Nonlinear Partial Differential Equations 1

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1 Preliminaries

1.1 Measure theory

1.2 Sobolev spaces

Definition 1.1. *A n-dimensional multi-index is a n-tuple in $\mathbb{N}_0^n$. And for such n-tuple α we define $|\alpha| = \sum_{i=1}^n \alpha_i$.*

Definition 1.2. *Let U be an open subset of n-dimensional real Euclidean space. A measurable function $f : U \to \mathbb{C}$ is said to be locally integrable over U if for any compact subset K of U, the integral $\int_K |f|dx$ is finite. The set of all such functions is denoted by $L^1_{\mathbf{loc}}(U)$.*

Definition 1.3. *Let X be a topological space. $f : X \to \mathbb{R}$ is said to be compactly supported if there is a compact subset K of X such that $f(X - K) = \{0\}$.*

Definition 1.4. *Let U be an open subset of $\mathbb{R}^n$. A test function $\phi : U \to \mathbb{R}$ is a function such that it is infinitely continuously differentiable and compactly supported. The set of all such functions over U is denoted by $\mathcal{C}^\infty_C(U)$.*

Definition 1.5. *Let $u : U \to \mathbb{R}$ be a $|\alpha|$ times continuously differentiable function over an open subset U of $\mathbb{R}^n$. Then the partial-derivative respect to α is*

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Definition 1.6. *Let $u, v \in L^1_{\mathbf{loc}}(U)$ where U is an open subset of $\mathbb{R}^n$ and α be a multi-index. We say that α-th weak derivative of u is v(denoted as $D^\alpha u = v$), if the following equality holds.*

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_v \phi dx$$

for any test function ϕ.

Remark 1.1. *In the definition above, if $|\alpha| = 1$, this means the integration by parts.*

Lemma 1.1. *Weak derivatives are unique almost everywhere. In other words, if v_1, v_2 are weak derivatives for u, then for any $\phi \in \mathcal{C}_C^\infty(U)$ we have*

$$\int_U (v_1 - v_2)\phi dx = 0.$$

Proof. The above equality is clear as the left-hand side equals to

$$\int_U uD^\alpha \phi dx - \int_U uD^\alpha \phi dx.$$

□

Definition 1.7. *Let U be an open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, $p \in [1, \infty]$. We define the Sobolev space with k and p over U such that*

$$W^{k,p}(U) = \{u \in L^1_{\mathbf{loc}}(U) \mid \forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq k \Rightarrow D^\alpha u \in L^p(U)\}.$$

Notation 1.1. *For the case $p = 2$ we denote $W^{k,2}(U) = H^k(U)$.*

Remark 1.2. *When $k = 0$ we have $W^{0,p}(U) = L^p(U)$.*

Definition 1.8. *Given an open subset U of $\mathbb{R}^n$, we define $H^k_{\mathbf{loc}}(U)$ to be such that*

$$H^k_{\mathbf{loc}}(U) = \{u : U \to \mathbb{R} \mid \textit{For any compact set } V \subset U, \textit{ we have } u \in H^2(V)\}.$$

Definition 1.9. *The essential supremum of a functional $f : U \to \mathbb{R}$ is*

$$\operatorname{esssup}_U(f) \inf\{c \in \mathbb{R} \mid \forall x \in U, f(x) \leq c\}.$$

Definition 1.10. *We define a norm on a Sobolev space $W^{k,p}(U)$ such that for $u \in W^{k,p}(U)$,*

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{\frac{1}{p}} & (1 \leq p < \infty), \\ \sum_{|\alpha| \leq k} \operatorname{esssup}_U(|D^\alpha u|) & (p = \infty). \end{cases}$$

Definition 1.11. *A sequence in the Sobolev space $W^{k,p}(U)$ converges if it converges in its Sobolev norm.*

Definition 1.12. *For a sequence $(u_m) \subset W^{k,p}(U)$, $u_m \to u$ in $W^{k,p}_{\mathbf{loc}}(U)$ if for any compact subset K of U, we have $u_m \to u$ in $W^{k,p}_{\mathbf{loc}}(K)$.*

Definition 1.13. *The Sobolev space $W^{k,p}_0(U)$ with boundary value zero at ∂U is the closure of $C_C^\infty(U)$ in the topology induced by the Sobolev norm.*

1.3 Example

Let $U = B(0,1)$ be a unit ball in $\mathbb{R}^n$ and $u : U \to \mathbb{R}$ be such that $u(x) = \|x\|^{-\alpha}$ for any x except 0. Then we have the following statement,

$$u \in W^{1,p}(U) \Leftrightarrow \alpha < \frac{n-p}{p}.$$

Let us consider $\{r_k\}_{k\in\mathbb{N}} \subset B(0,1) = U$ such that it is dense in U. The function

$$u(x) = \sum_{k=1}^{\infty} \frac{1}{2^l} |x - r_k|^{-\alpha},$$

is in $W^{1,p}(U)$ if and only if $\alpha = \frac{n-p}{p}$. This function is not bounded in any ball contained in U. (Use the fact that $p < n$) but $u \notin L^\infty(U)$.

Theorem 1.1. *Let U be an open and bounded set in $\mathbb{R}^n$. Suppose that $u \in W^{k,p}(U)$, then there exists a sequence $(u_m)_{m\in\mathbb{N}} \subset C^\infty(U) \cap W^{k,p}(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.*

1.4 Sobolev Inequalities

Definition 1.14. *Let $p \in [1, \infty)$. The Soolev conjugate of p is*

$$p^* = \frac{np}{n-p}.$$

Remark 1.3. *From the definition we see*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Thus $p^* > p$.

Theorem 1.2. *Let $U \subset \mathbb{R}^n$ be an open bounded set and $u \in W^{1,p}(U)$. Then there exists a constant C only depending on n and U such that*

$$\|u - (u)_U\|_{L^2(U)} \leq C(n,U)\|Du\||_{L^2(U)},$$

where

$$(u)_U = \frac{1}{\mu(U)} \int_U u dx.$$

Corollary 1.1. *In Theorem **??**, if we have $u \in H_0^1(U)$, then there exists a constant C only depending on n and U such that*

$$\|u\|_{L^2(U)} \leq C(n,U)\|Du\||_{L^2(U)},$$

Remark 1.4. *We may now replace the Sobolev norm in $H_0^1(U)$ with $\|Du\|_{L^2(U)}$. Recall that the norm defined on $H_0^1(U)$ is*

$$\|u\|_{H_0^1(U)} = \|u\|_{L^2(U)} + \|Du\|_{L^2(U)}.$$

By the corollary, we obtain the inequality,

$$\|Du\||_{L^2(U)} \leq \|u\|_{H_0^1(U)} \leq C(n,U)\|Du\||_{L^2(U)}.$$

Therefore, $\|Du\||_{L^2(U)}$ induces same topology as the Sobolev norm.

2 Second Order Elliptic Equations

2.1 Definitions

Notation 2.1. *For a function $u : U \to \mathbb{R}$ where U is an open subset of $\mathbb{R}^n$, we use the following notation.*

$$\frac{\partial u}{\partial x_i} = u_{x_i}.$$

Definition 2.1. *Give an open bounded subset U of $\mathbb{R}^n$, $f : U \to \mathbb{R}$. The second order elliptic equation is the problem to find functions $u : \overline{U} \to \mathbb{R}$ which satisfies the following equations,*

$$\begin{cases} Lu(x) = f(x) \quad (x \in U), \\ u(x) \equiv 0 \quad (x \in \partial U), \end{cases}$$

where L is called a second-order partial differential operator such that for any $u : U \to \mathbb{R}$,

$$Lu = -\sum_{i,j=1}^{n} (a^{i,j}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u, \tag{2.1}$$

where for each $i, j = 1, \cdots, n$, $a^{i,j}, b^i, c : U \to \mathbb{R}$.

Remark 2.1. *If $a^{i,j}$ are differentiable on U for each $i, j = 1, \cdots, n$, we can rewrite Equation* **??** *to be such that*

$$Lu = -\sum_{i,j=1}^{n} a^{i,j}(x)u_{x_i x_j} + \sum_{i=1}^{n} \overline{b}^i(x)u_{x_i} + c(x)u, \tag{2.2}$$

where for each $i = 1, \cdots, n$,

$$\overline{b}^i(x) = b_i(x) - \sum_{j=1}^{n} (a^{i,j}(x))_{x_j}(x).$$

This is due to the Leibniz rule.

Remark 2.2. *In the case $u \in \mathcal{C}^2(U)$, we may assume that $a_{i,j} = aj, i$ for each $i, j = 1, \cdots, n$ from now on. This is justified by the following procedure. Given $(a^{i,j})_{i,j=1\cdots,n}$ we define $(\tilde{a}^{i,j})_{i,j=1\cdots,n}$ in the following way,*

$$\tilde{a}^{i,j}(x) = \frac{1}{2}(a^{i,j}(x) + a^{j,i}(x)).$$

Because the first part of the equation **??** *can be rewritten as*

$$\sum_{i,j=1}^{n} \frac{1}{2}(a^{i,j}(x) + a^{j,i}(x))u_{x_i x_j} + \sum_{i,j=1}^{n} \frac{1}{2}(a^{i,j}(x) - a^{j,i}(x))u_{x_i x_j}.$$

Using Young's theorem we derive that

$$\sum_{i,j=1}^{n} \frac{1}{2}(a^{i,j}(x) - a^{j,i}(x))u_{x_i x_j} = 0.$$

Definition 2.2. *A second-order partial differential operator is said to be uniformly elliptic if there is $\theta > 0$ such that for any $\xi \in \mathbb{R}^n$*

$$\sum_{i,j=1}^{n} a^{i,j}(x)\xi_i\xi_j \geq \theta\|\xi\|^2$$

holds for almost everywhere on U.

Remark 2.3. *The above definition can be stated in a different manner. Given a quadratic form $A(x) = (a^{i,j}(x))_{i,j=1,\cdots,n}$. The problem is uniformly elliptic if and only if*

$$A(x) \geq \theta I$$

holds almost everywhere for a fixed constant $\theta > 0$.

Example 2.1. *If we take $a^{i,j} = \delta_{i,j}$, and $b^i, c \equiv 0$, the problem is*

$$Lu = -\Delta u.$$

2.2 Weak Solutions

In this subsection, we assume that $a^{i,j}, b^i, c \in L^\infty(U)$ for each $i, j = 1, \cdots, n$ and $f \in L^2(U)$. Suppose we have a second-order elliptic equation. Then by multiplying $v \in \mathcal{C}_0^\infty(U)$, we get

$$-\int_U \sum_{i,j=1}^{n} (a^{i,j}(x)u_{x_i})_{x_j} v dx = \int_U \sum_{i,j=1}^{n} a^{i,j}(x)u_{x_i}v_{x_j}dx$$

which is well-defined fi $\|Dv\| \in L^1(U)$.

Definition 2.3. *Given a second-order elliptic equation, we define a bilinear form $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ such that*

$$B(u, v) = \int_U \sum_{i,j=1}^{n} a^{i,j}(x)u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i(x)ux_iv + c(x)uvdx$$

Remark 2.4. *Such $B(u, v)$ is a well-defined continuous bilinear form.*

Definition 2.4. *Given a second-order elliptic equation*

$$\begin{cases} Lu(x) = f(x) & (x \in U), \\ u(x) = 0 & (x \in \partial U). \end{cases}$$

A function $u \in H_0^1(U)$ is called a weak solution to the problem if for any $v \in H_0^1(U)$, we have

$$B(u, v) = \langle f, v\rangle_{L^2(U)}.$$

Remark 2.5. *Suppose we have a classical solution u, (in other words $u \in \mathcal{C}^2(U)$ and $a^{i,j} \in C^1(U)$). Then such u is also a weak solution.*

Remark 2.6. *Suppose for $u \in H_0^1(U)$, we have that for any $v \in \mathcal{C}_0^\infty(U)$*

$$B(u, v) = \langle f, v \rangle_{L^2(U)}.$$

Then such u is a weak-solution, as $\mathcal{C}_0^\infty(U)$ is dense in $H_0^1(U)$.

We could also replace the condition on f which is that $f \in L^2(U)$ to $f \in H^{-1}(U)$.

Definition 2.5. *Given a second-order elliptic equation, we say that $u \in H_0^1(U)$ is a weak solution of the problem if*

$$B[u, v] = \langle f, v \rangle,$$

where

$$\langle f, v \rangle = \int_U f^0 v + \sum_{i=1}^n f^i v_{x_i} dx \quad (f^0, f^1, \cdots, f^n \in L^2(U)),$$

is the duality pairing of $H^{-1}(U)$ and $H_0^1(U)$.

Proposition 2.1. *Suppose we have a problem*

$$\begin{cases} Lu(x) = f(x) \quad (x \in U), \\ u(x) = g(x) \quad (x \in \partial U). \end{cases}$$

where ∂U is smoothly parametrizable. Furthermore, suppose there is $w \in H^1(U)$ such that

$$w(x) = g(x) \quad (x \in \partial U).$$

Then for the modified problem,

$$\begin{cases} L(u(x)) = f(x) - Lw(x) \quad (x \in U), \\ u(x) = 0 \quad (x \in \partial U). \end{cases}$$

we can get solutions of the original problem given a solution of the second one and adding w to it.

Proof. In order to show that such modified problem is indeed well-defined, we have to prove that

$$f - Lw \in H^{-1}(U).$$

By definition, $w \in H^1(U)$ thus $w_{x_i} \in L^2(U)$ for any $i = 1, \cdots, n$. For any $i, j = 1, \cdots, n, a^{i,j} \in L^\infty(U)$. We now deduce that $a^{i,j} w_{x_i} \in L^2(U)$. Later. $\square$

2.3 Existence of Weak Solutions

Theorem 2.1 (Lax-Milgram). *Let $(H, \langle \cdot, \cdot \rangle_H)$ be a real Hilbert space and H^* be the dual of it. Assume that for a bilinear form $B : H \times H \to \mathbb{R}$, there exists $\alpha, \beta > 0$ such that*

i). $|B(u, v)| \leq \alpha \|u\| \|v\|$,

ii). for any $v \in H$, $\beta \|u\|^2 \leq B(u, v)$.

Then for each $f \in H^*$, there is a unique $u \in H$ such that

$$B(u, v) = \langle f, v \rangle_H.$$

Proof. Given $u \in H$, the mapping $v \mapsto B(u, v)$ is a bounded linear operator by the first condition on B. By Riesz representation theorem, there is unique $w \in H$ such that

$$B(u, v) = \langle w, v \rangle_H.$$

Let us define $A : H \to H$ to be such that $A(u) = w$ where w is acquired through the above construction. We will show that A is a bounded linear operator.

We first show that it is linear. Given $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in H$,

$$\begin{aligned} \langle A(\lambda_1 u_1 + \lambda_2 u_2,)v \rangle_H &= B(\lambda_1 u_1 + \lambda_2 u_2, v) \\ &= \lambda_1 B(u_1, v) + \lambda_2 B(u_2, v) \\ &= \lambda_1 \langle A u_1, v \rangle_H + \lambda_2 \langle A u_2, v \rangle_H \\ &= \langle A \lambda_1 u_1 + A \lambda_2 u_2, v \rangle_H \end{aligned}$$

holds for any $v \in H$. Thus by uniqueness from Riesz representation theorem, we have proved that A is linear.

We then prove that A is bounded.

$$\begin{aligned} \|Au\|^2 &= \langle Au, Au \rangle_H, \\ &= B(u, Au), \\ &\leq \alpha \|u\| \|Au\|, \\ &\Rightarrow \|Au\| \leq \alpha \|u\|, \end{aligned}$$

by the first condition on B.

Such A has following two properties.

i). A is injective,

ii). $\mathcal{R}(A)$, the range of A is closed.

Suppose $Au = 0$ then by the second condition on B we derive

$$\begin{aligned}
\beta\|u\|^2 &\leq B(u,u),\\
&= \langle Au, u\rangle_H,\\
&\leq \|Au\|\|u\|\\
&\Rightarrow \beta\|u\| \leq \|Au\| = 0.
\end{aligned}$$

Let $(Au_n)_{n\in\mathbb{N}} \subset \mathcal{R}(A)$ be a convergent sequence with its limit w^*. Using the second condition on B and the previous argument on the norm of Au once again, we derive

$$\begin{aligned}
\beta\|u_n - u_m\| &\leq \|A(u_n - u_m)\|,\\
&= \|Au_n - Au_m\|.
\end{aligned}$$

Sincer $(Au_n)_{n\in\mathbb{N}}$ is a Cauchy sequence we derived that $(u_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in a complete space, thus convergent. We define the limit to be u^*. By continuity of A, we have $Au^* = w^*$.

We now prove that $\mathcal{R}(A) = H$. Suppose not, $\mathcal{R}(A) \neq H$, then we know that $\mathcal{R}(A)$ is closed therefore

$$(M^\perp)^\perp = M \neq H.$$

This shows that $M^\perp \neq \{0\}$. Take $u^\perp \in \mathcal{R}(A)^\perp$ and $u = Au^\perp$ which is in $\mathcal{R}(A)$. Then we have

$$\beta\|u^\perp\|^2 \leq B(u^\perp, u^\perp) = \langle Au^\perp, u^\perp\rangle_H = 0.$$

Therefore, we derived $\|u^\perp\| = 0$ which is a contradiction. We conclude $\mathcal{R}(A) = H$.

Using Riesz representation theorem, there exists $w \in H$ such that for any $v \in H$,

$$\langle f, v\rangle_H = \langle w, v\rangle_H.$$

By the subjectivity of A, there is $u \in H$ such that $Au = w$. Therefore we derive the formula,

$$B(u,v) = \langle Au, v\rangle_H = \langle w, v\rangle_H = \langle f, v\rangle_H.$$

Now we will prove the uniqueness of such u. Suppose $u, \overline{u} \in H$ are such that

$$\langle u, v\rangle_H = \langle f, u\rangle_H = \langle \overline{u}, v\rangle_H.$$

Then by the linearity of the scalar product, we obtain that for any $v \in H$,

$$B(u - \overline{u}, v) = 0.$$

In particular, when $v = u - \overline{u}$, we obtain,

$$\beta\|u - \overline{u}\|^2 = 0.$$

Thus the uniqueness is proven. □

Remark 2.7. *B does not have to be symmetric. In the case when B is symmetric, the theorem is trivial.*

This comes from by defining $(u, v) = B(u, v)$ *a new scalar product which is possible since* B *is symmetric and by the second condition we have*

$$(u, u) \geq \beta \|u\|^2 \Rightarrow ((u, u) = 0 \Leftrightarrow u = 0).$$

Using the Riesz representation theorem for $(\cdot, \cdot)$, *we obtain that, for any* $v \in H$,

$$(w, v) = \langle f, v \rangle_H.$$

Theorem 2.2. *Give an open bounded subset* U *of* $\mathbb{R}^n$, $f : U \to \mathbb{R}$. *We consider the second order elliptic equation.*

$$\begin{cases} Lu(x) = f(x) \quad (x \in U), \\ u(x) \equiv 0 \quad (x \in \partial U), \end{cases}$$

where L *is a second-order partial differential operator such that for any* $u : U \to \mathbb{R}$,

$$Lu = -\sum_{i,j=1}^{n} (a^{i,j}(x) u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x) u, \tag{2.3}$$

where for each $i, j = 1, \cdots, n$, $a^{i,j}, b^i, c : U \to \mathbb{R}$ *belong to* $L^\infty(U)$. *Furthermore, we pose the uniformly elliptic condition to* L.

Let us define $B(u, v) : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ *to be such that*

$$B(u, v) = \int_U \sum_{i,j=1}^{n} a^{i,j} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \, dx,$$

where $a^{i,j}, b^i, c \in L^\infty(U)$. *Then there exists* $\alpha, \beta > 0$ *and* $\gamma \geq 0$ *such that,*

1). $B(u, v) \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$.

2). $\beta \|u\|^2_{H_0^1(U)} \leq B(u, u) + \gamma \|u\|^2_{L^2(U)}$.

Proof.

$$\begin{aligned} |B(u, v)| \leq & \sum_{i,j=1}^{n} \|a^{i,j}\|_{L^2(U)} \int_U |Du||Dv| dx \\ & + \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} \int_U |Du||v| dx \\ & + \|c\|_{L^\infty(U)} \int_U |u||v| dx. \end{aligned}$$

We then use the Cauchy-Schwarz inequality to deduce that

$$\int_U |Du||Dv|dx \le \|Du\|_{L^2(U)}\|Dv\|_{L^2(U)},$$
$$\int_U |Du||v|dx \le \|Du\|_{L^2(U)}\|v\|_{L^2(U)},$$
$$\int_U |u||v|dx \le \|u\|_{L^2(U)}\|v\|_{L^2(U)}.$$

Using Poincaré inequality, we derive that

$$B(u,v) \le \alpha\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)}.$$

We now prove the second claim. First we note that, by the uniformly elliptic condition, there is $\theta > 0$ such that

$$\begin{aligned}\theta\int_U |Du|^2 &\le \int_U \sum_{i,j=1}^n a^{i,j}u_{x_i}u_{x_j}dx,\\ &\le B(u,u) - \int_U \sum_{i=1}^n b^i u_{x_i} u dx - \int_U cu^2 dx,\\ &\le B(u,u) + \sum_{i=1}^n \|b^i\|_{L^\infty(U)}\|Du\|_{L^2(U)}\|u\|_{L^2(U)} + \|c\|_{L^\infty(U)}\|u\|^2_{L^2(U)}.\end{aligned}$$

By AM-GM inequality, we obtain for any $x, y \le 0$ and $\epsilon > 0$, we have,

$$xy \le \varepsilon x^2 + \frac{1}{4\varepsilon}y^2.$$

Let us choose $\varepsilon > 0$ to be such that

$$\varepsilon = \frac{\theta}{2\sum_{i=1}^n \|b^i\|_{L^\infty(U)}}.$$

Substituting this to the inequality, we get

$$\theta\int_U |Du|^2 \le B(u,u) + \frac{\theta}{2}\|u\|^2_{H_0^1(U)} + \gamma\|u\|^2_{L^2(U)},$$
$$\frac{1}{2}\theta\|u\|^2_{H_0^1(U)} \le B(u,u) + \gamma\|u\|^2_{L^2(U)}.$$

Let $\beta = \frac{1}{2}$, we derived the claim. □

Remark 2.8. *In general the problem*

$$Lu = f, \quad u \in H_0^1(U),$$

is not solvable.

Theorem 2.3. *Let the second order elliptic operator L with boundaries. There exists $\gamma \geq 0$ such that for any $\mu \geq \gamma$ and each $f \in L^2(U)$, there exists a unique solution $u \in H_0^1(U)$ such that*

$$B(u,v) + \mu\langle u,v\rangle_{H_0^1(U)} = \langle f,v\rangle_{H_0^1(U)}$$

holds for any $v \in H_0^1(U)$.

Proof. Let $\gamma \geq 0$ be as in Theorem **??**. Now we define

$$B_\mu(u,v) = B(u,v) + \mu\langle u,v\rangle_{L^2(U)}.$$

Then we have

$$\begin{aligned} B_\mu(u,v) &\leq |B(u,v)| + \mu|\langle u,v\rangle_{L^2(U)}|, \\ &\leq \alpha\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)} + \mu\|u\|_{L^2(U)}\|v\|_{L^2(U)}, \\ &\leq (\alpha+\mu)\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)}. \end{aligned}$$

Thus satisfies the first condition for Theorem **??**. For the second part, we observe that

$$\begin{aligned} \beta\|u\|^2_{H_0^1(U)} \leq\ & B(u,v) + \gamma\|u\|_{H_0^1(U)}\|v\|^2_{H_0^1(U)}, \\ & B_\mu(u,u) - \mu\|u\|^2_{L^2(U)} + \gamma\|u\|_{H_0^1(U)}\|v\|^2_{H_0^1(U)}, \\ & \leq B_\mu(u,u). \end{aligned}$$

Therefore, there is a unique $u \in H_0^1(U)$, such that for any $v \in H_0^1(U)$, we have

$$B_\mu(u,v) = \langle f,v\rangle_{L^2(U)}.$$

(In other words, the problem has a weak solution). ☐

Remark 2.9. *We can have $\gamma = 0$ for some particular operators. For instance,*

$$Lu = \sum_{i,j=1}^{n}(a^{i,j}u_{x_i})_{x_j}, \quad (b^i, c = 0),$$

which we assume it to be uniformly elliptic.

We now use Lax-Milgram theorem, by defining

$$B(u,v) = \sum_{i,j=1}^{n}\int_U a^{i,j}u_{x_i}u_{x_j}.$$

By the uniformly elliptic condition, there is $\theta > 0$ such that

$$B(u,u) \geq \theta\|u\|^2_{H_0^1(U)},$$

for any $u \in H_0^1(U)$.

Remark 2.10. *In general, there might be a homogeneous solution to the given problem. (ie. a solution u such that $Lu = 0$). For example, in the case $n = 1$, we have*

$$Lu = \frac{\partial^2 u}{\partial x^2} - u \quad (0, \pi).$$

Then the problem

$$\begin{cases} Lu = f \quad ((0, \pi)), \\ u(0), u(\pi) = 0. \end{cases}$$

is not solvable. But when $f = 0$, there is a non-trivial homogeneous solution,

$$\psi(x) = K \sin x$$

2.4 Regularity theory (Basic Theory, $H^2(U)$ Theory)

Suppose we have a problem

$$-\Delta u = f,$$

where $u, f \in \mathcal{C}_C^\infty(\mathbb{R}^n)$ and also $f \in L^2(\mathbb{R}^n)$.

Lemma 2.1. *Let $U \subset \mathbb{R}^n$ be an open bounded set and L be a second order elliptic operator such that*

$$Lu = -\sum_{i,j=1}^{n} (a^{i,j} u_{x_i})_{x_j},$$

where each $a^{i,j} \in \mathcal{C}^1(U)$. Then we have the following implication.

Theorem 2.4 (Interior Regularity)**.** *Let $U \subset \mathbb{R}^n$ be an open bounded set and $u \in H_0^1(U)$. Given the second order elliptic operator*

$$Lu = -\sum_{i,j=1}^{n} (a^{i,j} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} + cu,$$

which is

i). uniformly elliptic with constant θ,

ii). each $a^{i,j} \in \mathcal{C}^1(U)$,

iii). each b^i and c are in $L^\infty(U)$,

iv). $f \in L^2(U)$.

Suppose that $u \in H^1(U)$ is a weak solution of the problem

$$Lu = f \quad (in\ U).$$

Then u satisfies the following.

1). $u \in H^2_{\mathbf{loc}}(U)$.

2). For any compact subset $V \subset U$, $\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$ *for some* $C > 0$.

3). If u *is bounded in* $L^2(V)$ *then there is a subset such that* Du, D^2u *are bounded.*