Sheet 3 Solutions

Toni Abi Aoun, So Murata

Exercise 3.1
Consider the one-dimensional wave equation for $(t, x) \in [0, \infty) \times \mathbb{R}$

$$\partial_t^2 u - \partial_x^2 u = 0, \quad u|_{t=0} = g, \quad \partial u_t|_{t=0} = f \tag{1}$$

where $g, f : \mathbb{R} \to \mathbb{R}$ are given.
(i) Show that for smooth g, f d'Alembert's formula

$$u(t, x) = \frac{1}{2}(g(x - t) + g(x + t)) + \frac{1}{2}\int_{x-t}^{x+t} f(y)dy$$

yields a solution to (**??**).

Proof. If $g \in C^2(\mathbb{R})$ and $f \in C^1(\mathbb{R})$, the above formula gives a solution. Indeed, in this case, we can differentiate the expression for u using ordinary rules of differentiation and we get for $(t, x) \in (0, \infty) \times \mathbb{R}$

$$\begin{aligned}
\partial_x u(t, x) &= \frac{1}{2}(g'(x - t) + g'(x + t)) + \frac{1}{2}(f(x + t) - f(x - t)) \\
\partial_x^2 u(t, x) &= \frac{1}{2}(g''(x - t) + g''(x + t)) + \frac{1}{2}(f'(x + t) - f'(x - t)) \\
\partial_t u(t, x) &= \frac{1}{2}(g'(x + t) - g'(x - t)) + \frac{1}{2}(f(x + t) + f(x - t)) \\
\partial_t^2 u(x, t) &= \frac{1}{2}(g''(x + t) + g''(x - t)) + \frac{1}{2}(f'(x + t) - f'(x - t))
\end{aligned}$$

We see therefore that u satisfies the one-dimensional wave equation in $(0, \infty) \times \mathbb{R}$ and its derivatives up to order 2 can be extended to $[0, \infty) \times \mathbb{R}$ since g, f are uniformly continuous on compact sets. By directly evaluating the formula for u and $\partial_t u$ at $(0, x)$, we see that u satisfies the intial conditions. □

(ii) Let $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ smooth and $v(t, x; s)$ be the solution to

$$\partial_t^2 v - \partial_x^2 v = 0, \quad v|_{t=s} = 0, \quad \partial_t v|_{t=s} = f(s, \cdot).$$

for all $s \geq 0$. Define

$$u(t, x) := \int_0^t v(t, x; s)ds.$$

Show that this provides a solution to the inhomogeneous wave equation

$$\partial_t^2 u - \partial_x^2 u = f, \quad u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0 \tag{2}$$

and compute a solution formula using (i).

Proof. Suppose $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is C^1.
For a fixed $s \geq 0$, the function $w_s : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ defined by

$$w_s(t,x) = v(s+t,x;s), \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}$$

solves the following one-dimensional wave equation on $[0,\infty) \times \mathbb{R}$

$$\partial_t^2 w_s - \partial_x^2 w_s = 0, \quad w_s|_{t=0} = 0, \quad \partial_t w_s|_{t=0} = f(s,\cdot).$$

From this using part (i), we deduce that

$$w_s(t,x) = \frac{1}{2}\int_{x-t}^{x+t} f(s,y)dy, \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}.$$

This gives the following formula for v on $\{(t,x;s) | 0 \leq s \leq t, x \in \mathbb{R}\}$

$$v(t,x;s) = w_s(t-s,x) = \frac{1}{2}\int_{x+s-t}^{x+t-s} f(s,y)dy.$$

In particular v is a C^2 function.
Therefore u is given by

$$u(t,x) = \int_0^t v(t,x;s)ds = \frac{1}{2}\int_0^t \int_{x+s-t}^{x+t-s} f(s,y)dyds.$$

To verify u solves the inhomogeneous one-dimensional wave equation, we compute its derivatives

$$\begin{aligned}
\partial_t u(t,x) &= v(t,x;t) + \int_0^t \partial_t v(t,x;s)ds = \int_0^t \partial_t v(t,x;s)ds \\
\partial_t^2 u(t,x) &= \partial_t v(t,x;t) + \int_0^t \partial_t^2 v(t,x;s)ds = f(t,x) + \int_0^t \partial_t^2 v(t,x;s)ds \\
\partial_x u(t,x) &= \int_0^t \partial_x v(t,x;s)ds \\
\partial_x^2 u(t,x) &= \int_0^t \partial_x^2 v(t,x;s)ds = \int_0^t \partial_t^2 v(t,x;s)ds.
\end{aligned}$$

From this we see that for $(t,x) \in (0,\infty) \times \mathbb{R}$, we have

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = f(t,x).$$

For $x \in \mathbb{R}$, we have

$$u(0,x) = 0 \text{ and } \partial_t u(0,x) = 0$$

□

(iii) Show that even if $f \in L^2_{\text{loc}}((0,\infty)\times\mathbb{R})$ the solution u to (**??**) is in general not in $H^2_{\text{loc}}((0,\infty)\times\mathbb{R})$.

Proof. □

3.2

By the assumption, we have

$$\int_{\mathbb{R}^n} Du \cdot Dv dx = \int_{\mathbb{R}^n} fv dx - \int_{\mathbb{R}^n} c(u)v dx, \tag{3}$$

for any $v \in H^1(\mathbb{R}^n)$. The above integration is defined since u is compactly supported and we can choose large enough ball that contains the support of u and integrate these expressions over it.

Let us now define

$$v = -D_k^{-h}(D_k^h u)$$

for sufficiently small h. Substitute this to Equation 1, we get

$$-\int_{\mathbb{R}^n} Du \cdot D(D_k^{-h}(D_k^h u))dx = -\int_{\mathbb{R}^n} fD_k^{-h}(D_k^h u)dx + \int_{\mathbb{R}^n} c(u)D_k^{-h}(D_k^h u)dx.$$

By applying the integration by parts of difference quotients, we derive

$$-\int_{\mathbb{R}^n} Du \cdot D(D_k^{-h}(D_k^h u))dx = \int_{\mathbb{R}^n} D_k^{-h} Du \cdot (D_k^h Du))dx = \|D_k^h(Du)\|^2_{L^2(\mathbb{R}^n)}.$$

By Cauchy inequality with ε and the inequality between difference quotients and weak-derivatives, we get

$$\begin{aligned}\left|-\int_{\mathbb{R}^n} fD_k^{-h}(D_k^h u)dx\right| &\leq \int_{\mathbb{R}^n} |f||D_k^{-h}(D_k^h u)|dx \\ &\leq (\int_{\mathbb{R}^n} |f|^2 dx)(\int_{\mathbb{R}^n} |D_k^{-h}(D_k^h u)|^2 dx) \\ &\leq \frac{C}{\varepsilon}(\int_{\mathbb{R}^n} |f|^2 dx) + \varepsilon(\int_{\mathbb{R}^n} |D_k^{-h}(D_k^h u)|^2 dx) \\ &\leq \frac{C}{\varepsilon}(\int_{\mathbb{R}^n} |f|^2 dx) + C_1\varepsilon(\int_{\mathbb{R}^n} |D_k^h(Du)|^2 dx)\end{aligned}$$

Also we observe that by the smoothness of c,

$$c(u)(x) = \int_0^{u(x)} c'(t)dt \Rightarrow |c(u)(x)| \leq |u(x)| \cdot \|c'\|_{L^\infty([0,u(x)])}.$$

With this we have the inequality and the same argument appeared previously we get

$$|\int_{\mathbb{R}^n} c(u)D_k^{-h}(D_k^h u)dx| \leq C_2\varepsilon \int_{\mathbb{R}^n} |D_k^h(Du)|^2 dx + \frac{C}{\varepsilon}\|c'\|_{L^\infty([0,u(x)])}\|u\|^2_{L^2}.$$

Combining these inequalities and the rewriting the expression of the left hand side we get

$$\|D_k^h(Du)\|^2_{L^2(\mathbb{R}^n)} \leq (C_1 + C_2)\varepsilon \int_{\mathbb{R}^n} |D_k^h(Du)|^2 dx + \frac{C}{\varepsilon}(\int_{\mathbb{R}^n} |f|^2 dx) + \frac{C}{\varepsilon}\|c'\|_{L^\infty([0,u(x)])}\|u\|^2_{L^2}$$

Let $\varepsilon = \frac{1}{2(C_1+C_2)}$, then we get

$$\frac{1}{2}\|D_k^h(Du)\|^2_{L^2(\mathbb{R}^n)} \leq \frac{C}{\varepsilon}(\int_{\mathbb{R}^n} |f|^2 dx) + \frac{C}{\varepsilon}\|c'\|_{L^\infty([0,u(x)])}\|u\|^2_{L^2}$$

This holds for each $k = 1, \cdots, n$, thus we conclude that for some constant K, the following inequality holds,

$$\|D^h u\|_{L^2(\mathbb{R}^n)} \leq K((\int_{\mathbb{R}^n} |f|^2 dx) + \|c'\|_{L^\infty([0,u(x)])} \|u\|^2_{L^2}).$$

Therefore $Du \in H^1(\mathbb{R}^n)$, therefore $u \in H^2(\mathbb{R}^n)$.

Exercise 3.3

Since ϕ is smooth, we have $\phi(u) \in H^1(U)$. Let us now define a bilinear form

$$B[u, v] = \sum_{i,j}^n \int_U A_{i,j} \partial_{x_i} u \partial_{x_j} v.$$

Let $v \in \mathcal{C}_C^\infty(U)$ and $v \geq 0$, then

$$\begin{aligned}
B[\phi(u), v] &= \sum_{i,j}^n \int_U A_{i,j} \partial_{x_i} \phi(u) \partial_{x_j} v \\
&= \sum_{i,j}^n \int_U A_{i,j} \phi'(u) \partial_{x_i} u \partial_{x_j} v \\
&= \sum_{i,j}^n \left(\int_U A_{i,j} \partial_{x_j} (\phi'(u) v) \partial_{x_i} u dx - \int_U A_{i,j} \phi''(u) \partial_{x_i} u \partial_{x_j} u \cdot v dx \right).
\end{aligned}$$

By uniform ellipticity and convexity of ϕ we get

$$\sum_{i,j}^n \int_U A_{i,j} \phi''(u) \partial_{x_i} u \partial_{x_j} u \cdot v dx \geq 0.$$

Also we have u is the weak-solution of the original problem therefore

$$\sum_{i,j}^n \left(\int_U A_{i,j} \partial_{x_j} (\phi'(u) v) \partial_{x_i} u dx \right) = \phi'(u) v \sum_{i,j}^n \left(\int_U \partial_{x_j} (A_{i,j} \partial_{x_i} u) dx \right) = 0.$$

Combining these we conclude that

$$B[\phi(u), v] \leq 0.$$

By the density of test functions, we arrived the conclusion.

Exercise 3.4

Let $U \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary. Consider the equation

$$\begin{cases} \Delta^2 u = f & \text{in } U, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U. \end{cases} \tag{4}$$

We say that $u \in H_0^2$ is a weak solution to (**??**) provided

$$\int_U \Delta u \Delta v = \int_U f v$$

for all $v \in H_0^2$. Given $f \in L^2(U)$, prove that there exists a unique weak solution to (**??**).

Proof. Consider the map $F : H_0^2(U) \to \mathbb{R}$ defined by

$$F(v) = \int_U fv, \quad \forall v \in H_0^2(U).$$

This map is well defined (Cauchy-Schwarz), linear (linearity of integral) and bounded. To prove boundedness, we argue as follow

$$|F(v)| = \left| \int_U fv \right| \leq \|f\|_{L^2} \|v\|_{L^2} \leq \|f\|_{L^2} \|v\|_{H_0^2}.$$

Now consider the map $B : H_0^2(U) \times H_0^2(U) \to \mathbb{R}$ defined by

$$B(u, v) = \int_U \Delta u \Delta v, \quad \forall u, v \in H_0^2(U).$$

This map is well defined (Cauchy-Schwarz), bilinear (linearity of weak derivatives and integral), bounded and coercive. To prove boundedness, we use Cauchy-Schwarz

$$|B(u, v)| = \left| \int_U \Delta u \Delta v \right| \leq \|\Delta u\|_{L^2} \|\Delta v\|_{L^2} \leq \|u\|_{H_0^2} \|v\|_{H_0^2}.$$

To prove coercivity, we need to prove that there exists $\alpha > 0$ such that

$$B(u, u) \geq \alpha \|u\|_{H_0^2}^2, \quad \forall u \in H_0^2(U).$$

This is equivalent to showing that there exists $C > 0$ such that

$$\|u\|_{H_0^2}^2 \leq C \int_U (\Delta u)^2, \quad \forall u \in H_0^2(U).$$

U has smooth boundary and $u \in H_0^1(U)$, then in particular we have $u \in H_0^2(U)$ so we can apply Poincare's inequality and get

$$\int_U |u|^2 \leq C \int_U |Du|^2.$$

As $u \in H_0^2(U)$ we know that $\forall i = 1, \ldots, n$ we have $D_i u \in H_0^1(U)$, so we can apply Poincare's inequality and get

$$\int_U |D_i u|^2 \leq C \int_U |D(D_i u)|^2.$$

Combining these two observations, we see that it suffices then to show that for any $i, j = 1, \ldots, n$ we have

$$\int_U |u_{x_i x_j}^2|^2 \leq C \int_U (\Delta u)^2.$$

Using the density of $C_c^\infty(U)$ in $H_0^2(U)$ (by definition), it suffices to prove such a constant C exists if $u \in C_c^\infty(U)$. In this case, we have

$$\begin{aligned}
\int_U (\Delta u)^2 &= \int_U \sum_{i,j=1}^n u_{x_i x_i} u_{x_j x_j} \\
&= -\int_U \sum_{i,j=1}^n u_{x_j x_i x_i} u_{x_j} \\
&= \int_U \sum_{i,j=1}^n |u_{x_i x_j}|^2
\end{aligned}$$

where we used integration by parts twice. This proves the coercivity bound for B and we can thus apply Lax-Milgram theorem to conclude the existence of a unique weak solution in H_0^2 to the problem. $\square$