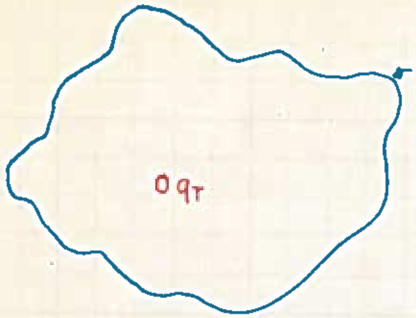
• In better notation.

$$\vec{E}_{TOTAL} = \langle \vec{E} \rangle + \delta\vec{E}$$ ~ rapidly varying field due to N.N.

EXAMPLE : Debye Screening of a static charge.



Continuum Plasma ("The Green Stuff")

• Consider Thermal Equilibrium. → ions and electrons have same temperature T.

$$n_i = n_0 \exp\left[-\frac{q\,\phi(\underline{r})}{kT}\right] \;;\quad n_e = n_0 \exp\left[+\frac{e\,\phi(\underline{r})}{kT}\right]$$

Let $\boxed{\nabla\cdot\vec{E} = 4\pi\rho \qquad \nabla\phi = -\vec{E}}$

$$\boxed{\nabla^2\phi = -4\pi\rho = 4\pi n_0\left[q\exp\left(-\frac{q\,\phi(\vec{r})}{kT}\right) - e\exp\left(\frac{e\,\phi}{T}\right)\right] - 4\pi q_T\,\delta(\vec{r})}$$

For simplicity, let $|q| = |e|$

$$\boxed{\nabla^2\phi - 8\pi n_0 e \sinh\left(\frac{e\,\phi}{T}\right) = -4\pi q_T\,\delta(\underline{r})}$$ NL - EQ in 3D.

* As of '90, exact solution is found only in 2-D.

• In a linear regime, $\boxed{\left(\frac{e\,\phi}{T}\right) \ll 1}$ →

$$\nabla^2\phi - \underline{\left(\frac{8\pi n_0 e^2}{T}\right)}\,\phi = -4\pi q_T\,\delta^3(\vec{r})$$

DEFINE $k_D^2 \equiv \dfrac{8\pi n_0 e^2}{T}$

* To solve (HW)

$$\boxed{\phi(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int \cancel{d^3k}\; d^3k\; e^{i\vec{k}\cdot\vec{r}}\;\tilde{\phi}(\vec{k})}$$

$$-(k^2 + k_D^2)\,\tilde{\phi}(k) = -4\pi q_T\, e^{i\vec{k}\cdot\vec{r}}$$

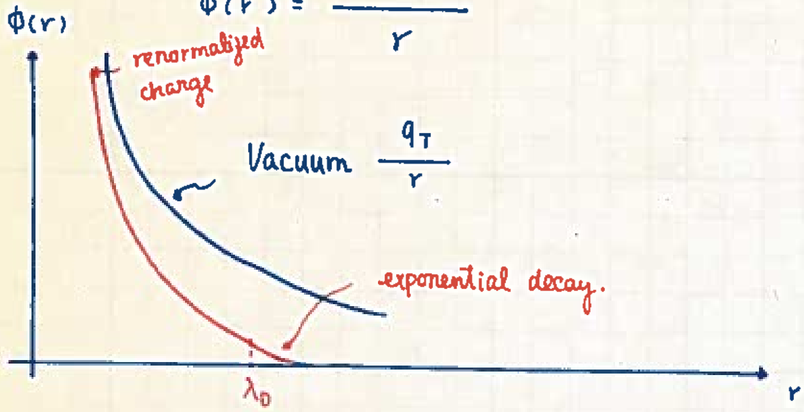
$$\boxed{\phi(\vec{r}) = \int\frac{d^3k}{(2\pi)^{3/2}}\;\frac{4\pi q_T}{k^2 + k_D^2}\; e^{i\vec{k}\cdot\vec{r}}}$$

$$\phi(\vec{r}) = \frac{q_T}{r} e^{-k_D r}$$

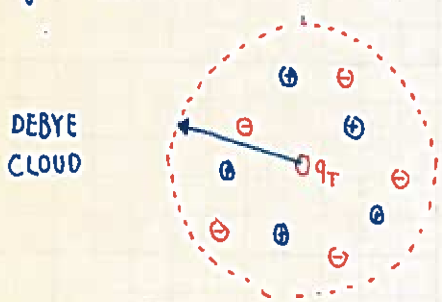
→ Consequence of full non-linear terms, i.e., $\boxed{\sinh\left(\frac{e\phi}{T}\right) \neq \frac{e\phi}{T}}$

- Charge becomes renormalized.
- as $r \to 0$

$$\phi(r) = \frac{f(q_T)}{r}$$



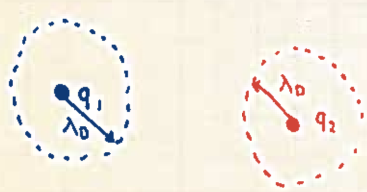
- for $q_T > 0$ (HW)



→ Cloud develops s.t. q_T is cancelled

→ In thermal equilibrium : $\frac{1}{2} q_T$ comes from electrons, $\frac{1}{2} q_T$ comes from ions

- In a plasma.



- if charges are within each others Debye Cloud. → behave like nearest neighbors.

- if charges are not within each others Debye cloud → particle sees the continuum

- for the picture to be self-consistent

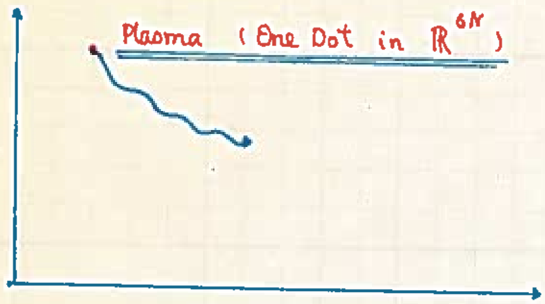
$$\boxed{\Gamma = n \lambda_D^3 >> 1}$$

Stat Mech of Plasma (N particles + classical)

(i) Exact + Useless

- $f(\underline{r}_1, \underline{v}_1 ; \underline{r}_2, \underline{v}_2 ; \ldots ; \underline{r}_N, \underline{v}_N, t)$

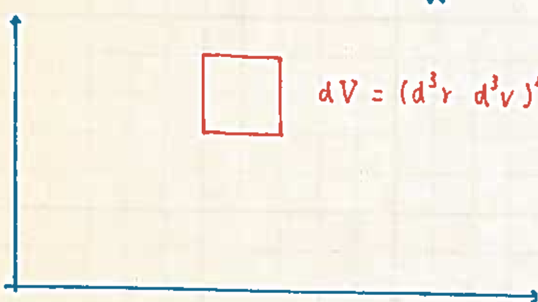
Liouville Function



6-N Dimensional Phase Space

- Gibb's Ensemble : Large # of replica's (identical in dynamics) but different in initial condition.

$\mathbb{R}^{6N}$



$dV = (d^3r \, d^3v)^N$ $\Leftarrow$ how many members of ensemble found in dV of the $6N$-Dimensional Phase Space. ; i.e., by normalizing. one find "probability of being in some region."

$$\boxed{\langle F \rangle = f^N(\Gamma^1, \Gamma^2, \ldots\ldots \Gamma^N ; t)}$$ $\longrightarrow$ N-particle distribution $\longrightarrow$ now continuous.

individual phase space

$$\frac{\partial}{\partial t} f^N + \{ f^N, H \} = 0$$

Today

Basic Equations of Plasma.

1. Given

- $f(\vec{r}_1, \vec{v}_1 ; \vec{r}_2, \vec{v}_2 ; \ldots\ldots ; \vec{r}_N, \vec{v}_N ; t)$
- $H = H(p_i, q_i)$

Evolution of f^N is the Hamiltonian flow

$$\boxed{\frac{\partial}{\partial t} f^N + \{ f^N, H \} = 0}$$

$$\{ f^N , H \} = \sum_i \left[\frac{\partial H}{\partial p_i} \frac{\partial f^N}{\partial q_i} - \frac{\partial f^N}{\partial p_i} \frac{\partial H}{\partial q_i} \right] \text{ —— } *$$

→ Hamilton - Jacoby Equation

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i}$$

* becomes

$$\boxed{\frac{\partial}{\partial t} f^N + \sum_i \dot{q}_i \frac{\partial f^N}{\partial q_i} + \dot{p}_i \frac{\partial f^N}{\partial p_i} = 0} \leftarrow \text{or source term}$$

Use Cartesian co-ordinate + NR.

$$\boxed{\dot{q}_i \to v_i \ ; \ \dot{p}_i = F_i \ ; \ p_i = m v_i} \longrightarrow \text{Particles now have meaning.}$$

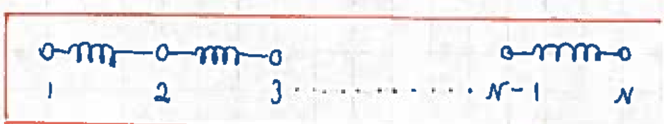
$$\frac{\partial}{\partial t} f^N + \sum_{i=1}^{N} v_i \cdot \frac{\partial}{\partial \underline{x}_i} f^N + \underline{a}_j \cdot \frac{\partial}{\partial \underline{v}_i} f^N = 0 \longleftarrow **$$

or source term.

$$\left[\text{e.g. } \underline{a}_j = \frac{q_j \underline{E}}{m_j} \right]$$

** Contains all the correlations ! ⟶ will separate it into NN terms and continuum terms

- example of a strongly correlated system ⟶ solid.



- plasma is weakly correlated material because it has a lot of KE.

- Easiest Example of $f^N \longrightarrow \boxed{q_j = 0} \longrightarrow$ i.e., ideal gas (N non-interacting particles...)

$$f^N(\Gamma_1, \Gamma_2 \ldots \Gamma_N) = \underbrace{f^{(1)}(\Gamma_1)\, f^{(1)}(\Gamma_2) \ldots\ldots f^{(1)}(\Gamma_N)}$$

single particle dist. function

- to project single-particle dist. function.

$$f^{(1)}(\Gamma_1) = \int d\Gamma_2\, d\Gamma_3 \ldots\ldots d\Gamma_N\, f^{(N)}(\Gamma_1, \Gamma_2, \ldots \Gamma_N)$$ exact....

$\longrightarrow$ Use a similar procedure to get 2-particle dist function.

$$f^{(2)}(\Gamma_1, \Gamma_2) = \int d\Gamma_3\, d\Gamma_4 \ldots d\Gamma_N\, f^{(N)}(\Gamma_1, \Gamma_2, \ldots \Gamma_N)$$

(Example: Myers cluster expansion)

$$\boxed{\cdot\ \text{Recall} \longrightarrow \text{Plasma parameter} \equiv \frac{\langle PE \rangle}{\langle KE \rangle} = \Gamma.}$$

- Perturbation Th to calculate $f^{(2)}(\Gamma_1, \Gamma_2)$

$$\boxed{f^{(2)}(\Gamma_1, \Gamma_2) = f^{(1)}(\Gamma_1)\, f^{(2)}(\Gamma_2) + \text{correlation terms.}}$$

2. find $f^{(1)}(\Gamma_1)$

- Given.

$$\boxed{\frac{\partial}{\partial t} f_N + \sum_{j=1}^{N} \underline{v}_j \cdot \frac{\partial f^{(N)}}{\partial \underline{x}_j} + \underline{a}_j \frac{\partial f^{(N)}}{\partial \underline{v}_j} = 0}$$

apply $\int d\Gamma_2\, d\Gamma_3\, d\Gamma_4 \ldots\ldots d\Gamma_N$

- 1st term $\longrightarrow \dfrac{\partial}{\partial t} f^{(1)}$

- 2nd term $\longrightarrow$

$$\sum_j \int (dx_2\, dy_2\, dz_2)(dv_{x_2}\, dv_{y_2}\, dv_{z_2}) \ldots\ldots \left[v_{x_1} \frac{\partial f^N}{\partial x_1} + \ldots + v_{x_2} \frac{\partial f^N}{\partial x_2} + v_{y_2} \frac{\partial f^N}{\partial y_2} + v_{z_2} \frac{\partial f^N}{\partial z_2} + \right.$$

Let $f^N(\Gamma_1, \Gamma_2 \ldots) \longrightarrow 0$ as $\Gamma_j \longrightarrow \infty$

2nd term : $\underline{v}_1 \cdot \dfrac{\partial}{\partial \underline{x}_1} f^{N(1)}(\Gamma_1)$

- 3rd term.

$$\boxed{\sum_{j=1}^{N} \int d\Gamma_2\, d\Gamma_3 \ldots d\Gamma_N\, \frac{q_j}{m_j}\, \underline{E} \cdot \frac{\partial}{\partial \underline{v}_j} f^N(\Gamma_1 \ldots \Gamma_N) = 0}$$

· What is $\underline{E}$?

$$\underline{E} = E(\underline{r}_1, \underline{r}_2, \underline{r}_3, \dots, \underline{r}_N)$$

$$\underline{E} = \langle \underline{E} \rangle + \delta \underline{E} \leftarrow \text{fluctuation due to \underline{correlations}.}$$

↑ average field

∴ 3rd term becomes

$$\sum_{j=1}^{N} \int d\Gamma_2\, d\Gamma_3 \dots d\Gamma_N \frac{q_j}{m_j} \underline{E} \cdot \frac{\partial}{\partial \underline{v}_j} f$$

$-$ $\langle \underline{E} \rangle = \langle \underline{E} \rangle (\underline{r}_j) \leftarrow$ averaged field experienced by $\underline{r}_j$

$-$ $\boxed{\delta \underline{E} = \delta \underline{E} (\underline{r}_1, \underline{r}_2 \dots \underline{r}_N)}$

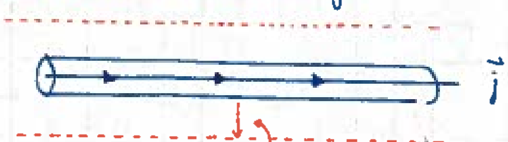
∴ the integral involving $\langle \underline{E} \rangle$ has only one surviving term.

$$\boxed{\frac{\partial}{\partial t} f^{(1)} + \underline{v}_1 \cdot \frac{\partial f^{(1)}}{\partial \underline{x}_1} + \frac{q_1}{m_1} \langle \underline{E} \rangle (\underline{r}_1) \cdot \frac{\partial}{\partial \underline{v}_1} f^{(1)} = - \sum_{j=1}^{N} \int d\Gamma_2\, d\Gamma_3 \dots d\Gamma_N\, \delta \underline{E} (\underline{r}_1, \underline{r}_2, \underline{r}_3 \dots \underline{r}_N) \cdot \frac{\partial f^N}{\partial \underline{v}_j}}$$

Kinetic Eq. For Plasma.

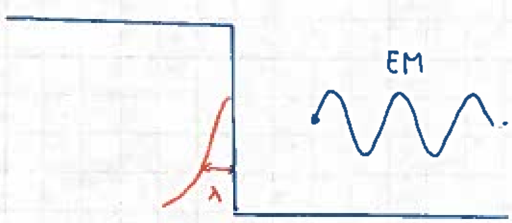
↑ Correlation term (Collision ?)

<u>Remark</u> : Shielded current filaments.



skin depth $\sim c/\omega_{pe}$

<u>Recall</u> :



$$k = \sqrt{\frac{\omega^2 - \omega_p^2}{c^2}} \simeq i \frac{\omega_p}{c} \longrightarrow \boxed{\lambda \simeq \frac{c}{\omega_p}}$$ for low frequency EM waves.

• Let's write $\delta \underline{E} (\underline{r}_1, \underline{r}_2 \dots \underline{r}_N)$ in terms of $f^{(2)}(\Gamma_i, \Gamma_j)$, and the term.

$$- \sum_{j=1}^{N} \int d\Gamma_2\, d\Gamma_3 \dots d\Gamma_N \frac{q_j}{m_j} \delta \underline{E} (\underline{r}_1, \underline{r}_2 \dots \underline{r}_N) \cdot \frac{\partial f^N}{\partial \underline{v}_j}$$

represents.

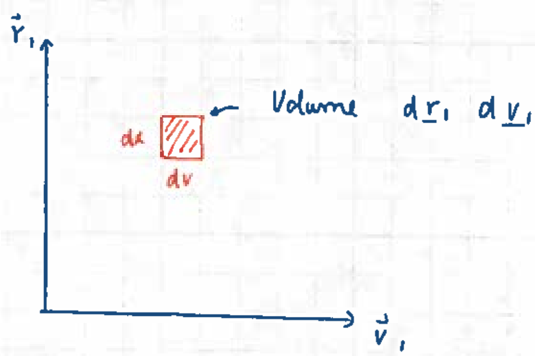
- 2 particles correlations
- N. N. collisions.

So, the Kinetic Equation becomes.

$$\boxed{\frac{\partial}{\partial t} f^{(1)} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} f^{(1)} + \frac{q_1}{m_1} \langle \underline{E} \rangle \cdot \frac{\partial}{\partial \underline{v}_1} f^{(1)} = -\left(\frac{\partial}{\partial t} f^{(1)}\right)_{\text{collision}}}$$

e.g. Fokker-Planck.

- $f^{(1)}(\underline{x}_1, \underline{v}_1)\, d^3 r_1\, d^3 v_1 \longrightarrow$ is the probability of finding "a particle" in a region of phase space $(\underline{x}_1, \underline{v}_1)$



- Generalize to
  - include $\vec{B}$ field.
  - drop (1) label.

$$\boxed{\begin{aligned}
&\bullet\ -\frac{df}{dt} + \vec{v}_1 \cdot \nabla f + \frac{q}{m}\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right] \cdot \nabla_v f = -\left(\frac{\partial}{\partial t} f\right)_{\text{coll.}} \\
&\bullet\ \nabla \cdot \vec{E} = -4\pi \sum_s \int d^3 v\, f \\
&\bullet\ \nabla \times \vec{B} = \frac{4\pi}{c} \sum_s \int d^3 v\, \vec{v} f + \frac{1}{c} \frac{\partial}{\partial t} \vec{E}
\end{aligned}}$$

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$$\frac{\partial}{\partial t} f_s(\vec{r},\vec{v};t) + \underline{v}\cdot\nabla_x f_s(\vec{r},\vec{v};t) + \frac{q_s}{m_s}\left[\underline{E}(\underline{r};t) + \frac{\underline{v}\times\underline{B}(\vec{r},t)}{c}\right]\cdot\nabla_v f_s(\vec{r},\vec{v};t)$$

$$= \underline{\left(\frac{\partial}{\partial t} f_s(\vec{r},\vec{v};t)\right)_{coll.}} \leftarrow \text{a complicated integral.}$$

Today.
Kinetic Equation.

1. $f_s(\underline{r},\underline{v};t) \longleftrightarrow f_s^{(1)}(\underline{r},\underline{v};t)$
 (Recall $\langle PE\rangle \ll \langle KE\rangle \longrightarrow f^{(2)}, f^{(3)}$ not important)

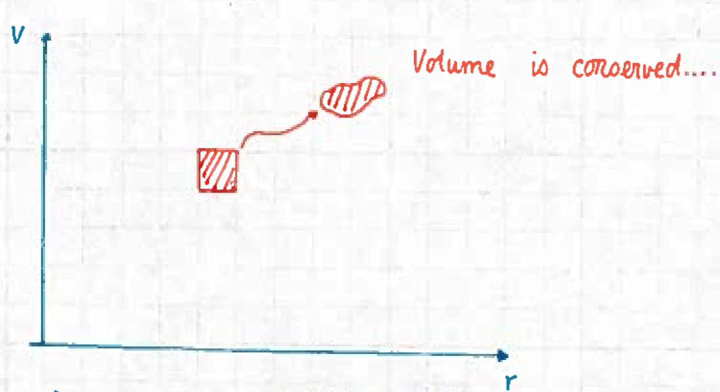
2. Collisionless plasma $\longrightarrow \left(\frac{\partial f_s^{(1)}}{\partial t}\right)_{coll} = 0$ [Vlasov Equation]

- Wave phenomena w/ time scale

$$\boxed{\tau \ll \tau_{COLL}}$$

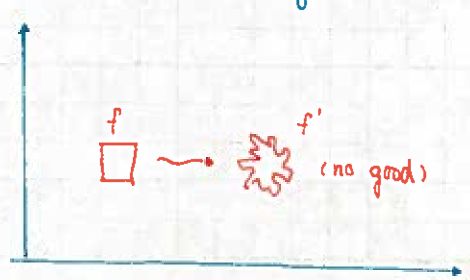
- V. E. is exact in the limit $n\lambda_D^3 \longrightarrow \infty$
- $f_{(s)}$ is a continuous fluid in phase space
- "Mean field" description
- No correlations.
- V. E. $\longrightarrow \frac{D}{Dt} f = 0 \longrightarrow$ incompressable fluid in phase space....

i. e.



- Can be used to calculate properties. $\longrightarrow$ "Water Bag Model"
 $\longrightarrow$ Calculate motion of boundaries.

Useful for simple boundaries



- Entropy is conserved in this system $\longrightarrow$ because $\nexists$ collisions.
  - role of collisions $\rightarrow$ discontinuous jump in phase space.
  - S is defined by

$$S = -\int d^3r\, d^3v \;\; f(\vec{v}, \vec{r}; t) \ln f(\vec{r}, \vec{v}; t)$$

$$\frac{dS}{dt} = \int d^3r\, d^3v \left[\ln f + 1 \right] \frac{\partial f}{\partial t}.$$

- unity term goes to 0 (continuity)
- ln term

$$\frac{\partial S}{\partial t} = \int d^3r\, d^3v \ln f \left[- \underline{v} \cdot \nabla_x f - \underline{a} \cdot \nabla_v f \right]$$

Use identity.

$$\frac{\partial}{\partial x} [f \ln f] = \ln f \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}.$$

$$\ln f \frac{\partial}{\partial x} f = \underbrace{\frac{\partial}{\partial x} [f \ln f] - \frac{\partial}{\partial x} f}_{}$$

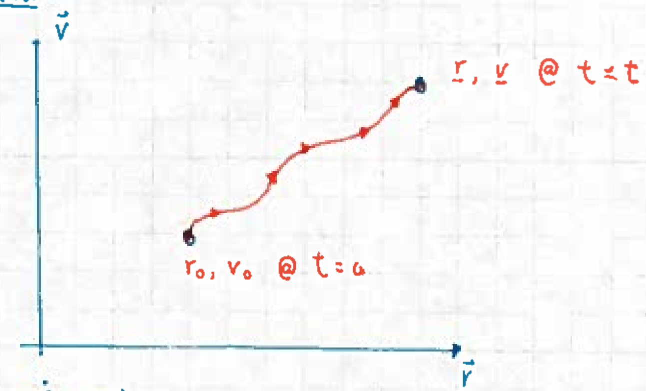
2 perfect differentials.

- This implies that plasma processes are reversible, i.e., Landau damping is ~~irreversible~~ reversible. $\longrightarrow$ plasma echo.

$$\frac{\partial S}{\partial t} = 0 \qquad (\tau \ll \tau_{collision})$$

$\longrightarrow$ plasma echo exists!

2. <u>Orbit functions</u>....



- Using $\dot{\vec{v}} = \vec{a}$, we can make connection between "a particle" and "a fluid element", therefore, we can make the statement.

Incompressible description

$$\boxed{f(\underline{r},\underline{v};t) = f(\underline{r}_0(\underline{r},\underline{v},t),\ \underline{v}_0(\underline{r},\underline{v},t);t)} \rightarrow \text{soln. along the characteristics}$$

$\therefore$ we have a mapping
→ if you know the orbits
→ you know f.

Remark : Example of characteristics → wave equation

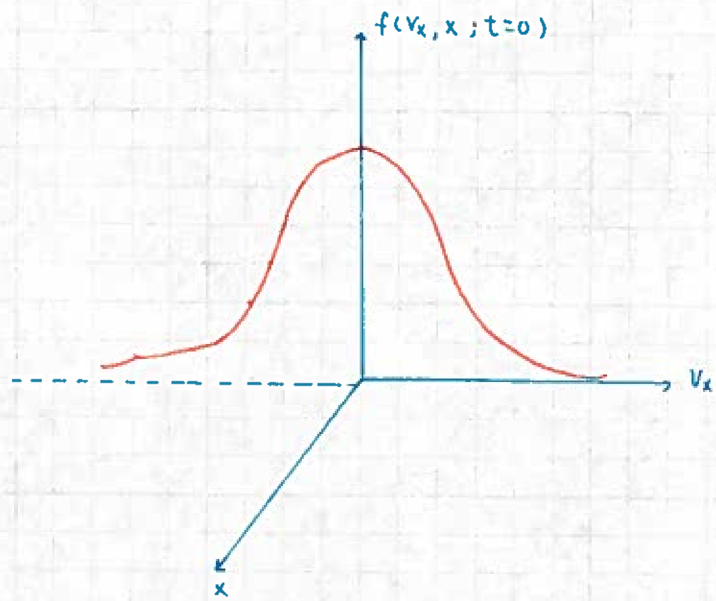
$$\boxed{\frac{\partial^2}{\partial x^2}\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0}$$

but $\psi = \psi(x-ct)$ is solution !!

Example :

- Say $q=0$, i.e., no charges → "free streaming"
- let @ $t=0$

$$\boxed{f(\underline{r},\underline{v};t=0) = \frac{\delta(x)\ e^{-\underline{v}^2/2\bar{v}^2}}{(2\pi\ \bar{v}^2)^{3/2}}}$$



Equation of motion

$$\begin{cases} \underline{v} = \underline{v}_0 \\ \underline{r} = \underline{r}_0 + \underline{v}_0 t \end{cases}$$

→ $\underline{r}_0 = \underline{r} - \underline{v}t.$

$\underline{r}_0(\underline{r},\underline{v};t) = \underline{r} - \underline{v}t.$

$$\boxed{x_0 = x - v_x t}$$

$\therefore$

$$f(\underline{v}, \underline{r}; t) = \frac{\delta(x - v_x t)}{(2\pi \bar{v}^2)^{3/2}} e^{-(v_x^2 + v_y^2 + v_z^2)/2\bar{v}^2}$$

$\rightarrow$ @ fixed t and x, only see $v_x = \frac{x}{t}$ particles.

$$\boxed{f(\underline{r}, \underline{v}; t) = \frac{\delta(x - v_x t)}{(2\pi \bar{v}^2)^{3/2}} e^{-\left(\frac{(x/t)^2}{2\bar{v}^2}\right)} e^{-(v_y^2 + v_z^2)/2\bar{v}^2}}$$

δ-function allows this substitution.

- ~~Vat~~ Vlasov - Picture $\rightarrow$ continuous perturbation due to free-streaming

Example of slightly perturbed orbits

- Say $q \neq 0$, but small change arises
  - either because perturbation is small or Δt is small
- Introduce a slight perturbation to the free streaming term

$$\begin{cases} \underline{r}_0(\underline{r}, \underline{v}; t) = \underline{r} - \cancel{\Delta t}\, \underline{v}\, t + \Delta \underline{r} \\ \underline{v}_0(\underline{r}, \underline{v}; t) = \underline{v} + \Delta \underline{v} \end{cases}$$

small quantity.

$$f(\underline{r}, \underline{v}; t) = f(\underline{r} - \underline{v} t + \Delta \underline{r}, \underline{v} + \Delta \underline{v}; t)$$

$$\boxed{\simeq f(\underline{r} - \underline{v} t, \underline{v}; t = 0) + \Delta \underline{r} \cdot \nabla_x f(t = 0) + \Delta \underline{v} \cdot \nabla_v f(t = 0)}$$

free streaming term.

Pushing elements of $f(\underline{r}, \underline{v}; t)$

3 How to specify a equilibrium plasma?

(a) for long time. $\rightarrow$ $T > T_{coll}$ $\rightarrow$ S is maximized.
$\rightarrow$ f reaches a maxwellian

$$\boxed{f(\vec{v}) \sim e^{-(\vec{v}^2/2\bar{v}^2)}}$$

i.e., thermal equilibrium.

(b) Vlasov equilibrium

$$\dot{f}_s(\underline{r}, \underline{v}; t) + \vec{v} \cdot \nabla_x f(\underline{r}, \underline{v}; t) + \vec{a} \cdot \nabla_v f(\vec{x}, \vec{v}; t) = 0$$

Equilibrium: $\boxed{\frac{\partial}{\partial t} f^{(s)} = 0}$

$\rightarrow$ Any function of the single particle invariant is a Vlasov equilibrium,