Representation Theory 1 V4A3

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2024/2025 Winter Semester - Uni Bonn

1 Overview of the material

1.1 Lie groups

Definition 1.1. *A Lie group is a group G whose underlying set is endowed with the structure of smooth manifolds such that multiplication and inversions are smooth maps.*

Definition 1.2. *A topological group is a group G whose underlying set is endowed with the structure of topological space such that multiplication and inversions are continuous.*

2 Preliminaries

2.1 Topology

Definition 2.1. *We have two axioms about the topological spaces*

1. *T_0(Komogolov) : Given any 2 points, there exists an open set such that it contains one of them but not both.*
2. *T_1(Hausdorff) : Given any 2 points, there exist disjoints open set that each contains one of them.*

Definition 2.2. *A topological space is second countable if it has a basis which contains at most countably many subsets.*

Definition 2.3. *Let X, Y be topological spaces. A continuous function $f : X \to Y$ is said to be proper if the preimage of arbitrary compact set in Y is again compact.*

2.2 Group Theory

Definition 2.4. *Let G be a group and X be a set. An action of group G on a set X is a mapping $l : G \times X \to X$ such that for any $g, h \in G$ and $x \in X$*

$$l(gh, x) = m(g, m(h, x)).$$

Definition 2.5. *A stabilizer of an element $x \in X$ by a group action of G is a subset of G such that*

$$G_x = \{g \in G \mid gx = x\}.$$

Definition 2.6. *A group action is said to be free if for any $x \in X$ we have $G_x = \{1\}$.*

3 Lie groups

3.1 Manifolds

Definition 3.1. *Let $f : X \to Y$ be a mapping between two topological spaces X, Y. f is called a homeomorphism if*

1. *f is a bijection,*
2. *f is continuous,*
3. *f^{-1} is also continuous.*

Definition 3.2. *Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open sets and $f : U \to V$ be a smooth map. Then the derivative of f at $p \in U$ is*

$$df(p) = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}.$$

Proposition 3.1. *Let $f : U \to V, g : V \to W$ be smooth maps. Then for $p \in U$ we have*

$$d(g \circ f) = dg(f(p))df(p).$$

Definition 3.3. *Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open sets. A map $f : U \to V$ is called a diffeomorphism if*

i). f is smooth. ($\Leftrightarrow$ arbitrary order of partial derivatives exists),

ii). f^{-1} is defined and is also a smooth map.

Definition 3.4. *Let X be a topological space. A chart on X is a homeomorphism $h : U \to V$ where $U \subseteq X$ is open and $V \subseteq \mathbb{R}^n$ is open.*

Definition 3.5. *An atlas $\mathscr{A}$ on a topological space X is a collection of charts $\{h_\lambda \mid h_\lambda : U_\lambda \to V_\lambda\}_{\lambda \in \Lambda}$ such that $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of X.*

Definition 3.6. *An atlas $\mathscr{A}$ of X is said to be smooth if for any two charts $h_1 : U_1 \to V_2, h_2 : U_2 \to V_2$. The following,*

$$h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2),$$

is a smooth map. Such map is called a transition map.

Definition 3.7. *Let X be a topological space and $\mathscr{A}_1, \mathscr{A}_2$ be smooth atlases. We say they are equivalent if $\mathscr{A}_1 \cup \mathscr{A}_2$ is also smooth.*

Proposition 3.2. *Above definition indeed defines an equivalence relation.*

Proof. For any $h_1 \in \mathscr{A}_1, h_2 \in \mathscr{A}_2, h_3 \in \mathscr{A}_3$,

$$h_3 \circ h_1^{-1} = h_3 \circ h_2^{-1} \circ h_2 \circ h_1^{-1}.$$

□

Definition 3.8. *A smooth manifold is a second countable Hausdorff topological space with equivalence classes of smooth atlases.*

Definition 3.9. *Let M, N be smooth manifolds, $f : M \to N$ be a map, and $p \in M$. f is said to be smooth at p if for one (hence any) pair of charts around p and $f(p)$,*

$$h_M : U_M \to V_M, h_N : U_N \to V_N,$$

the composed function

$$h_N \circ f \circ h_M^{-1} : V_M \to V_N$$

is smooth at $h_M(p)$.

Remark 3.1. *We can define a function* $\dim : M \to N$ *such that*

$$\dim(p) = \dim(V)_p,$$

for any chart $h : U \to V$ around p. And this function is locally constant. In particular, if M is connected then it has a well-defined dimensions.

Definition 3.10. *Let M, N be smooth manifold and $f : M \to N$ be a mapping which is smooth at $p \in M$. For any charts,*

$$h_N \circ f \circ h_M^{-1} : V_M \to V_N,$$

the rank of f at p is such that

$$\mathrm{rk}(f; p) = \mathrm{rank}(\mathbf{df}(h_M(p))(h_N \circ f \circ h_M^{-1})).$$

Definition 3.11. *Let M, N be smooth manifolds and $f : M \to N$ be a smooth map. A point p is said to be regular with respect to the map f. And a point $q \in N$ is called a regular value if all $p \in f^{-1}(q)$ are regular.*

Definition 3.12. *Let M be a manifold. A subset $N \subseteq M$ is called an embedded submanifold if for any point $p \in N$, there is a chart $h_M : U_M \to V_M$ around p such that*

$$h_M|_N : U_M \cap N \to V_M \cap \mathscr{R}^n,$$

is a diffeomorphism where n is the dimension of N.

In particular, an embedded submanifold of an euclidean space is called a embedded manifold.

Definition 3.13. *A map $f : M \to N$ of smooth manifolds is called a diffeomorphism if*

i). $f : M \to N$ is a bijection,

ii). f, f^{-1} are both smooth.

Theorem 3.1. *Let $f : M \to N$ be a smooth map between manifolds, and $q \in N$ be a regular value. Then $f^{-1}(q) \subset M$ is an embedded submanifold.*

Theorem 3.2. *Let $f : M \to N$ be a smooth map of manifolds $p \in M$ be a regular point, and $\dim(p) = \dim(f(p))$. Then f is a local diffeomorphism of p. In other words, there is a neighborhood U_M of p in M and $f(p) \in U_N \subset N$ such that*

$$f|_{U_M} : U_M \to U_N,$$

is a diffeomorphism.

Definition 3.14. *Let $M \subseteq \mathbb{R}^n$ be an embedded manifold such that for some open set $U \subset \mathbb{R}^n$, there is $V \subset \mathbb{R}^n$ such that*

$$h : U \to V, \quad h_M : U \cap M \to V \cap \mathbb{R}^m,$$

is a diffeomorphism where h_M is defined to be taking the first m coordinate of the points in V. (Thus $m \leq n$).

The tangent space T_pM of M at p is the subspace of $\mathbb{R}^n$ such that

$$(\mathbf{dh}(p))^{-1}(\mathbb{R}^m) \subset \mathbb{R}^n.$$

There are three definitions of tangent spaces and they are all equivalent. However, each of them has its own advantages.

Definition 3.15 (Coordinate tangent space)**.** *Given a smooth manifold M and a point $p \in M$. The coordinate tangent space of p is such that*

$$T_p^{\mathbf{Coo}}M = \{(h, v) \mid h : U \to V \text{ is a chart}, v \in \mathbb{R}^m\}/\sim.$$

Where $\sim$ is an equivalence relation such that

$$(h_1, v_1) \sim (h_2, v_2) \text{ if } (\mathbf{d}(h_2 \circ h_1^{-1})(h_1(p)))(v_1) = v_2.$$

Definition 3.16. *Given a smooth manifold M, a point $p \in M$, and a smooth map $\alpha : I \to M$ whose domain I is an open interval contains 0. α is called a smooth curve if $\alpha(0) = p$.*

Definition 3.17. *Two smooth curves $\alpha, \beta : I \to M$ through p are said to be tangentially equivalent if for one (hence any) charts $h : U \to V$ around p, we have*

$$d(h \circ \alpha)(0) = d(h \circ \beta)(0).$$

We denote such relation as $\sim_T$.

Definition 3.18 (Geometric tangent space)**.** *The geometric tangent space at p of a smooth manifold M is such that*

$$T_p^{\mathbf{Geo}} = \{\alpha : I \to M \mid \alpha \text{ is a smooth curve}\}/ \sim_T .$$

Definition 3.19. *A germ of smooth functions of manifolds M at p is an equivalence class of tuples (U, f) where*

i). $U \subset M$ is a neighborhood of p,

ii). $f : U \to \mathbb{R}$ is smooth,

and two tuples $(U_1, f_1), (U_2, f_2)$ are equivalent if there is a neighborhood V of p such that $V \in U_1 \cap U_2$ and $f_1|_V = f_2|_V$.

And we denote the set of germs at p as

$$\mathscr{C}^\infty(p).$$

Remark 3.2. *$\mathscr{C}^\infty(U, \mathbb{R})$ and $\mathscr{C}^\infty(p)$ are rings, in fact $\mathbb{R}$-algebras.*

Definition 3.20. *Let R be a ring and A be a bimodule over R. A R-derivation in A is an operator $X : A \to A$ such that the Leibniz rule holds. In other words,*

$$X(ab) = aX(b) + X(a)b,$$

holds for all $a, b \in A$.

Definition 3.21 (Algebraic tangent space)**.** *The algebraic tangent space $T_p^{\mathbf{Alg}}M$ of M at p is the set of $\mathbb{R}$-derivations $X : \mathscr{C}^\infty(p) \to \mathbb{R}$.*

Remark 3.3. *In the above definition, $\mathbb{R}$ is considered as a $\mathscr{C}^\infty(p)$-bimodule via the evaluation map $f \mapsto f(p)$.*

Theorem 3.3. *The following are isomorphisms of $\mathscr{R}$-vector spaces.*

$$\begin{aligned} T_p^{\mathbf{Geo}}M &\to T_p^{\mathbf{Alg}}M, \alpha \mapsto (f \mapsto (f \circ \alpha)'(0)), \\ T_p^{\mathbf{Alg}}M &\to T_p^{\mathbf{Coo}}M, X \mapsto (h, ((Xh_i)(p))_{i=1,\cdots,n}), \\ T_p^{\mathbf{Coo}}M &\to T_p^{\mathbf{Geo}}M, (h, v) \mapsto \alpha(t) = h^{-1}(h(p) + t \cdot v). \end{aligned}$$

Proposition 3.3. *$\mathscr{C}^\infty(p)$ is a local ring with its maximal ideal*

$$\mathfrak{m}_p = \{f \in \mathscr{C}^\infty(p) \mid f(p) = 0\}.$$

Moreover, if we have a derivation $X : \mathscr{C}^\infty(p) \to \mathbb{R}$, the restricted derivation $X|_{\mathfrak{m}_p}$ is in $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2)$. And by this restriction, we get an isomorphism between $T_p^{\mathbf{Alg}}M$ and $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R})$.

Remark 3.4. *In this way, a smooth manifold is recognized as a locally ringed space, locally isomorphic to $\mathbb{R}^n$.*

Remark 3.5. *Let V be a finite dimensional $\mathbb{R}$-vector space. It has a tautological smooth manifold structure by taking charts such that the sets of isomorphisms of V and $\mathbb{R}^n$ given by arbitrary basis of V.*

We claim that we have canonical isomorphisms

$$T_pV \to V,$$

for any $p \in V$,

$$\begin{aligned} V &\to T_p^{\mathbf{Coo}}V, v \mapsto (h, h(v)), \\ V &\to T_p^{\mathbf{Geo}}V, v \mapsto (t \mapsto p + tv), \\ V &\to T_p^{\mathbf{Alg}}V, v \mapsto \left(f \mapsto \left.\frac{d}{dt}\right|_{t=0} f(p+tv) \right) \end{aligned}$$

Definition 3.22. *Let $f : M \to N$ be a map of smooth manifolds which is smooth at $p \in M$. Its differential of p is the linear map*

$$\mathbf{d}f(p) = \mathbf{d}_p(f) : T_pM \to T_{f(p)}N,$$

defined as follows.

1). Geometric tangent space : $\mathbf{d}_p(f)(\alpha) = f \circ \alpha$ where α is a smooth curve.

2). Algebraic tangent space : $\mathbf{d}_p(f)(X)(\varphi) = X(\varphi \circ f)$ where $\varphi \in \mathscr{C}^\infty(f(p))$.

3). Coordinate tangent space : $\mathbf{d}_p(f)(h_M, v_M) = (h_N, d_{h_M(p)}(h_N)$.

Remark 3.6. *Given a chart $h : U \to V$ around $p \in M$. h consists of coordinate functions h_i where $1 \leq i \leq m$ for $V \subset \mathbb{R}^m$. We have for each i*

$$\mathbf{d}_p h_i : T_pM \to \mathbb{R},$$

and

$$B = \{d_p h_1, \cdots, d_p h_m\}$$

is a basis of the dual space $(T_pM)^*$.

Let

$$\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\}$$

be the dual basis of B. By definition, this means that for any $1 \leq i, j \leq m$, we have

$$\frac{\partial}{\partial x_i} h_j = d_p h_j(\frac{\partial}{\partial x_i}) = \delta_{ij}.$$

Proposition 3.4. *Let $f : M \to N$ be a map between smooth manifolds which is smooth and $q \in N$ be a regular value. For $p \in f^{-1}(q)$, we have*

$$T_p f^{-1}(q) = \mathbf{d}_p(f)^{-1}(0) \subset T_pM.$$

Proof. □

3.2 Immersions and Submersions

Definition 3.23. *Let $f : M \to N$ be a smooth map of smooth manifolds. f is called an*

1). immersion if $\mathbf{d}f : T_pM \to T_{f(p)}N$ is injective for any $p \in M$,

2). submersion, if $\mathbf{d}f(p) : T_pM \to T_{f(p)}N$ is surjective for any $p \in M$.

Remark 3.7. *An immersion need not be injective. The counter example is*

$$e^{ix} : \mathbb{R} \to S^1,$$

is an immersion.

Remark 3.8. *A submersion need not be injective. The counter example is*

$$i_U : U \to M,$$

an inclusion map is a submersion.

Remark 3.9. *We know that if f is a submersion, then $f^{-1}(q)$ is an embedded submanifold. However, if f is an immersion, even it is injective, $f(M)$ need not be an embedded submanifold of N.*

Definition 3.24. *An immersed submanifold is an image of an injective immersion.*

Remark 3.10. *We endow $f(M)$ with the transported topology and differential structure from M so that f becomes a diffeomorphism between M and $f(M)$. But this topology need not be the relative topology from N. It may be strictly finite.*

Example 3.1. *Let $T = S^1 \times S^1$ be a torus. Let $r \in \mathbb{R}$. We consider a map $f : \mathbb{R} \to T$ such that*

$$f(x) = (e^{2\pi tx}, e^{2\pi rix}).$$

This is an immersion for any r. We examine this by several cases.

First, when r is not a rational number then f is injective, the image is an immersed manifold. However, a copy of $\mathbb{R}$. But this image is a dense subset of the torus.

Second, if r is rational then f is not injective. It is going to factor through an injective immersion $\mathbb{R}/b\mathbb{Z} \to T$ where $r = \frac{a}{b}$, $a, b \in \mathbb{Z}$ are coprime. This image is not only immersed but also embedded.

Remark 3.11. *If $f : M \to N$ is an immersion, $\mathbf{d}f_(p)$ identifies T_pM with a linear subspace of $T_{f(p)}N$.*

Proposition 3.5. *If $f : M \to N$ is an injective immersion, that is also closed subset of N, then its image is an embedded submanifold.*

Remark 3.12. *Thus we have the notion of a closed submanifold.*

3.3 Multi-linear forms

Definition 3.25. *Let $\mathbb{V}$ be a vector space and $\varphi : \bigoplus_{i=1}^m V \to \mathbb{R}$ is called a m-multi-linear function if for any $i = 1, \cdots, m$ and $\{a_j\}_{j \neq i} \subset V$ we have*

$$\varphi(a_1, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_m) : V \to \mathbb{R}$$

is a linear function

Definition 3.26. *Let X be a smooth n-dimensional manifold and $m \in \mathbb{N}$. Then we define the followings*

1. *$\mathscr{L}_p^m = \{\varphi : \bigoplus_{i=1}^m T_pX \to \mathbb{R} | \varphi$ is a m-multi-linear function.$\}$*
2. *$\mathscr{L}^m = \bigcup_{p \in X} \mathscr{L}_p^m$*

Definition 3.27. *Let X be a smooth n-dimensional manifold. A map $V : X \to \mathscr{L}^m$ is called a m-tensorfield if*

i. *For any $p \in X$, $V(p) \in \mathscr{L}_p^m$.*

ii. *For any chart (U, φ) around p with a basis $\{e_1^\varphi, \cdots, e_n^\varphi\}$ and for any $i_1, \cdots, i_m \in \{1, \cdots, n\}$ we have a map $V_{(i_1, \cdots, i_m)} : X \to \mathbb{R}$ such that $V_{(i_1, \cdots, i_m)}(p) = V(p)(\underline{e}_{i_1}, \cdots, \underline{e}_{i_m})$ is smooth.*

Proposition 3.6. *For any m tensorfield V, we have*

Definition 3.28. *We define $\mathscr{V}^m(X)$ to be the set of all m-tensorfield.*

Proposition 3.7. *$\mathscr{V}^m(X)$ is a vector space over $\mathbb{R}$ and a module over $\mathscr{F}(X)$ with the common basis $\{E_{i_1, \cdots, i_m}\}_{i_1, \cdots, i_m \in \{1, \cdots, n\}}$*

Proposition 3.8. *Let X be a smooth n-dimensional manifold and $V : X \to \mathscr{L}^m$ be such that for any $p \in X, V(p) \in \mathscr{L}_p^m$ the followings are equivalent.*

1. *V is a m-tensorfield.*
2. *For any chart (U, φ) around p with basis $\{\underline{e}_1^\varphi, \cdots, \underline{e}_n^\varphi\}$ and for any $1 \leq i_1, \cdots, i_m \leq n$ there exist smooth mappings $\lambda_{i_1, \cdots, i_m} : X \to \mathbb{R}$ such that $V(p) = \sum_{1 \leq i_1, \cdots, i_m \leq n} \lambda_{i_1, \cdots, i_m}(p) E_{i_1, \cdots, i_m}^\varphi$.*
3. *For any vectorfields $v_1, \cdots, v_m : X \to TX$ we have a function $V : X \to \mathbb{R}$ such that $V_{v_1, \cdots, v_m}(p) = V(p)(v_1(p), \cdots, v_m(p))$ is smooth.*

Proof. 1.⇔2. is trivial. 1.⇒3. is clear by the multi-linearity, and 3.⇒1. is choosing $v_i = e_i^\varphi$ for each $i = 1, \cdots, n$.

□

Proposition 3.9. *Let $V : X \to \mathscr{L}^m$ then thne followings are equivalent.*

1. *V is a m-tensorfield.*
2. *For any $\{v_1, \cdots, v_m\} \in \mathscr{V}(X)$, $\Psi : \bigoplus_{i=1}^m \mathscr{V}(X) \to \mathscr{F}(X)$ such that $\Psi(v_1, \cdots, v_m)(p) = V(p)(v_1(p), \cdots, v_m(p))$ is smooth and $\mathscr{F}(X)$-linear.*

Proof. 1.⇒2. follows from the multilinearity and decompositions of tensors. 2.⇒1. follows by fixing all element except one we still have the linearity thus, the function is mutilinear. □

3.4 Tensor and Wedge products

Definition 3.29. *Let $V_1 : X \to \mathscr{L}^r, V_2 : X \to \mathscr{L}^s$ be tensorfield. Then We define the tensorproduct $V_1 \otimes V_2 : X \to \mathscr{L}^{r+s}$ of them to be*

$$(V_1 \otimes V_2)(p)(v_1, \cdots, v_r, v_{r+1}, \cdots, v_{r+s}) = V_1(p)(v_1, \cdots, v_r)V_2(p)(v_{r+1}, \cdots, v_{r+s})$$

Proposition 3.10. *The operation $\bigotimes$ is bilinear and associative.*

Proof. By substituting values, they are trivial. □

Proposition 3.11. *Let $U \subset X$ be an open set and $V_1, \cdots, V_n \in \mathscr{V}^1(U)$ be a basis in $\mathscr{V}^1(U)$ then $\{\bigotimes_{j=1}^r V_{i_j}\}_{1 \leq i_1, \cdots, i_r \leq r}$ is a basis in $\mathscr{V}^r(U)$.*

Proof. Since $\otimes$ is bilinear, this is a tensor product thus the set in the statement is indeed a basis. □

Definition 3.30. *Let $V \in \mathscr{V}^m(X)$ be a m-tensor. V is said to be alternating if for any $p \in X$, $(v_1, \cdots, v_m) \in \bigoplus_{i=1}^m T_pX$ and $\sigma \in \mathfrak{S}_m$ we have*

$$V(p)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) = \mathrm{sgn}(\sigma)V(p)(v_1, \cdots, v_m)$$

Furthermore, such V is called a m-form.

Notation 3.1. *The set of all m-forms is denoted by*

$$\mathscr{A}^m(X) = \{V \in \mathscr{V}^m(X) \mid V \text{ is a m-form.}\}$$

Definition 3.31. *Let $V_1 \in \mathscr{A}^r(X), V_2 \in \mathscr{A}^s(X)$ then the wedge product is*

$$(V_1 \wedge V_2)(p)(v_1, \cdots, v_{r+s}) = \frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \mathrm{sgn}(\sigma) V_1 \otimes V_2(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)})$$

Proposition 3.12. *Let $V_1, \cdots, V_n \in \mathscr{A}^1(X)$, $p \in X$ and $v_1, \cdots, v_n \in T_pX$ then we have*

$$(V_1 \wedge \cdots \wedge V_n)(p)(v_1, \cdots, v_n) = \det(V_i(p)(v_j))_{i,j}$$

Proof.

$$(V_1 \wedge \cdots \wedge V_n)(p)(v_1, \cdots, v_n) = \frac{1}{1! \cdots 1!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n V_i(p)(v_\sigma(i))$$

□

Proposition 3.13. *Similar to the case in tensorfields, we have the following statements.*

1. *$\mathscr{A}^m(X)$ is a subspace of $\mathscr{V}^m$ over $\mathbb{R}$.*
2. *$\mathscr{A}^m(X)$ is a module over $\mathscr{F}(X)$.*

Proof. Trivial. □

Proposition 3.14. *Let $V_1 \in \mathscr{A}^r, V_2 \in \mathscr{A}^s$, then $V_1 \wedge V_2 \in \mathscr{A}^{r+s}$ and such $\wedge : \mathscr{A}^r \times \mathscr{A}^s \to \mathscr{A}^{r+s}$ is bilinear.*

Proof. Bilinearity follows from the bilinearity of $\otimes$. We will show that this is indeed well-defined.

Let $\sigma \in \mathfrak{S}_{r+s}$. Then we have

$$\begin{aligned}(V_1 \wedge V_2)(p)(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)}) &= \frac{1}{r!s!} \sum_{\tau \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\tau) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)}) \\ &= \operatorname{sgn}(\sigma) \frac{1}{r!s!} \sum_{\tau\circ\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\tau\circ\sigma) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)}) \\ &= \operatorname{sgn}(\sigma)(V_1 \wedge V_2)(p)(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)})\end{aligned}$$

□

Proposition 3.15.

$$V_2 \wedge V_1 = (-1)^{rs}(V_1 \wedge V_2)$$

Proof. Let $\tau \in \mathfrak{S}_{r+s}$ to be such that

$$\tau(i) = \begin{cases} r+i & (1 \le i \le s) \\ i-s & (s+1 \le i \le r+s) \end{cases}$$

Then clearly the inversion number is $N(\tau) = rs$. It is also obvious that

$$V_2 \wedge V_1(p)(v_{\tau(1)}, \cdots, v_{\tau(r+s)}) = V_1 \wedge V_2(p)(v_1, \cdots, v_{r+s})$$

□

Proposition 3.16. *Let $V_1 \in \mathscr{A}^r, V_2 \in \mathscr{A}^s, V_3 \in \mathscr{A}^t$ then $(V_1 \wedge V_2) \wedge V_3 = V_1 \wedge (V_2 \wedge V_3)$.*

Proof.

$$\begin{aligned}(V_1 \wedge V_2) \wedge V_3(p)(v_1, \cdots, v_{r+s+t}) &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau)(V_1 \wedge V_2) \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\ &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau) \\ &\quad (\frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\sigma) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)})) \\ &\quad V_3(v_{\sigma(r+s+1)}, \cdots, v_{\sigma(r+s+t)})\end{aligned}$$

If for $\tau_1, \tau_2 \in \mathfrak{S}_{r+s+t}, \sigma_1, \sigma_2 \in \mathfrak{S}_{r+s}$ we have $\tau_1 \circ \sigma_1 = \tau_2 \circ \sigma_2$ then they satisfy the followings

i. For any $r+s+1 \leq i \leq r+s+t$ we have $\tau_1(i) = \tau_2(i)$.

ii. From above we get $\tau_2^{-1} \circ \tau_1 \in \mathfrak{S}_{r+s}$

Fixing σ_1, there exists $(r+s)!$ many such σ_2. This implies that we can choose σ_1 to be the identity. Thus we get

$$\begin{aligned}(V_1 \wedge V_2) \wedge V_3(p)(v_1, \cdots, v_{r+s+t}) &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau)(V_1 \wedge V_2) \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\ &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau) \frac{(r+s)!}{r!s!} V_1 \oplus V_2 \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\ &= \frac{1}{r!s!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau) V_1 \oplus V_2 \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)})\end{aligned}$$

From the previous proposition we get □

4 Integration

Definition 4.1. *A differential k-form ω on a smooth manifold M is a collection $\omega(p) \in A^k(T_pM)$ for all $p \in M$.*

Remark 4.1. *We can define what it means for ω to be continuous or smooth at some points $p \in M$ as follows.*

First, we pick a chart $h : U \to V$ around p and get the basis

$$\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\},$$

of T_pM that moves with $p \in U$.
We also have a basis $A^k(T_pM) = \bigwedge^k (T_pM)^*$. Hence we can express ω as p in terms of that basis and the scalars in this expression are functions on U.

$$\omega(p) = \sum f_{i_1, \cdots, i_k} \cdot d_{x_{i_1}} \wedge \cdots \wedge d_{x_{i_k}}.$$

And we can require $f_{i_1, \cdots, i_k} \cdots d_{x_{i_1}}$ to be smooth/continuous at p.

Example 4.1. *If $M = \mathbb{R}^n$, we have the canonical identification,*

$$T_pM = \mathbb{R}^n.$$

This gives us standard differential form of degree n. which is given by

$$e_1^* \wedge \cdots \wedge e_n^*,$$

where $e_1, \cdots, e_n$ is the standard basis of $\mathbb{R}^n$.

Definition 4.2. *Let $f : M \to N$ be a smooth map of manifolds and ω be a differential form of degree k on N. We define $f^*(\omega)$ of degree k on M by*

$$f^*(w)(p)(x_1, \cdots, x_k) = \omega(f(p))(\mathbf{d}f_p(x_1), \cdots, \mathbf{d}f_p(x_k)).$$

Definition 4.3. *A differential n-form ω on M is said to be locally integrable if for any point $p \in M$, if for any point $p \in M$, there is one (hence any) chart $h : U \to V$ such that $\omega|_U =$*

5 Lie Algebras

5.1 Important homomorphisms and their properties.

Recall if $f : M \to N$ is a smooth map of smooth manifolds and $p \in M$, we get $df(p) : T_pM \to T_{f(p)}N$ is linear.

Proposition 5.1. *Let $(G, \mu, \iota, 1)$ be a lie group and $\mathfrak{g} = T_1G$. We have*

$$d\mu(1,1) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (X, Y) \mapsto X + Y.$$
$$d\iota(1) : \mathfrak{g} \times \mathfrak{g}, X \mapsto -X$$

Definition 5.1. *A Lie group homomorphism is a smooth map of Lie groups that is a homomorphism.*

Remark 5.1. *If $f : G \to H$ is a Lie group homomorphism then*

$$df(1) : \mathfrak{g} \to \mathfrak{h}$$

is a linear map.

Definition 5.2. *Let G be a Lie group. The adjoint action of G on itself is*

$$\underline{\mathrm{Ad}}(g) : G \to G, h \mapsto ghg^{-1}$$

which is a group homomorphism.

Definition 5.3. *Let G be a Lie group and $\mathfrak{g} = T_1G$. Then we define*

$$\mathrm{Ad}(g) = d\underline{\mathrm{Ad}}(g)(1) : \mathfrak{g} \to \mathfrak{g}.$$

We call this the adjoint action of G on $\mathfrak{g}$.

Remark 5.2. *The term, "action" in the definition above is justified by the chain rule*

$$\mathrm{Ad}(g \cdot h) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h).$$

Definition 5.4. *Let G be a Lie group and $\mathfrak{g} = T_1G$. By regarding Ad as a function from G to $\mathrm{GL}(\mathfrak{g})$. Notice that by the definition of groups we have $\mathrm{Ad}(g)$ is injective.*

We now define the adjoint action of $\mathfrak{g}$ on itself to be

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), X \mapsto d\,\mathrm{Ad}(1)X.$$

Definition 5.5. *Let G be a Lie group and $\mathfrak{g} = T_1G$. The Lie bracket is $[\cdot|\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

$$[X|Y] = \mathrm{ad}(X)(Y).$$

Proposition 5.2. *Let $G = \mathrm{GL}_n(\mathbb{R})$ and $\mathfrak{g} = \mathbb{R}^{n\times n}$. Let $g \in G$ and $X, Y \in \mathfrak{g}$. We have*

$$[X|Y] = XY - YX.$$

Proof. Let $g \in G$,

$$\begin{aligned} \mathrm{Ad}(g)X &= d\underline{\mathrm{Ad}}(g)(1)X, \\ &= g(1+X)g^{-1} - g1g^{-1} \mod o(X), \\ &= gXg^{-1} \mod o(X), \\ &= gXg^{-1}. \end{aligned}$$

In particular $\underline{Ad}$ is a liner map. Now we compute the Lie bracket

$$[X|Y] = \mathrm{ad}(X)(Y) = [E_Y \circ \mathrm{ad}](X),$$

where E_Y is the evaluation map

$$E_Y : \mathrm{End}(\mathfrak{g}) \to \mathfrak{g}, \phi \mapsto \phi(Y).$$

$$\begin{aligned} [X|Y] &= [E_Y \circ \mathrm{ad}](X), \\ &= d[g \mapsto \mathrm{Ad}(g)Y](1)(X), \\ &= d[g \mapsto gYg^{-1}](1)(X). \end{aligned}$$

By the first computation we did, we see that

$$[X|Y] = (1+X)Y(1+X)^{-1} - Y \mod o(X).$$

We have the following identity

$$(1-X)^{-1} = 1 + X + X^2 + \cdots.$$

Substituting $-X$ we derive that

$$1 + X = \sum_{i=0}^{\infty} (-1)^i X^i.$$

And we only need at most degree 1 terms of X. We conclude that

$$[X|Y] = XY - YX.$$

∎

Remark 5.3. *This works for any matrix groups such as $\mathrm{SL}_n(\mathbb{R}), O(p,q)$.*

Proposition 5.3. *Let $f : G \to H$ be a Lie group homomorphism. For $g \in G$, we have*

$$df(1) \circ \mathrm{Ad}(g) = \mathrm{Ad}(f(g)) \circ df(1). \tag{5.1}$$

And for $X, Y \in \mathfrak{g}$, we have

$$df(1)([X|Y]_G) = [df(1)X, df(1)Y]_H. \tag{5.2}$$

Proof. Let us consider the composition of f and $\underline{\mathrm{Ad}}$. By definition, we see

$$f \circ \underline{\mathrm{Ad}}(g)(h) = f(g)f(h)f(g)^{-1} = \underline{\mathrm{Ad}}(f(g))(f(h)).$$

Since $\underline{\mathrm{Ad}}(1) = 1$ and by the chain rule we have Equation 5.1.

□

5.2 Lie Algebras

Definition 5.6. *A Lie algebra a (finite dimensional) vector space L over $\mathbb{R}$ or $\mathbb{C}$ together with a bilinear map $[\cdot|\cdot] : L \times L \to L$ such that*

i $[X|Y] = -[Y|X]$,

ii $[X|[Y|Z]] + [Y[Z|X]] + [Z|[X|Y]] = 0$ *which is called Jacobi identity.*

Proposition 5.4. *Let G be a Lie group and $\mathfrak{g} = T_1G$. Then $\mathfrak{g}$ equipped with $[X|Y] = \mathrm{ad}(X)(Y)$ is a $\mathbb{R}$-Lie algebra.*

Proof. Consider the commutator map $G \times G \to G, (x, y) \mapsto xyx^{-1}y^{-1}$. This is a smooth map as it is a composition of smooth maps $\mu(\mu(\cdot,\cdot), \mu(\iota(\cdot), \iota(\cdot)))$. Moreover, we can write this as

$$\underline{\mathrm{Ad}}(x)(y)\iota(y).$$

Differentiate this at $y = 1$ in the direction of Y, we get

$$d(\underline{\mathrm{Ad}}(x)(1)\iota(1))Y = \mathrm{Ad}(x)Y - Y,$$

since $d\iota(Y) = -Y$. Differentiate this again at $x = 1$ with respect to X we get $[X, Y]$.

Repeating the argument with

$$x\underline{\mathrm{Ad}}(y)(\iota(x)).$$

Differentiate this at $x = 1$ with the direction to X we get

$$X - \underline{\mathrm{Ad}}(y)X = X - yXy^{-1}.$$

Differentiate this again at $y = 1$ with the direction to Y, we get $-[Y|X]$. By smoothness, we get
$$[X|Y] = -[Y|X].$$
For the second property, we consider the Lie group homomorphism,
$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}).$$
By Proposition 5.3, we have
$$\mathrm{ad}[X|Y]_G = [\mathrm{ad}(X)|\,\mathrm{ad}(Y)]_{\mathrm{GL}(\mathfrak{g})} = \mathrm{ad}(X)\,\mathrm{ad}(Y) - \mathrm{ad}(Y)\,\mathrm{ad}(X).$$
Therefore, by definition of $[\cdot|\cdot]$, we get
$$[[X|Y]|Z] = [X|[Y|Z]] - [Y|[X|Z]]$$
By the first property, we get the Jacobi identity. □

Example 5.1. *If V is a finite dimensional $\mathbb{R}$-vector space then $\mathrm{End}(V)$ equipped with $[X|Y] = XY - YX$ is a Lie algebra. In fact, this coincides with the Lie algebra of the Lie group $\mathrm{GL}(V)$.*

Definition 5.7. *A homomorphism of Lie algebras is a linear map $f : L \to M$ such that for $X, Y \in L$*
$$f([X|Y]_L) = [f(X)|f(Y)]_M$$

Corollary 5.1. *If $f : G \to H$ is a homomorphism of Lie groups, then $df(1) : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras.*

5.3 The identity component

Lemma 5.1. *Let G be a topological group. If $H \subset G$ is an open subgroup, then it is also closed. Thus if G is connected we have $H = G$.*

Proof. Let $\{1\} \cup I$ be a set of representations of equivalence classes in G/H. In other words we have
$$G = H \cup \bigcup_{i \in I} iH.$$
Since $\bigcup_{i\in I} iH$ is open, thus its complement H is closed. □

Lemma 5.2. *Let G be a connected topological group and $U \subseteq G$ be a neighborhood of 1. Then U generates G.*

Proof. Since $U \cap U^{-1}$ is non-empty and open. We may assume with out the loss of generality that $U = U^{-1}$. Let us denote
$$U^n = \{g_1 \cdots g_n \mid g_1, \cdots, g_n \in U\}.$$

And for $g_1 \cdots g_n \in U^n$, we take $V \subset U$ an open subset and $g_1 \in V$. $Vg_2 \cdots g_n$ is open in U. We now conclude that

$$H = \bigcup_{n=1}^{\infty} U^n$$

is an open subset which is a subgroup of G since it is closed under multiplication and inversion. Since G is connected we conclude that $H = G$. $\square$

Definition 5.8. *A subgroup H of a group G is said to be characteristic if for any automorphism $\varphi : G \to G$, we have $\varphi(H) \subseteq H$.*

Definition 5.9. *Let X be a topological space. A connected component C of $x \in X$ is the largest connected set which contains x.*

Proposition 5.5. *If C is a connected component of the topological space X, then it is closed.*

Proof. $\square$

Proof. Let $f : \overline{C} \to \{0, 1\}$ be a continuous function where $\{0, 1\}$ is with the discrete topology. Then for any $x \in C$ we conclude $f(x) = 0$ without the loss of generality. By the continuity of f we conclude that $f(x) = 0$ for any $x \in \overline{C}$. $\square$

Definition 5.10. *A topological space $(X, \mathscr{T})$ is said to be locally connected if for any point $x \in X$ and its neighborhood U, there exists a connected neighborhood V such that $x \in V \subset U$.*

Proposition 5.6. *A component of locally connected topological space is open.*

Proposition 5.7. *Let G be a topological group and G^0 be the connected component of G containing* 1*.*

1) G^0 is a closed characteristic subgroup of G.

2) If G is locally connected then G^0 is open and contained in any open subgroup of G.

3) The connected component of G are precisely G^0-cosets.

Proof. By Proposition 5.5, G^0 is a closed set. Since continuous maps preserve connectedness and 1 is mapped to 1, we can conclude that G^0 is characteristic. Similarly, since multiplication and inversion are smooth, thus continuous, we conclude that G^0 is a subgroup of G. This proves the first statement.

If G is locally connected, by Proposition 5.6, G^0 is open. If $H \subset G$ is any open subgroup, then $H \cap G^0$ is an open subgroup of G^0. By Lemma 5.2, we have $H \cap G^0$ generates G^0. $H \cap G^0$ is a group, we conclude that it is equal to G^0. This shows that G^0 is contained in any open subgroup of G.

Let C be a connected component and $g \in C$. Since $\mu(\cdot, g^{-1} : G \to G$ is continuous, we conclude that $\mu(C, g^{-1})$ is contained in the connected component which contains 1. Hence $C = G^0 g$. $\square$

5.4 Invariant vector fields

Definition 5.11. *Let M be a manifold. A vector field v on M is an assignment that for each $p \in M$, we have $v(p) \in T_pM$. It is said to be smooth if locally around each point $p \in M$, it's coefficients in terms of local coordinates are smooth functions. In other words, given a chart $h : U \to V$, we can get a basis $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ of T_pM for all $p \in U$. And locally*

$$v(p) = \sum_{i=1}^{n} c_i(p) \cdot \frac{\partial}{\partial x_i}.$$

And each c_i is smooth.

Definition 5.12. *Let M be a manifold, v be a smooth vector field. An integral curve is a pair (I, γ) where*

i). I is an open interval,

ii). $\gamma : I \to M$ is a smooth map such that $\gamma'(t) = v(\gamma(t))$.

Proposition 5.8. *Let v be a smooth vector field on a manifold M, then we have the following statements.*

1). Given $p \in M$, there exists a integral curve (I, γ) such that $0 \in I$ and $\gamma(0) = p$.

2). $(I_1, \gamma_1), (I_2, \gamma_2)$ be integral curves with above properties. Then for any $t \in I_1 \cap I_2$, we have $\gamma_1(t) = \gamma_2(t)$.

3). In particular, we can splice such γ_1, γ_2.

Proof. Follows from existence and uniqueness of solutions of ordinary differential equations in $\mathbb{R}^n$ via charts. □

Remark 5.4. *There is a maximal integral curve through p.*

Definition 5.13. *Let G be a Lie group and $g \in G$. We define*

$$L_g : G \to G, L_g(x) = gx, \quad R_g : G \to G, R_g(x) = xg,$$

the left and the right translations. Obviously these are diffeomorphisms as the inverses are $L_{g^{-1}}, R_{g^{-1}}$, respectively.

Remark 5.5. *By differentiating these we get*

$$dL_g(1) : \mathfrak{g} \to T_gG, \quad dR_g(1) : \mathfrak{g} \to T_gG.$$

Proposition 5.9. *Let G be a Lie group and $g \in G$. Then $dL_g(1), dR_g(1)$ are isomorphisms between $\mathfrak{g}$ and T_gG. Therefore, we can naturally identify T_gG by $\mathfrak{g}$. Moreover, $dL_g(1), dR_g(1)$ are not the same in general and differ by the automorphism $\mathrm{Ad}(g)$.*

Proof. □

Definition 5.14. *A vector field v on the Lie group G is said to be*

1). left-invariant if $v(g) = dL_g(1)(v(1))$,

2). right-invariant if $v(g) = dR_g(1)(v(1))$.

Remark 5.6. *Such vector field is automatically smooth. And the assignments*

$$X^L = X \mapsto (g \to dL_g(1)(X)), \quad X^R = X \mapsto (g \to dR_g(1)(X))$$

identify the Lie algebra $\mathfrak{g}$ with the space of left/right-invariant vector fields on G.

Lemma 5.3. *Let v be a left-invariant vector field on G. The maximal integral curve γ with $\gamma(0) = 1$ is defined on all of $\mathbb{R}$ and is a group homomorphism.*

Proof. Let $\gamma : I \to G$ be an integral curve with v with $\gamma(0) = 1$.

Assume $I \neq \mathbb{R}$ thus, without the loss of generality I has an upper bound $t_0 \in \mathbb{R}$. We will have to show that γ is not maximal. To see this, we choose $0 < \varepsilon < t_0$ and $t_0 - \varepsilon < t_1 < t_0, t \in I$.

Consider $\delta(t) = \gamma(t_1) \cdot \gamma(t - t_1)$. Thus γ is a smooth curve defined an open neighborhood of t_0 and $\delta(t_1) = \gamma(t_1)$ and

$$\begin{aligned}
\delta'(t) &= d\delta(t)(1), \\
&= dL_{\gamma(t_1)}(\gamma(t - t_1))(dr(t - t_1)(1)), \\
&= dL_{\gamma(t_1)}(v(\gamma(t - t_1))), \\
&= dL_{\gamma(t_1)}(\gamma(t - t_1)(dL_{r(t-t_1)}(1)(v(1))), \\
&= dL_{\gamma(t_1)\gamma(t-t_1)}(1)(v(1)), \\
&= v(\gamma(t_1)\gamma(t - t_1)), \\
&= v(\delta(t)).
\end{aligned}$$

Thus δ is an integral curve for v defined on an open neighborhood of t_0 containing t_1 and $\delta(t_1) = \gamma(t_1)$. Therefore γ is not maximal.

Now we are going to show that γ is a homomorphism. For fixed $t \in \mathbb{R}$, note that the maps

$$s \mapsto \gamma(t + s), \quad s \mapsto \gamma(t)\gamma(s)$$

are both integral curves for v with equal value at $s = 0$, hence equal. □

5.5 The Exponential Maps

Proposition 5.10. *Let $X \in \mathfrak{g}$, there exists a unique group homomorphism $\gamma_X : \mathbb{R} \to G$ differentiable at 0 and $\gamma'_X(0) = X$. It is the maximal integral curve through 1 for both X^L and X^R. We have $\gamma_{tX}(s) = \gamma_X(ts)$ for $t \in \mathbb{R}$.*

Proof. By Lemma 5.3, there exist maximal integral curves for X^L and X^R , we denote them by $\gamma_{X^L}, \gamma_{X^R}$, respectively. By Lemma 5.8, we can assume $\gamma_{X^L}(0) = \gamma_{X^R}(0) = 1$, and these are defined on the whole $\mathbb{R}$.

For uniqueness, let $\gamma : \mathbb{R} \to G$ be a group homomorphism which is differentiable at 0 with $\gamma'(0) = X$. Then

$$\gamma(t)\gamma(s) = \gamma(t+s) = \gamma(s+t) = \gamma(s)\gamma(t). \tag{5.3}$$

Fix t and apply $\frac{d}{ds}|_{s=0}$ to see that γ is differentiable at any t in the following way

$$\begin{aligned} \frac{d}{ds}|_{s=0}\gamma(t)\gamma(s) &= \frac{d}{ds}|_{s=0}\gamma(t+s), \\ \Rightarrow \gamma(t)\gamma'(0) &= \gamma'(t). \end{aligned}$$

By construction, when $\gamma = \gamma_{X^L}$ we have

$$\begin{aligned} \gamma'_{X^L}(t) &= dL_{\gamma_{X^L}(t)}(1)(X^L(1)) \\ &= dL_{\gamma_{X^L}(t)}(1)X \\ &= L_{\gamma_{X^L}(t)}X. \end{aligned}$$

Similarly for $\gamma = \gamma_{X^R}$ we have

$$\begin{aligned} \gamma'_{X^R}(t) &= dR_{\gamma_{X^R}(t)}(1)(X^R(1)) \\ &= dR_{\gamma_{X^R}(t)}(1)X \\ &= R_{\gamma_{X^R}(t)}X. \end{aligned}$$

By the uniqueness of solutions of ordinary differential equations, we derive that $\gamma_{X^L} = \gamma_{X^R}$. This proves that $\gamma_{X^L}, \gamma_{X^R}$ are maximal as they are defined on all $t \in \mathbb{R}$.

For the second property, we only need to check that $\gamma_{tX}(s) = \gamma_X(ts)$ coincide at $s = 0$. □

Definition 5.15. *Let G be a Lie group and $\mathfrak{g} = T_1G$. Then we define the exponential map*

$$\exp_G : \mathfrak{g} \to G, \exp_G(X) = \gamma_X(1),$$

where γ_X is the integral curve of $v(g) = dL_g(1)X$.

Theorem 5.1. $\exp_G : \mathfrak{g} \to G$ *is smooth and has the following properties.*

1). $\underline{\mathrm{Ad}}(x) \circ \exp_G = \exp_G \circ \mathrm{Ad}(x)$ *for any $x \in G$.*

2). $\mathrm{Ad} \circ \exp_G = \exp_{\mathrm{GL}(\mathfrak{g})} \circ \mathrm{ad}$.

3). $d\exp_G(0) : \mathfrak{g} \to \mathfrak{g}$ *is an identity $id_{\mathfrak{g}}$.*

4). If $f : G \to H$ is a homomorphism of Lie groups, then $f \circ \exp_G = \exp_H \circ df(1)$.

5). $\gamma_X(t) = \exp_G(t \cdot X)$.

Proof. Look at the homework $\square$

Proposition 5.11. *Let V be a finite dimensional $\mathbb{R}$-vector space. Then*

$$\exp_{\mathrm{GL}(V)} : \mathfrak{gl}(V) \to \mathrm{GL}(V)$$

is given by

$$\exp_{\mathrm{GL}(V)}(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n.$$

Proof. Homework $\square$

Corollary 5.2. *Furthermore, we can derive the following properties of $\exp_G$,*

1). $\operatorname{Im} \exp_G \subseteq G^0$.

2). $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism locally around 0.

3). If $U \subseteq \mathfrak{g}$ is a neighborhood of 0 in $\mathfrak{g}$, then $\exp_G(U)$ generates G^0.

Proof. Note that $\exp_G$ is a smooth map.

By the smoothness, it is also continuous. Since $\mathfrak{g}$ is connected, it is mapped to a connected subset of G which contains 1. Thus we have the first property.

By the third property of Theorem 5.1, we have $d\exp_G(0)$ is invertible.

By the second property of the same theorem, $\exp_G(U)$ contains an open neighborhood of 1, thus generates G^0. $\square$

Definition 5.16. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. By the second statement of the corollary above, there exists a neighborhood U of 0 in $\mathfrak{g}$, such that $\exp_G |U$ is a diffeomorphism. We denote its inverse by $\log_G$.*

Corollary 5.3. *Let G be a connected Lie group and $g \in G$, we have the following*

$$g \in Z(G) \Leftrightarrow \mathrm{Ad}(g) = id_{\mathfrak{g}}.$$

Proof. If $g \in Z(G)$, then $\underline{\mathrm{Ad}}(g) = id_G$, therefore $\mathrm{Ad}(g) = id_{\mathfrak{g}}$. Conversely, if $\mathrm{Ad}(g) = id_{\mathfrak{g}}$, by the first property of Theorem 5.1 we have $\underline{\mathrm{Ad}}(g)$ is identity on the image of $\exp_G$. By the second statement of Corollary 5.2, this image generates G. Since $\underline{\mathrm{Ad}}(g)$ is a homomorphism, it is trivial on the entire group G. $\square$

Corollary 5.4. *Let G be a Lie group and $X, Y \in \mathfrak{g}$. We have*

$$[X|Y] = 0 \Rightarrow \exp_G(X)\exp_G(Y) = \exp_G(Y)\exp_G(X).$$

Proof. Let $x = \exp_G(X), y = \exp_G(Y)$. By the first and second statements of Theorem 5.1,

$$xyx^{-1} = \exp_G(\mathrm{Ad}(X)Y) = \exp_G(\exp_{\mathrm{GL}(\mathfrak{g})}(\mathrm{ad}(X)(Y))).$$

By Proposition 5.11 and the assumption, this is equal to

$$\exp_G(Y) = y.$$

□

Corollary 5.5. *Let $f_1, f_2 : H \to G$ be homomorphisms of Lie groups. If H is connected and $df_1(1) = df_2(1)$. Then $f_1 = f_2$.*

Proof. Using the forth statement of Theorem 5.1, we have $f_1 = f_2$ upon restriction to the image of $\exp_H$, and such image generates H. □

5.6 Differentials of $\exp_G$

Theorem 5.2. *Let $X \in \mathfrak{g}$. (Recall that we have the canonical identification $T_x\mathfrak{g} \to \mathfrak{g}$).*

Consider

$$d(\exp_G)(x) : \mathfrak{g} \to T_{\exp(X)}G, \quad dR_{\exp_G(x)}(1) : \mathfrak{g} \to T_{\exp_G(x)}G.$$

Then we have the following,

$$dR_{\exp_G(x)}(1)^{-1} \circ d(\exp_G)(x) : \mathfrak{g} \to \mathfrak{g}, X \to \int_0^1 \exp_{\mathrm{GL}(\mathfrak{g})}(s \cdot \mathrm{ad}(X))ds.$$

Proof. □

Corollary 5.6. *An element $X \in \mathfrak{g}$ is a singular point for $\exp_G$ if and only if $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ has an eigenvalue of the form $2\pi ik$ for some $k \in Z^\times$.*

Proof. Since both $\mathfrak{g}$ and G have the same dimension, X is singular if and only if $d(\exp_G)(X)$ is not invertible. By Theorem the equation

$$\int_0^1 \exp_{\mathrm{GL}(\mathfrak{g})}(s \cdot \mathrm{ad}(X))dx \tag{5.4}$$

is not invertible. In other words, it admits 0 as an eigenvalue. Using the formula

$$\int_0^1 \exp_{\mathrm{GL}(\mathfrak{g})}(s\lambda)dx = \begin{cases} \lambda^{-1}(e^\lambda - 1) & (\lambda \neq 0), \\ 1 \quad (\lambda = 0). \end{cases}$$

We see that the eigenvalues of the (5.4) are given by 1 if 0 is an eigenvalue of $\mathrm{ad}(X)$ and $\lambda^{-1}(e^\lambda - 1)$ if $\lambda \neq 0$ is an eigenvalue of $\mathrm{ad}(X)$. □

Remark 5.7. *The formula* (5.4) *generalizes to*

$$\int_0^1 e^{sA} ds = A^{-1}(e^A - 1) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} A^k.$$

for any $A \in \mathrm{GL}(V)$ *where* V *is a finite dimensional* $\mathbb{R}$*-vector space. If* A *is not invertible, we can define* $A^{-1}(e^A - 1)$ *by the above formula.*

This is particularly useful for $A = \mathrm{ad}(X)$*, for* $X \in \mathfrak{g} = V$*, which is never invertible since* $\mathrm{ad}(X)(X) = 0$*. Moreover, for* $A = \mathrm{ad}(X), A^{-1}(e^A - 1)$ *is invertible for* X *in a neighborhood of* 0 *by Corollary 5.6.*

5.7 The Product in Logarithmic Coordinates

Theorem 5.3. *Let* $U \subset \mathfrak{g}$ *be an open neighborhood of* 0*. For* $X, Y \in U$*, consider the differential equation for* $z : \mathbb{R} \to \mathfrak{g}$*, such that*

$$z(0) = Y, \quad \frac{dz}{dt}(t) = (\mathrm{ad}\, z(t))^{-1}(\exp_{\mathrm{GL}\,\mathfrak{g}}(\mathrm{ad}\, z(t)) - 1))^{-1}(X).$$

For U *sufficiently small, this differential equation has (a unique) solution for all* $X, Y \in U$ *and all* $t \in [0, 1]$*. Define* $\mu(X, Y) = z(1)$*. Then*

$$\exp_G(X) \exp_G(Y) = \exp_G(\mu(X, Y)).$$

Proof. □

Corollary 5.7. *The collection of maps* $\kappa_x : U \to G$*, where* $U \subset \mathfrak{g}$ *is an open neighborhood of* 0*.*

$$\kappa_x(Y) = x \cdot \exp_G(Y)$$

is a smooth, in fact real analytic, atlas for the manifold G*.*

Proof. We know that $\exp_G$ is smooth and a locally diffeomorphism around 0. So κ_x is a diffeomorphism onto its image . Thus $(\kappa_x)_{x \in G}$ is a smooth atlas.

The transition maps are expressible in terms of μ by Theorem 5.3. Since μ is real analytic in X, Y. we see that the atlas is real analytic. □

Definition 5.17 (Real analytic manifolds)**.** *A manifold is said to be*

Remark 5.8. *In particular, any Lie group is automatically real analytic.*

Theorem 5.4. *Let* $X, Y \in U$*, then*

$$\mu(X, Y) = X + Y + \sum_{k=1}^{\infty} \frac{(1)^k)}{k+1} \sum_{\substack{l_1, \cdots, l_k \geq 0, \\ m_1, \cdots, m_k \geq 0, \\ l_i + m_i > 0}} \frac{1}{\sum_{i=1}^k l_i + 1} \prod_{i=1}^k \frac{\mathrm{ad}(X)^{l_i}}{l_i!} \frac{\mathrm{ad}(X)^{m_i}}{m_i!}$$

Corollary 5.8.

$$\mu(X, Y) = X + Y + \frac{1}{2}[X, Y] + O(|(X, Y)|^3).$$

5.8 Lie Subgroups

Definition 5.18. *Let G be a Lie group. A Lie subgroup H of G is a immersive submanifold that is also a subgroup.*

Definition 5.19. *Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{h}$ of it is called a Lie subalgebra if it is closed under the Lie bracket operation $[\cdot|\cdot]$.*

Definition 5.20. *Let G be a Lie group, then we denote*

$$\mathrm{Lie}(G) = T_1 G.$$

Remark 5.9. *A tautological inclusion $i_H : H \to G$ is an injective immersion.*

Theorem 5.5. *Let G be a connected Lie group. Then there is a bijection between*

$$\{H \subseteq G \mid \textit{connected Lie subgroups.}\} \leftrightarrow \{\textit{Lie subalgebras of } \mathrm{Lie}(G).\},$$

And the bijection is given by $\mathrm{Lie}(\cdot)$*.*

Proof. Let H be a subgroup of G and $\mathfrak{h} = T_1 H$.

We first prove the injectivity of $\mathrm{Lie}(\cdot)$.

For the surjectivity, let us take $H \subseteq G$ to be a subgroup generated by the image of $\exp_G(\mathfrak{h})$ for a Lie subalgebra $\mathfrak{h}$. By Corollary 5.7, we have

$$(\kappa_x^{-1})_{x \in G}, \quad \kappa_x(Y) = x \exp_G(Y)$$

is an atlas for G. We will show that

$$(\kappa_x^{-1})_{x \in H}$$

is an atlas for H.

First, we claim that if

$$\kappa_x(U \cap \mathfrak{h}) \cap \kappa_y(U \cap \mathfrak{h}) \neq \emptyset,$$

then there exist neighborhood V_1, V_2 of 0 in $\mathfrak{g}$ such that

$$\kappa_y^{-1} \circ \kappa_x : V_1 \cap \mathfrak{h} \to V_2 \cap \mathfrak{h}$$

is a diffeomorphism.

Since κ_x is an atlas for G, there exist some open neighborhoods V_1, V_2 of 0 such that

$$\kappa_y^{-1} \circ \kappa_x : V_1 \to V_2$$

is a diffeomorphism. The above composition is given by

$$\kappa_y^{-1} \circ \kappa_x(Y) = \log_G(y^{-1}x \exp_G(Y)).$$

Let $z = y^{-1}x$ then since $x, y \in H$, $z \in H$. Let $X \in \mathfrak{h}$ be such that

$$z = \exp(X).$$

Thus by using Theorem 5.3, we obtain

$$y^{-1}x \exp_G(Y) = z \exp_G(Y) = \exp_G(X) \exp_G(Y) = \exp_G(\mu(X, Y)).$$

Thus if $X, Y \in \mathfrak{h}$, then $\mu(X, Y) \in \mathfrak{h}$

For each $x \in H$, through $\kappa_x : \mathfrak{h} \cap U \to H$, we get a basis of a neighborhood of 0 in $\mathfrak{h}$. This topologizes H and

$$(\kappa_x^{-1})_{x \in G}, \quad \kappa_x(Y) = x \exp_G(Y)$$

becomes an atlas. Therefore, the tautological inclusion $\iota : H \to G$ is now an immersion. □

Remark 5.10. *This map is a bijection from the set of connected Lie subalgebras of G to the set of Lie subalgebras of* $\mathrm{Lie}(G)$.

Definition 5.21. *A subset $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called an ideal if for any $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, we have*

$$[X|Y] \in \mathfrak{h}.$$

Lemma 5.4.

Lemma 5.5. *Given a Lie group G and its Lie algebra $\mathfrak{g}$. H is a normal connected Lie subgroup of G if and only if* $\mathrm{Lie}(H)$ *is an ideal.*

Proof. □

Lemma 5.6. *Let G be a topological group and $H \subset G$ be a locally closed subgroup. Then H is a closed as a subset of G.*

Proof. □

Lemma 5.7. *Let $H \subset G$ be a Lie subgroup of a Lie group G. Then H is embedded submanifold if and only if H is closed.*

Proof. Since H is an embedded submanifold, it is locally closed. By Lemma 5.6, H is closed. Conversely, if H is closed then a tautological injection $\iota : H \to G$ is a closed injective immersion. Hence H is an embedded submanifold. □

Theorem 5.6. *Let G be a Lie group and $H \subseteq G$ be a subgroup which is closed as a set. Then H is a closed Lie subgroup.*

Proof. Without the loss of generality, we may assume that H is a connected set.

Let us define a subset of the Lie algebra $\mathfrak{g}$ as

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R}, \exp_G(tX) \in H\}.$$

We will first show that $\mathfrak{h}$ is a subspace of $\mathfrak{g}$. The scalar multiplication is obvious from the definition. We need to show that this is closed under addition.

Fix $t \in \mathbb{R}$, for a large enough n, there is a neighborhood V of 0 in $\mathfrak{g}$ such that

$$n^{-1}tX, n^{-1}tY \in V.$$

Since V is sufficiently small, $\exp_G |V$ is a diffeomorphism and by Theorem 5.3, we have for any $H, K \in V$,

$$\exp_G(H)\exp_G(K) = \exp_G(\mu(H, K)).$$

Differentiate the above equation at $(0, 0)$, we get

$$d\mu(H, K) = H + K.$$

Therefore we have

$$n \cdot d\mu(n^{-1}tX, n^{-1}tY) = t(X + Y).$$

Proof of the closedness

Let $U \subseteq \mathfrak{h}$ be a neighborhood of 0. Then $\exp_G(U)$ is a neighborhood of 1 in H by the construction.

Suppose there is a sequence $(h_n)_{n\in\mathbb{N}} \subset H\backslash \exp_G(U)$ which converges to 1. Let $\mathfrak{k}$ be a complement of a vector space $\mathfrak{h}$, that is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}.$$

Consider the map

$$\Phi : \mathfrak{h} \oplus \mathfrak{k} \to G, \Phi(H, K) = \mu(\exp_G(H), \exp_G(K)).$$

Differentiating this at $(0, 0)$ we get

$$d\Phi(H, K) = H + K,$$

since $d\exp_G$ is identity at 0. We conclude that Φ is a local diffeomorphism. Thus we can find sequences $(H_n)_{n\in\mathbb{N}} \subset \mathfrak{h}, (K_n)_{n\in\mathbb{N}} \subset \mathfrak{k}$ such that for big enough N and $n \geq N$,

$$\exp_G(H_n)\exp_G(K_n) = h_n.$$

For such $(K_n)_{n\in\mathbb{N}}$ we have $K_n \to 0$.
Now define a sequence $K'_n = \frac{K_n}{|K_n|}$ in the Euclidean norm. This belongs to the unit sphere in $\mathfrak{k}$ which is a compact set. Thus, we find a subsequence $(K_{n_m})_{m\in\mathbb{N}}$ which is converging to some K.

Let $t > 0$ be fixed. Since $|K'_n| \o 0$, we can find $(k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$k_n \leq \frac{t}{|K'_n|} \leq k_{n+1}.$$

Therefore, as $n \to \infty$,

$$k_n |K'_n| \to t.$$

Passing these to $\exp_G$ we get

$$\exp_G(tK) = \lim_{m\to\infty} \exp_G(k_{n_m} K_{n_m}) = \lim_{m\to\infty} \exp_G(K_{n_m})^{k_{n_m}}.$$

Since $\exp_G(Y_n) = (\exp_G(H_n))^{-1} h_n$, it belongs to H, so does $\exp_G(tK)$. $\square$

Corollary 5.9. *Let $f : G \to H$ be a homomorphism of groups between two Lie groups which is continuous. Then f is a homomorphism of Lie groups.*

Proof. $\square$

Corollary 5.10. *Let $f : G \to H$ be a homomorphism of Lie groups. Then* $\operatorname{Ker} f$ *is a closed Lie subgroup with its Lie algebra* $\operatorname{Ker}(df(1))$*.*

Proof. Since H is Hausdorff by definition, $\{1\}$ is closed in H. f is continuous implies that $K = \operatorname{Ker} f$ is closed in G as it is an inverse image of a closed set. By Theorem 5.6, it is a Lie subgroup. By the forth statement of Theorem 5.1, we have

$$f \circ \exp_G = \exp_H \circ df(1).$$

Therefore we have an inclusion $\operatorname{Ker}(df(1)) \subseteq \operatorname{Lie}(K)$. Conversely, let $X \in \operatorname{Lie}(K)$ and define a map

$$t \mapsto f(\exp_G(tX)) = 1.$$

By differentiating with respect to t at 0 we see

$$df(1)(X) = 0.$$

Therefore $X \in K$. $\square$

Corollary 5.11. *Let $f : G \to H$ be a homomorphism of Lie groups. If f is an injection then f is an immersion. Furthermore, if f is a bijection then f is an isomorphism.*

Proof. f is injective if and only if $\operatorname{Ker} f = \{1\}$. By Corollary 5.10, we have

$$\operatorname{Ker}(df(1)) = \{0\}.$$

By translation by an arbitrary $g \in G$,

$$\operatorname{Ker}(df(g)) = \{0\}.$$

Moreover, if f is a bijection, then looking locally through charts, we see

$$\dim(G) = \dim(H).$$

Therefore we have

$$\dim \mathfrak{g} = \dim \mathfrak{h}.$$

Since $df(1)$ is an injective linear map between linear spaces with the same dimension, thus a bijection. Using translations again, we see that $df(g)$ is bijective for any $g \in G$.Therefore, f is a bijective local diffeomorphism everywhere, we conclude f is a diffeomorphism. □

5.9 Group Action of Lie Groups

Definition 5.22. *Let G be a Lie group and M be a manifold. A smooth action of G on M is a smooth map $l : G \times M \to M$ which is a group action.*

Definition 5.23. *A group action $l : G \times M \to M$ of a topological group G on a manifold M is said to be proper if the map*

$$(g, x) \mapsto (gx, x)$$

is proper.

Remark 5.11. *A group action $l : G \times M \to M$, we can define functions such as*

1). $l_x : G \to M$ for fixed $x \in X$, $l_x(g) = gx$,

2). $l_g : M \to M$ for fixed $g \in G$, $l_g(x) = gx$.

Clearly, both of them are continuous. Furthermore, we have

1). l_x is injective if and only if $G_x = \{1\}$.

2). l^g is always a diffeomorphism with the inverse $m_{g^{-1}}$.

Lemma 5.8. *Let $l : G \times M \to M$ be a smooth, free Lie group action. Then for any $x \in M$, $dl_x(1)$ is injective.*

Proof. Given $X \in \mathfrak{g}$, we have

$$l_x(\exp_G((t+h)X) = l_x(\exp_G(tX)\exp_G(hX)) = l^{\exp_G(tX)} l_x(\exp_G(hX)).$$

Suppose $X \in \operatorname{Ker}(dl_x(1))$ and fix $t_0 \in \mathbb{R}$. We have

$$\frac{d}{dt}\bigg|_{t=t_0} l_x(\exp_G(tX)) = \frac{d}{dh}\bigg|_{h=0} l_x(\exp_G((t_0+h)X)) = dl^{\exp_G(t_0X)}(X)(dl_x(1)(X)) = 0.$$

Therefore, $l_x(\exp(tX))$ is constant but l is a free group action. Therefore for any $t \in \mathbb{R}$, $\exp_G(tX) = 1$. We conclude that $X = 0$. $\square$

Theorem 5.7. *Let $m : G \times M \to M$ be a smooth, free, proper group action. We can embed the smooth manifold structure to the equivalence classes G/M by the orbits of the action with following properties.*

i). The topology on G/M is the quotient topology induced by the canonical map $\pi : M \to G/M$.

ii). For an arbitrary point $p \in G/M$, there is a neighborhood $V \subseteq G/M$ and a diffeomorphism $\pi^{-1}(V) \to G \times V$, which translates the G actions on $\pi^{-1}(V)$ given by the map l to the G action $G \times V$ by left multiplication on G.

Proof. Let S be a complement vector space of $dl_x(1)(\mathfrak{g})$ in T_xM, thus we have

$$T_xM = S \oplus dl_x(1)(\mathfrak{g}).$$

Choose a submanifold $N \subseteq M$ such tha $x \in N$ and t $T_xN = S$. (Such N is called a slice).We first show that for sufficiently small N, the restricted action

$$\bar{l} : G \times N \to M$$

is a diffeomorphism onto its image. Indeed, taking the derivative of $\bar{l}$ at $(1, x)$ we derive

$$d\bar{l}(1, x) : \mathfrak{g} \times T_xN \to T_xM.$$

By the construction of N and Lemma 5.8, this is a bijection. By translation, we have $d\bar{l}(g, y)$ is bijective for any $(g, y) \in G \times N$ close enough to $(1, x)$.
Take N small enough so that for each $y \in N$, $d\bar{l}(1, y)$ is bijective. Using the formula

$$d\bar{l}(g, y) = dl^g \circ d\bar{l}(1, y),$$

we see that $d\bar{l}(g, y)$ is bijective for any $(g, y) \in G \times N$.
Secondly, we prove that for sufficiently small enough N, $\bar{l}$ is injective. Suppose that there are sequences $(y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subset N$ and $(g_n)_{n\in\mathbb{N}}, (h_n)_{n\in\mathbb{N}} \subset G$ such that

$$g_iy_i = h_iz_i, (g_i, y_i) \neq (h_i, z_i).$$

If $g_i = h_i$ then by the definition of group actions, $y_i = z_i$, therefore, we may assume that $g_i \neq h_i$. Let $k_i = h_i^{-1} g_i \neq 1$. Then the sequence

$$(k_i y_i, y_i) = (z_i, y_i) \to (x, x),$$

is contained in a compact subset of $M \times M$.

By the properness of l, $(k_i)_{i \in \mathbb{N}}$ is contained in a compact subset of G. Thus it has a convergent subsequence which converges to some $k \in G$. Therefore, with that subsequence we have

$$x = \lim_{i \to \infty} z_i = \lim_{j \to \infty} k_{i_j} y_{i_j} = kx.$$

By the freeness of l, we conclude that $k = 1$. However, $(k_{i_j}, y_{i_j}), (1, z_{i_j}) \subset G \times N$ converge to $(1, x)$. This contradicts to the injectivity of $\bar{l}$ in a sufficiently small neighborhood of $(1, x)$.

Now we are ready to close our proof. Note that the properness of the action implies that the space G/M is Hausdorff as each equivalence class is closed.

We introduce an atlas by

$$\bar{l} : G \times N \to \pi^{-1}(\pi(N)).$$

This satisfies the desired property. □

Corollary 5.12. *Let $H \subset G$ be a closed Lie subgroup. Then the coset spaces $H \backslash G$ and G/H are both smooth manifold. Furthermore, $\pi_l : G \to H \backslash G$ and $\pi_r : G \to G/H$ are principal H-bundles.*

Proof. The left action map $l : H \times G \to G$ is a multiplication and $H \times G \to G \times G$ is proper. □

Corollary 5.13. *Let $H \subset G$ be a closed normal Lie subgroup. Then G/H is a Lie group and the canonical group homomorphism $\pi : G \to G/H$ is a Lie group homomorphism.*

Proof. □

Corollary 5.14. *Let $f : H \to G$ be a homomorphism of Lie groups. Then there exists a unique $\overline{f} : H/\operatorname{Ker} f \to G$ such that*

$$\overline{f} \circ \pi = f.$$

And such $\overline{f}$ is an injective immersion.

Proof. By elementary group theory, there exists a unique homomorphism $\overline{f}$. Since $\pi : H \to H/\operatorname{Ker} f$ is a fiber bundle, and f is smooth, $\overline{f}$ is also smooth, we conclude that $\overline{f}$ is an injective homomorphism of Lie groups. □

Corollary 5.15. *Let $f : H \to G$ be a homomorphism of Lie groups. If f is surjective, it is a principal fiber bundle with group $K = \operatorname{Ker} f$.*

Proof. □

5.10 Classification of abelian connected Lie groups.

Definition 5.24. *A Lie algebra L is abelian if $L = Z(L)$, in other words, the Lie bracket is everywhere* 0.

Proposition 5.12. *A connected Lie group G is abelian if and only if its Lie algebra* $\mathrm{Lie}(G)$ *is abelian.*

Proof. If G is abelian, then the adjoint action $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is constant and $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is everywhere 0. Conversely, if $\mathfrak{g} = \mathrm{Lie}(G)$ is abelian then for any $X, Y \in \mathfrak{g}$, we have

$$\exp_G(X + Y) = \exp_G(X) \exp_G(Y).$$

Thus generators of the Group G commutes, we conclude G is abelian. □

Proposition 5.13. *Let G be a connected abelian Lie group then $\exp_G : \mathfrak{g} \to G$ is a surjective homomorphism and its kernel is discrete.*

Proof. Let $X, Y \in \mathfrak{g}$ and consider two maps

$$t \mapsto \exp_G(t(X + Y)), \quad t \mapsto \exp_G(tX) \exp_G(tY).$$

These maps have the same derivatives at $t = 0$, namely $X + Y$.

Since G is abelian, we conclude these are homomorphisms of Lie groups. Therefore, they are equal to one another. We now have that $\exp_G$ is a group homomorphism, and $\exp_G(\mathfrak{g})$ generates G, therefore it is surjective.

We close the proof by stating that $\mathrm{Ker} \exp_G$ is a closed Lie subgroup with Lie algebra, $\mathrm{Ker}\, d\exp_G(0) = \{0\}$. □

Theorem 5.8. *Let G be a connected abelian then it is isomorphic to $\mathbb{R}^a \times (S^1)^b$ for some integers a, b.*

Proof. □

5.11 Connected Compact Lie groups

Proposition 5.14. *Let G be a connected topological group and $\alpha, \beta : [0, 1] \to G$ be loops base at the identity* 1. *Let us define another loop by*

$$\gamma(s) = \alpha(s)\beta(s), \quad s \in [0, 1].$$

Then in the fundamental group $\pi_1(G)$, we have

$$[\alpha] * [\beta] = [\gamma] = [\beta] * [\alpha].$$

In particular, $\pi_1(G)$ is abelian.

Proof. Let us consider the map $H : [0, 1] \times [0, 1] \to G, H(s, t) = \alpha(s)\beta(t)$. □

Proposition 5.15. *Let G be a connected, locally path connected (hence globally path connected) topological group. View $(G, 1)$ as a pointed space and*

$$p : (X, x) \to (G, 1)$$

be a covering space. There exists a unique topological group structure on (X, x) with x being the identity which makes p into a group homomorphism. Furthermore, such topological group and a homomorphism has the following properties.

1). $\mathrm{Ker}(p)$ *is central and discrete.*

2). The group of Deck transformations of p is identified with $\mathrm{Ker}(p)$ *acting by multiplication.*

Corollary 5.16. *In the above setting, if G i a Lie group, the above construction makes the group (X, x) into a Lie group. Such structure makes p into a Lie group homomorphism and* $\mathrm{Ker}(p)$ *turns out to be countable.*

Proposition 5.16. *Let G be a connected Lie group. Then the universal cover $\tilde{G}$ of G is a connected Lie group.*

Furthermore, the map $[\alpha] \mapsto \alpha(1) : \pi_1(G) \to \tilde{G}$ where

Proposition 5.17. *Let $f : H \to G$ be a homomorphism of connected Lie groups. Then the following statements are equivalent.*

1. *f is a covering,*
2. *$df(1)$ is an isomorphism.*

Proof. □

Proposition 5.18. *Let H be a connected and simply connected Lie group and G be a Lie group.*

For any homomorphism $\varphi : \mathrm{Lie}(H) \to \mathrm{Lie}(G)$ of Lie algebras, there exists a homomorphism $f : H \to G$ of Lie groups such that

$$df(1) = \varphi.$$

Proof. □

Lemma 5.9. *Connected Lie groups that have the same Lie algebra have the same universal cover.*

Proof. Let G_1, G_2 be Lie groups and $\varphi : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ is an isomorphism. □

5.12 Compact Lie algebras

Definition 5.25. *Let G be a topological group. A finite dimensional representation of G over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a pair (π, V) such that*

i). V is a finite dimensional vector space,

ii). $\pi : G \to \mathrm{GL}(V)$ is a continuous group homomorphism.

Remark 5.12. *If G is a Lie group then π is smooth and we can take $\varphi = d\pi(1) : \mathrm{Lie}(G) \to \mathrm{End}(V) = \mathfrak{gl}(V)$ a homomorphism of Lie algebras.*

Definition 5.26. *The adjoint representation of G is the homomorphism*

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

such that

$$\varphi = \mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is the adjoint representation of $\mathfrak{g}$ on itself.

Remark 5.13. *G is a connected Lie group then*

$$Z(G) = \Im(\mathrm{Ad}).$$

Thus it is a closed subgroup. We also have

$$\mathrm{ad}(\mathrm{Lie}(Z(G)) = \mathrm{Im}(\mathrm{ad}).$$

Definition 5.27. *Let L be a Lie algebra over a field k.*

A finite dimensional representation of L is a pair (φ, V) consisting of

i). V is a finite dimensional vector space,

ii). $\varphi : L \to \mathfrak{gl}(V)$ is a homomorphism of Lie algebras.

The adjoint representation is the homomorphism

$$\mathrm{ad} : L \to \mathfrak{gl}(L), X \to [X, \cdot]$$

and $Z(L) = \Im(\mathrm{ad})$.

Definition 5.28. *Let L be a Lie algebra over a filed k and (φ, V) be a finite dimensional representation of L.*

A bilinear form $b : V \times V \to k$ is said to be invariant with respect to the representation if for all $x \in L$ and $v, w \in V$ we have

$$b(\varphi(x)v, w) = -b(v, \varphi(x)w).$$

In particular, it is invariant if it is invariant with respect to ad.

Remark 5.14. *$b : L \times L \to k$ is invariant if and only if*

$$b([X, Y], Z) = b(X, [Y, Z]).$$

Lemma 5.10. *Let G be a connected Lie group and $L = \mathrm{Lie}(G)$ be a Lie algebra. Let us consider the representation (π, V) of G over $\mathbb{R}$ and a invariant bilinear form $b : V \times V \to \mathbb{R}$ with respect to π*