Algebraic Geometry 1

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1 Topology

1.1 Connected Sets

Definition 1.1. *Let $(X, \mathscr{T})$ be a topological space. A subset A of X is said to be connected if for any $U, V \in \mathscr{T}$, $U \cap V =, U \cup V \supset A$ then A is fully contained in one of U, V.*

Definition 1.2. *A connected component of a topological space is a maximal connected subset of a space.*

Proposition 1.1. *Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological space and $f : X \to Y$ be a continuous function. Then for any connected subset A of X, $f(A)$ is connected in Y.*

Proof.

$$U, V \in \mathscr{T}_Y, U \cup V \supset f(A), U \cap V = \emptyset,$$
$$\Rightarrow f^{-1}(U), f^{-1}(V) \in \mathscr{T}_X,$$
$$f^{-1}(U) \cup f^{-1}(V) \supset A,$$
$$f^{-1}(U) \cap f^{-1}(V) = \emptyset,$$
$$\Rightarrow f^{-1}(U) \supset A \vee f^{-1}(V) \supset A,$$
$$\Rightarrow U \supset f(A) \vee V \supset f(A).$$

$\square$

Definition 1.3. *Let X be a topological space. A point $\eta \in X$ is called a generic point if*
$$\overline{\{\eta\}} = X.$$

2 Category Theory

2.1 Categories

Definition 2.1. *A category $\mathscr{A}$ consists of*

- *a collection* $\operatorname{ob}(\mathscr{A})$ *of objects;*

- *for each* $A, B \in \operatorname{ob}(\mathscr{A})$, *a collection* $\mathscr{A}(A, B)$ *of morphisms from* A *to* B;

such that

    i). *for each* $A \in \operatorname{ob}(\mathscr{A})$, *the identity* $1_A \in \mathscr{A}(A, A)$;

    ii). *the composition* $\mathscr{A}(B, C) \times \mathscr{A}(A, B) \ni (g, f) \mapsto g \circ f \in \mathscr{A}(A, C)$ *is well-defined;*

and they satisfy the following axioms

    I). *Associativity* : $f \in \mathscr{A}(A, B), g \in \mathscr{A}(B, C), h \in \mathscr{A}(C, D)$, $(h \circ g) \circ f = h \circ (g \circ f)$.

    II). *Identity laws* : $f \in \mathscr{A}(A, B)$ *then* $f \circ 1_A = 1_B \circ f$.

Definition 2.2. *Let* $\mathscr{A}$ *be a category. A terminal object* $T \in \operatorname{ob}(\mathscr{A})$ *is an object such that for any* $A \in \operatorname{ob}(\mathscr{A})$, $\mathscr{A}(A, T)$ *is a single element set.*

Definition 2.3. *Given two categories* $\mathscr{A}, \mathscr{B}$, *we say* $\mathscr{A}$ *is a full-subcategory of* $\mathscr{B}$ *if*

    i). $\mathscr{A} \subset \mathscr{B}$,

    ii). $\operatorname{ob}(\mathscr{A}) = \operatorname{ob}(\mathscr{B})$.

Notation 2.1. *Here we give notations to some important categories.*

- (**Sets**) : *A category of sets equipped with set theoretic functions.*

- (**Ab**) : *A category of abelian groups with group homomorphisms.*

Example 2.1. *Given a partially ordered set* $(X, \leq)$. *This can be encoded to a category* $\mathscr{O}$ *by*

    i). $\operatorname{ob}(\mathscr{O}) = X$,

    ii). *For* $x, y \in X$, $x \leq y \Rightarrow \mathscr{O}(x, y) = \{*\}$ *otherwise the morphisms between* x, y *is an emptyset.*

Definition 2.4. *A opposite/dual category of a category* $\mathscr{A}$ *is* $\mathscr{A}^{op}$ *such that*

    i). $\operatorname{ob}(\mathscr{A}^{op}) = \operatorname{ob}(\mathscr{A})$,

    ii). $\mathscr{A}^{op}(B, A) = \mathscr{A}(A, B)$.

Definition 2.5. *Let* $\mathscr{A}$ *be a category and* $\varphi_1, \varphi_2 \in \mathscr{A}(M, N)$. *A morphism* $\varphi : K \to M$ *is called an equalizer of* (φ_1, φ_2) *if for any morphism* $\psi : P \to M$ *such that* $\varphi_1 \circ \psi = \varphi_2 \circ \psi$, *there is a unique morphism* $\tilde{\psi} : P \to K$ *such that* $\varphi \circ \tilde{\psi} = \psi$.

Proposition 2.1. *If an equalizer exists then it is unique up to unique isomorphism.*

Proof. Suppose $\varphi : K \to M, \psi : L \to M$ be equalizers of (φ_1, φ_2). Then we have

$$\varphi \circ \tilde{\psi} = \psi, \quad \psi \circ \tilde{\varphi} = \varphi$$

By the uniqueness, we have $\tilde{\varphi} \circ \tilde{\psi} = 1_L, \tilde{\psi} \circ \tilde{\varphi} = 1_K$. $\qquad\square$

Definition 2.6. *Let $\mathscr{A}, \mathscr{B}$ be categories. A functor $F : \mathscr{A} \to \mathscr{B}$ is a function such that for each $f \in \mathscr{A}(A, A')$, $F(f) : F(A) \to F(A')$. In other words, $f \mapsto F(f) : \mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$. Furthermore, F satisfies the following axioms.*

   I). $F(f' \circ f) = F(f') \circ F(f)$ whenever $f : A \to A', f' : A' \to A''$ in $\mathscr{A}$,

   II). $F(1_A) = 1_{F(A)}$ whenever $A \in \mathscr{A}$.

Definition 2.7. *Let F, G be functors between two categories $\mathscr{A}, \mathscr{B}$. A natural transformation $\alpha : F \to G$ is a family $(\alpha_A : F(A) \to G(A))_{A \in \mathscr{A}}$ such that*

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(A') \\
{\scriptstyle\alpha_A}\downarrow & & \downarrow{\scriptstyle\alpha_{A'}} \\
G(A) & \xrightarrow{G(f)} & G(A')
\end{array}
$$

is a commutative diagram. Each α_A is called a component of α.

2.2 Direct Limits

Definition 2.8. *A partially ordered set $(X, \leq)$ is directed if for any $x, y \in X$ there is $z \in X$ such that $x \leq c$ and $y \leq c$.*

Example 2.2. *Let $(X, \mathscr{T})$ be a topological space. A partially ordered set $(\mathscr{T}, \leq)$ such that*

$$V \subseteq U \Rightarrow U \leq V$$

is directed. Since for any $U \in \mathscr{T}$, $U \leq \emptyset$. As a category this is $\mathbf{Ouv}_X^{\mathbf{op}}$.

Example 2.3. *Let $(X, \mathscr{T})$ be a topological space. For $x \in X$, define $O_x = \{U \in \mathscr{T} \mid x \in U\}$. If we define an order as in the previous example, we get $(O_x, \leq)$ is directed. This follows from for any $U, V \in O_x$, $U, V \leq U \cap V$.*

Definition 2.9. *Let I be a directed partially ordered set and $\mathscr{A}$ be a category. A directed system of objects of $\mathscr{A}$ indexed by I is a collection of objects $(A_i)_{i \in I}$ and morphisms $(\rho_{ij})_{i \leq j}$ of $\mathscr{A}$ such that*

   i). $\rho_{ii} = \mathbf{id}_{A_i}$,

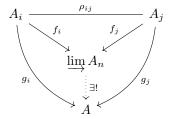ii). for $i, j, k \in I$, $i \leq j \leq k \Rightarrow \rho_{ik} = \rho_{jk} \circ \rho_{ij}$.

Remark 2.1. *Categorically, the directed system of objects of $\mathscr{A}$ indexed by I is a functor $\mathscr{O}^{op} \to \mathscr{C}$, where $\mathscr{O}$ is a category which encodes the ordered set I as a category by the same procedure as in Example* **??**. *Then a directed system if a functor $\mathscr{O}^{\mathbf{op}} \to \mathscr{A}$.*

Definition 2.10. *Given a directed system $((A_i)_{i \in I}, \{\rho_{ij}\}_{i \leq j})$ of objects in $\mathscr{A}$ indexed by I. A direct limit of the system is an object $\varinjlim A_n \in \mathbf{ob}(\mathscr{A})$ satisfying the following universal property.*

Given a collection of morphisms $(f_i)_{i \in I}$ such that

i). *$f_i : A_i \to \varinjlim A_n \in \mathscr{A}$,*

ii). *for any $i \leq j$, $f_j \circ \rho_{ij} = f_i$.*

For any $A \in \mathscr{A}$ where there is a collection of morphisms $(g_i)_{i \in I}$ satisfying the above condition, there is a unique map $\varphi : \varinjlim A_n \to A$ such that

$$
\begin{array}{ccc}
A_i & \xrightarrow{\ \rho_{ij}\ } & A_j \\
\end{array}
$$

is a commutative diagram.

Proposition 2.2. $\varinjlim$ *is an exact functor.*

Proposition 2.3. *In the cases where $\mathscr{A} = (\mathbf{Ab}), (\mathbf{Sets})$, there exist direct limits and for each category, such limit is constructed in the following ways.*

i). *$\varinjlim A_n = (\bigoplus_{i \in I} A_i)/N$ where $N = \{a_i - \rho_{ij}(a_i) \mid a_i, i \leq j\}$.*

ii). *$\varinjlim A_n = (\coprod_{i \in I} A_i)/ \sim$ where $a_i \sim a_j$ if there is k such that $i \leq k$ $j \leq k$, and $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.*

Furthermore, these two direct limits match as sets.

Proposition 2.4. $\varinjlim$ *is (left) exact in $(\mathbf{Ab})$. In other words, given a exact sequence of directed systems*

$$
0 \longrightarrow (M_i)_{i \in I} \longrightarrow (N_i)_{i \in I} \longrightarrow (P_i)_{i \in I} \longrightarrow 0
$$

in which we have

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_i & \longrightarrow & N_i & \longrightarrow & P_i & \longrightarrow & 0 \\
& & \rho_{ij}^M \downarrow & & \rho_{ij}^N \downarrow & & \rho_{ij}^P \downarrow & & \\
0 & \longrightarrow & M_j & \longrightarrow & N_j & \longrightarrow & P_j & \longrightarrow & 0
\end{array}
$$

There exists a short exact sequence

$$0 \longrightarrow \varinjlim M_n \longrightarrow \varinjlim N_n \longrightarrow \varinjlim P_n \longrightarrow 0$$

3 Commutative Algebra

3.1 Local Rings

Definition 3.1. *The total ring of fraction of a ring A is a localization of A by the set of all non-zero divisors. It is denoted as $Q(A)$.*

Definition 3.2. *A ring is said to be local if it has a unique maximal ideal.*

Definition 3.3. *A ring homomorphism $\phi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ of two local rings is said to be local if*
$$\mathfrak{m}_A = \phi^{-1}(\mathfrak{m}_B).$$

Example 3.1. *Let $i : \mathbb{Z}_{(p)} \to Q(\mathbb{Z}_{(p)})$ be an inclusion map. Then it is a homomorphism of local rings. However, If p is prime then $Q(\mathbb{Z}_{(p)})$ is a field thus its maximal ideal is (0). Obviously*

$$i^{-1}((0) = (0).$$

Therefore, i is not a local ring homomorphism.

Proposition 3.1. *Let $\phi : A \to B$ be a ring homomorphism. Recall that for any prime ideal $\mathfrak{q} \subseteq B$, we have $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ is a prime ideal in A. Thus ϕ induces a homomorphism between $A_\mathfrak{p}$ and $B_\mathfrak{q}$ which is a local ring homomorphism.*

Proof. If $a \in A$, $\phi(a) = 0$ then $a \in \mathfrak{p}$. Thus $\phi(s) \neq 0$ for any $s \notin \mathfrak{p}$. Since $\mathfrak{p}, \mathfrak{q}$ are unique maximal ideals of $A_\mathfrak{p}$ and $B_\mathfrak{q}$, respectively. We derived the claim. $\square$

Lemma 3.1. *Let k be an algebraically closed field and A be a k-algebra. A localization $A_\mathfrak{m}$ by a maximal(prime) ideal$\mathfrak{m} \subset A$, we have the following isomorphism.*

$$k \cong A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}.$$

Proof. Follows from the algebraically closedness of k. $\square$

3.2 Maximal Spec

Definition 3.4. *Let R be a commutative ring. We define the maximal spec of R as*
$$\mathrm{MaxSpec}(R) = \{\mathfrak{m}\, \mathrm{Spec}(R) \mid \mathfrak{m} \text{ is a maximal ideal.}\}.$$

Lemma 3.2. *Let k be an algebraically closed field. We have the following isomorphism*

$$\mathrm{MaxSpec}\, k[x_1, \cdots, x_n] \cong k^n, \quad (x_1 - a_1, \cdots, x_n - a - n) \leftrightarrow (a_1, \cdots, a_n).$$

Proof. Surjectivity follows from the algebraically closedness of k. $\square$

3.3 Zariski Topology

Definition 3.5. *Let k be a algebraically closed field. A subset X of k^n is called an affine algebraic set if there exists an ideal $\mathfrak{a} \subset k[x_1, \cdots, x_n]$ such that*

$$X = V(\mathfrak{a}) = \{(a_1, \cdots, a_n) \mid \forall f \in \mathfrak{a}, f(a_1, \cdots, a_n) = 0\}.$$

Definition 3.6. *Let k be an algebraically closed. The Zariski topologi on k^n is a topology generated by affine algebraic sets as closed subsets.*

Definition 3.7. *Let X be the Zariski topology on k^n. A function $f : X \supseteq U \to k$ is said to be regular if for any $a = (a_1, \cdots, c_n) \in U$, there exist a neighborhood $U_a \subseteq U$ and $f_1, f_2 \in k[x_1, \cdots, x_n]$ such that*

$$(b_1, \cdots, b_n) \in V_a \Rightarrow f(b_1, \cdots, b_n) = \frac{f_1(b_1, \cdots, b_n)}{f_2(b_1, \cdots, b_n)}.$$

Remark 3.1. *A regular function f on the Zariski topology on k^n is continuous as they are locally equivalent to quotients of polynomial functions.*

Lemma 3.3. *Let $\mathfrak{a} \subset A$ be an ideal. Then we have the following are homeomorphic.*
$$\mathrm{Spec}(A/\mathfrak{a}) \cong V(\mathfrak{a}).$$

4 Classical Algebraic Geometry

4.1 Affine Variety

Definition 4.1. *An affine algebraic set X is called an affine variety if there exists a prime ideal $\mathfrak{p} \subseteq k[x_1, \cdots, x_n]$ such that*

$$X = V(\mathfrak{p}).$$

Definition 4.2. *Let k be an algebraically closed field and $X \subseteq k^n$. The ideal of X is*

$$I(X) = \{f \in k[x_1, \cdots, x_n] \mid \forall (a_1, \cdots, a_n) \in X, f(a_1, \cdots, a_n) = 0\}.$$

Theorem 4.1. *For any ideal $\mathfrak{a} \subset k[x_1, \cdots, x_n]$, we have*

$$I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

Definition 4.3. *Let $X \subset k^n$ where k is an algebraically closed field. The affine coordinate ring with respect to X is*

$$A(X) = k[x_1, \cdots, x_n]/I(X).$$

5 Sheaf Theory

5.1 Presheaves

Definition 5.1. *Let $(X, \mathscr{T})$ be a topological space. We define the presheaf $\mathscr{F}$ of a category $\mathscr{A}$ on X such that*

- $U \in \mathscr{T}$, $\mathscr{F}(U) \in \mathrm{ob}(\mathscr{A})$,

- $U, V \in \mathscr{T}$, $V \subset U \Rightarrow$ there exists a map $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$

  such that

  i). *For any $U \in \mathscr{T}$, $\rho_{UU} = 1_{\mathscr{F}(U)}$.*

  ii). *$U, V, W \in \mathscr{T}, W \subset V \subset U \to \rho_{UW} = \rho_{VW} \circ \rho_{UW}$.*

Remark 5.1. *In the case $\mathscr{A} = (\mathbf{Sets}), (\mathbf{Ab})$, $\mathscr{F}(\emptyset) = \emptyset, \{1\}$, respectively.*

Definition 5.2. *An element of $\mathscr{F}(U)$ is called a local section of $\mathscr{F}$ and $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$ is called the space of sections over U. In particular $\Gamma(X, \mathscr{F})$ is called the space of global sections of $\mathscr{F}$.*

Definition 5.3. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{F}$ be a presheaf of a category $\mathscr{A}$ on X. Suppose we have two open sets $U, V \in \mathscr{T}$ such that $V \subset U$. Then for any section $s \in \mathscr{F}(U)$, $s|_V = \rho_{UV}(s)$ is called the restriction of s to V.*

Example 5.1. *Let $(X, \mathscr{T})$ be a topological space. We have a presheaf of continuous functions $\mathscr{C}_X(U) = \mathscr{C}^0(U, \mathbb{R})$. This is indeed a presheaf with restriction maps $\rho_{UV} : \mathscr{C}_X(U) \to \mathscr{C}_X(V)$. (Explicitly, $\rho_{UV}(f) = f \circ i_V$ where i_V is an inclusion map.) We note that we can introduce operations $+, \cdot$ to endow some algebraic structures (groups, rings, ...) on $\mathbb{R}$.*

Example 5.2. *Let $(X, \mathscr{T})$ be a topological space and suppose we have presheaves*

- $\mathscr{C}_X^{\mathit{diff}}(U) = \{f : U \to \mathbb{R} \mid f \text{ is differentiable.}\}.$

Then there is an inclusion relation $\mathscr{C}_X^{\mathit{diff}}(U) \subseteq \mathscr{C}_X(U)$ and this defines a presheaf.

Example 5.3. *Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological spaces. Define a presheaf on X by*

$$U \in \mathscr{T}_X, \mathscr{F}(U) = \mathscr{C}^0(X, Y).$$

And like the previous example, we define $\rho_{UV}(f) = f|_V$ for $U, V \in \mathscr{T}_X, V \subset U$. the restriction of f to V.
But this is a presheaf only of a set.

Example 5.4. *Let $(X, \mathscr{T})$ be a topological space and G be an abelian group. The constant presheaf $\mathbb{G}$ is such that*

$$U \in \mathscr{T}, \mathbb{G}(U) = G,$$

with $\rho_U V = id_G$ for any $U, V \in \mathscr{T}, V \subset U$.

5.2 Presheaves as Categories

Definition 5.4. *Let $(X, \mathscr{T})$ be a topological space then $(\mathbf{Ouv}_X)$ is the category such that its objects are the open sets of X and for any $U, V \in \mathscr{T}$ we have*

$$\mathbf{Ouv}_X(U, V) = \begin{cases} \emptyset & (V \not\subset U), \\ i_V & (V \subset U). \end{cases}$$

Definition 5.5. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{A}$ be a category. A presheaf of $\mathscr{A}$ on X is a functor $F : \mathbf{Ouv}_X \to \mathscr{A}$.*

Example 5.5. *For $\mathbf{Ouv}_X$, we can define a presheaf of F to be*

$$\mathrm{ob}(\mathbf{Ouv}_X) \ni U \mapsto F(U) = \mathscr{C}^0(U, \mathbb{R}).$$

Example 5.6. *Let A be a commutative ring with non-zero multiplicative identity and $X = \mathrm{Spec}(A)$. Let us consider the Zariski topology $(X, \mathscr{T})$. Let us consider a category $\mathscr{O}_X$ such that*

- $\mathrm{ob}(\mathscr{O}_X) = \mathscr{T}$,

- $\mathscr{O}_X(U) = \{s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}\}$,

where $s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ is a function such that for any $\mathfrak{p} \in U$,

   i). $s(p) \in A_{\mathfrak{p}}$,

   ii). there exists an open set $V \subset U$ such that $\mathfrak{p} \in V$ and for any $\mathfrak{q} \in V$, $s(\mathfrak{q}) = \frac{a}{b}$ for $b \notin \mathfrak{q}$.

Now we define a presheaf by the restrictions of maps such that

$$s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mapsto s|_V : V \to \coprod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

Definition 5.6. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{A}$ be a category. We define a set of presheaves of $\mathscr{A}$ on X as*

$$\mathrm{PreSh}_{\mathscr{A}}(X) = \mathrm{Fun}(\mathbf{Ouv}_X^{\mathbf{op}}, \mathscr{A}).$$

Definition 5.7. *A morphism of presheaves is a natural transformation $\varphi : \mathscr{F} \to \mathscr{G}$ where $\mathscr{F}, \mathscr{G} \in \mathrm{PreSh}_{\mathscr{A}}(X) = \mathrm{Fun}(\mathbf{Ouv}_X^{\mathbf{op}}, \mathscr{A})$.(See Definition **??**).*

   Such $\varphi : \mathscr{F} \to \mathscr{G}$ is

   i). injective if

Remark 5.2. $\mathrm{PreSh}(X)$ *can be regarded as a category with its objects presheaves and morphisms defined above.*

Notation 5.1. *In the case $\mathscr{A} = (\mathbf{Ab})$ then we denote $\mathrm{PreSh}(X) = \mathrm{PreSh}_{\mathbf{Ab}}(X)$.*

Example 5.7. *Let X be a differential manifold(eg. $X \subset \mathbb{R}^n$). Let us define*

$$\mathscr{C}^{\mathbf{diff}}(U) = \{f : U \to \mathbb{R} \mid f \text{ is differentiable.}\}.$$

Then the inclusions $\mathscr{C}^{\mathbf{diff}}_X(U) \subset \mathscr{C}_X(U)$ defines the natural transformation.

Example 5.8. *Let $X, Y = S^1$ be topological spaces and F be a presheaf such that for any open set $U \subset X$, $F(U) = \mathscr{C}^0(U, Y)$. Then we can introduce a natural transformation such that*

$$\mathscr{C}_X(U) \ni f \mapsto \exp(2\pi f i).$$

5.3   Sheaves

Definition 5.8. *A presheaf $\mathscr{F}$ on $(X, \mathscr{T})$ is called a sheaf if the following holds. For any collection of open sets $(U_i)_{i \in I} \subset \mathscr{T}, U = \bigcup_{i \in I} U_i$, the map $\varphi : \mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i)$ which is defined as*

$$\varphi(s) = (s|_{U_i})_{i \in I}.$$

is the equalizer of the following functions $\varphi_1, \varphi_2 : \prod_{i \in I} \mathscr{F}(U_i) \to \prod_{i,j \in I} \mathscr{F}(U_i \cap U_j)$,

$$\varphi_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})_{i,j \in I}, \quad \varphi_1((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})_{i,j \in I}.$$

Remark 5.3. *In the case $I = \{1, 2\}$, we have $U = U_1 \cup U_2$, and for any $U' \in \mathscr{T}$ such that $U \subset U'$, we have for $\mathscr{F}(U') \ni s : U' \to \mathbb{R}$,$\psi(s) = (s|_{U_1}, s|_{U_2})$, as in $\mathbf{Ouv}_X$, morphisms are inclusions. Let $\tilde{\psi}(s) = s|_U$, then this satisfies the condition for the equalizer (ie. $\varphi \circ \tilde{\psi} = \psi$).*

Remark 5.4. *A presheaf $\mathscr{O}_X$ with $X = \mathrm{Spec}(A)$ is a sheaf.*

Example 5.9. *Let $(X, \mathscr{T})$ be a topological space and G be a group. We define a constant presheaf $\mathbb{G}(U) = G$. In general, this is not a sheaf. Instead, we define a constant sheaf $\underline{\mathbb{G}}(U) = \mathscr{C}^0(U, G)$ where G is regarded as a topological space with the discrete topology. Then for any connected component of X is mapped to a single point set in G.*

Definition 5.9. *Let $\mathscr{F}_1, \mathscr{F}_2$ be sheaves. A mapping $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ is called a morphism of sheaves if it is a morphism of presheaves.*

Definition 5.10. *A set of sheaves of $\mathscr{A}$ on the topological space $(X, \mathscr{T})$ is denoted as $\mathrm{Sh}_{\mathscr{A}}(X)$.*

Remark 5.5. *As in the case of presheaves, $\mathrm{Sh}_{\mathscr{A}}(X)$ can be regarded as a category with sheaf morphisms.*

Remark 5.6. *$\mathrm{Sh}_{\mathscr{A}}(X)$ is a full-subcategory of $\mathrm{PreSh}_{\mathscr{A}}(X)$.*

Notation 5.2. *In the case $\mathscr{A} = (\mathbf{Ab})$, we denote $\mathrm{Sh}_{(\mathbf{Ab})}(X) = \mathrm{Sh}(X)$.*

5.4   Stalks

Definition 5.11. *Suppose we have a topological space $(X, \mathscr{T})$ and a category $\mathscr{A}$ which admits direct limits. For a presheaf $\mathscr{F} \in \mathrm{PreSh}_{\mathscr{A}}(X)$, by inheriting the notations from Example* **??**, *we define the stalk $\mathscr{F}_x$ of $\mathscr{F}$ at $x \in X$ by*

$$\mathscr{F}_x = \varinjlim_{U \in \mathscr{O}_x} \mathscr{F}(U) = \varinjlim_{x \in U, U \in \mathscr{T}} \mathscr{F}(U).$$

Example 5.10. *Let us assume that $\mathscr{A} = (\mathbf{Ab})$ in Definition* **??**. *Then stalks and germs can be constructed explicitly in the following way.*

$$\mathscr{F}_x = \{(s, U) \mid U \in \mathscr{O}_x, s \in \mathscr{F}(U)\}/\sim,$$

where $\sim$ is an equivalent relation such that for $(s, U), (t, V)$,

$$(s, U) \sim (t, V) \text{ if there is } W \in \mathscr{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t).$$

Definition 5.12. *Inheriting the notations from Definition* **??**, *suppose we have $(f_U : \mathscr{F}(U) \to \mathscr{F}_x)_{U \in \mathscr{O}_x}$ such that for f_U, f_V are compatible with ρ_{UV}. Then we define the germ of $s \in \mathscr{F}(U)$ to be $s_x = f_U(s)$. By the universal property of the direct limit, such s_x is unique up to images under isomorphisms.*

Example 5.11. *In the case of Remark* **??**, *we have for each $U \in \mathscr{T}$, $x \in U$, and $s \in \mathscr{F}(U)$,*

$$s_x = \{(t, V) \mid \text{ There is } W \in \mathscr{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t)\}.$$

Remark 5.7. *In the above definition, if a category $\mathscr{A}$ admits products, we get a map*

$$(s \mapsto (s_x)_{x \in U})) : \mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x. \tag{5.1}$$

This is neither surjective nor injective in general.

Proposition 5.1. *Suppose in the definition of stalks, $\mathscr{F}$ is a sheaf. Then the map defined by Equation* **??** *is injective.*

Proof. We prove the case when $\mathscr{A} = (\mathbf{Ab})$.

Suppose $s \in \mathscr{F}(U)$ is such that $s_x = 0$ in $\mathscr{F}_x$ for all $x \in U$. Since for any restriction maps are group homomorphisms. We have that there is $V_x \in \mathscr{O}_x$ such that

$$V_x \subseteq U, \quad \rho_{UV_x}(s) = 0.$$

Therefore $\{V_x\}_{x \in U}$ is an open covering of U. Since $\mathscr{F}$ is a sheaf, we derive that $s = 0$ in $\mathscr{F}(U)$. $\qquad\square$

Example 5.12. *Given $(X, \mathscr{F})$, a topological space and G, an abelian group. We will consider the constant presheaf $\mathbb{G}$ and the constant sheaf $\underline{\mathbb{G}}$ on X. For any open set U and $x \in U$ we have*

$$\mathbb{G}_x \cong \underline{\mathbb{G}}_x \cong G.$$

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For any U, V open such that $V \subset U$ we have, $\rho_{UV} = \mathbf{id}_G$. Thus by the construction, for $x \in U, V$, $(s, U) \sim (t, V)$ then $x \in U \cap V$ and $\rho_{UU \cap V}(s) = s = t = \rho_{VU \cap V}(t)$. Therefore, we proved the claim.

Definition 5.13. *Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. Then we define*

$$\varphi_x(s_x) = (\varphi(s)_U)_x.$$

This defines a morphism of presheaves.

Remark 5.8. *Categorically, taking stalks is a functor for each $x \in X$. Suppose we have $\mathscr{F}, \mathscr{G} \in \mathrm{PreSh}_{\mathscr{A}}(X)$ and a morphism $\varphi : \mathscr{F} \to \mathscr{G}$,*

Proposition 5.2. *Let $\mathscr{F}, \mathscr{G} \in \mathrm{Sh}_{(\mathbf{Ab})}(X)$ Then for any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ we have*

$$\varphi = 0 \Leftrightarrow \forall x \in X, \varphi_x = 0$$

Proof. $\Rightarrow$ is trivial by its construction. We will prove $\Leftarrow$.

We first note that $\varphi = 0$ means that for any $U \in \mathscr{T}$, we have $\varphi_U \equiv 0$ as a group homomorphism. Let $U \in \mathscr{T}$ and $s \in \mathscr{F}(U)$. Then by the assumption and Proposition **??**, we have proven the claim. $\square$

5.5 Sheafification

Definition 5.14. *Let $\mathscr{F} \in \mathrm{PreSh}_{\mathscr{A}}(X)$. The sheafification of $\mathscr{F}$ is a presheaf $\mathscr{F}^+$ which is a set of all $(s_x)_{x \in U} \in \prod_{x \in U} \mathscr{F}_x$ such that for any $x \in U$ there is $x \in V_x \subset U$, such that there is $t \in \mathscr{F}(V_x)$ satisfying for any $y \in V_x$, $s_y = t_y$. We give them restrictions such that*

$$\mathscr{F}^+(U) \ni (s_x)_{x \in U} \mapsto (s_x)_{x \in V} \in \mathscr{F}^+(V).$$

Proposition 5.3. *Such $\mathscr{F}^+$ is indeed a sheaf.*

Proof. later $\square$

Remark 5.9.
$$\mathscr{F} \mapsto \mathscr{F}^+ : \mathrm{PreSh}_{\mathscr{A}}(X) \to \mathrm{Sh}_{\mathscr{A}}(X)$$

is a functor. Indeed given $\varphi : \mathscr{F} \to \mathscr{G}$, a morphism of presheaves. We give

$$\varphi^+(U)((s_x)_{x \in U}) = (\varphi(s)_x)_{x \in U}.$$

later

Proposition 5.4. *A mapping $\varphi : \mathscr{F} \to \mathscr{F}^+$ such that for each $U \in \mathscr{T}$,*

$$\varphi_U : \mathscr{F}(U) \to \mathscr{F}^+(U), \quad \varphi(s) = (s_x)_{x \in U},$$

is a natural transformation thus a morphism of presheaves.

Proof. Later ◻

Proposition 5.5. *For any open set $U \in \mathscr{T}$ and a section $s \in \mathscr{F}^+(U)$, there is an open covering $(U_i)_{i \in I}$ which satisfies that there is a sequence $(s_i)_{i \in I} \in \prod_{i \in I} \mathscr{F}(U_i)$ and for each i, the following hodls.*

$$\rho_{UU_i}(s) = s_i.$$

Proof. Later. ◻

Proposition 5.6. *For each $x \in X$, there exists an isomorphism*

$$\mathscr{F}_x \cong (\mathscr{F}^+)_x,$$

as presheaves.

Proof. later ◻

Proposition 5.7. *Let $(X, \mathscr{T})$ be a topological group and $\mathscr{F}$ be a presheaf of a category $\mathscr{A}$ on X. Suppose for a sheaf $\mathscr{G}$ of a category $\mathscr{A}$ on X, there exists a morphism $\varphi : \mathscr{F} \to \mathscr{G}$. Then there exists a unique morphism $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}$, such that*

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{F}^+ \\ {\scriptstyle \varphi}\downarrow & \swarrow{\scriptstyle \exists!\varphi^+} & \\ \mathscr{G} & & \end{array}$$

is a commutative diagram.

Proof. Let $U \in \mathscr{T}$, then by Proposition **??**, for any $s \in \mathscr{F}^+$, there exists an open covering $(U_i)_{i \in I}$ and $(s_i)_{i \in I} \in \prod_{i \in I} \mathscr{F}(U_i)$ such that $\rho_{UU_i}(s) = s_i$ for any $i \in I$. We define

$$t_i = \varphi(s_i) \in \mathscr{G}(U_i),$$

for each $i \in I$. Using the definition of natural transformation we derive that

$$\rho^{\mathscr{G}}_{UU_i \cap U_j}(t_i) = \varphi^{\mathscr{F}}_{U_i \cap U_j}(\rho_{UU_i \cap U_j}(s)) = \rho^{\mathscr{G}}_{UU_i \cap U_j}(t_j).$$

Thus we can glue $(t_i)_{i \in I}$ to a section $t \in \mathscr{G}(U)$.

We now define $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}$. Given $(s_x)_{x \in U}$ which is the germ of s,

$$\varphi^+_U((s_x)_{x \in U}) = t.$$

Such φ^+ is unique since $\mathscr{G}$ is a sheaf. ◻

Corollary 5.1. *Let $i : \mathrm{Sh}_{\mathscr{A}}(X) \to \mathrm{PreSh}_{\mathscr{A}}(X)$ be a forgetful functor. Then we have*

$$\mathrm{PreSh}_{\mathscr{A}}(X)(\mathscr{F}, i(\mathscr{G})) \cong \mathrm{Sh}_{\mathscr{A}}(\mathscr{F}^+, \mathscr{G})$$

In other words, the sheafification is a left-adjoint functor of the inclusion map.

Proof. By Proposition **??**, we define two maps Φ, Ψ such that

$$\Phi : \mathrm{PreSh}_{\mathscr{A}}(X)(\mathscr{F}, i(\mathscr{G})) \to \mathrm{Sh}_{\mathscr{A}}(\mathscr{F}^+, \mathscr{G}),$$
$$\Phi(\varphi) = \varphi^+,$$
$$\Psi : \mathrm{Sh}_{\mathscr{A}}(\mathscr{F}^+, \mathscr{G}) \to \mathrm{PreSh}_{\mathscr{A}}(X)(\mathscr{F}, i(\mathscr{G})),$$
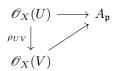$$\Psi(\varphi^+) = \varphi.$$

Then these two are inverses of each other. $\qquad\square$

Proposition 5.8. *Let $X = \mathrm{Spec}(A)$ and $\mathscr{O}_X$ be the structure sheaf defined in Example* **??***. Then we have the following.*

1). For any $\mathfrak{p} = x \in X$, $(\mathscr{O}_X)_x \cong A_{\mathfrak{p}}$.

2). For any $a \in A$, $\mathscr{O}_X(D(a)) \cong A_a$.

Proof. For a given $U \subset X$ open and $\mathfrak{p} \subset A$, there is $a, b \in A$ such that for $V \subset U$ open and $s \in \mathscr{O}_X(U), s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$.

$$s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}$$

holds for any $\mathfrak{q} \in V$.

$$
\begin{array}{ccc}
\mathscr{O}_X(U) & \longrightarrow & A_{\mathfrak{p}} \\
{\scriptstyle \rho_{UV}} \downarrow & \nearrow & \\
\mathscr{O}_X(V) & &
\end{array}
$$

$\qquad\square$

5.6   Morphisms in $\mathrm{PreSh}_{\mathbf{(Ab)}}(X)$

Definition 5.15. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism of presheaves $\mathrm{PreSh}_{\mathbf{(Ab)}}(X)$. Then we define the following.*

1). $\mathrm{Ker}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Ker}\,\varphi_U$,

2). $\mathrm{Im}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Im}\,\varphi_U$,

3). $\mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Coker}\,\varphi_U$.

Proposition 5.9. *Such $\mathrm{Ker}^{\mathbf{pre}}, \mathrm{Im}^{\mathbf{pre}}, \mathrm{Coker}^{\mathbf{pre}}$ are presheaves.*

Proof. For the case of kernels. Let $U, V \in \mathscr{T}$ and $V \subset U$. We define $\rho_U V : \mathrm{Ker}^{\mathbf{pre}}(\varphi)(U) \to \mathrm{Ker}^{\mathbf{pre}}(\varphi)(V)$ to be such that

$$\rho_U V(s) = \rho^{\mathscr{F}}(s).$$

Such construction is justified as the diagram below is commutative.

$$
\begin{array}{ccccc}
\mathscr{F}(U) & \xrightarrow{\rho^{\mathscr{F}}_{UV}} & \mathscr{F}(V) & \xrightarrow{\rho^{\mathscr{F}}_{UV}} & \mathscr{F}(W) \\
\varphi_U \downarrow & & \downarrow \varphi_V & & \downarrow \varphi_W \\
\mathscr{G}(U) & \xrightarrow[\rho^{\mathscr{G}}_{UV}]{} & \mathscr{G}(V) & \xrightarrow[\rho^{\mathscr{G}}_{UV}]{} & \mathscr{F}(W)
\end{array}
$$

Furthermore,

$$
\rho_U W(s) = \rho^{\mathscr{F}}_{UV}(s) = \rho^{\mathscr{F}}_{VW} \circ \rho^{\mathscr{F}}_{UV}(s) = \rho_{VW} \circ \rho_{UV}(s).
$$

Thus $\mathrm{Ker}^{\mathbf{pre}}(\varphi)(U)$ is a presheaf. $\qquad\square$

Corollary 5.2. *If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves. Then $\mathrm{Ker}^{\mathbf{pre}}$ is also a sheaf.*

Proof. Given $(s_i)_{i \in I} \in \prod_{i \in I} \mathrm{Ker}\, \varphi_{U_i}$ such that

$$
\rho(s_i)_{U_i U_i \cap U_j} = \rho(s_j)_{U_j U_i \cap U_j}
$$

for any $i, j \in I$. Then since $\mathscr{F}$ is a sheaf, we can glue $(s_i)_{i \in I}$ to $s \in \mathscr{F}(U)$. For such s we have

$$
\rho^{\mathscr{G}}_{UU_i}(\varphi_U(s)) = (\varphi_{U_i}(\rho^{\mathscr{F}}_{UU_i}(s))) = \varphi_{UU_i}(s_i) = 0.
$$

Therefore, since $\mathscr{G}$ is a sheaf, $\varphi_U(s) = 0$. $\qquad\square$

Remark 5.10. *Let $\varphi : \mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i), \varphi_1 : \prod_{i \in I} \mathscr{F}(U_i) \to \prod_{i,j \in I} \mathscr{F}(U_i \cap U_j), \varphi_2 : \prod_{i \in I} \mathscr{F}(U_j) \to \prod_{i,j \in I} \mathscr{F}(U_i \cap U_j)$. Then $\mathscr{F}$ is a sheaf if and only if*

$$
\mathrm{Ker}(\varphi_1 \circ \varphi - \varphi_2 \circ \varphi) = \mathscr{F}(U),
$$

holds for any open set U.

Remark 5.11. *$\mathrm{Im}^{\mathbf{pre}}\, \varphi, \mathrm{Coker}^{\mathbf{pre}}\, \varphi$ are not in general sheaves even tho $\varphi : \mathscr{F} \to \mathscr{G}$ is a homomorphism of sheaves.*

Example 5.13. *Let $X = \{x_1, x_2\}$ and we assign the discrete topology to it. Let G be an abelian group. We define a sheaf $\mathscr{F}, \mathscr{G} \in \mathrm{Sh}_{(\mathbf{Ab})}(X)$ by such that*

$$
\mathscr{F}(U) = \mathscr{G}(U) = \begin{cases} G \times G & (U = X), \\ G & (|U| = 1), \\ 0 & (U = \emptyset). \end{cases}
$$

Let us define a homomorphism of sheaves φ such that

$$
\varphi_U = \begin{cases} \mathbf{id}_{G \times G} & (U = X) \\ 0 & (U \neq X). \end{cases}
$$

Then we have

$$\text{Coker}^{\mathbf{pre}}(\varphi)(U) = \begin{cases} 0 & (U = X), \\ G & (U \neq X). \end{cases}$$

By **??**, *we observe that*

$$\text{Coker}^{\mathbf{pre}}(\varphi)(X) = G \times G / \mathbf{id}_{G \times G}(G \times G) = \{0\}.$$

However,

$$\text{later.}$$

Definition 5.16. *Given a morphism of sheaves* $\varphi : \mathscr{F} \to \mathscr{G}$, *we define the following.*

1). $\text{Ker}(\varphi) = \text{Ker}^{\mathbf{pre}}(\varphi)$,

2). $\text{Im}(\varphi) = (\text{Im}^{\mathbf{pre}}(\varphi))^+$,

3). $\text{Coker}(\varphi) = (\text{Coker}^{\mathbf{pre}}(\varphi))^+$.

Proposition 5.10 (Universal property of kernels)**.** *Given a sheaf homomorphism* $\varphi : \mathscr{F} \to \mathscr{G}$. *For any sheaf homomorphism* $\alpha : \mathscr{H} \to \mathscr{F}$, $\varphi \circ \alpha = 0$ *if and only if there is a unique* $\psi : \mathscr{H} \to \text{Ker}\,\varphi$ *such that*

$$
\begin{array}{ccc}
 & \mathscr{H} & \\
{\scriptstyle \exists!\psi} \swarrow \!\!\!\cdots & \downarrow {\scriptstyle \alpha} & \searrow {\scriptstyle \varphi_0 = 0} \\
\text{Ker}(\varphi) \longhookrightarrow & \mathscr{F} \xrightarrow{\ \varphi\ } & \mathscr{G}
\end{array}
$$

is a commutative diagram.

Proof. We argue by each open set of the space.

$$
\begin{array}{ccc}
 & \mathscr{H}(U) & \\
{\scriptstyle \exists!\psi_U} \swarrow \!\!\!\cdots & \downarrow {\scriptstyle \alpha_U} & \searrow {\scriptstyle (\varphi_0)_U = 0} \\
\text{Ker}(\varphi)(U) \longhookrightarrow & \mathscr{F}(U) \xrightarrow{\ \varphi_U\ } & \mathscr{G}(U)
\end{array}
$$

This is a universal property of the kernel in abelian groups. Thus the statement immediately follows from it. □

Proposition 5.11 (Universal property of Cokernels)**.** *Given a sheaf homomorphism* $\varphi : \mathscr{F} \to \mathscr{G}$. *For any sheaf homomorphism* $\alpha : \mathscr{G} \to \mathscr{H}$, $\alpha \circ \varphi = 0$ *if and only if there is a unique* $\psi : \text{Coker}\,\varphi \to \mathscr{H}$ *such that*

$$
\begin{array}{ccc}
\mathscr{F} \xrightarrow{\ \varphi\ } & \mathscr{G} \xrightarrow{\ \pi\ } & \text{Coker}(\varphi) \\
{\scriptstyle \varphi_0 = 0} \searrow & \downarrow {\scriptstyle \alpha} & \cdots \!\!\! \swarrow {\scriptstyle \exists!\psi} \\
 & \mathscr{H} &
\end{array}
$$

is a commutative diagram.

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Proof. We argue for each open set $U \subset X$.

$$\mathscr{F}(U) \xrightarrow{\varphi_U} \mathscr{G}(U) \longrightarrow \mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) \longrightarrow \mathrm{Coker}(\varphi)(U)$$

with $(\varphi_0)_U = 0$, α_U, $\exists! \psi_U^{\mathbf{pre}}$, $\mathscr{H}(U)$, $\exists! \psi_U$.

By the universal property of Cokernels of abelian groups, there is a unique $\varphi^{\mathbf{pre}}$. By the universal property of the sheafification operator, we derive a unique ψ. $\qquad\square$

Proposition 5.12. *Let $x \in X$, then we have the following.*

1). $\mathrm{Ker}(\varphi)_x = \mathrm{Ker}(\varphi_x),$

2). $\mathrm{Im}(\varphi)_x = \mathrm{Im}(\varphi_x),$

3). $\mathrm{Coker}(\varphi)_x = \mathrm{Coker}(\varphi_x).$

Proof. By Definition, **??** $\qquad\square$

Definition 5.17. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism. φ is called*

1). *a monomorphism if any morphism of sheaves $\varphi_0 : \mathscr{H} \to \mathscr{F}$, $\varphi \circ \varphi_0 = 0$ if and only if $\varphi_0 = 0$,*

Proposition 5.13. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of $(\mathbf{Ab})$. Then the following statements are equivalent.*

i). *φ is a monomorphism.*

ii). *$\mathrm{Ker}\, \varphi = 0$.*

iii). *For any open set $U \subset X$, φ_U is injective.*

iv). *For any $x \in X$, $\varphi_x :\to \mathscr{F}_x \to \mathscr{G}_x$ is injective.*

Proof. Here, I put the procedure of the proof.

$$
\begin{array}{ccc}
i) & & iv) \\
\downarrow & \nwarrow\searrow & \updownarrow \\
ii) & \longleftrightarrow & iii)
\end{array}
$$

$i) \Rightarrow ii)$,

$$
\begin{array}{c}
\mathrm{Ker}(\varphi) \\
\varphi_0 \downarrow \quad \searrow^{0} \\
\mathscr{F} \xrightarrow{\varphi} \mathscr{G}
\end{array}
$$

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Where $\varphi_0(U)$ is an inclusion map of abelian groups.

$ii) \Leftrightarrow iii)$,

$$\operatorname{Ker}\varphi = 0 \Leftrightarrow \forall U \in \mathscr{T}, \operatorname{Ker}\varphi(U) = 0 \Leftrightarrow \varphi_U \text{ is injective.}$$

$iii) \Rightarrow iv)$, Fix $x \in X$.

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\varphi_U} \mathscr{G}(U)$$

is an exact sequence as φ_U is injective for any $U \subset X$ open. Since $\varinjlim$ is left-exact we obtain,

$$0 \longrightarrow \mathscr{F}_x \xrightarrow{\varphi_x} \mathscr{G}_x$$
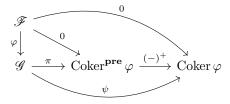
is also an exact sequence. $\qquad\qquad\square$

Proposition 5.14. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism in $\operatorname{Sh}(X)$. Then the following are equivalent.*

1). φ is an epimorphism (for any $\varphi_1, \varphi_2 : \mathscr{H} \to \mathscr{F}$, such that $\varphi \circ \varphi_1 = \varphi \circ \varphi_2$ implies $\varphi_1 = \varphi_2$).

2). $\operatorname{Coker}\varphi = 0$.

3). For any open set $U \subset X$,

4). For any $x \in X$, $\operatorname{Coker}\varphi_x = 0$, (in other words, φ_x is a surjection).

Proof. Recall the definition of epimorphisms is such that $\varphi : \mathscr{F} \to \mathscr{G}$ is an epimorphism if for any morphism $\psi : \mathscr{G} \to \mathscr{G}_0$, we have,

$$\psi \circ \varphi = 0 \Rightarrow \psi = 0.$$

$i) \Rightarrow iv)$. Suppose φ is an epimorphism, then we have

By the assumption $\psi = 0$.

Let $\mathscr{O}_x = \{U \in \mathscr{T} \mid x \in U\}$. We consider an exact sequence,

$$0 \longrightarrow \operatorname{Ker}(\varphi_U) \lhook\joinrel\longrightarrow \mathscr{F}(U) \xrightarrow{\varphi} \mathscr{G}(U) \xrightarrow{\pi} \operatorname{Coker}(\varphi_U) \longrightarrow 0,$$

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for each $U \in \mathcal{O}_x$. By Proposition **??**,

$$0 \longrightarrow \mathrm{Ker}(\varphi)_x \lhook\joinrel\longrightarrow \mathscr{F}_x \xrightarrow{\varphi_x} \mathscr{G}_x \xrightarrow{\pi_x} \mathrm{Coker}(\varphi)_x \longrightarrow 0$$

is also exact. Thus we conclude

$$\mathrm{Coker}^{pre}(\varphi)_x = \mathrm{Coker}(\varphi_x).$$

And we conclude that φ_x is surjective by the exactness of the sequence.

$iv) \Rightarrow ii)$. Assume For each $x \in X$, $\mathrm{Coker}(\varphi_x) = 0$. By applying Proposition. **??** to $\mathbf{id} : \mathscr{F} \to \mathscr{F}$, we obtain

$$\mathscr{F} = 0 \Leftrightarrow \forall x \in X, \mathscr{F}_x = 0.$$

Apply this to $\mathrm{Coker}\,\varphi$, we derive that

$$\mathrm{Coker}\,\varphi = 0.$$

$iv) \Rightarrow i)$. Assume $\mathrm{Coker}(\varphi_x) = 0$ for any $x \in X$. Consider a commutative diagram of sheaves

By assumption $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is a surjection. Thus $\psi_x = 0$ for any $x \in X$ which is equivalent to $\psi = 0$.

$ii) \Rightarrow i)$. Suppose $\mathrm{Coker}\,\varphi = 0$ if and only if $\mathrm{Coker}(\varphi)_x = \mathrm{Coker}(\varphi_x) = 0$ for any $x \in X$.

$iii) \Rightarrow iv)$. Assume $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is surjective for any $U \subset X$ open. By Proposition. **??**, we conclude that

$$\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$$

is also surjective. $\qquad\square$

Corollary 5.3. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then the following statements are equivalent.*

1). φ is an isomorphism.

2). For all $x \in X$, φ_x is an isomorphism.

Proof. $\qquad\square$

6 Scheme Theory

6.1 Ringed Spaces

Definition 6.1. *Let $(X, \mathscr{T})$ be a topological space. A ringed space is a sheaf $\mathscr{O}_X$ of rings on X.*

Definition 6.2. *A morphism of ringed spaces between $(X, \mathscr{O}_X), (Y, \mathscr{O}_Y)$ is a tuple $(f, f^{\#})$ where $f : X \to Y$ is a continuous map and $f : \mathscr{O}_Y \to f_* \mathscr{O}_X$ is a morphism of sheaves of rings.*

Example 6.1. *Let $(X, \mathscr{T})$ be a topological space. The sheaf of continuous functions $\mathscr{C}_X$ is a ringed space and any continuous map $f : X \to Y$ defines a morphism of ringed spaces.*

Example 6.2. *Let X is a differentiable manifold then the differentiable functions $\mathscr{C}_X^{\mathbf{diff}}$ is a ringed space. A morphism of ringed spaces $f : X \to Y$, for this case must satisfy the following condition.*

Example 6.3. *Let $X \subseteq \mathbb{C}^n$ be open subset. A sheaf of holomorphic functions $\mathscr{O}_X$ over X is a ringed space. And a morphism of such ringed spaces must be a holomorphic functions*

Example 6.4. *Given the Zariski topology on $X = k^n$ and the sheaf $\mathscr{O}_X(U) = \{f : U \to k \mid f \text{ is regular }\}$, $(X, \mathscr{O}_X)$ is a ringed space.*

Definition 6.3. *By Remark **??**, the sheaf of regular functions $\mathscr{O}_X$ is contained in the sheaf of continuous functions $\mathscr{C}_X$. Given two Zariski topologies X, Y, and a continuous function $f : X \to Y$, f is said to be regular if for any regular function $g : U \to k$ for an open set $U \subseteq Y$, $g \circ f : f^{-1}(U) \to k$ is also regular. In other words, f is said to be regular if it defines a morphism of ringed spaces between two ringed spaces of regular functions.*

Definition 6.4. *A locally ringed space is a ringed space $(X, \mathscr{O}_X)$ such that for any $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring.*

Example 6.5. *A sheaf of continuous functions on a topological space X is a locally ringed space. Indeed, for each $x \in X$ and the stalk $\mathscr{C}_{X,x}$, the ideal*

$$\mathfrak{m}_x = \{(f : U \to \mathbb{R}, U) \mid f(x) = 0\}$$

is a unique maximal ideal. In order to prove this, we recall that an ideal $\mathfrak{m}$ is a unique maximal ideal if any element not in $\mathfrak{m}$ is a unit.

For each $(f : U \to \mathbb{R}, U) \in \mathscr{C}_{X,x}$, $f(x) \neq 0$ implies that there exists a neighborhood $V \subset U$ such that $f(x) \neq 0$ for any $x \in V$. Thus $(f|_V : V \to \mathbb{R}, V)$ is invertible, therefore a unit.

Example 6.6. *In similar manner, the following are also locally ringed spaces.*

   1. X is a differentiable manifold and $(X, \mathscr{C}_X^{\mathbf{diff}})$.

2. $X \subseteq \mathbb{C}^n$ be an open set, and $(X, \mathcal{O}_X)$ be a sheaf of holomorphic functions.

3. A sheaf of regular functions on $X = k^n$.

Definition 6.5. *A morphism $(f, f^{\#}) : (X, \mathcal{O}_X \to (Y, \mathcal{O}_Y)$ between ringed spaces is a morphism of locally ringed space if $f^{\#}$ is local as a ring homomorphism.*

Example 6.7. *Let A be a commutative ring and consider the Zariski topology on $X = \mathrm{Spec}(A)$ and the structure sheaf $(X, \mathcal{O}_X)$. We have proven that*

$$\mathfrak{O}_{X, \mathfrak{p}} \cong A_{\mathfrak{p}}.$$

Therefore, $(X, \mathcal{O}_X)$ is a locally ringed space and for any ring homomorphism $\phi : A \to B$, it induces a morphism of locally ringed spaces $(f, f^{\#}) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ such that

$$\mathfrak{q} \in \mathrm{Spec}(B), f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q}) \in \mathrm{Spec}(A).$$

This is indeed a morphism of locally ringed spaces.

Proposition 6.1. *Let A, B be commutative rings. Then the map $\phi \mapsto (f, f^{\#})$ is a bijection between*

$$\mathrm{Hom}(A, B) \leftrightarrow \mathrm{Hom}_{\mathbf{loc}}(\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}), (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}))$$

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 6.6. *A category of ringed spaces is denoted by $(\mathbf{RingedSpaces})$ with morphisms $(f, f^{\#})$ morphisms of ringed spaces.*

Definition 6.7. *A category of ringed spaces is denoted by $(\mathbf{RingedSpaces})$ with morphisms $(f, f^{\#})$ morphisms of locally ringed spaces.*

Remark 6.1. *A composition of two morphisms locally ringed space is indeed a morphism of locally ringed spaces thus the above construction is justified.*

Definition 6.8. *Two locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are isomorphic if there exists morphisms $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ such that f and g are inverses of each other. (ie. there exists a morphism of locally ringed spaces $(f, f^{\#})$ where f is a homeomorphism).*

Example 6.8. *(A morphism of locally ringed spaces induced by homeomorphism but not an isomorphism of locally ringed spaces).*

Let $X = \mathbb{R}^n$ and consider the sheaf of continuous functionals $\mathcal{C}_X$ and the sheaf of smooth functionals $\mathcal{C}_X^{\mathbf{diff}}$. Furthermore, we consider $f = id_X$ then $f^{\#}$ is an inclusion as smooth functions are continuous. However, $(f, f^{\#})$ is not an isomorphism of locally ringed spaces.

Example 6.9. *Let us consider $X = \mathbb{C}^n$ and the sheaf of holomorphic functions $\mathscr{O}$ on X and the structure sheaf $\mathscr{O}_X$. Then consider the morphism of locally ringed spaces $(f, f^{\#})$ by the identity map. However, f is not continuous as the topology defined on the image is the Zariski topology.*

Definition 6.9. *Let $X = \mathbb{C}^n$ and $Y = \mathrm{MaxSpec}(\mathbb{C}[x_1, \cdots, x_n])$. Let $f : X \to Y$ be such that*
$$f(z_1, \cdots, z_n) = (x_1 - z_1, \cdots, x_n - z_n).$$

This is a bijection. Furthermore, f is continuous because polynomials are continuous functions.

We define $f^{\#}$ to be

6.2   Schemes

Definition 6.10. *An affine scheme is a locally ringed space $(X, \mathscr{O}_X)$ which is isomorphic to a structure sheaf $(\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)})$ for some commutative ring A.*

Example 6.10. *We consider the Zariski topology on $\mathrm{Spec}(\mathbb{Z})$ and a sheaf $\mathscr{O}$ such that*
$$\mathscr{O}(D(\mathfrak{a})) = \mathbb{Z}_{\mathfrak{a}}.$$

is an affine scheme.

Example 6.11. *Let k be a field. Then $\mathrm{Spec}(k)$ is a single point set. And we consider the sheaf $\mathscr{O}$ such that $\mathscr{O}(\mathrm{Spec}(k)) = k$.*

Definition 6.11. *For a field A be a commutative ring and n a natural number, we define*
$$\mathbb{A}_A^n = (\mathrm{Spec}(A[x_1, \cdots, x_n]), \mathscr{O}).$$

Example 6.12. *Let A be a discrete valuation ring in other words $k[t]_{(t)}$.*

Example 6.13. *Let k be a field and $A = k[x]/(x^2)$. Then $\mathrm{Spec}(A) = \{(x)\}$. Thus a single point set. However, this is not isomorphic to $(\mathrm{Spec}(k), \mathscr{O})$ introduced in Example **??**.*

Definition 6.12. *A scheme is a ringed space $(X, \mathscr{O}_X)$ which is locally isomorphic t an affine scheme. In other words, for any $x \in X$, there is a neighborhood U of X such that there exists a commutative ring A and $(U, \mathscr{O}|_U)$ is isomorphic to $(\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)})$.*

Definition 6.13. *A category of affine schemes is $(\mathbf{AffSch})$ where*

i). $\mathbf{ob}(\mathbf{AffSch}) = \{(\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)}) \,|\, A$ *is a commutative ring and $\mathscr{O}_{\mathrm{Spec}(A)}$ is a structure sheaf$\}$.*

ii). $(\mathbf{AffSch})((\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)}), (\mathrm{Spec}(B), \mathscr{O}_{\mathrm{Spec}(B)})) = \{$ *morphisms of locally ringed spaces$\}$.*

Definition 6.14. *A category of schemes is* (**Sch**) *where*

i). $\mathbf{ob}(\mathbf{Sch}) = \{(X, \mathscr{O}_X) \mid Schemes\}$.

ii). $(\mathbf{AffSch})((X, \mathscr{O}_X), (Y, \mathscr{O}_Y)) = \{$ *morphisms of locally ringed spaces*$\}$.

Remark 6.2. *We have the inclusion relations*

$$(\mathbf{AffSch}) \subset (\mathbf{Sch}) \subset (\mathbf{LocallyRingedSpaces})$$

which are all fullsubcategories however,

$$(\mathbf{LocallyRingedSpaces}) \subset (\mathbf{RingedSpaces})$$

is not a full subcategory

6.3   Connection with Classical Algebraic Geometry

Proposition 6.2. *Let* X *be an affine variety. The regular functions* $\mathscr{O}_X(U)$

$$\mathscr{O}_X(U) = \{h : U \to k \mid h \text{ is a regular function.}\}.$$

defined on open subset U *of* X *form a sheaf. Furthermore, it is a locally ringed space.*

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proposition 6.3. *Let* X *be an affine variety and* $Y = A(X)$ *be a coordinate ring. Let us consider the sheaf of regular functions* $(X, \mathscr{O}_X)$ *and an affine scheme* $(Y, \mathscr{O}_Y)$. *There exists a natural morphism of locally ringed spaces* $(f, f^{\#}) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$.

Proof. Notice that we have the following isomorphisms.

$$X \cong \operatorname{MaxSpec}(A(X)), \quad k^n \cong \operatorname{MaxSpec}(k[x_1, \cdots, x_n]).$$

For any maximal ideal $\mathfrak{m} \subset k[x_1, \cdots, x_n]$,

$$I(X) \subseteq \mathfrak{m} = (x_1 - a_1, \cdots, x_n - a_n) \Leftrightarrow \forall f \in I(X), f(a_1, \cdots, a_n) = 0.$$

Let $\pi : Y \to X$ to be the canonical map by $I(X)$, then the map $f : X \to Y, (\mathfrak{m}) = \pi^{-1}(\mathfrak{m})$ is an inclusion. Then f is continuous.

Let us define $f^{\#} : \mathscr{O}_Y \to f_*(\mathscr{O}_X)$. For an open set $U \subseteq Y$, we have

$$(s : U \to \coprod_{\mathfrak{p} \in U} A(x)_{\mathfrak{p}}) \mapsto (s : U \to \coprod_{\mathfrak{m} \in U \cap \operatorname{MaxSpec} A(x)} A(x)_{\mathfrak{m}}).$$

By Lemma **??** and applying canonical maps $\pi_{\mathfrak{m}} : A(X)_{\mathfrak{m}} \to A(X)_{\mathfrak{m}}/\mathfrak{m}A(X)_{\mathfrak{m}}$ locally, we get

$$s : U \to \coprod_{\mathfrak{m} \in U \cap \operatorname{MaxSpec} A(x)} \to \coprod_{\mathfrak{m} \in U \cap \operatorname{MaxSpec} A(x)} A(X)_{\mathfrak{m}}/\mathfrak{m}A(X)_{\mathfrak{m}} = k.$$

Thus we obtained a map $s : U \to k$. Locally, we have

$$s = \frac{g_1 + I(X)}{g_2 + I(X)},$$

for $g_1 + I(X), g_2 + I(X) \in A(X)$. We conclude, locally

$$t = \frac{g_1}{g_2}.$$

We now claim that $(f, f^\#)$ is a local morphism of ringed spaces. By the correspondence of a maximal ideal $\mathfrak{m}$ of $k[x_1, \cdots, x_n]$ and a point $(a_1, \cdots, a_n)$, we have the isomorphism

$$\mathscr{O}_{X,\mathfrak{m}} \xrightarrow{\sim} \mathscr{O}_{Y,\mathfrak{m}} = A(X)_{\mathfrak{m}}.$$

$\square$

Remark 6.3. *Since X is an algebraic variety, there is a prime ideal $\mathfrak{p}$ of $k[x_1, \cdots, x_n]$ such that*

$$X = V(\mathfrak{p}).$$

Let us define $(Y', \mathscr{O}_{Y'}) = (\mathrm{Spec}(k[x_1, \cdots, x_n]), \mathscr{O})$, where $I(X) = \mathfrak{a}$. Since k is field, $k[x_1, \cdots, x_n]$ is Noetherian, thus the primary decomposition exists for any ideal. Thus there is a bijection between

$$\mathrm{Spec}(k[x_1, \cdots, x_n]/\mathfrak{a}) \leftrightarrow \mathrm{Spec}(A(X)).$$

Example 6.14. *Let K be any field and $A = k[x]/(x^2)$. A is called the ring of dual numbers. Observe that*

$$(\mathrm{Spec}\, k, \mathscr{O}_{\mathrm{Spec}\, k}), (\mathrm{Spec}\, A, \mathscr{O}_{\mathrm{Spec}\, A}),$$

both consist of single points. Let us define $(f, f^\#) : (\mathrm{Spec}\, k, \mathscr{O}_{\mathrm{Spec}\, k}) \to (\mathrm{Spec}\, A, \mathscr{O}_{\mathrm{Spec}\, A})$. By the previous observation, the function $f : \mathrm{Spec}\, A \to \mathrm{Spec}\, k$ is unique map sending the unique point to the unique point.

6.4   Properties of Schemes

Theorem 6.1 (Topological properties of schemes)**.**

Definition 6.15. *A scheme is said to be locally Noetherian if there exists an open cover $(U_i)_{i \in I}$ such that for each $i \in I$,*

$$U_i \cong \mathrm{Spec}(A_i) \tag{6.1}$$

for some Noetherian ring A_i.

Lemma 6.1. *Let A, B be rings and $\varphi : A \to B$ be a ring homomorphism. Let $a \in A$ and $b = \varphi(a)$, then we have*

$$\mathrm{Spec}(A_a) = \mathrm{Spec}(B_b)$$

as sets.

Proof. Let $\mathfrak{q} \subset B$ be a prime ideal not containing b and $\mathfrak{p} \subset A$ be a contraction of $\mathfrak{q}$ by φ. φ^{-1} is the inclusion from $\mathrm{Spec}(B)$ to $\mathrm{Spec}(A)$. Also we have

$$\mathrm{Spec}(A_a) \cong D_A(a) \subset \mathrm{Spec}(A)$$

for an arbitrary ring and an element. And by φ^{-1} we have

$$D_B(b) \subset D_A(a).$$

Thus we have an inclusion

$$\mathrm{Spec}(B_b) \subset \mathrm{Spec}(A_a).$$

$\square$

Proposition 6.4. *Let A, B be rings and $\varphi : A \to B$ be a ring homomorphism. Let $a \in A$ and $b = \varphi(a)$, then we have*

$$\mathrm{Spec}(A_a) = \mathrm{Spec}(B_b)$$

as affine schemes.

Proof. By Lemma **??**, they are equal as sets. Since $\mathrm{Spec}(B)$ is open in $\mathrm{Spec}(A)$. Using the definition of structure sheaves, we have

$$\mathscr{O}_{\mathrm{Spec}(A)}|_{\mathrm{Spec}(B)} = \mathscr{O}_{\mathrm{Spec}(B)}.$$

Therefore, by the lemma we have

$$\mathscr{O}_{\mathrm{Spec}(A_a)} = \mathscr{O}_{\mathrm{Spec}(B_b)}.$$

$\square$

Lemma 6.2. *A scheme $(X, \mathscr{O}_X)$ is locally Noetherian if and only if for any open affine set $U \subset X, U = \mathrm{Spec}(A)$ for some Noetherian ring A.*

Proof. By definition, $\Leftarrow$ is trivially true. We will prove $\Rightarrow$ direction.

Let $X = \bigcup_{i \in I} \mathrm{Spec}(A_i)$ be an open affine Noetherian covering of X and $U = \mathrm{Spec}(A)$ be an open affine set. Let us define an open covering of U by

$$U_i = U \cap \mathrm{Spec}(A_i), \quad U = \bigcup_{i \in I} U_i.$$

By Theorem **??** and the assumption on U, U is quasi-compact. By arranging I, there exists a large enough $n \in \mathbb{N}$ such that

$$U = \bigcup_{i=1}^{n} U_i.$$

Since each $i = 1, \cdots, n$, U_i is open in $\mathrm{Spec}(A_i)$, thus there is $\{a_{ij}\}_{j=1\cdots,n_i} \subset A_i$ such that

$$U_i = \bigcup_{j=1}^{n_i} \mathrm{Spec}(A_{i,a_{ij}}).$$

Thus substituting this to (**??**), we get

$$U = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} \mathrm{Spec}(A_{i,a_{ij}}).$$

Again by the quasi-compactness of U, we conclude finitely many $A_{i,a_{ij}}$ cover U.

Since A_i is Noetherian for each $i \in I$, this means that any localization of it is also Noetherian. Thus we $\{\mathrm{Spec}(A_{i,a_{ij}})$ is an open Noetherian covering of U. By rearranging $\{a_{ij}\}$, we let $\{a_1, \cdots, a_n\}$ to be the elements which spectrums of their localizations cover U. We then show that

$$\mathfrak{a} = \bigcap_{i=1}^{n} \pi_i^{-1}(\pi_i(\mathfrak{a})A_{a_i}).$$

where $\pi_i : A \to A_{a_i}$ is the canonical inclusion for each $i = 1, \cdots, n$.

$\mathfrak{a} \subseteq \bigcap_{i=1}^{n} \pi_i^{-1}(\pi_i(\mathfrak{a})A_{a_i})$ is trivial, thus we prove $\mathfrak{a} \supset \bigcap_{i=1}^{n} \pi_i^{-1}(\pi_i(\mathfrak{a})A_{a_i})$. Let $b \in \bigcap_{i=1}^{n} \pi_i^{-1}(\pi_i(\mathfrak{a})A_{a_i})$ be arbitrary. By definition, for each $i = 1, \cdots, n$, there exists $b_i \in \mathfrak{a}$ and n_i such that

$$\pi_i(b) = \frac{b_i}{a_i^{n_i}}.$$

Using that $\mathfrak{a}$ is an ideal, we derive that for big enough N, we have

$$\pi_i(b) = \frac{b_i}{a_i^{N}}.$$

Using the definition of localization, for each $i = 1, \cdots, n$, there is m_i such that

$$(b_i - a_i^{N} b)a_i^{m_i} = 0.$$

Taking large enough M, for each $i = 1, \cdots, n$, we derive

$$(b_i - a_i^{N} b)a_i^{M} = 0.$$

Thus for all $i = 1, \cdots, n$, we know

$$a_i^{N+M} b \in \mathfrak{a}. \tag{6.2}$$

Since $\{D_A(a_i)\}_{i=1,\cdots,n}$ covers $\mathrm{Spec}\, A$, we have

$$\bigcap_{i=1}^{n} V(a_i) = \emptyset \Leftrightarrow V((a_1, \cdots, a_n)) = \emptyset \Leftrightarrow (a_1, \cdots, a_n) = (1).$$

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Therefore, for any $k \in \mathbb{N}_0$ we have $(a_1^k, \cdots, a_n^k) = (1)$. By Equation (**??**), we derive that for some $c_1, \cdots, c_n \in A$,

$$b = \sum_{i=1}^n c_i a_i^k \in \mathfrak{a}.$$

Finally, we prove that A is Noetherian. Given an ascending chain of ideal

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots .$$

We get an ascending chain of extended ideals

$$\pi(\mathfrak{a}_1)A_{a_i} \subset \pi(\mathfrak{a}_2)A_{a_i} \subset \cdots ,$$

for each $i = 1, \cdots, n$. Since each A_{a_i} is Noetherian, we conclude that there is large enough N such that

$$\pi_i(\mathfrak{a}_N)A_{a_i} = \pi_i(\mathfrak{a}_{N+1})A_{a_i}$$

for each $i = 1, \cdots, n$. By Equation (**??**), we conclude that

$$\mathfrak{a}_N = \bigcap_{i=1}^n \pi_i^{-1}(\pi_i(\mathfrak{a}_N)A_{a_i}) = \bigcap_{i=1}^n \pi_i^{-1}(\pi_i(\mathfrak{a}_{N+1})A_{a_i}) = \mathfrak{a}_{N+1}.$$

$\square$

Corollary 6.1. *An affine scheme* $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ *is a Noetherian scheme, then A is Noetherian.*

Remark 6.4. *A Sheaf is noetherian then its base space is Noetherian as topological space. The converse is not true.*

Definition 6.16. *A scheme* $(X, \mathscr{O}_X)$ *is said to be reduced if for any open subset $U \subseteq X$, $\mathscr{O}_X(U)$ is a reduced ring.*

Proposition 6.5. *A scheme* $(X, \mathscr{O}_X)$ *is reduced if and only if for any $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a reduced ring.*

Proof. $\square$

Definition 6.17. *A scheme is integral if every section of it is an integral domain.*

Proposition 6.6. *A scheme* $(X, \mathscr{O}_X)$ *is reduced then for any $x \in X$, the stalk $\mathscr{O}_{X,x}$ is an integral domain.*

Proof. $\square$

Remark 6.5. *The converse is not true.*

Let k be a field and R be a k-algebra.

Remark 6.6. *By the definition of reduced rings, it is obvious that integral schemes are reduced.*

Example 6.15. *An affine scheme on a field k is integral.*

Example 6.16. *Let k be a field. $(\operatorname{Spec} k[x]/(x^2), \mathscr{O}_{\operatorname{Spec} k[x]/(x^2)})$ is neither integral nor reduced.*

Example 6.17. *Let k be a field. $(\operatorname{Spec} k[x,y]/(x,y), \mathscr{O}_{\operatorname{Spec} k[x,y]/(x,y)})$ is reduced but not integral.*

Lemma 6.3. *Let $(X, \mathscr{O}_X)$ is a scheme and fix $s \in \mathscr{O}_X(U)$ for some open set U. For each $x \in U$, define $\mathfrak{m}_x$ to be a unique maximal ideal in the stalk. Then the set*

$$F = \{x \in U \mid s \in \mathfrak{m}_x\}$$

is a closed subset of X.

Proof. First, let us assume that $U = \operatorname{Spec}(A)$ for some ring A. We will prove that T^c is open $\qquad\square$

Lemma 6.4. *A scheme $(X, \mathscr{O}_X)$ is integral if and only if it is a reduced scheme on an irreducible topological space.*

Proof. Since integral schemes are reduced. We first prove that X is not irreducible then $(X, \mathscr{O}_X)$ is not integral.

Since X is not irreducible, there is non-empty disjoint open subsets U_1, U_2 of X. Then

$$\mathscr{O}_X(U_1 \cup U_2) \cong \mathscr{O}_X(U_1) \times \mathscr{O}_X(U_2).$$

Therefore, this is not an integral domain.

Suppose X is irreducible and $(X, \mathscr{O}_X)$ is reduced. Given an arbitrary open set $U \subset X$, and $s_1, s_2 \in \mathscr{O}_X(U)$, we will show that

$$s_1 s_2 = 0 \Rightarrow s_1 = 0 \lor s_2 = 0.$$

By Lemma **??**,

$$X_1 = \{x \in U \mid s_1 \in \mathfrak{m}_x\}, \quad X_2 = \{x \in U \mid s_2 \in \mathfrak{m}_x\},$$

are closed subsets.

By the sheaf property, we have $s_1 s_2 = 0$ implies for all $x \in U$,

$$(s_1 s_2)_x = s_{1,x} s_{2,x} = 0.$$

Since each $\mathfrak{m}_x$ is prime and $s_{1,x} s_{2,x} = 0 \in \mathfrak{m}_x$, $s_{1,x} \in \mathfrak{m}_x$ or $s_{2,x} \in \mathfrak{m}_x$. Therefore, this show that

$$U = X_1 \cup X_2.$$

Since X is irreducible, so is U, Without the loss of generality, we assume $U = X_1$. Let $\mathrm{Spec}(A) \subset U$ be an open affine set. Let us define

$$t = s_1|_{\mathrm{Spec}(A)} \in A.$$

for all $x = \mathfrak{p} \in \mathrm{Spec}(A) \subset U$, we have

$$t_x \in \mathfrak{m}_x \in \mathscr{O}_{X,x}.$$

In other words,

$$\frac{t}{1} \in \mathfrak{p}A_{\mathfrak{p}} \subset A_{\mathfrak{p}}.$$

Therefore, $t \in \mathfrak{p}$ for any prime ideal of A, thus t is a nilpotent. Furthermore, X is reduced, thus $t = 0$.

Thus any section $s_1|_{\mathrm{Spec}(A)} = 0$ for all $\mathrm{Spec}(A)$. By the sheaf property, we conclude $s_1 = 0$. $\qquad\square$

Corollary 6.2. *If X is integral, then there exists a unique generic point $\eta \in X$.*

Proof. For any $\mathrm{Spec}(A) \subseteq X$, A is an integral domain. Let $\eta = (0) \in \mathrm{Spec}(A)$. Then η is a generic point of $\mathrm{Spec}(A)$. By the irreducibility of X, we have η is a generic point of X.

For the uniqueness, assume ζ, η be generic points of X. Let us pick an open affine set $\mathrm{Spec}(A)$ containing η. By closedness of $X \backslash \mathrm{Spec}(A)$ and that ζ is also a generic point, we conclude that $\zeta \in \mathrm{Spec}(A)$.

Without the loss of generality, we assume that $\eta = (0) \subset A$. Since $\eta \neq \zeta$, $\zeta = \mathfrak{p} \subset A$ for some prime ideal. However

$$\eta \in V(\mathfrak{p}) \Leftrightarrow \mathfrak{p} = (0) \Rightarrow \eta = \zeta.$$

$\qquad\square$

6.5 Open and Closed Subschemes

Definition 6.18. *Let $(X, \mathscr{O}_X)$ be a scheme and $U \subseteq X$ be open. Then*

$$(U, \mathscr{O}_X|_U)$$

is called an open subscheme of X.

Remark 6.7. *If $\mathrm{Spec}(B) \subseteq \mathrm{Spec}(A)$ is an open subscheme then*

$$\mathrm{Spec}(B) = \bigcup_{i=1}^{n} \mathrm{Spec}(A_{a_i}) = \bigcup_{i=1}^{n} \mathrm{Spec}(B_{b_i}),$$

with morphisms $A \ni a_i \mapsto b_i \in B$.

Definition 6.19 (1st definition of closed subschemes). *Let $(X, \mathscr{O}_X)$ be a scheme and M be a set of morphisms of schemes such that for each $(i, i^\#) \in M$,*

    1. $i : Z \to X$, is a homeomorphism of Z and some closed subset $i(Z)$ of X.

    2. $i^\# : \mathscr{O}_X \to i_* \mathscr{O}_Z$ is surjective. In other words, for any $z \in Z$, $\mathscr{O}_{X,i(z)} \to \mathscr{O}_{Z,z}$ is a surjection.

We now define an equivalence relation $\sim$ such that $(i, i^\#) : Y \to X \sim (j, j^\#) : Z \to X$ if Y and Z are homeomorphic and the following diagram is commutative.

$$
\begin{array}{ccc}
(Y, \mathscr{O}_Y) & \xrightarrow{(i,i^\#)} & (X, \mathscr{O}_X) \\
\downarrow & \nearrow & \\
(Z, \mathscr{O}_Z) & {\scriptstyle (j,j^\#)} &
\end{array}
$$

Definition 6.20 (Second definition of closed subschemes). *A closed subscheme of a scheme $(X, \mathscr{O}_X)$ consists of a closed subset $i : Z \hookrightarrow X$ and a sheaf $\mathscr{O}_Z$ such that there is a sheaf of ideals $(Z, \mathscr{I}_Z)$ which is a subsheaf of $\mathscr{O}_X$ such that*

$$\mathscr{O}_X|_{\mathscr{I}_Z} \cong i_* \mathscr{O}_X.$$

Lemma 6.5. *Let $X = \operatorname{Spec}(A)$ for a commutative ring and consider a scheme $(X, \mathscr{O}_X)$. Consider a closed subscheme $(Z, \mathscr{O}_Z)$ of $(X, \mathscr{O}_X)$ and an ideal*

$$\mathfrak{a}_Z = \operatorname{Ker}(A \to Z).$$

Then we have the inclusions of set,

$$Z \subseteq V(\mathfrak{a}_Z) \subseteq \operatorname{Spec}(A).$$

Lemma 6.6. *The second claim of the preceding lemma.*

Lemma 6.7. *Let A be a ring then there exists a natural bijection between*

$$\{\mathfrak{a} \subseteq A \mid \mathfrak{a} \text{ is an ideal of } A\} \leftrightarrow \{Z \subseteq \operatorname{Spec}(A) \mid Z \text{ is a closed subscheme}\}.$$

Proof. □

6.6 Fiber Products

Definition 6.21. *Let X, S be schemes. X is called a S-scheme if there exists a morphism of schemes $\varphi : X \to S$.*

Definition 6.22. *Let S be a scheme. The category of S-schemes is $(\mathbf{Sch}/S)$ where*

    1. $\mathbf{ob}(\mathbf{Sch}/S) = \{\varphi : X \to S \mid \varphi \text{ is a morphism of schemes}\}.$

    2. $(\mathbf{Sch}/S)(\varphi : X \to S, \psi : Y \to S) = \{f : X \to Y \mid f \text{ is a morphism of schemes such that } \varphi = f \circ \psi\}.$

Remark 6.8. *Clearly we have*

$$(\mathbf{Sch}/\operatorname{Spec}(\mathbb{Z})) = (\mathbf{Sch}).$$

For a field k we have,