Nonlinear Partial Differential Equations Exercise Sheet 1 Solutions

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Exercise 1

Suppose the inequality does not hold. Then for each n, there is v_n such that

$$\|v_n\|_{L^p(\Omega)} > n\|Dv_n\|_{L^p(\Omega)}.$$

We can normalize v_n, we obtain a sequence $(u_k)_{k\in\mathbb{N}}$ in K such that

$$\|Du_k\|_{L^p(\Omega)} < \frac{1}{n}.$$

This procedure is justified since K is a cone.

As $(u_k)_{k\in\mathbb{N}}$ is bounded, there is a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ converging to some u in $L^p(\Omega)$. Since K is closed, $u \in K$. Then its derivative is 0 and by the assumption, $u = 0$ but $\|u\|_{L^p(\Omega)} = 1$. Thus a contradiction.

Exercise 4

(i)

Suppose if we have a weak solution, by integration by parts, we get

$$\begin{aligned}\int_U fv &= -\int_\Omega \Delta uvdx,\\ &= \int_\Omega \nabla u \cdot \nabla vdx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} vdx,\\ &= \int_\Omega \nabla u \cdot \nabla vdx - \int_{\partial\Omega} \beta uvdx.\end{aligned}$$

for any $v \in H^1(\Omega)$. We define a bilinear form $B(u,v) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ such that

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla vdx - \int_{\partial\Omega} \beta uvdx.$$

By trace theorem, we have

$$\begin{aligned}|B(u,v)| &\leq \|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + |\beta|\|u\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)},\\ &\leq \|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + |\beta|C^2\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},\end{aligned}$$

for some $C > 0$. By the definition, we have

$$\|u\|^2_{H^1(\Omega)} = \|Du\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}.$$

Thus we conclude

$$|B(u,v)| \leq (|\beta|C^2 + 1)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

, and this satisfies the first condition for Lax-Milgram theorem.

In the case $\beta \leq 0$, we will prove that B satisfies the second condition as well. In order to do so, we will derive a contradiction by assuming it does not satisfy the condition. If that is the case the it is equivalent to say that for any $n \in \mathbb{N}$, there is $u_n \in H^1(\Omega)$ such that

$$B(u_n, u_n) < \frac{1}{n}\|u_n\|^2_{H^1(\Omega)}.$$

We can normalize u_n and conclude that

$$B(u_n, u_n) < \frac{1}{n}.$$

Since $\partial\Omega$ is a unit circle around the center, it is Lipschitz. As the sequence is bounded, it contains a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ that converges to u in $L^2(\Omega)$ thus weak derivative $(Du_{n_k})_{k\in\mathbb{N}}$ converges weakly.

Now we derive a contradiction by

$$B(u_{n_k}, u_{n_k}) = \|Du_{n_k}\|^2_{L^2(\Omega)} - \beta\int_{\partial\Omega}(Tu_{n_k})^2 \to 0.$$

This implies

$$\|Du\|^2_{L^2(\Omega)} = 0$$

Therefore, u is a constant. On the other hands, we have that

$$u|_{\partial\Omega} = 0.$$

This is a contradiction as $\|u\|_{H^1(\Omega)} = 1$. We conclude that B satisfies the conditions for Lax-Milgram theorem, therefore has a unique solution.

(ii)

Let (r, θ) be the polar coordinate, then we have

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Suppose u is in the form $u(r, \theta) = R(r)A(\theta)$. Then we have

$$\Delta u = (R''(r) + \frac{1}{r}R'(r))A(\theta) + R(r)\frac{1}{r^2}A''(\theta) = 0.$$

Therefore, transforming the equation, we derive,

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{A''(\theta)}{A(\theta)}.$$

Each side has a different variable. Thus this is equal to a constant λ. First obviously

$$A(\theta) = a_1 e^{\sqrt{\lambda}} + a_2 e^{-\sqrt{\lambda}}.$$

for some constants, a_1, a_2. And for R,

$$r^2 R''(r) + rR'(r) = \lambda R(r)$$

By substituting $R(r) = r^\alpha$, we obtain

$$\alpha(\alpha - 1) + \alpha = \lambda.$$

Thus $\lambda = \alpha^2$.

First assumer $\lambda \geq 0$ then

$$R(r) = r^{\sqrt{\lambda}}, A(\theta) = a_1 \cos(\sqrt{\lambda}\theta) + a_2 \sin(\sqrt{\lambda}\theta).$$

For the boundary condition, we have $r = 1$ and

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

Substituting this to the condition we get,

$$R'(1)A(\theta) = \beta R(1)A(\theta).$$

Where $R(1) = 1, R'(1) = \sqrt{\lambda}$. Thus for $\beta > 0$,

$$R(r)A(\theta) = r^\beta(a_1 \cos(\beta\theta) + a_2 \sin \beta\theta).$$