V4A9 Homework 6

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(1) For each $w \in W$, denote f_w be such that

$$f_w(1) = w, f(g) = 0 \forall g \in H \backslash H'.$$

Then f_w is compactly supported modulo H' and is a smooth element since H' is open, and any compact open $K \subseteq H'$ is again compact open in H. Clearly, the set

$$\{\text{c-Ind}_{H'}^{H} \sigma(g) f_w \mid g \in H' \backslash H, w \in W\},$$

generates the whole representation. For each $\varphi \in \mathrm{Hom}_H(\sigma, \pi|_{H'})$, define,

$$\varphi^*(f_w) = \varphi(w).$$

Then by setting, for $g \in H' \backslash H$,

$$\varphi^*(g f_w) = \pi(g)\varphi(w),$$

we can extend it to an element $\varphi^* \in \mathrm{Hom}_H(\text{c-Ind}_{H'}^{H} \sigma, \pi)$ by observing for any $f \in \text{c-Ind}_{H'}^{H} W$,

$$f = \sum_{g \in H' \backslash H} g^{-1} f_{f(g)}.$$

Similarly, if we are given $\psi \in \mathrm{Hom}_H(\text{c-Ind}_{H'}^{H} \sigma, \pi)$, we can examine at each f_w. Since this is an induced representation which is compatible with actions σ, this induces a morphism $\psi|_{H'} : W \to V|_{H'}$. Orbits of f_w-s generate the whole $\text{c-Ind}_{H'}^{H} W$, we obtain this gives an isomorphism.

(2) Since (π_0, W) is irreducible it is generated by a single element $w \in W$. Since G° is normal in G, $G^\circ \backslash G = G/G^\circ$. Let $\{g_1, \cdots, g_n\}$ be the generator of G/G°. Define $f \in \text{c-Ind}_{G^\circ}^{G} W$ to be such that

$$f(1) = w, f(g_i) = 0 \forall g_i \notin G^\circ.$$

Then f is indeed compactly supported modulo G°. Set

$$\{\text{c-Ind}_{G^\circ}^{G}(g_i) f\}_{i=1,\cdots,n},$$

this set generates $\text{c-Ind}_{G^\circ}^{G} W$. Indeed, any $f \in \text{c-Ind}_{G^\circ}^{G} W$ is uniquely determined by the image of g_i since

$$f(g_i G^\circ) = f(G^\circ g_i) = \sigma(G^\circ) f(g_i).$$

Therefore, the set spans the whole.

(3) For $\phi \in \mathscr{H}(G, \pi^\circ)$ and $f \in \Pi$, define

$$t_\phi(f) = \left[G \ni x \mapsto \sum_{g \in G/G^\circ} \phi(g) f(g^{-1}x) \right].$$

This is clearly linear and defines an element in $\mathrm{End}(\Pi)$. We show this is an isomorphism. This preserves the multiplication since,

$$\begin{aligned} t_{\phi * \psi}(f)(x) &= \sum_{g \in G/G^\circ} (\phi * \psi)(g) f(g^{-1}x), \\ &= \sum_{g \in G/G^\circ} \sum_{h \in G/G^\circ} \phi(h)\psi(h^{-1}g) f(g^{-1}x), \quad (\text{set } g' = h^{-1}g) \\ &= \sum_{h \in G/G^\circ} \sum_{g' \in G/G^\circ} \psi(g') f((g')^{-1}h^{-1}x), \\ &= t_\phi(t_\psi(f))(x). \end{aligned}$$

The last equality is justified since each ϕ, ψ are supported in finite union of G°-cosets. Now define

$$\mathrm{End}(\Pi) \ni t \mapsto [G \ni [W \ni w \mapsto t(f_w)(g)]].$$

Then since t is a H-representation morphism, we have,

$$t(gx)(f_w) = t(f_w)(gx) = \pi^\circ(g) t(f_w)(x).$$

Then we have,

$$t_\phi(f_w)(g) = \phi(g),$$

since f_w is supported in G°. Also,

$$\sum_{g \in G/G^\circ} t(f_w)(g) f_v(g^{-1}x) = t(f_w)(1_G).$$

(4)

(a) Let us define the addition as the point-wise addition and the multiplication as the composition. Indeed, for $z^1, z^2 \in \mathfrak{z}(\mathscr{A})$ and an object $A \in \mathscr{A}$, since z^1_A, z^2_A are in $\mathrm{Mor}_{\mathscr{A}}(A, A)$, they indeed commute. The multiplicative identity $(\mathrm{id}_A)_{A \in \mathscr{A}}$.

(b) We have $\mathrm{Mor}_{(\mathrm{Sets})}(\{*\}, A) = A$ and $z_{\{*\}} = \mathrm{id}_{\{*\}}$. Therefore, $z_A = \mathrm{id}_A$. That is to say $\mathfrak{z}(\mathrm{Sets}) = \{*\}$.

(c) Note that we have for any abelian group A, $\mathrm{Hom}(\mathbb{Z}, A) \cong A$. Since $z \in \mathfrak{z}(\mathrm{Ab})$ is a natural transformation, we have,

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "z_{\mathbb{Z}}"'] & A \arrow[d, "z_A"] \\ \mathbb{Z} \arrow[r] & A \end{tikzcd}$$

The value of z_A evaluated at each $a \in A$, is $\varphi(z_{\mathbb{Z}}(1)) = z_{\mathbb{Z}}(1)$. Therefore, we conclude $\mathfrak{z}(\mathrm{Ab}) = \mathbb{Z}$.

(d) Since for any group G we have $\mathbb{Z} \to G, 1 \mapsto g$. Then we have,

$$z_G(g) = g^{z_{\mathbb{Z}}(1)}.$$

Consider $G = \langle x, y \rangle$. Then

(e) We have, $\mathrm{Hom}_G(\mathbb{C}[G], V) \cong V$. Therefore, by the similar argument we conclude $\mathfrak{z}(\mathrm{Rep}(G)) = \mathbb{C}[Z(G)]$.

(f) We have $\mathrm{Hom}(R, M) \cong M$ as R-modules. Thus z_R determines z, we have,

$$\begin{tikzcd} R \arrow[r, "z'_R"] \arrow[d, "z_R"'] & R \arrow[d, "z_R"] \\ R \arrow[r, "z'_R"'] & R \end{tikzcd}$$

Therefore the center of R-mod is the center of the ring.

(5)

(a)

$$\mathfrak{z}_\pi = [z \mapsto z_\pi \in \mathrm{End}(\pi, V)].$$

(b) Consider $(\sigma, V) = \text{c-Ind}_K^G(\text{triv})$. Then we have,

$$\mathrm{Hom}_G(V, V) \cong V^K, \chi \mapsto \chi(1).$$

Note that

$$V^K = (\text{c-Ind}_K^G \text{triv})^K \cong \mathscr{H}(G, K).$$

Since we have $f(kgk') = f(g)$ for any $g \in G$ and $k, k' \in K$. Now we want to show that $\mathscr{H}(G, K)$ is finitely generated as $Z(\mathscr{H}(G, K))$-module. Note that V is generated by id_K.

$$z(\mathfrak{z}(\mathrm{Rep}(G))) \subseteq \mathrm{End}_G(V)$$

is finitely generated as $\mathbb{C}$ algebra. $\mathrm{Hom}_G(V, V) = V^K$ is finitely generated as $\mathfrak{z}(\mathrm{Rep}(G))$-module. Note that $Z(\mathscr{H}(G, K)) \supseteq \mathfrak{z}(\mathrm{Rep}(G))$ in $\mathrm{End}_G(V)$. Given by

$$\mathscr{H}(G, K) \ni f \mapsto \sigma(f) \in \mathrm{End}_G(V),$$

and induces

$$\mathfrak{z}(\mathrm{Rep}(G)) \to \mathfrak{z}_\sigma(\mathfrak{z}(\mathrm{Rep}(G))) \subseteq \mathrm{End}_G(V).$$

We have

$$\mathscr{H}(G, K) = \mathrm{End}_G(V)$$

as $\mathbb{C}$-algebra. Thus,

$$Z(\mathscr{H}(G,K)) = Z(\mathrm{End}_G(V)).$$

We have,

$$\mathfrak{z}_\sigma(\mathfrak{z}(\mathrm{Rep}(G))) \subseteq Z(\mathrm{End}_G(V)) \subseteq \mathscr{H}(G,K).$$

Note that $\mathfrak{z}_\sigma(\mathfrak{z}(\mathrm{Rep}(G)))$ is finitely generated as $\mathbb{C}$ algebra therefore Noetherian. And $\mathscr{H}(G,K)$ is finitely generated as $\mathfrak{z}_\sigma(\mathfrak{z}(\mathrm{Rep}(G)))$-module. Thus $Z(\mathrm{End}_G(V))$ is also finitely generated.