Sheet 3 Solutions

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3.2

By the assumption, we have

$$\int_{\mathbb{R}^n} Du \cdot Dv dx = \int_{\mathbb{R}^n} f v dx - \int_{\mathbb{R}^n} c(u) v dx, \tag{1}$$

for any $v \in H^1(\mathbb{R}^n)$. The above integration is defined since u is compactly supported and we can choose large enough ball that contains the support of u and integrate these expressions over it.

Let us now define

$$v = -D_k^{-h}(D_k^h u)$$

for sufficiently small h. Substitute this to Equation 1, we get

$$-\int_{\mathbb{R}^n} Du \cdot D(D_k^{-h}(D_k^h u))dx = -\int_{\mathbb{R}^n} f D_k^{-h}(D_k^h u)dx + \int_{\mathbb{R}^n} c(u) D_k^{-h}(D_k^h u)dx.$$

By applying the integration by parts of difference quotients, we derive

$$-\int_{\mathbb{R}^n} Du \cdot D(D_k^{-h}(D_k^h u))dx = \int_{\mathbb{R}^n} D_k^{-h} Du \cdot (D_k^h Du))dx = \|D_k^h(Du)\|_{L^2(\mathbb{R}^n)}^2.$$

By Cauchy inequality with ε and the inequality between difference quotients and weak-derivatives, we get

$$\begin{aligned}
\left| -\int_{\mathbb{R}^n} f D_k^{-h}(D_k^h u)dx \right| &\leq \int_{\mathbb{R}^n} |f||D_k^{-h}(D_k^h u)|dx \\
&\leq (\int_{\mathbb{R}^n} |f|^2 dx)(\int_{\mathbb{R}^n} |D_k^{-h}(D_k^h u)|^2 dx) \\
&\leq \frac{C}{\varepsilon}(\int_{\mathbb{R}^n} |f|^2 dx) + \varepsilon(\int_{\mathbb{R}^n} |D_k^{-h}(D_k^h u)|^2 dx) \\
&\leq \frac{C}{\varepsilon}(\int_{\mathbb{R}^n} |f|^2 dx) + C_1\varepsilon(\int_{\mathbb{R}^n} |D_k^h(Du)|^2 dx)
\end{aligned}$$

Also we observe that by the smoothness of c,

$$c(u)(x) = \int_0^{u(x)} c'(t)dt \Rightarrow |c(u)(x)| \leq |u(x)| \cdot \|c'\|_{L^\infty([0,u(x)])}.$$

With this we have the inequality and the same argument appeared previously we get

$$|\int_{\mathbb{R}^n} c(u) D_k^{-h}(D_k^h u)dx| \leq C_2\varepsilon \int_{\mathbb{R}^n} |D_k^h(Du)|^2 dx + \frac{C}{\varepsilon}\|c'\|_{L^\infty([0,u(x)])}\|u\|_{L^2}^2.$$

Combining these inequalities and the rewriting the expression of the left hand side we get

$$\|D_k^h(Du)\|^2_{L^2(\mathbb{R}^n)} \le (C_1+C_2)\varepsilon \int_{\mathbb{R}^n} |D_k^h(Du)|^2 dx + \frac{C}{\varepsilon}(\int_{\mathbb{R}^n} |f|^2 dx) + \frac{C}{\varepsilon}\|c'\|_{L^\infty([0,u(x)])}\|u\|^2_{L^2}$$

Let $\varepsilon = \frac{1}{2(C_1+C_2)}$, then we get

$$\frac{1}{2}\|D_k^h(Du)\|^2_{L^2(\mathbb{R}^n)} \le \frac{C}{\varepsilon}(\int_{\mathbb{R}^n} |f|^2 dx) + \frac{C}{\varepsilon}\|c'\|_{L^\infty([0,u(x)])}\|u\|^2_{L^2}$$

This holds for each $k = 1, \cdots, n$, thus we conclude that for some constant K, the following inequality holds,

$$\|D^h u\|_{L^2(\mathbb{R}^n)} \le K((\int_{\mathbb{R}^n} |f|^2 dx) + \|c'\|_{L^\infty([0,u(x)])}\|u\|^2_{L^2}).$$

Therefore $Du \in H^1(\mathbb{R}^n)$, therefore $u \in H^2(\mathbb{R}^n)$.

Exercise 3.3

Since ϕ is smooth, we have $\phi(u) \in H^1(U)$. Let us now define a bilinear form

$$B[u,v] = \sum_{i,j}^n \int_U A_{i,j}\partial_{x_i} u \partial_{x_j} v.$$

Let $v \in \mathcal{C}_C^\infty(U)$ and $v \ge 0$, then

$$\begin{aligned} B[\phi(u),v] &= \sum_{i,j}^n \int_U A_{i,j}\partial_{x_i}\phi(u)\partial_{x_j} v \\ &= \sum_{i,j}^n \int_U A_{i,j}\phi'(u)\partial_{x_i} u\partial_{x_j} v \\ &= \sum_{i,j}^n \left(\int_U A_{i,j}\partial_{x_j}(\phi'(u)v)\partial_{x_i} u dx - \int_U A_{i,j}\phi''(u)\partial_{x_i} u \partial_{x_j} u \cdot v dx\right). \end{aligned}$$

By uniform ellipticity and convexity of ϕ we get

$$\sum_{i,j}^n \int_U A_{i,j}\phi''(u)\partial_{x_i} u\partial_{x_j} u \cdot v dx \ge 0.$$

Also we have u is the weak-solution of the original problem therefore

$$\sum_{i,j}^n \left(\int_U A_{i,j}\partial_{x_j}(\phi'(u)v)\partial_{x_i} u dx\right) = \phi'(u)v \sum_{i,j}^n \left(\int_U \partial_{x_j}(A_{i,j}\partial_{x_i} u) dx\right) = 0.$$

Combining these we conclude that

$$B[\phi(u),v] \le 0.$$

By the density of test functions, we arrived the conclusion.