

Algebraic Geometry 1 Week 3 Exercise Sheet Solutions

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Exercise 15

Let us consider a short exact sequence

$$0 \rightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi} \mathcal{O}_{S^1} \xrightarrow{\exp} \mathcal{O}_{S^1}^* \rightarrow 1.$$

Since each $s : S^1 \rightarrow \mathbb{Z}$ is continuous, $\bigcup_n \in \mathbb{Z}s^{-1}(n)$ is an open cover of S^1 and each of them are disjoint. By the connectedness of S^1 we conclude that s is constant on S^1 . In general, $s \in \underline{\mathbb{Z}}(U)$ is constant on each connected component in U . From the last exercise of the first week exercise sheet we know that $\text{Coker } f_* \exp = 0$, therefore by the definition of the exact sequence we conclude that $R^1 f_* \underline{\mathbb{Z}} = 0$.

For any connected open set U in S^1 , $f^{-1}(U) = (\frac{1}{2}U) \cup (\frac{1}{2}U + \pi)$. Thus it consists of two disjoint connected components. When taking direct limits, we can restrict the indexing open sets to be connected. Similarly from the 6th exercise, we conclude that

$$(f_* \underline{\mathbb{Z}})_x = (2\pi i \mathbb{Z})^{\pi_0((\frac{1}{2}U) \cup (\frac{1}{2}U + \pi))} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Exercise 16

We know that $\text{Sh}(\{x\}) = (\mathbf{Ab})$. And for such inclusion $i : \{x\} \rightarrow X$,

$$i_{x*}(G)(U) = \begin{cases} G & (x \in U), \\ \emptyset & (x \notin U). \end{cases}$$

Given an exact sequence of groups

$$0 \rightarrow G \rightarrow H \rightarrow J \xrightarrow{\pi} 0.$$

We know that i_{x*} is left-exact. Therefore, we will examine if i_{x*} sends $\pi : H \rightarrow J$ to an epimorphism. Indeed

$$i_{x*}(\pi)(U) = \begin{cases} \pi & (x \in U), \\ * & (x \notin U). \end{cases}$$

Therefore, for each $U \subset X$ we have $\text{Coker}(i_{x*}(\pi)(U)) = 0$. Thus i_{x*} is exact.