Sheet 7 V4A1

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(ii)

Let K/k be an arbitrary transcendental extension of fields. And $f : \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ be a natural mapping (ie. mapping the unique point to the unique point). This is not of finite type. But it has finite fibers, but it has a base change with infinite fibers. In particular, take $T \in K$ to be a transcendental element over k, and we claim that the base change by $\mathrm{Spec}(k(T)) \to \mathrm{Spec}(k)$ has infinite fibers. This base change is $\mathrm{Spec}(K \otimes_k k(T)) \to \mathrm{Spec}(k(T))$, which factors as

$$\mathrm{Spec}(K \otimes_k k(T)) \to \mathrm{Spec}(k(T) \otimes_k k(T)) \to \mathrm{Spec}(k(T))$$

whose first step is surjective (being a base change of the surjection $\mathrm{Spec}(K) \to \mathrm{Spec}(k(T))$, so it is enough that the second map $q : \mathrm{Spec}(k(T) \otimes_k k(T)) \to \mathrm{Spec}(k(T))$ has infinite fiber. But we described $k(T) \otimes_k k(T)$ above and its Spec is infinite.

This shows both that quasi-finite and injectivity are not preserved under morphisms.

(iii)

Let $f : Z \to X$ be a closed immersion. In other words, there is an open affine covering $\{U_i\}$, with $U_i = \mathrm{Spec}(A_i)$ of X such that $f^{-1}(U_i) = \mathrm{Spec}(A_i/I_i)$ for some ideal I_i of A_i.

Let $g : Y \to X$ be a morphism. To show that $q : Z \times_X Y \to Y$ is a closed immersion, we take an open affine covering $\{\mathrm{Spec}(B_i)\}$ of Y such that $g(\mathrm{Spec}(B_i)) \subset (\mathrm{Spec}(A_i))$.

By since $q(\mathrm{Spec}(B_i)) = (\mathrm{Spec}(B_i)) \times_X Z = (\mathrm{Spec}(B_i)) \times_{\mathrm{Spec}(A_i)} \mathrm{Spec}(A_i/I_i) = \mathrm{Spec}(B_i \otimes_{A_i} A_i/I_i) = \mathrm{Spec}(B_i/I_iB_i)$, therefore q is a closed immersion.

For open immersion, we have

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We have that $U \times_S V$ is an open subscheme of $X \times_S X$.

$$\begin{tikzcd}
X \arrow[r, bend left, "\Delta"] & X \times_S X \arrow[l, "p"'] \arrow[r, "q"'] & X \arrow[l, bend right, "\Delta"'] \\
U \arrow[u, "i_U"] & U \times_S V \arrow[l, "p_U"'] \arrow[r, "q_V"] \arrow[u, "i_{UV}"] & V \arrow[u, "i_V"'] \\
& U \cap V \arrow[ul, "\iota_U"] \arrow[ur, "\iota_V"'] \arrow[u, "\theta_{UV}"'] \arrow[uu, dashed, bend left] &
\end{tikzcd}$$

where $i_U, i_V, \iota_U, \iota_V$ are inclusions and θ_{UV} is an isomorphism of $U \times_S V$ and $U \cap V$, this follows from that $\Delta^{-1}(U \times_S V)$.

By the universal property of $X \times_S X$, we have

$$h = i_{UV}\theta_{UV} : U \cap V \to X \times_S X, \quad p \circ h = i_U \iota_U, q \circ h = i_v \iota_V.$$

On the other hands, there are $\delta i_U \iota_U, \delta i_V \iota_V$ satisfying the conditions as h. We conclude that

$$p_1 \circ (\Delta i_u \iota_U) = id_X i_U \iota_U = i_U \iota_U, p_2 \circ (\Delta i_V \iota_V) = i_V \iota_V.$$

Again using the universal property, we derive

$$h = \Delta i_U \iota_U = i_{UV}\theta_{UV} \Rightarrow \Delta(U \cap V) = \theta_{UV}(U\cap) \subseteq U \times_S V \Rightarrow U \cap V \supseteq \Delta^{-1}(U \times_S V).$$

Also we have

$$U \subseteq p(U \times_S V) \subseteq p_1 \Delta \Delta^{-1}(U \times_S V) = \Delta^{-1}(U \times_S V),$$

we derive $V \subseteq \Delta^{-1}(U \times_S V)$, we conclude $U \cap V \subseteq \Delta^{-1}(U \times_S V)$.

We have U, V, S are affine therefore, $U \times_S V$ is affine. Since $U \cap V = \Delta^{-1}(U \times_S V)$. $\Delta|_{U \cap V}$ is a restriction of a closed immersion, there fore a closed immersion. We conclude that $U \cap V$ is affine.