# V5A10 Analytic Number Theory

So Murata

WiSe 25/26, University of Bonn

## 1 Classical Number Theory

**Theorem 1.1** (Euclid)**.** *There are infinitely many prime numbers.*

**Definition 1.1.** $\pi : \mathbb{N} \to \mathbb{N}$ *be a function such that*

$$\pi(n) = \{\textit{prime numbers less than } n\}.$$

**Remark 1.1.**

$$\frac{\pi}{n \ln(n)} \approx 1.$$

**Definition 1.2.**

$$\mathrm{Li}(x) = \int_0^x \frac{1}{\ln(t)} dt.$$

**Notation 1.1.** *Given* $f, g : \mathbb{R} \to \mathbb{C}$,

$$f(x) = O(g(x0))$$

*means that*

$$\exists K \in (0, \infty), x_0 \in \mathbb{R}, \text{ s. t. } \forall x > x_0, |f(x)| \leq K|f(x)|.$$

**Notation 1.2.** *Let* $f, g : \mathbb{R} \to \mathbb{C}$ *be functions.* $f \sim g$ *denotes that*

$$\lim_{n \to \infty} \frac{f(x)}{g(x)}.$$

**Notation 1.3.** *Let* $f : \mathbb{R} \to \mathbb{C}$,

$$\mathrm{Li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k},$$

*denotes that*

$$\mathrm{Li}(x) = \frac{x}{\ln x} \sum_{k=1}^{N-1} \frac{k!}{(\ln x)^k} + O(\frac{x}{(\ln x)^{N+1}}).$$

*and as* $x \to \infty$*, this holds for any* $N \geq 1$.

**Remark 1.2.** *By the integration by parts, we see that it's asymptotic expansion is*

$$\mathrm{Li}(x) \approx \frac{x}{\ln(x)} \sum_{k=0}^{\infty} \frac{k!}{(ln(x))^k}.$$

**Theorem 1.2** (Prime Number Theorem)**.**

$$\lim_{x \to \infty} \frac{\pi(x)}{\mathrm{Li}(x)} = 1.$$

**Definition 1.3** (First Chebyschev Function)**.**

$$\vartheta(x) = \sum_{p \leq x} \ln p.$$

**Definition 1.4** (Second Chebyschev Function)**.**

$$\psi(x) = \sum_{\substack{m,p \\ p^m \leq x}} \ln p.$$

**Remark 1.3.** *We can rewrite the second Chebyschev function as follows.*

$$\psi(x) = \sum_{\substack{p \leq x \\ m = \max\{n \in \mathbb{N} \mid p^n \leq x\}}} m \ln p.$$

**Definition 1.5** (Möbius Function)**.** *Let $n \in \mathbb{N}$, we define,*

$$\mu(n) = \begin{cases} 1, & (n = 1), \\ (-1)^m, & (n \textit{ is square free and has } n \textit{ distinct prime divisors}), \\ 0, & (\text{otherwise}). \end{cases}$$

**Definition 1.6** (Möbius Function)**.**

$$\mu(n) = \begin{cases} 1 & (n = 1) \\ (-1)^k & (n = p_1 \cdots p_k, p_i = p_j \Rightarrow i = j) \\ 0 & (\exists p \text{ s. t. } p^2 | n). \end{cases}$$

**Remark 1.4.** *The prime number theorem is equivalent to the following statements.*

*1).* $\psi(x) \sim x$.

*2).* $\theta(x) \sim x$.

*3).* $\lim_{x \to \infty} \frac{\sum_n \leq x}{\frac{\mu(n)}{n}} x = 0$.

**Conjecture 1.1** (Twin Prime Conjecture)**.** *There exists infinitely many primes $p$ such that $p + 2$ is also prime.*

**Conjecture 1.2** (Goldbach's Conjecture)**.** *Let $n \in \mathbb{N}$ be an even number greater than 2, then there exists two primes $p, q$ such that $n = p + q$.*

**Conjecture 1.3** (Hardy-Littlewood Conjecture)**.**

$$\#\{\text{ prime numbers } p \text{ such that } 2p+1 \text{ is also a prime and } p < x\}$$

**Definition 1.7** (Riemann-Zeta Function)**.** *We define $\zeta : \mathbb{C} \to \mathbb{C}$ such that*

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}.$$

**Remark 1.5.** *Supposer $\mathrm{Re}(s) > 1$, then we have*

$$\begin{aligned} |\zeta(s)| &= \sum_{n \in \mathbb{N}} \frac{1}{|n|^s} \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n^{\mathrm{Re}(s)}} \end{aligned}$$

*By multiplying $\frac{1}{2^s}$, we obtain*

$$\frac{1}{2^s}\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{(2n)^s}.$$

*We get*

$$(1 - \frac{1}{2^s})\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{(2n+1)^s}.$$

*Continuing this procedure, we get the following proposition.*

**Proposition 1.1.**

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)$$

**Theorem 1.3** (Weierstrass)**.** *Let $A \subseteq \mathbb{C}$ and consider a sequence of functions $(f_n : A \to \mathbb{C})_{n \in \mathbb{N}}$ such that there exists a sequence of non-negative numbers $(M_n)_{n \in \mathbb{N}}$ such that*

*i).* $\forall x \in A, |f_n(x)| \leq M_n$.

*ii).* $\sum_{n \in \mathbb{N}} M_n$ *converges.*

*Then the sequence converges uniformly.*

**Theorem 1.4.** *Suppose the conditions in the previous theorem. If each function is analytic on a compact subset of $A$, then the limit is also analytic.*

**Corollary 1.1.** *Let $A$ be a compact subset of a complex plane where $\mathrm{Re}(s) > 1$. Then there exists $\delta > 0$ such that $\mathrm{Re}(s) > 1 + \delta$ and*

$$\sum_{n \in \mathbb{N}} \left| \frac{1}{n^s} \right| \leq \sum_{n \in \mathbb{N}} \frac{1}{n^{1+\delta}} < \infty.$$

**Fact 1.1.** *The Riemann zeta function can be analytically continued to the whole plane except for $s = 1$.*

**Definition 1.8** (Gamma Function)**.**

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

**Proposition 1.2.**

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx.$$

**Remark 1.6.** *$\zeta(1 + it) \neq 0$ if $t \in \mathbb{R}, t \neq 0$. $\zeta(s) \neq 0$ for $0 < s < 1$.*

**Definition 1.9** (Functional Equation)**.**

$$\zeta(1 - s) = 2(2\pi)^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$$

**Remark 1.7.** *$\Gamma(s)$ is defined for $\mathrm{Re}(s) > 0$ and can be analytically continued to the whole place except for $\mathbb{C}\backslash\{-2n \mid n \geq 0\}$.*

**Remark 1.8.** *For $s = -2m$ where $m \in \mathbb{N}$, we see $\zeta(s) = 0$.*

$$\begin{aligned}
\zeta(0) &= \frac{2}{2^\pi}\lim_{s\to1}\cos(\frac{\pi s}{2})\zeta(s) \\
&= \lim_{s\to1}\frac{\cos(\frac{\pi s}{2})}{s-1}\lim_{s\to1}(s-1)\zeta(s) \\
&= \frac{1}{\pi}\times\frac{-\pi}{2}\times 1 \\
&= -\frac{1}{2}.
\end{aligned}$$

**Definition 1.10.** *The subset of the complex plane with its real part between $0$ and $1$. The critical line is the line where $\mathrm{Re}(s) = \frac{1}{2}$.*

**Conjecture 1.4** (Riemann Hypothesis)**.** *Let $s$ be an element of the critical strip. If $\zeta(s) = 0$ then $\mathrm{Re}(s) = \frac{1}{2}$ (ie. it lies on the critical line).*

**Notation 1.4.** *Let $T > 0$. We denote $N(T)$ the number of zeros of $\zeta$ in the critical strip whose coefficient of the imaginary part is in $(0, T)$. That is*

$$N(T) = |\{\sigma + it \in \mathbb{C} \mid 0 < \sigma < 1, 0 < t < T\}|.$$

**Proposition 1.3.**

$$\lim_{T\to\infty}\frac{N(T)2\pi}{T\log(T)} = 1.$$

*Sketch of Proof (needs refinement).*

$$\psi(x) = \frac{1}{2\pi i}\int_l \frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds$$

where $l$ is the line $l = a$ for some $a > 1$.

$$\psi(s) = x - \sum_{\rho \text{ non-trivial zeros}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \log(1 - x^{-2}).$$

$\square$

**Definition 1.11.** *Let $q \in \mathbb{N}$ and $a$ be a natural number coprime to $q$. We define*

$$\pi(x; q, a) = |\{\textit{prime numbers } p \textit{ less than or equal to } x \textit{ such that } p \equiv a \mod q\}|$$

**Proposition 1.4.**

$$\pi(x; , q, a) \sim \frac{x}{\varphi(q)\log(x)}$$

*where $\varphi$ is a Euler phi-function.*

**Theorem 1.5** (Brun–Titchmarsh)**.** *For any $q < x$, we have*

$$\pi(x; , q, a) < \frac{2x}{\varphi(q)\log(\frac{x}{q})}.$$

# 2 Week 2

**Remark 2.1.**

$$\mathrm{Li}(x) \sim \frac{x}{\ln x}\sum_{k=0}^{\infty}\frac{k!}{(\ln x)^k}.$$

*Indeed we have*

$$\mathrm{Li}(x) = .\int_2^x \frac{dt}{\ln t}.$$

*Observe that*

$$\int_2^t \frac{1}{(\ln t)^N} \sim \frac{x}{(\ln x)^N}$$

*for all $N \geq 1$. Thus $\mathrm{Li}(x)$ can be expressed in terms of polynomials in $\frac{x}{\ln(x)}$, by keep replacing the greatest temr with the above approximation.*

**Definition 2.1.** *A function $f : \mathbb{N} \to \mathbb{C}$ is said to be*

*1). multiplicative if for any $(m, n) = 1$, we have $f(mn) = f(m)f(n)$,*

*2). completely multiplicative if for any natural numbers $m, n$, we have $f(mn) = f(m)f(n)$.*

**Example 2.1.** *Möbius function $\mu$ is multiplicative.*

**Definition 2.2** (Von-Mangoldt Function). *The Von-Mangoldt function* $\Lambda : \mathbb{N} \to \mathbb{C}$ *is defined as*

$$\Lambda(n) = \begin{cases} \log(p) & (n = p^k \text{ for some } k \geq 1), \\ 0 & (\text{otherwise}). \end{cases}$$

**Definition 2.3** (Euler Phi Function). *The Euler phi funtion is* $\varphi : \mathbb{N} \to \mathbb{C}$ *such that*

$$\varphi(n) = \{1 \leq a \leq n \mid (a, n) = 1\}.$$

**Example 2.2.** $\varphi$ *is multiplicative but* $\Lambda$ *is not.*

**Definition 2.4** (Dirichlet Characters Modulo $q$). *Let* $q \in \mathbb{N}$ *be a natural number and* $q \geq 2$.

$$\chi_1 : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$$

*be a group homomorphism. The Dirichlet character function modulo* $q$ *with respect to* $\chi_1$ *is such that*

$$\chi(n) = \begin{cases} \chi_1(\overline{n}) & ((n, q) = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

**Example 2.3.** *For* $q = 3$*, we have* $(\mathbb{Z}/3\mathbb{Z})^\times = \{\pm 1\}$*. The only possible character is* $\pm 1 \mapsto \pm 1$*. Therefore, we have*

$$\chi(1) = 1, \chi(2) = -1, \chi(0) = 0.$$

**Theorem 2.1.**

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1, \\ 0 & (\text{otherwise}). \end{cases}$$

*Proof.* When $n = 1$, this is trivial. Suppose $n \neq 1$. We factorize $n$ by

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

where $p_i$ is a prime and $\alpha_i \in \mathbb{N}$ for each $i = 1, \cdots, n$.

Observe that

$$\sum_{d|n} \mu(d) = \sum_{d|\prod_{i=1}^k p_i} \mu(d).$$

Now we see

$$\sum_{d|\prod_{i=1}^k p_i} \mu(d) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j = \sum_{j=0}^{k} \binom{k}{j} (1)^{k-j} (-1)^j = (1-1)^k = 0.$$

$\square$

**Proposition 2.1** (Möbius Inversion Formula)**.** *Let $f, g : \mathbb{N} \to \mathbb{C}$ be functions (we do not assume them to be multiplicative). If*

$$\sum_{d|n} g(d) = f(n),$$

*holds if and only if*

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = g(n).$$

*Proof.*

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu \sum_{e|\frac{n}{d}} g(e).$$

$e|\frac{n}{d}$ if and only if $de|n$ thus obtain,

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu \sum_{de|n} g(e).$$

In particular, we get the expression

$$= \sum_{de|n} \mu(d) g(e).$$

By reordering, we get

$$= \sum_{e|n} g(e) \sum_{d|\frac{n}{e}} \mu(d).$$

By Proposition 2.1, we get

$$\sum_{d|\frac{n}{e}} \mu(d) = 0$$

unless $e = n$. ◻

**Proposition 2.2.**

$$\sum_{d|n} \varphi(d) = n.$$

*Proof.* Consider $(\mathbb{Z}/n\mathbb{Z})^{\times}$. We know that

$$|(\mathbb{Z}/n\mathbb{Z})^{\times}| = \varphi(n).$$

◻

**Theorem 2.2.**

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

*Proof.* Using Proposition 2.2, we have,

$$\sum_{d|n} \mu(d)\frac{n}{d} = \varphi(n).$$

Dividing both sides by $n$ and observe that $\mu(d) \neq 0$ if and only if $d$ is a prime factor of $n$.

$$\begin{aligned}\frac{\varphi(n)}{n} &= \sum_{d|n} \frac{\mu(d)}{d}, \\ &= 1 \sum_{p|n} \frac{1}{p} + \sum_{p_1,p_2|n} \frac{1}{p_1 p_2} - \cdots .\end{aligned}$$

By the induction on the number of prime divisors of $n$, we get the statement. □

**Proposition 2.3.** *We have the following properties of $\varphi$.*

*1).* $n|m \Rightarrow \varphi(n)|\varphi(m)$.

*2).* $\varphi(n)$ *is even for* $n \geq 3$.

*3).* $\varphi(2n) = \begin{cases} 2\varphi(n), & (2|n) \\ \varphi(n), & (2 \not| n). \end{cases}$

*4).* $\varphi$ *is multiplicative.*

*5).* $\varphi(mn) = \varphi(m)\frac{\varphi(n)d}{\varphi(d)}$ *where* $d = (m, n)$.

*6).* $\varphi(n^m) = n^{m-1}\varphi(n)$.

*Proof.* Exercise. □

**Theorem 2.3.** *The following statements are equivalent.*

*1).* $\sum_{d|n} \Lambda(d) = \log n$

*2).* $\sum_{d|n} \mu(d) \log d = \Lambda(n)$.

*And in particular* $\sum_{d|n} \Lambda(d) = \log n$ *holds.*

*Proof.* The equivalence is a direct corollary of Möbius inversion formula. For the latter, Write

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}.$$

We have

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{k} \alpha_i \log p_i = \log(n).$$

□

**Notation 2.1.** *Let $n \in \mathbb{N}$, suppose a prime $p$ divides $n$. Then we denote $\alpha(p)$ to be the highest prime power factor of $n$.*

**Theorem 2.4.** *Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function. Then*

$$\sum_{d|n} f(d) = \prod_{p|n} \left( \sum_{i=0}^{\alpha(p)} f(p^i) \right).$$

*In particular $\sum_{d|n} f(d)$ is also multiplicative.*

*Proof.* Let $d|n$, then we have $d = \prod_{i=1}^{k} p_i^{\beta_i}$ for some $0 \leq \beta_i \leq \alpha(p_i)$. Since $f$ is multipliative we have

$$f(d) = \prod_{i=1}^{k} f(p_i^{\beta_i}).$$

The second part is a direct result of the first part. $\square$

**Remark 2.2.** *The Second Chebyschev Function $\psi$ can be written as*

$$\psi(x) = \sum_{d \leq x} \Lambda(d).$$

*This follows directly from the definition.*

**Definition 2.5** (Dirichlet Series)**.** *Let $f : \mathbb{N} \to \mathbb{C}$ be a function and $s \in \mathbb{C}$. We define*

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}.$$

*For another arithmetic function $g : \mathbb{N} \to \mathbb{C}$, we define*

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} + \sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} = \sum_{n \in \mathbb{N}} \frac{(f(n) + g(n))}{n^s}.$$

*and*

$$\left( \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \times \left( \sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} \right) = \sum_{n,m \in \mathbb{N}} \frac{(f(n)g(m))}{(nm)^s}.$$

$$= \sum_{t \in \mathbb{N}} \sum_{n|t} \frac{\left(f(n) g\left(\frac{n}{t}\right)\right)}{t^s}.$$

Recall the taylor expansion of $\ln x$ we get

$$\ln 2 = \sum_{n \in \mathbb{N}} \frac{(-1)^{n+1}}{n}.$$

Rearranging the following way

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots$$

we get this equals to $\frac{1}{2} \ln 2$.

**Theorem 2.5.** *Let $s \in \mathbb{C}$ be $\mathrm{Re}(s) > 1$, we have*

$$\frac{1}{\zeta(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^s}.$$

*Proof.*

$$\begin{aligned}
\zeta(s) \sum_{n\geq 1} \frac{\mu(n)}{n^s} &= \left(\sum_{n\in\mathbb{N}} \frac{1}{n^s}\right)\left(\sum_{n\in\mathbb{N}} \frac{\mu(n)}{n^s}\right) \\
&= \sum_{t\in\mathbb{N}} \frac{1}{t^s} \sum_{n|t} \mu(n) \\
&= 1.
\end{aligned}$$

$\square$

**Theorem 2.6.** *For $\mathrm{Re}(s) > 1$, we have*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\in\mathbb{N}} \frac{\Lambda(s)}{n^s}.$$

*From this we derive*

$$\lim_{n\to\infty} \frac{\log n}{n^{\varepsilon}} = 0.$$

*Proof.*

$$\begin{aligned}
\zeta(s) \sum_{n\in\mathbb{N}} \frac{\Lambda(n)}{n^s} &= \left(\sum_{m\in\mathbb{N}} \frac{1}{m^s}\right)\left(\sum_{n\in\mathbb{N}} \frac{\Lambda(n)}{n^s}\right), \\
&= \sum_{t\in\mathbb{N}} \frac{1}{t^s} \left(\sum_{n|t} \Lambda\left(\frac{t}{n}\right)\right), \\
&= \sum_{t\in\mathbb{N}} \frac{\log(t)}{t^s}, \\
&= -\zeta'(s).
\end{aligned}$$

$\square$

**Remark 2.3.**

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \left|\frac{\Lambda(s)}{n^s}\right| &\leq \sum_{n\in\mathbb{N}} \frac{\log(n)}{n^{\sigma}}, \\
&<< \sum_{n\in\mathbb{N}} \frac{n^{\varepsilon}}{n^{\sigma}}, \\
&= \sum_{n\in\mathbb{N}} \frac{1}{n^{\sigma-\varepsilon}}.
\end{aligned}$$

*We have* $\lim_{n\to\infty}\frac{\log(n)}{n^{\varepsilon}}=1$ *and the last equation is convergent if and only if* $\sigma-\varepsilon>1$*. Thus we have* $\sigma>1+\varepsilon$*.*

**Remark 2.4.** $\frac{\zeta'(s)}{\zeta(s)}$ *is a meromorphic functions except* $s=1$ *and where* $\zeta(s)$ *vanishes. Indeed, For general* $\frac{f}{g}$*, it is analytic if* $f,g$ *are analytic and* $g\neq 0$*.*

*1).* $\zeta(s)$ *is analytic except* $s=1$*.*

*2).* $\zeta'(s)$ *has a pole of order* $2$ *at* $s=1$*.*

*3).* $\zeta(s)$ *has a pole of order* $1$ *at* $s=1$*.*

Recall that for $|z|\geq 1$, we have,

1. $|z|\geq 1\Rightarrow\sum_{n\in\mathbb{Z}_{\geq 0}}z^n=\frac{1}{1-z}$,
2. $\prod_{n\in\mathbb{N}}(1+a_n)$ is convergent if $\sum_n a_n$ is absolutely convergent,
3. therefore $\prod_{n\in\mathbb{N}}(1+a_n)$ is convergent if and only if $\prod_{n\in\mathbb{N}}(1+|a_n|)$ is convergent.

**Theorem 2.7.** *Let* $f:\mathbb{N}\to\mathbb{C}$ *be a map.*

*If* $f$ *is multiplicative and for* $\mathrm{Re}(s)>r_0, r_0\in\mathbb{R}$ *then we have,*

$$\sum_{n\in\mathbb{N}}\frac{f(n)}{n^s}=\prod_p\left(\sum_{\nu\geq 0}f(p^\nu)p^{-\nu s}\right).$$

*If* $f$ *is completely multiplicative, then*

$$\sum_{n\in\mathbb{N}}\frac{f(n)}{n^s}=\prod_p(1-f(p)p^{-s})^{-1}.$$

*Proof.* Let $A(x)=\{n\in\mathbb{N}\mid \text{primes factors of } n \text{ are} \leq x\}$, then

$$\prod_{p\leq x}\sum_{\nu=0}^{\infty}f(p^\nu)p^{-\nu s}=\sum_{n\in A}\frac{f(n)}{n^s}.$$

Therefore,

$$\begin{aligned}\left|\prod_{x\leq x}\sum_{\nu=0}^{\infty}f(p^\nu)p^{-\nu x}-\sum_{n\in\mathbb{N}}\frac{f(n)}{n^s}\right| &= \left|\sum_{n\in A}\frac{f(n)}{n^s}-\sum_{n\in\mathbb{N}}\frac{f(n)}{n^s}\right|,\\ &=\left|\sum_{n\in\mathbb{N}\setminus A}\frac{f(n)}{n^s}\right|,\\ &\leq\sum_{n\notin A}\frac{|f(n)|}{n^{\mathrm{Re}(s)}},\\ &\leq\sum_{n>x}\frac{|f(n)|}{n^{\mathrm{Re}(s)}}\to 0.\end{aligned}$$

The last limit is due to that it is a tail of a an absolutely convergent series. Since $f$ is completely multipliative, we have

$$f(p^\nu) = (f(p))^\nu.$$

Therefore, we get,

$$\begin{aligned}\prod_p \left( \sum_{\nu \in \mathbb{Z}_{\geq 0}} (f(p^\nu) p^{-\nu s}) \right) &= \prod_p \left( \sum_{\nu \in \mathbb{Z}_{\geq 0}} (f(p) p^{-s})^{-\nu} \right), \\ &= \prod_p \left( \frac{1}{1 - f(p) p^{-s}} \right).\end{aligned}$$

□

**Example 2.4.** *Take $f(n) = 1$ as above we get,*

$$\sum_{n \in \mathbb{N}} \frac{1}{n^s} = \zeta(s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}, \mathrm{Re}(s) > 1.$$

**Example 2.5.**

$$\sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = \prod_p \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1}, \mathrm{Re}(s) > 1.$$

**Example 2.6.**

$$\begin{aligned}\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} &= \prod_p \left( 1 + \frac{\mu(p)}{p^s} \right), \\ &= \prod_p \left( 1 - \frac{1}{p^s} \right), \\ &= \frac{1}{\zeta(s)}.\end{aligned}$$

**Example 2.7.** *Note that $\phi(n) \leq n$. Thus for $\mathrm{Re}(s) > 2$, we have,*

$$\sum_{n \in \mathbb{N}} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

## 2.1 Order of arithmetic functions

**Definition 2.6.** *Let $f, g : \mathbb{N} \to \mathbb{C}$, we denote,*

$$f(n) = O(g(n)),$$

*if there is $K > 0$ and $n_0 \in \mathbb{N}$ such that*

$$n \geq n_0 \Rightarrow |f(n)| \leq K|g(n)|.$$

*An alternative notation for this is $f(n) = O(g(n))$.*

**Definition 2.7.** *We define following arithmetic functions,*

*1). $\nu(n) := \sum_{p|n} 1$, a number of primer divisors of $n$,*

*2). $d(n) := \sum_{d|n} 1$, the number of divisors of $n$,*

*3). $\sigma := \sum_{d|n} d$, the sum of all the divisors of $n$.*

**Lemma 2.1.**

$$\nu(n) << \log(n).$$

*Proof.* Let $n = \prod_{i=1}^{k} p_i^{\alpha(p_i)}$. Then $\nu(n) = k$. Since $p_i \geq 2$, we have,

$$\begin{aligned} \log(n) &= \sum_{i=1}^{k} \alpha(p_i) \log(p_i), \\ &\geq k \log(2). \end{aligned}$$

Therefore $\nu(n) \leq \frac{\log(n)}{\log 2}$. □

**Lemma 2.2.**

$$\sum_{k=2}^{n} \leq \log(n) + 1.$$

*Proof.* We know that

$$\int_1^n \frac{1}{t} dt = \log(n).$$

For $1 \leq k \leq t \leq k+1 \leq n$, we have,

$$\int_k^{k+1} \frac{1}{k+1} dt \leq \int_k^{k+1} \frac{1}{t} dt \leq \int_k^{k+1} \frac{1}{k} dt.$$

Thus we have,

$$\frac{1}{k+1} \leq \log(k+1) - \log(k) \leq \frac{1}{k}.$$

By telescopoing sum we get

$$\sum_{k=2}^{n} \frac{1}{k} \leq \log(n+1).$$

□

**Lemma 2.3.**

$$\sigma(n) << n(1 + \log(n)) \sim n \log(n).$$

*Proof.*

$$\begin{aligned}
\sigma(n) = \sum_{d|n} \frac{n}{d}&, \\
&= n\sum_{d|n} \frac{1}{d}, \\
&= n\left(1 + \sum_{d\geq 2}^{n} \frac{1}{d}\right), \\
&\leq n\left(1 + \sum_{d=2}^{n} \frac{1}{d}\right), \\
&\leq (1+\log(n)).
\end{aligned}$$

The last inequality is due to Lemma 2.2. □

**Exercise 2.1.** *Show that*

$$\sum_{k=1}^{n} \frac{1}{k} = \log(n) + O(1).$$

*That is*

$$\left|\sum_{k=1}^{n} \frac{1}{k} - \log(k)\right| << 1.$$

*Hint: Replace $\frac{1}{t}$ by an increasing function and derive the similar inequality to Lemma 2.2.*

**Lemma 2.4.**

$$d(n) \leq 2\sqrt{n}.$$

*Proof.* If $n = d_1 d_2$ then one of them must be less than or equal to $\sqrt{n}$. □

We have an improved inequality,

**Proposition 2.4.** *for $\varepsilon > 0$, we have,*

$$d(n) << n^{\varepsilon}.$$

*Proof.* Recall that for $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, we have $d(n) = \prod_{i=1}^{k}(\alpha_i+1)$. In particular, we have,

$$\frac{d(n)}{n^{\varepsilon}} = \prod_{i=1}^{k} \frac{(\alpha_i+1)}{p_i^{\varepsilon\alpha_i}}.$$

Let $A = \{i \mid p_i^{\varepsilon} \geq 2\}$. Rcall that for $x \geq 1$, $x+1 \leq 2^x$ that is

$$\frac{x+1}{2^x} \leq 1.$$

Then,

$$\prod_{i\in A}\frac{\alpha_i+1}{p_i^{\varepsilon\alpha_i}}\le\prod_{i=1}^{k}\frac{\alpha_i+1}{2^{\varepsilon\alpha_i}}\le 1.$$

For $p_i^{\varepsilon}<2$, we observe,

$$p_i^{\varepsilon\alpha_i}=e^{\varepsilon\alpha_i\log(p_i)}\ge\varepsilon\alpha_i\log(p_i).$$

Therefore,

$$\begin{aligned}\prod_{i\notin A}\frac{\alpha_i+1}{p_i^{\varepsilon\alpha_i}}&\le\prod_{i\notin A}\left(\frac{\alpha_i}{p_i^{\varepsilon\alpha_i}}+1\right),\\&\le\prod_{i\notin A}\left(\frac{\alpha_i}{\varepsilon\alpha_i\log(p_i)}+1\right),\\&\le\prod_{i\notin A}\left(\frac{1}{\varepsilon\log(2^{\frac{1}{\varepsilon}})}+1\right),\\&\le\prod_{i\notin A}\left(\frac{1}{\log(2)}+1\right).\end{aligned}$$

Combining two cases, we obtain the statement. $\square$

**Notation 2.2.** *Let $x\in\mathbb{R}$. We denote*

1. *the integer part $[x]\in\mathbb{Z}$ which is the greatest integer not exceeding $x$,*
2. *the fraction part $\{x\}=x-[x]$.*

**Proposition 2.5.**

$$\frac{\sum_{n\le x}d(n)}{x}\sim\log(x).$$

*Proof.* By definition, we have,

$$\sum_{n\le x}d(n)=\sum_{n\le x}\sum_{d|n}1.$$

$d|n$ if and only if there is $e$ such that $de=n$. Thus using this we obtain,

$$\sum_{n\le x}d(n)=\sum_{\substack{e,d\\de\le x}}1=\sum_{d\le x}\sum_{e\le\frac{x}{d}}1=\sum_{d\le x}\left[\frac{x}{d}\right].$$

Using the definition of $[x]$, we have,

$$\sum_{n\le x}d(n)=\sum_{d\le x}\frac{x}{d}-\left\{\frac{x}{d}\right\}=x\sum_{d\le x}\frac{1}{d}-\sum_{d\le x}\left\{\frac{x}{d}\right\}=x(\log(x)+O(1))+O(x).$$

Thus we obtain the statement. $\square$

**Definition 2.8.** *Let $f : \mathbb{N} \to \mathbb{C}$, we say the average order of $f$ is $g : \mathbb{R} \to \mathbb{C}$ if*

$$\frac{\sum_{n \leq x}}{f(n)} x \sim g(x).$$

Proposition 2.5 can be restated as follows.

**Proposition 2.6.** *The average order of $d$ is $\log(x)$.*

**Exercise 2.2.** *Examine the following statements.*

1. *Is it true that $d(n) << \log n$?*
2. *Do we have $d(n) = O(n^{\varepsilon})$ for any $\varepsilon > 0$?*
3. *What is the optimal bound for $d(n)$?*

**Theorem 2.8.** *There exists $c_1, c_2 > 0$ such that*

$$c_1 \leq \frac{\varphi(n)\sigma(n)}{n^2} \leq c_2.$$

*Proof.* Recall that

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

and

$$\sigma(n) = \prod_{p|n} \frac{p^{\alpha(p)+1} - 1}{p - 1}.$$

Thus we obtain,

$$\frac{\sigma(n)}{n} = \frac{\prod_{p|n}(1 + p + \cdots + p^{\alpha(p)})}{\prod_{p|n} p^{\alpha(p)}} = \prod_{p|n} \left(\frac{\frac{1}{p^{\alpha(p)+1}} - 1}{\frac{1}{p} - 1}\right)$$

By multiplying two we get,

$$\frac{\varphi(n)\sigma(n)}{n^2} = \prod_{p|n} \left(1 - \frac{1}{p^{\alpha(p)+1}}\right) \leq 1.$$

On the other hand,

$$\prod_{p} \left(1 - \frac{1}{p^2}\right) \leq \frac{\varphi(n)\sigma(n)}{n^2}.$$

The left hand side is equal to $\frac{1}{\zeta(2)}$ which is $\frac{6}{\pi^2}$. $\square$

**Theorem 2.9.** *The average order of $\varphi$ is $\frac{3n}{\pi^2}$.*

*Proof.*

$$\begin{aligned}
\sum_{n\leq x}\phi(n) &= \sum_{n\leq x} n \sum_{d|n}\frac{\mu(d)}{d}, \\
&= \sum_{\substack{d,e \\ de\leq x}} e\mu(d), \\
&= \sum_{d\leq x}\mu(d)\left(\sum_{e\leq\frac{x}{d}} e\right), \\
&= \frac{1}{2}\sum_{d\leq x}\mu(d)\left(\left[\frac{x}{d}\left(\left[\frac{x}{d}\right]+1\right)\right]\right),
\end{aligned}$$

Note that

$$\frac{x}{d} = \left[\frac{x}{d}\right] + \left\{\frac{x}{d}\right\} = \left[\frac{x}{d}\right] + o(1).$$

It is assined as an exercise to confirm that

$$\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right) = \frac{x^2}{d^2} + o\left(\frac{x}{d}\right).$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{d\leq x}\mu(d)\left(\frac{x^2}{d^x} + o\left(\frac{x}{d}\right)\right), \\
&= \frac{x^2}{2}\sum_{d\leq x}\frac{\mu(d)}{d^2} + o\left(x\sum_{d\leq x}\frac{\mu(d)}{d}\right), \\
&= \frac{x^2}{2}\left(\sum_{d\geq 1}\frac{\mu(d)}{d^2} - \sum_{d\geq x}\frac{\mu(d)}{d^2}\right) + o\left(x\sum_{d\geq x}\frac{\mu(d)}{d}\right), \\
&= \frac{x^2}{2}\frac{1}{\zeta(2)} - \frac{x^2}{2}\sum_{d\geq x}\frac{\mu(d)}{d}^2 + o\left(x\sum_{d\leq x}\frac{\mu(d)}{d}\right).
\end{aligned}$$

We have,

$$\begin{aligned}
\left|\sum_{d\geq x}\frac{\mu(d)}{d^2}\right| &\leq \sum_{d\geq x}\frac{1}{d^2} \\
&<< \int_x^\infty \frac{dt}{t^2}, \\
&<< \frac{1}{x}.
\end{aligned}$$

and,

$$\left| \sum_{d \leq x} \frac{\mu(d)}{d} \right| << \ln x.$$

Using these we have,

$$\begin{aligned} &= \frac{x^2}{2} \frac{1}{\zeta(2)} + o(x) + o(x \ln x), \\ &= \frac{x^2}{2\zeta(2)} + o(x \ln x). \end{aligned}$$

We conclude that

$$x \to \infty \Rightarrow \frac{\sum_{n \leq x} \phi(n)}{x^2} \to \frac{1}{2\zeta(2)}.$$

In particular,

$$\frac{\sum_{n \leq x} \phi(n)}{x} \sim \frac{x}{2\zeta(2)} = \frac{x \cdot 6}{2\pi^2}.$$

□

## 2.2 Abel's Summation Formula

Recall the harmonic series $\sum_{n \in \mathbb{N}} \frac{1}{n}$ is divergent. Our next goal is to find such $A_x$ that

$$\lim_{n \to \infty} \left( \sum_{n \leq x} \frac{1}{n} - A_x \right)$$

exists.

**Remark 2.5** (Euler-Mascheroni constant)**.** *By taking $A_x = \log(x)$, we have*

$$\lim_{n \to \infty} \left( \sum_{n \leq x} \frac{1}{n} - \log(x) \right) = \psi,$$

*exists. Such $\psi$ is called Euler-Mascheroni constant.*

**Remark 2.6** (Euler Kronecer constant)**.** *Take $A_x = \log(x)$, we have*

We can show that

$$\psi = \lim_{s \to 1^+} \left( \frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right).$$

Hint first show that

$$\zeta(s) = \frac{1}{s-1} + \psi + o(s-1).$$

**Proposition 2.7** (Abels' summation formula)**.** *Given* $(a_n)_{n\in\mathbb{N}}$ *in* $\mathbb{C}$ *and* $f(n)$ *is continuously differentiable in* $[1,x]$*. Set*

$$A(x) := \sum_{n\leq x} a_n.$$

*Then we have,*

$$\sum_{n\leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

*Proof.* Observe that

$$a_n = A(n) - A(n-1).$$

Assume $x \in \mathbb{N}$. We substitute this to $\sum_{n\leq x} a_n f(n)$, we get,

$$\begin{aligned}
\sum_{n\leq x} a_n f(n) &= \sum_{n\leq x} \left(A(n) - A(n-1)\right) f(n), \\
&= \sum_{n\leq x} A(n)f(n) - \sum_{n\leq x} A(n-1)f(n), \\
&= \sum_{n\leq x} A(n)f(n) - \sum_{n\leq x-1} A(n)f(n+1), \\
&= A(x)f(x) - \sum_{n\leq x-1} A(n)(f(n+1) - f(n)), \\
&= A(x)f(x) - \sum_{n\leq x-1} \int_n^{n+1} f'(t)dt, \\
&= A(x)f(x) - \sum_{n\leq x-1} \int_n^{n+1} A(t)f'(t)dt, \\
&= A(x)f(x) - \int_1^x A(t)f'(t)dt.
\end{aligned}$$

For the case when $n \notin \mathbb{N}$ and $n > 1$,

$$\sum_{n\leq x} a_n f(n) = \sum_{n\leq [x]} a_n f(n).$$

Using the previous case, we get,

$$\sum_{n\leq x} a_n f(n) = A([x])f([x]) - \int_1^{[x]} A(t)f'(t)dt.$$

Remains to show that we can remove the brackets. To do so,

$$\begin{aligned}\sum_{n\le x} a_n f(n) &= A([x])f([x]) - \int_1^x A(t)f'(t)dt + \int_{[x]}^x A(t)f'(t)dt,\\ &= A([x])f([x]) - \int_1^x A(t)f'(t)dt + A([x])\int_{[x]}^x f'(t)dt,\\ &= A([x])f([x]) - \int_1^x A(t)f'(t)dt + A([x])f(x) - A([x])(f[x]),\\ &= A([x])f(x) - \int_1^x A(t)f'(t)dt.\end{aligned}$$

□

**Corollary 2.1.**

1. $\sum_{n\le x}\frac{1}{n} = \ln x + \psi + o\left(\frac{1}{x}\right)$.

2. $\sum_{n\le x}\frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + o\left(\frac{1}{x^s}\right)$, *where* $\mathrm{Re}(s) > 0, s \neq 1$.

*We also have the following equivalent forms of prime number theorem when* $x \to \infty$.

$$\begin{aligned}\sum_{n\le x} s(n) \sim x&\\ &\Leftrightarrow \pi(x) \sim \frac{x}{\ln x},\\ &\Leftrightarrow \sum_{p\le x}\ln p \sim x,\\ &\Leftrightarrow \sum_{n\le x}\mu(n) = o(x).\end{aligned}$$

*Proof.* Consider $f(t) = \frac{1}{t^s}$ and $a_n = 1$ for all $n \in \mathbb{N}$.

$$\begin{aligned}\sum_{n\le x}\frac{1}{n} &= \frac{[x]}{x} + \int_1^x \frac{[x]}{t^2}dt,\\ &= \frac{x-\{x\}}{x} + s\int_1^x \frac{t-\{t\}}{t^{s+1}}dt.\end{aligned}$$

When $s = 1$, we have,

$$
\begin{aligned}
\sum_{n \le x} \frac{1}{n} &= 1 - \frac{\{x\}}{x} + \int_1^x \frac{t - \{t\}}{t^{s+1}} dt, \\
&= 1 + \ln x + o\left(\frac{1}{x}\right) - \int_1^x \frac{\{t\}}{t^2} dt, \\
\lim_{x \to \infty} \left( \sum_{n \le x} \frac{1}{n} - \ln x \right) &= 1 - \int_1^\infty \frac{\{t\}}{t^2} dt, \\
&= \psi. \\
&= x^{1-s} + o\left(\frac{1}{x^s}\right) + \frac{s x^{1-s}}{1-s} - \frac{s}{1-s} - s^2 \int_{\frac{\{t\}}{t^{s+1}}} dt.
\end{aligned}
$$

Recall that

$$
\left[ \int \frac{1}{t^s} = \frac{t^{-s+1}}{1-s} \right]_1^x = \frac{x^{1-s}}{1-s} - \frac{1}{1-s}.
$$

Using this we obtain,

$$
\begin{aligned}
&= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - x \int_1^x \frac{\{t\}}{t^{s+1}} dt, \\
x^{1-s}\left[1 + \frac{1}{1-s}\right] &= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - s \int_1^x \frac{\{t\}}{t^{s+1}} dt \\
\int_1^\infty \frac{\{t\}}{t^{s+1}} &< \infty, \\
&\le \int_1^\infty \frac{1}{t^{\mathrm{Re}(s)+1}} < \infty.
\end{aligned}
$$

As $x \to \infty$, the left hand side goes to $\zeta(s)$, for the right hand side, we get $= \frac{-s}{1-s} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt$, where $\mathrm{Re}(s) > 1$,

$$
\zeta(s) = \frac{-s}{1-s} - s \int \frac{\{t\}}{t^{s+1}} dt.
$$

Identity theorem for analytic function tells us the analytic continuation of Riemann zeta function is unique.

**Remark 2.7.** *It is an exercise that*

$$
\int_1^\infty \frac{\{t\}}{t^{s+1}} dt
$$

*where* $\mathrm{Re}(x) > 0$. *From Stein-Schakarchi 5.2, 5.3, we have*

$$
\sum f_n(z) \overset{unif.}{\to} f(z)
$$

*is analytic where* $\mathrm{Re}(x) > 0$ *and* $s \neq 1$*, also in this case,*

$$\zeta(s) = \frac{-s}{1-s} - s\int \frac{\{t\}}{t^{s+1}}dt.$$

*holds.*

**Remark 2.8** (Exercise)**.** *Let* $M \in \mathbb{N}$ *and*

$$\lim_{x\to\infty}\left(\sum_{\substack{n\leq x\\(n,M)=1}} \frac{1}{n} - \frac{\phi(M)}{M}\ln x\right)$$

*exists.*

Assume $\pi(x) \sim \frac{x}{\ln x}$, to show

$$\theta(x) := \sum_{p\leq x} \ln p \sim x,$$

Consider the following sequence

$$a_n = \begin{cases} 1 & n \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f(t) = \ln t.$$

Using Abel summation formula,

$$\sum_{p\leq x} \ln p = \pi(x)\ln(x) - \int_1^{\infty} \frac{\pi(t)}{t}dt.$$

can be written as

$$\frac{\theta(x)}{x} = \frac{\pi(x)\ln x}{x} - \frac{1}{x}\int_1^x \frac{\pi(t)}{t}dt.$$

**Remark 2.9** (Exercise)**.** *Use* $\frac{\pi(t)}{t} \sim \frac{1}{\ln t}$ *and*

$$\int_1^x \frac{dt}{\ln t} = o(x),$$

*prove that*

$$\lim_{x\to\infty} \frac{1}{x}\int_1^x \frac{\pi(t)}{t}dt = 0.$$

$$\begin{aligned}\int_2^{\sqrt{x}} \frac{dt}{\ln t} = \int_{\sqrt{x}}^x \frac{dt}{\ln t} &\leq \frac{1}{\ln 2}\int_2^{\sqrt{x}} dt + \frac{1}{\ln\sqrt{x}}\int_{\sqrt{x}}^x dt, \\ &= \frac{\sqrt{x}-2}{\ln 2} + \frac{x-\sqrt{x}}{\ln\sqrt{x}}, \\ &= o(x).\end{aligned}$$

**Remark 2.10.** 
$$\psi(x) = \sum_{n \le x} s(n) = \sum_{1 \le k, p^k \le x} \ln p.$$

*Thus we see,*

$$\psi(x) - \theta(x) = \sum_{2 \le k, p, p^k \le x} \ln p.$$

*Also,*

$$p^k \le x \Rightarrow k \le \frac{\ln x}{\ln p}.$$

*Using $k \ge 2$,*

$$p \le x^{\frac{1}{k}} \le \sqrt{x}, \forall k.$$

$$\begin{aligned}
&\le \sum_{p \le \sqrt{x}} \ln p \left( \sum_{2 \le k \le \frac{\ln x}{\ln p}} 1 \right), \\
&\le \sum_{p \le \sqrt{x}} \ln x, \\
&\le \ln(x) \sum_{n \le \sqrt{x}} 1, \\
&\le \sqrt{x} \ln x. \Rightarrow \qquad \frac{\psi(x)}{x} = \frac{\theta(x)}{x} + \frac{o(\sqrt{x}\ln x)}{x}.
\end{aligned}$$

Therefore we obtain,

$$\psi(x) \sim x \Leftrightarrow \theta(x) \sim x.$$

□

**Remark 2.11.** *As exercises, find the closed expressions for the following summations.*

$$\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n}, \sum_{p \le x} \frac{1}{p} = \ln \ln x + o(1).$$

## 2.3 Characters

**Definition 2.9.** *Let $G$ be a finite group. A character is a group homomorphism $f : G \to \mathbb{C}^\times$.*

**Remark 2.12.** *Let us denote*

$$\hat{G} := \{f : G \to \mathbb{C}^\times \mid \textit{characters}\}.$$

*If $G$ is finite abelian then $|\hat{G}| = |G|$. Furthermore, such characters are linearly independent over $\mathbb{C}$.*

**Definition 2.10.** *Let $q \in \mathbb{N}$ and $q \geq 3$. A Dirichlet character is a group homomorphism modulo $q$ is a group homomorphism*

$$\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times.$$

**Remark 2.13.** *Given a Dirichlet character $\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$. We ca define a character $\chi : \mathbb{Z} \to \mathbb{C}^\times$ as follows.*

$$\chi(a) := \begin{cases} \chi'(\overline{a}), & (a,q) = 1, \\ 0, & (a,q) \neq 1. \end{cases}$$

*From Remark 2.12, there are exactly $\varphi(q)$ many Dirichlet characters modulo $q$. Furthermore,*

$$\chi(a)^{\varphi(q)} = \chi'(\overline{a})^{\varphi(q)} = \chi'(\overline{a}^{\varphi(q)}) = \chi'(\overline{1}) = 1.$$

*In particular, images of $\chi$ are $\varphi(q)$-th roots of unity.*

**Example 2.8.** *For $q = 3$,*

$$(\mathbb{Z}/3\mathbb{Z})^\times = \{\overline{1}, \overline{2}\} \to \mathbb{C}^\times.$$

*We only have two characters, a trivial one and $\overline{2} \mapsto -1$.*

**Example 2.9.** *For $q = 5$,*

$$(\mathbb{Z}/5\mathbb{Z})^\times = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$$

| | $\overline{1}$ | $\overline{2}$ | $\overline{3}$ | $\overline{4}$ |
|---|---|---|---|---|
| $\chi_{1,5}(n)$ | 1 | 1 | 1 | 1 |
| $\chi_{2,5}(n)$ | 1 | $-1$ | $-1$ | 1 |
| $\chi_{3,5}(n)$ | 1 | $i$ | $-i$ | $-1$ |
| $\chi_{4,5}(n)$ | 1 | $-i$ | $i$ | $-1$ |

*$\chi_{1,5}$ is called a principle/trivial character.*

**Definition 2.11.** *A character $\chi : \mathbb{Z} \to \mathbb{C}^\times$ is called*

1. *trivial if $\chi(g) = 1$ for all $g \in G$,*
2. *even if $\chi(-1) = 1$,*
3. *odd if $\chi(-1) = -1$.*

*We also define these notions for characters $\chi_0 : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ accordingly if characters induces by $\chi : \mathbb{Z} \to \mathbb{C}^\times$ has these properties. Trivial characters are often denoted by $\chi_0$.*

**Theorem 2.10.**

$$\sum_{a \bmod q} \chi(a) = \begin{cases} \phi(q) & (\chi = \chi_0), \\ 0 & (otherwise). \end{cases}$$

*We also have,*

$$\sum_{\chi \bmod q} \chi(a) = \begin{cases} \phi(a) & (\overline{a} = 1), \\ 0 & (otherwise). \end{cases}$$

*Proof.* If $\chi$ is principle then the first assertion is clear. Suppose $\chi$ is not principle then there is $b \in \{1, \cdots, q\}$ such that $\chi(b) \neq 1$ and $(b, q) = 1$. Let

$$s = \sum_{a \bmod q} \chi(a).$$

Then by the definiion of group homomorphisms, we have,

$$\chi(b)s = s.$$

But $\chi(b) \in \mathbb{C}$, this means $s = 0$ as $\mathbb{C}$ is an integral domain.

For the second assertion, let $\overline{a} \neq 1$, then

$$\exists \chi' \bmod q, \text{ s.t. } \chi'(a) \neq 1.$$

Thus we get,

$$s = \sum_{\chi \bmod q} \chi(a), s \cdot \chi(a) = \sum_{\chi \bmod q} \chi\chi'(a) = s \Rightarrow s = 0.$$

The statement when $\overline{a} = 1$ follows from Remark 2.12. □

**Remark 2.14.** *One can check in the table of Example 2.9 that Theorem 2.10 indeed holds.*

**Exercise 2.3.**

$$\sum_{\substack{\chi \bmod q, \\ \chi(-1)=1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} & (\overline{a} = 1, -1), \\ 0 & (otherwise). \end{cases},$$

$$\sum_{\substack{\chi \bmod q, \\ \chi(-1)=-1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} & (\overline{a} = 1), \\ -\frac{\phi(a)}{2} & (\overline{a} = -1), \\ 0 & (otherwise). \end{cases},$$

Obviously we have the following equalities.

$$\sum_{n\le x}\chi(n)=\sum_{\substack{n\le x\\(n,q)=1}}\chi(n)=\sum_{n\le kq}\chi(n)+\sum_{n=kq+1}^{x}\chi(n),$$

where $k$ is the largest integer such that $kq\le x$. Then we observe from Theorem 2.10

$$\sum_{n\le kq}\chi(a)=k\left(\sum_{n=1}^{q}\chi(n)\right)=0,$$

unless $\chi$ is trivial. Also we have,

$$\left|\sum_{n\le x}\chi(n)\right|=\left|\sum_{\substack{kq+1\le n\le x\\(n,q)=1}}\chi(n)\right|\le\sum_{\substack{kq+1\le n\le x\\(n,q)=1}}1\le\sum_{\substack{kq+1\le n\le kq+q\\(n,q)=1}}1=\phi(q).$$

Thus we conclude,

$$\sum_{n\le x}\chi(n)\le\phi(n),$$

**Exercise 2.4.**

$$\sum_{n\le x}\chi_0(n)=?.$$

**Theorem 2.11** (Pólya–Vinogradov)**.**

$$\sum_{n\le x}\chi(n)<<\sqrt{q}\ln q,(\chi\ne\chi_0\mod q).$$

*notice that the above expression is bounded by $\sqrt{q}\ln\ln q$. Furthermore, this is uniform in $q$ that is the constant does not depend on $q$.*

$$\sum_{n\ge1}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}-\frac{s}{1-s}+O(x^{-s})-s\int_1^x\frac{\{t\}}{t^{s+1}}dt.$$

For $\mathrm{Re}(s)>1$, as $x\to\infty$ we have,

$$\sum_{n\ge1}^{\infty}\frac{1}{n^s}=\frac{-s}{1-s}-s\int_1^\infty\frac{\{t\}}{t^{s+1}}dt.$$

The last expression is analytic since,

$$\sum\int_n^{n+1}\frac{\{t\}}{t}^{s+1}dt\overset{\text{uniformly}}{\to}\int_1^\infty\frac{\{t\}}{t^{s+1}},\text{when }\mathrm{Re}(s)>0.$$

Suppose $\zeta(s)\ne0$, where $\mathrm{Re}(s)>0$, then Euler product exists.

**Theorem 2.12.** *Set*

$$A(n) := \sum_{n\in\mathbb{N}} a_n.$$

*Assume* $A(x) := \sum_{n\leq x} a_n = O(x^\delta)$, *then we have, for* $\mathrm{Re}(s) > \delta$,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s\int_1^\infty \frac{A(t)}{t^{s+1}}dt.$$

*Hence the Dirichlet series converges for* $\mathrm{Re}(s) > \delta$.

*Proof.*

$$\sum_{n\in\mathbb{N}} \frac{a_n}{n^s} = \frac{A(x)}{x^s} + s\int_1^x \frac{A(t)}{t^{s+1}}dt.$$

As $A(x) = O(x^\delta)$ and $\mathrm{Re}(s) > \delta$, $\frac{A(x)}{x^s} = O(x^{\delta-\mathrm{Re}(s)})$. Therefore, as $x \to \infty$, we have,

$$\sum_{n\in\mathbb{N}} \frac{a_n}{n^s} = s\int_1^\infty \frac{A(t)}{t^{s+1}}dt.$$

Again using the assumption, we have,

$$\int_1^\infty \left|\frac{A(t)}{t^{s+1}}\right| dt << \int_1^\infty t^{\delta-\mathrm{Re}(s)-1}dt = \left.\frac{t^{\delta-\mathrm{Re}(s)}}{\delta-\mathrm{Re}(s)}\right|_0^\infty = \frac{1}{\delta-\mathrm{Re}(s)}.$$

Thus the integral is convergent. □

**Definition 2.12.** *For* $\mathrm{Re}(s) > 1$, *we define*

$$L(s,\chi) := \sum_{n\in\mathbb{N}} \frac{\chi(n)}{n^s}.$$

**Remark 2.15.** *Since* $\mathrm{Re}(s) > 1$ *and for any character* $\chi : \mathbb{Z} \to \mathbb{C}^\times$ *we have* $|\chi(n)| \leq 1$, $L(s,\chi)$ *is uniformly absolutely convergent.*

**Example 2.10.** *Let* $\chi : \mathbb{Z} \to \mathbb{C}^\times$ *be a non-principal character modulo* $q$ *and set* $A(n) = \chi(n)$. *Recall that*

$$\sum_{n\leq x} \chi(n) \leq q.$$

*Taking* $A(n) = \chi(n)$ *and apply Theorem 2.12, we obtain, for* $\mathrm{Re}(s) > 0$,

$$L(s,\chi) = \sum_{n\in\mathbb{N}} \frac{\chi(n)}{n^s} = s\int_1^\infty \frac{\sum_{n\leq t}\chi(n)}{t^{s+1}}dt.$$

*Since for* $\mathrm{Re}(s) > 1, L(s,\chi)$ *is absolutely uniformly convergent. By Theorem 2.7, we have,*

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

*If $\chi = \chi_0$, a principal character, we have,*

$$L(s, \chi_0) = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p} \left(1 - \frac{1}{p^s}\right) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

*Using $\zeta$ function, we have,*

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

**Theorem 2.13.** *$\zeta$ has an analytic continuation for $\mathrm{Re}(s) > 0$ besides $s = 1$. For $s = 1$ we have a simple pole of residue $1$.*

*Proof.* Recall from the proof of Corollary 2.1. We have for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{-s}{1 - s} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

The right hand side of the equation is analytic when $\mathrm{Re}(s) > 0, s \neq 1$. When $s = 1$, we have,

$$\lim_{s \to 1^+} (s - 1)\zeta(s) = \lim_{s \to 1^+} \left(s - s(s - 1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt\right) = 1.$$

□

**Corollary 2.2.** *For $\mathrm{Re}(s) > 0$ and $s \neq 1$, we have an analytic continuation of $L(s, \chi_0)$ where $\chi_0$ is a principal character modulo $q$, which is*

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

*Obviously at $s = 1$, it has a simple pole of residue $\prod_{p|q} \left(1 - \frac{1}{p^s}\right)$, which we can write as*

$$\mathrm{Res}(L(s, \chi_0), 1) = \frac{\varphi(q)}{q}.$$

*Proof.* A direct corollary of Theorem 2.13. □

Suppose $\chi \neq \chi_0$, we have, analytic continuation of $L(s, \chi) \neq 0$is

**Theorem 2.14.** *Let $\chi$ be a non-principal character, then there is an analytic continuation of $L(s, \chi)$ for $\mathrm{Re}(s) > 0$.*

*Proof.* From Example 2.10, we have,

$$L(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = s \int_0^\infty \frac{\sum_{n \leq t} \chi(n)}{t^{s+1}} dt.$$

The right hand side is analytic. □

**Theorem 2.15.**
$$\sum_{n\in\mathbb{N}} \frac{a_n}{n^s}$$
*is analytic in its range of convergence.*

**Remark 2.16.** *We have the following conjecture.*
$$L\left(\frac{1}{2},\chi\right) = 0, \chi \neq \chi_0.?$$
$$\zeta\left(\frac{1}{2}\right) = \frac{1}{1-\sqrt{2}} \sum_{n\in\mathbb{N}} \frac{(-1)^{n-1}}{\sqrt{n}} \approx -1.46\cdots$$

**Definition 2.13.** *A character is said to be quadratic if its values are either* $\pm 1$*.*

**Remark 2.17.** *We have,*
$$\begin{aligned} L(s,\chi) &= 0 \quad \text{if } s = 0,-2,-4, \text{ when } \chi \text{ is an even character.}\\ L(s,\chi) &= 0 \quad \text{if } s = -1,-3,-5, \text{ when } \chi \text{ is an odd character.}\end{aligned}$$

**Lemma 2.5.** *For* $\sigma > 1$ *and* $t \in \mathbb{R}$*, we have,*
$$\mathrm{Re}(\ln(\zeta(\sigma+it))) = \sum \frac{\Lambda(n)}{n^\sigma \ln n} \ln(t\ln(n)).$$
*And also,*
$$\mathrm{Re}(3\ln(\zeta(\sigma)) + 4\ln(\zeta(\sigma+it)) + \ln(\zeta(\sigma+2it))) \geq 0.$$

*Proof.*
$$\begin{aligned} \zeta(s) &= \prod_p \left(1-\frac{1}{p^s}\right)^{-1}, \sigma > 0,\\ \ln(\zeta(s)) &= -\sum_p \ln(1-p^{-s})\\ &= \sum_{p,n} \frac{1}{np^{ns}}, \sigma > 1.\end{aligned}$$

We have,
$$\sum_{n\geq 2} \frac{\Lambda(n)}{n^s \ln n} = \sum_{p,k,k\geq 1} \frac{\ln p}{p^{ks}\ln p^k} = \sum_{p,k} \frac{1}{kp^{ks}} = \sum_{n\geq 2} \Lambda(n).$$

$$\mathrm{Re}(3\ln\zeta(\sigma) + 4\ln\zeta(\sigma+it) + \ln\zeta(\sigma+2it)) = \sum_{n\geq 2} \frac{\Lambda(n)}{n^\sigma \ln n}(3 + 4\cos(t\ln n) + \cos(2t\ln n)) \geq 0,$$

since

$$3 + 4\cos\theta + \cos 2\theta = 2(\cos\theta + 1)^2 \geq 0.$$

$$= \ln|\zeta(\sigma)^3\zeta(\sigma + it)^4\zeta(\sigma + 2it)| \geq 0.$$

Thus we have,

$$\mathrm{Re}(\ln(z)) \leq$$

□

**Theorem 2.16.** *For $t \in \mathbb{R}\backslash\{0\}$, we have*

$$\zeta(1 + it) \neq 0.$$

*Proof.* Using Lemma 2.5, we have, Thus we get,

$$|\zeta(\sigma)^3\zeta(\sigma + it)^4\zeta(\sigma + 2it)| \geq 1.$$

Suppose $\zeta(1 + it_0) = 0$, for $t_0 \in \mathbb{R}\backslash\{0\}$. Suppose further that the order of zero is $m \in \mathbb{N}$. Then by looking at, and taking $\sigma \to 1+$

$$\underbrace{\overset{3}{((\sigma - 1)\zeta(\sigma))}}_{\to\text{finite}} \underbrace{\overset{4}{\left(\frac{\zeta(\sigma + it_0)}{(\sigma - 1)^m}\right)}}_{\to\text{non-zero}} \underbrace{((\sigma - 1)^{4m-3}\zeta(\sigma + 2it_0))}_{\to 0}.$$

Contradicts to that the abosolute value of above expression is at least 1. □

**Theorem 2.17.** $\frac{\zeta'(s)}{\zeta(s)}$ *has an analytic continuation to* $\mathrm{Re}(s) = 1, s \neq 1$. *And for $s = 1$, we have a simple pole of residue $-1$.*

*Proof.* We have,

$$(s - 1)\zeta(s) = s - s(s - 1)\int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

Set

$$f(s) := 1 - (s - 1)\int_1^\infty \frac{\{t\}}{t^{s+1}} dt,$$

so that $(s - 1)\zeta(s) = sf(s)$. We already have $f(s)$ is analytic when $\mathrm{Re}(s) > 0$. Differentiating both sides we get,

$$(s - 1)\zeta'(s) + \zeta(s) = sf'(s) + f(s).$$

Dividing both sides by $(s - 1)\zeta(s)$ we get,

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s - 1} = \frac{sf'(s) + f(s)}{(s - 1)\zeta(s)}.$$

The right hand side is analytic when $\zeta$ does not vanish. From Theorem 2.16, by letting $s = 1 + it$ for some $t \in \mathbb{R}\backslash\{0\}$, we get the desired analytic continuation. For $s = 1$, we have

$$(s-1)\frac{\zeta'(s)}{\zeta(s)} = \frac{sf'(s) + f(s)}{\zeta(s)} - 1.$$

Recall that $\zeta(s)$ has a pole at 1 and observe that $f(1) = 1, f'(s)$ is finite thus the last statement follows. $\square$

**Theorem 2.18.** *Let $\chi$ be a non-trivial non-real Dirichlet character of modulo $q$, then*

$$L(1, \chi) \neq 0.$$

*Proof.* Recall that

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where $\mathrm{Re}(s) > 1$. Now consider,

$$\log L(s, \chi) = \sum_{n,p} \frac{\chi(p^s)}{np^{ns}} = \sum_{n \geq 2} \frac{\Lambda(n)\chi(n)}{n^s \log n}.$$

Set $\zeta_{\varphi(q)} = e^{\frac{2\pi i}{\varphi(q)}}$. Then for any $n \in (\mathbb{Z}/q\mathbb{Z})^\times$, there is $n' = n(n, q, \chi)$ (ie $n'$ depends on $n, q$, and $\chi$) such that

$$\chi(n) = \zeta_{\varphi(q)}^{n'}$$

Note that for a cyclic group $G = \langle g \rangle$, and a character $\chi : G \to \mathbb{C}^\times \in \hat{G}$, there is $a \in \{1, \cdots, \varphi(q)\}$ such that

$$\chi(g) = \zeta_{\varphi(q)}^a.$$

We have, for $\sigma > 1$

$$\mathrm{Re}(\log L(\sigma, \chi)) = \sum_{n \geq 2} \frac{\Lambda(n) \cos\left(\frac{2\pi i}{\varphi(q)}\right)}{n^\sigma \log(n)}.$$

By Lemma 2.5, we have,

$$\mathrm{Re}(3\zeta(\sigma) + 4L(\sigma, \chi) + L(\sigma, \chi^2)) \geq 0,$$

therefore,

$$|\zeta(\sigma)^3 L(\sigma, \chi)^4 L(\sigma, \chi^2)| \geq 1.$$

Note that $\chi^2 \neq \chi$ as it is non-real and $L(\sigma, \chi^2)$ is analytic by Theorem 2.14. If $L(\sigma, \chi) = 0$ then

$$\zeta(\sigma)^3 L(\sigma, \chi)^4$$

has a zero of order at least 1. Thus this is a contradiction. $\square$

**Lemma 2.6.** *For $k \in \mathbb{Z}_{\geq 0}$,*

$$\left(\frac{\sin\left(k+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}\right)^2 = 2k+1\sum_{j=1}^{2k} 2(2k+1-j)\cos j\theta.$$

**Theorem 2.19.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a fnction such that*

*i). $f$ is analytic,*

*ii). $f \not\equiv 0$,*

*iii). $\log f(s) = \sum_{n\in\mathbb{N}} \frac{a_n}{n^s}$ for $a_n \geq 0$ and $\mathrm{Re}(s) \geq 1$,*

*iv). $f$ is analytic on $\mathrm{Re}(s) = 1, s \neq 1$, and it has a pole at $s = 1$ of order $e$.*

*If $f(s) = 0$ on the line $\mathrm{Re}(s) = 1$, then the order of zero is at most $\frac{e}{2}$.*

*Proof.* For $e \leq 2k-1$, let us define,

$$g(s) := f(s)^{2k+1}\prod_{j=1}^{2k} f(s+ijt_0)^{2(2k+1-j)} = f(s)^{2k+1}f(s+it_0)^{4k}\cdots.$$

Then $f(s)^{2k+1}$ has a pole of order $e(2k+1)$ and $\prod_{j=1}^{2k} f(s+ijt_0)^{2(2k+1-j)}$ has a zero of order $2k(2k+1)$ Note that

$$2k(2k+1) - e(2k+1) \geq 1.$$

Thus $g$ has a zero at 1 of order at least 1, in particular, $g(1) = 0$.

Consider,

$$\begin{aligned}
\log g(\sigma) &= (2k+1)\log f(\sigma) + \sum_{j=1}^{2k} 2(2k+1-j)\log(f(\sigma+ijt_0)),\\
&= (2k+1)\sum_{n\in\mathbb{N}}\frac{a_n}{n^\sigma} + \sum_{j=1}^{2k} 2(2k+1-j)\left(\sum_{n\in\mathbb{N}}\frac{a_n}{a^{\sigma+ijt_0}}\right),\\
&= \sum_{n\in\mathbb{N}}\frac{a_n}{n^\sigma}\left(2k+1+\sum_{j=1}^{2k} 2(2k+i-j)e^{-ijt_0\log(n)}\right).\\
\mathrm{Re}(\log(g(\sigma))) &= \sum_{n\in\mathbb{N}}\frac{a_n}{n^\sigma}\left(2k+1+\sum_{j=1}^{2k} 2(2k+i-j)\cos(jt_0\log(n))\right).
\end{aligned}$$

Using Lemma 2.6, we get,

$$\mathrm{Re}\log(g(\sigma)) = \sum_{n\geq 1}\frac{a_n}{n^\sigma}\left(\frac{\sin\left(k+\frac{1}{2}\right)t_0\log(n)}{\sin\frac{t_0\log(n)}{2}}\right)^2 \geq 0.$$

Therefore $|g(\sigma)| \geq 1$ if $\mathrm{Re}\, g(\sigma) \geq 1$. ∎

**Corollary 2.3.** *For any Dirichlet character* $\chi$ *of modulo* $q$ *we have,* $L(s,\chi)$ *is analytic over* $\mathrm{Re}(s) > 1$. *Furthermore* $L(s,\chi) \neq 0$ *on* $\mathrm{Re}(s) = 1, s \neq 1$.

*Proof.* The first assertion is due to Theorem 2.14. Let

$$f(s) := \prod_{\chi \bmod q} L(s,\chi).$$

Then,

$$\log(f(s)) = \sum_{\chi \bmod q} \log L(s,\chi) = \sum_{n,p} \frac{\sum_{\chi \bmod q} \chi(p^n)}{np^{ns}}, \mathrm{Re}(s) > 1.$$

Using Theorem 2.10, we get,

$$\log(f(s)) = \sum_{\substack{n,p \\ p^n \equiv 1 \bmod q}} \frac{\varphi(q)}{np^{ns}}.$$

Note $f$ has a pole of order at most 1 at $\mathrm{Re}(s) = 1$ and its residue is

$$\mathrm{Res}(f(s),1) = \left(\lim_{s\to 1}(s-1)L(s,\chi_0)\right) \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} L(1,\chi) = \frac{\varphi(q)}{q} \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} L(1,\chi).$$

The right hand side is not 0 by Theorem 2.18. □

**Theorem 2.20.** *There are infinitely many primes.*

*Proof.* We have,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

$$\log \zeta(s) = \sum_{n,p} \frac{1}{np^{ns}} = \sum_p \frac{1}{p^s} + \sum_{n\geq 2} \frac{1}{np^{ns}}.$$

Since $\zeta$ has a pole at 1, so does its log. Observe that at $s = 1$,

$$\sum_p \frac{1}{p^s} + \sum_{n\geq 2} \frac{1}{np^{ns}} \leq \sum_p \sum_{n\geq 2} \frac{1}{p^n} = \sum_p \frac{1}{p^2} \frac{1}{1-\frac{1}{p}} = \sum_p \frac{1}{p(p-1)} << \sum_{n\in\mathbb{N}} \frac{1}{n^2}.$$

Thus $\lim_{s\to 1^+} \sum_p \frac{1}{p^s}$ must be infinity. □

**Corollary 2.4.** *For any Dirichlet character* $\chi$ *of modulo* $q$ *we have,* $L(s,\chi) \neq 0$ *for* $\mathrm{Re}(s) = 1, s \neq 1$.

*Proof.* Let

$$f(s) := \prod_{\chi \bmod q} L(s,\chi).$$

□

**Lemma 2.7.** *Let $p$ be a prime and $a, q$ be coprimes such that $p \equiv a \bmod q$. Then we have,*

$$\sum_{\chi \bmod q} \chi(p^n)\overline{\chi(a)} = \begin{cases} \varphi(q), & (p^n \equiv a \bmod q), \\ 0, & (\text{otherwise}). \end{cases}$$

*Proof.* Since $\chi(a)$ is a $\varphi(q)$-th root of unity, we have, $\overline{\chi(a)} = \chi(a^{-1})$. Therefore,

$$\begin{aligned} \sum_{\chi \bmod q} \chi(p^n)\overline{\chi(a)} &= \sum_{\chi \bmod q} \chi(p^n)\chi(a^{-1}), \\ &= \sum_{\chi \bmod q} \chi(p^n a^{-1}), \\ &= \begin{cases} \varphi(q), & (p^n a^{-1} \equiv 1 \bmod q), \\ 0, & (\text{otherwise}). \end{cases} \end{aligned}$$

□

**Theorem 2.21** (Dirichlet's Theorem)**.** *Let $a, q$ be coprime. Then $(a + nq)_{n \in \mathbb{N}}$ contains infinitely many primes.*

*Proof.* Motivated by the alternative proof of the existence of infinitely many primes, examine,

$$\sum_{\chi \bmod q} \log L(s, \chi) = \sum_{n,p} \left( \frac{\sum_{\chi \bmod q} \chi(p^n)}{np^{ns}} \right),$$

where $\mathrm{Re}(s) > 1$. Using Theorem 2.10, we have,

$$\sum_{\chi \bmod q} \log L(s, \chi) = \sum_{\substack{n,p \\ p^n \equiv 1 \bmod q}} \frac{\varphi(q)}{np^{ns}} = \sum_{p^n \equiv 1 \bmod q} \frac{\varphi(q)}{p^s} + \sum_{\substack{p, n \geq 2 \\ p^n \equiv 1 \bmod q}} \frac{\varphi(q)}{np^{ns}}.$$

Taking the log out we have,

$$\prod_{\chi \bmod q} L(s, \chi) = \exp\left( \varphi(q) \left( \sum_{p^n \equiv 1 \bmod q} \frac{1}{p^s} + \sum_{\substack{p, n \geq 2 \\ p^n \equiv 1 \bmod q}} \frac{1}{np^{ns}} \right) \right).$$

Since for $\chi \neq \chi_0, L(1, \chi) \neq 0$ and by Corollary 2.2, we have,

$$\lim_{s \to 1^+} (s - 1) \prod_{\chi \bmod q} L(s, \chi) = \frac{\varphi(q)}{q} \prod_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} L(1, \chi).$$

By the same argument from Theorem 2.20, we see,

$$\sum_{\substack{p,n\geq 2\\ p^n\equiv \bmod q}} \frac{1}{np^{ns}} \leq \sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

However,

$$\lim_{s\to 1^+}(s-1)\exp\left(\varphi(q)\left(\sum_{p^n\equiv 1 \bmod q}\frac{1}{p^s} + \sum_{\substack{p,n\geq 2\\ p^n\equiv 1 \bmod q}}\frac{1}{np^{ns}}\right)\right) \neq 0$$

which is only possible when

$$\lim_{s\to 1^+}\sum_{p^n\equiv 1 \bmod q}\frac{1}{p^s} = \infty.$$

Together with Lemma 2.7, we derived the general statement. □

**Theorem 2.22** (Bertrand's Postulate)**.** *For a sufficiently large $n\in\mathbb{N}$, there is a prime number inbetween $n$ and $2n$.*

*Proof.* Consider the second Chebyschev function, and by Remark 2.2, we have,

$$T(x) := \sum_{l\leq x}\psi\left(\frac{x}{l}\right) = \sum_{l\leq x}\sum_{n\leq \frac{x}{l}}\Lambda(n) = \sum_{\substack{l,n\\ ln\leq x}}\Lambda(n).$$

That is

$$\sum_{l\leq x}\psi\left(\frac{x}{l}\right) = \sum_{m\leq x}\sum_{d|m}\Lambda(d).$$

By Theorem 2.3,

$$T(x) = \sum_{m\leq x}\log(m).$$

Using the definition,

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= \sum_{l\leq x}\psi\left(\frac{x}{l}\right) - 2\sum_{2l\leq x}\psi\left(\frac{x}{2e}\right), \\ &= \sum_{l\leq x}(-1)^{l+1}\psi\left(\frac{x}{l}\right), \\ &= \psi(x) - \psi\left(\frac{x}{2}\right) + \cdots . \end{aligned}$$

Again by Remark 2.2

$$x\leq y \Rightarrow \psi(x)\leq\psi(y).$$

In particular,

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq \sum_{l \leq x} (-1)^{l+1} \psi\left(\frac{x}{l}\right) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right).$$

Recall Proposition 2.7, consider

$$A(x) = \sum_{n \leq x} 1 = \lfloor x \rfloor,$$

and since $\log(x)$ is continuously differentiable on $[1, x]$, we have,

$$T(x) = \lfloor x \rfloor \log(x) - \int_1^x \frac{\lfloor x \rfloor}{x} dx = x \log(x) - x + O(\log(x)).$$

Therefore,

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= x\log(x) - x + O(\log(x)) - 2\left(\frac{x}{2}\log\left(\frac{x}{2}\right) - \frac{x}{2} + O(\log(x))\right), \\ &= \log 2 \cdot x + O(\log(x)). \end{aligned}$$

Combining with the previous result, we get,

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq (\log 2)x + O(\log(x)) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right).$$

Generalizing this we obtain,

$$\psi\left(\frac{x}{2^{n-1}}\right) - \left(\frac{x}{2^n}\right) \leq \log(2)\frac{x}{2^{n-1}} + O(\log(x)).$$

By induction, we obtain,

$$\begin{aligned} \psi(x) - \psi\left(\frac{x}{2^n}\right) &\leq (\log 2)x\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) + O(n\log(x)), \\ &\leq (2\log 2)x + O(n\log(x)). \qquad (\psi 1) \end{aligned}$$

Take $n$ to be the maximal such that $2^n \leq x$ that is $\lfloor \frac{x}{2^n} \rfloor = 1$. Then, $\psi\left(\frac{x}{2^n}\right) = 0$, thus,

$$\psi(x) \leq (2\log 2)x + O((\log(x))^2).$$

On the other hand,

$$\begin{aligned} &\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \geq (\log 2)x + O(\log(x)), \\ &\Rightarrow \psi(x) - \psi\left(\frac{x}{2}\right) \geq (\log(2))x + O(\log(x)) - \psi\left(\frac{x}{3}\right) \end{aligned}$$

Using Inequality ($\psi 1$), we get,

$$\begin{aligned} \psi(x) - \psi\left(\frac{x}{2}\right) &\geq (\log 2)x + O(\log(x)) - (2\log 2)\frac{x}{3} + O((\log(x))^2), \\ &= (\log 2)\frac{x}{3} + O((\log(x))^2). \qquad (\psi 2) \end{aligned}$$

Using Remark 2.10, that is

$$\psi(x) - \theta(x) = O(\sqrt{x}\log(x)),$$

together with Inequalities ($\psi1$) and ($\psi2$), we obtain,

$$\theta(x) - \theta\left(\frac{x}{2}\right) \geq (\log 2)\frac{x}{3} + O(\sqrt{x}\log(x)).$$

Thus the right hand side is greater than 0 if $x$ is sufficiently large with coefficient, therefore by definition of $\theta$, we have,

$$\theta(x) - \theta\left(\frac{x}{2}\right) = \sum_{\frac{x}{2} \leq p \leq x} \log(p) > 0.$$

□

**Theorem 2.23** (Chebyschev)**.**

## 2.4 Ikehara-Wiener Theorem and Its Applications

**Theorem 2.24** (Ikehara-Wiener)**.** *Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of non-negative numbers and set,*

$$f(s) = \sum_{n\in\mathbb{N}} \frac{b_n}{n^s}.$$

*Suppose*

*i). the series converges abslutely for* $\mathrm{Re}(s) > 1$*,*

*ii). $f$ has an analytic continuation to* $\mathrm{Re}(s) = 1$ *except $s = 1$,*

*iii). $f$ has a simple pole at $s = 1$ with residue $R \geq 0$.*

*Then we have,*

$$\sum_{n\leq x} b_n = Rx + O(x).$$

*That is*

$$\lim_{x\to\infty} \frac{\sum_{n\leq x} b_n}{x} = R.$$

**Lemma 2.8.** *Let $\chi$ be a Dirichlet character modulo $q$, then*

$$L(s,\chi)^{-1} = \sum_{n\in\mathbb{N}} \frac{\chi(n)\mu(n)}{n^s}.$$

*Proof.* We immitate the proof of Theorem 2.5. Using Theorem 2.1,

$$L(s,\chi)\left(\sum_{n\in\mathbb{N}} \frac{\mu(n)\Lambda(n)}{n^s}\right) = \sum_{t\in\mathbb{N}}\sum_{n|t} \frac{\chi\left(\frac{t}{n}\right)\chi(n)\mu(n)}{t^s} = \sum_{t\in\mathbb{N}} \chi(t)\sum_{n|t}\mu(n) = 1.$$

□

**Theorem 2.25.** *Let $\chi$ be a character then, for* $\mathrm{Re}(s) > 1$,

$$-\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{n\in\mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s}.$$

*Proof.* By Lemma 2.8 and Proposition 2.1

$$\begin{aligned}
-\frac{L'(s,\chi)}{L(s,\chi)} &= \left(\sum_{t\in\mathbb{N}} \frac{\chi(n)\log(t)}{t^s}\right)\left(\sum_{n\in\mathbb{N}} \frac{\mu(n)\chi(n)}{n^s}\right), \\
&= \sum_{t\in\mathbb{N}} \chi(t) \sum_{n|t} \frac{\mu\left(\frac{t}{n}\right)\log(n)}{t^s}, \\
&= \sum_{n\in\mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s}.
\end{aligned}$$

□

**Definition 2.14.**

$$\psi(x,q,a) = \sum_{\substack{n\le x \\ n\equiv a \bmod q}} \Lambda(n).$$

**Proposition 2.8.** *As $x\to\infty$, we have,*

$$\psi(x,q,a) \sim \frac{x}{\varphi(q)}.$$

*Proof.* Consider

$$\sum_{\chi \bmod q} \overline{\chi}(a)\left(\sum_{n\in\mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s}\right) = \sum_{n\in\mathbb{N}} \frac{\Lambda(n)}{n^s}\left(\sum_{\chi \bmod q} \chi(a^{-1}n)\right)$$

Using Lemma 2.7, we get,

$$-\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a)\frac{L'(s,\chi)}{L(s,\chi)} = \varphi(q) \sum_{n\equiv a \bmod q} \frac{\Lambda(n)}{n^s}.$$

Recall from Corollary 2.2, we have,

$$\mathrm{Res}(L(s,\chi_0),1) = \frac{\varphi(q)}{q}.$$

And by Theorem 2.14, $L(s,\chi)$ is analytic at $s=1$ for $\chi\neq\chi_0$. Also using Theorem 2.16, we have

$$\chi\neq\chi_0 \Rightarrow \frac{L'(s,\chi)}{L(s,\chi)} \text{ is analytic for } \mathrm{Re}(s)\ge 1.$$

Furthermore, we have Theorem 2.17, we have

$$\operatorname{Res}\left(\frac{\zeta'(s)}{\zeta(s)}, 1\right) = -1.$$

Combining these, we get,

$$\lim_{s\to 1}(s-1)\left(-\frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\frac{L'(s,\chi)}{L(s,\chi)} - \frac{1}{\varphi(q)}\frac{L'(s,\chi_0)}{L(s,\chi_0)}\right) = \frac{1}{\varphi(q)}.$$

Now Using Theorem 2.24, and apply $b_n = \Lambda(n)$ $f = \varphi(q)\sum_{n\equiv a \bmod q}\frac{\Lambda(n)}{n^s}$, we get,

$$\psi(x,q,a) = \sum_{n\le x}\Lambda(n) = \frac{1}{\varphi(q)}x + O(x).$$

□

## 2.5 $L(s,\chi)\neq 0$ for Quadratic Characters

**Lemma 2.9.** *Let $f := \sum_{d|n}\chi(d)$, where $\chi$ is a character, then,*

$$\forall n\in\mathbb{N}, n \textit{ is a perfect square} \Rightarrow f(n)\ge 0, f(n)\ge 1.$$

*Proof.* Recall $n = \prod_{p|n} p^{\alpha(p)}$. Using this we have,

$$\begin{aligned}\sum_{d|n}\chi(d) &= \prod_{p|n}\left(\sum_{k=0}^{\alpha(p)}\chi(p)^k\right),\\ &= \begin{cases} 1 & \chi(p)=0,\\ \prod_{p|n}(1+\alpha(p)) & \chi(p)=1,\\ \prod_{p|n}\left(\frac{(1-(-1)^{\alpha(p)+1})}{2}\right) & \chi(p)=-1.\end{cases}\end{aligned}$$

Note that if $\alpha(p)$ are all even for $p|n$ (ie. $n$ is a perfect square), we have the last part of the cases equals to 1. Thus we have $f(n)\ge 1$. □

**Theorem 2.26.** *Let $f(n) = \sum_{d|n}\chi(d)$ for some character. Then we have,*

$$\sum_{n\le x}\frac{f(n)}{\sqrt{n}} = 2\sqrt{x}L(1,\chi) + o(1).$$

*Proof.*

$$\begin{aligned}
\sum_{n\le x}\frac{f(n)}{\sqrt{n}} &= \sum_{n\le x}\left(\frac{\sum_{d|n}\chi(d)}{\sqrt{n}}\right),\\
&= \sum_{\substack{d,e\\ de\le x}}\frac{\chi(d)}{\sqrt{de}},\\
&= \sum_{\substack{d,e\le x\\ d\le\sqrt{x}}}\frac{\chi(d)}{\sqrt{de}} + \sum_{\substack{de\le x\\ d>\sqrt{x}}}\frac{\chi(d)}{\sqrt{de}},\\
&= \sum_{d\le\sqrt{x}}\frac{\chi(d)}{\sqrt{d}}\left(\sum_{e\le\frac{x}{d}}\frac{1}{\sqrt{e}}\right) + \sum_{e\le\sqrt{x}}\frac{1}{\sqrt{e}}\left(\sum_{\sqrt{x}<d\le\frac{x}{e}}\frac{\chi(d)}{\sqrt{(d)}}\right).
\end{aligned}$$

Recall that from Proposition 2.7,

$$\sum_{m\le x}\frac{1}{\sqrt{m}} = 2\sqrt{x}+B+o\left(\frac{1}{\sqrt{x}}\right),$$

where $B$ is some constant as $x\to\infty$.

Let $x,y\in\mathbb{R}$, such that $x<y$, we have,

$$\begin{aligned}
\sum_{x<d\le y}\frac{\chi(d)}{\sqrt{d}} &= \sum_{d\le y}\frac{\chi(d)}{\sqrt{d}} - \sum_{d\le x}\frac{\chi(d)}{\sqrt{(d)}},\\
\sum_{d\le x}\frac{\chi(d)}{\sqrt{d}} &= \frac{\sum_{d\le x}\chi(d)}{\sqrt{x}} + \frac{1}{2}\int_1^x\frac{\sum_{d\le t}\chi(d)}{t^{\frac{1}{2}}},\\
&= o\left(\frac{1}{\sqrt{x}}\right).
\end{aligned}$$

Using these equations, we have,

$$\begin{aligned}
\sum_{e\le\sqrt{x}}\frac{1}{\sqrt{e}}\left(\sum_{\sqrt{x}<d\le\frac{x}{e}}\frac{\chi(d)}{\sqrt{(d)}}\right) &= \sum_{e\le x}\frac{1}{\sqrt{e}}\left(o\left(\frac{1}{x^{\frac{1}{r}}}\right)\right),\\
&= \left(o\left(\frac{1}{x^{\frac{1}{r}}}\right)\right)\sum_{e\le x}\frac{1}{\sqrt{e}},
\end{aligned}$$

$$\sum_{e\le\sqrt{x}}\frac{1}{\sqrt{x}} << \int_1^{\sqrt{x}}\frac{1}{\sqrt{t}}dt = x^{\frac{1}{4}}.$$

Thus we conclude,

$$\sum_{e\le\sqrt{x}}\frac{1}{\sqrt{e}}\left(\sum_{\sqrt{x}<d\le\frac{x}{e}}\frac{\chi(d)}{\sqrt{(d)}}\right) = o(1).$$

We also have,

$$\sum_{d\leq\sqrt{x}}\frac{\chi(d)}{\sqrt{d}}\left(\sum_{e\leq\frac{x}{d}}\frac{1}{\sqrt{e}}\right)=\sum_{d\leq\sqrt{x}}\frac{\chi(d)}{\sqrt{d}}\left(2\sqrt{\frac{x}{d}}+B+o\left(\sqrt{\frac{d}{x}}\right)\right),$$

$$=2\sqrt{x}\sum_{d\leq\sqrt{x}}\frac{\chi(d)}{d}+B\sum_{d\leq\sqrt{x}}\frac{\chi(d)}{\sqrt{d}}+o\left(\frac{1}{\sqrt{x}}\sum_{d\leq\sqrt{x}}\chi(d)\right).$$

$$2\sqrt{x}\sum_{d\leq\sqrt{x}}\frac{\chi(d)}{d}=2\sqrt{x}\left(\sum_{d\geq 1}\frac{\chi(d)}{d}-\sum_{d>\sqrt{x}}\frac{\chi(d)}{d}\right),$$

$$=2\sqrt{x}L(1,\chi)-2\sqrt{x}\sum_{d>\sqrt{x}}\frac{\chi(d)}{d},$$

$$=2\sqrt{x}L(1,\chi)-2\sqrt{x}o\left(\frac{1}{\sqrt{x}}\right),$$

$$=2\sqrt{x}L(1,\chi)+o(1).B\sum_{d\leq\sqrt{x}}\frac{\chi(d)}{\sqrt{d}}\leq o\left(\frac{B}{x^{\frac{1}{4}}}\right)=o(1).$$

$$\sum_{d\leq\sqrt{x}}\chi(d)\leq q?=o(1).$$

$\square$

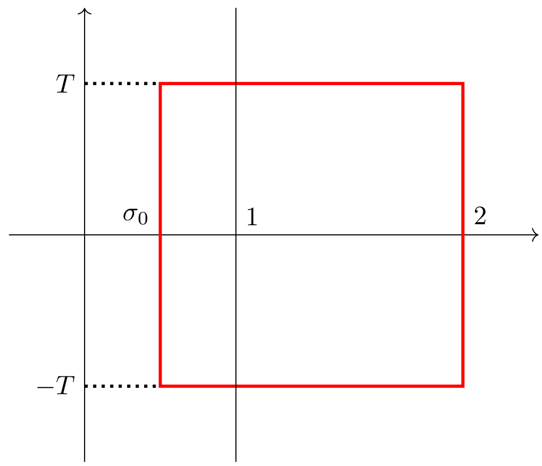
**Corollary 2.5.** *$L(1,\chi)\neq 0$ for a quadratic character $\chi$.*

*Proof.* To derive a contradiction, assume $L(1,\chi)=0$. Then, from Theorem 2.26 and Lemma 2.9

$$o(1)=\sum_{n\leq x}\frac{f(n)}{\sqrt{n}}\geq\sum_{\substack{n\leq x\\ n\text{ is a square}}}\frac{1}{\sqrt{n}}=\sum_{m\leq\sqrt{x}}\frac{1}{m}=\log\sqrt{x}+o(1).$$

Thus we obtain $o(1)\geq\log\sqrt{x}+o(1)$ which is a cotradiction. $\square$

**Lemma 2.10.** *Consider the following rectangle,*




*where $\sigma_0 = 1 - \log T$. On the boundary of the rectangle above, we have,*

1. $|\zeta(s)| = O(\log T), T \to \infty$,
2. $|\zeta'(s)| = O((\log T)^2)$.

*Proof.* The first statement is due to the first derivative of $\zeta$ and it is assgined as an exercise. For the second part, for $\mathrm{Re}(s), \sigma > 1$, we have,

$$\begin{aligned}
\zeta(s) &= \sum_{n\geq 1} \frac{1}{n^s}, \\
&= \sum_{n\leq T} \frac{1}{n^s} + \sum_{n>T} \frac{1}{n^s}, \\
&= \sum_{n\leq T} \frac{1}{n^s} - \frac{\lfloor T \rfloor}{T^s} + s\int_1^\infty \frac{\lfloor t \rfloor}{t^{s+1}} dt, \\
&= \sum_{n\leq T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} + \frac{\{T\}}{T^s} - s\int_T^\infty \frac{\{u\}}{u^{s+1}} du.
\end{aligned}$$

Note that the right hand side is analytic where $\mathrm{Re}(s) > 0$ and $s \neq 1$. Now we will estimate the last equation above on the boundary of the rectangle above.

$$\begin{aligned}
\left|\sum_{n\leq T} \frac{1}{n^s}\right| &<< \int_1^T \frac{du}{u^{\mathrm{Re}(s)}}, \\
&\leq \int_1^T \frac{du}{u^{\sigma_0}}, \\
&= \frac{T^{1-\sigma_0}}{1-\sigma_0}, \\
&<< \log T.
\end{aligned}$$

Observe that ,

$$1-\sigma_0 = \frac{1}{\log T}, T^{1-\sigma_0} = T^{\frac{1}{\log T}} = \exp(\log T^{\frac{1}{\log T}}) = \exp(1) = e.$$

We also have,

$$\begin{aligned}
\left|\frac{T^{1-s}}{s-1}\right| &= \frac{T^{1-\mathrm{Re}(s)}}{|s-1|}, \\
&\leq \frac{T^{1-\sigma_0}}{|s-1|}, \\
&\leq \frac{1}{|s-1|}, \\
&\leq \frac{1}{\sigma_0 - 1} = \log T.
\end{aligned}$$

Also consider,

$$\left|\frac{\{T\}}{T}\right| \leq \frac{1}{T^{\mathrm{Re}(s)}} \leq 1.$$

Finally we have,

$$\begin{aligned}
\left|-s\int_T^\infty \frac{\{u\}}{u^{s+1}} du\right| &\leq |s| \int_T^\infty \frac{du}{u^{\mathrm{Re}(s)+1}}, \\
&= \frac{|s|}{-\mathrm{Re}(s) u^{\mathrm{Re}(s)}}\Bigg|_T^\infty, \\
&= \frac{|s|}{\mathrm{Re}(s) T^{\mathrm{Re}(s)}}, \\
&\leq \frac{|s|}{T^{\mathrm{Re}(s)}}, \\
&\leq \frac{\sqrt{2^2+T^2}}{T^{\sigma_0}}, \\
&<< \frac{T}{T^{\sigma_0}} = T^{1-\sigma_0} = e.
\end{aligned}$$

For the second part, for $\mathrm{Re}(s) > 0, s \neq 1$,

$$\begin{aligned}
\zeta'(s) = &\sum_{n\leq T} \frac{-\log n}{n^s} + \frac{T^{1-s}(-\log T)}{s-1} + T^{1-s}\frac{-1}{(s-1)^2} \\
&+ \frac{\{T\}\log T}{T^s} - \int_T^\infty \frac{\{u\}}{u^{s+1}} du + s\int_T^\infty \frac{\{u\}\log u}{u^{s+1}} du.
\end{aligned}$$

This is obtained simply differentiating the equation,

$$\zeta(s) = \sum_{n\leq T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} - \frac{\{T\}}{T^s} - s\int_T^\infty \frac{\{u\}}{u^{s+1}} du.$$

The statement can be shown using all the estimates obtained to show the first part. ☐

**Theorem 2.27** (Complex Mean Value Theorem)**.** *Let $f : \Omega \to \mathbb{C}$ be an analytic, where $\Omega$ is a convex open set. Let $a, b \in \Omega$, then there exists $z_1, z_2 \in (a, b)$ such that*

$$\mathrm{Re}(f'(z_1)) = \mathrm{Re}\left(\frac{f(b) - f(a)}{b - a}\right), \mathrm{Im}(f'(z_2)) = \mathrm{Im}\left(\frac{f(b) - f(a)}{b - a}\right).$$

**Theorem 2.28.** *There exists constants $c_1, c_2$ such that*

$$1 - \frac{c_1}{(\log T)^9} \leq \sigma \leq 2, |\zeta(s)| > \frac{c_2}{(\log T)^7},$$

*where $1 < |\operatorname{Im}(s)| \leq T$.*

*Proof.* Recall Lemma 2.5,

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1, \sigma > 1.$$

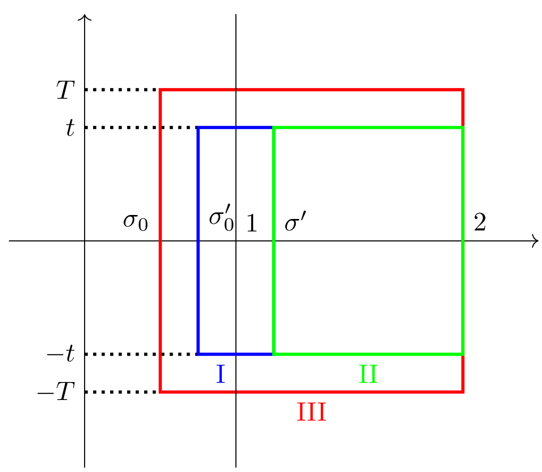
Thus we have,

$$|\zeta(\sigma + it)|^4 \geq |\zeta(\sigma)|^{-3} |\zeta(\sigma + 2it)|^{-1}, \tag{$*$}$$

Consider

$$\sigma_0' = 1 - \frac{c_1}{(\log T)^9}, \sigma' = 1 + \frac{c_1}{(\log T)^9}$$

$c_1$ is a positive constant which will be adjusted later.



Fix the domain II where $\sigma' \leq \sigma \leq 2$, $1 \leq |t| \leq T$.

$$\begin{aligned}
\zeta(s) &= \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}du, \mathrm{Re}(s) > 0, s \neq 1.\\
\zeta(\sigma) &= \frac{\sigma - 1 + 1}{\sigma - 1} - \sigma\int_1^\infty \frac{\{u\}}{u^{t+1}}dt.\\
\zeta(\sigma) &= 1 + \frac{1}{\sigma - 1} + o(1),\\
\zeta(\sigma) &<< \frac{1}{\sigma - 1} \text{ as } \sigma \to 1^+,\\
\Rightarrow & \zeta(\sigma)^{-1} >> (\sigma - 1), \sigma \to 1^+
\end{aligned}$$

On the domain II, we have, $\sigma_0' < \sigma \leq 2, 1 \leq |t| \leq T$, and thus

$$|\zeta(\sigma + 2it)| = O(\log T).$$

Substituting these to Equation $(*)$, we have,

$$\begin{aligned}
|\zeta(\sigma + it)|^4 &>> (1-\sigma)^3(\log T)^{-1},\\
&>> \frac{c_1^3}{(\log T)^{27}}(\log T)^{-1}.\\
\Rightarrow |\zeta(\sigma + it)| &>> \frac{c_1^{\frac{3}{4}}}{(\log T)^7},
\end{aligned} \tag{$D2$}$$

in the domain II.

On the domain I, similarly use Theorem 2.27 with

$$a = \sigma' + it, b = \sigma + it,$$

where,

$$1 - \frac{c_1}{(\log T)^9} \leq \sigma \leq \sigma'.$$

We have,

$$\frac{\zeta(\sigma' + it) - \zeta(\sigma + it)}{\sigma' - \sigma} = \mathrm{Re}(\zeta(z_1)) + \mathrm{Im}(\zeta(z_2)).$$

Furthermore,

$$\zeta(\sigma' + it) - \zeta(\sigma + it) = O((\sigma' - \sigma)(\log T)^2).$$

By the definition of $\sigma'$, we have,

$$\zeta(\sigma' + it) = \zeta(\sigma + it) + O\left(\frac{c_1}{(\log T)^7}\right). \tag{$D1$}$$

Combining Equations $(D2)$ and $(D1)$, there are some positive constants $A_1, A_2$ such that

$$\begin{aligned}
|\zeta(\sigma + it)| &\geq |\zeta(\sigma' + it)| - \frac{A_1 c_1}{(\log T)^7},\\
&\geq \frac{A_2 c_1^{\frac{3}{4}}}{(\log T)^7} - \frac{A_1 c_1}{(\log T)^7}.
\end{aligned}$$

Now take $c_1$ sufficiently small that in the region,

$$\sigma_0' \leq \sigma \leq \sigma', 1 \leq |t| \leq T,$$

we have,

$$|\zeta(\sigma + it)| \geq \frac{c_2}{(\log T)^7}.$$

Again combining this with Inequality $D1$, we obtain the statement. $\square$

**Corollary 2.6.** *There exists some constant $C$ such that*

$$\frac{\zeta'(s)}{\zeta(s)} = O((\log T)^9).$$

*For $1 - \frac{c}{(\log T)^9} \leq \mathrm{Re}(s) \leq 2$ and $1 \leq |\mathrm{Im}(s)| \leq T$, we have,*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}.$$

**Notation 2.3.** *Let $c \in \mathbb{R}$ and $f : \mathbb{C} \supseteq \Omega \to \mathbb{C}$. Then we denote,*

$$\int_{(c)} f(s)ds = \lim_{T \to \infty} \int_{c-iT}^{c+iT} f(s)ds.$$

**Lemma 2.11.**

$$\frac{1}{2\pi i} \int_{(c)} y^s \frac{ds}{s} = \begin{cases} 0, & (0 < y < 1), \\ \frac{1}{2}, & (y = 1), \\ 1, & (y > 1). \end{cases}$$

**Lemma 2.12.** *Set,*

$$f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

*Suppose this is absolutely convergence in $\mathrm{Re}(s) > c - \varepsilon$ for some $c \in \mathbb{R}, \varepsilon > 0$ and $x$ is not an integer, then*

$$\sum_{n<x} a_n = \frac{1}{2\pi i} \int_{(c)} f(s) \frac{x^s}{s} ds.$$

*Proof.* Since $f(s)$ is a limit of a continuous function series which is absolutely convergent on $\mathrm{Re}(s) > c - \varepsilon$, thus this is uniformly convergent in a compact subset of its domain of convergence. Thus we can exchange the integral and the sum namely,

$$\frac{1}{2\pi i} \int_{(c)} \left( \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right) \frac{x^s}{s} ds = \frac{1}{2\pi i} \sum_{n=1}^{\infty} a_n \int_{(c)} \left( \frac{x}{n} \right)^s \frac{ds}{s}.$$

By Lemma 2.11, and $\frac{x}{n} > 1 \Leftrightarrow n < x$, we derive the statement. $\square$

**Corollary 2.7.** *Let $c > 1$ and $x$ be a non-integer, then we have,*

$$\psi(x) = \frac{1}{2\pi i} \int_{(c)} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds.$$

*Proof.* Follows from Remark 2.2 and Theorem 2.6 and apply Lemma 2.12. $\square$

**Theorem 2.29.** *There is a positive constant $c' > 0$ such that*

$$\psi(x) = x + O(x \exp(-c' \log(x))).$$

*Proof.* Let $a = 1 + \frac{c}{\log T}$ for some $c, T > 1$ where $T$ is sufficiently large and will be chosen later. By Corollary 2.7, we have,

$$\psi(x) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + \frac{1}{2\pi i} \int_{(a)} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds.$$

**Claim 1.**

$$\left| \frac{1}{2\pi i} \int_{(a)} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \right| < \begin{cases} x^a \min\{1, T^{-1}, |\log x|^{-1}\}, & (x \neq 1), \\ \frac{a}{T}, & (x = 1). \end{cases}$$

Proof: $\blacksquare$ Let

$$\psi(x) = \underbrace{\frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds}_{=I_1} + \underbrace{\frac{1}{2\pi i} \int_{(a)} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds}_{=I_2}.$$

Again using Corollary 2.7 and the claim, we have,

$$\begin{aligned}
I_2 &= \frac{1}{2\pi i} \left( \int_{(a)} - \int_{a-iT}^{a+iT} \right) \left( \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \right) \frac{x^s}{s} ds, \\
&= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n) \left( \left( \int_{(a)} - \int_{a-iT}^{a+iT} \right) \left( \frac{x}{n} \right)^s \frac{ds}{s} \right), \\
&= \sum_{n=1}^{\infty} \Lambda(n) O\left( \left( \frac{x}{n} \right)^a \min\left(1, T^{-1} \left| \log\left( \frac{x}{n} \right) \right|^{-1} \right) \right), \\
&= O\left( \sum_{n=1}^{\infty} \Lambda(n) \left( \frac{x}{n} \right)^a \min\left(1, T^{-1} \left| \log\left( \frac{x}{n} \right) \right|^{-1} \right) \right).
\end{aligned}$$

Fix $x$ and let us consider two ranges $\left[\frac{x}{2}, \frac{3x}{2}\right]$ and $\left(-\infty, \frac{x}{2}\right) \cup \left(\frac{3x}{2}, \infty\right)$. If $n$ sits in the second range

$$\frac{\log \frac{n}{x}}{\frac{n}{x} - 1} \geq 2 \log \frac{3}{2}.$$

And if $n$ sits in the first interval, we have,

$$\left|\log \frac{x}{n}\right| << \left|\frac{n}{x} - 1\right|^{-1}.$$

Using this we have,

$$\begin{aligned}\sum_{n \in \left[\frac{x}{2}, \frac{3x}{2}\right]} \Lambda(n) \left(\frac{x}{n}\right)^a \min\left(1, T^{-1}\left|\log\left(\frac{x}{n}\right)\right|^{-1}\right) &<< \sum_{n \in \left[\frac{x}{2}, \frac{3x}{2}\right]} \Lambda(n) \left(\frac{x}{n}\right)^a \left|\frac{n}{x} - 1\right|^{-1} \\ &<< \frac{\log x}{T} \sum_{n \in \left[\frac{x}{2}, \frac{3x}{2}\right]} \frac{1}{\left|\frac{n}{x} - 1\right|}.\end{aligned}$$

Observe that

$$\sum_{n \in \left[\frac{x}{2}, x\right)} \frac{1}{\frac{n}{x} + 1} << \int_{\frac{x}{2}}^{\lfloor x \rfloor} \frac{dt}{1 - \frac{t}{x}},$$

$$\sum_{n \in \left(x, \frac{3x}{2}\right]} \frac{1}{\frac{n}{x} - 1} << \int_{\lceil x \rceil}^{\frac{3x}{2}} \frac{dt}{\frac{t}{x} - 1}.$$

$$\Rightarrow \sum_{n \in \left[\frac{x}{2}, x\right)} \frac{1}{\frac{n}{x} + 1} + \sum_{n \in \left(x, \frac{3x}{2}\right]} \frac{1}{\frac{n}{x} + 1} << x \log x.$$

Thus we conclude, as $x \to \infty$,

$$\sum_{n \in \left[\frac{x}{2}, \frac{3x}{2}\right]} \frac{1}{\left|\frac{n}{x} - 1\right|} << \frac{x(\log x)^2}{T}.$$

Let us turn our focus to the second range. We have $\left|\log \frac{x}{n}\right| > \log \frac{3}{2}$. Thus

$$\begin{aligned}\sum_{n < \frac{x}{2}, \frac{3x}{2} < n} \Lambda(n) \left(\frac{x}{n}\right)^a \min\left(1, T^{-1}\left|\log\left(\frac{x}{n}\right)\right|^{-1}\right) &<< \frac{x^a}{T} \sum_{n < \frac{x}{2}, \frac{3x}{2} < n} \frac{\Lambda(n)}{n^a}, \\ &\leq \frac{x^a}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^a}.\end{aligned}$$

Recall that we have,

$$-\frac{\zeta'(a)}{\zeta(a)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^a}.$$

Using Theorem 2.17, we see, as $a \to 1^+$,

$$\left|\frac{\zeta'(a)}{\zeta(a)}\right| << \frac{1}{a - 1}.$$

Using Corollary 2.6, we see,

$$\frac{x^a}{T}\frac{1}{a-1} << \frac{x^a}{T}(\log T)^9.$$

Combining all, we derive,

$$\psi(s) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} \frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds + O\left(\frac{x(\log x)^2}{T} + \frac{x^a}{T}(\log T)^9\right).$$

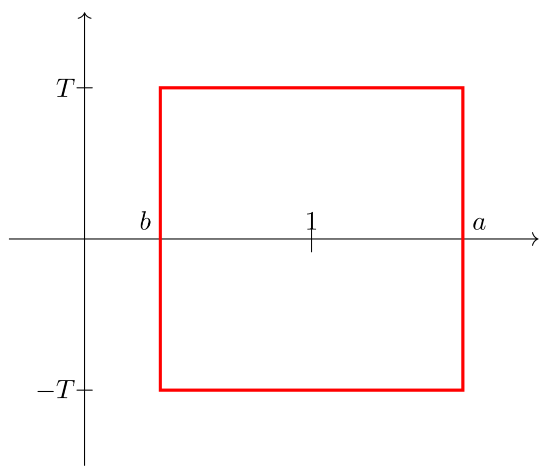
□

*Proof.* Recall Cauchy's Residue theorem, suppose $f : \Omega \to \mathbb{C}$ is meromorphic where $\Omega$ is simply connected.

**Example 2.11.** *Halfplane* $\mathbb{R}^n$ *and* $\mathbb{R}^n \backslash \{0\}$*, for* $n \geq 3$ *and convex.*

For $a_i, 1 \leq i \leq n$ poles in $U \subseteq \Omega$ simple closed curve. Then,

$$\frac{1}{2\pi i}\int_U f(s)ds = \sum_{i=1}^{n} \mathrm{Res}(f, a_i) + \text{constant}\,.$$




where $a = \frac{1}{+}\frac{c}{(\log(T))^9}, b = \frac{1}{-}\frac{c}{(\log(T))^9}$ $\mathrm{Re}(s) > 0, s \neq 1, \mathrm{Re}(s) > b$. We claim that $\frac{\zeta'(s)}{\zeta(s)}$ can be analytically continued to $\mathrm{Re}(s) \geq b$ except simple pole at $s = i$ with residue $-1$(See lecture 5 for the justification of such poles).

$$\begin{aligned}\frac{1}{2\pi i}\int_{R_T} \frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds &= \mathrm{Res}_{s=1}\left(\frac{-\zeta'(s)}{\zeta(s)}\right),\\ &= x.\end{aligned}$$

Now consider $R_T$ which is the closed path drew red in the graph.

$$\int_{R_T} = \int_{a-iT}^{a+iT} + \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT}.$$

We then have,

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds = x - \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT}$$

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{a+iT}^{b+iT}\frac{-\zeta(s)}{\zeta(s)}\frac{x^s}{s}ds\right| &= \left|\frac{1}{2\pi i}\int_a^b\frac{-\zeta(u+iT)}{\zeta(u+iT)}\frac{x^{(u+iT)}}{(u+iT)}du\right|, \\
&<< \int_a^b\left|\frac{-\zeta'(u+iT)}{\zeta(u+iT)}\right|\frac{x^u}{|u+iT|}du, \\
&\overset{\text{Using the previous theorem}}{<<} \frac{\log^9 T}{T}x^a\int_b^a du, \\
&<< \frac{x^a}{T}.
\end{aligned}$$

Now we have,

$$\begin{aligned}
\left|\int_{b-iT}^{b+iT}\frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds\right| &= \left|\frac{1}{2\pi}\int_{-T}^{T}\frac{\zeta'(b+iu)}{\zeta(b+iu)}\frac{x^{b+iu}}{b+iu}du\right|, \\
&<< \log^9 T\int_{-T}^{T}\frac{x^b}{\sqrt{b^2+u^2}}du, \\
&<< x^b\log^9 T\int_0^T\frac{du}{\sqrt{b^2+u^2}}du, \\
&<< x^b\log^9 T\int_1^T\frac{du}{u}, \\
&= x^b\log^{10} T.
\end{aligned}$$

Now back to the beginning,

$$\psi(x) = x + o(\frac{x^a}{T} + x^b\log^{10} T) + o(\frac{x\log^2 x}{T} + \frac{x^a}{T}\log^9 T).$$

Choose $T$ to be such that $2c\log x = \log^{10} T, x = e^{\frac{\log^{10} T}{x}}$.

$$x^{\frac{c}{\log^9 T}} = e^{\frac{\log T}{2}} = \sqrt{T}.$$

$$\begin{aligned}
&x^{1-\frac{c}{\log^9 T}}\cdot\log^{10} T + \frac{x\log^2 x}{T} + \frac{x^{1+\frac{c}{\log^9 T}}}{T}\log^9 T \\
&= x\cdot T^{-\frac{1}{2}}\log^1 0T + \frac{x\log^2 x}{T} + x\frac{\sqrt{T}}{T}\log^{10} T\left(\frac{x}{\sqrt{T}}\right)(\log^{10} T + \\
&<< \frac{x}{T^s} = xe^{-c(\log x)^{\frac{1}{10}}}.
\end{aligned}$$

□

**Definition 2.15** (Gamma function)**.** *We define the Gamma function* $\Gamma : \mathbb{C} \to \mathbb{C}$ *as*

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt, \sigma > 0.$$

**Remark 2.18.**

$$\begin{aligned}
|\Gamma(s)| &\le \int_0^\infty |t^{s-1}|e^{-t}dt,\\
&= \int_0^\infty t^{\sigma-1}e^{-t}dt,\\
&= \left(\int_0^1 + \int_1^\infty\right) t^{\sigma-1}e^{-t}dt,\\
&\le \int_0^1 t^{\sigma-1}dt + \int_1^\infty t^{\sigma-1}e^{-t}dt, \left(\frac{t^n}{e^t} \overset{t\to\infty}{\to} 0\right).\\
&<< \frac{1}{\sigma} + \int_1^\infty t^{\sigma-1}t^{-2\sigma}dt,\\
&<< \frac{1}{\sigma}.
\end{aligned}$$

**Theorem 2.30.**

$$F(s) = \int f(s,t)dt.$$

*$F : \Omega \to \mathbb{C}$ is analytic in $\Omega$ if*

*i). $f(s,t)$ is continuous in $(s,t)$,*

*ii). $f(s,t)$ is analytic in $s$,*

*iii). $\int f(s,t)$ is uniformly bounded on compact subsets of $\Omega$.*

In Remark 2.18, suppose $a \le \mathrm{Re}(s) \le b$ then the last two inequalities will be,

$$\begin{aligned}
\int_0^1 t^{\sigma-1}dt + \int_1^\infty t^{\sigma-1}e^{-t}dt \\
&<<_b \frac{1}{\sigma} + \int_1^\infty t^{b-1}t^{-2b}dt,\\
&<<_b \frac{1}{\sigma} + \frac{1}{b} = \frac{1}{a} + \frac{1}{b}.
\end{aligned}$$

Thus we observe that in $\sigma > 0$, using integration by parts,

$$\Gamma(s+1) = s\Gamma(s).$$

Thus for any $n \in \mathbb{N}$, we have $\Gamma(n) = n!$. We have,

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \sigma > 0.$$

Note $\Gamma(s+1)$ is analytic if $\sigma > -1$. Thus $\Gamma(s)$ is analytic in $\sigma > -1$ except simple pole at $s = 0$ with residue 1. That is

$$\lim_{s\to 0} s\Gamma(s) = 1.$$

We also have,

$$\Gamma(s) = \frac{\Gamma(s+2)}{\Gamma(s+1)}, \sigma > 0.$$

Note that in the numerator, we have $\sigma > -2$. $\Gamma(s)$ is analytic when $\sigma > -2$ except simple poles at $s = 0, -1$ with residue

$$\lim_{s\to -1}(s+1)\Gamma(s) = \frac{\Gamma(-1)}{-1} = -1.$$

Iterate the process, we have the following theorem,

**Theorem 2.31.** *$\Gamma(s)$ can be analytically continued to $\mathbb{C}$ except simple poles at $s = -k$ where $k \in \mathbb{Z}_{\geq 0}$ with residue $\frac{(-1)^k}{k!}$.*

**Remark 2.19.**

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, s \in \mathbb{C}.$$

$$\lim_{s\to m} \Gamma(s)\Gamma(1-s)\sin \pi s = \pi s.\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = \sqrt{\pi}2^{1-2s}\Gamma(2s), \forall s \in \mathbb{C}.$$

**Exercise 2.5.** *Show*

$$\sum_{n\in\mathbb{Z}} e^{-(n+\alpha)\frac{\pi}{x}} = \sqrt{x}\sum_{n\in\mathbb{Z}} e^{-n^2\pi x + 2\pi i n\alpha}, \forall \alpha \in i\mathbb{R}, x > 0.$$

*Note that $\sum_{n\in\mathbb{Z}} a_n$ is convergent if*

$$s_N := \sum_{|n|\leq N} a_n,$$

*is convergent.*

**Theorem 2.32.** $\Gamma(s) \neq 0, \forall s \in \mathbb{C}$.

*Proof.* Note that the case when $s \in \mathbb{Z}$ is already shown. Thus suppose $s \notin \mathbb{Z}$. If possible $\Gamma(s) = 0$. Then it will follow that $\Gamma(1-s)$ has a pole which is a contradiction. $\frac{1}{\Gamma(s)}$ has a simple zero at $s = 0, -1, -2, \cdots$.

Step 1

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t}s^{\frac{s}{2}-1}dt, \sigma > 0.$$

Replace $t = n^2\pi x, dt = n^2\pi dx$, we get,

$$\begin{aligned}
\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)n^{-s} &= \int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx, \sigma > 1.\\
\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \sum_{n\in\mathbb{N}}\int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx,\\
&= \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n\in\mathbb{N}}e^{-n^2\pi x}\right)dx,\\
&= \sum_{n\in\mathbb{N}}\int_0^\infty |x^{\frac{s}{2}-1}e^{-n^2\pi x}|dx,\\
&= \sum_{n\in\mathbb{N}}\left(\int_0^\infty x^{\frac{\sigma}{2}-1}e^{-n^2\pi x}dx\right),\\
&= \sum_{n=1}^{\infty}\pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right)n^{-\sigma},\\
&= \pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right)\zeta(s), \sigma > 1,\\
&< \infty.\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \qquad = \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n\in\mathbb{N}}e^{-n^2\pi x}\right).
\end{aligned}$$

Step 2, using Poisson summation formula and $\sigma > 1$, let $F \in L^1(\mathbb{R})$, ie $F : \mathbb{R} \to \mathbb{C}$ and

$$\int_{-\infty}^{\infty}|F(t)|dt < \infty.$$

We have,

$$\sum_{n\in\mathbb{Z}}F(n+u)$$

is absolutely and uniformly convergent in $u$. Also we have,

$$\sum_{n\in\mathbb{Z}}|\hat{F}(n)| \leq \infty,$$

then

$$\sum_{n\in\mathbb{Z}}F(n+u) = \sum_{n\in\mathbb{Z}}\hat{F}(n)e^{2\pi inu}.$$

Using Exercise 2.5 and set $\alpha = 0$, we have,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}e^{-n^2\frac{\pi}{2}} &= \sqrt{s}\sum_{n\in\mathbb{Z}}e^{-n^2\pi x}.\sum_{n\in\mathbb{Z}}e^{-n^2\pi x} \qquad << \sum_{n\in\mathbb{N}}e^{-n\pi x},\\
&<< \int_1^\infty e^{-\pi xt}dt,\\
&<< \frac{e^{-\pi x}}{x}, x > 0.
\end{aligned}$$

Set $\theta :== \sum_{n\in\mathbb{Z}} e^{n^2\pi x}$ then,

$$\theta(x) = 1 + 2\sum_{n\in\mathbb{N}} e^{-n^2\pi x}.$$

$$\sum_{n\in\mathbb{N}} e^{-n^2\pi x} = \frac{\theta(x)-1}{2}, x > 0.$$

Using the exercise, we have,

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\theta(x).$$

Set $w(x) := \frac{\theta(x)-1}{2}$. Thus write

$$\begin{aligned}
w\left(\frac{1}{x}\right) &= \frac{\theta\left(\frac{1}{x}\right)-1}{2}, \\
&= \frac{\sqrt{x}\theta(x)-1}{2}, \\
&= \frac{\sqrt{x}\theta(x)-1}{2}. \\
w\left(\frac{1}{x}\right) &= \sqrt{x}w(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}. \\
\pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1}w(x)dx, \sigma > 1.
\end{aligned}$$

Step 3

$$\int_0^1 x^{\frac{s}{2}-1}w(x)dx + \int_1^\infty x^{\frac{s}{2}-1}w(x)dx.$$

Taking $x = \frac{1}{y}$ and $dx = -\frac{1}{y^2}dy$ we have,

$$\begin{aligned}
\int_1^\infty \left(\frac{1}{y^2}\right)^{\frac{s}{2}-1} w\left(\frac{1}{y}\right)\frac{-1}{y^2}dy &= \int_1^\infty y^{-\frac{s}{2}} w\left(\frac{1}{y}\right)\frac{dy}{y}, \\
&= \int_1^\infty y^{-\frac{1}{2}}\left(\sqrt{y}w(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right)\frac{dy}{y}, \\
&= \int_1^\infty y^{-\frac{s}{2}+\frac{1}{2}}w(y)\frac{dy}{y} + \int_1^\infty \frac{y^{-\frac{s}{2}+\frac{1}{2}-1}}{2}dy - \frac{1}{2}\int_1^\infty y^{-\frac{s}{2}-1}dy, \\
&= \int_1^\infty y^{\frac{1-s}{2}}w(y)\frac{dy}{y} + \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}, \sigma > 1.
\end{aligned}$$

Step 4,

$$\int_1^\infty |x^{\frac{s}{2}}+x^{\frac{1-s}{2}}||w(x)|\frac{dx}{x} \le \int_1^\infty (x^{\frac{\sigma 2}{-}1}+x^{\frac{1-\sigma}{2}-1})|w(x)|dx,$$

$$<< \int_1^\infty \frac{(x^{\frac{\sigma}{2}-1}+x^{\frac{1-\sigma}{2}-1})}{e^{\pi x}}dx, \sigma \in \mathbb{R},$$

$$<< \int_1^\infty \frac{e^x}{e^{\pi x}}dx, \qquad << 1.$$

$\int_1^\infty |x^{\frac{s}{2}}+x^{\frac{1-s}{2}}||w(x)|\frac{dx}{x}$ is analytic in $\mathbb{C}$.

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = 1+s(s-1)\int_1^\infty (x^{\frac{s}{2}}+x^{\frac{1-s}{2}})w(x)\frac{dx}{x}.$$

Set $\xi(x) := 1+s(s-1)\int_1^\infty (x^{\frac{s}{2}}+x^{\frac{1-s}{2}})w(x)\frac{dx}{x}$, then it is entire. and for all $s$ we have,

$$\xi(1-s) = \xi(s).$$

Thus obtain,

$$(1-s)(1-s-1)\pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Using the construction of $w(s)$ we have,

$$|w(x)| << |\theta(x)| << \frac{e^{-\pi x}}{x}, \forall x > 0.$$

Also we have,

$$\zeta(1-s) = \pi^{-s+\frac{1}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s),$$

$$\zeta(1-s) = \pi^{-s}2^{1-s}\cos\left(\frac{\pi s}{s}\right)\Gamma(s)\zeta(s).$$

Note that $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ except simple pole at $s$.

$$\lim_{s\to 1}\zeta(1-s) = \pi^{-1}\lim_{s\to 1}\frac{\cos\frac{\pi s}{2}\zeta(s)(s-1)}{s-1},$$

$$\zeta(-2n) =$$

Recall

$$\zeta(s) = \sum_{n\in\mathbb{N}}\frac{1}{n^s},$$

$$= \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}dt, \sigma > 0, s \neq 1.$$

If $s$ is real,

$$|\zeta(s) - \frac{s}{s-1}| \leq |s| \int_1^\infty \frac{\{t\}}{t^{\sigma+1}} dt, \qquad < \frac{|s|}{\sigma} = \frac{\sigma}{\sigma} = 1.$$

Thus we obtain,

$$-1 + \frac{s}{s-1} < \zeta(s) < 1 + \frac{s}{s-1}$$
$$\frac{1}{s-1} < \zeta(s) < \frac{2s-1}{s-1},$$
$$-1 < (1-s)\zeta(s) < 1 - 2s < 0 \quad \text{if } \frac{1}{2} < s < 1$$
$$\Rightarrow \zeta(s) \neq 0, \quad \text{if } \frac{1}{2} < s < 1.$$

□

**Notation 2.4.** *Given $\chi \mod q$, we set,*

$$\tau(\chi) = \sum_{k=1}^{q} \chi(k) e^{\frac{2\pi i k}{q}}.$$

## 2.6 Primitive characters

**Definition 2.16.** *A character $\chi$ is called primitive if its conductor is*

**Example 2.12.** *Take $\chi : (\mathbb{Z}/8\mathbb{Z})^\times \to \mathbb{C}^times$.*

**Lemma 2.13.** *Suppose $x \in \left[0, \frac{1}{2}\right]$, then*

$$|\sin \pi x| \geq 2x.$$

*Proof.* Exercise. □

**Lemma 2.14.** *For $n \in \mathbb{Z}$, we have,*

$$\chi(n)\tau(\overline{\chi}) = \sum_{k=1}^{q} \overline{\chi}(k) e^{\frac{2\pi i k}{q}},$$

*where $\chi$ is a primitive character modulo $q$ and $\chi \neq \chi_0$.*

*Proof.*

$$\overline{\chi(n)\tau(\overline{\chi})} = \sum_{l=1}^{q} \chi(l) e^{-\frac{2\pi i l}{q}}.$$

Multiplying this equation with the one in the statement we have,

$$|\chi(n)|^2 |\tau(\overline{\chi})|^2 = \sum_{k,l=1}^{q} \overline{\chi(k)} \chi(l) e^{\frac{2\pi i (k-l)}{q}}.$$

Applying $\sum_{n \leq x}$ to the both sides of the above equation, we have,

$$|\tau(\overline{\chi})|^2 \sum_{n \leq x} |\chi(n)|^2 = \sum_{k,l=1}^{q} \overline{\chi(k)}\chi(l) \left( \sum_{n \leq x} \left( e^{\frac{2\pi i(k-l)}{q}} \right)^n \right).$$

Note that

$$x + x^2 + \cdots + x^q = \begin{cases} \frac{x(x^q-1)}{x-1}, (x \neq 1), \\ q, (x = 1), \end{cases}$$

$$= \begin{cases} 0, (x \neq 1, x^q = 1), \\ q, (x = 1), \end{cases}$$

Take $x = e^{\frac{2\pi i(k-l)}{q}} = \cos \frac{2\pi}{q}(k-l) + i \sin \frac{2\pi}{q}(k-l) = 1$. $x = 1$ if and only if $q|k-l$ thus

$$\sum_{n \leq x} \left( e^{\frac{2\pi i(k-l)}{q}} \right)^n = \begin{cases} 0, (q \not| k-l), \\ q, (\text{otherwise}). \end{cases}$$

Therefore, we get,

$$|\tau(\overline{\chi})|^@ \sum_{n \leq x} |\chi(n)|^2 = q \sum_{\substack{k,l=1 \\ q|k-l}}^{q} \overline{\chi}(k)\chi(l),$$

$$= q \sum_{k=1}^{q} \overline{\chi}(k)\chi(k),$$

$$= q \sum_{k=1}^{q} |\chi(k)|^2.$$

Therefore $|\tau(\overline{\chi})|^2 = q, |\tau| = \sqrt{q}$.

Consider $(n, q) = 1$, then

$$\chi(n)\tau(\overline{\chi}) = \chi(n) \sum_{k=1}^{q} \overline{\chi}(k) e^{\frac{2\pi ik}{q}},$$

$$= \chi(n) \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \overline{\chi}(k)\chi(k) e^{\frac{2\pi ik}{q}},$$

Set $k = nt$, we get,

$$\chi(n)\tau(\overline{\chi}) = \sum_{\substack{t=1 \\ (t,q)=1}}^{q} \overline{\chi}(nt)\chi(n)e^{\frac{2\pi int}{q}}.$$
$$\overline{\chi}(nt) = \overline{\chi}(n)\overline{\chi}(t).$$
$$\chi(n)\tau(\overline{\chi}) = \sum_{\substack{t=1 \\ (t,q)=1}}^{q} \overline{\chi}(t)e^{\frac{2\pi int}{q}}.$$

Observe that

$$\begin{aligned}
\tau(\overline{\chi})\left(\sum_{n\leq x}\chi(n)\right) &= \sum_{k=1}^{q-1}\overline{\chi}(k)\left(\sum_{n\leq x}e^{\frac{2\pi ikn}{q}}\right), \\
|\tau(\overline{\chi})|\cdot|\sum_{n\leq x}\chi(n)| &\leq \sum_{k=1}^{q-1}\left|\sum_{n\leq x}e^{\frac{2}{\pi}iknq}\right|, \\
&= \sum_{k=1}^{q-1}\left|\frac{e^{\frac{2\pi ik}{q}}\left(e^{\frac{2\pi in[x]}{q}}-1\right)}{e^{\frac{2}{\pi}ikq}-1}\right|, \\
&\leq \sum_{k=1}^{q-1}\frac{2}{|e^{\frac{2\pi k}{q}}-1|},
\end{aligned}$$

Note that for all $y \in i\mathbb{R}$,

$$2i\sin y = e^{-iy}(e^{2iy}-1),$$
$$|2\sin y| = |e^{2iy}-1|.$$

Apply this to the equation above we have,

$$\sum_{k=1}^{q-1}\frac{2}{|e^{\frac{2\pi k}{q}}-1|} = \sum_{k=1}^{q-1}\frac{1}{|\sin\frac{\pi k}{q}|}.$$

Using the lemma, we get,

$$\begin{aligned}
\sum_{k=1}^{q-1}\frac{1}{|\sin\frac{\pi k}{q}|} &= \sum_{1\leq k\leq\frac{q}{2}}\frac{1}{|\sin\frac{\pi k}{q}|} + \sum_{\frac{q}{2}<k}^{q-1}\frac{1}{|\sin\frac{\pi k}{q}|}. \qquad = 2\sum_{1\leq k\leq\frac{q}{2}}\frac{1}{|\sin\frac{\pi k}{q}|}, \\
&\leq \sum_{1\leq k\leq\frac{q}{2}}\frac{k}{q} << q\log\frac{q}{2} << q\log q.
\end{aligned}$$

Thus we conclude $|(\overline{\chi}) << \sqrt{q}$ if $\chi$ is primitive and $q > 1$, we have,

$$\left|\sum_{n\leq x}\chi(n)\right| << \sqrt{q}\log q,$$

uniformly in $q$ as $x \to \infty$. $\square$

$L(q,\chi)$ functional equation, where $\chi \neq \chi_0$ and $\chi$ is primitive. Suppose $\chi(-1) = 1$ that is it is an even character. From previous discussion, we have,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)n^{-s} = \int_0^\infty x^{\frac{s}{2}}e^{-n^2\pi x}\frac{dx}{x}.$$

Replace $x$ with $\frac{x}{q}$ and $\sigma > 0$, we habe,

$$\pi^{-\frac{s}{2}}q^{\frac{s}{2}}\Gamma\left(\frac{s}{1}\right)n^{-s} = \int_0^\infty x^{\frac{s}{2}}e{-}n^2\pi x\frac{dx}{x}, \sigma > 0.$$

For $\sigma > 1$, we have,

$$\begin{aligned}\pi^{-\frac{s}{2}}q^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi) &= \sum_{n=1}^{\infty}\int_0^\infty \chi(n)x^{\frac{s}{2}}e^{-n^2\frac{\pi x}{q}}\frac{dx}{x},\\ &= \int_0^\infty x^{\frac{s}{2}}\left(\sum_{n=1}^{\infty}\chi(n)e^{-n^2\frac{\pi x}{q}}\right)\frac{dx}{x}, \sigma > 1.\end{aligned}$$

Let

$$\theta(x,\chi) = \sum_{n\in\mathbb{Z}}\chi(n)e^{-\frac{n^2\pi x}{q}} = 2\sum_{n=1}^{\infty}\chi(n)e^{-\frac{n^2\pi x}{q}}, (x > 0).$$

Using this, we get,

$$\int_0^\infty x^{\frac{s}{2}}e{-}n^2\pi x\frac{dx}{x} = \frac{1}{2}\int_0^\infty x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}, (\sigma > 1).$$

Split the integral into $\int_0^1, \int_1^\infty$.

$$\begin{aligned}
\tau(\overline{\chi})\theta(x,\chi) &= \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1},\overline{\chi}), \\
&= \sum_{n\in\mathbb{Z}} (\tau(\overline{\chi})\chi(n)) e^{\frac{-n^2\pi x}{q}}, \\
&= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{q} \overline{\chi}(k) e^{\frac{2\pi i n k}{q}} e^{-\frac{n^2\pi x}{q}}, \\
&= \sum_{k=1}^{q} \overline{\chi(k)} \sum_{n\in\mathbb{Z}} e^{\frac{2\pi k n}{q} - n^2 \frac{\pi x}{q}}, \\
&\overset{\text{From Lecture 7 page 6}}{=} \sum_{k=1}^{q} \overline{\chi}(k) \left(\frac{x}{q}\right)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} e^{-\left(m+\frac{m}{q}\right)\frac{\pi q}{x}}, \\
&= \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^{q} \overline{\chi}(k) \sum_{n\in\mathbb{Z}} e^{-\left(m+\frac{m}{q}\right)\frac{\pi q}{x}}.
\end{aligned}$$

Put $qn+m=t$, then

$$\overline{\chi}(qn+m) = \overline{\chi}(m) = \overline{\chi}(t).$$

Thus,

$$\left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^{q} \overline{\chi}(k) \sum_{n\in\mathbb{Z}} e^{-\left(m+\frac{m}{q}\right)\frac{\pi q}{x}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{t\in\mathbb{Z}} \overline{\chi}(t) e^{-\frac{t^2\pi}{qx}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1},\overline{\chi}).$$

Now for the splitted integral, we have,

$$\frac{1}{2}\int_0^1 x^{\frac{s}{2}} \theta(x,\chi)\frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}\frac{dx}{x}.$$

Replacing $x$ with $\frac{1}{x}$, we get,

$$= \frac{\tau(\chi)}{2\sqrt{q}} \int_1^\infty x^{\frac{1-s}{2}} \theta(x,\overline{\chi})\frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}.$$

Set

$$\xi(s,\chi) := \frac{\tau(\chi)}{2\sqrt{q}} \int_1^\infty x^{\frac{1-s}{2}} \theta(x,\overline{\chi})\frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}.$$

Use that $\theta(x,\overline{\chi}) << e^{-\frac{\pi x}{q}}$ and $|\tau(\chi)| = \sqrt{q}$. It turns out that $\xi$ is uniformly bounded on compact subsets of $\mathbb{C}$ and in particular, this is entire.

**Lemma 2.15.**

$$\xi(1-s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\xi(s,\overline{\chi}).$$

*Proof.* Recall,

$$\tau(\overline{\chi})\theta(x,\chi) = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1},\overline{\chi}),$$

and

$$|\tau(\chi)|^2 = q.$$

We get,

$$\frac{\tau(\chi)}{\sqrt{q}} = \frac{\sqrt{q}}{\overline{\tau(\chi)}} = \frac{\sqrt{q}}{\tau(\overline{\chi})}.$$

Then,

$$L(1-s,\chi) = \frac{\tau(\chi)}{q^{1-s}}\pi^{-s}2^{1-s}\cos\frac{\pi s}{2}(\Gamma(s))L(s,\overline{\chi}).$$

Note that $\chi(-1) = 1$ and is primitive (in the case of odd integer replace cos with sin). Note also that

i). $L(s,\chi) = 0, s = 0,-2,-4,\cdots$ when $\chi$ is even,

ii). $L(s,\chi) = -1,-3,-5,\cdots$ when $\chi$ is odd.

When $0 < \sigma < 1$, □