# Algebraic Geometry 1 Week 2 Exercise Sheet Solutions

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## Exercise 8

Let $(X, \mathscr{T})$ be a topological space and $\mathscr{F} \in \mathrm{Sh}_{\mathscr{A}}(X)$ be a sheaf. We define

$$|\mathscr{F}| = \coprod_{x \in X} \mathscr{F}_x.$$

We first prove that $\mathscr{B} = \{\overline{s}(U) \mid U \in \mathscr{T}, \overline{s} : X \to |\mathscr{F}|\}$ defines a basis of the desired strongest topology on $|\mathscr{F}|$. In order to do so, we need $\overline{s}^{-1} \circ \overline{s}(U) = U$ and $\mathscr{B}$ is indeed a basis of a topological space. $\overline{s}^{-1} \circ \overline{s}(U) \supseteq U$ is obvious. Therefore, we will prove the other direction of inclusion.

If $\overline{s}(x) = \overline{s}(y)$, then for any open set which contains $x$ also contains $y$. In particular, $y \in U$. Thus we have the equality $\overline{s}^{-1} \circ \overline{s}(U) = U$. And for any $s_x \in \overline{s}(U) \cap \overline{s}(V)$, then $s_x \in \overline{s}(U \cap V)$. Therefore $\mathscr{B}$ is a basis. We denote such topology as $\mathscr{T}_M$.

We now show that there is an isomorphism between the sheaf of continuous functions $f : U \to |\mathscr{F}|$ and $\mathscr{F}^+$.

First for any $s \in \mathscr{F}(U)$, $x \mapsto s_x$ defines a continuous map on the topology $\mathscr{T}_M$. Let $V \in \mathscr{T}_M$ then $V$ is of the form,

$$V = \bigcup_{\lambda \in \Lambda} \overline{s}_\lambda(U_\lambda),$$

where $\overline{s}_\lambda$ is a map induced by $s \in \mathscr{F}(U_\lambda)$. Thus it is enough to check for some map $\bar{t} : V \to |\mathscr{F}|$, $\overline{s}^{-1}(\bar{t}(V))$ is open.

Indeed let $W = \{x \mid x, y \in U \cap V, s_x = t_y\}$, then this is an open map. This follows that take $W_x$ to be an open set such that $x \in W_x, s|_{W_x} = t|_{W_x}$. This is justified by the construction of stalks and germs in abelian groups. Then

$$W = \bigcup_{x \in W} W_x.$$

And this $W$ is exactly equal to $\overline{s}^{-1}(t(U))$.

On the other hand, we prove that for any continuous section $f : U \to |\mathscr{F}|$, there is $s \in \mathscr{F}(U)$ such that $f(x) = s_x$. Take $(t, V)$ to be such that $t \in \mathscr{F}(V)$, $x \in V, t_x = f(x)$. Then $V_x = f^{-1}(t(V))$ is an open set. This means for any $y \in V_x$, $f(y) = t_y, y \in V$. Since $(V_x)_{x \in U}$ is an open covering and every pair of terms $((t_y)_{y \in U_x})_{x \in U}$ coincide on the intersection of its domains, we can glue this to some $(s_x)_{x \in U}$.

For each $(s_x)_{x \in U}, (t_x)_{x \in U} \in \mathscr{F}^+(U)$,

$$\overline{s+t}(x) = (s+t)_x = s_x + t_x,$$

since restriction maps are group homomorphisms. This shows that $\mathscr{F}^+(U) \ni s \mapsto \overline{s}$ is a group homomorphism which has an inverse. Thus we have proven the statement.