V5A10 Analytic Number Theory

So Murata

WiSe 25/26, University of Bonn

1 Classical Number Theory

Theorem 1.1 (Euclid)**.** *There are infinitely many prime numbers.*

Definition 1.1. $\pi : \mathbb{N} \to \mathbb{N}$ *be a function such that*

$$\pi(n) = \{\textit{prime numbers less than } n\}.$$

Remark 1.1.

$$\frac{\pi}{n \ln(n)} \approx 1.$$

Definition 1.2.

$$\mathrm{Li}(x) = \int_0^x \frac{1}{\ln(t)} dt.$$

Notation 1.1. *Given* $f, g : \mathbb{R} \to \mathbb{C}$*,*

$$f(x) = O(g(x))$$

means that

$$\exists K \in (0, \infty), x_0 \in \mathbb{R}, \text{ s. t. } \forall x > x_0, |f(x)| \leq K|f(x)|.$$

Notation 1.2. *Let* $f, g : \mathbb{R} \to \mathbb{C}$ *be functions.* $f \sim g$ *denotes that*

$$\lim_{n \to \infty} \frac{f(x)}{g(x)} = c,$$

for some constant c*.*

Notation 1.3. *Let* $f : \mathbb{R} \to \mathbb{C}$*,*

$$\mathrm{Li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k},$$

denotes that

$$\mathrm{Li}(x) = \frac{x}{\ln x} \sum_{k=1}^{N-1} \frac{k!}{(\ln x)^k} + O\left(\frac{x}{(\ln x)^{N+1}}\right).$$

and as $x \to \infty$*, this holds for any* $N \geq 1$*.*

Remark 1.2. *By the integration by parts, we see that it's asymptotic expansion is*

$$\mathrm{Li}(x) \approx \frac{x}{\ln(x)} \sum_{k=0}^{\infty} \frac{k!}{(ln(x))^k}.$$

Theorem 1.2 (Prime Number Theorem)**.**

$$\lim_{x\to\infty} \frac{\pi(x)}{\mathrm{Li}(x)} = 1.$$

Definition 1.3 (First Chebyschev Function)**.**

$$\vartheta(x) = \sum_{p\leq x} \ln p.$$

Definition 1.4 (Second Chebyschev Function)**.**

$$\psi(x) = \sum_{\substack{m,p \\ p^m \leq x}} \ln p.$$

Remark 1.3. *We can rewrite the second Chebyschev function as follows.*

$$\psi(x) = \sum_{\substack{p\leq x \\ m=\max\{n\in\mathbb{N} \mid p^n \leq x\}}} m \ln p.$$

Definition 1.5 (Möbius Function)**.**

$$\mu(n) = \begin{cases} 1 \quad (n=1) \\ (-1)^k \quad (n = p_1 \cdots p_k, p_i = p_j \Rightarrow i = j) \\ 0 \quad (\exists p \text{ s.t. } p^2|n). \end{cases}$$

Remark 1.4. *The prime number theorem is equivalent to the following statements.*

1). $\psi(x) \sim x$.

2). $\theta(x) \sim x$.

3). $\lim_{x\to\infty} \frac{\sum_n \leq x}{\frac{\mu(n)}{n}} x = 0.$

Conjecture 1.1 (Twin Prime Conjecture)**.** *There exists infinitely many primes* p *such that* $p + 2$ *is also prime.*

Conjecture 1.2 (Goldbach's Conjecture)**.** *Let* $n \in \mathbb{N}$ *be an even number greater than* 2*, then there exists two primes* p, q *such that* $n = p + q$*.*

Conjecture 1.3 (Hardy-Littlewood Conjecture)**.**

$$\#\{ \textit{ prime numbers } p \textit{ such that } 2p+1 \textit{ is also a prime and } p < x\}$$

Definition 1.6 (Riemann-Zeta Function)**.** *We define* $\zeta : \mathbb{C} \to \mathbb{C}$ *such that*

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}.$$

Remark 1.5. *Supposer* $\mathrm{Re}(s) > 1$*, then we have*

$$\begin{aligned} |\zeta(s)| &= \sum_{n \in \mathbb{N}} \frac{1}{|n|^s} \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n^{\mathrm{Re}(s)}} \end{aligned}$$

By multiplying $\frac{1}{2^s}$*, we obtain*

$$\frac{1}{2^s}\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{(2n)^s}.$$

We get

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{(2n+1)^s}.$$

Continuing this procedure, we get the following proposition.

Proposition 1.1.

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)$$

Theorem 1.3 (Weierstrass)**.** *Let* $A \subseteq \mathbb{C}$ *and consider a sequence of functions* $(f_n : A \to \mathbb{C})_{n \in \mathbb{N}}$ *such that there exists a sequence of non-negative numbers* $(M_n)_{n \in \mathbb{N}}$ *such that*

i). $\forall x \in A, |f_n(x)| \leq M_n$*.*

ii). $\sum_{n \in \mathbb{N}} M_n$ *converges.*

Then the sequence converges uniformly.

Theorem 1.4. *Suppose the conditions in the previous theorem. If each function is analytic on a compact subset of* A*, then the limit is also analytic.*

Corollary 1.1. *Let* A *be a compact subset of a complex plane where* $\mathrm{Re}(s) > 1$*. Then there exists* $\delta > 0$ *such that* $\mathrm{Re}(s) > 1 + \delta$ *and*

$$\sum_{n \in \mathbb{N}} \left|\frac{1}{n^s}\right| \leq \sum_{n \in \mathbb{N}} \frac{1}{n^{1+\delta}} < \infty.$$

Fact 1.1. *The Riemann zeta function can be analytically continued to the whole plane except for* $s = 1$*.*

Definition 1.7 (Gamma Function)**.**

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

Proposition 1.2.

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx.$$

Remark 1.6. *$\zeta(1+it) \neq 0$ if $t \in \mathbb{R}, t \neq 0$. $\zeta(s) \neq 0$ for $0 < s < 1$.*

Definition 1.8 (Functional Equation)**.**

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$$

Remark 1.7. *$\Gamma(s)$ is defined for $\mathrm{Re}(s) > 0$ and can be analytically continued to the whole place except for $\mathbb{C}\backslash\{-2n \mid n \geq 0\}$.*

Remark 1.8. *For $s = -2m$ where $m \in \mathbb{N}$, we see $\zeta(s) = 0$.*

$$\begin{aligned}
\zeta(0) &= \frac{2}{2\pi}\lim_{s\to 1}\cos\left(\frac{\pi s}{2}\right)\zeta(s) \\
&= \frac{1}{\pi}\lim_{s\to 1}\frac{\cos(\frac{\pi s}{2})}{s-1}\lim_{s\to 1}(s-1)\zeta(s) \\
&= \frac{1}{\pi}\times\frac{-\pi}{2}\times 1 \\
&= -\frac{1}{2}.
\end{aligned}$$

Definition 1.9. *The critical strip is the subset of the complex plane with its real part between 0 and 1. The critical line is the line where $\mathrm{Re}(s) = \frac{1}{2}$.*

Conjecture 1.4 (Riemann Hypothesis)**.** *Let s be an element of the critical strip. If $\zeta(s) = 0$ then $\mathrm{Re}(s) = \frac{1}{2}$ (ie. it lies on the critical line).*

Notation 1.4. *Let $T > 0$. We denote $N(T)$ the number of zeros of ζ in the critical strip whose coefficient of the imaginary part is in $(0, T)$. That is*

$$N(T) = |\{\sigma + it \in \mathbb{C} \mid 0 < \sigma < 1, 0 < t < T\}|.$$

Proposition 1.3.

$$\lim_{T\to\infty}\frac{N(T)2\pi}{T\log(T)} = 1.$$

Sketch of Proof (needs refinement).

$$\psi(x) = \frac{1}{2\pi i}\int_l \frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds$$

where l is the line $l = a$ for some $a > 1$.

$$\psi(s) = x - \sum_{\rho \text{ non-trivial zeros}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \log(1 - x^{-2}).$$

$\square$

Definition 1.10. *Let $q \in \mathbb{N}$ and a be a natural number coprime to q. We define*

$\pi(x; q, a) = |\{$*prime numbers p less than or equal to x such that $p \equiv a \mod q$*$\}|$

Proposition 1.4.

$$\pi(x; , q, a) \sim \frac{x}{\varphi(q) \log(x)}$$

where φ is a Euler phi-function.

Theorem 1.5 (Brun–Titchmarsh)**.** *For any $q < x$, we have*

$$\pi(x; , q, a) < \frac{2x}{\varphi(q) \log(\frac{x}{q})}.$$

Remark 1.9.

$$\mathrm{Li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k}.$$

Indeed we have

$$\mathrm{Li}(x) = . \int_2^x \frac{dt}{\ln t}.$$

Observe that

$$\int_2^t \frac{1}{(\ln t)^N} \sim \frac{x}{(\ln x)^N}$$

for all $N \geq 1$. Thus $\mathrm{Li}(x)$ can be expressed in terms of polynomials in $\frac{x}{\ln(x)}$, by keep replacing the greatest temr with the above approximation.

2 Arithmetic Functions

2.1 Multiplicative Functions

Definition 2.1. *A function $f : \mathbb{N} \to \mathbb{C}$ is said to be*

1). multiplicative if for any $(m, n) = 1$, we have $f(mn) = f(m)f(n)$,

2). completely multiplicative if for any natural numbers m, n, we have $f(mn) = f(m)f(n)$.

Example 2.1. *Möbius function μ is multiplicative.*

Definition 2.2 (Von-Mangoldt Function)**.** *The Von-Mangoldt function* $\Lambda : \mathbb{N} \to \mathbb{C}$ *is defined as*

$$\Lambda(n) = \begin{cases} \log(p) & (n = p^k \text{ for some } k \geq 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Definition 2.3 (Euler Phi Function)**.** *The Euler phi funtion is* $\varphi : \mathbb{N} \to \mathbb{C}$ *such that*

$$\varphi(n) = \{1 \leq a \leq n \mid (a, n) = 1\}.$$

Example 2.2. φ *is multiplicative but* Λ *is not.*

Definition 2.4 (Dirichlet Characters Modulo q)**.** *Let* $q \in \mathbb{N}$ *be a natural number and* $q \geq 2$*.*

$$\chi_1 : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$$

be a group homomorphism. The Dirichlet character function modulo q *with respect to* χ_1 *is such that*

$$\chi(n) = \begin{cases} \chi_1(\overline{n}) & ((n, q) = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Example 2.3. *For* $q = 3$*, we have* $(\mathbb{Z}/3\mathbb{Z})^\times = \{\pm 1\}$*. The only possible character is* $\pm 1 \mapsto \pm 1$*. Therefore, we have*

$$\chi(1) = 1, \chi(2) = -1, \chi(0) = 0.$$

Theorem 2.1.

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1, \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. When $n = 1$, this is trivial. Suppose $n \neq 1$. We factorize n by

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

where p_i is a prime and $\alpha_i \in \mathbb{N}$ for each $i = 1, \cdots, n$.

Observe that

$$\sum_{d|n} \mu(d) = \sum_{d|\prod_{i=1}^k p_i} \mu(d).$$

Now we see

$$\sum_{d|\prod_{i=1}^k p_i} \mu(d) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j = \sum_{j=0}^{k} \binom{k}{j} (1)^{k-j} (-1)^j = (1-1)^k = 0.$$

$\square$

Proposition 2.1 (Möbius Inversion Formula)**.** *Let $f, g : \mathbb{N} \to \mathbb{C}$ be functions (we do not assume them to be multiplicative). If*

$$\sum_{d|n} g(d) = f(n),$$

holds if and only if

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = g(n).$$

Proof.

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \sum_{e|\frac{n}{d}} g(e).$$

$e|\frac{n}{d}$ if and only if $de|n$ thus obtain,

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \sum_{de|n} g(e).$$

In particular, we get the expression

$$= \sum_{de|n} \mu(d) g(e).$$

By reordering, we get

$$= \sum_{e|n} g(e) \sum_{d|\frac{n}{e}} \mu(d).$$

By Proposition 2.1, we get

$$\sum_{d|\frac{n}{e}} \mu(d) = 0$$

unless $e = n$. □

Proposition 2.2.

$$\sum_{d|n} \varphi(d) = n.$$

Proof. Consider $(\mathbb{Z}/n\mathbb{Z})^\times$. We know that

$$|(\mathbb{Z}/n\mathbb{Z})^\times| = \varphi(n).$$

Later □

Theorem 2.2.

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Proof. Using Proposition 2.2, we have,

$$\sum_{d|n} \mu(d) \frac{n}{d} = \varphi(n).$$

Dividing both sides by n and observe that $\mu(d) \neq 0$ if and only if d is a prime factor of n.

$$\begin{aligned} \frac{\varphi(n)}{n} &= \sum_{d|n} \frac{\mu(d)}{d}, \\ &= 1 \sum_{p|n} \frac{1}{p} + \sum_{p_1,p_2|n} \frac{1}{p_1 p_2} - \cdots . \end{aligned}$$

By the induction on the number of prime divisors of n, we get the statement. $\square$

Proposition 2.3. *We have the following properties of φ.*

1). $n|m \Rightarrow \varphi(n)|\varphi(m)$.

2). $\varphi(n)$ is even for $n \geq 3$.

3). $\varphi(2n) = \begin{cases} 2\varphi(n), & (2|n) \\ \varphi(n), & (2 \nmid n). \end{cases}$

4). φ is multiplicative.

5). $\varphi(mn) = \varphi(m)\frac{\varphi(n)d}{\varphi(d)}$ where $d = (m, n)$.

6). $\varphi(n^m) = n^{m-1}\varphi(n)$.

Proof. Exercise. $\square$

Theorem 2.3. *The following statements are equivalent.*

1). $\sum_{d|n} \Lambda(d) = \log n$

2). $\sum_{d|n} \mu(d) \log d = \Lambda(n)$.

And in particular $\sum_{d|n} \Lambda(d) = \log n$ holds.

Proof. The equivalence is a direct corollary of Möbius inversion formula (ie. Proposition 2.1). For the latter, Write

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}.$$

We have

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{k} \alpha_i \log p_i = \log(n).$$

$\square$

Notation 2.1. *Let $n \in \mathbb{N}$, suppose a prime p divides n. Then we denote $\alpha(p)$ to be the highest prime power factor of n.*

Theorem 2.4. *Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function. Then*

$$\sum_{d|n} f(d) = \prod_{p|n} \left(\sum_{i=0}^{\alpha(p)} f(p^i) \right).$$

In particular $\sum_{d|n} f(d)$ is also multiplicative.

Proof. Let $d|n$, then we have $d = \prod_{i=1}^{k} p_i^{\beta_i}$ for some $0 \leq \beta_i \leq \alpha(p_i)$. Since f is multipliative we have

$$f(d) = \prod_{i=1}^{k} f(p_i^{\beta_i}).$$

The second part is a direct result of the first part. □

Remark 2.1. *The Second Chebyschev Function ψ can be written as*

$$\psi(x) = \sum_{d \leq x} \Lambda(d).$$

This follows directly from the definition.

Definition 2.5 (Dirichlet Series)**.** *Let $f : \mathbb{N} \to \mathbb{C}$ be a function and $s \in \mathbb{C}$. We define*

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}.$$

For another arithmetic function $g : \mathbb{N} \to \mathbb{C}$, we define

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} + \sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} = \sum_{n \in \mathbb{N}} \frac{(f(n) + g(n))}{n^s}.$$

and

$$\left(\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \times \left(\sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} \right) = \sum_{n,m \in \mathbb{N}} \frac{(f(n)g(m))}{(nm)^s}.$$

$$= \sum_{t \in \mathbb{N}} \sum_{n|t} \frac{\left(f(n) g\left(\frac{n}{t}\right)\right)}{t^s}.$$

Recall the taylor expansion of $\ln x$ we get

$$\ln 2 = \sum_{n \in \mathbb{N}} \frac{(-1)^{n+1}}{n}.$$

Rearranging the following way

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots$$

we get this equals to $\frac{1}{2} \ln 2$.

Theorem 2.5. *Let $s \in \mathbb{C}$ be* $\mathrm{Re}(s) > 1$*, we have*

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}.$$

Proof.

$$\begin{aligned}
\zeta(s) \sum_{n \geq 1} \frac{\mu(n)}{n^s} &= \left(\sum_{n \in \mathbb{N}} \frac{1}{n^s} \right) \left(\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} \right) \\
&= \sum_{t \in \mathbb{N}} \frac{1}{t^s} \sum_{n | t} \mu(n) \\
&= 1.
\end{aligned}$$

□

Theorem 2.6. *For* $\mathrm{Re}(s) > 1$*, we have*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \in \mathbb{N}} \frac{\Lambda(s)}{n^s}.$$

From this we derive

$$\lim_{n \to \infty} \frac{\log n}{n^\varepsilon} = 0.$$

Proof.

$$\begin{aligned}
\zeta(s) \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} &= \left(\sum_{m \in \mathbb{N}} \frac{1}{m^s} \right) \left(\sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} \right), \\
&= \sum_{t \in \mathbb{N}} \frac{1}{t^s} \left(\sum_{n | t} \Lambda \left(\frac{t}{n} \right) \right), \\
&= \sum_{t \in \mathbb{N}} \frac{\log(t)}{t^s}, \\
&= -\zeta'(s).
\end{aligned}$$

□

Remark 2.2.

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \left| \frac{\Lambda(s)}{n^s} \right| &\leq \sum_{n \in \mathbb{N}} \frac{\log(n)}{n^\sigma}, \\
&<< \sum_{n \in \mathbb{N}} \frac{n^\varepsilon}{n^\sigma}, \\
&= \sum_{n \in \mathbb{N}} \frac{1}{n^{\sigma - \varepsilon}}.
\end{aligned}$$

We have $\lim_{n\to\infty} \frac{\log(n)}{n^{\varepsilon}} = 1$ *and the last equation is convergent if and only if* $\sigma - \varepsilon > 1$*. Thus we have* $\sigma > 1 + \varepsilon$*.*

Definition 2.6. *Let* $D \subseteq \mathbb{C}$ *be open. A meromorphic function is* $f : D \to \mathbb{C}$ *which is analytic on* D *except a discrete set of poles of* f*.*

Remark 2.3. $\frac{\zeta'(s)}{\zeta(s)}$ *is a meromorphic functions except* $s = 1$ *and where* $\zeta(s)$ *vanishes. Indeed, For general* $\frac{f}{g}$*, it is analytic if* f, g *are analytic and* $g \neq 0$*.*

1). $\zeta(s)$ *is analytic except* $s = 1$*.*

2). $\zeta'(s)$ *has a pole of order* 2 *at* $s = 1$*.*

3). $\zeta(s)$ *has a pole of order* 1 *at* $s = 1$*.*

Recall that for $|z| \geq 1$, we have,

1. $|z| \geq 1 \Rightarrow \sum_{n\in\mathbb{Z}_{\geq 0}} z^n = \frac{1}{1-z}$,
2. $\prod_{n\in\mathbb{N}}(1 + a_n)$ is convergent if $\sum_n a_n$ is absolutely convergent,
3. therefore $\prod_{n\in\mathbb{N}}(1 + a_n)$ is convergent if and only if $\prod_{n\in\mathbb{N}}(1 + |a_n|)$ is convergent.

Theorem 2.7. *Let* $f : \mathbb{N} \to \mathbb{C}$ *be a map.*
If f *is multiplicative and for* $\mathrm{Re}(s) > r_0, r_0 \in \mathbb{R}$ *then we have,*

$$\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s} = \prod_p \left(\sum_{\nu\geq 0} f(p^{\nu}) p^{-\nu s} \right).$$

If f *is completely multiplicative, then*

$$\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s} = \prod_p (1 - f(p)p^{-s})^{-1}.$$

Proof. Let $A(x) = \{n \in \mathbb{N} \mid \text{primes factors of } n \text{ are} \leq x\}$, then

$$\prod_{p\leq x} \sum_{\nu=0}^{\infty} f(p^{\nu}) p^{-\nu s} = \sum_{n\in A} \frac{f(n)}{n^s}.$$

Therefore,

$$\begin{aligned}
\left|\prod_{p\leq x}\sum_{\nu=0}^{\infty} f(p^{\nu})p^{-\nu s} - \sum_{n\in\mathbb{N}}\frac{f(n)}{n^s}\right| &= \left|\sum_{n\in A(x)}\frac{f(n)}{n^s} - \sum_{n\in\mathbb{N}}\frac{f(n)}{n^s}\right|, \\
&= \left|\sum_{n\in\mathbb{N}\setminus A(x)}\frac{f(n)}{n^s}\right|, \\
&\leq \sum_{n\notin A(x)}\frac{|f(n)|}{n^{\mathrm{Re}(s)}}, \\
&\leq \sum_{n>x}\frac{|f(n)|}{n^{\mathrm{Re}(s)}} \to 0.
\end{aligned}$$

The last limit is due to that it is a tail of a an absolutely convergent series. Since f is completely multipliative, we have

$$f(p^{\nu}) = (f(p))^{\nu}.$$

Therefore, we get,

$$\begin{aligned}
\prod_p\left(\sum_{\nu\in\mathbb{Z}_{\geq 0}}(f(p^{\nu})p^{-\nu s})\right) &= \prod_p\left(\sum_{\nu\in\mathbb{Z}_{\geq 0}}(f(p)p^{-s})^{-\nu}\right), \\
&= \prod_p\left(\frac{1}{1-f(p)p^{-s}}\right).
\end{aligned}$$

□

Example 2.4. *Take $f(n) = 1$ as above we get,*

$$\sum_{n\in\mathbb{N}}\frac{1}{n^s} = \zeta(s) = \prod_p\left(1-\frac{1}{p^s}\right)^{-1}, \mathrm{Re}(s) > 1.$$

Example 2.5.

$$\sum_{n\in\mathbb{N}}\frac{\chi(n)}{n^s} = \prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}, \mathrm{Re}(s) > 1.$$

Example 2.6.

$$\begin{aligned}
\sum_{n\in\mathbb{N}}\frac{\mu(n)}{n^s} &= \prod_p\left(1+\frac{\mu(p)}{p^s}\right), \\
&= \prod_p\left(1-\frac{1}{p^s}\right), \\
&= \frac{1}{\zeta(s)}.
\end{aligned}$$

Example 2.7. *Note that* $\phi(n) \leq n$. *Thus for* $\mathrm{Re}(s) > 2$, *we have,*

$$\sum_{n\in\mathbb{N}} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

2.2 Order of arithmetic functions

Definition 2.7. *Let* $f, g : \mathbb{N} \to \mathbb{C}$, *we denote,*

$$f(n) = O(g(n)),$$

if there is $K > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$n \geq n_0 \Rightarrow |f(n)| \leq K|g(n)|.$$

An alternative notation for this is $f(n) = O(g(n))$.

Definition 2.8. *We define following arithmetic functions,*

1). $\nu(n) := \sum_{p|n} 1$, *a number of primer divisors of* n,

2). $d(n) := \sum_{d|n} 1$, *the number of divisors of* n,

3). $\sigma(n) := \sum_{d|n} d$, *the sum of all the divisors of* n.

Lemma 2.1.

$$\nu(n) << \log(n).$$

Proof. Let $n = \prod_{i=1}^{k} p_i^{\alpha(p_i)}$. Then $\nu(n) = k$. Since $p_i \geq 2$, we have,

$$\begin{aligned}\log(n) &= \sum_{i=1}^{k} \alpha(p_i)\log(p_i),\\ &\geq k\log(2).\end{aligned}$$

Therefore $\nu(n) \leq \frac{\log(n)}{\log 2}$. □

Lemma 2.2.

$$\sum_{k=2}^{n} \frac{1}{k} \leq \log(n+1).$$

Proof. We know that

$$\int_1^n \frac{1}{t}dt = \log(n).$$

For $1 \leq k \leq t \leq k+1 \leq n$, we have,

$$\int_k^{k+1} \frac{1}{k+1}dt \leq \int_k^{k+1} \frac{1}{t}dt \leq \int_k^{k+1} \frac{1}{k}dt.$$

Thus we have,

$$\frac{1}{k+1} \leq \log(k+1) - \log(k) \leq \frac{1}{k}.$$

By telescopoing sum we get

$$\sum_{k=2}^{n} \frac{1}{k} \leq \log(n+1).$$

□

Lemma 2.3.

$$\sigma(n) << n(1+\log(n)) \sim n\log(n).$$

Proof.

$$\begin{aligned}
\sigma(n) = \sum_{d|n} \frac{n}{d}, \\
&= n\sum_{d|n} \frac{1}{d}, \\
&\leq n\left(1 + \sum_{d=2}^{n} \frac{1}{d}\right), \\
&\leq (1+\log(n)).
\end{aligned}$$

The last inequality is due to Lemma 2.2. □

Exercise 2.1. *Show that*

$$\sum_{k=1}^{n} \frac{1}{k} = \log(n) + O(1).$$

That is

$$\left|\sum_{k=1}^{n} \frac{1}{k} - \log(n)\right| << 1.$$

Hint: Replace $\frac{1}{t}$ by an increasing function and derive the similar inequality to Lemma 2.2.

Proof. By Lemma 2.2, we have,

$$\left|\sum_{k=1}^{n} \frac{1}{k} - \log(n)\right| \leq |1 + \log(n) - \log(n)| \leq 1.$$

□

Lemma 2.4.

$$d(n) \leq 2\sqrt{n}.$$

Proof. If $n = d_1 d_2$ then one of them must be less than or equal to $\sqrt{n}$. $\square$

We have an improved inequality,

Proposition 2.4. *for $\varepsilon > 0$, we have,*

$$d(n) << n^{\varepsilon}.$$

Proof. Recall that for $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, we have $d(n) = \prod_{i=1}^{k}(\alpha_i+1)$. In particular, we have,

$$\frac{d(n)}{n^{\varepsilon}} = \prod_{i=1}^{k} \frac{(\alpha_i+1)}{p_i^{\varepsilon\alpha_i}}.$$

Let $A = \{i \mid p_i^{\varepsilon} \geq 2\}$. Rcall that for $x \geq 1$, $x+1 \leq 2^x$ that is

$$\frac{x+1}{2^x} \leq 1.$$

Then,

$$\prod_{i \in A} \frac{\alpha_i+1}{p_i^{\varepsilon\alpha_i}} \leq \prod_{i \in A} \frac{\alpha_i+1}{2^{\alpha_i}} \leq 1.$$

For $p_i^{\varepsilon} < 2$, we observe,

$$p_i^{\varepsilon\alpha_i} = e^{\varepsilon\alpha_i \log(p_i)} \geq \varepsilon\alpha_i \log(p_i).$$

Therefore,

$$\begin{aligned}
\prod_{i \notin A} \frac{\alpha_i+1}{p_i^{\varepsilon\alpha_i}} &\leq \prod_{i \notin A} \left(\frac{\alpha_i}{p_i^{\varepsilon\alpha_i}} + 1\right), \\
&\leq \prod_{i \notin A} \left(\frac{\alpha_i}{\varepsilon\alpha_i \log(p_i)} + 1\right), \\
&\leq \prod_{i \notin A} \left(\frac{1}{\varepsilon \log(2^{\frac{1}{\varepsilon}})} + 1\right), \\
&\leq \prod_{i \notin A} \left(\frac{1}{\log(2)} + 1\right).
\end{aligned}$$

Combining two cases, we obtain the statement. $\square$

Notation 2.2. *Let $x \in \mathbb{R}$. We denote*

1. *the integer part $[x] \in \mathbb{Z}$ which is the greatest integer not exceeding x,*
2. *the fraction part $\{x\} = x - [x]$.*

Proposition 2.5.

$$\frac{\sum_{n \leq x} d(n)}{x} \sim \log(x).$$

Proof. By definition, we have,

$$\sum_{n\leq x} d(n) = \sum_{n\leq x}\sum_{d|n} 1.$$

$d|n$ if and only if there is e such that $de = n$. Using this we obtain,

$$\sum_{n\leq x} d(n) = \sum_{\substack{(e,d)\\ de\leq x}} 1 = \sum_{d\leq x}\sum_{e\leq \frac{x}{d}} 1 = \sum_{d\leq x}\left[\frac{x}{d}\right].$$

Using the definition of $[x]$, we have,

$$\sum_{n\leq x} d(n) = \sum_{d\leq x}\frac{x}{d} - \left\{\frac{x}{d}\right\} = x\sum_{d\leq x}\frac{1}{d} - \sum_{d\leq x}\left\{\frac{x}{d}\right\} = x(\log(x) + O(1)) + O(x).$$

Thus we obtain the statement. □

Definition 2.9. *Let $f : \mathbb{N} \to \mathbb{C}$, we say the average order of f is $g : \mathbb{R} \to \mathbb{C}$ if*

$$\frac{\sum_{n\leq x} f(n)}{x} \sim g(x).$$

Proposition 2.5 can be restated as follows.

Proposition 2.6. *The average order of d is $\log(x)$.*

Exercise 2.2. *Examine the following statements.*

1. *Is it true that $d(n) << \log n$?*
2. *Do we have $d(n) = O(n^{\varepsilon})$ for any $\varepsilon > 0$?*
3. *What is the optimal bound for $d(n)$?*

Theorem 2.8. *There exists $c_1, c_2 > 0$ such that*

$$c_1 \leq \frac{\varphi(n)\sigma(n)}{n^2} \leq c_2.$$

Proof. Recall that

$$\varphi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right),$$

and

$$\sigma(n) = \prod_{p|n}\frac{p^{\alpha(p)+1} - 1}{p - 1}.$$

Thus we obtain,

$$\frac{\sigma(n)}{n} = \frac{\prod_{p|n}(1 + p + \cdots + p^{\alpha(p)})}{\prod_{p|n} p^{\alpha(p)}} = \prod_{p|n}\left(\frac{\frac{1}{p^{\alpha(p)+1}} - 1}{\frac{1}{p} - 1}\right)$$

By multiplying two we get,

$$\frac{\varphi(n)\sigma(n)}{n^2} = \prod_{p|n}\left(1 - \frac{1}{p^{\alpha(p)+1}}\right) \leq 1.$$

On the other hand,

$$\prod_{p}\left(1 - \frac{1}{p^2}\right) \leq \frac{\varphi(n)\sigma(n)}{n^2}.$$

The left hand side is equal to $\frac{1}{\zeta(2)}$ which is $\frac{6}{\pi^2}$. □

Lemma 2.5. *For $x \geq 1$ and d an integer such that $d \leq x$, we have,*

$$\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right] + 1\right) = \frac{x^2}{d^2} + o\left(\frac{x}{d}\right).$$

Proof.

$$\begin{aligned}
\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right] + 1\right) &= \left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right)\left(\left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right) + 1\right), \\
&= \frac{x^2}{d^2} - 2\left\{\frac{x}{d}\right\}\frac{x}{d} + \left(\left\{\frac{x}{d}\right\}\right)^2 + \frac{x}{d} + \left\{\frac{x}{d}\right\}, \\
&= \frac{x^2}{d^2} + O(1)\cdot\frac{x}{d} + O(1).
\end{aligned}$$

□

Theorem 2.9. *The average order of φ is $\frac{3x}{\pi^2}$.*

Proof.

$$\begin{aligned}
\sum_{n\leq x}\varphi(n) &= \sum_{n\leq x} n \sum_{d|n}\frac{\mu(d)}{d}, \\
&= \sum_{\substack{d,e \\ de\leq x}} e\mu(d), \\
&= \sum_{d\leq x}\mu(d)\left(\sum_{e\leq\frac{x}{d}} e\right),
\end{aligned}$$

Since $\sum_{e\leq\frac{x}{d}} e$ is a sum of an arithmetic progression, we have,

$$\sum_{e\leq\frac{x}{d}} e = \frac{1}{2}\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right] + 1\right).$$

Substituting this, we obtain,

$$\sum_{n\leq x}\varphi(n)=\frac{1}{2}\sum_{d\leq x}\left(\mu(d)\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)\right).$$

Note that

$$\frac{x}{d}=\left[\frac{x}{d}\right]+\left\{\frac{x}{d}\right\}=\left[\frac{x}{d}\right]+O(1).$$

Using Lemma 2.5, that is,

$$\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)=\frac{x^2}{d^2}+o\left(\frac{x}{d}\right).$$

We then have,

$$\begin{aligned}\sum_{n\leq x}\varphi(n)&=\frac{1}{2}\sum_{d\leq x}\mu(d)\left(\frac{x^2}{d^x}+O\left(\frac{x}{d}\right)\right),\\&=\frac{x^2}{2}\sum_{d\leq x}\frac{\mu(d)}{d^2}+O\left(x\sum_{d\leq x}\frac{\mu(d)}{d}\right),\\&=\frac{x^2}{2}\left(\sum_{d\geq 1}\frac{\mu(d)}{d^2}-\sum_{d\geq x}\frac{\mu(d)}{d^2}\right)+O\left(x\sum_{d\geq x}\frac{\mu(d)}{d}\right),\\&=\frac{x^2}{2}\frac{1}{\zeta(2)}-\frac{x^2}{2}\sum_{d\geq x}\frac{\mu(d)}{d}^2+O\left(x\sum_{d\leq x}\frac{\mu(d)}{d}\right).\end{aligned}$$

We have,

$$\begin{aligned}\left|\sum_{d\geq x}\frac{\mu(d)}{d^2}\right|&\leq\sum_{d\geq x}\frac{1}{d^2}\\&<<\int_x^\infty\frac{dt}{t^2},\\&<<\frac{1}{x}.\end{aligned}$$

and,

$$\left|\sum_{d\leq x}\frac{\mu(d)}{d}\right|<<\ln x.$$

Using these we have,

$$\begin{aligned}\sum_{n\leq x}\varphi(n)&=\frac{x^2}{2}\frac{1}{\zeta(2)}+O(x)+O(x\ln x),\\&=\frac{x^2}{2\zeta(2)}+O(x\ln x).\end{aligned}$$

We conclude that
$$\frac{\sum_{n\leq x}\varphi(n)}{x^2} \overset{x\to\infty}{\to} \frac{1}{2\zeta(2)}.$$

In particular,
$$\frac{\sum_{n\leq x}\varphi(n)}{x} \sim \frac{x}{2\zeta(2)} = \frac{x\cdot 6}{2\pi^2}.$$

$\square$

2.3 Abel's Summation Formula

Recall the harmonic series $\sum_{n\in\mathbb{N}}\frac{1}{n}$ is divergent. Our next goal is to find such A_x that
$$\lim_{n\to\infty}\left(\sum_{n\leq x}\frac{1}{n} - A_x\right)$$
exists.

Remark 2.4 (Euler-Mascheroni constant)**.** *By taking $A_x = \log(x)$, we have*
$$\lim_{n\to\infty}\left(\sum_{n\leq x}\frac{1}{n} - \log(x)\right) = \psi,$$
exists. Such ψ is called Euler-Mascheroni constant.

Remark 2.5 (Euler Kronecer constant)**.** *Take $A_x = \log(x)$, we have*

We can show that
$$\psi = \lim_{s\to 1^+}\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right).$$

Hint first show that
$$\zeta(s) = \frac{1}{s-1} + \psi + o(s-1).$$

Proposition 2.7 (Abels' summation formula)**.** *Given $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ and $f(n)$ is continuously differentiable in $[1,x]$. Set*
$$A(x) := \sum_{n\leq x} a_n.$$

Then we have,
$$\sum_{n\leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

Proof. Observe that

$$a_n = A(n) - A(n-1).$$

Assume $x \in \mathbb{N}$. We substitute this to $\sum_{n \le x} a_n f(n)$, we get,

$$\begin{aligned}
\sum_{n \le x} a_n f(n) &= \sum_{n \le x} \left(A(n) - A(n-1)\right) f(n), \\
&= \sum_{n \le x} A(n) f(n) - \sum_{n \le x} A(n-1) f(n), \\
&= \sum_{n \le x} A(n) f(n) - \sum_{n \le x-1} A(n) f(n+1), \\
&= A(x) f(x) - \sum_{n \le x-1} A(n)(f(n+1) - f(n)), \\
&= A(x) f(x) - \sum_{n \le x-1} \int_n^{n+1} f'(t) dt, \\
&= A(x) f(x) - \sum_{n \le x-1} \int_n^{n+1} A(t) f'(t) dt, \\
&= A(x) f(x) - \int_1^x A(t) f'(t) dt.
\end{aligned}$$

For the case when $n \notin \mathbb{N}$ and $n > 1$,

$$\sum_{n \le x} a_n f(n) = \sum_{n \le [x]} a_n f(n).$$

Using the previous case, we get,

$$\sum_{n \le x} a_n f(n) = A([x]) f([x]) - \int_1^{[x]} A(t) f'(t) dt.$$

Remains to show that we can remove the brackets. To do so,

$$\begin{aligned}
\sum_{n \le x} a_n f(n) &= A([x]) f([x]) - \int_1^x A(t) f'(t) dt + \int_{[x]}^x A(t) f'(t) dt, \\
&= A([x]) f([x]) - \int_1^x A(t) f'(t) dt + A([x]) \int_{[x]}^x f'(t) dt, \\
&= A([x]) f([x]) - \int_1^x A(t) f'(t) dt + A([x]) f(x) - A([x])(f[x]), \\
&= A([x]) f(x) - \int_1^x A(t) f'(t) dt.
\end{aligned}$$

□

Corollary 2.1.

1. $\sum_{n\leq x} \frac{1}{n} = \ln x + \psi + o\left(\frac{1}{x}\right)$.

2. $\sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + o\left(\frac{1}{x^s}\right)$, *where* $\mathrm{Re}(s) > 0, s \neq 1$.

We also have the following equivalent forms of prime number theorem when $x \to \infty$.

$$\begin{aligned}\sum_{n\leq x} s(n) \sim x & \\ & \Leftrightarrow \pi(x) \sim \frac{x}{\ln x}, \\ & \Leftrightarrow \sum_{p\leq x} \ln p \sim x, \\ & \Leftrightarrow \sum_{n\leq x} \mu(n) = o(x).\end{aligned}$$

Proof. Consider $f(t) = \frac{1}{t^s}$ and $a_n = 1$ for all $n \in \mathbb{N}$.

$$\begin{aligned}\sum_{n\leq x} \frac{1}{n} &= \frac{[x]}{x} + \int_1^x \frac{[x]}{t^2} dt, \\ &= \frac{x - \{x\}}{x} + s\int_1^x \frac{t-\{t\}}{t^{s+1}} dt.\end{aligned}$$

When $s = 1$, we have,

$$\begin{aligned}\sum_{n\leq x} \frac{1}{n} &= 1 - \frac{\{x\}}{x} + \int_1^x \frac{t-\{t\}}{t^{s+1}} dt, \\ &= 1 + \ln x + o\left(\frac{1}{x}\right) - \int_1^x \frac{\{t\}}{t^2} dt, \\ \lim_{x\to\infty}\left(\sum_{n\leq x} \frac{1}{n} - \ln x\right) &= 1 - \int_1^\infty \frac{\{t\}}{t^2} dt, \\ &= \psi. \\ &= x^{1-s} + o\left(\frac{1}{x^s}\right) + \frac{sx^{1-s}}{1-s} - \frac{s}{1-s} - s^2 \int_{\frac{\{t\}}{t^{s+1}}} dt.\end{aligned}$$

Recall that

$$\left[\int \frac{1}{t^s} = \frac{t^{-s+1}}{1-s}\right]_1^x = \frac{x^{1-s}}{1-s} - \frac{1}{1-s}.$$

Using this we obtain,

$$
\begin{aligned}
&= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - x\int_1^x \frac{\{t\}}{t^{s+1}}dt, \\
x^{1-s}\left[1+\frac{1}{1-s}\right] &= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - s\int_1^x \frac{\{t\}}{t^{s+1}}dt \\
\int_1^\infty \frac{\{t\}}{t^{s+1}} &< \infty, \\
&\leq \int_1^\infty \frac{1}{t^{\mathrm{Re}(s)+1}} < \infty.
\end{aligned}
$$

As $x \to \infty$, the left hand side goes to $\zeta(s)$, for the right hand side, we get $= \frac{-s}{1-s} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}dt$, where $\mathrm{Re}(s) > 1$,

$$
\zeta(s) = \frac{-s}{1-s} - s\int \frac{\{t\}}{t^{s+1}}dt.
$$

Identity theorem for analytic function tells us the analytic continuation of Riemann zeta function is unique.

Remark 2.6. *It is an exercise that*

$$
\int_1^\infty \frac{\{t\}}{t^{s+1}}dt
$$

where $\mathrm{Re}(x) > 0$*. From Stein-Schakarchi 5.2, 5.3, we have*

$$
\sum f_n(z) \overset{unif.}{\to} f(z)
$$

is analytic where $\mathrm{Re}(x) > 0$ *and* $s \neq 1$*, also in this case,*

$$
\zeta(s) = \frac{-s}{1-s} - s\int \frac{\{t\}}{t^{s+1}}dt.
$$

holds.

Remark 2.7 (Exercise)**.** *Let* $M \in \mathbb{N}$ *and*

$$
\lim_{x\to\infty}\left(\sum_{\substack{n\leq x \\ (n,M)=1}} \frac{1}{n} - \frac{\phi(M)}{M}\ln x\right)
$$

exists.

Assume $\pi(x) \sim \frac{x}{\ln x}$, to show

$$
\theta(x) := \sum_{p\leq x} \ln p \sim x,
$$

Consider the following sequence

$$a_n = \begin{cases} 1 & n \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f(t) = \ln t.$$

Using Abel summation formula,

$$\sum_{p \le x} \ln p = \pi(x)\ln(x) - \int_1^\infty \frac{\pi(t)}{t} dt.$$

can be written as

$$\frac{\theta(x)}{x} = \frac{\pi(x)\ln x}{x} - \frac{1}{x}\int_1^x \frac{\pi(t)}{t} dt.$$

Remark 2.8 (Exercise)**.** *Use* $\frac{\pi(t)}{t} \sim \frac{1}{\ln t}$ *and*

$$\int_1^x \frac{dt}{\ln t} = o(x),$$

prove that

$$\lim_{x\to\infty} \frac{1}{x}\int_1^x \frac{\pi(t)}{t} dt = 0.$$

$$\begin{aligned} \int_2^{\sqrt{x}} \frac{dt}{\ln t} = \int_{\sqrt{x}}^x \frac{dt}{\ln t} &\le \frac{1}{\ln 2}\int_2^{\sqrt{x}} dt + \frac{1}{\ln\sqrt{x}}\int_{\sqrt{x}}^x dt, \\ &= \frac{\sqrt{x}-2}{\ln 2} + \frac{x-\sqrt{x}}{\ln\sqrt{x}}, \\ &= o(x). \end{aligned}$$

Remark 2.9.

$$\psi(x) = \sum_{n \le x} s(n) = \sum_{1 \le k, p^k \le x} \ln p.$$

Thus we see,

$$\psi(x) - \theta(x) = \sum_{2 \le k, p, p^k \le x} \ln p.$$

Also,

$$p^k \le x \Rightarrow k \le \frac{\ln x}{\ln p}.$$

Using $k \ge 2$*,*

$$p \le x^{\frac{1}{k}} \le \sqrt{x}, \forall k.$$

$$
\begin{aligned}
&\leq \sum_{p\leq\sqrt{x}} \ln p \left(\sum_{2\leq k\leq \frac{\ln x}{\ln p}} 1 \right), \\
&\leq \sum_{p\leq\sqrt{x}} \ln x, \\
&\leq \ln(x) \sum_{n\leq\sqrt{x}} 1, \\
&\leq \sqrt{x}\ln x. \Rightarrow \qquad \frac{\psi(x)}{x} = \frac{\theta(x)}{x} + \frac{o(\sqrt{x}\ln x)}{x}.
\end{aligned}
$$

Therefore we obtain,

$$\psi(x) \sim x \Leftrightarrow \theta(x) \sim x.$$

□

Remark 2.10. *As exercises, find the closed expressions for the following summations.*

$$\sum_{n\in\mathbb{N}} \frac{\mu(n)}{n}, \sum_{p\leq x} \frac{1}{p} = \ln\ln x + o(1).$$

2.4 Characters

Definition 2.10. *Let G be a finite group. A character is a group homomorphism $f : G \to \mathbb{C}^\times$.*

Remark 2.11. *Let us denote*

$$\hat{G} := \{f : G \to \mathbb{C}^\times \mid \textit{characters}\}.$$

If G is finite abelian then $|\hat{G}| = |G|$. Furthermore, such characters are linearly independent over $\mathbb{C}$.

Definition 2.11. *Let $q \in \mathbb{N}$ and $q \geq 3$. A Dirichlet character is a group homomorphism modulo q is a group homomorphism*

$$\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times.$$

Remark 2.12. *Given a Dirichlet character $\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$. We ca define a character $\chi : \mathbb{Z} \to \mathbb{C}^\times$ as follows.*

$$\chi(a) := \begin{cases} \chi'(\overline{a}), & (a,q) = 1, \\ 0, & (a,q) \neq 1. \end{cases}$$

From Remark 2.11, there are exactly $\varphi(q)$ many Dirichlet characters modulo q. Furthermore,

$$\chi(a)^{\varphi(q)} = \chi'(\overline{a})^{\varphi(q)} = \chi'(\overline{a}^{\varphi(q)}) = \chi'(\overline{1}) = 1.$$

In particular, images of χ are $\varphi(q)$-th roots of unity.

Example 2.8. *For $q = 3$,*

$$(\mathbb{Z}/3\mathbb{Z})^\times = \{\overline{1}, \overline{2}\} \to \mathbb{C}^\times.$$

We only have two characters, a trivial one and $\overline{2} \mapsto -1$.

Example 2.9. *For $q = 5$,*

$$(\mathbb{Z}/5\mathbb{Z})^\times = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$$

	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\chi_{1,5}(n)$	1	1	1	1
$\chi_{2,5}(n)$	1	-1	-1	1
$\chi_{3,5}(n)$	1	i	$-i$	-1
$\chi_{4,5}(n)$	1	$-i$	i	-1

$\chi_{1,5}$ *is called a principle/trivial character.*

Definition 2.12. *A character $\chi : \mathbb{Z} \to \mathbb{C}^\times$ is called*

1. *trivial if $\chi(g) = 1$ for all $g \in G$,*
2. *even if $\chi(-1) = 1$,*
3. *odd if $\chi(-1) = -1$.*

We also define these notions for characters $\chi_0 : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ accordingly if characters induces by $\chi : \mathbb{Z} \to \mathbb{C}^\times$ has these properties. Trivial characters are often denoted by χ_0.

Theorem 2.10.

$$\sum_{a \bmod q} \chi(a) = \begin{cases} \phi(q) & (\chi = \chi_0), \\ 0 & (\textit{otherwise}). \end{cases}$$

We also have,

$$\sum_{\chi \bmod q} \chi(a) = \begin{cases} \phi(a) & (\overline{a} = 1), \\ 0 & (\textit{otherwise}). \end{cases}$$

Proof. If χ is principle then the first assertion is clear. Suppose χ is not principle then there is $b \in \{1, \cdots, q\}$ such that $\chi(b) \neq 1$ and $(b, q) = 1$. Let

$$s = \sum_{a \bmod q} \chi(a).$$

Then by the definiion of group homomorphisms, we have,

$$\chi(b)s = s.$$

But $\chi(b) \in \mathbb{C}$, this means $s = 0$ as $\mathbb{C}$ is an integral domain.

For the second assertion, let $\overline{a} \neq 1$, then

$$\exists \chi' \bmod q, \text{ s.t. } \chi'(a) \neq 1.$$

Thus we get,

$$s = \sum_{\chi \bmod q} \chi(a), s \cdot \chi(a) = \sum_{\chi \bmod q} \chi\chi'(a) = s \Rightarrow s = 0.$$

The statement when $\overline{a} = 1$ follows from Remark 2.11. $\square$

Remark 2.13. *One can check in the table of Example 2.9 that Theorem 2.10 indeed holds.*

Exercise 2.3.

$$\sum_{\substack{\chi \bmod q, \\ \chi(-1)=1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} & (\overline{a} = 1, -1), \\ 0 & (\textit{otherwise}). \end{cases},$$

$$\sum_{\substack{\chi \bmod q, \\ \chi(-1)=-1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} & (\overline{a} = 1), \\ -\frac{\phi(a)}{2} & (\overline{a} = -1), \\ 0 & (\textit{otherwise}). \end{cases},$$

Obviously we have the following equalities.

$$\sum_{n \leq x} \chi(n) = \sum_{\substack{n \leq x \\ (n,q)=1}} \chi(n) = \sum_{n \leq kq} \chi(n) + \sum_{n=kq+1}^{x} \chi(n),$$

where k is the largest integer such that $kq \leq x$. Then we observe from Theorem 2.10

$$\sum_{n \leq kq} \chi(a) = k \left(\sum_{n=1}^{q} \chi(n) \right) = 0,$$

unless χ is trivial. Also we have,

$$\left| \sum_{n \leq x} \chi(n) \right| = \left| \sum_{\substack{kq+1 \leq n \leq x \\ (n,q)=1}} \chi(n) \right| \leq \sum_{\substack{kq+1 \leq n \leq x \\ (n,q)=1}} 1 \leq \sum_{\substack{kq+1 \leq n \leq kq+q \\ (n,q)=1}} 1 = \phi(q).$$

Thus we conclude,

$$\sum_{n \leq x} \chi(n) \leq \phi(n),$$

Exercise 2.4.

$$\sum_{n \leq x} \chi_0(n) = ?.$$

Theorem 2.11 (Pólya–Vinogradov)**.**

$$\sum_{n\leq x}\chi(n) << \sqrt{q}\ln q, (\chi \neq \chi_0 \mod q).$$

notice that the above expression is bounded by $\sqrt{q}\ln\ln q$*. Furthermore, this is uniform in* q *that is the constant does not depend on* q*.*

$$\sum_{n\geq 1}\frac{1}{n^s} = \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + O(x^{-s}) - s\int_1^x \frac{\{t\}}{t^{s+1}}dt.$$

For $\mathrm{Re}(s) > 1$, as $x \to \infty$ we have,

$$\sum_{n\geq 1}^{\infty}\frac{1}{n^s} = \frac{-s}{1-s} - s\int_1^{\infty}\frac{\{t\}}{t^{s+1}}dt.$$

The last expression is analytic since,

$$\sum\int_n^{n+1}\frac{\{t\}}{t}^{s+1}dt \overset{\text{uniformly}}{\to} \int_1^{\infty}\frac{\{t\}}{t^{s+1}}, \text{when } \mathrm{Re}(s) > 0.$$

Suppose $\zeta(s) \neq 0$, where $\mathrm{Re}(s) > 0$, then Euler product exists.

Theorem 2.12. *Set*

$$A(n) := \sum_{n\in\mathbb{N}} a_n.$$

Assume $A(x) := \sum_{n\leq x} a_n = O(x^{\delta})$*, then we have, for* $\mathrm{Re}(s) > \delta$*,*

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s} = s\int_1^{\infty}\frac{A(t)}{t^{s+1}}dt.$$

Hence the Dirichlet series converges for $\mathrm{Re}(s) > \delta$*.*

Proof.

$$\sum_{n\in\mathbb{N}}\frac{a_n}{n^s} = \frac{A(x)}{x^s} + s\int_1^x\frac{A(t)}{t^{s+1}}dt.$$

As $A(x) = O(x^{\delta})$ and $\mathrm{Re}(s) > \delta$, $\frac{A(x)}{x^s} = O(x^{\delta-\mathrm{Re}(s)})$. Therefore, as $x \to \infty$, we have,

$$\sum_{n\in\mathbb{N}}\frac{a_n}{n^s} = s\int_1^{\infty}\frac{A(t)}{t^{s+1}}dt.$$

Again using the assumption, we have,

$$\int_1^{\infty}\left|\frac{A(t)}{t^{s+1}}\right|dt << \int_1^{\infty} t^{\delta-\mathrm{Re}(s)-1}dt = \frac{t^{\delta-\mathrm{Re}(s)}}{\delta-\mathrm{Re}(s)}\Bigg|_0^{\infty} = \frac{1}{\delta-\mathrm{Re}(s)}.$$

Thus the integral is convergent. ∎

Definition 2.13. *For* $\mathrm{Re}(s) > 1$*, we define*

$$L(s,\chi) := \sum_{n\in\mathbb{N}} \frac{\chi(n)}{n^s}.$$

Remark 2.14. *Since* $\mathrm{Re}(s) > 1$ *and for any character* $\chi : \mathbb{Z} \to \mathbb{C}^\times$ *we have* $|\chi(n)| \leq 1$*,* $L(s,\chi)$ *is uniformly absolutely convergent.*

Example 2.10. *Let* $\chi : \mathbb{Z} \to \mathbb{C}^\times$ *be a non-principal character modulo* q *and set* $A(n) = \chi(n)$*. Recall that*

$$\sum_{n\leq x} \chi(n) \leq q.$$

Taking $A(n) = \chi(n)$ *and apply Theorem 2.12, we obtain, for* $\mathrm{Re}(s) > 0$*,*

$$L(s,\chi) = \sum_{n\in\mathbb{N}} \frac{\chi(n)}{n^s} = s\int_1^\infty \frac{\sum_{n\leq t}\chi(n)}{t^{s+1}}dt.$$

Since for $\mathrm{Re}(s) > 1, L(s,\chi)$ *is absolutely uniformly convergent. By Theorem 2.7, we have,*

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

If $\chi = \chi_0$*, a principal character, we have,*

$$L(s,\chi_0) = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 - \frac{1}{p^s}\right) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Using ζ *function, we have,*

$$L(s,\chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Theorem 2.13. ζ *has an analytic continuation for* $\mathrm{Re}(s) > 0$ *besides* $s = 1$*. For* $s = 1$ *we have a simple pole of residue* 1*.*

Proof. Recall from the proof of Corollary 2.1. We have for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{-s}{1-s} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}dt.$$

The right hand side of the equation is analytic when $\mathrm{Re}(s) > 0, s \neq 1$. When $s = 1$, we have,

$$\lim_{s\to 1^+} (s-1)\zeta(s) = \lim_{s\to 1^+} \left(s - s(s-1)\int_1^\infty \frac{\{t\}}{t^{s+1}}dt\right) = 1.$$

$\square$

Corollary 2.2. *For* $\mathrm{Re}(s) > 0$ *and* $s \neq 1$*, we have an analytic continuation of* $L(s, \chi_0)$ *where* χ_0 *is a principal character modulo* q*, which is*

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Obviously at $s = 1$*, it has a simple pole of residue* $\prod_{p|q} \left(1 - \frac{1}{p^s}\right)$*, which we can write as*

$$\mathrm{Res}(L(s, \chi_0), 1) = \frac{\varphi(q)}{q}.$$

Proof. A direct corollary of Theorem 2.13. $\square$

Suppose $\chi \neq \chi_0$, we have, analytic continuation of $L(s, \chi) \neq 0$is

Theorem 2.14. *Let* χ *be a non-principal character, then there is an analytic continuation of* $L(s, \chi)$ *for* $\mathrm{Re}(s) > 0$*.*

Proof. From Example 2.10, we have,

$$L(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = s \int_0^\infty \frac{\sum_{n \leq t} \chi(n)}{t^{s+1}} dt.$$

The right hand side is analytic. $\square$

Theorem 2.15.

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s}$$

is analytic in its range of convergence.

Remark 2.15. *We have the following conjecture.*

$$L\left(\frac{1}{2}, \chi\right) = 0, \chi \neq \chi_0.?$$

$$\zeta\left(\frac{1}{2}\right) = \frac{1}{1 - \sqrt{2}} \sum_{n \in \mathbb{N}} \frac{(-1)^{n-1}}{\sqrt{n}} \approx -1.46 \cdots$$

Definition 2.14. *A character is said to be quadratic if its values are either* ± 1*.*

Remark 2.16. *We have,*

$$\begin{aligned} L(s, \chi) &= 0 \quad \text{if } s = 0, -2, -4, \text{ when } \chi \text{ is an even character.} \\ L(s, \chi) &= 0 \quad \text{if } s = -1, -3, -5, \text{ when } \chi \text{ is an odd character.} \end{aligned}$$

Lemma 2.6. *For $\sigma > 1$ and $t \in \mathbb{R}$, we have,*

$$\mathrm{Re}(\ln(\zeta(\sigma + it))) = \sum \frac{\Lambda(n)}{n^{\sigma} \ln n} \ln(t \ln(n)).$$

And also,

$$\mathrm{Re}(3\ln(\zeta(\sigma)) + 4\ln(\zeta(\sigma + it)) + \ln(\zeta(\sigma + 2it))) \geq 0.$$

Proof.

$$\begin{aligned}
\zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \sigma > 0, \\
\ln(\zeta(s)) &= -\sum_p \ln(1 - p^{-s}) \\
&= \sum_{p,n} \frac{1}{np^{ns}}, \sigma > 1.
\end{aligned}$$

We have,

$$\sum_{n\geq 2} \frac{\Lambda(n)}{n^s \ln n} = \sum_{p,k,k\geq 1} \frac{\ln p}{p^{ks} \ln p^k} = \sum_{p,k} \frac{1}{kp^{ks}} = \sum_{n\geq 2} \Lambda(n).$$

$$\mathrm{Re}(3\ln\zeta(\sigma) + 4\ln\zeta(\sigma + it) + \ln\zeta(\sigma + 2it)) = \sum_{n\geq 2} \frac{\Lambda(n)}{n^{\sigma} \ln n}(3 + 4\cos(t\ln n) + \cos(2t\ln n)) \geq 0,$$

since

$$3 + 4\cos\theta + \cos 2\theta = 2(\cos\theta + 1)^2 \geq 0.$$

$$= \ln|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 0.$$

Thus we have,

$$\mathrm{Re}(\ln(z)) \leq$$

□

Theorem 2.16. *For $t \in \mathbb{R}\backslash\{0\}$, we have*

$$\zeta(1 + it) \neq 0.$$

Proof. Using Lemma 2.6, we have, Thus we get,

$$|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \geq 1.$$

Suppose $\zeta(1+it_0)=0$, for $t_0 \in \mathbb{R}\backslash\{0\}$. Suppose further that the order of zero is $m \in \mathbb{N}$. Then by looking at, and taking $\sigma \to 1+$

$$\underbrace{((\sigma-1)\zeta(\sigma))^3}_{\to\text{finite}}\underbrace{\left(\frac{\zeta(\sigma+it_0)}{(\sigma-1)^m}\right)^4}_{\to\text{non-zero}}\underbrace{((\sigma-1)^{4m-3}\zeta(\sigma+2it_0))}_{\to 0}.$$

Contradicts to that the abosolute value of above expression is at least 1. □

Theorem 2.17. $\frac{\zeta'(s)}{\zeta(s)}$ *has an analytic continuation to* $\mathrm{Re}(s)=1, s\neq 1$*. And for* $s=1$*, we have a simple pole of residue* -1*.*

Proof. We have,

$$(s-1)\zeta(s) = s - s(s-1)\int_1^\infty \frac{\{t\}}{t^{s+1}}dt.$$

Set

$$f(s) := 1-(s-1)\int_1^\infty \frac{\{t\}}{t^{s+1}}dt,$$

so that $(s-1)\zeta(s)=sf(s)$. We already have $f(s)$ is analytic when $\mathrm{Re}(s)>0$. Differentiating both sides we get,

$$(s-1)\zeta'(s)+\zeta(s) = sf'(s)+f(s).$$

Dividing both sides by $(s-1)\zeta(s)$ we get,

$$\frac{\zeta'(s)}{\zeta(s)}+\frac{1}{s-1}=\frac{sf'(s)+f(s)}{(s-1)\zeta(s)}.$$

The right hand side is analytic when ζ does not vanish. From Theorem 2.16, by letting $s=1+it$ for some $t\in\mathbb{R}\backslash\{0\}$, we get the desired analytic continuation. For $s=1$, we have

$$(s-1)\frac{\zeta'(s)}{\zeta(s)}=\frac{sf'(s)+f(s)}{\zeta(s)}-1.$$

Recall that $\zeta(s)$ has a pole at 1 and observe that $f(1)=1, f'(s)$ is finite thus the last statement follows. □

Theorem 2.18. *Let* χ *be a non-trivial non-real Dirichlet character of modulo* q*, then*

$$L(1,\chi)\neq 0.$$

Proof. Recall that

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where $\mathrm{Re}(s) > 1$. Now consider,

$$\log L(s,\chi) = \sum_{n,p} \frac{\chi(p^s)}{np^{ns}} = \sum_{n \geq 2} \frac{\Lambda(n)\chi(n)}{n^s \log n}.$$

Set $\zeta_{\varphi(q)} = e^{\frac{2\pi i}{\varphi(q)}}$. Then for any $n \in (\mathbb{Z}/q\mathbb{Z})^\times$, there is $n' = n(n,q,\chi)$ (ie n' depends on n, q, and χ) such that

$$\chi(n) = \zeta_{\varphi(q)}^{n'}$$

Note that for a cyclic group $G = \langle g \rangle$, and a character $\chi : G \to \mathbb{C}^\times \in \hat{G}$, there is $a \in \{1, \cdots, \varphi(q)\}$ such that

$$\chi(g) = \zeta_{\varphi(q)}^a.$$

We have, for $\sigma > 1$

$$\mathrm{Re}(\log L(\sigma,\chi)) = \sum_{n \geq 2} \frac{\Lambda(n) \cos\left(\frac{2\pi i}{\varphi(q)}\right)}{n^\sigma \log(n)}.$$

By Lemma 2.6, we have,

$$\mathrm{Re}(3\zeta(\sigma) + 4L(\sigma,\chi) + L(\sigma,\chi^2)) \geq 0,$$

therefore,

$$|\zeta(\sigma)^3 L(\sigma,\chi)^4 L(\sigma,\chi^2)| \geq 1.$$

Note that $\chi^2 \neq \chi$ as it is non-real and $L(\sigma,\chi^2)$ is analytic by Theorem 2.14. If $L(\sigma,\chi) = 0$ then

$$\zeta(\sigma)^3 L(\sigma,\chi)^4$$

has a zero of order at least 1. Thus this is a contradiction. □

Lemma 2.7. *For $k \in \mathbb{Z}_{\geq 0}$,*

$$\left(\frac{\sin\left(k+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}\right)^2 = 2k+1\sum_{j=1}^{2k} 2(2k+1-j)\cos j\theta.$$

Theorem 2.19. *Let $f : \mathbb{C} \to \mathbb{C}$ be a fnction such that*

i). f is analytic,

ii). $f \not\equiv 0$,

iii). $\log f(s) = \sum_{n\in\mathbb{N}} \frac{a_n}{n^s}$ *for* $a_n \geq 0$ *and* $\mathrm{Re}(s) \geq 1$*,*

iv). f *is analytic on* $\mathrm{Re}(s) = 1, s \neq 1$*, and it has a pole at* $s = 1$ *of order* e*.*

If $f(s) = 0$ *on the line* $\mathrm{Re}(s) = 1$*, then the order of zero is at most* $\frac{e}{2}$*.*

Proof. For $e \leq 2k-1$, let us define,

$$g(s) := f(s)^{2k+1}\prod_{j=1}^{2k} f(s+ijt_0)^{2(2k+1-j)} = f(s)^{2k+1}f(s+it_0)^{4k}\cdots.$$

Then $f(s)^{2k+1}$ has a pole of order $e(2k+1)$ and $\prod_{j=1}^{2k} f(s+ijt_0)^{2(2k+1-j)}$ has a zero of order $2k(2k+1)$ Note that

$$2k(2k+1) - e(2k+1) \geq 1.$$

Thus g has a zero at 1 of order at least 1, in particular, $g(1) = 0$.
Consider,

$$\begin{aligned}
\log g(\sigma) &= (2k+1)\log f(\sigma) + \sum_{j=1}^{2k} 2(2k+1-j)\log(f(\sigma+ijt_0)),\\
&= (2k+1)\sum_{n\in\mathbb{N}}\frac{a_n}{n^\sigma} + \sum_{j=1}^{2k} 2(2k+1-j)\left(\sum_{n\in\mathbb{N}}\frac{a_n}{a^{\sigma+ijt_0}}\right),\\
&= \sum_{n\in\mathbb{N}}\frac{a_n}{n^\sigma}\left(2k+1+\sum_{j=1}^{2k} 2(2k+i-j)e^{-ijt_0\log(n)}\right).\\
\mathrm{Re}(\log(g(\sigma))) &= \sum_{n\in\mathbb{N}}\frac{a_n}{n^\sigma}\left(2k+1+\sum_{j=1}^{2k} 2(2k+i-j)\cos(jt_0\log(n))\right).
\end{aligned}$$

Using Lemma 2.7, we get,

$$\mathrm{Re}\log(g(\sigma)) = \sum_{n\geq 1}\frac{a_n}{n^\sigma}\left(\frac{\sin\left(k+\frac{1}{2}\right)t_0\log(n)}{\sin\frac{t_0\log(n)}{2}}\right)^2 \geq 0.$$

Therefore $|g(\sigma)| \geq 1$ if $\mathrm{Re}\, g(\sigma) \geq 1$. □

Corollary 2.3. *For any Dirichlet character* χ *of modulo* q *we have,* $L(s,\chi)$ *is analytic over* $\mathrm{Re}(s) > 1$*. Furthermore* $L(s,\chi) \neq 0$ *on* $\mathrm{Re}(s) = 1, s \neq 1$*.*

Proof. The first assertion is due to Theorem 2.14. Let

$$f(s) := \prod_{\chi \bmod q} L(s,\chi).$$

Then,

$$\log(f(s)) = \sum_{\chi \bmod q} \log L(s,\chi) = \sum_{n,p} \frac{\sum_{\chi \bmod q} \chi(p^n)}{np^{ns}}, \mathrm{Re}(s) > 1.$$

Using Theorem 2.10, we get,

$$\log(f(s)) = \sum_{\substack{n,p \\ p^n \equiv 1 \bmod q}} \frac{\varphi(q)}{np^{ns}}.$$

Note f has a pole of order at most 1 at $\mathrm{Re}(s) = 1$ and its residue is

$$\mathrm{Res}(f(s),1) = \left(\lim_{s\to 1}(s-1)L(s,\chi_0)\right) \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} L(1,\chi) = \frac{\varphi(q)}{q} \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} L(1,\chi).$$

The right hand side is not 0 by Theorem 2.18. □

Theorem 2.20. *There are infinitely many primes.*

Proof. We have,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

$$\log \zeta(s) = \sum_{n,p} \frac{1}{np^{ns}} = \sum_p \frac{1}{p^s} + \sum_{n\geq 2} \frac{1}{np^{ns}}.$$

Since ζ has a pole at 1, so does its log. Observe that at $s = 1$,

$$\sum_p \frac{1}{p^s} + \sum_{n\geq 2} \frac{1}{np^{ns}} \leq \sum_p \sum_{n\geq 2} \frac{1}{p^n} = \sum_p \frac{1}{p^2} \frac{1}{1-\frac{1}{p}} = \sum_p \frac{1}{p(p-1)} << \sum_{n\in\mathbb{N}} \frac{1}{n^2}.$$

Thus $\lim_{s\to 1^+} \sum_p \frac{1}{p^s}$ must be infinity. □

Corollary 2.4. *For any Dirichlet character χ of modulo q we have, $L(s,\chi) \neq 0$ for* $\mathrm{Re}(s) = 1, s \neq 1$.

Proof. Let

$$f(s) := \prod_{\chi \bmod q} L(s,\chi).$$

□

Lemma 2.8. *Let p be a prime and a, q be coprimes such that $p \equiv a \bmod q$. Then we have,*

$$\sum_{\chi \bmod q} \chi(p^n)\overline{\chi(a)} = \begin{cases} \varphi(q), & (p^n \equiv a \bmod q), \\ 0, & (\text{otherwise}). \end{cases}$$

Proof. Since $\chi(a)$ is a $\varphi(q)$-th root of unity, we have, $\overline{\chi(a)} = \chi(a^{-1})$. Therefore,

$$\begin{aligned}\sum_{\chi \bmod q} \chi(p^n)\overline{\chi(a)} &= \sum_{\chi \bmod q} \chi(p^n)\chi(a^{-1}),\\ &= \sum_{\chi \bmod q} \chi(p^n a^{-1}),\\ &= \begin{cases}\varphi(q), & (p^n a^{-1} \equiv 1 \bmod q),\\ 0, & (\text{otherwise}).\end{cases}\end{aligned}$$

□

Theorem 2.21 (Dirichlet's Theorem)**.** *Let a, q be coprime. Then $(a+nq)_{n\in\mathbb{N}}$ contains infinitely many primes.*

Proof. Motivated by the alternative proof of the existence of infinitely many primes, examine,

$$\sum_{\chi \bmod q} \log L(s,\chi) = \sum_{n,p} \left(\frac{\sum_{\chi \bmod q} \chi(p^n)}{np^{ns}}\right),$$

where $\mathrm{Re}(s) > 1$. Using Theorem 2.10, we have,

$$\sum_{\chi \bmod q} \log L(s,\chi) = \sum_{\substack{n,p\\ p^n \equiv 1 \bmod q}} \frac{\varphi(q)}{np^{ns}} = \sum_{p^n \equiv 1 \bmod q} \frac{\varphi(q)}{p^s} + \sum_{\substack{p,n\geq 2\\ p^n \equiv 1 \bmod q}} \frac{\varphi(q)}{np^{ns}}.$$

Taking the log out we have,

$$\prod_{\chi \bmod q} L(s,\chi) = \exp\left(\varphi(q)\left(\sum_{p^n \equiv 1 \bmod q} \frac{1}{p^s} + \sum_{\substack{p,n\geq 2\\ p^n \equiv 1 \bmod q}} \frac{1}{np^{ns}}\right)\right).$$

Since for $\chi \neq \chi_0, L(1,\chi) \neq 0$ and by Corollary 2.2, we have,

$$\lim_{s\to 1^+} (s-1) \prod_{\chi \bmod q} L(s,\chi) = \frac{\varphi(q)}{q} \prod_{\substack{\chi \bmod q\\ \chi \neq \chi_0}} L(1,\chi).$$

By the same argument from Theorem 2.20, we see,

$$\sum_{\substack{p,n\geq 2\\ p^n \equiv \bmod q}} \frac{1}{np^{ns}} \leq \sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

However,

$$\lim_{s\to 1^+} (s-1) \exp\left(\varphi(q)\left(\sum_{p^n \equiv 1 \bmod q} \frac{1}{p^s} + \sum_{\substack{p,n\geq 2\\ p^n \equiv 1 \bmod q}} \frac{1}{np^{ns}}\right)\right) \neq 0$$

which is only possible when

$$\lim_{s\to 1^+} \sum_{p^n\equiv 1 \bmod q} \frac{1}{p^s} = \infty.$$

Together with Lemma 2.8, we derived the general statement. □

Theorem 2.22 (Bertrand's Postulate)**.** *For a sufficiently large $n \in \mathbb{N}$, there is a prime number inbetween n and $2n$.*

Proof. Consider the second Chebyschev function, and by Remark 2.1, we have,

$$T(x) := \sum_{l\le x} \psi\left(\frac{x}{l}\right) = \sum_{l\le x}\sum_{n\le \frac{x}{l}} \Lambda(n) = \sum_{\substack{l,n\\ ln\le x}} \Lambda(n).$$

That is

$$\sum_{l\le x} \psi\left(\frac{x}{l}\right) = \sum_{m\le x}\sum_{d|m} \Lambda(d).$$

By Theorem 2.3,

$$T(x) = \sum_{m\le x} \log(m).$$

Using the definition,

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= \sum_{l\le x} \psi\left(\frac{x}{l}\right) - 2\sum_{2l\le x} \psi\left(\frac{x}{2e}\right), \\ &= \sum_{l\le x} (-1)^{l+1}\psi\left(\frac{x}{l}\right), \\ &= \psi(x) - \psi\left(\frac{x}{2}\right) + \cdots . \end{aligned}$$

Again by Remark 2.1

$$x \le y \Rightarrow \psi(x) \le \psi(y).$$

In particular,

$$\psi(x) - \psi\left(\frac{x}{2}\right) \le \sum_{l\le x} (-1)^{l+1}\psi\left(\frac{x}{l}\right) \le \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right).$$

Recall Proposition 2.7, consider

$$A(x) = \sum_{n\le x} 1 = \lfloor x \rfloor,$$

and since $\log(x)$ is continuously differentiable on $[1, x]$, we have,

$$T(x) = \lfloor x \rfloor \log(x) - \int_1^x \frac{\lfloor x \rfloor}{x} dx = x\log(x) - x + O(\log(x)).$$

Therefore,

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= x\log(x) - x + O(\log(x)) - 2\left(\frac{x}{2}\log\left(\frac{x}{2}\right) - \frac{x}{2} + O(\log(x))\right), \\ &= \log 2 \cdot x + O(\log(x)). \end{aligned}$$

Combining with the previous result, we get,

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq (\log 2)x + O(\log(x)) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right).$$

Generalizing this we obtain,

$$\psi\left(\frac{x}{2^{n-1}}\right) - \left(\frac{x}{2^n}\right) \leq \log(2)\frac{x}{2^{n-1}} + O(\log(x)).$$

By induction, we obtain,

$$\begin{aligned} \psi(x) - \psi\left(\frac{x}{2^n}\right) &\leq (\log 2)x\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) + O(n\log(x)), \\ &\leq (2\log 2)x + O(n\log(x)). \end{aligned} \tag{$\psi 1$}$$

Take n to be the maximal such that $2^n \leq x$ that is $\lfloor \frac{x}{2^n} \rfloor = 1$. Then, $\psi\left(\frac{x}{2^n}\right) = 0$, thus,

$$\psi(x) \leq (2\log 2)x + O((\log(x))^2).$$

On the other hand,

$$\begin{aligned} \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) &\geq (\log 2)x + O(\log(x)), \\ \Rightarrow \psi(x) - \psi\left(\frac{x}{2}\right) &\geq (\log(2))x + O(\log(x)) - \psi\left(\frac{x}{3}\right) \end{aligned}$$

Using Inequality ($\psi 1$), we get,

$$\begin{aligned} \psi(x) - \psi\left(\frac{x}{2}\right) &\geq (\log 2)x + O(\log(x)) - (2\log 2)\frac{x}{3} + O((\log(x))^2), \\ &= (\log 2)\frac{x}{3} + O((\log(x))^2). \end{aligned} \tag{$\psi 2$}$$

Using Remark 2.9, that is

$$\psi(x) - \theta(x) = O(\sqrt{x}\log(x)),$$

together with Inequalities ($\psi 1$) and ($\psi 2$), we obtain,

$$\theta(x) - \theta\left(\frac{x}{2}\right) \geq (\log 2)\frac{x}{3} + O(\sqrt{x}\log(x)).$$

Thus the right hand side is greater than 0 if x is sufficiently large with coefficient, therefore by definition of θ, we have,

$$\theta(x) - \theta\left(\frac{x}{2}\right) = \sum_{\frac{x}{2} \leq p \leq x} \log(p) > 0.$$

$\square$

Theorem 2.23 (Chebyschev)**.**

2.5 Ikehara-Wiener Theorem and Its Applications

Theorem 2.24 (Ikehara-Wiener)**.** *Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of non-negative numbers and set,*

$$f(s) = \sum_{n\in\mathbb{N}} \frac{b_n}{n^s}.$$

Suppose

i). the series converges abslutely for $\mathrm{Re}(s) > 1$*,*

ii). f has an analytic continuation to $\mathrm{Re}(s) = 1$ *except $s = 1$,*

iii). f has a simple pole at $s = 1$ with residue $R \geq 0$.

Then we have,

$$\sum_{n\leq x} b_n = Rx + O(x).$$

That is

$$\lim_{x\to\infty} \frac{\sum_{n\leq x} b_n}{x} = R.$$

Lemma 2.9. *Let χ be a Dirichlet character modulo q, then*

$$L(s,\chi)^{-1} = \sum_{n\in\mathbb{N}} \frac{\chi(n)\mu(n)}{n^s}.$$

Proof. We immitate the proof of Theorem 2.5. Using Theorem 2.1,

$$L(s,\chi)\left(\sum_{n\in\mathbb{N}} \frac{\mu(n)\Lambda(n)}{n^s}\right) = \sum_{t\in\mathbb{N}}\sum_{n|t} \frac{\chi\left(\frac{t}{n}\right)\chi(n)\mu(n)}{t^s} = \sum_{t\in\mathbb{N}} \chi(t) \sum_{n|t} \mu(n) = 1.$$

□

Theorem 2.25. *Let χ be a character then, for* $\mathrm{Re}(s) > 1$*,*

$$-\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{n\in\mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s}.$$

Proof. By Lemma 2.9 and Proposition 2.1

$$\begin{aligned}
-\frac{L'(s,\chi)}{L(s,\chi)} &= \left(\sum_{t\in\mathbb{N}} \frac{\chi(n)\log(t)}{t^s}\right)\left(\sum_{n\in\mathbb{N}} \frac{\mu(n)\chi(n)}{n^s}\right), \\
&= \sum_{t\in\mathbb{N}} \chi(t) \sum_{n|t} \frac{\mu\left(\frac{t}{n}\right)\log(n)}{t^s}, \\
&= \sum_{n\in\mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s}.
\end{aligned}$$

□

Definition 2.15.
$$\psi(x,q,a) = \sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n).$$

Proposition 2.8. *As $x\to\infty$, we have,*
$$\psi(x,q,a) \sim \frac{x}{\varphi(q)}.$$

Proof. Consider
$$\sum_{\chi \bmod q} \overline{\chi}(a)\left(\sum_{n\in\mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s}\right) = \sum_{n\in\mathbb{N}} \frac{\Lambda(n)}{n^s}\left(\sum_{\chi \bmod q} \chi(a^{-1}n)\right)$$

Using Lemma 2.8, we get,
$$-\frac{1}{\varphi(q)}\sum_{\chi \bmod q} \overline{\chi}(a)\frac{L'(s,\chi)}{L(s,\chi)} = \varphi(q)\sum_{n\equiv a \bmod q} \frac{\Lambda(n)}{n^s}.$$

Recall from Corollary 2.2, we have,
$$\operatorname{Res}(L(s,\chi_0),1) = \frac{\varphi(q)}{q}.$$

And by Theorem 2.14, $L(s,\chi)$ is analytic at $s=1$ for $\chi\neq\chi_0$. Also using Theorem 2.16, we have
$$\chi\neq\chi_0 \Rightarrow \frac{L'(s,\chi)}{L(s,\chi)} \text{ is analytic for } \operatorname{Re}(s)\ge 1.$$

Furthermore, we have Theorem 2.17, we have
$$\operatorname{Res}\left(\frac{\zeta'(s)}{\zeta(s)},1\right) = -1.$$

Combining these, we get,
$$\lim_{s\to 1}(s-1)\left(-\frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} \frac{L'(s,\chi)}{L(s,\chi)} - \frac{1}{\varphi(q)}\frac{L'(s,\chi_0)}{L(s,\chi_0)}\right) = \frac{1}{\varphi(q)}.$$

Now Using Theorem 2.24, and apply $b_n = \Lambda(n)$ $f = \varphi(q)\sum_{n\equiv a \bmod q}\frac{\Lambda(n)}{n^s}$, we get,
$$\psi(x,q,a) = \sum_{n\le x}\Lambda(n) = \frac{1}{\varphi(q)}x + O(x).$$

□

2.6 $L(s,\chi) \neq 0$ for Quadratic Characters

Lemma 2.10. *Let $f := \sum_{d|n} \chi(d)$, where χ is a character, then,*

$$\forall n \in \mathbb{N}, n \textit{ is a perfect square} \Rightarrow f(n) \geq 0, f(n) \geq 1.$$

Proof. Recall $n = \prod_{p|n} p^{\alpha(p)}$. Using this we have,

$$\begin{aligned}
\sum_{d|n} \chi(d) &= \prod_{p|n} \left(\sum_{k=0}^{\alpha(p)} \chi(p)^k \right), \\
&= \begin{cases} 1 \quad \chi(p) = 0, \\ \prod_{p|n}(1+\alpha(p)) \quad \chi(p) = 1, \\ \prod_{p|n} \left(\frac{(1-(-1)^{\alpha(p)+1})}{2} \right) \quad \chi(p) = -1. \end{cases}
\end{aligned}$$

Note that if $\alpha(p)$ are all even for $p|n$ (ie. n is a perfect square), we have the last part of the cases equals to 1. Thus we have $f(n) \geq 1$. □

Theorem 2.26. *Let $f(n) = \sum_{d|n} \chi(d)$ for some character. Then we have,*

$$\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = 2\sqrt{x} L(1,\chi) + o(1).$$

Proof.

$$\begin{aligned}
\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} &= \sum_{n \leq x} \left(\frac{\sum_{d|n} \chi(d)}{\sqrt{n}} \right), \\
&= \sum_{\substack{d,e \\ de \leq x}} \frac{\chi(d)}{\sqrt{de}}, \\
&= \sum_{\substack{d,e \leq x \\ d \leq \sqrt{x}}} \frac{\chi(d)}{\sqrt{de}} + \sum_{\substack{de \leq x \\ d > \sqrt{x}}} \frac{\chi(d)}{\sqrt{de}}, \\
&= \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(\sum_{e \leq \frac{x}{d}} \frac{1}{\sqrt{e}} \right) + \sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left(\sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{(d)}} \right).
\end{aligned}$$

Recall that from Proposition 2.7,

$$\sum_{m \leq x} \frac{1}{\sqrt{m}} = 2\sqrt{x} + B + o\left(\frac{1}{\sqrt{x}} \right),$$

where B is some constant as $x \to \infty$.

Let $x, y \in \mathbb{R}$, such that $x < y$, we have,

$$\sum_{x<d\leq y} \frac{\chi(d)}{\sqrt{d}} = \sum_{d\leq y} \frac{\chi(d)}{\sqrt{d}} - \sum_{d\leq x} \frac{\chi(d)}{\sqrt{(}d)},$$

$$\sum_{d\leq x} \frac{\chi(d)}{\sqrt{d}} = \frac{\sum_{d\leq x}\chi(d)}{\sqrt{x}} + \frac{1}{2}\int_1^x \frac{\sum_{d\leq t}\chi(d)}{t^{\frac{1}{2}}},$$

$$= o\left(\frac{1}{\sqrt{x}}\right).$$

Using these equations, we have,

$$\sum_{e\leq\sqrt{x}} \frac{1}{\sqrt{e}} \left(\sum_{\sqrt{x}<d\leq\frac{x}{e}} \frac{\chi(d)}{\sqrt{(}d)} \right) = \sum_{e\leq x} \frac{1}{\sqrt{e}} \left(o\left(\frac{1}{x^{\frac{1}{r}}}\right)\right),$$

$$= \left(o\left(\frac{1}{x^{\frac{1}{r}}}\right)\right) \sum_{e\leq x} \frac{1}{\sqrt{e}},$$

$$\sum_{e\leq\sqrt{x}} \frac{1}{\sqrt{x}} << \int_1^{\sqrt{x}} \frac{1}{\sqrt{t}} dt = x^{\frac{1}{4}}.$$

Thus we conclude,

$$\sum_{e\leq\sqrt{x}} \frac{1}{\sqrt{e}} \left(\sum_{\sqrt{x}<d\leq\frac{x}{e}} \frac{\chi(d)}{\sqrt{(}d)} \right) = o(1).$$

We also have,

$$\begin{aligned}
\sum_{d\le\sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(\sum_{e\le \frac{x}{d}} \frac{1}{\sqrt{e}}\right) &= \sum_{d\le\sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(2\sqrt{\frac{x}{d}} + B + o\left(\sqrt{\frac{d}{x}}\right)\right), \\
&= 2\sqrt{x}\sum_{d\le\sqrt{x}} \frac{\chi(d)}{d} + B\sum_{d\le\sqrt{x}} \frac{\chi(d)}{\sqrt{d}} + o\left(\frac{1}{\sqrt{x}}\sum_{d\le\sqrt{x}} \chi(d)\right). \\
2\sqrt{x}\sum_{d\le\sqrt{x}} \frac{\chi(d)}{d} &= 2\sqrt{x}\left(\sum_{d\ge 1}\frac{\chi(d)}{d} - \sum_{d>\sqrt{x}}\frac{\chi(d)}{d}\right), \\
&= 2\sqrt{x}L(1,\chi) - 2\sqrt{x}\sum_{d>\sqrt{x}}\frac{\chi(d)}{d}, \\
&= 2\sqrt{x}L(1,\chi) - 2\sqrt{x}o\left(\frac{1}{\sqrt{x}}\right), \\
&= 2\sqrt{x}L(1,\chi) + o(1). B\sum_{d\le\sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \le o\left(\frac{B}{x^{\frac{1}{4}}}\right) = o(1). \\
\sum_{d\le\sqrt{x}} \chi(d) \le q? = o(1).
\end{aligned}$$

□

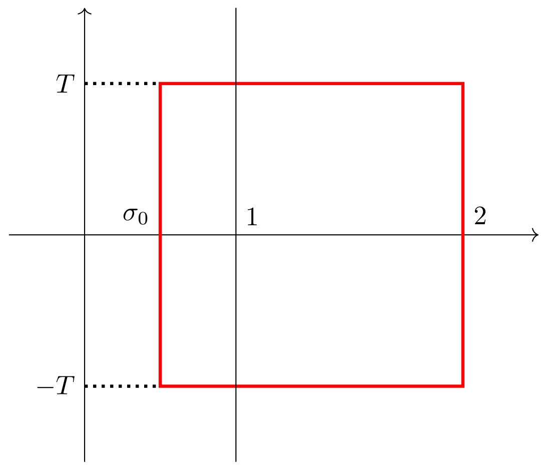
Corollary 2.5. *$L(1,\chi)\neq 0$ for a quadratic character χ.*

Proof. To derive a contradiction, assume $L(1,\chi)=0$. Then, from Theorem 2.26 and Lemma 2.10

$$o(1) = \sum_{n\le x}\frac{f(n)}{\sqrt{n}} \ge \sum_{\substack{n\le x \\ n \text{ is a square}}} \frac{1}{\sqrt{n}} = \sum_{m\le\sqrt{x}}\frac{1}{m} = \log\sqrt{x} + o(1).$$

Thus we obtain $o(1)\ge \log\sqrt{x} + o(1)$ which is a cotradiction. □

Lemma 2.11. *Consider the following rectangle,*



where $\sigma_0 = 1 - \log T$. On the boundary of the rectangle above, we have,

1. *$|\zeta(s)| = O(\log T), T \to \infty$,*
2. *$|\zeta'(s)| = O((\log T)^2)$.*

Proof. The first statement is due to the first derivative of ζ and it is assgined as an exercise. For the second part, for $\mathrm{Re}(s), \sigma > 1$, we have,

$$\begin{aligned}
\zeta(s) &= \sum_{n\geq 1} \frac{1}{n^s}, \\
&= \sum_{n\leq T} \frac{1}{n^s} + \sum_{n>T} \frac{1}{n^s}, \\
&= \sum_{n\leq T} \frac{1}{n^s} - \frac{\lfloor T \rfloor}{T^s} + s\int_1^\infty \frac{\lfloor t \rfloor}{t^{s+1}} dt, \\
&= \sum_{n\leq T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} + \frac{\{T\}}{T^s} - s\int_T^\infty \frac{\{u\}}{u^{s+1}} du.
\end{aligned}$$

Note that the right hand side is analytic where $\mathrm{Re}(s) > 0$ and $s \neq 1$. Now we will estimate the last equation above on the boundary of the rectangle above.

$$\begin{aligned}
\left| \sum_{n\leq T} \frac{1}{n^s} \right| &<< \int_1^T \frac{du}{u^{\mathrm{Re}(s)}}, \\
&\leq \int_1^T \frac{du}{u^{\sigma_0}}, \\
&= \frac{T^{1-\sigma_0}}{1-\sigma_0}, \\
&<< \log T.
\end{aligned}$$

Observe that ,

$$1-\sigma_0=\frac{1}{\log T}, T^{1-\sigma_0}=T^{\frac{1}{\log T}}=\exp(\log T^{\frac{1}{\log T}})=\exp(1)=e.$$

We also have,

$$\begin{aligned}\left|\frac{T^{1-s}}{s-1}\right| &= \frac{T^{1-\mathrm{Re}(s)}}{|s-1|},\\ &\leq \frac{T^{1-\sigma_0}}{|s-1|},\\ &\leq \frac{1}{|s-1|},\\ &\leq \frac{1}{\sigma_0-1}=\log T.\end{aligned}$$

Also consider,

$$\left|\frac{\{T\}}{T}\right| \leq \frac{1}{T^{\mathrm{Re}(s)}} \leq 1.$$

Finally we have,

$$\begin{aligned}\left|-s\int_T^\infty \frac{\{u\}}{u^{s+1}}du\right| &\leq |s|\int_T^\infty \frac{du}{u^{\mathrm{Re}(s)+1}},\\ &= \frac{|s|}{-\mathrm{Re}(s)u^{\mathrm{Re}(s)}}\Bigg|_T^\infty,\\ &= \frac{|s|}{\mathrm{Re}(s)T^{\mathrm{Re}(s)}},\\ &\leq \frac{|s|}{T^{\mathrm{Re}(s)}},\\ &\leq \frac{\sqrt{2^2+T^2}}{T^{\sigma_0}},\\ &<< \frac{T}{T^{\sigma_0}}=T^{1-\sigma_0}=e.\end{aligned}$$

For the second part, for $\mathrm{Re}(s)>0, s\neq 1$,

$$\begin{aligned}\zeta'(s)=&\sum_{n\leq T}\frac{-\log n}{n^s}+\frac{T^{1-s}(-\log T)}{s-1}+T^{1-s}\frac{-1}{(s-1)^2}\\ &+\frac{\{T\}\log T}{T^s}-\int_T^\infty\frac{\{u\}}{u^{s+1}}du+s\int_T^\infty\frac{\{u\}\log u}{u^{s+1}}du.\end{aligned}$$

This is obtained simply differentiating the equation,

$$\zeta(s)=\sum_{n\leq T}\frac{1}{n^s}+\frac{T^{1-s}}{s-1}-\frac{\{T\}}{T^s}-s\int_T^\infty\frac{\{u\}}{u^{s+1}}du.$$

The statement can be shown using all the estimates obtained to show the first part. ☐

Theorem 2.27 (Complex Mean Value Theorem)**.** *Let $f : \Omega \to \mathbb{C}$ be an analytic, where Ω is a convex open set. Let $a, b \in \Omega$, then there exists $z_1, z_2 \in (a, b)$ such that*

$$\mathrm{Re}(f'(z_1)) = \mathrm{Re}\left(\frac{f(b) - f(a)}{b - a}\right), \mathrm{Im}(f'(z_2)) = \mathrm{Im}\left(\frac{f(b) - f(a)}{b - a}\right).$$

Theorem 2.28. *There exists constants c_1, c_2 such that*

$$1 - \frac{c_1}{(\log T)^9} \leq \sigma \leq 2, |\zeta(s)| > \frac{c_2}{(\log T)^7},$$

where $1 < |\,\mathrm{Im}(s)| \leq T$.

Proof. Recall Lemma 2.6,

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1, \sigma > 1.$$

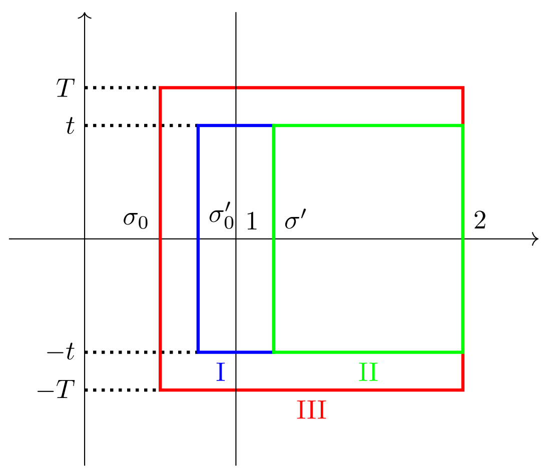
Thus we have,

$$|\zeta(\sigma + it)|^4 \geq |\zeta(\sigma)|^{-3} |\zeta(\sigma + 2it)|^{-1}, \tag{$*$}$$

Consider

$$\sigma_0' = 1 - \frac{c_1}{(\log T)^9}, \sigma' = 1 + \frac{c_1}{(\log T)^9}$$

c_1 is a positive constant which will be adjusted later.



Fix the domain II where $\sigma' \leq \sigma \leq 2$, $1 \leq |t| \leq T$.

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}du, \mathrm{Re}(s) > 0, s \neq 1.$$

$$\zeta(\sigma) = \frac{\sigma - 1 + 1}{\sigma - 1} - \sigma \int_1^\infty \frac{\{u\}}{u^{t+1}}dt.$$

$$\zeta(\sigma) = 1 + \frac{1}{\sigma - 1} + o(1),$$

$$\zeta(\sigma) << \frac{1}{\sigma - 1} \text{ as } \sigma \to 1^+,$$

$$\Rightarrow \zeta(\sigma)^{-1} >> (\sigma - 1), \sigma \to 1^+$$

On the domain II, we have, $\sigma_0' < \sigma \leq 2, 1 \leq |t| \leq T$, and thus

$$|\zeta(\sigma + 2it)| = O(\log T).$$

Substituting these to Equation $(*)$, we have,

$$|\zeta(\sigma + it)|^4 >> (1-\sigma)^3 (\log T)^{-1},$$

$$>> \frac{c_1^3}{(\log T)^{27}} (\log T)^{-1}.$$

$$\Rightarrow |\zeta(\sigma + it)| >> \frac{c_1^{\frac{3}{4}}}{(\log T)^7}, \tag{D2}$$

in the domain II.

On the domain I, similarly use Theorem 2.27 with

$$a = \sigma' + it, b = \sigma + it,$$

where,

$$1 - \frac{c_1}{(\log T)^9} \leq \sigma \leq \sigma'.$$

We have,

$$\frac{\zeta(\sigma' + it) - \zeta(\sigma + it)}{\sigma' - \sigma} = \mathrm{Re}(\zeta(z_1)) + \mathrm{Im}(\zeta(z_2)).$$

Furthermore,

$$\zeta(\sigma' + it) - \zeta(\sigma + it) = O((\sigma' - \sigma)(\log T)^2).$$

By the definition of σ', we have,

$$\zeta(\sigma' + it) = \zeta(\sigma + it) + O\left(\frac{c_1}{(\log T)^7}\right). \tag{D1}$$

Combining Equations $(D2)$ and $(D1)$, there are some positive constants A_1, A_2 such that

$$|\zeta(\sigma + it)| \geq |\zeta(\sigma' + it)| - \frac{A_1 c_1}{(\log T)^7},$$

$$\geq \frac{A_2 c_1^{\frac{3}{4}}}{(\log T)^7} - \frac{A_1 c_1}{(\log T)^7}.$$

Now take c_1 sufficiently small that in the region,

$$\sigma_0' \leq \sigma \leq \sigma', 1 \leq |t| \leq T,$$

we have,

$$|\zeta(\sigma + it)| \geq \frac{c_2}{(\log T)^7}.$$

Again combining this with Inequality $D1$, we obtain the statement. $\square$

Corollary 2.6. *There exists some constant C such that*

$$\frac{\zeta'(s)}{\zeta(s)} = O((\log T)^9).$$

For $1 - \frac{c}{(\log T)^9} \leq \mathrm{Re}(s) \leq 2$ and $1 \leq |\mathrm{Im}(s)| \leq T$, we have,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}.$$

Notation 2.3. *Let $c \in \mathbb{R}$ and $f : \mathbb{C} \supseteq \Omega \to \mathbb{C}$. Then we denote,*

$$\int_{(c)} f(s) ds = \lim_{T \to \infty} \int_{c-iT}^{c+iT} f(s) ds.$$

Lemma 2.12.

$$\frac{1}{2\pi i} \int_{(c)} y^s \frac{ds}{s} = \begin{cases} 0, & (0 < y < 1), \\ \frac{1}{2}, & (y = 1), \\ 1, & (y > 1). \end{cases}$$

Lemma 2.13. *Set,*

$$f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Suppose this is absolutely convergence in $\mathrm{Re}(s) > c - \varepsilon$ for some $c \in \mathbb{R}, \varepsilon > 0$ and x is not an integer, then

$$\sum_{n<x} a_n = \frac{1}{2\pi i} \int_{(c)} f(s) \frac{x^s}{s} ds.$$

Proof. Since $f(s)$ is a limit of a continuous function series which is absolutely convergent on $\mathrm{Re}(s) > c - \varepsilon$, thus this is uniformly convergent in a compact subset of its domain of convergence. Thus we can exchange the integral and the sum namely,

$$\frac{1}{2\pi i} \int_{(c)} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^s} \right) \frac{x^s}{s} ds = \frac{1}{2\pi i} \sum_{n=1}^{\infty} a_n \int_{(c)} \left(\frac{x}{n} \right)^s \frac{ds}{s}.$$

By Lemma 2.12, and $\frac{x}{n} > 1 \Leftrightarrow n < x$, we derive the statement. $\square$

Corollary 2.7. *Let $c > 1$ and x be a non-integer, then we have,*

$$\psi(x) = \frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds.$$

Proof. Follows from Remark 2.1 and Theorem 2.6 and apply Lemma 2.13. $\square$

Theorem 2.29. *There is a positive constant $c' > 0$ such that*

$$\psi(x) = x + O(x\exp(-c'\log(x))).$$

Proof. Let $a = 1 + \frac{c}{\log T}$ for some $c, T > 1$ where T is sufficiently large and will be chosen later. By Corollary 2.7, we have,

$$\psi(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds + \frac{1}{2\pi i}\int_{(a)} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds - \frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds.$$

Claim 1.

$$\left|\frac{1}{2\pi i}\int_{(a)} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds - \frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds\right| < \begin{cases} x^a \min\{1, T^{-1}, |\log x|^{-1}\}, & (x \neq 1), \\ \frac{a}{T}, & (x = 1). \end{cases}$$

Proof: $\blacksquare$

Let

$$\psi(x) = \underbrace{\frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds}_{=I_1} + \underbrace{\frac{1}{2\pi i}\int_{(a)} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds - \frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds}_{=I_2}.$$

Again using Corollary 2.7 and the claim, we have,

$$\begin{aligned} I_2 &= \frac{1}{2\pi i}\left(\int_{(a)} - \int_{a-iT}^{a+iT}\right)\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right)\frac{x^s}{s}ds, \\ &= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\Lambda(n)\left(\left(\int_{(a)} - \int_{a-iT}^{a+iT}\right)\left(\frac{x}{n}\right)^s\frac{ds}{s}\right), \\ &= \sum_{n=1}^{\infty}\Lambda(n)O\left(\left(\frac{x}{n}\right)^a \min\left(1, T^{-1}\left|\log\left(\frac{x}{n}\right)\right|^{-1}\right)\right), \\ &= O\left(\sum_{n=1}^{\infty}\Lambda(n)\left(\frac{x}{n}\right)^a \min\left(1, T^{-1}\left|\log\left(\frac{x}{n}\right)\right|^{-1}\right)\right). \end{aligned}$$

Fix x and let us consider two ranges $\left[\frac{x}{2}, \frac{3x}{2}\right]$ and $\left(-\infty, \frac{x}{2}\right) \cup \left(\frac{3x}{2}, \infty\right)$. If n sits in the second range

$$\frac{\log\frac{n}{x}}{\frac{n}{x} - 1} \geq 2\log\frac{3}{2}.$$

And if n sits in the first interval, we have,

$$\left|\log \frac{x}{n}\right| << \left|\frac{n}{x} - 1\right|^{-1}.$$

Using this we have,

$$\begin{aligned}\sum_{n\in\left[\frac{x}{2},\frac{3x}{2}\right]} \Lambda(n)\left(\frac{x}{n}\right)^{a} \min\left(1, T^{-1}\left|\log\left(\frac{x}{n}\right)\right|^{-1}\right) &<< \sum_{n\in\left[\frac{x}{2},\frac{3x}{2}\right]} \Lambda(n)\left(\frac{x}{n}\right)^{a}\left|\frac{n}{x}-1\right|^{-1} \\ &<< \frac{\log x}{T} \sum_{n\in\left[\frac{x}{2},\frac{3x}{2}\right]} \frac{1}{\left|\frac{n}{x}-1\right|}.\end{aligned}$$

Observe that

$$\sum_{n\in\left[\frac{x}{2},x\right)} \frac{1}{\frac{n}{x}+1} << \int_{\frac{x}{2}}^{\lfloor x\rfloor} \frac{dt}{1-\frac{t}{x}},$$

$$\sum_{n\in\left(x,\frac{3x}{2}\right]} \frac{1}{\frac{n}{x}-1} << \int_{\lceil x\rceil}^{\frac{3x}{2}} \frac{dt}{\frac{t}{x}-1}.$$

$$\Rightarrow \sum_{n\in\left[\frac{x}{2},x\right)} \frac{1}{\frac{n}{x}+1} + \sum_{n\in\left(x,\frac{3x}{2}\right]} \frac{1}{\frac{n}{x}+1} << x\log x.$$

Thus we conclude, as $x \to \infty$,

$$\sum_{n\in\left[\frac{x}{2},\frac{3x}{2}\right]} \frac{1}{\left|\frac{n}{x}-1\right|} << \frac{x(\log x)^2}{T}.$$

Let us turn our focus to the second range. We have $\left|\log \frac{x}{n}\right| > \log \frac{3}{2}$. Thus

$$\begin{aligned}\sum_{n<\frac{x}{2},\frac{3x}{2}<n} \Lambda(n)\left(\frac{x}{n}\right)^{a} \min\left(1, T^{-1}\left|\log\left(\frac{x}{n}\right)\right|^{-1}\right) &<< \frac{x^a}{T} \sum_{n<\frac{x}{2},\frac{3x}{2}<n} \frac{\Lambda(n)}{n^a}, \\ &\leq \frac{x^a}{T}\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^a}.\end{aligned}$$

Recall that we have,

$$-\frac{\zeta'(a)}{\zeta(a)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^a}.$$

Using Theorem 2.17, we see, as $a \to 1^+$,

$$\left|\frac{\zeta'(a)}{\zeta(a)}\right| << \frac{1}{a-1}.$$

Using Corollary 2.6, we see,

$$\frac{x^a}{T}\frac{1}{a-1} << \frac{x^a}{T}(\log T)^9.$$

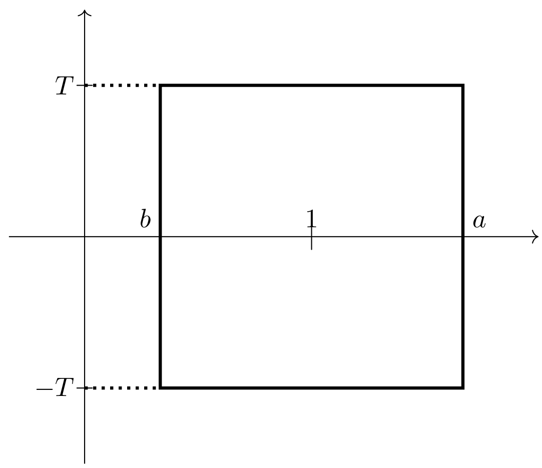
Combining all, we derive,

$$\psi(s) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds + O\left(\frac{x(\log x)^2}{T} + \frac{x^a}{T}(\log T)^9\right).$$

Recall Cauchy's Residue theorem, suppose $f : \Omega \to \mathbb{C}$ is meromorphic where Ω is simply connected. For $a_i, 1 \le i \le n$ poles in $U \subseteq \Omega$ simple closed curve. Then,

$$\frac{1}{2\pi i}\int_U f(s)ds = \sum_{i=1}^{n}\operatorname{Res}(f, a_i) + \text{constant}.$$

Consider the following closed path R_T which we traverse counterclockwise.



where $a = 1 + \frac{c}{(\log(T))^9}, b = 1 - \frac{c}{(\log(T))^9}$ $\operatorname{Re}(s) > 0, s \neq 1, \operatorname{Re}(s) > b$. We claim that $\frac{\zeta'(s)}{\zeta(s)}$ can be analytically continued to $\operatorname{Re}(s) \ge b$ except simple pole at $s = i$ with residue -1(See lecture 5 for the justification of such poles).

$$\begin{aligned}\frac{1}{2\pi i}\int_{R_T}\frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds &= \operatorname{Res}_{s=1}\left(\frac{-\zeta'(s)}{\zeta(s)}\right),\\ &= x.\end{aligned}$$

Now consider R_T which is the closed path.

$$\int_{R_T} = \int_{a-iT}^{a+iT} + \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT}.$$

We then have,

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds = x - \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT}$$

$$\left|\frac{1}{2\pi i}\int_{a+iT}^{b+iT}\frac{-\zeta(s)}{\zeta(s)}\frac{x^s}{s}ds\right| = \left|\frac{1}{2\pi i}\int_a^b\frac{-\zeta(u+iT)}{\zeta(u+iT)}\frac{x^{(u+iT)}}{(u+iT)}du\right|,$$

$$<< \int_a^b\left|\frac{-\zeta'(u+iT)}{\zeta(u+iT)}\right|\frac{x^u}{|u+iT|}du,$$

$$\overset{\text{Using the previous theorem}}{<<} \frac{\log^9 T}{T}x^a\int_b^a du,$$

$$<< \frac{x^a}{T}.$$

Now we have,

$$\left|\int_{b-iT}^{b+iT}\frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds\right| = \left|\frac{1}{2\pi}\int_{-T}^{T}\frac{\zeta'(b+iu)}{\zeta(b+iu)}\frac{x^{b+iu}}{b+iu}du\right|,$$

$$<< \log^9 T\int_{-T}^{T}\frac{x^b}{\sqrt{b^2+u^2}}du,$$

$$<< x^b\log^9 T\int_0^T\frac{du}{\sqrt{b^2+u^2}}du,$$

$$<< x^b\log^9 T\int_1^T\frac{du}{u},$$

$$= x^b\log^{10} T.$$

Now back to the beginning,

$$\psi(x) = x + o(\frac{x^a}{T} + x^b\log^{10} T) + o(\frac{x\log^2 x}{T} + \frac{x^a}{T}\log^9 T).$$

Choose T to be such that $2c\log x = \log^{10} T, x = e^{\frac{\log^{10} T}{x}}$.

$$x^{\frac{c}{\log^9 T}} = e^{\frac{\log T}{2}} = \sqrt{T}.$$

$$x^{1-\frac{c}{\log^9 T}}\cdot\log^{10} T + \frac{x\log^2 x}{T} + \frac{x^{1+\frac{c}{\log^9 T}}}{T}\log^9 T$$

$$= x\cdot T^{-\frac{1}{2}}\log^1 0T + \frac{x\log^2 x}{T} + x\frac{\sqrt{T}}{T}\log^{10} T\left(\frac{x}{\sqrt{T}}\right)(\log^{10} T + \frac{\log^2 x}{\sqrt{x}} + \log^9 T),$$

$$<< \frac{x}{T^s} = xe^{-c(\log x)^{\frac{1}{10}}}.$$

□

Definition 2.16 (Gamma function)**.** *We define the Gamma function* $\Gamma : \mathbb{C} \to \mathbb{C}$ *as*

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt, \sigma > 0.$$

Remark 2.17. *Suppose* $\mathrm{Re}(s) = \sigma$ *then,*

$$
\begin{aligned}
|\Gamma(s)| &\leq \int_0^\infty |t^{s-1}| e^{-t} dt, \\
&= \int_0^\infty t^{\sigma-1} e^{-t} dt, \\
&= \left(\int_0^1 + \int_1^\infty \right) t^{\sigma-1} e^{-t} dt, \\
&\leq \int_0^1 t^{\sigma-1} dt + \int_1^\infty t^{\sigma-1} e^{-t} dt, \left(\frac{t^n}{e^t} \overset{t\to\infty}{\to} 0 \right). \\
&<< \frac{1}{\sigma} + \int_1^\infty t^{\sigma-1} t^{-2\sigma} dt, \\
&<< \frac{1}{\sigma}.
\end{aligned}
$$

Theorem 2.30.

$$F(s) = \int f(s,t) dt.$$

$F : \Omega \to \mathbb{C}$ *is analytic in* Ω *if*

i). $f(s,t)$ *is continuous in* (s,t)*,*

ii). $f(s,t)$ *is analytic in* s*,*

iii). $\int f(s,t)$ *is uniformly bounded on compact subsets of* Ω*.*

In Remark 2.17, suppose $a \leq \mathrm{Re}(s) \leq b$ then the last two inequalities will be,

$$
\begin{aligned}
\int_0^1 t^{\sigma-1} dt + \int_1^\infty t^{\sigma-1} e^{-t} dt \\
&<<_b \frac{1}{\sigma} + \int_1^\infty t^{b-1} t^{-2b} dt, \\
&<<_b \frac{1}{\sigma} + \frac{1}{b} = \frac{1}{a} + \frac{1}{b}.
\end{aligned}
$$

Thus we observe that in $\sigma > 0$, using integration by parts,

$$\Gamma(s+1) = s\Gamma(s).$$

Thus for any $n \in \mathbb{N}$, we have $\Gamma(n) = n!$. We have,

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \sigma > 0.$$

Note $\Gamma(s+1)$ is analytic if $\sigma > -1$. Thus $\Gamma(s)$ is analytic in $\sigma > -1$ except simple pole at $s = 0$ with residue 1. That is

$$\lim_{s\to 0} s\Gamma(s) = 1.$$

We also have,

$$\Gamma(s) = \frac{\Gamma(s+2)}{\Gamma(s+1)}, \sigma > 0.$$

Note that in the numerator, we have $\sigma > -2$. $\Gamma(s)$ is analytic when $\sigma > -2$ except simple poles at $s = 0, -1$ with residue

$$\lim_{s\to -1}(s+1)\Gamma(s) = \frac{\Gamma(-1)}{-1} = -1.$$

Iterating the process, we have the following theorem,

Theorem 2.31. *$\Gamma(s)$ can be analytically continued to $\mathbb{C}$ except simple poles at $s = -k$ where $k \in \mathbb{Z}_{\geq 0}$ with residue $\frac{(-1)^k}{k!}$.*

Remark 2.18.

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, s \in \mathbb{C}.$$

$$\lim_{s\to m} \Gamma(s)\Gamma(1-s)\sin \pi s = \pi s.$$

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2s}\Gamma(2s), \forall s \in \mathbb{C}.$$

Exercise 2.5. *Show*

$$\sum_{n\in\mathbb{Z}} e^{-(n+\alpha)\frac{\pi}{x}} = \sqrt{x}\sum_{n\in\mathbb{Z}} e^{-n^2\pi x + 2\pi i n\alpha}, \forall \alpha \in i\mathbb{R}, x > 0.$$

Note that $\sum_{n\in\mathbb{Z}} a_n$ is convergent if

$$s_N := \sum_{|n|\leq N} a_n,$$

is convergent.

Theorem 2.32. $\Gamma(s) \neq 0, \forall s \in \mathbb{C}$.

Proof. Note that the case when $s \in \mathbb{Z}$ is already shown. Thus suppose $s \notin \mathbb{Z}$. If possible $\Gamma(s) = 0$. Then it will follow that $\Gamma(1-s)$ has a pole which is a contradiction. $\frac{1}{\Gamma(s)}$ has a simple zero at $s = 0, -1, -2, \cdots$.

Step 1

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t}s^{\frac{s}{2}-1}dt, \sigma > 0.$$

Replace $t = n^2\pi x, dt = n^2\pi dx$, we get,

$$
\begin{aligned}
\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)n^{-s} &= \int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx, \sigma > 1. \\
\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \sum_{n\in\mathbb{N}}\int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx, \\
&= \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n\in\mathbb{N}}e^{-n^2\pi x}\right)dx, \\
&= \sum_{n\in\mathbb{N}}\int_0^\infty |x^{\frac{s}{2}-1}e^{-n^2\pi x}|dx, \\
&= \sum_{n\in\mathbb{N}}\left(\int_0^\infty x^{\frac{\sigma}{2}-1}e^{-n^2\pi x}dx\right), \\
&= \sum_{n=1}^{\infty}\pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right)n^{-\sigma}, \\
&= \pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right)\zeta(s), \sigma > 1, \\
&< \infty. \\
\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n\in\mathbb{N}}e^{-n^2\pi x}\right).
\end{aligned}
$$

Step 2, using Poisson summation formula and $\sigma > 1$, let $F \in L^1(\mathbb{R})$, ie $F : \mathbb{R} \to \mathbb{C}$ and

$$\int_{-\infty}^{\infty}|F(t)|dt < \infty.$$

We have,

$$\sum_{n\in\mathbb{Z}}F(n+u)$$

is absolutely and uniformly convergent in u. Also we have,

$$\sum_{n\in\mathbb{Z}}|\hat{F}(n)| \leq \infty,$$

then

$$\sum_{n\in\mathbb{Z}}F(n+u) = \sum_{n\in\mathbb{Z}}\hat{F}(n)e^{2\pi inu}.$$

Using Exercise 2.5 and set $\alpha = 0$, we have,

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}}e^{-n^2\frac{\pi}{2}} &= \sqrt{s}\sum_{n\in\mathbb{Z}}e^{-n^2\pi x}.\sum_{n\in\mathbb{Z}}e^{-n^2\pi x} \qquad << \sum_{n\in\mathbb{N}}e^{-n\pi x}, \\
&<< \int_1^\infty e^{-\pi xt}dt, \\
&<< \frac{e^{-\pi x}}{x}, x > 0.
\end{aligned}
$$

Set $\theta := \sum_{n\in\mathbb{Z}} e^{n^2\pi x}$ then,

$$\theta(x) = 1 + 2\sum_{n\in\mathbb{N}} e^{-n^2\pi x}.$$

$$\sum_{n\in\mathbb{N}} e^{-n^2\pi x} = \frac{\theta(x)-1}{2}, x > 0.$$

Using the exercise, we have,

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\theta(x).$$

Set $w(x) := \frac{\theta(x)-1}{2}$. Thus write

$$\begin{aligned}
w\left(\frac{1}{x}\right) &= \frac{\theta\left(\frac{1}{x}\right)-1}{2}, \\
&= \frac{\sqrt{x}\theta(x)-1}{2}, \\
&= \frac{\sqrt{x}\theta(x)-1}{2}. \\
w\left(\frac{1}{x}\right) &= \sqrt{x}w(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}. \\
\pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1}w(x)dx, \sigma > 1.
\end{aligned}$$

Step 3

$$\int_0^1 x^{\frac{s}{2}-1}w(x)dx + \int_1^\infty x^{\frac{s}{2}-1}w(x)dx.$$

Taking $x = \frac{1}{y}$ and $dx = -\frac{1}{y^2}dy$ we have,

$$\begin{aligned}
\int_1^\infty \left(\frac{1}{y^2}\right)^{\frac{s}{2}-1} w\left(\frac{1}{y}\right)\frac{-1}{y^2}dy &= \int_1^\infty y^{-\frac{s}{2}} w\left(\frac{1}{y}\right)\frac{dy}{y}, \\
&= \int_1^\infty y^{-\frac{1}{2}}\left(\sqrt{y}w(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right)\frac{dy}{y}, \\
&= \int_1^\infty y^{-\frac{s}{2}+\frac{1}{2}}w(y)\frac{dy}{y} + \int_1^\infty \frac{y^{-\frac{s}{2}+\frac{1}{2}-1}}{2}dy - \frac{1}{2}\int_1^\infty y^{-\frac{s}{2}-1}dy, \\
&= \int_1^\infty y^{\frac{1-s}{2}}w(y)\frac{dy}{y} + \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}, \sigma > 1.
\end{aligned}$$

Step 4,

$$\begin{aligned}\int_1^\infty |x^{\frac{s}{2}} + x^{\frac{1-s}{2}}||w(x)|\frac{dx}{x} &\leq \int_1^\infty (x^{\overset{\sigma 2}{-}1} + x^{\frac{1-\sigma}{2}-1})|w(x)|dx,\\ &<< \int_1^\infty \frac{(x^{\frac{\sigma}{2}-1} + x^{\frac{1-\sigma}{2}-1})}{e^{\pi x}}dx, \sigma \in \mathbb{R},\\ &<< \int_1^\infty \frac{e^x}{e^{\pi x}}dx, \qquad\qquad << 1.\end{aligned}$$

$\int_1^\infty |x^{\frac{s}{2}} + x^{\frac{1-s}{2}}||w(x)|\frac{dx}{x}$ is analytic in $\mathbb{C}$.

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = 1 + s(s-1)\int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}.$$

Set $\xi(x) := 1 + s(s-1)\int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}$, then it is entire. and for all s we have,

$$\xi(1-s) = \xi(s).$$

Thus obtain,

$$(1-s)(1-s-1)\pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Using the construction of $w(s)$ we have,

$$|w(x)| << |\theta(x)| << \frac{e^{-\pi x}}{x}, \forall x > 0.$$

Also we have,

$$\begin{aligned}\zeta(1-s) &= \pi^{-s+\frac{1}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s),\\ \zeta(1-s) &= \pi^{-s}2^{1-s}\cos\left(\frac{\pi s}{s}\right)\Gamma(s)\zeta(s).\end{aligned}$$

Note that $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ except simple pole at s.

$$\begin{aligned}\lim_{s\to 1}\zeta(1-s) &= \pi^{-1}\lim_{s\to 1}\frac{\cos\frac{\pi s}{2}\zeta(s)(s-1)}{s-1}.\\ \lim_{s\to 1}\frac{\cos\frac{\pi s}{2}}{s-1} &= \lim_{s\to 0}\frac{\cos\left(\frac{\pi}{2}(1-s)\right)}{-s},\\ &= \lim_{s\to 0}\frac{\sin\frac{\pi}{2}s}{-s\cdot\frac{\pi}{2}}\frac{\pi}{2},\\ &= -\frac{\pi}{2}.\\ \Rightarrow \lim_{s\to 1}\zeta(1-s) &= -\frac{1}{2}.\end{aligned}$$

$$\zeta(-2n) = \pi^{-(2n+1)} 2^{1-2n-1} \cos\left(\frac{\pi}{2}(2n+1)\right) \Gamma(2n+1)\zeta(2n+1) = 0.$$

Recall

$$\begin{aligned}
\zeta(s) &= \sum_{n\in\mathbb{N}} \frac{1}{n^s}, \\
&= \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}} dt, \sigma > 0, s \neq 1.
\end{aligned}$$

If s is real,

$$\begin{aligned}
\left|\zeta(s) - \frac{s}{s-1}\right| &\leq |s| \int_1^\infty \frac{\{t\}}{t^{\sigma+1}} dt, \\
&< \frac{|s|}{\sigma} = \frac{\sigma}{\sigma} = 1.
\end{aligned}$$

Thus we obtain,

$$\begin{aligned}
-1 + \frac{s}{s-1} < \zeta(s) &< 1 + \frac{s}{s-1} \\
\frac{1}{s-1} < \zeta(s) &< \frac{2s-1}{s-1}, \\
-1 < (1-s)\zeta(s) < 1 - 2s < 0 \quad &\text{if } \frac{1}{2} < s < 1 \\
\Rightarrow \zeta(s) \neq 0, \quad &\text{if } \frac{1}{2} < s < 1.
\end{aligned}$$

□

2.7 Primitive characters

Suppose $q_0|q$. Then we have a natural map

$$\mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/q_0\mathbb{Z}$$

Definition 2.17. *Let $\chi \bmod q$ be a character and $q_0|q$. If there exists a character $\chi^* \bmod q_0$ making the following diagram commutative,*

$$\begin{tikzcd}
(\mathbb{Z}/q\mathbb{Z})^\times \arrow[r, "\chi"] \arrow[d, "f"'] & \mathbb{C}^\times \\
(\mathbb{Z}/q_0\mathbb{Z})^\times \arrow[ur, "\chi^*"'] &
\end{tikzcd}$$

we say χ factors through $(\mathbb{Z}/q_0\mathbb{Z})$ or χ^* induces χ. If q_0 is minimal among positive such numbers, we say q_0 is a conductor of χ.

Definition 2.18. *A character $\chi \bmod q$ is called primitive if its conductor is q only. Otherwise the character is imprimitive.*

Example 2.11. *Take* $\chi : (\mathbb{Z}/8\mathbb{Z})^\times \to \mathbb{C}^\times$ *such that*

n	$\chi(n)$
1	1
3	-1
5	1
7	-1

Then 4 *is a conductor with* $\chi^* : (\mathbb{Z}/4\mathbb{Z})^\times \to \mathbb{C}^\times$,

n	$\chi^*(n)$
1	1
3	-1

Example 2.12. *Take* $\chi : (\mathbb{Z}/6\mathbb{Z})^\times \to \mathbb{C}^\times$ *such that*

n	$\chi(n)$
1	1
5	-1

Then 4 *is a conductor with* $\chi^* : (\mathbb{Z}/3\mathbb{Z})^\times \to \mathbb{C}^\times$,

n	$\chi^*(n)$
1	1
2	-1

Remark 2.19. *All principal characters* $\bmod q$ *for* $q > 1$ *are imprimitive by construction.*

We set as usual $s = \sigma + it$ for $\sigma, t \in \mathbb{R}$.

Lemma 2.14. *Suppose* $\chi \bmod q$ *is induced by* $\chi^* \bmod q_0^*$. *We then have,*

$$\sigma > 1 \Rightarrow L(s,\chi) = L(s,\chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right).$$

Proof. We know that for $\sigma > 1$,

$$L(s,\chi) = \prod_{p\nmid q} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Also

$$\begin{aligned} L(s,\chi^*) &= \prod_{p\nmid q_0} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \\ &= \prod_{\substack{p\nmid q_0 \\ p\nmid q}} \left(1 - \frac{\chi^*(p)}{p^s}\right)^{-1} \prod_{\substack{p\nmid q_0 \\ p|q}} \left(1 - \frac{\chi^*(p)}{p^s}\right)^{-1}, \\ &= L(s,\chi) \prod_{\substack{p\nmid q_0 \\ p|q}} \left(1 - \frac{\chi^*(p)}{p^s}\right)^{-1}. \end{aligned}$$

$\square$

Remark 2.20. *Using Identity theorem, Lemma 2.14 holds for all $s \in \mathbb{C}$.*

Lemma 2.15. *Suppose $x \in \left[0, \frac{1}{2}\right]$, then*

$$|\sin \pi x| \geq 2x.$$

Proof. Exercise. $\square$

Lemma 2.16. *For $n \in \mathbb{Z}$, we have,*

$$\chi(n)\tau(\overline{\chi}) = \sum_{k=1}^{q} \overline{\chi}(k) e^{\frac{2\pi i k}{q}},$$

where χ is a primitive character modulo q and $\chi \neq \chi_0$.

Proof.

$$\overline{\chi(n)\tau(\overline{\chi})} = \sum_{l=1}^{q} \chi(l) e^{-\frac{2\pi i l}{q}}.$$

Multiplying this equation with the one in the statement we have,

$$|\chi(n)|^2 |\tau(\overline{\chi})|^2 = \sum_{k,l=1}^{q} \overline{\chi(k)} \chi(l) e^{\frac{2\pi i (k-l)}{q}}.$$

Applying $\sum_{n \leq x}$ to the both sides of the above equation, we have,

$$|\tau(\overline{\chi})|^2 \sum_{n \leq x} |\chi(n)|^2 = \sum_{k,l=1}^{q} \overline{\chi(k)} \chi(l) \left(\sum_{n \leq x} \left(e^{\frac{2\pi i (k-l)}{q}} \right)^n \right).$$

Note that

$$x + x^2 + \cdots + x^q = \begin{cases} \frac{x(x^q - 1)}{x-1}, (x \neq 1), \\ q, (x = 1), \end{cases}$$

$$= \begin{cases} 0, (x \neq 1, x^q = 1), \\ q, (x = 1), \end{cases}$$

Take $x = e^{\frac{2\pi i (k-l)}{q}} = \cos \frac{2\pi}{q}(k-l) + i \sin \frac{2\pi}{q}(k-l) = 1$. $x = 1$ if and only if $q | k - l$ thus

$$\sum_{n \leq x} \left(e^{\frac{2\pi i (k-l)}{q}} \right)^n = \begin{cases} 0, (q \not| k - l), \\ q, (\text{otherwise}). \end{cases}$$

Therefore, we get,

$$
\begin{aligned}
|\tau(\overline{\chi})|^{@} \sum_{n\leq x} |\chi(n)|^2 &= q \sum_{\substack{k,l=1\\ q|k-l}}^{q} \overline{\chi}(k)\chi(l), \\
&= q \sum_{k=1}^{q} \overline{\chi}(k)\chi(k), \\
&= q \sum_{k=1}^{q} |\chi(k)|^2.
\end{aligned}
$$

Therefore $|\tau(\overline{\chi})|^2 = q, |\tau| = \sqrt{q}$.

Consider $(n, q) = 1$, then

$$
\begin{aligned}
\chi(n)\tau(\overline{\chi}) &= \chi(n) \sum_{k=1}^{q} \overline{\chi}(k) e^{\frac{2\pi i k}{q}}, \\
&= \chi(n) \sum_{\substack{k=1\\ (k,q)=1}}^{q} \overline{\chi}(k)\chi(k) e^{\frac{2\pi i k}{q}},
\end{aligned}
$$

Set $k = nt$, we get,

$$
\begin{aligned}
\chi(n)\tau(\overline{\chi}) &= \sum_{\substack{t=1\\ (t,q)=1}}^{q} \overline{\chi}(nt)\chi(n) e^{\frac{2\pi i n t}{q}}. \\
\overline{\chi}(nt) &= \overline{\chi}(n)\overline{\chi}(t). \\
\chi(n)\tau(\overline{\chi}) &= \sum_{\substack{t=1\\ (t,q)=1}}^{q} \overline{\chi}(t) e^{\frac{2\pi i n t}{q}}.
\end{aligned}
$$

Observe that

$$\tau(\overline{\chi})\left(\sum_{n\le x}\chi(n)\right) = \sum_{k=1}^{q-1}\overline{\chi}(k)\left(\sum_{n\le x} e^{\frac{2\pi ikn}{q}}\right),$$

$$|\tau(\overline{\chi})|\cdot|\sum_{n\le x}\chi(n)| \le \sum_{k=1}^{q-1}\left|\sum_{n\le x} e^{\frac{2}{\pi}iknq}\right|,$$

$$= \sum_{k=1}^{q-1}\left|\frac{e^{\frac{2\pi ik}{q}}\left(e^{\frac{2\pi in[x]}{q}}-1\right)}{e^{\frac{2}{\pi}ikq}-1}\right|,$$

$$\le \sum_{k=1}^{q-1}\frac{2}{|e^{\frac{2\pi k}{q}}-1|},$$

Note that for all $y \in i\mathbb{R}$,

$$2i\sin y = e^{-iy}(e^{2iy}-1),$$
$$|2\sin y| = |e^{2iy}-1|.$$

Apply this to the equation above we have,

$$\sum_{k=1}^{q-1}\frac{2}{|e^{\frac{2\pi k}{q}}-1|} = \sum_{k=1}^{q-1}\frac{1}{|\sin\frac{\pi k}{q}|}.$$

Using the lemma, we get,

$$\sum_{k=1}^{q-1}\frac{1}{|\sin\frac{\pi k}{q}|} = \sum_{1\le k\le\frac{q}{2}}\frac{1}{|\sin\frac{\pi k}{q}|} + \sum_{\frac{q}{2}<k}^{q-1}\frac{1}{|\sin\frac{\pi k}{q}|}. \qquad = 2\sum_{1\le k\le\frac{q}{2}}\frac{1}{|\sin\frac{\pi k}{q}|},$$

$$\le \sum_{1\le k\le\frac{q}{2}}\frac{k}{q} << q\log\frac{q}{2} << q\log q.$$

Thus we conclude $|(\overline{\chi}) << \sqrt{q}$ if χ is primitive and $q > 1$, we have,

$$\left|\sum_{n\le x}\chi(n)\right| << \sqrt{q}\log q,$$

uniformly in q as $x\to\infty$. □

$L(q,\chi)$ functional equation, where $\chi \ne \chi_0$ and χ is primitive. Suppose $\chi(-1) = 1$ that is it is an even character. From previous discussion, we have,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)n^{-s} = \int_0^\infty x^{\frac{s}{2}}e^{-n^2\pi x}\frac{dx}{x}.$$

Replace x with $\frac{x}{q}$ and $\sigma > 0$, we habe,

$$\pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{1}\right) n^{-s} = \int_0^\infty x^{\frac{s}{2}} e{-}n^2\pi x \frac{dx}{x}, \sigma > 0.$$

For $\sigma > 1$, we have,

$$\begin{aligned}\pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s,\chi) &= \sum_{n=1}^{\infty} \int_0^\infty \chi(n) x^{\frac{s}{2}} e^{-n^2 \frac{\pi x}{q}} \frac{dx}{x}, \\ &= \int_0^\infty x^{\frac{s}{2}} \left(\sum_{n=1}^{\infty} \chi(n) e^{-n^2 \frac{\pi x}{q}} \right) \frac{dx}{x}, \sigma > 1.\end{aligned}$$

Let

$$\theta(x,\chi) = \sum_{n\in\mathbb{Z}} \chi(n) e^{-\frac{n^2\pi x}{q}} = 2\sum_{n=1}^{\infty} \chi(n) e^{-\frac{n^2\pi x}{q}}, (x > 0).$$

Using this, we get,

$$\int_0^\infty x^{\frac{s}{2}} e{-}n^2\pi x \frac{dx}{x} = \frac{1}{2} \int_0^\infty x^{\frac{s}{2}} \theta(x,\chi) \frac{dx}{x}, (\sigma > 1).$$

Split the integral into $\int_0^1, \int_1^\infty$.

$$\begin{aligned}\tau(\overline{\chi})\theta(x,\chi) &= \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \overline{\chi}), \\ &= \sum_{n\in\mathbb{Z}} (\tau(\overline{\chi})\chi(n)) e^{\frac{-n^2\pi x}{q}}, \\ &= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{q} \overline{\chi}(k) e^{\frac{2\pi i n k}{q}} e^{-\frac{n^2\pi x}{q}}, \\ &= \sum_{k=1}^{q} \overline{\chi(k)} \sum_{n\in\mathbb{Z}} e^{\frac{2\pi k n}{q} - n^2 \frac{\pi x}{q}}, \\ &\overset{\text{From Lecture 7 page 6}}{=} \sum_{k=1}^{q} \overline{\chi}(k) \left(\frac{x}{q}\right)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} e^{-\left(m+\frac{m}{q}\right)\frac{\pi q}{x}}, \\ &= \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^{q} \overline{\chi}(k) \sum_{n\in\mathbb{Z}} e^{-\left(m+\frac{m}{q}\right)\frac{\pi q}{x}}.\end{aligned}$$

Put $qn + m = t$, then

$$\overline{\chi}(qn+m) = \overline{\chi}(m) = \overline{\chi}(t).$$

Thus,

$$\left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^{q} \overline{\chi}(k) \sum_{n\in\mathbb{Z}} e^{-\left(m+\frac{m}{q}\right)\frac{\pi q}{x}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{t\in\mathbb{Z}} \overline{\chi}(t) e^{-\frac{t^2\pi}{qx}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \overline{\chi}).$$

Now for the splitted integral, we have,

$$\frac{1}{2}\int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}\frac{dx}{x}.$$

Replacing x with $\frac{1}{x}$, we get,

$$= \frac{\tau(\chi)}{2\sqrt{q}}\int_1^\infty x^{\frac{1-s}{2}}\theta(x,\overline{\chi})\frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}.$$

Set

$$\xi(s,\chi) := \frac{\tau(\chi)}{2\sqrt{q}}\int_1^\infty x^{\frac{1-s}{2}}\theta(x,\overline{\chi})\frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}.$$

Use that $\theta(x,\overline{\chi}) << e^{-\frac{\pi x}{q}}$ and $|\tau(\chi)| = \sqrt{q}$. It turns out that ξ is uniformly bounded on compact subsets of $\mathbb{C}$ and in particular, this is entire.

Lemma 2.17.

$$\xi(1-s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\xi(s,\overline{\chi}).$$

Proof. Recall,

$$\tau(\overline{\chi})\theta(x,\chi) = \left(\frac{q}{x}\right)^{\frac{1}{2}}\theta(x^{-1},\overline{\chi}),$$

and

$$|\tau(\chi)|^2 = q.$$

We get,

$$\frac{\tau(\chi)}{\sqrt{q}} = \frac{\sqrt{q}}{\overline{\tau(\chi)}} = \frac{\sqrt{q}}{\tau(\overline{\chi})}.$$

Then,

$$L(1-s,\chi) = \frac{\tau(\chi)}{q^{1-s}}\pi^{-s}2^{1-s}\cos\frac{\pi s}{2}(\Gamma(s))L(s,\overline{\chi}).$$

Note that $\chi(-1) = 1$ and is primitive (in the case of odd integer replace cos with sin). Note also that

i). $L(s,\chi) = 0, s = 0,-2,-4,\cdots$ when χ is even,

ii). $L(s,\chi) = -1,-3,-5,\cdots$ when χ is odd.

When $0 < \sigma < 1$, $\square$

Definition 2.19 (Entire function of finite order)**.** *An entire functio $nf(z)$ is saiad to be of finite order if there is $\alpha > 0$ such that*

$$|f(z)| << e^{|z|^\alpha},$$

as $|z| \to \infty$. The order of f is the infimum of such α.

Example 2.13. *We have the following examples,* *1.* $e^z \to 1$,

2. $\sin z \to 1$,

3. $\cos z \to 1$,

4. $e^{z^n} \to n$.

For the trigonometric functions use $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ *and similarly for* cos.

Remark 2.21. *Is* $|\sin z| < z, z \in \mathbb{C}$*? To show this consider,*

$$f(z) = \begin{cases} \frac{\sin z}{z}, z \neq 0, 1, z = 0. \end{cases}$$

And apply Liouville's theorem.

Theorem 2.33 (Special case of Hadamard's factorization theorem)**.** *Let* $f(z)$ *be an entire function of order* 1 *and* $(z_j)_{j\in\mathbb{N}}$ *be such that* $z_j \in \mathbb{C}\backslash\{0\}$*, and* $f(z_j) = 0$*.Then there are some constants* A, B *such that*

$$f(z) = e^{A+Bz} \prod_{j\geq 1} \left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j}}.$$

Theorem 2.34. *Let* f *be an entire function of order* $\alpha \in \mathbb{Z}$ *without any zeros. Then we have,*

$$f(z) = e^{p(z)},$$

where p *is some polynomial of degree* α*.*

Recall the completed zeta function,

Definition 2.20.

$$\xi(s) := (s-1)\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}+1\right)\zeta(s).$$

We know,

1. $\xi(s)$ *has an order* 1*,*

2. $\zeta(s)$ *has infinitely zeros in the critical strip.*

3. $\xi(s) = e^{As+B} \prod_{\substack{\rho, \xi(\rho)=0 \\ 0<\mathrm{Re}(\rho)<1}} \left(1 - \frac{s}{\rho}\right) e^{\frac{z}{\rho}}$*, where* A, B *ae some constant.*

Remark 2.22. *Zeros of* $\xi(s)$ *are precisely the non-trivials zeros of* $\zeta(s)$*.*

Theorem 2.35. *There eists a constant* $c > 0$ *such that for* $\sigma, t \in \mathbb{R}$ $s = \sigma + it$*,* $\sigma \geq 1 - \frac{c}{\log(|t|+2)}$*, we have,*

$$\zeta(s) \neq 0.$$

Taking derivatives, we obtain,

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{\Gamma'\left(\frac{s}{2}+1\right)}{2\Gamma\left(\frac{s}{2}+1\right)} + \frac{\zeta'(s)}{\zeta(s)},$$
$$\frac{\xi'(s)}{\xi(s)} = A + \sum_{\substack{\zeta(\rho)=0\\ 0<\mathrm{Re}(\rho)<1}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Note that $\frac{s}{2}\Gamma\left(\frac{s}{2}\right) = \Gamma\left(\frac{s}{2}+1\right)$. Thus we obtain,

$$-\frac{\zeta'(s)}{\zeta} = \frac{1}{s-1} - A - \frac{\log\pi}{2} + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

We then get,

$$\mathrm{Re}\left(-\frac{\zeta'(s)}{\zeta}\right) = \mathrm{Re}\left(\frac{1}{s-1}\right) - A - \frac{\log\pi}{2} + \frac{1}{2}\mathrm{Re}\left(\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)}\right) - \sum_{\rho}\mathrm{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Recall that for $s = \sigma + it, \sigma, t \in \mathbb{R}$, we have,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1}\frac{\Lambda(n)}{n^s}, \sigma \geq 1.$$

And also,

$$\frac{-3\zeta'(\sigma)}{\zeta(\sigma)} - \mathrm{Re}\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right) - \mathrm{Re}\left(\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right).$$

Check that

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}(3 + 4\cos(t\log n) + \cos(2t\log n)) > 0, \sigma > 1.$$

For $1 < \sigma < 2, t \geq 1$, we have,

$$\mathrm{Re}\left(\frac{1}{s-1}\right) = \frac{\sigma-1}{|s-1|^2} < \frac{1}{t^2} < 1.$$

$$\mathrm{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = \mathrm{Re}\left(\frac{\overline{s-\rho}}{|s-\rho|^2} + \frac{\overline{\rho}}{|\rho|^2}\right),$$
$$= \frac{\sigma-\beta}{|s-\rho|^2} + \frac{\beta}{|\rho|^2} > 0,$$

where $\beta = \mathrm{Re}(\rho)$. We have,

$$\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} < B\log\left(\frac{|t|}{2}+2\right), \forall t \in \mathbb{R}.$$

We have the following equations,

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z, |z| \to \infty.$$

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{2}{n}},$$

where γ is the Euler's constant.

Now consider $1 \leq \sigma \leq 2$, we have,

$$\left|\frac{\Gamma'(s)}{\Gamma(s)}\right| < B \log(|t| + 2),$$

where B is some constant for all $t \in\ R$. From above expressions, we obtain,

$$\operatorname{Re}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) < B' \log\left(\frac{|t|}{2} + 2\right), \forall t \geq 1.$$

Let $\rho = \beta + it$, be any non-trivial zero of $\zeta(s)$.

$$\operatorname{Re}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) < B' \log\left(\frac{|t|}{2} + 2\right) - \frac{1}{\sigma - \beta}, t \geq 1.$$

Recall for $\sigma > 0$,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + g(s),$$

where$g(s)$ is some analyti function. For $1 \leq \sigma \leq 2$,

$$\frac{-\zeta'(\sigma)}{\zeta(\sigma)} = \frac{1}{\sigma - 1} - g(\sigma),$$

$$< \frac{1}{\sigma - 1} + c,$$

where C is some constant.

$$0 < \frac{-3\zeta'(\sigma)}{\zeta(\sigma)} - 4\operatorname{Re}\left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)}\right) - \operatorname{Re}\left(\frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)}\right).$$

We then obtain,

$$0 < 3\left(\frac{1}{\sigma - 1} + C\right) + 4B' \log(|t| + 2) + B' \log(|t| + 2) - \frac{1}{\sigma - \beta}.$$

Note that

$$\frac{1}{\sigma - \beta} < \frac{3}{\sigma - 1} + B'' \log(|t| + 2), t \geq 1.$$

Take $\sigma = 1 + \frac{\varepsilon}{\log(|t|+2)}$, where $\varepsilon > 0$, sufficiently small which will be chosen later.

$$\beta < 1 - \frac{\varepsilon(1 - B''\varepsilon)}{\log(|t| + 2)}, t \geq 1.$$

So far we have shown that for $t \geq 1$ and

$$\sigma \geq 1 - \frac{c}{\log(|t|+2)},$$

we have $\zeta(s) \neq 0$. For some constant $C > 0$.

For $|t| \leq 1$, since $\zeta 9s)$ be analytic except $s = 1$, there exists a constant $c_1 > 0$ such that

$$\sigma > 1 - c_1 \Rightarrow \zeta(s) \neq 0.$$

If

$$1 - c_1 \leq 1 - \frac{c_2}{\log(|t|+2)},$$

then we have,

$$c_1 \geq \frac{c_2}{\log(|t|+2)}, |t| < 1.$$

Choose $c_2 = c_1 \log 2$. Take $c' = \min(c_1, c_1 \log 2)$.

Above proof can be generalized to $L(s, \chi)$ where χ is a non-trivial character.

Theorem 2.36. *Let χ be a non-trivial character. Then there si $c_1 > 0$ such that for $s = \sigma + it, \sigma, t \in \mathbb{R}$, we have,*

$$\sigma \geq 1_{\frac{c_1}{\log(q(|t|+2))}}.$$

Theorem 2.37 (Siegel zeros)**.** *There exists an absolute constant $c_2 > 0$ such that $L(s, \chi)$ where χ is quadratic $\bmod q$ has at most one zero in the region,*

$$\sigma > 1 - \frac{\delta}{\log q}, |t| < \frac{\delta}{\log q},$$

where $0 < \delta < c_2$. If such a zero exists, it is real, simple and often called Siegel zero.

Recall the completed Dirichlet L-function,

$$\xi(s, \chi) := \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi).$$

See the lecture note from 12/16. Where

$$a = \begin{cases} 0, & \chi(-1) = 1, \\ 1, & \chi(-1) = -1. \end{cases}$$

We have $\xi(s, \chi)$ is of order 1 and

$$\xi(s, \chi) = e^{A_1 + B_1 s} \prod_{\substack{\rho, L(\rho,\chi)=0 \\ 0 \leq \mathrm{Re}(\rho) < 1}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}.$$

Furthermore $L(s, \chi)$ has inifinitely many zero in the critical strip.

Remark 2.23.
$$L(s,\chi) = L(s,\chi^*)\prod_{p|q}\left(1\frac{\chi^*(p)}{p^s}\right), s \in \mathbb{C}$$

Then $L(s,\chi) = 0$ if and only if $L(s,\chi^*) = 0$ when $\sigma \neq 0$.

Recall

$$\xi(s) = \frac{s(s-1)}{2}\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Definition 2.21.

$$N(T) = |\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \beta \in (0,1), \gamma \in (0,T)\}|.$$

Theorem 2.38. *As $T \to \infty$, we have,*

$$N(T+1) - N(T) << \log T.$$

Proof. Enough to show that as $T \to \infty$.

$$\sum_{\substack{\rho=\beta+i\gamma \\ \zeta(\rho)=0 \\ \beta\in(0,1)}} \frac{1}{1+(T-\gamma)^2} << \log T$$

Observe that,

$$N(T+1) - N(T) = \sum_{\substack{\rho=\beta+i\gamma \\ \beta\in(0,1) \\ \zeta(\rho)=0 \\ T<\gamma<T+1}} \leq \sum.$$

Recall we have,

$$-\operatorname{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) < A\log(|t|+2) - \sum_{\rho=\beta+i\gamma}\operatorname{Re}\left(\frac{1}{s-p} - \frac{1}{p}\right).$$

where $s = \sigma + it$ for $\sigma \in [1,2], |t| > 1$ and A is some constant. Substitute $s = 2 + iT$ in the equation, we have,

$$\operatorname{Re}\left(\frac{1}{s-\rho}\right) = \operatorname{Re}\left(\frac{(\overline{s-\rho})}{|s-\rho|^2}\right).$$

Put $s = 2 + iT$ and $\rho = \beta + i\gamma$, we have,

$$\operatorname{Re}\left(\frac{(\overline{s-\rho})}{|s-\rho|^2}\right) = \frac{2-\beta}{(2-\beta)^2 + (T-\gamma)^2} \geq \frac{1}{4+(T-\gamma)^2}.$$

Also,

$$\operatorname{Re}\left(\frac{1}{\rho}\right) = \frac{\beta}{|\rho|^2} > 0.$$

By the previous equation, we get

$$\sum_{\substack{\rho=\beta+i\gamma\\ \beta\in(0,1)\\ \zeta(\rho)=0}} \mathrm{Re}\left(\frac{1}{2+iT-\rho}\right) < A\log(T+2)+\mathrm{Re}\left(\frac{\zeta'(2+iT)}{\zeta(2+iT)}\right) << \log T.$$

Recall that for $\sigma > 1$,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\in\mathbb{N}} \frac{\Lambda(n)}{n^s},$$
$$\left|\frac{\zeta'(2+iT)}{\zeta(2+iT)}\right| \leq \sum_{n\in\mathbb{N}} \frac{\Lambda(n)}{n^s} < \infty.$$

Therefore,

$$\sum_{\rho} \mathrm{Re}\left(\frac{1}{2+iT-\rho}\right) = \sum_{\rho} \frac{1}{4+(T-\gamma)^2},$$

and

$$4+(T-\gamma)^2 \leq 4(1+(T-\gamma)^2).$$

$\square$

Theorem 2.39. *As $T \to \infty$,*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log(T)).$$

That is, as $T \to \infty$,

$$N(T) \sim \frac{T}{2\pi}\log T.$$

Analogously, $\chi \bmod q$, as $T \to \infty$,

$$N(T,\chi) = \sum_{\substack{\rho=\beta+i\gamma\\ L(\rho,\chi)=0\\ \beta\in(0,1)\\ |\gamma|<T}} 1 \sim \frac{T}{\pi}\log qT.$$

Exercise 2.6.

$$N(T,\chi)' = \sum_{\substack{\rho=\beta+i\gamma\\ L(\rho,\chi)=0\\ \beta\in(0,1)\\ 0<gamma<T}} 1 \sim \frac{T}{2\pi}\log qT.$$

Recall by Remark 2.1, we have,

$$\psi(x) := \sum_{n\leq x} \Lambda(n).$$

Also recall that Theorem 2.17, $\frac{\zeta'(s)}{\zeta(s)}$ has a simple pole at $s = 1$ with residue -1.

Let $s = a$ be a zero of $\zeta(s)$ of order $n \in \mathbb{N}$. Then

$$\zeta(s) = (s-a)^n g(s),$$

where $g(s)$ is analytic and $g(a) \neq 0$. Taking log-derivative of both sides , we have,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{n}{s-a} + \frac{g'(a)}{g(a)}.$$

Resiue of $\frac{\zeta'(s)}{\zeta(s)}$ at 0 of $zeta(s)$ is precisely the multiplicity of that zero. Recall from the proof of Theorem **??**, we have,

$$\psi(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds + O\left(\sum_{n\in\mathbb{N}} \Lambda(n)\left(\frac{x}{n}\right)\min(1, T^{-1}\left|\log\frac{x}{n}\right|^{-1})\right),$$

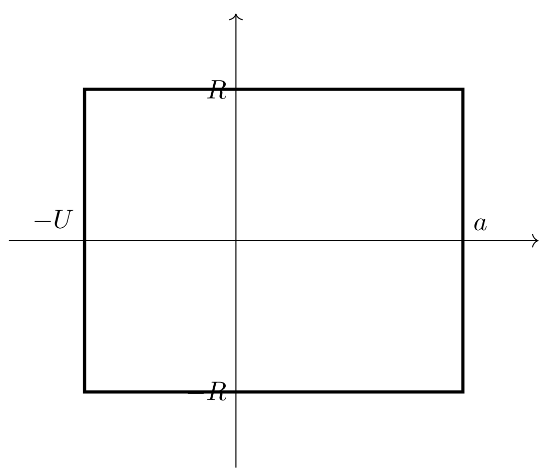
where x is not an integer and $T, a \geq 1$. Similarly to the proof of Theorem **??**, take

$$a = 1 + \frac{1}{\log x}.$$

we then get, as $x \to \infty$,

$$\psi(x) = \frac{1}{2\pi i}\int_{a-iR}^{a+iR} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}dx + O\left(\frac{x\log^2 x}{R}\right).$$

Now consider the following closed path



where U is some odd integer. By Cauchy's residue theorem, we have,

$$\frac{1}{2\pi i}\int_{C_R} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds = \text{sum of residues of } \frac{\zeta'(s)}{\zeta(s)} \text{ in } C_R,$$

$$= \overbrace{x}^{\text{contribution of poles of } \zeta(s)} - \underbrace{\frac{\zeta'(0)}{\zeta(0)}}_{\text{contribution of } s=0} + \overbrace{\sum_{0<2m<U}\frac{x^{-2m}}{2m}}_{\text{contribution of non-trivial zeros of } \zeta(s)} - \text{contributi}$$

Note that,

$$\int_{C_R} = \int_{a-iR}^{a+iR} + \underbrace{\int_{T_1,T_2,T_3}}_{=I}.$$

We then have

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{0<2m<U}\frac{x^{-2m}}{2m} - \sum_{\substack{\rho=\beta+i\gamma\\ \zeta(\rho)=0\\ \beta\in(0,1)\\ |\gamma|<R}}\frac{x^\rho}{\rho} - I + O\left(\frac{x\log^2 x}{R}\right).$$

Exercise 2.7. *Show that*

$$I = \frac{1}{2\pi}\int_{I_1+I_2+I_3} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds, \qquad = O\left(\frac{x\log^2 R}{R\log x} + \frac{R\log U}{Ux^R}\right).$$

Suppose we have above equality, then we have,

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{0<2m<U}\frac{x^{-2m}}{2m} - \sum\frac{x^\rho}{\rho} + O\left(\frac{x\log^2 R}{R\log x} + \frac{R\log U}{Ux^R} + \frac{x\log^2 x}{R}\right).$$

Recall U is an odd integer. Take $U\to\infty$, we have,

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \log(1-x^{-2}) - \sum_\rho\frac{x^\rho}{\rho} + O\left(\frac{x\log^2 R}{R\log x} + \frac{x\log^2 x}{R}\right).$$

Taking $R\to\infty$, we have,

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \log(1-x^{-2}) - \sum_\rho\frac{x^\rho}{\rho}.$$

Definition 2.22.

$$\psi(x,\chi) := \sum n \le x\Lambda(n)\chi(n).$$

Recall we had,

$$\psi(x,q,a) := \sum_{\substack{n\leq x\\ n=a \bmod q}} \Lambda(n),$$

where $(a,q)=1$. By Theorem **??**, we have, as $x\to\infty$,

$$\psi(x,q,a)\sim \frac{x}{\varphi(q)}.$$

Exercise 2.8. *Show that for $(a,q)=1$, we have,*

$$\psi(x,q,a)=\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\psi(x,\chi)\overline{\chi(a)}.$$

Theorem 2.40. *As $x\to\infty$, we have,*

$$\psi(x)=x+O(xe^{-c\sqrt{\log x}}),$$

where $c>0$ is some constant.

Proof. Recalling zero free regions of ζ, we have for $s=\sigma+it$,

$$\sigma>1-\frac{c}{\log R}\Rightarrow \zeta(s)\neq 0,$$

where c is some constant and $R=\mathrm{Im}(s)$. From this we have,

$$\left|\sum_{|\gamma|<R}\frac{x^\rho}{\rho}\right| \leq x^{1-\frac{c}{\log R}}\sum_{\substack{\rho=\beta+i\gamma\\ \zeta(\rho)=0\\ \beta\in(0,1)\\ |\gamma|<R}}\frac{1}{|\rho|} << x^{1-\frac{c}{\log R}}\sum_{\substack{\beta\in(0,1)\\ \gamma\in(0,R)\\ \zeta(\beta+i\gamma)=0}}\frac{1}{\sqrt{\beta^2+\gamma^2}}.$$

Set

$$g(t)=\frac{1}{\sqrt{\beta^2+t^2}},$$

and

$$a_t := \sum_{\substack{\beta\in(0,1)\\ \gamma\in(0,t)\\ \zeta(\beta+i\gamma)=0}} 1.$$

We now have,

$$\begin{aligned}\sum_{\substack{\beta\in(0,1)\\ \gamma\in(0,R)\\ \zeta(\beta+i\gamma)=0}}\frac{1}{\sqrt{\beta^2+\gamma^2}} &= \frac{N(R)}{\sqrt{\beta^2+\gamma^2}}+\int_0^R\frac{tN(R)}{(\beta^2+\gamma^2)^{\frac{3}{2}}}dt\\ &<< \frac{N(R)}{R}+\int_0^R\frac{N(t)}{t^2}dt,\\ &=\log R+\int_0^2\frac{\log t}{t}+\int_2^R\cdots,\\ &=\log^2 R.\end{aligned}$$

Combining all we get,

$$\left|\sum_{|\gamma|<R} \frac{x^\rho}{\rho}\right| << x^{1-\frac{c}{\log R}} \log^2 R.$$

Choose $\log R = c_2(\log \overline{x})^{\frac{1}{2}}$ we have,

$$\psi(x) = x + O(xe^{-c\sqrt{\log x}}).$$

Using Riemann hypotehsis, choose $R = \sqrt{x}$, we then have as $x \to \infty$,

$$\psi(x) = x + O(\sqrt{x}\log^2 x) + \frac{R\log U}{Ux^R} + \frac{x\log^2 x}{R}.$$

$\square$

Theorem 2.41 (Siegel-Walfisz)**.** *For any $N \in \mathbb{R}$ there is $c_N > 0$ such that as $x \to \infty$,*

$$\psi(x, q, a) = \frac{x}{\varphi(q)} + O(xe^{-c_N\sqrt{\log x}}),$$

where $q \leq (\log x)^N$. This is uniform in q.

Exercise 2.9. *Show that Rieman hypothesis is equivalent to that as $x \to \infty$,*

$$\psi(x) = x + O(\sqrt{x}\log^2 x).$$

Theorem 2.42. *Riemann hypothesis implies that for $(a, q) = 1$, we have, as $x \to \infty$,*

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O(\sqrt{x}\log^2 qx).$$

Remark 2.24. *For $(a, q) = 1$, fixed we have as $x \to \infty$.*

$$\pi(x, q, a) < \frac{2x}{\varphi \log \frac{x}{q}}, q < x.$$