# Compton Scattering with Relativistic Electrons

T. Susai Rajan

June 2021

## Contents



## 1 Introduction

Compton scattering occurs when a photon interacts with a stationary electron and imparts a portion of its kinetic energy and momentum to it. This effect is well-known and has its uses in medical imaging and radiation therapy. At the opposite end of the spectrum, the interaction of photons and relativistic electrons is common in astrophysical phenomena such as supernovae and active galactic nuclei. Low energy photons are scattered to high energies by relativistic electrons so that the photons gain and the electrons lose energy. This process is called inverse Compton scattering because the electrons lose energy rather than the photons, the opposite of the standard Compton effect. In both these effects, the particles energy and momenta are exchanged and their trajectories altered after collision as shown in Figure I.

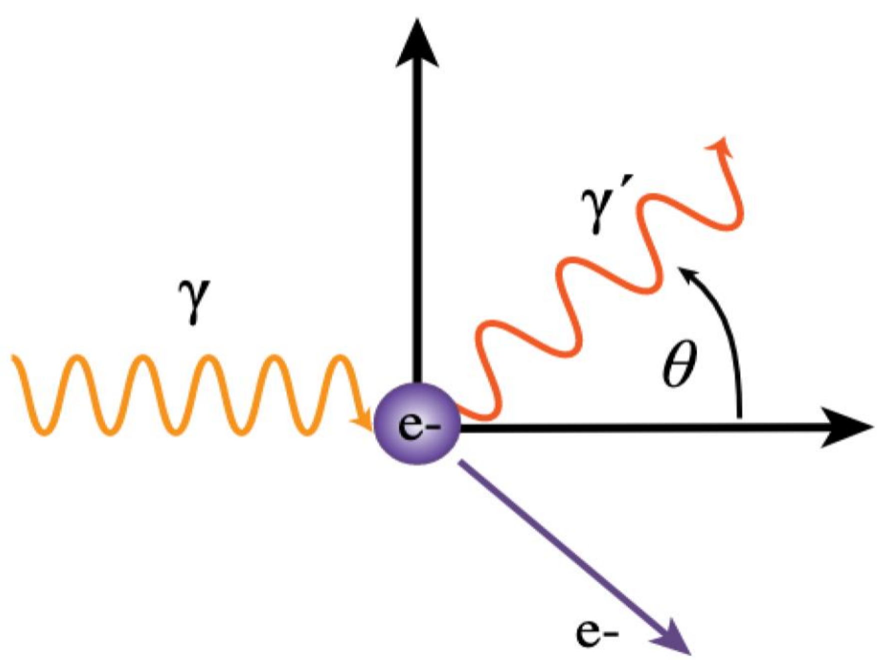


Figure I. Compton scattering of a photon off of an initially stationary electron. Adapted from Reference 1.

The wavelength and energy of the scattered photon are given by equations (1) and (2) respectively.

$$\Delta\lambda = \lambda' - \lambda = \left(\frac{h}{m_e c}\right)(1 - \cos\theta) \tag{1}$$

$$E_{\gamma'} = \frac{E_\gamma}{1 + \left(\frac{E_\gamma}{m_e c^2}\right)(1 - \cos\theta)} \tag{2}$$

In this paper, we derive equations for the energy and wavelength of a scattered photon for a more general case when the photon is collided at an arbitrary angle $\phi$ by an electron moving initially at relativistic speeds.

## 2 Derivation in Lab Frame

We start from the conservation of energy and momentum laws for the photon-electron system.

$$E_\gamma + E_e = E_{\gamma'} + E_{e'} \tag{3}$$

$$\mathbf{P}_\gamma + \mathbf{P}_e = \mathbf{P}_{\gamma'} + \mathbf{P}_{e'} \tag{4}$$

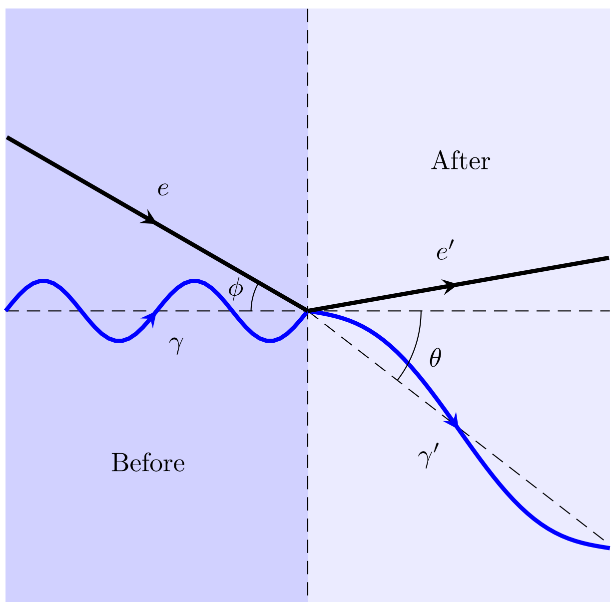


Figure II. Scattering of a photon off of an initially moving electron at an angle $\phi$.

Using the following substitutions

$$E_\gamma = hf$$
$$E_{\gamma'} = hf'$$
$$E_e = \sqrt{(P_e c)^2 + (m_e c^2)^2}$$
$$E_{e'} = \sqrt{(P_{e'} c)^2 + (m_e c^2)^2}$$

we can rewrite (3) as

$$hf + \sqrt{(P_e c)^2 + (m_e c^2)^2} = hf' + \sqrt{(P_{e'} c)^2 + (m_e c^2)^2}$$

which simplifies to

$$
\begin{aligned}
hf - hf' + \sqrt{(P_e c)^2 + (m_e c^2)^2} &= \sqrt{(P_{e'} c)^2 + (m_e c^2)^2} \\
(hf - hf' + \sqrt{(P_e c)^2 + (m_e c^2)^2})^2 &= (P_{e'} c)^2 + (m_e c^2)^2 \\
(P_{e'} c)^2 &= (hf - hf' + \sqrt{(P_e c)^2 + (m_e c^2)^2})^2 - (m_e c^2)^2 \\
&= (hf)^2 - 2h^2 ff' + 2hf\sqrt{(P_e c)^2 + (m_e c^2)^2} + (hf')^2 - \\
&\quad 2hf'\sqrt{(P_e c)^2 + (m_e c^2)^2} + (P_e c)^2 + (m_e c^2)^2 - (m_e c^2)^2
\end{aligned}
$$

$$
(P_{e'} c)^2 = (hf)^2 - 2h^2 ff' + 2hf\sqrt{(P_e c)^2 + (m_e c^2)^2} + (hf')^2 - 2hf'\sqrt{(P_e c)^2 + (m_e c^2)^2} + (P_e c)^2 \tag{5}
$$

We can simplify $\sqrt{(P_e c)^2 + (m_e c^2)^2}$ using $P_e = \gamma m_e v$.

$$
\begin{aligned}
\sqrt{(P_e c)^2 + (m_e c^2)^2} &= \sqrt{(P_e c)^2 \left(1 + \frac{(m_e c^2)^2}{(P_e c)^2}\right)} \\
&= P_e c\sqrt{1 + \left(\frac{m_e c^2}{P_e c}\right)^2} \\
&= P_e c\sqrt{1 + \left(\frac{m_e c^2}{\gamma m_e v c}\right)^2} \\
&= P_e c\sqrt{1 + \frac{c^2}{\gamma^2 v^2}} \\
&= P_e c\sqrt{1 + \frac{c^2\left(1 - \frac{v^2}{c^2}\right)}{v^2}} \\
&= P_e c\sqrt{1 + \frac{c^2}{v^2} - 1} \\
&= P_e \frac{c^2}{v}
\end{aligned}
$$

Thus, (5) can be rewritten as

$$
(P_{e'} c)^2 = (hf)^2 - 2h^2 ff' + 2hfP_e\frac{c^2}{v} + (hf')^2 - 2hf'P_e\frac{c^2}{v} + (P_e c)^2 \tag{6}
$$

From (4), we get

$$
\mathbf{P}_{e'} = \mathbf{P}_e + \mathbf{P}_\gamma - \mathbf{P}_{\gamma'}
$$

By dotting $\mathbf{P}_{e'}$ with itself, we get

$$
\begin{aligned}
P_{e'}^2 &= \mathbf{P}_{e'} \cdot \mathbf{P}_{e'} \\
&= (\mathbf{P}_e + \mathbf{P}_\gamma - \mathbf{P}_{\gamma'}) \cdot (\mathbf{P}_e + \mathbf{P}_\gamma - \mathbf{P}_{\gamma'}) \\
&= P_e^2 + 2P_e P_\gamma \cos\phi - 2P_e P_{\gamma'}\cos(\theta - \phi) + P_\gamma^2 - 2P_\gamma P_{\gamma'}\cos\theta + P_{\gamma'}^2
\end{aligned}
$$

Multiplying the equation by $c^2$ and using $P_\gamma = \dfrac{hf}{c}$, we get

$$
(P_{e'} c)^2 = P_e^2 c^2 + 2c^2\frac{(hf)}{c}P_e\cos\phi - 2c^2\frac{(hf')}{c}P_e\cos(\theta - \phi) + \frac{(hf)^2}{c^2}c^2 - 2c^2\frac{(hf)}{c}\frac{(hf')}{c}\cos\theta + \frac{(hf')^2}{c^2}c^2
$$

$$(P_{e'}c)^2 = (P_ec)^2 + 2chfP_e\cos\phi - 2chf'P_e\cos(\theta-\phi) + (hf)^2 - 2h^2ff'\cos\theta + (hf')^2 \tag{7}$$

By equating (6) and (7), we get

$$-2h^2ff' + 2hfP_e\frac{c^2}{v} - 2hf'P_e\frac{c^2}{v} = 2chfP_e\cos\phi - 2chf'P_e\cos(\theta-\phi) - 2h^2ff'\cos\theta$$

$$h^2ff' - hfP_e\frac{c^2}{v} + hf'P_e\frac{c^2}{v} = chf'P_e\cos(\theta-\phi) - chfP_e\cos\phi + h^2ff'\cos\theta \tag{8}$$

Using $E_\gamma = hf$ and multiplying by $\dfrac{1}{P_ec}$, we get

$$E_\gamma E_{\gamma'} - E_\gamma P_e\frac{c^2}{v} + E_{\gamma'}P_e\frac{c^2}{v} = cE_{\gamma'}P_e\cos(\theta-\phi) - cE_\gamma P_e\cos\phi + E_\gamma E_{\gamma'}\cos\theta$$

$$\frac{E_\gamma E_{\gamma'}}{P_ec}(1-\cos\theta) - E_{\gamma'}\cos(\theta-\phi) + E_{\gamma'}\frac{c}{v} = E_\gamma\frac{c}{v} - E_\gamma\cos\phi$$

Using $P_\gamma = \dfrac{E_\gamma}{c}$, we get

$$E_{\gamma'}\left(\frac{P_\gamma}{P_e}(1-\cos\theta) - \cos(\theta-\phi) + \frac{c}{v}\right) = E_\gamma\left(\frac{c}{v} - \cos\phi\right)$$

Rearranging the terms and substituting $\dfrac{c}{v}$ with $\dfrac{1}{\beta}$, we get

$$E_{\gamma'} = E_\gamma\left(\frac{\frac{1}{\beta} - \cos\phi}{\frac{P_\gamma}{P_e}(1-\cos\theta) - \cos(\theta-\phi) + \frac{1}{\beta}}\right)$$

Multiplying by $\dfrac{\beta}{\beta}$, we get

$$E_{\gamma'} = E_\gamma\left(\frac{1-\beta\cos\phi}{\beta\frac{P_\gamma}{P_e}(1-\cos\theta) - \beta\cos(\theta-\phi) + 1}\right)$$

Using $P_\gamma = \dfrac{E_\gamma}{c}$ and $P_e = \gamma m_e v$, we get

$$E_{\gamma'} = E_\gamma\left(\frac{1-\beta\cos\phi}{\left(\frac{v}{c}\right)\left(\frac{E_\gamma}{c}\right)\left(\frac{1}{\gamma m_e v}\right)(1-\cos\theta) - \beta\cos(\theta-\phi) + 1}\right)$$

$$E_{\gamma'} = E_\gamma\left(\frac{1-\beta\cos\phi}{1 + \left(\frac{E_\gamma}{\gamma m_e c^2}\right)(1-\cos\theta) - \beta\cos(\theta-\phi)}\right) \tag{9}$$

We can go further and derive an explicit equation for the wavelength of the scattered photon. Multiplying (8) by $\frac{1}{hff'P_e}$, we get

$$\frac{h}{P_e} - \frac{c^2}{vf'} + \frac{c^2}{vf} = \frac{c}{f}\cos(\theta - \phi) - \frac{c}{f'}\cos\phi + \frac{h}{P_e}\cos\theta$$

Using $\lambda = \frac{c}{f}$ and $\frac{1}{\beta} = \frac{c}{v}$, we get

$$\frac{h}{P_e} - \frac{\lambda'}{\beta} + \frac{\lambda}{\beta} = \lambda\cos(\theta - \phi) - \lambda'\cos\phi + \frac{h}{P_e}\cos\theta$$

Rearranging the terms and multiplying by $\beta$, we get

$$\begin{aligned}
\frac{h}{P_e}(1 - \cos\theta) + \lambda\left(\frac{1}{\beta} - \cos(\theta - \phi)\right) &= \lambda'\left(\frac{1}{\beta} - \cos\phi\right) \\
\beta\frac{h}{P_e}(1 - \cos\theta) + \lambda\left(1 - \beta\cos(\theta - \phi)\right) &= \lambda'\left(1 - \beta\cos\phi\right) \\
\beta\frac{h}{P_e}\left(\frac{1 - \cos\theta}{1 - \beta\cos\phi}\right) + \lambda\left(\frac{1 - \beta\cos(\theta - \phi)}{1 - \beta\cos\phi}\right) &= \lambda'
\end{aligned}$$

Using $\beta = \frac{v}{c}$ and $P_e = \gamma m_e v$, we get

$$\lambda' = \left(\frac{v}{c}\right)\left(\frac{h}{\gamma m_e v}\right)\left(\frac{1 - \cos\theta}{1 - \beta\cos\phi}\right) + \lambda\left(\frac{1 - \beta\cos(\theta - \phi)}{1 - \beta\cos\phi}\right)$$

$$\lambda' = \left(\frac{h}{\gamma m_e c}\right)\left(\frac{1 - \cos\theta}{1 - \beta\cos\phi}\right) + \lambda\left(\frac{1 - \beta\cos(\theta - \phi)}{1 - \beta\cos\phi}\right) \tag{10}$$

When $\beta = 0$, equations (9) and (10) reduce to their well-known counterparts (2) and (1) which describe Compton scattering for initially stationary electrons.

The quantity $\frac{h}{\gamma m_e c}$ is the relativistic analogue to the Compton wavelength of a stationary electron. This can be seen by equating the energies of a photon and a relativistic electron.

$$\begin{aligned}
E_e &= E_\gamma \\
\gamma m_e c^2 &= \frac{hc}{\lambda} \\
\lambda &= \frac{h}{\gamma m_e c}
\end{aligned}$$

# References


[1] "Scattering Calculators," Science Calculators. http://www.sciencecalculators.org/nuclear-physics/compton-scattering/. Accessed June 14, 2021.