

Algebraic Geometry 1

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1 Topology

1.1 Connected Sets

Definition 1.1. Let (X, \mathcal{T}) be a topological space. A subset A of X is said to be connected if for any $U, V \in \mathcal{T}$, $U \cap V = \emptyset$, $U \cup V \supset A$ then A is fully contained in one of U, V .

Definition 1.2. A connected component of a topological space is a maximal connected subset of a space.

Proposition 1.1. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological space and $f : X \rightarrow Y$ be a continuous function. Then for any connected subset A of X , $f(A)$ is connected in Y .

Proof.

$$\begin{aligned} U, V \in \mathcal{T}_Y, U \cup V \supset f(A), U \cap V = \emptyset, \\ \Rightarrow f^{-1}(U), f^{-1}(V) \in \mathcal{T}_X, \\ f^{-1}(U) \cup f^{-1}(V) \supset A, \\ f^{-1}(U) \cap f^{-1}(V) = \emptyset, \\ \Rightarrow f^{-1}(U) \supset A \vee f^{-1}(V) \supset A, \\ \Rightarrow U \supset f(A) \vee V \supset f(A). \end{aligned}$$

□

2 Category Theory

2.1 Categories

Definition 2.1. A category \mathcal{A} consists of

- a collection $\text{ob}(\mathcal{A})$ of objects;
- for each $A, B \in \text{ob}(\mathcal{A})$, a collection $\mathcal{A}(A, B)$ of morphisms from A to B ;

such that

- i). for each $A \in \text{ob}(\mathcal{A})$, the identity $1_A \in \mathcal{A}(A, A)$;
- ii). the composition $\mathcal{A}(B, C) \times \mathcal{A}(A, B) \ni (g, f) \mapsto g \circ f \in \mathcal{A}(A, C)$ is well-defined;

and they satisfy the following axioms

- I). Associativity : $f \in \mathcal{A}(A, B), g \in \mathcal{A}(B, C), h \in \mathcal{A}(C, D), (h \circ g) \circ f = h \circ (g \circ f)$.
- II). Identity laws : $f \in \mathcal{A}(A, B)$ then $f \circ 1_A = 1_B \circ f$.

Definition 2.2. Let \mathcal{A} be a category. A terminal object $T \in \text{ob}(\mathcal{A})$ is an object such that for any $A \in \text{ob}(\mathcal{A})$, $\mathcal{A}(A, T)$ is a single element set.

Definition 2.3. Given two categories \mathcal{A}, \mathcal{B} , we say \mathcal{A} is a full-subcategory of \mathcal{B} if

- i). $\mathcal{A} \subset \mathcal{B}$,
- ii). $\text{ob}(\mathcal{A}) = \text{ob}(\mathcal{B})$.

Notation 2.1. Here we give notations to some important categories.

- **(Sets)** : A category of sets equipped with set theoretic functions.
- **(Ab)** : A category of abelian groups with group homomorphisms.

Example 2.1. Given a partially ordered set (X, \leq) . This can be encoded to a category \mathcal{O} by

- i). $\text{ob}(\mathcal{O}) = X$,
- ii). For $x, y \in X$, $x \leq y \Rightarrow \mathcal{O}(x, y) = \{*\}$ otherwise the morphisms between x, y is an empty set.

Definition 2.4. A opposite/dual category of a category \mathcal{A} is \mathcal{A}^{op} such that

- i). $\text{ob}(\mathcal{A}^{op}) = \text{ob}(\mathcal{A})$,
- ii). $\mathcal{A}^{op}(B, A) = \mathcal{A}(A, B)$.

Definition 2.5. Let \mathcal{A} be a category and $\varphi_1, \varphi_2 \in \mathcal{A}(M, N)$. A morphism $\varphi : K \rightarrow M$ is called an equalizer of (φ_1, φ_2) if for any morphism $\psi : P \rightarrow M$ such that $\varphi_1 \circ \psi = \varphi_2 \circ \psi$, there is a unique morphism $\tilde{\psi} : P \rightarrow K$ such that $\varphi \circ \tilde{\psi} = \psi$.

Proposition 2.1. If an equalizer exists then it is unique up to unique isomorphism.

Proof. Suppose $\varphi : K \rightarrow M, \psi : L \rightarrow M$ be equalizers of (φ_1, φ_2) . Then we have

$$\varphi \circ \tilde{\psi} = \psi, \quad \psi \circ \tilde{\varphi} = \varphi$$

By the uniqueness, we have $\tilde{\varphi} \circ \tilde{\psi} = 1_L, \tilde{\psi} \circ \tilde{\varphi} = 1_K$. \square

Definition 2.6. Let \mathcal{A}, \mathcal{B} be categories. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is a function such that for each $f \in \mathcal{A}(A, A')$, $F(f) : F(A) \rightarrow F(A')$. In other words, $f \mapsto F(f) : \mathcal{A}(A, A') \rightarrow \mathcal{B}(F(A), F(A'))$. Furthermore, F satisfies the following axioms.

I). $F(f' \circ f) = F(f') \circ F(f)$ whenever $f : A \rightarrow A', f' : A' \rightarrow A''$ in \mathcal{A} ,

II). $F(1_A) = 1_{F(A)}$ whenever $A \in \mathcal{A}$.

Definition 2.7. Let F, G be functors between two categories \mathcal{A}, \mathcal{B} . A natural transformation $\alpha : F \rightarrow G$ is a family $(\alpha_A : F(A) \rightarrow G(A))_{A \in \mathcal{A}}$ such that

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ \alpha_A \downarrow & & \downarrow \alpha_{A'} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

is a commutative diagram. Each α_A is called a component of α .

2.2 Direct Limits

Definition 2.8. A partially ordered set (X, \leq) is directed if for any $x, y \in X$ there is $z \in X$ such that $x \leq z$ and $y \leq z$.

Example 2.2. Let (X, \mathcal{T}) be a topological space. A partially ordered set (\mathcal{T}, \leq) such that

$$V \subseteq U \Rightarrow U \leq V$$

is directed. Since for any $U \in \mathcal{T}$, $U \leq \emptyset$. As a category this is $\mathbf{Ouv}_X^{\text{op}}$.

Example 2.3. Let (X, \mathcal{T}) be a topological space. For $x \in X$, define $O_x = \{U \in \mathcal{T} \mid x \in U\}$. If we define an order as in the previous example, we get (O_x, \leq) is directed. This follows from for any $U, V \in O_x$, $U, V \leq U \cap V$.

Definition 2.9. Let I be a directed partially ordered set and \mathcal{A} be a category. A directed system of objects of \mathcal{A} indexed by I is a collection of objects $(A_i)_{i \in I}$ and morphisms $(\rho_{ij})_{i \leq j}$ of \mathcal{A} such that

i). $\rho_{ii} = \text{id}_{A_i}$,

ii). for $i, j, k \in I$, $i \leq j \leq k \Rightarrow \rho_{ik} = \rho_{jk} \circ \rho_{ij}$.

Remark 2.1. Categorically, the directed system of objects of \mathcal{A} indexed by I is a functor $\mathcal{O}^{\text{op}} \rightarrow \mathcal{C}$, where \mathcal{O} is a category which encodes the ordered set I as a category by the same procedure as in Example 2.1. Then a directed system is a functor $\mathcal{O}^{\text{op}} \rightarrow \mathcal{A}$.

Definition 2.10. Given a directed system $((A_i)_{i \in I}, \{\rho_{ij}\}_{i \leq j})$ of objects in \mathcal{A} indexed by I . A direct limit of the system is an object $\varinjlim A_n \in \mathbf{ob}(\mathcal{A})$ satisfying the following universal property.

Given a collection of morphisms $(f_i)_{i \in I}$ such that

- i). $f_i : A_i \rightarrow \varinjlim A_n \in \mathcal{A}$,
- ii). for any $i \leq j$, $f_j \circ \rho_{ij} = f_i$.

For any $A \in \mathcal{A}$ where there is a collection of morphisms $(g_i)_{i \in I}$ satisfying the above condition, there is a unique map $\varphi : \varinjlim A_n \rightarrow A$ such that

$$\begin{array}{ccccc}
 A_i & \xrightarrow{\rho_{ij}} & & A_j & \\
 & \searrow f_i & & \swarrow f_j & \\
 & & \varinjlim A_n & & \\
 & \swarrow g_i & \downarrow \exists! & \searrow g_j & \\
 & & A & &
 \end{array}$$

is a commutative diagram.

Proposition 2.2. In the cases where $\mathcal{A} = (\mathbf{Ab}), (\mathbf{Sets})$, there exist direct limits and for each category, such limit is constructed in the following ways.

- i). $\varinjlim A_n = (\bigoplus_{i \in I} A_i) / N$ where $N = \{a_i - \rho_{ij}(a_i) \mid a_i, i \leq j\}$.
- ii). $\varinjlim A_n = (\coprod_{i \in I} A_i) / \sim$ where $a_i \sim a_j$ if there is k such that $i \leq k \leq j$, and $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.

Furthermore, these two direct limits match as sets.

Proposition 2.3. \varinjlim is (left) exact in (\mathbf{Ab}) . In other words, given a exact sequence of directed systems

$$0 \longrightarrow (M_i)_{i \in I} \longrightarrow (N_i)_{i \in I} \longrightarrow (P_i)_{i \in I} \longrightarrow 0$$

in which we have

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M_i & \longrightarrow & N_i & \longrightarrow & P_i \longrightarrow 0 \\
 & & \rho_{ij}^M \downarrow & & \rho_{ij}^N \downarrow & & \rho_{ij}^P \downarrow \\
 0 & \longrightarrow & M_j & \longrightarrow & N_j & \longrightarrow & P_j \longrightarrow 0
 \end{array}$$

There exists a short exact sequence

$$0 \longrightarrow \varinjlim M_n \longrightarrow \varinjlim N_n \longrightarrow \varinjlim P_n \longrightarrow 0$$

3 Sheaf Theory

3.1 Presheaves

Definition 3.1. Let (X, \mathcal{T}) be a topological space. We define the presheaf \mathcal{F} of a category \mathcal{A} on X such that

- $U \in \mathcal{T}, \mathcal{F}(U) \in \text{ob}(\mathcal{A}),$
- $U, V \in \mathcal{T}, V \subset U \Rightarrow$ there exists a map $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$
such that
- i). For any $U \in \mathcal{T}, \rho_{UU} = 1_{\mathcal{F}(U)}.$
- ii). $U, V, W \in \mathcal{T}, W \subset V \subset U \rightarrow \rho_{UW} = \rho_{VW} \circ \rho_{UV}.$

Remark 3.1. In the case $\mathcal{A} = (\mathbf{Sets}), (\mathbf{Ab}), \mathcal{F}(\emptyset) = \emptyset, \{1\},$ respectively.

Definition 3.2. An element of $\mathcal{F}(U)$ is called a local section of \mathcal{F} and $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ is called the space of sections over U . In particular $\Gamma(X, \mathcal{F})$ is called the space of global sections of \mathcal{F} .

Definition 3.3. Let (X, \mathcal{T}) be a topological space and \mathcal{F} be a presheaf of a category \mathcal{A} on X . Suppose we have two open sets $U, V \in \mathcal{T}$ such that $V \subset U$. Then for any section $s \in \mathcal{F}(U)$, $s|_V = \rho_{UV}(s)$ is called the restriction of s to V .

Example 3.1. Let (X, \mathcal{T}) be a topological space. We have a presheaf of continuous functions $\mathcal{C}_X(U) = \mathcal{C}^0(U, \mathbb{R})$. This is indeed a presheaf with restriction maps $\rho_{UV} : \mathcal{C}_X(U) \rightarrow \mathcal{C}_X(V)$. (Explicitly, $\rho_{UV}(f) = f \circ i_V$ where i_V is an inclusion map.) We note that we can introduce operations $+, \cdot$ to endow some algebraic structures (groups, rings, ...) on \mathbb{R} .

Example 3.2. Let (X, \mathcal{T}) be a topological space and suppose we have presheaves

- $\mathcal{C}_X^{\text{diff}}(U) = \{f : U \rightarrow \mathbb{R} \mid f \text{ is differentiable}\}.$

Then there is an inclusion relation $\mathcal{C}_X^{\text{diff}}(U) \subseteq \mathcal{C}_X(U)$ and this defines a presheaf.

Example 3.3. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. Define a presheaf on X by

$$U \in \mathcal{T}_X, \mathcal{F}(U) = \mathcal{C}^0(X, Y).$$

And like the previous example, we define $\rho_{UV}(f) = f|_V$ for $U, V \in \mathcal{T}_X, V \subset U$. the restriction of f to V .

But this is a presheaf only of a set.

Example 3.4. Let (X, \mathcal{T}) be a topological space and G be an abelian group. The constant presheaf \mathbb{G} is such that

$$U \in \mathcal{T}, \mathbb{G}(U) = G,$$

with $\rho_U V = \text{id}_G$ for any $U, V \in \mathcal{T}, V \subset U$.

3.2 Presheaves as Categories

Definition 3.4. Let (X, \mathcal{T}) be a topological space then (\mathbf{Ouv}_X) is the category such that its objects are the open sets of X and for any $U, V \in \mathcal{T}$ we have

$$\mathbf{Ouv}_X(U, V) = \begin{cases} \emptyset & (V \not\subset U), \\ i_V & (V \subset U). \end{cases}$$

Definition 3.5. Let (X, \mathcal{T}) be a topological space and \mathcal{A} be a category. A presheaf of \mathcal{A} on X is a functor $F : \mathbf{Ouv}_X \rightarrow \mathcal{A}$.

Example 3.5. For \mathbf{Ouv}_X , we can define a presheaf of F to be

$$\text{ob}(\mathbf{Ouv}_X) \ni U \mapsto F(U) = \mathcal{C}^0(U, \mathbb{R}).$$

Example 3.6. Let A be a commutative ring with non-zero multiplicative identity and $X = \text{Spec}(A)$. Let us consider the Zariski topology (X, \mathcal{T}) . Let us consider a category \mathcal{O}_X such that

- $\text{ob}(\mathcal{O}_X) = \mathcal{T}$,
- $\mathcal{O}_X(U) = \{s : U \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}\}$,

where $s : U \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ is a function such that for any $\mathfrak{p} \in U$,

- i). $s(\mathfrak{p}) \in A_{\mathfrak{p}}$,
- ii). there exists an open set $V \subset U$ such that $\mathfrak{p} \in V$ and for any $\mathfrak{q} \in V$, $s(\mathfrak{q}) = \frac{a}{b}$ for $b \notin \mathfrak{q}$.

Now we define a presheaf by the restrictions of maps such that

$$s : U \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mapsto s|_V : V \rightarrow \coprod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

Definition 3.6. Let (X, \mathcal{T}) be a topological space and \mathcal{A} be a category. We define a set of presheaves of \mathcal{A} on X as

$$\text{PreSh}_{\mathcal{A}}(X) = \text{Fun}(\mathbf{Ouv}_X^{\text{op}}, \mathcal{A}).$$

Definition 3.7. A morphism of presheaves is a natural transformation $\alpha : F \rightarrow G$ where $F, G \in \text{Fun}(\mathbf{Ouv}_X^{\text{op}}, \mathcal{A})$. (See Definition 2.7).

Remark 3.2. $\text{PreSh}(X)$ can be regarded as a category with its objects presheaves and morphisms defined above.

Notation 3.1. In the case $\mathcal{A} = (\mathbf{Ab})$ then we denote $\text{PreSh}(X) = \text{PreSh}_{\mathbf{Ab}}(X)$.

Example 3.7. Let X be a differential manifold (eg. $X \subset \mathbb{R}^n$). Let us define

$$\mathcal{C}^{\text{diff}}(U) = \{f : U \rightarrow \mathbb{R} \mid f \text{ is differentiable}\}.$$

Then the inclusions $\mathcal{C}_X^{\text{diff}}(U) \subset \mathcal{C}_X(U)$ defines the natural transformation.

Example 3.8. Let $X, Y = S^1$ be topological spaces and F be a presheaf such that for any open set $U \subset X$, $F(U) = \mathcal{C}^0(U, Y)$. Then we can introduce a natural transformation such that

$$\mathcal{C}_X(U) \ni f \mapsto \exp(2\pi f i).$$

3.3 Sheaves

Definition 3.8. A presheaf \mathcal{F} on (X, \mathcal{T}) is called a sheaf if the following holds. For any collection of open sets $(U_i)_{i \in I} \subset \mathcal{T}$, $U = \bigcup_{i \in I} U_i$, the map $\varphi : \mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i)$ which is defined as

$$\varphi(s) = (s|_{U_i})_{i \in I}.$$

is the equalizer of the following functions $\varphi_1, \varphi_2 : \prod_{i \in I} \mathcal{F}(U_i) \rightarrow \prod_{i, j \in I} \mathcal{F}(U_i \cap U_j)$,

$$\varphi_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})_{i, j \in I}, \quad \varphi_2((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})_{i, j \in I}.$$

Remark 3.3. In the case $I = \{1, 2\}$, we have $U = U_1 \cup U_2$, and for any $U' \in \mathcal{T}$ such that $U \subset U'$, we have for $\mathcal{F}(U') \ni s : U' \rightarrow \mathbb{R}$, $\psi(s) = (s|_{U_1}, s|_{U_2})$, as in \mathbf{Ouv}_X , morphisms are inclusions. Let $\tilde{\psi}(s) = s|_U$, then this satisfies the condition for the equalizer (ie. $\varphi \circ \tilde{\psi} = \psi$).

Remark 3.4. A presheaf \mathcal{O}_X with $X = \text{Spec}(A)$ is a sheaf.

Example 3.9. Let (X, \mathcal{T}) be a topological space and G be a group. We define a constant presheaf $\mathbb{G}(U) = G$. In general, this is not a sheaf. Instead, we define a constant sheaf $\underline{\mathbb{G}}(U) = \mathcal{C}^0(U, G)$ where G is regarded as a topological space with the discrete topology. Then for any connected component of X is mapped to a single point set in G .

Definition 3.9. Let $\mathcal{F}_1, \mathcal{F}_2$ be sheaves. A mapping $\varphi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is called a morphism of sheaves if it is a morphism of presheaves.

Definition 3.10. A set of sheaves of \mathcal{A} on the topological space (X, \mathcal{T}) is denoted as $\text{Sh}_{\mathcal{A}}(X)$.

Remark 3.5. As in the case of presheaves, $\text{Sh}_{\mathcal{A}}(X)$ can be regarded as a category with sheaf morphisms.

Remark 3.6. $\text{Sh}_{\mathcal{A}}(X)$ is a full-subcategory of $\text{PreSh}_{\mathcal{A}}(X)$.

Notation 3.2. In the case $\mathcal{A} = (\mathbf{Ab})$, we denote $\text{Sh}_{(\mathbf{Ab})}(X) = \text{Sh}(X)$.

3.4 Stalks

Definition 3.11. Suppose we have a topological space (X, \mathcal{T}) and a category \mathcal{A} which admits direct limits. For a presheaf $\mathcal{F} \in \text{PreSh}_{\mathcal{A}}(X)$, by inheriting the notations from Example 2.3, we define the stalk \mathcal{F}_x of \mathcal{F} at $x \in X$ by

$$\mathcal{F}_x = \varinjlim_{U \in \mathcal{O}_x} \mathcal{F}(U) = \varinjlim_{x \in U, U \in \mathcal{T}} \mathcal{F}(U).$$

Example 3.10. Let us assume that $\mathcal{A} = (\mathbf{Ab})$ in Definition 3.11. Then stalks and germs can be constructed explicitly in the following way.

$$\mathcal{F}_x = \{(s, U) \mid U \in \mathcal{O}_x, s \in \mathcal{F}(U)\} / \sim,$$

where \sim is an equivalent relation such that for $(s, U), (t, V)$,

$$(s, U) \sim (t, V) \text{ if there is } W \in \mathcal{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t).$$

Definition 3.12. Inheriting the notations from Definition 3.11, suppose we have $(f_U : \mathcal{F}(U) \rightarrow \mathcal{F}_x)_{U \in \mathcal{O}_x}$ such that for f_U, f_V are compatible with ρ_{UV} . Then we define the germ of $s \in \mathcal{F}(U)$ to be $s_x = f_U(s)$. By the universal property of the direct limit, such s_x is unique up to images under isomorphisms.

Example 3.11. In the case of Remark 3.10, we have for each $U \in \mathcal{T}$, $x \in U$, and $s \in \mathcal{F}(U)$,

$$s_x = \{(t, V) \mid \text{There is } W \in \mathcal{O}_x \text{ such that } W \subseteq U \cap V, \rho_{UW}(s) = \rho_{VW}(t)\}.$$

Remark 3.7. In the above definition, if a category \mathcal{A} admits products, we get a map

$$(s \mapsto (s_x)_{x \in U}) : \mathcal{F}(U) \rightarrow \prod_{x \in U} \mathcal{F}_x. \quad (3.1)$$

This is neither surjective nor injective in general.

Proposition 3.1. Suppose in the definition of stalks, \mathcal{F} is a sheaf. Then the map defined by Equation 3.1 is injective.

Proof. We prove the case when $\mathcal{A} = (\mathbf{Ab})$.

Suppose $s \in \mathcal{F}(U)$ is such that $s_x = 0$ in \mathcal{F}_x for all $x \in U$. Since for any restriction maps are group homomorphisms. We have that there is $V_x \in \mathcal{O}_x$ such that

$$V_x \subseteq U, \quad \rho_{UV_x}(s) = 0.$$

Therefore $\{V_x\}_{x \in U}$ is an open covering of U . Since \mathcal{F} is a sheaf, we derive that $s = 0$ in $\mathcal{F}(U)$. \square

Example 3.12. Given (X, \mathcal{F}) , a topological space and G , an abelian group. We will consider the constant presheaf \mathbb{G} and the constant sheaf $\underline{\mathbb{G}}$ on X . For any open set U and $x \in U$ we have

$$\mathbb{G}_x \cong \underline{\mathbb{G}}_x \cong G.$$

For any U, V open such that $V \subset U$ we have, $\rho_{UV} = \text{id}_G$. Thus by the construction, for $x \in U, V$, $(s, U) \sim (t, V)$ then $x \in U \cap V$ and $\rho_{UU \cap V}(s) = s = t = \rho_{VU \cap V}(t)$. Therefore, we proved the claim.

Remark 3.8. Categorically, taking stalking is a functor for each $x \in X$. Suppose we have $\mathcal{F}, \mathcal{G} \in \text{PreSh}_{\mathcal{A}}(X)$ and a morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$, then

$$\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x, \varphi_x(s_x) = \varphi(s)_x.$$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{\varphi_x} & \mathcal{G}_x \end{array}$$

Proposition 3.2. *Let $\mathcal{F}, \mathcal{G} \in \text{Sh}_{(\mathbf{Ab})}(X)$. Then for any morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ we have*

$$\varphi = 0 \Leftrightarrow \forall x \in X, \varphi_x = 0$$

Proof. \Rightarrow is trivial by its construction. We will prove \Leftarrow .

We first note that $\varphi = 0$ means that for any $U \in \mathcal{T}$, we have $\varphi_U \equiv 0$ as a group homomorphism. Let $U \in \mathcal{T}$ and $s \in \mathcal{F}(U)$. Then by the assumption and Proposition 3.1, we have proven the claim. \square

3.5 Sheafification

Definition 3.13. *Let $\mathcal{F} \in \text{PreSh}_{\mathcal{A}}(X)$. The sheafification of \mathcal{F} is a presheaf \mathcal{F}^+ which is a set of all $(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$ such that for any $x \in U$ there is $x \in V_x \subset U$, such that there is $t \in \mathcal{F}(V_x)$ satisfying for any $y \in V_x$, $s_y = t_y$. We give them restriction such that*

$$\mathcal{F}^+(U) \ni (s_x)_{x \in U} \mapsto (s_x)_{x \in V} \in \mathcal{F}^+(V).$$

Proposition 3.3. *Such \mathcal{F}^+ is indeed a sheaf.*

Proof.

\square

Remark 3.9.

$$\mathcal{F} \mapsto \mathcal{F}^+ : \text{PreSh}_{\mathcal{A}}(X) \rightarrow \text{Sh}_{\mathcal{A}}(X)$$

is a functor. Indeed given $\varphi : \mathcal{F} \rightarrow \mathcal{G}$, a morphism of presheaves. We give

$$\varphi^+(U)((s_x)_{x \in U}) = (\varphi(s)_x)_{x \in U}.$$