

V4A9 Homework 3

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(1)

Consider $F = \mathbb{Q}_p$ and $G = \mathrm{SL}_2(F)$, we have,

$$K = \mathrm{SL}_2(\mathbb{Z}_p),$$

and in this case the parabolic subgroup is

$$P = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \ltimes \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

Consider

$$K \cap M = \left\{ \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix} \right\}, K \cap N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\} K \cap \overline{N} = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right\}.$$

However,

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} u + xyu^{-1} & xu^{-1} \\ u^{-1}y & u^{-1} \end{pmatrix}$$

But observe that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in K.$$

But this is not of the form above.

(2)

$$K_n = 1 + \varpi^n \mathrm{Mat}_{2 \times 2}(\mathcal{O}_F).$$

Take $G = \mathrm{GL}_2$ and we have ,

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \overline{N} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

Then we have for $p_{ij} \in \varpi^n \mathcal{O}_F$,

$$\begin{pmatrix} 1 + p_{11} & p_{12} \\ p_{21} & 1 + p_{22} \end{pmatrix}$$

We have,

$$K_r \cap N_0 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathfrak{p}^r \right\}, K_r \cap \overline{N}_0 = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| x \in \mathfrak{p}^r \right\},$$

and also,

$$K_r \cap M_0 = \left\{ \begin{pmatrix} 1+t_1 & 0 \\ 0 & 1+t_2 \end{pmatrix} \middle| t_1, t_2 \in \mathfrak{p}^r \right\}.$$

Thus we have,

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+t_1 & 0 \\ 0 & 1+t_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1+t_1+xy+xyt_2 & x+xt_2 \\ y+yt_2 & 1+t_2 \end{pmatrix}$$

Thus this gives an isomorphism by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \mapsto \begin{pmatrix} 1 & \frac{p_{12}}{1+p_{22}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+p_{11}-\frac{p_{21}}{1+p_{22}}p_{12} & 0 \\ 0 & 1+p_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{p_{21}}{1+p_{22}} & 1 \end{pmatrix}$$

Exercise 3 Take E/F to be a non-trivial extension. Take

$$\mathbb{G} = \text{Res}_{E/F} \mathbb{G}_{m,E} : R/F \mapsto (R \otimes_F E)^\times.$$

Then we have,

$$\mathbb{G} \otimes_F E = G_{m,E}^{[E:F]},$$

if \mathbb{G} is split then

$$\mathbb{G} = \mathbb{G}_{m,F}^n$$

but we have,

$$\mathbb{G}(F) = E^\times \neq (F^\times)^n = (\mathbb{G}_{m,F})(F).$$

(b) Take $F = \mathbb{Q}_p$ and $u \in \mathbb{Z}_p \setminus (\mathbb{Z}_p)^2$. Then, a \mathbb{Q}_p algebra D with basis $\{1, i, j, k\}$ such that

$$Di^2 = p, j^2 = u, ij = -ji = k,$$

easy to see that D/\mathbb{Q}_p is 4-dimensional. Consider, $\text{GL}_1(D) = D^\times$.

Remark 0.1. If we base change D as follows, we get,

$$D \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \xrightarrow{\sim} \text{Mat}_{2 \times 2}(\overline{\mathbb{Q}_p}),$$

such that

$$i \mapsto \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, k \mapsto \begin{pmatrix} 0 & \sqrt{p} \\ -u\sqrt{p} & 0 \end{pmatrix}, 1 \mapsto I_2$$

Thus $\text{GL}_1(D)$ is reductive ($\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \text{GL}_1(D) \cong \text{GL}_2(\overline{\mathbb{Q}_p})$). We also have,

$$\text{SL}_1(D) = [\text{GL}_1(D), \text{GL}_1(D)].$$

Claim 0.1. $\mathrm{SL}_1(D)(\mathbb{Q}_p)$ is compact.

Proof. Indeed D has a unique maximal order lattice generate the algebra?

$$\mathcal{O}_D = \mathbb{Z}_p[i, j, k] \subset D,$$

and

$$\mathrm{SL}_1(D)(\mathbb{Q}_p) \subseteq \mathcal{O}_D^\times$$

Suppose x is in the left hand side, its characteristic polynomial is $T^2 - \mathrm{tr}(x)T + \det(x) = 0$. So x is integral thus x is contained in some maximal order $X \in \mathcal{O}_D$. \square

From the claim, we have,

$$\mathrm{SL}_1(D) = M,$$

contains no split torus.

Exercise 4 Given a smooth H -representation (π, V) , we have, it is finite if and only if for any compact open subgroup $K \subseteq H$ and any vector $v \in V$, we have,

$$\mathrm{supp}(H \ni g \mapsto e_K(gv)),$$

is compactly supported.

Proof. \Leftarrow , let $\tilde{\lambda} \in \tilde{V}$, and $v \in V$, such that $\tilde{\lambda} = e_K \lambda$. for some $K \subset H$. compact open. We have,

$$\tilde{\lambda}(g, v) = (e_K \tilde{\lambda})(gv) = \tilde{\lambda}(e_K(gv)).$$

which is also compactly supported. Thus for $v \in V$ and $K \subseteq H$ compact open we have for a subgroup,

$$V_1 = Hv, \tilde{V} \twoheadrightarrow \tilde{V}_1,$$

thus \tilde{V}_1 is finite. Since V_1 is finite and finitely generated V_1 is admissible. In other words, \tilde{V}^K is finite dimensional with some basis $\lambda_1, \dots, \lambda_n$. Consider,

$$\mathrm{supp}(H \ni g \mapsto e_K(gv)).$$

If $e_K(gv)$ is not 0 then there exists some $\lambda_9(e_K(gv)) \neq 0$. Thus we have,

$$\mathrm{supp}(H \ni g \mapsto e_K(gv)) \subseteq \bigcup_{i=1}^n \mathrm{supp} m_{\tilde{\lambda}_i, v}.$$

$\tilde{\lambda}_i$ is the image of λ_i along $\tilde{V} \twoheadrightarrow \tilde{V}_1$. \square

Exercise 5 Suppose $V \in \mathrm{Rep}(H)$ which is irreducible and finite. we have,

$$\begin{array}{ccc} V_1 & \hookrightarrow & V_2 \\ f \downarrow & \nearrow \tilde{f} & \\ V & \hookrightarrow & \end{array} \quad \begin{array}{ccc} & & V \\ & \nearrow \tilde{f} & \downarrow \\ V_1 & \twoheadrightarrow & V_2 \end{array}$$

We have,

$$\text{Rep}(H) \cong \text{Rep}(H)^\pi \times \text{Rep}(H)_\pi.$$

Thus we get,

$$W \cong W^\pi \oplus W_\pi.$$

Where

$$W^\pi \cong \bigotimes_I (\pi, V).$$

Since we have

$$\begin{array}{ccc} V_1 & \xrightarrow{\cong} & V_1^\pi \oplus (V_1)_\pi \\ \iota \downarrow & & \downarrow \iota^\pi, \iota_\pi \\ V_2 & \xrightarrow{\cong} & V_2^\pi \oplus (V_2)_\pi \end{array}$$

And we have,

$$\text{Hom}_H((V_1)_\pi, V) = 0.$$

Without the loss of generality, we can assume that

$$V_i \cong \bigotimes_{J_i} V.$$

Since V_2 is semisimple, we have,

$$\begin{array}{ccccc} & & V_2 \cong V_1 \oplus W & & \\ & & \downarrow & & \\ V_1 & \hookrightarrow & V_1 \oplus W & \xrightarrow{\cong} & V_2 \\ f \downarrow & & \downarrow \text{pr}_1 & & \\ V & \xleftarrow{f} & V_1 & & \end{array}$$

For the second part we have,

$$0 \longrightarrow K \longrightarrow V_1 \longrightarrow V_2 \longrightarrow 0$$

$\quad \quad \quad \curvearrowleft \quad \quad \quad$
 $\quad \quad \quad r \quad \quad \quad$

Thus set $\tilde{f} = r \circ f$, and get,

$$\begin{array}{ccc} & & V \\ & \nearrow \tilde{f} & \downarrow \\ V_1 & \longrightarrow & V_2 \end{array}$$

Theorem 0.1. $e^\pi : \text{Rep}(H)^\pi \rightarrow \text{Rep}(H)$ is left and right adjoint thus e^π is exact.

Proof. Let $X \in \text{Rep}(H)^\pi$, then X is injective/projective in $\text{Rep}^\pi(H)$ if and only if X is injective/projective in $\text{Rep}(H)$. That is to say

X is injective in $\text{Rep}(H)$ if and only if $\text{Hom}_{\text{Rep}(H)}(\cdot, X) = \text{Hom}_{\text{Rep}(H)^\pi}(e^\pi(\cdot), X)$ is exact if and only if $\text{Hom}_{\text{Rep}(H)^\pi}(\cdot, X)$ is exact. \square

Corollary 0.1. *All finite dimensional H -representations are injective and projective.*

Exercise 6

(a) The isomorphism is given by

$$f \mapsto (f(g))_{[g]}.$$

This is indeed well-defined since

$$gkg^{-1} \in N \cap gKg^{-1} (k \in K).$$

We then have,

$$gkg^{-1}f(g) \stackrel{\text{left } N \text{ invariance}}{=} f(gk) \stackrel{f \text{ right } K \text{ invariant}}{=} f(g)$$

f is compactly supported thus only finitely many double cosets contained in $\text{supp}(f)$ Clearly it is K invariant. Consider

$$\bigoplus_{N \setminus W/K} W^{N \cap gKg^{-1}} \ni (w_g)_g \mapsto (ngk \mapsto nw_g) \in (c - \text{Ind}_M^H \pi)^K$$

This is well-defined. Get a compactly supported function as $N \setminus \text{supp}()$ is contained in finitely many right K -cosets.

$$ngk = n'gk'$$

where g are of the fixed representatives of $N \setminus H/K$. Then

$$n = n'gk'k^{-1}g^{-1} \in N \cap gKg^{-1}.$$

So $nw_g = n'w_g$. Easy to see that these are mutually inverse.

(b) (π, W) admits $N \setminus H$ compact, $c - \text{Ind}_N^H \pi$ is admissible.

Proof.

$$(c - \text{Ind}_N^H \pi)^K = \bigoplus_{N \setminus H/K} \overbrace{W^{N \cap gKg^{-1}}}^{\text{finite dimensional}}.$$

Furthermore, we have $N \setminus H/K$ is finite thus it is admissible. □

(c) Both $\text{Ind}, c - \text{Ind}$ are both exact functor from $\text{Rep}(N)$ to $\text{Rep}(H)$.

Proof. It suffices to show that for K compact open subgroup we have,

$$(c - \text{Ind}_N^H)^K, (\text{Ind}_N^H)^K$$

are both exact. But

$$(c - \text{Ind})^K = \bigoplus_{[g] \in N \setminus H/K} (\cdot)^{N \cap gKg^{-1}}, (\text{Ind})^K = \bigoplus_{[g] \in N \setminus H/K} (\cdot)^{N \cap gKg^{-1}},$$

are exact. □

(d) For (π, W) irreducible, N compact, $c\text{-Ind}_N^H \pi$ irreducible then $c\text{-Ind}_N^H \pi$ is finite.

Proof. We have $\widetilde{c\text{-Ind}_N^H \pi}$ is also irreducible and

$$(c\text{-Ind}_N^H \pi) \otimes (\widetilde{c\text{-Ind}_N^H \pi})$$

is also irreducible $H \times H$ representation. Thus it suffices to find one non-zero matrix coefficient which is compactly supported since $(h_1, h_2)f = h_1^{-1} \text{supp}(f)h_2$

Pick $v \in W$ and $\tilde{v} \in \tilde{V}$ such that $\tilde{v}(v) \neq 0$. Define,

$$f_v : H \rightarrow W, h \mapsto \begin{cases} hv, & h \in N, \\ 0, & \text{otherwise.} \end{cases}$$

and also,

$$(f_{\tilde{v}} : c\text{-Ind}_N^H W \ni \varphi \mapsto \tilde{v}(\varphi(1))) \in (\widetilde{c\text{-Ind}_N^H W}).$$

Then $f_{\tilde{v}}(f_v) = \tilde{v}(v) \neq 0$, we conclude,

$$m_{f_{\tilde{v}}f_v}(g) = \begin{cases} gv, & g \in N \\ 0 & \text{otherwise.} \end{cases}$$

We have $\text{supp}(m_{f_{\tilde{v}}f_v}) \subseteq N$ which is compact. □

(e)

$$\text{End}(c\text{-Ind}_K^H(\text{triv})) \ni t \xrightarrow{\cong} (\phi_t : H \ni g \mapsto t(e_K)(g))\mathcal{H}(H, K),$$

The inverse is given by

$$[t_\phi : f \mapsto (H \ni h \mapsto \sum_{g \in H/K} \phi(g)f(g^{-1}h))] \leftarrow \phi.$$