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THE SOLUTION OF THE RIEMANN PROBLEM FOR A HYPERBOLIC SYSTEM OF CONSERVATION LAWS MODELING POLYMER FLOODING*

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Abstract. The global Riemann problem for a nonstrictly hyperbolic system of conservation laws modeling polymer flooding is solved. In particular, the system contains a term that models adsorption effects.

Key words. Riemann problem, polymer flooding, adsorption

AMS(MOS) subject classifications. 35L65, 76S05

1. Introduction. In this paper we solve the global Riemann problem for the mathematical model

$$\begin{aligned} &s_t + f(s, c)_x = 0, \\ &[sc + a(c)]_t + [f(s, c)c]_x = 0, \end{aligned} \tag{1.1}$$

where $t \in \mathbf{R}^+$, $x \in \mathbf{R}$, the state vector $(s, c) \in I \times I$ and $f: I \times I \to \mathbf{R}$ and $a: I \to \mathbf{R}$ are smooth functions. Here I denotes the unit interval $I = [0, 1]$. More precise assumptions on the functions f and a will be given in § 2.

Our results generalize the results of Isaacson [6], where the Riemann problem for (1.1) is solved with the term $a(c)$ neglected. In this simplified case the solution of the Riemann problem can also be derived from the analysis given by Keyfitz and Kranzer [7].

We note that when c is constant, (1.1) reduces to the single equation

$$s_t + f(s)_x = 0, \tag{1.2}$$

which in the petroleum literature is known as the Buckley–Leverett equation.

The model (1.1) arises in connection with enhanced oil recovery, for example when oil is displaced in a porous rock by water containing dissolved polymer. The variable s is the saturation of the mixture of water and polymer, which we call the aqueous phase. The variable c is the concentration of polymer in the aqueous phase. Furthermore, f describes the fractional flow of the aqueous phase, which is assumed to be immiscible with oil. The function $a(c)$ models adsorption of the polymer on rock.

Some other processes governed by (1.1) are discussed by Pope in [9], where detailed physical assumptions for the model are also discussed.

The derivation of (1.1) is based on material balance considerations. We assume that the fluids and the rock are incompressible, and that volumes do not change when polymer is dissolved in water. Assuming one-dimensional flow in a homogeneous medium, the mass conservation of water, oil, and polymer, respectively, can be formulated as follows:

$$\phi s_t + v_x = 0, \tag{1.3a}$$

$$\phi s_t^0 + v_x^0 = 0, \tag{1.3b}$$

$$\phi[sc + a(c)]_t + [vc]_x = 0, \tag{1.3c}$$

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where ϕ is the rock porosity, s^0 is the oil saturation and v and v^0 denote the volumetric flow rates of the aqueous phase and oil, respectively. Neglecting gravity, capillarity and dispersion, we get v and v^0 by Darcy's law as follows:

$$v = -\lambda p_x, \qquad \lambda = -\frac{Kk}{\mu}, \tag{1.4a}$$

$$v^0 = -\lambda^0 p_x, \qquad \lambda^0 = -\frac{Kk^0}{\mu^0}. \tag{1.4b}$$

Here, p is the fluid pressure, K is the absolute permeability of the rock, $k = k(s, c)$ and $k^0 = k^0(s)$ are the relative permeabilities of the aqueous phase and oil, respectively, and μ, μ^0 are the corresponding viscosities.

Since $s^0 + s \equiv 1$, summation of (1.3a) and (1.3b) gives

$$v^T \equiv v^0 + v = \text{constant} \tag{1.5}$$

where the value of total volumetric flow rate v^T is determined from the boundary conditions. By elimination of the pressure p using (1.4a), (1.4b), and (1.5), the equations (1.3a) and (1.3b) can be rewritten in the form

$$\phi s_t + v^T f_x = 0, \tag{1.6a}$$

$$\phi[sc + a(c)]_t + v^T[fc]_x = 0, \tag{1.6b}$$

which together with (1.5) constitutes the model. Here $f = f(s, c)$ denotes the fractional flow function given by

$$f = \frac{\lambda}{\lambda + \lambda^0}. \tag{1.7}$$

The dependence on s is inherited from the relative permeability functions k and k^0 while the dependence on c is primarily introduced through the viscosity $\mu = \mu(c)$ of the aqueous phase.

If L denotes the length of the medium, a simple coordinate transformation

$$x' = \frac{x}{L} \quad \text{and} \quad t' = \frac{v^T t}{\phi L}$$

results in (1.1), when the primes on x', t' are dropped.

The model (1.1) is an example of a system of hyperbolic conservation laws. The main result of this paper is the construction of a unique global solution of the Riemann problem for the system (1.1); i.e., we construct a weak solution of the pure initial value problem for (1.1) with initial condition

$$(s, c)(x, 0) = \begin{cases} (s^L, c^L) & \text{if } x < 0, \\ (s^R, c^R) & \text{if } x > 0, \end{cases} \tag{1.8}$$

where the left and right states (s^L, c^L) and (s^R, c^R) in $I \times I$ are arbitrary. In order to distinguish the physically meaningful weak solutions we will also require that the solution satisfies an "entropy condition." This condition will be derived by demanding that the solution is evolutionary; i.e., any discontinuity of the solution is a limit of smooth solutions of associated "viscosity systems." The solution of the Riemann problem in general depends only on the ratio x/t and it will be constructed by connecting constant states, smooth solutions (or rarefaction waves) and discontinuous solutions (or shock waves).

In addition to being analytical solutions of the system, solutions of the Riemann problem can also be used as building blocks for the construction of numerical methods for (1.1). Examples of this are the Random Choice Method [1], [2], [4], and Godunov's method [5]. Unfortunately, for many hyperbolic systems no global solution of the Riemann problem has yet been found. A general theory for local existence and uniqueness of solution is described in [4], [8], or [11] under the condition of strict hyperbolicity of the system. However, this condition will not be satisfied in the analysis of (1.1) given below.

In the case of the single Buckley-Leverett equation (1.2) (i.e., when c is constant) the solution of the Riemann problem is well known (cf. [3], [8] and § 4 below for a precise formulation of the entropy condition in this case). Assume for example that $s^L < s^R$ and let $g(s) = (d/ds)f_L(s)$, where f_L is the lower convex envelope of f with respect to the interval $[s^L, s^R]$. The unique solution of the Riemann problem is then given by

$$s(x,t) = \begin{cases} s^L & \text{if } x/t < g(s^L), \\ s & \text{if } x/t = g(s), \\ s^R & \text{if } x/t > g(s^R), \end{cases} \tag{1.9}$$

where we adopt the convention that if $g(s) = \sigma$ on a maximal interval (s^1, s^2), with $s^1 < s^2$, then this corresponds to a discontinuity at $x = \sigma t$ with $s(\sigma t-, t) = s^1$ and $s(\sigma t+, t) = s^2$. Similarly, if $s^R < s^L$ the unique solution of the Riemann problem is again given by (1.9), where in this case $g(s) = (d/ds)f_U(s)$ is the derivative of the upper concave envelope f_U of f with respect to $[s^R, s^L]$. The solution of the Riemann problem for (1.2) given above will be a fundamental part of the solution of the Riemann problem for the system (1.1) constructed in this paper.

The main difference between the present paper and [6] is that the adsorption term $a(c)$ has been included in the model here. The effect of this term is that the linearly degenerate characteristic field appearing in the analysis in [6] (and [7]) is replaced by a nondegenerate field. The contact discontinuities in the Riemann solution given in [6] will therefore be replaced by either proper rarefaction waves or proper shock waves. As a consequence, both the state space solution and the x, t-space solution of the Riemann problem are unique. This is in contrast to [6], where the solution is unique in x, t-space, but not in state space. Also, in [6] the Lax entropy inequalities were used as the entropy condition for the system. In this paper we derive entropy conditions from traveling wave analysis and we conclude that, in general, the Lax inequalities are not the correct entropy condition for the system. In particular, we derive an admissible shock wave where both characteristics on both sides of the shock enter the shock. Such shock waves are referred to as overcompressive shocks by Schaeffer and Shearer [10]. However, this wave cannot be joined to any other wave in a Riemann solution.

The construction of the solution of the Riemann problem for the model (1.1) given below shows that if the solution is considered pointwise, it is discontinuous with respect to the left and right states for certain critical values of the left and right states. However, the solution is continuous in L^1 norm (cf. Example 8.2). This property is similar to properties of Riemann solutions of the nonstrictly hyperbolic systems studied in, e.g., [6], [7], and [10].

The precise assumptions on the system (1.1) are stated in § 2. The simple rarefaction waves are derived in § 3, while § 4 is devoted to shock waves and entropy conditions. In § 5 we formulate the general Riemann problem in terms of rarefaction waves and shock waves and state the main result of the paper. The proof is given in §§ 6 and 7.

In § 8 we present some numerical experiments. One of the purposes of these experiments is to show that the behavior of the exact solution of the Riemann problem is not easily detected from calculations done by standard finite difference schemes.

2. A precise formulation of the model. We recall that our model for the polymer process is the following 2×2 system of conservation laws:

$$\begin{aligned} &s_t+f(s,c)_x=0,\\ &(sc+a(c))_t+(cf(s,c))_x=0, \end{aligned} \tag{2.1}$$

where the unknown functions are $s=s(x,t)$ and $c=c(x,t)$. Throughout the paper we will assume that the real-valued function $f=f(s,c)$ is a smooth function for $(s,c)\in I\times I$, where $I=[0,1]$, with the following properties (cf. Fig. 2.1):

(i) $f(0,c)\equiv 0$, $f(1,c)\equiv 1$;

(ii) $f_s(s,c)>0$ for $0<s<1$, $0\leqq c\leqq 1$;

(iii) $f_c(s,c)<0$ for $0<s<1$, $0\leqq c\leqq 1$;

(iv) For each $c\in I$, $f(\cdot,c)$ has a unique point of inflection, $s^I=s^I(c)\in(0,1)$, such that $f_{ss}(s,c)>0$ for $0<s<s^I$ and $f_{ss}(s,c)<0$ for $s^I<s<1$.

The function $a=a(c)$, which models the adsorption effects of the process, is assumed to be a smooth function of $c\in I$ such that (cf. Fig. 2.2):

(i) $a(0)=0$;

(ii) $h(c)=(da/dc)(c)>0$ for $0<c<1$;

(iii) $(dh/dc)(c)=(d^2a/dc^2)(c)<0$ for $0<c<1$.

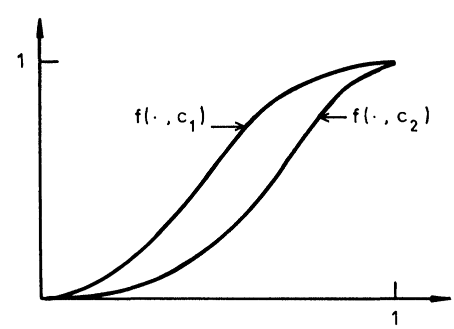


FIG. 2.1. $c_1<c_2$.

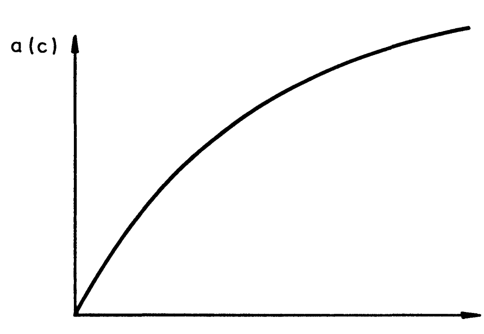


FIG. 2.2

Let u denote the state vector $u=(s, c)$. The system (2.1) may be reformulated in the form

$$u_t + A(u)u_x = 0,$$

where $A(u)=A(s, c)$ is the upper triangular 2×2 matrix

$$A(s, c) = \begin{bmatrix} f_s(s, c) & f_c(s, c) \\ 0 & \dfrac{f(s, c)}{s+h(c)} \end{bmatrix}. \tag{2.2}$$

The eigenvalues of A are $\lambda^s = f_s$ and $\lambda^c = f/(s+h)$, with corresponding eigenvectors $e^s=(1, 0)$, $e^c = (f_c, \lambda^c - \lambda^s)$ if $0<s<1$ and $e^c=(0,1)$ if $s=0, 1$.

We observe (cf. Figs. 2.3, 2.4) that for each $c\in I$ there is at most one $s^T = s^T(c)\in I$ such that

$$\lambda^c(s^T, c) = \lambda^s(s^T, c). \tag{2.3}$$

Throughout the paper we will assume that there exists a unique $c^T \in I$ such that (2.3) has a unique solution $s^T(c)$ for $0\leqq c \leqq c^T$ and that (2.3) has no solution for $c^T < c \leqq 1$ (cf. Figs. 2.5, 2.6). We let T denote the transition curve

$$T = \{(s, c)\,|\,0\leqq c\leqq c^T, s = s^T(c)\}$$

and $\mathscr{L}$ and $\mathscr{R}$ the regions

$$\mathscr{L} = \{(s, c)\in I\times I\,|\,\lambda^s > \lambda^c\}, \qquad \mathscr{R} = \{(s, c)\in I\times I\,|\,\lambda^c > \lambda^s\}.$$

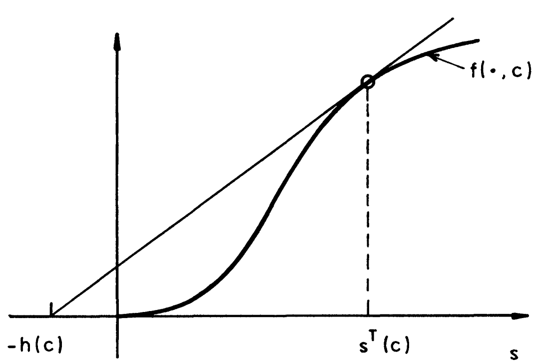


FIG. 2.3. $s^T(c)$ *exists.*

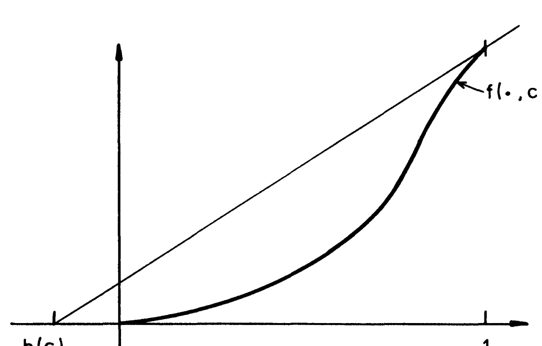


FIG. 2.4. *No* $s^T(c)$.

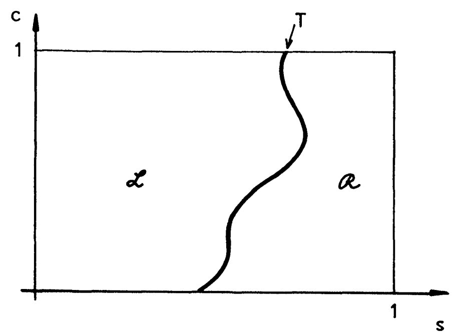


FIG. 2.5. $c^T = 1$.

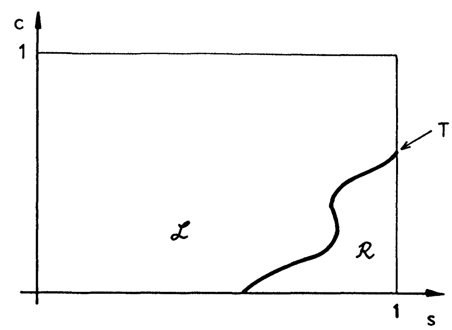


FIG. 2.6. $c^T < 1$.

We observe that if $(s, c) \in T$ and $0 < s < 1$, then $\lambda^s = \lambda^c$ and the two eigenvectors e^s and e^c become linearly dependent. Hence, in this case the matrix A given by (2.2) is not diagonalizable.

3. Rarefaction waves. The purpose of this section is to determine the simple rarefaction waves of the system (1.1). Hence, for two given states $u^L = (s^L, c^L)$ and $u^R = (s^R, c^R)$ we derive possible continuous solutions of the pure initial value problem for (1.1) with initial data

$$u(x, 0) = \begin{cases} u^L & \text{if } x < 0, \\ u^R & \text{if } x > 0. \end{cases} \tag{3.1}$$

Let λ be an eigenvalue of the matrix A given by (2.2) with corresponding eigenvector e. The simple rarefaction waves are continuous solutions of (1.1) and (3.1) of the form

$$u(x, t) = v(x/t),$$

where v corresponds to an integral curve of the vector field e. More precisely,

$$u(x, t) = \begin{cases} u^L & \text{if } x/t < \lambda(u^L), \\ v & \text{if } x/t = \lambda(v), \\ u^R & \text{if } x/t > \lambda(u^R), \end{cases} \tag{3.2}$$

where v is an integral curve of the vector field e connecting the states u^L and u^R with the additional property that the eigenvalue λ is increasing from u^L to u^R. Since the matrix A has two eigenvalues, λ^s and λ^c, there are two possible rarefaction curves through any given state u^L.

s-rarefaction waves. If $\lambda=\lambda^s$ and $e=e^s=(1,0)$ the integral curves of e are the curves where c is constant. Hence, a simple rarefaction wave of the form (3.2) exists if $c^L=c^R$ and if $\lambda^s=f_s(s,c^L)$ is increasing from s^L to s^R. This, of course, corresponds to a simple rarefaction wave of the Buckley-Leverett equation (1.2) with $f(s)=f(s,c^L)$. Such rarefaction waves will be referred to as s-rarefaction waves.

c-rarefaction waves. Next we consider the case when $\lambda=\lambda^c$. First assume that $0<s<1$. In this case $f_c<0$, the eigenvector e^c can be taken to be $(f_c, \lambda^c-\lambda^s)$, and the rarefaction curves are determined by the differential equation

$$f_c\frac{dc}{ds}=\lambda^c-\lambda^s, \tag{3.3}$$

where $c=c(s)$. Hence, $dc/ds>0$ in region $\mathscr{L}$, $dc/ds<0$ in $\mathscr{R}$ and $dc/ds=0$ on the transition curve T. Furthermore, by differentiating (3.3) with respect to s we also obtain

$$f_c\frac{d^2c}{ds^2}=-f_{ss}$$

when $(s,c(s))\in T$. Hence, $d^2c/ds^2<0$ on T. As a consequence, any rarefaction curve can at most intersect the transition curve T in one point (cf. Fig. 3.1).

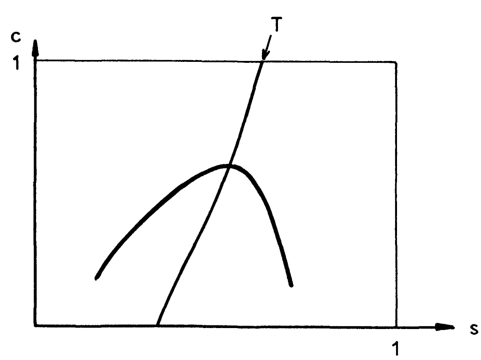


FIG. 3.1. *Rarefaction curve.*

Next consider $g(s)=\lambda^c(s,c(s))$, where $c(s)$ satisfies (3.3). A straightforward differentiation shows that

$$\frac{dg}{ds}=\frac{(f_s+f_c(dc/ds))(s+h)-f(1+(dh/dc)(dc/ds))}{(s+h)^2}$$

or

$$\frac{dg}{ds}(s)=-\frac{(dh/dc)(c(s))(dc/ds)(s)}{(s+h(c(s)))^2}f(s,c(s)).$$

Since $(dh/dc)(c)<0$ the eigenvalue $\lambda^c(s,c(s))$ is an increasing function of s in $\mathscr{L}$ and decreasing in $\mathscr{R}$. Hence, a simple rarefaction wave of the form (3.2) exists if and only if there is an integral curve of (3.3) connecting u^L and u^R with the additional property that c is increasing all the way from u^L and u^R (cf. Fig. 3.2). In particular this implies that such a simple c-rarefaction wave never exists when u^L and u^R are located on opposite sides of the transition curve T.

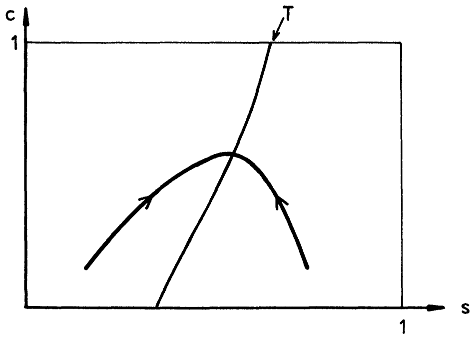


FIG. 3.2. *Rarefaction waves for* $0<s<1$.

Finally, consider the case when $s=0$ or $s=1$. Then $f_c=0$ and the eigenvector e^c can be taken to be $(0, 1)$. Hence, the rarefaction curves are given by $s\equiv 0$ or $s\equiv 1$. If $s\equiv 1$ the eigenvalue λ^c is given by

$$\lambda^c(1, c)=\frac{1}{1+h(c)},$$

which is a strictly increasing function of c. Therefore, if $c^L<c^R$ this corresponds to a simple rarefaction wave. Particularly, if $c^T<1$ (cf. § 2), such a c-rarefaction curve can connect states that are located on opposite sides of the transition curve T. If $s\equiv 0$ the eigenvalue $\lambda^c\equiv 0$. Hence, these "rarefaction curves" correspond to contact discontinuities with speed zero. In order to solve the general Riemann-problem for arbitrary states in $I\times I$ we will allow such discontinuities for arbitrary c^L and c^R in I. We remark however that these solutions are in some sense nonphysical, since states with $s=0$ and $c>0$ have no physical interpretation. We will refer to these discontinuities as c-rarefaction waves if $c^L<c^R$ and as c-shock waves if $c^L>c^R$ (cf. § 4).

4. Shock waves and entropy conditions. The purpose of this section is to determine the shock waves of the system (1.1); i.e., for given states $u^L=(s^L, c^L)$ and $u^R=(s^R, c^R)$ we derive possible discontinuous weak solutions of the system (1.1) of the form

$$u(x, t)=\begin{cases} u^L & \text{if } x/t<\sigma, \\ u^R & \text{if } x/t>\sigma. \end{cases} \tag{4.1}$$

Here σ denotes the shock speed. In order to distinguish the physically meaningful weak solutions of (1.1) we will also require that u satisfies an "entropy condition." This condition will be derived by requiring the shock waves to be evolutionary; i.e., any shock wave must be a limit of traveling wave solutions of associated "viscosity systems."

Any weak solution of (1.1) of the form (4.1) has to satisfy the Rankine–Hugoniot condition given by

$$\begin{aligned} &f(s^R, c^R)-f(s^L, c^L)=\sigma(s^R-s^L), \\ &c^Rf(s^R, c^R)-c^Lf(s^L, c^L)=\sigma(s^Rc^R+a(c^R)-s^Lc^L-a(c^L)). \end{aligned} \tag{4.2}$$

Let us first observe that if $c^R=c^L$ then the two equations of (4.2) are identical. Hence, in this case (4.2) reduces to

$$f(s^R, c^L)-f(s^L, c^L)=\sigma(s^R-s^L), \tag{4.3}$$

which is the Rankine–Hugoniot condition for the Buckley–Leverett equation (1.2) with $f(s)=f(s, c^L)$. From the theory for this equation (cf. [11] and § 1) shock waves of the form (4.1), with $c^L=c^R$ and with s^L and s^R satisfying (4.3), are said to satisfy an entropy condition if and only if

$$[f(s)-f(s^L)-\sigma(s-s^L)]\ \mathrm{sign}\,(s^R-s^L)\geqq 0 \tag{4.4}$$

for any s between s^L and s^R. These shock waves will be referred to as *s-shock waves* for the system (1.1). (We remark that the entropy condition (4.4) can be derived, in the same way as below, by requiring the shock wave to be evolutionary.)

The entropy condition (4.4) implies that

$$\lambda^s(u^L)>\sigma>\lambda^s(u^R),$$

and from the definition of λ^c it follows that either

$$\lambda^c(u^L),\ \lambda^c(u^R)\geqq\sigma$$

or

$$\lambda^c(u^L),\ \lambda^c(u^R)\leqq\sigma.$$

Hence, since exactly one characteristic enters the shock on both sides, the s-shock waves satisfy the celebrated Lax entropy condition (cf. [8] and [11]).

In the rest of this section we shall determine the shock waves with $c^L\neq c^R$. These shock waves will be referred to as *c-shock waves.*

By applying the first equation of (4.2), the second equation of (4.2) can be written in the form

$$(c^R-c^L)f(s^L, c^L)=\sigma(c^R-c^L)s^L+\sigma(a(c^R)-a(c^L)). \tag{4.5}$$

By introducing the quantity

$$h_L(c)=\begin{cases}\dfrac{a(c)-a(c^L)}{c-c^L} & \text{if } c\neq c_L,\\ h(c) & \text{if } c=c_L,\end{cases} \tag{4.6}$$

(4.5) can again be written as

$$\sigma=\frac{f(s^L, c^L)}{s^L+h_L(c_R)}.$$

By applying this final equality in the first equation of (4.2) we also obtain that

$$\sigma(s^R+h_L(c^R))=\sigma(s^L+h_L(c^R))+f(s^R, c^R)-f(s^L, c^L)=f(s^R, c^R).$$

Hence, when $c^L\neq c^R$ the Rankine–Hugoniot condition (4.2) can be written in the form

$$\frac{f(s^R, c^R)}{s^R+h_L(c^R)}=\frac{f(s^L, c^L)}{s^L+h_L(c^R)}=\sigma. \tag{4.7}$$

If $s^L=s^R=0$ this condition is satisfied with $\sigma=0$. As we have already seen in § 3 this corresponds to a contact discontinuity with speed zero. In the rest of the discussion we shall therefore assume that $\sigma>0$.

We observe that the value $h_L(c^R)$ is determined from values of c^L and c^R and that, if s^L, c^L and c^R are given, there are at most two values of s^R that satisfy the Rankine–Hugoniot condition (4.7) (cf. Fig. 4.1).

In order to distinguish the physically meaningful weak solutions of (1.1) we shall require that a shock wave be evolutionary. Hence, for any $\varepsilon>0$ we consider the perturbed system

$$\begin{aligned} &s_t+f(s, c)_x=\varepsilon s_{xx},\\ &(sc+a(c))_t+(cf(s, c))_x=\varepsilon(sc+a(c))_{xx}. \end{aligned} \tag{4.8}$$

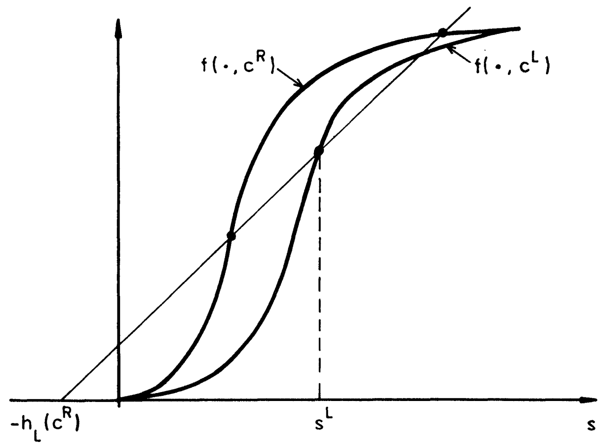


FIG. 4.1. $c^L > c^R$.

The entropy condition consists of requiring the shock wave to be a pointwise limit of traveling wave solutions of (4.8). In the same way as was done in, e.g., [7] or [11, Chap. 24], we obtain that two states $u^L = (s^L, c^L)$ and $u^R = (s^R, c^R)$, which satisfy the Rankine–Hugoniot condition (4.7) with shock speed σ, correspond to an evolutionary shock wave if and only if the 2×2 system

$$\begin{aligned} &\frac{ds}{d\xi} = f(s, c) - \sigma s - (f(s^L, c^L) - \sigma s^L), \\ (4.9) \qquad &\frac{d}{d\xi}(sc + a(c)) = cf(s, c) - \sigma(sc + a(c)) - (c^L f(s^L, c^L) - \sigma(s^L c^L + a(c^L))) \end{aligned}$$

has a solution $(s(\xi), c(\xi))$ with $(s(-\infty), c(-\infty)) = u^L$ and $(s(+\infty), c(+\infty)) = u^R$. By applying (4.7) and by performing the differentiation in the second equation of (4.9) we rewrite the system in the form

$$\begin{aligned} &\frac{ds}{d\xi} = f(s, c) - \sigma(s + h_L(c^R)), \\ (4.10) \qquad &(s + h(c)) \frac{dc}{d\xi} = \sigma(c - c^L)(h_L(c^R) - h_L(c)), \end{aligned}$$

where $h_L(c)$ is defined by (4.6).

We observe that the Rankine–Hugoniot condition (4.7) implies that $u^L = (s^L, c^L)$ and $u^R = (s^R, c^R)$ are equilibrium points of (4.10). Since $\sigma > 0$ the second equation of (4.10) implies, in particular, that $dc/d\xi < 0$ for any value of c strictly between c^L and c^R. Therefore $c^L > c^R$. For the rest of this discussion we therefore assume that $c^L > c^R$.

Let us first assume that u^R is a state such that

$$\lambda^s(u^R) < \sigma.$$

Then there are at most two possible values of s^L, s^L_- and s^L_+ such that the Rankine–Hugoniot condition (4.7) holds where

$$\lambda^s(s^L_-, c^L) \geqq \sigma \quad \text{and} \quad \lambda^s(s^L_+, c^L) \leqq \sigma$$

(cf. Fig. 4.2).

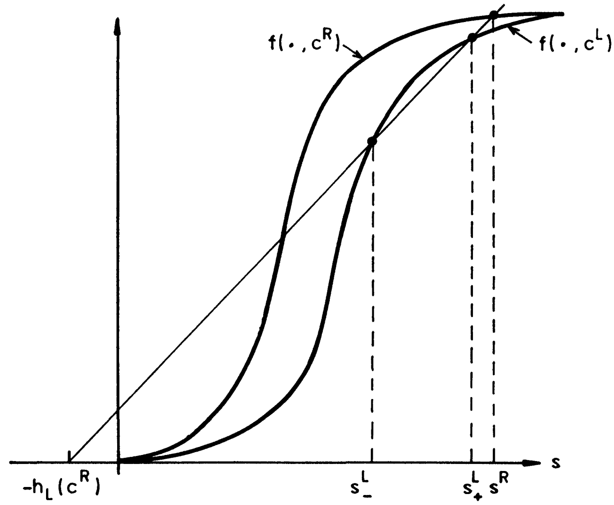


FIG. 4.2

Similarly, for any $c \in [c^R, c^L)$ let $s_-(c)$ and $s_+(c)$ be the two values determined by

$$\frac{f(s_-(c), c)}{s_-(c)+h_L(c^R)} = \sigma = \frac{f(s_+(c), c)}{s_+(c)+h_L(c^R)}$$

and

$$\lambda^s(s_-(c), c) > \sigma > \lambda^s(s_+(c), c).$$

Then $ds/d\xi < 0$ for $0 \leqq s < s_-(c)$ and $s_+(c) < s \leqq 1$, while $ds/d\xi > 0$ for $s_-(c) < s < s_+(c)$ (cf. Fig. 4.3).

It is now easy to see that any trajectory that passes through a point $(s_-(c), c)$, for some $c \in (c^R, c^L)$ has the properties

$$(s(-\infty), c(-\infty)) = (s_-^L, c^L) \quad \text{and} \quad (s(+\infty), c(+\infty)) = (s^R, c^R).$$

Furthermore, any trajectory through $(s_+(c), c)$, for some $c \in (c^R, c^L)$, is such that

$$(s(-\infty), c(-\infty)) = (+\infty, c^L) \quad \text{and} \quad (s(+\infty), c(+\infty)) = (s^R, c^R).$$

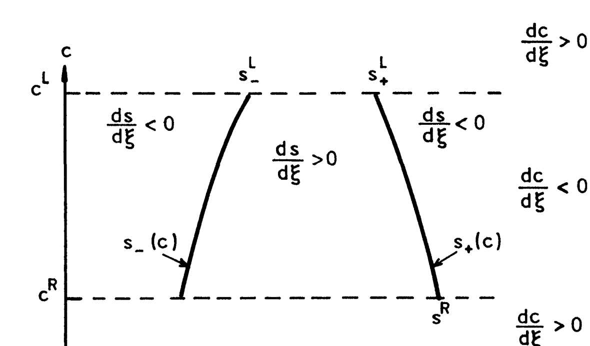


FIG. 4.3

Hence, by continuity, there must be at least one trajectory that separates the two classes and such a trajectory passes from (s_+^L, c^L) to u^R. Therefore we have seen that there exist trajectories from both the points (s_-^L, c^L) and (s_+^L, c^L) to u^R when $\lambda^s(u^R) < \sigma$.

Next consider the case when u^R satisfies

$$\lambda^s(u^R) > \sigma$$

(i.e., $s^R = s_-(c^R)$). By an argument similar to the one above we obtain the existence of a trajectory from (s_-^L, c^L) to u^R. However, if $u^L = (s_+^L, c^L)$ (i.e., $\lambda^s(u^L) < \sigma$) there exists no trajectory from u^L to u^R.

As a conclusion we obtain that two states $u^L = (s^L, c^L)$ and $u^R = (s^R, c^R)$, with $c^L \neq c^R$, correspond to a c-shock wave of speed σ if they satisfy the Rankine–Hugoniot condition (4.7) and if they satisfy the entropy conditions

$$c^L > c^R \tag{4.11}$$

and

$$\lambda^s(u^R) < \sigma \quad \text{or} \quad \lambda^s(u^L), \lambda^s(u^R) \geqq \sigma. \tag{4.12}$$

Since the function $h(c)$ is strictly decreasing, condition (4.7) implies that (4.11) is equivalent to the eigenvalue/shock speed relation

$$\lambda^c(u^L) > \sigma > \lambda^c(u^R).$$

Therefore, if

$$\lambda^s(u^L), \lambda^c(u^L) > \sigma > \lambda^s(u^R), \lambda^c(u^R),$$

we allow a shock wave where both characteristics on both sides of the shock enter the shock. This shock, which corresponds to $s^L = s_-^L$ and $s^R = s_+(c^R)$, is therefore a shock wave which does not satisfy the Lax entropy condition. We recall, that in the terminology of [10], this is an example of an overcompressive shock. In § 7 below we will discover that this shock cannot be joined by any other wave in a Riemann solution.

5. The Riemann problem. The purpose of the rest of this paper is to construct a unique global solution of the Riemann problem for the system (1.1); i.e., for arbitrary states $u^L, u^R \in I \times I$ we derive a solution of the pure initial value problem for (1.1) with initial condition

$$u(x, 0) = \begin{cases} u^L & \text{if } x < 0, \\ u^R & \text{if } x > 0, \end{cases} \tag{5.1}$$

which consists of a composition of a finite number of simple rarefaction waves and shock waves as described above.

If a left state u^1 can be connected to a right state u^2 by a simple rarefaction wave then the *initial speed* of the wave is $\lambda(u^1)$ and the *final speed* of the wave is $\lambda(u^2)$, where λ is the eigenvalue corresponding to the wave. The initial speed and the final speed of a simple shock wave is defined to be the shock speed σ.

By a *c-wave* we mean a simple c-rarefaction wave or a c-shock wave, while an *s-wave* is any composition of simple s-rarefaction waves and s-shock waves that corresponds to a solution of the Buckley–Leverett equation (1.2) with $f(s) = f(s, c)$ for some $c \in I$. We recall that for any left state $u^1 = (s^1, c)$ and right state $u^2 = (s^2, c)$, where $c \in I$, there is a unique s-wave that connects the two states (cf. the discussion in § 1).

Following [6] we will adopt the notation that $u^1 \xrightarrow{c} u^2$ means that the left state u^1 can be connected to the right state u^2 by a c-wave and we refer to this Riemann solution as $u^1 \xrightarrow{c} u^2$. The notation $u^1 \xrightarrow{s} u^2$ has the analogous meaning for s-waves.

Consider two waves $u^1 \xrightarrow{a} u^2$ and $u^2 \xrightarrow{b} u^3$. Let v_f^a denote the final wave speed of the a-wave and v_i^b the initial wave speed of the b-wave. The two waves are said to be *compatible* if they can be composed to solve the Riemann problem with left state u^1 and right state u^3. Hence the two waves are compatible if and only if

$$v_f^a \leqq v_i^b, \tag{5.2}$$

where we require a strict inequality if both v_f^a and v_i^b are shock speeds. Any solution of the Riemann problem will consist of a sequence of compatible s-waves and c-waves that connects the given left state u^L with the given right state u^R.

The main purpose of this paper is to give a constructive proof of the following existence/uniqueness theorem.

THEOREM 5.1. *For arbitrary states u^L, $u^R \in I \times I$ there exists a unique finite sequence of compatible s-waves and c-waves that generates a solution of the Riemann problem with left state u^L and right state u^R.*

The rest of this paper is devoted to the proof of Theorem 5.1. The following lemma will guarantee that the solution of the Riemann problem is monotone with respect to c; i.e., the solution $u=(s, c)$ of the Riemann problem has the property that the function $c(x, t)$ is a monotone function of x for any $t \geqq 0$.

LEMMA 5.1. *Assume that the three waves*

$$u^L \xrightarrow{c_1} u^1 \xrightarrow{s} u^2 \xrightarrow{c_2} u^R$$

are compatible. Then both the c-waves are rarefaction waves.

Proof. Since the three waves are all compatible we obtain from (5.2) that

$$v_f^1 \leqq v_i^s \leqq v_f^s \leqq v_i^2, \tag{5.3}$$

where v_i^s and v_f^s denote the initial and final speed of the s-wave, respectively, v_f^1 is the final speed of the c_1-wave and v_i^2 is the initial speed of the c_2-wave.

Let $u^1=(s^1, c)$ and $u^2=(s^2, c)$ for a suitable $c \in I$ and define

$$\alpha = \frac{f(s^1, c) - f(s^2, c)}{s^1 - s^2}.$$

From the structure of the s-waves (cf. § 1) it follows that

$$v_i^s \leqq \alpha \leqq v_f^s.$$

Therefore, (5.3) implies that

$$v_f^1 \leqq \alpha \leqq v_i^2. \tag{5.4}$$

From §§ 3 and 4 we recall that v_f^1 and v_i^2 are of the form

$$v_f^1 = \frac{f(s^1, c)}{s^1 + h_1} \quad \text{and} \quad v_i^2 = \frac{f(s^2, c)}{s^2 + h_2},$$

where h_1, $h_2 > 0$.

By applying this in (5.4) the inequality can be rewritten in the form

$$h_2 \leqq \frac{f(s^1, c)}{\alpha} - s^1 \leqq h_1. \tag{5.5}$$

However, from (5.5) we immediately obtain the desired result. Assume for example that the c_1-wave is a shock wave and that the c_2-wave is a rarefaction wave. If $s^1=0$ (i.e., the c_1-wave is a contact discontinuity), $h_2=h(c)>0$ while (5.5) implies that $h_2 \leqq 0$. We can therefore assume that $s^1>0$. In this case

$$h_1=h_L(c)=\frac{a(c)-a(c^L)}{c-c^L} \quad \text{and} \quad h_2=h(c),$$

which implies that $h_1<h_2$ since $c<c^L$ and h is strictly decreasing. But this contradicts (5.5). Similar arguments also show that the compositions where the c_1-wave is a rarefaction wave and the c_2-wave is a shock wave or where both c-waves are shock-waves are incompatible. Hence, the only possibility is that both the c-waves are rarefaction waves. In this case $h_1=h_2=h(c)$. □

Let $u^L=(s^L, c^L)$ and $u^R=(s^R, c^R)$ denote the left and right state, respectively, of the Riemann problem. The lemma above immediately implies that if $c^L<c^R$, any solution of the Riemann problem will be composed of s-waves and c-rarefaction waves and, if $c^L>c^R$, any solution will be composed of s-waves and a single c-shock wave. Furthermore, if $c^L=c^R$, any solution will consist of a single s-wave. Hence from the theory of the Buckley–Leverett equation (1.2) (cf. § 1) there is a unique solution of the Riemann problem for the system (1.1) when $c^L=c^R$. It is therefore enough to prove Theorem 5.1 when $c^L \neq c^R$.

In § 6 below we treat the case where $c^L<c^R$, while the case $c^L>c^R$ is covered in § 7.

6. The case $c^L<c^R$. Throughout this section we shall consider the Riemann problem for (1.1) with $c^L<c^R$. In this case the purpose is to prove Theorem 5.1.

For any state $u=(s, c)\in \mathcal{R}\cup T$ define the critical value $s^K=s^K(u)$ to be the unique value of s^K such that $(s^K, c)\in \mathcal{L}\cup T$ with the property (cf. Fig. 6.1)

$$\lambda^c(s^K, c)=\lambda^c(u). \tag{6.1}$$

Similarly, if $u=(s, c)\in \mathcal{L}$ let $s^K=s^K(u)$ be the unique s^K such that either $(s^K, c)\in \mathcal{R}$ and that (6.1) holds or, if no such s^K exists, $s^K=\infty$.

As we have already observed above, any solution of the Riemann problem is composed of s-waves and c-rarefaction waves. We first determine the compatible pairs of waves.

LEMMA 6.1. *Assume that* $c^L<c^R$.

(i) *The two waves*

$$u^L \xrightarrow{c} u^M \xrightarrow{s} u^R$$

are compatible if and only if $u^M \in \mathcal{L}\cup T$ *and* $0 \leqq s^R \leqq s^K(u^M)$.

(ii) *The two waves*

$$u^L \xrightarrow{s} u^M \xrightarrow{c} u^R$$

are compatible if and only if $u^M \in \mathcal{R}\cup T$ *and* $s^K(u^M) \leqq s^L \leqq 1$.

Proof. Consider part (i) above. The two waves are compatible if and only if

$$\lambda^c(u^M) \leqq v_i^s,$$

where v_i^s is the initial speed of the s-wave. Since $v_i^s \leqq \lambda^s(u^M)$ we therefore derive that $u^M \in \mathcal{L}\cup T$. Furthermore, from the structure of the s-waves it follows that $v_i^s \geqq \lambda^c(u^M)$ if and only if $0 \leqq s^R \leqq s^K(u^M)$.

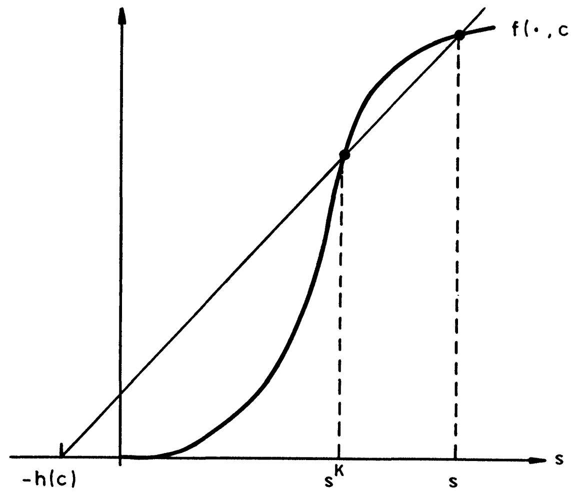


FIG. 6.1

Part (ii) follows by similar arguments. □

The above lemma implies that the three waves

$$u^L \xrightarrow{c_1} u^1 \xrightarrow{s} u^2 \xrightarrow{c_2} u^R \tag{6.2}$$

are compatible if and only if $u^1 \in \mathscr{L}$ and $u^2 \in \mathscr{R}$ with $s^K(u^1) = s^2$ (and $s^K(u^2) = s^1$). Hence, any solution of the Riemann problem will be composed of at most two c-rarefaction waves.

We are now in a position to construct the solution of the Riemann problem for arbitrary states u^L and u^R in $I \times I$. First we consider the case when $u^L \in \mathscr{R} \cup T$.

LEMMA 6.2. *Assume that $c^L < c^R$ and $u^L \in \mathscr{R} \cup T$. Then there exists a unique solution of the Riemann problem consisting of at most four states separated by s-waves and c-rarefaction waves.*

Proof. If $u^R \in \mathscr{R} \cup T$ the solution has the form

$$u^L \xrightarrow{s} u^1 \xrightarrow{c} u^R,$$

where $u^1 = (s^1, c^L) \in \mathscr{R}$ is determined by the c-rarefaction curve which connects u^R to the line $c = c^L$ (cf. Fig. 6.2). (Here and below it might, of course, occur that the s-wave is empty.)

This composition is compatible by part (ii) of Lemma 6.1.

Alternatively, consider the case when $u^R \in \mathscr{L}$. If $s^T(c^R)$ does not exist (cf. § 2), then a compatible composition is given by

$$u^L \xrightarrow{s} (1, c^L) \xrightarrow{c} (1, c^R) \xrightarrow{s} u^R.$$

Otherwise, the composition

$$u^L \rightarrow u^T \xrightarrow{s} u^R$$

where $u^T = (s^T(c^R), c^R)$ and $u^L \rightarrow u^T$ denotes the solution with left state u^L and right state u^T, is compatible (cf. Fig. 6.3).

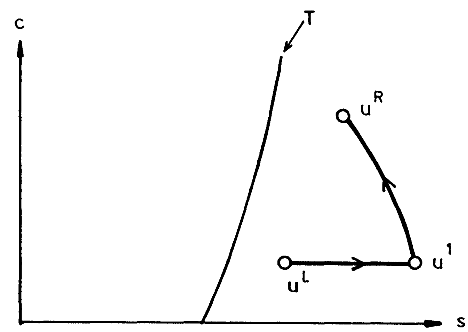


FIG. 6.2

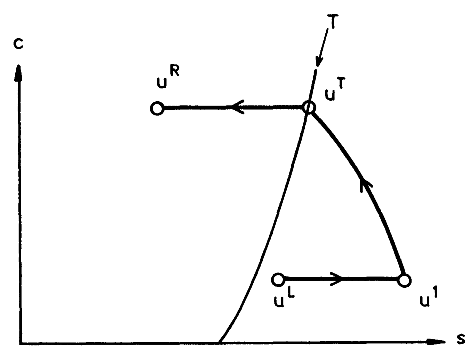


FIG. 6.3

Hence, the existence result of the lemma is established. Verification of the fact that all solutions are unique is straightforward from Lemma 6.1. □

When $u^L \in \mathscr{L}$ the situation is a little more complicated.

LEMMA 6.3. *Assume that $c^L < c^R$ and $u^L \in \mathscr{L}$. Then there exists a unique solution of the Riemann problem consisting of at most five states separated by s-waves and c-rarefaction waves.*

Proof. Let $\Gamma_R \in \mathscr{L} \cup T$ denote the c-rarefaction curve through u^L and let $u^* = (s^*, c^*) \in T$ be the state where Γ_R intersects T. (If no such u^* exists, let $u^* = (s^*, 1)$ be the state where Γ_R intersects the line $c = 1$.) Furthermore, let $\Gamma_K \subset \mathscr{R} \cup T$ be the associated critical curve defined by (cf. Fig. 6.4)

$$\Gamma_K = \{(s^K, c) \in \mathscr{R} \cup T \mid s^K = s^K(s, c) \text{ for } (s, c) \in \Gamma_R\}.$$

From (3.3), defining the c-rarefaction curves, and the definition of $s^K(u)$ it follows that, when $s^K(u) < \infty$,

$$(\lambda^c - \lambda^s)\frac{ds^K}{dc} = f_c + g, \tag{6.3}$$

where $g = g(c) = \lambda^c(s^K, c)(dh/dc)(c)((s^K - s)/(s + h(c)), (s, c) \in \Gamma_R$ and the functions λ^c, λ^s and f_c are all evaluated at (s^K, c). Since $dh/dc < 0$, it follows that $g(c) < 0$ for any $c \in [c^L, c^*)$. By comparing (6.3) with (3.3) we therefore obtain that any c-rarefaction

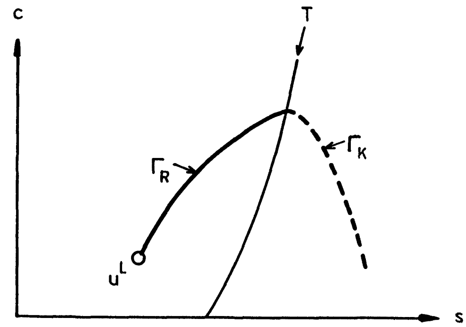


FIG. 6.4. Γ_R *and* Γ_K.

curve in $\mathscr{R}$ can, at most, intersect the critical curve Γ_K once, and that $ds^K/dc < ds/dc$, where $s(c)$ is a c-rarefaction curve (cf. Fig. 6.5).

After this introductory discussion we now construct the solution of the Riemann problem. Assume first that $u^R \in R_1$, where R_1 is the closed region in $I \times I$ bounded by the curves $c = c^L$, $c = c^*$ and Γ_K. In this case Lemma 6.1 implies that the composition

$$u^L \xrightarrow{c} u^1 \xrightarrow{s} u^R,$$

where $u^1 = (s^1, c^R) \in \Gamma_R$ is compatible (cf. Fig. 6.6)

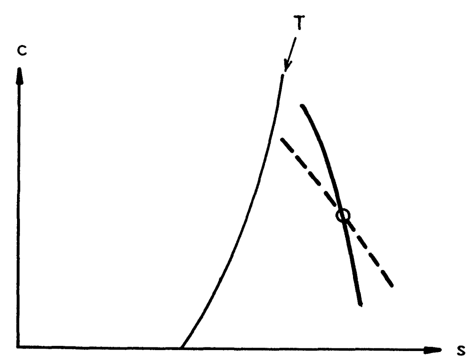


FIG. 6.5. ——— *c-rarefaction curve*; ------ Γ_K.

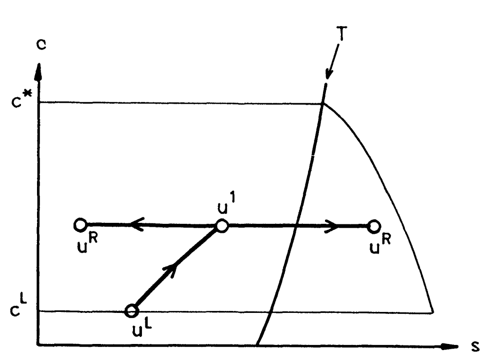


FIG. 6.6

Next assume that $u^R \in R_2$, where R_2 consists of all states in $\mathcal{R} \cup T$ that are not in R_1. Let $u^2 \in \mathcal{R}$ be the state where the c-rarefaction curve through u^R intersects the curve $\Gamma_K \cup \{c = c^L\}$.

If $c^2 = c^L$, Lemma 6.1 implies that the composition

$$u^L \xrightarrow{s} u^2 \xrightarrow{c} u^R$$

is compatible. If $c^2 > c^L$ the composition

$$u^L \xrightarrow{c_1} u^1 \xrightarrow{s} u^2 \xrightarrow{c_2} u^R,$$

where $u^1 = (s^1, c^2) \in \Gamma_R$, corresponds exactly to a compatible composition of the form (6.2). Hence we have constructed a solution for all $u^R \in R_2$ (cf. Fig. 6.7).

Finally, consider the case where

$$u^R \in R_3 = \{u = (s, c) \in \mathcal{L} \mid c > c^*\}.$$

If $s^T(c^R)$ exists the composition

$$u^L \to u^T \xrightarrow{s} u^R,$$

where $u^T = (s^T(c^R), c^R)$ and $u^L \to u^T$ denotes the solution with left state u^L and right state u^T, is compatible (cf. Fig. 6.8).

Otherwise, if $s^T(c^R)$ does not exist, a compatible composition is given by

$$u^L \to (1, c^T) \xrightarrow{c} (1, c^R) \xrightarrow{s} u^R$$

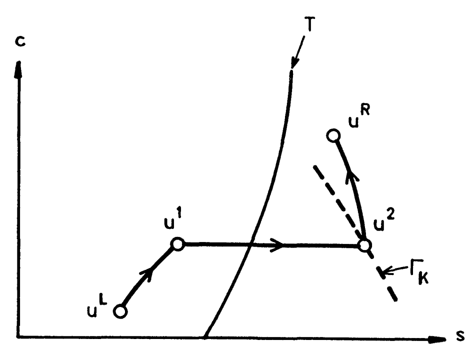


FIG. 6.7

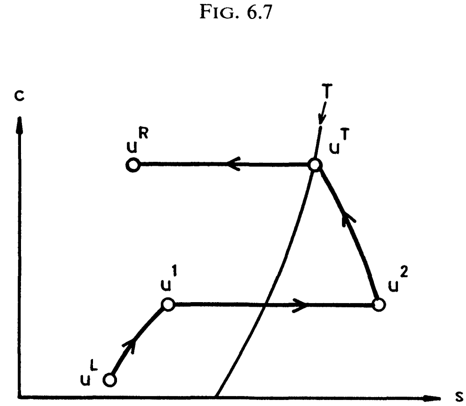


FIG. 6.8

where c^T is the unique value such that $(1, c^T) \in T$ (cf. § 2) and $u^L \to (1, c^T)$ is a solution of a Riemann problem. Hence, the desired existence result is established. Again uniqueness can be verified by applying Lemma 6.1. □

Lemmas 6.2 and 6.3 complete the proof of Theorem 5.1 in the case when $c^L < c^R$.

7. The case $c^L > c^R$. In this section we will complete the proof of Theorem 5.1 by constructing a unique solution of the Riemann problem when $c^L > c^R$. Throughout this section we shall therefore assume that the values of c^L and c^R are *fixed* with $c^L > c^R$.

For any state $u = (s, c) \in I \times I$ we define the "associated shock speed" $\sigma(u)$ by

$$\sigma(u) = \frac{f(s, c)}{s + h_L(c^R)}.$$

Hence, the Rankine–Hugoniot condition (4.7) can be written in the form

$$\sigma(u^L) = \sigma(u^R). \tag{7.1}$$

Similar to the definition given in § 6, for any state $u = (s, c) \in I \times I$, with $\lambda^s(u) \leqq \sigma(u)$, the critical value $s^K = s^K(u)$ is defined to be the unique value of s^K such that $\lambda^s(s^K, c) > \sigma(u)$ and such that (cf. Fig. 7.1)

$$\sigma(s^K, c) = \sigma(u). \tag{7.2}$$

If $\lambda^s(u) > \sigma(u)$, then $s^K(u)$ is either defined to be the unique value s^K such that $\lambda^s(s^K, c) < \sigma(u)$ and such that (7.2) holds or, if no such s^K exists, $s^K = \infty$.

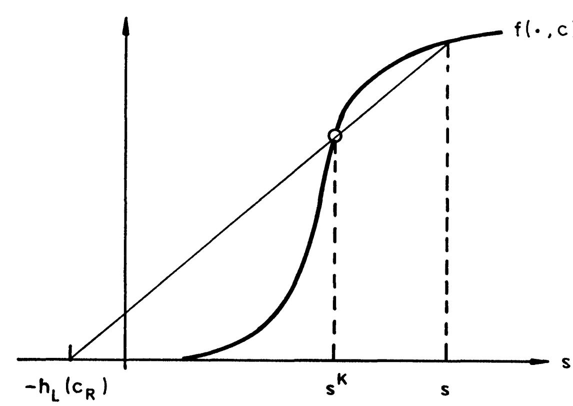


FIG. 7.1

As we have already observed in § 5, any solution of the Riemann problem will be composed of s-waves and one c-shock wave. We first characterize the compatible pairs of waves in the present case.

LEMMA 7.1. *Assume that $c^L > c^R$.*

(i) *The two waves*

$$u^L \xrightarrow{c} u^M \xrightarrow{s} u^R$$

are compatible if and only if $\lambda^s(u^M) \geqq \sigma(u^M)$ and $0 \leqq s^R < s^K(u^M)$.

(ii) *The two waves*

$$u^L \xrightarrow{s} u^M \xrightarrow{c} u^R$$

are compatible if and only if $\lambda^s(u^M) \leqq \sigma(u^M)$ and $s^K(u^M) < s^R \leqq 1$.

Proof. Consider part (i) above. The two waves are compatible if and only if

$$\sigma \leqq v_s^i, \tag{7.3}$$

where σ is the speed of the c-shock wave, v_s^i is the initial speed of the s-wave and where strict inequality is required if the s-wave starts with a shock wave. But since $\sigma = \sigma(u^M)$ and $v_s^i \leqq \lambda^s(u^M)$ it follows that $\lambda^s(u^M) \geqq \sigma(u^M)$. Furthermore, from the structure of the s-waves it follows that (7.3) holds if and only if $0 \leqq s^R < s^K(u^M)$. Part (ii) follows by similar arguments. □

As a consequence of the lemma above (and Lemma 5.1) we derive in particular that the admissible overcompressive shock discussed in § 4, where

$$\lambda^s(u^L), \lambda^c(u^L) > \sigma > \lambda^s(u^R), \lambda^c(u^R)$$

can never be joined by any other wave in a Riemann solution.

By applying the result of Lemma 7.1 we can now proceed to construct the solution of the Riemann problem.

LEMMA 7.2. *Assume that $c^L > c^R$. Then there exists a unique solution of the Riemann problem consisting of at most four states separated by s-waves and c-shock waves.*

Proof. Consider first the case when $\lambda^s(u^L) \geqq \sigma(u^L)$. Let $u^1 = (s^1, c^R)$ be the unique state such that (cf. Fig. 7.2)

$$\lambda^s(u^1) > \sigma(u^1) = \sigma(u^L).$$

Hence, from the Rankine–Hugoniot condition (7.1) and from the entropy conditions (4.10) and (4.11), the pair (u^L, u^1) corresponds to a c-shock wave with left state u^L, right state u^1 and shock speed $\sigma = \sigma(u^L)$. Furthermore, by part (i) of Lemma 7.1 the composition

$$u^L \xrightarrow{c} u^1 \xrightarrow{s} u^R$$

is compatible if $0 \leqq s^R < s^K(u^1)$. Alternatively, if $s^R > s^K(u^1)$ the composition

$$u^L \xrightarrow{s} u^2 \xrightarrow{c} u^R$$

is compatible, where $u^2 = (s^2, c^L)$ is the state such that

$$\lambda^s(u^2) \leqq \sigma(u^2) = \sigma(u^R).$$

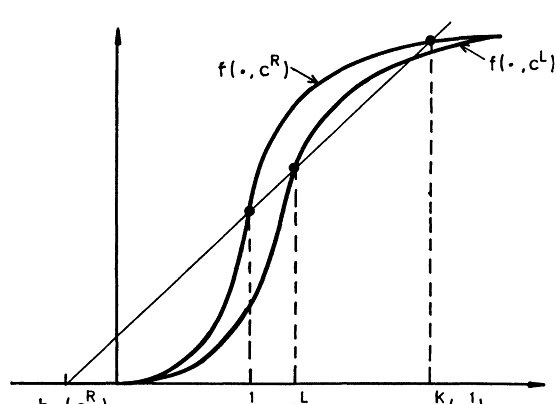


FIG. 7.2

Finally, if $s^R = s^K(u^1)$, the pair (u^L, u^R) corresponds to an admissible overcompressive c-shock (cf. § 4), where $\sigma = \sigma(u^L) = \sigma(u^R)$. Hence, we have constructed the solution of the Riemann problem in all cases when $\lambda^s(u^L) \geqq \sigma(u^L)$ (cf. Fig. 7.3).

Consider next the case when $\lambda^s(u^L) < \sigma(u^L)$ and let $u^* = (s^*, c^L)$ be the unique state such that

$$\sigma(u^*) = \lambda^s(u^*).$$

Furthermore, let $u^1 = (s^1, c^R)$ be determined by

$$\lambda^s(u^1) > \sigma(u^1) = \sigma(u^*)$$

and let $s^K = s^K(u^1)$ (cf. Fig. 7.4).

If $s^R \geqq s^K$ a solution of the Riemann problem is given by the composition

$$u^L \xrightarrow{s} u^2 \xrightarrow{c} u^R$$

where $u^2 = (s^2, c^L)$ is the unique state such that $\lambda^s(u^2) \leqq \sigma(u^2)$ and such that the pair (u^2, u^R) corresponds to a c-shock wave (cf. Fig. 7.5).

If $s^R < s^K$ a solution is given by the composition

$$u^L \xrightarrow{s} u^* \xrightarrow{c} u^1 \rightarrow u^R,$$

which is compatible by Lemma 7.1 (cf. Fig. 7.6).

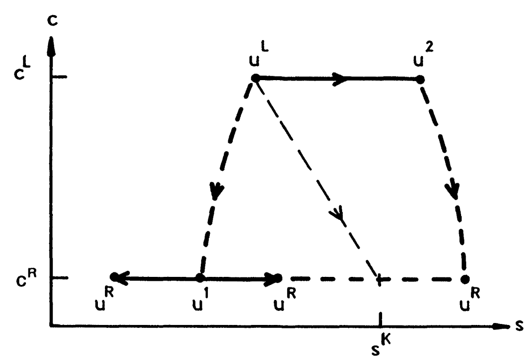


FIG. 7.3

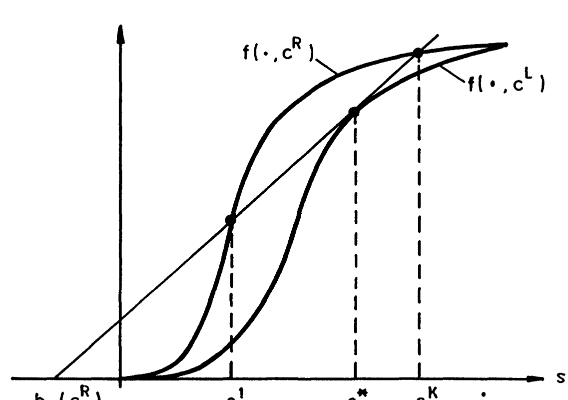


FIG. 7.4

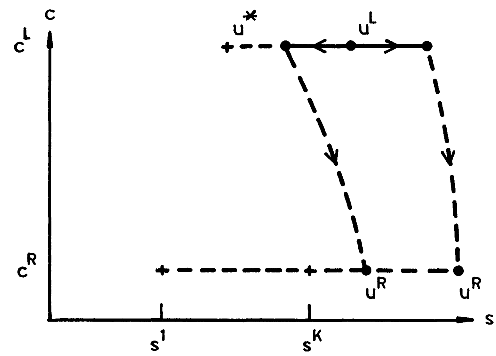


FIG. 7.5

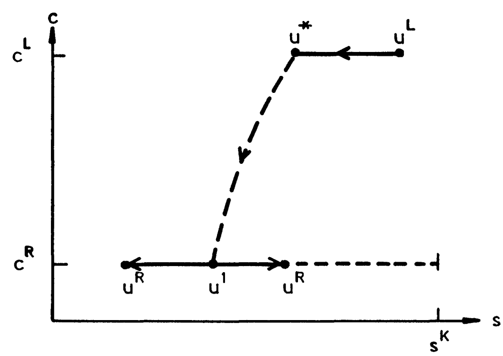


FIG. 7.6

Hence the desired existence result is established, and uniqueness can easily be verified by applying Lemma 7.1. □

As a consequence of the three Lemmas 6.2, 6.3, and 7.2 the proof of Theorem 5.1 is now completed. We observe that the existence proof is constructive; i.e., we have constructed an algorithm for the solution of the global Riemann problem for the system (1.1).

8. Numerical experiments. Based on the constructive proof of Theorem 5.1 a computer program is developed that solves the global Riemann problem for the model (1.1). In order to solve nonlinear equations obtained from the Rankine-Hugoniot condition (4.7) and the ordinary differential equations involved in the determination of the rarefaction waves, standard numerical methods are used.

The purpose of the numerical experiments presented below is to illustrate some typical behavior in the exact solution of the Riemann problem and to show that sometimes this behavior is not easily detected by means of finite difference schemes. In the examples presented below we have used the algebraic expressions

$$f(s,c)=\frac{s^2}{s^2+(0.5+100c)(1-s)^2} \tag{8.1}$$

and

$$a(c)=\frac{0.2c}{1+100c}. \tag{8.2}$$

The function f is graphed in Fig. 8.1 for different values of c uniformly distributed between 0.00 and 0.01.

Example 8.1. In this example we calculate the solution of the Riemann problem with $u^L=(s^L, c^L)=(1.0, 0.01)$ and $u^R=(s^R, c^R)=(0.0, 0.0)$. This models a situation where the porous medium is initially 100 percent saturated with oil. Then water containing polymer is injected in order to displace the oil.

The structure of the solution of the Riemann problem in this case is illustrated by Fig. 7.6. The exact solution is shown on Fig. 8.2.

We observe that a bank of polymer-free water separates the polymer from the region with 100 percent oil. We remark that this bank forms as a consequence of the adsorption term $a(c)$ in the system (1.1).

One of the purposes of polymer injection is to increase the viscosity of the aqueous phase in order to reduce viscous fingering. The bank of pure water observed above might therefore decrease the desired efficiency of the displacement process.

Example 8.2. Throughout this example we consider the Riemann problem with $u^L=(s^L, c^L)=(0.9, 0.007)$ and $c^R=0.003$ fixed and where s^R is close to the critical value s^K. These cases are illustrated in Fig. 7.5 ($s^R \geqq s^K$) and Fig. 7.6 ($s^R < s^K$). Examples of the s-component of the solution is shown in Fig. 8.3 and Fig. 8.4 below. Since the values s^* and s^1 are independent of s^R for $s^R < s^K$, this shows that, if the solution of the Riemann problem is considered pointwise, it is discontinuous with respect to s^R close to s^K. However, since the width of the constant solution s^1 is

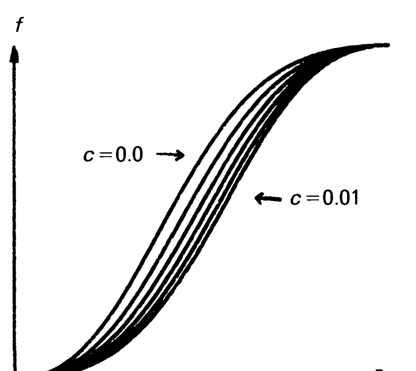


FIG. 8.1

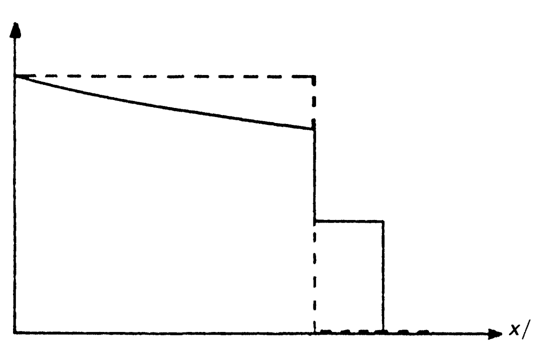
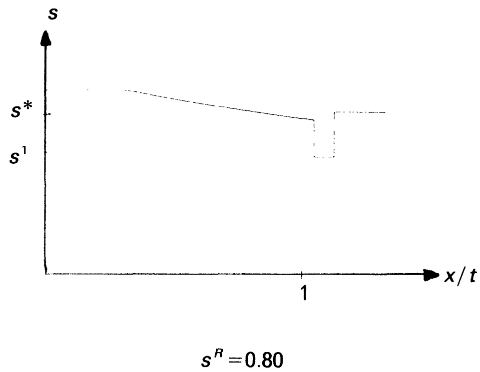


FIG. 8.2. ——— s; ------ $c \times 100$.



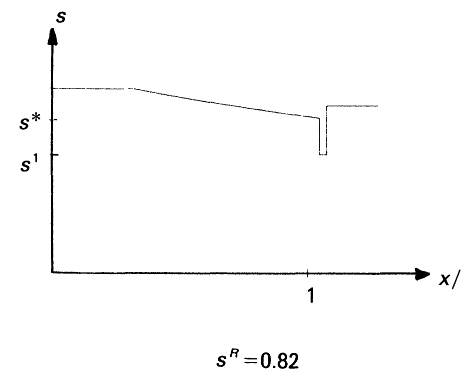


FIG. 8.3. $s^R < s^K$.

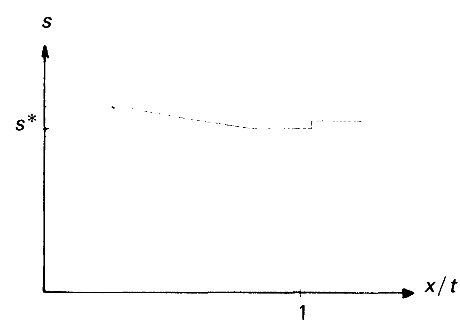


FIG. 8.4. $s^R = 0.84 > s^K$.

vanishing as s^R tends to s^K from below, the solution is continuous with respect to s^R in L^1 norm.

Example 8.3. Consider next the Riemann problem with $u^L = (s^L, c^L) = (0.45, 0.0)$ and $u^R = (s^R, c^R) = (0.20, 0.01)$. This corresponds to the case illustrated in Fig. 6.8 where the maximum number of intermediate states is present in the Riemann solution. The exact solution is shown in Fig. 8.5.

In order to test the efficiency of a standard difference method we have calculated the solution of this Riemann problem with the standard 1 order upwind scheme using a uniform grid. We have used $\Delta t/\Delta x = 12/25$ which satisfies the CFL stability condition.

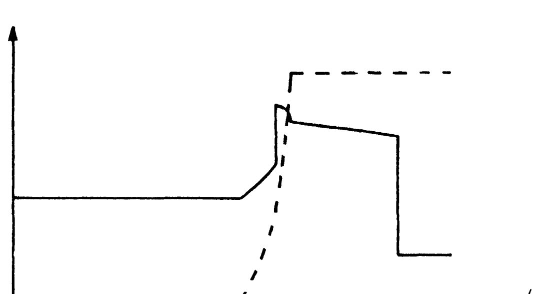


FIG. 8.5. ——— s; ------ $c \times 100$.

An approximation of the solution at $t=1$ is calculated with $\Delta t=0.04$ and the approximate saturation is compared with the exact saturation in Fig. 8.6.

As we observe the numerical method seems to neglect certain effects reflected in the exact solution. In order to investigate these phenomena further, similar numerical solutions, with decreasing time steps, have been calculated. In Fig. 8.7 we have plotted the different numerical paths in (s, c)-space together with the exact path. We have used $\Delta t=0.04$, 0.016, 0.008, 0.004, 0.0016.

The results seem indeed to indicate that the solutions obtained by the upwind scheme converge to the exact solution. However, it might be difficult to give an accurate physical interpretation of the process from the numerical results.

A comparison between the numerical solution of s with $\Delta t=0.0016$ and the exact solution (cf. Fig. 8.4) is shown in Fig. 8.8.

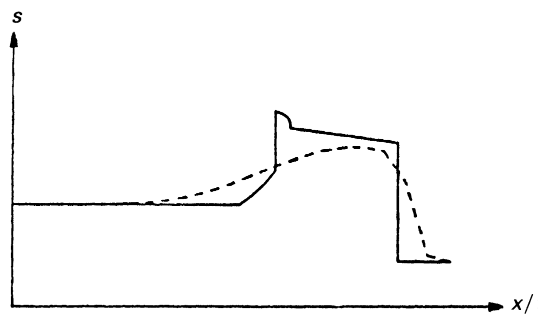


FIG. 8.6. ——— *exact solution of s*; - - - - - - *numerical solution of s.* $\Delta t=0.04$.

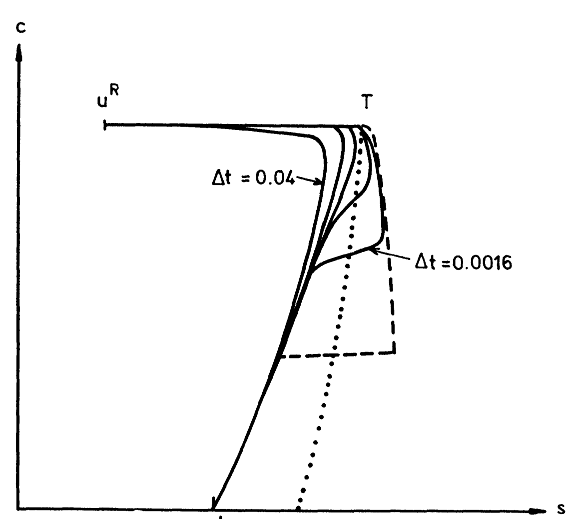


FIG. 8.7. ——— *numerical paths*; - - - - - - *exact path*; · · · · · *T.*

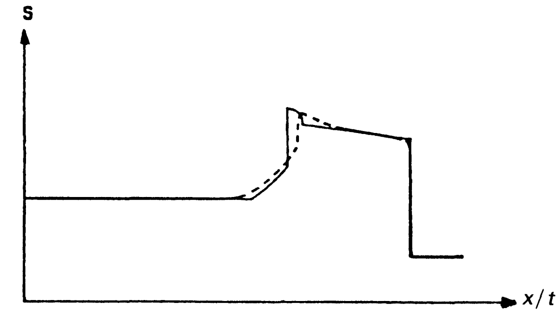


FIG. 8.8. ——— *exact solution of s*; ------ *numerical solution of s.* $\Delta t = 0.0016$.

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