V4A9 Homework 10

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1.
(a)

$$F : \mathrm{Hom}_G(V, \widetilde{W}) \ni f \mapsto [W \ni w \mapsto (v \mapsto (f(v)(w)))].$$

We check the smoothness, that is

$$\tilde{\pi}(g)(F(f))(w) = \tilde{\pi}(g)(v \mapsto f(v)(w)) = (v \mapsto f(\pi(g^{-1})(v))(w)) = (v \mapsto f(v)(\sigma(g)(w))).$$

For the G-equivariance, set $\Phi = F(f)$,

$$\begin{aligned}
\Phi(\sigma(g)w) &= (v \mapsto f(v)(\sigma(g)w)), \\
&= (v \mapsto (\tilde{\sigma}(g^{-1})f(v))(w)), \\
&= (v \mapsto (f(\pi(g^{-1})(v))(w))), \\
&= \tilde{\pi}(g)(v \mapsto f(v)(w)), \\
&= \tilde{\pi}(g)\Phi(w).
\end{aligned}$$

We then construct a similar map $G : \mathrm{Hom}_G(W, \tilde{V}) \to \mathrm{Hom}_G(V, \tilde{W})$.

(b)

$$\begin{aligned}
\mathrm{Hom}_P(\pi, \rho_{\overline{U}}(\tilde{\sigma})) &\cong \mathrm{Hom}_G(i_P^G(\pi), \tilde{\sigma}), \\
&\overset{\textbf{(a)}}{\cong} \mathrm{Hom}(\sigma, \widetilde{i_P^G(\pi)}), \\
&\overset{\text{HW9.3}}{\cong} \mathrm{Hom}(\sigma, i_P^G(\tilde{\pi})), \\
&\cong \mathrm{Hom}_L(\rho_U(\sigma), \tilde{\pi}), \\
&\cong \mathrm{Hom}_L(\pi, \widetilde{\rho_U(\sigma)}).
\end{aligned}$$

Using Yoneda Lemma, we have the isomorphism.

2. Suppose we have the result in **1**,

$$\begin{aligned}
\mathrm{Hom}_G(i_P^G(\sigma), \widetilde{\pi}) &= \mathrm{Hom}_G(\pi, \widetilde{i_P^G(\sigma)}), \\
&= \mathrm{Hom}_G(\pi, i_P^G(\tilde{\sigma})), \\
&= \mathrm{Hom}_L(\rho_U(\pi), \tilde{\sigma}), \\
&= \mathrm{Hom}_L(\rho_U(\pi), \tilde{\sigma}), \\
&= \mathrm{Hom}_L(\sigma, \widetilde{\rho_{\overline{U}}(\tilde{\pi})}).
\end{aligned}$$

We have a injection $\pi \hookrightarrow \tilde{\tilde{\pi}}$. Set $\pi_1 = \operatorname{coker}(\pi \hookrightarrow \tilde{\tilde{\pi}})$, then the second adjunction holds for $\tilde{\tilde{\pi}}$ and $\tilde{\tilde{\pi}}_1$. Considering the exact sequence,

$$0 \longrightarrow \pi \hookrightarrow \tilde{\tilde{\pi}} \longrightarrow \tilde{\tilde{\pi}}_1$$

By the naturality we have the second adjunction holds for $\ker f = \pi$. The last arrow is the composition of

$$\tilde{\tilde{\pi}} \longrightarrow \pi_1 \hookrightarrow \tilde{\tilde{\pi}}_1$$

As $\pi_1 \to \tilde{\tilde{\pi}}_1$ is injective it preserves the kernel.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_G(i_P^G(\sigma), \pi) \arrow[r] \arrow[d, dotted] & \mathrm{Hom}_G(i_P^G(\sigma), \tilde{\tilde{\pi}}) \arrow[r, "f_1"] \arrow[d, "\Phi,\sim"'] & \mathrm{Hom}_G(i_P^G(\sigma), \tilde{\tilde{\pi}}_1) \arrow[d, "\sim"] \\
0 \arrow[r] & \mathrm{Hom}_L(\sigma, \rho_{\overline{U}}(\pi)) \arrow[r] & \mathrm{Hom}_L(\sigma, \rho_{\overline{U}}(\tilde{\tilde{\pi}})) \arrow[r, "f_2"'] & \mathrm{Hom}_L(\sigma, \rho_U(\tilde{\tilde{\pi}}_1))
\end{tikzcd}$$

Note that $\mathrm{Hom}_G(i_P^G(\sigma), \cdot)$ is left exact and $\rho_{\overline{U}}(\cdot)$ is exact, thus the composition of two left exact functors are still left-exact. Thus Φ induces $\ker f_1 \cong \ker f_2$.

3.

$$\begin{aligned}
t_p(b) &= i_P^G(b), \\
V_{\mathrm{Ind}_P^G(\Sigma)} &\mapsto V_\Sigma, \\
f &\mapsto f(1).
\end{aligned}$$

Suppose $i_P^G(b) = 0$ then $b(f(g)) = 0 \forall g \in G$ where $f \in V_{\mathrm{Ind}_P^G(\Sigma)}$. For $v \in V_\Sigma$, then take

$$a_v : G \to V_\Sigma, g \mapsto \begin{cases} \Sigma(p)v, & (g = pk \in PK_0), \\ 0, & (\text{otherwise}) \end{cases}$$

Σ is smooth as of $(V_\Sigma)^K$ for same open compact $K \subseteq P$, such that $K \supset K_0 \cap P$.

4. Let

$$\Sigma_1 = \text{c-ind}_{H'}^H(\sigma) \otimes (\delta_G/\delta_H)^{-1}, \Sigma_2 = \text{c-ind}_{H'}^H(\sigma) \otimes (\delta(G)/\delta(H'))^{-1}.$$

$$\text{c-ind}_H^G(\text{c-ind}_{H'}^H(\sigma)) = \text{c-ind}_H^G(\Sigma_! \otimes (\delta_G/\delta_H)) \longleftrightarrow \mathscr{H}(G) \otimes_{\mathscr{H}(H)} \Sigma_1$$

$$\Sigma_1 = \text{c-ind}_{H'}^H(\Sigma_2 \otimes (\delta_G/\delta_H)) \longleftrightarrow \mathscr{H}(H) \otimes_{\mathscr{H}(H')} \Sigma_2$$

$$\begin{tikzcd}
\mathscr{H}(G) \otimes_{\mathscr{H}(H)} \Sigma_1 = \mathscr{H}(G) \otimes_{\mathscr{H}(H)} (\mathscr{H}(H) \otimes_{\mathscr{H}(H')} \Sigma_2) \arrow[r, "\cong"] & \mathscr{H}(G) \otimes_{\mathscr{H}(H')} \Sigma_2 \arrow[d, leftrightarrow] \\
& \text{c-ind}_{H'}^G(\Sigma_2 \otimes (\delta_G/\delta_{H'}))
\end{tikzcd}$$

Recall we have,

$$\text{c-ind}_H^G(\text{c-ind}_{H'}^G(\sigma \underset{1}{\otimes} /\delta_{H'})) \cong \text{c-ind}_H^G(\text{c-ind}_{H'}^H(\sigma \otimes (\delta_H/\delta_{H'})) \underset{1}{\otimes} /\delta_H) \cong \text{c-ind}_{H'}^G(\sigma \underset{1}{\otimes} /\delta_{H'}).$$