Summary of parallelized reservoir computing scheme

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My research topic

- Forecasting of stock price data and weather data, focusing on small-size datasets (less than 10,000 samples).
- See my GitHub repository, "**compare-LSTM-GRU-Reservoir**".

My conclusion

Reservoir computing performs best for the small-size datasets.

$\boxed{\text{Data structure for my research:}}$ at $u.shape[0]$ fixed observation points generate $u.shape[0]$-elements time series data.

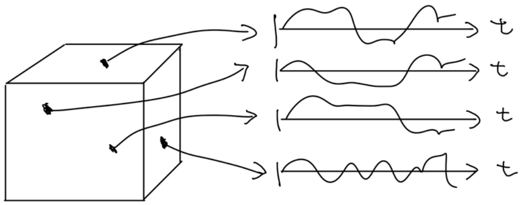


Figure: This case $u.shape[0] = 4$

Case 1: the set of time series data contains rich spatial information

- Apply FFT (fast Fourier transform)
- Example: in periodic box turbulence obtained from DNS
- Focus on time evolution of the amplitudes. See Nakai-Saiki (2018).
- The filter can be a low-pass or band-pass that directly cuts off high spatial frequencies.

Case 2: the set of time series data contains little spatial information

- Filters in the spatial direction may not be appropriate.
- Filtering can still be applied in the time direction, provided that the underlying systems generating each time series are considered to be nearly identical.

case 3: Intermediate case

- Graph Neural Networks might be one of them?

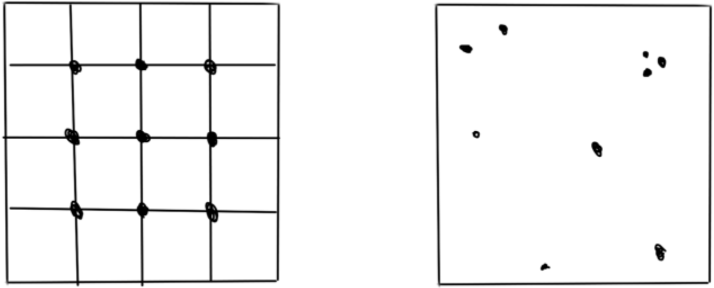


Figure: Left: suitable for FFT, RIGHT: try to apply GNN in the future

Here, we focus on the case 2, since wind-speed data seems having little spatial information, moreover, stock data clearly do not have such spatial information.

Overview of the Learning Scheme (Case 2)

Parallelized data can be written as:

$$\{u_{j,t}\}_{j\in[0,1,\cdots,u.shape[0]-1],t\in[0,1,\cdots,u.shape[1]-1]}$$

- the number of discrete data points: $u.shape[1]$
- the number of time series data: $u.shape[0]$

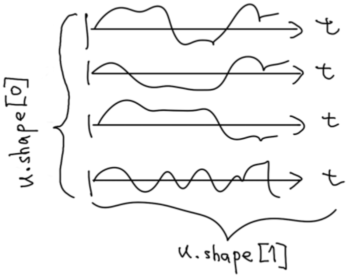


Figure: Image of data shape

Delay coordinate (see Nakai-Saiki 2021)

Rewrite the data in a form of *dim*-dimensional vector as the following:

$$U_{j,t} = \begin{pmatrix} u_{j,t} \\ u_{j,t-lag} \\ \vdots \\ u_{j,t-(dim-1)lag} \end{pmatrix}.$$

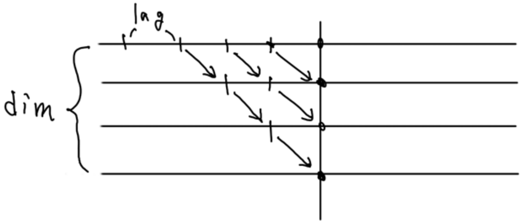


Figure: Image of $U_{j,t}$

Overview of our ML architecture

- M: the number of learnable parameters
- $\{w_i\}_{i=1}^{M}$: learnable parameters (reservoir's W_{in}, W, W_{out})

For a function $F_j : \mathbb{R}^M \times \mathbb{R}^{dim} \to \mathbb{R}^{dim}$, the prediction value $\widehat{U}_{j,t+1}$ is defined as

$$\widehat{U}_{j,t+1} = F_j\left(\{w_i\}_{i=0}^{M-1}, \{U_{j,t}\}_{j=0}^{u.shape[0]-1}\right).$$

We then select the $\{w_i\}_i$ that minimizes the energy with respect to the actual data $U_{j,t+1}$. That is,

$$\sum_{j=0}^{u.shape[0]-1} \sum_{t=0}^{T_{train}-1} |\widehat{U}_{j,t+1} - U_{j,t+1}|^2$$

is minimized (we omit regularization term here). Assume that underlying systems are almost identical, then F_j can be unified into a single F independent of j, that is,

$$\widehat{U}_{j,t+1} = F\left(\{w_i\}_{i=0}^{M-1}, U_{j,t}\right).$$

This significantly simplifies the learning model.

Reservoir Computing (overview)

Trainable parameters

- Input weight matrix: $W^{in} \in \mathbb{R}^{N_x \times dim}$
- Recurrent weight matrix: $W \in \mathbb{R}^{N_x \times N_x}$
- Output weight matrix: $W^{out} \in \mathbb{R}^{dim \times N_x}$

Substitute data $U_{j,t} \in \mathbb{R}^{dim}$ and feature vector $x_{j,t} \in \mathbb{R}^{N_x}$
Output predicted value $\widehat{U}_{j,t+1} \in \mathbb{R}^{dim}$ and feature vector $x_{j,t+1} \in \mathbb{R}^{N_x}$

$$(U_{j,t}, x_{j,t}) \mapsto (\widehat{U}_{j,t+1}, x_{j,t+1}),$$

$$\begin{cases} x_{j,t+1} & = (1-\alpha)x_{j,t} + \alpha \tanh(W^{in} U_{j,t} + W x_{j,t}), \\ \widehat{U}_{j,t+1} & = W^{out} x_{j,t+1}. \end{cases}$$

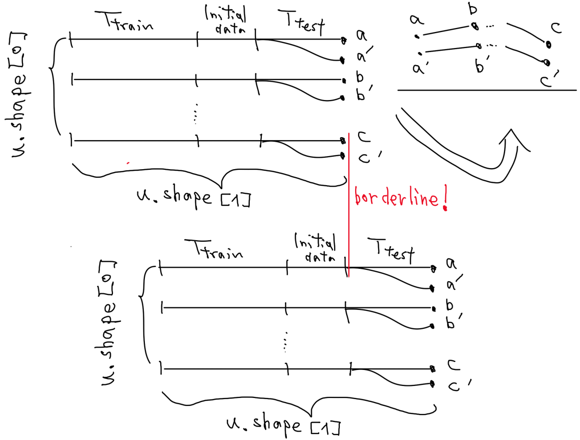
$\alpha \in (0, 1]$ is called the leaking rate, which is similar to the well-known concept of residual connections.

Conceptual diagram of the learning scheme

The scheme has two stages:

- Model selection
- Generalization performance evaluation

Each of them are including a training phase



Training phase

We select W^{out} that minimizes the following E:

Evaluation function

$$E = \frac{1}{2} \sum_{j=0}^{u.shape[0]-1} \sum_{t=0}^{T_{train}-1} |U_{j,t} - W^{out} x_t|^2 + \frac{\beta}{2} \sum_{i=0}^{dim-1} \sum_{j=0}^{N_x - 1} |W^{out}_{ij}|^2 .$$

- $\beta > 0$ is a regularization parameter
- T_{train} is the number of training data points
- due to convexity, W^{out} can be solved analytically

Model selection phase

- We forecast T_{test} steps ahead
- The initial value for model selection is $U_{j,T_{train}}$
- $x_{j,T_{train}}$ is already obtained during the training phase

Based on these, the next step is estimated as follows:

$$\begin{cases} x_{j,T_{train}+1} &= (1-\alpha)x_{j,T_{train}} + \alpha \tanh(W^{in}U_{j,T_{train}} + Wx_{j,T_{train}}), \\ \widehat{U}_{j,T_{train}+1} &= W^{out}x_{j,T_{train}+1}. \end{cases}$$

The resulting $\widehat{U}_{j,t}, x_{j,t}$ are then sequentially fed into the following recurrence relation to generate the predicted data:

$$\begin{cases} x_{j,t+1} &= (1-\alpha)x_{j,t} + \alpha \tanh(W^{in}\widehat{U}_{j,t} + Wx_{j,t}), \\ \widehat{U}_{j,t+1} &= W^{out}x_{j,t+1} \end{cases}$$

for $t = T_{train}+1, \cdots, T_{train}+T_{test}$.

Bayesian optimization phase

Apply **Bayesian opt (Optuna)** using the prediction data $\widehat{U}_{j,T_{train}+T_{test}}$.
More specifically, minimize the following MAE-based evaluation function:

Evaluation function

$$\frac{1}{u.shape[0]} \sum_{j=0}^{u.shape[0]-1} \left|\widehat{U}_{j,T_{train}+T_{test}} - U_{j,T_{train}+T_{test}}\right|$$

in the selection of the following parameters:

dim, lag, α, W, W^{in}.

Generalization Performance: re-training Phase

After selecting dim, lag, α, W, and W^{in} in the previous phase, we implement a new test to evaluate generalization performance. We re-select W^{out} that minimizes the following E:

Evaluation function

$$E = \frac{1}{2} \sum_{j=0}^{u.shape[0]-1} \sum_{t=T_{test}}^{T_{train}+T_{test}-1} |U_{j,t} - W^{out} x_{j,t}|^2 + \frac{\beta}{2} \sum_{i=0}^{dim-1} \sum_{j=0}^{N_x-1} |W_{ij}^{out}|^2 .$$

- We evaluate the generalization performance from $T_{train} + T_{test}$
- The initial value is $U_{j,T_{train}+T_{test}}$
- $x_{j,T_{train}+T_{test}}$ is already obtained during the re-training phase

Generalization Performance: evaluation phase

We generate the next step as follows.

$$\begin{cases} x_{j,T_{train}+T_{test}+1} & = (1-\alpha)x_{j,T_{train}+T_{test}} \\ & \qquad +\alpha\tanh(W^{in}U_{j,T_{train}+T_{test}} + Wx_{j,T_{train}+T_{test}}), \\ \widehat{U}_{j,T_{train}+T_{test}+1} & = W^{out}x_{j,T_{train}+T_{test}+1}. \end{cases}$$

The $(\widehat{U}_{j,t}, x_{j,t})$ obtained in this manner are then sequentially fed into the following recurrence relation to generate the prediction data:

$$\begin{cases} x_{j,t+1} & = (1-\alpha)x_{j,t} + \alpha\tanh(W^{in}\widehat{U}_{j,t} + Wx_{j,t}), \\ \widehat{U}_{j,t+1} & = W^{out}x_{j,t+1} \end{cases}$$

for $t = T_{train} + T_{test} + 1, \cdots, T_{train} + 2T_{test}$. Then, finally, the generalization performance is evaluated using the following MAE:

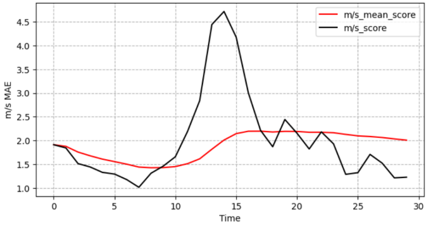
Evaluation function

$$\frac{1}{u.shape[0]} \sum_{j=0}^{u.shape[0]-1} |\widehat{U}_{j,T_{train}+2T_{test}} - U_{j,T_{train}+2T_{test}}|$$

Result (windspeed in Tokyo region)

- start date = "2024-Jan-01"   end date = "2024-Feb-15"
- predict 3 hours ahead ($T_{test} = 3$)
- most recent 1,000 hours of data are used to train($=: T_{train}$)
- $dim = 4$, $lag = 1$: fixed

Result MAE: 2.01 m/s (data-driven filter is applied: correlation 0.83, Jinno-Mitsui-Nakai-Saiki-Yoneda 2025)



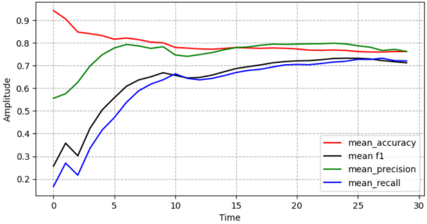
Cheng and Steenburgh (2005) CIRP, WRF (physical models)

MAE: 1.6 ∼ 1.9 m/s, but unclear whether a filter is applied...

Result (stock S&P500)

- start date = "2024-Jan-01" end date = "2025-May-10"
- predict 1 day ahead ($T_{test} = 1$)
- most recent 60 days of data are used to train($=: T_{train}$)
- $dim = 2$, $lag = 1$: fixed

Result Accuracy 0.762, F1 0.712, Recall 0.721 (data-driven filter is applied: correlation 0.985, Jinno-Mitsui-Nakai-Saiki-Yoneda 2025)



Gil, Duhamel-Sebline, McCarren (2024) xLSTM-TS

Accuracy 0.709, F1 0.730, Recall 0.768, wavelet filter is applied.