

# Algebraic Geometry 1

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## 1 Topology

### 1.1 Connected Sets

**Definition 1.1.** Let  $(X, \mathcal{T})$  be a topological space. A subset  $A$  of  $X$  is said to be connected if for any  $U, V \in \mathcal{T}$ ,  $U \cap V = \emptyset$ ,  $U \cup V \supset A$  then  $A$  is fully contained in one of  $U, V$ .

**Definition 1.2.** A connected component of a topological space is a maximal connected subset of a space.

**Proposition 1.1.** Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be topological space and  $f : X \rightarrow Y$  be a continuous function. Then for any connected subset  $A$  of  $X$ ,  $f(A)$  is connected in  $Y$ .

*Proof.*

$$\begin{aligned} U, V \in \mathcal{T}_Y, U \cup V \supset f(A), U \cap V = \emptyset, \\ \Rightarrow f^{-1}(U), f^{-1}(V) \in \mathcal{T}_X, \\ f^{-1}(U) \cup f^{-1}(V) \supset A, \\ f^{-1}(U) \cap f^{-1}(V) = \emptyset, \\ \Rightarrow f^{-1}(U) \supset A \vee f^{-1}(V) \supset A, \\ \Rightarrow U \supset f(A) \vee V \supset f(A). \end{aligned}$$

□

## 2 Category Theory

### 2.1 Categories

**Definition 2.1.** A category  $\mathcal{A}$  consists of

- a collection  $\text{ob}(\mathcal{A})$  of objects;
- for each  $A, B \in \text{ob}(\mathcal{A})$ , a collection  $\mathcal{A}(A, B)$  of morphisms from  $A$  to  $B$ ;

such that

- i). for each  $A \in \text{ob}(\mathcal{A})$ , the identity  $1_A \in \mathcal{A}(A, A)$ ;
- ii). the composition  $\mathcal{A}(B, C) \times \mathcal{A}(A, B) \ni (g, f) \mapsto g \circ f \in \mathcal{A}(A, C)$  is well-defined;

and they satisfy the following axioms

- I). Associativity :  $f \in \mathcal{A}(A, B), g \in \mathcal{A}(B, C), h \in \mathcal{A}(C, D), (h \circ g) \circ f = h \circ (g \circ f)$ .
- II). Identity laws :  $f \in \mathcal{A}(A, B)$  then  $f \circ 1_A = 1_B \circ f$ .

**Definition 2.2.** Let  $\mathcal{A}$  be a category. A terminal object  $T \in \text{ob}(\mathcal{A})$  is an object such that for any  $A \in \text{ob}(\mathcal{A})$ ,  $\mathcal{A}(A, T)$  is a single element set.

**Definition 2.3.** Given two categories  $\mathcal{A}, \mathcal{B}$ , we say  $\mathcal{A}$  is a full-subcategory of  $\mathcal{B}$  if

- i).  $\mathcal{A} \subset \mathcal{B}$ ,
- ii).  $\text{ob}(\mathcal{A}) = \text{ob}(\mathcal{B})$ .

**Notation 2.1.** Here we give notations to some important categories.

- **(Sets)** : A category of sets equipped with set theoretic functions.
- **(Ab)** : A category of abelian groups with group homomorphisms.

**Example 2.1.** Given a partially ordered set  $(X, \leq)$ . This can be encoded to a category  $\mathcal{O}$  by

- i).  $\text{ob}(\mathcal{O}) = X$ ,
- ii). For  $x, y \in X, x \leq y \Rightarrow \mathcal{O}(x, y) = \{*\}$  otherwise the morphisms between  $x, y$  is an empty set.

**Definition 2.4.** A opposite/dual category of a category  $\mathcal{A}$  is  $\mathcal{A}^{op}$  such that

- i).  $\text{ob}(\mathcal{A}^{op}) = \text{ob}(\mathcal{A})$ ,
- ii).  $\mathcal{A}^{op}(B, A) = \mathcal{A}(A, B)$ .

**Definition 2.5.** Let  $\mathcal{A}$  be a category and  $\varphi_1, \varphi_2 \in \mathcal{A}(M, N)$ . A morphism  $\varphi : K \rightarrow M$  is called an equalizer of  $(\varphi_1, \varphi_2)$  if for any morphism  $\psi : P \rightarrow M$  such that  $\varphi_1 \circ \psi = \varphi_2 \circ \psi$ , there is a unique morphism  $\tilde{\psi} : P \rightarrow K$  such that  $\varphi \circ \tilde{\psi} = \psi$ .

**Proposition 2.1.** If an equalizer exists then it is unique up to unique isomorphism.

*Proof.* Suppose  $\varphi : K \rightarrow M, \psi : L \rightarrow M$  be equalizers of  $(\varphi_1, \varphi_2)$ . Then we have

$$\varphi \circ \tilde{\psi} = \psi, \quad \psi \circ \tilde{\varphi} = \varphi$$

By the uniqueness, we have  $\tilde{\varphi} \circ \tilde{\psi} = 1_L, \tilde{\psi} \circ \tilde{\varphi} = 1_K$ .  $\square$

**Definition 2.6.** Let  $\mathcal{A}, \mathcal{B}$  be categories. A functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is a function such that for each  $f \in \mathcal{A}(A, A')$ ,  $F(f) : F(A) \rightarrow F(A')$ . In other words,  $f \mapsto F(f) : \mathcal{A}(A, A') \rightarrow \mathcal{B}(F(A), F(A'))$ . Furthermore,  $F$  satisfies the following axioms.

I).  $F(f' \circ f) = F(f') \circ F(f)$  whenever  $f : A \rightarrow A', f' : A' \rightarrow A''$  in  $\mathcal{A}$ ,

II).  $F(1_A) = 1_{F(A)}$  whenever  $A \in \mathcal{A}$ .

**Definition 2.7.** Let  $F, G$  be functors between two categories  $\mathcal{A}, \mathcal{B}$ . A natural transformation  $\alpha : F \rightarrow G$  is a family  $(\alpha_A : F(A) \rightarrow G(A))_{A \in \mathcal{A}}$  such that

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ \alpha_A \downarrow & & \downarrow \alpha_{A'} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

is a commutative diagram. Each  $\alpha_A$  is called a component of  $\alpha$ .

## 2.2 Direct Limits

**Definition 2.8.** A partially ordered set  $(X, \leq)$  is directed if for any  $x, y \in X$  there is  $z \in X$  such that  $x \leq z$  and  $y \leq z$ .

**Example 2.2.** Let  $(X, \mathcal{T})$  be a topological space. A partially ordered set  $(\mathcal{T}, \leq)$  such that

$$V \subseteq U \Rightarrow U \leq V$$

is directed. Since for any  $U \in \mathcal{T}$ ,  $U \leq \emptyset$ . As a category this is  $\mathbf{Ouv}_X^{\text{op}}$ .

**Example 2.3.** Let  $(X, \mathcal{T})$  be a topological space. For  $x \in X$ , define  $O_x = \{U \in \mathcal{T} \mid x \in U\}$ . If we define an order as in the previous example, we get  $(O_x, \leq)$  is directed. This follows from for any  $U, V \in O_x$ ,  $U, V \leq U \cap V$ .

**Definition 2.9.** Let  $I$  be a directed partially ordered set and  $\mathcal{A}$  be a category. A directed system of objects of  $\mathcal{A}$  indexed by  $I$  is a collection of objects  $(A_i)_{i \in I}$  and morphisms  $(\rho_{ij})_{i \leq j}$  of  $\mathcal{A}$  such that

i).  $\rho_{ii} = \text{id}_{A_i}$ ,

ii). for  $i, j, k \in I$ ,  $i \leq j \leq k \Rightarrow \rho_{ik} = \rho_{jk} \circ \rho_{ij}$ .

**Remark 2.1.** Categorically, the directed system of objects of  $\mathcal{A}$  indexed by  $I$  is a functor  $\mathcal{O}^{\text{op}} \rightarrow \mathcal{C}$ , where  $\mathcal{O}$  is a category which encodes the ordered set  $I$  as a category by the same procedure as in Example 2.1. Then a directed system is a functor  $\mathcal{O}^{\text{op}} \rightarrow \mathcal{A}$ .

**Definition 2.10.** Given a directed system  $((A_i)_{i \in I}, \{\rho_{ij}\}_{i \leq j})$  of objects in  $\mathcal{A}$  indexed by  $I$ . A direct limit of the system is an object  $\varinjlim A_n \in \mathbf{ob}(\mathcal{A})$  satisfying the following universal property.

Given a collection of morphisms  $(f_i)_{i \in I}$  such that

- i).  $f_i : A_i \rightarrow \varinjlim A_n \in \mathcal{A}$ ,
- ii). for any  $i \leq j$ ,  $f_j \circ \rho_{ij} = f_i$ .

For any  $A \in \mathcal{A}$  where there is a collection of morphisms  $(g_i)_{i \in I}$  satisfying the above condition, there is a unique map  $\varphi : \varinjlim A_n \rightarrow A$  such that

$$\begin{array}{ccccc}
 A_i & \xrightarrow{\rho_{ij}} & & A_j & \\
 & \searrow f_i & & \swarrow f_j & \\
 & & \varinjlim A_n & & \\
 & \swarrow g_i & \downarrow \exists! & \searrow g_j & \\
 & & A & & 
 \end{array}$$

is a commutative diagram.

**Proposition 2.2.** In the cases where  $\mathcal{A} = (\mathbf{Ab}), (\mathbf{Sets})$ , there exist direct limits and for each category, such limit is constructed in the following ways.

- i).  $\varinjlim A_n = (\bigoplus_{i \in I} A_i) / N$  where  $N = \{a_i - \rho_{ij}(a_i) \mid a_i, i \leq j\}$ .
- ii).  $\varinjlim A_n = (\coprod_{i \in I} A_i) / \sim$  where  $a_i \sim a_j$  if there is  $k$  such that  $i \leq k \leq j$ , and  $\rho_{ik}(a_i) = \rho_{jk}(a_j)$ .

Furthermore, these two direct limits match as sets.

**Proposition 2.3.**  $\varinjlim$  is (left) exact in  $(\mathbf{Ab})$ . In other words, given a exact sequence of directed systems

$$0 \longrightarrow (M_i)_{i \in I} \longrightarrow (N_i)_{i \in I} \longrightarrow (P_i)_{i \in I} \longrightarrow 0$$

in which we have

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M_i & \longrightarrow & N_i & \longrightarrow & P_i \longrightarrow 0 \\
 & & \rho_{ij}^M \downarrow & & \rho_{ij}^N \downarrow & & \rho_{ij}^P \downarrow \\
 0 & \longrightarrow & M_j & \longrightarrow & N_j & \longrightarrow & P_j \longrightarrow 0
 \end{array}$$

There exists a short exact sequence

$$0 \longrightarrow \varinjlim M_n \longrightarrow \varinjlim N_n \longrightarrow \varinjlim P_n \longrightarrow 0$$

## 3 Sheaf Theory

### 3.1 Presheaves

**Definition 3.1.** Let  $(X, \mathcal{T})$  be a topological space. We define the presheaf  $\mathcal{F}$  of a category  $\mathcal{A}$  on  $X$  such that

- $U \in \mathcal{T}, \mathcal{F}(U) \in \text{ob}(\mathcal{A}),$
- $U, V \in \mathcal{T}, V \subset U \Rightarrow$  there exists a map  $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$   
such that
  - i). For any  $U \in \mathcal{T}, \rho_{UU} = 1_{\mathcal{F}(U)}.$
  - ii).  $U, V, W \in \mathcal{T}, W \subset V \subset U \rightarrow \rho_{UW} = \rho_{VW} \circ \rho_{UV}.$

**Remark 3.1.** In the case  $\mathcal{A} = (\mathbf{Sets}), (\mathbf{Ab}), \mathcal{F}(\emptyset) = \emptyset, \{1\},$  respectively.

**Definition 3.2.** An element of  $\mathcal{F}(U)$  is called a local section of  $\mathcal{F}$  and  $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$  is called the space of sections over  $U$ . In particular  $\Gamma(X, \mathcal{F})$  is called the space of global sections of  $\mathcal{F}$ .

**Definition 3.3.** Let  $(X, \mathcal{T})$  be a topological space and  $\mathcal{F}$  be a presheaf of a category  $\mathcal{A}$  on  $X$ . Suppose we have two open sets  $U, V \in \mathcal{T}$  such that  $V \subset U$ . Then for any section  $s \in \mathcal{F}(U)$ ,  $s|_V = \rho_{UV}(s)$  is called the restriction of  $s$  to  $V$ .

**Example 3.1.** Let  $(X, \mathcal{T})$  be a topological space. We have a presheaf of continuous functions  $\mathcal{C}_X(U) = \mathcal{C}^0(U, \mathbb{R})$ . This is indeed a presheaf with restriction maps  $\rho_{UV} : \mathcal{C}_X(U) \rightarrow \mathcal{C}_X(V)$ . (Explicitly,  $\rho_{UV}(f) = f \circ i_V$  where  $i_V$  is an inclusion map.) We note that we can introduce operations  $+, \cdot$  to endow some algebraic structures (groups, rings, ...) on  $\mathbb{R}$ .

**Example 3.2.** Let  $(X, \mathcal{T})$  be a topological space and suppose we have presheaves

- $\mathcal{C}_X^{\text{diff}}(U) = \{f : U \rightarrow \mathbb{R} \mid f \text{ is differentiable}\}.$

Then there is an inclusion relation  $\mathcal{C}_X^{\text{diff}}(U) \subseteq \mathcal{C}_X(U)$  and this defines a presheaf.

**Example 3.3.** Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be topological spaces. Define a presheaf on  $X$  by

$$U \in \mathcal{T}_X, \mathcal{F}(U) = \mathcal{C}^0(X, Y).$$

And like the previous example, we define  $\rho_{UV}(f) = f|_V$  for  $U, V \in \mathcal{T}_X, V \subset U$ . the restriction of  $f$  to  $V$ .

But this is a presheaf only of a set.

**Example 3.4.** Let  $(X, \mathcal{T})$  be a topological space and  $G$  be an abelian group. The constant presheaf  $\mathbb{G}$  is such that

$$U \in \mathcal{T}, \mathbb{G}(U) = G,$$

with  $\rho_U V = \text{id}_G$  for any  $U, V \in \mathcal{T}, V \subset U$ .

### 3.2 Presheaves as Categories

**Definition 3.4.** Let  $(X, \mathcal{T})$  be a topological space then  $(\mathbf{Ouv}_X)$  is the category such that its objects are the open sets of  $X$  and for any  $U, V \in \mathcal{T}$  we have

$$\mathbf{Ouv}_X(U, V) = \begin{cases} \emptyset & (V \not\subset U), \\ i_V & (V \subset U). \end{cases}$$

**Definition 3.5.** Let  $(X, \mathcal{T})$  be a topological space and  $\mathcal{A}$  be a category. A presheaf of  $\mathcal{A}$  on  $X$  is a functor  $F : \mathbf{Ouv}_X \rightarrow \mathcal{A}$ .

**Example 3.5.** For  $\mathbf{Ouv}_X$ , we can define a presheaf of  $F$  to be

$$\text{ob}(\mathbf{Ouv}_X) \ni U \mapsto F(U) = \mathcal{C}^0(U, \mathbb{R}).$$

**Example 3.6.** Let  $A$  be a commutative ring with non-zero multiplicative identity and  $X = \text{Spec}(A)$ . Let us consider the Zariski topology  $(X, \mathcal{T})$ . Let us consider a category  $\mathcal{O}_X$  such that

- $\text{ob}(\mathcal{O}_X) = \mathcal{T}$ ,
- $\mathcal{O}_X(U) = \{s : U \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}\}$ ,

where  $s : U \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$  is a function such that for any  $\mathfrak{p} \in U$ ,

- i).  $s(p) \in A_{\mathfrak{p}}$ ,
- ii). there exists an open set  $V \subset U$  such that  $\mathfrak{p} \in V$  and for any  $\mathfrak{q} \in V$ ,  $s(\mathfrak{q}) = \frac{a}{b}$  for  $b \notin \mathfrak{q}$ .

Now we define a presheaf by the restrictions of maps such that

$$s : U \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mapsto s|_V : V \rightarrow \coprod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

**Definition 3.6.** Let  $(X, \mathcal{T})$  be a topological space and  $\mathcal{A}$  be a category. We define a set of presheaves of  $\mathcal{A}$  on  $X$  as

$$\text{PreSh}_{\mathcal{A}}(X) = \text{Fun}(\mathbf{Ouv}_X^{\text{op}}, \mathcal{A}).$$

**Definition 3.7.** A morphism of presheaves is a natural transformation  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  where  $\mathcal{F}, \mathcal{G} \in \text{PreSh}_{\mathcal{A}}(X) = \text{Fun}(\mathbf{Ouv}_X^{\text{op}}, \mathcal{A})$ . (See Definition 2.7).

Such  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is

- i). injective if

**Remark 3.2.**  $\text{PreSh}(X)$  can be regarded as a category with its objects presheaves and morphisms defined above.

**Notation 3.1.** In the case  $\mathcal{A} = (\mathbf{Ab})$  then we denote  $\text{PreSh}(X) = \text{PreSh}_{\mathbf{Ab}}(X)$ .

**Example 3.7.** Let  $X$  be a differential manifold (eg.  $X \subset \mathbb{R}^n$ ). Let us define

$$\mathcal{C}^{\text{diff}}(U) = \{f : U \rightarrow \mathbb{R} \mid f \text{ is differentiable}\}.$$

Then the inclusions  $\mathcal{C}_X^{\text{diff}}(U) \subset \mathcal{C}_X(U)$  defines the natural transformation.

**Example 3.8.** Let  $X, Y = S^1$  be topological spaces and  $F$  be a presheaf such that for any open set  $U \subset X$ ,  $F(U) = \mathcal{C}^0(U, Y)$ . Then we can introduce a natural transformation such that

$$\mathcal{C}_X(U) \ni f \mapsto \exp(2\pi f i).$$

### 3.3 Sheaves

**Definition 3.8.** A presheaf  $\mathcal{F}$  on  $(X, \mathcal{T})$  is called a sheaf if the following holds. For any collection of open sets  $(U_i)_{i \in I} \subset \mathcal{T}$ ,  $U = \bigcup_{i \in I} U_i$ , the map  $\varphi : \mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i)$  which is defined as

$$\varphi(s) = (s|_{U_i})_{i \in I}.$$

is the equalizer of the following functions  $\varphi_1, \varphi_2 : \prod_{i \in I} \mathcal{F}(U_i) \rightarrow \prod_{i, j \in I} \mathcal{F}(U_i \cap U_j)$ ,

$$\varphi_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})_{i, j \in I}, \quad \varphi_2((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})_{i, j \in I}.$$

**Remark 3.3.** In the case  $I = \{1, 2\}$ , we have  $U = U_1 \cup U_2$ , and for any  $U' \in \mathcal{T}$  such that  $U \subset U'$ , we have for  $\mathcal{F}(U') \ni s : U' \rightarrow \mathbb{R}$ ,  $\psi(s) = (s|_{U_1}, s|_{U_2})$ , as in  $\mathbf{Ouv}_X$ , morphisms are inclusions. Let  $\tilde{\psi}(s) = s|_U$ , then this satisfies the condition for the equalizer (ie.  $\varphi \circ \tilde{\psi} = \psi$ ).

**Remark 3.4.** A presheaf  $\mathcal{O}_X$  with  $X = \text{Spec}(A)$  is a sheaf.

**Example 3.9.** Let  $(X, \mathcal{T})$  be a topological space and  $G$  be a group. We define a constant presheaf  $\mathbb{G}(U) = G$ . In general, this is not a sheaf. Instead, we define a constant sheaf  $\underline{\mathbb{G}}(U) = \mathcal{C}^0(U, G)$  where  $G$  is regarded as a topological space with the discrete topology. Then for any connected component of  $X$  is mapped to a single point set in  $G$ .

**Definition 3.9.** Let  $\mathcal{F}_1, \mathcal{F}_2$  be sheaves. A mapping  $\varphi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is called a morphism of sheaves if it is a morphism of presheaves.

**Definition 3.10.** A set of sheaves of  $\mathcal{A}$  on the topological space  $(X, \mathcal{T})$  is denoted as  $\text{Sh}_{\mathcal{A}}(X)$ .

**Remark 3.5.** As in the case of presheaves,  $\text{Sh}_{\mathcal{A}}(X)$  can be regarded as a category with sheaf morphisms.

**Remark 3.6.**  $\text{Sh}_{\mathcal{A}}(X)$  is a full-subcategory of  $\text{PreSh}_{\mathcal{A}}(X)$ .

**Notation 3.2.** In the case  $\mathcal{A} = (\mathbf{Ab})$ , we denote  $\text{Sh}_{(\mathbf{Ab})}(X) = \text{Sh}(X)$ .

### 3.4 Stalks

**Definition 3.11.** Suppose we have a topological space  $(X, \mathcal{T})$  and a category  $\mathcal{A}$  which admits direct limits. For a presheaf  $\mathcal{F} \in \text{PreSh}_{\mathcal{A}}(X)$ , by inheriting the notations from Example 2.3, we define the stalk  $\mathcal{F}_x$  of  $\mathcal{F}$  at  $x \in X$  by

$$\mathcal{F}_x = \varinjlim_{U \in \mathcal{O}_x} \mathcal{F}(U) = \varinjlim_{x \in U, U \in \mathcal{T}} \mathcal{F}(U).$$

**Example 3.10.** Let us assume that  $\mathcal{A} = (\mathbf{Ab})$  in Definition 3.11. Then stalks and germs can be constructed explicitly in the following way.

$$\mathcal{F}_x = \{(s, U) \mid U \in \mathcal{O}_x, s \in \mathcal{F}(U)\} / \sim,$$

where  $\sim$  is an equivalent relation such that for  $(s, U), (t, V)$ ,

$$(s, U) \sim (t, V) \text{ if there is } W \in \mathcal{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t).$$

**Definition 3.12.** Inheriting the notations from Definition 3.11, suppose we have  $(f_U : \mathcal{F}(U) \rightarrow \mathcal{F}_x)_{U \in \mathcal{O}_x}$  such that for  $f_U, f_V$  are compatible with  $\rho_{UV}$ . Then we define the germ of  $s \in \mathcal{F}(U)$  to be  $s_x = f_U(s)$ . By the universal property of the direct limit, such  $s_x$  is unique up to images under isomorphisms.

**Example 3.11.** In the case of Remark 3.10, we have for each  $U \in \mathcal{T}$ ,  $x \in U$ , and  $s \in \mathcal{F}(U)$ ,

$$s_x = \{(t, V) \mid \text{There is } W \in \mathcal{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t)\}.$$

**Remark 3.7.** In the above definition, if a category  $\mathcal{A}$  admits products, we get a map

$$(s \mapsto (s_x)_{x \in U}) : \mathcal{F}(U) \rightarrow \prod_{x \in U} \mathcal{F}_x. \quad (3.1)$$

This is neither surjective nor injective in general.

**Proposition 3.1.** Suppose in the definition of stalks,  $\mathcal{F}$  is a sheaf. Then the map defined by Equation 3.1 is injective.

*Proof.* We prove the case when  $\mathcal{A} = (\mathbf{Ab})$ .

Suppose  $s \in \mathcal{F}(U)$  is such that  $s_x = 0$  in  $\mathcal{F}_x$  for all  $x \in U$ . Since for any restriction maps are group homomorphisms. We have that there is  $V_x \in \mathcal{O}_x$  such that

$$V_x \subseteq U, \quad \rho_{UV_x}(s) = 0.$$

Therefore  $\{V_x\}_{x \in U}$  is an open covering of  $U$ . Since  $\mathcal{F}$  is a sheaf, we derive that  $s = 0$  in  $\mathcal{F}(U)$ .  $\square$

**Example 3.12.** Given  $(X, \mathcal{T})$ , a topological space and  $G$ , an abelian group. We will consider the constant presheaf  $\mathbb{G}$  and the constant sheaf  $\underline{\mathbb{G}}$  on  $X$ . For any open set  $U$  and  $x \in U$  we have

$$\mathbb{G}_x \cong \underline{\mathbb{G}}_x \cong G.$$



For any  $U, V$  open such that  $V \subset U$  we have,  $\rho_{UV} = \mathbf{id}_G$ . Thus by the construction, for  $x \in U, V$ ,  $(s, U) \sim (t, V)$  then  $x \in U \cap V$  and  $\rho_{U \cap V}(s) = s = t = \rho_{V \cap V}(t)$ . Therefore, we proved the claim.

**Definition 3.13.** Suppose  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of presheaves. Then we define

$$\varphi_x(s_x) = (\varphi(s))_U.$$

This defines a morphism of presheaves.

**Remark 3.8.** Categorically, taking stalks is a functor for each  $x \in X$ . Suppose we have  $\mathcal{F}, \mathcal{G} \in \text{PreSh}_{\mathcal{A}}(X)$  and a morphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ ,

**Proposition 3.2.** Let  $\mathcal{F}, \mathcal{G} \in \text{Sh}_{(\mathbf{Ab})}(X)$  Then for any morphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  we have

$$\varphi = 0 \Leftrightarrow \forall x \in X, \varphi_x = 0$$

*Proof.*  $\Rightarrow$  is trivial by its construction. We will prove  $\Leftarrow$ .

We first note that  $\varphi = 0$  means that for any  $U \in \mathcal{T}$ , we have  $\varphi_U \equiv 0$  as a group homomorphism. Let  $U \in \mathcal{T}$  and  $s \in \mathcal{F}(U)$ . Then by the assumption and Proposition 3.1, we have proven the claim.  $\square$

### 3.5 Sheafification

**Definition 3.14.** Let  $\mathcal{F} \in \text{PreSh}_{\mathcal{A}}(X)$ . The sheafification of  $\mathcal{F}$  is a presheaf  $\mathcal{F}^+$  which is a set of all  $(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$  such that for any  $x \in U$  there is  $x \in V_x \subset U$ , such that there is  $t \in \mathcal{F}(V_x)$  satisfying for any  $y \in V_x$ ,  $s_y = t_y$ . We give them restrictions such that

$$\mathcal{F}^+(U) \ni (s_x)_{x \in U} \mapsto (s_x)_{x \in V} \in \mathcal{F}^+(V).$$

**Proposition 3.3.** Such  $\mathcal{F}^+$  is indeed a sheaf.

*Proof.* later  $\square$

**Remark 3.9.**

$$\mathcal{F} \mapsto \mathcal{F}^+ : \text{PreSh}_{\mathcal{A}}(X) \rightarrow \text{Sh}_{\mathcal{A}}(X)$$

is a functor. Indeed given  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ , a morphism of presheaves. We give

$$\varphi^+(U)((s_x)_{x \in U}) = (\varphi(s))_{x \in U}.$$

later

**Proposition 3.4.** A mapping  $\varphi : \mathcal{F} \rightarrow \mathcal{F}^+$  such that for each  $U \in \mathcal{T}$ ,

$$\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{F}^+(U), \quad \varphi(s) = (s_x)_{x \in U},$$

is a natural transformation thus a morphism of presheaves.

*Proof.* Later □

**Proposition 3.5.** *For any open set  $U \in \mathcal{T}$  and a section  $s \in \mathcal{F}^+(U)$ , there is an open covering  $(U_i)_{i \in I}$  which satisfies that there is a sequence  $(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i)$  and for each  $i$ , the following holds.*

$$\rho_{UU_i}(s) = s_i.$$

*Proof.* Later. □

**Proposition 3.6.** *For each  $x \in X$ , there exists an isomorphism*

$$\mathcal{F}_x \cong (\mathcal{F}^+)_x,$$

*as presheaves.*

*Proof.* later □

**Proposition 3.7.** *Let  $(X, \mathcal{T})$  be a topological group and  $\mathcal{F}$  be a presheaf of a category  $\mathcal{A}$  on  $X$ . Suppose for a sheaf  $\mathcal{G}$  of a category  $\mathcal{A}$  on  $X$ , there exists a morphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ . Then there exists a unique morphism  $\varphi^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$ , such that*

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ \varphi \downarrow & \nearrow \exists! \varphi^+ & \\ \mathcal{G} & & \end{array}$$

*is a commutative diagram.*

*Proof.* Let  $U \in \mathcal{T}$ , then by Proposition 3.5, for any  $s \in \mathcal{F}^+$ , there exists an open covering  $(U_i)_{i \in I}$  and  $(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i)$  such that  $\rho_{UU_i}(s) = s_i$  for any  $i \in I$ . We define

$$t_i = \varphi(s_i) \in \mathcal{G}(U_i),$$

for each  $i \in I$ . Using the definition of natural transformation we derive that

$$\rho_{UU_i \cap U_j}^{\mathcal{G}}(t_i) = \varphi_{U_i \cap U_j}^{\mathcal{F}}(\rho_{UU_i \cap U_j}(s)) = \rho_{UU_i \cap U_j}^{\mathcal{G}}(t_j).$$

Thus we can glue  $(t_i)_{i \in I}$  to a section  $t \in \mathcal{G}(U)$ .

We now define  $\varphi^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$ . Given  $(s_x)_{x \in U}$  which is the germ of  $s$ ,

$$\varphi_U^+((s_x)_{x \in U}) = t.$$

Such  $\varphi^+$  is unique since  $\mathcal{G}$  is a sheaf. □

**Corollary 3.1.** *Let  $i : \text{Sh}_{\mathcal{A}}(X) \rightarrow \text{PreSh}_{\mathcal{A}}(X)$  be a forgetful functor. Then we have*

$$\text{PreSh}_{\mathcal{A}}(X)(\mathcal{F}, i(\mathcal{G})) \cong \text{Sh}_{\mathcal{A}}(\mathcal{F}^+, \mathcal{G})$$

*In other words, the sheafification is a left-adjoint functor of the inclusion map.*

*Proof.* By Proposition 3.7, we define two maps  $\Phi, \Psi$  such that

$$\begin{aligned}\Phi : \text{PreSh}_{\mathcal{A}}(X)(\mathcal{F}, i(\mathcal{G})) &\rightarrow \text{Sh}_{\mathcal{A}}(\mathcal{F}^+, \mathcal{G}), \\ \Phi(\varphi) &= \varphi^+, \\ \Psi : \text{Sh}_{\mathcal{A}}(\mathcal{F}^+, \mathcal{G}) &\rightarrow \text{PreSh}_{\mathcal{A}}(X)(\mathcal{F}, i(\mathcal{G})), \\ \Psi(\varphi^+) &= \varphi.\end{aligned}$$

Then these two are inverses of each other.  $\square$

**Proposition 3.8.** *Let  $X = \text{Spec}(A)$  and  $\mathcal{O}_X$  be the structure sheaf defined in Example 3.6. Then we have the following.*

- 1). For any  $\mathfrak{p} = x \in X$ ,  $(\mathcal{O}_X)_x \cong A_{\mathfrak{p}}$ .
- 2). For any  $a \in A$ ,  $\mathcal{O}_X(D(a)) \cong A_a$ .

*Proof.* For a given  $U \subset X$  open and  $\mathfrak{p} \subset A$ , there is  $a, b \in A$  such that for  $V \subset U$  open and  $s \in \mathcal{O}_X(U)$ ,  $s : U \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ .

$$s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}$$

holds for any  $\mathfrak{q} \in V$ .

$$\begin{array}{ccc} \mathcal{O}_X(U) & \longrightarrow & A_{\mathfrak{p}} \\ \rho_{UV} \downarrow & \nearrow & \\ \mathcal{O}_X(V) & & \end{array}$$

$\square$

### 3.6 Morphisms in $\text{PreSh}_{(\mathbf{Ab})}(X)$

**Definition 3.15.** *Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a homomorphism of presheaves  $\text{PreSh}_{(\mathbf{Ab})}(X)$ . Then we define the following.*

- 1).  $\text{Ker}^{\mathbf{pre}}(\varphi)(U) = \text{Ker } \varphi_U$ ,
- 2).  $\text{Im}^{\mathbf{pre}}(\varphi)(U) = \text{Im } \varphi_U$ ,
- 3).  $\text{Coker}^{\mathbf{pre}}(\varphi)(U) = \text{Coker } \varphi_U$ .

**Proposition 3.9.** *Such  $\text{Ker}^{\mathbf{pre}}, \text{Im}^{\mathbf{pre}}, \text{Coker}^{\mathbf{pre}}$  are presheaves.*

*Proof.* For the case of kernels. Let  $U, V \in \mathcal{T}$  and  $V \subset U$ . We define  $\rho_{UV} : \text{Ker}^{\mathbf{pre}}(\varphi)(U) \rightarrow \text{Ker}^{\mathbf{pre}}(\varphi)(V)$  to be such that

$$\rho_{UV}(s) = \rho^{\mathcal{F}}(s).$$

Such construction is justified as the diagram below is commutative.

$$\begin{array}{ccccc}
\mathcal{F}(U) & \xrightarrow{\rho_{UV}^{\mathcal{F}}} & \mathcal{F}(V) & \xrightarrow{\rho_{UV}^{\mathcal{F}}} & \mathcal{F}(W) \\
\varphi_U \downarrow & & \downarrow \varphi_V & & \downarrow \varphi_W \\
\mathcal{G}(U) & \xrightarrow{\rho_{UV}^{\mathcal{G}}} & \mathcal{G}(V) & \xrightarrow{\rho_{UV}^{\mathcal{G}}} & \mathcal{F}(W)
\end{array}$$

Furthermore,

$$\rho_U W(s) = \rho_{UV}^{\mathcal{F}}(s) = \rho_{VW}^{\mathcal{F}} \circ \rho_{UV}^{\mathcal{F}}(s) = \rho_{VW} \circ \rho_{UV}(s).$$

Thus  $\text{Ker}^{\text{pre}}(\varphi)(U)$  is a presheaf.  $\square$

**Corollary 3.2.** *If  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism of sheaves. Then  $\text{Ker}^{\text{pre}}$  is also a sheaf.*

*Proof.* Given  $(s_i)_{i \in I} \in \prod_{i \in I} \text{Ker } \varphi_{U_i}$  such that

$$\rho(s_i)_{U_i U_i \cap U_j} = \rho(s_j)_{U_j U_i \cap U_j}$$

for any  $i, j \in I$ . Then since  $\mathcal{F}$  is a sheaf, we can glue  $(s_i)_{i \in I}$  to  $s \in \mathcal{F}(U)$ . For such  $s$  we have

$$\rho_{U U_i}^{\mathcal{G}}(\varphi_U(s)) = (\varphi_{U_i}(\rho_{U U_i}^{\mathcal{F}}(s))) = \varphi_{U U_i}(s_i) = 0.$$

Therefore, since  $\mathcal{G}$  is a sheaf,  $\varphi_U(s) = 0$ .  $\square$

**Remark 3.10.** *Let  $\varphi : \mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i)$ ,  $\varphi_1 : \prod_{i \in I} \mathcal{F}(U_i) \rightarrow \prod_{i, j \in I} \mathcal{F}(U_i \cap U_j)$ ,  $\varphi_2 : \prod_{i \in I} \mathcal{F}(U_j) \rightarrow \prod_{i, j \in I} \mathcal{F}(U_i \cap U_j)$ . Then  $\mathcal{F}$  is a sheaf if and only if*

$$\text{Ker}(\varphi_1 \circ \varphi - \varphi_2 \circ \varphi) = \mathcal{F}(U),$$

*holds for any open set  $U$ .*

**Remark 3.11.**  $\text{Im}^{\text{pre}} \varphi, \text{Coker}^{\text{pre}} \varphi$  are not in general sheaves even tho  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is a homomorphism of sheaves.

**Example 3.13.** *Let  $X = \{x_1, x_2\}$  and we assign the discrete topology to it. Let  $G$  be an abelian group. We define a sheaf  $\mathcal{F}, \mathcal{G} \in \text{Sh}_{(\mathbf{Ab})}(X)$  by such that*

$$\mathcal{F}(U) = \mathcal{G}(U) = \begin{cases} G \times G & (U = X), \\ G & (|U| = 1), \\ 0 & (U = \emptyset). \end{cases}$$

*Let us define a homomorphism of sheaves  $\varphi$  such that*

$$\varphi_U = \begin{cases} \text{id}_{G \times G} & (U = X) \\ 0 & (U \neq X). \end{cases}$$

Then we have

$$\mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) = \begin{cases} 0 & (U = X), \\ G & (U \neq X). \end{cases}$$

By 3.11, we observe that

$$\mathrm{Coker}^{\mathbf{pre}}(\varphi)(X) = G \times G / \mathrm{id}_{G \times G}(G \times G) = \{0\}.$$

However,

later.

**Definition 3.16.** Given a morphism of sheaves  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ , we define the following.

- 1).  $\mathrm{Ker}(\varphi) = \mathrm{Ker}^{\mathbf{pre}}(\varphi)$ ,
- 2).  $\mathrm{Im}(\varphi) = (\mathrm{Im}^{\mathbf{pre}}(\varphi))^+$ ,
- 3).  $\mathrm{Coker}(\varphi) = (\mathrm{Coker}^{\mathbf{pre}}(\varphi))^+$ .

**Proposition 3.10** (Universal property of kernels). *Given a sheaf homomorphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ . For any sheaf homomorphism  $\alpha : \mathcal{H} \rightarrow \mathcal{F}$ ,  $\varphi \circ \alpha = 0$  if and only if there is a unique  $\psi : \mathcal{H} \rightarrow \mathrm{Ker} \varphi$  such that*

$$\begin{array}{ccccc} & & \mathcal{H} & & \\ & \exists! \psi \swarrow & \downarrow \alpha & \searrow \varphi_0=0 & \\ \mathrm{Ker}(\varphi) & \hookrightarrow & \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \end{array}$$

is a commutative diagram.

*Proof.* We argue by each open set of the space.

$$\begin{array}{ccccc} & & \mathcal{H}(U) & & \\ & \exists! \psi_U \swarrow & \downarrow \alpha_U & \searrow (\varphi_0)_U=0 & \\ \mathrm{Ker}(\varphi)(U) & \hookrightarrow & \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \end{array}$$

This is a universal property of the kernel in abelian groups. Thus the statement immediately follows from it.  $\square$

**Proposition 3.11** (Universal property of Cokernels). *Given a sheaf homomorphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ . For any sheaf homomorphism  $\alpha : \mathcal{G} \rightarrow \mathcal{H}$ ,  $\alpha \circ \varphi = 0$  if and only if there is a unique  $\psi : \mathrm{Coker} \varphi \rightarrow \mathcal{H}$  such that*

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} & \xrightarrow{\pi} & \mathrm{Coker}(\varphi) \\ & \searrow \varphi_0=0 & \downarrow \alpha & \swarrow \exists! \psi & \\ & & \mathcal{H} & & \end{array}$$

is a commutative diagram.

*Proof.* We argue for each open set  $U \subset X$ .

$$\begin{array}{ccccccc}
 \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) & \longrightarrow & \text{Coker}^{\text{pre}}(\varphi)(U) & \longrightarrow & \text{Coker}(\varphi)(U) \\
 & \searrow (\varphi_0)_U = 0 & \downarrow \alpha_U & & \exists! \psi_U^{\text{pre}} & \nearrow & \\
 & & \mathcal{H}(U) & & & \nearrow \exists! \psi_U & 
 \end{array}$$

By the universal property of Cokernels of abelian groups, there is a unique  $\varphi^{\text{pre}}$ . By the universal property of the sheafification operator, we derive a unique  $\psi$ .  $\square$

**Proposition 3.12.** *Let  $x \in X$ , then we have the following.*

- 1).  $\text{Ker}(\varphi)_x = \text{Ker}(\varphi_x)$ ,
- 2).  $\text{Im}(\varphi)_x = \text{Im}(\varphi_x)$ ,
- 3).  $\text{Coker}(\varphi)_x = \text{Coker}(\varphi_x)$ .

*Proof.* By Definition, 3.13  $\square$

**Definition 3.17.** *Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a sheaf morphism.  $\varphi$  is called*

- 1). *a monomorphism if any morphism of sheaves  $\varphi_0 : \mathcal{H} \rightarrow \mathcal{F}$ ,  $\varphi \circ \varphi_0 = 0$  if and only if  $\varphi_0 = 0$ ,*

**Proposition 3.13.** *Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of sheaves of  $(\mathbf{Ab})$ . Then the following statements are equivalent.*

- i).  $\varphi$  is a monomorphism.
- ii).  $\text{Ker } \varphi = 0$ .
- iii). *For any open set  $U \subset X$ ,  $\varphi_U$  is injective.*
- iv). *For any  $x \in X$ ,  $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$  is injective.*

*Proof.* Here, I put the procedure of the proof.

$$\begin{array}{ccc}
 i) & & iv) \\
 \downarrow & \swarrow & \updownarrow \\
 ii) & \longleftrightarrow & iii)
 \end{array}$$

$$i) \Rightarrow ii),$$

$$\begin{array}{ccc}
 \text{Ker}(\varphi) & & \\
 \varphi_0 \downarrow & \searrow 0 & \\
 \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G}
 \end{array}$$

Where  $\varphi_0(U)$  is an inclusion map of abelian groups.

$$ii) \Leftrightarrow iii),$$

$$\text{Ker } \varphi = 0 \Leftrightarrow \forall U \in \mathcal{T}, \text{Ker } \varphi(U) = 0 \Leftrightarrow \varphi_U \text{ is injective.}$$

$$iii) \Rightarrow iv), \text{ Fix } x \in X.$$

$$\varphi_x(s_x) = (\varphi(s)_U)_x = 0.$$

This implies that

□