V4A9 Homework 10

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(1)

(a) Suppose (σ, W) is unitary with inner product $\langle\cdot,\cdot\rangle$, we then define for $f_1, f_2 \in \text{c-ind}_{ZK}^G(\sigma)$,

$$\int_{G/ZK} \langle f_1(g), f_2(g)\rangle dg.$$

This is well-defined as f_1, f_2 has compact support modulo ZK and thus equivariance and positive-definiteness follows from unitaricity of σ. Remains to show that σ is unitary.

k_F is finite, any representation of $\mathrm{GL}_2(k_F)$ is finite. Thus σ viewed as a representation of K is unitary by inheritting one from $\mathrm{GL}_2(k_F)$ as σ is irreducible by assumption. By the assumption of the way we extended the representation, we conclude (σ, W) is a semi-product of two unitary representations thus it is unitary.

(b) Consider $\mathrm{Hom}_G(W, \mathrm{Ind}_{ZK}^G \sigma)$. Using Frobenius reciprocity, we have,

$$\mathrm{Hom}_G(W, \mathrm{Ind}_{ZK}^G \sigma) \cong \mathrm{Hom}_{ZK}(W|_{ZK}, \sigma).$$

Since a restriction of unitary representation to a subgroup is still unitary, we have,

$$\mathrm{Hom}_{ZK}(W|_{ZK}, \sigma) \cong \mathrm{Hom}_{ZK}(\sigma, W|_{ZK}).$$

Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has a determinant 1.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & na+b \\ c & nc+d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -nac+1 & na^2 \\ -nc^2 & nac+1 \end{pmatrix}$$

I couldn't solve the rest.

1.

(a) We have

$$\pi : \mathcal{O}_F \to \mathcal{O}_F/\varpi\mathcal{O}_K.$$

By the irreducibility of σ, we have $V \cong C^n$ for some $n \in \mathbb{N}$. Thus (σ, V) is a unitary representation. Given a character $\omega : k_F^\times \to \mathbb{C}^\times$. Note we have an exact sequence,

$$0 \longrightarrow K(1) \longrightarrow \mathcal{O}_F^\times \longrightarrow k_F^\times \longrightarrow 0$$

where $K(1) = I_2 + \varpi \operatorname{Mat}_{2\times 2}(\mathcal{O}_F)$. Also note that

$$F^\times = \bigcup_{i\in\mathbb{Z}} \varpi^i \mathcal{O}_F^\times = (\mathcal{O}_F^\times)^{\mathbb{Z}}.$$

Given a character $\chi : F^\times \to \mathbb{C}^\times$ such that $\chi(\varpi) = 1$, we have

$$\forall a \in F^\times, a = \varpi^i u, u \in \mathcal{O}_F^\otimes.$$

Thus we define,

$$\chi(a) = \chi(u) = \omega(\overline{u}).$$

We now define,

$$\sigma : ZK \to \mathrm{GL}(V),$$

We now construct a Hermitian form on $\text{c-ind}_{ZK}^{\mathrm{GL}_2(F)}(\sigma)$.

$$\langle f_1, f_2 \rangle := \sum_{g\in ZK\backslash G} \langle f_1(g), f_2(g)\rangle.$$

This is well-defined that is it does not depend on the representative $g \in ZK\backslash G$. This is due to the equivariance of $\langle\cdot,\cdot\rangle$ under ZK-action. ZK is open since it is the union $\bigcup_{z\in Z} zK$. As K is open so is ZK. Furthermore $ZK\mathbb{G}$ is discrete. To see this every point is open in G/H as every point has an open preimage namagely $g \in G/H$ then the preimage is gH^{-1} which is open. Positive-definiteness is due to the positive definiteness of $\langle\cdot,\cdot\rangle$.

(b)

$$\operatorname{Hom}_{\mathbb{C}}(W, W) = \mathbb{C} \Leftrightarrow W \text{is irreducible.}$$

We have $W \hookrightarrow \operatorname{Ind}_{ZK}^G(V)$. Thus we have,

$$\operatorname{Hom}(W, W) \hookrightarrow \operatorname{Hom}_G(W, \operatorname{Ind}_{ZK}^G(V)).$$

Using Frobenius Reciprocity, we have,

$$\begin{aligned}
\operatorname{Hom}(W, W) \hookrightarrow \operatorname{Hom}_G(W, \operatorname{Ind}_{ZK}^G(V)) &= \operatorname{Hom}_{ZK}((\text{c-ind}_{ZK}^G(V))|_{ZK}, V),\\
&= \operatorname{Hom}_{ZK}\left(\bigoplus_{g\in ZK\backslash G/K} \text{c-ind}_{ZK\cap g^{-1}ZKg}^{ZK}({}^g\sigma)|_{ZK\cap {}^gZK}, V\right),\\
&= \prod_{\overline{g}\in ZK\backslash G/K} \operatorname{Hom}_{ZK}(\text{c-ind}_{ZK\cap {}^gZK}^{ZK}({}^g\sigma), V),\\
&= \prod_{\overline{g}\in ZK\backslash G/K} \operatorname{Hom}_{ZK\cap {}^gZK}({}^g\sigma|_{ZK\cap {}^gZK}, V|_{ZK\cap {}^gZK}).
\end{aligned}$$

For $\overline{g} = \overline{e}$ in $ZK\backslash G/ZK$, then for $g \in ZK$,

$$\operatorname{Hom}_{ZK}(V, V) = \mathbb{C},$$

Suppose $\overline{g_1} = \overline{g_2}$. As Z is the center we have $ZK\backslash G/K = K\backslash G/ZK$. Explicitly

$$z_1 k_1 g k_1' = z_2 k_2 g_2 k_2 \Rightarrow$$

We have Cartan decomposition,

$$\mathrm{GL}_2(F) = K\Delta K, \Delta = \{\mathrm{diag}(\varpi^a, \varpi^b) \mid a, b \in \mathbb{Z}, a \leq b\}.$$

For $T \in \mathrm{Hom}_{ZK \cap {}^g ZK}(\cong g^{-1}\sigma, \sigma), \sigma(h)T(v) = T(\sigma(ghg^{-1})v)$. Taking $h \in \mathbb{N}(k_F)$. That is

$$g = \mathrm{diag}(\varpi^a, 1), \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in ZK \cap \cong gZK, x \in \mathcal{O}_F.$$

$$g^{-1} \underbrace{\begin{pmatrix} 1 & \varpi^a x \\ & 1 \end{pmatrix}}_{\in ZK} g = \underbrace{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}}_{\in ZK}.$$

Take $h = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ for $x \in \mathcal{O}_F$, we get

$$ghg^{-1} = \begin{pmatrix} 1 & \varpi^a x \\ & 1 \end{pmatrix}$$

Furthermore $\sigma(ghg^{-1}) = \mathrm{id}$ as $\varpi^a x$ is modded out under the canonical map π. Thus this acts trivially. Thus

$$\sigma\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) T(v) = T(v).$$

However, no non-zero vector is fixed by elements of $N(k_F)$. Thus $T(v) = 0$.

(d) Provided **(c)**, such representation in **(b)** exists. Set $\pi := \text{c-ind}_{ZK}^G(\sigma$ is unitary and irreducible). By Theorem 51 of the lecture we have,

$$\beta(g) = \langle gv, w \rangle = \langle gv, w \rangle, v, w \in V.$$

Remark 0.1. *Let $K' \subseteq G$ be open such that $K'?Z$ is compact. Then* $\text{c-ind}_{K'}^G \sigma$ *is irreducible implies it is supercuspidal. This is from Problem 4(b) from Homework 6.*

2. We want to show that

$$V = \text{c-ind}_H^G(\pi_H)\nu \xrightarrow{\sim} \text{c-ind}_H^G(\pi_H \nu|_H) = V'.$$

Consider

$$F : V \to V', f \mapsto (g \mapsto (\nu(g) f(g))).$$

this is compactly supported, invariant under the right action by some open compact subgroup. Furthermore for all $h \in H, g \in G$,

$$F(f)(hg) = \nu(hg) f(hg) = \nu(h)\nu(g)\pi_H(g) f(g) = (\nu|_H \pi_H)(h)(F(f)(g)).$$

F is injective. Indeed if $f_1, f_2 \in V$, $F(f_1)(g) = F(f_2)(g)$ for all $g \in G$ then $f_1 = f_2$. This is surjective by sending $f \in V'$ to

$$g \mapsto \nu(g^{-1})f(g).$$

The G-equivariance is due to that $f \in V, g, s \in G$,

$$gF(f)(s) = \nu(sg)f(sg) = \nu(g)\nu(s)f(sg) = \nu(g)\nu(s)(gf)(s) = F(g \cdot f)(s).$$

(b) $H' \subseteq H \subseteq G$, We havea

$$\text{c-ind}_H^G(\sigma) \cong \mathbb{C}[G] \underset{\mathbb{C}[H]}{\otimes} (\sigma, W).$$

Thus we have,

$$\begin{array}{ccc} \mathbb{C}[G] \otimes_{\mathbb{C}[H]} (\sigma, W) & \mathbb{C}[H] \otimes_{\mathbb{C}[H']} (\sigma, W) & (\sigma, W) \\ & & \\ \mathbb{C}[G] & \mathbb{C}[H] & \mathbb{C}[H'] \end{array}$$

Therefore,

$$\begin{tikzcd} \mathbb{C}[G] \otimes_{C[H']} W \arrow[r] \arrow[d] & \mathbb{C}[G] \otimes_{\mathbb{C}[H]} (\mathbb{C}[H] \otimes_{\mathbb{C}[H']} W) \arrow[d] \\ \text{c-ind}_{H'}^G \sigma \arrow[r] & \text{c-ind}_H^G(\text{c-ind}_{H'}^H \sigma) \end{tikzcd}$$

(c)

$$T = \bigcap_{\nu \in X_{\mathrm{nr}}(G)_\pi}, H = \mathrm{Stab}_G(W).$$

Observe that $[H, T] < \infty$ and H/T is abelian. One way to see this is by observing $ZG^\circ \leq T \leq H \leq G$. $T \leq H$ comes from Lemma 95. By the previous part we have,

$$\text{c-ind}_T^H(\pi|_T) \cong \mathbb{C}[H] \underset{\mathbb{C}[T]}{\otimes} \pi|_T \cong \mathbb{C}[H/T] \underset{\mathbb{C}}{\otimes} (\pi)$$

The last isomorphism is constructed explicitly in the following way,

$$th_j \otimes w \mapsto \overline{h_i} \otimes \pi(t)w,$$

where $H/T = \{h_1, \cdots, h_m\}$.

Remark 0.2. *Let $H \leq G$ closed, and π be a rep of G σ be a rep of H then*

$$\text{c-ind}_H^G(\sigma \underset{\pi}{\otimes} |_G) \cong \text{c-ind}_H^G(\sigma) \otimes \pi.$$