V4A9 Homework 10

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(1)

(a) By Bruhats-decomposition, we have $G = B \cup BwB$. By the definition of induced representations, we have

$$f(1) = 0 \Rightarrow \forall b \in B, f(b) = \sigma(b)f(1) = 0.$$

Thereore f is supported in BwB. By definition, we have to show that

$$\int_N f(wn)dn,$$

is well-defined for any $f \in \mathrm{Ind}_B^G(W), f(1) = 0$. Observe that

$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}, \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} t_1 & xt_1 \\ & t_2 \end{pmatrix} = \begin{pmatrix} t_1 & xt_2 \\ & t_2 \end{pmatrix},$$

therefore we have $TN = NT$. Furthermore, elements of T and w commute therefore, we have,

$$BwB = TNwTN = BwN.$$

As f is a smooth element of a G-representation, we have a compact open subgroup K fixing f. By Iwahori decomposition, we may assume that,

$$K = (K \cap N)(K \cap T)(K \cap \overline{N}).$$

By the assumption on f, we conclude that f vanishes on $B(K \cap \overline{N})$. We have,

$$\overline{N}w = wN.$$

thus the support of f is contained in $Bw(K \cap N)$ which follows from that parabolic inductions are always compact. Observe that

$$f \mapsto f_N(1) = \int_{K_N} f(wn)dn = \int_{K_N} \mathrm{Ind}_B^G(\sigma)(n)f(w)dn.$$

Thus this is 0 if and only if $f \in V(N)$. Setting $f(w) = w_1$ for some $w_1 \in W$, we see the map is surjective. Thus the latter statement is proven.

(b). We have,

$$(tf)_N(x) = \int_N f(xwnt)dt.$$

Note that

$$xwnt = xwtt^{-1}nt = xwtwwt^{-1}nt.$$

Using the definition

$$\delta_B(t) = \left|\frac{tKt^{-1}}{tKt^{-1}\cap K}\right|\left|\frac{K}{tKt^{-1}\cap K}\right|^{-1},$$

and use $n \to t^{-1}nt$, we see

$$(tf)_N(x) = \delta_B(t^{-1})\int_N f(x\,{}^w twn)dn = \delta_B^{-1}(t)\int_N f(x\,{}^w twn)dn.$$

Therefore $f \mapsto f_N(1)$ induces a morphism of B-representation $V \mapsto ({}^w\sigma \otimes \delta_B, W)$.

(c). We have a short exact sequence,

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0$$

From part a), we have $V_N \cong W$. V is a subrepresntation of an induced representation and by definition N acts trivially on V. Therefore, we obtain the following,

$$0 \longrightarrow {}^w\sigma \otimes \delta_B \longrightarrow (\mathrm{Ind}_B^G\,\sigma) \longrightarrow \sigma \longrightarrow 0$$

Note that N acts trivially via ${}^w\sigma \otimes \delta_B, \sigma$ as they are inflated from T. Thus, taking the Jacquet functor, we obtain,

$$0 \longrightarrow {}^w\sigma \otimes \delta_B \longrightarrow (\mathrm{Ind}_B^G\,\sigma)_N \longrightarrow \sigma \longrightarrow 0$$

Setting $\sigma \otimes \delta_B^{\frac{1}{2}} \to \sigma$, we get,

$$0 \longrightarrow {}^w\sigma \otimes \delta_B^{\frac{1}{2}} \longrightarrow (\mathrm{Ind}_B^G\,\sigma \otimes \delta_B^{\frac{1}{2}})_N \longrightarrow \sigma \otimes \delta_B^{\frac{1}{2}} \longrightarrow 0$$

$$0 \subseteq {}^w\sigma \subseteq \rho_N(i_B^G(\sigma)).$$

2. $Z(G)G^\circ \subseteq T$ and $G^\circ \ker(\chi), \chi \in X_{\mathrm{nr}}(G)$ Thus we have,

$$G^\circ \subseteq T.$$

$$\begin{aligned}
\chi \otimes \pi \cong \pi &\Rightarrow W_{\chi\otimes\pi} = W_\pi,\\
&\Rightarrow \chi(z)W_\pi(z) = W_\pi(z)\forall z \in Z(G),\\
&\Rightarrow \chi|_{Z(G)} = 1, (\Rightarrow \forall\chi \in X_{\mathrm{nr}}(G), Z(G) \subseteq \ker\chi).
\end{aligned}$$

For part b). take $H = G/Z(G)G^\circ$ is finite abelian ths $H_T = T/Z(G)G^\circ \subseteq H$. Consider the map

$$\Phi : G \to (\widehat{X_{\mathrm{nr}}(G)_\pi}), g \mapsto (\chi \mapsto \chi(g)).$$

We have $\operatorname{Ker}\Phi = T$ and

$$\left|\frac{G}{T}\right| \leq |(\widehat{X_{\mathrm{nr}}(T)_\pi}) = |X_{\mathrm{nr}}(G)_\pi|.$$

We have for all $\chi \in X_{\mathrm{nr}}(G)_\pi, T \subseteq \ker\chi$,

$$X_{\mathrm{nr}}(G)_\pi \subseteq \{\nu \in X_{\mathrm{nr}}(G)_\pi \mid \nu|_T = 1\}.$$

Thus we have,

$$|X_{\mathrm{nr}}(G)_\pi| \geq \left|\frac{G}{T}\right|.$$

$$\{\nu \in X_{\mathrm{nr}}(G) \mid \nu|_T = 1\} = \operatorname{Ker} F.$$

where $F : \hat{H} \to \hat{H}_1, \chi \mapsto \chi|_H$.

$\{\nu \in X_{\mathrm{nr}}(G)|\nu|_T = 1\} = \{\nu : G \to \mathbb{C}^\times|\nu|_{G^\circ} = \nu|_T = 1\} = \{\nu : G/Z(G)G^\circ \to \mathbb{C}^\times\}.$

Furthermore,

$$|\hat{H}/\ker F| = |\operatorname{im} F|, \frac{|G/Z(G)G^\circ|}{|\ker F|} = |T/Z(G)G^\circ| \Rightarrow |G/T| = |\ker F|.$$

(c) We have $G/G^\circ \cong \mathbb{Z}^n$ and $S/G^\circ \cong \mathbb{Z}$,

3.Suppose F is faithful then we have an injection $\operatorname{Hom}(X, X) \to \operatorname{Hom}(F(X), F(X)) = \{0\}$. Thus $X = 0_{\mathscr{A}}$.

For the other direction, note that F induces a morphism of abelian groups $F : \operatorname{Hom}(X, Y) \to \operatorname{Hom}(F(X), F(Y))$. Observe that $F(\operatorname{Ker}(F)) = 0$, therefore $\operatorname{Ker}(F) = 0$. Since we have an exact sequence,

$$0 \longrightarrow \ker F \longrightarrow \operatorname{Hom}(X, Y) \longrightarrow \operatorname{im} F \longrightarrow 0$$

Therefore, we conclude F is faithful.