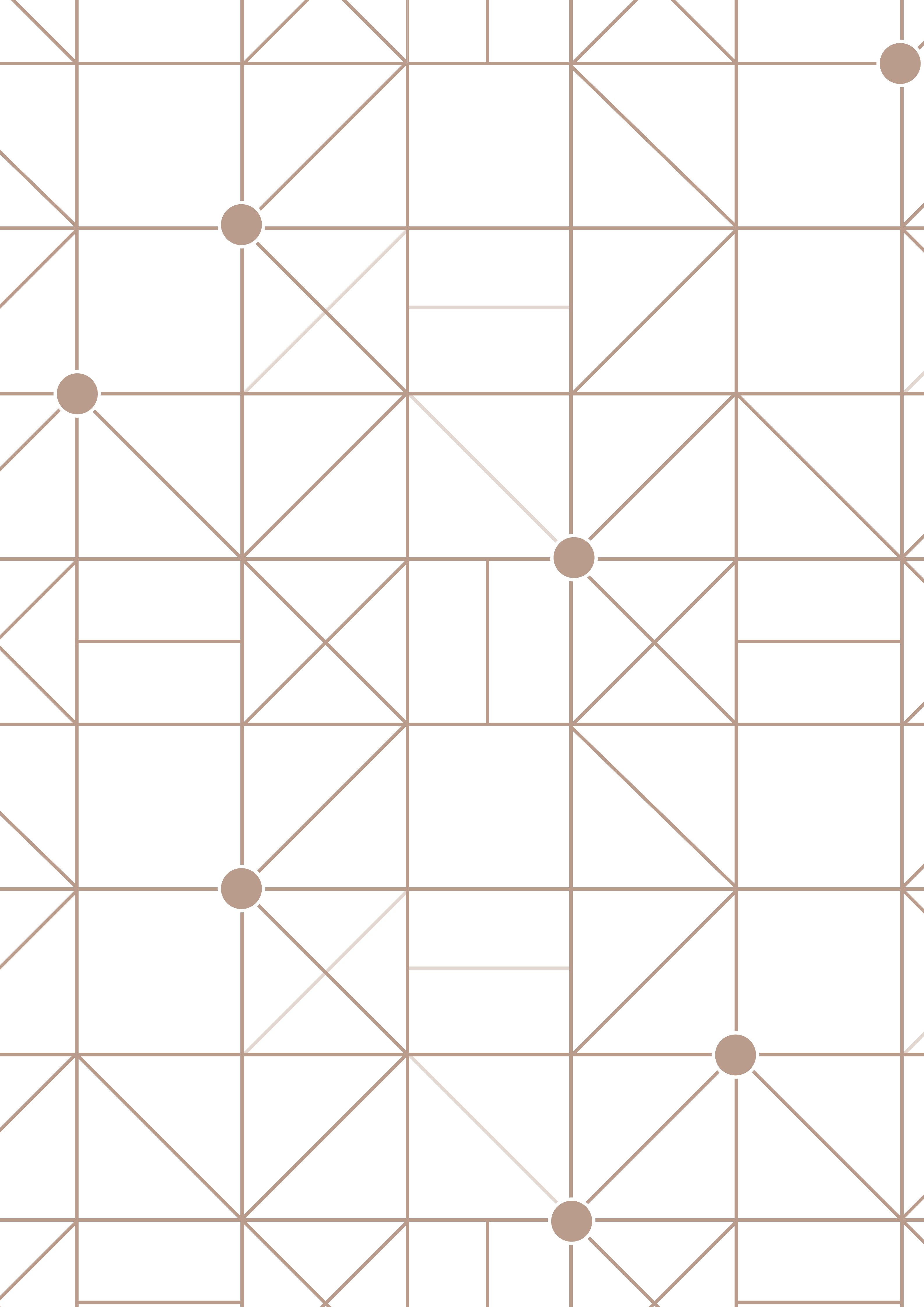


§ Introduction to Minimal Model Program and Singularities

Topic.

- Cone theorem
- lower dim'l singularities
  - ‥ rational and elliptic surface sing.
  - ‥ terminal, canonical 3-fold.
- Existence of 3-fold flip (after Shokurov)

Goal. X : a alg. var. . Find a good representation in the birational equiv. class of X.

Recall. · (MMP in $\dim = 2$) Castelnuovo's (-1)-contraction theorem

$\subsetneq$

(higher dim'l) Base point-free theorem (bpf)

· existence of (-1)-curve? Is K_S nef?

$\subsetneq$

Cone theorem. $\overline{NE}(X)$: Mori Cone

Thm. (Castelnuovo, cf. [Har, Thm V.5.7])

X : d-dim'l smooth proj. var. $/ k = \bar{k}$, $d \geq 2$. $\mathcal{O}_X(E)|_E$

$\cup$

E : divisor of X s.t. $E_{red} \simeq \mathbb{P}^{d-1}$ and $N_{E/X} = \mathcal{O}_X(E)|_E \simeq \mathcal{O}_{\mathbb{P}^{d-1}}(-1)$

$\Rightarrow \exists$ a sm. proj. var. Y and $p \in Y$ s.t.

$$\begin{array}{ccccc} Bl_pY & \simeq & X & \supseteq & E \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \ni & p \end{array}$$

(pf.) Choose a very ample H on X s.t. $H^1(X, \mathcal{O}_X(H)) = 0$.

$\because E \simeq \mathbb{P}^{d-1}$ $\therefore \exists a \in \mathbb{Z}$ s.t. $\mathcal{O}_X(H)|_E \simeq \mathcal{O}_{\mathbb{P}^{d-1}}(a)$.

1. $|H + aE|$ is bpf. :

$\because H$ is v.a. and $|H+aE| \supseteq |H|$ $\therefore |H+aE|$ is sep. point away from E.

It suffices to check $|H+aE|$ has no base points on E.

Fact. For $0 \leq i \leq a+1$, $H^1(X, \mathcal{O}_X(H+iE)) = 0$

$$\rightsquigarrow H^0(X,\mathcal{O}_X(H+aE)) \longrightarrow H^0(E,\underset{\substack{\text{SI}\\ \mathcal{O}_{\mathbb{P}^{d-1}}}}{\mathcal{O}_X(H+aE)|_E}) \longrightarrow \underset{\substack{\| \\ 0}}{H^1(X,\mathcal{O}_X(H+(a-1)E))}$$

$$s \longmapsto 1$$

Let $D=(s)_0$, then $\operatorname{Supp}D\cap\operatorname{Supp}E=\phi$, i.e. $\exists D\in|H+aE|$ s.t. $q\notin\operatorname{Supp}D$ $\forall q\in E$.

2. Get a morphism $\varphi=|H+aE|: X\longrightarrow\mathbb{P}^N$, $\varphi^*\mathcal{O}_{\mathbb{P}^N}(1)=\mathcal{O}_X(H+aE)$

$X \searrow \operatorname{Im}\varphi =: Y_1 \subset \mathbb{P}^N$

Since $\mathcal{O}_X(H+aE)|_E=\mathcal{O}_E$, $\varphi(E)=p_1$ is a point.

Since $|H+aE|$ sep. point and tangent vector on $X\setminus E$, $\varphi: X\setminus E\xrightarrow{\sim} Y_1\setminus\{p_1\}$.

Applying Stein factorization, we have

$$\begin{tikzcd} X \arrow[r, "\pi"', "\text{connected fiber}"] \arrow[rr, bend left, "\varphi"] & Y \arrow[r, "\text{finite}"] & Y_1 \end{tikzcd} \quad \text{with } \pi_*\mathcal{O}_X=\mathcal{O}_Y.$$

Since E is irr., $\pi(E)=p$ is a point. Hence $X\setminus E\overset{\pi}{\simeq}Y\setminus\{p\}\simeq Y_1\setminus\{p_1\}$

3. Y is smooth at p ($\Leftrightarrow\mathcal{O}_{Y,p}$ is reg. local ring $\Leftrightarrow\widehat{\mathcal{O}_{Y,p}}$ is reg. local ring)

$$\begin{tikzcd} E_n \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \operatorname{Spec}\mathcal{O}_Y/m_p^n \arrow[r] & Y \end{tikzcd}$$

$\xrightarrow{\text{formal function thm.}}$ $\widehat{\mathcal{O}_{Y,p}}\simeq\widehat{(\pi_*\mathcal{O}_X)_p}\simeq\varprojlim H^0(E_n,\mathcal{O}_{E_n})$ (as top. space $E_n \simeq E$; $\mathcal{O}_{E_n} = \mathcal{O}_X/m_p^n\mathcal{O}_X$: support on E)

$\simeq\varprojlim H^0(E,\mathcal{O}_X/\mathcal{J}_E^n)$

$\mathcal{O}_X/m_p^n\mathcal{O}_X$ and $\mathcal{O}_X/\mathcal{J}_E^n$ are cofinal

$\pi^{-1}(p)=E$ $(\sqrt{m_p}=\sqrt{\mathcal{J}_E})\Rightarrow\begin{cases} m_p\mathcal{O}_X\subseteq\mathcal{J}_E\\ \mathcal{J}_E^m\subseteq m_p\mathcal{O}_X \text{ for some } m\in\mathbb{N}\end{cases}$

Claim. $H^0(E,\mathcal{O}_X/\mathcal{J}_E^n)\simeq \underbrace{k[\![x_0,\cdots,x_{d-1}]\!]/(x_0,\cdots,x_{d-1})^n}_{=:A_n}$ for each $n\in\mathbb{N}$. ($\Rightarrow\widehat{\mathcal{O}_{Y,p}}=k[\![x_0,\cdots,x_{d-1}]\!]$: reg.)

$$0\to \mathcal{J}_E^n/\mathcal{J}_E^{n+1}\longrightarrow\mathcal{O}_{E_{n+1}}\longrightarrow\mathcal{O}_{E_n}\longrightarrow0$$

$$\mathcal{J}_E^n/\mathcal{J}_E^{n+1} \simeq \operatorname{Sym}^n(\mathcal{J}_E/\mathcal{J}_E^2)\simeq\mathcal{O}_{\mathbb{P}^{d-1}}(n)\quad(\because\ \mathcal{J}_E/\mathcal{J}_E^2\simeq N^\vee_{E/X}\simeq\mathcal{O}_{\mathbb{P}^{d-1}}(1))$$

$$\Rightarrow 0\to H^0(\mathcal{O}_{\mathbb{P}^{d-1}}(n))\longrightarrow H^0(\mathcal{O}_{E_{n+1}})\longrightarrow H^0(\mathcal{O}_{E_n})\longrightarrow H^1(\mathcal{O}_{\mathbb{P}^{d-1}}(n))=0$$

Since $H^0(\mathcal{O}_E)=k$ and $H^0(\mathcal{O}_{\mathbb{P}^{d-1}}(n))=\langle x_0^{i_0}\cdots x_{d-1}^{i_{d-1}}\mid\Sigma i_k=n\rangle_{v.s.}$, by induction $H^0(\mathcal{O}_{E_n})\simeq A_n$.

4. $X\simeq Bl_pY$: $m_p\mathcal{O}_X\subseteq\mathcal{J}_E$.

Since the image of $x_0,\cdots,x_{d-1}$ gen. $\mathcal{J}_E/\mathcal{J}_E^2\simeq\mathcal{O}_{\mathbb{P}^{d-1}}(1)$ and $\mathcal{O}_{\mathbb{P}^{d-1}}(1)$ gbgs,

$\rightsquigarrow x_0,\cdots,x_{d-1}$ gen. $\mathcal{J}_E$, i.e. $m_p\mathcal{O}_X=\mathcal{J}_E=\mathcal{O}_X(-E)$ is invertible.

By universal property of blow-up,

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\exists \rho"] & \\ Y & \arrow[l] Bl_p Y =: Y' \supseteq E' \end{tikzcd} \qquad Y \ni p \leftarrow E'$$

$Exc(\rho) := \{ x \in X \mid \rho^{-1} \text{ is not a morphism at } \rho(x) \}$.

Fact (cf. [KM Cor. 2.6.3.]) $\rho : X \xrightarrow{\text{bir}} Y'$, X: normal, Y': $\mathbb{Q}$-factorial (eg. sm.), $Exc(\rho) \neq \phi$.
$\Rightarrow Exc(\rho)$ is pure codim 1 in X and $\rho(Exc(\rho))$ has codim ≥ 2.

Since $X \setminus E \simeq Y \setminus \{p\} \simeq Y' \setminus E'$, $\rho(E) = E'$. By Fact, $Exc(\rho) = \phi$, i.e. ρ is isom. □

The category we work in the MMP

Objects = normal varieties

morphisms = $X \xrightarrow{\pi} Y$ with connected fibers (contraction morphism)

$\Updownarrow$ char $k = 0$

$\pi_* \mathcal{O}_X \simeq \mathcal{O}_Y$

Rmk. · MMP (or Mori theory) does NOT say much about finite morphisms

· $\forall$ morphism

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\pi"', "\text{connected fibers}"] & & Y' \\ & Y = \mathrm{Spec}\, \varphi_* \mathcal{O}_X \arrow[ur, "g\text{: finite}"'] & \end{tikzcd}$$

(Stein factorization)

·· In char $k = 0$, if the fibers of φ are connected, Y': normal, then g is an isom.
·· In char $k = p$, think of Frobenius morphism.

· In Mori theory, we focus on curves (not divisors)

$\{$ the curve contracted by $\varphi \} = \{$ the curve contracted by $\pi \}$

Def. X : proper variety

· $CDiv(X)$ = the gp. of Cartier div. of X

· $Z_1(X) = \{ \sum_{\text{finite}} a_i C_i \mid C_i : \text{integral curve}, a_i \in \mathbb{Z} \}$, $C \in Z_1(X)$ is effective 1-cycle if $a_i \geq 0 \ \forall i$.

· (numerically equivalent)

$D \equiv D'$ if $C.D = C.D'$ $\forall C \in Z_1(X)$.

$C \equiv C'$ if $C.D = C'.D$ $\forall D \in CDiv(X)$.

· $N^1(X)_{\mathbb{Z}} := CDiv(X)/\equiv$, $N_1(X)_{\mathbb{Z}} := Z_1(X)/\equiv$, $\rho(X) := \dim_{\mathbb{R}} N(X)_{\mathbb{R}} < \infty$ (by Lef. (1,1) if $/\mathbb{C}$)

$\overset{\shortparallel}{NS(X)}$: Néron - Severi gp.

$\Rightarrow N^1(X)_{\mathbb{Z}} \times N_1(X)_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ is nondegenerate

If $\pi : X \xrightarrow{\text{proper}} Y$ (↑ proper), then

$$\pi^* : N^1(Y)_{\mathbb{Z}} \longrightarrow N^1(X)_{\mathbb{Z}}, \quad [D] \longmapsto [\pi^* D]$$

$$\pi_* : N_1(X)_{\mathbb{Z}} \longmapsto N_1(Y)_{\mathbb{Z}}, \quad [C] \longmapsto \deg\left(C/\pi(C)\right)[\pi(C)]$$

and projection formula $C.\pi^* D = \pi_* C . D$.

Def. (Mori Cone)

$$N_1(X)_{\mathbb{Z}} \supseteq NE(X)_{\mathbb{Z}} = \{ \textstyle\sum a_i C_i \mid C_i : \text{integral curve}, \ a_i \in \mathbb{Z}_{\geq 0} \}$$

$\mathbb{Q}$ $\qquad\qquad$ $\mathbb{Q}$ $\qquad\qquad\qquad\qquad$ $\mathbb{Q}_{\geq 0}$

$\mathbb{R}$ $\qquad\qquad$ $\mathbb{R}$ $\qquad\qquad\qquad\qquad$ $\mathbb{R}_{\geq 0}$

$\overline{NE}(X) :=$ the closure of $NE(X)_{\mathbb{R}}$ in $N_1(X)_{\mathbb{R}}$.

Def. V : a $\mathbb{R}$-v.s.

$\ \ \cup\!\!\mid$

K : a cone ($rK \subseteq K \ \forall r \in \mathbb{R}_{>0}$)

a subcone $F \subseteq K$ is called extremal if $\begin{cases} u, v \in K \\ u + v \in F \end{cases} \Rightarrow u, v \in F$.

extremal face of K

$\qquad$ ray $\qquad$ (if $\dim F = 1$).

Fact. X, Y, Y' : proj. var., $\pi : X \to Y$ morphism.

(a) $NE(\pi) = NE(X/Y) := \ker(\pi_*) \cap NE(X)$ is an extremal face.

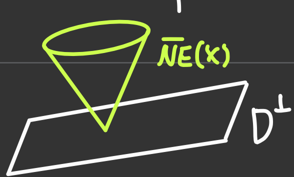
(b) Assume $\pi_* \mathcal{O}_X = \mathcal{O}_Y$ and let $\pi' : X \to Y'$ be another morphism.

If $NE(\pi) \subseteq NE(\pi') \Rightarrow \exists !$

$$\begin{tikzcd} Y \arrow[r] & Y' \\ X \arrow[u, "\pi"] \arrow[ur, "\pi'"'] & \end{tikzcd}$$

Thm. (Kleiman's ampleness criterion) X : proj. var., D : Cartier div.

Then D is ample $\Leftrightarrow D_{>0} := \{ x \in N_1(X)_{\mathbb{R}} \mid D.x > 0 \} \supseteq \overline{NE}(X) \setminus \{0\}$.



§ Nakai-Moishezon criterion

(assume separated sch. of finite type over k in this class)

Thm. (Nakai-Moishezon criterion) X ; a (proper) proj. scheme, D ; Cartier divisor on X. Then
D is ample $\Leftrightarrow D^{\dim Y}.Y > 0$ for every positive dim. closed subvariety $Y \subseteq X$.

Rmk. The same result holds when D is a $\mathbb{Q}$-Cartier divisor.

Def. (nef = numerically effective) X ; a proper scheme. A ($\mathbb{Q}$-) Cartier divisor D is nef if $D^{\dim Y}.\dim Y \geq 0$ for every subvar. $Y \subseteq X$.

Rmk. (Numerical nature) If $D_1 \equiv D_2$, the D_1 is ample (resp. nef) $\Leftrightarrow$ D_2 is ample (resp. nef)

* ample + nef = ample

Lemma. X ; proj. sch. of $\dim = n$, H ; ample Cartier. Fixed an integer $0 \leq r \leq n$.
D ; Cartier

If $D^r.Y \geq 0$ for every subvar. $Y \subseteq X$ of $\dim = r$, then $D^r.H^{n-r} \geq 0$.

(pf.) We proceed by induction on $\dim X = n$. WLOG X is an integral and $0 < r < n$.
(c.f. [Hart. III Ex5.7])

Since mH is very ample for $m >> 0$, $\exists$ an effective divisor $Y \in |mH|$. Then

$$D^r.H^{n-r} = \frac{1}{m} D^r.H^{n-r-1}.Y = \frac{1}{m}(D|_Y)^r.(H|_Y)^{n-r-1} \underset{\uparrow}{\geq} 0 \qquad \square$$

$\dim Y = n-1$ and by induction.

Now, for H ; ample Cartier on X, every subvar. $Y \subseteq X$ of $\dim Y = r$.
D ; nef

$$(H+D)^r.Y = \underbrace{H^r.Y}_{>0} + \sum_{k=1}^{r}\binom{r}{k} D^k.H^{r-k}.Y > \sum_{k=1}^{r}\binom{r}{k}(\underset{\text{nef}}{D|_Y})^k.(\underset{\text{ample}}{H|_Y})^{r-k} \geq 0.$$

By Nakai-Moishezon, $H+D$ is ample.

Def. $\mathrm{Amp}(X) :=$ the convex cone (may not contain 0) of all ample divisor class in $N^1(X)_{\mathbb{Q}}$ (which is open)
$\mathrm{Nef}(X)$ 〃 nef 〃 (which is closed)

Cor. (Kleiman) $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$, $\mathrm{int}(\mathrm{Nef}(X)) = \mathrm{Amp}(X)$.

$$D \xleftarrow{\varepsilon \to 0^+} D + \varepsilon H \qquad \overset{\Psi}{D} = \underbrace{(D - \varepsilon H)}_{\text{nef as } \varepsilon \text{ small}} + H$$

* A numerical chara of nefness

Thm. (Kleiman) X: proper scheme. A Cartier divisor D is nef $\Leftrightarrow$ $D.C \geq 0$ $\forall$ irr. curve $C \subseteq X$.

(pf.) ($\Leftarrow$) WLOG X is integral and proj. (by Chow lemma and projection formula).

We proceed by induction on $\dim X = n$. For $n=1$ is clear. For $n>1$, $D^{\dim Y}.Y \geq 0$ if $Y \subsetneq X$ by induction hypothesis. So it suffices to prove $D^n \geq 0$.

- Fix a very ample Cartier divisor H. Set $D_t = D + tH$ and deg n polynomial
$$P(t) := D_t^n = D^n + \binom{n}{1}(D^{n-1}.H)t + \cdots + \underbrace{H^n}_{>0} t^n.$$

Assume $P(0) = D^n < 0$, then $\exists$ largest $t_0 \in (0, \infty)$ s.t. $\begin{cases} P(t_0) = 0 \\ P(t) > 0 \quad \forall t > t_0 \end{cases}$

- D_t is ample $\forall t \in (t_0, \infty) \cap \mathbb{Q}$: For every subvar. $Y \subsetneq X$ of $0 < \dim Y = r < n$,
$$D_t^r.Y = \sum_{s=0}^{r-1} \binom{r}{s} \underbrace{D^s.H^{r-s}.Y}_{\substack{(D|_Y)^s.(H|_Y)^s \geq 0 \\ \text{by lemma}}} \cdot t^s + \underbrace{H^r.Y}_{\substack{>0 \\ \text{by ample}}} \cdot t^r > 0 \quad \text{as } t > 0.$$

Also $D_t^n = P(t) > 0$ for $t > t_0$. By N-M, D_t is ample for $t > t_0$.

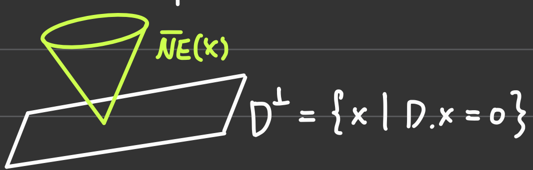
- Note $P(t) = D_t^{n-1}(D + tH) = \underbrace{D_t^{n-1}.D}_{Q(t)} + \underbrace{t \cdot D_t^{n-1}.H}_{R(t)}$.

Since $D.C \geq 0$ $\forall$ irr. curve $C \subseteq X$ and D_t is ample, by lemma $D.D_t^{n-1} \geq 0$ for rational $t > t_0$ (for all $t \geq t_0$ by limit). Also, $R(t) = t(\underbrace{D|_H}_{\text{nef}} + \underbrace{tH|_H}_{\text{ample}})^{n-1} > 0$ for $t > 0$.

Hence $P(t_0) = Q(t_0) + R(t_0) > 0$ $\rightarrow\!\leftarrow$. $\square$

Thm. (Kleiman's ampleness criterion) X: proj. var., D: Cartier div.

Then D is ample $\Leftrightarrow$ $D_{>0} := \{x \in N_1(X)_{\mathbb{R}} \mid D.x > 0\} \supseteq \overline{NE}(X) \setminus \{0\}$.



Rmk. D is called strictly nef if
$$D_{>0} \supseteq NE(X) \setminus \{0\}$$

(pf.) ($\Rightarrow$) Assume D is ample, clearly $D.z \geq 0$ $\forall z \in \overline{NE}(X)$.

Assume $D.z = 0$ for some $0 \neq z \in \overline{NE}(X)$. Since the intersection pairing is nondegenerate, $\exists$ a divisor $E.z < 0$ (or replace by $-E$) $\Rightarrow (D+tE).z = tE.z < 0$ for $t > 0$.

$\Rightarrow D+tE$ can't be ample $\forall t > 0$ $\rightarrow\leftarrow$ ($\because$ $Amp(X)$ is open)

($\Leftarrow$) Choose a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$, H: ample. Then $K = \{z \in \overline{NE}(X) : \|z\| = 1\}$ is compact. The functional $z \longmapsto D.z$ is positive on K by assumption.

$z \longmapsto H.z$ is bounded from above on $K^{\leftarrow \text{cpt}}$.

$\Rightarrow \exists\, a, b \in \mathbb{Q}_{>0}$ s.t. $D.z \geq a$, $H.z \leq b$ on K. Then $\forall z \in \overline{NE}(X) \setminus \{0\}$,

$$(D - \tfrac{a}{b}H).\frac{z}{\|z\|} \geq \frac{1}{\|z\|}(a - \tfrac{a}{b}\cdot b) = 0 \Rightarrow D - \tfrac{a}{b}H \text{ is nef.}$$

Hence $D = (D - \frac{a}{b}H) + \frac{a}{b}H$ is ample. □

<u>Proof of Nakai-Moishezon criterion.</u>

($\Rightarrow$) Take $m >> 0$ s.t. mD is very ample $\Rightarrow f = |mD| : X \hookrightarrow \mathbb{P}^N$ s.t. $f^*\mathcal{O}(1) = mD$.

Hence $(mD)^{\dim Y}.Y = (mD|_Y)^{\dim Y} = \deg f(Y) > 0$.

($\Leftarrow$) WLOG X is integral. Induction on $n = \dim X$. For $n=1$, $\deg D > 0 \Rightarrow D$: ample. For $n > 1$,

- $h^0(mD) > 0$ for $m >> 0$:

Write $D \sim A - B$ where A, B are very ample. Then

$$0 \to \mathcal{O}_X(mD-B) \xrightarrow{\cdot A} \mathcal{O}_X((m+1)D) \longrightarrow \underline{\mathcal{O}_A((m+1)D)} \to 0$$

$$0 \to \mathcal{O}_X(mD-B) \xrightarrow{\cdot B} \mathcal{O}_X(mD) \longrightarrow \underline{\mathcal{O}_B(mD)} \to 0$$

By induction, $D|_A$, $D|_B$ are ample.

By Serre vanishing, $h^i(\ \ )=0$ for $i>0$, $m>>0$.

For $i \geq 2$, $m >> 0$, $h^i(mD) = h^i(mD-B) = h^i((m+1)D)$ is constant. Hence

$$\frac{D^n}{n!}m^n + O(m^{n-1}) \overset{\text{Asympototic R-R}}{=} \chi(mD) = h^0(mD) - h^1(mD) + \text{constant}$$

$\Rightarrow h^0(mD) > 0$ as $m >> 0$.

- $\mathcal{O}_X(mD)$ is gbgs for $m >> 0$: Since "D is ample $\Leftrightarrow mD$ is ample for some $m \in \mathbb{N}$", we may assume D is effective (replace D by mD). Then $\mathcal{O}_X(mD)$ is bpf. away from $\mathrm{Supp}\, D$ ($\because$ $Bs|mD| \subseteq \mathrm{Supp}(mD) = \mathrm{Supp}\, D$).

It suffices to show $|mD|$ has no basepoint on $\mathrm{Supp}\, D$. Consider

$$0 \to \mathcal{O}_X((m-1)D) \xrightarrow{\cdot D} \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD) \longrightarrow 0$$

$$\Rightarrow H^0(mD) \longrightarrow H^0(mD|_D) \longrightarrow H^1((m-1)D) \longrightarrow H^1(mD) \longrightarrow \underset{\overset{\parallel}{0}}{H^1(mD|_D)}$$

stable (isom.) $\Leftarrow h^1(mD)$ decreasing $m >> 0 \Leftarrow$ $D|_D$ is ample by I.H. By Serre vanishing as $m>>0$

$$\overset{\text{gbgs}}{\downarrow}$$

Hence $H^0(mD) \twoheadrightarrow H^0(mD|_D)$ as $m>>0$ $\Rightarrow$ $|mD|$ has no base point on Supp D.

- For $m>>0$, we set a proj. morphism $\varphi = |mD| : X \longrightarrow \mathbb{P}^N$, $\varphi^*\mathcal{O}(1) = mD$. Since $D.C>0$ for any irr. curve on X, all fiber of φ are finite set, i.e. φ is projective quasi-finite morphism, which is finite. Hence $mD = \varphi^*\mathcal{O}(1)$ is ample. $\square$

Cor. X : proj. var., H : ample Cartier.

(1) $\overline{NE}(X)$ is a strongly convex cone, i.e. "$\pm z \in \overline{NE}(X) \Rightarrow z=0$".

(2) $\forall a \in \mathbb{R}_{>0}$, $W_a := \{ z \in \overline{NE}(X) ; H.z \leq a \}$ is compact. In particular, $\#(W_a \cap NE(X)_{\mathbb{Z}}) < \infty$.

(pf.) (1) If $\pm z \in \overline{NE}(X)$, then Kleiman's criterion $\Rightarrow$ $H.z>0$, $H.(-z)>0$ $\rightarrow\leftarrow$.

(2) Fix a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$. Assume that W_a is not cpt ($\Rightarrow$ not bounded)

$\Rightarrow$ $\exists$ a seq. $(z_n) \in \overline{NE}(X)$ in W_a s.t. $\|z_i\| \to \infty$. Since $\left(\frac{z_n}{\|z_n\|}\right)$ is a bounded seq. $\exists$ a convergent subseq. $z_{n_i}/\|z_{n_i}\| \longrightarrow y \in \overline{NE}(X) \setminus \{0\}$. But

$$H.y = \lim_{i\to\infty} \frac{\overset{\leq a}{H.z_{n_i}}}{\|z_{n_i}\|} = 0 \ \rightarrow\leftarrow.$$

$\square$

§ A rough introduction to Hilbert schemes and schemes of morphisms

S : Scheme

Def. $F : (Sch/S)^{op} \longrightarrow (Sets)$ is a (contravariant) functor.

F is called representable if $\exists$ a object $M \in (Sch/S)$ s.t. $\mathrm{Hom}_S(-, M) \xrightarrow{\sim} F(-)$.

In particular, $\exists\ \underset{F(M)}{U \in}\ \longleftrightarrow\ \underset{\mathrm{Hom}(M,M)}{id_M \in}$ s.t. $\mathrm{Hom}_S(T, M) \xrightarrow{\sim} F(T)$ $\forall T \in (Sch/S)$.

$$f \longmapsto \underbrace{f^* U}_{F(f) : F(M) \to F(T)}$$

Rmk. U is frequently called the universal element or universal family over M.
(M, U) is unique up to isomorphism.

Fact. (infinitesimal study) S/k. $F : (Sch/S) \longrightarrow (Sets)$ a repre. functor, represented by $M \in (Sch/S)$, $x_0 \in M$. Assume that we have an obstruction theory at the point x_0.
[Hart: Defor. p:145]

Then knowing its $\begin{cases} t_0 : \text{tangent space} \\ Ob_0 : \text{obstruction space} \end{cases}$, we have

$$\dim t_0 \geq \dim_{x_0} M \geq \underbrace{\dim t_0}_{=N} - \dim Ob_0$$

$\hat{\mathcal{O}}_{M,t_0}$ can be represented by $\dim Ob_0$ equations in $\hat{\mathcal{O}}_{\mathbb{A}^N,0}$
(cf. [Hart. III Thm 9.9])

Def. Fixed a closed subscheme (or quasi-proj.) $X \hookrightarrow \mathbb{P}^r \times S = \mathbb{P}^r_S$, $P \in \mathbb{Q}[m]$. The Hilbert functor

$$\mathcal{H}ilb^P_{X/S} : (Sch/S)^{op} \longrightarrow (Sets)$$
$$T \longmapsto \left\{ \begin{array}{l} \text{closed subschemes } Y \subseteq X \times_S T \text{, proper and flat over } T \\ \text{with } \chi(\mathcal{O}_{Y_t}(m)) = P(m) \end{array} \right\}$$

Thm. [Grothendieck] S : Noeth. sch., X : proj. S-scheme. $\mathcal{H}ilb^P_{X/S}$ is representable by a projective sch. $\mathrm{Hilb}^P_{X/S}$, a universal family $U^P_{X/S} \subseteq X \times_S \mathrm{Hilb}^P_{X/S}$.

$$U^P_{X/S} \downarrow \mathrm{Hilb}^P_{X/S}$$

(Hibert sch. of X) $\mathrm{Hilb}_{X/S} := \coprod_{P \in \mathbb{Q}[m]} \mathrm{Hilb}^P_{X/S}$.

eg. 1. Consider $P(m)=1$.

$\rightsquigarrow \mathrm{Hilb}^P_{X/S} = X$ and the universal family is $\Delta \xhookrightarrow{\text{diag.}} X\times_S X$

2. $S=\mathrm{Spec}\,k$. $X \hookrightarrow \mathbb{P}^r_k$ a hypersurface of degree d.

Let $V=H^0(\mathcal{O}_{\mathbb{P}^r}(d))$, $P(m)=\binom{m+r}{r}-\binom{m+r-d}{r} = \frac{d}{(r-1)!}m^{r-1}+\text{l.o.t.}$

$\rightsquigarrow \mathrm{Hilb}^P_X = \mathbb{P}(V^\vee)$ coord: C_I, the universal family is $(\sum C_I x^I = 0) \subseteq \mathbb{P}^r \times \mathbb{P}(V^\vee)$

$$\downarrow$$
$$\mathbb{P}(V^\vee)$$

Fact. $S=\mathrm{Spec}\,k$, $k=\bar{k}$. Assume that $Z\hookrightarrow X$ is regular embedding

$\Rightarrow$ the tangent space of Hilb_X at $[Z] = H^0(N_{Z/X})$

" obstruction " $= H^1(N_{Z/X})$

$\Rightarrow h^0(N_{Z/X}) \geq \dim_{[Z]}\mathrm{Hilb}_X \geq h^0(N_{Z/X}) - h^1(N_{Z/X})$.

Def. (Scheme of morphisms) $X,Y\in(\mathrm{Sch}/S)$. Define the functor of morphism from X to Y

$$\mathcal{M}or_S(X,Y) : (\mathrm{Sch}/S)^{op} \longrightarrow (\mathrm{Sets})$$
$$T \longmapsto \mathrm{Hom}_T(X\times_S T, Y\times_S T)$$

Thm. [FGA explained, p.133, Thm 5.23] S: noeth. sch., X: proj. sch. over S,

Y: quasi-proj. sch. over S ($X\times_S T \longrightarrow S$ quasi-proj.)

Assume moreover that X is flat over S $\Rightarrow \mathcal{M}or_S(X,Y)$ is represented by an open scheme $\mathrm{Mor}_S(X,Y)$ in $\mathrm{Hilb}_{X\times_S Y}$

Sketch of pf. $\forall T\in(\mathrm{Sch}/S)$, denoted $(-)_T = -\times_S T$. Consider $f\in \mathrm{Hom}_T(X_T,Y_T)$

$$\begin{tikzcd}
X_T \arrow[drr, bend left, "f"] \arrow[dr, "{(\mathrm{id}_{X_T}, f)}"] \arrow[ddr, bend right, "\mathrm{id}_{X_T}"'] & & \\
& \Gamma_T(f) \subseteq X_T\times_T Y_T = (X\times_S Y)_T \arrow[r] \arrow[d] & Y_T \arrow[d] \\
& X_T \arrow[r] & T
\end{tikzcd}$$

Since $Y\longrightarrow S$ is separated, $\Gamma_T(f)$ is closed in $X_T\times_T Y_T$.

Since X is proper and flat over S, $\Gamma_T(f)$ is proper and flat over S.

$\Rightarrow$ get a set map $\Gamma_T : \mathcal{M}or_S(X,Y)(T) \longrightarrow \mathrm{Hilb}_{X\times_S Y/S}(T)$ which is functorial in T.

$$f \longmapsto \Gamma_T(f)$$

$\Rightarrow \exists$ natural transformation of functors $\Gamma : \mathcal{M}or_S(X,Y) \to \mathcal{H}ilb_{X\times_S Y/S}$.
If G is a family of closed subscheme of $(X\times_S Y)_T$ proper and flat over T, then $\{t\in T ; \pi_t : G_t \xrightarrow{\sim} X_t\}$ is open, where $\pi : (X\times_S Y)_T \to X_T$. Hence $\exists$ open subscheme $Mor_S(X,Y)$ of $Hilb_{X\times_S Y/S}$ represents $\mathcal{M}or_S(X,Y)$. $\square$

Fact. $(S=\operatorname{Spec} k,\ k=\bar{k})$ $\ulcorner^{(\circledast)}$ X : proj. var., Y : quasi-proj. var., let $f: X\to Y$ be a morphism s.t. Y is smooth along $f(X)$.$\lrcorner$ Then

the tangent space of $Mor(X,Y)$ at point $[f] = H^0(f^*T_Y)$

" obstruction " $= H^1(f^*T_Y)$

$$\Rightarrow h^0(f^*T_Y) \geq \underset{\substack{\shortparallel \\ \dim_{[f]} Hilb_{X\times Y}}}{\dim_{[f]} Mor(X,Y)} \geq h^0(f^*T_Y) - h^1(f^*T_Y).$$

$$\dim_{[f]} Hilb_{X\times Y} \geq h^0(\underbrace{\mathcal{N}_{\Gamma(f)/X\times Y}}_{=f^*T_Y}) - h^1(\mathcal{N}_{\Gamma(f)/X\times Y})$$

$$\begin{tikzcd} \Gamma(f) \arrow[r] \arrow[d, phantom, "\wr"] & X\times Y \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

(Scheme of morphism with base scheme)

Fix a closed subscheme $B\subseteq X$ and $g: B\to Y$. We want to study $f: X\to Y$ in $(\circledast)$ with $f|_B = g$ $\rightsquigarrow$ the restriction $Mor(X,Y) \longrightarrow Mor(B,Y)$.

$$\begin{array}{ccc} Mor(X,Y) & \longrightarrow & Mor(B,Y) \\ \cup\text{I} & & \cup \\ Mor(X,Y;f|_B) & \longmapsto & [g] \end{array}$$

Consider $0\to \mathcal{I}_B \to \mathcal{O}_X \to \mathcal{O}_B \to 0 \xrightarrow{\otimes f^*T_Y} 0 \to \mathcal{I}_B\otimes f^*T_Y \to f^*T_Y \to \underset{\substack{g^*T_Y \\ \shortparallel}}{f^*T_Y|_B} \to 0$

$$\begin{aligned} \Rightarrow\ & 0 \to H^0(\mathcal{I}_B\otimes f^*T_Y) \to H^0(f^*T_Y) \to H^0(g^*T_Y) && \text{tangent} \\ & \to H^1(\mathcal{I}_B\otimes f^*T_Y) \to H^1(f^*T_Y) \to H^1(g^*T_Y) && \text{obstruction} \\ & \quad Mor(X,Y;f|_B) \qquad\quad Mor(X,Y) \qquad\quad Mor(B,Y) \end{aligned}$$

Fact. [Mori 79, Prop 2] $f: X\to Y$ are in $(\circledast)$, fix $g: B\to Y$, $f|_B = g$.

$$\Rightarrow h^0(f^*T_Y\otimes \mathcal{I}_B) \geq \dim_{[f]} Mor(X,Y;g) \geq h^0(f^*T_Y\otimes\mathcal{I}_B) - h^1(f^*T_Y\otimes \mathcal{I}_B)$$

§ Mori's Bend-and-Break technique ($k=\bar{k}$, char $k \geq 0$)

X: sm. proj. var., fix $f: C \longrightarrow X$ non-constant morphism

C: sm. proj. curve, fix a pt. $c_0 \in C$

Assume $\dim_{[f]} \mathrm{Mor}(C, X, f|_{\{c_0\}}) \geq 1$ (Bend)

$\Rightarrow \exists\, t_0 \in T$: sm. affine pointed curve, a non-trivial deformation family F of f fixing $\{c_0\}$:

$$F: C \times X \longrightarrow X \quad \text{s.t.} \begin{cases} F(c, t_0) = f(c) & \forall c \in C \\ F(c_0, t) = f(c_0) & \forall t \in T \\ F|_{C\times\{t\}} \not\equiv f & \text{for general.} \end{cases}$$

Let $\bar{T}$ be a sm. compactification (cf. [Hart. Cor I, 6.11. p.45]) $\Rightarrow$

$$\begin{tikzcd} C \times \bar{T} \arrow[r, dashed, "\bar{F}"] & X \\ C \times T \arrow[u, phantom, "\cup"] \arrow[ur, "F"'] & \end{tikzcd}$$

Claim. T is not proper, i.e. $\bar{F}$ is NOT defined at (c_0, t_1) for some $t_1 \in \bar{T} \setminus T$.

If so,

elimination of indeterminacy of $\bar{F}$ by blow-up points ✿(∵ $C \times T$ is surface)

$$\begin{tikzcd} & S \arrow[dl, "\varepsilon"'] \arrow[dr, "e"] & \\ C \times \bar{T} \arrow[rr, dashed, "\bar{F}"] & & X \\ C \times T \arrow[u, phantom, "\cup"] \arrow[urr, "F"'] & & \end{tikzcd}$$

$S_{t_1} = \pi^{-1}(t_1) \subseteq S \supseteq \pi^{-1}(t_0) = C \times \{t_0\}$

$$\begin{tikzcd} S \arrow[d, "\varepsilon"] \arrow[dd, bend left, "\pi"] \\ C \times \bar{T} \arrow[d] \\ t_1 \in \bar{T} \arrow[d] \\ T \ni t_0 \end{tikzcd}$$

$\Rightarrow \pi^{-1}(t_1) = \widetilde{C \times \{t_1\}} + \underset{\substack{\| \\ E}}{(\text{exceptional divisors for } \varepsilon)}$ is the union of (-1)-curves $\mathbb{P}^1$ ✿

Since $\{c_0\} \times \bar{T} \pitchfork C \times \{t_0\}$ intersect transversally, $\widetilde{\{c_0\} \times \bar{T}} \cap \widetilde{C \times \{t_0\}} = \phi$ (cf. [Hart. II Ex 7.12])

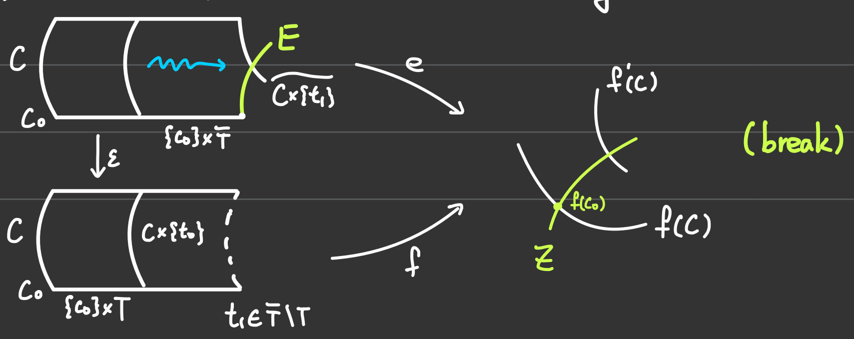
$\Rightarrow f(c_0) \in e(E) \subseteq X$ (∵ $e(\widetilde{\{c_0\} \times \bar{T}}) = \{f(c_0)\}$). Define $f': C \xleftarrow[\sim]{\varepsilon} \widetilde{C \times \{t_1\}} \subseteq S \xrightarrow{e} X$

and $Z := e_*[E]$.

<u>Upshot.</u> (bend and break with a fixed point) $\exists$ a (possibly constant) morphism $f': C \longrightarrow X$ and a nonzero effective 1-cycle Z of rational curve with $f(c_0) \subseteq \mathrm{Supp} Z$ s.t.

$$f_*[C] = e_*[S_{t_0}] \equiv e_*[S_{t_1}] = f'_*[C] + Z.$$

In particular, $\exists$ a rational curve on X through $f(c_0)$.



Proof of Claim. Assume $C\times\overline{T}\xrightarrow{\overline{F}} X$ is a morphism (proper: $\overline{T}$)

$$\begin{tikzcd} C\times\overline{T} \arrow[r, "\overline{F}"] \arrow[d, "p_1"'] & X \\ C \arrow[ur, "\exists g"'] & \end{tikzcd}$$

(Since $\overline{F}$ contracts $\{c_0\}\times\overline{T}$ to a point $f(c_0)\in X$
By rigidity lemma, $\exists g$)

But $g(c)=g(p_1(c,t_0))=F(c,t_0)=f(c)$ $\forall c\in C \rightsquigarrow F|_{C\times\{t\}}=f$ $\forall t\in T$ $\cancel{\rightarrow}$ ($\because F$ is nontrivial) $\square$

§ The existence of rational curve

Thm. [MM86] X : proj. normal variety $/k=\overline{k}$, $\dim_k X=n\geq 1$, char $k=p\geq 0$. H: ample Cartier divisor on X. If $C\subseteq X\setminus \operatorname{Sing}X$ a curve s.t. $K_X.C<0$, then $\forall c\in C$, $\exists$ a rational curve Γ on X (normalization is $\mathbb{P}^1$) through c with

$$0<H.\Gamma<2n\frac{H.C}{-K_X.C}$$

Rmk. C and Γ might have singularity, and Γ might pass through Sing X.

Prop. X,H,k are above, $\underset{\substack{\cup\!\!\mid \\ B=\{c_1,\cdots,c_b\}}}{C}\xrightarrow{f} X$, C : sm. proj. curve, f: non-constant. Assume $\dim_{[f]}\operatorname{Mor}(C,X,f|_B)\geq 1$, then $\exists$ a rational curve Γ on X s.t. $f(c_{i_0})\in\Gamma$ for some $1\leq i_0\leq b$ and $H.\Gamma\leq 2H.f_*C/b$.

(pf.)

$$\begin{tikzcd} C \arrow[r, "f:\text{finite}"] \arrow[d, "\exists f'"'] & f(C)\subseteq X \\ C' \arrow[ur] & \end{tikzcd}$$

$C' :=$ the normalization of $f(C)$

$\rightsquigarrow \deg f=[K(C):K(f(C))]=[K(C):K(C')][K(C'):\overset{\shortparallel}{K}(f(C))]=\deg f'$.

- If $C'\simeq\mathbb{P}^1$ and $\deg(C\xrightarrow{f}f(C))\geq\frac{b}{2}$, just take $\Gamma=C$. Then

$$H.[f(C)]=H.f_*C/\deg f\leq\frac{2}{b}H.f_*C$$

- From now on, we will assume that if $C'\simeq\mathbb{P}^1$, then $\deg(C\xrightarrow{f}f(C))<\frac{b}{2}$.
By assumption, $\exists\, t_0\in T$: sm. affine pointed curve, and a non-trivial deformation F of f fixing B, i.e. $F: C\times T\longrightarrow X$ s.t. $F|_{C\times\{t_0\}}=f$ and $F(\{c_i\}\times T)=\{f(c_i)\}$.
- $F(C\times T)\supsetneq F(C)$ (In particular, $F(C\times T)$ is a surface): Since

$$\dim_{[f']}\operatorname{Mor}(C,C';f'|_B)\leq h^0(C,\underbrace{f'^*\mathcal{T}_{C'}\otimes\mathcal{I}_B}_{\text{②}})=0$$

$$\left(\deg(f'^*\mathcal{T}_{C'}\otimes\mathcal{I}_B)=\deg f'\cdot\deg(\mathcal{T}_{C'})-b=2\deg f'(1-g(C'))-b\begin{cases}<0 & \text{if } g(C')\geq 1\\ <2\cdot\frac{b}{2}-b=0 & \text{if } g(C')=0\end{cases}\right)$$

If $C\times T \xrightarrow{F} f(C) \subseteq X \Rightarrow 1 = \dim T \leq \dim_{[f']} Mor(C, C', f'|_B) = 0 \rightarrow\leftarrow$.

$$\begin{tikzcd} C\times T \arrow[r, "F"] \arrow[d, "\exists F'"'] & f(C)\subseteq X \\ C' \arrow[ur] & \end{tikzcd}$$

- Claim T is not proper (by rigidty lemma).
- Let $\overline{T}$ be a sm. compactification of T, then

elimination of indeterminacy of $\overline{F}$ by blow-up points

$$\begin{tikzcd} & S \arrow[dl, "\varepsilon"'] \arrow[dr, "e"] & \\ C\times\overline{T} \arrow[rr, dashed, "\overline{F}"] & & X \\ C\times T \arrow[u, phantom, "\cup"] \arrow[urr, "F"'] & & \end{tikzcd}$$

For $i=1,\cdots,b$, we denote $E_{i,1},\cdots,E_{i,n_i}$ be the total transforms on S of the exceptional (-1)-curves obtained by blowing up an (infinitely near) point over $\{c_i\}\times\overline{T}$. Then

$$E_{i,j}.E_{i',j'} = -\delta_{i,i'}\cdot\delta_{j,j'}.$$

Let T_i be the proper transform of $\{c_i\}\times\overline{T}$ for $1\leq i\leq b$,

$\varepsilon^*\overline{T} := \varepsilon^*(\{p\}\times\overline{T})$ for a general point $p\in C$ (ε is isom. on $\{p\}\times\overline{T}$),

$\varepsilon^*C := \varepsilon^*(C\times\{t_0\})$.

Write $T_i = \varepsilon^*\overline{T} - \sum_{j=1}^{n_i}\varepsilon_{ij}E_{ij}$, then $\varepsilon_{ij} = T_i.E_{ij} = \begin{cases} 1 & \text{if the blown up point is on the (sm.) proper transform of } \{c_i\}\times\overline{T} \\ 0 & \text{otherwise} \end{cases}$.

Write also, $\underset{\substack{\| \\ e^*(H|_{e(S)})}}{e^*H} = a\,\varepsilon^*C + d\,\varepsilon^*\overline{T} - \sum_{i=1}^{b}\sum_{j=1}^{n_i} a_{ij}E_{ij} + G$,

where $G\perp \langle \varepsilon^*C, \varepsilon^*\overline{T}, E_{ij}\ \forall i,j\rangle_{\mathbb{R}}$ in $N^1(S)_{\mathbb{R}}$.

Since ε^*H is nef, $a_{ij} = e^*H.E_{ij}\geq 0$ $\left(\because\ \varepsilon^*\overline{T}.\varepsilon^*\overline{T}=0,\ \varepsilon^*\overline{T}.\varepsilon^*C = 1\right.$

$a = e^*H.\varepsilon^*\overline{T}\geq 0$ $\left.\quad E_{ij}.\varepsilon^*\overline{T} = \varepsilon_*E_{ij}.\overline{T} = 0,\ G\perp\langle\ \ \rangle\right)$

Since $e(T_i) = \{f(c_i)\}\ \forall i$, $0 = e^*H.T_i = a - \sum_{j=1}^{n_i}\varepsilon_{ij}a_{ij}$. Summing up over i, we get

$$ba = \sum_{i=1}^{b}\sum_{j=1}^{n_i}\varepsilon_{ij}a_{ij}. \qquad (1)$$

Claim. $G^2\leq 0$: Assume $G^2>0$. Since $\varepsilon^*C.G = 0 = (\varepsilon^*C)^2$, by Hodge index, $\varepsilon^*C\equiv 0$.

But $\varepsilon^*C.\varepsilon^*\overline{T} = 1$ $\rightarrow\leftarrow$. □

- Since H is ample and $e(S)$ is surface,

$$0 < (e^*H)^2 = 2ad - \sum_{i=1}^{b}\sum_{j=1}^{n_i}a_{ij}^2 + \underbrace{G^2}_{\leq 0} \overset{(1)}{\leq} \frac{2d}{b}\sum\sum\varepsilon_{ij}a_{ij} - \sum\sum a_{ij}^2$$

$$\underset{(\because\ \varepsilon_{ij}=0\text{ or }1)}{\leq} \frac{2d}{b}\sum\sum\varepsilon_{ij}a_{ij} - \sum\sum\varepsilon_{ij}a_{ij}^2 = \sum\sum\varepsilon_{ij}a_{ij}\left(\frac{2d}{b} - a_{ij}\right).$$

$0 < a_{i_0j_0} < \frac{2d}{b}$

Hence $\exists (i_0, j_0)$ s.t. (i_0,j_0)-term >0, i.e. $\varepsilon_{i_0j_0}=1$, $a_{i_0j_0}>0$, $2d/b - a_{i_0j_0} > 0$.
Since $H.f_*C = e^*H.\varepsilon^*C = d$ and $H.e_*E_{i_0j_0} = e^*H.E_{i_0j_0} = a_{i_0j_0}$, it is clear that $\forall$ irr. component of $e_*E_{i_0j_0}$ has degree $\le \frac{2d}{b}$. Since $E_{i_0j_0}.T_{i_0} = \varepsilon_{i_0j_0} = 1$, the rational 1-cycle $e_*E_{i_0j_0}$ passes through $f(C_{i_0})$.
$\hookrightarrow$ Pick an irr. comp. Γ of $e_*E_{i_0j_0}$ which passes though $f(C_{i_0})$ but is not contracted by e. $\square$

Proof of Thm.

- (Bend) = the existence of enough deformation in char $p>0$.

Define
$$\begin{tikzcd} C_m \arrow[r, "\text{Frob. of deg}"', "p^m"] \arrow[rr, bend right, "f"'] & C' \arrow[r, "\text{normalization}"] & C \subseteq X \end{tikzcd}, \quad \deg f = p^m, \quad g := g(C') = g(C_m).$$
(cf. [Kollar 96, Rmk 5.1.9])

Let $B_m \subseteq C_m$ be nonempty finite subset and $b_m = \# B_m$. Then

$$\begin{aligned} \dim_{[f]} \mathrm{Mor}(C_m, X; f|_{B_m}) &\ge h^0(C_m, f^*\mathcal{T}_X \otimes \mathcal{I}_{B_m}) - h^1(C_m, f^*\mathcal{T}_X \otimes \mathcal{I}_{B_m}) \\ &= \chi(f^*\mathcal{T}_m \otimes \mathcal{I}_{B_m}) \\ &\overset{\text{G-R-R}}{=} \deg f^*\mathcal{T}_m + \deg(\mathcal{O}_{C_m}(-B_m)^{\oplus n}) + n(1-g) \\ &= -p^m(K_X.C) - nb_m + n(1-g) \ge 1 \quad \text{if we take} \end{aligned}$$
$$b_m = \left\lfloor \frac{p^m(-K_X.C)-1}{n} \right\rfloor + 1 - g > 0 \quad \text{as } m \gg 0$$

- (Break) By Prop, $\exists$ a rational curve $\Gamma_m \subseteq X$ through some point of $f(B_m)$ s.t.

$$0 < H.\Gamma_m \le \frac{2H.f_*[C_m]}{b_m} = \frac{2p^m}{b_m}(H.C)$$

Since $p^m/b_m \longrightarrow n/{-K_X.C}$ as $m \to \infty$, for $m \gg 0$ we have $0 < H.\Gamma_m \le 2n \cdot \underbrace{\frac{H.C}{-K_X.C}}_{}$. $\overset{d}{\shortparallel}$

- M_d = the quasi-proj. scheme that para. $\mathbb{P}^1 \xrightarrow{g} X$ of $\deg_H^g < d$

$= \coprod_{\deg_H P(m) < d} \mathrm{Mor}^{P(m)}(\mathbb{P}^1, X)$ $\quad (P(m) = \chi(\mathbb{P}^1, mg^*H))$

$\rightsquigarrow$ the evaluation map $\mathbb{P}^1 \times M_d \xrightarrow{ev_\alpha} X$ has image

$$\mathrm{Im}(ev_\alpha) = \{ x \in X \mid \exists \text{ a rational curve } L \text{ in } X \text{ s.t. } L.H \le d \}$$

Fact. $\mathrm{Im}(ev_\alpha)$ is closed in X

(idea. a rational curve can only degenerate into a union of rat. curves of lower degrees
cf. [Debarré 01, Lemma 3.7])

• Prove Thm in char $k=0$ via reduction mod p.
(cf. [KM 98, Principle 1.11 p15], [Kollár 96, II 5.10, p.144])

Consider a f.g. $\mathbb{Z}$-alg. $R \subseteq k$ over which X, C, H are defined.

$C \simeq C_{\bar\eta}$, $H \simeq H_{\bar\eta}$
$c \simeq c_{\bar\eta}$

$$\begin{tikzcd} & C_R \ni c_R \\ X \simeq X_{\bar\eta} \arrow[r] \arrow[d] & X_R \arrow[d] \\ \operatorname{Spec} \underbrace{\overline{k(\eta)}}_{=k} \arrow[r, "\text{geom. generic}", "\text{fiber}"'] & \operatorname{Spec} R \end{tikzcd}$$

By shrinking Spec$R$, we may assume
- C_R is sm. over Spec$R$
- $X_R \longrightarrow \operatorname{Spec} R$ is sm. along C_R
- H_R : relative ample Cartier divisor

Let $g_R : \mathbb{P}^1_R \longrightarrow X_R$ be the constant morphism c_R,

$$\coprod_{0<\deg_{H_R} P \le d} \mathrm{Mor}^P_{\operatorname{Spec} R}(\mathbb{P}^1_R, X_R; g_R) \leftarrow \text{finite type } \left(P(m) = \chi(\mathbb{P}^1, mf^*H) = \underbrace{(H.f_*[\mathbb{P}^1])}_{\deg_H f} m + \chi(\mathcal{O}_{\mathbb{P}^1})\right)$$

$$\begin{tikzcd} \arrow[d, "\rho"] & \\ \operatorname{Spec} R & \operatorname{Spec} \overline{R/m} \arrow[l, "\text{geometric point}"'] \end{tikzcd} \quad \underset{\text{char}>0}{(\forall m \in \operatorname{Max} R,\ R/m \text{ is finite})}$$

which is constructible subset of R by Chevalley's thm

By Thm in char $=p>0$, $m \in \operatorname{Im}\rho \Rightarrow \operatorname{Im}\rho \supseteq \underset{\text{dense in Spec}R}{\operatorname{Max} R} \Rightarrow \eta \in \operatorname{Im}\rho$

by constructible & dense

□

§ The Covering Trick / $\mathbb{C}$

General strategy: To prove a vanishing for a certain $\mathbb{Q}$-divisor L, one will pull L back to a covering on which the problem simplifies in some ways.

Lemma. (Injectivity Lemma) $\pi : Y \longrightarrow X$ a finite, surjective morphism of var. X: normal, $\mathcal{E}$: locally free sheaf on X. Then the natural morphism
$$H^0(X,\mathcal{E}) \longrightarrow H^0(Y,\pi^*\mathcal{E})$$
induced by $\mathcal{E} \longrightarrow \pi_*\pi^*\mathcal{E}$ is injective.

$\mathrm{Hom}(\mathcal{E},\pi_*\pi^*\mathcal{E}) \simeq \mathrm{Hom}(\pi^*\mathcal{E},\pi^*\mathcal{E})$

(pf.)

- π: finite $\Rightarrow K(Y)/K(X)$: finite extension of $\deg\pi \Rightarrow \exists$ trace map $Tr_{Y/X} : K(Y) \longrightarrow K(X)$.

  Claim: $\exists\, Tr_{Y/X} : \pi_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ induced $Tr_{Y/X}$ (cf. [KM, Def 5.6 P.154])

  (subpf.) The statement is local, WLOG $\pi : \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$. Since π is surjective, $\pi^\# : A \hookrightarrow B$. $\forall \alpha \in K(Y)$, let $m_{\alpha,K(X)}(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_0$ be the minimal poly. of α over $K(X)$. Then $Tr_{Y/X}(\alpha) = -\frac{\deg\pi}{d} a_{d-1}$ (In particular, $Tr_{Y/X}(\alpha) = \deg\pi \cdot \alpha$ if $\alpha \in K(X)$)

  Since π is finite, B is integral over A. Since A is normal, $m_{\beta,K(X)}(t) \in A[t]$ $\forall \beta \in B$.

  $\Rightarrow Tr_{Y/X}(\beta) \in A$. Let $Tr := Tr_{Y/X}/\deg\pi : \pi_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$.

  $\Rightarrow 0 \to \mathcal{O}_X \xrightarrow{\pi^\#} \pi_*\mathcal{O}_Y \longrightarrow \underbrace{\ker \pi^\#}_{=:\mathcal{F}} \to 0 \Rightarrow \pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{F}$

  $(Tr \circ \pi^\# = id_{\mathcal{O}_X})$ Tr

- Hence
$$\begin{aligned} H^i(Y,\pi^*\mathcal{E}) &= H^i(X,\pi_*\pi^*\mathcal{E}) && (\pi : \text{finite}) \\ &= H^i(X,\mathcal{E}\otimes\pi_*\mathcal{O}_Y) && (\text{projection formula}) \\ &= H^i(X,\mathcal{E}) \oplus H^i(X,\mathcal{E}\otimes\mathcal{F}) \end{aligned}$$

□

m-fold cyclic covering

(local description) X: affine var., $0 \neq s \in \Gamma(X,\mathcal{O}_X) = \mathrm{Hom}(X,\mathbb{A}^1)$ (regular function). To define Y s.t. $s^{1/m}$ makes sense. Consider $\{t^m - s = 0\} =: Y \subseteq X \times \mathbb{A}^1 \rightsquigarrow \pi$ is cyclic covering branched along $D = \mathrm{div}\, s$.

$$\begin{tikzcd} Y \arrow[r, phantom, "\subseteq"] \arrow[dr, "\pi"'] & X\times\mathbb{A}^1 \arrow[d] \\ & X \end{tikzcd}$$

Setting $s' = t|_Y \in \mathbb{C}[Y]$, one has $(s')^m = \pi^* s$.

Prop. (Cyclic covering) X: var., $\mathcal{L}$: invertible sheaf on X. Fixed $m \in \mathbb{N}$, $0 \neq s \in H^0(X, \mathcal{L}^m)$, $D = \mathrm{div}(s) \Rightarrow \exists$ a finite surjective morphism $\pi: Y \longrightarrow X$ branched along D s.t. $\exists s' \in H^0(Y, \pi^*\mathcal{L})$ with $(s')^m = \pi^* s$. The eff. divisor $D' = \mathrm{div}\, s'$ maps isomorphically to D. Moreover, if X and D are smooth, then Y and D' are smooth.

$$\left(\begin{array}{llll} Y \setminus D' \longrightarrow X \setminus D \text{ étale} & \Rightarrow Y \setminus D' \text{ sm.} & \because D' \text{: sm. eff. Cartier} \\ D' \xrightarrow{\sim} D & \Rightarrow D' \text{ sm.} & \therefore \mathcal{O}_{Y,y} \text{: regular local ring } \forall y \in D' \end{array} \right)$$

(pf.)

- $X = \bigcup U_i$: affine Zarski open cover, $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i} \overset{\text{(locally)}}{\leadsto} \exists U_i' \longrightarrow U_i$ m-fold cyclic covering. $\because s \in H^0(\mathcal{L}^m) \leadsto$ glue together $\{U_i'\}$ to get Y.
- total space of $\mathcal{E} = V(\mathcal{E}^\vee) \supseteq Y = \{T^m - s = 0\} \xrightarrow{\pi} X$ □

where T is a global fiber coordinate.

Thm. (kodaira vanishing theorem). X: sm. proj. var., A: an ample divisor on X. Then $H^i(X, K_X + A) = 0$ $\forall i > 0$. (Rmk. cf. [KM, Principle 2.46])

(pf.) Equivalently, we prove $H^i(X, -A) = 0$ for $0 \le i < \dim X =: n$ by Serre's duality. Since A is ample, by Bertini theorem $\exists$ a sm. (irr.) divisor $D \in |mA|$ for $m >> 0$. By Prop., $\exists$ m-fold cyclic covering branched along D s.t.
$$\begin{array}{ccc} \pi: Y & \longrightarrow & X \\ \cup\!\! \mid & & \cup\!\! \mid \\ D' & \xrightarrow{\sim} & D \end{array}, \quad \pi^* D = mD'.$$

By injectivity lemma, it suffices to prove that $H^j(Y, -D') = 0$ $\forall j < n$.
(We are reduced to the case that given ample divisor is effective and smooth.)
By $0 \to \mathcal{O}_Y(-D') \to \mathcal{O}_Y \to \mathcal{O}_{D'} \to 0$,
$$H^{i-1}(\mathcal{O}_Y) \to H^{i-1}(\mathcal{O}_{D'}) \to H^i(\mathcal{O}_Y(-D')) \to H^i(\mathcal{O}_Y) \to H^i(\mathcal{O}_{D'}).$$

Fact. $r_{p,q}: H^q(Y, \Omega_Y^p) \longrightarrow H^q(Y, \Omega_{D'}^p)$ is $\begin{cases} \text{bijective} \\ \text{injective} \end{cases}$ for $\begin{cases} p+q \le n-2 \\ p+q = n-1 \end{cases}$

(Take $p = 0$, $q = j$ $\Rightarrow H^j(Y, -D') = 0$ for $j \le n-1$)

By Lefschetz hyperplane theorem,

$$\begin{array}{ccc} H^i(Y, \mathbb{C}) & \xrightarrow{\quad r_i \quad} & H^i(D', \mathbb{C}) \\ \| & \| & \| \\ \bigoplus_{p+q=i} H^q(\Omega_Y^p) & \xrightarrow{\oplus r_{p,q}} & \bigoplus_{p+q=i} H^q(\Omega_{D'}^p) \end{array} \quad \text{is } \begin{cases} \text{bijective for } i \le n-2 \\ \text{injective for } i = n-1 \end{cases}$$

Hodge decomp. is functorial

□

Def. (SNC divisor) X : sm. var. of $\dim = n$. $D = \sum D_i$, D_i : prime divisor.

D has simple normal crossing (SNC) if each D_i is smooth and $\forall p \in X$, $\exists$ open nbd. $U \ni p$ with local coordinate $z_1, \cdots, z_n$ ($\mathcal{O}_{X,p} \triangleright m_{X,p} = \langle z_1, \cdots, z_n \rangle$) s.t. $D|_U = (z_1 \cdots z_k = 0)$ for some $k \le n$, $\mathcal{O}_{D,p} \simeq \mathcal{O}_{X,p}/(z_1 \cdots z_k)$.

A $\mathbb{Q}$-divisor $\sum d_i D_i$ has SNC support if $D_{red} = \sum D_i$ is a SNC divisor.
($\mathbb{R}$)

fractional part of D

Thm. (Kawamata-Viehweg vanishing) X : sm. proj. var., D : an ample $\mathbb{Q}$-divisor s.t. $\lceil D \rceil - D$ has SNC support. Then $H^i(X, K_X + \lceil D \rceil) = 0$ $\forall i > 0$.

Rmk/Def. $D = \sum d_i D_i$, $\lceil D \rceil = \sum \lceil d_i \rceil D_i$: round up of D, $d_i \le \lceil d_i \rceil < d_i + 1$, $\lceil d_i \rceil \in \mathbb{Z}$.
($\mathbb{Q}$ or $\mathbb{R}$ divisor) Similarly for $\lfloor D \rfloor$.

Lemma. (Kawamata covering, [KMM, Thm 1-1-1]) Let X and D are in the above thm. Write $\lceil D \rceil - D = \sum_{i \in I} a_i D_i$, $0 \le a_i < 1$, D_i : prime divisor. For $m \gg 0$, $\exists$ a finite surjective morphism $Y \to X$ with $\mathbb{C}(Y)/\mathbb{C}(X)$ is Kummer extension.

(1) Y is smooth projective

(2) $\pi^* D$ is integral ($\pi^* D_i = m(\pi^* D_i)_{red}$ $\forall i \in I$)

(3) $K_Y = \pi^*(K_X + \sum \frac{m-1}{m} D_i + \sum \frac{m-1}{m} H_j)$ and $\sum D_i + \sum H_j$ has SNC.

(4) $\frac{m-1}{m} \ge a_i$ $\forall i$.

Rmk.
- The branch locus of $\pi = \bigcup D_i \cup \bigcup H_j \supseteq \bigcup D_i$. π preserves smoothness by adding branch locus artificially.
- "Y : smoothness" [EV, Lemma 3.19, P.31]
- $\because$ X, Y : smooth $\Rightarrow$ π : flat

Proof of K-V vanishing

- Take a finite Galois covering $\pi : Y \to X$ as in Kawamata covering Lemma. $G := \mathrm{Gal}(\mathbb{C}(Y)/\mathbb{C}(X))$.
  $\exists$ a natural G-action of $\pi_* \mathcal{O}_Y(K_Y + \pi^* D)$, which compatible with $G \curvearrowright \mathbb{C}(Y)$.
  Claim. $(\pi_* \mathcal{O}_Y(K_Y + \pi^* D))^G = \mathcal{O}_X(K_X + \lceil D \rceil)$
- Since $\Gamma(X, -)$ and $(-)^G$ commute for sheaves with G-action,

$$H^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = H^i(X, \pi_* \mathcal{O}_Y(K_Y + \pi^* D))^G \underset{\pi:\text{ finite}}{=} H^i(Y, \mathcal{O}_Y(K_Y + \underset{\text{ample Cartier}}{\pi^* D}))^G \underset{\text{Kawamata vanishing}}{=} 0$$

· Proof of Claim: $\forall U \subseteq X$ Zariski open subset,

$$\Gamma(U, (\pi_* \mathcal{O}_Y(K_Y + \pi^* D))^G) = \{ f \in \mathbb{C}(Y)^G \mid (\operatorname{div} f + K_Y + \pi^* D)|_{\pi^{-1}(U)} \geq 0 \}$$

$$= \{ f \in \mathbb{C}(X) \mid \operatorname{div} f + K_X + \underline{\tfrac{m-1}{m}(\Sigma D_i + \Sigma H_j) + D}|_U \geq 0 \}$$

$$\Sigma \underbrace{\left(\tfrac{m-1}{m} - a_i\right)}_{0 \leq \quad < 1} D_i + \underbrace{\tfrac{m-1}{m}}_{0 \leq \quad \leq 1} \Sigma H_j + \underbrace{\Sigma a_i D_i + D}_{= \lceil D \rceil}$$

$$= \{ f \in \mathbb{C}(X) \mid \operatorname{div} f + K_X + \lceil D \rceil |_U \geq 0 \}$$ □

Rmk. · [KM], Prop 2.67 = Bloch-Gieseker covering + cyclic covering ⇒ kawamata covering

· Compare Claim 2.65

Lemma. X: sm. var., $|V|$: a bpf. linear system, ΣD_i: a SNC divisor.

If $H \in |V|$ is a general divisor, then $H + \Sigma D_i$ again SNC divisor.

(pf.) Bertini theorem. □

Proof of kawamata covering.

· $n = \dim X$, fixed a very ample divisor A on X. Take $m \gg 0$, so that $ma_i \in \mathbb{N}$, $\frac{m-1}{m} \geq a_i$ $\forall i \in I$ and s.t. $mA - D_i$ is very ample $\forall i \in I$. By Lemma, $\forall i \in I$, we may take n general element

$$H_1^{(i)}, \cdots, H_n^{(i)} \in |mA - D_i| \text{ s.t. } \sum_{i \in I} D_i + \sum_{i \in I} \sum_{k=1}^{n} H_k^{(i)} \text{ has SNC.}$$

Let $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ be an affine open cover of X with the

transitions of $\mathcal{O}_X(A)$: $\{ a_{\lambda\mu} \in H^0(U_\lambda \cap U_\mu, \mathcal{O}_X^*) \}$ s.t. $(H_k^{(i)} + D_i)_{U_\lambda} = \operatorname{div}(\varphi_{k,\lambda}^{(i)})$ on U_λ

local section of $\mathcal{O}_X(mA)$: $\{ \varphi_{k,\lambda}^{(i)} \in H^0(U_\lambda, \mathcal{O}_X) \}$ $\quad \varphi_{k\lambda}^{(i)} = a_{\lambda\mu}^m \varphi_{k\mu}^{(i)}$ on $U_\lambda \cap U_\mu$.

· Claim. the normalization Y of X in $\mathbb{C}(X)[(\varphi_{k\lambda}^{(i)})^{1/m}]_{i,k}$ for some (any) $\lambda \in \Lambda$ provides the desired cover. ([Liu, Chap 4, Prop 1.25])

·· Y is smooth: $\forall$ closed point $x \in U_\lambda$, set $I_x = \{ i \in I : x \in D_i \}$. Since $\bigcap_{k=1}^{n} H_k^{(i)} \cap D_i = \emptyset$ (intersection of $(n+1)$ SNC) $\Rightarrow \forall i \in I_x$ $\exists 1 \leq k_i \leq n$ s.t. $x \notin H_{k_i}^{(i)}$. Now the set

$$R_x := \{ \varphi_{k\lambda}^{(i)} : i \in I_x \} \cup \{ \varphi_{k\lambda}^{(i)} : i \notin I_x, x \in H_k^{(i)} \} \cup \{ \varphi_{k\lambda}^{(i)} / \varphi_{k_i\lambda}^{(i)} : i \in I_x, x \in H_k^{(i)} \}$$

form a part of regular system of parameters of $\mathcal{O}_{X,x}$ (regular local ring), the set

$$T_x := \{ \varphi_{k\lambda}^{(i)} / \varphi_{k_i\lambda}^{(i)} : i \notin I_x, x \notin H_k^{(i)} \} \cup \{ \varphi_{k\lambda}^{(i)} : i \in I_x, x \notin H_k^{(i)} \}$$

are all units in $\mathcal{O}_{X,x}$.

sub-Lemma [KMM, Lemma 1-1-2, p.304]

$\mathcal{O}_{X,x} = (R, m_R)$: regular local ring $\mathbb{C}$-alg of $\dim = n$ with $R/m_R \simeq \mathbb{C}$.

$R_x = \{z_1, \cdots, z_\ell\} \subseteq \{z_1, \cdots, z_n\}$: a regular system of parameters, $T_x = \{u_1, \cdots, u_s\} \subseteq R^\times$: units.

Fix $m \in \mathbb{N}$, $1 \le \ell \le n$. Let "Y" $R' := R[z_1^{1/m}, \cdots, z_\ell^{1/m}, u_1^{1/m}, \cdots, u_s^{1/m}]$, $\forall m_{R'} \in \operatorname{Max} R'$, $y \in Y$, $\pi(y) = x$

$\mathcal{O}_{Y,y} = R_1 := R'_{m_{R'}}$ is a regular local ring with a regular system of parameters

$\{z_1^{1/m}, \cdots, z_\ell^{1/m}, z_{\ell+1}, \cdots, z_n\}$ (Hence Y is smooth)

(subpf.) Note that $\dim R_1 = n$, it suffices to show $\{z_1^{1/m}, \cdots, z_\ell^{1/m}, z_{\ell+1}, \cdots, z_n\}$ gen. $m_{R_1} = m_{R'} R'$.

- $m_{R'} = \langle z_1^{1/m}, \cdots, z_\ell^{1/m}, z_{\ell+1}, \cdots, z_n, u_1^{1/m} - \alpha_1, \cdots, u_s^{1/m} - \alpha_s \rangle_{R'}$ for some $\alpha_t \in \mathbb{C}^\times$:

  Since $m_{R'} \supseteq m_{R'} \cap R = m_R = \langle z_1, \cdots, z_n \rangle$ and $z_i = (z_i^{1/m})^m$ for $1 \le i \le n$, $z_i^{1/m} \in m_{R'}$ $\forall 1 \le i \le \ell$.

  On the other hand, $R'/m_{R'} \simeq \mathbb{C} \Rightarrow u_t^{1/m} - \alpha_t \in m_{R'}$. Hence RHS $\subseteq m_{R'}$.

  $u_t^{1/m} \mapsto \alpha_t \in \mathbb{C}^\times$ (by unit)

  Since $R'/\text{RHS} \simeq \mathbb{C}$, $m_{R'} = \text{RHS}$.

- It is suffices to show that $u_t^{1/m} - \alpha_t \in \langle z_1^{1/m}, \cdots, z_\ell^{1/m}, z_{\ell+1}, \cdots, z_n \rangle_{R_1}$. From

$$\underset{\substack{\Uparrow \\ m_R = \langle z_1, \cdots, z_n \rangle_R}}{u_t - \alpha_t^m} = (u_t^{1/m} - \alpha_t)\underline{(u_t^{m-1/m} + \cdots + \alpha_t^{m-1})} \quad \text{unit in } R_1$$

$$\equiv m\alpha_t^{m-1} \not\equiv 0 \mod m_{R'}.$$

$$\left(u_t^{1/m} \equiv \alpha_t \mod m_{R'} \Rightarrow u_t \equiv \alpha_t^m \mod \overset{m_R}{\overset{\shortparallel}{m_{R'} \cap R}}\right)$$

$\square$

Big divisors ("closed to ample")

Def. (Semigroup and exponent of divisor)

X : proj. var., D : Cartier divisor. Define $N(D) = N(X, D) := \{m \in \mathbb{N} \mid h^0(mD) \neq 0\}$ (is semigroup)

Assuming $N(D) \neq 0$, $\exists e \in \mathbb{N}$ s.t. $e\mathbb{Z} = \mathbb{Z}N(D)$. Define $e(D) := e$ called the exponent of D.

$e(D) = \gcd N(D) \Rightarrow \exists m' > 0$ s.t. $\forall m > m'$, $me \in N(D)$.

Lemma. X : proj. sch / a field, $\dim X = n$. B : Cartier divisor. $\exists C > 0$ s.t. $h^0(mB) \le Cm^n$ for all $m \in \mathbb{N}$.

(pf.) Let H be v.a. divisor s.t. $h^i(mH) = 0$ $\forall i > 0$, $m \in \mathbb{N}$ and $H - B \sim D \ge 0$. Then

$$0 \to \mathcal{O}_X(-mD) \to \mathcal{O}_X \xrightarrow{\otimes mH} 0 \to \mathcal{O}_X(m(H-D)) \to \mathcal{O}_X(mH)$$

$$\Rightarrow h^0(mB) \le h^0(mH) = \chi(mH) = \frac{H^n}{n!} m^n + O(m^{n-1}) \le Cm^n.$$

$\square$

Def. X : proper var. of $\dim = n$.

- A Cariter divisor D is big if $\exists c > 0$ s.t. $h^0(mD) > cm^n$ for $m \in N(X, D)$.
- A $\mathbb{Q}$-divisor D is big if $\exists k > 0$ s.t. kD is Cariter and big.

- A $\mathbb{R}$-divisor D is big if $D = \sum_{\text{finite}} a_i D_i$, where D_i : big Cartier and $a_i > 0$.

Def. (Iitaka dimension)

- X: normal var., the Iitaka dim. of a Cartier divisor D with $N(D) \neq \phi$ is
$$0 \le \kappa(D) = \kappa(X, D) := \max_{m \in N(D)} \{\dim \underbrace{\operatorname{Im} \phi_{|mD|}}_{\overline{\phi_{|mD|}(\xi)}}\} \le \dim X$$
where $\overline{\{\xi\}} = X$

- $\kappa(D) = -\infty$ if $N(D) = \phi$
- If X is non-normal, pass to it normalization $\nu : X' \longrightarrow X$ and set $\kappa(X, D) = \kappa(X', \nu^* D)$.

Rmk. (Iitaka conjecture $C_{n,m}$)

$f : X^n \longrightarrow Y^m$ surj. with connected fibers between sm. proj. var.
$\ \rotatebox{90}{$\subseteq$}$ — F : a general fiber of f

$\Rightarrow \kappa(X) \ge \kappa(Y) + \kappa(F)$, where $\kappa(\cdot) := \kappa(\cdot, K_\cdot)$.

Examples

1. Choose T: a proj. var. of $\dim = d \ge 1$ s.t. $\exists$ non-trivial $\eta \in \operatorname{Pic} T$ with $\eta^{\otimes e} = \mathcal{O}_T$.
Y: any proj. var. of $\dim = k$, B: v.a. on Y.
Let $X = Y \times T$, $\mathcal{O}_X(D) = pr_1^* \mathcal{O}_Y(B) \otimes pr_2^* \eta \rightsquigarrow e(D) = e$, $N(D) = e\mathbb{N}$.
$$\begin{cases} mD \text{ is bpf.} & \text{if } m \in N(D). \\ h^0(mD) = 0 & \text{if } m \notin N(D). \\ \kappa(D) = k \end{cases}$$

2. (non-normal var.) Let $T \subseteq \mathbb{P}^2$ be a nodal plane cubic curve.
$\mathcal{O}_T(D) \in \operatorname{Pic}^0(T) \simeq \mathbb{G}_m$, a non-torsion invertible sheaf of $\deg = 0 \rightsquigarrow H^0(T, mD) = 0 \quad \forall m > 0$.
But $\nu : T' \longrightarrow T$, $\mathcal{O}_{T'}(D') = \nu^* \mathcal{O}_T(D) = \mathcal{O}_{\mathbb{P}^1} \Rightarrow H^0(T', mT') = 1 \quad \forall m > 0$.
$T' \simeq \mathbb{P}^1$
By taking product as in 1, one get example with $\begin{cases} h^0(mD) = 0 \quad \forall m > 0 \\ \kappa(x', D') \in \mathbb{N} \text{ arbitray} \end{cases}$

Prop. (Characterization of big divisor) (cf. [KM, Lemma 2.60])

X: proj. var. of $\dim = n$, D: a Carifer divisor on X. TFAE

(1) D is big

(2) For any ample Cartier divisor A, $\exists m\in\mathbb{N}$, an eff. divisor E' s.t. $mD\sim A+E$

(2') For some ample Cartier divisor A, $\exists m\in\mathbb{N}$, an eff. divisor E' s.t. $mD\sim A+E$

(2'') For some ample Cartier divisor A, $\exists m\in\mathbb{N}$, an eff. divisor E' s.t. $mD\equiv A+E$

(3) $\kappa(X,D)=\dim X$, i.e. $\nu: X'\xrightarrow{\text{normalization}} X$, $D'=\nu^*D$, for some $m>0$ s.t. the rational map $\phi_{|mD|}: X' \dashrightarrow \mathbb{P}H^0(mD')$ is birational onto it image.

Lemma. (Kodaira's lemma)

X: $\overset{\text{(proj.)}}{\text{proper}}$ var., D: big divisor, $\forall$ eff. Cartier F on X, we have $H^0(X,mD-F)\neq 0$ for large $m\in N(X,D)$.

(pf.) Let $n=\dim X$. $0\to H^0(mD-F)\longrightarrow \underbrace{H^0(mD)}_{>cm^n}\longrightarrow \underbrace{H^0(mD|_F)}_{=O(m^{n-1})\text{ by Hw3.}}$ $\square$

for some $c>0$
for large $m\in N(D)$

Proof of Prop.

- $(1)\Rightarrow(2)$: Take $r\gg 0$ s.t. $rA\sim H_r\geq 0$, $(r+1)A\sim H_{r+1}\geq 0$. Applying Kodaira's lemma with $F=H_{r+1}$ to find $m\in\mathbb{N}$ and $0\leq F'\in|mD-H_{r+1}|$ with $mD\sim H_{r+1}+F'\sim A+\underbrace{H_r+F'}_{=E}$.

- $(2)\Rightarrow(2')\Rightarrow(2'')$: Trivial.
- $(2'')\Rightarrow(3)$: If $mD\equiv A+E$, then $mD-E$ is ample. WLOG $mD\sim H+E'$, where H: v.a. and $E'\geq 0$ (after replacing an even large mult. of D) $\rightsquigarrow \dim X\geq \kappa(X,D)\geq\kappa(X,H)=\dim X$.
- $(3)\Rightarrow(1)$: First we assume that X is normal and $Y:=\operatorname{Im}\phi_{|D|}\subseteq\mathbb{P}H^0(D)$ has $\dim=n$.

Then $h^0(\mathcal{O}_Y(m))+\underbrace{\sum_{i>0}(-1)^i h^i(\mathcal{O}_Y(m))}_{=O(m^{n-1})\text{ by Hw 3.}}=\chi(\mathcal{O}_Y(m))=\frac{\deg Y}{n!}m^n+O(m^{n-1})$.

$$\begin{tikzcd} X \arrow[r, dashed, "\phi:=\phi_{|D|}"] & \mathbb{P}H^0(D) \\ U \arrow[u, phantom, "\cup"] \arrow[r, "\text{dominant}"] & Y \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\rightsquigarrow \operatorname{codim} X\setminus U\geq 2$ by X is normal and $\mathbb{P}H^0(D)$ is proper.

largest Zariski open set ϕ is defined

$\Rightarrow \mathcal{O}_Y\hookrightarrow\phi_*\mathcal{O}_U$

$\Rightarrow \mathcal{O}_Y(m)\hookrightarrow\phi_*\mathcal{O}_U\otimes\mathcal{O}_Y(m)\overset{\text{proj. formula}}{=}\phi_*\phi^*\mathcal{O}_Y(m)\Rightarrow\phi^\#: H^0(Y,\mathcal{O}_Y(m))\hookrightarrow H^0(U,\mathcal{O}_X(mD)) \overset{X:\text{ normal, }\operatorname{codim}X\setminus U\geq 2.}{=} H^0(X,\mathcal{O}_X(mD))$

In general, let $\nu: X'\to X$, $D'=\nu^*D\rightsquigarrow\exists$ exact sequence $0\to\mathcal{O}_X\to\nu_*\mathcal{O}_{X'}\to\eta\to 0$, where η supported on a scheme of $\dim\leq n-1$. After $\otimes\,\mathcal{O}(mD)$ and long exact sequence,

$$h^0(X,\mathcal{O}_X(mD)) \le \overset{h^0(X',\mathcal{O}_{X'}(mD'))}{\overset{\parallel}{h^0(X,\underbrace{\nu_*\mathcal{O}_{X'}\otimes\mathcal{O}_X(mD)}_{\nu_*(\mathcal{O}_{X'}\otimes\nu^*\mathcal{O}_X(mD))})}} \le h^0(X,\mathcal{O}_X(mD)) + \overset{O(m^{n-1}) \text{ by Hw3}}{\overset{\parallel}{h^0(X,\eta\otimes\mathcal{O}_X(mD))}}$$

Hence $\kappa(X',D')=n \Leftrightarrow h^0(mD) > cm^n$ for some $c>0$, $N(D) \ni m \gg 0$ $\square$

Prop. (Bigness of nef divisor)

X : proj. var. of $\dim = n$. TFAE:

(1) D is big and nef.

(2) D is nef with $D^n > 0$.

(3) $\exists E$: eff. divisor s.t. $D - \frac{1}{k}E$ is ample for $k \gg 0$.

(pf.)

- $(1)\Leftrightarrow(2)$: Since D is nef, by Hw3 and asy. R-R, $h^0(mD) = \frac{D^n}{n!}m^n + O(m^{n-1})$.
  Hence D is big $\Leftrightarrow D^n > 0$.
- $(1)\Rightarrow(3)$: Fix an ample divisor A, $\exists\, m \gg 0$, $mD \sim A+E$ with E is effective.
  For $k \ge m$, $kD \sim \underbrace{(k-m)D + A}_{\text{nef + ample} \Rightarrow \text{ample}} + E$.
- $(3)\Rightarrow(1)$: $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)} \Rightarrow D \in \mathrm{Nef}(X)$. $kD = \underbrace{(kD-E)}_{\text{ample}} + \underbrace{E}_{\text{eff.}} \Rightarrow D$ is big. $\square$

Example. (Big divisor with negative self-intersection)

Choose X : sm. proj. surface, E : (-1)-curve, A : v.a. divisor on X. The $D_\ell = A + \ell E$ is big for $\ell \ge 0$.
But $D_\ell^2 = -\ell^2 + 2\ell(A.E) + A^2 < 0$ as $\ell \gg 0$.

§ Discrepancies

Def. (birational transform) $f: Y \dashrightarrow X$ a birational map of var.
(or strict, or proper) $\quad Z = \overline{\{\eta\}} \subseteq Y$ a closed subvar.

- The birational transform of Z is $f_* Z := \begin{cases} \overline{f(\eta)} \subseteq X & \text{if } \eta \in \mathrm{dom}(f) \\ 0 & \text{if } \eta \notin \mathrm{dom}(f) \end{cases}$

- If $g: X \longrightarrow Y$ is a birational morphism, we get $g^{-1}_* Z := (g^{-1})_* Z$.

Rmk. If f is not a morphism, then f_* need not preserve linear or alg. equivalence.

$*$ X : normal var. / $k = \bar{k}$, char $= 0$

<u>Weil divisors, revisited</u>

$D = \overline{\{\eta\}}$: a prime divisor $\Rightarrow \exists! \, v_D = v(D, X) : k(X) \setminus \{0\} \longrightarrow \mathbb{Z}$ ($\because X$ is sep. [Hart. II Ex 4.5])

(rank 1) - valuation $\quad (\mathcal{O}_{X,\eta}$: DVR with quotient field $k(X))$

Def. · Let $f: Y \longrightarrow X$ be a (not necessary proper) birational morphism from a normal variety Y.
Any prime divisor $E \subseteq Y$ is called a divisor over X.

- (center of E on X) $\mathrm{center}_X(E) := \overline{f(E)} \subseteq X$.
- A rk 1 valuation $v: k(X) \setminus \{0\} \longrightarrow \mathbb{Z}$ is geometric (or algebraic) if $\exists$ a divisor E over X s.t. $v = v_E$.

Rmk. · If $E \subseteq Y$, $E' \subseteq Y'$ are divisor over X,

$$\begin{tikzcd} Y \arrow[rr, dashed] \arrow[dr] & & Y' \arrow[dl, "\text{birational}"] \\ & X & \end{tikzcd}$$

then $\begin{cases} v(E, Y) = v(E', Y') \\ \mathrm{center}_X E = \mathrm{center}_X E' \end{cases} \iff Y \to X \dashrightarrow Y'$ is an isom. at the geometric points of $e \in E$ and $e' \in E'$ (i.e. $\mathcal{O}_{Y,e} \simeq \mathcal{O}_{Y',e'}$)

- $X \overset{\text{bir.}}{\dashrightarrow} Y \in [X]_{\text{bir.}} \iff k(X) \simeq k(Y) = K$

$\{f_* D : X \overset{f}{\underset{\text{bir.}}{\dashrightarrow}} Y, \eta \in \mathrm{dom}(f)\} \iff$ a geometric valuation $v(D, X) : K \setminus \{0\} \longrightarrow \mathbb{Z}$

Def. · $\mathrm{WDiv}(X)_{\mathbb{Q}, \mathbb{R}} (= Z^1(X)) = \{ D = \sum_{\text{finite}} a_i D_i \mid D_i : \text{prime divisor}, a_i \in \mathbb{Z}, \mathbb{Q}, \mathbb{R} \}$

- $f: Y \dashrightarrow X$ a birational map var. $\rightsquigarrow f_* : \mathrm{WDiv}(X) \longrightarrow \mathrm{WDiv}(Y)$ by extending the coeff.
- f is said to be isom. in codim 1 if f_* is bijective.

$(X:\text{integral})$

$D=\sum a_i D_i \in WDiv(X)$, define a sheaf $\mathcal{O}_X(D)\subseteq \mathcal{K}$ = the constant sheaf corresp. to $k(X)$ s.t.
$\forall U\subseteq X$, $\Gamma(U,\mathcal{O}_X(D)) := \{ f\in k(X) : \nu_{D_i}(f)\geq -a_i$ for all $U\cap D_i\neq\emptyset\}$

i.e. $\operatorname{div}(f)+D\geq 0$

Facts. • $\mathcal{O}_X(D)$ is a "reflexive sheaf of rank 1" (divisoral sheaf) (on generic point)
(coh. sheaf s.t. $\mathcal{F}\simeq\mathcal{F}^{\vee\vee}$)

• $D\sim D' \Leftrightarrow \mathcal{O}_X(D)\simeq\mathcal{O}_X(D')$

• $WDiv(X)\longleftrightarrow\{\text{divisoral sheaves}\}$
$Cl(X)\simeq$ " $/\text{isom.}$

• $\mathcal{O}_X(D+D') = (\mathcal{O}_X(D)\otimes\mathcal{O}_X(D'))^{\vee\vee}$ take reflexive hull
$\mathcal{O}_X(mD) =: \mathcal{O}_X(D)^{[m]}$.

• If $D\geq 0$, then $\mathcal{O}_X(-D)\simeq \mathcal{I}_D$.

(cf. [Reid.80] M.Reid, canonical threefolds, Appendix to §1, p281)

<u>Canonical divisor</u>

$U := X\setminus \operatorname{Sing}X \hookrightarrow X \xrightarrow{X:\text{normal}} \operatorname{codim}_X(X\setminus U)\geq 2$.

$\omega_U := \det(\Omega^1_X|_U)\in Pic(U) \rightsquigarrow j_*\omega_U$ is a divisoral sheaf $\Rightarrow j_*\omega_U = \mathcal{O}_X(K_X)$ for some $K_X\in WDiv(X)$.
(canonical sheaf) (canonical divisor)

Rmk. • By construction, K_X is defined up to linear equiv., $\mathcal{O}_X(K_X)$ is uniquely determined.
Usually, we will fix a divisor K_X.

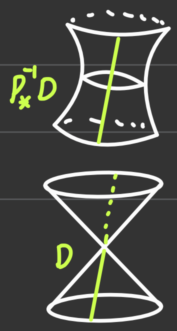
• [KM, P.184 Prop 5.75] X: normal proj. $\Rightarrow \mathcal{O}_X(K_X)\simeq\omega_X$: the dualizing sheaf of X.

Def. • $D\in WDiv(X)_{\mathbb{Q}}$ is $\mathbb{Q}$-Cartier if $mD\in CDiv(X)$ for some $m\in\mathbb{Z}$.

• X is $\mathbb{Q}$-Gorenstein if K_X is $\mathbb{Q}$-Cartier.

• X is $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier.

e.g. [Hart. II, Example 6.5.2, P.153]



$\mathbb{P}^1 = C \subseteq Bl_0X$
$\downarrow \quad \downarrow P$
$0=(0,0,0)\in X = (xy=z^2)\subseteq \mathbb{A}^3$
$\rotatebox{90}{$\subseteq$}$
$D=(x=z=0)$ is a prime divisor, NOT Cartier
$2D=(x=0)$ is Cartier.

$C.P_*^{-1}D = 1$, $C^2=-2$, $(f^*D.C)=(D.f_*C)=0$
$\rightsquigarrow p^*D = P_*^{-1}D+\frac{1}{2}C$ is not integral.

Def. (log resolution) X: normal var., $D \in \mathrm{WDiv}(X)_{\mathbb{Q}}$

A log resolution of (X, D) is a proper birational morphism $f: Y \longrightarrow X$ s.t.

- Y is smooth
- ✻ $\mathrm{Exc}(f)$ is a divisor
- $\mathrm{Exc}(f) + f_*^{-1}D$ has SNC support.

From Hironaka theorem, log resolution exists for var./k with $\mathrm{char}\, k = 0$.

Def. (Discrepancies) X: normal var., $\Delta = \sum a_i D_i$, $a_i \in \mathbb{Q}$, D_i: distinct prime divisors.

Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, i.e. $m(K_X+\Delta) \in \mathrm{CDiv}(X)$ for some $m \in \mathbb{Z}$.

- Suppose $\begin{array}{l} Y\text{: normal} \\ \downarrow f\text{: birational} \\ X \end{array}$, $f_*^{-1}\Delta = \sum a_i f_*^{-1} D_i$. Set $V := Y \setminus \mathrm{Exc}(f)$.

  Since $\mathcal{O}_Y(m(K_Y + f_*^{-1}\Delta))|_V \simeq f^*\mathcal{O}_X(m(K_X+\Delta))|_V \Rightarrow \forall$ prime divisor $E \subseteq \mathrm{Exc}(f)$, $\exists\, a(E,X,\Delta) \in \mathbb{Q}$ s.t. $m a(E,X,\Delta) \in \mathbb{Z}$ and $m(K_Y + f_*^{-1}\Delta) \sim m f^*(K_X+\Delta) +$ "$\sum_{E:\text{ exceptional}} m a(E,X,\Delta) E$"

  is uniquely determined

- (nonexceptional divisors) Set $a(D_i, X, \Delta) = -a_i$ and $a(D, X, \Delta) = 0$ for any prime divisor $D \subseteq X$, $D \neq D_i\ \forall i$

Rmk.
- $a(E,X,\Delta)$ depends only on the valution v_E but not on particular choice of f and Y.
- Some authors use log decrepancies, defined as $1 + a(E,X,\Delta)$ (≥ 0) later
- $\begin{array}{l} V \subseteq Y\text{: normal} \\ \quad \downarrow f\text{: birational} \\ \quad X \end{array}$, $V = Y \setminus \mathrm{Exc}(f)$, fix canonical divisors s.t. $f_* K_Y = K_X$

$$\begin{array}{ccccccc} \bigoplus_{E:\text{ exceptional}} \mathbb{Q}E & \longrightarrow & \mathrm{WDiv}(Y)_{\mathbb{Q}} & \longrightarrow & \mathrm{WDiv}(V)_{\mathbb{Q}} & \to & 0 \\ \sum a(E,X,\Delta)E & \longmapsto & (K_Y + f_*^{-1}\Delta) - f^*(K_X+\Delta) & \longmapsto & 0 & & \end{array}$$

$\Rightarrow \exists\, \Delta_Y \in \mathrm{WDiv}(Y)_{\mathbb{Q}}$ s.t. $K_Y + \Delta_Y = f^*(K_X + \Delta)$ in $\mathrm{WDiv}(Y)_{\mathbb{Q}}$. Here Δ_Y is uniquely determined as the sum of $f_*^{-1}\Delta$ and a $\mathbb{Q}$-div. supported on $\mathrm{Exc}(f)$, i.e. $f_*\Delta_Y = \Delta$ in $\mathrm{WDiv}(X)_{\mathbb{Q}}$

Def. X: normal, $\Delta = \sum a_i D_i \in \mathrm{WDiv}(X)_{\mathbb{Q}}$

$\mathrm{discrep}(X,\Delta) := \inf\{a(E,X,\Delta) : E \text{ is an exceptional (prime) divisor over } X\}$

$\rotatebox{90}{$\leq$}$

$\mathrm{totalcrep}(X,\Delta) := \inf\{a(E,X,\Delta) : E \text{ is a divisor over } X\}$

Prop 1. Either $\operatorname{discrep}(X,\Delta) = -\infty$ or $-1 \le \operatorname{totaldiscrep}(X,\Delta) \le \operatorname{discrep}(X,\Delta) \le 1$.

$$\operatorname{totaldiscrep}(X,\Delta) \le a(D_i, X, \Delta) = -a_i$$

Def. (boundary and subboundary)

If $a_i \in [0,1]$ $\forall i$, we call D a boundary divisor.

$a_i \in (-\infty, 1]$ $\forall i$, we call D a subboundary divisor.

Def. (multipicity) X : variety $\ni x$: a point (not necessary closed), $D = \{(U_i, f_i)\}$: eff. Cartier divisor. $\operatorname{mult}_x D := \underset{x \in U_i}{\operatorname{ord}}(f_i) = \max\{d \in \mathbb{N} \mid f_i \in m_x^d\}$.

If X is normal and $E = \overline{\{\eta\}}$ is a prime divisor, then $\operatorname{mult}_E D := \operatorname{mult}_\eta D = \operatorname{coeff}_E D$

Lemma. X : sm. var. $\supseteq Z$: a closed subvar.

$$\begin{tikzcd} \operatorname{Bl}_Z X \arrow[d, "p"'] & \supseteq & E \arrow[d] \\ X & \supseteq & Z = \overline{\{\eta\}} \end{tikzcd}$$

E = the irr. component of $p^{-1}(Z)$ which dominates Z.

$\Delta = \sum a_i D_i$, $a_i \in \mathbb{Q}$, D_i : distinct prime divisor.

$\Rightarrow a(E, X, \Delta) = (\operatorname{codim}_X Z - 1) - \sum a_i \operatorname{mult}_Z D_i$.

(pf.) Replacing X by $X \setminus \operatorname{Sing} Z \ni \eta$, we may assume Z is smooth. Then

$$\begin{cases} K_{\operatorname{Bl}_Z X} - p^* K_X = (\operatorname{codim}_X Z - 1) E \\ p^* D_i - p_*^{-1} D_i = \operatorname{mult}_Z D_i \cdot E \end{cases}$$

□

Lemma. Y : normal $\xrightarrow{f : \text{ proper, birational}}$ X : normal, $\Delta_X \in \operatorname{WDiv}(X)_{\mathbb{Q}}$.

Write $K_Y + \Delta_Y = f^*(K_X + \Delta_X)$, where $f_* K_Y = K_X$, $\Delta_Y \in \operatorname{WDiv}(Y)$ with $f_* \Delta_Y = \Delta_X$ in $\operatorname{WDiv}(X)$.

Then for any divisor F over X, $a(F, Y, \Delta_Y) = a(F, X, \Delta_X)$.

In particular, $\operatorname{discrep}(X, \Delta_X) = \min\{\operatorname{discrep}(Y, \Delta_Y), \underset{E : f\text{-excep.}}{a(E, X, \Delta_X)}\}$, $\operatorname{totaldiscrep}(X, \Delta_X) = \operatorname{totaldiscrep}(Y, \Delta_Y)$

(pf.)
- If $F \subseteq Y$, the equality follows from the definition of Δ_Y ($f_* \Delta_Y = \Delta_X$).
- Assume that F does not appear as a divisor on Y ($\rightsquigarrow$ not on X)

Consider

$$\begin{tikzcd} & Z \ni F \arrow[d, "p_h"] \arrow[ddl, bend right, "g"'] \arrow[ddr, bend left] & \\ & \Gamma_h \arrow[dl] \arrow[dr] & \\ Y \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & Y' \ni F \arrow[dl, "f' : \text{ birational}"] \\ & X & \end{tikzcd}$$

, we may assume $F \subseteq Z$: normal $\xrightarrow{g : \text{ birational}}$ Y

To compute $a(F,Y,\Delta_Y)$, we find $\Delta_Z \in WDiv(Z)_{\mathbb{Q}}$ s.t. $K_Z+\Delta_Z \overset{(1)}{=} g^*(K_Y+\Delta_Y)$ and $\begin{cases} g_*K_Z=K_Y \\ g_*\Delta_Z=\Delta_Y \end{cases}$

$\Rightarrow K_Z+\Delta_Z \overset{(2)}{=} (f\circ g)^*(K_X+\Delta_X)$ with $\begin{cases} (f\circ g)_*K_Z=K_X \\ (f\circ g)_*\Delta_Z=\Delta_X \end{cases} \Rightarrow -a(F,X,\Delta_X) \overset{\text{by }(2)}{=} \mathrm{mult}_F\Delta_Z \overset{\text{by }(1)}{=} -a(F,Y,\Delta_Y)$

□

Proof of Prop. 1.

(pf.) • $\mathrm{discrep}(X,\Delta)\le 1$: WLOG X is sm. ($\because$ $U=X\setminus \mathrm{Sing}X$ dense open $\Rightarrow \mathrm{discrep}(X,\Delta)\le \mathrm{discrep}(U,\Delta|_U)$)

Now let $Z=\overline{\{\eta\}}$ be any codim 2 subvar. of X s.t. $\eta\notin D_i\ \forall i$ ($Z\not\subseteq \mathrm{Supp}\Delta$)

$$\begin{array}{ccc} Bl_Z X & \supseteq & E \\ \downarrow & & \downarrow \\ X & \supseteq & Z \end{array} \quad , \ \mathrm{center}_X E = Z \Rightarrow a(E,X,\Delta)=1-0 \text{ by Lemma 1.}$$

• If $-1>\mathrm{totaldiscrep}(X,\Delta)$, then $\mathrm{discrep}(X,\Delta)=-\infty$: By assumption,

$\exists\ \overset{\leftarrow\text{normal}}{Y} \supseteq E=\overline{\{e\}}$: prime divisor s.t. $a(E,X,\Delta)=-1-c$ with $c>0$. WLOG Y is sm.

$\downarrow f$: birational

X (replace by $Y\setminus\mathrm{Sing}Y \ni e$)

Write $K_Y+\Delta_Y=f^*(K_X+\Delta)$ with $f_*K_Y=K_X$, $f_*\Delta_Y=\Delta$. $\Delta_Y=(1+c)E+\underbrace{(\text{other excep. divisor})+f_*^{-1}\Delta}_{(*)}$

Pick $Z_0\subseteq E$, $\mathrm{codim}_Y Z_0$ and not in $(*)$ $\Rightarrow \mathrm{mult}_{Z_0}E=1+c$. Consider

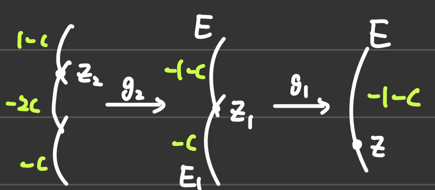
$$\begin{array}{ccccc} Y_1=Bl_{Z_0}Y & \supseteq & E_1 \\ g_1\downarrow & & \downarrow \\ E\subseteq Y & \supseteq & Z_0 \end{array} \Rightarrow a(E_1,X,\Delta) \underset{\text{Lemma 2}}{=} a(E_1,Y,\Delta_Y) \underset{\text{Lemma 1}}{=} 1-(1+c)=-c.$$

Let $Z_1:=g_{1*}^{-1}E\cap E_1$. On $\begin{array}{ccc} Y_2:=Bl_{Z_1}Y_1 & \longrightarrow & Y_1 \\ \cup & & \cup \\ E_2 & \longrightarrow & Z_1 \end{array}$, same computation gives

$a(E_2,X,\Delta)=a(E_2,Y,\Delta_Y)=-2c$. By induction, we construct $\begin{array}{ccc} Y_m:=Bl_{Z_{m-1}}Y_{m-1} & \xrightarrow{g_m} & Y_m \\ \cup & & \cup \\ E_m & \longrightarrow & Z_{m-1} \\ & & \parallel \\ & & E_{m-1}\cap(g_{m-1})^{-1}_*E \end{array}$ s.t.

excep. over Y (thus over X)

$a(E_m,X,\Delta)=-mc$



□

Prop2. (Computing discrep for a SNC sub-boundary)

X: sm. var., "$\Delta=\sum a_iD_i$ for SNC support with $\mathbb{Q}\ni a_i\le 1$"

$\Rightarrow \mathrm{discrep}(X,\Delta)=\min\{1,\ \min_i\{1-a_i\},\ \min_{\substack{D_i\cap D_j\neq\emptyset \\ i\neq j}}\{1-a_i-a_j\}\} =: r(X,\Delta)$

(pf.)

E : excep. div. with $\mathrm{center}_X E = Z$, $E \subset \mathrm{Bl}_Z X$

- LHS $\le$ RHS : Consider $\mathrm{Bl}_Z X \longrightarrow X$, $\mathrm{codim}_X Z = 2$. Then

$$a(E,X,\Delta) = \begin{cases} 1 & \text{if } Z \not\subseteq \mathrm{Supp}\,\Delta \\ 1-a_i & \text{if } Z \subseteq D_i,\ Z \not\subseteq D_j\ \forall j \neq i \\ 1-a_i-a_j & \text{if } Z \subseteq D_i \cap D_j,\ i \neq j \end{cases} \Rightarrow \mathrm{discrep}(X,\Delta) \le r(X,\Delta).$$

(by SNC, $Z \not\subseteq D_i \cap D_j \cap D_k$ for distinct i,j,k)

- RHS $\le$ LHS : For an exceptional divisor E over X, say $f : Y \xrightarrow{\text{bir.}} X$, $E \subset Y$: excep., we want to prove $a(E,X,\Delta) \ge r(X,\Delta)$.

Fact. (a result of Zarski and Abhyanker, cf. [KM, Lemma 2.4.5])
After possibly shrinking X, $\exists$ a seq. of blow-ups along sm. centers s.t.

$$\begin{tikzcd} E \subseteq Y = X_m \arrow[r] \arrow[rrr, bend right, "f"'] & X_{m-1} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r, "g_1"] & X_0 = X \end{tikzcd}$$

($r(X,\Delta)$ increase or remains the same after shrinking X)

Induction on m.

- $m=1$: $f : \mathrm{Bl}_Z X \longrightarrow X$, $E \longrightarrow Z$ ($E \subseteq \mathrm{Bl}_Z X$, $Z \subseteq X$), $\mathrm{codim}_X Z \ge 2$, s.t. $E + f_*^{-1}\Delta$ has SNC support (by shrinking X around a general point Z)

Set $I := \{ i : Z \subseteq D_i \}$. Since $\sum D_i$ SNC, $\# I \le \mathrm{codim}_X Z$. Also

$$a(E,X,\Delta) \overset{\text{Lm1.}}{=} (\mathrm{codim}_Z X - 1) - \mathrm{mult}_Z \Delta = (\mathrm{codim}_X Z - \# I) - 1 + \sum_{i \in I} (1-a_i)$$

$$\ge \begin{cases} 1 & \text{if } I = \emptyset \\ 1-a_0 & \text{if } I = \{i_0\} \\ 1-a_0-a_1 & \text{if } I = \{i_0, i_1\} \end{cases}$$

$\Rightarrow a(E,X,\Delta) \ge r(X,\Delta)$.

- $m>1$: Write $f : Y \xrightarrow{f_1} X_1 = \mathrm{Bl}_Z X \xrightarrow{g_1} X$, with $E \subseteq Y$, $E_1 \subseteq X_1$, $E_1 \longrightarrow Z \subseteq X$, and $K_{X_1} + \Delta_{X_1} = g_1^*(K_X + \Delta)$.

WLOG $\Delta_1 = -a(E_1,X,\Delta)E_1 + (g_1)_*^{-1}\Delta$ has SNC support. Note that we also have
$-a(E_1,X,\Delta) \le 1$ ($\because X$: sm., SNC support, Hw) (use for induction)

$$\Rightarrow r(X_1,\Delta_1) \ge \min\{ r(X,\Delta),\ 1 + a(E_1,X,\Delta) - \max a_i \ :\ E_1 \cap (g_1)_*^{-1} D_i \neq \emptyset \}$$

$$\ge \min\{ r(X,\Delta), \overset{\le}{;} a(E_1,X,\Delta)\} \ge r(X,\Delta).$$

(by $m=1$)

By induction hypothesis on f_1, one has $\overset{a(E,X,\Delta)}{\overset{\shortparallel}{a(E,X_1,\Delta_1)}} \ge r(X_1,\Delta_1)$ $\square$

§ log canonical and log terminal

Def. · (X,Δ) is called a pair if $\begin{cases} X: \text{normal var.} \\ \Delta: \text{boundary } (\text{coeff.} \in [0,1]) \end{cases}$ and $K_X+\Delta$ is $\mathbb{Q}$-Cartier.

· A pair $(X, \Delta = \sum a_i D_i)$ has SNC at a (not necessary closed) point $x \in X$ if $\mathcal{O}_{X,x}$ is regular local ring, $\exists$ an open nbd. $x \in U \subseteq X$ with local coordinate $z_1, \cdots, z_n \in m_{X,x}$ s.t. for each D_i, $\exists\, c(i)$ s.t. $D_i = (z_{c(i)} = 0)$ near x.

· $\mathrm{snc}(X,\Delta) :=$ the largest open subset $U \subseteq X$ s.t. $(U, \Delta|_U)$ is SNC. in X_{sm}.

non-snc$(X,\Delta) := X \setminus \mathrm{snc}(X,\Delta)$.

Next we define the 6 classes of singulaties that one most important for the MMP.

Idea: Recall $1 \geq \underset{\infty}{\overset{\#}{\mathrm{discrep}}}(X,\Delta) \geq \mathrm{totaldiscrep}(X,\Delta) \underline{\geq -1}$ the largest class where discrep still make sense.

			Δ: boundary		
(term.)	terminal	>0	$\Leftrightarrow$	>0	
(can.)	canonical	≥ 0		≥ 0	
(plt.)	purely log-terminal	>-1		>-1	(klt.) Kawamata log terminal
(lc.)	log-canonical	≥ -1	$\Leftrightarrow$	≥ -1	

Def. The pair (X,Δ) is

terminal / canonical / klt. / plt. / dlt. (divisorial) / lc. if $a(E,X,\Delta)$
$\begin{cases} >0 & \forall \text{ exception } E \text{ over } X \\ \geq 0 & \forall \text{ exception } E \text{ over } X \\ >-1 & \forall E \text{ over } X \\ >-1 & \forall \text{ exception } E \text{ over } X \\ >-1 & \text{if } \mathrm{center}_X E \subseteq \text{non-snc}(X,\Delta) \\ \geq -1 & \forall E \text{ over } X \end{cases}$

Rmk. · If Δ is only a sub-boundary, then (X,Δ) is called
$\begin{cases} \text{sub-klt} \\ \text{sub-plt} \\ \text{sub-lc} \end{cases}$ if $\mathrm{discrep}(X,\Delta) \begin{cases} >-1, \ \lfloor \Delta \rfloor \leq 0 \\ >-1 \\ \geq -1 \end{cases}$

· (X,Δ): (sub)-klt $\Leftrightarrow$ $\mathrm{discrep}(X,\Delta) > -1$, $\lfloor \Delta \rfloor = 0$ ($\leq$) $\Leftrightarrow$ $\mathrm{totaldiscrep}(X,\Delta) > -1$.

• If $\Delta=0$, we say X has $\begin{cases}\text{term.}\\ \text{can.}\\ \text{lt.}\end{cases}$ singularity if $(X,0)$ is $\begin{cases}\text{term.}\\ \text{can.}\\ \text{plt}=\text{dlt}=\text{klt}.\end{cases}$

$\text{discrep}(X):=\text{discrep}(X,\Delta)$.

• If $\dim X=1$, then $(X,\sum a_iD_i)$ is $\begin{cases}\text{terminal}=\text{klt}\\ \text{can.}=\text{plt}=\text{dlt}=\text{lc}\end{cases} \Longleftrightarrow \begin{cases}a_i<1 & \forall i\\ a_i\le 1\end{cases}$

• cf. [KM, P.57, (1)-(5)]

(terminal, see also Thm 1.3.2. p.28)

• term. $\Rightarrow$ klt $\Rightarrow$ plt $\Rightarrow$ dlt $\Rightarrow$ lc

$\Rightarrow$ can. $\Leftarrow$ (if $K_X+\Delta$ is Cartier, then all discrepancies are all integers)

Lemma 3. (monotonicity property) X: normal var., $\Delta,\Delta'\in WDiv(X)_{\mathbb{Q}}$, $(X,\Delta),(X,\Delta+\Delta')$ are pairs.
Let $f: Y \xrightarrow{\text{bir.}} X$, $E \subset Y$. Then $a(E,X,\Delta)=a(E,X,\Delta+\Delta')+\text{coeff}_E f^*\Delta'$.

In particular, if $\Delta'\ge 0$, then $a(E,X,\Delta)\ge a(E,X,\Delta+\Delta')$ for every divisor E over X.
and "$>$" $\Leftrightarrow$ $\text{center}_X E\subseteq \text{Supp}\,\Delta'$. Hence $\text{discrep}(X,\Delta)\ge\text{discrep}(X,\Delta+\Delta')$.
$\text{totaldiscrep}(X,\Delta)\ge\text{totaldiscrep}(X,\Delta+\Delta')$

(pf.) Write $K_Y+\Delta_1=f^*(K_X+\Delta)\Rightarrow K_Y+\Delta_1+f^*\Delta'=f^*(K_X+\Delta+\Delta')$. Hence
$a(E,X,\Delta+\Delta')=-\text{coeff}_E(\Delta_1+f^*\Delta')=a(E,X,\Delta)-\text{coeff}_E f^*\Delta'$. □

Cor 1. [KM. 2.3.2] Let $\Delta=\sum a_iD_i$ be a sub-boundary.
(1) $\exists$ a log. resol. f for (X,Δ) s.t. $\sum f_*^{-1}D_i$ is smooth.
(2) Let f be any such. Assume $\lambda:=\min\limits_{E:\,f\text{-excep.}}\{a(E,X,\Delta)\}\ge -1$, then $\text{discrep}(X,\Delta)=\min\{1,\min\limits_i\{1-a_i\},\lambda\}$

Cor. ($\Delta=0$) Given X, for any resol. of singularity $f:Y\to X$. Assume $\lambda:=\min\limits_{E:\,f\text{-excep.}}\{a(E,X,\Delta)\}\ge 0$,
then $\text{discrep}(X)=\min\{\lambda,1\}$.
In particular, X has can. (resp. term.) sing. $\Leftrightarrow$ $\exists$ a resol. $f:Y\to X$ with $K_Y=f^*K_X+\sum a_iE_i$
s.t. $a_i\ge 0$ (resp. $a_i>0$) $\forall i$.

(pf.) Write $K_Y+\Delta_Y=f^*\Delta_X$ with $f_*\Delta_Y=0$, $f_*K_Y=K_X$. Then $\Delta_Y=-\sum\limits_{E:\,f\text{-excep}} a(E,X,\Delta)E\le 0$.
Then $\text{discrep}(Y,\Delta_Y)\overset{\text{lm 3}}{\ge}\text{discrep}(Y,0)=1$ (Hw). By Lemma 2,
$\text{discrep}(X)=\min\limits_{E:\,f\text{-excep.}}\{\text{discrep}(Y,\Delta_Y),a(E,X)\}=\min\limits_{E:\,f\text{-excep.}}\{1,a(E,X)\}$ □

Proof of Cor 1. (char $k=0$)

(1) By Hironaka thm, $\exists$ a log resolution $g: Z \to X$ for (X,Δ) (where $\sum g_*^{-1} D_i$ is SNC). Then by induction, a seq. of blow-ups over the mutual intersection of $\sum g_*^{-1} D_i$'s gives f. [KM, p55, line 19~23]

(2) Write $K_Y + \Delta_Y = f^*(K_X + \Delta)$ with $f_* K_Y = K_X$, $f_* \Delta_Y = \Delta$. Then $\Delta_Y = f_*^{-1}\Delta_X - \sum_{E: f\text{-excep.}} a(E,X,\Delta)E$ is a sub-boundary. Set $b_E = a(E,X,\Delta)$. Since Y is sm. and Δ_Y is SNC sub-boundary, by Prop 2,

$$\operatorname{discrep}(Y,\Delta_Y) = \min_{\substack{E,E': f\text{-excep.}\\ E\cap E'\neq\emptyset}} \{ \underbrace{1-a_j-b_E}_{\geq 0}, \underbrace{1-b_{E'}-b_E}_{\geq 0}, \underbrace{1-a_i-a_j}_{\times \ \because \sum f_*^{-1}D_j \text{ sm.}}, 1-a_j, 1-b_E, 1 \}$$

$$\geq \min_{E: f\text{-excep.}} \{ \min_j \{1-a_j\}, -b_E, 1 \}$$

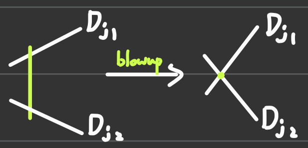
By Lemma 2, $\operatorname{discrep}(X,\Delta) = \min_{E: f\text{-excep.}} \{ \operatorname{discrep}(Y,\Delta_Y), -b_E \} = \min_{E: f\text{-excep.}} \{ \min_j \{1-a_j\}, -b_E, 1 \}$ $\square$

Cor. (continuity properties for pair) Read [KM, Cor 2.35, Cor 2.39]

§ klt pair admit the following special log resolution [KM, Prop 2.36 (1)(2)]

Prop. Let (X,Δ) be a sub-klt pair $\Rightarrow \exists$ a log resolution $f: Y \to X$ for (X,Δ) s.t. if we write $f^*(K_X+\Delta) = K_Y + A_Y - B_Y$, $A_Y, B_Y \geq 0$, $\mathrm{Supp}A_Y \cap \mathrm{Supp}B_Y = \phi$. Then $\mathrm{Supp}A_Y$ is smooth.

(pf.) • For a log resolution $f: Y \to X$ for (X,Δ), write $K_Y + \underbrace{\sum_j d_j D_j}_{\Delta_Y} = f^*(K_X+\Delta)$ with $f_*K_Y = K_X$, $f_*\Delta_Y = \Delta_X$.

⌜ idea: 

$\subset \Delta_Y$ (Question: Terminal?)

$f_1: Y_1 \longrightarrow Y \xrightarrow{f} X$ be the new resolution ⌟

Since (X,Δ) is sub-klt pair, $\exists m \in \mathbb{N}$ s.t. $\mathrm{totaldiscrep}(X,\Delta) > -1 + \frac{1}{m}$.

$\Rightarrow d_j \leq 1 - \frac{1}{m}$ and $0 < d_{j_1} + d_{j_2} \leq 2 - \frac{2}{m}$ $\forall j, j_1, j_2$. Consider the partition $(0, 2-\frac{2}{m}] = \bigsqcup_{i=1}^{2m-2} \underbrace{(\frac{i-1}{m}, \frac{i}{m}]}_{= I_i}$.

We define $r(f) := (r_i(f))_{i=1}^{2m-2} \in \mathbb{Z}_{\geq 0}^{2m-2}$ by $r_i(f) = \#\{(j_1,j_2) \mid D_{j_1} \cap D_{j_2} \neq \phi,\ j_1 < j_2,\ d_{j_1} + d_{j_2} \in I_i\}$.

Consider the inverse lexicographic order $>_{\mathrm{invlex}}$ on $\mathbb{Z}_{\geq 0}^{2m-2}$, which satisfies DCC.

• For a given f, take the maximal i s.t. $r_i(f) \neq 0$ and take (j_1,j_2) realizing it, i.e. $Z := D_{j_1} \cap D_{j_2} \neq \phi$ has codim 2 and $\mathrm{mult}_Z \Delta_Z = d_{j_1} + d_{j_2} \in I_i$. Consider

$f': Y' := \mathrm{Bl}_Z Y \xrightarrow{g} Y \xrightarrow{f} X$, write $K_{Y'} + \Delta_{Y'} = g^*(K_Y + \Delta_Y) = f'^*(K_X+\Delta)$ with

excep. div. $E \longrightarrow Z$ ($E \subset Y'$, $Z \subset Y$), $g_*K_{Y'} = K_Y$, $g_*\Delta_{Y'} = \Delta_Y$.

Then $e := \mathrm{coeff}_E \Delta_{Y'} = -a(E, Y, \Delta_Y) + d_{j_1} + d_{j_2} \Rightarrow -1 + \frac{i-1}{m} < e \leq -1 + \frac{i}{m}$. Since $d_{j_l} \leq 1 - \frac{1}{m}$ $(l=1,2)$, $e + d_{j_l} \leq \frac{i-1}{m}$. Note that (j_1,j_2) will not contribute in $r_k(f')$. So we conclude that $r_k(f') = r_k(f) = 0$ for $k > i$ and $r_i(f') = r_i(f) - 1$ i.e. $r_i(f') <_{\mathrm{invlex}} r_i(f)$. By DCC, one can get a log resol. f s.t. $r_i(f) = 0$ $\forall i$. □

$(K_Y + \overset{\mathrm{sm.}}{A_Y} - B_Y = f^*(K_X + \Delta))$

Cor. (X,Δ): sub klt, $f: Y \to X$ a special log resolution as in above Prop.. For divisor E over X s.t. $a(E,X,\Delta) \overset{(\star)}{<} 1 + \mathrm{totaldiscrep}(X,\Delta)$, then $\mathrm{center}_Y E$ is a divisor. In particular,

$\#\{$excep. divisor E over X with $(\star)\} \leq \#\{f\text{-excep. divsor}\} < \infty$.

$\rotatebox{90}{$\leq$}$

$\#\{$excep. divisor E over X with $\underline{a(E,X,\Delta) = 0}\}$

such E is call crepant divisor [KM, Def 6.22, P.195]

(pf.) For excep. divisor over Y, $a(E,X,\Delta) \overset{\mathrm{Lm}\,2}{=} a(E,Y,A_Y - B_Y) \overset{\mathrm{mono.}}{\geq} a(E,Y,A_Y) \geq \mathrm{discrep}(Y, A_Y)$.

Say $A_Y = \sum a_i A_i$ be sm. SNC. By Prop. 2,

$$\mathrm{discrep}(Y, A_Y) = \min\{1, 1-a_i, \min_{A_i \cap A_j \neq \phi} \{\cancel{1-a_i-a_j}\}\} \quad (a_i \geq 0)$$
(by sm.)
$$\geq 1 + \underbrace{\mathrm{totaldiscrep}(Y, A_Y - B_Y)}_{\leq -a_i}$$
$$= 1 + \mathrm{totaldiscrep}(X,\Delta).$$
□

§ Cone theorem.

Thm. (Basepoint-free theorem) (X,Δ) ; proj. klt pair, $\begin{cases} D \text{ ; nef Cartier divisor} \\ \exists a \in \mathbb{Q}_{>0} \text{ s.t. } aD-(K_X+\Delta) \text{ is big \& nef.} \end{cases}$

$\Rightarrow |mD|$ is bpf. for m divisible, i.e. D is semi-ample.

(bpf. $\Rightarrow$ nef)

Rmk. If mD is bpf for all $m \gg 0$, then $D = (m+1)D - mD$ is Cartier and D is nef, so these two condition in Bpf theorem are necessary.

Thm. (Non-vanishing thm) (Shokurov, 1985 pf, see [KM, §3.5. p91~93])

(X,Δ) ; proj. klt pair, $\begin{cases} D \text{: nef, Cartier divisor.} \\ \exists a \in \mathbb{Q}_{>0} \text{ and an eff. Cartier divisor s.t. } aD+A-(K_X+\Delta) \text{ is big \& nef.} \end{cases}$

$\Rightarrow H^0(X, mD+A) \neq 0$ for $m \gg 0$.

Lemma. (Alternative characterization of big & nef divisors) (cf. [KM, Prop 2.61, (4)])

X ; proj. var., M ; big & nef $\mathbb{Q}$-Cartier $\Rightarrow \exists$ a resol. $f: Y \to X$, ΣF_i SNC divisor on Y with $\cup F_i \supset \mathrm{exc}(f)$ s.t. $\forall \eta > 0$ $\exists p_i \in (0,\eta) \cap \mathbb{Q}$ s.t. $f^*M - \Sigma p_i F_i$ is ample.

Proof of Basepoint-free thm. (asuming non-vanishing)

By non-vanishing thm, $|mD| \neq \emptyset$ for $m \gg 0$. For any $b \geq 2$, let $B(b) :=$ the reduced base locus of $|bB|$.

Since X is noetherian, $(B(b^r))_{r=1}^{\infty}$ is stable (denoted by $B_\infty(b)$).

"$m = a \cdot 2^r + b \cdot 3^r$, $a,b \in \mathbb{Z}_{>0}$"

① If all $B_\infty(b) = \emptyset$, then $B(2^r) = B(3^r) = \emptyset$ for $r \gg 0$. Then for $m \gg 0$, $B(m) \subset B(2^r) \cup B(3^r) = \emptyset$

② From now on, we may assume that $B_\infty(b) = B(b^r) \neq \emptyset$ for some $b \geq 2$.

- $\exists$ a log resol. $f: Y \longrightarrow X$ and SNC divisors F_i on Y (not necessary f-excep.) s.t.
  - (a) $|b^r D - \Sigma r_i F_i|$; bpf, where $r_i \geq 0$ $\forall i \geq 0$ and $\bigcup_{r_i > 0} f(F_i) = B_\infty(b) = B(b^r)$.
  - (b) $K_Y \equiv f^*(K_X+\Delta) + \Sigma a_i F_i$ with $a_i > -1$ $\forall i$

    (Note that $a_i > 0$ only when F_i is f-excep. since $\Delta \geq 0$)

  - (c) $f^*(aD-(K_X+\Delta)) - \Sigma p_i F_i$; ample, for some $p_i \in (0, 1+a_i) \cap \mathbb{Q}$ $\forall i$.

⌜ idea : Since $f^{-1}(B(b^r)) = Bs(|b^r f^* D|)$, want to find some F_j with $r_j > 0$ s.t. $F_j \not\subset Bs|b^r f^* D|$.

$K_Y + N_{m,c} = mf^*D + \underset{\substack{\uparrow \\ \text{base locus}}}{B} - \underset{\substack{\uparrow \\ \text{sm. hypersurface}}}{F_{i_0}}$ ⌟

$\forall m \in \mathbb{N}$, $c \in \mathbb{Q}_{>0}$, define

$$N_{m,c} := mf^*D - K_Y + \sum(-cr_i + a_i - p_i)F_i$$

$$\equiv \underbrace{(m - cb^r - a)f^*D}_{\substack{\text{nef} \\ (\because D \text{ is nef, } f: \text{birational})}} + c\underbrace{(b^r f^*D - \sum r_i F_i)}_{\substack{\text{nef by (a)} \\ \Rightarrow \text{bpf}}} + \underbrace{f^*(aD - (K_X + \Delta)) - \sum p_i F_i}_{\text{ample by (c)}} \quad \text{is ample if } m > cb^r - a.$$

Since $B_\infty(b) \neq \emptyset$, not all r_i are zero. Since $1 + a_i - p_i > 0$ $\forall i$, take $c := \min_{r_i > 0} \frac{1 + a_i - p_i}{r_i} \in \mathbb{Q}_{>0}$, and thus $\min_i(-cr_i + a_i - p_i) = -1$ (for $r_i = 0$, $a_i - p_i > -1$).

By perturbling the p_i a little (this will not affect property (c)), $\exists!\, i_0$ s.t. $c = \frac{1 + a_{i_0} - p_{i_0}}{r_{i_0}}$, $r_{i_0} > 0$.

Let $F := F_{i_0}$ sm. divisor. We have then for $m \geq cb^r + a$,

$$\underset{\substack{\uparrow \\ \text{ample } \mathbb{Q}\text{-Cartier}}}{K_Y + N_{m,c}} \equiv mf^*D + \underbrace{\sum_{i \neq i_0} \underbrace{(-cr_i + a_i - p_i)}_{> -1} F_i}_{=:B} - F.$$

Notice that $\lceil B \rceil$ is effective and $\underline{f: \text{excep.}}$, $F \not\subseteq \operatorname{Supp} B$.

(F_i appears $\lceil B \rceil \Leftrightarrow a_i > cr_i + p_i > 0 \overset{(b)}{\Rightarrow} F_i : f\text{-excep.}$)

Pick $m =$ a power of $b > \max\{b^r, cb^r + a\}$. Since $\lceil N_{m,c} \rceil = mf^*D - K_Y + \lceil B \rceil - F$ and $0 \to \mathcal{O}_Y(-F) \to \mathcal{O}_Y \to \mathcal{O}_F \to 0$, we have

$$\underset{\substack{\text{SI} \\ H^0(X, mD)}}{H^0(Y, mf^*D + \underset{\substack{\uparrow \\ f: \text{excep.}}}{\lceil B \rceil})} \twoheadrightarrow H^0(F, (mf^*D + \lceil B \rceil)|_F) \longrightarrow \overset{=0}{H^1(Y, \overset{\substack{K_Y + \lceil N_{m,c} \rceil \\ \shortparallel}}{\underline{mf^*D + \lceil B \rceil - F}})} \text{ by K-V vanishing}$$

all element vanish on $F = F_{i_0}$, $r_{i_0} > 0$ since $f(F) \subset B_\infty(b) = B(m)$.

However, consider $\begin{cases} (F, (\lceil B \rceil - B)|_F) : \text{klt} \\ f^*D|_F : \text{nef Cartier} \\ \lceil B \rceil|_F : \text{effective Cartier} \end{cases}$ and $\lceil N_{m,c} \rceil|_F = mf^*D|_F + \lceil B \rceil|_F - K_F$: ample

By non-vanishing, $H^0(mf^*D + \lceil B \rceil|_F) \neq 0$. $\square$

Thm. (Rationality theorem)

(X,Δ) : proj. klt pair, $a = \min\{e \in \mathbb{N} : e(K_X+\Delta) \in \mathrm{CDiv}(X)\}$ (Gorenstein index of (X,Δ))

H: big & nef Cartier divisor on X. If $K_X+\Delta$ is NOT nef, then

$$r = r(H) := \sup\{t \in \mathbb{R} : H + t(K_X+\Delta) \text{ is nef}\} \in \mathbb{Q} \quad (\text{nef value of } H \text{ w.r.t. } (X,\Delta))$$

and $\frac{r}{a} = \frac{u}{v}$ with $u, v \in \mathbb{Z}$ coprime and $0 < v < (\dim X + 1)$.

Observation. $\{t \in \mathbb{R}_{\geq 0} : H + t(K_X+\Delta) \text{ is nef}\} = [0, r]$

- $0 \in$ LHS by H is nef
- $r < \infty$, otherwise $\frac{1}{t} + K_X + \Delta$ is nef $\longrightarrow$ $K_X+\Delta$ is NOT a nef $\rightarrow\!\leftarrow$.
- $r \in$ LHS by nef is closed condition.
- For $t \in [0,r) \cap \mathbb{Q}$, $H + t(K_X+\Delta) = \underbrace{\frac{t}{r}(H + r(K_X+\Delta))}_{\text{nef}} + \underbrace{(1-\frac{t}{r})H}_{\text{big \& nef}}$ is big & nef

Rougth idea. X: sm., $\Delta = 0$.

If $r \notin \mathbb{Q}$, consider $S = \{(p,q) \in \mathbb{N}^2 : \frac{p-1}{q} < r < \frac{p}{q}\} \rightsquigarrow \#S = \infty$.

For $(p,q) \in S$, $\begin{cases} pH + (q-1)K_X : \text{ big \& nef} \\ pH + qK_X = K_X + (\quad) : \text{can apply K-V vanishing, but not nef, hence has base points.} \end{cases}$

"base locus stabilizes for all $(p,q) \gg 0$", then follow the same idea of Bpf. thm.

Proof of rationality thm.

WLOG $r > 0$. Let $n = \dim X$ and $D(x,y) = xH + y(K_X+\Delta) \Rightarrow \begin{cases} D \text{ is "bilinear"} \\ D(x,y) \text{ is big \& nef if } \frac{y}{x} \in [0,r) \cap \mathbb{Q} \end{cases}$

Step 1. WLOG H and $H + a(K_X+\Delta)$ are bpf. (Using bpf thm.)

- Let $\mathbb{N} \ni t > \max\{\frac{a}{r}, \frac{a-1}{r}, \frac{a}{2r}, \frac{2a}{r}, \frac{a-1}{2r}, \frac{2a-1}{r}\}$.

  Then $D(t,a)$ and $D(t,a) - (K_X+\Delta) = D(t, a-1)$ are big & nef.

  By bpf. thm, $\underbrace{H' := mD(t,a)}$ is bpf for $m \gg 0$.

  still big & nef

  We hope that $H' + a(K_X+\Delta) = D(mt, (m+1)a) = \frac{m-1}{3}\underbrace{D(2t,a)}_{\text{big \& nef}} + \frac{m+2}{3}\underbrace{D(t,2a)}_{\text{big \& nef}}$ is bpf.

  Indeed, $\begin{cases} D(2t,a) - (K_X+\Delta) = D(2t, a-1) \\ D(t,2a) - (K_X+\Delta) = D(t, 2a-1) \end{cases}$ are big & nef.

  Hence by bpf thm again, for $m \gg 0$ with $m \equiv 1 \pmod 3$, H' and $H' + a(K_X+\Delta)$ are bpf.

• We solve the linear equation

$$H' + r'(K_X+\Delta) = mt\left(H + \frac{ma+r'}{mt}(K_X+\Delta)\right) \Rightarrow r = \frac{ma+r'}{mt}, \text{ i.e. } r\in\mathbb{Q} \Leftrightarrow r'\in\mathbb{Q}.$$

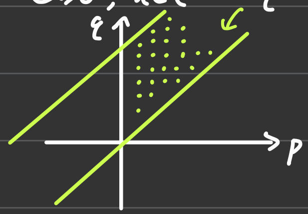
One we know $r/a = u/v$ with $\gcd(u,v)=1$, $r'/a = u'/v'$, $\gcd(u',v')=1$, we may choose m,t with $\begin{cases}\gcd(m,v)=\gcd(t,v)=1\\ m\equiv 1 \pmod 3\end{cases}$

$\Rightarrow v \mid v' \Rightarrow v \le v'$ ($\le a(n+1)$?)

Step2. For $\eta\in\mathbb{Q}_{>0}$, let $\Lambda_\eta := \left\{(p,q)\in\mathbb{N}^2 \;\middle|\; \frac{aq-\eta}{p} < r < \frac{aq}{p}\right\} \Rightarrow$ "$\eta<\varepsilon \Rightarrow \Lambda_\eta \subseteq \Lambda_\varepsilon$".

$\Updownarrow$

$0 < aq - rp < \eta$



• If $r\notin\mathbb{Q}$, then $\Lambda_\eta \ne \emptyset$ $\forall \eta\in\mathbb{Q}_{>0}$. If $\Lambda_\eta\ne\emptyset$, then Λ_η is infinite.

• If $(p,q)\in\Lambda_\eta$, then $D(p, aq-\eta)$: big & nef

whereas $D(p,aq)$: NOT nef $\Rightarrow$ not bpf.

$B(p,q) :=$ the reduced bases locus of $D(p,aq)$. Notice that $B(p,aq) = X \Leftrightarrow |D(p,aq)| = \emptyset$.

Step3. If $\exists (p_0,q_0)\in\Lambda_1$, then $B(p,q)\subseteq B(p_0,q_0)$ for all $(p,q)\in\Lambda_1$ and $q\gg 0$

(If so, then X is noetherian $\Rightarrow \exists$ a closed subset $B_\infty \subseteq X$ s.t. $B_\infty = B(p,q)$ $\forall (p,q)\in\Lambda_1$, $q\gg 0$)

(pf.) For $(p,q)\in\Lambda_1$, let $q = \alpha q_0 + \beta$ with $0\le\beta<q_0$, $\alpha,\beta\in\mathbb{N}$. Then

$$D(p,aq) = \alpha D(p_0, aq_0) + \beta\underbrace{D(1,a)}_{\text{bpf.}} + \underbrace{(p-\alpha p_0-\beta)}_{\text{need }>0}\underbrace{D(1,0)}_{=H:\text{ bpf.}}$$

Indeed,

$$p-\alpha p_0 - \beta > \frac{aq-1}{r} - \frac{q}{q_0}p_0 - q_0 = q\underbrace{\left(\frac{a}{r}-\frac{p_0}{q_0}\right)}_{>0\ \because (p_0,q_0)\in\Lambda_1} - \frac{1}{r} - q_0 > 0 \text{ if } q\gg 0.$$

Hence $B(p,q)\subseteq B(p_0,q_0)$ as $q\gg 0$.

We proceed by contradiction assuming that either $r\notin\mathbb{Q}$ or $\frac{r}{a} = \frac{u}{v}$, $\gcd(u,v)=1$, $u,v\in\mathbb{N}$ with $v > a(n+1)$.

Step4. $\Lambda_{\frac{1}{n+1}}$ is infinite, and $\exists$ sufficiently large $(p,q)\in\Lambda_1$ s.t. $|D(p,aq)|\ne\emptyset$ and thus $B_\infty \subsetneq X$

(pf.) • If $r\notin\mathbb{Q}$, then Λ_η is infinite $\forall\eta\in\mathbb{Q}_{>0}$ by Step2.

• For other case, $\exists\, p,q\in\mathbb{N}$ s.t. $0 < vq - up = 1 < \frac{v}{a(n+1)} \rightsquigarrow 0 < aq - rp < \frac{1}{n+1} \Rightarrow (p,q)\in\Lambda_{\frac{1}{n+1}}$

$\Rightarrow$ infinite (by Step2)

• For $(p,q)\in\Lambda_1$, $D(p,aq) - (K_X+\Delta) = D(p,aq-1)$ is big & nef. By K-V vanishing [KM-Thm 2.70], $H^i(X, D(p,aq)) = 0$ $\forall i>0$. It is therefore enough to $\underset{\uparrow\ \deg\le n}{P}(x,y) = \chi(X, D(x,ay))$ does not vanish at some point of Λ_1.

Lemma. Assume $\Lambda_{\frac{1}{n+1}}$ is infinite. If a poly. $R(x,y)$ of degree at most n that vanishes on all sufficiently large elements of Λ_η, then $R\equiv 0$.

(subpf.) For $0 \ll (p,q) \in \Lambda_{\frac{q}{n+1}}$, $(jp, jq) \in \Lambda_q$ $\forall 1 \le j \le n+1$, hence $R(jp, jq) = 0 \Rightarrow R|_{xq=yp} \equiv 0 \Rightarrow R \equiv 0$. ($\deg R \le n$; $\exists \infty$-such line)

Since $\Lambda_{\frac{1}{n+1}}$ is infinite and $P(x,0) = \chi(X, xH) = \frac{H^n}{n!}x^n + O(x^{n-1}) \neq 0$. ($H^n > 0$ by big & nef)

By Lemma, $\exists (p,q) \in \Lambda_1$ s.t. $P(p,q) \neq 0$.

Step 5. (Tie-breaking) $\Lambda_1^\infty := \{(p,q) \in \Lambda_1 \mid B(p,q) = B_\infty\}$. Fix $(p_0, q_0) \in \Lambda_1^\infty$ s.t. $(p,q) \in \Lambda_1^\infty$ if $q > q_0$.

Then $\exists$ a log resol. $f: Y \to X$ and a SNC divisor $\sum F_i$ on Y (not necess. excep.) s.t.

Similarly to pf. of bpf thm.

(a) $|f^*D(p_0, aq_0) - \sum r_i F_i|$: bpf., $r_i \ge 0$, $\bigcup_{r_i > 0} f(F_i) = B(p_0, q_0) = B_\infty$.

(b) $K_Y \equiv f^*(K_X + \Delta) + \sum a_i F_i$ with $a_i > -1$ $\forall i$.

Note that $a_i > 0$ only when F_i is f-excep. since $\Delta \ge 0$.

(c) $f^*D(p_0, aq_0) - \sum p_i F_i$ is ample for some $p_i \in (0, 1+a_i) \cap \mathbb{Q}$ $\forall i$.

Since $B_\infty \neq \emptyset$, r_i are NOT all zero. As in the proof of bpf. thm,

$$c := \min_{r_i > 0} \frac{1 + a_i - p_i}{r_i} \quad (\text{a kind of log canonical threshold})$$

and may assume $\exists!\, i_0$ attain minimum. Then $\min\{-cr_i + a_i - p_i\} = -1$. Let $F := F_{i_0}$ (sm. prime divisor) and $B := \sum_{i \neq i_0} (-cr_i + a_i - p_i) F_i$. Notice that $\lceil B \rceil \ge 0$, f-excep. and $F \not\subseteq \operatorname{Supp} B$.

- $N_{p,q} := f^*D(p, aq) - K_Y + \sum_i (-cr_i + a_i - p_i) F_i$

$$\begin{aligned} &\equiv f^*D(p - (c+1)p_0, a(q - (c+1)q_0)) - f^*(K_X + \overset{D(0,1)}{\overset{\shortparallel}{\Delta}}) + \sum(-cr_i - p_i)F_i + (c+1) f^*D(p_0, aq_0) \\ &= \underbrace{f^*D(p - (c+1)p_0, a(q - (c+1)q_0))}_{(\text{nef ?})} + c\underbrace{(f^*D(p_0, aq_0) - \sum r_i F_i)}_{\text{bpf} \Rightarrow \text{nef}} + \underbrace{f^*D(p_0, aq_0 - 1) - \sum p_i F_i}_{\text{ample}} \end{aligned}$$

Set $Q(x,y) = \chi(F, (f^*D(x,y) + \lceil B \rceil)|_F)$ and $\eta_0 = \min\{1, (c+1)(aq_0 - rp_0)\}$

- Claim. If $p > (c+1)p_0$, $q > (c+1)q_0$ and $(p,q) \in \Lambda_{\eta_0}$, then $N_{p,q}$ is ample and $Q(p,q) = 0$.

(subpf.) To let (?) be nef, we need $0 < p - (c+1)p_0$ and $0 < a(q - (c+1)q_0) \le r(p - (c+1)p_0)$, which follows by assumption. ("$aq - rp < (c+1)(aq_0 - rp_0)$") Since $\lceil N_{p,q} \rceil = f^*D(p, aq) - K_Y + \lceil B \rceil - F$,

$$\underset{\substack{\shortparallel \\ H^0(X, D(p,aq))}}{H^0(Y, f^*D(p,aq) + \underset{f\text{-excep.}}{\lceil B \rceil})} \twoheadrightarrow \underset{\substack{\shortparallel \\ 0}}{H^0(F, (f^*D(p,aq) + \lceil B \rceil)|_F)} \to H^1(Y, \underbrace{f^*D + \lceil B \rceil - F}_{\lceil N_{p,q} \rceil + K_Y}) \underset{\text{by K-V}}{=} 0$$

all element vanish on $F = F_{i_0}$ ($\because r_{i_0} > 0 \Rightarrow f(F) \subseteq B(p,q) = B_\infty$)

Since $(f^*D(p,aq) + B)|_F - K_F = N_{p,q}|_F$ is ample, $Q(p,q) \overset{\text{by K-V}}{=} h^0((f^*D(p,aq) + \lceil B \rceil)|_F) = 0$.

- For $p > (c+1)p_0$, $q > (c+1)q_0$ and $\frac{ap}{q} < r$, we have $\begin{cases} f^*D(p,aq)|_F \text{ is nef} & aq - rp < 0 < (c+1)(aq_0 - rp_0) \\ N_{p,q} = (f^*D(p,aq) + B)|_F - K_F \text{ is ample} \end{cases}$

Then for $m \gg 1$, $(f^*(mp, amq) + B)|_F - K_F = (m-1) f^*D(p,aq)|_F + N_{p,q}|_F$ is still ample. Hence $Q(mp, mq) \underset{\text{by K-V}}{=} h^0((f^*D(mp, amq) + \lceil B \rceil)|_F) \underset{\text{by non-vanishing}}{\neq} 0$ $\forall m \gg 1$. By Lemma and Claim, $\Lambda_{\frac{\eta_0}{n+1}} = \emptyset$, and thus $r \in \mathbb{Q}$.

(say $r = \frac{u}{v}$)

$\frac{aqv-up}{v}$ can take at most v values

Step 6. When $(p,q)\in\Lambda_1$, $0<aq-rp<1$. So we may choose $(p_0,q_0)\in\Lambda_1^\infty$ s.t. aq_0-rp_0 is max among elements of Λ_1^∞. If $(p,q)\in\Lambda_1^\infty$ with $p>(c+1)p_0$, $q>(c+1)q_0$, then

$$aq-rp\le aq_0-rp_0\le (c+1)(aq_0-rp_0)\Rightarrow (p,q)\in\Lambda_{q_0}\overset{\text{Claim.}}{\Rightarrow} Q(p,q)=0.$$

But Step 4 $\Rightarrow \Lambda_{\frac{1}{n+1}}$ is infinite $\overset{\text{Lemma}}{\Rightarrow} Q\equiv 0$ $\rightarrow\leftarrow$ with Step 5. □

Flow chart of proof of Cone. thm.

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Contraction thm.}^{(3)(4)} \longleftarrow \text{[KM, Thm 3.7]} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Uparrow \qquad\qquad\qquad\qquad\qquad \downarrow \\ \text{nonvanishing} \Rightarrow \text{bpf} \Rightarrow \text{rationality} \Rightarrow \left\{\begin{array}{l}\text{Cone thm. (weak-version)} \\ \text{length of extremal ray}\end{array}\right. \Rightarrow \text{Cone thm.}^{(1)(2)}\end{array}$$

Thm. (Cone theorem, weak version) [KM, Thm 3.7, (1),(2)]

(X,Δ): klt pair $\Rightarrow \mathcal{R}:=\{$ all $(K_X+\Delta)$-negative extermal rays in $\overline{NE}(X)\}$ is countable set and

$$\overline{NE}(X)=\overline{NE}(X)_{K_X+\Delta\ge 0}+\sum_{R\in\mathcal{R}}R.$$

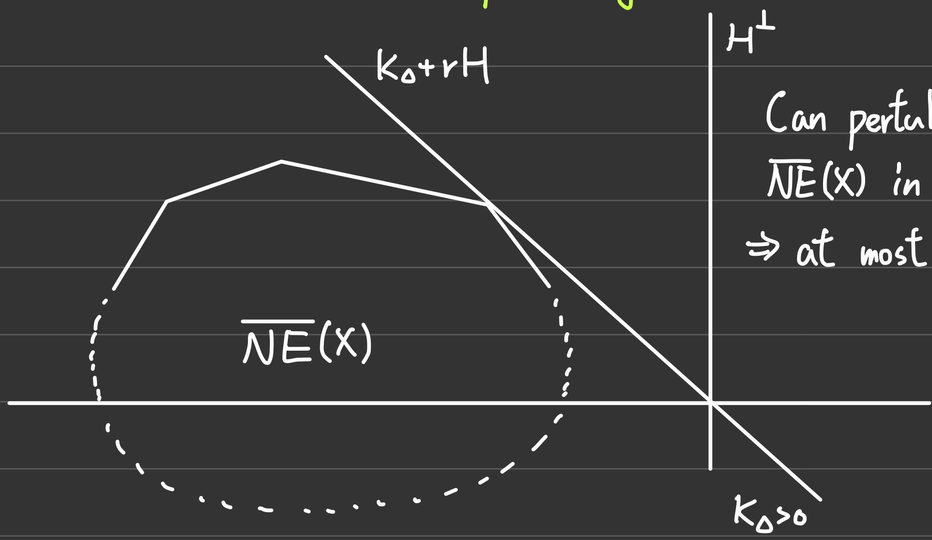
These rays are locally discrete in the half-space $\overline{NE}(X)_{K_X+\Delta<0}$.

$(\forall\varepsilon>0,\ H$: ample $\mathbb{Q}$-Cartier, $\#\{R\in\mathcal{R}\mid (K_X+\Delta+\varepsilon H).R<0\}<\infty)$

Moreover, by Kawamata's thm, we have: $\forall R\in\mathcal{R}$, $\exists$ a rational curve Γ spanning R s.t.

$$0<-(K_X+\Delta).\Gamma<2\dim X$$

Rmk. X: sm., $\Delta=0$. Cone thm. was proved by Mori and $-K_X.\Gamma<\dim X+1$.



$K_\Delta^\perp$, $K_\Delta:=K_X+\Delta$

Can pertub H s.t. K_X+rH meets $\overline{NE}(X)$ in an extermal ray $r(H)\in\mathbb{Q}$

$\Rightarrow$ at most countable many such rays

Thm. (Contraction thm.) [KM Thm 3.7 (3),(4), Cor 3.17, pf. P.84 Step 6~9]

(X,Δ): proj. klt pair, $\overline{NE}(X)\supset F$: a $(K_X+\Delta)$-negative extermal face $\Rightarrow \exists!$ (up to isom.)

$c_F=\mathrm{cont}_F: X\longrightarrow Z$ to a proj. var. Z s.t. $c_{F*}\mathcal{O}_X=\mathcal{O}_Z$ and for any irr. curve $C\subseteq X$,

$$c_F(C) \text{ is a point } \overset{(*)}{\Longleftrightarrow} [C]\in F.$$

In particular, $-(K_X+\Delta)$ is c_F-ample by relative Kleiman crieterion.

$c_{F*}\mathcal{O}_X = \mathcal{O}_Z \Rightarrow c_{F*}c_F^*L = L$

(b) $\mathrm{Im}(\mathrm{Pic}\, Z \xhookrightarrow{c_F^*} \mathrm{Pic}\, X) = \{L \in \mathrm{Pic}\, X \mid L.C = 0 \ \forall \text{irr. curve } C \text{ with } [C] \in F\}$.

In particular, we have the exact sequence

$$0 \longrightarrow N^1(Z)_{\mathbb{R}} \xrightarrow{c_F^*} N^1(X)_{\mathbb{R}} \longrightarrow \langle F \rangle^\vee_{v.s.} \longrightarrow 0$$

its dual exact sequence (✿)

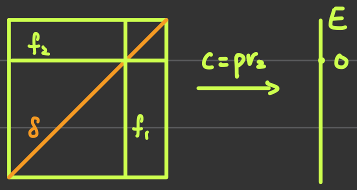
$$0 \to \langle F \rangle_{v.s} \longrightarrow N_1(X)_{\mathbb{R}} \xrightarrow{c_{F*}} N_1(Z)_{\mathbb{R}} \longrightarrow 0$$

and $\rho(Z) = \rho(X) - \dim F$.

Rmk. (1) $F = \overline{NE}(X/Z) = \overline{NE}(c_F)$ by (∗) and [KM, Prop. 3.27 (4)]

(2) The complex (✿) is not exact in general. (cf. [KM, p46, line 10, (1)(2)(3)])

e.g. (cf. [KM Exam. 1.46]) $X = E \times E$: abelian, $o \in E$: elliptic curve.



$f_1 = [\{o\} \times E] \quad \in N_1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{R}}$

$f_2 = [E \times \{o\}]$

$\delta = [\Delta]$, $\Delta \hookrightarrow E \times E$ diagonal

We have the complex (if it is exact)

$$0 \to \mathbb{R} \cdot f_2 \longrightarrow N_1(X)_{\mathbb{R}} \longrightarrow N_1(E)_{\mathbb{R}} \longrightarrow 0$$

$$\exists\, af_2 \longmapsto \delta - f_1 \neq 0 \longmapsto 0$$

$\deg N_{\Delta/X} = 2 - 2g(E) = 0$, where $T_E = N_{\Delta/X}$

$\Rightarrow \delta^2 = (af_1 + f_2)^2 = 2af_1.f_2 = 2a \neq 0 \ \rightarrow\leftarrow$

(3) Note that $\overline{NE}(X) = \mathrm{Nef}(X) = \left\{\alpha = xf_1 + yf_2 + z\delta \;\middle|\; \begin{matrix} x+y+z \geq 0 \\ xy+yz+zx \geq 0 \end{matrix}\right\}$ is a circular cone in $\mathbb{R}^3$ (NOT polyhedral)

If E is very general in the moduli, then $\{f_1, f_2, \delta\}$ is a basis of $N_1(X)_{\mathbb{R}}$ ($\rightsquigarrow \rho(X) = 3$).

Def. [KM, Def 3.22] $f: X \to Y$ a proper morphism of var. $/k = \bar{k}$ char $= 0$. D: Cartier div. on X.

- D is f-big if $\mathrm{rank}\, f_*\mathcal{O}_X(mD) > c \cdot m^n$ for some $c > 0$ and $m \gg 1$, where $n = \dim X - \dim f(X)$ is the dimension of the general fiber of f.
- D is f-free if $f^*f_*\mathcal{O}_X(D) \twoheadrightarrow \mathcal{O}_X(D)$, i.e. $\forall y \in Y$, $\exists$ affine open nbd. $y \in U \subseteq Y$ s.t. D is (base point) free on $f^{-1}U$.
  $\left(H^0(f^{-1}U, \mathcal{O}_X(D)) \otimes \mathcal{O}_{f^{-1}U} \twoheadrightarrow \mathcal{O}_X(D)|_{f^{-1}(U)}\right)$

[Kaw 91] Y. Kawamata, On the length of an extremal rational curve, 1991, Invent Math.

Thm. X, Y: normal var. $/\mathbb{C}$. $g: X \to Y$ a proj. morphism. Assume the (X, Δ): klt pair and $-(K_X + \Delta)$ is g-ample $\Rightarrow \forall$ irr. component $E \subseteq \mathrm{Exc}(g)$ is covered by rational curves Γ with $g(\Gamma) = a$: pt and s.t. $0 < -(K_X + \Delta).\Gamma \leq 2(\dim E - \dim g(E))$ (✿). ("Step 8" in [KM, p85])

Recall. Thm [MM 86]

$\begin{cases} Z: \text{proj. normal var.} \quad H: \text{ample on } Z \\ C \subseteq Z \setminus \mathrm{Sing}\, Z \text{ a curve with } K_Z.C<0 \end{cases} \Rightarrow \forall c \in C, \exists$ a rational curve Γ on Z through c with $0 < H.\Gamma \leq 2\dim X \cdot \frac{H.C}{-K_Z.C}$

(pf.). (Reduction step) WLOG g has connected fibers (by replacing g with its Stein factorization)

Since Y is normal, $E = g^{-1}(g(E))$. It suffices to show that, passing though a general point of E, $\exists$ a rational curve contract by g and satisfies (※).

(Reason. For a flat family of curves whose general fibers are rational curves, any irr. comp. of its special fiber it again a rational curve.
+ a rational curve can only degenerate into a union of rational curves of lower degree.)

We may replace Y with the $\bigcap^{\dim g(E)}$ general hyperplane sections and assume $g(E) = a$: pt.

$\left\{ \begin{array}{l} \text{the normality of } Y \\ -(K_X+\Delta): g\text{-ample} \\ (X,\Delta): \text{klt pair} \end{array} \right.$ are preserved (by [KM, Lemma 5.17 (1)])

- Let H be a very ample divisor on X, $e = \dim E$. $\nu: \widehat{E} \longrightarrow E$ be the normalization

$\widehat{E} \supseteq C = \bigcap_{i=1}^{e-1} D_i$ is a smooth curve in $\widehat{E} \setminus \mathrm{Sing}\,\widehat{E}$ (general $D_i \in |\nu^* H|$ by Bertini)

$\nu^*(K_X+\Delta).C = (K_X+\Delta)|_E.\nu_*C \quad (>) \quad K_{\widehat{E}}.C$ (if $E \neq X$)

- Claim. $(K_X+\Delta).H^{e-1}.E \geq K_{\widehat{E}}.(\nu^*H)^{e-1}$.

(subpf.) ‥ If $E = X$, $(K_X+\Delta).H^{\dim X-1} \geq K_X.H^{\dim X-1}$ (Δ: eff., H: ample)

WLOG $E \neq X$, $g(E) = a \rightsquigarrow g$ is birational.

‥ Reduce to $e=1$:

Assume $e \geq 2$. Take a general normal member $X_1 \in |H|$. $Y_1 :=$ the normalization of $g(X_1)$.

$E_1 = X_1 \cap E$ irr. comp. of $\dim = e-1$ of $\mathrm{Exc}(g_1)$, where $g_1 : X_1 \longrightarrow Y_1$ (by normal). $\Delta_1 := \Delta|_{X_1}$.

Note $\widehat{E}_1 = \nu^{-1}(X_1)$ is normal ($\widehat{E}_1 \in |\nu^*H|_E|$).

$$\begin{tikzcd} \widehat{E}_1 = \nu^{-1}(X_1) \arrow[r] \arrow[d] & \widehat{E} \arrow[d, "\nu"] \\ E_1 = E \cap X_1 \arrow[r, phantom, "\subseteq"] \arrow[dr] \arrow[d, phantom, "\cap"] & E \arrow[d, phantom, "\cap"] \\ & X_1 \subseteq X \arrow[d] \arrow[dl, "g_1"'] \arrow[r, phantom] & \\ Y_1 \arrow[r, "\text{normalization}"] & g(X_1) \subseteq Y \end{tikzcd}$$

$\because K_{X_1} \sim (K_X+H)|_{X_1}$, $K_{\widehat{E}_1} \sim (K_{\widehat{E}} + \nu^*H)|_{\widehat{E}_1}$

$\mathrm{LHS} = (K_{X_1}+\Delta_1).H|_H^{e-2}.E_1 = (K_{X_1}+H+\Delta)|_H \cdot H|_H^{e-2}.E|_H$
$= (K_X+\Delta).H^{e-1}.E + H^e.E$

$\mathrm{RHS} = K_{\widehat{E}_1}.(\nu^*H)^{e-2} = (K_{\widehat{E}}+\nu^*H)|_{\widehat{E}_1}.(\nu^*H)^{e-2} = K_{\widehat{E}}.(\nu^*H)^{e-1} + H^e.E$.

‥ $e=1$ (i.e. $(K_X+\Delta).E > \deg K_{\widehat{E}}$) : Assume to the contrary, $(K_X+\Delta).E \leq \deg K_{\widehat{E}}$.

$\Downarrow$ (※※) ($D_E - (K_X+\Delta)|_E$ has non-neg. degree)

Consider $\mathrm{Pic}(E) \xrightarrow{\nu^*} \mathrm{Pic}(\widehat{E}) \to 0$,

$\exists\ \mathcal{O}_E(D_E) \longmapsto \mathcal{O}(K_{\widehat{E}}) = \omega_{\widehat{E}}$ ($\widehat{E}$: normal curve $\Rightarrow$ smooth) (cf. [KM, Prop 5.77])

torsion free

$\exists$ an injective trace map $\nu_* \omega_{\tilde{E}} \hookrightarrow \omega_E$ (E proper $\overset{e=1}{\Rightarrow}$ proj., $\exists \omega_E$ [KM, Cor 5.69])

$\xrightarrow{\otimes \mathcal{O}(-D_E)} h^0(E, \omega_E \otimes \mathcal{O}_X(-D_E)) \geq h^0(E, \nu_*\omega_{\tilde{E}} \otimes \mathcal{O}(-D_E)) = h^0(\tilde{E}, \mathcal{O}_{\tilde{E}})$.

$\|$ Serre duality $\quad \nu_*(\omega_{\tilde{E}} \otimes \nu^*\mathcal{O}(-D_E)) \quad$ ν: finite

$h^1(E, \mathcal{O}_X(D_E))$ (Give E the reduced sch. str. $\Rightarrow$ NO embedded pt. $\Leftrightarrow$ E: C-M curve)

$\because$ After shrinking Y, WLOG Y is constractible and Stein.

(higher cohomo. of $\mathbb{Z}$ and any coh. sheaf vanish)

Moreover, we may assume g indues an isom. over $Y \setminus g(E) = Y \setminus \{0\}$. Set $X_0 = g^{-1}(\{0\}) \supset E$.

Since $\operatorname{Supp}(R^i g_* \mathbb{Z}) = \{0\} = \operatorname{Supp}(R^i g_* \mathcal{F})$ $\forall \mathcal{F}$: coh. $\forall i > 0$, by Leray spectral sequence,

we get $H^i(X, \mathbb{Z}) \simeq H^i(Y, R^i g_* \mathbb{Z}) = H^i(X_0, \mathbb{Z})$ and $H^i(X, \mathcal{F}) = H^i(Y, R^i g_* \mathcal{F})$. (☸)

[Debarre] P.212~213 footnote 20

$\because$ "We can extend D_E to Cartier divisor D on X if we replace X by a small analytic nbd. of E." Then $D - (K_X + \Delta)$ is g-ample (by (☸☸)) and g-big ($\because$ g is birational).

Since (X, Δ) is klt, $R^i g_* \mathcal{O}_X(D) = 0$ $\forall i > 0$ (K-V vanishing for cpx. analytic var. by Nakayama)

$\Rightarrow H^i(X, D) = 0 \quad \forall i > 0$ by (☸).

$0 \to \mathcal{J}_E \to \mathcal{O}_X \to \mathcal{O}_E \to 0 \xrightarrow{\otimes \mathcal{O}(D)} \underset{0}{\underset{\|}{H^1(X, D)}} \to \underset{\Rightarrow\ 0\ \to\leftarrow}{\underset{\|}{H^1(E, D_E)}} \to H^2(X, \mathcal{J}_E(D)) \simeq \underbrace{H^2(Y, \overset{\overset{0}{\|}}{R^i g_* \mathcal{J}_E(D)})}_{\text{by fiber dim} \leq 1}$

- Now, since $-(K_X + \Delta)$ is g-ample $\Rightarrow K_E . C \leq (K_X + \Delta) . H^{e-1} . E < 0$. Since $-\nu^*(K_X + \Delta)$ is ample on $\tilde{E}$, by [MM 86], we have $\forall x \in C$, $\exists$ a rational curve $\tilde{\Gamma} \subseteq \tilde{E}$ pass through x with

$$\underset{-(K_X+\Delta).\nu_*\tilde{\Gamma}}{\underset{\|}{-\nu^*(K_X + \Delta) . \tilde{\Gamma}}} \leq 2e \underbrace{\frac{-\nu^*(K_X + \Delta) . C}{-K_{\tilde{E}} . C}}_{\leq 1 \text{ by Claim}} \leq 2e = 2(\dim E - \dim g(E))$$

Therefore, E is covered by rational curve $\Gamma := \nu_* \tilde{\Gamma}$ with $0 < -(K_X + \Delta) . \Gamma \leq 2(\dim E - \dim g(E))$. $\square$

Lemmas before singularities

Lemma. Y : proper, $\downarrow f$: proper birational, X : normal. F : excep. Cartier div. on Y, D : Cartier div. on X $\Rightarrow \mathcal{O}_X(D) = f_*\mathcal{O}_Y(f^*D+F)$.

In particular, $H^0(X,D) = H^0(Y, f^*D+F)$ and $f_*\mathcal{O}_Y(F) = \mathcal{O}_X$.

(pf.)

- $\exists$ natural injection $\mathcal{O}_X(D) \overset{(\star)}{\hookrightarrow} f_*\mathcal{O}_Y(f^*D+F)$:

  projection formula

  f : dominant $\Rightarrow \mathcal{O}_X \hookrightarrow f_*\mathcal{O}_Y \xrightarrow{\otimes \mathcal{O}_X(D)} \mathcal{O}_X(D) \hookrightarrow f_*\mathcal{O}_Y \otimes \mathcal{O}_X(D) = f_*f^*\mathcal{O}_X(D)$

  $0 \to \mathcal{O}_Y \to \mathcal{O}_Y(F) \xrightarrow{\otimes f^*\mathcal{O}_X(D)} f^*\mathcal{O}_X(D) \hookrightarrow f^*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(F) \xrightarrow{f_*} f_*f^*\mathcal{O}_X(D) \hookrightarrow f_*\mathcal{O}_Y(f^*D+F)$.

- Since $\mathrm{codim}_X \underset{\underset{f(\mathrm{exc}(f))}{\shortparallel}}{\mathrm{dom} f^{-1}} \geq 2$ and $(\star)$ is isom. over $X \setminus \mathrm{dom}(f^{-1})$, $(\star)$ is an isom. by X is normal.

$\square$

Cor. Y : proper, factorial, $\downarrow f$: proper birational, X : normal.

(1) For an excep. div. E, E is eff. $\Leftrightarrow f_*\mathcal{O}_Y(E) = \mathcal{O}_X$.

(2) If D, D' : Cartier div. on X, F, F' : excep. div. on Y s.t. $f^*D+F \sim f^*D'+F'$, then $F = F'$ and $D \sim D'$.

(pf.)

(1) ($\Rightarrow$) By Lemma.

($\Leftarrow$) $\mathcal{O}_Y = f^*\mathcal{O}_X \simeq \underbrace{f^*f_*\mathcal{O}_Y(E) \longrightarrow \mathcal{O}_Y(E)}_{\text{is identify on } Y \setminus \mathrm{Exc}(f)}$ give a nonzero section in $H^0(Y,E)$.

$\Rightarrow E$ is eff.

(2) Write $F = F_1 - F_2$, $F_1, F_2 \geq 0$, $F' = F_1' - F_2'$, $F_1', F_2' \geq 0$ with no common component $\Rightarrow f^*(D-D') + \underbrace{F_1 + F_2'}_{\geq 0} \sim \underbrace{F_2 + F_1'}_{\geq 0}$

$\xrightarrow{f_*} D - D' \sim 0$ by Lemma. Hence $F_1 + F_2' \sim F_2 + F_1'$. By Lemma, $H^0(Y, F_1 + F_2') = H^0(X, \mathcal{O}_X) = k$.

$\Rightarrow F_1 + F_2' = F_2 + F_1' \Rightarrow F_1 = F_1', F_2 = F_2'$ by no common component.

focus on $\mathcal{O}_{X,x}$ or $\widehat{\mathcal{O}}_{X,x} \simeq \widehat{\mathcal{O}}^{an}_{X,x}$

§ Normal Surface Singularities

Thm. (Mumford) [KM, Lemma 3.40, p103]

Y : sm. surface $\Rightarrow (E_i . E_j)$ is negative definite.

$\downarrow f$: proper birational with excep. curves E_i

X : normal surface

(pf.) We only prove the case when X is projective (cf. [Ishii, Thm. 7.1.6]).

Let $H := f^*(\text{an ample Cartier div. on } X)$, then $H^2 > 0$ (by big & nef) and $H.E_i = 0$ (proj. formula)

For $D = \sum e_i E_i \neq 0$, $H.D = 0 \overset{\text{Hodge index}}{\Longrightarrow} D^2 \leq 0$ and "$D^2 = 0 \Leftrightarrow D \equiv 0$".

If D is effective, then $D \not\equiv 0$ since Y is projective $\left(\begin{array}{l} Y \text{ is proper surface} \Rightarrow \text{proj.} \\ \text{cf. [Hart. II Rmk 4.10.2]} \end{array} \right)$

In general, write $D = \underset{\underset{e_i>0}{\sum e_i E_i}}{D^+} - D^-$. Then $D^2 = (D^+)^2 + (D^-)^2 - 2D.D^- \leq (D^+)^2 + (D^-)^2 < 0$. □

Lemma. [KM, Lemma 3.41] S : sm. surface

$\cup$

$C = \sum_{\text{finite}} C_i$, C_i : distinct proper irr. curves with $(C_i . C_j)$ is negative definite.

Let $A = \sum a_i C_i$, $a_i \in \mathbb{R}$ such that $A.C_j \geq 0$ $\forall j$. We have

(1) $a_i \leq 0$ $\forall j$

(2) If C is connected and $A \neq 0$, then $a_i < 0$ $\forall i$.

(pf.) Write $A = \sum_{a_i>0} a_i C_i - \sum_{a_i<0} a_i C_i = A^+ - A^-$.

(1) If $A^+ \neq 0$, then $(A^+)^2 < 0$ by assumption $\Rightarrow \exists C_i \subseteq \operatorname{Supp} A^+$ s.t. $C_i . A^+ < 0$. Then $C_i \not\subseteq \operatorname{Supp} A^-$ implies $C_i . A^- \geq 0$. Hence $C_i . A = C_i . A^+ - C_i . A^- < 0$ →←.

(2) Assume that C is connected, $A \neq 0$ and $\phi = \operatorname{Supp} A^- \subsetneq \bigcup C_i = \operatorname{Supp} C$. Then $\exists C_i$ s.t. $C_i \not\subseteq \operatorname{Supp} A^-$ but $C_i \cap \operatorname{Supp} A^- \neq \phi \rightsquigarrow C_i . A = -C_i . A^- < 0$ →←. □

Cor. Under the same condition in the above Lemma, assume C is connected. Then

$\mathcal{Z}_{num} := \{ Z \in \oplus \mathbb{Z} C_i \setminus \{0\} \mid Z.C_i \leq 0 \ \forall i \}$ has a unique minimal element.

$\cap$ Eff(S) by Lemma (study by Lipman, 1969) $(Z_1 \geq Z_2 \Leftrightarrow Z_1 - Z \text{ is eff.})$

(pf.)

- $\mathcal{Z}_{num} \neq \phi$: Given $\alpha_j \in \mathbb{Z}_{<0}$ $\forall j$, $\exists m_i \in \mathbb{Z}$ s.t. $(\sum m_i C_i).C_j = m\alpha_j < 0$, where $m = |\det(C_i . C_j)| > 0$.
- Take any two elements $Z = \sum m_i C_i$, $Z' = \sum m_i' C_i$ and $n_i = \min\{m_i, m_i'\}$, define $Z \wedge Z' = \sum n_i C_i$.

  Fix any i, WLOG $m_i \leq m_i'$. Then $(Z \wedge Z').C_i = m_i C_i^2 + \sum_{j \neq i} n_j \underbrace{C_j . C_i}_{\geq 0} \leq m_i C_i^2 + \sum_{j \neq i} m_j C_j . C_i = Z.C_i \leq 0$.

  $\Rightarrow Z \wedge Z' \in \mathcal{Z}_{num}$.
- $\mathcal{Z}_{num}$ has unique minimal element since $\mathcal{Z}_{num} \subseteq \text{Eff}(S)$.

or Artin / numerical / minimal cycle

Def. (fundamental cycle) The fund. cycle of $f^{-1}(x)_{red}$ (or $f: Y \to X$) defined by

$$\begin{array}{ccc} Y & \supseteq & f^{-1}(x)_{red} = \sum E_i \\ f:\text{resol.} \downarrow & & \downarrow \\ X & \ni & x : \text{normal surface sing.} \end{array}$$

$$Z_f = Z_{num} := \min\{ Z \in \oplus \mathbb{Z}E_i \setminus \{0\} \mid Z.E_i \leq 0 \ \forall i \}$$

Prop. (Laufer's algorithm for finding Z_{num}. [Laufer, 1972] On rational singularities)

1. Choose any E_{i_1} and define $Z_1 := E_{i_1}$.
2. Assume that $E_1, \cdots, E_{\nu-1}$ are defined. If $\exists i_\nu$ s.t. $Z_{\nu-1}.E_{i_\nu} > 0$, then $Z_\nu := Z_{\nu-1} + E_{i_\nu}$.
3. If $Z_{\nu_0}.E_i \leq 0 \ \forall i$, then STOP and $Z_{\nu_0} = Z_{min}$.

This procedure will stop at a finite stage. The $\{Z_1, \cdots, Z_{\nu_0}\}$ reaching Z_{min} is called computation sequence. (idea. $Z_1 < Z_2 < \cdots \underset{\text{by induction}}{\leq} Z_{num}$, see [Ishii, Prop. 7.2.4])

Def. (arithmetic (virtual) genus) S: sm surface, D: divisor. Define $p_a(D) := \frac{1}{2}(D^2 + K_S.D) + 1$.

Prop. (i) If $D > 0$, then $p_a(D) = 1 - \chi(D, \mathcal{O}_D)$.

(ii) If D is a prime divisor, then $p_a(D) \geq 0$, and "$p_a(D) = 0 \Leftrightarrow D \simeq \mathbb{P}^1$".

(pf.)

(i) By Nagata's compactification thm, we may assume S is a complete smooth surface.

$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0 \Rightarrow \chi(\mathcal{O}_S) = \underline{\chi(\mathcal{O}_S(-D))} + \chi(\mathcal{O}_D)$

$= \frac{-D}{2}(-D-K_S) + \chi(\mathcal{O}_S)$ by RR.

(ii) • $p_a(D) = 1 - (h^0(\mathcal{O}_D) - h^1(\mathcal{O}_D)) = h^1(\mathcal{O}_D) \geq 0$

• $D \simeq \mathbb{P}^1 \Rightarrow p_a(D) = h^1(\mathcal{O}_{\mathbb{P}^1}) = 0$

• If $p_a(D) = 0$, take $\nu: \widetilde{D} \longrightarrow D$ normalization and $0 \to \mathcal{O}_D \to \nu_* \mathcal{O}_{\widetilde{D}} \to K \to 0$.

$\Rightarrow 0 \to \underset{\simeq k}{H^0(D, \mathcal{O}_D)} \to \underset{\simeq k \text{ by proper}}{H^0(\widetilde{D}, \mathcal{O}_{\widetilde{D}})} \to \underset{= 0}{H^0(D, K)} \to \underset{= 0 \text{ by } p_a(D)=0}{H^1(D, \mathcal{O}_D)}$. Hence ν is an isom., i.e.

D is a sm. proper curve with $H^1(\mathcal{O}_D) = 0 \Rightarrow D \simeq \mathbb{P}^1$.

Cor.

$$\begin{array}{ccc} Y & \supseteq & f^{-1}(x)_{red} = \sum E_i \\ f:\text{resol.} \downarrow & & \downarrow \\ X & \ni & x : \text{normal surface sing.} \end{array}$$

Z_{num} = the fund. cycle of $f^{-1}(x)_{red} \Rightarrow p_a(Z_{num}) \geq 0$.

(pf.) Let the computation seq. be $Z_1, \cdots, Z_{\nu_0} = Z_{num}$. By construction, $Z_\nu = Z_{\nu-1} + E_{i_\nu}$ and $Z_{\nu-1}.E_{i_\nu} > 0 \ \forall \nu$.

$p_a(Z_\nu) = p_a(Z_{\nu-1}) + \underbrace{p_a(E_{i_\nu})}_{\geq 0} + \underbrace{Z_{\nu-1}.E_{i_\nu} - 1}_{\geq 0} \geq p_a(Z_{\nu-1})$. Hence $p_a(Z_{num}) \geq p_a(Z_1) = p_a(E_{i_1}) \geq 0$.

$\because E_{i_\nu}$ is prime div.

Def. $Y \supseteq f^{-1}(x)_{red} = \sum E_i$. Define $p_a(f) := \sup\{p_a(Z) \mid Z > 0, \ \mathrm{Supp}\, Z \subseteq f^{-1}(x) = \cup E_i\} \geq p_a(Z_{num})$

f: resol. $\downarrow \quad \downarrow$

$X \ni x$: normal surface sing.

support on singularity

Fact. $\dim R^1 f_* \mathcal{O}_Y$, $p_a(f)$ and $p_a(Z_{num})$ are independent of choices of resolution f. ([Ishii, Prop 6.2.13, 7.2.6, 9, 12])

Consider common resolution and by Leray spectral seq.

analytic invariant

Def. $p_g(X, x) := \dim_{\mathbb{C}} R^1 f_* \mathcal{O}_Y$, geometric genus of (X, x)

$p_a(X, x) := p_a(f)$, arithmetic genus of (X, x) for a resol. $f: Y \to X$.

$p_f(X, x) := p_a(Z_{num})$, fundemental genus of (X, x)

topology invariant

Prop. (X, x): normal surface sing. $p_a(X, x) \geq p_g(X, x) \geq p_f(X, x) \geq 0$

(pf.) $Y \supseteq f^{-1}(x) \supseteq \mathrm{Supp}\, Z$. $\quad 0 \to \mathcal{O}_Y(-Z) \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0 \overset{f_*}{\Longrightarrow} R^1 f_* \mathcal{O}_Y \to \underset{H^1(Z, \mathcal{O}_Z)^\sim}{\underset{\|}{R^1 f_* \mathcal{O}_Z}} \to \underset{0}{\underset{\|}{R^2 f_* \mathcal{O}_Y(-Z)}}$. (fiber dim ≤ 1)

f: resol. $\downarrow \quad \downarrow \quad Z > 0$

$X \ni x$

Hence $p_a(Z) = 1 - \chi(\mathcal{O}_Z) = 1 - h^0(\mathcal{O}_Z) + h^1(\mathcal{O}_Z) \leq h^1(\mathcal{O}_Z) \leq \dim R^1 f_* \mathcal{O}_Y = p_g(X, x)$. □

Def. (X, x): normal surface singularity. It is called rational if $p_g(X, x) = 0$ (cf. [KM, Def 5.18])

strongly elliptic if $p_g(X, x) = 1$

weakly elliptic if $p_a(X, x) = 1$

(cf. [Wagreich] 1970 Elliptic singularities of surface)

Thm. (Artin) (X, x): a normal surface singularity. Then $p_g(X, x) = 0 \overset{\text{formal function thm.}}{\Longleftrightarrow} p_a(X, x) = 0 \overset{\text{computation seq.}}{\Longleftrightarrow} p_f(X, x) = 0$

(For a proof, see [Ishii, Thm 7.31])

Thm. [Wagreich, Laufer, $\cdots$] (X, x): a normal surface sing.. Then $p_a(X, x) = 1 \overset{\text{by Artin}}{\Longleftrightarrow} p_f(X, x) = 1$

[Laufer 77] On minimally elliptic singularities Cor 4.2.

Prop. (X,x) : a rational surface sing., $Z = Z_{num}$: a fund. cycle of a resol. $\Rightarrow$ $\mathrm{mult}_x \mathcal{O} = -Z^2$
$\mathrm{e.dim}(X,x) = -Z^2 + 1$.

(For a proof, see [Ishii, Cor. 7.55])

Def. A normal surface sing. (X,x) is called rational double point (RDP) if (X,x) is rational and $\mathrm{mult}_x X = 2$.

Rmk. (X,x) is RDP $\Rightarrow$ $-Z^2 = \mathrm{mult}_x X = 2$, $\mathrm{e.dim}\, X = 3$ $\Rightarrow$ (X,x) is rational hypersurface singularity $\Downarrow$ Gorenstein

$(X,x) \downarrow (\mathbb{C}^3, 0)$ $\mathrm{Res} f = \frac{dx \wedge dy}{\partial f/\partial z}$ is a local section of K_X

Def. (Mumford's $\mathbb{Q}$-valued pull-back) 1961 [KM, Notation 4.1]

$Y \xrightarrow{f} X$, f: resol. with excep. irr. curve E_i

X : normal surface, $D \in \mathrm{WDiv}(X)_{\mathbb{Q}}$. We define the pull-back $f^*D \in \mathrm{WDiv}(Y)_{\mathbb{Q}}$ as follows :

$f_*^{-1}D$: strict transform of D. Since $(E_i . E_j)$: negative definite, the system of linear equations

$(f_*^{-1}D + \sum a_i E_i) . E_j = 0 \quad \forall j$ has a unique solution $\{a_i\}$. Let $f^*D := f_*^{-1}D + \sum a_i E_i$.

Then $f^*D . E_j = 0$ and $D_1 . D_2 = f^*D_1 . f^*D_2$.

those data denoted by (※)

Prop. $\sum E_i = f^{-1}(x)_{red} \subseteq Y$
$\downarrow$ $\downarrow f$: a minimal resol.
$x \in X$: normal surface sing.

(i.e. $f^{-1}(x)$ doesn't contain (-1)-curves $\Leftrightarrow$ $E_i^2 \leq -2$
existence by [Ishii 7.11] or Cone thm.)

Write $K_Y = f^*K_X + \sum a_i E_i$, then $a_i \leq 0 \quad \forall i$.

(pf.) Note that K_Y is f-nef since f is minimal. Then $K_Y - f^*K_X = \sum a_i E_i$ is f-nef.

(If not, $K_Y . E_i < 0 \rightsquigarrow 0 > (K_Y + E_i) . E_i = \deg_{E_i} K_{E_i} = 2p_a(E_i) - 2 \Rightarrow p_a(E_i) = 0,\ E_i^2 = -1,\ K_Y . E_i = -1$ $\rightarrow\leftarrow$)

By [KM, Lemma 3.4.1], $a_i \leq 0$.

Cor. A normal surface singularity (X,x) is terminal $\Leftrightarrow$ it is smooth.

(pf.) $(\Leftarrow)$ Ok!

$(\Rightarrow)$ If not, take the minimal resolution $\overset{\text{Prop.}}{\Rightarrow}$ $a(X, E_i) \leq 0$ $\rightarrow\leftarrow$.

§ Du Val Singularities (cf. [Ishii, Thm 7.5.1] or [KM, Thm 4.20, Rmk. 4.21], [Dur 79])

Def. [KM, Def 4.4] (✲) is called a Du Val singularity if $K_Y.E_i = 0$ $\forall i$.

Rmk. $E_i^2 = -2$ and $E_i \simeq \mathbb{P}^1$, (-2)-curve

$\left(\because\ -2 \overset{\uparrow \text{ by minimal}}{\geq} E_i^2 = E_i(E_i + K_Y) = 2p_a(E_i) - 2 \Rightarrow p_a(E_i) = 0,\ E_i^2 = -2\right)$

Prop. (X, x) is Du Val singularity $\Leftrightarrow$ $(X, \Delta = 0)$ is (numerically) canonical.

(We will see that K_X is Cartier later)

(pf.) Consider (✲), $K_Y = f^*K_X + \sum a_i E_i$ with $a_i \leq 0$ by Prop.

($\Leftarrow$) $(X, \Delta = 0)$ is canonical $\Rightarrow a_i \geq 0\ \forall i \Rightarrow a_i = 0\ \forall i \Rightarrow K_Y.E_i = f^*K_X.E_i = 0$.

($\Rightarrow$) $0 = K_Y.E_j = \sum_i a_i E_i.E_j\ \forall j \Rightarrow a_i = 0\ \forall i$ by $(E_i.E_j)$ are negative definite.

Prop. (X, x) is RDP $\Leftrightarrow$ it is Du Val sing.

(pf.) Consider (✲) and let $Z = \sum r_i E_i$ be the fund. cycle of f.

($\Rightarrow$): RDP $\Leftrightarrow p_a(Z) = 0,\ Z^2 = -2 \Rightarrow Z.K_Y = 0$. By Laufer's algorithm, $\exists$ computation seq.

$\{Z_1 = \overset{\swarrow \text{any } i}{E_i}, \cdots, Z_{\nu_0} = Z\}$ and $0 = p_a(Z) \geq p_a(Z_{\nu_0 - 1}) \geq \cdots \geq p_a(E_i) \geq 0 \Rightarrow p_a(E_i) = 0 \Rightarrow E \simeq \mathbb{P}^1$.

$0 = Z.K_Y = \sum \underbrace{r_i}_{>0} \underbrace{(E_i.K_Y)}_{\geq 0 \text{ by } f\text{-nef}} \Rightarrow E_i.K_Y = 0\ \forall i$.

($\Leftarrow$) Since $K_Y.E_i = 0\ \forall i$, $Z.K_Y = 0 \Rightarrow 0 \leq \underset{\mathbb{Z}}{\underline{p_a(Z)}} = \frac{1}{2}\underset{\mathbb{Z}_{>0} \text{ by } (Z_i.Z_j) \text{ neg. definite}}{\underline{Z^2}} + \frac{1}{2}Z.K_Y + 1 \leq \frac{1}{2} \Rightarrow p_a(Z) = 0$ and $Z^2 = -2$.

$\Downarrow$ rational sing. $\Rightarrow -\mathrm{mult}_x X$

Def. A normal surface singularity $(X, 0)$ is called A-D-E if $\hat{\mathcal{O}}_{X,0} \simeq \mathbb{C}[[x,y,z]]/(f)$ and f is one of the equations $0 \in (f = 0) \subseteq (\mathbb{C}^3, 0)$

(A_n) $x^2 + y^3 + z^{n+1} = 0 \quad n \geq 1$

(D_n) $x^2 + y^2 z + z^{n-1} = 0 \quad n \geq 4$

(E_6) $x^2 + y^3 + z^4 = 0$

(E_7) $x^2 + y^3 + yz^3 = 0$

(E_8) $x^2 + y^3 + z^5 = 0$

Rmk. • "A_0" is smooth

• $A_n = (ab = c^{n+1})$ where $a = \sqrt{-1}(x+\sqrt{-1}y)$, $b = \sqrt{-1}(x-\sqrt{-1}y)$, $c = z$

• "D_3" $= A_3 = (a^2+b^2+c^4=0)$, where $a = x$, $b = z+\frac{1}{2}y^2$, $c = \frac{\sqrt{-1}+1}{2}y$

$\quad\ \ \|$

$(x^2+y^2z+z^2=0)$

• $D_4 = (a^2+b^3+c^3=0)$, where $a = x$, $b = 2^{1/3}z + 3^{-1/2}\cdot 2^{-1/3}y$, $c = 2^{1/3}z - 3^{-1/2}\cdot 2^{-1/3}y$.

Prop. (X,o) is a RDP $\Leftrightarrow$ it is A-D-E ($\Rightarrow K_X$ is Cartier)

(pf.) ($\Leftarrow$, Sketch)

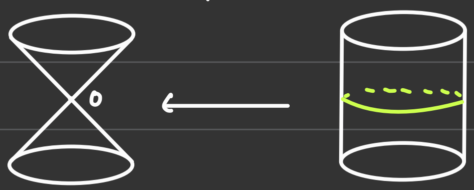
• Using the following two steps construct the minimal resol. of $(X,o) = ((f=0),o)$.

(i) If (X,o) is surface double point, then $K_{Bl_oX} = p^*K_X$, $p: Bl_oX \longrightarrow X$.

(ii) If (X,o) is surface, then every sing. point of Bl_oX is also A-D-E.

• Find the fund. cycle Z_{num} for the minimal. resol. f and conclude $Z_{num}^2 = -2$.

eg1.



$E = p^{-1}(o) = Z_{num} \Rightarrow E^2 = -2$.

$X = (xy - z^2)$ $\qquad$ Bl_oX

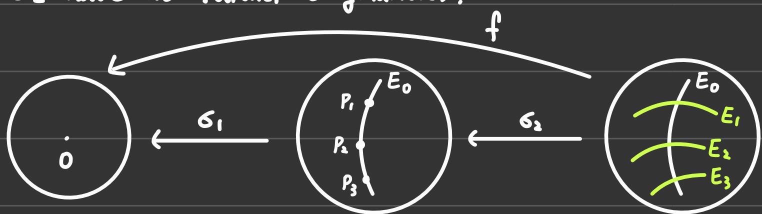
eg2. $D_4: o \in X = (x^2+y^3+z^3=0) \subseteq \mathbb{C}^3$, $Bl_oX = U_1 \cup U_2 \cup U_3$, $U_i \simeq \mathbb{C}^3$.

‥ Look in $U_3 = (z \neq 0)$ ($\mathbb{P}^2$ coord.) with coordinate (x_1, y_1, z_1), where $x_1 = \frac{x}{z}$, $y_1 = \frac{y}{z}$, $z_1 = z$ (from $\mathbb{C}^3$-coord.).

$f(xz, yz, z) = z^2(x_1^2 + z(y_1^3+1)) =: z^2g$. $\mathrm{Sing}(g=0) = \left(g = \frac{\partial g}{\partial x_1} = \frac{\partial g}{\partial y_1} = \frac{\partial g}{\partial z_1} = 0\right)$ All A_1 singularity

$= (g = 2x_1 = 3zy_1^3 = y_1^3+1 = 0) = \{(0, -\zeta_3^k, 0) : 0 \le k \le 2\}$

‥ Check U_1, U_2 have no further singularities!



X $\qquad$ $X_1 = Bl_oX$ $\qquad$ $X_2 = Bl_{\{P_1,P_2,P_3\}}X_1$

… E_1, E_2, E_3 are (-2)-curve.

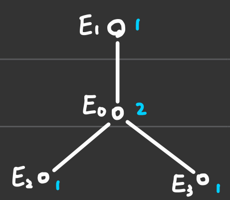
… $E_0^2 = -2$: $\mathrm{div} f^*y = 2E_0 + E_1 + E_2 + E_3 + C$ (strict transform of $y=0$) $\rightsquigarrow 0 = E_0.\mathrm{div} f^*y = 2E_0^2 + \underline{E_0.(E_1+E_2+E_3+C)}$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{4}{\|}$

… Fundemental cycle : $Z_{num} = 2E_0 + E_1 + E_2 + E_3 \rightsquigarrow Z_{num}^2 = -2$

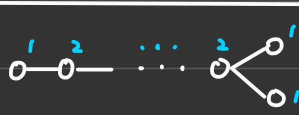
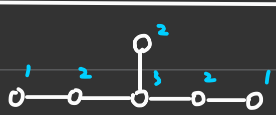
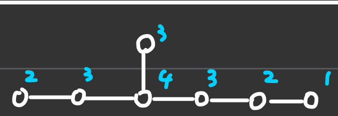
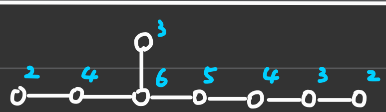
$\rightsquigarrow$ dual graph [KM, Def 4.6]

$E_i \longleftrightarrow$ vertiex v_i

$E_i.E_j \longleftrightarrow$ edge between v_i, v_j



The number express the coefficient of the fundamental cycle.

Type	Dynkin diagram
A_n	 "n vertices"
D_n	 "n vertices"
E_6	
E_7	
E_8	

($\Leftarrow$) $(X,0)$: RDP $\Rightarrow (X,0)\simeq(F=0,0)\subseteq(\mathbb{C}^3,0)$, $\mathrm{mult}_0 F=2$. We repeatedly use four methods:

(1) Weierstrass preparation thm.

$F\in\mathbb{C}[\![\underline{x_1,\cdots,x_n},y]\!]$ or a holo. function around $0\in\mathbb{C}^{n+1}$ s.t. $F(0,\cdots,0,y)=(\overset{\overset{0}{\#}}{\text{const.}})y^m$,
then $F=u(y^m+g_{m-1}(x)y^{m-1}+\cdots+g_0(x))$, where $u\in\mathbb{C}[\![x,y]\!]^\times$ and $g_i\in\mathbb{C}[\![x]\!]$.

(or g_i, u holo. around $0\in\mathbb{C}^{n+1}$ s.t. $u(0)=1$, $g_i(0)=0$)

(2) The elimination of the y^{n-1}-term from $a_ny^n+a_{n-1}y^{n-1}+\cdots+a_0$ by a coord. change $y\mapsto y-\frac{a_{n-1}}{na_n}$ when a_n is invertible.

(3) Hensel's lemma.

Let $f(y,z)$ be a power series with leading term $f_d(y,z)$. Assume $f_d=gh$, where g,h are relatively prime. Then $f=GH$, where g,h are leading term of G,H resp..

(4) M_1,M_2,M_3 : mult. indep. monomials in the var. x,y,z $\Rightarrow$ then any power series of the form $u_1M_1+u_2M_2+u_3M_3$ (where u_i are unit) is equiv. to $M_1+M_2+M_3$ by a suitable coordinate change $x\mapsto x\cdot(\text{unit})$, $y\mapsto y\cdot(\text{unit})$, $z\mapsto z\cdot(\text{unit})$.

Now apply (1), $F=x^2+a(y,z)x+b(y,z)\xrightarrow{\text{by }(2)}F=x^2+f(y,z)$ with $\mathrm{mult}_0 f\geq 2$.

Case1. If $\mathrm{mult}_0 f=2$: By (1),(2), $f(y,z)=u(y^2+vz^m)$ for some $m\geq 2$, u,v are units.

By (4), $F=uv\left(\frac{x^2}{uv}+\frac{y^2}{v}+z^m\right)=X^2+Y^2+Z^m$, $m\geq 2$.

Case2. If $\mathrm{mult}_0 f=3$, i.e. the cubic part $f_3\neq 0$.

- $f_3=l\cdot q\neq(\text{linear})^3$ with $l\nmid q$: By (3), $f=LQ$ and we can choose L as coordinate z, then $\mathrm{mult}_0 l=1$, $\mathrm{mult}_0 q=2$

$f=z(ay^2+\cdots)$ and $a\in\mathbb{C}^\times$ ($\because l\nmid q$). Apply (1),(2) to $(ay^2+\cdots)\rightsquigarrow F=x^2+uz(y^2+vz^m)$, $m\geq 2$.

$\xrightarrow{(4)}F=x^2+zy^2+z^{m+1}$, $m\geq 2$. (This gives the D cases)

• $f_3 = l^3$: By (1), (2) $\rightsquigarrow f(y,z) = u(y^3 + b(z)y + c(z)) = uy^3 + u_a z^a y + u_b z^b$, where u: unit, u_a, u_b either unit or zero at 0, $a \geq 3$, $b \geq 4$ ($\text{mult}_0 f = 3$).

Claim. $(X,0) = (x^2 + uy^3 + u_a z^a y + u_b z^b = 0, 0)$, $u(0) \neq 0$ is canonical $\Leftrightarrow$ "$a \leq 3$ and $u_a(0) \neq 0$" or "$b \leq 5$ and $u_b(0) \neq 0$"

(pf.)

Assume $a \geq 4$ and $b \geq 6$. Let Y defined as $(p^2 + u(qr^2, r)q^3 + u_a(qr^2, r)qr^{a-4} + u_b(qr^2, r)r^{b-6}, 0)$

Then
$$\begin{array}{ccc} \pi : \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3 \\ \cup\!\shortmid & & \cup\!\shortmid \\ E = (r=0) \subseteq Y & & X \end{array}, \quad (p,q,r) \mapsto (\underset{x}{\underset{\shortparallel}{pr^3}}, \underset{y}{\underset{\shortparallel}{qr^2}}, \underset{z}{\underset{\shortparallel}{r}}) \quad \text{and } K_X = \frac{dy \wedge dz}{\frac{\partial}{\partial x}(x^2 + f(y,z))} = \frac{dy \wedge dz}{2x}.$$

$$\Rightarrow \pi^*\left(\frac{dy \wedge dz}{2x}\right) = \frac{r^2 dq \wedge dr}{2pr^3} = \frac{1}{r} \overset{K_Y}{\frac{dq \wedge dr}{2p}} \Rightarrow \pi^* K_X = K_Y - E$$

and X is not canonical $\Leftrightarrow$ not Du Val $\Leftrightarrow$ not RDP. □

We remain two cases $\begin{cases} \overset{2\text{-}1}{a \geq b}, \ b = 4 \text{ or } 5 \\ \overset{2\text{-}2}{a = 3}, \ b \geq 5 \end{cases}$. and $(X,0) = (F := x^2 + \underbrace{uy^3 + u_a z^a y + u_b z^b}_{f} = 0)$

Case 2-1: If $a \geq b$, $f = uy^3 + u_a(\underbrace{z^{a-b}y + \text{unit}}_{\text{unit}})z^b \xrightarrow{(4)} F = x^2 + y^3 + z^b$, $b = 4, 5$ (E_6, E_8)

If $a = b-1$, then

$$f = uy^3 + u_a z^{b-1} y + u_b z^b = \underline{uy^3} + u_b\left(z + \frac{u_a}{bu_b}y\right)^b - * z^{b-2}y^2 - \underline{\sum_{i \geq 3} * z^{b-i}y^i}$$

$\xrightarrow{\text{by (2) for } z^b}$ $f = (\text{unit})y^3 + V_2 z^{b-2}y^2 + (\text{unit})z^b$

$\xrightarrow{\text{by (2) for } y^3}$ $f = (\text{unit})y^3 + (\text{unit})z^b$ $\quad (\because 2(b-2) \geq b$ for $b = 4,5) \rightsquigarrow (E_6$ or $E_8)$

Case 2-2: $f = (\text{unit})y^3 + (\text{unit})yz^3 + V_3 z^b$, $b \geq 5$ By blow-up the origin, we see in $\{z \neq 0\}$, $(y = y_1 z_1, z = z_1)$, we get $z_1^3((\text{unit})y_1^3 + (\text{unit})y_1 z_1 + V_3 z_1^{b-3}) =: z_1^3 \bar{f}$.

Since $\text{mult}_0 \bar{f} = 2$ and $\bar{f}$ is not a square, $\bar{f}$ is reducible and so is f.

Since $f_3 = (\text{unit})y^3$, one of the factor of f is $y + (\text{h.o.t.})$ (be new coordinate). By (1),

$f = y((\text{unit})y^2 + (\text{unit})z^3) \xrightarrow{(4)} f = y^3 + yz^3 \rightsquigarrow F = x^2 + y^3 + yz^3$ (E_7).

Case 3. If $\text{mult}_0 f \geq 4$, then we claim that $X = (x^2 + f(y,z) = 0)$ is not canonical sing..

Set $Y := (p^2 + \frac{1}{r^4} f(qr, r) = 0)$ and
$$\begin{array}{ccc} \pi : \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3 \\ \cup\!\shortmid & & \cup\!\shortmid \\ (r=0) = E \subseteq Y & \longrightarrow & X \end{array} \rightsquigarrow \pi^* K_X = K_Y - E.$$

Rmk. In Claim and Case 3, "Y" is defined by weighted blow-up of wt. $(3,2,1)$ and $(2,1,1)$ resp..

§ Elliptic surface singularities

Recall. $p_g \geq p_a \geq p_f \geq 0$. • rational : $p_g = p_a = p_f = 0$

• elliptic : $p_a = p_f = 1$

$p_g = 1$ (strongly)

$$\begin{array}{ccl} \sum E_i = f^{-1}(0)_{red} & \subseteq & Y \\ \downarrow & & \downarrow f : \text{minimal resol.} \\ 0 & \in & X : \text{normal surface singularity } (\hat{\mathcal{O}}_{X,0}) \end{array}$$

, WLOG X, Y : affine (contractible and Stein)

Def • (anti-canonical cycle) Write $K_Y = f^*K_X + \underbrace{\sum a_i E_i}_{\substack{\shortparallel \\ -Z_K \in WDiv(Y)_{\mathbb{Q}}}}$ (by Prop, $f^{-1}(0)$ is connected $\Rightarrow a_i < 0\ \forall i$ or $a_i = 0\ \forall i$)

[KM, Notation 4.44] $Z_K = \lfloor Z_K \rfloor + (Z_K - \lfloor Z_K \rfloor) =: Z + \Delta_Y$

• $(X,0)$ is called numerical Gorenstein if Z_K is an integral divisor, i.e. $\Delta_Y = 0$.

Rmk. • $Z_K = 0 \overset{\text{def.}}{\Longleftrightarrow} (X,0)$ is Du Val.

• $(X,0)$ is (num.) Goren. $\Leftrightarrow$ the cpx. line bundle $\Omega^2_{X \setminus 0}$ is topological (holomorphically) trivial.

$\begin{cases} Z_K : \text{topo. ob.} \\ K_X \ (\text{or } \Omega^2_X) : \text{analytic ob.} \end{cases}$

• If $(X,0)$ is num. Goren, then $Z_{num} \leq Z_K$.

($Z_K \equiv -K_Y$ is anti-f-nef)

Prop. L : f-nef line bundle on Y, $Z = \lfloor Z_K \rfloor \Rightarrow H^0(Y, L) \twoheadrightarrow H^0(Z, L|_Z)$

and $H^1(Y,L) \simeq H^1(Z, L|_Z)$.

(pf.)

$0 \to \mathcal{O}_Y(-Z) \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0 \xrightarrow{\otimes L} 0 \to L(-Z) \to L \to L|_Z \to 0$

$\xrightarrow{f_*} f_*L \to f_*L|_Z \to R^1f_*L(-Z) \to R^1f_*L \to R^1f_*L|_Z \to R^2f_*L(-Z)$

$R^1f_*L(-Z) = 0$ by rel. K-V, $R^2f_*L(-Z) = 0$

$\begin{cases} \because K_Y + Z + \Delta_Y \equiv 0 \\ \Rightarrow L(-Z) - (K_Y + \Delta_Y) \equiv_f L : f\text{-nef}, f\text{-big} \ (\Leftarrow \text{(def.)}) \\ (Y, \Delta_Y) : klt \end{cases}$

X, Y : affine

$\Rightarrow H^0(L) \twoheadrightarrow H^0(L|_Z)$ and $R^1f_*L \simeq R^1f_*L|_Z \Longleftrightarrow H^1(L) \simeq H^1(L|_Z)$.

Prop. $\omega_X / f_*\omega_Y$ is dual to $R^1f_*\mathcal{O}_Y$, where f could be any resol..

In particular, $R^1f_*\mathcal{O}_Y \simeq \mathbb{C} = \mathcal{O}_{X,0}/m_{X,0} \Leftrightarrow f_*\omega_Y = m_{X,0}\omega_X$.

(pf.) For a open neighborhood V of $0 \in X$, take $U = f^{-1}V$, $E = Exc(f)$.

$\Rightarrow 0 \to H^0_E(U,\omega_Y) \to H^0(U,\omega_Y) \to H^0(U \setminus E, \omega_Y) \to H^1_E(U,\omega_Y) \to H^1(U,\omega_Y)$

$\{s \in H^0(U,\omega_Y) : supp(s) \subseteq E\} = 0$ $\qquad$ $\forall U' \supset E$ $H^1_E(U',\omega_Y) = H^1_E(Y,\omega_Y)$

$\rightsquigarrow 0 \to f_*\omega_Y \to \omega_X|_{X\setminus\{0\}} \to H^1_E(\omega_Y) \to R^1f_*\omega_Y = 0$ by the Grauert-Riemenschneider Vanishing thm. ($/\mathbb{C}$) [KM, Cor. 2.68]

$\omega_X|_{X\setminus\{0\}} \simeq \omega_X$ (by X: normal)

$\Rightarrow \omega_X / f_*\omega_Y \simeq H^1_E(\omega_Y)$ is dual to $R^1f_*(\omega_Y \otimes \omega_Y^{-1})$ by local duality thm. (see [Ishii, Cor. 3.5.15])

Def. (Reid) [KM, Def. 4.48]

$(X,0)$ is elliptic Gorenstein surface singularity if $\begin{cases} K_X \text{ is Cartier} \\ R^1f_*\mathcal{O}_X \simeq \mathbb{C} \end{cases}$

[Laufer 77] On minimally elliptic singularities, Thm 3.4, 3.10.

Thm. [Laufer 77] Working with the minimal resol. of f, TFAE

(1) (minimal elliptic surface singularity)

$p_a(Z_{num}) = 1$ and any connected proper subdivisor of $\sum E_i = f^{-1}(0)_{red}$ contracts to a rational sing.

(2) $p_a(Z_{num}) = 1$ and "$p_a(D) < 1 \quad \forall\, 0 < D < Z_{num}$"

([Laufer 77, Def 3.1] Z_{num} is called a minimally elliptic cycle)

topo. (3) $Z_{num} = Z_K$ ($\Rightarrow$ num. Goren.)

analy. (4) $(X,0)$ is elliptic Gorenstein.

Rmk. $(X,0)$: Gorenstein. WLOG, $\mathcal{O}(K_X) \simeq \mathcal{O}_X \rightsquigarrow K_Y = f^*K_Y - Z_K = -Z_K$.

Lemma. (cf. [KM, 4.47])

The minimal resol. of a Gorenstein surface singularity $(X,0)$, we have "$h^1(D) \leq h^1(Z_K) \ \forall\, 0 < D < Z_K$."

Z_K in here is called coho. cycle Z_{coh}

Rmk. $Z_{coh} = Z$ and $(X,0)$: num. Goren. $\Leftrightarrow$ $(X,0)$: Goren.

(pf.) Notice that $K_Y = -Z_K$ (X: Gorenstein). By Serre duality,

$$H^1(\mathcal{O}_{Z_K})^\vee \simeq H^0(\omega_{Z_K}) \overset{\text{adjunction}}{=\!=\!=} H^0(Z_K, \mathcal{O}_{Z_K}(Z_K+K_Y)) = H^0(Z_K, \mathcal{O}_{Z_K}),$$

$$H^1(\mathcal{O}_D)^\vee \simeq H^0(\omega_D) = H^0(Z_K, \mathcal{O}_D(D-Z_K)).$$

Consider $0 \to \underset{\mathcal{O}_Y(-Z_K-D)\otimes\mathcal{O}_D}{\underset{\|}{\mathcal{O}_D(-Z_K-D)}} \to \mathcal{O}_{Z_K} \to \mathcal{O}_{Z_K-D} \to 0$ (Hw). Since $H^0(\mathcal{O}_{Z_K}) = H^0(\mathcal{O}_{Z_K-D})$ is nontrivial,
(constant section)

$h^1(\mathcal{O}_D(D-Z_K)) < h^1(\mathcal{O}_{Z_K})$.

Cor. [KM 4.49] Assume that $(X,0)$ is elliptic Gorenstein surface singularity. Set $Z = Z_{num} = Z_K$ for the minimal resol. $f: Y \to X$. Then either Z is an irreducible and reduced curve of $p_a(Z)=1$ or $\forall$ irr. comp. $E_i \subseteq f^{-1}(0)_{red}$ is a smooth rational curve with $E_i.(K_Y+E_i) = -2$.

(pf.) Since $\mathcal{O}_Y$ is f-nef, by Prop, $\underset{\underset{R^1f_*\mathcal{O}_Y}{\updownarrow\ X:\text{ affine}}}{H^1(Y,\mathcal{O}_Y)} \simeq H^1(Z,\mathcal{O}_Z) \Rightarrow h^1(\mathcal{O}_Z) = \dim_{\mathbb{C}} R^1f_*\mathcal{O}_Y = 1$.

If Z is irr. and reduced, then $p_a(Z) = h^1(\mathcal{O}_Z) = 1$.

Otherwise, $E_i < Z = Z_K$, hence by Lemma, $p_a(E_i) = h^1(E_i) < h^1(Z) = 1 \Rightarrow p_a(E_i) = 0$ i.e. $E_i \simeq \mathbb{P}^1$.

Since $K_Y = -Z$, $E_i(-Z+E_i) = E_i.(K_Y+E_i) = \deg_{E_i} K_{E_i} = -2$.

Example. (cf. [KM, Note 4.8])

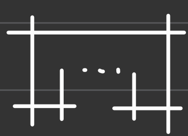
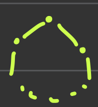
- (simple elliptic singularities) eg. $(X,0) = (x^3+y^3+z^3=0,\ 0) \subseteq (\mathbb{C}^3,0) \rightsquigarrow Z_{min} = E = Z_K$, $E^2 = -3$.

$$\left(\begin{tikzcd} Y \arrow[d, "f:\text{ min. resol.}"'] & \supseteq & E = f^{-1}(0)_{red} \arrow[d, "\text{sm. elliptic curve}"] \\ X & \ni & 0 \end{tikzcd}\right)$$

[K. Saito 1974]

- (cusp surface singularities)

$$\begin{tikzcd} Y \arrow[d, "f:\text{ min. resol.}"'] & \supseteq & f^{-1}(0)_{red} = \sum E_i = E = \arrow[d] \\ X & \ni & 0 \end{tikzcd}$$


node
I_1
or

I_r, $r \geq 2$
dual


$Z_{min} = \sum E_i = Z_K$

- (Brieskorn-Pham sing.) $(X,0) = (x^a+y^b+z^c=0,\ 0) \subseteq (\mathbb{C}^3,0)$

Fact. $p_g(X,0) = \#\{(i,j,k)\in\mathbb{N}^3 : \frac{i}{a}+\frac{j}{b}+\frac{k}{c} \leq 1\}$

eg. $(a,b,c) = (2,3,18k)$, $p_a = p_f = 1$, $p_g = 3k$.

Goal.

$$\begin{array}{ccl} & & Z_{red} = f^{-1}(0)_{red} \\ Y \supseteq f^{-1}(0)_{red} = \sum E_i & , & Z = Z_{min} = Z_K \Rightarrow K_Y = -Z \text{ . Set } L = \mathcal{O}_Y(-Z) \simeq \omega_Y \text{ : } f\text{-nef.} \\ \text{min. resol. } f \downarrow \quad \downarrow & & \\ X \ni 0 & & \text{elliptic Gorenstein surface sing. (WLOG } \mathcal{O}(K_X) \simeq \mathcal{O}_X) \end{array}$$

Understand $R(Y,L) := \bigoplus_{n=0}^{\infty} H^0(Y, L^{\otimes n})$ by reduce the problem first to Z and then to $\cup$ 0-dim'l subscheme

Lemma. V: a proper (possibly non-reduced) curve s.t. $H^1(\mathcal{O}_V) = 0$.

L: a nef line bundle on V

$\Rightarrow$ (1) L is globally generated, and

(2) $H^1(V, L) = 0$

(pf.) (1) Let $V_{red} = \bigcup V_i$, V_i: irr. comp. Pick general points $p_i \in V_i$ and Cartier divisor $D_i \subseteq V_i$ s.t. $D_i \cap V_i = \{p_i\}$. Set $m_i := \deg_{V_i}(L|_{V_i})$ and $L' := \mathcal{O}_V(\sum m_i D_i)$. Notice that the exponential sequence $0 \to \mathbb{Z}_V \to \mathcal{O}_V \xrightarrow{\exp} \mathcal{O}_V^* \to 1$ is exact even if $\mathcal{O}_V$ has nilpotent elements (cf. complex surfaces Chap II §2 p.63) Then

$$0 = H^1(\mathcal{O}_V) \longrightarrow H^1(\mathcal{O}_V^*) \xhookrightarrow{c_1} H^2(V, \mathbb{Z}) = \bigoplus \mathbb{Z}[V_i] \quad \Rightarrow \quad L = L'$$

$$L \longmapsto (\deg_{V_i} L|_{V_i})_i = (m_i)_i$$

$$L' \longmapsto (\deg_{V_i} L'|_{V_i})_i = (m_i)_i$$

Then L is g.b.g.s except possibly at the points p_i. By varying p_i, we get (1).

(2) By (1), $\underset{\mathcal{O}_V^{\oplus d},\ d = h^0(L)}{\underset{\text{S}|}{H^0(V,L) \otimes \mathcal{O}_V}} \xrightarrow{ev.} L \Rightarrow \underset{0}{\underset{\|}{H^1(\mathcal{O}_V^{\oplus d})}} \to H^1(L) \to \underset{0 \text{ (by curve)}}{\underset{\|}{H^2(\ker ev)}} \Rightarrow H^1(L) = 0$.

Prop. L: a nef line bundle on Z s.t. $\deg_Z L > 0$.

$\Rightarrow$ (1) $H^1(Z, L) = 0$

(2) $\exists s \in H^0(Z, L)$ s.t. $\begin{cases} (s=0) \text{ is a 0-dim'l subscheme} \\ (s=0) \cap \mathrm{Sing}(Z_{red}) = \emptyset \\ s|_{Z_{red}} \text{ has no multiple zeros.} \end{cases}$

(3) If $\exists C \subsetneq Z$ an irr. component of Z s.t. $\deg_C(L|_C) > 0$. Set $Z' = Z - C$, then $H^0(Z, L) \twoheadrightarrow H^0(Z', L|_{Z'})$ is surjective.

(pf.) Recall that $h^1(\mathcal{O}_Z) \overset{\text{by Prop [KM, 4.55 (2)]}}{=} h^1(\mathcal{O}_Y) = \dim R^1 f_* \mathcal{O}_Y = p_g(X,0) = 1$. Note that L is ample. (by deg > 0 & normalization)

If Z is irr. and reduced, then $p_a(Z)$ by Lemma [KM 4.44].

$\rightsquigarrow$ (1) by [Hart. III, Ex1.8, 1.9].

(2) by Bertini thm.

Otherwise, $\exists C$ as in (3), $0 < Z' = Z - C < Z$. Then $\underbrace{h^1(\mathcal{O}_{Z'})}_{=0} < h^1(\mathcal{O}_Z) = 1$ by Lemma [KM 4.47].

Since $L|_{Z'}$ is nef on Z', by Lemma, $L|_{Z'}$: gbgs and $H^1(Z', L|_{Z'}) = 0$.

Since $Z = Z'_{\geq 0} + C_{>0}$, by Dévissage of $\mathcal{O}_Z$, $0 \to \mathcal{O}_C(-Z') \to \mathcal{O}_Z \to \mathcal{O}_{Z'} \to 0$

$\overset{\otimes L}{\rightsquigarrow} 0 \to L(-Z')|_C \to L \to L|_{Z'} \to 0$.

Recall: By Lemma, $C \simeq \mathbb{P}^1$ and $-2 = C.(-Z+C) = -C.Z'$.

Since $\deg_C L(-Z')|_C = \deg_C L|_C - Z'.C \geq -1$,

$$
\begin{array}{ccccccccc}
H^0(Z,L) & \overset{(3)}{\twoheadrightarrow} & H^0(Z', L|_{Z'}) & \to & H^1(\overset{\mathbb{P}^1}{C}, \overset{\deg \geq -1}{L(-Z')|_C}) & \to & H^1(Z,L) & \to & H^1(Z', L'|_{Z'}) \\
 & & & & \parallel & & \parallel & & \parallel \\
\exists s & \underset{(2)}{\longmapsto} & \text{general section} & & 0 & \longrightarrow & 0 & \longleftarrow & 0
\end{array}
$$

by Bertini

[KM, 4.52]

Notation. $(X,0)$: elliptic Goren. surface sing., $Z = Z_{min} = Z_K$ for minimal resol. $f : Y \to X$.

- Assume that L : nef l.b. on Z with $s \in H^0(Z,L)$ s.t.

$$
\begin{cases}
(s=0) \text{ is } 0\text{-dim'l subsch.} \\
(s=0) \cap \mathrm{Sing}(Z_{red}) = \emptyset \\
s|_{Z_{red}} \text{ has no mult. zeros.}
\end{cases}
$$

- Set $V := (s=0)$. Then $A = \mathcal{O}_V$: a semilocal ring with (Jacobson) radical $\mathfrak{m}$.

Write $A = \bigoplus_{i=1}^{r} A_i$, $\mathfrak{m} = \bigoplus_{i=1}^{r} \mathfrak{m}_i$, $(A_i, \mathfrak{m}_i)$: local Artin $\mathbb{C}$-algebras.

Set $V_i := \mathrm{Spec}\, A_i \subseteq Z$, which is Cartier divisors s.t. $V = \sum V_i$.

$\mathrm{socle}(\mathfrak{m}) := \{ a \in \mathfrak{m} \mid \mathfrak{m} \cdot a = 0 \}$, the scale of $\mathfrak{m}$.

(Note : $A_i = A[t]/(t^a)$, $\mathrm{scale}(\mathfrak{m}) = (t^{a-1})$)

- $0 \to \underset{\underset{L^\vee}{\parallel}}{\mathcal{O}_Z(-V)} \xrightarrow{\otimes s} \mathcal{O}_Z \to \underset{\underset{A}{\parallel}}{\mathcal{O}_V} \to 0$

$\overset{\otimes L}{\rightsquigarrow} 0 \to \mathcal{O}_Z \to L \to \overset{\overset{A \otimes L}{\parallel}}{L|_V} \to 0$. Set $W_L = \mathrm{Im}(H^0(Z,L) \to H^0(V, L|_V)) \subseteq \overset{\overset{A \otimes L}{\parallel}}{H^0(V, L|_V)}$

Rmk. W_L is vector subspace of $A \otimes L$, which is not an A-submodule in general.

Lemma Notation and assumption as in Notation.

(1) If $A \neq 0$, then $\mathrm{codim}_{A\otimes L} W_L = 1$.

(2) For each j, the projection $W_L \hookrightarrow A\otimes L \longrightarrow A/A_j \otimes L$ is surjective.

(3) If $\mathfrak{m} \neq 0$, the projection $W_L \hookrightarrow A\otimes L \longrightarrow A/\mathrm{scale}(\mathfrak{m}) \otimes L$ is surjective.

(4) If $\underset{\deg_Z L}{\overset{}{\underset{\parallel}{\dim_{\mathbb{C}} A}}} \geq 2$, then W_L generates $A\otimes L$ as A-module.

(pf.)

(1)
$$\begin{tikzcd}[column sep=small] H^0(Z,L) \arrow[r] \arrow[dr] & A\otimes L \arrow[r] & H^1(\mathcal{O}_Z) \arrow[r] & H^1(Z,L) \\ & W_L \arrow[u] \arrow[dr] & & \\ 0 \arrow[ur] & & 0 & \end{tikzcd}$$
$H^1(\mathcal{O}_Z) = \mathbb{C}$, $H^1(Z,L) = 0$ ($\because \deg_Z L = \dim_{\mathbb{C}} A > 0$ & by Prop.), nef

(2) For each j, $\mathcal{O}_{V\setminus V_j}(V) = A/A_j \otimes L$. Then

$$0 \to \mathcal{O}_Z(-(V-V_j)) \to \mathcal{O}_Z \to \mathcal{O}_{V\setminus V_j} \to 0 \xrightarrow{\otimes L} 0 \to \mathcal{O}_Z(V_j) \to L \to A/A_j \otimes L \to 0.$$

$$\Rightarrow H^0(Z,L) \longrightarrow A/A_j \otimes L \longrightarrow H^1(Z, \underbrace{\mathcal{O}_Z(V_j)}_{\text{nef, deg}>0}) = 0$$
$$H^0(Z,L) \searrow W_L \subseteq A\otimes L \nearrow A/A_j\otimes L$$
, this show (2).

(3) Assume $\mathfrak{m} \neq 0$, then $\mathrm{scale}(\mathfrak{m}) \neq 0$. Let $C \subseteq Z$ be an irr. comp. s.t. $V = (s=0)$ has a non-reduced point on C $\Rightarrow \deg_C(L|_C) > 0$ and Z is not reduced along C ($\because s|_{Z_{red}}$: no mult. zero).

Set $Z' = Z - C > 0$, $s' = s|_{Z'}$ and $V' = (s'=0) \subseteq Z'$. Since $0 \to \mathcal{O}_{Z'} \xrightarrow{\otimes s'} L|_{Z'} \longrightarrow L|_{V'} \longrightarrow 0$,

$$\begin{tikzcd} H^0(Z,L) \arrow[r, two heads, "\text{by Prop.}"] \arrow[rr, bend right, two heads] & H^0(Z', L|_{Z'}) \arrow[r, two heads] & H^0(V', L|_{V'}) \arrow[r] & H^1(\mathcal{O}_{Z'}) = 0 \end{tikzcd}$$
$(h^1(\mathcal{O}_{Z'}) < h^1(\mathcal{O}_Z) = 1)$

Since $\mathcal{O}_{V'} \twoheadrightarrow A/\mathrm{scale}(\mathfrak{m})$, $A/\mathrm{scale}(\mathfrak{m}) \otimes L$ is a quotient of $H^0(V', L|_{V'})$, this prove (3).

(4) If (4) fails, then all elements of W_L vanish at a point $V = \mathrm{Spec}\, A$, but (2) and (3) show that this cannot happen.

Prop. [Laufer 77, Reid 76]

$(X,0)$: elliptic Goren. surface sing., $Z = Z_{min} = Z_K$ for the min. resol. $f: Y \to X$.

L : nef l.b. on Z, let $k = \deg_Z L$ and $R(Z,L) = \bigoplus_{n=0}^{\infty} H^0(Z, L^{\otimes n})$.

(1) If $k \geq 2$, then L is globally generated.

(2) If $k \geq 3$, then $R(Z,L)$ is generated by it elements of degree 1.

If $k \geq 2$, then $R(Z,L)$ is generated by it elements of degree 1, 2.

If $k \geq 1$, then $R(Z,L)$ is generated by it elements of degree 1, 2, 3.

More precisely, $R(Z,L)$ is

$\cdot (k \geq 3)$ $\mathbb{C}[x_1, \cdots, x_k] / I$, $\deg x_i = 1$, $I = \langle \deg 2, 3 \text{ elements} \rangle$.

$\cdot (k=2)$ $\mathbb{C}[x,y,z] / (z^2 + g_4(x,y))$, $\deg(x,y,z) = (1,1,2)$, g_4 is homo. of deg 4.

$\cdot (k=1)$ $\mathbb{C}[x,y,z] / (z^2 + y^3 + ayx + bx^6)$, $\deg(x,y,z) = (1,2,3)$, $a, b \in \mathbb{C}$.

(pf.)

(1) Let $V = (s=0)$, $A = \mathcal{O}_V$, $(A, \mathfrak{m}) = \bigoplus_{i=1}^{r} (A_i, \mathfrak{m}_i)$ as in Notation. We have $\dim_{\mathbb{C}} A = \deg_Z L = k \geq 1$.

Assume $k \geq 2$ ($\Rightarrow W_L$ gen. $A \otimes L$ as A-mod.)

$$\begin{tikzcd}
 & \mathcal{O}_Z \arrow[d, hook, "\cdot s"] & \\
H^0(Z,L) \otimes \mathcal{O}_Z \arrow[r, two heads] \arrow[d] & L \arrow[d] & \Rightarrow L \text{ is gbgs.} \\
W_L \otimes \mathcal{O}_V \arrow[r, two heads] & L|_V = A \otimes L & \\
H^0(L|_V) \otimes \mathcal{O}_V \arrow[u, phantom, "\cap"] \arrow[ur] & &
\end{tikzcd}$$

idea.

(2) Let T be any section of $L|_V$ generating $L|_V = A \otimes L$.

Note : $H^0(V, L|_V^{\otimes n}) = A \cdot T^n$ and $R(V, L|_V) = A[T]$.

From $0 \to \mathcal{O}_Z \xrightarrow{\otimes s} L \to L|_V \to 0$ and $\otimes L^{\otimes(n-1)}$,

$$0 \to H^0(L^{n-1}) \to H^0(L^n) \to \underset{\substack{\| \\ A \cdot T^n}}{H^0(L^n|_V)} \to H^1(Z, L^{n-1}) = 0$$

$\uparrow$ nef deg > 0

Let $R_Z = R(Z,L)$, then $R_Z / s \cdot R_Z (n) = \begin{cases} AT^n & \text{for } n \geq 2 \\ W_L & \text{for } n = 1 \\ \mathbb{C} & \text{for } n = 0 \end{cases}$.

(detail see [KM, p.141-142])

Def. (weighted blow-up of $\mathbb{A}^d_k$ at 0)

$w = (w_1, \cdots, w_d)$ where $w_1, \cdots, w_d \in \mathbb{N}$, $R = k[x_1, \cdots, x_d] \ni \prod x_i^{m_i} = x^M$, where $M = (m_1, \cdots, m_d)$.

Define the weight of x^M by $\sum m_i w_i$. For $f = \sum a_M x^M \in R$, $w(f) := \min\{ w(x^M) : a_M \neq 0 \}$.

Then we get the Ideal $\mathfrak{m}^w(n) = \{ f \in R : w(f) \geq n \}$. Define $B_0^w \mathbb{A}^d := \mathrm{Proj}_R \left(\bigoplus_{n=0}^{\infty} \mathfrak{m}^w(n) \right)$.

For any $X \subseteq \mathbb{A}^d$ this define $B_0^w X$ as the strict transform of X in $B_0^w \mathbb{A}^d$.

Thm. [Laufer 77, Reid 76]

(WLOG $\omega_X = \mathcal{O}_X$)

$L = \mathcal{O}_X(-Z)$

$\|$

(X, o) : elliptic Goren. surface sing., $Z = Z_{\min} = Z_K$ for the min. resol. $f : Y \to X$ with $\omega_Y = f^* \omega_X(-Z)$

Set $k = -Z^2 = \deg_Z L$.

(1) Assume $k \geq 3$, then $\mathrm{mult}_o X = k = \mathrm{edim}(X, o)$.

Choose any embedding $(X, o) \hookrightarrow (\mathbb{C}^k, o)$. Let x_i be the coordinate on $\mathbb{C}^k$, $w(x_1, \cdots, x_k) = (1, \cdots, 1)$.

(2) Assume $k \leq 2$, then $\mathrm{mult}_0 X = 2$ and $\mathrm{edim}(X,0) = 3$.

After an analytic coordinate change it can be given by equation

- $(k=2)$ $z^2 + q(x,y) = 0$, where $\mathrm{mult}_0 q = 4$ and $w(x,y,z) = (1,1,2)$.
- $(k=1)$ $z^2 + y^3 + y q_4(x) + q_6(x)$, where $\mathrm{mult}_0 q_i \geq i$ and $w(x,y,z) = (1,2,3)$.

(3)
$$\begin{tikzcd} Y \arrow[d] \arrow[dd, bend right, "f"', "\text{min. resol.}"'] \\ \overline{Y} := B_0^{\omega} X \arrow[d] \\ X \end{tikzcd}$$

§ Cohen-Macaulay and Duality.

Def. (R,m): a Noeth. local ring, M: f.g. R-module

- M is C-M if $\underset{\dim \operatorname{Supp} M}{\overset{\shortparallel}{\dim M}} = \operatorname{depth} M =$ the maximal length r of M-regular seq.
  i.e. $\exists x_1, \cdots, x_r \in m$ s.t. $M/(x_1,\cdots,x_{i-1})M \overset{\cdot x_i}{\hookrightarrow} M/(x_1,\cdots,x_{i-1})M$ $\forall i$.
- M satifies Serre's condition S_k $(k \geq 0)$ if $\operatorname{depth} M \geq \min\{k, \dim M\}$

(the module M studied in [KM] will satisfy $\dim M = \dim R$)

Def. X: noeth. scheme, $\mathcal{F}$: coh. sheaf on X.

- $\mathcal{F}$ is called C-M at closed point $x \in X$ if $\mathcal{F}_x$ is a C-M module over $\mathcal{O}_{X,x}$.
- $\mathcal{F}$ is called C-M if it is C-M at x for all closed point $x \in \operatorname{Supp} \mathcal{F}$.
- X is called C-M if $\mathcal{O}_X$ is C-M.
- $\mathcal{F}$ is called S_k if $\mathcal{F}_x$ is S_k for every $x \in X$.

Rmk. • normal $\Leftrightarrow$ $\overset{\text{regular in codim 1.}}{R_1} + S_2$

- $\mathcal{F}$ is C-M $\Leftrightarrow$ it is S_d $\Leftrightarrow$ it is S_k for all k.
  (with $d = \dim \operatorname{Supp} \mathcal{F} = \dim X$)

  $\because$ an isolated surface singularity is C-M $\Leftrightarrow$ it is normal.

Fact. [KM, Prop 5.3 (1)] (invariant under passing to a hyperplane section)
Let $h \in m$ be a nonzero divisor on M. Then M is C-M (resp. S_k) $\Leftrightarrow$ $M/h \cdot M$ is C-M (resp. S_k).

Rmk. $R = M = \mathcal{O}_X$, $x \in X$ is C-M $\Leftrightarrow$ $\exists h \in m_{X,x}$ which is a non-zero divisor on $\mathcal{O}_{X,x}$ s.t. $x \in H = (h=0)$ is C-M.
So a 3-fold sing. (X,x) which is reg. in codim 1 is C-M $\Leftrightarrow$ $\exists$ normal surface $x \in H \subseteq X$.

Prop. $f: X \longrightarrow Y$ a finite surjective morphism of varieties $/k = \bar{k}$, char $= 0$.
If X is C-M and Y is normal, then Y is also C-M.

(pf.) Since Y is normal, $0 \longrightarrow \mathcal{O}_Y \xrightarrow{f^\#} f_*\mathcal{O}_X$. Then a sequence from $\mathcal{O}_Y$ is $\mathcal{O}_Y$-regular if it is
$\exists \frac{1}{\deg f} \operatorname{Tr}_{X/Y} = \operatorname{Tr}$ s.t. $\operatorname{Tr} \circ f^\# = \operatorname{id}_{\mathcal{O}_Y}$

$f_*\mathcal{O}_X$-regular.

Thm. [KM, Thm 5.71] (Serre's duality for C-M sheaves)

X: proj. scheme of pure $\dim = n / k=\bar{k}$, char $k=0$. $\mathcal{F}$: a C-M sheaf on X s.t. $\operatorname{Supp}\mathcal{F}$ is of pure dimension n. Then $\exists$ a dualizing sheaf ω_X s.t.

still C-M (see [KM, Cor 5.70])

$$H^i(X,\mathcal{F}) \text{ is dual to } H^{n-i}(X, \text{"}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\omega_X)\text{"})$$

(sketch of pf.)

(idea) "coho. theory of C-M sheaves $\overset{\text{similar}}{\longleftrightarrow}$ coho. theory of locally free sheaves on a sm. proj. var. P" + "Serre duality on P".

- $\exists f: X \longrightarrow \mathbb{P}^n_k = P$, a finite morphism. Then
  - $H^i(X,\mathcal{F}) = H^i(P, f_*\mathcal{F})$.
  - $\mathcal{F}$ is C-M $\Leftrightarrow$ $f_*\mathcal{F}$ is locally free [KM, Cor 5.5]

(The existence of f from Noether normalization thm. Or more geometrically,

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] \arrow[ddr, bend right, "f"'] & \mathbb{P}^N_k \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\ & \mathbb{P}^N_k \setminus L \arrow[d] \arrow[u, dashed] \\ & P \end{tikzcd}$$

take a general linear space $L \subseteq \mathbb{P}^N_k$ of codim $= n+1$, i.e. $L \cap X = \emptyset$ and $P = \mathbb{P}^n_k$ s.t. $P \cap L = \emptyset$

Then f is proper + $\left\{\begin{matrix}\text{quasi-finite} \\ \text{or} \\ \text{affine}\end{matrix}\right.$ $\Rightarrow$ finite.)

- $\exists\, \omega_P$ by Hartshorne. Define $\omega_X = f^!\omega_P := \mathcal{H}om_{\mathcal{O}_P}(f_*\mathcal{O}_X, \omega_P)$ (see [Hart. III Ex 6.10, 7.2] or [KM, Prop 5.67, 5.68])

Then $H^i(P, f_*\mathcal{F})$ is dual to $H^{n-i}(P, \underline{\omega_P \otimes (f_*\mathcal{F})^\vee})$

$$\begin{matrix} H^i(P, f_*\mathcal{F}) & & \omega_P \otimes (f_*\mathcal{F})^\vee \\ \| & & \| \\ H^i(X,\mathcal{F}) & & \mathcal{H}om(f_*\mathcal{F},\omega_P) = \mathcal{H}om(\mathcal{F}, f^!\omega_P) = \mathcal{H}om(\mathcal{F},\omega_X) \end{matrix}$$

Cor. [KM Cor 5.72] (a coho chara. of C-M sheaves)

$X, \mathcal{F}$ as in Thm. D: an ample Cartier divisor on X. TFAE

(1) $\mathcal{F}$ is C-M.

(2) $H^i(X, \mathcal{F}(-rD)) = 0$ $\forall i<n$, $r \gg 0$.

(pf.)
- (1) $\Rightarrow$ (2): By Serre duality, $H^i(X,\mathcal{F}(-rD)) = H^{n-i}(X, \mathcal{H}om(\mathcal{F},\omega_X)(rD)) = 0$ as $r \gg 0$.
- (2) $\Rightarrow$ (1): By induction on $n = \dim X$. For $n=0$ is done. For $n>0$, take any $x \in X$. Since $h^0(\mathcal{F}(-rD)) = 0$, $\nexists$ subsheaf $\subseteq \mathcal{F}$ with $\operatorname{Supp} = \{x\}$. For $r' \gg 0$, $\exists s \in H^0(\mathcal{O}_X(r'D))$ s.t. "$s(x) = 0$ and s does not vanish at any associated point of $\mathcal{F}$" $\rightsquigarrow$ $\mathcal{F} \xhookrightarrow{\cdot s} \mathcal{F}(r'D)$ is injective.

($\mathcal{I}_x \otimes \mathcal{O}_X(r'D)$ gbgs as $r' \gg 0$ and D: ample to avoid ass. point
(cf. [Liu, chap 7. Def 1.6, Lemma 1.9] or Stack proj. Lemma. 31.2.10)

Set $Y=(s=0)\ni x \rightsquigarrow 0\to \mathcal{F}(-(r+r')D)\to \mathcal{F}(-rD)\to \mathcal{F}_Y(-rD)\to 0$

$\overset{\text{long-exact}}{\rightsquigarrow} H^i(Y,\mathcal{F}_Y(-rD))=0$ for $i<n-1$ and $r\gg 0$.

Then $\mathcal{F}_Y$ is C-M by induction, and thus $\mathcal{F}$ is C-M at x by Fact. □

§ Rational singularities

Def. X
$\downarrow f$: resol. of sing.
Y : a variety $/k$, char $k=0$

, we say $f: X\to Y$ is a rational resolution if (1) $f_*\mathcal{O}_X=\mathcal{O}_Y$ (i.e. Y normal)
(2) $R^if_*\mathcal{O}_X=0\ \ \forall i>0$.

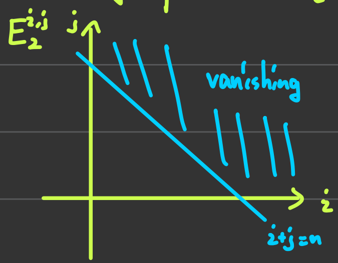
We say that Y has rational singularities if every resolution is rational.

Rmk. For char $k>0$, one need to assume also that $R^if_*\omega_X=0$ for $i>0$.
(This holds for char $k=0$ by Granert - Riemenschneider vanishing [KM, 2.6.8])

Thm. (Kempf) Y : var. $/k=\bar{k}$, char $k=0$. TFAE
(1) Y has rational singularities.
(2) $\exists$ a rational resolution of Y.
(3) Y is C-M and $\exists$ a resolution $f: X\to Y$ s.t. $\underset{\text{(relative trace map)}}{\underline{f_*\omega_X \xrightarrow{\sim} \omega_Y^\circ}}$.

Rmk 1. (Relative trace map) [KM, Prop 5.77]
X, Y : proj. sch. of pure dim $=n$ $/k=\bar{k}$, char $k=0$. $f: X\to Y$ generically finite morphism.
- $\forall \mathcal{F}$: coh. sheaf on $X \rightsquigarrow \dim \operatorname{Supp} R^jf_*\mathcal{F}\le n-j-1$ for $j>0$.

By Grothendieck vanishing, $H^i(Y,R^jf_*\mathcal{F})=0\ \ \forall i+j\ge n,\ ,j>0$. (✱)
- (Leray spectral seq.) $E_2^{i,j}=H^i(Y,R^jf_*\mathcal{F})\Rightarrow H^{i+j}=H^{i+j}(X,\mathcal{F})$. [Ishii, §3.6]



($\exists$ $H^n=F^0H^n\supseteq\cdots\supseteq F^nH^n\supseteq F^{n+1}H^n=0$

$F^iH^n/F^{i+1}H^n=\operatorname{Gr}_F^iH^n\simeq E_\infty^{i,n-i}=E_r^{i,n-i}$ as $r\gg 0$

at E_r page

$E_{r+1}^{i,j}=H\big(\cdot(i-r,\,j+(r-1))\longrightarrow\cdot(i,j)\longrightarrow\cdot(i+r,\,j-(r-1))\big)$)

By (✱), $F^iH^n/F^{i+1}H^n=\operatorname{Gr}_F^iH^n\simeq E_\infty^{i,n-i}=0$ for $i=0,\cdots,n-1\Rightarrow H^n=F^0H^n=\cdots=F^nH^n\supseteq F^{n+1}H^n=\{0\}$.

$\Rightarrow \underset{\substack{\parallel\\ H^n(Y,f_*\mathcal{F})}}{E_2^{n,0}}\longrightarrow\!\!\!\!\!\rightarrow E_\infty^{n,0}\simeq \operatorname{Gr}_F^nH^n=H^n=H^n(X,\mathcal{F})$.

· Apply this to $\mathcal{F} = \omega_X$ and by duality,

$$\begin{array}{ccc} H^0(X, \mathcal{H}om(\omega_X, \omega_X)) & \hookrightarrow & H^0(Y, \mathcal{H}om(f_*\omega_X, \omega_X)) \\ \| & & \\ \mathrm{Hom}(\omega_X, \omega_X) & & \\ \Psi & & \\ \mathrm{id}_{\omega_X} & \longmapsto & tr_{X/Y} : f_*\omega_X \longrightarrow \omega_X \end{array}$$ (relative trace map)

2. $R^p f_* \mathcal{O}_X$ and $f_*\omega_X$ are independent of resolution $f: X \to Y$ [Ishii, Prop 6.2.13]

$$\begin{tikzcd} & X'' \arrow[dl, "g"'] \arrow[dr, "g'"] & \\ X \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$ X'' : common resolution

$\because X, X', X''$ are sm. $\Rightarrow \begin{cases} g'_*\omega_{X''} = \omega_{X'} \\ g_*\omega_{X''} = \omega_X \end{cases}$

$\therefore f_*\omega_X = f_* g_* \omega_{X''} = f'_* g'_* \omega_{X''} = f'_*\omega_{X'}$.

Proof of Kempf's thm.

We prove only the case when Y is proj.. By Rmk 2, $(1) \Leftrightarrow (2)$

Let $f: X \to Y$ be a resolution and D an ample Cartier divisor on Y. By K-V vanishing,

$$H^i(X, \omega_X(rf^*D)) = 0 \quad \text{for } i > 0,\ r > 0.$$
$$\wr\|$$
$$H^i(X, \mathcal{O}_X(-rf^*D))^\vee$$

Consider Leray spectral seq., $E_2^{i,j} = H^i(Y, \underline{R^j f_* \mathcal{O}_X \otimes \mathcal{O}_Y(-rD)}) \Longrightarrow H^{i+j} = H^{i+j}(X, \mathcal{O}_X(-rf^*D))$.

$$\wr\| \text{ proj. formula}$$
$$R^j f_*(\mathcal{O}_X(-rf^*D))$$ (i.e. degenerate at $E_2^{0,0}$ (*))

· $(2) \Rightarrow (3)$: f : rational resolution $\Rightarrow R^j f_* \mathcal{O}_X = 0\ \forall j > 0 \Rightarrow E_2^{i,j} = 0$ for $j > 0$. This implies

$$E_2^{i,0} = H^i \quad ([\text{Ishii, Prop 3.6.3}])$$
$$H^i(Y, \mathcal{O}_Y(-rD)) \quad H^i(X, \mathcal{O}_X(-rf^*D)) = 0 \quad \text{for } i < n,\ r > 0.$$

By Cor, Y is C-M. For $i = n$,

$$\begin{array}{ccl} E_2^{n,0} = H^n(Y, \mathcal{O}_Y(-rD)) & \xrightarrow[\sim]{\mathcal{O}_Y : \text{C-M by duality}} & H^0(Y, \omega_Y \otimes \mathcal{O}_Y(rD))^\vee \\ \wr\| & & \\ H^n = H^n(X, \mathcal{O}_X(-rf^*D)) & \xrightarrow{\sim} & H^0(X, \omega_X \otimes \mathcal{O}_X(rf^*D))^\vee \\ & & \| \\ & & H^0(Y, \underline{f_*(\omega_X \otimes \mathcal{O}_X(rf^*D))})^\vee \\ & & \qquad \| \\ & & \qquad f_*\omega_X \otimes \mathcal{O}_Y(rD) \end{array}$$

Take r s.t. $\omega_Y \otimes \mathcal{O}_Y(rD)$ and $f_*\omega_X \otimes \mathcal{O}_Y(rD)$ are g.b.g.s $\Rightarrow tr_{X/Y} : f_*\omega_X \xrightarrow{\sim} \omega_Y$.

· $(3) \Rightarrow (2)$: By induction on $n = \dim Y$. Note that $R^n f_* \mathcal{O}_X = 0$.

Claim. $R^i f_* \mathcal{O}_X$ outside a zero dim'l set $\forall i > 0$.

$$\text{(subpf.)}\quad \begin{array}{ccc} H' = f^*H & \subseteq & X \\ {\scriptstyle \text{a resol. of } H}\downarrow & & \downarrow f \\ H & \subseteq & Y \end{array}$$

general hyperplane (CM-scheme)

, $f_*\omega_{H'} = f_*(\omega_X(H')\otimes\mathcal{O}_{H'}) = \underset{}{\overset{\omega_Y}{\overset{\shortparallel}{\underline{f_*\omega_X}}}} \otimes \mathcal{O}_H(H) = \omega_H$.

By induction, $\mathcal{O}_H \otimes R^i f_*\mathcal{O}_X = R^i f_*\mathcal{O}_{H'} = 0 \ \forall i>0$. Then

$$R^i f_*\mathcal{O}_Y \otimes \mathcal{O}_X(-H) \longrightarrow R^i f_*\mathcal{O}_Y \longrightarrow R^i f_*\mathcal{O}_{H'} = 0$$

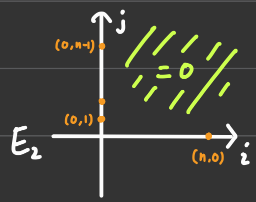
and by Nakayama lemma, $\operatorname{Supp} R^i f_*\mathcal{O}_Y \cap H = \emptyset$. □

By Claim, $E_2^{i,j} = H^i(Y, R^j f_*\mathcal{O}_X \otimes \mathcal{O}_Y(-rD)) = 0$ for $i,j>0$,

and $E_2^{i,0} = H^i(Y, \underset{\overset{\shortparallel}{\mathcal{O}_Y}\ (*)}{\underline{f_*\mathcal{O}_X}} \otimes \mathcal{O}_Y(-rD)) = 0$ $\forall i<n$ by Cor. ($\because$ Y is CM)

By Leray spectral sequence,

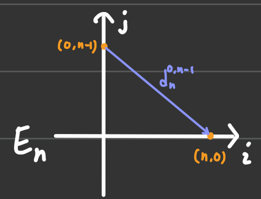
$$\begin{array}{ccc} E_2^{0,j} & \xrightarrow{\ \sim\ } & H^j \quad \text{for } j<n-1 \\ \shortparallel & & \shortparallel \\ H^0(Y, R^j f_*\mathcal{O}_X\otimes\mathcal{O}_Y(-rD)) & & H^j(X, \mathcal{O}_X(-rf^*D)) = 0 \text{ by K-V} \end{array}$$



$\because$ $\dim \operatorname{Supp} R^i f_*\mathcal{O}_X = 0 \xrightarrow{H^0=0} R^i f_*\mathcal{O}_X = 0$ for $0<i<n-1$.

On page n,

$$\begin{array}{ccccccc} & & & & \operatorname{coker} d_n^{0,n-1} & & \\ & & & & \shortparallel & & \\ E_n^{0,n-1} & \xrightarrow{d_n^{0,n-1}} & E_n^{n,0} & \overset{\alpha}{\twoheadrightarrow} & E_{n+1}^{n,0} & \longrightarrow & 0 \\ \shortparallel & & \shortparallel & & \shortparallel & & \\ E_2^{0,n-1} & & E_2^{n,0} & & E_\infty^{n,0} \simeq Gr_F^n H^n = H^n & & \\ \shortparallel & & \shortparallel & & \shortparallel & & \\ H^0(Y, R^{n-1}f_*\mathcal{O}_X\otimes\mathcal{O}_Y(-rD)) & & H^n(Y, \underset{}{\overset{\mathcal{O}_Y}{\underline{f_*\mathcal{O}_X}}}\otimes\mathcal{O}_Y(-rD)) & & H^n(X, \mathcal{O}_X(-rf^*D)) & & \\ \downarrow & & \wr\downarrow & & \wr\downarrow \text{ duality} & & \\ & & H^0(Y, \omega_Y(rD))^\vee & \xrightarrow{\ \sim\ } & H^0(X, \omega_X(rf^*D))^\vee & & \\ & & & & \shortparallel & & \\ H^0(Y, \underline{R^{n-1}f_*\mathcal{O}_X\otimes\mathcal{O}_Y(-rD)}) = 0 & & & & H^0(Y, \underline{f_*(\omega_X(rf^*D))})^\vee & & \\ \Downarrow & & & & \shortparallel & & \\ R^{n-1}f_*\mathcal{O}_X = 0. & & & & \underline{f_*\omega_X}\otimes\mathcal{O}_Y(rD) & & \\ & & & & \simeq \omega_Y & & \end{array}$$



support on pts

So we remain to prove $(*)$ $f_*\mathcal{O}_X = \mathcal{O}_Y$: Say $0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to Q \to 0$.

Pick $r \gg 0$ s.t. $H^1(\mathcal{O}_Y(rD)) = 0$ and $Q(rD)$: gbgs, then

$$0 \to H^0(\mathcal{O}_Y(rD)) \longrightarrow \underset{\overset{\shortparallel}{H^0(X, \mathcal{O}_X(rf^*D))}}{H^0(Y, f_*\mathcal{O}_X\otimes\mathcal{O}_Y(rD))} \longrightarrow H^0(Q(rD)) \longrightarrow 0$$

$$E_2^{p,q} = H^p(Y, R^q f_*\omega_X \otimes \mathcal{O}_Y(-rD)) \Rightarrow H^{p+q}(X, \omega_X(-rf^*D)) = H^{p+q}.$$

degenerate at E_2 since $R^i f_*\omega_X = 0$ $\forall i>0$ by G-R vanishing.

Hence $E_2^{n,0} \simeq H^n = H^n(X, \omega_X(-rf^*D)) \simeq H^0(X, \mathcal{O}_X(rf^*D))^\vee \Rightarrow H^0(Q(rD)) = 0$

$$\begin{array}{c} \shortparallel \\ H^n(Y, f_*\omega_X\otimes\mathcal{O}_Y(-rD)) \\ \wr\downarrow \\ H^0(Y, \mathcal{O}_Y(-rD))^\vee \end{array} \qquad \sim \qquad \Rightarrow Q(rD) = 0 \Rightarrow Q = 0$$

by $Q(rD)$: gbgs.

Prop. X: variety with rational singularity
$\downarrow f$: finite surj. $\Rightarrow Y$ has rational singularity.
Y: normal variety $/ k=\bar{k}$, char $=0$

(pf.) By [KM Prop 5.7], Y is Cohen Macauley.

$$\begin{tikzcd} X' \arrow[r, "g^X"] \arrow[d, "f'"', "\text{normalization in } k(X')"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g^Y : \text{a resol. of } Y"'] & Y \end{tikzcd}$$

s.t. $\begin{cases} f' : \text{finite} \\ X' : \text{normal} \\ g^X : \text{birational} \end{cases}$

Claim. $g^X_* \omega_{X'} = \omega_X$.

(subpf.) Consider $h: X'' \to X'$ a resolution, then $(g^X \circ h)_* \omega_{X''} = \omega_X$ (X has rational sing.).
Since h is birational, $tr_{X''/X'} : h_* \omega_{X''} \hookrightarrow \omega_{X'}$. Then

$$\omega_X \simeq g^X_* h_* \omega_{X''} \xhookrightarrow{g^X_*(tr_{X''/X'})} g^X_* \omega_{X'} \xhookrightarrow{tr_{X'/X}} \omega_X \Rightarrow g^X_* \omega_{X'} \xrightarrow{\sim} \omega_X .$$ $\square$

By Kempf's thm., it suffices to prove $g^Y_* \omega_{Y'} = \omega_Y$.

$$\begin{tikzcd} f_* g^X_* \omega_{X'} \arrow[r, "\text{by Claim}", "\simeq"'] \arrow[d, equal] & f_* \omega_X \arrow[r, hook, "tr_{X/Y}"] & \omega_Y \arrow[d, equal] \\ g^Y_* f'_* \omega_{X'} \arrow[r, "g^Y_* tr_{X'/Y'}"] & g^Y_* \omega_{Y'} \arrow[r, hook, "tr_{Y'/Y}"] & \omega_Y \end{tikzcd}$$

Since f is finite and char $=0$, $\mathcal{O}_Y$ (resp. ω_Y) is direct summand of $f_* \mathcal{O}_X$ (resp. $f_* \omega_X$).
Hence $tr_{X/Y}$ is surj. $\rightsquigarrow tr_{Y'/Y}$ is surj. $\rightsquigarrow g^Y_* \omega_{Y'} \simeq \omega_Y$.

Def. (Pullback and norm of Weil divisors via ramified covers)
$k=\bar{k}$, char $k=0$. $f: X \longrightarrow Y$ finite surj. between normal varieties. By generic flatness, $\exists Y_0 \overset{j}{\hookrightarrow} Y$ of codim ≥ 2 s.t. $f^\circ : X^\circ = X \times_Y Y^\circ \longrightarrow Y^\circ$ is finite flat.
(Rmk. $X : S_2 \Rightarrow j_* f^\circ_* \mathcal{O}_{X^\circ} = f_* \mathcal{O}_X \rightsquigarrow f$ is uniquely determined by f°)

- $$\begin{tikzcd} \mathrm{WDiv}(Y)/\sim \arrow[d, "\wr"] & & \mathrm{WDiv}(X)/\sim \arrow[d, "\wr"] \\ \mathrm{WDiv}(Y^\circ)/\sim \arrow[rr, "(f^\circ)^*", "\text{finite flat pull-back}"'] & & \mathrm{WDiv}(X^\circ)/\sim \\ D \arrow[rr, mapsto] & & f^{-1}(D) \end{tikzcd}$$
$\Rightarrow \mathrm{Cl}(Y) \longrightarrow \mathrm{Cl}(X)$
$D \longmapsto \overline{(f^\circ)^{-1}(D \cap Y_0)}$

- $\exists \mathrm{Nm}_f : f_* \mathcal{O}_X \longrightarrow \mathcal{O}_Y$ ([KM, Def 5.6])
For $D \in \mathrm{CDiv}(X)$, choose $Y = \cup Y_i$, open covering s.t. $D \cap f^{-1}(Y_i) = (\varepsilon_i = 0)$. Then $\mathrm{Nm}_f(D) = f_* D$ is defined by local equation $\mathrm{Nm}_f(\varepsilon_i) = 0$ on Y_i.
In general, for $D \in \mathrm{WDiv}(X)$, $\exists Z \underset{\text{closed}}{\subseteq} Y$ of codim ≥ 2 s.t.
$D|_{X \setminus f^{-1}Z}$ is Cartier, $f_* D := (f|_{X \setminus f^{-1}Z})_* (D_{X/Z}) \in \mathrm{CDiv}(X \setminus Z) \subseteq \mathrm{WDiv}(Y \setminus Z) \simeq \mathrm{WDiv}(Y)$.

Lemma. [KM, 5.16] $f: X \longrightarrow Y$ finite surj. morphism of normal varities $/ k=\bar{k}$, char $k=0$.

If X is $\mathbb{Q}$-factorial, then Y is $\mathbb{Q}$-factorial.

(pf.) By construction, $f_* f^* B = \deg f \cdot B \quad \forall B \in \mathrm{WDiv}(Y)$. Also, $f_*(\mathrm{CDiv}) \subset \mathrm{CDiv}(X)$ and f^* on Cartier is the usual pull-back. Hence the pull-back & norm take $\mathbb{Q}$-Cartier (resp. $\mathbb{Z}$ or $\mathbb{Q}$-divisors) to $\mathbb{Q}$-Cartier (resp. $\mathbb{Z}$ or $\mathbb{Q}$-divisors). For $B \in \mathrm{WDiv}(Y)$, $f^*B \in \mathrm{WDiv}(X)_{\mathbb{Q}} \subseteq \mathrm{CDiv}(X)_{\mathbb{Q}} \Rightarrow \deg f \cdot B = f_* f^* B \in \mathrm{CDiv}(Y)_{\mathbb{Q}}$. $\square$

Def. (Quotient singularity)

- $(x \in X)$: a germ of a complex analytic space (cf. [Ishii, §4.2]).

  It is a quotient singularity if $\exists$ a smooth germ $(0 \in Y)$ and a finite group G act on $(0 \in Y)$ s.t. $(x \in X) \simeq (0 \in Y)/G$.
- X : variety $/ \mathbb{C}$.

  We say that X has quotient singularity if X^{an} has (cf. Hartshorne or Redbook).

Prop. X : an algebraic or analytic variety $/ \mathbb{C}$ with only quotient singularity.

Then X has rational singularity and is $\mathbb{Q}$-factorial.

(pf.) GAGA principle : $f^{an}: X'^{an} \longrightarrow X^{an}$ is proper $\Leftrightarrow f: X' \longrightarrow X$ is proper [KM, Thm 2.4.3]

$$(R^i f_* \mathcal{O}_{X'})^{an} = R^i (f^{an})_* \mathcal{O}_{X'}^{an}$$

For a closed point $x \in X$, $\mathcal{O}_{X,x}$: alg. local ring

$\mathcal{O}_{X,x}^{an}$: an. local ring

$\hat{\mathcal{O}}_{X,x} = \hat{\mathcal{O}}_{X,x}^{an}$: completion

If $I \triangleleft \mathcal{O}_{X,x}$, the I is principle $\Leftrightarrow I\mathcal{O}_{X,x}^{an}$ is principle $\Leftrightarrow I\hat{\mathcal{O}}_{X,x}$ is principle [Matsumra 2.4E]

Thus it is sufficient to prove that analytic quotient singularities is rational and $\mathbb{Q}$-factorial.

$$(0 \in Y) \xrightarrow[\text{finite surj.}]{} (0 \in Y)/G$$

smooth $\Rightarrow$ rational + $\mathbb{Q}$-factorial, so is $(0 \in Y)/G$.

Prop. (Reid) [KM Prop 5.20]

$g: X' \longrightarrow X$ a finite morphism of normal varieties $/ k=\bar{k}$, char $=0$.

Let $\Delta \in \mathrm{WDiv}(X)_{\mathbb{Q}}$, $\Delta' \in \mathrm{WDiv}(X')_{\mathbb{Q}}$ s.t. $K_{X'} + \Delta' = g^*(K_X + \Delta)$ in $\mathrm{WDiv}(X')_{\mathbb{Q}}$. Then

(a) $K_X + \Delta$ is $\mathbb{Q}$-Cartier $\Leftrightarrow K_{X'} + \Delta'$ is $\mathbb{Q}$-Cartier.

(b) $\mathrm{discrep}(X', \Delta') + 1 \geq \mathrm{discrep}(X, \Delta) + 1 \geq \frac{1}{\deg g}(\mathrm{discrep}(X', \Delta') + 1)$.

(c) (X, Δ) is (sub) klt / plt / lc $\Leftrightarrow (X', \Delta')$ is (sub) klt / plt / lc (pf.) (a) It follows from $g_* g^* B = \deg g \cdot B \quad \forall B \in WDiv(X)$.

(c) $\Leftarrow$ (b)

(b) Let E be a exceptional divisor over X.

$$\begin{tikzcd} \overline{\{\eta\}} = F \subset Y' \arrow[r] \arrow[d] \arrow[d, "h"] & X' \arrow[d, "g"] \\ E \subset Y \arrow[r] & X \end{tikzcd}$$

, where Y' is normalization of Y in $K(X')$

$\rightsquigarrow h:$

§ Canonical and terminal singularities

Lemma. [KM, Lemma 5.17]

$(X, \Delta = \sum a_i D_i)$: $\begin{cases} \text{a sub-pair.} \left(\begin{array}{l} \text{i.e.} \\ X: \text{normal}, \ \Delta: \text{sub-boundary} \\ K_X + \Delta : \mathbb{Q}\text{-Cartier} \end{array} \right) \\ \dim X \geq 2 \text{ and satisfies Serre's condition } S_3. \end{cases}$

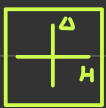
H : a general hyperplane section of X (quasi proj.). Then

(0) H is normal

(1) $\mathrm{discrep}(X, \Delta) \leq \mathrm{discrep}(H, \Delta|_H)$

(2) $\mathrm{discrep}(X, \Delta + H) = \min\{0, \mathrm{discrep}(X,\Delta), \min_{D_i \cap H \neq \emptyset}\{-a_i\}\}$.

Rmk. (Fujino) $X = \mathbb{P}^2$, $\Delta =$ a line, $H \in |\mathcal{O}_{\mathbb{P}^2}(1)|$.



$-1 \overset{\text{SNC}}{=} \text{LHS of (2)} \neq \min\{0, \underbrace{\mathrm{discrep}(X,\Delta)}_{1-1}\} = 0$

(pf.)

- H is normal:

  Since X is S_3, we have H is S_2.

  Since X is $R_1 \Leftrightarrow \mathrm{codim}_X \mathrm{Sing} X \geq 2$, we may take general H not contain any irr. comp. of $\mathrm{Sing} X \Rightarrow \mathrm{codim}_H H \cap \mathrm{Sing} X \geq 2$ WLOG H is smooth outside by Bertini's thm..

  Hence H has $S_2 + R_1 =$ normal by Serre's criterion.

- By Cor. [KM, Cor. 2.32 (2)], $\exists$ log resolution $f: X' \longrightarrow X$ of (X, Δ) s.t. $\sum f_*^{-1} D_i$ is sm..

  Set $H' := f^{-1}(H)$. As H is general, WLOG $f(\mathrm{Exc} f) \cap H = \emptyset \rightsquigarrow f_*^{-1} H = f^* H = H'$.

  f (resp. $f|_{H'}$) is also log resolution of $(X, \Delta + H)$ (resp. $(H, \Delta|_H)$).

  (1) Write $K_{X'} + \Delta' = f^*(K_X + \Delta)$, $f_* K_{X'} = K_X$, $f_* \Delta' = \Delta$.

  $\Rightarrow K_{X'} + \Delta' + H' \overset{\circledast}{=} f^*(K_X + \Delta + H) \overset{\cdot|_H}{\Rightarrow} K_{H'} + \Delta'|_H = f^*(K_H + \Delta|_H)$

  For E : f-excep. with $E \cap H' \neq \emptyset$,

  $a(E|_H, H, \Delta_H) = -\mathrm{coeff}_{E|_{H'}}(\Delta'|_{H'}) = -\mathrm{coeff}_E(\Delta') = a(E, X, \Delta) \rightsquigarrow$ We get (1).

  (2) By Lemma [KM, Lemma 2.30] and ($\circledast$),

  $$\mathrm{discrep}(X, \Delta + H) = \min_{E: f\text{-excep.}}\{\mathrm{discrep}(X', \Delta' + H'), a(E, X, \Delta + H)\}$$

  $$= \min\left\{0, \min_{E: f\text{-excep.}}\{a(E, X, \Delta + H), \min\{1 - a_i\}, 1\}, \min_{D_i \cap H \neq \emptyset}\{-a_i\}\right\}$$

Thm. [Elkik '81, Flenner] (X, Δ) : dlt pair $\Rightarrow$ X has rational sing.
(only proper)

[KM, Cor 5.18]

Cor. [KM. Cor 5.18] X : quasi-proj. var. of $\dim = n \geq 2$.

(1) X : terminal $\Rightarrow$ X is sm. in codim. 2, i.e. $\operatorname{codim}_X \operatorname{Sing} X \geq 3$.

(2) X : canonical $\Rightarrow$ K_X is Cartier in codim 2, i.e. $\operatorname{codim}_X \underbrace{\operatorname{Sing} K_X}_{} \geq 3$.

$\operatorname{Sing} K_X = \{x \in X,\ K_X \text{ is not free over } \mathcal{O}_{X,x}\}$

(pf.)

Set $Z := \operatorname{Sing} X$ or $\operatorname{Sing} K_X$, which is a closed subset of X. Take general hyperplane sections:

$$X \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-2},\quad \dim H_i = n-i.$$

Since X is $\begin{cases} \text{terminal} \\ \text{canonical} \end{cases} \xRightarrow[\text{sing.}]{\text{rational}} X : C\text{-}M \xRightarrow[S_2+R_1]{} H_i : \text{normal} \Rightarrow H_i : \begin{cases} \text{terminal} \\ \text{canonical} \end{cases}$

Since H is $\begin{cases} \text{terminal surface} \\ \underline{\text{canonical surface}} \end{cases} \Rightarrow \begin{cases} H_{n-2} \text{ is smooth} & \Rightarrow H_{n-2} \cap Z = \emptyset \rightsquigarrow \operatorname{codim}_X Z \geq 3. \\ K_{H_{n-2}} : \text{Cartier} \end{cases}$

$=$ Du Val $=$ ADE sing.

Def. [KM Def. 5.17] X : normal var., D : $\mathbb{Q}$-Cartier divisor

- r = the index of $D := \min\{e \in \mathbb{N} \mid eD \in \operatorname{CDiv}(X)\}$, $r = 1 \Leftrightarrow D \in \operatorname{CDiv}(X)$.
- index of $\underset{(X,\Delta)}{X}$ = index of $\underset{(K_X+\Delta)}{K_X}$

Prop. (index 1 cover) X: normal variety $/k = \bar{k}$, char $k = 0$. $D \in \operatorname{WDiv}(X)$, $\mathbb{Q}$-Cartier with index r. If $\mathcal{O}_X(rD) \simeq \mathcal{O}_X$,

then $\exists$
$$\begin{array}{c} Z \\ \downarrow p : \text{finite surj. morphism with } \operatorname{Gal}(k(Z)/k(X)) \simeq \mathbb{Z}/r\mathbb{Z} \\ X \end{array}$$

s.t. $\cdot$ p is étale in codim 1.

$\cdot$ $p^*D \in \operatorname{CDiv}(Z)$ linear equiv. to 0.

(pf.)

Fix a nowhere zero section $H^0(X, \mathcal{O}_X(rD)) \simeq \{h \in k(X) ; \operatorname{div}_X(h) + rD \geq 0\}$

$$1 \longmapsto h$$

Take $Z :=$ the normalization of X in $k(X)[h^{1/r}]$ (is a field $\because$ r is the index)

$\Rightarrow -p^*D = \operatorname{div}_Y(h^{1/r}) \sim 0$ is Cartier (by $rD \sim 0$)

p is étale over $X \setminus \operatorname{Sing}(\underset{\substack{\uparrow \\ \text{divisorial sheaf}}}{\mathcal{O}_X(D)}) \Rightarrow p$ is étale over $X \setminus \operatorname{Sing} X$

(If we have more smoothness, we can improve the statement
eg. X : canonical & $D = K_X \rightsquigarrow \operatorname{codim}_X \operatorname{Sing}_X K_X \geq 3$)

Cor. [KM Cor 5.20] $(x\in X)$ a germ of a normal sing.

$$(x\in X)\begin{cases}\text{klt} \\ \text{canonical} \\ \text{terminal}\end{cases} \overset{\Leftarrow \text{ by [KM, Prop. 5.20 (4)]}}{\Rightarrow} \text{it is a cyclic quotient of an index 1 } \begin{cases}\text{canonical} \\ \text{canonical} \\ \text{terminal}\end{cases} \text{singularity } (0\in Y) \text{ by}$$

$$\text{an action which is fixed point free in codim } \begin{cases}1 \\ 2 \\ 3\end{cases}$$

(pf.)

$(x\in X)$: klt $\Rightarrow K_X$: $\mathbb{Q}$-Cartier

$$\begin{array}{l} Y \\ \downarrow p: \text{the index 1 cover of } X\ (K_X) \\ X \end{array} \Rightarrow p \text{ is étale over } X\setminus \mathrm{Sing}(K_X),\ \mathrm{codim}_x \mathrm{Sing}\, K_X \geq \begin{cases} 2 & \text{klt.} \\ 3 \text{ for } (x\in X) \text{ being} & \text{can.} \\ 3 & \text{term.}\end{cases}$$

By [KM, Prop 5.20], $\underset{\overset{\Uparrow}{\mathrm{II}}\ (\because K_Y:\text{ Cartier})}{\mathrm{discrep}(Y)\geq \mathrm{discrep}(X)} \begin{cases} >-1 & \text{klt} \\ \geq 0 & \text{can.} \\ >0 & \text{term.}\end{cases} \Rightarrow \mathrm{discrep}(Y)\begin{cases}\geq 0 \text{ if } (x,X) \text{ is} & \text{klt} \\ \geq 0 & \text{can.} \\ >0 & \text{term}\end{cases}$

Prop. [KM, Cor 5.24] X : normal var.

X has at worst canonical singularities of index 1 $\Leftrightarrow$ X has rational Gorenstein singularities.

(pf.) Recall. Gorenstein = C-M + index 1

$(\Rightarrow)$ can. sing. $\overset{[\text{Elkik}]}{\Longrightarrow}$ rational sing. $\Rightarrow$ C-M.

$(\Leftarrow)$ $\forall$ resol. $f: Y\to X$, write $\underset{\omega_Y}{\underset{\mathbin{\rotatebox{90}{$\supseteq$}}}{\mathcal{O}_Y(K_Y)}} \simeq \underset{f^*\omega_X\otimes\mathcal{O}_Y(E)}{\underset{\mathbin{\rotatebox{90}{$\supseteq$}}}{\mathcal{O}_Y(f^*K_X)\otimes\mathcal{O}_Y(E)}}$, where E is f-excep. divisor.

$$\Rightarrow f_*\omega_Y = \underset{\omega_X}{\underset{\mathbin{\rotatebox{90}{$\supseteq$}}}{\omega_X}}\otimes f_*\mathcal{O}_Y(E) \Rightarrow f_*\mathcal{O}_Y(E)\simeq \mathcal{O}_X\ (\Leftrightarrow E \text{ is eff.}) \rightsquigarrow \mathrm{discrep}\, X\geq 0.$$

Def. (Reid) X : sch. of finite type $/k$. $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$

A finite dim'l k-vector subspace $V\subset \mathfrak{m}_x$ surj. $\mathfrak{m}_x/\mathfrak{m}_x^2$ (i.e. V generate $\mathfrak{m}_x$ by NAK.)

"a general hyperplane section H though x" = a subscheme $H\subset X_0$ ← suitable nbd. of x in X

$\mathcal{I}_H = \mathcal{O}_{X_0} h$ $(h=0)$

($\exists$ Zarski dense open subset $U\subseteq V$ s.t. $h\in U$)

Prop. (Reid) [KM, Lem. 5.30] $(o\in X)$: an index 1 canonical sing. with $\dim_{\mathbb{C}} X\geq 3$
$o\in H\subseteq X$: a general hyperplane through x, $g: H'\to H$ any resol.
$\Rightarrow \mathfrak{m}_{H,o}\omega_H \hookrightarrow g_*\omega_{H'} \xhookrightarrow{tr_{H'/H}} \omega_H$. $\quad (\omega_H = g_*\omega_{H'})$
In particular, the resol. g is either rational or $\mathfrak{m}_{H,o}\omega_H = g_*\omega_{H'}$

Rmk. If $\dim X\geq 3$, $(o\in H)$ is Du Val surface sing. $\quad 0$
or an elliptic Gorenstein sur. sing. $\quad 1$
minimal elliptic $\qquad \dim_{\mathbb{C}} \omega_H/g_*\omega_{H'} = \dim_{\mathbb{C}} R^1g_*\mathcal{O}_{H'}$

(pf.)
- Recall that
  - H is normal (X: CM $\Rightarrow$ H: CM $\Rightarrow$ S_2. Also H is R_1)
  - $g_*\omega_{H'} \xhookrightarrow{tr_{H'/H}} \omega_H$ (since g is birational), whose image is indep. of the choice of a resol. H'.

Thus we allowed to make a convenient choice of H'.

$$\begin{tikzcd}
H' := f_*^{-1}H \arrow[d, "f'=f|_{H'}"'] & \subseteq & Y \arrow[d] \arrow[dd, bend right, "f"'] \\
 & & Bl_oX = \mathrm{Proj}\left(\bigoplus_{i\geq 0}\mathfrak{m}_{x,o}^i\right) \arrow[d] \\
H & \subseteq & X
\end{tikzcd}$$

$\Rightarrow$
- $\mathfrak{m}_{x,o}\mathcal{O}_Y = \mathcal{O}_Y(-E)$ is invertibe, where E is an effective Cartier divisor on Y.
- $f^*H = H'+E$
- H' is smooth since $|H|$ is bpf. by Bertini.

Since $(o\in X)$ is an index 1 canonical sing.,
$\Rightarrow \omega_Y = f^*\omega_X\otimes\mathcal{O}_Y(F)$, where F is effective and exceptional
$\Rightarrow \omega_{H'} = \omega_Y(H')|_{H'} = \left(f^*(\omega_X(H))\otimes\mathcal{O}_Y(F-E)\right)|_{H'} = f'^*\omega_H\otimes\mathcal{O}_{H'}(F-E)$.
$\Rightarrow f'_*\omega_{H'} = \omega_H\otimes f'_*\mathcal{O}_{H'}(\underset{\text{eff.}}{F}-E) \geq \omega_H\otimes(\underbrace{f_*\mathcal{O}_{H'}(-E)|_H}_{=\ \mathfrak{m}_{x,o}})$

$\xrightarrow{tr_{H'/H}}\ \mathfrak{m}_{x,o}\cdot\omega_H \subseteq \omega_H$

Since $\dim_{\mathbb{C}} \omega_H/\mathfrak{m}_{x,o}\omega_H = 1$, either $f'_*\omega_{H'} = \mathfrak{m}_{x,o}\cdot\omega_H$ or ω_H holds.

Def. $(o\in X)$: 3-dim'l normal singularity. It is called a compound Du-Val (or cDV) singularity if a general hyperplane section $o\in H\subseteq X$ is a Du Val singularity.

Rmk. a cDV sing. $(o\in X) \overset{\text{analytically}}{\simeq} (f(x,y,z)+tg(x,y,z)=0)\subseteq (0,\overset{(x,y,z,t)}{\mathbb{C}^4})$

$\begin{cases} cA_n \\ cD_n \\ cE_n \end{cases} \qquad \begin{cases} f: A_n \text{ type surface sing.} \\ f: D_n \\ f: E_n \end{cases}$

More pricisely,

(cA_1) : $x^2+y^2+z^2+t^m=0$ for some $m\geq 2$

$(cA_{\geq 2})$: $x^2+y^2+g(z,t)=0$ with $\mathrm{mult}_0\, g\geq 3$

(cD_4) : $x^2+g(y,z,t)=0$ with $\mathrm{mult}_0\, g=3$ and with the cubic part of g having NO multiple factor.

$(cD_{\geq 5})$: $x^2+y^2z+g(y,z,t)=0$ with $\mathrm{mult}_0\, g\geq 4$

(cE) : $x^2+y^3+y\,g(z,t)+h(z,t)=0$ with $\mathrm{mult}_0\, g\geq 3$, $\mathrm{mult}_0\, h\geq 4$.

If a cD sing. is isolated, then it is $x^2+y^2z+\lambda yz^k+g(z,t)=0$.

§ Terminal Singularities of index one / $\mathbb{C}$

Thm. (Reid) [Rei 83b, Thm 1.1] [KM, Cor 5.38] $(0 \in X)$: a 3-dim'l normal singularity.
Then it is terminal of index $1 \Leftrightarrow$ it is an is isolated cDV singularity.
"hypersurface"

Cor. [Kaw 88, Lemma 5.1]
Every $\mathbb{Q}$-Cartier integral divisor D on a terminal 3-fold of index 1 is Cartier.

(pf.) (Due to Miles Reid and Masaski Ue)
$x \in \operatorname{Sing} X \overset{\text{Thm.}}{\Rightarrow} (x \in X)$ is an isolated hypersurface singularity of dim $= 3$. Let r be the index of D at $(x \in X) \Rightarrow \exists x \in U \subseteq X$ s.t. $\mathcal{O}_U(rD) \simeq \mathcal{O}_U$. Take index 1 cover with $\mathrm{Gal}(\mathbb{C}(V)/\mathbb{C}(U)) \simeq \mathbb{Z}/r\mathbb{Z}$,

$$\begin{array}{ccl} V \supset V \setminus p^{-1}(x) & & \\ \downarrow \quad \downarrow & \text{étale covering of deg} = r & \\ U \supset U \setminus \{x\} & & \end{array}$$

$\Rightarrow \mathbb{Z}/r\mathbb{Z} \simeq \pi_1(U \setminus \{x\}) = \{1\}$ by [Milnor 68, Singular points of complex hypersurfaces Thm 5.2]
$\Rightarrow r = 1$

by Hamm's Thm (?)

Rmk. The above proof works for "isolated $\underset{?}{\text{(hyper.)}}$ sing. of dim ≥ 3"

Prop. $(0 \in X)$: a normal singularity of dim ≥ 3. Assume that $\exists$ a hyperplane section $0 \in H \subseteq X$ s.t. $(0 \in H)$ is rational Gorenstein. Then

(1) [Rei 80] [Elle 78] [KM Thm 5.42] $(0 \in X)$ is also a rational Gorenstein singularity.

(2) (X, H) is a canonical pair.

(CM + 1-Goren)

(pf.) Note that $(0 \in X)$ is Gorenstein singularity $\Leftrightarrow (0 \in H)$ is a Gorenstein singularity.
(by [KM Prop 5.3 (1), Prop 5.73] or [Ishii. Prop 5.3.13])

$$\begin{array}{l} \text{Let } H' := f_*^{-1} H \subseteq Y \\ \qquad\qquad\qquad \downarrow f : \text{log resol. of } (X, H) \\ \qquad\qquad\qquad H \subseteq X \end{array}$$

By adjunction,

$$\begin{tikzcd} 0 \arrow[r] & \omega_Y \arrow[r] & \omega_Y(H') \arrow[r] & \omega_{H'} \arrow[r] & 0 & \\ 0 \arrow[r] & f_*\omega_Y \arrow[r] \arrow[d, "\alpha"', "\mathrm{tr}_{Y/X}"] & f_*\omega_Y(H') \arrow[r] \arrow[d, "\beta"'] & f_*\omega_{H'} \arrow[r] \arrow[d, "s"', "\gamma = \mathrm{tr}_{H'/H}"] & R^1 f_* \omega_Y = 0 & \\ 0 \arrow[r] & \omega_X \arrow[r, "\cdot h"] \arrow[d] & \omega_X(H) \arrow[r] \arrow[d] & \omega_H \arrow[r] & 0 & \\ & \mathcal{C} \arrow[r, two heads] & \mathcal{C}(H) & & & \end{tikzcd}$$

G-R vanishing

(by Kempf thm, since $(0 \in H)$: rational)

by snake lemma

$\mathcal{C} = \operatorname{coker} \alpha$

$\Rightarrow \mathcal{C} = 0$ by Nakayama

$\Rightarrow \mathrm{tr}_{Y/X} : f_*\omega_Y \xrightarrow{\sim} \omega_X \Rightarrow \begin{cases} (1) \\ \beta \text{ is isom.} \end{cases}$

Write $\omega_Y(H') = f^*\omega_X(H) \otimes \mathcal{O}_Y(E)$, where E is an excep. divisor.

$\Rightarrow \omega_X(H) \overset{\beta}{\simeq} f_*\omega_Y(H') = \omega_X(H) \otimes f_*\mathcal{O}_Y(E) \rightsquigarrow f_*\mathcal{O}_Y(E) = \mathcal{O}_X \rightsquigarrow E$ is effective

$\rightsquigarrow (X,H)$ is canonical.

Rmk. (1) Reid's original proof [Rei 80, Thm 2.6 II] use the observation that

$$\begin{tikzcd} \pi^{-1}(t) = X \arrow[d] & \subseteq \mathcal{X} \arrow[d, "\text{flat}"] & \supseteq H \times \mathbb{A}^1 = \pi^{-1}(0) \arrow[d] \\ 0 \neq t & \in \mathbb{A}^1_{\mathbb{C}} & \ni 0 \end{tikzcd}$$

and Elkik's thm.

(if $\pi^{-1}(0)$ has at worst rational sing., then $\pi^{-1}(t)$ is also
i.e. rationality of sing. is open condition)

(2) (inversion of adjunction) [KM, Thm 5.50]

$\because$ $(H, \Delta=0)$ is can. of index 1 $\Rightarrow$ (X,H) is plt. $\Rightarrow$ (X,H) is canonical

since K_X+H is Cartier.

Thm. D [KM, Thm. 5.34] $(0\in X)$: 3-dim'l normal singularity, $0\in H\subseteq X$: general hyperplane section. TFAE

(D1) $(0\in H)$ is Du Val, i.e. $(0\in X)$ is cDV.

(D2) (X,H) canonical pair

(D3) $(0\in X)$ is canonical of index 1 and $\forall$ resol. $f: Y \longrightarrow X \Rightarrow a(E,X)\geq 1$.

$\forall$ prime divisor $E \subseteq f^{-1}(0) \longrightarrow x$

Proof of Reid's thm.

(terminal surface $\Rightarrow$ sm.)

$(\Rightarrow)$ $(0\in X)$: terminal of index 1 $\Rightarrow \begin{cases} \text{sm. in codim } 2 \\ \text{(D3)} \end{cases} \Rightarrow \begin{cases} \text{isolated} \\ \text{(D1)} \end{cases}$

$(\Leftarrow)$ $(0\in X)$: $\begin{cases} \text{isolated} \\ \text{(D1)} \end{cases} \Rightarrow \begin{cases} \text{center}_X(E) = \{0\} \text{ for every excep. div. } E \text{ over } X \\ \underline{\text{Gorenstein}}, \text{(D3)} \end{cases}$

hyper. sing.

$\Rightarrow a(E,X)\geq 1 \Rightarrow$ terminal.

Thm E. [KM, Thm 5.35] $(0\in X)$: an index 1 canonical 3-fold singularity. TFAE

(E1) The general hypersection $0\in H\subseteq X$ is an elliptic Gorenstein singularity.

(E2) [Rei 80, Thm 2.11] $\exists f: Y \longrightarrow X$ is crepant birational projective morphism s.t.

- Y is normal, C-M (In particular, Y : index 1 can. 3-fold, $0 \leq \mathrm{discrep}(Y) = \mathrm{discrep}(X)$)
- $f^{-1}(0)$ is of pure dimension 1
- $Y \setminus f^{-1}(0) \xrightarrow{\sim} X \setminus \{0\}$.

Moreover, we can choose Y s.t. the $\mathrm{Aut}(o\in X)$-action on X lifts to an action on Y.

(E3) $\forall$ resol. $f: Y \longrightarrow X$, $\exists$ a prime divisor $E \subseteq f^{-1}(0)$ s.t. $a(E,X)=0$.

Plan of the proofs of Thm D and Thm E:

$\because$ $(o\in X)$ is index 1 canonical 3-fold singularity $\Rightarrow$ $\exists$ a general hyperplane section $(o\in H)$ is either Du Val or elliptic Gorenstein.

$\therefore$ "(D1) $\Rightarrow$ (D3)" $\Leftrightarrow$ "(E3) $\Rightarrow$ (E1)" and "(D3) $\Rightarrow$ (D1)" $\Leftrightarrow$ "(E1) $\Rightarrow$ (E3)".

Therefore we only need to prove $\begin{cases} (D1) \Rightarrow (D2) \Rightarrow (D3) \\ (E1) \Rightarrow (E2) \Rightarrow (E3) \end{cases}$.

(pf.)

- (D1) $\Rightarrow$ (D2): $(o\in H)$ is Du Val $\Leftrightarrow$ $(o\in H)$ is rational Gorenstein $\overset{\text{Prop 2.}}{\Rightarrow}$ (X,H) is can. pair.
- (D2) $\Rightarrow$ (D3): Let $H' := f_*^{-1}H \subseteq Y$ and write $K_Y + H' = f^*(K_X+H) + \sum a_i E_i$. (✻)

$$\begin{array}{ccc} H' \subseteq Y & & \\ \downarrow & \downarrow f: \text{log resol. of } (X,H) & \\ H \subseteq X & & \end{array} \qquad f^*H = H' + \sum b_i E_i$$

By assumption, $\begin{cases} K_X + H \text{ is } \mathbb{Q}\text{-Cartier, hence so is } K_X \Rightarrow K_Y = f^*K_X + \sum \underbrace{(a_i+b_i)}_{\geq 0} E_i. \\ a_i \geq 0 \ \forall i \quad \text{since } (X,H) \text{ is canonical} \\ b_i \in \mathbb{Z}_{\geq 0} \quad \text{since } H: \text{eff. Cartier} \end{cases}$

Hence $(X, \Delta=0)$ is canonical, and $a(E,X) = a_i + b_i \geq b_i \geq 1$ if $f(E_i) \subseteq H$.

$\cdot\cdot$ K_X is Cartier: By adjunction on $(✻)|_{H'}$, $K_{H'} = f^*K_H + \sum a_i (E_i|_H) \Rightarrow H$ is $\underset{\text{Du Val}}{\overset{}{\text{canonical}}}$ (Du Val $=$ canonical)

Hence $\mathrm{e.dim} H \leq 3$, and thus $\mathrm{e.dim} X \leq 4 \rightsquigarrow (o\in X)$: hypersurface singularity $\Rightarrow K_X$ is Cartier.

$\overset{\shortparallel}{(f=0)}$ has local gen. of K_X: $\mathrm{Res}_X \dfrac{dx\wedge dy\wedge dz\wedge dt}{f} = \dfrac{dx\wedge dy\wedge dz}{\partial f/\partial t}$

- (E2) $\Rightarrow$ (E3): By assumption, $\exists$ excep. divisor E over X s.t. $a(E,X)=0$ and $\mathrm{center}_X E = \{0\}$. By X is canonical, $0 = a(E,X) \geq \mathrm{discrep}(X) \geq 0$ and $\forall$ resol. $f: Y \to X$ we have $\min\limits_{E: f\text{-excep.}} \{a(E,X)\} \geq 0$

By [KM, Cor 2.32 (1)], $\underset{\overset{\shortparallel}{0}}{\mathrm{discrep}(X)} = \min\limits_{E: f\text{-excep.}} \{a(E,X), 1\} \Rightarrow \exists E \subseteq f^{-1}(0)$ s.t. $a(E,X)=0$

Prop. [Reid 80, Cor 2.10] (Reid invariant)

$(o\in X)$: a rational Gorenstein 3-fold sing. with a general hyperplane section $o\in H \subseteq X$.

$\Rightarrow \exists$ an invariant $k \in \mathbb{Z}_{\geq 0}$ associated with $(o\in X)$.

- $(k=0)$: $(o\in X)$ is cDV $\Leftrightarrow$ $(o\in H)$ is Du Val.

  $(k=1)$: $(o\in X) \simeq (x^2+y^3+yf_4(z,t)+f_6(z,t)=0)$ where $\mathrm{mult}_o f_i \geq i$.

  $(k=2)$: $(o\in X) \simeq (x^2+f(y,z,t)=0)$ where $\mathrm{mult}_o f \geq 4$.

$(k\geq 3)$: $\mathrm{mult}_o X = k$, $e.\dim(o\in X) = k+1$.

- For $k\geq 1$, $(o\in H)$ is elliptic Gorenstein and $k = -Z_{num}^2$, where Z_{num} is the Artin fund. cycle for the minimal resol. of $o\in H$.

(for a proof, see [KM, p.168] using Laufer-Reid's thm [KM, Thm 4.57])

Back to the proof of $(E1)\Rightarrow(E2)$.

Let $Y = Bl_o^w X$, where $w = wt(x,y,z,t) = \begin{cases} (3,2,1,1) & \text{if } k=1 \text{ in Prop.} \\ (2,1,1,1) & \text{if } k=2 \\ (1,1,1,1) & \text{if } k\geq 3 \end{cases}$ Then

$$\begin{array}{c} Y \\ \downarrow f \\ X \end{array}$$

(1) f^{-1} is isom. outside $\{o\}$.

(2) $o\in H\subseteq X$, general hyper. section. Write $f^*H = H' + \overset{\text{excep.}}{E}$ $\Rightarrow f' := f|_{H'}: H' \to H$ is isomorphic to the (weighted) blowup specified in [KM, 4.57] and $\mathcal{O}_Y(H')$ is f-ample.

In particular, H' has only Du-Val singularity and $\mathcal{O}_{H'}(K_{H'}) = \mathcal{O}_{H'}(1) = \mathcal{O}_{H'}(-E|_{H'})$.

(3) $Y = \mathrm{Proj}_X(\bigoplus_n \mathfrak{m}^w(n))$, $\mathfrak{m}^w(n)$ is indep. of the choice of the coordinates (pf. see

Fact. Y is normal and C-M (see [Rei 80, p270])

Check. f is crepant:

Write $K_Y = f^*K_X + F$ for some $F\geq 0$, excep.

WLOG $\underbrace{f^*K_X = 0}_{\text{Cartier at } (o\in X)}$ $\Rightarrow 0 = H' + E \Rightarrow (K_Y + H')|_{H'} = (F - E')|_{H'} \Rightarrow F|_{H'} = 0$.

$f^*H = 0 \qquad K_Y = F \qquad K_{H'} = -E|_{H'}$

Since H' is f-ample and $F\geq 0$, $F = 0$.

Cor. [KM Cor. 5.40] [Rei 80, Cor 1.14] X: a normal canonical 3-fold, then

$\#\{x\in X \mid x \text{ is a } \underline{\text{non-cDV point}}\} < \infty$

$\quad\mid\cap \qquad$ Reid called these dissident point

$\underset{\uparrow\ \mathrm{codim}\geq 3.}{\mathrm{Sing}\, K_X} \cup \{x\in X\setminus \mathrm{Sing}\, K_X \mid \exists E: \text{excep. div.}/X \text{ s.t. } a(E,X) = 0,\ \mathrm{center}_X E = \{x\}\}$

§ Terminalization of Canonical 3-folds

X : algebraic 3-fold with at worst canonical singularities
(resp. analytic)

$e(X) := \#\{\text{excep. divisor } E \text{ over } X \text{ with } a(E,X)=0\} < \infty$ by [KM, Prop. 2.36]

Thm. (Reid) [Rei 83 (b), (0.6) Main thm. II] [KM Thm 6.23]
$\exists$ a crepant projective partial resolution $\pi_X : X^{ter} \longrightarrow X$ s.t. X^{ter} has only terminal singularities.

Rmk. [KM Rmk 6.24] The Reid's terminalization construction is funtorial for open embedding, and compatiable with $(-)^{an}$ $\left((\pi_X)^{an} = \pi_{X^{an}} \right)$

Sketch of proof.
We use induction on $e(X)$. If $e(X)=0$, then X is terminal and $\pi_X = id_X$.
For $p \in \operatorname{Sing} X$, let $(\tilde{p} \in \tilde{X}) \xrightarrow[\text{cover}]{\text{index one}} (p \in X)$.

(a) $(\tilde{p} \in \tilde{X})$ is NOT a cDV point for some $p \in \operatorname{Sing} X$.
(b) $(\tilde{p} \in \tilde{X})$ is a cDV point for all $p \in \operatorname{Sing} X$ and $\dim \operatorname{Sing} X = 1$

$\Big\}$ $\exists f : Y \to X$ crepant proj. birat. s.t. $e(Y) < e(X)$

(c) $(\tilde{p} \in \tilde{X})$ is a cDV point for all $p \in \operatorname{Sing} X$ and $\dim \operatorname{Sing} X = 0 \rightsquigarrow e(X) = 0$

Indeed,
(a) [KM, Cor 5.41]

$$\begin{tikzcd} G \circlearrowright \tilde{Y} \arrow[r, "\tilde{f}"] \arrow[d] & \tilde{X} \circlearrowleft G = Gal(\tilde{X}/X) \arrow[d, "\text{index 1 cover of } (p \in X)"] \\ Y := \tilde{Y}/G \arrow[r, "f : \text{crep.}"] & X \end{tikzcd}$$

- $\tilde{f}$ is weighted blowup constructed in Thm (E2), [KM, Thm 5.35 (2)]

$$w = wt(x,y,z,t) = \begin{cases} (3,2,1,1) & \text{if } k=1 \ \ \leftarrow \text{Reid's invariant} \\ (2,1,1,1) & k=2 \\ (1,1,1,1) & k \geq 3 \end{cases}$$

$\circledast$ the ideals $\mathfrak{m}^w(n)$ are G-invariant, $Y = \operatorname{Proj}_X \left(\bigoplus_{n \geq 0} \mathfrak{m}^w(n) \right)$

(b) [KM, Thm 6.27] Let C be the 1-dim'l irr. comp. of $\operatorname{Sing} X$ with its reduced structure, and I be the defining ideal of C. For $\nu \in \mathbb{Z}_{\geq 0}$,
$I^{\nu} :=$ the ν-th symbolic power of I
$=$ the ideal sheaf consisting of germs of functions that have mult $\geq \nu$ at a generic pt of C

$\rightsquigarrow f : Y := \mathrm{Proj}_X\left(\bigoplus_{\nu\geq 0} I^{\nu}\right) \longrightarrow X$ s.t. $\begin{cases} Y \text{ is normal and canonical} \\ K_Y = f^*K_X \\ \text{every fiber } f^{-1}(x) \text{ are of } \dim \leq 1, \text{ and } =1 \text{ if } x \in C. \end{cases}$

(c) [KM, Lemma 6.31] If $(\tilde{p} \in X)$ is terminal $\Rightarrow$ $(p \in X)$ is terminal.

$$\begin{array}{ll} (\tilde{p} \in \tilde{X}) : & \text{can. sing. of index } 1 \\ {\scriptstyle \text{index 1 cover}} \downarrow & \qquad\Uparrow \\ (p \in X) : & \text{can. sing. of arb. dim.} \end{array}$$

§ Quotient singularities / $\mathbb{C}$

Recall. • A singularity $(x\in X)$ is a quotient sing. if $\exists$ a smooth germ $(0\in Y)$ and a finite group G acting on (an analytic nbd.) $(0\in Y)$ s.t. $(x\in X)\simeq (0\in Y)/G$.

• [KM, Cor 5.39]

fixed point free outside the origin $G \curvearrowright$ $G=\mathrm{Gal}(Y/X)\simeq \mu_r$

$(0\in Y)$: a terminal 3-fold of index 1 $\overset{\shortparallel}{}$ (an isolated cDV point $(x\in X)\hookrightarrow (0\in \mathbb{A}^4/\mu_r)$; or a smooth point $(x\in X)=(0\in\mathbb{A}^3/\mu_r)$)

$\downarrow$ index 1 cover

$(p\in X)$: a terminal 3-fold sing. of index $r\geq 1$

A group G giving a quotient sing. can be a linear group.

Thm. $(x\in X)$: a quotient sing. of $\dim = n \Rightarrow \exists$ a finite subgroup $G'\leq GL_n(\mathbb{C})$ s.t. $(x\in X)\simeq (0\in \mathbb{A}^n/G')$ as germs. (In particular, quotient sing. is alg.)

(pf.) Let $(x\in X)\simeq (y\in Y)/G$ (sm.), G : finite group.

WLOG the stabilizer $G_y=G$. (if $G_y\subsetneq G$, take an analytic nbd. Y' of y s.t. $Y'\cap G_y=\{y\}$, we can see $X\simeq Y'/G_y$ (or $Y\longrightarrow Y/G_y \xrightarrow{\text{étale at } y} Y/G=X$))

Let $\mathfrak{m}\lhd \mathcal{O}_{Y,y}$ be the maximal ideal.

$\because G=G_y \Rightarrow \mathfrak{m}$ is invariant under the action of G

$\Rightarrow$ this define a representation $\rho : G\longrightarrow GL(\mathfrak{m}/\mathfrak{m}^2)\simeq GL_n(\mathbb{C})$

$\uparrow$ by regular system of parameters $z_1,\cdots,z_n\in\mathfrak{m}$.

$$\Rightarrow \begin{pmatrix} g(z_1)\\ \vdots \\ g(z_n)\end{pmatrix} = \rho(g)\begin{pmatrix} z_1\\ \vdots\\ z_n\end{pmatrix},\quad g(z_i)=\rho(g)(z_i)\in \mathfrak{m}/\mathfrak{m}^2.$$

Define $y_i\in\mathcal{O}_{Y,y}$ by
$$\begin{pmatrix} y_1\\ \vdots\\ y_n\end{pmatrix} = \frac{1}{|G|}\sum_{g\in G}\rho(g^{-1})\begin{pmatrix} g\cdot z_1\\ \vdots \\ g\cdot z_n\end{pmatrix}$$

$\Rightarrow y_i - z_i\in\mathfrak{m}^2$ and hence $y_1,\cdots,y_n$ form a regular system of parameters in $\mathcal{O}_{Y,y}$.

Write $Z=\begin{pmatrix} z_1\\ \vdots\\ z_n\end{pmatrix}$ and $Y=\begin{pmatrix} y_1\\ \vdots\\ y_n\end{pmatrix}$ for briefy. $\forall h\in G$,

$$hY=\frac{1}{|G|}\sum\rho(g^{-1})\,h\cdot(gZ)=\frac{1}{|G|}\sum\rho(h\cdot(gh)^{-1})\cdot(gh\,Z)=\rho(h)\cdot Y$$ (G acts on $\mathcal{O}_Y$ from right)

i.e. $G\curvearrowright Y$ is linear wrt. $y_1,\cdots,y_n$. Set $G':=\rho(G)\curvearrowright\mathbb{A}^n$.

Rmk. Even if $(y\in Y)$ is singular with $\underset{\dim_{\mathbb{C}}\mathfrak{m}/\mathfrak{m}^2}{\overset{\shortparallel}{\text{e.dim}}}=e \Rightarrow \exists$ a finite subgp. $\rho(G)\leq GL_e(\mathbb{C})$ s.t.

$(x\in X)\simeq (0\in Y/G)\hookrightarrow (0\in\mathbb{A}^e/\rho(G))$

Def. • $\mathrm{id} \neq g \in GL_n(\mathbb{C})$ is pseudo-reflection (reflection) (p-rf.) if $\mathrm{ord}\, g < \infty$ ($\mathrm{ord}\, g = 2$) and $\mathrm{Fix}(g) := \{x \in \mathbb{C}^n : gx = g\}$ has codim $= 1$
(i.e. all but one of its eigenvalues are 1)
• A subgp. of $GL_n(\mathbb{C})$ generated by p-rfs is call a p-rf group.
• A finite subgp. of $GL_n(\mathbb{C})$ is small if it does not contain p-rfs.

Fact. (Chevalley - Shephard - Todd Theorem)
If $H \leq GL_n(\mathbb{C})$ is a finite p-rf group, then $\mathbb{A}^n/H \simeq \mathbb{A}^n$.

Cor. $(x \in X)$: quotient sing. of $\dim = n \Rightarrow \exists$ a small finite subgp. $G \leq GL_n(\mathbb{C})$ s.t. $(x \in X) \simeq (0 \in \mathbb{A}^n/G)$ as germs.
(pf.) Let H be the subgp. of G' generated by p-rfs in G'
• $H \triangleleft G'$: $\forall g \in G'$, $h \in H$ a p-rf, we have $\mathrm{Fix}(ghg^{-1}) = g(\mathrm{Fix}(h))$ is codim 1.
• Consider $G = G'/H \curvearrowright \mathbb{A}^n/H \underset{\uparrow}{\simeq} \mathbb{A}^n$ and $\mathbb{A}^n/G \simeq {}^{\mathbb{A}^n/H}/_{G'/H} \simeq \mathbb{A}^n/G'$.
by C-S-T thm.

• G doesn't contain p-rfs. For $H \neq \bar{g} = gH \in G$, $g \in G'$, we have $hg \notin H$ $\forall h \in H$, i.e. $\mathrm{codim}\,\mathrm{Fix}(hg) \geq 2$.
Let $\phi : \mathbb{A}^n \longrightarrow \mathbb{A}^n/H$ be the projection.
$\because$ $\bar{g} \cdot \phi(x) = \phi(x) \iff gx = hx$ for some $h \in H$.
$\therefore$ $\underset{\mathbb{A}^n/H}{\underset{\cap}{\mathrm{Fix}(\bar{g})}} = \bigcup_{h \in H} \phi(\mathrm{Fix}(hg))$ has codim ≥ 2 (since ϕ is finite & H is finite)

Cor. $(0 \in Y) \subseteq (0 \in \mathbb{A}^{n+1})$ be a n-dim'l hypersurface sing., G : a finite gp acting on Y.
$\Rightarrow \exists$ a small, finite $G' \leq GL_{n+1}(\mathbb{C})$ acting on Y s.t. $(0 \in Y/G') \simeq (0 \in Y/G) \subseteq (0 \in \mathbb{A}^{n+1}/G)$.
($\because$ $\mathrm{e.dim}\, Y = n+1$, this follows from Rmk and above Cor)

Cor. A quotient singularity is log terminal ($=$ klt $=$ plt $=$ dlt if $\Delta = 0$)
(pf.) By Cor., $\exists$ a small, finite group $G \leq GL_n(\mathbb{C})$ s.t.

$$\begin{array}{c} \mathbb{A}^n \\ \downarrow p \\ (x \in X) \simeq (0 \in \mathbb{A}^n/G) \end{array}$$

$\because$ G is small $\Rightarrow G$ acts on $\mathbb{A}^n$ freely in codim 1
$\Rightarrow K_{\mathbb{A}^n} = p^* K_{\mathbb{A}^n/G}$, i.e. p is crepant.
$\therefore$ $\mathbb{A}^n$ is sm. $\iff$

Def. • $(0 \in Y)$: a sm. germ admitting $\mu_r \curvearrowright Y$ which fixes 0. We fix a primitive character $\chi : \mu_r \hookrightarrow \mathbb{C}^*$.
We say $f \in \mathcal{O}_{Y,0}$ is semi-invariant (wrt. χ) if $\exists$ an integer $\underset{:= \mathrm{wt}(f) \in \mathbb{Z}/r\mathbb{Z}}{\underline{a \pmod r}}$ s.t. $g(f) = \chi(g)^a \cdot f$ $\forall g \in \mu_r$.
the weight of f wrt. χ

orbifold coordinate of $\mathbb{A}^n/\frac{1}{r}(a_1,\cdots,a_n)$

- We write $\mathbb{A}^n/\frac{1}{r}(a_1,\cdots,a_n)$ for the quotient $\mathbb{A}^n/\mu_r$ in which every coordinate of $\mathbb{A}^n$, x_i is semi-invariant with weight $\mathrm{wt}(x_i)=a_i$ (w.r.t. χ)

$$\left(\begin{array}{l}\mu_r=\left\langle g=\begin{pmatrix}\varepsilon^{a_1} & & \\ & \ddots & \\ & & \varepsilon^{a_n}\end{pmatrix}\right\rangle \le GL_n(\mathbb{C}),\ \varepsilon:\text{primitive } r\text{-th roots of unity},\ 0\le a_i\le n-1\ \ \forall i=1\sim n \\ g:(x_1,\cdots,x_n)\longmapsto(\varepsilon^{a_1}x_1,\cdots,\varepsilon^{a_n}x_n)\end{array}\right)$$

- A (small) cyclic quotient sing. of type $\frac{1}{r}(a_1,\cdots,a_n)$ is a singularity $\overset{\text{analytically}}{\simeq}$ $\left(0\in\mathbb{A}^n/\frac{1}{r}(a_1,\cdots,a_n)\right)$ (with $\mu_r\le GL_n(\mathbb{C})$ is small)

Rmk. If $\gcd(b,r)=1$, then $\mathbb{A}^n/\frac{1}{r}(a_1,\cdots,a_n)\simeq\mathbb{A}^n/\frac{1}{r}(ba_1,\cdots,ba_n)$ by taking χ^b instead of χ.

Lemma. Let $\overline{M}=\mathbb{Z}^n$, dual lattice $\overline{N}=\mathrm{Hom}_{\mathbb{Z}}(\overline{M},\mathbb{Z})\simeq\mathbb{Z}^n$, and $N=\overline{N}+\frac{1}{r}(a_1,\cdots,a_n)\mathbb{Z}$ $(r\in\mathbb{N})$. Let $e_1,\cdots,e_n$ be the standard basis of $N\otimes_{\mathbb{Z}}\mathbb{R}\simeq\mathbb{R}^n$.
$\Sigma :=$ the fan consisting faces of the cone $\sigma=\mathrm{Cone}(e_1,\cdots,e_n)\subseteq N\otimes_{\mathbb{Z}}\mathbb{R}$. Then $\mathbb{A}^n/\frac{1}{r}(a_1,\cdots,a_n)\simeq X_\Sigma$.

(pf.)
- $\overline{N}\subseteq N\ni\alpha\iff\alpha=\frac{k}{r}(a_1,\cdots,a_n)\pmod{\mathbb{Z}^n}$ for some $0\le k\le r-1$.
  $\overline{M}\supseteq M\ni m=(m_1,\cdots,m_n)\iff m:N\longrightarrow\mathbb{Z}\iff\frac{1}{r}\sum a_i m_i\in\mathbb{Z}$
- Since $\mathbb{A}^n=\mathrm{Spec}\,\mathbb{C}[\sigma^\vee\cap\overline{M}]\simeq\mathrm{Spec}\,\mathbb{C}[x_1,\cdots,x_n]$ and the affine coord. ring of $\mathbb{A}^n/\frac{1}{r}(a_1,\cdots,a_n)$ is $\mathbb{C}[x_1,\cdots,x_n]^G$, where $G=\langle g=\mathrm{diag}(\varepsilon^{a_1},\cdots,\varepsilon^{a_n})\rangle$.
- We only need to check $\mathbb{C}[\sigma^\vee\cap\overline{M}]^G=\mathbb{C}[\sigma\cap M]$.
  $\because$ $g:(x_1,\cdots,x_n)\longmapsto(\varepsilon^{a_1}x_1,\cdots,\varepsilon^{a_n}x_n)\Rightarrow\forall m=(m_1,\cdots,m_n)\in M$, $g\cdot\chi^m=\varepsilon^{\sum a_i m_i}\chi^m$.
  $\therefore$ χ^m is G-invariant $\iff r\mid\sum a_i m_i\iff m\in M$.

Recall. $(0\in X)$: cyclic quotient singularity, $\dim = n$.

$\rightsquigarrow X \underset{\text{analy.}}{\simeq} \mathbb{A}^n/\frac{1}{r}(a_1,\cdots,a_n)$ with $\underset{\text{small}}{\mu_r} \le GL_n(\mathbb{C})$, where $\overline{M}=\mathbb{Z}^n$, $\overline{N}=\mathrm{Hom}_{\mathbb{Z}}(\overline{M},\mathbb{Z})$

$\quad\quad \cong X_\Sigma$: toric $\quad\quad N=\overline{N}+\frac{1}{r}(a_1,\cdots,a_n)\mathbb{Z}$

Σ = the fan consisting of faces of cone $\sigma = \mathrm{Cone}(e_1,\cdots,e_n) \subseteq N\otimes_{\mathbb{Z}}\mathbb{R} \simeq \mathbb{R}^n$.

Def. • $0\ne e\in N$ is a primitive element if $\mathbb{Z}e = \mathbb{R}\cdot e \cap N$ in $N\otimes_{\mathbb{Z}}\mathbb{R}$

• Fix a primitive element $e=\frac{1}{r}(w_1,\cdots,w_n)\in N\cap\sigma$

$\Sigma^*(e)$ = the star subdivision of Σ at e

:= the fan in $N\otimes_{\mathbb{Z}}\mathbb{R}$ consisting of faces of the $\mathrm{Cone}(e_1,\cdots,e_{i-1},e,e_{i+1},\cdots,e_n)$ for $1\le i\le n$.

• The weighted blow-up of $X=\mathbb{A}^n/\frac{1}{r}(a_1,\cdots,a_n)$ of weights $wt(x_1,\cdots,x_n)=\frac{1}{r}(w_1,\cdots,w_n)=e$ is

$\pi: B = X_{\Sigma^*(e)} \longrightarrow X_\Sigma = X$. It is isomorphism outside $\bigcap_{w_i\ne 0}(x_i=0)$ in X.

$\quad\quad \supseteq E$: exceptional divisor

Fact. Suppose that $\mu_r\le GL_n(\mathbb{C})$ is small $\Rightarrow K_B = \pi^*K_X + (\frac{1}{r}\Sigma w_i - 1)E$, i.e. $a(E,X)=\frac{1}{r}\Sigma w_i - 1$.

(cf. [CLK, Lemma 11.4.10] or [Ishii, pf. of Prop. 8.3.11, Thm 8.3.4])

Notation. Fix $r\in\mathbb{N}$. For $k\in\mathbb{Z}$, let $\overline{k} := k - \lfloor\frac{k}{r}\rfloor r$.

Thm. (Reid-Tai criterion)

A small cyclic quotient singularity of type $\frac{1}{r}(a_1,\cdots,a_n)$ is terminal (resp. canonical)

$\Leftrightarrow \frac{1}{r}\Sigma\overline{ka_i} > 1$ (resp. ≥ 1) for all $k=1,\cdots,r-1$.

(pf.) Use the description of $X=\mathbb{A}^n/\frac{1}{r}(a_1,\cdots,a_n) = X_\Sigma$. $N=\overline{N}+\frac{1}{r}(a_1,\cdots,a_n)\mathbb{Z}$.

($\Rightarrow$) For $0<k<r$, take a primitive element $\frac{1}{r}(b_1,\cdots,b_n)$ in N from the ray $\mathbb{R}\cdot\frac{1}{r}(\overline{ka_1},\cdots,\overline{ka_n})$.

Let E_k = the excep. divisor obtained by the weighted blow-up of X_Σ with $wt(x_1,\cdots,x_n)=\frac{1}{r}(b_1,\cdots,b_n)$.

By Fact, $a(E,X)=\underbrace{\frac{1}{r}\Sigma b_i}_{\le\, \frac{1}{r}\Sigma\overline{ka_i}} - 1 \begin{cases} >0 & \text{if } X \text{ is terminal} \\ \ge 0 & \quad '' \quad \text{canonical}\end{cases}$.

($\Leftarrow$) Take a unimodular subdivison of X_Σ which provides a log resolution of X_Σ.

every excep. prime divisor E $\xleftrightarrow{1-1}$ a ray in $N\otimes\mathbb{R}$ gen. by a primitive element $\alpha=\frac{1}{r}(c_1,\cdots,c_n)\in\sigma\cap N$ s.t. at least two of $c_i>0$.

Recall. $\alpha\in\underset{\Downarrow\ \frac{c_i}{r}\ge 0}{\underline{\sigma\cap N}}\subseteq N \Leftrightarrow \alpha\equiv\frac{k}{r}(a_1,\cdots,a_n) \mod \mathbb{Z}^n \Leftrightarrow c_i\equiv ka_i \mod r\ \forall i \Rightarrow c_i\ge\overline{ka_i}\ \forall i$.

$$a(E,X)=\frac{1}{r}\sum c_i-1 \geq \begin{cases} 1 & \text{if } k=0 \\ \frac{1}{r}\sum \overline{ka_i}-1 & \text{if } 0<k<r \end{cases}.$$
($\because$ two $c_i>0 \to \geq 1$)

If $\frac{1}{r}\overline{ka_i}>1$ (resp. ≥ 1), then $a(E,X)>0$ (resp. ≥ 0) $\Rightarrow$ X is terminal (resp. canonical)

In dim=3, the condition of Reid-Tai criterion is well-understood.

Thm. [White] Let $r\in\mathbb{N}$, $a_1,a_2,a_3\in\mathbb{Z}$. If $\sum\overline{ka_i}>r$ for all $0<k<r$, then $r\mid a_i+a_j$ for some distinct $1\leq i,j\leq 3$.

Rmk. $r=1,2$ is trivial case. For $r\geq 3$, we explain an abstract approach due to
[MS] D.R Morrison and G. Stevens, Terimal quotient singularities in dim three or four, 1984.
write the proof by
[White] G.K White, Lattice tetrahedra, 1964 (See also, [YPG, Appendix to §5])

Def. · A (Dirichlet) character of $(\mathbb{Z}/r\mathbb{Z})^\times$ is a group homo. $\chi:(\mathbb{Z}/r\mathbb{Z})^\times\longrightarrow\mathbb{C}^\times$.

$f:=$ the conductor of $\chi=\min\{r'\in\mathbb{N}\mid r'|r$ and
$$\begin{tikzcd} (\mathbb{Z}/r\mathbb{Z})^\times \arrow[rr, "\chi"] \arrow[dr] & & \mathbb{C}^\times \\ & (\mathbb{Z}/r'\mathbb{Z})^\times \arrow[ur] & \end{tikzcd}$$
$\}$

· The character χ is $\begin{cases} \text{even} & \text{if } \chi(-1)=1 \\ \text{odd} & \text{if } \chi(-1)=-1 \end{cases}$

Def. · Define a function $B_1:\mathbb{Q}/\mathbb{Z}\longrightarrow\mathbb{Q}$ by $B_1(q)=\begin{cases} q-\lfloor q\rfloor-\frac{1}{2} & \text{if } q\notin\mathbb{Z} \\ 0 & \text{if } q\in\mathbb{Z} \end{cases}.$

Note that $B_1(-q)=B_1(q)$ since $\lfloor q\rfloor+\lfloor -q\rfloor=-1$ for $q\in\mathbb{Q}\setminus\mathbb{Z}$.

· We define the generalized Bernoulli number $B_{1,\chi}=\sum_{a\in(\mathbb{Z}/r\mathbb{Z})^\times}\chi(a)B_1(\frac{a}{f})\in\mathbb{C}$ by regard each $a\in\mathbb{Z}/r\mathbb{Z}$ as an integer $0\leq a<f=$ conductor of χ.

· (Dirichlet L-function) $L(s,\chi):=\sum_{n\geq 1}\frac{\chi(n)}{n^s}$ for $\mathrm{Re}\, s>1$ on $\mathbb{C}$. It may be continued analytically to $\mathbb{C}$, except for a simple pole at $s=1$ when $\chi\equiv 1$.

Fact. (Dirichlet) If χ is odd, then $B_{1,\chi}\neq 0$.
(cf. [GTM 85] L.C. Washington, Introduction to Cyclotomic Fields, 2nd, 1997])

Sketch of proof.

- Fact. $L(1,\chi)\neq 0$, $L(0,\chi)\neq 0$ if χ is odd.

(pf.) By regarding $\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})\simeq(\mathbb{Z}/r\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$, we let $K:=\mathbb{Q}(\zeta_r)^{\ker\chi}$. Then

$$\zeta_K(s)=\prod_{a=0}^{b-1}L(s,\chi^a)=\zeta(s)\prod_{a=1}^{b-1}L(s,\chi^a)\quad,\text{ where } b=\mathrm{ord}(\chi).$$

Dedekind zeta function $\qquad$ Riemann zeta

Since ζ_K and ζ have only a simple pole at $s=1$, none of factor $\underset{a>0}{L(s,\chi^a)}$ can vanish at $s=1$.

From functional equation, we have [GTM 83, pp. 30~31]

$$L(1-n,\chi)\neq 0 \text{ when } n \text{ is } \begin{cases}\text{even} & \text{if } \chi \text{ is even}\\ \text{odd} & \text{if } \chi \text{ is odd}.\end{cases}$$

- One can define $\displaystyle\sum_{a=1}^{f}\frac{\chi(a)te^{at}}{e^{ft}-1}=\sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!}$ (when $\chi\equiv 1$, $B_{n,1}$ is "ordinary" Bernoulli number B_n) (with $B_1=\frac{1}{2}$)

By [GTM84, Prop. 4.1], $B_{n,\chi}=f^{n-1}\displaystyle\sum_{a\in(\mathbb{Z}/f\mathbb{Z})^\times}\chi(a)B_n(\tfrac{a}{f})$, where $\displaystyle\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!}$ Bernoulli poly.

By a contour integration, one can see (cf. [GTM84, Thm 4.2]) $L(1-n,\chi)=-B_{n,\chi}/n$ for $n\geq 1$.

Hence when χ is odd, take $n=1 \Rightarrow B_{1,\chi}=-L(0,\chi)\neq 0$.

Def. • $V=\mathbb{C}[(\mathbb{Z}/r\mathbb{Z})^\times]$ the group alg. of $(\mathbb{Z}/r\mathbb{Z})^\times$ over $\mathbb{C}$ generated by σ_a, $a\in(\mathbb{Z}/r\mathbb{Z})^\times$.

- For $q\in\frac{1}{r}\mathbb{Z}/\mathbb{Z}\subseteq\mathbb{Q}/\mathbb{Z}$, we define $S(q):=\displaystyle\sum_{a\in(\mathbb{Z}/r\mathbb{Z})^\times}B_1(aq)\sigma_a\in V$.
- $W:=$ the vector subspace of V generated by $\{S(q): q\in\frac{1}{r}\mathbb{Z}/\mathbb{Z}\}$.

Let $W^\perp\subseteq V^\vee$ be the orthogonal complement w.r.t. the perfect pairing $V\times V^\vee\longrightarrow\mathbb{C}$.
- $\forall a\in(\mathbb{Z}/r\mathbb{Z})^\times$, let $\lambda_a:=\sigma_a^\vee+\sigma_{-a}^\vee\in V^\vee$

(Note that $\lambda_a(S(q))=B_1(aq)+B_1(-aq)=0\quad\forall q\in\frac{1}{r}\mathbb{Z}/\mathbb{Z}\Rightarrow\lambda_a\in W^\perp$)

Thm. [W-S] $W^\perp$ is generated by $\{\lambda_a \mid a\in(\mathbb{Z}/r\mathbb{Z})^\times\}$ as $\mathbb{C}$-v.s.. In particular, $\dim W^\perp=\dim W=\frac{\phi(r)}{2}$ for $r\geq 3$.

(pf.) Let $\chi:(\mathbb{Z}/r\mathbb{Z})^\times\longrightarrow\mathbb{C}^\times$ be an arbitrary character with conductor f. Using $\mathbb{Z}/f\mathbb{Z}\simeq\frac{1}{f}\mathbb{Z}/\mathbb{Z}\subseteq\mathbb{Q}/\mathbb{Z}$,
$a\longmapsto\frac{a}{f}$

we define $W\ni w_\chi:=\displaystyle\sum_{a\in(\mathbb{Z}/f\mathbb{Z})^\times}\chi(a)S(\tfrac{a}{f})=\sum_{b\in(\mathbb{Z}/f\mathbb{Z})^\times}\Big(\underbrace{\sum_{a\in(\mathbb{Z}/f\mathbb{Z})^\times}\chi(a)B_1(b\cdot\tfrac{a}{f})}\Big)\sigma_b$

coeff. of $\sigma_1=B_{1,\chi}\neq 0$ when χ is odd $\Rightarrow w_\chi\neq 0$

Define the action $\rho:(\mathbb{Z}/r\mathbb{Z})^\times\longrightarrow GL(V)$, then
$c\longmapsto(\sigma_b\mapsto\sigma_{cb})$

$\rho(c)(\omega_\chi) = \sum_a \sum_b \chi(a) B_1(\frac{ab}{r}) \chi_{cb} \xlongequal[b \mapsto c^{-1}b]{a \mapsto ac} \chi(a)\omega_\chi$, i.e. ω_χ lies in the χ-eigenspace of the $(\mathbb{Z}/r\mathbb{Z})^\times$-action.

$\Rightarrow$ all nonzero ω_χ are linearly indep. $\longrightarrow$ $\underline{\{\omega_\chi \mid \chi \text{ is odd character}\}}$ is l.i.d. subset of W.

$\# = \frac{\phi(r)}{2} \rightsquigarrow \dim_{\mathbb{C}} W \geq \frac{\phi(r)}{2}$

On the other hand, $\operatorname{codim}_{\mathbb{C}} W = \dim W^\perp \geq \dim_{\mathbb{C}} \langle \lambda_a : a \in (\mathbb{Z}/r\mathbb{Z})^\times \rangle_{v.s.} = \frac{\phi(r)}{2}$. Hence

$\phi(r) = \dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W + \dim_{\mathbb{C}} W^\perp \underset{\uparrow}{\geq} \frac{\phi(r)}{2} + \frac{\phi(r)}{2} \Rightarrow \dim_{\mathbb{C}} W = \dim_{\mathbb{C}} W^\perp = \frac{1}{2}\phi(r)$ and

"=" holds $\qquad W^\perp = \langle \lambda_a : a \in (\mathbb{Z}/r\mathbb{Z})^\times \rangle_{v.s.}$

Proof of White thm.

- Write $a(k) := \overline{ka_1} + \overline{ka_2} + \overline{ka_3}$ for brevity. Then
  - $\overline{kb} + \overline{(r-k)b} = \begin{cases} r & \text{if } \overline{kb} \neq 0 \\ 0 & \text{if } kb = 0 \end{cases}$
  - $a(k) + a(r-k) = r \cdot \#\{i \mid \overline{ka_i} \neq 0\} \in \{0, r, 2r, 3r\}$
- By assumption, $a(k) > r$ for all $0 < k < r \Rightarrow a(k) + a(r-k) = 3r$

  $\Rightarrow \gcd(a_i, r) = 1 \ \forall i \quad \left(\text{if } d = \gcd > 1, \text{ say } a_i = da_i', \ r = dr' \Rightarrow \underset{=0}{\underline{\overline{r'a_i}}} + \overline{(r-r')a_i} = r \right)$

  Further more, $r < a(k) < 2r \ \forall 0<k<r \Rightarrow a(k) \underset{\forall}{=} \overline{k(a_1+a_2+a_3)} + r$ and thus $\gcd(a_1+a_2+a_3, r) = 1$.

  $\left(\Leftrightarrow \sum (\overline{ka_i} - \frac{r}{2}) = \overline{k(a_1+a_2+a_3)} - \frac{r}{2}\right)$

  $\Rightarrow B_1(a_1 q) + B_1(a_2 q) + B_1(a_3 q) = B_1((a_1+a_2+a_3)q) \quad \forall q \in \frac{1}{r}\mathbb{Z}/\mathbb{Z}$.
- Consider $\mu = \sigma_{a_1}^\vee + \sigma_{a_2}^\vee + \sigma_{a_3}^\vee - \sigma_{a_1+a_2+a_3}^\vee \in V^\vee$, then $\mu(S(q)) = \sum B_1(a_i q) - B_1(q \sum a_i) = 0 \quad \forall q \in \frac{1}{r}\mathbb{Z}/\mathbb{Z}$.

  $\Rightarrow \mu \in W^\perp = \langle \sigma_a^\vee + \sigma_{-a}^\vee ; a \in (\mathbb{Z}/r\mathbb{Z})^\times \rangle \rightsquigarrow \sum a_i \equiv a_k \pmod r$ for some k.

Thm. (Terminal lemma) [YPG, (5.2)]

$(0 \in X)$: a cyclic quotient 3-fold singularity. It is terminal $\Leftrightarrow$ it is of type $\frac{1}{r}(1,-1,b)$ for some $\gcd(r,b) = 1$ and $\mu_r \curvearrowright \mathbb{C}^3$ is small.

(pf.) $(\Leftarrow)$ By Reid-Tai criterion,

$\mathbb{A}^3/\frac{1}{r}(1,-1,b)$ is terminal $\Leftrightarrow \underbrace{\overline{k} + \overline{-k}}_{= r} + \overline{kb} > r \quad \forall 0<k<r$

$\Leftrightarrow \gcd(r,b) = 1$.

$(\Rightarrow)$ Assume that $(0 \in X) = \mathbb{A}^3/\frac{1}{r}(a_1, a_2, a_3)$ and terminal.

$(0 \in X)$ is terminal 3-fold sing. $\Rightarrow (0 \in X)$: isolated $\Rightarrow \mu_r \curvearrowright \mathbb{C}^3$ is free outside 0

(i.e. the action is small)

$\Rightarrow$ all a_i are coprime to r

By Reid-Tai criterion and White thm., $r \mid a_1+a_2$ after permutation.
Take $a \in \mathbb{Z}$ s.t. $\overline{aa_1} = 1$, $(0 \in X)$ becomes of type $\frac{1}{r}(\overline{aa_1}, \overline{aa_2}, \overline{aa_3}) = \frac{1}{r}(1,-1,b)$ with $\gcd(b,r)=1$

§ Terminal singularities of higher index / $\mathbb{C}$

[CAT] : M. Kawakita. Complex algebraic threefolds (2023/09/19)

Notation. $\mathfrak{m} \in \mathcal{O}_{\mathbb{A}^4,0} = \mathbb{C}[x_1, x_2, x_3, x_4]$ the maximal ideal.

$g(x_3, x_4) \in \mathfrak{m}^2$ means $g \in \langle x_3, x_4 \rangle^2 \triangleleft \mathbb{C}[x_3, x_4] \hookrightarrow \mathcal{O}_{\mathbb{A}^4,0}$.

Write $f = f_2 + f_3 + \cdots$, where f_2 = the quadratic part of f

f_3 = the cubic part of f

We say $x_1 x_2 \in f$ if $x_1 x_2$ appear as monoimal part with nonzero coefficient in f.

[Mori 85] S. Mori, On 3-dimensional terminal singularities, 1985

(Thm. 12, 23, 25, Rmk. 12.2, 23.1, 25.1)

[YPG, (6.1)]

[CAT, Thm. 2.4.8]

[KM, Thm. 5.4.3]

Thm. (Mori)

$(o \in Y) \subseteq (o \in \mathbb{A}^4)$: a cDV singularity (not smooth) defined by $f = 0$.

An action $\mu_r \curvearrowright (o \in Y) \subseteq (o \in \mathbb{A}^4)$ with $wt(x_i) = (a_i)$ (i.e. $\mu_r \ni \varepsilon : x_i \longmapsto \varepsilon^{a_i} x_i$)

If $(o \in X) := (o \in Y/\mu_r) \subseteq \mathbb{A}^4/\frac{1}{r}(a_1, \cdots, a_4)$ is terminal singularity, then one of following holds after changing expression of type $\frac{1}{r}(a_1, \cdots, a_4)$ and orbifold coordinate x_1, x_2, x_3, x_4 :

	Name	index	Type of action	f: semi-inv. function	condition
(1)	cA/r	r	$\frac{1}{r}(1, -1, 0, b)$	$x_1 x_2 + g(x_3, x_4^r)$	$g \in \mathfrak{m}^2$, $\gcd(b, r) = 1$
(2)	$cAx/4$	4	$\frac{1}{4}(1, 3, 2, 1)$	$x_1^2 + x_2^2 + g(x_3, x_4^2)$	$g \in \mathfrak{m}^2$
(3)	$cAx/2$	2	$\frac{1}{2}(1, 0, 1, 1)$	$x_1^2 + x_2^2 + g(x_3, x_4)$	$g \in \mathfrak{m}^4$
(4)	$cD/3$	3	$\frac{1}{3}(0, 1, 2, 2)$	$x_1^2 + g(x_2, x_3, x_4)$	$g \in \mathfrak{m}^3$, $g_3 = x_2^3 + x_3^3 + x_4^3$ or $x_2^3 + x_3 x_4^2$
(5)	$cD/2$	2	$\frac{1}{2}(1, 1, 0, 1)$	$x_1^2 + g(x_2, x_3, x_4)$	$g \in \mathfrak{m}^3$, $x_2 x_3 x_4$ or $x_2^2 x_3 \in g$
(6)	$cE/2$	2	$\frac{1}{2}(1, 0, 1, 1)$	$x_1^2 + x_2^3 + x_2 g_3(x_3, x_4) + h(x_3, x_4)$	$g \in \mathfrak{m}^4$, $h \in \mathfrak{m}^4 \setminus \mathfrak{m}^5$ (i.e. $h_4 \neq 0$)

Rmk. (1) should be considered as the main series and (2) ~ (6) as the "exceptional" case.

We have an analogy of Reid-Tai's criterion for "the terminal hyperquotient"
[Mori 85, Thm 2], [CAT, Prop. 2.4.5], [YPG, (4.6)], [Ishii, Thm. 8.3.12]

Prop. (Rule I) $0 \in A := \mathbb{A}^n / \frac{1}{r}(a_1, \cdots, a_n)$ with orbi. coordinate $x_1, \cdots, x_n$.
$(0 \in X)$: the analytic subspace of A defined by a semi-inv. function $f \in \mathcal{O}_{\mathbb{A}^n, 0}$.
Suppose that $\mu_r \curvearrowright (f=0)$ freely in codim 1. $N = \mathbb{Z}^n + \frac{1}{r}(a_1, \cdots, a_n)\mathbb{Z}$, $\sigma = \mathrm{Cone}(e_1, \cdots, e_n) \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$
$\qquad (\mathbb{R}_{\geq 0})^n \subseteq \sigma$, $\mathbb{R}^n \cong N \otimes_{\mathbb{Z}} \mathbb{R}$

If $(0 \in X)$ is terminal (resp. canonical), then $\forall \frac{1}{r}(b_1, \cdots, b_n) \in N \cap \sigma$ with $\#\{i \mid b_i > 0\} \geq 3$,
the weighted order $W\text{-}\mathrm{ord}(f)$ of f w.r.t. $\mathrm{wt}(x_1, \cdots, x_n) = (b_1, \cdots, b_n)$ satisfies

$$W\text{-}\mathrm{ord}(f) + r < \sum b_i$$

(resp. $\leq$)

Rmk. In Reid's notation ([YPG, p.372] or [Ishii, Def. 8.3.10]), set $\alpha = \frac{1}{r}(b_1, \cdots, b_r) \in N \cap \sigma$,

$$\alpha\left(\prod x_i\right) = \frac{1}{r}\sum b_i, \quad \alpha(f) = \frac{1}{r} W\text{-}\mathrm{ord}(f),$$

where $\alpha(g) = \min\{\langle \alpha, m\rangle \mid x^m \in g\}$, $g \in \mathbb{C}[\sigma^\vee \cap M]$, $\langle\ ,\ \rangle : N \times M \longrightarrow \mathbb{Z}$.

We shall demonstrate Mori's thm. in the case when $(0 \in Y)$ (terminal) is an isolated cA sing. (cf. [Mori. 85, §2]).
$\mu_r \curvearrowright (f=0) \subseteq (0 \in \mathbb{A}^4)$ is free outside 0
$\Rightarrow$ Rule II [YPG, P.374]

(R_i) If $d = \gcd(a_i, r) > 1$, then some power of $x_i \in f$.
(The action of some element of μ_r fixes the x_i-axis pointwise
i.e. $a_i = a_i' d$, $r = r'd$, $0 < r' < r \rightsquigarrow \varepsilon^{r'} : x_i \longmapsto \varepsilon^{a_i r'} x_i = x_i$)
(R_{ij}) $\gcd(a_i, a_j, r) = 1$ for distinct $1 \leq i, j \leq 4$.
(Otherwise, the action is not free on $(f=0) \cap x_i x_j$-plane which have $\dim > 0$ at origion $0 \in \mathbb{A}^4$)

Rmk. (orbifold coordinate change) [CAT, Rmk 2.1.6]
If σ is a coordinate change of $(0 \in \mathbb{A}^n)$, i.e. $\sigma \in \mathrm{Aut}(\mathcal{O}_{\mathbb{A}^n, 0})$, then $\frac{1}{r}\sum_{i=0}^{r-1} \tau^{-i} \sigma \tau^i$ is an orbifold
coordinate change of $\mu_r \curvearrowright \mathbb{A}^n$, where $\tau \in \mathrm{Aut}(\mathcal{O}_{\mathbb{A}^n, 0})$ given by a generator of μ_r.

$\because$ $(0 \in Y) = (f=0)$ is sing. cA $\Rightarrow$ $f \in \mathfrak{m}^2$ and the f_2 has rank ≥ 2.
Lemma. After orbitfold coordinate change, either
(i) $f = x_1 x_2 + g(x_3, x_4)$
(ii) $f = x_1^2 + x_2^2 + g(x_3, x_4)$ with $x_3 x_4 \notin g_2$

diff. type if $a_1 \neq a_2$

(pf.) Claim. $f_2 \overset{(i)}{=} x_1x_2 + g_2(x_3,x_4)$

$\overset{(ii)}{=} x_1^2+x_2^2+h_2(x_3,x_4)$ with $x_3x_4 \notin h_2$

(The lemma follows from the claim and Tougeron's implicit function theorem (c.f. [CAT, 2.1.4~2.1.7]))

(subpf.) If $x_i^2 \in f$, then one can elimate the linear term in x_i. After orbifold coordinate change, write $f_2 = P_2(x_1,\cdots,x_i) + x_{i+1}^2+\cdots+x_4^2$ for some $0\le i\le 4$, so that $x_l^2 \notin P_2$ $\forall l=1\sim 4$.

- $P_2=0 \Rightarrow f_2=\sum x_i^2$, get (ii).
- $P_2\neq 0$, may assume $x_1x_2\in P_2$ and thus $i\ge 2$.

$\Rightarrow f_2 = (\text{unit})(x_1x_2 + a(x_3,x_4)x_1 + b(x_3,x_4)x_2 + c(x_3,x_4))$

$= (\text{unit})(\underbrace{(x_1+b)}_{}\underbrace{(x_2+a)}_{\text{new coord.}} + \underbrace{c-ab}_{g})$, get (i).

Proof of Mori's thm. in the case (i) of Lemma ($f = x_1x_2+g(x_3,x_4)$).

Goal. We shall derive the cA/r or $cAx/4$

For $0<k<r$, $w\text{-ord}_k(f) :=$ the weighted order of f wrt. $wt(x_1,\cdots,x_4) = (\overline{ka_1},\cdots,\overline{ka_4})$.

By (R_{ii}), $\#\{i \mid \overline{ka_i}>0\}\ge 3$. By Rule I, $w\text{-ord}_k(f)+r \overset{(*)}{<} \sum \overline{ka_i}$ $\forall\, 0<k<r$

Step. 1. Let $p := \gcd(a_1+a_2, r)$. x_1^n and $x_2^n \notin f \overset{(R_i)}{\Rightarrow} \gcd(a_i,r)=1$ for $i=1,2$.

- $p\mid a_3$ or $p\mid a_4$ (WLOG $p\mid a_3$):

Otherwise, $0<s:=\frac{r}{p}<r$ and all $\overline{sa_i}>0$. WLOG $\overline{sa_3}+\overline{sa_4}\le r$ $\left(\because \overline{-\alpha}+\overline{-\beta} = 2r-(\overline{\alpha}+\overline{\beta})\right.$
$(p\ge 2)$ $\left.\text{by change the character } \chi \text{ to } \chi^{-1}\right)$

Since $p\mid a_1+a_2$, $\overset{0<}{\overline{sa_1}+\overline{sa_2}} \overset{<2r}{=} \underset{s(a_1+a_2)\equiv 0\ (\text{mod } r)}{r}$. By $(*)$, $w\text{-ord}_s(f)+r<\sum\overline{sa_i}\le r+r \Rightarrow w\text{-ord}_s(f)<r$.

However, $w\text{-ord}_s(f)\equiv w\text{-ord}_s(x_1x_2) = \overline{sa_1}+\overline{sa_2}\equiv 0\ (\text{mod } r)$ $\rightarrow\leftarrow$.

- $\gcd(a_4,r)=1$: Since $p\mid a_3$, $p\mid r$, and $\gcd(a_3,a_4,r)=1$, we have $\gcd(a_4,p)=1$.

If $q=\gcd(a_4,r)>1$, then $\exists\, l\in\mathbb{N}$ s.t. $x_4^l\in f$ by $(R_i) \Rightarrow \underset{la_4}{wt(x_4^l)} \equiv \underset{a_1+a_2}{wt(x_1x_2)}\ (\text{mod } r)$

$\Rightarrow 0\equiv a_1+a_2\ (\text{mod } q) \Rightarrow q\mid \gcd(a_1+a_2,r)=p \Rightarrow \gcd(a_4,p)>1$ $\rightarrow\leftarrow$.

- $\gcd(a_3,r)=p$: Write $\gcd(a_3,r)=pg$ with $g\ge 2 \overset{(R_i)}{\Rightarrow}$ some power of $x_3\in f \Rightarrow pg\mid a_1+a_2$.
$(x_3^{l'}\in f \ni x_1x_2 \Rightarrow l'a_3\equiv a_1+a_2\ (r)\ \&\ pg\mid r)$

$\Rightarrow pg\mid\gcd(a_1+a_2)=p$ $\rightarrow\leftarrow$.

- $\because a_1,a_2,a_4$ are coprime to r and $p=\gcd(a_1+a_2,r)=\gcd(a_3,r)\mid a_3$

$\Rightarrow$ One can write $f = x_1x_2+g(x_3,x_4^p)$, $g\in\mathfrak{m}^2$

$(g(x_3,x_4)=g_1(x_3,x_4^p)$, replacing g by $g_1)$

• If $p=r$, then $(a_1,a_2,a_3,a_4)\equiv(a_1,-a_1,0,a_4)\pmod r$, where a_1,a_4 are coprime to r.
We get cA/r case.

Step2. Now we shall assume that $p<r$.

• If $k(a_1+a_2)\equiv\pm p \pmod r$ for some $0<k<r$ [Mori 85, (8.3)], then $\overline{ka_i}>0\ \forall i$:

$\because k(a_1+a_2)\equiv\pm p\not\equiv 0\pmod r \Rightarrow k\not\equiv 0 \pmod r \rightsquigarrow k\cdot\frac{a_1+a_2}{p}\equiv\pm1 \pmod{\frac{r}{p}} \Rightarrow \gcd(k,\frac{r}{p})=1$.

$\because a_1,a_2,a_4$ coprime to r $\Rightarrow ka_i\not\equiv 0\pmod r$ $i=1,2,4$

$p=\gcd(a_3,r)$ $\gcd(\frac{k}{p},r)=1 \Rightarrow ka_3\not\equiv 0\pmod r$

• Let $\mathcal{S}:=\left\{k\in\{1,\cdots,r-1\} \;\middle|\; \begin{array}{l} a.\ k(a_1+a_2)\equiv\pm p \pmod r \\ b.\ \overline{ka_3}+\overline{ka_4}\le r\end{array}\right\}$, then $\#\mathcal{S}\ge\begin{cases} p & \text{if } r>2p \\ p/2 & \text{if } r=2p\end{cases}$:

# of solution k of the linear congruence a. $=\begin{cases} 2p & \text{if } r>2p \\ p & \text{if } r=2p\end{cases}$.

For any such k, either k or $r-k$ satifies b..

• $\overline{ka_1}+\overline{ka_2}=w\text{-ord}_k(f)+r$ for $k\in\mathcal{S}$ ($\Rightarrow w\text{-ord}_k(f)<r$):

$\because w\text{-ord}_k(f)+r\le\sum_{i=1}^{4}\overline{ka_i}\overset{b.}{\le}\overline{ka_1}+\overline{ka_2}+r \Rightarrow w\text{-ord}_k(f)<\overline{ka_1}+\overline{ka_2}<2r$.

$\because w\text{-ord}_k(f)\equiv\overline{ka_1}+\overline{ka_2}\pmod r \Rightarrow \overline{ka_1}+\overline{ka_2}-w\text{-ord}_k(f)=r$.

• For $k\in\mathcal{S}$, $w\text{-ord}_k(f)\equiv\pm p\pmod r \Rightarrow w\text{-ord}_k(f)=p$ or $k-p$.

Let $\mathcal{S}_1:=\{k\in\mathcal{S} : w\text{-ord}_k(f)=p\}$
$\mathcal{S}_2:=\{k\in\mathcal{S} : w\text{-ord}_k(f)=r-p,\ r>2p\}$ $\Rightarrow \overline{ka_1}+\overline{ka_2}=\begin{cases} r+p & \text{if } k\in\mathcal{S}_1 \\ 2r-p & \text{if } k\in\mathcal{S}_2\end{cases}$

• $\#\mathcal{S}_1\le1$, $\#\mathcal{S}_2\le p-1$:

•• If $k\in\mathcal{S}_1$, then $w\text{-ord}_k(f)=p \underset{?}{\Rightarrow} x_4^p\in g$ with $\overline{ka_4}=w\text{-ord}_k(a_4)=1$, i.e. there is

($f=x_1x_2+g(x_3,x_4^p)$, $r+p\overset{\wedge}{=}\overline{ka_1}+\overline{ka_2} \Rightarrow \text{ord}_k(g(x_3,x_4^p))=p$, but $g\in m^2$ and $a_3\ge p$)

at most one $0<k<r$ s.t. $ka_4\equiv1\pmod r$.

•• $k\in\mathcal{S}_2 \Rightarrow \overline{ka_1}+\overline{ka_2}=2r-p \Rightarrow r-p\le\overline{ka_1}<r \underset{?}{\overset{\gcd(a_1,r)=1}{\Rightarrow}} \#\mathcal{S}_2\le\#((r-p,r)\cap\mathbb{N})=p-1$

($\overline{ka_1}=\overline{k'a_1} \Rightarrow r\mid(k-k')\cdot a_1 \Rightarrow r\mid k-k'$)

Step3.

• If $r=2p$, then $\frac{p}{2}\le\#\mathcal{S}=\#\mathcal{S}_1\le1 \Rightarrow p=2$ & $r=4$.

One can set $(a_1,a_2,a_3,a_4)=(1,1,2,a_4)$, where $a_4=1$ or 3.

By (✱), $k=1$, $w\text{-ord}_1(f)+4<\sum_{i=1}^{4}a_i=4+a_4 \Rightarrow a_4=3$.

$\because \mathcal{S}_1\ne\emptyset \Rightarrow x_4^2\in g$. Thus after orbfold coord. change (cf. [Mori. 85. Supplement 8.]), one can write $f=4x_1x_2+x_3^n+x_4^2=(x_1+x_2)^2+x_4^2+x_3^n-(x_1-x_2)^2$ (odd $n\ge3$) and the case $cAx/4$ holds for the orbfold coord. $x_1+x_2,x_4,x_3,\sqrt{-1}(x_1-x_2)$.

• Assume $r>2p$ (known $p\ge2$) $\Rightarrow p\le\#\mathcal{S}=\#\mathcal{S}_1+\#\mathcal{S}_2\le1+p-1$

$$\Rightarrow \mathcal{S}_2 \overset{(1\text{-}1)}{\longleftrightarrow} \{r-p+1,r-p+2,\cdots,r-1\}$$
$$k \longmapsto \overline{ka_1}$$

$\because \gcd(a_1,r)=1 \Rightarrow \forall i=1,2,\ \exists\, 0<k_i<r$ s.t. $k_i a_1 = r-i$.

$\Rightarrow 2k_1a_1 - k_2a_1 \equiv 2(r-1)-(r-2) \equiv 0 \pmod r \Rightarrow 2k_1 \equiv k_2 \pmod r$.

∴ If $p>2$, the bijective $(\div)$ implies $k_1,k_2 \in \mathcal{S}_2$. By definition of $\mathcal{S}_2$, $\overline{k_i a_2} = r-p+i$, $i=1,2$.
But then $2(r-p+1) = 2k_1a_2 \equiv k_2a_2 = (r-p+2) \pmod r \Rightarrow p \equiv 0 \pmod r$ $\rightarrow\leftarrow$ with $p<r$.

∴ When $p=2$, we have $\#\mathcal{S}_1 = \#\mathcal{S}_2 = 1$, say $\mathcal{S}_i = \{k_i\}$, $i=1,2$. Then $\overline{k_2a_1} = r-1$ and $\overline{k_2a_2} = r-1$.
$(2|r)$ $\quad (\Rightarrow a_1 = a_2 \ \because kx \equiv -1 \pmod r\ \exists!$ sol.)

$\because \mathcal{S}_1 = \{k_1\} \Rightarrow \overline{k_1a_1} + \overline{k_1a_2} = r+2 \Rightarrow \overline{k_1a_1} = \overline{k_1a_2} = \frac{r}{2}+1 \Rightarrow 2k_1a_1 \equiv 2k_1a_2 \equiv 2 \pmod r$.

Also, by $w\text{-}\mathrm{ord}_{k_1}(f) = 2 \Rightarrow x_4^2 \in g$ with $\overline{k_1a_4} = w\text{-}\mathrm{ord}_{k_1}(a_4) = 1$.

Then $(\overline{2k_1a_1}, \overline{2k_1a_2}, \overline{2k_1a_4}) = (2,2,2) \Rightarrow w\text{-}\mathrm{ord}_{2k_1}(x_1x_2) = 4$. By (σ),

$$4+r \underset{\uparrow}{\leq} w\text{-}\mathrm{ord}_{2k_1}(f) + r < \sum_{i=1}^{4} \overline{2k_1a_i} = 6 + \overline{2k_1a_3} \Rightarrow r-2 < \overline{2k_1a_3} < r \Rightarrow \underset{\text{even}}{\overline{2k_1a_3}} = \underset{\text{odd}}{r-1}$$

$(f = x_1x_2 + g(x_3,x_4^2)$ and $2|a_3)$

Rmk. By the above proof, we have seen in the case (i), $(o\in X)$ is cA/r or $cAx/4$,
and $\underline{x_4^2 \in f}$ if $(o\in X)$ is $cAx/4$.
$\Leftrightarrow \mathrm{rk} f_2 \geq 3$ (x_1^2, x_3^2, x_4^2)

Proof of Mori's thm in the case (ii), $f = x_1^2 + x_2^2 + g(x_3,x_4)$.
We shall derive the $cA/2$, $cAx/4$, $cAx/2$ with $x_3x_4 \notin g$.

- Claim: r is a power of 2 (cf. [Mori 85, (9.1)]).

(subpf.) Otherwise, $\exists$ odd $s|r \xRightarrow[\text{[KM, Prop 5.20 (3)]}]{} Y/\mu_s$ is also terminal.

$$Y/\mu_s \xrightarrow{\text{crepant}} Y/\mu_r = X : \text{term.}$$

WLOG r is odd by taking Y/μ_s instead of Y/μ_r. Since $x_1^2+x_2^2$ is semi-invariant,
$\underset{2a_1}{wt(x_1^2)} \equiv wt(f) \equiv \underset{2a_2}{wt(x_2^2)} \pmod r \underset{r:\text{odd}}{\Rightarrow} a_1 = a_2$. Hence $f = \underbrace{(x_1+\sqrt{-1}x_2)(x_1-\sqrt{-1}x_2)}_{\text{new orb. coord.}} + g(x_3,x_4)$

One can make f as in the case (i). $\because r$ is odd $\Rightarrow cA/r$ case, it is of type $\frac{1}{r}(1,-1,0,b) = \frac{1}{r}(a_1,-a_1,0,1)$

i.e. $\begin{cases} \gcd(a_1,r)=1 \\ a_2 \equiv -a_1 \pmod r \end{cases} \overset{a_1=a_2}{\Rightarrow} r|2a_1 \overset{\gcd(a_1,r)=1}{\Rightarrow} r|2$ i.e. $r=2$ $\rightarrow\leftarrow$.

- Claim: $r=2$ or 4.

(subpf.) Otherwise, taking Y/μ_8. WLOG $r=8 \overset{x_1^2+x_2^2 \text{ is semi-inv.}}{\Rightarrow} 2a_1 \equiv 2a_2 \pmod 8 \Rightarrow a_1 \equiv a_2 \pmod 4$.

One can apply to Y/μ_4 the result of the case (i) after replacing x_1, x_2 by $x_1+\sqrt{-1}x_2, x_1-\sqrt{-1}x_2$.

The case $cA/4$ never occurs since $\begin{cases} \gcd(a_1,4)=1 \\ a_1 \equiv a_2 \equiv -a_1 \pmod 4 \end{cases} \Rightarrow 2 \mid a_1 \Rightarrow 2 \mid \gcd(a_1,4)=1 \rightarrow\leftarrow$.

$Y \to Y/\mu_4 \to Y/\mu_8$

So $cAx/4$ would occur and $\mathrm{rk} f_2 \geq 3$, say $x_1^2, x_2^2, x_4^2 \in f \Rightarrow 2a_1 \equiv 2a_2 \equiv 2a_4 \pmod 8$

$\Rightarrow a_1 \equiv a_2 \equiv a_4 \pmod 4$

$\overset{(a_i,4)=1,\ i=1,2,4}{\Rightarrow} a_1 = a_2$ after permutation

- $r=4$: If $\underset{\text{is satified after permutation whenever } \mathrm{rk} f_2 \geq 3.}{a_1=a_2}$, one can apply Rmk as above derive the case $cAx/4$.

So we have only to deal with the case $f_2 = x_1^2+x_2^2$ with $a_1 \neq a_2$.

$\Rightarrow a_1 \equiv a_2 \pmod 2$ and $\gcd(a_1,a_2,a_4)=1$ by (R_{ii}). We may assume $(a_1,a_2)=(1,3)$.

Apply to Y/μ_2 $= cA/2$ the result of the case (i), we see that it is of type $\frac{1}{2}(1,1,0,b)$ with $2 \nmid b$, and thus exactly one of a_3, a_4 is even, say $2 \mid a_3$. Since $\gcd(a_3, \overset{4}{r})>1$, by $(R_i) \Rightarrow x_3^\ell \in f$ for some $\ell \in \mathbb{N} \Rightarrow wt(x_3^\ell) \equiv wt(x_1^2) \pmod 4$, i.e. $\ell \cdot \frac{a_3}{2} \equiv 1 \pmod 2 \Rightarrow \frac{a_3}{2}$ is odd $\Rightarrow a_3 = 2$.

Possibly permuting x_3 and x_4 and changing a_i with $4-a_i$ (depend on $a_4 = 1$ or 3), we can take $f_2 = x_1^2+x_2^2$ and $(a_1,a_2,a_3,a_4)=(1,3,2,1)$ which is $cAx/4$.

- $r=2$:
  - If $a_1=a_2 \rightsquigarrow cA/2$ by Rmk.
  - If $f_2 = x_1^2+x_2^2$ with $a_1 \neq a_2$. WLOG $(a_1,a_2)=(1,0) \overset{(\text{by } R_{ii})}{\Rightarrow} \gcd(a_2,a_j,r)=1$ for $j=3,4 \Rightarrow a_3=a_4=1$, which is $cAx/2$.

§ General elephant

Def. (Reid) [YPG]

X : normal var.. The element elephant of X is the general member of $-K_X$.

(General elephant conjecture / Question) [YPG, (6.5) Problem (1)]

X : terminal 3-fold s.t. $-K_X$ is ample. Does the general elephant of X have Du Val singularity?

Rmk. The "conj." holds if X is Gorenstein by Reid's Thm. (Thm D on [KM, Thm 5.14])

Thm. [KM, Thm. 5.4.3, table]

Under the same notation and assumption in Mori's thm,

$$\begin{array}{c} (f=0) = (0\in Y) \subseteq (0\in \mathbb{A}^4) \\ \downarrow \pi \\ (0\in H) \subseteq (0\in X) \subseteq (0\in \mathbb{A}^4/\tfrac{1}{r}(a_1,\cdots,a_4)) \end{array}$$

Then

(1) $(0\in X)$ is terminal $\Leftrightarrow$ the general elephant $(0\in H)$ of X is Du Val.

(2) $\widetilde{H} := \pi^{-1}H = \pi^* H$

name	Type of $\widetilde{H} \to H$	condition
cA/r	$A_{k-1} \xrightarrow{r:1} A_{kr-1}$	$k = \mathrm{mult}_0\, g(x_3,0)$, $f = x_1x_2 + g(x_3,x_4)$
$cAx/4$	$A_{2k-2} \xrightarrow{4:1} D_{2k+1}$	$2k-1 = \mathrm{mult}_0\, g(x_3,0)$
$cAx/2$	$A_{2k-1} \xrightarrow{2:1} D_{k+2}$	$2k = \mathrm{mult}_0\, g(x_3,0)$
$cD/3$	$D_4 \xrightarrow{3:1} E_6$	
$cD/2$	$D_{k+1} \xrightarrow{2:1} D_{2k}$	$k = \mathrm{mult}_0\, g(0,x_3,0)$
$cE/2$	$E_6 \xrightarrow{2:1} E_7$	

a complete list of all possible cyclic covers between Du Val sing, which are unramified outside 0

(3) [KSB] J. Kollar and N.I. Shepherd-Barron,

(Thm 6.5) Threefolds and deformations of surface singularities, 1998.

$\mu_r \curvearrowright (0\in Y) = (f=0)$: an isolated cDV sing. given by one of f in Mori's thm with
$\quad\quad\quad\quad\quad \text{in } (0\in\mathbb{A}^4)$

action of μ_r. If $\mu_r \curvearrowright (f=0)$ freely outside 0, the $(0\in Y/\mu_r)$ is terminal.

Rmk. $\mathbb{Q}$-smoothing, [CAT, Lemma 2.5.21]

$$(0\in X) = (0\in Y/\mu_r) \subseteq \left(0\in \mathbb{A}^4/\tfrac{1}{r}(a_1,\cdots,a_4)\right) \overset{\text{deform.}}{\rightsquigarrow} \mathbb{A}^3/\tfrac{1}{r}(a,-a,1)$$

Proof of Thm.

- (1) $\Rightarrow$ and (2): Use Mori's classification, [CAT, Thm. 2.4.15], [YPG, p.393].
- (1) $\Leftarrow$ and (3): For such f one see that $(0\in H)$ is canonical.

  $\because$ H is Cartier $\Rightarrow$ (X, H) is canonical by inverse of adjunction.

  (Use same pf. of [KM, p.166 (5.3)] or [Kollar, Sing of MMP] Prop. 2.56, p.76)

  Hence the isolated sing. $(0\in X)$ is a isolate cDV point.

§ Flips and Flops

Example (Francia's flips) (cf. [KM, Example 2.7])

[Fr 80] P. Francia, Some remarks on minimal models I, 1980.

 $= Q = \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$c = \frac{x}{v} = \frac{u}{y}$ $\qquad\qquad$ $c^+ = \frac{x}{u} = \frac{v}{y}$

$$\begin{tikzcd}
 & Q \arrow[dl] \arrow[dr] \arrow[d, phantom, "\cap" rotate=90] & \\
\mathbb{P}^1 \simeq C \arrow[d, phantom, "\cap" rotate=90] & \mathrm{Bl}_0 Y = \widetilde{X} \arrow[dl] \arrow[dr] & C^+ \simeq \mathbb{P}^1 \arrow[d, phantom, "\cap" rotate=90] \\
\mathrm{Bl}_{(x=v=0)} Y = X \arrow[rr, dashed, "\text{Atiyah flop}"] \arrow[dr] & & \mathrm{Bl}_{(x=u=0)} Y = X^+ \arrow[dl] \\
 & Y = (xy - uv = 0) \subseteq \mathbb{A}^4 &
\end{tikzcd}$$

$\xrightarrow[\text{cf. [KM, 7.7.2]}]{/\mu_r}$

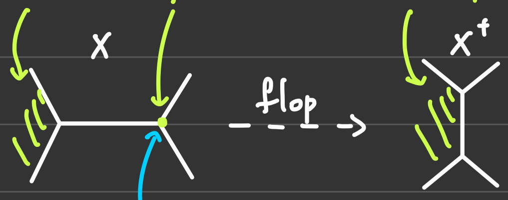
$$\begin{tikzcd}
 & \widetilde{X}_r \arrow[dl] \arrow[dr] & \\
C_r \subseteq X_r \arrow[dr] & & X_r^+ \supseteq C_r^+ \simeq \mathbb{P}^1 \arrow[dl] \\
 & Y_r = Y/\mu_r &
\end{tikzcd}$$

sing. pt $\frac{1}{r}(1,-1,1)$ $\qquad$ sm.

$\mu_r \curvearrowright Y \subseteq \mathbb{A}^4$

$\varepsilon : (x,y,u,v) \longmapsto (\varepsilon x, y, \varepsilon u, v)$

the codim ≥ 2 part of the fixed point set $= \emptyset$



$\longrightarrow \left\{ \begin{array}{l} \text{singularities} \\ \text{sign of } K \end{array} \right.$

$p =$ isolated fixed point $(y=v=0)$

$\frac{1}{r}(1,-1,1)$, local coord. $(x, v' = vx^{-1}, u)$, $\varepsilon : (x, v', u) \longmapsto (\varepsilon x, \varepsilon^{-1} v', \varepsilon u)$

$C = v'$-axis.

$\mu_r \curvearrowright -K_X$, $\quad \sigma = (v'^{r-1} - x)(dx \wedge dv' \wedge du)^{-1}$

$\quad \{ \mu_r$-inv.

$-K_{X_r}$, $\qquad$ local section σ_r of $-K_{X_r}$

(General elephant conj. holds)

$\underset{|-K_{X_r}|}{S_r \in} = (\sigma_r = 0) \simeq (v,u)\text{-plane}/\mu_r$, which is a Du-Val sing. of type A_{n-1}.

Rmk. · Historically, this example intended to claim that "the MMP is impossible in dim ≥ 3", but later it was included into the development of MMP, and become the simplest examples of flips.

· These examples are flips of toric varieties [CLS] § 15.4

[Rei 83a] Decomp. of toric morphism

· the general elephant conj. $\longrightarrow$ the existence of 3-fold flip.

(on an irr. extremal neighborhood)

see [CAT, Def 4.4.6] or [Kol 91b, §2.1]

[Mori 88] J.AMS

[Kau 88] Ann. of Math

[KM 92] J.AMS, Classification of flips

Def. (This is the algebraic analogue topological surgery)

X: normal var. (resp. cpx. analytic space), $D \in \mathrm{WDiv}(X)_{\mathbb{Q}}$ s.t. $K_X + D$ is $\mathbb{Q}$-Cartier.

(D: not necessarily effective)

- [KM, Def 3.33] A flipping contraction (X, D) is $f: X \longrightarrow Y$ proper birational morphism between normal varieties (resp. cpx. analytic space) s.t. $\begin{cases} \text{"}\mathrm{codim}_X \mathrm{Exc}(f) \geq 2\text{"} & (f \text{ is called a small contr.}) \\ -(K_X + D) \text{ is } f\text{-ample} \end{cases}$

  (In [KM], it called (K_X+D)-flipping contr.)

  $(\rightsquigarrow f_* \mathcal{O}_X = \mathcal{O}_Y)$

  (later will assume $\rho(X/Y) = 1$)

- A flip of f (wrt. (X, D)) is $f^+ : X^+ \longrightarrow Y$ proper birat. mor. between normal var (resp. cpx an. space) s.t. $\begin{cases} K_{X^+} + D^+ \text{ is } \mathbb{Q}\text{-Cartier, where } D^+ \text{ is strict transform of } D \text{ on } X^+. \\ K_{X^+} + D^+ \text{ is } f^+\text{-ample} \\ \mathrm{codim}_{X^+} \mathrm{Exc}(f^+) \geq 2 \end{cases}$

  (ϕ is called flip for (X, D))

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

$-(K_X+D)$: f-ample $\qquad (K_{X^+}+D^+)$: f^+-ample

- A flopping contraction for (X, D) is a flipping contraction $f: X \longrightarrow Y$ for another (X, B) s.t. $K_X + D \equiv_f 0$. We call $f^+ : X^+ \longrightarrow Y$ or $\phi: X \dashrightarrow X'$ a flop of (X, D).

  $\begin{pmatrix} B \in \mathrm{WDiv}(X)_{\mathbb{Q}} \\ K_X + B \text{ is } \mathbb{Q}\text{-Cart.} \end{pmatrix}$

  (f^+: a flip of (X, B))

Rmk. • The terminology in the literature is NOT uniform.

D-flop in [KM, Def 6.10] = a flop of $(X, \Delta = 0)$ which is a flip of (X, D).

- [Kaw 08] Flops connect minimal model
- Y is NOT $\mathbb{Q}$-factorial

  (Let $\Delta_Y = f_* D$. If $K_Y + \Delta_Y$ is $\mathbb{Q}$-Cartier, then $f^*(K_Y + \Delta_Y) = K_X + D$ is relative trivial, but K_X is rel. ample)

Lemma. (cf. [CAT, Lemma 1.5.20])

X: variety, $\mathcal{R} = \bigoplus_{i \geq 0} \mathcal{R}_i$: a sheaf of graded $\mathcal{O}_X$-alg. with $\mathcal{R}_0 = \mathcal{O}_X$.

(1) $\mathcal{R}$ is f.g. graded $\mathcal{O}_X$-alg. $\Leftrightarrow$ the ideal $\mathcal{R}_+ = \bigoplus_{i > 0} \mathcal{R}_i$ is a f.g. $\mathcal{R}$-module.

(2) Let $d \in \mathbb{N}$. If $\mathcal{R}$ is f.g. graded $\mathcal{O}_X$-alg., then the truncation of $\mathcal{R}$,

$$\mathcal{R}^{(d)} = \bigoplus_{i \geq 0} \mathcal{R}_{id} \quad \text{(Veronese subalg.)}$$

is a f.g. graded $\mathcal{O}_X$-alg. Moreover, the converse holds if $\mathcal{R}$ is an integral domain.

(pf.)

(1) the homo. gen. of graded $\mathcal{O}_X$-alg. $\mathcal{R}$ = the homo. gen. of the ideal $\mathcal{R}_+$.

(2) ($\Rightarrow$) Let $x_1, \cdots, x_t$ be homo. gen. of $\mathcal{R}$ $\Rightarrow$ $\mathcal{R}$ is generated by $\prod x_i^{a_i}$ $(0 \le a_i < d)$ as an $\mathcal{R}^{(d)}$-module. Let $M_j = \bigoplus_{i \ge 0} R_{id+j}$ $(1 \le j \le d)$, which is $\mathcal{R}^{(d)}$-module.

$\because \mathcal{R}_+ = \bigoplus_{j=1}^{d} M_j$ is f.g. $\mathcal{R}$-module $\Rightarrow$ it is f.g. $\mathcal{R}^{(d)}$-module.

$\Rightarrow (\mathcal{R}^{(d)})_+ = M_d$ is f.g. $\mathcal{R}^{(d)}$-module.

$\Rightarrow M_d$ is f.g. $\mathcal{O}_X$-module.

($\Leftarrow$) Suppose that $R^{(d)}$ is f.g. graded $\mathcal{O}_X$-alg. and $\mathcal{R}$ is integral domain.

$\Rightarrow (\mathcal{R}^{(d)})_+$ is f.g. $\mathcal{R}^{(d)}$-module, where $\mathcal{R}^{(d)}$ is Noetherian.

If for some $1 \le j \le d$, $\exists\, 0 \ne x \in M_j$, then $x^{d-1} M_j \hookrightarrow (\mathcal{R}^{(d)})_+$ $\Rightarrow$ M_j is f.g. $\mathcal{R}^{(d)}$-module.

$\Rightarrow \mathcal{R}_+$ is f.g. $\mathcal{R}$ module $\Rightarrow$ $\mathcal{R}$ is f.g. graded $\mathcal{O}_X$-alg.

Lemma. [KM Lemma 6.2] [Kaw 88 Lem. 3.1]

Y: a normal alg. (resp. analytic) variety. $B \in \mathrm{WDiv}(Y)$ not $\mathbb{Q}$-Cartier. TFAE

(1) $R(Y,B) := \bigoplus_{m \ge 0} \mathcal{O}_Y(mB)$ is a f.g. graded $\mathcal{O}_Y$-alg.

(2) $\exists$ a proj. (resp. proper) birational morphism $Z \to Y$ s.t. $\begin{cases} Z \text{ is normal} \\ \operatorname{codim}_X \operatorname{Exc}(g) \ge 2 \\ B' := g_*^{-1} B \text{ is } \mathbb{Q}\text{-Cartier and } g\text{-ample} \end{cases}$

(resp. over a suitable nbd of any cpt. subset of Y)

(pf.)

In the analytic case, we work near a cpt. subset of Y.

· (1) $\Rightarrow$ (2): $\exists\, m_0 \in \mathbb{N}$ st. $R(Y, m_0 B)$ is gen. by $\mathcal{O}_Y(m_0 B)$ over $\mathcal{O}_Y$. Hence, $R(Y, m_0 B) = R(Y,B)^{(m_0)}$ is also f.g. by Lemma. WLOG, $\mathcal{O}_Y(B)$ generates the graded alg. $R(Y,B)$.

Set $Z := \operatorname{Proj}_Y R(Y,B) \xrightarrow{g} Y$, then $\begin{cases} g_* \mathcal{O}_Z(m) = \mathcal{O}_Y(mD) \\ \mathcal{O}_Z(1) \text{ is } g\text{-ample} \end{cases}$.

∵ $\operatorname{codim}_Z \operatorname{Exc}(g) \ge 2$: If not, $\exists$ a prime divisor $E \le \operatorname{Exc}(g)$, then $0 \to \mathcal{O}_Z \to \mathcal{O}_Z(E) \to \overset{0 \ne}{Q} \to 0$.

By Serre's vanishing $R^1 f_* \mathcal{O}_Z(m) = 0$ $\forall m \gg 0$

$$\Rightarrow 0 \to \underset{\substack{\| \\ \mathcal{O}_Y(mD) \text{ is reflexive sheaf}}}{g_* \mathcal{O}_Z(m)} \xrightarrow[\subseteq]{\varphi} g_*(\mathcal{O}_Z(E)(m)) \to g_* \overset{0 \ne}{Q}(m) \to R^1 f_* \mathcal{O}_Z(m) = 0.$$

Since $\operatorname{codim} g(\operatorname{Exc}(g)) \ge 2$, $\forall U \subseteq Y$,

$$\begin{array}{ccc} H^0(U, g_* \mathcal{O}_Z(m)) & = & H^0(U \setminus g(E), g_* \mathcal{O}_Z(m)) \\ \downarrow \varphi_U & & \| \\ H^0(U, g_* \mathcal{O}_Z(E)(m)) & \hookrightarrow & H^0(U \setminus g(E), g_* \mathcal{O}_Z(E)(m)) \end{array}$$

·· We assume that $N := -B$ is effective.

(Work locally, one can assume $\exists$ ample divisor on Y (eg. Y: affine, $\mathcal{O}_Y$: ample)
cf. [Hart, p167, Step4 in pf. of Thm II 7.17], in general, $\underset{\text{fractional ideal}}{\mathcal{O}_Y(B)} \hookrightarrow \mathcal{K}_Y$)

$g: Z \to Y$ is the blow-up of Y along N and thus the strict transform of $B = -N$ is g-very ample and Cartier.

To see that Z is normal, we shall prove $R(Y,B)_{(f)}$ is normal $\forall f \overset{\neq 0}{\in} \mathcal{O}_Y(-N)$.

$\mathcal{K}_X$ = the fractional field of $R(Y,B)_{(f)}$ = the fractional field of $\mathcal{O}_Y$.

$\forall s \in \mathcal{K}_X$ s.t. $s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$ in $\mathcal{K}_Y$ with $a_i \in R(Y,B)_{(f)}$.

Write $a_i = b_i / f^{\ell}$ with $b_i \in \mathcal{O}_Y(-\ell N)$ for common $\ell \Rightarrow (f^{\ell}s)^n + c_1 (f^{\ell}s)^{n-1} + \cdots + c_n = 0$ with $c_i = b_i f^{(i-1)\ell} \in \mathcal{O}_Y(-i\ell N) \subseteq \mathcal{O}_Y \overset{Y:\text{ normal}}{\Rightarrow} f^{\ell}s \in \mathcal{O}_Y$ and moreover $f^{\ell}s \in \mathcal{O}_Y(-\ell N) \Rightarrow s \in R(Y,B)_{(f)}$.

· (2) ⇒ (1) : Replace $B' = g_*^{-1}B$ by suitable multiple, we may assume B' is Cartier.

Claim. $\mathcal{O}_Y(mB) = g_*\mathcal{O}_Z(mB')$ $\forall m$.

(If so, $R(Y,B) = \bigoplus_{m \geq 0} g_*\mathcal{O}_Z(mB')$ is f.g. since B' is g-ample.
Also, $Z = \mathrm{Proj}_Y R(Y,B)$, thus Z is unique)

(subpf.) $\exists$ an injective

$$\begin{tikzcd} g_*\mathcal{O}_Z(mB) \arrow[r, hook, "\psi"] \arrow[dr, hook] & \mathcal{O}_Y(mB) \arrow[d, hook] \\ & \mathcal{K}_Y \end{tikzcd}$$

To prove surj. of ψ, $\forall U \subseteq Y$,

$$\begin{tikzcd} H^0(g^{-1}U, \mathcal{O}_Z(mB')) \arrow[r, equal, "\text{Hartog}"] \arrow[d, "\varphi_U"] & H^0(g^{-1}(U) \setminus \mathrm{Exc}(g), \mathcal{O}_Z(mB')) \arrow[d, equal] \\ H^0(U, \mathcal{O}_Y(mB)) \arrow[r, hook] & H^0(U \setminus g(\mathrm{Exc}(g)), \mathcal{O}_Y(mB)) \end{tikzcd}$$

Z: normal + codim ≥ 2

g is isom. over $U \setminus g(\mathrm{Exc}(g))$

Rmk. [KM, Rmk 6.3] $\mathcal{O}_Y(B) = \mathcal{I}_N \subseteq \mathcal{O}_Y$, the m-th symbolic power of $\mathcal{I}_N$ is $\mathcal{I}_N^{(m)} = \mathcal{O}_Y(mB)$.
So $R(Y,B)$ is also called the symbolic power alg. of $\mathcal{I}_N$.

Cor. [KM, Cor 6.4. (2)-(4)]
$f: X \to Y$, a flipping contraction of (X,D), choose $r \in \mathbb{N}$ s.t. rD is an integral divisor. Then
(a) The flip of f wrt. (X,D) exists $\Leftrightarrow \bigoplus_{m \geq 0} \mathcal{O}_Y(mf_*(rK_X + rD))$ is sheaf of f.g. $\mathcal{O}_Y$-alg.
(b) The flip of f wrt. (X,D) is unique.
(c) Let $D' \in \mathrm{WDiv}(X)_{\mathbb{Q}}$ s.t. $K_X + D'$ is $\mathbb{Q}$-Cartier and $a(K_X + D) \overset{(*)}{\equiv_f} a'(K_X + D')$ for some $a, a' \in \mathbb{N}$

Note [KM, Def 3.24] extremal contraction.

Then the flip of f wrt. of (X,D) is also the flip of f wrt. of (X,D').

Rmk. (c) $\rightsquigarrow$ indep. of D, one call $f^+: X^+ \to Y$, the flip of f.

(pf.) (a), (b) follows from Lemma by setting $B = f_*(rK_X + rD)$.

For (c), $(\star) \Rightarrow a(K_{X^+}+D^+) \equiv_{f^+} a'(K_{X^+}+D'^+)$ is also f^+-ample.

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

Prop. [KM Prop. 6.6]

Y: normal var, $B \in WDiv(Y)$, $\hat{Y} := \operatorname{Spec} \hat{\mathcal{O}}_{Y,o}$ (the completion of $\mathcal{O}_{Y,o}$ at $\mathfrak{m}_{Y,o}$). TFAE.

$\overset{\Psi}{o}$: a closed point $\qquad$ still normal

(1) $R(\hat{Y}, B|_{\hat{Y}}) = \bigoplus_{m \geq 0} \hat{\mathcal{O}}_{Y,o}(mB|_{\hat{Y}})$ is a f.g. graded $\hat{\mathcal{O}}_{Y,o}$-alg.

(2 alg) $\exists$ a Zarski o.n.b.d. $o \in U \subseteq Y$ s.t. $R(U, B|_U)$ is a f.g. graded $\mathcal{O}_U$-alg.

(2 an) $\qquad$ Euclidean $\qquad U^{an} \qquad R(U^{an}, B|_{U^{an}}) \qquad \mathcal{O}_{U^{an}}$

(pf.)

- (2 alg) $\Rightarrow$ (2 an) : By GAGA!
- (2 an) $\Rightarrow$ (1) : Since $R(\hat{Y}, B|_{\hat{Y}}) \simeq \hat{\mathcal{O}}_{Y,o} \otimes R(U^{an}, B|_{U^{an}})$.
  (2 alg)
- (1) $\Rightarrow$ (2 alg): Assume that $R(\hat{Y}, B|_{\hat{Y}})$ is generated by $\bigoplus_{m=0}^{n} \hat{\mathcal{O}}_{Y,o}(mB|_{\hat{Y}})$ for some $n \in \mathbb{N}$.
  Let $R' \subseteq R(Y,B)$ be the subalg. gen. by $\bigoplus_{m=0}^{n} \mathcal{O}_Y(mB)$.

$$\begin{tikzcd} \operatorname{Proj}_{\hat{Y}} R(\hat{Y}, B|_{\hat{Y}}) = \operatorname{Spec} \hat{\mathcal{O}}_{Y,o} \times_Y Z \arrow[r] \arrow[d, "\text{small contraction}"'] & Z := \operatorname{Proj}_Y R' \arrow[d, "g"] \\ \hat{Y} \arrow[r] & Y \ni o \end{tikzcd}$$

$\Rightarrow$ no exceptional divisor of g intersect $g^{-1}(o)$. Let F = the union of all excep. divisors of g.

$\Rightarrow o \notin g(F)$ and so $U := Y \setminus g(F)$ is a Zarski o.n.b.d. of o.

Apply Lemma to $\begin{cases} g^{-1}U \xrightarrow{\text{small}} U \\ g_*^{-1}B|_U : \mathbb{Q}\text{-Cartier, } g\text{-ample} \end{cases}$, we see that $R(U, B|_U)$ is f.g..

(The existence of flips is a local problem in Euclidean topology)

Cor. X, Y normal var. (or cpx. analytic spaces), $f: X \to Y$, a flipping contraction for (X,D).

The flips of f wrt. (X,D) exists $\Leftrightarrow$ the following holds:

" $\forall y \in Y$, $\exists$ a (Zarski or Euclidean) o.n.b.d. at $y \in U_y \in Y$ s.t. the flip of
$f_y : f^{-1}U_y \to U_y$ wrt. $(f^{-1}U_y, D|_{f^{-1}U_y})$ exists "

(pf.) (⇒) Clear.

(⇐) Set $B = f_* D$ (WLOG it is integral divisor). For $y \in Y$, the flip of $f_y : f^{-1}U_y \longrightarrow U_y$ w.r.t. $(f^{-1}U_y, D|_{f^{-1}U_y})$ exists $\overset{\text{Lem.}}{\Longrightarrow} R(U_y, B|_{U_y})$ is f.g. $\overset{\text{Prop.}}{\Longrightarrow} R(\hat{Y}, B|_{\hat{Y}})$ is f.g.
$\Rightarrow R(Y,B)$ is f.g. in a Zarski o.n.b.d. of y. We can do this for every $y \in Y$, thus $R(Y,B)$ is f.g.

Prop. [KM Prop. 6.8] or [Kaw 88, Lem 3.2] $\quad$ ($\rightsquigarrow \mathcal{O}_{Y'}(B') = (h^* \mathcal{O}_Y(B))^{\vee\vee}$)

Y, Y': normal $/\mathbb{C}$, $h : Y' \longrightarrow Y$ finite surj. morphism. $B \in \mathrm{WDiv}(Y)$ and set $B' = h^* B$.
Then $R(Y,B)$ is f.g. $\Leftrightarrow R(Y',B')$ is f.g..

(pf.) (⇒) $\exists$ a small proj. contraction $g : \underset{\substack{\uparrow \\ \text{normal}}}{Z} = \mathrm{Proj}_Y R(Y,B) \longrightarrow Y$ s.t. $g_*^{-1} B$ is $\mathbb{Q}$-Cartier and g-ample.

Consider

$$\begin{tikzcd}
Y'' \arrow[d, "\text{normalization}"] \arrow[dd, bend right=60, "g'"'] & \\
Y' \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\
Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

$\Rightarrow$ g' is small and the pull-back to Y'' of $g_*^{-1} B$ is g'-ample
$\Rightarrow R(Y', B')$ is f.g..

(⇐) · WLOG h is Galois: Consider

$$\begin{array}{ccc}
\mathcal{A} \subseteq & L & \\
 & | \text{ finite} & \\
 & \mathbb{C}(Y') & \text{Gal.} \\
 & | & \\
\mathcal{O}_X \subseteq & \mathbb{C}(Y) &
\end{array}$$

int. closure of $\mathcal{O}_X$ in L

, and replacing Y' by $\mathrm{Spec}_Y \mathcal{A}$, we may assume $\mathbb{C}(Y')/\mathbb{C}(Y)$ is Galois with Galois gp. G acting on Y'.

· Construct

$$\begin{tikzcd}
G \curvearrowright \mathrm{Proj}_{Y'} R(Y',B') = Z' \arrow[r] \arrow[d, "g'"'] & Z'/G \arrow[d, "g : \text{small}"] \\
Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

$Z = Z'/G$: normal, since Z' is normal
Z'/G : geometric quoti.

cf. [KM Lem. 5.12]

The strict transform $g_*^{-1} B = \mathrm{Norm}_{Z'/Z}(g'^{-1}_* B')$ is $\mathbb{Q}$-Cartier and g-ample, since $g'^{-1}_* B'$ is $\mathbb{Q}$-Cartier and g'-ample. Hence $R(Y,B)$ is f.g.

Rmk. One can reduce the existence of certain flip to existence of flops, see [KM, Cor 6.9].

Mori's approach ⤹ Reid's approach ⤹

Thm. [KM, Thm 6.14] (cf. [Kaw 88, Thm. 4.1])

$f: X \to Y$, a flipping contraction for (X, D), $D \geq 0$. If X is terminal 3-fold and $K_X \equiv_f 0$, then the flip $f^+: X^+ \to Y$ w.r.t. (X, D) exists and $K_{X^+} \equiv_{f^+} 0$.

(pf.) • the pair $(Y, \Delta = 0)$ is terminal.

• Let $Q \in Y$ s.t. $f^{-1}(Q)$ is not a point. It is enough to treat Y as analytic germ $(Q \in Y)$ and prove $R(Y, K_Y + f_* D)$ is f.g.

• (Mori's approach). By classification,

involution $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$

$$\begin{tikzcd} (\tilde{Q} \in \tilde{Y}) = (x_1^2 + g(x_1, x_2, x_3)) \arrow[d, "\pi"', "\text{index 1 cover}"] \arrow[drr, "g", "\text{a double cover}"'] & \subseteq & (0 \in \mathbb{A}^4) \arrow[d] & (x_1, x_2, x_3, x_4) \arrow[d, mapsto] \\ (Q \in Y) & & (0 \in \mathbb{A}^3) & (x_2, x_3, x_4) \end{tikzcd}$$

For $F \in \mathrm{WDiv}(\tilde{Y})$, $F + \iota^* F = g^* g_* F \sim 0$ i.e. $\iota^* F \sim -F$. Thus $\iota^* R(\tilde{Y}, F) = R(\tilde{Y}, -F)$.

• $\because -(K_X + D)$ is f-ample $\Rightarrow R(Y, -(K_Y + f_* D))$ is f.g.

$\Rightarrow R(\tilde{Y}, -\pi^* f_* D)$ is f.g.

$\Rightarrow \iota^* R(\tilde{Y}, -\pi^* f_* D) = R(\tilde{Y}, \pi^* f_* D)$ is f.g.

$\Rightarrow R(Y, K_Y + f_* D)$ is f.g.

Thm. [KM, Thm 6.15] [Kol 89, Thm 2.4]

3-dim'l terminal <u>flops</u> preserve the analytic singularity type.
(K: rel. trivial)

More precisely, the flop is described in one of the following ways, where $r =$ index of $(Q \in Y)$.

$$\begin{tikzcd} X \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

(i) $(Q \in Y) \simeq (x_1^2 + f(x_2, x_3, x_4)) / \mu_r \subseteq (0 \in \mathbb{A}^4 / \tfrac{1}{r}(a_1, a_2, a_3, a_4))$ $(r = 2, 3, 4)$

and $f^+ = \iota \circ f$, where $\iota: (x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$.

(Note. X and X^+ are even isom.)

(ii) $(Q \in Y) \simeq (x_1 x_2 - f(x_3, x_4) = 0) / \mu_r \subseteq (0 \in \mathbb{A}^4 / \tfrac{1}{r}(1, -1, 0, a))$.

Let $\tilde{Y} = (x_1 x_2 - f(x_3, x_4) = 0)$ and $\tau: \tilde{Y} \to \tilde{Y}$ defined by $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_3, x_4)$.

The action on $\tilde{Y}$ by the $\tau \mu_r \tau^{-1}$ extends to an action regularly on
$$\begin{tikzcd} X \times_Y \tilde{Y} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Take the quotient $\tilde{X}^+ := X / \tau \mu_r \tau^{-1} \longrightarrow \tilde{Y}^+ := \tilde{Y} / \tau \mu_r \tau^{-1} \subseteq \mathbb{A}^4 / \tfrac{1}{r}(-1, 1, 0, a)$

§ 3-fold flips after Shokurov / $\mathbb{C}$

[C⁺07] A. Corti (ed.), Flips for 3-folds and 4-folds (Focus on Chap. 2)

[K⁺92] J. Kollar (ed.), Flips and Abundance for algebraic threefolds.

Recall. (X,Δ) : klt (or plt) pair

A klt (or plt) contraction for (X,Δ) is a <u>small</u> proj. birational $f: X \longrightarrow Z$ with
$(K_X+\Delta)$ $\quad$ $\mathrm{codim}_X \mathrm{Exc}(f) \geq 2$

$-(K_X+\Delta)$ being f-ample and $\rho(X/Z)=1$.

Fact. [KM, Prop. 5.51] (X,Δ): dlt pair. TFAE

(1) (X,Δ) is plt pair

(2) $\lfloor \Delta \rfloor$ is normal

(3) $\lfloor \Delta \rfloor$ is the disjoint union of its irr. components.

[Sho 03] Prelimiting flips

* (Reduction of klt flips to <u>pl.</u> flips)

Def. $(X, \underline{S+B})$: plt pair

means $\lfloor S+B \rfloor = S$ irr. & $S \not\subseteq \mathrm{Supp}\, B$ $\left(\begin{array}{l} \text{i.e.} \\ S: \text{a prime Weli divisor} \\ B = \sum b_i B_i,\ b_i \in (0,1)\cap \mathbb{Q},\ S \neq B_i\ \forall i \end{array}\right.$

A pl. (prelimiting) flipping contraction is a flipping contraction $f: X \longrightarrow Z$ for K_X+S+B s.t.
$\begin{cases} X \text{ is } \mathbb{Q}\text{-factorial} \\ S \text{ is } f\text{-negative (i.e. } f\text{-anti ample)} \end{cases}$

Rmk. [C⁺07, Rmk. 2.2.21] This def. is slightly more restrictive than [K⁺92, Def 18.6], [C⁺07, Def 4.3.1]
[Sho03, 1.1]

Fact. (Reduction Theorem) [K⁺92, 18.11], [C⁺07, Thm 4.3.1.]

klt flips exist in dim $= n$ provided that:

$(PL)_n$: pl. flips exists in dim $= n$.

$(ST)_n$: special termination holds in dim $= n$.

holds for $n \leq 4$ $\because$ MMP is ture in $n \leq 3$

In general, $(MMP)_{n-1} \Rightarrow (ST)_n$ (cf. [C⁺07, Thm 4.2.1], [K⁺92, 7.1])

$\ast$ (Goal: Flips of 3-fold pl. flipping contraction exist.)

$f: X \to Z$ a pl. flipping contraction for K_X+S+B.

Note. The existence of flips is local on Z in the Zarski topology, we always assume that Z is affine.

Set $A := H^0(Z, \mathcal{O}_Z)$, affine coordinate ring. So

$$\overset{\shortparallel}{H^0(X,\mathcal{O}_X)} \quad \because f_*\mathcal{O}_X = \mathcal{O}_Z$$

the pl flips exists $\Leftrightarrow R(X, K_X+S+B) := \bigoplus_{i\geq 0} H^0(X, \mathcal{O}_X(i(K_X+S+B)))$ is a f.g. A-alg.

Def. • A function algebra on X is a graded A-subalg. $V = \bigoplus_{i\geq 0} V_i$ of the polynomial alg. $\mathbb{C}(X)[T]$, where $V_0 = A$ and each $V_i \subseteq \mathbb{C}(X)$ is coherent A-module.

• V is bounded (by D) if $\exists D \in WDiv(X)$ s.t. $V_i \subseteq H^0(X, \mathcal{O}_X(iD))$ $\forall i$.

• $\overline{\{\eta\}} = S \subseteq X$ is irr. normal subvar. of $\operatorname{codim}_X = 1$.

V is regular along S if (1) $V_i \subseteq \mathcal{O}_{X,S} := \underbrace{\mathcal{O}_{X,\eta}}_{\text{DVR}} \subseteq \mathbb{C}(X)$ $\forall i$

(2) $V_1 \nsubseteq m_{X,S} = \{\varphi \in \mathbb{C}(X) \mid \underset{\substack{\shortparallel \\ \mathrm{mult}_S(\varphi)}}{v_S(\varphi)} > 0\}$

• If V is regular along S, the restricted algebra $V^\circ = \mathrm{res}_S V$ is $V^\circ = \bigoplus_{i\geq 0} V_i^\circ$, where $V_i := \mathrm{Im}(V \subseteq \mathcal{O}_{X,S} \longrightarrow \mathcal{O}_{X,S}/m_{X,S} = k(\eta) = \mathbb{C}(S)) \rightsquigarrow V_1^\circ \neq (0)$.

Rmk. If V is bounded by D, regular along S and $S \nsubseteq \operatorname{Supp} D$, then $V^\circ = \mathrm{res}_S V$ is also bounded.

Observation. Fix any f-negative $D \in \underset{\mathbb{Q}\text{-factorial}}{WDiv(X)}$

$\because \rho(X/Z) = 1$ and Z is affine $\Rightarrow D \sim r(K_X+S+B)$ for some $r \in \mathbb{Q}_{>0}$.

$\therefore R(X, K_X+S+B)$ is f.g. $\Leftrightarrow R(X,D)$ is f.g.

[KM, Cor 6.14 (4)]

Lemma. [C+07, Lem. 2.3.6.]

$f: X \to Z$, pl. flipping contraction for K_X+S+B. $0 \leq D \in WDiv(X)$ f-negative, $S \nsubseteq \operatorname{Supp} D$.

Then the flip of f exists $\Leftrightarrow R^\circ = \mathrm{res}_S R(X,D)$ is f.g.

(pf.) $(\Rightarrow)$ $\mathcal{R} := \mathcal{R}(X,D)$ is f.g. $\Rightarrow R^\circ$ is f.g.

$(\Leftarrow)$ $\because \rho(X/Z) = 1$, Z affine $\Rightarrow$ WLOG $D \sim S$.

$\Rightarrow \exists t \in \mathbb{C}(X)$ s.t. $\operatorname{div}(t) + D = S \geq 0$

$\Rightarrow t \in H^0(X, \mathcal{O}_X(D)) = R_1$

Claim. $\ker(R \to R^\circ) = (t)$. (If so, the lemma follows)

(subpf.) For $\varphi \in \ker(R \to R^\circ)$, we may assume $\varphi \in R_n = H^0(X, \mathcal{O}_X(nD))$, then $\operatorname{div}\varphi + nD \geq 0$ and φ has a zero along S $\Rightarrow$ $\underbrace{\operatorname{div}\varphi + nD - S}_{= \operatorname{div}\frac{\varphi}{t} + (n-1)D} \geq 0$ ($\because S \not\subseteq \operatorname{Supp} D$)

$\Rightarrow \varphi/t \in H^0(X, (n-1)D) = R_{n-1}$ and $\varphi = t \cdot \frac{\varphi}{t} \in (t)$.

Rmk. $X \supseteq S$: irr. divisor, f: flipping contraction $X \to Z$, Z: affine.

$\overset{(Z:\text{affine})}{\Rightarrow}$ $f_*\mathcal{O}_X(S)$ is globally generated. Let $\varphi = H^0(Z, f_*\mathcal{O}_X(S))$ $\big(= H^0(X, \mathcal{O}_X(S))\big)$ be a general section and $D := \operatorname{div}(\varphi) + S \geq 0$. Then $0 \leq D \sim S$ and $S \not\subseteq \operatorname{Supp} D$.

f-negative ($\because f$ is isom. in codim. 1)

Plan of proof. $f: X \to Z$ pl flipping contr. for $K_X + S + B$. (X: $\mathbb{Q}$-factorial, Z: affine)

Its flips of f exists $\Leftrightarrow$ $\mathcal{R}(X, D)$ is f.g. for some $0 \leq D \in \operatorname{WDiv}(X)$ f-negative with $S \not\subseteq \operatorname{Supp} D$

$\Leftrightarrow$ the restricted alg. $R^\circ = \operatorname{res}_S R(X, D)$ is f.g.

$\Leftrightarrow$ R_S, a pbd algebra associated with R°, is f.g.

$\Uparrow$ (Fano graded alg. (FGA) in Shokurov's own terminology)

- R_S is a <u>Shokurov algebra</u> (for all dim.)
- Shokurov algebra with mobile system on a surface is f.g. (f.g. conj. holds for surfaces)

$*$ (b-divisors) (b: birational)

X: normal variety (not necessary proper). Consider the following category:

- Obj.: a model proper over X: $\{Y \xrightarrow{\text{proper, birat.}} X\}$ (Y: normal var.) (a model of X)
- Mor.: $Y \xrightarrow{\text{morphism}} Y'$ over X, $K = \mathbb{C}(X) = \mathbb{C}(Y)$

Def. An (integral) Weil b-divisor on X is an element $\mathbb{D} = \mathbf{D} \in \operatorname{WDiv}(X)_{\mathbb{Q}} := \varprojlim \operatorname{WDiv}(Y)_{\mathbb{Q}}$ (b-$\mathbb{Q}$-divisor)

$\operatorname{WDiv}(X)_{\mathbb{Q}} = \mathbb{D}\text{iv}(X)$

[C+07]

where the (projective) limit is taken over all $f: Y \to X$ of X under $f_*: WDiv(Y)_{\mathbb{Q}} \longrightarrow WDiv(X)_{\mathbb{Q}}$.

Rmk. • If $f: Y \to X$ is a model of X, then $f_*: WDiv(Y) \xrightarrow{\sim} WDiv(X)$ is an isom.

• (Zarski - Riemann space of a subring k of a field K)

$\shortparallel$

a locally ringed space whose points are valuation ring $k \subseteq R \subseteq K$.

(see Z-S Com. Alg. II Thm 41 p122.)

$\mathfrak{X} := \varprojlim_{Y:\text{ model of } X} Y$ in the category of locally ring space

as sch.

($\mathfrak{X}$ is NOT a scheme anymore if $\dim X \geq 2$, cf. [Hart. Exc 4.12, p.108]
For $\dim X = 1$, $\mathfrak{X}$ is a scheme, cf. [Hart. Thm I 6.7])

$WDiv(\mathfrak{X}) = \varprojlim WDiv(Y) = WDiv(X)$.

Def. $\mathbb{D} = \sum d_\Gamma \Gamma$ a b-divisor on X.

• $\forall U \subseteq X$, $H^0(U, \underline{\mathcal{O}_X(\mathbb{D})}) := \{ \varphi \in \mathbb{C}(X) \mid v_\Gamma(\varphi) + d_\Gamma \geq 0 \ \ \forall \Gamma \text{ center on } U \}$

(In general, it is often NOT quasi-coherent.
However, it is a coherent sheaf in cases of interest to us.)

• $Y \to X$, a model of X. The trace of $\mathbb{D}$ on Y = $\underset{\substack{\shortparallel \\ tr_Y \mathbb{D}}}{\mathbb{D}_Y} = \sum_{\Gamma \in WDiv(Y)} d_\Gamma \Gamma$

Rmk. $|\mathbb{D}| = H^0(X, \mathcal{O}_X(\mathbb{D})) \setminus \{0\} / \mathbb{C}^\times = PH^0(X, \mathcal{O}_X(\mathbb{D}))$. $PH^0(X, \mathbb{D}) \subsetneq PH^0(X, \mathbb{D}_X)$ (in general)

b-divisors are a convenient language to track of a linear system with base conditions.

(Example of b-divisors)

• (principal b-divisors) $\varphi \in \mathbb{C}(X) \setminus \{0\}$, $\operatorname{\mathbf{div}}_X(\varphi) = \sum v_\Gamma(\varphi)\Gamma$, where we sum over all geometric valuation with center on X.

•• $\mathbb{D}_1 \sim \mathbb{D}_2$ if $\mathbb{D}_1 - \mathbb{D}_2 = \operatorname{\mathbf{div}}_X(\varphi)$ for some $\varphi \in \mathbb{C}(X)^\times$.

• ($\mathbb{Q}$-Cartier closure = pull-back to $\mathfrak{X} = \varprojlim Y$) $D \in CDiv(X)_{\mathbb{Q}}$

The $\mathbb{Q}$-Cartier closure of D is the b-divisor $\overline{D}$ with the trace $(\overline{D})_Y = f^*D$ on models $f: Y \longrightarrow X$ of X.

called a $\mathbb{Q}$-Cartier b-divisor

$\shortparallel$

$CDiv(\mathfrak{X})_{\mathbb{Q}} = \varprojlim_{Y:\text{ model of } X} CDiv(Y)_{\mathbb{Q}}$

e.g. $\varphi \in \mathbb{C}(X)^\times$, $\operatorname{\mathbf{div}}_X(\varphi) = \overline{\operatorname{div}_X(\varphi)}$

• (canonical b-divisor)

$\mathbb{K} = \mathrm{div}_X(\omega)$, a rational differential $0 \neq \omega \in \Omega_{\mathbb{C}(X)}$

It is a divisor in the canonical class $f: Y \to X$, $f_* \mathbb{K}_Y = K_X$.

• (discrepancy b-divisor)

$B \in \mathrm{WDiv}(X)_{\mathbb{Q}}$ with $K_X + B$: $\mathbb{Q}$-Cartier. The discrep. b-divisor $\mathbb{A} = \mathbb{A}(X,B)$ is the b-divisor with trace $\mathbb{A}_Y$ defined by $K_Y = f^*(K_X+B) + \mathbb{A}_Y$ on models $f: Y \to X$ of X.

• (strict transform b-divisor)

$D \in \mathrm{WDiv}(X)$, $\hat{D}$ is the b-divisor with trace $\hat{D}_Y = f_*^{-1} D$ on model $f: Y \to X$ of X.

(abuse notation, write D instead of $\hat{D}$)

[C+07, Def 1.7.4. (2), 2.4.5]

Rmk. $\mathbb{D}$: a b-divisor on X

If $\exists$ a model $f: Y \to X$ of X and $\exists\, 0 \le D_Y \in \mathrm{CDiv}(Y)_{\mathbb{Q}}$ s.t. $\underline{\mathbb{D} = \overline{D_Y}}$ (if descends to Y), then $\mathcal{O}_X(\mathbb{D})$ $\left(= f_* \mathcal{O}_Y(D_Y)\right)$ is coherent. In general, $\forall\, 0 \le \mathbb{D}' \le \overline{D_Y}$, $\mathcal{O}_X(\mathbb{D}')$ is also coherent.

Lemma. [C+07, Lem 2.3.14]

X : sm. var., $D \in \mathrm{WDiv}(X)_{\mathbb{Q}}$ has SNC support. $\mathbb{A} = \mathbb{A}(X,D)$: the discrep. b-divisor of (X,D). If $f: Y \to X$ is a model of X, then $\lceil \mathbb{A}_Y \rceil = f^* \lceil \mathbb{A}_X \rceil + \sum f_i E_i$, where the E_i's are f-excep. divisors and $f_i > 0$.

(pf.) By def., $D = -\mathbb{A}_X$.

$$K_Y = f^*(K_X + D) + \mathbb{A}_Y = f^*(K_X + \{-\mathbb{A}_X\}) + \mathbb{A}_Y - f^* \lceil \mathbb{A}_X \rceil$$

By assumption, $(X, \{-\mathbb{A}_X\})$ is klt pair $\Rightarrow 0 \le \lceil \mathbb{A}_Y - f^* \lceil \mathbb{A}_X \rceil \rceil = \lceil \mathbb{A}_Y \rceil - f^* \lceil \mathbb{A}_X \rceil$ $\underline{\text{is } f\text{-excep.}}$

$\left(\mathbb{A}_Y = \text{excep.} + f^* \lceil \mathbb{A}_X \rceil - f^* \{-\mathbb{A}_X\}\right)$

Rmk. Using this Lemma, one can show that if $\begin{cases} X : \text{normal},\ D \in \mathrm{WDiv}(X)_{\mathbb{Q}} \\ K_X + D : \mathbb{Q}\text{-Cartier} \end{cases}$, then

$\mathcal{O}_X(\lceil \mathbb{A}(X,D) \rceil)$ is coherent. (cf. [C+07, Lem. 2.3.15])

if $D \ge 0$, multiplier ideal sheaf $\mathcal{J}(D)$ [PAG II, Def. 9.2.1]

* (Saturated b-divisors)

Def. $D \in WDiv(X)_{\mathbb{Q}}$

- $\{0\} \neq V \subseteq H^0(X, \mathcal{O}_X(D)) = \{\varphi \in \mathbb{C}(X)^* \mid \mathrm{div}(\varphi) + D \geq 0\} \cup \{0\}$, vector subspace

  the mobile part of D wrt. V = $\mathrm{Mob}_V D = \sum_{\Gamma:\text{prime divisor on } X} (-\inf_{\varphi \in V} v_\Gamma(\varphi))\Gamma$

  (movable)

  When $V = H^0(X, \mathcal{O}_X(D))$, we simply write $\mathrm{Mob}\, D$.
- For $C \in WDiv(X)_{\mathbb{Q}}$, we say that D is C-saturated if $\mathrm{Mob}\lceil D+C \rceil \leq D$.

Rmk. • If D is not integral, the definition says $\mathrm{Mob}_V D = \mathrm{Mob}_V \llcorner D \lrcorner$.

- If D is integral, $\mathrm{Mob}\, D = D - \mathrm{Fix}|D|$, where $\mathrm{Fix}|D|$ is the biggest divisor $F \geq 0$ s.t. $F \leq D'$ $\forall D' \in |D|$. Moreover, $\mathrm{Supp}\,\mathrm{Fix}|D|$ is the divisorial pair of $\mathrm{Bs}|D|$.
- If $|\lceil D+C \rceil| = \phi$, then D is always C-saturated.

  Therefore, only "$|D| \neq \phi$, $|\lceil D+C \rceil| \neq \phi$" is useful.

Terminology. We say that a property $\mathcal{P}$ holds on high models ($Y \to X$ of X) if $\mathcal{P}$ holds on a particular model $Y \to X$ of X and on every higher models $Y' \to Y \to X$.

Def. • A b-divisor $\mathbb{D}$ on X is $\mathbb{C}$-saturated if $\mathbb{D}_Y$ is $\mathbb{C}_Y$-saturated on high model
(X,B): klt $\qquad$ $/\mathbb{A}(X,B)$

$Y \to X$ of Y. If $Y \to X$ is a (canonical) model of X and $\mathbb{D}_Y$ is $\mathbb{C}_Y$-saturated, we say
$/\mathbb{A}(X,B)$

that saturation holds on Y.

- $\mathbb{D}$ is exceptionally saturated over X if it is $\hat{E}$-saturated for all E: eff. and excep. over X.

Prop. [C⁺07, Prop. 2.3.27]

D: $\mathbb{Q}$-Cartier integral divisor $\Rightarrow$ the $\mathbb{Q}$-Cartier closure $\overline{D}$ is exceptional saturated over X.

(pf.) $\forall$ model $f: Y \to X$ of X, $f_* \mathcal{O}_Y \lceil f^*D + \sum a_i E_i \rceil = \mathcal{O}_Y(D)$ if all E_i are f-excep. and all $a_i \geq 0$.

both reflexive sheaf (cheak outside codim ≥ 2)

$\mathrm{Mob}\lceil f^*D + \sum a_i E_i \rceil = \mathrm{Mob}\, f^*D \leq f^*D$.

Lemma. [C⁺07, Lem. 2.3.28]

X: normal var., $B \in \mathrm{WDiv}(X)_{\mathbb{Q}}$ s.t. $K_X + B$: $\mathbb{Q}$-Cartier. $\mathbb{D}$: b-divisor on X, $\mathbb{A} = \mathbb{A}(X, B)$.

Let $Y \to X$ be a model of X satisfying

(1) Y : sm., $\mathbb{D}_Y + \mathbb{A}_Y$ has SNC support.

(2) $\overline{\mathbb{D}_Y} = \mathbb{D}$ (it descend to Y)

Then canonical saturation holds on Y $\Leftrightarrow$ canonical saturation holds on any higher model $f : Y' \to Y$.

That is, $\mathrm{Mob}\lceil \mathbb{D}_Y + \mathbb{A}_Y \rceil \leq \mathbb{D}_Y \Leftrightarrow \mathrm{Mob}\lceil \mathbb{D}_{Y'} + \mathbb{A}_{Y'} \rceil \leq \mathbb{D}_{Y'}$.

(pf.) (cf. [C⁺07, Lem 2.3.14])

$(2) \Rightarrow K_{Y'} = f^*(K_Y + \{-\mathbb{A}_Y - \mathbb{D}_Y\}) - f^*\lceil \mathbb{A}_Y + \mathbb{D}_Y \rceil + \mathbb{A}_{Y'} + \mathbb{D}_{Y'}$

$(1) \Rightarrow (Y, \{-\mathbb{A}_Y - \mathbb{D}_Y\})$: klt.

$\rightsquigarrow \lceil \mathbb{D}_{Y'} + \mathbb{A}_{Y'} \rceil = f^*\lceil \mathbb{D}_Y + \mathbb{A}_Y \rceil + E$, where E is f-excep. and effective.

$\Rightarrow f_* \mathcal{O}_{Y'}(\lceil \mathbb{D}_{Y'} + \mathbb{A}_{Y'} \rceil) = \mathcal{O}_Y(\lceil \mathbb{D}_Y + \mathbb{A}_Y \rceil) \longrightarrow$ we get $\Leftrightarrow$.

* (Shokurov algebra)

From now on, we always tacitly assume that the sheaf $\mathcal{O}_X(\mathbb{D})$ of b-divisor is coherent.

Def. • A sequence $\mathbb{D}_\bullet = \{\mathbb{D}_i\}_{i=1}^{\infty}$ of b-divisor is convex if $D_1 \gtrsim 0$ and

$$\mathbb{D}_{i+j} \geq \frac{i}{i+j}\mathbb{D}_i + \frac{j}{i+j}\mathbb{D}_j \quad \forall i, j \in \mathbb{N}$$

• $\mathbb{D}_\bullet$ is bounded if $\exists D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ s.t. $\mathbb{D}_i \leq \overline{D}$ $\forall i \in \mathbb{N}$.

Rmk. • $\mathbb{D}_\bullet$ is "increasing" in the sense that $\mathbb{D}_i \leq \mathbb{D}_j$ if $i \mid j$ (e.g. $\mathbb{D}_{2i} \geq \frac{i}{2i}\mathbb{D}_i + \frac{i}{2i}\mathbb{D}_i = \mathbb{D}_i$)

• If $\mathbb{D}_\bullet$ is bounded, convexity $\Rightarrow \lim_{i \to \infty} \mathbb{D}_i = \sup \mathbb{D}_i \in \mathrm{WDiv}(X)_{\mathbb{R}}$.

Def. • A pbd (= pesudo-b-divisoral) algebra is the functional algebra

$$\mathcal{R} = R(X, \mathbb{D}_\bullet) = \bigoplus_{i \geq 0} H^0(X, \mathcal{O}_X(i\mathbb{D}_i))$$

naturally asso. to a convex seq. $\mathbb{D}_\bullet = \{\mathbb{D}_i\}$ of b-divisors.

called the characteristic seq. of the pbd algebra.

(Note. $R_i \hookrightarrow \mathbb{C}(X)$, convexity of $\mathbb{D}_\bullet \Rightarrow R_i R_j \subset R_{i+j}$)

• We say that $R(X, \mathbb{D}_\bullet)$ is bounded if "it is bounded as a function algebra"

$\Leftrightarrow$ "$\mathbb{D}_\bullet$ is bounded"

Def. [C+07, Def. 2.3.57, 2.3.58]

- A convex seq. $\mathbb{D}_\cdot$ of effective b-divisors is $\mathbb{C}$-a-saturated if (asymptotically)

$$\mathrm{Mob}\lceil j\mathbb{D}_{iY} + \mathbb{C}_Y \rceil \leq j\mathbb{D}_{jY} \quad \text{for all } i, j \quad (\text{or } i \geq j > 0)$$

on higher models $Y \to X$, that is $\exists\, Y(i,j) \longrightarrow X$ depend on i, j s.t. above inequality holds for all $Y \longrightarrow Y(i,j)$.

  - We say it is uniformly saturated if $\exists\, X' \longrightarrow X$ s.t. inequality holds for all $Y \to X'$.
  - We say it is canonically if $\mathbb{C} = \mathbb{A}(X,B)$, where (X,B) is klt pair.
- A pbd alg. $R(X, \mathbb{D}_\cdot)$ is canonically a-sat. if $\mathbb{D}_\cdot$ is canonically a-sat.

Def. A Shokurov algebra is a bounded canonically a-saturated pbd-algebra.

<u>Finite generation conjecture</u>

(X,B) : klt, $-(K_X+B)$: big & nef $/Z$

weak (log) Fano contraction

$$\begin{array}{c} X \\ f \downarrow \text{ birational}, \ f_*\mathcal{O}_X = \mathcal{O}_Z \\ Z : \text{affine} \end{array}$$

$\Rightarrow$ All Shokurov algebras on X are finitely generated.

Rmk. • $\dim X = 1$ holds [C+07, §2.5.10]
• $\dim X = 2$ holds $\Rightarrow \exists$ of 3-fold flips
• $\dim X \geq 3$ unknown

Rmk. In general, the restricted algebra $R^0 = \mathrm{res}_S R$ is NOT a pbd algebra.

*(Reduction of restricted algebras to pbd algebras)

Def. (Mobile b-divisor)

- An integral b-divisor $\mathbb{D}$ is mobile if $\exists$ a model $Y \to X$ of X s.t.

$$\left.\begin{array}{l} (1)\ |\mathbb{D}_Y| : \text{free} \\ (2)\ \overline{\mathbb{D}_Y} = \mathbb{D} \end{array}\right\} \text{ such } \mathbb{D}_Y \text{ called b-free}$$

- $D \in \mathrm{WDiv}(X)_{\mathbb{Q}}$, $V \subseteq H^0(X, \mathcal{O}_X(D))$ a vector subspace. The mobile b-part of D wrt. V is
(=)

$$\mathbb{M}\mathrm{ob}_V D = \sum_{\substack{Y \to X \text{ model} \\ \cup\!\shortmid \\ \Gamma : \text{prime divisor}}} \left(-\inf_{\varphi \in \Gamma} \nu_\Gamma(\varphi)\right)\Gamma$$

$(\mathbb{M}\mathrm{ob}\, D)$

Rmk. If D is integral $\mathbb{Q}$-Cartier, and $f: Y \to X$ model of X,

$$(\mathbb{M}\mathrm{ov}_V D)_Y = f^*D - \mathrm{Fix}\, f^*|V|.$$

If, in addition, $V = H^0(X, \mathcal{O}_X(D))$,

$$(\mathbb{M}\mathrm{ov}\, D)_Y = \mathrm{Mob}\, f^*D = \mathrm{Mob}\ulcorner f^*D \urcorner.$$

Lemma. [C⁺07, Lem. 2.3.37]

The mobile b-part $\mathbb{M}\mathrm{ob}\, D$ of $\mathbb{Q}$-Cartier $D \in \mathrm{WDiv}(X)$ is exceptional saturated over X.

(pf.) $\forall f: Y \to X$ model of X,
$\quad E \geq 0$: eff. excep.

$$\mathrm{Mob}\ulcorner \underbrace{(\mathbb{M}\mathrm{ob}\, D)_Y}_{\mathrm{Mob}\, f^*D} + E \urcorner \leq \mathrm{Mob}\ulcorner f^*D + E \urcorner \overset{\downarrow}{=} \mathrm{Mob}\ulcorner f^*D \urcorner = (\mathbb{M}\mathrm{ob}\, D)_Y$$

$(\because H^0(Y, f^*D + E) = H^0(X, D))$

[C⁺07, Lem. 2.3.55] = [Sho03, Prop. 4.15]

Lemma. (a seq. of b-divisor ass. to $V = \oplus V_i$)

$V = \bigoplus_i V_i$: function algebra on X. We define the b-divisors

$$\mathbb{M}_i := \sum_{\substack{Y \to X \text{ model} \\ \Gamma \subseteq Y:\ \text{prime divisor}}} \left(-\inf_{\varphi \in V_i} v_\Gamma(\varphi)\right)\Gamma$$

(later will choose $V_i = H^0(X, iD)$, $\mathbb{M}_i = \mathbb{M}\mathrm{ob}(iD)$)

Then M_i has the properties:

(1) $V_i \subseteq H^0(X, \mathbb{M}_i)$

(2) $\mathbb{M}_i$ is mobile

(3) $\mathbb{M}_i + \mathbb{M}_j \leq \mathbb{M}_{i+j}$ (subadditivity)

(pf.) (1) $\forall s \in V_i$, $\quad \mathrm{div}(s) + \mathbb{M}_i = \sum_\Gamma \left(v_\Gamma(s) - \inf_{\varphi \in V_i} v_\Gamma(\varphi)\right)\Gamma \geq 0$.

(3) $$\mathbb{M}_i + \mathbb{M}_j = \sum \underline{\left(-\inf_{\varphi \in V_i} v_\Gamma(\varphi) - \inf_{\psi \in V_j} v_\Gamma(\psi)\right)}\Gamma = \sum \left(-\inf_{\substack{\varphi \in V_i \\ \psi \in V_j}} v_\Gamma(\varphi\psi)\right)\Gamma \underset{V_i V_j \subseteq V_{i+j}}{\leq} \mathbb{M}_{i+j}$$

(2) Fix any resol. $g: Y \to X$. By def,

$$(\mathbb{M}_i)_Y = \sum_{\Gamma \subseteq Y} \left(-\underline{\inf_{\varphi \in V_i}} v_\Gamma(\varphi)\right)\Gamma \quad \text{is (ordinary) integral divisor.}$$

may replace by min ($\because V_i$: f.g.)

Consider the linear system $\Lambda_i := \{(\mathbb{M}_i)_Y + \underbrace{\mathrm{div}(\varphi)_Y}_{\mathrm{div}(f^*\varphi)} \mid \varphi \in V_i\} \subseteq |(\mathbb{M}_i)_Y|$

and the blow-up $h : Y' \to Y$ along $\mathrm{Bs}\,\Lambda_i \Rightarrow h^*\Lambda_i$ is free , and

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cap\!\!\shortmid$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |h^*(\mathbb{M}_i)_Y|$ is also free

$$\underset{\text{b-free}}{(\mathbb{M}_i)_{Y'}} = h^*(\mathbb{M}_i)_Y - \text{“}\inf\{h^*(\mathbb{M}_i)_Y + \underbrace{\mathrm{div}(\varphi)_{Y'}}_{\mathrm{div}(h^* g^* \varphi)} \mid \varphi \in V_i\}\text{”}$$

$$\text{“}\inf\{\ldots\}\text{”} = \mathrm{Bs}\,h^*\Lambda_i = 0$$

$\Rightarrow \mathbb{M}_i$ is mobile.

Lemma. $V = \bigoplus_{i \geq 0} V_i$: function algebra on $X \Rightarrow \exists$ a pbd alg. $R^\vee = R(X, \mathbb{D}_\bullet) \geq V$ satisfies

$$R(X, \mathbb{D}_\bullet) = \bigoplus_{i \geq 0} H^0(X, i\mathbb{D}_i)$$

(0) $R^\vee$ is integral over V

(1) V is bounded $\Leftrightarrow R^\vee$ is bounded

(2) V is f.g. $\Leftrightarrow R^\vee$ is f.g.

(pf.) (Construction) By the above lemma, $\exists$ a seq. of mobile b-divisors $\mathbb{M}_\bullet = \{\mathbb{M}_i\}_{i=0}^\infty$. Multiplying by a suitable rational function, WLOG $\mathcal{O}_X \leq V_1$, i.e. the b-divisor $\mathbb{M}_1 > 0$. We take $\mathbb{D}_i = \frac{1}{i}\mathbb{M}_i$, by subadditivity, $\mathbb{D}_\bullet = \{D_i\}_{i=1}^\infty$ is convex. Hence we get a pbd alg.

$$R^\vee = R(X, \mathbb{D}_\bullet) = \bigoplus_{i \geq 0} \underset{\substack{\cup\!\shortmid \\ V_i}}{H^0(X, \mathbb{M}_i)} \geq V$$

(1) $(\Leftarrow)$ $R^\vee$: bounded $\Rightarrow \exists D$ s.t. $\underset{\substack{\cup\!\shortmid \\ V_i}}{H^0(X, \mathbb{M}_i)} \subseteq H^0(X, iD) \Rightarrow V$ is bounded.

$(\Rightarrow)$ If $V_i \subseteq H^0(X, iD)$ for some D, then by def of $\mathbb{M}_i = \sum(-\inf_{\varphi \in V_i} v_P(\varphi))\Gamma$, we have $H^0(X, \mathbb{M}_i) \subseteq H^0(X, i\overline{D}) \Rightarrow M_i \leq i\overline{D}$.

(0) For a proof, see [C'07, P.38] or
[Sho03, Prop 4.15 (6)] or
[Iskovskikh - Shokurov, Birational models and flips Lem 5.22]

Fact. $\bigoplus_{j \geq 0} V_i^j \subseteq \bigoplus_{j \geq 0} H^0(X, j\mathbb{M}_i)$ is an integral extension of algebra.

(2) $(\Leftarrow)$ $R^\vee$: f.g. $\xrightarrow{\text{by truncation}}$ WLOG $R^\vee$ is generated by $(R^\vee)_1 = H^0(X, \mathbb{M}_1)$ $\qquad \mathbb{M}_1 = \mathbb{D}_1$

$\xrightarrow{\text{Fact.}} R^\vee$ is integral over $V' := \bigoplus_{j \geq 0} V_1^j \leftarrow$ f.g. alg. $(V_1 : \text{f.g.})$

$\Rightarrow R^\vee$ is a f.g. V'- module

$\overset{V' \subseteq V \subseteq R^\vee}{\Rightarrow} V$ is also a f.g. V'- module $\Rightarrow V$ is a f.g. algebra.

$(\Rightarrow)$ By construction, V and $R^\vee$ are function alg. with same quotient algebra. By Noether's thm. on the finiteness of the integral closure. If V is f.g. alg, so is $R^\vee$.

Lemma. (Limiting criterion) [C+07, Lem 2.3.53]

Assume that $R = R(X, \mathbb{D}_\bullet)$ is a pbd alg. s.t. $i\mathbb{D}_i = \mathbb{M}_i$ is mobile.

Then R is f.g. $\Leftrightarrow$ $\exists i_0 \in \mathbb{N}$ s.t. $\mathbb{D}_{i_0} = \mathbb{D}_{i i_0}$ $\forall i \geq 1$.

(pf.) ($\Leftarrow$) Passing to a truncation $(R(X, \mathbb{D}_\bullet)^{(i_0)})$, WLOG $i_0 = 1$. Then $R = \bigoplus_{i \geq 0} H^0(X, i\mathbb{M}_1)$, $\mathbb{M}_1 = \mathbb{D}_1$.

Let $Y \to X$ be a model of X s.t. $|\mathbb{M}_{1Y}|$ is free and $\overline{\mathbb{M}_{1Y}} = \mathbb{M}_1$. Then

$$R = \bigoplus_{i \geq 0} H^0(Y, \mathbb{M}_{1Y}) = R(Y, \mathbb{M}_{1Y}) \text{ is f.g.}.$$

($\Rightarrow$) Assume that $R = \bigoplus_{i \geq 0} H^0(X, i\mathbb{D}_i)$ is f.g. $\Rightarrow \exists i_0 \in \mathbb{N}$ s.t. $R^{(i_0)}$ is f.g. by deg. 1 elements $H^0(X, \mathbb{M}_{i_0})$. Then $H^0(X, \mathbb{M}_{i i_0}) = H^0(X, \mathbb{M}_{i_0})^i \subseteq H^0(X, i\mathbb{M}_{i_0})$. Since $\mathbb{D}_\bullet$ is convex, it is "increasing", i.e. $\mathbb{D}_{i i_0} \geq \mathbb{D}_{i_0} \rightsquigarrow \mathbb{M}_{i i_0} \geq i\mathbb{M}_{i_0} \rightsquigarrow H^0(X, \mathbb{M}_{i i_0}) \geq H^0(X, i\mathbb{M}_{i_0})$.

Hence $\mathbb{M}_{i i_0} = i\mathbb{M}_{i_0} \rightsquigarrow \mathbb{D}_{i i_0} = \mathbb{D}_{i_0}$.

Rmk. Assume $\exists X \xrightarrow{\text{proper, birat.}} Z'$: affine.

Each pbd alg. arise from "a mobile sequence" [C+07, Lem. 2.3.52]

Def. (Mobile restriction, cf. [C+07, Def. 2.3.39 restriction])

$\mathbb{M}$: a mobile b-divisor on X. $X \supseteq S$: irr. normal subvar. of $\operatorname{codim}_X = 1$ with $S \not\subseteq \operatorname{Supp} \mathbb{M}_X$.

We define mobile restriction of $\mathbb{M}$ to S as follows:

Pick a model $Y \to X$ s.t. $|\mathbb{M}_Y|$ is free, $\overline{\mathbb{M}_Y} = \mathbb{M}$, and the strict transform $S' \subseteq Y$ is normal.

We define $\mathbb{M}^0 = \operatorname{res}_S \mathbb{M} = \overline{\mathbb{M}_Y|_{S'}} \in \operatorname{WDiv}(S') \simeq \operatorname{WDiv}(S)$.

Rmk. [C+07, Rmk 2.3.40]

- $(\mathbb{M}_1 + \mathbb{M}_2)^0 = \mathbb{M}_1^0 + \mathbb{M}_2^0$
- $\mathbb{M}_1 \geq \mathbb{M}_2 \Rightarrow \mathbb{M}_1^0 \geq \mathbb{M}_2^0$

Recall. $f : X \to Z$ (affine) pl. flipping contr. for $K_X + S + B$

$\boxed{\text{Want}}$

R^V : pbd alg. ass to V

$\cup$ integral extension

V : functional alg.

$$\underset{R(X,\mathbb{D}_\bullet)}{\overset{}{\bigoplus_{i \geq 0} H^0(X, \underset{\mathbb{M}ob(i\mathbb{D})}{\underline{\mathbb{M}_i}})}} = R^R \xrightarrow{\text{mobile restriction}} R(S, \mathbb{D}_\bullet^0) \underset{?}{=} R_S := \bigoplus_{i \geq 0} H^0(S, M_i^0) : \text{f.g.} \overset{\text{limiting criterion}}{\Longleftrightarrow} \exists i_0 \in \mathbb{N} \text{ s.t. } D_{i i_0}^0 = D_{i_0}^0 \ \forall i$$

$$\bigoplus_{i \geq 0} H^0(S, \mathbb{M}_i^0) = R^{R^0}$$

$\cup$ (f.g. $\Updownarrow$) $\qquad \cup$ (f.g. $\Updownarrow$)

$$R = R(X, D) : \text{f.g.} \overset{\text{res.}}{\Longleftrightarrow} R^0 = \operatorname{res}_S R : \text{f.g.}, \quad R_i^0 = \operatorname{Im}(H^0(X, iD) \longrightarrow \mathbb{C}(S))$$

$\left(0 \leq D \in \operatorname{WDiv}(X) \text{ f-neg. with } S \not\subseteq \operatorname{Supp} D \right)$

$H^0(X, iD) \subseteq \mathcal{O}_{X,S}$, $\quad \mathbb{C}(S) \subseteq \mathcal{O}_{X,S}/\mathfrak{m}_{X,S}$

cf. [C+07, Rmk. 2.3.42]

Lemma. [C+07, Lem. 2.4.2] Restriction of mobile b-divisor is compatible with restriction of rational function, i.e. $R_S = R(S, \mathbb{D}')$.

(pf.) By def., $\forall i \geq 0$, Pick a model (depend on i) $\begin{matrix} Y & \longrightarrow & X \\ \cup\!\! \mid & & \cup\!\! \mid \\ S' & \longrightarrow & S \end{matrix}$ as in the def. of mob. restr.

$$\mathbb{M}^{\circ}_{iS'} = \mathbb{M}_{iY}|_S = \Big(\sum_{\Gamma \subseteq Y} \big(-\inf_{\varphi \in H^0(X, iD)} v_\Gamma(\varphi)\big)\Gamma\Big)\Big|_S = \sum_{\Gamma' \subseteq S'} \big(-\inf v_{\Gamma'}(\varphi)\big)\Gamma' = \mathbb{M}'_{iS'}.$$

$\varphi \in \mathrm{Im}(H^0(X, iD) \longrightarrow \mathbb{C}(S))$

Lemma. [C+07, Lem. 2.4.3] R_S is a $\underset{\text{bounded a-saturated pbd alg.}}{\underline{\text{Shokurov algebra}}}$ on S.

[C+07, Lem. 2.4.4] $f : X \longrightarrow Z$: pl flipping contraction for $K_X + S + B$ $\quad (X, S+B)$: pl $\Updownarrow$ adjunction

$\quad\quad\quad\quad\ \ \, S \longrightarrow f(S)$

Weak (log) Fano contraction of $(S, \mathrm{Diff}_S B)$: klt

[C+07, §2.4.4.] $(K_X + S + B)|_S = K_S + \underbrace{\mathrm{Diff}_S B}_{B|_S + \underline{\mathrm{Diff}_S 0}_{\geq 0}}$

The finite generation conj. implies R_S is f.g.

$$R_S = R^{R^\circ} \underset{\text{int. ext.}}{\supseteq} R^\circ : \text{f.g.} \iff R = R(X, D) \text{ is f.g.}$$

<u>Proof of Lemma</u>

- Since $(R^\circ)_i \subseteq H^0(S, iD|_S)$, R_S is bounded pbd alg.
- (a-saturated) "excep. saturation on $X \Rightarrow$ canonical saturation on S"

By construction, $\mathbb{M}_i = \mathbb{M}\mathrm{ob}(iD)$ and thus $\mathbb{M}_{iY} = \mathrm{Mob}(f^*D)$ on models $f : Y \to X$ of X ($\mathbb{Q}$-factorial)

For $i, j \in \mathbb{N}$ and choose a model $f : Y \to X$ (depend on i, j) s.t.

(1) f is a log resol. of $(X, S + B + \mathbb{M}_{iX} + \mathbb{M}_{jX})$

(2) $|\mathbb{M}_{iY}|, |\mathbb{M}_{jY}|$ is free and $\mathbb{M}_i = \overline{\mathbb{M}_{iY}}$, $\mathbb{M}_j = \overline{\mathbb{M}_{jY}}$

Write

$$K_Y = f^*(K_X + S + B) \underbrace{- f_*^{-1}(S+B) + F}_{\mathbb{A}(X, S+B)_Y = \mathbb{A}_Y} \quad \text{and} \quad \mathbb{A}' := \mathbb{A} - \hat{S}.$$

(F: f-excep.; $\hat{S}$: strict transform b-divisor of S)

$$\mathbb{A}'_Y = -f_*^{-1}B + F \overset{\text{adjunction}}{\Longrightarrow} \mathbb{A}'_Y|_S = \mathbb{A}(S, \mathrm{Diff}_S B)_{S'}. \quad f : f_*^{-1}S \longrightarrow S$$

$((K_X + S + B)|_S = K_S + \mathrm{Diff}_S B) \quad\quad f_*^{-1}S = S' = \hat{S}_Y$

Recall. $R_S = R(S, \mathbb{D}^\circ_\cdot)$, $\mathbb{D}_i = \frac{1}{i}\mathbb{M}_i$.

Claim. $\mathrm{Mor}\ulcorner j\mathbb{D}^\circ_{iS'} + \mathbb{A}(S.\mathrm{Diff}_S B)_{S'}\urcorner \le j\mathbb{D}^\circ_{jS'}$ $\forall i \le j$.

(Therefore, by [C+07, Lem. 2.3.28], a-saturation holds on all models $S'' \longrightarrow S'$ higher than S')

(pf.) Consider $0 \to \mathcal{O}_Y(-S') \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{S'} \longrightarrow 0$ and $\otimes \ulcorner(j\mathbb{D}_i + \mathbb{A}')_Y\urcorner$,

$$0 \to \mathcal{O}_Y(\ulcorner(j\mathbb{D}_i + \mathbb{A})_Y\urcorner) \longrightarrow \mathcal{O}_Y(\ulcorner(j\mathbb{D}_i + \mathbb{A}')_Y\urcorner) \longrightarrow \mathcal{O}_{S'}(\underbrace{\ulcorner(j\mathbb{D}_i + \mathbb{A}')_Y|_S\urcorner}_{\text{SI adj.}}) \to 0$$

$$\ulcorner j\mathbb{D}^\circ_{iS'} + \mathbb{A}(S, \mathrm{Diff}_S B)_{S'}\urcorner$$

$$\Rightarrow H^0(Y, \ulcorner(j\mathbb{D}_i + \mathbb{A}')_Y\urcorner) \longrightarrow H^0(S', \ulcorner j\mathbb{D}^\circ_{iS'} + \mathbb{A}(S, \mathrm{Diff}_S B)_{S'}\urcorner)$$

$$\longrightarrow H^1(Y, \ulcorner(j\mathbb{D}_i + \mathbb{A})_Y\urcorner) = 0 \text{ by K.Y.}$$

$$K_Y + \underbrace{\tfrac{j}{i}\mathbb{M}_{iY}}_{\text{free}} - \underbrace{f^*(K_X + S + B)}_{\text{big \& nef}}$$

Therefore, to prove claim, we can chang to compare $\mathrm{Mob}\ulcorner(j\mathbb{D}_i + \mathbb{A}')_Y\urcorner$ with $j\mathbb{D}_{iY}$.
Then

$$\begin{aligned}
\mathrm{Mob}\ulcorner(j\mathbb{D}_i + \mathbb{A}')_Y\urcorner &= \mathrm{Mob}\ulcorner \tfrac{j}{i}\mathrm{Mob} f^*(iD) + \mathbb{A}'_Y\urcorner \\
&\le \mathrm{Mob}\ulcorner \tfrac{j}{i}\mathrm{Mob} f^*(iD) + F\urcorner \quad (\because \mathbb{A}'_Y = -f_*^{-1}B + F \text{ and } \ulcorner -f_*^{-1}B\urcorner = 0) \\
&\le \mathrm{Mob}\ulcorner \tfrac{j}{i} f^*(iD) + F\urcorner \\
&= \tfrac{j}{j}\mathrm{Mob}\ulcorner f^*(jD)\urcorner = j\mathbb{D}_{jY}.
\end{aligned}$$

$*$ (A Shokurov algebra with mobile system on surface is f.g.)

surface $R(X, \mathbb{D}_\cdot)$ with $i\mathbb{D}_i = \mathbb{M}_i$ is mobile $\forall i$

Fact. [C+07, Thm 2.4.7]

(X, B) : klt pair, $\dim X = n$. Assume that klt MMP holds in $\dim = n$.

$\Rightarrow \exists (X^{ter}, B^{ter})$: terminal pair and proj. birational morphism $\varphi : X^{ter} \longrightarrow X$ s.t.

$K_{X^{ter}} + B^{ter} = \varphi^*(K_X + B)$

Rmk. • We say that (X^{ter}, B^{ter}) is a terminal model of (X, B).

It is unique of $\dim X = 2$.

• For $(X, B = 0)$: canonical, X^{ter} = terminalization of X [KM, Thm 6.23].

- Idea of proof. $K_Y + B_+ + B_- = f^*(K_X + B)$

$$\begin{tikzcd}
(Y, B_+) \arrow[rr, dashed, "\text{run MMP}"] \arrow[dr, "f:\ \text{log resol.}"'] & & (X^{ter}, B^{ter}) \arrow[dl] \\
& (X, B)\ :\ \text{klt} &
\end{tikzcd}$$

- So if (X,B) is 2-dim'l klt pair and a birational <u>weak (log) Fano contraction</u>

$$\begin{tikzcd}
X^{ter} \arrow[r, "\varphi"] \arrow[rr, bend right, "\text{also weak (log) Fano contraction}"'] & X \arrow[r, "f"] & Z
\end{tikzcd} : \text{affine} \quad \text{with "}-(K_X+B): f\text{-big} \ \&\ f\text{-nef"}, \text{"} f_*\mathcal{O}_X = \mathcal{O}_Z \text{"}$$

So we may assume (X,B) is 2-dim'l terminal pair.

Thm. [C⁺07, Thm 2.4.6.]

(X,B) : 2-dim'l terminal pair (i.e. X: sm. and $\mathrm{mult}_x B < 1$ $\forall x \in X$ [KM Thm 4.5])

$\downarrow f$: birat. weak (log) Fano contra.

Z : affine

Let $\mathbb{M}$ be a mobile, canonically saturated b-divisor on X. Then

(1) $\mathbb{M}$ descends to X

(2) $\mathbb{M}_X$ is nef (This is NOT true in higher dim. and a naive generalization of the statement does not hold.)

(pf.) Let $g: Y \to X$ be a high enough log resolution of (X,B) s.t.

(a) canonical saturation holds on Y

(b) $\overline{\mathbb{M}_Y} = \mathbb{M}$ and $|\mathbb{M}_Y|$: free.

(2) By (1), $\mathbb{M}_Y = g^*\mathbb{M}_X$. Then $\forall$ irr. curve $C \subseteq X$, $\mathbb{M}_X . [C] = \underbrace{g^*\mathbb{M}_X}_{\text{nef.}} . g^*[C] \geq 0$

(1) Write $K_Y = g^*(K_X + B) - g_*^{-1}B + \sum a_i E_i$, where $a_i > 0$ $\forall i$ with E_i: g-excep.

$E := \ulcorner \mathbb{A}(X,B)_Y \urcorner = \ulcorner \sum a_i E_i \urcorner \overset{(a_i>0)}{\Rightarrow} \mathrm{Supp}\, E = \cup E_i = \mathrm{Exc}(g)$.

Claim. $E \cap D = \emptyset$ for a general member $D \in |\mathbb{M}_Y|$.

(If so, $|\mathbb{M}_Y|$ avoids the $\mathrm{Supp}\, E = \mathrm{Exc}(g)$ all together, and thus $\mathbb{M}_Y \overset{\text{local on } X, Y}{=} g^*\mathbb{M}_X \rightsquigarrow (1)$)

(subpf.) By (a), $\mathrm{Mob} \ulcorner \mathbb{M}_Y + E \urcorner \leq \mathbb{M}_Y \Rightarrow \mathbb{M}_Y + E - \mathrm{Mob} \ulcorner \mathbb{M}_Y + E \urcorner \geq \mathbb{M}_Y + E - \mathbb{M}_Y = E$.

$\Rightarrow \mathrm{Fix}\, |\mathbb{M}_Y + E| \geq E$.

Since $|\mathbb{M}_Y|$ is free, $\mathrm{Fix}\, |\mathbb{M}_Y + E| = E$. Consider

$$0 \to \mathcal{O}_Y(\mathbb{M}_Y + E - D) \to \mathcal{O}_Y(\mathbb{M}_Y + E) \to \mathcal{O}_D(\mathbb{M}_Y + E) \to 0$$
$$\mathcal{O}_Y(\mathbb{M}_Y + E - D) \cong \mathcal{O}_Y(E)$$

$(\because f$ is$)$
h is weak (log) Fano
$Y \to X \to Z$: affine

$\Rightarrow \quad h_*\mathcal{O}_Y(\mathbb{M}_Y+E) \to h_*\mathcal{O}_D(\mathbb{M}_Y+E) \to R^1h_*\mathcal{O}_Y(E) = 0$ by K-Y

$E = K_Y + \underline{\ulcorner -g^*(K_X+B) \urcorner}$

$= \mathbb{A}(X,B)_Y - K_Y$ is h-big & h-nef

Z: affine

$\Rightarrow \quad H^0(Y, \mathbb{M}_Y+E) \twoheadrightarrow H^0(D, (\mathbb{M}_Y+E)|_D)$,

therefore $E \cap D = \mathrm{Bs}|(\mathbb{M}_Y+E)|_D| = \phi$ $\left(\begin{array}{l}\because D \text{ is affine curve} \\ \therefore \text{every complete linear system on } D \text{ is free}\end{array}\right)$

Upshot [C$^+$07, Cor 2.4.9]

(X^{ter}, B^{ter}) : terminal model

$\downarrow$

(X, B) : klt surface

$\downarrow$ birat. weak (log) Fano contr.

Z : affine

, $\begin{array}{c} R(X^{ter}, \mathbb{D}.) \\ \| \\ R(X, \mathbb{D}.)\end{array}$: a $\underline{\text{Shokrov alg.}}$ with mobile system $\mathbb{M}. = \{\mathbb{M}_i\}$

bounded $\Rightarrow \exists$ a divisor $G = \Sigma G_i$ on X^{ter}

s.t. $\mathrm{Supp}\, \mathbb{M}_{iX^{ter}} \subseteq G \quad \forall i$

$= \mathrm{Supp}\, \mathbb{D}_{iX^{ter}}$

$\Rightarrow$ · all $\mathbb{M}_i$ descend to X^{ter}

· If $X'' \longrightarrow X^{ter}$ is a log resol. of $(X^{ter}, B^{ter}+G)$, then canonical a-saturated holds uniformly on models $Y \longrightarrow X''$ higher than X'', i.e.

$$\mathrm{Mob}\ulcorner j\mathbb{D}_{iY} + \mathbb{A}_Y \urcorner \le j\mathbb{D}_{jY} \quad \forall i \ge j \quad \text{, where } \mathbb{A} = \mathbb{A}(X,B) = \mathbb{A}(X^{ter}, B^{ter}).$$

Now, WLOG $\left\{\begin{array}{l}(X,B) : \text{terimal pair} \\ \exists \text{ a divisor } G = \Sigma G_j \text{ on } X \text{ s.t. } \mathrm{Supp}\, \mathbb{D}_{iX} \le G \quad \forall i.\end{array}\right.$

$\Rightarrow \mathbb{D} = \lim\limits_{i\to\infty} \mathbb{D}_i \in \mathrm{WDiv}(X)_{\mathbb{R}}$ and $\mathbb{D}_X \le G$.

[C$^+$07, Rmk 2.3.47]

Lemma. [C$^+$07, Lem 2.4.11] $\mathbb{D}_X$ is semi-ample

(pf.) $\mathbb{M}_i = i\mathbb{D}_i$ is mobile $\Rightarrow \mathbb{M}_{iX}$ is nef $\Rightarrow \mathbb{D}_X = \lim\limits_{i\to\infty} \mathbb{D}_{iX}$ is also nef.

X

$\downarrow f$: birat. weak (log) Fano contr.

Z : affine $\Rightarrow$ contain NO proj. curve $\Rightarrow$ every proj. curve $\subseteq X$ is in the fiber of f

$\Rightarrow \overline{NE}(X) = \overline{NE}(X/Z) = \overline{NE}(X/Z)_{(K_X+B+\varepsilon(\text{eff. ample}))<0}$ is a finite rational polyhedral.

$(\because -K_X+B$ is f-nef$)$

$\Rightarrow$ the dual cone $= \mathrm{Nef}(X)$ is gen. by the semi-ample divisors supporting the contractions of its extremal faces. Hence all nef divisor on X are semi-ample.

$*$ (Diophantine approximate) $\mathrm{Supp}\, \mathbb{D}_X \subseteq G = \sum G_j$, $N_{\mathbb{Z}}' := \bigoplus \mathbb{Z} G_j \subseteq \mathrm{WDiv}(X)$.
Since $\mathbb{D}_X$ is semi-ample $\Rightarrow$ choose eff. bpf. divisors $P_k \in N_{\mathbb{Z}}'$ s.t. $\mathbb{D}_X \in \sum \mathbb{R}_+ P_k \subseteq \sum \mathbb{R}_+ G_j \subseteq N_{\mathbb{Z}}' \otimes_{\mathbb{Z}} \mathbb{R}$

Fact. [C+07, Lem. 2.4.12]
If $\mathbb{D}_X$ is not rational, then $\forall \varepsilon > 0$, $\exists m \in \mathbb{N}$ and $M \in N_{\mathbb{Z}}'$ s.t.

(1) $|M|$ is free

(2) $\| m \mathbb{D}_X - M \| < \varepsilon$ sup norm wrt. $\{G_j\}$ of $N_{\mathbb{Z}}' \otimes \mathbb{R}$.

(3) $m \mathbb{D}_X - M$ is not effective.

Lemma. (Non-vanishing) [C+07, Lemma 2.4.13]
Choose $\mathbb{Q}^+ \ni r \ll 1$ s.t. $(X, B + rG)$: klt. Set $\mathbb{A} = \mathbb{A}(X, B)$. We have $\lceil \mathbb{A} - r\overline{G} \rceil \geq 0$.
Assume $\mathbb{D}$ is not rational, and let $M \in N_{\mathbb{Z}}'$ be as in Fact. If $0 < \varepsilon < r$, then on every model $f : Y \to X$ of X, then $\mathrm{Mob} \lceil m \mathbb{D}_Y + \mathbb{A}_Y \rceil \geq (\overline{M})_Y = f^* M$.

(pf.) Upshot $\Rightarrow \mathbb{D}_i = \overline{\mathbb{D}_{iX}}$ $\forall i$ $\Rightarrow \mathbb{D} = \overline{\mathbb{D}_X}$ and thus $\mathbb{D}_Y = f^* \mathbb{D}_X$. Let $F := m \mathbb{D}_X - M \rightsquigarrow \|F\| < \varepsilon$ implies $f^* F > -\varepsilon f^* G$. So

$$m \mathbb{D}_Y + \mathbb{A}_Y = f^* M + f^* F + \mathbb{A}_Y > f^* M - \varepsilon f^* G + \mathbb{A}_Y > f^* M - r f^* G + \mathbb{A}_Y .$$

$$\mathrm{Mob} \lceil m \mathbb{D}_Y + \mathbb{A}_Y \rceil \geq \mathrm{Mob} (f^* M + \lceil \mathbb{A}_Y - r f^* G \rceil) \geq \mathrm{Mob}\, f^* M = f^* M \text{ since } |M| : \text{free}.$$

Lemma. [C+07, Lemma 2.4.14] $\mathbb{D}_X$ is rational.
(pf.) Upshot $\Rightarrow \exists$ a log resol. $Y \to X$ of $(X, B + G)$ s.t. the canonical a-saturation holds unif. on Y, i.e. $\mathrm{Mob} \lceil m \mathbb{D}_{iY} + \mathbb{A}_Y \rceil \leq m \mathbb{D}_{iY} \leq m \mathbb{D}_Y$ $\forall i \overset{i \to \infty}{\Longrightarrow} \underset{\substack{\rotatebox{90}{$\leq$} \\ (\overline{M})_Y}}{\mathrm{Mob} \lceil m \mathbb{D}_Y + \mathbb{A}_Y \rceil} \leq m \mathbb{D}_{mY} \leq m \mathbb{D}_Y$

if $\mathbb{D}_X$ is NOT rational and by non-vanishing thm

$\Rightarrow (m \mathbb{D} - \overline{M})_Y \geq 0 \iff (m \mathbb{D} - \overline{M})_X \geq 0$ contradict to Dio. approx. (3).

Lemma. The characteristic seq. $\mathbb{D}_\bullet$ is eventually constant $\mathbb{D} = \mathbb{D}_m$ for some m.
(If so, by the limiting criterion, $R(X, \mathbb{D}_\bullet)$ is f.g.)

(pf.) Since $\mathbb{D}_X$ is rational and semi-ample, $\exists m \in \mathbb{N}$ s.t. $m\mathbb{D}_X$ is integral and free.

As before, $(X, B+rG)$: klt and $\lceil \mathbb{A} - r\bar{G} \rceil \geq 0$. By the same prove in above,

$$\mathrm{Mob}\lceil \underline{m\mathbb{D}_Y + \mathbb{A}_Y} \rceil \leq m\mathbb{D}_{mY} \leq m\mathbb{D}_Y \quad \Rightarrow \mathbb{D}_{mY} = \mathbb{D}_Y \text{ and thus } \mathbb{D}_m = \mathbb{D}.$$

$$\text{VI} \quad (\bar{M} + \mathbb{A})_Y$$

$$\mathrm{Mob}\lceil (\bar{M} + \mathbb{A} - rG)_Y \rceil \geq \mathrm{Mob}\lceil (\bar{M})_Y \rceil = (\bar{M})_Y = mD_Y$$