Representation Theory 1 V4A3 Exercise Sheet 1

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Exercise 1

1.1 $\mathcal{C}^\infty$ is a locally ringed space

Let us define a set such that

$$\mathfrak{m}_{X,p} = \{(U, f) \mid f \in \mathcal{C}^\infty(U), f(p) = 0\}.$$

Since $f : U \to \mathbb{R}$ is a smooth map and $f(p) \neq 0$, then the quotient $\frac{1}{f}$ is smooth at p, we derive that $\mathfrak{m}_{X,p}$ is a unique maximal ideal of $\mathcal{O}_{X,p}$.

1.2 A smooth map induces a map of locally ringed space.

For any $x \in g^{-1}(V)$, we have $g(x) \in V$ by the definition, we obtain that

$$f|_V \circ g|_{g^{-1}(V)} = (f \circ g)|_{g^{-1}(V)}.$$

This shows that $g^\#$ is a natural transformation. Furthermore, for any $f : U \to \mathbb{R}$ such that $f(p) = 0$, we obtain for any $q \in g^{-1}(\{p\})$, $f \circ g(q) = 0$. Thus the image of maximal ideal under g is contained a maximal idea. This proves the claim.

1.3

Composition of g and each component of a chart is smooth thus a composition of g and any chart is smooth. This concludes that g is smooth.

Exercise 2

2.1

Let us define the product topology on $M \times N$, since it is a direct product of finitely many topological space, it coincides with the box topology. And atlases of it is defined as the product of two atlases, ie. for any atlas

$$\mathcal{A}_{M\times N} = \{h_M \times h_N : U_M \times U_N \to V_M \times V_N \mid h_M \in \mathcal{A}_M, h_N \in \mathcal{A}_N\}$$

for some atlases $\mathcal{A}_M$ and $\mathcal{A}_N$. By the construction of the product topology, such set is indeed an atlas.

Let $h : U \to V$ and $(h_M, h_N) : U_M \times U_N \to V_M \times V_N$ be charts of $M, M \times N$, respectively. Then

$$h \circ \pi_1 \circ (h_M, h_N)^{-1} = h \circ \pi_1 \circ (h_M^{-1}, h_N^{-1}) = h \circ h_M^{-1}.$$

By the assumption, this is smooth, therefore π_1 is a smooth map.

Given a map of smooth manifolds $f : M' \to M \times N$. f is smooth if and only if

$$(h_M, h_N) \circ f \circ h'^{-1}$$

is smooth for any charts. By the elementary tool from analysis we know that this means, the function is coordinate-wise smooth thus the above function is smooth if and only if

$$h_M \circ \pi_1 \circ f \circ h'^{-1}, h_N \circ \pi_2 \circ f \circ h'^{-1}$$

are smooth.

Since π_1, π_2 are both smooth, any topology on $M \times N$ with these criterions would contain the product topology on it. Since identity map of $M \times N$ is smooth, we obtain that such $M \times N$ is unique.

2.2

Using the coordinate tangent space we obtain that

$$((h_M, h_N), (v_M, v_n)) \mapsto (h_M, v_M) \times (h_N, v_N).$$

By the construction of atlases in $M \times N$, this map is a bijection.

Exercise 3

3.1

Let $\alpha, \beta : I \to G$ be smooth curves such that $\alpha(0) = \beta(0) = e$ and α, β are representations of equivalent classes X, Y in $T^{\mathbf{Geo}}_{(e,e)}(G \times G)$. Since derivatives are linear maps we obtain

$$d\mu_{(e,e)}(X, Y) = d\mu_{(e,e)}(X, e) + d\mu_{(e,e)}(e, Y).$$

By using Geometric tangent space we get

$$d\mu_{(e,e)}(X, e) = \frac{d}{dt}|_0 \mu(\alpha(t), e) = \alpha'(0) = X.$$

By applying this to Y we obtain

$$d\mu_{(e,e)}(X, Y) = X + Y.$$

3.2

Trivially we have $\mu(e,e) = e$ and the derivative is surjective as it is proven in the previous problem.
Let $X = h, Y = 0$ for any $h \in T_{g,g^{-1}}G$ then this is a surjection thus full rank. We use the implicit function theorem and conclude there is ι such that

$$\mu(a, \iota(a)) = e$$

around a neighborhood of e. And this is smooth as μ is smooth. Let $\alpha : I \to G$ be a smooth curve.

$$\mu(\alpha(t), \iota(\alpha(t)) = e.$$

Therefore

$$d_{(e,e)}\mu(X, \iota(X)) = 0 = (\alpha'(0), \alpha'(0)\iota'(e)) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = X + d_e\iota X = 0.$$

3.3

Let $g \in G$ and consider $\mu(g, g^{-1}) = e$. Using similar arguments in the 3.1 we obtain that

$$d_{g,g^{-1}}\mu(X, Y) = d_{g,g^{-1}}(X, g^{-1}) + d_{g,g^{-1}}(g, Y).$$

We pick $\alpha, \beta : I \to G$ to be such that

$$\alpha(0) = g, \beta(0) = g^{-1}.$$

Then

$$d_{g,g^{-1}}\mu(X, Y) = Xd(g^{-1}) + d(g)Y.$$

Let $X = hg, Y = 0$ for any $h \in T_{g,g^{-1}}G$ then this is a surjection thus full rank. We can use the implicit function theorem and by the uniqueness of inverse, we conclude that G is a Lie group.