Representation Theory 1 V4A3

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1 Overview of the material

1.1 Lie groups

Definition 1.1. *A Lie group is a group G whose underlying set is endowed with the structure of smooth manifolds such that multiplication and inversions are smooth maps.*

Definition 1.2. *A topological group is a group G whose underlying set is endowed with the structure of topological space such that multiplication and inversions are continuous.*

2 Preliminaries

2.1 Topology

Definition 2.1. *We have two axioms about the topological spaces*

1. *T_0(Komogolov) : Given any 2 points, there exists an open set such that it contains one of them but not both.*
2. *T_1(Hausdorff) : Given any 2 points, there exist disjoints open set that each contains one of them.*

Definition 2.2. *A topological space is second countable if it has a basis which contains at most countably many subsets.*

3 Lie groups

3.1 Manifolds

Definition 3.1. *Let $f : X \to Y$ be a mapping between two topological spaces X, Y. f is called a homeomorphism if*

1. *f is a bijection,*
2. *f is continuous,*

3. f^{-1} is also continuous.

Definition 3.2. *Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open sets and $f : U \to V$ be a smooth map. Then the derivative of f at $p \in U$ is*

$$df(p) = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}.$$

Proposition 3.1. *Let $f : U \to V, g : V \to W$ be smooth maps. Then for $p \in U$ we have*

$$d(g \circ f) = dg(f(p))df(p).$$

Definition 3.3. *Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open sets. A map $f : U \to V$ is called a diffeomorphism if*

i). f is smooth. ($\Leftrightarrow$ arbitrary order of partial derivatives exists),

ii). f^{-1} is defined and is also a smooth map.

Definition 3.4. *Let X be a topological space. A chart on X is a homeomorphism $h : U \to V$ where $U \subseteq X$ is open and $V \subseteq \mathbb{R}^n$ is open.*

Definition 3.5. *An atlas $\mathscr{A}$ on a topological space X is a collection of charts $\{h_\lambda \mid h_\lambda : U_\lambda \to V_\lambda\}_{\lambda \in \Lambda}$ such that $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of X.*

Definition 3.6. *An atlas $\mathscr{A}$ of X is said to be smooth if for any two charts $h_1 : U_1 \to V_2, h_2 : U_2 \to V_2$. The following,*

$$h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2),$$

is a smooth map. Such map is called a transition map.

Definition 3.7. *Let X be a topological space and $\mathscr{A}_1, \mathscr{A}_2$ be smooth atlases. We say they are equivalent if $\mathscr{A}_1 \cup \mathscr{A}_2$ is also smooth.*

Proposition 3.2. *Above definition indeed defines an equivalence relation.*

Proof. For any $h_1 \in \mathscr{A}_1, h_2 \in \mathscr{A}_2, h_3 \in \mathscr{A}_3$,

$$h_3 \circ h_1^{-1} = h_3 \circ h_2^{-1} \circ h_2 \circ h_1^{-1}.$$

□

Definition 3.8. *A smooth manifold is a second countable Hausdorff topological space with equivalence classes of smooth atlases.*

Definition 3.9. *Let M, N be smooth manifolds, $f : M \to N$ be a map, and $p \in M$. f is said to be smooth at p if for one (hence any) pair of charts around p and $f(p)$,*

$$h_M : U_M \to V_M, h_N : U_N \to V_N,$$

the composed function

$$h_N \circ f \circ h_M^{-1} : V_M \to V_N$$

is smooth at $h_M(p)$.

Remark 3.1. *We can define a function* $\dim : M \to N$ *such that*

$$\dim(p) = \dim(V)_p,$$

for any chart $h : U \to V$ *around* p*. And this function is locally constant. In particular, if* M *is connected then it has a well-defined dimensions.*

Definition 3.10. *Let* M, N *be smooth manifold and* $f : M \to N$ *be a mapping which is smooth at* $p \in M$*. For any charts,*

$$h_N \circ f \circ h_M^{-1} : V_M \to V_N,$$

the rank of f *at* p *is such that*

$$\mathrm{rk}(f; p) = \mathrm{rank}(\mathbf{df}(h_M(p))(h_N \circ f \circ h_M^{-1})).$$

Definition 3.11. *Let* M, N *be smooth manifolds and* $f : M \to N$ *be a smooth map. A point* p *is said to be regular with respect to the map f. And a point* $q \in N$ *is called a regular value if all* $p \in f^{-1}(q)$ *are regular.*

Definition 3.12. *Let* M *be a manifold. A subset* $N \subseteq M$ *is called an embedded submanifold if for any point* $p \in N$*, there is a chart* $h_M : U_M \to V_M$ *around* p *such that*

$$h_M|_N : U_M \cap N \to V_M \cap \mathscr{R}^n,$$

is a diffeomorphism where n *is the dimension of* N*.*

In particular, an embedded submanifold of an euclidean space is called a embedded manifold.

Definition 3.13. *A map* $f : M \to N$ *of smooth manifolds is called a diffeomorphism if*

i). $f : M \to N$ *is a bijection,*

ii). f, f^{-1} *are both smooth.*

Theorem 3.1. *Let* $f : M \to N$ *be a smooth map between manifolds, and* $q \in N$ *be a regular value. Then* $f^{-1}(q) \subset M$ *is an embedded submanifold.*

Theorem 3.2. *Let* $f : M \to N$ *be a smooth map of manifolds* $p \in M$ *be a regular point, and* $\dim(p) = \dim(f(p))$*. Then* f *is a local diffeomorphism of* p*. In other words, there is a neighborhood* U_M *of* p *in* M *and* $f(p) \in U_N \subset N$ *such that*

$$f|_{U_M} : U_M \to U_N,$$

is a diffeomorphism.

Definition 3.14. *Let* $M \subseteq \mathbb{R}^n$ *be an embedded manifold such that for some open set* $U \subset \mathbb{R}^n$*, there is* $V \subset \mathbb{R}^n$ *such that*

$$h : U \to V, \quad h_M : U \cap M \to V \cap \mathbb{R}^m,$$

is a diffeomorphism where h_M is defined to be taking the first m coordinate of the points in V. (Thus $m \leq n$).

The tangent space T_pM of M at p is the subspace of $\mathbb{R}^n$ such that

$$(\mathbf{dh}(p))^{-1}(\mathbb{R}^m) \subset \mathbb{R}^n.$$

There are three definitions of tangent spaces and they are all equivalent. However, each of them has its own advantages.

Definition 3.15 (Coordinate tangent space)**.** *Given a smooth manifold M and a point $p \in M$. The coordinate tangent space of p is such that*

$$T_p^{\mathbf{Coo}}M = \{(h, v) \mid h : U \to V \text{ is a chart}, v \in \mathbb{R}^m\}/\sim .$$

Where $\sim$ is an equivalence relation such that

$$(h_1, v_1) \sim (h_2, v_2) \text{ if } (\mathbf{d}(h_2 \circ h_1^{-1})(h_1(p)))(v_1) = v_2.$$

Definition 3.16. *Given a smooth manifold M, a point $p \in M$, and a smooth map $\alpha : I \to M$ whose domain I is an open interval contains 0. α is called a smooth curve if $\alpha(0) = p$.*

Definition 3.17. *Two smooth curves $\alpha, \beta : I \to M$ through p are said to be tangentially equivalent if for one (hence any) charts $h : U \to V$ around p, we have*

$$d(h \circ \alpha)(0) = d(h \circ \beta)(0).$$

We denote such relation as $\sim_T$.

Definition 3.18 (Geometric tangent space)**.** *The geometric tangent space at p of a smooth manifold M is such that*

$$T_p^{\mathbf{Geo}} = \{\alpha : I \to M \mid \alpha \text{ is a smooth curve}\}/\sim_T .$$

Definition 3.19. *A germ of smooth functions of manifolds M at p is an equivalence class of tuples (U, f) where*

i). $U \subset M$ is a neighborhood of p,

ii). $f : U \to \mathbb{R}$ is smooth,

and two tuples $(U_1, f_1), (U_2, f_2)$ are equivalent if there is a neighborhood V of p such that $V \in U_1 \cap U_2$ and $f_1|_V = f_2|_V$.

And we denote the set of germs at p as

$$\mathscr{C}^\infty(p).$$

Remark 3.2. *$\mathscr{C}^\infty(U, \mathbb{R})$ and $\mathscr{C}^\infty(p)$ are rings, in fact $\mathbb{R}$-algebras.*

Definition 3.20. *Let R be a ring and A be a bimodule over R. A R-derivation in A is an operator $X : A \to A$ such that the Leibniz rule holds. In other words,*

$$X(ab) = aX(b) + X(a)b,$$

holds for all $a, b \in A$.

Definition 3.21 (Algebraic tangent space)**.** *The algebraic tangent space $T_p^{\mathbf{Alg}}M$ of M at p is the set of $\mathbb{R}$-derivations $X : \mathscr{C}^\infty(p) \to \mathbb{R}$.*

Remark 3.3. *In the above definition, $\mathbb{R}$ is considered as a $\mathscr{C}^\infty(p)$-bimodule via the evaluation map $f \mapsto f(p)$.*

Theorem 3.3. *The following are isomorphisms of $\mathscr{R}$-vector spaces.*

$$\begin{aligned}
T_p^{\mathbf{Geo}}M &\to T_p^{\mathbf{Alg}}M, \alpha \mapsto (f \mapsto (f \circ \alpha)'(0)),\\
T_p^{\mathbf{Alg}}M &\to T_p^{\mathbf{Coo}}M, X \mapsto (h, ((Xh_i)(p))_{i=1,\cdots,n}),\\
T_p^{\mathbf{Coo}}M &\to T_p^{\mathbf{Geo}}M, (h, v) \mapsto \alpha(t) = h^{-1}(h(p) + t \cdot v).
\end{aligned}$$

Proposition 3.3. *$\mathscr{C}^\infty(p)$ is a local ring with its maximal ideal*

$$\mathfrak{m}_p = \{f \in \mathscr{C}^\infty(p) \mid f(p) = 0\}.$$

Moreover, if we have a derivation $X : \mathscr{C}^\infty(p) \to \mathbb{R}$, the restricted derivation $X|_{\mathfrak{m}_p}$ is in $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2)$. And by this restriction, we get an isomorphism between $T_p^{\mathbf{Alg}}M$ and $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R})$.

Remark 3.4. *In this way, a smooth manifold is recognized as a locally ringed space, locally isomorphic to $\mathbb{R}^n$.*

Remark 3.5. *Let V be a finite dimensional $\mathbb{R}$-vector space. It has a tautological smooth manifold structure by taking charts such that the sets of isomorphisms of V and $\mathbb{R}^n$ given by arbitrary basis of V.*

We claim that we have canonical isomorphisms

$$T_pV \to V,$$

for any $p \in V$,

$$\begin{aligned}
V &\to T_p^{\mathbf{Coo}}V, v \mapsto (h, h(v)),\\
V &\to T_p^{\mathbf{Geo}}V, v \mapsto (t \mapsto p + tv),\\
V &\to T_p^{\mathbf{Alg}}V, v \mapsto \left(f \mapsto \frac{d}{dt}\bigg|_{t=0} f(p + tv)\right)
\end{aligned}$$

Definition 3.22. *Let $f : M \to N$ be a map of smooth manifolds which is smooth at $p \in M$. Its differential of p is the linear map*

$$\mathbf{d}f(p) = \mathbf{d}_p(f) : T_pM \to T_{f(p)}N,$$

defined as follows.

1). Geometric tangent space : $\mathbf{d}_p(f)(\alpha) = f \circ \alpha$ *where* α *is a smooth curve.*

2). Algebraic tangent space : $\mathbf{d}_p(f)(X)(\varphi) = X(\varphi \circ f)$ *where* $\varphi \in \mathscr{C}^\infty(f(p))$*.*

3). Coordinate tangent space : $\mathbf{d}_p(f)(h_M, v_M) = (h_N, d_{h_M(p)}(h_N)$*.*

Remark 3.6. *Given a chart* $h : U \to V$ *around* $p \in M$*.* h *consists of coordinate functions* h_i *where* $1 \leq i \leq m$ *for* $V \subset \mathbb{R}^m$*. We have for each* i

$$\mathbf{d}_p h_i : T_pM \to \mathbb{R},$$

and

$$B = \{d_p h_1, \cdots, d_p h_m\}$$

is a basis of the dual space $(T_pM)^*$*.*

Let

$$\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\}$$

be the dual basis of B*. By definition, this means that for any* $1 \leq i, j \leq m$*, we have*

$$\frac{\partial}{\partial x_i} h_j = d_p h_j(\frac{\partial}{\partial x_i}) = \delta_{ij}.$$

Proposition 3.4. *Let* $f : M \to N$ *be a map between smooth manifolds which is smooth and* $q \in N$ *be a regular value. For* $p \in f^{-1}(q)$*, we have*

$$T_p f^{-1}(q) = \mathbf{d}_p(f)^{-1}(0) \subset T_pM.$$

Proof. □

3.2 Immersions and Submersions

Definition 3.23. *Let* $f : M \to N$ *be a smooth map of smooth manifolds.* f *is called an*

1). immersion if $\mathbf{d}f : T_pM \to T_{f(p)}N$ *is injective for any* $p \in M$*,*

2). submersion, if $\mathbf{d}f(p) : T_pM \to T_{f(p)}N$ *is surjective for any* $p \in M$*.*

Remark 3.7. *An immersion need not be injective. The counter example is*

$$e^{ix} : \mathbb{R} \to S^1,$$

is an immersion.

Remark 3.8. *A submersion need not be injective. The counter example is*

$$i_U : U \to M,$$

an inclusion map is a submersion.

Remark 3.9. *We know that if f is a submersion, then $f^{-1}(q)$ is an embedded submanifold. However, if f is an immersion, even it is injective, $f(M)$ need not be an embedded submanifold of N.*

Definition 3.24. *An immersed submanifold is an image of an injective immersion.*

Remark 3.10. *We endow $f(M)$ with the transported topology and differential structure from M so that f becomes a diffeomorphism between M and $f(M)$. But this topology need not be the relative topology from N. It may be strictly finite.*

Example 3.1. *Let $T = S^1 \times S^1$ be a torus. Let $r \in \mathbb{R}$. We consider a map $f : \mathbb{R} \to T$ such that*

$$f(x) = (e^{2\pi tx}, e^{2\pi rix}).$$

This is an immersion for any r. We examine this by several cases.

First, when r is not a rational number then f is injective, the image is an immersed manifold. However, a copy of $\mathbb{R}$. But this image is a dense subset of the torus.

Second, if r is rational then f is not injective. It is going to factor through an injective immersion $\mathbb{R}/b\mathbb{Z} \to T$ where $r = \frac{a}{b}$, $a, b \in \mathbb{Z}$ are coprime. This image is not only immersed but also embedded.

Remark 3.11. *If $f : M \to N$ is an immersion, $\mathbf{d}f_(p)$ identifies T_pM with a linear subspace of $T_{f(p)}N$.*

Proposition 3.5. *If $f : M \to N$ is an injective immersion, that is also closed subset of N, then its image is an embedded submanifold.*

Remark 3.12. *Thus we have the notion of a closed submanifold.*

3.3 Multi-linear forms

Definition 3.25. *Let $\mathbb{V}$ be a vector space and $\varphi : \bigoplus_{i=1}^m V \to \mathbb{R}$ is called a m-multi-linear function if for any $i = 1, \cdots, m$ and $\{a_j\}_{j \neq i} \subset V$ we have*

$$\varphi(a_1, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_m) : V \to \mathbb{R}$$

is a linear function

Definition 3.26. *Let X be a smooth n-dimensional manifold and $m \in \mathbb{N}$. Then we define the followings*

1. $\mathscr{L}_p^m = \{\varphi : \bigoplus_{i=1}^m T_pX \to \mathbb{R} | \varphi$ *is a m-multi-linear function.*$\}$
2. $\mathscr{L}^m = \bigcup_{p \in X} \mathscr{L}_p^m$

Definition 3.27. *Let X be a smooth n-dimensional manifold. A map $V : X \to \mathscr{L}^m$ is called a m-tensorfield if*

i. For any $p \in X$, $V(p) \in \mathscr{L}^m_p$.

ii. For any chart (U, φ) around p with a basis $\{e^\varphi_1, \cdots, e^\varphi_n\}$ and for any $i_1, \cdots, i_m \in \{1, \cdots, n\}$ we have a map $V_{(i_1,\cdots,i_m)} : X \to \mathbb{R}$ such that $V_{(i_1,\cdots,i_m)}(p) = V(p)(\underline{e}_{i_1}, \cdots, \underline{e}_{i_m})$ is smooth.

Proposition 3.6. *For any m tensorfield V, we have*

Definition 3.28. *We define $\mathscr{V}^m(X)$ to be the set of all m-tensorfield.*

Proposition 3.7. *$\mathscr{V}^m(X)$ is a vector space over $\mathbb{R}$ and a module over $\mathscr{F}(X)$ with the common basis $\{E_{i_1,\cdots,i_m}\}_{i_1,\cdots,i_m \in \{1,\cdots,n\}}$*

Proposition 3.8. *Let X be a smooth n-dimensional manifold and $V : X \to \mathscr{L}^m$ be such that for any $p \in X, V(p) \in \mathscr{L}^m_p$ the followings are equivalent.*

1. V is a m-tensorfield.

2. For any chart (U, φ) around p with basis $\{\underline{e}^\varphi_1, \cdots, \underline{e}^\varphi_n\}$ and for any $1 \leq i_1, \cdots, i_m \leq n$ there exist smooth mappings $\lambda_{i_1,\cdots,i_m} : X \to \mathbb{R}$ such that $V(p) = \sum_{1\leq i_1,\cdots,i_m\leq n} \lambda_{i_1,\cdots,i_m}(p) E^\varphi_{i_1,\cdots,i_m}$.

3. For any vectorfields $v_1, \cdots, v_m : X \to TX$ we have a function $V : X \to \mathbb{R}$ such that $V_{v_1,\cdots,v_m}(p) = V(p)(v_1(p), \cdots, v_m(p))$ is smooth.

Proof. 1.⇔2. is trivial. 1.⇒3. is clear by the multi-linearity, and 3.⇒1. is choosing $v_i = e^\varphi_i$ for each $i = 1, \cdots, n$.

□

Proposition 3.9. *Let $V : X \to \mathscr{L}^m$ then thne followings are equivalent.*

1. V is a m-tensorfield.

2. For any $\{v_1, \cdots, v_m\} \in \mathscr{V}(X)$, $\Psi : \bigoplus_{i=1}^m \mathscr{V}(X) \to \mathscr{F}(X)$ such that $\Psi(v_1, \cdots, v_m)(p) = V(p)(v_1(p), \cdots, v_m(p))$ is smooth and $\mathscr{F}(X)$-linear.

Proof. 1.⇒2. follows from the multilinearity and decompositions of tensors. 2.⇒1. follows by fixing all element except one we still have the linearity thus, the function is mutilinear. □

3.4 Tensor and Wedge products

Definition 3.29. *Let $V_1 : X \to \mathscr{L}^r, V_2 : X \to \mathscr{L}^s$ be tensorfield. Then We define the tensorproduct $V_1 \otimes V_2 : X \to \mathscr{L}^{r+s}$ of them to be*

$$(V_1 \otimes V_2)(p)(v_1, \cdots, v_r, v_{r+1}, \cdots, v_{r+s}) = V_1(p)(v_1, \cdots, v_r)V_2(p)(v_{r+1}, \cdots, v_{r+s})$$

Proposition 3.10. *The operation $\bigotimes$ is bilinear and associative.*

Proof. By substituting values, they are trivial. $\square$

Proposition 3.11. *Let $U \subset X$ be an open set and $V_1, \cdots, V_n \in \mathscr{V}^1(U)$ be a basis in $\mathscr{V}^1(U)$ then $\{\bigotimes_{j=1}^{r} V_{i_j}\}_{1\le i_1,\cdots,i_r\le r}$ is a basis in $\mathscr{V}^r(U)$.*

Proof. Since $\otimes$ is bilinear, this is a tensor product thus the set in the statement is indeed a basis. $\square$

Definition 3.30. *Let $V \in \mathscr{V}^m(X)$ be a m-tensor. V is said to be alternating if for any $p \in X$, $(v_1, \cdots, v_m) \in \bigoplus_{i=1}^{m} T_pX$ and $\sigma \in \mathfrak{S}_m$ we have*

$$V(p)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) = \operatorname{sgn}(\sigma)V(p)(v_1, \cdots, v_m)$$

Furthermore, such V is called a m-form.

Notation 3.1. *The set of all m-forms is denoted by*

$$\mathscr{A}^m(X) = \{V \in \mathscr{V}^m(X) \mid V \text{ is a } m\text{-form.}\}$$

Definition 3.31. *Let $V_1 \in \mathscr{A}^r(X), V_2 \in \mathscr{A}^s(X)$ then the wedge product is*

$$(V_1 \wedge V_2)(p)(v_1, \cdots, v_{r+s}) = \frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\sigma) V_1 \otimes V_2(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)})$$

Proposition 3.12. *Let $V_1, \cdots, V_n \in \mathscr{A}^1(X)$, $p \in X$ and $v_1, \cdots, v_n \in T_pX$ then we have*

$$(V_1 \wedge \cdots \wedge V_n)(p)(v_1, \cdots, v_n) = \det(V_i(p)(v_j))_{i,j}$$

Proof.

$$(V_1 \wedge \cdots \wedge V_n)(p)(v_1, \cdots, v_n) = \frac{1}{1! \cdots 1!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} V_i(p)(v_\sigma(i))$$

$\square$

Proposition 3.13. *Similar to the case in tensorfields, we have the following statements.*

1. *$\mathscr{A}^m(X)$ is a subspace of $\mathscr{V}^m$ over $\mathbb{R}$.*
2. *$\mathscr{A}^m(X)$ is a module over $\mathscr{F}(X)$.*

Proof. Trivial. $\square$

Proposition 3.14. *Let $V_1 \in \mathscr{A}^r, V_2 \in \mathscr{A}^s$, then $V_1 \wedge V_2 \in \mathscr{A}^{r+s}$ and such $\wedge : \mathscr{A}^r \times \mathscr{A}^s \to \mathscr{A}^{r+s}$ is bilinear.*

Proof. Bilinearity follows from the bilinearity of $\otimes$. We will show that this is indeed well-defined.

Let $\sigma \in \mathfrak{S}_{r+s}$. Then we have

$$\begin{aligned}(V_1 \wedge V_2)(p)(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)}) &= \frac{1}{r!s!} \sum_{\tau \in \mathfrak{S}_{r+s}} \mathrm{sgn}(\tau) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)}) \\ &= \mathrm{sgn}(\sigma) \frac{1}{r!s!} \sum_{\tau\circ\sigma \in \mathfrak{S}_{r+s}} \mathrm{sgn}(\tau\circ\sigma) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)}) \\ &= \mathrm{sgn}(\sigma)(V_1 \wedge V_2)(p)(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)})\end{aligned}$$

□

Proposition 3.15.

$$V_2 \wedge V_1 = (-1)^{rs}(V_1 \wedge V_2)$$

Proof. Let $\tau \in \mathfrak{S}_{r+s}$ to be such that

$$\tau(i) = \begin{cases} r+i & (1 \le i \le s) \\ i-s & (s+1 \le i \le r+s) \end{cases}$$

Then clearly the inversion number is $N(\tau) = rs$. It is also obvious that

$$V_2 \wedge V_1(p)(v_{\tau(1)}, \cdots, v_{\tau(r+s)}) = V_1 \wedge V_2(p)(v_1, \cdots, v_{r+s})$$

□

Proposition 3.16. *Let $V_1 \in \mathscr{A}^r, V_2 \in \mathscr{A}^s, V_3 \in \mathscr{A}^t$ then $(V_1 \wedge V_2) \wedge V_3 = V_1 \wedge (V_2 \wedge V_3)$.*

Proof.

$$\begin{aligned}(V_1 \wedge V_2) \wedge V_3(p)(v_1, \cdots, v_{r+s+t}) &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \mathrm{sgn}(\tau)(V_1 \wedge V_2) \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\ &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \mathrm{sgn}(\tau) \\ &\quad (\frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \mathrm{sgn}(\sigma) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)})) \\ &\quad V_3(v_{\sigma(r+s+1)}, \cdots, v_{\sigma(r+s+t)})\end{aligned}$$

If for $\tau_1, \tau_2 \in \mathfrak{S}_{r+s+t}, \sigma_1, \sigma_2 \in \mathfrak{S}_{r+s}$ we have $\tau_1 \circ \sigma_1 = \tau_2 \circ \sigma_2$ then they satisfy the followings

i. For any $r+s+1 \le i \le r+s+t$ we have $\tau_1(i) = \tau_2(i)$.

ii. From above we get $\tau_2^{-1} \circ \tau_1 \in \mathfrak{S}_{r+s}$

Fixing σ_1, there exists $(r+s)!$ many such σ_2. This implies that we can choose σ_1 to be the identity. Thus we get

$$\begin{aligned}(V_1 \wedge V_2) \wedge V_3(p)(v_1, \cdots, v_{r+s+t}) &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau)(V_1 \wedge V_2) \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\ &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau) \frac{(r+s)!}{r!s!} V_1 \oplus V_2 \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\ &= \frac{1}{r!s!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau) V_1 \oplus V_2 \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)})\end{aligned}$$

From the previous proposition we get □

4 Integration

Definition 4.1. *A differential k-form ω on a smooth manifold M is a collection $\omega(p) \in A^k(T_pM)$ for all $p \in M$.*

Remark 4.1. *We can define what it means for ω to be continuous or smooth at some points $p \in M$ as follows.*

First, we pick a chart $h : U \to V$ around p and get the basis

$$\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\},$$

of T_pM that moves with $p \in U$.
We also have a basis $A^k(T_pM) = \bigwedge^k (T_pM)^*$. Hence we can express ω as p in terms of that basis and the scalars in this expression are functions on U.

$$\omega(p) = \sum f_{i_1, \cdots, i_k} \cdot d_{x_{i_1}} \wedge \cdots \wedge d_{x_{i_k}}.$$

And we can require $f_{i_1, \cdots, i_k} \cdots d_{x_{i_1}}$ to be smooth/continuous at p.

Example 4.1. *If $M = \mathbb{R}^n$, we have the canonical identification,*

$$T_pM = \mathbb{R}^n.$$

This gives us standard differential form of degree n. which is given by

$$e_1^* \wedge \cdots \wedge e_n^*,$$

where $e_1, \cdots, e_n$ is the standard basis of $\mathbb{R}^n$.

Definition 4.2. *Let $f : M \to N$ be a smooth map of manifolds and ω be a differential form of degree k on N. We define $f^*(\omega)$ of degree k on M by*

$$f^*(w)(p)(x_1, \cdots, x_k) = \omega(f(p))(\mathbf{d}f_p(x_1), \cdots, \mathbf{d}f_p(x_k)).$$

Definition 4.3. *A differential n-form ω on M is said to be locally integrable if for any point $p \in M$, if for any point $p \in M$, there is one (hence any) chart $h : U \to V$ such that $\omega|_U =$*