Sheet 1 Solutions

Toni Abi Aoun, So Murata

Exercise 1

Suppose the inequality does not hold. Then for each n, there is v_n such that

$$\|v_n\|_{L^p(\Omega)} > n\|Dv_n\|_{L^p(\Omega)}.$$

We can normalize v_n, we obtain a sequence $(u_k)_{k\in\mathbb{N}}$ in K such that

$$\|Du_k\|_{L^p(\Omega)} < \frac{1}{n}.$$

This procedure is justified since K is a cone.

As $(u_k)_{k\in\mathbb{N}}$ is bounded, there is a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ converging to some u in $L^p(\Omega)$.

$$\int_U \frac{\partial u_{n_k}}{\partial x_i}\varphi dx = -\int_U u_{n_k}\frac{\partial \varphi}{\partial x_i}dx,$$
$$\left|\int_U \frac{\partial u_{n_k}}{\partial x_i}\varphi dx\right| < \frac{A}{n_k},$$

for some constant A and a test function φ since U is bounded. By Fatou's lemma, we have that for each φ

$$\int_U u\frac{\partial \varphi}{\partial x_i}dx = 0.$$

Therefore, 0 is a weak derivative of u thus u is convergent in $W^{1,p}(U)$. Since K is closed we have $u \in K$. But $Du = 0$ implies that $u = 0$. This is a contradiction. Thus the inequality holds.

Problem 1.2

Let $\Omega \subset \mathbb{R}^n$ be open with smooth boundary.

(i) Let $f \in L^2(\Omega)$ and show that the problem

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$ and that this solution satisfies for some constant $C > 0$

$$\|\nabla u\|_{L^2} \leqslant C\|f\|_{L^2}.$$

Proof. To derive the weak formulation of this problem, consider $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega (-\Delta w)\varphi dx = \int_\Omega \nabla w \cdot \nabla\varphi dx = \int_\Omega f\varphi dx.$$

So the weak formulation of this problem is to find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

To this end consider the bilinear form $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ given by

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v dx \quad \forall u, v \in H_0^1(\Omega),$$

and the linear functional $f^* : H_0^1(\Omega) \to \mathbb{R}$ given by

$$\langle f^*, v \rangle = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega).$$

B is clearly bilinear and $\forall u, v \in H_0^1(\Omega)$, we have

$$|B(u,v)| = \left| \int_\Omega \nabla u \cdot \nabla v dx \right| \leqslant \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} = \|u\|_{H_0^1} \|v\|_{H_0^1}, \tag{1}$$

$$|B(u,u)| = \int_\Omega |\nabla u|^2 dx = \|u\|_{H_0^1}^2. \tag{2}$$

This proves that B is bounded and coercive.
Similarly, for all $v \in H_0^1(\Omega)$, we have

$$|\langle f^*, v \rangle| = \left| \int_\Omega f v dx \right| \leqslant \|f\|_{L^2} \|v\|_{L^2} \leqslant C_{Poi} \|f\|_{L^2} \|v\|_{H_0^1},$$

where C_{Poi} is the constant coming from Poincare's inequality, this proves that f^* is a continuous linear functional on $H_0^1(\Omega)$.
Since $H_0^1(\Omega)$ is a Hilbert space, we can apply Lax-Milgram's theorem to conclude that there exists a unique $u \in H_0^1(\Omega)$ such that

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx = \langle f^*, v \rangle \quad \forall v \in H_0^1(\Omega).$$

This weak solution further satisfies

$$\|\nabla u\|_{L^2}^2 = \int_\Omega |\nabla u|^2 dx = \int_\Omega f u dx \leqslant \|f\|_{L^2} \|u\|_{L^2} \leqslant C_{Poi} \|f\|_{L^2} \|\nabla u\|_{L^2},$$

which in turns yields

$$\|\nabla u\|_{L^2} \leqslant C_{Poi} \|f\|_{L^2}.$$

□

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Show that

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$ provided that the Lipschitz constant of f is sufficiently small.

Proof. Since f is Lipschtiz, for all $x \in \mathbb{R}$ we have

$$|f(x)| \leqslant |f(0)| + \mathrm{Lip}(f)|x|.$$

From this we deduce that for all $v \in L^2(\Omega)$, we have $f \circ v \in L^2(\Omega)$ since Ω is bounded (therefore of finite measure, so constant function are square integrable).
Now using question (i), we can define the map $T : L^2(\Omega) \to L^2(\Omega)$, where for any $v \in L^2(\Omega)$, we have $T(v)$ is the unique weak solution in $H_0^1(\Omega)$ to the problem

$$\begin{cases} -\Delta u = f(v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Given $v, w \in L^2(\Omega)$, we have that $T(v) - T(w)$ is the unique weak solution in $H_0^1(\Omega)$ to the problem

$$\begin{cases} -\Delta u = f(v) - f(w) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the estimate of question (i), we get

$$\|\nabla T(v) - \nabla T(w)\|_{L^2} \leqslant C_{Poi} \|f(v) - f(w)\|_{L^2} \leqslant C_{Poi} \,\mathrm{Lip}(f) \|v - w\|_{L^2}.$$

Using Poincare's inequality we thus get

$$\|T(v) - T(w)\|_{L^2} \leqslant C_{Poi} \|\nabla T(v) - \nabla T(w)\|_{L^2} \leqslant C_{Poi}^2 \,\mathrm{Lip}(f) \|v - w\|_{L^2}.$$

$L^2(\Omega)$ is a Hilbert space thus a complete metric space. If $C_{Poi}^2 \,\mathrm{Lip}(f) < 1$, we can apply Banach's fixed point theorem, to deduce that there is a unique fixed point $u \in L^2(\Omega)$. Since $T(v) \in H_0^1(\Omega)$ for all $v \in L^2(\Omega)$, we get that $u = T(u) \in H_0^1(\Omega)$. $\square$

Problem 1.3
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.
Recall the following version of the maximum/minimum principle: let $b, c \in C^0(\overline{\Omega}),\ c \geqslant 0$ and assume that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satsifies

$$-\Delta u + b \cdot \nabla u + cu \geqslant 0 \text{ in } \Omega.$$

Then

$$\min_{\overline{\Omega}} u \geqslant -\max_{\partial\Omega} u_-.$$

(i) Show that the unique solution of

$$\begin{cases} \Delta u = u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in $C^2(\Omega) \cap C^0(\overline{\Omega})$ is $u \equiv 0$.

Proof. Set $b = 0$ and $c = u^2 \geqslant 0$, then $b, c \in C^0(\overline{\Omega})$ and we have

$$-\Delta u + u^3 = -\Delta u + b \cdot \nabla u + cu = 0 \text{ in } \Omega.$$

Using the maximum/minimum principle, we get that

$$\min_{\overline{\Omega}} u \geqslant -\max_{\partial\Omega} u_- = 0.$$

By applying the same reasoning to $-u$, we deduce that

$$\max_{\overline{\Omega}} u = -\min_{\overline{\Omega}}(-u) \leqslant -\max_{\partial\Omega}(-u)_- = \min_{\partial\Omega} -u_+ = 0.$$

Combining these two inequalities gives that $u \equiv 0$. $\square$

(ii) Show that the problem

$$\begin{cases} \Delta u = u^2 & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution in $C^2(\Omega) \cap C^0(\overline{\Omega})$. Is the same true if the constraint $u \geqslant 0$ is dropped?

Proof. Set $b = 0$ and $c = u \geqslant 0$, then $b, c \in C^0(\overline{\Omega})$ and we have

$$-\Delta(-u) + u(-u) = -\Delta(-u) + b \cdot \nabla(-u) + c(-u) = 0 \text{ in } \Omega.$$

Using the maximum/minimum principle, we get that

$$\min_{\overline{\Omega}}(-u) \geqslant -\max_{\partial\Omega}(-u)_-,$$

which is equivalent to

$$\max_{\overline{\Omega}} u \leqslant \max_{\partial\Omega} u_+ = 0.$$

Combined with the constraint $u \geqslant 0$ in Ω, we get that the only solution in $C^2(\Omega) \cap C^0(\overline{\Omega})$ to the problem is $u \equiv 0$.
If the constraint $u \geqslant 0$ is dropped, we get the equation

$$\begin{cases} \Delta u = u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem does not always have a unique solution. Consider the case $n = 1$, then the equation reduces to

$$\begin{cases} u'' = u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

First consider the Cauchy problem with initial value

$$\begin{cases} u''(x) = u^2(x) & x \geqslant 0, \\ u(0) = 0, \ u'(0) = -1. \end{cases}$$

By the Cauchy-Lipschitz theorem, there is a maximal solution u which is C^2 on some open interval $I \subset \mathbb{R}$ containing 0.
Since $u''(x) = u^2(x) \geqslant 0$ for all $x \in I$, the solution u is convex. We know that $u(0) = 0$ and $u'(0) = -1$. This implies that u is decreasing on some interval $(0, \delta)$ for some $\delta > 0$. We claim that

there exists $x_0 > 0$ such that $u(x_0) = 0$. Suppose this were not the case, then since u is convex, it is necessarily non increasing on $I \cap (0, \infty)$ (if this were not the case, since u' is non decreasing, we would have $u'(x) > 0$ for some $x > 0$, which in turn would imply $\lim_{x \to \sup I} u(x) = +\infty$). Now there are two possibilities, either $\lim_{x \to \sup I} u(x) = -\infty$ or $\lim_{x \to \sup I} u(x) = l$ for some $l < 0$. In either case, using $u'' = u^2$ in I, this implies that $u''(x) > \epsilon$ for all $x \in I$ large enough and for some $\epsilon > 0$. But this would imply that $\lim_{x \to \sup I} u'(x) = +\infty$ which contradicts the fact that u is non increasing.
This proves that there exists $x_0 > 0$ such that $u(x_0) = 0$. By taking $\Omega = (0, x_0)$, the above u is a non zero solution to

$$\begin{cases} u''(x) = u^2(x) & x \in (0, x_0), \\ u(x) = 0 & x \in \{0, x_0\}. \end{cases}$$

$\square$

Exercise 4

(i)

Suppose if we have a weak solution, by integration by parts, we get

$$\begin{aligned} \int_U fv &= -\int_\Omega \Delta uv dx, \\ &= \int_\Omega \nabla u \cdot \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v dx, \\ &= \int_\Omega \nabla u \cdot \nabla v dx - \int_{\partial\Omega} \beta uv dx. \end{aligned}$$

for any $v \in H^1(\Omega)$. We define a bilinear form $B(u, v) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ such that

$$B(u, v) = \int_\Omega \nabla u \cdot \nabla v dx - \int_{\partial\Omega} \beta uv dx.$$

By trace theorem, we have

$$\begin{aligned} |B(u, v)| &\leq \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} + |\beta| \|u\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)}, \\ &\leq \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} + |\beta| C^2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \end{aligned}$$

for some $C > 0$. By the definition, we have

$$\|u\|^2_{H^1(\Omega)} = \|Du\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}.$$

Thus we conclude

$$|B(u, v)| \leq (|\beta| C^2 + 1) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

, and this satisfies the first condition for Lax-Milgram theorem.

In the case $\beta \leq 0$, we will prove that B satisfies the second condition as well. In order to do so, we will derive a contradiction by assuming it does not satisfy the condition. If that is the case the it is equivalent to say that for any $n \in \mathbb{N}$, there is $u_n \in H^1(\Omega)$ such that

$$B(u_n, u_n) < \frac{1}{n}\|u_n\|^2_{H^1(\Omega)}.$$

We can normalize u_n and conclude that

$$B(u_n, u_n) < \frac{1}{n}.$$

Since $\partial\Omega$ is a unit circle around the center, it is Lipschitz. As the sequence is bounded, it contains a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ that converges to u in $L^2(\Omega)$ thus weak derivative $(Du_{n_k})_{k\in\mathbb{N}}$ converges weakly.

Now we derive a contradiction by

$$B(u_{n_k}, u_{n_k}) = \|Du_{n_k}\|^2_{L^2(\Omega)} - \beta\int_{\partial\Omega}(Tu_{n_k})^2 \to 0.$$

This implies

$$\|Du\|^2_{L^2(\Omega)} = 0$$

Therefore, u is a constant. On the other hands, we have that

$$u|_{\partial\Omega} = 0.$$

This is a contradiction as $\|u\|_{H^1(\Omega)} = 1$. We conclude that B satisfies the conditions for Lax-Milgram theorem, therefore has a unique solution.

(ii)

Let (r, θ) be the polar coordinate, then we have

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Suppose u is in the form $u(r,\theta) = R(r)A(\theta)$. Then we have

$$\Delta u = (R''(r) + \frac{1}{r}R'(r))A(\theta) + R(r)\frac{1}{r^2}A''(\theta) = 0.$$

Therefore, transforming the equation, we derive,

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{A''(\theta)}{A(\theta)}.$$

Each side has a different variable. Thus this is equal to a constant λ. First obviously

$$A(\theta) = a_1e^{\sqrt{\lambda}} + a_2e^{-\sqrt{\lambda}}.$$

for some constants, a_1, a_2. And for R,

$$r^2R''(r) + rR'(r) = \lambda R(r)$$

By substituting $R(r) = r^\alpha$, we obtain

$$\alpha(\alpha - 1) + \alpha = \lambda.$$

Thus $\lambda = \alpha^2$.

First assumer $\lambda \geq 0$ then

$$R(r) = r^{\sqrt{\lambda}}, A(\theta) = a_1 \cos(\sqrt{\lambda}\theta) + a_2 \sin(\sqrt{\lambda}\theta).$$

For the boundary condition, we have $r = 1$ and

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos\theta + \frac{\partial u}{\partial y} \sin\theta.$$

Substituting this to the condition we get,

$$R'(1)A(\theta) = \beta R(1)A(\theta).$$

Where $R(1) = 1, R'(1) = \sqrt{\lambda}$. Thus for $\beta > 0$,

$$R(r)A(\theta) = r^\beta(a_1 \cos(\beta\theta) + a_2 \sin\beta\theta).$$