Multiplication of two Gaussian KDEs

A Gaussian KDE is a weighted sum of Gaussians centered at the data points $\{\mu_i\}$ and having the same covariance matrix Σ estimated based on the samples. Mathematically, it can be written as

$$\mathrm{KDE} = \sum_{i=0}^{N-1} w_i \mathcal{N}(\mu_i, \Sigma), \tag{1}$$

where $\{w_i\}$ are the weights for each sample.

The main concept to be noted is that the product of two Gaussians is also a Gaussian. In particular,

$$\mathcal{N}(\mu_1, \Sigma_1) \times \mathcal{N}(\mu_2, \Sigma_2) \propto \mathcal{N}(\mu_3, \Sigma_3), \tag{2}$$

where

$$\begin{aligned} \Sigma_3 &= \Sigma_1(\Sigma_1 + \Sigma_2)^{-1}\Sigma_2, \\ \mu_3 &= \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}\mu_1 + \Sigma_1(\Sigma_1 + \Sigma_2)^{-1}\mu_2. \end{aligned} \tag{3}$$

As a result, the product of two Gaussian KDEs can be computed as

$$\begin{aligned} &\mathrm{KDE}_1 \times \mathrm{KDE}_2 \\ &= \sum_{i=0}^{N-1} w_i \mathcal{N}(\mu_i, \Sigma_1) \times \sum_{j=0}^{M-1} w_j \mathcal{N}(\mu_j, \Sigma_2) \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} w_i w_j \mathcal{N}(\mu_i, \Sigma_1) \mathcal{N}(\mu_j, \Sigma_2) \\ &= \sum_{k=0}^{MN-1} w_k \mathcal{N}(\mu_k, \Sigma_3) \\ &\equiv \mathrm{KDE}_3 \end{aligned} \tag{4}$$

where

$$\begin{aligned} \Sigma_3 &= \Sigma_1(\Sigma_1 + \Sigma_2)^{-1}\Sigma_2, \\ \mu_k &= \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}\mu_i + \Sigma_1(\Sigma_1 + \Sigma_2)^{-1}\mu_j \\ w_k &\propto w_i \times w_j \times \exp\left[-\frac{1}{2}(\mu_i - \mu_j)^{\mathrm{T}}(\Sigma_1 + \Sigma_2)^{-1}(\mu_i - \mu_j)\right] \end{aligned} \tag{5}$$

with $k = Nj + i$.

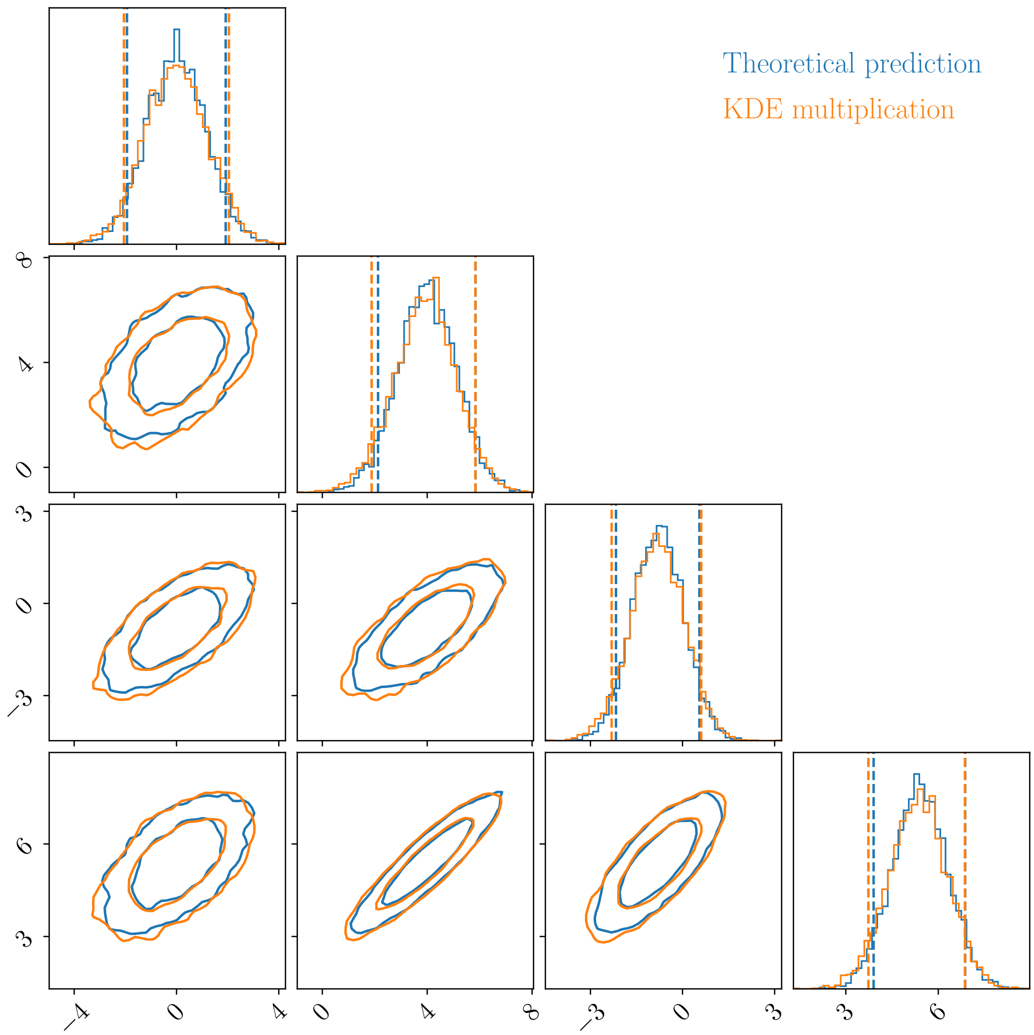

Figure 1: Comparison between the resulting distribution of multiply two Gaussians with analytical prediction (*blue*) and the previously described KDE multiplication method (*orange*).