V4A9 Homework 6

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(1)

(a) Taking the basis $\{e_1, \cdots, e_p\} \cup \{e_{p+1}, \cdots, e_{p+q}\}$ given in the problem, we see for any $X \in U(p,q)$,

$$h_A(Xe_i, Xe_j) = h_A(e_i, e_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

From this we see that,

$$\mathbb{T}(A) = \mathrm{diag}(e^{i\theta_1} \otimes 1_A, \cdots, e^{i\theta_{p+q}} \otimes 1_A).$$

In particular, we have,

$$\mathbb{T}(\mathbb{R}) \cong (S^1)^{p+q}.$$

Therefore, $A_0 = 0$. We conclude $\mathbb{T}$ is not split.

(b) We know that $U(n,n) \cong U(n,0) \times U(n,0)$. And for each $U(n)$, the borel subgroup is

$$\mathbb{B}(A) = U(n,0)(A) \times U(0,n) \cap \{\text{upper triangular matrices}\}.$$

In particular $\mathbb{B}(\mathbb{R}) = \mathrm{diag}(e^{i\theta_1}, \cdots, e^{i\theta_{2n}})$ since each column must be orthogonal to one another.