Algebraic Geometry 1 Sheet 4 Problem 22

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Exercise 22

Let us pick a short exact sequence of sheaves.

$$0 \to \mathscr{G}_1 \xrightarrow{i} \mathscr{G}_2 \xrightarrow{\pi} \mathscr{G}_3 \to 0.$$

Then we claim that

$$0 \to \mathrm{Hom}(\mathscr{F}, \mathscr{G}_1) \xrightarrow{i} \mathrm{Hom}(\mathscr{F}, \mathscr{G}_2) \xrightarrow{\pi} \mathrm{Hom}(\mathscr{F}, \mathscr{G}_3),$$

is an exact sequence. Indeed for any $\varphi, \psi \in \mathrm{Hom}(\mathscr{F}, \mathscr{G}_1)$, we have

$$i \circ \varphi = i \circ \psi$$

implies, $\varphi|_U = \psi|_U$ for any open set $U \subseteq X$ by the injectivity of i. Furthermore, for any $\varphi \in \operatorname{Ker} \pi$, we have $\operatorname{Im} \varphi|_U \subseteq \operatorname{Im} i_U$. Thus $\varphi|_U$ factors through $\operatorname{Im} i_U$, there is $\tilde{\varphi}|_U : \mathscr{F}(U) \to \mathscr{G}_1(U)$ such that $i|_U \circ \tilde{\varphi}|_U = \varphi|_U$. The other way is trivial as $\pi \circ i \circ \varphi = 0$ by the definition of sheaf morphisms. Thus we have proven the exactness at $\mathscr{G}_2$.

Now we show that $\mathrm{Hom}(\cdot, \mathscr{G})$ is a left exact covariant functor. Indeed, we are given

$$0 \to \mathscr{F}_1 \xrightarrow{i} \mathscr{F}_2 \xrightarrow{\pi} \mathscr{F}_3 \to 0,$$

we will show that

$$0 \to \mathrm{Hom}(\mathscr{F}_3, \mathscr{G}) \xrightarrow{\pi} \mathrm{Hom}(\mathscr{F}_2, \mathscr{G}) \xrightarrow{i} \mathrm{Hom}(\mathscr{F}_1, \mathscr{G})$$

is exact. By the definition of epimorphism, we have for any $\varphi, \psi \in \mathrm{Hom}(\mathscr{F}_3, \mathscr{G})$

$$\psi \circ \pi = \varphi \circ \pi$$

then $\psi = \varphi$. Furthermore, if $\varphi \circ i = 0$ then for any $U \subseteq X$ open we have

$$\operatorname{Im} i|_U \subseteq \operatorname{Ker} \varphi|_U.$$

And $\operatorname{Im} i|_U = \operatorname{Ker} \pi|_U$ by the exactness. Therefore, φ factors through $\operatorname{Ker} \pi|_U$, there is ψ such that $\psi\pi = \varphi$.