Exercise 1

(i)

For any $g \in G$ we have

$$b(\pi(g)x, \pi(g)y) = b(x, y).$$

Thus, if we regard $\Phi(g) \to b(\pi(g)x, \pi(g)y)$ as a function, this is a constant function mapping $g \in G$ to $b(x, y)$. Therefore, its derivative is 0.

Let $X \in L$ and calculate the derivative of Φ with respect to X, by the bilinearness of b we get

$$b(\pi(1+X)x, \pi(1+X)y) - b(x, y) = b(\pi(X)x, y) + b(x, \pi(X)y) + b(\pi(X)x, \pi(X)y) \mod o(X).$$

This is equal to 0, by taking at least second order terms we derive

$$b(d\pi(X)x, y) + b(x, d\pi(X)y) = 0.$$

(ii)

Since V is finite dimensional, $\mathfrak{gl}(V)$ is also finite dimensional. Therefore, trace is symmetric bilinear.

Observe that $GL(\mathfrak{gl}(V)) = GL(V)$ by linearlity of automorphisms as algebras and finite dimension arguments. Let

$$\pi : GL(V) \to GL(V), \mathfrak{gl}(V) \ni X \mapsto gXg^{-1}.$$

Then its derivative at 1 is

$$d\pi(1)(X) = [X, \cdot].$$

From the tools of linear algebra we have

$$\mathrm{tr}(gXg{-}1gYg^{-1}) = \mathrm{tr}(gXYg^{-1}) = \mathrm{tr}(gg^{-1}XY) = \mathrm{tr}(XY).$$

By using the previous problem, we conclude tr is invariant.

(iii)

Let $\mathrm{ad}_I(X)$ be the adjoint representation on the subalgebra I. Since L is finite dimensional, so is I. By extending the basis of I to L, we get a matrix

$$\begin{pmatrix} \mathrm{ad}_I(X) & * \\ O & O \end{pmatrix}$$

which is the matrix expression of $\mathrm{ad}_I(X)$ in L. The lower parts are O because I is an ideal. This equals to the matrix expression of $\mathrm{ad}(X)$ in L by the base change. By multiplying this for Y we get

$$\begin{pmatrix} \mathrm{ad}_I(X) \circ \mathrm{ad}_I(Y) & * \\ O & O \end{pmatrix}$$

which is equal to

$$\mathrm{ad}_L(X) \circ \mathrm{ad}_L(Y)$$

Therefore, we conclude the statement.

(iv)

Since $Z(L)$ is an ideal, we can extend a basis of it to get the basis in L. With respect to such basis we get

$$\mathrm{ad}_L(X) = \begin{pmatrix} \mathrm{ad}_J(X) & O \\ O & O \end{pmatrix}$$

Where J is the subspace of L generated by the basis elements not in the center. Thus by the projection we get

$$\pi \circ \mathrm{ad}_{L/Z(L)} = \mathrm{ad}_J \,.$$