V5A10 Analytic Number Theory

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1 Classical Number Theory

Theorem 1.1 (Euclid)**.** *There are infinitely many prime numbers.*

Definition 1.1. $\pi : \mathbb{N} \to \mathbb{N}$ *be a function such that*

$$\pi(n) = \{\textit{prime numbers less than } n\}.$$

Remark 1.1.

$$\frac{\pi}{n \ln(n)} \approx 1.$$

Definition 1.2.

$$\mathrm{Li}(x) = \int_0^x \frac{1}{\ln(t)} dt.$$

Notation 1.1. *Given* $f, g : \mathbb{R} \to \mathbb{C}$,

$$f(x) = O(g(x0))$$

means that

$$\exists K \in (0, \infty), x_0 \in \mathbb{R}, \text{ s. t. } \forall x > x_0, |f(x)| \leq K|f(x)|.$$

Notation 1.2. *Let* $f, g : \mathbb{R} \to \mathbb{C}$ *be functions.* $f \sim g$ *denotes that*

$$\lim_{n\to\infty} \frac{f(x)}{g(x)}.$$

Notation 1.3. *Let* $f : \mathbb{R} \to \mathbb{C}$,

$$\mathrm{Li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k},$$

denotes that

$$\mathrm{Li}(x) = \frac{x}{\ln x} \sum_{k=1}^{N-1} \frac{k!}{(\ln x)^k} + O(\frac{x}{(\ln x)^{N+1}}).$$

and as $x \to \infty$*, this holds for any* $N \geq 1$*.*

Remark 1.2. *By the integration by parts, we see that it's asymptotic expansion is*

$$\mathrm{Li}(x) \approx \frac{x}{\ln(x)} \sum_{k=0}^{\infty} \frac{k!}{(ln(x))^k}.$$

Theorem 1.2 (Prime Number Theorem)**.**

$$\lim_{x\to\infty} \frac{\pi(x)}{\mathrm{Li}(x)} = 1.$$

Definition 1.3 (First Chebyschev Function)**.**

$$\vartheta(x) = \sum_{p\leq x} \ln p.$$

Definition 1.4 (Second Chebyschev Function)**.**

$$\psi(x) = \sum_{\substack{m,p \\ p^m\leq x}} \ln p.$$

Definition 1.5 (Möbius Function)**.** *Let $n \in \mathbb{N}$, we define,*

$$\mu(n) = \begin{cases} 1, & (n = 1), \\ (-1)^m, & (n \textit{ is square free and has } n \textit{ distinct prime divisors}), \\ 0, & (\text{otherwise}). \end{cases}$$

Definition 1.6 (Möbius Function)**.**

$$\mu(n) = \begin{cases} 1 & (n = 1) \\ (-1)^k & (n = p_1 \cdots p_k, p_i = p_j \Rightarrow i = j) \\ 0 & (\exists p \text{ s.t. } p^2 | n). \end{cases}$$

Remark 1.3. *The prime number theorem is equivalent to the following statements.*

1). $\psi(x) \sim x$.

2). $\theta(x) \sim x$.

3). $\lim_{x\to\infty} \frac{\sum_{n\leq x}}{\frac{\mu(n)}{n}} x = 0$.

Conjecture 1.1 (Twin Prime Conjecture)**.** *There exists infinitely many primes p such that $p + 2$ is also prime.*

Conjecture 1.2 (Goldbach's Conjecture)**.** *Let $n \in \mathbb{N}$ be an even number greater than 2, then there exists two primes p, q such that $n = p + q$.*

Conjecture 1.3 (Hardy-Littlewood Conjecture)**.**

$$\#\{ \textit{ prime numbers } p \textit{ such that } 2p+1 \textit{ is also a prime and } p < x\}$$

Definition 1.7 (Riemann-Zeta Function)**.** *We define $\zeta : \mathbb{C} \to \mathbb{C}$ such that*

$$\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s}.$$

Remark 1.4. *Supposer $\Re(s) > 1$, then we have*

$$\begin{aligned} |\zeta(s)| &= \sum_{n\in\mathbb{N}} \frac{1}{|n|^s} \\ &= \sum_{n\in\mathbb{N}} \frac{1}{n^{\Re(s)}} \end{aligned}$$

By multiplying $\frac{1}{2^s}$, we obtain

$$\frac{1}{2^s}\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{(2n)^s}.$$

We get

$$(1-\frac{1}{2^s})\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{(2n+1)^s}.$$

Continuing this procedure, we get the following proposition.

Proposition 1.1.

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)$$

Theorem 1.3 (Weierstrass)**.** *Let $A \subseteq \mathbb{C}$ and consider a sequence of functions $(f_n : A \to \mathbb{C})_{n\in\mathbb{N}}$ such that there exists a sequence of non-negative numbers $(M_n)_{n\in\mathbb{N}}$ such that*

i). $\forall x \in A, |f_n(x)| \leq M_n$.

ii). $\sum_{n\in\mathbb{N}} M_n$ *converges.*

Then the sequence converges uniformly.

Theorem 1.4. *Suppose the conditions in the previous theorem. If each function is analytic on a compact subset of A, then the limit is also analytic.*

Corollary 1.1. *Let A be a compact subset of a complex plane where $\Re(s) > 1$. Then there exists $\delta > 0$ such that $\Re(s) > 1 + \delta$ and*

$$\sum_{n\in\mathbb{N}} \left|\frac{1}{n^s}\right| \leq \sum_{n\in\mathbb{N}} \frac{1}{n^{1+\delta}} < \infty.$$

Fact 1.1. *The Riemann zeta function can be analytically continued to the whole plane except for $s = 1$.*

Definition 1.8 (Gamma Function)**.**

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

Proposition 1.2.

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx.$$

Remark 1.5. $\zeta(1 + it) \neq 0$ *if* $t \in \mathbb{R}, t \neq 0$*.* $\zeta(s) \neq 0$ *for* $0 < s < 1$*.*

Definition 1.9 (Functional Equation)**.**

$$\zeta(1 - s) = 2(2\pi)^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$$

Remark 1.6. $\Gamma(s)$ *is defined for* $\Re(s) > 0$ *and can be analytically continued to the whole place except for* $\mathbb{C}\backslash\{-2n \mid n \geq 0\}$*.*

Remark 1.7. *For* $s = -2m$ *where* $m \in \mathbb{N}$*, we see* $\zeta(s) = 0$*.*

$$\begin{aligned}
\zeta(0) &= \frac{2}{2^\pi}\lim_{s\to 1}\cos(\frac{\pi s}{2})\zeta(s) \\
&= \lim_{s\to 1}\frac{\cos(\frac{\pi s}{2})}{s-1}\lim_{s\to 1}(s-1)\zeta(s) \\
&= \frac{1}{\pi}\times\frac{-\pi}{2}\times 1 \\
&= -\frac{1}{2}.
\end{aligned}$$

Definition 1.10. *The subset of the complex plane with its real part between* 0 *and* 1*. The critical line is the line where* $\Re(s) = \frac{1}{2}$*.*

Conjecture 1.4 (Riemann Hypothesis)**.** *Let s be an element of the critical strip. If* $\zeta(s) = 0$ *then* $\Re(s) = \frac{1}{2}$ *(ie. it lies on the critical line).*

Notation 1.4. *Let* $T > 0$*. We denote* $N(T)$ *the number of zeros of* ζ *in the critical strip whose coefficient of the imaginary part is in* $(0, T)$*. That is*

$$N(T) = |\{\sigma + it \in \mathbb{C} \mid 0 < \sigma < 1, 0 < t < T\}|.$$

Proposition 1.3.

$$\lim_{T\to\infty}\frac{N(T)2\pi}{T\log(T)} = 1.$$

Sketch of Proof (needs refinement).

$$\psi(x) = \frac{1}{2\pi i}\int_l \frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds$$

where l is the line $l = a$ for some $a > 1$.

$$\psi(s) = x - \sum_{\rho \text{ non-trivial zeros}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \log(1 - x^{-2}).$$

□

Definition 1.11. *Let $q \in \mathbb{N}$ and a be a natural number coprime to q. We define*

$$\pi(x; q, a) = |\{\textit{prime numbers } p \textit{ less than or equal to } x \textit{ such that } p \equiv a \mod q\}|$$

Proposition 1.4.

$$\pi(x; , q, a) \sim \frac{x}{\varphi(q)\log(x)}$$

where φ is a Euler phi-function.

Theorem 1.5 (Brun–Titchmarsh)**.** *For any $q < x$, we have*

$$\pi(x; , q, a) < \frac{2x}{\varphi(q)\log(\frac{x}{q})}.$$

2 Week 2

Remark 2.1.

$$\mathrm{Li}(x) \sim \frac{x}{\ln x}\sum_{k=0}^{\infty}\frac{k!}{(\ln x)^k}.$$

Indeed we have

$$\mathrm{Li}(x) = .\int_2^x \frac{dt}{\ln t}.$$

Observe that

$$\int_2^t \frac{1}{(\ln t)^N} \sim \frac{x}{(\ln x)^N}$$

for all $N \geq 1$. Thus $\mathrm{Li}(x)$ can be expressed in terms of polynomials in $\frac{x}{\ln(x)}$, by keep replacing the greatest temr with the above approximation.

Definition 2.1. *A function $f : \mathbb{N} \to \mathbb{C}$ is said to be*

1). multiplicative if for any $(m, n) = 1$, we have $f(mn) = f(m)f(n)$,

2). completely multiplicative if for any natural numbers m, n, we have $f(mn) = f(m)f(n)$.

Example 2.1. *Möbius function μ is multiplicative.*

Definition 2.2 (Von-Mangoldt Function)**.** *The Von-Mangoldt function* $\Lambda : \mathbb{N} \to \mathbb{C}$ *is defined as*

$$\Lambda(n) = \begin{cases} \log(p) & (n = p^k \textit{ for some } k \geq 1), \\ 0 & (\textit{otherwise}). \end{cases}$$

Definition 2.3 (Euler Phi Function)**.** *The Euler phi funtion is* $\varphi : \mathbb{N} \to \mathbb{C}$ *such that*

$$\varphi(n) = \{1 \leq a \leq n \mid (a, n) = 1\}.$$

Example 2.2. φ *is multiplicative but* Λ *is not.*

Definition 2.4 (Dirichlet Characters Modulo q)**.** *Let* $q \in \mathbb{N}$ *be a natural number and* $q \geq 2$*.*

$$\chi_1 : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$$

be a group homomorphism. The Dirichlet character function modulo q *with respect to* χ_1 *is such that*

$$\chi(n) = \begin{cases} \chi_1(\overline{n}) & ((n, q) = 1), \\ 0 & (\textit{otherwise}). \end{cases}$$

Example 2.3. *For* $q = 3$*, we have* $(\mathbb{Z}/3\mathbb{Z})^\times = \{\pm 1\}$*. The only possible character is* $\pm 1 \mapsto \pm 1$*. Therefore, we have*

$$\chi(1) = 1, \chi(2) = -1, \chi(0) = 0.$$

Theorem 2.1.

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1, \\ 0 & (\textit{otherwise}). \end{cases}$$

Proof. When $n = 1$, this is trivial. Suppose $n \neq 1$. We factorize n by

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

where p_i is a prime and $\alpha_i \in \mathbb{N}$ for each $i = 1, \cdots, n$.

Observe that

$$\sum_{d|n} \mu(d) = \sum_{d | \prod_{i=1}^k p_i} \mu(d).$$

Now we see

$$\sum_{d | \prod_{i=1}^k p_i} \mu(d) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j = \sum_{j=0}^{k} \binom{k}{j} (1)^{k-j} (-1)^j = (1-1)^k = 0.$$

$\square$

Proposition 2.1 (Möbius Inversion Formula)**.** *Let $f, g : \mathbb{N} \to \mathbb{C}$ be functions (we do not assume them to be multiplicative). If*

$$\sum_{d|n} g(d) = f(n),$$

holds if and only if

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = g(n).$$

Proof.

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu \sum_{e|\frac{n}{d}} g(e).$$

$e|\frac{n}{d}$ if and only if $de|n$ thus obtain,

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu \sum_{de|n} g(e).$$

In particular, we get the expression

$$= \sum_{de|n} \mu(d) g(e).$$

By reordering, we get

$$= \sum_{e|n} g(e) \sum_{d|\frac{n}{e}} \mu(d).$$

By Proposition 2.1, we get

$$\sum_{d|\frac{n}{e}} \mu(d) = 0$$

unless $e = n$. ∎

Proposition 2.2.

$$\sum_{d|n} \varphi(d) = n.$$

Proof. Consider $(\mathbb{Z}/n\mathbb{Z})^{\times}$. We know that

$$|(\mathbb{Z}/n\mathbb{Z})^{\times}| = \varphi(n).$$

∎

Theorem 2.2.

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Proof. Using Proposition 2.2, we have,

$$\sum_{d|n} \mu(d) \frac{n}{d} = \varphi(n).$$

Dividing both sides by n and observe that $\mu(d) \neq 0$ if and only if d is a prime factor of n.

$$\begin{aligned} \frac{\varphi(n)}{n} &= \sum_{d|n} \frac{\mu(d)}{d}, \\ &= 1 \sum_{p|n} \frac{1}{p} + \sum_{p_1,p_2|n} \frac{1}{p_1 p_2} - \cdots . \end{aligned}$$

By the induction on the number of prime divisors of n, we get the statement. □

Proposition 2.3. *We have the following properties of φ.*

1). $n|m \Rightarrow \varphi(n)|\varphi(m)$.

2). $\varphi(n)$ *is even for* $n \geq 3$.

3). $\varphi(2n) = \begin{cases} 2\varphi(n), & (2|n) \\ \varphi(n), & (2 \not| n). \end{cases}$

4). φ *is multiplicative.*

5). $\varphi(mn) = \varphi(m) \frac{\varphi(n) d}{\varphi(d)}$ *where* $d = (m, n)$.

6). $\varphi(n^m) = n^{m-1} \varphi(n)$.

Proof. Exercise. □

Theorem 2.3. *The following statements are equivalent.*

1). $\sum_{d|n} \Lambda(d) = \log n$

2). $\sum_{d|n} \mu(d) \log d = \Lambda(n)$.

And in particular $\sum_{d|n} \Lambda(d) = \log n$ *holds*

Proof. The equivalence is a direct corollary of Möbius inversion formula. For the latter, Write

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}.$$

We have

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{k} \alpha_i \log p_i = \log(n).$$

□

Notation 2.1. *Let $n \in \mathbb{N}$, suppose a prime p divides n. Then we denote $\alpha(p)$ to be the highest prime power factor of n.*

Theorem 2.4. *Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function. Then*

$$\sum_{d|n} f(d) = \prod_{p|n} \left(\sum_{i=0}^{\alpha(p)} f(p^i) \right).$$

In particular $\sum_{d|n} f(d)$ is also multiplicative.

Proof. Let $d|n$, then we have $d = \prod_{i=1}^{k} p_i^{\beta_i}$ for some $0 \leq \beta_i \leq \alpha(p_i)$. Since f is multipliative we have

$$f(d) = \prod_{i=1}^{k} f(p_i^{\beta_i}).$$

The second part is a direct result of the first part. □

Remark 2.2. *The Second Chebyschev Function ψ can be written as*

$$\psi(x) = \sum_{d \leq x} \Lambda(d).$$

Definition 2.5 (Dirichlet Series)**.** *Let $f : \mathbb{N} \to \mathbb{C}$ be a function and $s \in \mathbb{C}$. We define*

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}.$$

For another arithmetic function $g : \mathbb{N} \to \mathbb{C}$, we define

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} + \sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} = \sum_{n \in \mathbb{N}} \frac{(f(n) + g(n))}{n^s}.$$

and

$$\left(\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \times \left(\sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} \right) = \sum_{n,m \in \mathbb{N}} \frac{(f(n)g(m))}{(nm)^s}.$$

$$= \sum_{t \in \mathbb{N}} \sum_{n|t} \frac{\left(f(n)g\left(\frac{n}{t}\right)\right)}{t^s}.$$

Recall the taylor expansion of $\ln x$ we get

$$\ln 2 = \sum_{n \in \mathbb{N}} \frac{(-1)^{n+1}}{n}.$$

Rearranging the following way

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots$$

we get this equals to $\frac{1}{2} \ln 2$.

Theorem 2.5. *Let $s \in \mathbb{C}$ be $\Re(s) > 1$, we have*

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}.$$

Proof.

$$\begin{aligned}
\zeta(s) \sum_{n \geq 1} \frac{\mu(n)}{n^s} &= \left(\sum_{n \in \mathbb{N}} \frac{1}{n^s} \right) \left(\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} \right) \\
&= \sum_{t \in \mathbb{N}} \frac{1}{t^s} \sum_{n|t} \mu(n) \\
&= 1.
\end{aligned}$$

□

Theorem 2.6. *For $\Re(s) > 1$, we have*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \in \mathbb{N}} \frac{\Lambda(s)}{n^s}.$$

From this we derive

$$\lim_{n \to \infty} \frac{\log n}{n^{\varepsilon}} = 0.$$

Proof.

$$\begin{aligned}
\zeta(s) \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} &= \left(\sum_{m \in \mathbb{N}} \frac{1}{m^s} \right) \left(\sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} \right), \\
&= \sum_{t \in \mathbb{N}} \frac{1}{t^s} \left(\sum_{n|t} \Lambda\left(\frac{t}{n}\right) \right), \\
&= \sum_{t \in \mathbb{N}} \frac{\log(t)}{t^s}, \\
&= -\zeta'(s).
\end{aligned}$$

□

Remark 2.3.

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \left| \frac{\Lambda(s)}{n^s} \right| &\leq \sum_{n \in \mathbb{N}} \frac{\log(n)}{n^{\sigma}}, \\
&<< \sum_{n \in \mathbb{N}} \frac{n^{\varepsilon}}{n^{\sigma}}, \\
&= \sum_{n \in \mathbb{N}} \frac{1}{n^{\sigma - \varepsilon}}.
\end{aligned}$$

We have $\lim_{n\to\infty}\frac{\log(n)}{n^{\varepsilon}}=1$ *and the last equation is convergent if and only if* $\sigma-\varepsilon>1$. *Thus we have* $\sigma>1+\varepsilon$.

Remark 2.4. $\frac{\zeta'(s)}{\zeta(s)}$ *is a meromorphic functions except* $s=1$ *and where* $\zeta(s)$ *vanishes. Indeed, For general* $\frac{f}{g}$, *it is analytic if* f,g *are analytic and* $g\neq 0$.

1). $\zeta(s)$ *is analytic except* $s=1$.

2). $\zeta'(s)$ *has a pole of order* 2 *at* $s=1$.

3). $\zeta(s)$ *has a pole of order* 1 *at* $s=1$.

Recall that for $|z|\geq 1$, we have,

1. $|z|\geq 1\Rightarrow\sum_{n\in\mathbb{Z}_{\geq 0}}z^n=\frac{1}{1-z}$,
2. $\prod_{n\in\mathbb{N}}(1+a_n)$ is convergent if $\sum_n a_n$ is absolutely convergent,
3. therefore $\prod_{n\in\mathbb{N}}(1+a_n)$ is convergent if and only if $\prod_{n\in\mathbb{N}}(1+|a_n|)$ is convergent.

Theorem 2.7. *Let* $f:\mathbb{N}\to\mathbb{C}$ *be a map.*

If f *is multiplicative and for* $\Re(s)>r_0, r_0\in\mathbb{R}$ *then we have,*

$$\sum_{n\in\mathbb{N}}\frac{f(n)}{n^s}=\prod_p\left(\sum_{\nu\geq 0}f(p^{\nu})p^{-\nu s}\right).$$

If f *is completely multiplicative, then*

$$\sum_{n\in\mathbb{N}}\frac{f(n)}{n^s}=\prod_p(1-f(p)p^{-s})^{-1}.$$

Proof. Let $A(x)=\{n\in\mathbb{N}\mid \text{primes factors of } n \text{ are} \leq x\}$, then

$$\prod_{p\leq x}\sum_{\nu=0}^{\infty}f(p^{\nu})p^{-\nu s}=\sum_{n\in A}\frac{f(n)}{n^s}.$$

Therefore,

$$\begin{aligned}\left|\prod_{x\leq x}\sum_{\nu=0}^{\infty}f(p^{\nu})p^{-\nu x}-\sum_{n\in\mathbb{N}}\frac{f(n)}{n^s}\right| &= \left|\sum_{n\in A}\frac{f(n)}{n^s}-\sum_{n\in\mathbb{N}}\frac{f(n)}{n^s}\right|,\\ &=\left|\sum_{n\in\mathbb{N}\setminus A}\frac{f(n)}{n^s}\right|,\\ &\leq\sum_{n\notin A}\frac{|f(n)|}{n^{\Re(s)}},\\ &\leq\sum_{n>x}\frac{|f(n)|}{n^{\Re(s)}}\to 0.\end{aligned}$$

The last limit is due to that it is a tail of a an absolutely convergent series. Since f is completely multipliative, we have

$$f(p^\nu) = (f(p))^\nu.$$

Therefore, we get,

$$\begin{aligned}\prod_p \left(\sum_{\nu \in \mathbb{Z}_{\geq 0}} (f(p^\nu) p^{-\nu s}) \right) &= \prod_p \left(\sum_{\nu \in \mathbb{Z}_{\geq 0}} (f(p) p^{-s})^{-\nu} \right), \\ &= \prod_p \left(\frac{1}{1 - f(p) p^{-s}} \right).\end{aligned}$$

□

Example 2.4. *Take $f(n) = 1$ as above we get,*

$$\sum_{n \in \mathbb{N}} \frac{1}{n^s} = \zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \Re(s) > 1.$$

Example 2.5.

$$\sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \Re(s) > 1.$$

Example 2.6.

$$\begin{aligned}\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} &= \prod_p \left(1 + \frac{\mu(p)}{p^s}\right), \\ &= \prod_p \left(1 - \frac{1}{p^s}\right), \\ &= \frac{1}{\zeta(s)}.\end{aligned}$$

Example 2.7. *Note that $\phi(n) \leq n$. Thus for $\Re(s) > 2$, we have,*

$$\sum_{n \in \mathbb{N}} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

2.1 Order of arithmetic functions

Definition 2.6. *Let $f, g : \mathbb{N} \to \mathbb{C}$, we denote,*

$$f(n) = O(g(n)),$$

if there is $K > 0$ and $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow |f(n)| \leq K|g(n)|.$$

An alternative notation for this is $f(n) = O(g(n))$.

Definition 2.7. *We define following arithmetic functions,*

1). $\nu(n) := \sum_{p|n} 1$*, a number of primer divisors of* n*,*

2). $d(n) := \sum_{d|n} 1$*, the number of divisors of* n*,*

3). $\sigma := \sum_{d|n} d$*, the sum of all the divisors of* n*.*

Lemma 2.1.

$$\nu(n) << \log(n).$$

Proof. Let $n = \prod_{i=1}^{k} p_i^{\alpha(p_i)}$. Then $\nu(n) = k$. Since $p_i \geq 2$, we have,

$$\log(n) = \sum_{i=1}^{k} \alpha(p_i)\log(p_i),$$
$$\geq k\log(2).$$

Therefore $\nu(n) \leq \frac{\log(n)}{\log 2}$. □

Lemma 2.2.

$$\sum_{k=2}^{n} \leq \log(n) + 1.$$

Proof. We know that

$$\int_1^n \frac{1}{t}dt = \log(n).$$

For $1 \leq k \leq t \leq k+1 \leq n$, we have,

$$\int_k^{k+1} \frac{1}{k+1}dt \leq \int_k^{k+1} \frac{1}{t}dt \leq \int_k^{k+1} \frac{1}{k}dt.$$

Thus we have,

$$\frac{1}{k+1} \leq \log(k+1) - \log(k) \leq \frac{1}{k}.$$

By telescopoing sum we get

$$\sum_{k=2}^{n} \frac{1}{k} \leq \log(n+1).$$

□

Lemma 2.3.

$$\sigma(n) << n(1 + \log(n)) \sim n\log(n).$$

Proof.

$$\begin{aligned}
\sigma(n) = \sum_{d|n} \frac{n}{d}, \\
&= n \sum_{d|n} \frac{1}{d}, \\
&= n \left(1 + \sum_{d\geq 2}^{n} \frac{1}{d}\right), \\
&\leq n \left(1 + \sum_{d=2}^{n} \frac{1}{d}\right), \\
&\leq (1 + \log(n)).
\end{aligned}$$

The last inequality is due to Lemma 2.2. ☐

Exercise 2.1. *Show that*

$$\sum_{k=1}^{n} \frac{1}{k} = \log(n) + O(1).$$

That is

$$\left|\sum_{k=1}^{n} \frac{1}{k} - \log(k)\right| << 1.$$

Hint: Replace $\frac{1}{t}$ by an increasing function and derive the similar inequality to Lemma 2.2.

Lemma 2.4.

$$d(n) \leq 2\sqrt{n}.$$

Proof. If $n = d_1 d_2$ then one of them must be less than or equal to $\sqrt{n}$. ☐

We have an improved inequality,

Proposition 2.4. *for $\varepsilon > 0$, we have,*

$$d(n) << n^{\varepsilon}.$$

Proof. Recall that for $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, we have $d(n) = \prod_{i=1}^{k}(\alpha_i+1)$. In particular, we have,

$$\frac{d(n)}{n^{\varepsilon}} = \prod_{i=1}^{k} \frac{(\alpha_i + 1)}{p_i^{\varepsilon\alpha_i}}.$$

Let $A = \{i \mid p_i^{\varepsilon} \geq 2\}$. Rcall that for $x \geq 1$, $x + 1 \leq 2^x$ that is

$$\frac{x+1}{2^x} \leq 1.$$

Then,

$$\prod_{i\in A}\frac{\alpha_i+1}{p_i^{\varepsilon\alpha_i}} \le \prod_{i=1}^{k}\frac{\alpha_i+1}{2^{\varepsilon\alpha_i}} \le 1.$$

For $p_i^{\varepsilon} < 2$, we observe,

$$p_i^{\varepsilon\alpha_i} = e^{\varepsilon\alpha_i\log(p_i)} \ge \varepsilon\alpha_i\log(p_i).$$

Therefore,

$$\begin{aligned}\prod_{i\notin A}\frac{\alpha_i+1}{p_i^{\varepsilon\alpha_i}} &\le \prod_{i\notin A}\left(\frac{\alpha_i}{p_i^{\varepsilon\alpha_i}}+1\right),\\ &\le \prod_{i\notin A}\left(\frac{\alpha_i}{\varepsilon\alpha_i\log(p_i)}+1\right),\\ &\le \prod_{i\notin A}\left(\frac{1}{\varepsilon\log(2^{\frac{1}{\varepsilon}})}+1\right),\\ &\le \prod_{i\notin A}\left(\frac{1}{\log(2)}+1\right).\end{aligned}$$

Combining two cases, we obtain the statement. □

Notation 2.2. *Let $x\in\mathbb{R}$. We denote*

1. *the integer part $[x]\in\mathbb{Z}$ which is the greatest integer not exceeding x,*
2. *the fraction part $\{x\} = x-[x]$.*

Proposition 2.5.

$$\frac{\sum_{n\le x} d(n)}{x} \sim \log(x).$$

Proof. By definition, we have,

$$\sum_{n\le x} d(n) = \sum_{n\le x}\sum_{d|n} 1.$$

$d|n$ if and only if there is e such that $de=n$. Thus using this we obtain,

$$\sum_{n\le x} d(n) = \sum_{\substack{e,d\\ de\le x}} 1 = \sum_{d\le x}\sum_{e\le\frac{x}{d}} 1 = \sum_{d\le x}\left[\frac{x}{d}\right].$$

Using the definition of $[x]$, we have,

$$\sum_{n\le x} d(n) = \sum_{d\le x}\frac{x}{d} - \left\{\frac{x}{d}\right\} = x\sum_{d\le x}\frac{1}{d} - \sum_{d\le x}\left\{\frac{x}{d}\right\} = x(\log(x)+O(1))+O(x).$$

Thus we obtain the statement. □

Definition 2.8. *Let $f : \mathbb{N} \to \mathbb{C}$, we say the average order of f is $g : \mathbb{R} \to \mathbb{C}$ if*

$$\frac{\sum_{n \le x}}{f(n)} x \sim g(x).$$

Proposition 2.5 can be restated as follows.

Proposition 2.6. *The average order of d is* $\log(x)$.

Exercise 2.2. *Examine the following statements.*

1. *Is it true that $d(n) << \log n$?*
2. *Do we have $d(n) = O(n^{\varepsilon})$ for any $\varepsilon > 0$?*
3. *What is the optimal bound for $d(n)$?*

Theorem 2.8. *There exists $c_1, c_2 > 0$ such that*

$$c_1 \le \frac{\varphi(n)\sigma(n)}{n^2} \le c_2.$$

Proof. Recall that

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

and

$$\sigma(n) = \prod_{p|n} \frac{p^{\alpha(p)+1} - 1}{p - 1}.$$

Thus we obtain,

$$\frac{\sigma(n)}{n} = \frac{\prod_{p|n}(1 + p + \cdots + p^{\alpha(p)})}{\prod_{p|n} p^{\alpha(p)}} = \prod_{p|n} \left(\frac{\frac{1}{p^{\alpha(p)+1}} - 1}{\frac{1}{p} - 1} \right)$$

By multiplying two we get,

$$\frac{\varphi(n)\sigma(n)}{n^2} = \prod_{p|n} \left(1 - \frac{1}{p^{\alpha(p)+1}}\right) \le 1.$$

On the other hand,

$$\prod_{p} \left(1 - \frac{1}{p^2}\right) \le \frac{\varphi(n)\sigma(n)}{n^2}.$$

The left hand side is equal to $\frac{1}{\zeta(2)}$ which is $\frac{6}{\pi^2}$. □

Theorem 2.9. *The average order of φ is $\frac{3n}{\pi^2}$.*

Proof.

$$\begin{aligned}
\sum_{n\leq x}\phi(n) &= \sum_{n\leq x} n \sum_{d|n}\frac{\mu(d)}{d},\\
&= \sum_{\substack{d,e\\ de\leq x}} e\mu(d),\\
&= \sum_{d\leq x}\mu(d)\left(\sum_{e\leq \frac{x}{d}} e\right),\\
&= \frac{1}{2}\sum_{d\leq x}\mu(d)\left(\left[\frac{x}{d}\left(\left[\frac{x}{d}\right]+1\right)\right]\right),
\end{aligned}$$

Note that
$$\frac{x}{d} = \left[\frac{x}{d}\right] + \left\{\frac{x}{d}\right\} = \left[\frac{x}{d}\right] + o(1).$$
It is assined as an exercise to confirm that
$$\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right) = \frac{x^2}{d^2} + o\left(\frac{x}{d}\right).$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{d\leq x}\mu(d)\left(\frac{x^2}{d^x} + o\left(\frac{x}{d}\right)\right),\\
&= \frac{x^2}{2}\sum_{d\leq x}\frac{\mu(d)}{d^2} + o\left(x\sum_{d\leq x}\frac{\mu(d)}{d}\right),\\
&= \frac{x^2}{2}\left(\sum_{d\geq 1}\frac{\mu(d)}{d^2} - \sum_{d\geq x}\frac{\mu(d)}{d^2}\right) + o\left(x\sum_{d\geq x}\frac{\mu(d)}{d}\right),\\
&= \frac{x^2}{2}\frac{1}{\zeta(2)} - \frac{x^2}{2}\sum_{d\geq x}\frac{\mu(d)}{d}^2 + o\left(x\sum_{d\leq x}\frac{\mu(d)}{d}\right).
\end{aligned}$$

We have,

$$\begin{aligned}
\left|\sum_{d\geq x}\frac{\mu(d)}{d^2}\right| &\leq \sum_{d\geq x}\frac{1}{d^2}\\
&<< \int_x^\infty \frac{dt}{t^2},\\
&<< \frac{1}{x}.
\end{aligned}$$

and,

$$\left| \sum_{d \leq x} \frac{\mu(d)}{d} \right| << \ln x.$$

Using these we have,

$$\begin{aligned} &= \frac{x^2}{2} \frac{1}{\zeta(2)} + o(x) + o(x \ln x), \\ &= \frac{x^2}{2\zeta(2)} + o(x \ln x). \end{aligned}$$

We conclude that

$$x \to \infty \Rightarrow \frac{\sum_{n \leq x} \phi(n)}{x^2} \to \frac{1}{2\zeta(2)}.$$

In particular,

$$\frac{\sum_{n \leq x} \phi(n)}{x} \sim \frac{x}{2\zeta(2)} = \frac{x \cdot 6}{2\pi^2}.$$

□

2.2 Abel's Summation Formula

Recall the harmonic series $\sum_{n \in \mathbb{N}} \frac{1}{n}$ is divergent. Our next goal is to find such A_x that

$$\lim_{n \to \infty} \left(\sum_{n \leq x} \frac{1}{n} - A_x \right)$$

exists.

Remark 2.5 (Euler-Mascheroni constant)**.** *By taking $A_x = \log(x)$, we have*

$$\lim_{n \to \infty} \left(\sum_{n \leq x} \frac{1}{n} - \log(x) \right) = \psi,$$

exists. Such ψ is called Euler-Mascheroni constant.

Remark 2.6 (Euler Kronecer constant)**.** *Take $A_x = \log(x)$, we have*

We can show that

$$\psi = \lim_{s \to 1^+} \left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right).$$

Hint first show that

$$\zeta(s) = \frac{1}{s-1} + \psi + o(s-1).$$

Proposition 2.7 (Abels' summation formula). *Given $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{C}$ and $f(n)$ is continuously differentiable in $[1, x]$. Set*

$$A(x) := \sum_{n \leq x} a_n.$$

Then we have,

$$\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

Proof. Observe that

$$a_n = A(n) - A(n-1).$$

Assume $x \in \mathbb{N}$. We substitute this to $\sum_{n \leq x} a_n f(n)$, we get,

$$\begin{aligned}
\sum_{n \leq x} a_n f(n) &= \sum_{n \leq x} (A(n) - A(n-1)) f(n), \\
&= \sum_{n \leq x} A(n)f(n) - \sum_{n \leq x} A(n-1)f(n), \\
&= \sum_{n \leq x} A(n)f(n) - \sum_{n \leq x-1} A(n)f(n+1), \\
&= A(x)f(x) - \sum_{n \leq x-1} A(n)(f(n+1) - f(n)), \\
&= A(x)f(x) - \sum_{n \leq x-1} \int_n^{n+1} f'(t)dt, \\
&= A(x)f(x) - \sum_{n \leq x-1} \int_n^{n+1} A(t)f'(t)dt, \\
&= A(x)f(x) - \int_1^x A(t)f'(t)dt.
\end{aligned}$$

For the case when $n \notin \mathbb{N}$ and $n > 1$,

$$\sum_{n \leq x} a_n f(n) = \sum_{n \leq [x]} a_n f(n).$$

Using the previous case, we get,

$$\sum_{n \leq x} a_n f(n) = A([x])f([x]) - \int_1^{[x]} A(t)f'(t)dt.$$

Remains to show that we can remove the brackets. To do so,

$$\begin{aligned}
\sum_{n\leq x} a_n f(n) &= A([x])f([x]) - \int_1^x A(t)f'(t)dt + \int_{[x]}^x A(t)f'(t)dt, \\
&= A([x])f([x]) - \int_1^x A(t)f'(t)dt + A([x])\int_{[x]}^x f'(t)dt, \\
&= A([x])f([x]) - \int_1^x A(t)f'(t)dt + A([x])f(x) - A([x])(f[x]), \\
&= A([x])f(x) - \int_1^x A(t)f'(t)dt.
\end{aligned}$$

$\square$

Corollary 2.1.

1. $\sum_{n\leq x} \frac{1}{n} = \ln x + \psi + o\left(\frac{1}{x}\right)$.
2. $\sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + o\left(\frac{1}{x^s}\right)$, *where* $\Re(s) > 0, s \neq 1$.

We also have the following equivalent forms of prime number theorem when $x \to \infty$.

$$\begin{aligned}
\sum_{n\leq x} s(n) \sim x \\
&\Leftrightarrow \pi(x) \sim \frac{x}{\ln x}, \\
&\Leftrightarrow \sum_{p\leq x} \ln p \sim x, \\
&\Leftrightarrow \sum_{n\leq x} \mu(n) = o(x).
\end{aligned}$$

Proof. Consider $f(t) = \frac{1}{t^s}$ and $a_n = 1$ for all $n \in \mathbb{N}$.

$$\begin{aligned}
\sum_{n\leq x} \frac{1}{n} &= \frac{[x]}{x} + \int_1^x \frac{[x]}{t^2} dt, \\
&= \frac{x - \{x\}}{x} + s\int_1^x \frac{t - \{t\}}{t^{s+1}} dt.
\end{aligned}$$

When $s = 1$, we have,

$$\begin{aligned}
\sum_{n \le x} \frac{1}{n} &= 1 - \frac{\{x\}}{x} + \int_1^x \frac{t - \{t\}}{t^{s+1}} dt, \\
&= 1 + \ln x + o\left(\frac{1}{x}\right) - \int_1^x \frac{\{t\}}{t^2} dt, \\
\lim_{x\to\infty} \left(\sum_{n \le x} \frac{1}{n} - \ln x\right) &= 1 - \int_1^\infty \frac{\{t\}}{t^2} dt, \\
&= \psi. \\
&= x^{1-s} + o\left(\frac{1}{x^s}\right) + \frac{sx^{1-s}}{1-s} - \frac{s}{1-s} - s^2 \int_{\frac{\{t\}}{t^{s+1}}} dt.
\end{aligned}$$

Recall that

$$\left[\int \frac{1}{t^s} = \frac{t^{-s+1}}{1-s}\right]_1^x = \frac{x^{1-s}}{1-s} - \frac{1}{1-s}.$$

Using this we obtain,

$$\begin{aligned}
&= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - x \int_1^x \frac{\{t\}}{t^{s+1}} dt, \\
x^{1-s}\left[1 + \frac{1}{1-s}\right] &= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - s \int_1^x \frac{\{t\}}{t^{s+1}} dt \\
\int_1^\infty \frac{\{t\}}{t^{s+1}} &< \infty, \\
&\le \int_1^\infty \frac{1}{t^{\Re(s)+1}} < \infty.
\end{aligned}$$

As $x \to \infty$, the left hand side goes to $\zeta(s)$, for the right hand side, we get $= \frac{-s}{1-s} - s\int_1^\infty \frac{\{t\}}{t^{s+1}} dt$, where $\Re(s) > 1$,

$$\zeta(s) = \frac{-s}{1-s} - s \int \frac{\{t\}}{t^{s+1}} dt.$$

Identity theorem for analytic function tells us the analytic continuation of Riemann zeta function is unique.

Remark 2.7. *It is an exercise that*

$$\int_1^\infty \frac{\{t\}}{t^{s+1}} dt$$

where $\Re(x) > 0$. *From Stein-Schakarchi 5.2, 5.3, we have*

$$\sum f_n(z) \overset{unif.}{\to} f(z)$$

is analytic where $\Re(x) > 0$ *and* $s \neq 1$*, also in this case,*

$$\zeta(s) = \frac{-s}{1-s} - s\int \frac{\{t\}}{t^{s+1}}dt.$$

holds.

Remark 2.8 (Exercise)**.** *Let* $M \in \mathbb{N}$ *and*

$$\lim_{x\to\infty}\left(\sum_{\substack{n\le x\\(n,M)=1}} \frac{1}{n} - \frac{\phi(M)}{M}\ln x\right)$$

exists.

Assume $\pi(x) \sim \frac{x}{\ln x}$, to show

$$\theta(x) := \sum_{p\le x} \ln p \sim x,$$

Consider the following sequence

$$a_n = \begin{cases} 1 & n \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f(t) = \ln t.$$

Using Abel summation formula,

$$\sum_{p\le x} \ln p = \pi(x)\ln(x) - \int_1^\infty \frac{\pi(t)}{t}dt.$$

can be written as

$$\frac{\theta(x)}{x} = \frac{\pi(x)\ln x}{x} - \frac{1}{x}\int_1^x \frac{\pi(t)}{t}dt.$$

Remark 2.9 (Exercise)**.** *Use* $\frac{\pi(t)}{t} \sim \frac{1}{\ln t}$ *and*

$$\int_1^x \frac{dt}{\ln t} = o(x),$$

prove that

$$\lim_{x\to\infty} \frac{1}{x}\int_1^x \frac{\pi(t)}{t}dt = 0.$$

$$\begin{aligned}
\int_2^{\sqrt{x}} \frac{dt}{\ln t} = \int_{\sqrt{x}}^x \frac{dt}{\ln t} \le \frac{1}{\ln 2}\int_2^{\sqrt{x}} dt + \frac{1}{\ln\sqrt{x}}\int_{\sqrt{x}}^x dt,& \\
= \frac{\sqrt{x}-2}{\ln 2} + \frac{x-\sqrt{x}}{\ln\sqrt{x}},& \\
= o(x).&
\end{aligned}$$

Remark 2.10.
$$\psi(x) = \sum_{n \le x} s(n) = \sum_{1 \le k, p^k \le x} \ln p.$$

Thus we see,
$$\psi(x) - \theta(x) = \sum_{2 \le k, p, p^k \le x} \ln p.$$

Also,
$$p^k \le x \Rightarrow k \le \frac{\ln x}{\ln p}.$$

Using $k \ge 2$*,*
$$p \le x^{\frac{1}{k}} \le \sqrt{x}, \forall k.$$

$$\begin{aligned}
&\le \sum_{p \le \sqrt{x}} \ln p \left(\sum_{2 \le k \le \frac{\ln x}{\ln p}} 1 \right), \\
&\le \sum_{p \le \sqrt{x}} \ln x, \\
&\le \ln(x) \sum_{n \le \sqrt{x}} 1, \\
&\le \sqrt{x} \ln x. \Rightarrow \qquad \frac{\psi(x)}{x} = \frac{\theta(x)}{x} + \frac{o(\sqrt{x}\ln x)}{x}.
\end{aligned}$$

Therefore we obtain,
$$\psi(x) \sim x \Leftrightarrow \theta(x) \sim x.$$

□

Remark 2.11. *As exercises, find the closed expressions for the following summations.*
$$\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n}, \sum_{p \le x} \frac{1}{p} = \ln \ln x + o(1).$$

2.3 Characters

Definition 2.9. *Let G be a finite group. A character is a group homomorphism $f : G \to \mathbb{C}^\times$.*

Remark 2.12. *Let us denote*
$$\hat{G} := \{ f : G \to \mathbb{C}^\times \mid \textit{characters} \}.$$

If G is finite abelian then $|\hat{G}| = |G|$. Furthermore, such characters are linearly independent over $\mathbb{C}$.

Definition 2.10. *Let $q \in \mathbb{N}$ and $q \geq 3$. A Dirichlet character is a group homomorphism modulo q is a group homomorphism*

$$\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times.$$

Remark 2.13. *Given a Dirichlet character $\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$. We ca define a character $\chi : \mathbb{Z} \to \mathbb{C}^\times$ as follows.*

$$\chi(a) := \begin{cases} \chi'(\overline{a}), & (a, q) = 1, \\ 0, & (a, q) \neq 1. \end{cases}$$

From Remark 2.12, there are exactly $\varphi(q)$ many Dirichlet characters modulo q. Furthermore,

$$\chi(a)^{\varphi(q)} = \chi'(\overline{a})^{\varphi(q)} = \chi'(\overline{a}^{\varphi(q)}) = \chi'(\overline{1}) = 1.$$

In particular, images of χ are $\varphi(q)$-th roots of unity.

Example 2.8. *For $q = 3$,*

$$(\mathbb{Z}/3\mathbb{Z})^\times = \{\overline{1}, \overline{2}\} \to \mathbb{C}^\times.$$

We only have two characters, a trivial one and $\overline{2} \mapsto -1$.

Example 2.9. *For $q = 5$,*

$$(\mathbb{Z}/5\mathbb{Z})^\times = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$$

	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\chi_{1,5}(n)$	1	1	1	1
$\chi_{2,5}(n)$	1	-1	-1	1
$\chi_{3,5}(n)$	1	i	$-i$	-1
$\chi_{4,5}(n)$	1	$-i$	i	-1

$\chi_{1,5}$ is called a principle/trivial character.

Definition 2.11. *A character $\chi : \mathbb{Z} \to \mathbb{C}^\times$ is called*

1. *trivial if $\chi(g) = 1$ for all $g \in G$,*
2. *even if $\chi(-1) = 1$,*
3. *odd if $\chi(-1) = -1$.*

We also define these notions for characters $\chi_0 : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ accordingly if characters induces by $\chi : \mathbb{Z} \to \mathbb{C}^\times$ has these properties. Trivial characters are often denoted by χ_0.

Theorem 2.10.

$$\sum_{a \bmod q} \chi(a) = \begin{cases} \phi(q) \quad (\chi = \chi_0), \\ 0 \quad (\textit{otherwise}). \end{cases}$$

We also have,

$$\sum_{\chi \bmod q} \chi(a) = \begin{cases} \phi(a) \quad (\overline{a} = 1), \\ 0 \quad (\textit{otherwise}). \end{cases}$$

Proof. If χ is principle then the first assertion is clear. Suppose χ is not principle then there is $b \in \{1, \cdots, q\}$ such that $\chi(b) \neq 1$ and $(b, q) = 1$. Let

$$s = \sum_{a \bmod q} \chi(a).$$

Then by the definiion of group homomorphisms, we have,

$$\chi(b)s = s.$$

But $\chi(b) \in \mathbb{C}$, this means $s = 0$ as $\mathbb{C}$ is an integral domain.

For the second assertion, let $\overline{a} \neq 1$, then

$$\exists \chi' \bmod q, \text{ s.t. } \chi'(a) \neq 1.$$

Thus we get,

$$s = \sum_{\chi \bmod q} \chi(a), s \cdot \chi(a) = \sum_{\chi \bmod q} \chi\chi'(a) = s \Rightarrow s = 0.$$

The statement when $\overline{a} = 1$ follows from Remark 2.12. □

Remark 2.14. *One can check in the table of Example 2.9 that Theorem 2.10 indeed holds.*

Exercise 2.3.

$$\sum_{\substack{\chi \bmod q, \\ \chi(-1)=1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} \quad (\overline{a} = 1, -1), \\ 0 \quad (\textit{otherwise}). \end{cases},$$

$$\sum_{\substack{\chi \bmod q, \\ \chi(-1)=-1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} \quad (\overline{a} = 1), \\ -\frac{\phi(a)}{2} \quad (\overline{a} = -1), \\ 0 \quad (\textit{otherwise}). \end{cases},$$

Obviously we have the following equalities.

$$\sum_{n\le x}\chi(n)=\sum_{\substack{n\le x\\(n,q)=1}}\chi(n)=\sum_{n\le kq}\chi(n)+\sum_{n=kq+1}^{x}\chi(n),$$

where k is the largest integer such that $kq\le x$. Then we observe from Theorem 2.10

$$\sum_{n\le kq}\chi(a)=k\left(\sum_{n=1}^{q}\chi(n)\right)=0,$$

unless χ is trivial. Also we have,

$$\left|\sum_{n\le x}\chi(n)\right|=\left|\sum_{\substack{kq+1\le n\le x\\(n,q)=1}}\chi(n)\right|\le\sum_{\substack{kq+1\le n\le x\\(n,q)=1}}1\le\sum_{\substack{kq+1\le n\le kq+q\\(n,q)=1}}1=\phi(q).$$

Thus we conclude,

$$\sum_{n\le x}\chi(n)\le\phi(n),$$

Exercise 2.4.

$$\sum_{n\le x}\chi_0(n)=?.$$

Theorem 2.11 (Pólya–Vinogradov)**.**

$$\sum_{n\le x}\chi(n)<<\sqrt{q}\ln q,(\chi\ne\chi_0\mod q).$$

notice that the above expression is bounded by $\sqrt{q}\ln\ln q$. Furthermore, this is uniform in q that is the constant does not depend on q.

$$\sum_{n\ge1}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}-\frac{s}{1-s}+O(x^{-s})-s\int_1^x\frac{\{t\}}{t^{s+1}}dt.$$

For $\Re(s)>1$, as $x\to\infty$ we have,

$$\sum_{n\ge1}^{\infty}\frac{1}{n^s}=\frac{-s}{1-s}-s\int_1^\infty\frac{\{t\}}{t^{s+1}}dt.$$

The last expression is analytic since,

$$\sum\int_n^{n+1}\frac{\{t\}}{t}^{s+1}dt\overset{\text{uniformly}}{\to}\int_1^\infty\frac{\{t\}}{t^{s+1}},\text{when }\Re(s)>0.$$

Suppose $\zeta(s)\ne0$, where $\Re(s)>0$, then Euler product exists.

Theorem 2.12. *Set*

$$A(n) := \sum_{n \in \mathbb{N}} a_n.$$

Assume $A(x) := \sum_{n \leq x} a_n = O(x^\delta)$, *then we have, for* $\Re(s) > \delta$,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^\infty \frac{A(t)}{t^{s+1}} dt.$$

Hence the Dirichlet series converges for $\Re(s) > \delta$.

Proof.

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s} = \frac{A(x)}{x^s} + s \int_1^x \frac{A(t)}{t^{s+1}} dt.$$

As $A(x) = O(x^\delta)$ and $\Re(s) > \delta$, $\frac{A(x)}{x^s} = O(x^{\delta - \Re(s)})$. Therefore, as $x \to \infty$, we have,

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s} = s \int_1^\infty \frac{A(t)}{t^{s+1}} dt.$$

Again using the assumption, we have,

$$\int_1^\infty \left| \frac{A(t)}{t^{s+1}} \right| dt << \int_1^\infty t^{\delta - \Re(s) - 1} dt = \left. \frac{t^{\delta - \Re(s)}}{\delta - \Re(s)} \right|_0^\infty = \frac{1}{\delta - \Re(s)}.$$

Thus the integral is convergent. $\square$

Definition 2.12. *For* $\Re(s) > 1$, *we define*

$$L(s, \chi) := \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s}.$$

Remark 2.15. *Since* $\Re(s) > 1$ *and for any character* $\chi : \mathbb{Z} \to \mathbb{C}^\times$ *we have* $|\chi(n)| \leq 1$, $L(s, \chi)$ *is uniformly absolutely convergent.*

Example 2.10. *Let* $\chi : \mathbb{Z} \to \mathbb{C}^\times$ *be a non-principal character modulo* q *and set* $A(n) = \chi(n)$. *Recall that*

$$\sum_{n \leq x} \chi(n) \leq q.$$

Taking $A(n) = \chi(n)$ *and apply Theorem 2.12, we obtain, for* $\Re(s) > 0$,

$$L(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = s \int_1^\infty \frac{\sum_{n \leq t} \chi(n)}{t^{s+1}} dt.$$

Since for $\Re(s) > 1, L(s, \chi)$ *is absolutely uniformly convergent. By Theorem 2.7, we have,*

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

If $\chi = \chi_0$, a principal character, we have,

$$L(s, \chi_0) = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p} \left(1 - \frac{1}{p^s}\right) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Using ζ function, we have,

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Theorem 2.13. *ζ has an analytic continuation for $\Re(s) > 0$ besides $s = 1$. For $s = 1$ we have a simple pole of residue 1.*

Proof. Recall from the proof of Corollary 2.1. We have for $\Re(s) > 1$,

$$\zeta(s) = \frac{-s}{1-s} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

The right hand side of the equation is analytic when $\Re(s) > 0, s \neq 1$. When $s = 1$, we have,

$$\lim_{s \to 1^+} (s-1)\zeta(s) = \lim_{s \to 1^+} \left(s - s(s-1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt\right) = 1.$$

□

Corollary 2.2. *For $\Re(s) > 0$ and $s \neq 1$, we have an analytic continuation of $L(s, \chi_0)$ where χ_0 is a principal character modulo q, which is*

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Obviously at $s = 1$, it has a simple pole of residue $\prod_{p|q} \left(1 - \frac{1}{p^s}\right)$, which we can write as

$$\mathrm{Res}(L(s, \chi_0), 1) = \frac{\varphi(q)}{q}.$$

Proof. A direct corollary of Theorem 2.13. □

Suppose $\chi \neq \chi_0$, we have, analytic continuation of $L(s, \chi) \neq 0$is

Theorem 2.14. *Let χ be a non-principal character, then there is an analytic continuation of $L(s, \chi)$ for $\Re(s) > 0$.*

Proof. From Example 2.10, we have,

$$L(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = s \int_0^\infty \frac{\sum_{n \leq t} \chi(n)}{t^{s+1}} dt.$$

The right hand side is analytic. □

Theorem 2.15.
$$\sum_{n\in\mathbb{N}}\frac{a_n}{n^s}$$
is analytic in its range of convergence.

Remark 2.16. *We have the following conjecture.*
$$L\left(\frac{1}{2},\chi\right)=0,\chi\neq\chi_0.?$$
$$\zeta\left(\frac{1}{2}\right)=\frac{1}{1-\sqrt{2}}\sum_{n\in\mathbb{N}}\frac{(-1)^{n-1}}{\sqrt{n}}\approx -1.46\cdots$$

Definition 2.13. *A character is said to be quadratic if its values are either* ± 1.

Remark 2.17. *We have,*
$$\begin{aligned}&L(s,\chi)=0 \quad \textit{if } s=0,-2,-4, \textit{ when } \chi \textit{ is an even character.}\\ &L(s,\chi)=0 \quad \textit{if } s=-1,-3,-5, \textit{ when } \chi \textit{ is an odd character.}\end{aligned}$$

Lemma 2.5. *For* $\sigma>1$ *and* $t\in\mathbb{R}$*, we have,*
$$\Re(\ln(\zeta(\sigma+it)))=\sum\frac{\Lambda(n)}{n^\sigma\ln n}\ln(t\ln(n)).$$
And also,
$$\Re(3\ln(\zeta(\sigma))+4\ln(\zeta(\sigma+it))+\ln(\zeta(\sigma+2it)))\geq 0.$$

Proof.
$$\begin{aligned}\zeta(s)&=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},\sigma>0,\\ \ln(\zeta(s))&=-\sum_p\ln(1-p^{-s})\\ &=\sum_{p,n}\frac{1}{np^{ns}},\sigma>1.\end{aligned}$$
We have,
$$\sum_{n\geq 2}\frac{\Lambda(n)}{n^s\ln n}=\sum_{p,k,k\geq 1}\frac{\ln p}{p^{ks}\ln p^k}=\sum_{p,k}\frac{1}{kp^{ks}}=\sum_{n\geq 2}\Lambda(n).$$
$$\Re(3\ln\zeta(\sigma)+4\ln\zeta(\sigma+it)+\ln\zeta(\sigma+2it))=\sum_{n\geq 2}\frac{\Lambda(n)}{n^\sigma\ln n}(3+4\cos(t\ln n)+\cos(2t\ln n))\geq 0,$$

since

$$3 + 4\cos\theta + \cos 2\theta = 2(\cos\theta + 1)^2 \geq 0.$$

$$= \ln|\zeta(\sigma)^3\zeta(\sigma + it)^4\zeta(\sigma + 2it)| \geq 0.$$

Thus we have,

$$\Re(\ln(z)) \leq$$

Thus we get,

$$|\zeta(\sigma)^3\zeta(\sigma + it)^4\zeta(\sigma + 2it)| \geq 1.$$

Suppose $\zeta(1 + it_0) = 0$, for $t_0 \in \mathbb{R}\backslash\{0\}$, then,

$$\zeta(1 + it_0) \neq 0, t \neq 0.$$

$\square$

$$\zeta(s) = \frac{s}{s-1} + s\int_1^\infty \frac{\{t\}}{t}^{s+1} dt,$$

Theorem 2.16. *For $t \in \mathbb{R}\backslash\{0\}$, we have*

$$\zeta(1 + it) \neq 0.$$

And for non-trivial character χ we have,

$$L(1, \chi) \neq 0.$$

Proof. $\square$

Theorem 2.17. $\frac{\zeta'(s)}{\zeta(s)}$ *has an analytic continuation to $\Re(s) = 1, s \neq 1$. And for $s = 1$, we have a simple pole of residue -1.*

Proof. We have,

$$(s-1)\zeta(s) = s - s(s-1)\int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

Set

$$f(s) := 1 - (s-1)\int_1^\infty \frac{\{t\}}{t^{s+1}} dt,$$

so that $(s-1)\zeta(s) = sf(s)$. We already have $f(s)$ is analytic when $\Re(s) > 0$. Differentiating both sides we get,

$$(s-1)\zeta'(s) + \zeta(s) = sf'(s) + f(s).$$

Dividing both sides by $(s-1)\zeta(s)$ we get,

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \frac{sf'(s)+f(s)}{(s-1)\zeta(s)}.$$

The right hand side is analytic when ζ does not vanish. From Theorem 2.16, by letting $s = 1+it$ for some $t \in \mathbb{R}\backslash\{0\}$, we get the desired analytic continuation. For $s = 1$, we have

$$(s-1)\frac{\zeta'(s)}{\zeta(s)} = \frac{sf'(s)+f(s)}{\zeta(s)} - 1.$$

Recall that $\zeta(s)$ has a pole at 1 and observe that $f(1) = 1, f'(s)$ is finite thus the last statement follows. $\square$

Lemma 2.6. *Let $f := \sum_{d|n} \chi(d)$, where χ is a character. We have,*

$$\forall n \in \mathbb{N}, f(n) \geq 0, f(n) \geq 1 \text{ if } n \text{ is a perfect square.}$$

Proof. Recall $n = \prod_{p|n} p^{\alpha(p)}$. Using this we have,

$$\begin{aligned}
\sum_{d|n} \chi(d) &= \prod_{p|n} \left(\sum_{k=0}^{\alpha(p)} \chi(p)^k \right), \\
&= \begin{cases} 1 \quad \chi(p) = 0, \\ \prod_{p|n}(1+\alpha(p)) \quad \chi(p) = 1, \\ \prod_{p|n}\left(\frac{(1-(-1)^{\alpha(p)+1})}{2}\right) \quad \chi(p) = -1. \end{cases}
\end{aligned}$$

Note that if $\alpha(p)$ are all even for $p|n$, we have the last part of the cases equals to 1. $\square$

Theorem 2.18. *Let $f(n) = \sum_{d|n} \chi(d)$ for some character. Then we have,*

$$\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = 2\sqrt{x}L(1,\chi) + o(1).$$

Proof.

$$\begin{aligned}
\sum_{n\leq x} \frac{f(n)}{\sqrt{n}} &= \sum_{n \leq x} \left(\frac{\sum_{d|n} \chi(d)}{\sqrt{n}} \right), \\
&= \sum_{\substack{d,e \\ de \leq x}} \frac{\chi(d)}{\sqrt{de}}, \\
&= \sum_{\substack{d,e \leq x \\ d \leq \sqrt{x}}} \frac{\chi(d)}{\sqrt{de}} + \sum_{\substack{de \leq x \\ d > \sqrt{x}}} \frac{\chi(d)}{\sqrt{de}}, \\
&= \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(\sum_{e \leq \frac{x}{d}} \frac{1}{\sqrt{e}} \right) + \sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left(\sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{(d)}} \right).
\end{aligned}$$

Recall that

$$\sum_{m\leq x}\frac{1}{\sqrt{m}}=2\sqrt{x}+B+o\left(\frac{1}{\sqrt{x}}\right),$$

where B is some constant as $x\to\infty$.

Let $x,y\in\mathbb{R}$, such that $x<y$, we have,

$$\sum_{x<d\leq y}\frac{\chi(d)}{\sqrt{d}}=\sum_{d\leq y}\frac{\chi(d)}{\sqrt{d}}-\sum_{d\leq x}\frac{\chi(d)}{\sqrt{(d)}},$$

$$\sum_{d\leq x}\frac{\chi(d)}{\sqrt{d}}=\frac{\sum_{d\leq x}\chi(d)}{\sqrt{x}}+\frac{1}{2}\int_1^x\frac{\sum_{d\leq t}\chi(d)}{t^{\frac{1}{2}}},$$

$$=o\left(\frac{1}{\sqrt{x}}\right).$$

Using these equations, we have,

$$\sum_{e\leq\sqrt{x}}\frac{1}{\sqrt{e}}\left(\sum_{\sqrt{x}<d\leq\frac{x}{e}}\frac{\chi(d)}{\sqrt{(d)}}\right)=\sum_{e\leq x}\frac{1}{\sqrt{e}}\left(o\left(\frac{1}{x^{\frac{1}{r}}}\right)\right),$$

$$=\left(o\left(\frac{1}{x^{\frac{1}{r}}}\right)\right)\sum_{e\leq x}\frac{1}{\sqrt{e}},$$

$$\sum_{e\leq\sqrt{x}}\frac{1}{\sqrt{x}}<<\int_1^{\sqrt{x}}\frac{1}{\sqrt{t}}dt=x^{\frac{1}{4}}.$$

Thus we conclude,

$$\sum_{e\leq\sqrt{x}}\frac{1}{\sqrt{e}}\left(\sum_{\sqrt{x}<d\leq\frac{x}{e}}\frac{\chi(d)}{\sqrt{(d)}}\right)=o(1).$$

We also have,

$$\begin{aligned}
\sum_{d\le\sqrt{x}}\frac{\chi(d)}{\sqrt{d}}\left(\sum_{e\le\frac{x}{d}}\frac{1}{\sqrt{e}}\right) &= \sum_{d\le\sqrt{x}}\frac{\chi(d)}{\sqrt{d}}\left(2\sqrt{\frac{x}{d}}+B+o\left(\sqrt{\frac{d}{x}}\right)\right),\\
&= 2\sqrt{x}\sum_{d\le\sqrt{x}}\frac{\chi(d)}{d}+B\sum_{d\le\sqrt{x}}\frac{\chi(d)}{\sqrt{d}}+o\left(\frac{1}{\sqrt{x}}\sum_{d\le\sqrt{x}}\chi(d)\right).\\
2\sqrt{x}\sum_{d\le\sqrt{x}}\frac{\chi(d)}{d} &= 2\sqrt{x}\left(\sum_{d\ge1}\frac{\chi(d)}{d}-\sum_{d>\sqrt{x}}\frac{\chi(d)}{d}\right),\\
&= 2\sqrt{x}L(1,\chi)-2\sqrt{x}\sum_{d>\sqrt{x}}\frac{\chi(d)}{d},\\
&= 2\sqrt{x}L(1,\chi)-2\sqrt{x}o\left(\frac{1}{\sqrt{x}}\right),\\
&= 2\sqrt{x}L(1,\chi)+o(1).B\sum_{d\le\sqrt{x}}\frac{\chi(d)}{\sqrt{d}}\le o\left(\frac{B}{x^{\frac{1}{4}}}\right)=o(1).\\
\sum_{d\le\sqrt{x}}\chi(d)\le q? &= o(1).
\end{aligned}$$

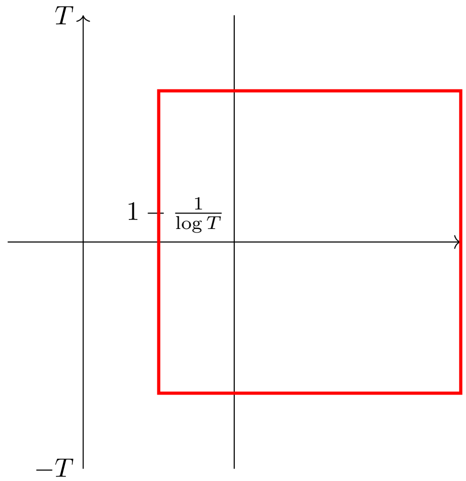
$\square$

Corollary 2.3. $L(1,\chi)\neq 0$ *for a quadratic character* χ.

Proof. To derive a contradiction, assume $L(1,\chi)=0$. Then, from the theorem

$$o(1)=\sum_{n\le x}\frac{f(n)}{\sqrt{n}}\ge\sum_{\substack{n\le x\\ n\text{ is a square}}}\frac{1}{\sqrt{n}}=\sum_{m\le\sqrt{x}}\frac{1}{m}=\log\sqrt{x}+o(1).$$

Thus we obtain $o(1)\ge\log\sqrt{x}+o(1)$ which is a cotradiction. $\square$



Lemma 2.7. $|\zeta(s)| = o(\log T), T \to \infty$. *That is there are* $K > 0$ *and* T_0 *such that for any* $T \geq T_0 \geq 1$*, we have,*

$$|\zeta(s)| \leq K \log T.$$

Proof. The first statement is due to the first derivative of ζ and it is assgined as an exercise. For the second part, for $\Re(s), \sigma > 1$, we have,

$$\begin{aligned}
\zeta(s) &= \sum_{n\geq 1} \frac{1}{n^s}, \\
&= \sum_{n\leq T} \frac{1}{n^s} + \sum_{n>T} \frac{1}{n^s}, \\
&= \sum_{n\leq T} \frac{1}{n^s} - \frac{\lfloor T \rfloor}{T^s} + s\int_1^\infty \frac{\lfloor t \rfloor}{t^{s+1}} dt, \\
&= \sum_{n\leq T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} + \frac{\{T\}}{T^s} - s\int_T^\infty \frac{\{u\}}{u^{s+1}} du.
\end{aligned}$$

Note that the right hand side is analytic where $\Re(s) > 0$ and $s \neq 1$. Now we will extimate the last equation above on the boundary of the red rectangle above. Let $\sigma_0 = 1 - \frac{1}{\log T}$.

$$\begin{aligned}
\left| \sum_{n\leq T} \frac{1}{n^s} \right| &<< \int_1^T \frac{du}{u^{\Re(s)}}, \\
&\leq \int_1^T \frac{du}{u^{\sigma_0}}, \\
&= \frac{T^{1-\sigma_0}}{1-\sigma_0}, \\
&<< \log T.
\end{aligned}$$

Observe that ,

$$\begin{aligned}
1-\sigma_0 &= \frac{1}{\log T},\\
T^{1-\sigma_0} &= T^{\frac{1}{\log T}},\\
&= e.\\
y = T^{\frac{1}{\log T}}, \log y &= \log T \frac{1}{\log T} = 1.\\
\Rightarrow y &= e.
\end{aligned}$$

We also have,

$$\begin{aligned}
\left|\frac{T^{1-s}}{s-1}\right| &= \frac{T^{1-\Re(s)}}{|s-1|},\\
&\leq \frac{T^{1-\sigma_0}}{|s-1|},\\
&\leq \frac{1}{|s-1|},\\
&\leq \frac{1}{\sigma_0-1} = \log T.
\end{aligned}$$

Also consider,

$$\left|\frac{\{T\}}{T}\right| \leq \frac{1}{T^{\Re(s)}} \leq 1.$$

Finally we have,

$$\begin{aligned}
&\leq |s| \int_T^\infty \frac{du}{u^{\Re(s)+1}},\\
&= \frac{|s|}{-\Re(s)u^{\Re(s)}}\Bigg|_T^\infty,\\
&= \frac{|s|}{\Re(s)T^{\Re(s)}},\\
&\leq \frac{|s|}{T^{\Re(s)}},\\
&\leq \frac{\sqrt{2^2+T^2}}{T^{\sigma_0}},\\
&<< \frac{T}{T^{\sigma_0}} = T^{1-\sigma_0} = e.
\end{aligned}$$

It is an exercise to check,

$$\begin{aligned}
|\zeta'(s)| &\leq o((\log T)^2),\\
\zeta'(s) &= \sum_{n\leq T} \frac{-\log n}{n^s} + \frac{T^{1-s}(-\log T)}{s-1} + T^{1-s}\frac{-1}{(s-1)^2} +
\end{aligned}$$

The rest is missing, check the lecture note. $\square$

Theorem 2.19. *Let $f : n \to \mathbb{C}$ be an analytic, Λ be a convex open set. Let $a, b \in \Lambda$, then ther eexists $z_1, z_2 \in (a, b)$ such that*

$$\Re(f'(z_1)) = \Re\left(\frac{f(b) - f(a)}{b - a}\right), \Im(f'(z_2)) = \Im\left(\frac{f(b) - f(a)}{b - a}\right).$$

Theorem 2.20. *Ther eexists constants c_1, c_2 such that*

$$1 - \frac{c_1}{(\log T)^9} \leq \sigma \leq 2, |\zeta(s)| > \frac{c_2}{(\log T)^7},$$

where $1 < |\Im(s) \leq T$.

Proof. Recall that

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1, \sigma > 1.$$

Thus we have,

$$|\zeta(\sigma + it)|^4 \geq |\zeta(\sigma)|^3 |\zeta(\sigma + 2it)|^{-1},$$

c_1 is some (missing) constant which will be chosen later.

Fix the domain II where $1 + \frac{c_1}{(\log T)^9} \leq \sigma \leq 2$, $1 \leq |t| \leq T$.

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{s+1}} du, \Re(s) > 0, s \neq 1. \zeta(\sigma) \quad = \frac{\sigma - 1 + 1}{\sigma - 1} - \sigma \int_1^\infty \frac{\{u\}}{u^{t+1}} dt.$$

$$\zeta(\sigma) = 1 + \frac{1}{\sigma - 1} + o(1),$$

$$\zeta(\sigma << \frac{1}{\sigma - 1}), \sigma \to 1^+,$$

$$\Rightarrow \zeta(\sigma)^{-1} >> (\sigma - 1), \sigma \to 1^+$$

We also have,

$$|\zeta(\sigma + 2it)| = o(\log T), 1 - \frac{1}{\log T}$$

□

Corollary 2.4. *There exists some constants C such that*

$$\frac{\zeta'(s)}{\zeta(s)} = o((\log T)^9).$$

For $1 - \frac{c}{(\log T)^9} \leq \Re(s) \leq 2$ and $1 \leq |\Im(s)| \leq T$, we have,

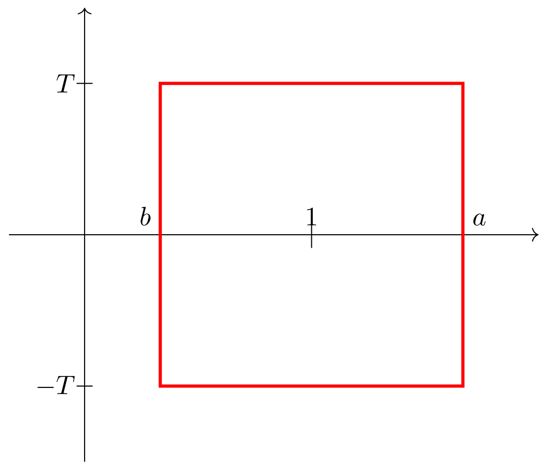
$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}.$$

Proof. Recall Cauchy's Residue theorem, suppose $f : \Omega \to \mathbb{C}$ is meromorphic where Ω is simply connected.

Example 2.11. *Halfplane* $\mathbb{R}^n$ *and* $\mathbb{R}^n \backslash \{0\}$, *for* $n \geq 3$ *and convex.*

For $a_i, 1 \leq i \leq n$ poles in $U \subseteq \Omega$ simple closed curve. Then,

$$\frac{1}{2\pi i} \int_U f(s) ds = \sum_{i=1}^{n} \operatorname{Res}(f, a_i) + \text{constant}\,.$$



where $a = \frac{1}{+} \frac{c}{(\log(T))^9}, b = \frac{1}{-} \frac{c}{(\log(T))^9}$ $\Re(s) > 0, s \neq 1, \Re(s) > b$. We claim that $\frac{\zeta'(s)}{\zeta(s)}$ can be analytically continued to $\Re(s) \geq b$ except simple pole at $s = i$ with residue -1(See lecture 5 for the justification of such poles).

$$\begin{aligned} \frac{1}{2\pi i} \int_{R_T} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds &= \operatorname{Res}_{s=1} \left(\frac{-\zeta'(s)}{\zeta(s)} \right), \\ &= x. \end{aligned}$$

Now consider R_T which is the closed path drew red in the graph.

$$\int_{R_T} = \int_{a-iT}^{a+iT} + \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT}.$$

We then have,

$$\frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = x - \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT}$$

$$
\begin{aligned}
\left|\frac{1}{2\pi i}\int_{a+iT}^{b+iT}\frac{-\zeta(s)}{\zeta(s)}\frac{x^s}{s}ds\right| &= \left|\frac{1}{2\pi i}\int_a^b\frac{-\zeta(u+iT)}{\zeta(u+iT)}\frac{x^{(u+iT)}}{(u+iT)}du\right|, \\
&<< \int_a^b\left|\frac{-\zeta'(u+iT)}{\zeta(u+iT)}\right|\frac{x^u}{|u+iT|}du, \\
&\overset{\text{Using the previous theorem}}{<<} \frac{\log^9 T}{T}x^a\int_b^a du, \\
&<< \frac{x^a}{T}.
\end{aligned}
$$

Now we have,

$$
\begin{aligned}
\left|\int_{b-iT}^{b+iT}\frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds\right| &= \left|\frac{1}{2\pi}\int_{-T}^{T}\frac{\zeta'(b+iu)}{\zeta(b+iu)}\frac{x^{b+iu}}{b+iu}du\right|, \\
&<< \log^9 T\int_{-T}^{T}\frac{x^b}{\sqrt{b^2+u^2}}du, \\
&<< x^b\log^9 T\int_0^T\frac{du}{\sqrt{b^2+u^2}}du, \\
&<< x^b\log^9 T\int_1^T\frac{du}{u}, \\
&= x^b\log^{10} T.
\end{aligned}
$$

Now back to the beginning,

$$
\psi(x) = x + o(\frac{x^a}{T} + x^b\log^{10} T) + o(\frac{x\log^2 x}{T} + \frac{x^a}{T}\log^9 T).
$$

Choose T to be such that $2c\log x = \log^{10} T, x = e^{\frac{\log^{10} T}{x}}$.

$$
x^{\frac{c}{\log^9 T}} = e^{\frac{\log T}{2}} = \sqrt{T}.
$$

$$
\begin{aligned}
x^{1-\frac{c}{\log^9 T}}\cdot\log^{10} T + \frac{x\log^2 x}{T} + \frac{x^{1+\frac{c}{\log^9 T}}}{T}\log^9 T \\
&= x\cdot T^{-\frac{1}{2}}\log^1 0T + \frac{x\log^2 x}{T} + x\frac{\sqrt{T}}{T}\log^{10} T\left(\frac{x}{\sqrt{T}}\right)(\log^{10} T + \\
&<< \frac{x}{T^s} = xe^{-c(\log x)^{\frac{1}{10}}}.
\end{aligned}
$$

□

Definition 2.14 (Gamma function)**.** *We define the Gamma function* $\Gamma : \mathbb{C} \to \mathbb{C}$ *as*

$$
\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt, \sigma > 0.
$$

Remark 2.18.

$$\begin{aligned}
|\Gamma(s)| &\leq \int_0^\infty |t^{s-1}|e^{-t}dt, \\
&= \int_0^\infty t^{\sigma-1}e^{-t}dt, \\
&= \left(\int_0^1 + \int_1^\infty\right) t^{\sigma-1}e^{-t}dt, \\
&\leq \int_0^1 t^{\sigma-1}dt + \int_1^\infty t^{\sigma-1}e^{-t}dt, \left(\frac{t^n}{e^t} \stackrel{t\to\infty}{\to} 0\right). \\
&<< \frac{1}{\sigma} + \int_1^\infty t^{\sigma-1}t^{-2\sigma}dt, \\
&<< \frac{1}{\sigma}.
\end{aligned}$$

Theorem 2.21.

$$F(s) = \int f(s,t)dt.$$

$F : \Omega \to \mathbb{C}$ is analytic in Ω if

i). $f(s,t)$ is continuous in (s,t),

ii). $f(s,t)$ is analytic in s,

iii). $\int f(s,t)$ is uniformly bounded on compact subsets of Ω.

In Remark 2.18, suppose $a \leq \Re(s) \leq b$ then the last two inequalities will be,

$$\begin{aligned}
&\int_0^1 t^{\sigma-1}dt + \int_1^\infty t^{\sigma-1}e^{-t}dt \\
&\qquad\qquad <<_b \frac{1}{\sigma} + \int_1^\infty t^{b-1}t^{-2b}dt, \\
&\qquad\qquad <<_b \frac{1}{\sigma} + \frac{1}{b} = \frac{1}{a} + \frac{1}{b}.
\end{aligned}$$

Thus we observe that in $\sigma > 0$, using integration by parts,

$$\Gamma(s+1) = s\Gamma(s).$$

Thus for any $n \in \mathbb{N}$, we have $\Gamma(n) = n!$. We have,

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \sigma > 0.$$

Note $\Gamma(s+1)$ is analytic if $\sigma > -1$. Thus $\Gamma(s)$ is analytic in $\sigma > -1$ except simple pole at $s = 0$ with residue 1. That is

$$\lim_{s\to 0} s\Gamma(s) = 1.$$

We also have,

$$\Gamma(s) = \frac{\Gamma(s+2)}{\Gamma(s+1)}, \sigma > 0.$$

Note that in the numerator, we have $\sigma > -2$. $\Gamma(s)$ is analytic when $\sigma > -2$ except simple poles at $s = 0, -1$ with residue

$$\lim_{s \to -1} (s+1)\Gamma(s) = \frac{\Gamma(-1)}{-1} = -1.$$

Iterate the process, we have the following theorem,

Theorem 2.22. $\Gamma(s)$ *can be analytically continued to* $\mathbb{C}$ *except simple poles at* $s = -k$ *where* $k \in \mathbb{Z}_{\geq 0}$ *with residue* $\frac{(-1)^k}{k!}$.

Remark 2.19.

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, s \in \mathbb{C}.$$

$$\lim_{s \to m} \Gamma(s)\Gamma(1-s) \sin \pi s = \pi s . \Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = \sqrt{\pi} 2^{1-2s} \Gamma(2s), \forall s \in \mathbb{C}.$$

Exercise 2.5. *Show*

$$\sum_{n \in \mathbb{Z}} e^{-(n+\alpha)\frac{\pi}{x}} = \sqrt{x} \sum_{n \in \mathbb{Z}} e^{-n^2 \pi x + 2\pi i n \alpha}, \forall \alpha \in i\mathbb{R}, x > 0.$$

Note that $\sum_{n \in \mathbb{Z}} a_n$ *is convergent if*

$$s_N := \sum_{|n| \leq N} a_n,$$

is convergent.

Theorem 2.23. $\Gamma(s) \neq 0, \forall s \in \mathbb{C}$.

Proof. Note that the case when $s \in \mathbb{Z}$ is already shown. Thus suppose $s \notin \mathbb{Z}$. If possible $\Gamma(s) = 0$. Then it will follow that $\Gamma(1-s)$ has a pole which is a contradiction. $\frac{1}{\Gamma(s)}$ has a simple zero at $s = 0, -1, -2, \cdots$.

Step 1

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t} s^{\frac{s}{2}-1} dt, \sigma > 0.$$

Replace $t = n^2\pi x, dt = n^2\pi dx$, we get,

$$
\begin{aligned}
\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)n^{-s} &= \int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx, \sigma > 1. \\
\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \sum_{n\in\mathbb{N}}\int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx, \\
&= \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n\in\mathbb{N}}e^{-n^2\pi x}\right)dx, \\
&= \sum_{n\in\mathbb{N}}\int_0^\infty |x^{\frac{s}{2}-1}e^{-n^2\pi x}|dx, \\
&= \sum_{n\in\mathbb{N}}\left(\int_0^\infty x^{\frac{\sigma}{2}-1}e^{-n^2\pi x}dx\right), \\
&= \sum_{n=1}^\infty \pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right)n^{-\sigma}, \\
&= \pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right)\zeta(s), \sigma > 1, \\
&< \infty . \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \qquad = \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n\in\mathbb{N}}e^{-n^2\pi x}\right).
\end{aligned}
$$

Step 2, using Poisson summation formula and $\sigma > 1$, let $F \in L^1(\mathbb{R})$, ie $F : \mathbb{R} \to \mathbb{C}$ and

$$\int_{-\infty}^\infty |F(t)|dt < \infty.$$

We have,

$$\sum_{n\in\mathbb{Z}} F(n+u)$$

is absolutely and uniformly convergent in u. Also we have,

$$\sum_{n\in\mathbb{Z}} |\hat{F}(n)| \leq \infty,$$

then

$$\sum_{n\in\mathbb{Z}} F(n+u) = \sum_{n\in\mathbb{Z}} \hat{F}(n)e^{2\pi inu}.$$

Using Exercise 2.5 and set $\alpha = 0$, we have,

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}} e^{-n^2\frac{\pi}{2}} &= \sqrt{s}\sum_{n\in\mathbb{Z}} e^{-n^2\pi x}.\sum_{n\in\mathbb{Z}} e^{-n^2\pi x} \qquad << \sum_{n\in\mathbb{N}} e^{-n\pi x}, \\
&<< \int_1^\infty e^{-\pi xt}dt, \\
&<< \frac{e^{-\pi x}}{x}, x > 0.
\end{aligned}
$$

Set $\theta :== \sum_{n\in\mathbb{Z}} e^{n^2\pi x}$ then,

$$\theta(x) = 1 + 2\sum_{n\in\mathbb{N}} e^{-n^2\pi x}.$$

$$\sum_{n\in\mathbb{N}} e^{-n^2\pi x} = \frac{\theta(x)-1}{2}, x > 0.$$

Using the exercise, we have,

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\theta(x).$$

Set $w(x) := \frac{\theta(x)-1}{2}$. Thus write

$$\begin{aligned} w\left(\frac{1}{x}\right) &= \frac{\theta\left(\frac{1}{x}\right)-1}{2}, \\ &= \frac{\sqrt{x}\theta(x)-1}{2}, \\ &= \frac{\sqrt{x}\theta(x)-1}{2}. \\ w\left(\frac{1}{x}\right) &= \sqrt{x}w(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}. \\ \pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1}w(x)dx, \sigma > 1. \end{aligned}$$

Step 3

$$\int_0^1 x^{\frac{s}{2}-1}w(x)dx + \int_1^\infty x^{\frac{s}{2}-1}w(x)dx.$$

Taking $x = \frac{1}{y}$ and $dx = -\frac{1}{y^2}dy$ we have,

$$\begin{aligned} \int_1^\infty \left(\frac{1}{y^2}\right)^{\frac{s}{2}-1} w\left(\frac{1}{y}\right)\frac{-1}{y^2}dy &= \int_1^\infty y^{-\frac{s}{2}} w\left(\frac{1}{y}\right)\frac{dy}{y}, \\ &= \int_1^\infty y^{-\frac{1}{2}}\left(\sqrt{y}w(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right)\frac{dy}{y}, \\ &= \int_1^\infty y^{-\frac{s}{2}+\frac{1}{2}}w(y)\frac{dy}{y} + \int_1^\infty \frac{y^{-\frac{s}{2}+\frac{1}{2}-1}}{2}dy - \frac{1}{2}\int_1^\infty y^{-\frac{s}{2}-1}dy, \\ &= \int_1^\infty y^{\frac{1-s}{2}}w(y)\frac{dy}{y} + \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}, \sigma > 1. \end{aligned}$$

Step 4,

$$\int_1^\infty |x^{\frac{s}{2}} + x^{\frac{1-s}{2}}||w(x)|\frac{dx}{x} \leq \int_1^\infty (x^{\frac{\sigma 2}{-}1} + x^{\frac{1-\sigma}{2}-1})|w(x)|dx,$$
$$<< \int_1^\infty \frac{(x^{\frac{\sigma}{2}-1} + x^{\frac{1-\sigma}{2}-1})}{e^{\pi x}}dx, \sigma \in \mathbb{R},$$
$$<< \int_1^\infty \frac{e^x}{e^{\pi x}}dx, \qquad << 1.$$

$\int_1^\infty |x^{\frac{s}{2}} + x^{\frac{1-s}{2}}||w(x)|\frac{dx}{x}$ is analytic in $\mathbb{C}$.

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = 1 + s(s-1)\int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}.$$

Set $\xi(x) := 1 + s(s-1)\int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}$, then it is entire. and for all s we have,

$$\xi(1-s) = \xi(s).$$

Thus obtain,

$$(1-s)(1-s-1)\pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Using the construction of $w(s)$ we have,

$$|w(x)| << |\theta(x)| << \frac{e^{-\pi x}}{x}, \forall x > 0.$$

Also we have,

$$\zeta(1-s) = \pi^{-s+\frac{1}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s),$$
$$\zeta(1-s) = \pi^{-s}2^{1-s}\cos\left(\frac{\pi s}{s}\right)\Gamma(s)\zeta(s).$$

Note that $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ except simple pole at s.

$$\lim_{s\to 1}\zeta(1-s) = \pi^{-1}\lim_{s\to 1}\frac{\cos\frac{\pi s}{2}\zeta(s)(s-1)}{s-1},$$

$$\zeta(-2n) =$$

Recall

$$\zeta(s) = \sum_{n\in\mathbb{N}}\frac{1}{n^s},$$
$$= \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}dt, \sigma > 0, s \neq 1.$$

If s is real,

$$|\zeta(s) - \frac{s}{s-1}| \leq |s| \int_1^\infty \frac{\{t\}}{t^{\sigma+1}} dt, \qquad < \frac{|s|}{\sigma} = \frac{\sigma}{\sigma} = 1.$$

Thus we obtain,

$$-1 + \frac{s}{s-1} < \zeta(s) < 1 + \frac{s}{s-1}$$
$$\frac{1}{s-1} < \zeta(s) < \frac{2s-1}{s-1},$$
$$-1 < (1-s)\zeta(s) < 1 - 2s < 0 \quad \text{if } \frac{1}{2} < s < 1$$
$$\Rightarrow \zeta(s) \neq 0, \quad \text{if } \frac{1}{2} < s < 1.$$

□

Notation 2.3. *Given $\chi \mod q$, we set,*

$$\tau(\chi) = \sum_{k=1}^{q} \chi(k) e^{\frac{2\pi i k}{q}}.$$

2.4 Primitive characters

Definition 2.15. *A character χ is called primitive if its conductor is*

Example 2.12. *Take $\chi : (\mathbb{Z}/8\mathbb{Z})^\times \to \mathbb{C}^times$.*

Lemma 2.8. *Suppose $x \in \left[0, \frac{1}{2}\right]$, then*

$$|\sin \pi x| \geq 2x.$$

Proof. Exercise. □

Lemma 2.9. *For $n \in \mathbb{Z}$, we have,*

$$\chi(n)\tau(\overline{\chi}) = \sum_{k=1}^{q} \overline{\chi}(k) e^{\frac{2\pi i k}{q}},$$

where χ is a primitive character modulo q and $\chi \neq \chi_0$.

Proof.

$$\overline{\chi(n)\tau(\overline{\chi})} = \sum_{l=1}^{q} \chi(l) e^{-\frac{2\pi i l}{q}}.$$

Multiplying this equation with the one in the statement we have,

$$|\chi(n)|^2 |\tau(\overline{\chi})|^2 = \sum_{k,l=1}^{q} \overline{\chi(k)} \chi(l) e^{\frac{2\pi i (k-l)}{q}}.$$

Applying $\sum_{n\le x}$ to the both sides of the above equation, we have,

$$|\tau(\overline{\chi})|^2 \sum_{n\le x} |\chi(n)|^2 = \sum_{k,l=1}^{q} \overline{\chi(k)}\chi(l) \left(\sum_{n\le x} \left(e^{\frac{2\pi i(k-l)}{q}} \right)^n \right).$$

Note that

$$x + x^2 + \cdots + x^q = \begin{cases} \frac{x(x^q-1)}{x-1}, (x \neq 1), \\ q, (x = 1), \end{cases}$$

$$= \begin{cases} 0, (x \neq 1, x^q = 1), \\ q, (x = 1), \end{cases}$$

Take $x = e^{\frac{2\pi i(k-l)}{q}} = \cos\frac{2\pi}{q}(k-l) + i\sin\frac{2\pi}{q}(k-l) = 1$. $x = 1$ if and only if $q|k-l$ thus

$$\sum_{n\le x} \left(e^{\frac{2\pi i(k-l)}{q}} \right)^n = \begin{cases} 0, (q \not| k-l), \\ q, (\text{otherwise}). \end{cases}$$

Therefore, we get,

$$|\tau(\overline{\chi})|^@ \sum_{n\le x} |\chi(n)|^2 = q \sum_{\substack{k,l=1 \\ q|k-l}}^{q} \overline{\chi}(k)\chi(l),$$

$$= q\sum_{k=1}^{q} \overline{\chi}(k)\chi(k),$$

$$= q\sum_{k=1}^{q} |\chi(k)|^2.$$

Therefore $|\tau(\overline{\chi})|^2 = q, |\tau| = \sqrt{q}$.

Consider $(n,q) = 1$, then

$$\chi(n)\tau(\overline{\chi}) = \chi(n)\sum_{k=1}^{q} \overline{\chi}(k)e^{\frac{2\pi ik}{q}},$$

$$= \chi(n) \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \overline{\chi}(k)\chi(k)e^{\frac{2\pi ik}{q}},$$

Set $k = nt$, we get,

$$\chi(n)\tau(\overline{\chi}) = \sum_{\substack{t=1 \\ (t,q)=1}}^{q} \overline{\chi}(nt)\chi(n)e^{\frac{2\pi int}{q}}.$$

$$\overline{\chi}(nt) = \overline{\chi}(n)\overline{\chi}(t).$$

$$\chi(n)\tau(\overline{\chi}) = \sum_{\substack{t=1 \\ (t,q)=1}}^{q} \overline{\chi}(t)e^{\frac{2\pi int}{q}}.$$

Observe that

$$\tau(\overline{\chi})\left(\sum_{n\le x}\chi(n)\right) = \sum_{k=1}^{q-1}\overline{\chi}(k)\left(\sum_{n\le x}e^{\frac{2\pi ikn}{q}}\right),$$

$$|\tau(\overline{\chi})|\cdot|\sum_{n\le x}\chi(n)| \le \sum_{k=1}^{q-1}\left|\sum_{n\le x}e^{\frac{2}{\pi}iknq}\right|,$$

$$= \sum_{k=1}^{q-1}\left|\frac{e^{\frac{2\pi ik}{q}}\left(e^{\frac{2\pi in[x]}{q}}-1\right)}{e^{\frac{2}{\pi}ikq}-1}\right|,$$

$$\le \sum_{k=1}^{q-1}\frac{2}{|e^{\frac{2\pi k}{q}}-1|},$$

Note that for all $y \in i\mathbb{R}$,

$$2i\sin y = e^{-iy}(e^{2iy}-1),$$

$$|2\sin y| = |e^{2iy}-1|.$$

Apply this to the equation above we have,

$$\sum_{k=1}^{q-1}\frac{2}{|e^{\frac{2\pi k}{q}}-1|} = \sum_{k=1}^{q-1}\frac{1}{|\sin\frac{\pi k}{q}|}.$$

Using the lemma, we get,

$$\sum_{k=1}^{q-1}\frac{1}{|\sin\frac{\pi k}{q}|} = \sum_{1\le k\le\frac{q}{2}}\frac{1}{|\sin\frac{\pi k}{q}|} + \sum_{\frac{q}{2}<k}^{q-1}\frac{1}{|\sin\frac{\pi k}{q}|}. \qquad = 2\sum_{1\le k\le\frac{q}{2}}\frac{1}{|\sin\frac{\pi k}{q}|},$$

$$\le \sum_{1\le k\le\frac{q}{2}}\frac{k}{q} << q\log\frac{q}{2} << q\log q.$$

Thus we conclude $|(\overline{\chi}) << \sqrt{q}$ if χ is primitive and $q > 1$, we have,

$$\left|\sum_{n\leq x} \chi(n)\right| << \sqrt{q}\log q,$$

uniformly in q as $x \to \infty$. $\square$

$L(q, \chi)$ functional equation, where $\chi \neq \chi_0$ and χ is primitive. Suppose $\chi(-1) = 1$ that is it is an even character. From previous discussion, we have,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)n^{-s} = \int_0^\infty x^{\frac{s}{2}}e^{-n^2\pi x}\frac{dx}{x}.$$

Replace x with $\frac{x}{q}$ and $\sigma > 0$, we habe,

$$\pi^{-\frac{s}{2}}q^{\frac{s}{2}}\Gamma\left(\frac{s}{1}\right)n^{-s} = \int_0^\infty x^{\frac{s}{2}}e{-}n^2\pi x\frac{dx}{x}, \sigma > 0.$$

For $\sigma > 1$, we have,

$$\begin{aligned}\pi^{-\frac{s}{2}}q^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi) &= \sum_{n=1}^\infty \int_0^\infty \chi(n)x^{\frac{s}{2}}e^{-n^2\frac{\pi x}{q}}\frac{dx}{x}, \\ &= \int_0^\infty x^{\frac{s}{2}}\left(\sum_{n=1}^\infty \chi(n)e^{-n^2\frac{\pi x}{q}}\right)\frac{dx}{x}, \sigma > 1.\end{aligned}$$

Let

$$\theta(x,\chi) = \sum_{n\in\mathbb{Z}} \chi(n)e^{-\frac{n^2\pi x}{q}} = 2\sum_{n=1}^\infty \chi(n)e^{-\frac{n^2\pi x}{q}}, (x > 0).$$

Using this, we get,

$$\int_0^\infty x^{\frac{s}{2}}e{-}n^2\pi x\frac{dx}{x} = \frac{1}{2}\int_0^\infty x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}, (\sigma > 1).$$

Split the integral into $\int_0^1, \int_1^\infty$.

$$\begin{aligned}
\tau(\overline{\chi})\theta(x,\chi) &= \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1},\overline{\chi}),\\
&= \sum_{n\in\mathbb{Z}} (\tau(\overline{\chi})\chi(n)) e^{\frac{-n^2\pi x}{q}},\\
&= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{q} \overline{\chi}(k) e^{\frac{2\pi i n k}{q}} e^{-\frac{n^2\pi x}{q}},\\
&= \sum_{k=1}^{q} \overline{\chi(k)} \sum_{n\in\mathbb{Z}} e^{\frac{2\pi k n}{q} - n^2 \frac{\pi x}{q}},\\
&\overset{\text{From Lecture 7 page 6}}{=} \sum_{k=1}^{q} \overline{\chi}(k) \left(\frac{x}{q}\right)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} e^{-\left(m+\frac{m}{q}\right)\frac{\pi q}{x}},\\
&= \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^{q} \overline{\chi}(k) \sum_{n\in\mathbb{Z}} e^{-\left(m+\frac{m}{q}\right)\frac{\pi q}{x}}.
\end{aligned}$$

Put $qn+m=t$, then

$$\overline{\chi}(qn+m) = \overline{\chi}(m) = \overline{\chi}(t).$$

Thus,

$$\left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^{q} \overline{\chi}(k) \sum_{n\in\mathbb{Z}} e^{-\left(m+\frac{m}{q}\right)\frac{\pi q}{x}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{t\in\mathbb{Z}} \overline{\chi}(t) e^{-\frac{t^2\pi}{qx}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1},\overline{\chi}).$$

Now for the splitted integral, we have,

$$\frac{1}{2}\int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}\frac{dx}{x}.$$

Replacing x with $\frac{1}{x}$, we get,

$$= \frac{\tau(\chi)}{2\sqrt{q}} \int_1^\infty x^{\frac{1-s}{2}}\theta(x,\overline{\chi})\frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}.$$

Set

$$\xi(s,\chi) := \frac{\tau(\chi)}{2\sqrt{q}} \int_1^\infty x^{\frac{1-s}{2}}\theta(x,\overline{\chi})\frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}}\theta(x,\chi)\frac{dx}{x}.$$

Use that $\theta(x,\overline{\chi}) << e^{-\frac{\pi x}{q}}$ and $|\tau(\chi)| = \sqrt{q}$. It turns out that ξ is uniformly bounded on compact subsets of $\mathbb{C}$ and in particular, this is entire.

Lemma 2.10.

$$\xi(1-s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\xi(s,\overline{\chi}).$$

Proof. Recall,

$$\tau(\overline{\chi})\theta(x,\chi) = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1},\overline{\chi}),$$

and

$$|\tau(\chi)|^2 = q.$$

We get,

$$\frac{\tau(\chi)}{\sqrt{q}} = \frac{\sqrt{q}}{\overline{\tau(\chi)}} = \frac{\sqrt{q}}{\tau(\overline{\chi})}.$$

Then,

$$L(1-s,\chi) = \frac{\tau(\chi)}{q^{1-s}}\pi^{-s}2^{1-s}\cos\frac{\pi s}{2}(\Gamma(s))L(s,\overline{\chi}).$$

Note that $\chi(-1) = 1$ and is primitive (in the case of odd integer replace cos with sin). Note also that

i). $L(s,\chi) = 0, s = 0,-2,-4,\cdots$ when χ is even,

ii). $L(s,\chi) = -1,-3,-5,\cdots$ when χ is odd.

When $0 < \sigma < 1$, □