V4A9 Homework 8

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(1) We have,

$$\mathscr{H}(G, G^\circ, \pi^\circ) = \mathrm{Span}_{\mathbb{C}}\{c\phi_1^{e_1} * \cdots * \phi_n^{e_n} \mid e_n \in \mathbb{Z}\}.$$

Define

$$\phi_1^{e_1} * \cdots * \phi_n^{e_n} \geq \phi_1^{e_1'} * \cdots * \phi_n^{e_n'}$$

in the dictionary order. Let $\psi_1\psi_2 \in \mathscr{H}(G, G^\circ, \pi^\circ)$ then $\psi_1 * \psi_2$ then $\psi_1 * \psi_2 \neq 0$, Then we have

$$(\text{leading term of } \psi_1) \times (\text{leading term of } \psi_2) = (\text{leading term of } \psi_1 * \psi_2).$$

Note that

$$\phi_i * \phi_j = c_{ij}\phi_j * \phi_i (c_{ij} \neq 0).$$

Thus above argument is justified.

(2) Let $P \subsetneq G$ be a proper parabolic subgroup. For $K \subseteq P$ a compact open subgroup, set

$$\delta_P(p) = |pKp^{-1}/pKp^{-1} \cap K||K/pKp^{-1} \cap K|^{-1}.$$

Let μ be a left Haar measure, over P, and the homeomorphism

$$g \mapsto gp^{-1},$$

induces another left Haar measure such that $\delta'(p)\mu = \mu_P$, then

$$\delta'(p)\mu(K) = \mu_P(K) = \mu(Kp^{-1}) = \mu(p^{-1}pKp^{-1}) = \mu(pKp^{-1})$$

Thus $\delta(p) = \frac{\mu(pKp^{-1})}{\mu(K)}$. If $K' \subset K$ be compact open subgroup then

$$K = \bigsqcup_{i=1}^{n} x_i K', n = [K : K'].$$

Thus we have $\mu(K) = n\mu(K')$ thus $[K : K'] = \frac{\mu(K)}{\mu(K')}$. THus $\delta'(p) = \delta_P(p)$ which is invariant. Note that left Haar measure is unique up to positive constant.

The triviality on a compact open subgroup follows from that we have proven the definition is invariant of the choice of K, thus for any compact open subgroup K, define δ_P using such K.

It is indeed a character indeed for $K \subseteq P$, compact,

$$\delta_P(pq)\mu(K) = \mu(K(pq)^{-1}) = \delta_P(p)\delta_P(q)\mu(K).$$

(4)

We know that

$$B = TN,$$

where T is the diagonal matrices and N is the unipotent subgroup whose diagonals are all 1. To determine the value of δ_B, it is enough to do it separately on T and N. We have $K = (1 + \varpi \operatorname{Mat}_{n\times n}(\mathcal{O}_F)) \cap B$ is a compact open subgroup of B.

Example 0.1 (Toy case, $n = 2$)**.**

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + a\varpi & b\varpi \\ c\varpi & 1 + d\varpi \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + a\varpi & dx\varpi + b\varpi - ax\varpi \\ 0 & 1 + d\varpi \end{pmatrix}$$

Also note that $\operatorname{ord}_\varpi(x) = \operatorname{ord}_\varpi(-x)$*, thus*

$$pKp = p^{-1}Kp,$$

where $p = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ *thus*

$$\delta_B(p) = \delta_B(p^{-1}).$$

Thus $\delta_B(p) = 1$*.*

Another argument is that

Remark 0.1. *The unipotent radical is exhausted by compact subgroups. Thus modulus character is trivial on the unipotent radical.*

For the diagonal part and let us suppose that $n = \operatorname{ord}_\varpi\left(\frac{t_i}{t_j}\right) \geq 0$.

$$\underbrace{\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}}_{=t\in T} K \begin{pmatrix} t_1^{-1} & & \\ & \ddots & \\ & & t_n^{-1} \end{pmatrix} = \begin{pmatrix} 1 + \varpi\mathcal{O}_F & & \\ & \ddots & \\ & & 1 + \varpi\mathcal{O}_F \end{pmatrix}$$

Thus $tKt^{-1} \subset K$, thus the problem amounts to show that

$$\delta_B(t) = |tKt^{-1}/K|.$$

Let $\lambda \in 1 + \underbrace{\varpi\mathcal{O}_F/1 + \varpi^{n+1}\mathcal{O}_F}_{q^n}$,

$$(I + \lambda E_{ij})tKt^{-1}$$

Thus we conclude,

$$\delta_B(t) = \prod_{i<j} \left| \frac{t_i}{t_j} \right|.$$

(5) Recall that we have, the set of roots is

$$\Phi = \{e_i - e_j \mid i \neq j\}, \Phi+ = \{e_i - e_j \mid i < j\}.$$

The set of simple roots are ,

$$\Delta = \{e_i - e_{i+1} \mid i = 1, \cdots, n-1\}, \Delta' \subset \Delta.$$

We claim that $P_{\Delta'}$ is determined by a partition $(n_1, \cdots, n_k)$ such that $n_1 + \cdots + n_k = n$.

$$P_{\Delta'} = \left\{ \begin{pmatrix} g_1 & * & * \\ O & \ddots & * \\ O & O & g_k \end{pmatrix} \middle| g_i \in \mathrm{GL}_{n_i} \right\}.$$

$e_i - e_{i+1} \in \Delta'$ if and only if i and $i+1$ belong to the same block. $P_\Delta = \mathrm{GL}_n$, $P_\emptyset = B$ where Δ corresponds to (n) and $\emptyset$ corersponds to $(1, \cdots, 1)$.

$$\delta(p) = \prod_{i=2}^{k} |\det g_i|^{-\sum_{j=1}^{i-1} n_j + \sum_{j=i+1}^{k} n_j}.$$

if $n_j =$ then

$$\delta(p) = \prod_{p=1}^{n} |t_i|^{2i+1-n} = \prod_{i<j} \left| \frac{t_i}{t_j} \right|.$$

From Exercise **(3)**, we have,

$$\int f(tn) dt dn$$

is a left Haar measure and

$$\int f(nt) dt dn = \int f(tn) \delta(tn) dt dn$$

is a right Haar measure. We compute the Jacobian of the following variable substitution,

$$\underbrace{\begin{pmatrix} u_1 & * & * \\ & \ddots & * \\ & & u_n \end{pmatrix}}_{\text{variable}} \underbrace{\begin{pmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_n \end{pmatrix}}_{\text{constant}} = \begin{pmatrix} v_1 & * & * \\ & \ddots & * \\ & & v_n \end{pmatrix}$$

For the detailed treatment see Goldfreld 14.3.6

(5) Dr. Dillery's solution

Let $P = MN$ where M is the levi factor and N is the unipotent part. As we have seen, it suffices to calculate

$$\delta_P(m), \forall m \in M.$$

Recall the Cartan decomposition $T_M = Z(M)^\circ(F) = \{t \in T_M \mid |\alpha(t)| \leq 1 \forall \alpha \in \Delta\}$. Use that there is K_0 compact open such that

$$G = \coprod_{t \in T_M^+} K_0 w t K_0.$$

For the exercise, in GL_n, w can be taken out from the equation. Assuming this we get,

$$G = \coprod_{t \in T_M^+} K_0 t K_0.$$

To compute $\delta_P(m)$ it suffices to compute $\delta_P(t)$ where $t \in T_m'$. Choose K small enough so that we have,

$$t \cdot K \cap P = t(K \cap M) \cdot (K \cap N) t^{-1}.$$

Since $t \in Z(M)(F)$, we have

$$t \cdot K \cap P = (K \cap M) t (K \cap N) t^{-1}.$$

Thus the explicit computation will be

$$|t(K \cap N)t^{-1} / t(K \cap N)t^{-1} \cap (K \cap N)| \cdot |K \cap N / t(K \cap N)t^{-1} \cap (K \cap N)|^{-1}.$$