Algebraic Geometry 1 Week 3 Exercise Sheet Solutions

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Exercise 15

Let us consider a short exact sequence

$$0 \to \underline{Z} \overset{2\pi}{\to} \mathscr{O}_{S^1} \overset{\exp}{\to} \mathscr{O}^*_{S^1} \to 1.$$

Since each $s : S^1 \to \mathbb{Z}$ is continuous, $\bigcup_n \in \mathbb{Z} s^{-1}(n)$ is an open cover of S^1 and each of them are disjoint. By the connectedness of S^1 we conclude that s is constant on S^1. In general, $s \in \underline{Z}(U)$ is constant on each connected component in U. From the last exercise of the first week exercise sheet we know that $\operatorname{Coker} f_* \exp = 0$, therefore by the definition of the exact sequence we conclude that $R^1 f_* \underline{Z} = 0$.

For any connected open set U in S^1, $f^{-1}(U) = \left(\frac{1}{2}U\right) \cup \left(\frac{1}{2}U + \pi\right)$. Thus it consists of two disjoint connected components. When taking direct limits, we can restrict the indexing open sets to be connected. Similarly from the 6th exercise, we conclude that

$$(f_* \underline{Z})_x = (2\pi i \mathbb{Z})^{\pi_0\left(\left(\frac{1}{2}U\right) \cup \left(\frac{1}{2}U + \pi\right)\right)} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

We conclude that $f_* \underline{Z}$ is isomorphic to $\underline{\mathbb{Z} \oplus \mathbb{Z}}$. Let $s \in \underline{\mathbb{Z} \oplus \mathbb{Z}}(U)$ then we conclude that there is connected component V in U such that $s(V) = \{(n, m)\}$, We assign $s \to f_* s$ such that

$$s(\frac{1}{2}V) = \{n\}, \quad s(\frac{1}{2}V + \pi) = \{m\}.$$

This is clearly a group homomorphism which is bijective by $s \to t$, $t(V) = \{(n, m)\}, s(\frac{1}{2}V) = \{n\}, \quad s(\frac{1}{2}V + \pi) = \{m\}$.

Exercise 16

We know that $\operatorname{Sh}(\{x\}) = (\mathbf{Ab})$. And for such inclusion $i : \{x\} \to X$,

$$i_{x*}(G)(U) = \begin{cases} G & (x \in U), \\ \emptyset & (x \notin U). \end{cases}$$

Given an exact sequence of groups

$$0 \to G \to H \to J \xrightarrow{\pi} 0.$$

We know that i_{x*} is left-exact. Therefore, we will examine if i_{x*} sends $\pi : H \to J$ to an epimorphism. Indeed

$$i_{x*}(\pi)(U) = \begin{cases} \pi & (x \in U), \\ * & (x \notin U). \end{cases}$$

Therefore, for each $U \subset X$ we have $\mathrm{Coker}(i_{x*}(\pi)(U)) = 0$. Thus i_{x*} is exact.