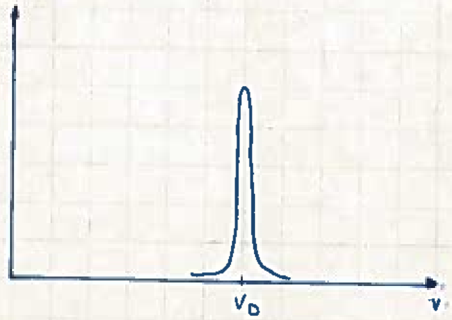
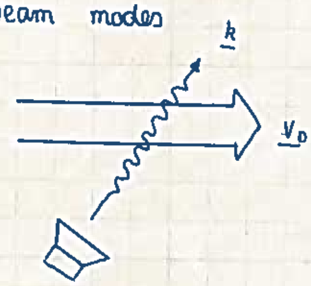
- <u>Beam Modes</u>

  - if $\bar{v}_D \ll v_D \longrightarrow f(v) \longrightarrow \delta^3(\underline{v} - \underline{v}_D)$



  - Beam modes



<u>Antenna</u> (Lab)

- <u>Linearized Equations</u>

  - $$m\left[\frac{\partial}{\partial t}\tilde{\underline{v}} + \underline{V}_0 \cdot \nabla_x \cdot \tilde{\underline{v}}\right] = -q\nabla\phi$$

  - $$\frac{\partial}{\partial t}\tilde{n} + \nabla \cdot \left[n_0 \tilde{\underline{v}} + \underline{v}_0 \tilde{n}\right] = 0$$

- Now, assume plane wave solution for. $\tilde{\underline{v}}, \tilde{n}, \phi \sim \exp\left[i(\underline{k}\cdot\underline{r} - \omega t)\right]$

- Cont. equation.

$$i\left[\underline{k}\cdot\underline{v}_0 - \omega\right]\tilde{n} = n_0\, i\, \underline{k}\cdot\tilde{\underline{v}}$$

$$\tilde{\underline{v}} = \frac{q\underline{k}\phi}{m\left[\omega - \underline{k}\cdot\underline{V}_0\right]}$$ (force equation)

$$\tilde{n} = \frac{n_0\, q k^2 \phi}{m\left[\omega - \underline{k}\cdot\underline{V}_0\right]^2}$$

- Poisson's eq.

$$\nabla^2\phi = 4\pi e\,\tilde{n}$$

$$[-4\pi q]$$

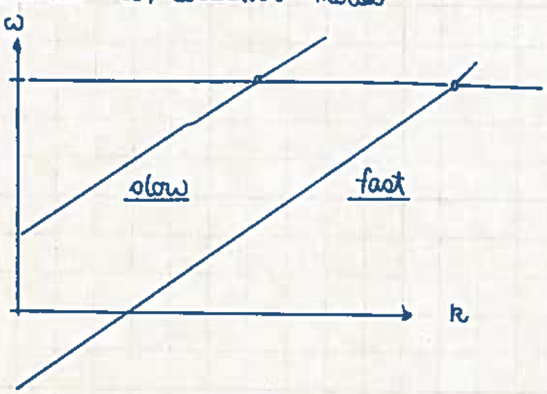
$$k^2 \phi = \frac{4\pi n_0 q^2 k^2 \phi}{m\left[\omega - \underline{k}\cdot\underline{v}_0\right]^2}$$

$$\left[1 - \frac{\omega_p^2}{(\omega - \underline{k}\cdot\underline{v}_0)^2}\right] k^2 \phi = 0$$

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{(\omega - \underline{k}\cdot\underline{v}_0)^2}$$

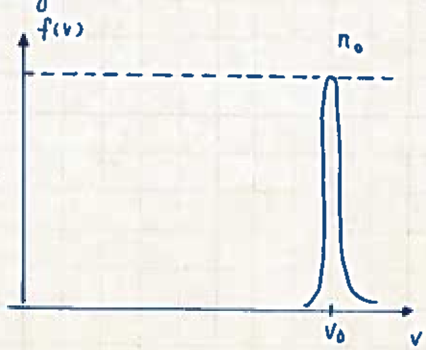
$$\omega = \pm\, \omega_p + \underline{k}\cdot\underline{v}_0$$

- Interaction w/ collective modes



∘ ⟶ mode coupling

- Buneman instability.



$$\boxed{\epsilon(\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega - \underline{k}\cdot\underline{v}_0)^2}}$$

- Resonance condition.

$$\underline{k}\cdot\underline{v}_0 - \omega_{pe} = \mathcal{O}(\omega_{pi})$$

$$\left[\underbrace{(\omega - \underline{k}\cdot\underline{v}_0)^2}_{\mathcal{O}(\omega_{pe}^2)} - \omega_{pe}^2\right] = \frac{\omega_{pi}^2}{\omega^2}(\omega - \underline{k}\cdot\underline{v}_0)^2$$

$$\underline{k}\cdot\underline{v}_o = \mathcal{O}(\omega_{pe})$$

$$\omega = \mathcal{O}(\omega_{pi})$$

$$-\underline{k}\cdot\underline{v}_o + \omega_{pe} \approx \omega_{pi}$$

$$-\left[(\omega - \underline{k}\cdot\underline{v}_o) \mp \omega_{pe}\right] \approx -2\omega_{pe} \longrightarrow \text{idler.}$$

$$-\left[\omega - (\underline{k}\cdot\underline{v}_o - \omega_{pe})\right] \longrightarrow \text{strongly Coupled Mode.}$$

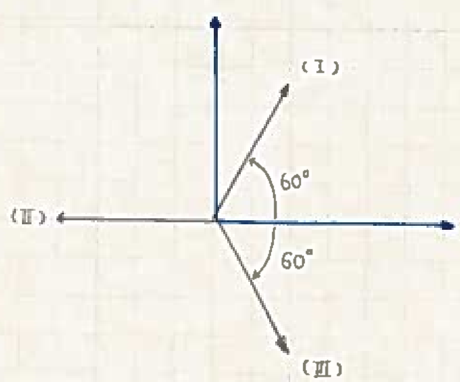
$$\omega^2\left[\omega - (\underline{k}\cdot\underline{V}_o - \omega_{pe})\right] = -\frac{\omega_{pi}^2\,\omega_{pe}}{2}$$

- Introduce Scaled freq.
  - $\omega_s = \left[\dfrac{\omega_{pi}^2\,\omega_{pe}}{2}\right]^{1/3}$
  - $W = \dfrac{\omega}{\omega_s}$
  - $y = \dfrac{(\underline{k}\cdot\underline{V}_o - \omega_{pe})}{\omega_s}$

$$W^2\left[W - y\right] = -1$$

- Consider. $y = 0 \longrightarrow$ Max. Growth Rate.

$$W^3 = -1$$



- (I) $\longrightarrow$ $\mathrm{Im}(\omega) > 0$, i.e., the root is unstable.
- (II) $\longrightarrow$ $\boxed{\omega = \omega_s}$ oscillatory root.
- (III) $\longrightarrow$ $\mathrm{Im}(\omega) < 0$, damped root.

- @ $y = 0$.
  - $Re(\omega) = \omega_s \cdot \cos\left(\frac{\pi}{3}\right)$
  - $Im(\omega) = \omega_s \sin\left(\frac{\pi}{3}\right)$

  $Im(\omega) > Re(\omega) \longrightarrow$ Purely Growing Mode.

- fastest growing mode.

$$W^2 [W - y] = -1$$

$$2W \frac{d}{dy} W \left[W - y \right] + W^2 \left[\frac{dW}{dy} - 1 \right] = 0$$

- Max Growth Rate.

$$\frac{d}{dy} (Im(W)) = 0$$

$$\longrightarrow \boxed{y = 0}$$

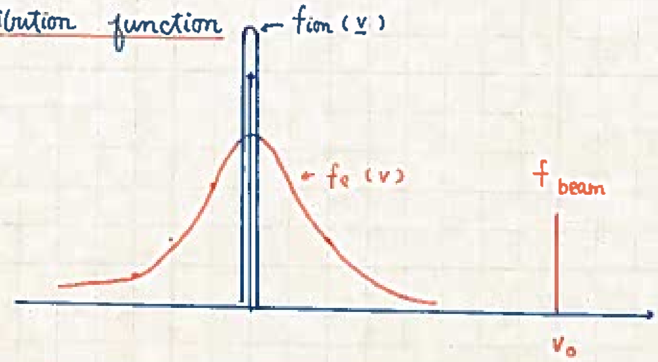
Ph 222 A

11/25

Today
- Hydrodynamic Instabilities

- Cold Beam Instabilities

  - distribution function



$$\boxed{\begin{array}{l} \bullet\ \text{fast wave} \longrightarrow \text{ions do not participate in this instability.} \\ \quad \longrightarrow \text{assume a } \delta\text{-function for the ions} \\ \bullet\ |\underline{v_b}| >> \bar{v}_e \longrightarrow \text{approximate background as "cold"} \\ \bullet\ \dfrac{n_b}{n_0} << 1 \quad (\text{order of } 10^{-2} \text{ to } 10^{-3}) \end{array}}$$

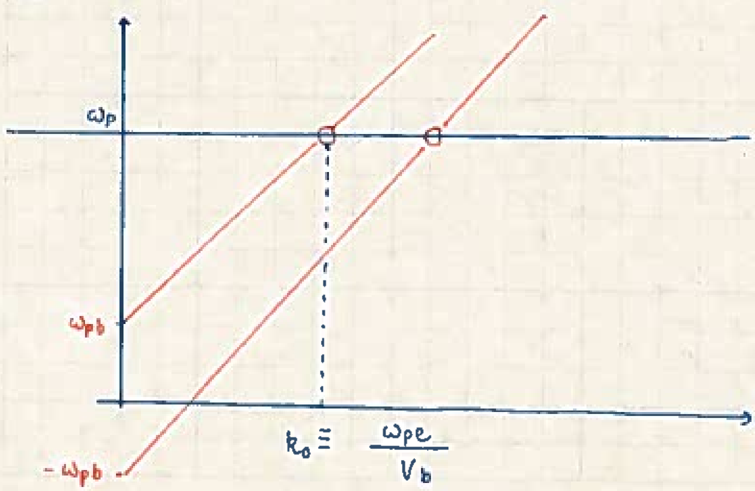
- 1-D geometry, i.e., beam travels parallel to the collective mode.

- Dispersion relation :

$$\boxed{1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{be}^2}{(\omega - kv)^2} = 0}$$

- Consider absolute instability.

- Physical Interpretation



In this case, the waves shifts "up" in order to interact with the electron plasma wave $\boxed{\text{(collective mode)}}$.

- Repeat resonance analysis.
  - Look @ physical picture, we assume.

$$\boxed{\begin{aligned} \omega &= \omega_{pe} + \delta\omega \\ k &= k_0 + \delta k \\ k_0 &= \frac{\omega_{pe}}{V_0} \end{aligned}}$$

- $1 - \dfrac{\omega_{pe}^2}{\omega^2} = \dfrac{\omega_{pb}^2}{(\omega - kv)^2}$ becomes

$$\boxed{\frac{(\omega - \omega_{pe})\,(\omega + \omega_{pe})}{\omega_{pe}^2} = \frac{\omega_{pb}^2}{(\delta\omega - \delta k\, v_0)^2}}$$

$$\rightarrow \quad \delta\omega \left[\delta\omega - \delta k\, v_0 \right]^2 = \frac{\omega_{pb}^2\,\omega_{pe}}{2}.$$

- Introduce scaling.
  - $\omega_s \equiv \left[\dfrac{\omega_{pb}^2\,\omega_{pe}}{2} \right]^{1/3} = \underbrace{\left[\dfrac{n_b}{\frac{\gamma}{2} n_0} \right]^{1/3}}_{\eta^{1/3}} \omega_{pe}^2$

$$\boxed{\omega_s = \eta^{1/3}\omega_{pe}}$$

  - Also define. $\boxed{k_s \equiv \eta^{1/3} k_0}$
  - $W = \dfrac{\delta\omega}{\omega_s}$, $y = \dfrac{\delta k}{k_s}$

$$\boxed{W\,(W - y)^2 = 1}$$

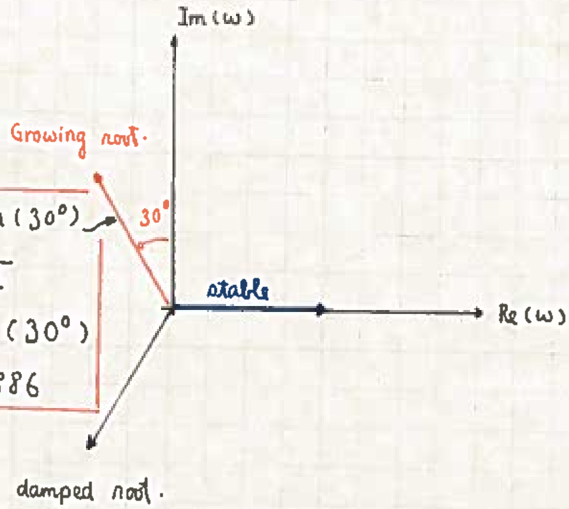
- Fastest growing root $\longrightarrow$ $y = 0$

$$W^3 = 1.$$

$$W = e^{[i2n/3 \cdot \pi]} \qquad n = 0, 1, 2$$

$Re(\omega) = -\sin(30^\circ)$
$= -\frac{1}{2}$

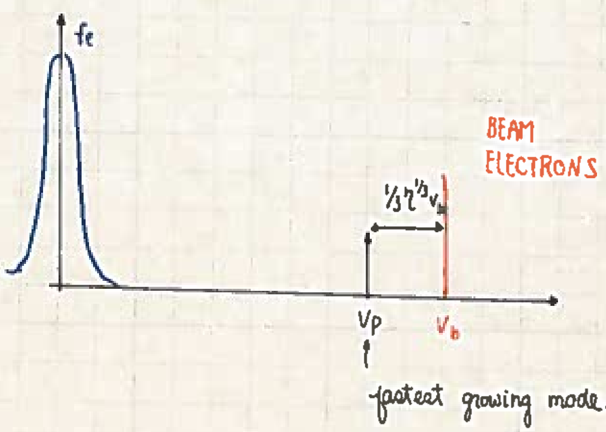
$Im(\omega) = \cos(30^\circ)$
≈ 0.886



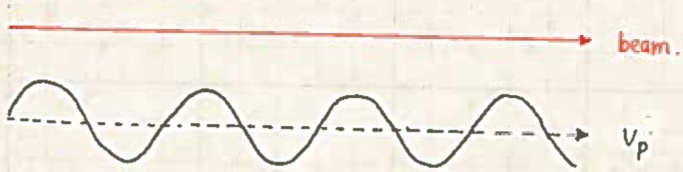
- What is the phase velocity of fastest growing root.
  - $U_p = \frac{Re(\omega)}{k}$.
  - $\delta k = 0$
  - $\delta\omega = \omega_s \cdot (-\sin(30^\circ))$

$\longrightarrow \quad v_p = (1 + \eta^{1/3} Re(W))\, v_b$

$$\boxed{\bullet \quad v_p = \left(1 - \frac{1}{2}\eta^{1/3}\right) v_b}$$



- How much energy is available to the wave?
  - In the wave frame



$$\frac{\text{Wave Energy}}{\text{Volume}} \lesssim \frac{1}{2} n_b m \underbrace{\left(\frac{1}{\sqrt[3]{2}}\eta^{1/3} v_b\right)^2}_{\Delta V^2} \lesssim \frac{1}{2} n_b m_e v_b^2 \left(\frac{\eta}{4}\right)^{2/3}$$

$$\boxed{\frac{\Delta V}{V} \lesssim \frac{\eta^{2/3}}{4}} \longrightarrow \text{efficiency of the instability.}$$

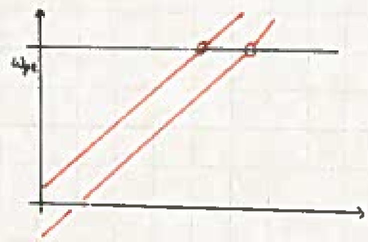
- So, even @ $\eta = 1$ (large beam)

$$\boxed{\frac{\Delta V}{V} < 25\ \%}$$

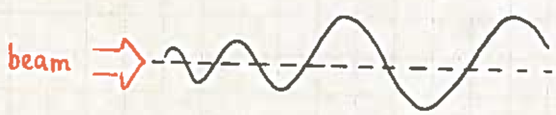
- gets into trouble when we speak of convective instability
  → relevent for experimental situations

- We've solved for absolute instability
  → what about convective instability

  - We get into trouble because the collective mode is not moving



  - Convective inst.



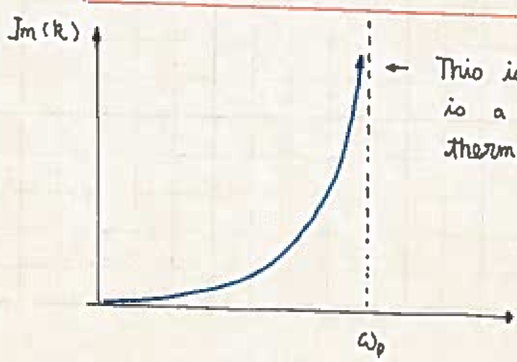
• real ω
• complex k

  - dispersion relation is.

  - $$1 - \frac{\omega_{pe}^2}{\omega^2} = \frac{\omega_{pb}^2}{(\omega - k v_b)^2}$$

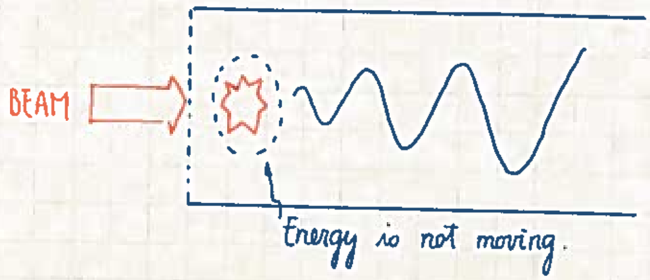
  - $$k v_b - \omega = \pm \frac{1}{\sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}}$$

  - $$\boxed{k = \frac{\omega}{V_b} \pm \frac{\omega_{pb}}{V_b} \cdot \frac{1}{\sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}}}$$

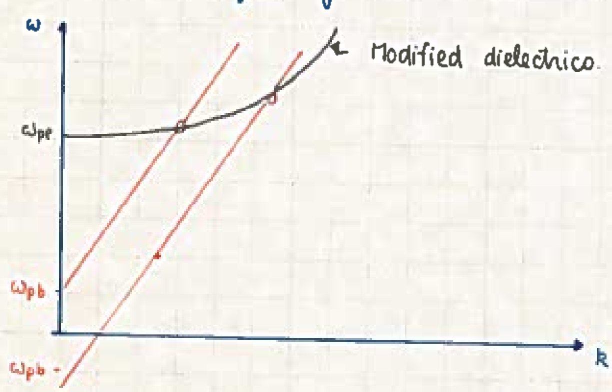


← This is nonsense because this is a failure to include thermal effects.

- Physically, we have



- Resonant analysis for dispersive modes.



- $$\boxed{\epsilon_b(k,\omega) = \frac{\omega_{pb}^2}{(\omega - kv)^2}}$$ ⟶ ignore "1" because vacuum is shared by plasma + beam.

- now, our problem becomes

$$\boxed{\begin{aligned} \omega &= \omega_0 + \delta\omega \\ k &= k_0 + \delta k \end{aligned}}$$

- $\epsilon(\omega, k) = \epsilon(\omega_0 + \delta\omega,\ k_0 + \delta k)$

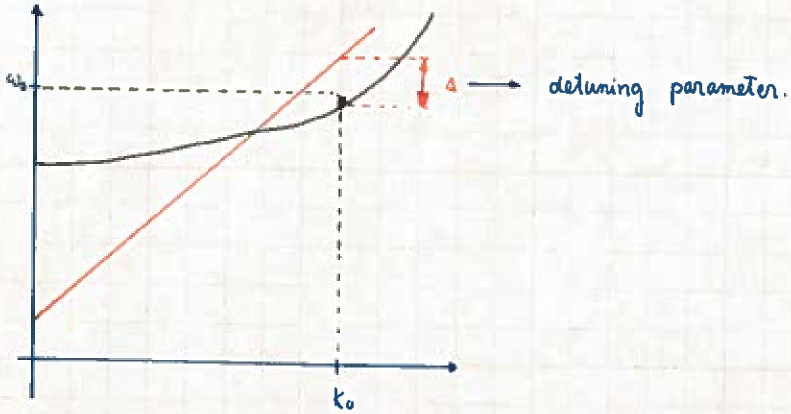
$$= \boxed{\cancel{\epsilon(\omega_0, k_0)}} + \left(\frac{\partial \epsilon}{\partial \omega}\right)\delta\omega + \left(\frac{\partial \epsilon}{\partial k}\right)\delta k + \underbrace{\mathcal{O}(\delta^2)}_{\text{neglect...}}$$

- DEFINES RESONANCE.

- $$\left[\left(\frac{\partial \epsilon}{\partial k}\right)\delta k + \left(\frac{\partial \epsilon}{\partial \omega}\right)\delta\omega\right] = \frac{\omega_{pb}^2}{\left[\omega_0 + \delta\omega - k_0 v_b - \delta k\, v_b\right]^2}$$

- define detuning parameter.

$$\boxed{\omega_0 - k_0 v = \Delta}$$



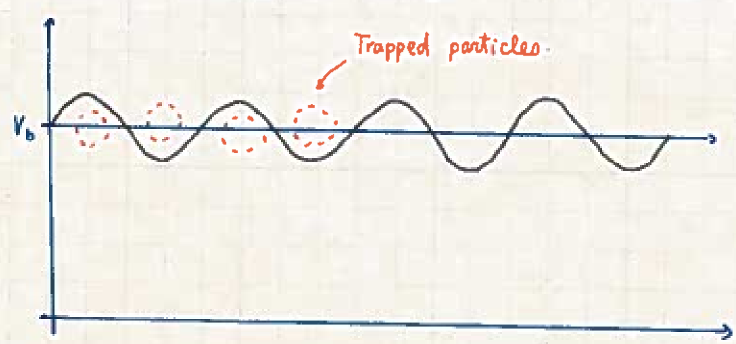
$$\left[\frac{\delta\omega}{V_b} - \delta k + \frac{\Delta}{V_b}\right]^2 \left[\frac{\partial\epsilon}{\partial k}\,\frac{\delta k}{V_{gb}} + \frac{\partial\epsilon}{\partial\omega}\,\frac{\delta\omega}{V_{gb}}\right] = \frac{\omega_{pb}^2}{V_b^3}$$

$$\boxed{\left[\frac{\delta\omega}{V_b} - \delta k + \frac{\Delta}{V_b}\right]^2 \cdot \left[\underbrace{\frac{\left(\frac{\partial\epsilon}{\partial\omega}\right)}{\left(\frac{\partial\epsilon}{\partial k}\right)}}_{V_g}\,\frac{\delta k}{V_b} + \frac{\delta\omega}{V_b}\right] = \frac{\omega_{pb}^2}{\left(\frac{\partial\epsilon}{\partial\omega}\right) V_b^3}}$$

- $$\left[\frac{\delta\omega}{V_b} - \delta k + \frac{\Delta}{V_b}\right]^2 \cdot \left[\frac{V_g}{V_b}\,\delta k + \delta\omega \cdot \frac{1}{V_b}\right] = \frac{\omega_{pb}^2}{\left[\frac{\partial\epsilon}{\partial\omega}\right] V_b^3}$$

- DEFINE : $$\boxed{\eta^3 = \frac{\omega_{pb}^2}{\left[\frac{\partial\epsilon}{\partial\omega}\right] V_b^3}}$$

- REF : Oneil, Winfrey, Malmberg. Phys. Fluid 14 (1204) (1971)



- REF : Gentle, Lohr 16, 1464 (1973)
  - convective inst.