V4A1 Sheet 5

So Murata

2024/2025 Winter Semester - Uni Bonn

Problem 4.1

Using the density argument, we prove the statement holds for any $u \in \mathscr{C}_C^\infty(\Omega)$. Since Φ is a smooth function on a bounded domain, its jacobian is also bounded. In other words, for each i, j $\frac{\partial_i \Phi}{\partial x_j}$ is bounded and continuous. Let $\varphi \in \mathscr{C}^\infty(B_1(0) \cap \mathbb{H}^n)$.

$$\int (u\eta) \circ \Phi \partial_{x_i} \varphi dx = \int \sum_{k=1}^{n} (\frac{\partial u}{\partial x_k} \circ \Phi) \frac{\partial \Phi_k}{\partial x_i} \varphi.$$

Since $\frac{\partial u}{\partial x_k}, \frac{\partial \Phi_k}{\partial x_i}$ are continuous on the bounded domain. We have proven the theorem.

Problem 4.2

By the Weak formulation of the problem we get that for any $\varphi \in \mathscr{C}_C^\infty(\Omega)$ we have

$$\int_\Omega \sum_{i,j=1}^{n} A^{i,j} D_i u D_j \varphi dx = \int_\Omega f \varphi dx. \tag{1}$$

Let $\eta : \omega \to \mathbb{R}$ be a cutoff function such that

$$\eta(x) = \Big\{ 1, \quad (x \in B(x_0, r)), 0, \quad (x \notin B(x_0, R)).$$

for some $0 < r < R$, with $|D\eta| \leq \frac{C}{R-r}$.

Without the loss of generality, we may assume $\lambda = 0$ as differentials of constants are 0. Let $\varphi = u\eta^2$. By substituting φ into Equation (**??**), we get

$$\begin{aligned}
\int_\Omega \sum_{i,j=1}^n A^{i,j} D_i u((D_j u)\eta^2 + 2(D_j\eta)u\eta)dx &= \int_\Omega fu\eta^2 dx, \\
&= \int_\Omega \left(\sum_{i,j=1}^n A^{i,j} D_i u\eta D_j u\eta \right) dx \\
&+ \int_\Omega \left(\sum_{i,j=1}^n A^{i,j} D_i u \cdot 2(D_j\eta) \right) u\eta dx \\
&= I_1 + I_2.
\end{aligned}$$

Since A is uniformly elliptic, we find a constant $\theta > 0$ such that

$$\theta \int_{B_r(x_0)} |\nabla u|^2 dx = \theta \int_\Omega |\nabla u\eta|^2 dx \leq \int_\Omega \left(\sum_{i,j=1}^n A^{i,j} D_i u\eta D_j u\eta \right) dx = I_1.$$

Since A is bounded, Cauchy-Schwarz we get,

$$|I_2| \leq 2\|A\|_{L^\infty} \left(\int_\Omega (\eta|Du|)^2 \right)^{\frac{1}{2}} \left(\int_\Omega (|D\eta|u)^2 dx \right)^{\frac{1}{2}}.$$

By using Young's inequality with $\varepsilon = \theta$ we get

$$|I_2| \leq \frac{\varepsilon}{2} \left(\int_\Omega (\eta|Du|)^2 \right) + \frac{\|A\|_{L^\infty}}{\varepsilon} \left(\int_\Omega (|D\eta|u)^2 dx \right).$$

By condition on $D\eta$ we get

$$|I_2| \leq \frac{\varepsilon}{2} \left(\int_\Omega (\eta|Du|)^2 \right) + \frac{\|A\|_{L^\infty}}{\varepsilon} \frac{C^2}{(R-r)^2} \left(\int_{B_R(x_0)\setminus B_r(x_0)} u^2 dx \right).$$

We observe that

$$f(x_1, \cdots, x_n) = \partial x_1 \int_{(x_0)_1}^{x_1} f(t, x_2, \cdots, x_n) dx$$

By letting $F(x) = \int_{(x_0)_1}^{x_1} f(t, x_2, \cdots, x_n)dx$ we obtain

$$\int_\Omega fu\eta^2 dx = \int_\Omega D_1 uF\eta^2 + 2D_1\eta Fu\eta dx.$$

By using Cauchy-Schwarz and Young's inequality, we derive

$$
\begin{aligned}
|\int_\Omega fu\eta^2 dx| \leq & \frac{\xi}{2}\int_{B_R(x_0)} |Du|^2 dx + \frac{1}{2\xi}\int_{B_R(x_0)} F^2 dx \\
& + \frac{\rho C^2}{(R-r)^2}\int_{B_R(x_0)\setminus B_r(x_0)} |u|^2 dx + \frac{1}{2\rho}\int_{B_R(x_0)\setminus B_r(x_0)} F^2 dx.
\end{aligned}
$$

Using Jensen's inequality we get

$$
\int_{B_R(x_0)} F^2 dx \leq cR^2 \int_{B_R(x_0)} f^2 dx.
$$

Combining these we get

$$
\theta\|\nabla u\|^2 \leq \frac{1}{2}(\varepsilon+\xi)\|\nabla u\|^2 + \frac{C^2}{(R-r)^2}(\rho + \frac{\|A\|_{L^\infty}}{\varepsilon})\int_{B_R(x_0)\setminus B_r(x_0)} |u|^2 dx + \frac{cR^2}{\min\{\xi,\rho\}}\int_{B_R(x_0)} f^2 dx.
$$

Let $\varepsilon = \xi = \rho = \frac{\theta}{2}$ and

$$
C = \sqrt{\frac{4}{\theta^2}c(1+\frac{4}{\theta^2}\|A\|_{L^\infty})^{-1}}, \quad C' = \frac{4}{\theta^2}c.
$$

We get the claim

$$
\|\nabla u\|^2 \leq \frac{C'}{(R-r)^2}\int_{B_R(x_0)\setminus B_r(x_0)} |u|^2 dx + C'R^2\int_{B_R(x_0)} f^2 dx.
$$