V4A9 Homework 2

So Murata

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Exercise (1) Let $v \in W^\perp$, then for any $w \in W$ and $g \in H$,,

$$(\pi(g)v, w) = (v, \pi(g^{-1})w) = 0,$$

since W is a subrepresentation. Thus $W^\perp$ is a subrepresentation.

Consider a compact open subgroup K and V^K. Since this is admissible, we know V^K is finite dimensional and W^K and $W^{\perp, K}$ are subspaces. We have

$$V^K = W^K \oplus (W^K)^\perp \cap V^K.$$

and clearly,

$$(W^K)^\perp \cap V^K \supseteq (W^\perp)^K.$$

Let $v \in (W^K)^\perp \cap V^K$, then there exists $v' \in (W^K)^\perp$ such that

$$e_K v' = v.$$

Take an arbitrary element $w \in W$, since a positive definite bilinear form restricted to some subspce is still positive definite, we observe,

$$\langle v, w \rangle = \langle e_K v', w \rangle,$$
$$= \left\langle \frac{1}{|K|} \int_K \pi(g) v' dg, w \right\rangle,$$
$$= \langle v', e_K w \rangle,$$
$$= 0.$$

The last part follows from that $e_K w \in W^K$. Here I used the H-invariance and the property of Haar measure, together with K being compact. Thus we conclude $v \in (W^\perp)^K$.

We now have $V^K = W^K \oplus (W^\perp)^K$. We conclude $V = W \oplus W^\perp$.

Exercise (2) Recall that V_1, V_2 are irreducible, so is $V_1 \otimes V_2$. $V_1 \otimes V_2$ consists of basis of the form

$$v_1 \otimes v_2$$

where $v_1 \in V_1, v_2 \in V_2$. A subspace fixed by a compact subgroup $K_1 \times K_2$ is, for $k_1 \in K_1, k_2 \in K_2$,

$$(\pi_1 \otimes \pi_2)(k_1, k_2) v_1 \otimes v_2 = (\pi_1(k_1)v_1) \otimes (\pi_2(k_2)v_2).$$

Therefore, we have,

$$(V_1 \otimes V_2)^{K_1 \times K_2} = V_1^{K_1} \otimes V_2^{K_2}.$$

In particular, if V_1, V_2 are admissible so is $V_1 \otimes V_2$.

Admissible and irreducible representations have their contragredient representations again admissible and irreducible. To see this observe, let W be a subrepresentation of $\tilde{V}$ then we have

$$0 \longrightarrow W \longrightarrow \tilde{V} \longrightarrow \tilde{V}/W \longrightarrow 0$$

as vectorspaces. Using V is admissible, and taking contragredient is a contravariant exact functor, we have

$$0 \longrightarrow \widetilde{\tilde{V}/W} \longrightarrow \tilde{\tilde{V}} \longrightarrow \tilde{W} \longrightarrow 0$$

Since V is irreducible, one of the arrows has to be isomorphism.

For our case, $\widetilde{V_1 \otimes V_2}$ is admissible and irreducible since $V_1 \otimes V_2$ is. $\tilde{V}_1, \tilde{V}_2$ are admissible and irreducible so is $\tilde{V}_1 \otimes \tilde{V}_2$.

The map

$$\lambda_1 \otimes \lambda_2 \mapsto [v_1 \otimes v_2 \mapsto \lambda_1(v_1)\lambda_2(v_2)]$$

is a morphism of H-representations which is not 0. By Schur's lemma, we have they are isomorphic.

(3)

(a) Since for the trace function in $\mathrm{Mat}_{n \times n}(\mathbb{C})$, we have for any $A, B \in \mathrm{Mat}_{n \times n}(\mathbb{C})$, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. From this and $\pi(g) \in \mathrm{GL}(V)$ where V is finite dimensional, we have

$$\mathrm{tr}(\pi(g^{-1})\pi(h)\pi(g)) = \mathrm{tr}(\pi(h)\pi(g)\pi(g^{-1})) = \mathrm{tr}(\pi(h)).$$

(b) Without loss of generality, we assume V has a basis $\{b_1, \cdots, b_n\}$. Since V is finite dimensional we have $V = \bigoplus_{i=1}^{k} V_i$ for some irreducible subrepresentations. On each subrepresentation, since it is irreducible and finite, there exists a Hermitian matrix A_i which corresponds to a positive-definite, H-invariant Hermitian form. Let us define a positive definite Hermitian matrix A to be such that

$$A := \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_k \end{pmatrix}.$$

Thus we conclude (π, V) is unitary. In the proof for the existence of the formal degree of irreducible and finite representations, the irreducibility was used to

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prove the admissibility and the unitariness of the contragredient representations besides that for a bilinear operator $f : V \times \tilde{V} \to \mathbb{C}$, there is c such that

$$f(v, \lambda) = c\lambda(v).$$

However, V is of finite dimension, thus $\tilde{V}$ is also finite dimensional. Thus its contragredient is admissible. Since we have

$$V = \bigoplus_{i=1}^{k} V_i. \tag{1}$$

For any $v \in V$, and $\lambda \in V^*$, we get

$$v = v_1 \oplus \cdots \oplus v_k, \quad \lambda = \lambda_1 \oplus \cdots \oplus \lambda_k.$$

Therefore,

$$m_{\lambda,v}(g) = \sum_{i=1}^{k} \lambda_i(\pi(g)v_i).$$

By using Schur's orthogonality, we get

$$\int_H m_{\lambda,v}(g) m_{\mu,u}(g^{-1}) dg = \sum_{i=1}^{k} \frac{1}{\deg(V_i)} \lambda_i(u_i)\mu_i(v_i).$$

Thus (π, V) admits a formal degree.

Furthermore, $V^* = \tilde{V}$ since there exists a compact open subgroup set K such that $V = V^K$. This follows from that each basis element is contained in some V^{K_i}. By taking the intersection of K_i and using H is Hausdorff, we conclude $K = \bigcap_i K_i$ is again a compact and open subgroup which fixes all the basis elements, so does any elements. We see $V^* = (V^*)^K$ for such K.

Take $\lambda \in V^*$ to be such that it maps each basis element to 1. Thus we have

$$\deg(V) \int_H m_{\lambda,\sum_{i=1}^n b_i} m_{\lambda,b_1} dg = \lambda(b_1 + \cdots + b_n)\lambda(b_1) \dim(V).$$

Thus scaling dg by $\int_H m_{\lambda,\sum_{i=1}^n b_i} m_{\lambda,b_1}$, we obtain the statement.
(4) Let $f_1, f_2 \in \mathcal{H}$, then

$$\tau(f_1 + f_2)w = \tau(f_1)w + \tau(f_2)w,$$

by the linearity of integrals. Now consider $f_1 * f_2$, indeed, we have

$$\int_H (f_1 * f_2)(h)\tau(h)w dh = \int_H \int_H f_1(g) f_2(g^{-1}h)\tau(h)w dg dh.$$

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Substitute $h \to gh$ and use the property of Haar measure, we have,

$$\int_H (f_1 * f_2)(h)\tau(h)wdh = \int_H \int_H f_1(g)f_2(h)\tau(gh)wdgdh,$$
$$= \int_H f_2(h)\tau(h) \int_H f_1(g)\tau(g)wdgdh,$$
$$= \tau(f_2)\tau(f_1)w.$$

For the $H \times H$ equivariance, we have using the property of Haar measure again,

$$\tau((h_1, h_2)f(g))w = \tau(f(h_1^{-1}gh_2))wdg,$$
$$= \int_H f(h_1^{-1}gh_2)\tau(g)wdg,$$
$$= \int_H f(g)\tau(h_1gh_2^{-1})wdg,$$
$$= \tau(h_1)\tau(f)\tau(h_2^{-1})w.$$