Prof. Dr. Daniel Huybrechts
Dr. Giacomo Mezzedimi

Winter term 2024/25

Exercises, Algebraic Geometry I – Week 1

Exercise 1. *Equalizers* (4 points)
Let $\mathcal{C}$ be a category, for example the category of sets $(Sets)$ or abelian groups (Ab), and let $\varphi_1, \varphi_2 \colon M \rightrightarrows N$ be two morphisms in $\mathcal{C}$. A morphism $\varphi\colon K \to M$ with $\varphi_1 \circ \varphi = \varphi_2 \circ \varphi$ is called an *equalizer* of (φ_1, φ_2) if for all $\psi\colon P \to M$ in $\mathcal{C}$ with $\varphi_1 \circ \psi = \varphi_2 \circ \psi$ there exists a unique morphism $\tilde{\psi}\colon P \to K$ with $\varphi \circ \tilde{\psi} = \psi$. If an equalizer exists it is unique up to unique isomorphism.

(i) Show that in $\mathcal{C} = (Sets)$ the inclusion $K := \{x \mid \varphi_1(x) = \varphi_2(x)\} \hookrightarrow M$ is an equalizer.

(ii) Show that in $\mathcal{C} = (Ab)$ the inclusion of the kernel $\operatorname{Ker}(\varphi_1 - \varphi_2) \hookrightarrow M$ is an equalizer.

Exercise 2. *Sheaf of continuous functions* (4 points)
Let X be a topological space. For any open subset $U \subset X$ set $\mathcal{C}(U) := \{f\colon U \to \mathbb{R} \mid \text{continuous}\}$ and for an inclusion of open subsets $V \subset U$ let $\rho_{UV}\colon \mathcal{C}(U) \to \mathcal{C}(V)$ be the restriction map $\rho_{UV}(f) := f|_V$. For the empty set define $\mathcal{C}(\emptyset) := \{*\}$, where $\{*\}$ is the one-element set. The restriction map $\rho_{U\emptyset}\colon \mathcal{C}(U) \to \mathcal{C}(\emptyset)$ is the unique map to the one-element set.

(i) Show that $\rho_{VW} \circ \rho_{UV} = \rho_{UW}\colon \mathcal{C}(U) \to \mathcal{C}(W)$ for open subsets $W \subset V \subset U$.

(ii) Let $U = \bigcup V_i$ be an open cover, set $V_{ij} := V_i \cap V_j$, and $\varphi_1, \varphi_2\colon \prod \mathcal{C}(V_i) \to \prod \mathcal{C}(V_{ij})$ be the two maps $(f_i)_i \mapsto (f_i|_{V_{ij}})_{ij}$ and $(f_i)_i \mapsto (f_j|_{V_{ij}})_{ij}$. Show that $\mathcal{C}(U) \to \prod \mathcal{C}(V_i)$, $f \mapsto (f|_{V_i})_i$ is an equalizer for (φ_1, φ_2).

Exercise 3. *Sheaf of maps to stalks* (4 points)
Consider a topological space X and fix an abelian group M_x for all $x \in X$. Define for any open set $U \subset X$ the abelian group $\mathcal{F}(U) := \prod_{x \in U} M_x$. Imitate the previous exercise, define natural restriction maps $\rho_{UV}\colon \mathcal{F}(U) \to \mathcal{F}(V)$, and prove (i) and (ii) as above.

Exercise 4. (4 points)
As a special case of Exercise 3 study $X = \operatorname{Spec}(\mathbb{Z})$ with $M_{(p)} := \mathbb{Z}_{(p)}$, for $p \in \mathbb{Z}$ prime, and $M_{(0)} := \mathbb{Q}$. Recall that the open sets are of the form $U = \operatorname{Spec}(\mathbb{Z}[1/n])$, with $n \in \mathbb{Z}$.

Modify the above construction and study $\mathcal{O}(U) := \mathbb{Z}[1/n]$ viewed as a subgroup of $\prod \mathbb{Z}_{(p)}$. Verify that the natural restriction maps satisfy (i) and (ii).

Exercise 5. *Sheaf $\mathcal{H}om$* (4 points)
For a (pre-)sheaf $\mathcal{F}$ on a topological space X and an open subset $U \subset X$ one defines the restriction $\mathcal{F}|_U$ to be the (pre-)sheaf on the topological space U given by $\mathcal{F}|_U(V) := \mathcal{F}(V)$ for any open subset $V \subset U$. Then for two sheaves $\mathcal{F}, \mathcal{G}$ of abelian groups, $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ denotes the abelian group of all sheaf homomorphisms $\mathcal{F}|_U \to \mathcal{G}|_U$. Show that this naturally defines a pre-sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})\colon U \mapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ which is in fact a sheaf.

Due Monday 14 October 2024.

Exercise 6. *Exponential map* (4 points)
Consider $X = \mathbb{C} \setminus \{0\}$ with its usual topology and let $\mathcal{O}_X$ be the sheaf of holomorphic functions, i.e. $\mathcal{O}_X(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}$. Similarly, let $\mathcal{O}_X^*$ be the sheaf of holomorphic functions without zeroes. (Throughout, you may work with differentiable function instead of holomorphic ones if you prefer.)

Show that the exponential map defines a morphism of sheaves (of abelian groups)

$$\exp : \mathcal{O}_X \to \mathcal{O}_X^*, f \in \mathcal{O}_X(U) \mapsto \exp(f) \in \mathcal{O}_X^*(U).$$

Find a basis of the topology such that $\exp_U : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$ is surjective for all U in this basis. Note that $\mathcal{O}_X(X) \to \mathcal{O}_X^*(X)$ is not surjective. Describe the kernel of $\exp_U$.

Registration for course on eCampus: Password: EGA

Registration for tutorials: 7 October 8pm - 21 October 8pm. Password: EGA. Please register only for one tutorial. Please do not forget to deregister should you later decide for another tutorial or to drop the whole class.

Solutions: Please hand in your solutions directly to your tutor (preferably electronically). In order to obtain the exercise credit at the end of the semester, you need to obtain at least half the total points of the exercise sheets.