V5A10 Analytic Number Theory

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1 Classical Number Theory

Theorem 1.1 (Euclid). *There are infinitely many prime numbers.*

Definition 1.1. $\pi : \mathbb{N} \to \mathbb{N}$ *be a function such that*

$$\pi(n) = \{prime \ numbers \ less \ than \ n\}.$$

Remark 1.1.

$$\frac{\pi}{n \ln(n)} \approx 1.$$

Definition 1.2.

$$\mathrm{Li}(x) = \int_0^x \frac{1}{\ln(t)} dt.$$

Notation 1.1. *Given* $f, g : \mathbb{R} \to \mathbb{C}$,

$$f(x) = O(g(x))$$

means that

$$\exists K \in (0, \infty), x_0 \in \mathbb{R}, \text{ s.t. } \forall x > x_0, |f(x)| \le K|f(x)|.$$

Notation 1.2. *Let* $f, g : \mathbb{R} \to \mathbb{C}$ *be functions.* $f \sim g$ *denotes that*

$$\lim_{n \to \infty} \frac{f(x)}{g(x)} = c,$$

for some constant c.

Notation 1.3. *Let* $f : \mathbb{R} \to \mathbb{C}$,

$$\mathrm{Li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k},$$

denotes that

$$\mathrm{Li}(x) = \frac{x}{\ln x} \sum_{k=1}^{N-1} \frac{k!}{(\ln x)^k} + O\left(\frac{x}{(\ln x)^{N+1}}\right).$$

and as $x \to \infty$, *this holds for any* $N \ge 1$.

Remark 1.2. *By the integration by parts, we see that it's asymptotic expansion is*

$$\mathrm{Li}(x) \approx \frac{x}{\ln(x)} \sum_{k=0}^{\infty} \frac{k!}{(ln(x))^k}.$$

Theorem 1.2 (Prime Number Theorem).

$$\lim_{x \to \infty} \frac{\pi(x)}{\mathrm{Li}(x)} = 1.$$

Definition 1.3 (First Chebyschev Function).

$$\vartheta(x) = \sum_{p \leq x} \ln p.$$

Definition 1.4 (Second Chebyschev Function).

$$\psi(x) = \sum_{\substack{m,p \\ p^m \leq x}} \ln p.$$

Remark 1.3. *We can rewrite the second Chebyschev function as follows.*

$$\psi(x) = \sum_{\substack{p \leq x \\ m = \max\{n \in \mathbb{N} \mid p^n \leq x\}}} m \ln p.$$

Definition 1.5 (Möbius Function).

$$\mu(n) = \begin{cases} 1 & (n = 1) \\ (-1)^k & (n = p_1 \cdots p_k, p_i = p_j \Rightarrow i = j) \\ 0 & (\exists p \text{ s.t. } p^2 | n). \end{cases}$$

Remark 1.4. *The prime number theorem is equivalent to the following statements.*

1). $\psi(x) \sim x$.

2). $\theta(x) \sim x$.

3). $\lim_{x \to \infty} \frac{\sum_{n \leq x}}{\frac{\mu(n)}{n}} x = 0$.

Conjecture 1.1 (Twin Prime Conjecture). *There exists infinitely many primes p such that $p + 2$ is also prime.*

Conjecture 1.2 (Goldbach's Conjecture). *Let $n \in \mathbb{N}$ be an even number greater than 2, then there exists two primes p, q such that $n = p + q$.*

Conjecture 1.3 (Hardy-Littlewood Conjecture).

$$\# \{ \text{ prime numbers } p \text{ such that } 2p + 1 \text{ is also a prime and } p < x\}$$

Definition 1.6 (Riemann-Zeta Function). *We define $\zeta : \mathbb{C} \to \mathbb{C}$ such that*

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}.$$

Remark 1.5. *Supposer $\mathrm{Re}(s) > 1$, then we have*

$$|\zeta(s)| = \sum_{n \in \mathbb{N}} \frac{1}{|n|^s}$$
$$= \sum_{n \in \mathbb{N}} \frac{1}{n^{\mathrm{Re}(s)}}$$

By multiplying $\frac{1}{2^s}$, we obtain

$$\frac{1}{2^s}\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{(2n)^s}.$$

We get

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{(2n+1)^s}.$$

Continuing this procedure, we get the following proposition.

Proposition 1.1.

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)$$

Theorem 1.3 (Weierstrass). *Let $A \subseteq \mathbb{C}$ and consider a sequence of functions $(f_n : A \to \mathbb{C})_{n \in \mathbb{N}}$ such that there exists a sequence of non-negative numbers $(M_n)_{n \in \mathbb{N}}$ such that*

i). $\forall x \in A, |f_n(x)| \leq M_n$.

ii). $\sum_{n \in \mathbb{N}} M_n$ converges.

Then the sequence converges uniformly.

Theorem 1.4. *Suppose the conditions in the previous theorem. If each function is analytic on a compact subset of A, then the limit is also analytic.*

Corollary 1.1. *Let A be a compact subset of a complex plane where $\mathrm{Re}(s) > 1$. Then there exists $\delta > 0$ such that $\mathrm{Re}(s) > 1 + \delta$ and*

$$\sum_{n \in \mathbb{N}} \left|\frac{1}{n^s}\right| \leq \sum_{n \in \mathbb{N}} \frac{1}{n^{1+\delta}} < \infty.$$

Fact 1.1. *The Riemann zeta function can be analytically continued to the whole plane except for $s = 1$.*

Definition 1.7 (Gamma Function).

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Proposition 1.2.

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Remark 1.6. $\zeta(1 + it) \neq 0$ *if* $t \in \mathbb{R}, t \neq 0$. $\zeta(s) \neq 0$ *for* $0 < s < 1$.

Remark 1.7 (Functional Equation).

$$\zeta(1 - s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s)\zeta(s)$$

Remark 1.8. $\Gamma(s)$ *is defined for* $\mathrm{Re}(s) > 0$ *and can be analytically continued to the whole place except for* $\mathbb{C}\backslash\{-2n \mid n \geq 0\}$.

Remark 1.9. *For* $s = -2m$ *where* $m \in \mathbb{N}$, *we see* $\zeta(s) = 0$.

$$\begin{aligned}
\zeta(0) &= \frac{2}{2\pi} \lim_{s\to 1} \cos\left(\frac{\pi s}{2}\right) \zeta(s) \\
&= \frac{1}{\pi} \lim_{s\to 1} \frac{\cos(\frac{\pi s}{2})}{s - 1} \lim_{s\to 1} (s - 1)\zeta(s) \\
&= \frac{1}{\pi} \times \frac{-\pi}{2} \times 1 \\
&= -\frac{1}{2}.
\end{aligned}$$

Definition 1.8. *The critical strip is the subset of the complex plane with its real part between* 0 *and* 1. *The critical line is the line where* $\mathrm{Re}(s) = \frac{1}{2}$.

Conjecture 1.4 (Riemann Hypothesis). *Let* s *be an element of the critical strip. If* $\zeta(s) = 0$ *then* $\mathrm{Re}(s) = \frac{1}{2}$ *(i.e. it lies on the critical line).*

Notation 1.4. *Let* $T > 0$. *We denote* $N(T)$ *the number of zeros of* ζ *in the critical strip whose coefficient of the imaginary part is in* $(0, T)$. *That is*

$$N(T) = |\{\sigma + it \in \mathbb{C} \mid 0 < \sigma < 1, 0 < t < T\}|.$$

Proposition 1.3.

$$\lim_{T\to\infty} \frac{N(T)2\pi}{T \log(T)} = 1.$$

Sketch of Proof (needs refinement).

$$\psi(x) = \frac{1}{2\pi i} \int_l \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds$$

where l is the line $l = a$ for some $a > 1$.

$$\psi(s) = x - \sum_{\rho \text{ non-trivial zeros}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \log(1 - x^{-2}).$$

$\square$

Definition 1.9. *Let $q \in \mathbb{N}$ and a be a natural number coprime to q. We define*

$\pi(x; q, a) = |\{prime\ numbers\ p\ less\ than\ or\ equal\ to\ x\ such\ that\ p \equiv a \mod q\}|$

Proposition 1.4.
$$\pi(x;, q, a) \sim \frac{x}{\varphi(q)\log(x)}$$

where φ is a Euler phi-function.

Theorem 1.5 (Brun–Titchmarsh)**.** *For any $q < x$, we have*

$$\pi(x;, q, a) < \frac{2x}{\varphi(q)\log(\frac{x}{q})}.$$

Remark 1.10.
$$\text{Li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k}.$$

Indeed we have
$$\text{Li}(x) = . \int_2^x \frac{dt}{\ln t}.$$

Observe that
$$\int_2^t \frac{1}{(\ln t)^N} \sim \frac{x}{(\ln x)^N}$$

for all $N \geq 1$. Thus $\text{Li}(x)$ can be expressed in terms of polynomials in $\frac{x}{\ln(x)}$, by keep replacing the greatest temr with the above approximation.

2 Arithmetic Functions

2.1 Multiplicative Functions

Definition 2.1. *A function $f : \mathbb{N} \to \mathbb{C}$ is said to be*

1). multiplicative if for any $(m, n) = 1$, we have $f(mn) = f(m)f(n)$,

2). completely multiplicative if for any natural numbers m, n, we have $f(mn) = f(m)f(n)$.

Example 2.1. *Möbius function μ is multiplicative.*

Definition 2.2 (Von-Mangoldt Function). *The Von-Mangoldt function* $\Lambda : \mathbb{N} \to \mathbb{C}$ *is defined as*

$$\Lambda(n) = \begin{cases} \log(p) & (n = p^k \text{ for some } k \geq 1), \\ 0 & (otherwise). \end{cases}$$

Definition 2.3 (Euler Phi Function). *The Euler phi funtion is* $\varphi : \mathbb{N} \to \mathbb{C}$ *such that*

$$\varphi(n) = \{1 \leq a \leq n \mid (a, n) = 1\}.$$

Example 2.2. φ *is multiplicative but* Λ *is not.*

Definition 2.4 (Dirichlet Characters Modulo q). *Let* $q \in \mathbb{N}$ *be a natural number and* $q \geq 2$.

$$\chi_1 : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$$

be a group homomorphism. The Dirichlet character function modulo q *with respect to* χ_1 *is such that*

$$\chi(n) = \begin{cases} \chi_1(\overline{n}) & ((n, q) = 1), \\ 0 & (otherwise). \end{cases}$$

Example 2.3. *For* $q = 3$, *we have* $(\mathbb{Z}/3\mathbb{Z})^\times = \{\pm 1\}$. *The only possible character is* $\pm 1 \mapsto \pm 1$. *Therefore, we have*

$$\chi(1) = 1, \chi(2) = -1, \chi(0) = 0.$$

Theorem 2.1.
$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1, \\ 0 & (otherwise). \end{cases}$$

Proof. When $n = 1$, this is trivial. Suppose $n \neq 1$. We factorize n by

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

where p_i is a prime and $\alpha_i \in \mathbb{N}$ for each $i = 1, \cdots, n$.

Observe that
$$\sum_{d|n} \mu(d) = \sum_{d | \prod_{i=1}^{k} p_i} \mu(d).$$

Now we see

$$\sum_{d | \prod_{i=1}^{k} p_i} \mu(d) = \sum_{j=0}^{k} \binom{k}{j}(-1)^j = \sum_{j=0}^{k} \binom{k}{j}(1)^{k-j}(-1)^j = (1-1)^k = 0.$$

$\square$

Proposition 2.1 (Möbius Inversion Formula). *Let $f, g : \mathbb{N} \to \mathbb{C}$ be functions (we do not assume them to be multiplicative). If*

$$\sum_{d|n} g(d) = f(n),$$

holds if and only if

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = g(n).$$

Proof.

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \sum_{e|\frac{n}{d}} g(e).$$

$e|\frac{n}{d}$ if and only if $de|n$ thus obtain,

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \sum_{de|n} g(e).$$

In particular, we get the expression

$$= \sum_{de|n} \mu(d) g(e).$$

By reordering, we get

$$= \sum_{e|n} g(e) \sum_{d|\frac{n}{e}} \mu(d).$$

By Proposition 2.1, we get

$$\sum_{d|\frac{n}{e}} \mu(d) = 0$$

unless $e = n$. $\qquad\qquad\square$

Proposition 2.2.

$$\sum_{d|n} \varphi(d) = n.$$

Proof. Consider $(\mathbb{Z}/n\mathbb{Z})^{\times}$. We know that

$$|(\mathbb{Z}/n\mathbb{Z})^{\times}| = \varphi(n).$$

Later $\qquad\qquad\square$

Theorem 2.2.

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Proof. Using Proposition 2.2, we have,

$$\sum_{d|n} \mu(d)\frac{n}{d} = \varphi(n).$$

Dividing both sides by n and observe that $\mu(d) \neq 0$ if and only if d is a prime factor of n.

$$\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d},$$

$$= 1\sum_{p|n} \frac{1}{p} + \sum_{p_1, p_2|n} \frac{1}{p_1 p_2} - \cdots .$$

By the induction on the number of prime divisors of n, we get the statement. $\quad\square$

Proposition 2.3. *We have the following properties of φ.*

1). $n|m \Rightarrow \varphi(n)|\varphi(m)$.

2). $\varphi(n)$ is even for $n \geq 3$.

3). $\varphi(2n) = \begin{cases} 2\varphi(n), & (2|n) \\ \varphi(n), & (2 \nmid n). \end{cases}$

4). φ is multiplicative.

5). $\varphi(mn) = \varphi(m)\frac{\varphi(n)d}{\varphi(d)}$ where $d = (m,n)$.

6). $\varphi(n^m) = n^{m-1}\varphi(n)$.

Proof. Exercise. $\quad\square$

Theorem 2.3. *The following statements are equivalent.*

1). $\sum_{d|n} \Lambda(d) = \log n$

2). $\sum_{d|n} \mu(d) \log d = \Lambda(n)$.

And in particular $\sum_{d|n} \Lambda(d) = \log n$ holds.

Proof. The equivalence is a direct corollary of Möbius inversion formula (ie. Proposition 2.1). For the latter, Write

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}.$$

We have

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{k} \alpha_i \log p_i = \log(n).$$

$\quad\square$

Notation 2.1. *Let $n \in \mathbb{N}$, suppose a prime p divides n. Then we denote $\alpha(p)$ to be the highest prime power factor of n.*

Theorem 2.4. *Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function. Then*

$$\sum_{d|n} f(d) = \prod_{p|n} \left(\sum_{i=0}^{\alpha(p)} f(p^i) \right).$$

In particular $\sum_{d|n} f(d)$ is also multiplicative.

Proof. Let $d|n$, then we have $d = \prod_{i=1}^{k} p_i^{\beta_i}$ for some $0 \leq \beta_i \leq \alpha(p_i)$. Since f is multipliative we have

$$f(d) = \prod_{i=1}^{k} f(p_i^{\beta_i}).$$

The second part is a direct result of the first part. $\qquad\qquad\square$

Remark 2.1. *The Second Chebyschev Function ψ can be written as*

$$\psi(x) = \sum_{d \leq x} \Lambda(d).$$

This follows directly from the definition.

Definition 2.5 (Dirichlet Series). *Let $f : \mathbb{N} \to \mathbb{C}$ be a function and $s \in \mathbb{C}$. We define*

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}.$$

For another arithmetic function $g : \mathbb{N} \to \mathbb{C}$, we define

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} + \sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} = \sum_{n \in \mathbb{N}} \frac{(f(n) + g(n))}{n^s}.$$

and

$$\left(\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \times \left(\sum_{n \in \mathbb{N}} \frac{g(n)}{n^s} \right) = \sum_{n,m \in \mathbb{N}} \frac{(f(n)g(m))}{(nm)^s}.$$

$$= \sum_{t \in \mathbb{N}} \sum_{n|t} \frac{\left(f(n)g\left(\frac{n}{t}\right) \right)}{t^s}.$$

Recall the taylor expansion of $\ln x$ we get

$$\ln 2 = \sum_{n \in \mathbb{N}} \frac{(-1)^{n+1}}{n}.$$

Rearranging the following way

$$\left(1 - \frac{1}{2} \right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6} \right) - \frac{1}{8} + \cdots$$

we get this equals to $\frac{1}{2} \ln 2$.

Theorem 2.5. *Let $s \in \mathbb{C}$ be $\mathrm{Re}(s) > 1$, we have*

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}.$$

Proof.

$$\zeta(s) \sum_{n \geq 1} \frac{\mu(n)}{n^s} = \left(\sum_{n \in \mathbb{N}} \frac{1}{n^s} \right) \left(\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} \right)$$

$$= \sum_{t \in \mathbb{N}} \frac{1}{t^s} \sum_{n | t} \mu(n)$$

$$= 1.$$

$\square$

Theorem 2.6. *For $\mathrm{Re}(s) > 1$, we have*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \in \mathbb{N}} \frac{\Lambda(s)}{n^s}.$$

From this we derive

$$\lim_{n \to \infty} \frac{\log n}{n^\varepsilon} = 0.$$

Proof.

$$\zeta(s) \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} = \left(\sum_{m \in \mathbb{N}} \frac{1}{m^s} \right) \left(\sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} \right),$$

$$= \sum_{t \in \mathbb{N}} \frac{1}{t^s} \left(\sum_{n | t} \Lambda \left(\frac{t}{n} \right) \right),$$

$$= \sum_{t \in \mathbb{N}} \frac{\log(t)}{t^s},$$

$$= -\zeta'(s).$$

$\square$

Remark 2.2.

$$\sum_{n \in \mathbb{N}} \left| \frac{\Lambda(s)}{n^s} \right| \leq \sum_{n \in \mathbb{N}} \frac{\log(n)}{n^\sigma},$$

$$<< \sum_{n \in \mathbb{N}} \frac{n^\varepsilon}{n^\sigma},$$

$$= \sum_{n \in \mathbb{N}} \frac{1}{n^{\sigma - \varepsilon}}.$$

We have $\lim_{n\to\infty} \frac{\log(n)}{n^\varepsilon} = 1$ and the last equation is convergent if and only if $\sigma - \varepsilon > 1$. Thus we have $\sigma > 1 + \varepsilon$.

Definition 2.6. *Let $D \subseteq \mathbb{C}$ be open. A meromorphic function is $f : D \to \mathbb{C}$ which is analytic on D except a discrete set of poles of f.*

Remark 2.3. $\frac{\zeta'(s)}{\zeta(s)}$ *is a meromorphic functions except $s = 1$ and where $\zeta(s)$ vanishes. Indeed, For general $\frac{f}{g}$, it is analytic if f, g are analytic and $g \neq 0$.*

1). $\zeta(s)$ is analytic except $s = 1$.

2). $\zeta'(s)$ has a pole of order 2 at $s = 1$.

3). $\zeta(s)$ has a pole of order 1 at $s = 1$.

Recall that for $|z| \geq 1$, we have,

1. $|z| \geq 1 \Rightarrow \sum_{n \in \mathbb{Z}_{\geq 0}} z^n = \frac{1}{1-z}$,

2. $\prod_{n \in \mathbb{N}}(1 + a_n)$ is convergent if $\sum_n a_n$ is absolutely convergent,

3. therefore $\prod_{n \in \mathbb{N}}(1 + a_n)$ is convergent if and only if $\prod_{n \in \mathbb{N}}(1 + |a_n|)$ is convergent.

Theorem 2.7. *Let $f : \mathbb{N} \to \mathbb{C}$ be a map.*
If f is multiplicative and for $\mathrm{Re}(s) > r_0, r_0 \in \mathbb{R}$ then we have,

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} = \prod_p \left(\sum_{\nu \geq 0} f(p^\nu) p^{-\nu s} \right).$$

If f is completely multiplicative, then

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} = \prod_p (1 - f(p)p^{-s})^{-1}.$$

Proof. Let $A(x) = \{n \in \mathbb{N} \,|\, \text{primes factors of } n \text{ are } \leq x\}$, then

$$\prod_{p \leq x} \sum_{\nu=0}^{\infty} f(p^\nu) p^{-\nu s} = \sum_{n \in A} \frac{f(n)}{n^s}.$$

Therefore,

$$\left| \prod_{p \le x} \sum_{\nu=0}^{\infty} f(p^\nu) p^{-\nu s} - \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right| = \left| \sum_{n \in A(x)} \frac{f(n)}{n^s} - \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right|,$$

$$= \left| \sum_{n \in \mathbb{N} \setminus A(x)} \frac{f(n)}{n^s} \right|,$$

$$\le \sum_{n \notin A(x)} \frac{|f(n)|}{n^{\mathrm{Re}(s)}},$$

$$\le \sum_{n > x} \frac{|f(n)|}{n^{\mathrm{Re}(s)}} \to 0.$$

The last limit is due to that it is a tail of a an absolutely convergent series. Since f is completely multipliative, we have

$$f(p^\nu) = (f(p))^\nu.$$

Therefore, we get,

$$\prod_p \left(\sum_{\nu \in \mathbb{Z}_{\ge 0}} (f(p^\nu) p^{-\nu s}) \right) = \prod_p \left(\sum_{\nu \in \mathbb{Z}_{\ge 0}} (f(p) p^{-s})^{-\nu} \right),$$

$$= \prod_p \left(\frac{1}{1 - f(p) p^{-s}} \right).$$

$\square$

Example 2.4. *Take $f(n) = 1$ as above we get,*

$$\sum_{n \in \mathbb{N}} \frac{1}{n^s} = \zeta(s) = \prod_p \left(1 - \frac{1}{p^s} \right)^{-1}, \mathrm{Re}(s) > 1.$$

Example 2.5.

$$\sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s} \right)^{-1}, \mathrm{Re}(s) > 1.$$

Example 2.6.

$$\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} = \prod_p \left(1 + \frac{\mu(p)}{p^s} \right),$$

$$= \prod_p \left(1 - \frac{1}{p^s} \right),$$

$$= \frac{1}{\zeta(s)}.$$

Example 2.7. *Note that $\phi(n) \leq n$. Thus for $\text{Re}(s) > 2$, we have,*

$$\sum_{n \in \mathbb{N}} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

2.2 Order of arithmetic functions

Definition 2.7. *Let $f, g : \mathbb{N} \to \mathbb{C}$, we denote,*

$$f(n) = O(g(n)),$$

if there is $K > 0$ and $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow |f(n)| \leq K|g(n)|.$$

An alternative notation for this is $f(n) = O(g(n))$.

Definition 2.8. *We define following arithmetic functions,*

1). $\nu(n) := \sum_{p|n} 1$, a number of primer divisors of n,

2). $d(n) := \sum_{d|n} 1$, the number of divisors of n,

3). $\sigma(n) := \sum_{d|n} d$, the sum of all the divisors of n.

Lemma 2.1.
$$\nu(n) << \log(n).$$

Proof. Let $n = \prod_{i=1}^{k} p_i^{\alpha(p_i)}$. Then $\nu(n) = k$. Since $p_i \geq 2$, we have,

$$\log(n) = \sum_{i=1}^{k} \alpha(p_i) \log(p_i),$$
$$\geq k \log(2).$$

Therefore $\nu(n) \leq \frac{\log(n)}{\log 2}$. $\square$

Lemma 2.2.
$$\sum_{k=2}^{n} \frac{1}{k} \leq \log(n+1).$$

Proof. We know that
$$\int_{1}^{n} \frac{1}{t} dt = \log(n).$$

For $1 \leq k \leq t \leq k+1 \leq n$, we have,

$$\int_{k}^{k+1} \frac{1}{k+1} dt \leq \int_{k}^{k+1} \frac{1}{t} dt \leq \int_{k}^{k+1} \frac{1}{k} dt.$$

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Thus we have,

$$\frac{1}{k+1} \le \log(k+1) - \log(k) \le \frac{1}{k}.$$

By telescopoing sum we get

$$\sum_{k=2}^{n} \frac{1}{k} \le \log(n+1).$$

$\square$

Lemma 2.3.

$$\sigma(n) << n(1 + \log(n)) \sim n\log(n).$$

Proof.

$$\sigma(n) = \sum_{d|n} \frac{n}{d},$$

$$= n \sum_{d|n} \frac{1}{d},$$

$$\le n \left(1 + \sum_{d=2}^{n} \frac{1}{d}\right),$$

$$\le (1 + \log(n)).$$

The last inequality is due to Lemma 2.2. $\square$

Exercise 2.1. *Show that*

$$\sum_{k=1}^{n} \frac{1}{k} = \log(n) + O(1).$$

That is

$$\left|\sum_{k=1}^{n} \frac{1}{k} - \log(n)\right| << 1.$$

Hint: Replace $\frac{1}{t}$ by an increasing function and derive the similar inequality to Lemma 2.2.

Proof. By Lemma 2.2, we have,

$$\left|\sum_{k=1}^{n} \frac{1}{k} - \log(n)\right| \le |1 + \log(n) - \log(n)| \le 1.$$

$\square$

Lemma 2.4.

$$d(n) \le 2\sqrt{n}.$$

Proof. If $n = d_1 d_2$ then one of them must be less than or equal to $\sqrt{n}$. $\qquad \square$

We have an improved inequality,

Proposition 2.4. *for $\varepsilon > 0$, we have,*

$$d(n) << n^\varepsilon.$$

Proof. Recall that for $n = \prod_{i=1}^k p_i^{\alpha_i}$, we have $d(n) = \prod_{i=1}^k (\alpha_i + 1)$. In particular, we have,

$$\frac{d(n)}{n^\varepsilon} = \prod_{i=1}^k \frac{(\alpha_i + 1)}{p_i^{\varepsilon \alpha_i}}.$$

Let $A = \{i \mid p_i^\varepsilon \geq 2\}$. Rcall that for $x \geq 1$, $x + 1 \leq 2^x$ that is

$$\frac{x+1}{2^x} \leq 1.$$

Then,

$$\prod_{i \in A} \frac{\alpha_i + 1}{p_i^{\varepsilon \alpha_i}} \leq \prod_{i \in A} \frac{\alpha_i + 1}{2^{\alpha_i}} \leq 1.$$

For $p_i^\varepsilon < 2$, we observe,

$$p_i^{\varepsilon \alpha_i} = e^{\varepsilon \alpha_i \log(p_i)} \geq \varepsilon \alpha_i \log(p_i).$$

Therefore,

$$\prod_{i \notin A} \frac{\alpha_i + 1}{p_i^{\varepsilon \alpha_i}} \leq \prod_{i \notin A} \left(\frac{\alpha_i}{p_i^{\varepsilon \alpha_i}} + 1 \right),$$

$$\leq \prod_{i \notin A} \left(\frac{\alpha_i}{\varepsilon \alpha_i \log(p_i)} + 1 \right),$$

$$\leq \prod_{i \notin A} \left(\frac{1}{\varepsilon \log(2^{\frac{1}{\varepsilon}})} + 1 \right),$$

$$\leq \prod_{i \notin A} \left(\frac{1}{\log(2)} + 1 \right).$$

Combining two cases, we obtain the statement. $\qquad \square$

Notation 2.2. *Let $x \in \mathbb{R}$. We denote*

1. *the integer part $[x] \in \mathbb{Z}$ which is the greatest integer not exceeding x,*

2. *the fraction part $\{x\} = x - [x]$.*

Proposition 2.5.

$$\frac{\sum_{n \leq x} d(n)}{x} \sim \log(x).$$

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Proof. By definition, we have,

$$\sum_{n \le x} d(n) = \sum_{n \le x} \sum_{d|n} 1.$$

$d|n$ if and only if there is e such that $de = n$. Using this we obtain,

$$\sum_{n \le x} d(n) = \sum_{\substack{(e,d) \\ de \le x}} 1 = \sum_{d \le x} \sum_{e \le \frac{x}{d}} 1 = \sum_{d \le x} \left[\frac{x}{d}\right].$$

Using the definition of $[x]$, we have,

$$\sum_{n \le x} d(n) = \sum_{d \le x} \frac{x}{d} - \left\{\frac{x}{d}\right\} = x \sum_{d \le x} \frac{1}{d} - \sum_{d \le x} \left\{\frac{x}{d}\right\} = x(\log(x) + O(1)) + O(x).$$

Thus we obtain the statement. $\qquad\square$

Definition 2.9. *Let $f : \mathbb{N} \to \mathbb{C}$, we say the average order of f is $g : \mathbb{R} \to \mathbb{C}$ if*

$$\frac{\sum_{n \le x} f(n)}{x} \sim g(x).$$

Proposition 2.5 can be restated as follows.

Proposition 2.6. *The average order of d is $\log(x)$.*

Exercise 2.2. *Examine the following statements.*

1. *Is it true that $d(n) << \log n$?*

2. *Do we have $d(n) = O(n^\varepsilon)$ for any $\varepsilon > 0$?*

3. *What is the optimal bound for $d(n)$?*

Theorem 2.8. *There exists $c_1, c_2 > 0$ such that*

$$c_1 \le \frac{\varphi(n)\sigma(n)}{n^2} \le c_2.$$

Proof. Recall that

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

and

$$\sigma(n) = \prod_{p|n} \frac{p^{\alpha(p)+1} - 1}{p - 1}.$$

Thus we obtain,

$$\frac{\sigma(n)}{n} = \frac{\prod_{p|n}(1 + p + \cdots + p^{\alpha(p)})}{\prod_{p|n} p^{\alpha(p)}} = \prod_{p|n} \left(\frac{\frac{1}{p^{\alpha(p)+1}} - 1}{\frac{1}{p} - 1}\right)$$

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By multiplying two we get,

$$\frac{\varphi(n)\sigma(n)}{n^2} = \prod_{p|n}\left(1 - \frac{1}{p^{\alpha(p)+1}}\right) \le 1.$$

On the other hand,

$$\prod_{p}\left(1 - \frac{1}{p^2}\right) \le \frac{\varphi(n)\sigma(n)}{n^2}.$$

The left hand side is equal to $\frac{1}{\zeta(2)}$ which is $\frac{6}{\pi^2}$. $\qquad\square$

Lemma 2.5. *For $x \ge 1$ and d an integer such that $d \le x$, we have,*

$$\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right] + 1\right) = \frac{x^2}{d^2} + o\left(\frac{x}{d}\right).$$

Proof.

$$\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right] + 1\right) = \left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right)\left(\left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right) + 1\right),$$

$$= \frac{x^2}{d^2} - 2\left\{\frac{x}{d}\right\}\frac{x}{d} + \left(\left\{\frac{x}{d}\right\}\right)^2 + \frac{x}{d} + \left\{\frac{x}{d}\right\},$$

$$= \frac{x^2}{d^2} + O(1)\cdot\frac{x}{d} + O(1).$$

$\qquad\square$

Theorem 2.9. *The average order of φ is $\frac{3x}{\pi^2}$.*

Proof.

$$\sum_{n\le x}\varphi(n) = \sum_{n\le x}n\sum_{d|n}\frac{\mu(d)}{d},$$

$$= \sum_{\substack{d,e \\ de \le x}}e\mu(d),$$

$$= \sum_{d\le x}\mu(d)\left(\sum_{e\le\frac{x}{d}}e\right),$$

Since $\sum_{e\le\frac{x}{d}}e$ is a sum of an arithmetic progression, we have,

$$\sum_{e\le\frac{x}{d}}e = \frac{1}{2}\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right] + 1\right).$$

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Substituting this, we obtain,

$$\sum_{n \leq x} \varphi(n) = \frac{1}{2} \sum_{d \leq x} \left(\mu(d) \left[\frac{x}{d} \right] \left(\left[\frac{x}{d} \right] + 1 \right) \right).$$

Note that

$$\frac{x}{d} = \left[\frac{x}{d} \right] + \left\{ \frac{x}{d} \right\} = \left[\frac{x}{d} \right] + O(1).$$

Using Lemma 2.5, that is,

$$\left[\frac{x}{d} \right] \left(\left[\frac{x}{d} \right] + 1 \right) = \frac{x^2}{d^2} + o \left(\frac{x}{d} \right).$$

We then have,

$$\sum_{n \leq x} \varphi(n) = \frac{1}{2} \sum_{d \leq x} \mu(d) \left(\frac{x^2}{d^x} + O \left(\frac{x}{d} \right) \right),$$

$$= \frac{x^2}{2} \sum_{d \leq x} \frac{\mu(d)}{d^2} + O \left(x \sum_{d \leq x} \frac{\mu(d)}{d} \right),$$

$$= \frac{x^2}{2} \left(\sum_{d \geq 1} \frac{\mu(d)}{d^2} - \sum_{d \geq x} \frac{\mu(d)}{d^2} \right) + O \left(x \sum_{d \geq x} \frac{\mu(d)}{d} \right),$$

$$= \frac{x^2}{2} \frac{1}{\zeta(2)} - \frac{x^2}{2} \sum_{d \geq x} \frac{\mu(d)}{d}^2 + O \left(x \sum_{d \leq x} \frac{\mu(d)}{d} \right).$$

We have,

$$\left| \sum_{d \geq x} \frac{\mu(d)}{d^2} \right| \leq \sum_{d \geq x} \frac{1}{d^2}$$

$$<< \int_x^\infty \frac{dt}{t^2},$$

$$<< \frac{1}{x}.$$

and,

$$\left| \sum_{d \leq x} \frac{\mu(d)}{d} \right| << \ln x.$$

Using these we have,

$$\sum_{n \leq x} \varphi(n) = \frac{x^2}{2} \frac{1}{\zeta(2)} + O(x) + O(x \ln x),$$

$$= \frac{x^2}{2\zeta(2)} + O(x \ln x).$$

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We conclude that

$$\frac{\sum_{n \leq x} \varphi(n)}{x^2} \xrightarrow{x \to \infty} \frac{1}{2\zeta(2)}.$$

In particular,

$$\frac{\sum_{n \leq x} \varphi(n)}{x} \sim \frac{x}{2\zeta(2)} = \frac{x \cdot 6}{2\pi^2}.$$

$\square$

2.3 Abel's Summation Formula

Recall the harmonic series $\sum_{n \in \mathbb{N}} \frac{1}{n}$ is divergent. Our next goal is to find such A_x that

$$\lim_{n \to \infty} \left(\sum_{n \leq x} \frac{1}{n} - A_x \right)$$

exists.

Remark 2.4 (Euler-Mascheroni constant). *By taking $A_x = \log(x)$, we have*

$$\lim_{n \to \infty} \left(\sum_{n \leq x} \frac{1}{n} - \log(x) \right) = \psi,$$

exists. Such ψ is called Euler-Mascheroni constant.

Remark 2.5 (Euler Kronecer constant). *Take $A_x = \log(x)$, we have*

We can show that

$$\psi = \lim_{s \to 1^+} \left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right).$$

Hint first show that

$$\zeta(s) = \frac{1}{s-1} + \psi + o(s-1).$$

Proposition 2.7 (Abels' summation formula). *Given $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{C}$ and $f(n)$ is continuously differentiable in $[1, x]$. Set*

$$A(x) := \sum_{n \leq x} a_n.$$

Then we have,

$$\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

Proof. Observe that
$$a_n = A(n) - A(n-1).$$
Assume $x \in \mathbb{N}$. We substitute this to $\sum_{n \leq x} a_n f(n)$, we get,

$$\sum_{n \leq x} a_n f(n) = \sum_{n \leq x} (A(n) - A(n-1)) f(n),$$

$$= \sum_{n \leq x} A(n) f(n) - \sum_{n \leq x} A(n-1) f(n),$$

$$= \sum_{n \leq x} A(n) f(n) - \sum_{n \leq x-1} A(n) f(n+1),$$

$$= A(x) f(x) - \sum_{n \leq x-1} A(n) (f(n+1) - f(n)),$$

$$= A(x) f(x) - \sum_{n \leq x-1} A(n) \int_n^{n+1} f'(t) dt,$$

$$= A(x) f(x) - \sum_{n \leq x-1} \int_n^{n+1} A(t) f'(t) dt,$$

$$= A(x) f(x) - \int_1^x A(t) f'(t) dt.$$

For the case when $n \notin \mathbb{N}$ and $n > 1$,

$$\sum_{n \leq x} a_n f(n) = \sum_{n \leq [x]} a_n f(n).$$

Using the previous case, we get,

$$\sum_{n \leq x} a_n f(n) = A([x]) f([x]) - \int_1^{[x]} A(t) f'(t) dt.$$

Remains to show that we can remove the brackets. To do so,

$$\sum_{n \leq x} a_n f(n) = A([x]) f([x]) - \int_1^x A(t) f'(t) dt + \int_{[x]}^x A(t) f'(t) dt,$$

$$= A([x]) f([x]) - \int_1^x A(t) f'(t) dt + A([x]) \int_{[x]}^x f'(t) dt,$$

$$= A([x]) f([x]) - \int_1^x A(t) f'(t) dt + A([x]) f(x) - A([x]) (f[x]),$$

$$= A([x]) f(x) - \int_1^x A(t) f'(t) dt.$$

$\square$

Corollary 2.1.

1. $\sum_{n \leq x} \frac{1}{n} = \ln x + \psi + o\left(\frac{1}{x}\right)$.

2. $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + o\left(\frac{1}{x^s}\right)$, where $\mathrm{Re}(s) > 0, s \neq 1$.

We also have the following equivalent forms of prime number theorem when $x \to \infty$.

$$\sum_{n \leq x} \Lambda(n) \sim x \Leftrightarrow \pi(x) \sim \frac{x}{\ln x},$$

$$\Leftrightarrow \sum_{p \leq x} \ln p \sim x,$$

$$\Leftrightarrow \sum_{n \leq x} \mu(n) = o(x).$$

Proof. Consider $f(t) = \frac{1}{t}$ and $a_n = 1$ for all $n \in \mathbb{N}$.

$$\sum_{n \leq x} \frac{1}{n} = \frac{[x]}{x} + \int_1^x \frac{[t]}{t^2} dt,$$

$$= \frac{x - \{x\}}{x} + s \int_1^x \frac{t - \{t\}}{t^2} dt,$$

$$= 1 + \log x - \frac{\{x\}}{x} - \int_1^\infty \frac{\{t\}}{t^2} dt.$$

For general $s, \mathrm{Re}(s) > 0, s \neq 1$, we have

$$\sum_{n \leq x} \frac{1}{n^s} = x^{1-s} + o\left(\frac{1}{x^s}\right) + \frac{sx^{1-s}}{1-s} - \frac{s}{1-s} - s \int \frac{\{t\}}{t^{s+1}} dt.$$

Recall that

$$\left[\int \frac{1}{t^s} = \frac{t^{-s+1}}{1-s}\right]_1^x = \frac{x^{1-s}}{1-s} - \frac{1}{1-s}.$$

Using this we obtain,

$$\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - x \int_1^x \frac{\{t\}}{t^{s+1}} dt,$$

$$x^{1-s}\left[1 + \frac{1}{1-s}\right] = \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + o\left(\frac{1}{x^s}\right) - s \int_1^x \frac{\{t\}}{t^{s+1}} dt$$

$$\int_1^\infty \frac{\{t\}}{t^{s+1}} < \infty,$$

$$\leq \int_1^\infty \frac{1}{t^{\mathrm{Re}(s)+1}} < \infty.$$

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As $x \to \infty$, the left hand side goes to $\zeta(s)$, for the right hand side, we get $= \frac{-s}{1-s} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt$, where $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{-s}{1-s} - s \int \frac{\{t\}}{t^{s+1}} dt.$$

Identity theorem for analytic function tells us the analytic continuation of Riemann zeta function is unique. □

Remark 2.6. *It is an exercise that*

$$\int_1^\infty \frac{\{t\}}{t^{s+1}} dt$$

where $\mathrm{Re}(x) > 0$. *From Stein-Schakarchi 5.2, 5.3, we have*

$$\sum f_n(z) \overset{unif.}{\to} f(z)$$

is analytic where $\mathrm{Re}(x) > 0$ *and* $s \neq 1$, *also in this case,*

$$\zeta(s) = \frac{-s}{1-s} - s \int \frac{\{t\}}{t^{s+1}} dt.$$

holds.

Remark 2.7 (Exercise). *Let* $M \in \mathbb{N}$ *and*

$$\lim_{x \to \infty} \left(\sum_{\substack{n \leq x \\ (n,M)=1}} \frac{1}{n} - \frac{\phi(M)}{M} \ln x \right)$$

exists.

Suppose we have, $\pi(x) \sim \frac{x}{\ln x}$. We want to show

$$\theta(x) := \sum_{p \leq x} \ln p \sim x,$$

Consider the following sequence

$$a_n = \begin{cases} 1, & (n \text{ is prime}), \\ 0, & (\text{otherwise}). \end{cases}$$

and

$$f(t) = \ln t.$$

Using Abel summation formula,

$$\sum_{p \leq x} \ln p = \pi(x) \ln(x) - \int_1^\infty \frac{\pi(t)}{t} dt.$$

Dividing both sides by x, we obtain,

$$\frac{\theta(x)}{x} = \frac{\pi(x) \ln x}{x} - \frac{1}{x} \int_1^x \frac{\pi(t)}{t} dt.$$

Remark 2.8 (Exercise). *Use $\frac{\pi(t)}{t} \sim \frac{1}{\ln t}$ and*

$$\int_1^x \frac{dt}{\ln t} = o(x),$$

prove that

$$\lim_{x \to \infty} \frac{1}{x} \int_1^x \frac{\pi(t)}{t} dt = 0.$$

$$\int_2^{\sqrt{x}} \frac{dt}{\ln t} = \int_{\sqrt{x}}^x \frac{dt}{\ln t} \leq \frac{1}{\ln 2} \int_2^{\sqrt{x}} dt + \frac{1}{\ln \sqrt{x}} \int_{\sqrt{x}}^x dt,$$

$$= \frac{\sqrt{x} - 2}{\ln 2} + \frac{x - \sqrt{x}}{\ln \sqrt{x}},$$

$$= o(x).$$

Remark 2.9.

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{\substack{1 \leq k \\ p^k \leq x}} \ln p.$$

Thus we see,

$$\psi(x) - \theta(x) = \sum_{\substack{2 \leq k \\ p, p^k \leq x}} \ln p.$$

Also,

$$p^k \leq x \Rightarrow k \leq \frac{\ln x}{\ln p}.$$

Using $k \geq 2$,

$$p \leq x^{\frac{1}{k}} \leq \sqrt{x}.$$

Thus we have the inequalities,

$$\psi(x) - \theta(x) \leq \sum_{p \leq \sqrt{x}} \ln p \left(\sum_{2 \leq k \leq \frac{\ln x}{\ln p}} 1 \right),$$

$$\leq \sum_{p \leq \sqrt{x}} \ln x,$$

$$\leq \ln(x) \sum_{n \leq \sqrt{x}} 1,$$

$$\leq \sqrt{x} \ln x.$$

$$\Rightarrow \frac{\psi(x)}{x} = \frac{\theta(x)}{x} + O\left(\frac{\sqrt{x} \ln x}{x} \right).$$

Therefore we obtain,

$$\psi(x) \sim x \Leftrightarrow \theta(x) \sim x.$$

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Remark 2.10. *As exercises, find the closed expressions for the following summations.*

$$\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n}, \sum_{p \leq x} \frac{1}{p} = \ln \ln x + o(1).$$

2.4 Characters

Definition 2.10. *Let G be a finite group. A character is a group homomorphism $f : G \to \mathbb{C}^\times$.*

Remark 2.11. *Let us denote*

$$\hat{G} := \{f : G \to \mathbb{C}^\times \mid characters\}.$$

If G is finite abelian then $|\hat{G}| = |G|$. Furthermore, such characters are linearly independent over $\mathbb{C}$.

Definition 2.11. *Let $q \in \mathbb{N}$ and $q \geq 3$. A Dirichlet character is a group homomorphism modulo q is a group homomorphism*

$$\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times.$$

Remark 2.12. *Given a Dirichlet character $\chi' : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$. We ca define a character $\chi : \mathbb{Z} \to \mathbb{C}^\times$ as follows.*

$$\chi(a) := \begin{cases} \chi'(\overline{a}), & (a, q) = 1, \\ 0, & (a, q) \neq 1. \end{cases}$$

From Remark 2.11, there are exactly $\varphi(q)$ many Dirichlet characters modulo q. Furthermore,

$$\chi(a)^{\varphi(q)} = \chi'(\overline{a})^{\varphi(q)} = \chi'(\overline{a}^{\varphi(q)}) = \chi'(\overline{1}) = 1.$$

In particular, images of χ are $\varphi(q)$-th roots of unity.

Example 2.8. *For $q = 3$,*

$$(\mathbb{Z}/3\mathbb{Z})^\times = \{\overline{1}, \overline{2}\} \to \mathbb{C}^\times.$$

We only have two characters, a trivial one and $\overline{2} \mapsto -1$.

Example 2.9. *For $q = 5$,*

$$(\mathbb{Z}/5\mathbb{Z})^\times = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$$

	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\chi_{1,5}(n)$	1	1	1	1
$\chi_{2,5}(n)$	1	-1	-1	1
$\chi_{3,5}(n)$	1	i	$-i$	-1
$\chi_{4,5}(n)$	1	$-i$	i	-1

$\chi_{1,5}$ *is called a principle/trivial character.*

Definition 2.12. *A character* $\chi : \mathbb{Z} \to \mathbb{C}^{\times}$ *is called*

1). trivial if $\chi(g) = 1$ *for all* $g \in G$,

2). even if $\chi(-1) = 1$,

3). odd if $\chi(-1) = -1$.

We also define these notions for characters $\chi_0 : (\mathbb{Z}/q\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$ *accordingly if the character* $\chi : \mathbb{Z} \to \mathbb{C}^{\times}$ *induced by it has these properties. Trivial characters are often denoted by* χ_0.

Theorem 2.10.
$$\sum_{a \bmod q} \chi(a) = \begin{cases} \phi(q) & (\chi = \chi_0), \\ 0 & (otherwise). \end{cases}$$

We also have,
$$\sum_{\chi \bmod q} \chi(a) = \begin{cases} \phi(a) & (\overline{a} = 1), \\ 0 & (otherwise). \end{cases}$$

Proof. If χ is principle then the first assertion is clear. Suppose χ is not principle then there is $b \in \{1, \cdots, q\}$ such that $\chi(b) \neq 1$ and $(b, q) = 1$. Let

$$s = \sum_{a \bmod q} \chi(a).$$

Then by the definiion of group homomorphisms, we have,

$$\chi(b)s = s.$$

But $\chi(b) \in \mathbb{C}$, this means $s = 0$ as $\mathbb{C}$ is an integral domain.

For the second assertion, let $\overline{a} \neq 1$, then

$$\exists \chi' \bmod q, \text{ s.t. } \chi'(a) \neq 1.$$

Thus we get,

$$s = \sum_{\chi \bmod q} \chi(a), s \cdot \chi(a) = \sum_{\chi \bmod q} \chi\chi'(a) = s \Rightarrow s = 0.$$

The statement when $\overline{a} = 1$ follows from Remark 2.11. $\qquad\square$

Remark 2.13. *One can check in the table of Example 2.9 that Theorem 2.10 indeed holds.*

Exercise 2.3.

$$\sum_{\substack{\chi \bmod q, \\ \chi(-1)=1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} & (\bar{a} = 1, -1), \\ 0 & (otherwise). \end{cases} \quad ,$$

$$\sum_{\substack{\chi \bmod q, \\ \chi(-1)=-1}} \chi(a) = \begin{cases} \frac{\phi(a)}{2} & (\bar{a} = 1), \\ -\frac{\phi(a)}{2} & (\bar{a} = -1), \\ 0 & (otherwise). \end{cases} \quad ,$$

Obviously we have the following equalities.

$$\sum_{n \leq x} \chi(n) = \sum_{\substack{n \leq x \\ (n,q)=1}} \chi(n) = \sum_{n \leq kq} \chi(n) + \sum_{n=kq+1}^{x} \chi(n),$$

where k is the largest integer such that $kq \leq x$. Then we observe from Theorem 2.10

$$\sum_{n \leq kq} \chi(a) = k \left(\sum_{n=1}^{q} \chi(n) \right) = 0,$$

unless χ is trivial. Also we have,

$$\left| \sum_{n \leq x} \chi(n) \right| = \left| \sum_{\substack{kq+1 \leq n \leq x \\ (n,q)=1}} \chi(n) \right| \leq \sum_{\substack{kq+1 \leq n \leq x \\ (n,q)=1}} 1 \leq \sum_{\substack{kq+1 \leq n \leq kq+q \\ (n,q)=1}} 1 = \phi(q).$$

Thus we conclude,

$$\sum_{n \leq x} \chi(n) \leq \varphi(q),$$

Exercise 2.4.

$$\sum_{n \leq x} \chi_0(n) = ?.$$

Theorem 2.11 (Pólya–Vinogradov)**.**

$$\sum_{n \leq x} \chi(n) << \sqrt{q} \ln q, (\chi \neq \chi_0 \mod q).$$

notice that the above expression is bounded by $\sqrt{q} \ln \ln q$. Furthermore, this is uniform in q that is the constant does not depend on q.

$$\sum_{n \geq 1} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} - \frac{s}{1-s} + O(x^{-s}) - s \int_1^x \frac{\{t\}}{t^{s+1}} dt.$$

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For $\mathrm{Re}(s) > 1$, as $x \to \infty$ we have,

$$\sum_{n \geq 1}^{\infty} \frac{1}{n^s} = \frac{-s}{1-s} - s \int_1^{\infty} \frac{\{t\}}{t^{s+1}} dt.$$

The last expression is analytic since,

$$\sum \int_n^{n+1} \frac{\{t\}}{t}^{s+1} dt \overset{\text{uniformly}}{\to} \int_1^{\infty} \frac{\{t\}}{t^{s+1}}, \text{when } \mathrm{Re}(s) > 0.$$

Suppose $\zeta(s) \neq 0$, where $\mathrm{Re}(s) > 0$, then Euler product exists.

Theorem 2.12. *Set*
$$A(n) := \sum_{n \in \mathbb{N}} a_n.$$

Assume $A(x) := \sum_{n \leq x} a_n = O(x^\delta)$, then we have, for $\mathrm{Re}(s) > \delta$,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt.$$

Hence the Dirichlet series converges for $\mathrm{Re}(s) > \delta$.

Proof.

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s} = \frac{A(x)}{x^s} + s \int_1^{x} \frac{A(t)}{t^{s+1}} dt.$$

As $A(x) = O(x^\delta)$ and $\mathrm{Re}(s) > \delta$, $\frac{A(x)}{x^s} = O(x^{\delta - \mathrm{Re}(s)})$. Therefore, as $x \to \infty$, we have,

$$\sum_{n \in \mathbb{N}} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt.$$

Again using the assumption, we have,

$$\int_1^{\infty} \left| \frac{A(t)}{t^{s+1}} \right| dt << \int_1^{\infty} t^{\delta - \mathrm{Re}(s) - 1} dt = \left. \frac{t^{\delta - \mathrm{Re}(s)}}{\delta - \mathrm{Re}(s)} \right|_0^{\infty} = \frac{1}{\delta - \mathrm{Re}(s)}.$$

Thus the integral is convergent. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Definition 2.13. *For $\mathrm{Re}(s) > 1$, we define*

$$L(s, \chi) := \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s}.$$

Remark 2.14. *Since $\mathrm{Re}(s) > 1$ and for any character $\chi : \mathbb{Z} \to \mathbb{C}^{\times}$ we have $|\chi(n)| \leq 1$, $L(s, \chi)$ is uniformly absolutely convergent.*

Example 2.10. *Let $\chi : \mathbb{Z} \to \mathbb{C}^\times$ be a non-principal character modulo q and set $A(n) = \chi(n)$. Recall that*

$$\sum_{n \leq x} \chi(n) \leq q.$$

Taking $A(n) = \chi(n)$ and apply Theorem 2.12, we obtain, for $\mathrm{Re}(s) > 0$,

$$L(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = s \int_1^\infty \frac{\sum_{n \leq t} \chi(n)}{t^{s+1}} dt.$$

Since for $\mathrm{Re}(s) > 1, L(s, \chi)$ is absolutely uniformly convergent. By Theorem 2.7, we have,

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s} \right)^{-1}.$$

If $\chi = \chi_0$, a principal character, we have,

$$L(s, \chi_0) = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s} \right)^{-1} = \prod_p \left(1 - \frac{1}{p^s} \right)^{-1} \prod_{p|q} \left(1 - \frac{1}{p^s} \right).$$

Using ζ function, we have,

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s} \right).$$

Theorem 2.13. *ζ has an analytic continuation for $\mathrm{Re}(s) > 0$ besides $s = 1$. For $s = 1$ we have a simple pole of residue 1.*

Proof. Recall from the proof of Corollary 2.1. We have for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{-s}{1-s} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

The right hand side of the equation is analytic when $\mathrm{Re}(s) > 0, s \neq 1$. When $s = 1$, we have,

$$\lim_{s \to 1^+} (s-1)\zeta(s) = \lim_{s \to 1^+} \left(s - s(s-1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt \right) = 1.$$

$\square$

Corollary 2.2. *For $\mathrm{Re}(s) > 0$ and $s \neq 1$, we have an analytic continuation of $L(s, \chi_0)$ where χ_0 is a principal character modulo q, which is*

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s} \right).$$

Obviously at $s = 1$, it has a simple pole of residue $\prod_{p|q} \left(1 - \frac{1}{p^s} \right)$, which we can write as

$$\mathrm{Res}(L(s, \chi_0), 1) = \frac{\varphi(q)}{q}.$$

Proof. A direct corollary of Theorem 2.13. For the residuce, write,

$$q = \prod_{p|q} p^{\alpha(p)},$$

then

$$\varphi(q) = \prod_{p|q}(p-1)p^{\alpha(p)-1} = q\prod_{p|q}(p-1)p^{-1}.$$

$\square$

Theorem 2.14. *Let χ be a non-principal character, then there is an analytic continuation of $L(s,\chi)$ for $\mathrm{Re}(s) > 0$.*

Proof. From Example 2.10, we have,

$$L(s,\chi) = \sum_{n\in\mathbb{N}}\frac{\chi(n)}{n^s} = s\int_0^\infty \frac{\sum_{n\leq t}\chi(n)}{t^{s+1}}dt.$$

The right hand side is analytic. $\square$

Theorem 2.15.
$$\sum_{n\in\mathbb{N}}\frac{a_n}{n^s}$$

is analytic in its range of convergence.

Remark 2.15. *We have the following conjecture.*

$$L\left(\frac{1}{2},\chi\right) = 0, \chi \neq \chi_0.?$$

$$\zeta\left(\frac{1}{2}\right) = \frac{1}{1-\sqrt{2}}\sum_{n\in\mathbb{N}}\frac{(-1)^{n-1}}{\sqrt{n}} \approx -1.46\cdots$$

Definition 2.14. *A character is said to be quadratic if its values are either ± 1.*

Remark 2.16. *We have,*

$$L(s,\chi) = 0 \quad \text{if } s = 0, -2, -4, \text{ when } \chi \text{ is an even character.}$$
$$L(s,\chi) = 0 \quad \text{if } s = -1, -3, -5, \text{ when } \chi \text{ is an odd character.}$$

Lemma 2.6. *For $\sigma > 1$ and $t \in \mathbb{R}$, we have,*

$$\mathrm{Re}(\ln(\zeta(\sigma + it))) = \sum \frac{\Lambda(n)}{n^\sigma \ln n}\ln(t\ln(n)).$$

And also,

$$\mathrm{Re}(3\ln(\zeta(\sigma)) + 4\ln(\zeta(\sigma + it)) + \ln(\zeta(\sigma + 2it))) \geq 0.$$

Proof.

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \sigma > 0,$$

$$\ln(\zeta(s)) = -\sum_p \ln(1 - p^{-s})$$

$$= \sum_{p,n} \frac{1}{np^{ns}}, \sigma > 1.$$

We have,

$$\sum_{n \geq 2} \frac{\Lambda(n)}{n^s \ln n} = \sum_{p,k,k \geq 1} \frac{\ln p}{p^{ks} \ln p^k} = \sum_{p,k} \frac{1}{kp^{ks}} = \sum_{n \geq 2} \Lambda(n).$$

$$\mathrm{Re}(3\ln\zeta(\sigma) + 4\ln\zeta(\sigma + it) + \ln\zeta(\sigma + 2it)) = \sum_{n \geq 2} \frac{\Lambda(n)}{n^\sigma \ln n}(3 + 4\cos(t\ln n) + \cos(2t\ln n)) \geq 0,$$

since

$$3 + 4\cos\theta + \cos 2\theta = 2(\cos\theta + 1)^2 \geq 0.$$

$$= \ln|\zeta(\sigma)^3\zeta(\sigma + it)^4\zeta(\sigma + 2it)| \geq 0.$$

Thus we have,

$$\mathrm{Re}(\ln(z)) \leq$$

$\square$

Theorem 2.16. *For $t \in \mathbb{R}\backslash\{0\}$, we have*

$$\zeta(1 + it) \neq 0.$$

Proof. Using Lemma 2.6, we have, Thus we get,

$$|\zeta(\sigma)^3\zeta(\sigma + it)^4\zeta(\sigma + 2it)| \geq 1.$$

Suppose $\zeta(1 + it_0) = 0$, for $t_0 \in \mathbb{R}\backslash\{0\}$. Suppose further that the order of zero is $m \in \mathbb{N}$. Then by looking at, and taking $\sigma \to 1+$

$$\underbrace{((\sigma - 1)\zeta(\sigma))^3}_{\to\text{finite}} \underbrace{\left(\frac{\zeta(\sigma + it_0)}{(\sigma - 1)^m}\right)^4}_{\to\text{non-zero}} \underbrace{((\sigma - 1)^{4m-3}\zeta(\sigma + 2it_0))}_{\to 0} \to 0.$$

Contradicts to that the abosolute value of above expression is at least 1. $\square$

Theorem 2.17. $\frac{\zeta'(s)}{\zeta(s)}$ *has an analytic continuation to* $\operatorname{Re}(s) = 1, s \neq 1$. *And for* $s = 1$, *we have a simple pole of residue* -1.

Proof. We have,

$$(s-1)\zeta(s) = s - s(s-1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt.$$

Set

$$f(s) := 1 - (s-1) \int_1^\infty \frac{\{t\}}{t^{s+1}} dt,$$

so that $(s-1)\zeta(s) = sf(s)$. We already have $f(s)$ is analytic when $\operatorname{Re}(s) > 0$. Differentiating both sides we get,

$$(s-1)\zeta'(s) + \zeta(s) = sf'(s) + f(s).$$

Dividing both sides by $(s-1)\zeta(s)$ we get,

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \frac{sf'(s) + f(s)}{(s-1)\zeta(s)}.$$

The right hand side is analytic when ζ does not vanish. From Theorem 2.16, by letting $s = 1 + it$ for some $t \in \mathbb{R}\backslash\{0\}$, we get the desired analytic continuation. For $s = 1$, we have

$$(s-1)\frac{\zeta'(s)}{\zeta(s)} = \frac{sf'(s) + f(s)}{\zeta(s)} - 1.$$

Recall that $\zeta(s)$ has a pole at 1 and observe that $f(1) = 1, f'(s)$ is finite thus the last statement follows. $\qquad\square$

Theorem 2.18. *Let* χ *be a non-trivial non-real Dirichlet character of modulo* q, *then*

$$L(1, \chi) \neq 0.$$

Proof. Recall that

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where $\operatorname{Re}(s) > 1$. Now consider,

$$\log L(s, \chi) = \sum_{n,p} \frac{\chi(p^s)}{np^{ns}} = \sum_{n \geq 2} \frac{\Lambda(n)\chi(n)}{n^s \log n}.$$

Set $\zeta_{\varphi(q)} = e^{\frac{2\pi i}{\varphi(q)}}$. Then for any $n \in (\mathbb{Z}/q\mathbb{Z})^\times$, there is $n' = n(n, q, \chi)$ (ie n' depends on n, q, and χ) such that

$$\chi(n) = \zeta_{\varphi(q)}^{n'}$$

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Note that for a cyclic group $G = \langle g \rangle$, and a character $\chi : G \to \mathbb{C}^\times \in \hat{G}$, there is $a \in \{1, \cdots, \varphi(q)\}$ such that

$$\chi(g) = \zeta_{\varphi(q)}^a.$$

We have, for $\sigma > 1$

$$\mathrm{Re}(\log L(\sigma, \chi)) = \sum_{n \geq 2} \frac{\Lambda(n) \cos\left(\frac{2\pi i}{\varphi(q)}\right)}{n^\sigma \log(n)}.$$

By Lemma 2.6, we have,

$$\mathrm{Re}(3 \ln \zeta(\sigma) + 4 \ln L(\sigma, \chi) + \ln L(\sigma, \chi^2)) \geq 0,$$

therefore,

$$|\zeta(\sigma)^3 L(\sigma, \chi)^4 L(\sigma, \chi^2)| \geq 1.$$

Note that $\chi^2 \neq \chi$ as it is non-real and $L(\sigma, \chi^2)$ is analytic by Theorem 2.14. If $L(\sigma, \chi) = 0$ then

$$\zeta(\sigma)^3 L(\sigma, \chi)^4$$

has a zero of order at least 1. Thus this is a contradiction. $\qquad \square$

Lemma 2.7. *For $k \in \mathbb{Z}_{\geq 0}$,*

$$\left(\frac{\sin\left(k + \frac{1}{2}\right)\theta}{\sin \frac{\theta}{2}}\right)^2 = 2k + 1 \sum_{j=1}^{2k} 2(2k + 1 - j) \cos j\theta.$$

Theorem 2.19. *Let $f : \mathbb{C} \to \mathbb{C}$ be a fnction such that*

i). f is analytic,

ii). $f \not\equiv 0$,

iii). $\log f(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s}$ for $a_n \geq 0$ and $\mathrm{Re}(s) > 1$,

iv). f is analytic on $\mathrm{Re}(s) = 1, s \neq 1$, and it has a pole at $s = 1$ of order e.

If $f(s) = 0$ on the line $\mathrm{Re}(s) = 1$, then the order of zero is at most $\frac{e}{2}$.

Proof. For $e \leq 2k - 1$, suppose f has a zero at $f(1 + it_0)$ of order k. Let us define,

$$g(s) := f(s)^{2k+1} \prod_{j=1}^{2k} f(s + ijt_0)^{2(2k+1-j)} = f(s)^{2k+1} f(s + it_0)^{4k} \cdots.$$

Then at $s = 1$, $f(s)^{2k+1}$ has a pole of order $e(2k+1)$ and $\prod_{j=1}^{2k} f(s+ijt_0)^{2(2k+1-j)}$ has a zero of order $4k$ Note that

$$4k^2 - e(2k + 1) \geq 1.$$

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Thus g has a zero at 1 of order at least 1, in particular, $g(1) = 0$.

Consider,

$$\log g(\sigma) = (2k+1)\log f(\sigma) + \sum_{j=1}^{2k} 2(2k+1-j)\log(f(\sigma + ijt_0)),$$

$$= (2k+1)\sum_{n\in\mathbb{N}} \frac{a_n}{n^\sigma} + \sum_{j=1}^{2k} 2(2k+1-j)\left(\sum_{n\in\mathbb{N}} \frac{a_n}{n^{\sigma+ijt_0}}\right),$$

$$= \sum_{n\in\mathbb{N}} \frac{a_n}{n^\sigma}\left(2k+1+\sum_{j=1}^{2k} 2(2k+i-j)e^{-ijt_0\log(n)}\right).$$

$$\mathrm{Re}(\log(g(\sigma))) = \sum_{n\in\mathbb{N}} \frac{a_n}{n^\sigma}\left(2k+1+\sum_{j=1}^{2k} 2(2k+i-j)\cos(jt_0\log(n))\right).$$

Using Lemma 2.7, we get,

$$\mathrm{Re}\log(g(\sigma)) = \sum_{n\geq 1} \frac{a_n}{n^\sigma}\left(\frac{\sin\left(k+\frac{1}{2}\right)t_0\log(n)}{\sin\frac{t_0\log(n)}{2}}\right)^2 \geq 0.$$

As $\mathrm{Re}(g(\sigma)) \geq 1$, $|g(\sigma)| \geq 1$. This contradicts that g has a zero of order at least 1 at $1 + it_0$. $\qquad\square$

Corollary 2.3. *For any Dirichlet character χ modulo q we have, $L(s,\chi)$ is analytic over $\mathrm{Re}(s) > 1$. Furthermore $L(s,\chi) \neq 0$ on $\mathrm{Re}(s) = 1, s \neq 1$.*

Proof. The first assertion is due to Theorem 2.14. Let

$$f(s) := \prod_{\chi \bmod q} L(s,\chi).$$

Then,

$$\log(f(s)) = \sum_{\chi \bmod q} \log L(s,\chi) = \sum_{n,p} \frac{\sum_{\chi \bmod q}\chi(p^n)}{np^{ns}}, \mathrm{Re}(s) > 1.$$

Using Theorem 2.10, we get,

$$\log(f(s)) = \sum_{\substack{n,p \\ p^n \equiv 1\bmod q}} \frac{\varphi(q)}{np^{ns}}.$$

Note f has a pole of order at most 1 at $\mathrm{Re}(s) = 1$ and its residue is

$$\mathrm{Res}(f(s),1) = \left(\lim_{s\to 1}(s-1)L(s,\chi_0)\right)\prod_{\substack{\chi\neq\chi_0 \\ \chi\bmod q}} L(1,\chi) = \frac{\varphi(q)}{q}\prod_{\substack{\chi\neq\chi_0 \\ \chi\bmod q}} L(1,\chi).$$

The right hand side is not 0 by Theorem 2.18. $\qquad\square$

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Theorem 2.20. *There are infinitely many primes.*

Proof. We have,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

$$\log \zeta(s) = \sum_{n,p} \frac{1}{np^{ns}} = \sum_p \frac{1}{p^s} + \sum_{n \geq 2} \frac{1}{np^{ns}}.$$

Since ζ has a pole at 1, so does its log. Observe that at $s = 1$,

$$\sum_p \frac{1}{p^s} + \sum_{n \geq 2} \frac{1}{np^{ns}} \leq \sum_p \sum_{n \geq 2} \frac{1}{p^n} = \sum_p \frac{1}{p^2} \frac{1}{1 - \frac{1}{p}} = \sum_p \frac{1}{p(p-1)} << \sum_{n \in \mathbb{N}} \frac{1}{n^2}.$$

Thus $\lim_{s \to 1^+} \sum_p \frac{1}{p^s}$ must be infinity. $\qquad\square$

Lemma 2.8. *Let p be a prime and a, q be coprimes such that $p \equiv a \bmod q$. Then we have,*

$$\sum_{\chi \bmod q} \chi(p^n)\overline{\chi(a)} = \begin{cases} \varphi(q), & (p^n \equiv a \bmod q), \\ 0, & (\text{otherwise}). \end{cases}$$

Proof. Since $\chi(a)$ is a $\varphi(q)$-th root of unity, we have, $\overline{\chi(a)} = \chi(a^{-1})$. Therefore,

$$\sum_{\chi \bmod q} \chi(p^n)\overline{\chi(a)} = \sum_{\chi \bmod q} \chi(p^n)\chi(a^{-1}),$$

$$= \sum_{\chi \bmod q} \chi(p^n a^{-1}),$$

$$= \begin{cases} \varphi(q), & (p^n a^{-1} \equiv 1 \bmod q), \\ 0, & (\text{otherwise}). \end{cases}$$

$\qquad\square$

Theorem 2.21 (Dirichlet's Theorem)**.** *Let a, q be coprime. Then $(a + nq)_{n \in \mathbb{N}}$ contains infinitely many primes.*

Proof. Motivated by the alternative proof of the existence of infinitely many primes, examine,

$$\sum_{\chi \bmod q} \log L(s, \chi) = \sum_{n,p} \left(\frac{\sum_{\chi \bmod q} \chi(p^n)}{np^{ns}}\right),$$

where $\text{Re}(s) > 1$. Using Theorem 2.10, we have,

$$\sum_{\chi \bmod q} \log L(s, \chi) = \sum_{\substack{n,p \\ p^n \equiv 1 \bmod q}} \frac{\varphi(q)}{np^{ns}} = \sum_{p^n \equiv 1 \bmod q} \frac{\varphi(q)}{p^s} + \sum_{\substack{p, n \geq 2 \\ p^n \equiv 1 \bmod q}} \frac{\varphi(q)}{np^{ns}}.$$

Taking the log out we have,

$$\prod_{\chi \bmod q} L(s, \chi) = \exp\left(\varphi(q)\left(\sum_{p^n \equiv 1 \bmod q} \frac{1}{p^s} + \sum_{\substack{p,n \geq 2 \\ p^n \equiv 1 \bmod q}} \frac{1}{np^{ns}}\right)\right).$$

Since for $\chi \neq \chi_0, L(1, \chi) \neq 0$ and by Corollary 2.2, we have,

$$\lim_{s \to 1^+} (s-1) \prod_{\chi \bmod q} L(s, \chi) = \frac{\varphi(q)}{q} \prod_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} L(1, \chi).$$

By the same argument from Theorem 2.20, we see,

$$\sum_{\substack{p,n \geq 2 \\ p^n \equiv \bmod q}} \frac{1}{np^{ns}} \leq \sum_{n \in \mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

However,

$$\lim_{s \to 1^+} (s-1) \exp\left(\varphi(q)\left(\sum_{p^n \equiv 1 \bmod q} \frac{1}{p^s} + \sum_{\substack{p,n \geq 2 \\ p^n \equiv 1 \bmod q}} \frac{1}{np^{ns}}\right)\right) \neq 0$$

which is only possible when

$$\lim_{s \to 1^+} \sum_{p^n \equiv 1 \bmod q} \frac{1}{p^s} = \infty.$$

Together with Lemma 2.8, we derived the general statement. $\square$

Theorem 2.22 (Bertrand's Postulate). *For a sufficiently large $n \in \mathbb{N}$, there is a prime number inbetween n and $2n$.*

Proof. Consider the second Chebyschev function, and by Remark 2.1, we have,

$$T(x) := \sum_{l \leq x} \psi\left(\frac{x}{l}\right) = \sum_{l \leq x} \sum_{n \leq \frac{x}{l}} \Lambda(n) = \sum_{\substack{l,n \\ ln \leq x}} \Lambda(n).$$

That is

$$\sum_{l \leq x} \psi\left(\frac{x}{l}\right) = \sum_{m \leq x} \sum_{d | m} \Lambda(d).$$

By Theorem 2.3,

$$T(x) = \sum_{m \leq x} \log(m).$$

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Using the definition,

$$T(x) - 2T\left(\frac{x}{2}\right) = \sum_{l \leq x} \psi\left(\frac{x}{l}\right) - 2\sum_{2l \leq x} \psi\left(\frac{x}{2l}\right),$$
$$= \sum_{l \leq x} (-1)^{l+1}\psi\left(\frac{x}{l}\right),$$
$$= \psi(x) - \psi\left(\frac{x}{2}\right) + \cdots.$$

Again by Remark 2.1

$$x \leq y \Rightarrow \psi(x) \leq \psi(y).$$

In particular,

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq \sum_{l \leq x} (-1)^{l+1}\psi\left(\frac{x}{l}\right) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right).$$

Recall Proposition 2.7, consider

$$A(x) = \sum_{n \leq x} 1 = \lfloor x \rfloor,$$

and since $\log(x)$ is continuously differentiable on $[1, x]$, we have,

$$T(x) = \lfloor x \rfloor \log(x) - \int_1^x \frac{\lfloor x \rfloor}{x} dx = x\log(x) - x + O(\log(x)).$$

Therefore,

$$T(x) - 2T\left(\frac{x}{2}\right) = x\log(x) - x + O(\log(x)) - 2\left(\frac{x}{2}\log\left(\frac{x}{2}\right) - \frac{x}{2} + O(\log(x))\right),$$
$$= \log 2 \cdot x + O(\log(x)).$$

Combining with the previous result, we get,

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq (\log 2)x + O(\log(x)) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right).$$

Generalizing this we obtain,

$$\psi\left(\frac{x}{2^{n-1}}\right) - \left(\frac{x}{2^n}\right) \leq \log(2)\frac{x}{2^{n-1}} + O(\log(x)).$$

By induction, we obtain,

$$\psi(x) - \psi\left(\frac{x}{2^n}\right) \leq (\log 2)x\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) + O(n\log(x)),$$
$$\leq (2\log 2)x + O(n\log(x)). \tag{ψ1}$$

Take n to be the maximal such that $2^n \leq x$ that is $\lfloor \frac{x}{2^n} \rfloor = 1$. Then, $\psi\left(\frac{x}{2^n}\right) = 0$, thus,

$$\psi(x) \leq (2\log 2)x + O((\log(x))^2).$$

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On the other hand,

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \geq (\log 2)x + O(\log(x)),$$

$$\Rightarrow \psi(x) - \psi\left(\frac{x}{2}\right) \geq (\log(2))x + O(\log(x)) - \psi\left(\frac{x}{3}\right)$$

Using Inequality ($\psi1$), we get,

$$\psi(x) - \psi\left(\frac{x}{2}\right) \geq (\log 2)x + O(\log(x)) - (2\log 2)\frac{x}{3} + O((\log(x))^2),$$

$$= (\log 2)\frac{x}{3} + O((\log(x))^2). \qquad (\psi2)$$

Using Remark 2.9, that is

$$\psi(x) - \theta(x) = O(\sqrt{x}\log(x)),$$

together with Inequalities ($\psi1$) and ($\psi2$), we obtain,

$$\theta(x) - \theta\left(\frac{x}{2}\right) \geq (\log 2)\frac{x}{3} + O(\sqrt{x}\log(x)).$$

Thus the right hand side is greater than 0 if x is sufficiently large with coeffi of the inequality, therefore by definition of θ, we have,

$$\theta(x) - \theta\left(\frac{x}{2}\right) = \sum_{\frac{x}{2} \leq p \leq x} \log(p) > 0.$$

$\square$

Theorem 2.23 (Chebyschev)**.**

1). $\theta(n) \leq 2n\log 2$,

2). $\psi(n) \geq 2n\log 2$,

3). there exists $A, B > 0$ such that for sufficiently large n, we have

$$\frac{Bx}{\log x} \leq \pi(x) \leq \frac{Ax}{\log x}.$$

Proof. $\square$

2.5 Ikehara-Wiener Theorem and Its Applications

Theorem 2.24 (Ikehara-Wiener)**.** *Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of non-negative numbers and set,*

$$f(s) = \sum_{n\in\mathbb{N}} \frac{b_n}{n^s}.$$

Suppose

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i). the series converges abslutely for $\mathrm{Re}(s) > 1$,

ii). f has an analytic continuation to $\mathrm{Re}(s) = 1$ except $s = 1$,

iii). f has a simple pole at $s = 1$ with residue $R \geq 0$.

Then we have,

$$\sum_{n \leq x} b_n = Rx + O(x).$$

That is

$$\lim_{x \to \infty} \frac{\sum_{n \leq x} b_n}{x} = R.$$

Lemma 2.9. *Let χ be a Dirichlet character modulo q, then*

$$L(s, \chi)^{-1} = \sum_{n \in \mathbb{N}} \frac{\chi(n)\mu(n)}{n^s}.$$

Proof. We immitate the proof of Theorem 2.5. Using Theorem 2.1,

$$L(s, \chi) \left(\sum_{n \in \mathbb{N}} \frac{\mu(n)\Lambda(n)}{n^s} \right) = \sum_{t \in \mathbb{N}} \sum_{n \mid t} \frac{\chi\left(\frac{t}{n}\right)\chi(n)\mu(n)}{t^s} = \sum_{t \in \mathbb{N}} \chi(t) \sum_{n \mid t} \mu(n) = 1.$$

$\square$

Theorem 2.25. *Let χ be a character then, for $\mathrm{Re}(s) > 1$,*

$$-\frac{L'(s, \chi)}{L(s, \chi)} = \sum_{n \in \mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s}.$$

Proof. By Lemma 2.9 and Proposition 2.1

$$-\frac{L'(s, \chi)}{L(s, \chi)} = \left(\sum_{t \in \mathbb{N}} \frac{\chi(n)\log(t)}{t^s} \right) \left(\sum_{n \in \mathbb{N}} \frac{\mu(n)\chi(n)}{n^s} \right),$$

$$= \sum_{t \in \mathbb{N}} \chi(t) \sum_{n \mid t} \frac{\mu\left(\frac{t}{n}\right)\log(n)}{t^s},$$

$$= \sum_{n \in \mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s}.$$

$\square$

Definition 2.15.

$$\psi(x, q, a) = \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n).$$

Proposition 2.8. *As $x \to \infty$, we have,*

$$\psi(x, q, a) \sim \frac{x}{\varphi(q)}.$$

Proof. Consider

$$\sum_{\chi \bmod q} \overline{\chi}(a) \left(\sum_{n \in \mathbb{N}} \frac{\chi(n)\Lambda(n)}{n^s} \right) = \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} \left(\sum_{\chi \bmod q} \chi(a^{-1}n) \right)$$

Using Lemma 2.8, we get,

$$-\sum_{\chi \bmod q} \overline{\chi}(a) \frac{L'(s, \chi)}{L(s, \chi)} = \varphi(q) \sum_{n \equiv a \bmod q} \frac{\Lambda(n)}{n^s}.$$

Recall from Corollary 2.2, we have,

$$\mathrm{Res}(L(s, \chi_0), 1) = \frac{\varphi(q)}{q}.$$

And by Theorem 2.14, $L(s, \chi)$ is analytic at $s = 1$ for $\chi \neq \chi_0$. Also using Theorem 2.16, we have

$$\chi \neq \chi_0 \Rightarrow \frac{L'(s, \chi)}{L(s, \chi)} \text{ is analytic for } \mathrm{Re}(s) \geq 1.$$

Furthermore, we have Theorem 2.17, we have

$$\mathrm{Res}\left(\frac{\zeta'(s)}{\zeta(s)}, 1 \right) = -1.$$

Combining these, we get,

$$\lim_{s \to 1}(s - 1) \left(-\frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \frac{L'(s, \chi)}{L(s, \chi)} - \frac{1}{\varphi(q)} \frac{L'(s, \chi_0)}{L(s, \chi_0)} \right) = \frac{1}{\varphi(q)}.$$

Now Using Theorem 2.24, and apply $b_n = \Lambda(n)$ $f = \varphi(q) \sum_{n \equiv a \bmod q} \frac{\Lambda(n)}{n^s}$, we get,

$$\psi(x, q, a) = \sum_{n \leq x} \Lambda(n) = \frac{1}{\varphi(q)} x + O(x).$$

$\square$

2.6 $L(s, \chi) \neq 0$ for Quadratic Characters

Lemma 2.10. *Let $f := \sum_{d|n} \chi(d)$, where χ is a character, then,*

$$\forall n \in \mathbb{N}, n \text{ is a perfect square} \Rightarrow f(n) \geq 0, f(n) \geq 1.$$

Proof. Recall $n = \prod_{p|n} p^{\alpha(p)}$. Using this we have,

$$\sum_{d|n} \chi(d) = \prod_{p|n} \left(\sum_{k=0}^{\alpha(p)} \chi(p)^k \right),$$

$$= \begin{cases} 1 & \chi(p) = 0, \\ \prod_{p|n}(1 + \alpha(p)) & \chi(p) = 1, \\ \prod_{p|n} \left(\frac{(1-(-1)^{\alpha(p)+1})}{2} \right) & \chi(p) = -1. \end{cases}$$

Note that if $\alpha(p)$ are all even for $p|n$ (ie. n is a perfect square), we have the last part of the cases equals to 1. Thus we have $f(n) \geq 1$. $\qquad\square$

Theorem 2.26. *Let $f(n) = \sum_{d|n} \chi(d)$ for some character. Then we have,*

$$\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = 2\sqrt{x}L(1, \chi) + o(1).$$

Proof.

$$\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = \sum_{n \leq x} \left(\frac{\sum_{d|n} \chi(d)}{\sqrt{n}} \right),$$

$$= \sum_{\substack{d,e \\ de \leq x}} \frac{\chi(d)}{\sqrt{de}},$$

$$= \sum_{\substack{d,e \leq x \\ d \leq \sqrt{x}}} \frac{\chi(d)}{\sqrt{de}} + \sum_{\substack{de \leq x \\ d > \sqrt{x}}} \frac{\chi(d)}{\sqrt{de}},$$

$$= \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(\sum_{e \leq \frac{x}{d}} \frac{1}{\sqrt{e}} \right) + \sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left(\sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{(d)}} \right).$$

Recall that from Proposition 2.7,

$$\sum_{m \leq x} \frac{1}{\sqrt{m}} = 2\sqrt{x} + B + o\left(\frac{1}{\sqrt{x}} \right),$$

where B is some constant as $x \to \infty$.

Let $x, y \in \mathbb{R}$, such that $x < y$, we have,

$$\sum_{x < d \leq y} \frac{\chi(d)}{\sqrt{d}} = \sum_{d \leq y} \frac{\chi(d)}{\sqrt{d}} - \sum_{d \leq x} \frac{\chi(d)}{\sqrt{(d)}},$$

$$\sum_{d \leq x} \frac{\chi(d)}{\sqrt{d}} = \frac{\sum_{d \leq x} \chi(d)}{\sqrt{x}} + \frac{1}{2} \int_1^x \frac{\sum_{d \leq t} \chi(d)}{t^{\frac{1}{2}}},$$

$$= O\left(\frac{1}{\sqrt{x}}\right).$$

Using these equations, we have,

$$\sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left(\sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{(d)}} \right) = \sum_{e \leq x} \frac{1}{\sqrt{e}} \left(O\left(\frac{1}{x^{\frac{1}{r}}}\right) \right),$$

$$= \left(O\left(\frac{1}{x^{\frac{1}{r}}}\right) \right) \sum_{e \leq x} \frac{1}{\sqrt{e}},$$

$$\sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{x}} << \int_1^{\sqrt{x}} \frac{1}{\sqrt{t}} dt = x^{\frac{1}{4}}.$$

Thus we conclude,

$$\sum_{e \leq \sqrt{x}} \frac{1}{\sqrt{e}} \left(\sum_{\sqrt{x} < d \leq \frac{x}{e}} \frac{\chi(d)}{\sqrt{(d)}} \right) = O(1).$$

We also have,

$$\sum_{d \le \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(\sum_{e \le \frac{x}{d}} \frac{1}{\sqrt{e}} \right) = \sum_{d \le \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(2\sqrt{\frac{x}{d}} + B + o\left(\sqrt{\frac{d}{x}} \right) \right),$$

$$= 2\sqrt{x} \sum_{d \le \sqrt{x}} \frac{\chi(d)}{d} + B \sum_{d \le \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} + o\left(\frac{1}{\sqrt{x}} \sum_{d \le \sqrt{x}} \chi(d) \right).$$

$$2\sqrt{x} \sum_{d \le \sqrt{x}} \frac{\chi(d)}{d} = 2\sqrt{x} \left(\sum_{d \ge 1} \frac{\chi(d)}{d} - \sum_{d > \sqrt{x}} \frac{\chi(d)}{d} \right),$$

$$= 2\sqrt{x} L(1, \chi) - 2\sqrt{x} \sum_{d > \sqrt{x}} \frac{\chi(d)}{d},$$

$$= 2\sqrt{x} L(1, \chi) - 2\sqrt{x} o\left(\frac{1}{\sqrt{x}} \right),$$

$$= 2\sqrt{x} L(1, \chi) + o(1).B \sum_{d \le \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \le o\left(\frac{B}{x^{\frac{1}{4}}} \right) = o(1).$$

$$\sum_{d \le \sqrt{x}} \chi(d) \le q? = o(1).$$

$\square$

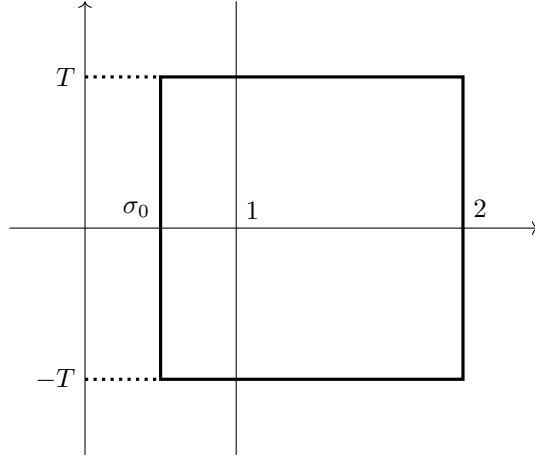**Corollary 2.4.** $L(1, \chi) \ne 0$ *for a quadratic character* χ.

Proof. To derive a contradiction, assume $L(1, \chi) = 0$. Then, from Theorem 2.26 and Lemma 2.10

$$o(1) = \sum_{n \le x} \frac{f(n)}{\sqrt{n}} \ge \sum_{\substack{n \le x \\ n \text{ is a square}}} \frac{1}{\sqrt{n}} = \sum_{m \le \sqrt{x}} \frac{1}{m} = \log \sqrt{x} + o(1).$$

Thus we obtain $o(1) \ge \log \sqrt{x} + o(1)$ which is a cotradiction. $\square$

Lemma 2.11. *Consider the following rectangle,*

where $\sigma_0 = 1 - \log T$. On the boundary of the rectangle above, we have,

1. $|\zeta(s)| = O(\log T), T \to \infty$,

2. $|\zeta'(s)| = O((\log T)^2)$.

Proof. Recall that

$$\zeta'(s) = \sum_{n=1}^{\infty} \frac{-\log n}{n^s}.$$

The first statement is due to the first derivative of ζ. and it is assgined as an exercise. For the second part, for $\mathrm{Re}(s), \sigma > 1$, we have,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s},$$

$$= \sum_{n \leq T} \frac{1}{n^s} + \sum_{n > T} \frac{1}{n^s},$$

Using Abel's summation formula by setting $a_n = 1, f(n) = \frac{1}{n^s}$, we have,

$$\zeta(s) = \sum_{n=1}^{\infty} = -s \int_1^{\infty} \frac{[t]}{t^{s+1}} dt,$$

$$\sum_{n \leq T} \frac{1}{n^s} = \frac{[T]}{T^s} - s \int_1^T \frac{[t]}{t^{s+1}} dt.$$

Using this, we get,

$$\zeta(s) = \sum_{n \leq T} \frac{1}{n^s} - \frac{\lfloor T \rfloor}{T^s} + s \int_1^{\infty} \frac{\lfloor t \rfloor}{t^{s+1}} dt,$$

$$= \sum_{n \leq T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} + \frac{\{T\}}{T^s} - s \int_T^{\infty} \frac{\{u\}}{u^{s+1}} du.$$

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Note that the right hand side is analytic where $\text{Re}(s) > 0$ and $s \neq 1$. Now we will estimate the last equation above on the boundary of the rectangle above.

$$\left| \sum_{n \leq T} \frac{1}{n^s} \right| << \int_1^T \frac{du}{u^{\text{Re}(s)}},$$

$$\leq \int_1^T \frac{du}{u^{\sigma_0}},$$

$$= \frac{T^{1-\sigma_0}}{1 - \sigma_0},$$

$$<< \log T.$$

Observe that ,

$$1 - \sigma_0 = \frac{1}{\log T}, T^{1-\sigma_0} = T^{\frac{1}{\log T}} = \exp(\log T^{\frac{1}{\log T}}) = \exp(1) = e.$$

We also have,

$$\left| \frac{T^{1-s}}{s-1} \right| = \frac{T^{1-\text{Re}(s)}}{|s-1|},$$

$$\leq \frac{T^{1-\sigma_0}}{|s-1|},$$

$$\leq \frac{e}{|s-1|},$$

$$\leq \frac{e}{\sigma_0 - 1} = e \log T.$$

Also consider,

$$\left| \frac{\{T\}}{T} \right| \leq \frac{1}{T^{\text{Re}(s)}} \leq 1.$$

Finally we have,

$$\left| -s \int_T^\infty \frac{\{u\}}{u^{s+1}} du \right| \leq |s| \int_T^\infty \frac{du}{u^{\text{Re}(s)+1}},$$

$$= -\frac{|s|}{\text{Re}(s)u^{\text{Re}(s)}} \Big|_T^\infty,$$

$$= \frac{|s|}{\text{Re}(s)T^{\text{Re}(s)}},$$

$$\leq \frac{|s|}{T^{\text{Re}(s)}},$$

$$\leq \frac{\sqrt{2^2 + T^2}}{T^{\sigma_0}},$$

$$<< \frac{T}{T^{\sigma_0}} = T^{1-\sigma_0} = e.$$

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For the second part, for $\text{Re}(s) > 0, s \neq 1$,

$$\zeta'(s) = \sum_{n \leq T} \frac{-\log n}{n^s} + \frac{T^{1-s}(-\log T)}{s-1} + T^{1-s}\frac{-1}{(s-1)^2}$$
$$+ \frac{\{T\}\log T}{T^s} - \int_T^\infty \frac{\{u\}}{u^{s+1}} du + s \int_T^\infty \frac{\{u\}\log u}{u^{s+1}} du.$$

This is obtained simply differentiating the equation,

$$\zeta(s) = \sum_{n \leq T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} - \frac{\{T\}}{T^s} - s \int_T^\infty \frac{\{u\}}{u^{s+1}} du.$$

The statement can be shown using all the estimates obtained to show the first part. $\qquad\square$

Theorem 2.27 (Complex Mean Value Theorem)**.** *Let $f : \Omega \to \mathbb{C}$ be an analytic, where Ω is a convex open set. Let $a, b \in \Omega$, then there exists $z_1, z_2 \in (a, b)$ such that*

$$\text{Re}(f'(z_1)) = \text{Re}\left(\frac{f(b) - f(a)}{b-a}\right), \text{Im}(f'(z_2)) = \text{Im}\left(\frac{f(b) - f(a)}{b-a}\right).$$

Theorem 2.28. *There exist some constants c_1, c_2 such that*

$$1 - \frac{c_1}{(\log T)^9} \leq \sigma \leq 2 \Rightarrow |\zeta(s)| > \frac{c_2}{(\log T)^7},$$

where $1 < |\text{Im}(s)| \leq T$.

Proof. We set $z = \sigma + it$ for $\sigma, t \in \mathbb{R}$. Recall Lemma 2.6,

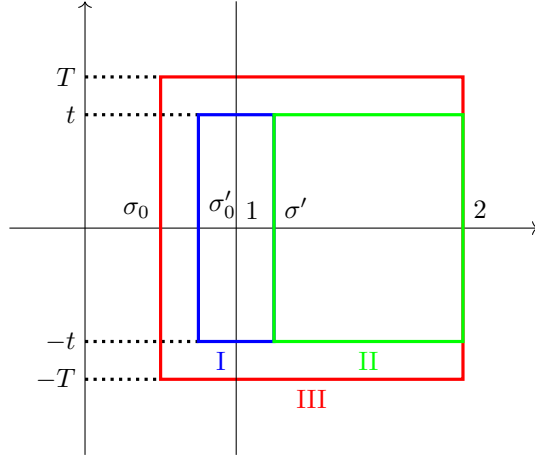$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1, \sigma > 1.$$

Thus we have,

$$|\zeta(\sigma + it)|^4 \geq |\zeta(\sigma)|^{-3}|\zeta(\sigma + 2it)|^{-1}, \qquad\qquad (*)$$

Consider

$$\sigma_0' = 1 - \frac{c_1}{(\log T)^9}, \sigma' = 1 + \frac{c_1}{(\log T)^9}$$

c_1 is a positive constant which will be adjusted later.

Fix the domain II where $\sigma' \le \sigma \le 2$, $1 \le |t| \le T$.

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{s+1}} du, \operatorname{Re}(s) > 0, s \ne 1.$$

$$\zeta(\sigma) = \frac{\sigma - 1 + 1}{\sigma - 1} - \sigma \int_1^\infty \frac{\{u\}}{u^{\sigma+1}} du.$$

$$<< \frac{1}{\sigma - 1} \text{ as } \sigma \to 1^+,$$

$$\Rightarrow \zeta(\sigma)^{-1} >> (\sigma - 1), \sigma \to 1^+$$

On the domain II, we have, $\sigma_0' < \sigma \le 2, 1 \le |t| \le T$. Using Lemma 2.11,

$$|\zeta(\sigma + 2it)| = O(\log T).$$

Substituting these to Equation $(*)$, we have,

$$|\zeta(\sigma + it)|^4 >> (1 - \sigma)^3 (\log T)^{-1},$$

$$>> \frac{c_1^3}{(\log T)^{27}} (\log T)^{-1}.$$

$$\Rightarrow |\zeta(\sigma + it)| >> \frac{c_1^{\frac{3}{4}}}{(\log T)^7}, \tag{D1}$$

in the domain II.

On the domain I, similarly use Theorem 2.27 with

$$a = \sigma' + it, b = \sigma + it,$$

where,

$$1 - \frac{c_1}{(\log T)^9} \le \sigma \le \sigma'.$$

We have,

$$\frac{\zeta(\sigma' + it) - \zeta(\sigma + it)}{\sigma' - \sigma} = \operatorname{Re}(\zeta(z_1)) + \operatorname{Im}(\zeta(z_2)).$$

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Furthermore,
$$\zeta(\sigma' + it) - \zeta(\sigma + it) = O((\sigma' - \sigma)(\log T)^2).$$

By the definition of σ', we have,
$$\zeta(\sigma' + it) = \zeta(\sigma + it) + O\left(\frac{c_1}{(\log T)^7}\right). \qquad (D2)$$

Combining Equations $(D1)$ and $(D2)$, there are some positive constants A_1, A_2 such that
$$|\zeta(\sigma + it)| \geq |\zeta(\sigma' + it)| - \frac{A_1 c_1}{(\log T)^7},$$

$$\geq \frac{A_2 c_1^{\frac{3}{4}}}{(\log T)^7} - \frac{A_1 c_1}{(\log T)^7}.$$

Now take c_1 sufficiently small so that in the region,
$$\sigma_0' \leq \sigma \leq \sigma', 1 \leq |t| \leq T,$$

we have,
$$|\zeta(\sigma + it)| \geq \frac{c_2}{(\log T)^7}.$$

Again combining this with Inequality $(D2)$, we obtain the statement. $\qquad \square$

Corollary 2.5. *There exists some constant c such that*
$$\frac{\zeta'(s)}{\zeta(s)} = O((\log T)^9),$$

for $1 - \frac{c}{(\log T)^9} \leq \mathrm{Re}(s) \leq 2$ and $1 \leq |\mathrm{Im}(s)| \leq T$.

Notation 2.3. *Let $c \in \mathbb{R}$ and $f : \mathbb{C} \supseteq \Omega \to \mathbb{C}$. Then we denote,*
$$\int_{(c)} f(s)ds = \lim_{T \to \infty} \int_{c-iT}^{c+iT} f(s)ds.$$

Lemma 2.12.
$$\frac{1}{2\pi i} \int_{(c)} y^s \frac{ds}{s} = \begin{cases} 0, & (0 < y < 1), \\ \frac{1}{2}, & (y = 1), \\ 1, & (y > 1). \end{cases}$$

Lemma 2.13. *Set,*
$$f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Suppose this is absolutely convergence in $\mathrm{Re}(s) > c - \varepsilon$ for some $c \in \mathbb{R}, \varepsilon > 0$ and x is not an integer, then
$$\sum_{n<x} a_n = \frac{1}{2\pi i} \int_{(c)} f(s) \frac{x^s}{s} ds.$$

Proof. Since $f(s)$ is a limit of a continuous function series which is absolutely convergent on $\mathrm{Re}(s) > c - \varepsilon$, thus this is uniformly convergent in a compact subset of its domain of convergence. Thus we can exchange the integral and the sum namely,

$$\frac{1}{2\pi i}\int_{(c)}\left(\sum_{n=1}^{\infty}\frac{a_n}{n^s}\right)\frac{x^s}{s}ds = \frac{1}{2\pi i}\sum_{n=1}^{\infty}a_n\int_{(c)}\left(\frac{x}{n}\right)^s\frac{ds}{s}.$$

By Lemma 2.12, and $\frac{x}{n} > 1 \Leftrightarrow n < x$, we derive the statement. $\qquad\square$

Corollary 2.6. *Let $c > 1$ and x be a non-integer, then we have,*

$$\psi(x) = \frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds.$$

Proof. From Remark 2.1,
$$\psi(x) = \sum_{d\leq x}\Lambda(d),$$

and by Theorem 2.6 set

$$f(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s},$$

Thus applying Lemma 2.13, we get the statement. $\qquad\square$

Theorem 2.29. *There is a positive constant $c' > 0$ such that*

$$\psi(x) = x + O(x\exp(-c'\log(x))).$$

Proof. Let $a = 1 + \frac{c}{\log T}$ for some c where $T > 1$ is sufficiently large and will be chosen later. By Corollary 2.6, we have,

$$\psi(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds + \frac{1}{2\pi i}\int_{(a)} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds - \frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds.$$

Claim 1.

$$\left|\frac{1}{2\pi i}\int_{(a)} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds - \frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds\right| < \begin{cases} x^a\min\{1, T^{-1}, |\log x|^{-1}\}, & (x \neq 1), \\ \frac{a}{T}, & (x = 1). \end{cases}$$

<u>Proof:</u> Exercise. $\qquad\blacksquare$

Let

$$\psi(x) = \underbrace{\frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds}_{=I_1} + \underbrace{\frac{1}{2\pi i}\int_{(a)} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds - \frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds}_{=I_2}.$$

Again using Corollary 2.6 and the claim, we have,

$$I_2 = \frac{1}{2\pi i} \left(\int_{(a)} - \int_{a-iT}^{a+iT} \right) \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \right) \frac{x^s}{s} ds,$$

$$= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n) \left(\left(\int_{(a)} - \int_{a-iT}^{a+iT} \right) \left(\frac{x}{n} \right)^s \frac{ds}{s} \right),$$

$$= \sum_{n=1}^{\infty} \Lambda(n) O \left(\left(\frac{x}{n} \right)^a \min \left(1, T^{-1} \left| \log \left(\frac{x}{n} \right) \right|^{-1} \right) \right),$$

$$= O \left(\sum_{n=1}^{\infty} \Lambda(n) \left(\frac{x}{n} \right)^a \min \left(1, T^{-1} \left| \log \left(\frac{x}{n} \right) \right|^{-1} \right) \right).$$

Fix x and let us consider two ranges $\left[\frac{x}{2}, \frac{3x}{2} \right]$ and $\left(-\infty, \frac{x}{2} \right) \cup \left(\frac{3x}{2}, \infty \right)$. If n sits in the second range

$$\frac{\log \frac{n}{x}}{\frac{n}{x} - 1} \geq 2 \log \frac{3}{2}.$$

And if n sits in the first interval, we have,

$$\left| \log \frac{x}{n} \right| << \left| \frac{n}{x} - 1 \right|^{-1}.$$

Using this we have,

$$\sum_{n \in \left[\frac{x}{2}, \frac{3x}{2} \right]} \Lambda(n) \left(\frac{x}{n} \right)^a \min \left(1, T^{-1} \left| \log \left(\frac{x}{n} \right) \right|^{-1} \right) << \sum_{n \in \left[\frac{x}{2}, \frac{3x}{2} \right]} \Lambda(n) \left(\frac{x}{n} \right)^a T^{-1} \left| \frac{n}{x} - 1 \right|^{-1}$$

$$<< \frac{\log x}{T} \sum_{n \in \left[\frac{x}{2}, \frac{3x}{2} \right]} \frac{1}{\left| \frac{n}{x} - 1 \right|}.$$

Observe that

$$\sum_{n \in \left[\frac{x}{2}, x \right)} \frac{1}{\frac{n}{x} + 1} << \int_{\frac{x}{2}}^{\lfloor x \rfloor} \frac{dt}{1 - \frac{t}{x}},$$

$$\sum_{n \in \left(x, \frac{3x}{2} \right]} \frac{1}{\frac{n}{x} - 1} << \int_{\lceil x \rceil}^{\frac{3x}{2}} \frac{dt}{\frac{t}{x} - 1}.$$

$$\Rightarrow \sum_{n \in \left[\frac{x}{2}, x \right)} \frac{1}{\frac{n}{x} + 1} + \sum_{n \in \left(x, \frac{3x}{2} \right]} \frac{1}{\frac{n}{x} + 1} << x \log x.$$

Thus we conclude, as $x \to \infty$,

$$\sum_{n \in \left[\frac{x}{2}, \frac{3x}{2} \right]} \frac{1}{\left| \frac{n}{x} - 1 \right|} << \frac{x (\log x)^2}{T}.$$

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Let us turn our focus to the second range. We have $\left|\log\frac{x}{n}\right| > \log\frac{3}{2}$. Thus

$$\sum_{n<\frac{x}{2},\frac{3x}{2}<n} \Lambda(n)\left(\frac{x}{n}\right)^a \min\left(1, T^{-1}\left|\log\left(\frac{x}{n}\right)\right|^{-1}\right) << \frac{x^a}{T}\sum_{n<\frac{x}{2},\frac{3x}{2}<n}\frac{\Lambda(n)}{n^a},$$

$$\leq \frac{x^a}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^a}.$$

Recall that we have,

$$-\frac{\zeta'(a)}{\zeta(a)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^a}.$$

Using Theorem 2.17, we see, as $a \to 1^+$,

$$\left|\frac{\zeta'(a)}{\zeta(a)}\right| << \frac{1}{a-1}.$$

Using Corollary 2.5, we see,
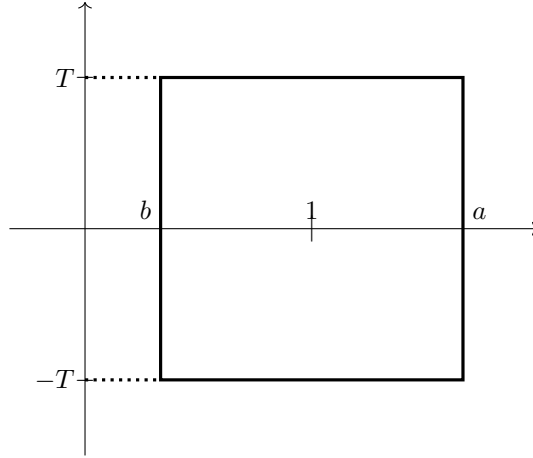
$$\frac{x^a}{T}\frac{1}{a-1} << \frac{x^a}{T}(\log T)^9.$$

Combining all, we derive,

$$\psi(s) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds + O\left(\frac{x(\log x)^2}{T} + \frac{x^a}{T}(\log T)^9\right).$$

Recall Cauchy's Residue theorem, suppose $f : \Omega \to \mathbb{C}$ is meromorphic where Ω is simply connected with poles $a_i, 1 \leq i \leq n$ poles in $U \subseteq \Omega$ where the boundary of U is a simple closed curve γ. Then,

$$\frac{1}{2\pi i}\int_{\gamma} f(s)ds = \sum_{i=1}^{n}\text{Res}(f, a_i) + \text{constant}.$$

Consider the following closed path R_T which we traverse counterclockwise.

where $a = 1 + \frac{c}{(\log(T))^9}, b = 1 - \frac{c}{(\log(T))^9}$ $\text{Re}(s) > 0, s \neq 1$. We claim that $\frac{\zeta'(s)}{\zeta(s)}$ can be analytically continued to $\text{Re}(s) \geq b$ except simple pole at $s = 1$ with residue -1. By Theorem 2.17 and Theorem 2.28, we obtain

$$\frac{1}{2\pi i} \int_{R_T} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = \text{Res}_{s=1} \left(\frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \right),$$
$$= x.$$

Now consider R_T which is the closed path.

$$\int_{R_T} = \int_{a-iT}^{a+iT} + \int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT}.$$

We then have,

$$\frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = x - \frac{1}{2\pi i} \left(\int_{a+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{a-iT} \right) \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds$$

$$\left| \frac{1}{2\pi i} \int_{a+iT}^{b+iT} \frac{-\zeta(s)}{\zeta(s)} \frac{x^s}{s} ds \right| = \left| \frac{1}{2\pi i} \int_a^b \frac{-\zeta(u+iT)}{\zeta(u+iT)} \frac{x^{(u+iT)}}{(u+iT)} du \right|,$$

$$<< \int_a^b \left| \frac{-\zeta'(u+iT)}{\zeta(u+iT)} \right| \frac{x^u}{|u+iT|} du, \quad (|u+iT|^2 \geq 1)$$

$$\overset{\text{Theorem 2.28}}{<<} \frac{\log^9 T}{T} x^a \int_b^a du,$$

$$<< \frac{x^a}{T}.$$

Now we have,

$$\left| \int_{b-iT}^{b+iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \right| = \left| \frac{1}{2\pi} \int_{-T}^T \frac{\zeta'(b+iu)}{\zeta(b+iu)} \frac{x^{b+iu}}{b+iu} du \right|,$$

$$<< \log^9 T \int_{-T}^T \frac{x^b}{\sqrt{b^2+u^2}} du,$$

$$<< x^b \log^9 T \int_0^T \frac{du}{\sqrt{b^2+u^2}} du,$$

$$<< x^b \log^9 T \int_1^T \frac{du}{u},$$

$$= x^b \log^{10} T.$$

Now back to the beginning,

$$\psi(x) = x + O\left(\frac{x^a}{T} + x^b \log^{10} T \right) + O\left(\frac{x \log^2 x}{T} + \frac{x^a}{T} \log^9 T \right).$$

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Choose T to be such that $2c\log x = \log^{10} T, x = e^{\frac{\log^{10} T}{2c}}$.

$$x^{\frac{c}{\log^9 T}} = e^{\frac{\log T}{2}} = \sqrt{T}.$$

$$x^{1-\frac{c}{\log^9 T}} \cdot \log^{10} T + \frac{x\log^2 x}{T} + \frac{x^{1+\frac{c}{\log^9 T}}}{T}\log^9 T$$

$$= x \cdot T^{-\frac{1}{2}}\log^{10} T + \frac{x\log^2 x}{T}$$

$$+ x\frac{\sqrt{T}}{T}\log^{10} T \left(\frac{x}{\sqrt{T}}\right)\left(\log^{10} T + \frac{\log^2 x}{\sqrt{x}} + \log^9 T\right),$$

$$\ll \frac{x}{T^s} = xe^{-c(\log x)^{\frac{1}{10}}}.$$

$\square$

Definition 2.16 (Gamma function)**.** *We define the Gamma function* $\Gamma : \mathbb{C} \to \mathbb{C}$ *as*

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt, \sigma > 0.$$

Remark 2.17. *Suppose* $\mathrm{Re}(s) = \sigma$ *then,*

$$|\Gamma(s)| \leq \int_0^\infty |t^{s-1}|e^{-t}dt,$$

$$= \int_0^\infty t^{\sigma-1}e^{-t}dt,$$

$$= \left(\int_0^1 + \int_1^\infty\right)t^{\sigma-1}e^{-t}dt,$$

$$\leq \int_0^1 t^{\sigma-1}dt + \int_1^\infty t^{\sigma-1}e^{-t}dt, \left(\frac{t^n}{e^t} \overset{t\to\infty}{\to} 0\right).$$

$$\ll \frac{1}{\sigma} + \int_1^\infty t^{\sigma-1}t^{-2\sigma}dt,$$

$$\ll \frac{1}{\sigma}.$$

Theorem 2.30.

$$F(s) = \int f(s,t)dt.$$

$F : \Omega \to \mathbb{C}$ *is analytic in* Ω *if*

i). $f(s,t)$ *is continuous in* (s,t),

ii). $f(s,t)$ *is analytic in* s,

iii). $\int f(s,t)$ *is uniformly bounded on compact subsets of* Ω.

In Remark 2.17, suppose $a \leq \operatorname{Re}(s) \leq b$ then the last two inequalities will be,

$$\int_0^1 t^{\sigma-1} dt + \int_1^\infty t^{\sigma-1} e^{-t} dt$$

$$<<_b \frac{1}{\sigma} + \int_1^\infty t^{b-1} t^{-2b} dt,$$

$$<<_b \frac{1}{\sigma} + \frac{1}{b} = \frac{1}{a} + \frac{1}{b}.$$

Thus we observe that in $\sigma > 0$, using integration by parts,

$$\Gamma(s+1) = s\Gamma(s).$$

Thus for any $n \in \mathbb{N}$, we have $\Gamma(n) = n!$. We have,

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \sigma > 0.$$

Note $\Gamma(s+1)$ is analytic if $\sigma > -1$. Thus $\Gamma(s)$ is analytic in $\sigma > -1$ except simple pole at $s = 0$ with residue 1. That is

$$\lim_{s \to 0} s\Gamma(s) = 1.$$

We also have,

$$\Gamma(s) = \frac{\Gamma(s+2)}{\Gamma(s+1)}, \sigma > 0.$$

Note that in the numerator, we have $\sigma > -2$. $\Gamma(s)$ is analytic when $\sigma > -2$ except simple poles at $s = 0, -1$ with residue

$$\lim_{s \to -1} (s+1)\Gamma(s) = \frac{\Gamma(-1)}{-1} = -1.$$

Iterating the process, we have the following theorem,

Theorem 2.31. $\Gamma(s)$ *can be analytically continued to* $\mathbb{C}$ *except simple poles at* $s = -k$ *where* $k \in \mathbb{Z}_{\geq 0}$ *with residue* $\frac{(-1)^k}{k!}$.

Remark 2.18.

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, s \in \mathbb{C}.$$

$$\lim_{s \to m} \Gamma(s)\Gamma(1-s)\sin \pi s = \pi.$$

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = \sqrt{\pi} 2^{1-2s} \Gamma(2s), \forall s \in \mathbb{C}.$$

Exercise 2.5. *Show*

$$\forall \alpha \in \mathbb{R}, x > 0, \sum_{n \in \mathbb{Z}} e^{-(n+\alpha)\frac{\pi}{x}} = \sqrt{x} \sum_{n \in \mathbb{Z}} e^{-n^2 \pi x + 2\pi i n \alpha}.$$

Note that $\sum_{n \in \mathbb{Z}} a_n$ *is convergent if*

$$s_N := \sum_{|n| \leq N} a_n,$$

is convergent.

Theorem 2.32. *Let* $\sigma > 1$ *and* $F \in L^1(\mathbb{R})$, *ie* $F : \mathbb{R} \to \mathbb{C}$ *and*

$$\int_{-\infty}^{\infty} |F(t)| dt < \infty.$$

We have,

$$\sum_{n \in \mathbb{Z}} F(n + u)$$

is absolutely and uniformly convergent in u. Also we have,

$$\sum_{n \in \mathbb{Z}} |\hat{F}(n)| \leq \infty,$$

then

$$\sum_{n \in \mathbb{Z}} F(n + u) = \sum_{n \in \mathbb{Z}} \hat{F}(n) e^{2\pi i n u}.$$

Proof. Use Exercise 2.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 2.33. $\Gamma(s) \neq 0, \forall s \in \mathbb{C}.$

Proof. Note that the case when $s \in \mathbb{Z}$ is already shown. Thus suppose $s \notin \mathbb{Z}$. Suppose $\Gamma(s) = 0$, then it will follow that $\Gamma(1 - s)$ by Remark 2.18. But this is a contradiction to Theorem 2.31 as it has a pole at non-integer point. Similarly from Remark 2.18, we see that $\frac{1}{\Gamma(s)}$ has a simple zero at $s = 0, -1, -2, \cdots$.

By definition, consider,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} e^{-t} t^{\frac{s}{2}-1} dt, \sigma > 0.$$

Replace $t = n^2\pi x, dt = n^2\pi dx$, we get,

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)n^{-s} = \int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx, \sigma > 1.$$

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n\in\mathbb{N}}\int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx,$$

$$= \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n\in\mathbb{N}}e^{-n^2\pi x}\right)dx,$$

$$= \sum_{n\in\mathbb{N}}\int_0^\infty |x^{\frac{s}{2}-1}e^{-n^2\pi x}|dx,$$

$$= \sum_{n\in\mathbb{N}}\left(\int_0^\infty x^{\frac{\sigma}{2}-1}e^{-n^2\pi x}dx\right),$$

$$= \sum_{n=1}^\infty \pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right)n^{-\sigma},$$

$$= \pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right)\zeta(s), \sigma > 1,$$

$$< \infty.$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n\in\mathbb{N}}e^{-n^2\pi x}\right).$$

Using Exercise 2.5 and set $\alpha = 0$, we have,

$$\sum_{n\in\mathbb{Z}}e^{-n^2\frac{\pi}{s}} = \sqrt{s}\sum_{n\in\mathbb{Z}}e^{-n^2\pi x}.$$

$$\sum_{n\in\mathbb{Z}}e^{-n^2\pi x} << \sum_{n\in\mathbb{N}}e^{-n\pi x},$$

$$<< \int_1^\infty e^{-\pi xt}dt,$$

$$<< \frac{e^{-\pi x}}{x}, x > 0.$$

Set $\theta(x) := \sum_{n\in\mathbb{Z}}e^{n^2\pi x}$ then,

$$\theta(x) = 1 + 2\sum_{n\in\mathbb{N}}e^{-n^2\pi x}.$$

$$\sum_{n\in\mathbb{N}}e^{-n^2\pi x} = \frac{\theta(x)-1}{2}, x > 0.$$

Using the exercise, we have,

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\theta(x).$$

Set $w(x) := \frac{\theta(x)-1}{2}$. Thus write

$$w\left(\frac{1}{x}\right) = \frac{\theta\left(\frac{1}{x}\right)-1}{2},$$

$$= \frac{\sqrt{x}\theta(x)-1}{2},$$

$$= \frac{\sqrt{x}\theta(x)-1}{2}.$$

$$w\left(\frac{1}{x}\right) = \sqrt{x}w(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}.$$

$$\pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}w(x)dx, \sigma > 1.$$

Step 3

$$\int_0^1 x^{\frac{s}{2}-1}w(x)dx + \int_1^\infty x^{\frac{s}{2}-1}w(x)dx.$$

Taking $x = \frac{1}{y}$ and $dx = -\frac{1}{y^2}dy$ we have,

$$\int_1^\infty \left(\frac{1}{y^2}\right)^{\frac{s}{2}-1} w\left(\frac{1}{y}\right)\frac{-1}{y^2}dy = \int_1^\infty y^{-\frac{s}{2}}w\left(\frac{1}{y}\right)\frac{dy}{y},$$

$$= \int_1^\infty y^{-\frac{1}{2}}\left(\sqrt{y}w(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right)\frac{dy}{y},$$

$$= \int_1^\infty y^{-\frac{s}{2}+\frac{1}{2}}w(y)\frac{dy}{y} + \int_1^\infty \frac{y^{-\frac{s}{2}+\frac{1}{2}-1}}{2}dy - \frac{1}{2}\int_1^\infty y^{-\frac{s}{2}-1}dy,$$

$$= \int_1^\infty y^{\frac{1-s}{2}}w(y)\frac{dy}{y} + \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}, \sigma > 1.$$

Step 4,

$$\int_1^\infty |x^{\frac{s}{2}} + x^{\frac{1-s}{2}}||w(x)|\frac{dx}{x} \le \int_1^\infty (x^{\frac{\sigma}{2}-1} + x^{\frac{1-\sigma}{2}-1})|w(x)|dx,$$

$$<< \int_1^\infty \frac{(x^{\frac{\sigma}{2}-1} + x^{\frac{1-\sigma}{2}-1})}{e^{\pi x}}dx, \sigma \in \mathbb{R},$$

$$<< \int_1^\infty \frac{e^x}{e^{\pi x}}dx, \qquad\qquad << 1.$$

$\int_1^\infty |x^{\frac{s}{2}} + x^{\frac{1-s}{2}}||w(x)|\frac{dx}{x}$ is analytic in $\mathbb{C}$.

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = 1 + s(s-1)\int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}.$$

Set $\xi(x) := 1 + s(s-1)\int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}$, then it is entire. and for all s we have,

$$\xi(1-s) = \xi(s).$$

Thus obtain,

$$(1-s)(1-s-1)\pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Using the construction of $w(s)$ we have,

$$|w(x)| << |\theta(x)| << \frac{e^{-\pi x}}{x}, \forall x > 0.$$

Also we have,

$$\zeta(1-s) = \pi^{-s+\frac{1}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s),$$

$$\zeta(1-s) = \pi^{-s}2^{1-s}\cos\left(\frac{\pi s}{s}\right)\Gamma(s)\zeta(s).$$

Note that $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ except simple pole at s.

$$\lim_{s\to1}\zeta(1-s) = \pi^{-1}\lim_{s\to1}\frac{\cos\frac{\pi s}{2}\zeta(s)(s-1)}{s-1}.$$

$$\lim_{s\to1}\frac{\cos\frac{\pi s}{2}}{s-1} = \lim_{s\to0}\frac{\cos\left(\frac{\pi}{2}(1-s)\right)}{-s},$$

$$= \lim_{s\to0}\frac{\sin\frac{\pi}{2}s}{-s\cdot\frac{\pi}{2}}\frac{\pi}{2},$$

$$= -\frac{\pi}{2}.$$

$$\Rightarrow \lim_{s\to1}\zeta(1-s) = -\frac{1}{2}.$$

$$\zeta(-2n) = \pi^{-(2n+1)}2^{1-2n-1}\cos\left(\frac{\pi}{2}(2n+1)\right)\Gamma(2n+1)\zeta(2n+1) = 0.$$

Recall

$$\zeta(s) = \sum_{n\in\mathbb{N}}\frac{1}{n^s},$$

$$= \frac{s}{s-1} - s\int_1^\infty\frac{\{t\}}{t^{s+1}}dt, \sigma > 0, s \neq 1.$$

If s is real,

$$\left|\zeta(s) - \frac{s}{s-1}\right| \leq |s|\int_1^\infty\frac{\{t\}}{t^{\sigma+1}}dt,$$

$$< \frac{|s|}{\sigma} = \frac{\sigma}{\sigma} = 1.$$

Thus we obtain,

$$-1 + \frac{s}{s-1} < \zeta(s) < 1 + \frac{s}{s-1}$$

$$\frac{1}{s-1} < \zeta(s) < \frac{2s-1}{s-1},$$

$$-1 < (1-s)\zeta(s) < 1 - 2s < 0 \quad \text{if } \frac{1}{2} < s < 1$$

$$\Rightarrow \zeta(s) \neq 0, \quad \text{if } \frac{1}{2} < s < 1.$$

$\square$

2.7 Primitive characters

Suppose $q_0 | q$. Then we have a natural map

$$\mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/q_0\mathbb{Z}$$

Definition 2.17. *Let $\chi \bmod q$ be a character and $q_0 | q$. If there exists a character $\chi^* \bmod q_0$ making the following diagram commutative,*

$$\begin{array}{ccc} (\mathbb{Z}/q\mathbb{Z})^\times & \xrightarrow{\chi} & \mathbb{C}^\times \\ f \downarrow & \nearrow_{\chi^*} & \\ (\mathbb{Z}/q_0\mathbb{Z})^\times & & \end{array}$$

we say χ factors through $(\mathbb{Z}/q_0\mathbb{Z})$ or χ^* induces χ. If q_0 is minimal among positive such numbers, we say q_0 is a conductor of χ.

Definition 2.18. *A character $\chi \bmod q$ is called primitive if its conductor is q only. Otherwise the character is imprimitive.*

Example 2.11. *Take $\chi : (\mathbb{Z}/8\mathbb{Z})^\times \to \mathbb{C}^\times$ such that*

n	$\chi(n)$
1	1
3	-1
5	1
7	-1

Then 4 is a conductor with $\chi^* : (\mathbb{Z}/4\mathbb{Z})^\times \to \mathbb{C}^\times$,

n	$\chi^*(n)$
1	1
3	-1

Example 2.12. *Take $\chi : (\mathbb{Z}/6\mathbb{Z})^\times \to \mathbb{C}^\times$ such that*

n	$\chi(n)$
1	1
5	-1

Then 3 is a conductor with $\chi^* : (\mathbb{Z}/3\mathbb{Z})^\times \to \mathbb{C}^\times$,

n	$\chi^*(n)$
1	1
2	-1

Remark 2.19. *All principal characters $\bmod\, q$ for $q > 1$ are imprimitive by construction.*

We set as usual $s = \sigma + it$ for $\sigma, t \in \mathbb{R}$.

Lemma 2.14. *Suppose $\chi \bmod q$ is induced by $\chi^* \bmod q_0$. We then have,*

$$\sigma > 1 \Rightarrow L(s, \chi) = L(s, \chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s} \right).$$

Proof. We know that for $\sigma > 1$,

$$L(s, \chi) = \prod_{p \nmid q} \left(1 - \frac{\chi(p)}{p^s} \right)^{-1}.$$

Also

$$L(s, \chi^*) = \prod_{p \nmid q_0} \left(1 - \frac{\chi(p)}{p^s} \right)^{-1},$$

$$= \prod_{\substack{p \nmid q_0 \\ p \nmid q}} \left(1 - \frac{\chi^*(p)}{p^s} \right)^{-1} \prod_{\substack{p \nmid q_0 \\ p | q}} \left(1 - \frac{\chi^*(p)}{p^s} \right)^{-1},$$

$$= L(s, \chi) \prod_{\substack{p \nmid q_0 \\ p | q}} \left(1 - \frac{\chi^*(p)}{p^s} \right)^{-1}.$$

$\square$

Remark 2.20. *Using Identity theorem, Lemma 2.14 holds for all $s \in \mathbb{C}$.*

Lemma 2.15. *Suppose $x \in \left[0, \frac{1}{2} \right]$, then*

$$|\sin \pi x| \geq 2x.$$

Proof. Exercise.

$\square$

Definition 2.19 (Gauss sum)**.** *Let* $\chi \bmod q$ *be a character. The Gauss sum is,*

$$\tau(\chi) := \sum_{k=1}^{q} \chi(k)e^{\frac{2\pi k i}{q}}.$$

Lemma 2.16. *For* $n \in \mathbb{Z}$, *we have,*

$$\chi(n)\tau(\overline{\chi}) = \sum_{k=1}^{q} \overline{\chi}(k)e^{\frac{2\pi i k n}{q}},$$

where χ *is a primitive character modulo* q *and* $\chi \neq \chi_0$.

Theorem 2.34. *If* $\chi \bmod q$ *is primitive then,*

$$|\tau(\chi)| = \sqrt{q}.$$

Proof. By Lemma 2.16, we have,

$$\chi(n)\tau(\overline{\chi}) = \sum_{k=1}^{q} \overline{\chi}(k)e^{\frac{2\pi i k}{q}},$$

$$\overline{\chi(n)\tau(\overline{\chi})} = \sum_{l=1}^{q} \chi(l)e^{-\frac{2\pi i l}{q}}.$$

Multiplying this equation with the one in the statement we have,

$$|\chi(n)|^2 |\tau(\overline{\chi})|^2 = \sum_{k,l=1}^{q} \overline{\chi(k)}\chi(l)e^{\frac{2\pi i(k-l)}{q}}.$$

Applying $\sum_{n=1}^{q}$ to the both sides of the above equation, we have,

$$|\tau(\overline{\chi})|^2 \sum_{n=1}^{q} |\chi(n)|^2 = \sum_{k,l=1}^{q} \overline{\chi(k)}\chi(l) \left(\sum_{n=1}^{q} \left(e^{\frac{2\pi i(k-l)}{q}} \right)^n \right).$$

Note that

$$x + x^2 + \cdots + x^q = \begin{cases} \frac{x(x^q - 1)}{x-1}, & (x \neq 1), \\ q, & (x = 1), \end{cases}$$

$$= \begin{cases} 0, & (x \neq 1, x^q = 1), \\ q, & (x = 1), \end{cases}$$

Take $x = e^{\frac{2\pi i(k-l)}{q}} = \cos \frac{2\pi}{q}(k-l) + i \sin \frac{2\pi}{q}(k-l) = 1$. $x = 1$ if and only if $q|k-l$ thus

$$\sum_{n=1}^{q} \left(e^{\frac{2\pi i(k-l)}{q}} \right)^n = \begin{cases} 0, & (q \nmid k-l), \\ q, & (\text{otherwise}). \end{cases}$$

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Therefore, we get,

$$|\tau(\overline{\chi})|^2 \sum_{n=1}^{q} |\chi(n)|^2 = q \sum_{\substack{k,l=1 \\ q|k-l}}^{q} \overline{\chi}(k)\chi(l),$$

$$= q \sum_{k=1}^{q} \overline{\chi}(k)\chi(k),$$

$$= q \sum_{k=1}^{q} |\chi(k)|^2.$$

$\square$

Theorem 2.35 (Pólya-Vinogradov).
Let $\chi \bmod q$ be a primitive character. We have,

$$\left| \sum_{n \leq x} \chi(n) \right| << \sqrt{q} \log q.$$

This inequality is uniform in q.

Proof. Consider $(n, q) = 1$, then

$$\chi(n)\tau(\overline{\chi}) = \chi(n) \sum_{k=1}^{q} \overline{\chi}(k) e^{\frac{2\pi i k}{q}},$$

$$= \chi(n) \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \overline{\chi}(k) e^{\frac{2\pi i k}{q}},$$

Set $k = nt$, we get,

$$\chi(n)\tau(\overline{\chi}) = \sum_{\substack{t=1 \\ (t,q)=1}}^{q} \overline{\chi}(nt)\chi(n) e^{\frac{2\pi i n t}{q}}.$$

$$\overline{\chi}(nt) = \overline{\chi}(n)\overline{\chi}(t).$$

$$\chi(n)\tau(\overline{\chi}) = \sum_{\substack{t=1 \\ (t,q)=1}}^{q} \overline{\chi}(t) e^{\frac{2\pi i n t}{q}}.$$

Observe that

$$\tau(\overline{\chi})\left(\sum_{n\leq x}\chi(n)\right)=\sum_{k=1}^{q-1}\overline{\chi}(k)\left(\sum_{n\leq x}e^{\frac{2\pi ikn}{q}}\right),$$

$$|\tau(\overline{\chi})|\cdot\left|\sum_{n\leq x}\chi(n)\right|\leq\sum_{k=1}^{q-1}\left|\sum_{n\leq x}e^{\frac{2\pi ikn}{q}}\right|,$$

$$=\sum_{k=1}^{q-1}\left|\frac{e^{\frac{2\pi ik}{q}}\left(e^{\frac{2\pi in[x]}{q}}-1\right)}{e^{\frac{2\pi ik}{q}}-1}\right|,$$

$$\leq\sum_{k=1}^{q-1}\frac{2}{\left|e^{\frac{2\pi k}{q}}-1\right|},$$

Note that for all $y\in\mathbb{R}$,

$$2i\sin y=e^{-iy}(e^{2iy}-1),$$
$$|2\sin y|=|e^{2iy}-1|.$$

Apply this to the equation above we have,

$$\sum_{k=1}^{q-1}\frac{2}{\left|e^{\frac{2\pi k}{q}}-1\right|}=\sum_{k=1}^{q-1}\frac{1}{\left|\sin\frac{\pi k}{q}\right|}.$$

Using Lemma 2.15, we get,

$$\sum_{k=1}^{q-1}\frac{1}{\left|\sin\frac{\pi k}{q}\right|}=\sum_{1\leq k\leq\frac{q}{2}}\frac{1}{\left|\sin\frac{\pi k}{q}\right|}+\sum_{\frac{q}{2}<k}^{q-1}\frac{1}{\left|\sin\frac{\pi k}{q}\right|},$$

$$=2\sum_{1\leq k\leq\frac{q}{2}}\frac{1}{\left|\sin\frac{\pi k}{q}\right|},$$

$$\leq\sum_{1\leq k\leq\frac{q}{2}}\frac{k}{q}<<q\log\frac{q}{2}<<q\log q.$$

Recall Theorem 2.34, we have $|\tau(\chi)|=\sqrt{q}$ as χ is primitive. Therefore, for $q>1$, we have,

$$\left|\sum_{n\leq x}\chi(n)\right|<<\sqrt{q}\log q,$$

uniformly in q as $x\to\infty$. $\qquad\qquad\square$

Consider $L(q, \chi)$ functional equation, where $\chi \neq \chi_0$ and χ is primitive. Suppose $\chi(-1) = 1$ (i.e it is an even character). From the proof of Theorem 2.33, we have,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty x^{\frac{s}{2}} e^{-n^2 \pi x} \frac{dx}{x}.$$

Replace x with $\frac{x}{q}$ then for $\mathrm{Re}(s) = \sigma > 0$, we have,

$$\pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty x^{\frac{s}{2}} e^{-n^2 \pi x} \frac{dx}{x}.$$

For $\sigma > 1$, we have,

$$\pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = \sum_{n=1}^\infty \int_0^\infty \chi(n) x^{\frac{s}{2}} e^{-n^2 \frac{\pi x}{q}} \frac{dx}{x},$$

$$= \int_0^\infty x^{\frac{s}{2}} \left(\sum_{n=1}^\infty \chi(n) e^{-n^2 \frac{\pi x}{q}}\right) \frac{dx}{x}.$$

Let

$$\theta(x, \chi) = \sum_{n \in \mathbb{Z}} \chi(n) e^{-\frac{n^2 \pi x}{q}} = 2 \sum_{n=1}^\infty \chi(n) e^{-\frac{n^2 \pi x}{q}}, (x > 0).$$

Using this, we get,

$$\int_0^\infty x^{\frac{s}{2}} e^{-n^2 \pi x} \frac{dx}{x} = \frac{1}{2} \int_0^\infty x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x}, (\sigma > 1).$$

Split the integral into $\int_0^1, \int_1^\infty$.

$$\tau(\overline{\chi}) \theta(x, \chi) = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \overline{\chi}),$$

$$= \sum_{n \in \mathbb{Z}} (\tau(\overline{\chi}) \chi(n)) e^{\frac{-n^2 \pi x}{q}},$$

$$= \sum_{n \in \mathbb{Z}} \sum_{k=1}^q \overline{\chi}(k) e^{\frac{2\pi i n k}{q}} e^{-\frac{n^2 \pi x}{q}},$$

$$= \sum_{k=1}^q \overline{\chi}(k) \sum_{n \in \mathbb{Z}} e^{\frac{2\pi k n}{q} - n^2 \frac{\pi x}{q}},$$

$$\stackrel{\text{From Lecture 7 page 6}}{=} \sum_{k=1}^q \overline{\chi}(k) \left(\frac{x}{q}\right)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} e^{-\left(m + \frac{m}{q}\right) \frac{\pi q}{x}},$$

$$= \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^q \overline{\chi}(k) \sum_{n \in \mathbb{Z}} e^{-\left(m + \frac{m}{q}\right) \frac{\pi q}{x}}.$$

Put $qn + m = t$, then

$$\overline{\chi}(qn + m) = \overline{\chi}(m) = \overline{\chi}(t).$$

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Thus,

$$\left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{k=1}^{q} \overline{\chi}(k) \sum_{n \in \mathbb{Z}} e^{-\left(m+\frac{m}{q}\right)\frac{\pi q}{x}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{t \in \mathbb{Z}} \overline{\chi}(t) e^{-\frac{t^2 \pi}{qx}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \overline{\chi}).$$

Now for the splitted integral, we have,

$$\frac{1}{2}\int_0^1 x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x} + \frac{1}{2}\int_1^\infty x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x}\frac{dx}{x}.$$

Replacing x with $\frac{1}{x}$, we get,

$$= \frac{\tau(\chi)}{2\sqrt{q}}\int_1^\infty x^{\frac{1-s}{2}} \theta(x, \overline{\chi}) \frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x}.$$

Set

$$\xi(s, \chi) := \frac{\tau(\chi)}{2\sqrt{q}}\int_1^\infty x^{\frac{1-s}{2}} \theta(x, \overline{\chi}) \frac{dx}{x} + \frac{1}{2}\int_1^\infty \int_0^1 x^{\frac{s}{2}} \theta(x, \chi) \frac{dx}{x}.$$

Use that $\theta(x, \overline{\chi}) << e^{-\frac{\pi x}{q}}$ and $|\tau(\chi)| = \sqrt{q}$. It turns out that ξ is uniformly bounded on compact subsets of $\mathbb{C}$ and in particular, this is entire.

Lemma 2.17.
$$\xi(1-s, \chi) = \frac{\tau(\chi)}{\sqrt{q}} \xi(s, \overline{\chi}).$$

Proof. Recall,

$$\tau(\overline{\chi})\theta(x, \chi) = \left(\frac{q}{x}\right)^{\frac{1}{2}} \theta(x^{-1}, \overline{\chi}),$$

and

$$|\tau(\chi)|^2 = q.$$

We get,

$$\frac{\tau(\chi)}{\sqrt{q}} = \frac{\sqrt{q}}{\tau(\chi)} = \frac{\sqrt{q}}{\tau(\overline{\chi})}.$$

Then,

$$L(1-s, \chi) = \frac{\tau(\chi)}{q^{1-s}}\pi^{-s}2^{1-s}\cos\frac{\pi s}{2}(\Gamma(s))L(s, \overline{\chi}).$$

Note that $\chi(-1) = 1$ and is primitive (in the case of odd integer replace cos with sin). Note also that

i). $L(s, \chi) = 0, s = 0, -2, -4, \cdots$ when χ is even,

ii). $L(s, \chi) = -1, -3, -5, \cdots$ when χ is odd.

When $0 < \sigma < 1$, $\hfill\square$

Definition 2.20 (Entire function of finite order)**.** *An entire functio n$f(z)$ is saiad to be of finite order if there is $\alpha > 0$ such that*

$$|f(z)| << e^{|z|^{\alpha}},$$

as $|z| \to \infty$. The order of f is the infimum of such α.

Example 2.13. *We have the following examples,*

1. $e^z \to 1$,

2. $\sin z \to 1$,

3. $\cos z \to 1$,

4. $e^{z^n} \to n$.

For the trigonometric functions use $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ and similarly for $\cos$.

Remark 2.21. *Is $|\sin z| < z, z \in \mathbb{C}$? To show this consider,*

$$f(z) = \begin{cases} \frac{\sin z}{z}, z \neq 0, 1, z = 0. \end{cases}$$

And apply Liouville's theorem.

Theorem 2.36 (Special case of Hadamard's factorization theorem)**.** *Let $f(z)$ be an entire function of order 1 and $(z_j)_{j \in \mathbb{N}}$ be such that $z_j \in \mathbb{C} \backslash \{0\}$, and $f(z_j) = 0.$Then there are some constants A, B such that*

$$f(z) = e^{A+Bz} \prod_{j \geq 1} \left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j}}.$$

Theorem 2.37. *Let f be an entire function of order $\alpha \in \mathbb{Z}$ without any zeros. Then we have,*
$$f(z) = e^{p(z)},$$

where p is some polynomial of degree α.

Recall the completed zeta function,

Definition 2.21.
$$\xi(s) := (s-1)\pi^{\frac{-s}{2}} \Gamma\left(\frac{s}{2} + 1\right) \zeta(s).$$

We know,

1. $\xi(s)$ has an order 1,

2. $\zeta(s)$ has infinitely zeros in the critical strip.

3. $\xi(s) = e^{As+B} \prod_{\substack{\rho,\xi(\rho)=0 \\ 0<\text{Re}(\rho)<1}} \left(1 - \frac{s}{\rho}\right) e^{\frac{z}{\rho}}$, *where A, B ae some constant.*

Remark 2.22. *Zeros of $\xi(s)$ are precisely the non-trivials zeros of $\zeta(s)$.*

Theorem 2.38. *There eists a constant $c > 0$ such that for $\sigma, t \in \mathbb{R}$ $s = \sigma + it$, $\sigma \geq 1 - \frac{c}{\log(|t|+2)}$, we have,*

$$\zeta(s) \neq 0.$$

Taking derivatives, we obtain,

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{\Gamma'\left(\frac{s}{2}+1\right)}{2\Gamma\left(\frac{s}{2}+1\right)} + \frac{\zeta'(s)}{\zeta(s)},$$

$$\frac{\xi'(s)}{\xi(s)} = A + \sum_{\substack{\zeta(\rho)=0 \\ 0<\mathrm{Re}(\rho)<1}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Note that $\frac{s}{2}\Gamma\left(\frac{s}{2}\right) = \Gamma\left(\frac{s}{2}+1\right)$. Thus we obtain,

$$-\frac{\zeta'(s)}{\zeta} = \frac{1}{s-1} - A - \frac{\log\pi}{2} + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} - \sum_\rho\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

We then get,

$$\mathrm{Re}\left(-\frac{\zeta'(s)}{\zeta}\right) = \mathrm{Re}\left(\frac{1}{s-1}\right) - A - \frac{\log\pi}{2} + \frac{1}{2}\mathrm{Re}\left(\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)}\right) - \sum_\rho\mathrm{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Recall that for $s = \sigma + it, \sigma, t \in \mathbb{R}$, we have,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1}\frac{\Lambda(n)}{n^s}, \sigma \geq 1.$$

And also,

$$\frac{-3\zeta'(\sigma)}{\zeta(\sigma)} - \mathrm{Re}\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right) - \mathrm{Re}\left(\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right).$$

Check that

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}(3 + 4\cos(t\log n) + \cos(2t\log n)) > 0, \sigma > 1.$$

For $1 < \sigma < 2, t \geq 1$, we have,

$$\mathrm{Re}\left(\frac{1}{s-1}\right) = \frac{\sigma-1}{|s-1|^2} < \frac{1}{t^2} < 1.$$

$$\mathrm{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = \mathrm{Re}\left(\frac{\overline{s-\rho}}{|s-\rho|^2} + \frac{\overline{\rho}}{|\rho|^2}\right),$$

$$= \frac{\sigma-\beta}{|s-\rho|^2} + \frac{\beta}{|\rho|^2} > 0,$$

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where $\beta = \text{Re}(\rho)$. We have,

$$\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} < B \log\left(\frac{|t|}{2}+2\right), \forall t \in \mathbb{R}.$$

We have the following equations,

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z, |z| \to \infty.$$

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}},$$

where γ is the Euler's constant.

Now consider $1 \le \sigma \le 2$, we have,

$$\left|\frac{\Gamma'(s)}{\Gamma(s)}\right| < B \log(|t|+2),$$

where B is some constant for all $t \in R$. From above expressions, we obtain,

$$\text{Re}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) < B' \log\left(\frac{|t|}{2}+2\right), \forall t \ge 1.$$

Let $\rho = \beta + it$, be any non-trivial zero of $\zeta(s)$.

$$\text{Re}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) < B' \log\left(\frac{|t|}{2}+2\right) - \frac{1}{\sigma-\beta}, t \ge 1.$$

Recall for $\sigma > 0$,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + g(s),$$

where $g(s)$ is some analyti function. For $1 \le \sigma \le 2$,

$$\frac{-\zeta'(\sigma)}{\zeta(\sigma)} = \frac{1}{\sigma-1} - g(\sigma),$$

$$< \frac{1}{\sigma-1} + c,$$

where C is some constant.

$$0 < \frac{-3\zeta'(\sigma)}{\zeta(\sigma)} - 4\text{Re}\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right) - \text{Re}\left(\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right).$$

We then obtain,

$$0 < 3\left(\frac{1}{\sigma-1}+C\right) + 4B' \log(|t|+2) + B' \log(|t|+2) - \frac{1}{\sigma-\beta}.$$

Note that
$$\frac{1}{\sigma - \beta} < \frac{3}{\sigma - 1} + B'' \log(|t| + 2), t \geq 1.$$

Take $\sigma = 1 + \frac{\varepsilon}{\log(|t|+2)}$, where $\varepsilon > 0$, sufficiently small which will be chosen later.

$$\beta < 1 - \frac{\varepsilon(1 - B''\varepsilon)}{\log(|t| + 2)}, t \geq 1.$$

So far we have shown that for $t \geq 1$ and

$$\sigma \geq 1 - \frac{c}{\log(|t| + 2)},$$

we have $\zeta(s) \neq 0$. For some constant $C > 0$.

For $|t| \leq 1$, since $\zeta(9s)$ be analytic except $s = 1$, there exists a constant $c_1 > 0$ such that

$$\sigma > 1 - c_1 \Rightarrow \zeta(s) \neq 0.$$

If

$$1 - c_1 \leq 1 - \frac{c_2}{\log(|t| + 2)},$$

then we have,

$$c_1 \geq \frac{c_2}{\log(|t| + 2)}, |t| < 1.$$

Choose $c_2 = c_1 \log 2$. Take $c' = \min(c_1, c_1 \log 2)$.

Above proof can be generalized to $L(s, \chi)$ where χ is a non-trivial character.

Theorem 2.39. *Let χ be a non-trivial character. Then there is $c_1 > 0$ such that for $s = \sigma + it, \sigma, t \in \mathbb{R}$, we have,*

$$\sigma \geq 1 + \frac{c_1}{\log(q(|t| + 2))}.$$

Theorem 2.40 (Siegel zeros). *There exists an absolute constant $c_2 > 0$ such that $L(s, \chi)$ where χ is quadratic $\mod q$ has at most one zero in the region,*

$$\sigma > 1 - \frac{\delta}{\log q}, |t| < \frac{\delta}{\log q},$$

where $0 < \delta < c_2$. If such a zero exists, it is real, simple and often called Siegel zero.

Recall the completed Dirichlet L-function,

$$\xi(s, \chi) := \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s + a}{2}\right) L(s, \chi).$$

See the lecture note from 12/16. Where

$$a = \begin{cases} 0, & \chi(-1) = 1, \\ 1, & \chi(-1) = -1. \end{cases}$$

We have $\xi(s, \chi)$ is of order 1 and

$$\xi(s, \chi) = e^{A_1 + B_1 s} \prod_{\substack{\rho, L(\rho, \chi)=0 \\ 0 \le \mathrm{Re}(\rho) < 1}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}.$$

Furthermore $L(s, \chi)$ has inifinitely many zero in the critical strip.

Remark 2.23.
$$L(s, \chi) = L(s, \chi^*) \prod_{p | q} \left(1_{\frac{\chi^*(p)}{p^s}}\right), s \in \mathbb{C}$$

Then $L(s, \chi) = 0$ if and only if $L(s, \chi^*) = 0$ when $\sigma \ne 0$.

Recall

$$\xi(s) = \frac{s(s - 1)}{2} \pi^{\frac{-s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Definition 2.22.

$$N(T) = |\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \beta \in (0, 1), \gamma \in (0, T)\}|.$$

Theorem 2.41. *As $T \to \infty$, we have,*

$$N(T + 1) - N(T) << \log T.$$

Proof. Enough to show that as $T \to \infty$.

$$\sum_{\substack{\rho = \beta + i\gamma \\ \zeta(\rho) = 0 \\ \beta \in (0,1)}} \frac{1}{1 + (T - \gamma)^2} << \log T$$

Observe that,

$$N(T + 1) - N(T) = \sum_{\substack{\rho = \beta + i\gamma \\ \beta \in (0,1) \\ \zeta(\rho) = 0 \\ T < \gamma < T+1}} \le \sum.$$

Recall we have,

$$-\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) < A \log(|t| + 2) - \sum_{\rho = \beta + i\gamma} \mathrm{Re}\left(\frac{1}{s - p} - \frac{1}{p}\right).$$

where $s = \sigma + it$ for $\sigma \in [1, 2], |t| > 1$ and A is some constant. Substitute $s = 2 + iT$ in the equation, we have,

$$\mathrm{Re}\left(\frac{1}{s - \rho}\right) = \mathrm{Re}\left(\frac{\overline{(s - \rho)}}{|s - \rho|^2}\right).$$

Put $s = 2 + iT$ and $\rho = \beta + i\gamma$, we have,

$$\text{Re}\left(\frac{\overline{(s - \rho)}}{|s - \rho|^2}\right) = \frac{2 - \beta}{(2 - \beta)^2 + (T - \gamma)^2} \geq \frac{1}{4 + (T - \gamma)^2}.$$

Also,

$$\text{Re}\left(\frac{1}{\rho}\right) = \frac{\beta}{|\rho|^2} > 0.$$

By the previous equation, we get

$$\sum_{\substack{\rho = \beta + i\gamma \\ \beta \in (0,1) \\ \zeta(\rho) = 0}} \text{Re}\left(\frac{1}{2 + iT - \rho}\right) < A\log(T + 2) + \text{Re}\left(\frac{\zeta'(2 + iT)}{\zeta(2 + iT)}\right) << \log T.$$

Recall that for $\sigma > 1$,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s},$$

$$\left|\frac{\zeta'(2 + iT)}{\zeta(2 + iT)}\right| \leq \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} < \infty.$$

Therefore,

$$\sum_\rho \text{Re}\left(\frac{1}{2 + iT - \rho}\right) = \sum_\rho \frac{1}{4 + (T - \gamma)^2},$$

and

$$4 + (T - \gamma)^2 \leq 4(1 + (T - \gamma)^2).$$

$\square$

Theorem 2.42. *As $T \to \infty$,*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log(T)).$$

That is, as $T \to \infty$,

$$N(T) \sim \frac{T}{2\pi} \log T.$$

Analogously, $\chi \bmod q$, as $T \to \infty$,

$$N(T, \chi) = \sum_{\substack{\rho = \beta + i\gamma \\ L(\rho, \chi) = 0 \\ \beta \in (0,1) \\ |\gamma| < T}} 1 \sim \frac{T}{\pi} \log qT.$$

Exercise 2.6.

$$N(T, \chi)' = \sum_{\substack{\rho = \beta + i\gamma \\ L(\rho, \chi) = 0 \\ \beta \in (0,1) \\ 0 < gamma < T}} 1 \sim \frac{T}{2\pi} \log qT.$$

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Recall by Remark 2.1, we have,

$$\psi(x) := \sum_{n \leq x} \Lambda(n).$$

Also recall that Theorem 2.17, $\frac{\zeta'(s)}{\zeta(s)}$ has a simple pole at $s = 1$ with residue -1.

Let $s = a$ be a zero of $\zeta(s)$ of order $n \in \mathbb{N}$. Then

$$\zeta(s) = (s - a)^n g(s),$$

where $g(s)$ is analytic and $g(a) \neq 0$. Taking log-derivative of both sides, we have,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{n}{s - a} + \frac{g'(a)}{g(a)}.$$

Resiue of $\frac{\zeta'(s)}{\zeta(s)}$ at 0 of $zeta(s)$ is precisely the multiplicity of that zero. Recall from the proof of Theorem **??**, we have,

$$\psi(x) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + O\left(\sum_{n \in \mathbb{N}} \Lambda(n) \left(\frac{x}{n}\right) \min(1, T^{-1} \left|\log \frac{x}{n}\right|^{-1}) \right),$$
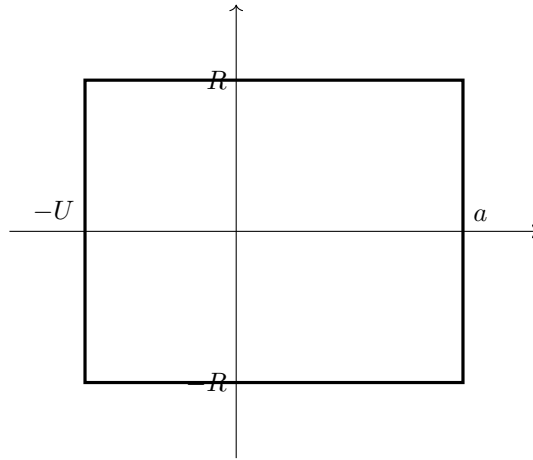
where x is not an integer and $T, a \geq 1$. Similarly to the proof of Theorem **??**, take

$$a = 1 + \frac{1}{\log x}.$$

we then get, as $x \to \infty$,

$$\psi(x) = \frac{1}{2\pi i} \int_{a-iR}^{a+iR} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} dx + O\left(\frac{x \log^2 x}{R} \right).$$

Now consider the following closed path

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where U is some odd integer. By Cauchy's residue theorem, we have,

$$\frac{1}{2\pi i} \int_{C_R} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = \text{sum of residues of } \frac{\zeta'(s)}{\zeta(s)} \text{ in } C_R,$$

$$= \overbrace{x}^{\text{contribution of poles of } \zeta(s)} - \underbrace{\frac{\zeta'(0)}{\zeta(0)}}_{\text{contribution of } s=0} + \overbrace{\sum_{0<2m<U} \frac{x^{-2m}}{2m}}^{\text{contribution of non-trivial zeros of } \zeta(s)} -$$

contributi

Note that,

$$\int_{C_R} = \int_{a-iR}^{a+iR} + \underbrace{\int_{T_1,T_2,T_3}}_{=I}.$$

We then have

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{0<2m<U} \frac{x^{-2m}}{2m} - \sum_{\substack{\rho=\beta+i\gamma \\ \zeta(\rho)=0 \\ \beta\in(0,1) \\ |\gamma|<R}} \frac{x^\rho}{\rho} - I + O\left(\frac{x\log^2 x}{R}\right).$$

Exercise 2.7. *Show that*

$$I = \frac{1}{2\pi} \int_{I_1+I_2+I_3} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds, \qquad = O\left(\frac{x\log^2 R}{R\log x} + \frac{R\log U}{Ux^R}\right).$$

Suppose we have above equality, then we have,

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{0<2m<U} \frac{x^{-2m}}{2m} - \sum_\rho \frac{x^\rho}{\rho} + O\left(\frac{x\log^2 R}{R\log x} + \frac{R\log U}{Ux^R} + \frac{x\log^2 x}{R}\right).$$

Recall U is an odd integer. Take $U \to \infty$, we have,

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \log(1-x^{-2}) - \sum_\rho \frac{x^\rho}{\rho} + O\left(\frac{x\log^2 R}{R\log x} + \frac{x\log^2 x}{R}\right).$$

Taking $R \to \infty$, we have,

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \log(1-x^{-2}) - \sum_\rho \frac{x^\rho}{\rho}.$$

Definition 2.23.
$$\psi(x,\chi) := \sum n \le x \Lambda(n)\chi(n).$$

Recall we had,

$$\psi(x, q, a) := \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n),$$

where $(a, q) = 1$. By Theorem **??**, we have, as $x \to \infty$,

$$\psi(x, q, a) \sim \frac{x}{\varphi(q)}.$$

Exercise 2.8. *Show that for $(a, q) = 1$, we have,*

$$\psi(x, q, a) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \psi(x, \chi)\overline{\chi(a)}.$$

Theorem 2.43. *As $x \to \infty$, we have,*

$$\psi(x) = x + O(xe^{-c\sqrt{\log x}}),$$

where $c > 0$ is some constant.

Proof. Recalling zero free regions of ζ, we have for $s = \sigma + it$,

$$\sigma > 1 - \frac{c}{\log R} \Rightarrow \zeta(s) \neq 0,$$

where c is some constant and $R = \mathrm{Im}(s)$. From this we have,

$$\left| \sum_{|\gamma| < R} \frac{x^\rho}{\rho} \right| \leq x^{1 - \frac{c}{\log R}} \sum_{\substack{\rho = \beta + i\gamma \\ \zeta(\rho) = 0 \\ \beta \in (0,1) \\ |\gamma| < R}} \frac{1}{|\rho|} << x^{1 - \frac{c}{\log R}} \sum_{\substack{\beta \in (0,1) \\ \gamma \in (0,R) \\ \zeta(\beta + i\gamma) = 0}} \frac{1}{\sqrt{\beta^2 + \gamma^2}}.$$

Set

$$g(t) = \frac{1}{\sqrt{\beta^2 + t^2}},$$

and

$$a_t := \sum_{\substack{\beta \in (0,1) \\ \gamma \in (0,t) \\ \zeta(\beta + i\gamma) = 0}} 1.$$

We now have,

$$\sum_{\substack{\beta \in (0,1) \\ \gamma \in (0,R) \\ \zeta(\beta + i\gamma) = 0}} \frac{1}{\sqrt{\beta^2 + \gamma^2}} = \frac{N(R)}{\sqrt{\beta^2 + \gamma^2}} + \int_0^R \frac{t N(R)}{(\beta^2 + \gamma^2)^{\frac{3}{2}}} dt$$

$$<< \frac{N(R)}{R} + \int_0^R \frac{N(t)}{t^2} dt,$$

$$= \log R + \int_0^2 \frac{\log t}{t} + \int_2^R \cdots,$$

$$= \log^2 R.$$

Combining all we get,

$$\left| \sum_{|\gamma| < R} \frac{x^\rho}{\rho} \right| << x^{1 - \frac{c}{\log R}} \log^2 R.$$

Choose $\log R = c_2 (\log \overline{x})^{\frac{1}{2}}$ we have,

$$\psi(x) = x + O(x e^{-c\sqrt{\log x}}).$$

Using Riemann hypotehsis, choose $R = \sqrt{x}$, we then have as $x \to \infty$,

$$\psi(x) = x + O(\sqrt{x} \log^2 x) + \frac{R \log U}{U x^R} + \frac{x \log^2 x}{R}.$$

$\square$

Theorem 2.44 (Siegel-Walfisz). *For any $N \in \mathbb{R}$ there is $c_N > 0$ such that as $x \to \infty$,*

$$\psi(x, q, a) = \frac{x}{\varphi(q)} + O(x e^{-c_N \sqrt{\log x}}),$$

where $q \leq (\log x)^N$. This is uniform in q.

Exercise 2.9. *Show that Rieman hypothesis is equivalent to that as $x \to \infty$,*

$$\psi(x) = x + O(\sqrt{x} \log^2 x).$$

Theorem 2.45. *Riemann hypothesis implies that for $(a, q) = 1$, we have, as $x \to \infty$,*

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O(\sqrt{x} \log^2 qx).$$

Remark 2.24. *For $(a, q) = 1$, fixed we have as $x \to \infty$.*

$$\pi(x, q, a) < \frac{2x}{\varphi \log \frac{x}{q}}, q < x.$$

3 Exercise Solutions

3.1 Lecture 1 ∼ 4

Exercise 3.1. *Show that*

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}.$$

Solution. From Theorem 2.5, we have,

$$\frac{1}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s}}.$$

Therefore, we examine,

$$\left(\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \left(|\mu(n)|\mu\left(\frac{n}{d}\right)\right)$$

Note that if $n \neq 1$ is not square free,

$$|\mu(n)|\mu\left(\frac{n}{d}\right) = 0,$$

unless $n = k^2$ and $d = k$. If n is square free then by Theorem 2.1, we have

$$\left(\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s}}.$$

$\square$

Exercise 3.2. *Let $\pi(x; z)$ be the number of $n \leq x$ such that it is coprime to all $p \leq z$. Then we have,*

$$\pi(x; z) = x \prod_{p \leq z} \left(1 - \frac{1}{p}\right) + O(2^z).$$

Solution. We have,

$$\pi(x; z) = \sum_{p_1, p_2, \cdots, p_k \leq z} (-1)^{k+1} \left[\frac{x}{p_1 p_2 \cdots p_k}\right],$$

$$= \sum_{p_1, p_2, \cdots, p_k \leq z} (-1)^k \left(\frac{x}{p_1 p_2 \cdots p_k} - \left\{\frac{x}{p_1 p_2 \cdots p_k}\right\}\right),$$

$$= x \prod_{p \leq z} \left(1 - \frac{1}{p}\right) + \sum_{p_1, \cdots, p_k \leq z} (-1)^{k+1} \left\{\frac{x}{p_1 p_2 \cdots p_k}\right\},$$

We have at most $[z]$ primes and there are $\binom{[z]}{k}$ many ways to pick k-many primes. Thus,

$$\left|\sum_{p_1, \cdots, p_k \leq z} (-1)^k \left\{\frac{x}{p_1 p_2 \cdots p_k}\right\}\right| \leq \sum_{k=0}^{[z]} \binom{[z]}{k} = 2^{[z]} \leq 2^z.$$

$\square$

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Exercise 3.3. *Show that*

$$\sum_{p \leq x} \frac{1}{p} \geq \log \log x + c,$$

for some constant c.

Solution. Let $\{p_1, \cdots, p_k\}$ be the primes less than x, then

$$\sum_{l \leq x} \frac{1}{l} \leq \sum_{\alpha \in \mathbb{Z}_{\geq 0}^k} \frac{1}{\prod p_i^{\alpha_i}}.$$

$$\log[x] \leq \sum_{l \leq x} \frac{1}{l}.$$

$$\Rightarrow \sum_{i=1}^k \log \left(\frac{1}{1 - \frac{1}{p}} \right) \geq \log \log[x].$$

$$\log p - \log(p-1) \leq \frac{1}{p}.$$

We then conclude that

$$\sum_{p \leq x} \frac{1}{p} \geq \log \log x - 1.$$

$\square$

Exercise 3.4. *Show that*

$$\pi(x) = O\left(\frac{x}{\log \log x} \right).$$

Solution.

Claim 1. *We have,*

$$-\sum_{p \leq z} \log \left(1 - \frac{1}{p} \right) = \sum_{p \leq z} \frac{1}{p} + O(1).$$

Proof: We have,

$$\log(1 - x) = -\sum_{n=0}^{\infty} \frac{x^n}{n}.$$

Thus substituting $x = \frac{1}{p}$, we obtain,

$$-\log \left(1 - \frac{1}{p} \right) = \frac{1}{p} + \sum_{n=2}^{\infty} \frac{1}{np^n} \leq \frac{1}{p} + \frac{\frac{1}{p^2}}{1 - \frac{1}{p}} = \frac{1}{p} + \frac{1}{p^2}.$$

Therefore,

$$-\sum_{p \leq z} \log \left(1 - \frac{1}{p} \right) \leq \sum_{p \leq z} \frac{1}{p} + \zeta(2).$$

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Using $\pi(x, z)$, we have,

$$\pi(x) \leq \pi(x, z) + z = x \prod_{p \leq z} \left(1 - \frac{1}{p}\right) + O(2^z) + z.$$

Setting $z = \log x$ and using Claim 1, we have,

$$\pi(x) \leq x \exp\left(-\sum_{x \leq \log x} \frac{1}{p} + O(1)\right) + O(x^{\log 2}).$$

Using Exercise 3.3, we get,

$$\pi(x) \leq x \exp\left(-\log \log \log(x) + O(1)\right) + O(x^{\log 2}).$$

This completes the proof. $\qquad\square$

Exercise 3.5. *Show that*

$$e^{\psi(2n+1)} \int_0^1 x^n(1-x)^n dx,$$

is a positive integer and deduce that

$$\psi(2n+1) \geq 2n \log 2.$$

Solution.

Claim 1. *For any $n \in \mathbb{N}$, we have,*

$$e^{\psi(n)} = \mathrm{lcm}(2, \cdots, n).$$

Using binomial coefficient, we have,

$$(1-x)^n = \sum_{k=0}^n \binom{n}{k}(-1)^k x^k.$$

Thus we have,

$$I = \int_0^1 x^n(1-x)^n = \int_0^1 \sum_{k=0}^n \binom{n}{k}(-1)^k x^{n+k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{n+k+1} x^{n+k+1}$$

Therefore, the first statement follows. For the latter observe that

$$x(1-x) \leq \frac{1}{4}.$$

Therefore,

$$I \leq 2^{-2n}.$$

Finally,

$$e^{\psi(2n+1)}I \geq 1 \Rightarrow e^{\psi(2n+1)} \geq 2^{2n}.$$

$\qquad\square$

Exercise 3.6. *Let $\chi \bmod q$ be a non-principal character. Then we have,*

$$\sum_{n>x} \frac{\chi(n)}{n} = O\left(\frac{1}{x}\right).$$

Proof. Using Proposition 2.7, we have,

$$\sum_{n \leq x} \frac{\chi(n)}{n} \leq \frac{\varphi(q)}{x} - \int_1^x \frac{\varphi(q)}{t}^2 \, dt.$$

Recall $L(s,\chi)$ is analytic for $\mathrm{Re}(s) > 0$, in particular,

$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

is bounded. Thus we have,

$$\sum_{n>x} \frac{\chi(n)}{n} \leq \int_x^{\infty} \frac{\varphi(q)}{t}^2 \, dt = O\left(\frac{1}{x}\right).$$

$\square$

Exercise 3.7. *Show that for a non-principal character $\chi \bmod q$, we have,*

$$\sum_{n \leq x} \sum_{d \mid n} \chi(d) = xL(1,\chi) + O(\sqrt{x}).$$

Solution.

$$\sum_{n \leq x} \sum_{d \mid n} \chi(d) = \sum_{de \leq x} \chi(d),$$

$$= \sum_{d \leq \sqrt{x}} \chi(d) \left[\frac{x}{d}\right] + \sum_{e \leq \sqrt{x}} \sum_{\sqrt{x} < d \leq \frac{x}{e}} \chi(d).$$

$$\sum_{\sqrt{x} < d \leq \frac{x}{e}} \chi(d) = \sum_{d \leq \frac{x}{e}} \chi(d) - \sum_{d \leq \sqrt{x}} \chi(d).$$

$$\sum_{e \leq \sqrt{x}} \sum_{\sqrt{x} < d \leq \frac{x}{e}} \chi(d) \leq 2\varphi(q)[\sqrt{x}] = O(\sqrt{x}).$$

$$\sum_{d \leq \sqrt{x}} \chi(d) \left[\frac{x}{d}\right] = x \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{d} + O(1),$$

$$= xL(1,\chi) - \sum_{n>x} \frac{\chi(n)}{n} + O(1).$$

By the previous exercise. We conclude the solution. $\square$

Exercise 3.8. *For non-principal real character $\chi \bmod q$, show that $L(1,\chi) \neq 0$.*

Solution. By Lemma 2.10, we have

$$a_n := \sum_{d|n} \chi(d) \geq 0.$$

Set

$$F(x) = \sum_{n \leq x} a_n \chi(d).$$

Then by the previous exercise, we have $F(x) = O(\sqrt{x})$. Let

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{F(t)}{t^{s+1}} dt.$$

Note that by the definition of Dirichlet series, we have,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = L(s, \chi)\zeta(s).$$

$\zeta(s)$ has a simple pole at $s = 1$ and L has zero of order at least one at $s = 1$, however,

$$\lim_{s \to 1} s \int_1^{\infty} \frac{F(t)}{t^{s+1}} dt = \infty.$$

This is a contradiction. □

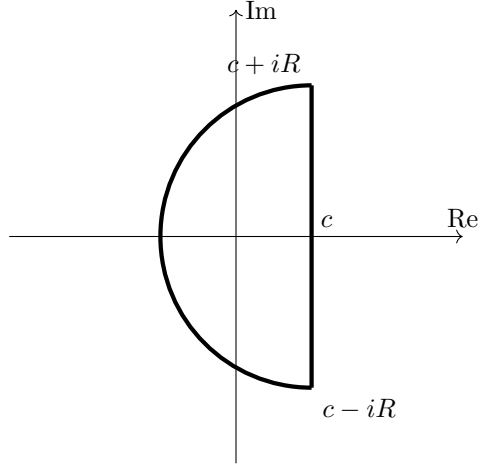**Exercise 3.9.** *Let $\chi \bmod q$ be a non-principal character then,*

$$\sum_{n \leq x} \sum_{d|n} \chi(d) = xL(1, \chi) + O(q\sqrt{x}).$$

Solution. Previous exercise and $\varphi(q) \leq q$. □

Exercise 3.10. *Show that for $x > 1$ and $c > 0$, we have*

$$\frac{1}{2\pi i} \int_{(c)} \frac{x^s}{s} ds = 1.$$

Solution. Let ζ_c be the contour

going the bold closed path in counterclockwise. Then

$$\frac{1}{2\pi i}\left(\int_{c-iR}^{c+iR}\frac{x^s}{s}\,ds + x^c\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{x^{e^{i\theta}}}{c+e^{i\theta}}\,d\theta\right) = \operatorname{Res}_s\left(\frac{x^s}{s},0\right) = 1.$$

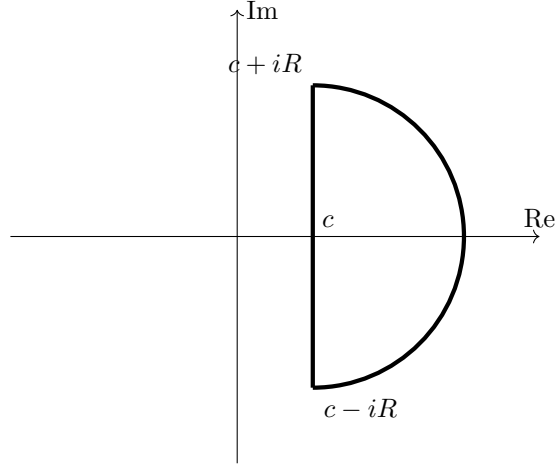Let us examine the second term, for some small $\delta > 0$,

$$\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{x^{Re^{i\theta}}}{c+Re^{i\theta}}\,d\theta\right| \leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x^{R\cos\theta}\theta,$$

$$= \left(\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\delta} + \int_{\frac{\pi}{2}+\delta}^{\frac{3\pi}{2}-\delta} + \int_{\frac{3\pi}{2}-\delta}^{\frac{3\pi}{2}}\right) x^{R\cos\theta}\,d\theta.$$

$$\delta \geq \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\delta} x^{R\cos\theta}\,d\theta, \int_{\frac{3\pi}{2}-\delta}^{\frac{3\pi}{2}+\delta} x^{R\cos\theta}\,d\theta.$$

$$\int_{\frac{\pi}{2}+\delta}^{\frac{3\pi}{2}-\delta} x^{R\cos\theta}\,d\theta \leq \pi x^{R\cos\left(\frac{\pi}{2}+\delta\right)} = \pi x^{-R\sin\delta} \overset{R\to\infty}{\to} 0.$$

As δ is arbitrary, we conclude the proof. □

Exercise 3.11. *Show that for $x \in (0,1)$ and $c > 0$, we have*

$$\frac{1}{2\pi i}\int_{(c)}\frac{x^s}{s}\,ds = 0.$$

Solution. As in the previous exercise, consider the following contour,

Then,

$$\frac{1}{2\pi i}\left(\int_{c-iR}^{c+iR}\frac{x^s}{s}ds + x^c\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{x e^{i\theta}}{c + e^{i\theta}}d\theta\right) = 0.$$

Similarly, we observe that

$$\int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}-\delta} x^{R\cos\theta}d\theta \leq \pi x^{R\cos\left(\frac{\pi}{2}-\delta\right)} = \pi x^{R\sin\delta} \overset{R\to\infty}{\to} 0.$$

$\square$

Exercise 3.12. *Show that for $c > 0$, we have*

$$\frac{1}{2\pi i}\int_{(c)}\frac{1}{s}ds = \frac{1}{2}.$$

Solution.

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{1}{s}ds = \frac{1}{2\pi i}\int_{-R}^{R}\frac{i}{c+it}dt,$$

$$= \frac{1}{2\pi}\int_{-R}^{R}\frac{c-it}{c^2+t^2}dt,$$

Since $\frac{t}{c^2+t^2}$ is odd, the imaginary part of the integral vanishes. As $\frac{1}{c^2+t^2}$ is even, we consider,

$$\frac{1}{\pi}\int_{0}^{R}\frac{c}{c^2+t^2}dt = \frac{1}{\pi}\int_{0}^{\frac{R}{c}}\frac{1}{1+u^2}du.$$

As R goes to infinity, the left hand side equals to $\frac{1}{\pi}\lim_{u\to\infty}\arctan u = \frac{1}{2}$. $\square$

Exercise 3.13. *Show that*

$$M(x) := \sum_{n \leq x} \mu(n) = O(x \exp(-c(\log x)^{\frac{1}{10}})),$$

for some constant $c > 0$.

Proof. Using Lemma 2.13, and recall that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

we get,

$$M(x) = \frac{1}{2\pi i} \int_{(c)} \frac{x^s}{s\zeta(s)} ds.$$

Thus we examine

$$\frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{x^s}{s\zeta(s)} ds,$$

and consider $R \to \infty$. □

Exercise 3.14. *Show that for any s with $\mathrm{Re}(s) = 1$, we have,*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

converges.

Solution. By the assumption, we set $s = 1 + it$ for $t \in \mathbb{R}$. Using Proposition 2.7, we have,

$$\sum_{n \leq N} \frac{\mu(n)}{n^{1+it}} = \frac{O(N \exp(-c(\log N)^{\frac{1}{10}}))}{N^{1+it}} + (1 + it) \int_1^x \frac{M(w)}{w^{2+it}} dw.$$

Examine the splitted integral,

$$\int_1^{\infty} \frac{M(w)}{w^{2+it}} dw - \int_N^{\infty} \frac{M(w)}{w^{2+it}} dw.$$

Ther first term conevrges as $\frac{M(w)}{w^{2+it}} = O(w^{-1-\delta})$ for some $\delta > 0$. Note that

$$\exp\left(-\frac{c}{2}(\log w)^{\frac{1}{10}}\right) \leq \exp\left(-\frac{c}{2}(\log w)^{\frac{1}{10}}\right).$$

Thus,

$$\left| \int_N^{\infty} \frac{M(w)}{w^{2+it}} dw \right| \leq \int_N^{\infty} \exp\left(-\frac{c}{2}(\log w)^{\frac{1}{10}}\right) \exp\left(-\frac{c}{2}(\log w)^{\frac{1}{10}}\right) \frac{dw}{w},$$

$$\leq c' \exp\left(-\frac{c}{2}(\log N)^{\frac{1}{10}}\right) \overset{N \to \infty}{\to} 0.$$

□

Exercise 3.15. *Show that*

$$\sum_{n \leq x} \frac{n}{\varphi(n)} << O(x).$$

Proof.

$$\sum_{n \leq x} \frac{n}{\varphi(n)} = \sum_{n \leq x} \frac{1}{\prod_{p|n} \left(1 - \frac{1}{p}\right)},$$

$$= \sum_{n \leq x} \sum_{\gamma(d)|n} \frac{1}{d},$$

$$\leq \sum_{\gamma(d) \leq x} \frac{1}{d} \frac{x}{\gamma(d)},$$

$$\leq x \prod_{p \leq x} \frac{1}{\left(1 - \frac{1}{p}\right)}.$$

$\square$

Exercise 3.16. *Show that*

$$\sum_{n \leq x} \frac{1}{\varphi(n)} << \log x.$$

Proof. Using the previous problem and Proposition 2.7, we get,

$$\sum_{n \leq x} \frac{n}{\varphi(n)} \frac{1}{n} \leq \frac{Cx}{x} + C \int_1^x \frac{t}{t^2} dt,$$

$$= C + C \log x.$$

$\square$

Exercise 3.17. *Prove that*

$$\sum_{n \leq x} \frac{1}{\varphi(n)} \sim c \log x.$$

Proof. Remains to show that $\log x << \sum_{n \leq x} \frac{1}{\varphi(n)}$. $\square$

Exercise 3.18. *Show that for a character $\chi \bmod q$ and $(n, q) = 1$, we have,*

$$\chi(n)\tau(\overline{\chi}) = \sum_{k=1}^{q} \overline{\chi}(n) e^{\frac{2\pi i k n}{q}}.$$

Solution. Note that we have $\chi(n) = \overline{\chi}(n)^{-1}$. And use

$$kn \equiv l \mod q \Rightarrow k \equiv ln^{-1} \mod q.$$

$\square$

Exercise 3.19. *Let $\chi \bmod q$ be a primitive character, then we have,*

$$L(1, \chi) = \sum_{n \le x} \frac{\chi(n)}{n} + O\left(\frac{\sqrt{q} \log q}{x}\right).$$

Proof. By Proposition 2.7,

$$L(1, \chi) = \int_1^\infty \frac{A(t)}{t^2} dt \Rightarrow \sum_{n > x} \frac{\chi(n)}{n} \le \int_x^\infty \frac{A(t)}{t^2} dt,$$

where

$$A(x) = \sum_{n \le x} \chi(n).$$

By Theorem 2.11, we have,

$$|A(x)| \le \sqrt{q} \log q.$$

Thus we obtain,

$$\int_x^\infty \frac{A(t)}{t^2} dt \le \frac{\sqrt{q} \log q}{x}.$$

$\square$

Exercise 3.20. *Show that*

$$\sum_{\chi \ne \chi_0} L(1, \chi) = \varphi(q) + O(\sqrt{q} \log q).$$

Proof. If χ is imprimitive, then use Theorem 2.11 to the primitive conductor of it, we obtain the same inequality. Thus

$$\sum_{\chi \ne \chi_0} L(1, \chi) = \sum_{n \le x} \left(\frac{\sum_{\chi \ne \chi_0} \chi(n)}{n} + O\left(\frac{\sqrt{q} \log q}{x}\right)\right).$$

Using Remark 2.12 and Theorem 2.10, furthermore choosing $x = q$ we obtain,

$$\sum_{\chi \ne \chi_0} L(1, \chi) = \varphi(q) \sum_{(n,q)=1} \frac{1}{n} - \sum_{n=1}^q \frac{1}{n} + O(\sqrt{q} \log q).$$

$\square$

Exercise 3.21. *Let $s \in \mathbb{C}$ be such that $\mathrm{Re}(s) = \sigma > 0$. Then for any x and a non-trivial character χ, show that*

$$L(s, \chi) = \sum_{n \le x} \frac{\chi(n)}{n^s} + O\left(\frac{|s| \sqrt{q} \log q}{\sigma x^\sigma}\right).$$

Solution. Using Proposition 2.7 and Theorem 2.11, we have,

$$\sum_{n>x} \frac{\chi(n)}{n^s} \leq |s| \int_x^\infty \frac{\sqrt{q}\log q}{t^{\sigma+1}} dt.$$

$\square$

Exercise 3.22. *For $\sigma > \frac{1}{2}$, show that*

$$\sum_{\chi \neq \chi_0} L(\sigma, \chi) = \varphi(q) + O(q^{\frac{3}{2}-\sigma}).$$

Solution. By previous exercises, by letting $x = q$, we have

$$\sum_{\chi \neq \chi_0} L(\sigma, \chi) = \varphi(q) + O(q^{1-\sigma}) + O(q^{\frac{3}{2}-\sigma}).$$

$\square$

Exercise 3.23. *For a prime p and a quadratic character χ, we have,*

$$\tau(\chi) = \begin{cases} \sqrt{p}, & (p \equiv 1 \mod p), \\ i\sqrt{p}, & (p \equiv 3 \mod p). \end{cases}$$

Proof. As $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic, there is a unique quadratic character χ for each p. Let $\zeta_p = e^{\frac{2\pi i}{p}}$. Then

$$\sum_{k=1}^{p-1} \zeta^k = -1.$$

Therefore,

$$\tau(\chi) - 1 = \sum_{k=1}^{p-1} (\chi(k) + 1)\zeta^k.$$

As $\mathbb{Z}/p\mathbb{Z}$ is a field, if an element has square roots then there are exactly two besides 0. And if k is a square $\chi(k) - 1 = 2$ otherwise 0. Therefore,

$$\tau(\chi) = 1 + \sum_{k=1}^{p-1} \zeta_p^{k^2}$$

Consider,

$$|\tau(\chi)|^2 = \sum_{k,l=0}^{p-1} \zeta_p^{k^2-l^2} = \sum_{a,b=0}^{p-1} \zeta_p^{ab} = p.$$

Remains to prove the parity of this. $\square$