Algebraic Geometry 1

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1 Topology

1.1 Connected Sets

Definition 1.1. *Let $(X, \mathscr{T})$ be a topological space. A subset A of X is said to be connected if for any $U, V \in \mathscr{T}$, $U \cap V =, U \cup V \supset A$ then A is fully contained in one of U, V.*

Definition 1.2. *A connected component of a topological space is a maximal connected subset of a space.*

Proposition 1.1. *Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological space and $f : X \to Y$ be a continuous function. Then for any connected subset A of X, $f(A)$ is connected in Y.*

Proof.

$$\begin{aligned}
U, V \in \mathscr{T}_Y, U \cup V \supset f(A), U \cap V = \emptyset, \\
&\Rightarrow f^{-1}(U), f^{-1}(V) \in \mathscr{T}_X, \\
&\quad f^{-1}(U) \cup f^{-1}(V) \supset A, \\
&\quad f^{-1}(U) \cap f^{-1}(V) = \emptyset, \\
&\Rightarrow f^{-1}(U) \supset A \vee f^{-1}(V) \supset A, \\
&\Rightarrow U \supset f(A) \vee V \supset f(A).
\end{aligned}$$

□

2 Category Theory

2.1 Categories

Definition 2.1. *A category $\mathscr{A}$ consists of*

- *a collection* $\mathrm{ob}(\mathscr{A})$ *of objects;*
- *for each $A, B \in \mathrm{ob}(\mathscr{A})$, a collection $\mathscr{A}(A, B)$ of morphisms from A to B;*

such that

- *i).* *for each* $A \in \mathrm{ob}(\mathscr{A})$*, the identity* $1_A \in \mathscr{A}(A,A)$*;*
- *ii).* *the composition* $\mathscr{A}(B,C) \times \mathscr{A}(A,B) \ni (g,f) \mapsto g \circ f \in \mathscr{A}(A,C)$ *is well-defined;*

and they satisfy the following axioms

- *I).* *Associativity :* $f \in \mathscr{A}(A,B), g \in \mathscr{A}(B,C), h \in \mathscr{A}(C,D)$*,* $(h \circ g) \circ f = h \circ (g \circ f)$*.*
- *II).* *Identity laws :* $f \in \mathscr{A}(A,B)$ *then* $f \circ 1_A = 1_B \circ f$*.*

Definition 2.2. *Let* $\mathscr{A}$ *be a category. A terminal object* $T \in \mathrm{ob}(\mathscr{A})$ *is an object such that for any* $A \in \mathrm{ob}(\mathscr{A})$*,* $\mathscr{A}(A,T)$ *is a single element set.*

Definition 2.3. *Given two categories* $\mathscr{A}, \mathscr{B}$*, we say* $\mathscr{A}$ *is a full-subcategory of* $\mathscr{B}$ *if*

- *i).* $\mathscr{A} \subset \mathscr{B}$*,*
- *ii).* $\mathrm{ob}(\mathscr{A}) = \mathrm{ob}(\mathscr{B})$.

Notation 2.1. *Here we give notations to some important categories.*

- (**Sets**) *: A category of sets equipped with set theoretic functions.*
- (**Ab**) *: A category of abelian groups with group homomorphisms.*

Example 2.1. *Given a partially ordered set* $(X, \leq)$*. This can be encoded to a category* $\mathscr{O}$ *by*

- *i).* $\mathrm{ob}(\mathscr{O}) = X$,
- *ii).* *For* $x, y \in X$*,* $x \leq y \Rightarrow \mathscr{O}(x,y) = \{*\}$ *otherwise the morphisms between* x, y *is an emptyset.*

Definition 2.4. *A opposite/dual category of a category* $\mathscr{A}$ *is* $\mathscr{A}^{\boldsymbol{op}}$ *such that*

- *i).* $\mathrm{ob}(\mathscr{A}^{\boldsymbol{op}}) = \mathrm{ob}(\mathscr{A})$,
- *ii).* $\mathscr{A}^{\boldsymbol{op}}(B,A) = \mathscr{A}(A,B)$.

Definition 2.5. *Let* $\mathscr{A}$ *be a category and* $\varphi_1, \varphi_2 \in \mathscr{A}(M,N)$*. A morphism* $\varphi : K \to M$ *is called an equalizer of* (φ_1, φ_2) *if for any morphism* $\psi : P \to M$ *such that* $\varphi_1 \circ \psi = \varphi_2 \circ \psi$*, there is a unique morphism* $\tilde{\psi} : P \to K$ *such that* $\varphi \circ \tilde{\psi} = \psi$*.*

Proposition 2.1. *If an equalizer exists then it is unique up to unique isomorphism.*

Proof. Suppose $\varphi : K \to M, \psi : L \to M$ be equalizers of (φ_1, φ_2). Then we have

$$\varphi \circ \tilde{\psi} = \psi, \quad \psi \circ \tilde{\varphi} = \varphi$$

By the uniqueness, we have $\tilde{\varphi} \circ \tilde{\psi} = 1_L, \tilde{\psi} \circ \tilde{\varphi} = 1_K$. $\square$

Definition 2.6. *Let $\mathscr{A}, \mathscr{B}$ be categories. A functor $F : \mathscr{A} \to \mathscr{B}$ is a function such that for each $f \in \mathscr{A}(A, A')$, $F(f) : F(A) \to F(A')$. In other words, $f \mapsto F(f) : \mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$. Furthermore, F satisfies the following axioms.*

I). $F(f' \circ f) = F(f') \circ F(f)$ whenever $f : A \to A', f' : A' \to A''$ in $\mathscr{A}$,

II). $F(1_A) = 1_{F(A)}$ whenever $A \in \mathscr{A}$.

Definition 2.7. *Let F, G be functors between two categories $\mathscr{A}, \mathscr{B}$. A natural transformation $\alpha : F \to G$ is a family $(\alpha_A : F(A) \to G(A))_{A \in \mathscr{A}}$ such that*

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_{A'}} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

is a commutative diagram. Each α_A is called a component of α.

2.2 Direct Limits

Definition 2.8. *A partially ordered set $(X, \leq)$ is directed if for any $x, y \in X$ there is $z \in X$ such that $x \leq c$ and $y \leq c$.*

Example 2.2. *Let $(X, \mathscr{T})$ be a topological space. A partially ordered set $(\mathscr{T}, \leq)$ such that*

$$V \subseteq U \Rightarrow U \leq V$$

is directed. Since for any $U \in \mathscr{T}$, $U \leq \emptyset$. As a category this is $\mathbf{Ouv}_X^{\mathbf{op}}$.

Example 2.3. *Let $(X, \mathscr{T})$ be a topological space. For $x \in X$, define $O_x = \{U \in \mathscr{T} \mid x \in U\}$. If we define an order as in the previous example, we get $(O_x, \leq)$ is directed. This follows from for any $U, V \in O_x$, $U, V \leq U \cap V$.*

Definition 2.9. *Let I be a directed partially ordered set and $\mathscr{A}$ be a category. A directed system of objects of $\mathscr{A}$ indexed by I is a collection of objects $(A_i)_{i \in I}$ and morphisms $(\rho_{ij})_{i \leq j}$ of $\mathscr{A}$ such that*

i). $\rho_{ii} = \mathbf{id}_{A_i}$,

ii). for $i, j, k \in I$, $i \leq j \leq k \Rightarrow \rho_{ik} = \rho_{jk} \circ \rho_{ij}$.

Remark 2.1. *Categorically, the directed system of objects of $\mathscr{A}$ indexed by I is a functor $\mathscr{O}^{\mathrm{op}} \to \mathscr{C}$, where $\mathscr{O}$ is a category which encodes the ordered set I as a category by the same procedure as in Example 2.1. Then a directed system if a functor $\mathscr{O}^{\mathbf{op}} \to \mathscr{A}$.*

Definition 2.10. *Given a directed system $((A_i)_{i\in I}, \{\rho_{ij}\}_{i\leq j})$ of objects in $\mathscr{A}$ indexed by I. A direct limit of the system is an object $\varinjlim A_n \in \mathbf{ob}(\mathscr{A})$ satisfying the following universal property.*

Given a collection of morphisms $(f_i)_{i\in I}$ such that

i). $f_i : A_i \to \varinjlim A_n \in \mathscr{A}$,

ii). for any $i \leq j$, $f_j \circ \rho_{ij} = f_i$.

For any $A \in \mathscr{A}$ where there is a collection of morphisms $(g_i)_{i\in I}$ satisfying the above condition, there is a unique map $\varphi : \varinjlim A_n \to A$ such that

$$\begin{tikzcd}
A_i \arrow[rr, dash, "\rho_{ij}"] \arrow[dr, "f_i"] \arrow[ddr, bend right, "g_i"'] & & A_j \arrow[dl, "f_j"'] \arrow[ddl, bend left, "g_j"] \\
& \varinjlim A_n \arrow[d, dotted, "\exists!"] & \\
& A &
\end{tikzcd}$$

is a commutative diagram.

Proposition 2.2. *$\varinjlim$ is an exact functor.*

Proposition 2.3. *In the cases where $\mathscr{A} = (\mathbf{Ab}), (\mathbf{Sets})$, there exist direct limits and for each category, such limit is constructed in the following ways.*

i). $\varinjlim A_n = (\bigoplus_{i\in I} A_i)/N$ where $N = \{a_i - \rho_{ij}(a_i) \mid a_i, i \leq j\}$.

ii). $\varinjlim A_n = (\coprod_{i\in I} A_i)/\sim$ where $a_i \sim a_j$ if there is k such that $i \leq k$ $j \leq k$, and $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.

Furthermore, these two direct limits match as sets.

Proposition 2.4. *$\varinjlim$ is (left) exact in $(\mathbf{Ab})$. In other words, given a exact sequence of directed systems*

$$0 \longrightarrow (M_i)_{i\in I} \longrightarrow (N_i)_{i\in I} \longrightarrow (P_i)_{i\in I} \longrightarrow 0$$

in which we have

$$\begin{tikzcd}
0 \arrow[r] & M_i \arrow[r] \arrow[d, "\rho_{ij}^M"'] & N_i \arrow[r] \arrow[d, "\rho_{ij}^N"'] & P_i \arrow[r] \arrow[d, "\rho_{ij}^P"'] & 0 \\
0 \arrow[r] & M_j \arrow[r] & N_j \arrow[r] & P_j \arrow[r] & 0
\end{tikzcd}$$

There exists a short exact sequence

$$0 \longrightarrow \varinjlim M_n \longrightarrow \varinjlim N_n \longrightarrow \varinjlim P_n \longrightarrow 0$$

3 Commutative Algebra

3.1 Local Rings

Definition 3.1. *The total ring of fraction of a ring A is a localization of A by the set of all non-zero divisors. It is denoted as $Q(A)$.*

Definition 3.2. *A ring is said to be local if it has a unique maximal ideal.*

Definition 3.3. *A ring homomorphism $\phi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ of two local rings is said to be local if*

$$\mathfrak{m}_A = \phi^{-1}(\mathfrak{m}_B).$$

Example 3.1. *Let $i : \mathbb{Z}_{(p)} \to Q(\mathbb{Z}_{(p)})$ be an inclusion map. Then it is a homomorphism of local rings. However, If p is prime then $Q(\mathbb{Z}_{(p)})$ is a field thus its maximal ideal is (0). Obviously*

$$i^{-1}((0) = (0).$$

Therefore, i is not a local ring homomorphism.

Proposition 3.1. *Let $\phi : A \to B$ be a ring homomorphism. Recall that for any prime ideal $\mathfrak{q} \subseteq B$, we have $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ is a prime ideal in A. Thus ϕ induces a homomorphism between $A_\mathfrak{p}$ and $B_\mathfrak{q}$ which is a local ring homomorphism.*

Proof. If $a \in A$, $\phi(a) = 0$ then $a \in \mathfrak{p}$. Thus $\phi(s) \neq 0$ for any $s \notin \mathfrak{p}$. Since $\mathfrak{p}, \mathfrak{q}$ are unique maximal ideals of $A_\mathfrak{p}$ and $B_\mathfrak{q}$, respectively. We derived the claim. □

Lemma 3.1. *Let k be an algebraically closed field and A be a k-algebra. A localization $A_\mathfrak{m}$ by a maximal(prime) ideal$\mathfrak{m} \subset A$, we have the following isomorphism.*

$$k \cong A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}.$$

Proof. Follows from the algebraically closedness of k. □

3.2 Maximal Spec

Definition 3.4. *Let R be a commutative ring. We define the maximal spec of R as*

$$\operatorname{MaxSpec}(R) = \{\mathfrak{m} \operatorname{Spec}(R) \mid \mathfrak{m} \textit{ is a maximal ideal.}\}.$$

Lemma 3.2. *Let k be an algebraically closed field. We have the following isomorphism*

$$\operatorname{MaxSpec} k[x_1, \cdots, x_n] \cong k^n, \quad (x_1 - a_1, \cdots, x_n - a - n) \leftrightarrow (a_1, \cdots, a_n).$$

Proof. Surjectivity follows from the algebraically closedness of k. □

3.3 Zariski Topology

Definition 3.5. *Let k be a algebraically closed field. A subset X of k^n is called an affine algebraic set if there exists an ideal $\mathfrak{a} \subset k[x_1, \cdots, x_n]$ such that*

$$X = V(\mathfrak{a}) = \{(a_1, \cdots, a_n) \mid \forall f \in \mathfrak{a}, f(a_1, \cdots, a_n) = 0\}.$$

Definition 3.6. *Let k be an algebraically closed. The Zariski topologi on k^n is a topology generated by affine algebraic sets as closed subsets.*

Definition 3.7. *Let X be the Zariski topology on k^n. A function $f : X \supseteq U \to k$ is said to be regular if for any $a = (a_1, \cdots, c_n) \in U$, there exist a neighborhood $U_a \subseteq U$ and $f_1, f_2 \in k[x_1, \cdots, x_n]$ such that*

$$(b_1, \cdots, b_n) \in V_a \Rightarrow f(b_1, \cdots, b_n) = \frac{f_1(b_1, \cdots, b_n)}{f_2(b_1, \cdots, b_n)}.$$

Remark 3.1. *A regular function f on the Zariski topology on k^n is continuous as they are locally equivalent to quotients of polynomial functions.*

4 Classical Algebraic Geometry

4.1 Affine Variety

Definition 4.1. *An affine algebraic set X is called an affine variety if there exists a prime ideal $\mathfrak{p} \subseteq k[x_1, \cdots, x_n]$ such that*

$$X = V(\mathfrak{p}).$$

Definition 4.2. *Let k be an algebraically closed field and $X \subseteq k^n$. The ideal of X is*

$$I(X) = \{f \in k[x_1, \cdots, x_n] \mid \forall (a_1, \cdots, a_n) \in X, f(a_1, \cdots, a_n) = 0\}.$$

Theorem 4.1. *For any ideal $\mathfrak{a} \subset k[x_1, \cdots, x_n]$, we have*

$$I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

Definition 4.3. *Let $X \subset k^n$ where k is an algebraically closed field. The affine coordinate ring with respect to X is*

$$A(X) = k[x_1, \cdots, x_n]/I(X).$$

5 Sheaf Theory

5.1 Presheaves

Definition 5.1. *Let $(X, \mathscr{T})$ be a topological space. We define the presheaf $\mathscr{F}$ of a category $\mathscr{A}$ on X such that*

- $U \in \mathscr{T}$, $\mathscr{F}(U) \in \mathrm{ob}(\mathscr{A})$,
- $U, V \in \mathscr{T}$, $V \subset U \Rightarrow$ *there exists a map* $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$

  such that

  i). For any $U \in \mathscr{T}$, $\rho_{UU} = 1_{\mathscr{F}(U)}$.

  ii). $U, V, W \in \mathscr{T}, W \subset V \subset U \to \rho_{UW} = \rho_{VW} \circ \rho_{UW}$.

Remark 5.1. *In the case* $\mathscr{A} = (\mathbf{Sets}), (\mathbf{Ab})$, $\mathscr{F}(\emptyset) = \emptyset, \{1\}$, *respectively.*

Definition 5.2. *An element of* $\mathscr{F}(U)$ *is called a local section of* $\mathscr{F}$ *and* $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$ *is called the space of sections over* U. *In particular* $\Gamma(X, \mathscr{F})$ *is called the space of global sections of* $\mathscr{F}$.

Definition 5.3. *Let* $(X, \mathscr{T})$ *be a topological space and* $\mathscr{F}$ *be a presheaf of a category* $\mathscr{A}$ *on* X. *Suppose we have two open sets* $U, V \in \mathscr{T}$ *such that* $V \subset U$. *Then for any section* $s \in \mathscr{F}(U)$, $s|_V = \rho_{UV}(s)$ *is called the restriction of* s *to* V.

Example 5.1. *Let* $(X, \mathscr{T})$ *be a topological space. We have a presheaf of continuous functions* $\mathscr{C}_X(U) = \mathscr{C}^0(U, \mathbb{R})$. *This is indeed a presheaf with restriction maps* $\rho_{UV} : \mathscr{C}_X(U) \to \mathscr{C}_X(V)$. *(Explicitly,* $\rho_{UV}(f) = f \circ i_V$ *where* i_V *is an inclusion map.) We note that we can introduce operations* $+, \cdot$ *to endow some algebraic structures (groups, rings, ...) on* $\mathbb{R}$.

Example 5.2. *Let* $(X, \mathscr{T})$ *be a topological space and suppose we have presheaves*

- $\mathscr{C}_X^{\boldsymbol{diff}}(U) = \{f : U \to \mathbb{R} \mid f \text{ is differentiable.}\}$.

Then there is an inclusion relation $\mathscr{C}_X^{\boldsymbol{diff}}(U) \subseteq \mathscr{C}_X(U)$ *and this defines a presheaf.*

Example 5.3. *Let* $(X, \mathscr{T}_X)$,$(Y, \mathscr{T}_Y)$ *be topological spaces. Define a presheaf on* X *by*

$$U \in \mathscr{T}_X, \mathscr{F}(U) = \mathscr{C}^0(X, Y).$$

And like the previous example, we define $\rho_{UV}(f) = f|_V$ *for* $U, V \in \mathscr{T}_X, V \subset U$. *the restriction of* f *to* V.
But this is a presheaf only of a set.

Example 5.4. *Let* $(X, \mathscr{T})$ *be a topological space and* G *be an abelian group. The constant presheaf* $\mathbb{G}$ *is such that*

$$U \in \mathscr{T}, \mathbb{G}(U) = G,$$

with $\rho_U V = id_G$ *for any* $U, V \in \mathscr{T}, V \subset U$.

5.2 Presheaves as Categories

Definition 5.4. *Let $(X, \mathscr{T})$ be a topological space then $(\mathbf{Ouv}_X)$ is the category such that its objects are the open sets of X and for any $U, V \in \mathscr{T}$ we have*

$$\mathbf{Ouv}_X(U,V) = \begin{cases} \emptyset & (V \not\subset U), \\ i_V & (V \subset U). \end{cases}$$

Definition 5.5. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{A}$ be a category. A presheaf of $\mathscr{A}$ on X is a functor $F : \mathbf{Ouv}_X \to \mathscr{A}$.*

Example 5.5. *For $\mathbf{Ouv}_X$, we can define a presheaf of F to be*

$$\mathrm{ob}(\mathbf{Ouv}_X) \ni U \mapsto F(U) = \mathscr{C}^0(U, \mathbb{R}).$$

Example 5.6. *Let A be a commutative ring with non-zero multiplicative identity and $X = \mathrm{Spec}(A)$. Let us consider the Zariski topology $(X, \mathscr{T})$. Let us consider a category $\mathscr{O}_X$ such that*

- $\mathrm{ob}(\mathscr{O}_X) = \mathscr{T}$,
- $\mathscr{O}_X(U) = \{s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}\}$,

where $s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ is a function such that for any $\mathfrak{p} \in U$,

i). $s(p) \in A_{\mathfrak{p}}$,

ii). there exists an open set $V \subset U$ such that $\mathfrak{p} \in V$ and for any $\mathfrak{q} \in V$, $s(\mathfrak{q}) = \frac{a}{b}$ for $b \notin \mathfrak{q}$.

Now we define a presheaf by the restrictions of maps such that

$$s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mapsto s|_V : V \to \coprod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

Definition 5.6. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{A}$ be a category. We define a set of presheaves of $\mathscr{A}$ on X as*

$$\mathrm{PreSh}_{\mathscr{A}}(X) = \mathrm{Fun}(\mathbf{Ouv}_X^{\mathbf{op}}, \mathscr{A}).$$

Definition 5.7. *A morphism of presheaves is a natural transformation $\varphi : \mathscr{F} \to \mathscr{G}$ where $\mathscr{F}, \mathscr{G} \in \mathrm{PreSh}_{\mathscr{A}}(X) = \mathrm{Fun}(\mathbf{Ouv}_X^{\mathbf{op}}, \mathscr{A})$.(See Definition 2.7).*

Such $\varphi : \mathscr{F} \to \mathscr{G}$ is

i). injective if

Remark 5.2. $\mathrm{PreSh}(X)$ *can be regarded as a category with its objects presheaves and morphisms defined above.*

Notation 5.1. *In the case $\mathscr{A} = (\mathbf{Ab})$ then we denote* $\mathrm{PreSh}(X) = \mathrm{PreSh}_{\mathbf{Ab}}(X)$.

Example 5.7. *Let X be a differential manifold(eg. $X \subset \mathbb{R}^n$). Let us define*

$$\mathscr{C}^{\mathbf{diff}}(U) = \{f : U \to \mathbb{R} \mid f \text{ is differentiable.}\}.$$

Then the inclusions $\mathscr{C}_X^{\mathbf{diff}}(U) \subset \mathscr{C}_X(U)$ defines the natural transformation.

Example 5.8. *Let $X, Y = S^1$ be topological spaces and F be a presheaf such that for any open set $U \subset X$, $F(U) = \mathscr{C}^0(U, Y)$.Then we can introduce a natural transformation such that*

$$\mathscr{C}_X(U) \ni f \mapsto \exp(2\pi f i).$$

5.3 Sheaves

Definition 5.8. *A presheaf $\mathscr{F}$ on $(X, \mathscr{T})$ is called a sheaf if the following holds. For any collection of open sets $(U_i)_{i\in I} \subset \mathscr{T}, U = \bigcup_{i\in I} U_i$, the map $\varphi : \mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i)$ which is defined as*

$$\varphi(s) = (s|_{U_i})_{i\in I}.$$

is the equalizer of the following functions $\varphi_1, \varphi_2 : \prod_{i\in I} \mathscr{F}(U_i) \to \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j)$,

$$\varphi_1((s_i)_{i\in I}) = (s_i|_{U_i\cap U_j})_{i,j\in I}, \quad \varphi_1((s_i)_{i\in I}) = (s_j|_{U_i\cap U_j})_{i,j\in I}.$$

Remark 5.3. *In the case $I = \{1, 2\}$, we have $U = U_1 \cup U_2$, and for any $U' \in \mathscr{T}$ such that $U \subset U'$, we have for $\mathscr{F}(U') \ni s : U' \to \mathbb{R}$,$\psi(s) = (s|_{U_1}, s|_{U_2})$, as in $\mathbf{Ouv}_X$, morphisms are inclusions. Let $\tilde{\psi}(s) = s|_U$, then this satisfies the condition for the equalizer (ie. $\varphi \circ \tilde{\psi} = \psi$).*

Remark 5.4. *A presheaf $\mathscr{O}_X$ with $X = \operatorname{Spec}(A)$ is a sheaf.*

Example 5.9. *Let $(X, \mathscr{T})$ be a topological space and G be a group. We define a constant presheaf $\mathbb{G}(U) = G$. In general, this is not a sheaf. Instead, we define a constant sheaf $\underline{\mathbb{G}}(U) = \mathscr{C}^0(U, G)$ where G is regarded as a topological space with the discrete topology. Then for any connected component of X is mapped to a single point set in G.*

Definition 5.9. *Let $\mathscr{F}_1, \mathscr{F}_2$ be sheaves. A mapping $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ is called a morphism of sheaves if it is a morphism of presheaves.*

Definition 5.10. *A set of sheaves of $\mathscr{A}$ on the topological space $(X, \mathscr{T})$ is denoted as $\operatorname{Sh}_{\mathscr{A}}(X)$.*

Remark 5.5. *As in the case of presheaves, $\operatorname{Sh}_{\mathscr{A}}(X)$ can be regarded as a category with sheaf morphisms.*

Remark 5.6. $\operatorname{Sh}_{\mathscr{A}}(X)$ *is a full-subcategory of* $\operatorname{PreSh}_{\mathscr{A}}(X)$.

Notation 5.2. *In the case $\mathscr{A} = (\mathbf{Ab})$, we denote $\operatorname{Sh}_{(\mathbf{Ab})}(X) = \operatorname{Sh}(X)$.*

5.4 Stalks

Definition 5.11. *Suppose we have a topological space* $(X, \mathscr{T})$ *and a category* $\mathscr{A}$ *which admits direct limits. For a presheaf* $\mathscr{F} \in \mathrm{PreSh}_{\mathscr{A}}(X)$*, by inheriting the notations from Example 2.3, we define the stalk* $\mathscr{F}_x$ *of* $\mathscr{F}$ *at* $x \in X$ *by*

$$\mathscr{F}_x = \varinjlim_{U \in \mathscr{O}_x} \mathscr{F}(U) = \varinjlim_{x \in U, U \in \mathscr{T}} \mathscr{F}(U).$$

Example 5.10. *Let us assume that* $\mathscr{A} = (\mathbf{Ab})$ *in Definition 5.11. Then stalks and germs can be constructed explicitly in the following way.*

$$\mathscr{F}_x = \{(s, U) \mid U \in \mathscr{O}_x, s \in \mathscr{F}(U)\}/\sim,$$

where $\sim$ *is an equivalent relation such that for* $(s, U), (t, V)$,

$$(s, U) \sim (t, V) \text{ if there is } W \in \mathscr{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t).$$

Definition 5.12. *Inheriting the notations from Definition 5.11, suppose we have* $(f_U : \mathscr{F}(U) \to \mathscr{F}_x)_{U \in \mathscr{O}_x}$ *such that for* f_U, f_V *are compatible with* ρ_{UV}*. Then we define the germ of* $s \in \mathscr{F}(U)$ *to be* $s_x = f_U(s)$*. By the universal property of the direct limit, such* s_x *is unique up to images under isomorphisms.*

Example 5.11. *In the case of Remark 5.10, we have for each* $U \in \mathscr{T}$*,* $x \in U$*, and* $s \in \mathscr{F}(U)$*,*

$$s_x = \{(t, V) \mid \text{ There is } W \in \mathscr{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t)\}.$$

Remark 5.7. *In the above definition, if a category* $\mathscr{A}$ *admits products, we get a map*

$$(s \mapsto (s_x)_{x \in U})) : \mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x. \tag{5.1}$$

This is neither surjective nor injective in general.

Proposition 5.1. *Suppose in the definition of stalks,* $\mathscr{F}$ *is a sheaf. Then the map defined by Equation 5.1 is injective.*

Proof. We prove the case when $\mathscr{A} = (\mathbf{Ab})$.

Suppose $s \in \mathscr{F}(U)$ is such that $s_x = 0$ in $\mathscr{F}_x$ for all $x \in U$. Since for any restriction maps are group homomorphisms. We have that there is $V_x \in \mathscr{O}_x$ such that

$$V_x \subseteq U, \quad \rho_{UV_x}(s) = 0.$$

Therefore $\{V_x\}_{x \in U}$ is an open covering of U. Since $\mathscr{F}$ is a sheaf, we derive that $s = 0$ in $\mathscr{F}(U)$. ☐

Example 5.12. *Given* $(X, \mathscr{F})$*, a topological space and* G*, an abelian group. We will consider the constant presheaf* $\mathbb{G}$ *and the constant sheaf* $\underline{\mathbb{G}}$ *on* X*. For any open set* U *and* $x \in U$ *we have*

$$\mathbb{G}_x \cong \underline{\mathbb{G}}_x \cong G.$$

For any U, V open such that $V \subset U$ we have, $\rho_{UV} = \mathbf{id}_G$. Thus by the construction, for $x \in U, V$, $(s, U) \sim (t, V)$ then $x \in U \cap V$ and $\rho_{UU\cap V}(s) = s = t = \rho_{VU\cap V}(t)$. Therefore, we proved the claim.

Definition 5.13. *Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. Then we define*

$$\varphi_x(s_x) = (\varphi(s)_U)_x.$$

This defines a morphism of presheaves.

Remark 5.8. *Categorically, taking stalks is a functor for each $x \in X$. Suppose we have $\mathscr{F}, \mathscr{G} \in \mathrm{PreSh}_{\mathscr{A}}(X)$ and a morphism $\varphi : \mathscr{F} \to \mathscr{G}$,*

Proposition 5.2. *Let $\mathscr{F}, \mathscr{G} \in \mathrm{Sh}_{(\mathbf{Ab})}(X)$ Then for any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ we have*

$$\varphi = 0 \Leftrightarrow \forall x \in X, \varphi_x = 0$$

Proof. $\Rightarrow$ is trivial by its construction. We will prove $\Leftarrow$.

We first note that $\varphi = 0$ means that for any $U \in \mathscr{T}$, we have $\varphi_U \equiv 0$ as a group homomorphism. Let $U \in \mathscr{T}$ and $s \in \mathscr{F}(U)$. Then by the assumption and Proposition 5.1, we have proven the claim. □

5.5 Sheafification

Definition 5.14. *Let $\mathscr{F} \in \mathrm{PreSh}_{\mathscr{A}}(X)$. The sheafification of $\mathscr{F}$ is a presheaf $\mathscr{F}^+$ which is a set of all $(s_x)_{x\in U} \in \prod_{x\in U} \mathscr{F}_x$ such that for any $x \in U$ there is $x \in V_x \subset U$, such that there is $t \in \mathscr{F}(V_x)$ satisfying for any $y \in V_x$, $s_y = t_y$. We give them restrictions such that*

$$\mathscr{F}^+(U) \ni (s_x)_{x\in U} \mapsto (s_x)_{x\in V} \in \mathscr{F}^+(V).$$

Proposition 5.3. *Such $\mathscr{F}^+$ is indeed a sheaf.*

Proof. later □

Remark 5.9.

$$\mathscr{F} \mapsto \mathscr{F}^+ : \mathrm{PreSh}_{\mathscr{A}}(X) \to \mathrm{Sh}_{\mathscr{A}}(X)$$

is a functor. Indeed given $\varphi : \mathscr{F} \to \mathscr{G}$, a morphism of presheaves. We give

$$\varphi^+(U)((s_x)_{x\in U}) = (\varphi(s)_x)_{x\in U}.$$

later

Proposition 5.4. *A mapping $\varphi : \mathscr{F} \to \mathscr{F}^+$ such that for each $U \in \mathscr{T}$,*

$$\varphi_U : \mathscr{F}(U) \to \mathscr{F}^+(U), \quad \varphi(s) = (s_x)_{x\in U},$$

is a natural transformation thus a morphism of presheaves.

Proof. Later $\square$

Proposition 5.5. *For any open set $U \in \mathscr{T}$ and a section $s \in \mathscr{F}^+(U)$, there is an open covering $(U_i)_{i\in I}$ which satisfies that there is a sequence $(s_i)_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i)$ and for each i, the following hodls.*

$$\rho_{UU_i}(s) = s_i.$$

Proof. Later. $\square$

Proposition 5.6. *For each $x \in X$, there exists an isomorphism*

$$\mathscr{F}_x \cong (\mathscr{F}^+)_x,$$

as presheaves.

Proof. later $\square$

Proposition 5.7. *Let $(X, \mathscr{T})$ be a topological group and $\mathscr{F}$ be a presheaf of a category $\mathscr{A}$ on X. Suppose for a sheaf $\mathscr{G}$ of a category $\mathscr{A}$ on X, there exists a morphism $\varphi : \mathscr{F} \to \mathscr{G}$. Then there exists a unique morphism $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}$, such that*

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[d, "\varphi"'] & \mathscr{F}^+ \arrow[dl, dotted, "\exists!\varphi^+"] \\
\mathscr{G} &
\end{tikzcd}$$

is a commutative diagram.

Proof. Let $U \in \mathscr{T}$, then by Proposition 5.5, for any $s \in \mathscr{F}^+$, there exists an open covering $(U_i)_{i\in I}$ and $(s_i)_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i)$ such that $\rho_{UU_i}(s) = s_i$ for any $i \in I$. We define

$$t_i = \varphi(s_i) \in \mathscr{G}(U_i),$$

for each $i \in I$. Using the definition of natural transformation we derive that

$$\rho^{\mathscr{G}}_{UU_i\cap U_j}(t_i) = \varphi^{\mathscr{F}}_{U_i\cap U_j}(\rho_{UU_i\cap U_j}(s)) = \rho^{\mathscr{G}}_{UU_i\cap U_j}(t_j).$$

Thus we can glue $(t_i)_{i\in I}$ to a section $t \in \mathscr{G}(U)$.

We now define $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}$. Given $(s_x)_{x\in U}$ which is the germ of s,

$$\varphi^+_U((s_x)_{x\in U}) = t.$$

Such φ^+ is unique since $\mathscr{G}$ is a sheaf. $\square$

Corollary 5.1. *Let $i : \mathrm{Sh}_{\mathscr{A}}(X) \to \mathrm{PreSh}_{\mathscr{A}}(X)$ be a forgetful functor. Then we have*

$$\mathrm{PreSh}_{\mathscr{A}}(X)(\mathscr{F}, i(\mathscr{G})) \cong \mathrm{Sh}_{\mathscr{A}}(\mathscr{F}^+, \mathscr{G})$$

In other words, the sheafification is a left-adjoint functor of the inclusion map.

Proof. By Proposition 5.7, we define two maps Φ, Ψ such that

$$\begin{aligned}\Phi : \mathrm{PreSh}_{\mathscr{A}}(X)(\mathscr{F}, i(\mathscr{G})) &\to \mathrm{Sh}_{\mathscr{A}}(\mathscr{F}^{+}, \mathscr{G}),\\ \Phi(\varphi) &= \varphi^{+},\\ \Psi : \mathrm{Sh}_{\mathscr{A}}(\mathscr{F}^{+}, \mathscr{G}) &\to \mathrm{PreSh}_{\mathscr{A}}(X)(\mathscr{F}, i(\mathscr{G})),\\ \Psi(\varphi^{+}) &= \varphi.\end{aligned}$$

Then these two are inverses of each other. □

Proposition 5.8. *Let $X = \mathrm{Spec}(A)$ and $\mathscr{O}_X$ be the structure sheaf defined in Example 5.6. Then we have the following.*

1). For any $\mathfrak{p} = x \in X$, $(\mathscr{O}_X)_x \cong A_{\mathfrak{p}}$.

2). For any $a \in A$, $\mathscr{O}_X(D(a)) \cong A_a$.

Proof. For a given $U \subset X$ open and $\mathfrak{p} \subset A$, there is $a, b \in A$ such that for $V \subset U$ open and $s \in \mathscr{O}_X(U), s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$.

$$s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}$$

holds for any $\mathfrak{q} \in V$.

$$\begin{tikzcd} \mathscr{O}_X(U) \arrow[r] \arrow[d, "\rho_{UV}"'] & A_{\mathfrak{p}} \\ \mathscr{O}_X(V) \arrow[ur] & \end{tikzcd}$$

□

5.6 Morphisms in $\mathrm{PreSh}_{(\mathbf{Ab})}(X)$

Definition 5.15. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism of presheaves $\mathrm{PreSh}_{(\mathbf{Ab})}(X)$. Then we define the following.*

1). $\mathrm{Ker}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Ker}\, \varphi_U$,

2). $\mathrm{Im}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Im}\, \varphi_U$,

3). $\mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) = \mathrm{Coker}\, \varphi_U$.

Proposition 5.9. *Such $\mathrm{Ker}^{\mathbf{pre}}, \mathrm{Im}^{\mathbf{pre}}, \mathrm{Coker}^{\mathbf{pre}}$ are presheaves.*

Proof. For the case of kernels. Let $U, V \in \mathscr{T}$ and $V \subset U$. We define $\rho_U V : \mathrm{Ker}^{\mathbf{pre}}(\varphi)(U) \to \mathrm{Ker}^{\mathbf{pre}}(\varphi)(V)$ to be such that

$$\rho_U V(s) = \rho^{\mathscr{F}}(s).$$

Such construction is justified as the diagram below is commutative.

$$\begin{array}{ccccc}
\mathscr{F}(U) & \xrightarrow{\rho^{\mathscr{F}}_{UV}} & \mathscr{F}(V) & \xrightarrow{\rho^{\mathscr{F}}_{UV}} & \mathscr{F}(W) \\
\varphi_U \downarrow & & \downarrow \varphi_V & & \downarrow \varphi_W \\
\mathscr{G}(U) & \xrightarrow[\rho^{\mathscr{G}}_{UV}]{} & \mathscr{G}(V) & \xrightarrow[\rho^{\mathscr{G}}_{UV}]{} & \mathscr{F}(W)
\end{array}$$

Furthermore,

$$\rho_U W(s) = \rho^{\mathscr{F}}_{UV}(s) = \rho^{\mathscr{F}}_{VW} \circ \rho^{\mathscr{F}}_{UV}(s) = \rho_{VW} \circ \rho_{UV}(s).$$

Thus $\mathrm{Ker}^{\mathbf{pre}}(\varphi)(U)$ is a presheaf. $\square$

Corollary 5.2. *If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves. Then $\mathrm{Ker}^{\mathbf{pre}}$ is also a sheaf.*

Proof. Given $(s_i)_{i\in I} \in \prod_{i\in I} \mathrm{Ker}\,\varphi_{U_i}$ such that

$$\rho(s_i)_{U_i U_i \cap U_j} = \rho(s_j)_{U_j U_i \cap U_j}$$

for any $i, j \in I$. Then since $\mathscr{F}$ is a sheaf, we can glue $(s_i)_{i\in I}$ to $s \in \mathscr{F}(U)$. For such s we have

$$\rho^{\mathscr{G}}_{UU_i}(\varphi_U(s)) = (\varphi_{U_i}(\rho^{\mathscr{F}}_{UU_i}(s))) = \varphi_{UU_i}(s_i) = 0.$$

Therefore, since $\mathscr{G}$ is a sheaf, $\varphi_U(s) = 0$. $\square$

Remark 5.10. *Let $\varphi : \mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i), \varphi_1 : \prod_{i\in I} \mathscr{F}(U_i) \to \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j), \varphi_2 : \prod_{i\in I} \mathscr{F}(U_j) \to \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j)$. Then $\mathscr{F}$ is a sheaf if and only if*

$$\mathrm{Ker}(\varphi_1 \circ \varphi - \varphi_2 \circ \varphi) = \mathscr{F}(U),$$

holds for any open set U.

Remark 5.11. $\mathrm{Im}^{\mathbf{pre}}\,\varphi, \mathrm{Coker}^{\mathbf{pre}}\,\varphi$ *are not in general sheaves even tho $\varphi : \mathscr{F} \to \mathscr{G}$ is a homomorphism of sheaves.*

Example 5.13. *Let $X = \{x_1, x_2\}$ and we assign the discrete topology to it. Let G be an abelian group. We define a sheaf $\mathscr{F}, \mathscr{G} \in \mathrm{Sh}_{(\mathbf{Ab})}(X)$ by such that*

$$\mathscr{F}(U) = \mathscr{G}(U) = \begin{cases} G \times G \quad (U = X), \\ G \quad (|U| = 1), \\ 0 \quad (U = \emptyset). \end{cases}$$

Let us define a homomorphism of sheaves φ such that

$$\varphi_U = \begin{cases} \mathbf{id}_{G\times G} \quad (U = X) \\ 0 \quad (U \neq X). \end{cases}$$

Then we have

$$\mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) = \begin{cases} 0 & (U = X), \\ G & (U \neq X). \end{cases}$$

By 5.11, we observe that

$$\mathrm{Coker}^{\mathbf{pre}}(\varphi)(X) = G \times G / \mathbf{id}_{G \times G}(G \times G) = \{0\}.$$

However,

later.

Definition 5.16. *Given a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, we define the following.*

1). $\mathrm{Ker}(\varphi) = \mathrm{Ker}^{\mathbf{pre}}(\varphi)$,

2). $\mathrm{Im}(\varphi) = (\mathrm{Im}^{\mathbf{pre}}(\varphi))^+$,

3). $\mathrm{Coker}(\varphi) = (\mathrm{Coker}^{\mathbf{pre}}(\varphi))^+$.

Proposition 5.10 (Universal property of kernels)**.** *Given a sheaf homomorphism $\varphi : \mathscr{F} \to \mathscr{G}$. For any sheaf homomorphism $\alpha : \mathscr{H} \to \mathscr{F}$, $\varphi \circ \alpha = 0$ if and only if there is a unique $\psi : \mathscr{H} \to \mathrm{Ker}\,\varphi$ such that*

$$\begin{tikzcd}
& \mathscr{H} \arrow[dl, dotted, "\exists!\psi"'] \arrow[d, "\alpha"] \arrow[dr, "\varphi_0=0"] & \\
\mathrm{Ker}(\varphi) \arrow[r, hook] & \mathscr{F} \arrow[r, "\varphi"'] & \mathscr{G}
\end{tikzcd}$$

is a commutative diagram.

Proof. We argue by each open set of the space.

$$\begin{tikzcd}
& \mathscr{H}(U) \arrow[dl, dotted, "\exists!\psi_U"'] \arrow[d, "\alpha_U"] \arrow[dr, "(\varphi_0)_U=0"] & \\
\mathrm{Ker}(\varphi)(U) \arrow[r, hook] & \mathscr{F}(U) \arrow[r, "\varphi_U"'] & \mathscr{G}(U)
\end{tikzcd}$$

This is a universal property of the kernel in abelian groups. Thus the statement immediately follows from it. $\square$

Proposition 5.11 (Universal property of Cokernels)**.** *Given a sheaf homomorphism $\varphi : \mathscr{F} \to \mathscr{G}$. For any sheaf homomorphism $\alpha : \mathscr{G} \to \mathscr{H}$, $\alpha \circ \varphi = 0$ if and only if there is a unique $\psi : \mathrm{Coker}\,\varphi \to \mathscr{H}$ such that*

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"] \arrow[dr, "\varphi_0=0"'] & \mathscr{G} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \mathrm{Coker}(\varphi) \arrow[dl, dotted, "\exists!\psi"] \\
& \mathscr{H} &
\end{tikzcd}$$

is a commutative diagram.

Proof. We argue for each open set $U \subset X$.

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[dr, "(\varphi_0)_U=0"'] & \mathscr{G}(U) \arrow[r] \arrow[d, "\alpha_U"] & \mathrm{Coker}^{\mathbf{pre}}(\varphi)(U) \arrow[r] \arrow[dl, dotted, "\exists ! \psi_U^{\mathbf{pre}}"'] & \mathrm{Coker}(\varphi)(U) \arrow[dll, dotted, "\exists ! \psi_U"] \\
& \mathscr{H}(U) & &
\end{tikzcd}$$

By the universal property of Cokernels of abelian groups, there is a unique $\varphi^{\mathbf{pre}}$. By the universal property of the sheafification operator, we derive a unique ψ. ☐

Proposition 5.12. *Let $x \in X$, then we have the following.*

1). $\mathrm{Ker}(\varphi)_x = \mathrm{Ker}(\varphi_x)$,

2). $\mathrm{Im}(\varphi)_x = \mathrm{Im}(\varphi_x)$,

3). $\mathrm{Coker}(\varphi)_x = \mathrm{Coker}(\varphi_x)$.

Proof. By Definition, 5.13 ☐

Definition 5.17. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism. φ is called*

1). a monomorphism if any morphism of sheaves $\varphi_0 : \mathscr{H} \to \mathscr{F}$, $\varphi \circ \varphi_0 = 0$ if and only if $\varphi_0 = 0$,

Proposition 5.13. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of $(\mathbf{Ab})$. Then the following statements are equivalent.*

i). φ is a monomorphism.

ii). $\mathrm{Ker}\,\varphi = 0$.

iii). For any open set $U \subset X$, φ_U is injective.

iv). For any $x \in X$, $\varphi_x :\to \mathscr{F}_x \to \mathscr{G}_x$ is injective.

Proof. Here, I put the procedure of the proof.

$$\begin{tikzcd}
i) \arrow[d] & iv) \arrow[d, leftrightarrow] \\
ii) \arrow[r, leftrightarrow] & iii) \arrow[ul]
\end{tikzcd}$$

$i) \Rightarrow ii)$,

$$\begin{tikzcd}
\mathrm{Ker}(\varphi) \arrow[d, "\varphi_0"'] \arrow[dr, "0"] & \\
\mathscr{F} \arrow[r, "\varphi"'] & \mathscr{G}
\end{tikzcd}$$

Where $\varphi_0(U)$ is an inclusion map of abelian groups.

$ii) \Leftrightarrow iii)$,

$$\operatorname{Ker}\varphi = 0 \Leftrightarrow \forall U \in \mathscr{T}, \operatorname{Ker}\varphi(U) = 0 \Leftrightarrow \varphi_U \text{ is injective.}$$

$iii) \Rightarrow iv)$, Fix $x \in X$.

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\varphi_U} \mathscr{G}(U)$$

is an exact sequence as φ_U is injective for any $U \subset X$ open. Since $\varinjlim$ is left-exact we obtain,

$$0 \longrightarrow \mathscr{F}_x \xrightarrow{\varphi_x} \mathscr{G}_x$$

is also an exact sequence. □

Proposition 5.14. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism in* $\mathrm{Sh}(X)$*. Then the following are equivalent.*

1). φ is an epimorphism (for any $\varphi_1, \varphi_2 : \mathscr{H} \to \mathscr{F}$, such that $\varphi \circ \varphi_1 = \varphi \circ \varphi_2$ implies $\varphi_1 = \varphi_2$).

2). $\operatorname{Coker}\varphi = 0$.

3). For any open set $U \subset X$,

4). For any $x \in X$, $\operatorname{Coker}\varphi_x = 0$, *(in other words, φ_x is a surjection).*

Proof. Recall the definition of epimorphisms is such that $\varphi : \mathscr{F} \to \mathscr{G}$ is an epimorphism if for any morphism $\psi : \mathscr{G} \to \mathscr{G}_0$, we have,

$$\psi \circ \varphi = 0 \Rightarrow \psi = 0.$$

$i) \Rightarrow iv)$. Suppose φ is an epimorphism, then we have

$$\begin{tikzcd}
\mathscr{F} \arrow[d, "\varphi"'] \arrow[dr, "0"] \arrow[drr, bend left, "0"] & & \\
\mathscr{G} \arrow[r, "\pi"] \arrow[rr, bend right, "\psi"'] & \operatorname{Coker}^{\mathbf{pre}}\varphi \arrow[r, "(-)^+"] & \operatorname{Coker}\varphi
\end{tikzcd}$$

By the assumption $\psi = 0$.

Let $\mathscr{O}_x = \{U \in \mathscr{T} \mid x \in U\}$. We consider an exact sequence,

$$0 \longrightarrow \operatorname{Ker}(\varphi_U) \hookrightarrow \mathscr{F}(U) \xrightarrow{\varphi} \mathscr{G}(U) \xrightarrow{\pi} \operatorname{Coker}(\varphi_U) \longrightarrow 0,$$

for each $U \in \mathscr{O}_x$. By Proposition 2.2,

$$0 \longrightarrow \mathrm{Ker}(\varphi)_x \hookrightarrow \mathscr{F}_x \xrightarrow{\varphi_x} \mathscr{G}_x \xrightarrow{\pi_x} \mathrm{Coker}(\varphi)_x \longrightarrow 0$$

is also exact. Thus we conclude

$$\mathrm{Coker}^{pre}(\varphi)_x = \mathrm{Coker}(\varphi_x).$$

And we conclude that φ_x is surjective by the exactness of the sequence.

iv) $\Rightarrow$ *ii)*. Assume For each $x \in X$, $\mathrm{Coker}(\varphi_x) = 0$. By applying Proposition. 5.2 to $\mathbf{id} : \mathscr{F} \to \mathscr{F}$, we obtain

$$\mathscr{F} = 0 \Leftrightarrow \forall x \in X, \mathscr{F}_x = 0.$$

Apply this to $\mathrm{Coker}\,\varphi$, we derive that

$$\mathrm{Coker}\,\varphi = 0.$$

iv) $\Rightarrow$ *i)*. Assume $\mathrm{Coker}(\varphi_x) = 0$ for any $x \in X$. Consider a commutative diagram of sheaves

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[dr, "0"'] & \mathscr{G} \arrow[d, "\psi"] \\ & \mathscr{G}_0 \end{tikzcd}$$

By assumption $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is a surjection. Thus $\psi_x = 0$ for any $x \in X$ which is equivalent to $\psi = 0$.

ii) $\Rightarrow$ *i)*. Suppose $\mathrm{Coker}\,\varphi = 0$ if and only if $\mathrm{Coker}(\varphi)_x = \mathrm{Coker}(\varphi_x) = 0$ for any $x \in X$.

iii) $\Rightarrow$ *iv)*. Assume $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is surjective for any $U \subset X$ open. By Proposition. 2.2, we conclude that

$$\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$$

is also surjective. □

Corollary 5.3. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then the following statements are equivalent.*

1). φ is an isomorphism.

2). For all $x \in X$, φ_x is an isomorphism.

Proof. □

6 Scheme Theory

6.1 Ringed Spaces

Definition 6.1. *Let $(X, \mathscr{T})$ be a topological space. A ringed space is a sheaf $\mathscr{O}_X$ of rings on X.*

Definition 6.2. *A morphism of ringed spaces between $(X, \mathscr{O}_X), (Y, \mathscr{O}_Y)$ is a tuple $(f, f^\#)$ where $f : X \to Y$ is a continuous map and $f : \mathscr{O}_Y \to f_* \mathscr{O}_X$ is a morphism of sheaves of rings.*

Example 6.1. *Let $(X, \mathscr{T})$ be a topological space. The sheaf of continuous functions $\mathscr{C}_X$ is a ringed space and any continuous map $f : X \to Y$ defines a morphism of ringed spaces.*

Example 6.2. *Let X is a differentiable manifold then the differentiable functions $\mathscr{C}_X^{\mathbf{diff}}$ is a ringed space. A morphism of ringed spaces $f : X \to Y$, for this case must satisfy the following condition.*

Example 6.3. *Let $X \subseteq \mathbb{C}^n$ be open subset. A sheaf of holomorphic functions $\mathscr{O}_X$ over X is a ringed space. And a morphism of such ringed spaces must be a holomorphic functions*

Example 6.4. *Given the Zariski topology on $X = k^n$ and the sheaf $\mathscr{O}_X(U) = \{f : U \to k \mid f \text{ is regular }\}$, $(X, \mathscr{O}_X)$ is a ringed space.*

Definition 6.3. *By Remark 3.1, the sheaf of regular functions $\mathscr{O}_X$ is contained in the sheaf of continuous functions $\mathscr{C}_X$. Given two Zariski topologies X, Y, and a continuous function $f : X \to Y$, f is said to be regular if for any regular function $g : U \to k$ for an open set $U \subseteq Y$, $g \circ f : f^{-1}(U) \to k$ is also regular. In other words, f is said to be regular if it defines a morphism of ringed spaces between two ringed spaces of regular functions.*

Definition 6.4. *A locally ringed space is a ringed space $(X, \mathscr{O}_X)$ such that for any $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring.*

Example 6.5. *A sheaf of continuous functions on a topological space X is a locally ringed space. Indeed, for each $x \in X$ and the stalk $\mathscr{C}_{X,x}$, the ideal*

$$\mathfrak{m}_x = \{(f : U \to \mathbb{R}, U) \mid f(x) = 0\}$$

is a unique maximal ideal. In order to prove this, we recall that an ideal $\mathfrak{m}$ is a unique maximal ideal if any element not in $\mathfrak{m}$ is a unit.

For each $(f : U \to \mathbb{R}, U) \in \mathscr{C}_{X,x}$, $f(x) \neq 0$ implies that there exists a neighborhood $V \subset U$ such that $f(x) \neq 0$ for any $x \in V$. Thus $(f|_V : V \to \mathbb{R}, V)$ is invertible, therefore a unit.

Example 6.6. *In similar manner, the following are also locally ringed spaces.*

1. *X is a differentiable manifold and $(X, \mathscr{C}_X^{\mathbf{diff}})$.*

2. $X \subseteq \mathbb{C}^n$ *be an open set, and* $(X, \mathscr{O}_X)$ *be a sheaf of holomorphic functions.*

3. *A sheaf of regular functions on* $X = k^n$.

Definition 6.5. *A morphism* $(f, f^{\#}) : (X, \mathscr{O}_X \to (Y, \mathscr{O}_Y)$ *between ringed spaces is a morphism of locally ringed space if* $f^{\#}$ *is local as a ring homomorphism.*

Example 6.7. *Let* A *be a commutative ring and consider the Zariski topology on* $X = \mathrm{Spec}(A)$ *and the structure sheaf* $(X, \mathscr{O}_X)$. *We have proven that*

$$\mathfrak{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}.$$

Therefore, $(X, \mathscr{O}_X)$ *is a locally ringed space and for any ring homomorphism* $\phi : A \to B$, *it induces a morphism of locally ringed spaces* $(f, f^{\#}) : (\mathrm{Spec}(B), \mathscr{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)})$ *such that*

$$\mathfrak{q} \in \mathrm{Spec}(B), f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q}) \in \mathrm{Spec}(A).$$

This is indeed a morphism of locally ringed spaces.

Proposition 6.1. *Let* A, B *be commutative rings. Then the map* $\phi \mapsto (f, f^{\#})$ *is a bijection between*

$$\mathrm{Hom}(A, B) \leftrightarrow \mathrm{Hom}_{\mathbf{loc}}(\mathrm{Spec}(B), \mathscr{O}_{\mathrm{Spec}(B)}), (\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)}))$$

Proof. $\square$

Definition 6.6. *A category of ringed spaces is denoted by* (**RingedSpaces**) *with morphisms* $(f, f^{\#})$ *morphisms of ringed spaces.*

Definition 6.7. *A category of ringed spaces is denoted by* (**RingedSpaces**) *with morphisms* $(f, f^{\#})$ *morphisms of locally ringed spaces.*

Remark 6.1. *A composition of two morphisms locally ringed space is indeed a morphism of locally ringed spaces thus the above construction is justified.*

Definition 6.8. *Two locally ringed spaces* $(X, \mathscr{O}_X)$ *and* $(Y, \mathscr{O}_Y)$ *are isomorphic if there exists morphisms* $(f, f^{\#}) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ *and* $(g, g^{\#}) : (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ *such that* f *and* g *are inverses of each other. (ie. there exists a morphism of locally ringed spaces* $(f, f^{\#})$ *where* f *is a homeomorphism).*

Example 6.8. *(A morphism of locally ringed spaces induced by homeomorphism but not an isomorphism of locally ringed spaces).*

Let $X = \mathbb{R}^n$ *and consider the sheaf of continuous functionals* $\mathscr{C}_X$ *and the sheaf of smooth functionals* $\mathscr{C}_X^{\mathbf{diff}}$. *Furthermore, we consider* $f = id_X$ *then* $f^{\#}$ *is an inclusion as smooth functions are continuous. However,* $(f, f^{\#})$ *is not an isomorphism of locally ringed spaces.*

Example 6.9. *Let us consider $X = \mathbb{C}^n$ and the sheaf of holomorphic functions $\mathscr{O}$ on X and the structure sheaf $\mathscr{O}_X$. Then consider the morphism of locally ringed spaces $(f, f^{\#})$ by the identity map. However, f is not continuous as the topology defined on the image is the Zariski topology.*

Definition 6.9. *Let $X = \mathbb{C}^n$ and $Y = \mathrm{MaxSpec}(\mathbb{C}[x_1, \cdots, x_n])$. Let $f : X \to Y$ be such that*

$$f(z_1, \cdots, z_n) = (x_1 - z_1, \cdots, x_n - z_n).$$

This is a bijection. Furthermore, f is continuous because polynomials are continuous functions.

We define $f^{\#}$ to be

6.2 Schemes

Definition 6.10. *An affine scheme is a locally ringed space $(X, \mathscr{O}_X)$ which is isomorphic to a structure sheaf $(\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)})$ for some commutative ring A.*

Example 6.10. *We consider the Zariski topology on $\mathrm{Spec}(\mathbb{Z})$ and a sheaf $\mathscr{O}$ such that*

$$\mathscr{O}(D(\mathfrak{a})) = \mathbb{Z}_{\mathfrak{a}}.$$

is an affine scheme.

Example 6.11. *Let k be a field. Then $\mathrm{Spec}(k)$ is a single point set. And we consider the sheaf $\mathscr{O}$ such that $\mathscr{O}(\mathrm{Spec}(k)) = k$.*

Definition 6.11. *For a field A be a commutative ring and n a natural number, we define*

$$\mathbb{A}^n_A = (\mathrm{Spec}(A[x_1, \cdots, x_n]), \mathscr{O}).$$

Example 6.12. *Let A be a discrete valuation ring in other words $k[t]_{(t)}$.*

Example 6.13. *Let k be a field and $A = k[x]/(x^2)$. Then $\mathrm{Spec}(A) = \{(x)\}$. Thus a single point set. However, this is not isomorphic to $(\mathrm{Spec}(k), \mathscr{O})$ introduced in Example 6.11.*

Definition 6.12. *A scheme is a ringed space $(X, \mathscr{O}_X)$ which is locally isomorphic t an affine scheme. In other words, for any $x \in X$, there is a neighborhood U of X such that there exists a commutative ring A and $(U, \mathscr{O}|_U)$ is isomorphic to $(\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)})$.*

Definition 6.13. *A category of affine schemes is* (**AffSch**) *where*

i). $\mathbf{ob}(\mathbf{AffSch}) = \{(\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)}) \,|\, A$ *is a commutative ring and* $\mathscr{O}_{\mathrm{Spec}(A)}$ *is a structure sheaf*$\}$.

ii). $(\mathbf{AffSch})((\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)}), (\mathrm{Spec}(B), \mathscr{O}_{\mathrm{Spec}(B)})) = \{$ *morphisms of locally ringed spaces*$\}$.

Definition 6.14. *A category of schemes is* (**Sch**) *where*

i). $\mathbf{ob}(\mathbf{Sch}) = \{(X, \mathscr{O}_X) \mid \textit{Schemes}\}$.

ii). $(\mathbf{AffSch})((X, \mathscr{O}_X), (Y, \mathscr{O}_Y)) = \{$ *morphisms of locally ringed spaces*$\}$.

Remark 6.2. *We have the inclusion relations*

$$(\mathbf{AffSch}) \subset (\mathbf{Sch}) \subset (\mathbf{LocallyRingedSpaces})$$

which are all fullsubcategories however,

$$(\mathbf{LocallyRingedSpaces}) \subset (\mathbf{RingedSpaces})$$

is not a full subcategory

6.3 Connection with Classical Algebraic Geometry

Proposition 6.2. *Let X be an affine variety. The regular functions $\mathscr{O}_X(U)$*

$$\mathscr{O}_X(U) = \{h : U \to k \mid h \textit{ is a regular function.}\}.$$

defined on open subset U of X form a sheaf. Furthermore, it is a locally ringed space.

Proof. □

Proposition 6.3. *Let X be an affine variety and $Y = A(X)$ be a coordinate ring. Let us consider the sheaf of regular functions $(X, \mathscr{O}_X)$ and an affine scheme $(Y, \mathscr{O}_Y)$. There exists a natural morphism of locally ringed spaces $(f, f^{\#}) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$.*

Proof. Notice that we have the following isomorphisms.

$$X \cong \operatorname{MaxSpec}(A(X)), \quad k^n \cong \operatorname{MaxSpec}(k[x_1, \cdots, x_n]).$$

For any maximal ideal $\mathfrak{m} \subset k[x_1, \cdots, x_n]$,

$$I(X) \subseteq \mathfrak{m} = (x_1 - a_1, \cdots, x_n - a_n) \Leftrightarrow \forall f \in I(X), f(a_1, \cdots, a_n) = 0.$$

Let $\pi : Y \to X$ to be the canonical map by $I(X)$, then the map $f : X \to Y, (\mathfrak{m}) = \pi^{-1}(\mathfrak{m})$ is an inclusion. Then f is continuous.

Let us define $f^{\#} : \mathscr{O}_Y \to f_*(\mathscr{O}_X)$. For an open set $U \subseteq Y$, we have

$$(s : U \to \coprod_{\mathfrak{p} \in U} A(x)_{\mathfrak{p}}) \mapsto (s : U \to \coprod_{\mathfrak{m} \in U \cap \operatorname{MaxSpec} A(x)} A(x)_{\mathfrak{m}}).$$

By Lemma 3.1 and applying canonical maps $\pi_{\mathfrak{m}} : A(X)_{\mathfrak{m}} \to A(X)_{\mathfrak{m}}/\mathfrak{m}A(X)_{\mathfrak{m}}$ locally, we get

$$s : U \to \coprod_{\mathfrak{m} \in U \cap \operatorname{MaxSpec} A(x)} \to \coprod_{\mathfrak{m} \in U \cap \operatorname{MaxSpec} A(x)} A(X)_{\mathfrak{m}}/\mathfrak{m}A(X)_{\mathfrak{m}} = k.$$

Thus we obtained a map $s : U \to k$. Locally, we have

$$s = \frac{g_1 + I(X)}{g_2 + I(X)},$$

for $g_1 + I(X), g_2 + I(X) \in A(X)$. We conclude, locally

$$t = \frac{g_1}{g_2}.$$

We now claim that $(f, f^{\#})$ is a local morphism of ringed spaces. By the correspondence of a maximal ideal $\mathfrak{m}$ of $k[x_1, \cdots, x_n]$ and a point $(a_1, \cdots, a_n)$, we have the isomorphism

$$\mathscr{O}_{X,\mathfrak{m}} \xrightarrow{\sim} \mathscr{O}_{Y,\mathfrak{m}} = A(X)_{\mathfrak{m}}.$$

$\square$

Remark 6.3. *Since X is an algebraic variety, there is a prime ideal $\mathfrak{p}$ of $k[x_1, \cdots, x_n]$ such that*

$$X = V(\mathfrak{p}).$$

Let us define $(Y', \mathscr{O}_{Y'}) = (\operatorname{Spec}(k[x_1, \cdots, x_n]), \mathscr{O})$, where $I(X) = \mathfrak{a}$. Since k is field, $k[x_1, \cdots, x_, n]$ is Noetherian, thus the primary decomposition exists for any ideal. Thus there is a bijection between

$$\operatorname{Spec}(k[x_1, \cdots, x_n]/\mathfrak{a}) \leftrightarrow \operatorname{Spec}(A(X)).$$

Example 6.14. *Let K be any field and $A = k[x]/(x^2)$. A is called the ring of dual numbers. Observe that*

$$(\operatorname{Spec} k, \mathscr{O}_{\operatorname{Spec} k}), (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}),$$

both consist of single points. Let us define $(f, f^{\#})$.