Darwin Plasma Model

Most complex is the Darwin (radiationless electromagnetic) model, where the force of interaction is determined by the Darwin subset of Maxwell's equation. The difference between the two is in the expression for Ampere's law. Maxwell's equation has:

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

whereas the Darwin subset has:

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}_L}{\partial t}$$

This small difference is significant because it turns the equations from hyperbolic form to elliptic form and eliminates light waves.

The main interaction loop is as follows:

1. Calculate charge, current and derivative of current density on a mesh from the particles:

$$\rho(\boldsymbol{x}) = \sum_i q_i S(\boldsymbol{x} - \boldsymbol{x}_i) \qquad \boldsymbol{j}(\boldsymbol{x},\boldsymbol{t}) = \sum_i q_i \boldsymbol{v}_i(t) S(\boldsymbol{x} - \boldsymbol{x}_i(t))$$

$$\frac{\partial \boldsymbol{j}(\boldsymbol{x})}{\partial t} = \sum_i q_i \left[\frac{d\boldsymbol{v}_i}{dt} S(\boldsymbol{x} - \boldsymbol{x}_i) - \boldsymbol{v}_i \nabla \cdot \boldsymbol{v}_i S(\boldsymbol{x} - \boldsymbol{x}_i) \right]$$

In the code, we actually deposit two quantities separately, an acceleration density and a velocity flux:

$$\boldsymbol{a}(\boldsymbol{x}) = \sum_i q_i \frac{d\boldsymbol{v}_i}{dt} S(\boldsymbol{x} - \boldsymbol{x}_i) \qquad \overleftrightarrow{\mathbf{M}}(\boldsymbol{x}) = \sum_i q_i \boldsymbol{v}_i \boldsymbol{v}_i S(\boldsymbol{x} - \boldsymbol{x}_i)$$

and then differentiate:

$$\frac{\partial \boldsymbol{j}(x)}{\partial t} = \boldsymbol{a} - \nabla \bullet \overleftrightarrow{\mathbf{M}}$$

2. Solve Maxwell's equation:

As in the electromagnetic code, we separate the electric field **E** into longitudinal and transverse parts, $\mathbf{E} = \mathbf{E}_L + \mathbf{E}_\perp$ and solve them separately:

$$\nabla \times \boldsymbol{E}_L = 0 \qquad \nabla \cdot \boldsymbol{E}_T = 0$$

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}_T = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}_L}{\partial t} \qquad \nabla^2 \boldsymbol{E}_T = \frac{1}{c}\nabla \times \frac{\partial \boldsymbol{B}}{\partial t} = \frac{4\pi}{c^2}\frac{\partial \boldsymbol{j}_T}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{E}_L = 4\pi\rho$$

3. Advance particle co-ordinates using the Lorentz Force:

$$m_i \frac{d\boldsymbol{v}_i}{dt} = q_i \int \left[\boldsymbol{E}(\boldsymbol{x}) + \boldsymbol{v}_i \times \boldsymbol{B}(\boldsymbol{x})/c\right] S(\boldsymbol{x}_i - \boldsymbol{x})\, d\boldsymbol{x} \qquad \frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i$$

For the Darwin case, the procedure for solving these equations for a gridless system is as follows:

1. Fourier Transform the charge, current, and derivative of current densities

$$\rho(\boldsymbol{k}) = \frac{1}{V}\int \sum_i q_i S(\boldsymbol{x} - \boldsymbol{x}_i(t)) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{x} = \sum_i q_i S(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_i}$$

$$\boldsymbol{j}(\boldsymbol{k}) = \frac{1}{V}\int \sum_i q_i \boldsymbol{v}_i S(\boldsymbol{x} - \boldsymbol{x}_i(t)) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} dx = \sum_i q_i \boldsymbol{v}_i S(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_i}$$

$$\frac{\partial \boldsymbol{j}(\boldsymbol{k})}{\partial t} = \sum_i q_i \left[\frac{d\boldsymbol{v}_i}{dt} - i(\boldsymbol{k}\cdot\boldsymbol{v}_i)\boldsymbol{v}_i\right] S(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_i}$$

2. Solve the Darwin subset of Maxwell's equation in Fourier space:

$$\boldsymbol{E}_L(\boldsymbol{k}) = \frac{-i\boldsymbol{k}}{k^2} 4\pi\rho(\boldsymbol{k}) \qquad \boldsymbol{B}(\boldsymbol{k}) = -\frac{4\pi}{c}\frac{i\boldsymbol{k}\times\boldsymbol{j}(\boldsymbol{k})}{k^2}$$

$$\frac{\partial \boldsymbol{j}_T(\boldsymbol{k})}{\partial t} = \frac{\partial \boldsymbol{j}(\boldsymbol{k})}{\partial t} - \frac{\boldsymbol{k}}{k^2}(\boldsymbol{k}\cdot\frac{\partial \boldsymbol{j}(\boldsymbol{k})}{\partial t}) \qquad \boldsymbol{E}_T(\boldsymbol{k}) = -\frac{4\pi}{k^2 c^2}\frac{\partial \boldsymbol{j}_T(\boldsymbol{k})}{\partial t}$$

3. Fourier Transform the Electric and Magnetic Fields to real space:

$$\boldsymbol{E}_S(\boldsymbol{x}_j) = V\sum_{\boldsymbol{k}=-\infty}^{\infty}\left[\boldsymbol{E}_T(\boldsymbol{k}) + \boldsymbol{E}_L(\boldsymbol{k})\right] S(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}_j} \qquad \boldsymbol{B}_S(\boldsymbol{x}_j) = V\sum_{\boldsymbol{k}=-\infty}^{\infty}\boldsymbol{B}(\boldsymbol{k}) S(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}_j}$$

Discretizing time for these field equations is much more complex than for the electromagnetic model, since one cannot use the leap-frog algorithm for $\mathbf{E}_T$. In fact, $\mathbf{E}_T$ depends on the acceleration $d\mathbf{v}_j/dt$ of all the particles, but the acceleration of a particle depends on $\mathbf{E}_T$, so we have a very large system of coupled equations!

A simple iterative scheme where one uses old values of $d\mathbf{v}_j/dt$ on the right hand side to find new values of $\mathbf{E}_T$:

$$\boldsymbol{E}_T(\boldsymbol{x}_j) = -\sum_{k=-\infty}^{\infty}\left[\frac{4\pi}{k^2c^2}\right]\left[\frac{\partial \boldsymbol{j}_T^o(t)}{\partial t}\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}_j}$$

is unstable when

$$kc < \omega_{pe}$$

To stabilize the iteration, one can modify the equation by subtracting a constant from both sides:

$$\nabla^2\boldsymbol{E}_T^n - \frac{\omega_{p0}^2}{c^2}\boldsymbol{E}_T^n = \frac{4\pi}{c^2}\frac{\partial \boldsymbol{j}_T}{\partial t} - \frac{\omega_{p0}^2}{c^2}\boldsymbol{E}_T^o$$

where the shift constant is the average plasma frequency:

$$\omega_{p0}^2 = \frac{4\pi}{V}\sum_i \frac{q_i^2}{m_i}$$

and the superscripts n and o refer to new and old values of the iteration. The solution to this new equation is:

$$\boldsymbol{E}_T^n(\boldsymbol{x}_j) = -\sum_{k=-\infty}^{\infty}\left[\frac{4\pi}{k^2c^2+\omega_{p0}^2}\right]\left[\frac{\partial \boldsymbol{j}_T(t)}{\partial t} - \frac{\omega_{p0}^2}{4\pi}\boldsymbol{E}_T^o\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}_j}$$

Note when the solution has converged, this equation reduces to the original one.

Solving this equation requires knowledge of the velocities and accelerations of the particles at time t. This is obtained from the leap-frog scheme follows:

$$\boldsymbol{v}_j(t) = \left[\frac{\boldsymbol{v}_j(t+\Delta t/2) + \boldsymbol{v}_j(t-\Delta t/2)}{2}\right] \qquad \frac{d\boldsymbol{v}_j(t)}{dt} = \left[\frac{\boldsymbol{v}_j(t+\Delta t/2) - \boldsymbol{v}_j(t-\Delta t/2)}{\Delta t}\right]$$

The iteration starts by first calculating $\mathbf{E}_L$(t) from $\mathbf{x}$(t), and setting

$$\boldsymbol{v}_j(t+\Delta t/2) = \boldsymbol{v}_j(t-\Delta t/2)$$

This is equivalent to assuming the forces are small and that changes in the currents are dominated by convection. Next solve for initial $\mathbf{E}_T$(t) and $\mathbf{B}$(t). The iteration loop then has two parts. First, advance particles, calculate $d\mathbf{v}_j(t)/dt$ and $\mathbf{v}_j(t)$, and deposit $d\mathbf{j}(t)/dt$ and $\mathbf{j}$(t). Do not update particles. Second, solve for improved $\mathbf{E}_T$(t) and $\mathbf{B}$(t). Repeat.

When converged, use the electromagnetic Boris Mover to update the particles.

This iteration scheme works well and converges in about 2 iterations so long as the plasma density does not vary too much, specifically if

$$\max(\omega_p^2(\boldsymbol{x})) < 1.5\omega_{p0}^2$$

Beyond that, the number of iterations needed increases, and eventually the algorithm becomes unstable again. It can be stabilized by modifying the shift constant as follows:

$$\omega_{po}^2 = \frac{1}{2}\left[\max(\omega_p^2(\boldsymbol{x})) + \min(\omega_p^2(\boldsymbol{x}))\right]$$

As the density variation becomes more extreme, the number of iterations increases, but it seems to remain stable.