Representation Theory 1 V4A3 Sheet 4 Exercise 1

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Exercise 2

(i)

Since the characteristic polynomial of M is of degree three in real coefficient, it has at least one real solution λ. Therefore M is similar to the matrix

$$\begin{pmatrix} R & \underline{o} \\ \underline{o}^T & \lambda \end{pmatrix}.$$

Since M is orthogonal, we have $M^T M = I$, this means that $\det R = 1$ and therefore $\lambda = 1$. By this construction, it obvious that if there are two linearly independent vectors which are the eigenvectors of M with eigenvalue 1, M is the identity matrix.

(ii)

Let us define a map

$$\Phi : B \to S^3, \Phi(\psi, \theta, \phi) = (\cos\psi, \sin\psi\cos\theta, \sin\psi\sin\theta\cos\phi, \sin\psi\sin\theta\sin\phi)$$

is the global diffeomorphism between B and S^3.

Let $\pi : B \to B/\sim$ to be a canonical mapping, then this is a quotient map since open set in U of B, we have

$$\pi(U) = U \sqcup (-U),$$

where $-U$ is the sets of points $(-x, -y, -z)$ in polar coordinate where there exists (ψ, θ, ϕ) represents (x, y, z) in U.

Let us now define a map

$$(\psi, \theta, \phi) \mapsto T_{E\to C} \begin{pmatrix} 1 & \underline{o}^T \\ \underline{o} & R_\psi \end{pmatrix} T_{C\to E}$$

where E is the standard basis of $\mathbb{R}^3$ and C is the matrix obtained by Gram-Schmidt orthonormalization started from the normalized vector of (ψ, θ, ϕ) in the cartesian coordinate.

Then this represents a rotation around a line passing through the origin and (ψ, θ, ϕ). This is a bijection between $B/\sim$ and SO(3) by the theorem of isometries in three dimensional space.

Let us define an operation by

$$(\psi_1, \theta_1, \phi_1) \cdot (\psi_2, \theta_2, \phi_2) = (\psi_1 + \psi_2 \mod \pi, \theta, \phi)$$

where the line passing through the origin and $(\psi_1+\psi_2 \mod \pi, \theta, \phi)$ is orthogonal to both lines passing through $(\psi_1, \theta_1, \phi_1), (\psi_2, \theta_2, \phi_2)$ and the origin if the two lines form a plane. Otherwise, define a rotation with the angle the sum of two rotations. This defines a group operation on $B/\sim$. Furthermore, this makes the bijection into an isomorphism since each rotation is represented as a product of two reflections.

(iii)

The map

$$\pi : B \to B/\sim$$

is a covering since for an arbitrary point $\underline{x} \in B$. For any ball U around x which does not contain the origin, take $\pi(U)$. Then we have

$$\pi^{-1}(\pi(U)) = U \sqcup -U.$$

And any ball U around the origin, we have

$$\pi^{-1}(\pi(U)) = U.$$

Therefore, π is a covering. Therefore, we can make B into a connected Lie group. We know this is diffeomorphic to S^3 by Φ. We claim the statement.