Polymer flooding:

Mass conservation equation: (see Lake for definitions).

$$\frac{\partial}{\partial t}\left(\phi \sum_{j=1}^{N_p} \rho_j S_j \omega_{ij} + (1-\phi)\rho_s \omega_{is}\right) + \vec{\nabla}\cdot\left(\sum_{j=1}^{N_p} \rho_j \omega_{ij} \vec{u}_j\right) = 0$$

Components i: 1 : H_2O
2 : HC
3 : polymer
4 : rock

Phases j: 1 : water
2 : oil
s : solid.

C: mass fraction polymer in water
C_s: mass fraction polymer in solid.

$\omega_{11} = 1-C$ $\quad \omega_{21} = 0$ $\quad \omega_{31} = C$ $\quad \omega_{41} = 0$

$\omega_{12} = 0$ $\quad \omega_{22} = 1$ $\quad \omega_{32} = 0$ $\quad \omega_{42} = 0$

$\omega_{1s} = 0$ $\quad \omega_{2s} = 0$ $\quad \omega_{3s} = C_s$ $\quad \omega_{4s} = 1-C_s$

for i=1

$$\frac{\partial}{\partial t}\left(\phi \rho_w S_w (1-C)\right) + \frac{\partial}{\partial x}\left(\rho_w (1-C) u_w\right) = 0$$

$1-C \triangleq 1$
density of water unchanged

i = 2

$$\frac{\partial}{\partial t}\left(\phi \rho_o S_o\right) + \frac{\partial}{\partial x}\left(\rho_o u_o\right) = 0$$

i = 3

$$\frac{\partial}{\partial t}\left(\phi \rho_w S_w C + (1-\phi)\rho_s C_s\right) + \frac{\partial}{\partial x}\left(\rho_w C u_w\right) = 0$$

i = 4

$$\frac{\partial}{\partial t}\left((1-\phi)\rho_s (1-C_s)\right) = 0$$

$1-C_s \triangleq 1$
density of rock unchanged.

Assumptions :

1. $1-C \simeq 1$, $1-C_s \simeq 1$: water and solid densities are not impacted by polymer
2. ρ_w, ρ_o, ρ_s = constant , incompressible. $\Rightarrow$ can be divided out.

Resulting eqs :

$$\begin{cases} \phi \dfrac{\partial S_w}{\partial t} + \dfrac{\partial}{\partial x} u_w = 0 \\ \phi \dfrac{\partial S_o}{\partial t} + \dfrac{\partial}{\partial x} u_o = 0 \\ \phi \dfrac{\partial}{\partial t} \left(S_w \cdot C + \dfrac{(1-\phi)}{\phi} \dfrac{\rho_s}{\rho_w} C_s \right) + \dfrac{\partial}{\partial x} (C u_w) = 0 \end{cases}$$

Because $S_w + S_o = 1$ and $u = u_w + u_o$ = constant, the (S_o, u_o) equation is not independent and can be left out.

In the above eqs : C : mass fraction of polymer in water
C_s : mass fraction of polymer in solid

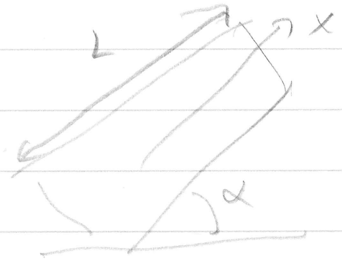
In the literature the polymer concentrations are switched to volume fractions. This can be achieved by multiplying the "polymer eq" by ρ_w/ρ_p where ρ_p is the polymer density :

$$C^{lit} = \frac{\rho_w}{\rho_p} C = \frac{\rho_w}{\rho_p} \frac{m_p}{m_w} = \frac{V_p}{V_w}$$

$$C_s^{lit} = \frac{\rho_w}{\rho_p} \frac{\rho_s}{\rho_w} C_s = \frac{\rho_s}{\rho_p} \frac{m_p}{m_s} = \frac{V_p}{V_s}$$

We will stick to the mass fraction.

Now use the Darcy eqs for water and oil to introduce a fractional flow curve: ($P_c = 0$)



$$\begin{cases} u_w = -\dfrac{k k_{rw}}{\mu_w}\left(\dfrac{\partial P}{\partial x} + \rho_w g \sin\alpha\right) \\ u_o = -\dfrac{k k_{ro}}{\mu_o}\left(\dfrac{\partial P}{\partial x} + \rho_o g \sin\alpha\right) \end{cases}$$

$$\frac{\mu_w}{k_{rw}} u_w - \frac{\mu_o}{k_{ro}} u_o = -k \, \Delta\rho \, g \sin\alpha \qquad \Delta\rho = \rho_w - \rho_o$$

$$\Rightarrow f_w = \frac{u_w}{u} = \frac{k_{rw}/\mu_w}{\dfrac{k_{rw}}{\mu_w} + \dfrac{k_{ro}}{\mu_o}} \underbrace{\left\{1 - \frac{k k_{ro}}{\mu_o} \frac{g \Delta\rho}{u} \sin\alpha\right\}}_{\text{we will ignore gravity, so this term will equal 1 } (g=0)}$$

where $u = u_o + u_w =$ constant.

So, the eqs become:

$$\begin{cases} \phi \dfrac{\partial S_w}{\partial t} + u \dfrac{\partial f_w}{\partial x} = 0 \\ \phi \dfrac{\partial}{\partial t}\left(S_w \cdot C + \dfrac{(1-\phi)}{\phi} \dfrac{\rho_s}{\rho_w} C_s\right) + u \dfrac{\partial}{\partial x}\left(f_w \cdot C\right) = 0 \end{cases}$$

Introduce dimensionless variables:

$$x_D = \frac{x}{L} \quad , \quad t_D = \frac{ut}{\phi L} = \frac{Q}{V_p} \begin{array}{l} \leftarrow \text{injected volume} \\ \leftarrow \text{pore volume} \end{array}$$

$$S^* = \frac{(S_w - S_{cw})}{(1 - S_{cw} - S_{orw})} \quad , \quad C^* = \frac{C}{C_{inj}} \quad , \quad a = \frac{(1-\phi)}{\phi} \frac{\rho_s}{\rho_w} \frac{C_s}{C_{inj}}$$

With the above the eqs become:

$$\begin{cases} \dfrac{\partial S_w}{\partial t_D} + \dfrac{\partial}{\partial x_D} f_w = 0 \\ \dfrac{\partial}{\partial t_D}\left(S_w \cdot C^* + a\right) + \dfrac{\partial}{\partial x_D}\left(f_w C^*\right) = 0 \end{cases}$$

Differentiate out the eqs:

assumption: $f_w = f_w(S, C^*)$
$a = a(C^*)$

$$\begin{cases} \dfrac{\partial S_w}{\partial t_D} + \dfrac{\partial f_w}{\partial S_w}\dfrac{\partial S_w}{\partial x_D} + \dfrac{\partial f_w}{\partial C^*}\dfrac{\partial C^*}{\partial x_D} = 0 \\ \\ \underline{C^* \dfrac{\partial S_w}{\partial t_D}} + S_w \dfrac{\partial C^*}{\partial t_D} + \dfrac{da}{dC^*}\dfrac{\partial C^*}{\partial t_D} + \underline{C^* \dfrac{\partial f_w}{\partial S_w}\dfrac{\partial S_w}{\partial x_D}} + \underline{C^* \dfrac{\partial f}{\partial C^*}\dfrac{\partial C^*}{\partial x_D}} + f_w \dfrac{\partial C^*}{\partial x_D} = \\ \qquad \text{add to zero (see above eq)} \\ = \left(S_w + \dfrac{da}{dC^*}\right)\dfrac{\partial C^*}{\partial t_D} + f_w \dfrac{\partial C^*}{\partial x_D} = 0 \end{cases}$$

Eq in matrix form:

$$\begin{pmatrix} S_w \\ C^* \end{pmatrix}_{t_D} + A(S_w, C^*) \cdot \begin{pmatrix} S_w \\ C^* \end{pmatrix}_{x_D} = 0 \quad ; \quad A = df = \begin{pmatrix} \dfrac{\partial f_w}{\partial S_w} & \dfrac{\partial f_w}{\partial C^*} \\ 0 & \dfrac{f_w}{S_w + \frac{da}{dC^*}} \end{pmatrix}$$

Parametrization of basic Functions:

Relperms: k_{rw}^{e}, S_{cw}, n_w; $k_{rw}(S_w) = k_{rw}^{e} (S^*)^{n_w}$

k_{ro}^{e}, S_{orw}, n_o; $k_{ro}(S_w) = k_{ro}^{e} (1-S^*)^{n_o}$

with $S^* = (S_w - S_{cw}) / (1 - S_{cw} - S_{orw})$

Viscosities: μ_o

$$\mu_w = \mu_w(C^*) = \mu_w \Big(1 + \underbrace{\frac{(\mu_p - \mu_w)}{\mu_w}}_{} C^* \Big) \quad \overset{\Delta\mu}{\shortparallel}$$

NB: $\mu_w(C^*=1) = \mu_p$ = polymer viscosity @ C_{inj}

$C^* = C / C_{inj}$; $\Delta\mu = \frac{(\mu_p - \mu_w)}{\mu_w}$

Adsorption:

$$a(C^*) = \frac{(1-\phi)}{\phi} \frac{\rho_s}{\rho_w} \frac{C_s}{C_{inj}} = \frac{A_1 C^*}{1 + A_2 C^*}$$

Langmuir: $C_s = \frac{aC}{1+bC} = \frac{C}{\frac{1}{a} + \frac{b}{a}C} = \frac{C}{\frac{1}{a} + C/C_{max}}$

$\Rightarrow$ $A_1 = \frac{(1-\phi)}{\phi} \frac{\rho_s}{\rho_w} a$; $A_2 = \frac{a C_{inj}}{C_{max}}$

$$a(C^*=1) = \frac{(1-\phi)}{\phi} \frac{\rho_s}{\rho_w} \frac{a}{1 + \frac{aC_{inj}}{C_{max}}} = \frac{(1-\phi)}{\phi} \frac{\rho_s}{\rho_w} \frac{C_s^{inj}}{C_{inj}}$$

C_s^{inj} = adsorption @ C_{inj}

End point mobility ratio : $M(C^*) = \frac{\mu_w(C^*)}{k_{rw}^e} \frac{k_{ro}^e}{\mu_o}$

N.B : different form used in BL code.

$$f(S_w, C^*) = \frac{(S^*)^{nw}}{(S^*)^{nw} + M(C^*)(1-S^*)^{no}}$$

$$\frac{\partial f}{\partial S} = \frac{\partial f}{\partial S^*}\frac{\partial S^*}{\partial S} = \frac{M(C^*)}{(1-S_{cw}-S_{orw})} \cdot \frac{(S^*)^{nw-1}(1-S^*)^{no-1} \cdot (n_w(1-S^*)+n_o S^*)}{((S^*)^{nw} + M(C^*)(1-S^*)^{no})^2}$$

$$\frac{\partial f}{\partial C} = \frac{-M'(C^*)(S^*)^{nw}(1-S^*)^{no}}{((S^*)^{nw} + M(C^*)(1-S^*)^{no})^2}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{M(C^*)}{(1-S_{cw}-S_{orw})^2} \frac{(1-S^*)^{no-2}(S^*)^{nw-2}}{((S^*)^{nw} + M(C^*)(1-S^*)^{no})^3} \cdot *$$

$$* \left\{ M(C^*)(1-S^*)^{no}\left[n_w^2(1-S^*)^2 + n_o(1+n_o)(S^*)^2 + n_w(1-S^*)(-1+(1+2n_o)S^*)\right] + \right.$$
$$\left. - (S^*)^{nw}\left[n_w^2(1-S^*)^2 + n_o(n_o-1)(S^*)^2 + n_w(1-S^*)(1+(2n_o-1)S^*)\right] \right\}$$

$$M(C^*) = \frac{\mu_w}{k_{rw}^e}\frac{k_{ro}^e}{\mu_o}(1+\Delta\mu C^*)$$

$$M'(C^*) = \frac{\mu_w}{k_{rw}^e}\frac{k_{ro}^e}{\mu_o}\Delta\mu$$

$$a(C^*) = \frac{A_1 C^*}{1+A_2 C^*} \qquad \frac{da}{dC} = \underbrace{\frac{A_1}{(1+A_2 C^*)^2}}_{>0} \qquad \frac{d^2a}{dC^2} = \underbrace{\frac{-2A_1 \cdot A_2}{(1+A_2 C)^3}}_{<0}$$

Polymer eqs:

$$\begin{pmatrix} S_w \\ c^* \end{pmatrix}_{t_D} + A(S_w, c^*) \begin{pmatrix} S_w \\ c^* \end{pmatrix}_{x_D} = 0$$

with

$$A(S_w, c^*) = df_w = \begin{pmatrix} \frac{\partial f_w}{\partial S_w} & \frac{\partial f_w}{\partial c^*} \\ 0 & \frac{f_w}{S_w + \frac{da}{dc^*}} \end{pmatrix}$$

Eigenvalues:

$$\lambda_1 = \frac{\partial f_w}{\partial S_w} \quad , \quad \lambda_2 = \frac{f_w}{S_w + \frac{da}{dc^*}}$$

Eigenvectors:

$$r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad r_2 = \begin{pmatrix} \frac{\partial f_w}{\partial c^*} \\ \frac{f_w}{S_w + \frac{da}{dc^*}} - \frac{\partial f_w}{\partial S_w} \end{pmatrix}$$

Note: From now on we will leave out the subscript w in S_w and the $*$ in c^*, unless this causes confusion.

Integral curves : smooth curve through state space (S,C) parametrized by scalar parameter ξ

$\tilde{q}'(\xi)$ is an integral curve of the vector field $r^p(\tilde{q}(\xi))$, if at each point $\tilde{q}(\xi)$, the tangent vector $\tilde{q}'(\xi)$ to the curve is an eigenvector of $df(\tilde{q}(\xi))$ corresponding to the eigen value $\lambda^p(\tilde{q}(\xi))$

So : $A\,\tilde{q}'(\xi) = \lambda^p(\tilde{q}(\xi))\,\tilde{q}'(\xi)$

$$\Rightarrow \left(A - \lambda^p(\tilde{q}(\xi))I\right)\tilde{q}'(\xi) = 0$$

For λ_1 :

$$\begin{pmatrix} 0 & \frac{\partial f_w}{\partial C} \\ 0 & \frac{f_w}{S+\frac{da}{dC}} - \frac{\partial f_w}{\partial S} \end{pmatrix}\begin{pmatrix} S'(\xi) \\ C'(\xi) \end{pmatrix} = 0$$

$$\Rightarrow C'(\xi) = 0 \qquad \Rightarrow C(\xi) = \text{constant}$$

For λ_2 :

$$\begin{pmatrix} \frac{\partial f_w}{\partial S} - \frac{f_w}{S+\frac{da}{dC}} & \frac{\partial f_w}{\partial C} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S'(\xi) \\ C'(\xi) \end{pmatrix} = 0$$

$$\left(\frac{\partial f_w}{\partial S} - \frac{f_w}{S+\frac{da}{dC}}\right)S'(\xi) + \frac{\partial f_w}{\partial C}C'(\xi) = 0$$

$$\Rightarrow \frac{dC}{dS} = \frac{\frac{f_w}{S+\frac{da}{dC}} - \frac{\partial f_w}{\partial S}}{\frac{\partial f_w}{\partial C}} \qquad \text{ODE}$$

$$\text{or} \quad \frac{dS}{dC} = \frac{\partial f_w/\partial C}{\frac{f_w}{S+\frac{da}{dC}} - \frac{\partial f}{\partial S}}$$

$$\frac{dS}{dC} = \frac{\frac{\partial f}{\partial C}}{\frac{f_w}{S_w+\frac{da}{dC}} - \frac{\partial f}{\partial S}} = \frac{\frac{\partial f}{\partial C}\left(S_w+\frac{da}{dC}\right)}{f_w - \frac{\partial f}{\partial S}\left(S_w+\frac{da}{dC}\right)}$$

$$D = \left((S^*)^{n_w} + M(C)(1-S^*)^{n_o}\right)$$

$$= \frac{\frac{-M'(C)(S^*)^{n_w}(1-S^*)^{n_o}\left(S_w+\frac{da}{dC}\right)}{D^2}}{\frac{(S^*)^{n_w}}{D} - \frac{M(C)}{(1-S_{cw}-S_{orw})}\frac{(S^*)^{n_w-1}(1-S^*)^{n_o-1}\left(n_w(1-S^*)+n_o S^*\right)\left(S_w+\frac{da}{dC}\right)}{D^2}}$$

extract $(S^*)^{n_w-1}$

note S_w, not S^*

$$= \frac{-M'(C)\,S^*(1-S^*)^{n_o}\left(S_w+\frac{da}{dC}\right)}{\underbrace{S^* D - \frac{M(C)}{1-S_{cw}-S_{orw}}(1-S^*)^{n_o-1}\left(n_w(1-S^*)+n_o S^*\right)\left(S_w+\frac{da}{dC}\right)}}$$

Denominator can be zero for some $0 < S^* < 1$, but not for $S^*=0$ or $S^*=1$ (for $n_o > 1$)

For $S_w = 1 - S_{orw}$, $C = 1$: $S_w = 1 - S_{orw}$ is a solution:

Check: $\left(\frac{\partial f_w}{\partial S} - \frac{f_w}{S+\frac{da}{dC}}\right)S'(\xi) + \frac{\partial f}{\partial C}C'(\xi) = 0$

for $n_o > 1$; $n_o > 0$

$\frac{\partial f_w}{\partial S} = 0$, $\frac{f_w}{S+\frac{da}{dC}} \neq 0$, $\frac{\partial f}{\partial C} = 0$

$\Rightarrow S'(\xi) = 0$ with $S_w = 1 - S_{orw}$ as boundary condition,

$\Rightarrow S_w = 1 - S_{orw}$, $C = 1$ is solution

Above also holds for $n_o = 1$

Rarefaction waves

$$R^P(x,t) = \begin{cases} g_l & x/t \le \xi_1 \\ \tilde{q}(x/t) & \xi_1 < x/t < \xi_2 \\ g_r & \frac{x}{t} \geq \xi_2 \end{cases}$$

g_l and g_r on a single integral curve with $\lambda^P(g_l) < \lambda^P(g_r)$
and

$$\begin{cases} \tilde{q}'(\xi) = \dfrac{r^P(\tilde{q}(\xi))}{\nabla\lambda^P(\tilde{q}(\xi)) \cdot r^P(\tilde{q}(\xi))} \\ \xi_1 = \lambda^P(g_l) \ \& \ \tilde{q}(\xi_1) = g_l \\ \xi_2 = \lambda^P(g_r) \ \& \ \tilde{q}(\xi_2) = g_r \end{cases}$$

ODE's

For λ_1:

$$\nabla\lambda^1 \cdot r^1 = \left(\frac{\partial^2 f_w}{\partial S^2}, \frac{\partial^2 f_w}{\partial S \partial C} \right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\partial^2 f_w}{\partial S^2}$$

$$\begin{cases} S'(\xi) = 1 \Big/ \frac{\partial^2 f_w}{\partial S^2} \\ C'(\xi) = 0 \end{cases}$$

(see also next page)

For λ_2:

$$\nabla\lambda^2 \cdot r^2 = \left(\frac{\frac{\partial f}{\partial S}}{(S + \frac{da}{dC})} - \frac{f_w}{(S+\frac{da}{dC})^2}, \frac{\frac{\partial f_w}{\partial C}}{(S+\frac{da}{dC})} - \frac{f_w}{(S+\frac{da}{dC})^2} \frac{d^2a}{dC^2} \right) \cdot r_2$$

$$= \left(\frac{\partial f_w}{\partial S} - \frac{f_w}{S + \frac{da}{dC}} \right) \cdot \frac{f_w}{(S+\frac{da}{dC})^2} \cdot \frac{d^2 a}{dC^2}$$

$$S'(\xi) = \frac{\frac{\partial f_w}{\partial C}}{\nabla\lambda^2 \cdot r^2}$$

$$C'(\xi) = \frac{\left(\frac{f_w}{S + da/dC} - \frac{\partial f_w}{\partial S} \right)}{\nabla\lambda^2 \cdot r^2}$$

Rarefaction solution:

1-wave: (from Leveque)

$$\xi_t + \lambda^1(\tilde{q}(\xi))\,\xi_x = 0 \qquad \xi = f(x,t)$$

$\tilde{q}(\xi)$ 1-integral curve : C = constant

$$\xi_t + \frac{\partial f}{\partial S}(\tilde{S}(\xi), C = \text{constant})\,\xi_x = 0$$

if: $\xi = \frac{x}{t}$ $\quad -\frac{x}{t^2} + \frac{\partial f}{\partial S}(\tilde{S}(\xi), C = \text{constant})\,\frac{1}{t} = 0$

$$\frac{\partial f}{\partial S}(\tilde{S}(\xi), C = \text{constant}) = \frac{x}{t} = \xi \implies \tilde{S}(\xi) \text{ can be found by inverting}$$

$$\begin{cases} S(x,t) = \begin{cases} S_l & x/t < \xi_1 \\ \tilde{S}(\xi) & \xi_1 < x/t < \xi_2 \\ S_r & \frac{x}{t} > \xi_2 \end{cases} \\ C = \text{constant} \end{cases}$$

$$\xi_1 = \lambda^1(q_l), \quad \xi_2 = \lambda^2(q_r)$$

Equivalent to:

$$\begin{cases} S'(\xi) = \dfrac{1}{\frac{\partial^2 f}{\partial S^2}(S(\xi), C = \text{constant})} \\ C'(\xi) = 0 \end{cases}$$

Integration in Python works, works better by integrating from S_r

"Shocks"

Back to the original eqs:
$$\begin{cases} \dfrac{\partial S}{\partial t_D} + \dfrac{\partial}{\partial x_D} f_w = 0 \\ \dfrac{\partial}{\partial t_D}\left(S \cdot C + a\right) + \dfrac{\partial}{\partial x_D}\left(f_w \cdot C\right) = 0 \end{cases}$$

Shock condition
$$\sigma = \frac{f^+ - f^-}{S^+ - S^-}$$
$$\sigma = \frac{f^+ C^+ - f^- C^-}{(S^+C^+ + a^+) - (S^-C^- + a^-)}$$

$$\Rightarrow (S^- f^+ - S^+ f^-)(C^+ - C^-) + (a^+ - a^-)(f^+ - f^-) = 0$$

$$\Rightarrow f^+ - f^- = \frac{(S^+ f^- - S^- f^+)(C^+ - C^-)}{(a^+ - a^-)}$$ if $a^+ \neq a^- \Rightarrow C^+ \neq C^-$

$$\sigma = \frac{(f^+ - f^-)}{(S^+ - S^-)} = \frac{(S^+ f^- - S^- f^+)(C^+ - C^-)}{(a^+ - a^-)(S^+ - S^-)} =$$

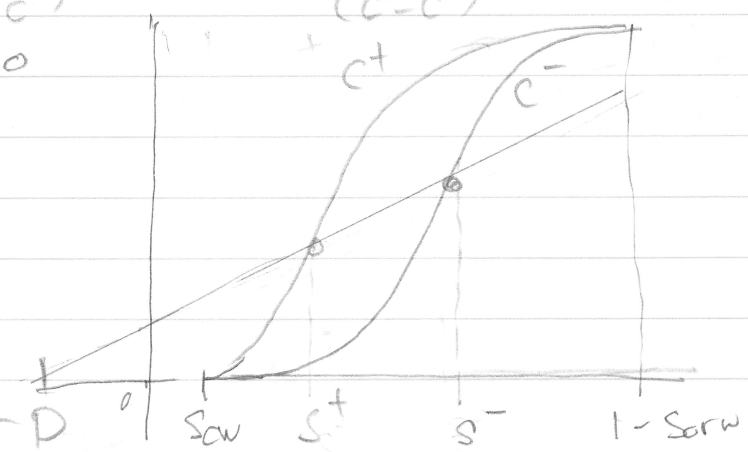
$f^+ = f^- + \sigma(S^+ - S^-)$ or $f^- = f^+ - \sigma(S^+ - S^-)$

$$= \frac{(f^- - \sigma S^-)(C^+ - C^-)}{(a^+ - a^-)} = \frac{(f^+ - \sigma S^+)(C^+ - C^-)}{a^+ - a^-}$$

Solving for σ:

$$\sigma = \frac{f^-}{S^- + \dfrac{(a^+ - a^-)}{(C^+ - C^-)}} = \frac{f^+}{S^+ + \dfrac{(a^+ - a^-)}{(C^+ - C^-)}} = \frac{f^+ - f^-}{S^+ - S^-}$$

$$\frac{da}{dC} > 0 \Rightarrow \frac{a^+ - a^-}{C^+ - C^-} \equiv D > 0$$



NB The Rankine-Hugoniot condition is based on the integral mass conservation.

$$f^+ - f^- = \sigma (s^+ - s^-)$$

$$f^+c^+ - f^-c^- = \sigma \{ (s^+c^+ + a^+) - (s^-c^- + a^-) \}$$

For $c^+ = c^-$; $a^+ = a^-$: The equations are equal

$$\Rightarrow \quad \sigma = \frac{f^+ - f^-}{s^+ - s^-} \quad \text{and}$$

Oleinik entropy condition : $\frac{f(s,c) - f(s_\ell, c)}{s - s_\ell} \geq \sigma \geq \frac{f(s,c) - f(s_r, c)}{s - s_r}$

for all S between S_ℓ and S_r

For $c^+ \neq c^-$:

$$f^+c^+ - f^-c^- \overset{1^{st}\,eq}{=} f^-c^+ - f^-c^- + \sigma c^+ (s^+ - s^-)$$

$$\Rightarrow \quad (c^+ - c^-) f^- = \sigma (a^+ - a^-) + \sigma \left[-c^+(\cancel{s^+} - s^-) + \cancel{s^+}c^+ - s^-c^- \right]$$

$$= \sigma \{ (c^+ - c^-) s^- + (a^+ - a^-) \}$$

$$\Rightarrow$$

$$\sigma = \frac{f^-}{s^- + \frac{(a^+ - a^-)}{(c^+ - c^-)}} = \frac{f^-}{s^- + h(c^+, c^-)}$$

$$\sigma (s^+ + h(c^+, c^-)) \underset{1^{st}\,eq}{=} \sigma (s^- + h(c^+, c^-)) + f^+ - f^- = f^+$$

Rankine-Hugoniot condition :

$$\boxed{\frac{f^-}{s^- + h(c^+, c^-)} = \frac{f^+}{s^+ + h(c^+, c^-)} = \sigma} \qquad h(c^+, c^-) = \begin{cases} \frac{a^+ - a^-}{c^+ - c^-} & c^+ \neq c^- \\ \frac{da}{dc} & c^+ = c^- \end{cases}$$

Condition : $c_L > c_R$ and $\lambda^1(u^R) < \sigma$ or $\lambda^1(u^R), \lambda^1(u_L) \geq \sigma$

Hugoniot loci

For 1-wave (λ_1): $c^+ = c^-$ $\Rightarrow$ $a^+ = a^-$

$\sigma_1 = \dfrac{f^+ - f^-}{s^+ - s^-}$, Hugoniot locus: C = constant

$\sigma_2 = \dfrac{f^+ - f^-}{s^+ - s^-}$

For 2-wave (λ_2): $c^+ \neq c^-$

Need to solve
$$\frac{f^+}{s^+ + \left(\frac{a^+ - a^-}{c^+ - c^-}\right)} = \frac{f^+ - f^-}{s^+ - s^-} \quad (1)$$

Hugoniot locus: given (s^+, c^+) find curve (s^-, c^-) that fulfills (1)

Equivalently solve: $f^+(s^+ - s^-) - (f^+ - f^-)\left(s^+ + \dfrac{(a^+ - a^-)}{(c^+ - c^-)}\right) = 0$

when solving we used $\dfrac{a^+ - a^-}{c^+ - c^-} \approx \dfrac{da(c^+)}{dc}$ for $c^+ - c^- < \varepsilon$

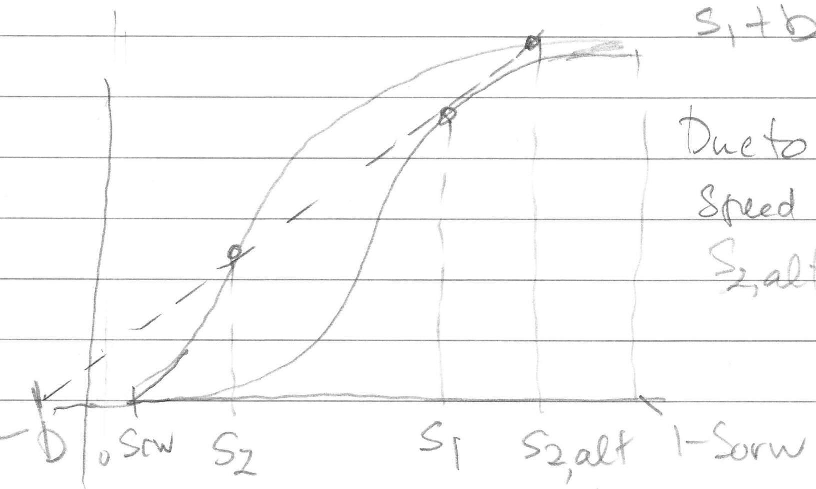
Note: $C_1 < C_2 \Rightarrow M(C_1) < M(C_2) \Rightarrow f(s, C_1) \geq f(s, C_2)$ for all S

→ Tangent requirement from mass balance (page 14 b&c)

For the solution one needs to find an $(S_1, C=1)$ and $(S_2, C=0)$ such that $\lambda_1(S_1, C=1) = \sigma_2(S_1, C=1; S_2, C=0)$.

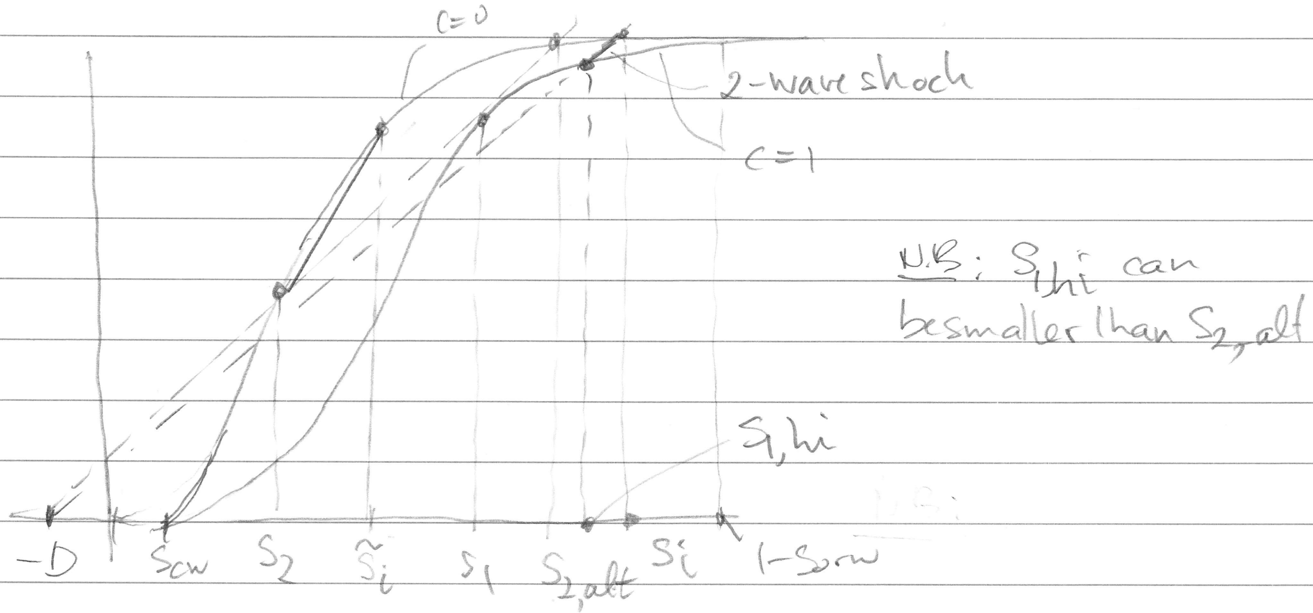
S_1 can be found from:
$$\frac{\partial f}{\partial s}(S_1, C=1) = \frac{f(S_1, C=1)}{S_1 + D} \quad ; \quad D = \frac{a(1) - a(0)}{1 - 0}$$

Two possible solutions for S_2: $\dfrac{f(S_1, C=1)}{S_1 + D} = \dfrac{f(S_1, C=1) - f(S, C=0)}{S_1 - S}$



Due to requirement of increasing speed on the solution, $S_{2,alt}$ is not a valid option.

The tangent solution breaks down if S_i is too large. In particular, if $S_i > S_{2,alt}$, the shock speed from S_2 to S_i is less than the shock speed from S_1 to S_2.



For this case the "initial" 1-wave rarefaction (C=1) only runs to $S_{1,hi}$ ($> S_1$) and it is followed by a 2-wave shock to S_i.

For $\tilde{S}_i < S_{2,alt}$, the solution consists of a 1-wave rarefaction (C=1), followed by by 2-wave shock from S_1 to S_2 and then a 1-wave shock (C=0) from S_2 to $\tilde{S}_i$.

Note that there are 3 possible solutions from S_2 to $\tilde{S}_i$:

- 1-wave shock (C=0)
- 1-wave rarefaction + shock (C=0)
- 1-wave rarefaction (C=0)

depending on the value of $\tilde{S}_i$ ($< S_{2,alt}$)

For 2-waves

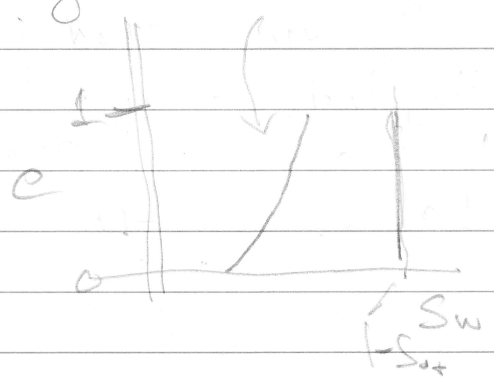
$$\begin{cases} S_w = 1 - S_{orw} \ (S^* = 1) \\ C = 1 \end{cases}$$ Hugoniot Locus

$$\frac{f(\overbrace{S^*=1, C=1}^{=1})}{1 - S_{orw} + \frac{a(1)-a(C)}{1-C}} = \frac{f(S_w, C)}{S_w + \frac{a(1)-a(C)}{1-C}}$$

- One can see that:

$S_w = 1 - S_{orw}$, $0 \le C \le 1$ is a solution as $f(1-S_{orw}, C) = 1$

- There is another branch that fulfills the Rankine-Hugoniot condition. Is determined numerically



$$\begin{cases} S_w = S_{cw} \ (S^* = 0) \\ C = 0 \end{cases}$$

$$\frac{f(\overbrace{S^*=0, C=0}^{=0})}{S_{cw} + \underbrace{\frac{a(0)-a(C)}{0-C}}_{=0}} = \frac{f(S_w, C)}{S_w + \frac{a(0)-a(C)}{0-C}}$$

$\Rightarrow$

$$\frac{0}{S_{cw} + a(C)/C} = \frac{f(S_w, C)}{S_w + \frac{a(C)}{C}}$$

Solution: $(S_w = S_{cw}, C)$

Contact discontinuity:

For the case without adsorption, we can have contact discontinuity for the 2-waves:

No adsorption $\Rightarrow a(c) = c$.

From page (10): $\nabla\lambda^2 \cdot r^2 = 0$ because $\frac{d^2a}{dC^2} = 0$

$\Rightarrow$ 2 wave is linearly degenerate, i.e. λ^2 is constant on integral curves

$\Rightarrow$ results in a contact disconti...

Because: integral curve: $(S(\xi), C(\xi)) = q(\xi)$

$$\frac{d}{d\xi}\lambda^2(S(\xi), C(\xi)) = \frac{\partial\lambda^2}{\partial S}\frac{dS}{d\xi} + \frac{\partial\lambda^2}{\partial C}\frac{dC}{d\xi} = \left(\frac{\partial\lambda^2}{\partial S}, \frac{\partial\lambda^2}{\partial C}\right)\cdot\left(\frac{dS}{d\xi}, \frac{dC}{d\xi}\right)$$

$$= \nabla\lambda^2 \cdot q'(\xi) = \alpha(\xi)\nabla\lambda^2 \cdot r^2 = 0$$

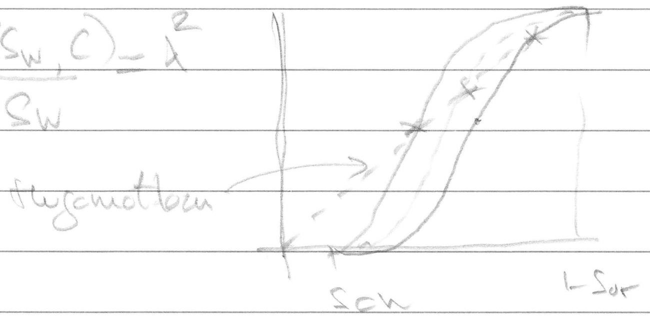
integral curve $q'(\xi) = \alpha(\xi) r^2(\xi)$

Note that the shock speeds are also constant on the Hugoniot locus and equal to the λ^2 speed:

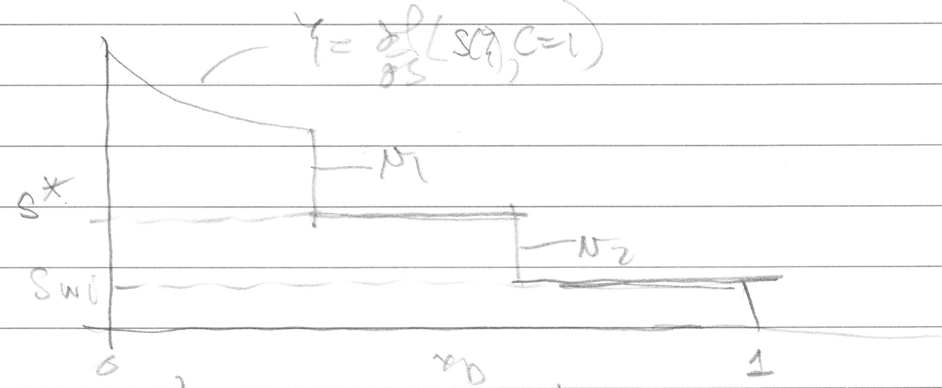
Hugoniot locus: $\frac{f(S_w^*, C^*)}{S_w^*} = \frac{f(S_w^*, C^*) - f(S_w, C)}{S_w^* - S_w}$
$(a = c)$

$$\Rightarrow f(S_w^*, C^*)(S_w^* - S_w) = S_w^*\left(f(S_w^*, C^*) - f(S_w, C)\right)$$

$$\Rightarrow \frac{f(S_w^*, C^*)}{S_w^*} = \frac{f(S_w, C)}{S_w} = \lambda^2$$



Polymer tangent condition.



$$\bar{S}(t) = \int_0^1 S(x,t)\,dx = \int_0^{\nu_1 t} S(x,t)\,dx + \int_{\nu_1 t}^{\nu_2 t} S^*\,dx + \int_{\nu_2 t}^{1} S_{wi}\,dx$$

$\xi = \frac{x}{t}$,

$$= t\int_{\xi=0}^{\xi=\nu_1} S(\xi)\,d\xi + S^*(\nu_2-\nu_1)t + S_{wi}(1-\nu_2 t)$$

$$= t\,S(\xi)\,\xi\Big|_{\xi=0}^{\xi=\nu_1} - t\int_{S(\xi=0)}^{S(\xi=\nu_1)} \xi\,dS + S^*(\nu_2-\nu_1)t + S_{wi}(1-\nu_2 t)$$

$$= t\,S(\xi=\nu_1)\cancel{\nu_1} + t\left[1 - f(S(\xi=\nu_1), C=1)\right] + S^*(\nu_2-\cancel{\nu_1})t + S_{wi}(\cancel{1}-\nu_2 t)$$

$$\nu_2 = \frac{f(S^*, C=0) - f(S_{wi}, C=0)}{S^* - S_{wi}}$$: Buckley Leverett knowledge

$$= S_{wi} + (S(\xi=\nu_1) - S^*)\nu_1 t + (S^* - S_{wi})\nu_2 t + \left[1 - f(S(\xi=\nu_1), C=1)\right] t$$

$$= S_{wi} + (S(\xi=\nu_1) - S^*)\nu_1 t + \left[f(S^*, C=0) - f(S_{wi}, C=0)\right]\cdot t\ +$$

$$+ \left[1 - f(S(\xi=\nu_1), C=1)\right]\cdot t$$

14d

$$V_p t$$
$$\parallel$$

Net injected water $= V_w(t) = \left[1 - f(S_{wi}, C=0)\right] Q\, tot$

$$\bar{S}_w(t) = \frac{S_{wi} V_p + V_w(t)}{V_p} = S_{wi} + \left[1 - f(S_{wi}, C=0)\right] \cdot t$$

Equating two expressions

$$S_{wi} + (S(\xi=\nu_1) - S^*)\,\nu_1 t + \left[f(S^*, C=0) - \cancel{f(S_{wi}, C=0)}\right] \cdot t$$
$$+ \left[\cancel{1} - f(S(\xi=\nu_1), C=1)\right] \cdot t = S_{wi} + \left[\cancel{1} - \cancel{f(S_{wi}, C=0)}\right] t$$

$$\Rightarrow (S(\xi=\nu_1) - S^*)\,\nu_1 = \left[f(S(\xi=\nu_1), C=1) - f(S^*, C=0)\right]$$

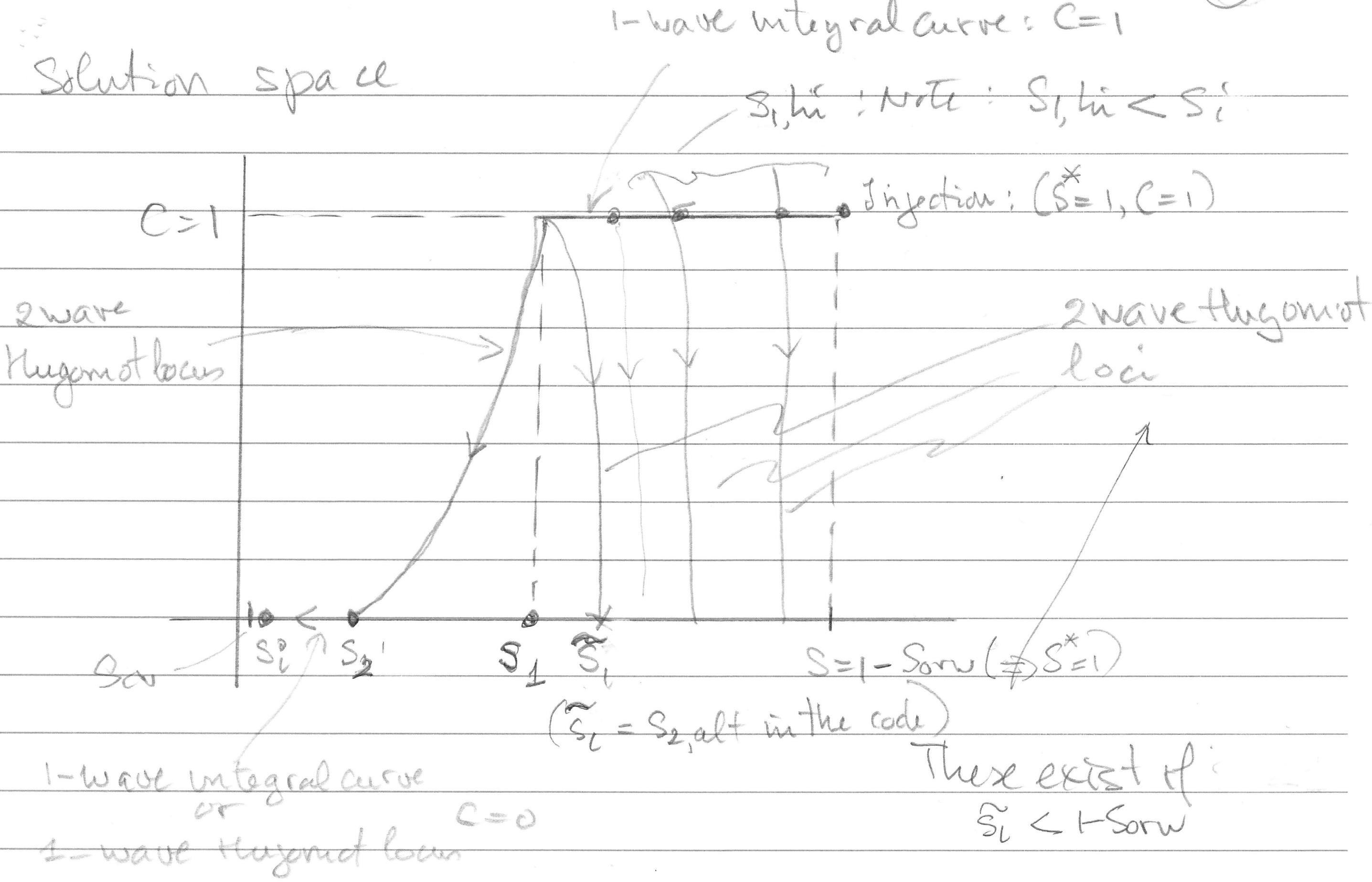
$$\left.\begin{aligned} \Rightarrow \nu_1 &= \frac{f(S(\xi=\nu_1), C=1) - f(S^*, C=0)}{S(\xi=\nu_1) - S^*} \\ \xi = \nu_1 &= \frac{\partial}{\partial S} f(S(\xi=\nu_1), C=1) \end{aligned}\right\}$$

$$\Rightarrow \frac{\partial f(S(\xi=\nu_1), C=1)}{\partial S} = \underbrace{\frac{f(S(\xi=\nu_1), C=1) - f(S^*, C=0)}{S(\xi=\nu_1) - S^*}}_{\text{2-wave shock}}$$

tangent condition

$$= \frac{f(S(\xi=\nu_1), C=1)}{S(\xi=\nu_1) + \frac{a(1) - a(0)}{1-0}}$$

Solution space



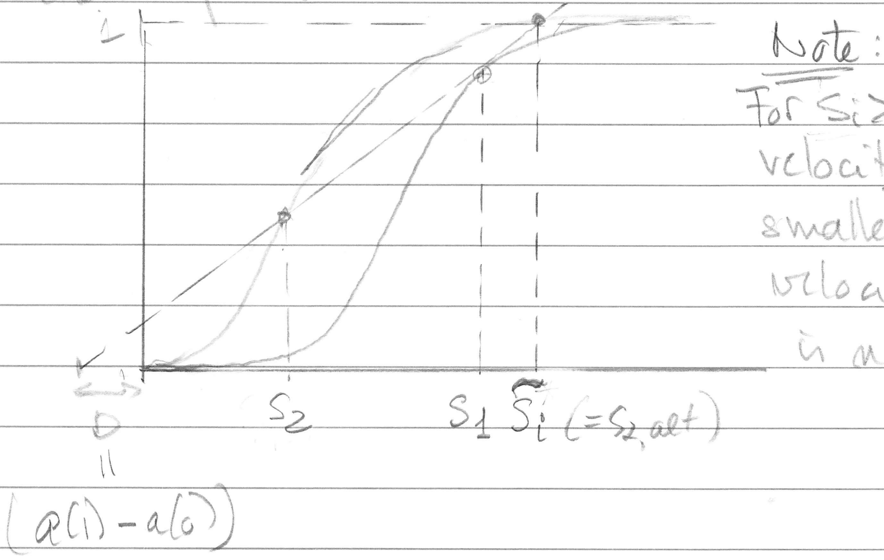
If $S_i < \tilde{S}_i$

① solution = 1-wave rarefaction + 2-wave shock + { 1wave-rarefaction or 1 wave-shock or 1wave rarefaction + shock }

If $S_i > \tilde{S}_i$

② solution = 1-wave rarefaction + 2 wave shock.

Position of $\tilde{S}_i$



Note:
For $S_i > \tilde{S}_i$ the 1-wave shock velocity in solution ① is smaller than the 2-wave shock velocity and solution ① is no longer valid

Special case : $n_o = 1$

If $n_o = 1$, Sw_1 can stop to exist depending in particular on the value of $M(C=1)$ (typically large value)

(N.B: as long as Sw_1 exists, the "general" solution can be followed)

This is caused by : $\lambda_1(1-S_{orw}, C=1) = \frac{\partial f}{\partial S_w}(1-S_{orw}, C=1) \neq 0$

In this case the solution does not start with an 1-wave rarefaction $(C=1)$, but with a 2-wave shock:

$(S_w = 1-S_{orw}, C=1) \longrightarrow (Sw_2, C=0)$

where Sw_2 is found by solving the Rankine-Hugoniot condition:

$$\frac{f(Sw_2, C=0)}{Sw_2 + a(1)} = \frac{f(1-S_{orw}, C=1)}{1-S_{orw} + a(1)}$$

Two scenarios:

① Two solution exists for the Rankine-Hugoniot condition:

a $1-S_{orw}$ and b $Sw_2 < 1-S_{orw}$.

The "correct" shock is for $Sw_2 < 1-S_{orw}$, due to speed ordering with the following 1-wave.

② Only one solution; $Sw_2 = 1-S_{orw}$.

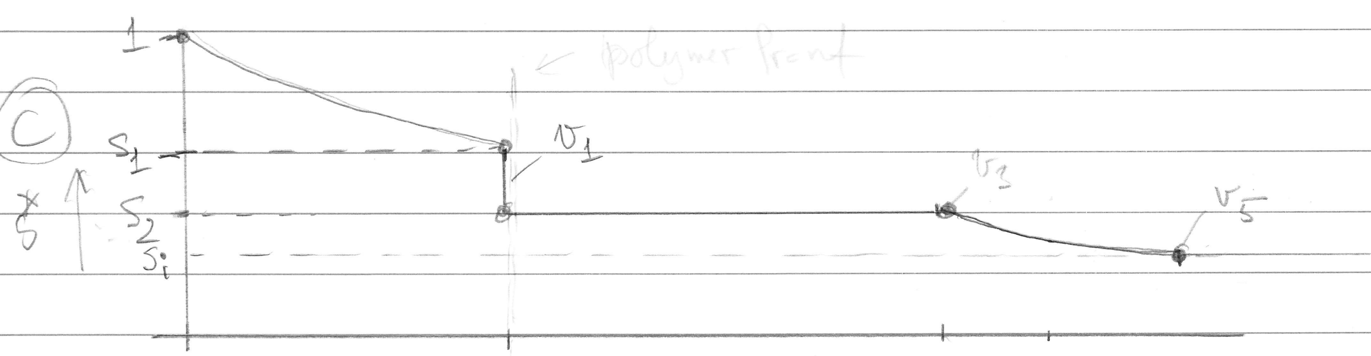
This is an admissible shock and the speed ordering is correct.

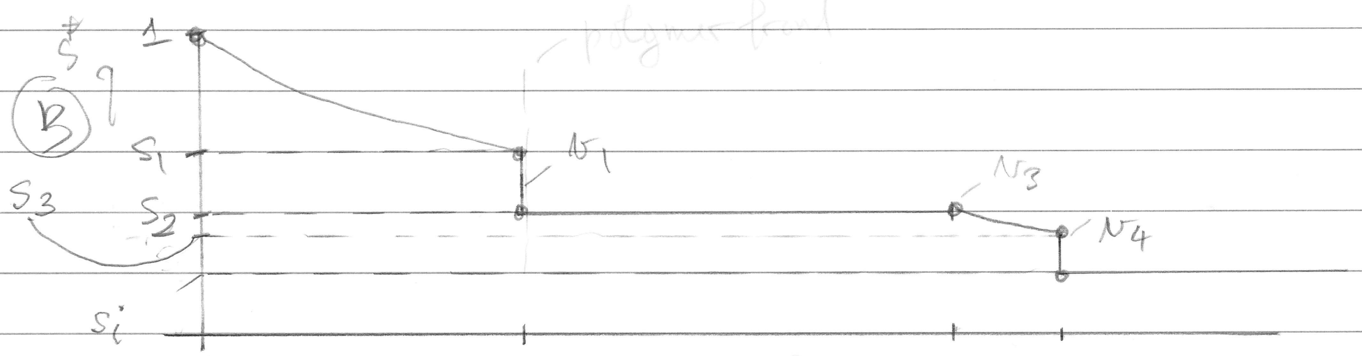
Note that solution is possible because the following 1-wave speed is > 0 as $n_o = 1$

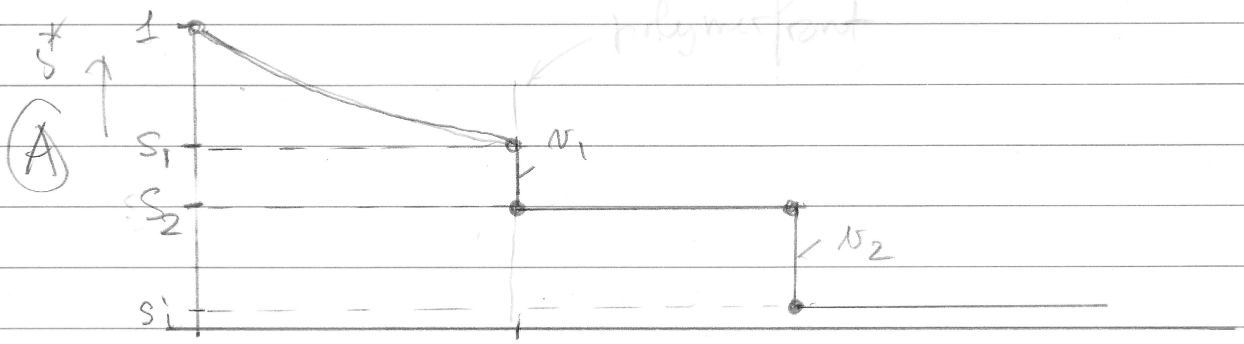
The above does not impact the analytical Sorg calculations in the following pages, because

$Sw_1 = 1-S_{orw}$, $f(Sw_1, C=1) = 1$ is used in the code.

Solution ① ($S_i < \tilde{S}_i$): 3 possibilities depending on S_i







$$v_1 = \mathrm{lam}_1(S_1, C=1) = \frac{f_w(S_1, C=1) - f_w(S_2, C=0)}{S_1 - S_2} = \sigma_2$$

$$v_2 = \frac{f_w(S_2, C=0) - f_w(S_i, C=0)}{S_2 - S_i}$$

$$v_3 = \mathrm{lam}_1(S_2, C=0)$$

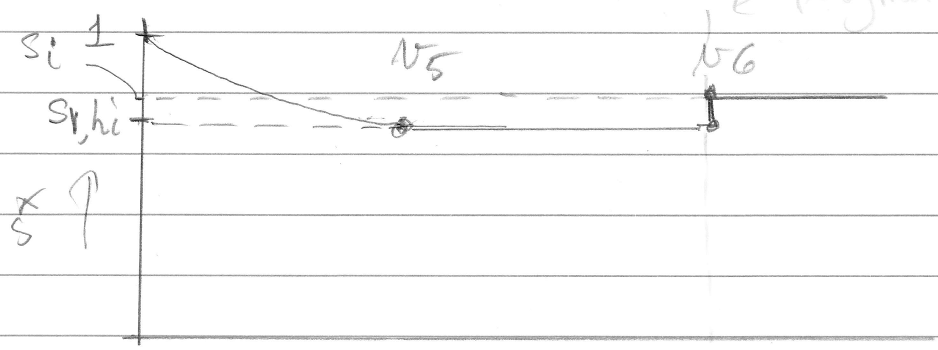
$$v_4 = \frac{f_w(S_3, C=0) - f_w(S_i, C=0)}{S_3 - S_i}$$

$$v_5 = \mathrm{lam}_1(S_i, C=0)$$

Define breakthrough times: $t_{B_i} = \dfrac{1}{v_i}$

N.B.: similar set of solutions exists for $S_i > S_{2,0}$ ($S_i < S_i^*$), saturation then rarefies, shocks upwards.

Solution 2 $(S_i > S_i^*)$:



Note $S_{1,hi} \leq S_i$

$v_5 = \text{lam1}(S_{1,hi}, C=1)$

$$v_6 = \frac{f_w(S_{1,hi}, C=1) - f_w(S_i, C=0)}{S_{1,hi} - S_i} = \sigma_2$$

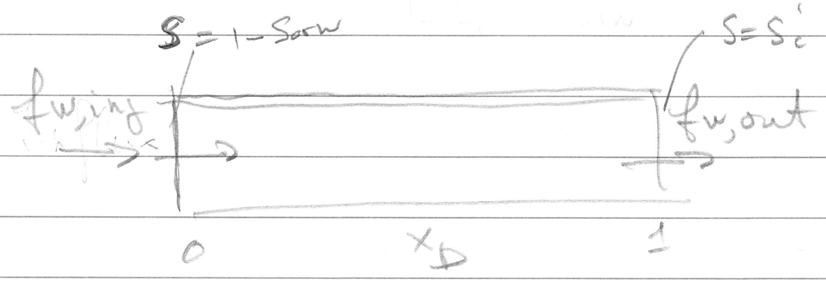
Again $t_{Bi} = \dfrac{1}{v_i}$

Calculation of average saturation $\bar{S}(t_D)$ in interval $0 \le x_D \le 1$

Before breakthrough (i.e $S(x=1, t_D) = S_i$)

Assumptions: $f_{w,inj} = 1$, $0 \le S_i \le \tilde{S}_i$

NB :



Influx : $Q_w^{in} = f_{w,inj} \cdot Q_{tot} = Q_{tot}$ ($f_{w,inj}=1$)

Outflux : $Q_w^{out} = f_w(S_i, C=0)\, Q_{tot}$

Net influx $= Q_w^{net} = Q_w^{in} - Q_w^{out} = (1 - f_w(S_i, C=0))\, Q_{tot}$

$Q_{tot} \overset{\text{seepage}}{=} V_p \cdot t_D$

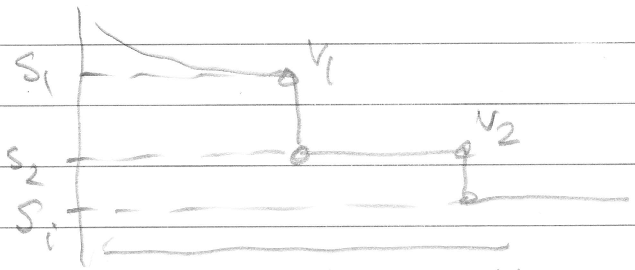
$V_w(t_D) = S_{wi} V_p + Q_w^{net}(t_D)$

$= S_{wi} V_p + (1 - f_w(S_i, C=0))\, V_p \cdot t_D$

$\bar{S}_w(t_D) = \dfrac{V_w(t_D)}{V_p} = S_{wi} + (1 - f_w(S_i, C=0))\, t_D$

After breakthrough:

Solution A



For $t_{B2} \le t_D \le t_{B1}$

$$\bar{S}(t) = \int_0^1 S(x,t)\,dx \overset{\xi = x/t}{=} t_D \int_0^{1/t_D} S(\xi)\,d\xi = t_D \left\{ \underbrace{\int_0^{v_1} S(\xi)\,d\xi}_{①} + \underbrace{\int_{v_1}^{1/t_D} \underbrace{S(\xi)}_{=S_2}\,d\xi}_{②} \right\}$$

$$① = \int_0^{v_1} S(\xi)\,d\xi = S(\xi)\,\xi \Big|_0^{v_1} - \int_{S(\xi=0)=1-S_{orw}}^{S(\xi=v_1)=S_1} \xi\,dS = S_1 v_1 - \int_{S=1-S_{orw}}^{S_1} \frac{\partial f}{\partial S}(S)\,dS \qquad \xi = \frac{\partial f}{\partial S}(S(\xi))$$

$$= S_1 v_1 - \Big[f(S_1, C=1) - \underbrace{f(1-S_{orw}, C=1)}_{=1} \Big] = S_1 v_1 + \big(1 - f(S_1, C=1)\big)$$

$$② = S_2 \left(1/t_D - v_1 \right)$$

$$\Rightarrow \bar{S}(t) = t_D \left\{ S_1 v_1 + \big(1 - f(S_1, C=1)\big) + S_2\left(1/t_D - v_1\right) \right\}$$

$$= S_2 + (S_1 - S_2)\, v_1\, t_D + t_D\big(1 - f(S_1, C=1)\big) =$$ use formula for v_1

$$= S_2 + t_D\big(1 - f_w(S_2, C=0)\big)$$

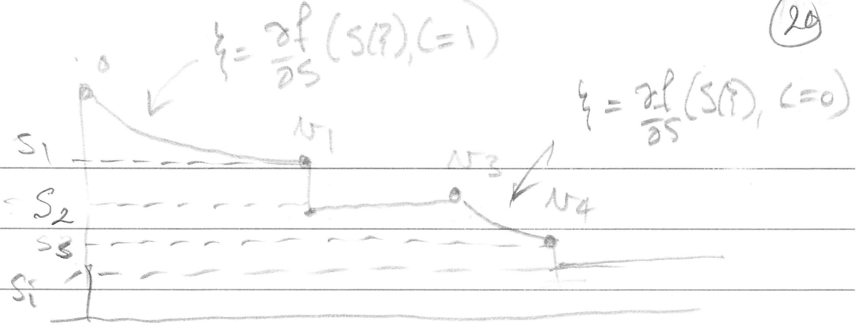
For $t_D \ge t_{B1}$:

$$\bar{S}(t) = \int_0^1 S(x,t)\,dx = t_D \int_0^{1/t_D} S(\xi)\,d\xi = t_D \left\{ S(\xi)\,\xi \Big|_0^{1/t_D} - \int_{S(\xi=0)=1-S_{orw}}^{S(\xi=1/t_D)} \xi\,dS \right\}$$

$$= t_D \left\{ S(\xi = 1/t_D)\,\frac{1}{t_D} - \Big[f\big(S(\xi = 1/t_D), C=1\big) - f(1-S_{orw}, C=1) \Big] \right\}$$

$$= S(x=1, t_D) + t_D\Big(1 - f\big(S(x=1, t_D), C=1\big)\Big)$$

Solution (B):



$t_{B4} \le t_D \le t_{B3}$

$$\bar{S}(t_D) = \int_0^1 S(x, t_D)\, dx \overset{\xi = \frac{x}{t}}{=} t_D \int_0^{1/t_D} S(\xi)\, d\xi$$

$$= t_D \left\{ \underbrace{\int_0^{\nu_1} S(\xi)\, d\xi}_{①} + \underbrace{\int_{\nu_1}^{\nu_3} S(\xi)\, d\xi}_{②} + \underbrace{\int_{\nu_3}^{1/t_D} S(\xi)\, d\xi}_{③} \right\}$$

$$① = \int_0^{\nu_1} S(\xi)\, d\xi = S(\xi)\cdot\xi \Big|_0^{\nu_1} - \int_{S(\xi=0)}^{S(\xi=\nu_1)} \xi\, dS = \underset{S_1}{S(\nu_1)}\,\nu_1 - \int_{S(\xi=0)}^{S(\xi=\nu_1)=S_1} \frac{\partial f}{\partial S}(S(\xi), C=1)\, d\xi$$

$$= S_1 \nu_1 - \left[f(S_1, C=1) - \underbrace{f(S(\xi=0), C=1)}_{1} \right] = S_1 \nu_1 - \left(f(S_1, C=1) - 1 \right)$$

$$② = \int_{\nu_1}^{\nu_3} S(\xi)\, d\xi = S_2 \cdot (\nu_3 - \nu_1)$$

$$③ = \int_{\nu_3}^{1/t_D} S(\xi)\, d\xi = S(\xi)\,\xi \Big|_{\nu_3}^{1/t_D} - \int_{S(\xi=\nu_3)=S_2}^{S(\xi=\frac{1}{t_D})} \xi\, d\xi =$$

$$= S(1/t_D)\cdot\frac{1}{t_D} - \underset{S_2}{S(\nu_3)}\cdot\nu_3 - \int_{S_2}^{S(\xi=1/t_D)} \frac{\partial}{\partial S} f(S(\xi), C=0)\, d\xi =$$

$$= S\left(\frac{1}{t_D}\right)\cdot\frac{1}{t_D} - S_2 \cdot \nu_3 - \left[f(S(1/t_D), C=0) - f(S_2, C=0) \right]$$

$t_{B4} \le t_D \le t_{B3}$

$$\bar{S}(t_D) = t_D \left(① + ② + ③ \right) =$$

$$= (S_1 - S_2)\nu_1 t_D + S\left(\frac{1}{t_D}\right) - t_D \left\{ f(S_1, C=1) - 1 + f\left(S\left(\frac{1}{t_D}\right), C=0\right) - f(S_2, C=0) \right\}$$

$$= (S_1 - S_2)\nu_1 t_D + S(x=1, t_D) - t_D \left\{ f(S_1, C=1) - 1 + f\left(S(x=1, t_D), C=0\right) - f(S_2, C=0) \right\}$$

$$= S(x=1, t_D) + t_D \left\{ 1 - f(S(x=1, t_D), C=0) \right\}$$

$t_{B3} \le t \le t_{B1}$

$$\bar{S}(t_D) = \int_0^1 S(x, t_D)\,dx \overset{\xi = \frac{x}{t}}{=} t_D \int_0^{1/t_D} S(\xi)\,d\xi = t_D \left\{ \int_0^{\nu_1} S(\xi)\,d\xi + \int_{\nu_1}^{1/t_D} \underbrace{S(\xi)}_{S_2}\,d\xi \right\}$$

①: see page 18

$$= t_D \left\{ S_1 \nu_1 - \left(f(S_1, C=1) - 1 \right) + S_2 \cdot \left(\frac{1}{t_D} - \nu_1 \right) \right\}$$

$$= S_2 + (S_1 - S_2)\nu_1 t_D + t_D \left(1 - f(S_1, C=1) \right)$$

$$= S_2 + t_D \left(1 - f_w(S_2, C=0) \right)$$

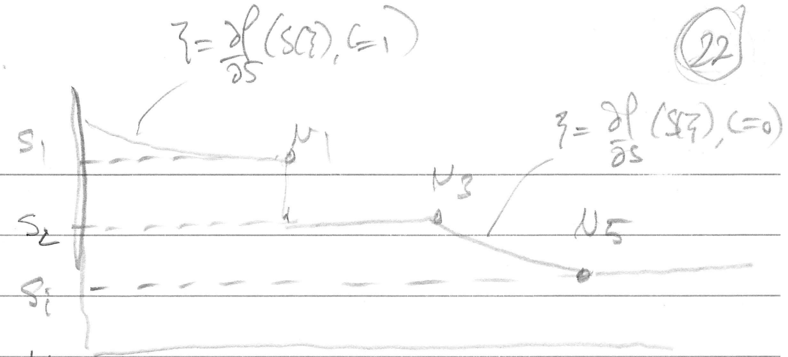
$t \ge t_{B1}$,

$$\bar{S}(t_D) = \int_0^1 S(x, t_D)\,dx = t_D \int_0^{1/t_D} S(\xi)\,d\xi = t_D \left\{ S(\xi)\xi \Big|_0^{1/t} - \int_{S(\xi=0)}^{S(\xi=1/t_D)} \xi\,dS \right\}$$

$$= S(1/t_D) - t_D \int_{S(\xi=0)}^{S(\xi=1/t_D)} \frac{\partial f}{\partial S}(S(\xi), C=1)\,dS =$$

$$= S(x=1, t_D) + t_D \left(1 - f(S(x=1, t_D), C=1) \right)$$

Solution (C):



$t_{B5} \le t_D \le t_{B3}$

$$\xi = \frac{x}{t}$$

$$\bar{S}(t_D) = \int_0^1 S(x,t_D)\,dx = t_D \int_0^{1/t_D} S(\xi)\,d\xi =$$

$$= t_D \cdot \left\{ \underbrace{\int_0^{\nu_1} S(\xi)\,d\xi}_{1} + \underbrace{\int_{\nu_1}^{\nu_3} S(\xi)\,d\xi}_{2} + \underbrace{\int_{\nu_3}^{1/t_D} S(\xi)\,d\xi}_{3} \right\}$$

see page 18 + 19

$$= (S_1 - S_2)\cdot \nu_1 \cdot t_D + S(x=1,t_D) - t_D \left\{ f(S_1, c=1) - 1 + f(S(x=1,t_D), c=0) - f(S_2, c=0) \right\}$$

$$= S(x=1,t_D) + t_D \left\{ 1 - f(S(x=1,t_D), c=0) \right\}$$

$t_{B3} \le t_D \le t_{D1}$

Same as (B) solution

$t_D \ge t_{B1}$

Same as (B) solution

Average saturation Summary Type ① solutions

- Before breakthrough

$$\bar{S}(t) = S_a(t_D) = S_i + (1 - f_w(S_i, C=0)) \cdot t_D$$

Solution (A).

$t_D \leq t_{B2}$: $\bar{S}(t_D) = S_a(t_D)$ (before breakthrough)

$t_{B2} \leq t_D \leq t_{B1}$: $\bar{S}(t_D) = S_b(t_D) = S_2 + t_D(1 - f_w(S_2, C=0))$

$t_D > t_{B1}$: $\bar{S}(t_D) = S_c(t_D) = S(x=1, t_D) + t_D(1 - f(S(x=1, t_D), C=1))$

Solution (B):

$t_D \leq t_{B4}$: $\bar{S}(t_D) = S_a(t_D)$ (before breakthrough)

$t_{B4} \leq t_D \leq t_{B3}$: $\bar{S}(t_D) = S_d(t_D) = S(x=1, t_D) + t_D(1 - f(S(x=1, t_D), C=0))$

$t_{B3} < t_D \leq t_{B1}$: $\bar{S}(t_D) = S_b(t_D)$

$t_D > t_{B1}$: $\bar{S}(t_D) = S_c(t_D)$

Solution (C):

$t_D \leq t_{B5}$; $\bar{S}(t_D) = S_a(t_D)$ (before breakthrough)

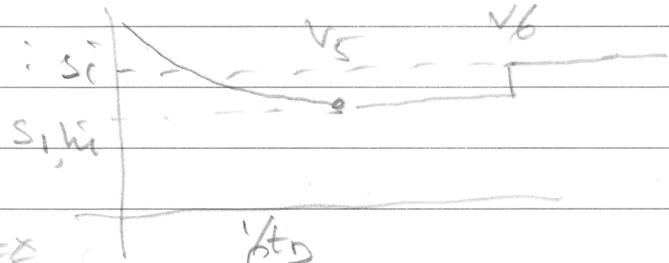
$t_{B5} \leq t_D \leq t_{B3}$: $\bar{S}(t_D) = S_d(t_D)$

$t_{B3} < t_D \leq t_{B1}$: $\bar{S}(t_D) = S_b(t_D)$

$t_D > t_{B1}$: $\bar{S}(t_D) = S_c(t_D)$

Average saturation solution ②

For $t_{B6} \leq t_D \leq t_{B5}$:



$$\bar{S}(t_D) = \int_0^1 S(x, t_D)\, dx \overset{\xi = \frac{x}{t}}{=} t_D \int_0^{1/t_D} S(\xi)\, d\xi$$

$$= t_D \left\{ \underbrace{\int_0^{v_5} S(\xi)\, d\xi}_{①} + \underbrace{\int_{v_5}^{1/t_D} \underbrace{S(\xi)}_{= S_{1,hi}} d\xi}_{②} \right\}$$

① = see page 19, replace v_1 by v_5 and S_1 by $S_{1,hi}$

$$= S_{1,hi} \cdot v_5 - \left(f(S_{1,hi}, C=1) - 1 \right)$$

$$② = S_{1,hi} \left(1/t_D - v_5 \right)$$

$$\bar{S}(t_D) = S_{1,hi} - t_D \left(f(S_{1,hi}, C=1) - 1 \right)$$

For $t_D \geq t_{B5}$

$$\bar{S}(t_D) = \int_0^1 S(x, t_D)\, dx \overset{\xi = \frac{x}{t}}{=} t_D \int_0^{1/t_D} S(\xi)\, d\xi$$

$$= S(x=1, t_D) - t_D \left(f(S(x=1, t_D), C=1) - 1 \right)$$

see page 20

Average saturation Summary Solution (2):

$t_D \le t_{B6}$: $\bar{S}(t_D) = S_a(t_D)$ (before breakthrough)

$t_{B6} \le t_D < t_{B5}$: $\bar{S}(t_D) = S_e(t_D) = S_{1,hi} + t_D(1 - f(S_{1,hi}, C=1))$

$t_D \ge t_{B5}$: $\bar{S}(t_D) = S_c(t_D)$

Recovery factors

(1) Normalized recovery factor

$$E_R = \frac{S_{oi} - \bar{S}_o}{S_{oi} - S_{orw}} = \frac{1 - S_{cw} - (1 - \bar{S}_w)}{1 - S_{cw} - S_{orw}} = \frac{\bar{S}_w - S_{cw}}{1 - S_{cw} - S_{orw}} = \bar{S}^*$$

(2) Normal recovery factor:

wrt S_{cw} : $$RF = \frac{S_{oi} - \bar{S}_o}{S_{oi}} = \frac{(S_{oi} - S_{orw})E_R}{S_{oi}} = \frac{(1 - S_{cw} - S_{orw}) \cdot \bar{S}^*}{1 - S_{cw}}$$

$$\parallel$$

$$\frac{\bar{S}_w - S_{cw}}{1 - S_{cw}}$$

wrt S_{wi} $$RF = \frac{S_{oi} - \bar{S}_o}{S_{oi}} = \frac{(1 - S_{wi}) - (1 - \bar{S}_w)}{1 - S_{wi}} = \frac{\bar{S}_w - S_{wi}}{1 - S_{wi}}$$

$$= \frac{(S_{cw} + (1 - S_{cw} - S_{orw})\bar{S}^* - S_{wi})}{1 - S_{wi}}$$

Adsorption : (see page 4)

$$a = \frac{(1-\phi)}{\phi} \frac{\rho_r}{\rho_w} \frac{C_s}{C_{inj}}$$

In the model : $a(C) = \dfrac{A_1 C}{1 + A_2 C}$

The relevant value : $a(1) = \dfrac{A_1}{1 + A_2} = \dfrac{(1-\phi)}{\phi} \dfrac{\rho_r}{\rho_w} \dfrac{C_s}{C_{inj}}$

Select : $A_2 = 1$ and $A_1 = 2 \times \dfrac{(1-\phi)}{\phi} \dfrac{\rho_r}{\rho_w} \dfrac{C_s}{C_{inj}}$

This should result in the same solution as standard in the literature.

NB: in the 'alternative' dimensionless formulation:

$$a = \frac{1}{(1 - S_{ca} - S_{orw})} \frac{(1-\phi)}{\phi} \frac{\rho_r}{\rho_w} \frac{C_s}{C_{inj}} = \frac{A_1}{1 + A_2}$$

$$\Rightarrow \quad A_2 = 1 \quad \& \quad A_1 = \frac{2}{(1 - S_{cw} - S_{orw})} \frac{(1-\phi)}{\phi} \frac{\rho_r}{\rho_w} \frac{C_s}{C_{inj}}.$$

Alternative dimensionless parameters

Introduce dimensionless variables:

$$x_D = \frac{x}{L}, \quad t_D = \frac{ut}{(1-S_{cw}-S_{orw})\phi L} = \frac{Q}{(1-S_{cw}-S_{orw})V_p}$$

Q ← injected volume, V_p ← porevolume

also: $S = \frac{(S_w - S_{cw})}{1-S_{cw}-S_{orw}}$, $c^* = \frac{c}{c_{inj}}$, $a = \frac{1}{(1-S_{cw}-S_{orw})} \frac{(1-\phi)}{\phi} \frac{\rho_s}{\rho_w} \frac{C_s}{c_{inj}}$

With the above the eqs are:

$$\begin{cases} \frac{\partial S}{\partial t_D} + \frac{\partial}{\partial x_D} f_w(S, c^*) = 0 \\ \frac{\partial}{\partial t_D}\left[\left(S + \underbrace{\frac{S_{cw}}{(1-S_{cw}-S_{orw})}}_{\tilde{S}_{cw}}\right) c^* + a\right] + \frac{\partial}{\partial x_D}\left(f_w c^*\right) = 0 \end{cases}$$

Assumption: $f_w = f_w(S, c^*)$ and $a = a(c^*)$

Differentiate out the eqs:

$$\begin{cases} \frac{\partial S}{\partial t_D} + \frac{\partial f}{\partial S}\frac{\partial S}{\partial x_D} + \frac{\partial f}{\partial c^*}\frac{\partial c^*}{\partial x_D} = 0 \\ c^* \frac{\partial S}{\partial t_D} + (S + \tilde{S}_{cw})\frac{\partial c^*}{\partial t_D} + \frac{da}{dc^*}\frac{\partial c^*}{\partial t_D} + f_w \frac{\partial c^*}{\partial x_D} + c^* \frac{\partial f_w}{\partial S}\frac{\partial S}{\partial x_D} + c^* \frac{\partial f}{\partial c^*}\frac{\partial c^*}{\partial x_D} = 0 \\ \qquad = \left(S + \tilde{S}_{cw} + \frac{da}{dc^*}\right)\frac{\partial c^*}{\partial t_D} + f_w \frac{\partial c^*}{\partial x_D} = 0 \end{cases}$$

The underlined terms $c^* \frac{\partial S}{\partial t_D}$, $c^* \frac{\partial f_w}{\partial S}\frac{\partial S}{\partial x_D}$, $c^* \frac{\partial f}{\partial c^*}\frac{\partial c^*}{\partial x_D}$ together $= 0$

Eqs in matrix form:

$$\begin{pmatrix} S \\ C^* \end{pmatrix}_{t_D} + A \begin{pmatrix} S \\ C^* \end{pmatrix}_{x_D} = 0 \qquad A = \begin{pmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial C^*} \\ 0 & \frac{f}{S + \tilde{S}_{cw} + \frac{da}{dC^*}} \end{pmatrix}$$

Eigenvalues:

$$\lambda_1 = \frac{\partial f_w}{\partial S} \qquad \lambda_2 = \frac{f_w}{S + \tilde{S}_{cw} + \frac{da}{dC^*}}$$

Eigenvectors

$$r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad r_2 = \begin{pmatrix} \frac{\partial f_w}{\partial C^*} \\ \frac{f_w}{S + \tilde{S}_{cw} + \frac{da}{dC^*}} - \frac{\partial f_w}{\partial S} \end{pmatrix}$$

Drop $*$ in C^* from now on.

Integral curves:

For λ_1 : $c(\xi) = \text{constant}$

For λ_2 :

$$\left(\frac{\partial f_w}{\partial S} - \frac{f_w}{S+\tilde{S}_{cw}+\frac{da}{dc}}\right) S'(\xi) + \frac{\partial f}{\partial c} c'(\xi) = 0$$

$$\Rightarrow \frac{dS}{dc} = \frac{-\frac{\partial f}{\partial c}}{\frac{f_w}{S+\tilde{S}_{cw}+\frac{da}{dc}} - \frac{\partial f}{\partial S}} = \frac{\frac{\partial f_w}{\partial C}\left(S+\tilde{S}_{cw}+\frac{da}{dc}\right)}{f_w - \frac{\partial f}{\partial S}\left(S+\tilde{S}_{cw}+\frac{da}{dc}\right)}$$

$D = (S^{n_w} + M(c)(1-S)^{n_o})$

$$\stackrel{!}{=} \frac{-\frac{M'(c)\,S^{n_w}(1-S)^{n_o}\left(S+\tilde{S}_{cw}+\frac{da}{dc}\right)}{D^2}}{\frac{S^{n_w}}{D} - \frac{M(c)\,S^{n_w-1}(1-S)^{n_o-1}\left(n_w(1-S)+n_o S\right)\left(S+\tilde{S}_{cw}+\frac{da}{dc}\right)}{D^2}}$$

extract S^{n_w-1}

$$\stackrel{!}{=} \frac{-M'(c)\,S\,(1-S)^{n_o}\left(S+\tilde{S}_{cw}+\frac{da}{dc}\right)}{\underbrace{S\cdot D}_{1} - M(c)(1-S)^{n_o-1}\left(n_w(1-S)+n_o S\right)\left(S+\tilde{S}_{cw}+\frac{da}{dc}\right)}$$

denominator can be zero for some $0 < S < 1$, but not for $S=0$ or $S=1$

Rarefaction waves: see other way of making dimensionless.
for λ_2 : need to replace $S+\frac{da}{dc} \Rightarrow S+\tilde{S}_{cw}+\frac{da}{dc}$,

However not relevant for solution as λ_2 rarefaction does not occur

Shocks:

Back to original eqs:
$$\begin{cases} \dfrac{\partial S}{\partial t_D} + \dfrac{\partial}{\partial x_D} f_w = 0 \\ \dfrac{\partial}{\partial t_D}\left[\left(S + \tilde{S}_{cw}\right) C + a\right] + \dfrac{\partial}{\partial x_D}(f_w C) = 0 \end{cases}$$

Shock condition:
$$\sigma_1 = \frac{f^+ - f^-}{S^+ - S^-}$$
$$\sigma_2 = \frac{f^+ C^+ - f^- C^-}{(S^+ + \tilde{S}_{cw})C^+ + a^+ - \left[(S^- + \tilde{S}_{cw})C^- + a^-\right]}$$

$\sigma_1 = \sigma_2$
$$\Rightarrow (f^+ S^- - f^- S^+)(C^+ - C^-) + (f^+ - f^-)\tilde{S}_{cw}(C^+ - C^-) + (f^+ - f^-)(a^+ - a^-) = 0$$

$$\Rightarrow f^+ - f^- = \frac{(S^+ f^- - S^- f^+)(C^+ - C^-)}{\tilde{S}_{cw}(C^+ - C^-) + (a^+ - a^-)} \quad \text{if } \begin{cases} a^+ \neq a^- \\ C^+ \neq C^- \end{cases}$$

$$\sigma = \frac{f^+ - f^-}{S^+ - S^-} = \frac{(S^+ f^- - S^- f^+)(C^+ - C^-)}{\left(\tilde{S}_w (C^+ - C^-) + (a^+ - a^-)\right)(S^+ - S^-)}$$

$f^+ = f^- + \sigma(S^+ - S^-)$

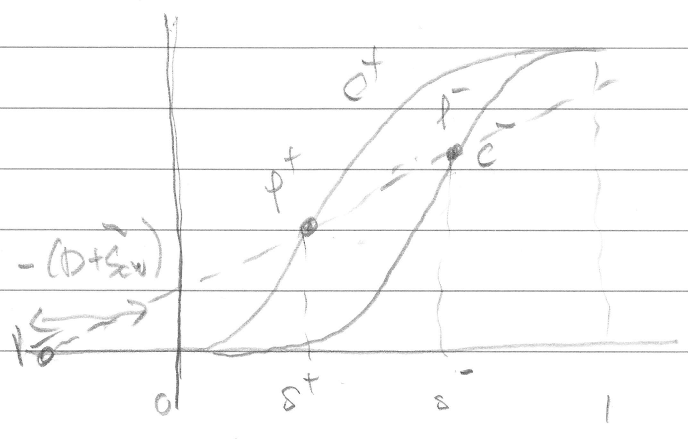
$f^- = f^+ - \sigma(S^+ - S^-)$

$$= \frac{(f^- - \sigma S^-)(C^+ - C^-)}{\tilde{S}_{cw}(C^+ - C^-) + (a^+ - a^-)} = \frac{(f^+ - \sigma S^+)(C^+ - C^-)}{\tilde{S}_{cw}(C^+ - C^-) + (a^+ - a^-)}$$

solving for σ,

$$\Rightarrow \sigma = \frac{f^-}{S^- + \tilde{S}_{cw} + \frac{(a^+ - a^-)}{(C^+ - C^-)}} = \frac{f^+}{S^+ + \tilde{S}_{cw} + \frac{(a^+ - a^-)}{(C^+ - C^-)}} = \frac{f^+ - f^-}{S^+ - S^-}$$

$\dfrac{da}{dC} > 0 : \dfrac{a^+ - a^-}{C^+ - C^-} \equiv D > 0$



Hugoniot loci

For 1-wave (λ_1) : $c^+ = c^- \Rightarrow a^+ = a^-$

$\sigma_1 = \frac{f^+ - f^-}{s^+ - s^-}$, Hugoniot locus : c = constant

$\sigma_2 = \frac{f^+ - f^-}{s^+ - s^-}$

For 2-wave (λ_2) : $c^+ \neq c^-$

Need to solve : $\frac{f^+}{s^+ + \tilde{s}_{cw} + \left(\frac{a^+ - a^-}{c^+ - c^-}\right)} = \frac{f^+ - f^-}{s^+ - s^-}$

Equivalently solve : $f^+(s^+ - s^-) - (f^+ - f^-)\left(s^+ + \tilde{s}_{cw} + \left(\frac{a^+ - a^-}{c^+ - c^-}\right)\right) = 0$

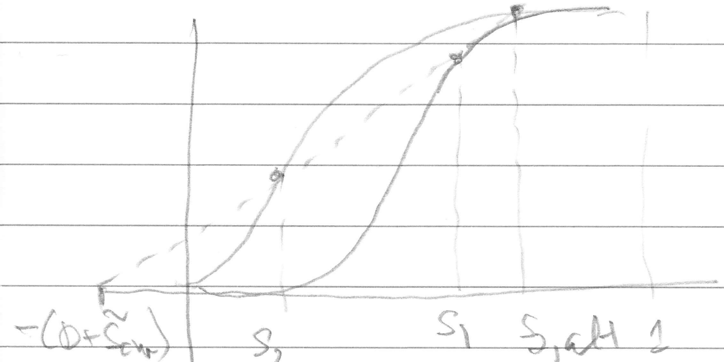
When solving we used $\frac{a^+ - a^-}{c^+ - c^-} \simeq \frac{da}{dc}(c^+)$ for $c^+ - c^- < \varepsilon$

Note : $c_1 < c_2 \Rightarrow M(c_1) < M(c_2) \Rightarrow f(s, c_1) \geq f(s, c_2)$ for all s

For the solution one needs to find an $(s_1, c=1)$ and $(s_2, c=0)$ such that $\lambda_1(s_1, c=1) = \sigma_2(s_1, c=1; s_2, c=0)$

s_1 can be found from : $\frac{\partial f(s_1, c=1)}{\partial s} = \frac{f(s_1, c=1)}{s_1 + \tilde{s}_{cw} + D}$; $D = \frac{a(1) - a(0)}{1 - 0}$

Two possible solution for s_2 : $\frac{f(s_1, c=1)}{s_1 + \tilde{s}_{cw} + D} = \frac{f(s_1, c=1) - f(s, c=0)}{s_1 - s}$



Due to requirement of increasing speed on the solution $s_{2,alt}$ is not a valid option

N.B. The tangent solution breaks down for $s_1 > s_{2,alt}$. See other dimensionless option for discussion.