

# V5A10 Analytic Number Theory

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## 1 Classical Number Theory

**Theorem 1.1** (Euclid). *There are infinitely many prime numbers.*

**Definition 1.1.**  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  be a function such that

$$\pi(n) = \{\text{prime numbers less than } n\}.$$

**Remark 1.1.**

$$\frac{\pi}{n \ln(n)} \approx 1.$$

**Definition 1.2.**

$$\text{Li}(x) = \int_0^x \frac{1}{\ln(t)} dt.$$

**Notation 1.1.** Given  $f, g : \mathbb{R} \rightarrow \mathbb{C}$ ,

$$f(x) = O(g(x))$$

means that

$$\exists K \in (0, \infty), x_0 \in \mathbb{R}, \text{ s. t. } \forall x > x_0, |f(x)| \leq K|g(x)|.$$

**Notation 1.2.** Let  $f, g : \mathbb{R} \rightarrow \mathbb{C}$  be functions.  $f \sim g$  denotes that

$$\lim_{n \rightarrow \infty} \frac{f(x)}{g(x)} = c,$$

for some constant  $c$ .

**Notation 1.3.** Let  $f : \mathbb{R} \rightarrow \mathbb{C}$ ,

$$\text{Li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k},$$

denotes that

$$\text{Li}(x) = \frac{x}{\ln x} \sum_{k=1}^{N-1} \frac{k!}{(\ln x)^k} + O\left(\frac{x}{(\ln x)^{N+1}}\right).$$

and as  $x \rightarrow \infty$ , this holds for any  $N \geq 1$ .

**Remark 1.2.** By the integration by parts, we see that it's asymptotic expansion is

$$\text{Li}(x) \approx \frac{x}{\ln(x)} \sum_{k=0}^{\infty} \frac{k!}{(\ln(x))^k}.$$

**Theorem 1.2** (Prime Number Theorem).

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{\text{Li}(x)} = 1.$$

**Definition 1.3** (First Chebyschev Function).

$$\vartheta(x) = \sum_{p \leq x} \ln p.$$

**Definition 1.4** (Second Chebyschev Function).

$$\psi(x) = \sum_{\substack{m,p \\ p^m \leq x}} \ln p.$$

**Remark 1.3.** We can rewrite the second Chebyschev function as follows.

$$\psi(x) = \sum_{\substack{p \leq x \\ m=\max\{n \in \mathbb{N} \mid p^n \leq x\}}} m \ln p.$$

**Definition 1.5** (Möbius Function).

$$\mu(n) = \begin{cases} 1 & (n = 1) \\ (-1)^k & (n = p_1 \cdots p_k, p_i = p_j \Rightarrow i = j) \\ 0 & (\exists p \text{ s. t. } p^2 | n). \end{cases}$$

**Remark 1.4.** The prime number theorem is equivalent to the following statements.

1).  $\psi(x) \sim x$ .

2).  $\theta(x) \sim x$ .

3).  $\lim_{x \rightarrow \infty} \frac{\sum_{\substack{n \leq x \\ \mu(n)}}}{n} x = 0$ .

**Conjecture 1.1** (Twin Prime Conjecture). There exists infinitely many primes  $p$  such that  $p + 2$  is also prime.

**Conjecture 1.2** (Goldbach's Conjecture). Let  $n \in \mathbb{N}$  be an even number greater than 2, then there exists two primes  $p, q$  such that  $n = p + q$ .

**Conjecture 1.3** (Hardy-Littlewood Conjecture).

$$\#\{ \text{prime numbers } p \text{ such that } 2p + 1 \text{ is also a prime and } p < x \}$$

**Definition 1.6** (Riemann-Zeta Function). We define  $\zeta : \mathbb{C} \rightarrow \mathbb{C}$  such that

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}.$$

**Remark 1.5.** Supposer  $\operatorname{Re}(s) > 1$ , then we have

$$\begin{aligned} |\zeta(s)| &= \sum_{n \in \mathbb{N}} \frac{1}{|n|^s} \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n^{\operatorname{Re}(s)}} \end{aligned}$$

By multiplying  $\frac{1}{2^s}$ , we obtain

$$\frac{1}{2^s} \zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{(2n)^s}.$$

We get

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{(2n+1)^s}.$$

Continuing this procedure, we get the following proposition.

**Proposition 1.1.**

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)$$

**Theorem 1.3** (Weierstrass). Let  $A \subseteq \mathbb{C}$  and consider a sequence of functions  $(f_n : A \rightarrow \mathbb{C})_{n \in \mathbb{N}}$  such that there exists a sequence of non-negative numbers  $(M_n)_{n \in \mathbb{N}}$  such that

i).  $\forall x \in A, |f_n(x)| \leq M_n$ .

ii).  $\sum_{n \in \mathbb{N}} M_n$  converges.

Then the sequence converges uniformly.

**Theorem 1.4.** Suppose the conditions in the previous theorem. If each function is analytic on a compact subset of  $A$ , then the limit is also analytic.

**Corollary 1.1.** Let  $A$  be a compact subset of a complex plane where  $\operatorname{Re}(s) > 1$ . Then there exists  $\delta > 0$  such that  $\operatorname{Re}(s) > 1 + \delta$  and

$$\sum_{n \in \mathbb{N}} \left| \frac{1}{n^s} \right| \leq \sum_{n \in \mathbb{N}} \frac{1}{n^{1+\delta}} < \infty.$$

**Fact 1.1.** The Riemann zeta function can be analytically continued to the whole plane except for  $s = 1$ .

**Definition 1.7** (Gamma function). *We define the Gamma function  $\Gamma : \mathbb{C} \rightarrow \mathbb{C}$  as*

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt, \operatorname{Re}(s) > 0.$$

**Proposition 1.2.**

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

**Remark 1.6.**  $\zeta(1+it) \neq 0$  if  $t \in \mathbb{R}, t \neq 0$ .  $\zeta(s) \neq 0$  for  $0 < s < 1$ .

**Remark 1.7** (Functional Equation).

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

**Remark 1.8.**  $\Gamma(s)$  is defined for  $\operatorname{Re}(s) > 0$  and can be analytically continued to the whole place except for  $\mathbb{C} \setminus \{-n \mid n \in \mathbb{Z}_{\geq 0}\}$ .

**Remark 1.9.** For  $s = -2m$  where  $m \in \mathbb{N}$ , we see  $\zeta(s) = 0$ .

$$\begin{aligned} \zeta(0) &= \frac{2}{2\pi} \lim_{s \rightarrow 1} \cos\left(\frac{\pi s}{2}\right) \zeta(s) \\ &= \frac{1}{\pi} \lim_{s \rightarrow 1} \frac{\cos(\frac{\pi s}{2})}{s-1} \lim_{s \rightarrow 1} (s-1)\zeta(s) \\ &= \frac{1}{\pi} \times \frac{-\pi}{2} \times 1 \\ &= -\frac{1}{2}. \end{aligned}$$

**Definition 1.8.** The critical strip is the subset of the complex plane with its real part between 0 and 1. The critical line is the line where  $\operatorname{Re}(s) = \frac{1}{2}$ .

**Conjecture 1.4** (Riemann Hypothesis). Let  $s$  be an element of the critical strip. If  $\zeta(s) = 0$  then  $\operatorname{Re}(s) = \frac{1}{2}$  (i.e. it lies on the critical line).

**Notation 1.4.** Let  $T > 0$ . We denote  $N(T)$  the number of zeros of  $\zeta$  in the critical strip whose coefficient of the imaginary part is in  $(0, T)$ . That is

$$N(T) = |\{\sigma + it \in \mathbb{C} \mid 0 < \sigma < 1, 0 < t < T\}|.$$

**Proposition 1.3.**

$$\lim_{T \rightarrow \infty} \frac{N(T)2\pi}{T \log(T)} = 1.$$

*Sketch of Proof (needs refinement).*

$$\psi(x) = \frac{1}{2\pi i} \int_l \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds$$

where  $l$  is the line  $l = a$  for some  $a > 1$ .

$$\psi(s) = x - \sum_{\rho \text{ non-trivial zeros}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \log(1 - x^{-2}).$$

□

**Definition 1.9.** Let  $q \in \mathbb{N}$  and  $a$  be a natural number coprime to  $q$ . We define

$$\pi(x; q, a) = |\{\text{prime numbers } p \text{ less than or equal to } x \text{ such that } p \equiv a \pmod{q}\}|$$

**Proposition 1.4.**

$$\pi(x; , q, a) \sim \frac{x}{\varphi(q) \log(x)}$$

where  $\varphi$  is a Euler phi-function.

**Theorem 1.5** (Brun–Titchmarsh). For any  $q < x$ , we have

$$\pi(x; , q, a) < \frac{2x}{\varphi(q) \log(\frac{x}{q})}.$$

**Remark 1.10.**

$$\text{Li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k}.$$

Indeed we have

$$\text{Li}(x) = \int_2^x \frac{dt}{\ln t}.$$

Observe that

$$\int_2^t \frac{1}{(\ln t)^N} \sim \frac{x}{(\ln x)^N}$$

for all  $N \geq 1$ . Thus  $\text{Li}(x)$  can be expressed in terms of polynomials in  $\frac{x}{\ln(x)}$ , by keep replacing the greatest term with the above approximation.

## 2 Arithmetic Functions

### 2.1 Multiplicative Functions

**Definition 2.1.** A function  $f : \mathbb{N} \rightarrow \mathbb{C}$  is said to be

- 1). multiplicative if for any  $(m, n) = 1$ , we have  $f(mn) = f(m)f(n)$ ,
- 2). completely multiplicative if for any natural numbers  $m, n$ , we have  $f(mn) = f(m)f(n)$ .

**Example 2.1.** Möbius function  $\mu$  is multiplicative.

**Definition 2.2** (Von-Mangoldt Function). *The Von-Mangoldt function  $\Lambda : \mathbb{N} \rightarrow \mathbb{C}$  is defined as*

$$\Lambda(n) = \begin{cases} \log(p) & (n = p^k \text{ for some } k \geq 1), \\ 0 & (\text{otherwise}). \end{cases}$$

**Definition 2.3** (Euler Phi Function). *The Euler phi function is  $\varphi : \mathbb{N} \rightarrow \mathbb{C}$  such that*

$$\varphi(n) = \{1 \leq a \leq n \mid (a, n) = 1\}.$$

**Example 2.2.**  $\varphi$  is multiplicative but  $\Lambda$  is not.

**Definition 2.4** (Dirichlet Characters Modulo  $q$ ). *Let  $q \in \mathbb{N}$  be a natural number and  $q \geq 2$ .*

$$\chi_1 : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$$

*be a group homomorphism. The Dirichlet character function modulo  $q$  with respect to  $\chi_1$  is such that*

$$\chi(n) = \begin{cases} \chi_1(\bar{n}) & ((n, q) = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

**Example 2.3.** For  $q = 3$ , we have  $(\mathbb{Z}/3\mathbb{Z})^\times = \{\pm 1\}$ . The only possible character is  $\pm 1 \mapsto \pm 1$ . Therefore, we have

$$\chi(1) = 1, \chi(2) = -1, \chi(0) = 0.$$

**Theorem 2.1.**

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1, \\ 0 & (\text{otherwise}). \end{cases}$$

*Proof.* When  $n = 1$ , this is trivial. Suppose  $n \neq 1$ . We factorize  $n$  by

$$n = \prod_{i=1}^k p_i^{\alpha_i}$$

where  $p_i$  is a prime and  $\alpha_i \in \mathbb{N}$  for each  $i = 1, \dots, n$ .

Observe that

$$\sum_{d|n} \mu(d) = \sum_{d|\prod_{i=1}^k p_i} \mu(d).$$

Now we see

$$\sum_{d|\prod_{i=1}^k p_i} \mu(d) = \sum_{j=0}^k \binom{k}{j} (-1)^j = \sum_{j=0}^k \binom{k}{j} (1)^{k-j} (-1)^j = (1 - 1)^k = 0.$$

□

**Proposition 2.1** (Möbius Inversion Formula). *Let  $f, g : \mathbb{N} \rightarrow \mathbb{C}$  be functions (we do not assume them to be multiplicative). If*

$$\sum_{d|n} g(d) = f(n),$$

*holds if and only if*

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = g(n).$$

*Proof.*

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \sum_{e|\frac{n}{d}} g(e).$$

$e|\frac{n}{d}$  if and only if  $de|n$  thus obtain,

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \sum_{de|n} g(e).$$

In particular, we get the expression

$$= \sum_{de|n} \mu(d) g(e).$$

By reordering, we get

$$= \sum_{e|n} g(e) \sum_{d|\frac{n}{e}} \mu(d).$$

By Proposition 2.1, we get

$$\sum_{d|\frac{n}{e}} \mu(d) = 0$$

unless  $e = n$ . □

**Proposition 2.2.**

$$\sum_{d|n} \varphi(d) = n.$$

*Proof.* Consider  $(\mathbb{Z}/n\mathbb{Z})^\times$ . We know that

$$|(\mathbb{Z}/n\mathbb{Z})^\times| = \varphi(n).$$

Later □

**Theorem 2.2.**

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

*Proof.* Using Proposition 2.2, we have,

$$\sum_{d|n} \mu(d) \frac{n}{d} = \varphi(n).$$

Dividing both sides by  $n$  and observe that  $\mu(d) \neq 0$  if and only if  $d$  is a prime factor of  $n$ .

$$\begin{aligned} \frac{\varphi(n)}{n} &= \sum_{d|n} \frac{\mu(d)}{d}, \\ &= 1 \sum_{p|n} \frac{1}{p} + \sum_{p_1, p_2|n} \frac{1}{p_1 p_2} - \dots \end{aligned}$$

By the induction on the number of prime divisors of  $n$ , we get the statement.  $\square$

**Proposition 2.3.** *We have the following properties of  $\varphi$ .*

1).  $n|m \Rightarrow \varphi(n)|\varphi(m)$ .

2).  $\varphi(n)$  is even for  $n \geq 3$ .

3).  $\varphi(2n) = \begin{cases} 2\varphi(n), & (2|n) \\ \varphi(n), & (2 \nmid n). \end{cases}$

4).  $\varphi$  is multiplicative.

5).  $\varphi(mn) = \varphi(m) \frac{\varphi(n)d}{\varphi(d)}$  where  $d = (m, n)$ .

6).  $\varphi(n^m) = n^{m-1}\varphi(n)$ .

*Proof.* Exercise.  $\square$

**Theorem 2.3.** *The following statements are equivalent.*

1).  $\sum_{d|n} \Lambda(d) = \log n$

2).  $\sum_{d|n} \mu(d) \log d = \Lambda(n)$ .

And in particular  $\sum_{d|n} \Lambda(d) = \log n$  holds.

*Proof.* The equivalence is a direct corollary of Möbius inversion formula (ie. Proposition 2.1). For the latter, Write

$$n = \prod_{i=1}^k p_i^{\alpha_i}.$$

We have

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^k \alpha_i \log p_i = \log(n).$$

$\square$

**Notation 2.1.** Let  $n \in \mathbb{N}$ , suppose a prime  $p$  divides  $n$ . Then we denote  $\alpha(p)$  to be the highest prime power factor of  $n$ .

**Theorem 2.4.** Let  $f : \mathbb{N} \rightarrow \mathbb{C}$  be a multiplicative function. Then

$$\sum_{d|n} f(d) = \prod_{p|n} \left( \sum_{i=0}^{\alpha(p)} f(p^i) \right).$$

In particular  $\sum_{d|n} f(d)$  is also multiplicative.

*Proof.* Let  $d|n$ , then we have  $d = \prod_{i=1}^k p_i^{\beta_i}$  for some  $0 \leq \beta_i \leq \alpha(p_i)$ . Since  $f$  is multiplicative we have

$$f(d) = \prod_{i=1}^k f(p_i^{\beta_i}).$$

The second part is a direct result of the first part.  $\square$

**Remark 2.1.** The Second Chebyschev Function  $\psi$  can be written as

$$\psi(x) = \sum_{d \leq x} \Lambda(d).$$

This follows directly from the definition.

**Definition 2.5** (Dirichlet Series). Let  $f : \mathbb{N} \rightarrow \mathbb{C}$  be a function and  $s \in \mathbb{C}$ . We define

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}.$$

For another arithmetic function  $g : \mathbb{N} \rightarrow \mathbb{C}$ , we define

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} + \sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} = \sum_{n \in \mathbb{N}} \frac{(f(n) + g(n))}{n^s}.$$

and

$$\begin{aligned} \left( \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \times \left( \sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} \right) &= \sum_{n, m \in \mathbb{N}} \frac{(f(n)g(m))}{(nm)^s}. \\ &= \sum_{t \in \mathbb{N}} \sum_{n|t} \frac{(f(n)g(\frac{n}{t}))}{t^s}. \end{aligned}$$

Recall the taylor expansion of  $\ln x$  we get

$$\ln 2 = \sum_{n \in \mathbb{N}} \frac{(-1)^{n+1}}{n}.$$

Rearranging the following way

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \dots$$

we get this equals to  $\frac{1}{2} \ln 2$ .

**Theorem 2.5.** Let  $s \in \mathbb{C}$  be  $\operatorname{Re}(s) > 1$ , we have

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}.$$

*Proof.*

$$\begin{aligned} \zeta(s) \sum_{n \geq 1} \frac{\mu(n)}{n^s} &= \left( \sum_{n \in \mathbb{N}} \frac{1}{n^s} \right) \left( \sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} \right) \\ &= \sum_{t \in \mathbb{N}} \frac{1}{t^s} \sum_{n|t} \mu(n) \\ &= 1. \end{aligned}$$

□

**Theorem 2.6.** For  $\operatorname{Re}(s) > 1$ , we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s}.$$

From this we derive

$$\lim_{n \rightarrow \infty} \frac{\log n}{n^\varepsilon} = 0.$$

*Proof.*

$$\begin{aligned} \zeta(s) \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} &= \left( \sum_{m \in \mathbb{N}} \frac{1}{m^s} \right) \left( \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} \right), \\ &= \sum_{t \in \mathbb{N}} \frac{1}{t^s} \left( \sum_{n|t} \Lambda\left(\frac{t}{n}\right) \right), \\ &= \sum_{t \in \mathbb{N}} \frac{\log(t)}{t^s}, \\ &= -\zeta'(s). \end{aligned}$$

□

**Remark 2.2.**

$$\begin{aligned} \sum_{n \in \mathbb{N}} \left| \frac{\Lambda(n)}{n^s} \right| &\leq \sum_{n \in \mathbb{N}} \frac{\log(n)}{n^\sigma}, \\ &\ll \sum_{n \in \mathbb{N}} \frac{n^\varepsilon}{n^\sigma}, \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n^{\sigma-\varepsilon}}. \end{aligned}$$

We have  $\lim_{n \rightarrow \infty} \frac{\log(n)}{n^\varepsilon} = 1$  and the last equation is convergent if and only if  $\sigma - \varepsilon > 1$ . Thus we have  $\sigma > 1 + \varepsilon$ .

**Definition 2.6.** Let  $D \subseteq \mathbb{C}$  be open. A meromorphic function is  $f : D \rightarrow \mathbb{C}$  which is analytic on  $D$  except a discrete set of poles of  $f$ .

**Remark 2.3.**  $\frac{\zeta'(s)}{\zeta(s)}$  is a meromorphic functions with a pole at  $s = 1$  and where  $\zeta(s)$  vanishes. Indeed, For general  $\frac{f}{g}$ , it is analytic if  $f, g$  are analytic and  $g \neq 0$ .

- 1).  $\zeta(s)$  is analytic except  $s = 1$ .
- 2).  $\zeta'(s)$  has a pole of order 2 at  $s = 1$ .
- 3).  $\zeta(s)$  has a pole of order 1 at  $s = 1$ .

Recall that for  $|z| \geq 1$ , we have,

1.  $|z| \geq 1 \Rightarrow \sum_{n \in \mathbb{Z}_{\geq 0}} z^n = \frac{1}{1-z}$ ,
2.  $\prod_{n \in \mathbb{N}} (1 + a_n)$  is convergent if  $\sum_n a_n$  is absolutely convergent,
3. therefore  $\prod_{n \in \mathbb{N}} (1 + a_n)$  is convergent if and only if  $\prod_{n \in \mathbb{N}} (1 + |a_n|)$  is convergent.

**Theorem 2.7.** Let  $f : \mathbb{N} \rightarrow \mathbb{C}$  be a map.

If  $f$  is multiplicative and for  $\operatorname{Re}(s) > r_0, r_0 \in \mathbb{R}$  then we have,

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} = \prod_p \left( \sum_{\nu \geq 0} f(p^\nu) p^{-\nu s} \right).$$

If  $f$  is completely multiplicative, then

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} = \prod_p (1 - f(p)p^{-s})^{-1}.$$

*Proof.* Let  $A(x) = \{n \in \mathbb{N} \mid \text{primes factors of } n \text{ are } \leq x\}$ , then

$$\prod_{p \leq x} \sum_{\nu=0}^{\infty} f(p^\nu) p^{-\nu s} = \sum_{n \in A} \frac{f(n)}{n^s}.$$

Therefore,

$$\begin{aligned}
\left| \prod_{p \leq x} \sum_{\nu=0}^{\infty} f(p^\nu) p^{-\nu s} - \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right| &= \left| \sum_{n \in A(x)} \frac{f(n)}{n^s} - \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right|, \\
&= \left| \sum_{n \in \mathbb{N} \setminus A(x)} \frac{f(n)}{n^s} \right|, \\
&\leq \sum_{n \notin A(x)} \frac{|f(n)|}{n^{\operatorname{Re}(s)}}, \\
&\leq \sum_{n > x} \frac{|f(n)|}{n^{\operatorname{Re}(s)}} \rightarrow 0.
\end{aligned}$$

The last limit is due to that it is a tail of a an absolutely convergent series. Since  $f$  is completely multiplicative, we have

$$f(p^\nu) = (f(p))^\nu.$$

Therefore, we get,

$$\begin{aligned}
\prod_p \left( \sum_{\nu \in \mathbb{Z}_{\geq 0}} (f(p^\nu) p^{-\nu s}) \right) &= \prod_p \left( \sum_{\nu \in \mathbb{Z}_{\geq 0}} (f(p) p^{-s})^{-\nu} \right), \\
&= \prod_p \left( \frac{1}{1 - f(p) p^{-s}} \right).
\end{aligned}$$

□

**Example 2.4.** Take  $f(n) = 1$  as above we get,

$$\sum_{n \in \mathbb{N}} \frac{1}{n^s} = \zeta(s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}, \operatorname{Re}(s) > 1.$$

**Example 2.5.**

$$\sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = \prod_p \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1}, \operatorname{Re}(s) > 1.$$

**Example 2.6.**

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} &= \prod_p \left( 1 + \frac{\mu(p)}{p^s} \right)^{-1}, \\
&= \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}, \\
&= \frac{1}{\zeta(s)}.
\end{aligned}$$

**Example 2.7.** Note that  $\phi(n) \leq n$ . Thus for  $\operatorname{Re}(s) > 2$ , we have,

$$\sum_{n \in \mathbb{N}} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

## 2.2 Order of arithmetic functions

**Definition 2.7.** Let  $f, g : \mathbb{N} \rightarrow \mathbb{C}$ , we denote,

$$f(n) = O(g(n)),$$

if there is  $K > 0$  and  $n_0 \in \mathbb{N}$  such that

$$n \geq n_0 \Rightarrow |f(n)| \leq K|g(n)|.$$

An alternative notation for this is  $f(n) = O(g(n))$ .

**Definition 2.8.** We define following arithmetic functions,

- 1).  $\nu(n) := \sum_{p|n} 1$ , a number of prime divisors of  $n$ ,
- 2).  $d(n) := \sum_{d|n} 1$ , the number of divisors of  $n$ ,
- 3).  $\sigma(n) := \sum_{d|n} d$ , the sum of all the divisors of  $n$ .

**Lemma 2.1.**

$$\nu(n) \ll \log(n).$$

*Proof.* Let  $n = \prod_{i=1}^k p_i^{\alpha(p_i)}$ . Then  $\nu(n) = k$ . Since  $p_i \geq 2$ , we have,

$$\begin{aligned} \log(n) &= \sum_{i=1}^k \alpha(p_i) \log(p_i), \\ &\geq k \log(2). \end{aligned}$$

Therefore  $\nu(n) \leq \frac{\log(n)}{\log 2}$ . □

**Lemma 2.2.**

$$\sum_{k=2}^n \frac{1}{k} \leq \log(n+1).$$

*Proof.* We know that

$$\int_1^n \frac{1}{t} dt = \log(n).$$

For  $1 \leq k \leq t \leq k+1 \leq n$ , we have,

$$\int_k^{k+1} \frac{1}{k+1} dt \leq \int_k^{k+1} \frac{1}{t} dt \leq \int_k^{k+1} \frac{1}{k} dt.$$

Thus we have,

$$\frac{1}{k+1} \leq \log(k+1) - \log(k) \leq \frac{1}{k}.$$

By telescoping sum we get

$$\sum_{k=2}^n \frac{1}{k} \leq \log(n+1).$$

□

**Lemma 2.3.**

$$\sigma(n) \ll n(1 + \log(n)) \sim n \log(n).$$

*Proof.*

$$\begin{aligned} \sigma(n) &= \sum_{d|n} \frac{n}{d}, \\ &= n \sum_{d|n} \frac{1}{d}, \\ &\leq n \left( 1 + \sum_{d=2}^n \frac{1}{d} \right), \\ &\leq (1 + \log(n)). \end{aligned}$$

The last inequality is due to Lemma 2.2.

□

**Exercise 2.1.** Show that

$$\sum_{k=1}^n \frac{1}{k} = \log(n) + O(1).$$

That is

$$\left| \sum_{k=1}^n \frac{1}{k} - \log(n) \right| \ll 1.$$

*Hint:* Replace  $\frac{1}{t}$  by an increasing function and derive the similar inequality to Lemma 2.2.

*Proof.* By Lemma 2.2, we have,

$$\left| \sum_{k=1}^n \frac{1}{k} - \log(n) \right| \leq |1 + \log(n) - \log(n)| \leq 1.$$

□

**Lemma 2.4.**

$$d(n) \leq 2\sqrt{n}.$$

*Proof.* If  $n = d_1 d_2$  then one of them must be less than or equal to  $\sqrt{n}$ .  $\square$

We have an improved inequality,

**Proposition 2.4.** *for  $\varepsilon > 0$ , we have,*

$$d(n) \ll n^\varepsilon.$$

*Proof.* Recall that for  $n = \prod_{i=1}^k p_i^{\alpha_i}$ , we have  $d(n) = \prod_{i=1}^k (\alpha_i + 1)$ . In particular, we have,

$$\frac{d(n)}{n^\varepsilon} = \prod_{i=1}^k \frac{(\alpha_i + 1)}{p_i^{\varepsilon \alpha_i}}.$$

Let  $A = \{i \mid p_i^\varepsilon \geq 2\}$ . Recall that for  $x \geq 1$ ,  $x + 1 \leq 2^x$  that is

$$\frac{x+1}{2^x} \leq 1.$$

Then,

$$\prod_{i \in A} \frac{\alpha_i + 1}{p_i^{\varepsilon \alpha_i}} \leq \prod_{i \in A} \frac{\alpha_i + 1}{2^{\alpha_i}} \leq 1.$$

For  $p_i^\varepsilon < 2$ , we observe,

$$p_i^{\varepsilon \alpha_i} = e^{\varepsilon \alpha_i \log(p_i)} \geq \varepsilon \alpha_i \log(p_i).$$

Therefore,

$$\begin{aligned} \prod_{i \notin A} \frac{\alpha_i + 1}{p_i^{\varepsilon \alpha_i}} &\leq \prod_{i \notin A} \left( \frac{\alpha_i}{p_i^{\varepsilon \alpha_i}} + 1 \right), \\ &\leq \prod_{i \notin A} \left( \frac{\alpha_i}{\varepsilon \alpha_i \log(p_i)} + 1 \right), \\ &\leq \prod_{i \notin A} \left( \frac{1}{\varepsilon \log(2^{\frac{1}{\varepsilon}})} + 1 \right), \\ &\leq \prod_{i \notin A} \left( \frac{1}{\log(2)} + 1 \right). \end{aligned}$$

Combining two cases, we obtain the statement.  $\square$

**Notation 2.2.** *Let  $x \in \mathbb{R}$ . We denote*

1. the integer part  $[x] \in \mathbb{Z}$  which is the greatest integer not exceeding  $x$ ,
2. the fraction part  $\{x\} = x - [x]$ .

**Proposition 2.5.**

$$\frac{\sum_{n \leq x} d(n)}{x} \sim \log(x).$$

*Proof.* By definition, we have,

$$\sum_{n \leq x} d(n) = \sum_{n \leq x} \sum_{d|n} 1.$$

$d|n$  if and only if there is  $e$  such that  $de = n$ . Using this we obtain,

$$\sum_{n \leq x} d(n) = \sum_{\substack{(e,d) \\ de \leq x}} 1 = \sum_{d \leq x} \sum_{e \leq \frac{x}{d}} 1 = \sum_{d \leq x} \left[ \frac{x}{d} \right].$$

Using the definition of  $[x]$ , we have,

$$\sum_{n \leq x} d(n) = \sum_{d \leq x} \frac{x}{d} - \left\{ \frac{x}{d} \right\} = x \sum_{d \leq x} \frac{1}{d} - \sum_{d \leq x} \left\{ \frac{x}{d} \right\} = x(\log(x) + O(1)) + O(x).$$

Thus we obtain the statement.  $\square$

**Definition 2.9.** Let  $f : \mathbb{N} \rightarrow \mathbb{C}$ , we say the average order of  $f$  is  $g : \mathbb{R} \rightarrow \mathbb{C}$  if

$$\frac{\sum_{n \leq x} f(n)}{x} \sim g(x).$$

Proposition 2.5 can be restated as follows.

**Proposition 2.6.** The average order of  $d$  is  $\log(x)$ .

**Exercise 2.2.** Examine the following statements.

1. Is it true that  $d(n) \ll \log n$ ?
2. Do we have  $d(n) = O(n^\varepsilon)$  for any  $\varepsilon > 0$ ?
3. What is the optimal bound for  $d(n)$ ?

**Theorem 2.8.** There exists  $c_1, c_2 > 0$  such that

$$c_1 \leq \frac{\varphi(n)\sigma(n)}{n^2} \leq c_2.$$

*Proof.* Recall that

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

and

$$\sigma(n) = \prod_{p|n} \frac{p^{\alpha(p)+1} - 1}{p - 1}.$$

Thus we obtain,

$$\frac{\sigma(n)}{n} = \frac{\prod_{p|n} (1 + p + \dots + p^{\alpha(p)})}{\prod_{p|n} p^{\alpha(p)}} = \prod_{p|n} \left( \frac{\frac{1}{p^{\alpha(p)+1}} - 1}{\frac{1}{p} - 1} \right)$$

By multiplying two we get,

$$\frac{\varphi(n)\sigma(n)}{n^2} = \prod_{p|n} \left(1 - \frac{1}{p^{\alpha(p)+1}}\right) \leq 1.$$

On the other hand,

$$\prod_p \left(1 - \frac{1}{p^2}\right) \leq \frac{\varphi(n)\sigma(n)}{n^2}.$$

The left hand side is equal to  $\frac{1}{\zeta(2)}$  which is  $\frac{6}{\pi^2}$ .  $\square$

**Lemma 2.5.** *For  $x \geq 1$  and  $d$  an integer such that  $d \leq x$ , we have,*

$$\left[\frac{x}{d}\right] \left(\left[\frac{x}{d}\right] + 1\right) = \frac{x^2}{d^2} + o\left(\frac{x}{d}\right).$$

*Proof.*

$$\begin{aligned} \left[\frac{x}{d}\right] \left(\left[\frac{x}{d}\right] + 1\right) &= \left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right) \left(\left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right) + 1\right), \\ &= \frac{x^2}{d^2} - 2\left\{\frac{x}{d}\right\} \frac{x}{d} + \left(\left\{\frac{x}{d}\right\}\right)^2 + \frac{x}{d} + \left\{\frac{x}{d}\right\}, \\ &= \frac{x^2}{d^2} + O(1) \cdot \frac{x}{d} + O(1). \end{aligned}$$

$\square$

**Theorem 2.9.** *The average order of  $\varphi$  is  $\frac{3x}{\pi^2}$ .*

*Proof.*

$$\begin{aligned} \sum_{n \leq x} \varphi(n) &= \sum_{n \leq x} n \sum_{d|n} \frac{\mu(d)}{d}, \\ &= \sum_{\substack{d,e \\ de \leq x}} e \mu(d), \\ &= \sum_{d \leq x} \mu(d) \left( \sum_{e \leq \frac{x}{d}} e \right), \end{aligned}$$

Since  $\sum_{e \leq \frac{x}{d}} e$  is a sum of an arithmetic progression, we have,

$$\sum_{e \leq \frac{x}{d}} e = \frac{1}{2} \left[\frac{x}{d}\right] \left(\left[\frac{x}{d}\right] + 1\right).$$

Substituting this, we obtain,

$$\sum_{n \leq x} \varphi(n) = \frac{1}{2} \sum_{d \leq x} \left( \mu(d) \left[ \frac{x}{d} \right] \left( \left[ \frac{x}{d} \right] + 1 \right) \right).$$

Note that

$$\frac{x}{d} = \left[ \frac{x}{d} \right] + \left\{ \frac{x}{d} \right\} = \left[ \frac{x}{d} \right] + O(1).$$

Using Lemma 2.5, that is,

$$\left[ \frac{x}{d} \right] \left( \left[ \frac{x}{d} \right] + 1 \right) = \frac{x^2}{d^2} + o\left(\frac{x}{d}\right).$$

We then have,

$$\begin{aligned} \sum_{n \leq x} \varphi(n) &= \frac{1}{2} \sum_{d \leq x} \mu(d) \left( \frac{x^2}{d^2} + O\left(\frac{x}{d}\right) \right), \\ &= \frac{x^2}{2} \sum_{d \leq x} \frac{\mu(d)}{d^2} + O\left(x \sum_{d \leq x} \frac{\mu(d)}{d}\right), \\ &= \frac{x^2}{2} \left( \sum_{d \geq 1} \frac{\mu(d)}{d^2} - \sum_{d \geq x} \frac{\mu(d)}{d^2} \right) + O\left(x \sum_{d \geq x} \frac{\mu(d)}{d}\right), \\ &= \frac{x^2}{2} \frac{1}{\zeta(2)} - \frac{x^2}{2} \sum_{d \geq x} \frac{\mu(d)^2}{d} + O\left(x \sum_{d \leq x} \frac{\mu(d)}{d}\right). \end{aligned}$$

We have,

$$\begin{aligned} \left| \sum_{d \geq x} \frac{\mu(d)}{d^2} \right| &\leq \sum_{d \geq x} \frac{1}{d^2} \\ &\ll \int_x^\infty \frac{dt}{t^2}, \\ &\ll \frac{1}{x}. \end{aligned}$$

and,

$$\left| \sum_{d \leq x} \frac{\mu(d)}{d} \right| \ll \ln x.$$

Using these we have,

$$\begin{aligned} \sum_{n \leq x} \varphi(n) &= \frac{x^2}{2} \frac{1}{\zeta(2)} + O(x) + O(x \ln x), \\ &= \frac{x^2}{2\zeta(2)} + O(x \ln x). \end{aligned}$$

We conclude that

$$\frac{\sum_{n \leq x} \varphi(n)}{x^2} \xrightarrow{x \rightarrow \infty} \frac{1}{2\zeta(2)}.$$

In particular,

$$\frac{\sum_{n \leq x} \varphi(n)}{x} \sim \frac{x}{2\zeta(2)} = \frac{x \cdot 6}{2\pi^2}.$$

□

### 2.3 Abel's Summation Formula

Recall the harmonic series  $\sum_{n \in \mathbb{N}} \frac{1}{n}$  is divergent. Our next goal is to find such  $A_x$  that

$$\lim_{n \rightarrow \infty} \left( \sum_{n \leq x} \frac{1}{n} - A_x \right)$$

exists.

**Remark 2.4** (Euler-Mascheroni constant). *By taking  $A_x = \log(x)$ , we have*

$$\lim_{n \rightarrow \infty} \left( \sum_{n \leq x} \frac{1}{n} - \log(x) \right) = \gamma,$$

*exists. Such  $\gamma$  is called Euler-Mascheroni constant.*

**Remark 2.5** (Euler Kronecer constant). *Take  $A_x = \log(x)$ , we have*

We can show that

$$\psi = \lim_{s \rightarrow 1^+} \left( \frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right).$$

Hint first show that

$$\zeta(s) = \frac{1}{s-1} + \psi + o(s-1).$$

**Proposition 2.7** (Abels' summation formula). *Given  $(a_n)_{n \in \mathbb{N}}$  in  $\mathbb{C}$  and  $f(n)$  is continuously differentiable in  $[1, x]$ . Set*

$$A(x) := \sum_{n \leq x} a_n.$$

*Then we have,*

$$\sum_{n \leq x} a_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t) dt.$$

*Proof.* Observe that

$$a_n = A(n) - A(n-1).$$

Assume  $x \in \mathbb{N}$ . We substitute this to  $\sum_{n \leq x} a_n f(n)$ , we get,

$$\begin{aligned} \sum_{n \leq x} a_n f(n) &= \sum_{n \leq x} (A(n) - A(n-1)) f(n), \\ &= \sum_{n \leq x} A(n)f(n) - \sum_{n \leq x} A(n-1)f(n), \\ &= \sum_{n \leq x} A(n)f(n) - \sum_{n \leq x-1} A(n)f(n+1), \\ &= A(x)f(x) - \sum_{n \leq x-1} A(n)(f(n+1) - f(n)), \\ &= A(x)f(x) - \sum_{n \leq x-1} A(n) \int_n^{n+1} f'(t) dt, \\ &= A(x)f(x) - \sum_{n \leq x-1} \int_n^{n+1} A(t)f'(t) dt, \\ &= A(x)f(x) - \int_1^x A(t)f'(t) dt. \end{aligned}$$

For the case when  $n \notin \mathbb{N}$  and  $n > 1$ ,

$$\sum_{n \leq x} a_n f(n) = \sum_{n \leq [x]} a_n f(n).$$

Using the previous case, we get,

$$\sum_{n \leq x} a_n f(n) = A([x])f([x]) - \int_1^{[x]} A(t)f'(t) dt.$$

Remains to show that we can remove the brackets. To do so,

$$\begin{aligned} \sum_{n \leq x} a_n f(n) &= A([x])f([x]) - \int_1^x A(t)f'(t) dt + \int_{[x]}^x A(t)f'(t) dt, \\ &= A([x])f([x]) - \int_1^x A(t)f'(t) dt + A([x]) \int_{[x]}^x f'(t) dt, \\ &= A([x])f([x]) - \int_1^x A(t)f'(t) dt + A([x])f(x) - A([x])(f[x]), \\ &= A([x])f(x) - \int_1^x A(t)f'(t) dt. \end{aligned}$$

□

**Corollary 2.1.**

1.  $\sum_{n \leq x} \frac{1}{n} = \ln x + \psi + o\left(\frac{1}{x}\right).$
2.  $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + o\left(\frac{1}{x^s}\right),$  where  $\operatorname{Re}(s) > 0, s \neq 1.$

We also have the following equivalent forms of prime number theorem when  $x \rightarrow \infty.$

$$\begin{aligned} \sum_{n \leq x} \Lambda(n) &\sim x \Leftrightarrow \pi(x) \sim \frac{x}{\ln x}, \\ &\Leftrightarrow \sum_{p \leq x} \ln p \sim x, \\ &\Leftrightarrow \sum_{n \leq x} \mu(n) = o(x). \end{aligned}$$

*Proof.* Consider  $f(t) = \frac{1}{t}$  and  $a_n = 1$  for all  $n \in \mathbb{N}.$

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n} &= \frac{\lfloor x \rfloor}{x} + \int_1^x \frac{[t]}{t^2} dt, \\ &= \frac{x - \{x\}}{x} + s \int_1^x \frac{t - \{t\}}{t^2} dt, \\ &= 1 + \log x - \frac{\{x\}}{x} - \int_1^\infty \frac{\{t\}}{t^2} dt. \end{aligned}$$

For general  $s, \operatorname{Re}(s) > 0, s \neq 1,$  we have

$$\sum_{n \leq x} \frac{1}{n^s} = x^{1-s} + o\left(\frac{1}{x^s}\right) + \frac{sx^{1-s}}{1-s} - \frac{s}{1-s} - s \int \frac{\{t\}}{t^{s+1}} dt.$$

Recall that

$$\left[ \int \frac{1}{t^s} = \frac{t^{-s+1}}{1-s} \right]_1^x = \frac{x^{1-s}}{1-s} - \frac{1}{1-s}.$$

Using this we obtain,

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n^s} &= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - x \int_1^x \frac{\{t\}}{t^{s+1}} dt, \\ x^{1-s} \left[ 1 + \frac{1}{1-s} \right] &= \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - s \int_1^x \frac{\{t\}}{t^{s+1}} dt \\ \int_1^\infty \frac{\{t\}}{t^{s+1}} &< \infty, \\ &\leq \int_1^\infty \frac{1}{t^{\operatorname{Re}(s)+1}} < \infty. \end{aligned}$$

As  $x \rightarrow \infty$ , the left hand side goes to  $\zeta(s)$ , for the right hand side, we get  
 $= \frac{-s}{1-s} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt$ , where  $\operatorname{Re}(s) > 1$ ,

$$\zeta(s) = \frac{-s}{1-s} - s \int \frac{\{t\}}{t^{s+1}} dt.$$

Identity theorem for analytic function tells us the analytic continuation of Riemann zeta function is unique.  $\square$

**Remark 2.6.** *It is an exercise that*

$$\int_1^\infty \frac{\{t\}}{t^{s+1}} dt$$

where  $\operatorname{Re}(x) > 0$ . From Stein-Schakarchi 5.2, 5.3, we have

$$\sum f_n(z) \xrightarrow{\text{unif.}} f(z)$$

is analytic where  $\operatorname{Re}(x) > 0$  and  $s \neq 1$ , also in this case,

$$\zeta(s) = \frac{-s}{1-s} - s \int \frac{\{t\}}{t^{s+1}} dt.$$

holds.

**Remark 2.7** (Exercise). Let  $M \in \mathbb{N}$  and

$$\lim_{x \rightarrow \infty} \left( \sum_{\substack{n \leq x \\ (n, M) = 1}} \frac{1}{n} - \frac{\phi(M)}{M} \ln x \right)$$

exists.

Suppose we have,  $\pi(x) \sim \frac{x}{\ln x}$ . We want to show

$$\theta(x) := \sum_{p \leq x} \ln p \sim x,$$

Consider the following sequence

$$a_n = \begin{cases} 1, & (n \text{ is prime}), \\ 0, & (\text{otherwise}). \end{cases}$$

and

$$f(t) = \ln t.$$

Using Abel summation formula,

$$\sum_{p \leq x} \ln p = \pi(x) \ln(x) - \int_1^\infty \frac{\pi(t)}{t} dt.$$

Dividing both sides by  $x$ , we obtain,

$$\frac{\theta(x)}{x} = \frac{\pi(x) \ln x}{x} - \frac{1}{x} \int_1^x \frac{\pi(t)}{t} dt.$$

**Remark 2.8** (Exercise). Use  $\frac{\pi(t)}{t} \sim \frac{1}{\ln t}$  and

$$\int_1^x \frac{dt}{\ln t} = o(x),$$

prove that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \int_1^x \frac{\pi(t)}{t} dt = 0.$$

$$\begin{aligned} \int_2^{\sqrt{x}} \frac{dt}{\ln t} &= \int_{\sqrt{x}}^x \frac{dt}{\ln t} \leq \frac{1}{\ln 2} \int_2^{\sqrt{x}} dt + \frac{1}{\ln \sqrt{x}} \int_{\sqrt{x}}^x dt, \\ &= \frac{\sqrt{x} - 2}{\ln 2} + \frac{x - \sqrt{x}}{\ln \sqrt{x}}, \\ &= o(x). \end{aligned}$$

**Remark 2.9.**

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{\substack{1 \leq k \\ p^k \leq x}} \ln p.$$

Thus we see,

$$\psi(x) - \theta(x) = \sum_{\substack{2 \leq k \\ p, p^k \leq x}} \ln p.$$

Also,

$$p^k \leq x \Rightarrow k \leq \frac{\ln x}{\ln p}.$$

Using  $k \geq 2$ ,

$$p \leq x^{\frac{1}{k}} \leq \sqrt{x}.$$

Thus we have the inequalities,

$$\begin{aligned} \psi(x) - \theta(x) &\leq \sum_{p \leq \sqrt{x}} \ln p \left( \sum_{2 \leq k \leq \frac{\ln x}{\ln p}} 1 \right), \\ &\leq \sum_{p \leq \sqrt{x}} \ln x, \\ &\leq \ln(x) \sum_{n \leq \sqrt{x}} 1, \\ &\leq \sqrt{x} \ln x. \\ \Rightarrow \frac{\psi(x)}{x} &= \frac{\theta(x)}{x} + O\left(\frac{\sqrt{x} \ln x}{x}\right). \end{aligned}$$

Therefore we obtain,

$$\psi(x) \sim x \Leftrightarrow \theta(x) \sim x.$$

**Remark 2.10.** As exercises, find the closed expressions for the following summations.

$$\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n}, \sum_{p \leq x} \frac{1}{p} = \ln \ln x + o(1).$$

## 2.4 Characters

**Definition 2.10.** Let  $G$  be a finite group. A character is a group homomorphism  $f : G \rightarrow \mathbb{C}^\times$ .

**Remark 2.11.** Let us denote

$$\hat{G} := \{f : G \rightarrow \mathbb{C}^\times \mid \text{characters}\}.$$

If  $G$  is finite abelian then  $|\hat{G}| = |G|$ . Furthermore, such characters are linearly independent over  $\mathbb{C}$ .

**Definition 2.11.** Let  $q \in \mathbb{N}$  and  $q \geq 3$ . A character is a group homomorphism modulo  $q$  is a group homomorphism

$$\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times.$$

Given a character  $\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ . We can define a Dirichlet character modulo  $q$ ,  $\chi : \mathbb{Z} \rightarrow \mathbb{C}^\times$  as follows.

$$\chi(a) := \begin{cases} \chi'(\bar{a}), & (a, q) = 1, \\ 0, & (a, q) \neq 1. \end{cases}$$

**Remark 2.12.** From Remark 2.11, there are exactly  $\varphi(q)$  many Dirichlet characters modulo  $q$ . Furthermore,

$$\chi(a)^{\varphi(q)} = \chi'(\bar{a})^{\varphi(q)} = \chi'(\bar{a}^{\varphi(q)}) = \chi'(\bar{1}) = 1.$$

In particular, images of  $\chi$  are  $\varphi(q)$ -th roots of unity.

**Example 2.8.** For  $q = 3$ ,

$$(\mathbb{Z}/3\mathbb{Z})^\times = \{\bar{1}, \bar{2}\} \rightarrow \mathbb{C}^\times.$$

We only have two characters, a trivial one and  $\bar{2} \mapsto -1$ .

**Example 2.9.** For  $q = 5$ ,

$$(\mathbb{Z}/5\mathbb{Z})^\times = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$$

	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\chi_{1,5}(n)$	1	1	1	1
$\chi_{2,5}(n)$	1	-1	-1	1
$\chi_{3,5}(n)$	1	$i$	$-i$	-1
$\chi_{4,5}(n)$	1	$-i$	$i$	-1

$\chi_{1,5}$  is called a principle/trivial character.

**Definition 2.12.** A character  $\chi : \mathbb{Z} \rightarrow \mathbb{C}^\times$  is called

- 1). trivial if  $\chi(g) = 1$  for all  $g \in G$ ,
- 2). even if  $\chi(-1) = 1$ ,
- 3). odd if  $\chi(-1) = -1$ .

We also define these notions for characters  $\chi_0 : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  accordingly if the character  $\chi : \mathbb{Z} \rightarrow \mathbb{C}^\times$  induced by it has these properties. Trivial characters are often denoted by  $\chi_0$ .

**Theorem 2.10.**

$$\sum_{a \bmod q} \chi(a) = \begin{cases} \phi(q) & (\chi = \chi_0), \\ 0 & (\text{otherwise}). \end{cases}$$

We also have,

$$\sum_{\chi \bmod q} \chi(a) = \begin{cases} \phi(a) & (\bar{a} = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

*Proof.* If  $\chi$  is principle then the first assertion is clear. Suppose  $\chi$  is not principle then there is  $b \in \{1, \dots, q\}$  such that  $\chi(b) \neq 1$  and  $(b, q) = 1$ . Let

$$s = \sum_{a \bmod q} \chi(a).$$

Then by the definition of group homomorphisms, we have,

$$\chi(b)s = s.$$

But  $\chi(b) \in \mathbb{C}$ , this means  $s = 0$  as  $\mathbb{C}$  is an integral domain.

For the second assertion, let  $\bar{a} \neq 1$ , then

$$\exists \chi' \bmod q, \text{ s.t. } \chi'(a) \neq 1.$$

Thus we get,

$$s = \sum_{\chi \bmod q} \chi(a), s \cdot \chi(a) = \sum_{\chi \bmod q} \chi \chi'(a) = s \Rightarrow s = 0.$$

The statement when  $\bar{a} = 1$  follows from Remark 2.11.  $\square$

**Remark 2.13.** One can check in the table of Example 2.9 that Theorem 2.10 indeed holds.

**Exercise 2.3.**

$$\sum_{\substack{\chi \text{ mod } q, \\ \chi(-1)=1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} & (\bar{a} = 1, -1), \\ 0 & (\text{otherwise}), \end{cases},$$

$$\sum_{\substack{\chi \text{ mod } q, \\ \chi(-1)=-1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} & (\bar{a} = 1), \\ -\frac{\phi(a)}{2} & (\bar{a} = -1), \\ 0 & (\text{otherwise}). \end{cases},$$

Obviously we have the following equalities.

$$\sum_{n \leq x} \chi(n) = \sum_{\substack{n \leq x \\ (n,q)=1}} \chi(n) = \sum_{n \leq kq} \chi(n) + \sum_{n=kq+1}^x \chi(n),$$

where  $k$  is the largest integer such that  $kq \leq x$ . Then we observe from Theorem 2.10

$$\sum_{n \leq kq} \chi(n) = k \left( \sum_{n=1}^q \chi(n) \right) = 0,$$

unless  $\chi$  is trivial. Also we have,

$$\left| \sum_{n \leq x} \chi(n) \right| = \left| \sum_{\substack{kq+1 \leq n \leq x \\ (n,q)=1}} \chi(n) \right| \leq \sum_{\substack{kq+1 \leq n \leq x \\ (n,q)=1}} 1 \leq \sum_{\substack{kq+1 \leq n \leq kq+q \\ (n,q)=1}} 1 = \phi(q).$$

Thus we conclude,

$$\sum_{n \leq x} \chi(n) \leq \varphi(q),$$

**Exercise 2.4.**

$$\sum_{n \leq x} \chi_0(n) = ?.$$

**Theorem 2.11** (Pólya–Vinogradov).

$$\sum_{n \leq x} \chi(n) \ll \sqrt{q} \ln q, (\chi \neq \chi_0 \pmod{q}).$$

notice that the above expression is bounded by  $\sqrt{q} \ln \ln q$ . Furthermore, this is uniform in  $q$  that is the constant does not depend on  $q$ .

$$\sum_{n \geq 1} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + O(x^{-s}) - s \int_1^x \frac{\{t\}}{t^{s+1}} dt.$$

For  $\operatorname{Re}(s) > 1$ , as  $x \rightarrow \infty$  we have,

$$\sum_{n \geq 1}^{\infty} \frac{1}{n^s} = \frac{-s}{1-s} - s \int_1^{\infty} \frac{\{t\}}{t^{s+1}} dt.$$

The last expression is analytic since,

$$\sum \int_n^{n+1} \frac{\{t\}}{t} dt \xrightarrow{\text{uniformly}} \int_1^{\infty} \frac{\{t\}}{t^{s+1}} dt, \text{ when } \operatorname{Re}(s) > 0.$$

Suppose  $\zeta(s) \neq 0$ , where  $\operatorname{Re}(s) > 0$ , then Euler product exists.

**Theorem 2.12.** Set

$$A(n) := \sum_{n \in \mathbb{N}} a_n.$$

Assume  $A(x) := \sum_{n \leq x} a_n = O(x^\delta)$ , then we have, for  $\operatorname{Re}(s) > \delta$ ,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt.$$

Hence the Dirichlet series converges for  $\operatorname{Re}(s) > \delta$ .

*Proof.*

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s} = \frac{A(x)}{x^s} + s \int_1^x \frac{A(t)}{t^{s+1}} dt.$$

As  $A(x) = O(x^\delta)$  and  $\operatorname{Re}(s) > \delta$ ,  $\frac{A(x)}{x^s} = O(x^{\delta-\operatorname{Re}(s)})$ . Therefore, as  $x \rightarrow \infty$ , we have,

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt.$$

Again using the assumption, we have,

$$\int_1^{\infty} \left| \frac{A(t)}{t^{s+1}} \right| dt \ll \int_1^{\infty} t^{\delta-\operatorname{Re}(s)-1} dt = \frac{t^{\delta-\operatorname{Re}(s)}}{\delta-\operatorname{Re}(s)} \Big|_0^{\infty} = \frac{1}{\delta-\operatorname{Re}(s)}.$$

Thus the integral is convergent.  $\square$

**Definition 2.13.** For  $\operatorname{Re}(s) > 1$ , we define

$$L(s, \chi) := \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s}.$$

**Remark 2.14.** Since  $\operatorname{Re}(s) > 1$  and for any character  $\chi : \mathbb{Z} \rightarrow \mathbb{C}^{\times}$  we have  $|\chi(n)| \leq 1$ ,  $L(s, \chi)$  is uniformly absolutely convergent.

**Example 2.10.** Let  $\chi : \mathbb{Z} \rightarrow \mathbb{C}^\times$  be a non-principal character modulo  $q$  and set  $A(n) = \chi(n)$ . Recall that

$$\sum_{n \leq x} \chi(n) \leq q.$$

Taking  $A(n) = \chi(n)$  and apply Theorem 2.12, we obtain, for  $\operatorname{Re}(s) > 0$ ,

$$L(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = s \int_1^\infty \frac{\sum_{n \leq t} \chi(n)}{t^{s+1}} dt.$$

Since for  $\operatorname{Re}(s) > 1$ ,  $L(s, \chi)$  is absolutely uniformly convergent. By Theorem 2.7, we have,

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

If  $\chi = \chi_0$ , a principal character, we have,

$$L(s, \chi_0) = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Using  $\zeta$  function, we have,

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

**Theorem 2.13.**  $\zeta$  has an analytic continuation for  $\operatorname{Re}(s) > 0$  besides  $s = 1$ . For  $s = 1$  we have a simple pole of residue 1.

*Proof.* Recall from the proof of Corollary 2.1. We have for  $\operatorname{Re}(s) > 1$ ,

$$\zeta(s) = \frac{-s}{1-s} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

The right hand side of the equation is analytic when  $\operatorname{Re}(s) > 0, s \neq 1$ . When  $s = 1$ , we have,

$$\lim_{s \rightarrow 1^+} (s-1)\zeta(s) = \lim_{s \rightarrow 1^+} \left( s - s(s-1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt \right) = 1.$$

□

**Corollary 2.2.** For  $\operatorname{Re}(s) > 0$  and  $s \neq 1$ , we have an analytic continuation of  $L(s, \chi_0)$  where  $\chi_0$  is a principal character modulo  $q$ , which is

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Obviously at  $s = 1$ , it has a simple pole of residue  $\prod_{p|q} \left(1 - \frac{1}{p^s}\right)$ , which we can write as

$$\operatorname{Res}(L(s, \chi_0), 1) = \frac{\varphi(q)}{q}.$$

*Proof.* A direct corollary of Theorem 2.13. For the residuum, write,

$$q = \prod_{p|q} p^{\alpha(p)},$$

then

$$\varphi(q) = \prod_{p|q} (p-1)p^{\alpha(p)-1} = q \prod_{p|q} (p-1)p^{-1}.$$

□

**Theorem 2.14.** Let  $\chi$  be a non-principal character, then there is an analytic continuation of  $L(s, \chi)$  for  $\operatorname{Re}(s) > 0$ .

*Proof.* From Example 2.10, we have,

$$L(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = s \int_0^\infty \frac{\sum_{n \leq t} \chi(n)}{t^{s+1}} dt.$$

The right hand side is analytic. □

**Theorem 2.15.**

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s}$$

is analytic in its range of convergence.

**Remark 2.15.** We have the following conjecture.

$$L\left(\frac{1}{2}, \chi\right) = 0, \chi \neq \chi_0.?$$

$$\zeta\left(\frac{1}{2}\right) = \frac{1}{1 - \sqrt{2}} \sum_{n \in \mathbb{N}} \frac{(-1)^{n-1}}{\sqrt{n}} \approx -1.46 \cdots$$

**Definition 2.14.** A character is said to be quadratic if its values are either  $\pm 1$ .

**Remark 2.16.** Suppose  $\chi \neq \chi_0$ . We have,

$$L(s, \chi) = 0 \quad \text{if } s = 0, -2, -4, \text{ when } \chi \text{ is an even character.}$$

$$L(s, \chi) = 0 \quad \text{if } s = -1, -3, -5, \text{ when } \chi \text{ is an odd character.}$$

**Lemma 2.6.** For  $\sigma > 1$  and  $t \in \mathbb{R}$ , we have,

$$\operatorname{Re}(\ln(\zeta(\sigma + it))) = \sum \frac{\Lambda(n)}{n^\sigma \ln n} \ln(t \ln(n)).$$

And also,

$$\operatorname{Re}(3 \ln(\zeta(\sigma)) + 4 \ln(\zeta(\sigma + it)) + \ln(\zeta(\sigma + 2it))) \geq 0.$$

*Proof.*

$$\begin{aligned}\zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \sigma > 0, \\ \ln(\zeta(s)) &= -\sum_p \ln(1 - p^{-s}) \\ &= \sum_{p,n} \frac{1}{np^{ns}}, \sigma > 1.\end{aligned}$$

We have,

$$\sum_{n \geq 2} \frac{\Lambda(n)}{n^s \ln n} = \sum_{p,k,k \geq 1} \frac{\ln p}{p^{ks} \ln p^k} = \sum_{p,k} \frac{1}{kp^{ks}} = \sum_{n \geq 2} \Lambda(n).$$

$$\operatorname{Re}(3 \ln \zeta(\sigma) + 4 \ln \zeta(\sigma + it) + \ln \zeta(\sigma + 2it)) = \sum_{n \geq 2} \frac{\Lambda(n)}{n^\sigma \ln n} (3 + 4 \cos(t \ln n) + \cos(2t \ln n)) \geq 0,$$

since

$$3 + 4 \cos \theta + \cos 2\theta = 2(\cos \theta + 1)^2 \geq 0.$$

$$= \ln |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 0.$$

Thus we have,

$$\operatorname{Re}(\ln(z)) \leq$$

□

**Theorem 2.16.** For  $t \in \mathbb{R} \setminus \{0\}$ , we have

$$\zeta(1 + it) \neq 0.$$

*Proof.* Using Lemma 2.6, we have, Thus we get,

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1.$$

Suppose  $\zeta(1 + it_0) = 0$ , for  $t_0 \in \mathbb{R} \setminus \{0\}$ . Suppose further that the order of zero is  $m \in \mathbb{N}$ . Then by looking at, and taking  $\sigma \rightarrow 1+$

$$\underbrace{((\sigma - 1)\zeta(\sigma))^3}_{\rightarrow \text{finite}} \underbrace{\left(\frac{\zeta(\sigma + it_0)}{(\sigma - 1)^m}\right)^4}_{\rightarrow \text{non-zero}} \underbrace{((\sigma - 1)^{4m-3} \zeta(\sigma + 2it_0))}_{\rightarrow 0} \rightarrow 0.$$

Contradicts to that the absolute value of above expression is at least 1. □

**Theorem 2.17.**  $\frac{\zeta'(s)}{\zeta(s)}$  has an analytic continuation to  $\operatorname{Re}(s) = 1, s \neq 1$ . And for  $s = 1$ , we have a simple pole of residue  $-1$ .

*Proof.* We have,

$$(s-1)\zeta(s) = s - s(s-1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

Set

$$f(s) := 1 - (s-1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt,$$

so that  $(s-1)\zeta(s) = sf(s)$ . We already have  $f(s)$  is analytic when  $\operatorname{Re}(s) > 0$ . Differentiating both sides we get,

$$(s-1)\zeta'(s) + \zeta(s) = sf'(s) + f(s).$$

Dividing both sides by  $(s-1)\zeta(s)$  we get,

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \frac{sf'(s) + f(s)}{(s-1)\zeta(s)}.$$

The right hand side is analytic when  $\zeta$  does not vanish. From Theorem 2.16, by letting  $s = 1 + it$  for some  $t \in \mathbb{R} \setminus \{0\}$ , we get the desired analytic continuation. For  $s = 1$ , we have

$$(s-1)\frac{\zeta'(s)}{\zeta(s)} = \frac{sf'(s) + f(s)}{\zeta(s)} - 1.$$

Recall that  $\zeta(s)$  has a pole at 1 and observe that  $f(1) = 1, f'(1)$  is finite thus the last statement follows.  $\square$

**Theorem 2.18.** Let  $\chi$  be a non-trivial non-real Dirichlet character of modulo  $q$ , then

$$L(1, \chi) \neq 0.$$

*Proof.* Recall that

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where  $\operatorname{Re}(s) > 1$ . Now consider,

$$\log L(s, \chi) = \sum_{n,p} \frac{\chi(p^s)}{np^{ns}} = \sum_{n \geq 2} \frac{\Lambda(n)\chi(n)}{n^s \log n}.$$

Set  $\zeta_{\varphi(q)} = e^{\frac{2\pi i}{\varphi(q)}}$ . Then for any  $n \in (\mathbb{Z}/q\mathbb{Z})^\times$ , there is  $n' = n(n, q, \chi)$  (ie  $n'$  depends on  $n, q$ , and  $\chi$ ) such that

$$\chi(n) = \zeta_{\varphi(q)}^{n'}$$

Note that for a cyclic group  $G = \langle g \rangle$ , and a character  $\chi : G \rightarrow \mathbb{C}^\times \in \hat{G}$ , there is  $a \in \{1, \dots, \varphi(q)\}$  such that

$$\chi(g) = \zeta_{\varphi(q)}^a.$$

We have, for  $\sigma > 1$

$$\operatorname{Re}(\log L(\sigma, \chi)) = \sum_{n \geq 2} \frac{\Lambda(n) \cos\left(\frac{2\pi i}{\varphi(q)}\right)}{n^\sigma \log(n)}.$$

By Lemma 2.6, we have,

$$\operatorname{Re}(3 \ln \zeta(\sigma) + 4 \ln L(\sigma, \chi) + \ln L(\sigma, \chi^2)) \geq 0,$$

therefore,

$$|\zeta(\sigma)^3 L(\sigma, \chi)^4 L(\sigma, \chi^2)| \geq 1.$$

Note that  $\chi^2 \neq \chi$  as it is non-real and  $L(\sigma, \chi^2)$  is analytic by Theorem 2.14. If  $L(\sigma, \chi) = 0$  then

$$\zeta(\sigma)^3 L(\sigma, \chi)^4$$

has a zero of order at least 1. Thus this is a contradiction.  $\square$

**Lemma 2.7.** *For  $k \in \mathbb{Z}_{\geq 0}$ ,*

$$\left( \frac{\sin(k + \frac{1}{2})\theta}{\sin \frac{\theta}{2}} \right)^2 = 2k + 1 \sum_{j=1}^{2k} 2(2k+1-j) \cos j\theta.$$

**Theorem 2.19.** *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a function such that*

- i).  $f$  is analytic,
- ii).  $f \not\equiv 0$ ,
- iii).  $\log f(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s}$  for  $a_n \geq 0$  and  $\operatorname{Re}(s) > 1$ ,
- iv).  $f$  is analytic on  $\operatorname{Re}(s) = 1, s \neq 1$ , and it has a pole at  $s = 1$  of order  $e$ .

If  $f(s) = 0$  on the line  $\operatorname{Re}(s) = 1$ , then the order of zero is at most  $\frac{e}{2}$ .

*Proof.* For  $e \leq 2k - 1$ , suppose  $f$  has a zero at  $f(1 + it_0)$  of order  $k$ . Let us define,

$$g(s) := f(s)^{2k+1} \prod_{j=1}^{2k} f(s + ijt_0)^{2(2k+1-j)} = f(s)^{2k+1} f(s + it_0)^{4k} \dots.$$

Then at  $s = 1$ ,  $f(s)^{2k+1}$  has a pole of order  $e(2k+1)$  and  $\prod_{j=1}^{2k} f(s + ijt_0)^{2(2k+1-j)}$  has a zero of order  $4k$ . Note that

$$4k^2 - e(2k+1) \geq 1.$$

Thus  $g$  has a zero at 1 of order at least 1, in particular,  $g(1) = 0$ .

Consider,

$$\begin{aligned}\log g(\sigma) &= (2k+1) \log f(\sigma) + \sum_{j=1}^{2k} 2(2k+1-j) \log(f(\sigma + ijt_0)), \\ &= (2k+1) \sum_{n \in \mathbb{N}} \frac{a_n}{n^\sigma} + \sum_{j=1}^{2k} 2(2k+1-j) \left( \sum_{n \in \mathbb{N}} \frac{a_n}{n^{\sigma+ijt_0}} \right), \\ &= \sum_{n \in \mathbb{N}} \frac{a_n}{n^\sigma} \left( 2k+1 + \sum_{j=1}^{2k} 2(2k+i-j) e^{-ijt_0 \log(n)} \right). \\ \operatorname{Re}(\log(g(\sigma))) &= \sum_{n \in \mathbb{N}} \frac{a_n}{n^\sigma} \left( 2k+1 + \sum_{j=1}^{2k} 2(2k+i-j) \cos(jt_0 \log(n)) \right).\end{aligned}$$

Using Lemma 2.7, we get,

$$\operatorname{Re} \log(g(\sigma)) = \sum_{n \geq 1} \frac{a_n}{n^\sigma} \left( \frac{\sin(k + \frac{1}{2}) t_0 \log(n)}{\sin \frac{t_0 \log(n)}{2}} \right)^2 \geq 0.$$

As  $\operatorname{Re}(g(\sigma)) \geq 1$ ,  $|g(\sigma)| \geq 1$ . This contradicts that  $g$  has a zero of order at least 1 at  $1 + it_0$ .  $\square$

**Corollary 2.3.** *For any Dirichlet character  $\chi$  modulo  $q$  we have,  $L(s, \chi)$  is analytic over  $\operatorname{Re}(s) > 1$ . Furthermore  $L(s, \chi) \neq 0$  on  $\operatorname{Re}(s) = 1, s \neq 1$ .*

*Proof.* The first assertion is due to Theorem 2.14. Let

$$f(s) := \prod_{\chi \bmod q} L(s, \chi).$$

Then,

$$\log(f(s)) = \sum_{\chi \bmod q} \log L(s, \chi) = \sum_{n,p} \frac{\sum_{\chi \bmod q} \chi(p^n)}{np^{ns}}, \operatorname{Re}(s) > 1.$$

Using Theorem 2.10, we get,

$$\log(f(s)) = \sum_{\substack{n,p \\ p^n \equiv 1 \pmod{q}}} \frac{\varphi(q)}{np^{ns}}.$$

Note  $f$  has a pole of order at most 1 at  $\operatorname{Re}(s) = 1$  and its residue is

$$\operatorname{Res}(f(s), 1) = \left( \lim_{s \rightarrow 1} (s-1)L(s, \chi_0) \right) \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} L(1, \chi) = \frac{\varphi(q)}{q} \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} L(1, \chi).$$

The right hand side is not 0 by Theorem 2.18.  $\square$

**Theorem 2.20.** *There are infinitely many primes.*

*Proof.* We have,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

$$\log \zeta(s) = \sum_{n,p} \frac{1}{np^{ns}} = \sum_p \frac{1}{p^s} + \sum_{n \geq 2} \frac{1}{np^{ns}}.$$

Since  $\zeta$  has a pole at 1, so does its log. Observe that at  $s = 1$ ,

$$\sum_p \frac{1}{p^s} + \sum_{n \geq 2} \frac{1}{np^{ns}} \leq \sum_p \sum_{n \geq 2} \frac{1}{p^n} = \sum_p \frac{1}{p^2} \frac{1}{1 - \frac{1}{p}} = \sum_p \frac{1}{p(p-1)} \ll \sum_{n \in \mathbb{N}} \frac{1}{n^2}.$$

Thus  $\lim_{s \rightarrow 1^+} \sum_p \frac{1}{p^s}$  must be infinity.  $\square$

**Lemma 2.8.** *Let  $p$  be a prime and  $a, q$  be coprimes such that  $p \equiv a \pmod{q}$ . Then we have,*

$$\sum_{\chi \pmod{q}} \chi(p^n) \overline{\chi(a)} = \begin{cases} \varphi(q), & (p^n \equiv a \pmod{q}), \\ 0, & (\text{otherwise}). \end{cases}$$

*Proof.* Since  $\chi(a)$  is a  $\varphi(q)$ -th root of unity, we have,  $\overline{\chi(a)} = \chi(a^{-1})$ . Therefore,

$$\begin{aligned} \sum_{\chi \pmod{q}} \chi(p^n) \overline{\chi(a)} &= \sum_{\chi \pmod{q}} \chi(p^n) \chi(a^{-1}), \\ &= \sum_{\chi \pmod{q}} \chi(p^n a^{-1}), \\ &= \begin{cases} \varphi(q), & (p^n a^{-1} \equiv 1 \pmod{q}), \\ 0, & (\text{otherwise}). \end{cases} \end{aligned}$$

$\square$

**Theorem 2.21** (Dirichlet's Theorem). *Let  $a, q$  be coprime. Then  $(a + nq)_{n \in \mathbb{N}}$  contains infinitely many primes.*

*Proof.* Motivated by the alternative proof of the existence of infinitely many primes, examine,

$$\sum_{\chi \pmod{q}} \log L(s, \chi) = \sum_{n,p} \left( \frac{\sum_{\chi \pmod{q}} \chi(p^n)}{np^{ns}} \right),$$

where  $\operatorname{Re}(s) > 1$ . Using Theorem 2.10, we have,

$$\sum_{\chi \pmod{q}} \log L(s, \chi) = \sum_{\substack{n,p \\ p^n \equiv 1 \pmod{q}}} \frac{\varphi(q)}{np^{ns}} = \sum_{p^n \equiv 1 \pmod{q}} \frac{\varphi(q)}{p^s} + \sum_{\substack{p,n \geq 2 \\ p^n \equiv 1 \pmod{q}}} \frac{\varphi(q)}{np^{ns}}.$$

Taking the log out we have,

$$\prod_{\chi \bmod q} L(s, \chi) = \exp \left( \varphi(q) \left( \sum_{p^n \equiv 1 \pmod{q}} \frac{1}{p^s} + \sum_{\substack{p, n \geq 2 \\ p^n \equiv 1 \pmod{q}}} \frac{1}{np^{ns}} \right) \right).$$

Since for  $\chi \neq \chi_0$ ,  $L(1, \chi) \neq 0$  and by Corollary 2.2, we have,

$$\lim_{s \rightarrow 1^+} (s-1) \prod_{\chi \bmod q} L(s, \chi) = \frac{\varphi(q)}{q} \prod_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} L(1, \chi).$$

By the same argument from Theorem 2.20, we see,

$$\sum_{\substack{p, n \geq 2 \\ p^n \equiv 1 \pmod{q}}} \frac{1}{np^{ns}} \leq \sum_{n \in \mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

However,

$$\lim_{s \rightarrow 1^+} (s-1) \exp \left( \varphi(q) \left( \sum_{p^n \equiv 1 \pmod{q}} \frac{1}{p^s} + \sum_{\substack{p, n \geq 2 \\ p^n \equiv 1 \pmod{q}}} \frac{1}{np^{ns}} \right) \right) \neq 0$$

which is only possible when

$$\lim_{s \rightarrow 1^+} \sum_{p^n \equiv 1 \pmod{q}} \frac{1}{p^s} = \infty.$$

Together with Lemma 2.8, we derived the general statement.  $\square$

**Theorem 2.22** (Bertrand's Postulate). *For a sufficiently large  $n \in \mathbb{N}$ , there is a prime number inbetween  $n$  and  $2n$ .*

*Proof.* Consider the second Chebyschev function, and by Remark 2.1, we have,

$$T(x) := \sum_{l \leq x} \psi \left( \frac{x}{l} \right) = \sum_{l \leq x} \sum_{n \leq \frac{x}{l}} \Lambda(n) = \sum_{\substack{l, n \\ ln \leq x}} \Lambda(n).$$

That is

$$\sum_{l \leq x} \psi \left( \frac{x}{l} \right) = \sum_{m \leq x} \sum_{d|m} \Lambda(d).$$

By Theorem 2.3,

$$T(x) = \sum_{m \leq x} \log(m).$$

Using the definition,

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= \sum_{l \leq x} \psi\left(\frac{x}{l}\right) - 2 \sum_{2l \leq x} \psi\left(\frac{x}{2l}\right), \\ &= \sum_{l \leq x} (-1)^{l+1} \psi\left(\frac{x}{l}\right), \\ &= \psi(x) - \psi\left(\frac{x}{2}\right) + \dots. \end{aligned}$$

Again by Remark 2.1

$$x \leq y \Rightarrow \psi(x) \leq \psi(y).$$

In particular,

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq \sum_{l \leq x} (-1)^{l+1} \psi\left(\frac{x}{l}\right) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right).$$

Recall Proposition 2.7, consider

$$A(x) = \sum_{n \leq x} 1 = \lfloor x \rfloor,$$

and since  $\log(x)$  is continuously differentiable on  $[1, x]$ , we have,

$$T(x) = \lfloor x \rfloor \log(x) - \int_1^x \frac{\lfloor x \rfloor}{x} dx = x \log(x) - x + O(\log(x)).$$

Therefore,

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= x \log(x) - x + O(\log(x)) - 2 \left( \frac{x}{2} \log\left(\frac{x}{2}\right) - \frac{x}{2} + O(\log(x)) \right), \\ &= \log 2 \cdot x + O(\log(x)). \end{aligned}$$

Combining with the previous result, we get,

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq (\log 2)x + O(\log(x)) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right).$$

Generalizing this we obtain,

$$\psi\left(\frac{x}{2^{n-1}}\right) - \left(\frac{x}{2^n}\right) \leq \log(2) \frac{x}{2^{n-1}} + O(\log(x)).$$

By induction, we obtain,

$$\begin{aligned} \psi(x) - \psi\left(\frac{x}{2^n}\right) &\leq (\log 2)x \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}}\right) + O(n \log(x)), \\ &\leq (2 \log 2)x + O(n \log(x)). \end{aligned} \tag{\psi 1}$$

Take  $n$  to be the maximal such that  $2^n \leq x$  that is  $\lfloor \frac{x}{2^n} \rfloor = 1$ . Then,  $\psi\left(\frac{x}{2^n}\right) = 0$ , thus,

$$\psi(x) \leq (2 \log 2)x + O((\log(x))^2).$$

On the other hand,

$$\begin{aligned}\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) &\geq (\log 2)x + O(\log(x)), \\ \Rightarrow \psi(x) - \psi\left(\frac{x}{2}\right) &\geq (\log(2))x + O(\log(x)) - \psi\left(\frac{x}{3}\right)\end{aligned}$$

Using Inequality  $(\psi 1)$ , we get,

$$\begin{aligned}\psi(x) - \psi\left(\frac{x}{2}\right) &\geq (\log 2)x + O(\log(x)) - (2 \log 2)\frac{x}{3} + O((\log(x))^2), \\ &= (\log 2)\frac{x}{3} + O((\log(x))^2).\end{aligned}\tag{\psi 2}$$

Using Remark 2.9, that is

$$\psi(x) - \theta(x) = O(\sqrt{x} \log(x)),$$

together with Inequalities  $(\psi 1)$  and  $(\psi 2)$ , we obtain,

$$\theta(x) - \theta\left(\frac{x}{2}\right) \geq (\log 2)\frac{x}{3} + O(\sqrt{x} \log(x)).$$

Thus the right hand side is greater than 0 if  $x$  is sufficiently large with coeffi of the inequality, therefore by definition of  $\theta$ , we have,

$$\theta(x) - \theta\left(\frac{x}{2}\right) = \sum_{\frac{x}{2} \leq p \leq x} \log(p) > 0.$$

□

**Theorem 2.23** (Chebyschev).

- 1).  $\theta(n) \leq 2n \log 2$ ,
- 2).  $\psi(n) \geq 2n \log 2$ ,
- 3). there exists  $A, B > 0$  such that for sufficiently large  $n$ , we have

$$\frac{Bx}{\log x} \leq \pi(x) \leq \frac{Ax}{\log x}.$$

*Proof.*

□

## 2.5 Ikehara-Wiener Theorem and Its Applications

**Theorem 2.24** (Ikehara-Wiener). Let  $(b_n)_{n \in \mathbb{N}}$  be a sequence of non-negative numbers and set,

$$f(s) = \sum_{n \in \mathbb{N}} \frac{b_n}{n^s}.$$

Suppose

- i). the series converges absolutely for  $\operatorname{Re}(s) > 1$ ,
- ii).  $f$  has an analytic continuation to  $\operatorname{Re}(s) = 1$  except  $s = 1$ ,
- iii).  $f$  has a simple pole at  $s = 1$  with residue  $R \geq 0$ .

Then we have,

$$\sum_{n \leq x} b_n = Rx + o(x).$$

That is

$$\lim_{x \rightarrow \infty} \frac{\sum_{n \leq x} b_n}{x} = R.$$

**Lemma 2.9.** Let  $\chi$  be a Dirichlet character modulo  $q$ , then

$$L(s, \chi)^{-1} = \sum_{n \in \mathbb{N}} \frac{\chi(n)\mu(n)}{n^s}.$$

*Proof.* We immitate the proof of Theorem 2.5. Using Theorem 2.1,

$$L(s, \chi) \left( \sum_{n \in \mathbb{N}} \frac{\mu(n)\Lambda(n)}{n^s} \right) = \sum_{t \in \mathbb{N}} \sum_{n|t} \frac{\chi\left(\frac{t}{n}\right) \chi(n)\mu(n)}{t^s} = \sum_{t \in \mathbb{N}} \chi(t) \sum_{n|t} \mu(n) = 1.$$

□

**Theorem 2.25.** Let  $\chi$  be a character then, for  $\operatorname{Re}(s) > 1$ ,

$$-\frac{L'(s, \chi)}{L(s, \chi)} = \sum_{n \in \mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s}.$$

*Proof.* By Lemma 2.9 and Proposition 2.1

$$\begin{aligned} -\frac{L'(s, \chi)}{L(s, \chi)} &= \left( \sum_{t \in \mathbb{N}} \frac{\chi(n) \log(t)}{t^s} \right) \left( \sum_{n \in \mathbb{N}} \frac{\mu(n)\chi(n)}{n^s} \right), \\ &= \sum_{t \in \mathbb{N}} \chi(t) \sum_{n|t} \frac{\mu\left(\frac{t}{n}\right) \log(n)}{t^s}, \\ &= \sum_{n \in \mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s}. \end{aligned}$$

□

**Definition 2.15.**

$$\psi(x, q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n).$$

**Proposition 2.8.** *As  $x \rightarrow \infty$ , we have,*

$$\psi(x, q, a) \sim \frac{x}{\varphi(q)}.$$

*Proof.* Consider

$$\sum_{\chi \bmod q} \bar{\chi}(a) \left( \sum_{n \in \mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s} \right) = \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} \left( \sum_{\chi \bmod q} \chi(a^{-1}n) \right)$$

Using Lemma 2.8, we get,

$$- \sum_{\chi \bmod q} \bar{\chi}(a) \frac{L'(s, \chi)}{L(s, \chi)} = \varphi(q) \sum_{n \equiv a \pmod q} \frac{\Lambda(n)}{n^s}.$$

Recall from Corollary 2.2, we have,

$$\text{Res}(L(s, \chi_0), 1) = \frac{\varphi(q)}{q}.$$

And by Theorem 2.14,  $L(s, \chi)$  is analytic at  $s = 1$  for  $\chi \neq \chi_0$ . Also using Theorem 2.16, we have

$$\chi \neq \chi_0 \Rightarrow \frac{L'(s, \chi)}{L(s, \chi)} \text{ is analytic for } \text{Re}(s) \geq 1.$$

Furthermore, we have Theorem 2.17, we have

$$\text{Res}\left(\frac{\zeta'(s)}{\zeta(s)}, 1\right) = -1.$$

Combining these, we get,

$$\lim_{s \rightarrow 1} (s-1) \left( -\frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \frac{L'(s, \chi)}{L(s, \chi)} - \frac{1}{\varphi(q)} \frac{L'(s, \chi_0)}{L(s, \chi_0)} \right) = \frac{1}{\varphi(q)}.$$

Now using Theorem 2.24, and apply  $b_n = \Lambda(n)$   $f = \varphi(q) \sum_{n \equiv a \pmod q} \frac{\Lambda(n)}{n^s}$ , we get,

$$\psi(x, q, a) = \sum_{n \leq x} \Lambda(n) = \frac{1}{\varphi(q)} x + o(x).$$

□

## 2.6 $L(s, \chi) \neq 0$ for Quadratic Characters

**Lemma 2.10.** Let  $f := \sum_{d|n} \chi(d)$ , where  $\chi$  is a character, then,

$$\forall n \in \mathbb{N}, n \text{ is a perfect square} \Rightarrow f(n) \geq 0, f(n) \geq 1.$$

*Proof.* Recall  $n = \prod_{p|n} p^{\alpha(p)}$ . Using this we have,

$$\begin{aligned} \sum_{d|n} \chi(d) &= \prod_{p|n} \left( \sum_{k=0}^{\alpha(p)} \chi(p)^k \right), \\ &= \begin{cases} 1 & \chi(p) = 0, \\ \prod_{p|n} (1 + \alpha(p)) & \chi(p) = 1, \\ \prod_{p|n} \left( \frac{(1 - (-1)^{\alpha(p)+1})}{2} \right) & \chi(p) = -1. \end{cases} \end{aligned}$$

Note that if  $\alpha(p)$  are all even for  $p|n$  (ie.  $n$  is a perfect square), we have the last part of the cases equals to 1. Thus we have  $f(n) \geq 1$ .  $\square$

**Theorem 2.26.** Let  $f(n) = \sum_{d|n} \chi(d)$  for some character. Then we have,

$$\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = 2\sqrt{x}L(1, \chi) + o(1).$$

*Proof.*

$$\begin{aligned} \sum_{n \leq x} \frac{f(n)}{\sqrt{n}} &= \sum_{n \leq x} \left( \frac{\sum_{d|n} \chi(d)}{\sqrt{n}} \right), \\ &= \sum_{\substack{d, e \\ de \leq x}} \frac{\chi(d)}{\sqrt{de}}, \\ &= \sum_{\substack{d, e \leq x \\ d \leq \sqrt{x}}} \frac{\chi(d)}{\sqrt{de}} + \sum_{\substack{de \leq x \\ d > \sqrt{x}}} \frac{\chi(d)}{\sqrt{de}}, \\ &= \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left( \sum_{e \leq \frac{x}{d}} \frac{1}{\sqrt{e}} \right) + \sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left( \sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{(d)}} \right). \end{aligned}$$

Recall that from Proposition 2.7,

$$\sum_{m \leq x} \frac{1}{\sqrt{m}} = 2\sqrt{x} + B + o\left(\frac{1}{\sqrt{x}}\right),$$

where  $B$  is some constant as  $x \rightarrow \infty$ .

Let  $x, y \in \mathbb{R}$ , such that  $x < y$ , we have,

$$\begin{aligned} \sum_{x < d \leq y} \frac{\chi(d)}{\sqrt{d}} &= \sum_{d \leq y} \frac{\chi(d)}{\sqrt{d}} - \sum_{d \leq x} \frac{\chi(d)}{\sqrt{(d)}}, \\ \sum_{d \leq x} \frac{\chi(d)}{\sqrt{d}} &= \frac{\sum_{d \leq x} \chi(d)}{\sqrt{x}} + \frac{1}{2} \int_1^x \frac{\sum_{d \leq t} \chi(d)}{t^{\frac{1}{2}}} dt, \\ &= O\left(\frac{1}{\sqrt{x}}\right). \end{aligned}$$

Using these equations, we have,

$$\begin{aligned} \sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left( \sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{(d)}} \right) &= \sum_{e \leq x} \frac{1}{\sqrt{e}} \left( O\left(\frac{1}{x^{\frac{1}{r}}}\right) \right), \\ &= \left( O\left(\frac{1}{x^{\frac{1}{r}}}\right) \right) \sum_{e \leq x} \frac{1}{\sqrt{e}}, \\ \sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{x}} &\ll \int_1^{\sqrt{x}} \frac{1}{\sqrt{t}} dt = x^{\frac{1}{4}}. \end{aligned}$$

Thus we conclude,

$$\sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left( \sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{(d)}} \right) = O(1).$$

We also have,

$$\begin{aligned}
\sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left( \sum_{e \leq \frac{x}{d}} \frac{1}{\sqrt{e}} \right) &= \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left( 2\sqrt{\frac{x}{d}} + B + o\left(\sqrt{\frac{d}{x}}\right) \right), \\
&= 2\sqrt{x} \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{d} + B \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} + o\left(\frac{1}{\sqrt{x}} \sum_{d \leq \sqrt{x}} \chi(d)\right). \\
2\sqrt{x} \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{d} &= 2\sqrt{x} \left( \sum_{d \geq 1} \frac{\chi(d)}{d} - \sum_{d > \sqrt{x}} \frac{\chi(d)}{d} \right), \\
&= 2\sqrt{x}L(1, \chi) - 2\sqrt{x} \sum_{d > \sqrt{x}} \frac{\chi(d)}{d}, \\
&= 2\sqrt{x}L(1, \chi) - 2\sqrt{x}o\left(\frac{1}{\sqrt{x}}\right), \\
&= 2\sqrt{x}L(1, \chi) + o(1).B \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \leq o\left(\frac{B}{x^{\frac{1}{4}}}\right) = o(1).
\end{aligned}$$

$$\sum_{d \leq \sqrt{x}} \chi(d) \leq q? = o(1).$$

□

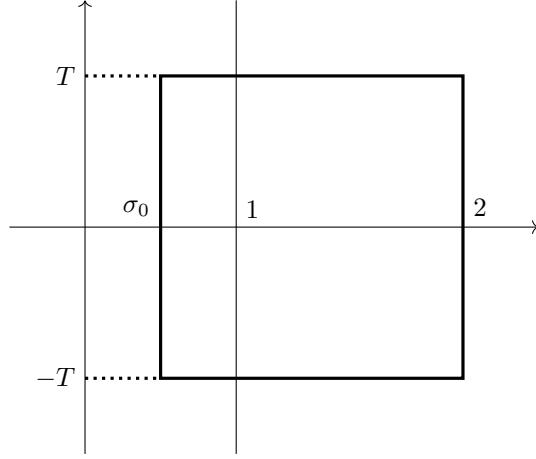
**Corollary 2.4.**  $L(1, \chi) \neq 0$  for a quadratic character  $\chi$ .

*Proof.* To derive a contradiction, assume  $L(1, \chi) = 0$ . Then, from Theorem 2.26 and Lemma 2.10

$$o(1) = \sum_{n \leq x} \frac{f(n)}{\sqrt{n}} \geq \sum_{\substack{n \leq x \\ n \text{ is a square}}} \frac{1}{\sqrt{n}} = \sum_{m \leq \sqrt{x}} \frac{1}{m} = \log \sqrt{x} + o(1).$$

Thus we obtain  $o(1) \geq \log \sqrt{x} + o(1)$  which is a contradiction. □

**Lemma 2.11.** Consider the following rectangle,



where  $\sigma_0 = 1 - \log T$ . On the boundary of the rectangle above, we have,

1.  $|\zeta(s)| = O(\log T), T \rightarrow \infty$ ,
2.  $|\zeta'(s)| = O((\log T)^2)$ .

*Proof.* Recall that

$$\zeta'(s) = \sum_{n=1}^{\infty} \frac{-\log n}{n^s}.$$

The first statement is due to the first derivative of  $\zeta$ . and it is assigned as an exercise. For the second part, for  $\operatorname{Re}(s), \sigma > 1$ , we have,

$$\begin{aligned} \zeta(s) &= \sum_{n \geq 1} \frac{1}{n^s}, \\ &= \sum_{n \leq T} \frac{1}{n^s} + \sum_{n > T} \frac{1}{n^s}, \end{aligned}$$

Using Abel's summation formula by setting  $a_n = 1, f(n) = \frac{1}{n^s}$ , we have,

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} = -s \int_1^{\infty} \frac{[t]}{t^{s+1}} dt, \\ \sum_{n \leq T} \frac{1}{n^s} &= \frac{[T]}{T^s} - s \int_1^T \frac{[t]}{t^{s+1}} dt. \end{aligned}$$

Using this, we get,

$$\begin{aligned} \zeta(s) &= \sum_{n \leq T} \frac{1}{n^s} - \frac{[T]}{T^s} + s \int_1^{\infty} \frac{[t]}{t^{s+1}} dt, \\ &= \sum_{n \leq T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} + \frac{\{T\}}{T^s} - s \int_T^{\infty} \frac{\{u\}}{u^{s+1}} du. \end{aligned}$$

Note that the right hand side is analytic where  $\operatorname{Re}(s) > 0$  and  $s \neq 1$ . Now we will estimate the last equation above on the boundary of the rectangle above.

$$\begin{aligned} \left| \sum_{n \leq T} \frac{1}{n^s} \right| &\ll \int_1^T \frac{du}{u^{\operatorname{Re}(s)}}, \\ &\leq \int_1^T \frac{du}{u^{\sigma_0}}, \\ &= \frac{T^{1-\sigma_0}}{1-\sigma_0}, \\ &\ll \log T. \end{aligned}$$

Observe that ,

$$1 - \sigma_0 = \frac{1}{\log T}, T^{1-\sigma_0} = T^{\frac{1}{\log T}} = \exp(\log T^{\frac{1}{\log T}}) = \exp(1) = e.$$

We also have,

$$\begin{aligned} \left| \frac{T^{1-s}}{s-1} \right| &= \frac{T^{1-\operatorname{Re}(s)}}{|s-1|}, \\ &\leq \frac{T^{1-\sigma_0}}{|s-1|}, \\ &\leq \frac{e}{|s-1|}, \\ &\leq \frac{e}{\sigma_0-1} = e \log T. \end{aligned}$$

Also consider,

$$\left| \frac{\{T\}}{T} \right| \leq \frac{1}{T^{\operatorname{Re}(s)}} \leq 1.$$

Finally we have,

$$\begin{aligned} \left| -s \int_T^\infty \frac{\{u\}}{u^{s+1}} du \right| &\leq |s| \int_T^\infty \frac{du}{u^{\operatorname{Re}(s)+1}}, \\ &= -\frac{|s|}{\operatorname{Re}(s) u^{\operatorname{Re}(s)}} \Big|_T^\infty, \\ &= \frac{|s|}{\operatorname{Re}(s) T^{\operatorname{Re}(s)}}, \\ &\leq \frac{|s|}{T^{\operatorname{Re}(s)}}, \\ &\leq \frac{\sqrt{2^2 + T^2}}{T^{\sigma_0}}, \\ &\ll \frac{T}{T^{\sigma_0}} = T^{1-\sigma_0} = e. \end{aligned}$$

For the second part, for  $\operatorname{Re}(s) > 0, s \neq 1$ ,

$$\begin{aligned}\zeta'(s) &= \sum_{n \leq T} \frac{-\log n}{n^s} + \frac{T^{1-s}(-\log T)}{s-1} + T^{1-s} \frac{-1}{(s-1)^2} \\ &\quad + \frac{\{T\} \log T}{T^s} - \int_T^\infty \frac{\{u\}}{u^{s+1}} du + s \int_T^\infty \frac{\{u\} \log u}{u^{s+1}} du.\end{aligned}$$

This is obtained simply differentiating the equation,

$$\zeta(s) = \sum_{n \leq T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} - \frac{\{T\}}{T^s} - s \int_T^\infty \frac{\{u\}}{u^{s+1}} du.$$

The statement can be shown using all the estimates obtained to show the first part.  $\square$

**Theorem 2.27** (Complex Mean Value Theorem). *Let  $f : \Omega \rightarrow \mathbb{C}$  be an analytic, where  $\Omega$  is a convex open set. Let  $a, b \in \Omega$ , then there exists  $z_1, z_2 \in (a, b)$  such that*

$$\operatorname{Re}(f'(z_1)) = \operatorname{Re}\left(\frac{f(b) - f(a)}{b-a}\right), \operatorname{Im}(f'(z_2)) = \operatorname{Im}\left(\frac{f(b) - f(a)}{b-a}\right).$$

**Theorem 2.28.** *There exist some constants  $c_1, c_2$  such that*

$$1 - \frac{c_1}{(\log T)^9} \leq \sigma \leq 2 \Rightarrow |\zeta(s)| > \frac{c_2}{(\log T)^7},$$

where  $1 < |\operatorname{Im}(s)| \leq T$ .

*Proof.* We set  $z = \sigma + it$  for  $\sigma, t \in \mathbb{R}$ . Recall Lemma 2.6,

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1, \sigma > 1.$$

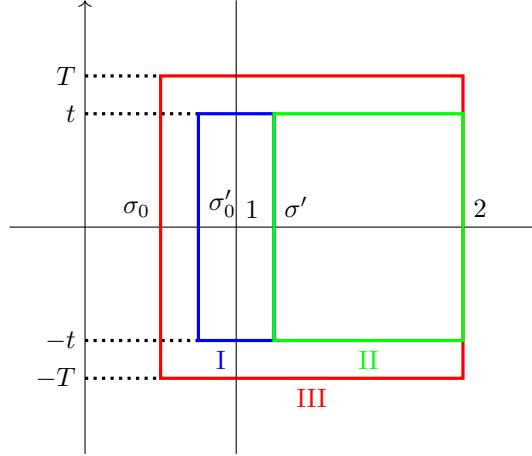
Thus we have,

$$|\zeta(\sigma + it)|^4 \geq |\zeta(\sigma)|^{-3} |\zeta(\sigma + 2it)|^{-1}, \quad (*)$$

Consider

$$\sigma'_0 = 1 - \frac{c_1}{(\log T)^9}, \sigma' = 1 + \frac{c_1}{(\log T)^9}$$

$c_1$  is a positive constant which will be adjusted later.



Fix the domain II where  $\sigma' \leq \sigma \leq 2$ ,  $1 \leq |t| \leq T$ .

$$\begin{aligned}\zeta(s) &= \frac{s}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{s+1}} du, \operatorname{Re}(s) > 0, s \neq 1. \\ \zeta(\sigma) &= \frac{\sigma-1+1}{\sigma-1} - \sigma \int_1^\infty \frac{\{u\}}{u^{\sigma+1}} du. \\ &\ll \frac{1}{\sigma-1} \text{ as } \sigma \rightarrow 1^+, \\ \Rightarrow \zeta(\sigma)^{-1} &>> (\sigma-1), \sigma \rightarrow 1^+\end{aligned}$$

On the domain II, we have,  $\sigma'_0 < \sigma \leq 2$ ,  $1 \leq |t| \leq T$ . Using Lemma 2.11,

$$|\zeta(\sigma + 2it)| = O(\log T).$$

Substituting these to Equation (\*), we have,

$$\begin{aligned}|\zeta(\sigma + it)|^4 &>> (1-\sigma)^3 (\log T)^{-1}, \\ &>> \frac{c_1^3}{(\log T)^{27}} (\log T)^{-1}. \\ \Rightarrow |\zeta(\sigma + it)| &>> \frac{c_1^{\frac{3}{4}}}{(\log T)^7},\end{aligned}\tag{D1}$$

in the domain II.

On the domain I, similarly use Theorem 2.27 with

$$a = \sigma' + it, b = \sigma + it,$$

where,

$$1 - \frac{c_1}{(\log T)^9} \leq \sigma \leq \sigma'.$$

We have,

$$\frac{\zeta(\sigma' + it) - \zeta(\sigma + it)}{\sigma' - \sigma} = \operatorname{Re}(\zeta(z_1)) + \operatorname{Im}(\zeta(z_2)).$$

Furthermore,

$$\zeta(\sigma' + it) - \zeta(\sigma + it) = O((\sigma' - \sigma)(\log T)^2).$$

By the definition of  $\sigma'$ , we have,

$$\zeta(\sigma' + it) = \zeta(\sigma + it) + O\left(\frac{c_1}{(\log T)^7}\right). \quad (D2)$$

Combining Equations (D1) and (D2), there are some positive constants  $A_1, A_2$  such that

$$\begin{aligned} |\zeta(\sigma + it)| &\geq |\zeta(\sigma' + it)| - \frac{A_1 c_1}{(\log T)^7}, \\ &\geq \frac{A_2 c_1^{\frac{3}{4}}}{(\log T)^7} - \frac{A_1 c_1}{(\log T)^7}. \end{aligned}$$

Now take  $c_1$  sufficiently small so that in the region,

$$\sigma'_0 \leq \sigma \leq \sigma', 1 \leq |t| \leq T,$$

we have,

$$|\zeta(\sigma + it)| \geq \frac{c_2}{(\log T)^7}.$$

Again combining this with Inequality (D2), we obtain the statement.  $\square$

**Corollary 2.5.** *There exists some constant  $c$  such that*

$$\frac{\zeta'(s)}{\zeta(s)} = O((\log T)^9),$$

for  $1 - \frac{c}{(\log T)^9} \leq \operatorname{Re}(s) \leq 2$  and  $1 \leq |\operatorname{Im}(s)| \leq T$ .

**Notation 2.3.** Let  $c \in \mathbb{R}$  and  $f : \mathbb{C} \supseteq \Omega \rightarrow \mathbb{C}$ . Then we denote,

$$\int_{(c)} f(s) ds = \lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT} f(s) ds.$$

**Lemma 2.12.**

$$\frac{1}{2\pi i} \int_{(c)} y^s \frac{ds}{s} = \begin{cases} 0, & (0 < y < 1), \\ \frac{1}{2}, & (y = 1), \\ 1, & (y > 1). \end{cases}$$

**Lemma 2.13.** Set,

$$f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Suppose this is absolutely convergence in  $\operatorname{Re}(s) > c - \varepsilon$  for some  $c \in \mathbb{R}, \varepsilon > 0$  and  $x$  is not an integer, then

$$\sum_{n < x} a_n = \frac{1}{2\pi i} \int_{(c)} f(s) \frac{x^s}{s} ds.$$

*Proof.* Since  $f(s)$  is a limit of a continuous function series which is absolutely convergent on  $\operatorname{Re}(s) > c - \varepsilon$ , thus this is uniformly convergent in a compact subset of its domain of convergence. Thus we can exchange the integral and the sum namely,

$$\frac{1}{2\pi i} \int_{(c)} \left( \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right) \frac{x^s}{s} ds = \frac{1}{2\pi i} \sum_{n=1}^{\infty} a_n \int_{(c)} \left( \frac{x}{n} \right)^s \frac{ds}{s}.$$

By Lemma 2.12, and  $\frac{x}{n} > 1 \Leftrightarrow n < x$ , we derive the statement.  $\square$

**Corollary 2.6.** *Let  $c > 1$  and  $x$  be a non-integer, then we have,*

$$\psi(x) = \frac{1}{2\pi i} \int_{(c)} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds.$$

*Proof.* From Remark 2.1,

$$\psi(x) = \sum_{d \leq x} \Lambda(d),$$

and by Theorem 2.6 set

$$f(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

Thus applying Lemma 2.13, we get the statement.  $\square$

**Theorem 2.29.** *There is a positive constant  $c' > 0$  such that*

$$\psi(x) = x + O(x \exp(-c' \log(x))).$$

*Proof.* Let  $a = 1 + \frac{c}{\log T}$  for some  $c$  where  $T > 1$  is sufficiently large and will be chosen later. By Corollary 2.6, we have,

$$\psi(x) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + \frac{1}{2\pi i} \int_{(a)} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds.$$

**Claim 1.**

$$\left| \frac{1}{2\pi i} \int_{(a)} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \right| < \begin{cases} x^a \min\{1, T^{-1}, |\log x|^{-1}\}, & (x \neq 1), \\ \frac{a}{T}, & (x = 1). \end{cases}$$

Proof: Exercise. ■

Let

$$\psi(x) = \underbrace{\frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds}_{=I_1} + \underbrace{\frac{1}{2\pi i} \int_{(a)} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds}_{=I_2} - \underbrace{\frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds}_{=I_2}.$$

Again using Corollary 2.6 and the claim, we have,

$$\begin{aligned}
I_2 &= \frac{1}{2\pi i} \left( \int_{(a)} - \int_{a-iT}^{a+iT} \right) \left( \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \right) \frac{x^s}{s} ds, \\
&= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n) \left( \left( \int_{(a)} - \int_{a-iT}^{a+iT} \right) \left( \frac{x}{n} \right)^s \frac{ds}{s} \right), \\
&= \sum_{n=1}^{\infty} \Lambda(n) O \left( \left( \frac{x}{n} \right)^a \min \left( 1, T^{-1} \left| \log \left( \frac{x}{n} \right) \right|^{-1} \right) \right), \\
&= O \left( \sum_{n=1}^{\infty} \Lambda(n) \left( \frac{x}{n} \right)^a \min \left( 1, T^{-1} \left| \log \left( \frac{x}{n} \right) \right|^{-1} \right) \right).
\end{aligned}$$

Fix  $x$  and let us consider two ranges  $[\frac{x}{2}, \frac{3x}{2}]$  and  $(-\infty, \frac{x}{2}) \cup (\frac{3x}{2}, \infty)$ . If  $n$  sits in the second range

$$\frac{\log \frac{n}{x}}{\frac{n}{x} - 1} \geq 2 \log \frac{3}{2}.$$

And if  $n$  sits in the first interval, we have,

$$\left| \log \frac{x}{n} \right| \ll \left| \frac{n}{x} - 1 \right|^{-1}.$$

Using this we have,

$$\begin{aligned}
\sum_{n \in [\frac{x}{2}, \frac{3x}{2}]} \Lambda(n) \left( \frac{x}{n} \right)^a \min \left( 1, T^{-1} \left| \log \left( \frac{x}{n} \right) \right|^{-1} \right) &\ll \sum_{n \in [\frac{x}{2}, \frac{3x}{2}]} \Lambda(n) \left( \frac{x}{n} \right)^a T^{-1} \left| \frac{n}{x} - 1 \right|^{-1} \\
&\ll \frac{\log x}{T} \sum_{n \in [\frac{x}{2}, \frac{3x}{2}]} \frac{1}{|\frac{n}{x} - 1|}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{n \in [\frac{x}{2}, x)} \frac{1}{\frac{n}{x} + 1} &\ll \int_{\frac{x}{2}}^{\lfloor x \rfloor} \frac{dt}{1 - \frac{t}{x}}, \\
\sum_{n \in (x, \frac{3x}{2}]} \frac{1}{\frac{n}{x} - 1} &\ll \int_{\lceil x \rceil}^{\frac{3x}{2}} \frac{dt}{\frac{t}{x} - 1}. \\
\Rightarrow \sum_{n \in [\frac{x}{2}, x)} \frac{1}{\frac{n}{x} + 1} + \sum_{n \in (x, \frac{3x}{2}]} \frac{1}{\frac{n}{x} + 1} &\ll x \log x.
\end{aligned}$$

Thus we conclude, as  $x \rightarrow \infty$ ,

$$\sum_{n \in [\frac{x}{2}, \frac{3x}{2}]} \frac{1}{|\frac{n}{x} - 1|} \ll \frac{x(\log x)^2}{T}.$$

Let us turn our focus to the second range. We have  $|\log \frac{x}{n}| > \log \frac{3}{2}$ . Thus

$$\begin{aligned} \sum_{n < \frac{x}{2}, \frac{3x}{2} < n} \Lambda(n) \left( \frac{x}{n} \right)^a \min \left( 1, T^{-1} \left| \log \left( \frac{x}{n} \right) \right|^{-1} \right) &\ll \frac{x^a}{T} \sum_{n < \frac{x}{2}, \frac{3x}{2} < n} \frac{\Lambda(n)}{n^a}, \\ &\leq \frac{x^a}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^a}. \end{aligned}$$

Recall that we have,

$$-\frac{\zeta'(a)}{\zeta(a)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^a}.$$

Using Theorem 2.17, we see, as  $a \rightarrow 1^+$ ,

$$\left| \frac{\zeta'(a)}{\zeta(a)} \right| \ll \frac{1}{a-1}.$$

Using Corollary 2.5, we see,

$$\frac{x^a}{T} \frac{1}{a-1} \ll \frac{x^a}{T} (\log T)^9.$$

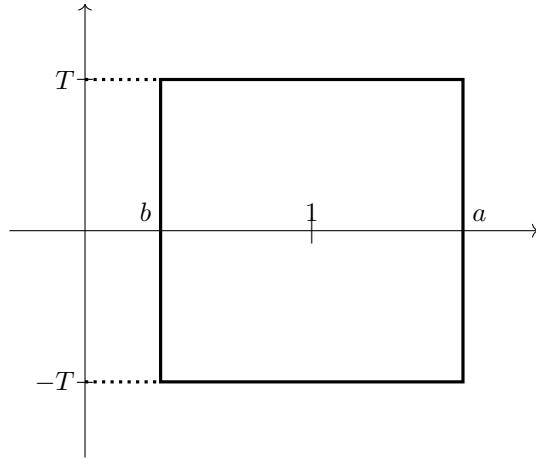
Combining all, we derive,

$$\psi(s) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + O \left( \frac{x(\log x)^2}{T} + \frac{x^a}{T} (\log T)^9 \right).$$

Recall Cauchy's Residue theorem, suppose  $f : \Omega \rightarrow \mathbb{C}$  is meromorphic where  $\Omega$  is simply connected with poles  $a_i, 1 \leq i \leq n$  poles in  $U \subseteq \Omega$  where the boundary of  $U$  is a simple closed curve  $\gamma$ . Then,

$$\frac{1}{2\pi i} \int_{\gamma} f(s) ds = \sum_{i=1}^n \text{Res}(f, a_i) + \text{constant}.$$

Consider the following closed path  $R_T$  which we traverse counterclockwise.



where  $a = 1 + \frac{c}{(\log(T))^9}$ ,  $b = 1 - \frac{c}{(\log(T))^9}$ . We claim that  $\frac{\zeta'(s)}{\zeta(s)}$  can be analytically continued to  $\operatorname{Re}(s) \geq b$  except simple pole at  $s = 1$  with residue  $-1$ . By Theorem 2.17 and Theorem 2.28, we obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_{R_T} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds &= \operatorname{Res}_{s=1} \left( \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \right), \\ &= x. \end{aligned}$$

Now consider  $R_T$  which is the closed path.

$$\int_{R_T} = \int_{a-iT}^{a+iT} + \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT}.$$

We then have,

$$\begin{aligned} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds &= x - \frac{1}{2\pi i} \left( \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT} \right) \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \\ \left| \frac{1}{2\pi i} \int_{a+iT}^{b+iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \right| &= \left| \frac{1}{2\pi i} \int_a^b \frac{-\zeta(u+iT)}{\zeta(u+iT)} \frac{x^{(u+iT)}}{(u+iT)} du \right|, \\ &\ll \int_a^b \left| \frac{-\zeta'(u+iT)}{\zeta(u+iT)} \right| \frac{x^u}{|u+iT|} du, \quad (|u+iT|^2 \geq 1) \\ &\stackrel{\text{Theorem 2.28}}{\ll} \frac{\log^9 T}{T} x^a \int_b^a du, \\ &\ll \frac{x^a}{T}. \end{aligned}$$

Now we have,

$$\begin{aligned} \left| \int_{b-iT}^{b+iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \right| &= \left| \frac{1}{2\pi} \int_{-T}^T \frac{\zeta'(b+iu)}{\zeta(b+iu)} \frac{x^{b+iu}}{b+iu} du \right|, \\ &\ll \log^9 T \int_{-T}^T \frac{x^b}{\sqrt{b^2+u^2}} du, \\ &\ll x^b \log^9 T \int_0^T \frac{du}{\sqrt{b^2+u^2}}, \\ &\ll x^b \log^9 T \int_1^T \frac{du}{u}, \\ &= x^b \log^{10} T. \end{aligned}$$

Now back to the beginning,

$$\psi(x) = x + O\left(\frac{x^a}{T} + x^b \log^{10} T\right) + O\left(\frac{x \log^2 x}{T} + \frac{x^a}{T} \log^9 T\right).$$

Choose  $T$  to be such that  $2c \log x = \log^{10} T, x = e^{\frac{\log^{10} T}{2c}}$ .

$$\begin{aligned}
x^{\frac{c}{\log^9 T}} &= e^{\frac{\log T}{2}} = \sqrt{T}. \\
x^{1-\frac{c}{\log^9 T}} \cdot \log^{10} T &+ \frac{x \log^2 x}{T} + \frac{x^{1+\frac{c}{\log^9 T}}}{T} \log^9 T \\
&= x \cdot T^{-\frac{1}{2}} \log^{10} T + \frac{x \log^2 x}{T} \\
&+ x \frac{\sqrt{T}}{T} \log^{10} T \left( \frac{x}{\sqrt{T}} \right) (\log^{10} T + \frac{\log^2 x}{\sqrt{x}} + \log^9 T), \\
&\ll \frac{x}{T^s} = xe^{-c(\log x)^{\frac{1}{10}}}.
\end{aligned}$$

□

**Remark 2.17.** Suppose  $\operatorname{Re}(s) = \sigma$  then,

$$\begin{aligned}
|\Gamma(s)| &\leq \int_0^\infty |t^{s-1}| e^{-t} dt, \\
&= \int_0^\infty t^{\sigma-1} e^{-t} dt, \\
&= \left( \int_0^1 + \int_1^\infty \right) t^{\sigma-1} e^{-t} dt, \\
&\leq \int_0^1 t^{\sigma-1} dt + \int_1^\infty t^{\sigma-1} e^{-t} dt, \left( \frac{t^n}{e^t} \xrightarrow[t \rightarrow \infty]{} 0 \right). \\
&\ll \frac{1}{\sigma} + \int_1^\infty t^{\sigma-1} t^{-2\sigma} dt, \\
&\ll \frac{1}{\sigma}.
\end{aligned}$$

**Theorem 2.30.**

$$F(s) = \int f(s, t) dt.$$

$F : \Omega \rightarrow \mathbb{C}$  is analytic in  $\Omega$  if

- i).  $f(s, t)$  is continuous in  $(s, t)$ ,
- ii).  $f(s, t)$  is analytic in  $s$ ,
- iii).  $\int f(s, t) dt$  is uniformly bounded on compact subsets of  $\Omega$ .

In Remark 2.17, suppose  $a \leq \operatorname{Re}(s) \leq b$  then the last two inequalities will

be,

$$\begin{aligned} & \int_0^1 t^{\sigma-1} dt + \int_1^\infty t^{\sigma-1} e^{-t} dt \\ & \ll_b \frac{1}{\sigma} + \int_1^\infty t^{b-1} t^{-2b} dt, \\ & \ll_b \frac{1}{\sigma} + \frac{1}{b} = \frac{1}{a} + \frac{1}{b}. \end{aligned}$$

Thus we observe that in  $\sigma > 0$ , using integration by parts,

$$\Gamma(s+1) = s\Gamma(s).$$

Thus for any  $n \in \mathbb{N}$ , we have  $\Gamma(n) = n!$ . We have,

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \sigma > 0.$$

Note  $\Gamma(s+1)$  is analytic if  $\sigma > -1$ . Thus  $\Gamma(s)$  is analytic in  $\sigma > -1$  except simple pole at  $s = 0$  with residue 1. That is

$$\lim_{s \rightarrow 0} s\Gamma(s) = 1.$$

We also have,

$$\Gamma(s) = \frac{\Gamma(s+2)}{\Gamma(s+1)}, \sigma > 0.$$

Note that in the numerator, we have  $\sigma > -2$ .  $\Gamma(s)$  is analytic when  $\sigma > -2$  except simple poles at  $s = 0, -1$  with residue

$$\lim_{s \rightarrow -1} (s+1)\Gamma(s) = \frac{\Gamma(-1)}{-1} = -1.$$

Iterating the process, we have the following theorem,

**Theorem 2.31.**  $\Gamma(s)$  can be analytically continued to  $\mathbb{C}$  except simple poles at  $s = -k$  where  $k \in \mathbb{Z}_{\geq 0}$  with residue  $\frac{(-1)^k}{k!}$ .

**Remark 2.18.**

$$\begin{aligned} \Gamma(s)\Gamma(1-s) &= \frac{\pi}{\sin \pi s}, s \in \mathbb{C}. \\ \lim_{s \rightarrow m} \Gamma(s)\Gamma(1-s) \sin \pi s &= \pi. \\ \Gamma(s)\Gamma\left(s + \frac{1}{2}\right) &= \sqrt{\pi} 2^{1-2s} \Gamma(2s), \forall s \in \mathbb{C}. \end{aligned}$$

**Exercise 2.5.** Show

$$\forall \alpha \in \mathbb{R}, x > 0, \sum_{n \in \mathbb{Z}} e^{-(n+\alpha)\frac{\pi}{x}} = \sqrt{x} \sum_{n \in \mathbb{Z}} e^{-n^2\pi x + 2\pi i n \alpha}.$$

Note that  $\sum_{n \in \mathbb{Z}} a_n$  is convergent if

$$s_N := \sum_{|n| \leq N} a_n,$$

is convergent.

**Theorem 2.32.** Let  $\sigma > 1$  and  $F \in L^1(\mathbb{R})$ , ie  $F : \mathbb{R} \rightarrow \mathbb{C}$  and

$$\int_{-\infty}^{\infty} |F(t)| dt < \infty.$$

We have,

$$\sum_{n \in \mathbb{Z}} F(n+u)$$

is absolutely and uniformly convergent in  $u$ . Also we have,

$$\sum_{n \in \mathbb{Z}} |\hat{F}(n)| \leq \infty,$$

then

$$\sum_{n \in \mathbb{Z}} F(n+u) = \sum_{n \in \mathbb{Z}} \hat{F}(n) e^{2\pi i n u}.$$

*Proof.* Use Exercise 2.5. □

**Theorem 2.33.**  $\Gamma(s) \neq 0, \forall s \in \mathbb{C}$ .

*Proof.* Note that the case when  $s \in \mathbb{Z}$  is already shown. Thus suppose  $s \notin \mathbb{Z}$ . Suppose  $\Gamma(s) = 0$ , then it will follow that  $\Gamma(1-s)$  by Remark 2.18. But this is a contradiction to Theorem 2.31 as it has a pole at non-integer point. Similarly from Remark 2.18, we see that  $\frac{1}{\Gamma(s)}$  has a simple zero at  $s = 0, -1, -2, \dots$ .

By definition, consider,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t} t^{\frac{s}{2}-1} dt, \sigma > 0.$$

Replace  $t = n^2\pi x$ ,  $dt = n^2\pi dx$ , we get,

$$\begin{aligned}
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} &= \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx, \sigma > 1. \\
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \sum_{n \in \mathbb{N}} \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx, \\
&= \int_0^\infty x^{\frac{s}{2}-1} \left( \sum_{n \in \mathbb{N}} e^{-n^2\pi x} \right) dx, \\
&= \sum_{n \in \mathbb{N}} \int_0^\infty |x^{\frac{s}{2}-1} e^{-n^2\pi x}| dx, \\
&= \sum_{n \in \mathbb{N}} \left( \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx \right), \\
&= \sum_{n=1}^\infty \pi^{-\frac{\sigma}{2}} \Gamma\left(\frac{\sigma}{2}\right) n^{-\sigma}, \\
&= \pi^{-\frac{\sigma}{2}} \Gamma\left(\frac{\sigma}{2}\right) \zeta(s), \sigma > 1, \\
&< \infty.
\end{aligned}$$

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} \left( \sum_{n \in \mathbb{N}} e^{-n^2\pi x} \right).$$

Using Exercise 2.5 and set  $\alpha = 0$ , we have,

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} e^{-n^2\frac{\pi}{s}} &= \sqrt{s} \sum_{n \in \mathbb{Z}} e^{-n^2\pi x}. \\
\sum_{n \in \mathbb{Z}} e^{-n^2\pi x} &\ll \sum_{n \in \mathbb{N}} e^{-n\pi x}, \\
&\ll \int_1^\infty e^{-\pi xt} dt, \\
&\ll \frac{e^{-\pi x}}{x}, x > 0.
\end{aligned}$$

Set  $\theta(x) := \sum_{n \in \mathbb{Z}} e^{n^2\pi x}$  then,

$$\begin{aligned}
\theta(x) &= 1 + 2 \sum_{n \in \mathbb{N}} e^{-n^2\pi x}. \\
\sum_{n \in \mathbb{N}} e^{-n^2\pi x} &= \frac{\theta(x) - 1}{2}, x > 0.
\end{aligned}$$

Using the exercise, we have,

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\theta(x).$$

Set  $w(x) := \frac{\theta(x)-1}{2}$ . Thus write

$$\begin{aligned} w\left(\frac{1}{x}\right) &= \frac{\theta\left(\frac{1}{x}\right) - 1}{2}, \\ &= \frac{\sqrt{x}\theta(x) - 1}{2}, \\ &= \frac{\sqrt{x}\theta(x) - 1}{2}. \\ w\left(\frac{1}{x}\right) &= \sqrt{x}w(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}. \\ \pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1}w(x)dx, \sigma > 1. \end{aligned}$$

Step 3

$$\int_0^1 x^{\frac{s}{2}-1}w(x)dx + \int_1^\infty x^{\frac{s}{2}-1}w(x)dx.$$

Taking  $x = \frac{1}{y}$  and  $dx = -\frac{1}{y^2}dy$  we have,

$$\begin{aligned} \int_1^\infty \left(\frac{1}{y^2}\right)^{\frac{s}{2}-1} w\left(\frac{1}{y}\right) \frac{-1}{y^2} dy &= \int_1^\infty y^{-\frac{s}{2}} w\left(\frac{1}{y}\right) \frac{dy}{y}, \\ &= \int_1^\infty y^{-\frac{1}{2}} \left(\sqrt{y}w(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right) \frac{dy}{y}, \\ &= \int_1^\infty y^{-\frac{s}{2}+\frac{1}{2}} w(y) \frac{dy}{y} + \int_1^\infty \frac{y^{-\frac{s}{2}+\frac{1}{2}-1}}{2} dy - \frac{1}{2} \int_1^\infty y^{-\frac{s}{2}-1} dy, \\ &= \int_1^\infty y^{\frac{1-s}{2}} w(y) \frac{dy}{y} + \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x) \frac{dx}{x}, \sigma > 1. \end{aligned}$$

Step 4,

$$\begin{aligned} \int_1^\infty |x^{\frac{s}{2}} + x^{\frac{1-s}{2}}| |w(x)| \frac{dx}{x} &\leq \int_1^\infty (x^{\frac{\sigma}{2}-1} + x^{\frac{1-\sigma}{2}-1}) |w(x)| dx, \\ &\ll \int_1^\infty \frac{(x^{\frac{\sigma}{2}-1} + x^{\frac{1-\sigma}{2}-1})}{e^{\pi x}} dx, \sigma \in \mathbb{R}, \\ &\ll \int_1^\infty \frac{e^x}{e^{\pi x}} dx, \quad \ll 1. \end{aligned}$$

$\int_1^\infty |x^{\frac{s}{2}} + x^{\frac{1-s}{2}}| |w(x)| \frac{dx}{x}$  is analytic in  $\mathbb{C}$ .

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = 1 + s(s-1) \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x) \frac{dx}{x}.$$

Set  $\xi(x) := 1 + s(s-1) \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x) \frac{dx}{x}$ , then it is entire. and for all  $s$  we have,

$$\xi(1-s) = \xi(s).$$

Thus obtain,

$$(1-s)(1-s-1)\pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Using the construction of  $w(s)$  we have,

$$|w(x)| \ll |\theta(x)| \ll \frac{e^{-\pi x}}{x}, \forall x > 0.$$

Also we have,

$$\begin{aligned} \zeta(1-s) &= \pi^{-s+\frac{1}{2}} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \zeta(s), \\ \zeta(1-s) &= \pi^{-s} 2^{1-s} \cos\left(\frac{\pi s}{s}\right) \Gamma(s) \zeta(s). \end{aligned}$$

Note that  $\zeta(s)$  has an analytic continuation to  $\mathbb{C}$  except simple pole at  $s$ .

$$\begin{aligned} \lim_{s \rightarrow 1} \zeta(1-s) &= \pi^{-1} \lim_{s \rightarrow 1} \frac{\cos \frac{\pi s}{2} \zeta(s)(s-1)}{s-1}. \\ \lim_{s \rightarrow 1} \frac{\cos \frac{\pi s}{2}}{s-1} &= \lim_{s \rightarrow 0} \frac{\cos\left(\frac{\pi}{2}(1-s)\right)}{-s}, \\ &= \lim_{s \rightarrow 0} \frac{\sin \frac{\pi}{2} s \frac{\pi}{2}}{-s \cdot \frac{\pi}{2}}, \\ &= -\frac{\pi}{2}. \\ \Rightarrow \lim_{s \rightarrow 1} \zeta(1-s) &= -\frac{1}{2}. \end{aligned}$$

$$\zeta(-2n) = \pi^{-(2n+1)} 2^{1-2n-1} \cos\left(\frac{\pi}{2}(2n+1)\right) \Gamma(2n+1) \zeta(2n+1) = 0.$$

Recall

$$\begin{aligned} \zeta(s) &= \sum_{n \in \mathbb{N}} \frac{1}{n^s}, \\ &= \frac{s}{s-1} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt, \quad \sigma > 0, s \neq 1. \end{aligned}$$

If  $s$  is real,

$$\begin{aligned} \left| \zeta(s) - \frac{s}{s-1} \right| &\leq |s| \int_1^\infty \frac{\{t\}}{t^{\sigma+1}} dt, \\ &< \frac{|s|}{\sigma} = \frac{\sigma}{\sigma} = 1. \end{aligned}$$

Thus we obtain,

$$\begin{aligned}
-1 + \frac{s}{s-1} &< \zeta(s) < 1 + \frac{s}{s-1} \\
\frac{1}{s-1} &< \zeta(s) < \frac{2s-1}{s-1}, \\
-1 < (1-s)\zeta(s) &< 1-2s < 0 \quad \text{if } \frac{1}{2} < s < 1 \\
\Rightarrow \zeta(s) &\neq 0, \quad \text{if } \frac{1}{2} < s < 1.
\end{aligned}$$

□

## 2.7 Primitive characters

Suppose  $q_0|q$ . Then we have a natural map

$$\mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{Z}/q_0\mathbb{Z}$$

**Definition 2.16.** Let  $\chi \pmod{q}$  be a character and  $q_0|q$ . If there exists a character  $\chi^* \pmod{q_0}$  making the following diagram commutative,

$$\begin{array}{ccc}
(\mathbb{Z}/q\mathbb{Z})^\times & \xrightarrow{\chi} & \mathbb{C}^\times \\
f \downarrow & \nearrow \chi^* & \\
(\mathbb{Z}/q_0\mathbb{Z})^\times & &
\end{array}$$

we say  $\chi$  factors through  $(\mathbb{Z}/q_0\mathbb{Z})^\times$  or  $\chi^*$  induces  $\chi$ . If  $q_0$  is minimal among positive such numbers, we say  $q_0$  is a conductor of  $\chi$ .

**Definition 2.17.** A character  $\chi \pmod{q}$  is called primitive if its conductor is  $q$  only. Otherwise the character is imprimitive.

**Example 2.11.** Take  $\chi : (\mathbb{Z}/8\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  such that

$n$	$\chi(n)$
1	1
3	-1
5	1
7	-1

Then 4 is a conductor with  $\chi^* : (\mathbb{Z}/4\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ ,

$n$	$\chi^*(n)$
1	1
3	-1

**Example 2.12.** Take  $\chi : (\mathbb{Z}/6\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  such that

$n$	$\chi(n)$
1	1
5	-1

Then 3 is a conductor with  $\chi^* : (\mathbb{Z}/3\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ ,

$n$	$\chi^*(n)$
1	1
2	-1

**Remark 2.19.** All principal characters mod  $q$  for  $q > 1$  are imprimitive by construction.

We set as usual  $s = \sigma + it$  for  $\sigma, t \in \mathbb{R}$ .

**Lemma 2.14.** Suppose  $\chi \bmod q$  is induced by  $\chi^* \bmod q_0$ . We then have,

$$\sigma > 1 \Rightarrow L(s, \chi) = L(s, \chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right).$$

*Proof.* We know that for  $\sigma > 1$ ,

$$L(s, \chi) = \prod_{p \nmid q} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Also

$$\begin{aligned} L(s, \chi^*) &= \prod_{p \nmid q_0} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \\ &= \prod_{\substack{p \nmid q_0 \\ p \nmid q}} \left(1 - \frac{\chi^*(p)}{p^s}\right)^{-1} \prod_{\substack{p \nmid q_0 \\ p|q}} \left(1 - \frac{\chi^*(p)}{p^s}\right)^{-1}, \\ &= L(s, \chi) \prod_{\substack{p \nmid q_0 \\ p|q}} \left(1 - \frac{\chi^*(p)}{p^s}\right)^{-1}. \end{aligned}$$

□

**Remark 2.20.** Using Identity theorem, Lemma 2.14 holds for all  $s \in \mathbb{C}$ .

**Lemma 2.15.** Suppose  $x \in [0, \frac{1}{2}]$ , then

$$|\sin \pi x| \geq 2x.$$

*Proof.* Exercise. □

**Definition 2.18** (Gauss sum). Let  $\chi \bmod q$  be a character. The Gauss sum is,

$$\tau(\chi) := \sum_{k=1}^q \chi(k) e^{\frac{2\pi k i}{q}}.$$

**Lemma 2.16.** For  $n \in \mathbb{Z}$ , we have,

$$\chi(n)\tau(\bar{\chi}) = \sum_{k=1}^q \bar{\chi}(k) e^{\frac{2\pi i k n}{q}},$$

where  $\chi$  is a primitive character modulo  $q$  and  $\chi \neq \chi_0$ .

**Theorem 2.34.** If  $\chi \bmod q$  is primitive then,

$$|\tau(\chi)| = \sqrt{q}.$$

*Proof.* By Lemma 2.16, we have,

$$\begin{aligned} \chi(n)\tau(\bar{\chi}) &= \sum_{k=1}^q \bar{\chi}(k) e^{\frac{2\pi i k}{q}}, \\ \overline{\chi(n)\tau(\bar{\chi})} &= \sum_{l=1}^q \chi(l) e^{-\frac{2\pi i l}{q}}. \end{aligned}$$

Multiplying this equation with the one in the statement we have,

$$|\chi(n)|^2 |\tau(\bar{\chi})|^2 = \sum_{k,l=1}^q \overline{\chi(k)} \chi(l) e^{\frac{2\pi i(k-l)}{q}}.$$

Applying  $\sum_{n=1}^q$  to the both sides of the above equation, we have,

$$|\tau(\bar{\chi})|^2 \sum_{n=1}^q |\chi(n)|^2 = \sum_{k,l=1}^q \overline{\chi(k)} \chi(l) \left( \sum_{n=1}^q \left( e^{\frac{2\pi i(k-l)}{q}} \right)^n \right).$$

Note that

$$\begin{aligned} x + x^2 + \cdots + x^q &= \begin{cases} \frac{x(x^q - 1)}{x - 1}, & (x \neq 1), \\ q, & (x = 1), \end{cases} \\ &= \begin{cases} 0, & (x \neq 1, x^q = 1), \\ q, & (x = 1), \end{cases} \end{aligned}$$

Take  $x = e^{\frac{2\pi i(k-l)}{q}} = \cos \frac{2\pi}{q}(k-l) + i \sin \frac{2\pi}{q}(k-l) = 1$ .  $x = 1$  if and only if  $q|k-l$  thus

$$\sum_{n=1}^q \left( e^{\frac{2\pi i(k-l)}{q}} \right)^n = \begin{cases} 0, & (q \nmid k-l), \\ q, & (\text{otherwise}). \end{cases}$$

Therefore, we get,

$$\begin{aligned}
|\tau(\bar{\chi})|^2 \sum_{n=1}^q |\chi(n)|^2 &= q \sum_{\substack{k,l=1 \\ q|k-l}}^q \bar{\chi}(k)\chi(l), \\
&= q \sum_{k=1}^q \bar{\chi}(k)\chi(k), \\
&= q \sum_{k=1}^q |\chi(k)|^2.
\end{aligned}$$

□

**Theorem 2.35** (Pólya-Vinogradov).

Let  $\chi \pmod{q}$  be a primitive character. We have,

$$\left| \sum_{n \leq x} \chi(n) \right| \ll \sqrt{q} \log q.$$

This inequality is uniform in  $q$ .

*Proof.* Consider  $(n, q) = 1$ , then

$$\begin{aligned}
\chi(n)\tau(\bar{\chi}) &= \chi(n) \sum_{k=1}^q \bar{\chi}(k) e^{\frac{2\pi i k}{q}}, \\
&= \chi(n) \sum_{\substack{k=1 \\ (k,q)=1}}^q \bar{\chi}(k) e^{\frac{2\pi i k}{q}},
\end{aligned}$$

Set  $k = nt$ , we get,

$$\begin{aligned}
\chi(n)\tau(\bar{\chi}) &= \sum_{\substack{t=1 \\ (t,q)=1}}^q \bar{\chi}(nt)\chi(n) e^{\frac{2\pi i nt}{q}}. \\
\bar{\chi}(nt) &= \bar{\chi}(n)\bar{\chi}(t). \\
\chi(n)\tau(\bar{\chi}) &= \sum_{\substack{t=1 \\ (t,q)=1}}^q \bar{\chi}(t) e^{\frac{2\pi i nt}{q}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\tau(\bar{\chi}) \left( \sum_{n \leq x} \chi(n) \right) &= \sum_{k=1}^{q-1} \bar{\chi}(k) \left( \sum_{n \leq x} e^{\frac{2\pi i k n}{q}} \right), \\
|\tau(\bar{\chi})| \cdot \left| \sum_{n \leq x} \chi(n) \right| &\leq \sum_{k=1}^{q-1} \left| \sum_{n \leq x} e^{\frac{2\pi i k n}{q}} \right|, \\
&= \sum_{k=1}^{q-1} \left| \frac{e^{\frac{2\pi i k}{q}} (e^{\frac{2\pi i n[x]}{q}} - 1)}{e^{\frac{2\pi i k}{q}} - 1} \right|, \\
&\leq \sum_{k=1}^{q-1} \frac{2}{|e^{\frac{2\pi i k}{q}} - 1|},
\end{aligned}$$

Note that for all  $y \in \mathbb{R}$ ,

$$\begin{aligned}
2i \sin y &= e^{-iy} (e^{2iy} - 1), \\
|2 \sin y| &= |e^{2iy} - 1|.
\end{aligned}$$

Apply this to the equation above we have,

$$\sum_{k=1}^{q-1} \frac{2}{|e^{\frac{2\pi i k}{q}} - 1|} = \sum_{k=1}^{q-1} \frac{1}{|\sin \frac{\pi k}{q}|}.$$

Using Lemma 2.15, we get,

$$\begin{aligned}
\sum_{k=1}^{q-1} \frac{1}{|\sin \frac{\pi k}{q}|} &= \sum_{1 \leq k \leq \frac{q}{2}} \frac{1}{|\sin \frac{\pi k}{q}|} + \sum_{\frac{q}{2} < k} \frac{1}{|\sin \frac{\pi k}{q}|}, \\
&= 2 \sum_{1 \leq k \leq \frac{q}{2}} \frac{1}{|\sin \frac{\pi k}{q}|}, \\
&\leq \sum_{1 \leq k \leq \frac{q}{2}} \frac{k}{q} \ll q \log \frac{q}{2} \ll q \log q.
\end{aligned}$$

Recall Theorem 2.34, we have  $|\tau(\chi)| = \sqrt{q}$  as  $\chi$  is primitive. Therefore, for  $q > 1$ , we have,

$$\left| \sum_{n \leq x} \chi(n) \right| \ll \sqrt{q} \log q,$$

uniformly in  $q$  as  $x \rightarrow \infty$ .  $\square$

Consider  $L(q, \chi)$  functional equation, where  $\chi \neq \chi_0$  and  $\chi$  is primitive. Suppose  $\chi(-1) = 1$  (i.e it is an even character). From the proof of Theorem 2.33, we have,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty x^{\frac{s}{2}} e^{-n^2 \pi x} \frac{dx}{x}.$$

Replace  $x$  with  $\frac{x}{q}$  then for  $\operatorname{Re}(s) = \sigma > 0$ , we have,

$$\pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty x^{\frac{s}{2}} e^{-n^2 \pi x} \frac{dx}{x}.$$

For  $\sigma > 1$ , we have,

$$\begin{aligned} \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi) &= \sum_{n=1}^\infty \int_0^\infty \chi(n) x^{\frac{s}{2}} e^{-n^2 \frac{\pi x}{q}} \frac{dx}{x}, \\ &= \int_0^\infty x^{\frac{s}{2}} \left( \sum_{n=1}^\infty \chi(n) e^{-n^2 \frac{\pi x}{q}} \right) \frac{dx}{x}. \end{aligned}$$

Let

$$\theta(x, \chi) = \sum_{n \in \mathbb{Z}} \chi(n) e^{-\frac{n^2 \pi x}{q}} = 2 \sum_{n=1}^\infty \chi(n) e^{-\frac{n^2 \pi x}{q}}, (x > 0).$$

Using this, we get,

$$\int_0^\infty x^{\frac{s}{2}} e^{-n^2 \pi x} \frac{dx}{x} = \frac{1}{2} \int_0^\infty x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x}, (\sigma > 1).$$

Split the integral into  $\int_0^1, \int_1^\infty$ .

$$\begin{aligned} \tau(\bar{\chi}) \theta(x, \chi) &= \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \bar{\chi}), \\ &= \sum_{n \in \mathbb{Z}} (\tau(\bar{\chi}) \chi(n)) e^{-\frac{n^2 \pi x}{q}}, \\ &= \sum_{n \in \mathbb{Z}} \sum_{k=1}^q \bar{\chi}(k) e^{\frac{2\pi i nk}{q}} e^{-\frac{n^2 \pi x}{q}}, \\ &= \sum_{k=1}^q \bar{\chi}(k) \sum_{n \in \mathbb{Z}} e^{\frac{2\pi i kn}{q} - n^2 \frac{\pi x}{q}}, \\ &\text{From Lecture 7 page 6 } \sum_{k=1}^q \bar{\chi}(k) \left(\frac{x}{q}\right)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} e^{-(m + \frac{m}{q}) \frac{\pi q}{x}}, \\ &= \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^q \bar{\chi}(k) \sum_{n \in \mathbb{Z}} e^{-(m + \frac{m}{q}) \frac{\pi q}{x}}. \end{aligned}$$

Put  $qn + m = t$ , then

$$\bar{\chi}(qn + m) = \bar{\chi}(m) = \bar{\chi}(t).$$

Thus,

$$\left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^q \bar{\chi}(k) \sum_{n \in \mathbb{Z}} e^{-(m+\frac{m}{q})\frac{\pi q}{x}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{t \in \mathbb{Z}} \bar{\chi}(t) e^{-\frac{t^2 \pi}{qx}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \bar{\chi}).$$

Now for the splitted integral, we have,

$$\frac{1}{2} \int_0^1 x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x} + \frac{1}{2} \int_1^\infty x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x} \frac{dx}{x}.$$

Replacing  $x$  with  $\frac{1}{x}$ , we get,

$$= \frac{\tau(\chi)}{2\sqrt{q}} \int_1^\infty x^{\frac{1-s}{2}} \theta(x, \bar{\chi}) \frac{dx}{x} + \frac{1}{2} \int_1^\infty \int_0^1 x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x}.$$

Set

$$\xi(s, \chi) := \frac{\tau(\chi)}{2\sqrt{q}} \int_1^\infty x^{\frac{1-s}{2}} \theta(x, \bar{\chi}) \frac{dx}{x} + \frac{1}{2} \int_1^\infty \int_0^1 x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x}.$$

Use that  $\theta(x, \bar{\chi}) \ll e^{-\frac{\pi x}{q}}$  and  $|\tau(\chi)| = \sqrt{q}$ . It turns out that  $\xi$  is uniformly bounded on compact subsets of  $\mathbb{C}$  and in particular, this is entire.

**Lemma 2.17.**

$$\xi(1-s, \chi) = \frac{\tau(\chi)}{\sqrt{q}} \xi(s, \bar{\chi}).$$

*Proof.* Recall,

$$\tau(\bar{\chi}) \theta(x, \chi) = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \bar{\chi}),$$

and

$$|\tau(\chi)|^2 = q.$$

We get,

$$\frac{\tau(\chi)}{\sqrt{q}} = \frac{\sqrt{q}}{\tau(\bar{\chi})} = \frac{\sqrt{q}}{\tau(\bar{\chi})}.$$

Then,

$$L(1-s, \chi) = \frac{\tau(\chi)}{q^{1-s}} \pi^{-s} 2^{1-s} \cos \frac{\pi s}{2} (\Gamma(s)) L(s, \bar{\chi}).$$

Note that  $\chi(-1) = 1$  and is primitive (in the case of odd integer replace cos with sin). Note also that

i).  $L(s, \chi) = 0, s = 0, -2, -4, \dots$  when  $\chi$  is even,

ii).  $L(s, \chi) = -1, -3, -5, \dots$  when  $\chi$  is odd.

When  $0 < \sigma < 1$ ,

□

**Definition 2.19** (Entire function of finite order). *An entire function  $f(z)$  is said to be of finite order if there is  $\alpha > 0$  such that*

$$|f(z)| \ll e^{|z|^\alpha},$$

as  $|z| \rightarrow \infty$ . The order of  $f$  is the infimum of such  $\alpha$ .

**Example 2.13.** We have the following examples,

1.  $e^z \rightarrow 1$ ,
2.  $\sin z \rightarrow 1$ ,
3.  $\cos z \rightarrow 1$ ,
4.  $e^{z^n} \rightarrow n$ .

For the trigonometric functions use  $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$  and similarly for  $\cos$ .

**Remark 2.21.** Is  $|\sin z| < z$ ,  $z \in \mathbb{C}$ ? To show this consider,

$$f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0, 1, z = 0. \\ 0, & z = 0. \end{cases}$$

And apply Liouville's theorem.

**Theorem 2.36** (Special case of Hadamard's factorization theorem). *Let  $f(z)$  be an entire function of order 1 and  $(z_j)_{j \in \mathbb{N}}$  be such that  $z_j \in \mathbb{C} \setminus \{0\}$ , and  $f(z_j) = 0$ . Then there are some constants  $A, B$  such that*

$$f(z) = e^{A+Bz} \prod_{j \geq 1} \left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j}}.$$

**Theorem 2.37.** *Let  $f$  be an entire function of order  $\alpha \in \mathbb{Z}$  without any zeros and assume  $f(1) \leq 0$ . Then we have,*

$$f(z) = e^{p(z)},$$

where  $p$  is some polynomial of degree  $\alpha$ .

Recall the completed zeta function,

**Definition 2.20.**

$$\xi(s) := (s-1)\pi^{\frac{-s}{2}} \Gamma\left(\frac{s}{2} + 1\right) \zeta(s).$$

We know,

1.  $\xi(s)$  has an order 1,
2.  $\zeta(s)$  has infinitely zeros in the critical strip.
3.  $\xi(s) = e^{As+B} \prod_{\substack{\rho, \xi(\rho)=0 \\ 0 < \operatorname{Re}(\rho) < 1}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$ , where  $A, B$  are some constant.

**Remark 2.22.** Zeros of  $\xi(s)$  are precisely the non-trivial zeros of  $\zeta(s)$ .

**Theorem 2.38.** There exists a constant  $c > 0$  such that for  $\sigma, t \in \mathbb{R}$   $s = \sigma + it$ ,  $\sigma \geq 1 - \frac{c}{\log(|t|+2)}$ , we have,

$$\zeta(s) \neq 0.$$

Taking derivatives, we obtain,

$$\begin{aligned} \frac{\xi'(s)}{\xi(s)} &= \frac{1}{s-1} - \frac{1}{2} \log \pi + \frac{\Gamma'(\frac{s}{2}+1)}{2\Gamma(\frac{s}{2}+1)} + \frac{\zeta'(s)}{\zeta(s)}, \\ \frac{\xi'(s)}{\xi(s)} &= A + \sum_{\substack{\zeta(\rho)=0 \\ 0 < \operatorname{Re}(\rho) < 1}} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right). \end{aligned}$$

Note that  $\frac{s}{2}\Gamma(\frac{s}{2}) = \Gamma(\frac{s}{2}+1)$ . Thus we obtain,

$$-\frac{\zeta'(s)}{\zeta} = \frac{1}{s-1} - A - \frac{\log \pi}{2} + \frac{1}{2} \frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} - \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

We then get,

$$\operatorname{Re} \left( -\frac{\zeta'(s)}{\zeta} \right) = \operatorname{Re} \left( \frac{1}{s-1} \right) - A - \frac{\log \pi}{2} + \frac{1}{2} \operatorname{Re} \left( \frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} \right) - \sum_{\rho} \operatorname{Re} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

Recall that for  $s = \sigma + it$ ,  $\sigma, t \in \mathbb{R}$ , we have,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}, \sigma \geq 1.$$

And also,

$$\frac{-3\zeta'(\sigma)}{\zeta(\sigma)} - \operatorname{Re} \left( \frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} \right) - \operatorname{Re} \left( \frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} \right).$$

Check that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} (3 + 4 \cos(t \log n) + \cos(2t \log n)) > 0, \sigma > 1.$$

For  $1 < \sigma < 2, t \geq 1$ , we have,

$$\operatorname{Re} \left( \frac{1}{s-1} \right) = \frac{\sigma-1}{|s-1|^2} < \frac{1}{t^2} < 1.$$

$$\begin{aligned} \operatorname{Re} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right) &= \operatorname{Re} \left( \frac{\overline{s-\rho}}{|s-\rho|^2} + \frac{\bar{\rho}}{|\rho|^2} \right), \\ &= \frac{\sigma-\beta}{|s-\rho|^2} + \frac{\beta}{|\rho|^2} > 0, \end{aligned}$$

where  $\beta = \operatorname{Re}(\rho)$ . We have,

$$\frac{\Gamma'(\frac{s}{2} + 1)}{\Gamma(\frac{s}{2} + 1)} < B \log\left(\frac{|t|}{2} + 2\right), \forall t \in \mathbb{R}.$$

We have the following equations,

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z, |z| \rightarrow \infty.$$

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}},$$

where  $\gamma$  is the Euler's constant.

Now consider  $1 \leq \sigma \leq 2$ , we have,

$$\left| \frac{\Gamma'(s)}{\Gamma(s)} \right| < B \log(|t| + 2),$$

where  $B$  is some constant for all  $t \in R$ . From above expressions, we obtain,

$$\operatorname{Re}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) < B' \log\left(\frac{|t|}{2} + 2\right), \forall t \geq 1.$$

Let  $\rho = \beta + it$ , be any non-trivial zero of  $\zeta(s)$ .

$$\operatorname{Re}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) < B' \log\left(\frac{|t|}{2} + 2\right) - \frac{1}{\sigma - \beta}, t \geq 1.$$

Recall for  $\sigma > 0$ ,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + g(s),$$

where  $g(s)$  is some analytic function. For  $1 \leq \sigma \leq 2$ ,

$$\begin{aligned} \frac{-\zeta'(\sigma)}{\zeta(\sigma)} &= \frac{1}{\sigma-1} - g(\sigma), \\ &< \frac{1}{\sigma-1} + c, \end{aligned}$$

where  $C$  is some constant.

$$0 < \frac{-3\zeta'(\sigma)}{\zeta(\sigma)} - 4 \operatorname{Re}\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right) - \operatorname{Re}\left(\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right).$$

We then obtain,

$$0 < 3 \left( \frac{1}{\sigma-1} + C \right) + 4B' \log(|t| + 2) + B' \log(|t| + 2) - \frac{1}{\sigma - \beta}.$$

Note that

$$\frac{1}{\sigma - \beta} < \frac{3}{\sigma - 1} + B'' \log(|t| + 2), t \geq 1.$$

Take  $\sigma = 1 + \frac{\varepsilon}{\log(|t| + 2)}$ , where  $\varepsilon > 0$ , sufficiently small which will be chosen later.

$$\beta < 1 - \frac{\varepsilon(1 - B''\varepsilon)}{\log(|t| + 2)}, t \geq 1.$$

So far we have shown that for  $t \geq 1$  and

$$\sigma \geq 1 - \frac{c}{\log(|t| + 2)},$$

we have  $\zeta(s) \neq 0$ . For some constant  $C > 0$ .

For  $|t| \leq 1$ , since  $\zeta(9s)$  be analytic except  $s = 1$ , there exists a constant  $c_1 > 0$  such that

$$\sigma > 1 - c_1 \Rightarrow \zeta(s) \neq 0.$$

If

$$1 - c_1 \leq 1 - \frac{c_2}{\log(|t| + 2)},$$

then we have,

$$c_1 \geq \frac{c_2}{\log(|t| + 2)}, |t| < 1.$$

Choose  $c_2 = c_1 \log 2$ . Take  $c' = \min(c_1, c_1 \log 2)$ .

Above proof can be generalized to  $L(s, \chi)$  where  $\chi$  is a non-trivial character.

**Theorem 2.39.** *Let  $\chi$  be a non-trivial character. Then there is  $c_1 > 0$  such that for  $s = \sigma + it, \sigma, t \in \mathbb{R}$ , we have,*

$$\sigma \geq 1 + \frac{c_1}{\log(q(|t| + 2))}.$$

**Theorem 2.40** (Siegel zeros). *There exists an absolute constant  $c_2 > 0$  such that  $L(s, \chi)$  where  $\chi$  is quadratic mod  $q$  has at most one zero in the region,*

$$\sigma > 1 - \frac{\delta}{\log q}, |t| < \frac{\delta}{\log q},$$

where  $0 < \delta < c_2$ . If such a zero exists, it is real, simple and often called Siegel zero.

Recall the completed Dirichlet  $L$ -function,

$$\xi(s, \chi) := \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi).$$

See the lecture note from 12/16. Where

$$a = \begin{cases} 0, & \chi(-1) = 1, \\ 1, & \chi(-1) = -1. \end{cases}$$

We have  $\xi(s, \chi)$  is of order 1 and

$$\xi(s, \chi) = e^{A_1 + B_1 s} \prod_{\substack{\rho, L(\rho, \chi)=0 \\ 0 \leq \operatorname{Re}(\rho) < 1}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}.$$

Furthermore  $L(s, \chi)$  has infinitely many zero in the critical strip.

**Remark 2.23.**

$$L(s, \chi) = L(s, \chi^*) \prod_{p|q} \left(1_{\frac{\chi^*(p)}{p^s}}\right), s \in \mathbb{C}$$

Then  $L(s, \chi) = 0$  if and only if  $L(s, \chi^*) = 0$  when  $\sigma \neq 0$ .

Recall

$$\xi(s) = \frac{s(s-1)}{2} \pi^{\frac{-s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

**Definition 2.21.**

$$N(T) = |\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \beta \in (0, 1), \gamma \in (0, T)\}|.$$

**Theorem 2.41.** As  $T \rightarrow \infty$ , we have,

$$N(T+1) - N(T) \ll \log T.$$

*Proof.* Enough to show that as  $T \rightarrow \infty$ .

$$\sum_{\substack{\rho = \beta + i\gamma \\ \zeta(\rho) = 0 \\ \beta \in (0, 1)}} \frac{1}{1 + (T - \gamma)^2} \ll \log T$$

Observe that,

$$N(T+1) - N(T) = \sum_{\substack{\rho = \beta + i\gamma \\ \beta \in (0, 1) \\ \zeta(\rho) = 0 \\ T < \gamma < T+1}} \leq \sum.$$

Recall we have,

$$-\operatorname{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) < A \log(|t| + 2) - \sum_{\rho = \beta + i\gamma} \operatorname{Re}\left(\frac{1}{s - \rho} - \frac{1}{\rho}\right).$$

where  $s = \sigma + it$  for  $\sigma \in [1, 2]$ ,  $|t| > 1$  and  $A$  is some constant. Substitute  $s = 2 + iT$  in the equation, we have,

$$\operatorname{Re}\left(\frac{1}{s - \rho}\right) = \operatorname{Re}\left(\frac{(s - \bar{\rho})}{|s - \rho|^2}\right).$$

Put  $s = 2 + iT$  and  $\rho = \beta + i\gamma$ , we have,

$$\operatorname{Re} \left( \frac{(s - \rho)}{|s - \rho|^2} \right) = \frac{2 - \beta}{(2 - \beta)^2 + (T - \gamma)^2} \geq \frac{1}{4 + (T - \gamma)^2}.$$

Also,

$$\operatorname{Re} \left( \frac{1}{\rho} \right) = \frac{\beta}{|\rho|^2} > 0.$$

By the previous equation, we get

$$\sum_{\substack{\rho = \beta + i\gamma \\ \beta \in (0,1) \\ \zeta(\rho) = 0}} \operatorname{Re} \left( \frac{1}{2 + iT - \rho} \right) < A \log(T + 2) + \operatorname{Re} \left( \frac{\zeta'(2 + iT)}{\zeta(2 + iT)} \right) \ll \log T.$$

Recall that for  $\sigma > 1$ ,

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s}, \\ \left| \frac{\zeta'(2 + iT)}{\zeta(2 + iT)} \right| &\leq \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} < \infty. \end{aligned}$$

Therefore,

$$\sum_{\rho} \operatorname{Re} \left( \frac{1}{2 + iT - \rho} \right) = \sum_{\rho} \frac{1}{4 + (T - \gamma)^2},$$

and

$$4 + (T - \gamma)^2 \leq 4(1 + (T - \gamma)^2).$$

□

**Theorem 2.42.** As  $T \rightarrow \infty$ ,

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log(T)).$$

That is, as  $T \rightarrow \infty$ ,

$$N(T) \sim \frac{T}{2\pi} \log T.$$

Analogously,  $\chi \bmod q$ , as  $T \rightarrow \infty$ ,

$$N(T, \chi) = \sum_{\substack{\rho = \beta + i\gamma \\ L(\rho, \chi) = 0 \\ \beta \in (0,1) \\ |\gamma| < T}} 1 \sim \frac{T}{\pi} \log qT.$$

**Exercise 2.6.**

$$N(T, \chi)' = \sum_{\substack{\rho = \beta + i\gamma \\ L(\rho, \chi) = 0 \\ \beta \in (0,1) \\ 0 < \text{gamma} < T}} 1 \sim \frac{T}{2\pi} \log qT.$$

Recall by Remark 2.1, we have,

$$\psi(x) := \sum_{n \leq x} \Lambda(n).$$

Also recall that Theorem 2.17,  $\frac{\zeta'(s)}{\zeta(s)}$  has a simple pole at  $s = 1$  with residue  $-1$ .

Let  $s = a$  be a zero of  $\zeta(s)$  of order  $n \in \mathbb{N}$ . Then

$$\zeta(s) = (s - a)^n g(s),$$

where  $g(s)$  is analytic and  $g(a) \neq 0$ . Taking log-derivative of both sides, we have,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{n}{s - a} + \frac{g'(a)}{g(a)}.$$

Residue of  $\frac{\zeta'(s)}{\zeta(s)}$  at 0 of  $\zeta(s)$  is precisely the multiplicity of that zero. Recall from the proof of Theorem ??, we have,

$$\psi(x) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + O\left(\sum_{n \in \mathbb{N}} \Lambda(n) \left(\frac{x}{n}\right) \min(1, T^{-1} |\log \frac{x}{n}|^{-1})\right),$$

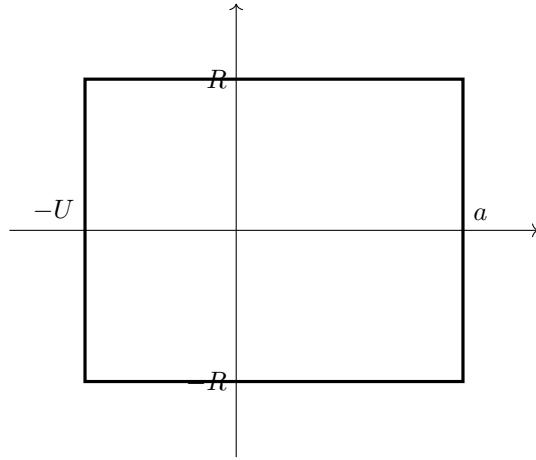
where  $x$  is not an integer and  $T, a \geq 1$ . Similarly to the proof of Theorem ??, take

$$a = 1 + \frac{1}{\log x}.$$

we then get, as  $x \rightarrow \infty$ ,

$$\psi(x) = \frac{1}{2\pi i} \int_{a-iR}^{a+iR} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + O\left(\frac{x \log^2 x}{R}\right).$$

Now consider the following closed path



where  $U$  is some odd integer. By Cauchy's residue theorem, we have,

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_R} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds &= \text{sum of residues of } \frac{\zeta'(s)}{\zeta(s)} \text{ in } C_R, \\ &= \underbrace{x}_{\substack{\text{contribution of poles of } \zeta(s) \\ \text{contribution of } s=0}} - \underbrace{\frac{\zeta'(0)}{\zeta(0)}}_{\substack{\text{contribution of } s=0}} + \underbrace{\sum_{0 < 2m < U} \frac{x^{-2m}}{2m}}_{\substack{\text{contribution of non-trivial zeros of } \zeta(s)}} - \end{aligned}$$

contribution of non-trivial zeros of  $\zeta(s)$

Note that,

$$\int_{C_R} = \int_{a-iR}^{a+iR} + \underbrace{\int_{T_1, T_2, T_3}}_{=I}.$$

We then have

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{0 < 2m < U} \frac{x^{-2m}}{2m} - \sum_{\substack{\rho = \beta + i\gamma \\ \zeta(\rho) = 0 \\ \beta \in (0, 1) \\ |\gamma| < R}} \frac{x^\rho}{\rho} - I + O\left(\frac{x \log^2 x}{R}\right).$$

**Exercise 2.7.** Show that

$$I = \frac{1}{2\pi} \int_{I_1 + I_2 + I_3} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds, \quad = O\left(\frac{x \log^2 R}{R \log x} + \frac{R \log U}{U x^R}\right).$$

Suppose we have above equality, then we have,

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{0 < 2m < U} \frac{x^{-2m}}{2m} - \sum_{\rho} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 R}{R \log x} + \frac{R \log U}{U x^R} + \frac{x \log^2 x}{R}\right).$$

Recall  $U$  is an odd integer. Take  $U \rightarrow \infty$ , we have,

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \log(1 - x^{-2}) - \sum_{\rho} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 R}{R \log x} + \frac{x \log^2 x}{R}\right).$$

Taking  $R \rightarrow \infty$ , we have,

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \log(1 - x^{-2}) - \sum_{\rho} \frac{x^\rho}{\rho}.$$

**Definition 2.22.**

$$\psi(x, \chi) := \sum n \leq x \Lambda(n) \chi(n).$$

Recall we had,

$$\psi(x, q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n),$$

where  $(a, q) = 1$ . By Theorem ??, we have, as  $x \rightarrow \infty$ ,

$$\psi(x, q, a) \sim \frac{x}{\varphi(q)}.$$

**Exercise 2.8.** Show that for  $(a, q) = 1$ , we have,

$$\psi(x, q, a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \psi(x, \chi) \overline{\chi(a)}.$$

**Theorem 2.43.** As  $x \rightarrow \infty$ , we have,

$$\psi(x) = x + O(xe^{-c\sqrt{\log x}}),$$

where  $c > 0$  is some constant.

*Proof.* Recalling zero free regions of  $\zeta$ , we have for  $s = \sigma + it$ ,

$$\sigma > 1 - \frac{c}{\log R} \Rightarrow \zeta(s) \neq 0,$$

where  $c$  is some constant and  $R = \text{Im}(s)$ . From this we have,

$$\left| \sum_{|\gamma| < R} \frac{x^\rho}{\rho} \right| \leq x^{1 - \frac{c}{\log R}} \sum_{\substack{\rho = \beta + i\gamma \\ \zeta(\rho) = 0 \\ \beta \in (0, 1) \\ |\gamma| < R}} \frac{1}{|\rho|} \ll x^{1 - \frac{c}{\log R}} \sum_{\substack{\beta \in (0, 1) \\ \gamma \in (0, R) \\ \zeta(\beta + i\gamma) = 0}} \frac{1}{\sqrt{\beta^2 + \gamma^2}}.$$

Set

$$g(t) = \frac{1}{\sqrt{\beta^2 + t^2}},$$

and

$$a_t := \sum_{\substack{\beta \in (0, 1) \\ \gamma \in (0, t) \\ \zeta(\beta + i\gamma) = 0}} 1.$$

We now have,

$$\begin{aligned} \sum_{\substack{\beta \in (0, 1) \\ \gamma \in (0, R) \\ \zeta(\beta + i\gamma) = 0}} \frac{1}{\sqrt{\beta^2 + \gamma^2}} &= \frac{N(R)}{\sqrt{\beta^2 + \gamma^2}} + \int_0^R \frac{t N(R)}{(\beta^2 + \gamma^2)^{\frac{3}{2}}} dt \\ &\ll \frac{N(R)}{R} + \int_0^R \frac{N(t)}{t^2} dt, \\ &= \log R + \int_0^2 \frac{\log t}{t} dt + \int_2^R \dots, \\ &= \log^2 R. \end{aligned}$$

Combining all we get,

$$\left| \sum_{|\gamma| < R} \frac{x^\rho}{\rho} \right| \ll x^{1 - \frac{c}{\log R}} \log^2 R.$$

Choose  $\log R = c_2(\log \bar{x})^{\frac{1}{2}}$  we have,

$$\psi(x) = x + O(xe^{-c\sqrt{\log x}}).$$

Using Riemann hypothesis, choose  $R = \sqrt{x}$ , we then have as  $x \rightarrow \infty$ ,

$$\psi(x) = x + O(\sqrt{x} \log^2 x) + \frac{R \log U}{Ux^R} + \frac{x \log^2 x}{R}.$$

□

**Theorem 2.44** (Siegel-Walfisz). *For any  $N \in \mathbb{R}$  there is  $c_N > 0$  such that as  $x \rightarrow \infty$ ,*

$$\psi(x, q, a) = \frac{x}{\varphi(q)} + O(xe^{-c_N \sqrt{\log x}}),$$

where  $q \leq (\log x)^N$ . This is uniform in  $q$ .

**Exercise 2.9.** Show that Riemann hypothesis is equivalent to that as  $x \rightarrow \infty$ ,

$$\psi(x) = x + O(\sqrt{x} \log^2 x).$$

**Theorem 2.45.** Riemann hypothesis implies that for  $(a, q) = 1$ , we have, as  $x \rightarrow \infty$ ,

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O(\sqrt{x} \log^2 qx).$$

**Remark 2.24.** For  $(a, q) = 1$ , fixed we have as  $x \rightarrow \infty$ .

$$\pi(x, q, a) < \frac{2x}{\varphi \log \frac{x}{q}}, q < x.$$

### 3 Exercise Solutions

#### 3.1 Lecture 1 ~ 4

**Exercise 3.1.** Show that

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}.$$

*Solution.* From Theorem 2.5, we have,

$$\frac{1}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s}}.$$

Therefore, we examine,

$$\left( \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} \right) \left( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \left( |\mu(n)| \mu\left(\frac{n}{d}\right) \right)$$

Note that if  $n \neq 1$  is not square free,

$$|\mu(n)| \mu\left(\frac{n}{d}\right) = 0,$$

unless  $n = k^2$  and  $d = k$ . If  $n$  is square free then by Theorem 2.1, we have

$$\left( \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} \right) \left( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s}}.$$

□

**Exercise 3.2.** Let  $\pi(x; z)$  be the number of  $n \leq x$  such that it is coprime to all  $p \leq z$ . Then we have,

$$\pi(x; z) = x \prod_{p \leq z} \left(1 - \frac{1}{p}\right) + O(2^z).$$

*Solution.* We have,

$$\begin{aligned} \pi(x; z) &= \sum_{p_1, p_2, \dots, p_k \leq z} (-1)^{k+1} \left[ \frac{x}{p_1 p_2 \cdots p_k} \right], \\ &= \sum_{p_1, p_2, \dots, p_k \leq z} (-1)^k \left( \frac{x}{p_1 p_2 \cdots p_k} - \left\{ \frac{x}{p_1 p_2 \cdots p_k} \right\} \right), \\ &= x \prod_{p \leq z} \left(1 - \frac{1}{p}\right) + \sum_{p_1, \dots, p_k \leq z} (-1)^{k+1} \left\{ \frac{x}{p_1 p_2 \cdots p_k} \right\}, \end{aligned}$$

We have at most  $[z]$  primes and there are  $\binom{[z]}{k}$  many ways to pick  $k$ -many primes. Thus,

$$\left| \sum_{p_1, \dots, p_k \leq z} (-1)^k \left\{ \frac{x}{p_1 p_2 \cdots p_k} \right\} \right| \leq \sum_{k=0}^{[z]} \binom{[z]}{k} = 2^{[z]} \leq 2^z.$$

□

**Exercise 3.3.** Show that

$$\sum_{p \leq x} \frac{1}{p} \geq \log \log x + c,$$

for some constant  $c$ .

*Solution.* Let  $\{p_1, \dots, p_k\}$  be the primes less than  $x$ , then

$$\begin{aligned} \sum_{l \leq x} \frac{1}{l} &\leq \sum_{\alpha \in \mathbb{Z}_{\geq 0}^k} \frac{1}{\prod p_i^{\alpha_i}}. \\ \log[x] &\leq \sum_{l \leq x} \frac{1}{l}. \\ \Rightarrow \sum_{i=1}^k \log \left( \frac{1}{1 - \frac{1}{p_i}} \right) &\geq \log \log[x]. \\ \log p - \log(p-1) &\leq \frac{1}{p}. \end{aligned}$$

We then conclude that

$$\sum_{p \leq x} \frac{1}{p} \geq \log \log x - 1.$$

□

**Exercise 3.4.** Show that

$$\pi(x) = O\left(\frac{x}{\log \log x}\right).$$

*Solution.*

**Claim 1.** We have,

$$-\sum_{p \leq z} \log \left(1 - \frac{1}{p}\right) = \sum_{p \leq z} \frac{1}{p} + O(1).$$

Proof: We have,

$$\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^n}{n}.$$

Thus substituting  $x = \frac{1}{p}$ , we obtain,

$$-\log \left(1 - \frac{1}{p}\right) = \frac{1}{p} + \sum_{n=2}^{\infty} \frac{1}{np^n} \leq \frac{1}{p} + \frac{\frac{1}{p^2}}{1 - \frac{1}{p}} = \frac{1}{p} + \frac{1}{p^2}.$$

Therefore,

$$-\sum_{p \leq z} \log \left(1 - \frac{1}{p}\right) \leq \sum_{p \leq z} \frac{1}{p} + \zeta(2).$$

■

Using  $\pi(x, z)$ , we have,

$$\pi(x) \leq \pi(x, z) + z = x \prod_{p \leq z} \left(1 - \frac{1}{p}\right) + O(2^z) + z.$$

Setting  $z = \log x$  and using Claim 1, we have,

$$\pi(x) \leq x \exp \left( - \sum_{x \leq \log x} \frac{1}{p} + O(1) \right) + O(x^{\log 2}).$$

Using Exercise 3.3, we get,

$$\pi(x) \leq x \exp(-\log \log \log(x) + O(1)) + O(x^{\log 2}).$$

This completes the proof. □

**Exercise 3.5.** Show that

$$e^{\psi(2n+1)} \int_0^1 x^n (1-x)^n dx,$$

is a positive integer and deduce that

$$\psi(2n+1) \geq 2n \log 2.$$

*Solution.*

**Claim 1.** For any  $n \in \mathbb{N}$ , we have,

$$e^{\psi(n)} = \text{lcm}(2, \dots, n).$$

Using binomial coefficient, we have,

$$(1-x)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k x^k.$$

Thus we have,

$$I = \int_0^1 x^n (1-x)^n dx = \int_0^1 \sum_{k=0}^n \binom{n}{k} (-1)^k x^{n+k} dx = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{n+k+1} x^{n+k+1}$$

Therefore, the first statement follows. For the latter observe that

$$x(1-x) \leq \frac{1}{4}.$$

Therefore,

$$I \leq 2^{-2n}.$$

Finally,

$$e^{\psi(2n+1)} I \geq 1 \Rightarrow e^{\psi(2n+1)} \geq 2^{2n}.$$

□

**Exercise 3.6.** Let  $\chi \bmod q$  be a non-principal character. Then we have,

$$\sum_{n>x} \frac{\chi(n)}{n} = O\left(\frac{1}{x}\right).$$

*Proof.* Using Proposition 2.7, we have,

$$\sum_{n \leq x} \frac{\chi(n)}{n} \leq \frac{\varphi(q)}{x} - \int_1^x \frac{\varphi(q)^2}{t} dt.$$

Recall  $L(s, \chi)$  is analytic for  $\operatorname{Re}(s) > 0$ , in particular,

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

is bounded. Thus we have,

$$\sum_{n>x} \frac{\chi(n)}{n} \leq \int_x^{\infty} \frac{\varphi(q)^2}{t} dt = O\left(\frac{1}{x}\right).$$

□

**Exercise 3.7.** Show that for a non-principal character  $\chi \bmod q$ , we have,

$$\sum_{n \leq x} \sum_{d|n} \chi(d) = xL(1, \chi) + O(\sqrt{x}).$$

*Solution.*

$$\begin{aligned} \sum_{n \leq x} \sum_{d|n} \chi(d) &= \sum_{de \leq x} \chi(d), \\ &= \sum_{d \leq \sqrt{x}} \chi(d) \left[ \frac{x}{d} \right] + \sum_{e \leq \sqrt{x}} \sum_{\sqrt{x} < d \leq \frac{x}{e}} \chi(d). \\ \sum_{\sqrt{x} < d \leq \frac{x}{e}} \chi(d) &= \sum_{d \leq \frac{x}{e}} \chi(d) - \sum_{d \leq \sqrt{x}} \chi(d). \\ \sum_{e \leq \sqrt{x}} \sum_{\sqrt{x} < d \leq \frac{x}{e}} \chi(d) &\leq 2\varphi(q)[\sqrt{x}] = O(\sqrt{x}). \\ \sum_{d \leq \sqrt{x}} \chi(d) \left[ \frac{x}{d} \right] &= x \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{d} + O(1), \\ &= xL(1, \chi) - \sum_{n>x} \frac{\chi(n)}{n} + O(1). \end{aligned}$$

By the previous exercise. We conclude the solution. □

**Exercise 3.8.** For non-principal real character  $\chi \bmod q$ , show that  $L(1, \chi) \neq 0$ .

*Solution.* By Lemma 2.10, we have

$$a_n := \sum_{d|n} \chi(d) \geq 0.$$

Set

$$F(x) = \sum_{n \leq x} a_n \chi(n).$$

Then by the previous exercise, we have  $F(x) = O(\sqrt{x})$ . Let

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{F(t)}{t^{s+1}} dt.$$

Note that by the definition of Dirichlet series, we have,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = L(s, \chi) \zeta(s).$$

$\zeta(s)$  has a simple pole at  $s = 1$  and  $L$  has zero of order at least one at  $s = 1$ , however,

$$\lim_{s \rightarrow 1} s \int_1^{\infty} \frac{F(t)}{t^{s+1}} dt = \infty.$$

This is a contradiction.  $\square$

**Exercise 3.9.** Let  $\chi \pmod{q}$  be a non-principal character then,

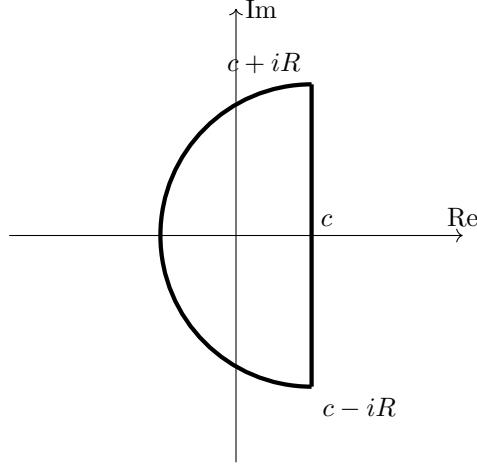
$$\sum_{n \leq x} \sum_{d|n} \chi(d) = xL(1, \chi) + O(q\sqrt{x}).$$

*Solution.* Previous exercise and  $\varphi(q) \leq q$ .  $\square$

**Exercise 3.10.** Show that for  $x > 1$  and  $c > 0$ , we have

$$\frac{1}{2\pi i} \int_{(c)} \frac{x^s}{s} ds = 1.$$

*Solution.* Let  $\zeta_c$  be the contour



going the bold closed path in counterclockwise. Then

$$\frac{1}{2\pi i} \left( \int_{c-iR}^{c+iR} \frac{x^s}{s} ds + x^c \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{x^{e^{i\theta}}}{c+e^{i\theta}} d\theta \right) = \text{Res}_s \left( \frac{x^s}{s}, 0 \right) = 1.$$

Let us examine the second term, for some small  $\delta > 0$ ,

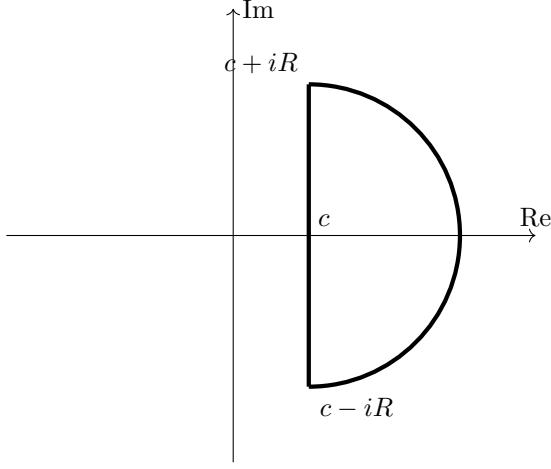
$$\begin{aligned} \left| \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{x^{Re^{i\theta}}}{c+Re^{i\theta}} d\theta \right| &\leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x^{R \cos \theta} \theta, \\ &= \left( \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\delta} + \int_{\frac{\pi}{2}+\delta}^{\frac{3\pi}{2}-\delta} + \int_{\frac{3\pi}{2}-\delta}^{\frac{3\pi}{2}} \right) x^{R \cos \theta} d\theta. \\ \delta &\geq \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\delta} x^{R \cos \theta} d\theta, \int_{\frac{3\pi}{2}-\delta}^{\frac{3\pi}{2}+\delta} x^{R \cos \theta} d\theta. \\ \int_{\frac{\pi}{2}+\delta}^{\frac{3\pi}{2}-\delta} x^{R \cos \theta} d\theta &\leq \pi x^{R \cos(\frac{\pi}{2}+\delta)} = \pi x^{-R \sin \delta} \xrightarrow{R \rightarrow \infty} 0. \end{aligned}$$

As  $\delta$  is arbitrary, we conclude the proof.  $\square$

**Exercise 3.11.** Show that for  $x \in (0, 1)$  and  $c > 0$ , we have

$$\frac{1}{2\pi i} \int_{(c)} \frac{x^s}{s} ds = 0.$$

*Solution.* As in the previous exercise, consider the following contour,



Then,

$$\frac{1}{2\pi i} \left( \int_{c-iR}^{c+iR} \frac{x^s}{s} ds + x^c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^{e^{i\theta}}}{c+e^{i\theta}} d\theta \right) = 0.$$

Similarly, we observe that

$$\int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}-\delta} x^{R \cos \theta} d\theta \leq \pi x^{R \cos(\frac{\pi}{2}-\delta)} = \pi x^{R \sin \delta} \xrightarrow{R \rightarrow \infty} 0.$$

□

**Exercise 3.12.** Show that for  $c > 0$ , we have

$$\frac{1}{2\pi i} \int_{(c)} \frac{1}{s} ds = \frac{1}{2}.$$

*Solution.*

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{1}{s} ds &= \frac{1}{2\pi i} \int_{-R}^R \frac{i}{c+it} dt, \\ &= \frac{1}{2\pi} \int_{-R}^R \frac{c-it}{c^2+t^2} dt, \end{aligned}$$

Since  $\frac{t}{c^2+t^2}$  is odd, the imaginary part of the integral vanishes. As  $\frac{1}{c^2+t^2}$  is even, we consider,

$$\frac{1}{\pi} \int_0^R \frac{c}{c^2+t^2} dt = \frac{1}{\pi} \int_0^{\frac{R}{c}} \frac{1}{1+u^2} du.$$

As  $R$  goes to infinity, the left hand side equals to  $\frac{1}{\pi} \lim_{u \rightarrow \infty} \arctan u = \frac{1}{2}$ . □

**Exercise 3.13.** Show that

$$M(x) := \sum_{n \leq x} \mu(n) = O(x \exp(-c(\log x)^{\frac{1}{10}})),$$

for some constant  $c > 0$ .

*Proof.* Using Lemma 2.13, and recall that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

we get,

$$M(x) = \frac{1}{2\pi i} \int_{(c)} \frac{x^s}{s\zeta(s)} ds.$$

Thus we examine

$$\frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{x^s}{s\zeta(s)} ds,$$

and consider  $R \rightarrow \infty$ .  $\square$

**Exercise 3.14.** Show that for any  $s$  with  $\operatorname{Re}(s) = 1$ , we have,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

converges.

*Solution.* By the assumption, we set  $s = 1 + it$  for  $t \in \mathbb{R}$ . Using Proposition 2.7, we have,

$$\sum_{n \leq N} \frac{\mu(n)}{n^{1+it}} = \frac{O(N \exp(-c(\log N)^{\frac{1}{10}}))}{N^{1+it}} + (1+it) \int_1^x \frac{M(w)}{w^{2+it}} dw.$$

Examine the splitted integral,

$$\int_1^{\infty} \frac{M(w)}{w^{2+it}} dw - \int_N^{\infty} \frac{M(w)}{w^{2+it}} dw.$$

The first term converges as  $\frac{M(w)}{w^{2+it}} = O(w^{-1-\delta})$  for some  $\delta > 0$ . Note that

$$\exp\left(-\frac{c}{2}(\log w)^{\frac{1}{10}}\right) \leq \exp\left(-\frac{c}{2}(\log w)^{\frac{1}{10}}\right).$$

Thus,

$$\begin{aligned} \left| \int_N^{\infty} \frac{M(w)}{w^{2+it}} dw \right| &\leq \int_N^{\infty} \exp\left(-\frac{c}{2}(\log w)^{\frac{1}{10}}\right) \exp\left(-\frac{c}{2}(\log w)^{\frac{1}{10}}\right) \frac{dw}{w}, \\ &\leq c' \exp\left(-\frac{c}{2}(\log N)^{\frac{1}{10}}\right) \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

$\square$

**Exercise 3.15.** Show that

$$\sum_{n \leq x} \frac{n}{\varphi(n)} \ll O(x).$$

*Proof.*

$$\begin{aligned} \sum_{n \leq x} \frac{n}{\varphi(n)} &= \sum_{n \leq x} \frac{1}{\prod_{p|n} \left(1 - \frac{1}{p}\right)}, \\ &= \sum_{n \leq x} \sum_{\gamma(d)|n} \frac{1}{d}, \\ &\leq \sum_{\gamma(d) \leq x} \frac{1}{d} \frac{x}{\gamma(d)}, \\ &\leq x \prod_{p \leq x} \frac{1}{\left(1 - \frac{1}{p}\right)}. \end{aligned}$$

□

**Exercise 3.16.** Show that

$$\sum_{n \leq x} \frac{1}{\varphi(n)} \ll \log x.$$

*Proof.* Using the previous problem and Proposition 2.7, we get,

$$\begin{aligned} \sum_{n \leq x} \frac{n}{\varphi(n)} \frac{1}{n} &\leq \frac{Cx}{x} + C \int_1^x \frac{t}{t^2} dt, \\ &= C + C \log x. \end{aligned}$$

□

**Exercise 3.17.** Prove that

$$\sum_{n \leq x} \frac{1}{\varphi(n)} \sim c \log x.$$

*Proof.* Remains to show that  $\log x \ll \sum_{n \leq x} \frac{1}{\varphi(n)}$ . □

**Exercise 3.18.** Show that for a character  $\chi \pmod{q}$  and  $(n, q) = 1$ , we have,

$$\chi(n)\tau(\bar{\chi}) = \sum_{k=1}^q \bar{\chi}(n)e^{\frac{2\pi i kn}{q}}.$$

*Solution.* Note that we have  $\chi(n) = \bar{\chi}(n)^{-1}$ . And use

$$kn \equiv l \pmod{q} \Rightarrow k \equiv ln^{-1} \pmod{q}.$$

□

**Exercise 3.19.** Let  $\chi \bmod q$  be a primitive character, then we have,

$$L(1, \chi) = \sum_{n \leq x} \frac{\chi(n)}{n} + O\left(\frac{\sqrt{q} \log q}{x}\right).$$

*Proof.* By Proposition 2.7,

$$L(1, \chi) = \int_1^\infty \frac{A(t)}{t^2} dt \Rightarrow \sum_{n > x} \frac{\chi(n)}{n} \leq \int_x^\infty \frac{A(t)}{t^2} dt,$$

where

$$A(x) = \sum_{n \leq x} \chi(n).$$

By Theorem 2.11, we have,

$$|A(x)| \leq \sqrt{q} \log q.$$

Thus we obtain,

$$\int_x^\infty \frac{A(t)}{t^2} dt \leq \frac{\sqrt{q} \log q}{x}.$$

□

**Exercise 3.20.** Show that

$$\sum_{\chi \neq \chi_0} L(1, \chi) = \varphi(q) + O(\sqrt{q} \log q).$$

*Proof.* If  $\chi$  is imprimitive, then use Theorem 2.11 to the primitive conductor of it, we obtain the same inequality. Thus

$$\sum_{\chi \neq \chi_0} L(1, \chi) = \sum_{n \leq x} \left( \frac{\sum_{\chi \neq \chi_0} \chi(n)}{n} + O\left(\frac{\sqrt{q} \log q}{x}\right) \right).$$

Using Remark 2.12 and Theorem 2.10, furthermore choosing  $x = q$  we obtain,

$$\sum_{\chi \neq \chi_0} L(1, \chi) = \varphi(q) \sum_{(n, q)=1} \frac{1}{n} - \sum_{n=1}^q \frac{1}{n} + O(\sqrt{q} \log q).$$

□

**Exercise 3.21.** Let  $s \in \mathbb{C}$  be such that  $\operatorname{Re}(s) = \sigma > 0$ . Then for any  $x$  and a non-trivial character  $\chi$ , show that

$$L(s, \chi) = \sum_{n \leq x} \frac{\chi(n)}{n^s} + O\left(\frac{|s| \sqrt{q} \log q}{\sigma x^\sigma}\right).$$

*Solution.* Using Proposition 2.7 and Theorem 2.11, we have,

$$\sum_{n>x} \frac{\chi(n)}{n^s} \leq |s| \int_x^\infty \frac{\sqrt{q} \log q}{t^{\sigma+1}} dt.$$

□

**Exercise 3.22.** For  $\sigma > \frac{1}{2}$ , show that

$$\sum_{\chi \neq \chi_0} L(\sigma, \chi) = \varphi(q) + O(q^{\frac{3}{2}-\sigma}).$$

*Solution.* By previous exercises, by letting  $x = q$ , we have

$$\sum_{\chi \neq \chi_0} L(\sigma, \chi) = \varphi(q) + O(q^{1-\sigma}) + O(q^{\frac{3}{2}-\sigma}).$$

□

**Exercise 3.23.** For a prime  $p$  and a quadratic character  $\chi$ , we have,

$$\tau(\chi) = \begin{cases} \sqrt{p}, & (p \equiv 1 \pmod{p}), \\ i\sqrt{p}, & (p \equiv 3 \pmod{p}). \end{cases}$$

*Proof.* As  $(\mathbb{Z}/p\mathbb{Z})^\times$  is cyclic, there is a unique quadratic character  $\chi$  for each  $p$ . Let  $\zeta_p = e^{\frac{2\pi i}{p}}$ . Then

$$\sum_{k=1}^{p-1} \zeta_p^k = -1.$$

Therefore,

$$\tau(\chi) - 1 = \sum_{k=1}^{p-1} (\chi(k) + 1) \zeta_p^k.$$

As  $\mathbb{Z}/p\mathbb{Z}$  is a field, if an element has square roots then there are exactly two besides 0. And if  $k$  is a square  $\chi(k) - 1 = 2$  otherwise 0. Therefore,

$$\tau(\chi) = 1 + \sum_{k=1}^{p-1} \zeta_p^{k^2}$$

Consider,

$$|\tau(\chi)|^2 = \sum_{k,l=0}^{p-1} \zeta_p^{k^2 - l^2} = \sum_{a,b=0}^{p-1} \zeta_p^{ab} = p.$$

Remains to prove the parity of this. □