doing that, however, we shall discuss some general properties of the full Vlasov equation which are useful in achieving a better understanding of the physics involved.

In Section A we relate solutions of the full Vlasov equation to the properties of single particle trajectories in (self-consistent) electromagnetic fields, and show how approximate trajectories can give rise to approximate solutions. In particular, this provides a different point of view regarding solutions of the lineorized Vlasov equation. In Section B, we derive the general energy-momentum conservation laws for the Vlasov equation and show how these lead to the concept of the energy associated with a single wave.

A. Solution in Terms of Particle Trajectories

In the 6-dimensional $(\underline{x},\underline{v})$ phase space, a particle trajectory or orbit is defined by the equations of motion

$$d\underline{x}/dt = \underline{v} \qquad d\underline{v}/dt = (q/m)(\underline{E} + \underline{v} \times \underline{B}/c) \equiv (q/m)\underline{\tilde{E}}$$

for given initial conditions on $(\underline{x},\underline{v})$ and given fields, $\underline{\tilde{E}}(\underline{x},\underline{v},t) \equiv [\underline{E}(\underline{x},t) + \underline{v} \times \underline{B}(\underline{x},t)/c]$. The Vlasov equation is simply a statement that $f(\underline{x},\underline{v},t)$ is constant along such a trajectory since $[\partial f/\partial t + \underline{v}.\nabla f + (q/m)\underline{\tilde{E}}.\nabla_v f]$ is just the convective rate of change of f for an observer moving through the $\underline{x},\underline{v}$ phase space along a trajectory.

It is convenient to introduce <u>trajectory functions</u> $\underline{X}(t'; \underline{x},\underline{v},t)$ and $\underline{V}(t'; \underline{x},\underline{v},t)$, defined as the position and velocity, at time t', of a particle whose position and velocity at time t are $\underline{x}$ and $\underline{v}$. Formally, $\underline{X}$ and $\underline{V}$ satisfy the differential equations in t'

$$d\underline{X}/dt' = \underline{V} \qquad d\underline{V}/dt' = (q/m)\underline{\tilde{E}}(\underline{X},\underline{V},t') \tag{6.1}$$

with boundary conditions

$$\underline{X}(t; \underline{x},\underline{v},t) = \underline{x} \qquad \underline{V}(t; \underline{x},\underline{v},t) = \underline{v} \tag{6.2}$$

The constancy of f along a trajectory then gives at once

$$f(\underline{x},\underline{v},t) = f[\underline{X}(t_0; \underline{x},\underline{v},t), \underline{V}(t_0; \underline{x},\underline{v},t), t_0] \tag{6.3}$$

i.e., the exact solution of the Vlasov equation at time t can be expressed in terms of the solution at an "initial" time t_0 provided we know the trajectory functions $\underline{X},\underline{V}$.

For notational convenience, we introduce a six-dimensional phase-space vector, r, to represent both $\underline{x}$ and $\underline{v}$, $r = \{\underline{x},\underline{v}\}$, and similarly define $R = \{\underline{X},\underline{V}\}$, so that (6.3) becomes

$$f(r,t) = f[R(t_0; r,t), t_0] \tag{6.4}$$

A different notation, used in some treatments of plasma turbulence, involves a unitary "time displacement" operator, $U(t,t_0)$, defined as follows: any function of phase-space, $f(r)$, is transformed by U into a new function $g = U(t,t_0)f$ given by

$$g(r) \equiv f[R(t_0; r,t)] \tag{6.5}$$

Then (6.3) takes the form

$$f(r,t) = U(t,t_0)\, f(r,t_0) \tag{6.6}$$

showing that the operator U advances the distribution function from time t_0 to time t.

No matter what notation we use, a formal solution is of little practical use, since finding the trajectory functions $\underline{X}, \underline{V}$ is difficult, even if we know $\underline{\tilde{E}}$. More to the point, as we have already seen, the essential feature of the Vlasov equation is self-consistency: $\underline{\tilde{E}}$ can only be determined from f, so a "solution" like (6.3) in which $\underline{\tilde{E}}$ is regarded as "known" would appear to be of little value. Nonetheless, such an exact, albeit formal, solution can be very useful as a basis for constructing explicit approximate solutions.

For example, it is sometimes convenient to split $\underline{\tilde{E}}$ into two parts

$$\underset{\sim}{\tilde{E}} = \underset{\sim}{\tilde{E}}_0 + \underset{\sim}{\tilde{E}}_1$$

and to write the Vlasov equation as

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{q}{m} \underline{\tilde{E}}_0 \cdot \frac{\partial f}{\partial \underline{v}} = -\frac{q}{m} \underline{\tilde{E}}_1 \cdot \frac{\partial f}{\partial \underline{v}} \equiv h(\underline{x}, \underline{v}, t) \quad (6.7)$$

Let $R_o = \{X_o, V_o\}$ specify the "unperturbed" trajectory functions, i.e., the set defined by (6.1) and (6.2) with $\underline{\tilde{E}}_0$ in place of $\underline{\tilde{E}}$, and define a function of t'

$$F(t'; r, t) \equiv f[R_o(t'; r, t), t']$$

Then using (6.7) with the independent variables X_o, V_o in place of $\underline{x}$, $\underline{v}$, we have

$$dF/dt' = h[R_o(t'; r, t), t']$$

which can be integrated immediately to give

$$F(t; \underline{x}, \underline{v}, t) - F(t_o; \underline{x}, \underline{v}, t) = \int_{t_o}^{t} dt' \, h[\underline{X}_o, \underline{V}_o, t']$$

From (6.2) it follows that

$$F(t; \underline{x}, \underline{v}, t) = f(\underline{x}, \underline{v}, t)$$

so we have, again, a formal solution of the Vlasov equation:

$$f(\underset{\sim}{x}, \underset{\sim}{v}, t) = f[\underline{X}_0(t_0; \underset{\sim}{x}, \underset{\sim}{v}, t), \underline{V}_0(t_0; \underset{\sim}{x}, \underset{\sim}{v}, t), t_0] +$$

$$+ \int_{t_0}^{t} dt' h[\underline{X}_0(t'; \underset{\sim}{x}, \underset{\sim}{v}, t), \underline{V}_0(t', \underset{\sim}{x}, \underset{\sim}{v}, t), t'] \qquad (6.8)$$

giving f at time t in terms of f at time t_0 and an integral of the inhomogeneous term, h, carried out along an unperturbed particle trajectory. Since h involves f itself, this is really an integral equation for f, rather than a solution, but if $\underline{\tilde{E}}_1$, and with it h, is in some sense small, then (6.8) provides a convenient iteration scheme provided $\underline{\tilde{E}}_0$ can be chosen so that the associated trajectory functions, $\underline{X}_0$, $\underline{V}_0$ can be found explicitly or at least have some simple property. Of course, the special case $\underline{E}_1 = 0$ reduces (6.8) to (6.3). For future reference we note that in terms of the "unperturbed" operator $U_0(t,t_0)$, defined by

$$U_0(t, t_0) F(r) = F[R_0(t_0; r, t)]$$

(6.8) can be written

$$f(r,t) = U_0(t, t_0) f(r, t_0) + \int_{t_0}^{t} dt' U_0(t, t') h(r, t')$$

An example of this use of the trajectory functions, consider the case where $\underline{\tilde{E}}_1$ is small. If we make the choice $\underline{\tilde{E}} = 0$, then the "unperturbed" trajectory functions, i.e., those given by (6.1) and (6.2) with $\underline{\tilde{E}} = 0$, will be just straight lines in configuration space:

$$\underline{V}_0(t';\underline{x},\underline{v},t) = \underline{v}$$

$$\underline{X}_0(t';\underline{x},\underline{v},t) = \underline{x} + \underline{v}\,(t'-t) \tag{6.9}$$

and all effects of the field $\underline{\tilde{E}} = \underline{\tilde{E}}_1$ are contained in

$$h = -\frac{q}{m}\,\underline{\tilde{E}}\cdot\frac{\partial f}{\partial \underline{v}}$$

From (6.9) we have, choosing $t_0 = 0$,

$$f(\underline{x},\underline{v},t) = f[\,\underline{x}-\underline{v}t,\,\underline{v},\,0] - \int_0^t dt'\,\frac{q}{m}\,\underline{\tilde{E}}[\,\underline{x}+\underline{v}(t'-t),t'\,]\cdot\frac{\partial f}{\partial \underline{v}} \tag{6.10}$$

an integral equation for f which is exact, irrespective of the size of $\underline{\tilde{E}}$. If f is independent of $\underline{x}$ at time t = 0,

$$f(\underline{x},\underline{v},0) = f_0(\underline{v})$$

and if we treat $\underline{\tilde{E}}$ as small, so that we can iterate (6.10), then we have to first order

$$f(\underline{x},\underline{v},t) = f_0(\underline{v}) - \tag{6.11}$$
$$-\,(q/m)\int_0^t dt'\,\underline{\tilde{E}}[\,\underline{x}+\underline{v}(t'-t),\,t'\,]\cdot\partial f_0/\partial\underline{v}$$

As is to be expected, the Fourier-Laplace transform of this reproduces the usual linear theory. However, (6.11) provides a different and useful physical picture of the linearized theory. In absence of the field $\tilde{\underline{E}}$, each particle follows its unperturbed (straight line) orbit. These orbits will, of course, be slightly changed by a small perturbing field, $\tilde{\underline{E}}$, but (6.11) shows that we can ignore this in computing the first order charge in the distribution function, i.e., the f_1 of linear theory. To find f_1 at a point $r = \{\underline{x},\underline{v}\}$ in phase space at time t, we simply trace the unperturbed trajectory backwards in time. For each earlier instant, $t' < t$, there is a contribution $(\tilde{\underline{E}} \cdot \partial f_0/\partial \underline{v})dt'$ to f_1, where $\underline{E}$ is evaluated at the past location, $\underline{x}-\underline{v}(t-t')$, corresponding to time t', and the sum of these contributions gives $f(\underline{x},\underline{v},t)$. A further virtue of (6.11) is that it gives f in physical $(\underline{x},t)$ space rather than in the more abstract $(\underline{k},\omega)$ space.

B. Energy and Momentum Conservation

1) The General Conservation Laws

It is clear, a priori, that the Vlasov equation, plus Maxwell's equations, will lead to energy and momentum conservation laws: this is certainly true for the equations (2.1) and (2.2) from which we started, and neither ensemble averaging nor the neglect of fluctuations, which leads to the Vlasov equation, can change this. (Of course, when, later, we include fluctuations, they will also contribute terms to the conservation equations which are of second order in the fluctuations.) For an explicit demonstration, we calculate first the rate of charge of particle kinetic energy density (energy per unit volume),

$$\frac{\partial K}{\partial t} = \frac{\partial}{\partial t}\int d\underline{v}\, n_0 \frac{mv^2}{2} f =$$

$$= -\int d\underline{v}\, n_0 \frac{mv^2}{2}\left(\underline{v}\cdot\nabla f + \frac{q}{m}\underline{\tilde{E}}\cdot\frac{\partial f}{\partial \underline{v}}\right) = -\nabla\cdot\underline{S}_M + \underline{j}\cdot\underline{E} \quad (6.12)$$

where

$$\underline{S}_M \equiv \int d\underline{v}\, n_0 (mv^2/2)\underline{v} f \quad (6.13)$$

is the flux of mechanical energy and, as usual, $\underline{j} = \int d\underline{v}\, n_0 q \underline{v} f$. From Maxwell's equations it follows that

$$\underline{j}\cdot\underline{E} = \frac{c}{4\pi}\left(\nabla\times\underline{B} - \frac{\dot{\underline{E}}}{c}\right)\cdot\underline{E} = -\frac{\partial}{\partial t}\frac{E^2}{8\pi} + \\ + \frac{c}{4\pi}\left[\nabla\cdot(\underline{B}\times\underline{E}) - \frac{\dot{\underline{B}}\cdot\underline{B}}{c}\right] \quad . (6.14)$$

Combining this with (6.12) gives the energy conservation law,

$$\frac{\partial}{\partial t}(K + W) + \nabla\cdot(\underline{S}_M + \underline{S}_E) = 0, \quad (6.15)$$

where

$$W = (E^2 + B^2)/8\pi \quad (6.16)$$

is the usual electromagnetic energy density and the Poynting vector,

$$\underline{S}_E = \frac{c}{4\pi}(\underline{E}\times\underline{B}),$$

gives the flux of electromagnetic energy. Similarly, we have the momentum conservation law,

$$\frac{\partial}{\partial t}\left(\underline{P}_M + \underline{P}_E\right) + \nabla\cdot\left(\underset{\sim}{T}_M + \underset{\sim}{T}_E\right) = 0, \tag{6.17}$$

where

$$\underline{P}_M = \int d\underline{v}\; n_o m \underline{v} f$$

is the mechanical momentum density;

$$\underline{P}_E = \underline{E} \times \underline{B} / 4\pi c$$

is the usual electromagnetic momentum density;

$$\underset{\sim}{T}_M = \sum n_o m \underline{u}\,\underline{u} + \underset{\sim}{P}$$

is the flux of mechanical mementum density, with

$$\underset{\sim}{P} = \int d\underline{v}\; n_o m (\underline{v}-\underline{u})(\underline{v}-\underline{u}) f\;;$$

and

$$\underset{\sim}{T}_E = \left(\underline{E}\,\underline{E} - \underset{\sim}{1} E^2/2 + \underline{B}\,\underline{B} - \underset{\sim}{1} B^2/2\right)/4\pi$$

is the usual Maxwell stress tensor.

2. Energy of a Linear Electrostatic Wave

Using the conservation laws derived above, it is instructive to examine, in detail, the energy of a linear wave, for example a plane electrostatic wave (Langmuir or ion acoustic),

$$\underline{E}(\underline{x},t) = \underline{E}_1 \sin(\underline{k}\cdot\underline{x} - \omega t). \qquad (6.18)$$

The electric field of the wave will give rise to an energy density $E^2/8\pi$, i.e., the W term of (6.15) and (6.16). In addition, the plasma wave involves a coherent motion of particles induced by $\underline{E}$ (this, in turn, leading to the charge density oscillations which serve as sources for $\underline{E}$ and hence sustain the wave). It is this kinetic energy part of the wave energy, i.e., the term K in (6.15), which we must now consider. Since K is a quadratic function of the particle velocity, a direct calculation of K requires solving the particle equations of motion (or, equivalently, the Vlasov equation) correct to second order in the wave amplitude, E_1. A less straightforward, but simpler, approach, which we shall follow here, is to calculate $\partial K/\partial t$ and then find K by integrating with respect to t, a produce which only requires the linear solution of the Vlasov equation (first order in E_1).

The periodic spatial variation of $\underline{E}$ (and hence of f) causes a variation in K over one wavelength which is of no physical interest, so we shall consider only the spatial average (over one wavelength), denoted by $\overline{K}$. Since mechanical energy flux $\underline{S}_M$ is likewise periodic, the spatial averaging removes the $\nabla.\underline{S}_M$ term in (6.12), leaving

$$\partial\overline{K}/\partial t = \overline{\underline{J}\cdot\underline{E}} = \overline{\underline{E}\cdot\int d\underline{v}\, n q \underline{v} f} \qquad (6.19)$$

This simply says that, on a space averaged basis, the rate of increase of kinetic energy equals the rate at which $\underline{E}$ does work on the plasma current, as we would expect. In evaluating the right hand side of (6.19) we must allow the amplitude, $\underline{E}_1$, of the plane wave to be a slowly varying function of time, $\underline{E}_1$ (t). If it were constant, then K would be time independent and this method of calculation could not be used. If $\underline{E}_1$ varied rapidly, then other effects would enter. We therefore allow it to vary with time, but so slowly that we can neglect time derivatives of $\underline{E}_1$ which are of second or higher order. We also assume that $\underline{E}_1 (0) = \dot{\underline{E}}_1 (0) = 0$.

In addition to the spatial averaging and the assumption of slow variation of $\underline{E}_1$, one other feature of the calculation must be mentioned, namely the necessity for considering separately the contributions of resonant and non-resonant particles. By resonant particles, we mean those velocity is almost equal to the phase velocity of the wave, so that

$$\omega - \underline{k} \cdot \underline{v} \approx 0.$$

In the rest frame of the wave, such particles are nearly at rest, and therefore see a nearly constant electric field, whose value depends upon the position of a particular particle relative to the wave. This is in sharp contrast to non-resonant particles, which have a velocity much different from the wave phase velocity and hence experience an oscillating electric field at the Doppler-shifted frequency $w-\underline{K}\cdot\underline{v}$. In an energy diagram, such as Fig. 1, which shows the potential energy of the electrostatic wave and also the energies of resonant and non-resonant particles, we see that the latter ride above the wave, so to speak, with a velocity which is nearly constant, decreasing slightly when they go over a crest of the wave and increasing slightly when they pass a trough.

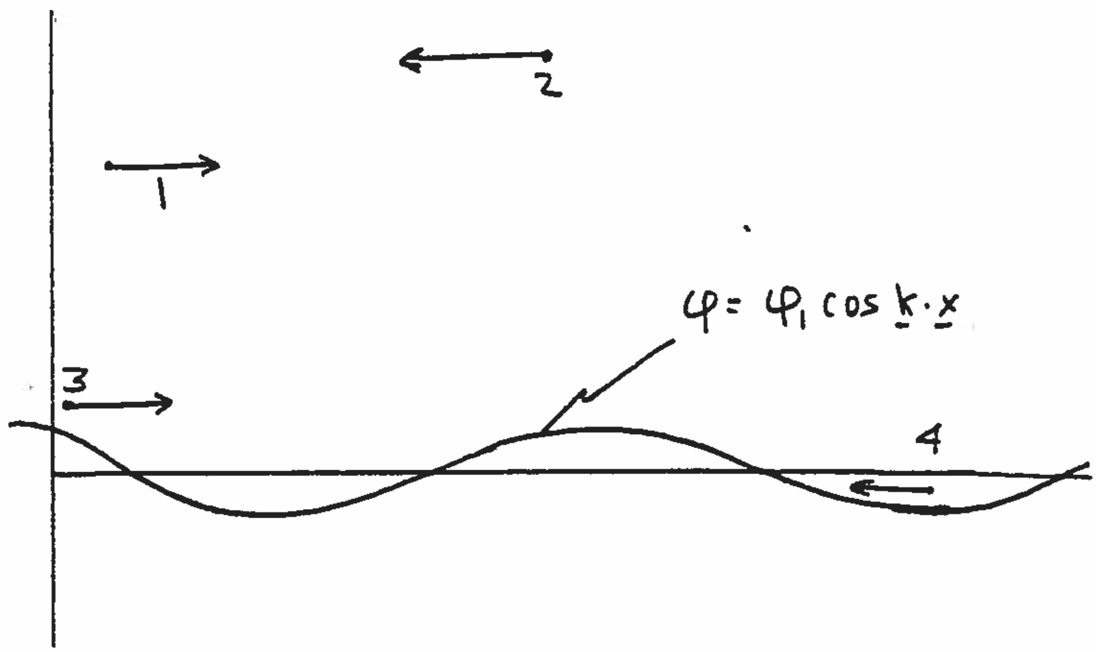

Fig. 1 Schematic representation of the wave potential energy in the rest frame of the wave and the kinetic energies of non-resonant (1,2) and resonant (3,4) particles. Particle 3 is resonant but untrapped, while particle 4 is trapped and, a fortiori, also resonant.

To make the distinction quantitative, let γ denote the order of magnitude of the fractional rate of charge of the wave amplitude, so that $|\dot{E}_1/E_1| \approx \gamma$. We define non-resonant particles as those which travel, in the wave frame, a distance much greater than $\lambda = 2\pi/k$ (in the direction of $\underline{k}$) during the time the wave changes amplitude appreciably; i.e., for non-resonant particles

$$|\omega - \underline{k}\cdot\underline{v}| >> \gamma \tag{6.20}$$

or, equivalently,

$$|v_{11} - u| \gtrsim \Delta v$$

where $u = \omega/k$ is the phase velocity of the wave, $v_{11} = \hat{\underline{k}}\cdot\underline{v}$, and ∇v is the width of the resonant particle portion of f_o, chosen to satisfy the condition $\Delta v >> \gamma/k$. Of course, this division of particles into resonant and non-resonant portions and the subsequent considerations of this section are useful only if also $\Delta v << a$, where a measures the width of f_o,

$$a^2 \sim \int d\underline{v}\, v^2 f_o \Big/ \int d\underline{v}\, f_o$$

When we later discuss nonlinear wave phenomena we shall also consider "trapped particles", i.e., those which have, in the wave frame, a total energy less than $2\phi_1$ and hence are trapped in the troughs of Fig. 1 (e.g. particle 4 of Fig. 1). Trapped particles oscillate in the potential well of a fixed amplitude wave with a frequency which is of order of the "bounce frequency"

$$\omega_b = (qkE_1/m)^{1/2}$$

We shall assume here that the wave amplitude is so small that $\omega_b \ll \gamma$. Since the variation in a particle's wave frame kinetic energy caused by the wave cannot exceed the charge in potential energy, $2E_1/k$, its velocity variation will be less than $(4qE_1/mk)^{\frac{1}{2}} = (2\omega_b/k)$. By assumption, this is small compared to γ/k and hence the distinction between resonant and non-resonant particles, given by (6.20), does not depend upon the phase of the particles relative to the wave. It also follows that none of the non-resonant particles, and only a small fraction of the resonant ones, are trapped. In later chapters we shall examine the behaviour of trapped particles in more detail, but so long as $\omega_b t \ll 1$, as we shall assume here, use of the unperturbed orbits, as in (6.11), will be a good approximation for all particles, trapped and untrapped, resonant and non-resonant.

We now return to (6.19). In calculating $\overline{f\underline{E}}$ we are justified in using linear theory for f, as explained above, and it is convenient to use (6.11), since it gives f in $(\underline{x},t)$ space:

$$\overline{f\underline{E}} = -(q/m)\,\underline{E}_1(t)\int_0^t dt'\,\underline{E}_1(t')\cdot(\partial f_0/\partial\underline{v})\cdot$$

$$\cdot\ \sin(\underline{k}\cdot\underline{x}-\omega t)\,\sin[\underline{k}\cdot\underline{x}-\omega t'+\underline{k}\cdot\underline{v}(t'-t)] =$$

$$= -(q/2m)\, \underline{E}_1(t) \int_0^t dt'\, \underline{E}_1(t')\cdot(\partial f_0/\partial \underline{v}) \cos[(\underline{k}\cdot\underline{v} - \omega)(t'-t)] \qquad (6.21)$$

This equation is valid for all $\underline{v}$, but the evaluation of (6.21) proceeds differently for resonant and non-resonant particles. For the latter, it is convenient to carry out two integrations by parts in (6.21). Using the condition $\underline{E}_1(0) = \dot{\underline{E}}_1(0) = 0$, we find

$$\overline{(f\,\underline{E})}_{NR} = -(q/2m)\, \underline{E}_1(t)\, \dot{\underline{E}}_1(t)\cdot(\partial f_0/\partial \underline{v})/(kw)^2 \qquad (6.22)$$

plus terms involving $\ddot{\underline{E}}_1$ and higher order derivatives, where

$$w \equiv (\underline{k}\cdot\underline{v} - \omega)/k = v_\parallel - u\,. \qquad (6.23)$$

The neglect of these terms implies an expansion in (γ/kw) which is, by definition, small for non-resonant particles.

On substituting (6.22) into (6.19) and remembering that (6.17) describes an electrostatic wave, $\underline{E}_1 = \hat{\underline{k}}E_1$, we find for the contribution of non-resonant particles to $\underline{j}\cdot\underline{E}$,

$$(\underline{j}\cdot\underline{E})_{NR} = -\int_{NR} d\underline{v}\,(nq^2/4mk^2)(\partial E_1^2/\partial t)\cdot \qquad (6.24)$$
$$\cdot\,(\underline{k}\cdot\partial f_0/\partial\underline{v})\,\underline{k}\cdot\underline{v}/(kw)^2$$

The velocity integral here is taken only over the non-resonant particles, $|v-u| > \Delta v \gg \gamma/k$. In the limit of small γ, it reduces to a principal value integral which is closely related to the real part, ε_R, of the dielectric function for electrostatic waves,

$$\epsilon_R(\underline{k},\omega) = 1 - P\int d\underline{v}\,(\omega_p^2/k^2)\,\underline{k}\cdot(\partial f_0/\partial\underline{v})/kw$$

and to its derivative,

$$\partial\epsilon_R/\partial\omega = -\int d\underline{v}\,(\omega_p^2/k^2)\,\underline{k}\cdot(\partial f_0/\partial\underline{v})/(kw)^2$$

Since $\overline{E^2} = E_1^2/2$, (6.24) becomes

$$(\underline{J}\cdot\underline{E})_{NR} = -\int d\underline{v}\,(\omega_p^2/k^2)\,\underline{k}\cdot(\partial f_0/\partial\underline{v})\cdot$$

$$\cdot\left[\omega/(kw)^2 + 1/kw\right]\partial W/\partial t =$$

$$= \left[\partial(\epsilon_R\omega)/\partial\omega - 1\right]\partial W/\partial t \qquad (6.25)$$

where $W = E^2/8\pi$ is the electrostatic part of the wave energy. It then follows that

$$\frac{\partial}{\partial t}\overline{(K_{NR} + W)} = (\underline{J}\cdot\underline{E})_{NR} + \frac{\partial \overline{W}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial\omega}(\epsilon_R \omega)\overline{W}\right] \tag{6.26}$$

and hence that

$$\overline{K_{NR} + W} = \frac{\partial}{\partial\omega}(\omega\epsilon_R)\overline{W} + K_o \tag{6.27}$$

where K_{NR} is the kinetic energy of non-resonant particles and K_o is a constant, independent of t. The left hand side of (6.27) is essentially the total energy of the wave, since the contribution, K_R, of the resonant particle kinetic energy will be small compared to K_{NR} if, as is usually the case, the width, Δv, of the resonant particle region is small compared to the width of $f_o(v)$, $\Delta v \ll a$. It is convenient to define the wave energy, W_ω, as

$$W_w = \overline{K_{NR} + W} = \frac{\partial}{\partial\omega}(\omega\epsilon_R)\overline{W} + K_o \tag{6.28}$$

In order that (6.28) be valid in the limit $E \to 0$, the constant K_o must be just the kinetic energy in absence of the wave,

$$K_o = \int d\underline{v}\,(mv^2/2)\,f_o(\underline{v}).$$

In order that (6.28) be valid in the limit $E \to 0$, the constant K_o must be just the kinetic energy in absence of the wave,

From the general energy conservation theorem, (6.15), we know that the total space-averaged energy is conserved,

$$\partial(\overline{K+W})/\partial t = \partial(\overline{K_R + W_w})/\partial t = 0$$

so that the wave energy, W_w, is not constant but changes at a rate

$$\partial W_w/\partial t = -\partial K_R/\partial t = -(\underline{J}\cdot\underline{E})_R \; . \tag{6.29}$$

To evaluate this, we return to (6.21) and integrate over the resonant particles, $|v_{11}-u| \leq \Delta v$. Integrating (6.21) over the components $\underline{v}_\perp$ of $\underline{v}$ perpendicular to $\underline{k}$ we have

$$(\underline{J}\cdot\underline{E})_R = \int_{-\Delta v}^{\Delta v} dw\, n q v_{\parallel} \overline{f E} =$$

$$= -\int_{-\Delta v}^{\Delta v} dw\,(nq^2/2m)\, v_{\parallel} \int_0^t d\tau\; E_1(t)\, E_1(t-\tau)\, \cdot$$

$$\cdot\, f_0'(v_{\parallel}) \cos kw\tau$$

where $\quad f_0'(v_{\parallel}) = \int d\underline{v}_\perp (\partial f_0/\partial v_{\parallel})\quad$ and we have replaced t' in (6.21) by $(t-\tau)$. If Δv is small compared to the region over which f'_0 varies, ($|f''_0 \Delta V/f'_0| \ll 1$) we can evaluate $f_0 v_{\parallel} = u$:

$$(\underline{J} \cdot \underline{E})_R = -\sum (\omega_p^2/4\pi) u f_0'(u) \cdot$$

$$\cdot \int_0^t d\tau \; E_1(t) \; E_1(t-\tau)[\sin(k\tau\Delta v)]/k\tau \doteq$$

$$\doteq -\sum (\omega_p^2 u/8k) \; f_0'(u) \; E_1^2(t) =$$

$$= -2\pi \sum (\omega_p^2 u/k) \; f_0' \; \overline{W} \quad . \qquad (6.30)$$

provided $kt\Delta v \gg 1$. (The sum here is over the species.)

Substituting (6.30) into (6.29) gives

$$\partial W_\omega / \partial t = 2\gamma_L W_\omega \qquad (6.31)$$

with

$$\gamma_L = \frac{\pi \sum(\omega_p^2 u f_0'/k)}{\partial(\epsilon_R \omega)/\partial\omega} \tag{6.32}$$

We can easily show that, as is to be expected, (6.32) agrees with the Landau damping rate for the field of an electrostatic wave, as calculated in Chap. V. From the general expression for

$$\epsilon(\underline{k},\omega) = 1 - \oint d\underline{v} \, \frac{\omega_p^2}{k^2} \, \frac{\underline{k}\cdot(\partial f_0/\partial \underline{v})}{\underline{k}\cdot\underline{v} - \omega} \, .$$

it follows that in the limit of real ω

$$\epsilon_I \equiv \mathrm{Im}\,\epsilon = -\pi \sum \int \frac{\omega_p^2}{k^2} \, \hat{\underline{k}} \cdot \frac{\partial f_0}{\partial \underline{v}} \, d\underline{v}_\perp \tag{6.33}$$

so (6.32) can be written

$$\gamma_L = \frac{-\omega \epsilon_I}{\partial(\omega \epsilon_R)/\partial\omega} = \frac{-\epsilon_I}{(\partial \epsilon_R/\partial\omega)} \tag{6.34}$$

if the plane wave, (6.17), is a normal mode of the system, i.e., if ϵ_R $(\underline{k},\omega) = 0$. This is the same as (5.65).

We note that the constant, K_o, portion of W_w, is of little interest, and so it is common to call $[\overline{W}\, \partial(\omega\varepsilon_R)/\partial\omega]$ the wave energy. Of course, this quantity may be positive or negative and in the latter case we speak of "negative energy" waves. Such "negative energy" waves have the property that dissipative effects, which decrease their total energy, correspond to an increase in the wave amplitude, E_1, i.e., to wave growth. Of course, the sum of $\overline{W}$ and $\overline{K}_R$ must be positive definite, and W_w, as defined by (6.28), has this property, so that "negative energy" waves do not really have negative energy.

In summary, the wave energy, W_w, defined as the electrostatic energy plus the kinetic energy of non-resonant particles, is given by (6.28). It changes with time due to energy exchange with the resonant particles, resulting in a growth rate γ_L for the field, or $2\gamma_L$ for the energy, with γ_L given by (6.32). The various approximations utilized are valid provided the wave amplitude is small, $\omega_b << \gamma_L$; the resonant particle width, Δv, is small, $\gamma_L/k << \Delta v << a$; and the time is small, $\omega_b t << 1$, but not too small, $tk\Delta v >> 1$, these two conditions being consistent since $\omega_b << k\Delta v$ is already guaranteed by the earlier inequalities.

3. General Expression for Energy of a Linear Wave

The expressions for the wave energy, W_w, (6.28), and the damping rate, γ_L, (6.32) or (6.34), were derived for a particular case (electrostatic waves) using the Vlasov equation. Since (6.28) and (6.34) involve only $\varepsilon = \varepsilon_R + i\varepsilon_I$, it can be expected that these expressions could also be obtained from a macroscopic or phenomenological treatment in which the plasma is simply characterized by a dielectric function, ε. Such an approach has the advantage that we can include electromagnetic waves in a magnetized plasma (provided we use a dielectric tensor, $\underline{\epsilon}$, to take account of the velocity space anisotropy resulting from a uniform magnetic field) but does not display the underlying physics as clearly as the microscopic treatment given above. For this more general, macroscopic treatment we follow the discussion of Landau and Lifschitz, "Electrodynamics of Continuous Media", Pergamon Press, N.Y. 1960, p. 253.

We write the Maxwell equations in the form

$$\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t}, \quad \nabla \times \underline{B} = \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t} \equiv \frac{1}{c}\frac{\partial \underline{D}}{\partial t} \tag{6.35}$$

where $\underline{D}$ is defined by the last equation. For the Fourier transforms, $\underline{E}(\underline{k},\omega)$, etc. we have, as usual,

$$-i\omega\, \underline{D}(\underline{k},\omega) = -i\omega\, \underline{E}(\underline{k},\omega) + 4\pi \underline{\sigma} \cdot \underline{E}(\underline{k},\omega)$$

or

$$\underline{D}(\underline{k},\omega) = \underset{\approx}{\epsilon}(\underline{k},\omega) \cdot \underline{E}(\underline{k},\omega), \quad \underset{\approx}{\epsilon} = 1 + \frac{4\pi i \underset{\approx}{\sigma}}{\omega} \tag{6.36}$$

It follows at once that

$$\nabla \cdot (\underline{E} \times \underline{B}) = -\frac{1}{c}\left(\underline{B} \cdot \frac{\partial \underline{B}}{\partial t} + \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}\right) \tag{6.37}$$

As in the previous discussion, we consider a wave packet but we now assume the amplitude to vary in both time and space, this variation being slow, i.e. occuring on a spatial (temporal) scale large compared to a wavelength (period) of the plane wave,

$$\underline{E} = \underline{\hat{E}}(\underline{x},t)\exp[i(\underline{k}\cdot\underline{x}-\omega t)] + c.c. \tag{6.38}$$

with similar expressions for $\underline{B}$ and

$$\underline{D} = \underline{\hat{D}}(\underline{x},t)\exp[i(\underline{k}\cdot\underline{x}-\omega t)] + c.c. \tag{6.39}$$

As before, it is convenient to deal with a spatial average, over one wavelength, denoted by a bar. Because of the slow spatial variation of the amplitudes, this spatial average only removes the rapid spatial variation (i.e., on the scale of $\lambda = 2\pi/k$) and yields quantities which are still slowly variation functions of $\underline{x}$. The spatial average of expressions quadratic in the fields, such as $\overline{\underline{E}\mathrm{x}\underline{B}}$, can be written as

$$\overline{(\underline{E}\times\underline{B})} = \underline{\hat{E}}\times\underline{\hat{B}}^{*} + \underline{\hat{E}}^{*}\times\underline{\hat{B}} \tag{6.40}$$

so (6.27) gives

$$\nabla\cdot(\underline{\hat{E}}\times\underline{\hat{B}}^{*}) + c.c. = -\frac{1}{c}\left[\frac{\partial|\underline{\hat{B}}|^2}{\partial t} + \right.$$
$$\left. + 2\,\mathrm{Re}\,\underline{\hat{E}}^{*}\cdot\partial\underline{\hat{D}}/\partial t + 2\omega\,\mathrm{Im}\,\underline{\hat{E}}^{*}\cdot\underline{\hat{D}}\right] \tag{6.41}$$

To cast this into the form of an energy conservation equation requires that the last two terms in (6.41) be transformed, in so far as possible, into space or time derivatives. We accomplish this by introducing Fourier transforms for $\underline{\hat{E}}$ and $\underline{\hat{D}}$:

$$\hat{\underline{E}}(x) = (2\pi)^{-4} \int dq \, \underline{E}_q \, e^{iq\cdot x}$$

$$\hat{\underline{D}}(x) = (2\pi)^{-4} \int dq \, \underline{D}_q \, e^{iq\cdot x} \qquad (6.42)$$

where, for notational convenience, we use a four vector notation: $x = (\underline{x},t)$ and $q = (\underline{q},q_0)$; $dq = d\underline{q}dq_0$; $q\cdot x = \underline{q}\cdot\underline{x} - q_0 t$. The assumption of "slowly varying" amplitudes, $\hat{\underline{E}}$, $\hat{\underline{D}}$ implies that $\underline{E}_q$, $\underline{D}_q$ are non-zero only for

$$|\underline{q}| << |\underline{k}|, \quad q_0 << \omega \qquad (6.43)$$

Since

$$\underline{D}(x) = (2\pi)^{-4} \int dq \, \underline{D}_q \, e^{i(k+q)\cdot x} + c.c. \qquad (6.44)$$

we have

$$\underline{D}_q = \underline{D}(k+q) = \underset{\approx}{\epsilon}(k+q)\cdot \underline{E}_q \qquad (6.45)$$

In view of (6.43), we can expand

$$\underset{\approx}{\epsilon}(k+q) = \underset{\approx}{\epsilon}(k) + q\cdot\frac{\partial}{\partial k}\underset{\approx}{\epsilon} + \cdots\cdot \qquad (6.46)$$

which, with (6.45) and (6.42) gives

$$\hat{\underline{D}}(x) = (2\pi)^{-4} \int dq\, e^{iq\cdot x} \{ \underset{\sim}{\epsilon}(k) \cdot \underline{E}_q +$$

$$+ q \cdot (\partial \underset{\sim}{\epsilon} / \partial k) \cdot \underline{E}_q + \cdots \} =$$

$$= \underset{\sim}{\epsilon}(k) \cdot \hat{\underline{E}} - i (\partial/\partial x) \cdot (\partial \underset{\sim}{\epsilon} / \partial k) \cdot \hat{\underline{E}} + \cdots \tag{6.47}$$

We now substitute (6.47) into (6.41), making the additional assumption that if we decompose $\underset{\sim}{\varepsilon}$ into Hermition and anti-Hermition parts,

$$\underset{\sim}{\epsilon}(k) = \underset{\sim}{\epsilon}_R + i \underset{\sim}{\epsilon}_I \tag{6.48}$$

the elements of $\underset{\sim}{\varepsilon}_I$ will be small compared to those of $\underset{\sim}{\varepsilon}_R$, i.e. the damping or dissipation is small. We have already assumed that the amplitude $\hat{\underline{E}}$ is slowly varying, and we shall drop all products of small terms like $\underset{\sim}{\varepsilon}_I \cdot \frac{\partial \hat{E}}{\partial x}$. (For example the second term in (6.47) is small and in it we can therefore replace $\underset{\sim}{\varepsilon}$ by $\underset{\sim}{\varepsilon}_R$.) Then

$$2 \mathrm{Re}\, \hat{\underline{E}}^* \cdot \partial \hat{\underline{D}} / \partial t = 2 \mathrm{Re}\, \hat{\underline{E}}^* \cdot \underset{\sim}{\epsilon}_R \cdot \partial \underline{E} / \partial t =$$

$$= \partial (\hat{\underline{E}}^* \cdot \underset{\sim}{\epsilon}_R \cdot \hat{\underline{E}}) \tag{6.49}$$

and

$$2 \mathrm{Im}\, \hat{\underline{E}}^* \cdot \hat{\underline{D}} = 2 \mathrm{Im} (\hat{\underline{E}}^* \cdot \underset{\sim}{\epsilon}_I \cdot \hat{\underline{E}}) -$$

$$- (\partial / \partial \underset{\sim}{x}) \cdot [\partial / \partial \underline{k} (\hat{\underline{E}}^* \cdot \underset{\sim}{\epsilon}_R \cdot \hat{\underline{E}})] +$$

$$+ (\partial / \partial t) (\partial / \partial \omega (\hat{\underline{E}}^* \cdot \underset{\sim}{\epsilon}_R \cdot \hat{\underline{E}}) \tag{6.50}$$

so (6.41) becomes

$$\nabla \cdot \underline{S} + \partial W_w / \partial t =$$

$$= -(\omega/4\pi)\,\mathrm{Im}\left(\hat{\underline{E}}^{*} \cdot \underset{\approx}{\epsilon} \cdot \hat{\underline{E}}\right) = -(\omega/4\pi)\,\mathrm{Im}\left(\hat{\underline{E}}^{*} \cdot \underline{D}\right) \tag{6.51}$$

where the energy density is

$$W_w = \left[|\hat{\underline{B}}|^2 + \hat{\underline{E}}^{*} \cdot \partial(\omega \underset{\approx}{\epsilon}_R)/\partial\omega \cdot \hat{\underline{E}} \right] / 8\pi +$$

$$+ K_0 \tag{6.52}$$

and

$$\underline{S} = (c/4\pi)\left(\hat{\underline{E}} \times \hat{\underline{B}}^{*} + c.c.\right) + \omega \frac{\partial}{\partial \underline{k}}\left(\hat{\underline{E}}^{*} \cdot \underset{\approx}{\epsilon}_R \cdot \hat{\underline{E}}\right) \tag{6.53}$$

We see that our earlier result, (6.28), for W_w is a special case of (6.52) and that for an electrostatic wave the right side of (6.51) reduces to $2\gamma_L W_w$ with γ_L given by (6.34).

The right-hand side of (6.41) takes into account the fact that the imaginary (or non-Hermitian) part of $\underset{\approx}{\epsilon}$ corresponds to the presence of dissipation, so instead of energy conservation we have an energy balance equation, expressing how the change of W_w with time, and the flux of energy, represented by $\underline{S}$, balance the dissipative loss. The $\underline{S}$ term can be important when we make Galilean transformations to moving frames, since it is only the combination $\partial W_w/\partial t + \nabla \cdot \underline{S}$ which is Galilean invariant.