V4A9 Homework 9

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(1) Following from the hint, we construct a G-equivariant surjective map,

$$\mathscr{H}(G) \to \mathrm{Ind}_P^G(\delta_P), f \mapsto [g \mapsto \int_P f(p)dp],$$

where dp is a left Haar measure on P. Then we lift the map to get the desired map I_P, explicitly,

$$\begin{tikzcd}
\mathrm{Ind}_P^G(\delta_P) \arrow[r, dashed, "I_P"] & G \\
\mathscr{H}(G) \arrow[u, two heads] \arrow[ur, "\exists"']
\end{tikzcd}$$

From a part of the solution from Homework 8, we have the following result,

$$\int_P f(xpg)dx = \delta_P(p) \int_P f(xg)dx.$$

That is

$$\delta_P(p)d(xp) = dx.$$

For the G-equivariance, we have,

$$(hf) \mapsto (g \mapsto \int f(pgh)dp = h \int f(pg)dp).$$

The equality follows from that we can write the right most hand side as a fintie sum. We note an important definition below,

Remark 0.1. *$\underline{P} \subseteq \underline{G}$ is parabolic if and only if $\underline{G}/\underline{P} \hookrightarrow \mathbb{P}^n_F$ that is $\underline{G}/\underline{P}$ is projective. We have,*

$$\underline{G}(k)/\underline{P}(k) \hookrightarrow (\underline{G}/\underline{P})(k),$$

which is compact. And the last morphism is actually an isomorphism. For the intution, we have, $\mathrm{GL}_2/B = \mathbb{P}^1$.

In Homework 4, we have proved that for compact induction

$$(\text{c-Ind}_N^H(\pi))^K = \bigoplus_{\overline{g} \in N \backslash H/K} W^{N \cap gKg^{-1}}.$$

$G = BK$ where B is minimal parabolic and K compact that is $G = PK = KP$ where P is parabolic containing B. Thus Ind_P^G is actually a compact induction. Motivated by this, we have,

$$(\mathrm{Ind}_P^G(\delta_P))^K = \bigoplus_{\overline{g} \in P\backslash G/K} (\delta, \mathbb{C})^{P \cap gKg^{-1}} = \bigoplus_{\overline{g} \in P\backslash G/K} \mathbb{C}.$$

Define

$$\phi_{s,1} = \begin{cases} \delta(p), g \in PsK, \\ 0, \text{otherwise}. \end{cases}$$

where S is the complete set of representatives of $P\backslash G/K$. Now set $f(g) = \chi_{sK}(g)$. Then,

$$\int f(xg)dx = \phi_{s,1}(g) = \int_{gKg^{-1}\cap P} dx.$$

Recall that

$$sKg^{-1} \cap P \neq \emptyset \Leftrightarrow sKg^{-1} = p \in P \Leftrightarrow g = p^{-1}sk \in PsK.$$

Therefore in that case for $g = psk$, $|sKg^{-1} \cap P| = |sKs^{-1}p^{-1} \cap P|$.

We now construct $\mathscr{H}(G) \to \mathbb{C}$. Take,

$$\mathscr{H}(G) \ni f \mapsto \int_G f(g)dg \in \mathbb{C}.$$

By the surjectivity, for any $\phi \in \mathrm{Ind}_P^G(\delta_P)$, there is f such that $f \mapsto \phi$. Since the surjective map is linear, in order to show the facrotization, it is enough to prove that any element killed by the map below,

$$\phi \mapsto \int_K \phi(k)dk$$

is contained in the kernel of $\mathscr{H}(G) \to \mathbb{C}$. Note that

$$\mathscr{H}(G)^K$$

is spanned by $\{1_{gK}\}_{g\in G/K}$. Furthermore, we have already seen that ,

$$\int_P \chi_{gsK} d\mu_P = \mu_(P \cap sKg^{-1})\chi_{psK}(g).$$

Note that the right hand side is positive. Thus in order to kill it, we need to multiply 0. Observe that,

$$\mathscr{H}(G) = \mathrm{Span}_{\mathbb{C}[G]}\{1_K \mid K \text{ is compact open}\}.$$

That is

$$f = \sum g\chi_{Kg}.$$

Take an arbitrary compact open K, we will show that $\lambda \in \mathrm{Hom}(\mathscr{H}(G), \mathbb{C})$ is determined by $\lambda(\chi_K)$. Consider two compact open K_1, K_2,

$$\lambda(K_1) = \lambda\left(\sum_{i=1}^{s} \chi_{K\cap Ksg_i}\right) = s\lambda(\chi_{K_1\cap K_2}),$$

where $s = [K_1 : K_1 \cap K_2]$.

(2)

Let $\phi \in \mathrm{Ind}_P^G(\sigma), \Phi \in \mathrm{Ind}_P^G(\sigma^\vee \cdot \delta_P)$. Consider $G \ni\mapsto g \mapsto \langle \phi(g), \Phi(g)\rangle$. Then for $p \in P$, we have,

$$\langle \phi(pg), \Phi(pg)\rangle = \delta_P(p)\Phi(g)(\sigma^\vee(p^{-1})\sigma(p)\phi(g)) = \delta_P(p)\langle \phi(g), \Phi(g)\rangle.$$

Since both slots take elements from induced representations we take

$$K(\phi) \cap K(\Phi) = K(g \mapsto \langle \phi(g), \Phi(g)\rangle).$$

Thus this lies in $\mathrm{Ind}_P^G(\delta_P)$. From exercise **(1)**, we have

$$[g \mapsto \langle \phi(g), \Phi(g)\rangle] \mapsto I_P(\langle \phi, \Phi\rangle),$$

gives a linear functional $\mathrm{Ind}_P^G(\delta_P) \to \mathbb{C}$.

Consider a map

$$\mathrm{Ind}_P^G(\sigma^\vee \cdot \delta_P) \ni \Phi \mapsto [\mathrm{Ind}_P^G(\sigma) \ni \phi \mapsto I_P(\langle \phi, \Phi\rangle)].$$

This is an element of $\mathrm{Ind}_P^G(\sigma)^\vee$. By the property **(a)** of I_P, we have,

$$I_P(\langle \phi, g \cdot \Phi\rangle) = I_P(\langle g^{-1} \cdot \phi, \Phi\rangle).$$

Therefore, the constructed map is indeed a G-homomorphism.

For the functoriality, it follows from that $\mathrm{Ind}_P^G, (\cdot)^\vee$ and multiplying by δ_P are all functorial.

Before moving on to the next problem, we note the following statement.

Proposition 0.1.

$$\mathrm{Ind}_P^G(\delta_P)^K = \bigoplus_{g_i \in S} W^{P\cap g_i K g_i^{-1}}.$$

Suppose $\mathcal{W}_i$ *is a basis of* $W^{P\cap g_i K g_i^{-1}}$. *Then*

$$\{\phi_{g_i,w}\}_{g_i\in S, w\in\mathcal{W}_i}$$

is a basis of $\mathrm{Ind}_P^G(\delta_P)^K$.

(3) Let K be an open compact subgroup. Take $\lambda \in (\mathrm{Ind}_P^G(\sigma)^\vee)^K = (\mathrm{Ind}_P^G(\sigma)^K)^\vee$ and $w \in W^K$, and

$$f_{w,g_i}(g) = \begin{cases} w, & g \in Pg_iK, \\ 0, & \text{otherwise}, \end{cases}$$

where $g_i \in \{g_1, \cdots, g_n\} = P\backslash G/K$. Note that

$$\mathrm{Ind}_P^G(\sigma)^K = \text{c-}\mathrm{Ind}_P^G(\sigma),$$

and $\{f_{w,g_i}\}$ spans it as it is locally constant and each support is a finite union of Pg_iK. Then $\lambda^\vee = w \mapsto \lambda(f_{w,g_i})$ gives an element of $(W^\vee)^K$. Similarly define,

$$f_{\lambda^\vee,g_i}(g) = \begin{cases} \lambda^\vee, & g \in Pg_iK, \\ 0, & \text{otherwise}, \end{cases}$$

Then this gives us that

$$I_P(\langle f_{w,g_i}, f_{\lambda^\vee,g_j}\rangle) = I_P(\chi_{Pg_iK})\lambda^\vee(w) = I_P(\chi_{Pg_iK})\lambda(f_{w,g_i}), \quad g_i = g_j.$$

Thus by scaling it, we see this is locally surjective for all K thus surjective globally. We also see that $\{f_{\lambda^\vee,g_i}\}$ is a basis for

$$(\mathrm{Ind}_P^G(\sigma^\vee \cdot \delta_P))^\vee = \text{c-}\mathrm{Ind}_P^G(\sigma^\vee \cdot \delta_P).$$

Therefore, an kernel of element must take zero for all f_{w,g_i} for each i. But for each i, it is a evaluation paring. We conclude this is injective. To prove the last statement take $\sigma \cdot \delta^{\frac{1}{2}}$ then $(\sigma \cdot \delta^{\frac{1}{2}})^\vee = \sigma^\vee \cdot \delta^{-\frac{1}{2}}$. Therefore, we have,

$$(\mathrm{Ind}_P^G(\sigma \cdot \delta_P^{\frac{1}{2}}))^\vee = \mathrm{Ind}_P^G(\sigma^\vee \cdot \delta_P^{-\frac{1}{2}} \cdot \delta_P) = \mathrm{Ind}_P^G(\sigma^\vee \cdot \delta_P^{\frac{1}{2}}).$$