V4A9 Homework 1

So Murata

WiSe 25/26, University of Bonn Number Theory 1

(1) Let $\varphi : H \times V \to V$ be such that $\varphi(g,v) = \pi(g)(v)$.

(a)⇒(b) For any $v \in V$, $\mathrm{Stab}_H(v)$ is open, thus it contains a basis consists of locally compact open subgroups. Pick one such subgroup K, we have $v \in V^K$.

(b)⇒(c) For any $v \in V$, there is K such that K fixes v. Clearly $K \times \{v\} \subseteq \varphi^{-1}(v)$. For arbitrary $(g,w) \in \varphi^{-1}(v)$, note that G is a topological group thus $gK \times \{w\} \subseteq \varphi^{-1}(v)$ is an open subset. Thus each basis element of the topology on V has an open preimage.

(c)⇒(a) For any $v \in V$, we have $H \times \{v\}$ is open by definition of Product topology and $\varphi^{-1}(v)$ is also open by assumption. Observe that $\mathrm{Stab}_H(v) = H \times \{v\} \cap \varphi^{-1}(v)$ which is an intersection of two open sets which is open.

(2) Using Iwasawa decomposition, we have,

$$\mathrm{SL}_2(F) = BK, \quad (K = \mathrm{SL}_2(\mathcal{O}_F)).$$

(3) The direction ⇐ is trivial. Suppose G has a proper parabolic subgroup. Consider a morphism of G-representation $\varphi : \mathrm{triv} \to \mathrm{Ind}_P^G \sigma$ such that $\varphi(1) = \mathrm{triv}$, note that $\mathrm{triv} : G \to \mathbb{C}$ itself is contained in $\mathrm{Ind}_P^G \mathbb{C}$. We can easily check

$$\varphi \circ \mathrm{triv}(g)(1) = \mathrm{Ind}_P^G \sigma(g) \circ \varphi(1) \Rightarrow \varphi(1) = \mathrm{Ind}_P^G \sigma(g) \circ \varphi(1).$$

In other words, f is right G invariant. This is of course injective. We conclude that $(\mathrm{triv}, \mathbb{C})$ can be embedded into any positive dimensional representation. Thus we have proven the contrapositive of the direction we claimed.

(4) We will prove the contraposition. Suppose $f \in \mathrm{Hom}_H(V_1, V_2) \backslash \{0\}$ exists, then by definition we have

$$\forall z \in Z(H), f \circ \pi_1(z) = \pi_2(z) \circ f.$$

Since f is F-linear, we obtain that

$$\forall z \in Z(H), \chi_1(z) f = \chi_2(z) f.$$

Thus the central characters coincide.

(5)
(a)+(b) For any $k \in K$, we have

$$\pi(k)e_K(v) = \frac{1}{|K|}\int_K \pi(k)\pi(g)(v)dg = \frac{1}{|K|}\int_K \pi(g)(v)dg = e_K(v),$$

as K is a subgroup. Thus $e_K : V \to V^K$. For any $v \in V^K$, obviously $e_K(v) = \frac{|K|}{|K|}v = v$. Thus this defines a projection.

(c) Clearly $V = V^K \oplus (1-e_K)V$ as a vectorspace. We have

$$\pi(k)e_K(v) = e_K(\pi(k)v)$$

as K-representations. Thus we conclude the statement to be true.

(d) By definition, K is a subgroup thus if g runs through K, so does g^{-1}. Since $\lambda \in \tilde{V}$, there is a compact open subgroup K' such that $\lambda \in (V^*)^{K'}$. Using H is Hausdorff, we conclude $K \cap K'$ is again a compact open subgrouop. This assures us that $e_K : \tilde{V} \to \tilde{V^K}$ is well-defined. Also using λ is linear, we get,

$$e_K\lambda(v) = \frac{1}{|K|}\int_K \pi^*(g)\lambda(v)dg = \frac{1}{|K|}\int_K \lambda(\pi(g^{-1})v)dg = \lambda(e_K(v)).$$

(e) By (c), there is a bijection between V^* and $(V^K)^* \oplus ((1-e_K)V)^*$. By (d), e_K defines a map from $\tilde{V}$ to $(V^K)^*$. For $\lambda \in \tilde{V}^K$, we have,

$$\lambda(v) = e_K\lambda(v) = \lambda(e_K v).$$

This is clearly injective and linear. For any $\lambda \in (V^K)^*$, we take $\nu : V \to k$ such that $\nu(v) = \lambda(e_K(v))$, then $\nu \in \tilde{V}^K$ and $e_K(\nu) = \lambda$. Therefore, surjective.

(6) a)$\Rightarrow$b) follows from Exercise 5 (e) together with the fact that a dual space of a finite dimensional vector space is of finite dimension. Note that by definition of dual representation we have,

$$\pi^{**}(g)\,\mathrm{ev}_v(\lambda) = \pi^*(g^{-1})\lambda(v) = \lambda(\pi(g)v) = \mathrm{ev}_{\pi(g)(v)}(f).$$

$a)b) \Leftrightarrow c)$, For any vectorspace V, we have the following inequality,

$$\dim V \leq \dim V^*.$$

In particular, the equality holds if and only if V is finite dimensional. Again using (e) of the previous problem, we have

$$\tilde{\tilde{V}}^K = (V^K)^{**},$$

which is of finite dimension. Thus have the same dimension as V^K. We conclude $a), b)$ are equivalent to $c)$ by passing the exact sequence

$$0 \longrightarrow V \longrightarrow \tilde{\tilde{V}} \longrightarrow 0$$

to the fixed part of arbitrary compact open subgroup K of H.