II. Fundamental Theoretical Formulation; the Microscopic Kinetic Equations

A. Maxwell's and Newton's Equations

We start from the simplest and most basic description of a collection of charged particles: Maxwell's equations for the electromagnetic field and Newton's equation (or its quantum mechanical equivalent, Schrodinger's equation) for the particle motion. The sources of the electromagnetic field are any external charges and currents plus those due to the plasma particles.

We shall adopt a classical point of view here, using Newton's equation for the mechanical motion and treating the electrons and ions as point particles. So long as the deBroglie wavelength, $(2\pi\hbar^2/mT)^{\frac{1}{2}}$, for each particle is much smaller than any other length in the problem, this is a good approximation. In the high density/low temperature regime where quantum mechanical effects become important, the details of the formulation must be appropriately modified for the mechanical motion and, more importantly, other expansion parameters replace ϵ_p. This regime is outside the scope of this text.

Our basic equations are then

$$m_i\dot{\underline{v}}_i = m_i\ddot{\underline{x}}_i = q_i(\underline{\mathcal{E}} + \underline{v}_i\times\underline{B}/c) \qquad i = 1,2, \ldots N_0 \tag{2.1}$$

$$\begin{aligned} \nabla\cdot\underline{\mathcal{E}} &= 4\pi(\rho_p + \rho_e) \qquad & \nabla\cdot\underline{\mathcal{E}} + \dot{\mathcal{B}}/c &= 0 \\ \nabla\times\underline{\mathcal{B}} &= 4\,(\underline{j}_p + \underline{j}_e + \dot{\underline{\mathcal{E}}}/c) \qquad & \nabla\cdot\mathcal{B} &= 0 \end{aligned} \tag{2.2}$$

where the subscripts p,e denote plasma particle and external source terms, respectively, and N_0 is the total number of particles. Since this is a microsopic theory, we need only one electric field vector, $\underline{\mathcal{E}}$ and one magnetic field vector, which we designate by $\underline{\mathcal{B}}$. All "material" effects are in the charge and current densities,

$$\rho_p(\underline{x},t) = \sum_1^{N_0} \delta[\underline{x} - \underline{x}_i(t)]\,q_i$$

and

$$\underline{j}_p(\underline{x},t) = \sum_1^{N_0} q_i\underline{v}_i(t)\delta[\underline{x} - \underline{x}_i(t)] \tag{2.3}$$

and since we shall be examining these in great detail, there is no particular advantage in introducing the auxiliary fields $\underline{D}$ and $\underline{H}$.

B. The Microscopic Distribution Function and Kinetic Equation

Our study of plasma physics is based entirely on the set of equations (2.1), (2.2), (2.3). They are exact, but, of course unsolvable, since the total number of particles, N_0, is of order 10^6 or larger. It is convenient to reformulate (2.1) and (2.3) by introducing the concept of the microscopic distribution function of Klimontovich[4]. For each species, α, we define

$$\mathcal{F}_\alpha(\underline{x},\underline{v},t) = \bar{n}_\alpha^{-1} \sum_{i=1}^{N_\alpha} \delta[\underline{x}-\underline{x}_i(t)]\delta[\underline{v}-\underline{v}_i(t)] \tag{2.4}$$

where N_α is the total number of particles of this species and n_α is their average density, $\bar{n}_\alpha = N_\alpha/V$, V being the volume of the system. In terms of the $\mathcal{F}_\alpha$, we have instead of (2.3)

$$\begin{aligned} \rho_p(\underline{x},t) &= ⨍ d\underline{v}\bar{n}\mathcal{F}(\underline{x},\underline{v},t)q \\ \underline{j}_p(\underline{x},t) &= ⨍ d\underline{v}\bar{n}\underline{v}\,\mathcal{F}(\underline{x},\underline{v},t)q \end{aligned} \tag{2.5}$$

where the symbol $⨍$ means integration over $\underline{v}$ and summation over the species index, α. (Note that in (5) we have suppressed the species index, as we shall generally do in order to minimize notational clutter.) Our choice of normalization, whose advantages will be apparent later, gives

$$\int d\underline{x}d\underline{v}\,\mathcal{F} = V \quad .$$

Using the equations of motion, (2.1) and the definition (2.4) of $\mathcal{F}$, we have

$$\frac{\partial \mathcal{f}}{\partial t}(\underline{x},\underline{v},t) = -\bar{n}^{-1}\sum_{1}^{N}[\underline{v}_i\cdot\nabla\delta(\underline{x}-\underline{x}_i)\delta(\underline{v}+\underline{v}_i) + \dot{\underline{v}}_i\cdot\nabla_v\delta(\underline{v}-\underline{v}_i)\delta(\underline{x}-\underline{x}_i)] =$$

$$= -\underline{v}\cdot\partial\mathcal{f}/\partial\underline{x} - (q/m)(\underline{\mathcal{E}} + \underline{v}\times\underline{\mathcal{B}}/c)\cdot\partial\mathcal{f}/\partial\underline{v}$$

where we take advantage of the delta functions to replace $\underline{v}_i$ by $\underline{v}$ in the first term and $\dot{\underline{v}}_i$ by $(q/m)(\underline{\mathcal{E}}+\underline{v}\times\underline{\mathcal{B}}/c)$ in the second, $\underline{\mathcal{E}}$ and $\underline{\mathcal{B}}$ being the fields at $\underline{x}$. (We have also suppressed the species indices.) Thus, we have replaced the equation of motion (2.1) by the microscopic kinetic equation

$$\boxed{\partial\mathcal{f}/\partial t + \underline{v}\cdot\partial\mathcal{f}/\partial\underline{x} + (q/m)(\underline{\mathcal{E}} + \underline{v}\times\underline{\mathcal{B}}/c)\cdot\partial\mathcal{f}/\partial\underline{v} = 0} \qquad (2.6)$$

This equation is deceptively simple in appearance, involving only 6 independent phase space variables plus the time. However, it is nonlinear and, even more important, $\mathcal{f}$ is stochastic, i.e. a wildly varying function of $\underline{x}$, $\underline{v}$, and t, due to the delta functions involved in its definition. Naturally, none of the many-body complexity of (2.1) has been eliminated by our reformulation.

C. Ensemble Averages and Fluctuations

As in all statistical mechanical problems, an exact formulation is not only insoluble, but, even if solved, would be of little value. Of principal physical interest are the average properties and the fluctuations about the mean values. We therefore introduce the usual statistical ensemble: many (conceptual) copies of the physical system, having the same observable properties but different microscopic parameters, e.g. the initial particle positions and velocities. We denote the ensemble averages by

$$f = \langle\mathcal{f}\rangle \qquad \underline{E} = \langle\underline{\mathcal{E}}\rangle \qquad \underline{B} = \langle\underline{\mathcal{B}}\rangle \qquad (2.7)$$

No special notation is required for the external sources, since these will be the same for each member of the ensemble. Thus

$$\begin{aligned} \langle\rho\rangle &= \rho_e + \int d\underline{v}\ \bar{n}f \\ \langle\underline{j}\rangle &= \underline{j}_e + \int d\underline{v}\ \bar{n}\underline{v}f \end{aligned} \tag{2.8}$$

The ensemble averages $\underline{E},\underline{B}$ are to be identified with the physical electromagnetic fields measured in an experiment. The averaged distribution function, f, has just the significance of the single particle distribution function of classical kinetic theory: $nf(\underline{x},\underline{v},t)$ is the number of particles, per unit phase space volume, which would be found at time t in the vicinity of the phase space point $(\underline{x},\underline{v})$. We shall sometimes use ξ to denote the set $(\underline{x},\underline{v})$. Thus, f, $\underline{E}$ and $\underline{B}$ are the principal physically significant dynamical variables. Next most important are fluctuations about these ensemble averages,

$$\delta\mathcal{F} = \mathcal{F} - f \qquad \delta\underline{\mathcal{E}} = \underline{\mathcal{E}} - \underline{E} \qquad \delta\underline{\mathcal{B}} = \underline{\mathcal{B}} - \underline{B} \tag{2.9}$$

where, by definition,

$$\langle\delta\mathcal{F}\rangle = \langle\delta\underline{\mathcal{E}}\rangle = \langle\delta\underline{\mathcal{B}}\rangle = 0 \tag{2.10}$$

It will often simplify the notation to denote the total electromagnetic force by

$$\underline{\tilde{\mathcal{E}}} \equiv \underline{\mathcal{E}} + \underline{v} \times \underline{\mathcal{B}}/c$$

with

$$\tilde{E} = \langle\tilde{\mathcal{E}}\rangle, \qquad \delta\tilde{E} = \underline{\tilde{\mathcal{E}}} - \underline{\tilde{E}} \tag{2.11}$$

Taking the ensemble average of (2.6) gives an equation for the single particle distribution function, f,

$$\partial f/\partial t + \underline{v}\cdot\nabla f + (q/m)\underline{\tilde{E}}\cdot\partial f/\partial\underline{v} = -(q/m)\ \langle\delta\underline{\tilde{\mathcal{E}}}\cdot\partial\delta\mathcal{F}/\partial\underline{v}\rangle \tag{2.12}$$

and subtracting this from (2.6) gives an equation for the fluctuations,

$$(\partial/\partial t + \underline{v}\cdot\nabla + (q/m)\underline{\tilde{E}}\cdot\nabla_v)\delta\mathcal{F} + (q/m)\delta\underline{\tilde{\mathcal{E}}}\cdot\nabla_v f = -(q/m)\nabla_v\cdot[\delta\underline{\tilde{\mathcal{E}}}\delta\mathcal{F} - \langle\delta\underline{\tilde{\mathcal{E}}}\ \delta\mathcal{F}\rangle] \tag{2.13}$$

Since Maxwell's equations are linear, their partition into average and fluctuation parts is trivial:

$$\begin{aligned} \nabla\cdot\underline{E} &= 4\pi(\rho_e + \int d\underline{v}\bar{n}qf) & \nabla\times\underline{E} + \dot{\underline{B}}/c &= 0 \\ \nabla\times\underline{B} &= 4\pi c^{-1}(\underline{j}_e + \int d\underline{v}\bar{n}q\underline{v}f) + \dot{E}/c & \nabla\cdot\underline{B} &= 0 \end{aligned} \tag{2.14}$$

$$\begin{aligned} \nabla\cdot\delta\underline{\mathcal{E}} &= 4\pi\int d\underline{v}\bar{n}q\delta \mathcal{f} & \nabla\times\delta\,\underline{\mathcal{E}} + \delta\,\dot{\underline{\mathcal{B}}}/c &= 0 \\ \nabla\times\delta\underline{\mathcal{B}} &= 4\pi c^{-1}(\int d\underline{v}\bar{n}q\underline{v}\delta \mathcal{f}) + \delta\,\dot{\underline{\mathcal{E}}}/c & \nabla\cdot\delta\underline{\mathcal{B}} &= 0 \end{aligned} \tag{2.15}$$

Again, we are simply making definitions, and the set (2.12), through (2.15) is identical, as regards both contents and difficulty, with the original set (2.1) through (2.3). However, the problem is now formulated in a way to facilitate the approximations necessary if we are to make any progress. We see that (2.12) and (2.14) would constitute a closed set of equations for the ensemble averages f, $\underline{E}$, $\underline{B}$ if only we knew enough about the fluctuation to compute the average value $\langle\delta \mathcal{f}\delta\,\tilde{\underline{\mathcal{E}}}\rangle$ which occurs on the right side of (2.12). On the other hand, if we knew f, $\underline{E}$, and $\underline{B}$ we need only solve the equations (2.13) and (2.15) for the fluctuations (a task made formidable, of course, by their nonlinear character). At the very least, some approximation scheme to decouple the fluctuations from the ensemble averages would be helpful.

D. The Expansion in Fluctuations

The rationale of the method of approximation which we shall use is very simple: we suppose the fluctuations to be, in some sense, "small" and therefore expand in the fluctuations. Specifically, to lowest order, we neglect terms of second order in the fluctuations, such as the r.h. side of (2.12); this completely decouples (2.12) and (2.14) from (2.13) and (2.15).

Thus, to lowest order (we shall call it first order, since the next order involves retention of terms quadratic in the fluctuations) we have the correlationless kinetic equation, plus the ensemble averaged Maxwell equations,

$$\boxed{\begin{array}{ll} \mathcal{L}f \equiv (\partial/\partial t + \underline{v}\cdot\nabla + (q/m)\tilde{\underline{E}}\cdot\nabla_v)f = 0 & \\ \nabla\cdot\underline{E} = 4\pi(⨍ d\underline{v}\bar{n}qf + \rho_e) & \nabla\times\underline{E} + \dot{\underline{B}}/c = 0 \\ \nabla\times\underline{B} = 4\pi c^{-1}(⨍ d\underline{v}\bar{n}q\underline{v}f + \underline{\ \ }_e) + \dot{\underline{E}}/c & \nabla\cdot\underline{B} = 0 \end{array}} \qquad \begin{array}{r}(2.16)\\ \\ (2.17)\end{array}$$

This set of equations was first written down, on phenomenological grounds, by A. Vlasov and is generally referred to by his name, although the misnomer "collisionless Boltzmann equation" is sometimes used. We shall refer to this lowest order of the expansion as the Vlasov approximation.

In second order we retain the quadratic terms, $\langle\delta\underline{\mathcal{E}}\delta\mathcal{F}\rangle$ on the right side of (2.12) but neglect terms of third order in the fluctuations. When (2.13) and (2.15) are solved for $\delta\mathcal{F}$ and $\delta\dot{\underline{\mathcal{E}}}$, the contributions of the right side of (2.13) will lead to terms of third order in (2.12), so in this order we can neglect the right hand side of (2.13) from the start. Thus to second order we have

$$\boxed{\begin{array}{l} \mathcal{L}f = -(q/m)\nabla_v \cdot \langle\delta\mathcal{F}\,\delta\,\tilde{\underline{\mathcal{E}}}\rangle \\ \mathcal{L}\delta\mathcal{F} + (q/m)\delta\,\tilde{\underline{\mathcal{E}}}\cdot\nabla_v f = 0 \end{array}} \qquad \begin{array}{r}(2.18)\\ (2.19)\end{array}$$

plus the Maxwell equations (2.14) and (2.15) for the self-consistent determination of $\tilde{\underline{E}}$ and $\delta\,\tilde{\underline{\mathcal{E}}}$. We shall designate this as the quasilinear approximation since (2.19) is linear in $\delta\mathcal{F}$, albeit nonlinear terms are retained in equation (2.18) for f. The fluctuations modify the average distribution, f, but interactions among the fluctuations, such as mode coupling, are neglected.

Finally, in third order we retain the right hand side of (2.13), so that the third order equations are formally identical with the exact equations, (2.12) through (2.15). As we shall see later, the approximation consists in solving (2.12) and (2.13) by a perturbation expansion, keeping only terms of third order in the fluctuations. (Similarly, we could go on to fourth or higher orders, but these remain largely unexplored at the present time.) Only in this order do we have mode coupling of fluctuations, nonlinear wave-particle interactions, self interaction of large amplitude waves and similar exotic phenomena, so we may describe it as the <u>nonlinear wave approximation</u>.

For a plasma in equilibrium, one can prove that this "expansion in fluctuations" is tantamount to an expansion in the plasma parameter, ϵ_p, and hence well justified if $\epsilon_p << 1$. However, the most interesting problems in plasma physics involve non-equilibrium phenomena, where this expansion procedure can really be justified only on an a posteriori basis, for each problem.

To avoid confusion with other treatments, we should emphasize that even within the Vlasov approximation one may make an expansion in the fields $\underline{E}$ and $\underline{B}$, (or in their deviation from the values characterizing some elementary solution) and hence encounter equations of "second order" or "third order" in the fields. These equations will be formally similar to those describing what we have called the quasilinear and nonlinear wave approximations, simply because of the obvious <u>formal</u> similarity between the Vlasov and Klimontovich equations. However, the physical interpretations are quite different, since within the Vlasov approximation we deal <u>only</u> with ensemble-averaged quantities, whereas the quasilinear and nonlinear wave approximations involve, in an essential way, stochastic variables. A crude designation of the difference is to describe the nonlinear Vlasov theory as one involving "coherent" waves. The confusion is compounded by the circumstance that in many problems the formal analysis (expansion in diagrams, etc.) may be quite similar, but the distinction between the physical significance is an important one, as we shall see in later chapters.)

E. An Overview

In subsequent chapters, we shall study systematically the consequences of these various orders of approximation. Before doing so, we have a few comments on their general properties.

1. The Vlasov equations show clearly the self-consistent aspect of plasma physics, with f determined by $\tilde{\underline{E}}$, and $\tilde{\underline{E}}$ having sources given partly by f.

2. Most of our understanding of the properties of the Vlasov system is based on a linearization of f about some time and space independent "equilibrium" or "unperturbed" function, $f_0(\underline{v})$, with only terms of first order in $(f-f_0)$ and $\tilde{\underline{E}}$ retained. Since <u>any</u> $f_0(\underline{v})$ satisfies (2.16) when $\tilde{\underline{E}} = 0$, we must look elsewhere for guidance in making a sensible choice of f_0. For this, we need to consider the second order effects. The neglect of fluctuations at the Vlasov level means that of the total force on a given particle we are including only the average part, $\tilde{\underline{E}}$, and ignoring the rapidly fluctuating portions which arise from the discrete, particulate character of the plasma. However, it is just the latter which determine the equilibrium f_0. In fact, the $\langle\delta \mathcal{f} \delta \tilde{\underline{\mathcal{E}}}\rangle$ terms in (2.13) correspond to two physical effects:

a) "Close" collisions (meaning those with impact parameter less than the Debye length, as we shall see later); and

b) "Quasilinear" modifications of the average distribution function by the fluctuations.

As we shall see, for a plasma in equilibrium, where the fluctuations are and remain, small, of order $\mathcal{E}_p$, only the first of these two effects is important and the right hand side of (2.13), which we write as

$$\delta f/\delta t \equiv -\ (q/m)\nabla_v \cdot <\delta f\, \delta \tilde{\underline{\mathcal{E}}}> \tag{2.20}$$

satisfies an H-theorem, i.e. tends to drive f_0 towards a Maxwellian distribution,

$$f_M(\underline{v}) = \exp(-v^2a^2)/a^3\pi^{3/2} \tag{2.21}$$

so we shall often make this choice for f_0.

3. For many purposes, even the Vlasov description is too difficult to solve and we deal instead with the moments of f, i.e. density, n, mean velocity, $\underline{v}$ and pressure tensor, p, as functions of $\underline{x}$ and t. It is easy to derive equations for these quantities from (2.12), albeit the set does not close without further approximations. This leads to "two fluid magnetohydrodynamics", so-called because there are equations and dependent variables for each of the two (or more) species in the plasma. A further approximation, valid at low frequencies and large wavelengths, reduces this to "one fluid mhd". These fluid approximations are clearly justified in the case of high collision frequencies, when the short mean free path tends to preserve the initial grouping of particles. However, they can often give a good account of many phenomena even when their use is not clearly justified, probably because they represent the basic conservations laws--mass, momentum and energy.

4. Within the Vlasov equations, it is often useful to make the "electrostatic approximation," neglecting the Lorentz force, $\underline{v} \times \underline{B}$, part of $\tilde{\underline{E}}$, and similarly for $\delta\tilde{\underline{\mathcal{E}}}$, when we go to second or third order. This represents an enormous simplification for the analysis, but it must be justified in each particular context.

5. So far as an external magnetic field, $\underline{B}_0$, is concerned, the simplest case is, of course, $\underline{B}_0 = 0$, and we shall consider that first in discussing the Vlasov equation. Next simplest is the case of very strong $\underline{B}_0$ (cyclotron frequency >> all other significant frequencies, cyclotron radius << all other significant lengths) when the Alfven guiding center approximation and related techniques which we shall discuss later, are applicable.

6. The relation amongst the various approximations or "models" of the plasma can be summarized in a block diagram:

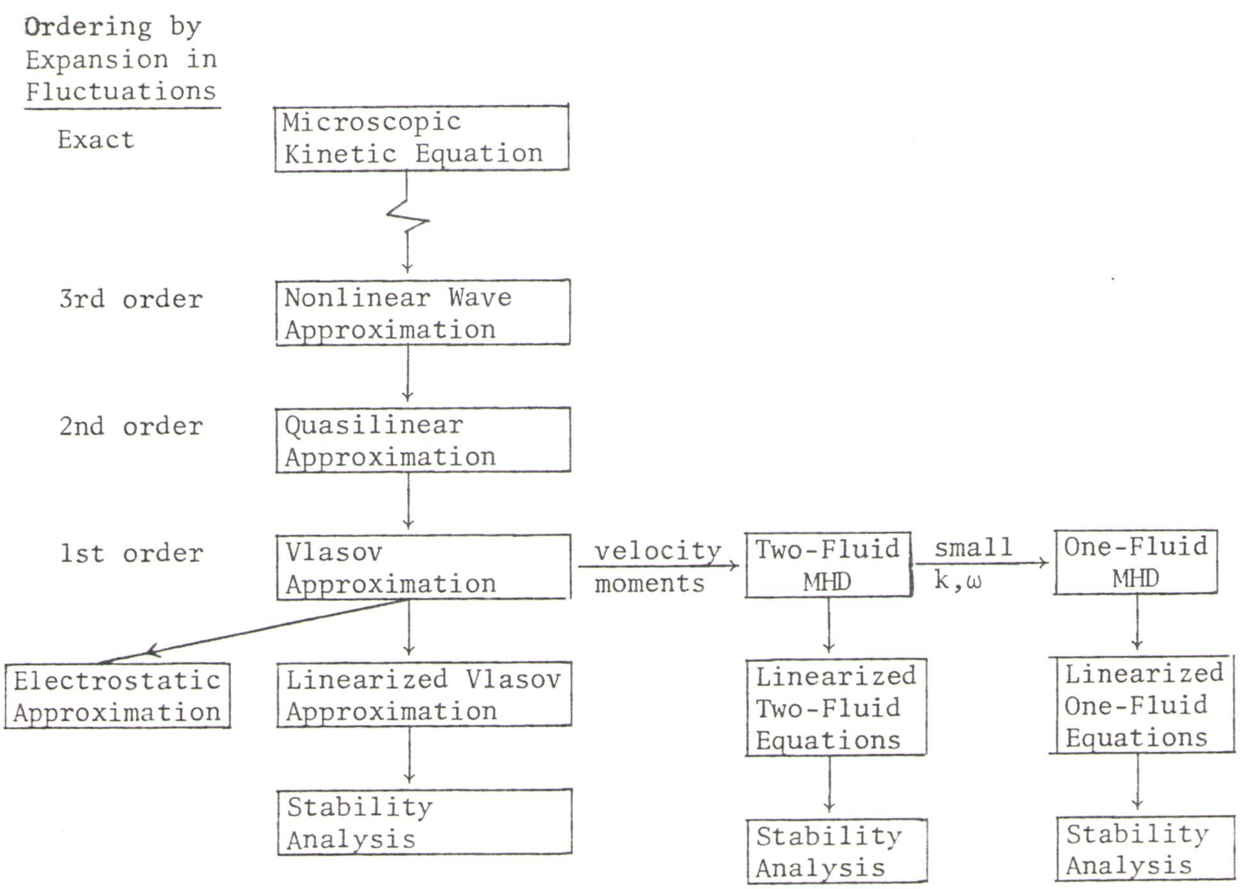

Fig. 1.1

7. Many of the approximations depicted here can be derived directly, on phenomenological grounds. For example, if we simply introduce a density function, $f(\underline{x},\underline{v},t)$ in the six dimensional phase space, then in absence of "collisions" between particles, conservation of particles gives a six dimensional continuity equation,

$$\partial f/\partial t + \nabla \cdot (\underline{v}f) + \nabla_v \cdot (\dot{\underline{v}}f) = 0$$

With external fields $\underline{E}_e$, $\underline{B}_e$,

$$\dot{\underline{v}} = (q/m)(\underline{E}_e + \underline{v} \times \underline{B}_e/c)$$

so

$$\partial f/\partial t + \underline{v}\cdot\nabla f + (q/m)(\underline{E}_e + \underline{v} \times \underline{B}_e/c)\cdot \nabla_v f = 0$$

If we allow the "external" fields to have as sources also the plasma charge and current densities described by f, i.e. replace $\underline{E}_e$ and $\underline{B}_e$ with $\underline{E}$ and $\underline{B}$ satisfying

$$\nabla \cdot \underline{E} = 4\pi(\rho_e + \int d\underline{v}\bar{n}qf)$$

$$\nabla \times \underline{B} = 4\pi c^{-1}(\underline{j}_e + \int d\underline{v}\bar{n}q\underline{v}f) + \dot{E}/c$$

$$\nabla \cdot \underline{B} = \nabla \times \underline{E} + \dot{\underline{B}}/c = 0$$

then we have just the Vlasov formulations. As we will see later, this includes, in the "self-consistent" fields, $\underline{E}-\underline{E}_e$, $\underline{B}-\underline{B}_e$, the particle interactions associated with impact parameter, b, greater than L_D and neglects the "close" collisions, $b < L_D$. It is the latter which are described by the $<\delta \not{f} \delta \underline{\tilde{\xi}}>$ terms neglected in the Vlasov approximation.

In Chapter III we shall derive the two-fluid equations and consider the linear theory in absence of a magnetic field in order to make contact with some elementary plasma phenomena of physical interest and also illustrate certain techniques which we will frequency employ. Following that, we derive in Chapter IV the one-fluid mhd equations and examine the linearized waves which they predict in the presence of an external magnetic field, thus getting our first taste of magnetic effects. We then return to the full Vlasov equations in Chapter V and reexamine some of the phenomena studied in Chapters III and IV in order to understand the differences between the Vlasov and fluid treatments.