Representation Theory 1 V4A3

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2024/2025 Winter Semester - Uni Bonn

1 Overview of the material

1.1 Lie groups

Definition 1.1. *A Lie group is a group G whose underlying set is endowed with the structure of smooth manifolds such that multiplication and inversions are smooth maps.*

Definition 1.2. *A topological group is a group G whose underlying set is endowed with the structure of topological space such that multiplication and inversions are continuous.*

2 Preliminaries

2.1 Topology

Definition 2.1. *We have two axioms about the topological spaces*

1. *T_0(Komogolov) : Given any 2 points, there exists an open set such that it contains one of them but not both.*
2. *T_1(Hausdorff) : Given any 2 points, there exist disjoints open set that each contains one of them.*

Definition 2.2. *A topological space is second countable if it has a basis which contains at most countably many subsets.*

Definition 2.3. *Let X, Y be topological spaces. A continuous function $f : X \to Y$ is said to be proper if the preimage of arbitrary compact set in Y is again compact.*

2.2 Linear Algebra

Definition 2.4. *Let V, W be finite dimensional $\mathbb{K}$-vector space. The duality pairing is a triplet $(V, W, \langle\cdot,\cdot\rangle)$ where*

$$\langle\cdot,\cdot\rangle : V \times W \to \mathbb{K}$$

is a bilinear form such that

$$\forall v \in W, \forall w \in W, \langle v, w\rangle = 0 \Rightarrow v = 0,$$

and similarly for $\langle \cdot, w\rangle$.

Proposition 2.1. *Let* $(V, W, \langle\cdot,\cdot\rangle)$ *be a duality pairing. Then we have the following isomorphisms*

$$V \cong W^*, \quad W \cong V^*.$$

Lemma 2.1. *Let* $(V, W, \langle\cdot,\cdot\rangle)$ *be a duality pairing over* $\mathbb{R}$. *Then we can extend this to a duality pairing over* $\mathbb{C}$ *by*

$$(V \underset{\mathbb{R}}{\otimes} \mathbb{C}) \times (W \underset{\mathbb{R}}{\otimes} \mathbb{C}) \to \mathbb{C},$$

Corollary 2.1. *Let* $(V, W, \langle\cdot,\cdot\rangle)$ *be a duality pairing over* $\mathbb{R}$ *then we have*

$$(V \underset{\mathbb{R}}{\otimes} \mathbb{C})^* \cong W \underset{\mathbb{R}}{\otimes} \mathbb{C} \cong V^* \underset{\mathbb{R}}{\otimes} \mathbb{C}.$$

Definition 2.5. *Let* V *be a finite dimensional* $\mathbb{R}$ *vector space. A subset* $R \subseteq V^*$ *is called a root system of rank* $\dim V$ *consisting of roots if it generates* V^**, for each* $\alpha, \beta \in R$*, there is a* $\alpha^\vee$ *such that*

1. $\langle \alpha^\vee, \alpha\rangle = 2$,
2. $s_\alpha : V^* \to V^*$, $s_\alpha(x) = x - \langle \alpha^\vee, x\rangle x$, $s_\alpha(R) \subseteq R$,
3. $\langle \alpha^\vee, \beta\rangle \in \mathbb{Z}$.

And such $\alpha^\vee$ *is called a coroot of* α.

Definition 2.6. *A root system* R *of a vector space* V *is reduced if for any* $\alpha \in R$*, we have*

$$\forall c \in \mathbb{R}, c\alpha \in R \Rightarrow c = \pm 1.$$

Lemma 2.2. *A root system* R *of a vector space* V. *Each root has a unique coroot.*

Remark 2.1. *Denote hyperplanes*

$$H_\alpha = \mathrm{Ker}(\langle\cdot, \alpha\rangle) \subseteq V, \quad H_{\alpha^\vee} = \mathrm{Ker}(\langle \alpha^\vee, \cdot\rangle) \subseteq V^*.$$

By the definition, $\alpha^\vee \notin H_\alpha, \alpha^\vee \notin H_{\alpha^\vee}$.

Observe that for any $x \in H_{\alpha^\vee}$*, we have*

$$s_\alpha(x) = x.$$

While for α *we have*

$$s_\alpha(\alpha) = -\alpha.$$

From this observation, we conclude that s_α *is a reflection onto the hyperplane* $H_{\alpha^\vee}$.

Definition 2.7. *Let R be a root system of a vector space V. For $\alpha \in V$ we denote,*

$$H_\alpha = \mathrm{Ker}(\langle \cdot, \alpha \rangle) \subseteq V, \quad H_{\alpha^\vee} = \mathrm{Ker}(\langle \alpha^\vee, \cdot \rangle) \subseteq V^*$$

which are called the root hyperplanes.

Definition 2.8. *Let R be a root system of a vector space V. The connected components of*

$$V \setminus \bigcup_{\alpha \in R} H_\alpha$$

are called the Weyl chambers.

Definition 2.9. *Let V_1, V_2 be finite dimensional vector spaces and R_1, R_2 be their root systems, respectively.*

An isomorphism of root systems $f : V_1 \to V_2$ is an isomorphism of vector spaces which satisfies that

$$f(R_1) = R_2, \quad \forall \alpha, \beta \in R_1, \langle f(\alpha)^\vee, \beta \rangle = \langle \alpha^\vee, \beta \rangle.$$

2.3 Group Theory

Definition 2.10. *Let G be a group and X be a set. An action of group G on a set X is a mapping $l : G \times X \to X$ such that for any $g, h \in G$ and $x \in X$*

$$l(gh, x) = m(g, m(h, x)).$$

Definition 2.11. *A stabilizer of an element $x \in X$ by a group action of G is a subset of G such that*

$$G_x = \{g \in G \mid gx = x\}.$$

Definition 2.12. *A group action is said to be free if for any $x \in X$ we have $G_x = \{1\}$.*

3 Lie groups

3.1 Manifolds

Definition 3.1. *Let $f : X \to Y$ be a mapping between two topological spaces X, Y. f is called a homeomorphism if*

1. *f is a bijection,*
2. *f is continuous,*
3. *f^{-1} is also continuous.*

Definition 3.2. *Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open sets and $f : U \to V$ be a smooth map. Then the derivative of f at $p \in U$ is*

$$df(p) = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}.$$

Proposition 3.1. *Let $f : U \to V, g : V \to W$ be smooth maps. Then for $p \in U$ we have*

$$d(g \circ f) = dg(f(p))df(p).$$

Definition 3.3. *Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open sets. A map $f : U \to V$ is called a diffeomorphism if*

i). f is smooth. ($\Leftrightarrow$ arbitrary order of partial derivatives exists),

ii). f^{-1} is defined and is also a smooth map.

Definition 3.4. *Let X be a topological space. A chart on X is a homeomorphism $h : U \to V$ where $U \subseteq X$ is open and $V \subseteq \mathbb{R}^n$ is open.*

Definition 3.5. *An atlas $\mathscr{A}$ on a topological space X is a collection of charts $\{h_\lambda \mid h_\lambda : U_\lambda \to V_\lambda\}_{\lambda \in \Lambda}$ such that $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of X.*

Definition 3.6. *An atlas $\mathscr{A}$ of X is said to be smooth if for any two charts $h_1 : U_1 \to V_2, h_2 : U_2 \to V_2$. The following,*

$$h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2),$$

is a smooth map. Such map is called a transition map.

Definition 3.7. *Let X be a topological space and $\mathscr{A}_1, \mathscr{A}_2$ be smooth atlases. We say they are equivalent if $\mathscr{A}_1 \cup \mathscr{A}_2$ is also smooth.*

Proposition 3.2. *Above definition indeed defines an equivalence relation.*

Proof. For any $h_1 \in \mathscr{A}_1, h_2 \in \mathscr{A}_2, h_3 \in \mathscr{A}_3$,

$$h_3 \circ h_1^{-1} = h_3 \circ h_2^{-1} \circ h_2 \circ h_1^{-1}.$$

□

Definition 3.8. *A smooth manifold is a second countable Hausdorff topological space with equivalence classes of smooth atlases.*

Definition 3.9. *Let M, N be smooth manifolds, $f : M \to N$ be a map, and $p \in M$. f is said to be smooth at p if for one (hence any) pair of charts around p and $f(p)$,*

$$h_M : U_M \to V_M, h_N : U_N \to V_N,$$

the composed function

$$h_N \circ f \circ h_M^{-1} : V_M \to V_N$$

is smooth at $h_M(p)$.

Remark 3.1. *We can define a function* $\dim : M \to N$ *such that*

$$\dim(p) = \dim(V)_p,$$

for any chart $h : U \to V$ *around* p*. And this function is locally constant. In particular, if* M *is connected then it has a well-defined dimensions.*

Definition 3.10. *Let* M, N *be smooth manifold and* $f : M \to N$ *be a mapping which is smooth at* $p \in M$*. For any charts,*

$$h_N \circ f \circ h_M^{-1} : V_M \to V_N,$$

the rank of f *at* p *is such that*

$$\mathrm{rk}(f; p) = \mathrm{rank}(\mathbf{df}(h_M(p))(h_N \circ f \circ h_M^{-1})).$$

Definition 3.11. *Let* M, N *be smooth manifolds and* $f : M \to N$ *be a smooth map. A point* p *is said to be regular with respect to the map f. And a point* $q \in N$ *is called a regular value if all* $p \in f^{-1}(q)$ *are regular.*

Definition 3.12. *Let* M *be a manifold. A subset* $N \subseteq M$ *is called an embedded submanifold if for any point* $p \in N$*, there is a chart* $h_M : U_M \to V_M$ *around* p *such that*

$$h_M|_N : U_M \cap N \to V_M \cap \mathscr{R}^n,$$

is a diffeomorphism where n *is the dimension of* N*.*

In particular, an embedded submanifold of an euclidean space is called a embedded manifold.

Definition 3.13. *A map* $f : M \to N$ *of smooth manifolds is called a diffeomorphism if*

i). $f : M \to N$ *is a bijection,*

ii). f, f^{-1} *are both smooth.*

Theorem 3.1. *Let* $f : M \to N$ *be a smooth map between manifolds, and* $q \in N$ *be a regular value. Then* $f^{-1}(q) \subset M$ *is an embedded submanifold.*

Theorem 3.2. *Let* $f : M \to N$ *be a smooth map of manifolds* $p \in M$ *be a regular point, and* $\dim(p) = \dim(f(p))$*. Then* f *is a local diffeomorphism of* p*. In other words, there is a neighborhood* U_M *of* p *in* M *and* $f(p) \in U_N \subset N$ *such that*

$$f|_{U_M} : U_M \to U_N,$$

is a diffeomorphism.

Definition 3.14. *Let* $M \subseteq \mathbb{R}^n$ *be an embedded manifold such that for some open set* $U \subset \mathbb{R}^n$*, there is* $V \subset \mathbb{R}^n$ *such that*

$$h : U \to V, \quad h_M : U \cap M \to V \cap \mathbb{R}^m,$$

is a diffeomorphism where h_M is defined to be taking the first m coordinate of the points in V. (Thus $m \leq n$).

The tangent space T_pM of M at p is the subspace of $\mathbb{R}^n$ such that

$$(\mathbf{dh}(p))^{-1}(\mathbb{R}^m) \subset \mathbb{R}^n.$$

There are three definitions of tangent spaces and they are all equivalent. However, each of them has its own advantages.

Definition 3.15 (Coordinate tangent space)**.** *Given a smooth manifold M and a point $p \in M$. The coordinate tangent space of p is such that*

$$T_p^{\mathbf{Coo}}M = \{(h, v) \mid h : U \to V \text{ is a chart}, v \in \mathbb{R}^m\} / \sim .$$

Where $\sim$ is an equivalence relation such that

$$(h_1, v_1) \sim (h_2, v_2) \text{ if } (\mathbf{d}(h_2 \circ h_1^{-1})(h_1(p)))(v_1) = v_2.$$

Definition 3.16. *Given a smooth manifold M, a point $p \in M$, and a smooth map $\alpha : I \to M$ whose domain I is an open interval contains 0. α is called a smooth curve if $\alpha(0) = p$.*

Definition 3.17. *Two smooth curves $\alpha, \beta : I \to M$ through p are said to be tangentially equivalent if for one (hence any) charts $h : U \to V$ around p, we have*

$$d(h \circ \alpha)(0) = d(h \circ \beta)(0).$$

We denote such relation as $\sim_T$.

Definition 3.18 (Geometric tangent space)**.** *The geometric tangent space at p of a smooth manifold M is such that*

$$T_p^{\mathbf{Geo}} = \{\alpha : I \to M \mid \alpha \text{ is a smooth curve}\} / \sim_T .$$

Definition 3.19. *A germ of smooth functions of manifolds M at p is an equivalence class of tuples (U, f) where*

i). $U \subset M$ is a neighborhood of p,

ii). $f : U \to \mathbb{R}$ is smooth,

and two tuples $(U_1, f_1), (U_2, f_2)$ are equivalent if there is a neighborhood V of p such that $V \in U_1 \cap U_2$ and $f_1|_V = f_2|_V$.

And we denote the set of germs at p as

$$\mathscr{C}^\infty(p).$$

Remark 3.2. *$\mathscr{C}^\infty(U, \mathbb{R})$ and $\mathscr{C}^\infty(p)$ are rings, in fact $\mathbb{R}$-algebras.*

Definition 3.20. *Let R be a ring and A be a bimodule over R. A R-derivation in A is an operator $X : A \to A$ such that the Leibniz rule holds. In other words,*

$$X(ab) = aX(b) + X(a)b,$$

holds for all $a, b \in A$.

Definition 3.21 (Algebraic tangent space)**.** *The algebraic tangent space $T_p^{\mathbf{Alg}}M$ of M at p is the set of $\mathbb{R}$-derivations $X : \mathscr{C}^\infty(p) \to \mathbb{R}$.*

Remark 3.3. *In the above definition, $\mathbb{R}$ is considered as a $\mathscr{C}^\infty(p)$-bimodule via the evaluation map $f \mapsto f(p)$.*

Theorem 3.3. *The following are isomorphisms of $\mathscr{R}$-vector spaces.*

$$\begin{aligned}
&T_p^{\mathbf{Geo}}M \to T_p^{\mathbf{Alg}}M, \alpha \mapsto (f \mapsto (f \circ \alpha)'(0)),\\
&T_p^{\mathbf{Alg}}M \to T_p^{\mathbf{Coo}}M, X \mapsto (h, ((Xh_i)(p))_{i=1,\cdots,n}),\\
&T_p^{\mathbf{Coo}}M \to T_p^{\mathbf{Geo}}M, (h, v) \mapsto \alpha(t) = h^{-1}(h(p) + t \cdot v).
\end{aligned}$$

Proposition 3.3. *$\mathscr{C}^\infty(p)$ is a local ring with its maximal ideal*

$$\mathfrak{m}_p = \{f \in \mathscr{C}^\infty(p) \mid f(p) = 0\}.$$

Moreover, if we have a derivation $X : \mathscr{C}^\infty(p) \to \mathbb{R}$, the restricted derivation $X|_{\mathfrak{m}_p}$ is in $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2)$. And by this restriction, we get an isomorphism between $T_p^{\mathbf{Alg}}M$ and $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R})$.

Remark 3.4. *In this way, a smooth manifold is recognized as a locally ringed space, locally isomorphic to $\mathbb{R}^n$.*

Remark 3.5. *Let V be a finite dimensional $\mathbb{R}$-vector space. It has a tautological smooth manifold structure by taking charts such that the sets of isomorphisms of V and $\mathbb{R}^n$ given by arbitrary basis of V.*

We claim that we have canonical isomorphisms

$$T_pV \to V,$$

for any $p \in V$,

$$\begin{aligned}
&V \to T_p^{\mathbf{Coo}}V, v \mapsto (h, h(v)),\\
&V \to T_p^{\mathbf{Geo}}V, v \mapsto (t \mapsto p + tv),\\
&V \to T_p^{\mathbf{Alg}}V, v \mapsto \left(f \mapsto \frac{d}{dt}\bigg|_{t=0} f(p + tv)\right)
\end{aligned}$$

Definition 3.22. *Let $f : M \to N$ be a map of smooth manifolds which is smooth at $p \in M$. Its differential of p is the linear map*

$$\mathbf{d}f(p) = \mathbf{d}_p(f) : T_pM \to T_{f(p)}N,$$

defined as follows.

1). Geometric tangent space : $\mathbf{d}_p(f)(\alpha) = f \circ \alpha$ *where* α *is a smooth curve.*

2). Algebraic tangent space : $\mathbf{d}_p(f)(X)(\varphi) = X(\varphi \circ f)$ *where* $\varphi \in \mathscr{C}^\infty(f(p))$*.*

3). Coordinate tangent space : $\mathbf{d}_p(f)(h_M, v_M) = (h_N, d_{h_M(p)}(h_N)$*.*

Remark 3.6. *Given a chart* $h : U \to V$ *around* $p \in M$*.* h *consists of coordinate functions* h_i *where* $1 \le i \le m$ *for* $V \subset \mathbb{R}^m$*. We have for each* i

$$\mathbf{d}_p h_i : T_pM \to \mathbb{R},$$

and

$$B = \{d_ph_1, \cdots, d_ph_m\}$$

is a basis of the dual space $(T_pM)^*$*.*

Let

$$\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\}$$

be the dual basis of B*. By definition, this means that for any* $1 \le i, j \le m$*, we have*

$$\frac{\partial}{\partial x_i} h_j = d_ph_j(\frac{\partial}{\partial x_i}) = \delta_{ij}.$$

Proposition 3.4. *Let* $f : M \to N$ *be a map between smooth manifolds which is smooth and* $q \in N$ *be a regular value. For* $p \in f^{-1}(q)$*, we have*

$$T_pf^{-1}(q) = \mathbf{d}_p(f)^{-1}(0) \subset T_pM.$$

Proof. □

3.2 Immersions and Submersions

Definition 3.23. *Let* $f : M \to N$ *be a smooth map of smooth manifolds.* f *is called an*

1). immersion if $\mathbf{d}f : T_pM \to T_{f(p)}N$ *is injective for any* $p \in M$*,*

2). submersion, if $\mathbf{d}f(p) : T_pM \to T_{f(p)}N$ *is surjective for any* $p \in M$*.*

Remark 3.7. *An immersion need not be injective. The counter example is*

$$e^{ix} : \mathbb{R} \to S^1,$$

is an immersion.

Remark 3.8. *A submersion need not be injective. The counter example is*

$$i_U : U \to M,$$

an inclusion map is a submersion.

Remark 3.9. *We know that if f is a submersion, then $f^{-1}(q)$ is an embedded submanifold. However, if f is an immersion, even it is injective, $f(M)$ need not be an embedded submanifold of N.*

Definition 3.24. *An immersed submanifold is an image of an injective immersion.*

Remark 3.10. *We endow $f(M)$ with the transported topology and differential structure from M so that f becomes a diffeomorphism between M and $f(M)$. But this topology need not be the relative topology from N. It may be strictly finite.*

Example 3.1. *Let $T = S^1 \times S^1$ be a torus. Let $r \in \mathbb{R}$. We consider a map $f : \mathbb{R} \to T$ such that*

$$f(x) = (e^{2\pi tx}, e^{2\pi rix}).$$

This is an immersion for any r. We examine this by several cases.

First, when r is not a rational number then f is injective, the image is an immersed manifold. However, a copy of $\mathbb{R}$. But this image is a dense subset of the torus.

Second, if r is rational then f is not injective. It is going to factor through an injective immersion $\mathbb{R}/b\mathbb{Z} \to T$ where $r = \frac{a}{b}$, $a, b \in \mathbb{Z}$ are coprime. This image is not only immersed but also embedded.

Remark 3.11. *If $f : M \to N$ is an immersion, $\mathbf{d}f_{(}p)$ identifies T_pM with a linear subspace of $T_{f(p)}N$.*

Proposition 3.5. *If $f : M \to N$ is an injective immersion, that is also closed subset of N, then its image is an embedded submanifold.*

Remark 3.12. *Thus we have the notion of a closed submanifold.*

3.3 Multi-linear forms

Definition 3.25. *Let $\mathbb{V}$ be a vector space and $\varphi : \bigoplus_{i=1}^m V \to \mathbb{R}$ is called a m-multi-linear function if for any $i = 1, \cdots, m$ and $\{a_j\}_{j\neq i} \subset V$ we have*

$$\varphi(a_1, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_m) : V \to \mathbb{R}$$

is a linear function

Definition 3.26. *Let X be a smooth n-dimensional manifold and $m \in \mathbb{N}$. Then we define the followings*

1. $\mathscr{L}_p^m = \{\varphi : \bigoplus_{i=1}^m T_pX \to \mathbb{R} | \varphi$ *is a m-multi-linear function.*$\}$
2. $\mathscr{L}^m = \bigcup_{p\in X} \mathscr{L}_p^m$

Definition 3.27. *Let X be a smooth n-dimensional manifold. A map $V : X \to \mathscr{L}^m$ is called a m-tensorfield if*

i. For any $p \in X$, $V(p) \in \mathscr{L}^m_p$.

ii. For any chart (U, φ) around p with a basis $\{e^\varphi_1, \cdots, e^\varphi_n\}$ and for any $i_1, \cdots, i_m \in \{1, \cdots, n\}$ we have a map $V_{(i_1,\cdots,i_m)} : X \to \mathbb{R}$ such that $V_{(i_1,\cdots,i_m)}(p) = V(p)(\underline{e}_{i_1}, \cdots, \underline{e}_{i_m})$ is smooth.

Proposition 3.6. *For any m tensorfield V, we have*

Definition 3.28. *We define $\mathscr{V}^m(X)$ to be the set of all m-tensorfield.*

Proposition 3.7. *$\mathscr{V}^m(X)$ is a vector space over $\mathbb{R}$ and a module over $\mathscr{F}(X)$ with the common basis $\{E_{i_1,\cdots,i_m}\}_{i_1,\cdots,i_m \in \{1,\cdots,n\}}$*

Proposition 3.8. *Let X be a smooth n-dimensional manifold and $V : X \to \mathscr{L}^m$ be such that for any $p \in X, V(p) \in \mathscr{L}^m_p$ the followings are equivalent.*

1. V is a m-tensorfield.

2. For any chart (U, φ) around p with basis $\{\underline{e}^\varphi_1, \cdots, \underline{e}^\varphi_n\}$ and for any $1 \leq i_1, \cdots, i_m \leq n$ there exist smooth mappings $\lambda_{i_1,\cdots,i_m} : X \to \mathbb{R}$ such that $V(p) = \sum_{1 \leq i_1,\cdots,i_m \leq n} \lambda_{i_1,\cdots,i_m}(p) E^\varphi_{i_1,\cdots,i_m}$.

3. For any vectorfields $v_1, \cdots, v_m : X \to TX$ we have a function $V : X \to \mathbb{R}$ such that $V_{v_1,\cdots,v_m}(p) = V(p)(v_1(p), \cdots, v_m(p))$ is smooth.

Proof. 1.⇔2. is trivial. 1.⇒3. is clear by the multi-linearity, and 3.⇒1. is choosing $v_i = e^\varphi_i$ for each $i = 1, \cdots, n$.

□

Proposition 3.9. *Let $V : X \to \mathscr{L}^m$ then thne followings are equivalent.*

1. V is a m-tensorfield.

2. For any $\{v_1, \cdots, v_m\} \in \mathscr{V}(X)$, $\Psi : \bigoplus_{i=1}^m \mathscr{V}(X) \to \mathscr{F}(X)$ such that $\Psi(v_1, \cdots, v_m)(p) = V(p)(v_1(p), \cdots, v_m(p))$ is smooth and $\mathscr{F}(X)$-linear.

Proof. 1.⇒2. follows from the multilinearity and decompositions of tensors. 2.⇒1. follows by fixing all element except one we still have the linearity thus, the function is mutilinear. □

3.4 Tensor and Wedge products

Definition 3.29. *Let $V_1 : X \to \mathscr{L}^r, V_2 : X \to \mathscr{L}^s$ be tensorfield. Then We define the tensorproduct $V_1 \otimes V_2 : X \to \mathscr{L}^{r+s}$ of them to be*

$$(V_1 \otimes V_2)(p)(v_1, \cdots, v_r, v_{r+1}, \cdots, v_{r+s}) = V_1(p)(v_1, \cdots, v_r)V_2(p)(v_{r+1}, \cdots, v_{r+s})$$

Proposition 3.10. *The operation $\bigotimes$ is bilinear and associative.*

Proof. By substituting values, they are trivial. □

Proposition 3.11. *Let $U \subset X$ be an open set and $V_1, \cdots, V_n \in \mathscr{V}^1(U)$ be a basis in $\mathscr{V}^1(U)$ then $\{\bigotimes_{j=1}^r V_{i_j}\}_{1 \leq i_1, \cdots, i_r \leq r}$ is a basis in $\mathscr{V}^r(U)$.*

Proof. Since $\otimes$ is bilinear, this is a tensor product thus the set in the statement is indeed a basis. □

Definition 3.30. *Let $V \in \mathscr{V}^m(X)$ be a m-tensor. V is said to be alternating if for any $p \in X$, $(v_1, \cdots, v_m) \in \bigoplus_{i=1}^m T_pX$ and $\sigma \in \mathfrak{S}_m$ we have*

$$V(p)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) = \operatorname{sgn}(\sigma)V(p)(v_1, \cdots, v_m)$$

Furthermore, such V is called a m-form.

Notation 3.1. *The set of all m-forms is denoted by*

$$\mathscr{A}^m(X) = \{V \in \mathscr{V}^m(X) \mid V \text{ is a } m\text{-form.}\}$$

Definition 3.31. *Let $V_1 \in \mathscr{A}^r(X), V_2 \in \mathscr{A}^s(X)$ then the wedge product is*

$$(V_1 \wedge V_2)(p)(v_1, \cdots, v_{r+s}) = \frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\sigma) V_1 \otimes V_2(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)})$$

Proposition 3.12. *Let $V_1, \cdots, V_n \in \mathscr{A}^1(X)$, $p \in X$ and $v_1, \cdots, v_n \in T_pX$ then we have*

$$(V_1 \wedge \cdots \wedge V_n)(p)(v_1, \cdots, v_n) = \det(V_i(p)(v_j))_{i,j}$$

Proof.

$$(V_1 \wedge \cdots \wedge V_n)(p)(v_1, \cdots, v_n) = \frac{1}{1! \cdots 1!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n V_i(p)(v_\sigma(i))$$

□

Proposition 3.13. *Similar to the case in tensorfields, we have the following statements.*

1. *$\mathscr{A}^m(X)$ is a subspace of $\mathscr{V}^m$ over $\mathbb{R}$.*
2. *$\mathscr{A}^m(X)$ is a module over $\mathscr{F}(X)$.*

Proof. Trivial. □

Proposition 3.14. *Let $V_1 \in \mathscr{A}^r, V_2 \in \mathscr{A}^s$, then $V_1 \wedge V_2 \in \mathscr{A}^{r+s}$ and such $\wedge : \mathscr{A}^r \times \mathscr{A}^s \to \mathscr{A}^{r+s}$ is bilinear.*

Proof. Bilinearity follows from the bilinearity of $\otimes$. We will show that this is indeed well-defined.

Let $\sigma \in \mathfrak{S}_{r+s}$. Then we have

$$\begin{aligned}(V_1 \wedge V_2)(p)(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)}) &= \frac{1}{r!s!} \sum_{\tau \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\tau) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)}) \\ &= \operatorname{sgn}(\sigma) \frac{1}{r!s!} \sum_{\tau\circ\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\tau \circ \sigma) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)}) \\ &= \operatorname{sgn}(\sigma)(V_1 \wedge V_2)(p)(v_{\sigma(1)}, \cdots, v_{\sigma(r+s)})\end{aligned}$$

$\square$

Proposition 3.15.

$$V_2 \wedge V_1 = (-1)^{rs}(V_1 \wedge V_2)$$

Proof. Let $\tau \in \mathfrak{S}_{r+s}$ to be such that

$$\tau(i) = \begin{cases} r+i & (1 \le i \le s) \\ i-s & (s+1 \le i \le r+s) \end{cases}$$

Then clearly the inversion number is $N(\tau) = rs$. It is also obvious that

$$V_2 \wedge V_1(p)(v_{\tau(1)}, \cdots, v_{\tau(r+s)}) = V_1 \wedge V_2(p)(v_1, \cdots, v_{r+s})$$

$\square$

Proposition 3.16. *Let* $V_1 \in \mathscr{A}^r, V_2 \in \mathscr{A}^s, V_3 \in \mathscr{A}^t$ *then* $(V_1 \wedge V_2) \wedge V_3 = V_1 \wedge (V_2 \wedge V_3)$.

Proof.

$$\begin{aligned}(V_1 \wedge V_2) \wedge V_3(p)(v_1, \cdots, v_{r+s+t}) &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau)(V_1 \wedge V_2) \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\ &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau) \\ &\quad (\frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\sigma) V_1 \otimes V_2(v_{\tau\circ\sigma(1)}, \cdots, v_{\tau\circ\sigma(r+s)})) \\ &\quad V_3(v_{\sigma(r+s+1)}, \cdots, v_{\sigma(r+s+t)})\end{aligned}$$

If for $\tau_1, \tau_2 \in \mathfrak{S}_{r+s+t}, \sigma_1, \sigma_2 \in \mathfrak{S}_{r+s}$ we have $\tau_1 \circ \sigma_1 = \tau_2 \circ \sigma_2$ then they satisfy the followings

i. For any $r+s+1 \le i \le r+s+t$ we have $\tau_1(i) = \tau_2(i)$.

ii. From above we get $\tau_2^{-1} \circ \tau_1 \in \mathfrak{S}_{r+s}$

Fixing σ_1, there exists $(r+s)!$ many such σ_2. This implies that we can choose σ_1 to be the identity. Thus we get

$$\begin{aligned}(V_1 \wedge V_2) \wedge V_3(p)(v_1, \cdots, v_{r+s+t}) &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau)(V_1 \wedge V_2) \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\ &= \frac{1}{(r+s)!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau) \frac{(r+s)!}{r!s!} V_1 \oplus V_2 \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)}) \\ &= \frac{1}{r!s!t!} \sum_{\tau \in \mathfrak{S}_{r+s+t}} \operatorname{sgn}(\tau) V_1 \oplus V_2 \oplus V_3(v_{\tau(1)}, \cdots, v_{\tau(r+s+t)})\end{aligned}$$

From the previous proposition we get □

4 Integration

Definition 4.1. *A differential k-form ω on a smooth manifold M is a collection $\omega(p) \in A^k(T_pM)$ for all $p \in M$.*

Remark 4.1. *We can define what it means for ω to be continuous or smooth at some points $p \in M$ as follows.*

First, we pick a chart $h : U \to V$ around p and get the basis

$$\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\},$$

of T_pM that moves with $p \in U$.
We also have a basis $A^k(T_pM) = \bigwedge^k(T_pM)^*$. Hence we can express ω as p in terms of that basis and the scalars in this expression are functions on U.

$$\omega(p) = \sum f_{i_1,\cdots,i_k} \cdot d_{x_{i_1}} \wedge \cdots \wedge d_{x_{i_k}}.$$

And we can require $f_{i_1,\cdots,i_k} \cdots d_{x_{i_1}}$ to be smooth/continuous at p.

Example 4.1. *If $M = \mathbb{R}^n$, we have the canonical identification,*

$$T_pM = \mathbb{R}^n.$$

This gives us standard differential form of degree n. which is given by

$$e_1^* \wedge \cdots \wedge e_n^*,$$

where $e_1, \cdots, e_n$ is the standard basis of $\mathbb{R}^n$.

Definition 4.2. *Let $f : M \to N$ be a smooth map of manifolds and ω be a differential form of degree k on N. We define $f^*(\omega)$ of degree k on M by*

$$f^*(w)(p)(x_1, \cdots, x_k) = \omega(f(p))(\mathbf{d}f_p(x_1), \cdots, \mathbf{d}f_p(x_k)).$$

Definition 4.3. *A differential n-form ω on M is said to be locally integrable if for any point $p \in M$, if for any point $p \in M$, there is one (hence any) chart $h : U \to V$ such that $\omega|_U =$*

5 Lie Algebras

5.1 Important homomorphisms and their properties.

Recall if $f : M \to N$ is a smooth map of smooth manifolds and $p \in M$, we get $df(p) : T_pM \to T_{f(p)}N$ is linear.

Proposition 5.1. *Let $(G, \mu, \iota, 1)$ be a lie group and $\mathfrak{g} = T_1G$. We have*

$$d\mu(1,1) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (X, Y) \mapsto X + Y.$$
$$d\iota(1) : \mathfrak{g} \times \mathfrak{g}, X \mapsto -X$$

Definition 5.1. *A Lie group homomorphism is a smooth map of Lie groups that is a homomorphism.*

Remark 5.1. *If $f : G \to H$ is a Lie group homomorphism then*

$$df(1) : \mathfrak{g} \to \mathfrak{h}$$

is a linear map.

Definition 5.2. *Let G be a Lie group. The adjoint action of G on itself is*

$$\underline{\mathrm{Ad}}(g) : G \to G, h \mapsto ghg^{-1}$$

which is a group homomorphism.

Definition 5.3. *Let G be a Lie group and $\mathfrak{g} = T_1G$. Then we define*

$$\mathrm{Ad}(g) = d\underline{\mathrm{Ad}}(g)(1) : \mathfrak{g} \to \mathfrak{g}.$$

We call this the adjoint action of G on $\mathfrak{g}$.

Remark 5.2. *The term, "action" in the definition above is justified by the chain rule*

$$\mathrm{Ad}(g \cdot h) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h).$$

Definition 5.4. *Let G be a Lie group and $\mathfrak{g} = T_1G$. By regarding Ad as a function from G to $\mathrm{GL}(\mathfrak{g})$. Notice that by the definition of groups we have $\mathrm{Ad}(g)$ is injective.*

We now define the adjoint action of $\mathfrak{g}$ on itself to be

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), X \mapsto d\,\mathrm{Ad}(1)X.$$

Definition 5.5. *Let G be a Lie group and $\mathfrak{g} = T_1 G$. The Lie bracket is $[\cdot|\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

$$[X|Y] = \mathrm{ad}(X)(Y).$$

Proposition 5.2. *Let $G = \mathrm{GL}_n(\mathbb{R})$ and $\mathfrak{g} = \mathbb{R}^{n\times n}$. Let $g \in G$ and $X, Y \in \mathfrak{g}$. We have*

$$[X|Y] = XY - YX.$$

Proof. Let $g \in G$,

$$\begin{aligned} \mathrm{Ad}(g)X &= d\underline{\mathrm{Ad}}(g)(1)X, \\ &= g(1+X)g^{-1} - g1g^{-1} \mod o(X), \\ &= gXg^{-1} \mod o(X), \\ &= gXg^{-1}. \end{aligned}$$

In particular $\underline{Ad}$ is a liner map. Now we compute the Lie bracket

$$[X|Y] = \mathrm{ad}(X)(Y) = [E_Y \circ \mathrm{ad}](X),$$

where E_Y is the evaluation map

$$E_Y : \mathrm{End}(\mathfrak{g}) \to \mathfrak{g}, \phi \mapsto \phi(Y).$$

$$\begin{aligned} [X|Y] &= [E_Y \circ \mathrm{ad}](X), \\ &= d[g \mapsto \mathrm{Ad}(g)Y](1)(X), \\ &= d[g \mapsto gYg^{-1}](1)(X). \end{aligned}$$

By the first computation we did, we see that

$$[X|Y] = (1+X)Y(1+X)^{-1} - Y \mod o(X).$$

We have the following identity

$$(1-X)^{-1} = 1 + X + X^2 + \cdots.$$

Substituting $-X$ we derive that

$$1 + X = \sum_{i=0}^{\infty} (-1)^i X^i.$$

And we only need at most degree 1 terms of X. We conclude that

$$[X|Y] = XY - YX.$$

$\square$

Remark 5.3. *This works for any matrix groups such as $\mathrm{SL}_n(\mathbb{R}), O(p,q)$.*

Proposition 5.3. *Let $f : G \to H$ be a Lie group homomorphism. For $g \in G$, we have*

$$df(1) \circ \mathrm{Ad}(g) = \mathrm{Ad}(f(g)) \circ df(1). \tag{5.1}$$

And for $X, Y \in \mathfrak{g}$, we have

$$df(1)([X|Y]_G) = [df(1)X, df(1)Y]_H. \tag{5.2}$$

Proof. Let us consider the composition of f and $\underline{\mathrm{Ad}}$. By definition, we see

$$f \circ \underline{\mathrm{Ad}}(g)(h) = f(g)f(h)f(g)^{-1} = \underline{\mathrm{Ad}}(f(g))(f(h)).$$

Since $\underline{\mathrm{Ad}}(1) = 1$ and by the chain rule we have Equation 5.1.

□

5.2 Lie Algebras

Definition 5.6. *A Lie algebra a (finite dimensional) vector space L over $\mathbb{R}$ or $\mathbb{C}$ together with a bilinear map $[\cdot|\cdot] : L \times L \to L$ such that*

i $[X|Y] = -[Y|X]$,

ii $[X|[Y|Z]] + [Y[Z|X]] + [Z|[X|Y]] = 0$ *which is called Jacobi identity.*

Proposition 5.4. *Let G be a Lie group and $\mathfrak{g} = T_1G$. Then $\mathfrak{g}$ equipped with $[X|Y] = \mathrm{ad}(X)(Y)$ is a $\mathbb{R}$-Lie algebra.*

Proof. Consider the commutator map $G \times G \to G, (x, y) \mapsto xyx^{-1}y^{-1}$. This is a smooth map as it is a composition of smooth maps $\mu(\mu(\cdot, \cdot), \mu(\iota(\cdot), \iota(\cdot)))$. Moreover, we can write this as

$$\underline{\mathrm{Ad}}(x)(y)\iota(y).$$

Differentiate this at $y = 1$ in the direction of Y, we get

$$d(\underline{\mathrm{Ad}}(x)(1)\iota(1))Y = \mathrm{Ad}(x)Y - Y,$$

since $d\iota(Y) = -Y$. Differentiate this again at $x = 1$ with respect to X we get $[X, Y]$.

Repeating the argument with

$$x\underline{\mathrm{Ad}}(y)(\iota(x)).$$

Differentiate this at $x = 1$ with the direction to X we get

$$X - \underline{\mathrm{Ad}}(y)X = X - yXy^{-1}.$$

Differentiate this again at $y = 1$ with the direction to Y, we get $-[Y|X]$. By smoothness, we get

$$[X|Y] = -[Y|X].$$

For the second property, we consider the Lie group homomorphism,

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}).$$

By Proposition 5.3, we have

$$\mathrm{ad}[X|Y]_G = [\mathrm{ad}(X)|\,\mathrm{ad}(Y)]_{\mathrm{GL}(\mathfrak{g})} = \mathrm{ad}(X)\,\mathrm{ad}(Y) - \mathrm{ad}(Y)\,\mathrm{ad}(X).$$

Therefore, by definition of $[\cdot|\cdot]$, we get

$$[[X|Y]|Z] = [X|[Y|Z]] - [Y|[X|Z]]$$

By the first property, we get the Jacobi identity. □

Example 5.1. *If V is a finite dimensional $\mathbb{R}$-vector space then $\mathrm{End}(V)$ equipped with $[X|Y] = XY - YX$ is a Lie algebra. In fact, this coincides with the Lie algebra of the Lie group $\mathrm{GL}(V)$.*

Definition 5.7. *A homomorphism of Lie algebras is a linear map $f : L \to M$ such that for $X, Y \in L$*

$$f([X|Y]_L) = [f(X)|f(Y)]_M$$

Corollary 5.1. *If $f : G \to H$ is a homomorphism of Lie groups, then $df(1) : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras.*

5.3 The identity component

Lemma 5.1. *Let G be a topological group. If $H \subset G$ is an open subgroup, then it is also closed. Thus if G is connected we have $H = G$.*

Proof. Let $\{1\} \cup I$ be a set of representations of equivalence classes in G/H. In other words we have

$$G = H \cup \bigcup_{i \in I} iH.$$

Since $\bigcup_{i \in I} iH$ is open, thus its complement H is closed. □

Lemma 5.2. *Let G be a connected topological group and $U \subseteq G$ be a neighborhood of 1. Then U generates G.*

Proof. Since $U \cap U^{-1}$ is non-empty and open. We may assume with out the loss of generality that $U = U^{-1}$. Let us denote

$$U^n = \{g_1 \cdots g_n \mid g_1, \cdots, g_n \in U\}.$$

And for $g_1 \cdots g_n \in U^n$, we take $V \subset U$ an open subset and $g_1 \in V$. $Vg_2 \cdots g_n$ is open in U. We now conclude that

$$H = \bigcup_{n=1}^{\infty} U^n$$

is an open subset which is a subgroup of G since it is closed under multiplication and inversion. Since G is connected we conclude that $H = G$. $\square$

Definition 5.8. *A subgroup H of a group G is said to be characteristic if for any automorphism $\varphi : G \to G$, we have $\varphi(H) \subseteq H$.*

Definition 5.9. *Let X be a topological space. A connected component C of $x \in X$ is the largest connected set which contains x.*

Proposition 5.5. *If C is a connected component of the topological space X, then it is closed.*

Proof. $\square$

Proof. Let $f : \overline{C} \to \{0, 1\}$ be a continuous function where $\{0, 1\}$ is with the discrete topology. Then for any $x \in C$ we conclude $f(x) = 0$ without the loss of generality. By the continuity of f we conclude that $f(x) = 0$ for any $x \in \overline{C}$. $\square$

Definition 5.10. *A topological space $(X, \mathscr{T})$ is said to be locally connected if for any point $x \in X$ and its neighborhood U, there exists a connected neighborhood V such that $x \in V \subset U$.*

Proposition 5.6. *A component of locally connected topological space is open.*

Proposition 5.7. *Let G be a topological group and G^0 be the connected component of G containing* 1.

1) G^0 is a closed characteristic subgroup of G.

2) If G is locally connected then G^0 is open and contained in any open subgroup of G.

3) The connected component of G are precisely G^0-cosets.

Proof. By Proposition 5.5, G^0 is a closed set. Since continuous maps preserve connectedness and 1 is mapped to 1, we can conclude that G^0 is characteristic. Similarly, since multiplication and inversion are smooth, thus continuous, we conclude that G^0 is a subgroup of G. This proves the first statement.

If G is locally connected, by Proposition 5.6, G^0 is open. If $H \subset G$ is any open subgroup, then $H \cap G^0$ is an open subgroup of G^0. By Lemma 5.2, we have $H \cap G^0$ generates G^0. $H \cap G^0$ is a group, we conclude that it is equal to G^0. This shows that G^0 is contained in any open subgroup of G.

Let C be a connected component and $g \in C$. Since $\mu(\cdot, g^{-1} : G \to G$ is continuous, we conclude that $\mu(C, g^{-1})$ is contained in the connected component which contains 1. Hence $C = G^0 g$. $\square$

5.4 Invariant vector fields

Definition 5.11. *Let M be a manifold. A vector field v on M is an assignment that for each $p \in M$, we have $v(p) \in T_pM$. It is said to be smooth if locally around each point $p \in M$, it's coefficients in terms of local coordinates are smooth functions. In other words, given a chart $h : U \to V$, we can get a basis $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ of T_pM for all $p \in U$. And locally*

$$v(p) = \sum_{i=1}^{n} c_i(p) \cdot \frac{\partial}{\partial x_i}.$$

And each c_i is smooth.

Definition 5.12. *Let M be a manifold, v be a smooth vector field. An integral curve is a pair (I, γ) where*

i). I is an open interval,

ii). $\gamma : I \to M$ is a smooth map such that $\gamma'(t) = v(\gamma(t))$.

Proposition 5.8. *Let v be a smooth vector field on a manifold M, then we have the following statements.*

1). Given $p \in M$, there exists a integral curve (I, γ) such that $0 \in I$ and $\gamma(0) = p$.

2). $(I_1, \gamma_1), (I_2, \gamma_2)$ be integral curves with above properties. Then for any $t \in I_1 \cap I_2$, we have $\gamma_1(t) = \gamma_2(t)$.

3). In particular, we can splice such γ_1, γ_2.

Proof. Follows from existence and uniqueness of solutions of ordinary differential equations in $\mathbb{R}^n$ via charts. □

Remark 5.4. *There is a maximal integral curve through p.*

Definition 5.13. *Let G be a Lie group and $g \in G$. We define*

$$L_g : G \to G, L_g(x) = gx, \quad R_g : G \to G, R_g(x) = xg,$$

the left and the right translations. Obviously these are diffeomorphisms as the inverses are $L_{g^{-1}}, R_{g^{-1}}$, respectively.

Remark 5.5. *By differentiating these we get*

$$dL_g(1) : \mathfrak{g} \to T_gG, \quad dR_g(1) : \mathfrak{g} \to T_gG.$$

Proposition 5.9. *Let G be a Lie group and $g \in G$. Then $dL_g(1), dR_g(1)$ are isomorphisms between $\mathfrak{g}$ and T_gG. Therefore, we can naturally identify T_gG by $\mathfrak{g}$. Moreover, $dL_g(1), dR_g(1)$ are not the same in general and differ by the automorphism* $\mathrm{Ad}(g)$.

Proof. □

Definition 5.14. *A vector field v on the Lie group G is said to be*

1). left-invariant if $v(g) = dL_g(1)(v(1))$,

2). right-invariant if $v(g) = dR_g(1)(v(1))$.

Remark 5.6. *Such vector field is automatically smooth. And the assignments*

$$X^L = X \mapsto (g \to dL_g(1)(X)), \quad X^R = X \mapsto (g \to dR_g(1)(X))$$

identify the Lie algebra $\mathfrak{g}$ with the space of left/right-invariant vector fields on G.

Lemma 5.3. *Let v be a left-invariant vector field on G. The maximal integral curve γ with $\gamma(0) = 1$ is defined on all of $\mathbb{R}$ and is a group homomorphism.*

Proof. Let $\gamma : I \to G$ be an integral curve with v with $\gamma(0) = 1$.

Assume $I \neq \mathbb{R}$ thus, without the loss of generality I has an upper bound $t_0 \in \mathbb{R}$. We will have to show that γ is not maximal. To see this, we choose $0 < \varepsilon < t_0$ and $t_0 - \varepsilon < t_1 < t_0, t \in I$.

Consider $\delta(t) = \gamma(t_1) \cdot \gamma(t - t_1)$. Thus γ is a smooth curve defined an open neighborhood of t_0 and $\delta(t_1) = \gamma(t_1)$ and

$$\begin{aligned}
\delta'(t) &= d\delta(t)(1), \\
&= dL_{\gamma(t_1)}(\gamma(t - t_1))(dr(t - t_1)(1)), \\
&= dL_{\gamma(t_1)}(v(\gamma(t - t_1))), \\
&= dL_{\gamma(t_1)}(\gamma(t - t_1)(dL_{r(t-t_1)}(1)(v(1))), \\
&= dL_{\gamma(t_1)\gamma(t-t_1)}(1)(v(1)), \\
&= v(\gamma(t_1)\gamma(t - t_1)), \\
&= v(\delta(t)).
\end{aligned}$$

Thus δ is an integral curve for v defined on an open neighborhood of t_0 containing t_1 and $\delta(t_1) = \gamma(t_1)$. Therefore γ is not maximal.

Now we are going to show that γ is a homomorphism. For fixed $t \in \mathbb{R}$, note that the maps

$$s \mapsto \gamma(t + s), \quad s \mapsto \gamma(t)\gamma(s)$$

are both integral curves for v with equal value at $s = 0$, hence equal. □

5.5 The Exponential Maps

Proposition 5.10. *Let $X \in \mathfrak{g}$, there exists a unique group homomorphism $\gamma_X : \mathbb{R} \to G$ differentiable at 0 and $\gamma'_X(0) = X$. It is the maximal integral curve through 1 for both X^L and X^R. We have $\gamma_{tX}(s) = \gamma_X(ts)$ for $t \in \mathbb{R}$.*

Proof. By Lemma 5.3, there exist maximal integral curves for X^L and X^R , we denote them by $\gamma_{X^L}, \gamma_{X^R}$, respectively. By Lemma 5.8, we can assume $\gamma_{X^L}(0) = \gamma_{X^R}(0) = 1$, and these are defined on the whole $\mathbb{R}$.

For uniqueness, let $\gamma : \mathbb{R} \to G$ be a group homomorphism which is differentiable at 0 with $\gamma'(0) = X$. Then

$$\gamma(t)\gamma(s) = \gamma(t+s) = \gamma(s+t) = \gamma(s)\gamma(t). \tag{5.3}$$

Fix t and apply $\frac{d}{ds}|_{s=0}$ to see that γ is differentiable at any t in the following way

$$\begin{aligned} \frac{d}{ds}|_{s=0}\gamma(t)\gamma(s) &= \frac{d}{ds}|_{s=0}\gamma(t+s), \\ \Rightarrow \gamma(t)\gamma'(0) &= \gamma'(t). \end{aligned}$$

By construction, when $\gamma = \gamma_{X^L}$ we have

$$\begin{aligned} \gamma'_{X^L}(t) &= dL_{\gamma_{X^L}(t)}(1)(X^L(1)) \\ &= dL_{\gamma_{X^L}(t)}(1)X \\ &= L_{\gamma_{X^L}(t)}X. \end{aligned}$$

Similarly for $\gamma = \gamma_{X^R}$ we have

$$\begin{aligned} \gamma'_{X^R}(t) &= dR_{\gamma_{X^R}(t)}(1)(X^R(1)) \\ &= dR_{\gamma_{X^R}(t)}(1)X \\ &= R_{\gamma_{X^R}(t)}X. \end{aligned}$$

By the uniqueness of solutions of ordinary differential equations, we derive that $\gamma_{X^L} = \gamma_{X^R}$. This proves that $\gamma_{X^L}, \gamma_{X^R}$ are maximal as they are defined on all $t \in \mathbb{R}$.

For the second property, we only need to check that $\gamma_{tX}(s) = \gamma_X(ts)$ coincide at $s = 0$. $\square$

Definition 5.15. *Let G be a Lie group and $\mathfrak{g} = T_1G$. Then we define the exponential map*

$$\exp_G : \mathfrak{g} \to G, \exp_G(X) = \gamma_X(1),$$

where γ_X is the integral curve of $v(g) = dL_g(1)X$.

Theorem 5.1. $\exp_G : \mathfrak{g} \to G$ *is smooth and has the following properties.*

1). $\underline{\mathrm{Ad}}(x) \circ \exp_G = \exp_G \circ \mathrm{Ad}(x)$ *for any* $x \in G$.

2). $\mathrm{Ad} \circ \exp_G = \exp_{\mathrm{GL}(\mathfrak{g})} \circ \mathrm{ad}$.

3). $d\exp_G(0) : \mathfrak{g} \to \mathfrak{g}$ *is an identity* $id_{\mathfrak{g}}$.

4). If $f : G \to H$ is a homomorphism of Lie groups, then $f \circ \exp_G = \exp_H \circ df(1)$.

5). $\gamma_X(t) = \exp_G(t \cdot X)$.

Proof. Look at the homework $\square$

Proposition 5.11. *Let V be a finite dimensional $\mathbb{R}$-vector space. Then*

$$\exp_{\mathrm{GL}(V)} : \mathfrak{gl}(V) \to \mathrm{GL}(V)$$

is given by

$$\exp_{\mathrm{GL}(V)}(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n.$$

Proof. Homework $\square$

Corollary 5.2. *Furthermore, we can derive the following properties of $\exp_G$,*

1). $\operatorname{Im} \exp_G \subseteq G^0$.

2). $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism locally around 0.

3). If $U \subseteq \mathfrak{g}$ is a neighborhood of 0 in $\mathfrak{g}$, then $\exp_G(U)$ generates G^0.

Proof. Note that $\exp_G$ is a smooth map.

By the smoothness, it is also continuous. Since $\mathfrak{g}$ is connected, it is mapped to a connected subset of G which contains 1. Thus we have the first property.

By the third property of Theorem 5.1, we have $d\exp_G(0)$ is invertible.

By the second property of the same theorem, $\exp_G(U)$ contains an open neighborhood of 1, thus generates G^0. $\square$

Definition 5.16. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. By the second statement of the corollary above, there exists a neighborhood U of 0 in $\mathfrak{g}$, such that $\exp_G |U$ is a diffeomorphism. We denote its inverse by $\log_G$.*

Corollary 5.3. *Let G be a connected Lie group and $g \in G$, we have the following*

$$g \in Z(G) \Leftrightarrow \mathrm{Ad}(g) = id_{\mathfrak{g}}.$$

Proof. If $g \in Z(G)$, then $\underline{\mathrm{Ad}}(g) = id_G$, therefore $\mathrm{Ad}(g) = id_{\mathfrak{g}}$. Conversely, if $\mathrm{Ad}(g) = id_{\mathfrak{g}}$, by the first property of Theorem 5.1 we have $\underline{\mathrm{Ad}}(g)$ is identity on the image of $\exp_G$. By the second statement of Corollary 5.2, this image generates G. Since $\underline{\mathrm{Ad}}(g)$ is a homomorphism, it is trivial on the entire group G. $\square$

Corollary 5.4. *Let G be a Lie group and $X, Y \in \mathfrak{g}$. We have*

$$[X|Y] = 0 \Rightarrow \exp_G(X)\exp_G(Y) = \exp_G(Y)\exp_G(X).$$

Proof. Let $x = \exp_G(X), y = \exp_G(Y)$. By the first and second statements of Theorem 5.1,

$$xyx^{-1} = \exp_G(\mathrm{Ad}(X)Y) = \exp_G(\exp_{\mathrm{GL}(\mathfrak{g})}(\mathrm{ad}(X)(Y))).$$

By Proposition 5.11 and the assumption, this is equal to

$$\exp_G(Y) = y.$$

□

Corollary 5.5. *Let $f_1, f_2 : H \to G$ be homomorphisms of Lie groups. If H is connected and $df_1(1) = df_2(1)$. Then $f_1 = f_2$.*

Proof. Using the forth statement of Theorem 5.1, we have $f_1 = f_2$ upon restriction to the image of $\exp_H$, and such image generates H. □

5.6 Differentials of $\exp_G$

Theorem 5.2. *Let $X \in \mathfrak{g}$. (Recall that we have the canonical identification $T_x\mathfrak{g} \to \mathfrak{g}$).*

Consider

$$d(\exp_G)(x) : \mathfrak{g} \to T_{\exp(X)}G, \quad dR_{\exp_G(x)}(1) : \mathfrak{g} \to T_{\exp_G(x)}G.$$

Then we have the following,

$$dR_{\exp_G(x)}(1)^{-1} \circ d(\exp_G)(x) : \mathfrak{g} \to \mathfrak{g}, X \to \int_0^1 \exp_{\mathrm{GL}(\mathfrak{g})}(s \cdot \mathrm{ad}(X))ds.$$

Proof. □

Corollary 5.6. *An element $X \in \mathfrak{g}$ is a singular point for $\exp_G$ if and only if $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ has an eigenvalue of the form $2\pi ik$ for some $k \in Z^{\times}$.*

Proof. Since both $\mathfrak{g}$ and G have the same dimension, X is singular if and only if $d(\exp_G)(X)$ is not invertible. By Theorem the equation

$$\int_0^1 \exp_{\mathrm{GL}(\mathfrak{g})}(s \cdot \mathrm{ad}(X))dx \tag{5.4}$$

is not invertible. In other words, it admits 0 as an eigenvalue. Using the formula

$$\int_0^1 \exp_{\mathrm{GL}(\mathfrak{g})}(s\lambda)dx = \begin{cases} \lambda^{-1}(e^{\lambda} - 1) & (\lambda \neq 0), \\ 1 \quad (\lambda = 0). \end{cases}$$

We see that the eigenvalues of the (5.4) are given by 1 if 0 is an eigenvalue of $\mathrm{ad}(X)$ and $\lambda^{-1}(e^{\lambda} - 1)$ if $\lambda \neq 0$ is an eigenvalue of $\mathrm{ad}(X)$. □

Remark 5.7. *The formula* (5.4) *generalizes to*

$$\int_0^1 e^{sA} ds = A^{-1}(e^A - 1) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} A^k.$$

for any $A \in \mathrm{GL}(V)$ *where* V *is a finite dimensional* $\mathbb{R}$*-vector space. If* A *is not invertible, we can define* $A^{-1}(e^A - 1)$ *by the above formula.*

This is particularly useful for $A = \mathrm{ad}(X)$, *for* $X \in \mathfrak{g} = V$, *which is never invertible since* $\mathrm{ad}(X)(X) = 0$. *Moreover, for* $A = \mathrm{ad}(X), A^{-1}(e^A - 1)$ *is invertible for* X *in a neighborhood of* 0 *by Corollary 5.6.*

5.7 The Product in Logarithmic Coordinates

Theorem 5.3. *Let* $U \subset \mathfrak{g}$ *be an open neighborhood of* 0. *For* $X, Y \in U$, *consider the differential equation for* $z : \mathbb{R} \to \mathfrak{g}$, *such that*

$$z(0) = Y, \quad \frac{dz}{dt}(t) = (\mathrm{ad}\, z(t))^{-1}(\exp_{\mathrm{GL}\,\mathfrak{g}}(\mathrm{ad}\, z(t)) - 1))^{-1}(X).$$

For U *sufficiently small, this differential equation has (a unique) solution for all* $X, Y \in U$ *and all* $t \in [0,1]$. *Define* $\mu(X,Y) = z(1)$. *Then*

$$\exp_G(X)\exp_G(Y) = \exp_G(\mu(X,Y)).$$

Proof. ☐

Corollary 5.7. *The collection of maps* $\kappa_x : U \to G$, *where* $U \subset \mathfrak{g}$ *is an open neighborhood of* 0.

$$\kappa_x(Y) = x \cdot \exp_G(Y)$$

is a smooth, in fact real analytic, atlas for the manifold G.

Proof. We know that $\exp_G$ is smooth and a locally diffeomorphism around 0. So κ_x is a diffeomorphism onto its image . Thus $(\kappa_x)_{x \in G}$ is a smooth atlas.

The transition maps are expressible in terms of μ by Theorem 5.3. Since μ is real analytic in X, Y. we see that the atlas is real analytic. ☐

Definition 5.17 (Real analytic manifolds)**.** *A manifold is said to be*

Remark 5.8. *In particular, any Lie group is automatically real analytic.*

Theorem 5.4. *Let* $X, Y \in U$, *then*

$$\mu(X,Y) = X + Y + \sum_{k=1}^{\infty} \frac{(1)^k)}{k+1} \sum_{\substack{l_1,\cdots,l_k \geq 0, \\ m_1,\cdots,m_k \geq 0, \\ l_i + m_i > 0}} \frac{1}{\sum_{i=1}^k l_i + 1} \prod_{i=1}^k \frac{\mathrm{ad}(X)^{l_i}}{l_i!} \frac{\mathrm{ad}(X)^{m_i}}{m_i!}$$

Corollary 5.8.

$$\mu(X,Y) = X + Y + \frac{1}{2}[X,Y] + O(|(X,Y)|^3).$$

5.8 Lie Subgroups

Definition 5.18. *Let G be a Lie group. A Lie subgroup H of G is a immersive submanifold that is also a subgroup.*

Definition 5.19. *Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{h}$ of it is called a Lie subalgebra if it is closed under the Lie bracket operation $[\cdot|\cdot]$.*

Definition 5.20. *Let G be a Lie group, then we denote*

$$\mathrm{Lie}(G) = T_1 G.$$

Remark 5.9. *A tautological inclusion $i_H : H \to G$ is an injective immersion.*

Theorem 5.5. *Let G be a connected Lie group. Then there is a bijection between*

$$\{H \subseteq G \mid \textit{connected Lie subgroups.}\} \leftrightarrow \{\textit{Lie subalgebras of } \mathrm{Lie}(G).\},$$

And the bijection is given by $\mathrm{Lie}(\cdot)$.

Proof. Let H be a subgroup of G and $\mathfrak{h} = T_1 H$.

We first prove the injectivity of $\mathrm{Lie}(\cdot)$.

For the surjectivity, let us take $H \subseteq G$ to be a subgroup generated by the image of $\exp_G(\mathfrak{h})$ for a Lie subalgebra $\mathfrak{h}$. By Corollary 5.7, we have

$$(\kappa_x^{-1})_{x \in G}, \quad \kappa_x(Y) = x \exp_G(Y)$$

is an atlas for G. We will show that

$$(\kappa_x^{-1})_{x \in H}$$

is an atlas for H.

First, we claim that if

$$\kappa_x(U \cap \mathfrak{h}) \cap \kappa_y(U \cap \mathfrak{h}) \neq \emptyset,$$

then there exist neighborhood V_1, V_2 of 0 in $\mathfrak{g}$ such that

$$\kappa_y^{-1} \circ \kappa_x : V_1 \cap \mathfrak{h} \to V_2 \cap \mathfrak{h}$$

is a diffeomorphism.

Since κ_x is an atlas for G, there exist some open neighborhoods V_1, V_2 of 0 such that

$$\kappa_y^{-1} \circ \kappa_x : V_1 \to V_2$$

is a diffeomorphism. The above composition is given by

$$\kappa_y^{-1} \circ \kappa_x(Y) = \log_G(y^{-1} x \exp_G(Y)).$$

Let $z = y^{-1}x$ then since $x, y \in H$, $z \in H$. Let $X \in \mathfrak{h}$ be such that

$$z = \exp(X).$$

Thus by using Theorem 5.3, we obtain

$$y^{-1} x \exp_G(Y) = z \exp_G(Y) = \exp_G(X) \exp_G(Y) = \exp_G(\mu(X, Y)).$$

Thus if $X, Y \in \mathfrak{h}$, then $\mu(X, Y) \in \mathfrak{h}$

For each $x \in H$, through $\kappa_x : \mathfrak{h} \cap U \to H$, we get a basis of a neighborhood of 0 in $\mathfrak{h}$. This topologizes H and

$$(\kappa_x^{-1})_{x \in G}, \quad \kappa_x(Y) = x \exp_G(Y)$$

becomes an atlas. Therefore, the tautological inclusion $\iota : H \to G$ is now an immersion. □

Remark 5.10. *This map is a bijection from the set of connected Lie subalgebras of G to the set of Lie subalgebras of* Lie(G).

Definition 5.21. *A subset $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called an ideal if for any $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, we have*

$$[X|Y] \in \mathfrak{h}.$$

Lemma 5.4.

Lemma 5.5. *Given a Lie group G and its Lie algebra $\mathfrak{g}$. H is a normal connected Lie subgroup of G if and only if* Lie(H) *is an ideal.*

Proof. □

Lemma 5.6. *Let G be a topological group and $H \subset G$ be a locally closed subgroup. Then H is a closed as a subset of G.*

Proof. □

Lemma 5.7. *Let $H \subset G$ be a Lie subgroup of a Lie group G. Then H is embedded submanifold if and only if H is closed.*

Proof. Since H is an embedded submanifold, it is locally closed. By Lemma 5.6, H is closed. Conversely, if H is closed then a tautological injection $\iota : H \to G$ is a closed injective immersion. Hence H is an embedded submanifold. □

Theorem 5.6. *Let G be a Lie group and $H \subseteq G$ be a subgroup which is closed as a set. Then H is a closed Lie subgroup.*

Proof. Without the loss of generality, we may assume that H is a connected set.

Let us define a subset of the Lie algebra $\mathfrak{g}$ as

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R}, \exp_G(tX) \in H\}.$$

We will first show that $\mathfrak{h}$ is a subspace of $\mathfrak{g}$. The scalar multiplication is obvious from the definition. We need to show that this is closed under addition.

Fix $t \in \mathbb{R}$, for a large enough n, there is a neighborhood V of 0 in $\mathfrak{g}$ such that

$$n^{-1}tX, n^{-1}tY \in V.$$

Since V is sufficiently small, $\exp_G |V$ is a diffeomorphism and by Theorem 5.3, we have for any $H, K \in V$,

$$\exp_G(H)\exp_G(K) = \exp_G(\mu(H, K)).$$

Differentiate the above equation at $(0, 0)$, we get

$$d\mu(H, K) = H + K.$$

Therefore we have

$$n \cdot d\mu(n^{-1}tX, n^{-1}tY) = t(X + Y).$$

Proof of the closedness

Let $U \subseteq \mathfrak{h}$ be a neighborhood of 0. Then $\exp_G(U)$ is a neighborhood of 1 in H by the construction.

Suppose there is a sequence $(h_n)_{n\in\mathbb{N}} \subset H \backslash \exp_G(U)$ which converges to 1. Let $\mathfrak{k}$ be a complement of a vector space $\mathfrak{h}$, that is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}.$$

Consider the map

$$\Phi : \mathfrak{h} \oplus \mathfrak{k} \to G, \Phi(H, K) = \mu(\exp_G(H), \exp_G(K)).$$

Differentiating this at $(0, 0)$ we get

$$d\Phi(H, K) = H + K,$$

since $d\exp_G$ is identity at 0. We conclude that Φ is a local diffeomorphism. Thus we can find sequences $(H_n)_{n\in\mathbb{N}} \subset \mathfrak{h}, (K_n)_{n\in\mathbb{N}} \subset \mathfrak{k}$ such that for big enough N and $n \geq N$,

$$\exp_G(H_n)\exp_G(K_n) = h_n.$$

For such $(K_n)_{n\in\mathbb{N}}$ we have $K_n \to 0$.
Now define a sequence $K'_n = \frac{K_n}{|K_n|}$ in the Euclidean norm. This belongs to the unit sphere in $\mathfrak{k}$ which is a compact set. Thus, we find a subsequence $(K_{n_m})_{m\in\mathbb{N}}$ which is converging to some K.

Let $t > 0$ be fixed. Since $|K'_n|\o 0$, we can find $(k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$k_n \leq \frac{t}{|K'_n|} \leq k_{n+1}.$$

Therefore, as $n \to \infty$,

$$k_n|K'_n| \to t.$$

Passing these to $\exp_G$ we get

$$\exp_G(tK) = \lim_{m\to\infty} \exp_G(k_{n_m}K_{n_m}) = \lim_{m\to\infty} \exp_G(K_{n_m})^{k_{n_m}}.$$

Since $\exp_G(Y_n) = (\exp_G(H_n))^{-1}h_n$, it belongs to H, so does $\exp_G(tK)$. □

Corollary 5.9. *Let $f : G \to H$ be a homomorphism of groups between two Lie groups which is continuous. Then f is a homomorphism of Lie groups.*

Proof. □

Corollary 5.10. *Let $f : G \to H$ be a homomorphism of Lie groups. Then* $\operatorname{Ker} f$ *is a closed Lie subgroup with its Lie algebra* $\operatorname{Ker}(df(1))$.

Proof. Since H is Hausdorff by definition, $\{1\}$ is closed in H. f is continuous implies that $K = \operatorname{Ker} f$ is closed in G as it is an inverse image of a closed set. By Theorem 5.6, it is a Lie subgroup. By the forth statement of Theorem 5.1, we have

$$f \circ \exp_G = \exp_H \circ df(1).$$

Therefore we have an inclusion $\operatorname{Ker}(df(1)) \subseteq \operatorname{Lie}(K)$. Conversely, let $X \in \operatorname{Lie}(K)$ and define a map

$$t \mapsto f(\exp_G(tX)) = 1.$$

By differentiating with respect to t at 0 we see

$$df(1)(X) = 0.$$

Therefore $X \in K$. □

Corollary 5.11. *Let $f : G \to H$ be a homomorphism of Lie groups. If f is an injection then f is an immersion. Furthermore, if f is a bijection then f is an isomorphism.*

Proof. f is injective if and only if $\operatorname{Ker} f = \{1\}$. By Corollary 5.10, we have

$$\operatorname{Ker}(df(1)) = \{0\}.$$

By translation by an arbitrary $g \in G$,

$$\operatorname{Ker}(df(g)) = \{0\}.$$

Moreover, if f is a bijection, then looking locally through charts, we see

$$\dim(G) = \dim(H).$$

Therefore we have

$$\dim \mathfrak{g} = \dim \mathfrak{h}.$$

Since $df(1)$ is an injective linear map between linear spaces with the same dimension, thus a bijection. Using translations again, we see that $df(g)$ is bijective for any $g \in G$.Therefore, f is a bijective local diffeomorphism everywhere, we conclude f is a diffeomorphism. $\square$

5.9 Group Action of Lie Groups

Definition 5.22. *Let G be a Lie group and M be a manifold. A smooth action of G on M is a smooth map $l : G \times M \to M$ which is a group action.*

Definition 5.23. *A group action $l : G \times M \to M$ of a topological group G on a manifold M is said to be proper if the map*

$$(g, x) \mapsto (gx, x)$$

is proper.

Remark 5.11. *A group action $l : G \times M \to M$, we can define functions such as*

1). $l_x : G \to M$ for fixed $x \in X$, $l_x(g) = gx$,

2). $l_g : M \to M$ for fixed $g \in G$, $l_g(x) = gx$.

Clearly, both of them are continuous. Furthermore, we have

1). l_x is injective if and only if $G_x = \{1\}$.

2). l^g is always a diffeomorphism with the inverse $m_{g^{-1}}$.

Lemma 5.8. *Let $l : G \times M \to M$ be a smooth, free Lie group action. Then for any $x \in M$, $dl_x(1)$ is injective.*

Proof. Given $X \in \mathfrak{g}$, we have

$$l_x(\exp_G((t+h)X) = l_x(\exp_G(tX)\exp_G(hX)) = l^{\exp_G(tX)} l_x(\exp_G(hX)).$$

Suppose $X \in \mathrm{Ker}(dl_x(1))$ and fix $t_0 \in \mathbb{R}$. We have

$$\frac{d}{dt}\bigg|_{t=t_0} l_x(\exp_G(tX)) = \frac{d}{dh}\bigg|_{h=0} l_x(\exp_G((t_0+h)X)) = dl^{\exp_G(t_0X)}(X)(dl_x(1)(X)) = 0.$$

Therefore, $l_x(\exp(tX))$ is constant but l is a free group action. Therefore for any $t \in \mathbb{R}$, $\exp_G(tX) = 1$. We conclude that $X = 0$. $\square$

Theorem 5.7. *Let $m : G \times M \to M$ be a smooth, free, proper group action. We can embed the smooth manifold structure to the equivalence classes G/M by the orbits of the action with following properties.*

i). The topology on G/M is the quotient topology induced by the canonical map $\pi : M \to G/M$.

ii). For an arbitrary point $p \in G/M$, there is a neighborhood $V \subseteq G/M$ and a diffeomorphism $\pi^{-1}(V) \to G \times V$, which translates the G actions on $\pi^{-1}(V)$ given by the map l to the G action $G \times V$ by left multiplication on G.

Proof. Let S be a complement vector space of $dl_x(1)(\mathfrak{g})$ in T_xM, thus we have

$$T_xM = S \oplus dl_x(1)(\mathfrak{g}).$$

Choose a submanifold $N \subseteq M$ such tha $x \in N$ and t $T_xN = S$. (Such N is called a slice).We first show that for sufficiently small N, the restricted action

$$\bar{l} : G \times N \to M$$

is a diffeomorphism onto its image. Indeed, taking the derivative of $\bar{l}$ at $(1, x)$ we derive

$$d\bar{l}(1, x) : \mathfrak{g} \times T_xN \to T_xM.$$

By the construction of N and Lemma 5.8, this is a bijection. By translation, we have $d\bar{l}(g, y)$ is bijective for any $(g, y) \in G \times N$ close enough to $(1, x)$.
Take N small enough so that for each $y \in N$, $d\bar{l}(1, y)$ is bijective. Using the formula

$$d\bar{l}(g, y) = dl^g \circ d\bar{l}(1, y),$$

we see that $d\bar{l}(g, y)$ is bijective for any $(g, y) \in G \times N$.
Secondly, we prove that for sufficiently small enough N, $\bar{l}$ is injective. Suppose that there are sequences $(y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subset N$ and $(g_n)_{n\in\mathbb{N}}, (h_n)_{n\in\mathbb{N}} \subset G$ such that

$$g_iy_i = h_iz_i, (g_i, y_i) \neq (h_i, z_i).$$

If $g_i = h_i$ then by the definition of group actions, $y_i = z_i$, therefore, we may assume that $g_i \neq h_i$. Let $k_i = h_i^{-1} g_i \neq 1$. Then the sequence

$$(k_i y_i, y_i) = (z_i, y_i) \to (x, x),$$

is contained in a compact subset of $M \times M$.

By the properness of l, $(k_i)_{i \in \mathbb{N}}$ is contained in a compact subset of G. Thus it has a convergent subsequence which converges to some $k \in G$. Therefore, with that subsequence we have

$$x = \lim_{i \to \infty} z_i = \lim_{j \to \infty} k_{i_j} y_{i_j} = kx.$$

By the freeness of l, we conclude that $k = 1$. However, $(k_{i_j}, y_{i_j}), (1, z_{i_j}) \subset G \times N$ converge to $(1, x)$. This contradicts to the injectivity of $\bar{l}$ in a sufficiently small neighborhood of $(1, x)$.

Now we are ready to close our proof. Note that the properness of the action implies that the space G/M is Hausdorff as each equivalence class is closed.

We introduce an atlas by

$$\bar{l} : G \times N \to \pi^{-1}(\pi(N)).$$

This satisfies the desired property. □

Corollary 5.12. *Let $H \subset G$ be a closed Lie subgroup. Then the coset spaces $H \backslash G$ and G/H are both smooth manifold. Furthermore, $\pi_l : G \to H \backslash G$ and $\pi_r : G \to G/H$ are principal H-bundles.*

Proof. The left action map $l : H \times G \to G$ is a multiplication and $H \times G \to G \times G$ is proper. □

Corollary 5.13. *Let $H \subset G$ be a closed normal Lie subgroup. Then G/H is a Lie group and the canonical group homomorphism $\pi : G \to G/H$ is a Lie group homomorphism.*

Proof. □

Corollary 5.14. *Let $f : H \to G$ be a homomorphism of Lie groups. Then there exists a unique $\overline{f} : H/\operatorname{Ker} f \to G$ such that*

$$\overline{f} \circ \pi = f.$$

And such $\overline{f}$ is an injective immersion.

Proof. By elementary group theory, there exists a unique homomorphism $\overline{f}$. Since $\pi : H \to H/\operatorname{Ker} f$ is a fiber bundle, and f is smooth, $\overline{f}$ is also smooth, we conclude that $\overline{f}$ is an injective homomorphism of Lie groups. □

Corollary 5.15. *Let $f : H \to G$ be a homomorphism of Lie groups. If f is surjective, it is a principal fiber bundle with group $K = \operatorname{Ker} f$.*

Proof. □

5.10 Classification of abelian connected Lie groups.

Definition 5.24. *A Lie algebra L is abelian if $L = Z(L)$, in other words, the Lie bracket is everywhere* 0.

Proposition 5.12. *A connected Lie group G is abelian if and only if its Lie algebra* $\mathrm{Lie}(G)$ *is abelian.*

Proof. If G is abelian, then the adjoint action $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is constant and $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is everywhere 0. Conversely, if $\mathfrak{g} = \mathrm{Lie}(G)$ is abelian then for any $X, Y \in \mathfrak{g}$, we have

$$\exp_G(X + Y) = \exp_G(X)\exp_G(Y).$$

Thus generators of the Group G commutes, we conclude G is abelian. $\square$

Proposition 5.13. *Let G be a connected abelian Lie group then $\exp_G : \mathfrak{g} \to G$ is a surjective homomorphism and its kernel is discrete.*

Proof. Let $X, Y \in \mathfrak{g}$ and consider two maps

$$t \mapsto \exp_G(t(X + Y)), \quad t \mapsto \exp_G(tX)\exp_G(tY).$$

These maps have the same derivatives at $t = 0$, namely $X + Y$.

Since G is abelian, we conclude these are homomorphisms of Lie groups. Therefore, they are equal to one another. We now have that $\exp_G$ is a group homomorphism, and $\exp_G(\mathfrak{g})$ generates G, therefore it is surjective.

We close the proof by stating that $\mathrm{Ker}\exp_G$ is a closed Lie subgroup with Lie algebra, $\mathrm{Ker}\, d\exp_G(0) = \{0\}$. $\square$

Theorem 5.8. *Let G be a connected abelian then it is isomorphic to $\mathbb{R}^a \times (S^1)^b$ for some integers a, b.*

Proof. $\square$

5.11 Connected Compact Lie groups

Proposition 5.14. *Let G be a connected topological group and $\alpha, \beta : [0,1] \to G$ be loops base at the identity* 1. *Let us define another loop by*

$$\gamma(s) = \alpha(s)\beta(s), \quad s \in [0,1].$$

Then in the fundamental group $\pi_1(G)$, we have

$$[\alpha] * [\beta] = [\gamma] = [\beta] * [\alpha].$$

In particular, $\pi_1(G)$ is abelian.

Proof. Let us consider the map $H : [0,1] \times [0,1] \to G, H(s,t) = \alpha(s)\beta(t)$. $\square$

Proposition 5.15. *Let G be a connected, locally path connected (hence globally path connected) topological group. View $(G, 1)$ as a pointed space and*

$$p : (X, x) \to (G, 1)$$

be a covering space. There exists a unique topological group structure on (X, x) with x being the identity which makes p into a group homomorphism. Furthermore, such topological group and a homomorphism has the following properties.

1). $\mathrm{Ker}(p)$ *is central and discrete.*

2). The group of Deck transformations of p is identified with $\mathrm{Ker}(p)$ *acting by multiplication.*

Corollary 5.16. *In the above setting, if G i a Lie group, the above construction makes the group (X, x) into a Lie group. Such structure makes p into a Lie group homomorphism and* $\mathrm{Ker}(p)$ *turns out to be countable.*

Proposition 5.16. *Let G be a connected Lie group. Then the universal cover $\tilde{G}$ of G is a connected Lie group.*

Furthermore, the map $[\alpha] \mapsto \alpha(1) : \pi_1(G) \to \tilde{G}$ where

Proposition 5.17. *Let $f : H \to G$ be a homomorphism of connected Lie groups. Then the following statements are equivalent.*

1. *f is a covering,*
2. *$df(1)$ is an isomorphism.*

Proof. □

Proposition 5.18. *Let H be a connected and simply connected Lie group and G be a Lie group.*

For any homomorphism $\varphi : \mathrm{Lie}(H) \to \mathrm{Lie}(G)$ of Lie algebras, there exists a homomorphism $f : H \to G$ of Lie groups such that

$$df(1) = \varphi.$$

Proof. □

Lemma 5.9. *Connected Lie groups that have the same Lie algebra have the same universal cover.*

Proof. Let G_1, G_2 be Lie groups and $\varphi : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ is an isomorphism. □

5.12 Compact Lie algebras

Definition 5.25. *Let G be a topological group. A finite dimensional representation of G over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a pair (π, V) such that*

i). V is a finite dimensional vector space,

ii). $\pi : G \to \mathrm{GL}(V)$ is a continuous group homomorphism.

Remark 5.12. *If G is a Lie group then π is smooth and we can take $\varphi = d\pi(1) : \mathrm{Lie}(G) \to \mathrm{End}(V) = \mathfrak{gl}(V)$ a homomorphism of Lie algebras.*

Definition 5.26. *The adjoint representation of G is the homomorphism*

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

such that

$$\varphi = \mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is the adjoint representation of $\mathfrak{g}$ on itself.

Remark 5.13. *G is a connected Lie group then*

$$Z(G) = \Im(\mathrm{Ad}).$$

Thus it is a closed subgroup. We also have

$$\mathrm{ad}(\mathrm{Lie}(Z(G)) = \mathrm{Im}(\mathrm{ad}).$$

Definition 5.27. *Let L be a Lie algebra over a field k.*

A finite dimensional representation of L is a pair (φ, V) consisting of

i). V is a finite dimensional vector space,

ii). $\varphi : L \to \mathfrak{gl}(V)$ is a homomorphism of Lie algebras.

The adjoint representation is the homomorphism

$$\mathrm{ad} : L \to \mathfrak{gl}(L), X \to [X, \cdot]$$

and $Z(L) = \Im(\mathrm{ad})$.

Definition 5.28. *Let G be a connected Lie group and k be a field. For the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and a finite dimensional representation (π, V) of $\mathfrak{g}$, a bilinear form bis said to be invariant with respect to π if for any $v, w \in$and for any $g \in G$ we have*

$$b(\pi(g)v, \pi(g)w) = b(v, w).$$

Definition 5.29. *Let L be a Lie algebra over a filed k and (φ, V) be a finite dimensional representation of L.*

A bilinear form $b : V \times V \to k$ is said to be invariant with respect to the representation if for all $x \in L$ and $v, w \in V$ we have

$$b(\varphi(x)v, w) = -b(v, \varphi(x)w).$$

In particular, it is invariant if it is invariant with respect to ad.

Remark 5.14. *$b : L \times L \to k$ is invariant if and only if*

$$b([X, Y], Z) = b(X, [Y, Z]).$$

Lemma 5.10. *Let G be a connected Lie group and $L = \mathrm{Lie}(G)$ be a Lie algebra. Let us consider the representation (π, V) of G over $\mathbb{R}$ and a invariant bilinear form $b : V \times V \to \mathbb{R}$ with respect to π. The b is invariant with respect ot $\varphi = d\pi(1)$.*

Proof. □

Lemma 5.11. *Let V be a finite dimensional k-vector space. We have that the bilinear form*

$$\mathrm{tr} : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to k, \quad (X, Y) \mapsto \mathrm{tr}(XY)$$

is a symmetric and invariant bilinear form.

Proof. □

Remark 5.15. *For map $\varphi : L \to \mathfrak{gl}(V)$ we can pull-back the trace form to get an invariant symmetric bilienar form on L.*

Definition 5.30. *Let L be a Lie algebra and k be a field. The killing form $\kappa : L \times L \to k$ is the pull-back of the trace form on the adjoint representation. In other words, for any $X, Y \in L$,*

$$\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

Corollary 5.17. *The killing form $\kappa : L \times L \to k$ is asymmetric bilinear form on L.*

Proof. □

5.13 Ideals of Lie Algebras and (Semi)Simplicity

Definition 5.31. *Let L be a Lie algebra. A vector subspace I of L is called an ideal if we have*

$$[I, L] \subseteq I.$$

Remark 5.16. *An ideal of a Lie algebra is a subalgebra but the converse, a subalgebra needs not be an ideal.*

Proposition 5.19. *Given a Lie algebra L and a representation (φ, V) of it , the following are ideals of L*

1. *The center $Z(L)$.*
2. $[L, L] = \mathrm{Span}([X, Y] \mid X, Y \in L)$.
3. $\mathrm{Ker}\, \varphi$.
4. *The left and right kernels of an invariant bilinear form $b : L \times L \to k$.*

Proof. □

Lemma 5.12. *Let L be a Lie algebra and I be its ideal. Then we can imbed the structure of Lie algebra on L/I with the bracket operator inherited from L.*

Proof. □

Lemma 5.13. *Let L be a Lie algebra and I be its ideal.*

Then the killing form on L/I is the restriction of the killing form of L. In particular, when $I = Z(L)$, the projection identifies the killing form.

Proof. □

Definition 5.32. *A Lie algebra L is said to be simple if L is not abelian and has non-trivial proper ideals.*

Definition 5.33. *A Lie algebra L is said to be semi-simple if it is a direct sum of simple ideals.*

Definition 5.34. *A Lie algebra L is said to be reductive if $L = Z(L) \oplus [L, L]$ and $[L, L]$ is semi-simple.*

Lemma 5.14. *Let L be a semi-simple Lie algebra. Then L is the direct sum of all of its simple ideals.*

Proof. □

Corollary 5.18. *If L is a semi-simple Lie algebra then we have the following statements.*

1. *Any ideal on quotient is semi-simple,*
2. $[L, L] = L$,
3. $Z(L) = 0$.

Proof. □

Proposition 5.20. *If the killing form of a Lie algebra is definite as a bilinear form, then the Lie algebra is semi-simple.*

Proof. □

Proposition 5.21. *If the killing form is definite then the Lie algebra is reductive.*

Definition 5.35. *Let k be a field and L be a Lie algebra over k.*

A derivative $d : L \to L$ on L is a linear map such that for any $X, Y \in L$ we have

$$d([X,Y]) = [d(X), Y] + [X, d(Y)].$$

We further define the set of all derivatives on L to be $\mathrm{Der}(L)$.

Lemma 5.15. *Such* $\mathrm{Der}(L)$ *is a subalgebra of the space* $\mathfrak{gl}$ *and contains the image of the adjoint representation.*

Proof. □

Lemma 5.16. *For any derivation d on L, we have*

$$d(Z(L)) \subseteq Z(L).$$

Proof. □

Definition 5.36. *Let L be a Lie algebra over $\mathbb{R}$. Then the group of automorphisms of L as a $\mathbb{R}$-vector space is denoted by* $\mathrm{GL}(L)$.

Lemma 5.17. *Let $M \subseteq L$ be a subalgebra of a Lie algebra. Then for any automorphisms $\varphi : M \to M$, there is an automorphism $\overline{\varphi} : L \to L$ which is an extension of φ.*

Proof. □

Lemma 5.18. *Let L be a Lie algebra and $\delta \in \mathrm{Der}(L)$. Then for the exponential map*

$$\exp : \mathfrak{gl}(L) \to \mathrm{GL}(L),$$

we have

$$\exp(\delta) \in \mathrm{GL}(L).$$

Proof. □

Lemma 5.19. *Let M be a subalgebra of a Lie algebra L and $\varphi \in \mathrm{Aut}(M)$. THen we hav*

$$d\varphi(1) \in \mathrm{Der}(L).$$

Proof. □

Lemma 5.20. *Let L be a Lie algebra. The natural action of* $\mathrm{Ad}(L)$ *on $Z(L)$ is trivial.*

Proof. Let $X \in \mathrm{Ad}(L)$ and $Z \in Z(L)$. The we have

$$\mathrm{ad}(X)(Z) = 0.$$

This shows that

$$\exp(\mathrm{ad}(X)) \cdot Z = Z.$$

$\square$

Lemma 5.21. *Let L be a Lie algebra and M be a subalgebra of L. Then $\mathrm{Aut}(M)$ is a closed subgroup of $\mathrm{GL}(L)$ with the following property.*

$$\mathrm{Aut}(M) \subseteq O(L) \cap \mathrm{Der}(L).$$

Proposition 5.22. *Let L be a Lie algebra over $\mathbb{R}$. Then $\mathrm{GL}(L)$ is a Lie group of automorphisms of L as a $\mathbb{R}$-vector space. Let L' be a subalgebra of L and denote $\mathrm{Aut}(L')$ to be the automorphisms of L'.*

$\mathrm{Aut}(L')$ is a closed subgroup of $\mathrm{GL}(L)$ and it lies in $O(k)$ with its Lie algebra $\mathrm{Der}(L)$.

Proof. $\square$

Lemma 5.22. *Let L be a Lie algebra. There is a natural homomorphism between $\mathrm{Aut}(L)$ and $\mathrm{Aut}(L/Z(L))$. Furthermore, if $L/Z(L)$ is semi-simple, the homomorphism is an isomorphism.*

Proof. $\square$

5.14 Compact Lie algebra

Definition 5.37. *A Lie algebra L over $\mathbb{R}$ is said to be compact if its killing form over $L/Z(L)$ is negative definite.*

Definition 5.38. *Let L be a Lie algebra. The radical of L is such that*

$$\mathrm{rad}(L) = \{X \in L \mid [X, \cdot] = 0\}.$$

Lemma 5.23. *Let L be a compact Lie algebra then*

$$Z(L) = \mathrm{rad}(L).$$

Lemma 5.24. *Let L be a compact Lie algebra. Then the following are equivalent.*

1. *$Z(L)$,*
2. *the killing form κ is non-degenerate (more negative definite),*
3. *L is semi-simple.*

Proof. □

Lemma 5.25. *Let L be a Lie algebra. For $X \in L$ and $\delta \in \mathrm{Der}(L)$, we have*

$$[X, \delta]_{\mathrm{Der}(L)} = -\delta(X).$$

Proof. □

Proposition 5.23. *Let L be a compact and semi-simple then we have the following statement.*

1. $\mathrm{ad} : L \to \mathrm{Der}(L)$ *is an isomorphism,*

2. $\mathrm{Ad}(L) = \mathrm{Aut}(L)^O$*, in particular, it is closed in* $\mathrm{Aut}(L)$*,*

3. *the adjoint action of* $\mathrm{Aut}(L)$ *on its Lie algebra* $\mathrm{Der}(L)$ *is translated under* 1*, which is the tautological action of* $\mathrm{Aut}(L)$ *on* L*,*

4. $\mathrm{Ad}(L)$ *has a trivial center.*

Proof. □

Proposition 5.24. *The following statements are equivalent for a Lie algebra L.*

1. $\mathrm{Ad}(L)$ *is a compact Lie group.*

2. *For $X \in L$, the curve $\gamma : \mathbb{R} \to \mathfrak{gl}(L), \gamma(t) = \exp(\mathrm{ad}(tX))$ is bounded.*

3. *For $X \in L$, $\mathrm{ad}(X) \in \mathfrak{gl}(L)$ is diagonalizable with purely imaginary eigenvalues.*

4. *L is a compact Lie algebra.*

Proof. □

Proposition 5.25. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra.*

If G is compact, then $\mathfrak{g}$ is compact.

Also if $\mathfrak{g}$ is compact and semi-simple, then G is compact.

Proof. □

5.15 Integration and Complete Reducibility

Definition 5.39. *A finite dimensional representation of a group of Lie algebra is called irreducible if it has no non-zero proper subspace. It is said to be completely irreducible if it is the direct sum of irreducibles, and indecomposable if it is not the direct sum of two proper non-zero invariant subgroups.*

Example 5.2. *Let us define a map* $\varphi : \mathbb{R} \to \mathrm{End}(\mathbb{R}^n)$,

$$\varphi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Theorem 5.9. *Let* G *be a compact topological group. Then there exists a unique linear functional*

$$\mathscr{C}(G, \mathbb{C}) \to \mathbb{C}, \quad f \mapsto \int_G f(g) dg$$

such that

1. *for any* $g \in G$, $f(g) > 0$ *then* $\int_G f dg > 0$,
2. *for each* $g \in G$, $\int_G f(gh) dh = \int_G f(g) dg = \int_G f(hg) dh$,
3. $\int_G 1_G dg = 1$.

Proof. □

Proposition 5.26. *Let* G *be a compact topological group and* H *be its closed subgroup. Then we have*

$$\int_G f(g) dg = \int_{G/H} \int_H f(gh) dh d(g+H).$$

Proof. □

Lemma 5.26. *Let* G *be a compact topological group and* (π, V) *be a finite dimensional representation of* G.

Then there exists a G*-invariant scalar product on* V.

Proof. □

Corollary 5.19. *Let* G *be a compact Lie group. There exists a* G*-invariant scalar product on* $\mathrm{Lie}(G)$.

Proof. □

Proposition 5.27. *Let* G *be a compact topological group and* (π, V) *be a finite dimensional representation of it. Then* G *is completely irreducible.*

Proof. □

5.16 Special Examples

Definition 5.40. *We have the following groups*

$$\mathrm{SL}_2(\mathbb{R}) = \{A \in \mathrm{GL}_2(\mathbb{R}) \mid \det(A) = 1\}, \quad \mathfrak{sl}_2(\mathbb{R}) = \{A \in R^{2\times 2} \mid \operatorname{tr} A = 0\}.$$

Lemma 5.27. *Let*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 01 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & -0 \end{pmatrix}.$$

Then $\{H, E, F\}$ forms a basis in $\mathfrak{sl}_2(\mathbb{R})$ with following property,

$$\kappa(E, F) = 4, \quad \kappa(E, E) = \kappa(F, F) = 0, \quad \kappa(H, H) = 8.$$

Furthermore $\mathfrak{sl}_2(\mathbb{R})$ is not compact if and only if κ is not negative definite.

Proof. □

Definition 5.41.

$$\mathrm{SU}_2(\mathbb{R}) = \{A \in \mathrm{SL}_2(\mathbb{C}) \mid v, w \in \mathbb{C}^2, \langle Av, Aw\rangle = \langle v, w\rangle\}$$

where

$$v = (v_1, v_2), w = (w_1, w_2), \langle v, w\rangle = v_1\overline{w_1} + v_2\overline{w_2}.$$

Lemma 5.28.

$$\mathrm{SU}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & -b \\ b & \overline{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\},$$

which is a connected and simply connected Lie group. And by immediate computation we see that

$$\mathrm{SU}_2(\mathbb{R}) \cong S^3 \subseteq \mathbb{R}^4.$$

Proof. □

Lemma 5.29.

$$\mathfrak{su}_2(\mathbb{R}) = \{A \in \mathbb{C}^{2\times 2} \mid A^* = -A, \operatorname{tr}(A) = 0\} = \left\{ \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \;\middle|\; a \in \mathbb{R}, b \in \mathbb{C} \right\}.$$

which is a vector space of dimension 3. We can take a basis

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, I = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

By calculating we derive

$$[I, J] = 2K, \quad [J, K] - 2J], \quad [I, K] = 2I.$$

And we also have for any $X \in \mathfrak{su}_2(\mathbb{R})$

$$\kappa(X, X) = -2(a^2 + b\bar{b}) < 0,$$

thus it is a negative definite. We conclude $\mathfrak{su}_2(\mathbb{R})$ is compact.

Proof. $\square$

Remark 5.17. *Let V be a finite dimensional $\mathbb{R}$-vector space. We denote*

$$V_{\mathbb{C}} = V \underset{\mathbb{R}}{\otimes} \mathbb{C},$$

which is a finite dimensional $\mathbb{C}$-vector space. In particular

$$\dim_{\mathbb{C}}(V_{\mathbb{C}}) = \dim_{\mathbb{R}}(V).$$

Let $W \subset V$ be a subspace. Then the map

$$W \mapsto W \underset{\mathbb{R}}{\otimes} \mathbb{C} = W_{\mathbb{C}}$$

defines a bijection between the subspaces of V and the subspaces of $V_{\mathbb{C}}$ which is σ-invariant where σ is the complex conjugation.

Definition 5.42. *Let L be a Lie algebra and $e, f, h \in L$ be elements such that*

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

We call such (e, f, h) a $\mathfrak{sl}_2$-triplet.

Definition 5.43. *Let L be a Lie algebra and $i, j, k \in L$ be elements such that*

$$[i, j] = 2k, \quad [i, h] = -2j, \quad [j, k] = 2i.$$

We call such (i, j, k) a SU_2-triplet.

Remark 5.18. *If (e, h, f) is a $\mathfrak{sl}_2$-triplet in L. We get a homomorphism, similarly for a SU_2-triplet (i, j, k).*

Example 5.3 (Representations of $\mathfrak{sl}_2(\mathbb{C})$)**.**

Proposition 5.28. *Every finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to a unique*

Lemma 5.30. *Let (π, V) be a representation of $\mathfrak{sl}_2(\mathbb{C})$ and let $\rho = d\pi(1)$. Then the following are equivalent for a subspace W of V.*

1. *W is ρ-invariant.*

2. *W is π-invariant.*

Theorem 5.10. *Every finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ id completely reducible.*

Definition 5.44. *Let V be a finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. An eigenvalue of h of the triplet is called a weight. The dimension of the eigenspace corresponding to the eigenvalue is called the multiplicity of the weight.*

Lemma 5.31. *Let (π, V) be a finite representation of $\mathfrak{sl}_2(\mathbb{C})$. Then we have the following statements*

1. *The weights of V are integers.*
2. *The isomorphic classes of V is determined by the function $m : \mathbb{Z} \to \mathbb{N}$ which sends a weight to its multiplicity.*
3. *$m(1)$ is equal to the number of irreducible constant of V with odd weights.*
4. *$m(0)$ is equal to the number of irreducible constant of V with even weights.*

Definition 5.45. *Let L be a semi-simple Lie algebra over $\mathbb{C}$.*

Definition 5.46. *Let L be a Lie algebra. $X \in L$ is said to be* ad*-semi-simple if $\mathrm{ad}(X) \in \mathrm{End}(L)$ is diagonalizable.*

Definition 5.47. *Let L bea. Lie algebra. $T \subseteq L$ is called a Cartan subalgebra if T is maximal abelian subalgebra consisting of* ad*-semi-simple elements.*

Lemma 5.32. *Let $\mathfrak{g}$ be a real compact semi-simple Lie algebra and $L = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Then the map*

$$\varphi : \mathfrak{g} \to L, \quad \varphi(t) = t \underset{\mathbb{R}}{\otimes} \mathbb{C} = T$$

is a bijection between

$$\{\textit{Maximal abelian subalgebras } \mathfrak{t} \subseteq \mathfrak{g}\} \leftrightarrow \{\textit{ Complex conjugate invariant maximal subalgebra } T \subseteq L\}.$$

Every such T consists of ad*-semi-simple elements.*

Definition 5.48. *Let L be a Lie algebra. For any subspace V of L, we define*

1. *the centralizer of V is $Z_L(V) = \{X \in L \mid [X, V] = 0\}$,*
2. *the normalizer of V is $N_L(V) = \{X \in L \mid [X, L] \subseteq V\}$*

Lemma 5.33. *Let L be a Lie algebra and $T \subseteq L$ be a Cartan subalgebra. Then we have*

$$Z_L(T) = T.$$

Definition 5.49. *Let L be a Lie algebra and T be a Cartan subalgebra. For any $\alpha \in T^* = \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$, we define*

$$L_\alpha = \{X \in L \mid \forall t \in T, \mathrm{ad}(t)(X) = d(t) \cdot X\}.$$

Theorem 5.11. *Let L be a Lie algebra and T be a Cartan subalgebra. Then we have*

$$L = \bigoplus_{\alpha \in T^*} L_\alpha.$$

Remark 5.19.

$$L_0 = Z_L(T) = T.$$

Definition 5.50. *Let L be a Lie algebra and T be its Cartan subalgebra. We define*
$$\Phi(T, L) = \{\alpha \in T^* \mid \alpha \neq 0, L_\alpha \neq 0\}.$$

Lemma 5.34. *Let L be a Lie algebra and T be its Cartan subalgebra. We have $\Phi(T, L)$ is finite and generates T^*.*

Remark 5.20. *By definition $\Phi(T, L) \subseteq T*$ is finite and*

$$L = T \oplus \bigoplus_{\alpha \in \Phi(T,L)} L_\alpha.$$

Lemma 5.35. *Let L be a Lie algebra and T be its Cartan subalgebra. For $\alpha, \beta \in T^*$, we have*

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$,
2. *if $X \in L_\alpha, \alpha \neq 0$, then* $\mathrm{ad}(X)$ *is nilpotent,*
3. *if $\alpha = -\beta$ then $L_\alpha \perp_\kappa L_\beta$.*

Corollary 5.20. *Let L be a Lie algebra and T be its Cartan subalgebra. We have*
$$\alpha \in \Phi(T, L) \Rightarrow -\alpha \in \Phi(T, L).$$
Furthermore, we have

$$L = T \oplus \bigoplus_{\pm\alpha \in \Phi(T,L)/I} (L_\alpha \oplus L_{-\alpha}).$$

Corollary 5.21. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{t} \subseteq \mathfrak{g}$, we set*

$$L = \mathfrak{g} \underset{\mathbb{R}}{\otimes} \mathbb{C}, \quad T = \mathfrak{t} \underset{\mathbb{R}}{\otimes} \mathbb{C}.$$

The restriction of κ to each summand in

$$L = T \oplus \bigoplus_{\pm\alpha \in \Phi(T,L)/I} (L_\alpha \oplus L_{-\alpha}).$$

is Lie algebra

And $\kappa : L_\alpha \to \mathbb{C}$ is a perfect pairing.

Lemma 5.36. *Let L be a Lie algebra and T be its Cartan subalgebra. For $X \in L_\alpha$ and $Y \in L_{-\alpha}$ we have*

$$[X, Y] = \kappa(X, Y) \cdot t_\alpha.$$

Remark 5.21. *Let σ denotes the complex conjugation. Then σ acts on a Lie algebra L over $\mathbb{C}$ which preserves a Cartan subalgebra T. In deed we have*

$$\alpha \in \Phi(T, L), \quad (\sigma\alpha)(t) = \sigma(\alpha(\sigma^{-1}(t)).$$

Lemma 5.37. *Let L be a Lie algebra, and T be its Cartan subalgebra, and σ be a complex conjugate. We have*

1. *For any $\alpha \in \Phi(T, L)$, $\sigma\alpha = -\alpha$.*
2. *$(i \cdot t_\alpha)\alpha \in \Phi(T, L)$ generates T^σ, and $(i \cdot \alpha)_\alpha$ generates $T^{*,\sigma} = T^*$.*
3. *$\kappa(t_\alpha, t_\alpha) > 0$.*
4. *$h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha}$ is well-defined, $\kappa(h_\alpha, h_\alpha) > 0$, $\alpha(h_\alpha) = 2$.*
5. *For any $\alpha \in \Phi(T, L)$ there is some $e \in L_\alpha$ such that $[e, \sigma(e)] = -h_\alpha$.*
6. *For such e and $f = -\sigma(e), h = h_\alpha$ is a $\mathfrak{sl}_2$-triplet.*

Remark 5.22. *There is not $e \in L_\alpha$ such that*

$$[e, \sigma(e)] = h_\alpha.$$

Proposition 5.29. *Let L be a Lie algebra and T be its Cartan subalgebra. For $\alpha, \beta \in \Phi(T, L)$ we have*

1. $\dim L_\alpha = 1$,
2. *If $c \in \mathbb{C}$ is such that $c\alpha \in \Phi(T, L)$ then $c = \pm 1$.*
3. *If $\alpha + \beta \in \Phi(T, L)$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.*
4. *$\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi(T, L)$.*
5. *Assume $\alpha \neq \pm\beta$, let $r, q \in \mathbb{N}$ be the largest such that $\beta - r\alpha, \beta + q\alpha \in \Phi(T, L)$, then $\beta + n\alpha \in \Phi(T, L) \Leftrightarrow n \in [-r, q] \cap \mathbb{Z}$ and $r - q = \beta(h_\alpha)$.*