Representation Theory 1 V4A3 Sheett 6

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Exercise 2

We have the homeormophism

$$\mathfrak{su}_2(\mathbb{R} \underset{\mathbb{R}}{\otimes} \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C}).$$

And the fact that for any vector space V and its subspace $W \subseteq V$,

$$W \text{ is } \mathfrak{sl}_2(\mathbb{C}) \text{ invariant.} \Leftrightarrow W \text{ is } \mathfrak{su}_2(\mathbb{R}) \text{ invariant.}$$

Since $\mathfrak{su}_2(\mathbb{R})$ is connected and simply connected, for any representation (ρ, V) there is a representation (π, V) of $\mathrm{SU}_2(\mathbb{R})$ such that

$$d\pi(1) = \rho.$$

For any $U \in \mathrm{SU}_2(\mathbb{R})$, there is $X \in \mathfrak{su}_2(\mathbb{R})$ and

$$\pi(U) = \exp(\rho(X)).$$

From the elementary linear algebra, we have U is diagonalizable. This is preserved under a group homomorphism thus $\pi(U)$ is diagonalizable. Again from the elementary linear algebra, we have

$$\exp(\rho(X)) \text{ is diagonalizable.} \Leftrightarrow \rho(x) \text{ is diagonalizable.}$$

By the previous argumentt $\rho(h)$ is diagonalizable. Let V_1 be an eigenspace of eigenvalues of λ_1 of e.

Since ρ preserves the bracket we have

$$[\rho(h), \rho(e)]V_1 = 2\rho(e)V_1 \Rightarrow \rho(h)\rho(e)V_1 = (2 + \lambda_1)\rho(e)V_1,$$
$$[\rho(h), \rho(f)]V_1 = -2\rho(f)V_1 \Rightarrow \rho(h)\rho(f)V_1 = (-2 + \lambda_1)\rho(f)V_1.$$

Solving the system of linear equation, we conclude that it is impossible unless $\rho(e)V_1, \rho(f)V_1$ are $\{o\}$.

Let us consider an action of $\mathrm{SU}_2(\mathbb{C})$ on $\mathbb{C}[x, y]$, which is by the base change such that for any $f \in \mathbb{C}[v_1, v_2]$ and $P \in \mathrm{SU}_2(\mathbb{C})$ we have

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad f(v_1, v_2)P = f(av_1 + cv_2, bv_1 + dv_2).$$

Denote V_n as the set of homogeneous polynomials of degree n. Then we claim this is an invariant subspace of the action. This is because such action is homogeneous linear transformation.

We learned that the basis of $V = \mathrm{Sym}^n(\mathbb{C}^2)$ is

$$\{v_1^i v_2^{n-i}\}_{i=0,\cdots,n}.$$

However, for each $v_1^i v_2^{n-i}$ we have

$$hv_1^i v_2^{n-i} = (n - 2i)v_1^i v_2^{n-i}.$$

Thus the eigenspace of $\rho(h)$ is the whole space, and thus $\rho(e)V, \rho(f)V = \{o\}$. Thus V is a irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$.

On the other hands, if V is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, then $\rho(h)$ is a multiple of an identity and $\rho(e), \rho(f) = O$. Thus we conclude the statement holds.