Algebraic Geometry 1

So Murata

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By taking the global section of

$$\tilde{M} \to \mathscr{F},$$

it is clear that the morphism

$$\alpha : \mathrm{Hom}_{\mathcal{O}_x}(\tilde{M}, \mathscr{F}) \to \mathrm{Hom}_A(M, \Gamma(X, \mathscr{F}))$$

exists. Let $\varphi : M \to \mathscr{F}(X)$ and for $X = \bigcup D(a)$, we define

$$\varphi_a : \tilde{M}(D(a)) = M_a \to \mathscr{F}(D(f)).,\ \frac{m}{a^i} \mapsto \frac{\varphi(m)|_{D(a)}}{a^i}.$$

Since $\mathscr{F}(D(a))$ i an A_a-module and $\frac{1}{a^i} \in A_a$, we have

$$\frac{\varphi(m)|_{D(a)}}{a^i} \in \mathscr{F}(D(a)).$$

In particular $\varphi_a(\frac{m}{1}) = \varphi(m)|_{D(a)}$.

By $D(a) \cap D(b) = D(ab)$ and the sheaf property if we have

$$(\varphi_a)_{ab} = (\varphi_b)_{ab},$$

then we can glue (φ_a) to $\varphi : \tilde{M} \to \mathscr{F}$.

$$\begin{aligned}
(\varphi_a)_{ab} : (M_a)_{ab} = M_{ab} \in \frac{m}{(ab)^i)} = \frac{\frac{1}{m}}{(ab)^i)} \\
&\mapsto \frac{\varphi_a(\frac{m}{1})|_{D(ab)}}{(ab)^i} \\
&= \frac{(\varphi(m)|_{D(a)})|_{D(ab)}}{(ab)^i} \\
&= \frac{\varphi(m)|_{D(ab)}}{(ab)^i} \in \mathscr{D}().
\end{aligned}$$

The last equality follows from the composition of restrictions. Therefore

$$(\varphi_a)_{ab}\left(\frac{m}{(ab)^i}\right) = \frac{\varphi(m)|_{D(ab)}}{(ab)^i} = (\varphi_b)_{ab}\left(\frac{m}{(ab)^i}\right).$$

Thus we can glue (φ_a) to get $\varphi : \tilde{M} \to \mathscr{F}$. Thus there exists

$$\beta : \mathrm{Hom}_A(M, \Gamma(X, \mathscr{F})) \to \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathscr{F}).$$

By the construction of $\beta(\varphi)$, taking the global section of it we get

$$\varphi : M \to \mathscr{F}(X).$$

Thus we get $\alpha\beta(\varphi) = \varphi$. On the other hands, by the uniqueness of glueing process, we have

$$\beta\alpha = \mathrm{id}_{\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathscr{F})}.$$

In conclusion, we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathscr{F}) \cong \mathrm{Hom}_A(M, \Gamma(X, \mathscr{F})).$$