Problem 1: [Sources Consulted: Jane(classmate), Zybooks]

10.1.3.c

Since Fiona must select vegetarian on Monday and Friday, then she has ten options to choose from on Tuesday, Wednesday, and Thursday. Since there are only three vegetarian options total, Fion has three options to choose from on both Monday and Friday. Then, the number of ways to select her lunches could be calculated with $3 \times 10 \times 10 \times 10 \times 3 = 9000$. Therefore, there are 9000 total ways for Fiona to select her lunches.

10.2.4.a

We know that the second project must be written by a senior and the third project must be written by a junior. Then there could only be 8 coders for project one (6 juniors, 2 seniors since 1 junior and 1 senior are assigned to each project 3 and 2) and so there are eight ways project 1 could be assigned. Now, there are three ways project 2 can be assigned since there are three seniors. There are seven ways project 3 to be assigned since there are seven juniors.

Therefore, there are $8 \times 3 \times 7 = 168\, ways$ for the projects to be assigned.

10.3.2.c

The set P_7 contains string palindromes that are 7-bit long. So, the first bit should equal the last bit (seventh bit). The second bit should equal the sixth bit. The third bit should equal the fifth bit. There are two ways each to fill the first, second, and third bit: 0 or 1. The fourth bit could be 0 or 1 so there are two ways to fill this position as well. So we could calculate the cardinality of P_7 as $|p_7| = 2 \times 2 \times 2 \times 2 = 16$. Hence, there are 16 ways to fill the first to fourth bits of the seven bit palindrome.

Now, we will show the bijection between P_7 and B^n. First, we will prove it is injective. To do this, we can set them equal to each other: $P_7 = B^n$. We know that we can as 2^n, then, we can rewrite the RHS of the equation as $P_7 = 2^n$. Since we know that $P_7 = 16$, we can rewrite the LHS of the equation as $2^4 = 2^n$. Then, using algebra, n must equal 4. Now, let us assign the four bits of any seven bit palindrome to function f(x). Then, any palindrome from P_7 must map to a four bit string. For example, let us assume the palindrome in question is 0000000. So, we get $f(0000000) = 0000$. Since $P_7 = B^n = 2^4 = 16$, and we know that if f(x) = f(y) then it must be true that x = y. Therefore, there is only one string for which f(x) = f(y) and x = y. Therefore, f(x) is one to one. We also know that the function is surjective since every palindrome maps to a four bit string. For example, f(1101011) maps to 1101 and etc. Therefore since the function is both injective and surjective, then there is a bijective from P_7 to B^n.

<u>10.4.2.b</u>

Numbers that start with 824: $10 \times 9 \times 8 \times 7 = 5040$

Numbers that start with 825: $10 \times 9 \times 8 \times 7 = 5040$

$5040 + 5040 = 1080$ different phone numbers in which the last four digits are all different.

<u>10.5.5.a</u>

Since the ordering of the selection does not matter, we will use combination. Then, we can say 30 choose 10 or $\frac{30!}{(30-10)!10!}$ for boys and 35 choose 10 or $\frac{35!}{(35-10)!10!}$ for girls. Then, we can multiply the two and get $\frac{30!}{(30-10)!10!} \times \frac{35!}{(35-10)!10!} = 5.51565 \times 10^{15}$. Therefore, there are 5.51565×10^{15} ways for the choir director to make his selection.

10.5.6.b

Since the subsets must contain the computers that have the file, there will be two remaining computers in the subset. Hence, we will choose two computers from the 37 that do not contain the file. Therefore, the number of 5-subsets that contain the three computers that have the file is 37 choose 2.

10.6.4.b

Since each kid gets four books and there are five kids, then the number of ways the books can be distributed could be calculated with: $\frac{20!}{4!4!4!4!4!}$. Therefore, there are $\frac{20!}{4!4!4!4!4!}$ ways to distribute 20 books such that each of the kids get 4 books.

10.7.3.c

Since there are only three kids who can play center, we multiply the different ways the rest of the students can be selected in the starting line-up by three. We then consider the three kids who can play center as one kid who can be interchanged in three ways to get the remaining students to be $P(11, 4)$. Therefore, we get $3 \times P(11, 4)$

10.8.2.b

To find how many 5-card hands have at least two cards with the same rank, we must find the number of ways that can be formed with 52 cards that have no card with the same rank and subtract that from the number of total combinations that can be formed with 52 cards and 5 cards hands. Since in a deck of 52 cards, there are 13 cards that are of the same rank, 5 card hands with 13 ranks combination would be $\frac{13!}{((5!)(13-5)!)} = \frac{13!}{5!8!}$. Therefore, the number of 5 card hands with no cards of the same rank is $\frac{13!}{5!8!}$. Now, we must find the total combinations that can be formed with 52 cards and 5 cards hands. Since there are 52 cards and 5 ways to distribute them, we would do $\frac{52!}{(5!)(52-5)!} - \frac{52!}{5!47!}$. Now we just subtract the number of ways that can be formed with 52 cards that have no card with the same rank from the number of total combinations that can be formed with 52 cards and 5 cards hands: $\frac{52!}{5!47!} - \frac{13!}{5!8!}$

<u>10.8.4.a</u>

To find the number of ways the president is not next to the VP, we must subtract the number of ways the president and the VP are next to each other from the total number of ways to line up 10 members. Since there are 10 members, the total number of ways to line them up is 10!. Next, to find the number of ways the VP and prescient are next to each other, we count them as 1 person who can be interchanged in 2 ways, so we would do 2! times the factorial of the people left over, which is $9!$, to get $2!\,9!$. Then, we subtract it from the number of ways to line up 10 people to get $10! - 2!\,9!$.

Problem 2: [Sources Consulted: Jane(classmate), Zybooks]

<u>10.9.2.a</u>

According to the pigeonhole principle if n items are placed into m containers in which n > m, there must be at least one container that contains more than one item. So, if 121.6×10^6 people have incomes that lie between $10,000 and $1,000,000, or 990000 different incomes, then there must be people earning the same income. We can use $(\frac{N}{M})$ in which N is total population and R is number of different incomes, if we map N to R by assigning each person to their annual income, then the Pigeonhole Principle says that there will be at least $(\frac{N}{M}) = (\frac{121600000}{990000}) = 123$. Therefore, 123 people are earning the same annual income.

10.9.3.b

According to the extended pigeonhole principle, a set of n must be at least $k(b - 1) + 1$ to ensure that there is an element y in the target to which f maps at least b elements from the domain. In this scenario, b represents the number of people and k represents the months. So $b - 1 = 20 - 1 = 19$ and $k = 12$ so $k(b - 1) + 1 = 12 \times 19 + 1 = 229$ people must be needed to ensure that 20 of them were born in the same month.

10.9.4.a

The pigeon principle states that if n items are placed into m containers in which $n > m$, there must be at least one container that contains more than one item. In other words, if $n > m$ at least one container will contain $p + 1$ or more items so $p = (\frac{m-1}{n}) = (\frac{15-1}{14}) = 1$. Therefore, $p = 1$ and $p + 1 = 2$ This proves that any two numbers selected from this set will always give a sum of 15

11.1.3.c

Since there are no restrictions, we can use the formula m^n and get $20^5 = 3200000$. Therefore, there are 3200000 ways to distribute the books to the children.

11.3.1.g

First, we must calculate the number of strings that have exactly 2 a's. Since of the 9 characters of the string, 2 must be "a" so we would do 9 choose 2. Then we would multiple 9 choose 2 by what letters are in the rest of the string. Since the rest of the string can be b or c (2 letters) and there are 7 letters left on the string, we would do 2^7. Therefore, the total number of strings that have exactly 2 a's is 9 choose 2 times 2^7.

Now, we will calculate the number of strings that have exactly 3 b's. Since 3 of the characters have to be b out of the 9 characters, we can just say C(9, 3). Then we would multiply C(9, 3) by what letters are in the rest of the string. Since the rest of the string can be a or c and there are 6 letters left of the string, we can do 2^6. Therefore, the total number of strings that has exactly 3 b's are C(9, 3) x 2^6.

Now, we must calculate the number of strings that have both 2 a's and 3 b's and subtract that value from the two values we have already collected. We do this so we do not have overlapping. To start, since 2 a's and 3 b's add up to $2 + 3 = 5$ total letters, then we have $9 - 5 = 4$ letters left, which will be filled by c's. Therefore to calculate the number of strings that have both 2 a's and 3 b's, we would get $\frac{9!}{2!3!4!}$.

Finally, we must subtract the number of strings that have both 2a's and 3b's from the sum of the number of strings that have exactly 3 b's and the number of strings that have exactly 2 a's:

$C(9, 2) \bullet 2^7 + C(9, 3) \bullet 2^6 - \frac{9!}{2!3!4!}$

Problem 3: [Sources Consulted: Jane(classmate), Zybooks]

11.2.8.a

The sum of all of the coefficients of each term is $9 + 2 + 5 + 7 + 2 = 25$, 9 v's, 2 w's, 5 x's, 7 y's, and 2' z's. Therefore, the coefficient is $\frac{25!}{9!2!5!7!2!}$.

11.2.8.b

Let $v^9w^2x^5y^7z^2$ be represented as $(v + w + x + y + z)^{25}$, then $a + b + c + d + e = 25$. Therefore, the total number of terms is calculated by the $C(n + m - 1, m - 1)$ number of ways we can distribute 25 into these 5 variables. Using the equation we get 29 choose 4 or 23751.

Therefore, there are 23751 ways.