

Analysis and Geometry on Manifolds

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1 Review of Structure Theorems for Differentiable Maps

Theorem 1.1 (Implicit Function Theorem). *Let $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$ be open sets and $F(x, y) : U \times V \rightarrow \mathbb{R}^q$ be of \mathcal{C}^∞ class. If $(a, b) \in U \times V$ satisfies $F(a, b) = 0$ and*

$$D_y F(a, b) = \left(\frac{\partial F_i}{\partial y_j}(a, b) \right) \in \mathrm{GL}_q(\mathbb{R})$$

then there exists

- i). neighborhoods $a \in U_1$ and $b \in V_1$,
- ii). $\varphi \in \mathcal{C}^\infty(U_1, V_1)$

such that

$$\forall (x, y) \in U_1 \times V_1, F(x, y) = 0 \Leftrightarrow \varphi(x) = y.$$

Furthermore, we have that

$$D\varphi(x) = -D_y F(x, \varphi(x))^{-1} \cdot D_x F(x, \varphi(x)).$$

Theorem 1.2 (Inverse Function Theorem). *Let $U \subseteq \mathbb{R}^p$ be an open subset and $f : U \rightarrow \mathbb{R}^q$ be smooth. Let $a \in U$ be such that $Df(a)$ is invertible. Then there are neighborhoods $a \in U_1 \subseteq U$ and $f(a) \subseteq V_1$ such that $f|_{U_1} : U_1 \rightarrow V_1$ is a diffeomorphism.*

Theorem 1.3 (Rank Theorem). *Let $U \subseteq \mathbb{R}^p$ be open and $f : U \rightarrow \mathbb{R}^q$ be smooth. Let $a \in U$ and $b = f(a)$. If $\mathrm{rk} Df(a) = r$ then there exists local diffeomorphisms*

- i). $\psi : U_\psi \subseteq U \rightarrow V_\psi \subseteq \mathbb{R}^p$ with $\psi(a) = 0$
- ii). $\varphi : U_\varphi \subseteq \mathbb{R}^q \rightarrow V_\varphi \subseteq \mathbb{R}^q$ with $\varphi(b) = 0$.

such that

$$\varphi \circ f \circ \psi^{-1}(x_1, \dots, x_p) = (x_1, \dots, x_r, \tilde{f}(x)).$$

Furthermore, if $\mathrm{rk} Df(x) = r$ in some neighborhood of a , then \tilde{f} can be chosen to be 0.

2 Differentiable Manifolds

2.1 Basics from Set Theoretic Topology

Definition 2.1. Let X be a topological space. X is said to be separated/Hausdorff if any two distinct points have open neighborhoods which are disjoint to one another.

Definition 2.2. Let X be a topological space and $(U_i)_{i \in I}$ be its open covering. A refinement of $(U_i)_{i \in I}$ is an open covering $(V_j)_{j \in J}$ such that

$$\forall j \in J, \exists i \in I \text{ s.t. } V_j \subseteq U_i.$$

It is locally finite if each point $x \in X$, there exists a neighborhood U_i such that

$$|\{j \in J \mid U \cap V_j \neq \emptyset\}| < \infty.$$

Definition 2.3. A topological space X is called paracompact if every open covering $(U_i)_{i \in I}$ has a locally finite refinement $(V_j)_{j \in J}$.

Example 2.1. The following spaces are paracompact.

1). Compact spaces.

2). Locally compact Hausdorff spaces which are first countable.

Definition 2.4. A subset $M \subseteq \mathbb{R}^p$ is said to be a submanifold of dimension m if M can be covered by open sets $(U_i)_{i \in I}$ such that there exists a smooth function $F_i : U_i \rightarrow \mathbb{R}^{p-m}$ with full rank and

$$M \cap U_i = \{x \in U_i \mid F_i(x) = 0\}.$$

In other words, M is locally a graph of a smooth map.

Definition 2.5. Let X be a separated and paracompact topological space. A chart is a pair (U, φ) where U is an open subset of X and $\varphi : U \rightarrow V \subseteq \mathbb{R}^n$ is a homeomorphism onto some open subset V of \mathbb{R}^n . An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ is a collection of charts such that $(U_i)_{i \in I}$ covers X .

Definition 2.6. Let $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ be an atlas. A transition map is a composition $\varphi_i \circ \varphi_j^{-1}|_{\varphi_j(U_i \cap U_j)} \rightarrow \varphi_i(U_i \cap U_j)$ for some $i, j \in I$.

Definition 2.7. An atlas \mathcal{A} is called smooth if all the transition maps are smooth.

Definition 2.8. A chart (U, φ) is smooth compatible with smooth atlas \mathcal{A} if for any $i \in I$, $\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ are smooth.

Definition 2.9. An atlas \mathcal{A} is smooth maximal if (U, φ) is smooth compatible with \mathcal{A} then $(U, \varphi) \in \mathcal{A}$.

Definition 2.10. A paracompact, smooth manifold is a pair (M, \mathcal{A}) such that

i). M is a paracompact separated topological space.

ii). \mathcal{A} is smooth maximal.

Remark 2.1. Above notions can be defined for C^k , analytic, continuous, algebraic, linear by simply replacing the word "smooth" with one of the formers.

Remark 2.2. It suffices to give one (smallest possible) atlas to define a smooth structure.

2.2 Examples of Smooth Manifolds

Definition 2.11. A topological space is second countable if it has a basis consists of at most countably many open subsets.

Remark 2.3. Instead of paracompactness, we assume smooth manifolds to be separated and second countable space. From this definition, we can also induce paracompactness.

Example 2.2. For $x, y \in \mathbb{R}$, $x \sim y \Leftrightarrow |x| = |y| > 1$. With this relation, we construct a quotient space $X = \mathbb{R}/\sim$.

We introduce its atlas by $U_1 = (-\infty, 1)/\sim = X - \{[1]\}$ with chart $\varphi_1(x) = [x]$, and $U_2 = (-1, \infty)/\sim = X - \{[-1]\}$ with chart $\varphi_2(x) = [x]$. However, this is non-separated as 1 and -1 cannot be separated.

Definition 2.12. A smooth submanifold of \mathbb{R}^n is a separated, second countable smooth manifold.

Take φ^{-1} , where φ runs through parametrization as charts.

Remark 2.4. $(\mathbb{R}^n, \{\text{id}\})$ is separated and second countable smooth manifold.

Definition 2.13. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$. Let $N \in S^n$ and call it the north pole. We define the stereographic projection with north pole N to be such that

$$\varphi_+ : S^n \setminus \{N\} \rightarrow \mathbb{R}^n, (x_1, \dots, x_n) \mapsto \frac{1}{1 - x_n} (x_1, \dots, x_{n-1}).$$

Similarly, we define

$$\varphi_- : S^n \setminus \{-N\} \rightarrow \mathbb{R}^n, (x_1, \dots, x_n) \mapsto \frac{1}{1 + x_n} (x_1, \dots, x_{n-1}).$$

Definition 2.14. A n -dimensional torus T^n is a direct product of n -many S^1 .

Definition 2.15. A lattice Γ in \mathbb{R}^n is a subgroup generated by a basis of \mathbb{R}^n over \mathbb{Z} .

Lemma 2.1.

$$T^n = \mathbb{R}^n / \Gamma.$$

Definition 2.16. Let $k = \mathbb{R}$ or \mathbb{C} . The projective space over the field K is

$$K\mathbb{P}^n = (K^{n+1} \setminus \{o\}) / K^\times.$$

Remark 2.5.

$$\mathbb{R}\mathbb{P}^n = S^n / x \sim -x.$$

And also we have,

$$\mathbb{C}\mathbb{P}^n = S^{2n+1} / S^1.$$

We can introduce atlases to them by

$$U_j = \{[x_0 : \cdots : x_n] \in K\mathbb{P}^n \mid x_j \neq 0\},$$

with chart,

$$\varphi_j : U_j \rightarrow K^n, \varphi([x_0 : \cdots : x_n]) = \left(\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j} \right).$$

Definition 2.17. Let \mathcal{M}, \mathcal{N} be smooth manifold. A function $f : \mathcal{M} \rightarrow \mathcal{N}$ is called smooth if for each $p \in \mathcal{M}$, there exist chart (U, φ) around p and (V, ψ) around $f(p)$ such that

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is smooth for some $m, n \in \mathbb{N}$.

Remark 2.6. Since transition maps are smooth, above map is well-defined and if f is smooth so is $\varphi' \circ f \circ \psi'$ for any charts φ', ψ' .

Proposition 2.1. Compositions of smooth maps are smooth.

Definition 2.18. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map. It is called a diffeomorphism if it is bijective and f^{-1} is also smooth.

2.3 Partitions of Unity

Proposition 2.2. Let $(\mathcal{M}, \mathcal{A})$ be a pair of a smooth manifold with its atlas which we assumed to be separated and second countable. Then it is paracompact. More precisely, given an open covering $(U_i)_{i \in I}$, there exists a countable locally finite refinement $(V_j)_{j \in J}$ together with a chart $\{\psi_j\}_{j \in J}$ which can be chosen such that

$$\psi_j : V_j \rightarrow B(o, 3)$$

such that

$$\mathcal{M} = \bigcup_{j \in J} \psi_j^{-1}(B(o, 1)).$$

Proof. \mathcal{M} is locally compact (look at the charts), hence there exists a compact subsets

$$K_1 \subset \subset K_2 \subset \subset K_3$$

such that

$$\mathcal{M} = \bigcup K_j$$

(ie. an exhaustion by compact sets). Note $K_{j+1} - \text{int } K_j$ is again compact. For $p \in K_{j+1} - \text{int } K_j$ choose a chart (V_p, ψ_p) such that

$$\psi_p(V_p) = B(o, 3), \psi_l(p) = 0.$$

Note that we can take V_p small enough so that there is $i \in I$ such that $V_p \subset U_i$.

By compactness, we can take $p_{j,1}, \dots, p_{j,r_j}$ such that

$$K_{j+1} - \text{int } K_j = \bigcup_{l=1}^{r_j} \varphi_{jl}^{-1}(B(0, 1)).$$

By making $V_{p_{j,l}}$ small enough we can assume

$$V_{p_{j,l}} \subset \text{int } K_{j+2} - K_{j-1}.$$

The union over all j gives the countable locally finite refinement. \square

From now on, a manifold refers to a smooth, separated, and second countable manifold.

Example 2.3. We define,

$$f_1(t) := \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Then f_1 smooth. We then define,

$$f_2(t) := \frac{f_1(t)}{f_1(t) + f_1(1-t)}.$$

f_2 is a function which is monotonically increasing and for $t \geq 1$, we have $f_2(t) = 1$.

$$f_3(t) = f_2(2+t) + f_2(2-t).$$

Again we define

$$f_4 : \mathbb{R}^n \rightarrow \mathbb{R}, f(x) = f_3(\|x\|).$$

Definition 2.19. A support of the function $f : X \rightarrow \mathbb{R}$ is

$$\text{supp}(f) := \overline{f^{-1}(\mathbb{R} \setminus \{0\})}.$$

Theorem 2.1. Let \mathcal{M} be a manifold and $(U_i)_{i \in I}$ be an open covering. Then there exist smooth functions $(\varphi_n : \mathcal{M} \rightarrow \mathbb{R})_{n \in \mathbb{N}}$ such that

- i). $\forall n \in \mathbb{N}, 0 \leq \varphi_n \leq 1$.
- ii). $(\text{supp } \varphi_n)_{n \in \mathbb{N}}$ is locally finite.
- iii). $\forall n \in \mathbb{N} \exists i \in I, \text{s.t. } \text{supp } \varphi_n \subset U_i$.
- iv). $\forall p \in \mathcal{M}, \sum_{n=1}^{\infty} \varphi_n(p) = 1$.

Such sequence of functions is called a partition of unity subordinated by the given covering $(U_i)_{i \in I}$.

Additionally, if $I \subset \mathbb{N}$, then $(\varphi_n)_{n \in \mathbb{N}}$ can be indexed by I as well in such a way that,

$$\forall i \in I, \text{supp } \varphi_i \subset U_i.$$

Proof. By Proposition 2.2, take a countable, locally finite refinement charts $((\tilde{\psi}_n, V_n))_{n \in \mathbb{N}}$. Borrowing the notation from Example 2.3, we set

$$\tilde{\varphi}_n(x) := \begin{cases} f_4(\tilde{\psi}_n(x)) & \forall x \in V_n, \\ 0, & x \notin V_n. \end{cases}$$

Then take $\tilde{\varphi}$ to be

$$\varphi := \sum_{n=1}^{\infty} \tilde{\varphi}_n \in \mathcal{C}^{\infty}(\mathcal{M}).$$

We observe that φ is nowhere 0. Thus we define

$$\varphi_n := \tilde{\varphi}_n / \tilde{\varphi},$$

we derived a desired family of functions.

For the second part, set

$$J_0 = \emptyset, \quad \varphi_0 = 0,$$

define inductively that

$$J_k = \left\{ i \in \mathbb{N} \setminus \bigcup_{i=0}^{k-1} J_i \mid \text{supp } \varphi_i \subseteq U_k \right\}.$$

We then take

$$\psi_k := \sum_{i \in J_k} \tilde{\varphi}_i / \tilde{\varphi}.$$

□

Proposition 2.3. Let $A \subset \mathcal{M}$ be a closed subset of a manifold such that there is an open set $A \subset G \subset \mathcal{M}$. Then there exists a smooth function $f \in \mathcal{C}^\infty(\mathcal{M})$ such that the image of f is contained in $[0, 1]$ and

$$\forall p \in A, f(p) = 1, \text{ and } \forall p \in G, f(p) = 0.$$

Proof. Observe that $\{\mathcal{M} - A, G\}$ is an open covering. By Theorem 2.1, we can take smooth functions φ, ψ such that

$$\text{supp } \varphi \subset \mathcal{M} - A, \text{supp } \psi \subset G, \varphi + \psi \equiv 1.$$

Take $f = \psi$. \square

2.4 Tangent Spaces

Example 2.4. Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a submanifold. Then we have a notion of tangent vector, such that for some $v \in \mathbb{R}^n$, we define a smooth curve γ such that

$$\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}, \quad \varphi'(0) = v.$$

Given $v \in \mathbb{R}^n$, $p \in \mathcal{M}$ and $f : \mathcal{M} \rightarrow \mathbb{R}^n$, we define

$$vf = (f \circ \gamma)'(0).$$

where $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}$ is a smooth curve such that $\gamma(0) = p, \gamma'(0) = v$. Exercise : show that this definition does not depend on the choice of γ satisfying the two conditions above. Furthermore, this defines a linear map

$$v : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathbb{R},$$

such that

$$v(f \cdot g) = (f \circ \gamma \cdot g \circ \gamma)'(0) = f(p) \cdot (vg) + (vf) \cdot g(p).$$

Definition 2.20. Let \mathcal{M} be a smooth manifold and $p \in \mathcal{M}$. The linear map $X_p : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathbb{R}$ is called a derivation at p if $X_p(fg) = f(p)X_p g + g(p)X_p f$.

We define the tangent space of \mathcal{M} at p to be

$$T_p \mathcal{M} = \{X_p : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathbb{R} \mid X_p \text{ is a derivation}\}.$$

Remark 2.7. Obviously $T_p \mathcal{M}$ is a \mathbb{R} -vectorspace.

Lemma 2.2. if $\varphi \in \mathcal{C}^\infty(\mathcal{M})$ is constant around $p \in \mathcal{M}$. Then $X \varphi = 0$ for any $X \in T_p \mathcal{M}$.

Proof. Let $\varphi \equiv 1$ around p . Choose $\chi \in \mathcal{C}^\infty(\mathcal{M})$ to be such that

1. $\chi \equiv 1$ in a neighborhood of p .
2. $\text{supp } \chi \subseteq \{q \in \mathcal{M} \mid \varphi(q) = 1\}$.

Then $\chi\varphi = \chi$. Thus we have

$$X(\chi) = X(\chi)\varphi(p) + \chi(p)X(\varphi). \quad (1)$$

Since $\varphi(p) = 1$ and $\chi(p) = 1$, thus we conclude $X(\varphi) = 0$. \square

Lemma 2.3. *If $X \in T_p\mathcal{M} \cap T_q\mathcal{M}$ then either $X = 0$ or $p = q$.*

Proof. Exercise. \square

Lemma 2.4. *Let (U, φ) be a chart of \mathcal{M} centered at $p \in \mathcal{M}$ (ie. $\varphi(p) = 0$) with the coordinate function x_1, \dots, x_n . Then for $f \in \mathcal{C}^\infty(U)$, there are functions $f_1, \dots, f_n \in \mathcal{C}^\infty(U)$, such that*

$$f = \sum_{j=1}^n f_j x_j + f(p).$$

Note that taking $\varphi(p) = 0$, justifies the $f_j x_j$ for each coordinate.

Analogously, given $f \in \mathcal{C}^\infty(\mathcal{M})$, we may choose $f_j \in \mathcal{C}^\infty(\mathcal{M})$ such that

$$f|_U = \sum_{j=1}^n f_j|_U x_j + f(p).$$

Proof. The proof for the second part is assigned as an exercise.

Consider $\mathcal{M} = \mathbb{R}^n$, $p = 0$, and $U = (-\varepsilon, \varepsilon)^n$. We have

$$f(x) = \sum_{j=1}^n (f(x_1, \dots, x_j, 0, \dots, 0) - f(x_1, \dots, x_{j-1}, 0, \dots, 0)) + f(0).$$

By the fundamental theorem of calculus, we obtain,

$$f(x) = \sum_{j=1}^n \int_0^1 \partial_j f(x_1, \dots, tx_j, 0, \dots, 0) dt x_j + f(0).$$

By setting $f_j = \int_0^1 \partial_j f(x_1, \dots, tx_j, 0, \dots, 0) dt$, we derive the statement. \square

Definition 2.21. *Given a chart (U, φ) of \mathcal{M} . For $f \in \mathcal{C}^\infty(U)$, we define*

$$\left. \frac{\partial}{\partial x_j} \right|_p f := \left. \frac{\partial}{\partial \gamma_j} \right|_{\varphi(p)} (f \circ \varphi^{-1})(\gamma) = D(f \circ \varphi^{-1})(\varphi(p))[e_j].$$

Proposition 2.4. *Let \mathcal{M} be a smooth manifold and (U, φ) be a n -dimensional chart at p . Then we have*

$$\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \dots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$$

forms a basis in $T_p\mathcal{M}$.

Proof. Write

$$f = \sum_{i=1}^n f_i x_i + f(p).$$

Choose $\chi \in \mathcal{C}^\infty(\mathcal{M})$ such that

- i). χ is compactly supported in U .
- ii). $\chi \equiv 1$ in a neighborhood of p .

Then

$$\chi^2 f = \sum_{j=1}^n (\chi f_j)(\chi x_j) + \chi^2 f(p),$$

in the neighborhood of p . Since χ is compactly supported. This is defined everywhere on U .

Let $X \in T_p \mathcal{M}$ be a derivation. Then observe that

$$X\chi^2 = 0, \quad Xf = X(\chi^2 f).$$

Therefore,

$$\begin{aligned} Xf &= \sum_{j=1}^n X((\chi f_j))(Xx_j)(p) + f_j(p)X(\chi x_j) \\ &= \sum_{j=1}^n f_j(p)X(\chi x_j) \\ &= \sum_{j=1}^n X(\chi x_j) \cdot \left. \frac{\partial}{\partial x_j} \right|_p f. \end{aligned}$$

This implies that

$$X = \sum_{j=1}^n X(\chi x_j) \left. \frac{\partial}{\partial x_j} \right|_p .$$

The set

$$\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \dots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$$

spans $T_p \mathcal{M}$. Remains to show the linearly independentness. To do so consider

$$\frac{\partial}{\partial x_j} x_i = \delta_{ij}.$$

We conclude the proof. \square

Example 2.5. For $\mathcal{M} = \mathbb{R}^n$ and $p \in \mathbb{R}^n$, the partial derivatives with respect to the standard coordinate at p is a basis of $T_p \mathbb{R}^n$. Explicitly each is of the form,

$$\frac{\partial}{\partial x_j} \Big|_p f = \frac{\partial f}{\partial x_j}(p).$$

Example 2.6. Let (U, φ) be a chart of an open subset of \mathbb{R}^n with coordinate function y_1, \dots, y_n , (ie. $\varphi = (y_1, \dots, y_n)$).

We have

$$\begin{aligned} \frac{\partial}{\partial x_j} \Big|_p f &= \frac{\partial}{\partial x_j} (f \circ \varphi^{-1} \circ \varphi)(0), \\ &= \sum_{j=1}^n \partial_j (f \circ \varphi^{-1})(\varphi(p)) \frac{\partial \varphi_j}{\partial x_i}(p). \end{aligned}$$

Let $\frac{\partial}{\partial y_j} \Big|_p f = \partial_j (f \circ \varphi^{-1})(\varphi(p))$, we get,

$$= \sum_{j=1}^n \frac{\partial y_j}{\partial x_i}(p) \frac{\partial}{\partial y_j} \Big|_p f.$$

In particular,

$$\frac{\partial}{\partial x_i} \Big|_p f = \sum_{j=1}^n \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j} \Big|_p f.$$

Definition 2.22. Let \mathcal{M}, \mathcal{N} be smooth manifold and $f : \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map. For $p \in \mathcal{M}$, we have a linear transform

$$T_p f : T_p \mathcal{M} \rightarrow T_p \mathcal{N}$$

such that given $X \in T_p \mathcal{M}$ and $h \in C^\infty(\mathcal{N})$, we define

$$T_p f(X)h = X(h \circ f).$$

Lemma 2.5. Let $\mathcal{M} \subseteq \mathbb{R}^m$ and $\mathcal{N} \subseteq \mathbb{R}^n$ be open. Then in the standard basis $T_p f$ is the Jacobi-matrix.

Proof. Exercise. □

Lemma 2.6 (Chain Rule). Let $f : \mathcal{M} \rightarrow \mathcal{N}$, $g : \mathcal{N} \rightarrow \mathcal{W}$ be smooth maps. Then we have

$$T_p(g \circ f) = T_{f(p)} g \circ T_p f.$$

Proof. Exercise. □

Lemma 2.7. Let (U, φ) be a chart. Viewing this as a locally smooth map from \mathcal{M} to \mathbb{R}^n , we obtain an isomorphism,

$$T_p\varphi : T_p\mathcal{M} \rightarrow T_{\varphi(p)}\mathbb{R}^n = \mathbb{R}^n.$$

If (V, ψ) is another chart then for $p \in U \cap V$, we have,

$$T_p\psi = T_p(\psi \circ \varphi^{-1} \circ \varphi) = D_{\varphi(p)}(\psi \circ \varphi^{-1})T_p\varphi.$$

Proof. Exercise. \square

Definition 2.23 (Physicists definition of tangent space). A tangent vector is a family $(\xi^\varphi)_\varphi$ where $\xi^\varphi \in \mathbb{R}^n$ where φ runs through charts containing p , together with a transformation rule

$$\xi^\psi = D_{\varphi(p)}(\psi \circ \varphi^{-1})[\xi^\varphi].$$

Definition 2.24. Let \mathcal{M} be a manifold. A curve $\gamma : I \rightarrow \mathcal{M}$ in \mathcal{M} is a smooth map such that from some interval $I \subset \mathbb{R}$, such that I is a manifold with the canonical charts. $d : I \rightarrow I$ where $\frac{d}{dt}$ a canonical tangent vector associated to this chart, we get a velocity vector of the curve

$$\dot{\gamma}(t_0) = T_{t_0}\gamma\left(\frac{d}{dt}\Big|_{t_0}\right) \in T_{\gamma(t_0)}\mathcal{M}.$$

Example 2.7. Given any $v \in T_p\mathcal{M}$, and a chart (U, φ) centered at p (ie. $\varphi(p) = 0$), we have

$$v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_j}\Big|_p$$

Define a curve

$$\gamma(t) = \varphi^{-1}(tv_1, \dots, tv_n).$$

Note that $\gamma(0) = \varphi^{-1}(0) = p$. We define

$$\dot{\gamma}(0)f = T_0\gamma\left(\frac{d}{dt}\Big|_0\right)f = \frac{d}{dt}\Big|_{t=0}f(\gamma(t)) = \sum_{i=1}^n v_i \frac{\partial f \circ \varphi^{-1}}{\partial x_j}(0) = vf.$$

2.5 Local properties of differentiable maps and submanifolds.

In this section, all manifolds are assumed to have pure dimension (ie. it has a constant dimension everywhere).

Notation 2.1. \mathcal{M}^m denotes that $\dim \mathcal{M} = \dim T_p\mathcal{M} = m$ for any point $p \in \mathcal{M}$.

Definition 2.25. Let $\mathcal{M}^m, \mathcal{N}^n$ be manifolds and $f : \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map.

- 1). $p \in \mathcal{M}$ is called a critical point for f if $\text{rank } T_p f < n = \dim \mathcal{N}$ and $f(p)$ is called the critical value.

2). $q \in \mathcal{N}$ is called a regular value if $\forall p \in f^{-1}(q)$ we have $T_q f = \dim \mathcal{N} = n$.

3). f is called a submersion if for any $p \in \mathcal{M}$, $T_p f = n$.

4). f is called an immersion if for any $p \in \mathcal{M}$, $T_p f = m$.

5). f is called a subimmersion if $\mathcal{M} \ni p \mapsto \text{rank } T_p f$ is a constant.

Notation 2.2. For $f : \mathcal{M} \rightarrow \mathcal{N}$, we denote the set of critical points to be C_f .

Definition 2.26 (Submanifold). Let $\mathcal{N} \subset \mathcal{M}^m$ is called a submanifold if for any chart (U, φ) of \mathcal{M} ,

$$\varphi(\mathcal{N} \cap U) \subseteq \mathbb{R}^m$$

is a submanifold of \mathbb{R}^m .

Remark 2.8. The above definition is equivalent to say that for each $p \in \mathcal{M}$, there exists a chart (U, φ) centered at p such that

$$\varphi(\mathcal{N} \cap U) = \mathbb{R}^k \times \{0\} \cap \varphi(U) \subset \varphi(U) \subset \mathbb{R}^m.$$

In other words, the intersection is diffeomorphic to some hyperspace with dimension k in \mathbb{R}^m .

Definition 2.27. A smooth map $f : \mathcal{M} \rightarrow \mathcal{N}$ is an embedding if

i). f is an injective immersion.

ii). $f : \mathcal{M} \rightarrow f(\mathcal{M}) \subset \mathcal{N}$ is a homeomorphism.

Definition 2.28. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be smooth. The rank of f is the map $\mathcal{M} \ni p \mapsto \text{rank } T_p f$.

Proposition 2.5. Let $f : \mathcal{M}^m \rightarrow \mathcal{N}^n$ be a subimmersion of rank k .

1). For $q \in \mathcal{N}$, the set $f^{-1}(\{q\}) \subset \mathcal{M}$ is empty or a submanifold of dimension $n - k$.

2). For $p \in \mathcal{M}, q := f(p)$ there exists a neighborhood U of p and V of q such that $S = f(U) \cap V$ is a submanifold of \mathcal{N} of dimension k .

Proof. Apply the rank theorem for $f(p) = q$. There exists a chart (U, φ) and (V, ψ) which are centered at p and q , respectively such that

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \varphi \downarrow & & \downarrow \psi \\ \mathbb{R}^m \supset U' & \xrightarrow{(x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0)} & V' \subset \mathbb{R}^n \end{array}$$

□

Corollary 2.1. $f : \mathcal{M}^m \rightarrow \mathcal{N}^n$ is smooth and $q \in \mathcal{N}$ is a regular value, then $f^{-1}(\{q\})$ is a submanifold of dimension $m - n$ or empty.

2.6 The Theorem of Morse-Sard

Notation 2.3. The Lebesgue measure on \mathbb{R}^m is denoted by λ^m .

Definition 2.29 (Null set). $A \subset \mathcal{M}^m$ is a nullset if for each chart (U, φ) , the set $\varphi(U \cap A)$ is a λ^m -nullset in \mathbb{R}^m .

Remark 2.9. A diffeomorphism maps nullsets to nullsets. Hence the above notion is well-defined.

Remark 2.10. A singleton of a manifold is a nullset. Countable unions of nullsets are again nullsets.

Remark 2.11. Let $A \subset \mathcal{M}^m$ where $m > 0$, $\mathcal{M} \setminus A$ is dense.

Remark 2.12. Suppose we have a smooth function $f : \mathbb{R}^m \supset U \rightarrow \mathbb{R}$. For the sake of simplicity, we assume $f(0) = 0$ and $\frac{\partial f}{\partial x_1}(0) \neq 0$. Consider

$$h(x) = (f(x), x_2, \dots, x_m).$$

Then we have

$$Jf(0) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(0) & \frac{\partial f}{\partial x_2}(0) & \cdots & \frac{\partial f}{\partial x_m}(0) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1. \end{pmatrix}$$

This is invertible, therefore we have h is a local diffeomorphism. Consider

$$g := f \circ h^{-1}.$$

We denote $(t, \xi) = h(x)$, then

$$g(t, \xi) = f \circ h^{-1} \circ h(x) = f(x) = t.$$

Theorem 2.2. Let $f : \mathcal{M}^m \rightarrow \mathcal{N}^n$ be smooth where $n \geq 1$. Then the set C_f of critical values of f is a nullset.

Proof. Suffices to prove that each $p \in \mathcal{M}$ has an open neighborhood U such that $f(C_f \cap U)$ is a nullset.

\Rightarrow) Without the loss of generality, we assume $\mathcal{M} = U \in \mathbb{R}^m$ and $\mathcal{N} = \mathbb{R}^n$ for $n \geq 1$. We will prove the theorem by induction on m .

If $m = 0$, then the image $f(\mathcal{M})$ is at most countable thus a nullset.

Assume the claim holds for all dimension less than m . Let us define

$$C := C_f, \quad C_1 := \left\{ x \in U \mid \forall |\alpha| \leq l, \frac{\partial^{|\alpha|} f}{\partial_x^\alpha}(x) = 0 \right\} l \geq 1. \quad (2)$$

Then we have $f(C \setminus C_l)$ is a nullset. Pick $\xi \in C \setminus C_l$ then there exists $\frac{\partial f_i}{\partial x_j}(\xi) \neq 0$ for some i, j . Without the loss of generality, we assume $i = j = 1$. We put

$$h(x) = (f_1(x), x_2, \dots, x_m).$$

By Remark 2.12, we have h is a local diffeomorphism and in particular,

$$\xi \mapsto (f_1(\xi), \xi_2, \xi_m).$$

Similarly to the remark, we define

$$g := f \circ h^{-1}, \quad g(t, x) = (t, \tilde{g}(t, x)).$$

(t, x) is critical for g if and only if x is critical for $\tilde{g}(t, \cdot)$. Now consider

$$\begin{aligned} \lambda^n(f(C_f \cap V)) &= \lambda^n(\{g(t, x) \mid (t, x) \text{ is critical for } g\}), \\ &= \lambda^n(\{(t, \tilde{g}(t, x)) \mid x \text{ is critical for } \tilde{g}(t, \cdot)\}), \\ &\stackrel{\text{Fubini}}{=} \int_I \lambda^{n-1}\{\tilde{g}(t, x) \mid x \text{ is a critical point of } \tilde{g}(t, \cdot)\} dt. \end{aligned}$$

The inside of the integral is 0 by the induction hypothesis.

Fix $1 \leq l < \infty$, then $f(C_l \setminus C_{l+1})$ is a nullset. Indeed, for $\xi \in C_l \setminus C_{l+1}$, without the loss of generality, we assume there is a multiindex β such that $|\beta| = l$ and

$$\frac{\partial^\beta f_1}{\partial x^\beta}(\xi) = 0.$$

Similarly for the previous case, we set

$$h(x) = \left(\frac{\partial^\beta f_1}{\partial x^\beta}(x), x_2, \dots, x_m \right).$$

By the similar argument as above, we get the claim.

Finally, let $d > 0$ and W be a cube of side length b . Consider $x \in C_k \cap W, y \in W$. From Taylor expansion at x implies that

$$|f(x) - f(y)| \leq L|x - y|^{k+1},$$

where L depends on the cube W , k and f . Also by fixing k , we can make it a locally uniform constant.

Subdivide each edge of W into r edges to make W into r^m many cubes (W_j) of edge length $\frac{d}{r}$. For $x \in C_k \cap W_j$, and $y \in W_j$, we have

$$|x - y| \leq \sqrt{m} \frac{d}{r}.$$

In particular, the constant \sqrt{m} only depends on the dimension. Back to the previous argument, we get

$$|f(x) - f(y)| \leq L \left(\frac{\sqrt{m}d}{r} \right)^{k+1}.$$

This means that $f(C_k \cap W_j)$ sits in a cube of edge length less than or equal to $2L \left(\frac{\sqrt{md}}{r}\right)^{k+1}$.

$$\lambda^n(f(C_k \cap W)) \leq r^m \lambda(\max_{1 \leq j \leq r^m} f(C_k \cap W_j)) \leq r^m \left\{ 2L \left(\frac{\sqrt{md}}{r}\right)^{k+1} \right\}^n = K \cdot r^{m-n(k+1)}.$$

Observe that $r \rightarrow \infty$ and $k \geq \frac{m}{n}$, we get $C_k \cap W$ has measure 0. \square

3 Vector fields and dynamical system

3.1 Definition

Definition 3.1. Let \mathcal{M} be a smooth manifold. A smooth vector field of \mathcal{M} is a map

$$X : \mathcal{M} \rightarrow T\mathcal{M},$$

such that

1. $\forall p \in \mathcal{M}, X(p) \in T_p\mathcal{M}$.
2. For any chart (U, φ) centered at p , we have $X|_U = \sum_{i=1}^m X_i^\varphi \frac{\partial}{\partial x_i}$ where $X_i^\varphi \in C^\infty(U)$ for each i .

Notation 3.1.

$$\Gamma(T\mathcal{M}) = \{\text{smooth vector fields on } \mathcal{M}\}.$$

Remark 3.1. The second condition can be restated as follow. Recall Definition 2.22. A chart $\varphi : U \rightarrow \mathbb{R}^m$ can be considered a smooth map between U and \mathbb{R}^m . We also have seen that $T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^n$. With this identification, we have,

$T\varphi : TU \rightarrow T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m$, $TU \ni [U \ni p \mapsto X_p] \mapsto [\mathbb{R}^m \ni x \mapsto X_{\varphi^{-1}(x)}]$, is smooth for all chart φ .

Example 3.1. Consider,

$$X : S^1 \rightarrow TS^1, (x, y) \mapsto (-y, x),$$

is a smooth vector field.

Definition 3.2. Let $I \subset \mathbb{R}$ be an interval and $X \in \Gamma(T\mathcal{M})$. A curve $\gamma : I \rightarrow \mathcal{M}$ is called an integral curve of X if

$$\forall t \in I, \gamma'(t) = X(\gamma(t)).$$

Remark 3.2. To find such curve is equivalent to solve the following autonomous initial value problem. Given a chart (U, φ) around a point $p \in \mathcal{M}$,

$$\begin{cases} (\varphi \circ \gamma)'(t) &= \begin{pmatrix} X_1^\varphi \circ \varphi^{-1} \\ \vdots \\ X_m^\varphi \circ \varphi^{-1} \end{pmatrix} (\varphi \circ \gamma(t)), \\ (\varphi \circ \gamma)(t_0) &= 0. \end{cases}$$

3.2 Flow-Box Theorem

Lemma 3.1 (Grönwall). *Let $f, g : [a, b] \rightarrow (0, \infty)$ be continuous functions such that*

$$f(t) \leq c + \int_0^t f(s)g(s)ds,$$

for some constant $c > 0$. Then we have,

$$f(t) \leq c \exp\left(\int_0^t g(s)ds\right).$$

Proof. Consider,

$$\tilde{f}(t) = c + \int_0^t f(s)g(s)dt.$$

Then by assumption $f(t) \leq \tilde{f}(t)$. Also set,

$$h(t) = \tilde{f}(t) \exp\left(-\int_0^t g(s)ds\right).$$

Then f, h are both differentiable thus we take,

$$\begin{aligned} h'(t) &= f(t)g(t) \exp\left(-\int_0^t g(s)ds\right) - g(t)\tilde{f}(t) \exp\left(-\int_0^t g(s)ds\right), \\ &= \underbrace{(f(t) - \tilde{f}(t))}_{\leq 0} g(t) \exp\left(-\int_0^t g(s)ds\right), \\ &\leq 0. \end{aligned}$$

We also have,

$$h(0) = c.$$

Thus we conclude,

$$h(t) \leq c \Rightarrow f(t) \leq \tilde{f}(t) = h(t) \exp\left(\int_0^t g(s)ds\right) \leq c \exp\left(\int_0^t g(s)ds\right).$$

□

Theorem 3.1. *Let \mathcal{M} be a smooth manifold and $X \in \Gamma(T\mathcal{M})$. Then for each $p \in \mathcal{M}$ there exists a neighborhood U , $\varepsilon > 0$, and a smooth map*

$$F : (-\varepsilon, \varepsilon) \times U \rightarrow \mathcal{M},$$

such that

- 1). $\forall x \in U, F(0, x) = x$,
- 2). $\partial_t F(t, x) = X(F(t, x))$.

Proof. Recall that a continuously differentiable function over a compact set is Lipschitz.

Consider a smooth function $f : \overline{B(y_0, r)}$ such that

$$a(y) < f(y) < b(y).$$

where $y_0 \in \mathbb{R}^n$, $r > 0$ and a, b are continuous function. For $p \in B(y_0, r)$, we let $F(t, y)$ to be the maximal solution of the initial value problem,

$$\begin{cases} \partial_t F(t, y) = f(F(t, y)), \\ F(0, y) = y_0. \end{cases}$$

By the construction, we get,

$$F(t, y) = y + \int_0^t f(F(s, y))ds.$$

Thus for $|t| \leq 1$, we get,

$$\begin{aligned} |F(t, y) - y_0| &\leq \left| y - y_0 + \int_0^t f(F(s, y))ds \right|, \\ &\leq |y - y_0| + |t||f(y_0)| + \int_0^t |f(F(s, y)) - f(y_0)|ds, \\ &\leq |y - y_0| + |f(y_0)| + \int_0^t L|F(s, y) - y_0|ds, \end{aligned}$$

for some $L > 0$. By applying Lemma 3.1, we get,

$$|F(t, y) - y_0| \leq (|y - y_0| + |f(y_0)|)e^{L|t|}.$$

□

Definition 3.3. Above $F(t, \cdot)$ is called the local flow of the vector field X .

3.3 Applications of Mrose-Sards THeorem

3.3.1 Embedding

Theorem 3.2 (Whitney). Each \mathcal{M}^m has an embedding into \mathbb{R}^{2m+1} .

Proof. We prove the case for \mathcal{M} compact. Note that if \mathcal{M} is compact and $f : \mathcal{M} \rightarrow \mathcal{N}$ is an injective immersion. Then f is an embedding. Since this gives the continuity of the inverse map.

Using Morse-Sard's theorem, \mathcal{M}^m compact, there exists an embedding into some \mathbb{R}^N , for $N \gg 0$. Indeed, choosing charts $(U_j, \varphi_j)_{j=1, \dots, r}$, (by the compactness, finitely many would cover the whole). Such that

$$\varphi_j(U_j) \supset B(0, 3),$$

and

$$\mathcal{M} = \bigcup_{j=1}^r \varphi^{-1}(B(0,1)).$$

Fix $g \in \mathcal{C}^\infty(\mathbb{R}^m)$ be such that

$$g(x) = \begin{cases} 1, & (|x| \leq \frac{4}{3}) \\ 0, & (|x| \geq \frac{5}{3}) \end{cases}.$$

Further defining the functions,

$$f_j(p) = \begin{cases} g(\varphi_j(p))\varphi_j(p), & (p \in U_j) \\ 0, & (\text{otherwise}) \end{cases}.$$

Then we have $f_j \in \mathcal{C}^\infty(\mathbb{R}^m)$. In particular

$$f_j|_{\varphi_j^{-1}(B(0,1))}$$

is a diffeomorphism.

Consider the tuple

$$(f_1, \dots, f_r, \dots, g \circ \varphi_1, \dots, g \circ \varphi_r) : \mathcal{M} \rightarrow \mathbb{R}^{(m+1)r},$$

is an injective immersion, hence it is an embedding.

Let $\mathcal{M} \subset \mathbb{R}^N$, compact. For $w \in S^{N-1}$, let π_w be the orthogonal projection onto $\langle w \rangle^\perp$ which is a hyperplane of dimension $N - 1$ which is given by

$$\pi_w(x) = x - \langle w, x \rangle w.$$

We have

$$\pi_w p = \pi_w q \Leftrightarrow \pi_w(p - q) = 0 \Leftrightarrow p - q \parallel w.$$

We construct a map,

$$\phi : \mathcal{M} \times \mathcal{M} \setminus \{(p, p) \mid p \in \mathcal{M}\} \rightarrow S^{N-1}, \phi(p, q) = \frac{p - q}{|p - q|}.$$

If $2m < N - 1$, using Morse-Sarde, the image of ϕ is of measure 0. Therefore,

$$\{w \in S^{N-1} \mid \exists p, q \in \mathcal{M}, p \neq q, p - q \parallel w\} = \text{Im } \phi \cup -\text{Im } \phi.$$

The right hand side is a nullset. More explicitly, the set of those w such that $\pi_w|_{\mathcal{M}}$ is not injective is a nullset.

Suppose π_w is an immersion if for $p \in \mathcal{M}$, $v \in T_p \mathcal{M} \setminus \{0\}$, we have $\pi_w(v) \neq 0$. π_w is an immersion, if

$$\forall p \in \mathcal{M}, w \notin T_p \mathcal{M} \Leftrightarrow \not\in \text{Im } \sigma$$

where

$$\sigma : T\mathcal{M} \setminus \{0\} \rightarrow S^{N-1}, v \mapsto \frac{v}{|v|}.$$

□

Definition 3.4. Let $X \in \Gamma(T\mathcal{M})$. We define a flow of X to be

$$\phi^X : A \rightarrow \mathcal{M},$$

where

$$A = \mathcal{D}(\phi^X) := \bigcup_{p \in \mathcal{M}} (a_p, b_p) \times \{p\} \subseteq \mathbb{R} \times \mathcal{M}$$

such that $\phi^X(p)$ is maximal curve through p . $\mathcal{D}(\phi^X)$ is called the flow domain of X .

Proposition 3.1. Let $X \in \Gamma(T\mathcal{M})$, and $\phi^X \mathcal{D}(\phi^X) = A \rightarrow \mathcal{M}$ be the flow of X . Then the following statements hold.

- 1). A is open subset in $\mathbb{R} \times \mathcal{M}$ and contains $\{0\} \times \mathcal{M}$.
- 2). $\phi^X \in \mathcal{C}^\infty(A)$.
- 3). For $t \in \mathbb{R}$, $\mathcal{D}(\phi_t^X) = \{p \in \mathcal{M} \mid (t, p) \in A\} \subset \mathcal{M}$ where $\phi_t^X(\cdot) = \phi^X(t, \cdot)$, is open.
- 4). $\mathcal{M} = \bigcup_{t>0} \mathcal{D}(\phi_t^X)$.
- 5). $\phi_t^X : \mathcal{D}(\phi_t) \rightarrow \mathcal{D}(\phi_{-t})$ is a diffeomorphism.
- 6). $\phi_s^X \circ \phi_t^X \subseteq \phi_{s+t}^X$ in other words if $p \in \mathcal{D}(\phi_t^X) \cap \phi_s^X(p) \in \mathcal{D}(\phi_s^X)$ then $p \in \mathcal{D}(\phi_{s+t}^X)$.
- 7). $\phi_s^X(\phi_t^X(p)) = \phi_{s+t}(p)$.

Proof.

For 4), let $s, t > 0$, and $p \in \mathcal{D}(\phi_t^X)$, then by assumption, $\phi_t^X(p) \in \mathcal{D}(\phi_{s+t}^X)$. Consider

$$c(u) = \begin{cases} \phi_u(p), & 0 \leq u \leq t, \\ \phi_{u-t}(\phi_t(p)), & t \leq u \leq t+s, \end{cases}$$

is an integral curve through p as the two expressions coincide at $u = t$, conclude using the uniqueness. Therefore,

$$\phi_{s+t}^X(p) = c(t+s) = \phi_s(\phi_t^X(p)).$$

The cases for $s \leq 0$ or $t \leq 0$ are exercises.

For 2), let $p \in \mathcal{D}(\phi_t^X)$, $t > 0$, we define,

$$B := \{ \phi_s^X(p) \mid 0 \leq s \leq t \}.$$

This is compact. By using Flow-box theorem, there is a neighborhood $W_0 \supseteq B$ and $\varepsilon > 0$ such that

$$(-\varepsilon, \varepsilon) \times W_0 \ni (u, q) \mapsto \phi_u^X(q) \mathcal{M}$$

is smooth. Then choose N such that $\frac{t}{N} < \varepsilon$. Define inductively W_j by

$$W_j = \left(\phi_{\frac{t}{N}}^X \Big|_{W_{j-1}} \right)^{-1} (W_{j-1}).$$

Claim 3.2.1. 1). $W_0 \supseteq W_1 \supseteq \cdots \supseteq W_N$ and they are all open.

2). For $q \in W_N$, $\phi_{t \frac{N-j}{N}}^X(q) \in W_j$.

3). For $0 \leq u \leq t \frac{N-j}{N}$, $\phi_u^X(p) \in W_j$. In particular $p \in W_N$.

Proof of the Claim. For Claim 2). $q \in W_N$, $\phi_{\frac{t}{N}}^X(q) \in W_{N-1}$ and inductively $\phi_{t \frac{j}{N}}^X(q) \in W_{N-j}$.

For Claim 3). $\varphi_u^X(p) \in B \subset W_0$, $0 \leq u \leq t$.

On the induction step $j \rightarrow j+1$, $0 \leq u \leq t \frac{N-j-1}{N}$, then

$$\frac{t}{N} + u \leq t \frac{N-j}{N},$$

so

$$\phi_{\frac{t}{N}}^X(\phi_u^X(p)) \in W_j, \quad \phi_u^X(p) \in W_j \Rightarrow \phi_u^X(p) \in W_{j+1}.$$

Let $q \in W_N$, then $\phi_t^X(q) = (\phi_{\frac{t}{N}}^X)^N(q)$, hence $q \in \mathcal{D}(\phi_t^X)$. \square

For 3). Let $p \in \mathcal{D}(\phi_t^X)$, set $c(u) := \phi_{t+u}^X(p)$ where $-t \leq u \leq 0$. This is an integral curve through $\phi_t^X(p)$. Thus $\phi_t^X(p) \in \mathcal{D}(\phi_{-t}^X)$ and $\phi_{-t}^X \circ \phi_t^X(p) = p$.

For 1). Let $(t_0, p) \in A$. Using 2). we can find an open neighborhood $\mathcal{D}(\phi_{t_0}^X) \subset W \ni p$. Using Flow-box theorem, there is $(-\varepsilon, \varepsilon) \times \tilde{W}$ open such that

$$(-\varepsilon, \varepsilon) \times \tilde{W} \ni (u, q) \mapsto \phi_u^X(q),$$

is smooth with image contained in W . In particular $p \in \tilde{W}$. Then for $q \in \tilde{W}$ with $|t - t_0| < \varepsilon$,

$$\phi_t^X(q) = \phi_{t_0}(\phi_{t-t_0}^X(q))$$

is smooth. Furthermore, $(t_0 - \varepsilon, t_0 + \varepsilon) \times \tilde{W} \subseteq A$. \square

Definition 3.5. A vector field $X \in \Gamma(T\mathcal{M})$ is called complete (or integrable) if $\mathcal{D}(\phi^X) = \mathbb{R} \times \mathcal{M}$.

Remark 3.3. Not all vector fields are complete.

Example 3.2. The following vector fields are not complete.

1). $\mathcal{M} = \mathbb{R}^2 \setminus \{0\}$ and $X(x, y) = \frac{\partial}{\partial x}$.

2). $\mathcal{M} = \mathbb{R}^2$ and $X(x, y) = x^2 \frac{\partial}{\partial x}$. Then we have $\phi^{(1,0)}(t) = \left(\frac{1}{1-t}, 0 \right)$.

Proposition 3.2. *Let $X \in \Gamma(T\mathcal{M})$ be a compactly supported vector field. Then X is complete. In particular, if \mathcal{M} is compact, then every vector field is complete.*

Proof. Set $K := \text{supp}(X)$. By the Flow-box theorem, there is an open set $K \subset W \subset \mathcal{M}$ and $\varepsilon > 0$, such that $(-\varepsilon, \varepsilon) \times W \subseteq \mathcal{D}(\phi^X)$. On the other hand, if $p \in \mathcal{M} \setminus K$, then for every $t \in \mathbb{R}$, $\phi_t^X(p) = p$. Hence, there is $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \times \mathcal{M} \subseteq \mathcal{D}(\phi^X)$. Letting $N \in \mathbb{N}$ be large enough such that $\frac{t}{N} < \varepsilon$, and using $\phi_t = (\phi_{\frac{t}{N}})^N$ shows X is integrable. \square

Proposition 3.3. *Let $X \in \Gamma(T\mathcal{M})$, and $p \in \mathcal{M}$, and assume $X_p \neq 0$. Then there is a chart (U, φ) centered around p , with $X = \frac{\partial}{\partial x_1}$ on U .*

Proof. Let ψ be a chart with $T_p\psi(X_p) = e_1$. This is possible because if for any chart $\tilde{\psi}$, we find $L \in \text{GL}_m(\mathbb{R})$ such that

$$LT_p\tilde{\psi}(X_p) = e_1.$$

Set $\psi = L \circ \tilde{\psi}$. Let $\varepsilon > 0, \delta > 0$ such that $(-\varepsilon, \varepsilon) \times \psi^{-1}(\overline{B(0, \delta)}) \subseteq \mathcal{D}(\phi^X)$.

Set for $|t| < \varepsilon$, $y \in \mathbb{R}^{m-1}$, $|y| < \varepsilon$.

$$\sigma(t, y) = \phi_t^X(\psi^{-1}(0, y)).$$

Then

$$D(\psi \circ \sigma)(0, 0) = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where the lower corner is the identity matrix.

$$\begin{aligned} \partial_t|_{(0,0)} \psi \circ \sigma(t, y) &= \partial_t|_{(0,0)} \psi(\phi_t^X(p)) = T_p\psi(X_p) = e_1, \\ \partial_y|_{(0,0)} \psi(\psi^{-1}(0, y)) &= I. \end{aligned}$$

So σ is locally invertible and $\varphi = \sigma^{-1}$ is a chart centered at p .

$$\frac{\partial}{\partial t} \sigma(t, y) = \frac{\partial}{\partial t} \phi_t^X(\psi^{-1}(0, y)) = X|_{\sigma(t, y)}.$$

So for $q \in \mathcal{D}(\sigma^{-1}) = \sigma((-\varepsilon, \varepsilon) \times B(0, \delta) \subset \mathcal{M})$. Thus,

$$X(q) = X(\sigma(\sigma^{-1}(q))) = T_{\sigma^{-1}(q)}\sigma(e_1) = D\sigma(\sigma^{-1}(q))e_1.$$

\square

Remark 3.4. *Given two vector field $X, Y \in \Gamma(T\mathcal{M})$, this is not in general false that*

$$XYf \neq YXf.$$

Furthermore, it can even happen that $XY \notin \Gamma(T\mathcal{M})$.

Definition 3.6. For $X, Y \in \Gamma(T\mathcal{M})$, and $f \in \mathcal{C}^\infty(\mathcal{M})$, set

$$[X, Y]f := X(Yf) - Y(Xf).$$

$[X, Y]$ is called the Lie bracket.

Definition 3.7. An algebra \mathfrak{g} is called a Lie algebra over \mathbb{R} together with Lie bracket $[\cdot, \cdot] : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M})$ if for any $X, Y, Z \in \mathfrak{g}$,

1. $[\cdot, \cdot]$ is \mathbb{R} -linear.
2. $[X, Y] = -[Y, X]$.
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

Proposition 3.4. Let $X, Y \in \Gamma(T\mathcal{M})$.

1. $[X, Y] \in \Gamma(T\mathcal{M})$.
2. $(\Gamma(T\mathcal{M}), [\cdot, \cdot])$ is a Lie algebra.

Proof. $[\cdot, \cdot]$ is \mathbb{R} -linear. consider,

$$\begin{aligned} [X, Y]|_p(fg) &= X|_p(Y(fg)) - Y|_p(X(fg)), \\ &= X|_p(Yf)g + X|_p(Yg)f - Y|_p(Xf)g - Y|_p(Xg)f, \\ &= X|_p(Yf)g + Yfg(p) + X|_pfYg + fX|_p(Yg) - Y|_p(Xf)g - XfY|_pg - Y|_pfXg - fY|_p(Xg), \\ &= ([X, Y]|_p f)g = f[X, Y]|_pg. \end{aligned}$$

The rest is an exercise. \square

Proposition 3.5. Let $X \in \Gamma(T\mathcal{M})$, such that $X_p \neq 0$, then there exists a chart (U, φ) centered at p such that

$$\frac{\partial}{\partial x_1} = X.$$

Definition 3.8. Let $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ be a diffeomorphism and $X \in \Gamma(T\mathcal{M})$. For $q \in \mathcal{N}$,

$$(\varphi_* X)(q) := T_{\varphi^{-1}(q)}\varphi[X(\varphi^{-1}(q))],$$

and

$$\varphi^* X := (\varphi^{-1})_*.$$

Definition 3.9. Let $\varphi : \mathcal{M} \rightarrow \mathcal{N}$, be a smooth map and $X \in \Gamma(TM)$ and $Y \in \Gamma(T\mathcal{N})$, we write

$$X \sim_\varphi Y, \text{ and say they are } \varphi\text{-related if } Y(\varphi(p)) = T_p\varphi(X(p)).$$

Remark 3.5. If φ is a diffeomorphism and $X \in \Gamma(T\mathcal{M})$, then clearly,

$$X \sim_\varphi \varphi_* X.$$

Proposition 3.6. Suppose $X_1, X_2 \in \Gamma(T\mathcal{M})$, $Y_1, Y_2 \in \Gamma(T\mathcal{N})$, and $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ be smooth.

If $X_j \sim_{\varphi} Y_j$ for each $j = 1, 2$, then $[X_1, X_2] \sim_{\varphi} [Y_1, Y_2]$.

Proof. Let $p \in \mathcal{M}$ and $f \in \mathcal{C}^i nfty(\mathcal{N})$.

$$\begin{aligned} T_p \varphi([X_1, X_2])(f) &= [X_1, X_2]_p(f \circ \varphi), \\ &= X_{1,p}(X_2 f \circ \varphi) - X_{2,p}(X_1 f \circ \varphi), \\ &= X_{1,p}((Y_2 f) \circ \varphi) - X_{2,p}((Y_1 f) \circ \varphi), \\ &= Y_{1,\varphi(p)}(Y_2 f) - Y_{2,\varphi(p)}(Y_1 f), \\ &= [Y_1, Y_2]|_{\varphi(p)} f. \end{aligned}$$

□

Lemma 3.2. Let $X \in \Gamma(T\mathcal{M})$ and $\phi = \phi^X$ be a flow of X . We have the following statements.

- 1). For $f \in \mathcal{C}^\infty(\mathcal{M})$, $\frac{d}{dt}(\phi_t^* f) = \phi_t^*(X f)$. In particular, $\frac{d}{dt}|_{t=0} \phi_t^* f = X f$.
- 2). For $Y \in \Gamma(T\mathcal{M})$, $\frac{d}{dt} \phi_t^* Y = \phi_t^*([X, Y])$, $\frac{d}{dt}|_{t=0} \phi_t^* Y = [X, Y]$.
- 3).

Proof. Refer to Lee's book. □

Proposition 3.7. Let $X, Y \in \Gamma(T\mathcal{M})$. Then the following statements are equivalent.

- 1). $[X, Y] = 0$,
- 2). $\forall t, \phi_t^* Y = Y$,
- 3). $\forall s, \psi_s^* X = X$,
- 4). $\psi_s \circ \phi_t = \phi_t \circ \psi_s$.

Proposition 3.8. Let $X_1, \dots, X_k \in \Gamma(T\mathcal{M})$ be commuting vector fields (ie, $[X_i, X_j] = 0$ for all $i, j = 1, \dots, k$). Suppose $X_1(p), \dots, X_k(p)$ are linearly independent. Then there exists a chart (U, φ) such that

$$X_j|_U = \frac{\partial}{\partial x_j}$$

for all $j = 1, \dots, k$.

Proof. Since the statement is about open sets, it suffices to show for the case where $\mathcal{M} = \mathbb{R}^n$ and $p = 0$. Consider,

$$\sigma(t, \xi) := \phi_{t_1}^1 \circ \dots \circ \phi_{t_k}^k(0\xi),$$

where $t \in \mathbb{R}^k$ and $\xi \in \mathbb{R}^{m-k}$. □

3.4 Vector Bundles

Definition 3.10. A (smooth, real) vector bundle over a manifold \mathcal{M} is a triple (E, π, \mathcal{M}) such that

- 1). E, \mathcal{M} are smooth manifolds.
- 2). $\pi : E \rightarrow \mathcal{M}$ is a surjective submersion.
- 3). For all $p \in \mathcal{M}$, the fiber $\pi^{-1}(\{p\}) = E_p$ is an \mathbb{R} -vectorspace with $\dim E_p < \infty$.

Furthermore, this satisfies the axiom of local triviality, namely,

For each $p \in \mathcal{M}$, there exists an open neighborhood $p \in U$ and a diffeomorphism

$$\varphi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^N,$$

such that

$$\forall q \in U, \varphi_{U|_{E_q}} : E_q \rightarrow \{q\} \times \mathbb{R}^N,$$

is a linear isomorphism.

E is called the total space of the bundle, \mathcal{M} is the base of the bundle, and φ_U is called the bundle chart.

Definition 3.11. A system $(\varphi_i)_{i \in I}$ of bundle charts

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N,$$

is a bundle atlas if

$$\bigcup_{i \in I} U_i = \mathcal{M}, \left(\text{or equivalently } \bigcup_{i \in I} \pi^{-1}(U_i) = E \right).$$

Example 3.3. Consider $(E = \mathcal{M} \times \mathbb{R}^N, \pi(p, v) = p, \mathcal{M})$. This is called a trivial vector bundle.

Definition 3.12. Consider (E, π_E, \mathcal{M}) and (F, π_F, \mathcal{M}) be vector bundles. A homomorphism of vector bundles between E and F is a smooth map $\varphi : E \rightarrow F$ such that for each point $p \in \mathcal{M}$, $\varphi|_{E_p} : E_p \rightarrow F_p$ is a linear map.

It is called an isomorphism if each $\varphi|_{E_p}$ is a linear isomorphism.

Definition 3.13. A vector bundle is said to be trivial if it is isomorphic to some trivial bundle $\mathcal{M} \times \mathbb{R}^N$.

Definition 3.14. Vector bundles with fibers being \mathbb{C} -vectorspaces are then called complex vector bundles.

Example 3.4. Consider the total space,

$$E = \{(z, w) \in \mathbb{C}^2 \cong \mathbb{R}^4 \mid |z| = 1, \frac{w^2}{z} \geq 0\},$$

and the projection map,

$$\pi : E \rightarrow S^1, (z, w) \mapsto z.$$

We observe that,

E is a submanifold of $\mathbb{C}^2 \cong \mathbb{R}^4$. Pick $z_0 \in S^1$ and a smooth square root function,

$$\xi : S^1 \setminus \{z_0\} \rightarrow S^1, \xi(x_1, x_2) = (\xi_1, \xi_2).$$

Let $w = (x_3, x_4)$ be a point such that $w^2 = \lambda z = \lambda \xi(z)^2$ for $\lambda \geq 0$.

It is equivalent to say that

$$\exists \mu \in \mathbb{R}, w = \mu \xi(z).$$

Reformulate this again, we get,

$$w = (x_3, x_4) \parallel (\xi_1(z), \xi_2(z)) \Leftrightarrow x_3 \xi_2(z) - x_4 \xi_1(z) = 0.$$

Thus we obtain,

$$E \cap \{(z, w) \mid z \in S^1 \setminus \{z_0\}\} = \{(z, w) \in (\mathbb{C} \setminus \mathbb{R}_+ z_0) \times \mathbb{C} \mid \begin{matrix} x_1^2 + x_2^2 - 1 = 0 \\ x_3 \xi_1(x_1, x_2) - x_4 \xi_2(x_1, x_2) = 0 \end{matrix}\}.$$

Examine the Jacobian of the conditions, we observe

$$\begin{pmatrix} 2x_1 & 2x_2 & 0 & 0 \\ * & * & \xi_2(x_1, x_2) & -\xi_1(x_1, x_2) \end{pmatrix}.$$

We know that at least one of these row is not 0, in other word the matrix is of full rank, thus E is a manifold.

Now we show that $\pi^{-1}(\{z\})$ is a one-dimensional \mathbb{R} -vectorspace.

Similar to the previous case, we get $w^2 = \lambda \xi(z)^2$ for all $z \in S^1 \setminus \{z_0\}$. We get $w = \pm \sqrt{\lambda} \xi(z)$. With this observation, we get,

$$\pi^{-1}(\{z\}) = \{(z, \mu \xi(z)) \mid \mu \in \mathbb{R}\}.$$

Bundle charts over $S^1 \setminus \{z_0\}$.

Consider

$$\varphi : \pi^{-1}(S^1 \setminus \{z_0\}) \rightarrow (S^1 \setminus \{z_0\}) \times \mathbb{R}, (z, w) \mapsto (z, \frac{w}{\xi(z)}).$$

Finally we show that the bundle $E \xrightarrow{\pi} S^1$ is not trivial.

To derive a contradiction, suppose there exists a bundle isomorphism $\varphi : E \rightarrow S^1 \times \mathbb{R}$. Consider

$$E \setminus \{(z, 0) \mid z \in S^1\} \xrightarrow{\varphi} S^1 \times \mathbb{R}^\times$$

which is disconnected. Indeed the map,

$$c : [0, 2\pi] \rightarrow E \setminus \{(z, 0) \mid z \in S^1\}, t \mapsto (e^{it}, e^{i\frac{t}{2}}).$$

Observe that

$$c(0) = (1, 1), c(2\pi) = (1, -1).$$

We derived a contradiction that two components get connected by a path.

This turns out to be one of the realization of a Möbius strip.

Construction of vector bundle from local charts Consider $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle and $(U_i, \varphi_i)_{i \in I}$ be a bundle chart. Consider

$$\begin{array}{ccccc} (U_i \cap U_j) \times \mathbb{R}^N & \xrightarrow{\varphi_j^{-1}} & \pi^{-1}(U_i \cap U_j) & \xrightarrow{\varphi_i} & (U_i \cap U_j) \times \mathbb{R}^N \\ & \searrow \text{pr}_1 & \downarrow \pi & \swarrow \text{pr}_1 & \\ & & U_i \cap U_j & & \end{array}$$

where $\text{pr}_1(p, v) = p$. Then we have,

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^N \rightarrow (U_i \cap U_j) \times \mathbb{R}^N, \varphi_{ij}(q, v) = (q, g_{ij}(q)v),$$

where

$$g_{ij} \in \mathcal{C}^\infty(U_i \cap U_j, \text{GL}_N(\mathbb{R})).$$

We have the following properties of $(g_{ij})_{ij}$.

- i). $g_{ij}g_{jk} = g_{ik}$ over $U_i \cap U_j \cap U_k$.
- ii). $g_{ii}(q) = I_N$.
- iii). $g_{ji}(q) = g_{ij}(q)^{-1}$.

Such collection $(g_{ij})_{ij}$ is called a co-cycle with values in $\text{GL}_N(\mathbb{R})$.

Remark 3.6. The last two properties follow from the first one but the proof is left to the readers.

Definition 3.15. Let \mathcal{M} be a smooth manifold and $(U_i)_{i \in I}$ be an open covering of \mathcal{M} and G be a Lie group (ie. a manifold with group structure such that both multiplication and inversion are smooth as functions). A family $(g_{ij})_{i,j \in I}$ of smooth map such that

$$g_{ij} : U_i \cap U_j \rightarrow G$$

is a cocycle with values in G if

1. $g_{ij}g_{jk} = g_{ik}$ over $U_i \cap U_j \cap U_k$.
2. $g_{ii} = e_G$.
3. $g_{ij} = g_{ji}^{-1}$

Definition 3.16. Suppose we are given a cocycle $(g_{ij})_{ij}$ in $G \subset \mathrm{GL}_N(\mathbb{R})$. We define a pre-bundle as follows,

$$E := \bigcup_{i \in I} i \times U_i \times \mathbb{R}^N / \sim,$$

where $(i, p, v) \sim (j, q, w)$ if

$$p = q \text{ and } v = g_{ij}(p) \cdot w.$$

We define the projection to be,

$$\pi(([i, p, v])) := p,$$

and the bundle chart,

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N, [i, p, v] \mapsto (p, v).$$

Remark 3.7. Observe that this has not topology defined yet.

Definition 3.17. Let E, F be vector bundles over \mathcal{M} . A smooth map $\varphi : E \rightarrow F$ is a bundle isomorphism such that

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ \searrow \pi_E & & \swarrow \pi_F \\ & \mathcal{M} & \end{array}$$

is a commutative diagram and for all $p \in \mathcal{M}$, φ_p is an isomorphism between E_p and F_p .

Example 3.5. Let \mathcal{M} be a smooth manifold. Then we have

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} \{p\} \times T_p\mathcal{M}.$$

Let $(U_i, \varphi_i)_{i \in I}$ be an atlas for \mathcal{M} . Now define a bundle atlas as follows.

$$\psi_i : TU_i = \bigcup_{p \in U_i} \{p\} \times T_p\mathcal{M} = \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^m, (p, X_p) \mapsto (p, T_p\varphi_i X_p).$$

Note that we have a canonical isomorphism $T_{\varphi_i(p)}\mathbb{R}^m \cong \mathbb{R}^m$.

Theorem 3.3. Given a pre-bundle (E, π, \mathcal{M}, I) . There exists a unique topology and C^∞ -structure on E such that (E, π, \mathcal{M}) is a vector bundle.

Moreover, given a cocycle $(g_{ij} : U_{ij} \rightarrow G \subseteq \mathrm{GL}_N(\mathbb{R}))_{i,j \in I}$, where $(U_{ij})_{i,j \in I}$ is an open covering, there is a unique vector bundle (E, π, \mathcal{M}) up to isomorphisms with atlas $(U_i, \varphi_i)_{i \in I}$ such that $(g_{ij})_{i,j \in I}$ is the corresponding cocycle.

Proof. Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle with bundle atlas $(\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N)$. Then for each open set $U \subseteq \mathcal{M}$ we have,

$$\pi^{-1}(U) \cap \pi^{-1}(U_i)$$

is open. Conversely, given a pre-vector bundle (E, π, \mathcal{M}, I) . Define a topology on E generated by base open sets of the form $\{\pi^{-1}(U) \cap \pi^{-1}(U_i)\}_{i \in I, U \subseteq \mathcal{M}}$, this gives a topology such that π is continuous. Similarly for the smooth structure. Define

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N,$$

to be such that

$$\varphi_i([(i, p, v)]) = (p, v).$$

This gives a smooth structure to $E \xrightarrow{\pi} \mathcal{M}$.

For the uniqueness, assume we have two vector bundles $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$ with corresponding atlases $(U_i, \varphi_i), (V_i, \psi_i)$, respectively, such that

$$\varphi_i \circ \varphi_j^{-1}(p, v) = \psi_i \circ \psi_j^{-1}(p, v) = (p, g_{ij}(p)v).$$

Define, $\Phi : E \rightarrow F$ as follows. For $p \in U_i$ and $\xi \in \mathbb{R}^N$, we put

$$\Phi(p, \xi) = \psi_i \circ \phi_i(p, \xi).$$

This gives us a bundle isomorphism. \square

Definition 3.18. Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle with bundle atlas $(U_i, \varphi_i)_{i \in I}$ and cocycle $(g_{ij})_{i,j \in I}$. The dual vector bundle of E is E^* which is constructed from pre-vector bundle $(E^*, \pi^*, \mathcal{M}, I)$ with cocycle $(g_{ji}^{-1})_{i,j \in I}$ and

$$E^* = \bigcup_{p \in \mathcal{M}} E_p^*.$$

Definition 3.19. Let $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$ be vector bundles with cocycle $(g_{ij})_{i,j \in I}, (h_{ij})_{i,j \in I}$ we define their direct sum to be

$$(E \oplus F)_p = E_p \oplus F_p$$

with cocycle $(g_{ij} \oplus h_{ij})_{i,j \in I}$.

Definition 3.20. Let $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$ be vector bundles with cocycle $(g_{ij})_{i,j \in I}, (h_{ij})_{i,j \in I}$ we define their tensor product to be

$$(E \otimes F)_p = E_p \otimes F_p$$

with cocycle $(g_{ij} \otimes h_{ij})_{i,j \in I}$.

Definition 3.21. Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle. A section of E is a smooth map $f : \mathcal{M} \rightarrow E$ such that $\pi \circ f = \text{id}_{\mathcal{M}}$.

Remark 3.8. Vector fields are exactly the sections of $T\mathcal{M}$.

Remark 3.9. A vector bundle $E \xrightarrow{\pi} \mathcal{M}$ is trivial of rank N if and only if there exists sections $s_1, \dots, s_N \in \Gamma(E)$ such that for all $p \in \mathcal{M}$, $\langle s_1(p), \dots, s_N(p) \rangle$ is a basis of E_p .

Given a bundle atlas $(U_\alpha, \varphi_\alpha)$ with cocycle $(g_{\alpha\beta})$, define,

$$f_\alpha : U_\alpha \rightarrow \mathbb{R}^n, f_\alpha(x) = \text{pr}_2(\varphi_\alpha(f(x))).$$

Of course, we have,

$$f_\alpha = g_{\alpha\beta}(x)f_\beta(x), x \in U_\alpha \cap U_\beta.$$

Conversely, given such f_α , we define,

$$f(x) = [(\alpha, x, f_\alpha(x))].$$

Proposition 3.9. There is a one-to-one correspondence between,

$$\{\text{sections } f \text{ of } E\} \leftrightarrow \{(f_\alpha)_{\alpha \in I} | \forall \alpha \in I, f_\alpha \in C^\infty(U_\alpha, \mathbb{R}^N), f_\alpha(x) = g_{\alpha\beta}(x)f_\beta(x)\}.$$

Proposition 3.10. Let $E, F \rightarrow \mathcal{M}$ be vector bundles. Let $\phi \in \text{Hom}(E, F)$ be surjective. Then there exists $j \in \text{Hom}(F, E)$ such that $\phi \circ j = \text{id}_F$. Furthermore, $\text{Ker } \phi$ is a vector bundle over \mathcal{M} and

$$E \simeq \text{Ker } \phi \otimes F.$$

That is to say, in the category of vector bundles of \mathcal{M} , a short exact sequence

$$0 \longrightarrow \text{Ker } \phi \hookrightarrow E \xrightarrow{\phi} F \longrightarrow 0$$

splits.

Proof. We will first show that locally such section exists. Let $p \in \mathcal{M}$. Choose sections of E (by using a chart) $s_1, \dots, s_n \in \Gamma(E)$ such that

$$\phi(s_1(p)), \dots, \phi(s_N(p)),$$

is linearly independent. Then there exists $p \in U$ an open neighborhood of p such that for all $q \in U$,

$$\phi(s_1(q)), \dots, \phi(s_N(q)),$$

is linearly independent. For $w \in F_q|_U$, we can now write,

$$w = \sum \xi_j \phi(s_j(q)).$$

Put

$$j(w) = \sum \xi_j s_j(q).$$

Clearlyk, we have that j is a section of ϕ over U .

For the global case, let $(U_\alpha)_{\alpha \in I}$ be an open covering with sections $j_\alpha : F|_{U_\alpha} \rightarrow E|_{U_\alpha}$. Let $(\rho_\alpha)_{\alpha \in I}$ be a subordinated partition of unity. Then,

$$j(p, w) := \left(p, \sum_\alpha \rho_\alpha j_\alpha(w) \right)$$

Now for $P \in \text{Hom}(E, E)$, we set,

$$P(v) := v - j(\varphi(v)),$$

then P is a projection $P^2 = P$ and the image is $\text{Im}(P) = \text{Ker } \phi$.

Fix p , and choose sections $\tilde{s}_1, \dots, \tilde{s}_r$ of E be such that

$$\tilde{s}_1(p), \dots, \tilde{s}_r(p)$$

is a basis of $\text{Ker } \phi|_P$. Set

$$s_j := P \circ \tilde{s}_j.$$

Then s_j are sections of $\text{Ker } \phi$ and in a neighborhoodof p , they are pairwise linearly independent hence a frame. \square

Proposition 3.11 (Swan's theorem). *Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle of a compact manifold \mathcal{M} . Then there exists a large enough natural number r such that there exists a surjective bundle homomorphism $\phi : \mathcal{M} \times \mathbb{R}^r \rightarrow E$.*

In parcitular, $E \oplus \text{Ker } \phi \simeq \mathcal{M} \times \mathbb{R}^r$ is trivial and hence there exit sections $s_1, \dots, s_r \in \Gamma(E)$, such that

$$\forall p \in \mathcal{M}, E_p = \text{Span}(s_1(p), \dots, s_r(p)).$$

Proof. Choose a finite bundle atlas $(U_1, \varphi_1), \dots, (U_l, \varphi_l)$ of E (this is justified as \mathcal{M} is compact) and a subordinated partition of unity ρ_1, \dots, ρ_l . We set,

$$s_j^i(p) := \rho_j(p) \varphi_j^{-1}(p, e_j),$$

where $N = \text{rank } E$ and e_1, \dots, e_N are the canonical basis of \mathbb{R}^N . Note that $s_j^i \in \Gamma(E)$.

Given $p \in \mathcal{M}$ there exists j such that $\rho_j(p) \neq 0$. Hence $s_j^i(p), \dots, s_j^N(p)$ is a basis of E_p , respectively

$$\forall p \in \mathcal{M}, E_p = \text{Span}(s_j^i(p))_{\substack{1 \leq i \leq N \\ 1 \leq j \leq l}}.$$

Re-enumerate them to $s_1, \dots, s_r \in \Gamma(E)$. Put

$$\phi : \mathcal{M} \times \mathbb{R}^r \rightarrow E(p, \xi) \mapsto (p, \sum \xi_j s_j(p)).$$

The rest follows from the previous propostion. \square

Definition 3.22. A R -module P is said to be projective if there exists another R -module Q such that $P \oplus Q$ is free.

Remark 3.10. Observe first that the sections $\Gamma(E)$ is a module over $\mathcal{C}^\infty(\mathcal{M})$ with the action defined by

$$\mathcal{C}^\infty(\mathcal{M}) \times \Gamma(E) \ni (f, s) \mapsto [\mathcal{M} \ni p \mapsto f(p) \cdot s(p)].$$

Proposition 3.11 tells us that $\Gamma(E)$ is finitely generated as a $\mathcal{C}^\infty(\mathcal{M})$ module. Furthermore, $\Gamma(E)$ is projective (ie. a direct summand of a free module).

4 Tensor Algebras and Exterior Algebras

In this section, we assume that K be a field of characteristic 0 and all vector spaces are finite dimensional.

4.1 Tensors

Let E be a vectorspace. We will denote the dual pairing as $\langle \cdot, \cdot \rangle_{E, E^*}$.

Definition 4.1. Let E_1, \dots, E_r be vectorspaces. Recall that a multilinear map $F : E_1 \times \dots \times E_r \rightarrow F$ is such that, for each $i = 1, \dots, r$, $F(a_1, \dots, a_{i-1}, \cdot, a_{i+1}, \dots, a_r)$ is linear. We denote,

$$L(E_1, \dots, E_r; F) = \{E_1 \times \dots \times E_r \rightarrow F \mid r\text{-multilinear maps}\}.$$

Remark 4.1.

$$\dim L(E_1, \dots, E_r; F) = \left(\prod_{i=1}^r \dim E_i \right) \times \dim F.$$

Definition 4.2. Let $\varphi \in E^*$ and $\psi \in F^*$. For $e \in E, f \in F$, we define,

$$(\varphi \otimes \psi)(e, f) := \varphi(e)\psi(f)$$

Remark 4.2. Obviously we have $\varphi \otimes \psi \in L(E, F; K)$.

Remark 4.3. A map $\cdot \otimes \cdot$ is a bilinear map $E^* \times F^* \rightarrow L(E, F; K)$.

Proposition 4.1. Let $(e_i)_{i \in I}, (f_j)_{j \in J}$ be bases of E, F . Then we have,

$$\varphi \otimes \psi = \sum \varphi(e_i)\psi(f_j)e_i^* \otimes f_j^*.$$

By definition, we have,

$$e_i^* \otimes f_j^*(e_k, f_l) = \delta_{ik}\delta_{jl}.$$

In particular, $(e_i^* \otimes f_j^*)_{(i,j) \in I \times J}$ is a basis of $L(E, F; K)$.

Definition 4.3. We define,

$$L(e_1, \dots, E_r, K) \times L(f_1, \dots, f_s; K) \rightarrow L(E_1, \dots, E_r, F_1, \dots, F_s; K)$$

by,

$$\varphi \otimes \psi(e_1, \dots, e_r, f_1, \dots, f_s) := \varphi(e_1, \dots, e_r) \psi(f_1, \dots, f_s).$$

It is left to the readers as an exercise to show this is bilinear and associative (ie $(\varphi \otimes \psi) \otimes \chi = \varphi \otimes (\psi \otimes \chi)$).

Notation 4.1. We denote

$$E^* \otimes F^* := L(E, F; K),$$

and

$$E \otimes F \cong E^{**} \otimes F^{**} = L(E^*, F^*, K).$$

Proposition 4.2 (Universality). The assigment $\otimes : E \times F \rightarrow E \otimes F$ solves the following universal property.

Let G be a vectorspace and $h : E \times F \rightarrow G$ be a bilinear map. Then there exists a unique linear map $\bar{h} : E \otimes F \rightarrow G$ making the diagram commutative.

$$\begin{array}{ccc} E \times F & \xrightarrow{h} & G \\ \otimes \downarrow & \nearrow \exists! \bar{h} & \\ E \otimes F & & \end{array}$$

Furthermore, the map

$$L(E, F; G) \ni h \mapsto \bar{h} \in L(E \otimes F; G),$$

is an isomorphism.

Proof. Check at the level of basis elements. \square

Definition 4.4. Let E be a vectorspace, we will define,

$$T^r E = \underbrace{E \otimes \cdots \otimes E}_{r \text{ times}},$$

and

$$T(E) = \bigoplus_{r=0}^{\infty} T^r(E),$$

which is called the tensor algebra over E . Furthermore, Setting \otimes as a multiplication of elements, $T(E)$ is a positively graded K -algebra.

Remark 4.4. Obviously, a basis of $T^r E$ is given by $(e_{i_1} \otimes \cdots \otimes e_{i_r})_{1 \leq i_1 \leq \cdots \leq i_r \leq n}$. Thus we have $\dim T^r E = (\dim E)^r$.

Lemma 4.1. The operation \otimes is associative.

$$\begin{array}{ccc} T^r E \times T^s E & \xrightarrow{(v_1 \otimes \cdots \otimes v_r, w_1 \otimes \cdots \otimes w_s) \mapsto v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s} & T^{r+s} E \\ \downarrow & \nearrow & \\ T^r E \otimes T^s E & & \end{array}$$

4.2 Totally antisymmetric(alternating) tensors

Remark 4.5.

$$T^r E \otimes T^r E^* \rightarrow K,$$

where

$$(v_1 \otimes \cdots \otimes v_r, v_1^* \otimes \cdots \otimes v_r^*) \mapsto \prod_{i=1}^r \langle v_i, v_i^* \rangle$$

is an dual pairing.

Let $\sigma \in S_r$ be the permutations of $\{1, \dots, r\}$. We have,

$$\begin{array}{ccc} E^r & \xrightarrow{\sigma} & E^r \\ \downarrow & & \downarrow \otimes \\ T^r E & \xrightarrow{T^r \sigma} & T^r E \end{array}$$

where $T^r \sigma$ is a linear isomorphism. and we have,

$$(T^r \sigma)^{-1} = T^r \sigma^{-1}.$$

Notation 4.2. $\text{sgn}(\sigma) = (-1)^\sigma$.

Definition 4.5. $T \in T^r E$ is called alternating or totally antisymmetric if for all $\sigma \in S_r$, we have,

$$T^r \sigma T = (-1)^\sigma T,$$

Remark 4.6. To check T is alternating, it suffices to check that $T^r \sigma T = -T$ for every transposition $\tau \in S_r$.

Notation 4.3. We denote the space of alternating tensors to be,

$$\bigwedge^r E \subseteq T^r E.$$

Definition 4.6. An anti-symmetrization map is ,

$$A_r : T^r \rightarrow T^r, A \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^\sigma (T^r \sigma)(T).$$

This is a linear map.

Lemma 4.2. $A_r \circ A_r = A_r$. In other words, it is idempotent.

Proof. Let $T \in T^r E$ and fix $\sigma \in S_r$. We have,

$$(T^r \tau) A_r(T) = \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^\sigma T^r (\tau \sigma)(T) (-1)^\tau (-1)^\tau = (-1)^\tau A_r(T).$$

In particular, the range of A_r is contained in $\bigwedge^r E$. If $T \in \bigwedge^r E$, we have,

$$A_r(T) = \frac{1}{r!} \sum_{\sigma \in S_r} T^r \sigma T = T.$$

Thus we have the range of A_r is exactly $\bigwedge^r E$. \square

4.3 Exterior Algebra

Notation 4.4.

$$\bigwedge E = \bigoplus_{r \in \mathbb{Z}_{\geq 0}} \bigwedge^r E.$$

Definition 4.7. Let $\psi : \mathbb{N} \rightarrow \mathbb{Z}_+$ be a sequence such that $\psi(0) = \psi(1) = 1$. We define,

$$\wedge_\psi : \bigwedge^p E \times \bigwedge^q E \rightarrow \bigwedge^{p+q} E, \xi \wedge_\psi \eta := \frac{\psi(p+q)}{\psi(p)\psi(q)} A_{p+q}(\xi \otimes \eta).$$

Definition 4.8. Let A be a graded ring. A mapping $(\cdot, \cdot) : A \rightarrow A$ is graded commutative if for any $a \in A^p, b \in A^q$, we have,

$$(a, b) = (-1)^{pq}(b, a).$$

Proposition 4.3. The map $\wedge = \wedge_\psi$ is bilinear, associated and graded commutative. Furthermore, for $v_1, \dots, v_r \in E = \bigwedge^1 E$,

$$v_1 \wedge \cdots \wedge v_r = \frac{\psi(r)}{r!} \sum_{\sigma \in S_r} \text{sgn } \sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

From above, we conclude,

$$(e_{i_1} \otimes \cdots \otimes e_{i_r})_{1 \leq i_1 < \cdots < i_r \leq \dim E}$$

is a basis of $\bigwedge^r E$. Thus $\dim \bigwedge^r E = \binom{\dim E}{r}$.

Proof. Let $\xi \in \bigwedge^p E, \eta \in \bigwedge^q E, \chi \in \bigwedge^r E$ and $\tau \in S_{p+q}$ such that

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & p & p+1 & \cdots & p+q \\ q+1 & q+2 & \cdots & p+q & 1 & \cdots & q. \end{pmatrix}$$

Observe that

$$\text{sgn}(\tau) = (-1)^{pq}, \tau(\xi \otimes \eta) = \eta \otimes \xi.$$

Thus we have,

$$A_{p+q}(\xi \otimes \eta) = (-1)^\tau A_{p+q}(\eta \otimes \xi).$$

We then obtain,

$$\xi \wedge \eta = (-1)^\tau \eta \wedge \xi = (-1)^{pq} \eta \wedge \xi.$$

We derived the graded-commutativity.

For the associativity, we have,

$$\begin{aligned}
(\xi \wedge \eta) \wedge \chi &= \frac{\psi(p+q+r)}{\psi(p+q)\psi(r)} A_{p+q+r}((\xi \wedge \eta) \otimes \chi), \\
&= \frac{\psi(p+q+r)}{\psi(p+q)\psi(r)} \frac{\psi(p+q)}{\psi(p)\psi(q)} \frac{1}{(p+q+r)!} \frac{1}{(p+q)!} \sum_{\tau \in S_{p+q}} \operatorname{sgn} \tau \\
&\quad \sum_{\sigma \in S_{p+q+r}} \operatorname{sgn}(\sigma \circ (\tau \times \text{id}))(\sigma \circ (\tau \times \text{id}))((\xi \wedge \eta) \otimes \chi), \\
&= \frac{\psi(p+q+r)}{\psi(p)\psi(q)\psi(r)} \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q+r}} \operatorname{sgn} \sigma \sigma(\xi \otimes \eta \otimes \chi), \\
&= \xi \wedge (\eta \wedge \chi).
\end{aligned}$$

By induction, for

$$\xi_1 \in \bigwedge^{p_1}, \dots, \xi_r \in \bigwedge^{p_r} E,$$

we see,

$$\xi_1 \wedge \dots \wedge \xi_r = \frac{\psi(p_1 + \dots + p_r)}{\psi(p_1) \dots \psi(p_r)} A_{p_1+\dots+p_r}(\xi_1 \otimes \dots \otimes \xi_r).$$

Take $p_1 = \dots = p_r = 1$, we obtain the formula.

□

Corollary 4.1. *Keeping the notation from the proposition above, we have,*

$$\langle v_1 \wedge \dots \wedge v_r, v_1^* \otimes \dots \otimes v_r^* \rangle_{T^r E, T^r E^*} = \frac{\psi(r)}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}),$$

and

$$\langle v_1 \wedge \dots \wedge v_r, v_1^* \wedge \dots \wedge v_r^* \rangle_{T^r E, T^r E^*} = \frac{\psi(r)^2}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}),$$

Proof.

$$\begin{aligned}
\langle v_1 \wedge \dots \wedge v_r, v_1^* \otimes \dots \otimes v_r^* \rangle_{T^r E, T^r E^*} &= \sum_{\sigma \in S_r} \operatorname{sgn} \sigma \langle v_{\sigma(1)} v_1^* \rangle \dots \langle v_{\sigma(r)} v_r^* \rangle, \\
&= \frac{\psi(r)}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}).
\end{aligned}$$

$$\begin{aligned}
\langle v_1 \wedge \dots \wedge v_r, v_1^* \wedge \dots \wedge v_r^* \rangle_{T^r E, T^r E^*} &= \frac{\psi(r)}{r!} \sum_{\tau \in S_r} \operatorname{sgn} \tau \langle v_1 \wedge \dots \wedge v_r, v_{\tau(1)}^* \otimes \dots \otimes v_{\tau(r)}^* \rangle_{T^r E, T^r E^*}, \\
&= \frac{\psi(r)}{r!} \sum_{\tau \in S_r} \operatorname{sgn} \tau \langle v_{\tau(1)} \wedge \dots \wedge v_{\tau(r)}, v_1^* \otimes \dots \otimes v_r^* \rangle_{T^r E, T^r E^*}, \\
&= \frac{\psi(r)}{(r!)^2} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}).
\end{aligned}$$

□

We will follow the convention from [2] which is $\psi \equiv 1$.

Definition 4.9. For $\xi \in \bigwedge^p E$, we define,

$$\text{ext}_\xi : \bigwedge E \rightarrow \bigwedge E, \eta \mapsto \xi \wedge \eta.$$

Respectively for $u \in E$, we define,

$$\text{int}_U : \bigwedge E^* \rightarrow \bigwedge E^*, \omega \mapsto \langle \omega, u \otimes \cdot \rangle = \langle \omega, u \wedge \cdot \rangle$$

Remark 4.7. For $\eta \in \bigwedge E$ and $\omega \in \bigwedge E^*$, we have,

$$\langle \text{ext}_u \eta, \omega \rangle = \langle u \wedge \eta, \omega \rangle = \langle \eta, \text{int}_U \omega \rangle. \quad (3)$$

Lemma 4.3. Suppose $\omega, \eta \in \bigwedge E^*$ are homogeneous elements. Then we have,

$$\text{int}_u(\omega \wedge \eta) = (\text{int}_u \omega) \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge \text{int}_u \eta.$$

In other words, int_u is an anti-derivation.

5 Orientations, volumes and the Hodge *-operator

5.1 Lines

In this section, we denote $K = \mathbb{R}$.

Definition 5.1. A line is a one-dimensional \mathbb{R} -vector space.

Definition 5.2. Fix $\alpha > 0$. An α -density on a n -dimensional vector space E is a map $\rho : \bigwedge^n E^* \rightarrow \mathbb{R}$ such that

$$\forall \omega \in \bigwedge^n E^*, \lambda \in \mathbb{R}, \rho(\lambda, \omega) = |\lambda|^\alpha \rho(\omega).$$

We will denote $|\bigwedge|^{\alpha} E^*$ to be the vector space of all α -densities.

Definition 5.3. A signed density on a n -dimensional vector space E is a map $\rho : \bigwedge^n E^* \rightarrow \mathbb{R}$ such that

$$\forall \omega \in \bigwedge^n E^*, \rho(\lambda, \omega) = \text{sgn } \lambda \rho(\omega).$$

We also define, the set of signed densities as, $\mathcal{O}(E)$ which we also call the orientation line.

Remark 5.1. We have a map,

$$(E^*)^n \rightarrow \bigwedge^n E.$$

With this we can define an α -density by,

$$\rho : (E^*)^n \rightarrow \mathbb{R}, \rho(\varphi(v_1^*), \dots, \varphi(v_n^*)) = |\det \varphi|^\alpha \rho(v_1^*, \dots, v_n^*),$$

for every $\varphi \in \mathrm{GL}(E)$. In particular, we have,

$$\varphi(v_1) \wedge \cdots \wedge \varphi(v_n) = (\det \varphi)v_1 \wedge \cdots \wedge v_n.$$

Similarly for the signed density case we put $\mathrm{sgn}(\det \varphi)$.

Remark 5.2.

$$\dim \left| \bigwedge E \right| = \dim \mathcal{O}(E) = 1.$$

Given $\xi \in \bigwedge^n E \setminus \{0\}$, we have,

$$\rho_\xi^\alpha(\omega) := |\xi(\omega)|^\alpha$$

which is an α -density. Also,

$$\rho_\xi^\alpha(\omega) := \mathrm{sgn}(\langle \xi, \omega \rangle)$$

which is a signed density. If ρ is an α -density,

$$\rho(\lambda\omega) = |\lambda|^\alpha \rho(\omega) = |\lambda|^\alpha \frac{\rho(\omega)}{\rho_\xi^\alpha(\omega)} \rho_\xi^\alpha(\omega) = \frac{\rho(\omega)}{\rho_\xi^\alpha(\omega)} \rho_\xi^\alpha(\lambda\omega).$$

Lemma 5.1. There are canonical isomorphisms between,

$$\begin{aligned} \left| \bigwedge \right|^\alpha E \otimes \left| \bigwedge \right|^\beta E \rho_1 \otimes \rho_2 &\mapsto \rho_1 \rho_2 \left| \bigwedge \right|^{\alpha+\beta} E. \\ \left| \bigwedge \right|^1 E \otimes \mathcal{O} &\rightarrow \bigwedge^n E \\ \bigwedge^n E &\rightarrow \left| \bigwedge \right|^1 E \otimes \mathcal{O}, \end{aligned}$$

and

$$\mathcal{O} \otimes \mathcal{O} \rightarrow \mathbb{R}.$$

Furthermore, \mathcal{O} has a canonical Euclidean metric, namely, for $\rho \in \mathcal{O}$, we have,

$$|\rho| := \sqrt{\rho(\omega)^2}, \forall \omega \neq 0.$$

Note that this is independent of choice of $\omega \neq 0$ since ρ is a signed-density and the square root will take the sign out.

5.2 Orientations

Definition 5.4. An orientation of E is given by a choice of a vector $\mathbf{o} \in \mathcal{O}$ of unit length.

Remark 5.3. We have an isometry by,

$$\mathcal{O} \ni \mathbf{o} \mapsto 1 \in \mathbb{R}.$$

Remark 5.4. We have an isomorphism such that,

$$\bigwedge^n E \ni \rho \mapsto \rho \cdot \mathbf{o} \in \left| \bigwedge^n E \right|.$$

Definition 5.5. Given a connected component $\bigwedge^n_+ E$ of $\bigwedge^n E \setminus \{0\}$. A basis e_1, \dots, e_n of E is called oriented if

$$e_1 \wedge \cdots \wedge e_n \in \bigwedge^n_+ E.$$

Or equivalently, we have,

$$e_1 \wedge \cdots \wedge e_n = \mathbf{o} \cdot |e_1 \wedge \cdots \wedge e_n|.$$

5.3 Volume elements

Let g be a symmetric bilinear form on E . Then g induces bilinear forms on TE and $\bigwedge E$ as follows.

$$(T^r g)(v_1 \otimes \cdots \otimes v_r, w_1 \otimes \cdots \otimes w_r) := \prod_{j=1}^r g(v_j, w_j).$$

Note that $\bigwedge E \subseteq TE$. Thus we have,

$$\begin{aligned} (T^r g)(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r) &:= \frac{1}{(r!)^2} \sum_{\sigma, \tau \in S_r} \operatorname{sgn} \sigma \tau \prod_{j=1}^r g(v_{\sigma(j)}, w_{\tau(j)}), \\ &= \frac{1}{(r!)^2} \sum_{\sigma \in S_r} \det((g(v_i, w_j))_{1 \leq i, j \leq r}), \\ &= \frac{1}{r!} \det((g(v_i, w_j))_{1 \leq i, j \leq r}). \end{aligned}$$

Lemma 5.2. If g is non-degenerate, so are $T^r g$, $\bigwedge^r g$. The latter is the restriction of $T^r g$ to $\bigwedge^r E \times \bigwedge^r E$. Also we have an isomorphism

$$(\cdot)^b : E \rightarrow E^*, v \mapsto g(v_j) = v^b,$$

which inverse $\#$.

g induces a non-degenerate bilinear form g^* on E^* , such that

$$g^*(v^*, w^*) := g(\#v^*, \#w^*).$$

Proof. Let e_1, \dots, e_n be a g -oriented basis of E that is $g(e_i, e_j) = \pm\delta_{ij}$.

$$(T^r g)(e_{i_1} \otimes \cdots \otimes e_{i_r}, e_{j_1} \otimes \cdots \otimes e_{j_r}) = \prod_{k=1}^r \pm\delta_{i_1 j_1}.$$

And we also have,

$$\left(\bigwedge^r g \right) = (e_{i_1} \wedge \cdots \wedge e_{i_r}, e_{j_1} \wedge \cdots \wedge e_{j_r}) = \frac{\pm 1}{r!} \prod_{k=1}^r \delta_{i_1 j_1}.$$

Remark 5.5. *Induced bases are orthogonal with respect to the induced forms. Therefore, they are non-degenerate.*

□

Proposition 5.1. *Let g be a non-degenerate bilinear form on E . Then g determines uniquely a positive 1-density $\text{vol}_g \in |\bigwedge|^1 E$.*

If e_1, \dots, e_n is a basis of E , then

$$\text{vol}_g = |\det((g(e_i, e_j))_{1 \leq i, j \leq n})^{\frac{1}{2}}| e_1 \wedge \cdots \wedge e_n|.$$

Furthermore, g induces for every $\alpha > 0$, a canonical isomorphism,

$$|\bigwedge|^\alpha E \rightarrow \mathbb{R}, \lambda \cdot |\text{vol}_g|^\alpha \mapsto \lambda.$$

Proof. Let us first prove that this is independent of the choice of bases. Let f_1, \dots, f_n be another basis. We can write,

$$f_i = \sum a_{ij} e_j.$$

Thus we have,

$$g(f_i, f_j) = \sum_{k,l} a_{ki} a_{lj} g(e_k, e_l).$$

Therefore, we have,

$$\det(g(f_i, f_j)) =$$

□

5.4 The Hodge $*$ -operator

Let g be a non-degenerate bilinear form on E where E is a real vector space of dimension n . Let $\text{vol}_g \in |\bigwedge|^1 E$.

Let $\mathcal{O}(E)$ be the orientation line and recall

Definition 5.6. *Let g be a non-degenerate bilinear form on a vector space V . A basis $\{e_1, \dots, e_n\}$ of V is said to be g -orthonormal if we have,*

$$g(e_i, e_j) = \pm\delta_{ij}.$$

Definition 5.7. Let g be a non-degenerate bilinear form on V and $\{e_1, \dots, e_n\}$ be a g -orthonormal basis of V , we define the index of g to be such that

$$\text{ind } g = |\{i \mid g(e_i, e_i) = -1\}|.$$

Notation 5.1. Let L be a one-dimensional \mathbb{R} -vector space and $f, g \in L \setminus \{0\}$. We have $f = \lambda g$ for some $\lambda \in \mathbb{R}$. We denote,

$$f/g = \lambda.$$

Definition 5.8. Let E be a n -dimensional \mathbb{R} -vectorspace. The Hodge $*$ -operator is a map

$$*_p : \bigwedge^p E \rightarrow \bigwedge^{n-p} E \otimes \mathcal{O},$$

satisfying the following properties.

i). It is a linear isomorphism.

ii). For g -orthonormal basis $\{e_1, \dots, e_n\}$ of E and $\sigma \in S_n$, we have

$$*e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(p)} = (-1)^{\text{sgn } \sigma} \prod_{i=1}^p g(e_{\sigma(i)}, e_{\sigma(i)}) e_{\sigma(p+1)} \wedge \cdots \wedge e_{\sigma(n)} \otimes e_1 \wedge \cdots \wedge e_n / \text{vol}_g.$$

iii). $*_{n-p} *_p = (-1)^{p(n-p)+\text{ind } g}$,

iv). For $\omega, \eta \in \bigwedge^p E$, we have,

$$\left(\bigwedge^{n-p} g \right) (*\omega, *\eta) = (-1)^{\text{ind } g} \left(\bigwedge^p g \right) (\omega, \eta).$$

v). Let $\omega \in \bigwedge^p E, \eta \in \bigwedge^{n-p} E \otimes \mathcal{O}$, then

$$\omega \wedge \eta = (-1)^{p(n-p)+\text{ind } g} (g)(\omega, *\eta) \cdot \text{vol}_g.$$

Theorem 5.1. Let E be a \mathbb{R} -vector space of dimension n and g be a non-degenerate bilinear form on E . Pick $\text{vol}_g \in \left| \bigwedge^1 \right|^1 E$. We have, there exists a unique bilinear pairing such that

$$\bigwedge^p E \times \left(\bigwedge^{n-p} E \otimes \mathcal{O} \right) \ni (\omega, \eta) \mapsto \omega \wedge \eta / \text{vol}_g \in \mathbb{R}.$$

In particular for any $\eta \in \bigwedge^p E$ there is a unique $\star \eta \in \bigwedge^{n-p} E \otimes \mathcal{O}$ such that for any $\omega \in \bigwedge^p E$,

$$\omega \wedge \star \eta = \bigwedge^p g(\omega, \eta) \text{vol}_g.$$

And such \star is the Hodge \star -operator \star_p .

Proof. The composition,

$$(\omega, \eta) \mapsto \omega \wedge \eta \mapsto \omega \wedge \eta / \text{vol}_g \in \mathbb{R}$$

is bilinear. Note that

$$\det(g(e_i, e_j))_{ij} = (-1)^{\text{ind } g}.$$

Let $I = \{i_1, \dots, i_p\} \subseteq \{1, \dots, n\}$. Then we denote,

$$e_I := e_{i_1} \wedge \dots \wedge e_{i_p}.$$

Note that an element $\omega \in \bigwedge^p E$ is of the form,

$$\omega = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=p}} \omega_I e_I, (\omega_I \in \mathbb{R}).$$

Fix $I_0 \subseteq \{1, \dots, n\}$ and denote $I_0^c = \{1, \dots, n\} \setminus I_0$. Using the expression of ω above, we have,

$$\omega \wedge e_{I_0^c} \otimes \underbrace{e_1 \wedge \dots \wedge e_n}_{\text{vol}_g} = \pm \omega_{I_0} \underbrace{|e_1 \wedge \dots \wedge e_n|}_{\text{vol}_g}.$$

This shows the non-degeneracy. The existence of $*\eta$ thus follows from non-degeneracy.

We now show that $*$ satisfies all the properties of the Hodge operator.

ii). Set $\omega = e_I$ for some $\{i_1, \dots, i_p\} \subseteq \{1, \dots, n\}$. Then

$$\mathfrak{o} = \left(\bigwedge^p g \right) (\omega, e_{I^c}).$$

We have,

$$\begin{aligned} \omega \wedge e_{I^c} &= (-1)^{\text{sgn } \sigma} \prod_{i \in I} g(e_i, e_i) \frac{e_1 \wedge \dots \wedge e_n}{|e_1 \wedge \dots \wedge e_n|}, \\ &= (-1)^{\text{sgn } \sigma} e_1 \wedge \dots \wedge e_n (-1)^{\text{sgn } \sigma} \prod_{i \in I} g(e_i, e_i) \frac{e_1 \wedge \dots \wedge e_n}{|e_1 \wedge \dots \wedge e_n|}, \\ &= \prod_{i \in I} g(e_i, e_i) |e_1 \wedge \dots \wedge e_n|, \\ &= \left(\bigwedge^p g \right) (e_I, e_I) |e_1 \wedge \dots \wedge e_n|. \end{aligned}$$

That is

$$*\epsilon_{\sigma(1)} \wedge \dots \wedge e_{\sigma(p)} = \text{sgn } \sigma e_1 \wedge \dots \wedge e_n (-1)^{\text{sgn } \sigma} \prod_{i \in I} g(e_i, e_i) e_{I^c} \frac{e_1 \wedge \dots \wedge e_n}{|e_1 \wedge \dots \wedge e_n|}.$$

iii). Applying the second property twice, and consider $\tau \in S_n$ such that

$$\tau = \begin{pmatrix} 1 & \cdots & n-p & n-p+1 & \cdots & n \\ p+1 & \cdots & n & 1 & \cdots & p \end{pmatrix}$$

Then we have $\text{sgn } \tau = (-1)^{p(n-p)}$. Rewrite e_{I^c} with τ , we obtain,

$$\star_{n-p} \star_p e_I = (-1)^{\text{ind } g} \text{sgn } \sigma \text{sgn}(\tau) e_{\tau(I^c)} = (-1)^{p(n-p)+\text{ind } g} e_I.$$

To show it satisfies the remaining properties is left to the readers as an exercise. \square

5.5 Tensor fields and differential forms

Remark 5.6. Given a manifold \mathcal{M} , we have a vectorbundle $T\mathcal{M} \xrightarrow{\pi} \mathcal{M}$, $T_p\mathcal{M} \mapsto p$. Let us denote $\{g_{ij}\}_{ij}$ be the corresponding cocycle, then we can define, $\bigwedge^p g_{ij}$ which corresponds to a vectorbundle $\bigwedge^p E$.

Definition 5.9. Let \mathcal{M} be a smooth manifold. We define,

$$T^{r,s}\mathcal{M} := \bigotimes^r T\mathcal{M} \otimes \bigotimes^s T^*\mathcal{M}.$$

Sections of $T^{r,s}\mathcal{M}$ are called the tensorfield of type (r, s) . We denote,

$$\Gamma(T^{r,s}\mathcal{M}) = \{\text{smooth sections of } T^{r,s}\mathcal{M}\}.$$

Proposition 5.2. For a map $f : \mathcal{M} \rightarrow T^*\mathcal{M}$ with

$$\forall p \in \mathcal{M}, f(p) \in T_p^*\mathcal{M},$$

the following are equivalent.

1). for vector fields, $X_1, \dots, X_n \in \Gamma(T\mathcal{M})$, and covector field $\omega_1, \dots, \omega_r \in \Gamma(T^*\mathcal{M})$, the function,

$$p \mapsto f(\omega_1, \dots, \omega_r, X_1, \dots, X_s) = f(p)(\omega_1(p), \dots, \omega_r(p), X_1(p), \dots, X_s(p)),$$

is smooth.

2). for any charts (U, φ) with coordinate functions x_1, \dots, x_m , we have,

$$f = \sum f_{j_1 \dots j_s}^{i_1 \dots i_r} \frac{\partial}{\partial x_{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{i_s}} \otimes dx_{j_1} \otimes \dots \otimes dx_{j_s},$$

where dx_1, \dots, dx_m are the dual (missing) to $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ with smooth functions $f_{j_1, \dots, j_s}^{i_1, \dots, i_r} \in C^\infty(U)$.

3). $f \in \Gamma(T^{r,s}\mathcal{M})$.

Proof. Exercise. □

Definition 5.10. The sections of the bundle $\bigwedge^p T^* \mathcal{M} \subseteq T^{0,p} \mathcal{M}$, are called differential forms. The set of differential forms are denoted by

$$\Omega^p(\mathcal{M}) = \Gamma\left(\bigwedge^p T^* \mathcal{M}\right).$$

Proposition 5.3. Let $f : \Gamma(T^* \mathcal{M})^r \times \Gamma(T \mathcal{M})^s \rightarrow \mathcal{C}^\infty(\mathcal{M})$, is induced by a section of $T^{r,s} \mathcal{M}$ if and only if it is $\mathcal{C}^\infty(\mathcal{M})$ -($r-s$) multilinear.

Proof. If $f \in \Gamma(T^{r,s}, \mathcal{M})$ then

$$(\omega_1, \dots, \omega_r, X_1, \dots, X_s) \mapsto (p \mapsto f(p(\omega_1(p), \dots, \omega_r(p), X_1(p), \dots, X_s(p))))$$

is clearly $\mathcal{C}^\infty(\mathcal{M})$ -($r+s$)-multilinear.

Conversely, given $\mathcal{C}^\infty(\mathcal{M})$ -multilinear map f , need to find $f(p)$. To do so we will show that for $p \in \mathcal{M}$, if there is i such that $\omega_i(p) = 0$ or there is j such that $X_j(p) = 0$, we have,

$$f(\omega_1, \dots, \omega_r, X_1, \dots, X_s)(p) = 0.$$

Indeed, we assume without loss of generality that $\omega_1(p) = 0$. Choose a chart (U, φ) centered around p . Choose $h \in \mathcal{C}^\infty(\mathcal{M})$ such that $h \equiv 1$ in small enough neighborhood of p . By assumption, we have,

$$\omega_1|_U = \sum_{j=1}^m a_j dx_j, a_j(p) = 0.$$

Combining these, we obtain,

$$\begin{aligned} \omega_1 &= (1 - h^2)\omega_1 + h^2\omega_1, \\ &= (1 - h^2)\omega_1 + \sum_{j=1}^m \underbrace{(a_j h)}_{\in \mathcal{C}^\infty(\mathcal{M})} \underbrace{(h dx_j)}_{\in \Gamma(T^* \mathcal{M})}. \end{aligned}$$

Using multilinearity of f , we get,

$$f(\omega_1, \dots)(p) = (1 - h^2)(p)f(\omega_1, \dots)(p) + \sum_{j=1}^m \underbrace{(ha_j)(p)}_{=0} f(h dx_j, \dots)(p) = 0.$$

Define $f(p)$ as follows. For $\theta_1, \dots, \theta_r \in T_p^* \mathcal{M}, v_1, \dots, v_s \in T_p \mathcal{M}$, choose $\omega_1, \dots, \omega_r \in \Gamma(T^* \mathcal{M}), X_1, \dots, X_s \in \Gamma(T \mathcal{M})$ with

$$\omega_j(p) = \theta_j, X_j(p) = v_j.$$

By what we have proved, we have,

$$f(p)(\theta_1, \dots, \theta_r, v_1, \dots, v_s) := f(\omega_1, \dots, \omega_r, X_1, \dots, X_s)(p),$$

which is independent of choices of ω_j, X_j . The rest follows from Proposition 5.2. □

Observe that given $f \in \mathcal{C}^\infty(\mathcal{M})$ and $X \in \Gamma(T\mathcal{M})$, the map,

$$\mathcal{M} \ni p \mapsto X_p f,$$

is a smooth function. In particular,

$$df : \Gamma(T\mathcal{M}) \rightarrow \mathcal{C}^\infty(\mathcal{M}), df(X) = Xf,$$

is a $\mathcal{C}^\infty(\mathcal{M})$ linear map.

From Proposition 5.3, it follows that $df \in \Gamma(T^*\mathcal{M})$. In particular, we have a linear map,

$$d : \mathcal{C}^\infty(\mathcal{M}) = \Omega^0 \mathcal{M} \rightarrow \Gamma(T^*\mathcal{M}) = \Omega^1 \mathcal{M}.$$

Definition 5.11 (Pullback). Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be smooth and $\xi \in T(T^{0,s}\mathcal{N})$, Put,

$$f^*\xi|_p(v_1, \dots, v_s) := \xi|_{f(p)}(T_p f v_1, \dots, T_p f v_s).$$

Proposition 5.4 (Properties of Pullbacks). Given $\mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{L}$ be smooth maps. We have,

1. $\varphi \in \mathcal{C}^\infty(\mathcal{N}) = \Gamma(T^{0,0}\mathcal{N})$, $f^*\varphi = \varphi \circ f$.
2. f^* is a linear map $\Gamma(T^{0,s}\mathcal{N}) \rightarrow \Gamma(T^{0,s}\mathcal{M})$ and $\Omega^p \mathcal{N} \rightarrow \Omega^p \mathcal{M}$.
3. $(g \circ f)^* = f^* \circ g^*$.
4. If f is a diffeomorphism then $(f^*)^{-1} = (f^{-1})^*$.
5. $\omega \in \Omega^p \mathcal{N}, \eta \in \Omega^q \mathcal{N}$ then $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Notation 5.2. Let $U \subseteq \mathcal{M}$ be an open set, we define,

$$\Omega^*(U) := \bigoplus_{p \in \mathbb{Z}_{\geq 0}} \Omega^p(U).$$

Theorem 5.2 (Cartan (exterior) derivative). Let \mathcal{M} be a smooth manifold and U be an open subset of \mathcal{M} , then there exists a unique family of linear maps

$$d : \Omega^*(U) \rightarrow \Omega^*(U),$$

which satisfies the following properties.

- 1). d is of degree 1 that is $d : \Omega^p(U) \rightarrow \Omega^{p+1}(U)$.
- 2). For $\omega \in \Omega^p(U), \eta \in \Omega^q(U)$, we have,

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d(\eta).$$

- 3). For $f \in \mathcal{C}^\infty(\mathcal{M}) = \Omega^0(\mathcal{M})$ and $X \in \Gamma(T\mathcal{M})$, we have,

$$\langle df, X \rangle = df(X) = Xf$$

4). $d \circ d = 0$.

5). For $U \subseteq V \subset \mathcal{M}$ open then we have a commutative diagram, where $\iota : U \hookrightarrow V$ is an inclusion and ι^* is the pullback of inclusion namely the restriction.s

$$\begin{array}{ccc} \Omega^*(V) & \xrightarrow{d} & \Omega^*(V) \\ \iota^* \downarrow & \circlearrowleft & \downarrow \iota^* \\ \Omega^*(U) & \xrightarrow{d} & \Omega^*(U) \end{array}$$

These five properties determine d uniquely, and satisfies furthermore,

6). For $\omega \in \Omega^p(\mathcal{M})$, $X_0, \dots, X_p \in \Gamma(T\mathcal{M})$, we have

$$\begin{aligned} d\omega(X_0, \dots, X_p) &= \sum_{i=0}^p (-1)^i X_i(\omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p)) \\ &\quad + \sum_{0 \leq i < j \leq m} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_{j-1}, X_{j+1}, \dots, X_p). \end{aligned}$$

7). $f : \mathcal{M} \rightarrow \mathcal{N}$ is smooth then $f^*(d\omega) = d(f^*\omega)$.

Proof. For the uniqueness, assume we had d, \tilde{d} satisfying the first five properties. Let (U, φ) be a chart. Denote for $I = \{i_1 < \dots < i_p\}$, $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_p}$. Using this notation, we have,

$$d(fd\varphi_I) = df \wedge d\varphi_I,$$

which follows from the second and the forth properties. Explicitly, this follows from that

$$dd\varphi_I = \underbrace{ddx_{i_1} \wedge \dots \wedge dx_{i_p}}_{=0} \pm dx_{i_1} \wedge d(dx_{i_2} \wedge \dots \wedge dx_{i_p}).$$

And using induction, we get $dd\varphi_I = 0$. Using the third property, we have,

$$d(fd\varphi_I) = \sum_{j \notin I} \frac{\partial f}{\partial x_j} dx_j \wedge d\varphi_I.$$

Cover \mathcal{M} by charts $\bigcup_i U_i$, by the fifth property, we have,

$$d\omega|_{U_i} = \tilde{\omega}|_{U_i}.$$

Therefore, we conclude $d = \tilde{d}$.

For the existence of d , taking 6). as definition. We need to show as per Proposition 5.3, for $\omega \in \Omega^p(U)$,

i). $d\omega$ is $\mathcal{C}^\infty(U)$ -multilinear,

ii). $d\omega$ is alternating.

These two will imply that $d\omega \in \Omega^{p+1}(U)$. For the second assertion, we see,

$$\begin{aligned} d\omega(X_1, X_0, X_2, \dots, X_p) &= X_1\omega(X_0, X_2, \dots, X_p) + \sum_{i=2}^p (-1)^i X_i\omega(X_1, X_0, \dots) + \dots, \\ &= -d\omega(X_0, X_1, \dots, X_p). \end{aligned}$$

For the first assertion, we have,

$$\begin{aligned} d\omega(fX_0, X_1, \dots, X_p) &= \sum_{i=1}^p (-1)^i \underbrace{X_i(f \cdot \omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p) + fX_i\omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p))}_{X_i(f \cdot \omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p))} \\ &\quad + f \cdot X_0\omega(X_1, \dots, X_p) \\ &\quad + \sum_{j=1}^p (-1)^j \omega(\underbrace{[fX_0, X_j]}_{fX_0 - X_j(fX_0) = f[X_0, X_j] - (X_j f)X_0}, X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_p) \\ &\quad + \sum_{1 \leq i < j \leq p} (-1)^{i+j} f \cdot \omega([X_i, X_j], X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_{j-1}, X_{j+1}, \dots, X_p), \\ &= f \cdot d\omega(X_0, \dots, X_p). \end{aligned}$$

We claim that the so-defined d satisfies 1). - 5). We have already seen that 1) and 5). Now it suffices to look at a coordinate system, x_1, \dots, x_n where

$$\forall 1 \leq i, j \leq n, [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

Then we have,

$$\begin{aligned} d(fd\omega_{i_1} \wedge \dots \wedge d\omega_{i_p}) &\left(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_p}} \right) \\ &= \frac{\partial^2}{\partial x_j \partial x_i} \left(fd\omega_{i_1} \wedge \dots \wedge \underbrace{d\omega_{i_p} \left(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_{i-1}}}, \frac{\partial}{\partial x_{j_{i+1}}}, \dots, \frac{\partial}{\partial x_{j_p}} \right)}_{0 \text{ or } \pm 1} \right), \\ &= \begin{cases} 0, & \{i_1, \dots, i_p\} \not\subseteq \{j_0, \dots, j_p\} \\ (-1)^\alpha \frac{\partial}{\partial x_\alpha} f & \{i_1 < \dots < i_p\} = \{j_0 < \dots < j_{\alpha-1} < j_{\alpha+1} < \dots < j_p\}, \end{cases} \\ &= \sum \frac{\partial f}{\partial x_k} dx_k \wedge d\omega_{i_1} \wedge \dots \wedge d\omega_{i_p} \left(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_p}} \right). \end{aligned}$$

Let

$$\begin{aligned} \omega &= f dx_1 \wedge \dots \wedge dx_p, \\ \eta &= g dx_{p+1} \wedge \dots \wedge dx_{p+q}. \end{aligned}$$

Using the concrete formula, we have,

$$\begin{aligned}
d(\omega \wedge \eta) &= d(f \cdot g dx_1 \wedge \cdots \wedge dx_{p+q}), \\
&= \sum_{j=p+q+1}^m \frac{\partial f g}{\partial x_j} (-1)^{p+q} dx_1 \wedge \cdots \wedge dx_{p+q} \wedge dx_j, \\
&= \underbrace{\sum_{j=p+q+1}^m \frac{\partial f}{\partial x_j} (-1)^p dx_1 \wedge \cdots \wedge dx_p \wedge dx_j \wedge \underbrace{g dx_{p+1} \wedge \cdots \wedge dx_{p+q} \wedge dx_j}_{=\eta}}_{=d\omega} \\
&\quad + \underbrace{\sum_{j=p+q+1}^m \underbrace{f dx_1 \wedge \cdots \wedge dx_p}_{=\omega} \wedge (-1)^{p+q} \frac{\partial g}{\partial x_j} dx_{p+1} \wedge \cdots \wedge dx_{p+q} \wedge dx_j}_{(-1)^p \omega \wedge d\eta}.
\end{aligned}$$

This proves 2). For the third, both slots are $\mathcal{C}^\infty(U)$ -linear in X . Hence without loss of generality, we assume $X = \frac{\partial}{\partial x_1}$.

$$\langle df, \frac{\partial}{\partial x_1} \rangle = \left\langle \sum_{j=1}^m \frac{\partial f}{\partial x_j} dx_j, \frac{\partial}{\partial x_1} \right\rangle = \frac{\partial f}{\partial x_1}.$$

For the fourth, we see,

$$\begin{aligned}
d \circ d(f dx_1 \wedge \cdots \wedge dx_p) &= d \left(\sum_{j=p+1}^m \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_p \right), \\
&= \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_1 \wedge \cdots \wedge dx_p, \\
&= \sum_{i < j} \frac{\partial^2 f}{\partial x_i \partial x_j} \underbrace{(dx_i \wedge dx_j + dx_j \wedge dx_i)}_{=0} \wedge dx_1 \wedge \cdots \wedge dx_p.
\end{aligned}$$

The seventh is obvious at charts. \square

5.6 Application : Classical vector analysis

Consider \mathcal{M}^m where $m \in \{2, 3, 4\}$. Let $g \in \Gamma(T^{0,2}\mathcal{M})$, be symmetric non-degenerate. May choose at orthonormal frame such that

$$g = \begin{pmatrix} -I_p & O \\ O & I_q \end{pmatrix}$$

where $m = p + q$. That is

$$g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \begin{cases} 0, & i \neq j, -1, \quad i = j \leq p, \\ 1, & i = j > p. \end{cases}$$

Interesting cases are

1. Riemannain metric when $p = 0$,
2. General relativity, when $p = 1, q = 3$.

Set the volume form as

$$\begin{aligned}\text{vol}_g &= \left| \det g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right|^{\frac{1}{2}} |dx_1 \wedge \cdots \wedge dx_m|, \\ \sigma &= \sigma(dx_1, \dots, dx_m) = \frac{dx_1 \wedge \cdots \wedge dx_m}{|dx_1 \wedge \cdots \wedge dx_m|}.\end{aligned}$$

Compute $*dx_j$ for a general coordinate system.

$$*dx_j = \sum_{k=1}^m (-1)^{k-1} g^{kj} \sqrt{g} dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_m \otimes \sigma,$$

$$g_{ij} = g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right), g^{ij} = g(dx_i, dx_j) = (\{(g_{kl})_{k,l}\}^{-1})_{i,j}.$$

Recall that $*dx_j$ is the unique twisted $(n-1)$ -form ω satisfying

$$\tau \wedge \omega = \bigwedge^1 g(\tau_1 dx_j) \text{vol } g,$$

where τ is an arbitrary 1-form.

$$\omega = \sum_{j=1}^m (-1)^{j-1} \omega_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma.$$

And we have,

$$\begin{aligned}dx_i \wedge \omega &= \left(\bigwedge^1 g \right) (dx_i, dx_j) \text{vol } g = g^{ij} \text{vol } g, \\ &= \sum_{j=1}^m (-1)^{j-1} \omega_j dx_i \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma, \\ &= \omega_i dx_1 \wedge \cdots \wedge dx_m \otimes \sigma, \\ &= \omega_i \sqrt{g}^{-1} \text{vol } g. \\ \omega_i &= g^{ij} \sqrt{g}.\end{aligned}$$

Proposition 5.5.

$$dx_i^\# = \sum_j g^{ij} \frac{\partial}{\partial x_j}, \left(\frac{\partial}{\partial x_i} \right)^b = \sum g_{ij} dx_j.$$

Proof.

$$\left\langle \left(\frac{\partial}{\partial x_i} \right)^b, \frac{\partial}{\partial x_k} \right\rangle = g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right) = g_{ik} \text{ gives the second formula.}$$

$$\langle (dx_i)^\#, dx_k \rangle = g(dx_i, dx_k) = g^{ik} \text{ gives the first formula}$$

□

Definition 5.12. We define the gradient to be such that

$$\text{grad}_g f = (def)^\# := \left(\sum (\partial_i f) dx_i \right)^\# = \sum_{ij} g^{ij} (\partial_i f) \frac{\partial}{\partial x_j}.$$

Definition 5.13. $\star dx_j$ is the unique $(n-1)$ form ω such that

$$(\tau \wedge \omega) = \left(\bigwedge^1 g \right) (tdx_j) \text{vol}_g$$

for all 1-form τ .

Remark 5.7.

$$\omega = \sum_{j=1}^m (-1)^{j-1} \omega_j dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_m \otimes \sigma.$$

In particular,

$$\begin{aligned} dx_j \wedge \omega &= w_i dx_1 \wedge \cdots \wedge dx_m \otimes \sigma, \\ &= \omega_i \sqrt{g}^{-1} \text{vol}_g, \quad \sqrt{g} = |\det(g_{ij})|, \\ &= \left(\bigwedge^1 g \right) (dx_i, dx_j) \text{vol}_g, \\ &= g_{ij} \text{vol}_g. \end{aligned}$$

Therefore, we obtain $\omega_i = g^{ij} \sqrt{g}$ and

$$\omega = \star(dx_j) = \sum_{k=1}^m (-1)^{k-1} g^{kj} \sqrt{g} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma.$$

We now identify vectorfields with $(m-1)$ -form by pairing,

$$X \mapsto \star(X^\flat).$$

Lemma 5.3.

$$\star X^\flat = \text{int}_X \text{vol}_g.$$

Proof. Since the formula we want to show is a pointwise identity, it suffices to check it on $X|_p = e_1|_p$, where e_1, \dots, e_m are g -orthonormal basis, since every small enough neighborhood admits a g -orthonormal basis. Observe that

$$\star e_1^\flat = \star(e_1)^* = c_1 e_2^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma(e_1^*, \dots, e_m^*).$$

Thus we have,

$$\text{int}_{e_1} \text{vol}_g = \text{int}_{e_1} e_1^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma = c_1 e_2^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma,$$

where $c_j = g_{jj} = g^{jj}$ for a g -orthonormal basis.

$$\begin{array}{ccc} \Omega^{m-1}(\mathcal{M}) & \xrightarrow{d} & \Omega^m(\mathcal{M}) \\ X \mapsto \star X^\flat \uparrow & & (-1)^p \star \downarrow \uparrow \star \\ \Gamma(\mathcal{M}) & \xrightarrow[\text{div}^g]{} & \mathcal{C}^\infty(\mathcal{M}) \end{array}$$

$$\text{div}^g(X) = (-1)^p \star d \star X^\flat,$$

thus

$$\begin{aligned} \text{div}^g(X) &= \text{div}^g \left(\sum_{j=1}^m X_j \frac{\partial}{\partial x_j} \right), \\ &= (-1)^p \star d \star \left(\sum_{jk} g_{jk} X_j dx_k \right), \\ &= (-1)^p \star d \left(\sum_{jkl} X_j d_{jk} (-1) dx_1 \wedge \cdots \wedge dx_{l-1} \wedge dx_{l+1} \wedge \cdots \wedge dx_m \right) \\ &= (-1)^p \star d \left(\sum (-1)^{j-1} x_j \sqrt{g} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma \right), \end{aligned}$$

Thus, we obtain,

$$\text{int}_X \text{vol}_g = \sum_j X_j \text{int}_{\partial_j} \sqrt{g} dx_1 \wedge \cdots \wedge dx_m \otimes \sigma =$$

□

Remark 5.8. On \mathcal{M}^2 , we have,

$$\begin{array}{ccccc} \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 \\ \downarrow & & \downarrow \# & & \downarrow \star \\ \mathcal{C}^\infty(\mathcal{M}) & \xrightarrow[\text{grad}^g]{} & \Gamma(T\mathcal{M}) & \xrightarrow[X \mapsto -\text{div}(\star X^\flat)^\#]{} & \mathcal{C}^\infty(\mathcal{M}) \end{array}$$

5.7 Integration of densities

In this section,

$$\mathcal{O} = \mathcal{O}(T^*\mathcal{M}).$$

In terms of the

Lemma 5.4. *If (U_i, φ_i) is an atlas of \mathcal{M} then the cocycle for $|\wedge|^\alpha$ is*

Proof.

$$\begin{aligned} g_{ij}(p) &= |\det((\varphi_i^{-1})^* \circ \varphi_j^*)|^\alpha, \\ &= |\det((\varphi_j \circ \varphi_i)^*)|^\alpha, \\ &= |\det(D(\varphi_j \circ \varphi_i^{-1})^t)|^\alpha, \\ &= |\det(D(\varphi_i \circ \varphi_j^{-1}))|^{-\alpha}|_{\varphi_j(p)}. \end{aligned}$$

Respectively for \mathcal{O} , we get,

$$g_{ij} = \operatorname{sgn} D(\varphi_i \circ \varphi_j^{-1})|_{\varphi_j(p)}.$$

□

A diffeomorphism $f : \mathcal{M} \rightarrow \mathcal{N}$ gives rise for pullback map

$$|\wedge|^\alpha(\mathcal{N}) \xrightarrow{f^*} |\wedge|(\mathcal{M}), \mathcal{O}(\mathcal{N}) \xrightarrow{f^*} \mathcal{O}(\mathcal{M}).$$

Proposition 5.6. *Let \mathcal{M}^m be a manifold. There is a unique linear form,*

$$\int_U : \Gamma_C(|\wedge \mathcal{M}|) \rightarrow \mathbb{R},$$

which is invariant under diffeomorphisms and in local coordinate cocycles with the Lebesgue integral,

$$\int_{\mathcal{M}} f(x)|dx| = \int_{\mathbb{R}^m} f(x)|dx_1 \wedge \cdots \wedge dx_m|.$$

Proof. Let $\omega \in \Gamma_C(|\wedge \mathcal{M}|)$ with support contained in some chart (U, φ) . We have to put

$$\int_{\mathcal{M}} \omega = \int_{\mathbb{R}^m} (\varphi^{-1})^* \omega.$$

If we have another chart (V, ψ) such that $\operatorname{supp} \omega \subseteq U \cap V$, then

$$\begin{aligned} (\varphi^{-1})^* \omega &= f|dx_1 \wedge \cdots \wedge dx_m|, \\ (\psi^{-1})^* \omega &= (\varphi^{-1} \circ \varphi \circ \psi^{-1})^* \omega = (\varphi) \\ &\Rightarrow \int_{\mathbb{R}^m} (\varphi^{-1})^* \omega = \int_{\mathbb{R}^m} (\psi^{-1})^* \omega. \end{aligned}$$

For the general case, choose $\rho_1, \dots, \rho_k \in \mathcal{C}_C^\infty(\mathcal{M})$ with,

$$\sum \rho_j|_{\text{supp } \omega} \equiv 1|_{\text{supp } \omega},$$

such that $\text{supp } \rho_j$ lies in a chart. Put

$$\int_{\mathcal{M}} \omega := \sum_{j=1}^k \int_{\mathcal{M}} \rho_j \omega.$$

If $\tilde{\rho}_1, \dots, \tilde{\rho}_l$ is a different partition of unity then,

$$\sum_{j=1}^k \int_{\mathcal{M}} \rho_j \omega = \sum_{j=1}^k \int_{\mathcal{M}} \sum_{i=1}^l \rho_j \tilde{\rho}_i \omega = \sum_{j,i=1}^{k,l} \int_{\mathcal{M}} \rho_j \tilde{\rho}_i \omega = \sum_{i=1}^l \int_{\mathcal{M}} \tilde{\rho}_i \omega_i.$$

□

Remark 5.9. Let $\omega = f|dx_1 \wedge \dots \wedge dx_m|$. Suppose we have a diffeomorphism,

$$(\mathbb{R}^m, y) \xrightarrow{\varphi} (\mathbb{R}^m, x).$$

Then we have,

$$\varphi^* \omega = f \circ \varphi | \det D\varphi | |dy_1 \wedge \dots \wedge dy_m|.$$

Then we have,

$$\begin{aligned} \int_{(\mathbb{R}^m, x)} f |dx_1 \wedge \dots \wedge dx_m| &= \int_{\mathbb{R}^m} f(x) dx_1 \wedge \dots \wedge dx_m, \\ &= \int_{(\mathbb{R}^m, y)} f(\varphi(y)) | \det D\varphi(y) | dy_1 \wedge \dots \wedge dy_m, \\ &= \int_{(\mathbb{R}^m, y)} f^* \omega. \end{aligned}$$

Corollary 5.1. Let $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ be a diffeomorphism between smooth manifolds and $\omega \in \Gamma_C(| \wedge |^1 \mathcal{N})$. Then

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{N}} \Phi^* \omega.$$

Proof. Write

$$\omega = \sum_{i=1}^k \omega_i,$$

such that $\text{supp } \omega \subseteq V_i$ for some coordinate chart $(V_i, \psi_i : V_i \rightarrow U_i \subseteq \mathbb{R}^n)$. Then

$$\text{supp } \Phi^* \omega_i \subseteq \Phi^{-1}(V_i),$$

which is a coordinate patch as well. To see this, we have,

$$\psi_i \circ \Phi|_{\Phi^{-1}(V_i)} : \Phi^{-1}(V_i) \rightarrow U_i \subseteq \mathbb{R}^n,$$

is again a chart. Then,

$$\int_{\mathcal{N}} \omega_i = \int_{V_i} \omega_i = \int_{\Phi^{-1}(V_i)} \Phi^* \omega_i = \int_{\mathcal{M}} \Phi^* \omega_i.$$

As ω is a finite sum of these ω_i , we have,

$$\int_N \omega = \sum_{i=1}^k \int_{\mathcal{N}} \omega_i = \sum_{i=1}^k \int_{\mathcal{M}} \Phi^* \omega_i = \int_{\mathcal{M}} \Phi^* \omega.$$

□

5.8 Orientations on manifolds

Definition 5.14. Let \mathcal{M} be a smooth manifold. An atlas \mathcal{A} oriented is called oriented if

$$\forall \varphi, \psi \in \mathcal{A}, x \in \mathcal{M}, \det D(\psi \circ \varphi^{-1})(x) > 0.$$

Definition 5.15. Two oriented atlases \mathcal{A}, \mathcal{B} are equivalent if $\mathcal{A} \cup \mathcal{B}$ is oriented.

Definition 5.16. An orientation is a choice of a maximal oriented atlas. In particular, a manifold is orientable if there is an orientation.

Clearly we have $(\mathbb{R}^m, \{\text{id}\})$ is oriented.

Example 5.1. The following is an example of non-oriented manifold.

$(S^n, \{\psi_{\pm}\})$ where

$$\psi_{\pm} : S^n \setminus \{\pm e_n\}, \psi_{\pm}(x, t) \mapsto \frac{x}{1 \mp t}$$

Then we have

$$\psi_{+} \circ \psi_{-}^{-1}(x) = \frac{x}{|x|^2}.$$

Thus $f(x) = \frac{x}{|x|^2}$ is orientation reversing. That is $\det Df(x) < 0$.

Remark 5.10. Fix an orientation reversing diffeomorphism of \mathbb{R}^n say $\Phi(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$. We have a new atlas,

$$\{\Phi \circ \psi_{+}, \psi_{-}\}$$

is oriented.

Example 5.2. A Möbius band cannot be oriented. It is an exercise for the readers to give it a proof.

Proposition 5.7. Let \mathcal{M} be a smooth manifold. \mathcal{M} is orientable if and only if one of following statements hold.

- i). There is a section s of the bundle $\mathcal{O}(\mathcal{M})$ with $|s(p)| = 1$ for all $p \in \mathcal{M}$. Note that \mathcal{O} comes with a canonical inner product, thus we can define a notion of length.

ii). There is a section $\omega \in \Omega^m(\mathcal{M})$ where $m = \dim \mathcal{M}$ with $\omega(p) \neq 0$ for all $p \in \mathcal{M}$.

Moreover, each of the conditions will determine an orientation.

Proof.

For the first one. Suppose \mathcal{M} is orientable. then consider $\mathcal{M} = \bigcup_n U_n$ where (U_n, φ_n) is an oriented chart and consider (ρ_n) be a subordinate partition of unity to the charts. We set,

$$\omega := \sum_{j=1}^{\infty} \rho_j dx_1^j \wedge \cdots \wedge dx_m^j. \quad (4)$$

We show that $\omega(p) \neq 0$ for all $p \in \mathcal{M}$. To see this for a fixed $p \in U_i$. In the small enough neighborhood around p , we have,

$$\omega = \sum_{j=1}^{\infty} \rho_j \circ \varphi \det D(\varphi^{-1} \varphi_j \circ \varphi_i) dx_1^i \wedge \cdots \wedge dx_m^i. \quad (5)$$

As we have assumed the manifold to be oriented. Each $\det D(\varphi_j \circ \varphi_i^{-1})$ is positive. And partition of unities are non-negative. We conclude that $\omega(p) > 0$.

Suppose the second statement holds. Fix $\omega \in \Omega^m \mathcal{M}$ such that $|\omega|(p) \neq 0$ for all $p \in \mathcal{M}$. Let (U, φ) be a connected chart. Then obviously, we have,

$$\varphi^* \omega = \rho_\varphi dx_1 \wedge \cdots \wedge dx_m.$$

with

$$\forall x \in \varphi(U), \rho_\varphi(x) \neq 0.$$

We call φ is oriented if $\rho_\varphi > 0$. This determines an oriented atlas. Thus proves that the second condition determines an atlas.

Furthermore, the first and the second conditions are equivalent. To show this, choose a Riemannian metric on \mathcal{M} . Given s as in the first condition. Set

$$\omega = \text{vol} \cdot s.$$

Conversely, given ω as in the second condition set $s = \frac{\omega}{\text{vol}}$. \square

6 Manifolds with boundaries

6.1 Basics

Definition 6.1 (Half-space). Let E be a real vector space of dimension $n < \infty$. A subset $H \subseteq E$ is called a half space if there is a linear form $\lambda \in E^* \setminus \{0\}$ such that

$$H = \{x \in E \mid \langle \lambda, x \rangle \leq 0\}.$$

We denote such H by E_λ^+ .

Example 6.1. $\{x \in \mathbb{R}^n \mid x_j \leq 0\}$ is a half space for $1 \leq j \leq n$.

Remark 6.1. Let $H_1, H_2 \subseteq E$ be two half-spaces. Then there is a linear isomorphism $T \in \text{GL}(E)$ such that $TH_1 = H_2$. That is if

$$H_j = E_{\lambda_j}^+,$$

then take $T^* \in \text{GL}(E^*)$ such that $T^*\lambda_1 = \lambda_2$. We have

$$\langle \lambda_2, x \rangle = \langle T^*\lambda_1, x \rangle = \langle \lambda_1, Tx \rangle.$$

Remark 6.2. The above remark is true in the oriented category only if $\dim E \geq 2$. That is $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$ are diffeomorphic but not oriented diffeomorphic.

Definition 6.2. Let $E_\lambda^+ \subseteq E$ be a half space. The normal space is $E / \ker \lambda$.

Lemma 6.1. A normal space $E / \ker \lambda$ is canonically oriented by defining a base of $E / \ker \lambda$ by $\lambda(v) > 0$.

Definition 6.3. If $v \in E$ with $\lambda(v) > 0$ then we say v points outward. Furthermore, we set

$$c(t) = tv.$$

Remark 6.3. The notion comes from the following case. Consider a curve $c : \mathbb{R} \rightarrow E$ such that for small enough $\varepsilon > 0$, we have $c((-\varepsilon, 0]) \subseteq E_\lambda^+$ for some $\lambda \in E^*$. And for $t > 0$, $c(t) \notin E_\lambda^+$.

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