

Analysis and Geometry on Manifolds

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1 Review of Structure Theorems for Differentiable Maps

Theorem 1.1 (Implicit Function Theorem). *Let $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$ be open sets and $F(x, y) : U \times V \rightarrow \mathbb{R}^q$ be of \mathcal{C}^∞ class. If $(a, b) \in U \times V$ satisfies $F(a, b) = 0$ and*

$$D_y F(a, b) = \left(\frac{\partial F_i}{\partial y_j}(a, b) \right) \in \mathrm{GL}_q(\mathbb{R})$$

then there exists

i). neighborhoods $a \in U_1$ and $b \in V_1$,

ii). $\varphi \in \mathcal{C}^\infty(U_1, V_1)$

such that

$$\forall (x, y) \in U_1 \times V_1, F(x, y) = 0 \Leftrightarrow \varphi(x) = y.$$

Furthermore, we have that

$$D\varphi(x) = -D_y F(x, \varphi(x))^{-1} \cdot D_x F(x, \varphi(x)).$$

Theorem 1.2 (Inverse Function Theorem). *Let $U \subseteq \mathbb{R}^p$ be an open subset and $f : U \rightarrow \mathbb{R}^q$ be smooth. Let $a \in U$ be such that $Df(a)$ is invertible. Then there are neighborhoods $a \in U_1 \subseteq U$ and $f(a) \in V_1$ such that $f|_{U_1} : U_1 \rightarrow V_1$ is a diffeomorphism.*

Theorem 1.3 (Rank Theorem). *Let $U \subseteq \mathbb{R}^p$ be open and $f : U \rightarrow \mathbb{R}^q$ be smooth. Let $a \in U$ and $b = f(a)$. If $\mathrm{rk} Df(a) = r$ then there exists local diffeomorphisms*

i). $\psi : U_\psi \subseteq U \rightarrow V_\psi \subseteq \mathbb{R}^p$ with $\psi(a) = 0$

ii). $\varphi : U_\varphi \subseteq \mathbb{R}^q \rightarrow V_\varphi \subseteq \mathbb{R}^q$ with $\varphi(b) = 0$.

such that

$$\varphi \circ f \circ \psi^{-1}(x_1, \dots, x_p) = (x_1, \dots, x_r, \tilde{f}(x)).$$

Furthermore, if $\text{rk } Df(x) = r$ in some neighborhood of a , then \tilde{f} can be chosen to be 0.

2 Differentiable Manifolds

2.1 Basics from Set Theoretic Topology

Definition 2.1. Let X be a topological space. X is said to be separated/Hausdorff if any two distinct points have open neighborhoods which are disjoint to one another.

Definition 2.2. Let X be a topological space and $(U_i)_{i \in I}$ be its open covering. A refinement of $(U_i)_{i \in I}$ is an open covering $(V_j)_{j \in J}$ such that

$$\forall j \in J, \exists i \in I \text{ s.t. } V_j \subseteq U_i.$$

It is locally finite if each point $x \in X$, there exists a neighborhood U_i such that

$$|\{j \in J \mid U \cap V_j \neq \emptyset\}| < \infty.$$

Definition 2.3. A topological space X is called paracompact if every open covering $(U_i)_{i \in I}$ has a locally finite refinement $(V_j)_{j \in J}$.

Example 2.1. The following spaces are paracompact.

- 1). Compact spaces.
- 2). Locally compact Hausdorff spaces which are first countable.

Definition 2.4. A subset $M \subseteq \mathbb{R}^p$ is said to be a submanifold of dimension m if M can be covered by open sets $(U_i)_{i \in I}$ such that there exists a smooth function $F_i : U_i \rightarrow \mathbb{R}^{p-m}$ with full rank and

$$M \cap U_i = \{x \in U_i \mid F_i(x) = 0\}.$$

In other words, M is locally a graph of a smooth map.

Definition 2.5. Let X be a separated and paracompact topological space. A chart is a pair (U, φ) where U is an open subset of X and $\varphi : U \rightarrow V \subseteq \mathbb{R}^n$ is a homeomorphism onto some open subset V of \mathbb{R}^n . An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ is a collection of charts such that $(U_i)_{i \in I}$ covers X .

Definition 2.6. Let $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ be an atlas. A transition map is a composition $\varphi_i \circ \varphi_j^{-1}|_{\varphi_j(U_i \cap U_j)} \rightarrow \varphi_i(U_i \cap U_j)$ for some $i, j \in I$.

Definition 2.7. An atlas \mathcal{A} is called smooth if all the transition maps are smooth.

Definition 2.8. A chart (U, φ) is smooth compatible with smooth atlas \mathcal{A} if for any $i \in I$, $\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ are smooth.

Definition 2.9. An atlas \mathcal{A} is smooth maximal if (U, φ) is smooth compatible with \mathcal{A} then $(U, \varphi) \in \mathcal{A}$.

Definition 2.10. A paracompact, smooth manifold is a pair (M, \mathcal{A}) such that

i). M is a paracompact separated topological space.

ii). \mathcal{A} is smooth maximal.

Remark 2.1. Above notions can be defined for C^k , analytic, continuous, algebraic, linear by simply replacing the word "smooth" with one of the formers.

Remark 2.2. It suffices to give one (smallest possible) atlas to define a smooth structure.

2.2 Examples of Smooth Manifolds

Definition 2.11. A topological space is second countable if it has a basis consists of at most countably many open subsets.

Remark 2.3. Instead of paracompactness, we assume smooth manifolds to be separated and second countable space. From this definition, we can also induce paracompactness.

Example 2.2. For $x, y \in \mathbb{R}$, $x \sim y \Leftrightarrow |x| = |y| > 1$. With this relation, we construct a quotient space $X = \mathbb{R} / \sim$.

We introduce its atlas by $U_1 = (-\infty, 1) / \sim = X - \{[1]\}$ with chart $\varphi_1(x) = [x]$, and $U_2 = (-1, \infty) / \sim = X - \{[-1]\}$ with chart $\varphi_2(x) = [x]$. However, this is non-separated as 1 and -1 cannot be separated.

Definition 2.12. A smooth submanifold of \mathbb{R}^n is a separated, second countable smooth manifold.

Take φ^{-1} , where φ runs through parametrization as charts.

Remark 2.4. $(\mathbb{R}^n, \{\text{id}\})$ is separated and second countable smooth manifold.

Definition 2.13. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$. Let $N \in S^n$ and call it the north pole. We define the stereographic projection with north pole N to be such that

$$\varphi_+ : S^n \setminus \{N\} \rightarrow \mathbb{R}^n, (x_1, \dots, x_n) \mapsto \frac{1}{1 - x_n}(x_1, \dots, x_{n-1}).$$

Similarly, we define

$$\varphi_- : S^n \setminus \{-N\} \rightarrow \mathbb{R}^n, (x_1, \dots, x_n) \mapsto \frac{1}{1 + x_n}(x_1, \dots, x_{n-1}).$$

Definition 2.14. A n -dimensional torus T^n is a direct product of n -many S^1 .

Definition 2.15. A lattice Γ in \mathbb{R}^n is a subgroup generated by a basis of \mathbb{R}^n over \mathbb{Z} .

Lemma 2.1.

$$T^n = \mathbb{R}^n / \Gamma.$$

Definition 2.16. Let $k = \mathbb{R}$ or \mathbb{C} . The projective space over the field K is

$$K\mathbb{P}^n = (K^{n+1} \setminus \{o\}) / K^\times.$$

Remark 2.5.

$$\mathbb{RP}^n = S^n / x \sim -x.$$

And also we have,

$$\mathbb{CP}^n = S^{2n+1} / S^1.$$

We can introduce atlases to them by

$$U_j = \{[x_0 : \cdots : x_n] \in K\mathbb{P}^n \mid x_j \neq 0\},$$

with chart,

$$\varphi_j : U_j \rightarrow K^n, \varphi_j([x_0 : \cdots : x_n]) = \left(\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j} \right).$$

Definition 2.17. Let \mathcal{M}, \mathcal{N} be smooth manifold. A function $f : \mathcal{M} \rightarrow \mathcal{N}$ is called smooth if for each $p \in \mathcal{M}$, there exist chart (U, φ) around p and (V, ψ) around $f(p)$ such that

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is smooth for some $m, n \in \mathbb{N}$.

Remark 2.6. Since transition maps are smooth, above map is well-defined and if f is smooth so is $\varphi' \circ f \circ \psi'$ for any charts φ', ψ' .

Proposition 2.1. Compositions of smooth maps are smooth.

Definition 2.18. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map. It is called a diffeomorphism if it is bijective and f^{-1} is also smooth.

2.3 Partitions of Unity

Proposition 2.2. Let $(\mathcal{M}, \mathcal{A})$ be a pair of a smooth manifold with its atlas which we assumed to be separated and second countable. Then it is paracompact. More precisely, given an open covering $(U_i)_{i \in I}$, there exists a countable locally finite refinement $(V_j)_{j \in J}$ together with a chart $\{\psi_j\}_{j \in J}$ which can be chosen such that

$$\psi_j : V_j \rightarrow B(o, 3)$$

such that

$$\mathcal{M} = \bigcup_{j \in J} \psi_j^{-1}(B(o, 1)).$$

Proof. \mathcal{M} is locally compact (look at the charts), hence there exists a compact subsets

$$K_1 \subset \subset K_2 \subset \subset K_3$$

such that

$$\mathcal{M} = \bigcup K_j$$

(ie. an exhaustion by compact sets). Note $K_{j+1} - \text{int } K_j$ is again compact. For $p \in K_{j+1} - \text{int } K_j$ choose a chart (V_p, ψ_p) such that

$$\psi_p(V_p) = B(o, 3), \psi_l(p) = 0.$$

Note that we can take V_p small enough so that there is $i \in I$ such that $V_p \subset U_i$.

By compactness, we can take $p_{j,1}, \dots, p_{j,r_j}$ such that

$$K_{j+1} - \text{int } K_j = \bigcup_{l=1}^{r_j} \varphi_{jl}^{-1}(B(0, 1)).$$

By making $V_{p_{j,l}}$ small enough we can assume

$$V_{p_{j,l}} \subset \text{int } K_{j+2} - K_{j-1}.$$

The union over all j gives the countable locally finite refinement. \square

From now on, a manifold refers to a smooth, separated, and second countable manifold.

Example 2.3. We define,

$$f_1(t) := \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Then f_1 smooth. We then define,

$$f_2(t) := \frac{f_1(t)}{f_1(t) + f_1(1-t)}.$$

f_2 is a function which is monotonically increasing and for $t \geq 1$, we have $f_2(t) = 1$.

$$f_3(t) = f_2(2+t) + f_2(2-t).$$

Again we define

$$f_4 : \mathbb{R}^n \rightarrow \mathbb{R}, f(x) = f_3(\|x\|).$$

Definition 2.19. A support of the function $f : X \rightarrow \mathbb{R}$ is

$$\text{supp}(f) := \overline{f^{-1}(\mathbb{R} \setminus \{0\})}.$$

Theorem 2.1. *Let \mathcal{M} be a manifold and $(U_i)_{i \in I}$ be an open covering. Then there exist smooth functions $(\varphi_n : \mathcal{M} \rightarrow \mathbb{R})_{n \in \mathbb{N}}$ such that*

- i). $\forall n \in \mathbb{N}, 0 \leq \varphi_n \leq 1.$
- ii). $(\text{supp } \varphi_n)_{n \in \mathbb{N}}$ is locally finite.
- iii). $\forall n \in \mathbb{N} \exists i \in I, \text{ s. t. } \text{supp } \varphi_n \subset U_i.$
- iv). $\forall p \in \mathcal{M}, \sum_{n=1}^{\infty} \varphi_n(p) = 1.$

Such sequence of functions is called a *partition of unity subordinated by the given covering* $(U_i)_{i \in I}$.

Additionally, if $I \subset \mathbb{N}$, then $(\varphi_n)_{n \in \mathbb{N}}$ can be indexed by I as well in such a way that,

$$\forall i \in I, \text{supp } \varphi_i \subset U_i.$$

Proof. By Proposition 2.2, take a countable, locally finite refinement charts $((\tilde{\psi}_n, V_n))_{n \in \mathbb{N}}$. Borrowing the notation from Exampple 2.3, we set

$$\tilde{\varphi}_n(x) := \begin{cases} f_4(\tilde{\psi}_n(x)) & \forall x \in V_n, \\ 0, & x \notin V_n. \end{cases}$$

Then take $\tilde{\varphi}$ to be

$$\varphi := \sum_{n=1}^{\infty} \tilde{\varphi}_n \in \mathcal{C}^{\infty}(\mathcal{M}).$$

We observe that φ is nowhere 0. Thus we define

$$\varphi_n := \tilde{\varphi}_n / \tilde{\varphi},$$

we derived a desired family of functions.

For the second part, set

$$J_0 = \emptyset, \quad \varphi_0 = 0,$$

define inductively that

$$J_k = \left\{ i \in \mathbb{N} \setminus \bigcup_{i=0}^{k-1} J_i \mid \text{supp } \varphi_i \subseteq U_k \right\}.$$

We then take

$$\psi_k := \sum_{i \in J_k} \tilde{\varphi}_i / \tilde{\varphi}.$$

□

Proposition 2.3. *Let $A \subset \mathcal{M}$ be a closed subset of a manifold such that there is an open set $A \subset G \subset \mathcal{M}$. Then there exists a smooth function $f \in \mathcal{C}^\infty(\mathcal{M})$ such that the image of f is contained in $[0, 1]$ and*

$$\forall p \in A, f(p) = 1, \text{ and } \forall p \in G, f(p) = 0.$$

Proof. Observe that $\{\mathcal{M} - A, G\}$ is an open covering. By Theorem 2.1, we can take smooth functions φ, ψ such that

$$\text{supp } \varphi \subset \mathcal{M} - A, \text{supp } \psi \subset G, \varphi + \psi \equiv 1.$$

Take $f = \psi$. □

2.4 Tangent Spaces

Example 2.4. *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a submanifold. Then we have a notion of tangent vector, such that for some $v \in \mathbb{R}^n$, we define a smooth curve γ such that*

$$\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}, \quad \gamma'(0) = v.$$

Given $v \in \mathbb{R}^n$, $p \in \mathcal{M}$ and $f : \mathcal{M} \rightarrow \mathbb{R}^n$, we define

$$vf = (f \circ \gamma)'(0).$$

where $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}$ is a smooth curve such that $\gamma(0) = p, \gamma'(0) = v$. Exercise : show that this definition does not depend on the choice of γ satisfying the two conditions above. Furthermore, this defines a linear map

$$v : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathbb{R},$$

such that

$$v(f \cdot g) = (f \circ \gamma \cdot g \circ \gamma)'(0) = f(p) \cdot (vg) + (vf) \cdot g(p).$$

Definition 2.20. *Let \mathcal{M} be a smooth manifold and $p \in \mathcal{M}$. The linear map $X_p : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathbb{R}$ is called a derivation at p if $X_p(fg) = f(p)X_p g + g(p)X_p f$.*

We define the tangent space of \mathcal{M} at p to be

$$T_p \mathcal{M} = \{X_p : \mathcal{C}^\infty \mathcal{M} \rightarrow \mathbb{R} \mid X_p \text{ is a derivation}\}.$$

Remark 2.7. *Obviously $T_p \mathcal{M}$ is a \mathbb{R} -vectorspace.*

Lemma 2.2. *if $\varphi \in \mathcal{C}^\infty(\mathcal{M})$ is constant around $p \in \mathcal{M}$. Then $X\varphi = 0$ for any $X \in T_p \mathcal{M}$.*

Proof. Let $\varphi \equiv 1$ around p . Choose $\chi \in \mathcal{C}^\infty(\mathcal{M})$ to be such that

1. $\chi \equiv 1$ in a neighborhood of p .
2. $\text{supp } \chi \subseteq \{q \in \mathcal{M} \mid \varphi(q) = 1\}$.

Then $\chi\varphi = \chi$. Thus we have

$$X(\chi) = X(\chi)\varphi(p) + \chi(p)X(\varphi). \quad (1)$$

Since $\varphi(p) = 1$ and $\chi(p) = 1$, thus we conclude $X(\varphi) = 0$. \square

Lemma 2.3. *If $X \in T_p\mathcal{M} \cap T_q\mathcal{M}$ then either $X = 0$ or $p = q$.*

Proof. Suppose $p \neq q$. Then take a smooth functional $\chi : \mathcal{M} \rightarrow \mathbb{R}$ such that $\chi(\mathcal{M}) \subseteq [0, 1]$ and

$$\begin{cases} \chi \equiv 1 & \text{around } p, \\ \chi \equiv 0 & \text{around } q. \end{cases}$$

Then for any $f \in \mathcal{C}^\infty(\mathcal{M})$,

$$Xf = X\chi f = 0.$$

\square

Lemma 2.4. *Let (U, φ) be a chart of \mathcal{M} centered at $p \in \mathcal{M}$ (ie. $\varphi(p) = 0$) with coordinate function x_1, \dots, x_n . Then for $f \in \mathcal{C}^\infty(U)$, there are functions $f_1, \dots, f_n \in \mathcal{C}^\infty(U)$, such that*

$$f = \sum_{j=1}^n f_j x_j + f(p).$$

Note that taking $\varphi(p) = 0$ justifies the $f_j x_j$ for each coordinate.

Analogously, given $f \in \mathcal{C}^\infty(\mathcal{M})$, we may choose $f_j \in \mathcal{C}^\infty(\mathcal{M})$ such that

$$f|_U = \sum_{j=1}^n f_j|_U x_j + f(p).$$

Proof. The proof for the second part is assigned as an exercise.

Consider $\mathcal{M} = \mathbb{R}^n$, $p = 0$, and $U = (-\varepsilon, \varepsilon)^n$. We have

$$f(x) = \sum_{j=1}^n (f(x_1, \dots, x_j, 0, \dots, 0) - f(x_1, \dots, x_{j-1}, 0, \dots, 0)) + f(0).$$

By the fundamental theorem of calculus, we obtain,

$$f(x) = \sum_{j=1}^n \int_0^1 \partial_j f(x_1, \dots, tx_j, 0, \dots, 0) dt x_j + f(0).$$

By setting $f_j = \int_0^1 \partial_j f(x_1, \dots, tx_j, 0, \dots, 0) dt$, we derive the statement. \square

Definition 2.21. *Given a chart (U, φ) of \mathcal{M} . For $f \in \mathcal{C}^\infty(U)$ we have $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}$. We define*

$$\left. \frac{\partial}{\partial x_j} \right|_p f := D(f \circ \varphi^{-1})(\varphi(p))[e_j].$$

Remark 2.8. Note that if we set $\gamma_i : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$ to be such that $\gamma(0) = \varphi(p)$ and $\gamma'(0) = e_i$, then

$$\left. \frac{\partial}{\partial x_i} \right|_p f = \left. \frac{d}{dt} \right|_{t=0} (f \circ \gamma_i)(t).$$

Proposition 2.4. Let \mathcal{M} be a smooth manifold and (U, φ) be a n -dimensional chart at p . Then we have

$$\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \dots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$$

forms a basis in $T_p \mathcal{M}$.

Proof. Write

$$f = \sum_{i=1}^n f_j x_j + f(p).$$

Choose $\chi \in \mathcal{C}^\infty(\mathcal{M})$ such that

- i). χ is compactly supported in U .
- ii). $\chi \equiv 1$ in a neighborhood of p .

Then

$$\chi^2 f = \sum_{j=1}^n (\chi f_j)(\chi x_j) + \chi^2 f(p),$$

in the neighborhood of p . Since χ is compactly supported. This is defined everywhere on U .

Let $X \in T_p \mathcal{M}$ be a derivation. Then observe that

$$X\chi^2 = 0, \quad Xf = X(\chi^2 f).$$

Since φ is centered at p , we have $x_j(p) = 0$ for all j . Therefore,

$$\begin{aligned} Xf &= \sum_{j=1}^n X((\chi f_j))(\chi x_j)(p) + f_j(p)X(\chi x_j) \\ &= \sum_{j=1}^n f_j(p)X(\chi x_j) \\ &= \sum_{j=1}^n X(\chi x_j) \cdot \left. \frac{\partial}{\partial x_j} \right|_p f. \end{aligned}$$

This implies that

$$X = \sum_{j=1}^n X(\chi x_j) \left. \frac{\partial}{\partial x_j} \right|_p.$$

The set

$$\left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\}$$

spans $T_p\mathcal{M}$. Remains to show the linearly independentness. To do so consider

$$\frac{\partial}{\partial x_j} x_i = \delta_{ij}.$$

We conclude the proof. \square

Example 2.5. For $\mathcal{M} = \mathbb{R}^n$ and $p \in \mathbb{R}^n$, the partial derivatives with respect to the standard coordinate at p is a basis of $T_p\mathbb{R}^n$. Explicitly each is of the form,

$$\frac{\partial}{\partial x_j} \Big|_p f = \frac{\partial f}{\partial x_j}(p).$$

Example 2.6. Let (U, φ) be a chart of an open subset of \mathbb{R}^n with coordinate function y_1, \dots, y_n , (ie. $\varphi = (y_1, \dots, y_n)$).

We have

$$\begin{aligned} \frac{\partial}{\partial x_j} \Big|_p f &= \frac{\partial}{\partial x_j} (f \circ \varphi^{-1} \circ \varphi)(0), \\ &= \sum_{j=1}^n \partial_j (f \circ \varphi^{-1})(\varphi(p)) \frac{\partial \varphi_j}{\partial x_i}(p). \end{aligned}$$

Let $\frac{\partial}{\partial y_j} \Big|_p f = \partial_j (f \circ \varphi^{-1})(\varphi(p))$, we get,

$$= \sum_{j=1}^n \frac{\partial y_j}{\partial x_i}(p) \frac{\partial}{\partial y_j} \Big|_p f.$$

In particular,

$$\frac{\partial}{\partial x_i} \Big|_p f = \sum_{j=1}^n \frac{\partial y_j}{\partial x_i} \Big|_{\varphi(p)} \frac{\partial}{\partial y_j} \Big|_p f.$$

Definition 2.22. Let \mathcal{M}, \mathcal{N} be smooth manifold and $f : \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map. For $p \in \mathcal{M}$, we have a linear transform

$$T_p f : T_p \mathcal{M} \rightarrow T_p \mathcal{N}$$

such that given $X \in T_p \mathcal{M}$ and $h \in C^\infty(\mathcal{N})$, we define

$$T_p f(X)h = X(h \circ f).$$

Lemma 2.5. Let $\mathcal{M} \subseteq \mathbb{R}^m$ and $\mathcal{N} \subseteq \mathbb{R}^n$ be open. Then in the standard basis $T_p f$ is the Jacobi-matrix.

Proof. Follows from the arguments in Example 2.6. \square

Lemma 2.6 (Chain Rule). Let $f : \mathcal{M} \rightarrow \mathcal{N}$, $g : \mathcal{N} \rightarrow \mathcal{W}$ be smooth maps. Then we have

$$T_p(g \circ f) = T_{f(p)}g \circ T_p f.$$

Proof. Since the statement is local we may assume $\mathcal{M}, \mathcal{N}, \mathcal{W}$ to be all open sets in some Euclidean spaces. Then using Lemma 2.5 we have the statement. \square

Lemma 2.7. Let (U, φ) be a chart. Viewing this as a locally smooth map from \mathcal{M} to \mathbb{R}^n , we obtain an isomorphism,

$$T_p \varphi : T_p \mathcal{M} \rightarrow T_{\varphi(p)} \mathbb{R}^n = \mathbb{R}^n.$$

If (V, ψ) is another chart then for $p \in U \cap V$, we have,

$$T_p \psi = T_p(\psi \circ \varphi^{-1} \circ \varphi) = D_{\varphi(p)}(\psi \circ \varphi^{-1}) T_p \varphi.$$

Proof. Exercise. \square

Definition 2.23 (Physicists definition of tangent space). A tangent vector is a family $(\xi^\varphi)_\varphi$ where $\xi^\varphi \in \mathbb{R}$ and φ runs through charts containing p , together with a transformation rule

$$\xi^\psi = D_{\varphi(p)}(\psi \circ \varphi^{-1}) [\xi^\varphi].$$

Definition 2.24. Let \mathcal{M} be a manifold. A curve $\gamma : I \rightarrow \mathcal{M}$ in \mathcal{M} is a smooth map such that from some interval $I \subseteq \mathbb{R}$, such that I is a manifold with the canonical charts. $d : I \rightarrow I$ where $\frac{d}{dt}$ a canonical tangent vector associated to this chart, we get a velocity vector of the curve

$$\overset{\circ}{\gamma}(t_0) = T_{t_0} \gamma \left(\frac{d}{dt} \Big|_{t_0} \right) \in T_{\gamma(t_0)} \mathcal{M}.$$

Example 2.7. Given any $v \in T_p \mathcal{M}$, and a chart (U, φ) centered at p (ie. $\varphi(p) = 0$), we have

$$v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_j} \Big|_p$$

Define a curve

$$\gamma(t) = \varphi^{-1}(tv_1, \dots, tv_n).$$

Note that $\gamma(0) = \varphi^{-1}(0) = p$. We define

$$\overset{\circ}{\gamma}(0)f = T_0 \gamma \left(\frac{d}{dt} \Big|_0 \right) f = \frac{d}{dt} \Big|_{t=0} f(\gamma(t)) = \sum_{i=1}^n v_i \frac{\partial f \circ \varphi^{-1}}{\partial x_j}(0) = v f.$$

2.5 Local properties of differentiable maps and submanifolds.

In this section, all manifolds are assumed to have pure dimension (ie. it has a constant dimension everywhere).

Notation 2.1. \mathcal{M}^m denotes that $\dim \mathcal{M} = \dim T_p \mathcal{M} = m$ for any point $p \in \mathcal{M}$.

Definition 2.25. Let $\mathcal{M}^m, \mathcal{N}^n$ be manifolds and $f : \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map.

- 1). $p \in \mathcal{M}$ is called a critical point for f if $\text{rank } T_p f < n = \dim \mathcal{N}$ and $f(p)$ is called the critical value.
- 2). $q \in \mathcal{N}$ is called a regular value if $\forall p \in f^{-1}(q)$ we have $T_p f = \dim \mathcal{N} = n$.
- 3). f is called a submersion if for any $p \in \mathcal{M}$, $T_p f = n$.
- 4). f is called an immersion if for any $p \in \mathcal{M}$, $T_p f = m$.
- 5). f is called a subimmersion if $\mathcal{M} \ni p \mapsto \text{rank } T_p f$ is a constant.

Notation 2.2. For $f : \mathcal{M} \rightarrow \mathcal{N}$, we denote the set of critical points to be C_f .

Definition 2.26 (Submanifold). Let $\mathcal{N} \subset \mathcal{M}^m$ is called a submanifold if for any chart (U, φ) of \mathcal{M} ,

$$\varphi(\mathcal{N} \cap U) \subseteq \mathbb{R}^m$$

is a submanifold of \mathbb{R}^m .

Remark 2.9. The above definition is equivalent to say that for each $p \in \mathcal{M}$, there exists a chart (U, φ) centered at p such that

$$\varphi(\mathcal{N} \cap U) = \mathbb{R}^k \times \{0\} \cap \varphi(U) \subset \varphi(U) \subset \mathbb{R}^m.$$

In other words, the intersection is diffeomorphic to some hyperspace with dimension k in \mathbb{R}^m .

Definition 2.27. A smooth map $f : \mathcal{M} \rightarrow \mathcal{N}$ is an embedding if

- i). f is an injective immersion.
- ii). $f : \mathcal{M} \rightarrow f(\mathcal{M}) \subset \mathcal{N}$ is a homeomorphism.

Definition 2.28. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be smooth. The rank of f is the map $\mathcal{M} \ni p \mapsto \text{rank } T_p f$.

Proposition 2.5. Let $f : \mathcal{M}^m \rightarrow \mathcal{N}^n$ be a subimmersion of rank k .

- 1). For $q \in \mathcal{N}$, the set $f^{-1}(\{q\}) \subset \mathcal{M}$ is empty or a submanifold of dimension $m - k$.
- 2). For $p \in \mathcal{M}, q := f(p)$ there exists a neighborhood U of p and V of q such that $S = f(U) \cap V$ is a submanifold of \mathcal{N} of dimension k .

Proof. Apply the rank theorem for $f(p) = q$. There exists a chart (U, φ) and (V, ψ) which are centered at p and q , respectively such that

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \varphi \downarrow & & \downarrow \psi \\ \mathbb{R}^m \supset U' & \xrightarrow{\quad} & V' \subset \mathbb{R}^n \\ (x_1, \dots, x_m) \mapsto & (x_1, \dots, x_k, 0, \dots, 0) \end{array}$$

This shows the second assertion. For the first assertion, note that if $f^{-1}(\{q\})$ is not empty then $k \leq m$

$$f^{-1}(\{q\}) = \varphi^{-1}(\{(x_1, \dots, x_m) \mid \forall i = 1, \dots, k, x_i = \psi(p)_i\}).$$

□

Corollary 2.1. $f : \mathcal{M}^m \rightarrow \mathcal{N}^n$ is smooth and $q \in \mathcal{N}$ is a regular value, then $f^{-1}(\{q\})$ is a submanifold of dimension $m - n$ or empty.

2.6 The Theorem of Morse-Sard

Notation 2.3. The Lebesgue measure on \mathbb{R}^m is denoted by λ^m .

Definition 2.29 (Null set). $A \subset \mathcal{M}^m$ is a nullset if for each chart (U, φ) , the set $\varphi(U \cap A)$ is a λ^m -nullset in \mathbb{R}^m .

Remark 2.10. A diffeomorphism maps nullsets to nullsets. Hence the above notion is well-defined.

Remark 2.11. A singleton of a manifold is a nullset. Countable unions of nullsets are again nullsets.

Remark 2.12. Let $A \subset \mathcal{M}^m$ be a nullset where $m > 0$, $\mathcal{M} \setminus A$ is dense.

Remark 2.13. Suppose we have a smooth function $f : \mathbb{R}^m \supset U \rightarrow \mathbb{R}$. For the sake of simplicity, we assume $f(0) = 0$ and $\frac{\partial f}{\partial x_1}(0) \neq 0$. Consider

$$h(x) = (f(x), x_2, \dots, x_m).$$

Then we have

$$Jf(0) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(0) & \frac{\partial f}{\partial x_2}(0) & \cdots & \frac{\partial f}{\partial x_m}(0) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

This is invertible, therefore we have h is a local diffeomorphism. Consider

$$g := f \circ h^{-1}.$$

We denote $(t, \xi) = h(x)$, then

$$g(t, \xi) = f \circ h^{-1} \circ h(x) = f(x) = t.$$

Theorem 2.2. *Let $f : \mathcal{M}^m \rightarrow \mathcal{N}^n$ be smooth where $n \geq 1$. Then the set C_f of critical values of f is a nullset.*

Proof. Suffices to prove that each $p \in \mathcal{M}$ has an open neighborhood U such that $f(C_f \cap U)$ is a nullset.

\Rightarrow) Without the loss of generality, we assume $\mathcal{M} = U \subseteq \mathbb{R}^m$ and $\mathcal{N} = \mathbb{R}^n$ for $n \geq 1$. We will prove the theorem by induction on m .

If $m = 0$, then the image $f(\mathcal{M})$ is at most countable thus a nullset.

Assume the claim holds for all dimension less than m . Let us define

$$C := C_f, \quad C_l := \left\{ x \in U \mid \forall |\alpha| \leq l, \frac{\partial^{|\alpha|} f}{\partial x^\alpha} = 0 \right\}, l \geq 1.$$

Claim 1. *$f(C \setminus C_l)$ is a nullset.*

Proof: Pick $\xi \in C \setminus C_l$ then there exists $\frac{\partial f_i}{\partial x_j}(\xi) \neq 0$ for some i, j . Without the loss of generality, we assume $i = j = 1$. We put

$$h(x) = (f_1(x), x_2, \dots, x_m).$$

By Remark 2.13, we have h is a local diffeomorphism and in particular,

$$\xi \mapsto (f_1(\xi), \xi_2, \dots, \xi_m).$$

Similarly to the remark, we define

$$g := f \circ h^{-1}, \quad g(t, x) = (t, \tilde{g}(t, x)).$$

(t, x) is critical for g if and only if x is critical for $\tilde{g}(t, \cdot)$. Now consider

$$\begin{aligned} \lambda^n(f(C_f \cap V)) &= \lambda^n(\{g(t, x) \mid (t, x) \text{ is critical for } g\}), \\ &= \lambda^n(\{(t, \tilde{g}(t, x)) \mid x \text{ is critical for } \tilde{g}(t, \cdot)\}), \\ &= \int_I \lambda^{n-1}\{\tilde{g}(t, x) \mid x \text{ is a critical point of } \tilde{g}(t, \cdot)\} dt. \end{aligned}$$

The last equality is due to Fubini. The inside of the integral is 0 by the induction hypothesis.

Fix $1 \leq l < \infty$, then $f(C_l \setminus C_{l+1})$ is a nullset. Indeed, for $\xi \in C_l \setminus C_{l+1}$, without the loss of generality, we assume there is a multiindex β such that $|\beta| = l$ and

$$\frac{\partial^\beta f_1}{\partial x^\beta}(\xi) = 0.$$

Similarly for the previous case, we set

$$h(x) = \left(\frac{\partial^\beta f_1}{\partial x^\beta}(x), x_2, \dots, x_m \right).$$

■

Finally, let $d > 0$ and W be a cube of side length b . Consider $x \in C_k \cap W, y \in W$. From Taylor expansion at x implies that

$$|f(x) - f(y)| \leq L|x - y|^{k+1},$$

where L depends on the cube W , k and f . Also by fixing k , we can make it a locally uniform constant.

Subdivide each edge of W into r edges to make W into r^m many cubes (W_j) of edge length $\frac{d}{r}$. For $x \in C_k \cap W_j$, and $y \in W_j$, we have

$$|x - y| \leq \sqrt{m} \frac{d}{r}.$$

In particular, the constant \sqrt{m} only depends on the dimension. Back to the previous argument, we get

$$|f(x) - f(y)| \leq L \left(\frac{\sqrt{m}d}{r} \right)^{k+1}.$$

This means that $f(C_k \cap W_j)$ sits in a cube of edge length less than or equal to $2L \left(\frac{\sqrt{m}d}{r} \right)^{k+1}$.

$$\lambda^n(f(C_k \cap W)) \leq r^m \lambda \left(\max_{1 \leq j \leq r^m} f(C_k \cap W_j) \right) \leq r^m \left\{ 2L \left(\frac{\sqrt{m}d}{r} \right)^{k+1} \right\}^n = K \cdot r^{m-n(k+1)}.$$

Observe that $r \rightarrow \infty$ and $k \geq \frac{m}{n}$, we get $C_k \cap W$ has measure 0. \square

3 Vector fields and dynamical system

3.1 Definition

Definition 3.1. Let \mathcal{M} be a smooth manifold. A smooth vector field of \mathcal{M} is a map

$$X : \mathcal{M} \rightarrow T\mathcal{M},$$

such that

1. $\forall p \in \mathcal{M}, X(p) \in T_p\mathcal{M}$.
2. For any chart (U, φ) centered at p , we have $X|_U = \sum_{i=1}^m X_j^\varphi \frac{\partial}{\partial x_i}$ where $X_i^\varphi \in C^\infty(U)$ for each i .

Notation 3.1.

$$\Gamma(T\mathcal{M}) = \{\text{smooth vector fields on } \mathcal{M}\}.$$

Remark 3.1. The second condition can be restated as follow. Recall Definition 2.22. A chart $\varphi : U \rightarrow \mathbb{R}^m$ can be considered as a smooth map between U and \mathbb{R}^n . We also have seen that $T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m$. With this identification, we have,

$$\begin{aligned} T\varphi : TU &\rightarrow T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m, \\ TU \ni [U \ni p \mapsto X_p] &\mapsto [\mathbb{R}^n \ni x \mapsto [\mathcal{C}^\infty(\mathcal{N}) \ni f \mapsto X_{\varphi^{-1}(x)}(f \circ \varphi)]], \end{aligned}$$

is smooth for all chart φ .

Example 3.1. Consider,

$$X : S^1 \rightarrow TS^1, (x, y) \mapsto (-y, x),$$

is a smooth vector field.

Definition 3.2. Let $I \subset \mathbb{R}$ be an interval and $X \in \Gamma(TM)$. A curve $\gamma : I \rightarrow \mathcal{M}$ is called an integral curve of X if

$$\forall t \in I, \gamma'(t) = X(\gamma(t)).$$

Remark 3.2. To find such curve is equivalent to solve the following autonomous initial value problem. Given a chart (U, φ) around a point $p \in \mathcal{M}$,

$$\begin{cases} (\varphi \circ \gamma)'(t) &= \begin{pmatrix} X_1^\varphi \circ \varphi^{-1} \\ \vdots \\ X_1^\varphi \circ \varphi^{-1} \end{pmatrix} (\varphi \circ \gamma(t)), \\ (\varphi \circ \gamma)(t_0) &= 0. \end{cases}$$

3.2 Flow-Box Theorem

Lemma 3.1 (Grönwall). Let $f, g : [a, b] \rightarrow (0, \infty)$ be continuous functions such that

$$f(t) \leq c + \int_0^t f(s)g(s)ds,$$

for some constant $c > 0$. Then we have,

$$f(t) \leq c \exp \left(\int_0^t g(s)ds \right).$$

Proof. Consider,

$$\tilde{f}(t) = c + \int_0^t f(s)g(s)dt.$$

Then by assumption $f(t) \leq \tilde{f}(t)$. Also set,

$$h(t) = \tilde{f}(t) \exp \left(- \int_0^t g(s)ds \right).$$

Then f, h are both differentiable thus we take,

$$\begin{aligned} h'(t) &= f(t)g(t) \exp\left(-\int_0^t g(s)ds\right) - g(t)\tilde{f}(t) \exp\left(-\int_0^t g(s)ds\right), \\ &= \underbrace{(f(t) - \tilde{f}(t))}_{\leq 0} g(t) \exp\left(-\int_0^t g(s)ds\right), \\ &\leq 0. \end{aligned}$$

We also have,

$$h(0) = c.$$

Thus we conclude,

$$h(t) \leq c \Rightarrow f(t) \leq \tilde{f}(t) = h(t) \exp\left(\int_0^t g(s)ds\right) \leq c \exp\left(\int_0^t g(s)ds\right).$$

□

Theorem 3.1. *Let \mathcal{M} be a smooth manifold and $X \in \Gamma(TM)$. Then for each $p \in \mathcal{M}$ there exists a neighborhood U , $\varepsilon > 0$, and a smooth map*

$$F : (-\varepsilon, \varepsilon) \times U \rightarrow \mathcal{M},$$

such that

- 1). $\forall x \in U, F(0, x) = x,$
- 2). $\partial_t F(t, x) = X(F(t, x)).$

Proof. Recall that a continuously differentiable function over a compact set is Lipschitz.

Consider a smooth function $f : \overline{B(y_0, r)}$ such that

$$a(y) < f(y) < b(y).$$

where $y_0 \in \mathbb{R}^n$, $r > 0$ and a, b are continuous function. For $p \in B(y_0, r)$, we let $F(t, y)$ to be the maximal solution of the initial value problem,

$$\begin{cases} \partial_t F(t, y) = f(F(t, y)), \\ F(0, y) = y_0. \end{cases} \quad (\text{P1})$$

By the construction, we get,

$$F(t, y) = y_0 + \int_0^t f(F(s, y))ds.$$

Thus for $|t| \leq 1$, we obtain,

$$\begin{aligned} |F(t, y) - y_0| &\leq \left| y - y_0 + \int_0^t f(F(s, y)) ds \right|, \\ &\leq |y - y_0| + |t| |f(y_0)| + \int_0^t |f(F(s, y)) - f(y_0)| ds, \\ &\leq |y - y_0| + |f(y_0)| + \int_0^t L |F(s, y) - y_0| ds, \end{aligned}$$

for some $L > 0$. By applying Lemma 3.1, we get,

$$|F(t, y) - y_0| \leq (|y - y_0| + |f(y_0)|) e^{L|t|}.$$

Set $|y - y_0| \leq \frac{r}{2}$ and take,

$$c := \min \left\{ 1, \frac{1}{L}, \frac{r}{\frac{r}{2} + |f(y_0)|} \right\},$$

then for $|t| < c$, we have,

$$|F(t, y_0)| \leq \dots < r.$$

Thus such F exists on a cylinder $(-\varepsilon, \varepsilon) \times \overline{B(y_0, \frac{r}{2})}$.

We now move on to examine the differentiability of such solutions. Let $0 < \rho < r$ and ε be such that F is a unique solution to the IVP on a cylinder $[-\varepsilon, \varepsilon] \times \overline{B(y_0, \rho)}$. Then we have,

$$\begin{aligned} |F(t, y) - F(t, \bar{y})| &\leq |y - \bar{y}| + \left| \int_0^t f(F(s, y)) - f(F(s, \bar{y})) ds \right|, \\ &\leq |y - \bar{y}| + L \int_0^{|t|} |F(s, y) - F(s, \bar{y})| ds. \end{aligned}$$

Using Lemma 3.1, we obtain,

$$|F(t, y) - F(t, \bar{y})| \leq |y - \bar{y}| e^{L|t|}.$$

By definition, F is continuously differentiable in t as f is continuous. For the differentiability in y , consider

$$\begin{cases} \partial_t D_2 F(t, y) = \underbrace{Df(F(t, y))}_{\in \text{Mat}_{m \times m}(\mathbb{R})} \overbrace{(D_2 F)(t, y)}^{\in \text{Mat}_{m \times m}(\mathbb{R})}, \\ (D_2 F)(0, y) = I_m \in \text{Mat}_{m \times m}(\mathbb{R}). \end{cases} \quad (\text{P2})$$

Equation (P2) is an IVP for a matrix valued functions. Let $\Phi : (-\varepsilon, \varepsilon) \times \overline{B(y_0, \rho)} \rightarrow \text{Mat}_{m \times m}(\mathbb{R})$ be a unique solution of

$$\begin{cases} \partial_t \phi(t, y) = Df(F(t, y)) \phi(t, y), \\ \phi(0, y) = I_m \in \text{Mat}_{m \times m}(\mathbb{R}). \end{cases} \quad (\text{P3})$$

Similar to the case of F we have Φ is (Lipschitz)-continuous.

Claim 1. $F(t, y)$ is partially differentiable in y and $D_2F(t, y) = \Phi(t, y)$.

Proof: Fix $(t, \bar{y}) \in (-\varepsilon, \varepsilon) \times B(y_0, \rho)$. Since f is continuously differentiable, we have,

$$f(x) - f(y) = (Df)(y)(x - y) + R(x, y)(x - y),$$

where R is an uniformly continuous function on $\overline{B(y, \rho)}$ such that $R(x, x) = 0$. Thus for $\tilde{\varepsilon} > 0$ there is $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |R(x, y)| < \tilde{\varepsilon}.$$

By the construction of F , we see F is locally uniform in s thus there is $\tilde{\delta} > 0$ such that

$$|x - y| < \tilde{\delta} \Rightarrow |F(s, x) - F(s, y)| < \delta.$$

Take y such that $|y - \bar{y}| < \tilde{\delta}$ and consider the equation.

$$\begin{aligned} f(F(s, y)) - f(F(s, \bar{y})) - (Df)(F(s, \bar{y}))\phi(s, \bar{y})(y - \bar{y}) \\ = (Df)(F(s, \bar{y}))(F(s, y) - F(s, \bar{y}) - \phi(s, \bar{y})(y - \bar{y})) \\ + R(F(s, y), F(s, \bar{y}))(F(s, y) - F(s, \bar{y})). \end{aligned}$$

We see,

$$|R(F(s, y), F(s, \bar{y}))| \cdot |F(s, y) - F(s, \bar{y})| \leq \tilde{\varepsilon}|y - \bar{y}|e^{|s|L}.$$

Finally,

$$\begin{aligned} |F(t, y) - F(t, \bar{y}) - \phi(s, \bar{y})(y - \bar{y})| &= \left| \int_0^t f(F(s, y)) - f(F(s, \bar{y})) - (Df)(F(s, \bar{y}))\phi(s, \bar{y})(y - \bar{y}) ds \right|, \\ &\leq \int_0^{|t|} |Df(F(s, \bar{y}))| |F(s, y) - F(s, \bar{y}) - \phi(s, \bar{y})(y - \bar{y})| ds \\ &\quad + \tilde{\varepsilon}|y - \bar{y}| \int_0^{|t|} e^{|s|L} ds, \\ &\leq c_1 \tilde{\varepsilon}|y - \bar{y}| + c_2 \int_0^{|t|} G(s, y, \bar{y}) ds. \end{aligned}$$

Using Lemma 3.1, we obtain,

$$G(t, y, \bar{y}) \leq c_1 \tilde{\varepsilon}|y - \bar{y}|e^{c_2|t|}.$$

Thus the solution Φ of Equation (P3) is equal to $D_2F(t, y)$. ■

Using the claim, we proved the regularity of F . □

Definition 3.3. Above $F(t, \cdot)$ is called the local flow of the vector field X .

Theorem 3.2. Let \mathcal{M} be a smooth manifold and $X \in \Gamma(T\mathcal{M})$, $p \in \mathcal{M}$. Then there exists continuous functions $a, b : \mathcal{M} \rightarrow \mathbb{R}$ such that

$$-\infty \leq a(p) < 0 < b(p) \leq \infty,$$

and an integral curve, $c_p : (a(p), b(p)) \rightarrow \mathcal{M}$ of X with $c_p(0) = p$ such that for any integral curve $\tilde{c}_p : I \rightarrow \mathcal{M}$ of X with $\tilde{c}_p(0) = p$, we have

1). $I \subseteq (a(p), b(p))$,

2). $c_p|_I = \tilde{c}_p$.

That is to say c_p is the maximal integral curve through p . By continuity the set,

$$A := \{t \in I_1 \cap I_2 \mid c_1(t) = c_2(t)\},$$

is closed. If $t_0 \in I_1 \cap I_2$ then c_1, c_2 are local solution of the IVP,

$$\begin{cases} \gamma'(t) = X(\gamma(t)), \\ \gamma(t_0) = c_1(t_0) = c_2(t_0). \end{cases}$$

Using Picard-Lindelöf, there is a neighborhood U of t_0 such that $U \subseteq A$. Since $A \neq \emptyset$, we conclude $A = I_1 \cap I_2$. That is

$$(c_1 \cup c_2) : I_1 \cup I_2 \rightarrow \mathcal{M}, I_1 \cup I_2 \ni t \mapsto \begin{cases} c_1(t), & (t \in I_1), \\ c_2(t), & (t \in I_2). \end{cases}$$

$c_1 \cup c_2$ is also an integral curve of X through p . Thus there is a maximal integral curve $c_{\max} : I_{\max} \rightarrow \mathcal{M}$. Set $c_{\max} = c_p$.

Proof. Let $c_1 : I_1 \rightarrow \mathcal{M}, c_2 : I_2 \rightarrow \mathcal{M}$ be two integral curves of X through p that is $c_1(0) = c_2(0) = p$. \square

3.3 Applications of Mrose-Sards Theorem

3.3.1 Embedding

Theorem 3.3 (Whiteney). *Each \mathcal{M}^m has an embedding into \mathbb{R}^{2m+1} .*

Proof. We prove the case for \mathcal{M} compact. Note that if \mathcal{M} is compact and $f : \mathcal{M} \rightarrow \mathcal{N}$ is an injective immersion. Then f is an embedding. Since this gives the continuity of the inverse map.

Using Morse-Sard's theorem, \mathcal{M}^m compact, there exists an embedding into some \mathbb{R}^N , for $N \gg 0$. Indeed, choosing charts $(U_j, \varphi_j)_{j=1, \dots, r}$, (by the compactness, finitely many would cover the whole). Such that

$$\varphi_j(U_j) \supset B(0, 3),$$

and

$$\mathcal{M} = \bigcup_{j=1}^r \varphi_j^{-1}(B(0, 1)).$$

Fix $g \in \mathcal{C}^\infty(\mathbb{R}^m)$ be such that

$$g(x) = \begin{cases} 1, & (|x| \leq \frac{4}{3}) \\ 0, & (|x| \geq \frac{5}{3}) \end{cases}.$$

Further defining the functions,

$$f_j(p) = \begin{cases} g(\varphi_j(p))\varphi_j(p), & (p \in U_j) \\ 0, & (\text{otherwise}) \end{cases}.$$

Then we have $f_j \in \mathcal{C}^\infty(\mathbb{R}^m)$. In particular

$$f_j|_{\varphi_j^{-1}(B(0,1))}$$

is a diffeomorphism.

Consider the tuple

$$(f_1, \dots, f_r, \dots, g \circ \varphi_1, \dots, g \circ \varphi_r) : \mathcal{M} \rightarrow \mathbb{R}^{(m+1)r},$$

is an injective immersion, hence it is an embedding.

Let $\mathcal{M} \subset \mathbb{R}^N$, compact. For $w \in S^{N-1}$, let π_w be the orthogonal projection onto $\langle w \rangle^\perp$ which is a hyperplane of dimension $N - 1$ which is given by

$$\pi_w(x) = x - \langle w, x \rangle w.$$

We have

$$\pi_w p = \pi_w q \Leftrightarrow \pi_w(p - q) = 0 \Leftrightarrow p - q \parallel w.$$

We construct a map,

$$\phi : \mathcal{M} \times \mathcal{M} \setminus \{(p, p) \mid p \in \mathcal{M}\} \rightarrow S^{N-1}, \phi(p, q) = \frac{p - q}{|p - q|}.$$

If $2m < N - 1$, using Morse-Sard's, the image of ϕ is of measure 0. Therefore,

$$\{w \in S^{N-1} \mid \exists p, q \in \mathcal{M}, p \neq q, p - q \parallel w\} = \text{Im } \phi \cup -\text{Im } \phi.$$

The right hand side is a nullset. More explicitly, the set of those w such that $\pi_w|_{\mathcal{M}}$ is not injective is a nullset.

Suppose π_w is an immersion if for $p \in \mathcal{M}$, $v \in T_p \mathcal{M} \setminus \{0\}$, we have $\pi_w(v) \neq 0$. π_w is an immersion, if

$$\forall p \in \mathcal{M}, w \notin T_p \mathcal{M} \Leftrightarrow \notin \text{Im } \sigma$$

where

$$\sigma : T\mathcal{M} \setminus \{0\} \rightarrow S^{N-1}, v \mapsto \frac{v}{|v|}.$$

□

Definition 3.4. Let $X \in \Gamma(TM)$. We define a flow of X to be

$$\phi^X : A \rightarrow \mathcal{M},$$

where

$$A = \mathcal{D}(\phi^X) := \bigcup_{p \in \mathcal{M}} (a_p, b_p) \times \{p\} \subseteq \mathbb{R} \times \mathcal{M}$$

such that $\phi^X(p)$ is maximal curve through p . $\mathcal{D}(\phi^X)$ is called the flow domain of X .

Proposition 3.1. *Let $X \in \Gamma(T\mathcal{M})$, and $\phi^X : \mathcal{D}(\phi^X) = A \rightarrow \mathcal{M}$ be the flow of X . Then the following statements hold.*

- 1). A is open subset in $\mathbb{R} \times \mathcal{M}$ and contains $\{0\} \times \mathcal{M}$.
- 2). $\phi^X \in \mathcal{C}^\infty(A)$.
- 3). For $t \in \mathbb{R}$, $\mathcal{D}(\phi_t^X) = \{p \in \mathcal{M} \mid (t, p) \in A\} \subset \mathcal{M}$ where $\phi_t^X(\cdot) = \phi^X(t, \cdot)$, is open.
- 4). $\mathcal{M} = \bigcup_{t>0} \mathcal{D}(\phi_t^X)$.
- 5). $\phi_t^X : \mathcal{D}(\phi_t^X) \rightarrow \mathcal{D}(\phi_{-t}^X)$ is a diffeomorphism.
- 6). $\phi_s^X \circ \phi_t^X \subseteq \phi_{s+t}^X$ in other words if $p \in \mathcal{D}(\phi_t^X)$ and $\phi_s^X(p) \in \mathcal{D}(\phi_s^X)$ then $p \in \mathcal{D}(\phi_{s+t}^X)$.
- 7). $\phi_s^X(\phi_t^X(p)) = \phi_{s+t}(p)$.

Proof.

For 6) and 7), let $s, t > 0$, and $p \in \mathcal{D}(\phi_t^X)$, then by assumption, $\phi_t^X(p) \in \mathcal{D}(\phi_{s+t}^X)$. Consider

$$c(u) = \begin{cases} \phi_u(p), & 0 \leq u \leq t, \\ \phi_{u-t}(\phi_t(p)), & t \leq u \leq t+s, \end{cases}$$

is an integral curve through p as the two expressions coincide at $u = t$, thus using the uniqueness we have $c = \phi_{s+t}^X$. That is,

$$\phi_{s+t}^X(p) = c(t+s) = \phi_s^X(\phi_t^X(p)).$$

The cases for $s \leq 0$ or $t \leq 0$ are exercises.

For 2), let $p \in \mathcal{D}(\phi_t^X)$, $t > 0$, we define,

$$B := \{\phi_s^X(p) \mid 0 \leq s \leq t\}.$$

This is compact. By using Theorem 3.1, there is a neighborhood $W_0 \supseteq B$ and $\varepsilon > 0$ such that

$$(-\varepsilon, \varepsilon) \times W_0 \ni (u, q) \mapsto \phi_u^X(q) \in \mathcal{M}$$

is smooth. Note that we take the maximum of ε from the theorem. This operation is justified by the compactness. Then choose N such that $\frac{t}{N} < \varepsilon$. Define inductively W_j by

$$W_j = \left(\phi_{\frac{t}{N}}^X \Big|_{W_{j-1}} \right)^{-1} (W_{j-1}).$$

Claim 1. *For such (W_j) we have,*

1). $W_0 \supseteq W_1 \supseteq \dots \supseteq W_N$ and they are all open.

2). For $q \in W_N$, $\phi_{t\frac{N-j}{N}}^X(q) \in W_j$.

3). For $0 \leq u \leq t\frac{N-j}{N}$, $\phi_u^X(p) \in W_j$. In particular $p \in W_N$.

Proof: The first assertion is clear as $\phi_{\frac{t}{N}}^X|_{W_{j-1}}$ is continuous and the constraint on N .

For 2). $q \in W_N$, $\phi_{\frac{t}{N}}^X(q) \in W_{N-1}$ and inductively $\phi_{t\frac{j}{N}}^X(q) \in W_{N-j}$.

For 3). $\phi_u^X(p) \in B \subset W_0$, $0 \leq u \leq t$.

On the induction step $j \rightarrow j+1$, $0 \leq u \leq t\frac{N-j-1}{N}$, then

$$\frac{t}{N} + u \leq t\frac{N-j}{N},$$

so

$$\phi_{\frac{t}{N}}^X(\phi_u^X(p)) \in W_j, \quad \phi_u^X(p) \in W_j \Rightarrow \phi_u^X(p) \in W_{j+1}.$$

Let $q \in W_N$, then $\phi_t^X(q) = (\phi_{\frac{t}{N}}^X)^N(q)$, hence $q \in \mathcal{D}(\phi_t^X)$. ■ For

3). Let $p \in \mathcal{D}(\phi_t^X)$, set $c(u) := \phi_{t+u}^X(p)$ where $-t \leq u \leq 0$. This is an integral curve through $\phi_t^X(p)$. Thus $\phi_t^X(p) \in \mathcal{D}(\phi_{-t}^X)$ and $\phi_{-t}^X \circ \phi_t^X(p) = p$.

For 1). Let $(t_0, p) \in A$. Using 2). we can find an open neighborhood $\mathcal{D}(\phi_{t_0}^X) \subset W \ni p$. Using Flow-box theorem, there is $(-\varepsilon, \varepsilon) \times \tilde{W}$ open such that

$$(-\varepsilon, \varepsilon) \times \tilde{W} \ni (u, q) \mapsto \phi_u^X(q),$$

is smooth with image contained in W . In particular $p \in \tilde{W}$. Then for $q \in \tilde{W}$ with $|t - t_0| < \varepsilon$,

$$\phi_t^X(q) = \phi_{t_0}^X(\phi_{t-t_0}^X(q))$$

is smooth. Furthermore, $(t_0 - \varepsilon, t_0 + \varepsilon) \times \tilde{W} \subseteq A$. □

Definition 3.5. A vector field $X \in \Gamma(TM)$ is called complete (or integrable) if $\mathcal{D}(\phi^X) = \mathbb{R} \times \mathcal{M}$.

Remark 3.3. Not all vector fields are complete.

Example 3.2. The following vector fields are not complete.

1). $\mathcal{M} = \mathbb{R}^2 \setminus \{0\}$ and $X(x, y) = \frac{\partial}{\partial x}$.

2). $\mathcal{M} = \mathbb{R}^2$ and $X(x, y) = x^2 \frac{\partial}{\partial x}$. Then we have $\phi^{(1,0)}(t) = \left(\frac{1}{1-t}, 0\right)$.

Proposition 3.2. Let $X \in \Gamma(TM)$ be a compactly supported vector field. Then X is complete. In particular, if \mathcal{M} is compact, then every vector field is complete.

Proof. Set $K := \text{supp}(X)$. By the Flow-box theorem, there is an open set $K \subset W \subset \mathcal{M}$ and $\varepsilon > 0$, such that $(-\varepsilon, \varepsilon) \times W \subseteq \mathcal{D}(\phi^X)$. On the other hand, if $p \in \mathcal{M} \setminus K$, then for every $t \in \mathbb{R}$, $\phi_t^X(p) = p$. Hence, there is $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \times \mathcal{M} \subseteq \mathcal{D}(\phi^X)$. Letting $N \in \mathbb{N}$ be large enough such that $\frac{t}{N} < \varepsilon$, and using $\phi_t = (\phi_{\frac{t}{N}})^N$ shows X is integrable. \square

Proposition 3.3. *Let $X \in \Gamma(T\mathcal{M})$, and $p \in \mathcal{M}$, and assume $X_p \neq 0$. Then there is a chart (U, φ) centered around p , with $X = \frac{\partial}{\partial x_1}$ on U .*

Proof. Let ψ be a chart with $T_p\psi(X_p) = e_1$. This is possible because if for any chart $\tilde{\psi}$, we find $L \in \text{GL}_m(\mathbb{R})$ such that

$$LT_p\tilde{\psi}(X_p) = e_1.$$

Set $\psi = L \circ \tilde{\psi}$. Let $\varepsilon > 0, \delta > 0$ such that $(-\varepsilon, \varepsilon) \times \psi^{-1}(\overline{B(0, \delta)}) \subseteq \mathcal{D}(\phi^X)$. Set for $|t| < \varepsilon, y \in \mathbb{R}^{m-1}, |y| < \delta$.

$$\sigma(t, y) = \phi_t^X(\psi^{-1}(0, y)).$$

Then

$$D(\psi \circ \sigma)(0, 0) = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where the lower coner is the identity matrix.

$$\begin{aligned} \partial_t|_{(0,0)} \psi \circ \sigma(t, y) &= \partial_t|_{(0,0)} \psi(\phi_t^X(p)) = T_p\psi(X_p) = e_1, \\ \partial_y|_{(0,0)} \psi(\psi^{-1}(0, y)) &= I. \end{aligned}$$

So σ is locally invertible and $\varphi = \sigma^{-1}$ is a chart centered at p .

$$\frac{\partial}{\partial t} \sigma(t, y) = \frac{\partial}{\partial t} \phi_t^X(\psi^{-1}(0, y)) = X|_{\sigma(t, y)}.$$

So for $q \in \mathcal{D}(\sigma^{-1}) = \sigma((-\varepsilon, \varepsilon) \times B(0, \delta)) \subset \mathcal{M}$. Thus,

$$X(q) = X(\sigma(\sigma^{-1}(q))) = T_{\sigma^{-1}(q)}\sigma(e_1) = D\sigma(\sigma^{-1}(q))e_1.$$

\square

Remark 3.4. *Given two vector field $X, Y \in \Gamma(T\mathcal{M})$, this is not in general false that*

$$XYf \neq YXf.$$

Furthermore, it can even happen that $XY \notin \Gamma(T\mathcal{M})$.

Definition 3.6. For $X, Y \in \Gamma(TM)$, and $f \in \mathcal{C}^\infty(\mathcal{M})$, set

$$[X, Y]f := X(Yf) - Y(Xf).$$

$[X, Y]$ is called the Lie bracket.

Definition 3.7. An algebra \mathfrak{g} is called a Lie algebra over \mathbb{R} together with Lie bracket $[\cdot, \cdot] : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$ if for any $X, Y, Z \in \mathfrak{g}$,

1. $[\cdot, \cdot]$ is \mathbb{R} -linear.
2. $[X, Y] = -[Y, X]$.
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

Proposition 3.4. Let $X, Y \in \Gamma(TM)$.

1. $[X, Y] \in \Gamma(TM)$.
2. $(\Gamma(TM), [\cdot, \cdot])$ is a Lie algebra.

Proof. $[\cdot, \cdot]$ is \mathbb{R} -linear. consider,

$$\begin{aligned} [X, Y]|_p(fg) &= X|_p(Y(fg)) - Y|_p(X(fg)), \\ &= X|_p(Yf)g + X|_p(Yg)f - Y|_p(Xf)g - Y|_p(Xg)f, \\ &= X|_p(Yf)g + Yfg(p) + X|_p fYg + fX|_p(Yg) - Y|_p(Xf)g - XfY|_p g - Y|_p fXg - fY|_p(Xg), \\ &= ([X, Y]|_p f)g = f[X, Y]|_p g. \end{aligned}$$

The rest is an exercise. □

Proposition 3.5. Let $X \in \Gamma(TM)$, such that $X_p \neq 0$, then there exists a chart (U, φ) centered at p such that

$$\frac{\partial}{\partial x_1} = X.$$

Definition 3.8. Let $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ be a diffeomorphism and $X \in \Gamma(TM)$. For $q \in \mathcal{N}$,

$$(\varphi_* X)(q) := T_{\varphi^{-1}(q)} \varphi[X(\varphi^{-1}(q))],$$

and for $Y \in \Gamma(T, \mathcal{N})$,

$$\varphi^* Y := (\varphi^{-1})_* Y.$$

Definition 3.9. Let $\varphi : \mathcal{M} \rightarrow \mathcal{N}$, be a smooth map and $X \in \Gamma(TM)$ and $Y \in \Gamma(T\mathcal{N})$, we write

$$X \sim_\varphi Y, \text{ and say they are } \varphi\text{-related if } Y(\varphi(p)) = T_p \varphi(X(p)).$$

Remark 3.5. If φ is a diffeomorphism and $X \in \Gamma(TM)$, then clearly,

$$X \sim_\varphi \varphi_* X.$$

Proposition 3.6 (Naturality of Lie Brackets). *Suppose $X_1, X_2 \in \Gamma(TM)$, $Y_1, Y_2 \in \Gamma(TN)$, and $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ be smooth.*

If $X_j \sim_\varphi Y_j$ for each $j = 1, 2$, then $[X_1, X_2] \sim_\varphi [Y_1, Y_2]$.

Proof. Let $p \in \mathcal{M}$ and $f \in \mathcal{C}^\infty(\mathcal{N})$.

$$\begin{aligned} T_p\varphi([X_1, X_2])(f) &= [X_1, X_2]_p(f \circ \varphi), \\ &= X_{1,p}(X_2 f \circ \varphi) - X_{2,p}(X_1 f \circ \varphi), \\ &= X_{1,p}((Y_2 f) \circ \varphi) - X_{2,p}((Y_1 f) \circ \varphi), \\ &= Y_{1,\varphi(p)}(Y_2 f) - Y_{2,\varphi(p)}(Y_1 f), \\ &= [Y_1, Y_2]_{\varphi(p)} f. \end{aligned}$$

□

Lemma 3.2. *Let $X \in \Gamma(TM)$ and $\phi = \phi^X$ be a flow of X . We have the following statements.*

- 1). *For $f \in \mathcal{C}^\infty(\mathcal{M})$, $\frac{d}{dt}(\phi_t^* f) = \phi_t^*(Xf)$. In particular, $\frac{d}{dt}|_{t=0} \phi_t^* f = Xf$.*
- 2). *For $Y \in \Gamma(TM)$, $\frac{d}{dt} \phi_t^* Y = \phi_t^*([X, Y])$, $\frac{d}{dt}|_{t=0} \phi_t^* Y = [X, Y]$.*
- 3).

Proof. Refer to Lee's book. □

Proposition 3.7. *Let $X, Y \in \Gamma(TM)$. Then the following statements are equivalent.*

- 1). $[X, Y] = 0$,
- 2). $\forall t, \phi_t^* Y = Y$,
- 3). $\forall s, \psi_s^* X = X$,
- 4). $\psi_s \circ \phi_t = \phi_t \circ \psi_s$.

Proposition 3.8. *Let $X_1, \dots, X_k \in \Gamma(TM)$ be commuting vector fields (ie, $[X_i, X_j] = 0$ for all $i, j = 1, \dots, k$). Suppose $X_1(p), \dots, X_k(p)$ are linearly independent. Then there exists a chart (U, φ) such that*

$$X_j|_U = \frac{\partial}{\partial x_j}$$

for all $j = 1, \dots, k$.

Proof. Since the statement is about open sets, it suffices to show for the case where $\mathcal{M} = \mathbb{R}^n$ and $p = 0$. Consider,

$$\sigma(t, \xi) := \phi_{t_1}^1 \circ \dots \circ \phi_{t_k}^k(0\xi),$$

where $t \in \mathbb{R}^k$ and $\xi \in \mathbb{R}^{m-k}$. □

3.4 Vector Bundles

Definition 3.10. A (smooth, real) vector bundle over a manifold \mathcal{M} is a triple (E, π, \mathcal{M}) such that

- 1). E, \mathcal{M} are smooth manifolds.
- 2). $\pi : E \rightarrow \mathcal{M}$ is a surjective submersion.
- 3). For all $p \in \mathcal{M}$, the fiber $\pi^{-1}(\{p\}) = E_p$ is an \mathbb{R} -vectorspace with $\dim E_p < \infty$.

Furthermore, this satisfies the axiom of local triviality, namely,

For each $p \in \mathcal{M}$, there exists an open neighborhood U and a diffeomorphism

$$\varphi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^N,$$

such that

$$\forall q \in U, \varphi_U|_{E_q} : E_q \rightarrow \{q\} \times \mathbb{R}^N,$$

is a linear isomorphism.

E is called the total space of the bundle, \mathcal{M} is the base of the bundle, and φ_U is called the bundle chart.

Definition 3.11. A system $(\varphi_i)_{i \in I}$ of bundle charts

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N,$$

is a bundle atlas if

$$\bigcup_{i \in I} U_i = \mathcal{M}, \left(\text{or equivalently } \bigcup_{i \in I} \pi^{-1}(U_i) = E \right).$$

Example 3.3. $(E = \mathcal{M} \times \mathbb{R}^N, \pi(p, v) = p, \mathcal{M})$ is called a trivial vector bundle.

Definition 3.12. Let (E, π_E, \mathcal{M}) and (F, π_F, \mathcal{M}) be vector bundles. A homomorphism of vector bundles between E and F is a smooth map $\varphi : E \rightarrow F$ such that for each point $p \in \mathcal{M}$, $\varphi|_{E_p} : E_p \rightarrow F_p$ is a linear map.

It is called an isomorphism if each $\varphi|_{E_p}$ is a linear isomorphism.

Definition 3.13. A vector bundle is said to be trivial if it is isomorphic to some trivial bundle $\mathcal{M} \times \mathbb{R}^N$.

Definition 3.14. Vector bundles with fibers being \mathbb{C} -vectorspaces are then called complex vector bundles.

Example 3.4. Consider the total space,

$$E = \left\{ (z, w) \in \mathbb{C}^2 \cong \mathbb{R}^4 \mid |z| = 1, \frac{w^2}{z} \geq 0 \right\},$$

and the projection map,

$$\pi : E \rightarrow S^1, (z, w) \mapsto z.$$

We observe that,

E is a submanifold of $\mathbb{C}^2 \cong \mathbb{R}^4$. Pick $z_0 \in S^1$ and a smooth square root function,

$$\xi : S^1 \setminus \{z_0\} \rightarrow S^1, \xi(x_1, x_2) = (\xi_1, \xi_2).$$

Let $w = (x_3, x_4)$ be a point such that $w^2 = \lambda z = \lambda \xi(z)^2$ for $\lambda \geq 0$.

It is equivalent to say that

$$\exists \mu \in \mathbb{R}, w = \mu \xi(z).$$

Reformulate this again, we get,

$$w = (x_3, x_4) \parallel (\xi_1(z), \xi_2(z)) \Leftrightarrow x_3 \xi_2(z) - x_4 \xi_1(z) = 0.$$

Thus we obtain,

$$E \cap \{(z, w) \mid z \in S^1 \setminus \{z_0\}\} = \left\{ (z, w) \in (\mathbb{C} \setminus \mathbb{R}_+ z_0) \times \mathbb{C} \mid \begin{matrix} x_1^2 + x_2^2 - 1 = 0 \\ x_3 \xi(x_1, x_2) - x_4 \xi(x_1, x_2) = 0 \end{matrix} \right\}.$$

Examine the Jacobian of the conditions, we observe

$$\begin{pmatrix} 2x_1 & 2x_2 & 0 & 0 \\ * & * & \xi_2(x_1, x_2) & -\xi_1(x_1, x_2) \end{pmatrix}.$$

We know that at least one of these row is not 0, in other word the matrix is of full rank, thus E is a manifold.

Now we show that $\pi^{-1}(\{z\})$ is a one-dimensional \mathbb{R} -vectorspace.

Similar to the previous case, we get $w^2 = \lambda \xi(z)^2$ for all $z \in S^1 \setminus \{z_0\}$. We get $w = \pm \sqrt{\lambda} \xi(z)$. With this observation, we get,

$$\pi^{-1}(\{z\}) = \{(z, \mu \xi(z)) \mid \mu \in \mathbb{R}\}.$$

Bundle charts over $S^1 \setminus \{z_0\}$.

Consider

$$\varphi : \pi^{-1}(S^1 \setminus \{z_0\}) \rightarrow (S^1 \setminus \{z_0\}) \times \mathbb{R}, (z, w) \mapsto (z, \frac{w}{\xi(z)}).$$

Finally we show that the bundle $E \xrightarrow{\pi} S^1$ is not trivial.

To derive a contradiction, suppose there exists a bundle isomorphism $\varphi : E \rightarrow S^1 \times \mathbb{R}$. Consider

$$E \setminus \{(z, 0) \mid z \in S^1\} \xrightarrow{\varphi} S^1 \times \mathbb{R}^\times$$

which is disconnected. Indeed the map,

$$c : [0, 2\pi] \rightarrow E \setminus \{(z, 0) \mid z \in S^1\}, t \mapsto (e^{it}, e^{i\frac{t}{2}}).$$

Observe that

$$c(0) = (1, 1), c(2\pi) = (1, -1).$$

We derived a contradiction that two components get connected by a path.

This turns out to be one of the realization of a Möbius strip.

Construction of vector bundle from local charts Consider $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle and $(U_i, \varphi_i)_{i \in I}$ be a bundle chart. Consider

$$\begin{array}{ccccc} (U_i \cap U_j) \times \mathbb{R}^N & \xrightarrow{\varphi_j^{-1}} & \pi^{-1}(U_i \cap U_j) & \xrightarrow{\varphi_i} & (U_i \cap U_j) \times \mathbb{R}^N \\ & \searrow \text{pr}_1 & \downarrow \pi & \swarrow \text{pr}_1 & \\ & & U_i \cap U_j & & \end{array}$$

where $\text{pr}_1(p, v) = p$. Then we have,

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^N \rightarrow (U_i \cap U_j) \times \mathbb{R}^N, \varphi_{ij}(q, v) = (q, g_{ij}(q)v),$$

where

$$g_{ij} \in \mathcal{C}^\infty(U_i \cap U_j, \text{GL}_N(\mathbb{R})).$$

We have the following properties of $(g_{ij})_{ij}$.

- i). $g_{ij}g_{jk} = g_{ik}$ over $U_i \cap U_j \cap U_k$.
- ii). $g_{ii}(q) = I_N$.
- iii). $g_{ji}(q) = g_{ij}(q)^{-1}$.

Such collection $(g_{ij})_{ij}$ is called a co-cycle with values in $\text{GL}_N(\mathbb{R})$.

Remark 3.6. The last two properties follow from the first one. Indeed the second property is obvious as $\varphi_{ii} = \text{id}_{U_i}$. Combining this with the first one, we derive the third.

Definition 3.15. Let \mathcal{M} be a smooth manifold and $(U_i)_{i \in I}$ be an open covering of \mathcal{M} and G be a Lie group (ie. a manifold with group structure such that both multiplication and inversion are smooth as functions). A family $(g_{ij})_{i,j \in I}$ of smooth map such that

$$g_{ij} : U_i \cap U_j \rightarrow G$$

is a cocycle with values in G if

1. $g_{ij}g_{jk} = g_{ik}$ over $U_i \cap U_j \cap U_k$.

2. $g_{ii} = e_G$.

3. $g_{ij} = g_{ji}^{-1}$

Definition 3.16. Suppose we are given a cocycle $(g_{ij})_{ij}$ in $G \subset \text{GL}_N(\mathbb{R})$. We define a pre-bundle as follows,

$$E := \bigcup_{i \in I} i \times U_i \times \mathbb{R}^N / \sim,$$

where $(i, p, v) \sim (j, q, w)$ if

$$p = q \text{ and } v = g_{ij}(p) \cdot w.$$

We define the projection to be,

$$\pi([i, p, v]) := p,$$

and the bundle chart,

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N, [i, p, v] \mapsto (p, v).$$

Remark 3.7. Observe that this does not have a topology defined yet.

Definition 3.17. Let E, F be vector bundles over \mathcal{M} . A smooth map $\varphi : E \rightarrow F$ is called a bundle isomorphism if

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ \pi_E \searrow & & \swarrow \pi_F \\ & \mathcal{M} & \end{array}$$

is a commutative diagram and for all $p \in \mathcal{M}$, φ_p is an isomorphism between E_p and F_p .

Example 3.5. Let \mathcal{M} be a smooth manifold. Then we have

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} \{p\} \times T_p\mathcal{M}.$$

Let $(U_i, \varphi_i)_{i \in I}$ be an atlas for \mathcal{M} . Now define a bundle atlas as follows.

$$\psi_i : TU_i = \bigcup_{p \in U_i} \{p\} \times T_p\mathcal{M} = \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^m, (p, X_p) \mapsto (p, T_p\varphi_i X_p).$$

Note that we have a canonical isomorphism $T_{\varphi_i(p)}\mathbb{R}^m \cong \mathbb{R}^m$.

Given two charts $(U_i, \varphi_i), (V_j, \psi_j)$ around a point $p \in \mathcal{M}$, and $v \in T_p\mathcal{M}$ then we have,

$$v = \sum_{k=1}^m \xi_k^i \frac{\partial}{\partial x_k} \Big|_p = \sum_{k=1}^m \xi_k^j \frac{\partial}{\partial y_k} \Big|_p.$$

Then we have

$$T_p \varphi_i(v) = \begin{pmatrix} \xi_1^i \\ \vdots \\ \xi_m^i \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_m}{\partial y_1} & \cdots & \frac{\partial x_m}{\partial y_m} \end{pmatrix} \begin{pmatrix} \xi_1^j \\ \vdots \\ \xi_m^j \end{pmatrix}$$

That is

$$\begin{aligned} \sum \xi_k^i \frac{\partial}{\partial x_k} \Big|_p &= \sum \xi_k^j \frac{\partial}{\partial y_k} \Big|_p, \\ &= \sum_k \left(\sum_l \sum \xi_k^j \frac{\partial x_k}{\partial y_l} \right) \frac{\partial}{\partial x_k} \Big|_p. \end{aligned}$$

Thus the cocycle is $(g_{kl}(p))_{kl} = \left(\frac{\partial x_k}{\partial y_l}(p) \right)$.

Theorem 3.4. *Given a pre-bundle (E, π, \mathcal{M}, I) . There exists a unique topology and \mathcal{C}^∞ -structure on E such that (E, π, \mathcal{M}) is a vector bundle.*

Moreover, given a cocycle $(g_{ij} : U_{ij} \rightarrow G \subseteq \text{GL}_N(\mathbb{R}))_{i,j \in I}$, where $(U_{ij})_{i,j \in I}$ is an open covering, there is a unique vector bundle (E, π, \mathcal{M}) up to isomorphisms with atlas $(U_i, \varphi_i)_{i \in I}$ such that $(g_{ij})_{i,j \in I}$ is the corresponding cocycle.

Proof. Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle with bundle atlas $(\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N)$. Then for each open set $U \subseteq \mathcal{M}$ we have,

$$\pi^{-1}(U) \cap \pi^{-1}(U_i)$$

is open. Conversely, given a pre-vector bundle (E, π, \mathcal{M}, I) . Define a topology on E generated by base open sets of the form $\{\pi^{-1}(U) \cap \pi^{-1}(U_i)\}_{i \in I, U \subseteq \mathcal{M}}$, this gives a topology such that π is continuous. Similarly for the smooth structure. Define

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^N,$$

to be such that

$$\varphi_i([(i, p, v)]) = (p, v).$$

This gives a smooth structure to $E \xrightarrow{\pi} \mathcal{M}$.

For the uniqueness, assume we have two vector bundles $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$ with corresponding atlases $(U_i, \varphi_i), (V_i, \psi_i)$, respectively, such that

$$\varphi_i \circ \varphi_j^{-1}(p, v) = \psi_i \circ \psi_j^{-1}(p, v) = (p, g_{ij}(p)v).$$

Define, $\Phi : E \rightarrow F$ as follows. For $p \in U_i$ and $\xi \in \mathbb{R}^N$, we put

$$\Phi(p, \xi) = \psi_i \circ \phi_i(p, \xi).$$

This gives us a bundle isomorphism. \square

Definition 3.18. Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle with bundle atlas $(U_i, \varphi_i)_{i \in I}$ and cocycle $(g_{ij})_{i,j \in I}$. The dual vector bundle of E is E^* which is constructed from pre-vector bundle $(E^*, \pi^*, \mathcal{M}, I)$ with cocycle $(g_{ji}^{-1})_{i,j \in I}$ and

$$E^* = \bigcup_{p \in \mathcal{M}} E_p^*.$$

Definition 3.19. Let $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$ be vector bundles with cocycle $(g_{ij})_{i,j \in I}, (h_{ij})_{i,j \in I}$ we define their direct sum to be

$$(E \oplus F)_p = E_p \oplus F_p$$

with cocycle $(g_{ij} \oplus h_{ij})_{i,j \in I}$.

Definition 3.20. Let $E \xrightarrow{\pi_E} \mathcal{M}, F \xrightarrow{\pi_F} \mathcal{M}$ be vector bundles with cocycle $(g_{ij})_{i,j \in I}, (h_{ij})_{i,j \in I}$ we define their tensor product to be

$$(E \otimes F)_p = E_p \otimes F_p$$

with cocycle $(g_{ij} \otimes h_{ij})_{i,j \in I}$.

Definition 3.21. Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle. A section of E is a smooth map $f : \mathcal{M} \rightarrow E$ such that $\pi \circ f = \text{id}_{\mathcal{M}}$.

Remark 3.8. Vector fields are exactly the sections of TM .

Remark 3.9. A vector bundle $E \xrightarrow{\pi} \mathcal{M}$ is trivial of rank N if and only if there exists sections $s_1, \dots, s_N \in \Gamma(E)$ such that for all $p \in \mathcal{M}$, $\langle s_1(p), \dots, s_N(p) \rangle$ is a basis of E_p .

Given a bundle atlas $(U_\alpha, \varphi_\alpha)$ with cocycle $(g_{\alpha\beta})$, define,

$$f_\alpha : U_\alpha \rightarrow \mathbb{R}^n, f_\alpha(x) = \text{pr}_2(\varphi_\alpha(f(x))).$$

Of course, we have,

$$f_\alpha = g_{\alpha\beta}(x)f_\beta(x), x \in U_\alpha \cap U_\beta.$$

Conversely, given such f_α , we define,

$$f(x) = [(\alpha, x, f_\alpha(x))].$$

Proposition 3.9. There is a one-to-one correspondence between,

$$\{\text{sections } f \text{ of } E\} \leftrightarrow \{(f_\alpha)_{\alpha \in I} | \forall \alpha \in I, f_\alpha \in \mathcal{C}^\infty(U_\alpha, \mathbb{R}^N), f_\alpha(x) = g_{\alpha\beta}(x)f_\beta(x)\}.$$

Proposition 3.10. Let $E, F \rightarrow \mathcal{M}$ be vector bundles. Let $\phi \in \text{Hom}(E, F)$ be surjective. Then there exists $j \in \text{Hom}(F, E)$ such that $\phi \circ j = \text{id}_F$. Furthermore, $\text{Ker } \phi$ is a vector bundle over \mathcal{M} and

$$E \simeq \text{Ker } \phi \oplus F.$$

That is to say, in the category of vector bundles of \mathcal{M} , a short exact sequence

$$0 \longrightarrow \text{Ker } \phi \hookrightarrow E \xrightarrow[\exists j]{\phi} F \longrightarrow 0$$

splits.

Proof. We will first show that locally such section exists. Let $p \in \mathcal{M}$. Choose sections of E (by using a chart) $s_1, \dots, s_n \in \Gamma(E)$ such that

$$\phi(s_1(p)), \dots, \phi(s_n(p)),$$

is linearly independent. Then there exists an open neighborhood U of p such that for all $q \in U$,

$$\phi(s_1(q)), \dots, \phi(s_n(q)),$$

is linearly independent. For $w \in F_q|_U$, we can now write,

$$w = \sum \xi_j \phi(s_j(q)).$$

Put

$$j(w) = \sum \xi_j s_j(q).$$

Clearly, we have that j is a section of ϕ over U .

For the global case, let $(U_\alpha)_{\alpha \in I}$ be an open covering with sections $j_\alpha : F|_{U_\alpha} \rightarrow E|_{U_\alpha}$. Let $(\rho_\alpha)_{\alpha \in I}$ be a subordinated partition of unity. Then,

$$j(p, w) := \left(p, \sum_{\alpha} \rho_\alpha j_\alpha(w) \right)$$

Now we define $P \in \text{Hom}(E, E)$, such that

$$P(v) := v - j(\phi(v)),$$

then P is a projection $P^2 = P$ and the image is $\text{Im}(P) = \text{Ker } \phi$.

Fix p , and choose sections $\tilde{s}_1, \dots, \tilde{s}_r$ of E be such that

$$\tilde{s}_1(p), \dots, \tilde{s}_1(p), \dots, \tilde{s}_r(p)$$

is a basis of $\text{Ker } \phi|_P$. Set

$$s_j := P \circ \tilde{s}_j.$$

Then s_j are sections of $\text{Ker } \phi$ and in a neighborhood of p , they are pairwise linearly independent hence a frame. \square

Proposition 3.11 (Swan's theorem). *Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle of a compact manifold \mathcal{M} . Then there exists a large enough natural number r such that there exists a surjective bundle homomorphism $\phi : \mathcal{M} \times \mathbb{R}^r \rightarrow E$.*

In particular, $E \oplus \text{Ker } \phi \simeq \mathcal{M} \times \mathbb{R}^r$ is trivial and hence there exists sections $s_1, \dots, s_r \in \Gamma(E)$, such that

$$\forall p \in \mathcal{M}, E_p = \text{Span}(s_1(p), \dots, s_r(p)).$$

Proof. Choose a finite bundle atlas $(U_1, \varphi_1), \dots, (U_l, \varphi_l)$ of E (this is justified as \mathcal{M} is compact) and a subordinated partition of unity ρ_1, \dots, ρ_l . We set,

$$s_j^i(p) := \rho_j(p) \varphi_j^{-1}(p, e_j),$$

where $N = \text{rank } E$ and e_1, \dots, e_N are the canonical basis of \mathbb{R}^N . Note that $s_j^i \in \Gamma(E)$.

Given $p \in \mathcal{M}$ there exists j such that $\rho_j(p) \neq 0$. Hence $s_j^1(p), \dots, s_j^N(p)$ is a basis of E_p , respectively

$$\forall p \in \mathcal{M}, E_p = \text{Span}(s_j^i(p))_{\substack{1 \leq j \leq l \\ 1 \leq i \leq N}}.$$

Re-enumerate them to $s_1, \dots, s_r \in \Gamma(E)$. Put

$$\phi : \mathcal{M} \times \mathbb{R}^r \rightarrow E(p, \xi) \mapsto (p, \sum \xi_j s_j(p)).$$

The rest follows from the previous proposition. \square

Definition 3.22. A R -module P is said to be projective if there exists another R -module Q such that $P \oplus Q$ is free.

Remark 3.10. Observe first that the sections $\Gamma(E)$ is a module over $\mathcal{C}^\infty(\mathcal{M})$ with the action defined by

$$\mathcal{C}^\infty(\mathcal{M}) \times \Gamma(E) \ni (f, s) \mapsto [\mathcal{M} \ni p \mapsto f(p) \cdot s(p)].$$

Proposition 3.11 tells us that $\Gamma(E)$ is finitely generated as a $\mathcal{C}^\infty(\mathcal{M})$ module. Furthermore, $\Gamma(E)$ is projective (ie. a direct summand of a free module).

4 Tensor Algebras and Exterior Algebras

In this section, we assume that K be a field of characteristic 0 and all vector spaces are finite dimensional.

4.1 Tensors

Let E be a vectorspace. We will denote the dual pairing as $\langle \cdot, \cdot \rangle_{E, E^*}$.

Definition 4.1. Let E_1, \dots, E_r be vectorspaces. Recall that a multilinear map $F : E_1 \times \dots \times E_r \rightarrow F$ is such that, for each $i = 1, \dots, r$,

$$E_i \ni a \mapsto F(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_r)$$

is linear. We denote,

$$\mathcal{L}(E_1, \dots, E_r; F) = \{E_1 \times \dots \times E_r \rightarrow F \mid r\text{-multilinear maps}\}.$$

Remark 4.1.

$$\dim \mathcal{L}(E_1, \dots, E_r; F) = \left(\prod_{i=1}^r \dim E_i \right) \times \dim F.$$

Definition 4.2. Let $\varphi \in E^*$ and $\psi \in F^*$. For $e \in E, f \in F$, we define,

$$(\varphi \otimes \psi)(e, f) := \varphi(e)\psi(f)$$

Remark 4.2. Obviously, $\varphi \otimes \psi \in L(E, F; K)$.

Remark 4.3. A map $\cdot \otimes \cdot$ is a bilinear map $E^* \times F^* \rightarrow \mathcal{L}(E, F; K)$.

Proposition 4.1. Let $(e_i)_{i \in I}, (f_j)_{j \in J}$ be bases of E, F . Then we have,

$$\varphi \otimes \psi = \sum \varphi(e_i)\psi(f_j)e_i^* \otimes f_j^*.$$

By definition, we have,

$$e_i^* \otimes f_j^*(e_k, f_l) = \delta_{ik}\delta_{jl}.$$

In particular, $(e_i^* \otimes f_j^*)_{(i,j) \in I \times J}$ is a basis of $\mathcal{L}(E, F; K)$.

Definition 4.3. We define,

$$\mathcal{L}(E_1, \dots, E_r, K) \times \mathcal{L}(F_1, \dots, F_s; K) \rightarrow \mathcal{L}(E_1, \dots, E_r, F_1, \dots, F_s; K)$$

by,

$$(\varphi, \psi) \mapsto \varphi \otimes \psi(e_1, \dots, e_r, f_1, \dots, f_s) := \varphi(e_1, \dots, e_r)\psi(f_1, \dots, f_s).$$

Proposition 4.2. The map defined in Definition 4.3, is a bilinear form which is associative (ie. $(\varphi \otimes \psi) \otimes \chi = \varphi \otimes (\psi \otimes \chi)$).

Proof. Bilinearity is clear from the construction. So is the associativity as $(\varphi \otimes \psi) \otimes \chi = \varphi \otimes \psi \otimes \chi$. \square

Notation 4.1. We denote

$$E^* \otimes F^* := \mathcal{L}(E, F; K),$$

and

$$E \otimes F \cong E^{**} \otimes F^{**} = \mathcal{L}(E^*, F^*, K).$$

Proposition 4.3 (Universality). The assignment $\otimes : E \times F \rightarrow E \otimes F$ solves the following universal property.

Let G be a vectorspace and $h : E \times F \rightarrow G$ be a bilinear map. Then there exists a unique linear map $\bar{h} : E \otimes F \rightarrow G$ making the diagram commutative.

$$\begin{array}{ccc} E \times F & \xrightarrow{h} & G \\ \otimes \downarrow & \nearrow \exists! \bar{h} & \\ E \otimes F & & \end{array}$$

Furthermore, the map

$$L(E, F; G) \ni h \mapsto \bar{h} \in L(E \otimes F; G),$$

is an isomorphism.

Proof. Can be checked at the level of basis elements. \square

Definition 4.4. Let E be a vectorspace, we will define,

$$T^r E = \underbrace{E \otimes \cdots \otimes E}_{r \text{ times}},$$

and

$$T(E) = \bigoplus_{r=0}^{\infty} T^r(E),$$

which is called the tensor algebra over E . Furthermore, Setting \otimes as a multiplication of elements, $T(E)$ is a positively graded K -algebra.

Remark 4.4. Obviously, a basis of $T^r E$ is given by $(e_{i_1} \otimes \cdots \otimes e_{i_r})_{1 \leq i_1 \leq \cdots \leq i_r \leq n}$. Thus we have $\dim T^r E = (\dim E)^r$.

Lemma 4.1. The operation \otimes is associative.

$$\begin{array}{ccc} (v_1 \otimes \cdots \otimes v_r, w_1 \otimes \cdots \otimes w_s) & \mapsto & (v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s) \\ T^r E \times T^s E & \xrightarrow{\quad} & T^{r+s} E \\ \downarrow & \nearrow & \\ T^r E \otimes T^s E & & \end{array}$$

4.2 Totally antisymmetric(alternating) tensors

Remark 4.5.

$$T^r E \otimes T^r E^* \rightarrow K,$$

where

$$(v_1 \otimes \cdots \otimes v_r, v_1^* \otimes \cdots \otimes v_r^*) \mapsto \prod_{i=1}^r \langle v_i, v_i^* \rangle$$

is an dual pairing.

Let $\sigma \in S_r$ be the permutations of $\{1, \dots, r\}$. We have,

$$\begin{array}{ccc} E^r & \xrightarrow{\sigma} & E^r \\ \otimes \downarrow & & \downarrow \otimes \\ T^r E & \xrightarrow{T^r \sigma} & T^r E \end{array}$$

where $T^r \sigma$ is a linear map. Observe that

$$(T^r \sigma)^{-1} = T^r \sigma^{-1}.$$

Thus it is an isomorphism. Explicitly, we have,

$$T^r \sigma(v_1 \otimes \cdots \otimes v_r) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

Notation 4.2. $\text{sgn}(\sigma) = (-1)^\sigma$.

Definition 4.5. $T \in T^r E$ is called *alternating* or *totally antisymmetric* if for all $\sigma \in S_r$, we have,

$$T^r \sigma T = (-1)^\sigma T.$$

Remark 4.6. To check T is alternating, it suffices to check that $T^r \sigma T = -T$ for every transposition $\tau \in S_r$.

Notation 4.3. We denote the space of alternating tensors as,

$$\bigwedge^r E \subseteq T^r E.$$

Definition 4.6. An anti-symmetrization map is ,

$$A_r : T^r \rightarrow T^r, A \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^\sigma (T^r \sigma)(T).$$

Remark 4.7. A_r is a linear map.

Lemma 4.2. $A_r \circ A_r = \text{id}$ In other words, it is idempotent.

Proof. Let $T \in T^r E$ and fix $\tau \in S_r$. We have,

$$(T^r \tau) A_r(T) = \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^\sigma T^r (\tau \sigma)(T) (-1)^\tau (-1)^\tau = (-1)^\tau A_r(T).$$

In particular, the range of A_r is contained in $\bigwedge^r E$. If $T \in \bigwedge^r E$, we have,

$$A_r(T) = \frac{1}{r!} \sum_{\sigma \in S_r} T^r \sigma T = T.$$

Thus we have the range of A_r is exactly $\bigwedge^r E$. □

4.3 Exterior Algebra

Notation 4.4.

$$\bigwedge E = \bigoplus_{r \in \mathbb{Z}_{\geq 0}} \bigwedge^r E.$$

Definition 4.7. Let $\psi : \mathbb{N} \rightarrow \mathbb{Z}_+$ be a sequence such that $\psi(0) = \psi(1) = 1$. We define,

$$\wedge_\psi : \bigwedge^p E \times \bigwedge^q E \rightarrow \bigwedge^{p+q} E, \xi \wedge_\psi \eta := \frac{\psi(p+q)}{\psi(p)\psi(q)} A_{p+q}(\xi \otimes \eta).$$

Definition 4.8. Let A be a graded ring. A mapping $(\cdot, \cdot) : A \rightarrow A$ is graded commutative if for any $a \in A^p, b \in A^q$, we have,

$$(a, b) = (-1)^{pq}(b, a).$$

Proposition 4.4. The map $\wedge = \wedge_\psi$ is bilinear, associated and graded commutative. Furthermore, for $v_1, \dots, v_r \in E = \bigwedge^1 E$,

$$v_1 \wedge \dots \wedge v_r = \frac{\psi(r)}{r!} \sum_{\sigma \in S_r} \text{sgn } \sigma v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(r)}.$$

From above, we conclude,

$$(e_{i_1} \otimes \dots \otimes e_{i_r})_{1 \leq i_1 < \dots < i_r \leq \dim E}$$

is a basis of $\bigwedge^r E$. Thus $\dim \bigwedge^r E = \binom{\dim E}{r}$.

Proof. Let $\xi \in \bigwedge^p E, \eta \in \bigwedge^q E, \chi \in \bigwedge^r E$ and $\tau \in S_{p+q}$ such that

$$\tau = \begin{pmatrix} 1 & 2 & \dots & p & p+1 & \dots & p+q \\ q+1 & q+2 & \dots & p+q & 1 & \dots & q \end{pmatrix}$$

Observe that

$$\text{sgn}(\tau) = (-1)^{pq}, \tau(\xi \otimes \eta) = \eta \otimes \xi.$$

Thus we have,

$$A_{p+q}(\xi \otimes \eta) = (-1)^\tau A_{p+q}(\eta \otimes \xi).$$

We then obtain,

$$\xi \wedge \eta = (-1)^\tau \eta \wedge \xi = (-1)^{pq} \eta \wedge \xi.$$

We derived the graded-commutativity.

For the associativity, we have,

$$\begin{aligned} (\xi \wedge \eta) \wedge \chi &= \frac{\psi(p+q+r)}{\psi(p+q)\psi(r)} A_{p+q+r}((\xi \wedge \eta) \otimes \chi), \\ &= \frac{\psi(p+q+r)}{\psi(p+q)\psi(r)} \frac{\psi(p+q)}{\psi(p)\psi(q)} \frac{1}{(p+q+r)!} \frac{1}{(p+q)!} \sum_{\tau \in S_{p+q}} \text{sgn } \tau \\ &\quad \sum_{\sigma \in S_{p+q+r}} \text{sgn}(\sigma \circ (\tau \times \text{id})) (\sigma \circ (\tau \times \text{id}))((\xi \wedge \eta) \otimes \chi), \\ &= \frac{\psi(p+q+r)}{\psi(p)\psi(q)\psi(r)} \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q+r}} \text{sgn } \sigma \sigma(\xi \otimes \eta \otimes \chi), \\ &= \xi \wedge (\eta \wedge \chi). \end{aligned}$$

By induction, for

$$\xi_1 \in \bigwedge^{p_1}, \dots, \xi_r \in \bigwedge^{p_r} E,$$

we see,

$$\xi_1 \wedge \dots \wedge \xi_r = \frac{\psi(p_1 + \dots + p_r)}{\psi(p_1) \dots \psi(p_r)} A_{p_1 + \dots + p_r}(\xi_1 \otimes \dots \otimes \xi_r).$$

Take $p_1 = \dots = p_r = 1$, we obtain the formula.

□

Corollary 4.1. *Keeping the notation from the proposition above, we have,*

$$\langle v_1 \wedge \dots \wedge v_r, v_1^* \otimes \dots \otimes v_r^* \rangle_{T^r E, T^r E^*} = \frac{\psi(r)}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}),$$

and

$$\langle v_1 \wedge \dots \wedge v_r, v_1^* \wedge \dots \wedge v_r^* \rangle_{T^r E, T^r E^*} = \frac{\psi(r)^2}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}),$$

Proof.

$$\begin{aligned} \langle v_1 \wedge \dots \wedge v_r, v_1^* \otimes \dots \otimes v_r^* \rangle_{T^r E, T^r E^*} &= \sum_{\sigma \in S_r} \text{sgn } \sigma \langle v_{\sigma(1)} v_1^* \rangle \dots \langle v_{\sigma(r)} v_r^* \rangle, \\ &= \frac{\psi(r)}{r!} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}). \end{aligned}$$

$$\begin{aligned} \langle v_1 \wedge \dots \wedge v_r, v_1^* \wedge \dots \wedge v_r^* \rangle_{T^r E, T^r E^*} &= \frac{\psi(r)}{r!} \sum_{\tau \in S_r} \text{sgn } \tau \langle v_1 \wedge \dots \wedge v_r, v_{\tau(1)}^* \otimes \dots \otimes v_{\tau(r)}^* \rangle_{T^r E, T^r E^*}, \\ &= \frac{\psi(r)}{r!} \sum_{\tau \in S_r} \text{sgn } \tau \langle v_{\tau(1)} \wedge \dots \wedge v_{\tau(r)}, v_1^* \otimes \dots \otimes v_r^* \rangle_{T^r E, T^r E^*}, \\ &= \frac{\psi(r)}{(r!)^2} \det((\langle v_i, v_j^* \rangle)_{1 \leq i, j \leq r}). \end{aligned}$$

□

We will follow the convention that $\psi \equiv 1$ which is used in [2].

Definition 4.9. *For $\xi \in \bigwedge^p E$, we define,*

$$\text{ext}_\xi : \bigwedge E \rightarrow \bigwedge E, \eta \mapsto \xi \wedge \eta.$$

Respectively for $u \in E$, we define,

$$\text{int}_U : \bigwedge E^* \mapsto \bigwedge E^*, \omega \mapsto \langle \omega, u \otimes \cdot \rangle = \langle \omega, u \wedge \cdot \rangle$$

Remark 4.8. For $\eta \in \bigwedge E$ and $\omega \in \bigwedge E^*$, we have,

$$\langle \text{ext}_u \eta, \omega \rangle = \langle u \wedge \eta, \omega \rangle = \langle \eta, \text{int}_U \omega \rangle. \quad (2)$$

Lemma 4.3. Suppose $\omega, \eta \in \bigwedge E^*$ are homogeneous elements. Then we have,

$$\text{int}_u(\omega \wedge \eta) = (\text{int}_u \omega) \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge \text{int}_u \eta.$$

In other words, int_u is an anti-derivation.

5 Orientations, volumes and the Hodge *-operator

5.1 Lines

In this section, we denote $K = \mathbb{R}$.

Definition 5.1. A line is a one-dimensional \mathbb{R} -vector space.

Definition 5.2. Fix $\alpha > 0$. An α -density on a n -dimensional vector space E is a map $\rho : \bigwedge^n E^* \rightarrow \mathbb{R}$ such that

$$\forall \omega \in \bigwedge^n E^*, \lambda \in \mathbb{R}, \rho(\lambda, \omega) = |\lambda|^\alpha \rho(\omega).$$

We will denote $|\bigwedge|^\alpha E^*$ to be the vector space of all α -densities.

Definition 5.3. A signed density on a n -dimensional vector space E is a map $\rho : \bigwedge^n E^* \rightarrow \mathbb{R}$ such that

$$\forall \omega \in \bigwedge^n E^*, \rho(\lambda, \omega) = \text{sgn } \lambda \rho(\omega).$$

We also define, the set of signed densities as, $\mathcal{O}(E)$ which we also call the orientation line.

Remark 5.1. We have a map,

$$(E^*)^n \rightarrow \bigwedge^n E.$$

With this we can define an α -density by,

$$\rho : (E^*)^n \rightarrow \mathbb{R}, \rho(\varphi(v_1^*), \dots, \varphi(v_n^*)) = |\det \varphi|^\alpha \rho(v_1^*, \dots, v_n^*),$$

for every $\varphi \in \text{GL}(E)$. In particular, we have,

$$\varphi(v_1) \wedge \dots \wedge \varphi(v_n) = (\det \varphi) v_1 \wedge \dots \wedge v_n.$$

Similarly for the signed density case we put $\text{sgn}(\det \varphi)$.

Remark 5.2.

$$\dim \left| \bigwedge \right| E = \dim \mathcal{O}(E) = 1.$$

Given $\xi \in \bigwedge^n E \setminus \{0\}$, we have,

$$\rho_\xi^\alpha(\omega) := |\xi(\omega)|^\alpha$$

which is an α -density. Also,

$$\rho_\xi^\alpha(\omega) := \text{sgn}(\langle \xi, \omega \rangle)$$

which is a signed density. If ρ is an α -density,

$$\rho(\lambda\omega) = |\lambda|^\alpha \rho(\omega) = |\lambda|^\alpha \frac{\rho(\omega)}{\rho_\xi^\alpha(\omega)} \rho_\xi^\alpha(\omega) = \frac{\rho(\omega)}{\rho_\xi^\alpha(\omega)} \rho_\xi^\alpha(\lambda\omega).$$

Lemma 5.1. There are canonical isomorphisms between,

$$\left| \bigwedge \right|^\alpha E \otimes \left| \bigwedge \right|^\beta E \rho_1 \otimes \rho_2 \mapsto \rho_1 \rho_2 \left| \bigwedge \right|^{\alpha+\beta} E.$$

$$\begin{aligned} \left| \bigwedge \right|^1 E \otimes \mathcal{O} &\rightarrow \bigwedge^n E \\ \bigwedge^n E &\rightarrow \left| \bigwedge \right|^1 E \otimes \mathcal{O}, \end{aligned}$$

and

$$\mathcal{O} \otimes \mathcal{O} \rightarrow \mathbb{R}.$$

Furthermore, \mathcal{O} has a canonical Euclidean metric, namely, for $\rho \in \mathcal{O}$, we have,

$$|\rho| := \sqrt{\rho(\omega)^2}, \forall \omega \neq 0.$$

Note that this is independent of choice of $\omega \neq 0$ since ρ is a signed-density and the square root will take the sign out.

5.2 Orientations

Definition 5.4. An orientation of E is given by a choice of a vector $\mathfrak{o} \in \mathcal{O}$ of unit length.

Remark 5.3. We have an isometry by,

$$\mathcal{O} \ni \mathfrak{o} \mapsto 1 \in \mathbb{R}.$$

Remark 5.4. We have an isomorphism such that,

$$\bigwedge^n E \ni \rho \mapsto \rho \cdot \mathfrak{o} \in \left| \bigwedge \right|^1 E.$$

Definition 5.5. Given a connected component $\bigwedge^+_n E$ of $\bigwedge^n E \setminus \{0\}$. A basis e_1, \dots, e_n of E is called oriented if

$$e_1 \wedge \dots \wedge e_n \in \bigwedge^+_n E.$$

Or equivalently, we have,

$$e_1 \wedge \dots \wedge e_n = \mathfrak{o} \cdot |e_1 \wedge \dots \wedge e_n|.$$

5.3 Volume elements

Let g be a symmetric bilinear form on E . Then g induces bilinear forms on TE and $\bigwedge E$ as follows.

$$(T^r g)(v_1 \otimes \cdots \otimes v_r, w_1 \otimes \cdots \otimes w_r) := \prod_{j=1}^r g(v_j, w_j).$$

Note that $\bigwedge E \subseteq TE$. Thus we have,

$$\begin{aligned} (T^r g)(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r) &:= \frac{1}{(r!)^2} \sum_{\sigma, \tau \in S_r} \operatorname{sgn} \sigma \tau \prod_{j=1}^r g(v_{\sigma(j)}, v_{\tau(j)}), \\ &= \frac{1}{(r!)^2} \sum_{\sigma \in S_r} \det((g(v_i, w_j))_{1 \leq i, j \leq r}), \\ &= \frac{1}{r!} \det((g(v_i, w_j))_{1 \leq i, j \leq r}). \end{aligned}$$

Lemma 5.2. *If g is non-degenerate, so are $T^r g, \bigwedge^r g$. The latter is the restriction of $T^r g$ to $\bigwedge^r E \times \bigwedge^r E$. Also we have an isomorphism*

$$(\cdot)^b : E \rightarrow E^*, v \mapsto g(v_j) = v^b,$$

which inverse $\#$.

g induces a non-degenerate bilinear form g^* on E^* , such that

$$g^*(v^*, w^*) := g(\#v^*, \#w^*).$$

Proof. Let e_1, \dots, e_n be a g -oriented basis of E that is $g(e_i, e_j) = \pm \delta_{ij}$.

$$(T^r g)(e_{i_1} \otimes \cdots \otimes e_{i_r}, e_{j_1} \otimes \cdots \otimes e_{j_r}) = \prod_{k=1}^r \pm \delta_{i_k j_k}.$$

And we also have,

$$\left(\bigwedge^r g \right) = (e_{i_1} \wedge \cdots \wedge e_{i_r}, e_{j_1} \wedge \cdots \wedge e_{j_r}) = \frac{\pm 1}{r!} \prod_{k=1}^r \delta_{i_k j_k}.$$

Remark 5.5. *Induced bases are orthogonal with respect to the induced forms. Therefore, they are non-degenerate.*

□

Proposition 5.1. *Let g be a non-degenerate bilinear form on E . Then g determines uniquely a positive 1-density $\operatorname{vol}_g \in |\bigwedge|^1 E$.*

If e_1, \dots, e_n is a basis of E , then

$$\operatorname{vol}_g = |\det((g(e_i, e_j))_{1 \leq i, j \leq n})^{\frac{1}{2}}| e_1 \wedge \cdots \wedge e_n|.$$

Furthermore, g induces for every $\alpha > 0$, a canonical isomorphism,

$$\left| \bigwedge \right|^\alpha E \rightarrow \mathbb{R}, \lambda \cdot |\operatorname{vol}_g|^\alpha \mapsto \lambda.$$

Proof. Let us first prove that this is independent of the choice of bases. Let f_1, \dots, f_n be another basis. We can write,

$$f_i = \sum a_{ij} e_j.$$

Thus we have,

$$g(f_i, f_j) = \sum_{k,l} a_{ki} a_{lj} g(e_k, e_l).$$

Therefore, we have,

$$\det(g(f_i, f_j)) =$$

□

5.4 The Hodge *-operator

Let g be a non-degenerate bilinear form on E where E is a real vector space of dimension n . Let $\text{vol}_g \in |\wedge|^1 E$.

Let $\mathcal{O}(E)$ be the orientation line and recall

Definition 5.6. Let g be a non-degenerate bilinear form on a vector space V . A basis $\{e_1, \dots, e_n\}$ of V is said to be g -orthonormal if we have,

$$g(e_i, e_j) = \pm \delta_{ij}.$$

Definition 5.7. Let g be a non-degenerate bilinear form on V and $\{e_1, \dots, e_n\}$ be a g -orthonormal basis of V , we define the index of g to be such that

$$\text{ind } g = |\{i \mid g(e_i, e_i) = -1\}|.$$

Notation 5.1. Let L be a one-dimensional \mathbb{R} -vector space and $f, g \in L \setminus \{0\}$. We have $f = \lambda g$ for some $\lambda \in \mathbb{R}$. We denote,

$$f/g = \lambda.$$

Definition 5.8. Let E be a n -dimensional \mathbb{R} -vectorspace. The Hodge *-operator is a map

$$*_p : \bigwedge^p E \rightarrow \bigwedge^{n-p} E \otimes \mathcal{O},$$

satisfying the following properties.

i). It is a linear isomorphism.

ii). For g -orthonormal basis $\{e_1, \dots, e_n\}$ of E and $\sigma \in S_n$, we have

$$*e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(p)} = (-1)^{\text{sgn } \sigma} \prod_{i=1}^p g(e_{\sigma(i)}, e_{\sigma(i)}) e_{\sigma(p+1)} \wedge \dots \wedge e_{\sigma(n)} \otimes e_1 \wedge \dots \wedge e_n / \text{vol}_g.$$

iii). $*_{n-p} * _p = (-1)^{p(n-p) + \text{ind } g}$,

iv). For $\omega, \eta \in \bigwedge^p E$, we have,

$$\left(\bigwedge^{n-p} g \right) (*\omega, *\eta) = (-1)^{\text{ind } g} \left(\bigwedge^p g \right) (\omega, \eta).$$

v). Let $\omega \in \bigwedge^p E, \eta \in \bigwedge^{n-p} E \otimes \mathcal{O}$, then

$$\omega \wedge \eta = (-1)^{p(n-p) + \text{ind } g} (g) (\omega, *\eta) \cdot \text{vol}_g.$$

Theorem 5.1. Let E be a \mathbb{R} -vector space of dimension n and g be a non-degenerate bilinear form on E . Pick $\text{vol}_g \in \left| \bigwedge^1 \right|^1 E$. We have, there exists a unique bilinear pairing such that

$$\bigwedge^p E \times \left(\bigwedge^{n-p} E \otimes \mathcal{O} \right) \ni (\omega, \eta) \mapsto \omega \wedge \eta / \text{vol}_g \in \mathbb{R}.$$

In particular for any $\eta \in \bigwedge^p E$ there is a unique $\star \eta \in \bigwedge^{n-p} E \otimes \mathcal{O}$ such that for any $\omega \in \bigwedge^p E$,

$$\omega \wedge \star \eta = \bigwedge^p g(\omega, \eta) \text{vol}_g.$$

And such \star is the Hodge \star -operator \star_p .

Proof. The composition,

$$(\omega, \eta) \mapsto \omega \wedge \eta \mapsto \omega \wedge \eta / \text{vol}_g \in \mathbb{R}$$

is bilinear. Note that

$$\det(g(e_i, e_j))_{ij} = (-1)^{\text{ind } g}.$$

Let $I = \{i_1, \dots, i_p\} \subseteq \{1, \dots, n\}$. Then we denote,

$$e_I := e_{i_1} \wedge \dots \wedge e_{i_p}.$$

Note that an element $\omega \in \bigwedge^p E$ is of the form,

$$\omega = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=p}} \omega_I e_I, (\omega_I \in \mathbb{R}).$$

Fix $I_0 \subseteq \{1, \dots, n\}$ and denote $I_0^c = \{1, \dots, n\} \setminus I_0$. Using the expression of ω above, we have,

$$\omega \wedge e_{I_0^c} \otimes \underbrace{e_1 \wedge \dots \wedge e_n}_{\text{vol}_g} = \pm \omega_{I_0} \underbrace{|e_1 \wedge \dots \wedge e_n|}_{\text{vol}_g}.$$

This shows the non-degeneracy. The existence of $\star\eta$ thus follows from non-degeneracy.

We now show that \star satisfies all the properties of the Hodge operator.

ii). Set $\omega = e_I$ for some $\{i_1, \dots, i_p\} \subseteq \{1, \dots, n\}$. Then

$$\mathfrak{o} = \left(\bigwedge^p g \right) (\omega, e_{I^c}).$$

We have,

$$\begin{aligned} \omega \wedge e_{I^c} &= (-1)^{\text{sgn } \sigma} \prod_{i \in I} g(e_i, e_i) \frac{e_1 \wedge \dots \wedge e_n}{|e_1 \wedge \dots \wedge e_n|}, \\ &= (-1)^{\text{sgn } \sigma} e_1 \wedge \dots \wedge e_n (-1)^{\text{sgn } \sigma} \prod_{i \in I} g(e_i, e_i) \frac{e_1 \wedge \dots \wedge e_n}{|e_1 \wedge \dots \wedge e_n|}, \\ &= \prod_{i \in I} g(e_i, e_i) |e_1 \wedge \dots \wedge e_n|, \\ &= \left(\bigwedge^p g \right) (e_I, e_I) |e_1 \wedge \dots \wedge e_n|. \end{aligned}$$

That is

$$\star e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(p)} = \text{sgn } \sigma e_1 \wedge \dots \wedge e_n (-1)^{\text{sgn } \sigma} \prod_{i \in I} g(e_i, e_i) e_{I^c} \frac{e_1 \wedge \dots \wedge e_n}{|e_1 \wedge \dots \wedge e_n|}.$$

iii). Applying the second property twice, and consider $\tau \in S_n$ such that

$$\tau = \begin{pmatrix} 1 & \dots & n-p & n-p+1 & \dots & n \\ p+1 & \dots & n & 1 & \dots & p \end{pmatrix}$$

Then we have $\text{sgn } \tau = (-1)^{p(n-p)}$. Rewrite e_{I^c} with τ , we obtain,

$$\star_{n-p} \star_p e_I = (-1)^{\text{ind } g} \text{sgn } \sigma \text{sgn } (\tau) e_{\tau(I^c)} = (-1)^{p(n-p) + \text{ind } g} e_I.$$

To show it satisfies the remaining properties is left to the readers as an exercise. \square

5.5 Tensor fields and differential forms

Remark 5.6. Given a manifold \mathcal{M} , we have a vectorbundle $T\mathcal{M} \xrightarrow{\pi} \mathcal{M}, T_p\mathcal{M} \mapsto p$. Let us denote $\{g_{ij}\}_{ij}$ be the corresponding cocycle, then we can define, $\bigwedge^p g_{ij}$ which corresponds to a vectorbundle $\bigwedge^p E$.

Definition 5.9. Let \mathcal{M} be a smooth manifold. We define,

$$T^{r,s}\mathcal{M} := \bigotimes^r T\mathcal{M} \otimes \bigotimes^s T^*\mathcal{M}.$$

Sections of $T^{r,s}\mathcal{M}$ are called the tensorfield of type (r, s) . We denote,

$$\Gamma(T^{r,s}\mathcal{M}) = \{\text{smooth sections of } T^{r,s}\mathcal{M}\}.$$

Proposition 5.2. For a map $f : \mathcal{M} \rightarrow T^*\mathcal{M}$ with

$$\forall p \in \mathcal{M}, f(p) \in T_p^{r,s}\mathcal{M},$$

the following are equivalent.

1). for vector fields, $X_1, \dots, X_n \in \Gamma(T\mathcal{M})$, and covector field $\omega_1, \dots, \omega_r \in \Gamma(T^*\mathcal{M})$, the function,

$$p \mapsto f(\omega_1, \dots, \omega_r, X_1, \dots, X_s) = f(p)(\omega_1(p), \dots, \omega_r(p), X_1(p), \dots, X_s(p)),$$

is smooth.

2). for any charts (U, φ) with coordinate functions x_1, \dots, x_m , we have,

$$f = \sum f_{j_1 \dots j_s}^{i_1 \dots i_r} \frac{\partial}{\partial x_{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{i_s}} \otimes dx_{j_1} \otimes \dots \otimes dx_{j_s},$$

where dx_1, \dots, dx_m are the dual (missing) to $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ with smooth functions $f_{j_1, \dots, j_s}^{i_1, \dots, i_r} \in \mathcal{C}^\infty(U)$.

3). $f \in \Gamma(T^{r,s}\mathcal{M})$.

Proof. Exercise. □

Definition 5.10. The sections of the bundle $\bigwedge^p T^*\mathcal{M} \subseteq T^{0,p}\mathcal{M}$, are called differential forms. The set of differential forms are denoted by

$$\Omega^p(\mathcal{M}) = \Gamma\left(\bigwedge^p T^*\mathcal{M}\right).$$

Proposition 5.3. Let $f : \Gamma(T^*\mathcal{M})^r \times \Gamma(T\mathcal{M})^s \rightarrow \mathcal{C}^\infty(\mathcal{M})$, is induced by a section of $T^{r,s}\mathcal{M}$ if and only if it is $\mathcal{C}^\infty(\mathcal{M})$ -($r-s$) multilinear.

Proof. If $f \in \Gamma(T^{r,s}, \mathcal{M})$ then

$$(\omega_1, \dots, \omega_r, X_1, \dots, X_s) \mapsto (p \mapsto f(p)(\omega_1(p), \dots, \omega_r(p), X_1(p), \dots, X_s(p))),$$

is clearly $\mathcal{C}^\infty(\mathcal{M})$ -($r+s$)-multilinear.

Conversely, given $\mathcal{C}^\infty(\mathcal{M})$ -multilinear map f , need to find $f(p)$. To do so we will show that for $p \in \mathcal{M}$, if there is i such that $\omega_i(p) = 0$ or there is j such that $X_j(p) = 0$, we have,

$$f(\omega_1, \dots, \omega_r, X_1, \dots, X_s)(p) = 0.$$

Indeed, we assume without loss of generality that $\omega_1(p) = 0$. Choose a chart (U, φ) centered around p . Choose $h \in \mathcal{C}^\infty(\mathcal{M})$ such that $h \equiv 1$ in small enough neighborhood of p . By assumption, we have,

$$\omega_1|_U = \sum_{j=1}^m a_j dx_j, a_j(p) = 0.$$

Combining these, we obtain,

$$\begin{aligned}\omega_1 &= (1 - h^2)\omega_1 + h^2\omega_1, \\ &= (1 - h^2)\omega_1 + \sum_{j=1}^m \underbrace{(a_j h)}_{\in \mathcal{C}^\infty(\mathcal{M})} \underbrace{(h dx_j)}_{\in \Gamma(T^*\mathcal{M})}.\end{aligned}$$

Using multilinearity of f , we get,

$$f(\omega_1, \dots)(p) = (1 - h^2)(p)f(\omega_1, \dots)(p) + \sum_{j=1}^m \underbrace{(ha_j)(p)}_{=0} f(h dx_j, \dots)(p) = 0.$$

Define $f(p)$ as follows. For $\theta_1, \dots, \theta_r \in T_p^*\mathcal{M}, v_1, \dots, v_s \in T_p\mathcal{M}$, choose $\omega_1, \dots, \omega_r \in \Gamma(T^*\mathcal{M}), X_1, \dots, X_s \in \Gamma(T\mathcal{M})$ with

$$\omega_j(p) = \theta_j, X_j(p) = v_j.$$

By what we have proved, we have,

$$f(p)(\theta_1, \dots, \theta_r, v_1, \dots, v_s) := f(\omega_1, \dots, \omega_r, X_1, \dots, X_s)(p),$$

which is independent of choices of ω_j, X_j . The rest follows from Proposition 5.2. \square

Observe that given $f \in \mathcal{C}^\infty(\mathcal{M})$ and $X \in \Gamma(T\mathcal{M})$, the map,

$$\mathcal{M} \ni p \mapsto X_p f,$$

is a smooth function. In particular,

$$df : \Gamma(T\mathcal{M}) \rightarrow \mathcal{C}^\infty(\mathcal{M}), df(X) = Xf,$$

is a $\mathcal{C}^\infty(\mathcal{M})$ linear map.

From Proposition 5.3, it follows that $df \in \Gamma(T^*\mathcal{M})$. In particular, we have a linear map,

$$d : \mathcal{C}^\infty(\mathcal{M}) = \Omega^0\mathcal{M} \rightarrow \Gamma(T^*\mathcal{M}) = \Omega^1\mathcal{M}.$$

Definition 5.11 (Pullback). *Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be smooth and $\xi \in T(T^{0,s}\mathcal{N})$, Put,*

$$f^*\xi|_p(v_1, \dots, v_s) := \xi|_{f(p)}(T_p f v_1, \dots, T_p f v_s).$$

Definition 5.12. *A wedge product $\wedge : \Omega^s(\mathcal{M}) \times \Omega^t(\mathcal{M}) \rightarrow \Omega^{s+t}(\mathcal{M})$ is defined pointwise that is, for $\eta \in \Omega^s(\mathcal{M})$ and $\zeta \in \Omega^t(\mathcal{M})$,*

$$\eta \wedge \zeta := [\mathcal{M} \ni p \mapsto \eta_p \wedge \zeta_p]. \quad (3)$$

Note that η, ζ are sections, therefore, we have $\eta_p \in \bigwedge^s T_p^\mathcal{M}$ and $\zeta_p \in \bigwedge^t T_p^*\mathcal{M}$.*

Proposition 5.4 (Properties of Pullbacks). *Given $\mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{L}$ be smooth maps. We have,*

1. $\varphi \in \mathcal{C}^\infty(\mathcal{N}) = \Gamma(T^{0,0}\mathcal{N})$, $f^*\varphi = \varphi \circ f$.
2. f^* is a linear map $\Gamma(T^{0,s}\mathcal{N}) \rightarrow \Gamma(T^{0,s}\mathcal{M})$ and $\Omega^p\mathcal{N} \rightarrow \Omega^p\mathcal{M}$.
3. $(g \circ f)^* = f^* \circ g^*$.
4. If f is a diffeomorphism then $(f^*)^{-1} = (f^{-1})^*$.
5. $\omega \in \Omega^p\mathcal{N}$, $\eta \in \Omega^q\mathcal{N}$ then $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Notation 5.2. Let $U \subseteq \mathcal{M}$ be an open set, we define,

$$\Omega^*(U) := \bigoplus_{p \in \mathbb{Z}_{\geq 0}} \Omega^p(U).$$

Theorem 5.2 (Cartan (exterior) derivative). Let \mathcal{M} be a smooth manifold and U be an open subset of \mathcal{M} , then there exists a unique family of linear maps

$$d : \Omega^*(U) \rightarrow \Omega^*(U),$$

which satisfies the following properties.

- 1). d is of degree 1 that is $d : \Omega^p(U) \rightarrow \Omega^{p+1}(U)$.
- 2). For $\omega \in \Omega^p(U)$, $\eta \in \Omega^q(U)$, we have,

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d(\eta).$$

- 3). For $f \in \mathcal{C}^\infty(\mathcal{M}) = \Omega^0(U)$ and $X \in \Gamma(T\mathcal{M})$, we have,

$$\langle df, X \rangle = df(X) = Xf$$

- 4). $d \circ d = 0$.

- 5). For $U \subseteq V \subset \mathcal{M}$ open then we have a commutative diagram, where $\iota : U \hookrightarrow V$ is an inclusion and ι^* is the pullback of inclusion namely the restriction.

$$\begin{array}{ccc} \Omega^*(V) & \xrightarrow{d} & \Omega^*(V) \\ \iota^* \downarrow & \circ & \downarrow \iota^* \\ \Omega^*(U) & \xrightarrow{d} & \Omega^*(U) \end{array}$$

These five properties determine d uniquely, and satisfies furthermore,

- 6). For $\omega \in \Omega^p(\mathcal{M})$, $X_0, \dots, X_p \in \Gamma(T\mathcal{M})$, we have

$$\begin{aligned} d\omega(X_0, \dots, X_p) &= \sum_{i=0}^p (-1)^i X_i(\omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p)) \\ &\quad + \sum_{0 \leq i < j \leq p} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_{j-1}, X_{j+1}, \dots, X_p). \end{aligned}$$

7). $f : \mathcal{M} \rightarrow \mathcal{N}$ is smooth then $f^*(d\omega) = d(f^*\omega)$.

Proof. For the uniqueness, assume we had d, \tilde{d} satisfying the first five properties. Let (U, φ) be a chart. Denote for $I = \{i_1 < \dots < i_p\}$, $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_p}$. Using this notation, we have,

$$d(fdx_I) = df \wedge dx_I,$$

which follows from the second and the forth properties. Explicitly, this follows from that

$$ddx_I = \underbrace{ddx_{i_1}}_{=0} \wedge \dots \wedge dx_{i_p} \pm dx_{i_1} \wedge d(dx_{i_2} \wedge \dots \wedge dx_{i_p}).$$

And using induction, we get $ddx_I = 0$. Using the third property, we have,

$$d(fdx_I) = \sum_{j \notin I} \frac{\partial f}{\partial x_j} dx_j \wedge dx_I.$$

Cover \mathcal{M} by charts $\bigcup_i U_i$, by the fifth property, we have,

$$d\omega|_{U_i} = \tilde{\omega}|_{U_i}.$$

Therefore, we conclude $d = \tilde{d}$.

For the existence of d , taking 6). as definition. We need to show as per Proposition 5.3, for $\omega \in \Omega^p(U)$,

- i). $d\omega$ is $\mathcal{C}^\infty(U)$ -multilinear,
- ii). $d\omega$ is alternating.

These two will imply that $d\omega \in \Omega^{p+1}(U)$. For the second assertion, we see,

$$\begin{aligned} d\omega(X_1, X_0, X_2, \dots, X_p) &= X_1\omega(X_0, X_2, \dots, X_p) + \sum_{i=2}^p (-1)^i X_i\omega(X_1, X_0, \dots) + \dots, \\ &= -d\omega(X_0, X_1, \dots, X_p). \end{aligned}$$

For the first assertion, we have,

$$\begin{aligned} d\omega(fX_0, X_1, \dots, X_p) &= \sum_{i=1}^p (-1)^i \overbrace{X_i(f \cdot \omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p))}^{X_i f \omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p) + f X_i \omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_p)} \\ &\quad + f \cdot X_0 \omega(X_1, \dots, X_p) \\ &\quad + \sum_{j=1}^p (-1)^j \omega(\underbrace{[fX_0, X_j]}_{fX_0 - X_j(fX_0) = f[X_0, X_j] - (X_j f)X_0}, X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_p) \\ &\quad + \sum_{1 \leq i < j \leq p} (-1)^{i+j} f \cdot \omega([X_i, X_j], X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_{j-1}, X_{j+1}, \dots, X_p), \\ &= f \cdot d\omega(X_0, \dots, X_p). \end{aligned}$$

We claim that the so-defined d satisfies 1). - 5). We have already seen that 1) and 5). Now it suffices to look at a coordinate system, x_1, \dots, x_n where

$$\forall 1 \leq i, j \leq n, \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0.$$

Then we have,

$$\begin{aligned} & d(f dx_{i_1} \wedge \dots \wedge dx_{i_p}) \left(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_p}} \right) \\ &= \frac{\partial^2}{\partial x_j \partial x_i} \left(f dx_{i_1} \wedge \dots \wedge dx_{i_p} \underbrace{\left(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_{i-1}}}, \frac{\partial}{\partial x_{j_{i+1}}}, \dots, \frac{\partial}{\partial x_{j_p}} \right)}_{0 \text{ or } \pm 1} \right), \\ &= \begin{cases} 0, \{i_1, \dots, i_p\} \not\subseteq \{j_0, \dots, j_p\} \\ (-1)^\alpha \frac{\partial}{\partial x_\alpha} f \quad \{i_1 < \dots < i_p\} = \{j_0 < \dots < j_{\alpha-1} < j_{\alpha+1} < \dots < j_p\}, \end{cases} \\ &= \sum \frac{\partial f}{\partial x_k} dx_k \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} \left(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_p}} \right). \end{aligned}$$

Let

$$\begin{aligned} \omega &= f dx_1 \wedge \dots \wedge dx_p, \\ \eta &= g dx_{p+1} \wedge \dots \wedge dx_{p+q}. \end{aligned}$$

Using the concrete formula, we have,

$$\begin{aligned} d(\omega \wedge \eta) &= d(f \cdot g dx_1 \wedge \dots \wedge dx_{p+q}), \\ &= \sum_{j=p+q+1}^m \frac{\partial f g}{\partial x_j} (-1)^{p+q} dx_1 \wedge \dots \wedge dx_{p+q} \wedge dx_j, \\ &= \underbrace{\sum_{j=p+q+1}^m \frac{\partial f}{\partial x_j} (-1)^p dx_1 \wedge \dots \wedge dx_p \wedge dx_j}_{=d\omega} \underbrace{\wedge g dx_{p+1} \wedge \dots \wedge dx_{p+q} \wedge dx_j}_{=\eta} \\ &+ \underbrace{\sum_{j=p+q+1}^m \underbrace{f dx_1 \wedge \dots \wedge dx_p}_{=\omega} \wedge (-1)^{p+q} \frac{\partial g}{\partial x_j} dx_{p+1} \wedge \dots \wedge dx_{p+q} \wedge dx_j}_{(-1)^p \omega \wedge d\eta}. \end{aligned}$$

This proves 2). For the third, both slots are $\mathcal{C}^\infty(U)$ -linear in X . Hence without loss of generality, we assume $X = \frac{\partial}{\partial x_1}$.

$$\left\langle df, \frac{\partial}{\partial x_1} \right\rangle = \left\langle \sum_{j=1}^m \frac{\partial f}{\partial x_j} dx_j, \frac{\partial}{\partial x_1} \right\rangle = \frac{\partial f}{\partial x_1}.$$

For the fourth, we see,

$$\begin{aligned}
d \circ d(f dx_1 \wedge \cdots \wedge dx_p) &= d \left(\sum_{j=p+1}^m \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_p \right), \\
&= \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_1 \wedge \cdots \wedge dx_p, \\
&= \sum_{i < j} \frac{\partial^2 f}{\partial x_i \partial x_j} \underbrace{(dx_i \wedge dx_j + dx_j \wedge dx_i)}_{=0} \wedge dx_1 \wedge \cdots \wedge dx_p.
\end{aligned}$$

The seventh is obvious at charts. \square

5.6 Application : Classical vector analysis

Consider \mathcal{M}^m where $m \in \{2, 3, 4\}$. Let $g \in \Gamma(T^{0,2}\mathcal{M})$, be symmetric non-degenerate. May choose an orthonormal frame such that

$$g = \begin{pmatrix} -I_p & O \\ O & I_q \end{pmatrix}$$

where $m = p + q$. That is

$$g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \begin{cases} 0, & i \neq j, -1, \quad i = j \leq p, \\ 1, & i = j > p. \end{cases}$$

Interesting cases are

1. Riemannian metric when $p = 0$,
2. General relativity, when $p = 1, q = 3$.

Set the volume form as

$$\begin{aligned}
\text{vol}_g &= \left| \det g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right|^{\frac{1}{2}} |dx_1 \wedge \cdots \wedge dx_m|, \\
\sigma &= \sigma(dx_1, \dots, dx_m) = \frac{dx_1 \wedge \cdots \wedge dx_m}{|dx_1 \wedge \cdots \wedge dx_m|}.
\end{aligned}$$

Compute $*dx_j$ for a general coordinate system.

$$*dx_j = \sum_{k=1}^m (-1)^{k-1} g^{kj} \sqrt{g} dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_m \otimes \sigma,$$

$$g_{ij} = g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right), g^{ij} = g(dx_i, dx_j) = (\{(g_{kl})_{k,l}\}^{-1})_{i,j}.$$

Recall that $*dx_j$ is the unique twisted $(n-1)$ -form ω satisfying

$$\tau \wedge \omega = \bigwedge^1 g(\tau_1 dx_j) \text{vol } g,$$

where τ is an arbitrary 1-form.

$$\omega = \sum_{j=1}^m (-1)^{j-1} \omega_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma.$$

And we have,

$$\begin{aligned} dx_i \wedge \omega &= \left(\bigwedge^1 g \right) (dx_i, dx_j) \text{vol } g = g^{ij} \text{vol } g, \\ &= \sum_{j=1}^m (-1)^{j-1} \omega_j dx_i \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma, \\ &= \omega_i dx_1 \wedge \cdots \wedge dx_m \otimes \sigma, \\ &= \omega_i \sqrt{g}^{-1} \text{vol } g. \\ \omega_i &= g^{ij} \sqrt{g}. \end{aligned}$$

Proposition 5.5.

$$dx_i^\# = \sum_j g^{ij} \frac{\partial}{\partial x_j}, \left(\frac{\partial}{\partial x_i} \right)^b = \sum g_{ij} dx_j.$$

Proof.

$$\begin{aligned} \left\langle \left(\frac{\partial}{\partial x_i} \right)^b, \frac{\partial}{\partial x_k} \right\rangle &= g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right) = g_{ik} \text{ gives the second formula.} \\ \langle (dx_i)^\#, dx_k \rangle &= g(dx_i, dx_k) = g^{ik} \text{ gives the first formula} \end{aligned}$$

□

Definition 5.13. We define the gradient to be such that

$$\text{grad}_g f = (df)^\# := \left(\sum (\partial_i f) dx_i \right)^\# = \sum_{ij} g^{ij} (\partial_i f) \frac{\partial}{\partial x_j}.$$

Definition 5.14. $\star dx_j$ is the unique $(n-1)$ form ω such that

$$(\tau \wedge \omega) = \left(\bigwedge^1 g \right) (tdx_j) \text{vol}_g$$

for all 1-form τ .

Remark 5.7.

$$\omega = \sum_{j=1}^m (-1)^{j-1} \omega_j dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_m \otimes \sigma.$$

In particular,

$$\begin{aligned} dx_j \wedge \omega &= w_i dx_1 \wedge \cdots \wedge dx_m \otimes \sigma, \\ &= \omega_i \sqrt{g}^{-1} \text{vol}_g, \quad \sqrt{g} = |\det(g_{ij})|, \\ &= \left(\bigwedge^1 g \right) (dx_i, dx_j) \text{vol}_g, \\ &= g_{ij} \text{vol}_g. \end{aligned}$$

Therefore, we obtain $\omega_i = g^{ij} \sqrt{g}$ and

$$\omega = \star(dx_j) = \sum_{k=1}^m (-1)^{k-1} g^{kj} \sqrt{g} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma.$$

We now identify vectorfields with $(m-1)$ -form by pairing,

$$X \mapsto \star(X^\flat).$$

Lemma 5.3.

$$\star X^\flat = \text{int}_X \text{vol}_g.$$

Proof. Since the formula we want to show is a pointwise identity, it suffices to check it on $X|_p = e_1|_p$, where e_1, \dots, e_m are g -orthonormal basis, since every small enough neighborhood admits a g -orthonormal basis. Observe that

$$\star e_1^\flat = \star(e_1)^* = c_1 e_2^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma(e_1^*, \dots, e_m^*).$$

Thus we have,

$$\text{int}_{e_1} \text{vol}_g = \text{int}_{e_1} e_1^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma = c_1 e_2^\flat \wedge \cdots \wedge e_m^\flat \otimes \sigma,$$

where $c_j = g_{jj} = g^{jj}$ for a g -orthonormal basis.

$$\begin{array}{ccc} \Omega^{m-1}(\mathcal{M}) & \xrightarrow{d} & \Omega^m(\mathcal{M}) \\ X \mapsto \star X^\flat \uparrow & & (-1)^p \star \downarrow \uparrow \star \\ \Gamma(\mathcal{M}) & \xrightarrow{\text{div}^g} & \mathcal{C}^\infty(\mathcal{M}) \end{array}$$

$$\text{div}^g(X) = (-1)^p \star d \star X^\flat,$$

thus

$$\begin{aligned}
\operatorname{div}^g(X) &= \operatorname{div}^g \left(\sum_{j=1}^m X_j \frac{\partial}{\partial x_j} \right), \\
&= (-1)^p \star d \star \left(\sum_{jk} g_{jk} X_j dx_k \right), \\
&= (-1)^p \star d \left(\sum_{jkl} X_j d_{jk} (-1) dx_1 \wedge \cdots \wedge dx_{l-1} \wedge dx_{l+1} \wedge \cdots \wedge dx_m \right) \\
&= (-1)^p \star d \left(\sum (-1)^{j-1} x_j \sqrt{g} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m \otimes \sigma \right),
\end{aligned}$$

Thus, we obtain,

$$\operatorname{int}_X \operatorname{vol}_g = \sum_j X_j \operatorname{int}_{\partial_j} \sqrt{g} dx_1 \wedge \cdots \wedge dx_m \otimes \sigma =$$

□

Remark 5.8. On \mathcal{M}^2 , we have,

$$\begin{array}{ccccc}
\Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 \\
= \downarrow & & \downarrow \# & & (-1)^p \star \uparrow \downarrow \star \\
\mathcal{C}^\infty(\mathcal{M}) & \xrightarrow{\operatorname{grad}^g} & \Gamma(T\mathcal{M}) & \xrightarrow{X \mapsto -\operatorname{div}(\star X^\flat)^\#} & \mathcal{C}^\infty(\mathcal{M})
\end{array}$$

5.7 Integration of densities

In this section,

$$\mathcal{O} = \mathcal{O}(T^*\mathcal{M}).$$

In terms of the

Lemma 5.4. If (U_i, φ_i) is an atlas of \mathcal{M} then the cocyle for $|\wedge|^\alpha$ is

Proof.

$$\begin{aligned}
g_{ij}(p) &= |\det((\varphi_i^{-1})^* \circ \varphi_j^*)|^\alpha, \\
&= |\det((\varphi_j \circ \varphi_i)^*)|^\alpha, \\
&= |\det(D(\varphi_j \circ \varphi_i^{-1})^t)|^\alpha, \\
&= |\det(D(\varphi_i \circ \varphi_j^{-1}))|^{-\alpha}|_{\varphi_j(p)}.
\end{aligned}$$

Respectively for \mathcal{O} , we get,

$$g_{ij} = \operatorname{sgn} D(\varphi_i \circ \varphi_j^{-1})|_{\varphi_j(p)}.$$

□

A diffeomorphism $f : \mathcal{M} \rightarrow \mathcal{N}$ gives rise for pullback map

$$\left| \bigwedge \right|^\alpha (\mathcal{N}) \xrightarrow{f^*} \left| \bigwedge \right| (\mathcal{M}), \mathcal{O}(\mathcal{N}) \xrightarrow{f^*} \mathcal{O}(\mathcal{M}).$$

Proposition 5.6. *Let \mathcal{M}^m be a manifold. There is a unique linear form,*

$$\int_U : \Gamma_C(| \wedge \mathcal{M} |) \rightarrow \mathbb{R},$$

which is invariant under diffeomorphisms and in local coordinate cocycles with the Lebesgue integral,

$$\int_{\mathcal{M}} f(x) |dx| = \int_{\mathbb{R}^m} f(x) |dx_1 \wedge \cdots \wedge dx_m|.$$

Proof. Let $\omega \in \Gamma_C(| \wedge \mathcal{M} |)$ with support contained in some chart (U, φ) . We have to put

$$\int_{\mathcal{M}} \omega = \int_{\mathbb{R}^m} (\varphi^{-1})^* \omega.$$

If we have another chart (V, ψ) such that $\text{supp } \omega \subseteq U \cap V$, then

$$\begin{aligned} (\varphi^{-1})^* \omega &= f |dx_1 \wedge \cdots \wedge dx_m|, \\ (\psi^{-1})^* \omega &= (\varphi^{-1} \circ \varphi \circ \psi^{-1})^* \omega = (\varphi) \\ &\Rightarrow \int_{\mathbb{R}^m} (\varphi^{-1})^* \omega = \int_{\mathbb{R}^m} (\psi^{-1})^* \omega. \end{aligned}$$

For the general case, choose $\rho_1, \dots, \rho_k \in \mathcal{C}_C^\infty(\mathcal{M})$ with,

$$\sum \rho_j |_{\text{supp } \omega} \equiv 1 |_{\text{supp } \omega},$$

such that $\text{supp } \rho_j$ lies in a chart. Put

$$\int_{\mathcal{M}} \omega := \sum_{j=1}^k \int_{\mathcal{M}} \rho_j \omega.$$

If $\tilde{\rho}_1, \dots, \tilde{\rho}_l$ is a different partition of unity then,

$$\sum_{j=1}^k \int_{\mathcal{M}} \rho_j \omega = \sum_{j=1}^k \int_{\mathcal{M}} \sum_{i=1}^l \rho_j \tilde{\rho}_i \omega = \sum_{j,i=1}^{k,l} \int_{\mathcal{M}} \rho_j \tilde{\rho}_i \omega = \sum_{i=1}^l \int_{\mathcal{M}} \tilde{\rho}_i \omega_i.$$

□

Remark 5.9. *Let $\omega = f |dx_1 \wedge \cdots \wedge dx_m|$. Suppose we have a diffeomorphism,*

$$(\mathbb{R}^m, y) \xrightarrow{\varphi} (\mathbb{R}^m, x).$$

Then we have,

$$\varphi^* \omega = f \circ \varphi |\det D\varphi| dy_1 \wedge \cdots \wedge dy_m.$$

Then we have,

$$\begin{aligned} \int_{(R^m, x)} f |dx_1 \wedge \cdots \wedge dx_m| &= \int_{\mathbb{R}^m} f(x) dx_1 \cdots dx_m, \\ &= \int_{(\mathbb{R}^m, y)} f(\varphi(y)) |\det D\varphi(y)| dy_1 \cdots dy_m, \\ &= \int_{(\mathbb{R}^m, y)} f^* \omega. \end{aligned}$$

Corollary 5.1. *Let $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ be a diffeomorphism between smooth manifolds and $\omega \in \Gamma_C(| \wedge |^1 \mathcal{N})$. Then*

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{N}} \Phi^* \omega.$$

Proof. Write

$$\omega = \sum_{i=1}^k \omega_i,$$

such that $\text{supp } \omega \subseteq V_i$ for some coordinate chart $(V_i, \psi_i : V_i \rightarrow U_i \subseteq \mathbb{R}^n)$. Then

$$\text{supp } \Phi^* \omega_i \subseteq \Phi^{-1}(V_i),$$

which is a coordinate patch as well. To see this, we have,

$$\psi_i \circ \Phi|_{\Phi^{-1}(V_i)} : \Phi^{-1}(V_i) \rightarrow U_i \subseteq \mathbb{R}^n,$$

is again a chart. Then,

$$\int_{\mathcal{N}} \omega_i = \int_{V_i} \omega_i = \int_{\Phi^{-1}(V_i)} \Phi^* \omega_i = \int_{\mathcal{M}} \Phi^* \omega_i.$$

As ω is a finite sum of these ω_i , we have,

$$\int_{\mathcal{N}} \omega = \sum_{i=1}^k \int_{\mathcal{N}} \omega_i = \sum_{i=1}^k \int_{\mathcal{M}} \Phi^* \omega_i = \int_{\mathcal{M}} \Phi^* \omega.$$

□

5.8 Orientations on manifolds

Definition 5.15. *Let \mathcal{M} be a smooth manifold. An atlas \mathcal{A} oriented is called oriented if*

$$\forall \varphi, \psi \in \mathcal{A}, x \in \mathcal{M}, \det D(\psi \circ \varphi^{-1}(x)) > 0.$$

Definition 5.16. Two oriented atlases \mathcal{A}, \mathcal{B} are equivalent if $\mathcal{A} \cup \mathcal{B}$ is oriented.

Definition 5.17. An orientation is a choice of a maximal oriented atlas. In particular, a manifold is orientable if there is an orientation.

Clearly we have $(\mathbb{R}^m, \{\text{id}\})$ is oriented.

Example 5.1. The following is an example of non-oriented manifold.

$(S^n, \{\psi_\pm\})$ where

$$\psi_\pm : S^n \setminus \{\pm e_n\}, \psi_\pm(x, t) \mapsto \frac{x}{1 \mp t}$$

Then we have

$$\psi_+ \circ \psi_-^{-1}(x) = \frac{x}{|x|^2}.$$

Thus $f(x) = \frac{x}{|x|^2}$ is orientation reversing. That is $\det Df(x) < 0$.

Remark 5.10. Fix an orientation reversing diffeomorphism of \mathbb{R}^n say $\Phi(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$. We have a new atlas,

$$\{\Phi \circ \psi_+, \psi_-\}$$

is oriented.

Example 5.2. A Möbius band cannot be oriented. It is an exercise for the readers to give it a proof.

Proposition 5.7. Let \mathcal{M} be a smooth manifold. \mathcal{M} is orientable if and only if one of following statements hold.

- i). There is a section s of the bundle $\mathcal{O}(\mathcal{M})$ with $|s(p)| = 1$ for all $p \in \mathcal{M}$. Note that \mathcal{O} comes with a canonical inner product, thus we can define a notion of length.
- ii). There is a section $\omega \in \Omega^m(\mathcal{M})$ where $m = \dim \mathcal{M}$ with $\omega(p) \neq 0$ for all $p \in \mathcal{M}$.

Moreover, each of the conditions will determine an orientation.

Proof.

For the first one. Suppose \mathcal{M} is orientable. then consider $\mathcal{M} = \bigcup_n U_n$ where (U_n, φ_n) is an oriented chart and consider (ρ_n) be a subordinate partition of unity to the charts. We set,

$$\omega := \sum_{j=1}^{\infty} \rho_j dx_1^j \wedge \dots \wedge dx_m^j. \quad (4)$$

We show that $\omega(p) \neq 0$ for all $p \in \mathcal{M}$. To see this for a fixed $p \in U_i$. In the small enough neighborhood around p , we have,

$$\omega = \sum_{j=1}^{\infty} \rho_j \circ \varphi \det D(\varphi^{-1} \varphi_j \circ \varphi_i) dx_1^i \wedge \dots \wedge dx_m^i. \quad (5)$$

As we have assumed the manifold to be oriented. Each $\det D(\varphi_j \circ \varphi_i^{-1})$ is positive. And partition of unities are non-negative. We conclude that $\omega(p) > 0$.

Suppose the second statement holds. Fix $\omega \in \Omega^m \mathcal{M}$ such that $\omega(p) \neq 0$ for all $p \in \mathcal{M}$. Let (U, φ) be a connected chart. Then obviously, we have,

$$\varphi^* \omega = \rho_\varphi dx_1 \wedge \cdots \wedge dx_m.$$

with

$$\forall x \in \varphi(U), \rho_\varphi(x) \neq 0.$$

We call φ is oriented if $\rho_\varphi > 0$. This determines an oriented atlas. Thus proves that the second condition determines an atlas.

Furthermore, the first and the second conditions are equivalent. To show this, choose a Riemannian metric on \mathcal{M} . Given s as in the first condition. Set

$$\omega = \text{vol} \cdot s.$$

Conversely, given ω as in the second condition set $s = \frac{\omega}{\text{vol}}$. □

6 Manifolds with boundaries

6.1 Basics

Definition 6.1 (Half-space). *Let E be a real vector space of dimension $n < \infty$. A subset $H \subseteq E$ is called a half space if there is a linear form $\lambda \in E^* \setminus \{0\}$ such that*

$$H = \{x \in E \mid \langle \lambda, x \rangle \leq 0\}.$$

We denote such H by E_λ^+ .

Example 6.1. $\{x \in \mathbb{R}^n \mid x_j \leq 0\}$ is a half space for $1 \leq j \leq n$.

Remark 6.1. *Let $H_1, H_2 \subseteq E$ be two half-spaces. Then there is a linear isomorphism $T \in \text{GL}(E)$ such that $TH_1 = H_2$. That is if*

$$H_j = E_{\lambda_j}^+,$$

then take $T^ \in \text{GL}(E^*)$ such that $T^* \lambda_1 = \lambda_2$. We have*

$$\langle \lambda_2, x \rangle = \langle T^* \lambda_1, x \rangle = \langle \lambda_1, Tx \rangle.$$

Remark 6.2. *The above remark is true in the oriented category only if $\dim E \geq 2$. That is $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$ are diffeomorphic but not oriented diffeomorphic.*

Definition 6.2. *Let $E_\lambda^+ \subseteq E$ be a half space. The normal space is $E / \ker \lambda$.*

Lemma 6.1. *A normal space $E / \ker \lambda$ is canonically oriented by defining a base of $E / \ker \lambda$ by $\lambda(v) > 0$.*

Definition 6.3. If $v \in E$ with $\lambda(v) > 0$ then we say v points outward. Furthermore, we set

$$c(t) = tv.$$

Remark 6.3. The notion comes from the following case. Consider a curve $c : \mathbb{R} \rightarrow E$ such that for small enough $\varepsilon > 0$, we have $c((-\varepsilon, 0]) \subseteq E_\lambda^+$ for some $\lambda \in E^*$. And for $t > 0$, $c(t) \notin E_\lambda^+$.

Definition 6.4. Let $A \subseteq \mathbb{R}^m$ be a set and $f : A \rightarrow \mathbb{R}^n$ be a map. f is differentiable if for any $a \in A$, there is a neighborhood $a \in U \subseteq \mathbb{R}^m$ (not necessarily contained in A) and a differentiable function $g : U \rightarrow \mathbb{R}^n$ such that

$$g|_{U \cap A} = f|_{U \cap A}.$$

Proposition 6.1. Let E_λ, E_μ be half spaces in $E \cong \mathbb{R}^n$, $U \subseteq E_\lambda, V \subseteq E_\mu$ be open, and $\phi : U \rightarrow V$ be a diffeomorphism. Then

$$\phi(\text{int } U) = \text{int } V, \partial U = \partial V.$$

Furthermore, for $x \in \partial U$, and any outward pointing vector v ,

$$D_v \mu \circ \phi(x) > 0.$$

Proof. Fix $x \in U$. Then there is a neighborhood \tilde{U}_x of x , and \tilde{V}_y of $y = \phi(x)$ in E and smooth maps

$$\tilde{\phi} : \tilde{U}_x \rightarrow \tilde{V}_y, \tilde{\psi} : \tilde{V}_y \rightarrow \tilde{U}_x,$$

extending f with respect to f^{-1} . Since E_λ, E_μ are half-spaces, we have,

$$D_x \phi^{-1} \circ \phi = D_{f(x)} \phi^{-1} \circ D_x \phi(x).$$

This implies that $D_{f(x)} \tilde{\psi} \circ D_x \tilde{\phi}(x) = \text{id}$.

Suppose $\phi(x) \in \partial V$. Then \tilde{V}_y contains an element outside of V .

Let $c : (-\varepsilon, 0] \rightarrow U$ with $c((-\varepsilon, 0)) \subseteq U^\circ, c(0) = x, c'(0) = v$. Then we have

$$D_v(\mu \circ \phi)(x) = \frac{d}{dt} \bigg|_{t=0} \underbrace{\mu(\phi(c(t)))}_{=\phi(t)}.$$

We have

$$t < 0 \Rightarrow f(t) \leq 0, f(0) = 0.$$

Therefore, we have $f'(x) \geq 0$. Suppose $f'(0) = 0$ then $(\phi \circ c)'(0) \in \ker \mu$. This is because that the outward pointing vectors and their negatives are (missing), this implies that range of $D_x(\phi) \subseteq \ker \mu$. This contradicts that the rank of $D_x \phi$ is full. \square

Definition 6.5. A smooth manifold with boundary is a second countable Hausdorff space \mathcal{M} together with maximal atlas of charts (U, φ) with $\varphi : U \rightarrow V \subseteq E_\lambda^+$ being a homeomorphism on to some open subset V in E_λ^+ .

Remark 6.4. Let \mathcal{M} be a smooth manifold with boundary. A transition map $\psi \circ \varphi^{-1}$ are diffeomorphism from an open subset of E_λ^+ to some open subset of E_μ^+ .

Definition 6.6. Let \mathcal{M} be a smooth manifold with boundary. We define,

- 1). the interior of \mathcal{M} to be $\mathcal{M}^\circ := \text{int } \mathcal{M} = \bigcup_{\varphi \in \mathcal{A}} \varphi^{-1}(\text{int } E_\lambda^+)$,
- 2). the boundary of \mathcal{M} to be $\partial\mathcal{M} := \varphi^{-1}(\partial E_\lambda^+)$.

Proposition 6.2. Let \mathcal{M}^m be a smooth boundaryless manifold and let $f : \mathcal{M} \rightarrow \mathbb{R}$ be a submersion that is $\forall p \in \mathcal{M}, df|_p \neq 0$. Then

$$\mathcal{N} = \{x \in \mathcal{M} \mid f(x) \leq 0\},$$

is a manifold with boundary and

$$\partial\mathcal{N} = \{x \in \mathcal{M} \mid f(x) = 0\}.$$

Proof. Given $p \in \mathcal{M}$, there exists a chart (U, φ) centered at p such that $\varphi_m = f$. Thus $\varphi : U \rightarrow V$ for $V \subseteq \mathbb{R}^m$. That is

$$U \cap \mathcal{N} \rightarrow V \cap \{x_m = 0\}.$$

All such charts give an atlas for a manifold with boundary \mathcal{N} . \square

Example 6.2. Let $B \subseteq \mathbb{R}^n$ be a unit ball. We know that $\partial B = S^{n-1}$. The function f above in this case is given by

$$f(x) = \|x\|^2 - 1 = \sum_{i=1}^n x_i^2 - 1.$$

Proposition 6.3. Let \mathcal{M}^m be a smooth manifold. A subset $A \subseteq \mathcal{M}$ is a submanifold with boundary if for each $a \in A$ there is an open neighborhood U in \mathcal{M} and a submersion $f : U \rightarrow \mathbb{R}$ such that $A \cap U = \{f \leq 0\}$. In that case $\partial \cap U = \{f = 0\}$.

6.2 Stokes theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be sufficiently differentiable. We have,

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Observe that $f'(x)dx \in \Omega^1([a, b])$. Obviously $[a, b]$ is a manifold with boundary and the above equation can be restated as,

$$\int_{\partial[a, b]} f d\#,$$

where $\#$ is the counting measure. Now consider $\{x_1 \leq 0\} \subseteq \mathbb{R}^m$. Take the canonical basis e_1, \dots, e_m and its dual basis e_1^*, \dots, e_m^* . For $\lambda = -e_1^*$,

$$\{x_1 \geq 0\} = E_\lambda^+.$$

Let $\omega \in \Omega_C^{m-1}(E_\lambda^+)$ such that

$$\omega = \sum_{j=1}^m (-1)^{j-1} f_j dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_m.$$

where $f_j \in C_C^\infty(E_\lambda^+)$. Thus

$$d\omega = \sum_{j=1}^m \frac{\partial f_j}{\partial x_j} dx_1 \wedge \dots \wedge dx_m = (\nabla f) \cdot dx_1 \wedge \dots \wedge dx_m.$$

Integrating this we get,

$$\begin{aligned} \int_{E_\lambda^+} d\omega &= \sum_{j=1}^m \int_{\{x_1 \leq 0\}} \frac{\partial f_j}{\partial x_j} dx_1 \dots dx_m, \\ &= \sum_{j=1}^m \int_{\{x_1 \leq 0\}} \underbrace{\left(\int_{\mathbb{R}} \frac{\partial f_j}{\partial x_j} dx_j \right)}_{=0} dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_m, \end{aligned}$$

where the last equality is due to Fubini. Except for $j = 1$, thus

$$\int_{E_\lambda^+} d\omega = \int_{\mathbb{R}^{m-1}} \left(\int_0^\infty \frac{\partial f_1}{\partial x_1} dx_1 \right) dx_2 \dots dx_m = \int_{\mathbb{R}^{m-1}} -f_1(0, x_2, \dots, x_m) dx_2 \dots dx_m.$$

Set $\iota : \partial E_\lambda^+ = \{x_1 = 0\} \hookrightarrow E_\lambda^+ \hookrightarrow \mathbb{R}^m$. We then have,

$$\iota^* \omega = f_1 dx_2 \wedge \dots \wedge dx_m.$$

And in particular,

$$\int_{E_\lambda^+} d\omega = - \int_{\partial E_\lambda^+} \omega.$$

By setting orientations on as e_1, \dots, e_m positive in E_λ^+ and e_2, \dots, e_m negative in ∂E_λ^+ , we have,

$$\int_{E_\lambda^+} d\omega = \int_{\partial E_\lambda^+} \omega.$$

Theorem 6.1 (Stokes). *Let \mathcal{M}^m be oriented manifold with boundary such that $\partial \mathcal{M}^m$ carrying the induced orientation, $\omega \in \Omega_C^{m-1}(\mathcal{M})$. Then*

$$\int_{\mathcal{M}} d\omega = \int_{\partial \mathcal{M}} \iota^* \omega.$$

Remark 6.5. Let E be a n -dimensional vector space and H be a half-space. Suppose E is oriented. Choose an oriented basis of E of the form

$$v, e_2, \dots, e_n,$$

where

$$H = E_\lambda^+ = \{x \in E \mid \langle \lambda, x \rangle \leq 0\},$$

and $e_2, \dots, e_n \in \ker \lambda$. This defines an induced orientation of $\ker \lambda$ and v points outward.

Definition 6.7. Let \mathcal{M}^m be an oriented manifold with boundary then $\partial \mathcal{M}^m$ is oriented as follows. For $p \in \partial \mathcal{M}^m$, let $\nu(p)$ be an outward pointing vector $\nu(p) \neq 0$. Take a basis v_1, \dots, v_m of $T_p \partial \mathcal{M}$ is oriented by definition if and only if

$$\nu(p), v_2, \dots, v_m$$

is an oriented basis of $T_p \mathcal{M}$. This is called the induced boundary orientation.

Proposition 6.4 (Stokes Theorem for Half Spaces). Let E be an oriented n -dimensional vector space and $H = E_\lambda^+ \subseteq E$ be a half space and equip $\partial H = \ker \lambda$ and ω the induced orientation. Let

$$i : \partial H \hookrightarrow H$$

be the inclusion. Then for $\omega \in \Omega_C^{n-1}(H)$, we have,

$$\int_H d\omega = \int_{\partial H} i^* \omega.$$

Proof. Choose oriented linear coordinate, x_1, \dots, x_n , such that $x_1 = \langle \lambda, x \rangle$. Then

$$H = \{x \in \mathbb{R}^n \mid x_1 \leq 0\}, \partial H = \{0\} \times \mathbb{R}^{n-1}.$$

Recall that for $j \geq 2$, we get,

$$\int_{\mathbb{R}} \frac{\partial \phi}{\partial x_j}(\dots, x_j, \dots) dx_j = 0,$$

for $j = 1$, we have,

$$\int_{-\infty}^0 \frac{\partial \phi}{\partial x_1}(x_1, 0, \dots, 0) dx_1 = \phi(0, a).$$

Note that

$$\omega = \sum_{j=1}^n (-1)^{j-1} \phi_j dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_n.$$

$$i^* \omega = \phi i^*(dx_2 \wedge \dots \wedge dx_m)?$$

Therefore,

$$d\omega = \sum_{j=1}^n \frac{\partial \phi_j}{\partial x_j} dx_1 \wedge \dots \wedge dx_n.$$

□

Theorem 6.2 (Stokes, note that we do not assume compactness). *Let \mathcal{M}^m be an oriented smooth manifold with boundary $\partial\mathcal{M}$. $\partial\mathcal{M}$ carries an induced orientation. Then for $\omega \in \Omega_C^{n-1}(\mathcal{M})$, we have,*

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} i^* \omega,$$

where $i : \partial\mathcal{M} \hookrightarrow \mathcal{M}$ is a natural inclusion.

Proof. The key is to use partition of unity. Choose $\rho_1, \dots, \rho_k \in \mathcal{C}_C^\infty(\mathcal{M})$ such that

$$\sum_{j=1}^k \rho_j = 1,$$

in the neighborhood of $\text{supp } \omega$. Furthermore, we may arrange that $\text{supp } \rho_j \subseteq U_j$ where (U_j, φ_j) is an oriented chart. Thus each φ_j maps $U_j \rightarrow V_j \subseteq H_j$ where H_j is a half space.

$$\begin{aligned} \int_{\mathcal{M}} d\omega &= \sum \int_{\mathcal{M}} d(\rho_j \omega), \\ &= \sum \int_{U_j} d(\rho_j \omega), \\ &= \sum \int_{H_j} d((\varphi_j^{-1})^* \rho_j \omega), \\ &= \sum \int_{\partial H_j} i^* (\varphi_j^{-1})^* \rho_j \omega, \\ &= \sum \int_{\partial H_j} (\tilde{\varphi}_j^{-1})^* i^* (\rho_j \omega), \\ &= \sum \int_{\partial\mathcal{M}} i^* (\rho_j \omega), \\ &= \int_{\partial\mathcal{M}} \omega, \end{aligned}$$

where $\tilde{\varphi}$ is such that

$$\begin{array}{ccc} \partial\mathcal{M} & \xhookrightarrow{i} & \mathcal{M} \\ \tilde{\varphi} \downarrow & & \downarrow \varphi \\ \partial H & \xhookrightarrow{i} & H \end{array}$$

□

Remark 6.6. Note if $\partial\mathcal{M} = \emptyset$ then $\int_{\mathcal{M}} d\omega = 0$. For example, consider a circle S_θ^1 , we have,

$$\int_{S^1} f'(\theta) d\theta = 0.$$

6.3 Classical Theorem of Gauss and Stokes

Theorem 6.3 (Divergence Theorem of Gauss). *Let (\mathcal{M}^m, g) be an oriented Riemannian manifold with boundary $\partial\mathcal{M}$. Let ν be an outer normal vector field. In other words,*

$$\nu : \partial\mathcal{M} \rightarrow T\mathcal{M},$$

such that for all $p \in \partial\mathcal{M}$, we have,

$$\nu(p) \perp T_p\partial\mathcal{M}, \|\nu(p)\| = 1,$$

and $\nu(p)$ points outward. Now $\text{vol}_{\mathcal{M}} \in \Omega^m(\mathcal{M}), \text{vol}_{\partial\mathcal{M}} \in \Omega^{m-1}(\partial\mathcal{M})$. Let $X \in \Gamma(T\mathcal{M})$ be a vector field. Then,

$$d(\star X^\flat) = d(i_X(\text{vol}_{\mathcal{M}})) = (\text{div } X) \text{vol}_{\mathcal{M}}.$$

Proof.

Claim 1.

$$i^*(i_X \text{vol}_{\mathcal{M}}) = \langle X, \nu \rangle \text{vol}_{\partial\mathcal{M}}.$$

Proof: For $p \in \partial\mathcal{M}$, e_2, \dots, e_m , be an orthonormal basis of $T_p\partial\mathcal{M}$ such that

$$X(p) = \langle X(p), \nu(p) \rangle \cdot \nu(p) + \sum_{j=2}^m \langle X(p), e_j \rangle e_j.$$

Then, by setting $e_1 = \nu(p)$, we get,

$$i_X \text{vol}_{\mathcal{M}}|_p = i_X e_1^\flat \wedge \dots \wedge e_m^\flat = \langle X(p), \nu(p) \rangle e_2^\flat \wedge \dots \wedge e_m^\flat = \text{vol}_{\partial\mathcal{M}}|_p. \quad \blacksquare$$

Apply Stokes, we have,

$$\int_{\mathcal{M}} \text{div } X \text{vol}_{\mathcal{M}} = \int_{\mathcal{M}} di_X \text{vol}_{\mathcal{M}} = \int_{\partial\mathcal{M}} i^*(i_X \text{vol}_{\mathcal{M}}) = \int_{\partial\mathcal{M}} \langle X(p), \nu(p) \rangle \text{vol}_{\partial\mathcal{M}}. \quad \square$$

Theorem 6.4 (Divergence Theorem). *Let (\mathcal{M}, g) be an oriented Riemannian manifold with boundary and $\nu : \partial\mathcal{M} \rightarrow T\mathcal{M}$ be a unit outer normal vector field. Then for any compactly supported C^1 -vector field, we have,*

$$\int_{\mathcal{M}} \text{div } X \text{vol}_{\mathcal{M}} = \int_{\partial\mathcal{M}} \langle X, \nu \rangle \text{vol}_{\partial\mathcal{M}}.$$

Theorem 6.5 (Stokes Theorem for Surface in 3-manifolds). *Let (\mathcal{M}, g) be oriented 3-dimensional Riemannian manifold. Let $F \subseteq \mathcal{M}$ be an oriented surface with boundary. Let $\nu_F : F \rightarrow T\mathcal{M}$ be such that for $p \in F$, there is $e_1, e_2 \in T_p F$, an oriented orthonormal basis of $T_p F$ such that*

$$\nu_F(p) := e_1 \times e_2.$$

Recall that for a vector field X over \mathcal{M} , we have

- 1). X^\flat is 1-form,
- 2). dX^\flat is 2-form,
- 3). $\star dX^\flat$ is 1-form.

Definition 6.8. Let X be a vector field on \mathcal{M} , then we define,

$$\text{curl } X := (\star dX^\flat)^\#.$$

Remark 6.7. $\text{curl } X$ coincides with the usual rotation in \mathbb{R}^3 .

Theorem 6.6 (Classical Stokes Theorem). *Inheritting the settings of Theorem 6.5, consider a vector field X on \mathcal{M} which is of class \mathcal{C}^1 . We have,*

$$\int_F \langle \text{curl } X, \nu_F \rangle \text{vol}_F = \int_{\partial F} \langle X, \tau_{\partial F} \rangle \text{vol}_{\partial F},$$

where $\tau_{\partial F} : \partial F \rightarrow T\partial F$ is the oriented unit tangent vectorfield to ∂F .

Proof.

$$\begin{aligned} \langle \text{curl } X, \nu_F \rangle \text{vol}_F &= (i_{\mathcal{M}}^F)^*(i_{\text{curl } X} \text{vol}_{\mathcal{M}}), \\ &= (i_{\mathcal{M}}^F)^*(\star(\text{curl } X)^\flat), \\ &= (i_{\mathcal{M}}^F)^*(\star \star dX^\flat), \\ &= (i_{\mathcal{M}}^F)^*(dX^\flat). \end{aligned}$$

Therefore,

$$\int_F \langle \text{curl } X, \nu \rangle \text{vol}_F = \int_F dX^\flat \stackrel{\text{Stokes}}{=} \int_{\partial F} (i_F^{\partial F})^* X^\flat.$$

□

6.4 Applications of Stokes Theorem

Let $A \subseteq \mathbb{R}^{m+1}$ be a compact set with smooth boundary. A is a compact manifold with boundary such that

$$\text{int } A = A \setminus \partial A$$

is open in \mathbb{R}^{m+1} . This is a compact hypersurface in \mathbb{R}^{m+1} . A Topological-boundary of the manifold boundary of A .

Remark 6.8. If A compact surface in \mathbb{R}^3 is the boundary of a compact set with smooth boundary, then it is orientable.

Consider in \mathbb{R}^{m+1} , define,

$$\eta := \sum_{j=1}^{m+1} (-1)^{j+1} x_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{m+1} \in \Omega^m(\mathbb{R}^{m+1}).$$

Then

$$d\eta = \sum_{j=1}^{m+1} (-1)^{j+1} dx_j \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{m+1}.$$

Therefore, we have ,

$$d\eta = (m+1) dx_1 \wedge \cdots \wedge dx_{m+1}.$$

This is a volume form in \mathbb{R}^{m+1} which is a volume for A as well. Therefore,

$$(m+1) \cdots dx_1 \wedge \cdots \wedge dx_{m+1}|_A = (m+1) \text{vol}_A.$$

Therefore, we have,

$$\text{vol}(A) = \int_A \frac{1}{m+1} d\eta = \frac{1}{m+1} \int_{\partial A} \eta.$$

Lemma 6.2. *Let $i : S^m \hookrightarrow \mathbb{R}^{m+1}$. Then $i^*\eta$ is the volume form on S^m .*

Corollary 6.1.

$$\text{vol}(S^m) = \int_{S^m} \eta = \int_{B(0,1) \subseteq \mathbb{R}^{m+1}} d\eta = (m+1) \text{vol}(B(0,1)).$$

Proof. Let $\omega = dx_1 \wedge \cdots \wedge dx_{m+1} = \text{vol}_{\mathbb{R}^{m+1}}$. We define a vector field,

$$X(x) = \sum_{j=1}^{m+1} x_j \frac{\partial}{\partial x_j},$$

which we will call it the radial vector field. Note that let $p \in S^m$, then $X(p)$ is the outward normal vector. We then have,

$$i_X \omega = i_X dx_1 \wedge \cdots \wedge dx_{m+1} = \sum_{i=1}^{m+1} (-1)^{i-1} dx_1 \wedge \cdots \wedge dx_{i-1} i_X(dx_i) \wedge dx_{i+1} \wedge \cdots \wedge dx_{m+1}.$$

But observe that $i_X(dx_j) = x_j$, thus

$$i_X \omega = \eta.$$

Let $x \in S^m$ and v_1, \dots, v_m be an oriented orthonormal basis of $T_x S^m$. Then

$$X(x), v_1, \dots, v_m$$

is an oriented orthonormal basis of $T_x \mathbb{R}^{m+1}$.

$$\begin{aligned} i_\eta^*(v_1, \dots, v_m) &= (i_X \omega)(v_1, \dots, v_m), \\ &= \omega(X(x), v_1, \dots, v_m), \\ &= 1. \end{aligned}$$

Thus we have shown that $i^* \eta$ is the volume form on S^m . \square

Example 6.3.

$$\text{vol}(A) = \frac{1}{2} \int_A x dy - y dx,$$

Proposition 6.5. *Consider the antipodal map,*

$$A : S^m \rightarrow S^m, x \mapsto -x.$$

This is orientation preserving/reversing, if and only if m is odd/even.

Proof. Consider η as in Corollary 6.1. We have,

$$A^* \eta = (-1)^{m+1} \eta.$$

This proves the statement. \square

Corollary 6.2. $\mathbb{RP}^m = S^m/A$ *is orientable if and only if m is odd.*

Let $\mathcal{M} \xrightarrow{\pi} \mathcal{M}/A$ be a 2-fold cover of an oriented connected manifold \mathcal{M} by an orientation preserving (respectively, reversing involution A).

Assume \mathcal{M}/A is orientable hence there is $\omega \in \Omega^m(\mathcal{M}/A)$ such that it is nowhere vanishing. Therefore, there is nowhere vanishing $f \in \mathcal{C}^\infty(\mathcal{M})$, (ie. $\forall p \in \mathcal{M}, f(p) \neq 0$), with

$$\pi^* \omega f \cdot \text{vol}_M.$$

We have,

$$A^* \pi^* \omega = \underbrace{(\pi \circ A)^*}_{=\pi} \omega = \pi^*$$

Thus we have $f \cdot \eta = A^*(f\eta) = -A^* f \eta$, if A is orientation reversing.

If A is orientation reversing, f is odd with respect to A . Using intermediate value theorem, we see that there is $p \in \mathcal{M}$ such that $f(p) = 0$ which is a contradiction.

Corollary 6.3. *Let us denote $A_m : S^m \rightarrow S^m$ to be the antipodal map of m -dimensional sphere. A_m is homotopic to the identity if and only if m is odd.*

Corollary 6.4 (Hairy Ball Theorem). *There is a non-vanishing vector field on S^m if and only if m is odd.*

Proof. For odd m , let us consider,

$$X(x) = (-x_2, x_1, -x_4, x_3, \dots, -x_{m+1}, x_m).$$

This is a non-vanishing vector field on S^{m+1} . Suppose we have $X \in \Gamma(TS^m)$ such that $\forall p \in S^m, X(p) \neq 0$. Replace,

$$X \rightarrow \left[S^m \ni p \mapsto \frac{X(p)}{|X(p)|} \right]$$

Without loss of generality, we assume $|X(p)| = 1$. Pick $\rho \in \mathcal{C}^\infty(\mathbb{R})$ to be such that

$$\rho(t) = \begin{cases} 0, & t \leq \frac{1}{3}, \\ 1, & t \geq \frac{2}{3}. \end{cases}$$

And $\rho(t) \in [0, 1]$ for all $t \in \mathbb{R}$. Now set $h : [0, 1] \times S^m \rightarrow S^m$ to be

$$h(t, p) := \cos(\pi \cdot \rho(t)) \cdot p + \sin(\pi \rho(t)) \cdot X(p).$$

$X(p) \perp p \Rightarrow \|h(t, p)\|^2 = 1$. Thus we obtain,

$$h(0, p) = p, h(1, p) = -p.$$

Therefore, if there exists a nowhere vanishing vector field, then antipodal map is homotopic to the identity.

Suppose m is even and A_m is homotopic. Then we have,

$$\int_{S^m} A^* \eta = - \int_{S^m} \eta = -\text{vol}(S^m) < 0.$$

Claim 2.

$$\int_{\mathcal{M}} h_t^* \omega$$

is independent of t .

Proof: Without loss of generality, we show

$$\int_{\mathcal{M}} h_0^* \omega = \int_{\mathcal{M}} h_1^* \omega.$$

Note that $[0, 1] \times \mathcal{M}$ is a manifold with oriented boundary $\mathcal{M} \cup (-\mathcal{M})$. By Stoke's theorem, we have,

$$\int_{\mathcal{M}} h_1^* \omega - \int_{\mathcal{M}} h_0^* \omega = \int_{\partial([0, 1] \times \mathcal{M})} h^* \omega = \int_{\partial([0, 1] \times \mathcal{M})} \text{int}_{\partial([0, 1] \times \mathcal{M})} d(h^* \omega) = \int_{[0, 1] \times \mathcal{M}} h^*(d\omega) = 0.$$

■

Using the claim, we have $\int_{S^m} A^* \eta$ is homotopy invariant. Therefore

$$\int_{S^m} A^* \eta = \int_{S^m} \eta = \text{vol}(S^m) < 0.$$

More general, let \mathcal{M}^m be a closed oriented manifold and $\omega \in \Omega^m(\mathcal{M})$. Furthermore, consider,

$$h : [0, 1] \times \mathcal{M} \rightarrow \mathcal{M}$$

be smooth. Note that it is a family of endomorphisms of \mathcal{M} thus we denote

$$h_t : \mathcal{M} \rightarrow \mathcal{M} = [\mathcal{M} \ni p \mapsto h(t, p) \in \mathcal{M}].$$

□

Example 6.4. *Using this corollary, we conclude that any vector field on S^2 vanishes somewhere.*

6.5 Brouwer's Fixed Point Theorem

Theorem 6.7. *Let $\overline{B(0, 1)}$ be a unit ball around the origin in \mathbb{R}^n and $f : \overline{B(0, 1)} \rightarrow \overline{B(0, 1)}$ be continuous. Then f has a fixed point.*

Proof. We first reduce to the case where f is smooth. Suppose a continuous function f does not have a fixed point. Then consider,

$$\delta := \inf_{x \in \overline{B(0, 1)}} \|f(x) - x\| > 0.$$

Let $g : \overline{B(0, 1)} \rightarrow \overline{B(0, 1)}$ be smooth such that

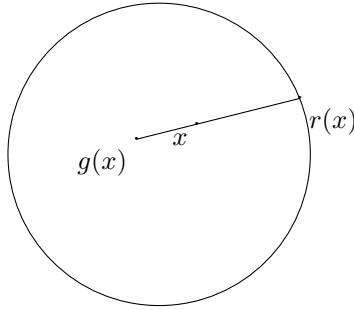
$$\|f - g\|_{L^\infty(\overline{B(0, 1)})} \leq \frac{\delta}{2}.$$

Then g is a smooth self map of $\overline{B(0, 1)}$ without fixed point.

Let $r : \overline{B(0, 1)} \rightarrow S^{n-1}$ to be such that for $x \in \overline{B(0, 1)}$ we define,

$$r(x) \in S^{n-1} \cap \{g(x) + t(x - g(x)) \mid t \geq 0\}.$$

This is smooth and $r|_{S^{m-1}} = \text{id}$. Intuitively, r is constructed by the following way shown in the diagram below.



That is if such g exists then there is a smooth retraction $r : \overline{B(0, 1)} \rightarrow S^{m-1}$. That is we have a following commutative diagram,

$$\begin{array}{ccc}
\overline{B(0,1)} & \xrightarrow{r} & S^{n-1} \\
\uparrow i & \nearrow \text{id} & \\
S^{n-1} & &
\end{array}$$

Let $\omega \in \Omega^{m-1}(S^{m-1})$ be volume form then $r^*\omega \in \Omega^{m-1}(\overline{B(0,1)})$ is exact by Poincaré's lemma. That is there is $\xi \in \Omega^{m-2}(B(0,1))$ such that

$$r^*\omega = d\xi.$$

We then have,

$$\omega = \text{id}^* \omega = i^* r^* \omega = i^* d\xi = d(i^* \xi).$$

We found ω is exact. By Stoke's theorem, we see,

$$\text{vol}(S^{m-1}) = \int \omega = \int di^* \xi = \int_{B(0,1)} dd\xi = 0.$$

□

7 De Rham Cohomology

7.1 Basics

Definition 7.1. Let \mathcal{M} be a smooth manifold. We define,

$$\Omega^\bullet(\mathcal{M}) = \bigoplus_{j=0}^{\dim \mathcal{M}} \Omega^j(\mathcal{M}).$$

Definition 7.2. A cochain complex is of k -vectorspace (we can also take R -modules) consists of a \mathbb{Z} -graded vector space $E := \bigoplus E^k$ and homomorphism $d : E^k \rightarrow E^{k+1}$ of degree 1 (ie: any homogeneous element of degree k is mapped to a homogeneous element of degree $k+1$) with $d \circ d = 0$.

A cochain complex is represented as a sequence,

$$\dots \longrightarrow E^{k-1} \xrightarrow{d_{k-1}} E^k \xrightarrow{d_k} E^{k+1} \longrightarrow \dots$$

This is not necessarily exact. We would like to measure the non-exactness of the series.

Definition 7.3. Given a cochain complex (E, d) , we define the following,

1. $Z^k E = \ker(E^k \xrightarrow{d} E^{k+1})$ which we call the cycles of (E, d) .
2. $B^k E = \text{im}(E^{k-1} \xrightarrow{d} E^k)$ which is the boundaries of (E, d) .
3. $H^k(E) = H^k(E, d) := Z^k E / B^k E$ which is called the k -th cohomology group of (E, d) .

Above taking the quotient $Z^k E / B^k E$ is justified as

$$d \circ d = 0 \Rightarrow \text{im } d_{k-1} \subseteq \ker d_k.$$

Definition 7.4. The k -th de Rham cohomology group of \mathcal{M} is

$$H_{\text{dR}}^k(\mathcal{M}) := H^k(\Omega^\bullet(\mathcal{M}), d).$$

Example 7.1. Let $f \in Z^0(\Omega^\bullet(\mathcal{M}))$, then f is a smooth function such that $df = 0$. That is f is a locally constant function. We conventionally set $B^0(\Omega^\bullet(\mathcal{M})) = \{0\}$. From the argument, we see,

$$H^0(\Omega^\bullet(\mathcal{M})) = \prod_{\text{components of } \mathcal{M}} \mathbb{R}.$$

Example 7.2. For $\mathcal{M} = S^1 = \{z \in \mathbb{C} \mid |z| = 1\} = \mathbb{R}/2\pi\mathbb{Z}$. The diffeomorphism is given by,

$$\mathbb{R}/2\pi\mathbb{Z} \ni \theta \mapsto e^{i\theta}.$$

From the example above we know

$$H^0(S^1) = \mathbb{R}.$$

Suppose $\omega \in \Omega^1(S^1)$. Then there is 2π periodic function $f \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\omega = f d\theta.$$

Thus $d\omega = 0$. This shows us that

$$Z^1(\Omega^\bullet(S^1)) = \Omega^1(S^1).$$

Suppose $\omega = dg$ for some 2π -periodic function $g \in \mathcal{C}^\infty(\mathbb{R})$ then,

$$\int_{S^1} \omega = \int_0^{2\pi} f(\theta) d\theta = g(2\pi) - g(0) = 0.$$

Remark 7.1. The above example can be further generalized as follows.

Suppose \mathcal{M} is compact and closed, $\omega \in \Omega^m(\mathcal{M})$ where $m = \dim \mathcal{M}$. If $\omega = d\eta$ then

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{M}} d\eta = \int_{\partial \mathcal{M} = \emptyset} \eta = 0.$$

Conversely, if $\int_{\mathcal{M}} \omega = 0$, put

$$g(\theta) := \int_0^\theta f(t) dt.$$

Then g is smooth and 2π -periodic. That is $dg = \omega$.

Proposition 7.1. Let us summarize what we have discussed. We have a commutative diagram,

$$\begin{array}{ccc}
\Omega^1(\mathbb{R}/2\pi\mathbb{Z}) & \xrightarrow{f} & \mathbb{R} \\
\downarrow & \nearrow \cong & \\
H^1(\mathbb{R}/2\pi\mathbb{Z}) & &
\end{array}$$

We will later see that for every compact closed oriented manifold, we have the above commutative diagram.

Definition 7.5. A homomorphism of cochain complexes is a linear map $f : (E, d^E) \rightarrow (F, d^F)$ of degree 0 such that

$$f \circ d^E = d^F \circ f.$$

In other words, we have a following commutative diagram,

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & E^{k-1} & \xrightarrow{d^{E,k-1}} & E^k & \xrightarrow{d^{E,k}} & E^{k+1} \longrightarrow \cdots \\
& & \downarrow f & & \downarrow f & & \downarrow f \\
\cdots & \longrightarrow & F^{k-1} & \xrightarrow{d^{F,k-1}} & F^k & \xrightarrow{d^{F,k}} & F^{k+1} \longrightarrow \cdots
\end{array}$$

Proposition 7.2. A cochain map $f \in \text{Hom}((E, d^E), (F, d^F))$ induces a homomorphism,

$$H^k(f) : H^k(E, d^E) \rightarrow H^k(F, d^F).$$

Proof. Clear from the above diagram. \square

Remark 7.2. Recall that for a smooth map $f : \mathcal{M} \rightarrow \mathcal{N}$, it induces a contravariant functor,

$$f^* : (\Omega^\bullet(\mathcal{N}), d) \rightarrow (\Omega^\bullet(\mathcal{M}), d).$$

Even though H^\bullet is a covariant functor from the category of cochain complexes to cochain complexes. The above functor is contravariant. This is due to that the pullback is contravariant.

Definition 7.6.

$$\Omega_C^k(\mathcal{M}) = \{\omega \in \Omega^k(\mathcal{M}) \mid \text{supp } \omega \text{ is compact.}\}.$$

Definition 7.7. Let us define a category $(\text{Mnfd}_{\text{OpEmb}})$ such that

i). $\text{Ob}(\text{Mnfd}_{\text{OpEmb}})$ consists of smooth manifolds.

ii). Morphisms are open embeddings.

Remark 7.3. Ω_C^\bullet is a covariant functor from $(\text{Mnfd}_{\text{OpEmb}})$ to the category of cochain complex.

Definition 7.8.

$$H_{\text{dR},C}^k(\mathcal{M}) = H^k(\Omega_C^\bullet(\mathcal{M}), d).$$

Proposition 7.3. If $f : \mathcal{M} \rightarrow \mathcal{N}$ is proper and $\omega \in \Omega_C^k(\mathcal{N})$ then $f^*\omega \in \Omega_C^k(\mathcal{M})$.

Remark 7.4. On the category of smooth manifolds with proper maps. Ω_C^\bullet and hence $H_{\text{dR},C}^\bullet$ are contravariant functors.

Definition 7.9.

7.2 Poincaré Lemma

Suppose we have,

$$f, g : (E^\bullet, d^E) \rightarrow (F^\bullet, d^F).$$

When do we have,

$$H^\bullet(f) = H^\bullet(g).$$

Definition 7.10. *Two cochain complex homomorphisms $f, g : (E^\bullet, d^E) \rightarrow (F^\bullet, d^F)$ are called cochain homotopic if there exists homomorphism $K_{k+1} : E^{k+1} \rightarrow F^k$ such that*

$$d_{k-1}^F K_k + K_{k+1} d_k^E = f - g.$$

Let $\omega \in Z^k(E)$. That is $\omega \in E^k, d_k \omega = 0$. Consider,

$$f(\omega) - g(\omega) = dK\omega + Kd\omega = dK\omega \in B^K(F).$$

Thus $H^k(f) = H^k(g)$.

Let \mathcal{M} be a smooth manifold (oriented). Consider,

$$\Omega^1(\mathcal{M}), \Omega^1(\mathcal{M} \times \mathbb{R}).$$

For fixed $t_0 \in \mathbb{R}$, let

$$j_k : \mathcal{M} \rightarrow \mathcal{M} \times \{t_0\} \subset \mathcal{M} \times \mathbb{R}, p \mapsto (p, t_0).$$

Put $K : \Omega^{k+1}(\mathcal{M} \times \mathbb{R}) \rightarrow \Omega^k(\mathcal{M} \times \mathbb{R})$ such that

$$K(\omega)|_{(p, t_0)} := \int_{t_0}^t (\text{int}_{\frac{\partial}{\partial t}} \omega)(p, t') dt'.$$

Take X_1, \dots, X_m be a coordinate in \mathcal{M} .

$$\begin{aligned} \int_{\frac{\partial}{\partial t}} (f(x, t) dt \wedge dx_{j_1} \wedge \dots \wedge dx_{j_p}) &= f(x, t) dx_{j_1} \wedge \dots \wedge dx_{j_p}, \\ \int_{\frac{\partial}{\partial t}} (g(x, t) dx_{j_1} \wedge \dots \wedge dx_{j_p}) &= 0. \end{aligned}$$

Proposition 7.4. *Let $\pi : \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M}$ be such that $(p, t) = p$. Then,*

$$dK + Kd = \text{id} - \pi^* \circ j_{t_0}^*.$$

Thus $H(\pi) \circ H(j_{t_0}) = \text{id}$. Furthermore,

$$H(\pi) = \pi^*, H^k(\mathcal{M}) \rightarrow H^k(\mathcal{M} \times \mathbb{R})$$

is an isomorphism.

Proof. For the last two assertions,

$$\begin{aligned}\pi \circ j_{t_0} &= \text{id}_{\mathcal{M}}, \\ j_{t_0}^* \circ \pi^* &= \text{id}_{\Omega^1(\Omega)}, \\ &\Rightarrow H(j_{t_0}) \circ H(\pi) = \text{id}.\end{aligned}$$

Let $\omega \in \Omega^{k+1}(\mathcal{M} \times \mathbb{R}_t)$ such that

$$\omega = \omega_1(t)dt + \omega_2(t).$$

Under this we have,

$$\begin{aligned}K\omega &= (-1)^K \int_{t_0}^t \omega_1(s)ds, \\ dK\omega &= \omega_1(t) \wedge dt + (-1)^K \int_{t_0}^t (d_{\mathcal{M}}\omega_1)(s)ds, \\ d\omega &= (d_{\mathcal{M}}\omega_1)(t)dt + (-1)^{K+1}\omega_2'(t) \wedge dt + (d_{\mathcal{M}}\omega_2)(t). \\ Kd\omega &= (-1)^{K+1} \int_{t_0}^t (d_{\mathcal{M}}\omega_1)(s)ds + \int_{t_0}^t \omega_2'(s)ds, \\ dK\omega + Kd\omega &= \underbrace{\omega_1(t) \wedge dt + \omega_2(t)}_{=\omega} - \overbrace{\omega_2(t_0)}^{\pi^* j_{t_0}^* \omega}, \\ &= \omega - \pi^* j_{t_0}^* \omega. \\ \pi^* j_{t_0}^* \omega &= \omega_2(t_0).\end{aligned}$$

□

Corollary 7.1. *For each $t_0 \in \mathbb{R}$, the map $j_{t_0}^* : \Omega^\bullet(\mathcal{M}) \rightarrow \Omega^\bullet(\mathcal{M})$, induces an isomorphism*

$$H_{\text{dR}}^\bullet(\mathcal{M} \times \mathbb{R}) \rightarrow H_{\text{dR}}^\bullet(\mathcal{M}),$$

with inverse

$$H(\pi) = \pi^*.$$

Remark 7.5. *For $t_0, t_1 \in \mathbb{R}$, we have*

$$j_{t_0}^* = j_{t_1}^*,$$

as maps from $H^\bullet(\mathcal{M} \times \mathbb{R}) \rightarrow H^\bullet(\mathcal{M})$.

Theorem 7.1. *Let $f, g : \mathcal{M} \rightarrow \mathcal{N}$ be smooth homotopic maps (the homotopy between them is smooth), then*

$$H(f) = H(g) : H^\bullet(\mathcal{N}) \rightarrow H^\bullet(\mathcal{M}).$$

In particular, if the manifolds \mathcal{M}, \mathcal{N} are smooth homotopic equivalent then

$$H^\bullet(\mathcal{M}) \cong H^\bullet(\mathcal{N}).$$

Proof. By assumption, there is a smooth map $F : \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{N}$ such that

$$F(\cdot, 0) = f, F(\cdot, 1) = g.$$

Let $j_t : \mathcal{M} \rightarrow \mathcal{M} \times \mathbb{R}$ be such that $j_t(p) = (p, t)$. Thus with this notation, we have,

$$F \circ j_0 = f, F \circ j_1 = g.$$

Thus we have,

$$\begin{aligned} H(f) &= H(F \circ j_0), \\ &= H(j_0) \circ H(F), \\ &= H(j_1) \circ H(F), \\ &= H(F \circ j_1), \\ &= H(g). \end{aligned}$$

\mathcal{M}, \mathcal{N} are smoothly homotopic equivalent if and only if there exists $f : \mathcal{M} \rightarrow \mathcal{N}, g : \mathcal{N} \rightarrow \mathcal{M}$ such that

$$f \circ g = \text{id}_{\mathcal{N}}, g \circ f = \text{id}_{\mathcal{M}}.$$

Then $H(f), H(g)$ are inverse to one another. \square

Theorem 7.2 (Poincaré Lemma). *Let $U \subseteq \mathbb{R}^m$ be a starshaped region (ie. an open set). Then,*

$$H^k(U) \cong \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

Proof. Choose $g \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$g(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t \geq 1. \end{cases}$$

Then set $F : U \times \mathbb{R} \rightarrow U$ by

$$F(x, t) = x_0 + g(t)(x - x_0),$$

where x_0 is a star-point of U . Thus we have,

$$F(x, 0) = x_0, F(x, 1) = x.$$

id_U is smooth homotopic to a constant map x_0 . Call it g . For $k \geq 1$,

$$\text{id}_{H^k(U)} = H(g^*)|_{H^k(U)} = 0.$$

\square

7.3 Mayer-Vietoris sequence

Definition 7.11. A short exact sequence of cochain complexes

$$0 \longrightarrow (A^\bullet, d) \xrightarrow{\varphi} (B^\bullet, d) \xrightarrow{\psi} (C^\bullet, d) \longrightarrow 0$$

is exact if each row of the following diagram is exact.

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & A^k & \xrightarrow{\varphi} & B^k & \xrightarrow{\psi} & C^k \longrightarrow 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d \\ 0 & \longrightarrow & A^{k+1} & \xrightarrow{\varphi} & B^{k+1} & \xrightarrow{\psi} & C^{k+1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

Lemma 7.1 (Snake Lemma). Suppose we have a short exact sequence

$$0 \longrightarrow (A^\bullet, d) \xrightarrow{\varphi} (B^\bullet, d) \xrightarrow{\psi} (C^\bullet, d) \longrightarrow 0$$

of cochain complexes. Then there exists δ such that

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^k(A) & \xrightarrow{H^k(\varphi)} & H^k(B) & \xrightarrow{H^k(\psi)} & H^k(C) \\ & & & & \searrow \delta & & \\ & & H^{k+1}(A) & \xrightarrow{H^{k+1}(\varphi)} & H^{k+1}(B) & \xrightarrow{H^{k+1}(\psi)} & H^{k+1}(C) \longrightarrow \dots \end{array}$$

is a long exact sequence.

Proof. Consider the following diagram, where we consider lifting from an element of $Z^k C$.

$$\begin{array}{ccccc} & A^{k+1} & \xrightarrow{d} & A^{k+2} & \\ & \varphi \downarrow & & \downarrow \varphi & \\ B^k & \xrightarrow{d} & B^{k+1}(B) & \xrightarrow{d} & B^{k+2} \\ \psi \downarrow & & \downarrow \psi & & \\ Z^k C & \xrightarrow{d} & C^{k+1} & & \end{array} \quad \begin{array}{ccccc} & a & \xrightarrow{d} & da & \\ & \varphi \downarrow & & \downarrow \varphi & \\ b & \xrightarrow{d} & db & \xrightarrow{d} & 0 \\ \psi \downarrow & & \downarrow \psi & & \\ c & \xrightarrow{d} & 0 & & \end{array}$$

We see that $\varphi(da) = 0$. By injectivity of φ , we have $da = 0$. Thus this defines a well-defined map from

$$Z^k C \ni c \mapsto a \in Z^{k+1}(A),$$

up to equivalence classes by cycles. We set,

$$\delta[c]_{H^k(C)} = [a]_{H^{k+1}(A)}.$$

We then need to check,

$$\text{i). } H^k(\psi) \circ \varphi^k = 0, \text{im } H^k(\varphi) = \ker H^k(\psi),$$

$$\text{ii). } \delta \circ H^k(\psi) = 0, \ker \delta = \text{im } \psi^k,$$

$$\text{iii). } H^k(\varphi) \circ \delta = 0, \ker H^k(\varphi) = \text{im } \delta.$$

Let $\tilde{c} \in Z^k C$ and $\tilde{c} = c + d\sigma$. Choose a lift $\tilde{b} \in B^k$ such that $\psi(\tilde{b}) = \tilde{c}$. Choose $\tilde{\sigma} \in B^{k-1}$ such that

$$\psi(\tilde{\sigma}) = \sigma.$$

Then,

$$\psi(\tilde{b} - b) = d\sigma = d\psi(\tilde{\sigma}) = \psi(d\tilde{\sigma}).$$

Therefore,

$$\psi(\tilde{b} - b - d\tilde{\sigma}) = 0.$$

Then there is $\tau \in A^k$ such that $\varphi(\tau) = \tilde{b} - b - d\tilde{\sigma}$. Furthermore, we have,

$$\varphi(d\tau) = d\varphi(\tau) = d\tilde{b} - db.$$

Let $\tilde{a}, a \in A^{k+1}$ be such that

$$\varphi(\tilde{a}) = d\tilde{b}, \varphi(a) = db.$$

That is

$$\varphi(\tilde{a} - a - d\tau) = 0 \Rightarrow \tilde{a} - a = d\tau.$$

That is $[\tilde{a}]_{H^{k+1}(A)} = [a]_{H^{k+1}(A)}$.

Now we examine the exactness, If $x \in H^k C$ is in $\text{im } H^k(\psi)$, then

$$\exists b \in Z^k B \text{ s.t. } x = [\psi(b)].$$

$db = 0$ hence $a = 0$.

Let $\delta[c] = 0$, then there is $a = d\alpha, \alpha \in A^k$, thus,

$$\psi(b - \varphi(\alpha)) = \psi(b) - c,$$

$$d(b - \varphi(\alpha)) = db - \varphi(d\alpha) = db - \varphi(\alpha) = 0.$$

Hence, $[c] = \psi_*[b - \varphi(\alpha)] \in \text{im } H^k(\psi)$.

Given $c \in Z^k A$, then $\varphi(a) = db$ hence $H^k(\varphi)[a] = 0$. Thus $H^k(\varphi) \circ \delta = 0$. Conversely, let $[a] \in H^{k+1}(A)$ with $H^{k+1}(\varphi)[a] = 0$. Then $da = 0, \varphi(a) = db$. Set $c := \psi(b)$. Thus

$$dc = d\psi(b) = \psi(db) = \psi \circ \varphi(a) = 0.$$

This proves that $[a] = \delta[c]$.

The rest is left to the readers. □

Theorem 7.3 (Mayer-Vietoris for deRham Theory). *Let $\mathcal{M} = U \cup V$ be a smooth manifold where $U, V \subseteq \mathcal{M}$ are open sets. Then,*

$$0 \longrightarrow \Omega^\bullet(\mathcal{M}) \xrightarrow{\omega \mapsto (\omega|_U, \omega|_V)} \Omega^\bullet(U) \xrightarrow{(\eta, \tau) \mapsto \eta|_{U \cap V} - \tau|_{U \cap V}} \Omega^\bullet(U \cap V) \longrightarrow 0$$

is exact. Hence there is the long exact cohomology sequence,

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0 \mathcal{M} & \longrightarrow & H^0 U \oplus H^0 V & \longrightarrow & H^0 U \cap V \\ & & & & & \searrow & \\ & & & & H^1 \mathcal{M} & \xleftarrow{\quad} & H^1 U \oplus H^1 V \longrightarrow H^1 U \cap V \longrightarrow \dots \\ & & & & & & \\ \dots & \longrightarrow & H^m \mathcal{M} & \longrightarrow & H^m U \oplus H^m V & \longrightarrow & H^m U \cap V \longrightarrow 0 \end{array}$$

Proof. Only the surjectivity at the right needs proof. Choose a smooth partition of unity $\{\varphi_U, \varphi_V\}$ subordinates to $\{U, V\}$. Given $\omega \in \Omega^k(U \cap V)$, put

$$s\omega = (\varphi_V \omega, -\varphi_U \omega) \in \Omega^k U \oplus \Omega^k V. \quad (*)$$

□

Remark 7.6. *The lifting map in Equation (*) allows to make δ explicit. That is given $\omega \in \Omega^k(U \cap V)$ representing $[\omega] \in H^k(U \cap V)$ that is $d\omega = 0$. Let*

$$\chi := s\omega = (\varphi_V \omega, -\varphi_U \omega), d\chi = (d(\varphi_V \omega), -d(\varphi_U \omega)).$$

Then we have,

$$\delta[\omega] = [d(\varphi_V \omega)]_{H^{k+1}(\mathcal{M})}.$$

We warn that $d(\varphi_j \omega)$ extends by 0 to a closed form on \mathcal{M} . It is not necessarily an exact on \mathcal{M} .

Lemma 7.2. *Let*

$$0 \longrightarrow E^0 \xrightarrow{d} E^1 \xrightarrow{d} \dots \xrightarrow{d} E^n \longrightarrow 0$$

be an exact sequence of finite dimensional vector spaces. Then

$$\sum_{j=0}^n (-1)^j \dim E^j = 0.$$

Proof. Exercise. □

Example 7.3. *Consider S^n and consider two stereographic projection ($U = S^n \setminus \{N\}, \sigma_N$), ($V = S^n \setminus \{-N\}, \sigma_{-N}$) where $N = (0, \dots, 0, 1)$. Then we have,*

$$U \cap V \simeq S^{n-1} \times \mathbb{R} \cong_n S^{n-1}.$$

Then consider Mayer-Vietoris sequence,

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0 S^n & \longrightarrow & H^0 \mathbb{R}^n \oplus H^0 \mathbb{R}^n & \longrightarrow & H^0 S^{n-1} \\
& & & & & \searrow & \\
& & H^1 S^n & \xleftarrow{\quad} & H^1 \mathbb{R}^n \oplus H^1 \mathbb{R}^n & \longrightarrow & H^1 S^{n-1} \longrightarrow \dots \\
& & & & & & \\
\dots & \longrightarrow & H^k S^n & \longrightarrow & H^k \mathbb{R}^n \oplus H^k \mathbb{R}^n & \longrightarrow & H^k S^{n-1} \longrightarrow \dots
\end{array}$$

Using Theorem 7.2 and Example 7.2, we obtain,

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \underbrace{\mathbb{R}}_{n \neq 1, n=1 \Rightarrow \mathbb{R} \oplus \mathbb{R}} \\
& & & & \searrow 0 & & \\
& & 0 & \xleftarrow{\quad} & 0 & \xrightarrow{0} & H^1 S^{n-1} \longrightarrow \dots \\
& & & & & & \\
\dots & \longrightarrow & H^k S^n & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \dots
\end{array}$$

For $n = 1$, we have,

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R} \oplus \mathbb{R} \\
& & & & \searrow & & \\
& & & & H^1(S^1) & \xleftarrow{\quad} & 0 \\
& & & & \underbrace{\hspace{1cm}}_{\cong \mathbb{R}} & &
\end{array}$$

In particular $H^{k-1}(S^{n-1}) \simeq H^k(S^n)$ for $k \geq 1$.

For $1 \leq k \leq n$,

$$H^k S^n = H^1(S^{n-k+1}) \cong \begin{cases} 0, & (k \neq n), \\ \mathbb{R}, & (k = n). \end{cases}$$

7.4 Compactly supported deRham Cohomology and Poincaré duality

Consider

$$0 \longrightarrow \Omega_C^\bullet(U \cap V) \xrightarrow{i_*^U - i_*^V = [\omega \mapsto (-i_*^U \omega, i_*^V \omega)]} \Omega_C^\bullet(U) \oplus \Omega_C^\bullet(V) \xrightarrow{(\tau, \eta) \mapsto (i_*^{\mathcal{M}} \tau + i_*^{\mathcal{M}} \eta)} \Omega_C^\bullet(\mathcal{M}) \longrightarrow 0$$

This turns out to be an exact sequence. The proof will be shown next week.

Proposition 7.5. *We have the following exact sequence.*

$$0 \longrightarrow \Omega_C^\bullet(U \cap V) \xrightarrow{\omega \mapsto (-i_*^U \omega, i_*^V \omega)} \Omega_C^\bullet(U) \oplus \Omega_C^\bullet(V) \xrightarrow{(\eta, \tau) \mapsto i_* \eta + i_* \tau} \Omega_C^\bullet(\mathcal{M}) \longrightarrow 0$$

Recall Theorem 7.2, we have,

$$H_{\text{dR}}^k(\mathbb{R}^N) = \begin{cases} 0, & k > 0, \\ \mathbb{R}, & k = 0. \end{cases}$$

We also have the following diagram,

$$\begin{array}{ccc} \Omega_C^n(\mathbb{R}^n) & \xrightarrow{f} & \mathbb{R} \\ \downarrow & \nearrow \eta & \\ H_C^n(\mathbb{R}^n) & & \end{array}$$

Proposition 7.6 (Poincaré Lemma with compact support).

$$H_C^k(\mathbb{R}^n) \cong \begin{cases} 0, & k \neq n, \\ \mathbb{R}, & k = n. \end{cases}$$

Proof. Only surjectivity at the right needs proof. Let $N \in S^n$ be the north pole and $U = S^n \setminus \{N\}, V = S^n \setminus \{-N\}$. Choose a smooth partition of unity $\{\varphi_U, \varphi_V\}$ subordinates to $\{U, V\}$. For $\omega \in \Omega_C^p(\mathcal{M})$, we have

$$\varphi_U \omega \in \Omega_C^p(U), \varphi_V \omega \in \Omega_C^p(V).$$

□

Proof. Recall

$$H^k S^n = \begin{cases} 0, & k \neq 0, n, \\ \mathbb{R}, & k = 0, n. \end{cases}$$

$$\begin{array}{ccc} \Omega^n S^n & \xrightarrow{f} & \mathbb{R} \\ \downarrow & \nearrow \cong & \\ H^n S^n & & \end{array}$$

Note that S^n is compact thus $H^k S^n = H_C^k S^n$. Let $\omega \in \Omega_C^k(S^n \setminus \{N\})$ such that $d\omega = 0, \int \omega = 0$. The goal is to show that

$$\exists \eta \in \Omega_C^{k-c}(S^n \setminus \{N\}), \begin{cases} \omega = \eta, & (k > 0), \\ \omega = 0, & (k = 0). \end{cases}$$

From this, the claim follows. For $k = 0$, as ω is smooth with compact support and $d\omega = 0$. This implies that ω is constant but the support is compact thus it must vanish.

For $k \geq 1$, we (missing) identify forms ω with compact support on $S^n \setminus \{N\}$ with forms on S^n by extension by 0. From the knowledge of $H^k(S^n)$, we infer that there exists $\eta \in \Omega^{k-1}(S^n)$ with $d\eta = \omega$. However, we do not know whether η vanishes near N . Choose $\rho \in \mathcal{C}_C^\infty(U)$ with $\rho|_{\overline{V}} = 1$.

For $k \geq 2$, we know,

$$d\eta|_U = \omega|_U = 0.$$

By Theorem 7.2, there is a $\xi \in \Omega^{k-2}(U)$ with $d\xi = \eta|_U$.

$$\eta_1 := \eta - d(\rho\xi) \in \Omega_C^{k-1}(S^n \setminus \{N\}),$$

since $\eta_1|_V = 0$. Thus,

$$d\eta_1 = d\eta - d^2(\rho\xi) = \omega.$$

For $k = 1$,

$$d\eta|_U = \omega|_U = 0 \Rightarrow \eta|_U = c,$$

for some constant c . Put,

$$\eta_1 := \eta - c \in \Omega_C^0(S^n \setminus \{N\}).$$

Thus $d\eta_1 = \omega$. □

Proposition 7.7. *Let \mathcal{M}^m be an oriented manifold. Consider,*

$$H_C^k(\mathcal{M}) \times H^{m-k}(\mathcal{M}) \ni ([\omega], [\tau]) \mapsto \int \omega \wedge \tau \in \mathbb{R}.$$

This is a well-defined bilinear pairing.

Proof. Let $\alpha \in \Omega_C^{k-1}(\mathcal{M})$, $\beta \in \Omega_C^{m-k-1}(\mathcal{M})$, and consider,

$$\omega' = \omega + d\alpha, \tau' = \tau + d\beta.$$

Then we have,

$$\begin{aligned} \omega' \wedge \tau' - \omega \wedge \tau &= (\omega + d\alpha) \wedge (\tau + d\beta) - \omega \wedge \tau + d\alpha \wedge d\beta, \\ &= d\alpha \wedge \tau' + \omega' \wedge d\beta, \\ &= d(\underbrace{\alpha \wedge \tau'}_{\in \Omega_C^{n-1}(\mathcal{M})}) + (-1)^k d(\underbrace{\omega' \wedge d\beta}_{\in \Omega_C^{m-1}(\mathcal{M})}) + d(\alpha \wedge d\beta). \end{aligned}$$

Using Stokes' theorem, we obtain,

$$\int \omega' \wedge \tau' = \int \omega \wedge \tau.$$

□

Notation 7.1. *We will denote such bilinear pairing by,*

$$\langle [\omega], [\tau] \rangle := \int \omega \wedge \tau.$$

Definition 7.12. *We set $\beta : H^{m-k}(\mathcal{M}) \rightarrow H_C^k(\mathcal{M})^*$, by*

$$\beta([\tau]) = \langle \cdot, [\tau] \rangle.$$

Definition 7.13. A manifold \mathcal{M}^m is a good manifold if

1. $H_{(C)}^k(\mathcal{M})$ are finite dimensional for all k .
2. β is an isomorphism.

Remark 7.7. From Poincaré Lemmas, we have that \mathbb{R}^m is a good manifold. More precisely,

$$H_C^k \mathbb{R}^m \times H^{m-k} \mathbb{R}^m \ni ([\omega], [\tau]) \mapsto \int \omega \wedge \tau \in \mathbb{R}.$$

If $k \neq m$, then this is $\mathcal{O} \times \mathcal{O} \rightarrow \mathbb{R}$ which is a dual pairing. For $k = m$,

$$H^m \mathbb{R}^m \times H^0 \mathbb{R}^m \rightarrow \mathbb{R}.$$

Note that for each element of $[\omega] \in H_C^m \mathbb{R}^m$, there is $\rho \in \mathcal{C}_C^\infty(\mathbb{R}^m)$ such that $\int \rho = 1$ and

$$[\omega] = [\rho dx_1 \wedge \cdots \wedge dx_m].$$

Recall that $H^0 \mathbb{R}^m = \{[1]\}$. Thus, we have,

$$\langle [\rho dx_1 \wedge \cdots \wedge dx_m], [1] \rangle = \int \rho dx_1 \wedge \cdots \wedge dx_m \wedge 1 = \int \rho = 1.$$

Lemma 7.3. Consider an exact sequence of vectorspaces,

$$\cdots \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow \cdots$$

If V_1, V_3 are finite dimensional then so is V_2 .

Lemma 7.4 (Five Lemma). Suppose the following diagram commutes and horizontal sequences are exact in

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \varepsilon \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Then $\alpha, \beta, \delta, \varepsilon$ are isomorphisms then so is γ .

Theorem 7.4 (Mayer-Vietoris Principle). For $\mathcal{M} = U \cup V$ for some open sets U, V . If $U, V, U \cap V$ are good then so is \mathcal{M} .

Proof.

Claim 1. Consider the sequence,

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^{m-k-1}U \cap V & \xrightarrow{\delta} & H^{m-k}\mathcal{M} & \longrightarrow & H^{m-k}U \oplus H^{m-k}V & \longrightarrow & H^{m-k}U \cap V & \xrightarrow{\delta} & H^{m-k+1}\mathcal{M} & \longrightarrow & \cdots \\ & & \downarrow \beta & & \downarrow \beta & & \downarrow \beta & & \downarrow \beta & & \downarrow \beta & & \\ \cdots & \longrightarrow & (H_C^{k+1}U \cap V)^* & \xrightarrow{\delta} & (H_C^k\mathcal{M})^* & \longrightarrow & (H_C^kU \oplus H_C^kV)^* & \longrightarrow & (H_C^kU \cap V)^* & \xrightarrow{\delta^*} & (H_C^{k-1}\mathcal{M})^* & \longrightarrow & \cdots \end{array}$$

Then each square commutes up to signs.

Proof: Follows from Lemma 7.4. ■

The detailed proof will be found in [3]. □

7.5 The Hodge deRham Theorem

Consider a Riemannian manifold \mathcal{M}^m and L be a flat line bundle (i.e. either $L = \mathbb{C}$ or $L = \mathcal{O}$) and we equip L with the Euclidean metric (imposing the flatness justifies such equipment). The Hodge \star -operator extends to an isometry.

$$\star : \Omega^p(\mathcal{M}, L) \mapsto \Omega^{m-p}(\mathcal{M}, \mathcal{O} \otimes L).$$

For $\omega, \tau \in \Omega^p(\mathcal{M}, L)$, we get a 1-density $\omega \wedge \star \tau$. Locally

$$\omega = \omega_1 \otimes s, \eta = \eta_1 \otimes t, s, t \in \Gamma(L).$$

We then get,

$$\omega \wedge \star \tau = \omega_1 \wedge \star \eta_1 \langle s, t \rangle.$$

In this way, $\Omega_C^p(\mathcal{M}, L)$ becomes an Euclidean (respectively Hermitian after tensoring with \mathbb{C}) space with scalar product

$$\langle \omega, \tau \rangle := \int_{\mathcal{M}} \omega \wedge \star \tau.$$

Proposition 7.8. *The adjoint of $d_p : \Omega_C^p(\mathcal{M}, L) \rightarrow \Omega_C^{p+1}(\mathcal{M}, L)$ is given by*

$$d_p^* = (-1)^{mp+1} *_{m-p-1} d_{m-p} *_{p-1} + 1.$$

Proof. For a p -form ω and $(p+1)$ -form τ , we have,

$$\begin{aligned} \langle d\omega, \tau \rangle &= \int_{\mathcal{M}} d\omega \wedge \star \tau = \int_{\mathcal{M}} [d(\omega \wedge \star \tau) - (-1)^p \omega \wedge d(\star \tau)], \\ &\stackrel{\text{Stokes}}{=} \int_{\mathcal{M}} (-1)^{p+1+p(m-p)} \omega \wedge \star \star d \star \tau, \\ &= \langle \omega, (-1)^{mp+1} \star d \star \tau \rangle, \quad (p^2 = p \pmod{2}). \end{aligned}$$

□

Definition 7.14. *Let $T : H \rightarrow H$ be a linear operator of Hilbert spaces. T is positive or $T \geq 0$ if for any $x \in H$,*

$$\langle Tx, x \rangle \geq 0. \tag{6}$$

Definition 7.15 (Hodge Laplace Operator).

$$\Delta_p := d_{p-1} d_p^* + d_p^* d_p = (d + d^*)_p. \tag{7}$$

Example 7.4. *For $p = 0$, this is the usual Laplace-Beltrami operator, that is*

$$\Delta_0 = -\operatorname{div} \operatorname{grad} = \Delta.$$

Definition 7.16. *A p -form ω is said to be harmonic if*

$$\Delta \omega = 0.$$

Theorem 7.5 (Hodge-deRham). *Let \mathcal{M} be a compact manifold then there is a orthogonal sum decomposition. That is*

$$\Omega^p(\mathcal{M}, L) = \ker \Delta_p \oplus \operatorname{im} d_{p-1} \oplus \operatorname{im} d_p^*. \quad (\text{HD})$$

In particular, $H_{\text{dR}}^p(\mathcal{M})$ is canonically isomorphic to $\mathcal{H}^p = \ker \Delta_p$. In other words, each deRham cohomology class has exactly one harmonic form.

Proof. We only prove the orthogonality of the three spaces in Equation (HD). The proof of equality requires stronger tools, (e.g. the elliptic regularity theorem).

Let $\omega \in \ker d_p, \eta \in \Omega^{p+1}(\mathcal{M}, L)$, then

$$(\omega, d_p^* \eta) = (d_p \omega, \eta) = 0.$$

Therefore, $\ker d_p \perp \operatorname{im} d_p^*$, in particular,

$$\operatorname{im} d_{p-1}, \mathcal{H}^p \perp \operatorname{im} d_p^*.$$

This follows from $d_p \circ d_{p-1} = 0$ and $\mathcal{H}^p \subseteq \ker d_p^*$. Similarly, $\ker d_{p-1}^* \perp \operatorname{im} d_{p-1}$, and hence,

$$\mathcal{H}^p \subseteq \ker d_{p-1}^* \perp \operatorname{im} d_{p-1}.$$

□

Corollary 7.2. *Poincaré duality is implemented by the Hodge \star -operator. That is \star gives an isomorphism of cohomology by*

$$\star : \mathcal{H}^p(\mathcal{M}, L) \rightarrow \mathcal{H}^{m-p}(\mathcal{M}, \mathcal{O} \otimes L).$$

7.6 DeRham Cohomology for Manifolds with Boundaries

For simplicity, all manifolds in this section are assumed to be oriented.

Let \mathcal{M} be a manifold with boundary $\partial\mathcal{M}$ and $i : \partial\mathcal{M} \hookrightarrow \mathcal{M}$ be the inclusion map. There are many variation of relative deRham cohomology which turns out to be equivalent. We put

$$\begin{aligned} \Omega^\bullet(\mathcal{M}) &= \{\text{differential forms smooth up to } \partial\mathcal{M}\}, \\ \Omega^\bullet(\mathcal{M}, \partial\mathcal{M}) &= \ker i^* : \Omega^\bullet(\mathcal{M}) \rightarrow \Omega^\bullet(\partial\mathcal{M}). \end{aligned}$$

We get an exact sequence of cochain complexes.

$$0 \longrightarrow \Omega^\bullet(\mathcal{M}, \partial\mathcal{M}) \longrightarrow \Omega^\bullet(\mathcal{M}) \xrightarrow{i^*} \Omega^\bullet(\partial\mathcal{M}) \longrightarrow 0$$

Proposition 7.9. *The inclusion map $\Omega^\bullet(\mathcal{M}) \rightarrow \Omega^\bullet(\mathcal{M} \setminus \partial\mathcal{M})$ induces an isomorphism,*

$$H^\bullet(\mathcal{M}) \rightarrow H^\bullet(\mathcal{M} \setminus \partial\mathcal{M}).$$

The inclusion map $\Omega_C^\bullet(\mathcal{M} \setminus \partial\mathcal{M}) \rightarrow \Omega^\bullet(\mathcal{M}, \partial\mathcal{M})$ induces an isomorphism

$$H_C^\bullet(\mathcal{M} \setminus \partial\mathcal{M}) \rightarrow H^\bullet(\mathcal{M}, \partial\mathcal{M}).$$

Proof. The homotopy operator shows that for any $t_0 \in (-\varepsilon, 0)$, the map

$$i_{t_0} : \partial\mathcal{M} \rightarrow (-\varepsilon, 0) \times \partial\mathcal{M},$$

induces an isomorphism,

$$i_{t_0}^* : H^\bullet((-\varepsilon, 0) \times \partial\mathcal{M}) \xrightarrow{p \mapsto (t_0, p)} H^\bullet(\partial\mathcal{M}). \quad (*)$$

For general \mathcal{M} , choose a cillinder $(-\varepsilon, 0] \times \partial\mathcal{M}$ of the boundary and a smooth cut-off function φ such that

$$\varphi(t) = \begin{cases} 0, & (t \leq -\varepsilon), \\ 1, & (t \geq -\frac{\varepsilon}{2}). \end{cases}$$

For the surjectivity, let $\omega \in Z^k(\mathcal{M}, \partial\mathcal{M})$. By Equation (*), there exists $\tau \in Z^k(\partial\mathcal{M})$ and $\xi \in \Omega^{k-1}((-\varepsilon, 0) \times \partial\mathcal{M})$ such that

$$\omega|_{(-\varepsilon, 0) \times \partial\mathcal{M}} = d\xi + \pi^* \tau.$$

Thn

$$\omega - d(\varphi\xi)|_{(-\varepsilon, 0) \times \partial\mathcal{M}} = d(\underbrace{(1 - \varphi)\xi}_{\Omega_C^{k-1}((-\varepsilon, 0) \times \partial\mathcal{M})}) + \pi^* \tau,$$

extends smoothly to the boundary.

For the injectivity, let $\omega \in Z^k(\mathcal{M})$ and $\omega|_{\mathcal{M} \setminus \partial\mathcal{M}} = d\eta$ for $\eta \in \Omega^{k-1}(\mathcal{M} \setminus \partial\mathcal{M})$. Then $\omega|_{(-\varepsilon, 0) \times \partial\mathcal{M}}$ is also exact and by Equation (*), there exists $\tau \in \Omega^{k-1}((-\varepsilon, 0] \times \partial\mathcal{M})$ with $\omega|_{\mathcal{M} \setminus \partial\mathcal{M}} = d\tau$.

$$\omega_1 = \omega - d(\varphi\tau) = d(\underbrace{\eta - \varphi\tau}_{=\eta}).$$

η_1 is closed on $(-\frac{\varepsilon}{2}, 0) \times \partial\mathcal{M}$ and by the surjectivity, we find $\xi \in \Omega^{k-2}((-\frac{\varepsilon}{2}, 0] \times \partial\mathcal{M})$ such that $\eta_2 = \eta_1 - d(\varphi\xi)$ extends smoothly to the boundary. Certainly

$$\omega_1 = d\eta_1 = d\eta_2.$$

Therefore,

$$\omega = d\eta = d(\eta_1 + \varphi\tau) = d(\eta_2 + \varphi\tau).$$

This proves the first part of the theorem. For the latter part, consider the short exact sequence,

$$0 \longrightarrow \Omega^\bullet((-\varepsilon, 0] \times \partial\mathcal{M}, \partial\mathcal{M}) \longrightarrow \Omega^\bullet((-\varepsilon, 0] \times \partial\mathcal{M}) \xrightarrow{i^*} \Omega^\bullet(\partial\mathcal{M}) \longrightarrow 0 \quad (C)$$

The homotopy operator shows that

$$i^* : H^\bullet((-\varepsilon, 0] \times \partial\mathcal{M}) \rightarrow H^\bullet(\partial\mathcal{M}),$$

is an isomorphism and hence the long exact cochain sequence shows that $H^\bullet((-\varepsilon, 0) \times \partial\mathcal{M}, \partial\mathcal{M})$ vanishes. Let φ be the cut-off function defined as before. For the surjectivity, given $\omega \in Z^k(\Omega^\bullet(\mathcal{M}, \partial\mathcal{M}))$,

$$\exists \xi \in \Omega^{k-1}((-\varepsilon, 0] \times \partial\mathcal{M}, \partial\mathcal{M}), \text{ s. t. } \omega|_{(-\varepsilon, 0] \times \partial\mathcal{M}} = d\xi.$$

Then,

$$\omega - d(\varphi\xi) \in Z^k(\Omega_C^\bullet(\mathcal{M} \setminus \partial\mathcal{M})).$$

For the injectivity, let $\omega \in Z^k(\Omega_C^\bullet(\mathcal{M} \setminus \partial\mathcal{M}))$, with $\omega = d\eta, \eta \in \Omega^\bullet(\mathcal{M}, \partial\mathcal{M})$. Then for a certain $\varepsilon > 0$,

$$\text{supp } \omega \subseteq \mathcal{M} \setminus (-\varepsilon, 0] \times \partial.$$

Hence,

$$\eta|_{(-\varepsilon, 0] \times \partial\mathcal{M}} \in Z^{k-1}((-\varepsilon, 0] \times \partial\mathcal{M}, \partial\mathcal{M}).$$

As before,

$$\eta|_{(-\varepsilon, 0] \times \partial\mathcal{M}} = d\xi,$$

then

$$\omega = d\left(\underbrace{\eta - d(\varphi\xi)}_{\in \Omega_C^\bullet(\mathcal{M} \setminus \partial\mathcal{M})} \right).$$

□

We come back to the short exact sequence of relative cohomology. There is a homological algebra realization of relative cohomologies.

Definition 7.17. A morphism of cochain complexes $\phi : (E^\bullet, d^E) \rightarrow (F^\bullet, d^F)$ is called a quasi-isomorphism if for all $n \in \mathbb{Z}_{\geq 0}$, we have,

$$H^n(f) : H^n(E^\bullet) \rightarrow H^n(F^\bullet)$$

is an isomorphism.

Proposition 7.10. Let

$$0 \longrightarrow (K^\bullet, d^K) \xrightarrow{f} (E^\bullet, d^E) \xrightarrow{g} (F^\bullet, d^F) \longrightarrow 0$$

be a short exact sequence of cochain complexes. Set,

$$Q^k := E^k \oplus F^{k-1}, d^Q := \begin{pmatrix} d^E & 0 \\ g & -d^F \end{pmatrix}$$

Then (Q^\bullet, d^Q) is a cochain complex and

$$\phi : K^\bullet \rightarrow Q^\bullet = [\omega \rightarrow (f(\omega), 0)]$$

is a quasiisomorphism.

Proof.

$$\begin{pmatrix} d^E & O \\ g & -d^F \end{pmatrix}^2 = \begin{pmatrix} (d^E)^2 & O \\ g d^E - d^F g & (d^F)^2 \end{pmatrix} = O.$$

ϕ is linear and respects boundaries that is

$$d\phi(\omega) = d(f(\omega), 0) = (df(\omega), gf(\omega)) = (f(d\omega), 0) = \phi(d\omega).$$

Thus ϕ is a morphism of cochain complexes.

For the injectivity, let $\omega \in Z^k(K^\bullet)$ with $\phi(\omega)$ exact. Then,

$$(f(\omega), 0) = d(\xi, \tau) = (d\xi, g(\xi) - d\tau).$$

Let $\tilde{\tau} \in E^{k-1}$ with $g(\tilde{\tau}) = \tau$, then,

$$\begin{aligned} d(\underbrace{(\xi, \tau) - d(\tilde{\tau}, 0)}_{=(\xi - d\tilde{\tau}, 0)}) &= (f(\omega), 0). \\ g(\xi - d\tilde{\tau}) &= g(\xi) - d\tau = 0. \end{aligned}$$

Therefore $\xi - d\tilde{\tau} \in \text{im } f$. Thus we conclude, $\omega \in B^k(K^\bullet)$.

For the surjectivity, let $(\xi, \tau) \in Z^k(Q^\bullet)$, then $d\xi = 0, g(\xi) = d\tau$. As before, choose $\tilde{\tau} \in E^{k-1}$ with $g(\tilde{\tau}) = \tau$. Then,

$$(\xi, \tau) - d(\tilde{\tau}, 0) = (\xi - d\tilde{\tau}, 0) \in \text{im } \phi.$$

Also,

$$g(\xi - d\tilde{\tau}) = g(\xi) - d\tau = 0, d(\xi - d\tilde{\tau}) = 0.$$

□

Definition 7.18 (Relative deRham Cohomology Using Q). *Motivated by Proposition 7.10, we consider $(\omega, \tau) \in \Omega^k(\mathcal{M}) \oplus \Omega^{k-1}(\partial\mathcal{M})$, with the boundary,*

$$d(\omega, \tau) = (d\omega, i^*\omega - d\tau) \mapsto \underbrace{(d^2\omega)}_{=0}, \underbrace{i^*d\omega - di^*\omega + d^2\tau}_{=0}.$$

Thus set

$$Q^k := \Omega^k(\mathcal{M}) \oplus \Omega^{k-1}(\partial\mathcal{M}) = \Omega^k(\mathcal{M}, \partial\mathcal{M}),$$

we have,

$$Q^k(\mathcal{M}, \partial\mathcal{M}) \ni (\omega, \tau) \mapsto \omega \Omega^k(\mathcal{M}),$$

which induces

$$\Omega_C^k(\mathcal{M} \setminus \partial\mathcal{M}) \hookrightarrow \Omega^k(\mathcal{M}, \partial\mathcal{M}) \xrightarrow{\omega \mapsto (\omega, 0)} Q^k(\mathcal{M}, \partial\mathcal{M})$$

We have the cycles (ω, τ) with $d\omega = 0, i^*\omega = d\tau$. An element of $H^k(\mathcal{M}, \partial\mathcal{M})$ can be represented by such a pair.

Theorem 7.6 (Poincaré duality of manifold with boundary). *Let \mathcal{M}^m be an oriented manifold with boundary with a finite good cover. Then*

$$H^p(\mathcal{M}) \times H^{m-p}(\mathcal{M}, \partial\mathcal{M}) \ni ([\omega], [(\tau, \eta)]) \mapsto \int_{\mathcal{M}} \omega \wedge \tau - (-1)^p \int_{\partial\mathcal{M}} i^* \omega \wedge \eta \in \mathbb{R}.$$

This is a well defined dual pairing extending the Poincaré duality pairing,

$$H^p(\mathcal{M} \setminus \partial\mathcal{M}) \times H_C^{m-p}(\mathcal{M} \setminus \partial\mathcal{M}) \ni ([\omega], [\tau]) \mapsto \int_{\mathcal{M}} \omega \wedge \tau \in \mathbb{R},$$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_C^k(\mathcal{M} \setminus \partial\mathcal{M}) & \longrightarrow & H^k(\mathcal{M}) & \longrightarrow & H^k(\partial\mathcal{M}) \longrightarrow H_C^{k+1}(\mathcal{M} \setminus \partial\mathcal{M}) \longrightarrow \cdots \\ & & \downarrow \cong & & & & \\ & & H^k(\mathcal{M}, \partial\mathcal{M}) & & & & \\ & & \downarrow \cong & & & & \\ & & H^k(Q^\bullet) & & & & \end{array}$$

(i.e. the first one is realized as the lift of the second one. If both are well-defined and we can show the former extends the latter, we automatically get that the first one is a dual pairing).

Proof. For well-definedness, suppose $(\tau, \eta) = d(\underbrace{\xi_1}_{\mathcal{M}}, \underbrace{\xi_2}_{\mathcal{M}}) = (d\xi_1, i^*\xi_1 - d\xi_2)$.

As ω is a closed form, we have,

$$\begin{aligned} \int_{\mathcal{M}} \omega \wedge \tau &= \int_{\mathcal{M}} \omega \wedge d\xi_1, \\ &= (-1)^p \int_{\mathcal{M}} d(\omega \wedge \xi_1), \\ &\stackrel{\text{Stokes'}}{=} (-1)^p \int_{\mathcal{M}} i^* \omega \wedge \underbrace{i^* \xi_1}_{= \eta + d\xi_2}, \\ &= (-1)^p \int_{\partial\mathcal{M}} i^* \omega \wedge \eta + i^* \omega \wedge d\xi_2. \end{aligned}$$

Observe that again using Stokes' $\partial\partial\mathcal{M} = \emptyset$, we obtain,

$$\int_{\partial\mathcal{M}} i^* \omega \wedge d\xi_2 = 0.$$

Although this does not necessarily mean that $i^* \omega \wedge d\xi_2 = 0$. We then have,

$$\int_{\mathcal{M}} \omega \wedge \tau - (-1)^p \int_{\partial\mathcal{M}} i^* \omega \wedge \eta = 0.$$

Similarly, one argues for $\omega = d\xi$. Now given $[\omega] \in H^p(\mathcal{M})$, $[(\tau, \eta)] \in H^{m-p}(\mathcal{M}, \partial\mathcal{M})$, there is a representative of $[(\tau, \eta)]$ of the form $[(\chi, 0)]$ with $\chi \in \Omega_C^{m-p-1}(\mathcal{M} \setminus \partial\mathcal{M})$, $d\chi =$

0, therefore, we have,

$$\langle [\omega], [(\chi, 0)] \rangle = \int_{\mathcal{M}} \omega \wedge \chi.$$

□

Remark 7.8. Let \mathcal{M} be a compact oriented manifold with boundary. Poincare duality tells us that

$$H^p(\mathcal{M}) \cong H^{m-p}(\mathcal{M}, \partial\mathcal{M})^*,$$

as vector spaces. The Euler characteristic is ,

$$\chi(\mathcal{M}) = \sum_{j=0}^m (-1)^j \dim H^j(\mathcal{M}) = \sum_{j=0}^m (-1)^j \dim H^{m-j}(\mathcal{M}, \partial\mathcal{M}) = (-1)^m \chi(\mathcal{M}, \partial\mathcal{M}).$$

From this observation and apply this for $\partial\mathcal{M}$, we derive,

Corollary 7.3.

$$\chi(\partial\mathcal{M}) = (-1)^{m-1} \chi(\mathcal{M}, \partial\mathcal{M}).$$

Thus if m is even then $\chi(\partial\mathcal{M}) = 0$. From exercise sheets, we have,

$$\chi(\mathcal{M}) = (-1)^{m-1} \chi(\mathcal{M}, \partial\mathcal{M}) + \chi(\partial\mathcal{M}).$$

Therefore, if m is even,

$$\chi(\mathcal{M}) = \chi(\mathcal{M}, \partial\mathcal{M}),$$

if m is odd,

$$\begin{aligned} \chi(\mathcal{M}) &= -\chi(\mathcal{M}, \partial\mathcal{M}), \\ &= \chi(\mathcal{M}, \partial\mathcal{M}) + \chi(\partial\mathcal{M}), \\ \chi(\partial\mathcal{M}) &= -2\chi(\mathcal{M}, \partial\mathcal{M}) = 2\chi(\mathcal{M}). \end{aligned}$$

We also have a deRham Hodge theorem for compact manifold with boundary. Let \mathcal{M} be an oriented Riemannian manifold with boundary. The following will be only fully understandable with the aid of the theory of elliptic boundary value problem.

$$\begin{aligned} \Omega_{\text{rel}}^p(\mathcal{M}) &:= \Omega^p(\mathcal{M}, \partial\mathcal{M}) = \{\omega \in \Omega^p(\mathcal{M}) \mid i^* \omega = 0\}, \\ \Omega_{\text{abs}}^p(\mathcal{M}) &:= \{\omega \in \Omega^p(\mathcal{M}) \mid i^* \star \omega = 0\}, \\ \mathcal{H}_{\text{abs}}^p(\mathcal{M}) &:= \{\omega \in \Omega^p(\mathcal{M}) \mid d_p \omega = 0, d_{p-1}^* \omega = 0\}, \\ \mathcal{H}_{\text{rel}}^p(\mathcal{M}) &:= \{\omega \in \Omega_{\text{rel}}^p(\mathcal{M}) \mid d_p \omega = 0, d_{p-1}^* \omega = 0\}. \end{aligned}$$

We have the following orthogonal sum decomposition,

$$\begin{aligned} \Omega^p(\mathcal{M}) &= \mathcal{H}_{\text{abs}}^p \oplus d_{p-1}(\Omega^{p-1}(\mathcal{M})) \oplus d_p^*(\Omega_{\text{rel}}^{p+1}(\mathcal{M})), \\ \Omega^p(\mathcal{M}) &= \mathcal{H}_{\text{rel}}^p \oplus d_{p-1}(\Omega_{\text{rel}}^{p-1}(\mathcal{M})) \oplus d_p^*(\Omega_{\text{abs}}^{p+1}(\mathcal{M})). \end{aligned}$$

In particular, we have natural isomorphisms,

$$\mathcal{H}_{\text{abs}}^p(\mathcal{M}) \cong H^p(\mathcal{M}), \mathcal{H}_{\text{rel}}^p = H^p(\mathcal{M}, \partial\mathcal{M}).$$

Poincaré duality is implemented by $\star : \mathcal{H}_{\text{abs}}^p(\mathcal{M}) \xrightarrow{\sim} \mathcal{H}_{\text{rel}}^{m-p}(\mathcal{M})$.

Proposition 7.11 (Künneth). *Let \mathcal{M}, \mathcal{N} be manifolds with finite good covers. Consider,*

$$\begin{array}{ccc} & \mathcal{M} \times \mathcal{N} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathcal{M} & & \mathcal{N} \end{array}$$

Then we have

$$H_C^p(\mathcal{M}) \times H_C^q(\mathcal{N}) \xrightarrow{(\omega_1, \omega_2) \mapsto \pi_1^* \omega_1 \wedge \pi_2^* \omega_2} H^{p+q}(\mathcal{M} \times \mathcal{N})$$

This induces an isomorphism,

$$\bigoplus_{p+q=k} H_C^p(\mathcal{M}) \times H_C^q(\mathcal{N}) \cong H_C^k(\mathcal{M} \times \mathcal{N}).$$

Sketch of the proof. This is proven by using Mayer-Vietoris sequence and induction by the cardinalities of the good covers. \square

Let $i : L^k \hookrightarrow M$ be a compact oriented submanifold. Then we have

$$\phi_L : H^h(\mathcal{M}) \rightarrow \mathbb{R}, [\omega] \mapsto \int_L i^* \omega,$$

is a linear functional. Thus there is a class,

$$\text{PD}(L) \in H_C^{m-k}(\mathcal{M}),$$

such that for each closed $\eta \in \Omega_C^{m-k}(\mathcal{M})$ with $[\eta] = \text{PD}(L)$, and each closed $\omega \in \Omega^k(\mathcal{M})$,

$$\phi_L([\omega]) = \int_{\mathcal{M}} \omega \wedge \eta.$$

Such $\text{PD}(L)$ is called the Poincaré dual of L .

Proposition 7.12. *Let $U \supseteq L$ be an open neighborhood of the submanifold. Then η representing $\text{PD}(L)$ can be chosen such that $\text{supp } \eta \subseteq U$.*

Proof. Let $\eta \in \Omega_C^{m-k}(U)$ be a representative of $\text{PD}^U(L)$ (i.e. the Poincaré dual of L in U), then for arbitrary closed $\omega \in \Omega^k(\mathcal{M})$, we have,

$$\int_L i^* \omega = \int_L i^* (\omega|_U) = \int_U \omega \wedge \eta = \int_{\mathcal{M}} \omega \wedge \eta.$$

Since η has a compact support and we might extend by 0 to \mathcal{M} . \square

Example 7.5. let \mathcal{M} be a compact oriented manifold. Let $L = \Delta = \{(p, p) \mid p \in \mathcal{M}\}$, the diagonal in $\mathcal{M} \times \mathcal{M}$.

Example 7.6. Let \mathcal{M} be an oriented manifold and $E \rightarrow \mathcal{M}$ be a vector bundle. \mathcal{M} sits inside E as the 0-section $i : \mathcal{M} \hookrightarrow E$. Then,

$$\Theta := \text{PD}(\mathcal{M}) \in H_C^m(E),$$

is called the Thom class of E . $e(E) := i^*\theta$ is called the Euler class of the bundle. If E is oriented then $e(E) \in H_C^m(\mathcal{M})$. The Euler class of the manifold \mathcal{M} is $e(\mathcal{M}) := e(T\mathcal{M})$.

Proposition 7.13. For any $s \in \Gamma(E)$, we have $s^*\Theta = e$. If E has a nowhere vanishing section s , then $e(E) = 0$.

Proof. Using homotopy invariance of cohomologies, we see that s is homotopic to 0-section $i : \mathcal{M} \hookrightarrow \mathcal{M}$. This shows the first assertion. For the second, choose a smooth metric on E and let

$$U = \{v \in E \mid |v| < |s(\pi(v))|\}.$$

Then U is an open neighborhood of $i(\mathcal{M})$ and by Proposition (ref missing), there is a form $\eta \in \Omega_C^m(U)$ representing $\Theta = \text{PD}(\mathcal{M})$. Thus $s^*\Theta = 0$. By the first assertion, we conclude $e(E) = 0$. \square

Now let \mathcal{M} be a compact oriented manifold. Choose a basis $([\omega_i])_{i=1, \dots, r}$ of $H^\bullet \mathcal{M}$ and let $[\tau_j]_{j=1, \dots, r}$ be the dual basis with respect to the Poincaré duality, that is,

$$\int_{\mathcal{M}} \omega_i \wedge \tau_j = \delta_{ij}.$$

Proposition 7.14. we have the following statements.

- 1). $\text{PD}(\Delta) = \sum_{i=1}^r (-1)^{|\omega_i|} \pi_1^* \omega_i \wedge \pi_2^* \tau_i$.
- 2). $\int_{\Delta} \text{PD}(\Delta) = \chi(\mathcal{M}) = \sum_{j \geq 0} (-1)^j \dim H^j(\mathcal{M})$.
- 3). $\int_{\mathcal{M}} e(\mathcal{M}) = \chi(\mathcal{M})$.

Proof. By Künneth theorem, we have,

$$\text{PD}(\Delta) = \sum_{ij} c_{ij} \pi_1^* \omega_i \wedge \pi_2^* \tau_j.$$

We use the defining relation for the Poincaré dual,

$$\begin{aligned}
(-1)^{|\omega_l||\tau|_k} \delta_{kl} &= \int_{\mathcal{M}} \tau_k \wedge \omega_l, \\
&= \int_{\mathcal{M}} i^* (\pi_1^* \tau_k \wedge \pi_2^* \omega_l), \\
&= \int_{\Delta} \pi_1^* \tau_k \wedge \pi_2^* \omega_l, \\
&= \int_{\mathcal{M} \times \mathcal{M}} \pi_1^* \tau_k \wedge \pi_2^* \omega_l \wedge \text{PD}(\Delta), \\
&= \sum_{ij} c_{ij} \int_{\mathcal{M} \times \mathcal{M}} \pi_1^* \tau_k \wedge \pi_2^* \omega_l \wedge \pi_1^* \omega_i \wedge \pi_2^* \tau_j, \\
&= \sum_{ij} c_{ij} (-1)^{|\omega_j||\omega_l|+|\tau_k||\omega_j|} \delta_{ik} \delta_{lj}, \\
&= c_{kl} (-1)^{|\tau_k||\omega_l|+|\omega_k||\omega_l|}. \\
&\Rightarrow c_{kl} = (-1)^{|\omega_k|} \delta_{kl}.
\end{aligned}$$

For the second statement, using the first, we have,

$$\int_{\Delta} \text{PD}(\Delta) = \sum_i (-1)^{|\omega_i|} \underbrace{\int_{\mathcal{M}} \omega_i \wedge \tau_i}_{=1} = \sum_i (-1)^{|\omega_i|} = \sum_j (-1)^j \dim H^j(\mathcal{M}) = \chi(\mathcal{M}).$$

For the third, a tubular neighborhood of Δ in $\mathcal{M} \times \mathcal{M}$ is $\mathcal{N}\Delta := T(\mathcal{M} \times \mathcal{M})/T\Delta$. Thus, as in Example (ref missing)

$$\text{PD}(\Delta) = i^*(\text{Thom class of } \mathcal{N}\Delta).$$

Since,

$$T(\mathcal{M} \times \mathcal{M})|_{\Delta} \cong T\Delta \oplus T\Delta,$$

we have $\mathcal{N}\Delta \cong T\Delta \cong T\mathcal{M}$, thus

$$\int_{\Delta} \text{PD}(\Delta) = \int_{\mathcal{M}} e(\mathcal{M}) = \chi(\mathcal{M}).$$

□

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