222 A 

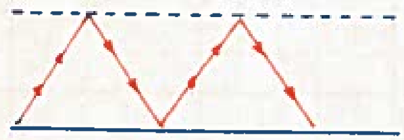
Today
Plasma Waves
- diagnostics.
- finite temperature effects.

1. Recall, from last time
- EM Dispersion

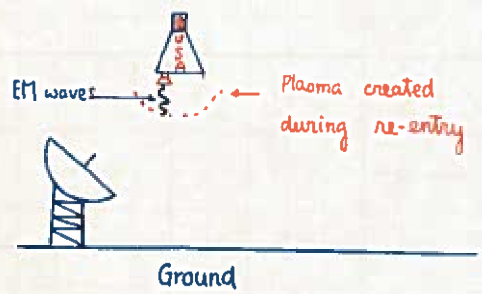
$$\boxed{\omega^2 = \omega_p^2 + k^2 c^2}$$

- Applications ⟶ Communication.

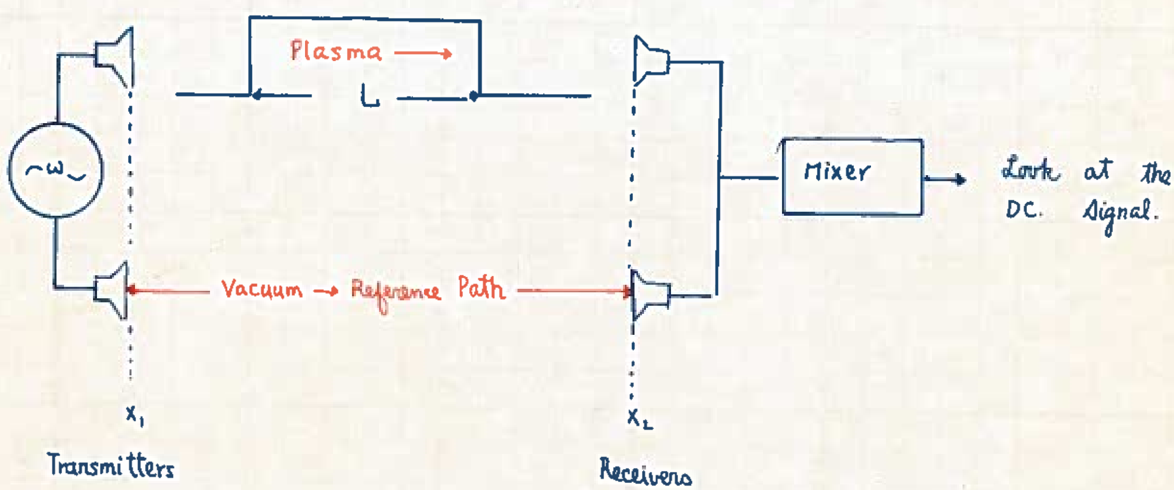
Ionisphere



- Also, plasma causes unexpected loss in communications, e.g.,




- Plasma Interferometer ⟶ Good for $\boxed{n \sim 10^{11} - 10^{15}\ cm^{-3}}$




- at position ②, plasma receiver signals look like.

$$- \ E_{Plasma} = E_p^0 \exp\left[ i \left( \phi_{12}^p - \omega t \right) \right]$$

$$- \ E_{Reference}(t) = E_R^0 \exp\left[ i \left( \phi_{21}^R - \omega t \right) \right]$$

Where

$$\boxed{\Phi_{21}^{R,P} = \int_{①}^{②} dx \quad k^{P,R}(x)}$$

$$\Phi_{21}^{R} = \int_{x_1}^{x_2} dx \quad \frac{\omega}{c}$$

$$\boxed{\begin{aligned}\Phi_{21}^{P} &= \int_{x_1}^{x_2} dx \cdot k^{P}(\omega) \\ &= \int_{x_1}^{x_2} dx \quad k_0 \cdot \sqrt{1 - \frac{\omega_p^2}{\omega^2}}\end{aligned}}$$

In experimental situations, we have.

- $\omega^2 >> \omega_p^2$

$$\Phi_{21}^{P} \approx k_0 (x_2 - x_1) - \frac{1}{2} \int_{x_1}^{x_2} k_0 \frac{\omega_p^2}{\omega^2} \, dx$$

- if plasma is uniform, then.

$$\int_{x_1}^{x_2} k_0 \frac{\omega_p^2}{\omega^2} \, dx = k_0 \frac{\omega_p^2}{\omega^2} L$$

- if plasma is non-uniform, then.

$$\int_{x_1}^{x_2} k_0 \frac{\omega_p^2(x)}{\omega^2} \, dx = k_0 \frac{\overline{\omega_p^2}}{\omega^2} L$$

where

- $\boxed{\overline{\omega_p^2} \longrightarrow \text{(line averaged plasma frequency (density)}}$

- D.C. output of the mixer. (Assume $E_P^0 = E_R^0 = E^0$)

$$\langle \text{Output} \rangle = \langle E_R \otimes E_P^* \rangle = E_0^2 \; \text{Re}\left[ e^{i[\Phi_{21}^{R} - \Phi_{21}^{P}]} \right]$$

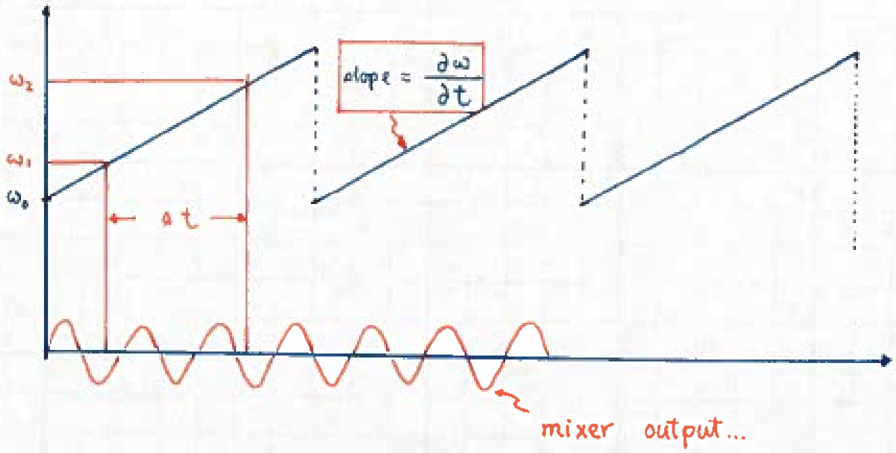
where (in the limit $\omega >> \omega_p$)

$$\boxed{\Phi_{21}^{R} - \Phi_{21}^{P} = \frac{1}{2} \int_{x_1}^{x_2} k_0 \frac{\omega_p^2(x)}{\omega^2} \, dx = \frac{1}{2} k_0 \frac{\overline{\omega_p^2}}{\omega^2} L}$$

hence, the mixer output is.

$$\langle \text{output} \rangle = \langle E_R \otimes E_P^* \rangle \sim \cos\left[ \frac{1}{2} k_0 L \cdot \left( \frac{\overline{\omega_p^2}}{\omega^2} \right) \right]$$

So, if we modulate the frequency, $\omega$. i.e.,




- So, if we let # of zero crossings be $N$, then

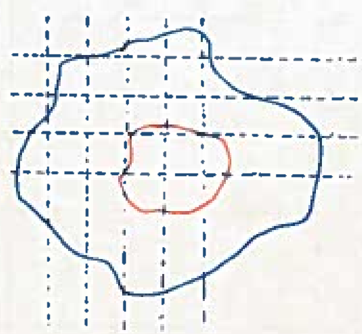
$$N\pi = \frac{1}{2}\,\omega_p^2 \cdot k_0 L \cdot \left[\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right]$$

$\boxed{\omega_1 \approx \omega_2, \quad \omega_2 = \omega_1 + \Delta\omega}$ yields

$$N\pi = \frac{1}{2}\,\omega_p^2 \cdot k_0 L \cdot \frac{\Delta\omega}{\omega_0^3}$$

$$\longrightarrow \boxed{\overline{\omega_p^2}\,L = \frac{1}{k_0}\cdot\frac{N\pi\,\omega_0^3}{\frac{\partial\omega}{\partial t}\cdot[\Delta t]}}$$

- $\overline{\omega_p^2}$ : line integrated plasma frequency.
- For a given plasma.



  - Take multiple cuts $\longrightarrow$ use a mirror.
  - Cut-off frequency can infer peak density.
  - Take Abel Transform $\longrightarrow$ get density contour.

2 Energy of an cold electrostatic wave.

- Electrostatic Energy

$$\langle u_E\rangle = \frac{\langle|\underline{E}|\rangle^2}{8\pi}$$

Kinetic Energy Density

$$\boxed{\langle u_K \rangle = \frac{1}{2} n m \langle |\underline{v}|^2 \rangle}$$

- If we write $\boxed{\underline{E} = \underline{\tilde{E}} e^{-i\omega t} + c.c.}$ → off by a factor of 2 from JDJ

  - $\langle |\underline{E}|^2 \rangle = \underline{\tilde{E}} \cdot \underline{\tilde{E}}^* + \underline{\tilde{E}}^* \cdot \underline{\tilde{E}} = 2 |\underline{\tilde{E}}|^2$

- Also, because

$$\underline{v} = \underline{\tilde{v}} e^{-i\omega t} + c.c.$$

$$\boxed{\begin{aligned} \langle u_K \rangle &= n m |\underline{\tilde{v}}^*|^2 \\ \langle u_E \rangle &= \frac{|\underline{\tilde{E}}|^2}{4\pi} \end{aligned}}$$

- from Newton's Law

$$\underline{\tilde{v}} = -\frac{e}{(-i\omega)} \cdot \frac{\underline{\tilde{E}}}{m} = \frac{e}{i\omega m} \underline{\tilde{E}}$$

$$\longrightarrow \langle u_T \rangle = \langle u_E \rangle + \langle u_K \rangle$$

$$= \frac{|\underline{\tilde{E}}|^2}{4\pi} + \frac{n m e^2}{\omega^2 m^2} |\underline{\tilde{E}}|^2$$

$$= \left[ 1 + \frac{4\pi e^2 n}{m \omega^2} \right] \frac{|\underline{\tilde{E}}|^2}{4\pi}.$$

$$\boxed{\langle u_T \rangle = \left[ 1 + \frac{\omega_p^2}{\omega^2} \right] \frac{|\underline{\tilde{E}}|^2}{4\pi}}$$

for self-consistent oscillations

$$\omega = \omega_p.$$

$$\langle u_T \rangle = [1 + 1] \frac{|\underline{\tilde{E}}|^2}{4\pi}.$$

→ Equally divided between Electric + Kinetic term.

Consider

$$1 + \frac{\omega_p^2}{\omega^2} = \frac{\partial}{\partial \omega} \left[ \omega - \frac{\omega_p^2}{\omega} \right]$$

$$= \frac{\partial}{\partial \omega} \left[ \omega \left[ 1 - \frac{\omega_p^2}{\omega^2} \right] \right]$$

- $$1 + \frac{\omega_p^2}{\omega^2} = \frac{\partial}{\partial\omega}\left[\omega\,\epsilon(\omega)\right]$$

  - In plasma, collective mode satisfies

$$\boxed{\epsilon(\omega) = 0}$$

$$\langle U_T \rangle = \cancel{\epsilon(\omega)\frac{|\underline{E}|^2}{4\pi}}^{\,0} + \omega\frac{\partial\epsilon(\omega)}{\partial\omega}\frac{|\underline{\tilde{E}}|^2}{4\pi}$$

(Ref. Landau + Lifschitz)

- Finite Te effects. (i.e., Langmuir waves)
  $\longrightarrow$ Cold plasma : $v_g = 0$ (NOT PHYSICAL!)

- From the moment description.

$$m\frac{\partial}{\partial t}\underline{\tilde{v}}_e = \underbrace{-e}_{q}\,\underline{\tilde{E}}(\underline{x}) - \frac{1}{n_0}\nabla P$$

scalar pressure.

  - $P = \gamma n k T$
    - $\frac{\omega}{k} \gg \overline{v_e} \longrightarrow \gamma = 3$
    - $\frac{\omega}{k} \ll \overline{v_e} \longrightarrow \gamma = 1$

  - for $\omega \sim \omega_p \rightarrow \gamma = 3$ (Bohm - Gross)

- Linearized Equations :

  - Force Equation.

$$\frac{\partial}{\partial t}\underline{\tilde{v}} = \frac{q}{m}\underline{E} - \frac{1}{n_0}(\gamma T)\nabla\tilde{n}$$

  - Cont.

$$\frac{\partial}{\partial t}\tilde{n} + n_0\,\nabla\cdot\underline{\tilde{v}} = 0$$

  - Poisson.

$$\nabla^2\phi = -4\pi\rho$$

$$\nabla\cdot\underline{E} = -4\pi\sum_s q_s\,\tilde{n}_s$$

  - Apply $\frac{\partial}{\partial t}$ to continuity.

$$\boxed{\frac{\partial^2}{\partial t^2}\tilde{n} + \left[\nabla\cdot\frac{\partial}{\partial t}\underline{\tilde{v}}\right]n_0 = 0}$$

$$\frac{\partial^2}{\partial t^2}\tilde{n} + \nabla\cdot\left(-\frac{e}{m}n_0\underline{E} - \frac{\gamma T}{m n_0}n_0\nabla\tilde{n}\right) = 0.$$

- Poisson's Equation.

$$\nabla\cdot\underline{E} = 4\pi(-e)\tilde{n}$$

$$\frac{\partial^2}{\partial t^2}\tilde{n} + \frac{4\pi n_0 e^2}{m}\tilde{n} - \underset{\gamma=3}{3}\frac{T}{m}\nabla^2\tilde{n} = 0$$

- Langmuir waves need not be plane waves.
- Uniform background $\longrightarrow$ not necessary.
- for plane wave, $\tilde{n} \sim \exp\left[i(\underline{k}\cdot\underline{r} - \omega t)\right]$

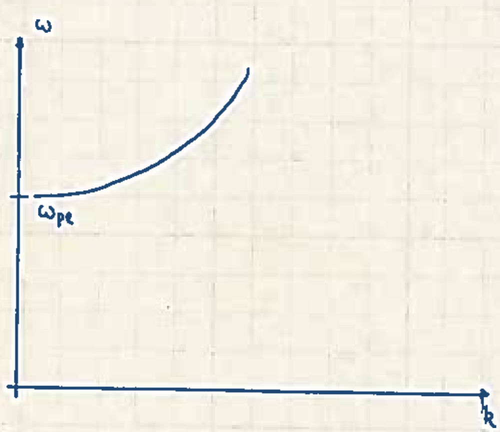
$$\omega^2 = \omega_{pe}^2 + 3k^2\bar{v}_e^2$$

$$\boxed{\omega = \sqrt{\omega_{pe}^2 + 3k^2\bar{v}_e^2}}$$

$$\frac{\partial\omega}{\partial k} = v_g = \frac{3k\,\bar{v}_e^2}{\omega},$$

$$= 3\bar{v}_e^2\cdot\frac{1}{v_\theta}$$

- $$\frac{v_g}{v_T} = 3\frac{\bar{v}_e}{v_\theta} = 3\frac{\bar{v}_e k}{\omega}$$



Last time : Langmuir Waves.

- We found

$$\boxed{\frac{\partial^2}{\partial t^2}\tilde{n} - 3\bar{v}_e^2\nabla^2\tilde{n} + \omega_p^2\tilde{n} = 0}$$

$\gamma$- factor

for plane wave solutions

$$\boxed{\tilde{n} \sim e^{i(\omega t - \vec{k}\cdot\vec{x})}} \longrightarrow \omega^2 = \omega_p^2 + 3k^2\bar{v}_e^2$$

- Bohm - Gross Relation

$$\omega = \sqrt{\omega_{pe}^2 + 3k^2\bar{v}_e^2}$$

exactly !

- Recall B-G dispersion is obtained thru moment eq. which requires.

$$\boxed{v_\theta >> \bar{v}_e}$$

from kinetic th., we'll find a more complicated expression.

$$\boxed{\omega \simeq \omega_p^2\left[1 + \frac{3}{2}\frac{k^2\bar{v}_e^2}{\omega_p^2}\right]}$$

i.e., identical to B-G in the lowest order

- Group velocity

$$\cancel{2}\omega\frac{\partial\omega}{\partial k} = \cancel{2}\cdot 3\, k\cdot\bar{v}_e^2$$

$$v_g \equiv \frac{\partial\omega}{\partial k} = 3\, k\left[\frac{\bar{v}_e^2}{\omega}\right]$$

$\omega \approx \omega_p$

$$\longrightarrow \boxed{v_g = 3\,\bar{v}_e\left[\frac{k}{k_D}\right]}$$

$\sim 0.1$

- A powerful description to solve important realistic problems
  $\longrightarrow$ "Modulation Description"

$$\boxed{\tilde{n}(\underline{r};t) = A(\underline{r};t)\,e^{-i\omega_p t}}$$

- Our approximation is

$$\left|\frac{1}{A}\frac{\partial}{\partial t}A\right| << \omega_p \longrightarrow \text{slow modulation.}$$