V4A1 Sheet 5

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Suppose the statement is false, then there exists u_k for each $k \in \mathbb{N}$ such that

$$[D^2 u_k]_{C^{0,\alpha}} > k[\Delta u_k]_{C^{0,\alpha}}.$$

We can assume $[D^2 u_k]_{C^{0,\alpha}} = 1$ by dividing u by $[D^2 u_k]_{C^{0,\alpha}}$. Thus we have the inequality

$$\frac{1}{k} > [\Delta u_k]_{C^{0,\alpha}}.$$

Using the pigeon hole principle on the definition of supremums, we find $i, j, k = 1, \cdots, n$ such that infinitely many $x_l \in \mathbb{R}^n$ and $h_l > 0$ we have

$$\frac{|D^2_{ij} u_l(x_l + h_l \underline{e}_k) - D^2_{ij} u_l(x_l)|}{h_l^\alpha} \geq \frac{1}{2n^3}.$$

We define a shifted u_l corresponding those x_l and h_l by

$$\tilde{u}_l(x) = h_l^{-2-\alpha} u_l(x_l + h_l x).$$

To summarize, we have

$$[D^2 \tilde{u}_l]_{C^{0,\alpha}} = 1, \quad [\Delta \tilde{u}_l]_{C^{0,\alpha}} < \frac{1}{l}, \quad |D^2_{ij} \tilde{u}_l(x) - D^2_{ij} \tilde{u}_l(0)| \geq \frac{1}{2n^3}.$$

By adding appropriate second order polynomial, we conclude that

$$\tilde{u}_l(0) = D\tilde{u}_l(0) = D^2 \tilde{u}_l(0) = 0.$$

Also we notice that at $\underline{e}_k$ we have

$$\tilde{u}_l(\underline{e}_k) \geq \frac{1}{2n^3}.$$

By the definition of $[\cdot]_{C^{0,\alpha}}$, it is clear that $\tilde{u}_l$ is bounded. Transforming the definition, we condlue

$$|\tilde{u}_l(x)| \leq C|h_l x|^{2+\alpha}, \quad |D\tilde{u}_l(x)| \leq C|h_l x|^{1+\alpha}, \quad |D^2 \tilde{u}_l(x)| \leq |h_l x|^\alpha.$$

The sequence is globally bounded and equicontinuous by the definition of Holder condition and the assumption on α. Furthermore, by appropriately large enough

R, and ArzelàAscoli theorem on $B_R(0)$, the definition of $[\cdot]_{C^{0,\alpha}}$ on Δu_l, we conclude that

$$\Delta_u \equiv 0, \quad |D^2\tilde{u}(x)| \leq |h_l x|^\alpha, \quad D^2_{ij}u_{(}\underline{e}_k) \geq \frac{1}{2n^3}.$$

$$\lim_{|x|\to\infty} \frac{|u(x)|}{|x|^3} = 0.$$

Thus, we conclude that u is a polynomial of at most degree 2. Therefore but at 0 all the derivatives vanish, we conclude $u \equiv 0$. This is a contradiction.