V4A1 Sheet 5

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Let $\varphi : A \to \Gamma(X, \mathscr{O}_X)$ be a homomorphism of rings.

By the definition of schemes, there exists an open covering $(U_i)_{i\in I}$ of X such that

$$\forall i \in I, \mathscr{O}(U_i) \cong (\operatorname{Spec} A_i, \mathscr{O}_{\operatorname{Spec} A-i})$$

for some ring A_i.

Let $\rho^X_{XU_i}$ be a restriction map. Then $\varphi \circ \rho_{XU_i} : \mathscr{O}_{\operatorname{Spec} A}(U_i) \to \mathscr{O}_X(U_i) = A_i$ is a ring homomorphism. Thus there exists a morphism of ringed spaces $(f_i, f_i^\#) : (\operatorname{Spec} A_i, \mathscr{O}_{\operatorname{Spec} A_i}) \to (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ corresponding to $9\varphi \circ \rho_{XU_i}$. We will show that this can be glued to a unique morphism of ringed spaces $(f, f^\#)$.

Indeed, by construction of structure sheaves and restrictions, we have that $\{(f_i, f_i^\#)\}_{i\in I}$ satisfies the sheaf condition. Thus can be glued to a unique morphism $(f, f^\#)$. The correspondence $\varphi\circ\rho_{XU_i} \leftrightarrow (f_i, f_i^\#)$ is a bijection. Therefore, $\varphi \to (f, f^\#)$ is an injection. By the same argument to the restrictions of $(f, f^\#)$ and the sheaf condition, we derive that this is a surjection.

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(i)

Let $(U_i)_{i\in I}$ be an open cover of X such that

$$\mathscr{O}_X \cong \operatorname{Spec} A_i.$$

Let $\iota_i : A_i \to \tilde{A}_i$ be an inclusion. Then we can define a morphism $\nu_i : \tilde{U}_i \to U_i$ by

$$\nu_i(\mathfrak{q}) = \iota_i^{-1}(\mathfrak{q}).$$

By the definition of affine schemes and its restrictions, $(\nu_i)_{i\in I}$ satisfies the sheaf condition, thus there exists $\tilde{X}$ and $\nu : \tilde{X} \to X$ extending each ν_i.

By the construction, $\mathscr{O}_{\tilde{X}}(\tilde{U}_i) \cong \operatorname{Spec} \tilde{A}_i$. Since $\tilde{A}_i$ is integrally closed, $\tilde{A}_{i,\mathfrak{p}}$ is also integrally closed for each $\mathfrak{p} \in \operatorname{Spec} \tilde{A}_i$. Therefore, $\tilde{X}$ is normal.

(ii)

Let U be a neighborhood of $x \in X$ such that $\mathscr{O}_X(U) \cong \operatorname{Spec} A$. Suppose $f : Z \to X$ be a dominant morphism of schemes then $f(Z)$ is dense in X.
Let $V = f^{-1}(U)$. Since f is dominant, we have that the ring homomorphism induced by f, $\varphi : A = \mathscr{O}_X(U) \to \mathscr{O}_Z(V) = B$ is an injection. Since A is an integral domain and B is integrally closed, there is a unique homomorphism $\tilde{\varphi}$ such that

$$\varphi = \tilde{\varphi} \circ \iota$$

where $\iota : A \to \tilde{A}$ is an injection and $\tilde{\varphi} : \tilde{A} \to B$. By using the previous problem, there is a morphism of scheme $(g, g^{\#}) : ((V, \mathscr{O}_Z|_V), (U, \operatorname{Spec} A)) \to$ corresponding to $\tilde{\varphi}$ such that $\nu \circ g^{\#} = f^{\#}$.