Algebraic Geometry 1

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1 Topology

1.1 Connected Sets

Definition 1.1. *Let $(X, \mathscr{T})$ be a topological space. A subset A of X is said to be connected if for any $U, V \in \mathscr{T}$, $U \cap V =, U \cup V \supset A$ then A is fully contained in one of U, V.*

Definition 1.2. *A connected component of a topological space is a maximal connected subset of a space.*

Proposition 1.1. *Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological space and $f : X \to Y$ be a continuous function. Then for any connected subset A of X, $f(A)$ is connected in Y.*

Proof.

$$
\begin{aligned}
U, V \in \mathscr{T}_Y, U \cup V \supset f(A), U \cap V = \emptyset, & \\
&\Rightarrow f^{-1}(U), f^{-1}(V) \in \mathscr{T}_X, \\
&\quad f^{-1}(U) \cup f^{-1}(V) \supset A, \\
&\quad f^{-1}(U) \cap f^{-1}(V) = \emptyset, \\
&\Rightarrow f^{-1}(U) \supset A \vee f^{-1}(V) \supset A, \\
&\Rightarrow U \supset f(A) \vee V \supset f(A).
\end{aligned}
$$

□

2 Category Theory

2.1 Categories

Definition 2.1. *A category $\mathscr{A}$ consists of*

- *a collection $\mathrm{ob}(\mathscr{A})$ of objects;*
- *for each $A, B \in \mathrm{ob}(\mathscr{A})$, a collection $\mathscr{A}(A, B)$ of morphisms from A to B;*

such that

i). for each $A \in \mathrm{ob}(\mathscr{A})$*, the identity* $1_A \in \mathscr{A}(A, A)$*;*

ii). the composition $\mathscr{A}(B, C) \times \mathscr{A}(A, B) \ni (g, f) \mapsto g \circ f \in \mathscr{A}(A, C)$ *is well-defined;*

and they satisfy the following axioms

I). Associativity : $f \in \mathscr{A}(A, B), g \in \mathscr{A}(B, C), h \in \mathscr{A}(C, D)$*,* $(h \circ g) \circ f = h \circ (g \circ f)$*.*

II). Identity laws : $f \in \mathscr{A}(A, B)$ *then* $f \circ 1_A = 1_B \circ f$*.*

Definition 2.2. *Let* $\mathscr{A}$ *be a category. A terminal object* $T \in \mathrm{ob}(\mathscr{A})$ *is an object such that for any* $A \in \mathrm{ob}(\mathscr{A})$*,* $\mathscr{A}(A, T)$ *is a single element set.*

Definition 2.3. *Given two categories* $\mathscr{A}, \mathscr{B}$*, we say* $\mathscr{A}$ *is a full-subcategory of* $\mathscr{B}$ *if*

i). $\mathscr{A} \subset \mathscr{B}$*,*

ii). $\mathrm{ob}(\mathscr{A}) = \mathrm{ob}(\mathscr{B})$.

Notation 2.1. *Here we give notations to some important categories.*

- (**Sets**) *: A category of sets equipped with set theoretic functions.*
- (**Ab**) *: A category of abelian groups with group homomorphisms.*

Example 2.1. *Given a partially ordered set* $(X, \leq)$*. This can be encoded to a category* $\mathscr{O}$ *by*

i). $\mathrm{ob}(\mathscr{O}) = X$,

ii). For $x, y \in X$*,* $x \leq y \Rightarrow \mathscr{O}(x, y) = \{*\}$ *otherwise the morphisms between* x, y *is an emptyset.*

Definition 2.4. *A opposite/dual category of a category* $\mathscr{A}$ *is* $\mathscr{A}^{\boldsymbol{op}}$ *such that*

i). $\mathrm{ob}(\mathscr{A}^{\boldsymbol{op}}) = \mathrm{ob}(\mathscr{A})$,

ii). $\mathscr{A}^{\boldsymbol{op}}(B, A) = \mathscr{A}(A, B)$.

Definition 2.5. *Let* $\mathscr{A}$ *be a category and* $\varphi_1, \varphi_2 \in \mathscr{A}(M, N)$*. A morphism* $\varphi : K \to M$ *is called an equalizer of* (φ_1, φ_2) *if for any morphism* $\psi : P \to M$ *such that* $\varphi_1 \circ \psi = \varphi_2 \circ \psi$*, there is a unique morphism* $\tilde{\psi} : P \to K$ *such that* $\varphi \circ \tilde{\psi} = \psi$*.*

Proposition 2.1. *If an equalizer exists then it is unique up to unique isomorphism.*

Proof. Suppose $\varphi : K \to M, \psi : L \to M$ be equalizers of (φ_1, φ_2). Then we have

$$\varphi \circ \tilde{\psi} = \psi, \quad \psi \circ \tilde{\varphi} = \varphi$$

By the uniqueness, we have $\tilde{\varphi} \circ \tilde{\psi} = 1_L, \tilde{\psi} \circ \tilde{\varphi} = 1_K$. □

Definition 2.6. *Let $\mathscr{A}, \mathscr{B}$ be categories. A functor $F : \mathscr{A} \to \mathscr{B}$ is a function such that for each $f \in \mathscr{A}(A, A')$, $F(f) : F(A) \to F(A')$. In other words, $f \mapsto F(f) : \mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$. Furthermore, F satisfies the following axioms.*

I). $F(f' \circ f) = F(f') \circ F(f)$ whenever $f : A \to A', f' : A' \to A''$ in $\mathscr{A}$,

II). $F(1_A) = 1_{F(A)}$ whenever $A \in \mathscr{A}$.

Definition 2.7. *Let F, G be functors between two categories $\mathscr{A}, \mathscr{B}$. A natural transformation $\alpha : F \to G$ is a family $(\alpha_A : F(A) \to G(A))_{A \in \mathscr{A}}$ such that*

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\
G(A) \arrow[r, "G(f)"] & G(A')
\end{tikzcd}$$

is a commutative diagram. Each α_A is called a component of α.

2.2 Direct Limits

Definition 2.8. *A partially ordered set $(X, \leq)$ is directed if for any $x, y \in X$ there is $z \in X$ such that $x \leq c$ and $y \leq c$.*

Example 2.2. *Let $(X, \mathscr{T})$ be a topological space. A partially ordered set $(\mathscr{T}, \leq)$ such that*

$$V \subseteq U \Rightarrow U \leq V$$

is directed. Since for any $U \in \mathscr{T}$, $U \leq \emptyset$. As a category this is $\mathbf{Ouv}_X^{\mathbf{op}}$.

Example 2.3. *Let $(X, \mathscr{T})$ be a topological space. For $x \in X$, define $O_x = \{U \in \mathscr{T} \mid x \in U\}$. If we define an order as in the previous example, we get $(O_x, \leq)$ is directed. This follows from for any $U, V \in O_x$, $U, V \leq U \cap V$.*

Definition 2.9. *Let I be a directed partially ordered set and $\mathscr{A}$ be a category. A directed system of objects of $\mathscr{A}$ indexed by I is a collection of objects $(A_i)_{i \in I}$ and morphisms $(\rho_{ij})_{i \leq j}$ of $\mathscr{A}$ such that*

i). $\rho_{ii} = \mathbf{id}_{A_i}$,

ii). for $i, j, k \in I$, $i \leq j \leq k \Rightarrow \rho_{ik} = \rho_{jk} \circ \rho_{ij}$.

Remark 2.1. *Categorically, the directed system of objects of $\mathscr{A}$ indexed by I is a functor $\mathscr{O}^{\text{op}} \to \mathscr{C}$, where $\mathscr{O}$ is a category which encodes the ordered set I as a category by the same procedure as in Example 2.1. Then a directed system if a functor $\mathscr{O}^{\mathbf{op}} \to \mathscr{A}$.*

Definition 2.10. *Given a directed system $((A_i)_{i\in I}, \{\rho_{ij}\}_{i\leq j})$ of objects in $\mathscr{A}$ indexed by I. A direct limit of the system is an object $\varinjlim A_n \in \mathbf{ob}(\mathscr{A})$ satisfying the following universal property.*

Given a collection of morphisms $(f_i)_{i\in I}$ such that

i). $f_i : A_i \to \varinjlim A_n \in \mathscr{A}$,

ii). for any $i \leq j$, $f_j \circ \rho_{ij} = f_i$.

For any $A \in \mathscr{A}$ where there is a collection of morphisms $(g_i)_{i\in I}$ satisfying the above condition, there is a unique map $\varphi : \varinjlim A_n \to A$ such that

$$
\begin{tikzcd}
A_i \arrow[rr, "\rho_{ij}"] \arrow[dr, "f_i"] \arrow[ddr, bend right, "g_i"'] & & A_j \arrow[dl, "f_j"'] \arrow[ddl, bend left, "g_j"] \\
& \varinjlim A_n \arrow[d, dotted, "\exists!"] & \\
& A &
\end{tikzcd}
$$

is a commutative diagram.

Proposition 2.2. *$\varinjlim$ is an exact functor.*

Proposition 2.3. *In the cases where $\mathscr{A} = (\mathbf{Ab}), (\mathbf{Sets})$, there exist direct limits and for each category, such limit is constructed in the following ways.*

i). $\varinjlim A_n = (\bigoplus_{i\in I} A_i)/N$ where $N = \{a_i - \rho_{ij}(a_i) \mid a_i, i \leq j\}$.

ii). $\varinjlim A_n = (\coprod_{i\in I} A_i)/\sim$ where $a_i \sim a_j$ if there is k such that $i \leq k$ $j \leq k$, and $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.

Furthermore, these two direct limits match as sets.

Proposition 2.4. *$\varinjlim$ is (left) exact in $(\mathbf{Ab})$. In other words, given a exact sequence of directed systems*

$$0 \longrightarrow (M_i)_{i\in I} \longrightarrow (N_i)_{i\in I} \longrightarrow (P_i)_{i\in I} \longrightarrow 0$$

in which we have

$$
\begin{tikzcd}
0 \arrow[r] & M_i \arrow[r] \arrow[d, "\rho_{ij}^M"'] & N_i \arrow[r] \arrow[d, "\rho_{ij}^N"'] & P_i \arrow[r] \arrow[d, "\rho_{ij}^P"'] & 0 \\
0 \arrow[r] & M_j \arrow[r] & N_j \arrow[r] & P_j \arrow[r] & 0
\end{tikzcd}
$$

There exists a short exact sequence

$$0 \longrightarrow \varinjlim M_n \longrightarrow \varinjlim N_n \longrightarrow \varinjlim P_n \longrightarrow 0$$

3 Sheaf Theory

3.1 Presheaves

Definition 3.1. *Let $(X, \mathscr{T})$ be a topological space. We define the presheaf $\mathscr{F}$ of a category $\mathscr{A}$ on X such that*

- $U \in \mathscr{T}$, $\mathscr{F}(U) \in \mathrm{ob}(\mathscr{A})$,
- $U, V \in \mathscr{T}$, $V \subset U \Rightarrow$ *there exists a map* $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$

  such that

i). For any $U \in \mathscr{T}$, $\rho_{UU} = 1_{\mathscr{F}(U)}$.

ii). $U, V, W \in \mathscr{T}, W \subset V \subset U \to \rho_{UW} = \rho_{VW} \circ \rho_{UW}$.

Remark 3.1. *In the case* $\mathscr{A} = (\mathbf{Sets}), (\mathbf{Ab})$, $\mathscr{F}(\emptyset) = \emptyset, \{1\}$, *respectively.*

Definition 3.2. *An element of $\mathscr{F}(U)$ is called a local section of $\mathscr{F}$ and $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$ is called the space of sections over U. In particular $\Gamma(X, \mathscr{F})$ is called the space of global sections of $\mathscr{F}$.*

Definition 3.3. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{F}$ be a presheaf of a category $\mathscr{A}$ on X. Suppose we have two open sets $U, V \in \mathscr{T}$ such that $V \subset U$. Then for any section $s \in \mathscr{F}(U)$, $s|_V = \rho_{UV}(s)$ is called the restriction of s to V.*

Example 3.1. *Let $(X, \mathscr{T})$ be a topological space. We have a presheaf of continuous functions $\mathscr{C}_X(U) = \mathscr{C}^0(U, \mathbb{R})$. This is indeed a presheaf with restriction maps $\rho_{UV} : \mathscr{C}_X(U) \to \mathscr{C}_X(V)$. (Explicitly, $\rho_{UV}(f) = f \circ i_V$ where i_V is an inclusion map.) We note that we can introduce operations $+, \cdot$ to endow some algebraic structures (groups, rings, ...) on $\mathbb{R}$.*

Example 3.2. *Let $(X, \mathscr{T})$ be a topological space and suppose we have presheaves*

- $\mathscr{C}_X^{\textit{\textbf{diff}}}(U) = \{f : U \to \mathbb{R} \mid f \text{ is differentiable.}\}$.

Then there is an inclusion relation $\mathscr{C}_X^{\textit{\textbf{diff}}}(U) \subseteq \mathscr{C}_X(U)$ and this defines a presheaf.

Example 3.3. *Let $(X, \mathscr{T}_X)$,$(Y, \mathscr{T}_Y)$ be topological spaces. Define a presheaf on X by*

$$U \in \mathscr{T}_X, \mathscr{F}(U) = \mathscr{C}^0(X, Y).$$

And like the previous example, we define $\rho_{UV}(f) = f|_V$ for $U, V \in \mathscr{T}_X, V \subset U$. the restriction of f to V.
But this is a presheaf only of a set.

Example 3.4. *Let $(X, \mathscr{T})$ be a topological space and G be an abelian group. The constant presheaf $\mathbb{G}$ is such that*

$$U \in \mathscr{T}, \mathbb{G}(U) = G,$$

with $\rho_U V = id_G$ for any $U, V \in \mathscr{T}, V \subset U$.

3.2 Presheaves as Categories

Definition 3.4. *Let* $(X, \mathscr{T})$ *be a topological space then* $(\mathbf{Ouv}_X)$ *is the category such that its objects are the open sets of* X *and for any* $U, V \in \mathscr{T}$ *we have*

$$\mathbf{Ouv}_X(U,V) = \begin{cases} \emptyset & (V \not\subset U), \\ i_V & (V \subset U). \end{cases}$$

Definition 3.5. *Let* $(X, \mathscr{T})$ *be a topological space and* $\mathscr{A}$ *be a category. A presheaf of* $\mathscr{A}$ *on* X *is a functor* $F : \mathbf{Ouv}_X \to \mathscr{A}$.

Example 3.5. *For* $\mathbf{Ouv}_X$*, we can define a presheaf of* F *to be*

$$\mathrm{ob}(\mathbf{Ouv}_X) \ni U \mapsto F(U) = \mathscr{C}^0(U, \mathbb{R}).$$

Example 3.6. *Let* A *be a commutative ring with non-zero multiplicative identity and* $X = \mathrm{Spec}(A)$*. Let us consider the Zariski topology* $(X, \mathscr{T})$*. Let us consider a category* $\mathscr{O}_X$ *such that*

- $\mathrm{ob}(\mathscr{O}_X) = \mathscr{T}$,
- $\mathscr{O}_X(U) = \{s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}\}$,

where $s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ *is a function such that for any* $\mathfrak{p} \in U$,

i). $s(p) \in A_{\mathfrak{p}}$,

ii). there exists an open set $V \subset U$ *such that* $\mathfrak{p} \in V$ *and for any* $\mathfrak{q} \in V$, $s(\mathfrak{q}) = \frac{a}{b}$ *for* $b \notin \mathfrak{q}$.

Now we define a presheaf by the restrictions of maps such that

$$s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mapsto s|_V : V \to \coprod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

Definition 3.6. *Let* $(X, \mathscr{T})$ *be a topological space and* $\mathscr{A}$ *be a category. We define a set of presheaves of* $\mathscr{A}$ *on* X *as*

$$\mathrm{PreSh}_{\mathscr{A}}(X) = \mathrm{Fun}(\mathbf{Ouv}_X^{\mathbf{op}}, \mathscr{A}).$$

Definition 3.7. *A morphism of presheaves is a natural transformation* $\varphi : \mathscr{F} \to \mathscr{G}$ *where* $\mathscr{F}, \mathscr{G} \in \mathrm{PreSh}_{\mathscr{A}}(X) = \mathrm{Fun}(\mathbf{Ouv}_X^{\mathbf{op}}, \mathscr{A})$*.(See Definition 2.7).*

Such $\varphi : \mathscr{F} \to \mathscr{G}$ *is*

i). injective if

Remark 3.2. $\mathrm{PreSh}(X)$ *can be regarded as a category with its objects presheaves and morphisms defined above.*

Notation 3.1. *In the case* $\mathscr{A} = (\mathbf{Ab})$ *then we denote* $\mathrm{PreSh}(X) = \mathrm{PreSh}_{\mathbf{Ab}}(X)$.

Example 3.7. *Let X be a differential manifold(eg. $X \subset \mathbb{R}^n$). Let us define*

$$\mathscr{C}^{\mathbf{diff}}(U) = \{f : U \to \mathbb{R} \mid f \text{ is differentiable.}\}.$$

Then the inclusions $\mathscr{C}_X^{\mathbf{diff}}(U) \subset \mathscr{C}_X(U)$ defines the natural transformation.

Example 3.8. *Let $X, Y = S^1$ be topological spaces and F be a presheaf such that for any open set $U \subset X$, $F(U) = \mathscr{C}^0(U, Y)$.Then we can introduce a natural transformation such that*

$$\mathscr{C}_X(U) \ni f \mapsto \exp(2\pi f i).$$

3.3 Sheaves

Definition 3.8. *A presheaf $\mathscr{F}$ on $(X, \mathscr{T})$ is called a sheaf if the following holds. For any collection of open sets $(U_i)_{i\in I} \subset \mathscr{T}, U = \bigcup_{i\in I} U_i$, the map $\varphi : \mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i)$ which is defined as*

$$\varphi(s) = (s|_{U_i})_{i\in I}.$$

is the equalizer of the following functions $\varphi_1, \varphi_2 : \prod_{i\in I} \mathscr{F}(U_i) \to \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j)$,

$$\varphi_1((s_i)_{i\in I}) = (s_i|_{U_i\cap U_j})_{i,j\in I}, \quad \varphi_1((s_i)_{i\in I}) = (s_j|_{U_i\cap U_j})_{i,j\in I}.$$

Remark 3.3. *In the case $I = \{1, 2\}$, we have $U = U_1 \cup U_2$, and for any $U' \in \mathscr{T}$ such that $U \subset U'$, we have for $\mathscr{F}(U') \ni s : U' \to \mathbb{R}$,$\psi(s) = (s|_{U_1}, s|_{U_2})$, as in $\mathbf{Ouv}_X$, morphisms are inclusions. Let $\tilde{\psi}(s) = s|_U$, then this satisfies the condition for the equalizer (ie. $\varphi \circ \tilde{\psi} = \psi$).*

Remark 3.4. *A presheaf $\mathscr{O}_X$ with $X = \mathrm{Spec}(A)$ is a sheaf.*

Example 3.9. *Let $(X, \mathscr{T})$ be a topological space and G be a group. We define a constant presheaf $\mathbb{G}(U) = G$. In general, this is not a sheaf. Instead, we define a constant sheaf $\underline{\mathbb{G}}(U) = \mathscr{C}^0(U, G)$ where G is regarded as a topological space with the discrete topology. Then for any connected component of X is mapped to a single point set in G.*

Definition 3.9. *Let $\mathscr{F}_1, \mathscr{F}_2$ be sheaves. A mapping $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ is called a morphism of sheaves if it is a morphism of presheaves.*

Definition 3.10. *A set of sheaves of $\mathscr{A}$ on the topological space $(X, \mathscr{T})$ is denoted as $\mathrm{Sh}_{\mathscr{A}}(X)$.*

Remark 3.5. *As in the case of presheaves, $\mathrm{Sh}_{\mathscr{A}}(X)$ can be regarded as a category with sheaf morphisms.*

Remark 3.6. *$\mathrm{Sh}_{\mathscr{A}}(X)$ is a full-subcategory of $\mathrm{PreSh}_{\mathscr{A}}(X)$.*

Notation 3.2. *In the case $\mathscr{A} = (\mathbf{Ab})$, we denote $\mathrm{Sh}_{(\mathbf{Ab})}(X) = \mathrm{Sh}(X)$.*

3.4 Stalks

Definition 3.11. *Suppose we have a topological space $(X, \mathscr{T})$ and a category $\mathscr{A}$ which admits direct limits. For a presheaf $\mathscr{F} \in \mathrm{PreSh}_{\mathscr{A}}(X)$, by inheriting the notations from Example 2.3, we define the stalk $\mathscr{F}_x$ of $\mathscr{F}$ at $x \in X$ by*

$$\mathscr{F}_x = \varinjlim_{U \in \mathscr{O}_x} \mathscr{F}(U) = \varinjlim_{x \in U, U \in \mathscr{T}} \mathscr{F}(U).$$

Example 3.10. *Let us assume that $\mathscr{A} = (\mathbf{Ab})$ in Definition 3.11. Then stalks and germs can be constructed explicitly in the following way.*

$$\mathscr{F}_x = \{(s, U) \mid U \in \mathscr{O}_x, s \in \mathscr{F}(U)\}/\sim,$$

where $\sim$ is an equivalent relation such that for $(s, U), (t, V)$,

$$(s, U) \sim (t, V) \text{ if there is } W \in \mathscr{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t).$$

Definition 3.12. *Inheriting the notations from Definition 3.11, suppose we have $(f_U : \mathscr{F}(U) \to \mathscr{F}_x)_{U \in \mathscr{O}_x}$ such that for f_U, f_V are compatible with ρ_{UV}. Then we define the germ of $s \in \mathscr{F}(U)$ to be $s_x = f_U(s)$. By the universal property of the direct limit, such s_x is unique up to images under isomorphisms.*

Example 3.11. *In the case of Remark 3.10, we have for each $U \in \mathscr{T}$, $x \in U$, and $s \in \mathscr{F}(U)$,*

$$s_x = \{(t, V) \mid \text{ There is } W \in \mathscr{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t)\}.$$

Remark 3.7. *In the above definition, if a category $\mathscr{A}$ admits products, we get a map*

$$(s \mapsto (s_x)_{x \in U})) : \mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x. \tag{3.1}$$

This is neither surjective nor injective in general.

Proposition 3.1. *Suppose in the definition of stalks, $\mathscr{F}$ is a sheaf. Then the map defined by Equation 3.1 is injective.*

Proof. We prove the case when $\mathscr{A} = (\mathbf{Ab})$.

Suppose $s \in \mathscr{F}(U)$ is such that $s_x = 0$ in $\mathscr{F}_x$ for all $x \in U$. Since for any restriction maps are group homomorphisms. We have that there is $V_x \in \mathscr{O}_x$ such that

$$V_x \subseteq U, \quad \rho_{UV_x}(s) = 0.$$

Therefore $\{V_x\}_{x \in U}$ is an open covering of U. Since $\mathscr{F}$ is a sheaf, we derive that $s = 0$ in $\mathscr{F}(U)$. □

Example 3.12. *Given $(X, \mathscr{F})$, a topological space and G, an abelian group. We will consider the constant presheaf $\mathbb{G}$ and the constant sheaf $\underline{\mathbb{G}}$ on X. For any open set U and $x \in U$ we have*

$$\mathbb{G}_x \cong \underline{\mathbb{G}}_x \cong G.$$

For any U, V open such that $V \subset U$ we have, $\rho_{UV} = \mathbf{id}_G$. Thus by the construction, for $x \in U, V$, $(s, U) \sim (t, V)$ then $x \in U \cap V$ and $\rho_{UU\cap V}(s) = s = t = \rho_{VU\cap V}(t)$. Therefore, we proved the claim.

Definition 3.13. *Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. Then we define*

$$\varphi_x(s_x) = (\varphi(s)_U)_x.$$

This defines a morphism of presheaves.

Remark 3.8. *Categorically, taking stalks is a functor for each $x \in X$. Suppose we have $\mathscr{F}, \mathscr{G} \in \mathrm{PreSh}_{\mathscr{A}}(X)$ and a morphism $\varphi : \mathscr{F} \to \mathscr{G}$,*

Proposition 3.2. *Let $\mathscr{F}, \mathscr{G} \in \mathrm{Sh}_{(\mathbf{Ab})}(X)$ Then for any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ we have*

$$\varphi = 0 \Leftrightarrow \forall x \in X, \varphi_x = 0$$

Proof. $\Rightarrow$ is trivial by its construction. We will prove $\Leftarrow$.

We first note that $\varphi = 0$ means that for any $U \in \mathscr{T}$, we have $\varphi_U \equiv 0$ as a group homomorphism. Let $U \in \mathscr{T}$ and $s \in \mathscr{F}(U)$. Then by the assumption and Proposition 3.1, we have proven the claim. □

3.5 Sheafification

Definition 3.14. *Let $\mathscr{F} \in \mathrm{PreSh}_{\mathscr{A}}(X)$. The sheafification of $\mathscr{F}$ is a presheaf $\mathscr{F}^+$ which is a set of all $(s_x)_{x\in U} \in \prod_{x\in U} \mathscr{F}_x$ such that for any $x \in U$ there is $x \in V_x \subset U$, such that there is $t \in \mathscr{F}(V_x)$ satisfying for any $y \in V_x$, $s_y = t_y$. We give them restrictions such that*

$$\mathscr{F}^+(U) \ni (s_x)_{x\in U} \mapsto (s_x)_{x\in V} \in \mathscr{F}^+(V).$$

Proposition 3.3. *Such $\mathscr{F}^+$ is indeed a sheaf.*

Proof. later □

Remark 3.9.

$$\mathscr{F} \mapsto \mathscr{F}^+ : \mathrm{PreSh}_{\mathscr{A}}(X) \to \mathrm{Sh}_{\mathscr{A}}(X)$$

is a functor. Indeed given $\varphi : \mathscr{F} \to \mathscr{G}$, a morphism of presheaves. We give

$$\varphi^+(U)((s_x)_{x\in U}) = (\varphi(s)_x)_{x\in U}.$$

later

Proposition 3.4. *A mapping $\varphi : \mathscr{F} \to \mathscr{F}^+$ such that for each $U \in \mathscr{T}$,*

$$\varphi_U : \mathscr{F}(U) \to \mathscr{F}^+(U), \quad \varphi(s) = (s_x)_{x\in U},$$

is a natural transformation thus a morphism of presheaves.

Proof. Later $\square$

Proposition 3.5. *For any open set $U \in \mathscr{T}$ and a section $s \in \mathscr{F}^+(U)$, there is an open covering $(U_i)_{i\in I}$ which satisfies that there is a sequence $(s_i)_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i)$ and for each i, the following hodls.*

$$\rho_{UU_i}(s) = s_i.$$

Proof. Later. $\square$

Proposition 3.6. *For each $x \in X$, there exists an isomorphism*

$$\mathscr{F}_x \cong (\mathscr{F}^+)_x,$$

as presheaves.

Proof. later $\square$

Proposition 3.7. *Let $(X, \mathscr{T})$ be a topological group and $\mathscr{F}$ be a presheaf of a category $\mathscr{A}$ on X. Suppose for a sheaf $\mathscr{G}$ of a category $\mathscr{A}$ on X, there exists a morphism $\varphi : \mathscr{F} \to \mathscr{G}$. Then there exists a unique morphism $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}$, such that*

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d, "\varphi"'] & \mathscr{F}^+ \arrow[dl, dotted, "\exists! \varphi^+"] \\ \mathscr{G} & \end{tikzcd}$$

is a commutative diagram.

Proof. Let $U \in \mathscr{T}$, then by Proposition 3.5, for any $s \in \mathscr{F}^+$, there exists an open covering $(U_i)_{i\in I}$ and $(s_i)_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i)$ such that $\rho_{UU_i}(s) = s_i$ for any $i \in I$. We define

$$t_i = \varphi(s_i) \in \mathscr{G}(U_i),$$

for each $i \in I$. Using the definition of natural transformation we derive that

$$\rho^{\mathscr{G}}_{UU_i\cap U_j}(t_i) = \varphi^{\mathscr{F}}_{U_i\cap U_j}(\rho_{UU_i\cap U_j}(s)) = \rho^{\mathscr{G}}_{UU_i\cap U_j}(t_j).$$

Thus we can glue $(t_i)_{i\in I}$ to a section $t \in \mathscr{G}(U)$.

We now define $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}$. Given $(s_x)_{x\in U}$ which is the germ of s,

$$\varphi^+_U((s_x)_{x\in U}) = t.$$

Such φ^+ is unique since $\mathscr{G}$ is a sheaf. $\square$

Corollary 3.1. *Let $i : \mathrm{Sh}_{\mathscr{A}}(X) \to \mathrm{PreSh}_{\mathscr{A}}(X)$ be a forgetful functor. Then we have*

$$\mathrm{PreSh}_{\mathscr{A}}(X)(\mathscr{F}, i(\mathscr{G})) \cong \mathrm{Sh}_{\mathscr{A}}(\mathscr{F}^+, \mathscr{G})$$

In other words, the sheafification is a left-adjoint functor of the inclusion map.

Proof. By Proposition 3.7, we define two maps Φ, Ψ such that

$$\begin{gathered}\Phi : \mathrm{PreSh}_{\mathscr{A}}(X)(\mathscr{F}, i(\mathscr{G})) \to \mathrm{Sh}_{\mathscr{A}}(\mathscr{F}^+, \mathscr{G}),\\ \Phi(\varphi) = \varphi^+,\\ \Psi : \mathrm{Sh}_{\mathscr{A}}(\mathscr{F}^+, \mathscr{G}) \to \mathrm{PreSh}_{\mathscr{A}}(X)(\mathscr{F}, i(\mathscr{G})),\\ \Psi(\varphi^+) = \varphi.\end{gathered}$$

Then these two are inverses of each other. □

Proposition 3.8. *Let $X = \mathrm{Spec}(A)$ and $\mathscr{O}_X$ be the structure sheaf defined in Example 3.6. Then we have the following.*

1). For any $\mathfrak{p} = x \in X$, $(\mathscr{O}_X)_x \cong A_{\mathfrak{p}}$.

2). For any $a \in A$, $\mathscr{O}_X(D(a)) \cong A_a$.

Proof. For a given $U \subset X$ open and $\mathfrak{p} \subset A$, there is $a, b \in A$ such that for $V \subset U$ open and $s \in \mathscr{O}_X(U), s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$.

$$s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}$$

holds for any $\mathfrak{q} \in V$.

$$\begin{tikzcd} \mathscr{O}_X(U) \arrow[r] \arrow[d, "\rho_{UV}"'] & A_{\mathfrak{p}} \\ \mathscr{O}_X(V) \arrow[ur] & \end{tikzcd}$$

□

3.6 Morphisms in $\mathrm{PreSh}_{(\mathbf{Ab})}(X)$

Definition 3.15. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism of presheaves $\mathrm{PreSh}_{(\mathbf{Ab})}(X)$. Then we define the following.*

1). $\mathrm{Ker}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Ker}\,\varphi_U$,

2). $\mathrm{Im}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Im}\,\varphi_U$,

3). $\mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Coker}\,\varphi_U$.

Proposition 3.9. *Such* $\mathrm{Ker}^{\mathbf{pre}}, \mathrm{Im}^{\mathbf{pre}}, \mathrm{Coker}^{\mathbf{pre}}$ *are presheaves.*

Proof. For the case of kernels. Let $U, V \in \mathscr{T}$ and $V \subset U$. We define $\rho_U V : \mathrm{Ker}^{\mathbf{pre}}(\varphi)(U) \to \mathrm{Ker}^{\mathbf{pre}}(\varphi)(V)$ to be such that

$$\rho_U V(s) = \rho^{\mathscr{F}}(s).$$

Such construction is justified as the diagram below is commutative.

$$\begin{array}{ccccc}
\mathscr{F}(U) & \xrightarrow{\rho^{\mathscr{F}}_{UV}} & \mathscr{F}(V) & \xrightarrow{\rho^{\mathscr{F}}_{UV}} & \mathscr{F}(W) \\
{\scriptstyle \varphi_U}\downarrow & & \downarrow{\scriptstyle \varphi_V} & & \downarrow{\scriptstyle \varphi_W} \\
\mathscr{G}(U) & \xrightarrow[\rho^{\mathscr{G}}_{UV}]{} & \mathscr{G}(V) & \xrightarrow[\rho^{\mathscr{G}}_{UV}]{} & \mathscr{F}(W)
\end{array}$$

Furthermore,

$$\rho_U W(s) = \rho^{\mathscr{F}}_{UV}(s) = \rho^{\mathscr{F}}_{VW} \circ \rho^{\mathscr{F}}_{UV}(s) = \rho_{VW} \circ \rho_{UV}(s).$$

Thus $\mathrm{Ker}^{\mathbf{pre}}(\varphi)(U)$ is a presheaf. $\square$

Corollary 3.2. *If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves. Then $\mathrm{Ker}^{\mathbf{pre}}$ is also a sheaf.*

Proof. Given $(s_i)_{i \in I} \in \prod_{i \in I} \mathrm{Ker}\, \varphi_{U_i}$ such that

$$\rho(s_i)_{U_i U_i \cap U_j} = \rho(s_j)_{U_j U_i \cap U_j}$$

for any $i, j \in I$. Then since $\mathscr{F}$ is a sheaf, we can glue $(s_i)_{i \in I}$ to $s \in \mathscr{F}(U)$. For such s we have

$$\rho^{\mathscr{G}}_{UU_i}(\varphi_U(s)) = (\varphi_{U_i}(\rho^{\mathscr{F}}_{UU_i}(s))) = \varphi_{UU_i}(s_i) = 0.$$

Therefore, since $\mathscr{G}$ is a sheaf, $\varphi_U(s) = 0$. $\square$

Remark 3.10. *Let $\varphi : \mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i), \varphi_1 : \prod_{i \in I} \mathscr{F}(U_i) \to \prod_{i,j \in I} \mathscr{F}(U_i \cap U_j), \varphi_2 : \prod_{i \in I} \mathscr{F}(U_j) \to \prod_{i,j \in I} \mathscr{F}(U_i \cap U_j)$. Then $\mathscr{F}$ is a sheaf if and only if*

$$\mathrm{Ker}(\varphi_1 \circ \varphi - \varphi_2 \circ \varphi) = \mathscr{F}(U),$$

holds for any open set U.

Remark 3.11. $\mathrm{Im}^{\mathbf{pre}}\,\varphi, \mathrm{Coker}^{\mathbf{pre}}\,\varphi$ *are not in general sheaves even tho $\varphi : \mathscr{F} \to \mathscr{G}$ is a homomorphism of sheaves.*

Example 3.13. *Let $X = \{x_1, x_2\}$ and we assign the discrete topology to it. Let G be an abelian group. We define a sheaf $\mathscr{F}, \mathscr{G} \in \mathrm{Sh}_{(\mathbf{Ab})}(X)$ by such that*

$$\mathscr{F}(U) = \mathscr{G}(U) = \begin{cases} G \times G \quad (U = X), \\ G \quad (|U| = 1), \\ 0 \quad (U = \emptyset). \end{cases}$$

Let us define a homomorphism of sheaves φ such that

$$\varphi_U = \begin{cases} \mathbf{id}_{G \times G} \quad (U = X) \\ 0 \quad (U \neq X). \end{cases}$$

Then we have

$$\mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) = \begin{cases} 0 & (U = X), \\ G & (U \neq X). \end{cases}$$

By 3.11, we observe that

$$\mathrm{Coker}^{\mathbf{pre}}(\varphi)(X) = G \times G / \mathbf{id}_{G\times G}(G \times G) = \{0\}.$$

However,

later.

Definition 3.16. *Given a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, we define the following.*

1). $\mathrm{Ker}(\varphi) = \mathrm{Ker}^{\mathbf{pre}}(\varphi)$,

2). $\mathrm{Im}(\varphi) = (\mathrm{Im}^{\mathbf{pre}}(\varphi))^{+}$,

3). $\mathrm{Coker}(\varphi) = (\mathrm{Coker}^{\mathbf{pre}}(\varphi))^{+}$.

Proposition 3.10 (Universal property of kernels)**.** *Given a sheaf homomorphism $\varphi : \mathscr{F} \to \mathscr{G}$. For any sheaf homomorphism $\alpha : \mathscr{H} \to \mathscr{F}$, $\varphi \circ \alpha = 0$ if and only if there is a unique $\psi : \mathscr{H} \to \mathrm{Ker}\,\varphi$ such that*

$$\begin{tikzcd}
& \mathscr{H} \arrow[dl, dotted, "\exists!\psi"'] \arrow[d, "\alpha"] \arrow[dr, "\varphi_0=0"] & \\
\mathrm{Ker}(\varphi) \arrow[r, hook] & \mathscr{F} \arrow[r, "\varphi"'] & \mathscr{G}
\end{tikzcd}$$

is a commutative diagram.

Proof. We argue by each open set of the space.

$$\begin{tikzcd}
& \mathscr{H}(U) \arrow[dl, dotted, "\exists!\psi_U"'] \arrow[d, "\alpha_U"] \arrow[dr, "(\varphi_0)_U=0"] & \\
\mathrm{Ker}(\varphi)(U) \arrow[r, hook] & \mathscr{F}(U) \arrow[r, "\varphi_U"'] & \mathscr{G}(U)
\end{tikzcd}$$

This is a universal property of the kernel in abelian groups. Thus the statement immediately follows from it. ☐

Proposition 3.11 (Universal property of Cokernels)**.** *Given a sheaf homomorphism $\varphi : \mathscr{F} \to \mathscr{G}$. For any sheaf homomorphism $\alpha : \mathscr{G} \to \mathscr{H}$, $\alpha \circ \varphi = 0$ if and only if there is a unique $\psi : \mathrm{Coker}\,\varphi \to \mathscr{H}$ such that*

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"] \arrow[dr, "\varphi_0=0"'] & \mathscr{G} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \mathrm{Coker}(\varphi) \arrow[dl, dotted, "\exists!\psi"] \\
& \mathscr{H} &
\end{tikzcd}$$

is a commutative diagram.

Proof. We argue for each open set $U \subset X$.

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[dr, "(\varphi_0)_U=0"'] & \mathscr{G}(U) \arrow[r] \arrow[d, "\alpha_U"] & \mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) \arrow[r] \arrow[dl, dotted, "\exists!\psi_U^{\mathbf{pre}}"'] & \mathrm{Coker}(\varphi)(U) \arrow[dll, dotted, "\exists!\psi_U"] \\
 & \mathscr{H}(U) & &
\end{tikzcd}$$

By the universal property of Cokernels of abelian groups, there is a unique $\varphi^{\mathbf{pre}}$. By the universal property of the sheafification operator, we derive a unique ψ. □

Proposition 3.12. *Let $x \in X$, then we have the following.*

1). $\mathrm{Ker}(\varphi)_x = \mathrm{Ker}(\varphi_x)$,

2). $\mathrm{Im}(\varphi)_x = \mathrm{Im}(\varphi_x)$,

3). $\mathrm{Coker}(\varphi)_x = \mathrm{Coker}(\varphi_x)$.

Proof. By Definition, 3.13 □

Definition 3.17. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism. φ is called*

1). a monomorphism if any morphism of sheaves $\varphi_0 : \mathscr{H} \to \mathscr{F}$, $\varphi \circ \varphi_0 = 0$ if and only if $\varphi_0 = 0$,

Proposition 3.13. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of* $(\mathbf{Ab})$*. Then the following statements are equivalent.*

i). φ is a monomorphism.

ii). $\mathrm{Ker}\,\varphi = 0$.

iii). For any open set $U \subset X$, φ_U is injective.

iv). For any $x \in X$, $\varphi_x :\to \mathscr{F}_x \to \mathscr{G}_x$ is injective.

Proof. Here, I put the procedure of the proof.

$$\begin{tikzcd}
i) \arrow[d] & iv) \arrow[d, leftrightarrow] \\
ii) \arrow[r, leftrightarrow] & iii) \arrow[ul]
\end{tikzcd}$$

$i) \Rightarrow ii)$,

$$\begin{tikzcd}
\mathrm{Ker}(\varphi) \arrow[d, "\varphi_0"'] \arrow[dr, "0"] & \\
\mathscr{F} \arrow[r, "\varphi"'] & \mathscr{G}
\end{tikzcd}$$

Where $\varphi_0(U)$ is an inclusion map of abelian groups.

ii) $\Leftrightarrow$ *iii)*,

$$\operatorname{Ker}\varphi = 0 \Leftrightarrow \forall U \in \mathscr{T}, \operatorname{Ker}\varphi(U) = 0 \Leftrightarrow \varphi_U \text{ is injective.}$$

iii) $\Rightarrow$ *iv)*, Fix $x \in X$.

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\varphi_U} \mathscr{G}(U)$$

is an exact sequence as φ_U is injective for any $U \subset X$ open. Since $\varinjlim$ is left-exact we obtain,

$$0 \longrightarrow \mathscr{F}_x \xrightarrow{\varphi_x} \mathscr{G}_x$$

is also an exact sequence. $\square$

Proposition 3.14. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism in $\operatorname{Sh}(X)$. Then the following are equivalent.*

1). φ is an epimorphism (for any $\varphi_1, \varphi_2 : \mathscr{H} \to \mathscr{F}$, such that $\varphi \circ \varphi_1 = \varphi \circ \varphi_2$ implies $\varphi_1 = \varphi_2$).

2). $\operatorname{Coker}\varphi = 0$.

3). For any open set $U \subset X$,

4). For any $x \in X$, $\operatorname{Coker}\varphi_x = 0$, *(in other words, φ_x is a surjection).*

Proof. Recall the definition of epimorphisms is such that $\varphi : \mathscr{F} \to \mathscr{G}$ is an epimorphism if for any morphism $\psi : \mathscr{G} \to \mathscr{G}_0$, we have,

$$\psi \circ \varphi = 0 \Rightarrow \psi = 0.$$

i) $\Rightarrow$ *iv)*. Suppose φ is an epimorphism, then we have

$$\begin{tikzcd}
\mathscr{F} \arrow[d, "\varphi"'] \arrow[dr, "0"] \arrow[drr, bend left, "0"] & & \\
\mathscr{G} \arrow[r, "\pi"] \arrow[rr, bend right, "\psi"'] & \operatorname{Coker}^{\mathbf{pre}}\varphi \arrow[r, "(-)^+"] & \operatorname{Coker}\varphi
\end{tikzcd}$$

By the assumption $\psi = 0$.

Let $\mathscr{O}_x = \{U \in \mathscr{T} \mid x \in U\}$. We consider an exact sequence,

$$0 \longrightarrow \operatorname{Ker}(\varphi_U) \hookrightarrow \mathscr{F}(U) \xrightarrow{\varphi} \mathscr{G}(U) \xrightarrow{\pi} \operatorname{Coker}(\varphi_U) \longrightarrow 0,$$

for each $U \in \mathscr{O}_x$. By Proposition 2.2,

$$0 \longrightarrow \mathrm{Ker}(\varphi)_x \hookrightarrow \mathscr{F}_x \xrightarrow{\varphi_x} \mathscr{G}_x \xrightarrow{\pi_x} \mathrm{Coker}(\varphi)_x \longrightarrow 0$$

is also exact. Thus we conclude

$$\mathrm{Coker}^{pre}(\varphi)_x = \mathrm{Coker}(\varphi_x).$$

□