Analysis and Geometry on Manifolds

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1 Review of Structure Theorems for Differentiable Maps

Theorem 1.1 (Implicit Function Theorem). *Let $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$ be open sets and $F(x,y) : U \times V \to \mathbb{R}^q$ be of $\mathcal{C}^\infty$ class. If $(a,b) \in U \times V$ satisfies $F(a,b) = 0$ and*

$$D_y F(a,b) = \left(\frac{\partial F_i}{\partial y_j}(a,b) \right) \in \mathrm{GL}_q(\mathbb{R})$$

then there exists

i). neighborhoods $a \in U_1$ and $b \in V_1$,

ii). $\varphi \in \mathcal{C}^\infty(U_1, V_1)$

such that

$$\forall (x,y) \in U_1 \times V_1, F(x,y) = 0 \Leftrightarrow \varphi(x) = y.$$

Furthermore, we have that

$$D\varphi(x) = -D_y F(x, \varphi(x))^{-1} \cdot D_x F(x, \varphi(x)).$$

Theorem 1.2 (Inverse Function Theorem). *Let $U \subseteq \mathbb{R}^p$ be an open subset and $f : U \to \mathbb{R}^q$ be smooth. Let $a \in U$ be such that $Df(a)$ is invertible. Then there are neighborhoods $a \in U_1 \subseteq U$ and $f(a) \subseteq V_1$ such that $f|_{U_1} : U_1 \to V_1$ is a diffeomorphism.*

Theorem 1.3 (Rank Theorem). *Let $U \subseteq \mathbb{R}^p$ be open and $f : U \to \mathbb{R}^q$ be smooth. Let $a \in U$ and $b = f(a)$. If $\operatorname{rk} Df(a) = r$ then there exists local diffeomorphisms*

i). $\psi : U_\psi \subseteq U \to V_\psi \subseteq \mathbb{R}^p$ with $\psi(a) = 0$

ii). $\varphi : U_\varphi \subseteq \mathbb{R}^q \to V_\varphi \subseteq \mathbb{R}^q$ with $\varphi(b) = 0$.

such that

$$\varphi \circ f \circ \psi^{-1}(x_1, \cdots, x_p) = (x_1, \cdots, x_r, \tilde{f}(x)).$$

Furthermore, if $\operatorname{rk} Df(x) = r$ in some neighborhood of a, then $\tilde{f}$ can be chosen to be 0.

2 Differentiable Manifolds

2.1 Basics from Set Theoretic Topology

Definition 2.1. *Let X be a topological space. X is said to be separated/Hausdorff if any two distinct points have open neighborhoods which are disjoint to one another.*

Definition 2.2. *Let X be a topological space and $(U_i)_{i\in I}$ be its open covering. A refinement of $(U_i)_{i\in I}$ is an open covering $(V_j)_{j\in J}$ such that*

$$\forall j \in J, \exists i \in I \,\mathrm{s.\,t.}\, V_j \subseteq U_i.$$

It is locally finite if each point $x \in X$, there exists a neighborhood U_i such that

$$|\{j \in J \mid U \cap V_j \neq \emptyset\}| < \infty.$$

Definition 2.3. *A topological space X is called paracompact if every open covering $(U_i)_{i\in I}$ has a locally finite refinement $(V_j)_{j\in J}$.*

Example 2.1. *The following spaces are paracompact.*

1). Compact spaces.

2). Locally compact Hausdorff spaces which are first countable.

Definition 2.4. *A subset $M \subseteq \mathbb{R}^p$ is said to be a submanifold of dimension m if M can be covered by open sets $(U_i)_{i\in I}$ such that there exists a smooth function $F_i : U_i \to \mathbb{R}^{p-m}$ with full rank and*

$$M \cap U_i = \{x \in U_i \mid F_i(x) = 0\}.$$

In other words, M is locally a graph of a smooth map.

Definition 2.5. *Let X be a separated and paracompact topological space. A chart is a pair (U, φ) where U is an open subset of X and $\varphi : U \to V \subseteq \mathbb{R}^n$ is a homeomorphism onto some open subset V of $\mathbb{R}^n$. An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ is a collection of charts such that $(U_i)_{i\in I}$ covers X.*

Definition 2.6. *Let $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ be an atlas. A transition map is a composition $\varphi_i \circ \varphi_j^{-1}|_{\varphi_j(U_i\cap U_j)} \to \varphi_i(U_i \cap U_j)$ for some $i, j \in I$.*

Definition 2.7. *An atlas $\mathcal{A}$ is called smooth if all the transition maps are smooth.*

Definition 2.8. *A chart (U, φ) is smooth compatible with smooth atlas $\mathcal{A}$ if for any $i \in I$, $\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ are smooth.*

Definition 2.9. *An atlas $\mathcal{A}$ is smooth maximal if (U, φ) is smooth compatible with $\mathcal{A}$ then $(U, \varphi) \in \mathcal{A}$.*

Definition 2.10. *A paracompact, smooth manifold is a pair $(M, \mathcal{A})$ such that* *i). M is a paracompact separated topological space.*

ii). $\mathcal{A}$ is smooth maximal.

Remark 2.1. *Above notions can be defined for $\mathcal{C}^k$, analytic, continuous, algebraic, linear by simply replacing the word "smooth" with one of the formers.*

Remark 2.2. *It suffices to give one (smallest possible) atlas to define a smooth structure.*

2.2 Examples of Smooth Manifolds

Definition 2.11. *A topological space is second countable if it has a basis consists of at most countably many open subsets.*

Remark 2.3. *Instead of paracompactness, we assume smooth manifolds to be separated and second countable space. From this definition, we can also induce paracompactness.*

Example 2.2. *For $x, y \in \mathbb{R}$, $x \sim y \Leftrightarrow |x| = |y| > 1$. With this relation, we construct a quotient space $X = \mathbb{R}/\sim$.*

We introduce its atlas by $U_1 = (-\infty, 1)/\sim = X - \{[1]\}$ with chart $\varphi_1(x) = [x]$, and $U_2 = (-1, \infty)/\sim = X - \{[-1]\}$ with chart $\varphi_2(x) = [x]$. However, this is non-separated as 1 and -1 cannot be separated.

Definition 2.12. *A smooth submanifold of $\mathbb{R}^n$ is a separated, second countable smooth manifold.*

Take φ^{-1}, where φ runs through parametrization as charts.

Remark 2.4. $(\mathbb{R}^n, \{\mathrm{id}\})$ *is separated and second countable smooth manifold.*

Definition 2.13. *Let $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$. Let $N \in S^n$ and call it the north pole. We define the stereographic projection with north pole N to be such that*

$$\varphi_+ : S^n \backslash \{N\} \to \mathbb{R}^n, (x_1, \cdots, x_n) \mapsto \frac{1}{1 - x_n}(x_1, \cdots, x_{n-1}).$$

Similarly, we define

$$\varphi_- : S^n \backslash \{-N\} \to \mathbb{R}^n, (x_1, \cdots, x_n) \mapsto \frac{1}{1 + x_n}(x_1, \cdots, x_{n-1}).$$

Definition 2.14. *A n-dimensional torus T^n is a direct product of n-many S^1.*

Definition 2.15. *A lattice Γ in $\mathbb{R}^n$ is a subgroup generated by a basis of $\mathbb{R}^n$ over $\mathbb{Z}$.*

Lemma 2.1.

$$T^n = \mathbb{R}^n/\Gamma.$$

Definition 2.16. *Let $k = \mathbb{R}$ or $\mathbb{C}$. The projective space over the field K is*

$$K\mathbb{P}^n = (K^{n+1} \backslash \{o\})/K^\times.$$

Remark 2.5.

$$\mathbb{R}\mathbb{P}^n = S^n/x \sim -x.$$

And also we have,

$$\mathbb{C}\mathbb{P}^n = S^{2n+1}/S^1.$$

We can introduce atlases to them by

$$U_j = \{[x_0 : \cdots : x_n] \in K\mathbb{P}^n \mid x_j \neq 0\},$$

with chart,

$$\varphi_j : U_j \to K^n, \varphi([x_0 : \cdots : x_n]) = \left(\frac{x_0}{x_j}, \cdots, \frac{x_n}{x_j}\right).$$

Definition 2.17. *Let $\mathcal{M}$,$\mathcal{N}$ be smooth manifold. A function $f : \mathcal{M} \to \mathcal{N}$ is called smooth if for each $p \in \mathcal{M}$, there exist chart (U, φ) around p and (V, ψ) around $f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$$

is smooth for some $m, n \in \mathbb{N}$.

Remark 2.6. *Since transition maps are smooth, above map is well-defined and if f is smooth so is $\varphi' \circ f \circ \psi'$ for any charts φ', ψ'.*

Proposition 2.1. *Compositions of smooth maps are smooth.*

Definition 2.18. *Let $f : \mathcal{M} \to \mathcal{N}$ be a smooth map. It is called a diffeomorphism if it is bijective and f^{-1} is also smooth.*

2.3 Partitions of Unity

Proposition 2.2. *Let $(\mathcal{M}, \mathcal{A})$ be a pair of a smooth manifold with its atlas which we assumed to be separated and second countable. Then it is paracompact. More precisely, given an open covering $(U_i)_{i \in I}$, there exists a countable locally finite refinement $(V_j)_{j \in J}$ together with a chart $\{\psi_j\}_{j \in J}$ which can be chosen such that*

$$\psi_j : V_j \to B(o, 3)$$

such that

$$\mathcal{M} = \bigcup_{j \in J} \psi_j^{-1}(B(o, 1)).$$

Proof. $\mathcal{M}$ is locally compact (look at the charts), hence there existsa acompact subsets

$$K_1 \subset\subset\subset K_2 \subset\subset K_3$$

such that

$$\mathcal{M} = \bigcup K_j$$

(ie. an exhaustion by compact sets). Note $K_{j+1} - \operatorname{int} K_j$ is again compact. For $p \in K_{j+1} - \operatorname{int} K_j$ choose a chart (V_p, ψ_p) such that

$$\psi_p(V_p) = B(o, 3), \psi_l(p) = 0.$$

Note that we can take V_p small enough so that there is $i \in I$ such that $V_p \subset U_i$.

By compactness, we can take $p_{j,1}, \cdots, p_{j,r_j}$ such that

$$K_{j+1} - \operatorname{int} K_j = \bigcup_{l=1}^{r_j} \varphi_{jl}^{-1}(B(0,1)).$$

By making $V_{p_{j,l}}$ small enough we can assume

$$V_{p_{j,l}} \subset \operatorname{int} K_{j+2} - K_{j-1}.$$

The union over all j gives the countable locally finite refinement. □

From now on, a manifold refers to a smooth, separated, and second countable manifold.

Example 2.3. *We define,*

$$f_1(t) := \begin{cases} e^{-\frac{1}{t}}, t > 0 \\ 0, t \leq 0. \end{cases}$$

Then f_1 smooth. We then define,

$$f_2(t) := \frac{f_1(t)}{f_1(t) + f_1(1-t)}.$$

f_2 is a function which is monotonically increasing and for $t \geq 1$, we have $f_2(t) =$ 1.

$$f_3(t) = f_2(2+t) + f_2(2-t).$$

Again we define

$$f_4 : \mathbb{R}^n \to \mathbb{R}, f(x) = f_3(\|x\|).$$

Definition 2.19. *A support of the function $f : X \to \mathbb{R}$ is*

$$\operatorname{supp}(f) := \overline{f^{-1}(\mathbb{R} \backslash \{0\})}.$$

Theorem 2.1. *Let $\mathcal{M}$ be a manifold and $(U_i)_{i\in I}$ be an open covering. Then there exist smooth functions $(\varphi_n : \mathcal{M} \to \mathbb{R})_{n\in\mathbb{N}}$ such that*

i). $\forall n \in \mathbb{N}, 0 \leq \varphi_n \leq 1$.

ii). $(\operatorname{supp}\varphi_n)_{n\in\mathbb{N}}$ *is locally finite.*

iii). $\forall n \in \mathbb{N} \exists i \in I, \mathrm{s.\,t.} \operatorname{supp}\varphi_n \subset U_i$.

iv). $\forall p \in \mathcal{M}, \sum_{n=1}^{\infty} \varphi_n(p) = 1$.

Such sequence of functions is called a partition of unity subordinated by the given covering $(U_i)_{i\in I}$.

Additionally, if $I \subset \mathbb{N}$, then $(\varphi_n)_{n\in\mathbb{N}}$ can be indexed by I as well in such a way that,

$$\forall i \in I, \operatorname{supp}\varphi_i \subset U_i.$$

Proof. By Proposition 2.2, take a countable, locally finite refinement charts $((\tilde{\psi}_n, V_n))_{n\in\mathbb{N}}$. Borrowing the notation from Exampple 2.3, we set

$$\tilde{\varphi_n}(x) := \begin{cases} f_4(\tilde{\psi}_n(x)) & \forall x \in V_n, \\ 0, & x \notin V_n. \end{cases}$$

Then take $\tilde{\varphi}$ to be

$$\varphi := \sum_{n=1}^{\infty} \tilde{\varphi_n} \in \mathcal{C}^{\infty}(\mathcal{M}).$$

We observe that φ is nowhere 0. Thus we define

$$\varphi_n := \tilde{\varphi_n}/\tilde{\varphi},$$

we derived a desired family of functions.

For the second part, set

$$J_0 = \emptyset, \quad \varphi_0 = 0,$$

define inductively that

$$J_k = \left\{ i \in \mathbb{N} \backslash \bigcup_{i=0}^{k-1} J_i \;\middle|\; \operatorname{supp}\varphi_i \subseteq U_k \right\}.$$

We then take

$$\psi_k := \sum_{i\in J_k} \tilde{\varphi}_i/\tilde{\varphi}.$$

□

Proposition 2.3. *Let $A \subset \mathcal{M}$ be a closed subset of a manifold such that there is an open set $A \subset G \subset \mathcal{M}$. Then there exists a smooth function $f \in \mathcal{C}^\infty(\mathcal{M})$ such that the image of f is contained in $[0,1]$ and*

$$\forall p \in A, f(p) = 1, \text{ and } \forall p \in G, f(p) = 0.$$

Proof. Observe that $\{\mathcal{M} - A, G\}$ is an open covering. By Theorem 2.1, we can take smooth functions φ, ψ such that

$$\operatorname{supp} \varphi \subset \mathcal{M} - A, \operatorname{supp} \psi \subset G, \varphi + \psi \equiv 1.$$

Take $f = \psi$. □

2.4 Tangent Spaces

Example 2.4. *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a submanifold. Then we have a notion of tangent vector, such that for some $v \in \mathbb{R}^n$, we define a smooth curve γ such that*

$$\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}, \quad \varphi'(0) = v.$$

Given $v \in \mathbb{R}^n$, $p \in \mathcal{M}$ and $f : \mathcal{M} \to \mathbb{R}^n$, we define

$$vf = (f \circ \gamma)'(0).$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ is a smooth curve such that $\gamma(0) = p, \gamma'(0) = v$. Exercise : show that this definition does not depend on the choice of γ satisfying the two conditions above. Furthermore, this defines a linear map

$$v : \mathcal{C}^\infty(\mathcal{M}) \to \mathbb{R},$$

such that

$$v(f \cdot g) = (f \circ \gamma \cdot g \circ \gamma)'(0) = f(p) \cdot (vg) + (vf) \cdot g(p).$$

Definition 2.20. *Let $\mathcal{M}$ be a smooth manifold and $p \in \mathcal{M}$. The linear map $X_p : \mathcal{C}^\infty(\mathcal{M}) \to \mathbb{R}$ is called a derivation at p if $X_p(fg) = f(p)X_p g + g(p)X_p f$.*

We define the tangent space of $\mathcal{M}$ at p to be

$$T_p\mathcal{M} = \{X_p : \mathcal{C}^\infty \mathcal{M} \to \mathbb{R} \mid X_p \text{ is a derivation}\}.$$

Remark 2.7. *Obviously $T_p\mathcal{M}$ is a $\mathbb{R}$-vectorspace.*

Lemma 2.2. *if $\varphi \in \mathcal{C}^\infty(\mathcal{M})$ is constant around $p \in \mathcal{M}$. Then $X\varphi = 0$ for any $X \in T_p\mathcal{M}$.*

Proof. Let $\varphi \equiv 1$ around p. Choose $\chi \in \mathcal{C}^\infty(\mathcal{M})$ to be such that

1. $\chi \equiv 1$ in a neighborhoodof p.
2. $\operatorname{supp} \chi \subseteq \{q \in \mathcal{M} \mid \varphi(q)) = 1\}$.

Then $\chi\varphi = \chi$. Thus we have

$$X(\chi) = X(\chi)\varphi(p) + \chi(p)X(\varphi). \tag{1}$$

Since $\varphi(p) = 1$ and $\chi(p) = 1$, thus we conclude $X(\varphi) = 0$. $\square$

Lemma 2.3. *If $X \in T_p\mathcal{M} \cap T_q\mathcal{M}$ then either $X = 0$ or $p = q$.*

Proof. Suppose $p \neq q$. Then take a smooth functional $\chi : \mathcal{M} \to \mathbb{R}$ such that $\chi(\mathcal{M}) \subseteq [0, 1]$ and

$$\begin{cases} \chi \equiv 1 \quad \text{around } p, \chi \equiv 0 \quad \text{around } q. \end{cases}$$

Then for any $f \in \mathcal{C}^\infty(\mathcal{M})$,

$$Xf = X\chi f = 0.$$

$\square$

Lemma 2.4. *Let (U, φ) be a chart of $\mathcal{M}$ centered at $p \in \mathcal{M}$ (ie. $\varphi(p) = 0$) with coordiante function $x_1, \cdots, x_n$. Then for $f \in \mathcal{C}^\infty(U)$, there are functions $f_1, \cdots, f_n \in \mathcal{C}^\infty(U)$, such that*

$$f = \sum_{j=1}^{n} f_j x_j + f(p).$$

Note that taking $\varphi(p) = 0$ justfies the $f_j x_j$ for each coordinate.

Analogously, given $f \in \mathcal{C}^\infty(\mathcal{M})$, we may choose $f_j \in \mathcal{C}^\infty(\mathcal{M})$ such that

$$f|_U = \sum_{j=1}^{n} f_j|_U x_j + f(p).$$

Proof. The proof for the second part is assgined as an exercise.

Consider $\mathcal{M} = \mathbb{R}^n$, $p = 0$, and $U = (-\varepsilon, \varepsilon)^n$. We have

$$f(x) = \sum_{j=1}^{n} (f(x_1, \cdots, x_j, 0, \cdots, 0) - f(x_1, \cdots, x_{j-1}, 0, \cdots, 0)) + f(0).$$

By the fundamental theorem of calculus, we obtain,

$$f(x) = \sum_{j=1}^{n} \int_0^1 \partial_j f(x_1, \cdots, tx_j, 0, \cdots, 0) dt x_j + f(0).$$

By setting $f_j = \int_0^1 \partial_j f(x_1, \cdots, tx_j, 0, \cdots, 0)dt$, we derive the statement. $\square$

Definition 2.21. *Given a chart (U, φ) of $\mathcal{M}$. For $f \in \mathcal{C}^\infty(U)$ we have $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$. We define*

$$\left.\frac{\partial}{\partial x_j}\right|_p f := D(f \circ \varphi^{-1})(\varphi(p))[e_j].$$

Remark 2.8. *Note that if we set* $\gamma_i : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ *to be such that* $\gamma(0) = \varphi(p)$ *and* $\gamma'(0) = e_i$*, then*

$$\left.\frac{\partial}{\partial x_i}\right|_p f = \left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma_i)(t).$$

Proposition 2.4. *Let* $\mathcal{M}$ *be a smooth manifold and* (U, φ) *be a* n*-dimensional chart at* p*. Then we have*

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p \right\}$$

forms a basis in $T_p\mathcal{M}$*.*

Proof. Write

$$f = \sum_{i=1}^{n} f_j x_j + f(p).$$

Choose $\chi \in \mathcal{C}^\infty(\mathcal{M})$ such that

i). χ is compactly supported in U.

ii). $\chi \equiv 1$ in a neighborhood of p.

Then

$$\chi^2 f = \sum_{j=1}^{n} (\chi f_j)(\chi x_j) + \chi^2 f(p),$$

in the neighborhood of p. Since χ is compactly supported. This is defined everywhere on U.

Let $X \in T_p\mathcal{M}$ be a derivation. Then observe that

$$X\chi^2 = 0, \quad Xf = X(\chi^2 f).$$

Since φ is centered at p, we have $x_j(p) = 0$ for all j. Therefore,

$$\begin{aligned} Xf &= \sum_{j=1}^{n} X((\chi f_j))(\chi x_j)(p) + f_j(p) X(\chi x_j) \\ &= \sum_{j=1}^{n} f_j(p) X(\chi x_j) \\ &= \sum_{j=1}^{n} X(\chi x_j) \cdot \left.\frac{\partial}{\partial x_j}\right|_p f. \end{aligned}$$

This implies that

$$X = \sum_{j=1}^{n} X(\chi x_j) \left.\frac{\partial}{\partial x_j}\right|_p.$$

The set

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \cdots, \frac{\partial}{\partial x_n}\bigg|_p \right\}$$

spans $T_p\mathcal{M}$. Remains to show the linearly independentness. To do so consider

$$\frac{\partial}{\partial x_j} x_i = \delta_{ij}.$$

We conclude the proof. □

Example 2.5. *For $\mathcal{M} = \mathbb{R}^n$ and $p \in \mathbb{R}^n$, the partial derivatives with respect to the standard coordinate at p is a basis of $T_p\mathbb{R}^n$. Explicitly each is of the form,*

$$\frac{\partial}{\partial x_j}\bigg|_p f = \frac{\partial f}{\partial x_j}(p).$$

Example 2.6. *Let (U, φ) be a chart of an open subset of $\mathbb{R}^n$ with coordinate function $y_1, \cdots, y_n$, (ie. $\varphi = (y_1, \cdots, y_n)$).*

We have

$$\begin{aligned} \frac{\partial}{\partial x_j}\bigg|_p f &= \frac{\partial}{\partial x_j}(f \circ \varphi^{-1} \circ \varphi)(0), \\ &= \sum_{j=1}^{n} \partial_j (f \circ \varphi^{-1})(\varphi(p)) \frac{\partial \varphi_j}{\partial x_i}(p). \end{aligned}$$

Let $\frac{\partial}{\partial y_j}\bigg|_p f = \partial_j (f \circ \varphi^{-1})(\varphi(p))$, we get,

$$= \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i}(p) \frac{\partial}{\partial y_j}\bigg|_p f.$$

In particular,

$$\frac{\partial}{\partial x_i}\bigg|_p f = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i}\bigg|_{\varphi(p)} \frac{\partial}{\partial y_j}\bigg|_p f.$$

Definition 2.22. *Let $\mathcal{M}, \mathcal{N}$ be smooth manifold and $f : \mathcal{M} \to \mathcal{N}$ be a smooth map. For $p \in \mathcal{M}$, we have a linear transform*

$$T_p f : T_p\mathcal{M} \to T_p\mathcal{N}$$

such that given $X \in T_p\mathcal{M}$ and $h \in \mathcal{C}^\infty(\mathcal{N})$, we define

$$T_p f(X)h = X(h \circ f).$$

Lemma 2.5. *Let $\mathcal{M} \subseteq \mathbb{R}^m$ and $\mathcal{N} \subseteq \mathbb{R}^n$ be open. Then in the standard basis $T_p f$ is the Jacobi-matrix.*

Proof. Follows from the arguments in Example 2.6. □

Lemma 2.6 (Chain Rule)**.** *Let $f : \mathcal{M} \to \mathcal{N}$, $g : \mathcal{N} \to \mathcal{W}$be smooth maps. Then we have*

$$T_p(g \circ f) = T_{f(p)} g \circ T_p f.$$

Proof. Since the statement is local we may assume $\mathcal{M}, \mathcal{N}, \mathcal{W}$ to be all open sets in some Euclidean spaces. Then using Lemma 2.5 we have the statement. □

Lemma 2.7. *Let (U, φ) be a chart. Viewing this as a locally smooth map from $\mathcal{M}$ to $\mathbb{R}^n$, we obtain an isomorphism,*

$$T_p\varphi : T_p\mathcal{M} \to T_{\varphi(p)}\mathbb{R}^n = \mathbb{R}^n.$$

If (V, ψ) is another chart then for $p \in U \cap V$, we have,

$$T_p\psi = T_p(\psi \circ \varphi^{-1} \circ \varphi) = D_{\varphi(p)}(\psi \circ \varphi^{-1}) T_p\varphi.$$

Proof. Exercise. □

Definition 2.23 (Physicists definition of tangent space)**.** *A tangent vector is a family $(\xi^\varphi)_\varphi$ where $\xi^\varphi \in \mathbb{R}$ and φ runs through charts containing p, together with a transformation rule*

$$\xi^\psi = D_{\varphi(p)} \left(\psi \circ \varphi^{-1}\right) [\xi^\varphi].$$

Definition 2.24. *Let $\mathcal{M}$ be a manifold. A curve $\gamma : I \to \mathcal{M}$ in $\mathcal{M}$ is a smooth map such that from some interval $I \in \mathbb{R}$, such that I is a manifold with the canonical charts. $d : I \to I$ where $\frac{d}{dt}$ a canonical tangent vector associated to this chart, we get a velocity vector of the curve*

$$\overset{\circ}{\gamma}(t_0) = T_{t_0}\gamma \left(\frac{d}{dt}\big|_{t_0}\right) \in T_{\gamma(t_0)}\mathcal{M}.$$

Example 2.7. *Given any $v \in T_p\mathcal{M}$, and a chart (U, φ) centered at p (ie. $\varphi(p) = 0$), we have*

$$v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_j}\bigg|_p$$

Define a curve

$$\gamma(t) = \varphi^{-1}(tv_1, \cdots, tv_n).$$

Note that $\gamma(0) = \varphi^{-1}(0) = p$. We define

$$\overset{\circ}{\gamma}(0) f = T_0\gamma \left(\frac{d}{dt}\bigg|_0\right) f = \frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) = \sum_{i=1}^{n} v_i \frac{\partial f \circ \varphi^{-1}}{\partial x_j}(0) = vf.$$

2.5 Local properties of differentiable maps and submanifolds.

In this section, all manifolds are assumed to have pure dimension (ie. it has a constant dimension everywhere).

Notation 2.1. *$\mathcal{M}^m$ denotes that $\dim \mathcal{M} = \dim T_p\mathcal{M} = m$ for any point $p \in \mathcal{M}$.*

Definition 2.25. *Let $\mathcal{M}^m, \mathcal{N}^n$ be manifolds and $f : \mathcal{M} \to \mathcal{N}$ be a smooth map.*

1). $p \in \mathcal{M}$ is called a critical point for f if $\operatorname{rank} T_pf < n = \dim \mathcal{N}$ and $f(p)$ is called the critical value.

2). $q \in \mathcal{N}$ is called a regular value if $\forall p \in f^{-1}(q)$ we have $T_qf = \dim \mathcal{N} = n$.

3). f is called a submersion if for any $p \in \mathcal{M}$, $T_pf = n$.

4). f is called an immersion if for any $p \in \mathcal{M}$, $T_pf = m$.

5). f is called a subimersion if $\mathcal{M} \ni p \mapsto \operatorname{rank} T_pf$ is a constant.

Notation 2.2. *For $f : \mathcal{M} \to \mathcal{N}$, we denote the set of critical points to be C_f.*

Definition 2.26 (Submanifold)**.** *Let $\mathcal{N} \subset \mathcal{M}^m$ is called a submanifold if for any chart (U, φ) of $\mathcal{M}$,*

$$\varphi(\mathcal{N} \cap U) \subseteq \mathbb{R}^m$$

is a submanifold of $\mathbb{R}^m$.

Remark 2.9. *The above definition is equivalent to say that for each $p \in \mathcal{M}$, there exists a chart (U, φ) centered at p such that*

$$\varphi(\mathcal{N} \cap U) = \mathbb{R}^k \times \{0\} \cap \varphi(U) \subset \varphi(U) \subset \mathbb{R}^m.$$

In other words, the intersection is diffeomorphic to some hyperspace with dimension k in $\mathbb{R}^m$.

Definition 2.27. *A smooth map $f : \mathcal{M} \to \mathcal{N}$ is an embedding if*

i). f is an injective immersion.

ii). $f : \mathcal{M} \to f(\mathcal{M}) \subset \mathcal{N}$ is a homeomorphism.

Definition 2.28. *Let $f : \mathcal{M} \to \mathcal{N}$ be smooth. The rank of f is the map $\mathcal{M} \ni p \mapsto \operatorname{rank} T_pf$.*

Proposition 2.5. *Let $f : \mathcal{M}^m \to \mathcal{N}^n$ be a subimmersion of rank k.*

1). For $q \in \mathcal{N}$, the set $f^{-1}(\{q\}) \subset \mathcal{M}$ is empty or a submanifold of dimension $m - k$.

2). For $p \in \mathcal{M}, q := f(p)$ there exists a neighborhood U of p and V of q such that $S = f(U) \cap V$ is a submanifold of $\mathcal{N}$ of dimension k.

Proof. Apply the rank theorem for $f(p) = q$. There exists a chart (U, φ) and (V, ψ) which are centered at p and q, respectively such that

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & V \\ \varphi\big\downarrow & & \big\downarrow\psi \\ \mathbb{R}^m \supset U' & \xrightarrow[(x_1,\cdots,x_m)\mapsto(x_1,\cdots,x_k,0,\cdots,0)]{} & V' \subset \mathbb{R}^n \end{array}$$

This shows the second assertion. For the first assertion, note that if $f^{-1}(\{q\})$ is not empty then $k \leq m$

$$f^{-1}(\{q\}) = \varphi^{-1}(\{(x_1, \cdots, x_m) \mid \forall i = 1, \cdots, k, x_i = \psi(p)_i\}).$$

$\square$

Corollary 2.1. *$f : \mathcal{M}^m \to \mathcal{N}^n$ is smooth and $q \in \mathcal{N}$ is a regular value, then $f^{-1}(\{q\})$ is a submanifold of dimension $m - n$ or empty.*

2.6 The Theorem of Morse-Sard

Notation 2.3. *The Lebesgue measure on $\mathbb{R}^m$ is denoted by λ^m.*

Definition 2.29 (Null set)**.** *$A \subset \mathcal{M}^m$ is a nullset if for each chart (U, φ), the set $\varphi(U \cap A)$ is a λ^m-nullset in $\mathbb{R}^m$.*

Remark 2.10. *A diffeomorphism maps nullsets to nullsets. Hence the above notion is well-defined.*

Remark 2.11. *A singleton of a manifold is a nullset. Countable unions of nullsets are again nullsets.*

Remark 2.12. *Let $A \subset \mathcal{M}^m$ be a nullset where $m > 0$, $\mathcal{M} \backslash A$ is dense.*

Remark 2.13. *Suppose we have a smooth function $f : \mathbb{R}^m \supset U \to \mathbb{R}$. For the sake of simplicity, we assume $f(0) = 0$ and $\frac{\partial f}{\partial x_1}(0) \neq 0$. Consider*

$$h(x) = (f(x), x_2, \cdots, x_m).$$

Then we have

$$Jf(0) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(0) & \frac{\partial f}{\partial x_2}(0) & \cdots & \frac{\partial f}{\partial x_m}(0) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0) & 0 & \cdots & 1. \end{pmatrix}$$

This is invertible, therefore we have h is a local diffeomorphism. COnsider

$$g := f \circ h^{-1}.$$

We denote $(t, \xi) = h(x)$, then

$$g(t, \xi) = f \circ h^{-1} \circ h(x) = f(x) = t.$$

Theorem 2.2. *Let $f : \mathcal{M}^m \to \mathcal{N}^n$ be smooth where $n \geq 1$. Then the set C_f of critical values of f is a nullset.*

Proof. Suffices to prove that each $p \in \mathcal{M}$ has an open neighborhood U such that $f(C_f \cap U)$ is a nullset.

$\Rightarrow$) Without the loss of generality, we assume $\mathcal{M} = U \subseteq \mathbb{R}^m$ and $\mathcal{N} = \mathbb{R}^n$ for $n \geq 1$. We will prove the theorem by induction on m.

If $m = 0$, then the image $f(\mathcal{M})$ is at most countable thus a nullset.

Assume the claim holds for all dimension less than m. Let us define

$$C := C_f, \quad C_l := \left\{ x \in U \mid \forall |\alpha| \leq l, \frac{\partial^{|\alpha|} f}{\partial x^\alpha} = 0 \right\}, l \geq 1.$$

Claim 1. *$f(C \backslash C_l)$ is a nullset.*

<u>Proof:</u> Pick $\xi \in C \backslash C_l$ then there exists $\frac{\partial f_i}{\partial x_j}(\xi) \neq 0$ for some i, j. Without the loss of genearlity, we assume $i = j = 1$. We put

$$h(x) = (f_1(x), x_2, \cdots, x_m).$$

By Remark 2.13, we have h is a local diffeomorphism and in particular,

$$\xi \mapsto (f_1(\xi), \xi_2, \cdots, \xi_m).$$

Similarly to the remark, we define

$$g := f \circ h^{-1}, \quad g(t, x) = (t, \tilde{g}(t, x)).$$

(t, x) is critical for g if and only if x is critical for $\tilde{g}(t, \cdot)$. Now consider

$$\begin{aligned}
\lambda^n(f(C_f \cap V)) &= \lambda^n(\{g(t, x) \mid (t, x) \text{ is critical for } g\}), \\
&= \lambda^n(\{(t, \tilde{g}(t, x)) \mid x \text{ is critical for } \tilde{g}(t, \cdot)\}), \\
&= \int_I \lambda^{n-1}\{\tilde{g}(t, x) \mid x \text{ is a critical point of } \tilde{g}(t, \cdot)\} dt.
\end{aligned}$$

The last equality is due to Fubini. The inside of the integral is 0 by the induction hypothesis.

Fix $1 \leq l < \infty$, then $f(C_l \backslash C_{l+1})$ is a nullset. Indeed, for $\xi \in C_l \backslash C_{l+1}$, without the loss of genearlity, we assume there is a multiindex β such that $|\beta| = l$ and

$$\frac{\partial^\beta f_1}{\partial x^\beta}(\xi) = 0.$$

Similarly for the previous case, we set

$$h(x) = \left(\frac{\partial^\beta f_1}{\partial x^\beta}(x), x_2, \cdots, x_m \right).$$

■

Finally, let $d > 0$ and W be a cube of side length b. Consider $x \in C_k \cap W, y \in W$. Fomr Taylor expansion at x implies that

$$|f(x) - f(y)| \leq L|x - y|^{k+1},$$

where L depends on the cube W, k and f. Also by fixing k, we can make it a locally uniform constant.

Subdivide each edge of W into r edges to make W into r^m many cubs (W_j) of edge length $\frac{d}{r}$. For $x \in C_k \cap W_j$, and $y \in W_j$, we have

$$|x - y| \leq \sqrt{m}\frac{d}{r}.$$

In particular, the constant $\sqrt{m}$ only depends on the dimension. Back to the previous arugment, we get

$$|f(x) - f(y)| \leq L\left(\frac{\sqrt{m}d}{r}\right)^{k+1}.$$

This means that $f(C_k \cap W_j)$ sits in a cube of edge length less than or equal to $2L\left(\frac{\sqrt{m}d}{r}\right)^{k+1}$.

$$\lambda^n(f(C_k \cap W)) \leq r^m \lambda(\max_{1 \leq j \leq r^m} f(C_k \cap W_j)) \leq r^m \left\{2L\left(\frac{\sqrt{m}d}{r}\right)\right\}^n = K \cdot r^{m-n(k+1)}.$$

Observe that $r \to \infty$ and $k \geq \frac{m}{n}$, we get $C_k \cap W$ has measure 0. $\square$

3 Vector fields and dynamical system

3.1 Definition

Definition 3.1. *Let $\mathcal{M}$ be a smooth manifold. A smooth vector field of $\mathcal{M}$ is a map*

$$X : \mathcal{M} \to T\mathcal{M},$$

such that

1. *$\forall p \in \mathcal{M}, X(p) \in T_p\mathcal{M}$.*
2. *For any chart (U, φ) centered at p, we have $X|_U = \sum_{i=1}^m X_j^\varphi \frac{\partial}{\partial x_i}$ where $X_i^\varphi \in \mathcal{C}^\infty(U)$ for each i.*

Notation 3.1.

$$\Gamma(T\mathcal{M}) = \{\textit{smooth vector fields on } \mathcal{M}\}.$$

Remark 3.1. *The second condition can be restated as follow. Recall Definition 2.22. A chart* $\varphi : U \to \mathbb{R}^m$ *can be considered as a smooth map between* U *and* $\mathbb{R}^n$*. We also have seen that* $T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^n$*. With this identification, we have,*

$$\begin{aligned} T\varphi :& TU \to T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m, \\ & TU \ni [U \ni p \mapsto X_p] \mapsto [\mathbb{R}^n \ni x \mapsto [\mathcal{C}^\infty(\mathcal{N}) \ni f \mapsto X_{\varphi^{-1}(x)}(f \circ \varphi)]], \end{aligned}$$

is smooth for all chart φ*.*

Example 3.1. *Consider,*

$$X : S^1 \to TS^1, (x, y) \mapsto (-y, x),$$

is a smooth vector field.

Definition 3.2. *Let* $I \subset \mathbb{R}$ *be an interval and* $X \in \Gamma(T\mathcal{M})$*. A curve* $\gamma : I :\to \mathcal{M}$ *is called an integral curve of* X *if*

$$\forall t \in I, \gamma'(t) = X(\gamma(t)).$$

Remark 3.2. *To find such curve is equivalent to solve the following autonomous initial value problem. Given a chart* (U, φ) *around a point* $p \in \mathcal{M}$*,*

$$\begin{cases} (\varphi \circ \gamma)'(t) &= \begin{pmatrix} X_1^\varphi \circ \varphi^{-1} \\ \vdots \\ X_1^\varphi \circ \varphi^{-1} \end{pmatrix} (\varphi \circ \gamma(t)), \\ (\varphi \circ \gamma)(t_0) &= 0. \end{cases}$$

3.2 Flow-Box Theorem

Lemma 3.1 (Grönwall)**.** *Let* $f, g : [a, b] \to (0, \infty)$ *be continuous functions such that*

$$f(t) \leq c + \int_0^t f(s)g(s)ds,$$

for some constant $c > 0$*. Then we have,*

$$f(t) \leq c \exp\left(\int_0^t g(s)ds\right).$$

Proof. Consider,

$$\tilde{f}(t) = c + \int_0^t f(s)g(s)dt.$$

Then by assumption $f(t) \leq \tilde{f}(t)$. Also set,

$$h(t) = \tilde{f}(t) \exp\left(-\int_0^t g(s)ds\right).$$

Then f, h are both differentiable thus we take,

$$\begin{aligned} h'(t) &= f(t)g(t)\exp\left(-\int_0^t g(s)ds\right) - g(t)\tilde{f}(t)\exp\left(-\int_0^t g(s)ds\right), \\ &= \underbrace{(f(t) - \tilde{f}(t))}_{\leq 0} g(t)\exp\left(-\int_0^t g(s)ds\right), \\ &\leq 0. \end{aligned}$$

We also have,

$$h(0) = c.$$

Thus we conclude,

$$h(t) \leq c \Rightarrow f(t) \leq \tilde{f}(t) = h(t)\exp\left(\int_0^t g(s)ds\right) \leq c\exp\left(\int_0^t g(s)ds\right).$$

$\square$

Theorem 3.1. *Let $\mathcal{M}$ be a smooth manifold and $X \in \Gamma(T\mathcal{M})$. Then for each $p \in \mathcal{M}$ there exists a neighborhood U, $\varepsilon > 0$, and a smooth map*

$$F : (-\varepsilon, \varepsilon) \times U \to \mathcal{M},$$

such that

1). $\forall x \in U, F(0, x) = x$,

2). $\partial_t F(t, x) = X(F(t, x))$.

Proof. Recall that a continuously differentiable function over a compact set is Lipschitz.

Consider a smooth function $f : \overline{B(y_0, r)}$ such that

$$a(y) < f(y) < b(y).$$

where $y_0 \in \mathbb{R}^n$, $r > 0$ and a, b are continuous function. For $p \in B(y_0, r)$, we let $F(t, y)$ to be the maximal solution of the initial value problem,

$$\begin{cases} \partial_t F(t, y) = f(F(t, y)), \\ F(0, y) = y_0. \end{cases} \tag{P1}$$

By the construction, we get,

$$F(t, y) = y + \int_0^t f(F(s, y))ds.$$

Thus for $|t| \leq 1$, we obtain,

$$\begin{aligned}
|F(t,y) - y_0| &\leq \left| y - y_0 + \int_0^t f(F(s,y))ds \right|, \\
&\leq |y - y_0| + |t||f(y_0)| + \int_0^t |f(F(s,y)) - f(y_0)|ds, \\
&\leq |y - y_0| + |f(y_0)| + \int_0^t L|F(s,y) - y_0|ds,
\end{aligned}$$

for some $L > 0$. By applying Lemma 3.1, we get,

$$|F(t,y) - y_0| \leq (|y - y_0| + |f(y_0)|)e^{L|t|}.$$

Set $|y - y_0| \leq \frac{r}{2}$ and take,

$$c := \min \left\{ 1, \frac{1}{L}, \frac{r}{\frac{r}{2} + |f(y_0)|} \right\},$$

then for $|t| < c$, we have,

$$|F(t, y_0)| \leq \cdots < r.$$

Thus such F exists on a cylinder $(-\varepsilon, \varepsilon) \times \overline{B\left(y_0, \frac{r}{2}\right)}$.

We now move on to examine the differentiability of such solutions. Let $0 < \rho < r$ and ε be such that F is a unique solution to the IVP on a cylinder $[-\varepsilon, \varepsilon] \times \overline{B(y_0, \rho)}$. Then we have,

$$\begin{aligned}
|F(t,y) - F(t,\overline{y})| &\leq |y - \overline{y}| + \left| \int_0^t f(F(s,y)) - f(F(s,\overline{y}))ds \right|, \\
&\leq |y - \overline{y}| + L \int_0^{|t|} |F(s,y) - F(s,\overline{y})|ds.
\end{aligned}$$

Using Lemma 3.1, we obtain,

$$|F(t,y) - F(t,\overline{y})| \leq |y - \overline{y}|e^{L|t|}.$$

By definition, F is continuously differentiable in t as f is continuous. For the differentiability in y, consider

$$\begin{cases} \partial_t D_2 F(t,y) = \underbrace{Df(F(t,y))}_{\in \mathrm{Mat}_{m\times m}(\mathbb{R})} \overbrace{(D_2 F)(t,y)}^{\in \mathrm{Mat}_{m\times m}(\mathbb{R})}, \\ (D_2 F)(0,y) = I_m \in \mathrm{Mat}_{m\times m}(\mathbb{R}). \end{cases} \tag{P2}$$

Equation (P2) is an IVP for a matrix valued functions. Let $\Phi : (-\varepsilon, \varepsilon) \times B(y_0, \rho) \to \mathrm{Mat}_{m\times m}(\mathbb{R})$ be a unique solution of

$$\begin{cases} \partial_t \phi(t,y) = Df(F(t,y))\phi(t,y), \\ \phi(0,y) = I_m \in \mathrm{Mat}_{m\times m}(\mathbb{R}). \end{cases} \tag{P3}$$

Similar to the case of F we have Φ is (Lipschitz)-continuous.

Claim 1. *$F(t,y)$ is partially differentiable in y and $D_2F(t,y) = \Phi(t,y)$.*

Proof: Fix $(t,\overline{y}) \in (-\varepsilon,\varepsilon) \times B(y_0,\rho)$. Since f is continuously differentiable, we have,

$$f(x) - f(y) = (Df)(y)(x-y) + R(x,y)(x-y),$$

where R is an uniformly continuous function on $\overline{B(y,\rho)}$ such that $R(x,x) = 0$. Thus for $\tilde{\varepsilon} > 0$ there is $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow |R(x,y)| < \tilde{\varepsilon}.$$

By the construction of F, we see F is locally uniform in s thus there is $\tilde{\delta} > 0$ such that

$$|x-y| < \tilde{\delta} \Rightarrow |F(s,x) - F(s,y)| < \delta.$$

Take y such that $|y - \overline{y}| < \tilde{\delta}$ and consider the equation.

$$\begin{aligned} f(F(s,y)) - f(F(s,\overline{y})) - (Df)(F(s,\overline{y}))\phi(s,\overline{y})(y-\overline{y}) \\ =&(Df)(F(s,\overline{y}))(F(s,y) - F(s,\overline{y}) - \phi(s,\overline{y})(y-\overline{y})) \\ &+ R(F(s,y),F(s,\overline{y}))(F(s,y) - F(s,\overline{y})). \end{aligned}$$

We see,

$$|R(F(s,y),F(s,\overline{y}))| \cdot |F(s,y) - F(s,y)| \leq \tilde{\varepsilon}|y-\overline{y}|e^{|s|L}.$$

Finally,

$$\begin{aligned} |F(t,y) - F(t,\overline{y}) - \phi(s,\overline{y})(y-\overline{y})| &= \left| \int_0^t f(F(t,y)) - f(F(s,\overline{y})) - (Df)(F(s,\overline{y}))\phi(s,\overline{y})(y-\overline{y})ds \right|, \\ &\leq \int_0^{|t|} |Df(F(s,\overline{y}))| F(s,y) - F(s,\overline{y}) - \phi(s,\overline{y})(y-\overline{y})|ds \\ &+ \tilde{\varepsilon}|y-\overline{y}| \int_0^{|t|} e^{|s|L} ds, \\ &\leq c_1\tilde{\varepsilon}|y-\overline{y}| + c_2 \int_0^{|t|} G(s,y,\overline{y})ds. \end{aligned}$$

Using Lemma 3.1, we obtain,

$$G(t,y,\overline{y}) \leq c_1\tilde{\varepsilon}|y-\overline{y}|e^{c_2|t|}.$$

Thus the solution Φ of Equation (P3) is equal to $D_2F(t,y)$. ■
Using the claim, we proved the regularity of F. □

Definition 3.3. *Above $F(t,\cdot)$ is called the local flow of the vector field X.*

Theorem 3.2. *Let $\mathcal{M}$ be a smooth manifold and $X \in \Gamma(T\mathcal{M}), p \in \mathcal{M}$. Then there exists continuous functions $a, b : \mathcal{M} \to \mathbb{R}$ such that*

$$-\infty \leq a(p) < 0 < b(p) \leq \infty,$$

and an integral curve, $c_p : (a(p), b(p)) \to \mathcal{M}$ of X with $c_0(0) = p$ such that for any integral curve $\tilde{c}_p : I \to \mathcal{M}$ of X with $\tilde{c}_p(0) = p$, we have

1). $I \subseteq (a(p), b(p))$,

2). $c_p|_I = \tilde{c}_p$.

That is to say c_p is the maximal integral curve through p. By continuity the set,

$$A := \{t \in I_1 \cap I_2 \mid c_1(t) = c_2(t)\},$$

is closed. If $t_0 \in I_1 \cap I_2$ then c_1, c_2 are local solution of the IVP,

$$\begin{cases} \gamma'(t) = X(\gamma(t)), \\ \gamma(t_0) = c_1(t_0) = c_2(t_0). \end{cases}$$

Using Picard-Lindelöf, there is a neighborhood U of t_0 such that $U \subseteq A$. Since $A \neq \emptyset$, we conclude $A = I_1 \cap I_2$. That is

$$(c_1 \cup c_2) : I_1 \cup I_2 \to \mathcal{M}, I_1 \cup I_2 \ni t \mapsto \begin{cases} c_1(t), & (t \in I_1), \\ c_2(t), & (t \in I_2). \end{cases}$$

$c_1 \cup c_2$ is also an integral curve of X through p. Thus there is a maximal integral curve $c_{\max} : I_{\max} \to \mathcal{M}$. Set $c_{\max} = c_p$.

Proof. Let $c_1 : I_1 \to \mathcal{M}, c_2 : I_2 \to \mathcal{M}$ be two integral curves of X through p that is $c_1(0) = c_2(0) = p$. □

3.3 Applications of Mrose-Sards Theorem

3.3.1 Embedding

Theorem 3.3 (Whiteney)**.** *Each $\mathcal{M}^m$ has an embedding into $\mathbb{R}^{2m+1}$.*

Proof. We prove the case for $\mathcal{M}$ compact. Note that if $\mathcal{M}$ is compact and $f : \mathcal{M} \to \mathcal{N}$ is an injective immersion. Then f is an embedding. Since this gives the continuity of the inverse map.

Using Morse-Sard's theorem, $\mathcal{M}^m$ compact, there exists an embedding into some $\mathbb{R}^N$, for $N >> 0$. Indeed, choosing charts $(U_j, \varphi_j)_{j=1,\cdots,r}$, (by the compactness, finitely many would cover the whole). Such that

$$\varphi_j(U_j) \supset B(0, 3),$$

and

$$\mathcal{M} = \bigcup_{j=1}^{r} \varphi^{-1}(B(0, 1)).$$

Fix $g \in \mathcal{C}^\infty(\mathbb{R}^m)$ be such that

$$g(x) = \begin{cases} 1, & (|x| \leq \frac{4}{3}) \\ 0, & (|x| \geq \frac{5}{3}) \end{cases} \quad .$$

Further defining the functions,

$$f_j(p) = \begin{cases} g(\varphi_j(p))\varphi_j(p), & (p \in U_j) \\ 0, & (\text{otherwise}) \end{cases}.$$

Then we have $f_j \in \mathcal{C}^\infty(\mathbb{R}^m)$. In particular

$$f_j|_{\varphi_j^{-1}(B(0,1))}$$

is a diffeomorphism.

Consider the tuple

$$(f_1, \cdots, f_r, \cdots, g \circ \varphi_1, \cdots, g \circ \varphi_r) : \mathcal{M} \to \mathbb{R}^{(m+1)r},$$

is an injective immersion, hence it is an embedding.

Let $\mathcal{M} \subset \mathbb{R}^N$, compact. For $w \in S^{N-1}$, let π_w be the orthogonal projection onto $\langle w \rangle^\perp$ which is a hyperplane of dimension $N-1$ which is given by

$$\pi_w(x) = x - \langle w, x \rangle w.$$

We have

$$\pi_w p = \pi_w q \Leftrightarrow \pi_w(p-q) = 0 \Leftrightarrow p - q \parallel w.$$

We construct a map,

$$\phi : \mathcal{M} \times \mathcal{M} \backslash \{(p,p) \mid p \in \mathcal{M}\} \to S^{N-1}, \phi(p,q) = \frac{p-q}{|p-q|}.$$

If $2m < N-1$, using Morse-Sard's, the image of ϕ is of measure 0. Therefore,

$$\{w \in S^{N-1} \mid \exists p, q \in \mathcal{M}, p \neq q, p - q \parallel w\} = \operatorname{Im}\phi \cup -\operatorname{Im}\phi.$$

The right hand side is a nullset. More explicitly, the set of those w such that $\pi_w|_{\mathcal{M}}$ is not injective is a nullset.

Suppose π_w is an immersion if for $p \in \mathcal{M}$, $v \in T_p\mathcal{M}\backslash\{0\}$, we have $\pi_w(v) \neq 0$.

π_w is an immersion, if

$$\forall p \in \mathcal{M}, w \notin T_p\mathcal{M} \Leftrightarrow \notin \operatorname{Im}\sigma$$

where

$$\sigma : T\mathcal{M}\backslash\{0\} \to S^{N-1}, v \mapsto \frac{v}{|v|}.$$

□

Definition 3.4. *Let $X \in \Gamma(T\mathcal{M})$. We define a flow of X to be*

$$\phi^X : A \to \mathcal{M},$$

where

$$A = \mathcal{D}(\phi^X) := \bigcup_{p \in \mathcal{M}} (a_p, b_p) \times \{p\} \subseteq \mathbb{R} \times \mathcal{M}$$

such that $\phi^X(p)$ is maximal curve through p. $\mathcal{D}(\phi^X)$ is called the flow domain of X.

Proposition 3.1. *Let $X \in \Gamma(T\mathcal{M})$, and $\phi^X : \mathcal{D}(\phi^X) = A \to \mathcal{M}$ be the flow of X. Then the following statements hold.*

1). A is open subset in $\mathbb{R} \times \mathcal{M}$ and contains $\{0\} \times \mathcal{M}$.

2). $\phi^X \in \mathcal{C}^\infty(A)$.

3). For $t \in \mathbb{R}$, $\mathcal{D}(\phi_t^X) = \{p \in \mathcal{M} \mid (t,p) \in A\} \subset \mathcal{M}$ where $\phi_t^X(\cdot) = \phi^X(t,\cdot)$, is open.

4). $\mathcal{M} = \bigcup_{t>0} \mathcal{D}(\phi_t^X)$.

5). $\phi_t^X : \mathcal{D}(\phi_t) \to \mathcal{D}(\phi_{-t}^X)$ is a diffeomorphism.

6). $\phi_s^X \circ \phi_t^X \subseteq \phi_{s+t}^X$ in other words if $p \in \mathcal{D}(\phi_t^X)$ and $\phi_s^X(p) \in \mathcal{D}(\phi_s^X)$ then $p \in \mathcal{D}(\phi_{s+t}^X)$.

7). $\phi_s^X(\phi_t^X(p)) = \phi_{s+t}(p)$.

Proof.
For 6) and 7), let $s, t > 0$, and $p \in \mathcal{D}(\phi_t^X)$, then by assumption, $\phi_t^X(p) \in \mathcal{D}(\phi_{s+t}^X)$. Consider

$$c(u) = \begin{cases} \phi_u(p), & 0 \le u \le t, \\ \phi_{u-t}(\phi_t(p)), & t \le u \le t+s, \end{cases}$$

is an integral curve through p as the two expressions coincide at $u = t$, thus using the uniqueness we have $c = \phi_{s+t}^X$. That is,

$$\phi_{s+t}^X(p) = c(t+s) = \phi_s^X(\phi_t^X(p)).$$

The cases for $s \le 0$ or $t \le 0$ are exercises.

For 2), let $p \in \mathcal{D}(\phi_t^X)$, $t > 0$, we define,

$$B := \{\phi_s^X(p) \mid 0 \le s \le t\}.$$

This is compact. By using Theorem 3.1, there is a neighborhood $W_0 \supseteq B$ and $\varepsilon > 0$ such that

$$(-\varepsilon, \varepsilon) \times W_0 \ni (u,q) \mapsto \phi_u^X(q) \in \mathcal{M}$$

is smooth. Note that we take the maximum of ε from the theorem. This operation is justified by the compactness. Then choose N such that $\frac{t}{N} < \varepsilon$. Define inductively W_j by

$$W_j = \left(\phi_{\frac{t}{N}}^X \Big|_{W_{j-1}} \right)^{-1} (W_{j-1}).$$

Claim 1. *For such (W_j) we have,* *1). $W_0 \supseteq W_1 \supseteq \cdots \supseteq W_N$ and they are all open.*

2). For $q \in W_N$, $\phi^X_{t\frac{N-j}{N}}(q) \in W_j$.

3). For $0 \leq u \leq t\frac{N-j}{N}$, $\phi^X_u(p) \in W_j$. In particular $p \in W_N$.

Proof: The first assertion is clear as $\phi^X_{\frac{t}{N}}|_{W_{j-1}}$ is continuous and the constraint on N.

For 2). $q \in W_N$, $\phi^X_{\frac{t}{N}}(q) \in W_{N-1}$ and inductively $\phi^X_{t\frac{j}{N}}(q) \in W_{N-j}$.

For 3). $\phi^X_u(p) \in B \subset W_0$, $0 \leq u \leq t$.

On the induction step $j \to j+1$, $0 \leq u \leq t\frac{N-j-1}{N}$, then

$$\frac{t}{N} + u \leq t\frac{N-j}{N},$$

so

$$\phi^X_{\frac{t}{N}}(\phi^X_u(p)) \in W_j, \quad \phi^X_u(p) \in W_j \Rightarrow \phi^X_u(p) \in W_{j+1}.$$

Let $q \in W_N$, then $\phi^X_t(q) = (\phi^X_{\frac{t}{N}})^N(q)$, hence $q \in \mathcal{D}(\phi^X_t)$. ■ For 3). Let $p \in \mathcal{D}(\phi^X_t)$, set $c(u) := \phi^X_{t+u}(p)$ where $-t \leq u \leq 0$. This is an integral curve through $\phi^X_t(p)$. Thus $\phi^X_t(p) \in \mathcal{D}(\phi^X_{-t})$ and $\phi^X_{-t} \circ \phi^X_t(p) = p$.

For 1). Let $(t_0, p) \in A$. Using 2). we can find an open neighborhood $\mathcal{D}(\phi^X_{t_0}) \subset W \ni p$. Using Flow-box theorem, there is $(-\varepsilon, \varepsilon) \times \tilde{W}$ open such that

$$(-\varepsilon, \varepsilon) \times \tilde{W} \ni (u, q) \mapsto \phi^X_u(q),$$

is smooth with image contained in W. In particular $p \in \tilde{W}$. Then for $q \in \tilde{W}$ with $|t - t_0| < \varepsilon$,

$$\phi^X_t(q) = \phi_{t_0}(\phi^X_{t-t_0}(q))$$

is smooth. Furthermore, $(t_0 - \varepsilon, t_0 + \varepsilon) \times \tilde{W} \subseteq A$. □

Definition 3.5. *A vector field $X \in \Gamma(T\mathcal{M})$ is called complete (or integrable) if $\mathcal{D}(\phi^X) = \mathbb{R} \times \mathcal{M}$.*

Remark 3.3. *Not all vector fields are complete.*

Example 3.2. *The following vector fields are not complete.*

1). $\mathcal{M} = \mathbb{R}^2 \backslash \{0\}$ and $X(x, y) = \frac{\partial}{\partial x}$.

2). $\mathcal{M} = \mathbb{R}^2$ and $X(x, y) = x^2 \frac{\partial}{\partial x}$. Then we have $\phi^{(1,0)}(t) = \left(\frac{1}{1-t}, 0\right)$.

Proposition 3.2. *Let $X \in \Gamma(T\mathcal{M})$ be a compactly supported vector field. Then X is complete. In particular, if $\mathcal{M}$ is compact, then every vector field is complete.*

Proof. Set $K := \mathrm{supp}(X)$.By the Flow-box theorem, there is an open set $K \subset W \subset \mathcal{M}$ and $\varepsilon > 0$, such that $(-\varepsilon, \varepsilon) \times W \subseteq \mathcal{D}(\phi^X)$. On the other hand, if $p \in \mathcal{M} \backslash K$, then for every $t \in \mathbb{R}$, $\phi_t^X(p) = p$. Henc, there is $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \times \mathcal{M} \subseteq \mathcal{D}(\phi^X)$. Letting $N \in \mathbb{N}$ be large enough such that $\frac{t}{N} < \varepsilon$, and using $\phi_t = (\phi_{\frac{t}{N}})^N$ shows X is integrable. $\square$

Proposition 3.3. *Let $X \in \Gamma(T\mathcal{M})$, and $p \in \mathcal{M}$, and assume $X_p \neq 0$. Then there is a chart (U, φ) centered around p, with $X = \frac{\partial}{\partial x_1}$ on U.*

Proof. Let ψ be a chart with $T_p\psi(X_p) = e_1$. This is possible because if for any chart $\tilde{\psi}$, we find $L \in \mathrm{GL}_m(\mathbb{R})$ such that

$$LT_p\tilde{\psi}(X_p) = e_1.$$

Set $\psi = L \circ \tilde{\psi}$. Let $\varepsilon > 0, \delta > 0$ such that $(-\varepsilon, \varepsilon) \times \psi^{-1}(\overline{B(0, \delta)}) \subseteq \mathcal{D}(\phi^X)$.

Set for $|t| < \varepsilon$, $y \in \mathbb{R}^{m-1}$, $|y| < \varepsilon$.

$$\sigma(t, y) = \phi_t^X(\psi^{-1}(0, y)).$$

Then

$$D(\psi \circ \sigma)(0,0) = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1. \end{pmatrix}$$

where the lower coner is the identity matrix.

$$\begin{aligned} &\partial_t\big|_{(0,0)}\psi \circ \sigma(t, y) = \partial_t\big|_{(0,0)}\psi(\phi_t^X(p)) = T_p\psi(X_p) = e_1, \\ &\partial_y\big|_{0,0)}\psi(\psi^{-1}(0, y)) = I. \end{aligned}$$

So σ is locally invertible and $\varphi = \sigma^{-1}$ is a chart centered at p.

$$\frac{\partial}{\partial t}\sigma(t, y) = \frac{\partial}{\partial t}\phi_t^X(\psi^{-1}(0, y)) = X\big|_{\sigma(t,y)}.$$

So for $q \in \mathcal{D}(\sigma^{-1}) = \sigma((-\varepsilon, \varepsilon) \times B(0, \delta) \subset \mathcal{M})$. Thus,

$$X(q) = X(\sigma(\sigma^{-1}(q))) = T_{\sigma^{-1}(q)}\sigma(e_1) = D\sigma(\sigma^{-1}(q))e_1.$$

$\square$

Remark 3.4. *Given two vector field $X, Y \in \Gamma(T\mathcal{M})$, this is not in general false that*

$$XYf \neq YXf.$$

Furthermore, it can even happen that $XY \notin \Gamma(T\mathcal{M})$.

Definition 3.6. *For $X, Y \in \Gamma(T\mathcal{M})$, and $f \in \mathcal{C}^\infty(\mathcal{M})$, set*

$$[X, Y]f := X(Yf) - Y(Xf).$$

$[X, Y]$ is called the Lie bracket.

Definition 3.7. *An algebra $\mathfrak{g}$ is called a Lie algebra over $\mathbb{R}$ together with Lie bracket $[\cdot, \cdot] : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ if for any $X, Y, Z \in \mathfrak{g}$,*

1. *$[\cdot, \cdot]$ is $\mathbb{R}$-linear.*
2. *$[X, Y] = -[Y, X]$.*
3. *$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$*

Proposition 3.4. *Let $X, Y \in \Gamma(T\mathcal{M})$.*

1. *$[X, Y] \in \Gamma(T\mathcal{M})$.*
2. *$(\Gamma(T\mathcal{M}), [\cdot, \cdot])$ is a Lie algebra.*

Proof. $[\cdot, \cdot]$ is $\mathbb{R}$-linear. consider,

$$\begin{aligned}
[X, Y]|_p(fg) &= X|_p(Y(fg)) - Y|_p(X(fg)),\\
&= X|_p(Yf)g + X|_p(Yg)f - Y|_p(Xf)g - Y|_p(Xg)f,\\
&= X|_p(Yf)g + Yfg(p) + X|_pfYg + fX|_p(Yg) - Y|_p(Xf)g - XfY|_pg - Y|_pfXg - fY|_p(Xg),\\
&= ([X, Y]|_pf)g = f[X, Y]|_pg.
\end{aligned}$$

The rest is an exercise. □

Proposition 3.5. *Let $X \in \Gamma(T\mathcal{M})$, such that $X_p \neq 0$, then there exists a chart (U, φ) centered at p such that*

$$\frac{\partial}{\partial x_1} = X.$$

Definition 3.8. *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism and $X \in \Gamma(T\mathcal{M})$. For $q \in \mathcal{N}$,*

$$(\varphi_* X)(q) := T_{\varphi^{-1}(q)}\varphi[X(\varphi^{-1}(q))],$$

and for $Y \in \Gamma(T, \mathcal{N})$,

$$\varphi^* Y := (\varphi^{-1})_* Y.$$

Definition 3.9. *Let $\varphi : \mathcal{M} \to \mathcal{N}$, be a smooth map and $X \in \Gamma(T\mathcal{M})$ and $Y \in \Gamma(T\mathcal{N})$, we write*

$$X \sim_\varphi Y, \textit{and say they are } \varphi\textit{-related if } Y(\varphi(p)) = T_p\varphi(X(p)).$$

Remark 3.5. *If φ is a diffeomorphism and $X \in \Gamma(T\mathcal{M})$, then clearly,*

$$X \sim_\varphi \varphi_* X.$$

Proposition 3.6 (Naturality of Lie Brackets)**.** *Suppose $X_1, X_2 \in \Gamma(T\mathcal{M})$, $Y_1, Y_2 \in \Gamma(T\mathcal{N})$, and $\varphi : \mathcal{M} \to \mathcal{N}$ be smooth.*

If $X_j \sim_\varphi Y_j$ for each $j = 1, 2$, then $[X_1, X_2] \sim_\varphi [Y_1, Y_2]$.

Proof. Let $p \in \mathcal{M}$ and $f \in \mathcal{C}^\infty(\mathcal{N})$.

$$\begin{aligned} T_p\varphi([X_1, X_2])(f) &= [X_1, X_2]_p(f \circ \varphi), \\ &= X_{1,p}(X_2 f \circ \varphi) - X_{2,p}(X_1 f \circ \varphi), \\ &= X_{1,p}((Y_2 f) \circ \varphi) - X_{2,p}((Y_1 f) \circ f), \\ &= Y_{1,\varphi(p)}(Y_2 f) - Y_{2,\varphi(p)}(Y_1 f), \\ &= [Y_1, Y_2]|_{\varphi(p)} f. \end{aligned}$$

□

Lemma 3.2. *Let $X \in \Gamma(T\mathcal{M})$ and $\phi = \phi^X$ be a flow of X. We have the following statements.*

1). For $f \in \mathcal{C}^\infty(\mathcal{M})$, $\frac{d}{dt}(\phi_t^* f) = \phi_t^*(Xf)$. In particular, $\frac{d}{dt}|_{t=0}\phi_t^* f = Xf$.

2). For $Y \in \Gamma(T\mathcal{M})$, $\frac{d}{dt}\varphi_t^* Y = \phi_t^*([X, Y])$, $\frac{d}{dt}|_{t=0}\phi_t^* Y = [X, Y]$.

3).

Proof. Refer to Lee's book. □

Proposition 3.7. *Let $X, Y \in \Gamma(T\mathcal{M})$. Then the following statements are equivalent.*

1). $[X, Y] = 0$,

2). $\forall t, \phi_t^* Y = Y$,

3). $\forall s, \psi_s^* X = X$,

4). $\psi_s \circ \phi_t = \phi_t \circ \psi_s$.

Proposition 3.8. *Let $X_1, \cdots, X_k \in \Gamma(T\mathcal{M})$ be commuting vector fields (ie, $[X_i, X_j] = 0$ for all $i, j = 1, \cdots, k$). Suppose $X_1(p), \cdots, X_k(p)$ are linearly independent. Then there exists a chart (U, φ) such that*

$$X_j|_U = \frac{\partial}{\partial x_j}$$

for all $j = 1, \cdots k$.

Proof. Since the statement is about open sets, it suffices to show for the case where $\mathcal{M} = \mathbb{R}^n$ and $p = 0$. Consider,

$$\sigma(t, \xi) := \phi^1_{t_1} \circ \cdots \circ \phi^k_{t_k}(0\xi),$$

where $t \in \mathbb{R}^k$ and $\xi \in \mathbb{R}^{m-k}$. □

3.4 Vector Bundles

Definition 3.10. *A (smooth, real) vector bundle over a manifold $\mathcal{M}$ is a triple $(E, \pi, \mathcal{M})$ such that*

1). $E, \mathcal{M}$ are smooth manifolds.

2). $\pi : E \to \mathcal{M}$ is a surjective submersion.

3). For all $p \in \mathcal{M}$, the fiber $\pi^{-1}(\{p\}) = E_p$ is an $\mathbb{R}$-vectorspace with $\dim E_p < \infty$.

Furthermore, this satisfies the axiom of local triviality, namely,

For each $p \in \mathcal{M}$, there exists an open neighborhood $p \in U$ and a diffeomorphism

$$\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^N,$$

such that

$$\forall q \in U, \varphi_{U|_{E_q}} : E_q \to \{q\} \times \mathbb{R}^N,$$

is a linear isomorphism.

E is called the total space of the bundle, $\mathcal{M}$ is the base of the bundle, and φ_U is called the bundle chart.

Definition 3.11. *A system $(\varphi_i)_{i \in I}$ of bundle charts*

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^N,$$

is a bundle atlas if

$$\bigcup_{i \in I} U_i = \mathcal{M}, \left(\textit{or equivalently} \bigcup_{i \in I} \pi^{-1}(U_i) = E \right).$$

Example 3.3. $(E = \mathcal{M} \times \mathbb{R}^N, \pi(p, v) = p, \mathcal{M})$ *is called a trivial vector bundle.*

Definition 3.12. *Let $(E, \pi_E, \mathcal{M})$ and $(F, \pi_F, \mathcal{M})$ be vector bundles. A homomorphism of vector bundles between E and F is a smooth map $\varphi : E \to F$ such that for each point $p \in \mathcal{M}$, $\varphi|_{E_p} : E_p \to F_p$ is a linear map.*

It is called an isomorphism if each $\varphi|_{E_p}$ is a linear isomorphism.

Definition 3.13. *A vector bundle is said to be trivial if it is isomorphic to some trivial bundle $\mathcal{M} \times \mathbb{R}^N$.*

Definition 3.14. *Vector bundles with fibers being $\mathbb{C}$-vectorspaces are then called complex vector bundles.*

Example 3.4. *Consider the total space,*

$$E = \left\{ (z, w) \in \mathbb{C}^2 \cong \mathbb{R}^4 \mid |z| = 1, \frac{w^2}{z} \geq 0 \right\},$$

and the projection map,

$$\pi : E \to S^1, (z, w) \mapsto z.$$

We observe that,

E is a submanifold of $\mathbb{C}^2 \cong \mathbb{R}^4$. Pick $z_0 \in S^1$ and a smooth square root function,

$$\xi : S^1 \backslash \{z_0\} \to S^1, \xi(x_1, x_2) = (\xi_1, \xi_2).$$

Let $w = (x_3, x_4)$ be a point such that $w^2 = \lambda z = \lambda \xi(z)^2$ for $\lambda \geq 0$.

It is equivalent to say that

$$\exists \mu \in \mathbb{R}, w = \mu \xi(z).$$

Reformulate this again, we get,

$$w = (x_3, x_4) \parallel (\xi_1(z), \xi_2(z)) \Leftrightarrow x_3 \xi_2(z) - x_4 \xi_1(z) = 0.$$

Thus we obtain,

$$E \cap \{(z, w) \mid z \in S^1 \backslash \{z_0\}\} = \left\{ (z, w) \in (\mathbb{C} \backslash \mathbb{R}_+ z_0) \times \mathbb{C} \mid {}_{x_3 \xi(x_1, x_2) - x_4 \xi(x_1, x_2) = 0}^{x_1^2 + x_2^2 - 1 = 0} \right\}.$$

Examine the Jacobian of the conditions, we observe

$$\begin{pmatrix} 2x_1 & 2x_2 & 0 & 0 \\ * & * & \xi_2(x_1, x_2) & -\xi_1(x_1, x_2) \end{pmatrix}.$$

We know that at least one of these row is not 0, in other word the matrix is of full rank, thus E is a manifold.

Now we show that $\pi^{-1}(\{z\})$ is a one-dimensional $\mathbb{R}$-vectorspace.

Similar to the previous case, we get $w^2 = \lambda \xi(z)^2$ for all $z \in S^1 \backslash \{z_0\}$. We get $w = \pm \sqrt{\lambda} \xi(z)$. With this observation, we get,

$$\pi^{-1}(\{z\}) = \{(z, \mu \xi(z)) \mid \mu \in \mathbb{R}\}.$$

Bundle charts over $S^1 \backslash \{z_0\}$.

Consider

$$\varphi : \pi^{-1}(S^1 \backslash \{z_0\}) \to (S^1 \backslash \{z_0\}) \times \mathbb{R}, (z, w) \mapsto (z, \frac{w}{\xi(z)}).$$

Finally we show that the bundle $E \xrightarrow{\pi} S^1$ is not trivial.

To derive a contradiction, suppose there exists a bundle isormophism $\varphi : E \to S^1 \times \mathbb{R}$. Consider

$$E \backslash \{(z, 0) \mid z \in S^1\} \overset{\varphi}{\cong} S^1 \times \mathbb{R}^\times$$

which is disconnected.Indeed the map,

$$c : [0, 2\pi] \to E\backslash\{(z, 0) \mid z \in S^1\}, t \mapsto (e^{it}, e^{i\frac{t}{2}}).$$

Observe that

$$c(0) = (1, 1), c(2\pi) = (1, -1).$$

We derived a contradiction that two components get connected by a path.

This turns out to be one of the realization of a Möbius strip.

Construction of vector bundle from local charts Consider $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle and $(U_i, \varphi_i)_{i\in I}$ be a bundle chart. Consider

$$\begin{tikzcd}
(U_i \cap U_j) \times \mathbb{R}^N \arrow[r, "\varphi_j^{-1}"] \arrow[dr, "\mathrm{pr}_1"'] & \pi^{-1}(U_i \cap U_j) \arrow[r, "\varphi_i"] \arrow[d, "\pi"] & (U_i \cap U_j) \times \mathbb{R}^N \arrow[dl, "\mathrm{pr}_1"] \\
& U_i \cap U_j &
\end{tikzcd}$$

where $\mathrm{pr}_1(p, v) = p$. Then we have,

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^N \to (U_i \cap U_j) \times \mathbb{R}^N, \varphi_{ij}(q, v) = (q, g_{ij}(q)v),$$

where

$$g_{ij} \in \mathcal{C}^\infty(U_i \cap U_j, \mathrm{GL}_N(\mathbb{R})).$$

We have the following properties of $(g_{ij})_{ij}$.

i). $g_{ij}g_{jk} = g_{ik}$ over $U_i \cap U_j \cap U_k$.

ii). $g_{ii}(q) = I_N$.

iii). $g_{ji}(q) = g_{ij}(q)^{-1}$.

Such collection $(g_{ij})_{ij}$ is called a co-cycle with values in $\mathrm{GL}_N(\mathbb{R})$.

Remark 3.6. *The last two properties follow from the first one. Indeed the second property is obvious as $\varphi_{ii} = \mathrm{id}_{U_i}$. Combining this with the first one, we derive the third.*

Definition 3.15. *Let $\mathcal{M}$ be a smooth manifold and $(U_i)_{i\in I}$ be an open covering of $\mathcal{M}$ and G be a Lie group(ie. a manifold with group structure such that both multiplication and inversion are smooth as functions). A family $(g_{ij})_{i,j\in I}$ of smooth map such that*

$$g_{ij} : U_i \cap U_j \to G$$

is a cocycle with values in G if

1. $g_{ij} g_{jk} = g_{ik}$ *over* $U_i \cap U_j \cap U_k$.
2. $g_{ii} = e_G$.
3. $g_{ij} = g_{ji}^{-1}$

Definition 3.16. *Suppose we are given a cocycle* $(g_{ij})_{ij}$ *in* $G \subset \mathrm{GL}_N(\mathbb{R})$. *We define a pre-bundle as follows,*

$$E := \bigcup_{i \in I} i \times U_i \times \mathbb{R}^N / \sim,$$

where $(i, p, v) \sim (j, q, w)$ *if*

$$p = q \text{ and } v = g_{ij}(p) \cdot w.$$

We define the projection to be,

$$\pi(([i, p, v])) := p,$$

and the bundle chart,

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^N, [i, p, v] \mapsto (p, v).$$

Remark 3.7. *Observe that this does not have a topology defined yet.*

Definition 3.17. *Let* E, F *be vector bundles over* $\mathcal{M}$. *A smooth map* $\varphi : E \to F$ *is called a bundle isomorphism if*

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & \mathcal{M} & \end{tikzcd}$$

is a commutative diagram and for all $p \in \mathcal{M}, \varphi_p$ *is an isomorphism between* E_p *and* F_p.

Example 3.5. *Let* $\mathcal{M}$ *be a smooth manifold. Then we have*

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} \{p\} \times T_p\mathcal{M}.$$

Let $(U_i, \varphi_i)_{i \in I}$ *be an atlas for* $\mathcal{M}$. *Now define a bundle atlas as follows.*

$$\psi_i : TU_i = \bigcup_{p \in U_i} \{p\} \times T_p\mathcal{M} = \pi^{-1}(U_i) \to U_i \times \mathbb{R}^m, (p, X_p) \mapsto (p, T_p\varphi_i X_p).$$

Note that we have a canonical isomorphism $T_{\varphi_i(p)}\mathbb{R}^m \cong \mathbb{R}^m$.

Given two charts $(U_i, \varphi_i), (V_j, \psi_j)$ *around a point* $p \in \mathcal{M}$, *and* $v \in T_p\mathcal{M}$ *then we have,*

$$v = \sum_{k=1}^{m} \xi_k^i \frac{\partial}{\partial x_k}\bigg|_p = \sum_{k=1}^{m} \xi_k^j \frac{\partial}{\partial y_k}\bigg|_p.$$

Then we have

$$T_p\varphi_i(v) = \begin{pmatrix} \xi_1^i \\ \vdots \\ \xi_m^i \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_m}{\partial y_1} & \cdots & \frac{\partial x_m}{\partial y_m} \end{pmatrix} \begin{pmatrix} \xi_1^j \\ \vdots \\ \xi_m^j \end{pmatrix}$$

That is

$$\begin{aligned} \sum \xi_k^i \frac{\partial}{\partial x_k}\bigg|_p &= \sum \xi_k^j \frac{\partial}{\partial y_k}\bigg|_p, \\ &= \sum_k \left(\sum_l \sum \xi_k^j \frac{\partial x_k}{\partial y_l} \right) \frac{\partial}{\partial x_k}\bigg|_p. \end{aligned}$$

Thus the cocycle is $(g_{kl}(p))_{kl} = \left(\frac{\partial x_k}{\partial y_k}(p)\right)$.

Theorem 3.4. *Given a pre-bundle $(E, \pi, \mathcal{M}, I)$. There exists a unique topology and $\mathcal{C}^\infty$-structure on E such that $(E, \pi, \mathcal{M})$ is a vector bundle.*

Moreover, given a cocycle $(g_{ij} : U_{ij} \to G \subseteq \mathrm{GL}_N(\mathbb{R}))_{i,j\in I}$, where $(U_{ij})_{i,j\in I}$ is an open covering, there is a unique vector bundle $(E, \pi, \mathcal{M})$ up to isomorphismsms with atlas $(U_i, \varphi_i)_{i\in I}$ such that $(g_{ij})_{i,j\in I}$ is the corresponding cocycle.

Proof. Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle with bundle atlas $(\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^N)$. Then for each open set $U \subseteq \mathcal{M}$ we have,

$$\pi^{-1}(U) \cap \pi^{-1}(U_i)$$

is open. Conversely, given a pre-vector bundle $(E, \pi, \mathcal{M}, I)$. Define a topology on E generated by base open sets of the form $\{\pi^{-1}(U) \cap \pi^{-1}(U_i)\}_{i\in I, U\subseteq\mathcal{M}}$, this gives a topology such that π is continuous. Similarly for the smooth structure. Define

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^N,$$

to be such that

$$\varphi_i([(i, p, v)]) = (p, v).$$

This gives a smooth structure to $E \xrightarrow{\pi} \mathcal{M}$.

For the uniqueness, assume we have two vector bundles $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$ with corresponding atlases $(U_i, \varphi_i), (V_i, \psi_i)$, respectively, such that

$$\varphi_i \circ \varphi_j^{-1}(p, v) = \psi_i \circ \psi_j^{-1}(p, v) = (p, g_{ij}(p)v).$$

Define, $\Phi : E \to F$ as follows. For $p \in U_i$ and $\xi \in \mathbb{R}^N$, we put

$$\Phi(p, \xi) = \psi_i \circ \phi_i(p, \xi).$$

This gives us a bundle isomorphism. $\square$

Definition 3.18. *Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle with bundle atlas $(U_i, \varphi_i)_{i\in I}$ and cocycle $(g_{ij})_{i,j\in I}$. The dual vector bundle of E is E^* which is constructed from pre-vector bundle $(E^*, \pi^*, \mathcal{M}, I)$ with cocycle $(g_{ji}^{-1})_{i,j\in I}$ and*

$$E^* = \bigcup_{p\in\mathcal{M}} E_p^*.$$

Definition 3.19. *Let $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$ be vector bundles with cocycle $(g_{ij})_{i,j\in I}, (h_{ij})_{i,j\in I}$ we define their direct sum to be*

$$(E \oplus F)_p = E_p \oplus F_p$$

with cocycle $(g_{ij} \oplus h_{ij})_{i,j\in I}$.

Definition 3.20. *Let $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$ be vector bundles with cocycle $(g_{ij})_{i,j\in I}, (h_{ij})_{i,j\in I}$ we define their tensor product to be*

$$(E \otimes F)_p = E_p \otimes F_p$$

with cocycle $(g_{ij} \otimes h_{ij})_{i,j\in I}$.

Definition 3.21. *Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle. A section of E is a smooth map $f : \mathcal{M} \to E$ such that $\pi \circ f = \mathrm{id}_{\mathcal{M}}$.*

Remark 3.8. *Vector fields are exactly the sections of $T\mathcal{M}$.*

Remark 3.9. *A vector bundle $E \xrightarrow{\pi} \mathcal{M}$ is trivial of rank N if and only if there exists sections $s_1, \cdots, s_N \in \Gamma(E)$ such that for all $p \in \mathcal{M}$, $\langle s_1(p), \cdots, s_N(p)\rangle$ is a basis of E_p.*

Given a bundle atlas $(U_\alpha, \varphi_\alpha)$ with cocycle $(g_{\alpha\beta})$, defne,

$$f_\alpha : U_\alpha \to \mathbb{R}^n, f_\alpha(x) = \mathrm{pr}_2(\varphi_\alpha(f(x))).$$

Of course, we have,

$$f_\alpha = g_{\alpha\beta}(x) f_\beta(x), x \in U_\alpha \cap U_\beta.$$

Conversely, given such f_α, we define,

$$f(x) = [(\alpha, x, f_\alpha(x))].$$

Proposition 3.9. *There is a one-to-one correspondence between,*

$$\{\text{sections } f \text{ of } E\} \leftrightarrow \{(f_\alpha)_{\alpha\in I} | \forall\alpha \in I, f_\alpha \in \mathcal{C}^\infty(U_\alpha, \mathbb{R}^N), f_\alpha(x) = g_{\alpha\beta}(x) f_\beta(x)\}.$$

Proposition 3.10. *Let $E, F \to \mathcal{M}$ be vector bundles. Let $\phi \in \mathrm{Hom}(E, F)$ be surjective. Then there exists $j \in \mathrm{Hom}(F, E)$ such that $\phi \circ j = \mathrm{id}_F$. Furthermore, $\mathrm{Ker}\,\phi$ is a vector bundle over $\mathcal{M}$ and*

$$E \simeq \mathrm{Ker}\,\phi \otimes F.$$

That is to say, in the category of vector bundles of $\mathcal{M}$, a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}\phi \arrow[r, hook] & E \arrow[r, two heads, "\phi"] & F \arrow[l, dotted, bend left, "\exists j"] \arrow[r] & 0 \end{tikzcd}$$

splits.

Proof. We will first show that locally such section exists. Let $p \in \mathcal{M}$. Choose sections of E (by using a chart) $s_1, \cdots, s_n \in \Gamma(E)$ such that

$$\phi(s_1(p)), \cdots, \phi(s_N(p)),$$

is linearly independent. Then there exists an open neighborhood U of p such that for all $q \in U$,

$$\phi(s_1(q)), \cdots, \phi(s_N(q)),$$

is linearly independent. For $w \in F_q|_U$, we can now write,

$$w = \sum \xi_j \phi(s_j(q)).$$

Put

$$j(w) = \sum \xi_j s_j(q).$$

Clearlyk, we have that j is a section of ϕ over U.

For the global case, let $(U_\alpha)_{\alpha \in I}$ be an open covering with sections $j_\alpha : F|_{U_\alpha} \to E|_{U_\alpha}$. Let $(\rho_\alpha)_{\alpha \in I}$ be a subordinated partition of unity. Then,

$$j(p, w) := \left(p, \sum_\alpha \rho_\alpha j_\alpha(w)\right)$$

Now we define $P \in \operatorname{Hom}(E, E)$, such that

$$P(v) := v - j(\varphi(v)),$$

then P is a projection $P^2 = P$ and the image is $\operatorname{Im}(P) = \operatorname{Ker}\phi$.

Fix p, and choose sections $\tilde{s}_1, \cdots, \tilde{s}_r$ of E be such that

$$\tilde{s}_1(p), \cdots \tilde{s}_1(p), \cdots, \tilde{s}_r(p)$$

is a basis of $\operatorname{Ker}\phi|_P$. Set

$$s_j := P \circ \tilde{s}_j.$$

Then s_j are sections of $\operatorname{Ker}\phi$ and in a neighborhoodof p, they are pairwise linearly independent hence a frame. □

Proposition 3.11 (Swan's theorem)**.** *Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle of a compact manifold $\mathcal{M}$. Then there exists a large enough natural number r such that there exists a surjective bundle homomorphism $\phi : \mathcal{M} \times \mathbb{R}^r \twoheadrightarrow E$.*

In parcitular, $E \oplus \operatorname{Ker}\phi \simeq \mathcal{M} \times \mathbb{R}^r$ is trivial and hence there exists sections $s_1, \cdots, s_r \in \Gamma(E)$, such that

$$\forall p \in \mathcal{M}, E_p = \operatorname{Span}(s_1(p), \cdots, s_r(p)).$$

Proof. Choose a finite bundle atlas $(U_1, \varphi_1), \cdots, (U_l, \varphi_l)$ of E (this is justified as $\mathcal{M}$ is compact) and a subordinated partition of unity $\rho_1, \cdots, \rho_l$. We set,

$$s_j^i(p) := \rho_j(p)\varphi_j^{-1}(p, e_j),$$

where $N = \operatorname{rank} E$ and $e_1, \cdots, e_N$ are the canonical basis of $\mathbb{R}^N$. Note that $s_j^i \in \Gamma(E)$.

Given $p \in \mathcal{M}$ there exists j such that $\rho_j(p) \neq 0$. Hence $s_j^i(p), \cdots, s_j^N(p)$ is a basis of E_p, respectively

$$\forall p \in \mathcal{M}, E_p = \operatorname{Span}(s_j^i(p))_{\substack{1 \leq j \leq l \\ 1 \leq i \leq N}}.$$

Re-enumerate them to $s_1, \cdots, s_r \in \Gamma(E)$. Put

$$\phi : \mathcal{M} \times \mathbb{R}^r \to E(p, \xi) \mapsto (p, \sum \xi_j s_j(p)).$$

The rest follows from the previous propostion. $\square$

Definition 3.22. *A R-module P is said to be projective if there exists another R-module Q such that $P \oplus Q$ is free.*

Remark 3.10. *Observe first that the sections $\Gamma(E)$ is a module over $\mathcal{C}^\infty(\mathcal{M})$ with the action defined by*

$$\mathcal{C}^\infty(\mathcal{M}) \times \Gamma(E) \ni (f, s) \mapsto [\mathcal{M} \ni p \mapsto f(p) \cdot s(p)].$$

Proposition 3.11 tells us that $\Gamma(E)$ is finitely generated as a $\mathcal{C}^\infty(\mathcal{M})$ module. Furthermore, $\Gamma(E)$ is projective (ie. a direct summand of a free module).

4 Tensor Algebras and Exteriror Algebras

In this section, we assume that K be a field of characteristic 0 and all vector spaces are finite dimensional.

4.1 Tensors

Let E be a vectorspace. We will denote the dual pairing as $\langle \cdot, \cdot \rangle_{E,E^*}$.

Definition 4.1. *Let $E_1, \cdots, E_r$ be vectorspaces. Recall that a multiliear map $F : E_1 \times \cdots E_r \to F$ is such that, for each $i = 1, \cdots, r$,*

$$E_i \ni a \mapsto F(a_1, \cdots, a_{i-1}, a, a_{i+1}, \cdots, a_r)$$

is linear. We denote,

$$L(E_1, \cdots, E_r; F) = \{E_1 \times \cdots \times E_r \to F \mid r\text{-multilinear maps}\}.$$

Remark 4.1.

$$\dim L(E_1, \cdots, E_r; F) = \left(\prod_{i=1}^{r} \dim E_i \right) \times \dim F.$$

Definition 4.2. *Let $\varphi \in E^*$ and $\psi \in F^*$. For $e \in E, f \in F$, we define,*

$$(\varphi \otimes \psi)(e, f) := \varphi(e)\psi(f)$$

Remark 4.2. *Obvisouly, $\varphi \otimes \psi \in L(E, F; K)$.*

Remark 4.3. *A map $\cdot \otimes \cdot$ is a bilinear map $E^* \times F^* \to L(E, F; K)$.*

Proposition 4.1. *Let $(e_i)_{i \in I}, (f_j)_{j \in J}$ be bases of E, F. Then we have,*

$$\varphi \otimes \psi = \sum \varphi(e_i)\psi(f_j) e_i^* \otimes f_j^*.$$

By definition, we have,

$$e_i^* \otimes f_j^*(e_k, f_l) = \delta_{ik}\delta_{jl}.$$

In particular, $(e_i^* \otimes f_j^*)_{(i,j) \in I \times J}$ is a basis of $L(E, F; K)$.

Definition 4.3. *We define,*

$$L(E_1, \cdots, E_r, K) \times L(F_1, \cdots, F_s; K) \to L(E_1, \cdots, E_r, F_1, \cdots, F_s; K)$$

by,

$$\varphi \otimes \psi(e_1, \cdots, e_r, f_1, \cdots, f_s) := \varphi(e_1, \cdots, e_r)\psi(f_1, \cdots, f_s).$$

Proposition 4.2. *The map defined in Definition 4.3, a bilinear form which is associative (ie. $(\varphi \otimes \psi) \otimes \chi = \varphi \otimes (\psi \otimes \chi)$).*

Proof. Bilinearlity is clear from the construction. So is the associativity as $(\varphi \otimes \psi) \otimes \chi = \varphi \otimes \psi \otimes \chi$. $\square$

Notation 4.1. *We denote*

$$E^* \otimes F^* := L(E, F; K),$$

and

$$E \otimes F \cong E^{**} \otimes F^{**} = L(E^*, F^*, K).$$

Proposition 4.3 (Universality)**.** *The assingment $\otimes : E \times F \to E \otimes F$ solves the following universal property.*

Let G be a vectorspace and $h : E \times F \to G$ be a bilinear map. Then there exists a unique linear map $\overline{h} : E \otimes F \to G$ making the diagram commutative.

$$\begin{tikzcd}
E \times F \arrow[r, "h"] \arrow[d, "\otimes"'] & G \\
E \otimes F \arrow[ur, "\exists! \overline{h}"'] &
\end{tikzcd}$$

Furthermore, the map

$$L(E, F; G) \ni h \mapsto \overline{h} \in L(E \otimes F; G),$$

is an isomorphism.

Proof. Check at the level of basis elements. $\square$

Definition 4.4. *Let E be a vectorspace, we will define,*

$$T^r E = \underbrace{E \otimes \cdots \otimes E}_{r \ times},$$

and

$$T(E) = \bigoplus_{r=0}^{\infty} T^r(E),$$

which is called the tensor algebra over E. Furthermore, Setting $\otimes$ as a multiplication of elements, $T(E)$ is a positively graded K-algebra.

Remark 4.4. *Obviously, a basis of $T^r E$ is given by $(e_{i_1} \otimes \cdots \otimes e_{i_r})_{1 \le i_1 \le \cdots \le i_r \le n}$. Thus we have $\dim T^r E = (\dim E)^r$.*

Lemma 4.1. *The operation $\otimes$ is associative.*

$$\begin{tikzcd}
T^r E \times T^s E \arrow[r, "(v_1 \otimes \cdots \otimes v_r, w_1 \otimes \cdots \otimes w_s) \mapsto (v_1 \otimes \cdots \otimes v_r \otimes w_1 \cdots \otimes w_s)"] \arrow[d] & T^{r+s} E \\
T^r E \otimes T^s E \arrow[ur] &
\end{tikzcd}$$

4.2 Totally antisymmetric(alternating) tensors

Remark 4.5.

$$T^r E \otimes T^r E^* \to K,$$

where

$$(v_1 \otimes \cdots \otimes v_r, v_1^* \otimes \cdots \otimes v_r^*) \mapsto \prod_{i=1}^{r} \langle v_i, v_i^* \rangle$$

is an dual pariing.

Let $\sigma \in S_r$ be the permutations of $\{1, \cdots, r\}$. We have,

$$\begin{tikzcd}
E^r \arrow[r, "\sigma"] \arrow[d, "\otimes"'] & E^r \arrow[d, "\otimes"] \\
T^r E \arrow[r, "T^r \sigma"'] & T^r E
\end{tikzcd}$$

where $T^r\sigma$ is a linear map. Observe that

$$(T^r\sigma)^{-1} = T^r\sigma^{-1}.$$

Thus it is an isomorphism. Explicitly, we have,

$$T^r\sigma(v_1 \otimes \cdots \otimes v_r) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

Notation 4.2. $\operatorname{sgn}(\sigma) = (-1)^\sigma$.

Definition 4.5. *$T \in T^rE$ is called alternating or totally antisymmetric if for all $\sigma \in S_r$, we have,*

$$T^r\sigma T = (-1)^\sigma T.$$

Remark 4.6. *To check T is alternating, it suffices to check that $T^r\sigma T = -T$ for every transposition $\tau \in S_r$.*

Notation 4.3. *We denote the space of alternating tensors as,*

$$\bigwedge^r E \subseteq T^rE.$$

Definition 4.6. *An anti-symmetrization map is ,*

$$A_r : T^r \to T^r, A \mapsto \frac{1}{r!}\sum_{\sigma\in S_r}(-1)^\sigma(T^r\sigma)(T).$$

Remark 4.7. *A_r is a linear map.*

Lemma 4.2. *$A_r \circ A_r = \mathrm{id}$ In other words, it is idempotent.*

Proof. Let $T \in T^rE$ and fix $\tau \in S_r$. We have,

$$(T^r\tau)A_r(T) = \frac{1}{r!}\sum_{\sigma\in S_r}(-1)^\sigma T^r(\tau\sigma)(T)(-1)^\tau(-1)^\tau = (-1)^\tau A_r(T).$$

In particular, the range of A_r is contained in $\bigwedge^r E$. If $T \in \bigwedge^r E$, we have,

$$A_r(T) = \frac{1}{r!}\sum_{\sigma\in S_r} T^r\sigma T = T.$$

Thus we have the range of A_r is exactly $\bigwedge^r E$. □

4.3 Exterior Algebra

Notation 4.4.

$$\bigwedge E = \bigoplus_{r\in\mathbb{Z}_{\geq 0}} \bigwedge^r E.$$

Definition 4.7. *Let $\psi : \mathbb{N} \to \mathbb{Z}_+$ be a sequence such that $\psi(0) = \psi(1) = 1$. We define,*

$$\wedge_\psi : \bigwedge^p E \times \bigwedge^q E \to \bigwedge^{p+q} E, \xi \wedge_\psi \eta := \frac{\psi(p+q)}{\psi(p)\psi(q)} A_{p+q}(\xi \otimes \eta).$$

Definition 4.8. *Let A be a graded ring. A mapping $(\cdot,\cdot) : A \to A$ is graded commutative if for any $a \in A^p, b \in A^q$, we have,*

$$(a, b) = (-1)^{pq}(b, a).$$

Proposition 4.4. *The map $\wedge = \wedge_\psi$ is bilinear, associated and graded commutative. Furthermore, for $v_1, \cdots, v_r \in E = \bigwedge^1 E$,*

$$v_1 \wedge \cdots \wedge v_r = \frac{\psi(r)}{r!} \sum_{\sigma \in S_r} \operatorname{sgn} \sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

From above, we conclude,

$$(e_{i_1} \otimes \cdots \otimes e_{i_r})_{1 \le i_1 < \cdots < i_r \le \dim E}$$

is a basis of $\bigwedge^r E$. Thus $\dim \bigwedge^r E = \binom{\dim E}{r}$.

Proof. Let $\xi \in \bigwedge^p E, \eta \in \bigwedge^q E, \chi \in \bigwedge^r E$ and $\tau \in S_{p+q}$ such that

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & p & p+1 & \cdots & p+q \\ q+1 & q+2 & \cdots & p+q & 1 & \cdots & q. \end{pmatrix}$$

Observe that

$$\operatorname{sgn}(\tau) = (-1)^{pq}, \tau(\xi \otimes \eta) = \eta \otimes \xi.$$

Thus we have,

$$A_{p+q}(\xi \otimes \eta) = (-1)^\tau A_{p+q}(\eta \otimes \xi).$$

We then obtain,

$$\xi \wedge \eta = (-1)^\tau \eta \wedge \xi = (-1)^{pq} \eta \wedge \xi.$$

We derived the graded-commutativity.

For the associativity, we have,

$$\begin{aligned}
(\xi \wedge \eta) \wedge \chi &= \frac{\psi(p+q+r)}{\psi(p+q)\psi(r)} A_{p+q+r}((\xi \wedge \eta) \otimes \chi), \\
&= \frac{\psi(p+q+r)}{\psi(p+q)\psi(r)} \frac{\psi(p+q)}{\psi(p)\psi(q)} \frac{1}{(p+q+r)!} \frac{1}{(p+q)!} \sum_{\tau \in S_{p+q}} \operatorname{sgn} \tau \\
&\sum_{\sigma \in S_{p+q+r}} \operatorname{sgn}(\sigma \circ (\tau \times \mathrm{id}))(\sigma \circ (\tau \times \mathrm{id}))((\xi \wedge \eta) \otimes \chi), \\
&= \frac{\psi(p+q+r)}{\psi(p)\psi(q)\psi(r)} \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q+r}} \operatorname{sgn} \sigma \sigma(\xi \otimes \eta \otimes \chi), \\
&= \xi \wedge (\eta \wedge \chi).
\end{aligned}$$

By induction, for

$$\xi_1 \in \bigwedge^{p_1}, \cdots, \xi_r \in \bigwedge^{p_r} E,$$

we see,

$$\xi_1 \wedge \cdots \wedge \xi_r = \frac{\psi(p_1 + \cdots + p_r)}{\psi(p_1) \cdots \psi(p_r)} A_{p_1 + \cdots + p_r}(\xi_1 \otimes \cdots \otimes \xi_r).$$

Take $p_1 = \cdots = p_r = 1$, we obtain the formula.

□

Corollary 4.1. *Keeping the notation from the proposition above, we have,*

$$\langle v_1 \wedge \cdots \wedge v_r, v_1^* \otimes \cdots \otimes v_r^* \rangle_{T^r E, T^r E^*} = \frac{\psi(r)}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i,j \leq r}),$$

and

$$\langle v_1 \wedge \cdots \wedge v_r, v_1^* \wedge \cdots \wedge v_r^* \rangle_{T^r E, T^r E^*} = \frac{\psi(r)^2}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i,j \leq r}),$$

Proof.

$$\begin{aligned}
\langle v_1 \wedge \cdots \wedge v_r, v_1^* \otimes \cdots \otimes v_r^* \rangle_{T^r E, T^r E^*} &= \sum_{\sigma \in S_r} \operatorname{sgn} \sigma \langle v_{\sigma(1)} v_1^* \rangle \cdots \langle v_{\sigma(r)} v_r^* \rangle, \\
&= \frac{\psi(r)}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i,j \leq r}).
\end{aligned}$$

$$\begin{aligned}
\langle v_1 \wedge \cdots \wedge v_r, v_1^* \wedge \cdots \wedge v_r^* \rangle_{T^r E, T^r E^*} &= \frac{\psi(r)}{r!} \sum_{\tau \in S_r} \operatorname{sgn} \tau \langle v_1 \wedge \cdots \wedge v_r, v_{\tau(1)}^* \otimes \cdots \otimes v_{\tau(r)}^* \rangle_{T^r E, T^r E^*}, \\
&= \frac{\psi(r)}{r!} \sum_{\tau \in S_r} \operatorname{sgn} \tau \langle v_{\tau(1)} \wedge \cdots \wedge v_{\tau(r)}, v_1^* \otimes \cdots \otimes v_r^* \rangle_{T^r E, T^r E^*}, \\
&= \frac{\psi(r)}{(r!)^2} \det((\langle v_i, v_j^* \rangle)_{1 \leq i,j \leq r}).
\end{aligned}$$

□

We will follow the convention that $\psi \equiv 1$ which is used in [2].

Definition 4.9. *For $\xi \in \bigwedge^p E$, we define,*

$$\operatorname{ext}_\xi : \bigwedge E \to \bigwedge E, \eta \mapsto \xi \wedge \eta.$$

Respectively for $u \in E$, we define,

$$\operatorname{int}_U : \bigwedge E^* \mapsto \bigwedge E^*, \omega \mapsto \langle \omega, u \otimes \cdot \rangle = \langle \omega, u \wedge \cdot \rangle$$

Remark 4.8. *For $\eta \in \bigwedge E$ and $\omega \in \bigwedge E^*$, we have,*

$$\langle \operatorname{ext}_u \eta, \omega \rangle = \langle u \wedge \eta, \omega \rangle = \langle \eta, \operatorname{int}_U \omega \rangle. \tag{2}$$

Lemma 4.3. *Suppose $\omega, \eta \in \bigwedge E^*$ are homogeneous elements. Then we have,*

$$\operatorname{int}_u(\omega \wedge \eta) = (\operatorname{int}_u \omega) \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge \operatorname{int}_u \eta.$$

In other words, int_u *is an anti-derivation.*

5 Orientations, volumes and the Hodge $*$-operator

5.1 Lines

In this section, we denote $K = \mathbb{R}$.

Definition 5.1. *A line is a one-dimensional $\mathbb{R}$-vector space.*

Definition 5.2. *Fix $\alpha > 0$. An α-density on a n-dimensional vector space E is a map $\rho : \bigwedge^n E^* \to \mathbb{R}$ such that*

$$\forall \omega \in \bigwedge^n E^*, \lambda \in \mathbb{R}, \rho(\lambda, \omega) = |\lambda|^\alpha \rho(\omega).$$

We will denote $|\bigwedge|^\alpha E^*$ to be the vector space of all α-densities.

Definition 5.3. *A signed density on a n-dimensional vector space E is a map $\rho : \bigwedge^n E^* \to \mathbb{R}$ such that*

$$\forall \omega \in \bigwedge^n E^*, \rho(\lambda, \omega) = \operatorname{sgn} \lambda \rho(\omega).$$

We also define, the set of sgined densities as, $\mathcal{O}(E)$ which we also call the orientation line.

Remark 5.1. *We have a map,*

$$(E^*)^n \to \bigwedge^n E.$$

With this we can define an α-density by,

$$\rho : (E^*)^n \to \mathbb{R}, \rho(\varphi(v_1^*), \cdots, \varphi(v_n^*)) = |\det \varphi|^\alpha \rho(v_1^*, \cdots, v_n^*),$$

for every $\varphi \in \mathrm{GL}(E)$. In particular, we have,

$$\varphi(v_1) \wedge \cdots \wedge \varphi(v_n) = (\det \varphi) v_1 \wedge \cdots \wedge v_n.$$

Similarly for the signed density case we put $\operatorname{sgn}(\det \varphi)$.

Remark 5.2.

$$\dim \left|\bigwedge\right| E = \dim \mathcal{O}(E) = 1.$$

Given $\xi \in \bigwedge^n E\backslash\{0\}$, we have,

$$\rho_\xi^\alpha(\omega) := |\xi(\omega)|^\alpha$$

which is an α-density. Also,

$$\rho_\xi^\alpha(\omega) := \operatorname{sgn}(\langle \xi, \omega\rangle)$$

which is a signed density. If ρ is an α-density,

$$\rho(\lambda\omega) = |\lambda|^\alpha \rho(\omega) = |\lambda|^\alpha \frac{\rho(\omega)}{\rho_\xi^\alpha(\omega)} \rho_\xi^\alpha(\omega) = \frac{\rho(\omega)}{\rho_\xi^\alpha(\omega)} \rho_\xi^\alpha(\lambda\omega).$$

Lemma 5.1. *THere are canonical isomorphisms between,*

$$\left|\bigwedge\right|^\alpha E \otimes \left|\bigwedge\right|^\beta E \rho_1 \otimes \rho_2 \mapsto \rho_1\rho_2 \left|\bigwedge\right|^{\alpha+\beta} E.$$

$$\left|\bigwedge\right|^1 E \otimes \mathcal{O} \to \bigwedge^n E$$

$$\bigwedge^n E \to \left|\bigwedge\right|^1 E \otimes \mathcal{O},$$

and

$$\mathcal{O} \otimes \mathcal{O} \to \mathbb{R}.$$

Furthermore, $\mathcal{O}$ has a canonical Euclidean metric, namely, for $\rho \in \mathcal{O}$, we have,

$$|\rho| := \sqrt{\rho(\omega)^2}, \forall \omega \neq 0.$$

Note that this is independent of choice of $\omega \neq 0$ since ρ is a signed-density and the square root will take the sign out.

5.2 Orientations

Definition 5.4. *An orientation of E is given by a choice of a vector $\mathfrak{o} \in \mathcal{O}$ of unit length.*

Remark 5.3. *We have an isometry by,*

$$\mathcal{O} \ni \mathfrak{o} \mapsto 1 \in \mathbb{R}.$$

Remark 5.4. *We have an isomorphism such that,*

$$\bigwedge^n E \ni \rho \mapsto \rho \cdot \mathfrak{o} \in \left|\bigwedge\right|^1 E.$$

Definition 5.5. *Given a connected component ${\bigwedge^n}_+ E$ of $\bigwedge^n E\backslash\{0\}$. A basis $e_1, \cdots, e_n$ of E is called oriented if*

$$e_1 \wedge \cdots \wedge e_n \in \bigwedge^n{}_+ E.$$

Or equivalently, we have,

$$e_1 \wedge \cdots \wedge e_n = \mathfrak{o} \cdot |e_1 \wedge \cdots \wedge e_n|.$$

5.3 Volume elements

Let g be a symmetric bilinear form on E. Then g induces bilinear forms on TE and $\bigwedge E$ as follows.

$$(T^r g)(v_1 \otimes \cdots \otimes v_r, w_1 \otimes \cdots \otimes w_r) := \prod_{j=1}^{r} g(v_j, w_j).$$

Note that $\bigwedge E \subseteq TE$. Thus we have,

$$\begin{aligned}(T^r g)(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r) &:= \frac{1}{(r!)^2} \sum_{\sigma,\tau \in S_r} \operatorname{sgn} \sigma\tau \prod_{j=1}^{r} g(v_{\sigma(j)}, v_{\tau(j)}),\\ &= \frac{1}{(r!)^2} \sum_{\sigma \in S_r} \det((g(v_i, w_j))_{1 \leq i,j \leq r}),\\ &= \frac{1}{r!} \det((g(v_i, w_j))_{1 \leq i,j \leq r}).\end{aligned}$$

Lemma 5.2. *If g is non-degenerate, so are $T^r g, \bigwedge^r g$. The latter is the restriction of $T^r g$ to $\bigwedge^r E \times \bigwedge^r E$. Also we have an isomorphism*

$$(\cdot)^b : E \to E^*, v \mapsto g(v_j) = v^b,$$

which inverse $\#$.

g induces a non-degenerate bilienar form g^* on E^*, such that

$$g^*(v^*, w^*) := g(\#v^*, \#w^*).$$

Proof. Let $e_1, \cdots, e_n$ be a g-oriented basis of E that is $g(e_i, e_j) = \pm\delta_{ij}$.

$$(T^r g)(e_{i_1} \otimes \cdots \otimes e_{i_r}, e_{j_1} \otimes \cdots \otimes e_{j_r}) = \prod_{k=1}^{r} \pm\delta_{i_1 j_1}.$$

And we also have,

$$\left(\bigwedge^r g\right) = (e_{i_1} \wedge \cdots \wedge e_{i_r}, e_{j_1} \wedge \cdots \wedge e_{j_r}) = \frac{\pm 1}{r!} \prod_{k=1}^{r} \delta_{i_1 j_1}.$$

Remark 5.5. *Induced bases are orthogonal with respect to the induced forms. Therefore, they are non-degenerate.*

□

Proposition 5.1. *Let g be a non-degenerate bilinear form on E. Then g determines uniquely a positive 1-density $\operatorname{vol}_g \in |\bigwedge|^1 E$.*

If $e_1, \cdots, e_n$ is a basis of E, then

$$\operatorname{vol}_g = |\det((g(e_i, e_j))_{1 \leq i,j \leq n})^{\frac{1}{2}} |e_1 \wedge \cdots \wedge e_n|.$$

Furthermore, g induces for every $\alpha > 0$, a canonical isomorphism,

$$\left|\bigwedge\right|^{\alpha} E \to \mathbb{R}, \lambda \cdot |\operatorname{vol}_g|^{\alpha} \mapsto \lambda.$$

Proof. Let us first prove that this is independent of the choice of bases. Let $f_1, \cdots, f_n$ be another basis. We can write,

$$f_i = \sum a_{ij} e_j.$$

Thus we have,

$$g(f_i, f_j) = \sum_{k,l} a_{ki} a_{lj} g(e_k, e_l).$$

Therefore, we have,

$$\det(g(f_i, f_j)) =$$

□

5.4 The Hodge $*$-operator

Let q be a non-degenerate biliear form on E where E is a real vector space of dimension n. Let $\mathrm{vol}_g \in |\bigwedge|^1 E$.

Let $\mathcal{O}(E)$ be the orientation line and recall

Definition 5.6. *Let g be a non-degenerate bilinear form on a vector space V. A basis $\{e_1, \cdots, e_n\}$ of V is said to be g-orthonormal if we have,*

$$g(e_i, e_j) = \pm\delta_{ij}.$$

Definition 5.7. *Let g be a non-degenerate biliear form on V and $\{e_1, \cdots, e_n\}$ be a g-orthonormal basis of V, we define the index of g to be such that*

$$\operatorname{ind} g = |\{i \mid g(e_i, e_i) = -1\}|.$$

Notation 5.1. *Let L be a one-dimensional $\mathbb{R}$-vector space and $f, g \in L \backslash \{0\}$. We have $f = \lambda g$ for some $\lambda \in \mathbb{R}$. We denote,*

$$f/g = \lambda.$$

Definition 5.8. *Let E be a n-dimensional $\mathbb{R}$-vectorspace. The Hodge $*$-operator is a map*

$$*_p : \bigwedge^p E \to \bigwedge^{n-p} E \otimes \mathcal{O},$$

satisfying the following properties.

i). It is a linear isomorphism.

ii). For g-orthonormal basis $\{e_1, \cdots, e_n\}$ of E and $\sigma \in S_n$, we have

$$*e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(p)} = (-1)^{\operatorname{sgn}\sigma} \prod_{i=1}^{p} g(e_{\sigma(i)}, e_{\sigma(i)}) e_{\sigma(p+1)} \wedge \cdots \wedge e_{\sigma(n)} \otimes e_1 \wedge \cdots \wedge e_n / \mathrm{vol}_g\,.$$

iii). $*_{n-p} *_p = (-1)^{p(n-p)+\operatorname{ind} g}$,

iv). *For* $\omega, \eta \in \bigwedge^p E$, *we have,*

$$\left(\bigwedge^{n-p} g\right)(*\omega, *\eta) = (-1)^{\operatorname{ind} g} \left(\bigwedge^{p} g\right)(\omega, \eta).$$

v). *Let* $\omega \in \bigwedge^p E, \eta \in \bigwedge^{n-p} E \otimes \mathcal{O}$, *then*

$$\omega \wedge \eta = (-1)^{p(n-p)+\operatorname{ind} g} (g)(\omega, *\eta) \cdot \operatorname{vol}_g .$$

Theorem 5.1. *Let* E *be a* $\mathbb{R}$*-vector space of dimension* n *and* g *be a non-degenerate bilinear form on* E. *Pick* $\operatorname{vol}_g \in \left|\bigwedge^1\right|^1 E$. *We have, there exists a unique bilinear pairing such that*

$$\bigwedge^p E \times \left(\bigwedge^{n-p} E \otimes \mathcal{O}\right) \ni (\omega, \eta) \mapsto \omega \wedge \eta / \operatorname{vol}_g \in \mathbb{R}.$$

In particular for any $\eta \in \wedge^p E$ *there is a unique* $\star\eta \in \bigwedge^{n-p} E \otimes \mathcal{O}$ *such that for any* $\omega \in \bigwedge^p E$,

$$\omega \wedge \star\eta = \bigwedge^p g(\omega, \eta) \operatorname{vol}_g .$$

And such $\star$ *is the Hodge* $\star$*-operator* $\star_p$.

Proof. The composition,

$$(\omega, \eta) \mapsto \omega \wedge \eta \mapsto \omega \wedge \eta / \operatorname{vol}_g \in \mathbb{R}$$

is bilinear. Note that

$$\det(g(e_i, e_j))_{ij} = (-1)^{\operatorname{ind} g}.$$

Let $I = \{i_1, \cdots, i_p\} \subseteq \{1, \cdots, n\}$. Then we denote,

$$e_I := e_{i_1} \wedge \cdots \wedge e_{i_p}.$$

Note that an element $\omega \in \bigwedge^p E$ is of the form,

$$\omega = \sum_{\substack{I \subseteq \{1, \cdots, n\} \\ |I| = p}} \omega_I e_I, (\omega_I \in \mathbb{R}).$$

Fix $I_0 \subseteq \{1, \cdots, n\}$ and denote $I_0^c = \{1, \cdots, n\} \backslash I_0$. Using the expression of ω above, we have,

$$\omega \wedge e_{I_0^c} \otimes \underbrace{e_1 \wedge \cdots \wedge e_n}_{\operatorname{vol}_g} = \pm \omega_{I_0} |\underbrace{e_1 \wedge \cdots \wedge e_n}_{\operatorname{vol}_g}|.$$

This shows the non-degeneracy. The existence of $*\eta$ thus follows from non-degeneracy.

We now show that $\star$ satisfies all the properties of the Hodge operator.

ii). Set $\omega = e_I$ for some $\{i_1, \cdots, i_p\} \subseteq \{1, \cdots, n\}$. Then

$$\mathfrak{o} = \left(\bigwedge^p g\right)(\omega, e_{I^c}).$$

We have,

$$\begin{aligned}
\omega \wedge e_{I^c} &= (-1)^{\operatorname{sgn}\sigma} \prod_{i\in I} g(e_i, e_i) \frac{e_1 \wedge \cdots \wedge e_n}{|e_1 \wedge \cdots \wedge e_n|}, \\
&= (-1)^{\operatorname{sgn}\sigma} e_1 \wedge \cdots \wedge e_n (-1)^{\operatorname{sgn}\sigma} \prod_{i\in I} g(e_i, e_i) \frac{e_1 \wedge \cdots \wedge e_n}{|e_1 \wedge \cdots \wedge e_n|}, \\
&= \prod_{i\in I} g(e_i, e_i) |e_1 \wedge \cdots \wedge e_n|, \\
&= \left(\bigwedge^p g\right)(e_I, e_I) |e_1 \wedge \cdots \wedge e_n|.
\end{aligned}$$

That is

$$\star e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(p)} = \operatorname{sgn}\sigma e_1 \wedge \cdots \wedge e_n (-1)^{\operatorname{sgn}\sigma} \prod_{i\in I} g(e_i, e_i) e_{I^c} \frac{e_1 \wedge \cdots \wedge e_n}{|e_1 \wedge \cdots \wedge e_n|}.$$

iii). Applying the second propert twice, and consider $\tau \in S_n$ such that

$$\tau = \begin{pmatrix} 1 & \cdots & n-p & n-p+1 & \cdots n & \\ p+1 & \cdots & n & 1 & \cdots & p \end{pmatrix}$$

Then we have $\operatorname{sgn}\tau = (-1)^{p(n-p)}$. Rewrite e_{I^c} with τ, we obtain,

$$\star_{n-p} \star_p e_I = (-1)^{\operatorname{ind} g} \operatorname{sgn}\sigma \operatorname{sgn}(\tau) e_{\tau(I^c)} = (-1)^{p(n-p)+\operatorname{ind} g} e_I.$$

To show it satisfies the remaining properties is left to the readers as an exercise.

□

5.5 Tensor fields and differential forms

Remark 5.6. *Given a manifold $\mathcal{M}$, we have a vectorbundle $T\mathcal{M} \xrightarrow{\pi} \mathcal{M}, T_p\mathcal{M} \mapsto p$. Let us denote $\{g_{ij}\}_{ij}$ be the corresponding cocycle, then we can define, $\bigwedge^p g_{ij}$ which corresponds to a vectorbundle $\bigwedge^p E$.*

Definition 5.9. *Let $\mathcal{M}$ be a smooth manifold. We define,*

$$T^{r,s}\mathcal{M} := \bigotimes^r T\mathcal{M} \otimes \bigotimes^s T^*\mathcal{M}.$$

Sections of $T^{r,s}\mathcal{M}$ are called the tensorfield of type (r, s). We denote,

$$\Gamma(T^{r,s}\mathcal{M}) = \{\textit{smooth sections of } T^{r,s}\mathcal{M}\}.$$

Proposition 5.2. *For a map $f : \mathcal{M} \to T^*\mathcal{M}$ with*

$$\forall p \in \mathcal{M}, f(p) \in T_p^{r,s}\mathcal{M},$$

the following are equivalent.

1). for vector fields, $X_1, \cdots, X_n \in \Gamma(T\mathcal{M})$, and covector field $\omega_1, \cdots, \omega_r \in \Gamma(T\mathcal{M})$, the function,

$$p \mapsto f(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s) = f(p)(\omega_1(p), \cdots, \omega_r(p), X_1(p), \cdots, X_s(p)),$$

is smooth.

2). for any charts (U, φ) with coordiate functions $x_1, \cdots, x_m$, we have,

$$f = \sum f_{j_1 \cdots j_s}^{i_1 \cdots i_r} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_s}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s},$$

where $dx_1, \cdots, dx_m$ are the dual (missing) to $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}$ with smooth functions $f_{j_1, \cdots, j_s}^{i_1, \cdots, i_r} \in \mathcal{C}^\infty(U)$.

3). $f \in \Gamma(T^{r,s}\mathcal{M})$.

Proof. Exercise. $\square$

Definition 5.10. *THe sections of the bundle $\bigwedge^p T^*\mathcal{M} \subseteq T^{0,p}\mathcal{M}$, are called differential forms. The set of differential forms are denoted by*

$$\Omega^p(\mathcal{M}) = \Gamma\left(\bigwedge^p T^*\mathcal{M}\right).$$

Proposition 5.3. *Let $f : \Gamma(T^*\mathcal{M})^r \times \Gamma(T\mathcal{M})^s \to \mathcal{C}^\infty(\mathcal{M})$, is indueced by a section of $T^{r,s}\mathcal{M}$ if and onlly if it is $\mathcal{C}^\infty(\mathcal{M})$-$(r-s)$ multilinear.*

Proof. If $f \in \Gamma(T^{r,s}, \mathcal{M})$ then

$$(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s) \mapsto (p \mapsto f(p(\omega_1(p), \cdots, \omega_r(p), X_1(p), \cdots, X_s(p)))),$$

is clearly $\mathcal{C}^\infty(\mathcal{M})$-$(r+s)$-multilinear.

Conversely, given $\mathcal{C}^\infty(\mathcal{M})$-multilinear map f, need to find $f(p)$. To do so we will show that for $p \in \mathcal{M}$, if there is i such that $\omega_i(p) = 0$ or there is j such that $X_j(p) = 0$, we have,

$$f(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s)(p) = 0.$$

Indeed, we assume without loss of generality that $\omega_1(p) = 0$. Choose a chart (U, φ) centered around p. Choose $h \in \mathcal{C}^\infty(\mathcal{M})$ such that $h \equiv 1$ in small enough neighborhood of p. By assumption, we have,

$$\omega_1|_U = \sum_{j=1}^m a_j dx_j, a_j(p) = 0.$$

Combining these, we obtain,

$$\begin{aligned}\omega_1 &= (1-h^2)\omega_1 + h^2\omega_1,\\ &= (1-h^2)\omega_1 + \sum_{j=1}^{m} \underbrace{(a_j h)}_{\in \mathcal{C}^\infty(\mathcal{M})} \underbrace{(h dx_j)}_{\in \Gamma(T^*\mathcal{M})}.\end{aligned}$$

Using multilinearlity of f, we get,

$$f(\omega_1, \cdots)(p) = (1-h^2)(p) f(\omega_1, \cdots)(p) + \sum_{j=1}^{m} \underbrace{(ha_j)(p)}_{=0} f(h dx_j, \cdots)(p) = 0.$$

Define $f(p)$ as follows. For $\theta_1, \cdots, \theta_r \in T_p^*\mathcal{M}, v_1, \cdots, v_s \in T_p\mathcal{M}$, choose $\omega_1, \cdots, \omega_r \in \Gamma(T^*\mathcal{M}), X_1, \cdots, X_s \in \Gamma(T\mathcal{M})$ with

$$\omega_j(p) = \theta_j, X_j(p) = v_j.$$

By what we have proved, we have,

$$f(p)(\theta_1, \cdots, \theta_r, v_1, \cdots, v_s) := f(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s)(p),$$

which is independent of choices of ω_j, X_j. The rest follows from Proposition 5.2. $\square$

Observe that given $f \in \mathcal{C}^\infty(\mathcal{M})$ and $X \in \Gamma(T\mathcal{M})$, the map,

$$\mathcal{M} \ni p \mapsto X_p f,$$

is a smooth function. In particular,

$$df : \Gamma(T\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M}), df(X) = Xf,$$

is a $\mathcal{C}^\infty(\mathcal{M})$ linear map.

From Proposition 5.3, it follows that $df \in \Gamma(T^*\mathcal{M})$. In particular, we have a linear map,

$$d : \mathcal{C}^\infty(\mathcal{M}) = \Omega^0\mathcal{M} \to \Gamma(T^*\mathcal{M}) = \Omega^1\mathcal{M}.$$

Definition 5.11 (Pullback)**.** *Let $f : \mathcal{M} \to \mathcal{N}$ be smooth and $\xi \in T(T^{0,s}\mathcal{N})$, Put,*

$$f^*\xi|_p(v_1, \cdots, v_s) := \xi|_{f(p)}(T_p f v_1, \cdots, T_p f v_s).$$

Definition 5.12. *A wedge product $\wedge : \Omega^s(\mathcal{M}) \times \Omega^t(\mathcal{M}) \to \Omega^{s+t}(\mathcal{M})$ is defined pointwise that is, for $\eta \in \Omega^s(\mathcal{M})$ and $\zeta \in \Omega^t(\mathcal{M})$,*

$$\eta \wedge \zeta := [\mathcal{M} \ni p \mapsto \eta_p \wedge \zeta_p]. \tag{3}$$

Note that η, ζ are sections, therefore, we have $\eta_p \in \bigwedge^s T^*\mathcal{M}$ and $\zeta_p \in \bigwedge^t T^*\mathcal{M}$.

Proposition 5.4 (Properties of Pullbacks)**.** *Given $\mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{L}$ be smooth maps. We have,*

1. $\varphi \in \mathcal{C}^\infty(\mathcal{N}) = \Gamma(T^{0,0}\mathcal{N}), f^*\varphi = \varphi \circ f$.

2. f^* is a linear map $\Gamma(T^{0,s}\mathcal{N}) \to \Gamma(T^{0,s}\mathcal{M})$ and $\Omega^p\mathcal{N} \to \Omega^p\mathcal{M}$.

3. $(g \circ f)^* = f^* \circ g^*$.

4. If f is a diffeomorphism then $(f^*)^{-1} = (f^{-1})^*$.

5. $\omega \in \Omega^p\mathcal{N}, \eta \in \Omega^q\mathcal{N}$ then $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Notation 5.2. *Let $U \subseteq \mathcal{M}$ be an open set, we define,*

$$\Omega^*(U) := \bigoplus_{p \in \mathbb{Z}_{\geq 0}} \Omega^p(U).$$

Theorem 5.2 (Cartan (exterior) derivative)**.** *Let $\mathcal{M}$ be a smooth manifold and U be an open subset of $\mathcal{M}$, then there exists aunique family of linear maps*

$$d : \Omega^*(U) \to \Omega^*(U),$$

which satisfies the following properties.

1). d is of degree 1 that is $d : \Omega^p(U) \to \Omega^{p+1}(U)$.

2). For $\omega \in \Omega^p(U), \eta \in \Omega^q(U)$, we have,

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d(\eta).$$

3). For $f \in \mathcal{C}^\infty(\mathcal{M}) = \Omega^0(U)$ and $X \in \Gamma(T\mathcal{M})$, we have,

$$\langle df, X \rangle = df(X) = Xf$$

4). $d \circ d = 0$.

5). For $U \subseteq V \subset \mathcal{M}$ open then we have a commutative diagram, where $\iota : U \hookrightarrow V$ is an inclusion and ι^* is the pullback of inclusion namely the restriction.s

$$\begin{tikzcd}
\Omega^*(V) \arrow[r, "d"] \arrow[d, "\iota^*"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^*(V) \arrow[d, "\iota^*"] \\
\Omega^*(U) \arrow[r, "d"'] & \Omega^*(U)
\end{tikzcd}$$

These five properties determine d uniquely, and satisfies furthermore,

6). For $\omega \in \Omega^p(\mathcal{M}), X_0, \cdots, X_p \in \Gamma(T\mathcal{M})$, we have

$$\begin{aligned} d\omega(X_0, \cdots, X_p) = &\sum_{i=0}^{p} (-1)^i X_i(\omega(X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_p)) \\ &+ \sum_{0 \leq i < j \leq m} (-1)^{i+j} \omega([X_i, X_j], X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_{j-1}, X_{j+1}, \cdots, X_p). \end{aligned}$$

7). $f : \mathcal{M} \to \mathcal{N}$ is smooth then $f^*(d\omega) = d(f^*\omega)$.

Proof. For the uniqueness, assume we had $d, \tilde{d}$ satisfying the first five properties. Let (U, φ) be a chart. Denote for $I = \{i_1 < \cdots < i_p\}$, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Using this notation, we have,

$$d(f dx_I) = df \wedge dx_I,$$

which follows from the second and the forth properties. Explicitly, this follows from that

$$ddx_I = \underbrace{ddx_{i_1}}_{=0} \wedge \cdots \wedge dx_{i_p} \pm dx_{i_1} \wedge d(dx_{i_2} \wedge \cdots \wedge dx_{i_p}).$$

And using induction, we get $ddx_I = 0$. Using the third property, we have,

$$d(f dx_I) = \sum_{j \notin I} \frac{\partial f}{\partial xj} dx_j \wedge dx_I.$$

Cover $\mathcal{M}$ by charts $\bigcup_i U_i$, by the fifth property, we have,

$$d\omega|_{U_i} = \tilde{\omega}|_{U_i}.$$

Therefore, we conclude $d = \tilde{d}$.

For the existence of d, taking 6). as definition. We need to show as per Proposition 5.3, for $\omega \in \Omega^p(U)$,

i). $d\omega$ is $\mathcal{C}^\infty(U)$-multilinear,

ii). $d\omega$ is alternating.

These two will imply that $d\omega \in \Omega^{p+1}(U)$. For the second assertion, we see,

$$\begin{aligned} d\omega(X_1, X_0, X_2, \cdots, X_p) &= X_1\omega(X_0, X_2, \cdots, X_p) + \sum_{i=2}^{p} (-1)^i X_i \omega(X1, X_0, \cdots) + \cdots, \\ &= -d\omega(X_0, X_1, \cdots, X_p). \end{aligned}$$

For the first assertion, we have,

$$\begin{aligned} d\omega(fX_0, X_1, \cdots, X_p) = &\sum_{i=1}^{p} (-1)^i \overbrace{X_i(f \cdot \omega(X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_p))}^{X_i f \omega(X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_p) + f X_i \omega(X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_p)} \\ &+ f \cdot X_0 \omega(X_1, \cdots, X_p) \\ &+ \sum_{j=1}^{p} (-1)^j \omega(\underbrace{[fX_0, X_j]}_{fX_0 - X_j(fX_0) = f[X_0, X_j] - (X_j f)X_0}, X_1, \cdots, X_{j-1}, X_{j+1}, \cdots, X_p) \\ &+ \sum_{1 \le i < j \le p} (-1)^{i+j} f \cdot \omega([X_i, X_j], X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_{j-1}, X_{j+1}, \cdots, X_p), \\ &= f \cdot d\omega(X_0, \cdots, X_p). \end{aligned}$$

We claim that the so-defined d satisfies 1). - 5). We have already seen that 1) and 5). Now it suffies to look at a coordinate system, $x_1, \cdots, x_n$ where

$$\forall 1 \leq i, j \leq n, [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

Then we have,

$$\begin{aligned}
&d(f dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \left(\frac{\partial}{\partial x_{j_0}}, \cdots, \frac{\partial}{\partial x_{j_p}} \right) \\
&= \frac{\partial^2}{\partial x_j \partial x_i} \left(f dx_{i_1} \wedge \cdots \wedge \underbrace{dx_{i_p} \left(\frac{\partial}{\partial x_{j_0}}, \cdots, \frac{\partial}{\partial x_{j_{i-1}}}, \frac{\partial}{\partial x_{j_{i+1}}}, \cdots, \frac{\partial}{\partial x_{j_p}} \right)}_{0 \text{ or } \pm 1} \right), \\
&= \begin{cases} 0, \{i_1, \cdots, i_p\} \not\subseteq \{j_0, \cdots, j_p\} \\ (-1)^{\alpha} \frac{\partial}{\partial x_\alpha} f \quad \{i_1 < \cdots < i_p\} = \{j_0 < \cdots < j_{\alpha-1} < j_{\alpha+1} < \cdots < j_p\}, \end{cases} \\
&= \sum \frac{\partial f}{\partial x_k} dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \left(\frac{\partial}{\partial x_{j_0}}, \cdots, \frac{\partial}{\partial x_{j_p}} \right).
\end{aligned}$$

Let

$$\begin{aligned}
\omega &= f dx_1 \wedge \cdots \wedge dx_p, \\
\eta &= g dx_{p+1} \wedge \cdots \wedge dx_{p+q}.
\end{aligned}$$

Using the concrete formula, we have,

$$\begin{aligned}
d(\omega \wedge \eta) &= d(f \cdot g dx_1 \wedge \cdots \wedge dx_{p+q}), \\
&= \sum_{j=p+q+1}^{m} \frac{\partial fg}{\partial x_j} (-1)^{p+q} dx_1 \wedge \cdots \wedge dx_{p+q} \wedge dx_j, \\
&= \underbrace{\sum_{j=p+q+1}^{m} \frac{\partial f}{\partial x_j} (-1)^p dx_1 \wedge \cdots \wedge dx_p \wedge dx_j}_{=d\omega} \wedge \underbrace{g dx_{p+1} \wedge \cdots \wedge dx_{p+q} \wedge dx_j}_{=\eta} \\
&+ \underbrace{\sum_{j=p+q+1}^{m} \underbrace{f dx_1 \wedge \cdots \wedge dx_p}_{=\omega} \wedge (-1)^{p+q} \frac{\partial g}{\partial x_j} dx_{p+1} \wedge \cdots \wedge dx_{p+q} \wedge dx_j}_{(-1)^p \omega \wedge d\eta}.
\end{aligned}$$

This proves 2). For the third, both slots are $\mathcal{C}^\infty(U)$-linear in X. Hence without loss of genearlity, we assume $X = \frac{\partial}{\partial x_1}$.

$$\langle df, \frac{\partial}{\partial x_1} \rangle = \left\langle \sum_{j=1}^{m} \frac{\partial f}{\partial x_j} dx_j, \frac{\partial}{\partial x_1} \right\rangle = \frac{\partial f}{\partial x_1}.$$

For the fourth, we see,

$$
\begin{aligned}
d \circ d(f dx_1 \wedge \cdots \wedge dx_p) &= d\left(\sum_{j=p+1}^{m} \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_p\right), \\
&= \sum_{\substack{i,j=1 \\ i\neq j}}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_1 \wedge \cdots \wedge dx_p, \\
&= \sum_{i<j} \frac{\partial^2 f}{\partial x_i \partial x_j} \underbrace{(dx_i \wedge dx_j + dx_j \wedge dx_i)}_{=0} \wedge dx_1 \wedge \cdots \wedge dx_p.
\end{aligned}
$$

The seventh is obvious at charts. $\square$

5.6 Application : Classical vector analysis

Consider $\mathcal{M}^m$ where $m \in \{2,3,4\}$. Let $g \in \Gamma(T^{0,2}\mathcal{M})$, be symmetric non-degenerate. May choose at orthonormal frame such that

$$g = \begin{pmatrix} -I_p & O \\ O & I_q \end{pmatrix}$$

where $m = p + q$. That is

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \begin{cases} 0, & i \neq j, -1, \quad i = j \leq p, \\ 1, & i = j > p. \end{cases}$$

Interesting cases are

1. Riemannain metric when $p = 0$,
2. General relativity, when $p = 1, q = 3$.

Set the volume form as

$$
\begin{aligned}
\mathrm{vol}_g &= \left|\det g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right|^{\frac{1}{2}} |dx_1 \wedge \cdots \wedge dx_m|, \\
\sigma = \sigma(dx_1, \cdots, dx_m) &= \frac{dx_1 \wedge \cdots \wedge dx_m}{|dx_1 \wedge \cdots \wedge dx_m|}.
\end{aligned}
$$

Compute $*dx_j$ for a general coordinate system.

$$*dx_j = \sum_{k=1}^{m} (-1)^{k-1} g^{kj} \sqrt{g} dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_m \otimes \sigma,$$

$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), g^{ij} = g(dx_i, dx_j) = (\{(g_{kl})_{k,l}\}^{-1})_{i,j}.$

Recall that $*dx_j$ is the unique twisted $(n-1)$-form ω satisfying

$$\tau \wedge \omega = \bigwedge^1 g(\tau_1 dx_j) \operatorname{vol} g,$$

where τ is an arbitrary 1-form.

$$\omega = \sum_{j=1}^{m} (-1)^{j-1} \omega_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge dx_m \otimes \sigma.$$

And we have,

$$\begin{aligned}
dx_i \wedge \omega &= \left(\bigwedge^1 g \right) (dx_i, dx_j) \operatorname{vol} g = g^{ij} \operatorname{vol} g, \\
&= \sum_{j=1}^{m} (-1)^{j-1} \omega_j dx_i \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots dx_m \otimes \sigma, \\
&= \omega_i . dx_1 \wedge \cdots \wedge dx_m \otimes \sigma, \\
&= \omega_i \sqrt{g}^{-1} \operatorname{vol} g. \\
\omega_i &= g^{ij} \sqrt{g}.
\end{aligned}$$

Proposition 5.5.

$$dx_i^{\#} = \sum_j g^{ij} \frac{\partial}{\partial x_j}, \left(\frac{\partial}{\partial x_i} \right)^b = \sum g_{ij} dx_j.$$

Proof.

$$\left\langle \left(\frac{\partial}{\partial x_i} \right)^b, \frac{\partial}{\partial x_k} \right\rangle = g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right) = g_{ik} \text{ gives the second formula.}$$

$$\langle (dx_i)^{\#}, dx_k \rangle = g(dx_i, dx_k) = g^{ik} \text{ gives the first formula}$$

□

Definition 5.13. *We define the gradient to be such that*

$$\operatorname{grad}_g f = (def)^{\#} := \left(\sum (\partial_i f) dx_i \right)^{\#} = \sum_{ij} g^{ij} (\partial_i f) \frac{\partial}{\partial x_j}.$$

Definition 5.14. $\star dx_j$ *is the unique* $(n-1)$ *form* ω *such that*

$$(\tau \wedge \omega) = \left(\bigwedge^1 g \right) (t dx_j) \operatorname{vol}_g$$

for all 1*-form* τ.

Remark 5.7.

$$\omega = \sum_{j=1}^{m} (-1)^{j-1} \omega_j dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots dx_m \otimes \sigma.$$

In particular,

$$\begin{aligned} dx_j \wedge \omega &= w_i dx_1 \wedge \cdots \wedge dx_m \otimes \sigma, \\ &= \omega_i \sqrt{g}^{-1} \operatorname{vol}_g, \quad \sqrt{g} = |\det(g_{ij})|, \\ &= \left(\bigwedge^1 g \right) (dx_i, dx_j) \operatorname{vol}_g, \\ &= g_{ij} \operatorname{vol}_g . \end{aligned}$$

Therefore, we obtain $\omega_i = g^{ij} \sqrt{g}$ *and*

$$\omega = \star(dx_j) = \sum_{k=1}^{m} (-1)^{k-1} g^{kj} \sqrt{g} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma.$$

We now identify vectorfields with $(m-1)$-form by pairing,

$$X \mapsto \star(X^\flat).$$

Lemma 5.3.

$$\star X^\flat = \operatorname{int}_X \operatorname{vol}_g .$$

Proof. Since the formula we want to show is a pointwise identity, it suffices to check it on $X|_p = e_1|_p$, where $e_1, \cdots, e_m$ are g-orthonormal basis, since every small enough neighborhood admits a g-orthonormal basis. Observe that

$$\star e_1^\flat = \star(e_1)^* = c_1 e_2^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma(e_1^*, \cdots, e_m^*).$$

Thus we have,

$$\operatorname{int}_{e_1} \operatorname{vol}_g = \operatorname{int}_{e_1} e_1^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma = c_1 e_2^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma,$$

where $c_j = g_{jj} = g^{jj}$ for a g-orthonormal basis.

$$\begin{tikzcd} \Omega^{m-1}(\mathcal{M}) \arrow[r, "d"] & \Omega^m(\mathcal{M}) \arrow[d, shift right, "(-1)^p\star"'] \\ \Gamma(\mathcal{M}) \arrow[u, "X \mapsto \star X^\flat"] \arrow[r, "\operatorname{div}^g"'] & \mathcal{C}^\infty(\mathcal{M}) \arrow[u, shift right, "\star"'] \end{tikzcd}$$

$$\operatorname{div}^g(X) = (-1)^p \star d \star X^\flat,$$

thus

$$
\begin{aligned}
\operatorname{div}^g(X) &= \operatorname{div}^g \left(\sum_{j=1}^{m} X_j \frac{\partial}{\partial x_j} \right), \\
&= (-1)^p \star d \star \left(\sum_{jk} g_{jk} X_j dx_k \right), \\
&= (-1)^p * d \left(\sum_{jkl} X_j d_{jk} (-1) dx_1 \wedge \cdots \wedge dx_{l-1} \wedge dx_{l+1} \wedge \cdots \wedge dx_m \right) \\
&= (-1)^p * d \left(\sum (-1)^{j-1} x_j \sqrt{g} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma \right),
\end{aligned}
$$

Thus, we obtain,

$$\operatorname{int}_X \operatorname{vol}_g = \sum_j X_j \operatorname{int}_{\partial_j} \sqrt{g} dx_1 \wedge \cdots \wedge dx_m \otimes \sigma =$$

$\square$

Remark 5.8. *On $\mathcal{M}^2$, we have,*

$$
\begin{tikzcd}
\Omega^0 \arrow[r, "d"] \arrow[d, "="'] & \Omega^1 \arrow[r, "d"] \arrow[d, "\#"] & \Omega^2 \arrow[d, shift left, "\star"] \\
\mathcal{C}^\infty(\mathcal{M}) \arrow[r, "\operatorname{grad}^g"'] & \Gamma(T\mathcal{M}) \arrow[r, "X \mapsto -\operatorname{div}(\star X^\flat)^\#"'] & \mathcal{C}^\infty(\mathcal{M}) \arrow[u, shift left, "(-1)^p \star"]
\end{tikzcd}
$$

5.7 Integration of densities

In this section,

$$\mathcal{O} = \mathcal{O}(T^*\mathcal{M}).$$

In terms of the

Lemma 5.4. *If (U_i, φ_i) is an atlas of $\mathcal{M}$ then the cocyle for $|\bigwedge|^\alpha$ is*

Proof.

$$
\begin{aligned}
g_{ij}(p) &= |\det((\varphi_i^{-1})^* \circ \varphi_j^*)|^\alpha, \\
&= |\det((\varphi_j \circ \varphi_i)^*)|^\alpha, \\
&= |\det(D(\varphi_j \circ \varphi_i^{-1})^t)|^\alpha, \\
&= |\det(D(\varphi_i \circ \varphi_j^{-1}))|^{-\alpha}|_{\varphi_j(p)}.
\end{aligned}
$$

Respectively for $\mathcal{O}$, we get,

$$g_{ij} = \operatorname{sgn} D(\varphi_i \circ \varphi_j^{-1})|_{\varphi_j(p)}.$$

$\square$

A diffeomorphism $f : \mathcal{M} \to \mathcal{N}$ gives rise for pullback map

$$\left|\bigwedge\right|^{\alpha}(\mathcal{N}) \overset{f^*}{to} \left|\bigwedge\right|(\mathcal{M}), \mathcal{O}(\mathcal{N}) \xrightarrow{f^*} \mathcal{O}(\mathcal{M}).$$

Proposition 5.6. *Let $\mathcal{M}^m$ be a manifold. There is a unique linear form,*

$$\int_U : \Gamma_C(|\wedge \mathcal{M}|) \to \mathbb{R},$$

which is invariant under diffeomorhpisms and in local coordinate cocycles with the Lebesgue integral,

$$\int_{\mathcal{M}} f(x)|dx| = \int_{\mathbb{R}^m} f(x)|dx_1 \wedge \cdots \wedge dx_m|.$$

Proof. Let $\omega \in \Gamma_C(|\wedge \mathcal{M}|)$ with support contained in some chart (U, φ). We have to put

$$\int_{\mathcal{M}} \omega = \int_{\mathbb{R}^m} (\varphi^{-1})^* \omega.$$

If we have another chart (V, ψ) such that $\operatorname{supp} \omega \subseteq U \cap V$, then

$$\begin{aligned} (\varphi^{-1})^*\omega &= f|dx_1 \wedge \cdots \wedge dx_m|, \\ (\psi^{-1})^*\omega &= (\varphi^{-1} \circ \varphi \circ \psi^{-1})^* \omega \qquad\qquad = (\varphi) \end{aligned}$$

$$\Rightarrow \int_{\mathbb{R}^m} (\varphi^{-1})^*\omega = \int_{\mathbb{R}^m} (\psi^{-1})^*\omega.$$

For the general case, choose $\rho_1, \cdots, \rho_k \in \mathcal{C}_C^\infty(\mathcal{M})$ with,

$$\sum \rho_j|_{\operatorname{supp}\omega} \equiv 1|_{\operatorname{supp}\omega},$$

such that $\operatorname{supp} \rho_j$ lies in a chart. Put

$$\int_{\mathcal{M}} \omega := \sum_{j=1}^k \int_{\mathcal{M}} \rho_j \omega.$$

If $\tilde{\rho}_1, \cdots, \tilde{\rho}_l$ is a different partition of unity then,

$$\sum_{j=1}^k \int_{\mathcal{M}} \rho_j \omega = \sum_{j=1}^k \int_{\mathcal{M}} \sum_{i=1}^l \rho_j \tilde{\rho}_i \omega = \sum_{j,i=1}^{k,l} \int_{\mathcal{M}} \rho_j \tilde{\rho}_i \omega = \sum_{i=1}^l \int_{\mathcal{M}} \tilde{\rho}_i \omega_i.$$

□

Remark 5.9. *Let $\omega = f|dx_1 \wedge \cdots \wedge dx_m|$. Suppose we have a diffeomorphism,*

$$(\mathbb{R}^m, y) \xrightarrow{\varphi} (\mathbb{R}^m, x).$$

Then we have,

$$\varphi^*\omega = f \circ \varphi |\det D\varphi||dy_1 \wedge \cdots \wedge dy_m|.$$

Then we have,

$$\begin{aligned}
\int_{(R^m,x)} f|dx_1 \wedge \cdots \wedge dx_m| &= \int_{\mathbb{R}^m} f(x)dx_1 \cdots dx_m, \\
&= \int_{(\mathbb{R}^m,y)} f(\varphi(y))|\det D\varphi(y)|dy_1 \cdots dy_m, \\
&= \int_{(\mathbb{R}^m,y)} f^*\omega.
\end{aligned}$$

Corollary 5.1. *ALet $\Phi : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism between smooth manifolds and $\omega \in \Gamma_C(|\wedge|^1\mathcal{N})$. Then*

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{N}} \Phi^*\omega.$$

Proof. Write

$$\omega = \sum_{i=1}^{k} \omega_i,$$

such that $\operatorname{supp} \omega \subseteq V_i$ for some coordinate chart $(V_i, \psi_i : V_i \to U_i \subseteq \mathbb{R}^n)$. Then

$$\operatorname{supp} \Phi^*\omega_i \subseteq \Phi^{-1}(V_i),$$

which is a coordinate patch as well. To see this, we have,

$$\psi_i \circ \Phi|_{\Phi^{-1}(V_i)} : \Phi^{-1}(V_i) \to U_i \subseteq \mathbb{R}^n,$$

is again a chart. Then,

$$\int_{\mathcal{N}} \omega_i = \int_{V_i} \omega_i = \int_{\Phi^{-1}(V_i)} \Phi^*\omega_i = \int_{\mathcal{M}} \Phi^*\omega_i.$$

As ω is a finite sum of these ω_i, we have,

$$\int_N \omega = \sum_{i=1}^{k} \int_{\mathcal{N}} \omega_i = \sum_{i=1}^{k} \int_{\mathcal{M}} \Phi^*\omega_i = \int_{\mathcal{M}} \Phi^*\omega.$$

$\square$

5.8 Orientations on manifolds

Definition 5.15. *Let $\mathcal{M}$ be a smooth manifold. An atlas $\mathscr{A}$ oriented is called oriented if*

$$\forall \varphi, \psi \in \mathscr{A}, x \in \mathcal{M}, \det D(\psi \circ \varphi^{-1}(x)) > 0.$$

Definition 5.16. *Two oriented atlases $\mathscr{A}, \mathscr{B}$ are equivalent if $\mathscr{A} \cup \mathscr{B}$ is oriented.*

Definition 5.17. *An orientation is a choice of a maximal oriented atlas. In particular, a manifold is orientable if there is an orientation.*

Clearly we have $(\mathbb{R}^m, \{\mathrm{id}\})$ is oriented.

Example 5.1. *The following is an example of non-oriented manifold.*
$(S^n, \{\psi_\pm\})$ where

$$\psi_\pm : S^n \backslash \{\pm e_n\}, \psi_\pm(x,t) \mapsto \frac{x}{1 \mp t}$$

Then we have

$$\psi + \circ \psi_-^{-1}(x) = \frac{x}{|x|^2}.$$

Thus $f(x) = \frac{x}{|x|^2}$ is orientation reversing. That is $\det Df(x) < 0$.

Remark 5.10. *Fix an orientation reversing diffeomorphism of $\mathbb{R}^n$ say $\Phi(x_1, \cdots, x_n) = (-x_1, x_2, \cdots, x_n)$. We have a new atlas,*

$$\{\Phi \circ \psi_+, \psi_-\}$$

is oriented.

Example 5.2. *A Möbius band cannot be oriented. It is an exercise for the readers to give it a proof.*

Proposition 5.7. *Let $\mathcal{M}$ be a smooth manifold. $\mathcal{M}$ is orientable if and only if one of following statements hold.*

i). There is a section s of the bundle $\mathcal{O}(\mathcal{M})$ with $|s(p)| = 1$ for all $p \in \mathcal{M}$. Note that $\mathcal{O}$ comes with a canonical inner product, thus we can define a notion of length.

ii). There is a section $\omega \in \Omega^m(\mathcal{M})$ where $m = \dim \mathcal{M}$ with $\omega(p) \neq 0$ for all $p \in \mathcal{M}$.

Moreover, each of the conditions will determine an orientation.

Proof.
For the first one. Suppose $\mathcal{M}$ is orientable. then consider $\mathcal{M} = \bigcup_n U_n$ where (U_n, φ_n) is an oriented chart and consider (ρ_n) be a subordinate partition of unity to the charts. We set,

$$\omega := \sum_{j=1}^{\infty} \rho_j dx_1^j \wedge \cdots \wedge dx_m^j. \tag{4}$$

We show that $\omega(p) \neq 0$ for all $p \in \mathcal{M}$. To see this for a fixed $p \in U_i$. In the small enough neighborhood around p, we have,

$$\omega = \sum_{j=1}^{\infty} \rho_j \circ \varphi \det D(\varphi^{-1} \varphi_j \circ \varphi_i) dx_1^i \wedge \cdots \wedge dx_m^i. \tag{5}$$

As we have assumed the manifold to be oriented. Each $\det D(\varphi_j \circ \varphi_i^{-1})$ is positive. And partition of unities are non-negative. We conclude that $\omega(p) > 0$.

Suppose the second statement holds. Fix $\omega \in \Omega^m \mathcal{M}$ such that $|omega(p) \neq 0$ for all $p \in \mathcal{M}$. Let (U, φ) be a connected chart. Then obviously, we have,

$$\varphi^* \omega = \rho_\varphi dx_1 \wedge \cdots dx_m.$$

with

$$\forall x \in \varphi(U), \rho_\varphi(x) \neq 0.$$

We call φ is oriented if $\rho_\varphi > 0$. This determines an oriented atlas. Thus proves that the second condition determines an atlas.

Furthermore, the first and the second conditions are equivalent. To show this, choose a Riemannian metric on $\mathcal{M}$. Given s as in the first condition. Set

$$\omega = \mathrm{vol} \cdot s.$$

Conversely, given ω as in the second condition set $s = \frac{\omega}{\mathrm{vol}}$. $\square$

6 Manifolds with boundaries

6.1 Basics

Definition 6.1 (Half-space)**.** *Let E be a real vector space of dimension $n < \infty$. A subset $H \subseteq E$ is called a half space if there is a linear form $\lambda \in E^* \backslash \{0\}$ such that*

$$H = \{x \in E \mid \langle \lambda, x \rangle \leq 0\}.$$

We denote such H by E_λ^+.

Example 6.1. *$\{x \in \mathbb{R}^n \mid x_j \leq 0\}$ is a half space for $1 \leq j \leq n$.*

Remark 6.1. *Let $H_1, H_2 \subseteq E$ be two half-spaces. Then there is a linear isomorphism $T \in \mathrm{GL}(E)$ such that $TH_1 = H_2$. That is if*

$$H_j = E_{\lambda_j}^+,$$

then take $T^* \in \mathrm{GL}(E^*)$ such that $T^* \lambda_1 = \lambda_2$. We have

$$\langle \lambda_2, x \rangle = \langle T^* \lambda_1, x \rangle = \langle \lambda_1, Tx \rangle.$$

Remark 6.2. *The above remark is true in the oriented category only if $\dim E \geq 2$. That is $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$ are diffeomorphic but not oriented diffeomorphic.*

Definition 6.2. *Let $E_\lambda^+ \subseteq E$ be a half space. The normal space is $E/\ker \lambda$.*

Lemma 6.1. *A normal space $E/\ker \lambda$ is canonically oriented by defining a base of $E/\ker \lambda$ by $\lambda(v) > 0$.*

Definition 6.3. *If $v \in E$ with $\lambda(v) > 0$ then then we say v points outward. Furthermore, we set*

$$c(t) = tv.$$

Remark 6.3. *The notion comes from the following case. Consider a curve $c : \mathbb{R} \to E$ such that for small enough $\varepsilon > 0$, we have $c((-\varepsilon, 0]) \subseteq E_\lambda^+$ for some $\lambda \in E^*$. And for $t > 0, c(t) \notin E_\lambda^+$.*

Definition 6.4. *Let $A \subseteq \mathbb{R}^m$ be a set and $f : A \to \mathbb{R}^n$ be a map. f is differentiable if for any $a \in A$, there is a neighborhood $a \in U \subseteq \mathbb{R}^m$ (not necessarily contained in A) and a differentiable function $g : U \to \mathbb{R}^n$ such that*

$$g|_{U\cap A} = f|_{U\cap A}.$$

Proposition 6.1. *Let E_λ, E_μ be half spaces in $E \cong \mathbb{R}^n$, $U \subseteq E_\lambda, V \subseteq E_\mu$ be open, and $\phi : U \to V$ be a diffeomorphism. Then*

$$\phi(\operatorname{int} U) = \operatorname{int} V, \partial U = \partial V.$$

Furthermore, for $x \in \partial U$, and any outward pointing vector v,

$$D_v\mu \circ \phi(x) > 0.$$

Proof. Fix $x \in U$. Then there is a neighborhood $\tilde{U}_x$ of x, and $\tilde{V}_y$ of $y = \phi(x)$ in E and smooth maps

$$\tilde{\phi} : \tilde{U}_x \to \tilde{V}_y, \tilde{\psi} : \tilde{V}_y \to \tilde{U}_x,$$

extening f with respect to f^{-1}. Since E_λ, E_μ are half-spaces, we have,

$$D_x\phi^{-1} \circ \phi = D_{f(x)}\phi^{-1} \circ D_x\phi(x).$$

This implies that $D_{f(x)}\tilde{\psi} \circ D_x\tilde{\phi}(x) = \mathrm{id}$.

Suppose $\phi(x) \in \partial V$. Then $\tilde{V}_y$ contains an element outside of V.

Let $c : (-\varepsilon, 0] \to U$ with $c((-\varepsilon, 0)) \subseteq U^\circ, c(0) = x, c'(0) = v$. Then we have

$$D_v(\mu \circ \phi)(x) = \frac{d}{dt}\bigg|_{t=0} \underbrace{\mu(\phi(c(t)))}_{=\phi(t)}.$$

We have

$$t < 0 \Rightarrow f(t) \leq 0, f(0) = 0.$$

Therefore, we have $f'(x) \geq 0$. Suppose $f'(0) = 0$ then $(\phi \circ c)'(0) \in \ker \mu$. This is because that the outward pointing vectors and their negatives are (missing), this implies that range of $D_x(\phi) \subseteq \ker \mu$. This contradicts that the rank of $D_x\phi$ is full. $\square$

Definition 6.5. *A smooth manifold with boundary is a second countable Hausdorff space $\mathcal{M}$ together with maximal atlas of charts (U, φ) with $\varphi : U \to V \subseteq E_\lambda^+$ being a homeomorphism on to some open subset V in E_λ^+.*

Remark 6.4. *Let $\mathcal{M}$ be a smooth manifold with boundary. A transition map $\psi \circ \varphi^{-1}$ are diffeomorhpism from an open subset of E_λ^+ to some open subset of E_μ^+.*

Definition 6.6. *Let $\mathcal{M}$ be a smooth manifold with boundary. We define,*

1). the interior of $\mathcal{M}$ to be $\mathcal{M}^\circ := \operatorname{int} \mathcal{M} = \bigcup_{\varphi \in \mathcal{A}} \varphi^{-1}(\operatorname{int} E_\lambda^+)$,

2). the boundary of $\mathcal{M}$ to be $\partial \mathcal{M} := \varphi^{-1}(\partial E_\lambda^+)$.

Proposition 6.2. *Let $\mathcal{M}^m$ be a smooth boundaryless manifold and let $f : \mathcal{M} \to \mathbb{R}$ be a submersion that is $\forall p \in \mathcal{M}, df|_p \neq 0$. Then*

$$\mathcal{N} = \{x \in \mathcal{M} \mid f(x) \leq 0\},$$

is a manifold with boundary and

$$\partial \mathcal{N} = \{x \in \mathcal{M} \mid f(x) = 0\}.$$

Proof. Given $p \in \mathcal{M}$, there exists a chart (U, φ) centered at p such that $\varphi_m = f$. Thus $\varphi : U \to V$ for $V \subseteq \mathbb{R}^m$. That is

$$U \cap \mathcal{N} \to V \cap \{x_m = 0\}.$$

All such charts give an atlas for a manifold with boundary $\mathcal{N}$. □

Example 6.2. *Let $B \subseteq \mathbb{R}^n$ be a unit ball. We know that $\partial B = S^{m-1}$. The function f above in this case is given by*

$$f(x) = \|x\|^2 - 1 = \sum_{i=1} x_i^2 - 1.$$

Proposition 6.3. *Let $\mathcal{M}^m$ be a smooth manifold. A subset $A \subseteq \mathcal{M}$ is a submanifold with boundary if for each $a \in A$ there is an open neighborhood U in $\mathcal{M}$ and a submersion $f : U \to \mathbb{R}$ such that $A \cap U = \{f \leq 0\}$. In that case $\partial \cap U = \{f = 0\}$.*

6.2 Stokes theorem

Let $f : [a, b] \to \mathbb{R}$ be sufficiently differentiable. We have,

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Observe that $f'(x)dx \in \Omega^1([a, b])$. Obviously $[a, b]$ is a manifold with boundary and the above equation can be restated as,

$$\int_{\partial[a,b]} f d\#,$$

where $\#$ is the counting measure. Now consider $\{x_1 \leq 0\} \subseteq \mathbb{R}^m$. Take the canonical basis $e_1, \cdots, e_m$ and its dual basis $e_1^*, \cdots, e_m^*$. For $\lambda = -e_1^*$,

$$\{x_1 \geq 0\} = E_\lambda^+.$$

Let $\omega \in \Omega_C^{m-1}(E_\lambda^+)$ such that

$$\omega = \sum_{j=1}^{m} (-1)^{j-1} f_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m.$$

where $f_j \in \mathcal{C}_C^\infty(E_\lambda^+)$. Thus

$$d\omega = \sum_{j=1}^{m} \frac{\partial f_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_m = (\nabla f) \cdot dx_1 \wedge \cdots \wedge dx_m.$$

Integrating this we get,

$$\begin{aligned}
\int_{E_\lambda^+} d\omega &= \sum_{j=1}^{m} \int_{\{x_1 \leq 0\}} \frac{\partial f_j}{\partial x_j} dx_1 \cdots dx_m, \\
&= \sum_{j=1}^{m} \int_{\{x_1 \leq 0\}} \underbrace{\left(\int_{\mathbb{R}} \frac{\partial f_j}{\partial x_j} dx_j \right)}_{=0} dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_m,
\end{aligned}$$

where the last equality is due to Fubini. Except for $j = 1$, thus

$$\int_{E_\lambda^+} d\omega = \int_{\mathbb{R}^{m-1}} \left(\int_0^\infty \frac{\partial f_1}{\partial x_1} dx_1 \right) dx_2 \cdots dx_m = \int_{\mathbb{R}^{m-1}} -f_1(0, x_2, \cdots, x_m) dx_2 \cdots dx_m.$$

Set $\iota : \partial E_\lambda^+ = \{x_1 = 0\} \hookrightarrow E_\lambda^+ \hookrightarrow \mathbb{R}^m$. We then have,

$$\iota^* \omega = f_1 dx_2 \wedge \cdots \wedge dx_m.$$

And in particular,

$$\int_{E_\lambda^+} d\omega = -\int_{\partial E_\lambda^+} \omega.$$

By setting orientations on as $e_1, \cdots, e_m$ positive in E_λ^+ and $e_2, \cdots, e_m$ negative in ∂E_λ^+, we have,

$$\int_{E_\lambda^+} d\omega = \int_{\partial E_\lambda^+} \omega.$$

Theorem 6.1 (Stokes)**.** *Let $\mathcal{M}^m$ be oriented manifold with boundary such that $\partial \mathcal{M}^m$ carrying the induced orientation,$\omega \in \Omega_C^{m-1}(\mathcal{M})$. Then*

$$\int_{\mathcal{M}} d\omega = \int_{\partial \mathcal{M}} \iota^* \omega.$$

Remark 6.5. *Let E be a n-dimensional vector space and H be a half-space. Suppose E is oriented. Choose an oriented basis of E of the form*

$$v, e_2, \cdots, e_n,$$

where

$$H = E_\lambda^+ = \{x \in E \mid \langle \lambda, x \rangle \leq 0\},$$

and $e_2, \cdots, e_n \in \ker \lambda$. This defines an induced orientation of $\ker \lambda$ and v points outward.

Definition 6.7. *Let $\mathcal{M}^m$ be an oriented manifold with boundary then $\partial \mathcal{M}^m$ is oriented as follows. For $p \in \partial \mathcal{M}^m$, let $\nu(p)$ be an outward pointing vector $\nu(p) \neq 0$. Take a basis $v_1, \cdots, v_m$ of $T_p \partial \mathcal{M}$ is oriented by definition if and only if*

$$\nu(p), v_2, \cdots, v_m$$

is an oriented basis of $T_p \mathcal{M}$. This is called the induced boundary orientation.

Proposition 6.4 (Stokes Theorem for Half Spaces)**.** *Let E be an oriented n-dimensional vector space and $H = E_\lambda^+ \subseteq E$ be a half space and equip $\partial H = \ker \lambda$ and ω the induced orientation. Let*

$$i : \partial H \hookrightarrow H$$

be the inclusion. Then for $\omega \in \Omega_C^{n-1}(H)$, we have,

$$\int_H d\omega = \int_{\partial H} i^* \omega.$$

Proof. Choose oriented linear coordinate, $x_1, \cdots, x_n$, such that $x_1 = \langle \lambda, x \rangle$. Then

$$H = \{x \in \mathbb{R}^n \mid x_1 \leq 0\}, \partial H = \{0\} \times \mathbb{R}^{n-1}.$$

Recall that for $j \geq 2$, we get,

$$\int_{\mathbb{R}} \frac{\partial \phi}{\partial x_j}(\cdots, x_j, \cdots) dx_j = 0,$$

for $j = 1$, we have,

$$\int_{-\infty}^0 \frac{\partial \phi}{\partial x_1}(x_1, 0, \cdots, 0) dx_1 = \phi(0, a).$$

Note that

$$\omega = \sum_{j=1}^n (-1)^{j-1} \phi_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n.$$

$$i^* \omega = \phi i^* (dx_2 \wedge \cdots \wedge dx_m)?$$

Therefore,

$$d\omega = \sum_{j=1}^n \frac{\partial \phi_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n.$$

□

Theorem 6.2 (Stokes, note that we do not assume compactness)**.** *Let $\mathcal{M}^m$ be an oriented smooth manifold with boundary $\partial\mathcal{M}$. $\partial\mathcal{M}$ carries an induced orientation. Then for $\omega \in \Omega_C^{n-1}(\mathcal{M})$, we have,*

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} i^*\omega,$$

where $i : \partial\mathcal{M} \hookrightarrow \mathcal{M}$ is a natural inclusion.

Proof. The key is to use partition of unity. Choose $\rho_1, \cdots, \rho_k \in \mathcal{C}_C^\infty(\mathcal{M})$ such that

$$\sum_{i=1}^{k} \rho_j = 1,$$

in the neighborhood of $\operatorname{supp}\omega$. Furthermore, we may arrange that $\operatorname{supp}\rho_j \subseteq U_j$ where (U_j, φ_j) is an oriented chart. Thus each φ_j maps $U_j \to V_j \subseteq H_j$ where H_j is a half space.

$$\begin{aligned}
\int_{\mathcal{M}} d\omega &= \sum \int_{\mathcal{M}} d(\rho_j\omega), \\
&= \sum \int_{U_j} d(\rho_\omega), \\
&= \sum \int_{H_j} d((\varphi_j^{-1})^*\rho_j\omega), \\
&= \sum \int_{\partial H_j} i^*(\varphi_j^{-1})^*\rho_j\omega, \\
&= \sum \int_{\partial H_j} (\tilde{\varphi_j}^{-1})^* i^*(\rho_j\omega), \\
&= \sum \int_{\partial\mathcal{M}} i^*(\rho_j\omega), \\
&= \int_{\partial\mathcal{M}} \omega,
\end{aligned}$$

where $\tilde{\varphi}$ is such that

$$\begin{tikzcd}
\partial\mathcal{M} \arrow[r, hook, "i"] \arrow[d, "\tilde{\varphi}"'] & \mathcal{M} \arrow[d, "\varphi"] \\
\partial H \arrow[r, hook, "i"'] & H
\end{tikzcd}$$

□

Remark 6.6. *Note if $\partial\mathcal{M} = \emptyset$ then $\int_{\mathcal{M}} d\omega = 0$. For example, consider a circle S^1_θ, we have,*

$$\int_{S^1} f'(\theta)d\theta = 0.$$

6.3 Classical Theorem of Gauss and Stokes

Theorem 6.3 (Divergence Theorem of Gauss)**.** *Let $(\mathcal{M}^m, g)$ be an oriented Riemannian manifold with boundary $\partial\mathcal{M}$. Let ν be an outer normal vector field. In other words,*

$$\nu : \partial\mathcal{M} \to T\mathcal{M},$$

such that for all $p \in \partial\mathcal{M}$, we have,

$$\nu(p) \perp T_P\partial\mathcal{M}, \|\nu(p)\| = 1,$$

and $\nu(p)$ points outward. Now $\mathrm{vol}_{\mathcal{M}} \in \Omega^m(\mathcal{M}), \mathrm{vol}_{\partial\mathcal{M}} \in \Omega^{m-1}(\partial\mathcal{M})$. Let $X \in \Gamma(T\mathcal{M})$ be a vector field. Then,

$$d(\star X^\flat) = d(i_X(\mathrm{vol}_{\mathcal{M}})) = (\mathrm{div}\, X)\, \mathrm{vol}_{\mathcal{M}}\, .$$

Proof.

Claim 1.

$$i^*(i_X\, \mathrm{vol}_{\mathcal{M}}) = \langle X, \nu \rangle\, \mathrm{vol}_{\mathcal{M}}\, .$$

Proof: For $p \in \partial\mathcal{M}, e_2, \cdots, e_m$, be an orthonormal basis of $T_p\partial\mathcal{M}$ such that

$$X(p) = \langle X(p), \nu(p)\rangle \cdot \nu(p) + \sum_{j=2}^{m} \langle X(p), e_j\rangle e_j.$$

Then, by setting $e_1 = \nu(p)$, we get,

$$i_X\, \mathrm{vol}_{\mathcal{M}} |p = i_X e_1^\flat \wedge \cdots \wedge e_m^\flat = \langle X(p), \nu(p)\rangle e_2^\flat \wedge \cdots \wedge e_m^\flat = \mathrm{vol}_{\partial\mathcal{M}} |_p.$$

■

Apply Stokes, we have,

$$\int_{\mathcal{M}} \mathrm{div}\, X\, \mathrm{vol}_{\mathcal{M}} = \int_{\mathcal{M}} d i_X\, \mathrm{vol}_{\mathcal{M}} = \int_{\partial\mathcal{M}} i^*(i_X\, \mathrm{vol}_{\mathcal{M}}) = \int_{\partial\mathcal{M}} \langle X(p), \nu(p)\rangle\, \mathrm{vol}_{\partial\mathcal{M}}\, .$$

□

Theorem 6.4 (Divergence Theorem)**.** *Let $(\mathcal{M}, g)$ be an oriented Riemannian manifold with boundary and $\nu : \partial\mathcal{M} \to T\mathcal{M}$ be a unit outer normal vector field. Then for aany compactly supported $\mathcal{C}^1$-vector field, we have,*

$$\int_{\mathcal{M}} \mathrm{div}\, X\, \mathrm{vol}_{\mathcal{M}} = \int_{\partial\mathcal{M}} \langle X, \nu\rangle\, \mathrm{vol}_{\partial\mathcal{M}}\, .$$

Theorem 6.5 (Stokes Theorem for Surface in 3-manifolds)**.** *Let $(\mathcal{M}, g)$ be oriented 3-dimensional Riemannian manifold. Let $F \subseteq \mathcal{M}$ be an oriented surface with boundary. Let $\nu_F : F \to T\mathcal{M}$ be such that for $p \in F$, there is $e_1, e_2 \in T_pF$, an oriented orthonormal basis of T_pF such that*

$$\nu_F(p) := e_1 \times e_2.$$

Recall that for a vector field X over $\mathcal{M}$, we have

1). $X^\flat$ is 1-form,

2). $dX^\flat$ is 2-form,

3). $\star dX^\flat$ is 1-form.

Definition 6.8. *Let X be a vector field on $\mathcal{M}$, then we define,*

$$\operatorname{curl} X := (\star dX^\flat)^\#.$$

Remark 6.7. $\operatorname{curl} X$ *coincides with the usual rotation in* $\mathbb{R}^3$*.*

Theorem 6.6 (Classical Stokes Theorem)**.** *Inheritting the settings of Theorem 6.5, consider a vector field X on $\mathcal{M}$ which is of class $\mathcal{C}^1$. We have,*

$$\int_F \langle \operatorname{curl} X, \nu_F \rangle \operatorname{vol}_F = \int_{\partial F} \langle X, \tau_{\partial F} \rangle \operatorname{vol}_{\partial F},$$

where $\tau_{\partial F} : \partial F \to T\partial F$ is the oriented unit tangent vectorfield to ∂F.

Proof.

$$\begin{aligned}
\langle \operatorname{curl} X, \nu_F \rangle \operatorname{vol}_F &= (i^F_{\mathcal{M}})^*(i_{\operatorname{curl} X} \operatorname{vol}_{\mathcal{M}}), \\
&= (i^F_{\mathcal{M}})^*(\star(\operatorname{curl} X)^\flat), \\
&= (i^F_{\mathcal{M}})^*(\star \star dX^\flat), \\
&= (i^F_{\mathcal{M}})^*(dX^\flat).
\end{aligned}$$

Therefore,

$$\int_F \langle \operatorname{curl} X, \nu \rangle \operatorname{vol}_F = \int_F dX^\flat \overset{\text{Stokes}}{=} \int_{\partial F} (i^{\partial F}_F)^* X^\flat.$$

□

6.4 Applications of Stokes Theorem

Let $A \subseteq \mathbb{R}^{m+1}$ be a compact set with smooth boundary. A is a compact manifold with boundary such that

$$\operatorname{int} A = A \backslash \partial A$$

is open in $\mathbb{R}^{m+1}$. THis is a compact hypersurface in $\mathbb{R}^{m+1}$. A Topological-boundary of the anifold boundary of A.

Remark 6.8. *If A compact surface in $\mathbb{R}^3$ is the boundary of a compact set with smooth boundary, then it is orientable.*

Consider in $\mathbb{R}^{m+1}$, define,

$$\eta := \sum_{j=1}^{m+1} (-1)^{j+1} x_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{m+1} \in \Omega^m(\mathbb{R}^{m+1}).$$

Then

$$d\eta = \sum_{j=1}^{m+1} (-1)^{j+1} dx_j \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{m+1}.$$

Therefore, we have ,

$$d\eta = (m+1) dx_1 \wedge \cdots \wedge dx_{m+1}.$$

This is a volume form in $\mathbb{R}^{m+1}$ which is a volume for A as well. Therefore,

$$(m+1) \cdots dx_1 \wedge \cdots \wedge dx_{m+1}|_A = (m+1) \operatorname{vol}_A .$$

Thereore, we have,

$$\operatorname{vol}(A) = \int_A \frac{1}{m+1} d\eta = \frac{1}{m+1} \int_{\partial A} \eta.$$

Lemma 6.2. *Let $i : S^m \hookrightarrow \mathbb{R}^{m+1}$. Then $i^*\eta$ is the volume form on S^m.*

Corollary 6.1.

$$\operatorname{vol}(S^m) = \int_{S^m} \eta = \int_{B(0,1) \subseteq \mathbb{R}^{m+1}} d\eta = (m+1) \operatorname{vol}(B(0,1)).$$

Proof. Let $\omega = dx_1 \wedge \cdots dx_{m+1} = \operatorname{vol}_{\mathbb{R}^{m+1}}$. We define a vector field,

$$X(x) = \sum_{j=1}^{m+1} x_j \frac{\partial}{\partial x_j},$$

which we will call it the radial vector field. Note that let $p \in S^m$, then $X(p)$ is the outward normal vector. We then have,

$$i_X \omega = i_X dx_1 \wedge \cdots \wedge dx_{m+1} = \sum_{i=1}^{m+1} (-1)^{i-1} dx_1 \wedge \cdots \wedge dx_{i-1} i_X(dx_j) \wedge dx_{j+1} \wedge \cdots \wedge dx_{m+1}.$$

But observe that $i_X(dx_j) = x_j$, thus

$$i_X \omega = \eta.$$

Let $x \in S^m$ and $v_1, \cdots, v_m$ be an oriented orthonormal basis of $T_x S^m$. Then

$$X(x), v_1, \cdots, v_m$$

is an oriented orthonormal basis of $T_x\mathbb{R}^{m+1}$.

$$\begin{aligned} i_\eta^*(v_1, \cdots, v_m) &= (i_X\omega)(v_1, \cdots, v_m), \\ &= \omega(X(x), v_1, \cdots, v_m), \\ &= 1. \end{aligned}$$

Thus we have shown that $i^*\eta$ is the volume form on S^m. □

Example 6.3.

$$\mathrm{vol}(A) = \frac{1}{2}\int_A xdy - ydx,$$

Proposition 6.5. *Consider the antipodal map,*

$$A : S^m \to S^m, x \mapsto -x.$$

This is orientation preserving/reversing, if and only if m is odd/even.

Proof. Consider η as in Corollay 6.1. We have,

$$A^*\eta = (-1)^{m+1}\eta.$$

This proves the statement. □

Corollary 6.2. $\mathbb{RP}^m = S^m/A$ *is orientable if and only if m is odd.*

Let $\mathcal{M} \xrightarrow{\pi} \mathcal{M}/A$ be a 2-fold cover of an oriented connected manifold $\mathcal{M}$ by an orientation preserving(respectively, reversing involution A).

Assume $\mathcal{M}/A$ is orientable hence there is $\omega \in \Omega^m(\mathcal{M}/A)$ such that it is nowhere vanishing. Therefore, there is nowhere vanishing $f \in \mathcal{C}^\infty(\mathcal{M})$, (ie. $\forall p \in \mathcal{M}, f(p) \neq 0$), with

$$\pi^*\omega f \cdot \mathrm{vol}_M .$$

We have,

$$A^*\pi^*\omega = \underbrace{(\pi \circ A)}_{=\pi}{}^*\omega = \pi^*$$

Thus we have $f \cdot \eta = A^*(f\eta) = -A^*f\eta$, if A is orientation reversing.

If A is orientation reversin the, f is odd with respect to A. Using intermediate value theorem, we see that there is $p \in \mathcal{M}$ such that $f(p) = 0$ which is a contradiction.

Corollary 6.3. *Let us denote $A_m : S^m \to S^m$ to be the antipodal map of m-dimensional sphere. A_m is homotopic to the identity if and only if m is odd.*

Corollary 6.4 (Hairly Ball Theorem)**.** *There is a non-vanishing vector field on S^m if and only if m is odd.*

Proof. For odd m, let us consider,

$$X(x) = (-x_2, x_1, -x_4, x_3, \cdots, -x_{m+1}, x_m).$$

This is a non-vanishing vector field on S^{m+1}. Suppose we have $X \in \Gamma(TS^m)$ such that $\forall p \in S^m, X(p) \neq 0$. Replace,

$$X \to \left[S^m \ni p \mapsto \frac{X(p)}{|X(p)|} \right]$$

Without loss of generality, we assume $|X(p)| = 1$. Pick $\rho \in \mathcal{C}^\infty(\mathbb{R})$ to be such that

$$\rho(t) = \begin{cases} 0, & t \leq \frac{1}{3}, \\ 1, & t \geq \frac{2}{3}. \end{cases}$$

And $rho(t) \in [0, 1]$ for all $t \in \mathbb{R}$. Now set $h : [0, 1] \times S^m \to S^m$ to be

$$h(t, p) := \cos(\pi \cdot \rho(t)) \cdot p + \sin(\pi \rho(t)) \cdot X(p).$$

$X(p) \perp p \Rightarrow \|h(t, p)\|^2 = 1$. Thus we obtain,

$$h(0, p) = p, h(1, p) = -p.$$

Therefore, if there exists a norwhere vanishing vector field, then antipodal map is homotopic to the identity.

Suppose m is even and A_m is homotopic. Then we have,

$$\int_{S^m} A^* \eta = -\int_{S^m} \eta = -\operatorname{vol}(S^m) < 0.$$

Claim 2.

$$\int_{\mathcal{M}} h_t^* \omega$$

is independent of t.

<u>Proof:</u> Without loss of generality, we show

$$\int_{\mathcal{M}} h_0^* \omega = \int_{\mathcal{M}} h_1^* \omega.$$

Note that $[0, 1] \times \mathcal{M}$ is a manifold with oriented boundary $\mathcal{M} \cup (-\mathcal{M})$. By Stoke's theorem, we have,

$$\int_{\mathcal{M}} h_1^* \omega - \int_{\mathcal{M}} h_0^* \omega = \int_{\partial([0,1] \times \mathcal{M})} h_\cdot^* \omega = \ int_{\partial([0,1] \times \mathcal{M})} d(h_\cdot^* \omega) = \int_{[0,1] \times \mathcal{M}} h^*(d\omega) = 0.$$

■

Using the claim, we have $\int_{S^m} A^* \eta$ is homotopy invariant. Therefore

$$\int_{S^m} A^* \eta = \int_{S^m} \eta = \operatorname{vol}(S^m) < 0.$$

More general, let $\mathcal{M}^m$ be a closed oriented manifold and $\omega \in \Omega^m(\mathcal{M})$. Furthermore, consider,

$$h : [0,1] \times \mathcal{M} \to \mathcal{M}$$

be smooth. Note that it is a family of endomorphisms of $\mathcal{M}$ thus we denote

$$h_t : \mathcal{M} \to \mathcal{M} = [\mathcal{M} \ni p \mapsto h(t,p) \in \mathcal{M}].$$

□

Example 6.4. *Using this corollary, we conclude that any vector field on S^2 vanishes somewhere.*

6.5 Brouwer's Fixed Point Theorem

Theorem 6.7. *Let $\overline{B(0,1)}$ be a unit ball around the origin in $\mathbb{R}^n$ and $f : \overline{B(0,1)} \to \overline{B(0,1)}$ be continuous. Then f has a fixed point.*

Proof. We first reduce to the case where f is smooth. Suppose a continuous function f does not have a fixed point. Then consider,

$$\delta := \inf_{x \in \overline{B(0,1)}} \|f(x) - x\| > 0.$$

Let $g : \overline{B(0,1)} \to \overline{B(0,1)}$ be smooth such that

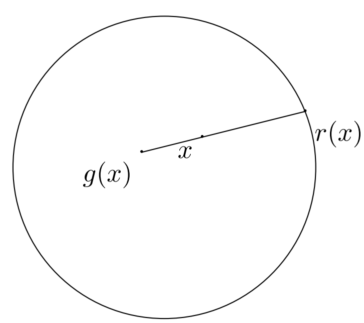
$$\|f - g\|_{L^\infty(\overline{B(0,1)})} \le \frac{\delta}{2}.$$

Then g is a smooth self map of $\overline{B(0,1)}$ without fixed point.

Let $r : \overline{B(0,1)} \to S^{n-1}$ to be such that for $x \in \overline{B(0,1)}$ we define,

$$r(x) \in S^{n-1} \cap \{g(x) + t(x - g(x)) \mid t \ge 0\}.$$

This is smooth and $r|_{S^{m-1}} = \mathrm{id}$. Intuitively, r is constructed by the following way shown in the diagram below.



That is if such g exists then there is a smooth retraction $r : \overline{B(0,1)} \to S^{m-1}$. That is we have a following commutative diagram,

$$\begin{tikzcd}
\overline{B(0,1)} \arrow[r, "r"] & S^{n-1} \\
S^{n-1} \arrow[u, hook, "i"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

Let $\omega \in \Omega^{m-1}(S^{m-1})$ be volume form then $r^*\omega \in \Omega^{m-1}(\overline{B(0,1)})$ is exact by Poincaré's lemma. That is there is $\xi \in \Omega^{m-2}(B(0,1))$ such that

$$r^*\omega = d\xi.$$

We then have,

$$\omega = \mathrm{id}^* \omega = i^* r^* \omega = i^* d\xi = d(i^*\xi).$$

We found ω is exact. By Stoke's theorem, we see,

$$\mathrm{vol}(S^{m-1}) = \int \omega = \int di^*\xi = \int_{B(0,1)} dd\xi = 0.$$

$\square$

7 De Rham Cohomology

7.1 Basics

Definition 7.1. *Let $\mathcal{M}$ be a smooth manifold. We define,*

$$\Omega^\bullet(\mathcal{M}) = \bigoplus_{j=0}^{\dim \mathcal{M}} \Omega^j(\mathcal{M}).$$

Definition 7.2. *A cochain complex is of k-vectorspace (we can also take R-modules) consists of a $\mathbb{Z}$-graded vector space $E := \bigoplus E^k$ and homomorphism $d : E^k \to E^{k+1}$ of degree 1 (ie: any homogeneous element of degree k is mapped to a homogeneous element of degree $k+1$) with $d \circ d = 0$.*

A cochain complex is represented as a sequence,

$$\cdots \longrightarrow E^{k-1} \xrightarrow{d_{k-1}} E^k \xrightarrow{d_k} E^{k+1} \longrightarrow \cdots$$

This is not necessarily exact. We would like to measure the non-exactness of the series.

Definition 7.3. *Given a cochain complex (E, d), we define the following,*

1. $Z^k E = \ker(E^k \xrightarrow{d} E^{k+1})$ *which we call the cycles of (E, d).*
2. $B^k E = \mathrm{im}(E^{k-1} \xrightarrow{d} E^k)$ *which is the boundaries of (E, d).*
3. $H^k(E) = H^k(E, d) := Z^k E / B^k E$ *which is called the k-th cohomology group of (E, d).*

Above taking the quotient Z^kE/B^kE is justified as

$$d \circ d = 0 \Rightarrow \operatorname{im} d_{k-1} \subseteq \ker d_k.$$

Definition 7.4. *The k-th de Rham cohomology gruop of $\mathcal{M}$ is*

$$H^k_{\mathrm{dR}}(\mathcal{M}) := H^k(\Omega^\bullet(\mathcal{M}), d).$$

Example 7.1. *Let $f \in Z^0(\Omega^\bullet(\mathcal{M}))$, then f is a smooth function such that $df = 0$. That is f is a locally constant function. We conventionally set $B^0(\Omega^\bullet(\mathcal{M})) = \{0\}$. From the argument, we see,*

$$H^0(\Omega^\bullet(\mathcal{M})) = \prod_{\text{components of } \mathcal{M}} \mathbb{R}.$$

Example 7.2. *For $\mathcal{M} = S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\} = \mathbb{R}/2\pi\mathbb{Z}$. The diffeomorphism is given by,*

$$\mathbb{R}/2\pi\mathbb{Z} \ni \theta \mapsto e^{i\theta}.$$

From the example above we know

$$H^0(S^1) = \mathbb{R}.$$

Suppose $\omega \in \Omega^1(S^1)$. Then there is 2π periodic function $f \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\omega = f d\theta.$$

Thus $d\omega = 0$. This shows us that

$$Z^1(\Omega^\bullet(S^1)) = \Omega^1(S^1).$$

Suppose $\omega = dg$ for some 2π-periodic function $g \in \mathcal{C}^\infty(\mathbb{R})$ then,

$$\int_{S^1} \omega = \int_0^{2\pi} f(\theta) d\theta = g(2\pi) - g(0) = 0.$$

Remark 7.1. *The above example can be furthere generalized as follows.*

Suppose $\mathcal{M}$ is compact and closed, $\omega \in \Omega^m(\mathcal{M})$ where $m = \dim \mathcal{M}$. If $\omega = d\eta$ then

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{M}} d\eta = \int_{\partial\mathcal{M} = \emptyset} \eta = 0.$$

Conversely, if $\int_{\mathcal{M}} \omega = 0$, put

$$g(\theta) := \int_0^\theta f(t) dt.$$

Then g is smooth and 2π-periodic. That is $dg = \omega$.

Proposition 7.1. *Let us surmarize what we have discussed. We have a commutative diagram,*

$$\begin{tikzcd} \Omega^1(\mathbb{R}/2\pi\mathbb{Z}) \arrow[r, "\int"] \arrow[d] & \mathbb{R} \\ H^1(\mathbb{R}/2\pi\mathbb{Z}) \arrow[ur, "\cong"'] & \end{tikzcd}$$

We will later see that for every compact closed oriented manifold, we have the above commutative diagram.

Definition 7.5. *A homomorphism of cochain complexes is a linear map* $f : (E, d^E) \to (F, d^F)$ *of degree* 0 *such that*

$$f \circ d^E = d^F \circ f.$$

In other words, we have a following commutative diagram,

$$\begin{tikzcd} \cdots \arrow[r] & E^{k-1} \arrow[r, "d^{E,k-1}"] \arrow[d, "f"] & E^k \arrow[r, "d^{E,k}"] \arrow[d, "f"] & E^{k+1} \arrow[r] \arrow[d, "f"] & \cdots \\ \cdots \arrow[r] & F^{k-1} \arrow[r, "d^{F,k-1}"'] & F^k \arrow[r, "d^{F,k}"'] & F^{k+1} \arrow[r] & \cdots \end{tikzcd}$$

Proposition 7.2. *A cochain map* $f \in \mathrm{Hom}((E, d^E), (F, d^F))$ *induces a homomorphism,*

$$H^k(f) : H^k(E, d^E) \to H^k(F, d^F).$$

Proof. Clear from the above diagram. □

Remark 7.2. *Recall that for a smooth map* $f : \mathcal{M} \to \mathcal{N}$*, it induces a contravariant functor,*

$$f^* : (\Omega^\bullet(\mathcal{N}), d) \to (\Omega^\bullet(\mathcal{M}), d).$$

Even though $H^\bullet$ *is a covariant functor from the category of cochain complexes to cochain complexes. The above functor is contravariant. This is due to that the pullback is contravariant.*

Definition 7.6.

$$\Omega^k_C(\mathcal{M}) = \{\omega \in \Omega^k(\mathcal{M}) \mid \operatorname{supp}\omega \text{ is compact.}\}.$$

Definition 7.7. *Let us define a category* $(\mathrm{Mnfd}_{\mathrm{OpEmb}})$ *such that*

i). $\mathrm{Ob}(\mathrm{Mnfd}_{\mathrm{OpEmb}})$ *consists of smooth manifolds.*

ii). Morphisms are open embeddings.

Remark 7.3. $\Omega^\bullet_C$ *is a covariant functor from* $(\mathrm{Mnfd}_{\mathrm{OpEmb}})$ *to the category of cochain complex.*

Definition 7.8.

$$H^k_{\mathrm{dR},C}(\mathcal{M}) = H^k(\Omega^\bullet_C(\mathcal{M}), d).$$

Proposition 7.3. *If* $f : \mathcal{M} \to \mathcal{N}$ *is proper and* $\omega \in \Omega^k_C(\mathcal{N})$ *then* $f^*\omega \in \Omega^k_C(\mathcal{M})$*.*

Remark 7.4. *On the category of smooth manifolds with proper maps.* $\Omega^\bullet_C$ *and hence* $H^\bullet_{\mathrm{dR},C}$ *are contravariant functors.*

Definition 7.9.

7.2 Poincaré Lemma

Suppose we have,

$$f, g : (E^\bullet, d^E) \to (F^\bullet, d^F).$$

When do we have,

$$H^\bullet(f) = H^\bullet(g).$$

Definition 7.10. *Two cochain complex homomorphisms $f, g : (E^\bullet, d^E) \to (F^\bullet, d^F)$ are called cochain homotopic if there exists homomorphism $K_{k+1} : E^{k+1} \to F^k$ such that*

$$d^F_{k-1} K_k + K_{k+1} d^E_k = f - g.$$

Let $\omega \in Z^k(E)$. That is $\omega \in E^k, d_k \omega = 0$. Consider,

$$f(\omega) - g(\omega) = dK\omega + Kd\omega = dK\omega \in B^K(F).$$

Thus $H^k(f) = H^k(g)$.
Let $\mathcal{M}$ be a smooth manifold (oriented). Consider,

$$\Omega^1(\mathcal{M}), \Omega^1(\mathcal{M} \times \mathbb{R}).$$

For fixed $t_0 \in R$, let

$$j_k : \mathcal{M} \to \mathcal{M} \times \{t_0\} \subset \mathcal{M} \times \mathbb{R}, p \mapsto (p, t_0).$$

Put $K : \Omega^{k+1}(\mathcal{M} \times \mathbb{R}) \to \Omega^k(\mathcal{M} \times \mathbb{R})$ such that

$$K(\omega)|_{(p,t_0)} := \int_{t_0}^{t} (\mathrm{int}_{\frac{\partial}{\partial t}} \omega)(p, t') dt'.$$

Take $X_1, \cdots, X_m$ be a coordinate in $\mathcal{M}$.

$$\int_{\frac{\partial}{\partial t}} (f(x,t) dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_p}) = f(x,t) dx_{j_1} \wedge \cdots \wedge dx_{j_p},$$
$$\int_{\frac{\partial}{\partial t}} (g(x,t) dx_{j_1} \wedge \cdots \wedge dx_{j_p}) = 0.$$

Proposition 7.4. *Let $\pi : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ be such that $(p,t) = p$. Then,*

$$dK + Kd = \mathrm{id} - \pi^* \circ j^*_{t_0}.$$

Thus $H(\pi) \circ H(j_{t_0}) = \mathrm{id}$. Furthermore,

$$H(\pi) = \pi^*, H^k(\mathcal{M}) \to H^K(\mathcal{M} \times \mathbb{R})$$

is an isomorphism.

Proof. For the last two assertions,

$$\begin{aligned}\pi \circ j_{t_0} &= \mathrm{id}_{\mathcal{M}},\\ j_{t_0}^* \circ \pi^* &= \mathrm{id}_{\Omega^1(\Omega)},\\ &\Rightarrow H(j_{t_0}) \circ H(\pi) = \mathrm{id}\,.\end{aligned}$$

Let $\omega \in \Omega^{k+1}(\mathcal{M} \times \mathbb{R}_t)$ such that

$$\omega = \omega_1(t)dt + \omega_2(t).$$

Under this we have,

$$\begin{aligned}K\omega &= (-1)^K \int_{t_0}^{t} \omega_1(s)ds,\\ dK\omega &= \omega_1(t) \wedge dt + (-1)^K \int_{t_0}^{t} (d_{\mathcal{M}}\omega_1)(s)ds,\\ d\omega &= (d_{\mathcal{M}}\omega_1)(t)dt + (-1)^{K+1}\omega_2'(t) \wedge dt + (d_{\mathcal{M}}\omega_2)(t).\\ Kd\omega &= (-1)^{K+1} \int_{t_0}^{t} (d_{\mathcal{M}}\omega_1)(s)ds + \int_{t_0}^{t} \omega_2'(s)ds,\\ dK\omega + Kd\omega &= \underbrace{\omega_1(t) \wedge dt + \omega_2(t)}_{=\omega} - \overbrace{\omega_2(t_0)}^{\pi^* j_{t_0}^* \omega},\\ &= \omega - \pi^* j_{t_0}^* \omega.\\ \pi^* j_{t_0}^* \omega = \omega_2(t_0).\end{aligned}$$

$\square$

Corollary 7.1. *For each $t_0 \in \mathbb{R}$, the map $j_{t_0}^* : \Omega^\bullet(\mathcal{M}) \to \Omega^\bullet(\mathcal{M})$, induces an isomorphism*

$$H_{\mathrm{dR}}^\bullet(\mathcal{M} \times \mathbb{R}) \to H_{\mathrm{dR}}^\bullet(\mathcal{M}),$$

with inverse

$$H(\pi) = \pi^*.$$

Remark 7.5. *For $t_0, t_1 \in \mathbb{R}$, we have*

$$j_{t_0}^* = j_{t_1}^*,$$

as maps from $H^\bullet(\mathcal{M} \times \mathbb{R}) \to H^\bullet(\mathcal{M})$.

Theorem 7.1. *Let $f, g : \mathcal{M} \to \mathcal{N}$ be smooth homotopic maps(the homotopy between them is smooth), then*

$$H(f) = H(g) : H^\bullet(\mathcal{N}) \to H^\bullet(\mathcal{M}).$$

In particular, if the manifolds $\mathcal{M}, \mathbb{N}$ are smooth homotopic equivalent then

$$H^\bullet(\mathcal{M}) \cong H^\bullet(\mathcal{N}).$$

Proof. By assumption, there is a smooth map $F : \mathcal{M} \times \mathbb{R} \to \mathcal{N}$ such that

$$F(\cdot, 0) = f, F(\cdot, 1) = g.$$

Let $j_t : \mathcal{M} \to \mathcal{M} \times \mathbb{R}$ be such that $j_t(p) = (p, t)$. Thus with this notation, we have,

$$F \circ j_0 = f, F \circ j_1 = g.$$

Thus we have,

$$\begin{aligned}
H(f) &= H(F \circ j_0), \\
&= H(j_0) \circ H(F), \\
&= H(j_1) \circ H(F), \\
&= H(F \circ j_1), \\
&= H(g).
\end{aligned}$$

$\mathcal{M}, \mathcal{N}$ are smoothly hotomipic equivalent if and only if there exists $f : \mathcal{M} \to \mathbb{N}, g : \mathcal{N} \to \mathcal{M}$ such that

$$f \circ g = \mathrm{id}_{\mathbb{N}}, g \circ f = \mathrm{id}_{\mathcal{M}}.$$

Then $H(f), H(g)$ are inverse to one another. □

Theorem 7.2 (Poincaré Lemma)**.** *Let $U \subseteq \mathbb{R}^m$ be a starshaped region(ie. an open set). Then,*

$$H^k(U) \cong \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

Proof. Choose $g \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$g(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t \geq 1. \end{cases}$$

Then set $F : U \times \mathbb{R} \to U$ by

$$F(x, t) = x_0 + g(t)(x - x_0),$$

where x_0 is a star-point of U. Thus we have,

$$F(x, 0) = x_0, F(x, 1) = x.$$

id_U is smooth homotopic to a constant map x_0. Call it g. For $k \geq 1$,

$$\mathrm{id}_{H^k(U)} = H(g^*)|_{H^k(U)} = 0.$$

□

7.3 Mayer-Vietoris sequence

Definition 7.11. *A short exact sequence of cochain complexes*

$$0 \longrightarrow (A^\bullet, d) \xrightarrow{\varphi} (B^\bullet, d) \xrightarrow{\psi} (C^\bullet, d) \longrightarrow 0$$

is exact if each row of the following diagram is exact.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A^k \arrow[r, "\varphi"] \arrow[d, "d"] & B^k \arrow[r, "\psi"] \arrow[d, "d"] & C^k \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{k+1} \arrow[r, "\varphi"] \arrow[d] & B^{k+1} \arrow[r, "\psi"] \arrow[d] & C^{k+1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Lemma 7.1 (Snake Lemma)**.** *Suppose we have a short exact sequence*

$$0 \longrightarrow (A^\bullet, d) \xrightarrow{\varphi} (B^\bullet, d) \xrightarrow{\psi} (C^\bullet, d) \longrightarrow 0$$

of cochain complexes. Then there exists δ such that

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(A) \arrow[r, "H^k(\varphi)"] & H^k(B) \arrow[r, "H^k(\psi)"] & H^k(C) \arrow[dll, "\delta"'] & \\
 & H^{k+1}(A) \arrow[r, "H^{k+1}(\varphi)"] & H^{k+1}(B) \arrow[r, "H^{k+1}(\psi)"] & H^{k+1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Consider the following diagram, where we consider lifting from an element of $Z^k C$.

$$\begin{tikzcd}
 & A^{k+1} \arrow[r, "d"] \arrow[d, "\varphi"'] & A^{k+2} \arrow[d, "\varphi"] \\
B^k \arrow[r, "d"] \arrow[d, "\psi"'] & B^{k+1}(B) \arrow[r, "d"'] \arrow[d, "\psi"] & B^{k+2} \\
Z^k C \arrow[r, "d"'] & C^{k+1} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & a \arrow[r, mapsto, "d"] \arrow[d, mapsto, "\varphi"'] & da \arrow[d, mapsto, "\varphi"] \\
b \arrow[r, mapsto, "d"] \arrow[d, mapsto, "\psi"'] & db \arrow[r, mapsto, "d"'] \arrow[d, mapsto, "\psi"] & 0 \\
c \arrow[r, mapsto, "d"'] & 0 &
\end{tikzcd}$$

We see that $\varphi(da) = 0$. By injectivity of φ, we have $da = 0$. Thus this defines a well-defined map from

$$Z^k C \ni c \mapsto a \in Z^{k+1}(A),$$

up to equivalence classes by cycles. We set,

$$\delta[c]_{H^k(C)} = [a]_{H^{k+1}(A)}.$$

We then need to check,

i). $H^k(\psi) \circ \varphi^k = 0, \operatorname{im} H^k(\varphi) = \ker H^k(\psi)$,

ii). $\delta \circ H^k(\psi) = 0, \ker \delta = \operatorname{im} \psi^k$,

iii). $H^k(\varphi) \circ \delta = 0, \ker H^k(\varphi) = \operatorname{im} \delta$.

Let $\tilde{c} \in Z^k C$ and $\tilde{c} = c + d\sigma$. Choose a lift $\tilde{b} \in B^k$ such that $\psi(\tilde{b}) = \tilde{c}$. Choose $\tilde{\sigma} \in B^{k-1}$ such that

$$\psi(\tilde{\sigma}) = \sigma.$$

Then,

$$\psi(\tilde{b} - b) = d\sigma = d\psi(\tilde{\sigma}) = \psi(d\tilde{\sigma}).$$

Therefore,

$$\psi(\tilde{b} - b - d\tilde{\sigma}) = 0.$$

Then there is $\tau \in A^k$ such that $\varphi(\tau) = \tilde{b} - b - d\tilde{\sigma}$. Furthermore, we have,

$$\varphi(d\tau) = d\varphi(\tau) = d\tilde{b} - db.$$

Let $\tilde{a}, a \in A^{k+1}$ be such that

$$\varphi(\tilde{a}) = d\tilde{b}, \varphi(a) = db.$$

That is

$$\varphi(\tilde{a} - a - d\tau) = 0 \Rightarrow \tilde{a} - a = d\tau.$$

That is $[\tilde{a}]_{H^{k+1}(A)} = [a]_{H^{k+1}(A)}$.

Now we examine the exactness, If $x \in H^k C$ is in $\operatorname{im} H^k(\psi)$, then

$$\exists b \in Z^k B \text{ s.t. } x = [\psi(b)].$$

$db = 0$ hence $a = 0$.

Let $\delta[c] = 0$, then there is $a = d\alpha, \alpha \in A^k$, thus,

$$\begin{aligned} \psi(b - \varphi(\alpha)) &= \psi(b) - c, \\ d(b - \varphi(\alpha)) = db - \varphi(d\alpha) &= db - \varphi(\alpha) = 0. \end{aligned}$$

Hence, $[c] = \psi_*[b - \varphi(\alpha)] \in \operatorname{im} H^k(\psi)$.

Given $c \in Z^k A$, then $\varphi(a) = db$ hence $H^k(\varphi)[a] = 0$. Thus $H^k(\varphi) \circ \delta = 0$. Conversely, let $[a] \in H^{k+1}(A)$ with $H^{k+1}(\varphi)[a] = 0$. Then $da = 0, \varphi(a) = db$. Set $c := \psi(b)$. Thus

$$dc = d\psi(b) = \psi(db) = \psi \circ \varphi(a) = 0.$$

This proves that $[a] = \delta[c]$.

The rest is left to the readers. □

Theorem 7.3 (Mayer-Vietoris for deRham Theory)**.** *Let $\mathcal{M} = U \cup V$ be a smooth manifold where $U, V \subseteq \mathcal{M}$ are open sets. Then,*

$$0 \longrightarrow \Omega^\bullet(\mathcal{M}) \xrightarrow{\omega \mapsto (\omega|_U, \omega|_V)} \Omega^\bullet(U) \xrightarrow[(\eta,\tau)\mapsto \eta|_{U\cap V} - \tau|_{U\cap V}]{} \Omega^\bullet(U \cap V) \longrightarrow 0$$

is exact. Hence there is the long exact cohomology sequence,

$$\begin{array}{l} 0 \longrightarrow H^0\mathcal{M} \longrightarrow H^0U \oplus H^0V \longrightarrow H^0U \cap V \\ \qquad H^1\mathcal{M} \longrightarrow H^1U \oplus H^1V \longrightarrow H^1U \cap V \longrightarrow \cdots \\ \cdots \longrightarrow H^m\mathcal{M} \longrightarrow H^mU \oplus H^mV \longrightarrow H^mU \cap V \longrightarrow 0 \end{array}$$

Proof. Only the surjectivity at the right needs proof. Choose a smooth partition of unity $\{\varphi_U, \varphi_V\}$ subordinates to $\{U, V\}$. Given $\omega \in \Omega^k(U \cap V)$, put

$$s\omega = (\varphi_V\omega, -\varphi_U\omega) \in \Omega^kU \oplus \Omega^k. \tag{$*$}$$

□

Remark 7.6. *The lifting map in Equation $(*)$ allows to make δ explicit. That is given $\omega \in \Omega^k(U \cap V)$ representing $[\omega] \in H^k(U \cap V)$ that is $d\omega = 0$. Let*

$$\chi := s\omega = (\varphi_V\omega, -\varphi_U\omega), d\chi = (d(\varphi_V\omega), -d(\varphi_U\omega)).$$

Then we have,

$$\delta[\omega] = [d(\varphi_V\omega)]_{H^{k+1}(\mathcal{M})}.$$

We warn that $d(\varphi_j\omega)$ extends by 0 to a closed form on $\mathcal{M}$. It is not necessarily an exact on $\mathcal{M}$.

Lemma 7.2. *Let*

$$0 \longrightarrow E^0 \xrightarrow{d} E^1 \xrightarrow{d} \cdots \xrightarrow{d} E^n \longrightarrow 0$$

be an exact sequence of finite dimensional vector spaces. Then

$$\sum_{j=0}^n (-1)^j \dim E^j = 0.$$

Proof. Exercise. □

Example 7.3. *Consider S^n and consider two stereographic projection $(U = S^n \backslash \{N\}, \sigma_N), (V = S^n \backslash \{-N\}, \sigma_{-N})$ where $N = (0, \cdots, 0, 1)$. Then we have,*

$$U \cap V \simeq S^{n-1} \times \mathbb{R} \cong_n S^{n-1}.$$

Then consider Mayer-Vietoris sequence,

$$\begin{tikzcd}
0 \arrow[r] & H^0S^n \arrow[r] & H^0\mathbb{R}^n \oplus H^0\mathbb{R}^n \arrow[r] & H^0S^{n-1} \arrow[dll] & \\
& H^1S^n \arrow[r] & H^1\mathbb{R}^n \oplus H^1\mathbb{R}^n \arrow[r] & H^1S^{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\cdots \longrightarrow H^kS^n \longrightarrow H^k\mathbb{R}^n \oplus H^k\mathbb{R}^n \longrightarrow H^kS^{n-1} \longrightarrow \cdots$$

Using Theorem 7.2 and Example 7.2, we obtain,

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}\oplus\mathbb{R} \arrow[r] & \underbrace{\mathbb{R}}_{n\neq 1, n=1\Rightarrow \mathbb{R}\oplus\mathbb{R}} \arrow[dll, "0"'] & \\
& 0 \arrow[r] & 0 \arrow[r, "0"] & H^1S^{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\cdots \longrightarrow H^kS^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

For $n = 1$, we have,

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}\oplus\mathbb{R} \arrow[r] & \mathbb{R}\oplus\mathbb{R} \arrow[dll] \\
& \underbrace{H^1(S^1)}_{\cong\mathbb{R}} \arrow[r] & 0 &
\end{tikzcd}$$

In particular $H^{k-1}(S^{n-1}) \simeq H^k(S^n)$ for $k \geq 1$.

For $1 \leq k \leq n$,

$$H^kS^n = H^1(S^{n-k+1}) \cong \begin{cases} 0, & (k \neq n), \\ \mathbb{R}, & (k = n). \end{cases}$$

7.4 Compactly supported deRham Cohomology and Poincaré duality

Consider

$$0 \longrightarrow \Omega^\bullet_C(U\cap V) \xrightarrow{i^U_* - i^V_* = [\omega \mapsto (-i^U_*\omega, i^V_*\omega)]} \Omega^\bullet_C(U) \oplus \Omega^\bullet_C(V) \xrightarrow[(\tau,\eta)\mapsto(i^{\mathcal{M}}_*\tau + i^{\mathcal{M}}_*\eta)]{} \Omega^\bullet_C(\mathcal{M}) \longrightarrow 0$$

This turns out to be an exact sequence. The proof will be shown next week.

References

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