Algebraic Geometry 1 Week 2 Exercise Sheet Solutions

So Murata

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Exercise 7

$$f \mapsto (f(x))_{x \in U}$$

This is the sheafification of the presheaf since by the construction of germs in abelian groups, we have for $s \in \mathscr{F}(U)$,

$s_x = \{(t, V) | x \in V, \text{There exists } x \in W \text{ such that } , W \subset U, V, \rho_{UW}(s) = \rho_{VW}(t)\}.$

Since W is open we can take an arbitrary small open ball $B(x, \frac{1}{n})$ and any functions in s_x as an equivalence class must coincide in the ball. As n is arbitrary, we conclude that $f_x = f(x)$.

Exercise 8

Let $(X, \mathscr{T})$ be a topological space and $\mathscr{F} \in \mathrm{Sh}_{\mathscr{A}}(X)$ be a sheaf. We define

$$|\mathscr{F}| = \coprod_{x \in X} \mathscr{F}_x.$$

We first prove that $\mathscr{B} = \{\overline{s}(U) \mid U \in \mathscr{T}, \overline{s} : X \to |\mathscr{F}|\}$ defines a basis of the desired strongest topology on $|\mathscr{F}|$. In order to do so, we will prove the following claims.

Claim 1. *For any $s \in \mathscr{F}(U)$ and open subset V of U, we have*

$$\overline{s}^{-1} \circ \overline{s}(V) = V.$$

Proof. $\overline{s}^{-1} \circ \overline{s}(V) \supseteq V$ is obvious. Therefore, we will prove the other direction of inclusion.

Let $y \in \overline{s}^{-1} \circ \overline{s}(V)$. Then $\overline{s}(y) = \overline{s}(x)$ for some $x \in V$ as an equivalence class of a pair of a section and an open sets. If $\overline{s}(x) = \overline{s}(y)$, then for any open set which contains x also contains y. In particular, $y \in V$. Thus we have the equality $\square$

Claim 2. *$\mathscr{B} = \{\overline{s}(U) \mid U \in \mathscr{T}, \overline{s} : X \to |\mathscr{F}|\}$ is a basis.*

Proof. For any $s_x \in \overline{s}(U) \cap \overline{t}(V)$, then $(s, U), (t, V) \in s_x$. Therefore there is $W \in U \cap V$ open such that $\rho_{UW}(s) = \rho_{VW}(t)$. In particular $s_x \in \overline{s}(W) \in \mathscr{B}$. Therefore, $\mathscr{B}$ is a basis. □

We denote the topology generated by $\mathscr{B}$ as $\mathscr{T}_M$.

We now show that there is an isomorphism between the sheaf of continuous sections $f : U \to |\mathscr{F}|$ and $\mathscr{F}^+$.

First for any $s \in \mathscr{F}(U)$, $x \mapsto s_x$ defines a continuous map on the topology $\mathscr{T}_M$. Since we have a basis, it is enough to check that for each $\overline{t} : V \to |\mathscr{F}|$, $\overline{s}^{-1}(\overline{t}(V))$ is open.

Indeed let $W = \{x \mid x, y \in U \cap V, s_x = t_y\}$, then this is an open map. Since for each $x \in W$, we can take an open set W_x such that $x \in W_x, s|_{W_x} = t|_{W_x}$. Then

$$W = \bigcup_{x \in W} W_x.$$

And this W is exactly equal to

$$\overline{s}^{-1}(t(V)) = \bigcup_{x \in W} \overline{s}^{-1}(\overline{s}(W_x)) = \bigcup_{x \in W} W_x = W.$$

by Claim. 1.

On the other hand, we must prove that for any continuous section $f : U \to |\mathscr{F}|$, there is $s \in \mathscr{F}(U)$ such that $f(x) = s_x$. Take (t, V) to be such that $t \in \mathscr{F}(V)$, $x \in V, t_x = f(x)$. Then $V_x = f^{-1}(t(V))$ is an open set. This means for any $y \in V_x$, $f(y) = t_y, y \in V$. Since $(V_x)_{x \in U}$ is an open covering of U and every pair of terms $((t_y)_{y \in U_x})_{x \in U}$ coincide on the intersection of its domains, we can glue this to some $s = (s_x)_{x \in U}$ and f coincides with the section induced by s. Thus we have proven that there is a one-to-one correspondence between

$$\{\text{Sections of } \mathscr{F}^+\} \leftrightarrow \{\text{Continuous sections } f : U \to |\mathscr{F}|\}.$$

Suppose for $U \in |\mathscr{F}|$, we have $\overline{s}^{-1}(U)$ is open for any s, then for any $s_x \in U$, there is (s^x, U_x), such that $(s^x, U_x) \in s_x, x \in U$. Therefore

$$U^x \cap \overline{s}^{-1}(U)$$

is open and

$$U = \bigcup_{s_x \in U} s(U^x \cap \overline{s}^{-1}(U)).$$

Therefore U is contained in the topology $\mathscr{T}_M$. This shows that $\mathscr{T}_M$ is the strongest topology among all topology where all $\overline{s}$ is continuous.

For each $(s_x)_{x\in U}, (t_x)_{x\in U} \in \mathscr{F}^+(U)$,

$$\overline{s+t}(x) = (s+t)_x = s_x + t_x,$$

by definition. This shows that $\mathscr{F}^+(U) \ni s \mapsto \overline{s}$ is a group homomorphism which has an inverse. Thus we have proven the statement.

Exercise 9

(i)

Let $s, t \in \mathscr{F}(U)$. Then we let $W = \{x \in U \mid s_x = t_x\}$. By the construction, there exists $U_s, U_t \subset U$ open such that $x \in U_s, U_t$ and there is $x \in W_x \in U_s, U_t$ such that

$$s|_{W_x} = t|_{W_x}.$$

Since they coincide on W, we have $s_y = t_y$ for any $y \in W_x$, therefore $W_x \subset W$. Furthermore, this W_x can be defined for each $x \in W$. We obtain

$$W = \bigcup_{x\in W} W_x,$$

which is an arbitrary union of open sets, thus open.