V4A9 Homework 3

So Murata, Heijing Shi

WiSe 25/26, University of Bonn Number Theory 1

(1)

(a) Observe that for any $n \in N$, we have,

$$v = \nu(n)^{-1}\pi(n)v - \nu(n)^{-1}(\pi(n)v - \nu(n)v).$$

For $\Leftarrow$, we see that

$$v = \frac{1}{|N_K|}\int_{N_K} v d\mu_N = \underbrace{\frac{1}{|N_K|}\int_{N_K} \nu(n)^{-1}\pi(n)v d\mu_N}_{=0} - \frac{1}{|N_K|}\int_{N_K} \nu(n)^{-1}(\pi(n)v - \nu(n)v)d\mu_N.$$

Thus $v \in V_\nu$.

(b) The functor preserves injectivity since for any $\varphi : (\tau, U) \to (\pi, V)$ a morphism of N-representation commutes with τ, π by definition and ν as it is $\mathbb{C}$-linear. Since taking a quotient vector space is right-exact, this completes the proof.

(c)

Since $(\cdot)_\nu$ is exact, the injectivity is clear. In $V(N)_\nu$, each representative is of the form,

$$\nu(n)v - v$$

for some $n \in N$.

We also have,

$$\begin{tikzcd}
0 \arrow[r] & V(N) \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & V_N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V(N)_\nu \arrow[r] & V_\nu \arrow[r] & (V_N)_\nu \arrow[r] & 0
\end{tikzcd}$$

as $(\cdot)_\nu$ is exact. Note that N acts trivially on V_N, thus elements of $(V_N)_\nu$ is of the form,

$$v - \nu(n)v,$$

which is in $V(N)_\nu$. Thus we conclude $V(N)_\nu \to V_\nu$ is an isomorphism.

(d)

By considering a trivial character ν, we obtain,

$$V(N) = V(\nu).$$

From **(a)**, we have,

$$V(N) = V(\nu) = \{v \in V \mid \exists K_N \text{ open compact, subgroup}, e_{K_N} v = 0\}.$$

(2)
(a) Let $v \in V \backslash \{0\}$, consider

$$W = \mathrm{Span}_{\mathbb{C}}\, \pi(g)v \mid g \in N.$$

Then this is irreducible representation. Thus the subrepresentation (π, W) has a central character ν. This extends to (π, V). Then the character $-\nu$, $v \notin V(-\nu)$.

(b) From the first part and considering the exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(N) & \longrightarrow & V & \longrightarrow & V_N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V(N)_\nu & \longrightarrow & V_\nu & \longrightarrow & (V_N)_\nu & \longrightarrow & 0 \end{array}$$

we have,

$$V(N) \simeq V.$$

Note that $N = \mathbb{G}_a$ thus $N = Z(N)$. From this, observe that V has a central character, namely the trivial representation from the exercise 1. But $V_\nu = 0$ tells that V also has a central character which is ν. But $\nu \neq \text{triv}$, thus we conclude $V = \{0\}$.