Algebraic Geometry 1

So Murata

2024/2025 Winter Semester - Uni Bonn

1 Topology

1.1 Connected Sets

Definition 1.1. *Let $(X, \mathcal{T})$ be a topological space. A subset A of X is said to be connected if for any $U, V \in \mathcal{T}$, $U \cap V = \emptyset, U \cup V \supset A$ then A is fully contained in one of U, V.*

Definition 1.2. *A connected component of a topological space is a maximal connected subset of a space.*

Proposition 1.1. *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological space and $f : X \to Y$ be a continuous function. Then for any connected subset A of X, $f(A)$ is connected in Y.*

Proof.

$$
\begin{aligned}
U, V \in \mathcal{T}_Y, U \cup V \supset f(A), U \cap V = \emptyset,& \\
&\Rightarrow f^{-1}(U), f^{-1}(V) \in \mathcal{T}_X, \\
&\quad f^{-1}(U) \cup f^{-1}(V) \supset A, \\
&\quad f^{-1}(U) \cap f^{-1}(V) = \emptyset, \\
&\Rightarrow f^{-1}(U) \supset A \vee f^{-1}(V) \supset A, \\
&\Rightarrow U \supset f(A) \vee V \supset f(A).
\end{aligned}
$$

□

Definition 1.3. *Let X be a topological space. A point $\eta \in X$ is called a generic point if*

$$\overline{\{\eta\}} = X.$$

2 Category Theory

2.1 Categories

Definition 2.1. *A category $\mathcal{A}$ consists of*

- *a collection* $\mathrm{ob}(\mathcal{A})$ *of objects;*
- *for each* $A, B \in \mathrm{ob}(\mathcal{A})$*, a collection* $\mathcal{A}(A, B)$ *of morphisms from* A *to* B*;*

such that

i). for each $A \in \mathrm{ob}(\mathcal{A})$*, the identity* $1_A \in \mathcal{A}(A, A)$*;*

ii). the composition $\mathcal{A}(B, C) \times \mathcal{A}(A, B) \ni (g, f) \mapsto g \circ f \in \mathcal{A}(A, C)$ *is well-defined;*

and they satisfy the following axioms

I). Associativity : $f \in \mathcal{A}(A, B), g \in \mathcal{A}(B, C), h \in \mathcal{A}(C, D)$, $(h \circ g) \circ f = h \circ (g \circ f)$.

II). Identity laws : $f \in \mathcal{A}(A, B)$ *then* $f \circ 1_A = 1_B \circ f$.

Definition 2.2. *Let* $\mathcal{A}$ *be a category. A terminal object* $T \in \mathrm{ob}(\mathcal{A})$ *is an object such that for any* $A \in \mathrm{ob}(\mathcal{A})$*,* $\mathcal{A}(A, T)$ *is a single element set.*

Definition 2.3. *Given two categories* $\mathcal{A}, \mathcal{B}$*, we say* $\mathcal{A}$ *is a full-subcategory of* $\mathcal{B}$ *if*

i). $\mathcal{A} \subset \mathcal{B}$,

ii). $\mathrm{ob}(\mathcal{A}) = \mathrm{ob}(\mathcal{B})$.

Notation 2.1. *Here we give notations to some important categories.*

- (**Sets**) *: A category of sets equipped with set theoretic functions.*
- (**Ab**) *: A category of abelian groups with group homomorphisms.*

Example 2.1. *Given a partially ordered set* $(X, \leq)$*. This can be encoded to a category* $\mathcal{O}$ *by*

i). $\mathrm{ob}(\mathcal{O}) = X$,

ii). For $x, y \in X$, $x \leq y \Rightarrow \mathcal{O}(x, y) = \{*\}$ *otherwise the morphisms between* x, y *is an emptyset.*

Definition 2.4. *A opposite/dual category of a category* $\mathcal{A}$ *is* $\mathcal{A}^{op}$ *such that*

i). $\mathrm{ob}(\mathcal{A}^{op}) = \mathrm{ob}(\mathcal{A})$,

ii). $\mathcal{A}^{op}(B, A) = \mathcal{A}(A, B)$.

Definition 2.5. *Let* $\mathcal{A}$ *be a category and* $\varphi_1, \varphi_2 \in \mathcal{A}(M, N)$*. A morphism* $\varphi : K \to M$ *is called an equalizer of* (φ_1, φ_2) *if for any morphism* $\psi : P \to M$ *such that* $\varphi_1 \circ \psi = \varphi_2 \circ \psi$*, there is a unique morphism* $\tilde{\psi} : P \to K$ *such that* $\varphi \circ \tilde{\psi} = \psi$.

Proposition 2.1. *If an equalizer exists then it is unique up to unique isomorphism.*

Proof. Suppose $\varphi : K \to M, \psi : L \to M$ be equalizers of (φ_1, φ_2). Then we have

$$\varphi \circ \tilde{\psi} = \psi, \quad \psi \circ \tilde{\varphi} = \varphi$$

By the uniqueness, we have $\tilde{\varphi} \circ \tilde{\psi} = 1_L, \tilde{\psi} \circ \tilde{\varphi} = 1_K$. □

Definition 2.6. *Let $\mathcal{A}, \mathcal{B}$ be categories. A functor $F : \mathcal{A} \to \mathcal{B}$ is a function such that for each $f \in \mathcal{A}(A, A')$, $F(f) : F(A) \to F(A')$. In other words, $f \mapsto F(f) : \mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A'))$. Furthermore, F satisfies the following axioms.*

I). $F(f' \circ f) = F(f') \circ F(f)$ whenever $f : A \to A', f' : A' \to A''$ in $\mathcal{A}$,

II). $F(1_A) = 1_{F(A)}$ whenever $A \in \mathcal{A}$.

Definition 2.7. *Let F, G be functors between two categories $\mathcal{A}, \mathcal{B}$. A natural transformation $\alpha : F \to G$ is a family $(\alpha_A : F(A) \to G(A))_{A \in \mathcal{A}}$ such that*

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_{A'}} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

is a commutative diagram. Each α_A is called a component of α.

2.2 Direct Limits

Definition 2.8. *A partially ordered set $(X, \leq)$ is directed if for any $x, y \in X$ there is $z \in X$ such that $x \leq z$ and $y \leq z$.*

Example 2.2. *Let $(X, \mathcal{T})$ be a topological space. A partially ordered set $(\mathcal{T}, \leq)$ such that*

$$V \subseteq U \Rightarrow U \leq V$$

is directed. Since for any $U \in \mathcal{T}$, $U \leq \emptyset$. As a category this is $\mathbf{Ouv}_X^{\mathbf{op}}$.

Example 2.3. *Let $(X, \mathcal{T})$ be a topological space. For $x \in X$, define $O_x = \{U \in \mathcal{T} \mid x \in U\}$. If we define an order as in the previous example, we get $(O_x, \leq)$ is directed. This follows from for any $U, V \in O_x$, $U, V \leq U \cap V$.*

Definition 2.9. *Let I be a directed partially ordered set and $\mathcal{A}$ be a category. A directed system of objects of $\mathcal{A}$ indexed by I is a collection of objects $(A_i)_{i \in I}$ and morphisms $(\rho_{ij})_{i \leq j}$ of $\mathcal{A}$ such that*

i). $\rho_{ii} = \mathbf{id}_{A_i}$,

ii). for $i,j,k \in I$, $i \leq j \leq k \Rightarrow \rho_{ik} = \rho_{jk} \circ \rho_{ij}$.

Remark 2.1. *Categorically, the directed system of objects of $\mathcal{A}$ indexed by I is a functor $\mathcal{O}^{op} \to \mathcal{C}$, where $\mathcal{O}$ is a category which encodes the ordered set I as a category by the same procedure as in Example 2.1. Then a directed system if a functor $\mathcal{O}^{\mathbf{op}} \to \mathcal{A}$.*

Definition 2.10. *Given a directed system $((A_i)_{i\in I}, \{\rho_{ij}\}_{i\leq j})$ of objects in $\mathcal{A}$ indexed by I. A direct limit of the system is an object $\varinjlim A_n \in \mathbf{ob}(\mathcal{A})$ satisfying the following universal property.*

Given a collection of morphisms $(f_i)_{i\in I}$ such that

i). $f_i : A_i \to \varinjlim A_n \in \mathcal{A}$,

ii). for any $i \leq j$, $f_j \circ \rho_{ij} = f_i$.

For any $A \in \mathcal{A}$ where there is a collection of morphisms $(g_i)_{i\in I}$ satisfying the above condition, there is a unique map $\varphi : \varinjlim A_n \to A$ such that

$$
\begin{tikzcd}
A_i \arrow[rr, dash, "\rho_{ij}"] \arrow[dr, "f_i"] \arrow[ddr, bend right, "g_i"'] & & A_j \arrow[dl, "f_j"'] \arrow[ddl, bend left, "g_j"] \\
& \varinjlim A_n \arrow[d, dotted, "\exists !"] & \\
& A &
\end{tikzcd}
$$

is a commutative diagram.

Proposition 2.2. *$\varinjlim$ is an exact functor.*

Proof. $\square$

Proposition 2.3. *In the cases where $\mathcal{A} = (\mathbf{Ab}), (\mathbf{Sets})$, there exist direct limits and for each category, such limit is constructed in the following ways.*

i). $\varinjlim A_n = (\bigoplus_{i\in I} A_i)/N$ where $N = \{a_i - \rho_{ij}(a_i) \mid a_i, i \leq j\}$.

ii). $\varinjlim A_n = (\coprod_{i\in I} A_i)/\sim$ where $a_i \sim a_j$ if there is k such that $i \leq k$ $j \leq k$, and $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.

Furthermore, these two direct limits coincide as sets.

Proof. $\square$

Proposition 2.4. *$\varinjlim$ is (left) exact in $(\mathbf{Ab})$. In other words, given a exact sequence of directed systems*

$$0 \longrightarrow (M_i)_{i\in I} \longrightarrow (N_i)_{i\in I} \longrightarrow (P_i)_{i\in I} \longrightarrow 0$$

in which we have

$$\begin{tikzcd}
0 \arrow[r] & M_i \arrow[r] \arrow[d, "\rho_{ij}^M"'] & N_i \arrow[r] \arrow[d, "\rho_{ij}^N"'] & P_i \arrow[r] \arrow[d, "\rho_{ij}^P"'] & 0 \\
0 \arrow[r] & M_j \arrow[r] & N_j \arrow[r] & P_j \arrow[r] & 0
\end{tikzcd}$$

There exists a short exact sequence

$$0 \longrightarrow \varinjlim M_n \longrightarrow \varinjlim N_n \longrightarrow \varinjlim P_n \longrightarrow 0$$

Proof. □

2.3 Abelian Categories

Definition 2.11. *A category $\mathscr{A}$ is said to be preadditive if for any objects $A, B \in \mathscr{A}$, $\mathscr{A}(A,B)$ is forms an abelian group and the map*

$$(f,g) \in \mathscr{A}(A,B) \times \mathscr{A}(A,B) \to \mathscr{A}(A,B), \quad (f,g) \mapsto g \circ f$$

is bilinear.

Definition 2.12. *Let $\mathscr{A}$ be a category and $A, B \in \mathbf{ob}(\mathscr{A})$. The coproduct $A \coprod B$ of A and B is an object equipped with two morphisms $\iota_A : A \to A \coprod B, \iota_B : B \to A \coprod B$ such that we have a following commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[dr, "f"'] & A \coprod B \arrow[d, dotted, "\exists!(f,g)"] & B \arrow[l, "\iota_B"'] \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

Definition 2.13. *A preadditive category $\mathscr{A}$ is said to be additive if any pair of objects admits a coproduct.*

Definition 2.14. *Let $\mathscr{A}$ be an additive category and $A_1, \cdots, A_n \in \mathbf{ob}(\mathscr{A})$. A biproduct of these objects is an object $\bigoplus_{i=1}^n A_i$ together with morphisms*

$$\begin{tikzcd}
\bigoplus_{i=1}^n A_i \arrow[r, shift left, "\pi_k"] & A_k \arrow[l, shift left, "\iota_k"]
\end{tikzcd}$$

for each $k = 1, \cdots, n$ such that

$$\iota_k \circ \pi_l = \delta_{kl}, \quad \sum_{k=1}^n \pi \circ \iota_k = \mathrm{id}_{\bigoplus_{i=1}^n A_i}.$$

Definition 2.15. *Let $\mathscr{A}, \mathscr{B}$ be additive categories. A functor $F : \mathscr{A} \to \mathscr{B}$ is said to be additive if it preserves finite biproduct. That is to say for a zero object*

$0 \in \mathscr{A}$, $F(0)$ *is a zero object in* $\mathscr{B}$ *and for given any two objects* $A, B \in \mathscr{A}$, *there is an isomorphism*

$$F(A \oplus B) \cong F(A) \oplus F(B),$$

which respects the inclusion and projection maps of the direct sum.

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"'] & & B \arrow[dl, "\iota_B"] & & F(A) \arrow[dr, "F(\iota_A)"] & & F(B) \arrow[dl, "F(\iota_B)"'] \\
& A \oplus B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] \arrow[rrrr, mapsto, "F"] & & & & F(A \oplus B) \cong F(A) \oplus F(B) \arrow[dl, "F(\pi_A)"'] \arrow[dr, "F(\pi_B)"] & \\
A & & B & & F(A) & & F(B)
\end{tikzcd}$$

Definition 2.16. *Let* $\mathscr{A}$ *be an abelian category. An object* $I \in \mathbf{ob}(\mathscr{A})$ *is said to be injective if for any monomorphism* $\iota \in \mathscr{A}(A_1, A_2)$ *and a morphism* $\varphi \in \mathscr{A}(A_1, I)$, *there is* $\psi \in \mathscr{A}(A_2, I)$ *making the following diagram commutative.*

$$\begin{tikzcd}
A_1 \arrow[r, hook, "\iota"] \arrow[d, "\varphi"'] & A_2 \arrow[dl, dotted, "\exists\psi"] \\
I &
\end{tikzcd}$$

Remark 2.2. *Clearly, a zero object is injective by the definition.*

Definition 2.17. *Let* $\mathscr{A}$ *be an abelian category. A short exact sequence*

$$0 \longrightarrow A_1 \xrightarrow{\ \iota\ } A_2 \xrightarrow{\ \pi\ } A_3 \longrightarrow 0$$

in $\mathscr{A}$ *is said to split if there exists a morphism* $\varphi : A_2 \to A_1$ *such that* $\varphi \circ \iota = \mathrm{id}_{A_1}$.

Definition 2.18.

Lemma 2.1. *Let* $G \in \mathbf{ob}(\mathbf{Ab})$. *Then the functor*

$$\mathrm{Hom}(G, \cdot) : (\mathbf{Ab}) \to (\mathbf{Ab}),$$

is left exact.

Proof. □

Lemma 2.2. *Let* $\mathscr{A}$ *be an abelian category and*

$$0 \longrightarrow A_1 \xrightarrow{\ \iota\ } A_2 \xrightarrow{\ \pi\ } A_3 \longrightarrow 0$$

be a short exact sequence in $\mathscr{A}$.

The sequence splits if and only if $A_2 \cong A_1 \oplus A_3$ *and under such isomorphism,* ι, π *are mapped to an injection and a projection from* A_1 *and to* A_3, *respectively.*

Proof.

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, "\iota"] \arrow[d, "\mathrm{id}_A"] & A_2 \arrow[r, "\pi"] \arrow[dl, dotted, "\varphi"] & A_3 \arrow[r] & 0 \\ & A_1 & & & \end{tikzcd}$$

□

Lemma 2.3. *Let $\mathscr{A}$ be an abelian category and*

$$0 \longrightarrow A_1 \xrightarrow{\iota} A_2 \xrightarrow{\pi} A_3 \longrightarrow 0$$

be a short exact sequence in $\mathscr{A}$.

If A_1 is injective then the sequence splits.

Proof. By the definition of injective objects, such $\varphi : A_2 \to A_1$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, "\iota"] \arrow[d, "\mathrm{id}_A"] & A_2 \arrow[r, "\pi"] \arrow[dl, dotted, "\varphi"] & A_3 \arrow[r] & 0 \\ & A_1 & & & \end{tikzcd}$$

commutative exists. □

Definition 2.19. *Let $\mathscr{A}$ be an abelian category and $A \in \mathbf{ob}(\mathscr{A})$.*

An injective resolution of A is a long exact sequence in $\mathscr{A}$

$$0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

, where all $(I_i)_{i \in \mathbb{N}_0}$ are injective objects

Definition 2.20. *An abelian category is said to have enough injective objects if any objects admits an injective resolution.*

Example 2.4. *The category $(\mathbf{Ab})$ of abelian groups has enough injectives. (eg. $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}$).*

3 Commutative Algebra

3.1 Local Rings

Definition 3.1. *The total ring of fraction of a ring A is a localization of A by the set of all non-zero divisors. It is denoted as $Q(A)$.*

Definition 3.2. *A ring is said to be local if it has a unique maximal ideal.*

Definition 3.3. *A ring homomorphism $\phi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ of two local rings is said to be local if*

$$\mathfrak{m}_A = \phi^{-1}(\mathfrak{m}_B).$$

Example 3.1. *Let $i : \mathbb{Z}_{(p)} \to Q(\mathbb{Z}_{(p)})$ be an inclusion map. Then it is a homomorphism of local rings. However, If p is prime then $Q(\mathbb{Z}_{(p)})$ is a field thus its maximal ideal is (0). Obviously*

$$i^{-1}((0) = (0).$$

Therefore, i is not a local ring homomorphism.

Proposition 3.1. *Let $\phi : A \to B$ be a ring homomorphism. Recall that for any prime ideal $\mathfrak{q} \subseteq B$, we have $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ is a prime ideal in A. Thus ϕ induces a homomorphism between $A_\mathfrak{p}$ and $B_\mathfrak{q}$ which is a local ring homomorphism.*

Proof. If $a \in A$, $\phi(a) = 0$ then $a \in \mathfrak{p}$. Thus $\phi(s) \neq 0$ for any $s \notin \mathfrak{p}$. Since $\mathfrak{p}, \mathfrak{q}$ are unique maximal ideals of $A_\mathfrak{p}$ and $B_\mathfrak{q}$, respectively. We derived the claim. □

Lemma 3.1. *Let k be an algebraically closed field and A be a k-algebra. A localization $A_\mathfrak{m}$ by a maximal(prime) ideal$\mathfrak{m} \subset A$, we have the following isomorphism.*

$$k \cong A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}.$$

Proof. Follows from the algebraically closedness of k. □

3.2 Maximal Spec

Definition 3.4. *Let R be a commutative ring. We define the maximal spec of R as*

$$\mathrm{MaxSpec}(R) = \{\mathfrak{m}\,\mathrm{Spec}(R) \mid \mathfrak{m} \textit{ is a maximal ideal.}\}.$$

Lemma 3.2. *Let k be an algebraically closed field. We have the following isomorphism*

$$\mathrm{MaxSpec}\, k[x_1, \cdots, x_n] \cong k^n, \quad (x_1 - a_1, \cdots, x_n - a - n) \leftrightarrow (a_1, \cdots, a_n).$$

Proof. Surjectivity follows from the algebraically closedness of k. □

3.3 Zariski Topology

Definition 3.5. *Let k be a algebraically closed field. A subset X of k^n is called an affine algebraic set if there exists an ideal $\mathfrak{a} \subset k[x_1, \cdots, x_n]$ such that*

$$X = V(\mathfrak{a}) = \{(a_1, \cdots, a_n) \mid \forall f \in \mathfrak{a}, f(a_1, \cdots, a_n) = 0\}.$$

Definition 3.6. *Let k be an algebraically closed. The Zariski topologi on k^n is a topology generated by affine algebraic sets as closed subsets.*

Definition 3.7. *Let X be the Zariski topology on k^n. A function $f : X \supseteq U \to k$ is said to be regular if for any $a = (a_1, \cdots, c_n) \in U$, there exist a neighborhood $U_a \subseteq U$ and $f_1, f_2 \in k[x_1, \cdots, x_n]$ such that*

$$(b_1, \cdots, b_n) \in V_a \Rightarrow f(b_1, \cdots, b_n) = \frac{f_1(b_1, \cdots, b_n)}{f_2(b_1, \cdots, b_n)}.$$

Remark 3.1. *A regular function f on the Zariski topology on k^n is continuous as they are locally equivalent to quotients of polynomial functions.*

Lemma 3.3. *Let $\mathfrak{a} \subset A$ be an ideal. Then we have the following are homeomorphic.*

$$\operatorname{Spec}(A/\mathfrak{a}) \cong V(\mathfrak{a}).$$

Lemma 3.4. *Let A be a commutative ring and $a \in A$. For elements $\frac{b_1}{a^{n_1}}, \frac{b_2}{a^{n_2}}$ of A_a, assume that for any $\mathfrak{p} \in D(a)$,*

$$\frac{b_1}{a^{n_1}}, \frac{b_2}{a^{n_2}} \in A_{\mathfrak{p}}, \quad \frac{b_1}{a^{n_1}} = \frac{b_2}{a^{n_2}}$$

Let us define an ideal

$$\mathfrak{q} = \operatorname{Ann}(b_1 a^{n_2} - b_2 a^{n_1}).$$

Then for any $\mathfrak{p} \in D(a)$, we have $\mathfrak{q} \not\subseteq \mathfrak{p}$.

4 Classical Algebraic Geometry

4.1 Affine Variety

Definition 4.1. *An affine algebraic set X is called an affine variety if there exists a prime ideal $\mathfrak{p} \subseteq k[x_1, \cdots, x_n]$ such that*

$$X = V(\mathfrak{p}).$$

Definition 4.2. *Let k be an algebraically closed field and $X \subseteq k^n$. The ideal of X is*

$$I(X) = \{f \in k[x_1, \cdots, x_n] \mid \forall (a_1, \cdots, a_n) \in X, f(a_1, \cdots, a_n) = 0\}.$$

Theorem 4.1. *For any ideal $\mathfrak{a} \subset k[x_1, \cdots, x_n]$, we have*

$$I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

Definition 4.3. *Let $X \subset k^n$ where k is an algebraically closed field. The affine coordinate ring with respect to X is*

$$A(X) = k[x_1, \cdots, x_n]/I(X).$$

5 Sheaf Theory

5.1 Presheaves

Definition 5.1. *Let $(X, \mathcal{T})$ be a topological space. We define a presheaf $\mathcal{F}$ of a category $\mathcal{A}$ on X to be such that*

- $U \in \mathcal{T}$, $\mathcal{F}(U) \in \mathrm{ob}(\mathcal{A})$,
- $U, V \in \mathcal{T}$, $V \subset U \Rightarrow$ *there exists a map* $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$

  satisfying,

i). For any $U \in \mathcal{T}$, $\rho_{UU} = 1_{\mathcal{F}(U)}$.

ii). $U, V, W \in \mathcal{T}, W \subset V \subset U \to \rho_{UW} = \rho_{VW} \circ \rho_{UW}$.

Notation 5.1. *In the case* $\mathcal{A} = (\mathbf{Sets}), (\mathbf{Ab})$, $\mathcal{F}(\emptyset) = \emptyset, \{1\}$, *respectively.*

Definition 5.2. *An element of $\mathcal{F}(U)$ is called a local section of $\mathcal{F}$ and $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ is called the space of sections over U. In particular $\Gamma(X, \mathcal{F})$ is called the space of global sections of $\mathcal{F}$.*

Definition 5.3. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{F}$ be a presheaf of a category $\mathcal{A}$ on X. Suppose we have two open sets $U, V \in \mathcal{T}$ such that $V \subset U$. Then for any section $s \in \mathcal{F}(U)$, $s|_V = \rho_{UV}(s)$ is called the restriction of s to V.*

Example 5.1. *Let $(X, \mathcal{T})$ be a topological space. We have a presheaf of continuous functions $\mathcal{C}_X(U) = \mathcal{C}^0(U, \mathbb{R})$. This is indeed a presheaf with restriction maps $\rho_{UV} : \mathcal{C}_X(U) \to \mathcal{C}_X(V)$. (Explicitly, $\rho_{UV}(f) = f \circ i_V$ where i_V is an inclusion map.) We note that we can introduce operations $+, \cdot$ to endow some algebraic structures (groups, rings, ...) on $\mathbb{R}$.*

Example 5.2. *Let $(X, \mathcal{T})$ be a topological space and suppose we have presheaves*

- $\mathcal{C}_X^{\textit{\textbf{diff}}}(U) = \{f : U \to \mathbb{R} \mid f \text{ is differentiable.}\}$.

Then there is an inclusion relation $\mathcal{C}_X^{\textit{\textbf{diff}}}(U) \subseteq \mathcal{C}_X(U)$ and this defines a presheaf.

Example 5.3. *Let $(X, \mathcal{T}_X)$,$(Y, \mathcal{T}_Y)$ be topological spaces. Define a presheaf on X by*

$$U \in \mathcal{T}_X, \mathcal{F}(U) = \mathcal{C}^0(X, Y).$$

And like the previous example, we define $\rho_{UV}(f) = f|_V$ for $U, V \in \mathcal{T}_X, V \subset U$. the restriction of f to V.
But this is a presheaf only of a set.

Example 5.4. *Let $(X, \mathcal{T})$ be a topological space and G be an abelian group. The constant presheaf $\mathbb{G}$ is such that*

$$U \in \mathcal{T}, \mathbb{G}(U) = G,$$

with $\rho_U V = id_G$ for any $U, V \in \mathcal{T}, V \subset U$.

5.2 Presheaves as Categories

Definition 5.4. *Let $(X,\mathcal{T})$ be a topological space then $(\mathbf{Ouv}_X)$ is the category such that its objects are the open sets of X and for any $U, V \in \mathcal{T}$ we have*

$$\mathbf{Ouv}_X(U,V) = \begin{cases} \emptyset & (V \not\subset U), \\ i_V & (V \subset U). \end{cases}$$

Definition 5.5. *Let $(X,\mathcal{T})$ be a topological space and $\mathcal{A}$ be a category. A presheaf of $\mathcal{A}$ on X is a functor $F : \mathbf{Ouv}_X \to \mathcal{A}$.*

Example 5.5. *For $\mathbf{Ouv}_X$, we can define a presheaf of F to be*

$$\mathrm{ob}(\mathbf{Ouv}_X) \ni U \mapsto F(U) = \mathcal{C}^0(U,\mathbb{R}).$$

Example 5.6. *Let A be a commutative ring with non-zero multiplicative identity and $X = \mathrm{Spec}(A)$. Let us consider the Zariski topology $(X,\mathcal{T})$. Let us consider a category $\mathcal{O}_X$ such that*

- $\mathrm{ob}(\mathcal{O}_X) = \mathcal{T}$,
- $\mathcal{O}_X(U) = \{s : U \to \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}}\}$,

where $s : U \to \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}}$ is a function such that for any $\mathfrak{p} \in U$,

i). $s(p) \in A_{\mathfrak{p}}$,

ii). there exists an open set $V \subset U$ such that $\mathfrak{p} \in V$ and for any $\mathfrak{q} \in V$, $s(\mathfrak{q}) = \frac{a}{b}$ for $b \notin \mathfrak{q}$.

Now we define a presheaf by the restrictions of maps such that

$$s : U \to \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}} \mapsto s|_V : V \to \coprod_{\mathfrak{q}\in V} A_{\mathfrak{q}}.$$

Definition 5.6. *Let $(X,\mathcal{T})$ be a topological space and $\mathcal{A}$ be a category. We define a set of presheaves of $\mathcal{A}$ on X as*

$$\mathrm{PreSh}_{\mathcal{A}}(X) = \mathrm{Fun}(\mathbf{Ouv}_X^{\mathbf{op}}, \mathcal{A}).$$

Definition 5.7. *A morphism of presheaves is a natural transformation $\varphi : \mathcal{F} \to \mathcal{G}$ where $\mathcal{F}, \mathcal{G} \in \mathrm{PreSh}_{\mathcal{A}}(X) = \mathrm{Fun}(\mathbf{Ouv}_X^{\mathbf{op}}, \mathcal{A})$.(See Definition 2.7).*

Such $\varphi : \mathcal{F} \to \mathcal{G}$ is

i). injective if

Remark 5.1. $\mathrm{PreSh}(X)$ *can be regarded as a category with its objects presheaves and morphisms defined above.*

Notation 5.2. *In the case $\mathcal{A} = (\mathbf{Ab})$ then we denote* $\mathrm{PreSh}(X) = \mathrm{PreSh}_{\mathbf{Ab}}(X)$.

Example 5.7. *Let X be a differential manifold(eg. $X \subset \mathbb{R}^n$). Let us define*

$$\mathcal{C}^{\mathbf{diff}}(U) = \{f : U \to \mathbb{R} \mid f \text{ is differentiable.}\}.$$

Then the inclusions $\mathcal{C}_X^{\mathbf{diff}}(U) \subset \mathcal{C}_X(U)$ defines a morphism of presheaves.

Example 5.8. *Let $X, Y = S^1$ be topological spaces and F be a presheaf such that for any open set $U \subset X$, $F(U) = \mathcal{C}^0(U, Y)$.Then we can introduce a natural transformation such that*

$$\mathcal{C}_X(U) \ni f \mapsto \exp(2\pi f i)$$

which is indeed a morphism of presheaves $\mathcal{C}_X(U)$ and $F(U)$

5.3 Sheaves

Definition 5.8. *A presheaf $\mathcal{F}$ on $(X, \mathcal{T})$ is called a sheaf if the following holds. For any collection of open sets $(U_i)_{i\in I} \subset \mathcal{T}, U = \bigcup_{i\in I} U_i$, the map $\varphi : \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$ which is defined as*

$$\varphi(s) = (s|_{U_i})_{i\in I}.$$

is the equalizer of the following functions $\varphi_1, \varphi_2 : \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$,

$$\varphi_1((s_i)_{i\in I}) = (s_i|_{U_i\cap U_j})_{i,j\in I}, \quad \varphi_1((s_i)_{i\in I}) = (s_j|_{U_i\cap U_j})_{i,j\in I}.$$

Remark 5.2. *In the case $I = \{1, 2\}$, we have $U = U_1 \cup U_2$, and for any $U' \in \mathcal{T}$ such that $U \subset U'$, we have for $\mathcal{F}(U') \ni s : U' \to \mathbb{R}$,$\psi(s) = (s|_{U_1}, s|_{U_2})$, as in $\mathbf{Ouv}_X$, morphisms are inclusions. Let $\tilde{\psi}(s) = s|_U$, then this satisfies the condition for the equalizer (ie. $\varphi \circ \tilde{\psi} = \psi$).*

Remark 5.3. *A presheaf $\mathcal{O}_X$ with $X = \mathrm{Spec}(A)$ is a sheaf.*

Example 5.9. *Let $(X, \mathcal{T})$ be a topological space and G be a group. We define a constant presheaf $\mathbb{G}(U) = G$. In general, this is not a sheaf. Instead, we define a constant sheaf $\underline{\mathbb{G}}(U) = \mathcal{C}^0(U, G)$ where G is regarded as a topological space with the discrete topology. Then for any connected component of X is mapped to a single point set in G.*

Definition 5.9. *Let $\mathcal{F}_1, \mathcal{F}_2$ be sheaves. A mapping $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ is called a morphism of sheaves if it is a morphism of presheaves.*

Notation 5.3. *A set of sheaves of $\mathcal{A}$ on the topological space $(X, \mathcal{T})$ is denoted as $\mathrm{Sh}_{\mathcal{A}}(X)$.*

Remark 5.4. *As in the case of presheaves, $\mathrm{Sh}_{\mathcal{A}}(X)$ can be regarded as a category with sheaf morphisms.*

Remark 5.5. $\mathrm{Sh}_{\mathcal{A}}(X)$ *is a full-subcategory of* $\mathrm{PreSh}_{\mathcal{A}}(X)$.

Notation 5.4. *In the case $\mathcal{A} = (\mathbf{Ab})$, we denote $\mathrm{Sh}_{(\mathbf{Ab})}(X) = \mathrm{Sh}(X)$.*

5.4 Stalks

Notation 5.5. *Let $(X, \mathcal{T})$ be a topological space. For a point $x \in X$,we denote the collection of all open sets which contain x as*

$$\mathcal{O}_x = \{U \in \mathcal{T} \mid x \in U\}.$$

Definition 5.10. *Suppose we have a topological space $(X, \mathcal{T})$ and a category $\mathcal{A}$ which admits direct limits. For a presheaf $\mathcal{F} \in \mathrm{PreSh}_{\mathcal{A}}(X)$, by inheriting the notations from Example 2.3, we define the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at $x \in X$ by*

$$\mathcal{F}_x = \varinjlim_{U \in \mathcal{O}_x} \mathcal{F}(U).$$

Example 5.10. *Let us assume that $\mathcal{A} = (\mathbf{Ab})$ in Definition 5.10. Then stalks and germs can be constructed explicitly in the following way.*

$$\mathcal{F}_x = \{(s, U) \mid U \in \mathcal{O}_x, s \in \mathcal{F}(U)\}/\sim,$$

where $\sim$ is an equivalent relation such that for $(s, U), (t, V)$,

$$(s, U) \sim (t, V) \text{ if there is } W \in \mathcal{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t).$$

Definition 5.11. *Inheriting the notations from Definition 5.10, suppose we have $(f_U : \mathcal{F}(U) \to \mathcal{F}_x)_{U \in \mathcal{O}_x}$ such that for f_U, f_V are compatible with ρ_{UV}. Then we define the germ of $s \in \mathcal{F}(U)$ to be $s_x = f_U(s)$. By the universal property of the direct limit, such s_x is unique up to images under isomorphisms.*

Example 5.11. *In the case of Remark 5.10, we have for each $U \in \mathcal{T}$, $x \in U$, and $s \in \mathcal{F}(U)$,*

$$s_x = \{(t, V) \mid \text{ There is } W \in \mathcal{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t)\}.$$

Remark 5.6. *In the above definition, if a category $\mathcal{A}$ admits products, we get a map*

$$(s \mapsto (s_x)_{x \in U})) : \mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x. \tag{5.1}$$

This is neither surjective nor injective in general.

Proposition 5.1. *Suppose in the definition of stalks, $\mathcal{F}$ is a sheaf. Then the map defined by Equation 5.1 is injective.*

Proof. We prove the case when $\mathcal{A} = (\mathbf{Ab})$.

Suppose $s \in \mathcal{F}(U)$ is such that $s_x = 0$ in $\mathcal{F}_x$ for all $x \in U$. Since for any restriction maps are group homomorphisms. We have that there is $V_x \in \mathcal{O}_x$ such that

$$V_x \subseteq U, \quad \rho_{UV_x}(s) = 0.$$

Therefore $\{V_x\}_{x \in U}$ is an open covering of U. Since $\mathcal{F}$ is a sheaf, we derive that $s = 0$ in $\mathcal{F}(U)$. □

Example 5.12. *Given $(X, \mathcal{F})$, a topological space and G, an abelian group. We will consider the constant presheaf $\mathbb{G}$ and the constant sheaf $\underline{\mathbb{G}}$ on X. For any open set U and $x \in U$ we have*

$$\mathbb{G}_x \cong \underline{\mathbb{G}}_x \cong G.$$

For any U, V open such that $V \subset U$ we have, $\rho_{UV} = \mathbf{id}_G$. Thus by the construction, for $x \in U, V$, $(s, U) \sim (t, V)$ then $x \in U \cap V$ and $\rho_{UU\cap V}(s) = s = t = \rho_{VU\cap V}(t)$. Therefore, we proved the claim.

Definition 5.12. *Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Then we define*

$$\varphi_x(s_x) = (\varphi(s)_U)_x.$$

This defines a morphism of presheaves.

Remark 5.7. *Categorically, taking stalks is a functor for each $x \in X$. Suppose we have $\mathcal{F}, \mathcal{G} \in \mathrm{PreSh}_{\mathcal{A}}(X)$ and a morphism $\varphi : \mathcal{F} \to \mathcal{G}$,*

Proposition 5.2. *Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}_{(\mathbf{Ab})}(X)$ Then for any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ we have*

$$\varphi = 0 \Leftrightarrow \forall x \in X, \varphi_x = 0$$

Proof. $\Rightarrow$ is trivial by its construction. We will prove $\Leftarrow$.

We first note that $\varphi = 0$ means that for any $U \in \mathcal{T}$, we have $\varphi_U \equiv 0$ as a group homomorphism. Let $U \in \mathcal{T}$ and $s \in \mathcal{F}(U)$. Then by the assumption and Proposition 5.1, we have proven the claim. □

5.5 Sheafification

Definition 5.13. *Let $\mathcal{F} \in \mathrm{PreSh}_{\mathcal{A}}(X)$. The sheafification of $\mathcal{F}$ is a presheaf $\mathcal{F}^+$ which is a set of all $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$ such that for any $x \in U$ there is $x \in V_x \subset U$, such that there is $t \in \mathcal{F}(V_x)$ satisfying for any $y \in V_x$, $s_y = t_y$. We give them restrictions such that*

$$\mathcal{F}^+(U) \ni (s_x)_{x\in U} \mapsto (s_x)_{x\in V} \in \mathcal{F}^+(V).$$

Proposition 5.3. *Such $\mathcal{F}^+$ is indeed a sheaf.*

Proof. later □

Remark 5.8.

$$\mathcal{F} \mapsto \mathcal{F}^+ : \mathrm{PreSh}_{\mathcal{A}}(X) \to \mathrm{Sh}_{\mathcal{A}}(X)$$

is a functor. Indeed given $\varphi : \mathcal{F} \to \mathcal{G}$, a morphism of presheaves. We give

$$\varphi^+(U)((s_x)_{x\in U}) = (\varphi(s)_x)_{x\in U}.$$

later

Proposition 5.4. *A mapping $\varphi : \mathcal{F} \to \mathcal{F}^+$ such that for each $U \in \mathcal{T}$,*

$$\varphi_U : \mathcal{F}(U) \to \mathcal{F}^+(U), \quad \varphi(s) = (s_x)_{x \in U},$$

is a natural transformation thus a morphism of presheaves.

Proof. Later $\square$

Proposition 5.5. *For any open set $U \in \mathcal{T}$ and a section $s \in \mathcal{F}^+(U)$, there is an open covering $(U_i)_{i \in I}$ which satisfies that there is a sequence $(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i)$ and for each i, the following hodls.*

$$\rho_{UU_i}(s) = s_i.$$

Proof. Later. $\square$

Proposition 5.6. *For each $x \in X$, there exists an isomorphism*

$$\mathcal{F}_x \cong (\mathcal{F}^+)_x,$$

as presheaves.

Proof. later $\square$

Proposition 5.7. *Let $(X, \mathcal{T})$ be a topological group and $\mathcal{F}$ be a presheaf of a category $\mathcal{A}$ on X. Suppose for a sheaf $\mathcal{G}$ of a category $\mathcal{A}$ on X, there exists a morphism $\varphi : \mathcal{F} \to \mathcal{G}$. Then there exists a unique morphism $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$, such that*

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "\varphi"'] & \mathcal{F}^+ \arrow[dl, dotted, "\exists ! \varphi^+"] \\
\mathcal{G} &
\end{tikzcd}$$

is a commutative diagram.

Proof. Let $U \in \mathcal{T}$, then by Proposition 5.5, for any $s \in \mathcal{F}^+$, there exists an open covering $(U_i)_{i\in I}$ and $(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i)$ such that $\rho_{UU_i}(s) = s_i$ for any $i \in I$. We define

$$t_i = \varphi(s_i) \in \mathcal{G}(U_i),$$

for each $i \in I$. Using the definition of natural transformation we derive that

$$\rho^{\mathcal{G}}_{UU_i \cap U_j}(t_i) = \varphi^{\mathcal{F}}_{U_i \cap U_j}(\rho_{UU_i \cap U_j}(s)) = \rho^{\mathcal{G}}_{UU_i \cap U_j}(t_j).$$

Thus we can glue $(t_i)_{i\in I}$ to a section $t \in \mathcal{G}(U)$.

We now define $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$. Given $(s_x)_{x\in U}$ which is the germ of s,

$$\varphi^+_U((s_x)_{x\in U}) = t.$$

Such φ^+ is unique since $\mathcal{G}$ is a sheaf. $\square$

Corollary 5.1. *Let $i : \mathrm{Sh}_{\mathcal{A}}(X) \to \mathrm{PreSh}_{\mathcal{A}}(X)$ be a forgetful functor. Then we have*

$$\mathrm{PreSh}_{\mathcal{A}}(X)(\mathcal{F}, i(\mathcal{G})) \cong \mathrm{Sh}_{\mathcal{A}}(\mathcal{F}^+, \mathcal{G})$$

In other words, the sheafification is a left-adjoint functor of the inclusion map.

Proof. By Proposition 5.7, we define two maps Φ, Ψ such that

$$\begin{aligned}
\Phi : \mathrm{PreSh}_{\mathcal{A}}(X)(\mathcal{F}, i(\mathcal{G})) &\to \mathrm{Sh}_{\mathcal{A}}(\mathcal{F}^+, \mathcal{G}), \\
\Phi(\varphi) &= \varphi^+, \\
\Psi : \mathrm{Sh}_{\mathcal{A}}(\mathcal{F}^+, \mathcal{G}) &\to \mathrm{PreSh}_{\mathcal{A}}(X)(\mathcal{F}, i(\mathcal{G})), \\
\Psi(\varphi^+) &= \varphi.
\end{aligned}$$

Then these two are inverses of each other. □

Proposition 5.8. *Let $X = \mathrm{Spec}(A)$ and $\mathcal{O}_X$ be the structure sheaf defined in Example 5.6. Then we have the following.*

1). For any $\mathfrak{p} = x \in X$, $(\mathcal{O}_X)_x \cong A_{\mathfrak{p}}$.

2). For any $a \in A$, $\mathcal{O}_X(D(a)) \cong A_a$.

Proof. For a given $U \subset X$ open and $\mathfrak{p} \subset A$, there is $a, b \in A$ such that for $V \subset U$ open and $s \in \mathcal{O}_X(U), s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$.

$$s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}$$

holds for any $\mathfrak{q} \in V$.

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[r] \arrow[d, "\rho_{UV}"'] & A_{\mathfrak{p}} \\
\mathcal{O}_X(V) \arrow[ur] &
\end{tikzcd}$$

□

5.6 Morphisms in $\mathrm{PreSh}_{(\mathbf{Ab})}(X)$

Definition 5.14. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of presheaves $\mathrm{PreSh}_{(\mathbf{Ab})}(X)$. Then we define the following presheaves.*

1). $\mathrm{Ker}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Ker}\, \varphi_U$,

2). $\mathrm{Im}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Im}\, \varphi_U$,

3). $\mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Coker}\, \varphi_U$.

Proposition 5.9. *Such* $\mathrm{Ker}^{\mathbf{pre}}, \mathrm{Im}^{\mathbf{pre}}, \mathrm{Coker}^{\mathbf{pre}}$ *are presheaves.*

Proof. For the case of kernels. Let $U, V \in \mathcal{T}$ and $V \subset U$. We define $\rho_U V : \mathrm{Ker}^{\mathbf{pre}}(\varphi)(U) \to \mathrm{Ker}^{\mathbf{pre}}(\varphi)(V)$ to be such that

$$\rho_U V(s) = \rho^{\mathcal{F}}(s).$$

Such construction is justified as the diagram below is commutative.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\rho^{\mathcal{F}}_{UV}"] \arrow[d, "\varphi_U"'] & \mathcal{F}(V) \arrow[r, "\rho^{\mathcal{F}}_{UV}"] \arrow[d, "\varphi_V"] & \mathcal{F}(W) \arrow[d, "\varphi_W"] \\
\mathcal{G}(U) \arrow[r, "\rho^{\mathcal{G}}_{UV}"'] & \mathcal{G}(V) \arrow[r, "\rho^{\mathcal{G}}_{UV}"'] & \mathcal{F}(W)
\end{tikzcd}$$

Furthermore,

$$\rho_U W(s) = \rho^{\mathcal{F}}_{UV}(s) = \rho^{\mathcal{F}}_{VW} \circ \rho^{\mathcal{F}}_{UV}(s) = \rho_{VW} \circ \rho_{UV}(s).$$

Thus $\mathrm{Ker}^{\mathbf{pre}}(\varphi)(U)$ is a presheaf. $\square$

Corollary 5.2. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Then $\mathrm{Ker}^{\mathbf{pre}}$ is also a sheaf.*

Proof. Given $(s_i)_{i \in I} \in \prod_{i \in I} \mathrm{Ker}\, \varphi_{U_i}$ such that

$$\rho(s_i)_{U_i U_i \cap U_j} = \rho(s_j)_{U_j U_i \cap U_j}$$

for any $i, j \in I$. Then since $\mathcal{F}$ is a sheaf, we can glue $(s_i)_{i \in I}$ to $s \in \mathcal{F}(U)$. For such s we have

$$\rho^{\mathcal{G}}_{UU_i}(\varphi_U(s)) = (\varphi_{U_i}(\rho^{\mathcal{F}}_{UU_i}(s))) = \varphi_{UU_i}(s_i) = 0.$$

Therefore, since $\mathcal{G}$ is a sheaf, $\varphi_U(s) = 0$. $\square$

Remark 5.9. *Let $\varphi : \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i), \varphi_1 : \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j), \varphi_2 : \prod_{i \in I} \mathcal{F}(U_j) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$. Then $\mathcal{F}$ is a sheaf if and only if*

$$\mathrm{Ker}(\varphi_1 \circ \varphi - \varphi_2 \circ \varphi) = \mathcal{F}(U),$$

holds for any open set U.

Remark 5.10. $\mathrm{Im}^{\mathbf{pre}}\, \varphi, \mathrm{Coker}^{\mathbf{pre}}\, \varphi$ *are not in general sheaves even tho $\varphi : \mathcal{F} \to \mathcal{G}$ is a homomorphism of sheaves.*

Example 5.13. *Let $X = \{x_1, x_2\}$ and we assign the discrete topology to it. Let G be an abelian group. We define a sheaf $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}_{(\mathbf{Ab})}(X)$ by such that*

$$\mathcal{F}(U) = \mathcal{G}(U) = \begin{cases} G \times G \quad (U = X), \\ G \quad (|U| = 1), \\ 0 \quad (U = \emptyset). \end{cases}$$

Let us define a homomorphism of sheaves φ such that

$$\varphi_U = \begin{cases} \mathbf{id}_{G\times G} & (U = X) \\ 0 \quad (U \neq X). \end{cases}$$

Then we have

$$\mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) = \begin{cases} 0 & (U = X), \\ G & (U \neq X). \end{cases}$$

By 5.10, we observe that

$$\mathrm{Coker}^{\mathbf{pre}}(\varphi)(X) = G \times G/\mathbf{id}_{G\times G}(G \times G) = \{0\}.$$

However,

later.

Definition 5.15. *Given a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, we define the following.*

1). $\mathrm{Ker}(\varphi) = \mathrm{Ker}^{\mathbf{pre}}(\varphi)$,

2). $\mathrm{Im}(\varphi) = (\mathrm{Im}^{\mathbf{pre}}(\varphi))^+$,

3). $\mathrm{Coker}(\varphi) = (\mathrm{Coker}^{\mathbf{pre}}(\varphi))^+$.

Proposition 5.10 (Universal property of kernels)**.** *Given a sheaf homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$. For any sheaf homomorphism $\alpha : \mathcal{H} \to \mathcal{F}$, $\varphi \circ \alpha = 0$ if and only if there is a unique $\psi : \mathcal{H} \to \mathrm{Ker}\,\varphi$ such that*

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, dotted, "\exists!\psi"'] \arrow[d, "\alpha"] \arrow[dr, "\varphi_0=0"] & \\ \mathrm{Ker}(\varphi) \arrow[r, hook] & \mathcal{F} \arrow[r, "\varphi"'] & \mathcal{G} \end{tikzcd}$$

is a commutative diagram.

Proof. We argue by each open set of the space.

$$\begin{tikzcd} & \mathcal{H}(U) \arrow[dl, dotted, "\exists!\psi_U"'] \arrow[d, "\alpha_U"] \arrow[dr, "(\varphi_0)_U=0"] & \\ \mathrm{Ker}(\varphi)(U) \arrow[r, hook] & \mathcal{F}(U) \arrow[r, "\varphi_U"'] & \mathcal{G}(U) \end{tikzcd}$$

This is a universal property of the kernel in abelian groups. Thus the statement immediately follows from it. $\square$

Proposition 5.11 (Universal property of Cokernels)**.** *Given a sheaf homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$. For any sheaf homomorphism $\alpha : \mathcal{G} \to \mathcal{H}$, $\alpha \circ \varphi = 0$ if and only if there is a unique $\psi : \operatorname{Coker} \varphi \to \mathcal{H}$ such that*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[dr, "\varphi_0=0"'] & \mathcal{G} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \operatorname{Coker}(\varphi) \arrow[dl, dotted, "\exists!\psi"] \\
& \mathcal{H} &
\end{tikzcd}$$

is a commutative diagram.

Proof. We argue for each open set $U \subset X$.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[dr, "(\varphi_0)_U=0"'] & \mathcal{G}(U) \arrow[r] \arrow[d, "\alpha_U"] & \operatorname{Coker}^{\mathbf{pre}}(\varphi)(U) \arrow[r] \arrow[dl, dotted, "\exists!\psi_U^{\mathbf{pre}}"'] & \operatorname{Coker}(\varphi)(U) \arrow[dll, dotted, "\exists!\psi_U"] \\
& \mathcal{H}(U) & &
\end{tikzcd}$$

By the universal property of Cokernels of abelian groups, there is a unique $\varphi^{\mathbf{pre}}$. By the universal property of the sheafification operator, we derive a unique ψ. $\square$

Proposition 5.12. *Let $x \in X$, then we have the following.*

1). $\operatorname{Ker}(\varphi)_x = \operatorname{Ker}(\varphi_x)$,

2). $\operatorname{Im}(\varphi)_x = \operatorname{Im}(\varphi_x)$,

3). $\operatorname{Coker}(\varphi)_x = \operatorname{Coker}(\varphi_x)$.

Proof. By Definition, 5.12 $\square$

Definition 5.16. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a sheaf morphism. φ is called*

1). a monomorphism if any morphism of sheaves $\varphi_0 : \mathcal{H} \to \mathcal{F}$, $\varphi \circ \varphi_0 = 0$ if and only if $\varphi_0 = 0$,

Proposition 5.13. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of* (**Ab**)*. Then the following statements are equivalent.*

i). φ is a monomorphism.

ii). $\operatorname{Ker} \varphi = 0$.

iii). For any open set $U \subset X$, φ_U is injective.

iv). For any $x \in X$, $\varphi_x :\to \mathcal{F}_x \to \mathcal{G}_x$ is injective.

Proof. Here, I put the procedure of the proof.

$$\begin{tikzcd} i) \arrow[d] & iv) \arrow[d, leftrightarrow] \\ ii) \arrow[r, leftrightarrow] & iii) \arrow[ul] \end{tikzcd}$$

$i) \Rightarrow ii)$,

$$\begin{tikzcd} \mathrm{Ker}(\varphi) \arrow[d, "\varphi_0"'] \arrow[dr, "0"] & \\ \mathcal{F} \arrow[r, "\varphi"'] & \mathcal{G} \end{tikzcd}$$

Where $\varphi_0(U)$ is an inclusion map of abelian groups.

$ii) \Leftrightarrow iii)$,

$$\mathrm{Ker}\,\varphi = 0 \Leftrightarrow \forall U \in \mathcal{T}, \mathrm{Ker}\,\varphi(U) = 0 \Leftrightarrow \varphi_U \text{ is injective.}$$

$iii) \Rightarrow iv)$, Fix $x \in X$.

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U)$$

is an exact sequence as φ_U is injective for any $U \subset X$ open. Since $\varinjlim$ is left-exact we obtain,

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x$$

is also an exact sequence. □

Proposition 5.14. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism in* $\mathrm{Sh}(X)$*. Then the following are equivalent.*

1). φ is an epimorphism (for any $\varphi_1, \varphi_2 : \mathcal{H} \to \mathcal{F}$, such that $\varphi \circ \varphi_1 = \varphi \circ \varphi_2$ implies $\varphi_1 = \varphi_2$).

2). $\mathrm{Coker}\,\varphi = 0$.

3). For any open set $U \subset X$,

4). For any $x \in X$, $\mathrm{Coker}\,\varphi_x = 0$, *(in other words, φ_x is a surjection).*

Proof. Recall the definition of epimorphisms is such that $\varphi : \mathcal{F} \to \mathcal{G}$ is an epimorphism if for any morphism $\psi : \mathcal{G} \to \mathcal{G}_0$, we have,

$$\psi \circ \varphi = 0 \Rightarrow \psi = 0.$$

i) $\Rightarrow$ *iv)*. Suppose φ is an epimorphism, then we have

$$\begin{tikzcd} \mathcal{F} \arrow[d, "\varphi"'] \arrow[dr, "0"] \arrow[drr, bend left, "0"] & & \\ \mathcal{G} \arrow[r, "\pi"] \arrow[rr, bend right, "\psi"'] & \mathrm{Coker}^{\mathbf{pre}}\,\varphi \arrow[r, "(-)^+"] & \mathrm{Coker}\,\varphi \end{tikzcd}$$

By the assumption $\psi = 0$.

Let $\mathcal{O}_x = \{U \in \mathcal{T} \mid x \in U\}$. We consider an exact sequence,

$$0 \longrightarrow \mathrm{Ker}(\varphi_U) \hookrightarrow \mathcal{F}(U) \xrightarrow{\varphi} \mathcal{G}(U) \xrightarrow{\pi} \mathrm{Coker}(\varphi_U) \longrightarrow 0,$$

for each $U \in \mathcal{O}_x$. By Proposition 2.2,

$$0 \longrightarrow \mathrm{Ker}(\varphi)_x \hookrightarrow \mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\pi_x} \mathrm{Coker}(\varphi)_x \longrightarrow 0$$

is also exact. Thus we conclude

$$\mathrm{Coker}^{pre}(\varphi)_x = \mathrm{Coker}(\varphi_x).$$

And we conclude that φ_x is surjective by the exactness of the sequence.

iv) $\Rightarrow$ *ii)*. Assume For each $x \in X$, $\mathrm{Coker}(\varphi_x) = 0$. By applying Proposition. 5.2 to $\mathbf{id} : \mathcal{F} \to \mathcal{F}$, we obtain

$$\mathcal{F} = 0 \Leftrightarrow \forall x \in X, \mathcal{F}_x = 0.$$

Apply this to $\mathrm{Coker}\,\varphi$, we derive that

$$\mathrm{Coker}\,\varphi = 0.$$

iv) $\Rightarrow$ *i)*. Assume $\mathrm{Coker}(\varphi_x) = 0$ for any $x \in X$. Consider a commutative diagram of sheaves

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[dr, "0"'] & \mathcal{G} \arrow[d, "\psi"] \\ & \mathcal{G}_0 \end{tikzcd}$$

By assumption $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is a surjection. Thus $\psi_x = 0$ for any $x \in X$ which is equivalent to $\psi = 0$.

ii) $\Rightarrow$ *i)*. Suppose $\mathrm{Coker}\,\varphi = 0$ if and only if $\mathrm{Coker}(\varphi)_x = \mathrm{Coker}(\varphi_x) = 0$ for any $x \in X$.

iii) $\Rightarrow$ *iv)*. Assume $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for any $U \subset X$ open. By Proposition. 2.2, we conclude that

$$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$$

is also surjective. $\square$

Corollary 5.3. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then the following statements are equivalent.*

1). φ is an isomorphism.

2). For all $x \in X$, φ_x is an isomorphism.

Proof. $\square$

5.7 Exact Sequences of Sheaves

Definition 5.17. *A sequence*

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

of sheaves with morphisms φ, ψ of sheaves is said to be exact at $\mathcal{G}$ if

$$\operatorname{Im} \varphi = \operatorname{Ker} \psi.$$

Definition 5.18. *A short exact sequence of sheaves is a sequence*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$$

which is exact at all terms. In other words,

i). φ is injective,

ii). ψ is surjective,

iii). $\operatorname{Im} \varphi = \operatorname{Ker} \psi$.

Proposition 5.15. *Given a exact sequence*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H},$$

of sheaves on a topological space X. Then for each open set $U \subseteq X$, we derive an exact sequence of abelian groups

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi} \mathcal{G}(U) \xrightarrow{\psi} \mathcal{H}(U),$$

Proof. $\square$

Corollary 5.4. *Taking stalks in* $\mathrm{Sh}_{(\mathbf{Ab})}(X)$ *is an exact functor.*

Theorem 5.1. *The category of sheaves of abelian groups is an abelian category.*

5.8 Direct Image Functors

Definition 5.19. *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ be a continuous map. The direct image functor is the functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ such that,*

$$\mathcal{F} \in \mathrm{Sh}(X), U \in \mathcal{T}_Y \{ \quad f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

Proposition 5.16. *Such $f_*\mathcal{F}$ is indeed a sheaf.*

Proof. □

Definition 5.20. *Let $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ be a direct image functor. Given a morphism of sheaves, $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Sh}(X)$. For an open set $U \in Y$, the image of φ under f_* is*

$$(f_*\varphi)_U = \varphi_{f^{-1}(U)} : f_*\mathcal{F}(U) \to f_*\mathcal{G}(U).$$

Definition 5.21. *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ be a continuous map. The inverse image functor*

$$f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$$

is defined as follow.

For a sheaf $\mathcal{G} \in \mathrm{Sh}(Y)$, we set $f^{-1}\mathcal{G}$ to be the sheafification of the presheaf

$$V \in \mathcal{T}_X, \quad f^{-1}\mathcal{G}(V) = \varinjlim_{U \in \mathcal{O}_V} \mathcal{G}(U),$$

where $\mathcal{O}_V = \{U \in \mathcal{T}_X \mid V \subseteq U\}$.

Definition 5.22. *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ be a continuous map. For a sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ and a point $y \in Y$, we define the stalk of $f_*\mathcal{F}$ at y as*

$$(f_*\mathcal{F})_y = \varinjlim_{U \in \mathcal{O}_y} \mathcal{F}(f^{-1}(U)).$$

Proposition 5.17. *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ be a continuous map. For $\mathcal{F} \in \mathrm{Sh}(Y), x \in X$, there exists an isomorphism such that*

$$(f^{-1}\mathcal{F})_x \cong \mathcal{F}_{f(x)}.$$

Example 5.14. *Given a topological space X and a single point set $\{y\}$. A unique map $f : X \to \{y\}$ is continuous. We know that*

$$\mathrm{Sh}(\{y\}) = (\mathbf{Ab}).$$

Thus the direct image functor f_* and $\mathcal{F} \in \mathrm{Sh}(\{y\})$, we have

$$f_*\mathcal{F} = \mathcal{F}(X) \in (\mathbf{Ab}).$$

Example 5.15. *Given a topological space X and a single point set $\{y\}$ and a unique continuous map $f : X \to \{y\}$. We have for $G \in \mathrm{Sh}(\{y\}) = (\mathbf{Ab})$,*

$$f^{-1}G = G.$$

Example 5.16. *Given a topological space X and a single point set $\{y\}$ and a continuous map $f : \{y\} \to X$. We have for $G \in \mathrm{Sh}(\{y\}) = (\mathbf{Ab})$,*

$$f_*G(U) = \begin{cases} G & (f(y) \in U), \\ 0 & (f(y) \notin U). \end{cases}$$

Example 5.17. *Given a topological space X and a single point set $\{y\}$ and a continuous map $f : \{y\} \to X$. For $\mathcal{F} \in \mathrm{Sh}(X)$,*

$$f^{-1}\mathcal{F}(\{y\}) = \mathcal{F}_f(y).$$

Lemma 5.1. *Given a topological spaces X and Y, $f : X \to Y$ a continuous function. The inverse image functor f^{-1} is exact.*

Lemma 5.2. *Given a topological spaces X and Y, f a continuous function. The direct image functor f_* is (only) left-exact.*

Lemma 5.3. *Given a topological spaces X and Y, f a continuous function. We have seen that f_* induces a group homomorphism between $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and $\mathrm{Hom}(f_*\mathcal{F}, f_*\mathcal{G})$. Allowing the abuse of notation, we denote this homomorphism by*

$$f_* : \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(f_*\mathcal{F}, f_*\mathcal{G}).$$

For such homomorphism we have

$$f_*(\mathcal{F} \oplus \mathcal{G}) \cong f_*\mathcal{F} \oplus f_*\mathcal{G}.$$

6 Cohomology

6.1 Basics of Cohomology

Example 6.1. *Sheaf of holomorphic functions.*

6.2 δ-Functors

Definition 6.1. *Let $\mathcal{A}, \mathcal{B}$ be categories. A δ=functor from $\mathcal{A} \to \mathcal{B}$ consist of additive functors*

$$H^i : \mathcal{A} \to \mathcal{B} \quad (i = 0, 1, \cdots),$$

such that for any short exact sequence

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

in $\mathcal{A}$, we have mapping

$$\delta^i : H^i(A_3) \to H^{i+1}(A_1),$$

which make the sequence

$$0 \longrightarrow H_0(A_1) \longrightarrow H_0(A_2) \longrightarrow H_0(A_3) \xrightarrow{\delta^0} H_1(A_1) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_i(A_3) \xrightarrow{\delta^i} H_{i+1}(A_1) \longrightarrow \cdots$$

a long exact sequence.

Furthermore, given a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \end{array}$$

we get a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^i(A_1) & \longrightarrow & H^i(A_2) & \longrightarrow & H^i(A_3) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^i(B_1) & \longrightarrow & H^i(B_2) & \longrightarrow & H^i(B_3) & \longrightarrow & \cdots \end{array}$$

Definition 6.2. *A δ-functor $(H^i : \mathcal{A} \to \mathcal{B}, \delta^i)_i$ is universal if for any other δ-functor $(\tilde{H}^i : \mathcal{A} \to \mathcal{B}, \tilde{\delta}^i)_i$ together with a morphism (functor transforms) $H^i \to \tilde{H}^i$ such that for any exact sequence*

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

in $\mathcal{A}$, we get a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^i(A_2) & \longrightarrow & H^i(A_3) & \xrightarrow{\delta^i} & H^i(A_1) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \tilde{H}^i(A_2) & \longrightarrow & \tilde{H}^i(A_3) & \xrightarrow{\tilde{\delta}^i} & \tilde{H}^i(A_1) & \longrightarrow & \cdots \end{array}$$

Lemma 6.1. *A universal δ-functor $(H^i : \mathscr{A} \to \mathscr{B}, \delta^i)$ is uniquely determined by fixing H^0.*

Proof. $\square$

Definition 6.3. *Let $F : \mathscr{A} \to \mathscr{B}$ be a functor. F is said to be erasable if for any $A \in \mathbf{ob}(\mathscr{A})$, there exists a monomorphism $\varphi : A \hookrightarrow A'$ for some $A' \in \mathbf{ob}(\mathscr{A})$ such that $F(\varphi) = 0$.*

Theorem 6.1. *Supposer $(H^i : \mathscr{A} \to \mathscr{B}, \delta^i)$ be a δ-functor such that for any $i \in \mathbb{N}$, H^i is erasable. Then (H^i, δ^i) is universal.*

Proof. □

Definition 6.4. *A δ-functor $(H^i : \mathscr{A} \to \mathscr{B}, \delta^i)$such that for any $i \in \mathbb{N}$, H^i is erasable. Let us denote $H^0 = F$, then $(H^i : \mathscr{A} \to \mathscr{B}, \delta^i)$ is called higher derived functors of F and each H^i is denoted as R^iF.*

Definition 6.5. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories and $F : \mathscr{A} \to \mathscr{B}$ be a functor such that*

i). $\mathscr{A}$ has enough injectives,

ii). F is additive left exact.

Given an injective resolution of A

$$0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

we define

$$R^iF(A) = \frac{\mathrm{Ker}(FI_i \to FI_{i+1})}{\mathrm{Im}(FI_{i-1} \to FI_i)}.$$

Lemma 6.2. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories with $\mathscr{A}$ having enough injectives, $F : \mathscr{A} \to \mathscr{B}$ be an additive left exact functor.*

Then for any $A \in \mathbf{ob}(\mathscr{A})$, we have

$$R^0F(A) \cong F(A).$$

Proof. By the left exactness of F, we see

$$R^0FA = \mathrm{Ker}(FI_0 \to FI_1) = FA.$$

□

Lemma 6.3. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories with $\mathscr{A}$ having enough injectives, $F : \mathscr{A} \to \mathscr{B}$ be an additive left exact functor.*

Each $R^iF(A)$ is independent of the choice of injective resolution, thus Definition 6.5 is well-defined.

Proof. □

Corollary 6.1. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories with $\mathscr{A}$ having enough injectives, $F : \mathscr{A} \to \mathscr{B}$ be an additive left exact functor.*

For any injective object $A \in \mathscr{A}$, we have

$$\forall i \in \mathbb{N}, \quad R^iF(A) = 0$$

Proof. ☐

Lemma 6.4. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories with $\mathscr{A}$ having enough injectives, $F : \mathscr{A} \to \mathscr{B}$ be an additive left exact functor.*

Proof. ☐

Lemma 6.5. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories with $\mathscr{A}$ having enough injectives, $F : \mathscr{A} \to \mathscr{B}$ be an additive left exact functor.*

For any $i \in \mathbb{N}$, $R^i F$ is erasable.

Proof. ☐

Corollary 6.2. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories with $\mathscr{A}$ having enough injectives, $F : \mathscr{A} \to \mathscr{B}$ be an additive left exact functor.*

(R^F, δ^i) is a universal δ-functor where

$$\delta$$

with $H^0 = F$.

Proof. ☐

Example 6.2. *Let $G \in \mathbf{ob}(\mathbf{Ab})$ be a fixed abelian group. Since*

$$H \in \mathbf{ob}(\mathbf{Ab}), \quad H \mapsto \operatorname{Hom}(G, H),$$

is an additive left exact functor, we can define the following

$$\operatorname{Ext}^i(G, \cdot) = R^i \operatorname{Hom}(G, \cdot)$$

which is called the Ext-functor.

Definition 6.6. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories with $\mathscr{A}$ having enough injectives, $F : \mathscr{A} \to \mathscr{B}$ be an additive left exact functor.*

An object $A \in \mathbf{ob}(\mathscr{A})$ is said to be F-acyclic if we have

$$\forall i \in \mathbb{N} \quad R^i F(A) = 0.$$

Remark 6.1. *For any such functor $F : \mathscr{A} \to \mathscr{B}$, by Corollary **??**, every injective object is F-acyclic.*

Definition 6.7. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories with $\mathscr{A}$ having enough injectives, $F : \mathscr{A} \to \mathscr{B}$ be an additive left exact functor.*

A F-acyclic resolution of an object $A \in \mathbf{ob}(\mathscr{A})$ is the complex

$$0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdot$$

which is exact and each $I_0, I_1, \cdots$ are F-acyclic.

6.3 Cohomology of Sheaves

Lemma 6.6. *For a topological space X, the category of sheaves $\mathrm{Sh}(X)$ on X has enough injectives.*

Proof. $\square$

Example 6.3. *Let X be a topological space. Then the global section functor*

$$\Gamma(X, \cdot) : \mathrm{Sh}(X) \to (\mathbf{Ab}), \quad \mathcal{F} \mapsto \mathcal{F}(X).$$

is an additive left exact functor.

Example 6.4. *Let X, Y be topological spaces and $f : X \to Y$ be a continuous function. Then the direct image functor f_* is an additive left exact functor.*

7 Scheme Theory

7.1 Ringed Spaces

Definition 7.1. *Let $(X, \mathcal{T})$ be a topological space. A ringed space is a sheaf $\mathcal{O}_X$ of rings on X.*

Definition 7.2. *A morphism of ringed spaces between $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ is a tuple $(f, f^\#)$ where $f : X \to Y$ is a continuous map and $f : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings.*

Example 7.1. *Let $(X, \mathcal{T})$ be a topological space. The sheaf of continuous functions $\mathcal{C}_X$ is a ringed space and any continuous map $f : X \to Y$ defines a morphism of ringed spaces.*

Example 7.2. *Let X is a differentiable manifold then the differentiable functions $\mathcal{C}_X^{\mathbf{diff}}$ is a ringed space. A morphism of ringed spaces $f : X \to Y$, for this case must satisfy the following condition.*

Example 7.3. *Let $X \subseteq \mathbb{C}^n$ be open subset. A sheaf of holomorphic functions $\mathcal{O}_X$ over X is a ringed space. And a morphism of such ringed spaces must be a holomorphic functions*

Example 7.4. *Given the Zariski topology on $X = k^n$ and the sheaf $\mathcal{O}_X(U) = \{f : U \to k \mid f \text{ is regular }\}$, $(X, \mathcal{O}_X)$ is a ringed space.*

Definition 7.3. *By Remark 3.1, the sheaf of regular functions $\mathcal{O}_X$ is contained in the sheaf of continuous functions $\mathcal{C}_X$. Given two Zariski topologies X, Y, and a continuous function $f : X \to Y$, f is said to be regular if for any regular function $g : U \to k$ for an open set $U \subseteq Y$, $g \circ f : f^{-1}(U) \to k$ is also regular. In other words, f is said to be regular if it defines a morphism of ringed spaces between two ringed spaces of regular functions.*

Definition 7.4. *A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that for any $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.*

Example 7.5. *A sheaf of continuous functions on a topological space X is a locally ringed space. Indeed, for each $x \in X$ and the stalk $\mathcal{C}_{X,x}$, the ideal*

$$\mathfrak{m}_x = \{(f : U \to \mathbb{R}, U) \mid f(x) = 0\}$$

is a unique maximal ideal. In order to prove this, we recall that an ideal $\mathfrak{m}$ is a unique maximal ideal if any element not in $\mathfrak{m}$ is a unit.

For each $(f : U \to \mathbb{R}, U) \in \mathcal{C}_{X,x}$, $f(x) \neq 0$ implies that there exists a neighborhood $V \subset U$ such that $f(x) \neq 0$ for any $x \in V$. Thus $(f|_V : V \to \mathbb{R}, V)$ is invertible, therefore a unit.

Example 7.6. *In similar manner, the following are also locally ringed spaces.*

1. *X is a differentiable manifold and $(X, \mathcal{C}_X^{\mathbf{diff}})$.*
2. *$X \subseteq \mathbb{C}^n$ be an open set, and $(X, \mathcal{O}_X)$ be a sheaf of holomorphic functions.*
3. *A sheaf of regular functions on $X = k^n$.*

Definition 7.5. *A morphism $(f, f^{\#}) : (X, \mathcal{O}_X \to (Y, \mathcal{O}_Y)$ between ringed spaces is a morphism of locally ringed space if $f^{\#}$ is local as a ring homomorphism.*

Example 7.7. *Let A be a commutative ring and consider the Zariski topology on $X = \mathrm{Spec}(A)$ and the structure sheaf $(X, \mathcal{O}_X)$. We have proven that*

$$\mathfrak{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}.$$

Therefore, $(X, \mathcal{O}_X)$ is a locally ringed space and for any ring homomorphism $\phi : A \to B$, it induces a morphism of locally ringed spaces $(f, f^{\#}) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ such that

$$\mathfrak{q} \in \mathrm{Spec}(B), f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q}) \in \mathrm{Spec}(A).$$

This is indeed a morphism of locally ringed spaces.

Proposition 7.1. *Let A, B be commutative rings. Then the map $\phi \mapsto (f, f^{\#})$ is a bijection between*

$$\mathrm{Hom}(A, B) \leftrightarrow \mathrm{Hom}_{\mathbf{loc}}(\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}), (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}))$$

Proof. □

Definition 7.6. *A category of ringed spaces is denoted by (**RingedSpaces**) with morphisms $(f, f^{\#})$ morphisms of ringed spaces.*

Definition 7.7. *A category of ringed spaces is denoted by (**RingedSpaces**) with morphisms $(f, f^{\#})$ morphisms of locally ringed spaces.*

Remark 7.1. *A composition of two morphisms locally ringed space is indeed a morphism of locally ringed spaces thus the above construction is justified.*

Definition 7.8. *Two locally ringed spaces* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *are isomorphic if there exists morphisms* $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *and* $(g, g^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ *such that* f *and* g *are inverses of each other. (ie. there exists a morphism of locally ringed spaces* $(f, f^{\#})$ *where* f *is a homeomorphism).*

Example 7.8. *(A morphism of locally ringed spaces induced by homeomorphism but not an isomorphism of locally ringed spaces).*

Let $X = \mathbb{R}^n$ *and consider the sheaf of continuous functionals* $\mathcal{C}_X$ *and the sheaf of smooth functionals* $\mathcal{C}_X^{\mathbf{diff}}$*. Furthermore, we consider* $f = id_X$ *then* $f^{\#}$ *is an inclusion as smooth functions are continuous. However,* $(f, f^{\#})$ *is not an isomorphism of locally ringed spaces.*

Example 7.9. *Let us consider* $X = \mathbb{C}^n$ *and the sheaf of holomorphic functions* $\mathcal{O}$ *on* X *and the structure sheaf* $\mathcal{O}_X$*. Then consider the morphism of locally ringed spaces* $(f, f^{\#})$ *by the identity map. However,* f *is not continuous as the topology defined on the image is the Zariski topology.*

Definition 7.9. *Let* $X = \mathbb{C}^n$ *and* $Y = \mathrm{MaxSpec}(\mathbb{C}[x_1, \cdots, x_n])$*. Let* $f : X \to Y$ *be such that*

$$f(z_1, \cdots, z_n) = (x_1 - z_1, \cdots, x_n - z_n).$$

This is a bijection. Furthermore, f *is continuous because polynomials are continuous functions.*

We define $f^{\#}$ *to be*

7.2 Schemes

Definition 7.10. *An affine scheme is a locally ringed space* $(X, \mathcal{O}_X)$ *which is isomorphic to a structure sheaf* $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ *for some commutative ring* A*.*

Example 7.10. *We consider the Zariski topology on* $\mathrm{Spec}(\mathbb{Z})$ *and a sheaf* $\mathcal{O}$ *such that*

$$\mathcal{O}(D(\mathfrak{a})) = \mathbb{Z}_{\mathfrak{a}}.$$

is an affine scheme.

Example 7.11. *Let* k *be a field. Then* $\mathrm{Spec}(k)$ *is a single point set. And we consider the sheaf* $\mathcal{O}$ *such that* $\mathcal{O}(\mathrm{Spec}(k)) = k$*.*

Definition 7.11. *For a field* A *be a commutative ring and* n *a natural number, we define*

$$\mathbb{A}_A^n = (\mathrm{Spec}(A[x_1, \cdots, x_n]), \mathcal{O}).$$

Example 7.12. *Let* A *be a discrete valuation ring in other words* $k[t]_{(t)}$*.*

Example 7.13. *Let* k *be a field and* $A = k[x]/(x^2)$*. Then* $\mathrm{Spec}(A) = \{(x)\}$*. Thus a single point set. However, this is not isomorphic to* $(\mathrm{Spec}(k), \mathcal{O})$ *introduced in Example 7.11.*

Definition 7.12. *A scheme is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic t an affine scheme. In other words, for any $x \in X$, there is a neighborhood U of X such that there exists a commutative ring A and $(U, \mathcal{O}|_U)$ is isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$.*

Definition 7.13. *A category of affine schemes is* $(\mathbf{AffSch})$ *where*

i). $\mathbf{ob}(\mathbf{AffSch}) = \{(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) | A \textit{ is a commutative ring and } \mathcal{O}_{\mathrm{Spec}(A)} \textit{ is a structure sheaf}\}$.

ii). $(\mathbf{AffSch})((\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}), (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)})) = \{ \textit{ morphisms of locally ringed spaces}\}$.

Definition 7.14. *A category of schemes is* $(\mathbf{Sch})$ *where*

i). $\mathbf{ob}(\mathbf{Sch}) = \{(X, \mathcal{O}_X) \mid \textit{Schemes}\}$.

ii). $(\mathbf{AffSch})((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) = \{ \textit{ morphisms of locally ringed spaces}\}$.

Remark 7.2. *We have the inclusion relations*

$$(\mathbf{AffSch}) \subset (\mathbf{Sch}) \subset (\mathbf{LocallyRingedSpaces})$$

which are all fullsubcategories however,

$$(\mathbf{LocallyRingedSpaces}) \subset (\mathbf{RingedSpaces})$$

is not a full subcategory

7.3 Connection with Classical Algebraic Geometry

Proposition 7.2. *Let X be an affine variety. The regular functions $\mathcal{O}_X(U)$*

$$\mathcal{O}_X(U) = \{h : U \to k \mid h \textit{ is a regular function.}\}.$$

defined on open subset U of X form a sheaf. Furthermore, it is a locally ringed space.

Proof. □

Proposition 7.3. *Let X be an affine variety and $Y = A(X)$ be a coordinate ring. Let us consider the sheaf of regular functions $(X, \mathcal{O}_X)$ and an affine scheme $(Y, \mathcal{O}_Y)$. There exists a natural morphism of locally ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.*

Proof. Notice that we have the following isomorphisms.

$$X \cong \mathrm{MaxSpec}(A(X)), \quad k^n \cong \mathrm{MaxSpec}(k[x_1, \cdots, x_n]).$$

For any maximal ideal $\mathfrak{m} \subset k[x_1, \cdots, x_n]$,

$$I(X) \subseteq \mathfrak{m} = (x_1 - a_1, \cdots, x_n - a_n) \Leftrightarrow \forall f \in I(X), f(a_1, \cdots, a_n) = 0.$$

Let $\pi : Y \to X$ to be the canonical map by $I(X)$, then the map $f : X \to Y, (\mathfrak{m}) = \pi^{-1}(\mathfrak{m})$ is an inclusion. Then f is continuous.

Let us define $f^{\#} : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$. For an open set $U \subseteq Y$, we have

$$(s : U \to \coprod_{\mathfrak{p}\in U} A(x)_{\mathfrak{p}}) \mapsto (s : U \to \coprod_{\mathfrak{m}\in U\cap \operatorname{MaxSpec} A(x)} A(x)_{\mathfrak{m}}).$$

By Lemma 3.1 and applying canonical maps $\pi_{\mathfrak{m}} : A(X)_{\mathfrak{m}} \to A(X)_{\mathfrak{m}}/\mathfrak{m}A(X)_{\mathfrak{m}}$ locally, we get

$$s : U \to \coprod_{\mathfrak{m}\in U\cap \operatorname{MaxSpec} A(x)} \to \coprod_{\mathfrak{m}\in U\cap \operatorname{MaxSpec} A(x)} A(X)_{\mathfrak{m}}/\mathfrak{m}A(X)_{\mathfrak{m}} = k.$$

Thus we obtained a map $s : U \to k$. Locally, we have

$$s = \frac{g_1 + I(X)}{g_2 + I(X)},$$

for $g_1 + I(X), g_2 + I(X) \in A(X)$. We conclude, locally

$$t = \frac{g_1}{g_2}.$$

We now claim that $(f, f^{\#})$ is a local morphism of ringed spaces. By the correspondence of a maximal ideal $\mathfrak{m}$ of $k[x_1, \cdots, x_n]$ and a point $(a_1, \cdots, a_n)$, we have the isomorphism

$$\mathcal{O}_{X,\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{Y,\mathfrak{m}} = A(X)_{\mathfrak{m}}.$$

□

Remark 7.3. *Since X is an algebraic variety, there is a prime ideal $\mathfrak{p}$ of $k[x_1, \cdots, x_n]$ such that*

$$X = V(\mathfrak{p}).$$

Let us define $(Y', \mathcal{O}_{Y'}) = (\operatorname{Spec}(k[x_1, \cdots, x_n]), \mathcal{O})$, where $I(X) = \mathfrak{a}$. Since k is field, $k[x_1, \cdots, x_, n]$ is Noetherian, thus the primary decomposition exists for any ideal. Thus there is a bijection between

$$\operatorname{Spec}(k[x_1, \cdots, x_n]/\mathfrak{a}) \leftrightarrow \operatorname{Spec}(A(X)).$$

Example 7.14. *Let K be any field and $A = k[x]/(x^2)$. A is called the ring of dual numbers. Observe that*

$$(\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k}), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}),$$

both consist of single points. Let us define $(f, f^{\#}) : (\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. By the previous observation, the function $f : \operatorname{Spec} A \to \operatorname{Spec} k$ is unique map sending the unique point to the unique point.

7.4 Properties of Schemes

Theorem 7.1 (Topological properties of schemes)**.**

Definition 7.15. *A scheme is said to be locally Noetherian if there exists an open cover $(U_i)_{i\in I}$ such that for each $i \in I$,*

$$U_i \cong \mathrm{Spec}(A_i) \tag{7.1}$$

for some Noetherian ring A_i.

Lemma 7.1. *Let A, B be rings and $\varphi : A \to B$ be a ring homomorphism. Let $a \in A$ and $b = \varphi(a)$, then we have*

$$\mathrm{Spec}(A_a) = \mathrm{Spec}(B_b)$$

as sets.

Proof. Let $\mathfrak{q} \subset B$ be a prime ideal not containing b and $\mathfrak{p} \subset A$ be a contraction of $\mathfrak{q}$ by φ. φ^{-1} is the inclusion from $\mathrm{Spec}(B)$ to $\mathrm{Spec}(A)$. Also we have

$$\mathrm{Spec}(A_a) \cong D_A(a) \subset \mathrm{Spec}(A)$$

for an arbitrary ring and an element. And by φ^{-1} we have

$$D_B(b) \subset D_A(a).$$

Thus we have an inclusion

$$\mathrm{Spec}(B_b) \subset \mathrm{Spec}(A_a).$$

□

Proposition 7.4. *Let A, B be rings and $\varphi : A \to B$ be a ring homomorphism. Let $a \in A$ and $b = \varphi(a)$, then we have*

$$\mathrm{Spec}(A_a) = \mathrm{Spec}(B_b)$$

as affine schemes.

Proof. By Lemma 7.1, they are equal as sets. Since $\mathrm{Spec}(B)$ is open in $\mathrm{Spec}(A)$. Using the definition of structure sheaves, we have

$$\mathcal{O}_{\mathrm{Spec}(A)}|_{\mathrm{Spec}(B)} = \mathcal{O}_{\mathrm{Spec}(B)}.$$

Therefore, by the lemma we have

$$\mathcal{O}_{\mathrm{Spec}(A_a)} = \mathcal{O}_{\mathrm{Spec}(B_b)}.$$

□

Lemma 7.2. *A scheme $(X, \mathcal{O}_X)$ is locally Noetherian if and only if for any open affine set $U \subset X, U = \mathrm{Spec}(A)$ for some Noetherian ring A.*

Proof. By definition, $\Leftarrow$ is trivially true. We will prove $\Rightarrow$ direction.

Let $X = \bigcup_{i \in I} \mathrm{Spec}(A_i)$ be an open affine Noetherian covering of X and $U = \mathrm{Spec}(A)$ be an open affine set. Let us define an open covering of U by

$$U_i = U \cap \mathrm{Spec}(A_i), \quad U = \bigcup_{i \in I} U_i.$$

By Theorem 7.1 and the assumption on U, U is quasi-compact. By arranging I, there exists a large enough $n \in \mathbb{N}$ such that

$$U = \bigcup_{i=1}^{n} U_i.$$

Since each $i = 1, \cdots, n$, U_i is open in $\mathrm{Spec}(A_i$, thus there is $\{a_{ij}\}_{j=1\cdots,n_i} \subset A_i$ such that

$$U_i = \bigcup_{j=1}^{n_i} \mathrm{Spec}(A_{i,a_{ij}}).$$

Thus substituting this to (7.1), we get

$$U = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} \mathrm{Spec}(A_{i,a_{ij}}).$$

Again by the quasi-compactness of U, we conclude finitely many $A_{i,a_{ij}}$ cover U.

Since A_i is Noetherian for each $i \in I$, this means that any localization of it is also Noetherian. Thus we $\{\mathrm{Spec}(A_{i,a_{ij}})$ is an open Noetherian covering of U. By rearranging $\{a_{ij}\}$, we let $\{a_1, \cdots, a_n\}$ to be the elements which spectrums of their localizations cover U. We then show that

$$\mathfrak{a} = \bigcap_{i=1}^{n} \pi_i^{-1}(\pi_i(\mathfrak{a})A_{a_i}).$$

where $\pi_i : A \to A_{a_i}$ is the canonical inclusion for each $i = 1, \cdots, n$.

$\mathfrak{a} \subseteq \bigcap_{i=1}^{n} \pi_i^{-1}(\pi_i(\mathfrak{a})A_{a_i})$ is trivial, thus we prove $\mathfrak{a} \supset \bigcap_{i=1}^{n} \pi_i^{-1}(\pi_i(\mathfrak{a})A_{a_i})$. Let $b \in \bigcap_{i=1}^{n} \pi_i^{-1}(\pi_i(\mathfrak{a})A_{a_i})$ be arbitrary. By definition, for each $i = 1, \cdots, n$, there exists $b_i \in \mathfrak{a}$ and n_i such that

$$\pi_i(b) = \frac{b_i}{a_i^{n_i}}.$$

Using that $\mathfrak{a}$ is an ideal, we derive that for big enough N, we have

$$\pi_i(b) = \frac{b_i}{a_i^{N}}.$$

Using the definition of localization, for each $i = 1, \cdots, n$, there is m_i such that

$$(b_i - a_i^N b)a_i^{m_i} = 0.$$

Taking large enough M, for each $i = 1, \cdots, n$, we derive

$$(b_i - a_i^N b)a_i^M = 0.$$

Thus for all $i = 1, \cdots, n$, we know

$$a_i^{N+M} b \in \mathfrak{a}. \tag{7.2}$$

Since $\{D_A(a_i)\}_{i=1,\cdots,n}$ covers $\operatorname{Spec} A$, we have

$$\bigcap_{i=1}^{n} V(a_i) = \emptyset \Leftrightarrow V((a_1, \cdots, a_n)) = \emptyset \Leftrightarrow (a_1, \cdots, a_n) = (1).$$

Therefore, for any $k \in \mathbb{N}_0$ we have $(a_1^k, \cdots, a_n^k) = (1)$. By Equation (7.2), we derive that for some $c_1, \cdots, c_n \in A$,

$$b = \sum_{i=1}^{n} c_i a_i^k \in \mathfrak{a}.$$

Finally, we prove that A is Noetherian. Given an ascending chain of ideal

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots.$$

We get an ascending chain of extended ideals

$$\pi(\mathfrak{a}_1)A_{a_i} \subset \pi(\mathfrak{a}_2)A_{a_i} \subset \cdots,$$

for each $i = 1, \cdots, n$. Since each A_{a_i} is Noetherian, we conclude that there is large enough N such that

$$\pi_i(\mathfrak{a}_N)A_{a_i} = \pi_i(\mathfrak{a}_{N+1})A_{a_i}$$

for each $i = 1, \cdots, n$. By Equation (7.4), we conclude that

$$\mathfrak{a}_N = \bigcap_{i=1}^{n} \pi_i^{-1}(\pi_i(\mathfrak{a}_N)A_{a_i}) = \bigcap_{i=1}^{n} \pi_i^{-1}(\pi_i(\mathfrak{a}_{N+1})A_{a_i}) = \mathfrak{a}_{N+1}.$$

□

Corollary 7.1. *An affine scheme* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *is a Noetherian scheme, then* A *is Noetherian.*

Remark 7.4. *A Sheaf is noetherian then its base space is Noetherian as topological space. The converse is not true.*

Definition 7.16. *A scheme $(X, \mathcal{O}_X)$ is said to be reduced if for any open subset $U \subseteq X$, $\mathcal{O}_X(U)$ is a reduced ring.*

Proposition 7.5. *A scheme $(X, \mathcal{O}_X)$ is reduced if and only if for any $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a reduced ring.*

Proof. □

Definition 7.17. *A scheme is integral if every section of it is an integral domain.*

Proposition 7.6. *A scheme $(X, \mathcal{O}_X)$ is reduced then for any $x \in X$, the stalk $\mathcal{O}_{X,x}$ is an integral domain.*

Proof. □

Remark 7.5. *The converse is not true.*

Let k be a field and R be a k-algebra.

Remark 7.6. *By the definition of reduced rings, it is obvious that integral schemes are reduced.*

Example 7.15. *An affine scheme on a field k is integral.*

Example 7.16. *Let k be a field.* $(\operatorname{Spec} k[x]/(x^2), \mathcal{O}_{\operatorname{Spec} k[x]/(x^2)})$ *is neither integral nor reduced.*

Example 7.17. *Let k be a field.* $(\operatorname{Spec} k[x,y]/(x,y), \mathcal{O}_{\operatorname{Spec} k[x,y]/(x,y)})$ *is reduced but not integral.*

Lemma 7.3. *Let $(X, \mathcal{O}_X)$ is a scheme and fix $s \in \mathcal{O}_X(U)$ for some open set U. For each $x \in U$, define $\mathfrak{m}_x$ to be a unique maximal ideal in the stalk. Then the set*

$$F = \{x \in U \mid s \in \mathfrak{m}_x\}$$

is a closed subset of X.

Proof. First, let us assume that $U = \operatorname{Spec}(A)$ for some ring A. We will prove that T^c is open □

Lemma 7.4. *A scheme $(X, \mathcal{O}_X)$ is integral if and only if it is a reduced scheme on an irreducible topological space.*

Proof. Since integral schemes are reduced. We first prove that X is not irreducible then $(X, \mathcal{O}_X)$ is not integral.

Since X is not irreducible, there is non-empty disjoint open subsets U_1, U_2 of X. Then

$$\mathcal{O}_X(U_1 \cup U_2) \cong \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2).$$

Therefore, this is not an integral domain.

Suppose X is irreducible and $(X, \mathcal{O}_X)$ is reduced. Given an arbitrary open set $U \subset X$, and $s_1, s_2 \in \mathcal{O}_X(U)$, we will show that

$$s_1 s_2 = 0 \Rightarrow s_1 = 0 \vee s_2 = 0.$$

By Lemma 7.3,

$$X_1 = \{x \in U \mid s_1 \in \mathfrak{m}_x\}, \quad X_2 = \{x \in U \mid s_2 \in \mathfrak{m}_x\},$$

are closed subsets.

By the sheaf property, we have $s_1 s_2 = 0$ implies for all $x \in U$,

$$(s_1 s_2)_x = s_{1,x} s_{2,x} = 0.$$

Since each $\mathfrak{m}_x$ is prime and $s_{1,x} s_{2,x} = 0 \in \mathfrak{m}_x$, $s_{1,x} \in \mathfrak{m}_x$ or $s_{2,x} \in \mathfrak{m}_x$. Therefore, this show that

$$U = X_1 \cup X_2.$$

Since X is irreducible, so is U, Without the loss of generality, we assume $U = X_1$. Let $\mathrm{Spec}(A) \subset U$ be an open affine set. Let us define

$$t = s_1|_{\mathrm{Spec}(A)} \in A.$$

for all $x = \mathfrak{p} \in \mathrm{Spec}(A) \subset U$, we have

$$t_x \in \mathfrak{m}_x \in \mathcal{O}_{X,x}.$$

In other words,

$$\frac{t}{1} \in \mathfrak{p} A_{\mathfrak{p}} \subset A_{\mathfrak{p}}.$$

Therefore, $t \in \mathfrak{p}$ for any prime ideal of A, thus t is a nilpotent. Furthermore, X is reduced, thus $t = 0$.

Thus any section $s_1|_{\mathrm{Spec}(A)} = 0$ for all $\mathrm{Spec}(A)$. By the sheaf property, we conclude $s_1 = 0$. □

Corollary 7.2. *If X is integral, then there exists a unique generic point $\eta \in X$.*

Proof. For any $\mathrm{Spec}(A) \subseteq X$, A is an integral domain. Let $\eta = (0) \in \mathrm{Spec}(A)$. Then η is a generic point of $\mathrm{Spec}(A)$. By the irreducibility of X, we have η is a generic point of X.

For the uniqueness, assume ζ, η be generic points of X. Let us pick an open affine set $\mathrm{Spec}(A)$ containing η. By closedness of $X \setminus \mathrm{Spec}(A)$ and that ζ is also a generic point, we conclude that $\zeta \in \mathrm{Spec}(A)$.

Without the loss of generality, we assume that $\eta = (0) \subset A$. Since $\eta \neq \zeta$, $\zeta = \mathfrak{p} \subset A$ for some prime ideal. However

$$\eta \in V(\mathfrak{p}) \Leftrightarrow \mathfrak{p} = (0) \Rightarrow \eta = \zeta.$$

□

7.5 Open and Closed Subschemes

Definition 7.18. *Let $(X, \mathcal{O}_X)$ be a scheme and $U \subseteq X$ be open. Then*

$$(U, \mathcal{O}_X|_U)$$

is called an open subscheme of X.

Remark 7.7. *If* $\mathrm{Spec}(B) \subseteq \mathrm{Spec}(A)$ *is an open subscheme then*

$$\mathrm{Spec}(B) = \bigcup_{i=1}^{n} \mathrm{Spec}(A_{a_i}) = \bigcup_{i=1}^{n} \mathrm{Spec}(B_{b_i}),$$

with morphisms $A \ni a_i \mapsto b_i \in B$.

Definition 7.19 (1st definition of closed subschemes)**.** *Let $(X, \mathcal{O}_X)$ be a scheme and M be a set of morphisms of schemes such that for each $(i, i^{\#}) \in M$,*

1. *$i : Z \to X$, is a homeomorphism of Z and some closed subset $i(Z)$ of X.*
2. *$i^{\#} : \mathcal{O}_X \to i_* \mathcal{O}_Z$ is surjective. In other words, for any $z \in Z$, $\mathcal{O}_{X,i(z)} \to \mathcal{O}_{Z,z}$ is a surjection.*

We now define an equivalence relation $\sim$ such that $(i, i^{\#}) : Y \to X \sim (j, j^{\#}) : Z \to X$ if Y and Z are homeomorphic and the following diagram is commutative.

$$\begin{tikzcd}
(Y, \mathcal{O}_Y) \arrow[r, "{(i,i^{\#})}"] \arrow[d] & (X, \mathcal{O}_X) \\
(Z, \mathcal{O}_Z) \arrow[ur, "{(j,j^{\#})}"'] &
\end{tikzcd}$$

Definition 7.20 (Second definition of closed subschemes)**.** *A closed subscheme of a scheme $(X, \mathcal{O}_X)$ consists of a closed subset $i : Z \hookrightarrow X$ and a sheaf $\mathcal{O}_Z$ such that there is a sheaf of ideals $(Z, \mathcal{I}_Z)$ which is a subsheaf of $\mathcal{O}_X$ such that*

$$\mathcal{O}_X|_{\mathcal{I}_Z} \cong i_* \mathcal{O}_X.$$

Lemma 7.5. *Let $X = \mathrm{Spec}(A)$ for a commutative ring and consider a scheme $(X, \mathcal{O}_X)$. Consider a closed subscheme $(Z, \mathcal{O}_Z)$ of $(X, \mathcal{O}_X)$ and an ideal*

$$\mathfrak{a}_Z = \mathrm{Ker}(A \to Z).$$

Then we have the inclusions of set,

$$Z \subseteq V(\mathfrak{a}_Z) \subseteq \mathrm{Spec}(A).$$

Lemma 7.6. *The second claim of the preceding lemma.*

Lemma 7.7. *Let A be a ring then there exists a natural bijection between*

$$\{\mathfrak{a} \subseteq A \mid \mathfrak{a} \textit{ is an ideal of } A\} \leftrightarrow \{Z \subseteq \mathrm{Spec}(A) \mid Z \textit{ is a closed subscheme}\}.$$

Proof. □

7.6 Fiber Products

Definition 7.21. *Let X, S be schemes. X is called a S-scheme if there exists a morphism of schemes $\varphi : X \to S$.*

Definition 7.22. *Let S be a scheme. The category of S-schemes is $(\mathbf{Sch}/S)$ where*

1. $\mathbf{ob}(\mathbf{Sch}/S) = \{\varphi : X \to S \mid \varphi$ *is a morphism of schemes*$\}$.
2. $(\mathbf{Sch}/S)(\varphi : X \to S, \psi : Y \to S) = \{f : X \to Y \mid f$ *is a morphism of schemes such that* $\varphi = f \circ \psi\}$.

Remark 7.8. *Clearly we have*

$$(\mathbf{Sch}/\operatorname{Spec}(\mathbb{Z})) = (\mathbf{Sch}).$$

For a field k we have,

A given scheme can have many k-scheme structure.

Definition 7.23. *A fiber product of S-schemes X, Y with morphisms $f : X \to S, g : Y \to S$ is a scheme $X \underset{S}{\times} Y$ together with a morphisms $p : X \underset{S}{\times} Y \to X, q : X \underset{S}{\times} Y \to Y$ such that*

$$f \circ p = g \circ q$$

and for any scheme F with such pair of morphisms $p_F : F \to X, q_F : F \to Y$, there is a morphisms of scheme $\varphi : F \to X \underset{S}{\times} Y$ such that we have

$$p \circ \varphi = p_F, \quad q \circ \varphi = q_F.$$

$$\begin{tikzcd}
F \arrow[dr, dotted, "\exists!\varphi"] \arrow[drr, "q_F"] \arrow[ddr, "p_F"'] & & \\
& X \underset{S}{\times} Y \arrow[r, "q"'] \arrow[d, "p"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & S
\end{tikzcd}$$

Proposition 7.7. *A fiber product of S-schemes X, Y is unique up to isomorphisms if it exists.*

Proof. $\square$

Proposition 7.8. *Let X, Y be S-schemes. Then we have an isomorphism*

$$X \underset{S}{\times} Y \cong Y \underset{S}{\times} X.$$

Proof. $\square$

Proposition 7.9. *Let X be a S-scheme, Z be a T-scheme, and Y be both a S and T-scheme. Then we have an isomorphism*

$$(X \underset{S}{\times} Y) \underset{T}{\times} Z \cong X \underset{S}{\times} (Y \underset{T}{\times} Z).$$

Proof. $\square$

Example 7.18. *Let $X, Y, S \in \mathbf{ob}(\mathbf{Sch})$ and $f : X \to S, g : Y \to S$ be mappings. Define*

$$X \underset{S}{\times} Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

Then this is a fiber product together with restriction of projections from $X \times Y \to X, Y$ to $X \underset{S}{\times} Y$, denoted by p, q.

Example 7.19. *There does not exist a fiber product for $\mathbb{A}_k^1, \mathbb{A}_k^1, \mathrm{Spec}(k)$.*

Lemma 7.8. *Fiber products exist in $(\mathbf{AffSch})$.*

Proof. $\square$

Corollary 7.3. *Let k be a field and $m, n \in \mathbb{N}$ then we have*

$$\mathbb{A}_k^m \times \mathbb{A}_k^m \cong \mathbb{A}_k^{m+n}.$$

Proof. $\square$

Remark 7.9. *For a scheme X, write $|X|$ be its underling topological space. Then as topological spaces we have*

$$|\mathbb{A}_k^1| \times |\mathbb{A}_k^1| \ncong |\mathbb{A}_k^2|.$$

Since

Lemma 7.9. *Let X, Y be S-schemes and assume the fiber product $X \underset{S}{\times} Y$ exists with projections $p : X \underset{S}{\times} Y \to X$ and $q : X \underset{S}{\times} Y \to Y$. Then for open sets $U \subseteq X, V \subseteq Y$, we have*

$$p^{-1}(U) \cong U \underset{S}{\times} Y, \quad q^{-1}(V) \cong X \underset{S}{\times} V.$$

Proof. $\square$

Lemma 7.10. *Let X, Y be S-schemes and $X = \bigcup_{i \in I} U_i$ be an open covering of X. If for each i, the fiber product $U_i \underset{S}{\times} Y$ exists, then the fiber product $X \underset{S}{\times} Y$ exists.*

Proof. $\square$

Corollary 7.4. *Let X, Y be S-schemes and assume Y, S are affine. Then $X \underset{S}{\times} Y$ exists.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an open covering such that for each $i \in I$, $\Gamma(U_i, \mathcal{O}_X)$ is affine. Then by Lemma 7.8, fiber products $\{U_i \underset{S}{\times} Y\}_{i \in I}$ exist. By Lemma 7.10, we conclude the fiber product $X \underset{S}{\times} Y$ exists. $\square$

Corollary 7.5. *Let X, Y be S-schemes and assume X, S are affine. Then $X \underset{S}{\times} Y$ exists.*

Corollary 7.6. *Let X, Y be S-schemes and assume S is affine. Then $X \underset{S}{\times} Y$ exists.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an open covering such that for each $i \in I$, $\Gamma(U_i, \mathcal{O}_X)$ is affine. They by Corollary 7.5, fiber products $\{U_i \underset{S}{\times} Y\}_{i \in I}$ exist. By Lemma 7.10, $X \underset{S}{\times} Y$ exists. $\square$

Proposition 7.10. *Fiber products exist in* (**Sch**).

Proof. $\square$

Definition 7.24. *Let Y be a topological group and $(Y, \mathcal{O}_Y)$ be a locally ringed space. We denote the residue field of y to be*

$$k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$$

where $\mathfrak{m}_y$ is the maximal ideal of the stalk $\mathcal{O}_{Y,y}$.

Definition 7.25. *Let $f : X \to Y$ be a scheme morphism and $y \in Y$. The fiber of f over y is the scheme X_y defined as*

$$X_y = X \underset{Y}{\times} \operatorname{Spec}(k(y)).$$

$$\begin{array}{ccc} \operatorname{Spec}(k(y)) \underset{Y}{\times} X & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec}(k(y)) & \xrightarrow[i]{} & Y \end{array}$$

Definition 7.26. *Let $|X_y|$ be an underlying topological space of X_y. Then we have the homeomorphism between*

$$|X_y| \cong f^{-1}(y)$$

with the topology on $f^{-1}(y)$ is the induced topology.

Proof. $\square$

Definition 7.27. *Let k be a field. We define a category $(\mathbf{Sch}/k)$ such that its object consists of morphisms of schemes $f : X \to \mathrm{Spec}(k)$ and morphisms between two objects $\varphi : X \to \mathrm{Spec}(k)$ and $\psi : Y \to \mathrm{Spec}(k)$ consists of morphisms $f : X \to Y$ such that*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[dl, "\psi"] \\ \mathrm{Spec}(k) & \end{tikzcd}$$

is a commutative diagram.

Proposition 7.11. *Let K/k be an extension of fields. Then we have a morphism such that*

$$(\mathbf{Sch}/k) \ni [f : X \to \mathrm{Spec}(k)] \mapsto [X \underset{k}{\times} K \to \mathrm{Spec}(K)] \in (\mathbf{Sch}/K).$$

Definition 7.28. *Let K/k be an extension of fields. Let X be a $\mathrm{Spec}(k)$ scheme then the base change X_K of X to K is*

$$X_K = X \underset{k}{\times} K.$$

Proposition 7.12. *Let K/k be an extension of fields. Let X, Y be $\mathrm{Spec}(k)$ scheme. A morphism $f : X \to Y$ between two objects $\varphi : X \to \mathrm{Spec}(k), \psi : Y \to \mathrm{Spec}(k)$ of $(\mathbf{Sch}/k)$, there is a unique morphism $f_K : X_K \to Y_K$ such that it is a morphism between X_K and Y_K in $(\mathbf{Sch}/K)$.*

Definition 7.29. *Let K/k be a field extension and X be a $\mathrm{Spec}(k) -$ scheme. Then we define the set of K-rational points of X by*

$$X(K) =$$

Remark 7.10. *There is one to one correspondence between*

$$X(K) \leftrightarrow X_K(K).$$

Definition 7.30. *Let $f : X \to Y$ be a morphism of schemes and $y \in Y$. Consider the residue field $k(y)$ and its algebraic closure $\overline{k(y)}$.*

The geometric fiber over y is

$$X_{\overline{y}} = \mathrm{Spec}(\overline{k(y)}) \underset{\mathrm{Spec}(k(y))}{\times} X_y.$$

Proposition 7.13. *We have the isomorphism,*

$$X_{\overline{y}} \cong \mathrm{Spec}(\overline{k(y)}) \underset{Y}{\times} X.$$

Proof. ☐

Example 7.20.

$$\begin{tikzcd}
\mathbb{A}^1_{\overline{\mathbb{Q}}} \arrow[r] \arrow[d] & \mathbb{A}^1_{\mathbb{Q}} \arrow[d] \\
\operatorname{Spec}(\overline{\mathbb{Q}}) \arrow[r] & \operatorname{Spec}(\mathbb{Q}) = \{y\}
\end{tikzcd}$$

where

$$\mathbb{A}^1_{\mathbb{Q}} = \{(f) \mid f \in \mathbb{Q}[x] \text{ is irreducible}\} \cup \{(0)\}$$

and

$$\mathbb{A}^1_{\overline{\mathbb{Q}}} = \{(x - \lambda) \mid \lambda \in \mathbb{Q}\} \cup \{(0)\}.$$

Example 7.21. *Consider $X \to Y = \operatorname{Spec}(\mathbb{Z}_{(p)})$. Notice that $Y = \{\eta, t\}$ where $\eta = (0)$ a generic point and $t =$We have*

$$X_\eta \cong \operatorname{Spec}(\mathbb{Q}), \quad X_t \cong \operatorname{Spec}(\mathbb{F}_p).$$

And geometric fibers

$$X_{\overline{\eta}} \cong \operatorname{Spec}(\overline{\mathbb{Q}}), \quad X_{\overline{t}} \cong \operatorname{Spec}(\overline{\mathbb{F}_p}).$$

Example 7.22. *There is a bijection between $\mathbb{A}^n_k(\overline{k})$ and $\overline{k}^n$ by*

$$(x_1 - \lambda_1, \cdots, x_n - \lambda_n) \leftrightarrow (\lambda_1, \cdots, \lambda_n).$$

Remark 7.11. *The base change does not preserve the topology in general.*

$$\begin{tikzcd}
X_K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(K) \arrow[r] & \operatorname{Spec}(k)
\end{tikzcd}$$

And X is connected does not imply X_K being connected. Indeed, we have a counter example that

$$X = \operatorname{Spec}(\mathbb{C}), \quad k = \mathbb{R}, \quad K = \mathbb{C}.$$

By Proposition 7.13 we have

$$X_K = \operatorname{Spec}(\mathbb{C}) \underset{\operatorname{Spec}(\mathbb{R})}{\times} \operatorname{Spec}(\mathbb{C}).$$

We observe that

$$|X_K| = \{(x - i), (x + i)\}$$

which is disconnected however,

$$|X| = \{(0)\}$$

which is connected.

Proposition 7.14. *Let K/k be a field extension where both k, K are algebraically closed. Then the topology on X_K coincides with the topology on X.*

Proof. $\square$

Example 7.23.

$$\begin{tikzcd}
X_{\overline{\mathbb{Q}}} \arrow[r] \arrow[d] & X = \operatorname{Spec}(\overline{\mathbb{Q}}) \arrow[d] & \\
\operatorname{Spec}(\overline{\mathbb{Q}}) \arrow[r] & \operatorname{Spec}(\mathbb{Q}) & \\
& & \operatorname{Spec}(\overline{\mathbb{Q}}) \arrow[uul] \arrow[ull, "id"]
\end{tikzcd}$$

Then $X_{\overline{\mathbb{Q}}} = X(\overline{\mathbb{Q}}) = \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Definition 7.31 (Conjugate of k-schemes)**.** *Given a k-scheme $X \to \operatorname{Spec}(k)$, and $\sigma \in \operatorname{Aut}(k)$. A conjugate k-scheme is defined as the fiber product.*

$$\begin{tikzcd}
X^\sigma \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(k) \arrow[r, "\sigma"'] & \operatorname{Spec}(k)
\end{tikzcd}$$

We call the morphism $X^\sigma \to \operatorname{Spec}(k)$ the k-scheme structure of X^σ.

Remark 7.12. *As schemes we have $X^\sigma \cong X$, but typically not as k-schemes. We could have taken another isomorphism of schemes*

$$\begin{tikzcd}
X^\sigma \arrow[r, "\varphi"] \arrow[dr] & X \arrow[d] \\
& \operatorname{Spec}(k)
\end{tikzcd}$$

Example 7.24. *Let $X = \operatorname{Spec}(\mathbb{R})$ and consider $\operatorname{Spec}(X \to \mathbb{Q}(\sqrt{2}))$ and $\sigma : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(-\sqrt{2})$. Such that $\sigma(\sqrt{2}) = -\sqrt{2}$.*

Proposition 7.15. *Let $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be a morphism of affine schemes corresponding to a ring homomorphism $\varphi : A \to B$. For $s \in A$ and $t = \varphi(s) \in B$ we have*

$$f^{-1}(\operatorname{Spec}(A_s)) = \operatorname{Spec}(B_t).$$

Proposition 7.16. *Suppose $\operatorname{Spec}(A), \operatorname{Spec}(B) \subseteq X$ be schemes which are open affine. Then the intersection $\operatorname{Spec}(A) \cap \operatorname{Spec}(B)$ can be covered by open affine sets $U \subseteq \operatorname{Spec}(A) \cap \operatorname{Spec}(B)$ of the form*

$$U = D(s) = \operatorname{Spec}(A_s)$$

for some element $s \in A$ and

$$U = D(t) = \mathrm{Spec}(B_t)$$

for some element $t \in B$.

Proposition 7.17. *Suppose we have a morphism $f : X \to Y$ of schemes where*

$$X = \bigcup_{i \in I} \mathrm{Spec}(B_i), \quad Y = \mathrm{Spec}(A).$$

Given homomorphisms of rings $\varphi_i : A \to B_i, \varphi(s) = t_i$ for some fixed $s \in A$. We have

$$f^{-1}(\mathrm{Spec}(A_s)) = \mathrm{Spec}((B_i)_{t_i}).$$

Proposition 7.18. *Suppose we have a ring homomorphism $\varphi : A \to B$ and $\{b_i\}_{i \in I} \subset B$ be such that*

$$(b_1, \cdots, b_i, \cdots)_{i \in I} = (1) = B.$$

If each B_{b_i} is of finite type over A for all $i \in I$, then B is also of finite type over A.

Proposition 7.19. *Given a ring homomorphism $\varphi : A \to B$ and $\{s_i\}_{i \in I} \subseteq A$ such that*

$$(a_i)_{i \in I} = (1) = A.$$

Denote $\{t_i = \varphi(s_i)\}_{i \in I}$. If for each B_{t_i} is of finite type over A_{s_i} then B is of finite type over A.

Proposition 7.20. *Let X be a scheme and denote its global section by $A = \Gamma(X, \mathcal{O}_X)$. Then X is affine (ie. $X = \mathrm{Spec}\, 9A$) if and only if there exists finitely many $a_1, \cdots, a_n \in A$ such that*

1. *$(a_1, \cdots, a_n) = (1) = A$,*
2. *$X_{a_i} = \{x \in X \mid (a_i)_x \in \mathcal{O}^*_{X,x}\}$ is affine.*

Proposition 7.21. *Let $f : X \to Y$ be a morphism of schemes. Then f is locally of finite type if and only if for any open affine set $V = \mathrm{Spec}(A) \subseteq Y$ and an open affine subset $U = \mathrm{Spec}(B)$ of $f^{-1}(V)$, B is of finite type over A.*

Proposition 7.22. *Let $f : X \to Y$ be a morphism of scheme. f is locally of finite type if and only if for any open affine set $V = \mathrm{Spec}(A) \subseteq Y$, and for any open affine subset $U = \mathrm{Spec}(B) \subseteq f^{-1}(V)$, B is of finite type over A.*

Proposition 7.23. *Let $f : X \to Y$ be a morphism of schemes. Then f is affine if and only if for any open set $V = \mathrm{Spec}(A) \subseteq Y$, $f^{-1}(V)$ is affine.*

Corollary 7.7. *If a morphism of schemes $f : X \to \mathrm{Spec}(A)$ is affine then X is affine.*

Corollary 7.8. *A morphism of scheme f is finite if and only if for any open set $V = \mathrm{Spec}(A) \subseteq Y$, and for all $f^{-1}(\mathrm{Spec}(B))$, B is finite over A.*

Lemma 7.11. *Let $f : X \to Y$ be a morphism of affine schemes where $X = \mathrm{Spec}(B), Y = \mathrm{Spec}(A)$. Then if f is finite then f is closed and quasi-finite.*

Proposition 7.24. *Let $f : X \to Y$ be a morphism of schemes. If f is finite then f is closed and quasi-fintie.*

7.7 Separated and Proper Morphisms

Proposition 7.25. *Any morphism of affine schemes is separated.*

Proposition 7.26. *Any affine morphism is separated.*

Proposition 7.27. *Any open or closed immersions are separated.*

Lemma 7.12. *Let $f : X \to Y$ be a morphism of schemes, then we have*

$$f \text{ is separated.} \Leftrightarrow \Delta(X) \subseteq X \underset{Y}{\times} X.$$

Lemma 7.13. *Let Z be a topological space. Z is Hausdorff if and only if $\Delta(Z) = \{(z, z) \mid z \in Z\} \subseteq Z \times Z$ is closed.*

Proposition 7.28. *A morphism $f : X \to Y$ of schemes is separated if and only if $\Delta(X) \subseteq X \underset{Y}{\times} X$ is closed.*

Remark 7.13. *The Zariski topology on $X \underset{Y}{\times} X$ is not the product topology.*

Definition 7.32 (Universally closed)**.** *A morphism of scheme $f : X \to Y$ is said to be universally closed if,*

Definition 7.33. *A morphism $f : X \to Y$ of scheme such that*

1. *f is separated,*
2. *f is of finite type,*
3. *f is universally closed,*

is said to be proper.

Example 7.25. *A morphism $\mathbb{A}^1_k \to \mathrm{Spec}(k)$ is closed but not universally closed. thus not a proper morphism.*

Remark 7.14. *A morphism $\mathbb{P}^1_k \to \mathrm{Spec}(k)$ is proper (in fact this is projective.)*

Proposition 7.29. *Every finite morphisms of scheme is proper.*

Example 7.26. *A morphism of scheme begin*

1. *quasi-finite,*

2. *separated,*
3. *finite-type,*
4. *surjective*

does not imply that it is proper.

Proposition 7.30 (Nagata compactification)**.** *If a morphism $f : X \to Y$ of schemes is separated and finite type between Noetherian schemes, then there exists a scheme Z and morphisms $h : X \to Z, g : Z \to Y$ of schemes such that*

$$f = g \circ h$$

with h is an open immersion, g is proper.

Proposition 7.31. *Let $f : X \to Y$ be a morphism of schemes. Then we have*

$$f \text{ is finite.} \Leftrightarrow f \text{ is quasi-finite and proper.}$$

7.8 Magic Squares

Definition 7.34. *Given morphisms of schemes*

$$\begin{tikzcd} X_1 \arrow[dr, "f_1"] & & \\ & Y \arrow[r, "g"] & Z \\ X_2 \arrow[ur, "f_2"'] & & \end{tikzcd}$$

We define a magic square of it to be

$$\begin{tikzcd} X_1 \underset{Y}{\times} X_2 \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_{Y/Z}"] \\ X_1 \underset{Z}{\times} X_2 \arrow[r] & Y \underset{Z}{\times} Y \end{tikzcd}$$

which is a fiber product.

Proposition 7.32. *Given morphisms of schems*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

We have

1. *f, g are both separated $\Rightarrow$ $g \circ f$ is separated.*
2. *f, g are both proper $\Rightarrow, g \circ f$ is proper.*

Corollary 7.9. *Given morphisms of schemes*

$$\begin{tikzcd} X_1 \arrow[dr, "f_1"] & & \\ & Y \arrow[r, "g"] & Z \\ X_2 \arrow[ur, "f_2"'] & & \end{tikzcd}$$

where g is separated. We have the graph

$$\Gamma_f : X \to X \underset{Z}{\times} Y$$

is a closed immersion.

Corollary 7.10. *Given morphisms of schemes*

$$\begin{tikzcd} X_1 \arrow[dr, "f_1"] & & \\ & Y \arrow[r, "g"] & Z \\ X_2 \arrow[ur, "f_2"'] & & \end{tikzcd}$$

where g is separated. Then we have, for the magic square

$$\begin{tikzcd} X_1 \underset{Y}{\times} X_2 \arrow[r] \arrow[d, "\psi"] & Y \arrow[d, "\Delta_{Y/Z}"] \\ X_1 \underset{Z}{\times} X_2 \arrow[r] & Y \underset{Z}{\times} Y \end{tikzcd}$$

Δ *is a closed immersion. Hence ψ is also a closed immersion.*