Buckley Leverett :

$$\begin{cases} \dfrac{\partial S}{\partial t_D} + \dfrac{\partial f}{\partial x_D} = 0 \\ S(x_D, 0) = S_I \\ S(0, t_D) = S_J \end{cases}$$

$$f(S) = \frac{\tilde{S}^{n_w}}{\tilde{S}^{n_w} + M_{wo} (1-\tilde{S})^{n_o}}$$

$$\tilde{S} = \frac{(S_w - S_{cw})}{(1 - S_{cw} - S_{or})}$$

$$M_{wo} = \frac{\mu_w}{k_{rw}} \frac{k_{ro}}{\mu_o}$$

$$x_D = \frac{x}{L}, \quad t_D = \frac{u \cdot t}{\phi \cdot L}$$

$$\boxed{\frac{\partial S}{\partial t} + \frac{\partial f}{\partial S} \frac{\partial S}{\partial x} = 0}$$

$$\frac{\partial s}{\partial t} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} = 0$$

<u>Self similarity</u> : $s(x,t) \longrightarrow s(cx,ct)$ also solution.

Find solution of the type $s(x,t) = s(\xi)$ $\xi = \frac{x}{t}$

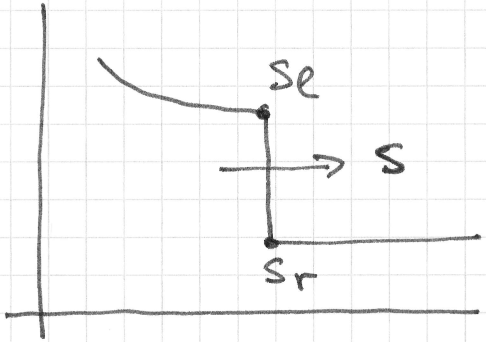
$$-\frac{x}{t^2}\frac{ds}{d\xi} + \frac{\partial f}{\partial s}\frac{1}{t}\frac{ds}{d\xi} = 0$$

$$\Rightarrow \frac{1}{t}\left(\frac{\partial f}{\partial s} - \xi\right)\frac{ds}{d\xi} = 0$$

$\Rightarrow \frac{ds}{d\xi} = 0$ or $\xi = \frac{\partial f}{\partial s}$ "speed of saturations"

Can be solved : $1 = \frac{\partial^2 f}{\partial s^2}\frac{ds}{d\xi} \Rightarrow \frac{ds}{d\xi} = \left(\frac{\partial^2 f}{\partial s^2}\right)^{-1}$

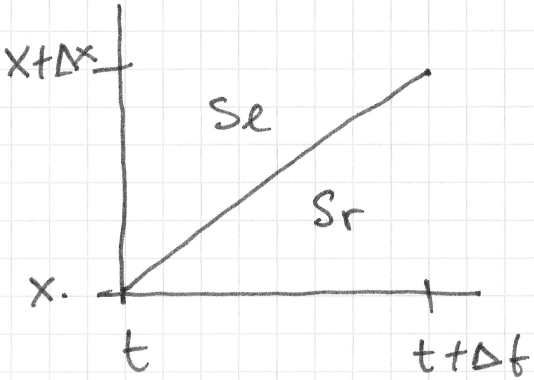
Shock



$$\frac{\partial S}{\partial t} + \frac{\partial f}{\partial x} = 0$$

Integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} S(x,t)\,dx = f(S(x_1,t)) - f(S(x_2,t))$$



$$\int_{x}^{x+\Delta x} S(x,t+\Delta t)dx - \int_{x}^{x+\Delta x} S(x,t)\,dt =$$

$$= \int_{t}^{t+\Delta t} f(S(x_1,t))dt - \int_{t}^{t+\Delta t} f(S(x+\Delta x,t))dt$$

$$\Rightarrow \quad S_r \Delta x - S_\ell \Delta x = f(S_r)\Delta t - f(S_\ell)\Delta t$$

$$\Rightarrow \quad \sigma = \frac{\Delta x}{\Delta t} = \frac{f(S_r) - f(S_\ell)}{S_r - S_\ell}$$ "Shock speed"

Solution for :

$$\begin{cases} S_J = 1 - S_{or} \\ S_I = S_{cw} \end{cases}$$

$$\xi_J = \frac{\partial f}{\partial S}(S = 1 - S_{or})$$

$$\xi_s = \frac{\partial f}{\partial S}(S = S_h)$$

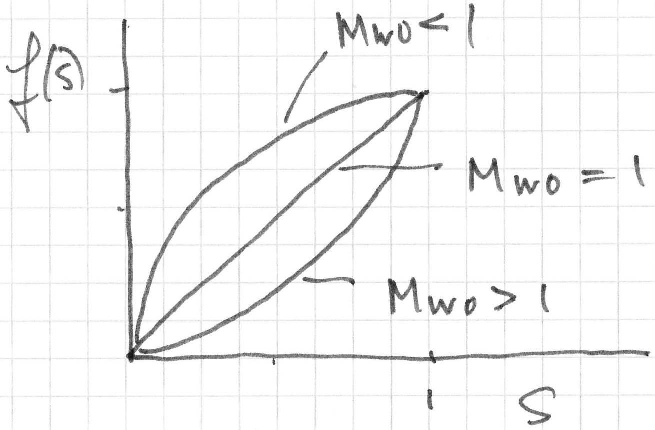
$$\sigma = \frac{f(S_h) - f(S = S_{cw})}{S_h - S_{cw}} = \xi_s$$

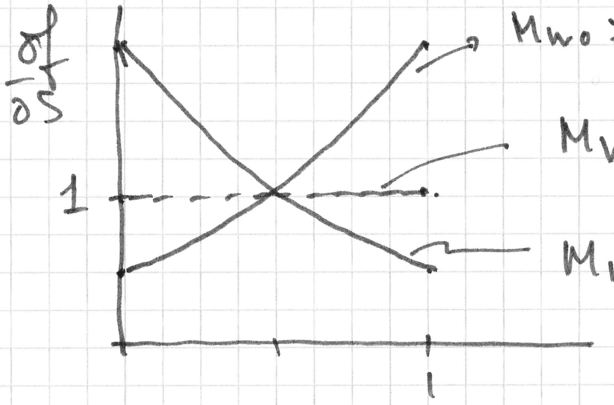
$$S(\xi) = \begin{cases} R(\xi) & \xi_J \le \xi \le \xi_s \\ S_{cw} & \xi > \xi_s \end{cases}$$

$R(\xi)$ is solution of $\dfrac{dS}{d\xi} = \left(\dfrac{\partial^2 f}{\partial S^2}\right)^{-1}$ $S(\xi_J) = 1 - S_{or}$

Special case : $n_w = n_o = 1$, $S_{cw} = S_{or} = 0$

$$f(S) = \frac{S}{S + M_{wo}(1-S)} \qquad \frac{\partial f}{\partial S} = \frac{M_{wo}}{(S + M_{wo}(1-S))^2}$$





Contact discontinuity :

$$\begin{cases} \dfrac{\partial S}{\partial t} + \dfrac{\partial S}{\partial x} = 0 \\ S(x,0) = S_I \\ S(0,t) = S_J \end{cases} \Longrightarrow S(x,t) = \begin{cases} S_J & x < t \\ S_I & x > t \end{cases}$$

2 dimensional problem

$$u = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \qquad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad f_i = f_i(u)$$

$$\begin{cases} u_t + F_x = 0 \\ u(x,0) = u_I \\ u(0,t) = u_J \end{cases}$$

$$u_t + A\, u_x = 0$$

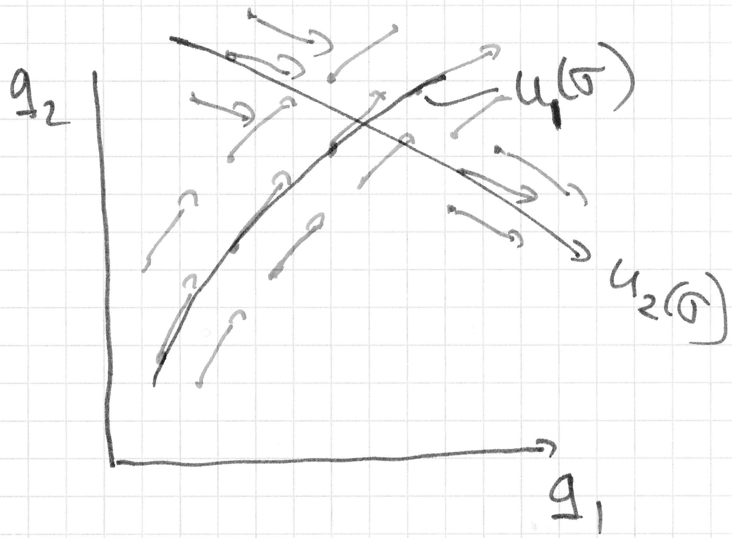
with

$$A = \left(\frac{\partial F}{\partial u}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial g_1} & \frac{\partial f_1}{\partial g_2} \\ \frac{\partial f_2}{\partial g_1} & \frac{\partial f_2}{\partial g_2} \end{pmatrix}$$

Eigen values, eigen vectors & integral curves

$$| A(u) - \lambda(u) I | = 0$$

$$\lambda_p(u), \quad r_p(u) \qquad p = 1,2$$



Ordering : $\lambda_1(u) < \lambda_2(u)$

Integral curves:

$u(\sigma)$ is an integral curve of the eigen vector field $r_p(u)$, if at each point $u(\sigma)$ the tangent vector $\frac{du}{d\sigma}$ is an eigenvector of $A(u(\sigma))$:

$$A(u) \cdot \frac{du}{d\sigma} = \lambda_p(u) \frac{du}{d\sigma}$$

$$\frac{du}{d\sigma} = \alpha(\sigma) \cdot r_p(u(\sigma))$$

Centered rarefaction wave:

Self similar solution: $u(x,t) \to u(cx, ct)$ also solution.

Solutions of the type: $U(x,t) = U(\xi)$ $\quad \xi = \frac{x}{t}$

$$-\frac{x}{t^2}\frac{dU}{d\xi} + A\frac{1}{t}\frac{dU}{d\xi} = 0$$

$$\frac{1}{t}\left(A(u) - \xi I\right)\frac{dU}{d\xi} = 0$$

ξ: "characteristic speed" of "state" U

Two solutions:

$$\xi = \lambda_p(u(\xi)) \quad p = 1,2 \quad , \quad \frac{dU_p}{d\xi} \text{ on integral curve } r_p$$

$$1 = \nabla\lambda_p(u(\xi)) \cdot \frac{du}{d\xi} = \alpha(\xi)\nabla\lambda_p(u(\xi)) \cdot r_p(u(\xi))$$

$$\Rightarrow \alpha(\xi) = \frac{1}{\nabla\lambda_p(u(\xi)) \cdot r_p(u(\xi))}$$

$$\Rightarrow \frac{du}{d\xi}p = \frac{r_p(u(\xi))}{\nabla\lambda_p(u(\xi)) \cdot r_p(u(\xi))}$$

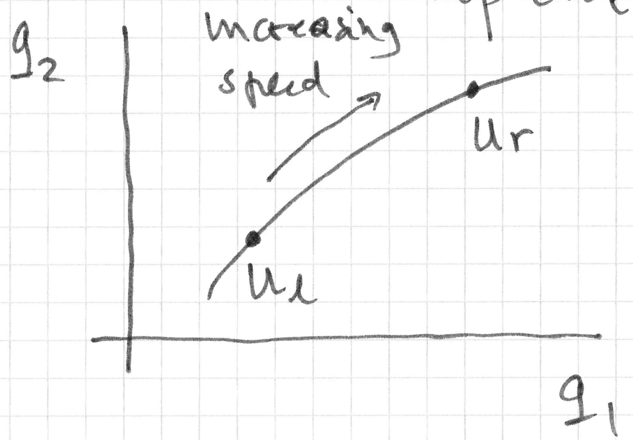
Centered rarefaction wave can connect two "states" u_ℓ and u_r on a single integral curve of r_p

$$u_p(x,t) = \begin{cases} u_\ell & x/t \le \xi_1 \\ u_p(\xi) & \xi_1 \le \frac{x}{t} \le \xi_2 \\ u_r & \frac{x}{t} \ge \xi_2 \end{cases}$$

where $\xi_1 = \lambda_p(u_\ell)$, $\xi_2 = \lambda_p(u_r)$

Important ! above only makes sense if

$$\lambda_p(u_\ell) < \lambda_p(u_r)$$



Shock:

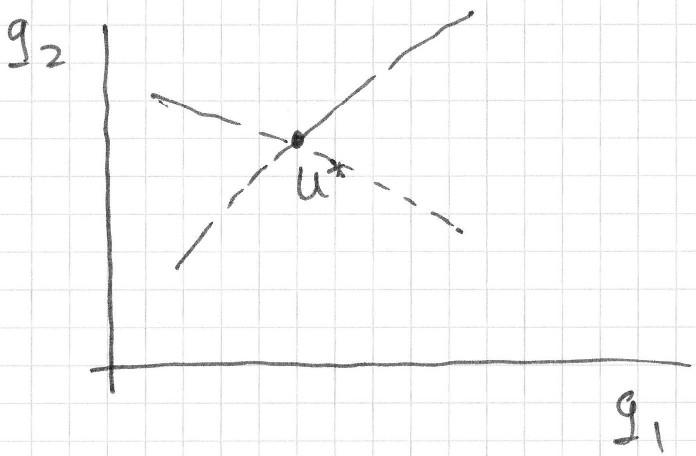
Rankine – Hugoniot shock condition:

$$\sigma \, (u_\ell - u_r) = F(u_\ell) - F(u_r)$$

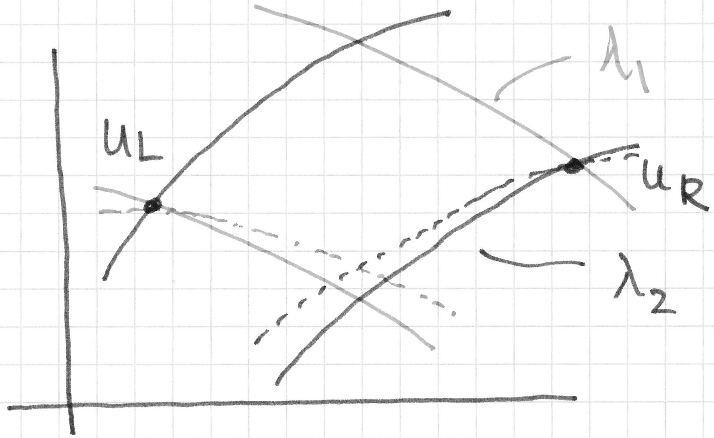
Hugoniot locus of state u^*:

collection of states u that can be connected to u^* by a shock

Two set of loci corresponding to ℓ_p, r_p. $p = 1, 2$



General solution by combining rarefactions and shocks



First "slow" wave λ_1, then "fast" wave λ_2

Choose rarefaction or shock based on "velocity profile"

Polymer:

$$\begin{cases} \dfrac{\partial}{\partial t_D} S + \dfrac{\partial}{\partial x_D} f(S,C) = 0 \\[2ex] \dfrac{\partial}{\partial t_D}(S \cdot C + a) + \dfrac{\partial}{\partial x_D}(f \cdot c) = 0 \\[2ex] S(x,0) = S_I \quad C(x,0) = C_I \\ S(0,t) = S_J \quad C(0,t) = C_J \end{cases}$$

$$x_D = \frac{x}{L} \qquad t_D = \frac{u \cdot t}{\phi \cdot L}$$

We look at:

$C_I = 0, \; S_J = 1 - S_{or}, \; C_J = 1$

$$a(c) = \frac{A_1 C}{1 + A_2 C}$$ or more importantly $a''(c) \leq 0$

$$f(S,C) = \frac{\tilde{S}^{nw}}{\tilde{S}^{nw} + M(c)(1-\tilde{S})^{no}} \qquad S^* = \frac{(S_w - S_{cw})}{1 - S_{or} - S_{cw}}$$

$$M(c) = \frac{\mu_w(c)}{k_{rw}} \frac{k_{ro}}{\mu_o}$$, $\mu_w(c)$ = some function of polymer concentration.

Rewrite :

$$\begin{cases} \dfrac{\partial S}{\partial t} + \dfrac{\partial f}{\partial S}\dfrac{\partial S}{\partial x} + \dfrac{\partial f}{\partial C}\dfrac{\partial C}{\partial x} = 0 \\[2ex] \dfrac{\partial C}{\partial t} + \dfrac{f}{S + \frac{da}{dC}}\dfrac{\partial C}{\partial x} = 0 \end{cases}$$

with $u = \begin{pmatrix} S \\ C \end{pmatrix}$

$$u_t + A\, u_x = 0$$

with

$$A = \begin{pmatrix} \dfrac{\partial f}{\partial S} & \dfrac{\partial f}{\partial C} \\[2ex] 0 & \dfrac{f}{S + \frac{da}{dC}} \end{pmatrix}$$

Eignvalues, eignvectors of A

$$\lambda_1 = \frac{\partial f}{\partial s} \,, \qquad r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = \frac{f}{s + \frac{da}{dc}} \,, \qquad r_2 = \begin{pmatrix} \frac{\partial f}{\partial c} \\ \frac{f}{s + \frac{da}{dc}} - \frac{\partial f}{\partial s} \end{pmatrix}$$

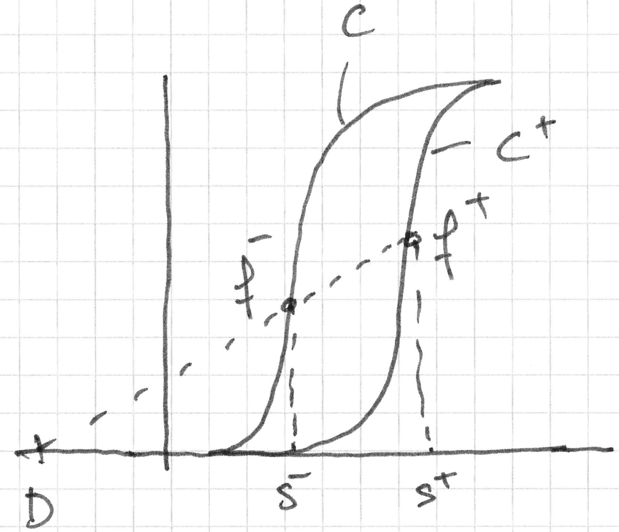
<u>Integral curves</u>:

λ_1 : C = constant

λ_2 : $$\frac{dc}{ds} = \frac{\frac{f}{s + da/dc} - \frac{\partial f}{\partial s}}{\frac{\partial f}{\partial c}}$$

Hugoniot - loci & shock speeds :

Shock condition :
$$\begin{cases} \dfrac{\partial s}{\partial t} + \dfrac{\partial f}{\partial x} = 0 \\[2ex] \dfrac{\partial (sc + a)}{\partial t} + \dfrac{\partial (f \cdot c)}{\partial x} = 0 \end{cases}$$



$$D = \frac{a^+ - a^-}{c^+ - c^-}$$

$$\Rightarrow \quad \sigma (s^+ - s^-) = f^+ - f^-$$

$$\sigma (s^+ c^+ + a^+ - s^- c^- - a^-) = f^+ c^+ - f^- c^-$$

Two loci :

① $C =$ constant : $\sigma = \dfrac{f(s^+, c) - f(s^-, c)}{s^+ - s^-}$: λ_1 shock

② $$\frac{f(s^-, c^-)}{s^- + \dfrac{(a^+ - a^-)}{(c^+ - c^-)}} = \frac{f(s^+, c^+) - f(s^-, c^-)}{s^+ - s^-} \quad : \lambda_2 \text{ shock}$$

$$\sigma = \frac{f^-}{s^- + \dfrac{a^+ - a^-}{c^+ - c^-}} = \frac{f^+}{s^+ + \dfrac{(a^+ - a^-)}{c^+ - c^-}} = \frac{f(s^+, c^+) - f(s^-, c^-)}{s^+ - s^-}$$

Rarefaction solutions:

$$\frac{dU}{d\xi} = \frac{r_p(U(\xi))}{\nabla\lambda_p(U(\xi))\cdot r_p(U(\xi))}$$

$$\lambda_1: \begin{cases} \dfrac{ds}{d\xi} = \left(\dfrac{\partial^2 f}{\partial s^2}\right)^{-1} \\ C = \text{constant} \end{cases}$$

$$\lambda_2: \quad \frac{dU}{d\xi} = \begin{pmatrix} s'(\xi) \\ c'(\xi) \end{pmatrix} = \frac{1}{\nabla\lambda_2\cdot r_p} \cdot \begin{pmatrix} \dfrac{\partial f}{\partial c} \\ \dfrac{f}{s+\frac{da}{dc}} - \dfrac{\partial f}{\partial s} \end{pmatrix}$$

Do not need λ_2 rarefaction for polymer solutions if $a''(c) \leq 0$.