V4A3 Sheet 7

So Murata

(i)

Given two roots $\alpha, \beta \in R$, we have

$$\langle \alpha^\vee, \beta \rangle = \frac{2(a,b)}{(a,a)} = \frac{2\|\beta\|}{\|\alpha\|} \cos\theta \in \mathbb{Z}.$$

Similarly for β, α, we derive

$$\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle = 4\cos^2\theta \in \mathbb{Z}.$$

Thus the possibilities of $\cos\theta$ are

$$\pm 1, \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}$$

Thus angles corresponding them in $0 \leq \theta \leq \pi$ are

$$\frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi, \frac{3}{4}\pi, \frac{5}{6}\pi.$$

(ii)

Let $\Delta = \{\alpha, \beta\}$ be its simple roots. Then the angle between α, β are only the angles listed above. If the root system R is reducible then $\operatorname{Span} R = \operatorname{Span} R_1 \oplus \operatorname{Span} R_2$. $\operatorname{Span} R$ has a dimension 2, therefore $\operatorname{Span} R_1, \operatorname{Span} R_2$ both have dimension 1 and correspond to the standard basis. This is only possible when the angle between them are $\frac{1}{2}\pi$ with $\Delta = \{2e_1, 2e_2\}, R = \{\pm 2e_1, \pm 2e_2\}$.

(iii)

We now have to show that each irreducible root systems have angles corresponding to

$$\frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{2}{3}\pi, \frac{3}{4}\pi, \frac{5}{6}\pi.$$

As we are talking about the reduced and irreducible root system, we can examine the Dynkin diagram to get all the possibilities.

1. For $\frac{2}{3}\pi$, it corresponds to A_2,

2. For $\frac{5}{6}\pi$, it corresponds to G_2,
3. For $\frac{3}{4}\pi$, it corresponds to either B_2, C_2 depending on the direction of the arrow.