Algebraic Geometry 1

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1 Topology

1.1 Connected Sets

Definition 1.1. *Let $(X, \mathscr{T})$ be a topological space. A subset A of X is said to be connected if for any $U, V \in \mathscr{T}$, $U \cap V =, U \cup V \supset A$ then A is fully contained in one of U, V.*

Definition 1.2. *A connected component of a topological space is a maximal connected subset of a space.*

Proposition 1.1. *Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological space and $f : X \to Y$ be a continuous function. Then for any connected subset A of X, $f(A)$ is connected in Y.*

Proof.

$$\begin{aligned}
U, V \in \mathscr{T}_Y, U \cup V \supset f(A), U \cap V = \emptyset, \\
&\Rightarrow f^{-1}(U), f^{-1}(V) \in \mathscr{T}_X, \\
&\quad f^{-1}(U) \cup f^{-1}(V) \supset A, \\
&\quad f^{-1}(U) \cap f^{-1}(V) = \emptyset, \\
&\Rightarrow f^{-1}(U) \supset A \vee f^{-1}(V) \supset A, \\
&\Rightarrow U \supset f(A) \vee V \supset f(A).
\end{aligned}$$

□

2 Category Theory

2.1 Categories

Definition 2.1. *A category $\mathscr{A}$ consists of*

- *a collection* $\mathrm{ob}(\mathscr{A})$ *of objects;*
- *for each $A, B \in \mathrm{ob}(\mathscr{A})$, a collection $\mathscr{A}(A, B)$ of morphisms from A to B;*

such that

i). *for each* $A \in \mathrm{ob}(\mathscr{A})$, *the identity* $1_A \in \mathscr{A}(A, A)$;

ii). *the composition* $\mathscr{A}(B, C) \times \mathscr{A}(A, B) \ni (g, f) \mapsto g \circ f \in \mathscr{A}(A, C)$ *is well-defined;*

and they satisfy the following axioms

I). *Associativity :* $f \in \mathscr{A}(A, B), g \in \mathscr{A}(B, C), h \in \mathscr{A}(C, D)$, $(h \circ g) \circ f = h \circ (g \circ f)$.

II). *Identity laws :* $f \in \mathscr{A}(A, B)$ *then* $f \circ 1_A = 1_B \circ f$.

Definition 2.2. *Let* $\mathscr{A}$ *be a category. A terminal object* $T \in \mathrm{ob}(\mathscr{A})$ *is an object such that for any* $A \in \mathrm{ob}(\mathscr{A})$, $\mathscr{A}(A, T)$ *is a single element set.*

Definition 2.3. *Given two categories* $\mathscr{A}, \mathscr{B}$, *we say* $\mathscr{A}$ *is a full-subcategory of* $\mathscr{B}$ *if*

i). $\mathscr{A} \subset \mathscr{B}$,

ii). $\mathrm{ob}(\mathscr{A}) = \mathrm{ob}(\mathscr{B})$.

Notation 2.1. *Here we give notations to some important categories.*

- (**Sets**) *: A category of sets equipped with set theoretic functions.*
- (**Ab**) *: A category of abelian groups with group homomorphisms.*

Example 2.1. *Given a partially ordered set* $(X, \leq)$. *This can be encoded to a category* $\mathscr{O}$ *by*

i). $\mathrm{ob}(\mathscr{O}) = X$,

ii). *For* $x, y \in X$, $x \leq y \Rightarrow \mathscr{O}(x, y) = \{*\}$ *otherwise the morphisms between* x, y *is an emptyset.*

Definition 2.4. *A opposite/dual category of a category* $\mathscr{A}$ *is* $\mathscr{A}^{\boldsymbol{op}}$ *such that*

i). $\mathrm{ob}(\mathscr{A}^{\boldsymbol{op}}) = \mathrm{ob}(\mathscr{A})$,

ii). $\mathscr{A}^{\boldsymbol{op}}(B, A) = \mathscr{A}(A, B)$.

Definition 2.5. *Let* $\mathscr{A}$ *be a category and* $\varphi_1, \varphi_2 \in \mathscr{A}(M, N)$. *A morphism* $\varphi : K \to M$ *is called an equalizer of* (φ_1, φ_2) *if for any morphism* $\psi : P \to M$ *such that* $\varphi_1 \circ \psi = \varphi_2 \circ \psi$, *there is a unique morphism* $\tilde{\psi} : P \to K$ *such that* $\varphi \circ \tilde{\psi} = \psi$.

Proposition 2.1. *If an equalizer exists then it is unique up to unique isomorphism.*

Proof. Suppose $\varphi : K \to M, \psi : L \to M$ be equalizers of (φ_1, φ_2). Then we have

$$\varphi \circ \tilde{\psi} = \psi, \quad \psi \circ \tilde{\varphi} = \varphi$$

By the uniqueness, we have $\tilde{\varphi} \circ \tilde{\psi} = 1_L, \tilde{\psi} \circ \tilde{\varphi} = 1_K$. □

Definition 2.6. *Let $\mathscr{A}, \mathscr{B}$ be categories. A functor $F : \mathscr{A} \to \mathscr{B}$ is a function such that for each $f \in \mathscr{A}(A, A')$, $F(f) : F(A) \to F(A')$. In other words, $f \mapsto F(f) : \mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$. Furthermore, F satisfies the following axioms.*

I). $F(f' \circ f) = F(f') \circ F(f)$ whenever $f : A \to A', f' : A' \to A''$ in $\mathscr{A}$,

II). $F(1_A) = 1_{F(A)}$ whenever $A \in \mathscr{A}$.

Definition 2.7. *Let F, G be functors between two categories $\mathscr{A}, \mathscr{B}$. A natural transformation $\alpha : F \to G$ is a family $(\alpha_A : F(A) \to G(A))_{A \in \mathscr{A}}$ such that*

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle \alpha_A}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_{A'}} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

is a commutative diagram. Each α_A is called a component of α.

2.2 Direct Limits

Definition 2.8. *A partially ordered set $(X, \leq)$ is directed if for any $x, y \in X$ there is $z \in X$ such that $x \leq c$ and $y \leq c$.*

Example 2.2. *Let $(X, \mathscr{T})$ be a topological space. A partially ordered set $(\mathscr{T}, \leq)$ such that*

$$V \subseteq U \Rightarrow U \leq V$$

is directed. Since for any $U \in \mathscr{T}$, $U \leq \emptyset$. As a category this is $\mathbf{Ouv}_X^{\mathbf{op}}$.

Example 2.3. *Let $(X, \mathscr{T})$ be a topological space. For $x \in X$, define$O_x = \{U \in \mathscr{T} \mid x \in U\}$. If we define an order as in the previous example, we get $(O_x, \leq)$ is directed. This follows from for any $U, V \in O_x$, $U, V \leq U \cap V$.*

Definition 2.9. *Let I be a directed partially ordered set and $\mathscr{A}$ be a category.*

3 Sheaf Theory

3.1 Presheaves

Definition 3.1. *Let $(X, \mathscr{T})$ be a topological space. We define the presheaf $\mathscr{F}$ of a category $\mathscr{A}$ on X such that*

- $U \in \mathscr{T}$, $\mathscr{F}(U) \in \mathrm{ob}(\mathscr{A})$,
- $U, V \in \mathscr{T}$, $V \subset U \Rightarrow$ *there exists a map* $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$

  such that

i). For any $U \in \mathscr{T}$, $\rho_{UU} = 1_{\mathscr{F}(U)}$.

ii). $U, V, W \in \mathscr{T}, W \subset V \subset U \to \rho_{UW} = \rho_{VW} \circ \rho_{UW}$.

Remark 3.1. *In the case* $\mathscr{A} = (\mathbf{Sets}), (\mathbf{Ab})$, $\mathscr{F}(\emptyset) = \emptyset, \{1\}$, *respectively.*

Definition 3.2. *An element of* $\mathscr{F}(U)$ *is called a local section of* $\mathscr{F}$ *and* $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$ *is called the space of sections over* U. *In particular* $\Gamma(X, \mathscr{F})$ *is called the space of global sections of* $\mathscr{F}$.

Definition 3.3. *Let* $(X, \mathscr{T})$ *be a topological space and* $\mathscr{F}$ *be a presheaf of a category* $\mathscr{A}$ *on* X. *Suppose we have two open sets* $U, V \in \mathscr{T}$ *such that* $V \subset U$. *Then for any section* $s \in \mathscr{F}(U)$, $s|_V = \rho_{UV}(s)$ *is called the restriction of* s *to* V.

Example 3.1. *Let* $(X, \mathscr{T})$ *be a topological space. We have a presheaf of continuous functions* $\mathscr{C}_X(U) = \mathscr{C}^0(U, \mathbb{R})$. *This is indeed a presheaf with restriction maps* $\rho_{UV} : \mathscr{C}_X(U) \to \mathscr{C}_X(V)$. *(Explicitly,* $\rho_{UV}(f) = f \circ i_V$ *where* i_V *is an inclusion map.) We note that we can introduce operations* $+, \cdot$ *to endow some algebraic structures (groups, rings, ...) on* $\mathbb{R}$.

Example 3.2. *Let* $(X, \mathscr{T})$ *be a topological space and suppose we have presheaves*

- $\mathscr{C}_X^{\boldsymbol{diff}}(U) = \{f : U \to \mathbb{R} \mid f \text{ is differentiable.}\}$.

Then there is an inclusion relation $\mathscr{C}_X^{\boldsymbol{diff}}(U) \subseteq \mathscr{C}_X(U)$ *and this defines a presheaf.*

Example 3.3. *Let* $(X, \mathscr{T}_X)$,$(Y, \mathscr{T}_Y)$ *be topological spaces. Define a presheaf on* X *by*

$$U \in \mathscr{T}_X, \mathscr{F}(U) = \mathscr{C}^0(X, Y).$$

And like the previous example, we define $\rho_{UV}(f) = f|_V$ *for* $U, V \in \mathscr{T}_X, V \subset U$. *the restriction of* f *to* V.
But this is a presheaf only of a set.

Example 3.4. *Let* $(X, \mathscr{T})$ *be a topological space and* G *be an abelian group. The constant presheaf* $\mathbb{G}$ *is such that*

$$U \in \mathscr{T}, \mathbb{G}(U) = G,$$

with $\rho_U V = id_G$ *for any* $U, V \in \mathscr{T}, V \subset U$.

3.2 Presheaves as Categories

Definition 3.4. *Let* $(X, \mathscr{T})$ *be a topological space then* $(\mathbf{Ouv}_X)$ *is the category such that its objects are the open sets of* X *and for any* $U, V \in \mathscr{T}$ *we have*

$$\mathbf{Ouv}_X(U,V) = \begin{cases} \emptyset & (V \not\subset U), \\ i_V & (V \subset U). \end{cases}$$

Definition 3.5. *Let* $(X, \mathscr{T})$ *be a topological space and* $\mathscr{A}$ *be a category. A presheaf of* $\mathscr{A}$ *on* X *is a functor* $F : \mathbf{Ouv}_X \to \mathscr{A}$.

Example 3.5. *For* $\mathbf{Ouv}_X$, *we can define a presheaf of* F *to be*

$$\mathrm{ob}(\mathbf{Ouv}_X) \ni U \mapsto F(U) = \mathscr{C}^0(U, \mathbb{R}).$$

Example 3.6. *Let* A *be a commutative ring with non-zero multiplicative identity and* $X = \mathrm{Spec}(A)$. *Let us consider the Zariski topology* $(X, \mathscr{T})$. *Let us consider a category* $\mathscr{O}_X$ *such that*

- $\mathrm{ob}(\mathscr{O}_X) = \mathscr{T}$,
- $\mathscr{O}_X(U) = \{s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}\}$,

where $s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ *is a function such that for any* $\mathfrak{p} \in U$,

i). $s(p) \in A_{\mathfrak{p}}$,

ii). *there exists an open set* $V \subset U$ *such that* $\mathfrak{p} \in V$ *and for any* $\mathfrak{q} \in V$, $s(\mathfrak{q}) = \frac{a}{b}$ *for* $b \notin \mathfrak{q}$.

Now we define a presheaf by the restrictions of maps such that

$$s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mapsto s|_V : V \to \coprod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

Definition 3.6. *Let* $(X, \mathscr{T})$ *be a topological space and* $\mathscr{A}$ *be a category. We define a set of presheaves of* $\mathscr{A}$ *on* X *as*

$$\mathrm{PreSh}_{\mathscr{A}}(X) = \mathrm{Fun}(\mathbf{Ouv}_X^{\mathbf{op}}, \mathscr{A}).$$

Definition 3.7. *A morphism of presheaves is a natural transformation* $\alpha : F \to G$ *where* $F, G \in \mathrm{Fun}(\mathbf{Ouv}_X^{\mathbf{op}}, \mathscr{A})$.*(See Definition* **??***).*

Remark 3.2. $\mathrm{PreSh}(X)$ *can be regarded as a category with its objects presheaves and morphisms defined above.*

Notation 3.1. *In the case* $\mathscr{A} = (\mathbf{Ab})$ *then we denote* $\mathrm{PreSh}(X) = \mathrm{PreSh}_{\mathbf{Ab}}(X)$.

Example 3.7. *Let* X *be a differential manifold(eg.* $X \subset \mathbb{R}^n$*). Let us define*

$$\mathscr{C}^{\mathbf{diff}}(U) = \{f : U \to \mathbb{R} \mid f \text{ is differentiable.}\}.$$

Then the inclusions $\mathscr{C}_X^{\mathbf{diff}}(U) \subset \mathscr{C}_X(U)$ *defines the natural transformation.*

Example 3.8. *Let* $X, Y = S^1$ *be topological spaces and* F *be a presheaf such that for any open set* $U \subset X$, $F(U) = \mathscr{C}^0(U, Y)$.*Then we can introduce a natural transformation such that*

$$\mathscr{C}_X(U) \ni f \mapsto \exp(2\pi f i).$$

3.3 Sheaves

Definition 3.8. *A presheaf* $\mathscr{F}$ *on* $(X, \mathscr{T})$ *is called a sheaf if the following holds. For any collection of open sets* $(U_i)_{i\in I} \subset \mathscr{T}, U = \bigcup_{i\in I} U_i$*, the map* $\varphi : \mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i)$ *which is defined as*

$$\varphi(s) = (s|_{U_i})_{i\in I}.$$

is the equalizer of the following functions $\varphi_1, \varphi_2 : \prod_{i\in I} \mathscr{F}(U_i) \to \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j)$*,*

$$\varphi_1((s_i)_{i\in I}) = (s_i|_{U_i\cap U_j})_{i,j\in I}, \quad \varphi_1((s_i)_{i\in I}) = (s_j|_{U_i\cap U_j})_{i,j\in I}.$$

Remark 3.3. *In the case* $I = \{1, 2\}$*, we have* $U = U_1 \cup U_2$*, and for any* $U' \in \mathscr{T}$ *such that* $U \subset U'$*, we have for* $\mathscr{F}(U') \ni s : U' \to \mathbb{R}$ *,*$\psi(s) = (s|_{U_1}, s|_{U_2})$*, as in* $\mathbf{Ouv}_X$*, morphisms are inclusions. Let* $\tilde{\psi}(s) = s|_U$*, then this satisfies the condition for the equalizer (ie.* $\varphi \circ \tilde{\psi} = \psi$*).*

Remark 3.4. *A presheaf* $\mathscr{O}_X$ *with* $X = \mathrm{Spec}(A)$ *is a sheaf.*

Example 3.9. *Let* $(X, \mathscr{T})$ *be a topological space and* G *be a group. We define a constant presheaf* $\mathbb{G}(U) = G$*. In general, this is not a sheaf. Instead, we define a constant sheaf* $\underline{\mathbb{G}}(U) = \mathscr{C}^0(U, G)$ *where* G *is regarded as a topological space with the discrete topology. Then for any connected component of* X *is mapped to a single point set in* G*.*

Definition 3.9. *Let* $\mathscr{F}_1, \mathscr{F}_2$ *be sheaves. A mapping* $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ *is called a morphism of sheaves if it is a morphism of presheaves.*

Definition 3.10. *A set of sheaves of* $\mathscr{A}$ *on the topological space* $(X, \mathscr{T})$ *is denoted as* $\mathrm{Sh}_{\mathscr{A}}(X)$*.*

Remark 3.5. *As in the case of presheaves,* $\mathrm{Sh}_{\mathscr{A}}(X)$ *can be regarded as a category with sheaf morphisms.*

Remark 3.6. $\mathrm{Sh}_{\mathscr{A}}(X)$ *is a full-subcategory of* $\mathrm{PreSh}_{\mathscr{A}}(X)$*.*

Notation 3.2. *In the case* $\mathscr{A} = (\mathbf{Ab})$*, we denote* $\mathrm{Sh}_{(\mathbf{Ab})}(X) = \mathrm{Sh}(X)$*.*