

V5A10 Analytic Number Theory

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1 Classical Number Theory

Theorem 1.1 (Euclid). *There are infinitely many prime numbers.*

Definition 1.1. $\pi : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that

$$\pi(n) = \{\text{prime numbers less than } n\}.$$

Remark 1.1.

$$\frac{\pi}{n \ln(n)} \approx 1.$$

Definition 1.2.

$$\text{Li}(x) = \int_0^x \frac{1}{\ln(t)} dt.$$

Notation 1.1. Given $f, g : \mathbb{R} \rightarrow \mathbb{C}$,

$$f(x) = O(g(x))$$

means that

$$\exists K \in (0, \infty), x_0 \in \mathbb{R}, \text{ s. t. } \forall x > x_0, |f(x)| \leq K|g(x)|.$$

Notation 1.2. Let $f, g : \mathbb{R} \rightarrow \mathbb{C}$ be functions. $f \sim g$ denotes that

$$\lim_{n \rightarrow \infty} \frac{f(x)}{g(x)}.$$

Notation 1.3. Let $f : \mathbb{R} \rightarrow \mathbb{C}$,

$$\text{Li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k},$$

denotes that

$$\text{Li}(x) = \frac{x}{\ln x} \sum_{k=1}^{N-1} \frac{k!}{(\ln x)^k} + O\left(\frac{x}{(\ln x)^{N+1}}\right).$$

and as $x \rightarrow \infty$, this holds for any $N \geq 1$.

Remark 1.2. By the integration by parts, we see that it's asymptotic expansion is

$$\text{Li}(x) \approx \frac{x}{\ln(x)} \sum_{k=0}^{\infty} \frac{k!}{(\ln(x))^k}.$$

Theorem 1.2 (Prime Number Theorem).

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{\text{Li}(x)} = 1.$$

Definition 1.3 (First Chebyschev Function).

$$\vartheta(x) = \sum_{p \leq x} \ln p.$$

Definition 1.4 (Second Chebyschev Function).

$$\psi(x) = \sum_{\substack{m,p \\ p^m \leq x}} \ln p.$$

Remark 1.3. We can rewrite the second Chebyschev function as follows.

$$\psi(x) = \sum_{\substack{p \leq x \\ m=\max\{n \in \mathbb{N} \mid p^n \leq x\}}} m \ln p.$$

Definition 1.5 (Möbius Function). Let $n \in \mathbb{N}$, we define,

$$\mu(n) = \begin{cases} 1, & (n = 1), \\ (-1)^m, & (n \text{ is square free and has } n \text{ distinct prime divisors}), \\ 0, & (\text{otherwise}). \end{cases}$$

Definition 1.6 (Möbius Function).

$$\mu(n) = \begin{cases} 1 & (n = 1) \\ (-1)^k & (n = p_1 \cdots p_k, p_i = p_j \Rightarrow i = j) \\ 0 & (\exists p \text{ s.t. } p^2 | n). \end{cases}$$

Remark 1.4. The prime number theorem is equivalent to the following statements.

1). $\psi(x) \sim x$.

2). $\theta(x) \sim x$.

3). $\lim_{x \rightarrow \infty} \frac{\sum_{\substack{n \leq x \\ \mu(n)}}}{n} x = 0$.

Conjecture 1.1 (Twin Prime Conjecture). There exists infinitely many primes p such that $p + 2$ is also prime.

Conjecture 1.2 (Goldbach's Conjecture). *Let $n \in \mathbb{N}$ be an even number greater than 2, then there exists two primes p, q such that $n = p + q$.*

Conjecture 1.3 (Hardy-Littlewood Conjecture).

$$\#\{\text{ prime numbers } p \text{ such that } 2p + 1 \text{ is also a prime and } p < x\}$$

Definition 1.7 (Riemann-Zeta Function). *We define $\zeta : \mathbb{C} \rightarrow \mathbb{C}$ such that*

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}.$$

Remark 1.5. *Supposer $\operatorname{Re}(s) > 1$, then we have*

$$\begin{aligned} |\zeta(s)| &= \sum_{n \in \mathbb{N}} \frac{1}{|n|^s} \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n^{\operatorname{Re}(s)}} \end{aligned}$$

By multiplying $\frac{1}{2^s}$, we obtain

$$\frac{1}{2^s} \zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{(2n)^s}.$$

We get

$$(1 - \frac{1}{2^s}) \zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{(2n+1)^s}.$$

Continuing this procedure, we get the following proposition.

Proposition 1.1.

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)$$

Theorem 1.3 (Weierstrass). *Let $A \subseteq \mathbb{C}$ and consider a sequence of functions $(f_n : A \rightarrow \mathbb{C})_{n \in \mathbb{N}}$ such that there exists a sequence of non-negative numbers $(M_n)_{n \in \mathbb{N}}$ such that*

i). $\forall x \in A, |f_n(x)| \leq M_n$.

ii). $\sum_{n \in \mathbb{N}} M_n$ converges.

Then the sequence converges uniformly.

Theorem 1.4. *Suppose the conditions in the previous theorem. If each function is analytic on a compact subset of A , then the limit is also analytic.*

Corollary 1.1. *Let A be a compact subset of a complex plane where $\operatorname{Re}(s) > 1$. Then there exists $\delta > 0$ such that $\operatorname{Re}(s) > 1 + \delta$ and*

$$\sum_{n \in \mathbb{N}} \left| \frac{1}{n^s} \right| \leq \sum_{n \in \mathbb{N}} \frac{1}{n^{1+\delta}} < \infty.$$

Fact 1.1. *The Riemann zeta function can be analytically continued to the whole plane except for $s = 1$.*

Definition 1.8 (Gamma Function).

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Proposition 1.2.

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Remark 1.6. $\zeta(1 + it) \neq 0$ if $t \in \mathbb{R}, t \neq 0$. $\zeta(s) \neq 0$ for $0 < s < 1$.

Definition 1.9 (Functional Equation).

$$\zeta(1 - s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$$

Remark 1.7. $\Gamma(s)$ is defined for $\operatorname{Re}(s) > 0$ and can be analytically continued to the whole place except for $\mathbb{C} \setminus \{-2n \mid n \geq 0\}$.

Remark 1.8. For $s = -2m$ where $m \in \mathbb{N}$, we see $\zeta(s) = 0$.

$$\begin{aligned} \zeta(0) &= \frac{2}{2\pi} \lim_{s \rightarrow 1} \cos\left(\frac{\pi s}{2}\right) \zeta(s) \\ &= \lim_{s \rightarrow 1} \frac{\cos\left(\frac{\pi s}{2}\right)}{s - 1} \lim_{s \rightarrow 1} (s - 1) \zeta(s) \\ &= \frac{1}{\pi} \times \frac{-\pi}{2} \times 1 \\ &= -\frac{1}{2}. \end{aligned}$$

Definition 1.10. The subset of the complex plane with its real part between 0 and 1. The critical line is the line where $\operatorname{Re}(s) = \frac{1}{2}$.

Conjecture 1.4 (Riemann Hypothesis). Let s be an element of the critical strip. If $\zeta(s) = 0$ then $\operatorname{Re}(s) = \frac{1}{2}$ (ie. it lies on the critical line).

Notation 1.4. Let $T > 0$. We denote $N(T)$ the number of zeros of ζ in the critical strip whose coefficient of the imaginary part is in $(0, T)$. That is

$$N(T) = |\{\sigma + it \in \mathbb{C} \mid 0 < \sigma < 1, 0 < t < T\}|.$$

Proposition 1.3.

$$\lim_{T \rightarrow \infty} \frac{N(T)2\pi}{T \log(T)} = 1.$$

Sketch of Proof (needs refinement).

$$\psi(x) = \frac{1}{2\pi i} \int_l \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds$$

where l is the line $l = a$ for some $a > 1$.

$$\psi(s) = x - \sum_{\rho \text{ non-trivial zeros}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \log(1 - x^{-2}).$$

□

Definition 1.11. Let $q \in \mathbb{N}$ and a be a natural number coprime to q . We define

$$\pi(x; q, a) = |\{\text{prime numbers } p \text{ less than or equal to } x \text{ such that } p \equiv a \pmod{q}\}|$$

Proposition 1.4.

$$\pi(x; , q, a) \sim \frac{x}{\varphi(q) \log(x)}$$

where φ is a Euler phi-function.

Theorem 1.5 (Brun–Titchmarsh). For any $q < x$, we have

$$\pi(x; , q, a) < \frac{2x}{\varphi(q) \log(\frac{x}{q})}.$$

2 Week 2

Remark 2.1.

$$\text{Li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k}.$$

Indeed we have

$$\text{Li}(x) = \int_2^x \frac{dt}{\ln t}.$$

Observe that

$$\int_2^t \frac{1}{(\ln t)^N} \sim \frac{x}{(\ln x)^N}$$

for all $N \geq 1$. Thus $\text{Li}(x)$ can be expressed in terms of polynomials in $\frac{x}{\ln(x)}$, by keep replacing the greatest temr with the above approximation.

Definition 2.1. A function $f : \mathbb{N} \rightarrow \mathbb{C}$ is said to be

- 1). multiplicative if for any $(m, n) = 1$, we have $f(mn) = f(m)f(n)$,
- 2). completely multiplicative if for any natural numbers m, n , we have $f(mn) = f(m)f(n)$.

Example 2.1. Möbius function μ is multiplicative.

Definition 2.2 (Von-Mangoldt Function). *The Von-Mangoldt function $\Lambda : \mathbb{N} \rightarrow \mathbb{C}$ is defined as*

$$\Lambda(n) = \begin{cases} \log(p) & (n = p^k \text{ for some } k \geq 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Definition 2.3 (Euler Phi Function). *The Euler phi function is $\varphi : \mathbb{N} \rightarrow \mathbb{C}$ such that*

$$\varphi(n) = \{1 \leq a \leq n \mid (a, n) = 1\}.$$

Example 2.2. φ is multiplicative but Λ is not.

Definition 2.4 (Dirichlet Characters Modulo q). *Let $q \in \mathbb{N}$ be a natural number and $q \geq 2$.*

$$\chi_1 : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$$

be a group homomorphism. The Dirichlet character function modulo q with respect to χ_1 is such that

$$\chi(n) = \begin{cases} \chi_1(\bar{n}) & ((n, q) = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Example 2.3. For $q = 3$, we have $(\mathbb{Z}/3\mathbb{Z})^\times = \{\pm 1\}$. The only possible character is $\pm 1 \mapsto \pm 1$. Therefore, we have

$$\chi(1) = 1, \chi(2) = -1, \chi(0) = 0.$$

Theorem 2.1.

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1, \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. When $n = 1$, this is trivial. Suppose $n \neq 1$. We factorize n by

$$n = \prod_{i=1}^k p_i^{\alpha_i}$$

where p_i is a prime and $\alpha_i \in \mathbb{N}$ for each $i = 1, \dots, n$.

Observe that

$$\sum_{d|n} \mu(d) = \sum_{d|\prod_{i=1}^k p_i} \mu(d).$$

Now we see

$$\sum_{d|\prod_{i=1}^k p_i} \mu(d) = \sum_{j=0}^k \binom{k}{j} (-1)^j = \sum_{j=0}^k \binom{k}{j} (1)^{k-j} (-1)^j = (1 - 1)^k = 0.$$

□

Proposition 2.1 (Möbius Inversion Formula). *Let $f, g : \mathbb{N} \rightarrow \mathbb{C}$ be functions (we do not assume them to be multiplicative). If*

$$\sum_{d|n} g(d) = f(n),$$

holds if and only if

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = g(n).$$

Proof.

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu \sum_{e|\frac{n}{d}} g(e).$$

$e|\frac{n}{d}$ if and only if $de|n$ thus obtain,

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu \sum_{de|n} g(e).$$

In particular, we get the expression

$$= \sum_{de|n} \mu(d) g(e).$$

By reordering, we get

$$= \sum_{e|n} g(e) \sum_{d|\frac{n}{e}} \mu(d).$$

By Proposition 2.1, we get

$$\sum_{d|\frac{n}{e}} \mu(d) = 0$$

unless $e = n$. □

Proposition 2.2.

$$\sum_{d|n} \varphi(d) = n.$$

Proof. Consider $(\mathbb{Z}/n\mathbb{Z})^\times$. We know that

$$|(\mathbb{Z}/n\mathbb{Z})^\times| = \varphi(n).$$

□

Theorem 2.2.

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Proof. Using Proposition 2.2, we have,

$$\sum_{d|n} \mu(d) \frac{n}{d} = \varphi(n).$$

Dividing both sides by n and observe that $\mu(d) \neq 0$ if and only if d is a prime factor of n .

$$\begin{aligned} \frac{\varphi(n)}{n} &= \sum_{d|n} \frac{\mu(d)}{d}, \\ &= 1 \sum_{p|n} \frac{1}{p} + \sum_{p_1, p_2|n} \frac{1}{p_1 p_2} - \dots. \end{aligned}$$

By the induction on the number of prime divisors of n , we get the statement. \square

Proposition 2.3. *We have the following properties of φ .*

- 1). $n|m \Rightarrow \varphi(n)|\varphi(m)$.
- 2). $\varphi(n)$ is even for $n \geq 3$.
- 3). $\varphi(2n) = \begin{cases} 2\varphi(n), & (2|n) \\ \varphi(n), & (2 \nmid n). \end{cases}$
- 4). φ is multiplicative.
- 5). $\varphi(mn) = \varphi(m) \frac{\varphi(n)d}{\varphi(d)}$ where $d = (m, n)$.
- 6). $\varphi(n^m) = n^{m-1}\varphi(n)$.

Proof. Exercise. \square

Theorem 2.3. *The following statements are equivalent.*

- 1). $\sum_{d|n} \Lambda(d) = \log n$
- 2). $\sum_{d|n} \mu(d) \log d = \Lambda(n)$.

And in particular $\sum_{d|n} \Lambda(d) = \log n$ holds.

Proof. The equivalence is a direct corollary of Möbius inversion formula. For the latter, Write

$$n = \prod_{i=1}^k p_i^{\alpha_i}.$$

We have

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^k \alpha_i \log p_i = \log(n).$$

\square

Notation 2.1. Let $n \in \mathbb{N}$, suppose a prime p divides n . Then we denote $\alpha(p)$ to be the highest prime power factor of n .

Theorem 2.4. Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be a multiplicative function. Then

$$\sum_{d|n} f(d) = \prod_{p|n} \left(\sum_{i=0}^{\alpha(p)} f(p^i) \right).$$

In particular $\sum_{d|n} f(d)$ is also multiplicative.

Proof. Let $d|n$, then we have $d = \prod_{i=1}^k p_i^{\beta_i}$ for some $0 \leq \beta_i \leq \alpha(p_i)$. Since f is multiplicative we have

$$f(d) = \prod_{i=1}^k f(p_i^{\beta_i}).$$

The second part is a direct result of the first part. \square

Remark 2.2. The Second Chebyschev Function ψ can be written as

$$\psi(x) = \sum_{d \leq x} \Lambda(d).$$

This follows directly from the definition.

Definition 2.5 (Dirichlet Series). Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be a function and $s \in \mathbb{C}$. We define

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}.$$

For another arithmetic function $g : \mathbb{N} \rightarrow \mathbb{C}$, we define

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} + \sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} = \sum_{n \in \mathbb{N}} \frac{(f(n) + g(n))}{n^s}.$$

and

$$\begin{aligned} \left(\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \times \left(\sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} \right) &= \sum_{n, m \in \mathbb{N}} \frac{(f(n)g(m))}{(nm)^s}. \\ &= \sum_{t \in \mathbb{N}} \sum_{n|t} \frac{(f(n)g(\frac{n}{t}))}{t^s}. \end{aligned}$$

Recall the taylor expansion of $\ln x$ we get

$$\ln 2 = \sum_{n \in \mathbb{N}} \frac{(-1)^{n+1}}{n}.$$

Rearranging the following way

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \dots$$

we get this equals to $\frac{1}{2} \ln 2$.

Theorem 2.5. Let $s \in \mathbb{C}$ be $\operatorname{Re}(s) > 1$, we have

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}.$$

Proof.

$$\begin{aligned} \zeta(s) \sum_{n \geq 1} \frac{\mu(n)}{n^s} &= \left(\sum_{n \in \mathbb{N}} \frac{1}{n^s} \right) \left(\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} \right) \\ &= \sum_{t \in \mathbb{N}} \frac{1}{t^s} \sum_{n|t} \mu(n) \\ &= 1. \end{aligned}$$

□

Theorem 2.6. For $\operatorname{Re}(s) > 1$, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s}.$$

From this we derive

$$\lim_{n \rightarrow \infty} \frac{\log n}{n^\varepsilon} = 0.$$

Proof.

$$\begin{aligned} \zeta(s) \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} &= \left(\sum_{m \in \mathbb{N}} \frac{1}{m^s} \right) \left(\sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} \right), \\ &= \sum_{t \in \mathbb{N}} \frac{1}{t^s} \left(\sum_{n|t} \Lambda\left(\frac{t}{n}\right) \right), \\ &= \sum_{t \in \mathbb{N}} \frac{\log(t)}{t^s}, \\ &= -\zeta'(s). \end{aligned}$$

□

Remark 2.3.

$$\begin{aligned} \sum_{n \in \mathbb{N}} \left| \frac{\Lambda(n)}{n^s} \right| &\leq \sum_{n \in \mathbb{N}} \frac{\log(n)}{n^\sigma}, \\ &<< \sum_{n \in \mathbb{N}} \frac{n^\varepsilon}{n^\sigma}, \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n^{\sigma-\varepsilon}}. \end{aligned}$$

We have $\lim_{n \rightarrow \infty} \frac{\log(n)}{n^\varepsilon} = 1$ and the last equation is convergent if and only if $\sigma - \varepsilon > 1$. Thus we have $\sigma > 1 + \varepsilon$.

Remark 2.4. $\frac{\zeta'(s)}{\zeta(s)}$ is a meromorphic functions except $s = 1$ and where $\zeta(s)$ vanishes. Indeed, For general $\frac{f}{g}$, it is analytic if f, g are analytic and $g \neq 0$.

- 1). $\zeta(s)$ is analytic except $s = 1$.
- 2). $\zeta'(s)$ has a pole of order 2 at $s = 1$.
- 3). $\zeta(s)$ has a pole of order 1 at $s = 1$.

Recall that for $|z| \geq 1$, we have,

1. $|z| \geq 1 \Rightarrow \sum_{n \in \mathbb{Z}_{\geq 0}} z^n = \frac{1}{1-z}$,
2. $\prod_{n \in \mathbb{N}} (1 + a_n)$ is convergent if $\sum_n a_n$ is absolutely convergent,
3. therefore $\prod_{n \in \mathbb{N}} (1 + a_n)$ is convergent if and only if $\prod_{n \in \mathbb{N}} (1 + |a_n|)$ is convergent.

Theorem 2.7. Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be a map.

If f is multiplicative and for $\operatorname{Re}(s) > r_0, r_0 \in \mathbb{R}$ then we have,

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} = \prod_p \left(\sum_{\nu \geq 0} f(p^\nu) p^{-\nu s} \right).$$

If f is completely multiplicative, then

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} = \prod_p (1 - f(p)p^{-s})^{-1}.$$

Proof. Let $A(x) = \{n \in \mathbb{N} \mid \text{primes factors of } n \text{ are } \leq x\}$, then

$$\prod_{p \leq x} \sum_{\nu=0}^{\infty} f(p^\nu) p^{-\nu s} = \sum_{n \in A} \frac{f(n)}{n^s}.$$

Therefore,

$$\begin{aligned} \left| \prod_{x \leq p \leq x} \sum_{\nu=0}^{\infty} f(p^\nu) p^{-\nu s} - \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right| &= \left| \sum_{n \in A} \frac{f(n)}{n^s} - \sum_{n \in \mathbb{N} \setminus A} \frac{f(n)}{n^s} \right|, \\ &= \left| \sum_{n \in \mathbb{N} \setminus A} \frac{f(n)}{n^s} \right|, \\ &\leq \sum_{n \notin A} \frac{|f(n)|}{n^{\operatorname{Re}(s)}}, \\ &\leq \sum_{n > x} \frac{|f(n)|}{n^{\operatorname{Re}(s)}} \rightarrow 0. \end{aligned}$$

The last limit is due to that it is a tail of a an absolutely convergent series. Since f is completely multipliative, we have

$$f(p^\nu) = (f(p))^\nu.$$

Therefore, we get,

$$\begin{aligned} \prod_p \left(\sum_{\nu \in \mathbb{Z}_{\geq 0}} (f(p^\nu)p^{-\nu s}) \right) &= \prod_p \left(\sum_{\nu \in \mathbb{Z}_{\geq 0}} (f(p)p^{-s})^{-\nu} \right), \\ &= \prod_p \left(\frac{1}{1 - f(p)p^{-s}} \right). \end{aligned}$$

□

Example 2.4. Take $f(n) = 1$ as above we get,

$$\sum_{n \in \mathbb{N}} \frac{1}{n^s} = \zeta(s) = \prod_p \left(1 - \frac{1}{p^s} \right)^{-1}, \operatorname{Re}(s) > 1.$$

Example 2.5.

$$\sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s} \right)^{-1}, \operatorname{Re}(s) > 1.$$

Example 2.6.

$$\begin{aligned} \sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} &= \prod_p \left(1 + \frac{\mu(p)}{p^s} \right), \\ &= \prod_p \left(1 - \frac{1}{p^s} \right), \\ &= \frac{1}{\zeta(s)}. \end{aligned}$$

Example 2.7. Note that $\phi(n) \leq n$. Thus for $\operatorname{Re}(s) > 2$, we have,

$$\sum_{n \in \mathbb{N}} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

2.1 Order of arithmetic functions

Definition 2.6. Let $f, g : \mathbb{N} \rightarrow \mathbb{C}$, we denote,

$$f(n) = O(g(n)),$$

if there is $K > 0$ and $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow |f(n)| \leq K|g(n)|.$$

An alternative notation for this is $f(n) = O(g(n))$.

Definition 2.7. We define following arithmetic functions,

- 1). $\nu(n) := \sum_{p|n} 1$, a number of prime divisors of n ,
- 2). $d(n) := \sum_{d|n} 1$, the number of divisors of n ,
- 3). $\sigma := \sum_{d|n} d$, the sum of all the divisors of n .

Lemma 2.1.

$$\nu(n) << \log(n).$$

Proof. Let $n = \prod_{i=1}^k p_i^{\alpha(p_i)}$. Then $\nu(n) = k$. Since $p_i \geq 2$, we have,

$$\begin{aligned} \log(n) &= \sum_{i=1}^k \alpha(p_i) \log(p_i), \\ &\geq k \log(2). \end{aligned}$$

Therefore $\nu(n) \leq \frac{\log(n)}{\log 2}$. □

Lemma 2.2.

$$\sum_{k=2}^n \leq \log(n) + 1.$$

Proof. We know that

$$\int_1^n \frac{1}{t} dt = \log(n).$$

For $1 \leq k \leq t \leq k+1 \leq n$, we have,

$$\int_k^{k+1} \frac{1}{k+1} dt \leq \int_k^{k+1} \frac{1}{t} dt \leq \int_k^{k+1} \frac{1}{k} dt.$$

Thus we have,

$$\frac{1}{k+1} \leq \log(k+1) - \log(k) \leq \frac{1}{k}.$$

By telescoping sum we get

$$\sum_{k=2}^n \frac{1}{k} \leq \log(n+1).$$

□

Lemma 2.3.

$$\sigma(n) << n(1 + \log(n)) \sim n \log(n).$$

Proof.

$$\begin{aligned}
\sigma(n) &= \sum_{d|n} \frac{n}{d}, \\
&= n \sum_{d|n} \frac{1}{d}, \\
&= n \left(1 + \sum_{d \geq 2} \frac{1}{d} \right), \\
&\leq n \left(1 + \sum_{d=2}^n \frac{1}{d} \right), \\
&\leq (1 + \log(n)).
\end{aligned}$$

The last inequality is due to Lemma 2.2. \square

Exercise 2.1. Show that

$$\sum_{k=1}^n \frac{1}{k} = \log(n) + O(1).$$

That is

$$\left| \sum_{k=1}^n \frac{1}{k} - \log(k) \right| << 1.$$

Hint: Replace $\frac{1}{t}$ by an increasing function and derive the similar inequality to Lemma 2.2.

Lemma 2.4.

$$d(n) \leq 2\sqrt{n}.$$

Proof. If $n = d_1 d_2$ then one of them must be less than or equal to \sqrt{n} . \square

We have an improved inequality,

Proposition 2.4. for $\varepsilon > 0$, we have,

$$d(n) << n^\varepsilon.$$

Proof. Recall that for $n = \prod_{i=1}^k p_i^{\alpha_i}$, we have $d(n) = \prod_{i=1}^k (\alpha_i + 1)$. In particular, we have,

$$\frac{d(n)}{n^\varepsilon} = \prod_{i=1}^k \frac{(\alpha_i + 1)}{p_i^{\varepsilon \alpha_i}}.$$

Let $A = \{i \mid p_i^\varepsilon \geq 2\}$. Recall that for $x \geq 1$, $x + 1 \leq 2^x$ that is

$$\frac{x+1}{2^x} \leq 1.$$

Then,

$$\prod_{i \in A} \frac{\alpha_i + 1}{p_i^{\varepsilon \alpha_i}} \leq \prod_{i=1}^k \frac{\alpha_i + 1}{2^{\varepsilon \alpha_i}} \leq 1.$$

For $p_i^\varepsilon < 2$, we observe,

$$p_i^{\varepsilon \alpha_i} = e^{\varepsilon \alpha_i \log(p_i)} \geq \varepsilon \alpha_i \log(p_i).$$

Therefore,

$$\begin{aligned} \prod_{i \notin A} \frac{\alpha_i + 1}{p_i^{\varepsilon \alpha_i}} &\leq \prod_{i \notin A} \left(\frac{\alpha_i}{p_i^{\varepsilon \alpha_i}} + 1 \right), \\ &\leq \prod_{i \notin A} \left(\frac{\alpha_i}{\varepsilon \alpha_i \log(p_i)} + 1 \right), \\ &\leq \prod_{i \notin A} \left(\frac{1}{\varepsilon \log(2^{\frac{1}{\varepsilon}})} + 1 \right), \\ &\leq \prod_{i \notin A} \left(\frac{1}{\log(2)} + 1 \right). \end{aligned}$$

Combining two cases, we obtain the statement. \square

Notation 2.2. Let $x \in \mathbb{R}$. We denote

1. the integer part $[x] \in \mathbb{Z}$ which is the greatest integer not exceeding x ,
2. the fraction part $\{x\} = x - [x]$.

Proposition 2.5.

$$\frac{\sum_{n \leq x} d(n)}{x} \sim \log(x).$$

Proof. By definition, we have,

$$\sum_{n \leq x} d(n) = \sum_{n \leq x} \sum_{d|n} 1.$$

$d|n$ if and only if there is e such that $de = n$. Thus using this we obtain,

$$\sum_{n \leq x} d(n) = \sum_{\substack{e,d \\ de \leq x}} 1 = \sum_{d \leq x} \sum_{e \leq \frac{x}{d}} 1 = \sum_{d \leq x} \left[\frac{x}{d} \right].$$

Using the definition of $[x]$, we have,

$$\sum_{n \leq x} d(n) = \sum_{d \leq x} \frac{x}{d} - \left\{ \frac{x}{d} \right\} = x \sum_{d \leq x} \frac{1}{d} - \sum_{d \leq x} \left\{ \frac{x}{d} \right\} = x(\log(x) + O(1)) + O(x).$$

Thus we obtain the statement. \square

Definition 2.8. Let $f : \mathbb{N} \rightarrow \mathbb{C}$, we say the average order of f is $g : \mathbb{R} \rightarrow \mathbb{C}$ if

$$\frac{\sum_{n \leq x} f(n)}{f(n)} x \sim g(x).$$

Proposition 2.5 can be restated as follows.

Proposition 2.6. The average order of d is $\log(x)$.

Exercise 2.2. Examine the following statements.

1. Is it true that $d(n) << \log n$?
2. Do we have $d(n) = O(n^\varepsilon)$ for any $\varepsilon > 0$?
3. What is the optimal bound for $d(n)$?

Theorem 2.8. There exists $c_1, c_2 > 0$ such that

$$c_1 \leq \frac{\varphi(n)\sigma(n)}{n^2} \leq c_2.$$

Proof. Recall that

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

and

$$\sigma(n) = \prod_{p|n} \frac{p^{\alpha(p)+1} - 1}{p - 1}.$$

Thus we obtain,

$$\frac{\sigma(n)}{n} = \frac{\prod_{p|n} (1 + p + \dots + p^{\alpha(p)})}{\prod_{p|n} p^{\alpha(p)}} = \prod_{p|n} \left(\frac{\frac{1}{p^{\alpha(p)+1}} - 1}{\frac{1}{p} - 1} \right)$$

By multiplying two we get,

$$\frac{\varphi(n)\sigma(n)}{n^2} = \prod_{p|n} \left(1 - \frac{1}{p^{\alpha(p)+1}}\right) \leq 1.$$

On the other hand,

$$\prod_p \left(1 - \frac{1}{p^2}\right) \leq \frac{\varphi(n)\sigma(n)}{n^2}.$$

The left hand side is equal to $\frac{1}{\zeta(2)}$ which is $\frac{6}{\pi^2}$. \square

Theorem 2.9. The average order of φ is $\frac{3n}{\pi^2}$.

Proof.

$$\begin{aligned}
\sum_{n \leq x} \phi(n) &= \sum_{n \leq x} n \sum_{d|n} \frac{\mu(d)}{d}, \\
&= \sum_{\substack{d,e \\ de \leq x}} e \mu(d), \\
&= \sum_{d \leq x} \mu(d) \left(\sum_{e \leq \frac{x}{d}} e \right), \\
&= \frac{1}{2} \sum_{d \leq x} \mu(d) \left(\left[\frac{x}{d} \right] \left(\left[\frac{x}{d} \right] + 1 \right) \right),
\end{aligned}$$

Note that

$$\frac{x}{d} = \left[\frac{x}{d} \right] + \left\{ \frac{x}{d} \right\} = \left[\frac{x}{d} \right] + o(1).$$

It is assigned as an exercise to confirm that

$$\begin{aligned}
&\left[\frac{x}{d} \right] \left(\left[\frac{x}{d} \right] + 1 \right) = \frac{x^2}{d^2} + o\left(\frac{x}{d}\right). \\
&= \frac{1}{2} \sum_{d \leq x} \mu(d) \left(\frac{x^2}{d^x} + o\left(\frac{x}{d}\right) \right), \\
&= \frac{x^2}{2} \sum_{d \leq x} \frac{\mu(d)}{d^2} + o\left(x \sum_{d \leq x} \frac{\mu(d)}{d}\right), \\
&= \frac{x^2}{2} \left(\sum_{d \geq 1} \frac{\mu(d)}{d^2} - \sum_{d \geq x} \frac{\mu(d)}{d^2} \right) + o\left(x \sum_{d \geq x} \frac{\mu(d)}{d}\right), \\
&= \frac{x^2}{2} \frac{1}{\zeta(2)} - \frac{x^2}{2} \sum_{d \geq x} \frac{\mu(d)^2}{d} + o\left(x \sum_{d \leq x} \frac{\mu(d)}{d}\right).
\end{aligned}$$

We have,

$$\begin{aligned}
\left| \sum_{d \geq x} \frac{\mu(d)}{d^2} \right| &\leq \sum_{d \geq x} \frac{1}{d^2} \\
&<< \int_x^\infty \frac{dt}{t^2}, \\
&<< \frac{1}{x}.
\end{aligned}$$

and,

$$\left| \sum_{d \leq x} \frac{\mu(d)}{d} \right| << \ln x.$$

Using these we have,

$$\begin{aligned} &= \frac{x^2}{2} \frac{1}{\zeta(2)} + o(x) + o(x \ln x), \\ &= \frac{x^2}{2\zeta(2)} + o(x \ln x). \end{aligned}$$

We conclude that

$$x \rightarrow \infty \Rightarrow \frac{\sum_{n \leq x} \phi(n)}{x^2} \rightarrow \frac{1}{2\zeta(2)}.$$

In particular,

$$\frac{\sum_{n \leq x} \phi(n)}{x} \sim \frac{x}{2\zeta(2)} = \frac{x \cdot 6}{2\pi^2}.$$

□

2.2 Abel's Summation Formula

Recall the harmonic series $\sum_{n \in \mathbb{N}} \frac{1}{n}$ is divergent. Our next goal is to find such A_x that

$$\lim_{n \rightarrow \infty} \left(\sum_{n \leq x} \frac{1}{n} - A_x \right)$$

exists.

Remark 2.5 (Euler-Mascheroni constant). *By taking $A_x = \log(x)$, we have*

$$\lim_{n \rightarrow \infty} \left(\sum_{n \leq x} \frac{1}{n} - \log(x) \right) = \psi,$$

exists. Such ψ is called Euler-Mascheroni constant.

Remark 2.6 (Euler Kronecer constant). *Take $A_x = \log(x)$, we have*

We can show that

$$\psi = \lim_{s \rightarrow 1^+} \left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right).$$

Hint first show that

$$\zeta(s) = \frac{1}{s-1} + \psi + o(s-1).$$

Proposition 2.7 (Abels' summation formula). *Given $(a_n)_{n \in \mathbb{N}}$ in \mathbb{C} and $f(n)$ is continuously differentiable in $[1, x]$. Set*

$$A(x) := \sum_{n \leq x} a_n.$$

Then we have,

$$\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

Proof. Observe that

$$a_n = A(n) - A(n-1).$$

Assume $x \in \mathbb{N}$. We substitute this to $\sum_{n \leq x} a_n f(n)$, we get,

$$\begin{aligned} \sum_{n \leq x} a_n f(n) &= \sum_{n \leq x} (A(n) - A(n-1)) f(n), \\ &= \sum_{n \leq x} A(n)f(n) - \sum_{n \leq x} A(n-1)f(n), \\ &= \sum_{n \leq x} A(n)f(n) - \sum_{n \leq x-1} A(n)f(n+1), \\ &= A(x)f(x) - \sum_{n \leq x-1} A(n)(f(n+1) - f(n)), \\ &= A(x)f(x) - \sum_{n \leq x-1} \int_n^{n+1} f'(t)dt, \\ &= A(x)f(x) - \sum_{n \leq x-1} \int_n^{n+1} A(t)f'(t)dt, \\ &= A(x)f(x) - \int_1^x A(t)f'(t)dt. \end{aligned}$$

For the case when $n \notin \mathbb{N}$ and $n > 1$,

$$\sum_{n \leq x} a_n f(n) = \sum_{n \leq [x]} a_n f(n).$$

Using the previous case, we get,

$$\sum_{n \leq x} a_n f(n) = A([x])f([x]) - \int_1^{[x]} A(t)f'(t)dt.$$

Remains to show that we can remove the brackets. To do so,

$$\begin{aligned}
\sum_{n \leq x} a_n f(n) &= A([x])f([x]) - \int_1^x A(t)f'(t)dt + \int_{[x]}^x A(t)f'(t)dt, \\
&= A([x])f([x]) - \int_1^x A(t)f'(t)dt + A([x]) \int_{[x]}^x f'(t)dt, \\
&= A([x])f([x]) - \int_1^x A(t)f'(t)dt + A([x])f(x) - A([x])(f[x]), \\
&= A([x])f(x) - \int_1^x A(t)f'(t)dt.
\end{aligned}$$

□

Corollary 2.1.

1. $\sum_{n \leq x} \frac{1}{n} = \ln x + \psi + o\left(\frac{1}{x}\right)$.
2. $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + o\left(\frac{1}{x^s}\right)$, where $\operatorname{Re}(s) > 0, s \neq 1$.

We also have the following equivalent forms of prime number theorem when $x \rightarrow \infty$.

$$\begin{aligned}
\sum_{n \leq x} s(n) &\sim x \\
\Leftrightarrow \pi(x) &\sim \frac{x}{\ln x}, \\
\Leftrightarrow \sum_{p \leq x} \ln p &\sim x, \\
\Leftrightarrow \sum_{n \leq x} \mu(n) &= o(x).
\end{aligned}$$

Proof. Consider $f(t) = \frac{1}{t^s}$ and $a_n = 1$ for all $n \in \mathbb{N}$.

$$\begin{aligned}
\sum_{n \leq x} \frac{1}{n} &= \frac{[x]}{x} + \int_1^x \frac{[x]}{t^2} dt, \\
&= \frac{x - \{x\}}{x} + s \int_1^x \frac{t - \{t\}}{t^{s+1}} dt.
\end{aligned}$$

When $s = 1$, we have,

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n} &= 1 - \frac{\{x\}}{x} + \int_1^x \frac{t - \{t\}}{t^{s+1}} dt, \\ &= 1 + \ln x + o\left(\frac{1}{x}\right) - \int_1^x \frac{\{t\}}{t^2} dt, \\ \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{1}{n} - \ln x \right) &= 1 - \int_1^\infty \frac{\{t\}}{t^2} dt, \\ &= \psi. \\ &= x^{1-s} + o\left(\frac{1}{x^s}\right) + \frac{sx^{1-s}}{1-s} - \frac{s}{1-s} - s^2 \int_{\frac{\{t\}}{t^{s+1}}} dt. \end{aligned}$$

Recall that

$$\left[\int \frac{1}{t^s} = \frac{t^{-s+1}}{1-s} \right]_1^x = \frac{x^{1-s}}{1-s} - \frac{1}{1-s}.$$

Using this we obtain,

$$\begin{aligned} &= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - x \int_1^x \frac{\{t\}}{t^{s+1}} dt, \\ x^{1-s} \left[1 + \frac{1}{1-s} \right] &= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - s \int_1^x \frac{\{t\}}{t^{s+1}} dt \\ &\int_1^\infty \frac{\{t\}}{t^{s+1}} dt < \infty, \\ &\leq \int_1^\infty \frac{1}{t^{\operatorname{Re}(s)+1}} dt < \infty. \end{aligned}$$

As $x \rightarrow \infty$, the left hand side goes to $\zeta(s)$, for the right hand side, we get $= \frac{-s}{1-s} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt$, where $\operatorname{Re}(s) > 1$,

$$\zeta(s) = \frac{-s}{1-s} - s \int \frac{\{t\}}{t^{s+1}} dt.$$

Identity theorem for analytic function tells us the analytic continuation of Riemann zeta function is unique.

Remark 2.7. *It is an exercise that*

$$\int_1^\infty \frac{\{t\}}{t^{s+1}} dt$$

where $\operatorname{Re}(x) > 0$. From Stein-Schakarchi 5.2, 5.3, we have

$$\sum f_n(z) \xrightarrow{\text{unif.}} f(z)$$

is analytic where $\operatorname{Re}(x) > 0$ and $s \neq 1$, also in this case,

$$\zeta(s) = \frac{-s}{1-s} - s \int \frac{\{t\}}{t^{s+1}} dt.$$

holds.

Remark 2.8 (Exercise). Let $M \in \mathbb{N}$ and

$$\lim_{x \rightarrow \infty} \left(\sum_{\substack{n \leq x \\ (n, M)=1}} \frac{1}{n} - \frac{\phi(M)}{M} \ln x \right)$$

exists.

Assume $\pi(x) \sim \frac{x}{\ln x}$, to show

$$\theta(x) := \sum_{p \leq x} \ln p \sim x,$$

Consider the following sequence

$$a_n = \begin{cases} 1 & n \text{ is prime}, \\ 0 & \text{otherwise}. \end{cases}$$

and

$$f(t) = \ln t.$$

Using Abel summation formula,

$$\sum_{p \leq x} \ln p = \pi(x) \ln(x) - \int_1^\infty \frac{\pi(t)}{t} dt.$$

can be written as

$$\frac{\theta(x)}{x} = \frac{\pi(x) \ln x}{x} - \frac{1}{x} \int_1^x \frac{\pi(t)}{t} dt.$$

Remark 2.9 (Exercise). Use $\frac{\pi(t)}{t} \sim \frac{1}{\ln t}$ and

$$\int_1^x \frac{dt}{\ln t} = o(x),$$

prove that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \int_1^x \frac{\pi(t)}{t} dt = 0.$$

$$\begin{aligned} \int_2^{\sqrt{x}} \frac{dt}{\ln t} &= \int_{\sqrt{x}}^x \frac{dt}{\ln t} \leq \frac{1}{\ln 2} \int_2^{\sqrt{x}} dt + \frac{1}{\ln \sqrt{x}} \int_{\sqrt{x}}^x dt, \\ &= \frac{\sqrt{x} - 2}{\ln 2} + \frac{x - \sqrt{x}}{\ln \sqrt{x}}, \\ &= o(x). \end{aligned}$$

Remark 2.10.

$$\psi(x) = \sum_{n \leq x} s(n) = \sum_{1 \leq k, p^k \leq x} \ln p.$$

Thus we see,

$$\psi(x) - \theta(x) = \sum_{2 \leq k, p, p^k \leq x} \ln p.$$

Also,

$$p^k \leq x \Rightarrow k \leq \frac{\ln x}{\ln p}.$$

Using $k \geq 2$,

$$p \leq x^{\frac{1}{k}} \leq \sqrt{x}, \forall k.$$

$$\begin{aligned} &\leq \sum_{p \leq \sqrt{x}} \ln p \left(\sum_{2 \leq k \leq \frac{\ln x}{\ln p}} 1 \right), \\ &\leq \sum_{p \leq \sqrt{x}} \ln x, \\ &\leq \ln(x) \sum_{n \leq \sqrt{x}} 1, \\ &\leq \sqrt{x} \ln x. \Rightarrow \quad \frac{\psi(x)}{x} = \frac{\theta(x)}{x} + \frac{o(\sqrt{x} \ln x)}{x}. \end{aligned}$$

Therefore we obtain,

$$\psi(x) \sim x \Leftrightarrow \theta(x) \sim x.$$

□

Remark 2.11. As exercises, find the closed expressions for the following summations.

$$\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n}, \sum_{p \leq x} \frac{1}{p} = \ln \ln x + o(1).$$

2.3 Characters

Definition 2.9. Let G be a finite group. A character is a group homomorphism $f : G \rightarrow \mathbb{C}^\times$.

Remark 2.12. Let us denote

$$\hat{G} := \{f : G \rightarrow \mathbb{C}^\times \mid \text{characters}\}.$$

If G is finite abelian then $|\hat{G}| = |G|$. Furthermore, such characters are linearly independent over \mathbb{C} .

Definition 2.10. Let $q \in \mathbb{N}$ and $q \geq 3$. A Dirichlet character is a group homomorphism modulo q is a group homomorphism

$$\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times.$$

Remark 2.13. Given a Dirichlet character $\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$. We can define a character $\chi : \mathbb{Z} \rightarrow \mathbb{C}^\times$ as follows.

$$\chi(a) := \begin{cases} \chi'(\bar{a}), & (a, q) = 1, \\ 0, & (a, q) \neq 1. \end{cases}$$

From Remark 2.12, there are exactly $\varphi(q)$ many Dirichlet characters modulo q . Furthermore,

$$\chi(a)^{\varphi(q)} = \chi'(\bar{a})^{\varphi(q)} = \chi'(\bar{a}^{\varphi(q)}) = \chi'(\bar{1}) = 1.$$

In particular, images of χ are $\varphi(q)$ -th roots of unity.

Example 2.8. For $q = 3$,

$$(\mathbb{Z}/3\mathbb{Z})^\times = \{\bar{1}, \bar{2}\} \rightarrow \mathbb{C}^\times.$$

We only have two characters, a trivial one and $\bar{2} \mapsto -1$.

Example 2.9. For $q = 5$,

$$(\mathbb{Z}/5\mathbb{Z})^\times = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$$

	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\chi_{1,5}(n)$	1	1	1	1
$\chi_{2,5}(n)$	1	-1	-1	1
$\chi_{3,5}(n)$	1	i	$-i$	-1
$\chi_{4,5}(n)$	1	$-i$	i	-1

$\chi_{1,5}$ is called a principle/trivial character.

Definition 2.11. A character $\chi : \mathbb{Z} \rightarrow \mathbb{C}^\times$ is called

1. trivial if $\chi(g) = 1$ for all $g \in G$,
2. even if $\chi(-1) = 1$,
3. odd if $\chi(-1) = -1$.

We also define these notions for characters $\chi_0 : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ accordingly if characters induces by $\chi : \mathbb{Z} \rightarrow \mathbb{C}^\times$ has these properties. Trivial characters are often denoted by χ_0 .

Theorem 2.10.

$$\sum_{a \bmod q} \chi(a) = \begin{cases} \phi(q) & (\chi = \chi_0), \\ 0 & (\text{otherwise}). \end{cases}$$

We also have,

$$\sum_{\chi \bmod q} \chi(a) = \begin{cases} \phi(a) & (\bar{a} = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. If χ is principle then the first assertion is clear. Suppose χ is not principle then there is $b \in \{1, \dots, q\}$ such that $\chi(b) \neq 1$ and $(b, q) = 1$. Let

$$s = \sum_{a \bmod q} \chi(a).$$

Then by the definition of group homomorphisms, we have,

$$\chi(b)s = s.$$

But $\chi(b) \in \mathbb{C}$, this means $s = 0$ as \mathbb{C} is an integral domain.

For the second assertion, let $\bar{a} \neq 1$, then

$$\exists \chi' \bmod q, \text{ s.t. } \chi'(a) \neq 1.$$

Thus we get,

$$s = \sum_{\chi \bmod q} \chi(a), s \cdot \chi(a) = \sum_{\chi \bmod q} \chi \chi'(a) = s \Rightarrow s = 0.$$

The statement when $\bar{a} = 1$ follows from Remark 2.12. \square

Remark 2.14. One can check in the table of Example 2.9 that Theorem 2.10 indeed holds.

Exercise 2.3.

$$\sum_{\substack{\chi \bmod q, \\ \chi(-1)=1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} & (\bar{a} = 1, -1), \\ 0 & (\text{otherwise}). \end{cases},$$

$$\sum_{\substack{\chi \bmod q, \\ \chi(-1)=-1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} & (\bar{a} = 1), \\ -\frac{\phi(a)}{2} & (\bar{a} = -1), \\ 0 & (\text{otherwise}). \end{cases},$$

Obviously we have the following equalities.

$$\sum_{n \leq x} \chi(n) = \sum_{\substack{n \leq x \\ (n,q)=1}} \chi(n) = \sum_{n \leq kq} \chi(n) + \sum_{n=kq+1}^x \chi(n),$$

where k is the largest integer such that $kq \leq x$. Then we observe from Theorem 2.10

$$\sum_{n \leq kq} \chi(n) = k \left(\sum_{n=1}^q \chi(n) \right) = 0,$$

unless χ is trivial. Also we have,

$$\left| \sum_{n \leq x} \chi(n) \right| = \left| \sum_{\substack{kq+1 \leq n \leq x \\ (n,q)=1}} \chi(n) \right| \leq \sum_{\substack{kq+1 \leq n \leq x \\ (n,q)=1}} 1 \leq \sum_{\substack{kq+1 \leq n \leq kq+q \\ (n,q)=1}} 1 = \phi(q).$$

Thus we conclude,

$$\sum_{n \leq x} \chi(n) \leq \phi(n),$$

Exercise 2.4.

$$\sum_{n \leq x} \chi_0(n) = ?.$$

Theorem 2.11 (Pólya–Vinogradov).

$$\sum_{n \leq x} \chi(n) << \sqrt{q} \ln q, (\chi \neq \chi_0 \pmod{q}).$$

notice that the above expression is bounded by $\sqrt{q} \ln \ln q$. Furthermore, this is uniform in q that is the constant does not depend on q .

$$\sum_{n \geq 1} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + O(x^{-s}) - s \int_1^x \frac{\{t\}}{t^{s+1}} dt.$$

For $\operatorname{Re}(s) > 1$, as $x \rightarrow \infty$ we have,

$$\sum_{n \geq 1} \frac{1}{n^s} = \frac{-s}{1-s} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

The last expression is analytic since,

$$\sum \int_n^{n+1} \frac{\{t\}}{t} dt \xrightarrow{\text{uniformly}} \int_1^\infty \frac{\{t\}}{t} dt, \text{ when } \operatorname{Re}(s) > 0.$$

Suppose $\zeta(s) \neq 0$, where $\operatorname{Re}(s) > 0$, then Euler product exists.

Theorem 2.12. Set

$$A(n) := \sum_{n \in \mathbb{N}} a_n.$$

Assume $A(x) := \sum_{n \leq x} a_n = O(x^\delta)$, then we have, for $\operatorname{Re}(s) > \delta$,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt.$$

Hence the Dirichlet series converges for $\operatorname{Re}(s) > \delta$.

Proof.

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s} = \frac{A(x)}{x^s} + s \int_1^x \frac{A(t)}{t^{s+1}} dt.$$

As $A(x) = O(x^\delta)$ and $\operatorname{Re}(s) > \delta$, $\frac{A(x)}{x^s} = O(x^{\delta - \operatorname{Re}(s)})$. Therefore, as $x \rightarrow \infty$, we have,

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt.$$

Again using the assumption, we have,

$$\int_1^{\infty} \left| \frac{A(t)}{t^{s+1}} \right| dt << \int_1^{\infty} t^{\delta - \operatorname{Re}(s) - 1} dt = \frac{t^{\delta - \operatorname{Re}(s)}}{\delta - \operatorname{Re}(s)} \Big|_0^{\infty} = \frac{1}{\delta - \operatorname{Re}(s)}.$$

Thus the integral is convergent. \square

Definition 2.12. For $\operatorname{Re}(s) > 1$, we define

$$L(s, \chi) := \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s}.$$

Remark 2.15. Since $\operatorname{Re}(s) > 1$ and for any character $\chi : \mathbb{Z} \rightarrow \mathbb{C}^{\times}$ we have $|\chi(n)| \leq 1$, $L(s, \chi)$ is uniformly absolutely convergent.

Example 2.10. Let $\chi : \mathbb{Z} \rightarrow \mathbb{C}^{\times}$ be a non-principal character modulo q and set $A(n) = \chi(n)$. Recall that

$$\sum_{n \leq x} \chi(n) \leq q.$$

Taking $A(n) = \chi(n)$ and apply Theorem 2.12, we obtain, for $\operatorname{Re}(s) > 0$,

$$L(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = s \int_1^{\infty} \frac{\sum_{n \leq t} \chi(n)}{t^{s+1}} dt.$$

Since for $\operatorname{Re}(s) > 1$, $L(s, \chi)$ is absolutely uniformly convergent. By Theorem 2.7, we have,

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s} \right)^{-1}.$$

If $\chi = \chi_0$, a principal character, we have,

$$L(s, \chi_0) = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 - \frac{1}{p^s}\right) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Using ζ function, we have,

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Theorem 2.13. ζ has an analytic continuation for $\operatorname{Re}(s) > 0$ besides $s = 1$. For $s = 1$ we have a simple pole of residue 1.

Proof. Recall from the proof of Corollary 2.1. We have for $\operatorname{Re}(s) > 1$,

$$\zeta(s) = \frac{-s}{1-s} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

The right hand side of the equation is analytic when $\operatorname{Re}(s) > 0, s \neq 1$. When $s = 1$, we have,

$$\lim_{s \rightarrow 1^+} (s-1)\zeta(s) = \lim_{s \rightarrow 1^+} \left(s - s(s-1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt \right) = 1.$$

□

Corollary 2.2. For $\operatorname{Re}(s) > 0$ and $s \neq 1$, we have an analytic continuation of $L(s, \chi_0)$ where χ_0 is a principal character modulo q , which is

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Obviously at $s = 1$, it has a simple pole of residue $\prod_{p|q} \left(1 - \frac{1}{p^s}\right)$, which we can write as

$$\operatorname{Res}(L(s, \chi_0), 1) = \frac{\varphi(q)}{q}.$$

Proof. A direct corollary of Theorem 2.13. □

Suppose $\chi \neq \chi_0$, we have, analytic continuation of $L(s, \chi) \neq 0$ is

Theorem 2.14. Let χ be a non-principal character, then there is an analytic continuation of $L(s, \chi)$ for $\operatorname{Re}(s) > 0$.

Proof. From Example 2.10, we have,

$$L(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = s \int_0^\infty \frac{\sum_{n \leq t} \chi(n)}{t^{s+1}} dt.$$

The right hand side is analytic. □

Theorem 2.15.

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s}$$

is analytic in its range of convergence.

Remark 2.16. We have the following conjecture.

$$L\left(\frac{1}{2}, \chi\right) = 0, \chi \neq \chi_0.?$$

$$\zeta\left(\frac{1}{2}\right) = \frac{1}{1 - \sqrt{2}} \sum_{n \in \mathbb{N}} \frac{(-1)^{n-1}}{\sqrt{n}} \approx -1.46 \dots$$

Definition 2.13. A character is said to be quadratic if its values are either ± 1 .

Remark 2.17. We have,

$$L(s, \chi) = 0 \quad \text{if } s = 0, -2, -4, \text{ when } \chi \text{ is an even character.}$$

$$L(s, \chi) = 0 \quad \text{if } s = -1, -3, -5, \text{ when } \chi \text{ is an odd character.}$$

Lemma 2.5. For $\sigma > 1$ and $t \in \mathbb{R}$, we have,

$$\operatorname{Re}(\ln(\zeta(\sigma + it))) = \sum \frac{\Lambda(n)}{n^\sigma \ln n} \ln(t \ln(n)).$$

And also,

$$\operatorname{Re}(3 \ln(\zeta(\sigma)) + 4 \ln(\zeta(\sigma + it)) + \ln(\zeta(\sigma + 2it))) \geq 0.$$

Proof.

$$\begin{aligned} \zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \sigma > 0, \\ \ln(\zeta(s)) &= - \sum_p \ln(1 - p^{-s}) \\ &= \sum_{p,n} \frac{1}{np^{ns}}, \sigma > 1. \end{aligned}$$

We have,

$$\sum_{n \geq 2} \frac{\Lambda(n)}{n^s \ln n} = \sum_{p,k,k \geq 1} \frac{\ln p}{p^{ks} \ln p^k} = \sum_{p,k} \frac{1}{kp^{ks}} = \sum_{n \geq 2} \Lambda(n).$$

$$\operatorname{Re}(3 \ln \zeta(\sigma) + 4 \ln \zeta(\sigma + it) + \ln \zeta(\sigma + 2it)) = \sum_{n \geq 2} \frac{\Lambda(n)}{n^\sigma \ln n} (3 + 4 \cos(t \ln n) + \cos(2t \ln n)) \geq 0,$$

since

$$3 + 4 \cos \theta + \cos 2\theta = 2(\cos \theta + 1)^2 \geq 0.$$

$$= \ln |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 0.$$

Thus we have,

$$\operatorname{Re}(\ln(z)) \leq$$

□

Theorem 2.16. For $t \in \mathbb{R} \setminus \{0\}$, we have

$$\zeta(1 + it) \neq 0.$$

Proof. Using Lemma 2.5, we have, Thus we get,

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1.$$

Suppose $\zeta(1 + it_0) = 0$, for $t_0 \in \mathbb{R} \setminus \{0\}$. Suppose further that the order of zero is $m \in \mathbb{N}$. Then by looking at, and taking $\sigma \rightarrow 1+$

$$\underbrace{((\sigma - 1)^3 \zeta(\sigma))}_{\substack{\rightarrow \text{finite} \\ \rightarrow \text{non-zero}}} \underbrace{\left(\frac{\zeta(\sigma + it_0)}{(\sigma - 1)^m} \right)}_{\substack{\rightarrow 0}} \underbrace{((\sigma - 1)^{4m-3} \zeta(\sigma + 2it_0))}_{\rightarrow 0}.$$

Contradicts to that the absolute value of above expression is at least 1. □

Theorem 2.17. $\frac{\zeta'(s)}{\zeta(s)}$ has an analytic continuation to $\operatorname{Re}(s) = 1, s \neq 1$. And for $s = 1$, we have a simple pole of residue -1 .

Proof. We have,

$$(s - 1)\zeta(s) = s - s(s - 1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

Set

$$f(s) := 1 - (s - 1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt,$$

so that $(s - 1)\zeta(s) = sf(s)$. We already have $f(s)$ is analytic when $\operatorname{Re}(s) > 0$. Differentiating both sides we get,

$$(s - 1)\zeta'(s) + \zeta(s) = sf'(s) + f(s).$$

Dividing both sides by $(s - 1)\zeta(s)$ we get,

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s - 1} = \frac{sf'(s) + f(s)}{(s - 1)\zeta(s)}.$$

The right hand side is analytic when ζ does not vanish. From Theorem 2.16, by letting $s = 1 + it$ for some $t \in \mathbb{R} \setminus \{0\}$, we get the desired analytic continuation. For $s = 1$, we have

$$(s-1) \frac{\zeta'(s)}{\zeta(s)} = \frac{sf'(s) + f(s)}{\zeta(s)} - 1.$$

Recall that $\zeta(s)$ has a pole at 1 and observe that $f(1) = 1, f'(1)$ is finite thus the last statement follows. \square

Theorem 2.18. *Let χ be a non-trivial non-real Dirichlet character of modulo q , then*

$$L(1, \chi) \neq 0.$$

Proof. Recall that

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where $\operatorname{Re}(s) > 1$. Now consider,

$$\log L(s, \chi) = \sum_{n,p} \frac{\chi(p^s)}{np^{ns}} = \sum_{n \geq 2} \frac{\Lambda(n)\chi(n)}{n^s \log n}.$$

Set $\zeta_{\varphi(q)} = e^{\frac{2\pi i}{\varphi(q)}}$. Then for any $n \in (\mathbb{Z}/q\mathbb{Z})^\times$, there is $n' = n(n, q, \chi)$ (ie n' depends on n, q , and χ) such that

$$\chi(n) = \zeta_{\varphi(q)}^{n'}$$

Note that for a cyclic group $G = \langle g \rangle$, and a character $\chi : G \rightarrow \mathbb{C}^\times \in \hat{G}$, there is $a \in \{1, \dots, \varphi(q)\}$ such that

$$\chi(g) = \zeta_{\varphi(q)}^a.$$

We have, for $\sigma > 1$

$$\operatorname{Re}(\log L(\sigma, \chi)) = \sum_{n \geq 2} \frac{\Lambda(n) \cos\left(\frac{2\pi i}{\varphi(q)}\right)}{n^\sigma \log(n)}.$$

By Lemma 2.5, we have,

$$\operatorname{Re}(3\zeta(\sigma) + 4L(\sigma, \chi) + L(\sigma, \chi^2)) \geq 0,$$

therefore,

$$|\zeta(\sigma)^3 L(\sigma, \chi)^4 L(\sigma, \chi^2)| \geq 1.$$

Note that $\chi^2 \neq \chi$ as it is non-real and $L(\sigma, \chi^2)$ is analytic by Theorem 2.14. If $L(\sigma, \chi) = 0$ then

$$\zeta(\sigma)^3 L(\sigma, \chi)^4$$

has a zero of order at least 1. Thus this is a contradiction. \square

Lemma 2.6. For $k \in \mathbb{Z}_{\geq 0}$,

$$\left(\frac{\sin(k + \frac{1}{2})\theta}{\sin \frac{\theta}{2}} \right)^2 = 2k + 1 \sum_{j=1}^{2k} 2(2k+1-j) \cos j\theta.$$

Theorem 2.19. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a function such that

i). f is analytic,

ii). $f \not\equiv 0$,

iii). $\log f(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s}$ for $a_n \geq 0$ and $\operatorname{Re}(s) \geq 1$,

iv). f is analytic on $\operatorname{Re}(s) = 1, s \neq 1$, and it has a pole at $s = 1$ of order e .

If $f(s) = 0$ on the line $\operatorname{Re}(s) = 1$, then the order of zero is at most $\frac{e}{2}$.

Proof. For $e \leq 2k - 1$, let us define,

$$g(s) := f(s)^{2k+1} \prod_{j=1}^{2k} f(s + ijt_0)^{2(2k+1-j)} = f(s)^{2k+1} f(s + it_0)^{4k} \dots.$$

Then $f(s)^{2k+1}$ has a pole of order $e(2k+1)$ and $\prod_{j=1}^{2k} f(s + ijt_0)^{2(2k+1-j)}$ has a zero of order $2k(2k+1)$. Note that

$$2k(2k+1) - e(2k+1) \geq 1.$$

Thus g has a zero at 1 of order at least 1, in particular, $g(1) = 0$.

Consider,

$$\begin{aligned} \log g(\sigma) &= (2k+1) \log f(\sigma) + \sum_{j=1}^{2k} 2(2k+1-j) \log(f(\sigma + ijt_0)), \\ &= (2k+1) \sum_{n \in \mathbb{N}} \frac{a_n}{n^\sigma} + \sum_{j=1}^{2k} 2(2k+1-j) \left(\sum_{n \in \mathbb{N}} \frac{a_n}{a^{\sigma + ijt_0}} \right), \\ &= \sum_{n \in \mathbb{N}} \frac{a_n}{n^\sigma} \left(2k+1 + \sum_{j=1}^{2k} 2(2k+i-j) e^{-ijt_0 \log(n)} \right). \\ \operatorname{Re}(\log(g(\sigma))) &= \sum_{n \in \mathbb{N}} \frac{a_n}{n^\sigma} \left(2k+1 + \sum_{j=1}^{2k} 2(2k+i-j) \cos(jt_0 \log(n)) \right). \end{aligned}$$

Using Lemma 2.6, we get,

$$\operatorname{Re} \log(g(\sigma)) = \sum_{n \geq 1} \frac{a_n}{n^\sigma} \left(\frac{\sin(k + \frac{1}{2}) t_0 \log(n)}{\sin \frac{t_0 \log(n)}{2}} \right)^2 \geq 0.$$

Therefore $|g(\sigma)| \geq 1$ if $\operatorname{Re} g(\sigma) \geq 1$. \square

Corollary 2.3. *For any Dirichlet character χ of modulo q we have, $L(s, \chi)$ is analytic over $\operatorname{Re}(s) > 1$. Furthermore $L(s, \chi) \neq 0$ on $\operatorname{Re}(s) = 1, s \neq 1$.*

Proof. The first assertion is due to Theorem 2.14. Let

$$f(s) := \prod_{\chi \bmod q} L(s, \chi).$$

Then,

$$\log(f(s)) = \sum_{\chi \bmod q} \log L(s, \chi) = \sum_{n,p} \frac{\sum_{\chi \bmod q} \chi(p^n)}{np^{ns}}, \operatorname{Re}(s) > 1.$$

Using Theorem 2.10, we get,

$$\log(f(s)) = \sum_{\substack{n,p \\ p^n \equiv 1 \pmod{q}}} \frac{\varphi(q)}{np^{ns}}.$$

Note f has a pole of order at most 1 at $\operatorname{Re}(s) = 1$ and its residue is

$$\operatorname{Res}(f(s), 1) = \left(\lim_{s \rightarrow 1} (s-1)L(s, \chi_0) \right) \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} L(1, \chi) = \frac{\varphi(q)}{q} \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} L(1, \chi).$$

The right hand side is not 0 by Theorem 2.18. \square

Theorem 2.20. *There are infinitely many primes.*

Proof. We have,

$$\begin{aligned} \zeta(s) &= \prod_p \left(1 - \frac{1}{p^s} \right)^{-1}. \\ \log \zeta(s) &= \sum_{n,p} \frac{1}{np^{ns}} = \sum_p \frac{1}{p^s} + \sum_{n \geq 2} \frac{1}{np^{ns}}. \end{aligned}$$

Since ζ has a pole at 1, so does its log. Observe that at $s = 1$,

$$\sum_p \frac{1}{p^s} + \sum_{n \geq 2} \frac{1}{np^{ns}} \leq \sum_p \sum_{n \geq 2} \frac{1}{p^n} = \sum_p \frac{1}{p^2} \frac{1}{1 - \frac{1}{p}} = \sum_p \frac{1}{p(p-1)} \ll \sum_{n \in \mathbb{N}} \frac{1}{n^2}.$$

Thus $\lim_{s \rightarrow 1^+} \sum_p \frac{1}{p^s}$ must be infinity. \square

Corollary 2.4. *For any Dirichlet character χ of modulo q we have, $L(s, \chi) \neq 0$ for $\operatorname{Re}(s) = 1, s \neq 1$.*

Proof. Let

$$f(s) := \prod_{\chi \bmod q} L(s, \chi).$$

\square

Lemma 2.7. Let p be a prime and a, q be coprimes such that $p \equiv a \pmod{q}$. Then we have,

$$\sum_{\chi \pmod{q}} \chi(p^n) \overline{\chi(a)} = \begin{cases} \varphi(q), & (p^n \equiv a \pmod{q}), \\ 0, & (\text{otherwise}). \end{cases}$$

Proof. Since $\chi(a)$ is a $\varphi(q)$ -th root of unity, we have, $\overline{\chi(a)} = \chi(a^{-1})$. Therefore,

$$\begin{aligned} \sum_{\chi \pmod{q}} \chi(p^n) \overline{\chi(a)} &= \sum_{\chi \pmod{q}} \chi(p^n) \chi(a^{-1}), \\ &= \sum_{\chi \pmod{q}} \chi(p^n a^{-1}), \\ &= \begin{cases} \varphi(q), & (p^n a^{-1} \equiv 1 \pmod{q}), \\ 0, & (\text{otherwise}). \end{cases} \end{aligned}$$

□

Theorem 2.21 (Dirichlet's Theorem). Let a, q be coprime. Then $(a + nq)_{n \in \mathbb{N}}$ contains infinitely many primes.

Proof. Motivated by the alternative proof of the existence of infinitely many primes, examine,

$$\sum_{\chi \pmod{q}} \log L(s, \chi) = \sum_{n,p} \left(\frac{\sum_{\chi \pmod{q}} \chi(p^n)}{np^{ns}} \right),$$

where $\operatorname{Re}(s) > 1$. Using Theorem 2.10, we have,

$$\sum_{\chi \pmod{q}} \log L(s, \chi) = \sum_{\substack{n,p \\ p^n \equiv 1 \pmod{q}}} \frac{\varphi(q)}{np^{ns}} = \sum_{\substack{p^n \equiv 1 \pmod{q}}} \frac{\varphi(q)}{p^s} + \sum_{\substack{p,n \geq 2 \\ p^n \equiv 1 \pmod{q}}} \frac{\varphi(q)}{np^{ns}}.$$

Taking the log out we have,

$$\prod_{\chi \pmod{q}} L(s, \chi) = \exp \left(\varphi(q) \left(\sum_{\substack{p^n \equiv 1 \pmod{q}}} \frac{1}{p^s} + \sum_{\substack{p,n \geq 2 \\ p^n \equiv 1 \pmod{q}}} \frac{1}{np^{ns}} \right) \right).$$

Since for $\chi \neq \chi_0$, $L(1, \chi) \neq 0$ and by Corollary 2.2, we have,

$$\lim_{s \rightarrow 1^+} (s-1) \prod_{\chi \pmod{q}} L(s, \chi) = \frac{\varphi(q)}{q} \prod_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} L(1, \chi).$$

By the same argument from Theorem 2.20, we see,

$$\sum_{\substack{p,n \geq 2 \\ p^n \equiv 1 \pmod{q}}} \frac{1}{np^{ns}} \leq \sum_{n \in \mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

However,

$$\lim_{s \rightarrow 1^+} (s-1) \exp \left(\varphi(q) \left(\sum_{p^n \equiv 1 \pmod{q}} \frac{1}{p^s} + \sum_{\substack{p,n \geq 2 \\ p^n \equiv 1 \pmod{q}}} \frac{1}{np^{ns}} \right) \right) \neq 0$$

which is only possible when

$$\lim_{s \rightarrow 1^+} \sum_{p^n \equiv 1 \pmod{q}} \frac{1}{p^s} = \infty.$$

Together with Lemma 2.7, we derived the general statement. \square

Theorem 2.22 (Bertrand's Postulate). *For a sufficiently large $n \in \mathbb{N}$, there is a prime number inbetween n and $2n$.*

Proof. Consider the second Chebyshev function, and by Remark 2.2, we have,

$$T(x) := \sum_{l \leq x} \psi\left(\frac{x}{l}\right) = \sum_{l \leq x} \sum_{n \leq \frac{x}{l}} \Lambda(n) = \sum_{\substack{l,n \\ ln \leq x}} \Lambda(n).$$

That is

$$\sum_{l \leq x} \psi\left(\frac{x}{l}\right) = \sum_{m \leq x} \sum_{d|m} \Lambda(d).$$

By Theorem 2.3,

$$T(x) = \sum_{m \leq x} \log(m).$$

Using the definition,

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= \sum_{l \leq x} \psi\left(\frac{x}{l}\right) - 2 \sum_{2l \leq x} \psi\left(\frac{x}{2l}\right), \\ &= \sum_{l \leq x} (-1)^{l+1} \psi\left(\frac{x}{l}\right), \\ &= \psi(x) - \psi\left(\frac{x}{2}\right) + \dots. \end{aligned}$$

Again by Remark 2.2

$$x \leq y \Rightarrow \psi(x) \leq \psi(y).$$

In particular,

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq \sum_{l \leq x} (-1)^{l+1} \psi\left(\frac{x}{l}\right) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right).$$

\square

Lemma 2.8. Let $f := \sum_{d|n} \chi(d)$, where χ is a character, then,

$$\forall n \in \mathbb{N}, n \text{ is a perfect square} \Rightarrow f(n) \geq 0, f(n) \geq 1.$$

Proof. Recall $n = \prod_{p|n} p^{\alpha(p)}$. Using this we have,

$$\begin{aligned} \sum_{d|n} \chi(d) &= \prod_{p|n} \left(\sum_{k=0}^{\alpha(p)} \chi(p)^k \right), \\ &= \begin{cases} 1 & \chi(p) = 0, \\ \prod_{p|n} (1 + \alpha(p)) & \chi(p) = 1, \\ \prod_{p|n} \left(\frac{(1 - (-1)^{\alpha(p)+1})}{2} \right) & \chi(p) = -1. \end{cases} \end{aligned}$$

Note that if $\alpha(p)$ are all even for $p|n$, we have the last part of the cases equals to 1. \square

Theorem 2.23. Let $f(n) = \sum_{d|n} \chi(d)$ for some character. Then we have,

$$\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = 2\sqrt{x}L(1, \chi) + o(1).$$

Proof.

$$\begin{aligned} \sum_{n \leq x} \frac{f(n)}{\sqrt{n}} &= \sum_{n \leq x} \left(\frac{\sum_{d|n} \chi(d)}{\sqrt{n}} \right), \\ &= \sum_{\substack{d, e \\ de \leq x}} \frac{\chi(d)}{\sqrt{de}}, \\ &= \sum_{\substack{d, e \leq x \\ d \leq \sqrt{x}}} \frac{\chi(d)}{\sqrt{de}} + \sum_{\substack{de \leq x \\ d > \sqrt{x}}} \frac{\chi(d)}{\sqrt{de}}, \\ &= \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(\sum_{e \leq \frac{x}{d}} \frac{1}{\sqrt{e}} \right) + \sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left(\sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{(d)}} \right). \end{aligned}$$

Recall that

$$\sum_{m \leq x} \frac{1}{\sqrt{m}} = 2\sqrt{x} + B + o\left(\frac{1}{\sqrt{x}}\right),$$

where B is some constant as $x \rightarrow \infty$.

Let $x, y \in \mathbb{R}$, such that $x < y$, we have,

$$\begin{aligned} \sum_{x < d \leq y} \frac{\chi(d)}{\sqrt{d}} &= \sum_{d \leq y} \frac{\chi(d)}{\sqrt{d}} - \sum_{d \leq x} \frac{\chi(d)}{\sqrt{d}}, \\ \sum_{d \leq x} \frac{\chi(d)}{\sqrt{d}} &= \frac{\sum_{d \leq x} \chi(d)}{\sqrt{x}} + \frac{1}{2} \int_1^x \frac{\sum_{d \leq t} \chi(d)}{t^{\frac{1}{2}}} dt, \\ &= o\left(\frac{1}{\sqrt{x}}\right). \end{aligned}$$

Using these equations, we have,

$$\begin{aligned} \sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left(\sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{d}} \right) &= \sum_{e \leq x} \frac{1}{\sqrt{e}} \left(o\left(\frac{1}{x^{\frac{1}{r}}}\right) \right), \\ &= \left(o\left(\frac{1}{x^{\frac{1}{r}}}\right) \right) \sum_{e \leq x} \frac{1}{\sqrt{e}}, \\ \sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} &<< \int_1^{\sqrt{x}} \frac{1}{\sqrt{t}} dt = x^{\frac{1}{4}}. \end{aligned}$$

Thus we conclude,

$$\sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left(\sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{d}} \right) = o(1).$$

We also have,

$$\begin{aligned}
\sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(\sum_{e \leq \frac{x}{d}} \frac{1}{\sqrt{e}} \right) &= \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(2\sqrt{\frac{x}{d}} + B + o\left(\sqrt{\frac{d}{x}}\right) \right), \\
&= 2\sqrt{x} \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{d} + B \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} + o\left(\frac{1}{\sqrt{x}} \sum_{d \leq \sqrt{x}} \chi(d)\right). \\
2\sqrt{x} \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{d} &= 2\sqrt{x} \left(\sum_{d \geq 1} \frac{\chi(d)}{d} - \sum_{d > \sqrt{x}} \frac{\chi(d)}{d} \right), \\
&= 2\sqrt{x}L(1, \chi) - 2\sqrt{x} \sum_{d > \sqrt{x}} \frac{\chi(d)}{d}, \\
&= 2\sqrt{x}L(1, \chi) - 2\sqrt{x}o\left(\frac{1}{\sqrt{x}}\right), \\
&= 2\sqrt{x}L(1, \chi) + o(1).B \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \leq o\left(\frac{B}{x^{\frac{1}{4}}}\right) = o(1). \\
\sum_{d \leq \sqrt{x}} \chi(d) &\leq q? = o(1).
\end{aligned}$$

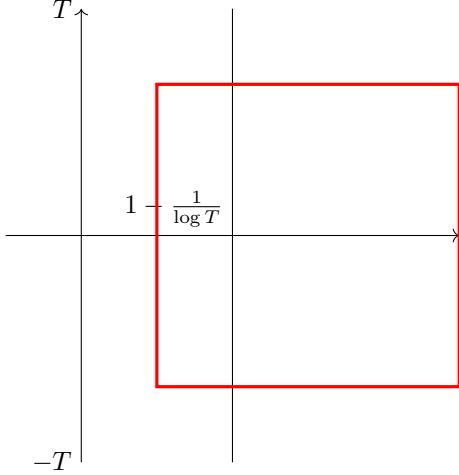
□

Corollary 2.5. $L(1, \chi) \neq 0$ for a quadratic character χ .

Proof. To derive a contradiction, assume $L(1, \chi) = 0$. Then, from the theorem

$$o(1) = \sum_{n \leq x} \frac{f(n)}{\sqrt{n}} \geq \sum_{\substack{n \leq x \\ n \text{ is a square}}} \frac{1}{\sqrt{n}} = \sum_{m \leq \sqrt{x}} \frac{1}{m} = \log \sqrt{x} + o(1).$$

Thus we obtain $o(1) \geq \log \sqrt{x} + o(1)$ which is a contradiction. □



Lemma 2.9. $|\zeta(s)| = o(\log T)$, $T \rightarrow \infty$. That is there are $K > 0$ and T_0 such that for any $T \geq T_0 \geq 1$, we have,

$$|\zeta(s)| \leq K \log T.$$

Proof. The first statement is due to the first derivative of ζ and it is assigned as an exercise. For the second part, for $\operatorname{Re}(s), \sigma > 1$, we have,

$$\begin{aligned} \zeta(s) &= \sum_{n \geq 1} \frac{1}{n^s}, \\ &= \sum_{n \leq T} \frac{1}{n^s} + \sum_{n > T} \frac{1}{n^s}, \\ &= \sum_{n \leq T} \frac{1}{n^s} - \frac{\lfloor T \rfloor}{T^s} + s \int_1^\infty \frac{\lfloor t \rfloor}{t^{s+1}} dt, \\ &= \sum_{n \leq T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} + \frac{\{T\}}{T^s} - s \int_T^\infty \frac{\{u\}}{u^{s+1}} du. \end{aligned}$$

Note that the right hand side is analytic where $\operatorname{Re}(s) > 0$ and $s \neq 1$. Now we will estimate the last equation above on the boundary of the red rectangle above. Let $\sigma_0 = 1 - \frac{1}{\log T}$.

$$\begin{aligned} \left| \sum_{n \leq T} \frac{1}{n^s} \right| &<< \int_1^T \frac{du}{u^{\operatorname{Re}(s)}}, \\ &\leq \int_1^T \frac{du}{u^{\sigma_0}}, \\ &= \frac{T^{1-\sigma_0}}{1-\sigma_0}, \\ &<< \log T. \end{aligned}$$

Observe that ,

$$\begin{aligned}
1 - \sigma_0 &= \frac{1}{\log T}, \\
T^{1-\sigma_0} &= T^{\frac{1}{\log T}}, \\
&= e. \\
y &= T^{\frac{1}{\log T}}, \log y &= \log T \frac{1}{\log T} = 1. \\
\Rightarrow y &= e.
\end{aligned}$$

We also have,

$$\begin{aligned}
\left| \frac{T^{1-s}}{s-1} \right| &= \frac{T^{1-\operatorname{Re}(s)}}{|s-1|}, \\
&\leq \frac{T^{1-\sigma_0}}{|s-1|}, \\
&\leq \frac{1}{|s-1|}, \\
&\leq \frac{1}{\sigma_0 - 1} = \log T.
\end{aligned}$$

Also consider,

$$\left| \frac{\{T\}}{T} \right| \leq \frac{1}{T^{\operatorname{Re}(s)}} \leq 1.$$

Finally we have,

$$\begin{aligned}
&\leq |s| \int_T^\infty \frac{du}{u^{\operatorname{Re}(s)+1}}, \\
&= \frac{|s|}{-\operatorname{Re}(s) u^{\operatorname{Re}(s)}} \Big|_T^\infty, \\
&= \frac{|s|}{\operatorname{Re}(s) T^{\operatorname{Re}(s)}}, \\
&\leq \frac{|s|}{T^{\operatorname{Re}(s)}}, \\
&\leq \frac{\sqrt{2^2 + T^2}}{T^{\sigma_0}}, \\
&<< \frac{T}{T^{\sigma_0}} = T^{1-\sigma_0} = e.
\end{aligned}$$

It is an exercise to check,

$$|\zeta'(s)| \leq o((\log T)^2),$$

$$\zeta'(s) = \sum_{n \leq T} \frac{-\log n}{n^s} + \frac{T^{1-s}(-\log T)}{s-1} + T^{1-s} \frac{-1}{(s-1)^2} +$$

The rest is missing, check the lecture note. \square

Theorem 2.24. Let $f : n \rightarrow \mathbb{C}$ be an analytic, Λ be a convex open set. Let $a, b \in \Lambda$, then there exists $z_1, z_2 \in (a, b)$ such that

$$\operatorname{Re}(f'(z_1)) = \operatorname{Re}\left(\frac{f(b) - f(a)}{b - a}\right), \operatorname{Im}(f'(z_2)) = \operatorname{Im}\left(\frac{f(b) - f(a)}{b - a}\right).$$

Theorem 2.25. There exists constants c_1, c_2 such that

$$1 - \frac{c_1}{(\log T)^9} \leq \sigma \leq 2, |\zeta(s)| > \frac{c_2}{(\log T)^7},$$

where $1 < |\operatorname{Im}(s)| \leq T$.

Proof. Recall that

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1, \sigma > 1.$$

Thus we have,

$$|\zeta(\sigma + it)|^4 \geq |\zeta(\sigma)|^3 |\zeta(\sigma + 2it)|^{-1},$$

c_1 is some (missing) constant which will be chosen later.

Fix the domain II where $1 + \frac{c_1}{(\log T)^9} \leq \sigma \leq 2, 1 \leq |t| \leq T$.

$$\begin{aligned} \zeta(s) &= \frac{s}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{s+1}} du, \operatorname{Re}(s) > 0, s \neq 1. \zeta(\sigma) &= \frac{\sigma-1+1}{\sigma-1} - \sigma \int_1^\infty \frac{\{u\}}{u^{s+1}} dt. \\ \zeta(\sigma) &= 1 + \frac{1}{\sigma-1} + o(1), \\ \zeta(\sigma) &<< \frac{1}{\sigma-1}, \sigma \rightarrow 1^+, \\ \Rightarrow \zeta(\sigma)^{-1} &>> (\sigma-1), \sigma \rightarrow 1^+ \end{aligned}$$

We also have,

$$|\zeta(\sigma + 2it)| = o(\log T), 1 - \frac{1}{\log T}$$

□

Corollary 2.6. There exists some constants C such that

$$\frac{\zeta'(s)}{\zeta(s)} = o((\log T)^9).$$

For $1 - \frac{c}{(\log T)^9} \leq \operatorname{Re}(s) \leq 2$ and $1 \leq |\operatorname{Im}(s)| \leq T$, we have,

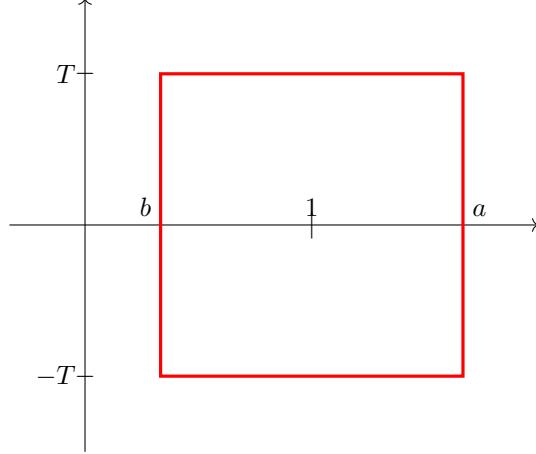
$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}.$$

Proof. Recall Cauchy's Residue theorem, suppose $f : \Omega \rightarrow \mathbb{C}$ is meromorphic where Ω is simply connected.

Example 2.11. Halfplane \mathbb{R}^n and $\mathbb{R}^n \setminus \{0\}$, for $n \geq 3$ and convex.

For $a_i, 1 \leq i \leq n$ poles in $U \subseteq \Omega$ simple closed curve. Then,

$$\frac{1}{2\pi i} \int_U f(s) ds = \sum_{i=1}^n \text{Res}(f, a_i) + \text{constant}.$$



where $a = \frac{1}{+\frac{c}{(\log(T))^9}}$, $b = \frac{1}{-\frac{c}{(\log(T))^9}}$. $\text{Re}(s) > 0, s \neq 1, \text{Re}(s) > b$. We claim that $\frac{\zeta'(s)}{\zeta(s)}$ can be analytically continued to $\text{Re}(s) \geq b$ except simple pole at $s = 1$ with residue -1 (See lecture 5 for the justification of such poles).

$$\begin{aligned} \frac{1}{2\pi i} \int_{R_T} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds &= \text{Res}_{s=1} \left(\frac{-\zeta'(s)}{\zeta(s)} \right), \\ &= x. \end{aligned}$$

Now consider R_T which is the closed path drew red in the graph.

$$\int_{R_T} = \int_{a-iT}^{a+iT} + \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT}.$$

We then have,

$$\frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = x - \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT}$$

$$\begin{aligned}
\left| \frac{1}{2\pi i} \int_{a+iT}^{b+iT} \frac{-\zeta(s)}{\zeta(s)} \frac{x^s}{s} ds \right| &= \left| \frac{1}{2\pi i} \int_a^b \frac{-\zeta(u+iT)}{\zeta(u+iT)} \frac{x^{(u+iT)}}{(u+iT)} du \right|, \\
&<< \int_a^b \left| \frac{-\zeta'(u+iT)}{\zeta(u+iT)} \right| \frac{x^u}{|u+iT|} du, \\
&\stackrel{\text{Using the previous theorem}}{<<} \frac{\log^9 T}{T} x^a \int_b^a du, \\
&<< \frac{x^a}{T}.
\end{aligned}$$

Now we have,

$$\begin{aligned}
\left| \int_{b-iT}^{b+iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \right| &= \left| \frac{1}{2\pi} \int_{-T}^T \frac{\zeta'(b+iu)}{\zeta(b+iu)} \frac{x^{b+iu}}{b+iu} du \right|, \\
&<< \log^9 T \int_{-T}^T \frac{x^b}{\sqrt{b^2+u^2}} du, \\
&<< x^b \log^9 T \int_0^T \frac{du}{\sqrt{b^2+u^2}} du, \\
&<< x^b \log^9 T \int_1^T \frac{du}{u}, \\
&= x^b \log^{10} T.
\end{aligned}$$

Now back to the beginning,

$$\psi(x) = x + o\left(\frac{x^a}{T} + x^b \log^{10} T\right) + o\left(\frac{x \log^2 x}{T} + \frac{x^a}{T} \log^9 T\right).$$

Choose T to be such that $2c \log x = \log^{10} T$, $x = e^{\frac{\log^{10} T}{x}}$.

$$\begin{aligned}
x^{\frac{c}{\log^9 T}} &= e^{\frac{\log T}{2}} = \sqrt{T}. \\
x^{1-\frac{c}{\log^9 T}} \cdot \log^{10} T + \frac{x \log^2 x}{T} + \frac{x^{1+\frac{c}{\log^9 T}}}{T} \log^9 T &= x \cdot T^{-\frac{1}{2}} \log^1 0T + \frac{x \log^2 x}{T} + x \frac{\sqrt{T}}{T} \log^{10} T \left(\frac{x}{\sqrt{T}} \right) (\log^{10} T + \\
&<< \frac{x}{T^s} = x e^{-c(\log x)^{\frac{1}{10}}} \cdot
\end{aligned}$$

□

Definition 2.14 (Gamma function). *We define the Gamma function $\Gamma : \mathbb{C} \rightarrow \mathbb{C}$ as*

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt, \sigma > 0.$$

Remark 2.18.

$$\begin{aligned}
|\Gamma(s)| &\leq \int_0^\infty |t^{s-1}|e^{-t}dt, \\
&= \int_0^\infty t^{\sigma-1}e^{-t}dt, \\
&= \left(\int_0^1 + \int_1^\infty \right) t^{\sigma-1}e^{-t}dt, \\
&\leq \int_0^1 t^{\sigma-1}dt + \int_1^\infty t^{\sigma-1}e^{-t}dt, \left(\frac{t^n}{e^t} \xrightarrow{t \rightarrow \infty} 0 \right). \\
&<< \frac{1}{\sigma} + \int_1^\infty t^{\sigma-1}t^{-2\sigma}dt, \\
&<< \frac{1}{\sigma}.
\end{aligned}$$

Theorem 2.26.

$$F(s) = \int f(s, t)dt.$$

$F : \Omega \rightarrow \mathbb{C}$ is analytic in Ω if

- i). $f(s, t)$ is continuous in (s, t) ,
- ii). $f(s, t)$ is analytic in s ,
- iii). $\int f(s, t) dt$ is uniformly bounded on compact subsets of Ω .

In Remark 2.18, suppose $a \leq \operatorname{Re}(s) \leq b$ then the last two inequalities will be,

$$\begin{aligned}
&\int_0^1 t^{\sigma-1}dt + \int_1^\infty t^{\sigma-1}e^{-t}dt \\
&<<_b \frac{1}{\sigma} + \int_1^\infty t^{b-1}t^{-2b}dt, \\
&<<_b \frac{1}{\sigma} + \frac{1}{b} = \frac{1}{a} + \frac{1}{b}.
\end{aligned}$$

Thus we observe that in $\sigma > 0$, using integration by parts,

$$\Gamma(s+1) = s\Gamma(s).$$

Thus for any $n \in \mathbb{N}$, we have $\Gamma(n) = n!$. We have,

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \sigma > 0.$$

Note $\Gamma(s+1)$ is analytic if $\sigma > -1$. Thus $\Gamma(s)$ is analytic in $\sigma > -1$ except simple pole at $s = 0$ with residue 1. That is

$$\lim_{s \rightarrow 0} s\Gamma(s) = 1.$$

We also have,

$$\Gamma(s) = \frac{\Gamma(s+2)}{\Gamma(s+1)}, \sigma > 0.$$

Note that in the numerator, we have $\sigma > -2$. $\Gamma(s)$ is analytic when $\sigma > -2$ except simple poles at $s = 0, -1$ with residue

$$\lim_{s \rightarrow -1} (s+1)\Gamma(s) = \frac{\Gamma(-1)}{-1} = -1.$$

Iterate the process, we have the following theorem,

Theorem 2.27. $\Gamma(s)$ can be analytically continued to \mathbb{C} except simple poles at $s = -k$ where $k \in \mathbb{Z}_{\geq 0}$ with residue $\frac{(-1)^k}{k!}$.

Remark 2.19.

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, s \in \mathbb{C}.$$

$$\lim_{s \rightarrow m} \Gamma(s)\Gamma(1-s) \sin \pi s = \pi s \cdot \Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = \sqrt{\pi} 2^{1-2s} \Gamma(2s), \forall s \in \mathbb{C}.$$

Exercise 2.5. Show

$$\sum_{n \in \mathbb{Z}} e^{-(n+\alpha)\frac{\pi}{x}} = \sqrt{x} \sum_{n \in \mathbb{Z}} e^{-n^2 \pi x + 2\pi i n \alpha}, \forall \alpha \in i\mathbb{R}, x > 0.$$

Note that $\sum_{n \in \mathbb{Z}} a_n$ is convergent if

$$s_N := \sum_{|n| \leq N} a_n,$$

is convergent.

Theorem 2.28. $\Gamma(s) \neq 0, \forall s \in \mathbb{C}$.

Proof. Note that the case when $s \in \mathbb{Z}$ is already shown. Thus suppose $s \notin \mathbb{Z}$. If possible $\Gamma(s) = 0$. Then it will follow that $\Gamma(1-s)$ has a pole which is a contradiction. $\frac{1}{\Gamma(s)}$ has a simple zero at $s = 0, -1, -2, \dots$.

Step 1

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t} t^{\frac{s}{2}-1} dt, \sigma > 0.$$

Replace $t = n^2\pi x$, $dt = n^2\pi dx$, we get,

$$\begin{aligned}
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} &= \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx, \sigma > 1. \\
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \sum_{n \in \mathbb{N}} \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx, \\
&= \int_0^\infty x^{\frac{s}{2}-1} \left(\sum_{n \in \mathbb{N}} e^{-n^2\pi x} \right) dx, \\
&= \sum_{n \in \mathbb{N}} \int_0^\infty |x^{\frac{s}{2}-1} e^{-n^2\pi x}| dx, \\
&= \sum_{n \in \mathbb{N}} \left(\int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx \right), \\
&= \sum_{n=1}^\infty \pi^{-\frac{\sigma}{2}} \Gamma\left(\frac{\sigma}{2}\right) n^{-\sigma}, \\
&= \pi^{-\frac{\sigma}{2}} \Gamma\left(\frac{\sigma}{2}\right) \zeta(s), \sigma > 1, \\
&< \infty. \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1} \left(\sum_{n \in \mathbb{N}} e^{-n^2\pi x} \right).
\end{aligned}$$

Step 2, using Poisson summation formula and $\sigma > 1$, let $F \in L^1(\mathbb{R})$, ie $F : \mathbb{R} \rightarrow \mathbb{C}$ and

$$\int_{-\infty}^\infty |F(t)| dt < \infty.$$

We have,

$$\sum_{n \in \mathbb{Z}} F(n+u)$$

is absolutely and uniformly convergent in u . Also we have,

$$\sum_{n \in \mathbb{Z}} |\hat{F}(n)| \leq \infty,$$

then

$$\sum_{n \in \mathbb{Z}} F(n+u) = \sum_{n \in \mathbb{Z}} \hat{F}(n) e^{2\pi i n u}.$$

Using Exercise 2.5 and set $\alpha = 0$, we have,

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} e^{-n^2 \frac{\pi}{2}} &= \sqrt{s} \sum_{n \in \mathbb{Z}} e^{-n^2 \pi x} \cdot \sum_{n \in \mathbb{Z}} e^{-n^2 \pi x} && \ll \sum_{n \in \mathbb{N}} e^{-n^2 \pi x}, \\
&\ll \int_1^\infty e^{-\pi x t} dt, \\
&\ll \frac{e^{-\pi x}}{x}, x > 0.
\end{aligned}$$

Set $\theta := \sum_{n \in \mathbb{Z}} e^{n^2 \pi x}$ then,

$$\begin{aligned}\theta(x) &= 1 + 2 \sum_{n \in \mathbb{N}} e^{-n^2 \pi x}. \\ \sum_{n \in \mathbb{N}} e^{-n^2 \pi x} &= \frac{\theta(x) - 1}{2}, x > 0.\end{aligned}$$

Using the exercise, we have,

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\theta(x).$$

Set $w(x) := \frac{\theta(x)-1}{2}$. Thus write

$$\begin{aligned}w\left(\frac{1}{x}\right) &= \frac{\theta\left(\frac{1}{x}\right) - 1}{2}, \\ &= \frac{\sqrt{x}\theta(x) - 1}{2}, \\ &= \frac{\sqrt{x}\theta(x) - 1}{2}. \\ w\left(\frac{1}{x}\right) &= \sqrt{x}w(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}. \\ \pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1}w(x)dx, \sigma > 1.\end{aligned}$$

Step 3

$$\int_0^1 x^{\frac{s}{2}-1}w(x)dx + \int_1^\infty x^{\frac{s}{2}-1}w(x)dx.$$

Taking $x = \frac{1}{y}$ and $dx = -\frac{1}{y^2}dy$ we have,

$$\begin{aligned}\int_1^\infty \left(\frac{1}{y^2}\right)^{\frac{s}{2}-1} w\left(\frac{1}{y}\right) \frac{-1}{y^2} dy &= \int_1^\infty y^{-\frac{s}{2}} w\left(\frac{1}{y}\right) \frac{dy}{y}, \\ &= \int_1^\infty y^{-\frac{1}{2}} \left(\sqrt{y}w(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right) \frac{dy}{y}, \\ &= \int_1^\infty y^{-\frac{s}{2}+\frac{1}{2}} w(y) \frac{dy}{y} + \int_1^\infty \frac{y^{-\frac{s}{2}+\frac{1}{2}-1}}{2} dy - \frac{1}{2} \int_1^\infty y^{-\frac{s}{2}-1} dy, \\ &= \int_1^\infty y^{\frac{1-s}{2}} w(y) \frac{dy}{y} + \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x) \frac{dx}{x}, \sigma > 1.\end{aligned}$$

Step 4,

$$\begin{aligned}
\int_1^\infty |x^{\frac{s}{2}} + x^{\frac{1-s}{2}}| |w(x)| \frac{dx}{x} &\leq \int_1^\infty (x^{\frac{\sigma}{2}-1} + x^{\frac{1-\sigma}{2}-1}) |w(x)| dx, \\
&<< \int_1^\infty \frac{(x^{\frac{\sigma}{2}-1} + x^{\frac{1-\sigma}{2}-1})}{e^{\pi x}} dx, \quad \sigma \in \mathbb{R}, \\
&<< \int_1^\infty \frac{e^x}{e^{\pi x}} dx, \quad << 1.
\end{aligned}$$

$\int_1^\infty |x^{\frac{s}{2}} + x^{\frac{1-s}{2}}| |w(x)| \frac{dx}{x}$ is analytic in \mathbb{C} .

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = 1 + s(s-1) \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}}) w(x) \frac{dx}{x}.$$

Set $\xi(x) := 1 + s(s-1) \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}}) w(x) \frac{dx}{x}$, then it is entire. and for all s we have,

$$\xi(1-s) = \xi(s).$$

Thus obtain,

$$(1-s)(1-s-1)\pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Using the construction of $w(s)$ we have,

$$|w(x)| << |\theta(x)| << \frac{e^{-\pi x}}{x}, \forall x > 0.$$

Also we have,

$$\begin{aligned}
\zeta(1-s) &= \pi^{-s+\frac{1}{2}} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \zeta(s), \\
\zeta(1-s) &= \pi^{-s} 2^{1-s} \cos\left(\frac{\pi s}{s}\right) \Gamma(s) \zeta(s).
\end{aligned}$$

Note that $\zeta(s)$ has an analytic continuation to \mathbb{C} except simple pole at s .

$$\lim_{s \rightarrow 1} \zeta(1-s) = \pi^{-1} \lim_{s \rightarrow 1} \frac{\cos \frac{\pi s}{2} \zeta(s)(s-1)}{s-1},$$

$$\zeta(-2n) =$$

Recall

$$\begin{aligned}
\zeta(s) &= \sum_{n \in \mathbb{N}} \frac{1}{n^s}, \\
&= \frac{s}{s-1} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt, \quad \sigma > 0, s \neq 1.
\end{aligned}$$

If s is real,

$$|\zeta(s) - \frac{s}{s-1}| \leq |s| \int_1^\infty \frac{\{t\}}{t^{\sigma+1}} dt, \quad < \frac{|s|}{\sigma} = \frac{\sigma}{\sigma} = 1.$$

Thus we obtain,

$$\begin{aligned} -1 + \frac{s}{s-1} &< \zeta(s) < 1 + \frac{s}{s-1} \\ \frac{1}{s-1} &< \zeta(s) < \frac{2s-1}{s-1}, \\ -1 < (1-s)\zeta(s) &< 1-2s < 0 \quad \text{if } \frac{1}{2} < s < 1 \\ \Rightarrow \zeta(s) &\neq 0, \quad \text{if } \frac{1}{2} < s < 1. \end{aligned}$$

□

Notation 2.3. Given $\chi \pmod{q}$, we set,

$$\tau(\chi) = \sum_{k=1}^q \chi(k) e^{\frac{2\pi i k}{q}}.$$

2.4 Primitive characters

Definition 2.15. A character χ is called primitive if its conductor is

Example 2.12. Take $\chi : (\mathbb{Z}/8\mathbb{Z})^\times \rightarrow \mathbb{C}^t$ times.

Lemma 2.10. Suppose $x \in [0, \frac{1}{2}]$, then

$$|\sin \pi x| \geq 2x.$$

Proof. Exercise. □

Lemma 2.11. For $n \in \mathbb{Z}$, we have,

$$\chi(n)\tau(\bar{\chi}) = \sum_{k=1}^q \bar{\chi}(k) e^{\frac{2\pi i k}{q}},$$

where χ is a primitive character modulo q and $\chi \neq \chi_0$.

Proof.

$$\overline{\chi(n)\tau(\bar{\chi})} = \sum_{l=1}^q \chi(l) e^{-\frac{2\pi i l}{q}}.$$

Multiplying this equation with the one in the statement we have,

$$|\chi(n)|^2 |\tau(\bar{\chi})|^2 = \sum_{k,l=1}^q \overline{\chi(k)} \chi(l) e^{\frac{2\pi i (k-l)}{q}}.$$

Applying $\sum_{n \leq x}$ to the both sides of the above equation, we have,

$$|\tau(\bar{\chi})|^2 \sum_{n \leq x} |\chi(n)|^2 = \sum_{k,l=1}^q \overline{\chi(k)} \chi(l) \left(\sum_{n \leq x} \left(e^{\frac{2\pi i (k-l)}{q}} \right)^n \right).$$

Note that

$$\begin{aligned} x + x^2 + \cdots + x^q &= \begin{cases} \frac{x(x^q - 1)}{x - 1}, & (x \neq 1), \\ q, & (x = 1), \end{cases} \\ &= \begin{cases} 0, & (x \neq 1, x^q = 1), \\ q, & (x = 1), \end{cases} \end{aligned}$$

Take $x = e^{\frac{2\pi i (k-l)}{q}} = \cos \frac{2\pi}{q}(k-l) + i \sin \frac{2\pi}{q}(k-l) = 1$. $x = 1$ if and only if $q|k-l$ thus

$$\sum_{n \leq x} \left(e^{\frac{2\pi i (k-l)}{q}} \right)^n = \begin{cases} 0, & (q \nmid k-l), \\ q, & (\text{otherwise}). \end{cases}$$

Therefore, we get,

$$\begin{aligned} |\tau(\bar{\chi})|^2 \sum_{n \leq x} |\chi(n)|^2 &= q \sum_{\substack{k,l=1 \\ q|k-l}}^q \overline{\chi(k)} \chi(l), \\ &= q \sum_{k=1}^q \overline{\chi(k)} \chi(k), \\ &= q \sum_{k=1}^q |\chi(k)|^2. \end{aligned}$$

Therefore $|\tau(\bar{\chi})|^2 = q$, $|\tau| = \sqrt{q}$.

Consider $(n, q) = 1$, then

$$\begin{aligned} \chi(n) \tau(\bar{\chi}) &= \chi(n) \sum_{k=1}^q \overline{\chi(k)} e^{\frac{2\pi i k}{q}}, \\ &= \chi(n) \sum_{\substack{k=1 \\ (k,q)=1}}^q \overline{\chi(k)} \chi(k) e^{\frac{2\pi i k}{q}}, \end{aligned}$$

Set $k = nt$, we get,

$$\begin{aligned}\chi(n)\tau(\bar{\chi}) &= \sum_{\substack{t=1 \\ (t,q)=1}}^q \bar{\chi}(nt)\chi(n)e^{\frac{2\pi i nt}{q}}. \\ \bar{\chi}(nt) &= \bar{\chi}(n)\bar{\chi}(t). \\ \chi(n)\tau(\bar{\chi}) &= \sum_{\substack{t=1 \\ (t,q)=1}}^q \bar{\chi}(t)e^{\frac{2\pi i nt}{q}}.\end{aligned}$$

Observe that

$$\begin{aligned}\tau(\bar{\chi}) \left(\sum_{n \leq x} \chi(n) \right) &= \sum_{k=1}^{q-1} \bar{\chi}(k) \left(\sum_{n \leq x} e^{\frac{2\pi i kn}{q}} \right), \\ |\tau(\bar{\chi})| \cdot \left| \sum_{n \leq x} \chi(n) \right| &\leq \sum_{k=1}^{q-1} \left| \sum_{n \leq x} e^{\frac{2\pi i kn}{q}} \right|, \\ &= \sum_{k=1}^{q-1} \left| \frac{e^{\frac{2\pi i k}{q}} (e^{\frac{2\pi i n[x]}{q}} - 1)}{e^{\frac{2\pi i k}{q}} - 1} \right|, \\ &\leq \sum_{k=1}^{q-1} \frac{2}{|e^{\frac{2\pi i k}{q}} - 1|},\end{aligned}$$

Note that for all $y \in i\mathbb{R}$,

$$\begin{aligned}2i \sin y &= e^{-iy}(e^{2iy} - 1), \\ |2 \sin y| &= |e^{2iy} - 1|.\end{aligned}$$

Apply this to the equation above we have,

$$\sum_{k=1}^{q-1} \frac{2}{|e^{\frac{2\pi i k}{q}} - 1|} = \sum_{k=1}^{q-1} \frac{1}{|\sin \frac{\pi k}{q}|}.$$

Using the lemma, we get,

$$\begin{aligned}\sum_{k=1}^{q-1} \frac{1}{|\sin \frac{\pi k}{q}|} &= \sum_{1 \leq k \leq \frac{q}{2}} \frac{1}{|\sin \frac{\pi k}{q}|} + \sum_{\frac{q}{2} < k} \frac{1}{|\sin \frac{\pi k}{q}|}. &= 2 \sum_{1 \leq k \leq \frac{q}{2}} \frac{1}{|\sin \frac{\pi k}{q}|}, \\ &\leq \sum_{1 \leq k \leq \frac{q}{2}} \frac{k}{q} << q \log \frac{q}{2} << q \log q.\end{aligned}$$

Thus we conclude $|(\bar{\chi})| << \sqrt{q}$ if χ is primitive and $q > 1$, we have,

$$\left| \sum_{n \leq x} \chi(n) \right| << \sqrt{q} \log q,$$

uniformly in q as $x \rightarrow \infty$. \square

$L(q, \chi)$ functional equation, where $\chi \neq \chi_0$ and χ is primitive. Suppose $\chi(-1) = 1$ that is it is an even character. From previous discussion, we have,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty x^{\frac{s}{2}} e^{-n^2 \pi x} \frac{dx}{x}.$$

Replace x with $\frac{x}{q}$ and $\sigma > 0$, we have,

$$\pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty x^{\frac{s}{2}} e^{-n^2 \pi x} \frac{dx}{x}, \sigma > 0.$$

For $\sigma > 1$, we have,

$$\begin{aligned} \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi) &= \sum_{n=1}^\infty \int_0^\infty \chi(n) x^{\frac{s}{2}} e^{-n^2 \frac{\pi x}{q}} \frac{dx}{x}, \\ &= \int_0^\infty x^{\frac{s}{2}} \left(\sum_{n=1}^\infty \chi(n) e^{-n^2 \frac{\pi x}{q}} \right) \frac{dx}{x}, \sigma > 1. \end{aligned}$$

Let

$$\theta(x, \chi) = \sum_{n \in \mathbb{Z}} \chi(n) e^{-\frac{n^2 \pi x}{q}} = 2 \sum_{n=1}^\infty \chi(n) e^{-\frac{n^2 \pi x}{q}}, (x > 0).$$

Using this, we get,

$$\int_0^\infty x^{\frac{s}{2}} e^{-n^2 \pi x} \frac{dx}{x} = \frac{1}{2} \int_0^\infty x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x}, (\sigma > 1).$$

Split the integral into \int_0^1, \int_1^∞ .

$$\begin{aligned}
\tau(\bar{\chi})\theta(x, \chi) &= \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \bar{\chi}), \\
&= \sum_{n \in \mathbb{Z}} (\tau(\bar{\chi})\chi(n)) e^{\frac{-n^2 \pi x}{q}}, \\
&= \sum_{n \in \mathbb{Z}} \sum_{k=1}^q \bar{\chi}(k) e^{\frac{2\pi i n k}{q}} e^{-\frac{n^2 \pi x}{q}}, \\
&= \sum_{k=1}^q \overline{\chi(k)} \sum_{n \in \mathbb{Z}} e^{\frac{2\pi i k n}{q} - n^2 \frac{\pi x}{q}}, \\
&\stackrel{\text{From Lecture 7 page 6}}{=} \sum_{k=1}^q \bar{\chi}(k) \left(\frac{x}{q}\right)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} e^{-(m + \frac{m}{q}) \frac{\pi q}{x}}, \\
&= \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^q \bar{\chi}(k) \sum_{n \in \mathbb{Z}} e^{-(m + \frac{m}{q}) \frac{\pi q}{x}}.
\end{aligned}$$

Put $qn + m = t$, then

$$\bar{\chi}(qn + m) = \bar{\chi}(m) = \bar{\chi}(t).$$

Thus,

$$\left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^q \bar{\chi}(k) \sum_{n \in \mathbb{Z}} e^{-(m + \frac{m}{q}) \frac{\pi q}{x}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{t \in \mathbb{Z}} \bar{\chi}(t) e^{-\frac{t^2 \pi}{qx}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \bar{\chi}).$$

Now for the splitted integral, we have,

$$\frac{1}{2} \int_0^1 x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x} + \frac{1}{2} \int_1^\infty \int_0^1 x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x} \frac{dx}{x}.$$

Replacing x with $\frac{1}{x}$, we get,

$$= \frac{\tau(\chi)}{2\sqrt{q}} \int_1^\infty x^{\frac{1-s}{2}} \theta(x, \bar{\chi}) \frac{dx}{x} + \frac{1}{2} \int_1^\infty \int_0^1 x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x}.$$

Set

$$\xi(s, \chi) := \frac{\tau(\chi)}{2\sqrt{q}} \int_1^\infty x^{\frac{1-s}{2}} \theta(x, \bar{\chi}) \frac{dx}{x} + \frac{1}{2} \int_1^\infty \int_0^1 x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x}.$$

Use that $\theta(x, \bar{\chi}) \ll e^{-\frac{\pi x}{q}}$ and $|\tau(\chi)| = \sqrt{q}$. It turns out that ξ is uniformly bounded on compact subsets of \mathbb{C} and in particular, this is entire.

Lemma 2.12.

$$\xi(1-s, \chi) = \frac{\tau(\chi)}{\sqrt{q}} \xi(s, \bar{\chi}).$$

Proof. Recall,

$$\tau(\bar{\chi})\theta(x, \chi) = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \bar{\chi}),$$

and

$$|\tau(\chi)|^2 = q.$$

We get,

$$\frac{\tau(\chi)}{\sqrt{q}} = \frac{\sqrt{q}}{\tau(\chi)} = \frac{\sqrt{q}}{\tau(\bar{\chi})}.$$

Then,

$$L(1-s, \chi) = \frac{\tau(\chi)}{q^{1-s}} \pi^{-s} 2^{1-s} \cos \frac{\pi s}{2} (\Gamma(s)) L(s, \bar{\chi}).$$

Note that $\chi(-1) = 1$ and is primitive (in the case of odd integer replace cos with sin). Note also that

- i). $L(s, \chi) = 0, s = 0, -2, -4, \dots$ when χ is even,
- ii). $L(s, \chi) = -1, -3, -5, \dots$ when χ is odd.

When $0 < \sigma < 1$,

□