Nonlinear Partial Differential Equations Exercise Sheet 1 Solutions

So Murata

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Exercise 1

Suppose the inequality does not hold. Then for each n, there is v_n such that

$$\|v_n\|_{L^p(\Omega)} > n\|Du\|_{L^p(\Omega)}.$$

Normalizing v_n, we obtain a sequence $(v_k)_{k\in\mathbb{N}}$ such that

$$\|Du\|_{L^p(\Omega)} < \frac{1}{n}.$$

Since $(v_k)_{k\in\mathbb{N}}$ is bounded, there is a subsequence $(v_{k_j})_{j\in\mathbb{N}}$ converging to some u. Then its derivative is 0 by the assumption. Then $u = 0$ but $\|u\|_{L^p(\Omega)} = 1$. Thus a contradiction.

Exercise 2

(i)

Suppose if we have a weak solution, by integration by parts, we get

$$\begin{aligned}\int_U fv &= -\int_\Omega \Delta uvdx,\\ &= \int_\Omega \nabla u\cdot\nabla vdx - \int_{\partial\Omega}\frac{\partial u}{\partial\nu}vdx,\\ &= \int_\Omega \nabla u\cdot\nabla vdx - \int_{\partial\Omega}\beta uvdx.\end{aligned}$$

for any $v \in H^1(\Omega)$. We define a bilinear form $B(u,v) : H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ such that

$$B(u,v) = \int_\Omega \nabla u\cdot\nabla vdx - \int_{\partial\Omega}\beta uvdx.$$

By trace theorem, we have

$$
\begin{aligned}
|B(u,v)| &\leq \|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + |\beta|\|u\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)}, \\
&\leq \|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + |\beta|C^2\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},
\end{aligned}
$$

for some $C > 0$. By the definition, we have

$$\|u\|^2_{H^1(\Omega)} = \|Du\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}.$$

Thus we conclude

$$|B(u,v)| \leq (|\beta|C^2 + 1)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

, and this satisfies the first condition for Lax-Milgram theorem.

In the case $\beta \leq 0$, we will prove that B satisfies the second condition as well. In order to do so, we will derive a contradiction by assuming it does not satisfy the condition. If that is the case the it is equivalent to say that for any $n \in \mathbb{N}$, there is $u_n \in H^1(\Omega)$ such that

$$B(u_n, u_n) < \frac{1}{n}\|u_n\|^2_{H^1(\Omega)}.$$

We can normalize u_n and conclude that

$$B(u_n, u_n) < \frac{1}{n}.$$

Since $\partial\Omega$ is a unit circle around the center, it is Lipschitz. As the sequence is bounded, it contains a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ that converges to u in $L^2(\Omega)$ thus weak derivative $(Du_{n_k})_{k\in\mathbb{N}}$ converges weakly.

Now we derive a contradiction by

$$B(u_{n_k}, u_{n_k}) = \|Du_{n_k}\|^2_{L^2(\Omega)} - \beta\int_{\partial\Omega}(Tu_{n_k})^2 \to 0.$$

This implies

$$\|Du\|^2_{L^2(\Omega)} = 0$$

Therefore, u is a constant. On the other hands, we have that

$$u|_{\partial\Omega} = 0.$$

This is a contradiction as $\|u\|_{H^1(\Omega)} = 1$. We conclude that B satisfies the conditions for Lax-Milgram theorem, therefore has a unique solution.