Representation Theory 1 V4A3 Sheet 3 Exercise 1

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1.

By the property of the exponential map we have

$$\underline{\mathrm{Ad}}(g) \circ \exp_H = \exp_H \circ \mathrm{Ad}(g).$$

We have $\exp_H(\mathfrak{h})$ generates H. Thus $\underline{\mathrm{Ad}}(g) \circ \exp_H(\mathfrak{h})$ generates gHg^{-1}. With the equation above, we conclude the statement.

2.

Let $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. Then $\exp_G(X) \in G$ and $\exp_G(Y) \in H$. By using properties of exp,

$$\begin{aligned}\underline{\mathrm{Ad}}(\exp_G X) \exp_G Y &= \exp_G \mathrm{Ad}(\exp_G X)Y, \\ &= \exp_G \exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(X)(Y), \\ &= \exp_G(X) \exp_G(Y) \exp_G(X)^{-1}, \\ &= \exp_G(X) \exp_G(Y) \exp_G(-X).\end{aligned}$$

We let

$$Y' = \exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(X)(Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{ad}(X)^n Y$$

If $\mathfrak{h}$ is an ideal, then $Y' \in \mathfrak{h}$. Since $\exp_G(\mathfrak{g})$ and $\exp_G(\mathfrak{h})$ generate G and H, respectively. We conclude H for $\mathfrak{h} = \mathrm{Lie}(H)$ is normal.

On the other hand, we have

$$\begin{aligned}\frac{d}{dt}|_{t=0} \exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(tX)(Y) &= \exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(tX)(Y) - exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(0)(Y) \mod o(t) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{ad}(tX)^n Y - 1 \mod o(t) \\ &= \mathrm{ad}(X)(Y).\end{aligned}$$

Since H is normal for any

$$\exp_G \exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(tX)(Y) = \exp_G(tX) \exp_G(Y) \exp_G(-tX) \in H.$$

for any t thus, $\mathrm{ad}(X)(Y) \in \mathfrak{h}$.