Sheet 2 Solutions

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Exercise 2.1

2.1.i

For a smooth function $u \in C^\infty(\mathbb{R}^n)$ which also belongs to $H^k(\mathbb{R}^n)$, we derive

$$\widehat{D^\alpha u} = (iy)^\alpha \hat{u}.$$

Since $D^\alpha u \in L^2(\mathbb{R}^n)$, we derive the right hand side $(iy)^\alpha \hat{u}$ is also in $L^2(\mathbb{R}^n)$ for each α. by choosing $\alpha = (k, 0, \cdots, 0), (0, k, \cdots, 0), \cdots, (0, 0, \cdots, k)$, we derive

$$\int_{\mathbb{R}^n} |y|^{2k} |\hat{u}|^2 dy \leq \|D^k u\|^2_{L^2(\mathbb{R}^n)}.$$

Since Fourier transform is an isometry we derive and there are other norms of derivatives added,

$$\int_{\mathbb{R}^n} 1 + |y|^{2k} |\hat{u}|^2 dx \leq \|u\|^2_{H^k}.$$

We have for $a, b > 0$ and $s > 0$, $(a+b)^s \leq 2^s(a^s + b^s)$. We conclude that $\|(1 + |y|^k)\hat{u}\|^2 \leq \|u\|^2_{H^k}$ thus $(1 + |y|^k)\hat{u}$is in $L^2(\mathbb{R}^n)$.

On the other hand $(1 + |y|^k)\hat{u} \in L^2(\mathbb{R}^n)$ then for $|\alpha| \leq k$, we have

$$\|(iy)^\alpha \hat{u}\|_{L^2} \leq \int_{\mathbb{R}^n} \|y\|^{2|\alpha|} |\hat{u}|^2 dy \leq C\|(1 + |y|^k)^2 \hat{u}\|_{L^2(\mathbb{R}^n)}. \tag{1}$$

Let us denote $u_\alpha = \frac{1}{2\pi} \int_{\mathbb{R}^n} ((iy)^\alpha \hat{u}) e^{iyx} dy$ be the image of the inverse fourier transform of $((iy)^\alpha \hat{u})$. Then

$$\int_{\mathbb{R}^n} (D^\alpha \varphi) \overline{u} dx = \int_{\mathbb{R}^n} \widehat{D^\alpha \varphi} \overline{\hat{u}} dx = \int_{\mathbb{R}^n} (iy)^\alpha \hat{\varphi} \overline{\hat{u}} dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \varphi u_\alpha dx.$$

By Equation 1, u_α is in L^2, therefore this is a weak derivative of u and u is in H^k.

Exercise 2.2

$$Lu = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}.$$

We let $B_0[u, v] = -\sum_{i=1}^n \partial_{x_i} u \partial_{x_i} v$.

0.1 Exercise 2.2.i

With Poincare inequality, we see that

$$\|u\|_{H_0^1(\mathbb{R})^n} \leq C\|Du\|_{H_0^1(\mathbb{R})^n} \Rightarrow \frac{1}{C^2}\|u\|^2_{H_0^1(\mathbb{R})^n} \leq \|Du\|^2_{H_0^1(\mathbb{R})^n} = B_0[u,u].$$

also

$$|B_0[u,v]| \leq \|u\|_{H_0^1(\mathbb{R})^n}\|v\|_{H_0^1(\mathbb{R})^n}$$

follows from Cauchy-Schwarz inequality. Thus $\gamma = 0$ for the existence of weak solutions by Lax-Milgram. We derived from $c > 0$ that the equation stated in the sheet has a solution in $H_0^1(\mathbb{R}^n)$ thus in $H^1(\mathbb{R}^n)$.

2.2.ii

By taking the fourier transform of the equation we get

$$\sum_{i=1}^n y_i^2 \hat{u} + c\hat{u} = \hat{f}.$$

Since $u \in L^2(\mathbb{R}^2)$ so $(1-c)u$ is in $L^2(\mathbb{R}^n)$. We conclude that $(1+|y|^2)\hat{u}$ is in $L^2(\mathbb{R}^n)$. Therefore, u is in $H^2(\mathbb{R}^n)$.

2.2.iii

Suppose the statement is true for $k-1$.

By assumption we have that $(1+|y|^k)\hat{f} \in L^2(\mathbb{R}^n)$. This means that

$$(1+|y|^k)(c+|y|^2)\hat{u} \in L^2(\mathbb{R}^n).$$

By the induction hypothesis, $|y|^k\hat{u}, |y|^2\hat{u}$ are both in $L^2(\mathbb{R}^n)$. Thus we conclude that $(1+|y|^{k+2})\hat{u}$ is in $L^2(\mathbb{R}^n)$, which completes the claim.

Exercise 2.3

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary. We denote the dual space of $H_0^1(\Omega)$ by $H^{-1}(\Omega)$. Recall that the standard norm on $H^{-1}(\Omega)$ is given by

$$\|f\|_{H^{-1}} := \sup\left\{\langle f, u\rangle : u \in H_0^1(\Omega), \|u\|_{H_0^1} \leq 1\right\}.$$

(i) Show that $L^2(\Omega)$ is continuously embedded in $H^{-1}(\Omega)$ by means of the following identification: for any $v \in L^2(\Omega)$ we can define the functional $\hat{v} : H_0^1(\Omega) \to \mathbb{R}$ via

$$\hat{v} : u \mapsto \int_\Omega uv dx.$$

Proof. For any $v \in L^2(\Omega)$, it is clear that $\hat{v}$ is well defined (using Cauchy-Schwarz) and linear. To prove that $\hat{v} \in H^{-1}(\Omega)$ is remains to prove that it is bounded. For this, take $u \in H_0^1(\Omega)$ then

$$|\hat{v}(u)| = \left|\int_\Omega uv dx\right| \leq \|v\|_{L^2}\|u\|_{L^2} \leq \|v\|_{L^2}\|u\|_{H_0^1}.$$

This proves that $\hat{v}$ is a bounded linear operator (therefore continuous) with operator norm

$$\|\hat{v}\| \leq \|v\|_{L^2}.$$

□

(ii) Let $F \in H^{-1}(\Omega)$. Show that there are $f_0, f_1, \ldots, f_n \in L^2(\Omega)$ such that

$$\langle F, u\rangle = \int_\Omega \left(f_0 u + \sum_{i=1}^n f_i \frac{\partial u}{\partial x_i} \right) dx \tag{2}$$

for all $u \in H_0^1(\Omega)$.

Proof. We know that $H^1(\Omega)$ is a Hilbert space with inner product

$$(u, v) = \int_\Omega uv + Du \cdot Dv dx \quad \forall u, v \in H^1(\Omega).$$

By definition, $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$, in particular it is closed. This implies that $H_0^1(\Omega)$ equipped with the same scalar product is also a Hilbert space. Given $F \in H^{-1}(\Omega)$ a continuous linear functional on $H_0^1(\Omega)$, we can apply Riesz's representation theorem to obtain $f \in H_0^1(\Omega)$ such that

$$\langle F, u\rangle = (f, u) \quad \forall u \in H_0^1(\Omega).$$

Set $f_0 = f$ and $f_i = \partial_{x_i} f$ for $i = 1, \ldots, n$ then $f_0, f_1, \ldots, f_n \in L^2(\Omega)$ and we have

$$\langle F, u\rangle = (f, u) = \int_\Omega fu + Df \cdot Du dx = \int_\Omega f_0 u + \sum_{i=1}^n f_i \frac{\partial u}{\partial x_i} dx \quad \forall u \in H_0^1(\Omega).$$

□

(iii) Show that for $F \in H^{-1}(\Omega)$

$$\|F\|_{H^{-1}} = \inf \left\{ \left(\sum_{i=0}^n \int_\Omega |f_i(x)|^2 dx \right)^{\frac{1}{2}} : f_0, f_1, \ldots, f_n \in L^2(\Omega) \text{ satisfy } (2) \right\}$$

Proof. If $f_0, f_1, \dots, f_n \in L^2(\Omega)$ satisfy (2) then for any $u \in H_0^1(\Omega)$ with $\|u\|_{L^2} \leq 1$, we have

$$\begin{aligned}
|\langle F, u\rangle| &= \left| \int_\Omega f_0 u + \sum_{i=1}^n f_i \frac{\partial u}{\partial x_i} dx \right| \\
&= \int_\Omega \left| f_0 u + \sum_{i=1}^n f_i \frac{\partial u}{\partial x_i} \right| dx \\
&\leq \int_\Omega \left(f_0^2 + \sum_{i=1}^n f_i^2 \right)^{\frac{1}{2}} \left(u^2 + \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{\frac{1}{2}} dx \\
&\leq \left(\int_\Omega f_0^2 + \sum_{i=1}^n f_i^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega u^2 + \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^n \int_\Omega |f_i(x)|^2 dx \right)^{\frac{1}{2}} \|u\|_{H_0^1} \\
&\leq \left(\sum_{i=0}^n |f_i(x)|^2 dx \right)^{\frac{1}{2}}.
\end{aligned}$$

This gives the following bound for the norm of F

$$\|F\|_{H^{-1}} \leq m.$$

where m denotes the infimum in the statement.
To get the converse inequality, let $f_0, f_1, \dots, f_n$ be given by Riesz's representation theorem as in the previous question, then by taking $u = f$, we get

$$|\langle F, u\rangle| = |(f, f)| = \|f\|_{H_0^1} \|u\|_{H_0^1}.$$

In the case $F = 0$, by the uniqueness in Riesz's representation theorem, we would have $f = 0$. If not, we can renormalize so that $f/\|f\|_{H_0^1}$ is of norm 1 from which we deduce

$$\|F\|_{H^{-1}} \geq m.$$

□

(iv) Let $n = 1$, $\Omega = (-1, 1)$. Show that the Delta-distribution δ_0 lies in $H^{-1}(\Omega)$ and find a representation as in (2). Recall that

$$\langle \delta_0, u\rangle = u(0).$$

Proof. Clearly, this operator is linear and for any $u \in H_0^1(\Omega)$, we have

$$|u(0)| = \left| \int_{-1}^0 u'(x) dx \right| \leq \left(\int_{-1}^0 |u'(x)|^2 dx \right)^{\frac{1}{2}} \leq \|u\|_{H_0^1}$$

which proves that $\delta_0 \in H^{-1}(\Omega)$.
To obtain a representation as in (2), notice that

$$u(0) = \int_{-1}^0 u'(x) dx = \int_{-1}^1 f_0(x) u(x) + f_1(x) u'(x) dx$$

where $f_0 = 0$ and $f_1 = 1_{(0,1)}$. Since $f_0, f_1 \in L^2(\Omega)$, this gives us the desired representation. □

Exercise 2.4
Consider the half space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ and let $p \in [1, \infty)$.
(i) Show that for some constant $C > 0$ we have

$$\int_{R^{n-1}} |u(x_1, \ldots, x_{n-1}, 0)|^p dx_1 \ldots dx_{n-1} \leq C \|u\|^p_{W^{1,p}(\mathbb{H}^n)}$$

for all $u \in W^{1,p}(\mathbb{H}^n) \cap C^1(\overline{\mathbb{H}^n})$.

Proof. Using

$$\begin{aligned} u(x_1, \ldots, x_{n-1}, 0) &= -\int_0^\infty \partial_{x_n} \left(e^{-x_n} u(x_1, \ldots, x_{n-1}, x_n)\right) dx_n \\ &= -\int_0^\infty e^{-x_n} \partial_{x_n} u(x_1, \ldots, x_n) - e^{-x_n} u(x_1, \ldots, x_n) dx_n. \end{aligned}$$

Using Jensen's inequality for the probability measure $e^{-x_n} dx_n$ on $(0, \infty)$, we get

$$\begin{aligned} |u(x_1, \ldots, x_{n-1}, 0)|^p &= \left| \int_0^\infty (\partial_{x_n} u(x_1, \ldots, x_n) - u(x_1, \ldots, x_n)) e^{-x_n} dx_n \right|^p \\ &\leq \int_0^\infty |\partial_{x_n} u(x_1, \ldots, x_n) - u(x_1, \ldots, x_n)|^p e^{-x_n} dx_n \\ &\leq \int_0^\infty 2^{p-1} (|u(x_1, \ldots, x_n)|^p + |\partial_{x_n} u(x_1, \ldots, x_n)|^p) dx_n, \end{aligned}$$

where we used $e^{-x} \leq 1$ for $x > 0$ and $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \geq 0$ by convexity of $x \mapsto x^p$ on $(0, \infty)$.
From this we get

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} |u(x_1, \ldots, x_{n-1}, 0)|^p dx_1 \ldots dx_{n-1} &\leq 2^{p-1} \int_{\mathbb{R}^{n-1}} \int_0^\infty |u(x_1, \ldots, x_n)|^p + |\partial_{x_n} u(x_1, \ldots, x_n)|^p dx_n dx_1 \ldots dx_{n-1} \\ &= 2^{p-1} \int_{\mathbb{H}^n} |u(x_1, \ldots, x_n)|^p + |\partial_{x_n} u(x_1, \ldots, x_n)|^p dx_1 \ldots dx_n \\ &= 2^{p-1} \|u\|^p_{W^{1,p}(\mathbb{H}^n)} \end{aligned}$$

□

(ii) Show that there is a linear bounded map

$$T : W^{1,p}(\mathbb{H}^n) \to L^p(\partial \mathbb{H}^n)$$

such that $Tu = u|_{\partial \mathbb{H}^n}$ for $u \in C(\overline{\mathbb{H}^n}) \cap W^{1,p}(\mathbb{H}^n)$.

Proof. For every $u \in W^{1,p}(\mathbb{H}^n) \cap C^1(\overline{\mathbb{H}^n})$, define $Tu = u|_{\partial \mathbb{H}^n}$. This map is clearly linear, and by the previous question, it is bounded as a map $W^{1,p}(\mathbb{H}^n) \cap C^1(\overline{\mathbb{H}^n}) \to L^p(\partial \mathbb{H}^n)$ for the $W^{1,p}(\mathbb{H}^n)$ norm. Since $W^{1,p}(\mathbb{H}^n) \cap C^1(\overline{\mathbb{H}^n})$ is dense in $W^{1,p}(\mathbb{H}^n)$, the linear operator T extends uniquely to a bounded linear operator $W^{1,p}(\mathbb{H}^n) \to L^p(\partial \mathbb{H}^n)$. To prove that $Tu = u|_{\partial \mathbb{H}^n}$ for all $u \in W^{1,p}(\mathbb{H}^n) \cap C(\overline{\mathbb{H}^n})$, note that for any $u \in W^{1,p}(\mathbb{H}^n) \cap C(\overline{\mathbb{H}^n})$, we can always find a sequence $(u_k) \subset W^{1,p}(\mathbb{H}^n) \cap C^1(\overline{\mathbb{H}^n})$ such that $u_k \to u$ in $W^{1,p}(\mathbb{H}^n)$ and $u_k \to u$ pointwise (by taking convolution with good kernels for example). In particular we have $\lim u_k|_{\partial \mathbb{H}^n} = u|_{\partial \mathbb{H}^n}$ pointwise and in $L^p(\partial \mathbb{H}^n)$, so we get

$$Tu = \lim_{k \to \infty} Tu_k = \lim_{k \to \infty} u_k|_{\partial \mathbb{H}^n} = u|_{\partial \mathbb{H}^n}.$$

□