Algebraic Geometry 1 Week 1 Exercise Sheet Solutions

So Murata

2024/2025 Winter Semester - Uni Bonn

Exercise 1

(i)

For any $\psi : P \to M$ such that $\varphi_1 \circ \psi = \varphi_2 \circ \psi$, thus the image of P under ψ is contained in K. Therefore, we can define a mapping $\tilde{\psi} = x \mapsto \psi(x) : P \to K$. As φ is an inclusion, $\varphi \circ \tilde{\psi} = \psi$. The uniqueness follows from the fact that φ is inclusion and any ψ_1, ψ_2 such that $\varphi \circ \psi_1 = \varphi \circ \psi_2$ we have $\psi_1 = \psi_2$.

(ii)

$x \in \mathrm{Ker}(\varphi_1 - \varphi_2) \Leftrightarrow \varphi_1(x) - \varphi_2(x) = 0 \Leftrightarrow \varphi_1(x) = \varphi_2(x)$. Therefore, we can use the previous argument and conclude that the inclusion is an equalizer.

Exercise 2

0.1 (i)

Let $f \in \mathscr{C}(U)$. Then for any $x \in W$, we have $x \in V$ and thus $x \in U$. Therefore, $(f|_V)|_W(x) = f|_V(x) = f(x) = f|_W(x)$. In the case $W = \emptyset$ we have $\rho_U W(f) = *$, and $\rho_{VW} \circ \rho_{UV}(f) = *$ for any $f \in \mathscr{C}(U)$.

0.2 (ii)

Let $\psi : \mathscr{C}(U') \to \prod \mathscr{C}(V_i)$ be such that $\varphi_1 \circ \psi = \varphi_2 \circ \psi$ where $U \subset U'$. Let us define $\tilde{\psi} : \mathscr{C}(U') \to \mathscr{C}(U)$ to be such that

$$\tilde{\psi}(f)|_{V_i} = \psi(f)_i.$$

Then

$$\varphi \circ \tilde{\psi}(f)|_{V_i} = \psi(f)_i.$$

which is equal to ψ. And this $\tilde{\psi}$ is unique as φ is injective. The injectivity comes from that $(V_i)_i$ is a covering of U, thus for any function $f \in \mathscr{C}(U')$ and any point x has an index i such that $x \in V_i$. Therefore, any image of morphism $\tilde{\psi}' : \mathscr{C}(U') \to \mathscr{C}(U)$, $\tilde{\psi}(f) \in \mathscr{C}(U)$ equals to $\tilde{\psi}(f)$.

Exercise 3

For $(m_x)_{x \in U} \in \mathscr{F}(U)$, we define the map $\rho_{UV}((m_x)_{x\in U}) = (m_y)_{y \in V}$. i) follows from the inclusion relations for $W \subset V \subset U$.

(ii)

Exercise 4

Construction

Let us define a presheaf such that

$$\mathscr{O}(U) = \bigcap_{(p) \in U} \mathbb{Z}_{(p)}.$$

Then this is isomorphic to

$$\mathbb{Z}\left[\frac{1}{n}\right] \quad \text{where } n = \prod_{i=1}^{n} p_i,$$

such that each $\{p_i\}_{i=1,\cdots,n}$ is the set of all primes not contained in any of $(p) \in U$. We can assure that there are finitely many such primes. By the definition, the closed set of the Zariski topology is of the form $V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subset \mathbb{Z}$. $\mathbb{Z}$ is a principle ideal domain, there is $a \in \mathbb{Z}$ such that $\mathfrak{a} = (a)$. Since a has finitely many prime divisors, we now conclude that there are finitely many primes belong to $X \backslash U$.

We will now show that $\mathscr{O}(U) = \mathbb{Z}\left[\frac{1}{n}\right]$. Any $\frac{a}{b} \in \mathscr{O}(U)$, $b \in \mathbb{Z}\backslash(p)$ for each $(p) \in U$. Therefore, the primes that appear in the prime decomposition of b is the subset of $\{p_i\}_{i=1,\cdots,n}$. As $\frac{1}{p_i} = \frac{\prod_{j \neq i} p_j}{n}$, $\frac{1}{p_i} \in \mathbb{Z}[\frac{1}{n}]$. This implies that $\bigcap_{(p)\in U} \mathbb{Z}_{(p)} \subseteq \mathbb{Z}[\frac{1}{n}]$. By the construction of n, we have $\mathbb{Z}[\frac{1}{n}] \subseteq \bigcap_{(p)\in U} \mathbb{Z}_{(p)}$. Hence they are isomorphic.

(i)

Let us now define $\rho_{UV}(\frac{a}{b}) = \frac{a}{b}$, then this is well-defined when $V \subseteq U$ since such inclusion implies $\bigcap_{(p)\in U} \mathbb{Z}_{(p)} \subseteq \bigcap_{(q)\in V} \mathbb{Z}_{(q)}$.

Thus for any $W \subset V \subset U \subset X$ open,

$$\rho_{UW}\left(\frac{a}{b}\right) = \frac{a}{b} = \rho_{VW}\left(\frac{a}{b}\right) = \rho_{VW}\left(\rho_{UV}\left(\frac{a}{b}\right)\right).$$

Therefore, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

As in the previous exercise, for an open covering $(V_i)_{i\in I}$ with $U = \bigcup_{i\in I} V_i$, we define $\varphi_1, \varphi_2 : \prod_{i\in I} \mathscr{O}(V_i) \to \prod_{i,j\in I} \mathscr{O}(V_i \cap V_j)$ such that

$$\varphi_1\left(\frac{a_i}{b_i}\right) = \left(\rho_{UV_i\cap V_j}\left(\frac{a_i}{b_i}\right)\right)_{i,j\in I}, \varphi_2\left(\frac{a_i}{b_i}\right) = \left(\rho_{UV_i\cap V_j}\left(\frac{a_j}{b_j}\right)\right)_{i,j\in I}.$$

(ii)

We will show that

$$\varphi : \mathscr{O}(U) \to \prod_{i\in I} \mathscr{O}(V_i), \quad \varphi\left(\frac{a}{b}\right) = \left(\rho_{UV_i}\left(\frac{a}{b}\right)\right)$$

is an equalizer of (φ_1, φ_2).

Suppose for $U' \subset X$ open and there is $\psi : \mathscr{O}(U') \to \prod_{i\in I} \mathscr{O}(V_i)$ such that

$$\varphi_1 \circ \psi = \varphi_2 \circ \psi. \tag{1}$$

We define $\tilde{\psi}\left(\frac{a}{b}\right) = \psi\left(\frac{a}{b}\right)_1$. By Equation 1, we get that for each $\psi\left(\frac{a}{b}\right)_i = \psi\left(\frac{a}{b}\right)_j$ in $V_i \cap V_j$ for any $i, j \in I$. Hence for each $i \in I$,

$$\varphi(\tilde{\psi})\left(\frac{a}{b}\right)_i = \psi\left(\frac{a}{b}\right)_1 = \psi\left(\frac{a}{b}\right)_i. \tag{2}$$

We've proven that $\varphi \circ \tilde{\psi} = \psi$.

Conversely, suppose $\tilde{\psi}$ is such that $\varphi \circ \tilde{\psi} = \psi$. Then for any $i \in I$.

$$\psi\left(\frac{a}{b}\right)_i = \rho_{UV_i}\left(\tilde{\psi}\left(\frac{a}{b}\right)\right) = \tilde{\psi}\left(\frac{a}{b}\right).$$

In particular,

$$\psi\left(\frac{a}{b}\right)_1 = \tilde{\psi}\left(\frac{a}{b}\right).$$

1 Exercise 5

Let $\mathscr{F}, \mathscr{G}$ be presheaves on a topological space X. We need to find homomorphisms $\rho_{UV} : \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U) \to \mathrm{Hom}(\mathscr{F}|_V, \mathscr{G}|_V)$ such that $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ holds for any $W \subset V \subset U \subset X$ open. Let $\alpha : \mathscr{F}_U \to \mathscr{G}_U$ be a presheaf homomorphism. We define $\rho_{UV}(\alpha)$ to be such that for any $W \subset V$, $\rho_{UV}(\alpha)_W = \alpha_W$. By this construction, $\rho_{UU} = \mathbf{id}_{\mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)}$. And this satisfies the inclusion criterion, thus a presheaf.

Let us denote the presheaf by $\mathscr{H}(U) = \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$. Suppose $\{U_i\}_{i\in I}$ is an open covering of U. Then we define $\varphi, \varphi_1, \varphi_2$ as in the definition of sheaves.

We will now show that φ is an equalizer of (φ_1, φ_2).

Suppose there is $\psi : M \to \prod_{i\in I} \mathscr{H}(U_i)$ such that $\varphi_1 \circ \psi = \varphi_2 \circ \psi$ for some $M \in (\mathbf{Ab})$. We are going to construct a suitable $\tilde{\psi}$.

For each $i \in I$ and an open subset V of U, we define $\{V_i\}_{i\in I}$ to be a collection where $V_i = V \cap U_i$ for each $i \in I$. Then this is an open covering of V.

$$\begin{tikzcd}
\mathscr{F}(V_i) \arrow[r, "\rho^{\mathscr{F}}_{V_iV_i\cap V_j}"] \arrow[d, "(\psi(\alpha)_i)_{V_i}"'] & \mathscr{F}(V_i\cap V_j) \arrow[d, "(\psi(\alpha)_i)_{V_i\cap V_j}"', shift right] \arrow[d, "(\psi(\alpha)_j)_{V_i\cap V_j}", shift left] & \mathscr{F}(V_j) \arrow[l, "\rho^{\mathscr{F}}_{V_iV_i\cap V_i}"'] \arrow[d, "(\psi(\alpha)_j)_{V_j}"] \\
\mathscr{G}(V_i) \arrow[r, "\rho^{\mathscr{G}}_{V_iV_i\cap V_j}"'] & \mathscr{G}(V_i\cap V_j) & \mathscr{G}(V_j) \arrow[l, "\rho^{\mathscr{G}}_{V_jV_i\cap V_j}"]
\end{tikzcd}$$

By the assumption of ψ, we have the equality

$$(\psi(\alpha)_i)_{V_i\cap U_j} = \rho_{V_iV_i\cap V_j}(\psi(\alpha)_i)_{V_i\cap V_j} = \rho_{V_jV_i\cap V_j}(\psi(\alpha)_j)_{V_i\cap V_j} = (\psi(\alpha)_j)_{V_i\cap V_j}.$$

Therefore, the diagram above is well-define and commutative.

For each $s \in \mathscr{F}(V)$ and $i \in I$, we define $s_i = \rho_{VV_i}(s)$ and $t_i \in \mathscr{G}(V_i)$ to be a section such that

$$t_i = (\psi(\alpha)_i)_{V_i}(s_i).$$

$$\begin{tikzcd}
 & \mathscr{F}(V) \arrow[dl, "\rho^{\mathscr{F}}_{VV_i}"'] \arrow[dr, "\rho^{\mathscr{F}}_{VV_j}"] & \\
\mathscr{F}(V_i) \arrow[r, "\rho^{\mathscr{F}}_{V_iV_i\cap V_j}"] \arrow[d, "(\psi(\alpha)_i)_{V_i}"'] & \mathscr{F}(V_i\cap V_j) \arrow[d, "(\psi(\alpha)_i)_{V_i\cap V_j}"', shift right] \arrow[d, "(\psi(\alpha)_j)_{V_i\cap V_j}", shift left] & \mathscr{F}(V_j) \arrow[l, "\rho^{\mathscr{F}}_{V_jV_i\cap V_j}"'] \arrow[d, "(\psi(\alpha)_j)_{V_j}"] \\
\mathscr{G}(V_i) \arrow[r, "\rho^{\mathscr{G}}_{V_iV_i\cap V_j}"'] & \mathscr{G}(V_i\cap V_j) & \mathscr{G}(V_j) \arrow[l, "\rho^{\mathscr{G}}_{V_jV_i\cap V_j}"]
\end{tikzcd}$$

This diagram is commutative since $\mathscr{F}$ is a presheaf and the upper-triangle part of the diagram is commutative. Furthermore, we have $\mathscr{G}$ is a sheaf. Thus combining with the commutativity of the diagram, we deduce that there exists a section $t \in \mathscr{G}(U)$ such that $\rho^{\mathscr{G}}_{VV_i}(t) = t_i$. We now define the natural transformation $\tilde{\psi}(\alpha)$ to be

$$\tilde{\psi}(\alpha)_V(s) = t.$$

We now check such $\tilde{\psi}$ satisfies the conditions.

First we will show that such $\tilde{\psi}$ is indeed a sheaf homomorphism. For any open subset V, W of U such that $W \subset V$, we define $\{W_i\}_{i\in I}$ to be $W_i = W \cap U_i$

for each $i \in I$. Then this is an open cover of W. Therefore, since $\mathscr{G}$ is a sheaf, it is sufficient to show that for each $i \in$I, the following equality holds.

$$\rho^{\mathscr{G}}_{WW_i}(\rho^{\mathscr{G}}_{VW} \circ (\tilde{\psi}(\alpha))_V(s)) = \rho^{\mathscr{G}}_{WW_i}(\tilde{\psi}(\alpha)_W(\rho^{\mathscr{F}}_{VW}(s))).$$

Indeed, as $\psi(\alpha)$ is an natural transformation, the left-hand side is equal to

$$\begin{aligned}
\rho^{\mathscr{G}}_{VW_i}(\tilde{\psi}(\alpha)_V(s)) &= \rho^{\mathscr{G}}_{V_iW_i} \circ \rho^{\mathscr{G}}_{VV_i}(\tilde{\psi}(\alpha)_V(s)), \\
&= \rho^{\mathscr{G}}_{V_iW_i}((\psi(\alpha)_i)_{V_i}(\rho^{\mathscr{F}}_{VV_i}(s))), \\
&= (\psi(\alpha)_i)_{W_i}(\rho^{\mathscr{F}}_{VW_i}(s)), \\
&= (\psi(\alpha)_i)_{W_i}(\rho^{\mathscr{F}}_{WW_i} \circ \rho^{\mathscr{F}}_{VW}(s)), \\
&= \rho^{\mathscr{F}}_{WW_i}((\psi(\alpha)_i)_W(\rho^{\mathscr{F}}_{VW}(s))), \\
&= \rho^{\mathscr{G}}_{WW_i}(\tilde{\psi}(\alpha)_W(\rho^{\mathscr{F}}_{VW}(s))).
\end{aligned}$$

Thus such $\tilde{\psi}$ defines a natural transformation at each $\alpha \in M$. We then prove that

$$\varphi \circ \tilde{\psi} = \psi.$$

Let us define $\{V_i\}_{i \in I}$ as before. Then we have $V_i = V$ for i such that $V \subset U_i$. By the construction of $\tilde{\psi}$,

$$\tilde{\psi}(\alpha)_V(s) = t = \rho^{\mathscr{G}}_{VV}(t) = \rho^{\mathscr{G}}_{VV_i}(t) = t_i = (\psi(\alpha)_i)_{V_i}(s) = (\psi(\alpha)_i)_V(s).$$

Thus we proved the claim. Finally we prove that such $\tilde{\psi}$ is unique. This is due to the fact that $\mathscr{G}$ is a sheaf as for any open subset V of U and $s \in \mathscr{F}(V)$, let $t = \tilde{\psi}(\alpha)_V(s)$,

$$\rho_{VV_i}(t) = (\psi(\alpha)_i)_{V_i}(\rho_{VV_i}(s)).$$

Thus such t is uniquely determined for each s.

2 Exercise 6

Let us define a collection of open sets $U_{(x,n)} = B(x, \frac{|x|}{n})$. Then $\mathscr{B} = \{U_{(x,n)}\}_{X \times \mathbb{N}}$ forms a basis of the space which each basis element is simply connected as they do not contain the origin.

In order to show the subjectivity, we first note that by Cauchy's integral formula, we have that for any holomorphic function $f : U \to \mathbb{C}$ over a simply connected domain U, there exists a anti-derivative $F : U \to \mathbb{C}$ of f on U.

For $U \in \mathscr{B}$ and $f \in \mathscr{O}(U)^*_X$, let us define a function $h : \mathscr{U} \to \mathbb{C}$, such that

$$h(z) = \frac{f'(z)}{f(z)}.$$

Since f is nowhere 0, this is a well-defined holomorphic function on a simply connected open set U. Thus it has an anti-derivative $H : U \to \mathbb{C}$.

We know show that $f = \exp(H)$.

$$\begin{aligned}(f\exp(-H))' &= f'\exp(-H) + f(-H')\exp(-H),\\ &= f'\exp(-H) - f'\exp(-H),\\ &= 0.\end{aligned}$$

From this we conclude that $f(z) = c\exp(H) = \exp(H + C)$, for some constants $c, C \in \mathbb{C}$. Thus the map $\exp : \mathscr{O}_X(U) \to \mathscr{O}_X^*(U)$ is a surjection.

We now examine the kernel of the morphism. Let $z \in U$,

$$e^{f(z)} = e^{g(z)} \Rightarrow \frac{e^{f(z)}}{e^{g(z)}} = 1 = e^{f(z)-g(z)} = e^{2\pi k i}$$

for some $k \in \mathbb{Z}$. By the continuity of f, g, we have that $f - g$ is also continuous and the values $f - g$ can take is the discrete set $2\pi i\mathbb{Z}$. Therefore $f - g$ is a fixed value $2\pi k i$ for some $k \in \mathbb{Z}$. We conclude $\mathrm{Ker}(\exp) = 2\pi i\mathbb{Z}$.