

# Analysis and Geometry on Manifolds

So Murata

WiSe 25/26, University of Bonn

## 1 Review of Structure Theorems for Differentiable Maps

**Theorem 1.1** (Implicit Function Theorem). *Let  $U \subseteq \mathbb{R}^p$ ,  $V \subseteq \mathbb{R}^q$  be open sets and  $F(x, y) : U \times V \rightarrow \mathbb{R}^q$  be of  $\mathcal{C}^\infty$  class. If  $(a, b) \in U \times V$  satisfies  $F(a, b) = 0$  and*

$$D_y F(a, b) = \left( \frac{\partial F_i}{\partial y_j}(a, b) \right) \in \mathrm{GL}_q(\mathbb{R})$$

*then there exists*

- i). neighborhoods  $a \in U_1$  and  $b \in V_1$ ,
- ii).  $\varphi \in \mathcal{C}^\infty(U_1, V_1)$

*such that*

$$\forall (x, y) \in U_1 \times V_1, F(x, y) = 0 \Leftrightarrow \varphi(x) = y.$$

*Furthermore, we have that*

$$D\varphi(x) = -D_y F(x, \varphi(x))^{-1} \cdot D_x F(x, \varphi(x)).$$

**Theorem 1.2** (Inverse Function Theorem). *Let  $U \subseteq \mathbb{R}^p$  be an open subset and  $f : U \rightarrow \mathbb{R}^q$  be smooth. Let  $a \in U$  be such that  $Df(a)$  is invertible. Then there are neighborhoods  $a \in U_1 \subseteq U$  and  $f(a) \subseteq V_1$  such that  $f|_{U_1} : U_1 \rightarrow V_1$  is a diffeomorphism.*

**Theorem 1.3** (Rank Theorem). *Let  $U \subseteq \mathbb{R}^p$  be open and  $f : U \rightarrow \mathbb{R}^q$  be smooth. Let  $a \in U$  and  $b = f(a)$ . If  $\mathrm{rk} Df(a) = r$  then there exists local diffeomorphisms*

- i).  $\psi : U_\psi \subseteq U \rightarrow V_\psi \subseteq \mathbb{R}^p$  with  $\psi(a) = 0$
- ii).  $\varphi : U_\varphi \subseteq \mathbb{R}^q \rightarrow V_\varphi \subseteq \mathbb{R}^q$  with  $\varphi(b) = 0$ .

*such that*

$$\varphi \circ f \circ \psi^{-1}(x_1, \dots, x_p) = (x_1, \dots, x_r, \tilde{f}(x)).$$

*Furthermore, if  $\mathrm{rk} Df(x) = r$  in some neighborhood of  $a$ , then  $\tilde{f}$  can be chosen to be 0.*

## 2 Differentiable Manifolds

### 2.1 Basics from Set Theoretic Topology

**Definition 2.1.** Let  $X$  be a topological space.  $X$  is said to be separated/Hausdorff if any two distinct points have open neighborhoods which are disjoint to one another.

**Definition 2.2.** Let  $X$  be a topological space and  $(U_i)_{i \in I}$  be its open covering. A refinement of  $(U_i)_{i \in I}$  is an open covering  $(V_j)_{j \in J}$  such that

$$\forall j \in J, \exists i \in I \text{ s.t. } V_j \subseteq U_i.$$

It is locally finite if each point  $x \in X$ , there exists a neighborhood  $U_i$  such that

$$|\{j \in J \mid U \cap V_j \neq \emptyset\}| < \infty.$$

**Definition 2.3.** A topological space  $X$  is called paracompact if every open covering  $(U_i)_{i \in I}$  has a locally finite refinement  $(V_j)_{j \in J}$ .

**Example 2.1.** The following spaces are paracompact.

1). Compact spaces.

2). Locally compact Hausdorff spaces which are first countable.

**Definition 2.4.** A subset  $M \subseteq \mathbb{R}^p$  is said to be a submanifold of dimension  $m$  if  $M$  can be covered by open sets  $(U_i)_{i \in I}$  such that there exists a smooth function  $F_i : U_i \rightarrow \mathbb{R}^{p-m}$  with full rank and

$$M \cap U_i = \{x \in U_i \mid F_i(x) = 0\}.$$

In other words,  $M$  is locally a graph of a smooth map.

**Definition 2.5.** Let  $X$  be a separated and paracompact topological space. A chart is a pair  $(U, \varphi)$  where  $U$  is an open subset of  $X$  and  $\varphi : U \rightarrow V \subseteq \mathbb{R}^n$  is a homeomorphism onto some open subset  $V$  of  $\mathbb{R}^n$ . An atlas  $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$  is a collection of charts such that  $(U_i)_{i \in I}$  covers  $X$ .

**Definition 2.6.** Let  $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$  be an atlas. A transition map is a composition  $\varphi_i \circ \varphi_j^{-1}|_{\varphi_j(U_i \cap U_j)} \rightarrow \varphi_i(U_i \cap U_j)$  for some  $i, j \in I$ .

**Definition 2.7.** An atlas  $\mathcal{A}$  is called smooth if all the transition maps are smooth.

**Definition 2.8.** A chart  $(U, \varphi)$  is smooth compatible with smooth atlas  $\mathcal{A}$  if for any  $i \in I$ ,  $\varphi_i \circ \varphi^{-1}$  and  $\varphi \circ \varphi_i^{-1}$  are smooth.

**Definition 2.9.** An atlas  $\mathcal{A}$  is smooth maximal if  $(U, \varphi)$  is smooth compatible with  $\mathcal{A}$  then  $(U, \varphi) \in \mathcal{A}$ .

**Definition 2.10.** A paracompact, smooth manifold is a pair  $(M, \mathcal{A})$  such that

i).  $M$  is a paracompact separated topological space.

ii).  $\mathcal{A}$  is smooth maximal.

**Remark 2.1.** Above notions can be defined for  $C^k$ , analytic, continuous, algebraic, linear by simply replacing the word "smooth" with one of the formers.

**Remark 2.2.** It suffices to give one (smallest possible) atlas to define a smooth structure.

## 2.2 Examples of Smooth Manifolds

**Definition 2.11.** A topological space is second countable if it has a basis consists of at most countably many open subsets.

**Remark 2.3.** Instead of paracompactness, we assume smooth manifolds to be separated and second countable space. From this definition, we can also induce paracompactness.

**Example 2.2.** For  $x, y \in \mathbb{R}$ ,  $x \sim y \Leftrightarrow |x| = |y| > 1$ . With this relation, we construct a quotient space  $X = \mathbb{R}/\sim$ .

We introduce its atlas by  $U_1 = (-\infty, 1)/\sim = X - \{[1]\}$  with chart  $\varphi_1(x) = [x]$ , and  $U_2 = (-1, \infty)/\sim = X - \{[-1]\}$  with chart  $\varphi_2(x) = [x]$ . However, this is non-separated as 1 and -1 cannot be separated.

**Definition 2.12.** A smooth submanifold of  $\mathbb{R}^n$  is a separated, second countable smooth manifold.

Take  $\varphi^{-1}$ , where  $\varphi$  runs through parametrization as charts.

**Remark 2.4.**  $(\mathbb{R}^n, \{\text{id}\})$  is separated and second countable smooth manifold.

**Definition 2.13.** Let  $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$ . Let  $N \in S^n$  and call it the north pole. We define the stereographic projection with north pole  $N$  to be such that

$$\varphi_+ : S^n \setminus \{N\} \rightarrow \mathbb{R}^n, (x_1, \dots, x_n) \mapsto \frac{1}{1 - x_n} (x_1, \dots, x_{n-1}).$$

Similarly, we define

$$\varphi_- : S^n \setminus \{-N\} \rightarrow \mathbb{R}^n, (x_1, \dots, x_n) \mapsto \frac{1}{1 + x_n} (x_1, \dots, x_{n-1}).$$

**Definition 2.14.** A  $n$ -dimensional torus  $T^n$  is a direct product of  $n$ -many  $S^1$ .

**Definition 2.15.** A lattice  $\Gamma$  in  $\mathbb{R}^n$  is a subgroup generated by a basis of  $\mathbb{R}^n$  over  $\mathbb{Z}$ .

**Lemma 2.1.**

$$T^n = \mathbb{R}^n / \Gamma.$$

**Definition 2.16.** Let  $k = \mathbb{R}$  or  $\mathbb{C}$ . The projective space over the field  $K$  is

$$K\mathbb{P}^n = (K^{n+1} \setminus \{o\}) / K^\times.$$

**Remark 2.5.**

$$\mathbb{R}\mathbb{P}^n = S^n / x \sim -x.$$

And also we have,

$$\mathbb{C}\mathbb{P}^n = S^{2n+1} / S^1.$$

We can introduce atlases to them by

$$U_j = \{[x_0 : \cdots : x_n] \in K\mathbb{P}^n \mid x_j \neq 0\},$$

with chart,

$$\varphi_j : U_j \rightarrow K^n, \varphi([x_0 : \cdots : x_n]) = \left( \frac{x_0}{x_j}, \dots, \frac{x_n}{x_j} \right).$$

**Definition 2.17.** Let  $\mathcal{M}, \mathcal{N}$  be smooth manifold. A function  $f : \mathcal{M} \rightarrow \mathcal{N}$  is called smooth if for each  $p \in \mathcal{M}$ , there exist chart  $(U, \varphi)$  around  $p$  and  $(V, \psi)$  around  $f(p)$  such that

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is smooth for some  $m, n \in \mathbb{N}$ .

**Remark 2.6.** Since transition maps are smooth, above map is well-defined and if  $f$  is smooth so is  $\varphi' \circ f \circ \psi'$  for any charts  $\varphi', \psi'$ .

**Proposition 2.1.** Compositions of smooth maps are smooth.

**Definition 2.18.** Let  $f : \mathcal{M} \rightarrow \mathcal{N}$  be a smooth map. It is called a diffeomorphism if it is bijective and  $f^{-1}$  is also smooth.

### 2.3 Partitions of Unity

**Proposition 2.2.** Let  $(\mathcal{M}, \mathcal{A})$  be a pair of a smooth manifold with its atlas which we assumed to be separated and second countable. Then it is paracompact. More precisely, given an open covering  $(U_i)_{i \in I}$ , there exists a countable locally finite refinement  $(V_j)_{j \in J}$  together with a chart  $\{\psi_j\}_{j \in J}$  which can be chosen such that

$$\psi_j : V_j \rightarrow B(o, 3)$$

such that

$$\mathcal{M} = \bigcup_{j \in J} \psi_j^{-1}(B(o, 1)).$$

*Proof.*  $\mathcal{M}$  is locally compact (look at the charts), hence there exists a compact subsets

$$K_1 \subset \subset K_2 \subset \subset K_3$$

such that

$$\mathcal{M} = \bigcup K_j$$

(ie. an exhaustion by compact sets). Note  $K_{j+1} - \text{int } K_j$  is again compact. For  $p \in K_{j+1} - \text{int } K_j$  choose a chart  $(V_p, \psi_p)$  such that

$$\psi_p(V_p) = B(o, 3), \psi_l(p) = 0.$$

Note that we can take  $V_p$  small enough so that there is  $i \in I$  such that  $V_p \subset U_i$ .

By compactness, we can take  $p_{j,1}, \dots, p_{j,r_j}$  such that

$$K_{j+1} - \text{int } K_j = \bigcup_{l=1}^{r_j} \varphi_{jl}^{-1}(B(0, 1)).$$

By making  $V_{p_{j,l}}$  small enough we can assume

$$V_{p_{j,l}} \subset \text{int } K_{j+2} - K_{j-1}.$$

The union over all  $j$  gives the countable locally finite refinement.  $\square$

From now on, a manifold refers to a smooth, separated, and second countable manifold.

**Example 2.3.** We define,

$$f_1(t) := \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Then  $f_1$  smooth. We then define,

$$f_2(t) := \frac{f_1(t)}{f_1(t) + f_1(1-t)}.$$

$f_2$  is a function which is monotonically increasing and for  $t \geq 1$ , we have  $f_2(t) = 1$ .

$$f_3(t) = f_2(2+t) + f_2(2-t).$$

Again we define

$$f_4 : \mathbb{R}^n \rightarrow \mathbb{R}, f(x) = f_3(\|x\|).$$

**Definition 2.19.** A support of the function  $f : X \rightarrow \mathbb{R}$  is

$$\text{supp}(f) := \overline{f^{-1}(\mathbb{R} \setminus \{0\})}.$$

**Theorem 2.1.** Let  $\mathcal{M}$  be a manifold and  $(U_i)_{i \in I}$  be an open covering. Then there exist smooth functions  $(\varphi_n : \mathcal{M} \rightarrow \mathbb{R})_{n \in \mathbb{N}}$  such that

- i).  $\forall n \in \mathbb{N}, 0 \leq \varphi_n \leq 1$ .
- ii).  $(\text{supp } \varphi_n)_{n \in \mathbb{N}}$  is locally finite.
- iii).  $\forall n \in \mathbb{N} \exists i \in I, \text{s.t. } \text{supp } \varphi_n \subset U_i$ .
- iv).  $\forall p \in \mathcal{M}, \sum_{n=1}^{\infty} \varphi_n(p) = 1$ .

Such sequence of functions is called a partition of unity subordinated by the given covering  $(U_i)_{i \in I}$ .

Additionally, if  $I \subset \mathbb{N}$ , then  $(\varphi_n)_{n \in \mathbb{N}}$  can be indexed by  $I$  as well in such a way that,

$$\forall i \in I, \text{supp } \varphi_i \subset U_i.$$

*Proof.* By Proposition 2.2, take a countable, locally finite refinement charts  $((\tilde{\psi}_n, V_n))_{n \in \mathbb{N}}$ . Borrowing the notation from Example 2.3, we set

$$\tilde{\varphi}_n(x) := \begin{cases} f_4(\tilde{\psi}_n(x)) & \forall x \in V_n, \\ 0, & x \notin V_n. \end{cases}$$

Then take  $\tilde{\varphi}$  to be

$$\varphi := \sum_{n=1}^{\infty} \tilde{\varphi}_n \in \mathcal{C}^{\infty}(\mathcal{M}).$$

We observe that  $\varphi$  is nowhere 0. Thus we define

$$\varphi_n := \tilde{\varphi}_n / \tilde{\varphi},$$

we derived a desired family of functions.

For the second part, set

$$J_0 = \emptyset, \quad \varphi_0 = 0,$$

define inductively that

$$J_k = \left\{ i \in \mathbb{N} \setminus \bigcup_{i=0}^{k-1} J_i \mid \text{supp } \varphi_i \subseteq U_k \right\}.$$

We then take

$$\psi_k := \sum_{i \in J_k} \tilde{\varphi}_i / \tilde{\varphi}.$$

□

**Proposition 2.3.** Let  $A \subset \mathcal{M}$  be a closed subset of a manifold such that there is an open set  $A \subset G \subset \mathcal{M}$ . Then there exists a smooth function  $f \in \mathcal{C}^\infty(\mathcal{M})$  such that the image of  $f$  is contained in  $[0, 1]$  and

$$\forall p \in A, f(p) = 1, \text{ and } \forall p \in G, f(p) = 0.$$

*Proof.* Observe that  $\{\mathcal{M} - A, G\}$  is an open covering. By Theorem 2.1, we can take smooth functions  $\varphi, \psi$  such that

$$\text{supp } \varphi \subset \mathcal{M} - A, \text{supp } \psi \subset G, \varphi + \psi \equiv 1.$$

Take  $f = \psi$ .  $\square$

## 2.4 Tangent Spaces

**Example 2.4.** Let  $\mathcal{M} \subseteq \mathbb{R}^n$  be a submanifold. Then we have a notion of tangent vector, such that for some  $v \in \mathbb{R}^n$ , we define a smooth curve  $\gamma$  such that

$$\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}, \quad \varphi'(0) = v.$$

Given  $v \in \mathbb{R}^n$ ,  $p \in \mathcal{M}$  and  $f : \mathcal{M} \rightarrow \mathbb{R}^n$ , we define

$$vf = (f \circ \gamma)'(0).$$

where  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}$  is a smooth curve such that  $\gamma(0) = p, \gamma'(0) = v$ . Exercise : show that this definition does not depend on the choice of  $\gamma$  satisfying the two conditions above. Furthermore, this defines a linear map

$$v : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathbb{R},$$

such that

$$v(f \cdot g) = (f \circ \gamma \cdot g \circ \gamma)'(0) = f(p) \cdot (vg) + (vf) \cdot g(p).$$

**Definition 2.20.** Let  $\mathcal{M}$  be a smooth manifold and  $p \in \mathcal{M}$ . The linear map  $X_p : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathbb{R}$  is called a derivation at  $p$  if  $X_p(fg) = f(p)X_p g + g(p)X_p f$ .

We define the tangent space of  $\mathcal{M}$  at  $p$  to be

$$T_p \mathcal{M} = \{X_p : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathbb{R} \mid X_p \text{ is a derivation}\}.$$

**Remark 2.7.** Obviously  $T_p \mathcal{M}$  is a  $\mathbb{R}$ -vectorspace.

**Lemma 2.2.** if  $\varphi \in \mathcal{C}^\infty(\mathcal{M})$  is constant around  $p \in \mathcal{M}$ . Then  $X \varphi = 0$  for any  $X \in T_p \mathcal{M}$ .

*Proof.* Let  $\varphi \equiv 1$  around  $p$ . Choose  $\chi \in \mathcal{C}^\infty(\mathcal{M})$  to be such that

1.  $\chi \equiv 1$  in a neighborhood of  $p$ .
2.  $\text{supp } \chi \subseteq \{q \in \mathcal{M} \mid \varphi(q) = 1\}$ .

Then  $\chi\varphi = \chi$ . Thus we have

$$X(\chi) = X(\chi)\varphi(p) + \chi(p)X(\varphi). \quad (1)$$

Since  $\varphi(p) = 1$  and  $\chi(p) = 1$ , thus we conclude  $X(\varphi) = 0$ .  $\square$

**Lemma 2.3.** *If  $X \in T_p\mathcal{M} \cap T_q\mathcal{M}$  then either  $X = 0$  or  $p = q$ .*

*Proof.* Exercise.  $\square$

**Lemma 2.4.** *Let  $(U, \varphi)$  be a chart of  $\mathcal{M}$  centered at  $p \in \mathcal{M}$  (ie.  $\varphi(p) = 0$ ) with the coordinate function  $x_1, \dots, x_n$ . Then for  $f \in \mathcal{C}^\infty(U)$ , there are functions  $f_1, \dots, f_n \in \mathcal{C}^\infty(U)$ , such that*

$$f = \sum_{j=1}^n f_j x_j + f(p).$$

Note that taking  $\varphi(p) = 0$ , justifies the  $f_j x_j$  for each coordinate.

Analogously, given  $f \in \mathcal{C}^\infty(\mathcal{M})$ , we may choose  $f_j \in \mathcal{C}^\infty(\mathcal{M})$  such that

$$f|_U = \sum_{j=1}^n f_j|_U x_j + f(p).$$

*Proof.* The proof for the second part is assigned as an exercise.

Consider  $\mathcal{M} = \mathbb{R}^n$ ,  $p = 0$ , and  $U = (-\varepsilon, \varepsilon)^n$ . We have

$$f(x) = \sum_{j=1}^n (f(x_1, \dots, x_j, 0, \dots, 0) - f(x_1, \dots, x_{j-1}, 0, \dots, 0)) + f(0).$$

By the fundamental theorem of calculus, we obtain,

$$f(x) = \sum_{j=1}^n \int_0^1 \partial_j f(x_1, \dots, tx_j, 0, \dots, 0) dt x_j + f(0).$$

By setting  $f_j = \int_0^1 \partial_j f(x_1, \dots, tx_j, 0, \dots, 0) dt$ , we derive the statement.  $\square$

**Definition 2.21.** *Given a chart  $(U, \varphi)$  of  $\mathcal{M}$ . For  $f \in \mathcal{C}^\infty(U)$ , we define*

$$\left. \frac{\partial}{\partial x_j} \right|_p f := \left. \frac{\partial}{\partial \gamma_j} \right|_{\varphi(p)} (f \circ \varphi^{-1})(\gamma) = D(f \circ \varphi^{-1})(\varphi(p))[e_j].$$

**Proposition 2.4.** *Let  $\mathcal{M}$  be a smooth manifold and  $(U, \varphi)$  be a  $n$ -dimensional chart at  $p$ . Then we have*

$$\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \dots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$$

*forms a basis in  $T_p\mathcal{M}$ .*

*Proof.* Write

$$f = \sum_{i=1}^n f_i x_i + f(p).$$

Choose  $\chi \in \mathcal{C}^\infty(\mathcal{M})$  such that

- i).  $\chi$  is compactly supported in  $U$ .
- ii).  $\chi \equiv 1$  in a neighborhood of  $p$ .

Then

$$\chi^2 f = \sum_{j=1}^n (\chi f_j)(\chi x_j) + \chi^2 f(p),$$

in the neighborhood of  $p$ . Since  $\chi$  is compactly supported. This is defined everywhere on  $U$ .

Let  $X \in T_p \mathcal{M}$  be a derivation. Then observe that

$$X\chi^2 = 0, \quad Xf = X(\chi^2 f).$$

Therefore,

$$\begin{aligned} Xf &= \sum_{j=1}^n X((\chi f_j))(Xx_j)(p) + f_j(p)X(\chi x_j) \\ &= \sum_{j=1}^n f_j(p)X(\chi x_j) \\ &= \sum_{j=1}^n X(\chi x_j) \cdot \left. \frac{\partial}{\partial x_j} \right|_p f. \end{aligned}$$

This implies that

$$X = \sum_{j=1}^n X(\chi x_j) \left. \frac{\partial}{\partial x_j} \right|_p .$$

The set

$$\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \dots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$$

spans  $T_p \mathcal{M}$ . Remains to show the linearly independentness. To do so consider

$$\frac{\partial}{\partial x_j} x_i = \delta_{ij}.$$

We conclude the proof.  $\square$

**Example 2.5.** For  $\mathcal{M} = \mathbb{R}^n$  and  $p \in \mathbb{R}^n$ , the partial derivatives with respect to the standard coordinate at  $p$  is a basis of  $T_p \mathbb{R}^n$ . Explicitly each is of the form,

$$\frac{\partial}{\partial x_j} \Big|_p f = \frac{\partial f}{\partial x_j}(p).$$

**Example 2.6.** Let  $(U, \varphi)$  be a chart of an open subset of  $\mathbb{R}^n$  with coordinate function  $y_1, \dots, y_n$ , (ie.  $\varphi = (y_1, \dots, y_n)$ ).

We have

$$\begin{aligned} \frac{\partial}{\partial x_j} \Big|_p f &= \frac{\partial}{\partial x_j} (f \circ \varphi^{-1} \circ \varphi)(0), \\ &= \sum_{j=1}^n \partial_j (f \circ \varphi^{-1})(\varphi(p)) \frac{\partial \varphi_j}{\partial x_i}(p). \end{aligned}$$

Let  $\frac{\partial}{\partial y_j} \Big|_p f = \partial_j (f \circ \varphi^{-1})(\varphi(p))$ , we get,

$$= \sum_{j=1}^n \frac{\partial y_j}{\partial x_i}(p) \frac{\partial}{\partial y_j} \Big|_p f.$$

In particular,

$$\frac{\partial}{\partial x_i} \Big|_p f = \sum_{j=1}^n \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j} \Big|_p f.$$

**Definition 2.22.** Let  $\mathcal{M}, \mathcal{N}$  be smooth manifold and  $f : \mathcal{M} \rightarrow \mathcal{N}$  be a smooth map. For  $p \in \mathcal{M}$ , we have a linear transform

$$T_p f : T_p \mathcal{M} \rightarrow T_p \mathcal{N}$$

such that given  $X \in T_p \mathcal{M}$  and  $h \in C^\infty(\mathcal{N})$ , we define

$$T_p f(X)h = X(h \circ f).$$

**Lemma 2.5.** Let  $\mathcal{M} \subseteq \mathbb{R}^m$  and  $\mathcal{N} \subseteq \mathbb{R}^n$  be open. Then in the standard basis  $T_p f$  is the Jacobi-matrix.

*Proof.* Exercise. □

**Lemma 2.6** (Chain Rule). Let  $f : \mathcal{M} \rightarrow \mathcal{N}$ ,  $g : \mathcal{N} \rightarrow \mathcal{W}$  be smooth maps. Then we have

$$T_p(g \circ f) = T_{f(p)} g \circ T_p f.$$

*Proof.* Exercise. □

**Lemma 2.7.** Let  $(U, \varphi)$  be a chart. Viewing this as a locally smooth map from  $\mathcal{M}$  to  $\mathbb{R}^n$ , we obtain an isomorphism,

$$T_p\varphi : T_p\mathcal{M} \rightarrow T_{\varphi(p)}\mathbb{R}^n = \mathbb{R}^n.$$

If  $(V, \psi)$  is another chart then for  $p \in U \cap V$ , we have,

$$T_p\psi = T_p(\psi \circ \varphi^{-1} \circ \varphi) = D_{\varphi(p)}(\psi \circ \varphi^{-1})T_p\varphi.$$

*Proof.* Exercise.  $\square$

**Definition 2.23** (Physicists definition of tangent space). A tangent vector is a family  $(\xi^\varphi)_\varphi$  where  $\xi^\varphi \in \mathbb{R}^n$  where  $\varphi$  runs through charts containing  $p$ , together with a transformation rule

$$\xi^\psi = D_{\varphi(p)}(\psi \circ \varphi^{-1})[\xi^\varphi].$$

**Definition 2.24.** Let  $\mathcal{M}$  be a manifold. A curve  $\gamma : I \rightarrow \mathcal{M}$  in  $\mathcal{M}$  is a smooth map such that from some interval  $I \subset \mathbb{R}$ , such that  $I$  is a manifold with the canonical charts.  $d : I \rightarrow I$  where  $\frac{d}{dt}$  a canonical tangent vector associated to this chart, we get a velocity vector of the curve

$$\dot{\gamma}(t_0) = T_{t_0}\gamma\left(\frac{d}{dt}\Big|_{t_0}\right) \in T_{\gamma(t_0)}\mathcal{M}.$$

**Example 2.7.** Given any  $v \in T_p\mathcal{M}$ , and a chart  $(U, \varphi)$  centered at  $p$  (ie.  $\varphi(p) = 0$ ), we have

$$v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_j}\Big|_p$$

Define a curve

$$\gamma(t) = \varphi^{-1}(tv_1, \dots, tv_n).$$

Note that  $\gamma(0) = \varphi^{-1}(0) = p$ . We define

$$\dot{\gamma}(0)f = T_0\gamma\left(\frac{d}{dt}\Big|_0\right)f = \frac{d}{dt}\Big|_{t=0}f(\gamma(t)) = \sum_{i=1}^n v_i \frac{\partial f \circ \varphi^{-1}}{\partial x_j}(0) = vf.$$

## 2.5 Local properties of differentiable maps and submanifolds.

In this section, all manifolds are assumed to have pure dimension (ie. it has a constant dimension everywhere).

**Notation 2.1.**  $\mathcal{M}^m$  denotes that  $\dim \mathcal{M} = \dim T_p\mathcal{M} = m$  for any point  $p \in \mathcal{M}$ .

**Definition 2.25.** Let  $\mathcal{M}^m, \mathcal{N}^n$  be manifolds and  $f : \mathcal{M} \rightarrow \mathcal{N}$  be a smooth map.

- 1).  $p \in \mathcal{M}$  is called a critical point for  $f$  if  $\text{rank } T_p f < n = \dim \mathcal{N}$  and  $f(p)$  is called the critical value.

2).  $q \in \mathcal{N}$  is called a regular value if  $\forall p \in f^{-1}(q)$  we have  $T_q f = \dim \mathcal{N} = n$ .

3).  $f$  is called a submersion if for any  $p \in \mathcal{M}$ ,  $T_p f = n$ .

4).  $f$  is called an immersion if for any  $p \in \mathcal{M}$ ,  $T_p f = m$ .

5).  $f$  is called a subimmersion if  $\mathcal{M} \ni p \mapsto \text{rank } T_p f$  is a constant.

**Notation 2.2.** For  $f : \mathcal{M} \rightarrow \mathcal{N}$ , we denote the set of critical points to be  $C_f$ .

**Definition 2.26** (Submanifold). Let  $\mathcal{N} \subset \mathcal{M}^m$  is called a submanifold if for any chart  $(U, \varphi)$  of  $\mathcal{M}$ ,

$$\varphi(\mathcal{N} \cap U) \subseteq \mathbb{R}^m$$

is a submanifold of  $\mathbb{R}^m$ .

**Remark 2.8.** The above definition is equivalent to say that for each  $p \in \mathcal{M}$ , there exists a chart  $(U, \varphi)$  centered at  $p$  such that

$$\varphi(\mathcal{N} \cap U) = \mathbb{R}^k \times \{0\} \cap \varphi(U) \subset \varphi(U) \subset \mathbb{R}^m.$$

In other words, the intersection is diffeomorphic to some hyperspace with dimension  $k$  in  $\mathbb{R}^m$ .

**Definition 2.27.** A smooth map  $f : \mathcal{M} \rightarrow \mathcal{N}$  is an embedding if

i).  $f$  is an injective immersion.

ii).  $f : \mathcal{M} \rightarrow f(\mathcal{M}) \subset \mathcal{N}$  is a homeomorphism.

**Definition 2.28.** Let  $f : \mathcal{M} \rightarrow \mathcal{N}$  be smooth. The rank of  $f$  is the map  $\mathcal{M} \ni p \mapsto \text{rank } T_p f$ .

**Proposition 2.5.** Let  $f : \mathcal{M}^m \rightarrow \mathcal{N}^n$  be a subimmersion of rank  $k$ .

1). For  $q \in \mathcal{N}$ , the set  $f^{-1}(\{q\}) \subset \mathcal{M}$  is empty or a submanifold of dimension  $n - k$ .

2). For  $p \in \mathcal{M}, q := f(p)$  there exists a neighborhood  $U$  of  $p$  and  $V$  of  $q$  such that  $S = f(U) \cap V$  is a submanifold of  $\mathcal{N}$  of dimension  $k$ .

*Proof.* Apply the rank theorem for  $f(p) = q$ . There exists a chart  $(U, \varphi)$  and  $(V, \psi)$  which are centered at  $p$  and  $q$ , respectively such that

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \varphi \downarrow & & \downarrow \psi \\ \mathbb{R}^m \supset U' & \xrightarrow{(x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0)} & V' \subset \mathbb{R}^n \end{array}$$

□

**Corollary 2.1.**  $f : \mathcal{M}^m \rightarrow \mathcal{N}^n$  is smooth and  $q \in \mathcal{N}$  is a regular value, then  $f^{-1}(\{q\})$  is a submanifold of dimension  $m - n$  or empty.

## 2.6 The Theorem of Morse-Sard

**Notation 2.3.** The Lebesgue measure on  $\mathbb{R}^m$  is denoted by  $\lambda^m$ .

**Definition 2.29** (Null set).  $A \subset \mathcal{M}^m$  is a nullset if for each chart  $(U, \varphi)$ , the set  $\varphi(U \cap A)$  is a  $\lambda^m$ -nullset in  $\mathbb{R}^m$ .

**Remark 2.9.** A diffeomorphism maps nullsets to nullsets. Hence the above notion is well-defined.

**Remark 2.10.** A singleton of a manifold is a nullset. Countable unions of nullsets are again nullsets.

**Remark 2.11.** Let  $A \subset \mathcal{M}^m$  where  $m > 0$ ,  $\mathcal{M} \setminus A$  is dense.

**Remark 2.12.** Suppose we have a smooth function  $f : \mathbb{R}^m \supset U \rightarrow \mathbb{R}$ . For the sake of simplicity, we assume  $f(0) = 0$  and  $\frac{\partial f}{\partial x_1}(0) \neq 0$ . Consider

$$h(x) = (f(x), x_2, \dots, x_m).$$

Then we have

$$Jf(0) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(0) & \frac{\partial f}{\partial x_2}(0) & \cdots & \frac{\partial f}{\partial x_m}(0) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1. \end{pmatrix}$$

This is invertible, therefore we have  $h$  is a local diffeomorphism. Consider

$$g := f \circ h^{-1}.$$

We denote  $(t, \xi) = h(x)$ , then

$$g(t, \xi) = f \circ h^{-1} \circ h(x) = f(x) = t.$$

**Theorem 2.2.** Let  $f : \mathcal{M}^m \rightarrow \mathcal{N}^n$  be smooth where  $n \geq 1$ . Then the set  $C_f$  of critical values of  $f$  is a nullset.

*Proof.* Suffices to prove that each  $p \in \mathcal{M}$  has an open neighborhood  $U$  such that  $f(C_f \cap U)$  is a nullset.

$\Rightarrow$ ) Without the loss of generality, we assume  $\mathcal{M} = U \in \mathbb{R}^m$  and  $\mathcal{N} = \mathbb{R}^n$  for  $n \geq 1$ . We will prove the theorem by induction on  $m$ .

If  $m = 0$ , then the image  $f(\mathcal{M})$  is at most countable thus a nullset.

Assume the claim holds for all dimension less than  $m$ . Let us define

$$C := C_f, \quad C_1 := \left\{ x \in U \mid \forall |\alpha| \leq l, \frac{\partial^{|\alpha|} f}{\partial_x^\alpha}(x) = 0 \right\} l \geq 1. \quad (2)$$

Then we have  $f(C \setminus C_l)$  is a nullset. Pick  $\xi \in C \setminus C_l$  then there exists  $\frac{\partial f_i}{\partial x_j}(\xi) \neq 0$  for some  $i, j$ . Without the loss of generality, we assume  $i = j = 1$ . We put

$$h(x) = (f_1(x), x_2, \dots, x_m).$$

By Remark 2.12, we have  $h$  is a local diffeomorphism and in particular,

$$\xi \mapsto (f_1(\xi), \xi_2, \xi_m).$$

Similarly to the remark, we define

$$g := f \circ h^{-1}, \quad g(t, x) = (t, \tilde{g}(t, x)).$$

$(t, x)$  is critical for  $g$  if and only if  $x$  is critical for  $\tilde{g}(t, \cdot)$ . Now consider

$$\begin{aligned} \lambda^n(f(C_f \cap V)) &= \lambda^n(\{g(t, x) \mid (t, x) \text{ is critical for } g\}), \\ &= \lambda^n(\{(t, \tilde{g}(t, x)) \mid x \text{ is critical for } \tilde{g}(t, \cdot)\}), \\ &\stackrel{\text{Fubini}}{=} \int_I \lambda^{n-1}\{\tilde{g}(t, x) \mid x \text{ is a critical point of } \tilde{g}(t, \cdot)\} dt. \end{aligned}$$

The inside of the integral is 0 by the induction hypothesis.

Fix  $1 \leq l < \infty$ , then  $f(C_l \setminus C_{l+1})$  is a nullset. Indeed, for  $\xi \in C_l \setminus C_{l+1}$ , without the loss of generality, we assume there is a multiindex  $\beta$  such that  $|\beta| = l$  and

$$\frac{\partial^\beta f_1}{\partial x^\beta}(\xi) = 0.$$

Similarly for the previous case, we set

$$h(x) = \left( \frac{\partial^\beta f_1}{\partial x^\beta}(x), x_2, \dots, x_m \right).$$

By the similar argument as above, we get the claim.

Finally, let  $d > 0$  and  $W$  be a cube of side length  $b$ . Consider  $x \in C_k \cap W, y \in W$ . From Taylor expansion at  $x$  implies that

$$|f(x) - f(y)| \leq L|x - y|^{k+1},$$

where  $L$  depends on the cube  $W$ ,  $k$  and  $f$ . Also by fixing  $k$ , we can make it a locally uniform constant.

Subdivide each edge of  $W$  into  $r$  edges to make  $W$  into  $r^m$  many cubes ( $W_j$ ) of edge length  $\frac{d}{r}$ . For  $x \in C_k \cap W_j$ , and  $y \in W_j$ , we have

$$|x - y| \leq \sqrt{m} \frac{d}{r}.$$

In particular, the constant  $\sqrt{m}$  only depends on the dimension. Back to the previous argument, we get

$$|f(x) - f(y)| \leq L \left( \frac{\sqrt{m}d}{r} \right)^{k+1}.$$

This means that  $f(C_k \cap W_j)$  sits in a cube of edge length less than or equal to  $2L \left(\frac{\sqrt{md}}{r}\right)^{k+1}$ .

$$\lambda^n(f(C_k \cap W)) \leq r^m \lambda(\max_{1 \leq j \leq r^m} f(C_k \cap W_j)) \leq r^m \left\{ 2L \left(\frac{\sqrt{md}}{r}\right)^{k+1} \right\}^n = K \cdot r^{m-n(k+1)}.$$

Observe that  $r \rightarrow \infty$  and  $k \geq \frac{m}{n}$ , we get  $C_k \cap W$  has measure 0.  $\square$

### 3 Vector fields and dynamical system

#### 3.1 Definition

**Definition 3.1.** Let  $\mathcal{M}$  be a smooth manifold. A smooth vector field of  $\mathcal{M}$  is a map

$$X : \mathcal{M} \rightarrow T\mathcal{M},$$

such that

1.  $\forall p \in \mathcal{M}, X(p) \in T_p\mathcal{M}$ .
2. For any chart  $(U, \varphi)$  centered at  $p$ , we have  $X|_U = \sum_{i=1}^m X_i^\varphi \frac{\partial}{\partial x_i}$  where  $X_i^\varphi \in C^\infty(U)$  for each  $i$ .

**Notation 3.1.**

$$\Gamma(T\mathcal{M}) = \{ \text{smooth vector fields on } \mathcal{M} \}.$$

**Remark 3.1.** The second condition can be restated as follow. Recall Definition 2.22. A chart  $\varphi : U \rightarrow \mathbb{R}^m$  can be considered a smooth map between  $U$  and  $\mathbb{R}^m$ . We also have seen that  $T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^n$ . With this identification, we have,

$T\varphi : TU \rightarrow T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m$ ,  $TU \ni [U \ni p \mapsto X_p] \mapsto [\mathbb{R}^m \ni x \mapsto [C^\infty(\mathcal{N}) \ni f \mapsto X_{\varphi^{-1}(x)} f \circ \varphi]]$ , is smooth for all chart  $\varphi$ .

**Example 3.1.** Consider,

$$X : S^1 \rightarrow TS^1, (x, y) \mapsto (-y, x),$$

is a smooth vector field.

**Definition 3.2.** Let  $I \subset \mathbb{R}$  be an interval and  $X \in \Gamma(T\mathcal{M})$ . A curve  $\gamma : I \rightarrow \mathcal{M}$  is called an integral curve of  $X$  if

$$\forall t \in I, \gamma'(t) = X(\gamma(t)).$$

**Remark 3.2.** To find such curve is equivalent to solve the following autonomous initial value problem. Given a chart  $(U, \varphi)$  around a point  $p \in \mathcal{M}$ ,

$$\begin{cases} (\varphi \circ \gamma)'(t) &= \begin{pmatrix} X_1^\varphi \circ \varphi^{-1} \\ \vdots \\ X_m^\varphi \circ \varphi^{-1} \end{pmatrix} (\varphi \circ \gamma(t)), \\ (\varphi \circ \gamma)(t_0) &= 0. \end{cases}$$

### 3.2 Flow-Box Theorem

**Lemma 3.1** (Grönwall). *Let  $f, g : [a, b] \rightarrow (0, \infty)$  be continuous functions such that*

$$f(t) \leq c + \int_0^t f(s)g(s)ds,$$

*for some constant  $c > 0$ . Then we have,*

$$f(t) \leq c \exp\left(\int_0^t g(s)ds\right).$$

*Proof.* Consider,

$$\tilde{f}(t) = c + \int_0^t f(s)g(s)dt.$$

Then by assumption  $f(t) \leq \tilde{f}(t)$ . Also set,

$$h(t) = \tilde{f}(t) \exp\left(-\int_0^t g(s)ds\right).$$

Then  $f, h$  are both differentiable thus we take,

$$\begin{aligned} h'(t) &= f(t)g(t) \exp\left(-\int_0^t g(s)ds\right) - g(t)\tilde{f}(t) \exp\left(-\int_0^t g(s)ds\right), \\ &= \underbrace{(f(t) - \tilde{f}(t))}_{\leq 0} g(t) \exp\left(-\int_0^t g(s)ds\right), \\ &\leq 0. \end{aligned}$$

We also have,

$$h(0) = c.$$

Thus we conclude,

$$h(t) \leq c \Rightarrow f(t) \leq \tilde{f}(t) = h(t) \exp\left(\int_0^t g(s)ds\right) \leq c \exp\left(\int_0^t g(s)ds\right).$$

□

**Theorem 3.1.** *Let  $\mathcal{M}$  be a smooth manifold and  $X \in \Gamma(T\mathcal{M})$ . Then for each  $p \in \mathcal{M}$  there exists a neighborhood  $U$ ,  $\varepsilon > 0$ , and a smooth map*

$$F : (-\varepsilon, \varepsilon) \times U \rightarrow \mathcal{M},$$

*such that*

- 1).  $\forall x \in U, F(0, x) = x$ ,
- 2).  $\partial_t F(t, x) = X(F(t, x))$ .

*Proof.* Recall that a continuously differentiable function over a compact set is Lipschitz.

Consider a smooth function  $f : \overline{B(y_0, r)}$  such that

$$a(y) < f(y) < b(y).$$

where  $y_0 \in \mathbb{R}^n$ ,  $r > 0$  and  $a, b$  are continuous function. For  $p \in B(y_0, r)$ , we let  $F(t, y)$  to be the maximal solution of the initial value problem,

$$\begin{cases} \partial_t F(t, y) = f(F(t, y)), \\ F(0, y) = y_0. \end{cases} \quad (\text{P1})$$

By the construction, we get,

$$F(t, y) = y + \int_0^t f(F(s, y))ds.$$

Thus for  $|t| \leq 1$ , we get,

$$\begin{aligned} |F(t, y) - y_0| &\leq \left| y - y_0 + \int_0^t f(F(s, y))ds \right|, \\ &\leq |y - y_0| + |t||f(y_0)| + \int_0^t |f(F(s, y)) - f(y_0)|ds, \\ &\leq |y - y_0| + |f(y_0)| + \int_0^t L|F(s, y) - y_0|ds, \end{aligned}$$

for some  $L > 0$ . By applying Lemma 3.1, we get,

$$|F(t, y) - y_0| \leq (|y - y_0| + |f(y_0)|)e^{L|t|}.$$

Set  $|y - y_0| \leq \frac{r}{2}$  and take,

$$c := \min \left\{ 1, \frac{1}{L}, \frac{r}{\frac{r}{2} + |f(y_0)|} \right\},$$

then for  $|t| < c$ , we have,

$$|F(t, y_0)| \leq \cdot < r.$$

Thus such  $F$  exists on a cylinder  $(-\varepsilon, \varepsilon) \times \overline{B(y_0, \frac{r}{2})}$ .

We now move on to examine the differentiability of such solutions. Let  $0 < \rho < r$  and  $\varepsilon$  be such that  $F$  is a unique solution to the IVP on a cylinder  $[-\varepsilon, \varepsilon] \times \overline{B(y_0, \rho)}$ . Then we have,

$$\begin{aligned} |F(t, y) - F(t, \bar{y})| &\leq |y - \bar{y}| + \left| \int_0^t f(F(s, y)) - f(F(s, \bar{y}))ds \right|, \\ &\leq |y - \bar{y}| + L \int_0^{|t|} |F(s, y) - F(s, \bar{y})| ds. \end{aligned}$$

Using Lemma 3.1, we obtain,

$$|F(t, y) - F(t, \bar{y})| \leq |y - \bar{y}| e^{L|t|}.$$

By definition,  $F$  is continuously differentiable in  $t$  as  $f$  is continuous. For the differentiability in  $y$ , consider

$$\begin{cases} \partial_t D_2 F(t, y) = \underbrace{Df(F(t, y))}_{\in \text{Mat}_{m \times m}(\mathbb{R})} \overbrace{(D_2 F)(t, y)}^{\in \text{Mat}_{m \times m}(\mathbb{R})}, \\ (D_2 F)(0, y) = I_m \in \text{Mat}_{m \times m}(\mathbb{R}). \end{cases} \quad (\text{P2})$$

Equation (P2) is an IVP for a matrix valued functions. Let  $\Phi : (-\varepsilon, \varepsilon) \times B(y_0, \rho) \rightarrow \text{Mat}_{m \times m}(\mathbb{R})$  be a unique solution of

$$\begin{cases} \partial_t \phi(t, y) = Df(F(t, y))\phi(t, y), \\ \phi(0, y) = I_m \in \text{Mat}_{m \times m}(\mathbb{R}). \end{cases} \quad (\text{P3})$$

Similar to the case of  $F$  we have  $\Phi$  is (Lipschitz)-continuous.

**Claim 1.**  $F(t, y)$  is partially differentiable in  $y$  and  $D_2 F(t, y) = \Phi(t, y)$ .

*Proof of Claim.* Fix  $(t, \bar{y}) \in (-\varepsilon, \varepsilon) \times B(y_0, \rho)$ . Since  $f$  is continuously differentiable, we have,

$$f(x) - f(y) = (Df)(y)(x - y) + R(x, y)(x - y),$$

where  $R$  is an uniformly continuous function on  $\overline{B(y, \rho)}$  such that  $R(x, x) = 0$ . Thus for  $\tilde{\varepsilon} > 0$  there is  $\delta > 0$  such that

$$|x - y| < \delta \Rightarrow |R(x, y)| < \tilde{\varepsilon}.$$

By the construction of  $F$ , we see  $F$  is locally uniform in  $s$  thus there is  $\tilde{\delta} > 0$  such that

$$|x - y| < \tilde{\delta} \Rightarrow |F(s, x) - F(s, y)| < \delta.$$

Take  $y$  such that  $|y - \bar{y}| < \tilde{\delta}$  and consider the equation.

$$\begin{aligned} f(F(s, y)) - f(F(s, \bar{y})) - (Df)(F(s, \bar{y}))\phi(s, \bar{y})(y - \bar{y}) \\ = (Df)(F(s, \bar{y}))(F(s, y) - F(s, \bar{y}) - \phi(s, \bar{y})(y - \bar{y})) \\ + R(F(s, y), F(s, \bar{y}))(F(s, y) - F(s, \bar{y})). \end{aligned}$$

We see,

$$|R(F(s, y), F(s, \bar{y}))| \cdot |F(s, y) - F(s, \bar{y})| \leq \tilde{\varepsilon} |y - \bar{y}| e^{|s|L}.$$

Finally,

$$\begin{aligned}
|F(t, y) - F(t, \bar{y}) - \phi(s, \bar{y})(y - \bar{y})| &= \left| \int_0^t f(F(s, y)) - f(F(s, \bar{y})) - (Df)(F(s, \bar{y}))\phi(s, \bar{y})(y - \bar{y}) ds \right|, \\
&\leq \int_0^{|t|} |Df(F(s, \bar{y}))| |F(s, y) - F(s, \bar{y}) - \phi(s, \bar{y})(y - \bar{y})| ds \\
&\quad + \tilde{\varepsilon} |y - \bar{y}| \int_0^{|t|} e^{|s|L} ds, \\
&\leq c_1 \tilde{\varepsilon} |y - \bar{y}| + c_2 \int_0^{|t|} G(s, y, \bar{y}) ds.
\end{aligned}$$

Using Lemma 3.1, we obtain,

$$G(t, y, \bar{y}) \leq c_1 \tilde{\varepsilon} |y - \bar{y}| e^{c_2 |t|}.$$

Thus the solution  $\Phi$  of Equation (P3) is equal to  $D_2 F(t, y)$ .  $\square$

Using the claim, we proved the regularity of  $F$ .  $\square$

**Definition 3.3.** Above  $F(t, \cdot)$  is called the local flow of the vector field  $X$ .

**Theorem 3.2.** Let  $\mathcal{M}$  be a smooth manifold and  $X \in \Gamma(T\mathcal{M})$ ,  $p \in \mathcal{M}$ . Then there exists continuous functions  $a, b : \mathcal{M} \rightarrow \mathbb{R}$  such that

$$-\infty \leq a(p) < 0 < b(p) \leq \infty,$$

and an integral curve,  $c_p : (a(p), b(p)) \rightarrow \mathcal{M}$  of  $X$  with  $c_p(0) = p$  such that for any integral curve  $\tilde{c}_p : I \rightarrow \mathcal{M}$  of  $X$  such that  $\tilde{c}_p(0) = p$ , we have

i).  $I \subseteq (a(p), b(p))$ ,

ii).  $c_p|_I = \tilde{c}_p$ .

That is to say  $c_p$  is the maximal integral curve through  $p$ . By continuity the set,

$$A := \{t \in I_1 \cap I_2 \mid c_1(t) = c_2(t)\},$$

is closed. If  $t_0 \in I_1 \cap I_2$  then  $c_1, c_2$  are local solution of the IVP,

$$\begin{cases} \gamma'(t) = X(\gamma(t)), \\ \gamma(t_0) = c_1(t_0) = c_2(t_0). \end{cases}$$

Using Picard-Lindelöf, there is a neighborhood  $U$  of  $t_0$  such that  $U \subseteq A$ . Since  $A \neq \emptyset$ , we conclude  $A = I_1 \cap I_2$ . That is

$$(c_1 \cup c_2) : I_1 \cup I_2 \rightarrow \mathcal{M}, I_1 \cup I_2 \ni t \mapsto \begin{cases} c_1(t), & (t \in I_1), \\ c_2(t), & (t \in I_2). \end{cases}$$

$c_1 \cup c_2$  is also an integral curve of  $X$  through  $p$ . Thus there is a maximal integral curve  $c_{\max} : I_{\max} \rightarrow \mathcal{M}$ . Set  $c_{\max} = c_p$ .

*Proof.* Let  $c_1 : I_1 \rightarrow \mathcal{M}$ ,  $c_2 : I_2 \rightarrow \mathcal{M}$  be two integral curves of  $X$  through  $p$  that is  $c_1(0) = c_2(0) = p$ .  $\square$

### 3.3 Applications of Mrose-Sards THeorem

#### 3.3.1 Embedding

**Theorem 3.3** (Whitney). *Each  $\mathcal{M}^m$  has an embedding into  $\mathbb{R}^{2m+1}$ .*

*Proof.* We prove the case for  $\mathcal{M}$  compact. Note that if  $\mathcal{M}$  is compact and  $f : \mathcal{M} \rightarrow \mathcal{N}$  is an injective immersion. Then  $f$  is an embedding. Since this gives the continuity of the inverse map.

Using Morse-Sard's theorem,  $\mathcal{M}^m$  compact, there exists an embedding into some  $\mathbb{R}^N$ , for  $N >> 0$ . Indeed, choosing charts  $(U_j, \varphi_j)_{j=1, \dots, r}$ , (by the compactness, finitely many would cover the whole). Such that

$$\varphi_j(U_j) \supset B(0, 3),$$

and

$$\mathcal{M} = \bigcup_{j=1}^r \varphi^{-1}(B(0, 1)).$$

Fix  $g \in \mathcal{C}^\infty(\mathbb{R}^m)$  be such that

$$g(x) = \begin{cases} 1, & (|x| \leq \frac{4}{3}) \\ 0, & (|x| \geq \frac{5}{3}) \end{cases}.$$

Further defining the functions,

$$f_j(p) = \begin{cases} g(\varphi_j(p))\varphi_j(p), & (p \in U_j) \\ 0, & (\text{otherwise}) \end{cases}.$$

Then we have  $f_j \in \mathcal{C}^\infty(\mathbb{R}^m)$ . In particular

$$f_j|_{\varphi_j^{-1}(B(0, 1))}$$

is a diffeomorphism.

Consider the tuple

$$(f_1, \dots, f_r, \dots, g \circ \varphi_1, \dots, g \circ \varphi_r) : \mathcal{M} \rightarrow \mathbb{R}^{(m+1)r},$$

is an injective immersion, hence it is an embedding.

Let  $\mathcal{M} \subset \mathbb{R}^N$ , compact. For  $w \in S^{N-1}$ , let  $\pi_w$  be the orthogonal projection onto  $\langle w \rangle^\perp$  which is a hyperplane of dimension  $N - 1$  which is given by

$$\pi_w(x) = x - \langle w, x \rangle w.$$

We have

$$\pi_w p = \pi_w q \Leftrightarrow \pi_w(p - q) = 0 \Leftrightarrow p - q \parallel w.$$

We construct a map,

$$\phi : \mathcal{M} \times \mathcal{M} \setminus \{(p, p) \mid p \in \mathcal{M}\} \rightarrow S^{N-1}, \phi(p, q) = \frac{p - q}{|p - q|}.$$

If  $2m < N - 1$ , using Morse-Sarde, the image of  $\phi$  is of measure 0. Therefore,

$$\{w \in \S^{N-1} \mid \exists p, q \in \mathcal{M}, p \neq q, p - q \parallel w\} = \text{Im } \phi \cup -\text{Im } \phi.$$

The right hand side is a nullset. More explicitly, the set of those  $w$  such that  $\pi_w|_{\mathcal{M}}$  is not injective is a nullset.

Suppose  $\pi_w$  is an immersion if for  $p \in \mathcal{M}$ ,  $v \in T_p \mathcal{M} \setminus \{0\}$ , we have  $\pi_w(v) \neq 0$ .  $\pi_w$  is an immersion, if

$$\forall p \in \mathcal{M}, w \notin T_p \mathcal{M} \Leftrightarrow w \notin \text{Im } \sigma$$

where

$$\sigma : T\mathcal{M} \setminus \{0\} \rightarrow S^{N-1}, v \mapsto \frac{v}{|v|}.$$

□

**Definition 3.4.** Let  $X \in \Gamma(T\mathcal{M})$ . We define a flow of  $X$  to be

$$\phi^X : A \rightarrow \mathcal{M},$$

where

$$A = \mathcal{D}(\phi^X) := \bigcup_{p \in \mathcal{M}} (a_p, b_p) \times \{p\} \subseteq \mathbb{R} \times \mathcal{M}$$

such that  $\phi^X(p)$  is maximal curve through  $p$ .  $\mathcal{D}(\phi^X)$  is called the flow domain of  $X$ .

**Proposition 3.1.** Let  $X \in \Gamma(T\mathcal{M})$ , and  $\phi^X : A \rightarrow \mathcal{M}$  be the flow of  $X$ . Then the following statements hold.

- 1).  $A$  is open subset in  $\mathbb{R} \times \mathcal{M}$  and contains  $\{0\} \times \mathcal{M}$ .
- 2).  $\phi^X \in C^\infty(A)$ .
- 3). For  $t \in \mathbb{R}$ ,  $\mathcal{D}(\phi_t^X) = \{p \in \mathcal{M} \mid (t, p) \in A\} \subset \mathcal{M}$  where  $\phi_t^X(\cdot) = \phi^X(t, \cdot)$ , is open.
- 4).  $\mathcal{M} = \bigcup_{t>0} \mathcal{D}(\phi_t^X)$ .
- 5).  $\phi_t^X : \mathcal{D}(\phi_t) \rightarrow \mathcal{D}(\phi_{-t})$  is a diffeomorphism.
- 6).  $\phi_s^X \circ \phi_t^X \subseteq \phi_{s+t}^X$  in other words if  $p \in \mathcal{D}(\phi_t^X) \cap \phi_s^X(p) \in \mathcal{D}(\phi_s^X)$  then  $p \in \mathcal{D}(\phi_{s+t}^X)$ .
- 7).  $\phi_s^X(\phi_t^X(p)) = \phi_{s+t}(p)$ .

*Proof.*

For 4), let  $s, t > 0$ , and  $p \in \mathcal{D}(\phi_t^X)$ , then by assumption,  $\phi_t^X(p) \in \mathcal{D}(\phi_{s+t}^X)$ . Consider

$$c(u) = \begin{cases} \phi_u(p), & 0 \leq u \leq t, \\ \phi_{u-t}(\phi_t(p)), & t \leq u \leq t+s, \end{cases}$$

is an integral curve through  $p$  as the two expressions coincide at  $u = t$ , conclude using the uniqueness. Therefore,

$$\phi_{s+t}^X(p) = c(t+s) = \phi_s(\phi_t^X(p)).$$

The cases for  $s \leq 0$  or  $t \leq 0$  are exercises.

For 2), let  $p \in \mathcal{D}(\phi_t^X)$ ,  $t > 0$ , we define,

$$B := \{ \phi_s^X(p) \mid 0 \leq s \leq t \}.$$

This is compact. By using Flow-box theorem, there is a neighborhood  $W_0 \supseteq B$  and  $\varepsilon > 0$  such that

$$(-\varepsilon, \varepsilon) \times W_0 \ni (u, q) \mapsto \phi_u^X(q) \mathcal{M}$$

is smooth. Then choose  $N$  such that  $\frac{t}{N} < \varepsilon$ . Define inductively  $W_j$  by

$$W_j = \left( \phi_{\frac{t}{N}}^X \Big|_{W_{j-1}} \right)^{-1} (W_{j-1}).$$

**Claim 2.** 1).  $W_0 \supseteq W_1 \supseteq \cdots \supseteq W_N$  and they are all open.

2). For  $q \in W_N$ ,  $\phi_{t \frac{N-j}{N}}^X(q) \in W_j$ .

3). For  $0 \leq u \leq t \frac{N-j}{N}$ ,  $\phi_u^X(p) \in W_j$ . In particular  $p \in W_N$ .

*Proof of the Claim.* For Claim 2).  $q \in W_N$ ,  $\phi_{\frac{t}{N}}^X(q) \in W_{N-1}$  and inductively  $\phi_{t \frac{j}{N}}^X(q) \in W_{N-j}$ .

For Claim 3).  $\phi_u^X(p) \in B \subset W_0$ ,  $0 \leq u \leq t$ .

On the induction step  $j \rightarrow j+1$ ,  $0 \leq u \leq t \frac{N-j-1}{N}$ , then

$$\frac{t}{N} + u \leq t \frac{N-j}{N},$$

so

$$\phi_{\frac{t}{N}}^X(\phi_u^X(p)) \in W_j, \quad \phi_u^X(p) \in W_j \Rightarrow \phi_u^X(p) \in W_{j+1}.$$

Let  $q \in W_N$ , then  $\phi_t^X(q) = (\phi_{\frac{t}{N}}^X)^N(q)$ , hence  $q \in \mathcal{D}(\phi_t^X)$ .  $\square$

For 3). Let  $p \in \mathcal{D}(\phi_t^X)$ , set  $c(u) := \phi_{t+u}^X(p)$  where  $-t \leq u \leq 0$ . This is an integral curve through  $\phi_t^X(p)$ . Thus  $\phi_t^X(p) \in \mathcal{D}(\phi_{-t}^X)$  and  $\phi_{-t}^X \circ \phi_t^X(p) = p$ .

For 1). Let  $(t_0, p) \in A$ . Using 2). we can find an open neighborhood  $\mathcal{D}(\phi_{t_0}^X) \subset W \ni p$ . Using Flow-box theorem, there is  $(-\varepsilon, \varepsilon) \times \tilde{W}$  open such that

$$(-\varepsilon, \varepsilon) \times \tilde{W} \ni (u, q) \mapsto \phi_u^X(q),$$

is smooth with image contained in  $W$ . In particular  $p \in \tilde{W}$ . Then for  $q \in \tilde{W}$  with  $|t - t_0| < \varepsilon$ ,

$$\phi_t^X(q) = \phi_{t_0}(\phi_{t-t_0}^X(q))$$

is smooth. Furthermore,  $(t_0 - \varepsilon, t_0 + \varepsilon) \times \tilde{W} \subseteq A$ .  $\square$

**Definition 3.5.** A vector field  $X \in \Gamma(T\mathcal{M})$  is called complete (or integrable) if  $\mathcal{D}(\phi^X) = \mathbb{R} \times \mathcal{M}$ .

**Remark 3.3.** Not all vector fields are complete.

**Example 3.2.** The following vector fields are not complete.

1).  $\mathcal{M} = \mathbb{R}^2 \setminus \{0\}$  and  $X(x, y) = \frac{\partial}{\partial x}$ .

2).  $\mathcal{M} = \mathbb{R}^2$  and  $X(x, y) = x^2 \frac{\partial}{\partial x}$ . Then we have  $\phi^{(1,0)}(t) = \left(\frac{1}{1-t}, 0\right)$ .

**Proposition 3.2.** Let  $X \in \Gamma(T\mathcal{M})$  be a compactly supported vector field. Then  $X$  is complete. In particular, if  $\mathcal{M}$  is compact, then every vector field is complete.

*Proof.* Set  $K := \text{supp}(X)$ . By the Flow-box theorem, there is an open set  $K \subset W \subset \mathcal{M}$  and  $\varepsilon > 0$ , such that  $(-\varepsilon, \varepsilon) \times W \subseteq \mathcal{D}(\phi^X)$ . On the other hand, if  $p \in \mathcal{M} \setminus K$ , then for every  $t \in \mathbb{R}$ ,  $\phi_t^X(p) = p$ . Hence, there is  $\varepsilon > 0$  such that  $(-\varepsilon, \varepsilon) \times \mathcal{M} \subseteq \mathcal{D}(\phi^X)$ . Letting  $N \in \mathbb{N}$  be large enough such that  $\frac{t}{N} < \varepsilon$ , and using  $\phi_t = (\phi_{\frac{t}{N}})^N$  shows  $X$  is integrable.  $\square$

**Proposition 3.3.** Let  $X \in \Gamma(T\mathcal{M})$ , and  $p \in \mathcal{M}$ , and assume  $X_p \neq 0$ . Then there is a chart  $(U, \varphi)$  centered around  $p$ , with  $X = \frac{\partial}{\partial x_1}$  on  $U$ .

*Proof.* Let  $\psi$  be a chart with  $T_p\psi(X_p) = e_1$ . This is possible because if for any chart  $\tilde{\psi}$ , we find  $L \in \text{GL}_m(\mathbb{R})$  such that

$$LT_p\tilde{\psi}(X_p) = e_1.$$

Set  $\psi = L \circ \tilde{\psi}$ . Let  $\varepsilon > 0, \delta > 0$  such that  $(-\varepsilon, \varepsilon) \times \psi^{-1}(\overline{B(0, \delta)}) \subseteq \mathcal{D}(\phi^X)$ .

Set for  $|t| < \varepsilon$ ,  $y \in \mathbb{R}^{m-1}$ ,  $|y| < \varepsilon$ .

$$\sigma(t, y) = \phi_t^X(\psi^{-1}(0, y)).$$

Then

$$D(\psi \circ \sigma)(0, 0) = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1. \end{pmatrix}$$

where the lower corner is the identity matrix.

$$\begin{aligned}\partial_t|_{(0,0)}\psi \circ \sigma(t,y) &= \partial_t|_{(0,0)}\psi(\phi_t^X(p)) = T_p\psi(X_p) = e_1, \\ \partial_y|_{(0,0)}\psi(\psi^{-1}(0,y)) &= I.\end{aligned}$$

So  $\sigma$  is locally invertible and  $\varphi = \sigma^{-1}$  is a chart centered at  $p$ .

$$\frac{\partial}{\partial t}\sigma(t,y) = \frac{\partial}{\partial t}\phi_t^X(\psi^{-1}(0,y)) = X|_{\sigma(t,y)}.$$

So for  $q \in \mathcal{D}(\sigma^{-1}) = \sigma((-\varepsilon, \varepsilon) \times B(0, \delta) \subset \mathcal{M})$ . Thus,

$$X(q) = X(\sigma(\sigma^{-1}(q))) = T_{\sigma^{-1}(q)}\sigma(e_1) = D\sigma(\sigma^{-1}(q))e_1.$$

□

**Remark 3.4.** Given two vector fields  $X, Y \in \Gamma(T\mathcal{M})$ , this is not in general false that

$$XYf \neq YXf.$$

Furthermore, it can even happen that  $XY \notin \Gamma(T\mathcal{M})$ .

**Definition 3.6.** For  $X, Y \in \Gamma(T\mathcal{M})$ , and  $f \in C^\infty(\mathcal{M})$ , set

$$[X, Y]f := X(Yf) - Y(Xf).$$

$[X, Y]$  is called the Lie bracket.

**Definition 3.7.** An algebra  $\mathfrak{g}$  is called a Lie algebra over  $\mathbb{R}$  together with Lie bracket  $[\cdot, \cdot] : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M})$  if for any  $X, Y, Z \in \mathfrak{g}$ ,

1.  $[\cdot, \cdot]$  is  $\mathbb{R}$ -linear.
2.  $[X, Y] = -[Y, X]$ .
3.  $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

**Proposition 3.4.** Let  $X, Y \in \Gamma(T\mathcal{M})$ .

1.  $[X, Y] \in \Gamma(T\mathcal{M})$ .
2.  $(\Gamma(T\mathcal{M}), [\cdot, \cdot])$  is a Lie algebra.

*Proof.*  $[\cdot, \cdot]$  is  $\mathbb{R}$ -linear. consider,

$$\begin{aligned}[X, Y]|_p(fg) &= X|_p(Y(fg)) - Y|_p(X(fg)), \\ &= X|_p(Yf)g + X|_p(Yg)f - Y|_p(Xf)g - Y|_p(Xg)f, \\ &= X|_p(Yf)g + Yfg(p) + X|_pfYg + fX|_p(Yg) - Y|_p(Xf)g - XfY|_pg - Y|_pfXg - fY|_p(Xg), \\ &= ([X, Y]|_p f)g = f[X, Y]|_pg.\end{aligned}$$

The rest is an exercise. □

**Proposition 3.5.** Let  $X \in \Gamma(T\mathcal{M})$ , such that  $X_p \neq 0$ , then there exists a chart  $(U, \varphi)$  centered at  $p$  such that

$$\frac{\partial}{\partial x_1} = X.$$

**Definition 3.8.** Let  $\varphi : \mathcal{M} \rightarrow \mathcal{N}$  be a diffeomorphism and  $X \in \Gamma(T\mathcal{M})$ . For  $q \in \mathcal{N}$ ,

$$(\varphi_* X)(q) := T_{\varphi^{-1}(q)}\varphi[X(\varphi^{-1}(q))],$$

and

$$\varphi^* X := (\varphi^{-1})_*.$$

**Definition 3.9.** Let  $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ , be a smooth map and  $X \in \Gamma(TM)$  and  $Y \in \Gamma(T\mathcal{N})$ , we write

$$X \sim_{\varphi} Y, \text{ and say they are } \varphi\text{-related if } Y(\varphi(p)) = T_p\varphi(X(p)).$$

**Remark 3.5.** If  $\varphi$  is a diffeomorphism and  $X \in \Gamma(T\mathcal{M})$ , then clearly,

$$X \sim_{\varphi} \varphi_* X.$$

**Proposition 3.6.** Suppose  $X_1, X_2 \in \Gamma(T\mathcal{M})$ ,  $Y_1, Y_2 \in \Gamma(T\mathcal{N})$ , and  $\varphi : \mathcal{M} \rightarrow \mathcal{N}$  be smooth.

If  $X_j \sim_{\varphi} Y_j$  for each  $j = 1, 2$ , then  $[X_1, X_2] \sim_{\varphi} [Y_1, Y_2]$ .

*Proof.* Let  $p \in \mathcal{M}$  and  $f \in \mathcal{C}^i\text{nfty}(\mathcal{N})$ .

$$\begin{aligned} T_p\varphi([X_1, X_2])(f) &= [X_1, X_2]_p(f \circ \varphi), \\ &= X_{1,p}(X_2 f \circ \varphi) - X_{2,p}(X_1 f \circ \varphi), \\ &= X_{1,p}((Y_2 f) \circ \varphi) - X_{2,p}((Y_1 f) \circ \varphi), \\ &= Y_{1,\varphi(p)}(Y_2 f) - Y_{2,\varphi(p)}(Y_1 f), \\ &= [Y_1, Y_2]|_{\varphi(p)} f. \end{aligned}$$

□

**Lemma 3.2.** Let  $X \in \Gamma(T\mathcal{M})$  and  $\phi = \phi^X$  be a flow of  $X$ . We have the following statements.

- 1). For  $f \in \mathcal{C}^\infty(\mathcal{M})$ ,  $\frac{d}{dt}(\phi_t^* f) = \phi_t^*(X f)$ . In particular,  $\frac{d}{dt}|_{t=0}\phi_t^* f = X f$ .
- 2). For  $Y \in \Gamma(T\mathcal{M})$ ,  $\frac{d}{dt}\varphi_t^* Y = \phi_t^*([X, Y])$ ,  $\frac{d}{dt}|_{t=0}\phi_t^* Y = [X, Y]$ .
- 3).

*Proof.* Refer to Lee's book. □

**Proposition 3.7.** Let  $X, Y \in \Gamma(T\mathcal{M})$ . Then the following statements are equivalent.

- 1).  $[X, Y] = 0$ ,

- 2).  $\forall t, \phi_t^* Y = Y,$
- 3).  $\forall s, \psi_s^* X = X,$
- 4).  $\psi_s \circ \phi_t = \phi_t \circ \psi_s.$

**Proposition 3.8.** *Let  $X_1, \dots, X_k \in \Gamma(T\mathcal{M})$  be commuting vector fields (ie,  $[X_i, X_j] = 0$  for all  $i, j = 1, \dots, k$ ). Suppose  $X_1(p), \dots, X_k(p)$  are linearly independent. Then there exists a chart  $(U, \varphi)$  such that*

$$X_j|_U = \frac{\partial}{\partial x_j}$$

for all  $j = 1, \dots, k$ .

*Proof.* Since the statement is about open sets, it suffices to show for the case where  $\mathcal{M} = \mathbb{R}^n$  and  $p = 0$ . Consider,

$$\sigma(t, \xi) := \phi_{t_1}^1 \circ \dots \circ \phi_{t_k}^k(0\xi),$$

where  $t \in \mathbb{R}^k$  and  $\xi \in \mathbb{R}^{m-k}$ .  $\square$

### 3.4 Vector Bundles

**Definition 3.10.** *A (smooth, real) vector bundle over a manifold  $\mathcal{M}$  is a triple  $(E, \pi, \mathcal{M})$  such that*

- 1).  $E, \mathcal{M}$  are smooth manifolds.
- 2).  $\pi : E \rightarrow \mathcal{M}$  is a surjective submersion.
- 3). For all  $p \in \mathcal{M}$ , the fiber  $\pi^{-1}(\{p\}) = E_p$  is an  $\mathbb{R}$ -vectorspace with  $\dim E_p < \infty$ .

Furthermore, this satisfies the axiom of local triviality, namely,

For each  $p \in \mathcal{M}$ , there exists an open neighborhood  $p \in U$  and a diffeomorphism

$$\varphi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^N,$$

such that

$$\forall q \in U, \varphi_{U|_{E_q}} : E_q \rightarrow \{q\} \times \mathbb{R}^N,$$

is a linear isomorphism.

$E$  is called the total space of the bundle,  $\mathcal{M}$  is the base of the bundle, and  $\varphi_U$  is called the bundle chart.

**Definition 3.11.** A system  $(\varphi_i)_{i \in I}$  of bundle charts

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N,$$

is a bundle atlas if

$$\bigcup_{i \in I} U_i = \mathcal{M}, \left( \text{or equivalently } \bigcup_{i \in I} \pi^{-1}(U_i) = E \right).$$

**Example 3.3.** Consider  $(E = \mathcal{M} \times \mathbb{R}^N, \pi(p, v) = p, \mathcal{M})$ . This is called a trivial vector bundle.

**Definition 3.12.** Consider  $(E, \pi_E, \mathcal{M})$  and  $(F, \pi_F, \mathcal{M})$  be vector bundles. A homomorphism of vector bundles between  $E$  and  $F$  is a smooth map  $\varphi : E \rightarrow F$  such that for each point  $p \in \mathcal{M}$ ,  $\varphi|_{E_p} : E_p \rightarrow F_p$  is a linear map.

It is called an isomorphism if each  $\varphi|_{E_p}$  is a linear isomorphism.

**Definition 3.13.** A vector bundle is said to be trivial if it is isomorphic to some trivial bundle  $\mathcal{M} \times \mathbb{R}^N$ .

**Definition 3.14.** Vector bundles with fibers being  $\mathbb{C}$ -vectorspaces are then called complex vector bundles.

**Example 3.4.** Consider the total space,

$$E = \{(z, w) \in \mathbb{C}^2 \cong \mathbb{R}^4 \mid |z| = 1, \frac{w^2}{z} \geq 0\},$$

and the projection map,

$$\pi : E \rightarrow S^1, (z, w) \mapsto z.$$

We observe that,

$E$  is a submanifold of  $\mathbb{C}^2 \cong \mathbb{R}^4$ . Pick  $z_0 \in S^1$  and a smooth square root function,

$$\xi : S^1 \setminus \{z_0\} \rightarrow S^1, \xi(x_1, x_2) = (\xi_1, \xi_2).$$

Let  $w = (x_3, x_4)$  be a point such that  $w^2 = \lambda z = \lambda \xi(z)^2$  for  $\lambda \geq 0$ .

It is equivalent to say that

$$\exists \mu \in \mathbb{R}, w = \mu \xi(z).$$

Reformulate this again, we get,

$$w = (x_3, x_4) \parallel (\xi_1(z), \xi_2(z)) \Leftrightarrow x_3 \xi_2(z) - x_4 \xi_1(z) = 0.$$

Thus we obtain,

$$E \cap \{(z, w) \mid z \in S^1 \setminus \{z_0\}\} = \{(z, w) \in (\mathbb{C} \setminus \mathbb{R}_+ z_0) \times \mathbb{C} \mid \begin{array}{l} x_1^2 + x_2^2 - 1 = 0 \\ x_3 \xi_1(z) - x_4 \xi_2(z) = 0 \end{array}\}.$$

Examine the Jacobian of the conditions, we observe

$$\begin{pmatrix} 2x_1 & 2x_2 & 0 & 0 \\ * & * & \xi_2(x_1, x_2) & -\xi_1(x_1, x_2) \end{pmatrix}.$$

We know that at least one of these row is not 0, in other word the matrix is of full rank, thus  $E$  is a manifold.

Now we show that  $\pi^{-1}(\{z\})$  is a one-dimensional  $\mathbb{R}$ -vectorspace.

Similar to the previous case, we get  $w^2 = \lambda \xi(z)^2$  for all  $z \in S^1 \setminus \{z_0\}$ . We get  $w = \pm \sqrt{\lambda} \xi(z)$ . With this observation, we get,

$$\pi^{-1}(\{z\}) = \{(z, \mu \xi(z)) \mid \mu \in \mathbb{R}\}.$$

Bundle charts over  $S^1 \setminus \{z_0\}$ .

Consider

$$\varphi : \pi^{-1}(S^1 \setminus \{z_0\}) \rightarrow (S^1 \setminus \{z_0\}) \times \mathbb{R}, (z, w) \mapsto (z, \frac{w}{\xi(z)}).$$

Finally we show that the bundle  $E \xrightarrow{\pi} S^1$  is not trivial.

To derive a contradiction, suppose there exists a bundle isomorphism  $\varphi : E \rightarrow S^1 \times \mathbb{R}$ . Consider

$$E \setminus \{(z, 0) \mid z \in S^1\} \xrightarrow{\varphi} S^1 \times \mathbb{R}^\times$$

which is disconnected. Indeed the map,

$$c : [0, 2\pi] \rightarrow E \setminus \{(z, 0) \mid z \in S^1\}, t \mapsto (e^{it}, e^{it/2}).$$

Observe that

$$c(0) = (1, 1), c(2\pi) = (1, -1).$$

We derived a contradiction that two components get connected by a path.

This turns out to be one of the realization of a Möbius strip.

**Construction of vector bundle from local charts** Consider  $E \xrightarrow{\pi} \mathcal{M}$  be a vector bundle and  $(U_i, \varphi_i)_{i \in I}$  be a bundle chart. Consider

$$\begin{array}{ccc} (U_i \cap U_j) \times \mathbb{R}^N & \xrightarrow{\varphi_j^{-1}} & \pi^{-1}(U_i \cap U_j) & \xrightarrow{\varphi_i} & (U_i \cap U_j) \times \mathbb{R}^N \\ & \searrow \text{pr}_1 & \downarrow \pi & \swarrow \text{pr}_1 & \\ & & U_i \cap U_j & & \end{array}$$

where  $\text{pr}_1(p, v) = p$ . Then we have,

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^N \rightarrow (U_i \cap U_j) \times \mathbb{R}^N, \varphi_{ij}(q, v) = (q, g_{ij}(q)v),$$

where

$$g_{ij} \in \mathcal{C}^\infty(U_i \cap U_j, \text{GL}_N(\mathbb{R})).$$

We have the following properties of  $(g_{ij})_{ij}$ .

- i).  $g_{ij}g_{jk} = g_{ik}$  over  $U_i \cap U_j \cap U_k$ .
- ii).  $g_{ii}(q) = I_N$ .
- iii).  $g_{ji}(q) = g_{ij}(q)^{-1}$ .

Such collection  $(g_{ij})_{ij}$  is called a co-cycle with values in  $\mathrm{GL}_N(\mathbb{R})$ .

**Remark 3.6.** The last two properties follow from the first one but the proof is left to the readers.

**Definition 3.15.** Let  $\mathcal{M}$  be a smooth manifold and  $(U_i)_{i \in I}$  be an open covering of  $\mathcal{M}$  and  $G$  be a Lie group (ie. a manifold with group structure such that both multiplication and inversion are smooth as functions). A family  $(g_{ij})_{i,j \in I}$  of smooth map such that

$$g_{ij} : U_i \cap U_j \rightarrow G$$

is a cocycle with values in  $G$  if

- 1.  $g_{ij}g_{jk} = g_{ik}$  over  $U_i \cap U_j \cap U_k$ .
- 2.  $g_{ii} = e_G$ .
- 3.  $g_{ji} = g_{ij}^{-1}$

**Definition 3.16.** Suppose we are given a cocycle  $(g_{ij})_{ij}$  in  $G \subset \mathrm{GL}_N(\mathbb{R})$ . We define a pre-bundle as follows,

$$E := \bigcup_{i \in I} i \times U_i \times \mathbb{R}^N / \sim,$$

where  $(i, p, v) \sim (j, q, w)$  if

$$p = q \text{ and } v = g_{ij}(p) \cdot w.$$

We define the projection to be,

$$\pi(([i, p, v])) := p,$$

and the bundle chart,

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N, [i, p, v] \mapsto (p, v).$$

**Remark 3.7.** Observe that this has not topology defined yet.

**Definition 3.17.** Let  $E, F$  be vector bundles over  $\mathcal{M}$ . A smooth map  $\varphi : E \rightarrow F$  is a bundle isomorphism such that

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ \searrow \pi_E & & \swarrow \pi_F \\ & \mathcal{M} & \end{array}$$

is a commutative diagram and for all  $p \in \mathcal{M}$ ,  $\varphi_p$  is an isomorphism between  $E_p$  and  $F_p$ .

**Example 3.5.** Let  $\mathcal{M}$  be a smooth manifold. Then we have

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} \{p\} \times T_p\mathcal{M}.$$

Let  $(U_i, \varphi_i)_{i \in I}$  be an atlas for  $\mathcal{M}$ . Now define a bundle atlas as follows.

$$\psi_i : TU_i = \bigcup_{p \in U_i} \{p\} \times T_p\mathcal{M} = \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^m, (p, X_p) \mapsto (p, T_p\varphi_i X_p).$$

Note that we have a canonical isomorphism  $T_{\varphi_i(p)}\mathbb{R}^m \cong \mathbb{R}^m$ .

Given two charts  $(U_i, \varphi_i), (V_j, \psi_j)$  around a point  $p \in \mathcal{M}$ , and  $v \in T_p\mathcal{M}$  then we have,

$$v = \sum_{k=1}^m \xi_k^i \frac{\partial}{\partial x_k} \Big|_p = \sum_{k=1}^m \xi_k^j \frac{\partial}{\partial y_k} \Big|_p.$$

Then we have

$$T_p\varphi_i(v) = \begin{pmatrix} \xi_1^i \\ \vdots \\ \xi_m^i \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_m}{\partial y_1} & \cdots & \frac{\partial x_m}{\partial y_m} \end{pmatrix} \begin{pmatrix} \xi_1^j \\ \vdots \\ \xi_m^j \end{pmatrix}$$

That is

$$\begin{aligned} \sum \xi_k^i \frac{\partial}{\partial x_k} \Big|_p &= \sum \xi_k^j \frac{\partial}{\partial y_k} \Big|_p, \\ &= \sum_k \left( \sum_l \xi_k^j \frac{\partial x_k}{\partial y_l} \right) \frac{\partial}{\partial x_k} \Big|_p. \end{aligned}$$

Thus the cocycle is  $(g_{kl}(p))_{kl} = \left( \frac{\partial x_k}{\partial y_l}(p) \right)$ .

**Theorem 3.4.** Given a pre-bundle  $(E, \pi, \mathcal{M}, I)$ . There exists a unique topology and  $C^\infty$ -structure on  $E$  such that  $(E, \pi, \mathcal{M})$  is a vector bundle.

Moreover, given a cocycle  $(g_{ij} : U_{ij} \rightarrow G \subseteq \mathrm{GL}_N(\mathbb{R}))_{i,j \in I}$ , where  $(U_{ij})_{i,j \in I}$  is an open covering, there is a unique vector bundle  $(E, \pi, \mathcal{M})$  up to isomorphisms with atlas  $(U_i, \varphi_i)_{i \in I}$  such that  $(g_{ij})_{i,j \in I}$  is the corresponding cocycle.

*Proof.* Let  $E \xrightarrow{\pi} \mathcal{M}$  be a vector bundle with bundle atlas  $(\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N)$ . Then for each open set  $U \subseteq \mathcal{M}$  we have,

$$\pi^{-1}(U) \cap \pi^{-1}(U_i)$$

is open. Conversely, given a pre-vector bundle  $(E, \pi, \mathcal{M}, I)$ . Define a topology on  $E$  generated by base open sets of the form  $\{\pi^{-1}(U) \cap \pi^{-1}(U_i)\}_{i \in I, U \subseteq \mathcal{M}}$ , this

gives a topology such that  $\pi$  is continuous. Similarly for the smooth structure.  
Define

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N,$$

to be such that

$$\varphi_i([(i, p, v)]) = (p, v).$$

This gives a smooth structure to  $E \xrightarrow{\pi} \mathcal{M}$ .

For the uniqueness, assume we have two vector bundles  $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$  with corresponding atlases  $(U_i, \varphi_i), (V_i, \psi_i)$ , respectively, such that

$$\varphi_i \circ \varphi_j^{-1}(p, v) = \psi_i \circ \psi_j^{-1}(p, v) = (p, g_{ij}(p)v).$$

Define,  $\Phi : E \rightarrow F$  as follows. For  $p \in U_i$  and  $\xi \in \mathbb{R}^N$ , we put

$$\Phi(p, \xi) = \psi_i \circ \phi_i(p, \xi).$$

This gives us a bundle isomorphism.  $\square$

**Definition 3.18.** Let  $E \xrightarrow{\pi} \mathcal{M}$  be a vector bundle with bundle atlas  $(U_i, \varphi_i)_{i \in I}$  and cocycle  $(g_{ij})_{i,j \in I}$ . The dual vector bundle of  $E$  is  $E^*$  which is constructed from pre-vector bundle  $(E^*, \pi^*, \mathcal{M}, I)$  with cocycle  $(g_{ji}^{-1})_{i,j \in I}$  and

$$E^* = \bigcup_{p \in \mathcal{M}} E_p^*.$$

**Definition 3.19.** Let  $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$  be vector bundles with cocycle  $(g_{ij})_{i,j \in I}, (h_{ij})_{i,j \in I}$  we define their direct sum to be

$$(E \oplus F)_p = E_p \oplus F_p$$

with cocycle  $(g_{ij} \oplus h_{ij})_{i,j \in I}$ .

**Definition 3.20.** Let  $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$  be vector bundles with cocycle  $(g_{ij})_{i,j \in I}, (h_{ij})_{i,j \in I}$  we define their tensor product to be

$$(E \otimes F)_p = E_p \otimes F_p$$

with cocycle  $(g_{ij} \otimes h_{ij})_{i,j \in I}$ .

**Definition 3.21.** Let  $E \xrightarrow{\pi} \mathcal{M}$  be a vector bundle. A section of  $E$  is a smooth map  $f : \mathcal{M} \rightarrow E$  such that  $\pi \circ f = \text{id}_{\mathcal{M}}$ .

**Remark 3.8.** Vector fields are exactly the sections of  $T\mathcal{M}$ .

**Remark 3.9.** A vector bundle  $E \xrightarrow{\pi} \mathcal{M}$  is trivial of rank  $N$  if and only if there exists sections  $s_1, \dots, s_N \in \Gamma(E)$  such that for all  $p \in \mathcal{M}$ ,  $\langle s_1(p), \dots, s_N(p) \rangle$  is a basis of  $E_p$ .

Given a bundle atlas  $(U_\alpha, \varphi_\alpha)$  with cocycle  $(g_{\alpha\beta})$ , define,

$$f_\alpha : U_\alpha \rightarrow \mathbb{R}^n, f_\alpha(x) = \text{pr}_2(\varphi_\alpha(f(x))).$$

Of course, we have,

$$f_\alpha = g_{\alpha\beta}(x)f_\beta(x), x \in U_\alpha \cap U_\beta.$$

Conversely, given such  $f_\alpha$ , we define,

$$f(x) = [(\alpha, x, f_\alpha(x))].$$

**Proposition 3.9.** *There is a one-to-one correspondence between,*

$$\{\text{sections } f \text{ of } E\} \leftrightarrow \{(f_\alpha)_{\alpha \in I} | \forall \alpha \in I, f_\alpha \in C^\infty(U_\alpha, \mathbb{R}^N), f_\alpha(x) = g_{\alpha\beta}(x)f_\beta(x)\}.$$

**Proposition 3.10.** *Let  $E, F \rightarrow \mathcal{M}$  be vector bundles. Let  $\phi \in \text{Hom}(E, F)$  be surjective. Then there exists  $j \in \text{Hom}(F, E)$  such that  $\phi \circ j = \text{id}_F$ . Furthermore,  $\text{Ker } \phi$  is a vector bundle over  $\mathcal{M}$  and*

$$E \simeq \text{Ker } \phi \otimes F.$$

*That is to say, in the category of vector bundles of  $\mathcal{M}$ , a short exact sequence*

$$0 \longrightarrow \text{Ker } \phi \hookrightarrow E \xrightarrow{\phi} F \longrightarrow 0$$

*splits.*

*Proof.* We will first show that locally such section exists. Let  $p \in \mathcal{M}$ . Choose sections of  $E$  (by using a chart)  $s_1, \dots, s_n \in \Gamma(E)$  such that

$$\phi(s_1(p)), \dots, \phi(s_N(p)),$$

is linearly independent. Then there exists  $p \in U$  an open neighborhood of  $p$  such that for all  $q \in U$ ,

$$\phi(s_1(q)), \dots, \phi(s_N(q)),$$

is linearly independent. For  $w \in F_q|_U$ , we can now write,

$$w = \sum \xi_j \phi(s_j(q)).$$

Put

$$j(w) = \sum \xi_j s_j(q).$$

Clearly, we have that  $j$  is a section of  $\phi$  over  $U$ .

For the global case, let  $(U_\alpha)_{\alpha \in I}$  be an open covering with sections  $j_\alpha : F|_{U_\alpha} \rightarrow E|_{U_\alpha}$ . Let  $(\rho_\alpha)_{\alpha \in I}$  be a subordinated partition of unity. Then,

$$j(p, w) := \left( p, \sum_\alpha \rho_\alpha j_\alpha(w) \right)$$

Now for  $P \in \text{Hom}(E, E)$ , we set,

$$P(v) := v - j(\varphi(v)),$$

then  $P$  is a projection  $P^2 = P$  and the image is  $\text{Im}(P) = \text{Ker } \phi$ .

Fix  $p$ , and choose sections  $\tilde{s}_1, \dots, \tilde{s}_r$  of  $E$  be such that

$$\tilde{s}_1(p), \dots, \tilde{s}_1(p), \dots, \tilde{s}_r(p)$$

is a basis of  $\text{Ker } \phi|_P$ . Set

$$s_j := P \circ \tilde{s}_j.$$

Then  $s_j$  are sections of  $\text{Ker } \phi$  and in a neighborhood of  $p$ , they are pairwise linearly independent hence a frame.  $\square$

**Proposition 3.11** (Swan's theorem). *Let  $E \xrightarrow{\pi} \mathcal{M}$  be a vector bundle of a compact manifold  $\mathcal{M}$ . Then there exists a large enough natural number  $r$  such that there exists a surjective bundle homomorphism  $\phi : \mathcal{M} \times \mathbb{R}^r \twoheadrightarrow E$ .*

In particular,  $E \oplus \text{Ker } \phi \simeq \mathcal{M} \times \mathbb{R}^r$  is trivial and hence there exist sections  $s_1, \dots, s_r \in \Gamma(E)$ , such that

$$\forall p \in \mathcal{M}, E_p = \text{Span}(s_1(p), \dots, s_r(p)).$$

*Proof.* Choose a finite bundle atlas  $(U_1, \varphi_1), \dots, (U_l, \varphi_l)$  of  $E$  (this is justified as  $\mathcal{M}$  is compact) and a subordinated partition of unity  $\rho_1, \dots, \rho_l$ . We set,

$$s_j^i(p) := \rho_j(p)\varphi_j^{-1}(p, e_j),$$

where  $N = \text{rank } E$  and  $e_1, \dots, e_N$  are the canonical basis of  $\mathbb{R}^N$ . Note that  $s_j^i \in \Gamma(E)$ .

Given  $p \in \mathcal{M}$  there exists  $j$  such that  $\rho_j(p) \neq 0$ . Hence  $s_j^i(p), \dots, s_j^N(p)$  is a basis of  $E_p$ , respectively

$$\forall p \in \mathcal{M}, E_p = \text{Span}(s_j^i(p))_{\substack{1 \leq i \leq N \\ 1 \leq j \leq l}}.$$

Re-enumerate them to  $s_1, \dots, s_r \in \Gamma(E)$ . Put

$$\phi : \mathcal{M} \times \mathbb{R}^r \rightarrow E(p, \xi) \mapsto (p, \sum \xi_j s_j(p)).$$

The rest follows from the previous proposition.  $\square$

**Definition 3.22.** A  $R$ -module  $P$  is said to be projective if there exists another  $R$ -module  $Q$  such that  $P \oplus Q$  is free.

**Remark 3.10.** Observe first that the sections  $\Gamma(E)$  is a module over  $\mathcal{C}^\infty(\mathcal{M})$  with the action defined by

$$\mathcal{C}^\infty(\mathcal{M}) \times \Gamma(E) \ni (f, s) \mapsto [\mathcal{M} \ni p \mapsto f(p) \cdot s(p)].$$

Proposition 3.11 tells us that  $\Gamma(E)$  is finitely generated as a  $\mathcal{C}^\infty(\mathcal{M})$  module. Furthermore,  $\Gamma(E)$  is projective (ie. a direct summand of a free module).

## 4 Tensor Algebras and Exterior Algebras

In this section, we assume that  $K$  be a field of characteristic 0 and all vector spaces are finite dimensional.

### 4.1 Tensors

Let  $E$  be a vectorspace. We will denote the dual pairing as  $\langle \cdot, \cdot \rangle_{E, E^*}$ .

**Definition 4.1.** Let  $E_1, \dots, E_r$  be vectorspaces. Recall that a multilinear map  $F : E_1 \times \dots \times E_r \rightarrow F$  is such that, for each  $i = 1, \dots, r$ ,  $F(a_1, \dots, a_{i-1}, \cdot, a_{i+1}, \dots, a_r)$  is linear. We denote,

$$L(E_1, \dots, E_r; F) = \{E_1 \times \dots \times E_r \rightarrow F \mid r\text{-multilinear maps}\}.$$

**Remark 4.1.**

$$\dim L(E_1, \dots, E_r; F) = \left( \prod_{i=1}^r \dim E_i \right) \times \dim F.$$

**Definition 4.2.** Let  $\varphi \in E^*$  and  $\psi \in F^*$ . For  $e \in E, f \in F$ , we define,

$$(\varphi \otimes \psi)(e, f) := \varphi(e)\psi(f)$$

**Remark 4.2.** Obviously we have  $\varphi \otimes \psi \in L(E, F; K)$ .

**Remark 4.3.** A map  $\cdot \otimes \cdot$  is a bilinear map  $E^* \times F^* \rightarrow L(E, F; K)$ .

**Proposition 4.1.** Let  $(e_i)_{i \in I}, (f_j)_{j \in J}$  be bases of  $E, F$ . Then we have,

$$\varphi \otimes \psi = \sum \varphi(e_i)\psi(f_j)e_i^* \otimes f_j^*.$$

By definition, we have,

$$e_i^* \otimes f_j^*(e_k, f_l) = \delta_{ik}\delta_{jl}.$$

In particular,  $(e_i^* \otimes f_j^*)_{(i,j) \in I \times J}$  is a basis of  $L(E, F; K)$ .

**Definition 4.3.** We define,

$$L(e_1, \dots, E_r, K) \times L(f_1, \dots, f_s; K) \rightarrow L(E_1, \dots, E_r, F_1, \dots, F_s; K)$$

by,

$$\varphi \otimes \psi(e_1, \dots, e_r, f_1, \dots, f_s) := \varphi(e_1, \dots, e_r)\psi(f_1, \dots, f_s).$$

It is left to the readers as an exercise to show this is bilinear and associative (ie  $(\varphi \otimes \psi) \otimes \chi = \varphi \otimes (\psi \otimes \chi)$ ).

**Notation 4.1.** We denote

$$E^* \otimes F^* := L(E, F; K),$$

and

$$E \otimes F \cong E^{**} \otimes F^{**} = L(E^*, F^*, K).$$

**Proposition 4.2** (Universality). *The assignment  $\otimes : E \times F \rightarrow E \otimes F$  solves the following universal property.*

Let  $G$  be a vectorspace and  $h : E \times F \rightarrow G$  be a bilinear map. Then there exists a unique linear map  $\bar{h} : E \otimes F \rightarrow G$  making the diagram commutative.

$$\begin{array}{ccc} E \times F & \xrightarrow{h} & G \\ \otimes \downarrow & \nearrow \exists! \bar{h} & \\ E \otimes F & & \end{array}$$

Furthermore, the map

$$L(E, F; G) \ni h \mapsto \bar{h} \in L(E \otimes F; G),$$

is an isomorphism.

*Proof.* Check at the level of basis elements.  $\square$

**Definition 4.4.** Let  $E$  be a vectorspace, we will define,

$$T^r E = \underbrace{E \otimes \cdots \otimes E}_{r \text{ times}},$$

and

$$T(E) = \bigoplus_{r=0}^{\infty} T^r(E),$$

which is called the tensor algebra over  $E$ . Furthermore, Setting  $\otimes$  as a multiplication of elements,  $T(E)$  is a positively graded  $K$ -algebra.

**Remark 4.4.** Obviously, a basis of  $T^r E$  is given by  $(e_{i_1} \otimes \cdots \otimes e_{i_r})_{1 \leq i_1 \leq \cdots \leq i_r \leq n}$ . Thus we have  $\dim T^r E = (\dim E)^r$ .

**Lemma 4.1.** The operation  $\otimes$  is associative.

$$\begin{array}{ccc} T^r E \times T^s E & \xrightarrow{(v_1 \otimes \cdots \otimes v_r, v_s \otimes \cdots \otimes w_s) \mapsto v_1 \otimes \cdots \otimes v_r \otimes w_s} & T^{r+s} E \\ \downarrow & \nearrow & \\ T^r E \otimes T^s E & & \end{array}$$

## 4.2 Totally antisymmetric(alternating) tensors

**Remark 4.5.**

$$T^r E \otimes T^r E^* \rightarrow K,$$

where

$$(v_1 \otimes \cdots \otimes v_r, v_1^* \otimes \cdots \otimes v_r^*) \mapsto \prod_{i=1}^r \langle v_i, v_i^* \rangle$$

is an dual pairing.

Let  $\sigma \in S_r$  be the permutations of  $\{1, \dots, r\}$ . We have,

$$\begin{array}{ccc} E^r & \xrightarrow{\sigma} & E^r \\ \downarrow & & \downarrow \otimes \\ T^r E & \xrightarrow{T^r \sigma} & T^r E \end{array}$$

where  $T^r \sigma$  is a linear isomorphism. and we have,

$$(T^r \sigma)^{-1} = T^r \sigma^{-1}.$$

**Notation 4.2.**  $\text{sgn}(\sigma) = (-1)^\sigma$ .

**Definition 4.5.**  $T \in T^r E$  is called alternating or totally antisymmetric if for all  $\sigma \in S_r$ , we have,

$$T^r \sigma T = (-1)^\sigma T,$$

**Remark 4.6.** To check  $T$  is alternating, it suffices to check that  $T^r \sigma T = -T$  for every transposition  $\tau \in S_r$ .

**Notation 4.3.** We denote the space of alternating tensors to be,

$$\bigwedge^r E \subseteq T^r E.$$

**Definition 4.6.** An anti-symmetrization map is ,

$$A_r : T^r \rightarrow T^r, A \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^\sigma (T^r \sigma)(A).$$

This is a linear map.

**Lemma 4.2.**  $A_r \circ A_r = A_r$  In other words, it is idempotent.

*Proof.* Let  $T \in T^r E$  and fix  $\sigma \in S_r$ . We have,

$$(T^r \tau) A_r(T) = \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^\sigma T^r(\tau \sigma)(T) (-1)^\tau (-1)^\tau = (-1)^\tau A_r(T).$$

In particular, the range of  $A_r$  is contained in  $\bigwedge^r E$ . If  $T \in \bigwedge^r E$ , we have,

$$A_r(T) = \frac{1}{r!} \sum_{\sigma \in S_r} T^r \sigma T = T.$$

Thus we have the range of  $A_r$  is exactly  $\bigwedge^r E$ .  $\square$

### 4.3 Exterior Algebra

**Notation 4.4.**

$$\bigwedge E = \bigoplus_{r \in \mathbb{Z}_{\geq 0}} \bigwedge^r E.$$

**Definition 4.7.** Let  $\psi : \mathbb{N} \rightarrow \mathbb{Z}_+$  be a sequence such that  $\psi(0) = \psi(1) = 1$ . We define,

$$\wedge_\psi : \bigwedge^p E \times \bigwedge^q E \rightarrow \bigwedge^{p+q} E, \xi \wedge_\psi \eta := \frac{\psi(p+q)}{\psi(p)\psi(q)} A_{p+q}(\xi \otimes \eta).$$

**Definition 4.8.** Let  $A$  be a graded ring. A mapping  $(\cdot, \cdot) : A \rightarrow A$  is graded commutative if for any  $a \in A^p, b \in A^q$ , we have,

$$(a, b) = (-1)^{pq}(b, a).$$

**Proposition 4.3.** The map  $\wedge = \wedge_\psi$  is bilinear, associated and graded commutative. Furthermore, for  $v_1, \dots, v_r \in E = \bigwedge^1 E$ ,

$$v_1 \wedge \cdots \wedge v_r = \frac{\psi(r)}{r!} \sum_{\sigma \in S_r} \text{sgn } \sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

From above, we conclude,

$$(e_{i_1} \otimes \cdots \otimes e_{i_r})_{1 \leq i_1 < \cdots < i_r \leq \dim E}$$

is a basis of  $\bigwedge^r E$ . Thus  $\dim \bigwedge^r E = \binom{\dim E}{r}$ .

*Proof.* Let  $\xi \in \bigwedge^p E, \eta \in \bigwedge^q E, \chi \in \bigwedge^r E$  and  $\tau \in S_{p+q}$  such that

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & p & p+1 & \cdots & p+q \\ q+1 & q+2 & \cdots & p+q & 1 & \cdots & q. \end{pmatrix}$$

Observe that

$$\text{sgn}(\tau) = (-1)^{pq}, \tau(\xi \otimes \eta) = \eta \otimes \xi.$$

Thus we have,

$$A_{p+q}(\xi \otimes \eta) = (-1)^\tau A_{p+q}(\eta \otimes \xi).$$

We then obtain,

$$\xi \wedge \eta = (-1)^\tau \eta \wedge \xi = (-1)^{pq} \eta \wedge \xi.$$

We derived the graded-commutativity.

For the associativity, we have,

$$\begin{aligned}
(\xi \wedge \eta) \wedge \chi &= \frac{\psi(p+q+r)}{\psi(p+q)\psi(r)} A_{p+q+r}((\xi \wedge \eta) \otimes \chi), \\
&= \frac{\psi(p+q+r)}{\psi(p+q)\psi(r)} \frac{\psi(p+q)}{\psi(p)\psi(q)} \frac{1}{(p+q+r)!} \frac{1}{(p+q)!} \sum_{\tau \in S_{p+q}} \operatorname{sgn} \tau \\
&\quad \sum_{\sigma \in S_{p+q+r}} \operatorname{sgn}(\sigma \circ (\tau \times \text{id}))(\sigma \circ (\tau \times \text{id}))((\xi \wedge \eta) \otimes \chi), \\
&= \frac{\psi(p+q+r)}{\psi(p)\psi(q)\psi(r)} \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q+r}} \operatorname{sgn} \sigma \sigma(\xi \otimes \eta \otimes \chi), \\
&= \xi \wedge (\eta \wedge \chi).
\end{aligned}$$

By induction, for

$$\xi_1 \in \bigwedge^{p_1}, \dots, \xi_r \in \bigwedge^{p_r} E,$$

we see,

$$\xi_1 \wedge \dots \wedge \xi_r = \frac{\psi(p_1 + \dots + p_r)}{\psi(p_1) \dots \psi(p_r)} A_{p_1+\dots+p_r}(\xi_1 \otimes \dots \otimes \xi_r).$$

Take  $p_1 = \dots = p_r = 1$ , we obtain the formula.

□

**Corollary 4.1.** *Keeping the notation from the proposition above, we have,*

$$\langle v_1 \wedge \dots \wedge v_r, v_1^* \otimes \dots \otimes v_r^* \rangle_{T^r E, T^r E^*} = \frac{\psi(r)}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}),$$

and

$$\langle v_1 \wedge \dots \wedge v_r, v_1^* \wedge \dots \wedge v_r^* \rangle_{T^r E, T^r E^*} = \frac{\psi(r)^2}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}),$$

*Proof.*

$$\begin{aligned}
\langle v_1 \wedge \dots \wedge v_r, v_1^* \otimes \dots \otimes v_r^* \rangle_{T^r E, T^r E^*} &= \sum_{\sigma \in S_r} \operatorname{sgn} \sigma \langle v_{\sigma(1)} v_1^* \rangle \dots \langle v_{\sigma(r)} v_r^* \rangle, \\
&= \frac{\psi(r)}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}).
\end{aligned}$$

$$\begin{aligned}
\langle v_1 \wedge \dots \wedge v_r, v_1^* \wedge \dots \wedge v_r^* \rangle_{T^r E, T^r E^*} &= \frac{\psi(r)}{r!} \sum_{\tau \in S_r} \operatorname{sgn} \tau \langle v_1 \wedge \dots \wedge v_r, v_{\tau(1)}^* \otimes \dots \otimes v_{\tau(r)}^* \rangle_{T^r E, T^r E^*}, \\
&= \frac{\psi(r)}{r!} \sum_{\tau \in S_r} \operatorname{sgn} \tau \langle v_{\tau(1)} \wedge \dots \wedge v_{\tau(r)}, v_1^* \otimes \dots \otimes v_r^* \rangle_{T^r E, T^r E^*}, \\
&= \frac{\psi(r)}{(r!)^2} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}).
\end{aligned}$$

□

We will follow the convention from [2] which is  $\psi \equiv 1$ .

**Definition 4.9.** For  $\xi \in \bigwedge^p E$ , we define,

$$\text{ext}_\xi : \bigwedge E \rightarrow \bigwedge E, \eta \mapsto \xi \wedge \eta.$$

Respectively for  $u \in E$ , we define,

$$\text{int}_U : \bigwedge E^* \rightarrow \bigwedge E^*, \omega \mapsto \langle \omega, u \otimes \cdot \rangle = \langle \omega, u \wedge \cdot \rangle$$

**Remark 4.7.** For  $\eta \in \bigwedge E$  and  $\omega \in \bigwedge E^*$ , we have,

$$\langle \text{ext}_u \eta, \omega \rangle = \langle u \wedge \eta, \omega \rangle = \langle \eta, \text{int}_U \omega \rangle. \quad (3)$$

**Lemma 4.3.** Suppose  $\omega, \eta \in \bigwedge E^*$  are homogeneous elements. Then we have,

$$\text{int}_u(\omega \wedge \eta) = (\text{int}_u \omega) \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge \text{int}_u \eta.$$

In other words,  $\text{int}_u$  is an anti-derivation.

## 5 Orientations, volumes and the Hodge \*-operator

### 5.1 Lines

In this section, we denote  $K = \mathbb{R}$ .

**Definition 5.1.** A line is a one-dimensional  $\mathbb{R}$ -vector space.

**Definition 5.2.** Fix  $\alpha > 0$ . An  $\alpha$ -density on a  $n$ -dimensional vector space  $E$  is a map  $\rho : \bigwedge^n E^* \rightarrow \mathbb{R}$  such that

$$\forall \omega \in \bigwedge^n E^*, \lambda \in \mathbb{R}, \rho(\lambda, \omega) = |\lambda|^\alpha \rho(\omega).$$

We will denote  $|\bigwedge|^{\alpha} E^*$  to be the vector space of all  $\alpha$ -densities.

**Definition 5.3.** A signed density on a  $n$ -dimensional vector space  $E$  is a map  $\rho : \bigwedge^n E^* \rightarrow \mathbb{R}$  such that

$$\forall \omega \in \bigwedge^n E^*, \rho(\lambda, \omega) = \text{sgn } \lambda \rho(\omega).$$

We also define, the set of signed densities as,  $\mathcal{O}(E)$  which we also call the orientation line.

**Remark 5.1.** We have a map,

$$(E^*)^n \rightarrow \bigwedge^n E.$$

With this we can define an  $\alpha$ -density by,

$$\rho : (E^*)^n \rightarrow \mathbb{R}, \rho(\varphi(v_1^*), \dots, \varphi(v_n^*)) = |\det \varphi|^\alpha \rho(v_1^*, \dots, v_n^*),$$

for every  $\varphi \in \mathrm{GL}(E)$ . In particular, we have,

$$\varphi(v_1) \wedge \cdots \wedge \varphi(v_n) = (\det \varphi)v_1 \wedge \cdots \wedge v_n.$$

Similarly for the signed density case we put  $\mathrm{sgn}(\det \varphi)$ .

**Remark 5.2.**

$$\dim \left| \bigwedge E \right| = \dim \mathcal{O}(E) = 1.$$

Given  $\xi \in \bigwedge^n E \setminus \{0\}$ , we have,

$$\rho_\xi^\alpha(\omega) := |\xi(\omega)|^\alpha$$

which is an  $\alpha$ -density. Also,

$$\rho_\xi^\alpha(\omega) := \mathrm{sgn}(\langle \xi, \omega \rangle)$$

which is a signed density. If  $\rho$  is an  $\alpha$ -density,

$$\rho(\lambda\omega) = |\lambda|^\alpha \rho(\omega) = |\lambda|^\alpha \frac{\rho(\omega)}{\rho_\xi^\alpha(\omega)} \rho_\xi^\alpha(\omega) = \frac{\rho(\omega)}{\rho_\xi^\alpha(\omega)} \rho_\xi^\alpha(\lambda\omega).$$

**Lemma 5.1.** There are canonical isomorphisms between,

$$\begin{aligned} \left| \bigwedge \right|^\alpha E \otimes \left| \bigwedge \right|^\beta E \rho_1 \otimes \rho_2 &\mapsto \rho_1 \rho_2 \left| \bigwedge \right|^{\alpha+\beta} E. \\ \left| \bigwedge \right|^1 E \otimes \mathcal{O} &\rightarrow \bigwedge^n E \\ \bigwedge^n E &\rightarrow \left| \bigwedge \right|^1 E \otimes \mathcal{O}, \end{aligned}$$

and

$$\mathcal{O} \otimes \mathcal{O} \rightarrow \mathbb{R}.$$

Furthermore,  $\mathcal{O}$  has a canonical Euclidean metric, namely, for  $\rho \in \mathcal{O}$ , we have,

$$|\rho| := \sqrt{\rho(\omega)^2}, \forall \omega \neq 0.$$

Note that this is independent of choice of  $\omega \neq 0$  since  $\rho$  is a signed-density and the square root will take the sign out.

## 5.2 Orientations

**Definition 5.4.** An orientation of  $E$  is given by a choice of a vector  $\mathbf{o} \in \mathcal{O}$  of unit length.

**Remark 5.3.** We have an isometry by,

$$\mathcal{O} \ni \mathbf{o} \mapsto 1 \in \mathbb{R}.$$

**Remark 5.4.** We have an isomorphism such that,

$$\bigwedge^n E \ni \rho \mapsto \rho \cdot \mathbf{o} \in \left| \bigwedge^n E \right|.$$

**Definition 5.5.** Given a connected component  $\bigwedge^n_+ E$  of  $\bigwedge^n E \setminus \{0\}$ . A basis  $e_1, \dots, e_n$  of  $E$  is called oriented if

$$e_1 \wedge \cdots \wedge e_n \in \bigwedge^n_+ E.$$

Or equivalently, we have,

$$e_1 \wedge \cdots \wedge e_n = \mathbf{o} \cdot |e_1 \wedge \cdots \wedge e_n|.$$

## 5.3 Volume elements

Let  $g$  be a symmetric bilinear form on  $E$ . Then  $g$  induces bilinear forms on  $TE$  and  $\bigwedge E$  as follows.

$$(T^r g)(v_1 \otimes \cdots \otimes v_r, w_1 \otimes \cdots \otimes w_r) := \prod_{j=1}^r g(v_j, w_j).$$

Note that  $\bigwedge E \subseteq TE$ . Thus we have,

$$\begin{aligned} (T^r g)(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r) &:= \frac{1}{(r!)^2} \sum_{\sigma, \tau \in S_r} \operatorname{sgn} \sigma \tau \prod_{j=1}^r g(v_{\sigma(j)}, w_{\tau(j)}), \\ &= \frac{1}{(r!)^2} \sum_{\sigma \in S_r} \det((g(v_i, w_j))_{1 \leq i, j \leq r}), \\ &= \frac{1}{r!} \det((g(v_i, w_j))_{1 \leq i, j \leq r}). \end{aligned}$$

**Lemma 5.2.** If  $g$  is non-degenerate, so are  $T^r g$ ,  $\bigwedge^r g$ . The latter is the restriction of  $T^r g$  to  $\bigwedge^r E \times \bigwedge^r E$ . Also we have an isomorphism

$$(\cdot)^b : E \rightarrow E^*, v \mapsto g(v_j) = v^b,$$

which inverse  $\#$ .

$g$  induces a non-degenerate bilinear form  $g^*$  on  $E^*$ , such that

$$g^*(v^*, w^*) := g(\#v^*, \#w^*).$$

*Proof.* Let  $e_1, \dots, e_n$  be a  $g$ -oriented basis of  $E$  that is  $g(e_i, e_j) = \pm\delta_{ij}$ .

$$(T^r g)(e_{i_1} \otimes \cdots \otimes e_{i_r}, e_{j_1} \otimes \cdots \otimes e_{j_r}) = \prod_{k=1}^r \pm\delta_{i_1 j_1}.$$

And we also have,

$$\left( \bigwedge^r g \right) = (e_{i_1} \wedge \cdots \wedge e_{i_r}, e_{j_1} \wedge \cdots \wedge e_{j_r}) = \frac{\pm 1}{r!} \prod_{k=1}^r \delta_{i_1 j_1}.$$

**Remark 5.5.** *Induced bases are orthogonal with respect to the induced forms. Therefore, they are non-degenerate.*

□

**Proposition 5.1.** *Let  $g$  be a non-degenerate bilinear form on  $E$ . Then  $g$  determines uniquely a positive 1-density  $\text{vol}_g \in |\bigwedge|^1 E$ .*

*If  $e_1, \dots, e_n$  is a basis of  $E$ , then*

$$\text{vol}_g = |\det((g(e_i, e_j))_{1 \leq i, j \leq n})^{\frac{1}{2}}| e_1 \wedge \cdots \wedge e_n|.$$

*Furthermore,  $g$  induces for every  $\alpha > 0$ , a canonical isomorphism,*

$$|\bigwedge|^\alpha E \rightarrow \mathbb{R}, \lambda \cdot |\text{vol}_g|^\alpha \mapsto \lambda.$$

*Proof.* Let us first prove that this is independent of the choice of bases. Let  $f_1, \dots, f_n$  be another basis. We can write,

$$f_i = \sum a_{ij} e_j.$$

Thus we have,

$$g(f_i, f_j) = \sum_{k,l} a_{ki} a_{lj} g(e_k, e_l).$$

Therefore, we have,

$$\det(g(f_i, f_j)) =$$

□

## 5.4 The Hodge $*$ -operator

Let  $g$  be a non-degenerate bilinear form on  $E$  where  $E$  is a real vector space of dimension  $n$ . Let  $\text{vol}_g \in |\bigwedge|^1 E$ .

Let  $\mathcal{O}(E)$  be the orientation line and recall

**Definition 5.6.** *Let  $g$  be a non-degenerate bilinear form on a vector space  $V$ . A basis  $\{e_1, \dots, e_n\}$  of  $V$  is said to be  $g$ -orthonormal if we have,*

$$g(e_i, e_j) = \pm\delta_{ij}.$$

**Definition 5.7.** Let  $g$  be a non-degenerate bilinear form on  $V$  and  $\{e_1, \dots, e_n\}$  be a  $g$ -orthonormal basis of  $V$ , we define the index of  $g$  to be such that

$$\text{ind } g = |\{i \mid g(e_i, e_i) = -1\}|.$$

**Notation 5.1.** Let  $L$  be a one-dimensional  $\mathbb{R}$ -vector space and  $f, g \in L \setminus \{0\}$ . We have  $f = \lambda g$  for some  $\lambda \in \mathbb{R}$ . We denote,

$$f/g = \lambda.$$

**Definition 5.8.** Let  $E$  be a  $n$ -dimensional  $\mathbb{R}$ -vectorspace. The Hodge  $*$ -operator is a map

$$*_p : \bigwedge^p E \rightarrow \bigwedge^{n-p} E \otimes \mathcal{O},$$

satisfying the following properties.

i). It is a linear isomorphism.

ii). For  $g$ -orthonormal basis  $\{e_1, \dots, e_n\}$  of  $E$  and  $\sigma \in S_n$ , we have

$$*e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(p)} = (-1)^{\text{sgn } \sigma} \prod_{i=1}^p g(e_{\sigma(i)}, e_{\sigma(i)}) e_{\sigma(p+1)} \wedge \cdots \wedge e_{\sigma(n)} \otimes e_1 \wedge \cdots \wedge e_n / \text{vol}_g.$$

iii).  $*_{n-p} *_p = (-1)^{p(n-p)+\text{ind } g}$ ,

iv). For  $\omega, \eta \in \bigwedge^p E$ , we have,

$$\left( \bigwedge^{n-p} g \right) (*\omega, *\eta) = (-1)^{\text{ind } g} \left( \bigwedge^p g \right) (\omega, \eta).$$

v). Let  $\omega \in \bigwedge^p E, \eta \in \bigwedge^{n-p} E \otimes \mathcal{O}$ , then

$$\omega \wedge \eta = (-1)^{p(n-p)+\text{ind } g} (g)(\omega, *\eta) \cdot \text{vol}_g.$$

**Theorem 5.1.** Let  $E$  be a  $\mathbb{R}$ -vector space of dimension  $n$  and  $g$  be a non-degenerate bilinear form on  $E$ . Pick  $\text{vol}_g \in \left| \bigwedge^1 \right|^1 E$ . We have, there exists a unique bilinear pairing such that

$$\bigwedge^p E \times \left( \bigwedge^{n-p} E \otimes \mathcal{O} \right) \ni (\omega, \eta) \mapsto \omega \wedge \eta / \text{vol}_g \in \mathbb{R}.$$

In particular for any  $\eta \in \bigwedge^p E$  there is a unique  $\star \eta \in \bigwedge^{n-p} E \otimes \mathcal{O}$  such that for any  $\omega \in \bigwedge^p E$ ,

$$\omega \wedge \star \eta = \bigwedge^p g(\omega, \eta) \text{vol}_g.$$

And such  $\star$  is the Hodge  $\star$ -operator  $\star_p$ .

*Proof.* The composition,

$$(\omega, \eta) \mapsto \omega \wedge \eta \mapsto \omega \wedge \eta / \text{vol}_g \in \mathbb{R}$$

is bilinear. Note that

$$\det(g(e_i, e_j))_{ij} = (-1)^{\text{ind } g}.$$

Let  $I = \{i_1, \dots, i_p\} \subseteq \{1, \dots, n\}$ . Then we denote,

$$e_I := e_{i_1} \wedge \dots \wedge e_{i_p}.$$

Note that an element  $\omega \in \bigwedge^p E$  is of the form,

$$\omega = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=p}} \omega_I e_I, (\omega_I \in \mathbb{R}).$$

Fix  $I_0 \subseteq \{1, \dots, n\}$  and denote  $I_0^c = \{1, \dots, n\} \setminus I_0$ . Using the expression of  $\omega$  above, we have,

$$\omega \wedge e_{I_0^c} \otimes \underbrace{e_1 \wedge \dots \wedge e_n}_{\text{vol}_g} = \pm \omega_{I_0} \underbrace{|e_1 \wedge \dots \wedge e_n|}_{\text{vol}_g}.$$

This shows the non-degeneracy. The existence of  $*\eta$  thus follows from non-degeneracy.

We now show that  $*$  satisfies all the properties of the Hodge operator.

**ii).** Set  $\omega = e_I$  for some  $\{i_1, \dots, i_p\} \subseteq \{1, \dots, n\}$ . Then

$$\mathfrak{o} = \left( \bigwedge^p g \right) (\omega, e_{I^c}).$$

We have,

$$\begin{aligned} \omega \wedge e_{I^c} &= (-1)^{\text{sgn } \sigma} \prod_{i \in I} g(e_i, e_i) \frac{e_1 \wedge \dots \wedge e_n}{|e_1 \wedge \dots \wedge e_n|}, \\ &= (-1)^{\text{sgn } \sigma} e_1 \wedge \dots \wedge e_n (-1)^{\text{sgn } \sigma} \prod_{i \in I} g(e_i, e_i) \frac{e_1 \wedge \dots \wedge e_n}{|e_1 \wedge \dots \wedge e_n|}, \\ &= \prod_{i \in I} g(e_i, e_i) |e_1 \wedge \dots \wedge e_n|, \\ &= \left( \bigwedge^p g \right) (e_I, e_I) |e_1 \wedge \dots \wedge e_n|. \end{aligned}$$

That is

$$*\epsilon_{\sigma(1)} \wedge \dots \wedge e_{\sigma(p)} = \text{sgn } \sigma e_1 \wedge \dots \wedge e_n (-1)^{\text{sgn } \sigma} \prod_{i \in I} g(e_i, e_i) e_{I^c} \frac{e_1 \wedge \dots \wedge e_n}{|e_1 \wedge \dots \wedge e_n|}.$$

iii). Applying the second property twice, and consider  $\tau \in S_n$  such that

$$\tau = \begin{pmatrix} 1 & \cdots & n-p & n-p+1 & \cdots & n \\ p+1 & \cdots & n & 1 & \cdots & p \end{pmatrix}$$

Then we have  $\text{sgn } \tau = (-1)^{p(n-p)}$ . Rewrite  $e_{I^c}$  with  $\tau$ , we obtain,

$$\star_{n-p} \star_p e_I = (-1)^{\text{ind } g} \text{sgn } \sigma \text{sgn}(\tau) e_{\tau(I^c)} = (-1)^{p(n-p)+\text{ind } g} e_I.$$

To show it satisfies the remaining properties is left to the readers as an exercise.  $\square$

## 5.5 Tensor fields and differential forms

**Remark 5.6.** Given a manifold  $\mathcal{M}$ , we have a vectorbundle  $T\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ ,  $T_p\mathcal{M} \mapsto p$ . Let us denote  $\{g_{ij}\}_{ij}$  be the corresponding cocycle, then we can define,  $\bigwedge^p g_{ij}$  which corresponds to a vectorbundle  $\bigwedge^p E$ .

**Definition 5.9.** Let  $\mathcal{M}$  be a smooth manifold. We define,

$$T^{r,s}\mathcal{M} := \bigotimes^r T\mathcal{M} \otimes \bigotimes^s T^*\mathcal{M}.$$

Sections of  $T^{r,s}\mathcal{M}$  are called the tensorfield of type  $(r, s)$ . We denote,

$$\Gamma(T^{r,s}\mathcal{M}) = \{\text{smooth sections of } T^{r,s}\mathcal{M}\}.$$

**Proposition 5.2.** For a map  $f : \mathcal{M} \rightarrow T^*\mathcal{M}$  with

$$\forall p \in \mathcal{M}, f(p) \in T_p^*\mathcal{M},$$

the following are equivalent.

1). for vector fields,  $X_1, \dots, X_n \in \Gamma(T\mathcal{M})$ , and covector field  $\omega_1, \dots, \omega_r \in \Gamma(T^*\mathcal{M})$ , the function,

$$p \mapsto f(\omega_1, \dots, \omega_r, X_1, \dots, X_s) = f(p)(\omega_1(p), \dots, \omega_r(p), X_1(p), \dots, X_s(p)),$$

is smooth.

2). for any charts  $(U, \varphi)$  with coordinate functions  $x_1, \dots, x_m$ , we have,

$$f = \sum f_{j_1 \dots j_s}^{i_1 \dots i_r} \frac{\partial}{\partial x_{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{i_s}} \otimes dx_{j_1} \otimes \dots \otimes dx_{j_s},$$

where  $dx_1, \dots, dx_m$  are the dual (missing) to  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$  with smooth functions  $f_{j_1, \dots, j_s}^{i_1, \dots, i_r} \in C^\infty(U)$ .

3).  $f \in \Gamma(T^{r,s}\mathcal{M})$ .

*Proof.* Exercise. □

**Definition 5.10.** The sections of the bundle  $\bigwedge^p T^* \mathcal{M} \subseteq T^{0,p} \mathcal{M}$ , are called differential forms. The set of differential forms are denoted by

$$\Omega^p(\mathcal{M}) = \Gamma\left(\bigwedge^p T^* \mathcal{M}\right).$$

**Proposition 5.3.** Let  $f : \Gamma(T^* \mathcal{M})^r \times \Gamma(T \mathcal{M})^s \rightarrow \mathcal{C}^\infty(\mathcal{M})$ , is induced by a section of  $T^{r,s} \mathcal{M}$  if and only if it is  $\mathcal{C}^\infty(\mathcal{M})$ - $(r-s)$  multilinear.

*Proof.* If  $f \in \Gamma(T^{r,s}, \mathcal{M})$  then

$$(\omega_1, \dots, \omega_r, X_1, \dots, X_s) \mapsto (p \mapsto f(p(\omega_1(p), \dots, \omega_r(p), X_1(p), \dots, X_s(p))))$$

is clearly  $\mathcal{C}^\infty(\mathcal{M})$ - $(r+s)$ -multilinear.

Conversely, given  $\mathcal{C}^\infty(\mathcal{M})$ -multilinear map  $f$ , need to find  $f(p)$ . To do so we will show that for  $p \in \mathcal{M}$ , if there is  $i$  such that  $\omega_i(p) = 0$  or there is  $j$  such that  $X_j(p) = 0$ , we have,

$$f(\omega_1, \dots, \omega_r, X_1, \dots, X_s)(p) = 0.$$

Indeed, we assume without loss of generality that  $\omega_1(p) = 0$ . Choose a chart  $(U, \varphi)$  centered around  $p$ . Choose  $h \in \mathcal{C}^\infty(\mathcal{M})$  such that  $h \equiv 1$  in small enough neighborhood of  $p$ . By assumption, we have,

$$\omega_1|_U = \sum_{j=1}^m a_j dx_j, a_j(p) = 0.$$

Combining these, we obtain,

$$\begin{aligned} \omega_1 &= (1 - h^2)\omega_1 + h^2\omega_1, \\ &= (1 - h^2)\omega_1 + \sum_{j=1}^m \underbrace{(a_j h)}_{\in \mathcal{C}^\infty(\mathcal{M})} \underbrace{(h dx_j)}_{\in \Gamma(T^* \mathcal{M})}. \end{aligned}$$

Using multilinearity of  $f$ , we get,

$$f(\omega_1, \dots)(p) = (1 - h^2)(p)f(\omega_1, \dots)(p) + \sum_{j=1}^m \underbrace{(ha_j)(p)}_{=0} f(h dx_j, \dots)(p) = 0.$$

Define  $f(p)$  as follows. For  $\theta_1, \dots, \theta_r \in T_p^* \mathcal{M}, v_1, \dots, v_s \in T_p \mathcal{M}$ , choose  $\omega_1, \dots, \omega_r \in \Gamma(T^* \mathcal{M}), X_1, \dots, X_s \in \Gamma(T \mathcal{M})$  with

$$\omega_j(p) = \theta_j, X_j(p) = v_j.$$

By what we have proved, we have,

$$f(p)(\theta_1, \dots, \theta_r, v_1, \dots, v_s) := f(\omega_1, \dots, \omega_r, X_1, \dots, X_s)(p),$$

which is independent of choices of  $\omega_j, X_j$ . The rest follows from Proposition 5.2. □

Observe that given  $f \in \mathcal{C}^\infty(\mathcal{M})$  and  $X \in \Gamma(T\mathcal{M})$ , the map,

$$\mathcal{M} \ni p \mapsto X_p f,$$

is a smooth function. In particular,

$$df : \Gamma(T\mathcal{M}) \rightarrow \mathcal{C}^\infty(\mathcal{M}), df(X) = Xf,$$

is a  $\mathcal{C}^\infty(\mathcal{M})$  linear map.

From Proposition 5.3, it follows that  $df \in \Gamma(T^*\mathcal{M})$ . In particular, we have a linear map,

$$d : \mathcal{C}^\infty(\mathcal{M}) = \Omega^0 \mathcal{M} \rightarrow \Gamma(T^*\mathcal{M}) = \Omega^1 \mathcal{M}.$$

**Definition 5.11** (Pullback). Let  $f : \mathcal{M} \rightarrow \mathcal{N}$  be smooth and  $\xi \in T(T^{0,s}\mathcal{N})$ , Put,

$$f^*\xi|_p(v_1, \dots, v_s) := \xi|_{f(p)}(T_p f v_1, \dots, T_p f v_s).$$

**Proposition 5.4** (Properties of Pullbacks). Given  $\mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{L}$  be smooth maps. We have,

1.  $\varphi \in \mathcal{C}^\infty(\mathcal{N}) = \Gamma(T^{0,0}\mathcal{N})$ ,  $f^*\varphi = \varphi \circ f$ .
2.  $f^*$  is a linear map  $\Gamma(T^{0,s}\mathcal{N}) \rightarrow \Gamma(T^{0,s}\mathcal{M})$  and  $\Omega^p \mathcal{N} \rightarrow \Omega^p \mathcal{M}$ .
3.  $(g \circ f)^* = f^* \circ g^*$ .
4. If  $f$  is a diffeomorphism then  $(f^*)^{-1} = (f^{-1})^*$ .
5.  $\omega \in \Omega^p \mathcal{N}, \eta \in \Omega^q \mathcal{N}$  then  $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$ .

**Notation 5.2.** Let  $U \subseteq \mathcal{M}$  be an open set, we define,

$$\Omega^*(U) := \bigoplus_{p \in \mathbb{Z}_{\geq 0}} \Omega^p(U).$$

**Theorem 5.2** (Cartan (exterior) derivative). Let  $\mathcal{M}$  be a smooth manifold and  $U$  be an open subset of  $\mathcal{M}$ , then there exists a unique family of linear maps

$$d : \Omega^*(U) \rightarrow \Omega^*(U),$$

which satisfies the following properties.

- 1).  $d$  is of degree 1 that is  $d : \Omega^p(U) \rightarrow \Omega^{p+1}(U)$ .
- 2). For  $\omega \in \Omega^p(U), \eta \in \Omega^q(U)$ , we have,

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d(\eta).$$

- 3). For  $f \in \mathcal{C}^\infty(\mathcal{M}) = \Omega^0(\mathcal{M})$  and  $X \in \Gamma(T\mathcal{M})$ , we have,

$$\langle df, X \rangle = df(X) = Xf$$

4).  $d \circ d = 0$ .

5). For  $U \subseteq V \subset \mathcal{M}$  open then we have a commutative diagram, where  $\iota : U \hookrightarrow V$  is an inclusion and  $\iota^*$  is the pullback of inclusion namely the restriction.s

$$\begin{array}{ccc} \Omega^*(V) & \xrightarrow{d} & \Omega^*(V) \\ \iota^* \downarrow & \circlearrowleft & \downarrow \iota^* \\ \Omega^*(U) & \xrightarrow{d} & \Omega^*(U) \end{array}$$

These five properties determine  $d$  uniquely, and satisfies furthermore,

6). For  $\omega \in \Omega^p(\mathcal{M})$ ,  $X_0, \dots, X_p \in \Gamma(T\mathcal{M})$ , we have

$$\begin{aligned} d\omega(X_0, \dots, X_p) &= \sum_{i=0}^p (-1)^i X_i(\omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p)) \\ &\quad + \sum_{0 \leq i < j \leq m} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_{j-1}, X_{j+1}, \dots, X_p). \end{aligned}$$

7).  $f : \mathcal{M} \rightarrow \mathcal{N}$  is smooth then  $f^*(d\omega) = d(f^*\omega)$ .

*Proof.* For the uniqueness, assume we had  $d, \tilde{d}$  satisfying the first five properties. Let  $(U, \varphi)$  be a chart. Denote for  $I = \{i_1 < \dots < i_p\}$ ,  $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_p}$ . Using this notation, we have,

$$d(fd\varphi_I) = df \wedge d\varphi_I,$$

which follows from the second and the forth properties. Explicitly, this follows from that

$$dd\varphi_I = \underbrace{ddx_{i_1} \wedge \dots \wedge dx_{i_p}}_{=0} \pm dx_{i_1} \wedge d(dx_{i_2} \wedge \dots \wedge dx_{i_p}).$$

And using induction, we get  $dd\varphi_I = 0$ . Using the third property, we have,

$$d(fd\varphi_I) = \sum_{j \notin I} \frac{\partial f}{\partial x_j} dx_j \wedge d\varphi_I.$$

Cover  $\mathcal{M}$  by charts  $\bigcup_i U_i$ , by the fifth property, we have,

$$d\omega|_{U_i} = \tilde{\omega}|_{U_i}.$$

Therefore, we conclude  $d = \tilde{d}$ .

For the existence of  $d$ , taking 6). as definition. We need to show as per Proposition 5.3, for  $\omega \in \Omega^p(U)$ ,

i).  $d\omega$  is  $\mathcal{C}^\infty(U)$ -multilinear,

ii).  $d\omega$  is alternating.

These two will imply that  $d\omega \in \Omega^{p+1}(U)$ . For the second assertion, we see,

$$\begin{aligned} d\omega(X_1, X_0, X_2, \dots, X_p) &= X_1\omega(X_0, X_2, \dots, X_p) + \sum_{i=2}^p (-1)^i X_i\omega(X_1, X_0, \dots) + \dots, \\ &= -d\omega(X_0, X_1, \dots, X_p). \end{aligned}$$

For the first assertion, we have,

$$\begin{aligned} d\omega(fX_0, X_1, \dots, X_p) &= \sum_{i=1}^p (-1)^i \underbrace{X_i(f \cdot \omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p) + fX_i\omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p))}_{X_i(f \cdot \omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p))} \\ &\quad + f \cdot X_0\omega(X_1, \dots, X_p) \\ &\quad + \sum_{j=1}^p (-1)^j \omega(\underbrace{[fX_0, X_j]}_{fX_0 - X_j(fX_0) = f[X_0, X_j] - (X_j f)X_0}, X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_p) \\ &\quad + \sum_{1 \leq i < j \leq p} (-1)^{i+j} f \cdot \omega([X_i, X_j], X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_{j-1}, X_{j+1}, \dots, X_p), \\ &= f \cdot d\omega(X_0, \dots, X_p). \end{aligned}$$

We claim that the so-defined  $d$  satisfies 1). - 5). We have already seen that 1) and 5). Now it suffices to look at a coordinate system,  $x_1, \dots, x_n$  where

$$\forall 1 \leq i, j \leq n, [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

Then we have,

$$\begin{aligned} d(fd\omega_{i_1} \wedge \dots \wedge d\omega_{i_p}) &\left( \frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_p}} \right) \\ &= \frac{\partial^2}{\partial x_j \partial x_i} \left( fd\omega_{i_1} \wedge \dots \wedge \underbrace{d\omega_{i_p} \left( \frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_{i-1}}}, \frac{\partial}{\partial x_{j_{i+1}}}, \dots, \frac{\partial}{\partial x_{j_p}} \right)}_{0 \text{ or } \pm 1} \right), \\ &= \begin{cases} 0, & \{i_1, \dots, i_p\} \not\subseteq \{j_0, \dots, j_p\} \\ (-1)^\alpha \frac{\partial}{\partial x_\alpha} f & \{i_1 < \dots < i_p\} = \{j_0 < \dots < j_{\alpha-1} < j_{\alpha+1} < \dots < j_p\}, \end{cases} \\ &= \sum \frac{\partial f}{\partial x_k} dx_k \wedge d\omega_{i_1} \wedge \dots \wedge d\omega_{i_p} \left( \frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_p}} \right). \end{aligned}$$

Let

$$\begin{aligned} \omega &= f dx_1 \wedge \dots \wedge dx_p, \\ \eta &= g dx_{p+1} \wedge \dots \wedge dx_{p+q}. \end{aligned}$$

Using the concrete formula, we have,

$$\begin{aligned}
d(\omega \wedge \eta) &= d(f \cdot g dx_1 \wedge \cdots \wedge dx_{p+q}), \\
&= \sum_{j=p+q+1}^m \frac{\partial f g}{\partial x_j} (-1)^{p+q} dx_1 \wedge \cdots \wedge dx_{p+q} \wedge dx_j, \\
&= \underbrace{\sum_{j=p+q+1}^m \frac{\partial f}{\partial x_j} (-1)^p dx_1 \wedge \cdots \wedge dx_p \wedge dx_j \wedge \underbrace{g dx_{p+1} \wedge \cdots \wedge dx_{p+q} \wedge dx_j}_{=\eta}}_{=d\omega} \\
&\quad + \underbrace{\sum_{j=p+q+1}^m \underbrace{f dx_1 \wedge \cdots \wedge dx_p}_{=\omega} \wedge (-1)^{p+q} \frac{\partial g}{\partial x_j} dx_{p+1} \wedge \cdots \wedge dx_{p+q} \wedge dx_j}_{(-1)^p \omega \wedge d\eta}.
\end{aligned}$$

This proves 2). For the third, both slots are  $\mathcal{C}^\infty(U)$ -linear in  $X$ . Hence without loss of generality, we assume  $X = \frac{\partial}{\partial x_1}$ .

$$\langle df, \frac{\partial}{\partial x_1} \rangle = \left\langle \sum_{j=1}^m \frac{\partial f}{\partial x_j} dx_j, \frac{\partial}{\partial x_1} \right\rangle = \frac{\partial f}{\partial x_1}.$$

For the fourth, we see,

$$\begin{aligned}
d \circ d(f dx_1 \wedge \cdots \wedge dx_p) &= d \left( \sum_{j=p+1}^m \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_p \right), \\
&= \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_1 \wedge \cdots \wedge dx_p, \\
&= \sum_{i < j} \frac{\partial^2 f}{\partial x_i \partial x_j} \underbrace{(dx_i \wedge dx_j + dx_j \wedge dx_i)}_{=0} \wedge dx_1 \wedge \cdots \wedge dx_p.
\end{aligned}$$

The seventh is obvious at charts.  $\square$

## 5.6 Application : Classical vector analysis

Consider  $\mathcal{M}^m$  where  $m \in \{2, 3, 4\}$ . Let  $g \in \Gamma(T^{0,2}\mathcal{M})$ , be symmetric non-degenerate. May choose at orthonormal frame such that

$$g = \begin{pmatrix} -I_p & O \\ O & I_q \end{pmatrix}$$

where  $m = p + q$ . That is

$$g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \begin{cases} 0, & i \neq j, -1, \quad i = j \leq p, \\ 1, & i = j > p. \end{cases}$$

Interesting cases are

1. Riemannain metric when  $p = 0$ ,
2. General relativity, when  $p = 1, q = 3$ .

Set the volume form as

$$\begin{aligned}\text{vol}_g &= \left| \det g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right|^{\frac{1}{2}} |dx_1 \wedge \cdots \wedge dx_m|, \\ \sigma &= \sigma(dx_1, \dots, dx_m) = \frac{dx_1 \wedge \cdots \wedge dx_m}{|dx_1 \wedge \cdots \wedge dx_m|}.\end{aligned}$$

Compute  $*dx_j$  for a general coordinate system.

$$*dx_j = \sum_{k=1}^m (-1)^{k-1} g^{kj} \sqrt{g} dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_m \otimes \sigma,$$

$$g_{ij} = g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right), g^{ij} = g(dx_i, dx_j) = (\{(g_{kl})_{k,l}\}^{-1})_{i,j}.$$

Recall that  $*dx_j$  is the unique twisted  $(n-1)$ -form  $\omega$  satisfying

$$\tau \wedge \omega = \bigwedge^1 g(\tau_1 dx_j) \text{vol } g,$$

where  $\tau$  is an arbitrary 1-form.

$$\omega = \sum_{j=1}^m (-1)^{j-1} \omega_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma.$$

And we have,

$$\begin{aligned}dx_i \wedge \omega &= \left( \bigwedge^1 g \right) (dx_i, dx_j) \text{vol } g = g^{ij} \text{vol } g, \\ &= \sum_{j=1}^m (-1)^{j-1} \omega_j dx_i \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma, \\ &= \omega_i dx_1 \wedge \cdots \wedge dx_m \otimes \sigma, \\ &= \omega_i \sqrt{g}^{-1} \text{vol } g. \\ \omega_i &= g^{ij} \sqrt{g}.\end{aligned}$$

**Proposition 5.5.**

$$dx_i^\# = \sum_j g^{ij} \frac{\partial}{\partial x_j}, \left( \frac{\partial}{\partial x_i} \right)^b = \sum g_{ij} dx_j.$$

*Proof.*

$$\left\langle \left( \frac{\partial}{\partial x_i} \right)^b, \frac{\partial}{\partial x_k} \right\rangle = g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right) = g_{ik} \text{ gives the second formula.}$$

$$\langle (dx_i)^\#, dx_k \rangle = g(dx_i, dx_k) = g^{ik} \text{ gives the first formula}$$

□

**Definition 5.12.** We define the gradient to be such that

$$\text{grad}_g f = (def)^\# := \left( \sum (\partial_i f) dx_i \right)^\# = \sum_{ij} g^{ij} (\partial_i f) \frac{\partial}{\partial x_j}.$$

**Definition 5.13.**  $\star dx_j$  is the unique  $(n-1)$  form  $\omega$  such that

$$(\tau \wedge \omega) = \left( \bigwedge^1 g \right) (tdx_j) \text{vol}_g$$

for all 1-form  $\tau$ .

**Remark 5.7.**

$$\omega = \sum_{j=1}^m (-1)^{j-1} \omega_j dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_m \otimes \sigma.$$

In particular,

$$\begin{aligned} dx_j \wedge \omega &= w_i dx_1 \wedge \cdots \wedge dx_m \otimes \sigma, \\ &= \omega_i \sqrt{g}^{-1} \text{vol}_g, \quad \sqrt{g} = |\det(g_{ij})|, \\ &= \left( \bigwedge^1 g \right) (dx_i, dx_j) \text{vol}_g, \\ &= g_{ij} \text{vol}_g. \end{aligned}$$

Therefore, we obtain  $\omega_i = g^{ij} \sqrt{g}$  and

$$\omega = \star(dx_j) = \sum_{k=1}^m (-1)^{k-1} g^{kj} \sqrt{g} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma.$$

We now identify vectorfields with  $(m-1)$ -form by pairing,

$$X \mapsto \star(X^\flat).$$

**Lemma 5.3.**

$$\star X^\flat = \text{int}_X \text{vol}_g.$$

*Proof.* Since the formula we want to show is a pointwise identity, it suffices to check it on  $X|_p = e_1|_p$ , where  $e_1, \dots, e_m$  are  $g$ -orthonormal basis, since every small enough neighborhood admits a  $g$ -orthonormal basis. Observe that

$$\star e_1^\flat = \star(e_1)^* = c_1 e_2^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma(e_1^*, \dots, e_m^*).$$

Thus we have,

$$\text{int}_{e_1} \text{vol}_g = \text{int}_{e_1} e_1^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma = c_1 e_2^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma,$$

where  $c_j = g_{jj} = g^{jj}$  for a  $g$ -orthonormal basis.

$$\begin{array}{ccc} \Omega^{m-1}(\mathcal{M}) & \xrightarrow{d} & \Omega^m(\mathcal{M}) \\ X \mapsto \star X^\flat \uparrow & & (-1)^p \star \downarrow \uparrow \star \\ \Gamma(\mathcal{M}) & \xrightarrow[\text{div}^g]{} & \mathcal{C}^\infty(\mathcal{M}) \end{array}$$

$$\text{div}^g(X) = (-1)^p \star d \star X^\flat,$$

thus

$$\begin{aligned} \text{div}^g(X) &= \text{div}^g \left( \sum_{j=1}^m X_j \frac{\partial}{\partial x_j} \right), \\ &= (-1)^p \star d \star \left( \sum_{jk} g_{jk} X_j dx_k \right), \\ &= (-1)^p \star d \left( \sum_{jkl} X_j d_{jk} (-1) dx_1 \wedge \cdots \wedge dx_{l-1} \wedge dx_{l+1} \wedge \cdots \wedge dx_m \right) \\ &= (-1)^p \star d \left( \sum (-1)^{j-1} x_j \sqrt{g} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma \right), \end{aligned}$$

Thus, we obtain,

$$\text{int}_X \text{vol}_g = \sum_j X_j \text{int}_{\partial_j} \sqrt{g} dx_1 \wedge \cdots \wedge dx_m \otimes \sigma =$$

□

**Remark 5.8.** On  $\mathcal{M}^2$ , we have,

$$\begin{array}{ccccc} \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 \\ \downarrow & & \downarrow \# & & \downarrow \star \\ \mathcal{C}^\infty(\mathcal{M}) & \xrightarrow[\text{grad}^g]{} & \Gamma(T\mathcal{M}) & \xrightarrow[X \mapsto -\text{div}(\star X^\flat)^\#]{} & \mathcal{C}^\infty(\mathcal{M}) \end{array}$$

## 5.7 Integration of densities

In this section,

$$\mathcal{O} = \mathcal{O}(T^*\mathcal{M}).$$

In terms of the

**Lemma 5.4.** *If  $(U_i, \varphi_i)$  is an atlas of  $\mathcal{M}$  then the cocycle for  $|\wedge|^\alpha$  is*

*Proof.*

$$\begin{aligned} g_{ij}(p) &= |\det((\varphi_i^{-1})^* \circ \varphi_j^*)|^\alpha, \\ &= |\det((\varphi_j \circ \varphi_i)^*)|^\alpha, \\ &= |\det(D(\varphi_j \circ \varphi_i^{-1})^t)|^\alpha, \\ &= |\det(D(\varphi_i \circ \varphi_j^{-1}))|^{-\alpha}|_{\varphi_j(p)}. \end{aligned}$$

Respectively for  $\mathcal{O}$ , we get,

$$g_{ij} = \operatorname{sgn} D(\varphi_i \circ \varphi_j^{-1})|_{\varphi_j(p)}.$$

□

A diffeomorphism  $f : \mathcal{M} \rightarrow \mathcal{N}$  gives rise for pullback map

$$|\wedge|^\alpha(\mathcal{N}) \xrightarrow{f^*} |\wedge|(\mathcal{M}), \mathcal{O}(\mathcal{N}) \xrightarrow{f^*} \mathcal{O}(\mathcal{M}).$$

**Proposition 5.6.** *Let  $\mathcal{M}^m$  be a manifold. There is a unique linear form,*

$$\int_U : \Gamma_C(|\wedge \mathcal{M}|) \rightarrow \mathbb{R},$$

*which is invariant under diffeomorphisms and in local coordinate cocycles with the Lebesgue integral,*

$$\int_{\mathcal{M}} f(x)|dx| = \int_{\mathbb{R}^m} f(x)|dx_1 \wedge \cdots \wedge dx_m|.$$

*Proof.* Let  $\omega \in \Gamma_C(|\wedge \mathcal{M}|)$  with support contained in some chart  $(U, \varphi)$ . We have to put

$$\int_{\mathcal{M}} \omega = \int_{\mathbb{R}^m} (\varphi^{-1})^* \omega.$$

If we have another chart  $(V, \psi)$  such that  $\operatorname{supp} \omega \subseteq U \cap V$ , then

$$\begin{aligned} (\varphi^{-1})^* \omega &= f|dx_1 \wedge \cdots \wedge dx_m|, \\ (\psi^{-1})^* \omega &= (\varphi^{-1} \circ \varphi \circ \psi^{-1})^* \omega = (\varphi) \\ &\Rightarrow \int_{\mathbb{R}^m} (\varphi^{-1})^* \omega = \int_{\mathbb{R}^m} (\psi^{-1})^* \omega. \end{aligned}$$

For the general case, choose  $\rho_1, \dots, \rho_k \in \mathcal{C}_C^\infty(\mathcal{M})$  with,

$$\sum \rho_j|_{\text{supp } \omega} \equiv 1|_{\text{supp } \omega},$$

such that  $\text{supp } \rho_j$  lies in a chart. Put

$$\int_{\mathcal{M}} \omega := \sum_{j=1}^k \int_{\mathcal{M}} \rho_j \omega.$$

If  $\tilde{\rho}_1, \dots, \tilde{\rho}_l$  is a different partition of unity then,

$$\sum_{j=1}^k \int_{\mathcal{M}} \rho_j \omega = \sum_{j=1}^k \int_{\mathcal{M}} \sum_{i=1}^l \rho_j \tilde{\rho}_i \omega = \sum_{j,i=1}^{k,l} \int_{\mathcal{M}} \rho_j \tilde{\rho}_i \omega = \sum_{i=1}^l \int_{\mathcal{M}} \tilde{\rho}_i \omega_i.$$

□

**Remark 5.9.** Let  $\omega = f|dx_1 \wedge \dots \wedge dx_m|$ . Suppose we have a diffeomorphism,

$$(\mathbb{R}^m, y) \xrightarrow{\varphi} (\mathbb{R}^m, x).$$

Then we have,

$$\varphi^* \omega = f \circ \varphi | \det D\varphi | |dy_1 \wedge \dots \wedge dy_m|.$$

Then we have,

$$\begin{aligned} \int_{(\mathbb{R}^m, x)} f |dx_1 \wedge \dots \wedge dx_m| &= \int_{\mathbb{R}^m} f(x) dx_1 \wedge \dots \wedge dx_m, \\ &= \int_{(\mathbb{R}^m, y)} f(\varphi(y)) | \det D\varphi(y) | dy_1 \wedge \dots \wedge dy_m, \\ &= \int_{(\mathbb{R}^m, y)} f^* \omega. \end{aligned}$$

**Corollary 5.1.** Let  $\Phi : \mathcal{M} \rightarrow \mathcal{N}$  be a diffeomorphism between smooth manifolds and  $\omega \in \Gamma_C(| \wedge |^1 \mathcal{N})$ . Then

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{N}} \Phi^* \omega.$$

*Proof.* Write

$$\omega = \sum_{i=1}^k \omega_i,$$

such that  $\text{supp } \omega \subseteq V_i$  for some coordinate chart  $(V_i, \psi_i : V_i \rightarrow U_i \subseteq \mathbb{R}^n)$ . Then

$$\text{supp } \Phi^* \omega_i \subseteq \Phi^{-1}(V_i),$$

which is a coordinate patch as well. To see this, we have,

$$\psi_i \circ \Phi|_{\Phi^{-1}(V_i)} : \Phi^{-1}(V_i) \rightarrow U_i \subseteq \mathbb{R}^n,$$

is again a chart. Then,

$$\int_{\mathcal{N}} \omega_i = \int_{V_i} \omega_i = \int_{\Phi^{-1}(V_i)} \Phi^* \omega_i = \int_{\mathcal{M}} \Phi^* \omega_i.$$

As  $\omega$  is a finite sum of these  $\omega_i$ , we have,

$$\int_N \omega = \sum_{i=1}^k \int_{\mathcal{N}} \omega_i = \sum_{i=1}^k \int_{\mathcal{M}} \Phi^* \omega_i = \int_{\mathcal{M}} \Phi^* \omega.$$

□

## 5.8 Orientations on manifolds

**Definition 5.14.** Let  $\mathcal{M}$  be a smooth manifold. An atlas  $\mathcal{A}$  oriented is called oriented if

$$\forall \varphi, \psi \in \mathcal{A}, x \in \mathcal{M}, \det D(\psi \circ \varphi^{-1})(x) > 0.$$

**Definition 5.15.** Two oriented atlases  $\mathcal{A}, \mathcal{B}$  are equivalent if  $\mathcal{A} \cup \mathcal{B}$  is oriented.

**Definition 5.16.** An orientation is a choice of a maximal oriented atlas. In particular, a manifold is orientable if there is an orientation.

Clearly we have  $(\mathbb{R}^m, \{\text{id}\})$  is oriented.

**Example 5.1.** The following is an example of non-oriented manifold.

$(S^n, \{\psi_{\pm}\})$  where

$$\psi_{\pm} : S^n \setminus \{\pm e_n\}, \psi_{\pm}(x, t) \mapsto \frac{x}{1 \mp t}$$

Then we have

$$\psi_{+} \circ \psi_{-}^{-1}(x) = \frac{x}{|x|^2}.$$

Thus  $f(x) = \frac{x}{|x|^2}$  is orientation reversing. That is  $\det Df(x) < 0$ .

**Remark 5.10.** Fix an orientation reversing diffeomorphism of  $\mathbb{R}^n$  say  $\Phi(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$ . We have a new atlas,

$$\{\Phi \circ \psi_{+}, \psi_{-}\}$$

is oriented.

**Example 5.2.** A Möbius band cannot be oriented. It is an exercise for the readers to give it a proof.

**Proposition 5.7.** Let  $\mathcal{M}$  be a smooth manifold.  $\mathcal{M}$  is orientable if and only if one of following statements hold.

- i). There is a section  $s$  of the bundle  $\mathcal{O}(\mathcal{M})$  with  $|s(p)| = 1$  for all  $p \in \mathcal{M}$ . Note that  $\mathcal{O}$  comes with a canonical inner product, thus we can define a notion of length.

ii). There is a section  $\omega \in \Omega^m(\mathcal{M})$  where  $m = \dim \mathcal{M}$  with  $\omega(p) \neq 0$  for all  $p \in \mathcal{M}$ .

Moreover, each of the conditions will determine an orientation.

*Proof.*

For the first one. Suppose  $\mathcal{M}$  is orientable. then consider  $\mathcal{M} = \bigcup_n U_n$  where  $(U_n, \varphi_n)$  is an oriented chart and consider  $(\rho_n)$  be a subordinate partition of unity to the charts. We set,

$$\omega := \sum_{j=1}^{\infty} \rho_j dx_1^j \wedge \cdots \wedge dx_m^j. \quad (4)$$

We show that  $\omega(p) \neq 0$  for all  $p \in \mathcal{M}$ . To see this for a fixed  $p \in U_i$ . In the small enough neighborhood around  $p$ , we have,

$$\omega = \sum_{j=1}^{\infty} \rho_j \circ \varphi \det D(\varphi^{-1} \varphi_j \circ \varphi_i) dx_1^i \wedge \cdots \wedge dx_m^i. \quad (5)$$

As we have assumed the manifold to be oriented. Each  $\det D(\varphi_j \circ \varphi_i^{-1})$  is positive. And partition of unities are non-negative. We conclude that  $\omega(p) > 0$ .

Suppose the second statement holds. Fix  $\omega \in \Omega^m \mathcal{M}$  such that  $|\omega|(p) \neq 0$  for all  $p \in \mathcal{M}$ . Let  $(U, \varphi)$  be a connected chart. Then obviously, we have,

$$\varphi^* \omega = \rho_\varphi dx_1 \wedge \cdots \wedge dx_m.$$

with

$$\forall x \in \varphi(U), \rho_\varphi(x) \neq 0.$$

We call  $\varphi$  is oriented if  $\rho_\varphi > 0$ . This determines an oriented atlas. Thus proves that the second condition determines an atlas.

Furthermore, the first and the second conditions are equivalent. To show this, choose a Riemannian metric on  $\mathcal{M}$ . Given  $s$  as in the first condition. Set

$$\omega = \text{vol} \cdot s.$$

Conversely, given  $\omega$  as in the second condition set  $s = \frac{\omega}{\text{vol}}$ .  $\square$

## 6 Manifolds with boundaries

### 6.1 Basics

**Definition 6.1** (Half-space). Let  $E$  be a real vector space of dimension  $n < \infty$ . A subset  $H \subseteq E$  is called a half space if there is a linear form  $\lambda \in E^* \setminus \{0\}$  such that

$$H = \{x \in E \mid \langle \lambda, x \rangle \leq 0\}.$$

We denote such  $H$  by  $E_\lambda^+$ .

**Example 6.1.**  $\{x \in \mathbb{R}^n \mid x_j \leq 0\}$  is a half space for  $1 \leq j \leq n$ .

**Remark 6.1.** Let  $H_1, H_2 \subseteq E$  be two half-spaces. Then there is a linear isomorphism  $T \in \mathrm{GL}(E)$  such that  $TH_1 = H_2$ . That is if

$$H_j = E_{\lambda_j}^+,$$

then take  $T^* \in \mathrm{GL}(E^*)$  such that  $T^*\lambda_1 = \lambda_2$ . We have

$$\langle \lambda_2, x \rangle = \langle T^*\lambda_1, x \rangle = \langle \lambda_1, Tx \rangle.$$

**Remark 6.2.** The above remark is true in the oriented category only if  $\dim E \geq 2$ . That is  $\mathbb{R}_+ = [0, \infty)$  and  $\mathbb{R}_- = (-\infty, 0]$  are diffeomorphic but not oriented diffeomorphic.

**Definition 6.2.** Let  $E_\lambda^+ \subseteq E$  be a half space. The normal space is  $E/\ker \lambda$ .

**Lemma 6.1.** A normal space  $E/\ker \lambda$  is canonically oriented by defining a base of  $E/\ker \lambda$  by  $\lambda(v) > 0$ .

**Definition 6.3.** If  $v \in E$  with  $\lambda(v) > 0$  then we say  $v$  points outward. Furthermore, we set

$$c(t) = tv.$$

**Remark 6.3.** The notion comes from the following case. Consider a curve  $c : \mathbb{R} \rightarrow E$  such that for small enough  $\varepsilon > 0$ , we have  $c((-\varepsilon, 0]) \subseteq E_\lambda^+$  for some  $\lambda \in E^*$ . And for  $t > 0$ ,  $c(t) \notin E_\lambda^+$ .

**Definition 6.4.** Let  $A \subseteq \mathbb{R}^m$  be a set and  $f : A \rightarrow \mathbb{R}^n$  be a map.  $f$  is differentiable if for any  $a \in A$ , there is a neighborhood  $a \in U \subseteq \mathbb{R}^m$  (not necessarily contained in  $A$ ) and a differentiable function  $g : U \rightarrow \mathbb{R}^n$  such that

$$g|_{U \cap A} = f|_{U \cap A}.$$

**Proposition 6.1.** Let  $E_\lambda, E_\mu$  be half spaces in  $E \cong \mathbb{R}^n$ ,  $U \subseteq E_\lambda, V \subseteq E_\mu$  be open, and  $\phi : U \rightarrow V$  be a diffeomorphism. Then

$$\phi(\mathrm{int} U) = \mathrm{int} V, \partial U = \partial V.$$

Furthermore, for  $x \in \partial U$ , and any outward pointing vector  $v$ ,

$$D_v \mu \circ \phi(x) > 0.$$

*Proof.* Fix  $x \in U$ . Then there is a neighborhood  $\tilde{U}_x$  of  $x$ , and  $\tilde{V}_y$  of  $y = \phi(x)$  in  $E$  and smooth maps

$$\tilde{\phi} : \tilde{U}_x \rightarrow \tilde{V}_y, \tilde{\psi} : \tilde{V}_y \rightarrow \tilde{U}_x,$$

extending  $f$  with respect to  $f^{-1}$ . Since  $E_\lambda, E_\mu$  are half-spaces, we have,

$$D_x \phi^{-1} \circ \phi = D_{f(x)} \phi^{-1} \circ D_x \phi(x).$$

This implies that  $D_{f(x)}\tilde{\psi} \circ D_x\tilde{\phi}(x) = \text{id}$ .

Suppose  $\phi(x) \in \partial V$ . Then  $\tilde{V}_y$  contains an element outside of  $V$ .

Let  $c : (-\varepsilon, 0] \rightarrow U$  with  $c((-\varepsilon, 0)) \subseteq U^\circ, c(0) = x, c'(0) = v$ . Then we have

$$D_v(\mu \circ \phi)(x) = \frac{d}{dt} \Big|_{t=0} \underbrace{\mu(\phi(c(t)))}_{=\phi(t)}.$$

We have

$$t < 0 \Rightarrow f(t) \leq 0, f(0) = 0.$$

Therefore, we have  $f'(x) \geq 0$ . Suppose  $f'(0) = 0$  then  $(\phi \circ c)'(0) \in \ker \mu$ . This is because that the outward pointing vectors and their negatives are (missing), this implies that range of  $D_x(\phi) \subseteq \ker \mu$ . This contradicts that the rank of  $D_x\phi$  is full.  $\square$

**Definition 6.5.** A smooth manifold with boundary is a second countable Hausdorff space  $\mathcal{M}$  together with maximal atlas of charts  $(U, \varphi)$  with  $\varphi : U \rightarrow V \subseteq E_\lambda^+$  being a homeomorphism on to some open subset  $V$  in  $E_\lambda^+$ .

**Remark 6.4.** Let  $\mathcal{M}$  be a smooth manifold with boundary. A transition map  $\psi \circ \varphi^{-1}$  are diffeomorphism from an open subset of  $E_\lambda^+$  to some open subset of  $E_\mu^+$ .

**Definition 6.6.** Let  $\mathcal{M}$  be a smooth manifold with boundary. We define,

- 1). the interior of  $\mathcal{M}$  to be  $\mathcal{M}^\circ := \text{int } \mathcal{M} = \bigcup_{\varphi \in \mathcal{A}} \varphi^{-1}(\text{int } E_\lambda^+)$ ,
- 2). the boundary of  $\mathcal{M}$  to be  $\partial \mathcal{M} := \varphi^{-1}(\partial E_\lambda^+)$ .

**Proposition 6.2.** Let  $\mathcal{M}^m$  be a smooth boundaryless manifold and let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be a submersion that is  $\forall p \in \mathcal{M}, df|_p \neq 0$ . Then

$$\mathcal{N} = \{x \in \mathcal{M} \mid f(x) \leq 0\},$$

is a manifold with boundary and

$$\partial \mathcal{N} = \{x \in \mathcal{M} \mid f(x) = 0\}.$$

*Proof.* Given  $p \in \mathcal{M}$ , there exists a chart  $(U, \varphi)$  centered at  $p$  such that  $\varphi_m = f$ . Thus  $\varphi : U \rightarrow V$  for  $V \subseteq \mathbb{R}^m$ . That is

$$U \cap \mathcal{N} \rightarrow V \cap \{x_m = 0\}.$$

All such charts give an atlas for a manifold with boundary  $\mathcal{N}$ .  $\square$

**Example 6.2.** Let  $B \subseteq \mathbb{R}^n$  be a unit ball. We know that  $\partial B = S^{m-1}$ . The function  $f$  above in this case is given by

$$f(x) = \|x\|^2 - 1 = \sum_{i=1}^n x_i^2 - 1.$$

**Proposition 6.3.** Let  $\mathcal{M}^m$  be a smooth manifold. A subset  $A \subseteq \mathcal{M}$  is a submanifold with boundary if for each  $a \in A$  there is an open neighborhood  $U$  in  $\mathcal{M}$  and a submersion  $f : U \rightarrow \mathbb{R}$  such that  $A \cap U = \{f \leq 0\}$ . In that case  $\partial \cap U = \{f = 0\}$ .

## 6.2 Stokes theorem

Let  $f : [a, b] \rightarrow \mathbb{R}$  be sufficiently differentiable. We have,

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Observe that  $f'(x)dx \in \Omega^1([a, b])$ . Obviously  $[a, b]$  is a manifold with boundary and the above equation can be restated as,

$$\int_{\partial[a,b]} fd\#,$$

where  $\#$  is the counting measure. Now consider  $\{x_1 \leq 0\} \subseteq \mathbb{R}^m$ . Take the canonical basis  $e_1, \dots, e_m$  and its dual basis  $e_1^*, \dots, e_m^*$ . For  $\lambda = -e_1^*$ ,

$$\{x_1 \geq 0\} = E_\lambda^+.$$

Let  $\omega \in \Omega_C^{m-1}(E_\lambda^+)$  such that

$$\omega = \sum_{j=1}^m (-1)^{j-1} f_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m.$$

where  $f_j \in \mathcal{C}_C^\infty(E_\lambda^+)$ . Thus

$$d\omega = \sum_{j=1}^m \frac{\partial f_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_m = (\nabla f) \cdot dx_1 \wedge \cdots \wedge dx_m.$$

Integrating this we get,

$$\begin{aligned} \int_{E_\lambda^+} d\omega &= \sum_{j=1}^m \int_{\{x_1 \leq 0\}} \frac{\partial f_j}{\partial x_j} dx_1 \cdots dx_m, \\ &\stackrel{\text{Fubini}}{=} \sum_{j=1}^m \int_{\{x_1 \leq 0\}} \underbrace{\left( \int_{\mathbb{R}} \frac{\partial f_j}{\partial x_j} dx_j \right)}_{=0} dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_m. \end{aligned}$$

Except for  $j = 1$ , thus

$$\int_{E_\lambda^+} d\omega = \int_{\mathbb{R}^{m-1}} \left( \int_0^\infty \frac{\partial f_1}{\partial x_1} dx_1 \right) dx_2 \cdots dx_m = \int_{\mathbb{R}^{m-1}} -f_1(0, x_2, \dots, x_m) dx_2 \cdots dx_m.$$

Set  $\iota : \partial E_\lambda^+ = \{x_1 = 0\} \hookrightarrow E_\lambda^+ \hookrightarrow \mathbb{R}^m$ . We then have,

$$\iota^* \omega = f_1 dx_2 \wedge \cdots \wedge dx_m.$$

And in particular,

$$\int_{E_\lambda^+} d\omega = - \int_{\partial E_\lambda^+} \omega.$$

By setting orientations on  $e_1, \dots, e_m$  positive in  $E_\lambda^+$  and  $e_2, \dots, e_m$  negative in  $\partial E_\lambda^+$ , we have,

$$\int_{E_\lambda^+} d\omega = \int_{\partial E_\lambda^+} \omega.$$

**Theorem 6.1** (Stokes). *Let  $\mathcal{M}^m$  be oriented manifold with boundary such that  $\partial \mathcal{M}^m$  carrying the induced orientation,  $\omega \in \Omega_C^{m-1}(\mathcal{M})$ . Then*

$$\int_{\mathcal{M}} d\omega = \int_{\partial \mathcal{M}} \iota^* \omega.$$

## References

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