Representation Theory 1 V4A3 Sheet 3 Exercise 1

So Murata

2024/2025 Winter Semester - Uni Bonn

1.

By the property of the exponential map we have

$$\underline{\mathrm{Ad}}(g) \circ \exp_H = \exp_H \circ \mathrm{Ad}(g).$$

We have $\exp_H(\mathfrak{h})$ generates H. Thus $\underline{\mathrm{Ad}}(g) \circ \exp_H(\mathfrak{h})$ generates gHg^{-1}. With the equation above, we conclude the statement.

2.

Let $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. Then $\exp_G(X) \in G$ and $\exp_G(Y) \in H$. By using properties of exp,

$$\begin{aligned}\underline{\mathrm{Ad}}(\exp_G X) \exp_G Y &= \exp_G \mathrm{Ad}(\exp_G X) Y,\\ &= \exp_G \exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(X)(Y),\\ &= \exp_G(X) \exp_G(Y) \exp_G(X)^{-1},\\ &= \exp_G(X) \exp_G(Y) \exp_G(-X).\end{aligned}$$

We let

$$Y' = \exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(X)(Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{ad}(X)^n Y$$

If $\mathfrak{h}$ is an ideal, then $Y' \in \mathfrak{h}$. Since $\exp_G(\mathfrak{g})$ and $\exp_G(\mathfrak{h})$ generate G and H, respectively. We conclude H for $\mathfrak{h} = \mathrm{Lie}(H)$ is normal.

On the other hand, we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(tX)(Y) &= \exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(tX)(Y) - exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(0)(Y) \mod o(t)\\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{ad}(tX)^n Y - 1 \mod o(t)\\ &= \mathrm{ad}(X)(Y).\end{aligned}$$

Since H is normal for any

$$\exp_G \exp_{\mathrm{GL}(\mathfrak{g})} \mathrm{ad}(tX)(Y) = \exp_G(tX)\exp_G(Y)\exp_G(-tX) \in H.$$

for any t thus, $\mathrm{ad}(X)(Y) \in \mathfrak{h}$.

Exercise 3

Since H is 1-dimensional, each chart has its terminal set a 1-dimensional vector space. We know that we can have an atlas such that

$$(\kappa_x(Y) = x\exp_G(Y))_{x\in H}.$$

where $Y \in U$ some open neighborhood of 0. We conclude that U is 1-dimensional thus generated by a single element X.

Now we prove that if H is closed, one of three properties will hold. Since $X \in \mathfrak{gl}(V)$, we can use the explicit form of the exponential map. X is similar to a matrix where diagonals are e^λ for an eigenvalue λ of X. Suppose X is diagonalizable in $\mathbb{C}$. If there is $t \neq 0$ such that $\exp_G(tX) = 1$ then for any imaginary eigenvalue b_1, b_2 we have

$$tb_j = 2k_j\pi i \quad (j = 1, 2).$$

Thus $\frac{b_1}{b_2}$ is a rational number. If such t doesn't exist then there exists an eigenvalue which is not purely imaginary.