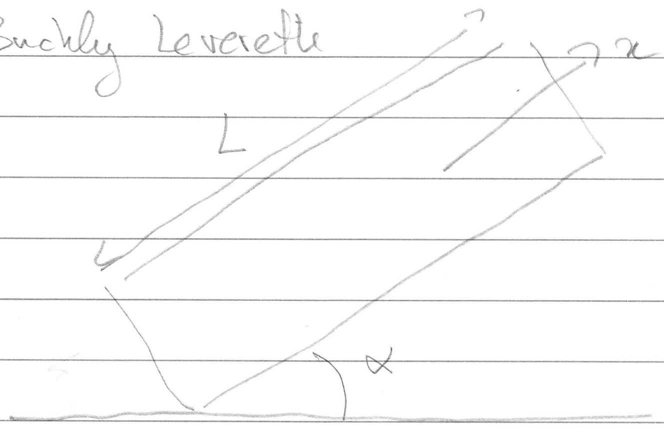
Buckley Leverette



Darcy's equations:

$$q_o = -\frac{k k_{ro} A}{\mu_o}\left(\frac{\partial P_o}{\partial x} + \rho_o g \sin\alpha\right)$$

$$q_w = -\frac{k k_{rw} A}{\mu_w}\left(\frac{\partial P_w}{\partial x} + \rho_w g \sin\alpha\right)$$

$P_w = P_o - P_{cow}$

$$= -\frac{k k_{rw} A}{\mu_w}\left(\frac{\partial (P_o - P_{cow})}{\partial x} + \rho_w g \sin\alpha\right)$$

After some rearranging:

$$\begin{cases} -\dfrac{q_o \mu_o}{k k_{ro} A} = \dfrac{\partial P_o}{\partial x} + \rho_o g \sin\alpha \\[2ex] -\dfrac{q_w \mu_w}{k k_{rw} A} = \dfrac{\partial P_o}{\partial x} - \dfrac{\partial P_{cow}}{\partial x} + \rho_w g \sin\alpha \end{cases}$$

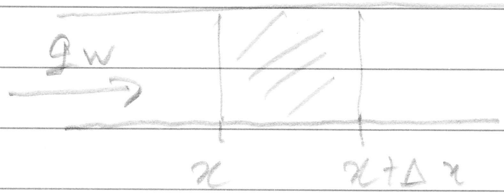
$$\Rightarrow -\frac{1}{kA}\left(q_w \frac{\mu_w}{k_{rw}} - q_o \frac{\mu_o}{k_{ro}}\right) = -\frac{\partial P_{cow}}{\partial x} + \Delta\rho g \sin\alpha$$

where $\Delta\rho = \rho_w - \rho_o$

Introducing $q = q_o + q_w$

$$f_w = \frac{q_w}{q} = \frac{k_{rw}/\mu_w}{\frac{k_{rw}}{\mu_w} + \frac{k_{ro}}{\mu_o}} \left\{ 1 + \frac{k k_{ro} A}{q \mu_o} \left(\frac{\partial P_{cow}}{\partial x} - \Delta \rho g \sin\alpha \right) \right\}$$

Derivation Buckley Leverett equation:



Mass balance:

$$\left[(q_w \rho_w)_x - (q_w \rho_w)_{x+\Delta x} \right] \cdot \Delta t$$

$$= A \Delta x \phi \left[(S_w \rho_w)_{t+\Delta t} - (S_w \rho_w)_t \right]$$

$$\Rightarrow \quad A \phi \frac{\partial}{\partial t} (S_w \rho_w) + \frac{\partial}{\partial x} (q_w \rho_w) = 0$$

Assuming incompressible fluids: $\frac{\partial \rho_w}{\partial x} = 0$; $\frac{\partial \rho_w}{\partial t} = 0$

$$\Rightarrow \quad \left. \begin{aligned} & A \phi \frac{\partial}{\partial t} S_w + \frac{\partial}{\partial x} q_w = 0 \\ & q_w = f_w \cdot q \ ; \ q = \text{constant} \end{aligned} \right\}$$

$$\Rightarrow \quad \frac{A\phi}{q} \frac{\partial S_w}{\partial t} + \frac{\partial f_w(S_w)}{\partial x} = 0$$

$$\Rightarrow \quad \boxed{\frac{A\phi}{q} \frac{\partial S_w}{\partial t} + \frac{\partial f_w}{\partial S_w} \frac{\partial S_w}{\partial x} = 0}$$

Buckley Leverett (assuming $\frac{\partial P_{cow}}{\partial x} = 0$)

$$\begin{cases} \frac{A\phi}{q}\frac{\partial S_w}{\partial t} + \frac{\partial f_w}{\partial S_w}\frac{\partial S_w}{\partial x} = 0 & x \in [0, L] \\ S_w(x,0) = S_{wi} & x \in (0, L] \\ S_w(0,t) = S_{winj} & \text{such that } f_w(S_{winj}) = \text{injected water cut} \end{cases}$$

with $f_w(S_w) = \frac{q_w}{q} = \frac{k_{rw}/\mu_w}{\frac{k_{rw}}{\mu_w} + \frac{k_{ro}}{\mu_o}} \left\{ 1 - \frac{k k_{ro} A}{q \mu_o} \Delta\rho \sin\alpha \right\}$

$\Delta\rho = \rho_w - \rho_o$

N.B: q_w and q are subsurface water and total rates!

· Parametrization of relperms:

$$\begin{cases} k_{rw}(S_w) = k_{rw}^e S^{n_w} \;;\; k_{ro} = k_{ro}^e (1-S)^{n_o} \\ \text{with } S = \frac{S_w - S_{cw}}{1 - S_{cw} - S_{orw}} \end{cases}$$

Introduce:

Endpoint mobility ratio: $M = \frac{k_{rw}^e}{\mu_w} \Big/ \left(\frac{k_{ro}^e}{\mu_o}\right)$

Dimensionless gravity number: $N_G = \frac{k \cdot k_{ro}^e \, g \Delta\rho}{\mu_o} \frac{A}{q}$

$f_w(S_w) = \frac{M S^{n_w}}{(1-S)^{n_o} + M S^{n_w}} \left\{ 1 - N_G (1-S)^{n_o} \sin\alpha \right\}$

with $S = \frac{S_w - S_{cw}}{1 - S_{cw} - S_{orw}}$

Introducing dimensionless variables:

$$x_D = \frac{x}{L}, \quad t_D = \frac{q \cdot t}{\phi A L} = \frac{Q}{V_p} \; ; \quad V_p = \phi A L$$

$= \text{pore volume}$

$Q = \text{injected volume}$

$$\begin{cases} \dfrac{\partial S_w}{\partial t_D} + \dfrac{\partial f_w}{\partial S_w} \dfrac{\partial S_w}{\partial x_D} = 0 \\ S_w(x_D, 0) = S_{wi} \\ S_w(0, t_D) = S_{winj} \end{cases}$$

We absorb $\sin\alpha$ into N_G : $\tilde{N}_G = N_G \cdot \sin\alpha$

$$f_w(S_w) = \frac{M S^{n_w}}{(1-S)^{n_o} + M S^{n_w}} \left\{ 1 - \tilde{N}_G (1-S)^{n_o} \right\}$$

$$\frac{\partial f_w(S_w)}{\partial S_w} = \frac{\partial S}{\partial S_w} \frac{\partial f_w}{\partial S} = \frac{1}{1 - S_{cw} - S_{orw}} \frac{\partial f_w}{\partial S}$$

$$\frac{\partial f_w}{\partial S} = \frac{M S^{n_w - 1} (1-S)^{n_o - 1} \left\{ n_w (1-S) + n_o S \right\} \left\{ 1 - \tilde{N}_G (1-S)^{n_o} \right\}}{\left\{ (1-S)^{n_o} + M S^{n_w} \right\}^2}$$

$$+ \frac{M S^{n_w}}{(1-S)^{n_o} + M S^{n_w}} \tilde{N}_G \, n_o (1-S)^{n_o - 1}$$

$$\frac{\partial^2 f_w}{\partial S_w^2} = \left(\frac{\partial S}{\partial S_w}\right)^2 \frac{\partial^2 f_w}{\partial S^2} = \frac{1}{(1-S_{cw}-S_{orw})^2} \frac{\partial^2 f_w}{\partial S^2}$$

$$\frac{\partial^2 f_w}{\partial S^2} = \frac{M}{\{(1-S)^{n_o} + M S^{n_w}\}^3} \cdot (1-S)^{n_o-2} S^{n_w-2} \times \{1 - \tilde{N}_G (1-S)^{n_o}\}$$

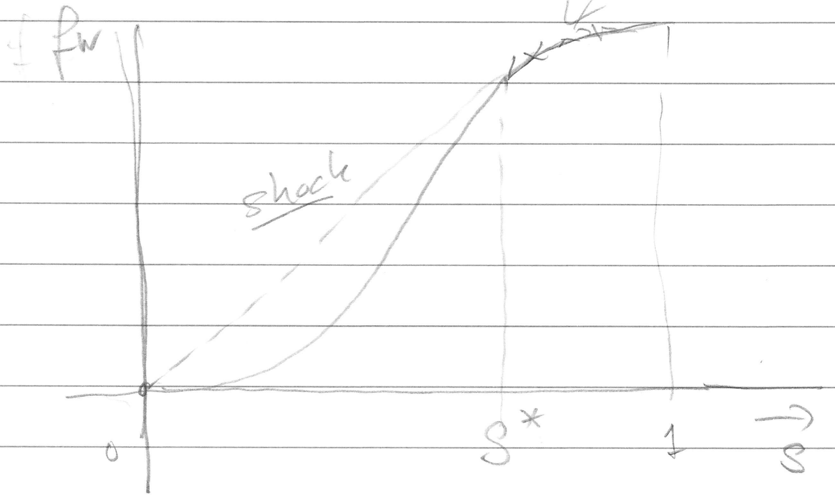
$$\times \Big\{ (1-S)^{n_o} \left[n_w^2 (1-S)^2 + n_o(1+n_o) S^2 + n_w (1-S)(-1+(1+2n_o)S) \right]$$

$$- M S^{n_w} \left[n_w^2 (1-S)^2 + n_o(n_o-1) S^2 + n_w(1-S)(1+(2n_o-1)S) \right] \Big\}$$

$$+ \frac{2 M S^{n_w-1} (1-S)^{n_o-1} \{ n_w(1-S) + n_o S \}}{\{(1-S)^{n_o} + M S^{n_w}\}^2} \cdot \tilde{N}_G \cdot n_o (1-S)^{n_o-1}$$

$$- \frac{M S^{n_w}}{(1-S)^{n_o} + M S^{n_w}} \tilde{N}_G n_o (n_o - 1)(1-S)^{n_o-2}$$

Solution : ($\nu_G = 0$)



$S_{wiy} > S_{wi}$

Tangent approach

(see next two pages)

Shock: $\dfrac{\partial f(S^*)}{\partial S} = \dfrac{f(S^*) - f(S_{wi})}{S^* - S_{wi}} = \nu$: shock velocity.

Rarefaction: $S_w(x,t) = R(\xi)$, $\xi = \dfrac{x}{t}$

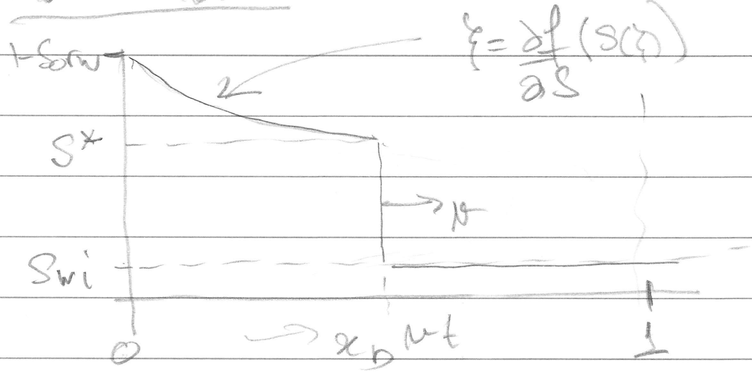
can be found by inverting the relation : $\dfrac{\partial f}{\partial S_w}(R(\xi)) = \xi$

or by solving

$$\frac{dS_w(\xi)}{d\xi} = \left(\frac{\partial^2 f_w}{\partial S_w^2}\right)^{-1}$$

Why tangent condition for Buckley Leverett?

Mass balance:



$$\bar{S}_w(t) = \int_0^1 S(x,t)\,dx = \int_0^{vt} S(x,t)\,dx + \int_{vt}^1 S_{wi}\,dx$$

$$= t\int_{\xi=0}^{\xi=v} S(\xi)\,d\xi + S_{wi}(1-vt)$$

$\xi = \frac{x}{t}$

$$= tS(\xi=v)v - t\int_{S(\xi=0)}^{S(\xi=v)} \underbrace{\xi}_{=\frac{\partial f}{\partial S}}\,dS + S_{wi}(1-vt) =$$

$$= tS(\xi=v)\cdot v - t\int_{S(\xi=0)=1-S_{orw}}^{S(\xi=v)} \frac{\partial f}{\partial S}\,dS + S_{wi}(1-vt)$$

$$= tS(\xi=v)\,v + t\left[1 - f(S(\xi=v))\right] + S_{wi}(1-vt)$$

Net injected water: $V_w = \left[1 - f(S_{wi})\right]\cdot Q_{tot} = \left[1-f(S_{wi})\right]V_p t$

$$\bar{S}_w(t) = \frac{S_{wi}V_p + V_w}{V_p} = S_{wi} + \left[1 - f(S_{wi})\right]t$$

Equating the two expressions:

$$S_{wi} + (S(\xi = v) - S_{wi})\, v \cdot t + [1 - f(S(\xi = v))] \cdot t$$

$$= S_{wi} + [1 - f(S_{wi})] \cdot t$$

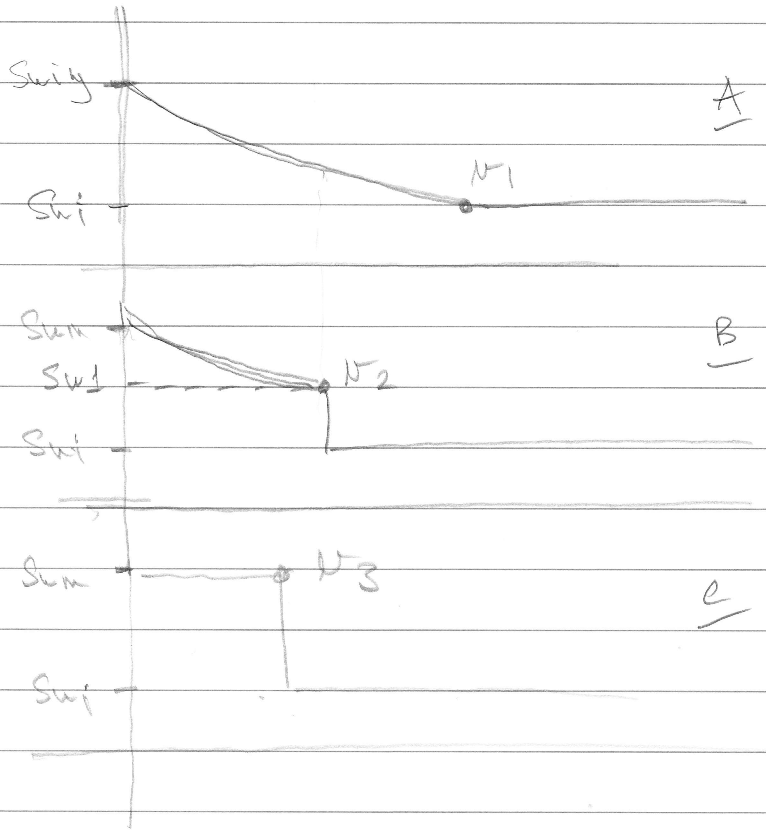
$$\Rightarrow (S(\xi = v) - S_{wi})\, v = f(S(\xi = v)) - f(S_{wi})$$

$$\Rightarrow v = \frac{f(S(\xi = v)) - f(S_{wi})}{S(\xi = v) - f(S_{wi})}$$

we also have $\xi = v = \dfrac{\partial f(S(\xi = v))}{\partial S}$

$$\Rightarrow \frac{\partial f(S)}{\partial S} = \frac{f(S) - f(S_{wi})}{S - S_{wi}} \quad : \text{tangent condition}$$

Types of possible solutions



A: $v_1 = \frac{\partial f}{\partial S_w}(S_{wi})$

B: $v_2 = \frac{f(S_{w1}) - f(S_{wi})}{S_{w1} - S_{wi}} = \frac{\partial f}{\partial S_w}(S_{w1})$

C: $v_3 = \frac{f(S_{winj}) - f(S_{wi})}{S_{winj} - S_{wi}}$

Breakthrough times: $t_{B_i} = \frac{1}{v_i}$

Constrain to cases with $f_w(S_{winj}) = 1$ and $S_{wi} < S_{winj}$

Average water saturation.

Before breakthrough:

$f_w(S_{winj}) = 1$

Influx : $Q_w^{in} = f_{w,inj} \cdot Q_{tot} = Q_{tot}$

Outflux : $Q_w^{out} = f_w(S_{wi}) \cdot Q_{tot}$.

Net influx $= Q_w^{net} = Q_w^{in} - Q_w^{out} = (1 - f_w(S_{wi})) \cdot Q_{tot}$.

$$= (1 - f_w(S_{wi})) \cdot V_p \cdot t_D$$

$$V_w(t_D) = S_{wi} \cdot V_p + Q_w^{net}(t_D)$$

$$= \left[S_{wi} + (1 - f_w(S_{wi}))\, t_D\right] \cdot V_p$$

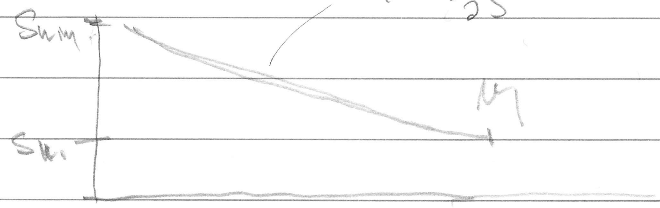
$$\Rightarrow \overline{S_w}(t_D) = \frac{V_w(t_D)}{V_p} = S_{wi} + (1 - f_w(S_{wi})) \cdot t_D$$

NB: if $S_{wi} = S_{cw}$: $\overline{S_w}(t_D) = S_{wi} + t_D$

After breakthrough

$\xi = \frac{\partial f}{\partial S}(S(\xi))$, $\xi = \frac{x}{t}$

Case A :



$v_1 = \frac{\partial f}{\partial S_w}(S_{wi})$

$$\bar{S}_w(t_D) = \int_0^1 S_w(x,t)\,dx \overset{\xi=\frac{x}{t}}{=} t_D \int_0^{1/t_D} S_w(\xi)\,d\xi =$$

$$= t_D \left[S_w(\xi)\cdot\xi \Big|_0^{1/t_D} - \int_{S(\xi=0)}^{S(\xi=1/t_D)} \xi\, dS \right]$$

$$= t_D \left[S_w(\xi=\tfrac{1}{t_D})\cdot\frac{1}{t_D} - \int_{S(\xi=0)}^{S(\xi=1/t_D)} \frac{\partial f}{\partial S}\, dS \right]$$

$$= t_D \left[\frac{1}{t_D} S_w(\xi=\tfrac{1}{t_D}) - f\left(S_w(\xi=\tfrac{1}{t_D})\right) + f\left(S_w(\xi=0)\right) \right]$$

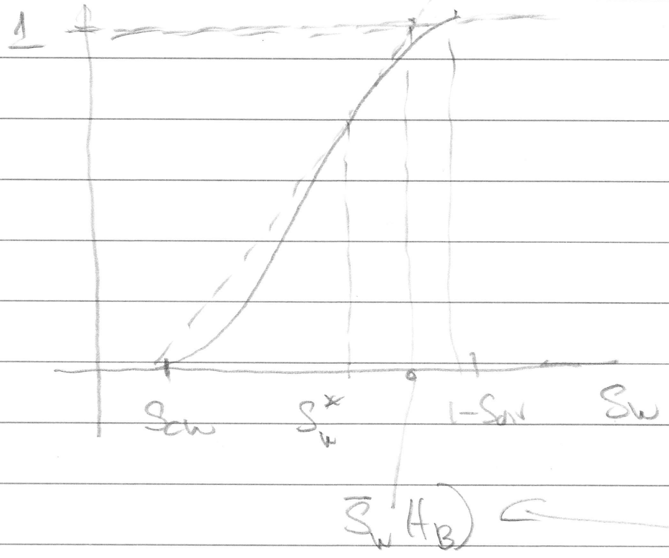
$$= S_w(x=1,t_D) + t_D \Big(\underbrace{f(S_w(x=0))}_{=1} - f(S_w(x=1,t_D)) \Big)$$

$$= S_w(x_D=1,t_D) + t_D \left[1 - f(S_w(x=1,t_D)) \right]$$

Case B: same as case A

Case C: $\bar{S}_w(t_D) = S_{winj}$

Graphical technique



For shock saturation S_w^*:

$$\frac{\partial f_w(S_w^*)}{\partial S_w} = \frac{f_w(S_w^*) - f_w(S_{cw})}{S_w^* - S_{cw}}$$

Breakthrough time $t_B = 1 \Big/ \frac{\partial f}{\partial S_w}(S_w^*)$

From p 10 : $\bar{S}_w(t_B) = S_{cw} + \frac{(1 - f_w(S_{cw}))}{\frac{\partial f}{\partial S_w}(S_w^*)}$

$$\Rightarrow \quad \frac{\partial f}{\partial S_w}(S_w^*) = \frac{(1 - f_w(S_{cw}))}{(\bar{S}_w(t_B) - S_{cw})}$$

Generalizing the average S_w calculation to $f_{winj} \neq 1$

Before breakthrough: (see p 10)

Influx: $Q_w^{in} = f_w(S_{winj}) \cdot Q\,t$

Outflux: $Q_w^{out} = f_w(S_{wi}) \cdot Q\,t$

$$V_w(t_D) = \left[S_{wi} + \left(f_w(S_{winj}) - f_w(S_{wi}) \right) t_D \right] V_p$$

$$\Rightarrow \bar{S}_w(t_D) = \frac{V_w(t_D)}{V_p} = S_{wi} + \left(f_w(S_{winj}) - f_w(S_{wi}) \right) t_D$$

After breakthrough. (p 11)

Cases A + B:

$$\bar{S}_w(t_D) = S_w(x_D = 1, t_D) + t_D \left[f_w(S_{winj}) - f(S_w(x=1, t_D)) \right]$$

Case C:

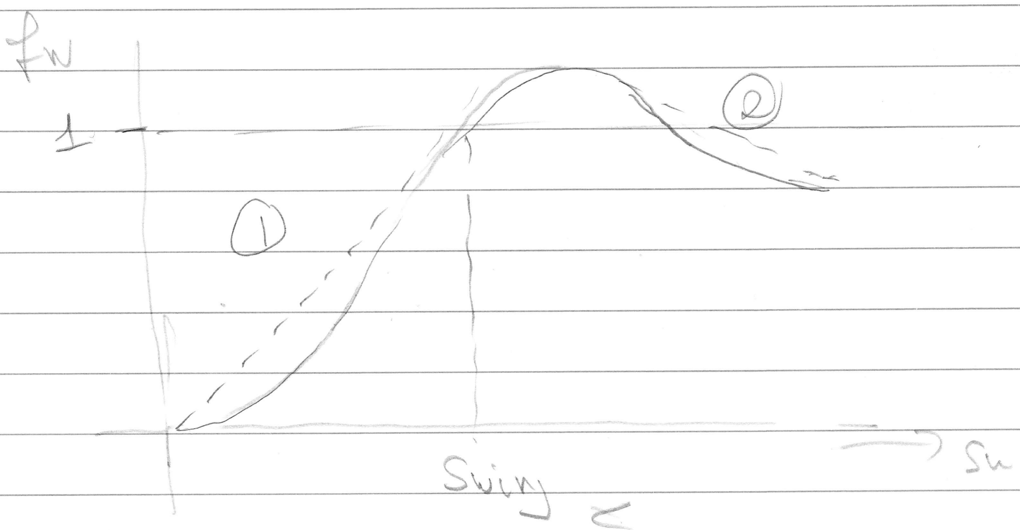
$\bar{S}_w(t_D) = S_{winj}.$

Alternative to tangent approach is the convex hull approach:

Determine the convex hull of:

$$\{ (S_w, y) : S_{wr} \leq S_w \leq S_{wl} \text{ and } y \leq f_w(S_w) \}$$

E.g. for large N_G and negative α:



Two potential shocks.

However: for shock ② we have a negative velocity
⇒ outflow happens at $x=0$
⇒ not allowed.

In the code we set S_{winj} such that $f_w(S_{winj}) = 1$

We have implemented the convex hull approach using ConvexHull from SciPy.