V4A9 Homework 3

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(1) Let $v \otimes \lambda, u \otimes \mu \in V \otimes \tilde{V}$. Then their composition is

$$[V \ni w \mapsto \mu(w)u] \circ [V \ni w \mapsto \lambda(w)v] = [V \ni w \mapsto \lambda(w)\mu(v)u].$$

Thus the corresponding element in hecke algebra is

$$f_{\mu(v)\lambda,u}(h) = \deg \pi \mu(v)\lambda(\pi(h^{-1})(u)).$$

Consider the convolution,

$$f_{u,\mu} * f_{v,\lambda}(h) = (\deg \pi)^2 \int_H \mu(\pi(g^{-1})(u))\lambda(\pi(h^{-1}g)(v))dg.$$

Let us denote $\lambda' = \tilde{\pi}(g)\lambda$. Since (V, π) is finite and irreducible, by Schur orthogonality, we obtain

$$f_{u,\mu} * f_{v,\lambda}(h) = (\deg \pi)^2 \int_H m_{u,\mu}(g^{-1})m_{v,\lambda'}(g)dg = \deg \pi \lambda'(u)\mu(v) = \deg \pi \mu(v)\lambda(\pi(h^{-1})(v)).$$

Thus ϕ preserves products.

(2)

(b)

$$(e^\pi)(\tau(g)(w)) = \tau(e^\pi_K)(\tau(g)(w)) = \tau(e^\pi_{gKg^{-1}})(w) = \tau((g \times g)e^\pi_K)(w) = g \cdot e^\pi(w).$$

(c) By Lemma 25 from the class, since $K \subseteq K$, therefore,

$$e^\pi \circ e^\pi(w) = \tau(e^\pi_K) \circ \tau(e^\pi_K)(w) = \tau(e^\pi_K * e^\pi_K)(w) = \tau(e^\pi_K)(w) = e^\pi(w).$$

(d) Consider a H-equivariant morphism $\alpha : W_1 \to W_2$. Then for any $w \in W_1$, we have

$$w \in W_1^K \Rightarrow \alpha(w) \in W_2^K.$$

Furthermore, e^π_K is compactly supported by the definition, thus this commutes with a linear map α, thus,

$$\tau(e^\pi_K)(\alpha(w)) = \alpha(\tau(e^\pi_K)(w)).$$

3. Let W be a subrepresentation of (π, V) for $V = \bigoplus_{i\in I} V_i$ for irreducible subrepresentations V_i. Then

$$V = W \oplus W'$$

for another subrepresentation W'. Let $U \subseteq W$ be also a subrepresentation then again,

$$V = U \oplus U',$$

Then

$$W = W \cap U \oplus W \cap U' = U \oplus (W \cap U').$$

Thus W is semisimple. Remains to show the quotient is semisimple. Let $W = \bigoplus_{j\in J} W_j$ where W_j is an irreducible subrepresentaion. Consider an inclusion $\iota_j : W_j \hookrightarrow V$. Since W_j is irreducible, we have $\iota_j(W_j)$ is an irreducible subrepresentation of V thus there is $i \in I$ such that

$$W_j = V_i.$$

This measn that

$$V/W = \bigoplus_{\substack{i\in I \\ V_i \not\subseteq W}} V_i.$$

Thus a quotient is semisimple if V is semisimple. We conclude a subquotient is semisimple.