equation of the form of Eq. (173) for every species of particle. Also there will be a collision term for each species of particle with which a collision takes place.

XIV. Transfer Equations

A. General Equations

Let Q be some function of the velocities of the particles. For example, it might be one of the components of the momentum, or energy. The average value of Q at any spatial point is given by

$$\overline{Q(\vec{r},t)} = \frac{1}{n(\vec{r},t)} \iiint_{\text{ALL VELOCITIES}} Q\, f(x,y,z,u,v,w,t)\, du\, dv\, dw. \tag{174}$$

Although Q is not a function of $\vec{r}$ and t, and is only a function of $\vec{v}$, $\overline{Q}$ is only a function of $\vec{r}$ and t because f is a function of $\vec{r}$ and t.

We are now interested in the time rate of change of $\overline{Q}$. We have, from Eqs. (174) and (173),

$$\frac{\partial (n\overline{Q})}{\partial t} = \iiint Q \frac{\partial f}{\partial t}\, du\, dv\, dw =$$

$$-\iiint Q\left[(\vec{v}\cdot\vec{\nabla}_r) f + \left(\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]\cdot\vec{\nabla}_v\right) f - \left(\frac{\partial f}{\partial t}\right)_{\text{COLL}}\right] du\, dv\, dw. \tag{175}$$

The first term on the right is

$$\vec{\nabla}_r \cdot \iiint \vec{v}\, Q\, f\, d^3v = \vec{\nabla}_r \cdot n\overline{\vec{v}Q} \tag{176}$$

since $\vec{\nabla}_r$ does not operate on $\vec{v}$. The second term on the right-hand side,

$$-\iiint Q\left(\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]\cdot\vec{\nabla}_v\right)f\,d^3v, \tag{177}$$

can be integrated by parts. Consider the contribution to the integral which comes from the acceleration in the x direction, i.e.,

$$-\iiint Q\left(\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]_x\cdot\frac{\partial f}{\partial u}\,d^3v. \tag{178}$$

Integrating by parts with respect to u, keeping v and w fixed, gives

$$-\iint Q\left[\frac{q}{m}\left[\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right]+\frac{\vec{F}}{m}\right]_x f\Big|_{u=-\infty}^{u=\infty} dv\,dw$$

$$+\iiint\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]_x\frac{\partial Q}{\partial u}f\,du\,dv\,dw$$

$$=\left(\frac{q}{m}\vec{E}+\frac{\vec{F}}{m}\right)_x\cdot n\,\overline{\frac{\partial Q}{\partial u}}+\frac{q}{m}\,n\overline{\left[\left[\frac{\vec{V}\times\vec{B}}{c}\right]_x\frac{\partial Q}{\partial u}\right]} \tag{179}$$

since, as we have seen, $\frac{\partial}{\partial u}\left(\frac{\vec{V}\times\vec{B}}{c}\right)_x = 0.$

Thus we can write Eq. (177) in the form

$$-\iiint Q\left[\frac{q}{m}\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)+\frac{\vec{F}}{m}\right]\cdot\vec{\nabla}_v f\,d^3v$$

$$=\left[\frac{q\vec{E}}{m}+\frac{\vec{F}}{m}\right]\cdot n\,\overline{\nabla_v Q}+\frac{qn}{m}\overline{\left[\left(\frac{\vec{V}\times\vec{B}}{c}\right)\cdot\vec{\nabla}_v Q\right]} \tag{180}$$

the dot product $\left(\frac{\vec{v}\times\vec{B}}{c}\right)\cdot\vec{\nabla}_v Q$ being defined by comparing Eqs. (179) and (180). Then Eq. (175) takes the form

$$\frac{\partial(n\overline{Q})}{\partial t} = -\vec{\nabla}_r \cdot n(\overline{\vec{v}Q}) + \left[\frac{e}{m}\vec{E} + \frac{\vec{F}}{m}\right] \cdot n\overline{\vec{\nabla}_v Q}$$

$$+ \frac{e\,n}{m}\overline{\left[\left(\frac{\vec{v}\times\vec{B}}{c}\right)\cdot\vec{\nabla}_v Q\right]} + \left[\frac{\partial(n\overline{Q})}{\partial t}\right]_{\text{coll.}} \qquad (181)$$

B. Specific Examples of Q

(1) Q = 1, The Continuity Equation

As the simplest nontrivial example of Q we consider $Q = 1$. For this case, Eq. (181) becomes

$$\frac{\partial n}{\partial t} = -\vec{\nabla}_r \cdot (n\overline{\vec{v}}) \qquad (182)$$

if collisions conserve particles. If particles are not conserved in a collision, then Eq. (181) takes the form

$$\frac{\partial n}{\partial t} = -\vec{\nabla}_r \cdot (n\overline{\vec{v}}) + \left(\frac{\partial n}{\partial t}\right)_{\text{coll.}} \qquad (183)$$

This last case arises when we have ionization or recombination. Eq. (182) is the familiar continuity equation, while Eq. (183) is the form this equation takes when we have sources or sinks for particles.

From now on, unless otherwise stated, we shall assume that no ionization or recombination takes place.

(2) $Q = \vec{v}$, Conservation of Momentum

The next simplest function which Q can be is $\vec{v}$. For this case Eq. (181) becomes

$$\frac{\partial(n\overline{\vec{v}})}{\partial t} = -\vec{\nabla}_r \cdot (n\overline{\vec{v}\vec{v}}) - \left[\frac{e}{m}\vec{E} + \frac{\vec{F}}{m}\right] n$$

$$- \frac{e}{m}\, n\, \frac{\overline{\vec{v}}\times\vec{B}}{c} + \left[\frac{\partial}{\partial t}(n\overline{\vec{v}})\right]_{\text{coll.}} \qquad (184)$$

The second term involves the gradient of the momentum transfer dyadic. The physical meaning of this term is described in the next section, where we have made use of the fact that $\vec{A}\,\vec{\nabla}_v \vec{v} = \vec{A}$, or, in terms of dyadics

$$\vec{\nabla}_v \vec{v} = \underline{I} \tag{185}$$

where $\underline{I}$ is the unit dyadic. It should be emphasized that there is one equation of the form (184) for each species of particle. By writing

$$\sigma = \varrho n \tag{186}$$

and

$$\vec{J} = \varrho n \overline{\vec{v}} \tag{187}$$

and by noting that from conservation of momentum collisions with particles of the same species can make no contribution to

$$\left[\frac{\partial(n\overline{\vec{v}})}{\partial t}\right]_{coll}$$

we can write Eq. (184) in the form

$$m\left[\frac{\partial(n\overline{\vec{v}})}{\partial t} + \nabla_r \cdot n\overline{\vec{v}\vec{v}} - \sigma\vec{E} - n\vec{F} - \frac{\vec{J}\times\vec{B}}{c}\right] = m\left[\frac{\partial(n\overline{\vec{v}})}{\partial t}\right]_{\substack{coll\ with \\ other\ species}} \tag{188}$$

If we further write

$$\vec{v} = \vec{c} + \overline{\vec{v}} \tag{189}$$

where $\vec{c}$ is the deviation of $\vec{v}$ from its average value, then we have for $\overline{\vec{v}\,\vec{v}}$

$$\overline{\vec{v}\vec{v}} = \overline{(\vec{c}+\overline{\vec{v}})(\vec{c}+\overline{\vec{v}})} = \overline{\vec{c}\,\vec{c}} + \overline{\vec{v}}\,\overline{\vec{v}} \tag{190}$$

since $\overline{\vec{v}\,\vec{c}}$ and $\overline{\vec{c}\,\vec{v}}$ are zero.

The first term of Eq. (188) may be written

$$\frac{\partial(n\overline{\vec{v}})}{\partial t} = n\frac{\partial\overline{\vec{v}}}{\partial t} + \overline{\vec{v}}\frac{\partial n}{\partial t}. \tag{191}$$

Using the continuity equation, this becomes

$$\frac{\partial (n\overline{V})}{\partial t} = n\frac{\partial \overline{V}}{\partial t} + \overline{V}\left(-\overline{V}\cdot\vec{\nabla}_r n - n\,\vec{\nabla}_r\cdot\overline{V}\right). \tag{192}$$

Furthermore, the second term in Eq. (188), when Eq. (190) is applied, has the term

$$\vec{\nabla}_r \cdot n\overline{V}\,\overline{V} = \overline{V}(\overline{V}\cdot\vec{\nabla}_r n) + \overline{V}\,n(\vec{\nabla}_r\cdot\overline{V}) + n(\overline{V}\cdot\vec{\nabla}_r)\overline{V} \tag{193}$$

Using Eqs. (190), (192), and (193), Eq. (188) becomes

$$mn\left[\frac{\partial}{\partial t} + \overline{V}\cdot\vec{\nabla}_r\right]\overline{V} + \vec{\nabla}_r\cdot(nm\overline{CC}) - \sigma\vec{E}$$

$$-n\vec{F} - \frac{\vec{J}\times\vec{B}}{C} = \left[\frac{\partial \vec{P}}{\partial t}\right]_{\text{COLL. WITH OTHER SPECIES}}. \tag{194}$$

Eq. (194) is the force equation or the equation for conservation of momentum. The quantity $\underset{\sim}{\Pi}$

$$n\,m\,\overline{CC} = \underset{\sim}{\Pi} \tag{195}$$

is called the stress tensor. The right-hand side of Eq. (194) is the momentum transferred to the species under consideration, by collisions with other species of particles. If the mean velocities of the particles are small compared to their thermal velocities, then this term will be proportional to the difference in mean velocities. It can be written as a resistivity to the relative flow times the relative velocity.

The stress tensor contains not only the pressure but also the viscous stresses as well. For a general case, the stress tensor is hard to compute. However, for the case of a very high collision rate the velocity distribution will be isotropic and $\underset{\sim}{\Pi}$ will have only diagonal terms. All the diagonal terms will be equal and $\underset{\sim}{\Pi}$ can be replaced by

$$\underset{\sim}{\Pi} = p\underset{\sim}{I} \tag{196}$$

where $\underset{\sim}{I}$ is the unit dyadic and p is the scalar pressure.

When the collision rate is large, but not so large that it can be considered infinite, then the off-diagonal terms give the viscous stresses and can be written down in terms of the shear viscosity. In addition, the diagonal terms are no longer all equal, the difference arising from the bulk viscosity.

If the collision rate is small, then in general we must compute the stress tensor from the Boltzmann equation. However, there are a few cases here when $\underset{\sim}{\Pi}$ takes on a particularly simple form. If time variations are slow compared to the cyclotron frequency and spatial variations are much larger than a Larmor radius, then the velocity distribution must be symmetric in the two directions perpendicular to the magnetic field. In this case the stress tensor is again diagonal in a coordinate system with one axis along the direction of the magnetic field, and the two terms which come from the velocities perpendicular to the magnetic field are equal. That is, if we take $\vec{B}$ to be in the z direction, then $\underset{\sim}{\Pi}$ has the form

$$\underset{\sim}{\Pi} = \hat{x}\hat{x}\,\Pi_{\perp} + \hat{y}\hat{y}\,\Pi_{\perp} + \hat{z}\hat{z}\,\Pi_{\parallel}. \tag{197}$$

Here again one must, in general, solve the Boltzmann equation to find $\Pi_\perp$ and $\Pi_{\parallel}$. However, there are certain simple cases when they can be obtained simply. In particular, when the gradients along the field direction are small, so that variations in this direction can be neglected, then $\Pi_\perp$ can be obtained from the adiabatic law for a two-dimensional gas

$$\Pi_\perp = \Pi_{\perp 0}\, \rho^2/\rho_0^2 \tag{198}$$

as we have already seen in our treatment of the two-dimensional motions of a plasma. In certain cases $\Pi_{\parallel}$ is also given by an adiabatic law, that for a one-dimensional gas

$$\Pi_{\parallel} = \Pi_{\parallel 0}\, \rho^3/\rho_0^3. \tag{199}$$

We saw an example of this earlier when we treated the longitudinal invatiant. The adiabatic conditions are usually obtained when mixing of particles or equivalently energy flow along the lines can be neglected. This happens when the system is closed in the z direction, or for the case of wave propagation when the phase velocity along the field is much faster than the root mean square particle velocity in that direction.

When both these adiabatic laws can be employed, we say that the gas obeys a double adiabatic equation of state.

(3) <u>The Momentum Transfer Dyadic</u>

We consider a small cube with edges $\vec{e}_x dx$, $\vec{e}_y dy$, and $\vec{e}_z dz$ (Fig. 42). In one second all the particles within a distance v_x from face 1 will cross this area, each carrying a momentum $m\vec{v}$. The

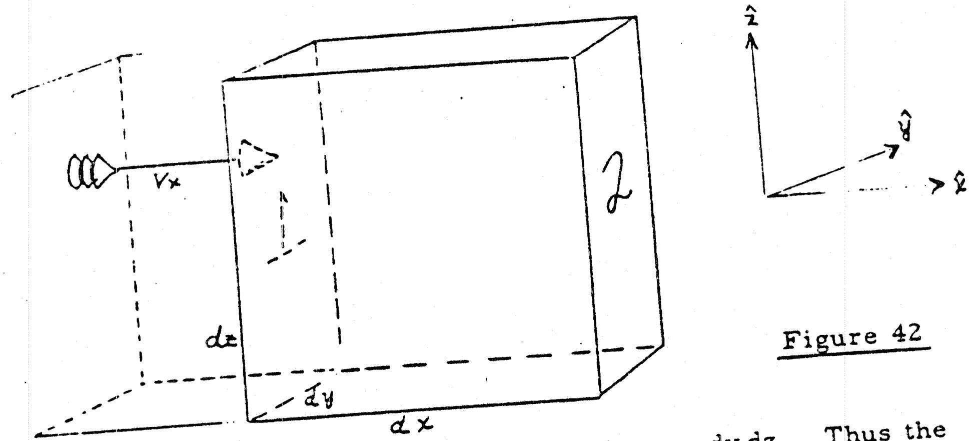

Figure 42

number of particles in this volume is $n v_x \, dy \, dz$. Thus the momentum entering face 1 per second is

$$m\vec{v} \, (n v_x \, dy \, dz) = mn \, v_x \vec{v} \, dy \, dz. \tag{200}$$

The momentum leaving face 2 per second is

$$mn v_x \vec{v} \, dy \, dz + \frac{\partial}{\partial x}(mn v_x \vec{v} \, dy \, dz) \, dx. \tag{201}$$

Thus the net momentum gain per second through these two surfaces is

$$-\frac{\partial}{\partial x}(mn \, v_x \vec{v} \, dx \, dy \, dz). \tag{202}$$

The other surfaces similarly contribute

$$-\frac{\partial}{\partial y}(mn \, v_y \vec{v} \, dx \, dy \, dz) - \frac{\partial}{\partial z}(mn \, v_z \vec{v} \, dx \, dy \, dz). \tag{203}$$

The increase of momentum per unit time per unit volume is just the sum of Eqs. (202) and (203), which, converting to dyadic notation, is simply

$$\vec{\nabla}_r \cdot \underline{\Pi} = \vec{\nabla}_r \cdot (mn \vec{v}\vec{v}). \tag{204}$$

(4) $Q = \vec{v} \cdot \vec{v}$, Conservation of Energy

Let us now look at $Q = \vec{v} \cdot \vec{v}$. This will give us an equation for the time development of the kinetic energy density. Substituting in Eq. (181) gives

$$\frac{\partial(n\overline{\vec{V}\cdot\vec{V}})}{\partial t} = -\vec{\nabla}_r\cdot(n\overline{\vec{V}(\vec{V}\cdot\vec{V})})$$

$$+\left[\frac{q}{m}\vec{E}+\frac{\vec{F}}{m}\right]\cdot 2n\overline{\vec{V}}+\left[\frac{\partial}{\partial t}(n\overline{\vec{V}\cdot\vec{V}})\right]_{coll} \qquad (205)$$

Here we have made use of the fact that

$$\vec{\nabla}_v(\vec{V}\cdot\vec{V}) = 2\vec{V} \qquad (206)$$

and that

$$(\vec{V}\times\vec{B})\cdot\vec{V} = 0. \qquad (207)$$

Again, writing

$$\vec{V} = \vec{C}+\overline{\vec{V}} \qquad (208)$$

gives

$$\overline{\vec{V}\cdot\vec{V}} = \overline{\vec{C}\cdot\vec{C}} + \overline{\vec{V}}\cdot\overline{\vec{V}} \qquad (209)$$

and

$$\nabla_r\cdot(n\,\overline{V(V\cdot V)}) =$$

$$\vec{\nabla}_r\cdot\left\{n\overline{\left[(\vec{C}+\overline{\vec{V}})[(\vec{C}+\overline{\vec{V}})\cdot(\vec{C}+\overline{\vec{V}})]\right]}\right\} =$$

$$\nabla_r\cdot\left\{n\,\overline{\vec{C}[\vec{C}\cdot\vec{C}]}+2n\overline{\vec{C}\vec{C}}\cdot\overline{\vec{V}}+n\,\overline{\vec{V}}\;\overline{\vec{C}\cdot\vec{C}}\right.$$

$$\left.+n\,\overline{\vec{V}}(\overline{\vec{V}}\cdot\overline{\vec{V}})+2\overline{\overline{\vec{V}}(\vec{C}\cdot\overline{\vec{V}})}+\overline{\vec{C}}\;\overline{\vec{V}}\cdot\overline{\vec{V}}\right\} \qquad (210)$$

where the last two terms are zero, since $\overline{\vec{V}}$ is a constant which may be taken outside the average.

Substituting Eq. (210) in Eq. (205) and making use of the continuity equation (182) gives

$$n\left[\frac{\partial}{\partial t}\left(\frac{m}{2}\overline{\vec{V}}\cdot\overline{\vec{V}}\right)+(\overline{\vec{V}}\cdot\vec{\nabla}_r)\left(\frac{m}{2}\overline{\vec{V}}\cdot\overline{\vec{V}}\right)\right]$$

$$+n\left[\frac{\partial}{\partial t}\left(\frac{m}{2}\overline{\vec{c}\cdot\vec{c}}\right)+(\overline{\vec{V}}\cdot\nabla_r)\left(\frac{m}{2}\overline{\vec{c}\cdot\vec{c}}\right)\right]$$

$$+\nabla_r\cdot\left[nm\overline{\vec{c}\vec{c}}\cdot\overline{\vec{V}}+\frac{nm}{2}\overline{\vec{V}}\,\overline{\vec{c}\cdot\vec{c}}\right]$$

$$+\nabla_r\cdot\left[\frac{nm}{2}\overline{\vec{c}(\vec{c}\cdot\vec{c})}\right]=$$

$$=\vec{E}\cdot\vec{j}+\vec{F}\cdot n\overline{\vec{V}}+\left[\frac{\partial(nK)}{\partial t}\right]_{\substack{\text{COLL WITH}\\ \text{OTHER SPECIES}}} \tag{211}$$

where we have written

$$\vec{j}=q\,n\,\overline{\vec{V}} \tag{212}$$

and K is the total mean kinetic energy per particle, and from conservation of energy K for the species of particles under consideration does not change for collisions with itself.

Problem: Verify Eq. (211).

The terms appearing in Eq. (211) have the following meanings: The first two terms on the left are the convective time derivative of the energy of mean motion; the second two terms on the left form the convective time derivative of the energy of the random motion about the mean — i.e., the heat energy. The fifth term is the divergence of the stress tensor dotted with the mean velocity. The stress tensor dotted with the mean velocity gives the rate at which the fluid stresses are doing work, so this is the divergence of this rate of doing work. The sixth term is the divergence of the flux of random energy due to the mean motion and the seventh term is the divergence of the flux of random motion due to the random motions. This seventh term is called the heat flow. The first term on the right-hand side is the rate at which the electric field does work on the current; the second term on the right-hand side is the rate at which the external force does work on the fluid. Finally, the last term on the right-hand side is the rate at which the species of particles under consideration lose energy to other types of particles.

Again, if we have large collision rates the stress tensor can be written in terms of the viscous stresses which are proportional to the gradient of $\vec{v}$, while the heat flow in this case is proportional to the temperature gradient. In general, the thermal conductivity may be a tensor, particularly for a plasma in a magnetic field, and the heat flow will be given by

$$\vec{H} = \underset{\sim}{K} \cdot \vec{\nabla}_r T. \qquad (213)$$

The energy exchange between different species of particles will have two terms — one proportional to the mean relative drifts squared and the other proportional to the temperature differences (i.e., differences in mean random energy.

Here, as in the case of the momentum equation when collisions are infrequent, one cannot write simple expressions for these terms. If collisions are negligible, then we are in general forced to solve the full collisionless Boltzmann equation along with Maxwell's equations to find these quantities.

XV. The Basic Fluid Equations for a Two-Component Plasma

Let us look at the problem of a plasma composed of electrons and one species of ion with charge ze. We have just found the conservation equations for particles, momentum, and energy for a single species. We may apply these equations to our present problem. First, consider the momentum equation (194). There is one such equation for the electrons and one for the ions. We will drop the bar from these equations, since everything will be understood to stand for averages from now on.

$$m_e n_e \left(\frac{\partial \vec{v}_e}{\partial t} + \vec{v}_e \cdot \nabla_r \vec{v}_e \right) + \vec{\nabla}_r \cdot \underset{\sim}{\Pi_e} + n_e e \vec{E} - n_e \vec{F}_e$$

$$+ n_e e \frac{\vec{v}_e \times \vec{B}}{c} = \left[\frac{\partial \vec{P}_e}{\partial t} \right]_{i.e.} \tag{214}$$

$$m_i n_i \left(\frac{\partial \vec{v}_i}{\partial t} + (\vec{v}_i \cdot \vec{\nabla}_r) \vec{v}_i \right) + \vec{\nabla}_r \cdot \underset{\sim}{\Pi_i} - n_i z e \vec{E} - n_i \vec{F}_i$$

$$- n_i z e \frac{\vec{v}_i \times \vec{B}}{c} = \left[\frac{\partial \vec{P}_i}{\partial t} \right]_{e.i.} = - \left[\frac{\partial \vec{P}_e}{\partial t} \right]_{i.e} \tag{215}$$

It is possible to work with these equations directly in investigating the behavior of a plasma. This is particularly useful if only one species of particle (for example, electrons) plays a significant role in the motion. However, often the two species move together as a fluid, and it is then useful to work with two new variables — the mean velocity and the current. We therefore define the mass velocity $\vec{v}$ by

$$\vec{V} = \frac{n_i m_i \vec{v}_i + n_e m_e \vec{v}_e}{n_i m_i + n_e m_e}, \tag{216}$$