Representation Theory 1 V4A1 Sheet 6 Exercise 30,32

So Murata

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Let $f : X \to k$ be a regular function. Since $\mathbb{P}^1_k$ is irreducible, any open sets in X intersects with any other open sets. Therefore, let $x, y \in X$ and U_x, U_y be open sets such that

$$f|_{U_x} = \frac{p_x}{q_x}, f|_{U_y} = \frac{p_y}{q_y}.$$

Then $U_x \cap U_y \neq \emptyset$, thus two rational functions agree on an open set, thus we conclude

$$f = \frac{p}{q}.$$

Since k is algebraically closed, q has a pole unless it is a constant. If the former is the case f is not regular thus q is a constant. Since this is a regular function on a projective line, p and q have the same degree. We conclude f is constant.

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Since Noetherian scheme is quasi compact, there exists a open covering consisting only of distinguished open sets $\{X_{a_i}\}_{i=1,\cdots,n}$. This follows that X is locally Noetherian. Let f_i be a restriction of f to X_{a_i}. We have that

$$X_a \cap X_{a_i} = X_{aa_i}.$$

Since the restriction of f to X_a is 0, the restriction of f to X_{aa_i} is also 0. Therefore for some $n_i \in \mathbb{N}$, we have

$$a^{n_i} f_i = 0$$

in the localization map. Since we have only finitely many a_i, we conclude that for some n

$$a^n f = 0$$

together with the sheaf property.

(ii)

We again argue by the finite cover by distinguished open sets and the sheaf property. Let $h \in \Gamma(X_a, \mathscr{O}_{X_a})$. By restricting h to some X_{a_i}, we get

$$h_i = a^{-n_i} s_i$$

for some $s_i \in \mathscr{O}_{X_{a_i}}(X_{a_i})$. Argue again by the finiteness, we conclude there is n such that

$$a^n h_i \in \mathscr{O}_{X_{a_i}}(X_{a_i})$$

for all i. Each s_i can be glued since for each i, j

$$s_i - s_j = 0$$

in $X_{aa_ia_j}$. This means that

$$f^{m_{ij}}(s_i - s_j) = 0$$

for sufficiently large enough m_{ij} in $X_{a_ia_j}$. Again by the finiteness argument, we conclude that $\{s_i\}$ can be glued to some t in the global section. And the restriction of t to X_a is $a^n h$ for some n.

(iii)

Let $f, g \in A$ be such that the restriction of f and g are equal in X_a. Then

$$a^n(f - g) = 0.$$

But this means that f and g are the same in the localization A_a. On the other hand for any $h \in \mathscr{O}_{X_a}(X_a)$ is a restriction of some $t \in A$. Thus $\mathscr{O}_{X_a}(X_a)$ satisfies the properties of localization. By the uniqueness of localizations up to isomorphisms, we conclude the statement.