Representation Theory 1 V4A3

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1 Overview of the material

1.1 Lie groups

Definition 1.1. *A Lie group is a group G whose underlying set is endowed with the structure of smooth manifolds such that multiplication and inversions are smooth maps.*

Definition 1.2. *A topological group is a group G whose underlying set is endowed with the structure of topological space such that multiplication and inversions are continuous.*

2 Preliminaries

2.1 Topology

Definition 2.1. *We have two axioms about the topological spaces*

1. *T_0(Komogolov) : Given any 2 points, there exists an open set such that it contains one of them but not both.*
2. *T_1(Hausdorff) : Given any 2 points, there exist disjoints open set that each contains one of them.*

Definition 2.2. *A topological space is second countable if it has a basis which contains at most countably many subsets.*

3 Lie groups

3.1 Manifolds

Definition 3.1. *Let $f : X \to Y$ be a mapping between two topological spaces X, Y. f is called a homeomorphism if*

1. *f is a bijection,*
2. *f is continuous,*

3. f^{-1} *is also continuous.*

Definition 3.2. *Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open sets and $f : U \to V$ be a smooth map. Then the derivative of f at $p \in U$ is*

$$df(p) = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}.$$

Proposition 3.1. *Let $f : U \to V, g : V \to W$ be smooth maps. Then for $p \in U$ we have*

$$d(g \circ f) = dg(f(p))df(p).$$

Definition 3.3. *Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open sets. A map $f : U \to V$ is called a diffeomorphism if*

i). f is smooth. ($\Leftrightarrow$ arbitrary order of partial derivatives exists),

ii). f^{-1} is defined and is also a smooth map.

Definition 3.4. *Let X be a topological space. A chart on X is a homeomorphism $h : U \to V$ where $U \subseteq X$ is open and $V \subseteq \mathbb{R}^n$ is open.*

Definition 3.5. *An atlas $\mathscr{A}$ on a topological space X is a collection of charts $\{h_\lambda \mid h_\lambda : U_\lambda \to V_\lambda\}_{\lambda \in \Lambda}$ such that $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of X.*

Definition 3.6. *An atlas $\mathscr{A}$ of X is said to be smooth if for any two charts $h_1 : U_1 \to V_2, h_2 : U_2 \to V_2$. The following,*

$$h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2),$$

is a smooth map. Such map is called a transition map.

Definition 3.7. *Let X be a topological space and $\mathscr{A}_1, \mathscr{A}_2$ be smooth atlases. We say they are equivalent if $\mathscr{A}_1 \cup \mathscr{A}_2$ is also smooth.*

Proposition 3.2. *Above definition indeed defines an equivalence relation.*

Proof. For any $h_1 \in \mathscr{A}_1, h_2 \in \mathscr{A}_2, h_3 \in \mathscr{A}_3$,

$$h_3 \circ h_1^{-1} = h_3 \circ h_2^{-1} \circ h_2 \circ h_1^{-1}.$$

□

Definition 3.8. *A smooth manifold is a second countable Hausdorff topological space with equivalence classes of smooth atlases.*

Definition 3.9. *Let M, N be smooth manifolds, $f : M \to N$ be a map, and $p \in M$. f is said to be smooth at p if for one (hence any) pair of charts around p and $f(p)$,*

$$h_M : U_M \to V_M, h_N : U_N \to V_N,$$

the composed function

$$h_N \circ f \circ h_M^{-1} : V_M \to V_N$$

is smooth at $h_M(p)$.

Remark 3.1. *We can define a function* $\dim : M \to N$ *such that*

$$\dim(p) = \dim(V)_p,$$

for any chart $h : U \to V$ *around* p*. And this function is locally constant. In particular, if* M *is connected then it has a well-defined dimensions.*

Definition 3.10. *Let* M, N *be smooth manifold and* $f : M \to N$ *be a mapping which is smooth at* $p \in M$*. For any charts,*

$$h_N \circ f \circ h_M^{-1} : V_M \to V_N,$$

the rank of f *at* p *is such that*

$$\mathrm{rk}(f; p) = \mathrm{rank}(\mathbf{df}(h_M(p))(h_N \circ f \circ h_M^{-1})).$$

Definition 3.11. *Let* M, N *be smooth manifolds and* $f : M \to N$ *be a smooth map. A point* p *is said to be regular with respect to the map f. And a point* $q \in N$ *is called a regular value if all* $p \in f^{-1}(q)$ *are regular.*

Definition 3.12. *Let* M *be a manifold. A subset* $N \subseteq M$ *is called an embedded submanifold if for any point* $p \in N$*, there is a chart* $h_M : U_M \to V_M$ *around* p *such that*

$$h_M|_N : U_M \cap N \to V_M \cap \mathscr{R}^n,$$

is a diffeomorphism where n *is the dimension of* N*.*

In particular, an embedded submanifold of an euclidean space is called a embedded manifold.

Definition 3.13. *A map* $f : M \to N$ *of smooth manifolds is called a diffeomorphism if*

i). $f : M \to N$ *is a bijection,*

ii). f, f^{-1} *are both smooth.*

Theorem 3.1. *Let* $f : M \to N$ *be a smooth map between manifolds, and* $q \in N$ *be a regular value. Then* $f^{-1}(q) \subset M$ *is an embedded submanifold.*

Theorem 3.2. *Let* $f : M \to N$ *be a smooth map of manifolds* $p \in M$ *be a regular point, and* $\dim(p) = \dim(f(p))$*. Then* f *is a local diffeomorphism of* p*. In other words, there is a neighborhood* U_M *of* p *in* M *and* $f(p) \in U_N \subset N$ *such that*

$$f|_{U_M} : U_M \to U_N,$$

is a diffeomorphism.

Definition 3.14. *Let* $M \subseteq \mathbb{R}^n$ *be an embedded manifold such that for some open set* $U \subset \mathbb{R}^n$*, there is* $V \subset \mathbb{R}^n$ *such that*

$$h : U \to V, \quad h_M : U \cap M \to V \cap \mathbb{R}^m,$$

is a diffeomorphism where h_M is defined to be taking the first m coordinate of the points in V. (Thus $m \leq n$).

The tangent space T_pM of M at p is the subspace of $\mathbb{R}^n$ such that

$$(\mathbf{dh}(p))^{-1}(\mathbb{R}^m) \subset \mathbb{R}^n.$$

There are three definitions of tangent spaces and they are all equivalent. However, each of them has its own advantages.

Definition 3.15 (Coordinate tangent space)**.** *Given a smooth manifold M and a point $p \in M$. The coordinate tangent space of p is such that*

$$T_p^{\mathbf{Coo}}M = \{(h, v) \mid h : U \to V \textit{ is a chart}, v \in \mathbb{R}^m\}/\sim .$$

Where $\sim$ is an equivalence relation such that

$$(h_1, v_1) \sim (h_2, v_2) \textit{ if } (\mathbf{d}(h_2 \circ h_1^{-1})(h_1(p)))(v_1) = v_2.$$

Definition 3.16. *Given a smooth manifold M, a point $p \in M$, and a smooth map $\alpha : I \to M$ whose domain I is an open interval contains 0. α is called a smooth curve if $\alpha(0) = p$.*

Definition 3.17. *Two smooth curves $\alpha, \beta : I \to M$ through p are said to be tangentially equivalent if for one (hence any) charts $h : U \to V$ around p, we have*

$$d(h \circ \alpha)(0) = d(h \circ \beta)(0).$$

We denote such relation as $\sim_T$.

Definition 3.18 (Geometric tangent space)**.** *The geometric tangent space at p of a smooth manifold M is such that*

$$T_p^{\mathbf{Geo}} = \{\alpha : I \to M \mid \alpha \textit{ is a smooth curve}\}/\sim_T .$$

Definition 3.19. *A germ of smooth functions of manifolds M at p is an equivalence class of tuples (U, f) where*

i). $U \subset M$ is a neighborhood of p,

ii). $f : U \to \mathbb{R}$ is smooth,

and two tuples $(U_1, f_1), (U_2, f_2)$ are equivalent if there is a neighborhood V of p such that $V \in U_1 \cap U_2$ and $f_1|_V = f_2|_V$.

And we denote the set of germs at p as

$$\mathscr{C}^\infty(p).$$

Remark 3.2. *$\mathscr{C}^\infty(U, \mathbb{R})$ and $\mathscr{C}^\infty(p)$ are rings, in fact $\mathbb{R}$-algebras.*

Definition 3.20. *Let R be a ring and A be a bimodule over R. A R-derivation in A is an operator $X : A \to A$ such that the Leibniz rule holds. In other words,*

$$X(ab) = aX(b) + X(a)b,$$

holds for all $a, b \in A$.

Definition 3.21 (Algebraic tangent space)**.** *The algebraic tangent space $T_p^{\mathbf{Alg}}M$ of M at p is the set of $\mathbb{R}$-derivations $X : \mathscr{C}^\infty(p) \to \mathbb{R}$.*

Remark 3.3. *In the above definition, $\mathbb{R}$ is considered as a $\mathscr{C}^\infty(p)$-bimodule via the evaluation map $f \mapsto f(p)$.*

Theorem 3.3. *The following are isomorphisms of $\mathscr{R}$-vector spaces.*

$$\begin{aligned}
&T_p^{\mathbf{Geo}}M \to T_p^{\mathbf{Alg}}M, \alpha \mapsto (f \mapsto (f \circ \alpha)'(0)),\\
&T_p^{\mathbf{Alg}}M \to T_p^{\mathbf{Coo}}M, X \mapsto (h, ((Xh_i)(p))_{i=1,\cdots,n}),\\
&T_p^{\mathbf{Coo}}M \to T_p^{\mathbf{Geo}}M, (h, v) \mapsto \alpha(t) = h^{-1}(h(p) + t \cdot v).
\end{aligned}$$

Proposition 3.3. *$\mathscr{C}^\infty(p)$ is a local ring with its maximal ideal*

$$\mathfrak{m}_p = \{f \in \mathscr{C}^\infty(p) \mid f(p) = 0\}.$$

Moreover, if we have a derivation $X : \mathscr{C}^\infty(p) \to \mathbb{R}$, the restricted derivation $X|_{\mathfrak{m}_p}$ is in $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2)$. And by this restriction, we get an isomorphism between $T_p^{\mathbf{Alg}}M$ and $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R})$.

Remark 3.4. *In this way, a smooth manifold is recognized as a locally ringed space, locally isomorphic to $\mathbb{R}^n$.*

Remark 3.5. *Let V be a finite dimensional $\mathbb{R}$-vector space. It has a tautological smooth manifold structure by taking charts such that the sets of isomorphisms of V and $\mathbb{R}^n$ given by arbitrary basis of V.*

We claim that we have canonical isomorphisms

$$T_pV \to V,$$

for any $p \in V$,

$$\begin{aligned}
&V \to T_p^{\mathbf{Coo}}V, v \mapsto (h, h(v)),\\
&V \to T_p^{\mathbf{Geo}}V, v \mapsto (t \mapsto p + tv),\\
&V \to T_p^{\mathbf{Alg}}V, v \mapsto \left(f \mapsto \frac{d}{dt}\bigg|_{t=0} f(p + tv)\right)
\end{aligned}$$

Definition 3.22. *Let $f : M \to N$ be a map of smooth manifolds which is smooth at $p \in M$. Its differential of p is the linear map*

$$\mathbf{d}f(p) = \mathbf{d}_p(f) : T_pM \to T_{f(p)}N,$$

defined as follows.

1). Geometric tangent space : $\mathbf{d}_p(f)(\alpha) = f \circ \alpha$ *where* α *is a smooth curve.*

2). Algebraic tangent space : $\mathbf{d}_p(f)(X)(\varphi) = X(\varphi \circ f)$ *where* $\varphi \in \mathscr{C}^\infty(f(p))$.

3). Coordinate tangent space : $\mathbf{d}_p(f)(h_M, v_M) = (h_N, d_{h_M(p)}(h_N)$.

Remark 3.6. *Given a chart* $h : U \to V$ *around* $p \in M$. h *consists of coordinate functions* h_i *where* $1 \le i \le m$ *for* $V \subset \mathbb{R}^m$. *We have for each* i

$$\mathbf{d}_p h_i : T_pM \to \mathbb{R},$$

and

$$B = \{d_p h_1, \cdots, d_p h_m\}$$

is a basis of the dual space $(T_pM)^*$.

Let

$$\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\}$$

be the dual basis of B. *By definition, this means that for any* $1 \le i, j \le m$, *we have*

$$\frac{\partial}{\partial x_i} h_j = d_p h_j(\frac{\partial}{\partial x_i}) = \delta_{ij}.$$

Proposition 3.4. *Let* $f : M \to N$ *be a map between smooth manifolds which is smooth and* $q \in N$ *be a regular value. For* $p \in f^{-1}(q)$, *we have*

$$T_p f^{-1}(q) = \mathbf{d}_p(f)^{-1}(0) \subset T_pM.$$

Proof. □

3.2 Immersions and Submersions

Definition 3.23. *Let* $f : M \to N$ *be a smooth map of smooth manifolds.* f *is called an*

1). immersion if $\mathbf{d}f : T_pM \to T_{f(p)}N$ *is injective for any* $p \in M$,

2). submersion, if $\mathbf{d}f(p) : T_pM \to T_{f(p)}N$ *is surjective for any* $p \in M$.

Remark 3.7. *An immersion need not be injective. The counter example is*

$$e^{ix} : \mathbb{R} \to S^1,$$

is an immersion.

Remark 3.8. *A submersion need not be injective. The counter example is*

$$i_U : U \to M,$$

an inclusion map is a submersion.

Remark 3.9. *We know that if f is a submersion, then $f^{-1}(q)$ is an embedded submanifold. However, if f is an immersion, even it is injective, $f(M)$ need not be an embedded submanifold of N.*

Definition 3.24. *An immersed submanifold is an image of an injective immersion.*

Remark 3.10. *We endow $f(M)$ with the transported topology and differential structure from M so that f becomes a diffeomorphism between M and $f(M)$. But this topology need not be the relative topology from N. It may be strictly finite.*

Example 3.1. *Let $T = S^1 \times S^1$ be a torus. Let $r \in \mathbb{R}$. We consider a map $f : \mathbb{R} \to T$ such that*

$$f(x) = (e^{2\pi tx}, e^{2\pi rix}).$$

This is an immersion for any r. We examine this by several cases.

First, when r is not a rational number then f is injective, the image is an immersed manifold. However, a copy of $\mathbb{R}$. But this image is a dense subset of the torus.

Second, if r is rational then f is not injective. It is going to factor through an injective immersion $\mathbb{R}/b\mathbb{Z} \to T$ where $r = \frac{a}{b}$, $a, b \in \mathbb{Z}$ are coprime. This image is not only immersed but also embedded.

Remark 3.11. *If $f : M \to N$ is an immersion, $\mathbf{d}f_(p)$ identifies T_pM with a linear subspace of $T_{f(p)}N$.*

Proposition 3.5. *If $f : M \to N$ is an injective immersion, that is also closed subset of N, then its image is an embedded submanifold.*

Remark 3.12. *Thus we have the notion of a closed submanifold.*

3.3 Multi-linear forms

Definition 3.25. *Let $\mathbb{V}$ be a vector space and $\varphi : \bigoplus_{i=1}^m V \to \mathbb{R}$ is called a m-multi-linear function if for any $i = 1, \cdots, m$ and $\{a_j\}_{j \neq i} \subset V$ we have*

$$\varphi(a_1, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_m) : V \to \mathbb{R}$$

is a linear function

Definition 3.26. *Let X be a smooth n-dimensional manifold and $m \in \mathbb{N}$. Then we define the followings*

1. $\mathscr{L}_p^m = \{\varphi : \bigoplus_{i=1}^m T_pX \to \mathbb{R} | \varphi$ *is a m-multi-linear function.*$\}$
2. $\mathscr{L}^m = \bigcup_{p \in X} \mathscr{L}_p^m$

Definition 3.27. *Let X be a smooth n-dimensional manifold. A map $V : X \to \mathscr{L}^m$ is called a m-tensorfield if*

i. For any $p \in X$, $V(p) \in \mathscr{L}^m_p$.

ii. For any chart (U, φ) around p with a basis $\{e^\varphi_1, \cdots, e^\varphi_n\}$ and for any $i_1, \cdots, i_m \in \{1, \cdots, n\}$ we have a map $V_{(i_1,\cdots,i_m)} : X \to \mathbb{R}$ such that $V_{(i_1,\cdots,i_m)}(p) = V(p)(\underline{e}_{i_1}, \cdots, \underline{e}_{i_m})$ is smooth.

Proposition 3.6. *For any m tensorfield V, we have*

Definition 3.28. *We define $\mathscr{V}^m(X)$ to be the set of all m-tensorfield.*

Proposition 3.7. *$\mathscr{V}^m(X)$ is a vector space over $\mathbb{R}$ and a module over $\mathscr{F}(X)$ with the common basis $\{E_{i_1,\cdots,i_m}\}_{i_1,\cdots,i_m \in \{1,\cdots,n\}}$*

Proposition 3.8. *Let X be a smooth n-dimensional manifold and $V : X \to \mathscr{L}^m$ be such that for any $p \in X, V(p) \in \mathscr{L}^m_p$ the followings are equivalent.*

1. *V is a m-tensorfield.*
2. *For any chart (U, φ) around p with basis $\{\underline{e}^\varphi_1, \cdots, \underline{e}^\varphi_n\}$ and for any $1 \leq i_1, \cdots, i_m \leq n$ there exist smooth mappings $\lambda_{i_1,\cdots,i_m} : X \to \mathbb{R}$ such that $V(p) = \sum_{1\leq i_1,\cdots,i_m \leq n} \lambda_{i_1,\cdots,i_m}(p) E^\varphi_{i_1,\cdots,i_m}$.*
3. *For any vectorfields $v_1, \cdots, v_m : X \to TX$ we have a function $V : X \to \mathbb{R}$ such that $V_{v_1,\cdots,v_m}(p) = V(p)(v_1(p), \cdots, v_m(p))$ is smooth.*

Proof. 1.⇔2. is trivial. 1.⇒3. is clear by the multi-linearity, and 3.⇒1. is choosing $v_i = e^\varphi_i$ for each $i = 1, \cdots, n$.

□

Proposition 3.9. *Let $V : X \to \mathscr{L}^m$ then thne followings are equivalent.*

1. *V is a m-tensorfield.*
2. *For any $\{v_1, \cdots, v_m\} \in \mathscr{V}(X)$, $\Psi : \bigoplus_{i=1}^m \mathscr{V}(X) \to \mathscr{F}(X)$ such that $\Psi(v_1, \cdots, v_m)(p) = V(p)(v_1(p), \cdots, v_m(p))$ is smooth and $\mathscr{F}(X)$-linear.*

Proof. 1.⇒2. follows from the multilinearity and decompositions of tensors. 2.⇒1. follows by fixing all element except one we still have the linearity thus, the function is mutilinear. □

3.4 Tensor and Wedge products

Definition 3.29. *Let $V_1 : X \to \mathscr{L}^r, V_2 : X \to \mathscr{L}^s$ be tensorfield. Then We define the tensorproduct $V_1 \otimes V_2 : X \to \mathscr{L}^{r+s}$ of them to be*

$$(V_1 \otimes V_2)(p)(v_1, \cdots, v_r, v_{r+1}, \cdots, v_{r+s}) = V_1(p)(v_1, \cdots, v_r)V_2(p)(v_{r+1}, \cdots, v_{r+s})$$

Proposition 3.10. *The operation $\bigotimes$ is bilinear and associative.*

Proof. By substituting values, they are trivial. □

Proposition 3.11. *Let $U \subset X$ be an open set and $V_1, \cdots, V_n \in \mathscr{V}^1(U)$ be a basis in $\mathscr{V}^1(U)$ then $\{\bigotimes_{j=1}^r V_{i_j}\}_{1 \le i_1, \cdots, i_r \le r}$ is a basis in $\mathscr{V}^r(U)$.*

Proof. Since $\otimes$ is bilinear, this is a tensor product thus the set in the statement is indeed a basis. □

Definition 3.30. *Let $V \in \mathscr{V}^m(X)$ be a m-tensor. V is said to be alternating if for any $p \in X$, $(v_1, \cdots, v_m) \in \bigoplus_{i=1}^m T_pX$ and $\sigma \in \mathfrak{S}_m$ we have*

$$V(p)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) = \mathrm{sgn}(\sigma)V(p)(v_1, \cdots, v_m)$$

Furthermore, such V is called a m-form.

Notation 3.1. *The set of all m-forms is denoted by*

$$\mathscr{A}^m(X) = \{V \in \mathscr{V}^m(X) \mid V \text{ is a } m\text{-form.}\}$$

Definition 3.31. *Let $V_1 \in \mathscr{A}^r(X), V_2 \in \mathscr{A}^s(X)$ then the wedge product is*

$$(V_1 \wedge V_2)(p)(v_1, \cdots, v_{r+s}) = \frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \mathrm{sgn}(\sigma) V_1 \otimes V_2(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)})$$

Proposition 3.12. *Let $V_1, \cdots, V_n \in \mathscr{A}^1(X)$, $p \in X$ and $v_1, \cdots, v_n \in T_pX$ then we have*

$$(V_1 \wedge \cdots \wedge V_n)(p)(v_1, \cdots, v_n) = \det(V_i(p)(v_j))_{i,j}$$

Proof.

$$(V_1 \wedge \cdots \wedge V_n)(p)(v_1, \cdots, v_n) = \frac{1}{1! \cdots 1!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n V_i(p)(v_\sigma(i))$$

□

Proposition 3.13. *Similar to the case in tensorfields, we have the following statements.*

1. *$\mathscr{A}^m(X)$ is a subspace of $\mathscr{V}^m$ over $\mathbb{R}$.*
2. *$\mathscr{A}^m(X)$ is a module over $\mathscr{F}(X)$.*

Proof. Trivial. □

Proposition 3.14. *Let $V_1 \in \mathscr{A}^r, V_2 \in \mathscr{A}^s$, then $V_1 \wedge V_2 \in \mathscr{A}^{r+s}$ and such $\wedge : \mathscr{A}^r \times \mathscr{A}^s \to \mathscr{A}^{r+s}$ is bilinear.*

Proof. Bilinearity follows from the bilinearity of $\otimes$. We will show that this is indeed well-defined.

Let $\sigma \in \mathfrak{S}_{r+s}$. Then we have

$$\begin{aligned}
(V_1 \wedge V_2)(p)(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)}) &= \frac{1}{r!s!} \sum_{\tau \in \mathfrak{S}_{r+s}} \mathrm{sgn}(\tau) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)}) \\
&= \mathrm{sgn}(\sigma) \frac{1}{r!s!} \sum_{\tau\circ\sigma \in \mathfrak{S}_{r+s}} \mathrm{sgn}(\tau\circ\sigma) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)}) \\
&= \mathrm{sgn}(\sigma)(V_1 \wedge V_2)(p)(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)})
\end{aligned}$$

□

Proposition 3.15.

$$V_2 \wedge V_1 = (-1)^{rs}(V_1 \wedge V_2)$$

Proof. Let $\tau \in \mathfrak{S}_{r+s}$ to be such that

$$\tau(i) = \begin{cases} r+i & (1 \le i \le s) \\ i-s & (s+1 \le i \le r+s) \end{cases}$$

Then clearly the inversion number is $N(\tau) = rs$. It is also obvious that

$$V_2 \wedge V_1(p)(v_{\tau(1)}, \cdots, v_{\tau(r+s)}) = V_1 \wedge V_2(p)(v_1, \cdots, v_{r+s})$$

□

Proposition 3.16. *Let $V_1 \in \mathscr{A}^r, V_2 \in \mathscr{A}^s, V_3 \in \mathscr{A}^t$ then $(V_1 \wedge V_2) \wedge V_3 = V_1 \wedge (V_2 \wedge V_3)$.*

Proof.

$$\begin{aligned}
(V_1 \wedge V_2) \wedge V_3(p)(v_1, \cdots, v_{r+s+t}) &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \mathrm{sgn}(\tau)(V_1 \wedge V_2) \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\
&= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \mathrm{sgn}(\tau) \\
&\quad (\frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \mathrm{sgn}(\sigma) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)})) \\
&\quad V_3(v_{\sigma(r+s+1)}, \cdots, v_{\sigma(r+s+t)})
\end{aligned}$$

If for $\tau_1, \tau_2 \in \mathfrak{S}_{r+s+t}, \sigma_1, \sigma_2 \in \mathfrak{S}_{r+s}$ we have $\tau_1 \circ \sigma_1 = \tau_2 \circ \sigma_2$ then they satisfy the followings

i. For any $r+s+1 \le i \le r+s+t$ we have $\tau_1(i) = \tau_2(i)$.

ii. From above we get $\tau_2^{-1} \circ \tau_1 \in \mathfrak{S}_{r+s}$

Fixing σ_1, there exists $(r+s)!$ many such σ_2. This implies that we can choose σ_1 to be the identity. Thus we get

$$\begin{aligned}
(V_1 \wedge V_2) \wedge V_3(p)(v_1, \cdots, v_{r+s+t}) &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau)(V_1 \wedge V_2) \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\
&= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau) \frac{(r+s)!}{r!s!} V_1 \oplus V_2 \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\
&= \frac{1}{r!s!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau) V_1 \oplus V_2 \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)})
\end{aligned}$$

From the previous proposition we get □

4 Integration

Definition 4.1. *A differential k-form ω on a smooth manifold M is a collection $\omega(p) \in A^k(T_pM)$ for all $p \in M$.*

Remark 4.1. *We can define what it means for ω to be continuous or smooth at some points $p \in M$ as follows.*

First, we pick a chart $h : U \to V$ around p and get the basis

$$\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\},$$

of T_pM that moves with $p \in U$.
We also have a basis $A^k(T_pM) = \bigwedge^k (T_pM)^*$. Hence we can express ω as p in terms of that basis and the scalars in this expression are functions on U.

$$\omega(p) = \sum f_{i_1, \cdots, i_k} \cdot d_{x_{i_1}} \wedge \cdots \wedge d_{x_{i_k}}.$$

And we can require $f_{i_1, \cdots, i_k} \cdots d_{x_{i_1}}$ to be smooth/continuous at p.

Example 4.1. *If $M = \mathbb{R}^n$, we have the canonical identification,*

$$T_pM = \mathbb{R}^n.$$

This gives us standard differential form of degree n. which is given by

$$e_1^* \wedge \cdots \wedge e_n^*,$$

where $e_1, \cdots, e_n$ is the standard basis of $\mathbb{R}^n$.

Definition 4.2. *Let $f : M \to N$ be a smooth map of manifolds and ω be a differential form of degree k on N. We define $f^*(\omega)$ of degree k on M by*

$$f^*(w)(p)(x_1, \cdots, x_k) = \omega(f(p))(\mathbf{d}f_p(x_1), \cdots, \mathbf{d}f_p(x_k)).$$

Definition 4.3. *A differential n-form ω on M is said to be locally integrable if for any point $p \in M$, if for any point $p \in M$, there is one (hence any) chart $h : U \to V$ such that $\omega|_U =$*

5 Lie Algebras

5.1 Important homomorphisms and their properties.

Recall if $f : M \to N$ is a smooth map of smooth manifolds and $p \in M$, we get $df(p) : T_pM \to T_{f(p)}N$ is linear.

Proposition 5.1. *Let $(G, \mu, \iota, 1)$ be a lie group and $\mathfrak{g} = T_1G$. We have*

$$d\mu(1,1) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (X, Y) \mapsto X + Y.$$
$$d\iota(1) : \mathfrak{g} \times \mathfrak{g}, X \mapsto -X$$

Definition 5.1. *A Lie group homomorphism is a smooth map of Lie groups that is a homomorphism.*

Remark 5.1. *If $f : G \to H$ is a Lie group homomorphism then*

$$df(1) : \mathfrak{g} \to \mathfrak{h}$$

is a linear map.

Definition 5.2. *Let G be a Lie group. The adjoint action of G on itself is*

$$\underline{\mathrm{Ad}}(g) : G \to G, h \mapsto ghg^{-1}$$

which is a group homomorphism.

Definition 5.3. *Let G be a Lie group and $\mathfrak{g} = T_1G$. Then we define*

$$\mathrm{Ad}(g) = d\underline{\mathrm{Ad}}(g)(1) : \mathfrak{g} \to \mathfrak{g}.$$

We call this the adjoint action of G on $\mathfrak{g}$.

Remark 5.2. *The term, "action" in the definition above is justified by the chain rule*

$$\mathrm{Ad}(g \cdot h) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h).$$

Definition 5.4. *Let G be a Lie group and $\mathfrak{g} = T_1G$. By regarding* Ad *as a function from G to* $\mathrm{GL}(\mathfrak{g})$. *Notice that by the definition of groups we have* $\mathrm{Ad}(g)$ *is injective.*

We now define the adjoint action of $\mathfrak{g}$ on itself to be

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), X \mapsto d\,\mathrm{Ad}(1)X.$$

Definition 5.5. *Let G be a Lie group and $\mathfrak{g} = T_1 G$. The Lie bracket is $[\cdot|\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

$$[X|Y] = \mathrm{ad}(X)(Y).$$

Proposition 5.2. *Let $G = \mathrm{GL}_n(\mathbb{R})$ and $\mathfrak{g} = \mathbb{R}^{n \times n}$. Let $g \in G$ and $X, Y \in \mathfrak{g}$. We have*

$$[X|Y] = XY - YX.$$

Proof. Let $g \in G$,

$$\begin{aligned}
\mathrm{Ad}(g)X &= d\underline{\mathrm{Ad}}(g)(1)X, \\
&= g(1+X)g^{-1} - g1g^{-1} \mod o(X), \\
&= gXg^{-1} \mod o(X), \\
&= gXg^{-1}.
\end{aligned}$$

In particular $\underline{Ad}$ is a liner map. Now we compute the Lie bracket

$$[X|Y] = \mathrm{ad}(X)(Y) = [E_Y \circ \mathrm{ad}](X),$$

where E_Y is the evaluation map

$$E_Y : \mathrm{End}(\mathfrak{g}) \to \mathfrak{g}, \phi \mapsto \phi(Y).$$

$$\begin{aligned}
[X|Y] &= [E_Y \circ \mathrm{ad}](X), \\
&= d[g \mapsto \mathrm{Ad}(g)Y](1)(X), \\
&= d[g \mapsto gYg^{-1}](1)(X).
\end{aligned}$$

By the first computation we did, we see that

$$[X|Y] = (1+X)Y(1+X)^{-1} - Y \mod o(X).$$

We have the following identity

$$(1-X)^{-1} = 1 + X + X^2 + \cdots.$$

Substituting $-X$ we derive that

$$1 + X = \sum_{i=0}^{\infty} (-1)^i X^i.$$

And we only need at most degree 1 terms of X. We conclude that

$$[X|Y] = XY - YX.$$

∎

Remark 5.3. *This works for any matrix groups such as $\mathrm{SL}_n(\mathbb{R}), O(p,q)$.*

Proposition 5.3. *Let $f : G \to H$ be a Lie group homomorphism. For $g \in G$, we have*

$$df(1) \circ \mathrm{Ad}(g) = \mathrm{Ad}(f(g)) \circ df(1). \tag{5.1}$$

And for $X, Y \in \mathfrak{g}$, we have

$$df(1)([X|Y]_G) = [df(1)X, df(1)Y]_H. \tag{5.2}$$

Proof. Let us consider the composition of f and $\underline{\mathrm{Ad}}$. By definition, we see

$$f \circ \underline{\mathrm{Ad}}(g)(h) = f(g)f(h)f(g)^{-1} = \underline{\mathrm{Ad}}(f(g))(f(h)).$$

Since $\underline{\mathrm{Ad}}(1) = 1$ and by the chain rule we have Equation 5.1.

□

5.2 Lie Algebras

Definition 5.6. *A Lie algebra a (finite dimensional) vector space L over $\mathbb{R}$ or $\mathbb{C}$ together with a bilinear map $[\cdot|\cdot] : L \times L \to L$ such that*

i $[X|Y] = -[Y|X]$,

ii $[X|[Y|Z]] + [Y[Z|X]] + [Z|[X|Y]] = 0$ *which is called Jacobi identity.*

Proposition 5.4. *Let G be a Lie group and $\mathfrak{g} = T_1 G$. Then $\mathfrak{g}$ equipped with $[X|Y] = \mathrm{ad}(X)(Y)$ is a $\mathbb{R}$-Lie algebra.*

Proof. Consider the commutator map $G \times G \to G, (x, y) \mapsto xyx^{-1}y^{-1}$. This is a smooth map as it is a composition of smooth maps $\mu(\mu(\cdot, \cdot), \mu(\iota(\cdot), \iota(\cdot)))$. Moreover, we can write this as

$$\underline{\mathrm{Ad}}(x)(y)\iota(y).$$

Differentiate this at $y = 1$ in the direction of Y, we get

$$d(\underline{\mathrm{Ad}}(x)(1)\iota(1))Y = \mathrm{Ad}(x)Y - Y,$$

since $d\iota(Y) = -Y$. Differentiate this again at $x = 1$ with respect to X we get $[X, Y]$.

Repeating the argument with

$$x\underline{\mathrm{Ad}}(y)(\iota(x)).$$

Differentiate this at $x = 1$ with the direction to X we get

$$X - \underline{\mathrm{Ad}}(y)X = X - yXy^{-1}.$$

Differentiate this again at $y = 1$ with the direction to Y, we get $-[Y|X]$. By smoothness, we get
$$[X|Y] = -[Y|X].$$
For the second property, we consider the Lie group homomorphism,
$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}).$$
By Proposition 5.3, we have
$$\mathrm{ad}[X|Y]_G = [\mathrm{ad}(X)|\,\mathrm{ad}(Y)]_{\mathrm{GL}(\mathfrak{g})} = \mathrm{ad}(X)\,\mathrm{ad}(Y) - \mathrm{ad}(Y)\,\mathrm{ad}(X).$$
Therefore, by definition of $[\cdot|\cdot]$, we get
$$[[X|Y]|Z] = [X|[Y|Z]] - [Y|[X|Z]]$$
By the first property, we get the Jacobi identity. $\square$

Example 5.1. *If V is a finite dimensional $\mathbb{R}$-vector space then $\mathrm{End}(V)$ equipped with $[X|Y] = XY - YX$ is a Lie algebra. In fact, this coincides with the Lie algebra of the Lie group $\mathrm{GL}(V)$.*

Definition 5.7. *A homomorphism of Lie algebras is a linear map $f : L \to M$ such that for $X, Y \in L$*
$$f([X|Y]_L) = [f(X)|f(Y)]_M$$

Corollary 5.1. *If $f : G \to H$ is a homomorphism of Lie groups, then $df(1) : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras.*

5.3 The identity component

Lemma 5.1. *Let G be a topological group. If $H \subset G$ is an open subgroup, then it is also closed. Thus if G is connected we have $H = G$.*

Proof. Let $\{1\} \cup I$ be a set of representations of equivalence classes in G/H. In other words we have
$$G = H \cup \bigcup_{i \in I} iH.$$
Since $\bigcup_{i \in I} iH$ is open, thus its complement H is closed. $\square$

Lemma 5.2. *Let G be a connected topological group and $U \subseteq G$ be a neighborhood of 1. Then U generates G.*

Proof. Since $U \cap U^{-1}$ is non-empty and open. We may assume with out the loss of generality that $U = U^{-1}$. Let us denote
$$U^n = \{g_1 \cdots g_n \mid g_1, \cdots, g_n \in U\}.$$

And for $g_1 \cdots g_n \in U^n$, we take $V \subset U$ an open subset and $g_1 \in V$. $Vg_2 \cdots g_n$ is open in U. We now conclude that

$$H = \bigcup_{n=1}^{\infty} U^n$$

is an open subset which is a subgroup of G since it is closed under multiplication and inversion. Since G is connected we conclude that $H = G$. $\square$

Definition 5.8. *A subgroup H of a group G is said to be characteristic if for any automorphism $\varphi : G \to G$, we have $\varphi(H) \subseteq H$.*

Definition 5.9. *Let X be a topological space. A connected component C of $x \in X$ is the largest connected set which contains x.*

Proposition 5.5. *If C is a connected component of the topological space X, then it is closed.*

Proof. $\square$

Proof. Let $f : \overline{C} \to \{0, 1\}$ be a continuous function where $\{0, 1\}$ is with the discrete topology. Then for any $x \in C$ we conclude $f(x) = 0$ without the loss of generality. By the continuity of f we conclude that $f(x) = 0$ for any $x \in \overline{C}$. $\square$

Definition 5.10. *A topological space $(X, \mathscr{T})$ is said to be locally connected if for any point $x \in X$ and its neighborhood U, there exists a connected neighborhood V such that $x \in V \subset U$.*

Proposition 5.6. *A component of locally connected topological space is open.*

Proposition 5.7. *Let G be a topological group and G^0 be the connected component of G containing* 1.

1) G^0 is a closed characteristic subgroup of G.

2) If G is locally connected then G^0 is open and contained in any open subgroup of G.

3) The connected component of G are precisely G^0-cosets.

Proof. By Proposition 5.5, G^0 is a closed set. Since continuous maps preserve connectedness and 1 is mapped to 1, we can conclude that G^0 is characteristic. Similarly, since multiplication and inversion are smooth, thus continuous, we conclude that G^0 is a subgroup of G. This proves the first statement.

If G is locally connected, by Proposition 5.6, G^0 is open. If $H \subset G$ is any open subgroup, then $H \cap G^0$ is an open subgroup of G^0. By Lemma 5.2, we have $H \cap G^0$ generates G^0. $H \cap G^0$ is a group, we conclude that it is equal to G^0. This shows that G^0 is contained in any open subgroup of G.

Let C be a connected component and $g \in C$. Since $\mu(\cdot, g^{-1} : G \to G$ is continuous, we conclude that $\mu(C, g^{-1})$ is contained in the connected component which contains 1. Hence $C = G^0 g$. $\square$

5.4 Invariant vector fields

Definition 5.11. *Let M be a manifold. A vector field v on M is an assignment that for each $p \in M$, we have $v(p) \in T_pM$. It is said to be smooth if locally around each point $p \in M$, it's coefficients in terms of local coordinates are smooth functions. In other words, given a chart $h : U \to V$, we can get a basis $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ of T_pM for all $p \in U$. And locally*

$$v(p) = \sum_{i=1}^{n} c_i(p) \cdot \frac{\partial}{\partial x_i}.$$

And each c_i is smooth.

Definition 5.12. *Let M be a manifold, v be a smooth vector field. An integral curve is a pair (I, γ) where*

i). I is an open interval,

ii). $\gamma : I \to M$ is a smooth map such that $\gamma'(t) = v(\gamma(t))$.

Proposition 5.8. *Let v be a smooth vector field on a manifold M, then we have the following statements.*

1). Given $p \in M$, there exists a integral curve (I, γ) such that $0 \in I$ and $\gamma(0) = p$.

2). $(I_1, \gamma_1), (I_2, \gamma_2)$ be integral curves with above properties. Then for any $t \in I_1 \cap I_2$, we have $\gamma_1(t) = \gamma_2(t)$.

3). In particular, we can splice such γ_1, γ_2.

Proof. Follows from existence and uniqueness of solutions of ordinary differential equations in $\mathbb{R}^n$ via charts. ☐

Remark 5.4. *There is a maximal integral curve through p.*

Definition 5.13. *Let G be a Lie group and $g \in G$. We define*

$$L_g : G \to G, L_g(x) = gx, \quad R_g : G \to G, R_g(x) = xg,$$

the left and the right translations. Obviously these are diffeomorphisms as the inverses are $L_{g^{-1}}, R_{g^{-1}}$, respectively.

Remark 5.5. *By differentiating these we get*

$$dL_g(1) : \mathfrak{g} \to T_gG, \quad dR_g(1) : \mathfrak{g} \to T_gG.$$

Proposition 5.9. *Let G be a Lie group and $g \in G$. Then $dL_g(1), dR_g(1)$ are isomorphisms between $\mathfrak{g}$ and T_gG. Therefore, we can naturally identify T_gG by $\mathfrak{g}$. Moreover, $dL_g(1), dR_g(1)$ are not the same in general and differ by the automorphism* $\mathrm{Ad}(g)$.

Proof. □

Definition 5.14. *A vector field v on the Lie group G is said to be*

1). left-invariant if $v(g) = dL_g(1)(v(1))$,

2). right-invariant if $v(g) = dR_g(1)(v(1))$.

Remark 5.6. *Such vector field is automatically smooth. And the assignments*

$$X^L = X \mapsto (g \to dL_g(1)(X)), \quad X^R = X \mapsto (g \to dR_g(1)(X))$$

identify the Lie algebra $\mathfrak{g}$ with the space of left/right-invariant vector fields on G.

Lemma 5.3. *Let v be a left-invariant vector field on G. The maximal integral curve γ with $\gamma(0) = 1$ is defined on all of $\mathbb{R}$ and is a group homomorphism.*

Proof. Let $\gamma : I \to G$ be an integral curve with v with $\gamma(0) = 1$.

Assume $I \neq \mathbb{R}$ thus, without the loss of generality I has an upper bound $t_0 \in \mathbb{R}$. We will have to show that γ is not maximal. To see this, we choose $0 < \varepsilon < t_0$ and $t_0 - \varepsilon < t_1 < t_0, t \in I$.

Consider $\delta(t) = \gamma(t_1) \cdot \gamma(t - t_1)$. Thus γ is a smooth curve defined an open neighborhood of t_0 and $\delta(t_1) = \gamma(t_1)$ and

$$\begin{aligned}
\delta'(t) &= d\delta(t)(1), \\
&= dL_{\gamma(t_1)}(\gamma(t - t_1))(dr(t - t_1)(1)), \\
&= dL_{\gamma(t_1)}(v(\gamma(t - t_1))), \\
&= dL_{\gamma(t_1)}(\gamma(t - t_1)(dL_{r(t-t_1)}(1)(v(1))), \\
&= dL_{\gamma(t_1)\gamma(t-t_1)}(1)(v(1)), \\
&= v(\gamma(t_1)\gamma(t - t_1)), \\
&= v(\delta(t)).
\end{aligned}$$

Thus δ is an integral curve for v defined on an open neighborhood of t_0 containing t_1 and $\delta(t_1) = \gamma(t_1)$. Therefore γ is not maximal.

Now we are going to show that γ is a homomorphism. For fixed $t \in \mathbb{R}$, note that the maps

$$s \mapsto \gamma(t + s), \quad s \mapsto \gamma(t)\gamma(s)$$

are both integral curves for v with equal value at $s = 0$, hence equal. □

5.5 The Exponential Maps

Proposition 5.10. *Let $X \in \mathfrak{g}$, there exists a unique group homomorphism $\gamma_X : \mathbb{R} \to G$ differentiable at 0 and $\gamma'_X(0) = X$. It is the maximal integral curve through 1 for both X^L and X^R. We have $\gamma_{tX}(s) = \gamma_X(ts)$ for $t \in \mathbb{R}$.*

Proof. By Lemma 5.3, there exist maximal integral curves for X^L and X^R , we denote them by $\gamma_{X^L}, \gamma_{X^R}$, respectively. By Lemma 5.8, we can assume $\gamma_{X^L}(0) = \gamma_{X^R}(0) = 1$, and these are defined on the whole $\mathbb{R}$.

For uniqueness, let $\gamma : \mathbb{R} \to G$ be a group homomorphism which is differentiable at 0 with $\gamma'(0) = X$. Then

$$\gamma(t)\gamma(s) = \gamma(t+s) = \gamma(s+t) = \gamma(s)\gamma(t). \tag{5.3}$$

Fix t and apply $\frac{d}{ds}|_{s=0}$ to see that γ is differentiable at any t in the following way

$$\begin{aligned} \frac{d}{ds}|_{s=0}\gamma(t)\gamma(s) &= \frac{d}{ds}|_{s=0}\gamma(t+s), \\ \Rightarrow \gamma(t)\gamma'(0) &= \gamma'(t). \end{aligned}$$

By construction, when $\gamma = \gamma_{X^L}$ we have

$$\begin{aligned} \gamma'_{X^L}(t) &= dL_{\gamma_{X^L}(t)}(1)(X^L(1)) \\ &= dL_{\gamma_{X^L}(t)}(1)X \\ &= L_{\gamma_{X^L}(t)}X. \end{aligned}$$

Similarly for $\gamma = \gamma_{X^R}$ we have

$$\begin{aligned} \gamma'_{X^R}(t) &= dR_{\gamma_{X^R}(t)}(1)(X^R(1)) \\ &= dR_{\gamma_{X^R}(t)}(1)X \\ &= R_{\gamma_{X^R}(t)}X. \end{aligned}$$

By the uniqueness of solutions of ordinary differential equations, we derive that $\gamma_{X^L} = \gamma_{X^R}$. This proves that $\gamma_{X^L}, \gamma_{X^R}$ are maximal as they are defined on all $t \in \mathbb{R}$.

For the second property, we only need to check that $\gamma_{tX}(s) = \gamma_X(ts)$ coincide at $s = 0$. $\square$

Definition 5.15. *Let G be a Lie group and $\mathfrak{g} = T_1G$. Then we define the exponential map*

$$\exp_G : \mathfrak{g} \to G, \exp_G(X) = \gamma_X(1),$$

where γ_X is the integral curve of $v(g) = dL_g(1)X$.

Theorem 5.1. $\exp_G : \mathfrak{g} \to G$ *is smooth and has the following properties.*

1). $\underline{\mathrm{Ad}}(x) \circ \exp_G = \exp_G \circ \mathrm{Ad}(x)$ *for any $x \in G$.*

2). $\mathrm{Ad} \circ \exp_G = \exp_{\mathrm{GL}(\mathfrak{g})} \circ \mathrm{ad}$.

3). $d\exp_G(0) : \mathfrak{g} \to \mathfrak{g}$ *is an identity $id_{\mathfrak{g}}$.*

4). If $f : G \to H$ is a homomorphism of Lie groups, then $f \circ \exp_G = \exp_H \circ df(1)$.

5). $\gamma_X(t) = \exp_G(t \cdot X)$.

Proof. Look at the homework $\square$

Proposition 5.11. *Let V be a finite dimensional $\mathbb{R}$-vector space. Then*

$$\exp_{\mathrm{GL}(V)} : \mathfrak{gl}(V) \to \mathrm{GL}(V)$$

is given by

$$\exp_{\mathrm{GL}(V)}(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n.$$

Proof. Homework $\square$

Corollary 5.2. *Furthermore, we can derive the following properties of $\exp_G$,*

1). $\operatorname{Im} \exp_G \subseteq G^0$.

2). $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism locally around 0.

3). If $U \subseteq \mathfrak{g}$ is a neighborhood of 0 in $\mathfrak{g}$, then $\exp_G(U)$ generates G^0.

Proof. Note that $\exp_G$ is a smooth map.

By the smoothness, it is also continuous. Since $\mathfrak{g}$ is connected, it is mapped to a connected subset of G which contains 1. Thus we have the first property.

By the third property of Theorem 5.1, we have $d\exp_G(0)$ is invertible.

By the second property of the same theorem, $\exp_G(U)$ contains an open neighborhood of 1, thus generates G^0. $\square$

Definition 5.16. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. By the second statement of the corollary above, there exists a neighborhood U of 0 in $\mathfrak{g}$, such that $\exp_G |U$ is a diffeomorphism. We denote its inverse by $\log_G$.*

Corollary 5.3. *Let G be a connected Lie group and $g \in G$, we have the following*

$$g \in Z(G) \Leftrightarrow \mathrm{Ad}(g) = id_{\mathfrak{g}}.$$

Proof. If $g \in Z(G)$, then $\underline{\mathrm{Ad}}(g) = id_G$, therefore $\mathrm{Ad}(g) = id_{\mathfrak{g}}$. Conversely, if $\mathrm{Ad}(g) = id_{\mathfrak{g}}$, by the first property of Theorem 5.1 we have $\underline{\mathrm{Ad}}(g)$ is identity on the image of $\exp_G$. By the second statement of Corollary 5.2, this image generates G. Since $\underline{\mathrm{Ad}}(g)$ is a homomorphism, it is trivial on the entire group G. $\square$

Corollary 5.4. *Let G be a Lie group and $X, Y \in \mathfrak{g}$. We have*

$$[X|Y] = 0 \Rightarrow \exp_G(X)\exp_G(Y) = \exp_G(Y)\exp_G(X).$$

Proof. Let $x = \exp_G(X), y = \exp_G(Y)$. By the first and second statements of Theorem 5.1,

$$xyx^{-1} = \exp_G(\mathrm{Ad}(X)Y) = \exp_G(\exp_{\mathrm{GL}(\mathfrak{g})}(\mathrm{ad}(X)(Y))).$$

By Proposition 5.11 and the assumption, this is equal to

$$\exp_G(Y) = y.$$

□

Corollary 5.5. *Let $f_1, f_2 : H \to G$ be homomorphisms of Lie groups. If H is connected and $df_1(1) = df_2(1)$. Then $f_1 = f_2$.*

Proof. Using the forth statement of Theorem 5.1, we have $f_1 = f_2$ upon restriction to the image of $\exp_H$, and such image generates H. □

5.6 Differentials of $\exp_G$

Theorem 5.2. *Let $X \in \mathfrak{g}$. (Recall that we have the canonical identification $T_x\mathfrak{g} \to \mathfrak{g}$).*

Consider

$$d(\exp_G)(x) : \mathfrak{g} \to T_{\exp(X)}G, \quad dR_{\exp_G(x)}(1) : \mathfrak{g} \to T_{\exp_G(x)}G.$$

Then we have the following,

$$dR_{\exp_G(x)}(1)^{-1} \circ d(\exp_G)(x) : \mathfrak{g} \to \mathfrak{g}, X \to \int_0^1 \exp_{\mathrm{GL}(\mathfrak{g})}(s \cdot \mathrm{ad}(X))ds.$$

Proof. □

Corollary 5.6. *An element $X \in \mathfrak{g}$ is a singular point for $\exp_G$ if and only if $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ has an eigenvalue of the form $2\pi ik$ for some $k \in Z^{\times}$.*

Proof. Since both $\mathfrak{g}$ and G have the same dimension, X is singular if and only if $d(\exp_G)(X)$ is not invertible. By Theorem the equation

$$\int_0^1 \exp_{\mathrm{GL}(\mathfrak{g})}(s \cdot \mathrm{ad}(X))dx \tag{5.4}$$

is not invertible. In other words, it admits 0 as an eigenvalue. Using the formula

$$\int_0^1 \exp_{\mathrm{GL}(\mathfrak{g})}(s\lambda)dx = \begin{cases} \lambda^{-1}(e^{\lambda} - 1) & (\lambda \neq 0), \\ 1 \quad (\lambda = 0). \end{cases}$$

We see that the eigenvalues of the (5.4) are given by 1 if 0 is an eigenvalue of $\mathrm{ad}(X)$ and $\lambda^{-1}(e^{\lambda} - 1)$ if $\lambda \neq 0$ is an eigenvalue of $\mathrm{ad}(X)$. □

Remark 5.7. *The formula* (5.4) *generalizes to*

$$\int_0^1 e^{sA} ds = A^{-1}(e^A - 1) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} A^k.$$

for any $A \in \mathrm{GL}(V)$ where V is a finite dimensional $\mathbb{R}$-vector space. If A is not invertible, we can define $A^{-1}(e^A - 1)$ by the above formula.

This is particularly useful for $A = \mathrm{ad}(X)$, for $X \in \mathfrak{g} = V$, which is never invertible since $\mathrm{ad}(X)(X) = 0$. Moreover, for $A = \mathrm{ad}(X), A^{-1}(e^A - 1)$ is invertible for X in a neighborhood of 0 by Corollary 5.6.

5.7 The Product in Logarithmic Coordinates

Theorem 5.3. *Let $U \subset \mathfrak{g}$ be an open neighborhood of 0. For $X, Y \in U$, consider the differential equation for $z : \mathbb{R} \to \mathfrak{g}$, such that*

$$z(0) = Y, \quad \frac{dz}{dt}(t) = (\mathrm{ad}\, z(t))^{-1}(\exp_{\mathrm{GL}\,\mathfrak{g}}(\mathrm{ad}\, z(t)) - 1))^{-1}(X).$$

For U sufficiently small, this differential equation has (a unique) solution for all $X, Y \in U$ and all $t \in [0, 1]$. Define $\mu(X, Y) = z(1)$. Then

$$\exp_G(X) \exp_G(Y) = \exp_G(\mu(X, Y)).$$

Proof. □

Corollary 5.7. *The collection of maps $\kappa_x : U \to G$, where $U \subset \mathfrak{g}$ is an open neighborhood of 0.*

$$\kappa_x(Y) = x \cdot \exp_G(Y)$$

is a smooth, in fact real analytic, atlas for the manifold G.

Proof. We know that $\exp_G$ is smooth and a locally diffeomorphism around 0. So κ_x is a diffeomorphism onto its image . Thus $(\kappa_x)_{x \in G}$ is a smooth atlas.

The transition maps are expressible in terms of μ by Theorem 5.3. Since μ is real analytic in X, Y. we see that the atlas is real analytic. □

Definition 5.17 (Real analytic manifolds)**.** *A manifold is said to be*

Remark 5.8. *In particular, any Lie group is automatically real analytic.*

Theorem 5.4. *Let $X, Y \in U$, then*

$$\mu(X, Y) = X + Y + \sum_{k=1}^{\infty} \frac{(1)^k)}{k+1} \sum_{\substack{l_1, \cdots, l_k \geq 0, \\ m_1, \cdots, m_k \geq 0, \\ l_i + m_i > 0}} \frac{1}{\sum_{i=1}^k l_i + 1} \prod_{i=1}^k \frac{\mathrm{ad}(X)^{l_i}}{l_i!} \frac{\mathrm{ad}(X)^{m_i}}{m_i!}$$

Corollary 5.8.

$$\mu(X, Y) = X + Y + \frac{1}{2}[X, Y] + O(|(X, Y)|^3).$$

5.8 Lie Subgroups

Definition 5.18. *Let G be a Lie group. A Lie subgroup H of G is a immersive submanifold that is also a subgroup.*

Definition 5.19. *Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{h}$ of it is called a Lie subalgebra if it is closed under the Lie bracket operation $[\cdot|\cdot]$.*

Definition 5.20. *Let G be a Lie group, then we denote*

$$\mathrm{Lie}(G) = T_1 G.$$

Remark 5.9. *A tautological inclusion $i_H : H \to G$ is an injective immersion.*

Theorem 5.5. *Let G be a connected Lie group. Then there is a bijection between*

$$\{H \subseteq G \mid \textit{Lie subgroups.}\} \leftrightarrow \{\textit{Lie subalgebras of } \mathrm{Lie}(G)\textit{.}\},$$

And the bijection is given by $\mathrm{Lie}(\cdot)$.

Proof. Let H be a subgroup of G and $\mathfrak{h} = T_1 H$.

We first prove the injectivity of $\mathrm{Lie}(\cdot)$.

Fo the surjectivity, let us take $H \subseteq G$ to be a subgroup generated by the image of $\exp_G(\mathfrak{h})$ for a Lie subalgebra $\mathfrak{h}$. By Corollary 5.7, we have

$$(\kappa_x^{-1})_{x \in G}, \quad \kappa_x(Y) = x \exp_G(Y)$$

is an atlas for G. We will show that

$$(\kappa_x^{-1})_{x \in H}$$

is an atlas for H.

First, we claim that if

$$\kappa_x(U\mathfrak{h}) \cap \kappa_y(U \cap \mathfrak{h}) \neq,$$

then there exist neighborhood V_1, V_2 of 0 in $\mathfrak{g}$ such that

$$\kappa_y^{-1} \circ \kappa_x : V_1 \cap \mathfrak{h} \to V_2 \cap \mathfrak{h}$$

is a diffeomorphism.

Since κ_x is an atlas for G, there exist some open neighborhoods V_1, V_2 of 0 such that

$$\kappa_y^{-1} \circ \kappa_x : V_1 \to V_2$$

is a diffeomorphism. The above composition is given by

$$\kappa_y^{-1} \circ \kappa_x(Y) = \log_G(y^{-1} x \exp_G(Y)).$$

∎

Remark 5.10. *This map is a bijection from the set of connected Lie subalgebras of G to the set of Lie subalgebras of* $\mathrm{Lie}(G)$.