Sheet 2 Solutions

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Exercise 2.1

2.1.i

For a smooth function $u \in C^\infty(\mathbb{R}^n)$ which also belongs to $H^k(\mathbb{R}^n)$, we derive

$$\widehat{D^\alpha u} = (iy)^\alpha \hat{u}.$$

Since $D^\alpha u \in L^2(\mathbb{R}^n)$, we derive the right hand side $(iy)^\alpha \hat{u}$ is also in $L^2(\mathbb{R}^n)$ for each α. by choosing $\alpha = (k, 0, \cdots, 0), (0, k, \cdots, 0), \cdots, (0, 0, \cdots, k)$, we derive

$$\int_{\mathbb{R}^n} |y|^{2k} |\hat{u}|^2 dy \leq \|D^k u\|^2_{L^2(\mathbb{R}^n)}.$$

Since Fourier transform is an isometry we derive and there are other norms of derivatives added,

$$\int_{\mathbb{R}^n} 1 + |y|^{2k} |\hat{u}|^2 dx \leq \|u\|^2_{H^k}.$$

We have for $a, b > 0$ and $s > 0$, $(a+b)^s \leq 2^s(a^s + b^s)$. We conclude that $(1 + |y|^k)\hat{u}$ is in $L^2(\mathbb{R}^n)$.

On the other hand $(1 + |y|^k)\hat{u} \in L^2(\mathbb{R}^n)$ then for $|\alpha| \leq k$, we have

$$\|(iy)^\alpha \hat{u}\|_{L^2} \leq \int_{\mathbb{R}^n} \|y\|^{2|\alpha|} |\hat{u}|^2 dy \leq C \|(1 + |y|^k)^2 \hat{u}\|_{L^2(\mathbb{R}^n)}. \tag{1}$$

Let us denote $u_\alpha = \frac{1}{2\pi} \int_{\mathbb{R}^n} ((iy)^\alpha \hat{u}) e^{iyx} dy$ be the image of the inverse fourier transform of $((iy)^\alpha \hat{u})$. Then

$$\int_{\mathbb{R}^n} (D^\alpha \varphi) \overline{u} dx = \int_{\mathbb{R}^n} \widehat{D^\alpha \varphi} \overline{\hat{u}} dx = \int_{\mathbb{R}^n} (iy)^\alpha \hat{\varphi} \overline{\hat{u}} dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \varphi u_\alpha dx.$$

By Equation 1, u_α is in L^2, therefore this is a weak derivative of u and u is in H^k.

Exercise 2.2

$$Lu = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}.$$

We let $B_0[u, v] = -\sum_{i=1}^n \partial_{x_i} u \partial_{x_i} v$.

0.1 Exercise 2.2.i

With Poincare inequality, we see that

$$\|u\|_{H_0^1(\mathbb{R})^n} \leq C\|Du\|_{H_0^1(\mathbb{R})^n} \Rightarrow \frac{1}{C^2}\|u\|^2_{H_0^1(\mathbb{R})^n} \leq \|Du\|^2_{H_0^1(\mathbb{R})^n} = B_0[u,u].$$

also

$$|B_0[u,v]| \leq \|u\|_{H_0^1(\mathbb{R})^n}\|v\|_{H_0^1(\mathbb{R})^n}$$

follows from Cauchy-Schwarz inequality. Thus $\gamma = 0$ for the existence of weak solutions by Lax-Milgram. We derived from $c > 0$ that the equation stated in the sheet has a solution in $H_0^1(\mathbb{R}^n)$ thus in $H^1(\mathbb{R}^n)$.

2.2.ii

By taking the fourier transform of the equation we get

$$\sum_{i=1}^{n} y_i^2 \hat{u} + c\hat{u} = \hat{f}.$$

Since $u \in L^2(\mathbb{R}^2)$ so $(1-c)u$ is in $L^2(\mathbb{R}^n)$. We conclude that $(1+|y|^2)\hat{u}$ is in $L^2(\mathbb{R}^n)$. Therefore, u is in $H^2(\mathbb{R}^n)$.

2.2.iii

Suppose the statement is true for $k-1$.

By assumption we have that $(1+|y|^k)\hat{f} \in L^2(\mathbb{R}^n)$. This means that

$$(1+|y|^k)(c+|y|^2)\hat{u} \in L^2(\mathbb{R}^n).$$

By the induction hypothesis, $|y|^k\hat{u}, |y|^2\hat{u}$ are both in $L^2(\mathbb{R}^n)$.