

Algebraic Geometry 1

So Murata

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1 Preliminaries

1.1 Topology

Definition 1.1. Let (X, \mathcal{T}) be a topological space. A subset A of X is said to be connected if for any $U, V \in \mathcal{T}$, $U \cap V = \emptyset$, $U \cup V \supset A$ then A is fully contained in one of U, V .

Definition 1.2. A connected component of a topological space is a maximal connected subset of a space.

Proposition 1.1. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological space and $f : X \rightarrow Y$ be a continuous function. Then for any connected subset A of X , $f(A)$ is connected in Y .

Proof.

$$\begin{aligned} U, V \in \mathcal{T}_Y, U \cup V \supset f(A), U \cap V = \emptyset, \\ \Rightarrow f^{-1}(U), f^{-1}(V) \in \mathcal{T}_X, \\ f^{-1}(U) \cup f^{-1}(V) \supset A, \\ f^{-1}(U) \cap f^{-1}(V) = \emptyset, \\ \Rightarrow f^{-1}(U) \supset A \vee f^{-1}(V) \supset A, \\ \Rightarrow U \supset f(A) \vee V \supset f(A). \end{aligned}$$

□

1.2 Category theory

Definition 1.3. A category \mathcal{A} consists of

- a collection $\text{ob}(\mathcal{A})$ of objects;
 - for each $A, B \in \text{ob}(\mathcal{A})$, a collection $\mathcal{A}(A, B)$ of morphisms from A to B ;
- such that

- i). for each $A \in \text{ob}(\mathcal{A})$, the identity $1_A \in \mathcal{A}(A, A)$;
- ii). the composition $\mathcal{A}(B, C) \times \mathcal{A}(A, B) \ni (g, f) \mapsto g \circ f \in \mathcal{A}(A, C)$ is well-defined;

and they satisfy the following axioms

I). Associativity : $f \in \mathcal{A}(A, B), g \in \mathcal{A}(B, C), h \in \mathcal{A}(C, D), (h \circ g) \circ f = h \circ (g \circ f)$.

II). Identity laws : $f \in \mathcal{A}(A, B)$ then $f \circ 1_A = 1_B \circ f$.

Definition 1.4. Let \mathcal{A} be a category. A terminal object $T \in \text{ob}(\mathcal{A})$ is an object such that for any $A \in \text{ob}(\mathcal{A})$, $\mathcal{A}(A, T)$ is a single element set.

Definition 1.5. Given two categories \mathcal{A}, \mathcal{B} , we say \mathcal{A} is a full-subcategory of \mathcal{B} if

- i). $\mathcal{A} \subset \mathcal{B}$,
- ii). $\text{ob}(\mathcal{A}) = \text{ob}(\mathcal{B})$.

Notation 1.1. Here we give notations to some important categories.

- (Sets) : A category of sets equipped with set theoretic functions.
- (Ab) : A category of abelian groups with group homomorphisms.

Example 1.1. Given a partially ordered set (X, \leq) . This can be encoded to a category \mathcal{O} by

- i). $\text{ob}(\mathcal{O}) = X$,
- ii). For $x, y \in X, x \leq y \Rightarrow \mathcal{O}(x, y) = \{*\}$ otherwise the morphisms between x, y is an empty set.

Definition 1.6. A opposite/dual category of a category \mathcal{A} is \mathcal{A}^{op} such that

- i). $\text{ob}(\mathcal{A}^{\text{op}}) = \text{ob}(\mathcal{A})$,
- ii). $\mathcal{A}^{\text{op}}(B, A) = \mathcal{A}(A, B)$.

Definition 1.7. Let \mathcal{A} be a category and $\varphi_1, \varphi_2 \in \mathcal{A}(M, N)$. A morphism $\varphi : K \rightarrow M$ is called an equalizer of (φ_1, φ_2) if for any morphism $\psi : P \rightarrow M$ such that $\varphi_1 \circ \psi = \varphi_2 \circ \psi$, there is a unique morphism $\tilde{\psi} : P \rightarrow K$ such that $\varphi \circ \tilde{\psi} = \psi$.

Proposition 1.2. If an equalizer exists then it is unique up to unique isomorphism.

Proof. Suppose $\varphi : K \rightarrow M, \psi : L \rightarrow M$ be equalizers of (φ_1, φ_2) . Then we have

$$\varphi \circ \tilde{\psi} = \psi, \quad \psi \circ \tilde{\varphi} = \varphi$$

By the uniqueness, we have $\tilde{\varphi} \circ \tilde{\psi} = 1_L, \tilde{\psi} \circ \tilde{\varphi} = 1_K$. □

Definition 1.8. Let \mathcal{A}, \mathcal{B} be categories. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is a function such that for each $f \in \mathcal{A}(A, A')$, $F(f) : F(A) \rightarrow F(A')$. In other words, $f \mapsto F(f) : \mathcal{A}(A, A') \rightarrow \mathcal{B}(F(A), F(A'))$. Furthermore, F satisfies the following axioms.

I). $F(f' \circ f) = F(f') \circ F(f)$ whenever $f : A \rightarrow A', f' : A' \rightarrow A''$ in \mathcal{A} ,

II). $F(1_A) = 1_{F(A)}$ whenever $A \in \mathcal{A}$.

Definition 1.9. Let F, G be functors between two categories \mathcal{A}, \mathcal{B} . A natural transformation $\alpha : F \rightarrow G$ is a family $(\alpha_A : F(A) \rightarrow G(A))_{A \in \mathcal{A}}$ such that

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ \alpha_A \downarrow & & \downarrow \alpha_{A'} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

is a commutative diagram. Each α_A is called a component of α .

1.3 Zariski Topology

Definition 1.10. Let A be a ring. Then we define $\text{Spec}(A) = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal}\}$

Definition 1.11. Let A be a ring and $\mathfrak{a} \subset A$ be an ideal $V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{a} \subset \mathfrak{p}\}$ and $D(\mathfrak{a}) = \text{Spec}(A) - V(\mathfrak{a})$.

Definition 1.12. The Zariski topology on a commutative ring A is a topology generated by closed sets $\{V(\mathfrak{a})\}_{\mathfrak{a} \subset A}$.

Theorem 1.1. Let $B = \{D(\mathfrak{a}) \mid \mathfrak{a} \subset A \text{ is an ideal}\}$. Then B is a basis for the Zariski topology.

2 Sheaf theory

2.1 Presheaves

Definition 2.1. Let (X, \mathcal{T}) be a topological space. We define the presheaf \mathcal{F} of a category \mathcal{A} on X such that

- $U \in \mathcal{T}, \mathcal{F}(U) \in \text{ob}(\mathcal{A})$,
- $U, V \in \mathcal{T}, V \subset U \Rightarrow$ there exists a map $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$

such that

i). For any $U \in \mathcal{T}, \rho_{UU} = 1_{\mathcal{F}(U)}$.

ii). $U, V, W \in \mathcal{T}, W \subset V \subset U \rightarrow \rho_{UV} = \rho_{VW} \circ \rho_{UW}$.

Remark 2.1. In the case $\mathcal{A} = (\mathbf{Sets}), (\mathbf{Ab}), \mathcal{F}(\emptyset) = \emptyset, \{1\}$, respectively.

Definition 2.2. An element of $\mathcal{F}(U)$ is called a local section of \mathcal{F} and $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ is called the space of sections over U . In particular $\Gamma(X, \mathcal{F})$ is called the space of global sections of \mathcal{F} .

Definition 2.3. Let (X, \mathcal{T}) be a topological space and \mathcal{F} be a presheaf of a category \mathcal{A} on X . Suppose we have two open sets $U, V \in \mathcal{T}$ such that $V \subset U$. Then for any section $s \in \mathcal{F}(U)$, $s|_V = \rho_{UV}(s)$ is called the restriction of s to V .

Example 2.1. Let (X, \mathcal{T}) be a topological space. We have a presheaf of continuous functions $\mathcal{C}_X(U) = \mathcal{C}^0(U, \mathbb{R})$. This is indeed a presheaf with restriction maps $\rho_{UV} : \mathcal{C}_X(U) \rightarrow \mathcal{C}_X(V)$. (Explicitly, $\rho_{UV}(f) = f \circ i_V$ where i_V is an inclusion map.) We note that we can introduce operations $+$, \cdot to endow some algebraic structures (groups, rings, ...) on \mathbb{R} .

Example 2.2. Let (X, \mathcal{T}) be a topological space and suppose we have presheaves

- $\mathcal{C}_X^{\text{diff}}(U) = \{f : U \rightarrow \mathbb{R} \mid f \text{ is differentiable}\}.$

Then there is an inclusion relation $\mathcal{C}_X^{\text{diff}}(U) \subseteq \mathcal{C}_X(U)$ and this defines a presheaf.

Example 2.3. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. Define a presheaf on X by

$$U \in \mathcal{T}_X, \mathcal{F}(U) = \mathcal{C}^0(X, Y).$$

And like the previous example, we define $\rho_{UV}(f) = f|_V$ for $U, V \in \mathcal{T}_X, V \subset U$. the restriction of f to V .

But this is a presheaf only of a set.

Example 2.4. Let (X, \mathcal{T}) be a topological space and G be an abelian group. The constant presheaf \mathbb{G} is such that

$$U \in \mathcal{T}, \mathbb{G}(U) = G,$$

with $\rho_U V = \text{id}_G$ for any $U, V \in \mathcal{T}, V \subset U$.

2.2 Presheaves as Categories

Definition 2.4. Let (X, \mathcal{T}) be a topological space then (\mathbf{Ouv}_X) is the category such that its objects are the open sets of X and for any $U, V \in \mathcal{T}$ we have

$$\mathbf{Ouv}_X(U, V) = \begin{cases} \emptyset & (V \not\subset U), \\ i_V & (V \subset U). \end{cases}$$

Definition 2.5. Let (X, \mathcal{T}) be a topological space and \mathcal{A} be a category. A presheaf of \mathcal{A} on X is a functor $F : \mathbf{Ouv}_X \rightarrow \mathcal{A}$.

Example 2.5. For \mathbf{Ouv}_X , we can define a presheaf of F to be

$$\text{ob}(\mathbf{Ouv}_X) \ni U \mapsto F(U) = \mathcal{C}^0(U, \mathbb{R}).$$

Example 2.6. Let A be a commutative ring with non-zero multiplicative identity and $X = \text{Spec}(A)$. Let us consider the Zariski topology (X, \mathcal{T}) . Let us consider a category \mathcal{O}_X such that

- $\text{ob}(\mathcal{O}_X) = \mathcal{T}$,
- $\mathcal{O}_X(U) = \{s : U \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}\}$,

where $s : U \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ is a function such that for any $\mathfrak{p} \in U$,

- i). $s(\mathfrak{p}) \in A_{\mathfrak{p}}$,
- ii). there exists an open set $V \subset U$ such that $\mathfrak{p} \in V$ and for any $\mathfrak{q} \in V$, $s(\mathfrak{q}) = \frac{a}{b}$ for $b \notin \mathfrak{q}$.

Now we define a presheaf by the restrictions of maps such that

$$s : U \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mapsto s|_V : V \rightarrow \coprod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

Definition 2.6. Let (X, \mathcal{T}) be a topological space and \mathcal{A} be a category. We define a set of presheaves of \mathcal{A} on X as

$$\text{PreSh}_{\mathcal{A}}(X) = \text{Fun}(\mathbf{Ouv}_X^{\text{op}}, \mathcal{A}).$$

Definition 2.7. A morphism of presheaves is a natural transformation $\alpha : F \rightarrow G$ where $F, G \in \text{Fun}(\mathbf{Ouv}_X^{\text{op}}, \mathcal{A})$. (See Definition 1.9).

Remark 2.2. $\text{PreSh}(X)$ can be regarded as a category with its objects presheaves and morphisms defined above.

Notation 2.1. In the case $\mathcal{A} = (\mathbf{Ab})$ then we denote $\text{PreSh}(X) = \text{PreSh}_{\mathbf{Ab}}(X)$.

Example 2.7. Let X be a differential manifold (eg. $X \subset \mathbb{R}^n$). Let us define

$$\mathcal{C}^{\text{diff}}(U) = \{f : U \rightarrow \mathbb{R} \mid f \text{ is differentiable}\}.$$

Then the inclusions $\mathcal{C}_X^{\text{diff}}(U) \subset \mathcal{C}_X(U)$ defines the natural transformation.

Example 2.8. Let $X, Y = S^1$ be topological spaces and F be a presheaf such that for any open set $U \subset X$, $F(U) = \mathcal{C}^0(U, Y)$. Then we can introduce a natural transformation such that

$$\mathcal{C}_X(U) \ni f \mapsto \exp(2\pi f i).$$

2.3 Sheaf

Definition 2.8. A presheaf \mathcal{F} on (X, \mathcal{T}) is called a sheaf if the following holds. For any collection of open sets $(U_i)_{i \in I} \subset \mathcal{T}$, $U = \bigcup_{i \in I} U_i$, the map $\varphi : \mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i)$ which is defined as

$$\varphi(s) = (s|_{U_i})_{i \in I}.$$

is the equalizer of the following functions $\varphi_1, \varphi_2 : \prod_{i \in I} \mathcal{F}(U_i) \rightarrow \prod_{i, j \in I} \mathcal{F}(U_i \cap U_j)$,

$$\varphi_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})_{i, j \in I}, \quad \varphi_2((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})_{i, j \in I}.$$

Remark 2.3. In the case $I = \{1, 2\}$, we have $U = U_1 \cup U_2$, and for any $U' \in \mathcal{T}$ such that $U \subset U'$, we have for $\mathcal{F}(U') \ni s : U' \rightarrow \mathbb{R}$, $\psi(s) = (s|_{U_1}, s|_{U_2})$, as in \mathbf{Ouv}_X , morphisms are inclusions. Let $\tilde{\psi}(s) = s|_U$, then this satisfies the condition for the equalizer (ie. $\varphi \circ \tilde{\psi} = \psi$).

Remark 2.4. A presheaf \mathcal{O}_X with $X = \text{Spec}(A)$ is a sheaf.

Example 2.9. Let (X, \mathcal{T}) be a topological space and G be a group. We define a constant presheaf $\mathbb{G}(U) = G$. In general, this is not a sheaf. Instead, we define a constant sheaf $\underline{\mathbb{G}}(U) = \mathcal{C}^0(U, G)$ where G is regarded as a topological space with the discrete topology. Then for any connected component of X is mapped to a single point set in G .

Definition 2.9. Let $\mathcal{F}_1, \mathcal{F}_2$ be sheaves. A mapping $\varphi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is called a morphism of sheaves if it is a morphism of presheaves.

Definition 2.10. A set of sheaves of \mathcal{A} on the topological space (X, \mathcal{T}) is denoted as $\text{Sh}_{\mathcal{A}}(X)$.

Remark 2.5. As in the case of presheaves, $\text{Sh}_{\mathcal{A}}(X)$ can be regarded as a category with sheaf morphisms.

Remark 2.6. $\text{Sh}_{\mathcal{A}}(X)$ is a full-subcategory of $\text{PreSh}_{\mathcal{A}}(X)$.

Notation 2.2. In the case $\mathcal{A} = (\mathbf{Ab})$, we denote $\text{Sh}_{(\mathbf{Ab})}(X) = \text{Sh}(X)$.