Algebraic Geometry 1

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46

Universally closedness is stable under the base change by definition. We will prove both separated and finite-type are stable under any base change.

Being of separated is stable under the base change.

Let $f : X \to Y$ be a morphism of schemes over a base S. Let $S' \to S$ be a morphism of schemes. Let $f' : X_{S'} \to Y_{S'}$ be the base change of f. Then the diagonal morphism of f' is

$$\Delta_{f'} : X_{S'} = S' \underset{S}{\times} X \to X_{S'} \underset{Y_{S'}}{\times} X_{S'} = S' \underset{S}{\times} (X \underset{Y}{\times} X)$$

which can be seen as the base change of Δ_f. Since closed immersion is stable under base change from the previous sheet. We need to show that quasi-compactness is stable under the base change.

We see that it is enough to prove the case for affine schemes. Let $Y = \mathrm{Spec}(B), Y' = \mathrm{Spec}(B')$ be affine. And let $g : X \to Y$ be quasi-compact. Then X can be covered by a finite collection of affine open subschemes $\mathrm{Spec}(A_i)$ and then

$$X' = X \underset{Y}{\times} Y'$$

is covered by the finite collection of affine open subschemes $\mathrm{Spec}(A_i \underset{B}{\otimes} B')$ by the definition of fiber products. Thus X' is quasi-compact with Y' being quasi-compact.

Being of finite type is stable under the base change.

Let $f : X \to Y$ be of finite type and $g : S' \to S$ be a scheme morphism.

$$\begin{tikzcd}
X' = X \underset{S}{\times} S' \arrow[r, "f'"] \arrow[d] & S' \arrow[d, "g"] \\
X \arrow[r, "f"] & S
\end{tikzcd}$$

Where $S = \bigcup_{i \in I} S_i, S_i = \mathrm{Spec}(B_i)$. We have

$$f^{-1}(S_i) = \bigcup_{j \in J} U_{i_j}, U_{i_j} = \mathrm{Spec}(A_{i_j})$$

where J is finite. We can regard A_{i_j} as a finitely generated B_i algebra.

$g^{-1}(S_i)$ is an open subscheme, thus $g^{-1}(S_i) = \bigcup_{k \in K} W_{i_k}, W_{i_k} = \mathrm{Spec}(C_{i_k})$. We derive that

$$S' = g^{-1}(S) = \bigcup_{i,k} W_{i_k}.$$

We have

$$f'^{-1}(W_{i_j}) = X \underset{S}{\times} W_{i_j} = f^{-1}(S_i) \underset{S_i}{\times} W_{i_j} = \bigcup_{k \in K} U_{i_k} \underset{S_i}{\times} W_{i_j} = \bigcup_{k \in K} \mathrm{Spec}(A_{i_k} \underset{B_i}{\otimes} C_{i_j}).$$

Therefore each $f'^{-1}(W_{i_j})$ can be covered by finitely many affine sets. We have A_{i_k} is finite over B_i, there are $a_1, \cdots, a_n \in A_{i_k}$ such that

$$A_{i_k} = B_i[a_1, \cdots, a_n].$$

We also have C_{i_j} is an B_i algebra, thus by scalar multiplication we have

$$A_{i_k} \underset{B_i}{\otimes} C_{i_j} = C_{i_j}[a_1 \underset{B_i}{\otimes} 1, \cdots, a_n \underset{B_i}{\otimes} 1].$$

We derived that base change preserves finite type.

In conclusion, properness is stable under base changes.

Exercise 45

First let us assume that we have $X = \mathrm{Spec}(A), S = \mathrm{Spec}(\mathbb{F}_p)$ with the morphism $\mathrm{Fr}_X : X \to S$ induced by a homomorphism of rings of finite type $\varphi : \mathbb{F}_p \to A$. Thus A is automatically of characteristic p. Since φ is of finite type we have

$$A = \mathbb{F}_p[a_1, \cdots, a_n]$$

for some $a_1, \cdots, a_n \in A$. Under this condition, we also have that $F_S = id$. Then

$$X^{(p)} = X \underset{S}{\times} S = \mathrm{Spec}(A \underset{\mathbb{F}_p}{\otimes} \mathbb{F}_p) = \mathrm{Spec}(A) = X.$$

And $F_{X/S}$ is an identity.

In the case where $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(\mathbb{F}_{p^r}$. From Commutative Algebra, we know that

$$A \cong \mathbb{F}_{p^r}[(X_i)_{i \in I}]/((f_j)_{j \in J}),$$

for some variables and polynomials. Then $A^{(p)}$ is such that

$$A^{(p)} \cong \mathbb{F}_{p^r}[(X_i)_{i \in I}]/((f_j^p)_{j \in J}),$$

And $F_{X/S}$ is induced by the inclusion from $\varphi(x) = x^p$.