V4A9 Homework 10

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1. Clearly if all $\phi_i \in Z(\mathscr{H}(G, G^\circ, \pi^\circ))$ then $\phi \in Z(\mathscr{H}(G, G^\circ, \pi^\circ))$. Let $\{\phi_1, \cdots, \phi_n\}$ be elements of $\mathscr{H}(G, G^\circ, \pi^\circ)$ such that supported in $x_i G^\circ$ and

$$\{\phi_1^{e_1} * \cdots * \phi_n^{e_n} \mid e_1, \cdots, e_n \in \mathbb{Z}\}$$

forms the basis of the Hecke algebra. By assumption, we have,

$$\phi * \phi_k = \phi_k * \phi,$$

for all $k = 1, \cdots, n$. But we have,

$$\phi * \phi_k = \sum_{i=1}^{r} \phi_i * \phi_k = \sum_{i=1}^{r} c_{ik} \phi_k * \phi_i = \sum_{i=1}^{r} \phi_k * \phi_i.$$

As they are basis we have $c_{ik} = 1$. Thus the statement is proven.

2.

The map

$$[tG^\circ] \mapsto \varphi_{[t^{-1}]},$$

respects the addition furthermore for $x \in stG^\circ$, $s^{-1}x \in tG^\circ$ therefore,

$$[stG^\circ] \mapsto \varphi_{[t^{-1}s^{-1}]} = [x \mapsto \sigma_T(s)\sigma_T(s^{-1}x)] = \varphi_{[s^{-1}]} * \varphi_{[t^{-1}]}.$$

Therefore this is an algebra homomorphism. The algebra $\mathbb{C}[T/G^\circ]$ is well-defined by Lemma 95 from the lecture. Remains to show that $\{\varphi_{[t^{-1}]}\}_{t \in T/G^\circ}$. From Lemma 70 and and the construction in the proof, $\{\phi_{[t^{-1}]}\}_{t \in T/G^\circ}$ is a generator of the algebra. Thus we have a one-to-one correspondence between generators, this induces an isomorphism.

3.

(a) First we have $(gk)K(gk)^{-1} = gKg^{-1}$. Let $\phi \in \mathrm{Hom}_{K \cap gKg^{-1}}({}^g\rho, \rho)$ then set

$$\phi' := \phi \circ \rho(k).$$

Then we have,

$$\begin{aligned}\phi'({}^{gk}\rho(x)(w)) &= \phi({}^g\rho(x)\rho(k)(w)),\\ &= \rho(x)\phi(\rho(k)(w)),\\ &= \rho(x)\phi'(w).\end{aligned}$$

Similarly, for $\phi \in \mathrm{Hom}_{K\cap gKg^{-1}}({}^g\rho, \rho)$, set

$$\phi' := \rho(k) \circ \phi.$$

Then we have,

$$\begin{aligned}
\phi'(\rho^{kg}(h)(w)) &= \rho(k) \circ \phi(\rho^g(k^{-1}hk)(w)), \\
&= \rho(k)\rho(k^{-1}hk) \circ \phi(w), \\
&= \rho(h)\phi'(w).
\end{aligned}$$

(b)

Remark 0.1. *Two characters intertwines if and only if they are the same.*

Suppose g intertwines ψ if and only if there is $f : \mathbb{C} \to \mathbb{C}$ linear such that

$$f(\psi(g^{-1}xg)(v)) = \psi(x)f(v). \Leftrightarrow \psi(g^{-1}xg) = \psi(x)$$

holds on $x \in ZI_1 \cap gZI_1g^{-1}$. From previous part, we would like to look at the representatives of $I_1\backslash \mathrm{SL}(F)/I_1$. Note that we have,

$$\mathrm{SL}_2(F) = \bigsqcup_{w\in N_G(T)/T\cap I} IwI$$

where

$$\begin{aligned}
T &= \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| a \in F^\times \right\} \\
, I \cap T &= \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| a \in \mathcal{O}_F^\times \right\}, \\
N_G(T) &= \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \begin{pmatrix} & a \\ -a^{-1} & \end{pmatrix} \middle| a \in F^\times \right\}, \\
\tilde{W} &= \left\{ \begin{pmatrix} \varpi^k & \\ & \varpi^{-k} \end{pmatrix}, \begin{pmatrix} & \varpi^k \\ -\varpi^{-k} & \end{pmatrix} \middle| k \in \mathbb{Z} \right\}.
\end{aligned}$$

As I is a preimage of $B(k_F)$, we have

$$\mathrm{SL}_2(\mathcal{O}_F) = K \xrightarrow{r} \mathrm{SL}_2(k_F),$$

We then have,

$$\begin{aligned}
I &:= r^{-1}(B(k_F)), \\
I_1 &:= r^{-1}(U(k_F)), \\
I/I_1 &\cong B(k_F)/U(k_F) = T(k_F). \\
T(\mathcal{O}_F) &\to T(k_F).
\end{aligned}$$

Note that we have,

$$I = \bigcup_{t,\text{diagonal}} I_1 t I_1, IwI = \bigcup_t I_1 wt I_1.$$

Thus we need to check

$$g = \begin{pmatrix} m & \\ & m^{-1} \end{pmatrix}, \begin{pmatrix} & m \\ -m^{-1} & \end{pmatrix}$$

We only show the case for the latter. Consider $x \in g^{-1} Z I_1 g$, such that

$$g^{-1} x g = \begin{pmatrix} 1 & 0 \\ -m^2 b & 1 \end{pmatrix} \in Z I_1.$$

This means that $\mathrm{val}(m^{-2} b) = 1$, thus $\mathrm{val}(b) > 2\,\mathrm{val}(m) + 1$. We have,

$$\varpi(-\varpi^{-1} m^{-2} b).$$

Therefore,

$$\psi(g^{-1} x g) = \phi(x) \Leftrightarrow \forall b \in \mathcal{O}_F, \mathrm{val}(m^{-2} b) \geq 1 \Leftrightarrow \mathrm{val}(\varpi^{-1} m^{-2} b) \geq 0 \psi(\overline{b}) = \phi(\overline{-\varpi^{-1} m^{-2} b})$$

Suppose $\mathrm{val}(m) = n$ and $2n + 1 > 0$. Thus the valuation of b is positive thus $\psi(\overline{b}) = 1$. Take b_0 be such that $\psi(\overline{b}_0) \neq 0$, we have $b = -\varpi m^2 b_0$ is a counter example.

If $2n + 1 < 0$, then $b \in \mathcal{O}_F$. Therefore,

$$\psi(\overline{\varpi^{-1} m^2 b}) = 1.$$

Take b_0 be such that $\psi(b_0) \neq 1$, we will have a contradiction.

(c) From Homework 6 (b), if K is an ope group compact modulo center of G and ψ is irreducible we have, $\text{c-ind}_K^G \psi$ is irreducible implies it is supercuspidal. The strategy is to use the Schur's lemma

$$\mathrm{End}_G(\text{c-ind}_{ZI_1}^G(\psi)) = \mathbb{C}.$$

The above isomorphism holds if and only if ψ is irreducible under the assumption it is unitary. From previous arguments, we have,

$$\mathrm{Hom}_G(\text{c-ind}_{ZI_1}^G(\psi), \mathrm{Ind}_{ZI_1}^G(\psi)) \cong \prod_{ZI_1 \backslash G / ZI_1} [\mathrm{Hom}_{ZI_1 \cap g^{-1}(ZI_1) g}({}^{g^{-1}}\psi, \psi)]$$

From previous argument, we have

$$\dim \mathrm{Hom}_{ZI_1 \cap g^{-1}(ZI_1) g}({}^{g^{-1}}\psi, \psi) = \begin{cases} 1, & (g \in ZI_1), \\ 0, & (\text{otherwise}). \end{cases}$$

Since $\mathrm{End}_G(\text{c-ind}_{ZI_1}^G(\psi))$ can be embedded to $\mathrm{Hom}_G(\text{c-ind}_{ZI_1}^G(\psi), \mathrm{Ind}_{ZI_1}^G(\psi))$, and the latter is one dimensional the former has to be one dimensional.

Remark 0.2. *Let π be an irreducible representation of G and $(\pi^\circ, W) \subseteq (\pi|_{G^\circ}, V)$. Take,*

$$T := \bigcap_{\chi \in X_{\mathrm{nr}}(G)_\pi} \ker(\chi)$$

Then we have,

$$\bigcap_{\substack{\chi \in X_{\mathrm{nr}}(G) \\ \chi|_H = 1}} \ker \chi = H = N_G(W) \subseteq N_G(\pi^\circ).$$

$T \subseteq H$. The key fact is that $\pi = \text{c-ind}_H^G(\pi^\circ)$ and $\nu \in X_{\mathrm{nr}}(G)$, then,

$$\pi \cdot \nu = \text{c-ind}_H^G(\pi^\circ)\nu = \text{c-ind}_H^G(\pi^\circ \nu|_H).$$

If $\nu|_H = 1$, then $\nu \in X_{\mathrm{nr}}(G)_\pi$.

4. Suppose g intertwines ρ, then pick

$$\phi \in \mathrm{Hom}_{K \cap gKg^{-1}}({}^g\rho, \rho).$$

Set,

$$f(h) = \begin{cases} \rho(k_1) \circ \phi\rho(k_2) & (h = k_1 g k_2), \\ 0 \quad \text{(otherwise)}. \end{cases}$$

This is well-defined and has a compact support by the equivalent condition about intertwining. For the other direction, suppose $f(g) = \phi \neq 0$ then for $x \in K \cap gKg^{-1}$,

$$\begin{aligned} \rho(x^{-1})f(g)\rho(g^{-1}xg) &= f(x^{-1}gg^{-1}xg), \\ &= f(g). \\ \Rightarrow \rho(x)\phi &= \phi({}^g\rho(x)(w)). \end{aligned}$$

Thus g intertwines ρ.