TOMAS BJÖRK

Arbitrage Theory in Continuous Time

fourth edition

Now including Equilibrium Theory

OXFORD
FINANCE

Arbitrage Theory in Continuous Time

Fourth Edition

Arbitrage Theory in Continuous Time

FOURTH EDITION

TOMAS BJÖRK
Stockholm School of Economics

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford, OX2 6DP,
United Kingdom

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide. Oxford is a registered trade mark of
Oxford University Press in the UK and in certain other countries

© Tomas Björk 2020

The moral rights of the author have been asserted

First Edition published in 1998
Fourth Edition published in 2020

Impression: 1

All rights reserved. No part of this publication may be reproduced, stored in
a retrieval system, or transmitted, in any form or by any means, without the
prior permission in writing of Oxford University Press, or as expressly permitted
by law, by licence or under terms agreed with the appropriate reprographics
rights organization. Enquiries concerning reproduction outside the scope of the
above should be sent to the Rights Department, Oxford University Press, at the
address above

You must not circulate this work in any other form
and you must impose this same condition on any acquirer

Published in the United States of America by Oxford University Press
198 Madison Avenue, New York, NY 10016, United States of America

British Library Cataloguing in Publication Data
Data available

Library of Congress Control Number: 2019949441

ISBN 978–0–19–885161–5

DOI: 10.1093/oso/9780198851615.001.0001

Printed and bound in Great Britain by
Clays Ltd, Elcograf S.p.A.

Links to third party websites are provided by Oxford in good faith and
for information only. Oxford disclaims any responsibility for the materials
contained in any third party website referenced in this work.

To Agneta, Kajsa, and Stefan

PREFACE TO THE FOURTH EDITION

The fourth edition differs from the third edition by the fact that I have added chapters on the following subjects.

- Incomplete markets. This includes the theory of Esscher transforms, minimal martingale measures, f-divergences, portfolio optimization in incomplete markets, indifference pricing, and good deal bounds.
- Equilibrium theory. This is an introduction to the (vast) area of dynamic equilibrium theory. It includes the Cox–Ingersoll–Ross production factor model as well as the basic theory for unit net supply endowment models.

Due to space limitations I have deleted a chapter on the Black–Scholes approach to multi asset models, and the chapter on barrier options. The interested reader can find an excellent exposition of barrier options (and many other topics) in Joshi (2008).

Prerequisites. For Chapters 1–10 the necessary mathematical background is a good knowledge of advanced calculus and elementary probability theory. From Chapter 11 and onwards, the theoretical level is higher, and in order to read most of the following chapters the reader should be familiar with basic measure and integration theory, as well as with abstract probability theory. A self-contained introduction to these topics can be found in Appendices A–C.

I have corrected a large number of typos and other errors from the third edition, and I am very grateful for comments from Nicholas Amuyedo, Andrey Ashikhmin, Adnan Buyukbilgin, Jiakai Chen, Joseph Clark, Arrigo Coen Coria, Anders Dahlner, Martin Jönsson, Edward Kao, Alex Karpenko, Yavor Kovachev, Katka Lucivjanska, Andrey Lizyayev, Glenn Mickelsson, Asad Munir, Kevin Schmid, Alexander Szimayer, Bill Thygerson, A.M. Underwood, Sebastian Wagner, and Russell Yang Gao. In all probability there are still several typos left. If you find any of these, I would be very grateful if you could inform me by e-mail < tomas.bjork@hhs.se >.

I am much indebted to Mariana Khapko, Ema Iancu, and Simon Wehrmüller. They persuaded me to start writing lecture notes on equilibrium theory and they took a very active part in an informal seminar series on the subject. Special thanks are due to Mariana Khapko, who has discussed most of the equilibrium topics in detail with me.

Tomas Björk

Stockholm
March 1, 2019

PREFACE TO THE FIRST EDITION

The purpose of this book is to present arbitrage theory and its applications to pricing problems for financial derivatives. It is intended as a textbook for graduate and advanced undergraduate students in finance, economics, mathematics, and statistics and I also hope that it will be useful for practitioners.

Because of its intended audience, the book does not presuppose any previous knowledge of abstract measure theory. The only mathematical prerequisites are advanced calculus and a basic course in probability theory. No previous knowledge in economics or finance is assumed.

The book starts by contradicting its own title, in the sense that the second chapter is devoted to the binomial model. After that, the theory is exclusively developed in continuous time.

The main mathematical tool used in the book is the theory of stochastic differential equations (SDEs), and instead of going into the technical details concerning the foundations of that theory I have focused on *applications*. The object is to give the reader, as quickly and painlessly as possible, a solid working knowledge of the powerful mathematical tool known as Itô calculus. We treat basic SDE techniques, including Feynman–Kač representations and the Kolmogorov equations. Martingales are introduced at an early stage. Throughout the book there is a strong emphasis on concrete computations, and the exercises at the end of each chapter constitute an integral part of the text.

The mathematics developed in the first part of the book is then applied to arbitrage pricing of financial derivatives. We cover the basic Black–Scholes theory, including delta hedging and "the Greeks", and we extend it to the case of several underlying assets (including stochastic interest rates) as well as to dividend-paying assets. Barrier options, as well as currency and quanto products, are given separate chapters. We also consider, in some detail, incomplete markets.

American contracts are treated only in passing. The reason for this is that the theory is complicated and that few analytical results are available. Instead I have included a chapter on stochastic optimal control and its applications to optimal portfolio selection.

Interest rate theory constitutes a large part of the book, and we cover the basic short rate theory, including inversion of the yield curve and affine term structures. The Heath–Jarrow–Morton theory is treated, both under the objective measure and under a martingale measure, and we also present the Musiela parametrization. The basic framework for most chapters is that of a multifactor model, and this allows us, despite the fact that we do not formally use measure theory, to give a fairly complete treatment of the general change of numeraire technique which is so essential to modern interest rate theory.

In particular we treat forward neutral measures in some detail. This allows us to present the Geman–El Karoui–Rochet formula for option pricing, and we apply it to the general Gaussian forward rate model, as well as to a number of particular cases.

Concerning the mathematical level, the book falls between the elementary text by Hull (2003) and more advanced texts such as Duffie (2001) or Musiela and Rutkowski (1997). These books are used as canonical references in the present text.

In order to facilitate using the book for shorter courses, the pedagogical approach has been that of first presenting and analyzing a simple (typically one-dimensional) model, and then to derive the theory in a more complicated (multidimensional) framework. The drawback of this approach is of course that some arguments are being repeated, but this seems to be unavoidable, and I can only apologize to the technically more advanced reader.

Notes to the literature can be found at the end of most chapters. I have tried to keep the reference list on a manageable scale, but any serious omission is unintentional, and I will be happy to correct it. For more bibliographic information the reader is referred to Duffie (1996) and to Musiela and Rutkowski (1997) which both contain encyclopedic bibliographies.

On the more technical side the following facts can be mentioned. I have tried to present a reasonably honest picture of SDE theory, including Feynman–Kač representations, while avoiding the explicit use of abstract measure theory. Because of the chosen technical level, the arguments concerning the construction of the stochastic integral are thus forced to be more or less heuristic. Nevertheless I have tried to be as precise as possible, so even the heuristic arguments are the "correct" ones in the sense that they can be completed to formal proofs. In the rest of the text I try to give full proofs of all mathematical statements, with the exception that I have often left out the checking of various integrability conditions.

Since the Girsanov theory for absolutely continuous changes of measures is outside the scope of this text, martingale measures are introduced by the use of locally riskless portfolios, partial differential equations (PDEs) and the Feynman–Kač representation theorem. Still, the approach to arbitrage theory presented in the text is basically a probabilistic one, emphasizing the use of martingale measures for the computation of prices.

The integral representation theorem for martingales adapted to a Wiener filtration is also outside the scope of the book. Thus we do not treat market completeness in full generality, but restrict ourselves to a Markovian framework. For most applications this is, however, general enough.

Tomas Björk

Stockholm
July 1998

ACKNOWLEDGEMENTS

Bertil Näslund, Staffan Viotti, Peter Jennergren, and Ragnar Lindgren persuaded me to start studying financial economics, and they have constantly and generously shared their knowledge with me.

Hans Bühlman, Paul Embrechts, and Hans Gerber gave me the opportunity to give a series of lectures for a summer school at Monte Verita in Ascona 1995. This summer school was for me an extremely happy and fruitful time, as well as the start of a partially new career. The set of lecture notes produced for that occasion is the basis for the present book.

Over the years of writing, I have received valuable comments and advice from a large number of people. My greatest debt is to Camilla Landén, who has given me more good advice (and pointed out more errors) than I thought was humanly possible. I am also highly indebted to Flavio Angelini, Pia Berg, Nick Bingham, Samuel Cox, Darrell Duffie, Otto Elmgart, Malin Engström, Jan Ericsson, Damir Filipović, Andrea Gombani, Stefano Herzel, David Lando, Angus MacDonald, Alexander Matros, Ragnar Norberg, Joel Reneby, Wolfgang Runggaldier, Per Sjöberg, Patrik Säfvenblad, Nick Webber, and Anna Vorwerk.

The main part of this book has been written while I have been at the Finance Department of the Stockholm School of Economics. I am deeply indebted to the school, the department, and the staff working there for support and encouragement.

Parts of the book were written while I was still at the mathematics department of KTH, Stockholm. It is a pleasure to acknowledge the support I got from the department, and from the persons within it.

Finally I would like to express my deeply felt gratitude to Andrew Schuller, James Martin, and Kim Roberts, all at Oxford University Press, and Neville Hankins, the freelance copy-editor who worked on the book. The help given (and patience shown) by these people has been remarkable and invaluable.

This book would never have come into existence without the hard work of Judith Acevedo, Kumar Anbazhagan, Katie Bishop, and Kathleen Gill. All of these have worked on the book through the production process with care, professionalism, and efficiency. For this I am truly grateful.

CONTENTS

Part V Incomplete Markets

1

INTRODUCTION

1.1 Problem Formulation

The main project in this book consists in studying theoretical pricing models for those financial assets which are known as **financial derivatives**. Before we give the formal definition of the concept of a financial derivative we will, however, by means of a concrete example, introduce the single most important example: the European call option.

Let us thus consider the Swedish company *C&H*, which today (denoted by $t = 0$) has signed a contract with an American counterpart *ACME Ltd.* The contract stipulates that *ACME* will deliver 1,000 computers that *C&H* will pay 1000 US dollars per computer to *ACME* at the time of delivery (denoted by $t = T$). For the sake of the argument we assume that the present spot currency rate between the Swedish krona (SEK) and the US dollar is 8.00 SEK/$.

One of the problems with this contract from the point of view of *C&H* is that it involves a considerable **currency risk**. Since *C&H* does not know the currency rate prevailing six months from now, this means that it does not know how many SEK it will have to pay at $t = T$. If the currency rate at $t = T$ is still 8.00 SEK/$ it will have to pay 8,000,000 SEK, but if the rate rises to, say, 8.50 it will face a cost of 8,500,000 SEK. Thus *C&H* faces the problem of how to guard itself against this currency risk, and we now list a number of natural strategies.

1. The most naive strategy for *C&H* is perhaps that of buying one $1,000,000 **today** at the price of 8,000,000 SEK, and then keeping this money (in a Eurodollar account) for six months. The advantage of this procedure is of course that the currency risk is completely eliminated, but there are also some drawbacks. First of all the strategy above has the consequence of tying up a substantial amount of money for a long period of time, but an even more serious objection may be that *C&H* perhaps does not have access to 8,000,000 SEK today.
2. A more sophisticated arrangement, which does not require any outlays at all today, is that *C&H* goes to the forward market and buys a **forward contract** for $1,000,000 with delivery six months from now. Such a contract may, for example, be negotiated with a commercial bank, and in the contract two things will be stipulated.
   - The bank will, at $t = T$, deliver $1,000,000 to *C&H*.
   - *C&H* will, at $t = T$, pay for this delivery at the rate of K SEK/$.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

The exchange rate K, which is called the **forward price** (or forward exchange rate), at $t = 0$, for delivery at $t = T$, is determined at $t = 0$. By the definition of a forward contract, the cost of entering the contract equals zero, and the forward rate K is thus determined by supply and demand on the forward market. Observe, however, that even if the price of entering the forward contract (at $t = 0$) is zero, the contract may very well fetch a non-zero price during the interval $[0, T]$.

Let us now assume that the forward rate today for delivery in six months equals 8.10 SEK/\$. If *C&H* enters the forward contract this simply means that there are no outlays today, and that in six months it will get one \$1,000,000 at the predetermined total price of 8,100,000 SEK. Since the forward rate is determined today, *C&H* has again completely eliminated the currency risk.

However, the forward contract also has some drawbacks, which are related to the fact that a forward contract is a **binding** contract. To see this let us look at two scenarios.

- Suppose that the spot currency rate at $t = T$ turns out to be 8.20. Then *C&H* can congratulate itself, because it can now buy dollars at the rate 8.10 despite the fact that the market rate is 8.20. In terms of the million dollars at stake *C&H* has thereby made an indirect profit of $8,200,000 - 8,100,000 = 100,000$ SEK.
- Suppose on the other hand that the spot exchange rate at $t = T$ turns out to be 7.90. Because of the forward contract this means that *C&H* is forced to buy dollars at the rate of 8.10 despite the fact that the market rate is 7.90, which implies an indirect loss of $8,100,000 - 7,900,000 = 200,000$ SEK.

3. What *C&H* would like to have of course is a contract which guards it against a high spot rate at $t = T$, while still allowing it to take advantage of a low spot rate at $t = T$. Such contracts do in fact exist, and they are called **European call options**. We will now go on to give a formal definition of such an option.

Definition 1.1 *A* **European call option** *on the amount of X US dollars, with* **strike price** *(exercise price) K SEK/\$ and* **exercise date** *T is a contract written at $t = 0$ with the following properties.*

- *The holder of the contract has, exactly at the time $t = T$, the* **right** *to buy X US dollars at the price K SEK/\$.*
- *The holder of the option has no obligation to buy the dollars.*

Concerning the nomenclature, the contract is called an option precisely because it gives the holder the option (as opposed to the obligation) of buying some **underlying** asset (in this case US dollars). A **call** option gives the holder the right to buy, wheareas a **put** option gives the holder the right to sell the underlying object at a prespecified price. The prefix **European** means that the option can only be exercised at exactly the date of expiration. There also exist

American options, which give the holder the right to exercise the option at any time before the date of expiration.

Options of the type above (and with many variations) are traded on options markets all over the world, and the underlying objects can be anything from foreign currencies to stocks, oranges, timber, orange juice, or pig stomachs. For a given underlying object there are typically a large number of options with different dates of expiration and different strike prices.

We now see that *C&H* can insure itself against the currency risk very elegantly by buying a European call option, expiring six months from now, on a million dollars with a strike price of, for example, 8.00 SEK/$. If the spot exchange rate at T exceeds the strike price, say that it is 8.20, then *C&H* exercises the option and buys at 8.00 SEK/$. Should the spot exchange rate at T fall below the strike price, it simply abstains from exercising the option.

Note, however, that in contrast to a forward contract, which by definition has the price zero at the time at which it is entered, an option will always have a non-negative price, which is determined on the existing options market. This means that our friends in *C&H* will have the rather delicate problem of determining exactly which option they wish to buy, since a higher strike price (for a call option) will reduce the price of the option.

One of the main problems in this book is to see what can be said from a theoretical point of view about the market price of an option like the one above. In this context it is worth noting that the European call has some properties which turn out to be fundamental.

- Since the value of the option (at T) depends on the future level of the spot exchange rate, the holding of an option is equivalent to a **future stochastic claim**.
- The option is a **derivative asset** in the sense that it is **defined** in terms of some **underlying** financial asset.

Since the value of the option is contingent on the evolution of the exchange rate, the option is often called a **contingent claim**. Later on we will give a precise mathematical definition of this concept, but for the moment the informal definition above will do. An option is just one example of a financial derivative, and a far from complete list of commonly traded derivatives is given below:

- European calls and puts
- American options
- Forward rate agreements
- Convertibles
- Futures
- Bonds and bond options
- Caps and floors
- Interest rate swaps

Later on we will give precise definitions of (most of) these contracts, but at the moment the main point is the fact that financial derivatives exist in a great variety and are traded in huge volumes. We can now formulate the two main problems which concern us in the rest of the book.

Main Problems: Take a fixed derivative as given.

- What is a "fair" price for the contract?
- Suppose that we have sold a derivative, such as a call option. Then we have exposed ourselves to a certain amount of financial risk at the date of expiration. How do we protect ("hedge") ourselves against this risk?

Let us look more closely at the pricing question above. There exist two natural and mutually contradictory answers.

Answer 1: "Using standard principles of operations research and/or actuarial mathematics a reasonable price for the derivative is obtained by computing the expected value of the discounted future stochastic payoff."

Answer 2: "Using standard economic reasoning, the price of a contingent claim, like the price of any other commodity, will be determined by market forces. In particular it will be determined by the supply and demand curves for the market for derivatives. Supply and demand will in their turn be influenced by such factors as aggregate risk aversion, liquidity preferences, etc., so it is impossible to say anything concrete about the theoretical price of a derivative."

The reason that there is such a thing as a theory for derivatives lies in the following fact.

Main Result: *Both answers above are incorrect! It* **is** *possible (given, of course, some assumptions) to talk about the "correct" price of a derivative, and this price is* **not** *computed by the method given in Answer 1 above.*

In the succeeding chapters we will analyze these problems in detail, but we can already state the basic philosophy here. The main ideas are as follows.

Main Ideas:

- A financial derivative is **defined in terms of** some underlying asset which already exists on the market.
- The derivative cannot therefore be priced arbitrarily **in relation to the underlying prices** if we want to **avoid mispricing between the derivative and the underlying price**.
- We thus want to price the derivative in a way that is **consistent** with the underlying prices given by the market.
- We are **not** trying to compute the price of the derivative in some "absolute" sense. The idea instead is to determine the price of the derivative **in terms of the market prices of the underlying assets**.

PART I

DISCRETE TIME MODELS

2

THE BINOMIAL MODEL

In this chapter we will study, in some detail, the simplest possible nontrivial model of a financial market—the binomial model. This is a discrete time model, but despite the fact that the main purpose of the book concerns continuous time models, the binomial model is well worth studying. The model is very easy to understand, almost all important concepts which we will study later on already appear in the binomial case, the mathematics required to analyze it is at high school level, and last but not least the binomial model is often used in practice.

2.1 The One Period Model

We start with the one period version of the model. In Section 2.2 we will (easily) extend the model to an arbitrary number of periods.

2.1.1 *Model Description*

Running time is denoted by the letter t, and by definition we have two points in time, $t=0$ ("today") and $t=1$ ("tomorrow"). In the model we have two assets: a **bond** and a **stock**. At time t the price of a bond is denoted by B_t, and the price of one share of the stock is denoted by S_t. Thus we have two price processes B and S.

The bond price process is deterministic and given by

$$\begin{aligned} B_0 &= 1, \\ B_1 &= 1+R. \end{aligned}$$

The constant R is the spot rate for the period, and we can also interpret the existence of the bond as the existence of a bank with R as its rate of interest.

The stock price process is a stochastic process, and its dynamical behaviour is described as follows:

$$\begin{aligned} S_0 &= s, \\ S_1 &= \begin{cases} s\cdot u, & \text{with probability } p_u. \\ s\cdot d, & \text{with probability } p_d. \end{cases} \end{aligned}$$

It is often convenient to write this as

$$\begin{cases} S_0 = s, \\ S_1 = s\cdot Z, \end{cases}$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

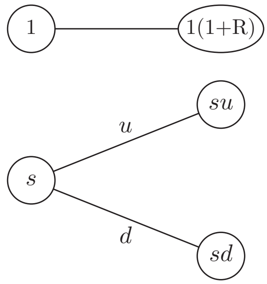

FIG. 2.1. Price dynamics

where Z is a stochastic variable defined as

$$Z = \begin{cases} u, & \text{with probability } p_u. \\ d, & \text{with probability } p_d. \end{cases}$$

We assume that today's stock price s is known, as are the positive constants u, d, p_u and p_d. We assume that $d < u$, and we have of course $p_u + p_d = 1$. We can illustrate the price dynamics using the tree structure in Fig. 2.1.

2.1.2 *Portfolios and Arbitrage*

We will study the behaviour of various **portfolios** on the (B, S) market, and to this end we define a portfolio as a vector $h = (x, y)$. The interpretation is that x is the number of bonds we hold in our portfolio, whereas y is the number of units of the stock held by us. Note that it is quite acceptable for x and y to be positive as well as negative. If, for example, $x = 3$, this means that we have bought three bonds at time $t = 0$. If on the other hand $y = -2$, this means that we have sold two shares of the stock at time $t = 0$. In financial jargon we have a **long** position in the bond and a **short** position in the stock. It is an important assumption of the model that short positions are allowed.

Assumption 2.1.1 *We assume the following institutional facts*:

- *Short positions, as well as fractional holdings, are allowed. In mathematical terms this means that every $h \in R^2$ is an allowed portfolio.*
- *There is no bid-ask spread, i.e. the selling price is equal to the buying price of all assets.*
- *There are no transactions costs of trading.*
- *The market is completely liquid, i.e. it is always possible to buy and/or sell unlimited quantities on the market. In particular it is possible to borrow unlimited amounts from the bank (by selling bonds short).*

Consider now a fixed portfolio $h = (x, y)$. This portfolio has a deterministic market value at $t = 0$ and a stochastic value at $t = 1$.

Definition 2.1 *The* **value process** *of the portfolio h is defined by*

$$V_t^h = xB_t + yS_t, \ t = 0, 1,$$

or, in more detail,

$$V_0^h = x + ys,$$
$$V_1^h = x(1+R) + ysZ.$$

Everyone wants to make a profit by trading on the market, and in this context a so-called arbitrage portfolio is a dream come true; this is one of the central concepts of the theory.

Definition 2.2 *An* **arbitrage** *portfolio is a portfolio h with the properties*

$$V_0^h = 0,$$
$$V_1^h > 0, \ \textit{with probability } 1.$$

An arbitrage portfolio is thus basically a deterministic money-making machine, and we interpret the existence of an arbitrage portfolio as equivalent to a serious case of mispricing on the market. It is now natural to investigate when a given market model is arbitrage free, i.e. when there are no arbitrage portfolios.

Proposition 2.3 *The model above is free of arbitrage if and only if the following conditions hold*:

$$d \leq (1+R) \leq u. \tag{2.1}$$

Proof The condition (2.1) has an easy economic interpretation. It simply says that the return on the stock is not allowed to dominate the return on the bond and vice versa. To show that absence of arbitrage implies (2.1), we assume that (2.1) does in fact not hold, and then we show that this implies an arbitrage opportunity. Let us thus assume that one of the inequalities in (2.1) does not hold, so that we have, say, the inequality $s(1+R) > su$. Then we also have $s(1+R) > sd$ so it is always more profitable to invest in the bond than in the stock. An arbitrage strategy is now formed by the portfolio $h = (s, -1)$, i.e. we sell the stock short and invest all the money in the bond. For this portfolio we obviously have $V_0^h = 0$, and as for $t = 1$ we have

$$V_1^h = s(1+R) - sZ,$$

which by assumption is positive.

Now assume that (2.1) is satisfied. To show that this implies absence of arbitrage let us consider an arbitrary portfolio such that $V_0^h = 0$. We thus have $x + ys = 0$, i.e. $x = -ys$. Using this relation we can write the value of the portfolio at $t = 1$ as

$$V_1^h = \begin{cases} ys\left[u - (1+R)\right], & \text{if } Z = u. \\ ys\left[d - (1+R)\right], & \text{if } Z = d. \end{cases}$$

Assume now that $y > 0$. Then h is an arbitrage strategy if and only if we have the inequalities

$$u > 1 + R,$$
$$d > 1 + R,$$

but this is impossible because of the condition (2.1). The case $y < 0$ is treated similarly. □

At first glance this result is perhaps only moderately exciting, but we may write it in a more suggestive form. To say that (2.1) holds is equivalent to saying that $1 + R$ is a convex combination of u and d, i.e.

$$1 + R = q_u \cdot u + q_d \cdot d,$$

where $q_u, q_d \geq 0$ and $q_u + q_d = 1$. In particular we see that the weights q_u and q_d can be interpreted as probabilities for a new probability measure Q with the property $Q(Z = u) = q_u,\ Q(Z = d) = q_d$. Denoting expectation w.r.t. this measure by E^Q we now have the following easy calculation

$$\frac{1}{1+R} E^Q [S_1] = \frac{1}{1+R} [q_u su + q_d sd] = \frac{1}{1+R} \cdot s(1+R) = s.$$

We thus have the relation

$$s = \frac{1}{1+R} E^Q [S_1],$$

which to an economist is a well known relation. It is in fact a **risk neutral** valuation formula, in the sense that it gives today's stock price as the discounted expected value of tomorrow's stock price. Of course we do not assume that the agents in our market are risk neutral—what we have shown is only that if we use the Q-probabilities instead of the objective probabilities then we have in fact a risk neutral valuation of the stock (given absence of arbitrage). A probability measure with this property is called a **risk neutral measure**, or alternatively a **risk adjusted measure** or a **martingale measure**. Martingale measures will play a dominant role in the sequel so we give a formal definition.

Definition 2.4 *A probability measure Q is called a* **martingale measure** *if the following condition holds*:

$$S_0 = \frac{1}{1+R} E^Q [S_1].$$

We may now state the condition of no arbitrage in the following way.

Proposition 2.5 *The market model is arbitrage free if and only if there exists a martingale measure Q.*

For the binomial model it is easy to calculate the martingale probabilities. The proof is left to the reader.

Proposition 2.6 *For the binomial model above, the martingale probabilities are given by*

$$\begin{cases} q_u = \dfrac{(1+R)-d}{u-d}, \\ q_d = \dfrac{u-(1+R)}{u-d}. \end{cases}$$

2.1.3 *Contingent Claims*

Let us now assume that the market in the preceding section is arbitrage free. We go on to study pricing problems for contingent claims.

Definition 2.7 *A* **contingent claim** *(financial derivative) is any stochastic variable X of the form $X = \Phi(Z)$, where Z is the stochastic variable driving the stock price process above.*

We interpret a given claim X as a contract which pays X SEK to the holder of the contract at time $t = 1$. See Fig. 2.2, where the value of the claim at each node is given within the corresponding box. The function Φ is called the **contract function**. A typical example would be a European call option on the stock with strike price K. For this option to be interesting we assume that $sd < K < su$. If $S_1 > K$ then we use the option, pay K to get the stock and then sell the stock on the market for su, thus making a net profit of $su - K$. If $S_1 < K$ then the option is obviously worthless. In this example we thus have

$$X = \begin{cases} su - K, & \text{if } Z = u, \\ 0, & \text{if } Z = d, \end{cases}$$

and the contract function is given by

$$\Phi(u) = su - K,$$
$$\Phi(d) = 0.$$

Our main problem is now to determine the "fair" price, if such an object exists at all, for a given contingent claim X. If we denote the price of X at

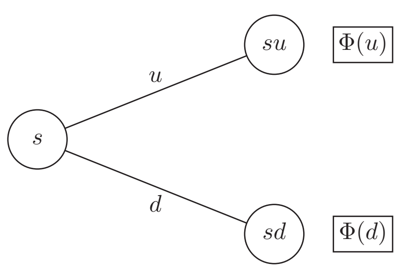

FIG. 2.2. Contingent claim

time t by $\Pi_t[X]$, then it can be seen that at time $t = 1$ the problem is easy to solve. In order to avoid arbitrage we must (why?) have

$$\Pi_1[X] = X,$$

and the hard part of the problem is to determine $\Pi_0[X]$. To attack this problem we make a slight detour.

Since we have assumed absence of arbitrage we know that we cannot make money out of nothing, but it is interesting to study what we **can** achieve on the market.

Definition 2.8 *A given contingent claim X can be* **replicated***, or is said to be* **reachable** *if there exists a portfolio h such that*

$$V_1^h = X,$$

with probability 1. In that case we say that the portfolio h is a **hedging** *portfolio or a* **replicating** *portfolio. If all claims can be replicated we say that the market is* **complete***.*

If a certain claim X is reachable with replicating portfolio h, then, from a financial point of view, there is no difference between holding the claim and holding the portfolio. No matter what happens on the stock market, the value of the claim at time $t = 1$ will be exactly equal to the value of the portfolio at $t = 1$. Thus the price of the claim should equal the market value of the portfolio, and we have the following basic pricing principle.

Pricing principle 1 *If a claim X is reachable with replicating portfolio h, then the only reasonable price process for X is given by*

$$\Pi_t[X] = V_t^h, \quad t = 0, 1.$$

The word "reasonable" above can be given a more precise meaning as in the following proposition. We leave the proof to the reader.

Proposition 2.9 *Suppose that a claim X is reachable with replicating portfolio h. Then any price at $t = 0$ of the claim X, other than V_0^h, will lead to an arbitrage possibility.*

We see that in a complete market we can in fact price all contingent claims, so it is of great interest to investigate when a given market is complete. For the binomial model we have the following result.

Proposition 2.10 *Assume that the general binomial model is free of arbitrage. Then it is also complete.*

Proof We fix an arbitrary claim X with contract function Φ, and we want to show that there exists a portfolio $h = (x, y)$ such that

$$\begin{aligned} V_1^h &= \Phi(u), \quad \text{if } Z = u, \\ V_1^h &= \Phi(d), \quad \text{if } Z = d. \end{aligned}$$

If we write this out in detail we want to find a solution (x, y) to the following system of equations

$$(1+R)x + suy = \Phi(u),$$

$$(1+R)x + sdy = \Phi(d).$$

Since by assumption $d < u$, this linear system has a unique solution, and a simple calculation shows that it is given by

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d}, \tag{2.2}$$

$$y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}. \tag{2.3}$$

□

2.1.4 *Risk Neutral Valuation*

Since the binomial model is shown to be complete we can now price any contingent claim. According to the pricing principle of the preceding section the price at $t = 0$ is given by

$$\Pi_0[X] = V_0^h,$$

and using the explicit formulas (2.2)–(2.3) we obtain, after some reshuffling of terms,

$$\begin{aligned}\Pi_0[X] &= x + sy \\ &= \frac{1}{1+R}\left\{\frac{(1+R)-d}{u-d} \cdot \Phi(u) + \frac{u-(1+R)}{u-d} \cdot \Phi(d)\right\}.\end{aligned}$$

Here we recognize the martingale probabilities q_u and q_d of Proposition 2.6. If we assume that the model is free of arbitrage, these are true probabilities (i.e. they are non-negative), so we can write the pricing formula above as

$$\Pi_0[X] = \frac{1}{1+R}\{\Phi(u) \cdot q_u + \Phi(d) \cdot q_d\}.$$

The right-hand side can now be interpreted as an expected value under the martingale probability measure Q, so we have proved the following basic pricing result, where we also add our old results about hedging.

Proposition 2.11 *If the binomial model is free of arbitrage, then the arbitrage free price of a contingent claim X is given by*

$$\Pi_0[X] = \frac{1}{1+R} E^Q[X]. \tag{2.4}$$

Here the martingale measure Q is uniquely determined by the relation

$$S_0 = \frac{1}{1+R} E^Q[S_1], \tag{2.5}$$

and the explicit expressions for q_u and q_d are given in Proposition 2.6. Furthermore the claim can be replicated using the portfolio

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d}, \tag{2.6}$$

$$y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}. \tag{2.7}$$

We see that the formula (2.4) is a "risk neutral" valuation formula, and that the probabilities which are used are just those for which the stock itself admits a risk neutral valuation. The main economic moral can now be summarized.

Moral:

- The only role played by the objective probabilities is that they determine which events are possible and which are impossible. In more abstract probabilistic terminology they thus determine the class of *equivalent probability measures*. See Chapter 11.
- When we compute the arbitrage free price of a financial derivative we carry out the computations **as if** we live in a risk neutral world.
- This does **not** mean that we de facto live (or believe that we live) in a risk neutral world.
- The valuation formula holds for all investors, regardless of their attitude towards risk, as long as they prefer more deterministic money to less.
- The formula above is therefore often referred to as a "preference free" valuation formula.

We end by studying a concrete example.

Example 2.12 We set $s = 100$, $u = 1.2$, $d = 0.8$, $p_u = 0.6$, $p_d = 0.4$ and, for computational simplicity, $R = 0$. By convention, the monetary unit is the US dollar. Thus we have the price dynamics

$$S_0 = 100,$$

$$S_1 = \begin{cases} 120, & \text{with probability } 0.6 \\ 80, & \text{with probability } 0.4. \end{cases}$$

If we compute the discounted expected value (under the objective probability measure P) of tomorrow's price we get

$$\frac{1}{1+R} E^P[S_1] = 1 \cdot [120 \cdot 0.6 + 80 \cdot 0.4] = 104.$$

This is higher than the value of today's stock price of 100, so the market is risk averse. Since condition (2.1) obviously is satisfied we know that the market is arbitrage free. We consider a European call with strike price $K = 110$, so the claim X is given by

$$X = \begin{cases} 10, & \text{if } S_1 = 120. \\ 0, & \text{if } S_1 = 80. \end{cases}$$

Using the method of computing the price as the discounted expected values under the objective probabilities, i.e. "Answer 1" in Section 1.1, this would give the price as

$$\Pi_0[X] = \frac{1}{1+0}[10 \cdot 0.6 + 0 \cdot 0.4] = 6.$$

Using the theory above it is easily seen that the martingale probabilities are given by $q_u = q_d = 0.5$, thus giving us the theoretical price

$$\Pi_0[X] = \frac{1}{1+0}[10 \cdot 0.5 + 0 \cdot 0.5] = 5.$$

We thus see that the theoretical price differs from the naive approach above. If our theory is correct we should also be able to replicate the option, and from the proposition above the replicating portfolio is given by

$$\begin{aligned} x &= \frac{1.2 \cdot 0 - 0.8 \cdot 10}{1.2 - 0.8} = -20, \\ y &= \frac{1}{100} \cdot \frac{10 - 0}{1.2 - 0.8} = \frac{1}{4}. \end{aligned}$$

In everyday terms this means that the replicating portfolio is formed by borrowing \$20 from the bank, and investing this money in a quarter of a share in the stock. Thus the net value of the portfolio at $t = 0$ is five dollars, and at $t = 1$ the value is given by

$$\begin{aligned} V_1^h &= -20 + \frac{1}{4} \cdot 120 = 10, \quad \text{if } S_1 = 120, \\ V_1^h &= -20 + \frac{1}{4} \cdot 80 = 0, \quad \text{if } S_1 = 80, \end{aligned}$$

so we see that we have indeed replicated the option. We also see that if anyone is foolish enough to buy the option from us for the price \$6, then we can make a riskless profit. We sell the option, thereby obtaining six dollars. Out of these six we invest five in the replicating portfolio and invest the remaining one in the bank. At time $t = 1$ the claims of the buyer of the option are completely balanced by the value of the replicating portfolio, and we still have one dollar invested in the bank. We have thus made an arbitrage profit. If someone is willing to sell the option to us at a price lower than five dollars, we can also make an arbitrage profit by selling the portfolio short.

We end this section by making some remarks.

First of all we have seen that in a complete market, like the binomial model above, there is indeed a unique price for any contingent claim. The price is given by the value of the replicating portfolio, and a negative way of expressing this is as follows. There exists a theoretical price for the claim precisely because of the fact that, strictly speaking, the claim is superfluous—it can equally well be replaced by its hedging portfolio.

Secondly we see that the structural reason for the completeness of the binomial model is the fact that we have two financial instruments at our disposal (the bond and the stock) in order to solve two equations (one for each possible outcome in the sample space). This fact can be generalized. A model is complete (in the generic case) if the number of underlying assets (including the bank account) equals the number of outcomes in the sample space.

If we would like to make a more realistic multiperiod model of the stock market, then the last remark above seems discouraging. If we make a (non-recombining) tree with 20 time steps this means that we have $2^{20} \sim 10^6$ elementary outcomes, and this number exceeds by a large margin the number of assets on any existing stock market. It would therefore seem that it is impossible to construct an interesting complete model with a reasonably large number of time steps. Fortunately the situation is not at all as bad as that; in a multiperiod model we will also have the possibility of considering **intermediary trading**, i.e. we can allow for portfolios which are rebalanced over time. This will give us many more degrees of freedom, and in Section 2.2 we will in fact study a complete multiperiod model.

2.2 The Multiperiod Model

2.2.1 *Portfolios and Arbitrage*

The multiperiod binomial model is a discrete time model with the time index t running from $t = 0$ to $t = T$, where the horizon T is fixed. As before we have two underlying assets, a bond with price process B_t and a stock with price process S_t.

We assume a constant deterministic short rate of interest R, which is interpreted as the simple period rate. This means that the bond price dynamics are given by

$$\begin{aligned} B_{n+1} &= (1+R)B_n, \\ B_0 &= 1. \end{aligned}$$

The dynamics of the stock price are given by

$$\begin{aligned} S_{n+1} &= S_n \cdot Z_n, \\ S_0 &= s. \end{aligned}$$

here $Z_0, \ldots, Z_{T-1}$ are assumed to be i.i.d. (independent and identically distributed) stochastic variables, taking only the two values u and d with probabilities

$$\begin{aligned} P(Z_n = u) &= p_u, \\ P(Z_n = d) &= p_d. \end{aligned}$$

We can illustrate the stock dynamics by means of a tree, as in Fig. 2.3. Note that the tree is **recombining** in the sense that an "up"-move followed by a "down"-move gives the same result as a "down"-move followed by an "up"-move.

We now go on to define the concept of a dynamic portfolio strategy.

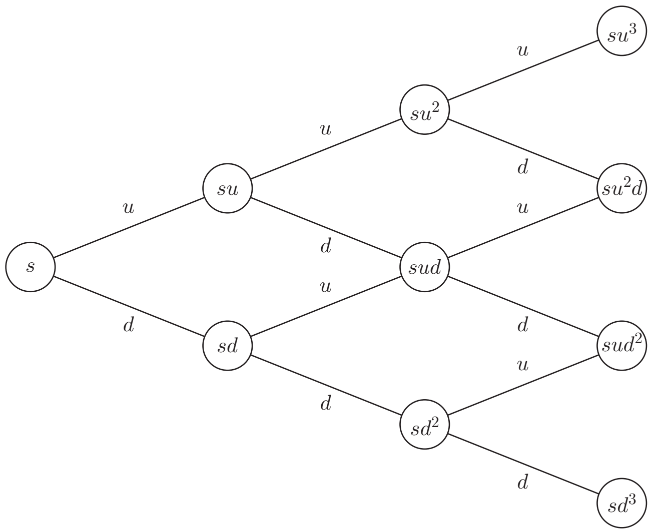

FIG. 2.3. Price dynamics

Definition 2.13 *A* **portfolio strategy** *is a stochastic process*

$$\{h_t = (x_t, y_t); \quad t = 1, \ldots, T\}$$

such that h_t is a function of $S_0, S_1, \ldots, S_{t-1}$. For a given portfolio strategy h we set $h_0 = h_1$ by convention. The **value process** *corresponding to the portfolio h is defined by*

$$V_t^h = x_t(1+R) + y_t S_t.$$

The interpretation of the formal definition is that x_t is the amount of money which we invest in the bank at time $t-1$ and keep until time t. We interpret y_t as the number of shares that we buy at time $t-1$ and keep until time t. We allow the portfolio strategy to be a contingent strategy, i.e. the portfolio we buy at t is allowed to depend on all information we have collected by observing the evolution of the stock price up to time t. We are, however, not allowed to look into the future. The entity V_t^h above is of course the market value of the portfolio (x_t, y_t) (which has been held since $t-1$) at time t.

The portfolios which primarily interest us are the **self-financing** portfolios, i.e. portfolios without any exogenous infusion or withdrawal of money. In practical terms this means that in a self-financing portfolio strategy the accession of a new asset has to be financed through the sale of some other asset. The mathematical definition is as follows.

Definition 2.14 *A portfolio strategy h is said to be* **self-financing** *if the following condition holds for all $t = 0, \dots, T-1$:*

$$x_t(1+R) + y_t S_t = x_{t+1} + y_{t+1} S_t.$$

The condition above is simply a budget equation. It says that, at each time t, the market value of the "old" portfolio (x_t, y_t) (which was created at $t-1$) equals the purchase value of the new portfolio (x_{t+1}, y_{t+1}), which is formed at t (and held until $t+1$).

We can now define the multiperiod version of an arbitrage possibility.

Definition 2.15 *An* **arbitrage** *possibility is a self-financing portfolio h with the properties*

$$\begin{aligned} V_0^h &= 0, \\ P\left(V_T^h \geq 0\right) &= 1, \\ P\left(V_T^h > 0\right) &> 0. \end{aligned}$$

We immediately have the following necessary condition for absence of arbitrage.

Lemma 2.16 *If the model is free of arbitrage then the following conditions necessarily must hold*:

$$d \leq (1+R) \leq u. \tag{2.8}$$

The condition above is in fact also sufficient for absence of arbitrage, but this fact is a little harder to show, and we will prove it later. In any case we assume that the condition holds.

Assumption 2.2.1 *Henceforth we assume that $d < u$, and that the condition (2.8) holds.*

As in the one period model we will have use for "martingale probabilities" which are defined and computed exactly as before.

Definition 2.17 *The martingale probabilities q_u and q_d are defined as the probabilities for which the relation*

$$s = \frac{1}{1+R} E^Q \left[S_{t+1} \middle| S_t = s \right]$$

holds.

Proposition 2.18 *The martingale probabilities are given by*

$$\begin{cases} q_u = \dfrac{(1+R) - d}{u-d}, \\ q_d = \dfrac{u - (1+R)}{u-d}. \end{cases}$$

2.2.2 *Contingent Claims*

We now give the formal definition of a contingent claim in the model.

Definition 2.19 *A* **contingent claim** *is a stochastic variable X of the form*

$$X = \Phi(S_T),$$

where the **contract function** Φ *is some given real valued function.*

The interpretation is that the holder of the contract receives the stochastic amount X at time $t = T$. Notice that we are only considering claims that are "simple", in the sense that the value of the claim only depends on the value S_T of the stock price at the final time T. It is also possible to consider stochastic payoffs which depend on the entire path of the price process during the interval $[0, T]$, but then the theory becomes a little more complicated, and in particular the event tree will become nonrecombining.

Our main problem is that of finding a "reasonable" price process

$$\{\Pi_t[X];\ t = 0, \ldots, T\}$$

for a given claim X, and as in the one period case we attack this problem by means of replicating portfolios.

Definition 2.20 *A given contingent claim X is said to be* **reachable** *if there exists a self-financing portfolio h such that*

$$V_T^h = X,$$

with probability 1*. In that case we say that the portfolio h is a* **hedging** *portfolio or a* **replicating** *portfolio. If all claims can be replicated we say that the market is (dynamically)* **complete**.

Again we have a natural pricing principle for reachable claims.

Pricing principle 2 *If a claim X is reachable with replicating (self-financing) portfolio h, then the only reasonable price process for X is given by*

$$\Pi_t[X] = V_t^h, \quad t = 0, 1, \ldots, T$$

Let us go through the argument in some detail. Suppose that X is reachable using the self-financing portfolio h. Fix t and suppose that at time t we have access to the amount V_t^h. Then we can invest this money in the portfolio h, and since the portfolio is self-financing we can rebalance it over time without any extra cost so as to have the stochastic value V_T^h at time T. By definition $V_T^h = X$ with probability one, so regardless of the stochastic movements of the stock price process the value of our portfolio will, at time T, be equal to the value of the claim X. Thus, from a financial point of view, the portfolio h and the claim X are equivalent so they should fetch the same price.

The "reasonableness" of the pricing formula above can be expressed more formally as follows. The proof is left to the reader.

Proposition 2.21 *Suppose that X is reachable using the portfolio h. Suppose furthermore that, at some time t, it is possible to buy X at a price cheaper than (or to sell it at a price higher than) V_t^h. Then it is possible to make an arbitrage profit.*

We now turn to the completeness of the model.

Proposition 2.22 *The multiperiod binomial model is complete, i.e. every claim can be replicated by a self-financing portfolio.*

It is possible, and not very hard, to give a formal proof of the proposition, using mathematical induction. The formal proof will, however, look rather messy with lots of indices, so instead we prove the proposition for a concrete example, using a binomial tree. This should (hopefully) convey the idea of the proof, and the mathematically inclined reader is then invited to formalize the argument.

Example 2.23 We set $T = 3$, $S_0 = 80$, $u = 1.5$, $d = 0.5$, $p_u = 0.6$, $p_d = 0.4$, and, for computational simplicity, $R = 0$.

The dynamics of the stock price can now be illustrated using the binomial tree in Fig. 2.4, where in each node we have written the value of the stock price.

We now consider a particular contingent claim, namely a European call on the underlying stock. The date of expiration of the option is $T = 3$, and the strike price is chosen to be $K = 80$. Formally this claim can be described as

$$X = \max[S_T - K, 0].$$

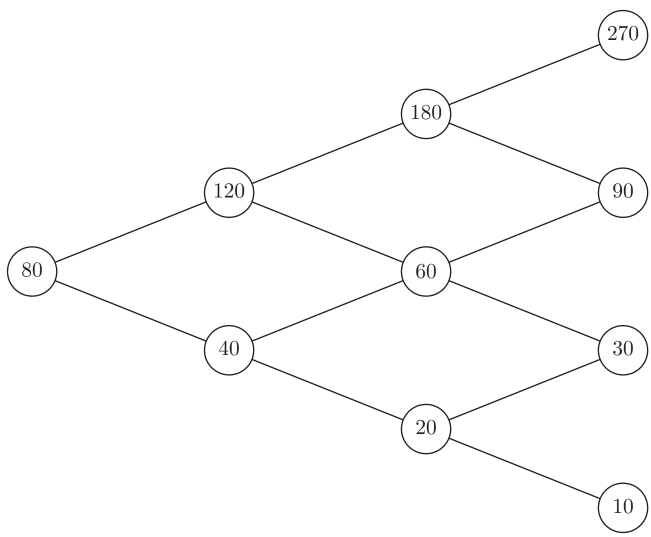

FIG. 2.4. Price dynamics

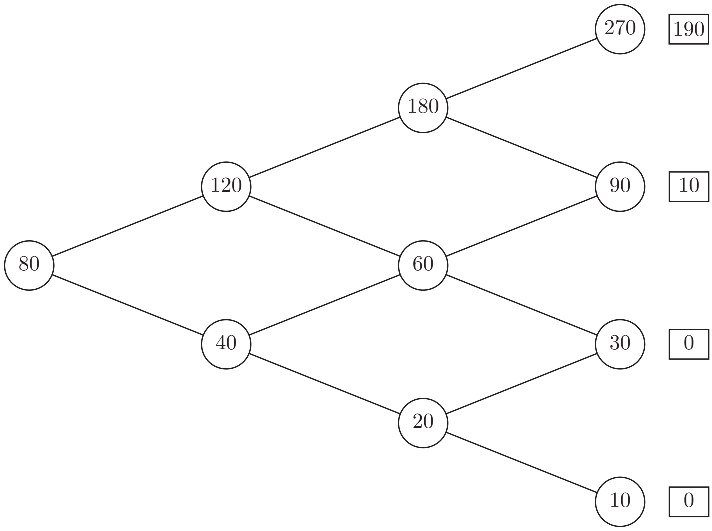

FIG. 2.5.

We will now show that this particular claim can be replicated, and it will be obvious from the argument that the result can be generalized to any binomial model and any claim.

The idea is to use induction on the time variable and to work backwards in the tree from the leaves at $t = T$ to the root at $t = 0$. We start by computing the price of the option at the date of expiration. This is easily done since obviously (why?) we must have, for any claim X, the relation

$$\Pi_T[X] = X.$$

This result is illustrated in Fig. 2.5, where the boxed numbers indicate the price of the claim. Just to check, we see that if $S_3 = 90$, then we exercise the option, pay 80 to obtain the stock, and then immediately sell the stock at market price 90, thus making a profit of 10.

Our problem is thus that of replicating the boxed payoff structure at $t = 3$. Imagine for a moment that we are at some node at $t = 2$, e.g. at the node $S_2 = 180$. What we then see in front of us, from this particular node, is a simple one period binomial model, given in Fig. 2.6, and it now follows directly from the one period theory that the payoff structure in Fig. 2.6 can indeed be replicated from the node $S_2 = 180$. We can in fact compute the cost of this replicating portfolio by risk neutral valuation, and since the martingale probabilities for this example are given by $q_u = q_d = 0.5$ the cost of the replicating portfolio is

$$\frac{1}{1+0}\left[190 \cdot 0.5 + 10 \cdot 0.5\right] = 100.$$

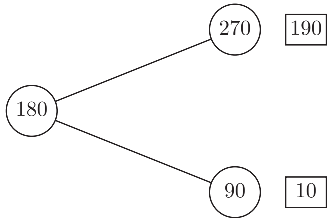

FIG. 2.6.

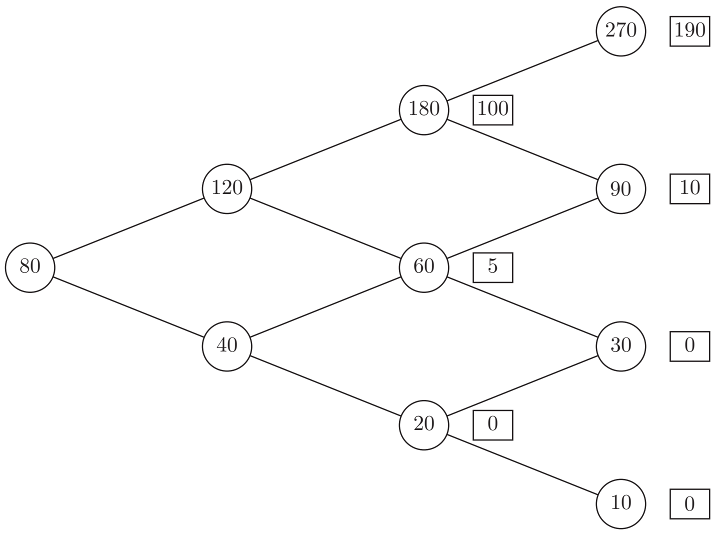

FIG. 2.7.

In the same way we can consider all the other nodes at $t = 2$, and compute the cost of the corresponding replicating portfolios. The result is the set of boxed numbers at $t = 2$ in Fig. 2.7.

What we have done by this procedure is to show that if we can find a self-financing portfolio which replicates the boxed payoff structure at $t = 2$, then it is in fact possible to replicate the original claim at $t = 3$. We have thus reduced the problem in the time variable, and from now on we simply reproduce the construction above, but this time at $t = 1$. Take, for example, the node $S_1 = 40$. From the point of view of this node we have a one period model given by Fig. 2.8, and by risk neutral valuation we can replicate the payoff structure using a portfolio, which at the node $S_1 = 40$ will cost

$$\frac{1}{1+0}\left[5 \cdot 0.5 + 0 \cdot 0.5\right] = 2.5.$$

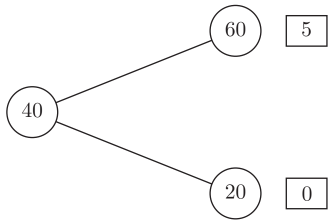

FIG. 2.8.

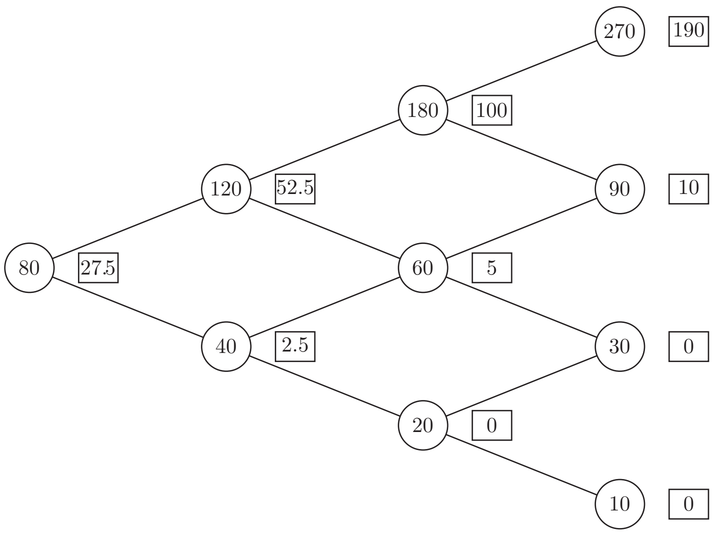

FIG. 2.9.

In this manner we fill the nodes at $t = 1$ with boxed portfolio costs, and then we carry out the same construction again at $t = 0$. The result is given in Fig. 2.9.

We have thus proved that it is in fact possible to replicate the European call option at an initial cost of 27.5. To check this let us now follow a possible price path forward through the tree.

We start at $t = 0$, and since we want to reproduce the boxed claim $(52.5, 2.5)$ at $t = 1$, we can use Proposition 2.4 to compute the hedging portfolio as $x_1 = -22.5$, $y_1 = 5/8$. The reader should check that the cost of this portfolio is exactly 27.5.

Suppose that the price now moves to $S_1 = 120$. Then our portfolio is worth

$$-22.5 \cdot (1 + 0) + \frac{5}{8} \cdot 120 = 52.5.$$

Since we now are facing the claim $(100, 5)$ at $t = 2$ we can again use Proposition 2.4 to calculate the hedging portfolio as $x_2 = -42.5$, $y_2 = 95/120$, and the reader should again check that the cost of this portfolio equals the value of our old portfolio, i.e. 52.5. Thus it is really possible to rebalance the portfolio in a self-financing manner.

We now assume that the price falls to $S_2 = 60$. Then our portfolio is worth

$$-42.5 \cdot (1+0) + \frac{95}{120} \cdot 60 = 5.$$

Facing the claim $(10, 0)$ at $t = 3$ we use Proposition 2.4 to calculate the hedging portfolio as $x_3 = -5, y_3 = 1/6$, and again the cost of this portfolio equals the value of our old portfolio.

Now the price rises to $S_3 = 90$, and we see that the value of our portfolio is given by

$$-5 \cdot (1+0) + \frac{1}{6} \cdot 90 = 10,$$

which is exactly equal to the value of the option at that node in the tree. In Fig. 2.10 we have computed the hedging portfolio at each node.

If we think a bit about the computational effort we see that all the value computations, i.e. all the boxed values, have to be calculated off-line. Having done this we have of course not only computed the arbitrage free price at $t = 0$ for the claim, but also computed the arbitrage free price, at every node in the tree.

The dynamic replicating portfolio does not have to be computed off-line. As in the example above, it can be computed on-line as the price process evolves

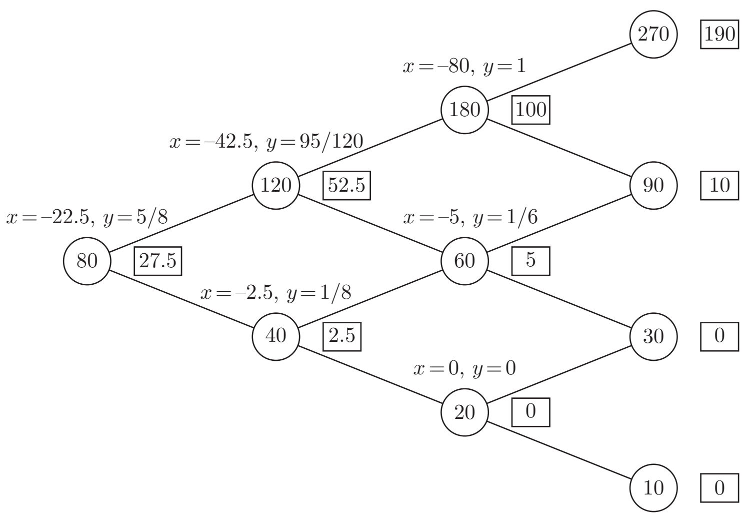

FIG. 2.10.

over time. In this way we only have to compute the portfolio for those nodes that we actually visit.

We now go on to give the general binomial algorithm. In order to do this we need to introduce some more notation to help us keep track of the price evolution. It is clear from the construction that the value of the price process at time t can be written as

$$S_t = su^k d^{t-k}, \; k = 0, \ldots, t$$

where k denotes the number of up-moves that have occurred. Thus each node in the binomial tree can be represented by a pair (t, k) with $k = 0, \ldots, t$.

Proposition 2.24 (Binomial algorithm) *Consider a T-claim $X = \Phi(S_T)$. Then this claim can be replicated using a self-financing portfolio. If $V_t(k)$ denotes the value of the portfolio at the node (t, k) then $V_t(k)$ can be computed recursively by the scheme*

$$\begin{cases} V_t(k) = \dfrac{1}{1+R}\{q_u V_{t+1}(k+1) + q_d V_{t+1}(k)\}, \\ V_T(k) = \Phi\left(su^k d^{T-k}\right). \end{cases}$$

where the martingale probabilities q_u and q_d are given by

$$\begin{cases} q_u = \dfrac{(1+R)-d}{u-d}, \\ \\ q_d = \dfrac{u-(1+R)}{u-d}. \end{cases}$$

With the notation as above, the hedging portfolio is given by

$$\begin{cases} x_t(k) = \dfrac{1}{1+R} \cdot \dfrac{uV_t(k) - dV_t(k+1)}{u-d}, \\ \\ y_t(k) = \dfrac{1}{S_{t-1}} \cdot \dfrac{V_t(k+1) - V_t(k)}{u-d}. \end{cases}$$

In particular, the arbitrage free price of the claim at $t = 0$ is given by $V_0(0)$.

From the algorithm above it is also clear that we can obtain a risk neutral valuation formula.

Proposition 2.25 *The arbitrage free price at $t = 0$ of a T-claim X is given by*

$$\Pi_0[X] = \frac{1}{(1+R)^T} \cdot E^Q[X],$$

where Q denotes the martingale measure, or more explicitly

$$\Pi_0[X] = \frac{1}{(1+R)^T} \cdot \sum_{k=0}^{T} \binom{T}{k} q_u^k q_d^{T-k} \Phi\left(su^k d^{T-k}\right).$$

Proof The first formula follows directly from the algorithm above. If we let Y denote the number of up-moves in the tree we can write

$$X = \Phi(S_T) = \Phi\big(su^Y d^{T-Y}\big),$$

and now the second formula follows from the fact that Y has a binomial distribution. □

We end this section by proving absence of arbitrage.

Proposition 2.26 *The condition*

$$d < (1+R) < u$$

is a necessary and sufficient condition for absence of arbitrage.

Proof The necessity follows from the corresponding one period result. Assume that the condition is satisfied. We want to prove absence of arbitrage, so let us assume that h (a potential arbitrage portfolio) is a self-financing portfolio satisfying the conditions

$$P(V_T^h \geq 0) = 1,$$
$$P(V_T^h > 0) > 0.$$

From these conditions, and from the risk neutral valuation formula, it follows that

$$V_0^h = \frac{1}{(1+R)^T} \cdot E^Q\left[V_T^h\right] > 0,$$

which shows that h is not an arbitrage portfolio. □

2.3 Exercises

Exercise 2.1

(a) Prove Proposition 2.6.
(b) Show, in the one period binomial model, that if $\Pi_1[X] \neq X$ with probability 1, then you can make a riskless profit.

Exercise 2.2 Prove Proposition 2.21.

Exercise 2.3 Consider the multiperiod example in the text. Suppose that at time $t = 1$ the stock price has gone up to 120, and that the market price of the option turns out to be 50.0. Show explictly how you can make an arbitrage profit.

Exercise 2.4 Prove Proposition 2.24, by using induction on the time horizon T.

2.4 Notes

For the origins of the binomial model, see Cox et al. (1979) and Rendleman and Bartter (1979). The textbook Cox and Rubinstein (1992) has become a standard reference.

3

A MORE GENERAL ONE PERIOD MODEL

In this chapter we will investigate absence of arbitrage and completeness in slightly more general terms than in the binomial model. To keep things simple we will be content with a one period model, but the financial market and the underlying sample space will be more general than for the binomial model. The point of this investigation of a simple case is that it highlights some very basic and important ideas, and our main results will in fact be valid for much more general models.

3.1 The Model

We consider a financial market with N different financial assets. These assets could in principle be almost anything, like bonds, stocks, options, or whatever financial instrument that is traded on a liquid market. The market only exists at the two points in time $t=0$ and $t=1$, and the price per unit of asset No. i at time t will be denoted by S_t^i. We thus have a price vector process $S_t,\ t=0,1$ and we will view the price vector as a *column* vector, i.e.

$$S_t = \begin{bmatrix} S_t^1 \\ \vdots \\ S_t^N \end{bmatrix}$$

The randomness in the system is modeled by assuming that we have a finite sample space $\Omega = \{\omega_1, \dots, \omega_M\}$ and that the probabilities $p_j = P(\omega_j),\ j=1,\dots,M$ are all strictly positive. The price vector S_0 is assumed to be deterministic and known to us, but the price vector at time $t=1$ depends upon the outcome $\omega \in \Omega$, and $S_1^i(\omega_j)$ denotes the price per unit of asset No. i at time $t=1$ if ω_j has occured.

We may therefore define the matrix D by

$$D = \begin{bmatrix} S_1^1(\omega_1) & S_1^1(\omega_2) \cdots S_1^1(\omega_M) \\ S_1^2(\omega_1) & S_1^2(\omega_2) \cdots S_1^2(\omega_M) \\ \vdots & \vdots \qquad \vdots \\ S_1^N(\omega_1) & S_1^N(\omega_2) \cdots S_1^N(\omega_M) \end{bmatrix}$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

We can also write D as

$$D = S_t = \begin{bmatrix} | & & | \\ d_1 & \cdots & d_M \\ | & & | \end{bmatrix}$$

where $d_1, \ldots, d_M$ are the columns of D. We will need one important but very mild assumption.

Assumption 3.1.1 *We assume that asset price process S^1 is strictly positive or, more precisely,*

$$\begin{aligned} S_0^1 &> 0, \\ S_1^1(\omega_j) &> 0, \quad \jmath = 1, \ldots, M. \end{aligned}$$

3.2 Absence of Arbitrage

We now define a **portfolio** as an N dimensional row vector $h = \left[h^1, \ldots, h^N\right]$ with the interpretation that h^i is the number of units of asset No. i that we buy at time $t = 0$ and keep until time $t = 1$.

Since we are buying the assets with deterministic prices at time $t = 0$ and selling them at time $t = 1$ at stochastic prices, the **value process** of our portfolio will be a stochastic process V_t^h defined by

$$V_t^h = \sum_{i=1}^{N} h^i S_t^i = hS_t, \quad t = 0, 1, \tag{3.1}$$

and in more detail we can write this as

$$V_t^h(\omega_i) = hS_t(\omega_i) = hd_i = (hD)_i.$$

There are various similar, but not equivalent, variations of the concept of an arbitrage portfolio. The standard one is the following.

Definition 3.1 *The portfolio h is an* **arbitrage portfolio** *if it satisfies the conditions*

$$\begin{aligned} V_0^h &= 0, \\ P\left(V_1^h \geq 0\right) &= 1, \\ P\left(V_1^h > 0\right) &> 0. \end{aligned}$$

In more detail we can write this as

$$\begin{aligned} V_0^h &< 0, \\ V_1^h(\omega_i) &\geq 0, \quad \textit{for all } i = 1, \ldots, M, \\ V_1^h(\omega_i) &> 0, \quad \textit{for some } i = 1, \ldots, M. \end{aligned}$$

We now go on to investigate when the market model above is free of arbitrage possibilities, and the main technical tool for this investigation is Farkas' Lemma.

Lemma 3.2 (Farkas' Lemma) *Suppose that $d_0, d_1, \ldots d_K$ are column vectors in R^N. Then exactly one of the two following problems possesses a solution.*

Problem 1: *Find* **non-negative** *numbers $\lambda_1, \ldots, \lambda_K$ such that*

$$d_0 = \sum_{j=1}^{K} \lambda_j d_j.$$

Problem 2: *Find a row vector $h \in R^N$ such that*

$$\begin{aligned} &hd_0 < 0, \\ &hd_j \geq 0, \quad j = 1, \ldots, M. \end{aligned}$$

Proof Let K be the set of all non-negative linear combinations of $d_1, \ldots, d_M$. It is easy to see that K is a closed convex cone containing the origin. Then exactly one of the following cases can hold:

- The vector d_0 belongs to K. This means that Problem 1 above has a solution.
- The vector d_0 does not belong to K. Then, by the separation theorem for convex sets, there exists a hyperplane H such that d_0 is strictly on one side of H whereas K is on the other side. Letting h be defined as a normal vector to H pointing in the direction where K lies, this means that Problem 2 has a solution. □

We now go on to investigate absence of arbitrage for our model, and in order to see more clearly what is going on we will consider not only the **nominal price system** $S^1, \ldots, S^N$, but also the **normalized price system** $Z^1, \ldots, Z^N$**, under the numeraire** S^1, defined by

$$Z_t^i(\omega_j) = \frac{S_t^i(\omega_j)}{S_t^1(\omega_j)}, \quad i = 1, \ldots, N, \quad j = 1, \ldots, M.$$

The economist will immediately recognize the Z price vector as the vector of relative prices under the numeraire S^1, so if the nominal prices are in dollars and the numeraire is the ACME stock, then the relative prices are given in terms of ACME. More precisely, if a given asset TBINC is quoted at the price 150 dollars, and ACME is quoted at 100 dollars, then the price of TBINC in terms of ACME will be 1.5. The reason for changing from the nominal prices to relative prices is simply that the relative price system is so much easier to analyze. If we obtain a number of results for the relative price system we can then easily translate these results back to the nominal system. To make this translation between the two price systems, we note that a give portfolio h can be viewed as a portfolio in the S system and as a portfolio in the Z system. It will thus generate two different (but equivalent) value processes: The S value process, defined by $V_t^h = hS_t$, and the Z value process, defined by $V_t^{h,Z} = hZ_t$.

We now note the following simple results.

Lemma 3.3 *With notation as above, the following hold.*

1. *The Z value process is related to the S value process by*
$$V_t^{h,Z} = \frac{1}{S_t^1} V_t^h.$$
2. *A portfolio is an arbitrage in the S economy if and only if it is an arbitrage in the Z economy.*
3. *In the Z price system, the numeraire asset Z^1 has unit constant price, i.e.*
$$Z_0^1 = 1,$$
$$Z_1^1(\omega_j) = 1, \quad j = 1, \ldots, M.$$

Proof Obvious. □

The economic interpretation of the fact that $Z^1 \equiv 1$ is that **in the normalized price system, the numeraire asset is risk free** corresponding to a bank with **zero interest rate**. This is the main reason why the normalized prices system is so much easier to analyze than the nominal one.

We can now formulate our first main result.

Proposition 3.4 *The market is arbitrage free if and only if there exists strictly positive real numbers $q_1, \ldots, q_M$ with*

$$q_1 + \ldots + q_j = 1, \tag{3.2}$$

such that the following vector equality holds

$$Z_0 = \sum_{j=1}^{M} Z_1(\omega_j) q_j, \tag{3.3}$$

or, on component form,

$$Z_0^i = \sum_{j=1}^{M} Z_1^i(\omega_j) q_j, \quad i = 1, \ldots, N.$$

Proof Define the matrix D^Z by

$$D^Z = \begin{bmatrix} Z_1^1(\omega_1) & Z_1^1(\omega_2) \cdots Z_1^1(\omega_M) \\ Z_1^2(\omega_1) & Z_1^2(\omega_2) \cdots Z_1^2(\omega_M) \\ \vdots & \vdots \qquad \vdots \\ Z_1^N(\omega_1) & Z_1^N(\omega_2) \cdots Z_1^N(\omega_M) \end{bmatrix}$$

which we can also write as

$$D^Z = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & & & \\ Z_1^2(\omega_1) & Z_1^2(\omega_2) & \cdots & Z_1^2(\omega_M) \\ \vdots & \vdots & & \vdots \\ Z_1^N(\omega_1) & Z_1^N(\omega_2) & \cdots & Z_1^N(\omega_M) \end{bmatrix} \tag{3.4}$$

From the definition of arbitrage, and from Lemma 3.3 it is clear that the market is arbitrage free if and only if the following system of equations has no solution, $h \in R^N$, where $(D^Z)_j$ is component No. j of the row vector hD^Z.

$$\begin{aligned} hZ_0 &= 0, \\ (hD^Z)_j &\geq 0, \quad \text{for all } j = 1, \ldots, M \\ (hD^Z)_j &> 0, \quad \text{for some } j = 1, \ldots, M. \end{aligned}$$

We now want to apply Farkas' Lemma to this system and in order to do this we define the column vector p in R^M by

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_M \end{bmatrix}$$

We can now rewrite the system above as

$$\begin{aligned} hZ_0 &= 0, \\ hD^Z &\geq 0, \\ hD^Z p &> 0, \end{aligned}$$

where the second inequality is interpreted component wise. (We remark that we could in fact have replaced the vector p by any vector in R^M with strictly positive components.) We finally rewrite the first equality as a double inequality to obtain

$$\begin{aligned} hZ_0 &\geq 0, \\ h(-Z_0) &\geq 0, \\ hD^Z &\geq 0, \\ -hD^Z p &< 0. \end{aligned}$$

The point of all this is that if we define the vector $\hat{d}_0$ and the block matrix $\hat{D}$ by

$$\hat{d}_0 = -D^Z p,$$
$$\hat{D} = \left[D^Z,\ Z_0,\ -Z_0\right]$$

we see that the market is free of arbitrage if and only if the system

$$h\hat{d}_0 < 0,$$
$$h\hat{D} \geq 0,$$

has no solution $h \in R^N$, where the last inequality is interpreted component wise. Applying Farkas' Lemma to this system we thus see that absence of arbitrage is equivalent to the existence of non-negative real numbers $\lambda_1, \lambda_2, \ldots, \lambda_M, \lambda_{M+1}, \lambda_{M+2}$ such that (with obvious notation)

$$\hat{d}_0 = \hat{D}\lambda.$$

If we now define the vector $\beta \in R^M$ by

$$\beta = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_M \end{bmatrix}$$

and define the real number α by $\alpha = \lambda_{M+2} - \lambda_{M+1}$ we can write the relation $\hat{d}_0 = \hat{D}\lambda$ as

$$-D^Z p = D^Z \beta - \alpha Z_0$$

or as

$$\alpha Z_0 = D^Z (p + \beta). \tag{3.5}$$

Here $\beta \geq 0$ but we do not know the sign of α.

If we focus on the first component of the vector equality (3.5), and recall that $Z_0^1 = 1$ and that the first row of D^Z has the unit 1 in all positions, we obtain

$$\alpha = \sum_{j=1}^{M} (p_j + \beta_j).$$

From this we see that $\alpha > 0$, and if we define the column vector $q \in R^M$ by

$$q = \frac{1}{\alpha}(p + \beta)$$

we see that

$$q_j > 0, \quad j = 1, \ldots, M. \tag{3.6}$$

and that

$$\sum_{i=1}^{M} q_j = 1. \tag{3.7}$$

Rewriting eqn (3.5) as

$$Z_0 = D^Z q,$$

gives us (3.3). □

3.3 Martingale Measures

In this section we will discuss the result from Section 3.2 from a more probabilistic and economic perspective, and also express the results in a way which can be extended to a much more general situation.

Definition 3.5 *Given the objective probability measure P above, we say that another probability measure Q defined on the same on Ω is* **equivalent** *to P if*

$$P(A) = 0 \Longleftrightarrow Q(A) = 0,$$

or equivalently

$$P(A) = 1 \Longleftrightarrow Q(A) = 1.$$

Two probability measures are thus equivalent if they agree on the events with probability zero and the events which have probability one. In our setting above where $P(\omega_j) > 0$ for $j = 1, \ldots, M$, it is clear that $Q \sim P$ if and only if $Q(\omega_j) > 0$ for $j = 1, \ldots, M$.

We now come to a very important probabilistic concept.

Definition 3.6 *A discrete time random process $\{X_n : \ n = 0, 1, \ldots\}$ is a* **martingale** *if*

$$E\left[X_m | \mathcal{F}_n\right] = X_n, \quad \textit{for all } n \leq m,$$

where $\mathcal{F}_n$ denotes the information available at time n.

A martingale is thus a process which is constant in conditional mean. See Appendix C for details. We note that in the rather trivial setting when we have a process X defined only for time $t = 0, 1$, then the martingale definition trivializes to the condition

$$E\left[X_1\right] = X_0.$$

We now go on to define one of the central concepts of arbitrage pricing.

Definition 3.7 *Consider the market model above, and fix the asset S^1 as the numeraire asset. We say that a probability measure Q defined on Ω is a* **martingale measure** *if it satisfies the following conditions:*

1. *Q is equivalent ot P, i.e.*

$$Q \sim P$$

2. *For every $i = 1, \ldots, N$, the normalized asset price process*

$$Z_t^i = \frac{S_t^i}{S_t^1},$$

is a martingale under the measure Q.

We can now restate Proposition 3.4 as the following result, which in its far-reaching generalizations is known as "the first fundamental theorem of mathematical finance".

Theorem 3.8 (First Fundamental Theorem) *Given a fixed numeraire, the market is free of arbitrage possibilities if and only if there exists a martingale measure Q.*

Proof From Proposition 3.4 we know that absence of arbitrage is equivalent to the existence of strictly positive constants $q_1, \ldots, q_M$ with $q_1 + \ldots + q_N = 1$ such that (3.3) is satisfied. We may thus define a probability measure Q by setting $Q(\omega_j) = q_j$ and since $q_j > 0$ for all j we see that $Q \sim P$. With this definition of Q, the relation (3.3) takes the form

$$Z_0 = E^Q [Z_1], \tag{3.8}$$

which is precisely the stated martingale condition. □

The martingale condition above will allow for a particularly nice economic interpretation if we assume that the numeraire asset S^1 is a **risk free** asset, in the sense that the asset price S_1^1 at time $t = 1$ is constant as a function of ω. By scaling we may assume that $S_0^1 = 1$, and since S_1^1 does not depend on ω we may write

$$S_1^1(\omega_j) = 1 + R, \quad \text{for all } j = 1, \ldots, M,$$

where we can interpret R as the **short interest rate**. We now have the following variation of Theorem 3.8.

Theorem 3.9 (First Fundamental Theorem) *Assume that there exists a risk free asset, and denote the corresponding risk free interest rate by R. Then the market is arbitrage free if and only if there exists a measure $Q \sim P$ such that*

$$S_0^i = \frac{1}{1+R} E^Q \left[S_1^i \right], \quad \textit{for all } i = 1, \ldots, N. \tag{3.9}$$

The economic interpretation is thus that today's asset prices are obtained as the expected value of tomorrow's asset prices, discounted with the risk free rate. The formula is also referred to as a "risk neutral pricing formula". Note that the expectation above is **not** taken under the objective probability measure P but under the martingale measure Q.

3.4 Martingale Pricing

In this section we will study how to price *financial derivatives* or, in technical terms, *contingent claims*. We take the previously studied market model as given and we assume for simplicity that there exists a risk free asset. In order to highlight the role of the risk free asset we denote its price process by B_t and we may thus regard B_t as a bank account, where our money (or our debts) grow at the risk free rate. (In Section 3.3 we thus had $B = S^1$.)

Definition 3.10 *A* **contingent claim** *is any random variable X, defined on the sample space Ω.*

The interpretation is that a contingent claim X represents a stochastic amount of money which we will obtain at time $t = 1$. Our main problem is now to determine a "reasonable" price $\Pi_0[X]$, at time $t = 0$ for a given claim X, and in order to do this we must give a more precise meaning to the word "reasonable" above.

More precisely we would like to price the claim X **consistently** with the underlying a priori given assets $S^1, \ldots, S^N$, or put in other words, we would like to price the claim X in such a way that there are **no arbitrage opportunities** on the extended market consisting of $\Pi, S^1, \ldots, S^N$. This problem is, however, easily solved by applying the First Fundamental Theorem to the extended market. We thus see that the extended market is arbitrage free if and only if there exists some martingale measure Q such that

$$\Pi_0[X] = \frac{1}{1+R} E^Q[\Pi_1[X]],$$

and

$$S_0 = \frac{1}{1+R} E^Q[S_1].$$

Thus, in particular, Q is a martingale measure for the underlying assets. At time $t = 1$ the value of the claim X is known, so in order to avoid arbitrage we must have $\Pi_1[X] = X$. Plugging this into the equation above we have the following result.

Proposition 3.11 *Consider a given claim X. In order to avoid arbitrage, X must then be priced according to the formula*

$$\Pi_0[X] = \frac{1}{1+R} E^Q[X], \tag{3.10}$$

where Q is a martingale measure for the underlying market.

We see that this formula extends the corresponding risk neutral pricing formula (3.9) for the underlying assets. Again we have the economic interpretation that the price at $t = 0$ of the claim X is obtained by computing the expected value of X, discounted by the risk free interest rate, and we again emphasize that the expectation is **not** taken under the objective measure P but under the martingale measure Q.

The pricing formula (3.10) looks very nice, but there is a problem: If there exists several different martingale measures then we will have several possible arbitrage free prices for a given claim X. This has to do with the (possible lack of) **completeness** of the market.

3.5 Completeness

In this section we will discuss how it is possible to generate payment streams at $t=1$ by forming portfolios in the underlying assets.

Assumption 3.5.1 *We assume that the market $S^1,\ldots,S^N$ is arbitrage free and that there exists a risk free asset.*

Definition 3.12 *Consider a contingent claim X. If there exists a portfolio h, based on the underlying assets, such that*

$$V_1^h = X, \quad \textit{with probability } 1. \tag{3.11}$$

i.e.

$$V_1^h(\omega_j) = X(\omega_j), \quad j=1,\ldots,M, \tag{3.12}$$

then we say that X is **replicated**, *or* **hedged** *by h. Such a portfolio h is called a replicating, or hedging portfolio. If every contingent claim can be replicated, we say that the market is* **complete**.

It is easy to characterize completeness in our market, and we have the following result.

Proposition 3.13 *The market is complete if and only if the rows of the matrix D span R^M, i.e. if and only if D has rank M.*

Proof For any portfolio h, we view the random variable V_1^h as a row vector $\left[V_1^h(\omega_1),\ldots,V_1^h(\omega_M)\right]$ and with this notation we have

$$V_1^h = hD.$$

The market is thus complete if and only if, for every random variable X (viewed as a row vector in R^M), the equation

$$hD = X$$

has a solution. But hD is exactly a linear combination of the rows of D with the components of h as coefficients. $\square$

The concept of a replicating portfolio gives rise to an alternative way of pricing contingent claims. Assume that the claim X can be replicated by the portfolio h. Then there is an obvious candidate as the price (at time $t=0$) for X, namely the market price, at $t=0$, of the replicating portfolio. We thus propose the natural pricing formula

$$\Pi_0[X] = V_0^h. \tag{3.13}$$

We also note that any other price would produce an arbitrage possibility, since if for example $V_0^h = 7$ and the price of the claim is given by $\Pi_0[X] = 10$ then we would sell the claim, obtain 10 units of money, and use 7 of these to buy the replicating portfolio. The remaining 3 units of money would be invested safely in the bank. At time $t = 1$ our obligations $(-X)$ would be exactly matched by our assets $V_1^h = X$ and we would still have our money in the bank.

Here there is a possibility that may get us into trouble. There may very well exist two different hedging portfolios f and g, and it could in principle happen that $V_0^f \neq V_0^g$. It is, however, easy to see that this would lead to an arbitrage possibility (how?) so we may disregard that possibility.

The pricing formula (3.13) can also be written in another way, so let us assume that h replicates X. Then, by definition, we have

$$X = hS_1, \tag{3.14}$$

and from (3.13) we obtain

$$\Pi_0[X] = hS_0. \tag{3.15}$$

However, on our arbitrage free market we also have the pricing formula (3.9)

$$S_0 = \frac{1}{1+R} E^Q [S_1]. \tag{3.16}$$

combining this with (3.14)–(3.15) we obtain the pricing formula

$$\Pi_0[X] = \frac{1}{1+R} E^Q [X]. \tag{3.17}$$

which is exactly the formula given by Proposition 3.11. Thus the two pricing approaches coincide on the set of hedgeable claims.

In Proposition 3.13 we obtained one characterization of complete markets. There is another characterization which connects completeness to martingale measures. This result, which we give below in our simple setting, is known as "the second fundamental theorem of mathematical finance".

Proposition 3.14 (Second Fundamental Theorem) *Assume that the model is arbitrage free. Then the market is complete if and only if the martingale measure is unique.*

Proof From Proposition 3.13 we know that the market is complete if and only if the rows of D span the whole of R^M, i.e. if and only if

$$Im\,[D^\star] = R^M,$$

where we view the transposed matrix $D^\star$ as a mapping from R^N to R^M. On the other hand, from Proposition 3.4 and the assumption of absence of arbitrage we know that there exists a solution (even a strictly positive one) to the equation

$$Z_0 = D^Z q.$$

This solution is unique if and only if the kernel (null space) of D^Z is trivial, i.e. if and only if

$$Ker\left[D^Z\right] = 0,$$

and it is easy to see that this is equivalent to the condition

$$Ker\,[D] = 0.$$

We now recall the following well known duality result:

$$(Im\,[D^{\star}])^{\perp} = Ker\,[D]\,.$$

Thus $Ker\,[D] = 0$ if and only if $Im\,[D^{\star}] = R^M$, i.e. the market is complete if and only if the martingale measure is unique. □

We may now summarize our findings.

Proposition 3.15 *Using the risk free asset as the numeraire, the following hold:*

- *The market is arbitrage free if and only if there exists a martingale measure Q.*
- *The market is complete if and only if the martingale measure is unique.*
- *For any claim X, the only prices which are consistent with absence of arbitrage are of the form*

$$\Pi_0[X] = \frac{1}{1+R} E^Q\,[X]\,, \tag{3.18}$$

  where Q is a martingale measure for the underlying market.
- *If the market is incomplete, then different choices of martingale measures Q in the formula (3.18) will generically give rise to different prices.*
- *If X is replicable then, even in an incomplete market, the price in (3.18) will not depend upon the particular choice of martingale measure Q. If X is replicable, then*

$$V_0^h = \frac{1}{1+R} E^Q\,[X]\,,$$

  for all martingale measures Q and for all replicating portfolios h.

3.6 Stochastic Discount Factors

In the previous sections we have seen that we can price financial derivatives by using martingale measures and the formula

$$\Pi_0[X] = \frac{1}{1+R} E^Q\,[X]\,.$$

In some applications of the theory (in particular in asset pricing) it is common to write this expected value directly under the objective probability measure P instead of under Q.

Recalling the notation $p_i = P(\omega_i)$ and $q_i = Q(\omega_i)$, $i = 1, \ldots, M$, and the assumption that $p_i > 0$ for $i = 1, \ldots, M$, we may define a new random variable on Ω.

Definition 3.16 *The random variable L on Ω is defined by*

$$L(\omega_i) = \frac{q_i}{p_i}, \quad i = 1, \ldots, M.$$

Thus L gives us the *likelihood ratio* between the measures P and Q, and in more general situations it is known as the *Radon–Nikodym derivative* of Q w.r.t. P.

Definition 3.17 *Assume absence of arbitrage, and fix a martingale measure Q. With notations as above, the* **stochastic discount factor** *(or "state price deflator") is the random variable Λ on Ω defined by*

$$\mathbf{M}(\omega) = \frac{1}{1+R} \cdot L(\omega). \tag{3.19}$$

We can now express our arbitrage free pricing formulas in a slightly different way.

Proposition 3.18 *The arbitrage free price of any claim X is given by the formula*

$$\Pi_0[X] = E^P\left[\mathbf{M} \cdot X\right] \tag{3.20}$$

where $\mathbf{M}$ is a stochastic discount factor.

Proof Exercise for the reader. $\square$

We see that there is a one-to-one correspondence between stochastic discount factors and martingale measures, and it is largely a matter of taste if you want to work with $\mathbf{M}$ or with Q. The advantage of working with $\mathbf{M}$ is that you formally stay with the objective measure P. The advantage with working under Q is that the decomposition of $\mathbf{M}$ in (3.19) gives us important structural information, and in more complicated situations there exists a deep theory (see "Girsanov transformations" later in the text) which allows us to have complete control over the class of martingale measures.

From an economic point of view, the stochastic discount factor is precisely an Arrow-Debreu state price system, which gives the price $\mathbf{M}(\omega_i)$ to the primitive claim X_i which pays 1 if ω_i occurs, and zero otherwise.

3.7 Exercises

Exercise 3.1 Prove that Q in Proposition 3.11 is a martingale measure also for the price process $\Pi_t[X]$, i.e. show that

$$\frac{\Pi_0[X]}{B_0} = E^Q\left[\frac{\Pi_1[X]}{B_1}\right].$$

where B is the risk free asset.

Exercise 3.2 Prove the last item in Proposition 3.15.

Exercise 3.3 Prove Proposition 3.18.

PART II

STOCHASTIC CALCULUS

4

STOCHASTIC INTEGRALS

4.1 Introduction

The purpose of this book is to study asset pricing on financial markets in continuous time. We thus want to model asset prices as continuous time stochastic processes, and the most complete and elegant theory is obtained if we use **diffusion processes** and **stochastic differential equations** as our building blocks. What, then, is a diffusion?

Loosely speaking we say that a stochastic process X is a diffusion if its local dynamics can be approximated by a stochastic difference equation of the following type:

$$X_{t+\Delta t} - X_t = \mu(t, X_t)\Delta t + \sigma(t, X_t) Z_t. \tag{4.1}$$

Here Z_t is a normally distributed disturbance term which is independent of everything which has happened up to time t, while μ and σ are given deterministic functions. The intuitive content of (4.1) is that, over the time interval $[t, t+\Delta t]$, the X-process is driven by two separate terms.

- A locally deterministic velocity $\mu(t, X_t)$
- A Gaussian disturbance term, amplified by the factor $\sigma(t, X_t)$.

The function μ is called the (local) **drift** term of the process, whereas σ is called the **diffusion** term.

4.1.1 *The Wiener Process*

In order to model the Gaussian disturbance terms we need the concept of a Wiener process.

Definition 4.1 *A stochastic process W is called a* **Wiener process** *if the following conditions hold.*

1. $W_0 = 0$.
2. *The process W has independent increments, i.e. if $r < s \le t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent random variables.*
3. *For $s < t$ the random variable $W_t - W_s$ has the Gaussian distribution $N[0, t-s\ ]$.*
4. *W has continuous trajectories.*

Remark 4.1.1 *Note that we use the notation, where $N\left[\mu, \sigma^2\right]$ denotes a Gaussian distribution with expected value μ and variance σ^2.*

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

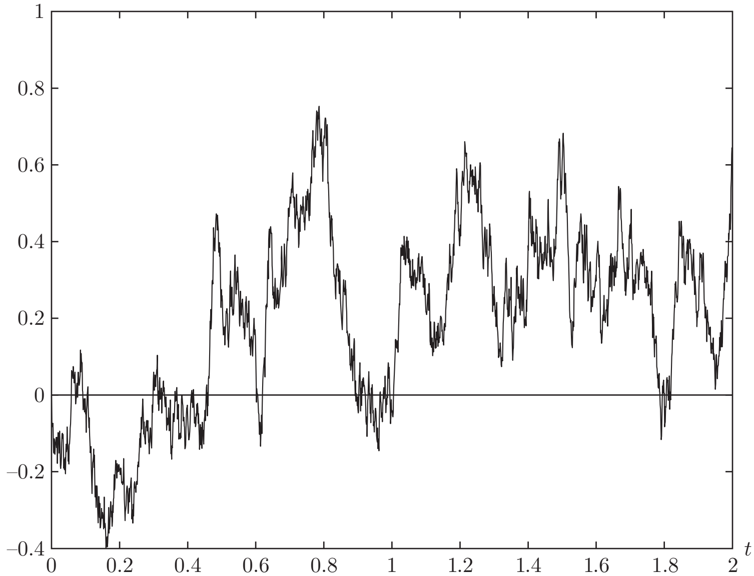

FIG. 4.1. A Wiener trajectory

In Fig. 4.1 a computer-simulated Wiener trajectory is shown. As can be seen from the figure, the Wiener trajectory is very "kinky", and one can in fact prove the following deep result.

Theorem 4.2 *A Wiener trajectory is with probability one nowhere differentiable, and it has locally infinite total variation.*

We may now use a Wiener process in order to write (4.1) as

$$X_{t+\Delta t} - X_t = \mu(t, X_t)\,\Delta t + \sigma(t, X_t)\,\Delta W_t, \tag{4.2}$$

where ΔW_t is defined by

$$\Delta W_t = W_{t+\Delta t} - W_t.$$

Let us now try to make (4.2) a bit more precise. It is then tempting to divide the equation by Δt and let Δt tend to zero. Formally we would obtain

$$\dot{X}_t = \mu(t, X_t) + \sigma(t, X_t)\,v_t, \tag{4.3}$$

$$X_0 = a, \tag{4.4}$$

where $\dot{}$ denotes time derivative, we have added an initial condition, and where

$$v_t = \frac{dW_t}{dt}$$

is the formal time derivative of the Wiener trajectory $t \longmapsto W_t$.

If v were an ordinary (and well-defined) process we would now in principle be able to solve (4.3) as a standard ordinary differential equation (ODE) for each v-trajectory. From Theorem 4.2 we know, however, that a Wiener trajectory is nowhere differentiable (cf. Fig. 4.1), so the process v cannot even be defined. Thus this is a dead end.

Another possibility of making eqn (4.2) more precise is to let Δt tend to zero without first dividing the equation by Δt. Formally we will then obtain the expression

$$\begin{cases} dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \\ X_0 = a, \end{cases} \tag{4.5}$$

and it is now natural to interpret (4.5) as a shorthand version of the following integral equation

$$X_t = a + \int_0^t \mu(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s. \tag{4.6}$$

In eqn (4.6) we may interpret the ds-integral as an ordinary Riemann integral. The natural interpretation of the dW-integral is to view it as a Riemann–Stieltjes integral for each W-trajectory, but unfortunately this is not possible since we know from Theorem 4.2 that the W-trajectories are of locally unbounded variation. Thus the stochastic dW-integral cannot be defined in a naive way.

As long as we insist on giving a precise meaning to eqn (4.2) **for each W-trajectory separately**, we thus seem to be in a hopeless situation. If, however, we relax our demand that the dW-integral in eqn (4.6) should be defined trajectorywise we can still proceed. It is in fact possible to give a global (L^2-)definition of integrals of the form

$$\int_0^t g_s dW_s \tag{4.7}$$

for a large class of processes g. This new integral concept—the so-called Itô integral—will then give rise to a very powerful type of stochastic differential calculus—the Itô calculus. Our program for the future thus consists of the following steps:

1. Define integrals of the type
$$\int_0^t g_s dW_s.$$
2. Develop the corresponding differential calculus.
3. Analyze stochastic differential equations of the type (4.5) using the stochastic calculus above.

4.2 Information

Let X be any given stochastic process. In the sequel it will be important to define "the information generated by X" as time goes by. To do this in a rigorous fashion is outside the scope of the elementary parts of this book. See Appendixes A and B for precise definitions and results. However, for most practical purposes, and until we reach Chapter 11, the following heuristic definitions will do nicely.

Definition 4.3 *The symbol $\mathcal{F}_t^X$ denotes "the information generated by X on the interval $[0,t]$", or alternatively "what has happened to X over the interval $[0,t]$".*

1. *If, based upon observations of the trajectory $\{X_s;\ 0 \le s \le t\}$, it is possible to decide whether a given event A has occurred or not, then we write this as*
$$A \in \mathcal{F}_t^X,$$
or say that "A is $\mathcal{F}_t^X$-measurable".
2. *If the value of a given random variable Z can be completely determined given observations of the trajectory $\{X_s;\ 0 \le s \le t\}$, then we also write*
$$Z \in \mathcal{F}_t^X.$$
3. *If Y is a stochastic process such that we have*
$$Y_t \in \mathcal{F}_t^X$$
for all $t \ge 0$ then we say that Y is **adapted** *to the* **filtration** $\left\{\mathcal{F}_t^X\right\}_{t\ge 0}$. *For brevity of notation, we will sometimes write the filtration as $\left\{\mathcal{F}_t^X\right\}_{t\ge 0} = \mathbf{F}$.*

The above definition is only intended to have an intuitive content, since a precise definition would take us into the realm of abstract measure theory (see Appendix B for details). Nevertheless it is usually extremely simple to use the definition, and we now give some fairly typical examples.

1. If we define the event A by $A = \{X_s \le 3.14, \ \text{ for all } s \le 9\}$ then we have $A \in \mathcal{F}_9^X$.
2. For the event $A = \{X_{10} > 8\}$ we have $A \in \mathcal{F}_{10}^X$. Note, however, that we do not have $A \in \mathcal{F}_9^X$, since it is impossible to decide if A has occurred or not on the basis of having observed the X-trajectory only over the interval $[0,9]$.
3. For the random variable Z, defined by
$$Z = \int_0^5 X_s ds,$$
we have $Z \in \mathcal{F}_5^X$.
4. If W is a Wiener process and if the process X is defined by
$$X_t = \sup_{s \le t} W_s$$
then X is adapted to the W-filtration.
5. With W as above, but with X defined as
$$X_t = \sup_{s \le t+1} W_s,$$
then X is not adapted to the W-filtration.

4.3 Stochastic Integrals

We now turn to the construction of the stochastic integral. For that purpose we consider as given a Wiener process W, and another stochastic process g. In order to guarantee the existence of the stochastic integral we have to impose some kind of integrability conditions on g, and the class $\mathcal{L}^2$ turns out to be natural.

Definition 4.4

(i) We say that the process g belongs to the class $\mathcal{L}^2[a,b]$ if the following conditions are satisfied.
- $\int_a^b E\left[g_s^2\right] ds < \infty$.
- *The process g is adapted to the $\mathcal{F}_t^W$-filtration.*

(ii) We say that the process g belongs to the class $\mathcal{L}^2$ if $g \in \mathcal{L}^2[0,t]$ for all $t > 0$.

Our object is now to define the stochastic integral $\int_a^b g_s dW_s$, for a process $g \in \mathcal{L}^2[a,b]$, and this is carried out in two steps.

Suppose to begin with that the process $g \in \mathcal{L}^2[a,b]$ is **simple**, i.e. that there exist deterministic points in time $a = t_0 < t_1 < \cdots < t_n = b$, such that g is constant on each subinterval. In other words we assume that $g_s = g_{t_k}$ for $s \in [t_k, t_{k+1})$. Then we define the stochastic integral by the obvious formula

$$\int_a^b g_s dW_s = \sum_{k=0}^{n-1} g_{t_k}\left[W_{t_{k+1}} - W_{t_k}\right]. \tag{4.8}$$

Remark 4.3.1 *Note that in the definition of the stochastic integral we take so-called* **forward** *increments of the Wiener process. More specifically, in the generic term $g_{t_k}\left[W_{t_{k+1}} - W_{t_k}\right]$ of the sum the process g is evaluated at the* **left** *end t_k of the interval $[t_k, t_{k+1}]$ over which we take the W-increment. This is essential to the following theory both from a mathematical and (as we shall see later) from an economics point of view.*

For a general process $g \in \mathcal{L}^2[a,b]$ which is not simple we may schematically proceed as follows:

1. Approximate g with a sequence $\{g^n\}$ of simple processes such that

$$\int_a^b E\left[\{g_s^n - g_s\}^2\right] ds \to 0.$$

2. For each n the integral $\int_a^b g_s^n dW_s$ is a well defined random variable Z_n, and it is possible to prove that there exists a random variable Z such that $Z_n \to Z$ (in L^2) as $n \to \infty$.
3. We now define the stochastic integral by

$$\int_a^b g_s dW_s = \lim_{n \to \infty} \int_a^b g_s^n dW_s. \tag{4.9}$$

The most important properties of the stochastic integral are given by the following proposition. In particular we will use the property (4.12) over and over again.

Proposition 4.5 *Let g be a process satisfying the conditions*

$$\int_a^b E\left[g_s^2\right] ds < \infty, \tag{4.10}$$

$$g \text{ is adapted to the } \mathcal{F}_t^W\text{-filtration.} \tag{4.11}$$

Then the following relations hold

$$E\left[\int_a^b g_s dW_s\right] = 0. \tag{4.12}$$

$$E\left[\left(\int_a^b g_s dW_s\right)^2\right] = \int_a^b E\left[g_s^2\right] ds. \tag{4.13}$$

$$\int_a^b g_s dW_s \quad \text{is } \mathcal{F}_b^W\text{-measurable.} \tag{4.14}$$

Proof A full proof is outside the scope of this book, but the general strategy is to start by proving all the assertions above in the case when g is simple. This is fairly easily done, and then it "only" remains to go to the limit in the sense of (4.9). We illustrate the technique by proving (4.12) in the case of a simple g. We obtain

$$\begin{aligned} E\left[\int_a^b g_s dW_s\right] &= E\left[\sum_{k=0}^{n-1} g_{t_k}\left[W_{t_{k+1}} - W_{t_k}\right]\right] \\ &= \sum_{k=0}^{n-1} E\left[g_{t_k}\left[W_{t_{k+1}} - W_{t_k}\right]\right]. \end{aligned}$$

Since g is adapted, the value $g(t_k)$ only depends on the behavior of the Wiener process on the interval $[0, t_k]$. Now, by definition W has independent increments, so $\left[W_{t_{k+1}} - W_{t_k}\right]$ (which is a **forward** increment) is independent of g_{t_k}. Thus we have

$$\begin{aligned} E\left[g_{t_k}\left[W_{t_{k+1}} - W_{t_k}\right]\right] &= E\left[g_{t_k}\right] \cdot E\left[W_{t_{k+1}} - W_{t_k}\right] \\ &= E\left[g_{t_k}\right] \cdot 0 = 0. \end{aligned}$$

□

Remark 4.3.2 *It is possible to define the stochastic integral for a process g satisfying only the weak condition*

$$P\left(\int_a^b g_s^2 ds < \infty\right) = 1. \tag{4.15}$$

For such a general g we have no guarantee that the properties (4.12) and (4.13) hold. Property (4.14) is, however, still valid.

4.4 Martingales

The theory of stochastic integration is intimately connected to the theory of martingales, and the modern theory of financial derivatives is in fact based mainly on martingale theory. Martingale theory, however, requires some basic knowledge of abstract measure theory, and a formal treatment is thus outside the scope of the more elementary parts of this book.

Because of its great importance for the field, however, it would be unreasonable to pass over this important topic entirely, and the object of this section is to (informally) introduce the martingale concept. The more advanced reader is referred to Appendix B for details.

Let us therefore consider a given filtration ("flow of information") $\{\mathcal{F}_t\}_{t\geq 0}$, where, as before, the reader can think of $\mathcal{F}_t$ as the information generated by all observed events up to time t. For any random variable Y we now let the symbol

$$E\left[Y \middle| \mathcal{F}_t\right]$$

denote the "expected value of Y, given the information available at time t". A precise definition of this object requires measure theory, so we have to be content with this informal description. Note that for a fixed t, the object $E\left[Y \middle| \mathcal{F}_t\right]$ is a random variable. If, for example, the filtration is generated by a single observed process X, then the information available at time t will of course depend upon the behaviour of X over the interval $[0,t]$, so the conditional expectation $E\left[Y \middle| \mathcal{F}_t\right]$ will in this case be a function of all past X-values $\{X_s : s \leq t\}$. We will need the following two rules of calculation.

Proposition 4.6

- *If Y and Z are random variables, and Z is $\mathcal{F}_t$-measurable, then*

$$E\left[Z \cdot Y \middle| \mathcal{F}_t\right] = Z \cdot E\left[Y \middle| \mathcal{F}_t\right].$$

- *If Y is a random variable, and if $s < t$, then*

$$E\left[E\left[Y \middle| \mathcal{F}_t\right] \middle| \mathcal{F}_s\right] = E\left[Y \middle| \mathcal{F}_s\right].$$

The first of these results should be obvious: in the expected value $E\left[Z \cdot Y \middle| \mathcal{F}_t\right]$ we condition upon all information available at time t. If now $Z \in \mathcal{F}_t$, this means that, given the information $\mathcal{F}_t$, we know exactly the value of Z, so in the conditional expectation Z can be treated as a constant, and thus it can be taken outside the expectation. The second result is called the "law of iterated expectations", and it is basically a version of the law of total probability.

We can now define the martingale concept.

Definition 4.7 *A stochastic process X is called an $(\mathcal{F}_t)$-***martingale** *if the following conditions hold:*

- *X is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.*
- *For all t we have*

$$E\left[|X_t|\right] < \infty.$$

- *For all s and t with $s \leq t$ the following relation holds*

$$E\left[X_t \middle| \mathcal{F}_s\right] = X_s.$$

A process satisfying, for all s and t with $s \leq t$, the inequality

$$E\left[X_t \middle| \mathcal{F}_s\right] \leq X_s,$$

is called a **supermartingale***, and a process satisfying*

$$E\left[X_t \middle| \mathcal{F}_s\right] \geq X_s,$$

is called a **submartingale***.*

The first condition says that we can observe the value X_t at time t, and the second condition is just a technical condition. The really important condition is the third one, which says that the expectation of a future value of X, given the information available today, equals today's observed value of X. Another way of putting this is to say that a martingale has no systematic drift.

It is possible to prove the following extension of Proposition 4.5.

Proposition 4.8 *For any process $g \in \pounds^2[s,t]$ the following hold:*

$$E\left[\int_s^t g_u dW_u \middle| \mathcal{F}_s^W\right] = 0.$$

As a corollary we obtain the following important fact.

Corollary 4.9 *For any process $g \in \pounds^2$, the process X, defined by*

$$X_t = \int_0^t g_s dW_s,$$

is an $(\mathcal{F}_t^W)$-martingale. In other words, modulo an integrability condition, **every stochastic integral is a martingale***.*

Proof Fix s and t with $s < t$. We have

$$\begin{aligned} E\left[X_t \middle| \mathcal{F}_s^W\right] &= E\left[\int_0^t g_u dW_u \middle| \mathcal{F}_s^W\right] \\ &= E\left[\int_0^s g_u dW_u \middle| \mathcal{F}_s^W\right] + E\left[\int_s^t g_u dW_u \middle| \mathcal{F}_s^W\right]. \end{aligned}$$

The integral in the first expectation is, by Proposition 4.5, measurable w.r.t. $\mathcal{F}_s^W$, so by Proposition 4.6 we have

$$E\left[\int_0^s g_u dW_u \middle| \mathcal{F}_s^W\right] = \int_0^s g_u dW_u,$$

From Proposition 4.5 we also see that $E\left[\int_s^t g_u dW_u \middle| \mathcal{F}_s^W\right] = 0$, so we obtain

$$E\left[X_t | \mathcal{F}_s^W\right] = \int_0^s g_u dW_u + 0 = X_s.$$

□

We have in fact the following stronger (and very useful) result.

Lemma 4.10 *Within the framework above, and assuming enough integrability, a stochastic process X (having a stochastic differential) is a martingale if and only if the stochastic differential has the form*

$$dX_t = g_t dW_t,$$

i.e. X has no dt-term.

Proof We have already seen that if dX has no dt-term then X is a martingale. The reverse implication is much harder to prove, and the reader is referred to the literature cited in the notes below. □

4.5 Stochastic Calculus and the Itô Formula

Let X be a stochastic process and suppose that there exists a real number a and two adapted processes μ and σ such that the following relation holds for all $t \geq 0$.

$$X_t = a + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, \tag{4.16}$$

where a is some given real number. As usual W is a Wiener process. To use a less cumbersome notation we will often write eqn (4.16) in the following form:

$$dX_t = \mu_t dt + \sigma_t dW_t, \tag{4.17}$$

$$X(0) = a. \tag{4.18}$$

In this case we say that X has a **stochastic differential** given by (4.17) with an **initial condition** given by (4.18). It is important to note that the formal string $dX_t = \mu_t dt + \sigma_t dW_t$ has no independent meaning. It is simply a shorthand version of the expression (4.16) above. From an intuitive point of view the stochastic differential is, however, a much more natural object to consider than the corresponding integral expression. This is because the stochastic differential gives us the "infinitesimal dynamics" of the X-process, and as we have seen in Section 4.1 both the drift term μ_t and the diffusion term σ_t have a natural intuitive interpretation.

Let us assume that X indeed has the stochastic differential above. Loosely speaking we thus see that the infinitesimal increment dX_t consists of a locally deterministic drift term $\mu_t dt$ plus an **additive Gaussian** noise term $\sigma_t dW_t$. Assume furthermore that we are given a $C^{1,2}$-function

$$f : R_+ \times R \to R$$

and let us now define a new process Z by

$$Z_t = f(t, X_t).$$

We may now ask what the local dynamics of the Z-process look like, and at first it seems fairly obvious that, except for the case when f is linear in x, Z will not have a stochastic differential. Consider, for example, a discrete time example where X satisfies a stochastic difference equation with additive Gaussian noise in each step, and suppose that $f(t,x) = e^x$. Then it is clear that Z will not be driven by additive Gaussian noise—the noise will in fact be multiplicative and log-normal. It is therefore extremely surprising that for continuous time models the stochastic differential structure with a drift term plus additive Gaussian noise will in fact be preserved even under nonlinear transformations. Thus the process Z **will** have a stochastic differential, and the form of dZ_t is given explicitly by the famous Itô formula below. Before turning to the Itô formula we have to take a closer look at some rather fine properties of the trajectories of the Wiener process.

As we saw earlier the Wiener process is defined by a number of very simple probabilistic properties. It is therefore natural to assume that a typical Wiener trajectory is a fairly simple object, but this is not at all the case. On the contrary—one can show that, with probability one, the Wiener trajectory will be a continuous function of time (see the definition above) which is nondifferentiable at every point. Thus a typical trajectory is a continuous curve consisting entirely of corners and it is of course quite impossible to draw a figure of such an object (it is in fact fairly hard to prove that such a curve actually exists). This lack of smoothness gives rise to an odd property of the quadratic variation of the Wiener trajectory, and since the entire theory to follow depends on this particular property we now take some time to study the Wiener increments a bit closer.

Let us therefore fix two points in time, t with $t + \Delta t$, and let us use the handy notation

$$\Delta W_t = W_{t+\Delta t} - W_t. \tag{4.19}$$

Using well known properties of the normal distribution it is fairly easy to obtain the following results, which we will use frequently:

$$E[\Delta W_t] = 0, \tag{4.20}$$

$$E\left[(\Delta W_t)^2\right] = \Delta t, \tag{4.21}$$

$$Var[\Delta W_t] = \Delta t, \tag{4.22}$$

$$Var[(\Delta W_t)^2] = 2(\Delta t)^2. \tag{4.23}$$

We see that the squared Wiener increment $(\Delta W_t)^2$ has an expected value which equals the time increment Δt. The really important fact, however, is that, according to (4.23), the variance of $[\Delta W_t]^2$ is negligible compared to its expected value. In other words, as Δt tends to zero $[\Delta W_t]^2$ will of course also tend to zero, but the variance will approach zero much faster than the expected value. Thus $[\Delta W_t]^2$ will look more and more "deterministic" and we are led to believe that in the limit we have the purely formal equality

$$[dW_t]^2 = dt. \tag{4.24}$$

The reasoning above is purely heuristic. It requires a lot of hard work to turn the relation (4.24) into a mathematically precise statement, and it is of course even harder to prove it. We will not attempt either a precise formulation or a precise proof. In order to give the reader a flavor of the full theory we will, however, give another argument for the relation (4.24).

Let us therefore fix a point in time t and subdivide the interval $[0,t]$ into n equally large subintervals of the form $\left[k\frac{t}{n},(k+1)\frac{t}{n}\right]$, where $k=0,1,\ldots,n-1$. Given this subdivision, we now define the quadratic variation of the Wiener process by S_n, i.e.

$$S_n = \sum_{i=1}^{n}\left[W\left(i\frac{t}{n}\right) - W_{(i-1)\frac{t}{n}}\right]^2, \tag{4.25}$$

and our goal is to see what happens to S_n as the subdivision becomes finer, i.e. as $n \to \infty$. We immediately see that

$$\begin{aligned} E\left[S_n\right] &= \sum_{i=1}^{n} E\left[\left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right)\right]^2\right] \\ &= \sum_{i=1}^{n}\left[i\frac{t}{n} - (i-1)\frac{t}{n}\right] = t. \end{aligned}$$

Using the fact that W has independent increments we also have

$$\begin{aligned} Var[S_n] &= \sum_{i=1}^{n} Var[\left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right)\right]^2] \\ &= \sum_{i=1}^{n} 2\left[\frac{t^2}{n^2}\right] = \frac{2t^2}{n}. \end{aligned}$$

Thus we see that $E\left[S_n\right] = t$ whereas $Var[S_n] \to 0$ as $n \to \infty$. In other words, as $n \to \infty$ we see that S_n tends to the **deterministic** limit t. This motivates us to write

$$\int_0^t (dW_t)^2 = t, \tag{4.26}$$

or, equivalently,

$$(dW_t)^2 = dt. \tag{4.27}$$

Note again that all the reasoning above has been purely motivational. In this text we will have to be content with accepting (4.27) as a dogmatic truth, and now we can give the main result in the theory of stochastic calculus—the Itô formula.

Theorem 4.11 (Itô's formula) *Assume that the process X has a stochastic differential given by*

$$dX_t = \mu_t dt + \sigma_t dW_t, \tag{4.28}$$

where μ and σ are adapted processes, and let f be a $C^{1,2}$-function. Define the process Z by $Z(t) = f(t, X_t)$. Then Z has a stochastic differential given by

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + \sigma \frac{\partial f}{\partial x}(t, X_t) dW_t. \tag{4.29}$$

Proof A full formal proof is outside the scope of this text, so we only give a heuristic proof. (See Remark 4.5.1 below.) If we make a Taylor expansion including second-order terms we obtain

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x} dt dX_t. \tag{4.30}$$

By definition we have

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

so, at least formally, we obtain

$$(dX_t)^2 = \mu_t^2 (dt)^2 + 2\mu_t \sigma_t (dt)(dW_t) + \sigma_t^2 (dW_t)^2.$$

The term containing $(dt)^2$ above is negligible compared to the dt-term in (4.28), and it can also be shown that the $(dt)(dW)$-term is negligible compared to the dt-term. Furthermore we have $(dW_t)^2 = dt$ from (4.24), and plugging all this into the Taylor expansion (4.30) gives us the result. □

It may be hard to remember the Itô formula, so for practical purposes it is often easier to copy our "proof" above and make a second-order Taylor expansion.

Proposition 4.12 (Itô's formula) *With assumptions as in Theorem 4.11, df is given by*

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 \tag{4.31}$$

where we use the following formal multiplication table:

$$\begin{cases} (dt)^2 = 0, \\ dt \cdot dW_t = 0, \\ (dW_t)^2 = dt. \end{cases}$$

Remark 4.5.1 *As we have pointed out, the "proof" of the Itô formula above does not at all constitute a formal proof. We end this section by giving an outline of the full proof. What we have to prove is that, for all t, the following relation holds with probability one:*

$$f(t,X_t)-f(0,X_0)=\int_0^t\left(\frac{\partial f}{\partial t}+\mu\frac{\partial f}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)ds$$
$$+\int_0^t\sigma\frac{\partial f}{\partial x}dW_s. \tag{4.32}$$

We therefore divide the interval $[0,t]$ as $0=t_0<t_1<\ldots<t_n=t$ into n equal subintervals. Then we have

$$f(t,X_t)-f(0,X_0)=\sum_{k=0}^{n-1}f\left(t_{k+1},X_{t_{k+1}}\right)-f\left(t_k,X_{t_k}\right). \tag{4.33}$$

Using Taylor's formula we obtain, with subscripts for f denoting partial derivatives and hopefully obvious notation,

$$f\left(t_{k+1},X_{t_{k+1}}\right)-f\left(t_k,X_{t_k}\right)=f_t\left(t_k,X_{t_k}\right)\Delta t+f_x\left(t_k,X_{t_k}\right)\Delta X_k$$
$$+\frac{1}{2}f_{xx}\left(t_k,X_{t_k}\right)\left(\Delta X_k\right)^2+Q_k, \tag{4.34}$$

where Q_k is the remainder term. Furthermore we have

$$\Delta X_k=X_{t_{k+1}}-X_{t_k}=\int_{t_k}^{t_{k+1}}\mu_s ds+\int_{t_k}^{t_{k+1}}\sigma_s dW_s$$
$$=\mu_{t_k}\Delta t+\sigma_{t_k}\Delta W_k+S_k, \tag{4.35}$$

where S_k is a remainder term. From this we obtain

$$\left(\Delta X_k\right)^2=\mu_{t_k}^2\left(\Delta t\right)^2+2\mu_{t_k}\sigma_{t_k}\Delta t\Delta W_k+\sigma_{t_k}^2\left(\Delta W_k\right)^2+P_k, \tag{4.36}$$

where P_k is a remainder term. If we now substitute (4.34)–(4.36) into (4.33) we obtain, in shorthand notation,

$$f(t,X_t)-f(0,X_0)=I_1+I_2+I_3+\frac{1}{2}I_4+\frac{1}{2}K_1+K_2+R,$$

where

$$I_1=\sum_k f_t(t_k,X_{t_k})\Delta t, \qquad I_2=\sum_k f_x(t_k,X_{t_k})\mu_{t_k}\Delta t,$$
$$I_3=\sum_k f_x(t_k,X_{t_k})\sigma_{t_k}\Delta W_k, \qquad I_4=\sum_k f_{xx}(t_k,X_{t_k})\sigma_{t_k}^2\left(\Delta W_k\right)^2,$$
$$K_1=\sum_k f_{xx}(t_k,X_{t_k})\mu_{t_k}^2\left(\Delta t\right)^2, \qquad K_2=\sum_k f_{xx}(t_k,X_{t_k})\mu_{t_k}\sigma_{t_k}\Delta t\Delta W_k,$$
$$R=\sum_k\left\{Q_k+S_k+P_k\right\}.$$

Letting $n\to\infty$ we have, more or less by definition,

$$I_1\to\int_0^t f_t\left(s,X_s\right)ds, \qquad I_2\to\int_0^t f_x\left(s,X_s\right)\mu(s)ds,$$
$$I_3\to\int_0^t f_x\left(s,X_s\right)\sigma(s)dW_s.$$

Very much as when we proved earlier that $\sum\left(\Delta W_k\right)^2\to t$, it is possible to show that

$$I_4 \to \int_0^t f_{xx}(s, X_s)\,\sigma^2(s)ds,$$

and it is fairly easy to show that K_1 and K_2 converge to zero. The really hard part is to show that the term R, which is a large sum of individual remainder terms, also converges to zero. This can, however, also be done and the proof is finished.

4.6 Examples

In order to illustrate how to use the Itô formula we now give some examples. Before going onto the examples, however, we make a very important note.

Note 4.6.1 *Let us recall the Itô formula*

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + \sigma \frac{\partial f}{\partial x}(t, X_t) dW_t.$$

When you apply this formula it is very important to realize that all partial derivatives are partial derivatives of the **deterministic function** $f(t, x)$*. For example, the term*

$$\frac{\partial f}{\partial t}(t, X_t)$$

must be interpreted in the following way:

- **First** *you compute the usual* **deterministic derivative** $\frac{\partial f}{\partial t}(t, x)$.
- **Then** *you plug in the random variable* X_t *to obtain* $\frac{\partial f}{\partial t}(t, X_t)$.

Example 4.13 We start with a purely technical five-finger exercise. The task is to compute

$$d(t \cdot e^{aW_t}).$$

We can (pedantically) define X by $X_t = W_t$ and set $f(t, x) = te^{ax}$. We then have

$$dX_t = 0dt + 1dW_t$$

so in terms of the Itô formula, we have $\mu_t = 0$ and $\sigma_t = 1$. We also have

$$\frac{\partial f}{\partial t}(t, x) = e^{ax}, \quad \frac{\partial f}{\partial x}(t, x) = ate^{ax}, \quad \frac{\partial^2 f}{\partial t^2}(t, x) = a^2 te^{ax}.$$

The Itô formula then gives us

$$df(tX_t) = \left\{ e^{aX_t} + 0 \cdot ate^{aX_t} + \frac{1}{2} \cdot 1 \cdot a^2 te^{aX_t} \right\} dt + ate^{aX_t} dW_t,$$

which, using the equality $X_t = W_t$, gives us

$$d(te^{aW_t}) = \left\{ e^{aW_t} + \frac{1}{2} \cdot a^2 te^{aW_t} \right\} dt + ate^{aW_t} dW_t.$$

Example 4.14 Compute $d(W_t^2)$.

Solution: We again define X by $X_t = W_t$ and set $f(t,x) = x^2$. We then have

$$dX_t = 0dt + 1dW_t$$

so in terms of the Itô formula, we have $\mu_t = 0$ and $\sigma_t = 1$. We also have

$$\frac{\partial f}{\partial t}(t,x) = 0, \quad \frac{\partial f}{\partial x}(t,x) = 2x, \quad \frac{\partial^2 f}{\partial t^2}(t,x) = 2.$$

The Itô formula then gives us

$$df(tX_t) = \left\{0 + 0 \cdot 2X_t + \frac{1}{2} \cdot 1 \cdot 2\right\} dt + 1 \cdot 2X_t dW_t,$$

which, using the equality $X_t = W_t$, gives us

$$d(W_t^2) = dt + 2W_t dW_t.$$

We note how this differs from the usual deterministic result $d(x^2) = 2xdx$.

Example 4.15 It is natural to ask whether one can "compute" (in some sense) the value of a stochastic integral. This is a fairly vague question, but regardless of how it is interpreted, the answer is generally no. There are just a few examples where the stochastic integral can be computed in a fairly explicit way. Here is the most famous one. Compute

$$\int_0^t W_s dW_s.$$

Solution: A natural guess is perhaps that $\int_0^t W_s dW_s = \frac{W_t^2}{2}$. Since Itô calculus does not coincide with ordinary calculus this guess cannot possibly be true, but nevertheless it seems natural to start by investigating the process $Z_t = W_t^2$. From the previous example we have

$$d(W_t^2) = dt + 2W_t dW_t.$$

In integrated form this reads

$$W_t^2 = t + 2\int_0^t W_s dW_s,$$

so we get our answer

$$\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}.$$

We now move on to two examples which illustrate a useful technique for computing expected values in situations involving Wiener processes. Since arbitrage pricing to a large extent consists of precisely the computation of certain expected values this technique will be used repeatedly in the sequel. These examples are quite simple, and the results could have been obtained as well by using standard techniques from elementary probability theory. The full force of the Itô calculus will be seen in the following chapters.

Suppose that we want to compute the expected value $E[Y]$ where Y is some random variable. Schematically we will then proceed as follows:

1. Try to write Y as $Y = Z_T$ where T is some point in time and Z is a random process having an Itô differential.
2. Use the Itô formula to compute dZ as, for example,

$$\begin{aligned} dZ_t &= \mu_t dt + \sigma_t dW_t, \\ Z_0 &= z_0. \end{aligned}$$

3. Write this expression in integrated form as

$$Z_t = z_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s.$$

4. Take expected values. Using Proposition 4.5 we see that the dW-integral will vanish. For the ds-integral we may move the expectation operator inside the integral sign (an integral is "just" a sum), and we thus have

$$E[Z_t] = z_0 + \int_0^t E[\mu_s] ds.$$

   Now two cases can occur.
   (a) We may, by skill or pure luck, be able to calculate the expected value $E[\mu_s]$ explicitly. Then we only have to compute an ordinary Riemann integral to obtain $E[Z_t]$, and thus to read off $E[Y] = E[Z_T]$.
   (b) If we cannot compute $E[\mu_s]$ directly we have a harder problem, but in some cases we may convert our problem to that of solving an ordinary differential equation (ODE).

Example 4.16 Compute $E\left[W_t^4\right]$.

Solution: Define Z by $Z_t = W_t^4$. Then we have $Z_t = f(t, X_t)$ where $X = W$ and f is given by $f(t,x) = x^4$. Thus the stochastic differential of X is trivial, namely $dX_t = dW_t$, which, in the notation of the Itô formula (4.29), means that $\mu = 0$ and $\sigma = 1$. Furthermore we have $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial x} = 4x^3$, and $\frac{\partial^2 f}{\partial x^2} = 12x^2$. Thus the Itô formula gives us

$$\begin{aligned} dZ_t &= 6W_t^2 dt + 4W_t^3 dW_t, \\ Z_0 &= 0. \end{aligned}$$

Written in integral form this reads

$$Z_t = 0 + 6\int_0^t W_s^2 ds + 4\int_0^t W_s^3 dW_s.$$

Now we take the expected values of both members of this expression. Then, by Proposition 4.5, the stochastic integral will vanish. Furthermore we may move the expectation operator inside the ds-integral, so we obtain

$$E\left[Z_t\right] = 6\int_0^t E\left[W_s^2\right] ds.$$

Now we recall that $E\left[W_s^2\right] = s$, so in the end we have our desired result

$$E\left[W_t^4\right] = E\left[Z_t\right] = 6\int_0^t s ds = 3t^2.$$

□

Example 4.17 Compute $E\left[e^{\alpha W_t}\right]$.

Solution: Define Z by $Z(t) = e^{\alpha W_t}$. The Itô formula gives us

$$dZ(t) = \frac{1}{2}\alpha^2 e^{\alpha W_t} dt + \alpha e^{\alpha W_t} dW_t,$$

so we see that Z satisfies the **stochastic differential equation** (SDE)

$$dZ_t = \frac{1}{2}\alpha^2 Z_t dt + \alpha Z_t dW_t,$$

$$Z_0 = 1.$$

In integral form this reads

$$Z_t = 1 + \frac{1}{2}\alpha^2 \int_0^t Z_s ds + \alpha \int_0^t Z_s dW_s.$$

Taking expected values will make the stochastic integral vanish. After moving the expectation within the integral sign in the ds-integral and defining m by $m_t = E\left[Z_t\right]$ we obtain the equation

$$m_t = 1 + \frac{1}{2}\alpha^2 \int_0^t m_s ds.$$

This is an integral equation, but if we take the t-derivative we obtain the ODE

$$\begin{cases} \dot{m}_t = \frac{\alpha^2}{2} m_t, \\ m_0 = 1. \end{cases}$$

Solving this standard equation gives us the answer

$$E\left[e^{\alpha W_t}\right] = E\left[Z_t\right] = m_t = e^{\frac{\alpha^2}{2}t}.$$

We end with a useful lemma.

□

Lemma 4.18 *Let $\sigma(t)$ be a given* **deterministic** *function of time and define the process X by*

$$X_t = \int_0^t \sigma(s) dW_s. \tag{4.37}$$

Then X_t has a normal distribution with zero mean, and variance given by

$$Var[X_t] = \int_0^t \sigma^2(s)ds.$$

This is of course an expected result because the integral is "merely" a linear combination of the normally distributed Wiener increments with deterministic coefficients. See the exercises for a hint of the proof.

4.7 The Multidimensional Itô Formula

Let us now consider a vector process $X = (X^1, \ldots, X^n)^\star$, where the component X^i has a stochastic differential of the form

$$dX_t^i = \mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j$$

and $W^1, \ldots, W^d$ are d **independent** Wiener processes.

Defining the drift vector process μ by

$$\mu_t = \begin{bmatrix} \mu^1 \\ \vdots \\ \mu^n \end{bmatrix},$$

the d-dimensional vector Wiener process W by

$$W = \begin{bmatrix} W^1 \\ \vdots \\ W^d \end{bmatrix},$$

and the $n \times d$-dimensional **diffusion matrix** process σ_t by

$$\sigma = \begin{bmatrix} \sigma^{11} & \sigma^{12} \ldots \sigma^{1d} \\ \sigma^{21} & \sigma^{22} \ldots \sigma^{2d} \\ \vdots & \vdots \ \ddots \ \vdots \\ \sigma^{n1} & \sigma^{n2} \ldots \sigma^{nd} \end{bmatrix},$$

we may write the X-dynamics as

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

Let us furthermore define the process Z by

$$Z_t = f(t, X_t),$$

where $f : R_+ \times R^n \to R$ is a $C^{1,2}$ mapping. Then, using arguments as above, it can be shown that the stochastic differential df is given by

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial x^i} dX_t^i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j} dX_t^i dX_t^j, \tag{4.38}$$

with the extended multiplication rule (see the exercises)

$$dW_t^i \cdot dW_t^j = 0, \quad \text{for } i \neq j.$$

Written out in full (see the exercises) this gives us the following result.

Theorem 4.19 (Itô's formula) *Let the n-dimensional process X have dynamics given by*

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

with notation as above. Then the following hold:

- *The process $f(t, X_t)$ has a stochastic differential given by*

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_t^i \frac{\partial f}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{n} C_t^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} \right\} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \sigma_t^i dW_t.$$

  Here the row vector σ_i is the i:th row of the matrix σ, i.e.

$$\sigma_i = [\sigma^{i1}, \ldots, \sigma^{id}],$$

  and the matrix C is defined by

$$C = \sigma\sigma^\star,$$

  where * denotes transpose.
- *Alternatively, the differential is given by the formula*

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j} dX^i dX^j,$$

  with the formal multiplication table

$$\begin{cases} (dt)^2 = 0, & \\ dt \cdot dW_t^i = 0, & i = 1, \ldots, d, \\ \left(dW_t^i\right)^2 = dt, & i = 1, \ldots, d, \\ dW_t^i \cdot dW_t^j = 0, & i \neq j. \end{cases}$$

Remark 4.7.1 (Itô's formula) *The Itô formula can also be written as*

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_t^i \frac{\partial f}{\partial x^i} + \frac{1}{2} tr\left[\sigma_t^\star H \sigma_t\right] \right\} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \sigma_t^i dW_t,$$

where H denotes the Hessian matrix

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j},$$

and tr denotes the **trace** *of a matrix. The trace is defined, for any square matrix A, as the sum of the diagonal elements, i.e.*

$$tr(A) = \sum_i A_{ii}.$$

See the exercises for details.

4.8 Correlated Wiener Processes

Up to this point we have only considered independent Wiener processes, but sometimes it is convenient to build models based upon Wiener processes which are correlated. In order to define such objects, let us therefore consider d independent standard (i.e. unit variance) Wiener processes $\bar{W}^1, \ldots, \bar{W}^d$. Let furthermore a (deterministic and constant) matrix

$$\delta = \begin{bmatrix} \delta_{11} & \delta_{12} \ldots \delta_{1d} \\ \delta_{21} & \delta_{22} \ldots \delta_{2d} \\ \vdots & \vdots \ \ddots \ \vdots \\ \delta_{n1} & \delta_{n2} \ldots \delta_{nd} \end{bmatrix}$$

be given, and consider the n-dimensional processes W, defined by

$$W = \delta \bar{W}$$

where

$$W = \begin{bmatrix} W^1 \\ \vdots \\ W^n \end{bmatrix}.$$

In other words

$$W_t^i = \sum_{j=1}^{d} \delta_{ij} \bar{W}_t^j, \quad i = 1, \ldots, n.$$

Let us now **assume that the rows of δ have unit length**, i.e.

$$||\delta_i|| = 1, \quad i = 1, \ldots, n,$$

where the Euclidean norm is defined as usual by

$$||x|| = \sqrt{\sum_{i=1}^{d} x_i^2}.$$

Then it is easy to see (how?) that each of the components $W_1, \ldots, W_n$ separately are standard (i.e. unit variance) Wiener processes. Let us now define the (instantaneous) **correlation matrix** ρ of W by

$$\rho_{ij} dt = Cov[dW_t^i, dW_t^j].$$

We then obtain

$$\begin{aligned}\rho_{ij}dt &= E[dW_t^i \cdot dW_t^j] - E[dW_t^i] \cdot E[dW_t^j] = E[dW_t^i \cdot dW_t^j] \\ &= E\left[\sum_{k=1}^{d} \delta_{ik} d\bar{W}_t^k \cdot \sum_{l=1}^{d} \delta_{jl} d\bar{W}_t^l\right] = \sum_{kl} \delta_{ik}\delta_{jl} E[d\bar{W}_t^k \cdot d\bar{W}_t^l] \\ &= \sum_{k=1}^{d} \delta_{ik}\delta_{jk} = \delta_i \delta_j^\star dt,\end{aligned}$$

i.e.

$$\rho = \delta\delta^\star.$$

Definition 4.20 *The process W, constructed as above, is called a vector of* **correlated** *Wiener processes, with* **correlation matrix** *ρ.*

Using this definition we have the following Itô formula for correlated Wiener processes.

Proposition 4.21 (Itô's formula) *Consider a vector Wiener process $W = (W^1, \ldots, W^n)$ with correlation matrix ρ, and assume that the vector process $X = (X^1, \ldots, X^k)^\star$ has a stochastic differential. Then the following hold:*

- *For any $C^{1,2}$ function f, the stochastic differential of the process $f(t, X_t)$ is given by*

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j} dX_t^i dX_t^j,$$

with the formal multiplication table

$$\begin{cases} (dt)^2 &= 0, \\ dt \cdot dW_t^i &= 0, \quad i = 1, \ldots, n, \\ dW_t^i \cdot dW_t^j &= \rho_{ij} dt. \end{cases}$$

- *If, in particular, $k = n$ and dX_t has the structure*

$$dX_t^i = \mu_t^i dt + \sigma_t^i dW_t^i, \quad i = 1, \ldots, n,$$

where $\mu^1, \ldots, \mu^n$ and $\sigma^1, \ldots, \sigma^n$ are scalar processes, then the stochastic differential of the process $f(t, X_t)$ is given by

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_t^i \frac{\partial f}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{n} \sigma_t^i \sigma_t^j \rho_{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} \right\} dt + \sum_{i=1}^{n} \sigma_t^i \frac{\partial f}{\partial x^i} dW_t^i.$$

4.9 Exercises

Exercise 4.1 Compute the stochastic differential dZ_t when

(a) $Z_t = e^{\alpha t}$

(b) $Z_t = \int_0^t g_s dW_s$, where g is an adapted stochastic process

(c) $Z_t = e^{\alpha W_t}$

(d) $Z_t = e^{\alpha X_t}$, where X has the stochastic differential

$$dX_t = \mu dt + \sigma dW_t$$

where μ and σ are constants.

(e) $Z_t = X_t^2$, where X has the stochastic differential

$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

Exercise 4.2 Compute the stochastic differential for Z when $Z_t = \frac{1}{X_t}$ and X has the stochastic differential

$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

By using the definition $Z = X^{-1}$ you can in fact express the right-hand side of dZ entirely in terms of Z itself (rather than in terms of X). Thus Z satisfies a stochastic differential equation. Which one?

Exercise 4.3 Let $\sigma(t)$ be a given deterministic function of time and define the process X by

$$X_t = \int_0^t \sigma(s) dW_s. \tag{4.39}$$

Use the technique described in Example 4.17 in order to show that the characteristic function of X_t (for a fixed t) is given by

$$E\left[e^{iuX_t}\right] = \exp\left\{-\frac{u^2}{2}\int_0^t \sigma^2(s)ds\right\}, \quad u \in R, \tag{4.40}$$

thus showing that X_t is normally distibuted with zero mean and a variance given by

$$Var[X_t] = \int_0^t \sigma^2(s)ds.$$

Exercise 4.4 Suppose that X has the stochastic differential

$$dX_t = \alpha X_t dt + \sigma_t dW_t,$$

where α is a real number whereas $\sigma(t)$ is any (sufficiently integrable) adapted random process. Use the technique in Example 4.17 in order to determine the function $m(t) = E\left[X_t\right]$.

Exercise 4.5 Suppose that the process X has a stochastic differential

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

and that $\mu_t \geq 0$ with probability one for all t. Show that this implies that X is a submartingale.

Exercise 4.6 A function $h(x_1, \ldots, x_n)$ is said to be **harmonic** if it satisfies the condition

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} = 0.$$

It is **subharmonic** if it satisfies the condition

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Let $W_1, \ldots, W_n$ be independent standard Wiener processes, and define the process X by $X_t = h(W_1(t), \ldots, W_n(t))$. Show that X is a martingale (submartingale) if h is harmonic (subharmonic).

Exercise 4.7 The object of this exercise is to give an argument for the formal identity

$$dW_1(t) \cdot dW_2(t) = 0,$$

when W_1 and W_2 are independent Wiener processes. Let us therefore fix a time t, and divide the interval $[0,t]$ into equidistant points $0 = t_0 < t_1 < \cdots < t_n = t$, where $t_i = \frac{i}{n} \cdot t$. We use the notation

$$\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1}), \quad i = 1, 2.$$

Now define Q_n by

$$Q_n = \sum_{k=1}^{n} \Delta W_1(t_k) \cdot \Delta W_2(t_k).$$

Show that $Q_n \to 0$ in L^2, i.e. show that

$$E[Q_n] = 0,$$
$$Var[Q_n] \to 0.$$

Exercise 4.8 Let X and Y be given as the solutions to the following system of stochastic differential equations.

$$dX_t = \alpha X_t dt - Y_t dW_t, \quad X_0 = x_0,$$
$$dY_t = \alpha Y_t dt + X_t dW_t, \quad Y_0 = y_0.$$

Note that the initial values x_0, y_0 are deterministic constants.

(a) Prove that the process R defined by $R_t = X_t^2 + Y_t^2$ is deterministic.
(b) Compute $E[X_t]$.

Exercise 4.9 For a $n \times n$ matrix A, the **trace** of A is defined as

$$tr(A) = \sum_{i=1}^{n} A_{ii}.$$

(a) If B is $n \times d$ and C is $d \times n$, then BC is $n \times n$. Show that

$$tr(BC) = \sum_{ij} B_{ij} C_{ji}.$$

(b) With assumptions as above, show that

$$tr(BC) = tr(CB).$$

(c) Show that the Itô formula in Theorem 4.19 can be written

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu^i \frac{\partial f}{\partial x^i} + \frac{1}{2} tr\left[\sigma^\star H \sigma\right] \right\} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \sigma^i dW^i$$

where H denotes the Hessian matrix

$$H_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}.$$

Exercise 4.10 Prove all claims in Section 4.8.

4.10 Notes

As (far-reaching) introductions to stochastic calculus and its applications, Øksendal (1998) and Steele (2001) can be recommended. Standard references on a more advanced level are Karatzas and Shreve (2008), and Revuz and Yor (1991). The theory of stochastic integration can be extended from the Wiener framework to allow for semimartingales as integrators, and a classic in this field is Meyer (1976). Standard references are Jacod and Shiryaev (1987), Elliott (1982), Dellacherie and Meyer (1972), and Protter (2004).

5

STOCHASTIC DIFFERENTIAL EQUATIONS

5.1 Stochastic Differential Equations

Let $M(n,d)$ denote the class of $n \times d$ matrices, and consider as given the following objects:

- A d-dimensional (column-vector) Wiener process W
- A (column-vector valued) function $\mu : R_+ \times R^n \to R^n$
- A function $\sigma : R_+ \times R^n \to M(n,d)$
- A real (column) vector $x_0 \in R^n$.

We now want to investigate whether there exists a stochastic process X which satisfies the **stochastic differential equation** (SDE)

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \tag{5.1}$$

$$X_0 = x_0. \tag{5.2}$$

To be more precise we want to find a process X satisfying the integral equation

$$X_t = x_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s, \quad \text{for all } t \geq 0. \tag{5.3}$$

The standard method for proving the existence of a solution to the SDE above is to construct an iteration scheme of Cauchy–Picard type. The idea is to define a sequence of processes $X^0, X^1, X^2, \ldots$ according to the recursive definition

$$X_t^0 \equiv x_0, \tag{5.4}$$

$$X_t^{n+1} = x_0 + \int_0^t \mu(s, X_s^n)\, ds + \int_0^t \sigma(s, X_s^n)\, dW_s. \tag{5.5}$$

Having done this one expects that the sequence $\{X^n\}_{n=1}^{\infty}$ will converge to some limiting process X, and that this X is a solution to the SDE. This construction can in fact be carried out, but as the proof requires some rather hard inequalities we only give the result.

Proposition 5.1 *Suppose that there exists a constant K such that the following conditions are satisfied for all x, y and t.*

$$\|\mu(t,x) - \mu(t,y)\| \leq K\|x - y\|, \tag{5.6}$$

$$\|\sigma(t,x) - \sigma(t,y)\| \leq K\|x - y\|, \tag{5.7}$$

$$\|\mu(t,x)\| + \|\sigma(t,x)\| \leq K(1 + \|x\|). \tag{5.8}$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

Then there exists a unique solution to the SDE (5.1)–(5.2). The solution has the properties

1. *X is $\mathcal{F}_t^W$-adapted.*
2. *X has continuous trajectories.*
3. *X is a Markov process.*
4. *There exists a constant C such that*

$$E\left[\|X_t\|^2\right] \le Ce^{Ct}\left(1+\|x_0\|^2\right). \tag{5.9}$$

The fact that the solution X is $\mathcal{F}_t^W$-adapted means that for each fixed t the process value X_t is a functional of the Wiener trajectory on the interval $[0,t]$, and in this way an SDE induces a transformation of the space $C[0,\infty)$ into itself, where a Wiener trajectory $W.(\omega)$ is mapped to the corresponding solution trajectory $X.(\omega)$. Generically this transformation, which takes a Wiener trajectory into the corresponding X-trajectory, is enormously complicated and it is extremely rare that one can "solve" an SDE in some "explicit" manner. There are, however, a few nontrivial interesting cases where it is possible to solve an SDE, and the most important example for us is the equation below, describing the so-called Geometric Brownian Motion (GBM).

5.2 Geometric Brownian Motion

Geometric Brownian Motion will be one of our fundamental building blocks for the modeling of asset prices, and it also turns up naturally in many other places. The equation is one of two natural generalizations of the simplest linear ODE and looks as follows:

Geometric Brownian Motion:

$$dX_t = \alpha X_t dt + \sigma X_t dW_t, \tag{5.10}$$

$$X_0 = x_0. \tag{5.11}$$

Written in a slightly sloppy form we can write the equation as

$$\dot{X}_t = \left(\alpha + \sigma \dot{W}_t\right) X_t$$

where $\dot{W}$ is "white noise", i.e. the (formal) time derivative of the Wiener process. Thus we see that GBM can be viewed as a linear ODE, with a stochastic coefficient driven by white noise. See Fig. 5.1, for a computer simulation of GBM with $\alpha = 1$, $\sigma = 0.2$, and $X(0) = 1$. The smooth line is the graph of the expected value function $E[X_t] = 1 \cdot e^{\alpha t}$. For small values of σ, the trajectory will (at least initially) stay fairly close to the expected value function, whereas a large value of σ will give rise to large random deviations. This can clearly be seen when we compare the simulated trajectory in Fig. 5.1 to the three simulated trajectories in Fig. 5.2 where we have $\sigma = 0.4$.

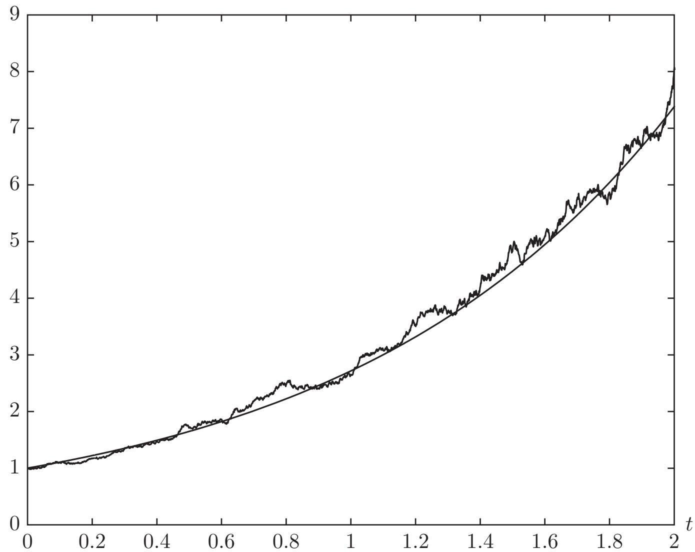

Fig. 5.1. Geometric Brownian Motion: $\alpha = 1,\ \sigma = 0.2$

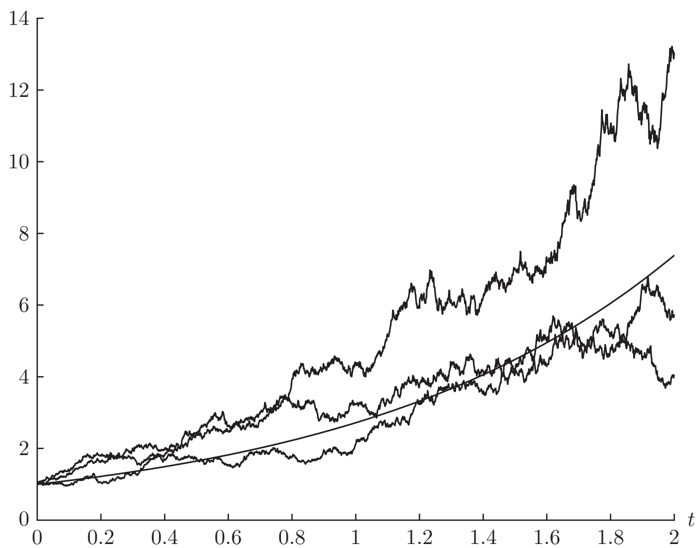

Fig. 5.2. Geometric Brownian Motion: $\alpha = 1,\ \sigma = 0.4$

Inspired by the fact that the solution to the corresponding deterministic linear equation is an exponential function of time we are led to investigate the process Z, defined by $Z_t = \ln X_t$, where we **assume** that X is a solution and that X is strictly positive (see below). The Itô formula gives us

$$\begin{aligned} dZ_t &= \frac{1}{X_t} dX_t + \frac{1}{2}\left\{-\frac{1}{X_t^2}\right\}[dX_t]^2 \\ &= \frac{1}{X_t}\{\alpha X_t dt + \sigma X_t dW_t\} + \frac{1}{2}\left\{-\frac{1}{X_t^2}\right\}\sigma^2 X_t^2 dt \\ &= \{\alpha dt + \sigma dW_t\} - \frac{1}{2}\sigma^2 dt. \end{aligned}$$

Thus we have the equation

$$\begin{aligned} dZ_t &= \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma dW_t, \\ Z_0 &= \ln x_0. \end{aligned}$$

This equation, however, is extremely simple: since the right-hand side does not contain Z it can be integrated directly to

$$Z_t = \ln x_0 + \left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t,$$

which means that X is given by

$$X_t = x_0 \cdot \exp\left\{\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t\right\}. \tag{5.12}$$

Strictly speaking there is a logical flaw in the reasoning above. In order for Z to be well defined we have to assume that there actually exists a solution X to eqn (5.10) and we also have to assume that the solution is positive. As for the existence, this is covered by Proposition 5.1, but the positivity seems to present a bigger problem. We may actually avoid both these problems by regarding the calculations above as purely heuristic. Instead we **define** the process X by the formula (5.12). Then it is an easy exercise to show that X thus defined actually satisfies the SDE (5.10)–(5.11). Thus we really have proved the first part of the following result, which will be used repeatedly in the sequel. The result about the expected value is an easy exercise, which is left to the reader.

Proposition 5.2 *The solution to the equation*

$$dX_t = \alpha X_t dt + \sigma X_t dW_t, \tag{5.13}$$

$$X_0 = x_0, \tag{5.14}$$

is given by

$$X(t) = x_0 \cdot \exp\left\{\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right\}. \tag{5.15}$$

The expected value is given by

$$E[X_t] = x_0 e^{\alpha t}. \tag{5.16}$$

5.3 The Linear SDE

In this section we will study the linear SDE, which in the scalar case has the form

$$\begin{cases} dX_t = aX_t dt + \sigma dW_t, \\ X_0 = x_0. \end{cases} \tag{5.17}$$

This equation turns up in various physics applications, and we will also meet it below in connection with interest rate theory.

In order to get some feeling for how to solve this equation we recall that the linear ODE

$$\frac{dx_t}{dt} = ax_t + u_t,$$

where u is a deterministic function of time, has the solution

$$x_t = e^{at}x_0 + \int_0^t e^{a(t-s)}u_s ds. \tag{5.18}$$

If we, for a moment, reason heuristically, then it is tempting to formally divide eqn (5.17) by dt. This would (formally) give us

$$\frac{dX_t}{dt} = aX_t + \sigma\frac{dW_t}{dt},$$

and, by analogy with the ODE above, one is led to conjecture the formal solution

$$X_t = e^{at}X_0 + \sigma\int_0^t e^{a(t-s)}\frac{dW_s}{ds}ds = e^{at}X_0 + \sigma\int_0^t e^{a(t-s)}dW_s.$$

Generally speaking, tricks like this will not work, since the solution of the ODE is based on ordinary calculus, whereas we have to use Itô calculus when dealing with SDEs. In this case, however, we have a linear structure, which means that the second-order term in the Itô formula does not come into play. Thus the solution of the linear SDE is indeed given by the heuristically derived formula above. We formulate the result for a slightly more general situation, where we allow X to be vector-valued.

Proposition 5.3 *Consider the n-dimensional linear SDE*

$$\begin{cases} dX_t = (AX_t + b_t)\,dt + \sigma_t dW_t, \\ X_0 = x_0 \end{cases} \tag{5.19}$$

where A is an $n \times n$ matrix, b is an R^n-valued deterministic function (in column vector form), σ is a deterministic function taking values in $M(n,d)$, and W a d-dimensional Wiener process. The solution of this equation is given by

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)}b_s ds + \int_0^t e^{A(t-s)}\sigma_s dW_s. \tag{5.20}$$

Here we have used the matrix exponential e^{At}, defined by

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!}t^k.$$

Proof Defining the process X by (5.20) and using the Itô formula, it is easily seen that X satisfies the SDE (5.19). See the exercises for some details. □

In the exercises you will find results about the moments of X_t as well as details about the matrix exponential.

5.4 The Infinitesimal Operator

Consider, as in Section 5.1, the n-dimensional SDE

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t. \tag{5.21}$$

Through the Itô formula, the process above is closely connected to a partial differential operator $\mathcal{A}$, defined below. The next two sections are devoted to investigating the connections between, on the one hand, the analytical properties of the operator $\mathcal{A}$, and on the other hand the probabilistic properties of the process X above.

Definition 5.4 *Given the SDE in (5.21), the partial differential operator $\mathcal{A}$, referred to as the* **infinitesimal operator** *of X, is defined, for any function $h(x)$ with $h \in C^2(R^n)$, by*

$$\mathcal{A}h(t,x) = \sum_{i=1}^{n} \mu_i(t,x)\frac{\partial h}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x)\frac{\partial^2 h}{\partial x_i \partial x_j}(x),$$

where as before

$$C(t,x) = \sigma(t,x)\sigma^{\star}(t,x).$$

This operator is also known as the **Dynkin operator**, the **Itô operator**, or the **Kolmogorov backward operator**. We note that, in terms of the infinitesimal generator, the Itô formula takes the form

$$df(t,X_t) = \left\{\frac{\partial f}{\partial t} + \mathcal{A}f\right\} dt + [\nabla_x f]\sigma dW_t$$

where the gradient ∇_x is defined for $h \in C^1(R^n)$ as

$$\nabla_x h = \left[\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}\right].$$

5.5 Partial Differential Equations

In this section we will explore the intimate connection which exists between stochastic differential equations and certain parabolic partial differential equations. Consider for example the following so-called **Cauchy problem**.

We are given three scalar functions $\mu(t,x)$, $\sigma(t,x)$, and $\Phi(x)$. Our task is to find a function F which satisfies the following **boundary value problem** on $[0,T] \times R$:

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) = 0, \tag{5.22}$$

$$F(T,x) = \Phi(x). \tag{5.23}$$

Now, instead of attacking this problem using purely analytical tools, we will produce a so-called **stochastic representation formula**, which gives the solution to (5.22)–(5.23) in terms of the solution to an SDE which is associated to (5.22)–(5.23) in a natural way. Thus we assume that there actually exists a solution F to (5.22)–(5.23).

Let us now fix a point in time t and a point in space x. Having fixed these we **define** the stochastic process X on the time interval $[t,T]$ as the solution to the SDE

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s, \tag{5.24}$$

$$X_t = x, \tag{5.25}$$

and the point is that the infinitesimal generator $\mathcal{A}$ for this process is given by

$$\mathcal{A} = \mu(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2},$$

which is exactly the operator apearing in the PDE above. Thus we may write the boundary value problem as

$$\frac{\partial F}{\partial t}(t,x) + \mathcal{A}F(t,x) = 0, \tag{5.26}$$

$$F(T,x) = \Phi(x). \tag{5.27}$$

Applying the Itô formula to the process $F(s,X(s))$ gives us

$$\begin{aligned} F(T,X_T) = F(t,X_t) & \\ &+ \int_t^T \left\{ \frac{\partial F}{\partial t}(s,X_s) + \mathcal{A}F(s,X_s) \right\} ds \\ &+ \int_t^T \sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)dW_s. \end{aligned} \tag{5.28}$$

Since, by assumption, F actually satisfies eqn (5.26), the time integral above will vanish. If furthermore the process $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$ is sufficiently integrable and we take expected values, the stochastic integral will also vanish. The initial value $X_t = x$ and the boundary condition $F(T,x) = \Phi(x)$ will eventually leave us with the formula

$$F(t,x) = E_{t,x}\left[\Phi\left(X_T\right)\right],$$

where we have indexed the expectation operator in order to emphasize that the expected value is to be taken given the initial value $X_t = x$. Thus we have proved the following result, which is known as the **Feynman–Kač stochastic representation formula**.

Proposition 5.5 (Feynman–Kač) *Assume that F is a solution to the boundary value problem*

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) = 0,$$
$$F(T,x) = \Phi(x).$$

Assume furthermore that the process

$$\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$$

is in $\mathcal{L}^2$ (see Definition 4.4), where X is defined below. Then F has the representation

$$F(t,x) = E_{t,x}\left[\Phi\left(X_T\right)\right], \tag{5.29}$$

where X satisfies the SDE

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s, \tag{5.30}$$
$$X_t = x. \tag{5.31}$$

Note that we need the integrability assumption $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s) \in \mathcal{L}^2$ in order to guarantee that the expected value of the stochastic integral in (5.28) equals zero. In fact the generic situation is that a boundary value problem of the type above—a so-called **parabolic** problem—will have infinitely many solutions, (see John 1982). It will, however, only have one “nice” solution, the others being rather “wild”, and the proposition above will only give us the “nice” solution.

We may also consider the closely related boundary value problem

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0, \tag{5.32}$$
$$F(T,x) = \Phi(x), \tag{5.33}$$

where r is a given real number. Equations of this type appear over and over again in the study of pricing problems for financial derivatives. Inspired by the ODE technique of integrating factors we are led to multiply the entire eqn (5.32) by the factor e^{rs}, and if we then consider the process $Z(s) = e^{-rs}F(s,X(s))$, where X as before is defined by (5.30)–(5.31), we obtain the following result.

Proposition 5.6 (Feynman–Kač) *Assume that F is a solution to the boundary value problem*

$$\frac{\partial F}{\partial t}(t,x)+\mu(t,x)\frac{\partial F}{\partial x}(t,x)+\frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x)-rF(t,x)=0, \tag{5.34}$$

$$F(T,x)=\Phi(x). \tag{5.35}$$

Assume furthermore that the process $e^{-rs}\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$ is in $\mathcal{L}^2$, where X is defined below. Then F has the representation

$$F(t,x)=e^{-r(T-t)}E_{t,x}\left[\Phi\left(X_T\right)\right], \tag{5.36}$$

where X satisfies the SDE

$$dX_s=\mu(s,X_s)ds+\sigma(s,X_s)dW_s, \tag{5.37}$$

$$X_t=x. \tag{5.38}$$

Example 5.7 Solve the PDE

$$\frac{\partial F}{\partial t}(t,x)+\frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(t,x)=0,$$

$$F(T,x)=x^2,$$

where σ is a constant.

Solution: From Proposition 5.5 we immediately have

$$F(t,x)=E_{t,x}\left[X_T^2\right],$$

where

$$dX_s=0\cdot ds+\sigma dW_s,$$

$$X_t=x.$$

This equation can easily be solved, and we have

$$X_T=x+\sigma\left[W_T-W_t\right],$$

so X_T has the distribution $N[x,\sigma\sqrt{T-t}]$. Thus we have the solution

$$F(t,x)=E\left[X_T^2\right]=Var[X_T]+\left\{E\left[X_T\right]\right\}^2$$

$$=\sigma^2(T-t)+x^2.$$

Up to now we have only treated the scalar case, but exactly the same arguments as above will give us the following result.

Proposition 5.8 *Take as given*

- *A (column-vector valued) function $\mu:R_+\times R^n\to R^n$.*
- *A function $C:R_+\times R^n\to M(n,n)$, which can be written in the form*

$$C(t,x) = \sigma(t,x)\sigma^{\star}(t,x),$$

for some function $\sigma : R_{+} \times R^{n} \to M(n,d)$.
- *A real valued function* $\Phi : R^{n} \to R$.
- *A real number* r.

Assume that $F : R_{+} \times R^{n} \to R$ *is a solution to the boundary value problem*

$$\frac{\partial F}{\partial t}(t,x) + \sum_{i=1}^{n} \mu_i(t,x)\frac{\partial F}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x)\frac{\partial^2 F}{\partial x_i \partial x_j}(t,x) - rF(t,x) = 0,$$
$$F(T,x) = \Phi(x).$$

Assume furthermore that the process

$$e^{-rs}\sum_{i=1}^{n} \sigma_i(s,X_s)\frac{\partial F}{\partial x_i}(s,X_s)$$

is in $\pounds^2$ *(see Definition 4.4), where* X *is defined below. Then* F *has the representation*

$$F(t,x) = e^{-r(T-t)}E_{t,x}\left[\Phi\left(X_T\right)\right], \tag{5.39}$$

where X *satisfies the SDE*

$$dX_s = \mu(s,X_s)dt + \sigma(s,X_s)dW_s, \tag{5.40}$$
$$X_t = x. \tag{5.41}$$

We end this section with a useful result. Given Lemma 4.10 the proof is easy and left to the reader.

Proposition 5.9 *Consider as given a vector process* X *with generator* $\mathcal{A}$, *and a function* $F(t,x)$. *Then, modulo some integrability conditions, the following hold:*

- *The process* $F(t,X_t)$ *is a martingale relative to the filtration* $\mathcal{F}^X$ *if and only if* F *satisfies the PDE*

$$\frac{\partial F}{\partial t} + \mathcal{A}F = 0.$$

- *The process* $F(t,X_t)$ *is a martingale relative to the filtration* $\mathcal{F}^X$ *if and only if, for every* (t,x) *and* $T \geq t$, *we have*

$$F(t,x) = E_{t,x}\left[F(T,X_T)\right].$$

5.6 The Kolmogorov Equations

We will now use the results of the previous section in order to derive some classical results concerning the transition probabilities for the solution to an SDE. The discussion has the nature of an overview, so we allow ourselves some latitude as to technical details.

Suppose that X is a solution to the equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{5.42}$$

with infinitesimal generator $\mathcal{A}$ given by

$$(\mathcal{A}f)(s,y) = \sum_{i=1}^{n} \mu_i(s,y)\frac{\partial f}{\partial y_i}(s,y) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(s,y)\frac{\partial^2 f}{\partial y_i \partial y_j}(s,y),$$

where as usual

$$C(t,x) = \sigma(t,x)\sigma^{\star}(t,x).$$

Now consider the boundary value problem

$$\left(\frac{\partial u}{\partial s} + \mathcal{A}u\right)(s,y) = 0, \qquad (s,y) \in (0,T) \times R^n,$$

$$u(T,y) = I_B(y), \quad y \in R^n,$$

where I_B is the indicator function of the set B. From Proposition 5.8 we immediately have

$$u(s,y) = E_{s,y}\left[I_B(X_T)\right] = P\left(X_T \in B \,|\, X_s = y\right),$$

where X is a solution of (5.42). This argument can also be turned around, and we have thus (more or less) proved the following result.

Proposition 5.10 (Kolmogorov backward equation) *Let X be a solution to eqn (5.42). Then the transition probabilities $P(s,y;t,B) = P\left(X_t \in B \,|\, X_s = y\right)$ are given as the solution to the equation*

$$\left(\frac{\partial P}{\partial s} + \mathcal{A}P\right)(s,y;t,B) = 0, \quad (s,y) \in (0,t) \times R^n, \tag{5.43}$$

$$P(t,y;t,B) = I_B(y). \tag{5.44}$$

Using basically the same reasoning one can also prove the following corresponding result for transition densities.

Proposition 5.11 (Kolmogorov backward equation) *Let X be a solution to eqn (5.42). Assume that the measure $P(s,y;t,dx)$ has a density $p(s,y;t,x)dx$. Then we have*

$$\left(\frac{\partial p}{\partial s} + \mathcal{A}p\right)(s,y;t,x) = 0, \quad (s,y) \in (0,t) \times R^n, \tag{5.45}$$

$$p(s,y;t,x) \to \delta_x, \text{ as } s \to t. \tag{5.46}$$

The reason that eqns (5.43) and (5.45) are called backward equations is that the differential operator is working on the "backward variables" (s,y). We will now derive a corresponding "forward" equation, where the action of the differential operator is on the "forward" variables (t,x). For simplicity we consider only the scalar case.

We assume that X has a transition density. Let us then fix two points in time s and T with $s < T$. Now consider an arbitrary "test function", i.e. an infinite differentiable function $h(t,x)$ with compact support in the set $(s,T) \times R$. From the Itô formula we have

$$h(T,X_T) = h(s,X_s) + \int_s^T \left(\frac{\partial h}{\partial t} + \mathcal{A}h\right)(t,X_t)dt + \int_s^T \frac{\partial h}{\partial x}(t,X_t)dW_t.$$

Applying the expectation operator $E_{s,y}[\cdot]$, and using the fact that, because of the compact support, $h(T,x) = h(s,x) = 0$, we obtain

$$\int_{-\infty}^{\infty}\int_s^T p(s,y;t,x)\left(\frac{\partial}{\partial t} + \mathcal{A}\right)h(t,x)dxdt = 0.$$

Partial integration with respect to t (for $\frac{\partial}{\partial t}$) and with respect to x (for $\mathcal{A}$) gives us

$$\int_{-\infty}^{\infty}\int_s^T h(t,x)\left(-\frac{\partial}{\partial t} + \mathcal{A}^\star\right)p(s,y;t,x)dxdt = 0,$$

where the adjoint operator $\mathcal{A}^\star$ is defined by

$$(\mathcal{A}^\star f)(t,x) = -\frac{\partial}{\partial x}[\mu(t,x)f(t,x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(t,x)f(t,x)\right].$$

Since this equation holds for all test functions we have shown the following result.

Proposition 5.12 (Kolmogorov forward equation) *Assume that the solution X of eqn (5.42) has a transition density $p(s,y;t,x)$. Then p will satisfy the* **Kolmogorov forward equation**

$$\frac{\partial}{\partial t}p(s,y;t,x) = \mathcal{A}^\star p(s,y;t,x), \quad (t,x) \in (0,T) \times R, \tag{5.47}$$

$$p(s,y;t,x) \to \delta_y, \text{ as } t \downarrow s. \tag{5.48}$$

This equation is also known as the **Fokker–Planck equation**. The multidimensional version is readily obtained as

$$\frac{\partial p}{\partial t}p(s,y;t,x) = \mathcal{A}^\star p(s,y;t,x),$$

where the adjoint operator $\mathcal{A}^\star$ is defined by

$$(\mathcal{A}^\star f)(t,x) = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}[\mu_i(t,x)f(t,x)] + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}[C_{ij}(t,x)f(t,x)].$$

Example 5.13 Let us consider a standard Wiener process with constant diffusion coefficient σ, i.e. the SDE

$$dX_t = \sigma dW_t.$$

The Fokker–Planck equation for this process is

$$\frac{\partial p}{\partial t}(s,y;t,x) = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2}(s,y;t,x),$$

and it is easily checked that the solution is given by the Gaussian density

$$p(s,y;t,x) = \frac{1}{\sigma\sqrt{2\pi(t-s)}} \exp\left[-\frac{1}{2}\frac{(x-y)^2}{\sigma^2(t-s)}\right].$$

Example 5.14 Consider the GBM process

$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

The Fokker–Planck equation for this process is

$$\frac{\partial p}{\partial t}(s,y;t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2 x^2 p(s,y;t,x)\right] - \frac{\partial}{\partial x}\left[\alpha x p(s,y;t,x)\right],$$

i.e.

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 p}{\partial x^2} + \left(2\sigma^2 - \alpha\right) x \frac{\partial p}{\partial x} + \left(\sigma^2 - \alpha\right) p.$$

A change of variables of the form $x = e^y$ reduces this equation to an equation with constant coefficients, which can be solved by Fourier methods. For us it is perhaps easier to get the transition density directly by solving the SDE above. See the exercises below.

5.7 Exercises

Exercise 5.1 Show that the scalar SDE

$$\begin{aligned} dX_t &= \alpha X_t dt + \sigma dW_t, \\ X_0 &= x_0, \end{aligned}$$

has the solution

$$X(t) = e^{\alpha t} \cdot x_0 + \sigma \int_0^t e^{\alpha(t-s)} dW_s, \tag{5.49}$$

by differentiating X as defined by eqn (5.49) and showing that X so defined actually satisfies the SDE.

Hint: Write eqn (5.49) as

$$X_t = Y_t + Z_t \cdot R_t,$$

where

$$\begin{aligned} Y_t &= e^{\alpha t} \cdot x_0, \\ Z_t &= e^{\alpha t} \cdot \sigma, \\ R_t &= \int_0^t e^{-\alpha s} dW_s, \end{aligned}$$

and first compute the differentials dZ, dY, and dR. Then use the multidimensional Itô formula on the function $f(y,z,r) = y + z \cdot r$.

Exercise 5.2 Let A be an $n \times n$ matrix, and define the matrix exponential e^A by the series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

This series can be shown to converge uniformly.

(a) Show, by taking derivatives under the summation sign, that

$$\frac{de^{At}}{dt} = Ae^{At}.$$

(b) Show that

$$e^0 = I,$$

where 0 denotes the zero matrix, and I denotes the identity matrix.

(c) Convince yourself that if A and B commute, i.e. $AB = BA$, then

$$e^{A+B} = e^A \cdot e^B = e^B \cdot e^A.$$

Hint: Write the series expansion in detail.

(d) Show that e^A is invertible for every A, and that in fact

$$\left[e^A\right]^{-1} = e^{-A}.$$

(e) Show that for any A, t, and s

$$e^{A(t+s)} = e^{At} \cdot e^{As}.$$

(f) Show that

$$\left(e^A\right)^{\star} = e^{A^{\star}}.$$

Exercise 5.3 Use the exercise above to complete the details of the proof of Proposition 5.3.

Exercise 5.4 Consider again the linear SDE (5.19). Show that the expected value function $m(t) = E[X(t)]$, and the covariance matrix $C(t) = \{Cov(X_i(t), X_j(t)\}_{i,j}$ are given by

$$m(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} b(s)ds,$$

$$C(t) = \int_0^t e^{A(t-s)} \sigma(s)\sigma^{\star}(s) e^{A^{\star}(t-s)} ds,$$

where * denotes transpose.

Hint: Use the explicit solution above, and the fact that

$$C(t) = E\left[X_t X_t^{\star}\right] - m(t)m^{\star}(t).$$

Geometric Brownian Motion (GBM) constitutes a class of processes which is closed under a number of nice operations. Here are some examples.

Exercise 5.5 Suppose that X satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

Now define Y by $Y_t = X_t^\beta$, where β is a real number. Then Y is also a GBM process. Compute dY and find out which SDE Y satisfies.

Exercise 5.6 Suppose that X satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and Y satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dV_t,$$

where V is a Wiener process which is independent of W. Define Z by $Z = \frac{X}{Y}$ and derive an SDE for Z by computing dZ and substituting Z for $\frac{X}{Y}$ in the right-hand side of dZ. If X is nominal income and Y describes inflation then Z describes real income.

Exercise 5.7 Suppose that X satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and Y satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dW_t.$$

Note that now both X and Y are driven by the same Wiener process W. Define Z by $Z = \frac{X}{Y}$ and derive an SDE for Z.

Exercise 5.8 Suppose that X satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and Y satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dV_t,$$

where V is a Wiener process which is independent of W. Define Z by $Z = X \cdot Y$ and derive an SDE for Z. If X describes the price process of, for example, IBM in US$ and Y is the currency rate SEK/US$ then Z describes the dynamics of the IBM stock expressed in SEK.

Exercise 5.9 Use a stochastic representation result in order to solve the following boundary value problem in the domain $[0, T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} &= 0, \\ F(T, x) &= \ln(x^2). \end{aligned}$$

Here μ and σ are assumed to be known constants.

Exercise 5.10 Consider the following boundary value problem in the domain $[0, T] \times R$.

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + k(t,x) = 0,$$
$$F(T,x) = \Phi(x).$$

Here μ, σ, k, and Φ are assumed to be known functions.

Prove that this problem has the stochastic representation formula

$$F(t,x) = E_{t,x}\left[\Phi(X_T)\right] + \int_t^T E_{t,x}\left[k(s,X_s)\right] ds,$$

where as usual X has the dynamics

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s,$$
$$X_t = x.$$

Hint: Define X as above, assume that F actually solves the PDE and consider the process $Z_s = F(s, X_s)$.

Exercise 5.11 Use the result of the previous exercise in order to solve

$$\frac{\partial F}{\partial t} + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2} + x = 0,$$
$$F(T,x) = \ln(x^2).$$

Exercise 5.12 Consider the following boundary value problem in the domain $[0, T] \times R$.

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + r(t,x)F = 0,$$
$$F(T,x) = \Phi(x).$$

Here $\mu(t,x)$, $\sigma(t,x)$, $r(t,x)$, and $\Phi(x)$ are assumed to be known functions. Prove that this problem has a stochastic representation formula of the form

$$F(t,x) = E_{t,x}\left[\Phi(X_T)e^{\int_t^T r(s,X_s)ds}\right],$$

by considering the process $Z_s = F(s, X_s) \times \exp\left[\int_t^s r(u,X_u)du\right]$ on the time interval $[t, T]$.

Exercise 5.13 Solve the boundary value problem

$$\frac{\partial F}{\partial t}(t,x,y) + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(t,x,y) + \frac{1}{2}\delta^2\frac{\partial^2 F}{\partial y^2}(t,x,y) = 0,$$
$$F(T,x,y) = xy.$$

Exercise 5.14 Go through the details in the derivation of the Kolmogorov forward equation.

Exercise 5.15 Consider the SDE

$$dX_t = \alpha dt + \sigma dW_t,$$

where α and σ are constants.

(a) Compute the transition density $p(s,y;t,x)$, by solving the SDE.
(b) Write down the Fokker–Planck equation for the transition density and check the equation is indeed satisfied by your answer in (a).

Exercise 5.16 Consider the standard GBM

$$dX_t = \alpha X_t dt + \sigma X_t dW_t$$

and use the representation

$$X_t = X_s \exp\left\{\left[\alpha - \frac{1}{2}\sigma^2\right](t-s) + \sigma\,[W_t - W_s]\right\}$$

in order to derive the transition density $p(s,y;t,x)$ of GBM. Check that this density satisfies the Fokker–Planck equation in Example 5.14.

5.8 Notes

All the results in this chapter are standard and can be found in Karatzas and Shreve (2008), Revuz and Yor (1991) and Øksendal (1998). For an encyclopedic treatment of the probabilistic approach to parabolic PDEs see Doob (1984).

PART III

ARBITRAGE THEORY

6

PORTFOLIO DYNAMICS

6.1 Introduction

Let us consider a financial market consisting of different assets such as stocks, bonds with different maturities, or various kinds of financial derivatives. In this chapter we will take the price dynamics of the various assets as given, and the main objective is that of deriving the dynamics of (the value of) a so-called **self-financing** portfolio. In continuous time this turns out to be a fairly delicate task, so we start by studying a model in discrete time. We will then let the length of the time step tend to zero, thus obtaining the continuous time analogs.

6.2 Self-financing Portfolios in Discrete Time

We consider a financial market living in discrete time on a filtered probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$, so we are allowed trade at discrete points in time $t = 0, 1, 2, \ldots$.

6.2.1 *Basic Definitions*

On the market we can trade in N different assets with (adapted) price processes $S^1, \ldots, S^N$. We will use the following notation.

Definition 6.1

$$
\begin{aligned}
S_n^i &= \textit{the price of one unit of asset No. } i \textit{ at time } n, \\
h_n^i &= \textit{ number of units of asset No. } i \textit{ bought at time } n, \\
d_n^i &= \textit{dividends from asset No. } i \textit{ at time } n, \\
h_n &= \textit{the portfolio } \left[h_n^1, \ldots, h_n^N\right], \\
c_n &= \textit{consumption at time } n, \\
V_n &= \textit{the value of the portfolio } h_{n-1} \textit{ at time } n.
\end{aligned}
$$

The interpretation of the dividend process d is that if you are holding one unit of asset No. i during the interval $[n-1, n]$, then you obtain the amount d_n at time n.

Remark 6.2.1 *The processes S and d are exogenously given and adapted. The processes c and h are decision processes, and we* **require** *that they shall be adapted, i.e. the portfolio an consumption choices h_n and c_n at time n are only allowed to depend on the information $\mathcal{F}_n$ that is available at time n. In all*

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

honesty we should also require some simple integrability properties of h and c, but we sweep these under the carpet and/or leave it to the reader.

In all discrete time models you have to be very careful about the timing on the micro level, i.e. you have to specify very carefully the exact order in which you make decisions at a given time n. For our model it works as follows:

1. At time n we buy the portfolio h_n at the price S_n, and the portfolio value V_n is defined as the value of h_n at time n. We then keep the portfolio until time $n+1$.
2. We enter time $n+1$ carrying our old portfolio h_n with us.
3. At time $n+1$ we get our share of the dividend d_{n+1} and decide on the amount c_{n+1} to be consumed.
4. After consuming c_{n+1}, we re-balance the old portfolio h_n into the new portfolio h_{n+1}.
5. We keep the portfolio h_{n+1} until time $n+2$, etc.

We start our analysis by giving a formula for V_n. Since, by definition, V_n is the value of the portfolio h_n at the price vector S_n, we see that $h_n^i S_n^i$ is the dollar value, at time n, of our holdings in asset No. i. Summing over all i we obtain the basic formula

$$V_n = \sum_{i=1}^{N} h_n^i S_n^i. \tag{6.1}$$

In vector formulation where $h_n S_n$ denotes scalar product we can write this as

$$V_n = h_n S_n. \tag{6.2}$$

Remark 6.2.2 *Note that, with our definitions, the portfolio value V_n is the value of the "outgoing portfolio" at time n, i.e. the portfolio we have created* **after** *having consumed c_n and received dividends d_n. You could of course instead define V_n as the value of the "ingoing portfolio" at time n, i.e. $V_n = h_{n-1}S_n$. If you choose to make this second definition, nothing will of course change in real terms, but the formal appearance of some formulas below will be different.*

6.2.2 *Self-financing Portfolios*

Our main goal is to study the dynamics of the value process V for a **self-financing** portfolio, so we must now define this concept.

Definition 6.2 *A* **self-financing portfolio supporting the consumption stream c** *is a portfolio with no exogenous infusion or withdrawal of money (apart from dividends and consumption). In other words, the purchase of a new portfolio, as well as all consumption, must be financed solely by the dividends obtained and/or by selling assets already in the portfolio.*

We now need to formulate the self-financing condition in mathematical terms, and this is done as follows:

1. At time $n+1$ the value of our old portfolio is $h_n S_{n+1}$.
2. At time $n+1$ we get the dividend d^i_{n+1} per unit of asset No. i. The total amount obtained from dividends is thus given by $h_n d_{n+1}$. We also choose to consume c_{n+1}.
3. We then buy the new portfolio h_{n+1} at the price S_{n+1}, so the cost of this new portfolio is $h_{n+1}S_{n+1}$.
4. The self-financing condition is the condition that, at time $n+1$, the cost of the new portfolio plus the cost of consumption equals the value of the old portfolio plus dividends.
5. The self-financing **budget constraint** is thus given by the formula

$$h_{n+1}S_{n+1} + c_{n+1} = h_n S_{n+1} + h_n d_{n+1}. \tag{6.3}$$

In order to obtain the V-dynamics we introduce the following notation.

Definition 6.3 *For any sequence $\{x_n\}_{n=0}^{\infty}$ of real numbers, we define the operator Δ by*

$$\Delta x_n = x_{n+1} - x_n. \tag{6.4}$$

Remark 6.2.3 *We note the following*

- *The expression Δx_n is a discrete time version of our informal interpretation of a stochastic differential dX_t as*

$$dX_t = X_{t+dt} - X_t.$$

- *We can obviously extend the definition to the case where x and y are vector sequences.*
- *The expression $x_n \Delta y_n$ corresponds to a stochastic differential of the form $X_t dY_t$, but an expression of the form $x_{n+1}\Delta y_n$ does not.*

We also need a small result on discrete time differentiation by parts.

Lemma 6.4 *For any pair of sequences of real numbers $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ we have the relations*

$$\Delta(xy)_n = x_n \Delta y_n + y_{n+1}\Delta x_n, \tag{6.5}$$

$$\Delta(xy)_n = y_n \Delta x_n + x_{n+1}\Delta y_n, \tag{6.6}$$

$$\Delta(xy)_n = x_n \Delta y_n + y_n \Delta x_n + \Delta x_n \Delta y_n. \tag{6.7}$$

These results are also valid if $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are sequences of vectors in R^N, in which case the products above are interpreted as scalar products.

Proof The trivial proof is left to the reader. □

We now want to compute ΔV_n for a self-financing portfolio, and we then recall that by definition

$$V_n = h_n S_n,$$

so from (6.5) we obtain

$$\Delta V_n = h_n \Delta S_n + S_{n+1}\Delta h_n. \tag{6.8}$$

We now note that we can write the budget constraint (6.3) as

$$S_{n+1}\Delta h_n = h_n d_{n+1} - c_{n+1}. \tag{6.9}$$

Substituting this into the V dynamics (6.8) gives us the following result.

Proposition 6.5 *The dynamics of a self-financing portfolio supporting the consumption stream c are given by*

$$\Delta V_n = h_n \Delta S_n + h_n d_{n+1} - c_{n+1}. \tag{6.10}$$

Substituting the self-financing budget constraint (6.3) into (6.4) gives us the following result.

Proposition 6.6 *For a self-financing portfolio supporting the consumption stream c, the dynamics are*

$$\Delta V_n = h_n \Delta S_n + h_n d_{n+1} - c_{n+1} \tag{6.11}$$

or, in more detail

$$\Delta V_n = \sum_{i=1}^{N} h_n^i \left\{ \Delta S_n^i + d_{n+1}^i \right\} - c_{n+1}. \tag{6.12}$$

6.2.3 *The Cumulative Dividend Process*

Up to now we have specified the exogenously given dividend process d, where d_{n+1}^i is the amount you get at time $n+1$ if you are holding one unit of asset No. i from time n to time $n+1$. This way of describing a dividend process is very natural and it works well in discrete time. If you are working in continuous time and you have discrete dividends, i.e. that you get a lump sum of money at discrete points in time, then this formalism still works well. If, however, you would like to describe a continuous flow of dividends, then you will get zero dollars at any fixed point in time, so the d-formalism does not work.

We will now describe a different, but equivalent, formalism for the dividend process, and the new formalism **can** in fact easily be extended to continuous time.

Definition 6.7 *We define the* **cumulative dividend process** D^i *by*

$$D_n^i = \sum_{k=1}^{n} d_k^i. \tag{6.13}$$

We see that D_n^i is the raw sum of all dividends paid out over the time interval $[0, n]$ from one unit of asset No. i. Note that there is no discounting or capitalization involved. We also see that we can recover the d process from D by the formula

$$d_{n+1}^i = \Delta D_n^i. \quad i = 1, \ldots, N. \tag{6.14}$$

The point of all this is that the concept of a cumulative dividend process can easily be extended to continuous time, and we now reformulate Proposition 6.5 in the new terms as follows.

Proposition 6.8 *For a self-financing portfolio supporting the consumption stream c, the dynamics are*

$$\Delta V_n = h_n \Delta S_n + h_n d\Delta D_n - c_{n+1} \tag{6.15}$$

or, in more detail

$$\Delta V_n = \sum_{i=1}^{N} h_n^i \left\{ \Delta S_n^i + \Delta D_n^i \right\} - c_{n+1}, \tag{6.16}$$

where $D = (D^1, \ldots D^N)$ *is the cumulative vector dividend process.*

We now note that the individual term $h_n \Delta S_n$ has the form

$$h_n \left[S_{n+1} - S_n \right],$$

and it is important to notice that the term h_n is evaluated at time n and multiplied by the term $S_{n+1} - S_n$, which is a **forward increment** over the time interval $[n, n+1]$. The moral of all this is that the term $h_n \Delta S_n$ is a discrete time version of the Itô stochastic differential $h_t dS_t = h_t \left[S_{t+dt} - S_t \right]$. Note that a term of the form $h_{n+1} \Delta S_n$ would **not** correspond to an Itô differential. We can thus easily extend these formulas to continuous time.

6.3 Self-financing Portfolios in Continuous Time

We now move to continuous time and consider a financial market with N assets.

Definition 6.9

$$\begin{aligned}
S_t^i &= \textit{the price of one unit of asset No. } i \textit{ at time } t, \\
h_t^i &= \textit{number of units of asset No. } i \textit{ held at time } t, \\
h_t &= \textit{the portfolio } \left[h_t^1, \ldots, h_t^N \right], \\
D_t^i &= \textit{the cumulative dividend process for asset No. } i, \\
c_t &= \textit{consumption rate at time } n, \\
V_t &= \textit{the value of the portfolio } h_t \textit{ at time } tn.
\end{aligned}$$

Remark 6.3.1 *There are two important differences from the discrete time definitions, and these concern consumption and dividends:*

1. *In discrete time,* c_n *denotes consumption at time* n*, so it is measured in dollars. The consumption* **rate** c_t *above, on the other hand, is measured in dollars per unit time, so in continuous time we have a* **continuous flow** *of consumption, where the dollar value of consumption over an*

infinitesimal interval $[t, t+dt]$ *is given by* $c_t dt$. *It is possible to include "impulse consumption" where you get a lump sum of money at a specific point in time, but this would require a more general stochastic integration theory, so we do not include this.*

2. *We assume that the cumulative dividend process* D^i *has a stochastic Itô differential, so in particular it has* **continuous trajectories**. *We thus exclude discrete dividends. To include discrete dividends would require a more general stochastic integration theory involving jump processes, and we would also have to change some of the definitions above.*

We now (ruthlessly) go to the formal continuous time limit with the discrete time theory. In more detail this means that we make the identifications

$$n \sim t,$$

$$n+1 \sim t+dt,$$

$$\Delta V_n \sim dV_t,$$

$$\Delta S_n \sim dS_t,$$

$$\Delta D_n \sim dD_t.$$

Plugging all this into the discrete time formulas leads to the following formal definition.

Definition 6.10 *Consider an* $\mathbf{F}$*-adapted* N*-dimensional price process* S.

1. *A* **portfolio strategy** *is any* $\mathbf{F}$*-adapted* N*-dimensional process* h.
2. *The* **value process** V^h *corresponding to the portfolio* h *is given by*

$$V_t^h = \sum_{i=1}^{N} h_t^i S_t^i. \tag{6.17}$$

3. *A* **consumption process** *is any* $\mathcal{F}_t$*-adapted one-dimensional process* c.
4. *A portfolio–consumption pair* (h, c) *is called* **self-financing** *if the value process* V^h *defined by (6.17) satisfies the condition*

$$dV_t^h = \sum_{i=1}^{N} h_t^i \left\{ dS_t^i + dD_t^i \right\} - c_t dt, \tag{6.18}$$

i.e. if

$$dV_t^h = h_t dS_t + h_t dD_t - c_t dt.$$

5. *The* **gain process** G *is defined by*

$$G_t = S_t + D_t, \tag{6.19}$$

so we can write the self-financing condition as

$$dV_t = h_t dG_t - c_t dt. \tag{6.20}$$

6. *The portfolio h is said to be* **Markovian** *if it is of the form*

$$h_t = h(t, S_t),$$

for some function $h : R_+ \times R^N \to R^N$.

At this point there are some natural questions:

1. In which sense (L^2, P-a.s., etc.) is the limiting procedure of from discrete to continuous time supposed to be interpreted?
2. Equation (6.18) is supposed to be describing the dynamics of a self-financing portfolio in continuous time, but what is "continuous time trading" supposed to mean "in real life"?

The answer to these questions is simply that there is no such thing as "continuous trading" in "real life", and that the continuous time arguments above has only been of a motivating nature The continuous time theory developed above is an approximation of a market where one can trade "very often". This is why equations (6.17) and (6.18) are definitions instead of being propositions.

Remark 6.3.2 *The careful reader may have noticed that formally we should have the term $c_{t+dt}dt$ instead of the term $c_t dt$ above. However, since we are integrating against Lebesgue measure dt, this does not matter.*

6.4 Portfolio Weights

For computational purposes it is often convenient to describe a portfolio in relative terms instead of in absolute terms as above. In other words, instead of specifying the absolute number of shares held of a certain stock h_t^i , we specify the relative proportion w_t^i of the total portfolio value which is invested in the stock.

Definition 6.11 *For a given portfolio h the corresponding* **relative portfolio** *or* **portfolio weights** *w are defined by*

$$w_t^i = \frac{h_t^i S_t^i}{V_t^h}, \quad i = 1, \ldots, N, \tag{6.21}$$

so, in particular, we have

$$\sum_{i=1}^{N} w_t^i = 1.$$

The self-financing condition can now easily be given in terms of the relative portfolio.

Lemma 6.12 *A portfolio–consumption pair (h, c) is self-financing if and only if*

$$dV_t^h = V_t^h \cdot \sum_{i=1}^{N} w_t^i \frac{dS_t^i + dD_t^i}{S_t^i} - c_t dt. \tag{6.22}$$

In the future we will need the following slightly technical result which roughly says that if a process **looks** as if it is the value process of a self-financing portfolio, then it actually **is** such a value process. For simplicity we only consider the non-dividend case.

Lemma 6.13 *Consider the case of no dividends. Let c be a consumption process, and assume that there exist a scalar process Z and a vector process $q = \left(q^1, \ldots, q^N\right)$ such that*

$$dZ_t = Z_t \sum_{i=1}^{N} q_t^i \frac{dS_t^i}{S_t^i} - c_t dt, \tag{6.23}$$

$$\sum_{i=1}^{N} q_t^i = 1. \tag{6.24}$$

Now define a portfolio h by

$$h_t^i = \frac{q_t^i Z_t}{S_t^i}. \tag{6.25}$$

Then the value process V^h is given by $V^h = Z$, the pair (h, c) is self-financing, and the corresponding relative portfolio w is given by $w = q$.

Proof By definition the value process V^h is given by $V_t^h = h_t S_t$, so eqns (6.24) and (6.25) give us

$$V_t^h = \sum_{i=1}^{N} h_t^i S_t^i = \sum_{i=1}^{N} q_t^i Z_t = Z_t \sum_{i=1}^{N} q_t^i = Z_t. \tag{6.26}$$

Inserting (6.26) into (6.25) we see that the relative portfolio w corresponding to h is given by $w = q$. Inserting (6.26) and (6.25) into (6.23) we obtain

$$dV_t^h = \sum_{i=1}^{N} h_t^i dS_t^i - c_t dt,$$

which shows that (h, c) is self-financing. □

7

ARBITRAGE PRICING

7.1 Introduction

In this chapter we will study a special case of the general model set out in Chapter 6. We will basically follow the arguments of Merton (1973), which only require the mathematical machinery presented in the previous chapters. For the full story see Chapter 11.

Let us therefore consider a financial market consisting of only two assets: a risk free asset with price process B, and a stock with price process S. What, then, is a risk free asset?

Definition 7.1 *The price process B is the price of a* **risk free** *asset if it has the dynamics*

$$dB_t = r_t B_t dt, \tag{7.1}$$

where r is any adapted process.

The defining property of a risk free asset is thus that it has no driving dW-term. We see that we also can write the B-dynamics as

$$\frac{dB_t}{dt} = r_t B_t,$$

so the B-process is given by the expression

$$B_t = B_0 e^{\int_0^t r_s ds}.$$

and as a notational convention we put

$$B_0 = 1.$$

A natural interpretation of a risk free asset is that it corresponds to a bank with the (possibly stochastic) **short interest rate** r. An important special case appears when r is a deterministic constant, in which case we can interpret B as the price of a bond.

We assume that the stock price S is given by

$$dS_t = S_t \mu(t, S_t)\, dt + S_t \sigma(t, S_t)\, dW_t, \tag{7.2}$$

where W is a Wiener process and μ and σ are given deterministic functions. The reason for the notation W, instead of the simpler W, will become clear below. The function σ is known as the **volatility** of S, while μ is the **local mean rate of return** of S.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

Remark 7.1.1 *Note the difference between the risky stock price S, as modeled above, and the riskless asset B. The rate of return of B is formally given by*

$$\frac{dB_t}{B_t} = r_t dt.$$

This object is **locally deterministic** *in the sense that, at time t, we have complete knowledge of the return by simply observing the prevailing short rate r_t. Compare this to the return on the stock S. This is formally given by*

$$\frac{dS_t}{S_t} = \mu(t, S_t)\, dt + \sigma(t, S_t)\, dW_t,$$

and this is **not** *observable at time t. It consists of the terms $\mu(t, S_t)\, dt$ and $\sigma(t, S_t)$, which both are observable at time t, plus the Wiener noise term dW_t, which is random. Thus, as opposed to the risk free asset, the stock has a* **stochastic return**, *even on the infinitesimal scale.*

The most important special case of the above model occurs when r, μ, and σ are deterministic constants. This is the famous **Black–Scholes model**.

Definition 7.2 *The* **Black–Scholes model** *consists of two assets with dynamics given by*

$$dB_t = rB_t dt, \tag{7.3}$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{7.4}$$

where r, μ, and σ are deterministic constants.

7.2 More on the Bank Account

In this section we will discuss and interpret the bank account concept in some detail. We will mainly do this in discrete time, and at the end we go to the formal continuous time limit. First we need the concept of a zero coupon bond.

Definition 7.3 *A* **zero coupon bond** *with maturity T (henceforth "T-bond") is an asset which pays the holder the face value 1 dollar at time T. The price at time n of a T-bond is denoted by $p(n, T)$.*

We assume that there exists a liquid market for bonds of all maturities, and we have the obvious no arbitrage condition

$$p(n, n) = 1, \quad n = 0, 1, \ldots \tag{7.5}$$

We can now define the discrete time short rate.

Definition 7.4 *The (possibly stochastic) discrete* **short rate** *r_n, for the period $[n, n+1]$, is defined as*

$$p(n, n+1) = \frac{1}{1 + r_n}. \tag{7.6}$$

The interpretation is that the factor $(1+r_n)^{-1}$ acts as a discount factor to discount the face value 1 of the bond maturing at $n+1$ back to time n. It is important to notice that r_n is known already at time n.

We can also view the existing bond market as an investment opportunity. Suppose that we have x dollars to invest at time n. For these dollars we can afford to buy exactly $x/p(n,n+1)$ bonds maturing at $n+1$. We do this and at time $n+1$ we get 1 dollar for each bond, giving us a total of $x/p(n,n+1)$ dollars. Summing up we see that an investment of x dollars at time n has provided us with $x/p(n,n+1)$ dollars a time $n+1$, and the point is that this amount was known already at time n. In other words we have made a **risk free** investment, and if we denote the value of this particular portfolio with V, that we have

$$V_{n+1} = \frac{1}{p(n,n+1)} V_n$$

and using (7.6) we have

$$V_{n+1} = (1+r_n)V_n.$$

To sum up: The short rate r_n is a **risk free** interest rate from n to $n+1$.

We can now extend this idea further by considering the following self-financing portfolio:

- We start at time 0 with 1 dollar.
- At any time n we invest the entire portfolio value in bonds maturing at time $n+1$.
- At time $n+1$ the bonds mature, and we invest everything in bonds maturing at $n+2$ etc. etc.
- The value process B_n of this portfolio is the **bank account**.

This type of portfolio is known as a **roll over** portfolio. Typically we use the convention $B_0 = 1$, and we now go on to derive the dynamics of the bank account B. By h_n we denote the number of bonds, bought at time n, and maturing at $n+1$. We then have (why?)

$$h_n = \frac{B_n}{p(n,n+1)}.$$

At $n+1$ the bonds mature giving us the amount

$$B_{n+1} = h_n \cdot 1.$$

We thus have

$$B_{n+1} = \frac{B_n}{p(n,n+1)} = B_n(1+r_n),$$

which can be written as

$$B_{n+1} - B_n = B_n r_n,$$

and we have our result.

Proposition 7.5 *The dynamics of the bank account are given by*

$$\Delta B_n = r_n B_n. \tag{7.7}$$

We thus see that the bank account is **locally risk free**, in the sense that, even if the short rate is a random process, the return r_n of the bank account over the interval $[n, n+1]$ is risk free (i.e. deterministic given the information available at time n). Note, however, that the return of B over a **longer** interval such as $[n, n+2]$ is typically stochastic.

We now go to the formal continuous time limit in (7.7), but in continuous time we do not have a one-period short rate r_n, since there is no "next point in time". Instead we have an annualized short rate, meaning that r_n must be replaced by $r_n \Delta n$ where Δn is the real time step size between n and $n+1$. In the continuous limit we thus replace r_n with $r_t dt$ to obtain the B dynamics

$$dB_t = r_t B_t dt. \tag{7.8}$$

We thus have an informal interpretation of the continuous time bank account as a portfolio where, at every point in time, we are rolling over just-maturing bonds. This informal interpretation can in fact be given a precise mathematical formulation, but the technical cost is high.

7.3 Contingent Claims and Arbitrage

We take as given the model of a financial market given by eqns (7.1)–(7.2), and we now approach the main problem to be studied in this book, namely the pricing of financial derivatives. Later we will give a mathematical definition, but let us at once present the single most important derivative—the European call option.

Definition 7.6 *A* **European call option** *with* **exercise price** *(or strike price) K and* **time of maturity** *(exercise date) T on the* **underlying asset** *S is a contract defined by the following clauses*:

- *The holder of the option has, at time T, the right to buy one share of the underlying stock at the price K dollars from the underwriter of the option.*
- *The holder of the option is in no way obliged to buy the underlying stock.*
- *The right to buy the underlying stock at the price K can only be exercised at the precise time T.*

Note that the exercise price K and the time of maturity T are determined at the time when the option is written, which for us typically will be at $t = 0$. A **European put option** is an option which in the same way gives the holder the right to **sell** a share of the underlying asset at a predetermined strike price. For an **American call option** the right to buy a share of the underlying asset can be exercised at any time before the given time of maturity. The common factor of all these contracts is that they all are completely defined in terms of the underlying asset S, which makes it natural to call them **derivative**

instruments or **contingent claims**. We will now give the formal definition of a contingent claim.

Definition 7.7 *Consider a financial market with vector price process S. A* **contingent claim** *with* **date of maturity** *(exercise date) T, also called a T-claim, is any random variable $\mathcal{X} \in \mathcal{F}_T^S$. A contingent claim $\mathcal{X}$ is called a* **simple** *claim if it is of the form $\mathcal{X} = \Phi(S_t)$. The function Φ is called the* **contract function***.*

The interpretation of this definition is that a contingent claim is a contract that stipulates that the holder of the contract will obtain $\mathcal{X}$ dollars (which can be positive or negative) at the time of maturity T. The requirement that $\mathcal{X} \in \mathcal{F}_T^S$ simply means that, at time T, it will actually be possible to determine the amount of money to be paid out. We see that the European call is a simple contingent claim, for which the contract function is given by

$$\Phi(x) = \max[x - K, 0].$$

The graphs of the contract functions for European calls and puts can be seen in Figs 7.1–7.2. It is obvious that a contingent claim, e.g. like a European call option, is a financial asset which will fetch a price on the market. Exactly how much the option is worth on the market will of course depend on the time t and on the price S_t of the underlying stock. Our main problem is to determine a "fair" (in some sense) price for the claim, and we will use the standard notation

$$\Pi_t[\mathcal{X}], \tag{7.9}$$

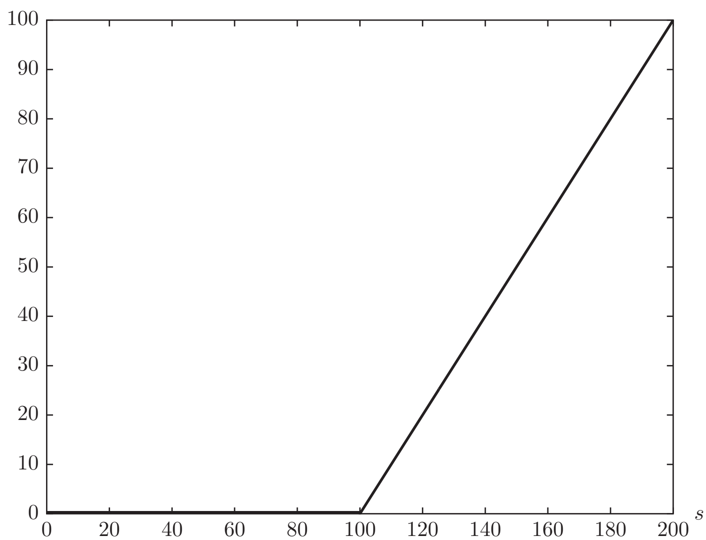

FIG. 7.1. Contract function. European call, $K = 100$

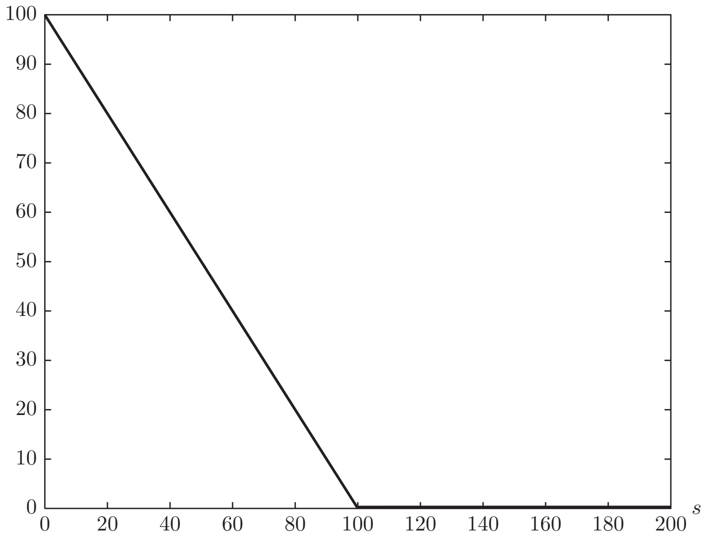

FIG. 7.2. Contract function. European put, $K = 100$

for the price process of the claim $\mathcal{X}$, where we sometimes suppress the $\mathcal{X}$. In the case of a simple claim we will sometimes write $\Pi_t[\Phi]$.

If we start at time T the situation is simple. Let us first look at the particular case of a European call.

1. If $S_T \geq K$ we can make a certain profit by exercising the option in order to buy one share of the underlying stock. This will cost us K dollars. Then we immediately sell the asset on the stock exchange at the price S_T, thus giving us a net profit of $S_T - K$ dollars.
2. If $S_T < K$ the option has no value whatsoever.

Thus we see that the only reasonable price $\Pi_T[\mathcal{X}]$ for the option at time T is given by

$$\Pi_T[\mathcal{X}] = \max[S_T - K, 0]. \tag{7.10}$$

In exactly the same way we see that for a more general contingent claim $\mathcal{X}$ we have the relation

$$\Pi_T[\mathcal{X}] = \mathcal{X}, \tag{7.11}$$

and in the particular case of a simple claim $\mathcal{X} = \Phi(S_T)$

$$\Pi_T[\mathcal{X}] = \Phi(S_T). \tag{7.12}$$

For any time $t < T$ it is, however, far from obvious what the correct price is for a claim $\mathcal{X}$. On the contrary it seems to be obvious that there is no such thing as a "correct" or "fair" price. The price of an option, like the price of any other asset, is of course determined on the (option) market, and should therefore be an extremely complex aggregate of, for example, the various attitudes to risk on the market and expectations about the future stock prices. It is therefore an extremely surprising fact that, given some fairly mild assumptions, there is a formula (the Black–Scholes formula) which gives the unique price of the option. The main assumption we will make is that the market is **efficient** in the sense that it is **free of arbitrage possibilities**. We now define this new and central concept.

Definition 7.8 *An* **arbitrage** *possibility on a financial market is a self-financed portfolio h such that*

$$V^h(0) = 0, \tag{7.13}$$

$$P\left(V_T^h \geq 0\right) = 1, \tag{7.14}$$

$$P\left(V_T^h > 0\right) > 0. \tag{7.15}$$

We say that the market is **arbitrage free** *if there are no arbitrage possibilities.*

An arbitrage possibility is thus essentially equivalent to the possibility of making a positive amount of money out of nothing without taking any risk. It is thus essentially a riskless money-making machine or, if you will, a free lunch on the financial market. We interpet an arbitrage possibility as a serious case of mispricing in the market, and our main assumption is that the market is efficient in the sense that no arbitrage is possible.

Assumption 7.3.1 *We assume that the price process $\Pi_t[\mathcal{X}]$ is such that there are no arbitrage possibilities on the market consisting of $(B_t,\ S_t,\ \Pi_t[\mathcal{X}])$.*

A natural question now is how we can identify an arbitrage possibility. The general answer to this question requires quite a lot of fairly heavy probabilistic machinery that the more advanced reader will find in Chapter 11. Happily enough there is a partial result which is sufficient for our present purposes.

Proposition 7.9 *Suppose that there exists a self-financing portfolio h, such that the value process V^h has the dynamics*

$$dV_t^h = k_t V_t^h dt, \tag{7.16}$$

where k is an adapted process. Then it must hold that $k_t = r_t$ for all t, or there exists an arbitrage possibility.

Proof We sketch the argument, and assume for simplicity that k and r are constant and that $k > r$. Then we can borrow money from the bank at the rate r. This money is immediately invested in the portfolio strategy h where it will grow at the rate k with $k > r$. Thus the net investment at $t = 0$ is zero, whereas our wealth at any time $t > 0$ will be positive. In other words we have an arbitrage. If on the other hand $r > k$, we sell the portfolio h short and invest this money in the bank, and again there is an arbitrage. The cases with non-constant and non-deterministic r and k are handled in the same way. □

The main point of the above is that if a portfolio has a value process whose dynamics contain no driving Wiener process, i.e. a **locally risk free portfolio**, then the rate of return of that portfolio must equal the short rate of interest. To put it in another way, the existence of such a portfolio h is for practical purposes equivalent to the existence of a bank with k as its short rate. We can then paraphrase the lemma above by saying that on an arbitrage free market there can only be one risk free interest rate.

We now return to the question of how the price process $\Pi_t[\mathcal{X}]$ for a contingent claim $\mathcal{X}$ can behave, and the main idea is the following:

- Since the claim is **defined** entirely in terms of the underlying asset(s), we ought to be able to **price** it in terms of the price of the underlying asset(s) if arbitrage possibilities are to be avoided.
- Thus we are looking for a way to price the derivative in a way which is **consistent** with the price process of the underlying asset.

To take a simple example, it is quite obvious that for a European call we must have the relation $\Pi_t \leq S_t$ in an arbitrage free market, because no one in their right mind will buy an option to buy a share at a later date at price K if the share itself can be bought cheaper than the option. For a more formal argument, suppose that at some time t we actually have the relation $\Pi_t > S_t$. Then we simply sell one option. A part of that money can be used for buying the underlying stock and the rest is invested in the bank (i.e. we buy the riskless asset). Then we sit down and do nothing until time T. In this way we have created a self-financed portfolio with zero net investment at time t. At time T we will owe $\max[S_T - K, 0]$ to the holder of the option, but this money can be paid by selling the stock. Our net wealth at time T will thus be $S_T - \max[S_T - K, 0]$, which is positive, plus the money invested in the bank. Thus we have an arbitrage.

It is thus clear that the requirement of an arbitrage free market will impose some restrictions on the behavior of the price process $\Pi_t[\mathcal{X}]$. This in itself is not terribly surprising. What is surprising is the fact that in the market specified by eqns (7.1)–(7.2) these restrictions are so strong as to completely specify, for any given claim $\mathcal{X}$, a **unique** price process $\Pi_t[\mathcal{X}]$ which is consistent with absence of arbitrage. For the case of **simple** contingent claims the formal argument will be given in Section 7.4, but we will now give the general idea.

To start with, it seems reasonable to assume that the price $\Pi_t[\mathcal{X}]$ at time t in some way is determined by expectations about the future stock price S_t. Since S is a Markov process such expectations are in their turn based on the present value of the price process (rather than on the entire trajectory on $[0, t]$). We thus make the following assumption.

Assumption 7.3.2 *We assume that:*

1. *The derivative instrument in question can be bought and sold on a market.*
2. *The market is free of arbitrage.*

3. *The price process for the derivative asset is of the form*

$$\Pi_t[\mathcal{X}] = F(t, S_t), \tag{7.17}$$

where F is some smooth function.

Our task is to determine what F might look like if the market consisting of S_t, B_t, and $\Pi_t[\mathcal{X}]$ is arbitrage free. Schematically we will proceed in the following manner:

1. Consider μ, σ, Φ, F, and r as exogenously given.
2. Use the general results from Chapter 6 to describe the dynamics of the value of a hypothetical self-financed portfolio based on the derivative instrument and the underlying stock (nothing will actually be invested in or borrowed from the bank).
3. It turns out that, by a clever choice, we can form a self-financing portfolio whose value process has a stochastic differential without any driving Wiener process. It will thus be of the form (7.16) above.
4. Since we have assumed absence of arbitrage we must have $k = r$.
5. The condition $k = r$ will in fact have the form of a partial differential equation with F as the unknown function. In order for the market to be efficient F must thus solve this PDE.
6. The equation has a unique solution, thus giving us the unique pricing formula for the derivative, which is consistent with absence of arbitrage.

7.4 The Black–Scholes Equation

In this section we will carry through the schematic argument given in the previous section. We assume that the a priori given market consists of two assets with dynamics given by

$$dB_t = rB_t dt, \tag{7.18}$$

$$dS_t = S_t\mu\,(t, S_t)\,dt + S_t\sigma\,(t, S_t)\,dW_t, \tag{7.19}$$

where the short rate or is a deterministic constant. We consider a simple contingent claim of the form

$$\mathcal{X} = \Phi(S_t), \tag{7.20}$$

and we assume that this claim can be traded on a market and that its price process $\Pi_t[\Phi]$ has the form

$$\Pi_t[\Phi] = F(t, S_t), \tag{7.21}$$

for some smooth function F. Our problem is to find out what F must look like in order for the market $[S_t, B_t, \Pi_t[\Phi]]$ to be free of arbitrage possibilities.

We start by computing the price dynamics of the derivative asset, and the Itô formula applied to (7.21) and (7.19) gives us

$$dF = \mu_F F dt + \sigma_F F dW_t, \tag{7.22}$$

where we have suppressed the variables (t, S_t) and where the processes μ^F and σ^F are defined by

$$\mu_F = \frac{F_t + \mu S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{F}, \tag{7.23}$$

$$\sigma_F = \frac{\sigma S F_s}{F}. \tag{7.24}$$

Here subscripts denote partial derivatives, and we have used a shorthand notation of the form

$$\frac{\sigma S F_s}{F} = \frac{\sigma(t, S_t) S_t F_s(t, S_t)}{F(t, S_t)},$$

and similarly for the other terms above.

Let us now form a portfolio based on two assets: the underlying stock and the derivative asset. Denoting the relative portfolio by (w^S, w^F) and using eqn (6.22) we obtain the following dynamics for the value V of the portfolio:

$$dV = V\left\{w^S\left[\mu dt + \sigma dW\right] + w^F\left[\mu_F dt + \sigma_F dW\right]\right\} \tag{7.25}$$

where we have suppressed t. We now collect dt- and dW-terms to obtain

$$dV = V\left[w^S \mu + w^F \mu_F\right] dt + V\left[w^S \sigma + w^F \sigma_F\right] dW. \tag{7.26}$$

The point to notice here is that both brackets above are linear in the arguments w^S and w^F. Recall furthermore that the only restriction on the relative portfolio is that we must have

$$w^S + w^F = 1$$

for all t. Let us thus define the relative portfolio by the linear system of equations

$$w^S + w^F = 1, \tag{7.27}$$

$$w^S \sigma + w^F \sigma_F = 0. \tag{7.28}$$

Using this portfolio we see that by its very definition the driving dW-term in the V-dynamics of eqn (7.26) vanishes completely, leaving us with the equation

$$dV = V\left[w^S \mu + w^F \mu_F\right] dt. \tag{7.29}$$

Thus we have obtained a locally riskless portfolio, and because of the requirement that the market is free of arbitrage, we may now use Proposition 7.9 to deduce that we must have the relation

$$w^S \mu + w^F \mu_F = r. \tag{7.30}$$

This is thus the condition for absence of arbitrage, and we will now look more closely at this equation.

It is easily seen that the system (7.27)–(7.28) has the solution

$$w^S = \frac{\sigma_F}{\sigma_F - \sigma}, \tag{7.31}$$

$$w^F = \frac{-\sigma}{\sigma_F - \sigma}, \tag{7.32}$$

which, using (7.24), gives us the portfolio more explicitly as

$$w_t^S = \frac{S_t F_s(t, S_t)}{S_t F_s(t, S_t) - F(t, S_t)}, \tag{7.33}$$

$$w_t^F = \frac{-F(t, S_t)}{S_t F_s(t, S_t) - F(t, S_t)}. \tag{7.34}$$

Now we substitute (7.23), (7.33), and (7.34) into the absence of arbitrage condition (7.30). Then, after some calculations, we obtain the equation

$$F_t\left(t, S_t\right) + rS_t F_s\left(t, S_t\right) + \frac{1}{2}\sigma^2\left(t, S_t\right) S_t^2 F_{ss}\left(t, S_t\right) - rF\left(t, S_t\right) = 0.$$

Furthermore, from the previous section we must have the relation

$$\Pi_T[\mathcal{X}] = \Phi(S_t).$$

so, since $\Pi_t[\Phi] = F(t, S_t)$, we must have $F(t, S_T) = \Phi(S_T)$ $P - a.s$ These two equations have to hold with probability 1 for each fixed t. Furthermore it can be shown that under very weak assumptions (which trivially are satisfied in the Black–Scholes model) the distribution of S_t for every fixed $t > 0$ has support on the entire positive real line. Thus S_t can take any value whatsoever, so F has to satisfy the following (deterministic) PDE:

$$F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2(t, s)F_{ss}(t, s) - rF(t, s) = 0,$$

$$F(T, s) = \Phi(s).$$

Summing up these results we have proved the following proposition, which is in fact one of the most central results in the book.

Theorem 7.10 (Black–Scholes Equation) *Assume that the market is specified by eqns (7.18)–(7.19) and that we want to price a contingent claim of the form (7.20). Then the only pricing function of the form (7.21) which is consistent with the absence of arbitrage is when F is the solution of the following boundary value problem in the domain $[0, T] \times R_+$:*

$$F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2(t, s)F_{ss}(t, s) - rF(t, s) = 0, \tag{7.35}$$

$$F(T, s) = \Phi(s). \tag{7.36}$$

Before we go on to a closer study of the pricing equation (7.35) let us make a few comments.

Firstly it is important to stress the fact that we have obtained the price of the claim $\mathcal{X}$ in the form $\Pi_t[\mathcal{X}] = F(t, S_t)$, i.e. the price of the claim is given as a function of the price of the underlying asset S. This is completely in line with the basic idea explained earlier, that the pricing of derivative assets is a question of pricing the derivative in a way which is **consistent** with the price of the underlying asset. We are thus **not** presenting an **absolute** pricing formula for $\mathcal{X}$. On the contrary, derivative pricing is all about **relative** pricing, i.e. pricing the derivative asset **in terms of** the price of the underlying asset. In particular this means that in order to use the technique of arbitrage pricing at all we must have one or several underlying price processes given a priori.

Secondly a word of criticism. At a first glance our derivation of the pricing equation (7.35) seems to be fairly convincing, but in fact it contains some rather weak points. The logic of the argument was that we **assumed** that the price of the derivative was a function F of t and S_t. Using this assumption we then showed that in an arbitrage free market F had to satisfy the Black–Scholes equation. The question now is if we really have good reasons to assume that the price is of the form $F(t, S_t)$. The Markovian argument given above sounds good, but it is not totally convincing.

A much more serious objection is that we assume that there actually exists a market for the derivative asset, and in particular that there exists a price process for the derivative. This assumption of an existing market for the derivative is crucial for the argument since we are actually constructing a portfolio based on the derivative (and the underlying asset). If the derivative is not traded then the portfolio cannot be formed and our argument breaks down. The assumption of an existing price for the derivative is of course innocent enough in the case of a standard derivative, like a European call option, which *de facto* is traded in large volumes. If, however, we want to price an OTC ("over the counter") instrument, i.e. an instrument which is not traded on a regular basis, then we seem to be in big trouble.

Happily enough there is an alternative argument for the derivation of the pricing equation (7.35), and this argument (which will be given below) is not open to the criticism above. The bottom line is that the reader can feel safe: equation (7.35) really is the "correct" equation.

Let us end by noting an extremely surprising fact about the pricing equation, namely that it does not contain the local mean rate of return $\mu(t,s)$ of the underlying asset. In particular this means that when it comes to pricing derivatives, the local rate of return of the underlying asset plays no role whatsoever. The only aspect of the underlying price process which is of any importance is the volatility $\sigma(t,s)$. Thus, for a given volatility, the price of a fixed derivative (like a European call option) will be exactly the same regardless of whether the underlying stock has a 10%, a 50%, or even a -50% rate of return. At a first glance this sounds highly counter-intuitive and one is tempted to doubt the whole procedure of arbitrage pricing. There is, however, a natural explanation for this phenomenon, and we will come back to it later. At this point we can only say that the phenomenon is closely connected to the fact that we are pricing the derivative in terms of the price of the underlying asset.

7.5 Risk Neutral Valuation

Let us again consider a market given by the equations

$$dB_t = rB_t dt, \tag{7.37}$$

$$dS_t = S_t\mu(t, S_t)\,dt + S_t\sigma(t, S_t)\,dW_t, \tag{7.38}$$

and a contingent claim of the form $\mathcal{X} = \Phi(S_T)$. Then we know that the arbitrage free price is given by $\Pi_t[\Phi] = F(t, S_t)$ where the function F is the solution of the pricing equation (7.35)–(7.36). We now turn to the question of actually solving the pricing equation and we notice that this equation is precisely of the form which can be solved using a stochastic representation formula à la Feynman–Kač. Using the results from Section 5.5 we see that the solution is given by

$$F(t,s) = e^{-r(T-t)} E_{t,s}\left[\Phi(X_T)\right], \tag{7.39}$$

where the X process is defined by the dynamics

$$dX_u = rX_u du + X_u \sigma(u, X_u) dW_u, \tag{7.40}$$

$$X_t = s, \tag{7.41}$$

where W is a Wiener process. The important point to note here is that the SDE (7.40) is of precisely the same form as that of the price process S. The only, but important, change is that whereas S has the local rate of return μ, the X-process has the short rate r as its local rate of return.

The X-process above is logically just a technical tool, defined for the moment, and in particular we can name it as we please. In view of the resemblance between X and S it is rather tempting to call it S instead of X. This is perfectly acceptable as long as we do not confuse the "real" S-process of (7.38) with the "new" S-process, and one way to achieve this goal is by the following procedure.

Let us agree to denote the "objective" probability measure which governs our real model (7.37)–(7.38) by the letter P. Thus we say that the P-dynamics of the S-process are that of (7.38). We now define another probability measure Q under which the S-process has a different probability distribution. This is done by defining the Q-dynamics of S as

$$dS_t = rS_t dt + S_t \sigma\left(t, S_t\right) dW_t^Q, \tag{7.42}$$

where W^Q is a Q-Wiener process. In order to distinguish the measure under which we take expectations we introduce some notational conventions.

Notational convention 7.5.1 *For the rest of the text, the following conventions will be used:*

- *We identify the expectation operator by letting E denote expectations taken under the P-measure whereas E^Q denotes expectations taken under the Q-measure.*
- *We identify the Wiener process. Thus W will denote a P-Wiener process, whereas W^Q will denote a Q-Wiener process.*

The convention on W has the advantage that it is possible, at a glance, to decide under which measure a certain SDE is given. Using this notation we may now state the following central result for derivative pricing.

Theorem 7.11 (Risk Neutral Valuation) *The arbitrage free price of the claim $\Phi(S_t)$ is given by $\Pi_t[\Phi] = F(t, S_t)$, where F is given by the formula*

$$F(t,s) = e^{-r(T-t)} E^Q_{t,s}[\Phi(S_T)], \tag{7.43}$$

where the Q-dynamics of S are those of (7.42).

There is a natural economic interpretation of the formula (7.43). We see that the price of the derivative, given today's date t and today's stock price s, is computed by taking the expectation of the final payment $E^Q_{t,s}[\Phi(S_t)]$ and then discounting this expected value to present value using the discount factor $e^{-r(T-t)}$. The important point to note is that when we take the expected value we are **not** to do this using the objective probability measure P. Instead we shall use the Q-measure defined in (7.42). This Q-measure is sometimes called the **risk adjusted measure** but most often it is called the **martingale measure**, and this will be our terminology. The reason for the name is that under Q the normalized process $\frac{S_t}{B_t}$ turns out to be a Q-martingale. In the deeper investigation of arbitrage pricing, which will be undertaken in Chapter 11, the Q-measure is the fundamental object of study. We formulate the martingale property as a separate result.

Proposition 7.12 (The Martingale Property) *In the Black–Scholes model, the price process Π_t for every traded asset, be it the underlying or derivative asset, has the property that the normalized price process*

$$Z_t = \frac{\Pi_t}{B_t}$$

is a martingale under the measure Q.

Proof See the exercises. □

The formula (7.43) is sometimes referred to as the formula of **risk neutral valuation**. Suppose that all agents are risk neutral. Then all assets will command a rate of return equal to the short rate of interest, i.e. in a risk neutral world the stock price will actually have the Q-dynamics above (more precisely, in this case we will have $Q = P$). Furthermore, in a risk neutral world the present value of a future stochastic payout will equal the expected value of the net payments discounted to present value using the short rate of interest. Thus formula (7.43) is precisely the kind of formula which would be used for valuing a contingent claim in a risk neutral world. Observe, however, that we do **not** assume that the agents in our model are risk neutral. The formula only says that the value of the contingent claim can be calculated **as if** we live in a risk neutral world. In particular the agents are allowed to have any attitude to risk whatsoever, as long as they all prefer a larger amount of (certain) money to a lesser amount. Thus the valuation formula above is **preference free** in the sense that it is valid regardless of the specific form of the agents' preferences.

7.6 The Black–Scholes Formula

In this section we specialize the model of the previous section to the case of the Black–Scholes model,

$$dB_t = rB_t dt, \tag{7.44}$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{7.45}$$

where μ and σ are constants. From the results of the previous section we know that the arbitrage free price of a simple claim $\Phi(S_t)$ is given by

$$F(t,s) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S_T)\right], \tag{7.46}$$

where the Q-dynamics of S are given by

$$dS_u = rS_u du + \sigma S_u dW^Q_u, \tag{7.47}$$
$$S_t = s. \tag{7.48}$$

In this SDE we recognize our old friend Geometric Brownian Motion from Section 5.2. Using the results from Section 5.2 we can thus write S_T explicitly as

$$S_T = s \cdot \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W^Q_T - W^Q_t\right)\right\}. \tag{7.49}$$

Thus we have the pricing formula

$$F(t,s) = e^{-r(T-t)} \int_{-\infty}^{\infty} \Phi\left(se^z\right) f(z)dz, \tag{7.50}$$

where f is the density of a random variable Z with the distribution

$$N\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma\sqrt{T-t}\right].$$

Formula (7.50) is an integral formula which, for a general choice of contract function Φ, must be evaluated numerically. There are, however, a few particular cases where we can evaluate (7.50) more or less analytically, and the best known of these is the case of a European call option, where Φ has the form $\Phi(x) = \max\left[x - K, 0\right]$. In this case we obtain

$$E^Q_{t,s}\left[\max\left[se^Z - K, 0\right]\right] = 0 \cdot Q\left(se^Z \le K\right) + \int_{\ln\left(\frac{K}{s}\right)}^{\infty} \left(se^z - K\right) f(z)dz. \tag{7.51}$$

After some standard calculations we are left with the following famous result which is known as the **Black–Scholes Formula**.

Proposition 7.13 (Black–Scholes Formula) *The price of a European call option with strike price K and time of maturity T is given by the formula $\Pi(t) = F(t, S_t)$, where*

$$F(t,s) = sN\left[d_1(t,s)\right] - e^{-r(T-t)} KN\left[d_2(t,s)\right]. \tag{7.52}$$

Here N is the cumulative distribution function for the $N[0,1]$ distribution and

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \tag{7.53}$$

$$d_2(t,s) = d_1(t,s) - \sigma\sqrt{T-t}. \tag{7.54}$$

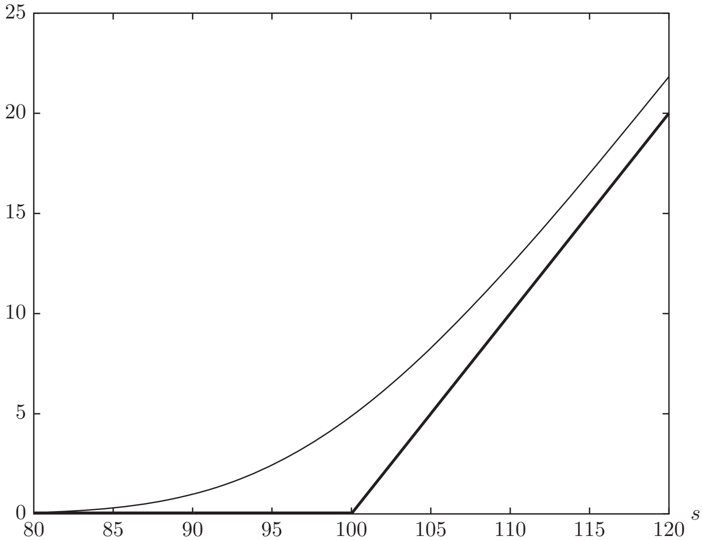

FIG. 7.3. The Black–Scholes price of a call option: $K = 100$, $\sigma = 0.2$, $T - t = 0.25$

Shown in Fig. 7.3 is the graph of the Black–Scholes pricing function (the unit of time is chosen to be one year).

7.7 Forward and Futures Contracts

The purpose of this section is to briefly discuss forward and futures contracts, and to derive the Black formula for options written on a futures contract. The reader is referred to Chapter 17 (and the Notes), where theses topics are discussed in much more detail.

7.7.1 *Forward Contracts*

Consider a standard Black–Scholes model, a simple T-claim $\mathcal{X} = \Phi(S_T)$, and assume that we are standing at time t. A **forward contract** on $\mathcal{X}$, made at t, is a contract which stipulates that the holder of the contract pays the deterministic amount K at the **delivery date** T, and receives the stochastic amount $\mathcal{X}$ at T. Nothing is paid or received at the time t, when the contract is made. Note that **forward price** K is determined already at time t. It is customary to use the notation $K = f(t; T, \mathcal{X})$, and our primary concern is to compute $f(t; T, \mathcal{X})$.

This is, however, easily done. We see that the entire forward contract is a contingent T-claim Y of the form

$$Y = \mathcal{X} - K,$$

and, by definition, the value of Y at the time t when the contract is made equals zero. Thus we have

$$\Pi_t[\mathcal{X} - K] = 0,$$

which leads to

$$\Pi_t[\mathcal{X}] = \Pi_t[K].$$

Using risk neutral valuation we immediately have $\Pi_t[K] = e^{-r(T-t)}K$ and $\Pi_t[\mathcal{X}] = e^{-r(T-t)} \times E^Q_{t,s}[\mathcal{X}]$, so we have proved the first part of the following result. The second part is left as an exercise.

Proposition 7.14 *Assuming a constant short rate, the forward price $f(t;T,\mathcal{X})$, contracted at t, on the T-claim $\mathcal{X}$ is given by*

$$f(t;T,\mathcal{X}) = E^Q_{t,s}[\mathcal{X}]. \tag{7.55}$$

In particular, if $\mathcal{X} = S_T$ the corresponding forward price, denoted by $f(t;T)$, is given by

$$f(t;T) = e^{r(T-t)}S_t. \tag{7.56}$$

Remark 7.7.1 *Note the difference between the forward price $f(t;T,\mathcal{X})$ which is a sum to be paid at T, for a forward contract entered at time t, and the spot price of the entire forward contract. This latter price is zero at the time t when the contract is made, but at any subsequent time $s > t$ it will typically have a non-zero value.*

7.7.2 *Futures Contracts and the Black Formula*

With the same setup as the previous section we will now discuss a **futures contract** on $\mathcal{X}$. This contract is very close to the corresponding forward contract in the sense that it is still a contract for the delivery of $\mathcal{X}$ at T. The difference is that all the payments, from the holder of the contract to the underwriter, are no longer made at T. Let us denote the futures price by $F(t;T,\mathcal{X})$; the payments are delivered continuously over time, such that the holder of the contract over the time interval $[s, s+\Delta s]$ receives the amount

$$F(s+\Delta s;T,\mathcal{X}) - F(s;T,\mathcal{X})$$

from the underwriter. Finally the holder will receive $\mathcal{X}$, and pay $F(T;T,\mathcal{X})$, at the **delivery date** T. By definition, the (spot) price (at any time) of the entire futures contract equals zero. Thus the cost of entering or leaving a futures contract is zero, and the only contractual obligation is the payment stream described above. See Chapter 17 for more details, and for a proof of the following result.

Proposition 7.15 *If the short rate is deterministic, then the forward and the futures price processes coincide, and we have*

$$F(t;T,\mathcal{X}) = E^Q_{t,s}[\mathcal{X}]. \tag{7.57}$$

We will now study the problem of pricing a European call option, with exercise date T, and exercise price K, on an underlying futures contract. The futures contract is a future on S with delivery date T_1, with $T < T_1$. Options of this kind are traded frequently, and by definition the holder of this option will, at the exercise time T, obtain a long position in the futures contract, plus the stochastic amount

$$\mathcal{X} = \max[F(T;T_1) - K, 0]. \tag{7.58}$$

Since the spot price of the futures contract equals zero, we may, for pricing purposes, forget about the long futures position embedded in the option, and identify the option with the claim in (7.58).

We now go on to price the futures option, and we start by using Proposition 7.15 and eqn (7.56) in order to write

$$\mathcal{X} = e^{r(T_1-T)}\max[S_T - e^{-r(T_1-T)}K, 0].$$

Thus we see that the futures option consists of $e^{r(T_1-T)}$ call options on the underlying asset S, with exercise date T and exercise price $e^{-r(T_1-T)}K$. Denoting the price at T of the futures option by c, the stock price at t by s, and the futures price $F(t;T_1)$ by F, we thus have, from the Black–Scholes formula

$$c = e^{r(T_1-T)}\left[sN[d_1] - e^{-r(T-t)}e^{-r(T_1-T)}KN[d_2]\right]$$

where d_1 and d_2 are obtained from the Black–Scholes d_1 and d_2 by replacing k with $e^{-r(T_1-T)}K$. Finally we may substitute $s = Fe^{-r(T_1-t)}$, and simplify, to obtain the so-called Black-76 formula.

Proposition 7.16 (Black's formula) *The price, at t, of a European call option, with exercise date T and exercise price K, on a futures contract (on an underlying asset price S) with delivery date T_1 is given by*

$$c = e^{-r(T-t)}\left[FN[d_1] - KN[d_2]\right], \tag{7.59}$$

where F is the futures price $F = F(t;T_1)$, and

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

7.8 Volatility

In order to be able to use the theory derived above in a concrete situation, we need to have numerical estimates of all the input parameters. In the Black–Scholes model the input data consists of the string s, r, T, t, and σ. Out of these five parameters, s, r, T, and t can be observed directly, which leaves us with the problem of obtaining an estimate of the volatility σ. Here there are two basic approaches, namely to use "historic volatility" or "implied volatility".

7.8.1 *Historic Volatility*

Suppose that we want to value a European call with six months left to maturity. An obvious idea is to use historical stock price data in order to estimate σ. Since, in real life, the volatility is not constant over time, one standard practice is to use historical data for a period of the same length as the time to maturity, which in our case means that we use data for the last six months.

In order to obtain an estimate of σ we assume that we have the standard Black–Scholes GBM model (7.4) under the objective measure P. We sample (observe) the stock price process S at $n+1$ discrete equidistant points $t_0, t_1, \ldots, t_n$, where Δt denotes the length of the sampling interval, i.e. $\Delta t = t_i - t_{i-1}$.

We thus observe $S(t_0), \ldots, S(t_n)$, and in order to estimate σ we use the fact that S has a log-normal distribution. Let us therefore define $\xi_1, \ldots, \xi_n$ by

$$\xi_i = \ln\left(\frac{S(t_i)}{S(t_{i-1})}\right).$$

From (5.15) we see that $\xi_1, \ldots, \xi_n$ are independent, normally distributed random variables with

$$E[\xi_i] = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t,$$
$$Var[\xi_i] = \sigma^2 \Delta t.$$

Using elementary statistical theory we see that an estimate of σ is given by

$$\sigma^\star = \frac{S_\xi}{\sqrt{\Delta t}},$$

where the sample variance S_ξ^2 is given by

$$S_\xi^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\xi_i - \bar{\xi}\right)^2,$$
$$\bar{\xi} = \frac{1}{n}\sum_{i=1}^{n}\xi_i.$$

The standard deviation, D, of the estimate $\sigma^\star$ is approximatively given by

$$D(\sigma^\star) \approx \frac{\sigma^\star}{\sqrt{2n}}.$$

7.8.2 *Implied Volatility*

Suppose again that we want to value a European call with six months left to maturity. An argument against the use of historical volatility is that in real life volatility is not constant, but changes over time, and thus we want an estimate of the volatility for the coming six months. Using historical volatility we will, however, only obtain an estimate for the volatility over the past six months. If, furthermore, our objective is to price our option **consistently** with respect to other assets which are already priced by the market, then we really should use the **market expectation** of the volatility for the next six months.

One way of finding the market expectation of the volatility is by getting market price data for another six-month "benchmark" option, written on the same underlying stock as the option which we want to value. Denoting the price of the benchmark option by p, the strike price by K, today's observed value of the underlying stock by s, and writing the Black–Scholes pricing formula for European calls by $c(s,t,T,r,\sigma,K)$, we then solve the following equation for σ

$$p = c(s,t,T,r,\sigma,K).$$

In other words, we try to find the value of σ which the market has implicitly used for valuing the benchmark option. Ths value of σ is called the **implied volatility**, and we then use the implied volatility for the benchmark in order to price our original option. Put another way, we price the original option in terms of the benchmark.

We note that implied volatilities can be used to test (in a non-standard way) the Black–Scholes model. Suppose, for example, that we observe the market prices of a number of European calls with the same exercise date on a single underlying stock. If the model is correct (with a constant volatility) then, if we plot implied volatility as a function of the exercise price, we should obtain a horizontal straight line. Contrary to this, it is often empirically observed that options far out of the money or deep into the money are traded at higher implied volatilities than options at the money. The graph of the observed implied volatility function thus often looks like the smile of the Cheshire cat, and for this reason the implied volatility curve is termed the **volatility smile**.

Remark 7.8.1 *A call option is said to be "in the money" at time t if $S_t > K$, and "out of the money" if $S_t < K$. For put options the inequalities are reversed. If $S_t = K$ the option is said to be "at the money".*

7.9 American Options

Up to now we have assumed that a contract, like a call option, can only be exercised exactly at the exercise time T. In real life a large number of options can in fact be exercised at **any time prior to T**. The choice of exercise time is thus left to the holder of the contract, and a contract with this feature is called an **American** contract. This topic will be discussed in detail in Chapter 28, but we will already at this point give a flavor of the theory.

To put it more formally, let us fix a final exercise date T and a contract function Φ. The European version of this contract will, as usual, pay the amount $\Phi(S_T)$ at time T to the holder of the contract. If the contract, on the other hand, is of the American type, then the holder will obtain the amount $\Phi(S_t)$ if he/she chooses to exercise the contract at time t. The situation is complicated further by the fact that the exercise time t does not have to be chosen a priori (i.e. at $t=0$). It can be chosen on the basis of the information generated by the stock price process, and thus the holder will in fact choose a **random exercise time** τ. The exercise time (or rather exercise strategy) τ has to be chosen such that the decision on whether to exercise the contract at time t or not, depends only upon the information generated by the price process up to time t. The mathematical formulation of this property is in terms of so-called stopping times, but for the moment we will not go further into this subject. See Appendix C.4 for details.

American contracts are thus more complicated to analyze than their European counterparts, since the holder of the contract has to decide on an **optimal exercise strategy**. Mathematically this means that we have to solve the "optimal stopping problem"

$$\max_{\tau} E^Q\left[e^{-r\tau}\Phi(S_\tau)\right],$$

where τ is allowed to vary over the class of stopping times. Problems of this kind are quite hard to solve, and analytically they lead to so-called free boundary value problems (or variational inequalities) instead of the corresponding parabolic PDEs for the European counterparts. For details, see Chapter 28 where we give an introduction to the theory of optimal stopping. For the moment we note that for American contracts practically no analytical formulas are at hand. See the Notes of Chapter 28 for references.

One situation, however, is very easy to analyze, even for American contracts, and that is the case of an American call option on a non-dividend-paying underlying stock. Let us consider an American call option with final exercise date T and exerise price K. We denote the pricing function for the American option by $C(t,s)$ and the pricing function for the corresponding European option (with the same T and K) by $c(t,s)$.

Firstly we note that we have (why?) the trivial inequality

$$C(t,s) \geq c(t,s). \tag{7.60}$$

Secondly we, have for all $t<T$, the less obvious inequality

$$c(t,s) \geq s - Ke^{-r(T-t)}. \tag{7.61}$$

To see why this inequality holds it is sufficient to consider two portfolios, A and B. A consists of a long position in the European option, whereas B consists of a long position in the underlying stock and a loan expiring at T, with face value K. Denoting the price of A and B at any time t by A_t and B_t respectively, it is easily seen that $A_T \geq B_T$ regardless of the value of S_T (analyze the two cases $S_T \geq K$ and $S_T < K$). In order to avoid arbitrage possibilities we then must have $A_t \geq B_t$ for all $t \leq T$, which is precisely the content of (7.61).

Furthermore, assuming a positive rate of interest, we have the trivial inequality

$$s - Ke^{-r(T-t)} > s - K, \quad \forall t < T,$$

so we end up with the inequality

$$C(t,s) > s - K, \quad \forall t < T. \tag{7.62}$$

On the left-hand side we have the value of the American option at time t, whereas the right-hand side gives us the value of actually exercising the option at time t. Since the value of the option is strictly greater than the value of exercising the option, it can thus not be optimal to exercise the option at time t. Since this holds for all $t < T$, we see that it is in fact never optimal to exercise the option before T, and we have the following result.

Proposition 7.17 *Assume that $r > 0$. For an American call option, written on an underlying stock without dividends, the optimal exercise time τ is given by $\tau = T$. Thus the price of the American option coincides with the price of the corresponding European option.*

For American call options with discrete dividends, the argument above can be extended to show that it can only be optimal to exercise the option either at the final time T or at one of the dividend times. The American put option (even without dividends) presents a hard problem without an analytical solution. See the Notes below.

7.10 Exercises

Exercise 7.1 Consider the standard Black–Scholes model and a T-claim $\mathcal{X}$ of the form $\mathcal{X} = \Phi(S_t)$. Denote the corresponding arbitrage free price process by Π_t.

(a) Show that, under the martingale measure Q, Π_t has a local rate of return equal to the short rate or. In other words show that Π_t has a differential of the form

$$d\Pi_t = r \cdot \Pi_t dt + g_t dW_t^Q.$$

Hint: Use the Q-dynamics of S together with the fact that F satisfies the pricing PDE.

(b) Show that, under the martingale measure Q, the process $Z_t = \frac{\Pi_t}{B_t}$ is a **martingale**. More precisely, show that the stochastic differential for Z has zero drift term, i.e. it is of the form

$$dZ_t = Z_t \sigma_t^Z dW_t^Q.$$

Determine also the diffusion process σ_t^Z (in terms of the pricing function F and its derivatives).

Exercise 7.2 Consider the standard Black–Scholes model. An innovative company, *F&H INC*, has produced the derivative "the Golden Logarithm", henceforth abbreviated as the *GL*. The holder of a *GL* with maturity time T, denoted as GL_t, will, at time T, obtain the sum $\Phi(S_T) = \ln S_T$. Note that if $S_T < 1$ this means that the holder has to pay a positive amount to *F&H INC*. Determine the arbitrage free price process for the GL_t.

Exercise 7.3 Consider the standard Black–Scholes model. Derive the Black–Scholes formula for the European call option.

Exercise 7.4 Consider the standard Black–Scholes model. Derive the arbitrage free price process for the T-claim $\mathcal{X}$ where $\mathcal{X}$ is given by $\mathcal{X} = S_T^\beta$. Here β is a known constant.

Hint: For this problem you may find Exercises 5.5 and 4.4 useful.

Exercise 7.5 A so-called **binary option** is a claim which pays a certain amount if the stock price at a certain date falls within some pre-specified interval. Otherwise nothing will be paid out. Consider a binary option which pays K dollars to the holder at date T if the stock price at time T is in the interval $[\mu, \beta]$. Determine the arbitrage free price. The pricing formula will involve the standard Gaussian cumulative distribution function N.

Exercise 7.6 Consider the standard Black–Scholes model. Derive the arbitrage free price process for the claim $\mathcal{X}$ where $\mathcal{X}$ is given by $\mathcal{X} = \frac{S(T_1)}{S(T_0)}$. The times T_0 and T_1 are given and the claim is paid out at time T_1.

Exercise 7.7 Consider the American corporation *ACME INC*. The price process S for *ACME* is of course denoted in dollars and has the P-dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t^1,$$

where μ and σ are known constants. The currency ratio euro/dollar is denoted by Y and Y has the dynamics

$$dY_t = \beta Y_t dt + \delta Y_t dW_t^2,$$

where W^2 is independent of W^1. The broker firm *F&H* has invented the derivative "Euler". The holder of a T-Euler will, at the time of maturity T, obtain the sum

$$\mathcal{X} = \ln\left[Z_T^2\right]$$

in euros. Here Z_t is the price at time t in euros of the *ACME* stock.

Compute the arbitrage free price (in euros) at time t of a T-Euler, given that the price (in euros) of the *ACME* stock is z. The euro short rate is denoted by r.

Exercise 7.8 Prove formula (7.56).

Exercise 7.9 Derive a formula for the value, at s, of a forward contract on the T-claim X, where the forward contract is made at t, and $t < s < T$.

7.11 Notes

The classics in the field are Black and Scholes (1973), and Merton (1973). A very good textbook treatment can be found in Joshi (2008) where, apart from the theory, the reader also can find lots of valuable information concerning numerical methods and other practical issues.

For a wealth of information on forward and futures contracts, see Hull (2003) and Duffie (1989). Black's formula was derived in Black (1976). For American options see Barone-Adesi and Elliott (1991), Geske and Johnson (1984), and Musiela and Rutkowski (1997).

8

COMPLETENESS AND HEDGING

8.1 Introduction

In Chapter 7 we noticed that our derivation of the pricing equation (7.35) was somewhat unsatisfactory, and a major criticism was that we were forced to assume that the derivative asset a priori possessed a price process and actually was traded on the market. In this chapter we will look at arbitrage pricing from a somewhat different point of view, and this alternative approach will have two benefits. Firstly it will allow us to dispose of the annoying assumption above that the derivative is actually traded, and secondly it will provide us with an explanation of the surprising fact that the simple claims investigated earlier can be given a unique price. For a more detailed discussion see Chapters 11, 13, and 29.

We start with a fairly general situation by considering a financial market with a price vector process $S = \left(S^1, \ldots, S^N\right)$, governed by an objective probability measure P. The process S is as usual interpreted as the price process of the exogenously given underlying assets and we now want to price a contingent T-claim $\mathcal{X}$. We assume that all the underlying assets are traded on the market, but we do not assume that there exists an a priori market (or a price process) for the derivative. To avoid trivialities we also assume that the underlying market is arbitrage free.

Definition 8.1 *We say that a T-claim $\mathcal{X}$ can be* **replicated***, alternatively that it is* **reachable** *or* **hedgeable***, if there exists a self-financing portfolio h such that*

$$V_T^h = \mathcal{X}, \quad P-a.s. \tag{8.1}$$

In this case we say that h is a **hedge** *against $\mathcal{X}$. Alternatively, h is called a* **replicating** *or* **hedging** *portfolio. If every contingent claim is reachable we say that the market is* **complete** *.*

Let us now consider a fixed T-claim $\mathcal{X}$ and let us assume that $\mathcal{X}$ can be replicated by a portfolio h. Then we can make the following mental experiment.

1. Fix a point in time t with $t \leq T$.
2. Suppose that we, at time t, possess V_t^h dollars.
3. We can then use this money to buy the portfolio h_t. If, furthermore, we follow the portfolio strategy h on the time interval $[t, T]$ this will cost us nothing, since h is self-financing. At time T the value of our portfolio will then be V_T^h dollars.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

4. By the replication assumption the value, at time T, of our portfolio will thus be exactly $\mathcal{X}$ dollars, regardless of the stochastic price movements over the interval $[t,T]$.
5. From a purely financial point of view, holding the portfolio h is thus equivalent to the holding of the contract $\mathcal{X}$.
6. The "correct" price of $\mathcal{X}$ at time t is thus given by $\Pi_t[\mathcal{X}] = V_t^h$.

For a hedgeable claim we thus have a natural price process, $\Pi_t[\mathcal{X}] = V_t^h$, and we may now ask if this has anything to do with absence of arbitrage.

Proposition 8.2 *Suppose that the claim $\mathcal{X}$ can be hedged using the portfolio h. Then the only price process $\Pi_t[\mathcal{X}]$ which is consistent with no arbitrage is given by $\Pi_t[\mathcal{X}] = V_t^h$. Furthermore, if $\mathcal{X}$ can be hedged by g as well as by h then $V_t^g = V_t^h$ holds for all t with probability* 1.

Proof If at some time t we have $\Pi_t[\mathcal{X}] < V_t^h$ then we can make an arbitrage by selling the portfolio short and buying the claim, and vice versa if $\Pi_t[\mathcal{X}] > V_t^h$. A similar argument shows that we must have $V_t^g = V_t^h$. □

8.2 Completeness in the Black–Scholes Model

We will now investigate completeness for the generalized Black–Scholes model given by

$$dS_t = S_t\mu(t,S_t)\,dt + S_t\sigma(t,S_t)\,dW_t, \tag{8.2}$$

$$dB_t = rB_t dt, \tag{8.3}$$

where we assume that $\sigma(t,s) > 0$ for all (t,s). The main result is the following.

Theorem 8.3 *The model (8.3)–(8.2) is complete.*

The proof of this theorem requires some fairly deep results from probability theory and is thus outside the scope of this book. We will prove a weaker version of the theorem, namely that every **simple** claim can be hedged. This is often quite sufficient for practical purposes, and our proof of the restricted completeness also has the advantage that it gives the replicating portfolio in explicit form. We will use the notational convention $h_t = \left[h_t^B, h_t^S\right]$ where h^B is the number of bonds in the portfolio, whereas h^S denotes the number of shares in the underlying stock. We thus fix a simple T-claim of the form $\mathcal{X} = \Phi(S_t)$ and we now want to show that this claim can be hedged. Since the formal proof is of the form "consider the following odd construction", we will instead start by presenting a purely heuristic (but good) argument. This argument is, from a formal point of view, only of motivational nature and the logic of it is rather unclear. Since the argument is only heuristic the logical flaws do not matter, since in the end we will in fact present a rigorous statement and a rigorous proof. Before we start the heuristics, let us make more precise what we are looking for. Using Lemma 6.13 we immediately have the following result.

Lemma 8.4 *Suppose that there exists an adapted process V and an adapted process $w = \left[w^B, w^S\right]$ with*

$$w_t^B + w_t^S = 1, \tag{8.4}$$

such that

$$\begin{cases} dV_t = V_t \left\{ w_t^B r + w_t^S \mu(t, S_t) \right\} dt + V_t w_t^S \sigma(t, S_t) dW_t, \\ V_t = \Phi(S_t). \end{cases} \tag{8.5}$$

Then the claim $\mathcal{X} = \Phi(S_t)$ can be replicated using u as the relative portfolio. The corresponding value process is given by the process V and the absolute portfolio h is given by

$$h_t^B = \frac{w_t^B V_t}{B_t}, \tag{8.6}$$

$$h_t^S = \frac{w_t^S V_t}{S_t}. \tag{8.7}$$

Our strategy now is to look for a process V and a process w satisfying the conditions above.

Begin Heuristics
We assume what we want to prove, namely that $\mathcal{X} = \Phi(S_t)$ is indeed replicable, and then we ponder on what the hedging strategy w might look like. Since the S-process and (trivially) the B-process are Markov processes it seems reasonable to assume that the hedging portfolio is of the form $h_t = h(t, S_t)$ where, with a slight misuse of notation, the h in the right member of the equality is a deterministic function. Since, furthermore, the value process V (we suppress the superscript h) is defined as $V_t = h_t^B B_t + h_t^S S_t$ it will also be a function of time and stock price as

$$V_t = F(t, S_t), \tag{8.8}$$

where F is some real valued deterministic function which we would like to know more about.

Assume therefore that (8.8) actually holds. Then we may apply the Itô formula to V in order to obtain the V-dynamics as

$$dV = \left\{ F_t + \mu S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss} \right\} dt + \sigma S F_s dW, \tag{8.9}$$

where we have suppressed the fact that V and S are to be evaluated at time t, whereas μ, σ, and F are to be evaluated at (t, S_t). Now, in order to make (8.9) look more like (8.5) we rewrite (8.9) as

$$dV = V \left\{ \frac{F_t + \mu S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{V} \right\} dt + V \frac{S F_s}{V} \sigma dW. \tag{8.10}$$

Since we have assumed that $\mathcal{X}$ is replicated by V we see from (8.10) and (8.5) that w^S must be given by

$$w_t^S = \frac{S_t F_s(t, S_t)}{F(t, S_t)}, \tag{8.11}$$

(remember that we have assumed that $V = F$), and if we substitute (8.11) into (8.10) we get

$$dV = V \left\{ \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} r + w^S \mu \right\} dt + V w^S \sigma dW. \tag{8.12}$$

Comparing this expression to (8.5) we see that the natural choice for w^B is given by

$$w^B = \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF}, \tag{8.13}$$

but we also have to satisfy the requirement $w^B + w^S = 1$ of (8.4). Using (8.11) and (8.13) this gives us the relation

$$\frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} = \frac{F - SF_s}{F}, \tag{8.14}$$

which, after some manipulation, turns out to be the familiar Black–Scholes equation

$$F_t + rSF_s + \frac{1}{2}\sigma^2 S^2 F_{ss} - rF = 0. \tag{8.15}$$

Furthermore, in order to satisfy the relation $F(T, S_t) = \Phi(S_t)$ of (8.5) (remember that we assume that $V = F$) we must have the boundary condition

$$F(T, s) = \Phi(s), \quad \text{for all } s \in R_+. \tag{8.16}$$

End Heuristics

Since at this point the reader may well be somewhat confused as to the logic of the reasoning, let us try to straighten things out. The logic of the reasoning above is basically as follows:

- We **assumed** that the claim $\mathcal{X}$ was replicable.
- Using this and some further (reasonable) assumptions we showed that they **implied** that the value process of the replicating portfolio was given as $V_t = F(t, S_t)$ where F is a solution of the Black–Scholes equation.

This is of course not at all what we wish to achieve. What we want to do is to **prove** that $\mathcal{X}$ really can be replicated. In order to do this we put the entire argument above within a logical parenthesis and formally disregard it. We then have the following result.

Theorem 8.5 *Consider the market (8.3)–(8.2), and a contingent claim of the form $\mathcal{X} = \Phi(S_t)$. Define F as the solution to the boundary value problem*

$$\begin{cases} F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF = 0, \\ F(T,s) = \Phi(s). \end{cases} \tag{8.17}$$

Then $\mathcal{X}$ can be replicated by the relative portfolio

$$w_t^B = \frac{F(t,S_t) - S_t F_s(t,S_t)}{F(t,S_t)}, \tag{8.18}$$

$$w_t^S = \frac{S_t F_s(t,S_t)}{F(t,S_t)}. \tag{8.19}$$

The corresponding absolute portfolio is given by

$$h_t^B = \frac{F(t,S_t) - S_t F_s(t,S_t)}{B_t}, \tag{8.20}$$

$$h_t^S = F_s(t,S_t), \tag{8.21}$$

and the value process V^h is given by

$$V_t^h = F(t,S_t). \tag{8.22}$$

Proof Applying the Itô formula to the process V_t defined by (8.22) and performing exactly the same calculations as in the heuristic argument above, will show that we can apply Lemma 8.4. □

The result above gives us an explanation of the surprising facts that there actually exists a **unique** price for a derivative asset in the Black–Scholes model and that this price does not depend on any particular assumptions about individual preferences. The arbitrage free price of a derivative asset is uniquely determined simply because in this model the derivative is superfluous. It can always be replaced by a corresponding "synthetic" derivative in terms of a replicating portfolio.

Since the replication is done with P-probability 1, we also see that if a contingent claim $\mathcal{X}$ is replicated under P by a portfolio h and if $P^\star$ is some other probability measure such that P and $P^\star$ assign probability 1 to exactly the same events (such measures P and $P^\star$ are said to be **equivalent**), then h will replicate $\mathcal{X}$ also under the measure $P^\star$. Thus the pricing formula for a certain claim will be exactly the same for all measures which are equivalent to P. It is a well known fact (the Girsanov Theorem) in the theory of SDEs that if we change the measure from P to some other equivalent measure, this will change the drift in the SDE, but the diffusion term will be unaffected. Thus the drift will play no part in the pricing equation, which explains why μ does not appear in the Black–Scholes equation.

Let us now list some popular claims and see which of them will fall into the framework above:

$$\mathcal{X} = \max[S_T - K, 0] \quad \text{(European call option)} \tag{8.23}$$

$$\mathcal{X} = S_T - K \quad \text{(Forward contract)} \tag{8.24}$$

$$\mathcal{X} = \max\left[\frac{1}{T}\int_0^T S_T dt - K, 0\right] \quad \text{(Asian option)} \tag{8.25}$$

$$\mathcal{X} = S_T - \inf_{0 \le t \le T} S_t \quad \text{(Lookback contract)} \tag{8.26}$$

We know from Theorem 8.3 that in fact all of the claims above can be replicated. For general claims this is, however, only an abstract existence result and we have no guarantee of obtaining the replicating portfolio in an explicit form. The point of Theorem 8.5 is precisely that, by restricting ourselves to **simple** claims, i.e. claims of the form $\mathcal{X} = \Phi(S_T)$, we obtain an explicit formula for the hedging portfolio.

It is clear that the European call as well as the forward contract above are simple claims, and we may thus apply Theorem 8.5. The Asian option (also called a mean value option) as well as the lookback present harder problems since neither of these claims is simple. Instead of just being functions of the value of S at time T we see that the claims depend on the entire S-trajectory over the interval $[0, T]$. Thus, while we know that there exist hedging portfolios for both these claims, we have presently no obvious way of determining the shape of these portfolios.

It is in fact quite hard to determine the hedging portfolio for the lookback, but the Asian option belongs to a class of contracts for which we can give a fairly explicit representation of the replicating portfolio, using very much the same technique as in Theorem 8.5.

Proposition 8.6 *Consider the model*

$$dS_t = S_t\mu(t, S_t)\,dt + S_t\sigma(t, S_t)\,dW_t, \tag{8.27}$$

$$dB_t = rB_t dt, \tag{8.28}$$

and let $\mathcal{X}$ be a T-claim of the form

$$\mathcal{X} = \Phi(S_T, Z_T), \tag{8.29}$$

where the process Z is defined by

$$Z_t = \int_0^t g(u, S_u)du, \tag{8.30}$$

for some choice of the deterministic function g. Then $\mathcal{X}$ can be replicated using a relative portfolio given by

$$w_t^B = \frac{F(t, S_t, Z_t) - S_t F_s(t, S_t, Z_t)}{F(t, S_t, Z_t)}, \tag{8.31}$$

$$w_t^S = \frac{S_t F_s(t, S_t, Z_t)}{F(t, S_t, Z_t)}, \tag{8.32}$$

where F is the solution to the boundary value problem

$$\begin{cases} F_t + srF_s + \frac{1}{2}s^2\sigma^2 F_{ss} + gF_z - rF = 0, \\ F(T,s,z) = \Phi(s,z). \end{cases} \tag{8.33}$$

The corresponding value process V is given by $V_t = F(t, S_t, Z_t)$, and F has the stochastic representation

$$F(t,s,z) = e^{-r(T-t)} E^Q_{t,s,z}\left[\Phi(S_T, Z_T)\right], \tag{8.34}$$

where the Q-dynamics are given by

$$dS_u = rS_u du + S_u \sigma(u, S_u) dW^Q_u, \tag{8.35}$$

$$S_t = s, \tag{8.36}$$

$$dZ_u = g(u, S_u) du, \tag{8.37}$$

$$Z_t = z. \tag{8.38}$$

Proof The proof is left as an exercise for the reader. Use the same technique as in the proof of Theorem 8.5. □

Again we see that the arbitrage free price of a contingent claim is given as the expected value of the claim discounted to the present time. Here, as before, the expected value is to be calculated using the martingale measure Q instead of the objective probability measure P. As we have said before, this general structure of arbitrage free pricing holds in much more general situations, and as a rule of thumb one can view the martingale measure Q as being defined by the property that all **traded** underlying assets have r as the rate of return under Q. It is important to stress that it is only traded assets which will have sr as the rate of return under Q. For models with non-traded underlying objects we have a completely different situation, which we will encounter below.

8.3 Completeness—Absence of Arbitrage

In this section we will give some general rules of thumb for quickly determining whether a certain model is complete and/or free of arbitrage. The arguments will be purely heuristic.

Let us consider a model with N traded underlying assets **plus** the risk free asset (i.e. totally $N+1$ assets). We assume that the price processes of the underlying assets are driven by R "random sources". We cannot give a precise definition of what constitutes a "random source" here, but the typical example is a driving Wiener process. If, for example, we have five independent Wiener processes driving our prices, then $R = 5$. Another example of a random source would be a counting process such as a Poisson process. In this context it is important to note that if the prices are driven by a point process with different jump sizes then the appropriate number of random sources equals the number of different jump sizes.

When discussing completeness and absence of arbitrage it is important to realize that these concepts work in opposite directions. Let the number of random sources R be fixed. Then every new underlying asset added to the model (without increasing R) will of course give us a potential opportunity of creating an arbitrage portfolio, so in order to have an arbitrage free market the number N of underlying assets must be small in comparison to the number of random sources R.

On the other hand we see that every new underlying asset added to the model gives us new possibilities of replicating a given contingent claim, so completeness requires N to be great in comparison to R.

We cannot formulate and prove a precise result here, but the following rule of thumb, or "meta-theorem", is nevertheless extremely useful. In concrete cases it can in fact be given a precise formulation and a precise proof. See Chapters 11 and 14. We will later use the meta-theorem when dealing with problems connected with non-traded underlying assets in general and interest rate theory in particular.

Meta-theorem 8.3.1 *Let N denote the number of underlying* **traded** *assets in the model* **excluding** *the risk free asset, and let R denote the number of random sources. Generically we then have the following relations*:

1. *The model is arbitrage free if and only if $N \leq R$.*
2. *The model is complete if and only if $N \geq R$.*
3. *The model is complete and arbitrage free if and only if $N = R$.*

As an example we take the Black–Scholes model, where we have one underlying asset S plus the risk free asset so $N = 1$. We have one driving Wiener process, giving us $R = 1$, so in fact $M = R$. Using the meta-theorem above we thus expect the Black–Scholes model to be arbitrage free as well as complete and this is indeed the case.

8.4 Exercises

Exercise 8.1 Consider a model for the stock market where the short rate of interest r is a deterministic constant. We focus on a particular stock with price process S. Under the objective probability measure P we have the following dynamics for the price process:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + \delta S(t-) dN_t.$$

Here W is a standard Wiener process whereas N is a Poisson process with intensity λ. We assume that μ, σ, δ, and λ are known to us. The dN term is to be interpreted in the following way:

1. Between the jump times of the Poisson process N, the S-process behaves just like ordinary Geometric Brownian Motion.
2. If N has a jump at time t this induces S to have a jump at time t. The size of the S-jump is given by

$$S_t - S_{t-} = \delta \cdot S_{t-}.$$

Discuss the following questions:

(a) Is the model free of arbitrage?
(b) Is the model complete?
(c) Is there a unique arbitrage free price for, say, a European call option?
(d) Suppose that you want to replicate a European call option maturing in January 1999. Is it posssible (theoretically) to replicate this asset by a portfolio consisting of bonds, the underlying stock and European call option maturing in December 2001?

Exercise 8.2 Use the Feynman–Kač technique in order to derive a risk neutral valuation formula in connection with Proposition 8.6.

Exercise 8.3 The fairly unknown company *F&H INC.* has blessed the market with a new derivative, "the Mean". With "effective period" given by $[T_1, T_2]$ the holder of a Mean contract will, at the date of maturity T_2, obtain the amount

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S_u du.$$

Determine the arbitrage free price, at time t, of the Mean contract. Assume that you live in a standard Black–Scholes world, and that $t < T_1$.

Exercise 8.4 Consider the standard Black–Scholes model, and n different simple contingent claims with contract functions $\Phi_1, \dots, \Phi_n$. Let

$$V = \sum_{i=1}^{n} h_t^i S_t^i$$

denote the value process of a self-financing, Markovian (see Definition 6.10) portfolio. Because of the Markovian assumption, V will be of the form $V(t, S_t)$. Show that V satisfies the Black–Scholes equation.

8.5 Notes

Completeness is mathematically closely related to rather deep results about the possibility of representing martingales as sums of stochastic integrals. Using this connection, it can be shown that the market is complete if and only if the martingale measure is unique. This is developed in some detail in Chapters 11 and 14. See also Harrison and Pliska (1981) and Musiela and Rutkowski (1997).

9

A PRIMER ON INCOMPLETE MARKETS

9.1 Introduction

In this chapter we will give a brief introduction to some aspects of derivative pricing in incomplete markets. We will use the classical delta hedging technique of Black–Scholes. A much more detailed discussion using martingale techniques can be found in Chapters 29–34.

We know from Theorem 8.3.1 that markets generically are incomplete when there are more random sources than there are traded assets, and this can occur in an infinite number of ways, so there is no "canonical" way of writing down a model of an incomplete market. We will confine ourselves to study a particular type of incomplete market, namely a "factor model", i.e. a market where there are some non-traded underlying objects. Before we go on to the formal description of the models let us briefly recall what we may expect in an incomplete model:

- Since, by assumption, the market is incomplete we will not be able to hedge a generic contingent claim.
- In particular there will not be a unique price for a generic derivative.

9.2 A Scalar Non-priced Underlying Asset

We will start by studying the simplest possible incomplete market, namely a market where the only randomness comes from a scalar stochastic process which is **not** the price of a traded asset. We will then discuss the problems which arise when we want to price derivatives which are written in terms of the underlying object. The model is as follows.

Assumption 9.2.1 *The only objects which are a priori given are the following:*

- *An empirically observable stochastic process X, which is* **not** *assumed to be the price process of a traded asset, with P-dynamics given by*

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t. \tag{9.1}$$

  Here W is a standard scalar P-Wiener process.
- *A risk free asset (money account) with the dynamics*

$$dB_t = rB_t dt, \tag{9.2}$$

  where r as usual is the deterministic short rate.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

We now consider a given contingent claim, written in terms of the process X. More specifically we define the T-claim $\mathcal{Y}$ by

$$\mathcal{Y} = \Phi(X_T), \tag{9.3}$$

where Φ is some given deterministic function, and our main problem is that of studying the price process $\Pi_t[\mathcal{Y}]$ for this claim.

In order to give some substance to the discussion, and to understand the difference between the present setting and that of the previous chapters, let us consider a specific concrete, interpretation of the model. We may, for example, interpret the process X as the temperature at some specific point on the earth, say the end of the Palace Pier in Brighton. Thus X_t is the temperature (in centigrade) at time t at the Palace Pier. Suppose now that you want to go to Brighton for a holiday, but that you fear that it will be unpleasantly cold at the particular time T when you visit Brighton. Then it may be wise to buy "holiday insurance", i.e. a contract which pays you a certain amount of money if the weather is unpleasant at a prespecified time in a prespecified place. If the monetary unit is the pound sterling, the contract function Φ above may have the form

$$\Phi(x) = \begin{cases} 100, & \text{if } x \leq 20, \\ 0, & \text{if } x > 20. \end{cases}$$

In other words, if the temperature at time T is below 20^oC (degrees centigrade) you will obtain 100 pounds from the insurance company, whereas you will get nothing if the temperature exceeds 20^oC.

The problem is now that of finding a "reasonable" price for the contract above, and as usual we interpret the word "reasonable" in the sense that there should be no arbitrage possibilities if we are allowed to trade the contract. This last sentence contains a hidden assumption which we now formalize.

Assumption 9.2.2 *There is a liquid market for every contingent claim.*

If we compare this model with the standard Black–Scholes model, we see many similarities. In both cases we have the money account B, and in both cases we have an a priori given underlying process. For the Black–Scholes model the underlying process is the stock price S, whereas we now have the underlying process X, and in both models the claim to be priced is a deterministic function Φ of the underlying process, evaluated at time T.

In view of these similarities it is now natural to assume that the results from the Black–Scholes analysis will carry over to the present case, i.e. we are (perhaps) led to believe that the price process for the claim $\mathcal{Y}$ is uniquely determined by the P-dynamics of the underlying process X. It is, however, very important to understand that this is, most emphatically, **not** the case, and the reasons are as follows:

1. If we consider the a priori given market, which only consists of the money account B, we see that the number R of random sources in this case

equals one (one driving Wiener process), while the number M of traded assets (always excluding the money account B) equals zero. From the meta-theorem 8.3.1 it now follows that the market is incomplete. The incompleteness can also be seen from the obvious fact that in the a priori given market there are no interesting ways of forming self-financing portfolios. The only strategy that is allowed is to invest all our money in the bank B, and then we can only sit down and passively watch our money grow at the rate r. In particular we have no possibility to replicate any interesting derivative of the form $\Phi(X_T)$. We thus conclude that, since we cannot replicate our claim, we cannot expect to obtain a unique arbitrage free price process.

2. One natural strategy to follow in order to obtain a unique price for the claim X is of course to imitate the scheme for the Black–Scholes model. We would assume that the price process $\Pi_t[\mathcal{Y}]$ is of the form $\Pi_t[\mathcal{Y}] = F(t, X_t)$, and then we would form a portfolio based on the derivative F and the underlying X. Choosing the portfolio weights such that the portfolio has no driving Wiener process would give us a risk free asset, the rate of return on which would have to equal the short rate r, and this last equality would finally have the form of a PDE for the pricing function F. This approach is, however, completely nonsensical, since in the present setting the process X is (by assumption) **not the price of a traded asset**, and thus it is meaningless to talk about a "portfolio based on X". In our concrete interpretation this is eminently clear. Obviously you can buy any number of insurance contracts and put them in your portfolio, but it is also obvious that you cannot meaningfully add, for example, 15^oC to that portfolio.

We can summarize the situation as follows:

- The price of a particular derivative will **not** be completely determined by the specification (9.1) of the X-dynamics and the requirement that the market $(B_t, \Pi_t[\mathcal{Y}])$ is free of arbitrage.
- The reason for this fact is that arbitrage pricing is always a case of pricing a derivative **in terms of** the price of some underlying assets. In our market we do not have sufficiently many underlying assets.

Thus we will not obtain a unique price of a particular derivative. This fact does not mean, however, that prices of various derivatives can take any form whatsoever. From the discussion above we see that the reason for the incompleteness is that we do not have enough underlying assets, so if we adjoin one more asset to the market, without introducing any new Wiener processes, then we expect the market to be complete. This idea can be expressed in the following ways.

Idea 9.2.1

- We **cannot** say anything about the price of any **particular** derivative.

- The requirement of an arbitrage free derivative market implies that **prices of different derivatives** (i.e. claims with different contract functions or different times of expiration) will have to satisfy certain **internal consistency relations** in order to avoid arbitrage possibilities on the derivatives market. In terms of our concrete interpretation this means that even if we are unable to produce a unique price for a fixed weather insurance contract, say the "20^oC contract"

$$\Phi(x) = \begin{cases} 100, & \text{if } x \leq 20, \\ 0, & \text{if } x > 20, \end{cases}$$

for the fixed date T, there must be internal consistency requirements between the price of this contract and the price of the following "25^oC contract"

$$\Gamma(x) = \begin{cases} 100, & \text{if } x \leq 25, \\ 0, & \text{if } x > 25, \end{cases}$$

with some expiration date T' (where of course we may have $T = T'$).
- In particular, if we take the price of **one** particular "benchmark" derivative as a priori given, then the prices of all other derivatives will be uniquely determined by the price of the benchmark. This fact is in complete agreement with the meta-theorem 8.3.1, since in an a priori given market consisting of one benchmark derivative plus the risk free asset we will have $R = M = 1$, thus guaranteeing completeness. In our concrete interpretation we expect that the prices of all insurance contracts should be determined by the price of any fixed benchmark contract. If, for example, we choose the 20^oC contract above as our benchmark, and take its price as given, then we expect the 25^oC contract to be priced uniquely in terms of the benchmark price.

To put these ideas into action we now take as given two fixed T-claims, $\mathcal{Y}$ and $\mathcal{Z}$, of the form

$$\mathcal{Y} = \Phi\left(X_T\right),$$
$$\mathcal{Z} = \Gamma\left(X_T\right),$$

where Φ and Γ are given deterministic real valued functions. The project is to find out how the prices of these two derivatives must be related to each other in order to avoid arbitrage possibilities on the derivative market. As above we assume that the contracts are traded on a frictionless market, and as in the Black–Scholes analysis we make an assumption about the structure of the price processes.

Assumption 9.2.3 *We assume that*

- *There is a liquid, frictionless market for each of the contingent claims* $\mathcal{Y}$ *and* $\mathcal{Z}$.

- *The market prices of the claims are of the form*

$$\Pi_t[\mathcal{Y}] = F(t, X_t),$$
$$\Pi_t[\mathcal{Z}] = G(t, X_t),$$

where F and G are smooth real valued functions.

We now proceed exactly as in the Black–Scholes case. We form a portfolio based on F and G, and choose the weights so as to make the portfolio locally riskless. The rate of return of this riskless portfolio has to equal the short rate, and this relation will give us some kind of equation, which we then have to analyze in detail.

From the assumption above, the Itô formula, and the X-dynamics we obtain the following price dynamics for the price processes $F(t, X_t)$ and $G(t, X_t)$:

$$dF = \mu_F F dt + \sigma_F F dW, \tag{9.4}$$

$$dG = \mu_G G dt + \sigma_G G dW. \tag{9.5}$$

Here the processes μ_F and σ_F are given by

$$\mu_F = \frac{F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}}{F},$$
$$\sigma_F = \frac{\sigma F_x}{F},$$

and correspondingly for μ_G and σ_G. As usual we have suppressed most arguments, so in more detail we have, for example,

$$\mu F_x = \mu(t, X_t)\frac{\partial F}{\partial x}(t, X_t).$$

We now form a self-financing portfolio based on F and G, with portfolio weights denoted by w^F and w_t^G respectively. According to (6.22), the portfolio dynamics are given by

$$dV_t = V_t\left\{w_t^F \cdot \frac{dF}{F} + w_t^G \cdot \frac{dG}{G}\right\},$$

and using the expressions above we get

$$dV_t = V_t\left\{w_t^F \cdot \mu_F + w_t^G \cdot \mu_G\right\}dt + V_t\left\{w_t^F \cdot \sigma_F + w_t^G \cdot \sigma_G\right\}dW_t. \tag{9.6}$$

In order to make this portfolio locally risk free we must choose w_t^F and w_t^G such that $w_t^F \cdot \sigma_F + w_t^G \cdot \sigma_G = 0$, and we must also remember that they must add to unity. Thus we define w_t^F and w_t^G as the solution to the following system of equations:

$$\begin{cases} w_t^F + w_t^G = 1, \\ w_t^F \cdot \sigma_F + w_t^G \cdot \sigma_G = 0. \end{cases}$$

The solution to this system is given by

$$w_t^F = \frac{-\sigma_G}{\sigma_F - \sigma_G}, \qquad w_t^G = \frac{\sigma_F}{\sigma_F - \sigma_G},$$

and inserting this into the portfolio dynamics equation (9.6) gives us

$$dV_t = V_t \cdot \left\{ \frac{\mu_G \cdot \sigma_F - \mu_F \cdot \sigma_G}{\sigma_F - \sigma_G} \right\} dt.$$

We have thus created a locally risk free asset, so using Proposition 7.9, absence of arbitrage must imply the equation

$$\frac{\mu_G \cdot \sigma_F - \mu_F \cdot \sigma_G}{\sigma_F - \sigma_G} = r.$$

After some reshuffling we can rewrite this equation as

$$\frac{\mu_F - r}{\sigma_F} = \frac{\mu_G - r}{\sigma_G}.$$

The important fact to notice about this equation is that the left-hand side does not depend on the choice of G, while the right-hand side does not depend upon the choice of F. The common quotient will thus depend neither on the choice of F nor on the choice of G, and we have proved the following central result.

Proposition 9.1 *Assume that the market for derivatives is free of arbitrage. Then there exists a universal process $\lambda(t, X_t)$ such that, with probability 1, and for all t, we have*

$$\frac{\mu_F(t, X_t) - r}{\sigma_F(t, X_t)} = \lambda(t, X_t), \tag{9.7}$$

regardless of the specific choice of the derivative F.

There is a natural economic interpretation of this result, and of the process λ. In eqn (9.7) the numerator is given by $\mu_F - r$, and from (9.4) we recognize μ_F as the local mean rate of return on the derivative F. The numerator $\mu_F - r$ is thus the local mean excess return on the derivative F over the riskless rate of return r, i.e. the risk premium of F. In the denominator we find the volatility σ_F of the F process, so we see that λ has the dimension "risk premium per unit of volatility". This is a concept well known from CAPM theory, so λ is commonly called "the market price of risk". Proposition 9.1 can now be formulated in the following, slightly more flashy, form:

- In a no arbitrage market all derivatives will, regardless of the specific choice of contract function, have the same market price of risk.

We can obtain more explicit information from eqn (9.7) by substituting our earlier formulas for μ_F and σ_F into it. After some algebraic manipulations we then end up with the following PDE:

$$F_t + \{\mu - \lambda\sigma\} F_x + \frac{1}{2}\sigma^2 F_{xx} - rF = 0.$$

This is really shorthand notation for an equation which must hold with probability 1, for each t, when all terms are evaluated at the point (t, X_t). Assuming for the moment that the support of X is the entire real line, we can then draw the conclusion that the equation must also hold identically when we evaluate it at an arbitrary deterministic point (t, x). Furthermore it is clear that we must have the boundary condition

$$F(T, x) = \Phi(x), \quad \forall x \in R,$$

so we finally end up with the following result.

Proposition 9.2 (Pricing equation) *Assuming absence of arbitrage, the pricing function $F(t, x)$ of the T-claim $\Phi(X_T)$ solves the following boundary value problem.*

$$F_t(t, x) + \mathcal{A}F(t, x) - rF(t, x) = 0, \qquad (t, x) \in (0, T) \times R, \tag{9.8}$$

$$F(T, x) = \Phi(x), \quad x \in R, \tag{9.9}$$

where

$$\mathcal{A}F(t, x) = \{\mu(t, x) - \lambda(t, x)\sigma(t, x)\} F_x(t, x) + \frac{1}{2}\sigma^2(t, x)F_{xx}(t, x).$$

At first glance this result may seem to contradict the moral presented above. We have stressed earlier the fact that, because of the incompleteness of the market, there will be **no** unique arbitrage free price for a particular derivative. In Proposition 9.2, on the other hand, we seem to have arrived at a PDE which, when solved, will give us precisely the unique pricing function for any simple claim. The solution to this conundrum is that the pricing equation above is indeed very nice, but in order to solve it we have to know the short rate r, as well as the functions $\mu(t, x)$, $\sigma(t, x)$, $\Phi(x)$, and $\lambda(t, x)$. Of these, only r, $\mu(t, x)$, $\sigma(t, x)$, and $\Phi(x)$ are specified exogenously. The market price of risk λ, on the contrary, is **not** specified within the model. We can now make Idea 9.2.1 above more precise.

Firstly we see that, even though we cannot determine a unique price for a particular derivative, prices of different derivatives must satisfy internal consistency requirements. This requirement is formulated precisely in Proposition 9.1, which says that all derivatives must have the same market price of risk. Thus, if we consider two different derivative assets with price processes F and G, these may have completely different local mean rates of return (μ_F and μ_G), and they may also have completely different volatilities (σ_F and σ_G). The consistency relation which has to be satisfied in order to avoid arbitrage between the derivatives is that, at all times, the quotient

$$\frac{\mu_F - r}{\sigma_F}$$

must equal

$$\frac{\mu_G - r}{\sigma_G}.$$

Secondly, let us assume that we take the price process of one particular derivative as given. To be concrete let us fix the "benchmark" claim $\Gamma(X_T)$ above and assume that the pricing function $G(t,x)$, for $\Gamma(X_T)$, is specified exogenously. Then we can compute the market price of risk by the formula

$$\lambda(t,x) = \frac{\mu_G(t,x) - r}{\sigma_G(t,x)}. \tag{9.10}$$

Let us then consider an arbitrary pricing function, say the function F for the claim $\Phi(X_T)$. Since the market price of risk is the same for all derivatives we may now take the expression for λ obtained from (9.10) and insert this into the pricing equation (9.8)–(9.9) for the function F. Now everything in this equation is well specified, so we can (in principle) solve it, in order to obtain F. Thus we see that the price F of an arbitrary claim is indeed uniquely determined by the price G of any exogenously specified benchmark claim.

We can obtain more information from the pricing equation by applying the Feynman–Kač representation. The result can be read off immediately and is as follows.

Proposition 9.3 (Risk neutral valuation) *Assuming absence of arbitrage, the pricing function $F(t,x)$ of the T-claim $\Phi(X_T)$ is given by the formula*

$$F(t,x) = e^{-r(T-t)} E^Q_{t,x}\left[\Phi(X_T)\right]. \tag{9.11}$$

The dynamics of X under the martingale measure Q are given by

$$dX_t = \{\mu(t,X_t) - \lambda(t,X_t)\sigma(t,X_t)\}\, dt + \sigma(t,X_t)\, dW_t^Q,$$

where W^Q is a Q-Wiener process.

9.3 Summing Up

We may now sum up our experiences from the preceding sections.

Result 9.3.1

- In an arbitrage free market, regardless of whether the market is complete or incomplete, there will exist a market price of risk process, λ_t, which is common to all assets in the market. More precisely, let S_t be any price process in the market, with P-dynamics

  $$dS_t = S_t \mu_t^S dt + S_t \sigma_t^S dW_t.$$

  Then the following holds, for all t, and P-a.s.

  $$\mu_t^S - r = \sigma_t^S \lambda_t.$$

- In a complete market the price of any derivative will be **uniquely** determined by the requirement of absence of arbitrage. In hedging terms this means that the price is unique because the derivative can equally well be replaced by its replicating portfolio. Phrased in terms of pricing PDEs and risk neutral valuation formulas, the price is unique because a complete market has the property that the martingale measure Q, or equivalently the market price of risk λ, is uniquely determined within the model.

- In an incomplete market the requirement of no arbitrage is no longer sufficient to determine a unique price for the derivative. We have several possible martingale measures, and several market prices of risk. The reason that there are several possible martingale measures simply means that there are several different price systems for the derivatives, all of which are consistent with absence of arbitrage.

Schematically speaking the price of a derivative is thus determined by two major factors.

- We require that the derivative should be priced in such a way so as to not introduce arbitrage possibilities into the market. This requirement is reflected by the fact that all derivatives must be priced by formula (9.11) where the same Q is used for all derivatives, or equivalently by the pricing PDE (9.8)–(9.9), where the same λ is used for all derivatives.
- In an incomplete market the price is also partly determined, in a nontrivial way, by aggregate supply and demand on the market. Supply and demand for a specific derivative are in turn determined by the aggregate risk aversion on the market, as well as by liquidity considerations and other factors. All these aspects are aggregated into the particular martingale measure used by the market.

When dealing with derivative pricing in an incomplete market we thus have to fix a specific martingale measure Q, or equivalently a λ, and the question arises as to how this is to be done.

Question:

Who chooses the martingale measure?

From the discussions above the answer should by now be fairly clear.

Answer:

The market!

The main implication of this message is that, within our framework, it is **not** the job of the theorist to determine the "correct" market price of risk. The market price of risk is determined on the market, by the agents in the market, and in particular this means that if we assume a particular structure of the market price of risk, then we have implicitly made an assumption about the preferences on the market.

9.4 Exercises

Exercise 9.1 Consider a claim $\Phi(X_T)$ with pricing function $F(t, x)$. Prove that the process $F(t, X_T)/B_t$ is a Q martingale.
Hint: Use Itô's formula on F, using the Q-dynamics of X. Then use the fact that F satisfies the pricing PDE.

Exercise 9.2 Convince yourself that the market price of risk process λ really is of the form

$$\lambda = \lambda(t, X_t).$$

Exercise 9.3 Consider the scalar model in Section 9.2 and a fixed claim $\Gamma(X_T)$. Choose an arbitrary market price of risk of the form $\lambda(t, x)$ and define the pricing function $G(t, x)$ as the solution to the corresponding pricing PDE. Assume that the volatility function $\sigma_G(t, x)$ is non-zero. We now expect the market (B, G) to be complete. Show that this is indeed the case, i.e. show that every simple claim of the form $\Phi(X_T)$ can be replicated by a portfolio based on B and G.

9.5 Notes

See Chapters 29–34 for a much more complete exposition of the theory of incomplete markets.

10

PARITY RELATIONS AND DELTA HEDGING

10.1 Parity Relations

Consider the standard Black–Scholes model. As we know from general theory (Theorem 8.3) this model allows us to replicate any contingent claim using a portfolio based on the underlying asset and the risk free asset. For a non-trivial claim the structure of the hedging portfolio is typically quite complicated, and in particular it is a portfolio which is continuously rebalanced. For practical purposes this continuous rebalancing presents a problem because in real life trading does have a cost. For managerial purposes it would be much nicer if we could replicate a given claim with a portfolio which did not have to be rebalanced, in other words a portfolio which is **constant** over time. Such a portfolio is known as a **buy-and-hold** portfolio.

If we insist on using only B and S in our replicating portfolio we cannot replicate any interesting claims using constant portfolios, but if we allow ourselves to include some derivative, like a European call option, in our hedging portfolio, then life becomes much simpler, and the basic result we will use is the following trivial linear property of pricing.

Proposition 10.1 *Let Φ and Ψ be contract functions for the T-claims $\mathcal{X} = \Phi(S(T))$ and $Y = \Psi(S(T))$. Then for any real numbers α and β we have the following price relation*:

$$\Pi_t[\alpha\Phi + \beta\Psi] = \alpha\Pi_t[\Phi] + \beta\Pi_t[\Psi]\,. \tag{10.1}$$

Proof This follows immediately from the risk neutral valuation formula (7.43) and the linear property of mathematical expectation. □

To set notation let $c(t,s;K,T,r,\sigma)$ and $p(t,s;K,T,r,\sigma)$ denote the price at time t given $S(t) = s$ of a European call option and a European put option respectively. In both cases T denotes the time of maturity, K the strike price, whereas r and σ indicate the dependence on model parameters. From time to time we will freely suppress one or more of the variables (t,s,K,T,r,σ). Let us furthermore consider the following "basic" contract functions:

$$\Phi_S(x) = x, \tag{10.2}$$

$$\Phi_B(x) \equiv 1, \tag{10.3}$$

$$\Phi_{C,K}(x) = \max\,[x - K, 0]\,. \tag{10.4}$$

The corresponding claims at the time of maturity give us one share of the stock, \$1, and one European call with strike price K respectively. For these claims the prices are given by

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

$$\Pi_t[\Phi_S] = S(t), \tag{10.5}$$

$$\Pi_t[\Phi_B] = e^{-r(T-t)}, \tag{10.6}$$

$$\Pi_t[\Phi_{C,K}] = c(t, S(t); K, T). \tag{10.7}$$

Let us now fix a time of maturity T and a T-claim $\mathcal{X}$ of the form $\mathcal{X} = \Phi(S(T))$, i.e. a simple claim. It is now clear that if Φ is a linear combination of the basic contracts above, i.e. if we have

$$\Phi = \alpha\Phi_S + \beta\Phi_B + \sum_{i=1}^{n} \gamma_i \Phi_{C,K_i}, \tag{10.8}$$

then we may price Φ in terms of the prices of the basic contracts as

$$\Pi_t[\Phi] = \alpha\Pi_t[\Phi_S] + \beta\Pi_t[\Phi_B] + \sum_{i=1}^{n} \gamma_i \Pi_t[\Phi_{C,K_i}]. \tag{10.9}$$

Note also that in this case we may replicate the claim Φ using a portfolio consisting of basic contracts that is **constant** over time, i.e. a "buy-and-hold" portfolio. More precisely the replicating portfolio consists of:

- α shares of the underlying stock,
- β zero coupon T-bonds with face value \$1,
- γ_i European call options with strike price K_i, all maturing at T.

The result above is of course interesting only if there is a reasonably large class of contracts which in fact can be written as linear combinations of the basic contracts given by (10.2), (10.3), and (10.4). This is indeed the case, and as a first example we consider the European put option with strike price K, for which the contract function $\Phi_{P,K}$ is defined by

$$\Phi_{P,K}(x) = \max[K - x, 0]. \tag{10.10}$$

It is now easy to see (draw a figure!) that

$$\Phi_{P,K} = K\Phi_B + \Phi_{C,K} - \Phi_S,$$

so we have the following so-called put–call parity relation.

Proposition 10.2 (Put–call parity) *Consider a European call and a European put, both with strike price K and time of maturity T. Denoting the corresponding pricing functions by $c(t,s)$ and $p(t,s)$, we have the following relation*:

$$p(t,s) = Ke^{-r(T-t)} + c(t,s) - s. \tag{10.11}$$

In particular the put option can be replicated with a constant (over time) portfolio consisting of a long position in a zero coupon T-bond with face value K, a long position in a European call option and a short position in one share of the underlying stock.

It is now natural to pose the following more general question. Which contracts can be replicated in this way using a constant portfolio consisting of bonds, call options and the underlying stock? The answer is very pleasing.

Proposition 10.3 *Fix an arbitrary continuous contract function Φ with compact support. Then the corresponding contract can be replicated with arbitrary precision (in sup-norm) using a constant portfolio consisting only of bonds, call options and the underlying stock.*

Proof It is easily seen that any affine function can be written as a linear combination of the basic contract functions. The result now follows from the fact that any continuous function with compact support can be approximated uniformly by a piecewise linear function. □

10.2 The Greeks

Let $P(t, s)$ denote the pricing function at time t for a portfolio based on a **single underlying asset** with price process S_t. The portfolio can thus consist of a position in the underlying asset itself, as well as positions in various options written on the underlying asset. For practical purposes it is often of vital importance to have a grip on the sensitivity of P with respect to the following:

1. Price changes of the underlying asset.
2. Changes in the model parameters.

In case 1 above we want to obtain a measure of our risk exposure, i.e. how the value of our portfolio (consisting of stock and derivatives) will change given a certain change in the underlying price. At first glance case 2 seems self-contradictory, since a model parameter is by definition a given constant, and thus it cannot possibly change within the given model. This case is therefore not one of risk exposure but rather one of sensitivity with respect to misspecifications of the model parameters.

We introduce some standard notation.

Definition 10.4

$$\Delta = \frac{\partial P}{\partial s}, \tag{10.12}$$

$$\Gamma = \frac{\partial^2 P}{\partial s^2}, \tag{10.13}$$

$$\rho = \frac{\partial P}{\partial r}, \tag{10.14}$$

$$\Theta = \frac{\partial P}{\partial t}, \tag{10.15}$$

$$\mathcal{V} = \frac{\partial P}{\partial \sigma}. \tag{10.16}$$

All these sensitivity measures are known as "the Greeks". This includes $\mathcal{V}$, which in this case is given the Anglo-Hellenic pronounciation "vega". A portfolio which is insensitive w.r.t. small changes in one of the parameters above is said to be **neutral**, and formally this means that the corresponding Greek equals zero. A portfolio with zero delta is said to be **delta neutral**, and correspondingly for the other Greeks. In Section 10.3 we will study various hedging schemes, based upon the Greeks, but first we present the basic formulas for the case of a call option. See Figs 10.1–10.5 for graphs of the Greeks as functions of the underlying stock price.

Proposition 10.5 *For a European call with strike price K and time of maturity T we have the following relations, with notation as in the Black–Scholes formula. The letter φ denotes the density function of the $N[0,1]$ distribution.*

$$\Delta = N(d_1), \tag{10.17}$$

$$\Gamma = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}, \tag{10.18}$$

$$\rho = K(T-t)e^{-r(T-t)}N(d_2), \tag{10.19}$$

$$\Theta = -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2), \tag{10.20}$$

$$\mathcal{V} = s\varphi(d_1)\sqrt{T-t}. \tag{10.21}$$

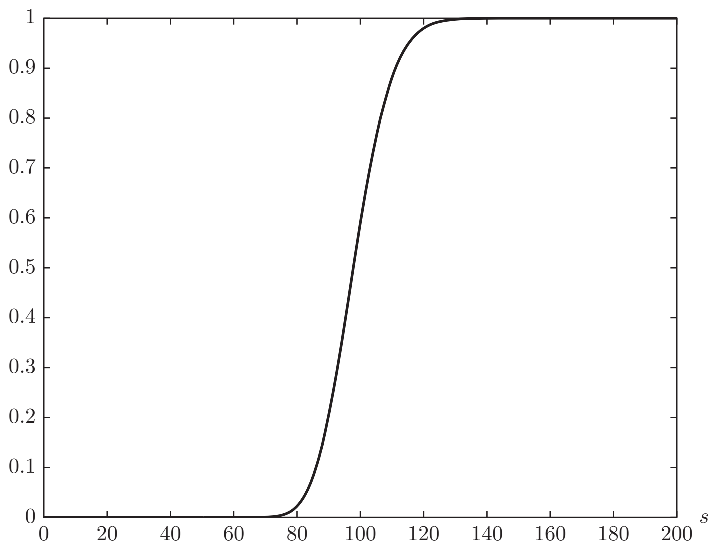

FIG. 10.1. Delta for a European call

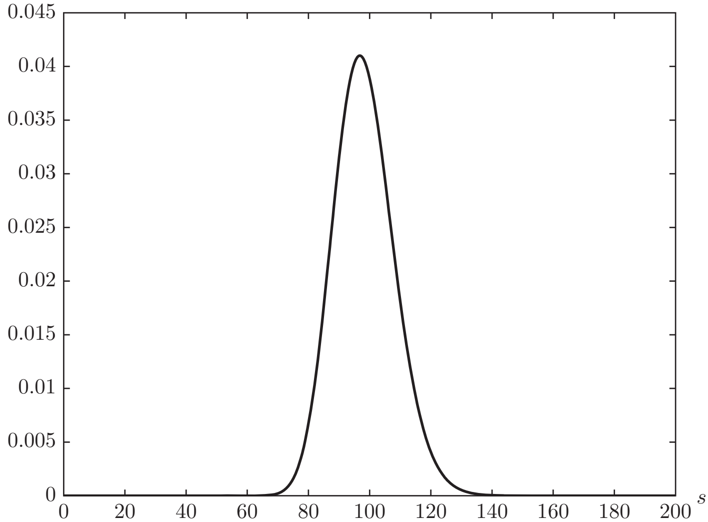

Fig. 10.2. Gamma for a European call

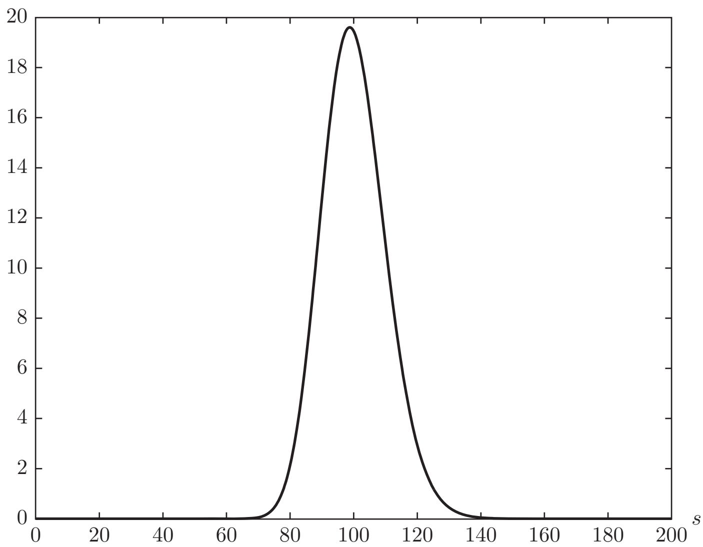

Fig. 10.3. Vega for a European call

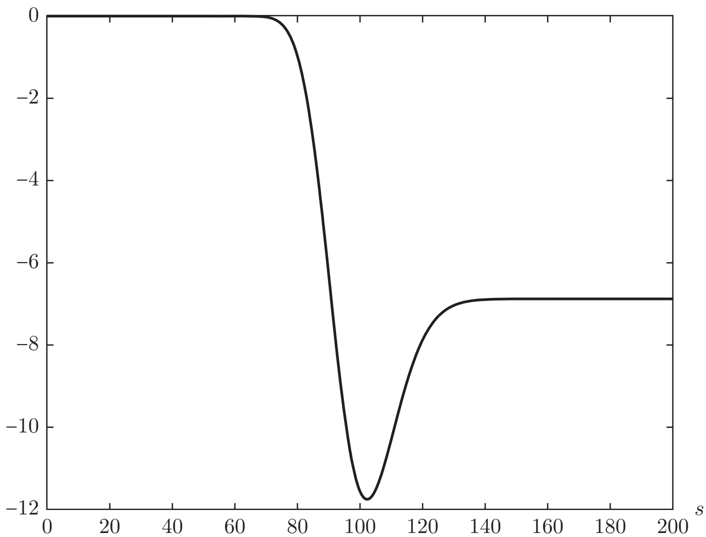

Fig. 10.4. Theta for a European call

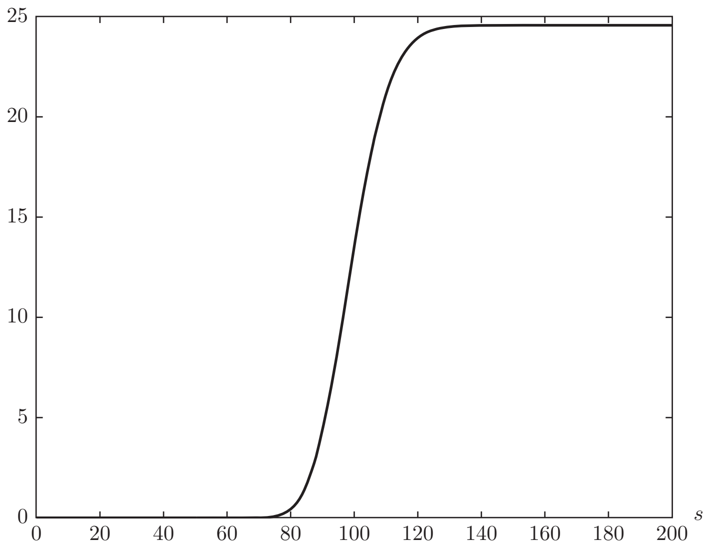

Fig. 10.5. Rho for a European call

Proof Use the Black–Scholes formula (7.52) and take derivatives. The (brave) reader is invited to carry this out in detail. The calculations are sometimes quite messy. □

10.3 Delta and Gamma Hedging

As in Section 10.2, let us consider a given portfolio with pricing function $P(t,s)$. The object is to immunize this portfolio against small changes in the underlying asset price s. If the portfolio already is delta neutral, i.e. if

$$\Delta_P = \frac{\partial P}{\partial s} = 0,$$

then we are done, but what can we do in the more interesting case when $\Delta_P \neq 0$? One possibility is of course to sell the entire portfolio, and invest the sum thus obtained in the bank, but this is in most cases neither practically feasible, nor preferable.

A more interesting idea is to add a derivative (e.g. an option or the underlying asset itself) to the portfolio. Since the price of a derivative is perfectly correlated with the underlying asset price, we should be able to balance the derivative against the portfolio in such a way that the adjusted portfolio becomes delta neutral. The reader will recognize this argument from the derivation of the Black–Scholes PDE, and the formal argument is as follows.

We denote the pricing function of the chosen derivative by $F(t,s)$, and x denotes the number of units of the derivative which we will add to the a priori given portfolio. The value V of the adjusted portfolio is then given by

$$V(t,s) = P(t,s) + x \cdot F(t,s). \tag{10.22}$$

In order to make this portfolio delta neutral we have to choose x such that $\frac{\partial V}{\partial s} = 0$, and this gives us the equation

$$\frac{\partial P}{\partial s} + x\frac{\partial F}{\partial s} = 0,$$

which, with obvious notation, has the solution

$$x = -\frac{\Delta_P}{\Delta_F}. \tag{10.23}$$

Example 10.6 Let us assume that we have sold a particular derivative with pricing function $F(t,s)$, and that we wish to hedge it using the underlying asset itself. In (10.22) we now have $P = -1 \cdot F$, whereas F is replaced by s, and we get the equation

$$\frac{\partial}{\partial s}\left[-F(t,s) + x \cdot s\right] = 0,$$

with the solution

$$x = \Delta_F = \frac{\partial F(t,s)}{\partial s}.$$

We thus see that the delta of a derivative gives us the number of units of the underlying stock that is needed in order to hedge the derivative.

It is important to note that a delta hedge only works well for small changes in the underlying price, and thus only for a short time. In Example 10.6 above, what we did was to approximate the pricing function $F(t,s)$ with its tangent, and in Fig. 10.6 this is illustrated for the case when F is the pricing function of a European call option. Δ_F equals the slope of the tangent. In Fig. 10.1 we have a graph of the delta of a European call, as a function of the underlying stock price. As time goes by the value of s (and t) will change, and thus we will be using an old, incorrect value of Δ. What is done in practice is to perform a **discrete rebalanced delta hedge**, which for the example above can be done along the following lines:

- Sell one unit of the derivative at time $t=0$ at the price $F(0,s)$.
- Compute Δ and buy Δ shares. Use the income from the sale of the derivative, and if necessary borrow money from the bank.
- Wait one day (week, minute, second). The stock price has now changed and your old Δ is no longer correct.
- Compute the new value of Δ and adjust your stock holdings accordingly. Balance your account by borrowing from or lending to the bank.
- Repeat this procedure until the exercise time T.
- In this way the value of your stock and money holdings will approximately equal the value of the derivative.

It is in fact not hard to prove (see the exercises) the following asymptotic result.

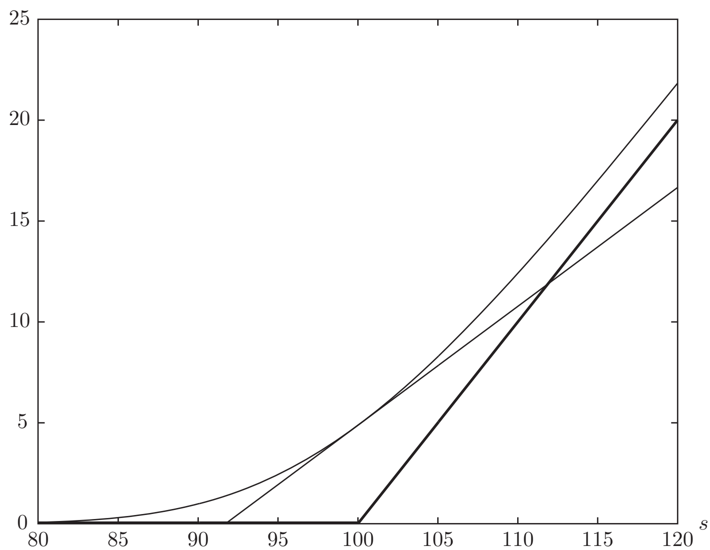

FIG. 10.6. Linear approximation of a European call

Proposition 10.7 *In a continuously rebalanced delta hedge, the value of the stock and money holdings will replicate the value of the derivative.*

In a (discrete) scheme of the kind above we face a dilemma concerning the frequency of the rebalancing points in time. If we rebalance often, we will have a very good hedge, but we will also suffer from high transaction costs. The reason why we have to rebalance is that delta changes as the underlying price changes, and a measure of the sensitivity of Δ with respect to s is of course given by $\Gamma = \frac{\partial \Delta}{\partial s} = \frac{\partial^2 P}{\partial s^2}$. See Fig. 10.2 for a graph of the gamma of a European call. If the gamma is high we have to rebalance often, whereas a low gamma will allow us to keep the delta hedge for a longer period. It is thus preferable to form a portfolio which, apart from being delta neutral, is also **gamma neutral**.

In order to analyze this in some generality, let us again consider an a priori given portfolio with price function $P(t, s)$. For future use we state the following trivial but important facts.

Lemma 10.8 *For the underlying stock, the delta and gamma are given by*

$$\Delta_S = 1,$$
$$\Gamma_S = 0.$$

From the fact that the gamma of the underlying stock equals zero, it follows that we cannot use the stock itself in order to change the gamma of the portfolio. Since we want the adjusted portfolio to be both delta and gamma neutral, it is also obvious that we need two different derivatives in the hedge. Let us thus fix two derivatives, e.g. two call options with different exercise prices or different times of maturity, with pricing functions F and G. We denote the number of units of the derivatives by x_F and x_G respectively, and the value of the hedged portfolio is now given by

$$V = P(t, s) + x_F \cdot F(t, s) + x_G \cdot G(t, s).$$

In order to make this portfolio both delta and gamma neutral we have to choose x_F and x_G such that the equations

$$\frac{\partial V}{\partial s} = 0,$$
$$\frac{\partial^2 V}{\partial s^2} = 0,$$

are satisfied. With obvious notation we thus obtain the system

$$\Delta_P + x_F \cdot \Delta_F + x_G \cdot \Delta_G = 0, \tag{10.24}$$
$$\Gamma_P + x_F \cdot \Gamma_F + x_G \cdot \Gamma_G = 0, \tag{10.25}$$

which can easily be solved.

It is natural, and very tempting, to construct a delta and gamma neutral hedge by the following two step procedure:

1. Choose x_F such that the portfolio consisting of P and F is delta neutral. This portfolio will generally not be gamma neutral.
2. Now add the derivative G in order to make the portfolio gamma neutral.

The problem with this scheme is that the second step in general will destroy the delta neutrality obtained by the first step. In this context we may, however, use the fact that the stock itself has zero gamma and we can thus modify the scheme as follows.

1. Choose x_F such that the portfolio consisting of P and F is gamma neutral. This portfolio will generally not be delta neutral.
2. Now add the underlying stock in order to make the portfolio delta neutral.

Formally the value of the hedged portfolio will now be given by

$$V = P + x_F \cdot F + x_S \cdot s$$

and, using the lemma above, we obtain the following system.

$$\Delta_P + x_F \cdot \Delta_F + x_S = 0, \tag{10.26}$$

$$\Gamma_P + x_F \cdot \Gamma_F = 0. \tag{10.27}$$

This system is triangular, and thus much simpler than the system (10.24)–(10.25). The solution is given by

$$x_F = -\frac{\Gamma_P}{\Gamma_F},$$

$$x_S = \frac{\Delta_F \cdot \Gamma_P}{\Gamma_F} - \Delta_P.$$

Using the technique described above one can easily derive hedging schemes in order to make a given portfolio neutral with respect to any of the Greeks above. This is, however, left to the reader.

10.4 Exercises

Exercise 10.1 Consider the standard Black–Scholes model. Fix the time of maturity T and consider the following T-claim $\mathcal{X}$:

$$\mathcal{X} = \begin{cases} K & \text{if } S(T) \le A \\ K + A - S(T) & \text{if } A < S(T) < K + A \\ 0 & \text{if } S(T) > K + A. \end{cases} \tag{10.28}$$

This contract can be replicated using a portfolio, consisting solely of bonds, stock, and European call options, which is constant over time. Determine this portfolio as well as the arbitrage free price of the contract.

Exercise 10.2 The setup is the same as in the previous exercise. Here the contract is a so-called **straddle**, defined by

$$\mathcal{X} = \begin{cases} K - S(T) & \text{if } 0 < S(T) \leq K \\ S(T) - K & \text{if } K < S(T). \end{cases} \tag{10.29}$$

Determine the constant replicating portfolio as well as the arbitrage free price of the contract.

Exercise 10.3 The setup is the same as in the previous exercises. We will now study a so-called bull spread (see Fig. 10.7). With this contract we can, to a limited extent, take advantage of an increase in the market price while being protected from a decrease. The contract is defined by

$$\mathcal{X} = \begin{cases} B & \text{if } S(T) > B \\ S(T) & \text{if } A \leq S(T) \leq B \\ A & \text{if } S(T) < A. \end{cases} \tag{10.30}$$

We have of course the relation $A < B$. Determine the constant replicating portfolio as well as the arbitrage free price of the contract.

Exercise 10.4 The setup and the problem are the same as in the previous exercises. The contract is defined by

$$\mathcal{X} = \begin{cases} 0 & \text{if } S(T) < A \\ S(T) - A & \text{if } A \leq S(T) \leq B \\ C - S(T) & \text{if } B \leq S(T) \leq C \\ 0 & \text{if } S(T) > C. \end{cases} \tag{10.31}$$

By definition the point C divides the interval $[A, C]$ in the middle, i.e $B = \frac{A+C}{2}$.

Exercise 10.5 Suppose that you have a portfolio P with $\Delta_P = 2$ and $\Gamma_P = 3$. You want to make this portfolio delta and gamma neutral by using two

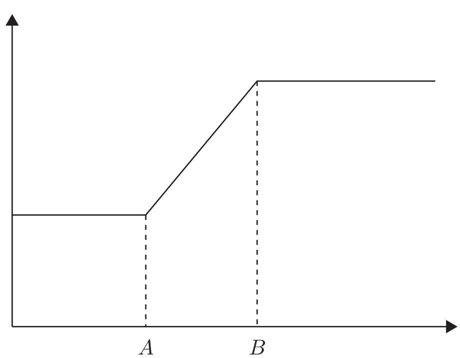

FIG. 10.7. Illustration for Exercise 10.3

derivatives F and G, with $\Delta_F = -1$, $\Gamma_F = 2$, $\Delta_G = 5$ and $\Gamma_G = -2$. Compute the hedge.

Exercise 10.6 Consider the same situation as above, with the difference that now you want to use the underlying S instead of G. Construct the hedge according to the two-step scheme descibed in Section 10.3.

Exercise 10.7 Prove Proposition 10.7 by comparing the stock holdings in the continuously rebalanced portfolio to the replicating portfolio in Theorem 8.5 of the previous chapter.

Exercise 10.8 Consider a self-financing Markovian portfolio (in continuous time) containing various derivatives of the single underlying asset in the Black–Scholes model. Denote the value (pricing function) of the portfolio by $P(t, s)$. Show that the following relation must hold between the various Greeks of P:

$$\Theta_P + rs\Delta_P + \frac{1}{2}\sigma^2 s^2 \Gamma_P = rP.$$

Hint: Use Exercise 8.4.

Exercise 10.9 Use the result in the previous exercise to show that if the portfolio is both delta and gamma neutral, then it replicates the risk free asset, i.e. it has a risk free rate of return which is equal to the short rate r.

Exercise 10.10 Show that for a European put option the delta and gamma are given by

$$\Delta = N[d_1] - 1,$$
$$\Gamma = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}.$$

Hint: Use put–call parity.

Exercise 10.11 Consider the usual portfolio P, and investigate how you can hedge it in order to make it both delta and vega neutral.

11

THE MARTINGALE APPROACH TO ARBITRAGE THEORY

This chapter is the start of the more advanced part of the book, and in order to read most of the following chapters the reader should be familiar with basic measure and integration theory, as well as with measure theoretic based probability theory. A self-contained short introduction to these topics can be found in Appendices A–C.

In this chapter we consider a market model consisting of $N+1$ a priori given asset price processes $S^0, S^1, \ldots, S^N$. Typically we specify the model by giving the dynamics of the asset price processes under the objective probability measure P, and the main problems are as follows.

Fundamental Problems 11.1

1. *Under what conditions is the market arbitrage free?*
2. *Under what conditions is the market complete?*

We attack the fundamental problems above by presenting the "martingale approach" to financial derivatives. This is, so far, the most general approach existing for arbitrage pricing, and it is also extremely efficient from a computational point of view. The answers to the problems above are given by the famous so-called **First** and **Second Fundamental Theorems** of Mathematical Finance, which will be treated below. However; while these results are extremely general and powerful, they are also quite deep, necessarily involving hard results from functional analysis, so at some points we only present the main structural ideas of the proofs. For full proofs the reader is directed to the references in the Notes. The very ambitious reader will read the entire chapter, but there are also two perfectly acceptable alternatives:

1. The reader who does not want to go deeply into the theory can read Sections 11.1–11.2, and then go to the summary of results given in Section 11.9. These results will give you the necessary and sufficient prerequisites for reading the rest of the book.
2. The reader who is unfamiliar with hard functional analysis, but who still wants to get a solid feeling for the Wiener driven theory, is recommended to read Sections 11.1–11.3.1. Then skip the entire Section 11.3.2, apart from reading the statement of Theorem 11.10 while replacing the term "NFLVR" by "No Arbitrage", and ignoring the condition"locally bounded" (this is always satisfied for Wiener-driven models). Then read the rest of the chapter.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

11.1 The Case of Zero Interest Rate

We will start by considering the special case when one of the assets on the market is a risk free asset with zero rate of return. This may sound very restrictive, but we will later show how the general case easily can be reduced to this special case.

As the basic setup we thus consider a financial market consisting of N exogenously given risky traded assets, and the asset price vector is as usual denoted by

$$S_t = \begin{bmatrix} S_t^1 \\ \vdots \\ S_t^N \end{bmatrix} \tag{11.1}$$

We also assume that there exists a risk free asset with price process S_t^0. This will be our numeraire, and in this section we assume that in fact it is constant, i.e. it has zero rate of return.

Assumption 11.1.1 *We assume that*

$$S_t^0 = 1, \quad \textit{for all } t \geq 0. \tag{11.2}$$

The S^0 asset can thus be interpreted as a money account in a bank with zero short rate. In the most general version of the theory, the risky price processes are allowed to be general semimartingales, but for our purposes it will be enough to assume that all price processes possess stochastic differentials with a finite number of driving Wiener processes. Our fundamental problems are to find out under what conditions the market described above is free of arbitrage possibilities, and under what conditions it is complete.

Before starting a formal discussion of this project we have to be a bit more precise about the set of admissible portfolios. Let us on a preliminary basis define a **naive** portfolio process as *any* adapted process $h_t = \left[h_t^0, h_t^1, \ldots, h_t^N\right]$. It then turns out that in order to construct a reasonable theory, the class of naive self-financing portfolios is simply too big, and we have in fact the following strongly negative result.

Theorem 11.1 *If at least one of the assets $S^1, \ldots S^N$ has a diffusion term which is non-zero at all times, and if naive portfolio strategies are admitted, then the model admits arbitrage.*

Proof The idea of the proof is based upon the so-called doubling strategy for the roulette. In this strategy you start by investing one dollar on black. If you win you stop, having won one dollar. If you lose, you bet another two dollars, and if you win in this bet your total gain is again one dollar. If you lose again you bet another four dollars, etc. Thus as soon as you win you stop, and as long as you lose you double your bet. In this way, and as long as the roulette has positive probability for black coming up (it does not have to be evenly balanced), you will eventually (i.e. with probability one) win, and your net profit will be one dollar.

This is an arbitrage on the roulette, and the reason that this does not work well in practice, like in Monte Carlo, is that it requires you to have unlimited credit, since at some points in the game you will have lost an enormous amount of money before eventually winning one dollar. Also, the time spent until you win is a priori unbounded although it is finite with probability one. Thus the probability is high that the sun (and you) has died until you get your dollar.

In real life you do not have unlimited credit, but within our theoretical framework credit **is** unlimited, and it is in fact quite simple to use our market model to imitate the Monte Carlo roulette wheel and the doubling strategy above in *finite time*. If you want the play to be over at $t = 1$ you simply invest at the discrete times $1 - 1/n$; $n = 1, 2, \ldots$. You start by investing one dollar in the risky asset, financing by a bank loan, and then you stop as soon as you gain on the investment and you double your investment as long as you lose (all the time financing by a bank loan). It is then easy to see that you can in fact repeat this arbitrage strategy an infinite number of times on any bounded interval, so with probability one you will become infinitely rich. □

In order to have a reasonable theory we must thus restrict the class of admissible strategies to a smaller class where these doubling strategies are excluded. There are many ways of doing this and a commonly used one is given below. In order to have a compact notation we will use h_t^S as shorthand for the part of the portfolio which is connected to the risky assets, i.e. $h_t^S = \left[h_t^1, \ldots, h_t^N\right]$, and we can thus write the entire portfolio h as $h = \left[h^0, h^S\right]$.

Definition 11.2

- *For any process $h = \left[h^0, h^S\right]$, its* **value process** V_t^h *is defined by*

$$V_t^h = h_t^0 \cdot 1 + \sum_{i=1}^{N} h_t^i S_t^i, \tag{11.3}$$

or in compact form

$$V_t^h = h_t^0 \cdot 1 + h_t^S S_t. \tag{11.4}$$

- *An adapted process h^S is called* **admissible** *if there exists a non-negative real number α (which may depend on the choice of h^S) such that*

$$\int_0^t h_u^S dS_u \geq -\alpha, \quad \textit{for all } t \in [0, T]. \tag{11.5}$$

A process $h^t = \left[h_t^0, h_t^S\right]$, is called an **admissible portfolio** *process if h^S is admissible.*

- *An admissible portfolio is said to be* **self-financing**, *if*

$$V_t^h = V_0^h + \int_0^t h_u^S dS_u, \tag{11.6}$$

i.e. if

$$dV_t^h = h_t^S dS_t. \tag{11.7}$$

Comparing with Definition 6.10, we note that formally the self-financing condition should be

$$dV_t^h = h_t^0 dS_t^0 + h_t^S dS_t,$$

but since in our case $S^0 \equiv 1$, we have $dS_t^0 \equiv 0$ so the self-financing condition reduces to (11.7). This is a simple but important fact, which is highlighted by the following result.

Lemma 11.3 *For any adapted process h^S satisfying the admissibility condition (11.5), and for any real number x, there exists a unique adapted process h^0, such that:*

- *The portfolio h defined by $h = \left[h^0, h^S\right]$ is self-financing.*
- *The value process is given by*

$$V_t^h = x + \int_0^t h_u^S dS_u. \tag{11.8}$$

In particular, the space $\mathcal{K}_0$ of portfolio values, reachable at time T by means of a self-financing portfolio with zero initial cost is given by

$$\mathcal{K}_0 = \left\{ \int_0^T h_t^S dS_t : \ h^S \text{ is admissible} \right\} \tag{11.9}$$

Proof Define h^0 by

$$h_t^0 = x + \int_0^t h_u^S dS_u - h_t^S S_t.$$

Then, by the definition of the value process, we obviously have

$$V_t^h = h_t^0 + h_t^S S_t = x + \int_0^t h_u^S dS_u$$

and from this we obtain directly

$$dV_t^h = h_t^S dS_t,$$

which shows that h is self-financing. The last item is now obvious. □

We stress the fact that the simple characterization of the zero cost reachable claims in (11.9) depends crucially on our assumption that $S^0 \equiv 1$.

11.2 Absence of Arbitrage and Martingale Measures

We consider the market model (11.1) over the finite time interval $[0, T]$, still with the assumption that $S^0 \equiv 1$.

We now give the formal definition of a martingale measure.

Definition 11.4 *A probability measure Q on $\mathcal{F}_T$ is called an* **equivalent martingale measure** *for the market model (11.1), the numeraire S^0, and the time interval $[0,T]$, if it has the following properties:*

- *$Q \sim P$ on $\mathcal{F}_T$, so P and Q are equivalent.*
- *All price processes $S^0, S^1, \ldots, S^N$ are martingales under Q on the time interval $[0,T]$.*

An equivalent martingale measure will often be referred to as just "a martingale measure" or as "an EMM". If $Q \sim P$ has the property that $S^0, S^1, \ldots, S^N$ are local martingales, then Q is called a **local martingale measure***.*

We note that by our assumption above, S^0 is trivially always a martingale. From an informal point of view, the main result of the entire arbitrage theory is the following not very precisely formulated theorem.

Theorem 11.5 (The First Fundamental Theorem)
The model is arbitrage free essentially if and only if there exists a (local) martingale measure Q.

This widely quoted result has the nature of a "Folk Theorem" in the sense that it is known to everyone and that, apart from the diffuse term "essentially", it is correct. Below we will discuss exactly what we mean with "essentially" in the formulation above, and we will also give more exact formulations of it. A full proof of a precise version of the First Fundamental Theorem is very hard and technical. It is to a large extent outside the scope of the book, and you can without serious problems do research in arbitrage theory without having read the full proof. The main ideas, however, are quite simple and straightforward. We will now present these ideas and we will also point out where the technical problems appear. The reader interested in the full story is referred to the Notes.

11.3 A Rough Sketch of the Proof

In this section we will informally discuss the main ideas of the proof of the First Fundamental Theorem, and we will also point out the problems encountered. The proof consists of two parts:

- Existence of an EMM implies absence of arbitrage.
- Absence of arbitrage implies existence of an EMM.

The first part, presented in Section 11.3.1, is rather easy. The second part, presented in Section 11.3.2, is very hard. Depending on the level of ambition, you can, as we noted above, read the entire chapter, or simply skip Section 11.3.2.

11.3.1 *Existence of an EMM Implies Absence of Arbitrage*

This part is in fact surprisingly easy. To see this, let us assume that there does indeed exist a martingale measure Q. In our Wiener-driven world this implies that all price processes have zero drift under Q, i.e. their Q dynamics are of the form

$$dS_t^i = S_t^i \sigma_t^i dW_t^Q, \quad i = 1, \dots, N, \tag{11.10}$$

where W^Q is some multidimensional Q-Wiener process and σ^i is some adapted row vector process.

We now want to prove that there exist no arbitrage possibilities, so we assume that for some self-financing process h, which we for the moment assume to be uniformly bounded, the corresponding value process satisfies the relations

$$P\left(V_T^h \geq 0\right) = 1, \tag{11.11}$$

$$P\left(V_T^h > 0\right) > 0. \tag{11.12}$$

We are thus viewing h as a potential arbitrage portfolio, and in order to prove absence of arbitrage we thus want to show that $V_0^h > 0$.

Since $Q \sim P$ we see that we also have the relations

$$Q\left(V_T^h \geq 0\right) = 1, \tag{11.13}$$

$$Q\left(V_T^h > 0\right) > 0. \tag{11.14}$$

Since h is self-financing we have (remember that $dS^0 = 0$)

$$dV_t^h = \sum_{i=1}^{N} h_t^i S_t^i \sigma_t^i dW_t^Q,$$

and thus (by the boundedness assumptions) we see that V_t^h is a Q-martingale. In particular we then have

$$V_0^h = E^Q\left[V_T^h\right].$$

However, (11.13)–(11.14) imply that $E^Q\left[V_T^h\right] > 0$, so $V(0;h) > 0$. We have thus shown that (11.11)–(11.12) implies $V_0^h > 0$, thereby proving the nonexistence of a bounded arbitrage portfolio.

For the case of a possibly unbounded, but of course still admissible, portfolio we have to resort to more delicate arguments. One can then show that, since the value process is bounded from below it is in fact a supermartingale. Thus

$$V_0^h \geq E^Q\left[V_T^h\right] > 0,$$

and the proof of this part is finished.

11.3.2 *Absence of Arbitrage Implies Existence of an EMM*

This is the really difficult part of the First Fundamental Theorem. It requires several hard results from functional analysis, but the basic ideas are as follows.

In order to avoid integrability problems we assume that all asset price processes are bounded and we interpret "arbitrage" as "bounded arbitrage". We thus assume absence of arbitrage and we want to prove the existence of an EMM, or in more technical terms we would like to prove the existence of a

Radon–Nikodym derivative L on $\mathcal{F}_T$ which will transform the P-measure into a martingale measure Q. Inspired from the simple one period model discussed in Chapter 3 it is natural to look for some sort of convex separation theorem, and to this end we need to put our problem within a more functional analytical setting. Since the Radon–Nikodym derivative L should be in L^1, it is natural to try to utilize duality between L^∞ and L^1 so therefore we define the following sets, with L^1 denoting $L^1(\Omega, \mathcal{F}_T, P)$ and L^∞ denoting $L^\infty(\Omega, \mathcal{F}_T, P)$. (Recall that $\mathcal{K}_0$ is the space of all claims which can be reached by a self-financed portfolio at zero initial cost.)

$$\mathcal{K} = \mathcal{K}_0 \cap L^\infty, \tag{11.15}$$

$$L_+^\infty = \text{the non-negative random variables in } L^\infty, \tag{11.16}$$

$$\mathcal{C} = \mathcal{K} - L_+^\infty. \tag{11.17}$$

The space $\mathcal{K}$ thus consists of all bounded claims which are reachable by a self-financing portfolio at zero initial cost. The set $\mathcal{C}$ are those claims which are dominated by the claims in $\mathcal{K}$, so every claim in $\mathcal{C}$ can be reached by self-financing portfolio with zero initial cost if you also allow yourself to throw away money.

Since we have assumed absence of arbitrage we deduce that

$$\mathcal{C} \cap L_+^\infty = \{0\}. \tag{11.18}$$

Now, both $\mathcal{C}$ and L_+^∞ are convex sets in L^∞ with only one point in common, so at this point (which is the crucial point of the argument, see below) one would like to refer to a convex separation theorem to guarantee the existence of a non-zero random variable $L \in L^1$ such that

$$E^P[LX] \geq 0, \quad \text{for all } X \in L_+^\infty, \tag{11.19}$$

$$E^P[LX] \leq 0, \quad \text{for all } X \in \mathcal{C}. \tag{11.20}$$

Assume for the moment that this part of the argument can be carried out. From (11.19) we can then deduce that in fact $L \geq 0$, and by scaling we can choose L such that $E^P[L] = 1$. We can thus use L as a Radon–Nikodym derivative to define a new measure Q by $dQ = LdP$ on $\mathcal{F}_T$, and Q is now our natural candidate as a martingale measure.

Although the main ideas above are good, there are two hard technical problems which must be dealt with:

- Since L^1 is **not** the dual of L^∞ (in the norm topologies) we cannot use a standard convex separation theorem. An application of a standard Banach space separation theorem would provide us with a linear functional $L \in (L^\infty)^\star$ such that $\langle X, L\rangle \geq 0$ for all $X \in L_+^\infty$ and $\langle X, L\rangle \leq 0$ for all X in $\mathcal{C}$, but since L^1 is strictly included in $(L^\infty)^\star$ we have no guarantee that L can be represented by an element in L^1. We thus need a stronger separation theorem than the standard one.

- Supposing that the duality problem above can be resolved, it remains to prove that L is strictly positive (not only non-negative), since otherwise we may only have $Q << P$ but not $Q \sim P$.

We now move on to a more formal discussion of the various versions the First Fundamental Theorem. For the main proof we follow Delbaen and Schachermayer (1994). This will force us to use some results and concepts from functional analysis which are outside the present text, and the reader is referred to Rudin (1991) for general information. The new ingredients of the full proof are as follows:

- We introduce a variation of the concept of no arbitrage, namely "No Free Lunch with Vanishing Risk".
- In order to obtain a duality between L^1 and L^∞ we consider the weak* topologies instead of the norm topologies.
- We use the Kreps–Yan Separation Theorem.

As a first step, it turns out that the standard definition of absence of arbitrage is a bit too restrictive to allow us to deduce the existence of an EMM, so we need to modify this concept slightly.

Definition 11.6 *With notation as in the previous section, we say that the model admits*

- **No Arbitrage** *(NA) if*
$$\mathcal{C} \cap L_+^\infty = \{0\}, \tag{11.21}$$
- **No Free Lunch with Vanishing Risk** *(NFLVR) if*
$$\bar{\mathcal{C}} \cap L_+^\infty = \{0\}, \tag{11.22}$$
where $\bar{\mathcal{C}}$ denotes the closure of $\mathcal{C}$ in L^∞.

The no arbitrage condition is the same as before, whereas NFLVR is a slightly wider concept. If NFLVR does not hold then there will exist a non-zero claim $X \in L_+^\infty$ and a sequence $X_n \in \mathcal{C}$ such that $|X_n - X| < 1/n$ for all n, so in particular $X_n > -1/n$. Thus, for each n there exists a self-financing (zero initial cost) portfolio generating a claim which is closer then $1/n$ to the arbitrage claim X, while the downside risk is less than $1/n$. This is almost an arbitrage.

As a second step we consider the weak* topology on L^∞ generated by L^1. It is well known (see Rudin (1991)) that with the weak* topology, the dual of L^∞ is L^1 so we are now in a nice position to apply a separation theorem. More precisely we will need the following deep result.

Theorem 11.7 (Kreps–Yan Separation Theorem)
If $\mathcal{C}$ is weak closed, and if*
$$\mathcal{C} \cap L_+^\infty = \{0\},$$

then there exists a random variable $L \in L^1$ such that L is P almost surely strictly positive, and

$$E^P[L \cdot X] \leq 0, \quad \text{for all } X \in \mathcal{C}.$$

Proof For a proof and references see Schachermayer (1994). □

We are now almost in business, and we see that in order for the Kreps–Yan Theorem to work we need to assume No Arbitrage and we also need assumptions which guarantee that $\mathcal{C}$ is weak* closed. Happily enough we have the following surprising result from Delbaen–Schachermayer (1994) which shows that the closedness of $\mathcal{C}$ in fact follows from NFLVR. The proof is very hard and therefore omitted.

Proposition 11.8 *If the asset price processes are uniformly bounded, then the condition NFLVR implies that $\mathcal{C}$ is weak* closed.*

We can now state and prove the main result.

Theorem 11.9 (First Fundamental Theorem) *Assume that the asset price process S is bounded. Then there exists an equivalent martingale measure if and only if the model satisfies NFLVR.*

Proof The *only if* part is the easy one, and the proof is already given in Section 11.3. Before going on we recall the definitions of $\mathcal{K}$ and $\mathcal{C}$ and from (11.15)–(11.17). For the *if* part we assume NFLVR. This implies that $\mathcal{C}$ is weak* closed and it also (trivially) implies No Arbitrage, i.e. $\mathcal{C} \cap L_+^\infty = \{0\}$. We may thus apply the Kreps–Yan Separation to deduce the existence of a random variable $L \in L^1$ such that L is P almost surely strictly positive, and

$$E^P[L \cdot X] \leq 0, \quad \text{for all } X \in \mathcal{C}. \tag{11.23}$$

By scaling we can choose L such that $E^P[L] = 1$. We may thus use L as a Radon–Nikodym derivative to define a new measure Q by $dQ = LdP$ on $\mathcal{F}_T$, and Q is now our natural candidate as a martingale measure. It follows from (11.23) and the definition of $\mathcal{K}$ that $E^P[LX] \leq 0$ for all $X \in \mathcal{K}$. Since $\mathcal{K}$ is a linear subspace this implies that in fact $E^Q[X] = E^P[LX] = 0$ for all $X \in \mathcal{K}$. In order to prove the martingale property of S^i for a fixed i, we choose fixed s and t with $s \leq t$, as well as an arbitrary event $A \in \mathcal{F}_s$. Now consider the following self-financing portfolio strategy:

- Start with zero wealth and do nothing until time s.
- At time s buy I_A units of asset No. i. Finance this by a loan in the bank.
- At time t sell the holdings of asset No. i and repay the loan. Put any surplus in the bank and keep it there until time T.

Since the short rate equals zero, the initial loan (at time s) in the bank is payed back (at time t) by the same amount, so at time t the value of our portfolio is given by $V_t^h = I_A\left[S_t^i - S_s^i\right]$. Since the short rate equals zero this will also be the value of our portfolio at time T. Thus we have $I_A\left(S_t^i - S_s^i\right) \in \mathcal{K}$ so we must have $E^Q\left[I_A\left(S_t^i - S_s^i\right)\right] = 0$, and since this holds for all s, t and $A \in \mathcal{F}_s$ we have proved that S^i is a Q martingale. $\square$

In most applications, the assumption of a bounded S process is far too restrictive. The Delbaen–Schachermayer Theorem can, however, easily be extended to a more general case.

Theorem 11.10 *Assume that the asset price process S is locally bounded. Then there exists an equivalent local martingale measure if and only if the model satisfies NFLVR.*

Remark 11.3.1 *We note that in particular the result above will hold if the S process has continuous trajectories. It will also hold for an S process with jumps as long as the jumps are bounded. The case of an S process which is not locally bounded such as for example a process with lognormally distributed jumps at exponentially distributed times is much more difficult, and in such a case NFLVR is only equivalent to the existence of an equivalent measure Q such that S becomes a so-called sigma-martingale under Q. See the Notes for references.*

11.4 The General Case

We now relax the assumption that $S^0 \equiv 1$, and go on to consider a market model consisting of the price processes

$$S^0, S^1, \ldots, S^N,$$

where we make the following assumption.

Assumption 11.4.1 *We assume that $S_t^0 > 0$ P-a.s. for all $t \geq 0$.*

The main problem is to give conditions for absence of arbitrage in this model, and these are easily obtained by moving to the "normalized" economy where we use S^0 as a numeraire.

Thus, instead of looking at the price vector process $S = \left[S^0, S^1, \ldots, S^N\right]$ we look at the *relative price vector process* S_t/S_t^0, where we have used S^0 as the numeraire price. This object will be studied in more detail in Chapter 15.

Definition 11.11 *The* **normalized economy** *(also referred to as the "Z-economy") is defined by the price vector process Z, where*

$$Z_t = \frac{S_t}{S_t^0},$$

i.e.

$$Z_t = \left[Z_t^0, \ldots, Z_t^N\right] = \left[1, \frac{S_t^1}{S_t^0}, \frac{S_t^2}{S_t^0}, \ldots, \frac{S_t^N}{S_t^0}\right]. \tag{11.24}$$

The point of this is that in the Z economy we have a risk free asset $Z^0 \equiv 1$, with zero rate of return, so the simple idea is to apply the results from the previous sections to the Z economy.

Note, however, that we have now two price systems to keep track of: The S-system and the Z-system, and before going on we have to clarify the relations between these systems. In particular, for any portfolio process h there will be associated two value processes, one in the S system and one in the Z system, and we thus need to introduce some notation.

Definition 11.12

- *A* **portfolio strategy** *is any adapted* $(N+1)$*-dimensional process*

$$h_t = \left[h_t^0, h_t^1, \ldots, h_t^N\right].$$

- *The* **S-value process** V_t^S *corresponding to the portfolio* h *is given by*

$$V_t^S = \sum_{i=0}^{N} h_t^i S_t^i. \tag{11.25}$$

- *The* **Z-value process** V_t^Z *corresponding to the portfolio* h *is given by*

$$V_t^Z = \sum_{i=0}^{N} h_t^i Z_t^i. \tag{11.26}$$

- *A portfolio is said to be* **admissible** *if it is admissible in the sense of Definition 11.2 as a* Z *portfolio.*
- *An admissible portfolio is said to be* **S-self-financing** *if*

$$dV_t^S = \sum_{i=0}^{N} h_t^i dS_t^i. \tag{11.27}$$

- *An admissible portfolio is said to be* **Z-self-financing** *if*

$$dV_t^Z = \sum_{i=0}^{N} h_t^i dZ_t^i. \tag{11.28}$$

We can also make the obvious definitions of a given T-claim being S-reachable and Z-reachable respectively.

The intuitive feeling is that the concept of a self-financing portfolio should not depend upon the particular choice of numeraire. That this is indeed the case is shown by the following "Invariance Lemma".

Lemma 11.13 (Invariance Lemma)
With assumptions and notation as above, the following hold.

(i) A portfolio h *is* S*-self-financing if and only if it is* Z*-self-financing.*

(ii) The value processes V^S and V^Z are connected by

$$V_t^Z = \frac{1}{S_t^0} \cdot V_t^S.$$

(iii) A claim $\mathcal{Y}$ is S-replicable if and only if the claim

$$\frac{\mathcal{Y}}{S_T^0}$$

is Z-replicable.

- *The model is S arbitrage free if and only if it is Z arbitrage free.*

Proof Items (ii) and (iii) are obvious. Thus it only remains to prove the self-financing result, and for simplicity we assume that all processes possess stochastic differentials driven by a finite number of Wiener processes. Assume therefore that the portfolio h is S-self-financing. Denoting the scalar product between vectors by the "scalar dot" $\cdot$, using the notation $\beta_t = S_t^0$, and we have from this assumption that

$$Z_t = \beta_t^{-1} S_t, \tag{11.29}$$

$$V_t^S = h_t \cdot S_t, \tag{11.30}$$

$$V_t^Z = \beta_t^{-1} V_t^S, \tag{11.31}$$

$$dV_t^S = h_t \cdot dS_t. \tag{11.32}$$

We now want to prove that in fact

$$dV_t^Z = h_t \cdot dZ_t.$$

Using the Itô formula on $Z = \beta^{-1} S$, we thus want to prove that

$$dV_t^Z = \beta_t^{-1} h_t \cdot dS_t + h_t \cdot S_t d\beta_t^{-1} + h_t \cdot dS_t d\beta_t^{-1}. \tag{11.33}$$

Now, from (11.31) we have

$$dV_t^Z = \beta_t^{-1} dV_t^S + V_t^S d\beta_t^{-1} + d\beta_t^{-1} dV_t^S.$$

Substituting (11.30) and (11.32) into this equation gives

$$dV_t^Z = \beta_t^{-1} h_t \cdot dS_t + h_t \cdot S_t d\beta_t^{-1} + d\beta_t^{-1} h_t \cdot dS_t,$$

which is what we wanted to prove. □

We may now formulate and prove the main result concerning absence of arbitrage.

Theorem 11.14 (The First Fundamental Theorem)
Consider the market model $S^0, S^1, \ldots, S^N$ where we assume that $S_t^0 > 0$, P-a.s. for all $t \geq 0$. Assume furthermore that $S^0, S^1, \ldots, S^N$ are locally bounded. Then the following conditions are equivalent:

- *The model satisfies NFLVR.*
- *There exists a measure $Q \sim P$ such that the processes*

$$Z^0, Z^1, \dots, Z^N,$$

defined through (11.24), are local martingales under Q.

Proof This follows directly from the Invariance Lemma and from Theorem 11.10. □

Remark 11.4.1 *It is important to note that the martingale measure Q above is connected to the particular choice of numeraire S^0, and that different numeraires will produce different martingale measures. A more precise notation for the martingale measure should therefore perhaps be Q^0, in order to emphasize the dependence upon the chosen numeraire.*

From now on we will, for a given numeraire, use the term "martingale measure" to denote the (not necessarily unique) local martingale measure of Theorem 11.14.

11.5 Completeness

In this section we assume absence of arbitrage, i.e. we assume that there exists a (local) martingale measure. We now turn to the possibility of replicating a given contingent claim in terms of a portfolio based on the underlying assets. This problem is most conveniently carried out in terms of normalized prices, and we have the following useful lemma, which shows that hedging is equivalent to the existence of a stochastic integral representation of the normalized claim.

Lemma 11.15 *Consider a given T-claim X. Fix a martingale measure Q and assume that the normalized claim $X/S^0(T)$ is integrable. If the Q-martingale M, defined by*

$$M_t = E^Q \left[\left. \frac{X}{S_T^0} \right| \mathcal{F}_t \right], \tag{11.34}$$

admits an integral representation of the form

$$M_t = x + \sum_{i=1}^{N} \int_0^t h_s^i dZ_s^i, \tag{11.35}$$

then X can be hedged in the S-economy. Furthermore, the replicating portfolio $(h^0, h^1, \cdots, h^N)$ is given by (11.35) for $(h^1, \cdots, h^K)$, whereas h^0 is given by $h_t^0 = M_t - \sum_{i=1}^N h_t^i Z_t^i$.

Proof We want to hedge X in the S economy, i.e. we want to hedge $X/S^0(T)$ in the Z economy. In terms of normalized prices, and using the Invariance Lemma, we are thus looking for a process $h = (h^0, h^1, \cdots, h^N)$ such that

$$V_T^Z = \frac{X}{S_T^0}, \ P - a.s. \tag{11.36}$$

$$dV_t^Z = \sum_{i=1}^{N} h_t^i dZ_t^i, \tag{11.37}$$

where the normalized value process is given by

$$V_t^Z = h_t^0 \cdot 1 + \sum_{i=1}^{N} h_t^i Z_t^i. \tag{11.38}$$

A reasonable guess is that $M = V^Z$, so we let M be defined by (11.34). Furthermore we define $(h^1, \cdots, h^N)$ by (11.35), and we define h^0 by

$$h_t^0 = M_t - \sum_{i=1}^{N} h_t^i Z_t^i. \tag{11.39}$$

Now, from (11.38) we obviously have $M = V^Z$, and from (11.35) we get

$$dV_t^Z = dM_t = \sum_{i=1}^{N} h_t^i dZ_t^i,$$

which shows that the portfolio is self-financing. Furthermore we have

$$V_T^Z = M_T = E^Q\left[\left.\frac{X}{S_T^0}\right| \mathcal{F}_T\right] = \frac{X}{S_T^0},$$

which shows that X is replicated by h. □

We thus see that, modulo some integrability considerations, completeness is equivalent to the existence of a Martingale Representation Theorem for the discounted price process. Thus we may draw on the deep results of Jacod (1979) from semimartingale theory which connect martingale representation properties for Z with the extremal points of the set of martingale measures.

Theorem 11.16 (Jacod) *Let $\mathcal{M}$ denote the (convex) set of equivalent martingale measures. Then, for any fixed $Q \in \mathcal{M}$, the following statements are equivalent*:

- *Every Q local martingale M has dynamics of the form*

$$dM_t = \sum_{i=1}^{N} h_t^i dZ_t^i.$$

- *Q is an extremal point of $\mathcal{M}$.*

We then have the second fundamental theorem of mathematical finance.

Theorem 11.17 (The Second Fundamental Theorem) *Assume that the market is arbitrage free and consider a fixed numeraire asset S^0. Then the market is complete if and only if the martingale measure Q, corresponding to the numeraire S^0, is unique.*

Proof If the martingale measure Q is unique then $\mathcal{M}$ is a singleton $\mathcal{M} = \{Q\}$ so Q is trivially an extremal point of $\mathcal{M}$. Thus the Jacod Theorem provides us with a stochastic integral representation of every Q martingale, and it then follows from Lemma 11.15 that the model is complete. The other implication follows easily from (11.61) of Proposition 11.27 below. $\square$

Remark 11.5.1 *The reader may find the proof given above rather abstract, and we provide two alternatives:*

- *A more functional analytic proof of the Second Fundamental Theorem would be roughly as follows: The market is complete if and only of the set $\mathcal{C}$ of reachable claims at zero initial cost has codimension one, i.e. if*

$$L^\infty = \mathcal{C} \oplus R \cdot Y$$

for some $Y \in L^\infty$. This implies that the separating hyperplane implied by the Kreps–Yan Theorem 11.7 is unique and thus that the martingale measure is unique.
- *In Section 14.3 we provide a self-contained and complete proof of the Second Fundamental Theorem for the special case of purely Wiener-driven models.*

11.6 Pricing Contingent Claims

We now turn to the pricing problem for contingent claims. In order to do this, we consider the "primary" market $S^0, S^1, \ldots, S^N$ as given a priori, and we fix a T-claim X. Our task is that of determining a "reasonable" price process $\Pi_t[X]$ for X, and we assume that the primary market is arbitrage free. There are two main approaches:

- The derivative should be priced in a way that is **consistent** with the prices of the underlying assets. More precisely we should demand that the extended market $(\Pi\,[X], S^0, S^1, \cdots, S^N)$ is free of arbitrage possibilities.
- If the claim can be replicated, with hedging portfolio h, then the only reasonable price is given by $\Pi_t[X] = V_t^h$.

In the first approach above, we thus demand that there should exist a martingale measure Q for the extended market $(\Pi[X], S^0, S^1, \cdots, S^N)$. This obviously also implies that Q is a martingale measure for the underlying market $(S^0, S^1, \cdots, S^N)$. Letting Q denote such a measure, assuming enough integrability, and applying the definition of a martingale measure we obtain

$$\frac{\Pi_t[X]}{S_t^0} = E^Q\left[\left.\frac{\Pi_T[X]}{S_T^0}\right|\mathcal{F}_t\right] = E^Q\left[\left.\frac{X}{S_T^0}\right|\mathcal{F}_t\right]. \tag{11.40}$$

We thus have the following result.

Theorem 11.18 (General Pricing Formula) *The arbitrage free price process for the T-claim X is given by*

$$\Pi_t[X] = S_t^0 E^Q \left[\frac{X}{S_T^0} \middle| \mathcal{F}_t \right], \tag{11.41}$$

where Q is the (not necessarily unique) martingale measure for the a priori given market $S^0, S^1, \cdots, S^N$, with S^0 as the numeraire.

Note that different choices of Q will generically give rise to different price processes.

In particular we note that if we assume that if S^0 is the bank account

$$S_t^0 = S_0^0 \cdot e^{\int_0^t r(s)ds},$$

where r is the short rate, then (11.41) reduced to the familiar "risk neutral valuation formula".

Theorem 11.19 (Risk Neutral Valuation Formula)
Assuming the existence of a short rate, the pricing formula takes the form

$$\Pi_t[X] = E^Q \left[e^{-\int_t^T r(s)ds} X \middle| \mathcal{F}_t \right] \tag{11.42}$$

where Q is a (not necessarily unique) martingale measure with the money account as the numeraire.

An important particular case occurs when we study zero coupon bonds. Bond markets and interest rate theory will be studied in detail in Chapters 19–24, but we will need some elementary concepts and results already at this point.

Definition 11.20 *A* **zero coupon bond** *with* **maturity date** T*, also called a T-bond, is a contract which guarantees the holder* 1 *dollar to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $p(t,T)$.*

A T-bond is thus a derivative on the trivial T-claim $X \equiv 1$, and we have the following obvious result.

Proposition 11.21 *The price of a zero coupon T-bond is given by*

$$p(t,T) = E^Q \left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right], \tag{11.43}$$

and in particular we have

$$p(T,T) = 1, \quad \textit{for all} \;\; T \geq 0. \tag{11.44}$$

11.7 Pricing by Replication

For the second approach to pricing let us assume that X can be replicated by h. Since the holding of the derivative contract and the holding of the replicating portfolio are equivalent from a financial point of view, we see that price of the derivative must be given by the formula

$$\Pi_t[X] = V_t^h. \tag{11.45}$$

One problem here is what will happen in a case when X can be replicated by two different portfolios, and one would also like to know how this formula is connected to (11.41).

Defining $\Pi_t[X]$ by (11.45) see that $\Pi_t[X]/S_t^0 = V_t^Z$ and since, assuming enough integrability, V^Z is a Q-martingale, we see that also $\Pi_t[X]/S_t^0$ is a Q-martingale. Thus we again obtain the formula (11.41) and for a attainable claim we have in particular the formula

$$V_t^h = S_t^0 E^Q \left[\left. \frac{X}{S_T^0} \right| \mathcal{F}_t \right], \tag{11.46}$$

which will hold for any replicating portfolio and for any martingale measure Q. Thus we see that the two pricing approaches above do in fact coincide on the set of attainable claims. In Section 11.9 we will summarize our results.

11.8 Stochastic Discount Factors

In the previous sections we have seen that we can price a contingent T-claim X by using the formula

$$\Pi_t[X] = E^Q \left[\left. e^{-\int_t^T r_s ds} X \right| \mathcal{F}_t \right], \tag{11.47}$$

where Q is a martingale measure with the money account as a numeraire. In some applications of the theory (in particular in asset pricing) it is common to write this expected value directly under the objective probability measure P instead of under Q. This can easily be obtained by using the likelihood process L, where as usual L is defined on the interval $[0, T]$ through

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t. \tag{11.48}$$

We can now write (11.47), for $t = 0$, as

$$\Pi_0[X] = E^P \left[e^{-\int_0^T r_s ds} L_T X \right],$$

which naturally leads us to the following definition.

Definition 11.22 *Assume the existence of a short rate r. For any fixed martingale measure Q, let the likelihood process L be defined by (11.48). The* **stochastic discount factor** *(SDF) process $\mathbf{M}$, corresponding to Q, is defined as*

$$\mathbf{M}_t = e^{-\int_0^t r(s) ds} L_t, \tag{11.49}$$

i.e. as

$$\mathbf{M}_t = \frac{1}{B_t} \cdot L_t. \tag{11.50}$$

We thus see that there is a one-to-one correspondence between martingale measures and stochastic discount factors. We have now more or less proved the following result.

Proposition 11.23 *Assume absence of arbitrage. With notation as above, the following hold*:

- *For any sufficiently integrable T-claim X, the arbitrage free price is given by*

$$\Pi_t[X] = E^P\left[\left.\frac{\mathbf{M}_T}{\mathbf{M}_t}X\right|\mathcal{F}_t\right]. \tag{11.51}$$

- *For any arbitrage free asset price process S(derivative or underlying) the process*

$$\mathbf{M}_t S_t \tag{11.52}$$

is a (local) P-martingale.

- *The P-dynamics of $\mathbf{M}$ are given by*

$$d\mathbf{M}_t = -r_t\mathbf{M}_t dt + \frac{1}{B_t}dL_t. \tag{11.53}$$

Proof From the usual risk neutral pricing formula, the Abstract Bayes' Formula (Theorem B.41), and the fact that L is P-martingale we obtain

$$\Pi_t[X] = E^Q\left[\left.e^{-\int_t^T r_s ds}X\right|\mathcal{F}_t\right] = \frac{E^Q\left[\left.e^{-\int_t^T r_s ds}L_T X\right|\mathcal{F}_t\right]}{E^Q\left[L_T\middle|\mathcal{F}_t\right]}$$
$$= E^Q\left[\left.e^{-\int_t^T r_s ds}\frac{L_T}{L_t}X\right|\mathcal{F}_t\right] = E^P\left[\left.\frac{\mathbf{M}_T}{\mathbf{M}_t}X\right|\mathcal{F}_t\right],$$

which proves (11.51). The martingale property of $\mathbf{M}_t S_t$ follows from Bayes' Formula, and the remaining details of the proof are left to the reader. □

From the discussion above it is clear that SDFs and martingale measures are logically equivalent objects. It is thus up to the individual researcher to decide which language to use—that of SDFs or that of EMMs—and it is often convenient to be able to switch from one to the other.

Experience seems to indicate (at least to the author of this book) that in equilibrium theory (see Chapters 35–38) the preferable concept is that of stochastic discount factors, while the formalism of martingale measures is preferable in the context of pricing financial derivatives.

An alternative approach to SDFs is to **define** an SDF as any non-negative random process $\mathbf{M}$ possessing the property that $S_t\mathbf{M}_t$ is a (local) P-martingale for every asset price process S. The First Fundamental Theorem can then be restated as the equivalence between absence of arbitrage and the existence of a stochastic discount factor.

11.9 Summary for the Working Economist

In this section we summarize the results for the martingale approach. We consider a market model consisting of the asset price processes $S^0, S^1, \ldots, S^N$ on the time interval $[0,T]$. The "numeraire process" S^0 is assumed to be strictly positive. Modulo some technicalities we then have the following results. The first provides conditions for absence of arbitrage.

Theorem 11.24 (First Fundamental Theorem)
The market model is free of arbitrage if and only if there exists a **martingale measure**, *i.e. a measure $Q \sim P$ such that the processes*

$$\frac{S_t^0}{S_t^0}, \frac{S_t^1}{S_t^0}, \ldots, \frac{S_t^N}{S_t^0}$$

are (local) martingales under Q.

For the case when the numeraire is the money account we have an alternative characterization of a martingale measure. The proof is a simple application of the Itô formula.

Proposition 11.25 *If the numeraire S^0 is the money account, i.e.*

$$S_t^0 = e^{\int_0^t r_s ds},$$

where r is the (possibly stochastic) short rate, and if we assume that all processes are Wiener driven, then a measure $Q \sim P$ is a martingale measure if and only if all assets $S^0, S^1, \ldots, S^N$ have the short rate as their local rates of return, i.e. if the Q-dynamics are of the form

$$dS_t^i = S_t^i r_t dt + S_t^i \sigma_t^i dW_t^Q, \tag{11.54}$$

where W^Q is a (multidimensional) Q-Wiener process.

The second result gives us conditions for market completeness.

Theorem 11.26 (Second Fundamental Theorem)
Assuming absence of arbitrage, the market model is complete if and only if the martingale measure Q is unique.

As far as pricing of contingent claims is concerned the theory can be summarized as follows.

Proposition 11.27

1. *In order to avoid arbitrage, X must be priced according to the formula*

$$\Pi_t[X] = S_t^0 E^Q \left[\left. \frac{X}{S_T^0} \right| \mathcal{F}_t \right], \tag{11.55}$$

where Q is a martingale measure for $\left[S^0, S^1, \cdots, S^N\right]$, with S^0 as the numeraire.

2. *In particular, we can choose the bank account B_t as the numeraire. Then B has the dynamics*

$$dB_t = r_t B_t dt, \tag{11.56}$$

where r is the (possibly stochastic) short rate process. In this case the pricing formula above reduces to

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r(s)ds} X \middle| \mathcal{F}_t\right]. \tag{11.57}$$

3. *As a special case, the price of a zero coupon T-bond is given by*

$$p(t,T) = E^Q\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t\right]. \tag{11.58}$$

4. *Defining the stochastic discount factor* $\mathbf{M}$ *by* $\mathbf{M}_t = B_t^{-1}L_t$ *we also have the pricing formula.*

$$\Pi_t[X] = E^P\left[\frac{\mathbf{M}_T}{\mathbf{M}_t} X \middle| \mathcal{F}_t\right]. \tag{11.59}$$

5. *Different choices of Q will generically give rise to different price processes for a fixed claim X. However, if X is attainable then all choices of Q will produce the same price process, which then is given by*

$$\Pi_t[X] = V_t^h, \tag{11.60}$$

where h is the hedging portfolio. Different choices of hedging portfolios (if such exist) will produce the same price process.

6. *In particular, for every replicable claim X it holds that*

$$V_t^h = E^Q\left[e^{-\int_t^T r_s ds} X \middle| \mathcal{F}_t\right]. \tag{11.61}$$

Summing up we see that in a complete market the price of any derivative will be **uniquely** determined by the requirement of absence of arbitrage. The price is unique precisely because the derivative is in a sense superfluous—it can equally well be replaced by its replicating portfolio. In particular we see that the price does not depend on any assumptions made about the risk-preferences of the agents in the market. The agents can have any attitude towards risk, as long as they prefer more (deterministic) money to less.

In an incomplete market the requirement of no arbitrage is no longer sufficient to determine a unique price for the derivative. We have several martingale measures, all of which can be used to price derivatives in a way consistent with no arbitrage. The question which martingale measure one should use for pricing has a very simple answer: The martingale measure is chosen by the market.

Schematically speaking the price of a derivative is thus determined by two major factors:

1. We require that the derivative should be priced in such a way as to not introduce arbitrage possibilities into the market. This requirement is reflected by the fact that all derivatives must be priced by formula (11.55) where the same Q is used for all derivatives.
2. In an incomplete market the price is also partly determined by aggregate supply and demand on the market. Supply and demand for a specific derivative are in turn determined by the aggregate risk aversion on the market, as well as by liquidity considerations and other factors. All these

aspects are aggregated into the particular martingale measure used by the market.

Let us now assume that we have specified some model under the objective probability measure P. This means that we have specified the P-dynamics of all asset prices in the primary market. We may also have specified the P-dynamics of some processes which are not price processes, like the inflation rate, the unemployment rate, or the outside temperature (which influences the demand for electric energy).

In order to be able to apply the theory developed above, it is then clear that we need the following tools:

- We need to have full control of the class of equivalent measure transformations that can be made from a given objective measure P.
- Given an equivalent measure Q (a potential martingale measure), we must be able to write down the Q-dynamics of all processes under consideration.
- We need theorems which allow us to write certain stochastic variables (typically contingent claims) as stochastic integrals of some given processes (typically normalized asset prices).

All these tools are in fact provided by the following mathematical results which are the objects under study in the next chapter.

- The Martingale Representation Theorem for Wiener processes.
- The Girsanov Theorem.

11.10 Notes

The martingale approach to arbitrage pricing was developed in Harrison and Kreps (1979), Kreps (1981), and Harrison and Pliska (1981). It was then extended by, among others, Duffie and Huang (1986), Delbaen (1992), Schachermayer (1994), and Delbaen and Schachermayer (1994). In this chapter we follow closely Delbaen and Schachermayer (1994) for the case of locally bounded price processes. The general case of unbounded price processes and its connection to sigma-martingales was finally resolved in Delbaen and Schachermayer (1998), which also contains further bibliographic information on this subject. Rudin (1991) is a standard reference on functional analysis, which is also treated in Royden (1988).

Stochastic discount factors are treated in Duffie (2001), and in most modern textbooks on asset pricing such as Cochrane (2001).

12

THE MATHEMATICS OF THE MARTINGALE APPROACH

In this chapter we will present the two main workhorses of the martingale approach to arbitrage theory. These are:

- The Martingale Representation Theorem, which shows that in a Wiener world every martingale can be written as a stochastic integral w.r.t. the underlying Wiener process.
- The Girsanov Theorem, which gives us complete control of all absolutely continuous measure transformations in a Wiener world.

12.1 Stochastic Integral Representations

Let us consider a fixed time interval $[0,T]$, a probability space $(\Omega, \mathcal{F}, P)$, with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and an adapted vector Wiener process $W = (W^1, \dots, W^d)^\star$. Now fix a vector process $h = (h^1, \dots, h^d)$ which is "integrable enough" (for example $h \in \mathcal{L}^2$ is enough) and a real number x_0. If we now define the process M by

$$M_t = x_0 + \sum_{i=1}^{d} \int_0^t h_s^i dW_s^i, \quad t \in [0,T], \tag{12.1}$$

then we know that M is a martingale. In other words: under mild integrability conditions, every stochastic integral w.r.t a Wiener process is an $\mathcal{F}_t$-martingale. A very natural and important question is now whether the converse holds, i.e. if every $\mathcal{F}_t$-adapted martingale M can be written in the form (12.1). If this is indeed the case, then we say that M has a **stochastic integral representation** w.r.t the Wiener process W.

It is not hard to see that in the completely general case, there is no hope for a stochastic integral representation w.r.t W for a general martingale M. As a counter example, let W be scalar (i.e. $d = 1$) and consider, apart from W, also a Poisson process N, with constant intensity λ, where N is independent of W. Now define the filtration by $\mathcal{F}_t = \mathcal{F}_t^{W,N}$, i.e. $\mathcal{F}_t$ contains all the information generated by W and N over the interval $[0,t]$.

It is now very easy to see that the process M defined by

$$M_t = N_t - \lambda t,$$

is an $\mathcal{F}_t$-martingale. If we look at the trajectories of M, they consist of straight lines with downward slope λ, interrupted at exponentially distributed points in time by positive jumps of unit size. From this it is obvious that M can possess no stochastic integral representation of the form (12.1), since any such

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

representation implies that M has continuous trajectories. The intuitive reason is of course that since M is independent of W, we cannot use W in order to represent M.

From this example it is clear that we can only hope for a stochastic integral representation result in the case when $\{\mathcal{F}_t\}_{t \geq 0}$ is the *internal filtration* generated by the Wiener process W itself. We start with the following basic representation for Wiener functionals, which in turn will give us our martingale representation result.

Theorem 12.1 (Representation of Wiener Functionals)
Let W be a d dimensional Wiener process, and let X be a random variable such that

- $X \in \mathcal{F}_T^W$,
- $E[|X|] < \infty$.

Then there exist uniquely determined $\mathcal{F}_t^W$-adapted processes $h^1, \ldots, h^d$, such that X has the representation

$$X = E[X] + \sum_{i=1}^{d} \int_0^T h_s^i dW_s^i. \tag{12.2}$$

Under the additional assumption

$$E\left[X^2\right] < \infty,$$

then $h^1, \ldots, h^d$ are in $\mathcal{L}^2$.

Proof We only give the proof for the L^2 case, where we present the main ideas of a particularly nice proof from Steele (2001). For notational simplicity we only consider the scalar case.

We start by recalling that the GBM equation

$$\begin{aligned} dX_t &= \sigma X_t dW_t, \\ X_0 &= 1, \end{aligned}$$

has the solution

$$X_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}. \tag{12.3}$$

Writing the SDE on integral form as

$$X_t = 1 + \int_0^t \sigma X_s dW_s \tag{12.4}$$

and plugging (12.3) into (12.4) we obtain, after some reshuffling of terms,

$$e^{\sigma W_t} = e^{\frac{1}{2}\sigma^2 t} + \sigma \int_0^t e^{-\frac{1}{2}\sigma^2 (u-t) + \sigma W_u} dW_u.$$

Using the same argument we easily obtain, for $s \leq t$,

$$e^{\sigma(W_t - W_s)} = e^{\frac{1}{2}\sigma^2 (t-s)} + \sigma \int_s^t e^{-\frac{1}{2}\sigma^2 (u-t+s) + \sigma W_u} dW_u, \tag{12.5}$$

where the important point is that the integral is only over the interval $[s,t]$. Thus any random variable Z of the form

$$Z = \exp\{\sigma(W_t - W_s)\}$$

will have a representation of the form

$$Z = E[Z] + \int_0^T h_u dW_u,$$

where $h \equiv 0$ outside $[s,t]$. From this it follows easily (see the exercises) that any variable Z of the form

$$Z = \prod_{k=1}^{n} \exp\left\{\sigma_k \left(W_{t_k} - W_{t_{k-1}}\right)\right\} \tag{12.6}$$

where $0 \le t_0 \le t_1 \le \ldots \le t_n \le T$, has a representation of the form

$$Z = E[Z] + \int_0^T h_u dW_u. \tag{12.7}$$

It is now fairly straightforward to see that also any variable of the form

$$Z = \prod_{k=1}^{n} \exp\left\{i\sigma_k \left(W_{t_k} - W_{t_{k-1}}\right)\right\} \tag{12.8}$$

where i is the imaginary unit, has a representation of the form (12.7). At this point we may use Fourier techniques to see that the set of variables of the form (12.8) is dense in the (complex) space $L^2(\mathcal{F}_T)$, and from this one can deduce that in fact every $Z \in L^2(\mathcal{F}_T)$ has a representation of the form (12.7). See Steele (2001) for the details. □

From this result we now easily obtain the Martingale Representation Theorem.

Theorem 12.2 (The Martingale Representation Theorem)
Let W be a d-dimensional Wiener process, and assume that the filtration $\mathbf{F}$ *is defined as*

$$\mathcal{F}_t = \mathcal{F}_t^W, \quad t \in [0,T].$$

Let M be any $\mathcal{F}_t$-adapted martingale. Then there exist uniquely determined $\mathcal{F}_t$-adapted processes $h^1, \ldots, h^d$ such that M has the representation

$$M_t = M_0 + \sum_{i=1}^{d} \int_0^t h_s^i dW_s^i, \quad t \in [0,T]. \tag{12.9}$$

If the martingale M is square integrable, then $h^1, \ldots, h^d$ are in $\mathcal{L}^2$.

Proof From Theorem 12.1 we have

$$M_T = M_0 + \sum_{i=1}^{d} \int_0^T h_s^i dW_s^i.$$

The result now follows by taking conditional expectations and using the fact that M as well as the stochastic integral is a martingale. □

It is worth noticing that the Martingale Representation Theorem above is an abstract existence result. It guarantees the existence of the processes $h^1, \dots, h^d$, but it does not tell us what the h process looks like. In fact, in the general case we know very little about what exact form of h. The most precise description of h obtained so far is via the so-called Clarc-Ocone formula, but that requires the use and language of Malliavin calculus so it is outside the present text.

In one special case, however, we have a rather explicit description of the integrand h. Let us therefore assume that we have some *a priori* given n-dimensional process X with dynamics of the form

$$dX_t = \mu_t dt + \sigma_t dW_t, \tag{12.10}$$

where W is as above, whereas μ and σ are adapted processes taking values in R^n and $M(n,d)$ respectively. Let us now assume that the martingale M is of the very particular form $M_t = f(t, X_t)$ for some deterministic smooth function $f(t,x)$. From the Itô formula we then have

$$df(t,X_t) = \left\{ \frac{\partial f}{\partial t}(t,X_t) + \mathcal{A}f(t,X_t) \right\} dt + (\nabla_x f)(t,X_t)\sigma_t dW_t$$

where $\mathcal{A}$ is the usual Itô operator. Now, since $f(t,X_t)$ was assumed to be a martingale, the drift must vanish, so in fact we have

$$df(t,X_t) = (\nabla_x f)(t,X_t)\sigma_t dW_t.$$

Written out in more detail this becomes

$$df(t,X_t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(t,X_t)\sigma_t^i dW_t^i,$$

where σ^i is the $i:th$ row of σ. In this particular case we thus have the explicit description of the integrand h as

$$h_t^i = \frac{\partial f}{\partial x^i}(t,X_t)\sigma_t^i, \quad i = 1, \dots d.$$

12.2 The Girsanov Theorem: Heuristics

We now start a discussion of the effect that an absolutely continuous measure transformation will have upon a Wiener process. This discussion will lead us to the Girsanov Theorem which is the central result of the next section.

Assume therefore that our space $(\Omega, \mathcal{F}, P, \mathbf{F})$ carries a scalar P-Wiener process W, and that for some fixed T we have changed to a new measure Q on $\mathcal{F}_T$ by choosing a non-negative random variable $L_T \in \mathcal{F}_T$ and defining Q by

$$dQ = L_T dP, \quad \text{on } \mathcal{F}_T.$$

This measure transformation will generate a likelihood process (see Section C.3) $\{L_t;\ t \geq 0\}$ defined by

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t,$$

and from Proposition C.12 we know that L is a P-martingale.

Since L is a non-negative P-martingale, and since any (suitably integrable) stochastic integral w.r.t. W is a martingale, it is natural to define L as the solution of the SDE

$$dL_t = \varphi_t L_t dW_t, \tag{12.11}$$

$$L_0 = 1 \tag{12.12}$$

for some choice of the process φ.

It thus seems that we can generate a large class of natural measure transformations from P to a new measure Q by the following prescription:

- Choose an arbitrary adapted process φ.
- Define a likelihood process L by

$$dL_t = \varphi_t L_t dW_t, \tag{12.13}$$

$$L_0 = 1. \tag{12.14}$$

- Define a new measure Q by setting

$$dQ = L_t dP, \tag{12.15}$$

  on $\mathcal{F}_t$ for all $t \in [0,T]$.

By applying the Itô formula we easily see that we can express L as

$$L_t = e^{\int_0^t \varphi_s dW_s - \frac{1}{2}\int_0^t \varphi_s^2 ds},$$

so L is non-negative, which is necessary if it is going to act as a likelihood process. If φ is integrable enough (see the Novikov condition below) it is also clear that L is a martingale and the initial condition $L_0 = 1$ guarantees that $E^P[L_T] = 1$.

To see what the dynamics of W are under Q, let us first recall that if a process X has the dynamics

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

then the drift μ and (squared) diffusion σ has the interpretation of being the conditional drift and quadratic variation processes respectively. A bit more precisely, but still heuristically, we have

$$E^P[dX_t | \mathcal{F}_t] = \mu_t dt,$$

$$E^P\left[(dX_t)^2 \middle| \mathcal{F}_t\right] = \sigma_t^2 dt,$$

where we have the informal interpretation $dX_t = X_{t+dt} - X_t$. Let us now define the process X by $X = W^p$, i.e. we have $\mu = 0$ and $\sigma = 1$ under P. Our task is to compute the drift and diffusion under Q and for that we will use the Abstract

Bayes' Theorem B.41. Using the fact that L is a P-martingale, and recalling that $dX_t \in \mathcal{F}_{t+dt}$ (see definition above), we obtain

$$E^Q\left[dX_t\right|\mathcal{F}_t] = \frac{E^P\left[L_{t+dt}dX_t\right|\mathcal{F}_t]}{E^P\left[L_{t+dt}\right|\mathcal{F}_t]} = \frac{E^P\left[L_{t+dt}dX_t\right|\mathcal{F}_t]}{L_t}$$
$$= \frac{E^P\left[L_t dX_t + dL_t dX_t\right|\mathcal{F}_t]}{L_t} = \frac{E^P\left[L_t dX_t\right|\mathcal{F}_t]}{L_t} + \frac{E^P\left[dL_t dX_t\right|\mathcal{F}_t]}{L_t}.$$

Since L is adapted (so $L_t \in \mathcal{F}_t$) and X has zero drift under P, we have

$$\frac{E^P\left[L_t dX_t\right|\mathcal{F}_t]}{L_t} = L_t \cdot \frac{E^P\left[dX_t\right|\mathcal{F}_t]}{L_t} = E^P\left[dX_t\right|\mathcal{F}_t] = 0 \cdot dt.$$

Furthermore we have

$$dL_t dX_t = L_t \varphi_t dW_t \left(0 \cdot dt + 1 \cdot dW_t\right) = L_t \varphi_t \left(dW_t\right)^2 = L_t \varphi_t dt.$$

Using this and the fact that $L_t \varphi_t \in \mathcal{F}_t$ we get

$$\frac{E^P\left[dL_t dX_t\right|\mathcal{F}_t]}{L_t} = \frac{L_t \varphi_t}{L_t} dt = \varphi_t dt.$$

Using the fact that under P we have $dX_t^2 = dt$ we can also easily compute the quadratic variation of X under Q as

$$E^Q\left[\left(dX_t\right)^2\Big|\mathcal{F}_t\right] = \frac{E^P\left[L_{t+dt} \cdot \left(dX_t\right)^2\Big|\mathcal{F}_t\right]}{L_t} = \frac{E^P\left[L_{t+dt} \cdot dt\right|\mathcal{F}_t]}{L_t}$$
$$= \frac{E^P\left[L_{t+dt}\right|\mathcal{F}_t]}{L_t} dt = \frac{L_t}{L_t} dt = dt.$$

Summing up we have thus obtained the formal relations

$$E^Q\left[dX_t\right|\mathcal{F}_t] = \varphi_t dt,$$
$$E^Q\left[\left(dX_t\right)^2\Big|\mathcal{F}_t\right] = 1 \cdot dt,$$

or in other words:

- The process $X = W$ was, under P, a standard Wiener process with unit diffusion term and zero drift.
- Under the probability measure Q defined above, the drift process of X has changed from zero to φ, while the diffusion term remains the same as under P (i.e. unit diffusion).

12.3 The Girsanov Theorem

Rephrasing the results of the previous discussion, we thus see that we should be able to write the P-Wiener process W as

$$dW_t = \varphi_t dt + dW_t^Q,$$

where W^Q is a Q-Wiener process. This is precisely the content of the Girsanov Theorem, which we now formulate.

Theorem 12.3 (The Girsanov Theorem) *Let W be a d-dimensional standard P-Wiener process on $(\Omega, \mathcal{F}, P, \mathbf{F})$ and let φ be any d-dimensional adapted column vector process. Choose a fixed T and define the process L on $[0,T]$ by*

$$dL_t = \varphi_t^\star L_t dW_t, \tag{12.16}$$

$$L_0 = 1, \tag{12.17}$$

i.e.

$$L_t = e^{\int_0^t \varphi_s^\star dW_s - \frac{1}{2}\int_0^t \|\varphi_s\|^2 ds}.$$

Assume that

$$E^P[L_T] = 1, \tag{12.18}$$

and define the new probability measure Q on $\mathcal{F}_T$ by

$$L_T = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_T. \tag{12.19}$$

Then

$$dW_t = \varphi_t dt + dW_t^Q, \tag{12.20}$$

where W^Q is a Q-Wiener process.

Remark 12.3.1 *An equivalent, but perhaps less suggestive, way of formulating the conclusion (12.20) of the Girsanov Theorem is to say that the process W^Q, defined by*

$$W_t^Q = W_t - \int_0^t \varphi_s ds \tag{12.21}$$

is a standard Q-Wiener process.

Proof We only give the proof in the scalar case, the multidimensional case being a straightforward extension. Using the formulation in Remark 12.3.1 we thus have to show that, for $s < t$ and under Q, the increment $W_t^Q - W_s^Q$ is independent of $\mathcal{F}_s$, and normally distributed with zero mean and variance $t-s$. We start by considering the special case when $s = 0$ and we thus want to show that, for any t, W_t^Q is normal with zero mean and variance t under Q. Using characteristic functions it is thus enough to show that for all $t \in R_+$ and $u \in R$ we have

$$E^Q\left[e^{iuW_t^Q}\right] = e^{-\frac{u^2}{2}t},$$

i.e.

$$E^P\left[L_t \cdot e^{iuW_t^Q}\right] = e^{-\frac{u^2}{2}t}.$$

To show this, let us choose any fixed u, and define the process Z by

$$Z_t = L_t \cdot e^{iuW_t^Q}.$$

The dynamics of Z are given by

$$dZ_t = L_t \cdot d\left(e^{iuW_t^Q}\right) + e^{iuW_t^Q} dL_t + d\left(e^{iuW_t^Q}\right) \cdot dL_t. \tag{12.22}$$

From the definitions we have

$$\begin{aligned} dL_t &= \varphi_t L_t dW_t, \\ dW_t^Q &= dW_t - \varphi_t dt, \end{aligned}$$

so, remembering that W is P-Wiener, the Itô formula gives us

$$\begin{aligned} d\left(e^{iuW_t^Q}\right) &= iue^{iuW_t^Q} dW_t^Q - \frac{u^2}{2} e^{iuW_t^Q} \left(dW_t^Q\right)^2 \\ &= iue^{iuW_t^Q} dW_t - iue^{iuW_t^Q} \varphi_t dt - \frac{u^2}{2} e^{iuW_t^Q} dt. \end{aligned}$$

Plugging this, and the L-dynamics above into (12.22), we obtain

$$\begin{aligned} dZ_t &= iuZ_t dW_t - iuZ_t\varphi_t dt - \frac{u^2}{2} Z_t dt + \varphi_t Z_t dW_t + iu\varphi_t Z_t dt \\ &= \{iuZ_t + \varphi_t Z_t\} dW_t - \frac{u^2}{2} Z_t dt. \end{aligned}$$

Since W is P-Wiener, standard technique gives us

$$E^P[Z_t] = e^{-\frac{u^2}{2} \cdot t},$$

which finishes the proof in the special case when $s = 0$.

In the general case we want to prove that for any $s \le t$

$$E^Q\left[e^{iu(W_t^Q - W_s^Q)} \middle| \mathcal{F}_s \right] = e^{-\frac{u^2}{2} \cdot (t-s)}$$

and this is equivalent (why?) to proving that

$$E^Q\left[I_A \cdot e^{iu(W_t^Q - W_s^Q)} \right] = Q(A) e^{-\frac{u^2}{2} \cdot (t-s)}, \tag{12.23}$$

for every $A \in \mathcal{F}_s$. To prove (12.23) we define, for fixed s and $A \in \mathcal{F}_s$, the process $\{Z_t; t \ge s\}$ by

$$Z_t = L_t \cdot I_A \cdot e^{iu(W_t^Q - W_s^Q)},$$

and then we can proceed exactly as above. □

Remark 12.3.2 *The process φ above will often be referred to as the* **Girsanov kernel** *of the measure transformation.*

Remark 12.3.3 *In the formulation above we have used vector notation. Written on component form, and with obvious notation, the L dynamics will have the form*

$$dL_t = L_t \sum_{i=1}^{d} \varphi_t^i dW_t^i,$$

and L will have the explicit form

$$L_t = \exp\left\{ \sum_{i=1}^{d} \int_0^t \varphi_s^i dW_s^i - \frac{1}{2} \int_0^t \sum_{i=1}^{d} (\varphi_s^i)^2 ds \right\}.$$

The conclusion of the Girsanov Theorem is then that we can write

$$dW_t^i = \varphi_t^i dt + dW_t^{Qi}, \quad i = 1, \ldots, d, \tag{12.24}$$

where $W^{Q1}, \ldots, W^{Qd}$ are independent standard Wiener processes under Q.

Since the process L above is so important it has a name of its own:

Definition 12.4 *For any Wiener process W and any kernel process φ, the* **Doleans exponential** *process $\mathcal{E}$ is defined by*

$$\mathcal{E}\left(\varphi \bullet W\right)_t = \exp\left\{ \int_0^t \varphi_s^\star dW_s - \frac{1}{2} \int_0^t \|\varphi\|_s^2 ds \right\}. \tag{12.25}$$

With notation as above we thus have

$$L_t = \mathcal{E}\left(\varphi \bullet W\right)_t. \tag{12.26}$$

Remark 12.3.4 *Note that in the Girsanov Theorem we have to assume ad hoc that φ is such that $E^P[L_T] = 1$ or, in other words, that L is a martingale. The problem is one of integrability on the process $L\varphi$, since otherwise we have no guarantee that L will be a true martingale and in the general case it could in fact happen that $E^P[L_T] < 1$. A sufficient condition for $E^P[L_T] = 1$ is of course that the process $L \cdot \varphi$ is in $\pounds^2$ but it is not easy to give a general* **a priori** *condition on φ only, which guarantees the martingale property of L. This problem used to occupy a minor industry, and the most general result so far is the "Novikov Condition" below.*

Lemma 12.5 (The Novikov Condition) *Assume that the Girsanov kernel φ is such that*

$$E^P\left[e^{\frac{1}{2}\int_0^T \|\varphi_t\|^2 dt}\right] < \infty. \tag{12.27}$$

Then L is a martingale and in particular $E^P[L_T] = 1$.

There are counter examples which show that the exponent $\frac{1}{2}$ in the Novikov condition cannot be improved.

12.4 The Converse of the Girsanov Theorem

If we start with a measure P and perform a Girsanov transformation, according to (12.16)–(12.21), to define a new measure Q, then we know that $Q << P$. A natural question to ask is now whether **all** absolutely continuous measure transformations are obtained in this way, i.e. by means of a Girsanov transformation.

It is clear that in a completely general situation, this cannot possibly be true, since the Girsanov transformation above is completely defined in terms of the Wiener process W whereas there could be many other processes living on $(\Omega, \mathcal{F}, P, \mathbf{F})$. However, in the case where the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is the one generated by the Wiener process itself, i.e. in the case when we have no other sources of randomness apart from W, then we have a converse result of the Girsanov Theorem.

Theorem 12.6 (The Converse of the Girsanov Theorem) *Let W be a d-dimensional standard (i.e. zero drift and unit variance independent components) P-Wiener process on $(\Omega, \mathcal{F}, P, \mathbf{F})$ and assume that*

$$\mathcal{F}_t = \mathcal{F}_t^W, \forall t.$$

Assume that there exists a probability measure Q such that $Q << P$ on $\mathcal{F}_T$. Then there exists an adapted process φ such that the likelihood process L has the dynamics

$$\begin{aligned} dL_t &= L_t \varphi_t^\star dW_t, \\ L_0 &= 1. \end{aligned}$$

Proof We know from Theorem C.12 that the likelihood process L is a P martingale. Since the filtration is the one generated by W we deduce from the Martingale Representation Theorem 12.2 that there exists a process g such that

$$dL_t = g_t^\star dW_t.$$

Now we simply define φ by

$$\varphi_t = \frac{1}{L_t} \cdot g_t$$

and the proof is basically finished. There remains a small problem, namely what happens when $L_t = 0$ but also this can be handled and we omit it. $\square$

This converse result is very good news, since it implies that for the case of a Wiener filtration we have complete control of the class of absolutely continuous measure transformations.

12.5 Girsanov Transformations and Stochastic Differentials

We will now discuss the effect that a Girsanov transformation has on the dynamics of a more general Itô process. Suppose therefore that, under the original measure P, we have a process X with P-dynamics

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

where W is a (possible multidimentional) standard P-Wiener process, and where μ and σ are adapted and suitably integrable. Suppose furthermore that we perform a Girsanov transformation with kernel process φ and transform from P to a new measure Q. The problem is to find out what the Q dynamics of X look like.

This problem is easily solved, since from the Girsanov Theorem we know that we can write

$$dW_t = \varphi_t dt + dW_t^Q$$

where W^Q is Q-Wiener. We now simply plug this expression into the X dynamics above, collect the dt-terms and obtain

$$dX_t = \{\mu_t + \sigma_t \varphi_t\} dt + \sigma_t dW_t^Q dt.$$

The moral of this is as follows:

- The diffusion term is unchanged.
- The drift term is changed from μ to $\mu + \sigma\varphi$.

12.6 Maximum Likelihood Estimation

In this section we give a brief introduction to maximum likelihood (ML) estimation for Itô processes. It is a bit outside the main scope of the book, but since ML theory is such an important topic and we already have developed most of the necessary machinery, we include it.

We need the concept of a statistical model.

Definition 12.7 *A dynamic* **statistical model** *over a finite time interval* $[0, T]$ *consists of the following objects.*

- *A measurable space* $(\Omega, \mathcal{F})$.
- *A flow of information on the space, formalized by a filtration* $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$.
- *An indexed family of probability measures* $\{P_\alpha;\ \alpha \in A\}$, *defined on the space* $(\Omega, \mathcal{F})$, *where* A *is some index set and where all measures are assumed to be absolutely continuous on* $\mathcal{F}_T$ *w.r.t. some base measure* P_{α_0}.

In most concrete applications (see examples below) the parameter α will be a real number or a finite dimensional vector, i.e. A will be the real line or some finite dimensional Euclidian space. The filtration will typically be generated by some observation process X.

The interpretation of all this is that the probability distribution is governed by some measure P_α, but we do no know which. We do have, however, access to a flow of information over time, and this is formalized by the filtration above, so at time t we have the information contained in $\mathcal{F}_t$. Our problem is to try to estimate α given this flow of observations, or more precisely: for every t we want an estimate α_t of α, based upon the information contained in $\mathcal{F}_t$, i.e. based on the observations over the time interval $[0, t]$. The last requirement is formalized by requiring that the estimation process should be adapted to $\underline{\mathcal{F}}$, i.e. that $\alpha_t \in \mathcal{F}_t$.

One of the most common techniques used in this context is that of finding, for each t, the **maximum likelihood** estimate of α. Formally the procedure works as follows.

- Compute, for each α the corresponding likelihood process L^α defined by

$$L_t^\alpha = \frac{dP_\alpha}{dP_{\alpha_0}}, \quad \text{on } \mathcal{F}_t.$$

- For each fixed t, find the value of α which maximizes the likelihood L_t^α.
- The optimal α is denoted by $\hat{\alpha}_t$ and is called the **maximum likelihood estimate** of α based on the information gathered over $[0,t]$.

As the simplest possible example let us consider the problem of estimating the constant but unknown drift of a scalar Wiener process. In elementary terms we could naively formulate the model by saying that we can observe a process X with dynamics given by

$$\begin{aligned} dX_t &= \alpha dt + dW_t, \\ X_0 &= 0. \end{aligned}$$

Here W is assumed to be Wiener under some given measure P and the drift α is some unknown real number. Since this example is so simple, we do in fact have an obvious candidate (why?) for the estimator process, namely

$$\widehat{\alpha}_t = \frac{X_t}{t}.$$

In a naive formulation like this, we have a single underlying Wiener process, W under a single given probability measure P, and we see that for different choices of α we have different X-processes. In order to apply the ML techniques we must reformulate our problem, so that we instead have a single X process and a family of measures. This is done as follows:

- Fix a process X which is Wiener under some probability measure P_0. In other words, under P_0, the process X has the dynamics

$$dX_t = 0 \cdot dt + dW_t^0,$$

  where W^0 is P_0-Wiener.
- We assume that the information flow is the one generated by observations of X, so we define the filtration by setting $\mathcal{F}_t = \mathcal{F}_t^X$. For every real number α, we then define a Girsanov transformation to a new measure P_α by defining the likelihood process $L(\alpha)$ through

$$dL_t^\alpha = \alpha L_t^\alpha dX_t, \tag{12.28}$$
$$L_0^\alpha = 1. \tag{12.29}$$

- From Girsanov's Theorem it now follows immediately that we can write $dW_t^0 = \alpha dt + dW_t^\alpha$, where W^α is a P_α Wiener process. Thus X will have the P_α dynamics

$$dX_t = \alpha dt + dW_t^\alpha.$$

We now have a statistical model along the general lines above, and we notice that, as opposed to the case in the naive formulation, we have a single process X, but the driving Wiener processes are different for different values of α.

To obtain the ML estimation process for α, we need to compute the likelihood process explicitly, i.e. we have to solve (12.28)–(12.29). This is easily done and we have

$$L_t^{\alpha} = e^{\alpha \cdot X_t - \frac{1}{2}\alpha^2 \cdot t}.$$

We may of course maximize $\ln[L_t(\alpha)]$ instead of maximize L_t^{α} so our problem is to maximize (over α) the expression

$$\alpha \cdot X_t - \frac{1}{2}\alpha^2 \cdot t.$$

This trivial quadratic optimization problem can be solved by setting the α derivative equal to zero, and we obtain the optimal α as

$$\widehat{\alpha}_t = \frac{X_t}{t}.$$

Thus we see that in this example the ML estimator actually coincides with our naive guess above. The point of using the ML technique is of course that in a more complicated situation (see the exercises) we may have no naive candidate, whereas the ML technique in principle is always applicable.

12.7 Exercises

Exercise 12.1 Complete an argument in the proof of Theorem 12.1 by proving that if X and Y are random variables of the form

$$X = x_0 + \int_0^T g_s dW_s,$$
$$Y = y_0 + \int_0^T h_s dW_s,$$

and if g and h have disjoint support on the time axis, i.e. if

$$g_t h_t = 0, \quad P - a.s. \quad 0 \le t \le T$$

then

$$XY = x_0 y_0 + \int_0^T [X_s h_s + Y_s g_s] \, dW_s.$$

Hint: Define the processes X_t and Y_t by $X_t = x_0 + \int_0^t g_s dW_s$ and correspondingly for Y and use the Itô formula.

Exercise 12.2 Consider the following SDE:

$$dX_t = \alpha f(X_t)dt + \sigma(X_t)dW_t,$$
$$X_0 = x_0.$$

Here f and σ are known functions, whereas α is an unknown parameter. We assume that the SDE possesses a unique solution for every fixed choice of α.

Construct a dynamical statistical model for this problem and compute the ML estimator process $\hat{\alpha}_t$ for α, based upon observations of X.

12.8 Notes

The results in this chapter can be found in any textbook on stochastic analysis such as Karatzas and Shreve (2008), Øksendal (1998), and Steele (2001).

13

BLACK–SCHOLES FROM A MARTINGALE POINT OF VIEW

In this chaper we will discuss the standard Black–Scholes model from the martingale point of view. We thus choose a probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$ carrying a P-Wiener proces W, where the filtration $\mathbf{F}$ is the one generated by W, i.e. $\mathcal{F}_t = \mathcal{F}_t^W$. On this space we define the model by

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{13.1}$$

$$dB_t = r B_t dt. \tag{13.2}$$

13.1 Absence of Arbitrage

We now want to see whether the model is arbitrage free on a finite interval $[0, T]$, and for that purpose we use the First Fundamental Theorem 11.24 which says that we have absence of arbitrage if and only if there exists a martingale measure Q for our model. We then use the Girsanov Theorem to look for a Girsanov kernel process φ such that the induced measure Q is a martingale measure. Defining, as usual, the likelihood process L by

$$dL_t = \varphi_t L_t dW_t,$$

and setting $dQ = L_T dP$ on $\mathcal{F}_T$, we know from Girsanov's Theorem that

$$dW_t = \varphi_t dt + dW_t^Q,$$

where W^Q is Q-Wiener. Inserting the above expression into the stock price dynamics we obtain, after a collection of terms, the Q-dynamics of S as

$$dS_t = S_t \{\mu + \sigma \varphi_t\} dt + \sigma S_t dW_t^Q.$$

In order for Q to be a martingale measure, we know from (11.54) that the local rate of return under Q must equal the short rate. Thus we want to determine the process φ such that

$$\mu + \sigma \varphi_t = r. \tag{13.3}$$

This equation has the simple solution

$$\varphi_t = -\frac{\mu - r}{\sigma},$$

and we see that the Girsanov kernel process φ is in fact deterministic and constant.

Furthermore, φ has an important economic interpretation: In the quotient

$$\frac{\mu - r}{\sigma},$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

the numerator $\mu - r$, commonly known as the "risk premium" of the stock, denotes the excess rate return of the stock over the risk free rate of return on the market. In the denominator we have the volatility of the stock, so the quotient above has an interpretation as "risk premium per unit volatility" or "risk premium per unit risk". This important concept will be discussed in some detail later on, and it is known in the literature as "the market price of risk". It is commonly denoted by λ, so we have the following result.

Lemma 13.1 *The Girsanov kernel φ is given by*

$$\varphi = -\lambda$$

where the market price of risk λ is defined by

$$\lambda = \frac{\mu - r}{\sigma}.$$

We have thus proved the existence of a martingale measure and from the First Fundamnetal Theorem we then have the following basic result for the Black–Scholes model.

Theorem 13.2 *The Black–Scholes model above is arbitrage free.*

We note in passing that instead of the standard Black–Scholes model above we could have considered a much more general model of the form

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \tag{13.4}$$

$$dB_t = r_t B_t dt, \tag{13.5}$$

where μ, σ, and r are allowed to be arbitrary adapted (but suitably integrable) processes with $\sigma_t \neq 0$ P-a.s. and for all t. The analysis of this more complicated model would be completely parallel to the one carried out above, with the only difference that the Girsanov kernel φ would now be a stochastic process given by the formula

$$\varphi_t = -\frac{\mu_t - r_t}{\sigma_t}. \tag{13.6}$$

As long as this φ satisfies the Novikov condition, the market would still be arbitrage free.

Remark 13.1.1 *The formal reason for the condition $\sigma_t \neq 0$ is that otherwise the quotient in (13.6) is undefined. Being a bit more precise, and going back to the fundamental equation*

$$\mu_t + \varphi_t \sigma_t = r_t,$$

we see that we can in fact solve this equation (and thus guarantee absence of arbitrage) as long as the condition

$$\sigma_t = 0 \quad \Rightarrow \quad \mu_t = r_t$$

is valid. The economic interpretation of this condition is that if $\sigma_t = 0$, then the stock price is locally riskless with dynamics $dS_t = S_t \mu_t dt$, so in order to avoid arbitrage with the money account B we must have $\mu_t = r_t$.

13.2 Pricing

Consider the standard Black–Scholes model and a fixed T-claim X. From Proposition 11.27 we immediately have the usual "risk neutral" pricing formula

$$\Pi_t[X] = e^{-r(T-t)} E^Q\left[X \mid \mathcal{F}_t\right], \tag{13.7}$$

where the Q dynamics of S are give as usual by

$$dS_t = rS_t dt + \sigma S_t dW_t^Q.$$

For a general claim we cannot say so much more, but for the case of a simple claim of the form

$$X = \Phi(S_T),$$

we can of course, write down the Kolmogorov backward equation for the expectation and express the price as

$$\Pi_t[X] = F(t, S_t),$$

where the pricing function F solves the Black–Scholes equation:

$$\begin{cases} \dfrac{\partial F}{\partial t} + rs\dfrac{\partial F}{\partial s} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 F}{\partial s^2} - rF = 0, \\ F(T,s) = \Phi(s). \end{cases} \tag{13.8}$$

The moral of all this is that the fundamental object is the risk neutral valuation formula (13.7), which is valid for all possible claims, whereas the Black–Scholes PDE is only valid for the case of simple claims.

13.3 Completeness

We now go on to investigate the completeness of the Black–Scholes model, and to this end we will use the Second Fundamental Theorem 11.26 which says that the market is complete if and only if the martingale measure is unique. We have seen in Section 13.2 that there exists a martingale measure and the remaining question is whether this is the only martingale measure.

In the present setting, where the filtration is the one generated by W we know from Theorem 12.6 that **every** absolutely continuous measure transformation is obtained from a Girsanov transformation, and since the basic equation (13.3) has a unique solution, we see that the martingale measure is in fact unique. The same argument is valid for the more general model above, and we have thus proved the following result.

Theorem 13.3 *The Black–Scholes model (13.1)–(13.2) is complete. This also holds for the more general model (13.4)–(13.5).*

From an abstract point of view, the theorem above settles the completeness question, but since it is based on the Second Fundamental Theorem, which in turn relies on rather abstract martingale theory, the argument is perhaps not overly instructive. We will therefore provide a more self-contained completeness proof, which more clearly shows the use and central importance of the Martingale Representation Theorem 12.2.

We will carry out the argument for the standard Black–Scholes model (13.1)–(13.2), but the argument goes through with very small changes also for the more general model above. We will use the technique in Lemma 11.15 and in terms of the notation of that lemma we identify the numeraire S^0 with the money account B, and S^1 with our the stock price S. We then define the normalized processes Z^0 and Z^1 by

$$Z_t^0(t) = \frac{B_t}{B_t}, \quad Z_t^1 = \frac{S_t}{B_t}.$$

Let Q be the (unique) martingale measure derived above, and consider an arbitrary T-claim X with

$$E^Q\left[\frac{X}{B_T}\right] < \infty.$$

(For the standard Black–Scholes model we may of course take the factor $1/B_T$ out of the expectation.) We then define the Q-martingale M by

$$M_t = E^Q\left[\left.\frac{X}{B_T}\right|\mathcal{F}_t\right], \tag{13.9}$$

and it now follows from Lemma 11.15 that the model is complete if we can find a process h_t^1 such that

$$dM_t = h_t^1 dZ_t^1. \tag{13.10}$$

In order to prove the existence of such a process h^1 we use the Martingale Representation Theorem 12.2 (under Q), which says that there exists a process g_t such that

$$dM_t = g_t dW_t^Q \tag{13.11}$$

where W^Q is the Q Wiener process defined earlier. With the purpose of connecting (13.11) to (13.10) we now use the Itô formula and the fact that Q is a martingale measure for the numeraire B to derive the Q dynamics of Z^1 as

$$dZ_t^1 = Z_t^1 \sigma dW_t^Q. \tag{13.12}$$

We thus have

$$dW_t = \frac{1}{Z_t^1 \sigma} dZ_t^1,$$

and plugging this into (13.11) we see that we in fact have (13.10) satisfied with h_1 defined by

$$h_t^1 = \frac{g_t}{\sigma Z_t^1}.$$

Again using Lemma 11.15, we have thus proved the following result.

Theorem 13.4 *In the Black–Scholes model (standard as well as extended), every T-claim X satisfying*

$$E^Q\left[\frac{X}{B_T}\right] < \infty$$

can be replicated. The replicating portfolio is given by

$$h_t^1 = \frac{g_t}{\sigma Z_t^1}, \tag{13.13}$$

$$h_t^0 = M_t - h_t^1 Z_t^1, \tag{13.14}$$

where M is defined by (13.9) and g is defined by (13.11).

This completeness result is much more general than the one derived in Chapter 8. The price that we have to pay for the increased generality is that we have to rely on the Martingale Representation Theorem which is an abstract existence result. Thus, for a general claim it is very hard (or virtually impossible) to compute the hedging portfolio in a reasonably explicit way. However, for the case of a simple claim of the form

$$X = \Phi(S_T),$$

the situation is of course more manageable. In this case we have

$$M_t = E^Q\left[e^{-rT}\Phi(S_T)\middle|\mathcal{F}_t\right],$$

and from the Kolmogorov backward equation (or from a Feynman–Kač representation) we have $M_t = f(t, S_t)$ where f solves the boundary value problem

$$\begin{cases} \dfrac{\partial f}{\partial t}(t,s) + rs\dfrac{\partial f}{\partial s}(t,s) + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 f}{\partial s^2}(t,s) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad f(T,s) = e^{-rT}\Phi(s). \end{cases}$$

Itô's formula now gives us

$$dM_t = \sigma S_t \frac{\partial f}{\partial s}(t, S_t)\, dW_t,$$

so in terms of the notation above we have

$$g_t = \sigma S_t \cdot \frac{\partial f}{\partial s}(t, S_t),$$

which gives us the replicating portfolio h as

$$\begin{aligned} h_t^0 &= f(t, S_t) - S_t \frac{\partial f}{\partial s}(t, S_t), \\ h_t^1 &= B_t \frac{\partial f}{\partial s}(t, S_t). \end{aligned}$$

We have the interpretation $f(t, S_t) = V^Z(t)$, i.e. f is the value of the normalized hedging portfolio, but it is natural to express everything in terms of the unnormalized value process $V(t)$ rather than in terms of V^Z. Therefore we define $F(t,s)$ by $F(t,s) = e^{rt} f(t,s)$ which gives us the following result which we recognize from Chapter 8.

Proposition 13.5 *Consider the Black–Scholes model and a T-claim of the form $X = \Phi(S(T))$. Then X can be replicated by the portfolio*

$$\begin{cases} h_t^0 = \dfrac{F(t, S_t) - S_t \frac{\partial F}{\partial s}(t, S_t)}{B_t}, \\ h_t^1 = \dfrac{\partial F}{\partial s}(t, S_t), \end{cases} \tag{13.15}$$

where F solves the **Black–Scholes equation**

$$\begin{cases} \dfrac{\partial F}{\partial t} + rs\dfrac{\partial F}{\partial s} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 F}{\partial s^2} - rF = 0, \\ F(T, s) = \Phi(s). \end{cases} \tag{13.16}$$

Furthermore the value process for the replicating portfolio is given by

$$V(t) = F(t, S_t).$$

14

MULTIDIMENSIONAL MODELS: MARTINGALE APPROACH

In this chapter we will use the martingale machinery of Chapter 11 to analyze, within a Wiener framework, a reasonably general multidimensional model. In particular we will produce a self-contained proof of the Second Fundamental Theorem for the special case of Wiener-driven models.

Let us thus consider a filtered probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$ carrying an N-dimensional standard Wiener process W. The basic setup is as follows.

Assumption 14.0.1 *We assume the following:*

- *There are n risky assets $S^1, \ldots, S^n$.*
- *Under the objective probability measure P, the S-dynamics are given by*

$$dS_t^i = \mu_t^i S_t^i dt + S_t^i \sum_{j=1}^{N} \sigma_t^{ij} dW_t^j, \tag{14.1}$$

  for $i = 1, \ldots, n$.
- *The coefficient processes μ^i and σ^{ij} above are assumed to be adapted.*
- *We have a standard risk free asset with price process B with dynamics*

$$dB_t = r_t B_t dt, \tag{14.2}$$

  where the short rate process r is assumed to be an adapted stochastic process.

In order to write the model on more compact form we view the vector processes S, and W as column vectors and we define the column vector process μ by

$$\mu = \begin{bmatrix} \mu^1 \\ \vdots \\ \mu^n \end{bmatrix}.$$

We also define the $n \times N$ matrix function σ by the component processes σ_t^{ij} above, and we define $D(S)$ as the diagonal matrix

$$D(S) = diag[S^1, \ldots, S^n].$$

We can finally write our model on compact form as

$$dS_t = D(S_t)\mu_t dt + D(S_t)\sigma_t dW_t, \tag{14.3}$$

$$dB_t = r_t B_t dt. \tag{14.4}$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

Note that at this point we do **not** assume that we have the same number of driving Wiener processes as the number of risky assets. Also note that the probability space is allowed to carry also other processes than the Wiener process W, so in general we allow that $\mathcal{F}_t^W \subseteq \mathcal{F}_t$, implying that the filtration could be generated by other processes beside W. For example, we make no assumptions about the distribution of the short rate process r or of the processes μ^i and σ^{ij}—we only assume that they are adapted. In particular this allows for models where these processes are path dependent upon the Wiener process, or that they are driven by some other "hidden" state variable processes.

14.1 Absence of Arbitrage

Our first task is to investigate when our model (14.3)–(14.4) is free of arbitrage, and to this end we use the First Fundamental Theorem 11.24 and look for a martingale measure Q with B as the numeraire. Using the Girsanov Theorem 12.3 we define a prospective likelihood process L by

$$dL_t = L_t \varphi_t^\star dW_t, \tag{14.5}$$

$$L(0) = 1, \tag{14.6}$$

where φ is an adapted N-dimensional (column-vector) process.

We now define our candidate martingale measure Q by setting $dQ = L_t dP$ on $\mathcal{F}_t$, and the Girsanov Theorem implies that we can write

$$dW_t = \varphi_t dt + dW_t^Q, \tag{14.7}$$

where W^Q is a standard Q-Wiener process. Plugging (14.7) into the P-dynamics (14.3) we obtain the following Q-dynamics of S:

$$dS_t = D\left(S_t\right)\left[\mu_t + \sigma_t \varphi_t\right] dt + D\left(S_t\right) \sigma_t dW_t^Q. \tag{14.8}$$

From (11.54) we know that, disregarding integrability problems, Q is a martingale measure if and only if the local rate of return of each asset equals the short rate, i.e. if and only if the equality

$$\mu_t + \sigma_t \varphi_t = \mathbf{r}_t \tag{14.9}$$

holds with probability one for each t, where $\mathbf{r} \in R^n$ is defined by

$$\mathbf{r} = \begin{bmatrix} r \\ \vdots \\ r \end{bmatrix}. \tag{14.10}$$

In this equation, henceforth referred to as **the martingale equation** which we write as

$$\sigma_t \varphi_t = \mathbf{r}_t - \mu_t, \tag{14.11}$$

the processes μ, σ, and r are given a priori and we want to solve for φ. Thus, for each t, and P-a.s., the n-dimensional vector $\mathbf{r}_t - \mu_t$ must be in the image of the diffusion matrix σ_t so we have the following result.

Proposition 14.1 *A necessary condition for absence of arbitrage is that*

$$\mathbf{r}_t - \mu_t \in Im\,[\sigma_t] \tag{14.12}$$

with probability one for each t. A sufficient condition for absence of arbitrage is that there exists a process φ which solves (14.11) and such that L is a martingale.

Note that it is not enough for φ to solve (14.11). We also need enough integrability to ensure that L is a true martingale (and not just a local one). Hence the following definition.

Definition 14.2 *A Girsanov kernel φ is said to be* **admissible** *if it generates a martingale measure, i.e. it solves (14.11) and L is a martingale.*

The integrability problem for L will be discussed further below, as in Proposition 30.11. However, the main focus will be on equation (14.11) and we will thus often carry out the arguments "modulo integrability problems".

Proposition 14.1 is quite general, but it also covers "pathological" models, such as those where all assets $S^1, \ldots, S^n$ are identical. In order not to be distracted by silly models like that we make the following definition which will guarantee that the concept of no arbitrage is structurally stable.

Definition 14.3 *The model above is said to be* **generically arbitrage free** *if it is arbitrage free for every (sufficiently integrable) choice of μ.*

We then have the following central result.

Proposition 14.4 *Disregarding integrability problems the model is generically arbitrage free if and only if, for each $t \leq T$ and P-a.s., the mapping*

$$\sigma_t : R^N \to R^n$$

is surjective, i.e. if and only if the volatility matrix σ_t has rank n.

From this result it follows in particular that for absence of arbitrage we must in the generic case necessarily have $n \leq N$, i.e. we must have at least as many independent Wiener processes as we have risky assets. This is quite in accordance with the informal reasoning in the meta-theorem 8.3.1.

We note that in the case of a factor model of the form (29.7)–(29.9) the martingale equation (14.11) only involves the S-dynamics, and **not** the dynamics of the extra factor process Y. This is of course because the no arbitrage restrictions only concern **prices of traded assets**.

14.2 Completeness

We now go on to obtain conditions for the model to be complete, and in order to avoid pathological cases we assume that the model is generically arbitrage free. From the Second Fundamental Theorem 11.26 we know that the model is complete if and only if the martingale measure is unique, so it is tempting to draw the conclusion that we have completeness if and only if equation (14.11) has a unique solution, i.e. if and only if the condition

$$Ker\,[\sigma_t] = \{0\} \tag{14.13}$$

is satisfied for all t and with probability one. This is, however, not quite true and the reason is that, in case of a general filtered probability space, there is no guarantee that *all* equivalent measure transformations are of the Girsanov type above. In a general situation, where there are other sources of randomness beside the Wiener process W, like say an independent Poisson process N, the Girsanov transformation above will only change the measure for the Wiener process, but it will not affect the Poisson process. Thus, even if equation (14.11) has a unique solution we do not have a unique martingale measure, since we have no restriction on how we are allowed to change the measure for the Poisson process. Put into more economic terms it is fairly obvious that if we consider a claim $\mathcal{X}$ of the type $\mathcal{X} = \Phi(N_T)$, then it is impossible to hedge this claim if the asset prices are driven by the Wiener process alone.

In order to obtain sharp results we are therefore forced to make the assumption that all randomness in our model is generated by the Wiener process W. We then have the following basic result.

Proposition 14.5 *Assume that the model is generically arbitrage free and that the filtration* $\mathbf{F}$ *is defined by*

$$\mathcal{F}_t = \mathcal{F}_t^W. \tag{14.14}$$

Then, disregarding integrability problems, the model is complete if and only if $n = N$ *and the volatility matrix* σ_t *is invertible* P*-a.s. for each* $t \leq T$.

Proof From the Converse of the Girsanov Theorem 12.6, we know that under the assumption that the filtration is the one generated by the Wiener process, *every* equivalent measure transformation is obtained by a Girsanov transformation of the type above. Hence the martingale measure is unique if and only if the solution of the martingale measure equation (14.11) is unique, and this occurs if and only if σ_t is injective, which implies $n \geq N$. Since we have assumed generic absence of arbitrage, we know that $n \leq N$ and that σ_t is surjective. Thus $n = N$ and σ_t is invertible. □

14.3 Hedging

In this section we will discuss the completeness question from the more concrete perspective of actually producing hedging strategies for an arbitrary T-claim $\mathcal{X}$. This has independent interest and it will also provide us with a new proof (within the framework of the present chapter) for the Second Fundamental Theorem. The advantage of this alternative proof is that it is much more concrete than the rather abstract one given in Section 11.5. The drawback is that we only provide the proof for Wiener-driven models, whereas the Second Fundamental Theorem holds in much more general situations.

Let us thus again consider the model from (14.3)

$$dS_t = D\,(S_t)\,\mu_t dt + D\,(S_t)\,\sigma_t dW_t. \tag{14.15}$$

Assumption 14.3.1 *We assume that the model is generically free of arbitrage, i.e. that*

$$Im\left[\sigma_t\right] = R^n, \tag{14.16}$$

for all t and with probability one. We also assume that the model is purely Wiener driven, i.e. that $\mathcal{F}_t = \mathcal{F}_t^W$.

Since we have assumed absence of arbitrage there exists some (not necessarily unique) martingale measure and we choose a particular one, denote it by Q and keep it fixed for the rest of the argument.

We then choose an arbitrary T-claim $\mathcal{X}$ in $L^1(Q)$, and the problem is to find a hedging portfolio for $\mathcal{X}$. From Lemma 11.15 we know that $\mathcal{X}$ can be hedged if and only if the martingale M, defined by

$$M_t = E^Q\left[\left.\frac{\mathcal{X}}{B_T}\right|\mathcal{F}_t\right], \tag{14.17}$$

admits dynamics of the form

$$dM_t = h_t dZ_t, \tag{14.18}$$

where as usual $Z = S/B$, and where h is an adapted (row vector) process in R^n. Given our assumption of a purely Wiener-driven system it follows from the Martingale Representation Theorem 12.2 that there exists an adapted N-dimensional (row vector) process g such that

$$dM_t = g_t dW_t^Q. \tag{14.19}$$

On the other hand, since the Z process is a (local) martingale under Q, it follows easily from Itô that the Q-dynamics are given by

$$dZ_t = D\left(Z_t\right)\sigma_t dW_t^Q. \tag{14.20}$$

Plugging (14.20) into (14.18) gives us

$$dM_t = h_t D\left(Z_t\right)\sigma_t dW_t^Q. \tag{14.21}$$

Comparing (14.21) with (14.19) we see that we can hedge $\mathcal{X}$ if and only if we can solve (at each t and for every ω) the equation

$$h_t D\left(Z_t\right)\sigma_t = g_t,$$

or, alternatively the equation

$$\sigma_t^\star D\left(Z_t\right)h_t^\star = g_t^\star. \tag{14.22}$$

In this equation, g is generated by the claim $\mathcal{X}$ and we want to solve the equation for h. Since $D\left(Z_t\right)$ is nondegenerate diagonal, the equation can be solved if and only if

$$g_t^\star \in Im\left[\sigma_t^\star\right],$$

and we have the following result.

Proposition 14.6 *Under Assumption 14.3.1 the model is complete if and only if*

$$Im\,[\sigma_t^\star] = R^N. \tag{14.23}$$

If the model is complete then, using the notation of Chapter 11, the replicating portfolio $[h^0, h^S]$ *is given by*

$$h_t^S = g_t \sigma_t^{-1} D^{-1}(Z_t), \tag{14.24}$$

$$h_t^0 = M_t - h_t Z_t. \tag{14.25}$$

Proof Follows immediately from Lemma 11.15. □

Note that $D^{-1}(Z_t)$ is just a diagonal matrix with $(Z_t^1)^{-1}, \ldots, (Z_t^n)^{-1}$ on the diagonal.

We can now easily provide an alternative proof of a restricted version of the Second Fundamental Theorem.

Theorem 14.7 (The Second Fundamental Theorem)
Under Assumption 14.3.1 the model is complete if and only if the martingale measure is unique.

Proof Using Proposition 14.6 and the standard duality result

$$\{Im\,[\sigma_t^\star]\}^\perp = Ker\,[\sigma_t]$$

we see that the model is complete if and only if

$$Ker\,[\sigma_t] = \{0\}. \tag{14.26}$$

This is, however, precisely the condition for the uniqueness of the martingale measure obtained earlier. □

It is instructive to compare these duality arguments with those given in the simple setting of Chapter 3, and in particular to the proof of Theorem 3.14, to see how much of the structure is carried over from the simple one-period model to the present general setting. From the discussion above we see that the "martingale measure equation" (14.11) and the "hedging equation" (14.22) are adjoint equations. Thus absence of arbitrage and completeness are truly dual concepts from a functional analytical point of view.

14.4 Pricing

Assuming absence of arbitrage, the general pricing formula for a T-claim $\mathcal{X}$ is, as always, given by the risk neutral valuation formula

$$\Pi_t[\mathcal{X}] = E^Q\left[e^{-\int_t^T r_u du}\mathcal{X} \middle| \mathcal{F}_t \right], \tag{14.27}$$

where Q is some choice of martingale measure, and where the Q dynamics of S are given by

$$dS_t = D(S_t)\mathbf{r}_t dt + D(S_t)\sigma_t dW_t^Q, \tag{14.28}$$

and where W^Q is Q-Wiener. Alternatively we can write the price as

$$\Pi_t[\mathcal{X}] = E^Q\left[\left.\frac{\mathbf{M}_T}{\mathbf{M}_t}\mathcal{X}\right|\mathcal{F}_t\right], \tag{14.29}$$

where $\mathbf{M}$ is the stochastic discount factor, defined by

$$\mathbf{M}_t = \frac{1}{B_t}L_t.$$

14.5 Markovian Models and PDEs

We now apply the martingale approach to a simple multidimensional version of the Black–Scholes model. We assume that $N = n$, that the short rate r, the vector of returns μ as well as the volatility matrix σ are deterministic and constant over time, that σ is invertible and that the filtration is the one generated by the Wiener process. Thus S will be a (lognormal) Markov process with P-dynamics

$$dS_t = D(S_t)\mu dt + D(S_t)\sigma dW_t.$$

The general pricing formula is, as always,

$$\Pi_t[\mathcal{X}] = e^{-r(T-t)}E^Q[\mathcal{X}|\mathcal{F}_t], \tag{14.30}$$

where the Q dynamics of S are given by

$$dS_t = D(S_t)\mathbf{r}dt + D(S_t)\sigma dW_t^Q.$$

If we now assume that $\mathcal{X}$ is a *simple* claim, i.e. of form $\mathcal{X} = \Phi(S_t)$, then, since S is Markovian we have

$$e^{-r(T-t)}E^Q[\Phi(S_t)|\mathcal{F}_t] = e^{-r(T-t)}E^Q[\Phi(S_t)|S_t],$$

and thus (exactly why?) the pricing process must be of the form $\Pi_t[\Phi] = F(t, S_t)$ for some pricing function F. We can then apply the Kolmogorov backward equation to the expectation above and we immediately see that the pricing function must solve the PDE

$$\begin{cases} F_t(t,s) + \sum_{i=1}^{n} rs_iF_i(t,s) + \frac{1}{2}tr\{\sigma^\star D[S]F_{ss}D[S]\sigma\} - rF(t,s) = 0, \\ F(T,s) = \Phi(s), \end{cases} \tag{14.31}$$

where $F_i = \frac{\partial F}{\partial s_i}$ and F_{ss} denotes the Hessian matrix.

Turning to hedging, we know from the uniqueness of the martingale measure that the market is complete, and thus that there exists a hedging portfolio $h = (h^0, h^1, \ldots, h^n)$. The value process dynamics are of course given by

$$dV_t^h = h_t^0 dB_t + \sum_{i=1}^{n} h_t^i dS_t^i,$$

but we also have $V_t^h = F(t, S_t)$ where F solves the PDE above. Applying the Itô formula this gives us

$$dV_t^h = \sum_{i=1}^{n} F_i(t, S_t) dS_t^i + (\text{second order terms})dt.$$

Comparing these two equations we can identify $h^1, \ldots, h^n$ from the dS_i terms and we see that the hedging portfolio is given by

$$h_t^i = \frac{\partial F}{\partial s_i}(t, S_t), \quad i = 1, \ldots, n, \tag{14.32}$$

$$h_t^0 = \frac{1}{B_t}\left\{F(t, S_t) - \sum_{i=1}^{n} \frac{\partial F}{\partial s_i}(t, S_t) S_t^i\right\}. \tag{14.33}$$

We may of course also express the portfolio in terms of relative weights.

14.6 Market Prices of Risk

Going back to the general model of Section 14.1 let us assume that the model is generically free of arbitrage possibilities, then we know that the martingale equation

$$\sigma_t \varphi_t = \mathbf{r}_t - \mu_t \tag{14.34}$$

always possesses a (not necessarily unique) solution $\varphi = (\varphi^1, \ldots, \varphi^N)^\star$, where φ is the Girsanov kernel used in the transition from P to Q. If we now define the vector process λ by $\lambda = -\varphi$, then we can write (14.34) as

$$\mu_t - \mathbf{r}_t = \sigma_t \lambda_t, \tag{14.35}$$

and on component form this becomes

$$\mu_t^i - r_t = \sum_{j=1}^{k} \sigma_t^{ij} \lambda_t^j, \quad i = 1, \ldots, n. \tag{14.36}$$

We have an economic interpretation of this equation. On the left-hand side we have the excess rate of return over the risk free rate for asset No. i, and on the right-hand side we have a linear combination of the volatilities σ^{ij} of asset No. i with respect to the individual Wiener processes $W^1, \ldots, W^N$. Thus λ^j is the "factor loading" for the individual risk factor W^j, and this object is often referred to as the "market price of risk for risk factor No. j". Roughly speaking one can then say that λ^j gives us a measure of the aggregate risk aversion in the market towards risk factor No. j. The main point to notice here is that *the same* λ is used for *all* assets. We can summarize the situation as follows:

- Under absence of arbitrage there will exist a market price of risk vector process λ satisfying (14.35)–(14.36).

- The market price of risk λ is related to the Girsanov kernel φ through
$$\varphi_t = -\lambda_t.$$
- In a complete market the market price of risk, or alternatively the martingale measure Q, is uniquely determined and there is thus a unique price for every derivative.
- In an incomplete market there are several possible market prices of risk processes and several possible martingale measures, all of which are consistent with no arbitrage.

We formulate this simple but important insight as a separate result.

Result 14.6.1 *In an incomplete market, φ, λ, and Q are not determined by absence of arbitrage alone. Instead they will also be partly determined by supply, demand, and aggregate risk aversion on the actual market. The martingale measure Q is thus determined by* **the market**.

14.7 The Stochastic Discount Factor

Consider again the model in Section 14.1, assume absence of arbitrage, and choose a fixed martingale measure Q with corresponding Girsanov kernel φ. In Section 11.8 we defined the stochastic discount factor $\mathbf{M}$ by
$$\mathbf{M}_t = e^{-\int_0^t r_s ds} L_t,$$
where L is the likelihood process for the measure transformation from P to Q. Recalling the L-dynamics as
$$dL_t = L_t \varphi_t^\star dW_t, \tag{14.37}$$
in the special case of a Wiener-driven model we can compute L explicitly as
$$L_t = e^{\int_0^t \varphi_s^\star dW_s - \frac{1}{2}\int_0^t \|\varphi_s\|^2 ds},$$
so in this model we have an explicit expression for $\mathbf{M}$ as
$$\mathbf{M}_t = e^{\int_0^t \varphi_s^\star dW_s - \int_0^t \left\{\frac{1}{2}\|\varphi_s\|^2 + r_s\right\} ds}, \tag{14.38}$$
We also have the following simple but important result.

Proposition 14.8 *With L-dynamics as in (14.37), the $\mathbf{M}$-dynamics are*
$$d\mathbf{M}_t = -r_t \mathbf{M}_t dt + \mathbf{M}_t \varphi_t^* dW_t, \tag{14.39}$$
or alternatively
$$d\mathbf{M}_t = -r_t \mathbf{M}_t dt - \mathbf{M}_t \lambda_t^\star dW_t, \tag{14.40}$$
where $\lambda = -\varphi$ is the market price of risk vector process.

The point of this result is that we can recover the short rate process r and the Girsanov kernel φ from $\mathbf{M}$-dynamics. This is very valuable and we will use it repeatedly in Chapters 35–38.

14.8 The Hansen–Jagannathan Bounds

Assume that we have generic absence of arbitrage, i.e. that σ_t is surjective, so the martingale measure equation

$$\sigma_t \varphi_t = \mathbf{r}_t - \mu_t \tag{14.41}$$

always possesses a solution. We now move on to derive what, in discrete time asset pricing, is known as the "Hansen–Jagannathan bounds" (see the Notes). To obtain these bounds we consider the price process p_t of any asset in the model (underlying or derivative), write its P-dynamics on the form

$$dp_t = p_t \mu_t^p dt + p_t \sigma_t^p dW_t$$

and define its **Sharpe ratio** process SR by the expression

$$SR = \frac{\mu_t^p - r_t}{\|\sigma_t^p\|}.$$

Denoting the P-variance by Var^P we have the informal interpretation

$$\|\sigma_t^p\|^2 dt = Var^P \left[dp_t/p_t | \mathcal{F}_t \right]$$

so the Sharpe ratio gives us the conditional mean excess rate of return per unit of total volatility.

We now have the following simple but interesting result.

Proposition 14.9 (The Hansen–Jagannathan Bounds)
Assume generic absence of arbitrage. Then the following holds for all assets, underlying or derivative, and for all admissible Girsanov kernels φ, and market prices of risk λ.

$$\frac{|\mu_t^p - r_t|}{\|\sigma_t^p\|} \le \|\varphi_t\|, \quad \frac{|\mu_t^p - r_t|}{\|\sigma_t^p\|} \le \|\lambda_t\|. \tag{14.42}$$

Proof Fix any λ generating a martingale measure. We then have

$$\mu_t^p - r_t = \sigma_t^p \lambda_t,$$

so from the Cauchy–Schwartz inequality in finite dimensional space we obtain

$$|\mu_t^p - r_t| = |\sigma_t^p \lambda_t| \le \|\sigma_t^p\| \cdot \|\lambda_t\|$$

which proves the result. □

We can now connect this result to the stochastic discount factor $\mathbf{M}$ by recalling that from Proposition 14.8 we have the equivalent formulas

$$d\mathbf{M}_t = -r_t \mathbf{M}_t dt + \mathbf{M}_t \varphi_t^\star dW_t, \tag{14.43}$$

$$d\mathbf{M}_t = -r_t \mathbf{M}_t dt - \mathbf{M}_t \lambda_t^\star dW_t. \tag{14.44}$$

The point of this from an economic perspective is that we can either view the Hansen–Jagannathan bounds as a lower bound for the stochastic discount factor volatility (i.e. the market price of risk) for a given or observed Sharpe ratio, or as an upper bound on the Sharpe ratio for a given or observed stochastic discount factor volatility.

14.9 Exercises

Exercise 14.1 Derive (14.43).

14.10 Notes

The results in this chapter are fairly standard. The Hansen–Jagannathan bounds were first derived (in discrete time) in Hansen and Jagannathan (1991). They have since then become the subject of a large literature. See the textbook Cochrane (2001) for an exposition of (mostly discrete time) asset pricing, including a detailed discussion of the discrete time HJ bounds, connections to the "equity premium puzzle" and an extensive bibliography on the subject.

15

CHANGE OF NUMERAIRE

15.1 Introduction

Consider a given financial market with the usual locally risk free asset B, and a risk neutral martingale measure Q. As noted in Chapter 11 a measure is a martingale measure only relative to some chosen numeraire asset, and we recall that the risk neutral martingale measure Q, with the money account B as numeraire, has the property of martingalizing all processes of the form S_t/B_t where S is the arbitrage free price process of any (non-dividend-paying) traded asset.

In many concrete situations the computational work needed for the determination of arbitrage free prices can be drastically reduced by a clever change of numeraire, from the bank acount B to some other numeraire S, and the purpose of the present chapter, which to a large extent follows and is inspired by Geman et al. (1995), is to analyze such changes. See the Notes for the more bibliographic information.

15.2 Generalities

We now proceed to the formal discussion of numeraire changes, and we start by setting the scene.

Assumption 15.2.1 *We consider an arbitrage free market model with asset prices $S^0, S^1, \ldots, S^n$ where S^0 is assumed to be strictly positive.*

Sometimes, but not always, we will need to assume that all prices are Wiener driven.

Condition 15.2.1 *Under P, the S-dynamics are of the form*

$$dS_t^i = \mu_t^i S_t^i dt + S_t^i \sigma_t^i dW_t, \quad i = 0, \ldots, n, \tag{15.1}$$

where the coefficient processes are adapted and W is a multidimensional standard P-Wiener process.

Remark 15.2.1 *We do not necessarily assume the existence of a short rate and a money account. If the model admits a short rate and a money account they will as usual be denoted by r and B respectively. We denote by Q the usual risk neutral measure where we use B as numeraire.*

From a mathematical point of view, most of the results concerning changes of numeraire are really special cases of the First Fundamental Theorem and the associated pricing formulas. Thus the difference between the present chapter

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

and Chapter 11 is more one of perspective than one of essence. We now recall some facts from Chapter 11 and start with the Invariance Lemma.

Lemma 15.1 (Invariance Lemma)
Let β be any strictly positive Itô process, and define the normalized process Z with numeraire β, by $Z = S/\beta$. Then h is S-self-financing if and only if h is Z-self-financing, i.e. with notation as in Chapter 11 we have

$$dV_t^S = h_t dS_t \tag{15.2}$$

if and only if

$$dV_t^Z = h_t dZ_t. \tag{15.3}$$

Proof Follows immediately from the Itô formula. □

We make two remarks on the Invariance Lemma:

- A process β satisfying the assumptions above is sometimes called a "deflator process".
- We have assumed that S and β are Itô processes. This is not important, and the Invariance Lemma does in fact hold also in a general semimartingale setting.
- Observe that at this point we do **not** assume that the deflator process β is the price process for a traded asset. The Invariance Lemma will hold for any positive process β satisfying the assumptions above.

From Chapter 11 (see summary in Section 11.9) we now recall the First Fundamental Theorem and the corresponding pricing formula.

Theorem 15.2 *Under the assumptions above, the following hold with S^0 as the numeraire*:

- *The market model is free of arbitrage if and only if there exists a* **martingale measure**, *$Q^0 \sim P$ such that the processes*

$$\frac{S_t^0}{S_t^0}, \frac{S_t^1}{S_t^0}, \ldots, \frac{S_t^n}{S_t^0}$$

are (local) martingales under Q^0.
- *An arbitrage free price system for all T-claims X is given by the formula*

$$\Pi_t[X] = S_t^0 E^0 \left[\left. \frac{X}{S_T^0} \right| \mathcal{F}_t \right], \tag{15.4}$$

where E^0 denotes expectation under Q^0.

Remark 15.2.2 *Note that we do not have to calculate the term S_t^0 in (15.4). We simply observe it on the market.*

In most of our applications earlier in the book we have used the money account B as the numeraire, and the corresponding risk netural measure Q, but in many applications the choice of another asset as the numeraire asset can greatly facilitate computations.

We will give several other concrete examples below, but first we will investigate how we change from one choice of numeraire to another, i.e. how we determine the appropriate Girsanov transformation. This will be done in Section 15.3.

Remark 15.2.3 *Since there sometimes seems to be confusion around what is a bona fide choice of numeraire, let us recall some points in the derivation of the First Fundamental Theorem*:

- *In the basic version of the theorem (Theorem 11.9) we assumed that S^0 was a risk free* **traded asset** *with zero rate of return. It was a crucial ingredient in the proof that we were allowed to invest in this risk free asset.*
- *For the general case we used the* **traded asset** *S^0 as the numeraire. In the normalized economy this provided us with a* **traded asset** *Z_0 which was risk free with zero rate of return. The Invariance Lemma then allowed us to use the basic version of Theorem 11.9 to complete the proof.*
- *The point of these comments is that the Invariance Lemma is true for any deflator process β, but when it comes to the existence of martingale measures and pricing, we* **must** *use a numeraire which is the price process of a* **traded asset without dividends**.
- *In particular, if we want to use numeraires like*
  - *a non-financial index,*
  - *a forward or futures price process,*
  - *the price process of a traded asset with dividends,*

  then we must carry out a careful separate analysis, since in these cases we do **not** *have access to a standard version of the First Fundamental Theorem.*

15.3 Changing the Numeraire

Suppose that for a specific numeraire S^0 we have determined a corresponding (not necessarily unique) martingale measure Q^0, and the associated dynamics of the asset prices (and possibly also the dynamics of other factors within the model). In the case of a complete market the measure Q^0 will be unique, but for an incomplete market there will typically exist infinitely many measures Q^0. We assume, however, that we have chosen one specific Q^0 and that we keep this choice of measure fixed during the discussion below.

This particular choice of martingale measure Q^0 will generate an arbitrage free **price system** for all contingent claims by the formula (15.1), i.e. by

$$\Pi_t[X] = S_t^0 E^0 \left[\frac{X}{S_T^0} \middle| \mathcal{F}_t \right], \tag{15.5}$$

and we note that in the case of a complete market, the price system will be unique. For an incomplete market there will exist many arbitrage free price systems—one for each choice of Q^0—but since we have fixed one specific Q^0 we have also chosen a fixed price system.

Suppose now that we want to change the numeraire from S^0 to, say, S^1. Our immediate problem is then the following.

Problem 15.3.1 *Find a martingale measure Q^1 corresponding to the numeraire S^1,* **while keeping the fixed price system defined by** *(15.5).*

Having solved the problem above we also want to find the appropriate Girsanov transformation which will take us from Q^0 to Q^1.

This problem is in fact quite easily solved, and to see this, let us use the pricing part of Theorem 15.2 for an arbitrary choice of T-claim X. We have

$$\Pi_t[X] = S_t^0 E^0\left[\left.\frac{X}{S_T^0}\right|\mathcal{F}_t\right], \tag{15.6}$$

and since we are keeping the price system fixed, we should also have

$$\Pi_0[X] = S_0^1 E^1\left[\frac{X}{S_T^1}\right]. \tag{15.7}$$

Denoting by L_t^{01} the Radon–Nikodym derivative

$$L_t^{01} = \frac{dQ^1}{dQ^0}, \quad \text{on } \mathcal{F}_t, \tag{15.8}$$

we can write (15.7) as

$$\Pi_0[X] = S_0^1 E^0\left[\frac{X}{S_T^1}\cdot L_T^{01}\right], \tag{15.9}$$

and we thus have

$$S_0^0 E^0\left[\frac{X}{S_T^0}\right] = S_0^1 E^0\left[\frac{X}{S_T^1}\cdot L_T^{01}\right] \tag{15.10}$$

for all (sufficiently integrable) T-claims X. We thus deduce that

$$\frac{S_0^0}{S_T^0} = \frac{S_0^1}{S_T^1}\cdot L_T^{01},$$

so we obtain

$$L_T^{01} = \frac{S_T^1}{S_T^0}\cdot\frac{S_0^0}{S_0^1},$$

which is our candidate as a Radon–Nikodym derivative. Since T was arbitrary, the obvious choice of the induced likelihood process is of course given by

$$L_t^{01} = \frac{S_t^1}{S_t^0}\cdot\frac{S_0^0}{S_0^1}, \quad 0 \le t \le T,$$

where S_0^0/S_0^1 is a normalizing constant to ensure that $L_0^{01}=1$. This looks promising, since the process S_t^1/S_t^0 is a Q^0-martingale (why?), and we know that any likelihood process for the transition from Q^0 to Q^1 has to be a Q^0-martingale. In more formal terms we have the following proposition.

Proposition 15.3 *Assume that Q^0 is a martingale measure for the numeraire S^0 (on $\mathcal{F}_T$) and assume that S^1 is a positive asset price process such that S_t^1/S_t^0 is a true Q^0-martingale (and not just a local one). Define Q^1 on $\mathcal{F}_T$ by the likelihood process*

$$L_t^{01} = \frac{S_t^1}{S_t^0} \cdot \frac{S_0^0}{S_0^1}, \quad 0 \le t \le T. \tag{15.11}$$

Then Q^1 is a martingale measure for S^1, and it generates the same price system as Q^0.

Proof We have to show that for every (sufficiently integrable) arbitrage free price process Π, the normalized process Π_t/S_t^1 is a Q^1-martingale. Now, if Π is an arbitrage free price process then we know that Π/S^0 is a Q^0-martingale and for $s \le t$ we have the following calculation, where we use the Abstract Bayes' Formula.

$$\begin{aligned} E^1\left[\left.\frac{\Pi_t}{S_t^1}\right|\mathcal{F}_s\right] &= \frac{E^0\left[\left. L_t^{01}\frac{\Pi_t}{S_t^1}\right|\mathcal{F}_s\right]}{L_s^{01}} = \frac{E^0\left[\left.\frac{S_0^0}{S_0^1}\cdot\frac{S_t^1}{S_t^0}\cdot\frac{\Pi_t}{S_t^1}\right|\mathcal{F}_s\right]}{L_s^{01}} \\ &= \frac{\frac{S_0^0}{S_0^1}\cdot E^0\left[\left.\frac{\Pi_t}{S_t^0}\right|\mathcal{F}_s\right]}{L_s^{01}} = \frac{\frac{S_0^0}{S_0^1}\cdot\frac{\Pi_s}{S_s^0}}{L_s^{01}} = \frac{\Pi_s}{S_s^1}. \end{aligned}$$

$\square$

We have an easy corollary to this result, when $S^0 = B$, and since this case occurs so frequently we formulate it as a separate result.

Proposition 15.4 *Let Q denote the usual risk neutral measure with B as numeraire, and consider a numeraire asset S. Denote the relevant martingale measure by Q^S. Then the likelihood process*

$$L_t^S = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t \tag{15.12}$$

is given by

$$L_t^S = \frac{S_t}{B_t}\cdot\frac{1}{S_0}. \tag{15.13}$$

Since we have determined the relevant likelihood process, we can identify the Girsanov kernel.

Proposition 15.5 *Assume absence of arbitrage and a model of the form (15.1). Denote the corresponding Q^0-Wiener process by W^0. Then the Q^0-dynamics of the likelihood process L^{01} are given by*

$$dL_t^{01} = L_t^{01}\left\{\sigma_t^1 - \sigma_t^0\right\}dW_t^0. \tag{15.14}$$

Thus the Girsanov kernel φ^{01} for the transition from Q^0 to Q^1 is given by the volatility difference

$$\varphi_t^{01} = \sigma_t^1 - \sigma_t^0. \tag{15.15}$$

Proof The result follows immediately from applying the Itô formula to (15.11) while using the dynamics (15.1). □

We again have an easy corollary for the transition from Q to Q^S.

Proposition 15.6 *Assume that the P-dynamics for S are of the form*

$$dS_t = S_t\mu_t dt + S_t\sigma_t dW_t \tag{15.16}$$

and define L^S by (15.12). We then have

$$dL_t^S = L_t^S \sigma_t dW_t^Q, \tag{15.17}$$

where W^Q is Q-Wiener.

We thus see that the Girsanov kernel for the transformation from Q to Q^S is precisely the volatility process of the numeraire asset S.

15.4 Some Examples

We will now present some examples to illustrate the usefulness of the change of numeraire technique. In all these examples we will change from the risk neutral measure Q to the measure Q^S which uses a (cleverly chosen) asset S as the numeraire. To this end we recall the pricing formula (15.4)

$$\Pi_t[X] = S_t E^S\left[\left.\frac{X}{S_T}\right| \mathcal{F}_t\right], \tag{15.18}$$

where E^S denotes expectation under Q^S. From this formula it is clear that if we can write

$$X = S_T \cdot Y, \tag{15.19}$$

where Y is easier (in some reasonable sense) to handle than X, then a change of numeraire may be useful, because we will then have

$$\Pi_t[X] = S_t E^S\left[Y \middle| \mathcal{F}_t\right]. \tag{15.20}$$

Example 15.7 (An Asset-or-Nothing Option) In this example we consider a standard Black–Scholes model with the usual P-dynamics

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dB_t &= rB_t dt. \end{aligned}$$

The claim X under consideration is an "asset-or-nothing call option", defined by

$$X = S_T \cdot I\left\{S_T > K\right\}$$

where the strike price K is a positive constant, and I is the indicator function. This option will give us one unit of the underlying asset if $S_T > K$, and nothing if $S_T \leq K$.

We restrict ourselves to computing the price at time $t = 0$ (see below for the general case). The form of X clearly suggests that we should use the underlying asset S as the numeraire, and from (15.18) we have

$$\begin{aligned}\Pi_0[X] &= S_0 E^S\left[I\left\{S_T > K\right\}\right] = S_0\left(1 - E^S\left[I\left\{S_T > K\right\}\right]\right) \\ &= S_0 - S_0 Q^S(S_T \leq K).\end{aligned}$$

It thus remains to compute $Q^S(S_T \leq K)$, and to this end we use Proposition 15.6. The Q-dynamics of S are, as always,

$$dS_t = rS_t dt + S_t \sigma dW_t^Q.$$

Proposition 15.6 and the Girsanov Theorem implies that we can write

$$dW_t^Q = \sigma dt + dW_t^S,$$

where W^S is Q^S-Wiener. We thus obtain the Q^S dynamics of S as

$$dS_t = (r + \sigma^2)S_t dt + S_t \sigma dW_t^S.$$

This is GBM, so we have

$$S_T = S_0^{(r+\frac{1}{2}\sigma^2)T + \sigma W_T^S}$$

which we can write as

$$S_T = S_0 e^Y$$

where Y has the distribution $N\left[(r + \frac{1}{2}\sigma^2)T, \sigma^2 T\right]$ under Q^S. A standard calculation shows that

$$Q^S(S_T \leq K) = N[d]$$

where N is the cdf of a standard normal distribution and

$$d = \frac{\ln(K/S) - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

We have thus computed the price at $t = 0$, and for a general t, we simply replace T by $T - t$.

Result 15.4.1 *The price of the asset-or-nothing call option is given by*

$$\Pi_t[X] = S_0\left(1 - N[d]\right).$$

Example 15.8 (Linearly Homogeneous Contracts) A typical example when a change of numeraire is useful occurs when dealing with derivatives defined in terms of several underlying assets. Assume for example that we are given two asset prices S^1 and S_2, and that the contract X to be priced is of the form $X = \Phi\left(S_t^1, S_t^2\right)$, where Φ is a given **linearly homogeneous** function, i.e.

$$\Phi(\lambda x, \lambda y) = \lambda \cdot \Phi(x, y)$$

for all x, y, and all $\lambda > 0$. Using the standard risk neutral machinery with B as the numeraire, and denoting the risk neutral martingale measure by Q we would have to compute the price as

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds}\Phi\left(S_t^1, S_t^2\right)\Big|\mathcal{F}_t\right],$$

which essentially amounts to the calculation of a triple integral. If we instead use S^1 as numeraire, with martingale measure Q^1, we have

$$\Pi_t[X] = S_t^1 E^{Q^1}\left[\frac{\Phi\left(S_T^1, S_T^2\right)}{S_T^1}\middle|\mathcal{F}_t\right] \tag{15.21}$$

$$= S_t^1 E^{Q^1}\left[\varphi\left(Z_T^2\right)\big|\mathcal{F}_t\right], \tag{15.22}$$

where $\varphi(z) = \Phi(1, z)$ and $Z_t^2 = S_t^2/S_t^1$. In this formula we note that the factor S_t^1 is the price of the traded asset S^1 at time t, so this quantity does not have to be computed—it can be directly observed on the market. Thus the computational work is reduced to computing a single integral. We also note the important fact that in the Z economy we have **zero short rate**. We now go on to study a concrete example in this setting.

Example 15.9 (An Exchange Option) As an example of the reasoning above, assume that we have two stocks, S^1 and S_2, with price processes of the following form under the objective probability measure P:

$$dS_t^1 = \mu_1 S_t^1 dt + S_t^1 \sigma_1 dW_t, \tag{15.23}$$

$$dS_t^2 = \mu_2 S_t^2 dt + S_t^2 \sigma_2 dW_t. \tag{15.24}$$

Here $\mu_1, \mu_2 \in R$ are deterministic real numbers, whereas $\sigma_1, \sigma_2 \in R^2$ are two-dimensional row vectors assumed to be deterministic, W is a two-dimensional standard Wiener process under P, and we assume absence of arbitrage.

The T-claim to be priced is an **exchange option**, which gives the holder the right, but not the obligation, to exchange one S^2 share for one S^1 share at time T. Formally this means that the claim is given by $X = \max\left[S_t^2 - S_t^1,\ 0\right]$, and we note that we have a linearly homogeneous contract function. It is thus natural to use one of the assets as the numeraire, and we choose S^1. From Theorem 15.2, and using homogeneity, the price is given by

$$\Pi_t[X] = S_t^1 E^1\left[\max\left[Z_t^2 - 1,\ 0\right]\big|\mathcal{F}_t\right],$$

with $Z_t^2 = S_t^2/S_t^1$ and with E^1 denoting expectation under Q^1. Note that the upper case indices for Z^2, S^2 and S^1 are not powers but merely indices. We now see that the expectation above is in fact the value of a European call option on Z^2, with strike price $K = 1$ in the Z^2-economy, where (as in all normalized asset price systems) the short rate is zero.

We now have to compute the Q^1 dynamics of Z^2, but this turns out to be very easy. From Itô, the P-dynamics of Z^2 are of the form

$$dZ_t^2 = Z_t^2 (\ldots) dt + Z_t^2 \{\sigma_2 - \sigma_1\} dW_t$$

where we do not care about the precise form of the dt-terms. Under Q^1 we know that Z^2 is a martingale, and since the volatility terms do not change under a Girsanov transformation we obtain directly the Q^1 dynamics as

$$dZ_t^2 = Z_t^2 \{\sigma_2 - \sigma_1\} dW_t^1 \tag{15.25}$$

where W^1 is Q^1-Wiener. We can write this as

$$dZ_t^2 = Z_t^2 \sigma d\bar{W}_t^1$$

where $\bar{W}^1$ is a scalar Q^1-Wiener process and the scalar volatility σ is given by

$$\sigma = \|\sigma_2 - \sigma_1\|.$$

Using the Black–Scholes formula with zero short rate, unit strike price and volatility σ, we have the following result.

Result 15.4.2 *The price of the exchange option is thus given by the formula*

$$\Pi_t[X] = S_t^1 \left\{ Z_t^2 N[d_1] - N[d_2] \right\} \tag{15.26}$$

$$= S_t^2 N[d_1] - S_t^1 N[d_2], \tag{15.27}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{S_t^2}{S_t^1}\right) + \frac{1}{2}\sigma^2 (T-t) \right\}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

If, instead of using a two-dimensional standard Wiener process, we model the stock price dynamics as

$$dS_t^1 = \mu_1 S_t^1 dt + S_t^1 \sigma_1 dW_t^1,$$

$$dS_t^2 = \mu_2 S_t^2 dt + S_t^2 \sigma_2 dW_t^2.$$

Where W^1 and W^2 are scalar P-Wiener with local correlation ρ, and thus σ_1 and σ_2 are scalar constants, then it is easy to see that the relevant volatility to use in the formula above is given by

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

Note that we made no assumption whatsoever about the dynamics of the short rate. The result above thus holds for every possible specification of the short rate process.

15.5 Forward Measures

In this section we specialize the theory developed in the previous section to the case when the new numeraire chosen is a bond maturing at time T. As can be expected this choice of numeraire is particularly useful when dealing with interest rate derivatives.

15.5.1 *Using the T-bond as Numeraire*

For this section the reader may want to recall the concept of a zero coupon bond from Definition 11.20.

Suppose that we are given a specified bond market model with a fixed risk neutral martingale measure Q. For a fixed time of maturity T we now choose the zero coupon bond maturing at T as our new numeraire, and as usual we denote the price at t of a T-bond by $p(t,T)$.

Definition 15.10 *For a fixed T, the T-**forward measure** Q^T is defined as the martingale measure for the numeraire process $p(t,T)$.*

In interest rate theory we often have our models specified under the risk neutral martingale measure Q with the money account B as the numeraire. We then have the following explicit description Q^T.

Proposition 15.11 *If Q denotes the risk neutral martingale measure, then the following hold:*

1. *The likelihood process*

$$L_t^T = \frac{dQ^T}{dQ}, \quad \text{on } \mathcal{F}_t,\ 0 \le t \le T$$

is given by

$$L_t^T = \frac{p(t,T)}{B_t}\frac{1}{p(0,T)}. \tag{15.28}$$

2. *In particular, if the Q-dynamics of the T-bond are Wiener driven, i.e. of the form*

$$dp(t,T) = r_t p(t,T)dt + p(t,T)v(t,T)dW_t, \tag{15.29}$$

where W is a (possibly multidimensional) Q Wiener process, then the L^T dynamics are given by

$$dL_t^T = L_t^T v(t,T)dW_t, \tag{15.30}$$

i.e. the Girsanov kernel for the transition from Q to Q^T is given by the T-bond volatility $v(t,T)$.

Proof The result follows immediately from Proposition 15.3 with $Q^T = Q^1$ and $Q^0 = Q$. □

Observing that $P(T,T) = 1$ we have the following useful pricing formula as an immediate corollary of Proposition 15.2.

Proposition 15.12 *For any T-claim X we have*

$$\Pi_t[X] = p(t,T)E^T[X|\mathcal{F}_t], \tag{15.31}$$

where E^T denotes integration w.r.t. Q^T.

Note again that the price $p(t,T)$ does not have to be computed. It can be observed directly on the market at time t.

A natural question to ask is when Q and Q^T coincide.

Lemma 15.13 *The relation $Q = Q^T$ holds if and only if r is deterministic.*

Proof Exercise for the reader. □

15.6 A General Option Pricing Formula

The object of this section is to give a fairly general formula for the pricing of European call options, and for the rest of the section we basically follow the seminal paper Geman *et al.* (1995).

15.6.1 *General Theory*

We consider a financial market with a (possibly stochastic) short rate r, and a strictly positive asset price process S_t.

The option under consideration is a European call on S with date of maturity T and strike price K. We are thus considering the T-claim

$$\mathcal{X} = \max[S_T - K,\ 0], \tag{15.32}$$

and, for readability reasons, we confine ourselves to computing the option price $\Pi_t[\mathcal{X}]$ at time $t = 0$.

The main reason for the existence of a large number of explicit option pricing formulas is that the contract function for an option is piecewise linear. We can capitalize on this fact by using the following trick with indicator functions. Write the option as

$$\mathcal{X} = [S_T - K] \cdot I\{S_T \geq K\},$$

where I is the indicator function. Denoting the option value at $t = 0$ by c_0 we can thus write

$$c_0 = \Pi_0[A] - K\Pi_0[B],$$

where the claims A and B are given by

$$A = S_T \cdot I\{S_T \geq K\}, \qquad B = I\{S_T \geq K\}.$$

For the A-claim we use the measure Q^S having S as numeraire, and for the B-claim we use the T-forward measure Q^T. From Theorem 15.2 formula (15.4) and Proposition 15.12 we then obtain the following basic option pricing formula, which is a substantial extension of the standard Black–Scholes formula. The extension to an arbitrary initial time t is obvious.

Proposition 15.14 (General Option Pricing Formula) *Given the assumptions above, the option price at time t is given by*

$$c_t = S_t Q_t^S(S_T \geq K) - Kp(t,T)Q_t^T(S_T \geq K). \tag{15.33}$$

Here Q_t^T denotes the T-forward measure conditional on $\mathcal{F}_t$, whereas Q_t^S denotes the martingale measure for the numeraire process S conditional on $\mathcal{F}_t$.

15.6.2 *The Case of Deterministic Volatility*

In order to use the general option formula (15.33) in a real situation we have to be able to compute the probabilities above, and the standard condition which ensures computability turns out to be that volatilities should be deterministic. Hence we have the following assumption.

Assumption 15.6.1 *Assume that the process $Z_{S,T}$ defined by*

$$Z_{S,T}(t) = \frac{S_t}{p(t,T)} \tag{15.34}$$

has a stochastic differential under P of the form

$$dZ_{S,T}(t) = Z_{S,T}(t)m_{S,T}(t)dt + Z_{S,T}(t)\sigma_{S,T}(t)dW_t, \tag{15.35}$$

where the volatility process $\sigma_{S,T}(t)$ is **deterministic**.

The crucial point here is of course the assumption that the row-vector process $\sigma_{S,T}$ is deterministic. Note that the volatility process as always is unaffected by a change of measure, so in fact we do not have to specify under which measure we check the condition. It is the same under P, Q, Q^S, and Q^T.

We start the computations by writing the probability in the second term of (15.33) as

$$Q^T\left(S_T \geq K\right) = Q^T\left(\frac{S(T)}{p(T,T)} \geq K\right) = Q^T\left(Z_{S,T}(T) \geq K\right). \tag{15.36}$$

Since $Z_{S,T}$ defined in (15.34) is an asset price, normalized by the price of a T-bond, it is a Q^T martingale, so its Q^T-dynamics are given by

$$dZ_{S,T}(t) = Z_{S,T}(t)\sigma_{S,T}(t)dW_t^T. \tag{15.37}$$

This is basically GBM, driven by a multidimensional Wiener process, and it is easy to see that the solution is given by

$$Z_{S,T}(T) = \frac{S_0}{p(0,T)} \exp\left\{-\frac{1}{2}\int_0^T \|\sigma_{S,T}\|^2(t)dt + \int_0^T \sigma_{S,T}(t)dW_t^T\right\}. \tag{15.38}$$

In the exponent we have a stochastic integral and a deterministic time integral. Since the integrand in the stochastic integral is deterministic, an easy extension of Lemma 4.18 shows that the stochastic integral has a Gaussian distribution with zero mean and variance

$$\Sigma_{S,T}^2(T) = \int_0^T \|\sigma_{S,T}(t)\|^2 dt. \tag{15.39}$$

The entire exponent is thus normally distributed, and we can write the probability in the second term in (15.33) as

$$Q^T\left(S_T \geq K\right) = N[d_2],$$

where

$$d_2 = \frac{\ln\left(\frac{S_0}{Kp(0,T)}\right) - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}. \tag{15.40}$$

Since the first probability term in (15.33) is a Q^S-probability, it is natural to write the event under consideration in terms of a quotient with S in the denominator. Thus we write

$$Q^S\left(S_T \geq K\right) = Q^S\left(\frac{p(T,T)}{S_T} \leq \frac{1}{K}\right) = Q^S\left(Y_{S,T}(T) \leq \frac{1}{K}\right), \tag{15.41}$$

where $Y_{S,T}$ is defined by

$$Y_{S,T}(t) = \frac{p(t,T)}{S_t} = \frac{1}{Z_{S,T}(t)}.$$

Under Q^S the process $Y_{S,T}$ has zero drift, so its Q^S-dynamics are of the form

$$dY_{S,T}(t) = Y_{S,T}(t)\delta_{S,T}(t)dW^S_t.$$

Since $Y_{S,T} = Z^{-1}_{S,T}$, an easy application of Itô's formula gives us $\delta_{S,T}(t) = -\sigma_{S,T}(t)$. Thus we have

$$Y_{S,T}(T) = \frac{p(0,T)}{S(0)} \exp\left\{-\frac{1}{2}\int_0^T \|\sigma_{S,T}\|^2(t)dt - \int_0^T \sigma_{S,T}(t)dW^S\right\},$$

and again we have a normally distributed exponent. Thus, after some simplification,

$$Q^S\left(S_T \geq K\right) = N[d_1],$$

where

$$d_1 = d_2 + \sqrt{\Sigma^2_{S,T}(T)}. \tag{15.42}$$

We have thus proved the following result.

Proposition 15.15 (Geman–El Karoui–Rochet) *Under the conditions given in Assumption 15.6.1, the price of the call option defined in (15.32) is given by the formula*

$$c_0 = S_0 N[d_1] - K \cdot p(0,T)N[d_2]. \tag{15.43}$$

Here d_2 and d_1 are given in (15.40) and (15.42) respectively, whereas $\Sigma^2_{S,T}(T)$ is defined by (15.39).

15.7 The Numeraire Portfolio

We have seen above that if we choose the price process S^0 of any asset without dividends, then there exists a martingale measure $Q^0 \sim P$ such that the normalized asset price

$$\frac{\Pi_t}{S_t^0}$$

is a Q^0 martingale for every arbitrage free asset price Π. The measure Q^0 will obviously depend on the numeraire asset S^0 and a natural question to ask is whether this machine can be made to run backwards. Assume, for simplicity, that there exists a bank account B and that we have chosen a fixed risk neutral martingale measure Q (i.e. a fixed price system). We then have the following problem.

Problem 15.7.1 *Suppose we are given a measure $Q^* \sim P$. Can we then find a numeraire N such that Q^* is the martingale measure for the numeraire N?*

15.7.1 *General Theory*

The general solution to the problem above turns out to be very simple. We start by noting that since we are given Q^* exogenously we can in principle compute the likelihood process $L^* = dQ^*/dQ$, so we can view L^* as exogenously given. Let us now provisionally assume that N exists, and without loss of generality we may assume that $N_0 = 1$. It then follows from (15.12) that we have

$$L_t^* = \frac{N_t}{B_t}$$

and we can now simply solve this for N. We then have the following result.

Proposition 15.16 *Assume that there exists a bank account B, and consider a fixed risk neutral martingale measure Q. Assume furthermore that we are given a measure $Q^* \sim P$, and define L^* by*

$$L_t^* = \frac{dQ^*}{dQ}, \quad \text{on } \mathcal{F}_t. \tag{15.44}$$

Define the process N by

$$N_t = B_t L_t^*. \tag{15.45}$$

Then N is an arbitrage free price process, and N is a numeraire process for Q^*, i.e. we have $Q^* = Q^N$.

Proof To show that N is an arbitrage free price process it is enough to show that N/B is a Q-martingale. By definition $N/B = L^*$, and since $L^* = dQ^*/dQ$, we know from general theory that L^* is a Q-martingale. Thus N/B is indeed a Q-martingale. The second part of the proposition follows from the definition (15.45) and (15.12). □

Note that, in an incomplete market, the price process N above is not unique since it depends on the risk neutral measure Q (and thus on the particular price system) that we have chosen. It is also important to understand that the price process N is merely an arbitrage free price process in the price system generated by our particular choice of Q. There is thus no guarantee that N can be realized as a portfolio based on the underlying assets in the market. For a complete market, however, the situation is simpler.

Proposition 15.17 *If the market is complete, then the price process N in (15.45) is unique, and N is in fact the value process on a portfolio based on the underlying assets.*

15.7.2 *The Objective Measure P as a Martingale Measure*

An interesting special case of the theory above occurs when we set $Q^* = P$. We are thus looking for a numeraire N with the property that the objective measure P is the martingale measure for N, i.e. that S_t/N_t is a P-martingale for every asset S. This is a famous object, so we give it a name.

Definition 15.18 *Given a choice of risk neutral measure Q, a* **numeraire portfolio** *is any asset price process N such that $Q^N = P$.*

Proposition 15.16 implies immediately that N is given by the formula

$$N_t = B_t L_t^P,$$

where we use the notation

$$L^P = dP/dQ.$$

This can be formulated in more familiar terms if we use the standard notation

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t.$$

We then have $L_t^P = L_t^{-1}$ and we can write

$$N_t = B_t L_t^{-1},$$

and we see that in fact

$$N_t = \mathbf{M}_t^{-1}$$

where $\mathbf{M}$ is the stochastic discount factor defined by $\mathbf{M}_t = B_t^{-1} L_t$. We sum up the results as a proposition.

Proposition 15.19 *Given a fixed choice of Q, the numeraire portfolio is given by any of the formulas*

$$N_t = B_t L_t^{-1}, \tag{15.46}$$

$$N_t = \mathbf{M}_t^{-1}. \tag{15.47}$$

In an incomplete market, N is merely an arbitrage free price process, but in a complete market N is the value process of a self-financing portfolio based on the underlying assets.

In Section 27.6 below we will in fact see that in a complete market, the numeraire portfolio is identical with the log optimal portfolio.

It now seems that we have two pricing formulas for a T-claim X, using the objective measure P. We have the numeraire pricing formula

$$\Pi_t[X] = N_t E^P\left[\left.\frac{X}{N_T}\right|\mathcal{F}_t\right]$$

and the stochastic discount factor pricing formula from (11.51)

$$\Pi_t[X] = \frac{1}{\mathbf{M}_t} E^P\left[\mathbf{M}_T X \right|\mathcal{F}_t].$$

From the theory above, we see that these formulas are in fact identical. In other words: Using a stochastic discount factor is equivalent to a change of numeraire to the numeraire portfolio.

15.8 Exercises

Exercise 15.1 Consider the model (15.23)–(15.24) and derive the pricing formula for the **maximum option**, defined by

$$X = \max\left[S_T^1, S_T^2\right].$$

Exercise 15.2 Consider the model (15.23)–(15.24) and derive the pricing formula for the **minimum option**, defined by

$$X = \min\left[S_T^1, S_T^2\right].$$

Exercise 15.3 Consider the model (15.23)–(15.24) and derive the pricing formula for the claim defined by

$$X = S_T^1 I\left\{S_T^2 \le K\right\}.$$

Exercise 15.4 Derive a pricing formula for European bond options in the Ho–Lee model.

Exercise 15.5 Prove Proposition 15.17.

15.9 Notes

The first usage of a numeraire different from the risk free asset B was probably in Merton (1973) where, however, the technique is not explicitly discussed. The first to explicitly use a change of numeraire was Margrabe (1978), who (referring to a discussion with S. Ross) used an underlying stock as numeraire in order to value an exchange option. The numeraire change is also used in

Harrison and Kreps (1979), Harrison and Pliska (1981) and basically in all later works on the existence of martingale measures in order to reduce (as we did in Chapter 11) the general case to the basic case of zero short rate. In these papers the numeraire change as such is however not put to systematic use as an instrument for facilitating the computation of option prices in complicated models. In the context of interest rate theory, changes of numeraire were then used and discussed independently by Geman (1989) and (in a Gaussian framework) Jamshidian (1989), who both used a bond maturing at a fixed time T as numeraire. A systematic study of general changes of numeraire has been carried out by Geman, El Karoui and Rochet in a series of papers, and many of the results above can be found in Geman et al. (1995). For further examples of the change of numeraire technique, see Benninga et al. (2002). The numeraire portfolio was introduced in Long (1990), and has since then been studied intensively.

16

DIVIDENDS

The object of the present chapter is to study pricing problems for contingent claims which are written on dividend-paying underlying assets, so at this point it may be a good idea to recall the results from Section 6.1. In real life the vast majority of all traded options are written on stocks having at least one dividend left before the date of expiration of the option. Thus the study of dividends is important from a practical point of view. Furthermore, it turns out that the theory developed in this chapter will be of use in the study of currency derivatives and we will also need it in connection with futures contracts. We start the analysis using the classical delta hedging arguments, but in Section 16.3 we will also study dividend-paying price processes using the martingale approach.

16.1 Discrete Dividends

In real life, all dividends are of course discrete: At some distinct points in time you get a lump sum of money. The *theory* of discrete dividends, on the other hand, is notoriously messy. A proper treatment would require a stochastic integral theory for jump processes, involving some rather delicate measure theoretic issues, and this is outside the scope of the present text. Nevertheless it seems unreasonable to skip the subject completely, so the section below is a light introduction using delta hedging arguments.

16.1.1 *Dividend Structure*

We consider an underlying asset ("the stock") with price process S, over a fixed time interval $[0, T]$. We consider a number of exogenously given deterministic points in time, $T_1, \ldots, T_K$, where

$$0 < T_1 < T_2 < \ldots < T_K < T.$$

The interpretation is that at these points in time dividends are paid out to the holder of the stock.

The first conceptual issue that we have to deal with concerns the interpretation of S_t as "the price of the stock at time t", and the problem to be handled is the following: Do we regard S_t as the price immediately **after**, or immediately **before**, the payment of a dividend? From a logical point of view we can choose any interpretation—nothing is affected in real terms, but our choice of interpretation will affect the notation below. We will in fact choose the first interpretation, i.e. we view the stock price as the price **ex dividend**. If we

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

think of the model on an infinitesimal time scale we have the interpretation that, if T_n is a dividend point, then dividends are paid out, not at the time point T_i, but rather at $T_n - dt$ or, if you will, at T_n-.

Next we go on to model the size of the dividends.

Assumption 16.1.1 *We assume that at the dividend time T_n the stock pays the amount δ_n in dividends. The dividend δ_n is allowed to be random, but we require that the condition*

$$\delta_n \in \mathcal{F}_{T_n-} \tag{16.1}$$

holds for each n. In other words: We know exactly the size of the dividend δ_n just before the dividend time T_n. We also assume that

$$\delta_n \le S_{T_n-}. \tag{16.2}$$

Our next problem concerns the behavior of the stock price at a dividend point t, and an easy (but slightly heuristic) arbitrage argument (see the exercises) gives us the following result.

Proposition 16.1 (Jump Condition) *In order to avoid arbitrage possibilities the following jump condition must hold at every dividend point T_n*

$$S_{T_n} = S_{T_n-} - \delta_n. \tag{16.3}$$

Proof The proof is left to the reader. □

We now see why we needed the condition (16.2). If that condition is not satisfied, it would by (16.3) imply that $S_{T_n} < 0$. This, however, cannot happen, since it would in fact be against corporate law concerning limited companies.

We finish this part with a small but important fact concerning options written on dividend-paying stocks.

Institutional Fact 16.1.1 *Suppose that you hold an option, say a European call with strike K, on a dividend-paying stock. At the exercise date you will then only receive the contractual value $max[S_T - K, 0]$. You will* **not** *receive any of the dividends that have been paid out.*

16.1.2 *The Price Structure*

We now go on to construct a model for the stock price process as well as for the dividend structure, and we assume that, under the objective probability measure P, the stock price has the following dynamics, **between dividends**:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \tag{16.4}$$

where μ and σ are adapted processes. To be quite precise we assume that the S-process satisfies the SDE above on each half-open interval of the form $[T_n, T_{n+1})$,

$i = 1, \ldots, K-1$, as well as on the intervals $[0, T_1)$ and $[T_K, T]$. The stock price structure can now be summarized as follows:

- Between dividend points the stock price process satisfies the SDE

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t.$$

- Immediately before a dividend time t, i.e. at $t- = t - dt$, we observe the stock price $S_{t-} = \lim_{u \uparrow t} S_u$.
- Given the stock price above, the size of the dividend is determined as δ_t.
- "Between" $t - dt$ and t the dividend is paid out.
- At time t the stock price has a jump, determined by

$$S_t = S_{t-} - \delta_t.$$

16.1.3 *A Black–Scholes Model with a Discrete Dividend*

In this section we will discuss the simplest possible case of derivatives pricing for a dividend-paying stock. The claim we wish to price is of the form

$$X = \Phi(S_T),$$

and we now need a dynamic model for the stock price process. We choose the simplest extension of the standard Black–Scholes model.

Assumption 16.1.2 *We assume the following*:

1. *The stock price has the following dynamics* **between dividends**:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{16.5}$$

$$dB_t = r B_t dt. \tag{16.6}$$

2. *We assume that there is exactly one dividend point* T_1 *with* $0 < T_1 < T$, *and that the dividend size is given by*

$$\delta_{T_1} = \delta \cdot S_{T_1-}, \tag{16.7}$$

where δ *in the right hand is a constant with* $0 \leq \delta \leq 1$.

The model above implies that on the interval $[0, T_1)$ the S-process will evolve according to GBM. At time T_1 there will be a jump given by

$$S_{T_1} = S_{T_1-} - \delta S_{T_1-},$$

and on the interval $[T_1, T]$ we will again have GBM. Note that if the constant $\delta = 1$, then we will get $S_{T_1} = 0$, so $S_t = 0$ for all t with $T_1 \leq t \leq T$.

Before going on to the formal theory let us, in order to get some feeling for the effect of dividends on derivatives pricing, give the following qualitative result for a European call option.

Result 16.1.1 *Assume that we have price dynamics as in Assumption 16.1.2, and that we want to price, at* $t = 0$, *a European call with strike prices* K *and* *exercise date T. Assume furthermore that there is a dividend date T_1 with $0 < T_1 < T$. Denoting the call price for the nondividend case by c^0 and the call price for the dividend case by c^d we then have*

$$c^d < c^0.$$

Proof Denoting the dividend stock price process by S_t^d and the non-dividend stock price by S_t^0 we have $S_t^d = S_t^0$ on the interval $[0, T_1)$. At T_1 there is a downward jump in S^d but not in S, and it follows that we have $S_T^d < S_T^0$. Since the contract function Φ for a call option is increasing in s, the result follows from risk neutral valuation. $\square$

16.1.4 *Option Pricing*

We now go on to price the T-claim $X = \Phi(S_T)$ above, and the idea is to do this recursively in backward time. We thus divide the interval $[0, T]$ into the half-open interval $[0, T_1)$ and the closed interval $[T_1, T]$, and we discuss the pricing problem for each of these intervals separately. We will denote the pricing function for the interval $[T_1, T]$ by G^0, and the pricing function for the interval $[0, T_1)$ by G^1. We will see below why we need the slightly complicated notation G^0 and G^1 (instead of the usual F) for the pricing function.

1. We start by computing $\Pi_t[X]$ for $t \in [T_1, T]$. Since our interpretation of the stock price is ex dividend, this means that we are actually facing a problem without dividends over this interval. Thus, for $T_1 \le t \le T$, we have $\Pi_t[X] = G^0(t, S_t)$ where G^0 solves the usual Black–Scholes equation

$$\begin{cases} \dfrac{\partial G^0}{\partial t} + rs\dfrac{\partial G^0}{\partial s} + \dfrac{1}{2}s^2\sigma^2\dfrac{\partial^2 G^0}{\partial s^2} - rG^0 &= 0, \\ G^0(T, s) = \Phi(s). \end{cases} \tag{16.8}$$

   In particular, the pricing function at T_1 is given by $G^0(T_1, s)$.

2. Now we go on to compute the price of the claim for $0 \le t < T_1$, and for this interval we use the notation G^1 to denote the pricing function. Suppose therefore that we are holding one unit of the contingent claim, and let us assume that the price at time T_1- is given by $S_{T_1-} = \lim_{t \uparrow T_1} = s$. By definition of G^1 we thus have the contract value $G^1(T_1-, s) = \lim_{t \uparrow T_1} G^1(t, s)$. The price s is the price cum dividend, so in the next infinitesimal interval the following will happen:
   - The dividend δs will be paid out to the shareholder (but not to the holder of the contract Φ).
   - At time T_1 the stock price will have dropped to $s - \delta s$.
   - At time T_1 we have no dividends left, so the value of contract is then given by $G^0(T_1, s - \delta s)$, where G^0 is determined by (16.8). We thus have the **jump condition**

$$G^1(T_1-, s) = G^0(T_1, s - \delta s). \tag{16.9}$$

3. It now remains to compute G^1 for $0 \le t < T_1$, but this turns out to be quite easy. We are holding a contingent claim on an underlying asset which over the interval $[0, T_1)$ is not paying dividends. Thus the standard Black–Scholes argument applies, which means that G^1 has to solve the usual Black–Scholes equation

$$\frac{\partial G^1}{\partial t} + rs\frac{\partial G^1}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 G^1}{\partial s^2} - rG^1 = 0$$

over this interval. The boundary value is now given by the jump condition (16.9) above.

We may summarize our results as follows.

Proposition 16.2 *Denote the pricing function on the entire interval* $[0, T]$ *by* F. *Then the following hold*:

- *On the interval* $[T_1, T]$ *we have* $F(t,s) = G^0(t,s)$, *where* G^0 *solves the boundary value problem*

$$\begin{cases} \dfrac{\partial G^0}{\partial t} + rs\dfrac{\partial G^0}{\partial s} + \dfrac{1}{2}s^2\sigma^2\dfrac{\partial^2 G^0}{\partial s^2} - rG^0 = 0, \\ G^0(T,s) = \Phi(s). \end{cases} \tag{16.10}$$

- *On the half-open interval* $[0, T_1)$ *we have* $F(t,s) = G^1(t,s)$ *where* G^1, *over the closed interval* $[0, T_1]$, *solves the boundary value problem*

$$\begin{cases} \dfrac{\partial G^1}{\partial t} + rs\dfrac{\partial G^1}{\partial s} + \dfrac{1}{2}s^2\sigma^2\dfrac{\partial^2 G^1}{\partial s^2} - rG^1 = 0, \\ G^1(T_i,s) = G^0(T_i, s - \delta s). \end{cases} \tag{16.11}$$

Throughout the entire section we have assumed the standard Black–Scholes price dynamics (16.4) between dividends and the simple dividend structure (16.7). It is easy to see that the same type of arguments can be applied to a situation with dividend points $0 < T_1 < \cdots < T_K < T$, stock price dynamics between dividend points

$$dS_t = S_t\mu(t, S_t)dt + S_t\sigma(t, S_t)dW_t \tag{16.12}$$

and dividend structure

$$\delta_t = \delta(S_{t-}),$$

where δ on the right-hand side denotes a deterministic function.

16.1.5 *Risk Neutral Valuation*

We now turn to the possibility of obtaining a probabilistic "risk neutral valuation" formula for the contingent claim above, and as in the PDE approach this can be in a recursive manner, using Feynman–Kač. After a number of easy but slightly messy calculations one can prove the following (very much expected) result.

Proposition 16.3 (Risk Neutral Valuation) *Consider a T-claim of the form $\Phi(S_T)$ as above. Assume that the price dynamics between dividends are given by*

$$dS_t = \alpha(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t,$$

and that the dividend size at a dividend point t is given by

$$\delta = \delta(S_{t-}).$$

Then the arbitrage free pricing function $F(t,s)$ has the representation

$$F(t,s) = e^{-r(T-t)} E_{t,s}^{Q}[\Phi(S_T)], \tag{16.13}$$

where the Q-dynamics of S between dividends are given by

$$dS_t = rS_t dt + \sigma(t, S_t)S_t dW_t^Q, \tag{16.14}$$

with the jump condition

$$S_t = S_{t-} - \delta(S_{t-}) \tag{16.15}$$

at each dividend point, i.e. at $t = T_1, T_2, \ldots, T_K$.

16.1.6 *An Example*

In this section we specialize to the previously studied case when we have the standard Black–Scholes dynamics

$$dS_t = \mu S_t dt + \sigma S dW_t \tag{16.16}$$

between dividend times $T_1, \cdots, T_K$, and the dividend structure has the particularly simple form

$$\delta(s) = \delta s, \tag{16.17}$$

where δ on the right-hand side denotes a positive constant.

As usual we consider the T-claim $\Phi(S_T)$ and, in order to emphasize the role of the parameter δ, we let $F^{\delta}(t,s)$ denote the pricing function for the claim Φ. In particular we observe that F^0 is our standard pricing function for Φ in a model with no dividends at all. Using the risk neutral valuation formula above, it is not hard to prove the following result.

Proposition 16.4 *Assume that the P-dynamics of the stock price and the dividend structure are given by (16.16)–(16.17). Then the following relation holds*:

$$F^{\delta}(t,s) = F^0(t, (1-\delta)^n \cdot s), \tag{16.18}$$

where n is the number of dividend points in the interval $(t, T]$.

Proof We will prove this using risk neutral valuation, and for simplicity of notation we confine ourselves to prove the result for the special case when $t = 0$, and thus $n = K$. On the interval $[0, T_1)$ the price process S is GBM so we have

$$S_t = se^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t^Q}$$

and, using the fact that the exponent is continuous in the t variable, we have in particular

$$S_{T_1-} = se^{(r-\frac{1}{2}\sigma^2)T_1+\sigma W_{T_1}^Q}.$$

At T_1 we have a dividend of size δS_{T_1-} and a corresponding jump in the S process, given by

$$S_{T_1} = S_{T_1-} - \delta S_{T_1-} = (1-\delta)S_{T_1-},$$

so we obtain

$$S_{T_1} = (1-\delta)se^{(r-\frac{1}{2}\sigma^2)T_1+\sigma W_{T_1}^Q}.$$

We now face GBM dynamics on the interval $[T_1, T_2)$ and a jump condition at T_2, and then the story repeats itself. At each dividend point we will get a multiplicative factor $(1-\delta)$, so it is clear from the calculations above that we will obtain

$$S_T = \left[(1-\delta)^K s\right] e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T^Q}. \tag{16.19}$$

This is the price at time T of a dividend-paying asset with initial price s and K dividends. From direct inspection and comparison with a standard Black–Scholes model it is however clear that, as far as the probability distribution is concerned, we can also view equation (16.19) as the price at T of a non-dividend-paying asset with initial price $s(1-\delta)^K$. The result now follows from risk neutral valuation. □

The point of this result is of course that in the simple setting of (16.16)–(16.17) we may use our "old" formulas for no-dividend models in order to price contingent claims in the presence of dividends. In particular we may use the standard Black–Scholes formula for European call options in order to price call options on a dividend-paying stock.

Proposition 16.5 (European Call Option) *Assume that the P-dynamics of the stock price and the dividend structure are given by (16.16)–(16.17). The price of a European call option with strike price K and time of maturity T is given by the formula $\Pi(t) = F(t, S_t)$, where*

$$F(t,s) = s(1-\delta)^n N[d_1(t,s)] - e^{-r(T-t)}KN[d_2(t,s)]. \tag{16.20}$$

Here n is the number of remaining dividend points, N is the cumulative distribution function for the $N[0,1]$ distribution, and

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + n\ln(1-\delta) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\},$$

$$d_2(t,s) = d_1(t,s) - \sigma\sqrt{T-t}.$$

16.2 Continuous Dividends I: Classical Methods

In this section and Section 16.3 we consider the case when dividends are paid out continuously in time. As usual S_t denotes the price of the stock at time t, and by D_t we denote the cumulative dividends over the interval $[0,t]$. Put in differential form this means that over the infinitesimal interval $(t,t+dt]$ the holder of the stock receives the amount $dD_t = D_{t+dt} - D_t$. See Chapter 6 for the necessary background on the cumulative dividend process and self-financing portfolios on dividend-paying assets. From Remark 6.3.1 we recall the following standing assumption.

Assumption 16.2.1 *We assume that the cumulative dividend process D_t has an Itô stochastic differential, implying in particular that it has continuous trajectories.*

We will start by analyzing an important special case—the case of a continuous dividend yield—using the classical Black–Scholes arguments, but in Section 16.3 we provide the much more general martingale-based theory.

16.2.1 *Continuous Dividend Yield*

We start by analyzing the simplest case of continuous dividends, which is when we have a **continuous dividend yield**. For simplicity we assume a standard Black–Scholes setting.

Assumption 16.2.2 *The price dynamics, under the objective probability measure, are given by*

$$dS_t = S_t\mu dt + S_t\sigma dW_t, \tag{16.21}$$

$$dB_t = rB_t dt. \tag{16.22}$$

The dividend structure is assumed to be of the form

$$dD_t = S_t \cdot q dt, \tag{16.23}$$

where q is a deterministic constant. We also assume a constant short rate r.

We note that, since we have no discrete dividends, we do not have to worry about the interpretation of the stock price as being ex dividend or cum dividend.

The problem to be solved is that of determining the arbitrage free price for a T-claim of the form $\Phi(S_T)$. This turns out to be quite easy, and we can in fact follow the strategy of Chapter 7. More precisely we recall the following scheme:

1. Assume that the pricing function is of the form $F(t, S_t)$.
2. Consider μ, σ, Φ, F, q, and r as exogenously given.
3. Use the general results from Section 6.2.3 to describe the dynamics of the value of a hypothetical self-financed portfolio based on the derivative instrument and the underlying stock.

4. Form a self-financed portfolio whose value process V has a stochastic differential without any driving Wiener process, i.e. it is of the form
$$dV_t = V_t k_t dt.$$
5. Since we have assumed absence of arbitrage we must have $k = r$.
6. The condition $k = r$ will in fact have the form of a partial differential equation with F as the unknown function. In order for the market to be efficient F must thus solve this PDE.
7. The equation has a unique solution, thus giving us the unique pricing formula for the derivative, which is consistent with absence of arbitrage.

We now carry out this scheme and, since the calculations are very close to those in Chapter 7, we will be rather brief.

Denoting the relative weights of the portfolio invested in the stock and in the derivative by w^S and w^F respectively we obtain (see Section 6.2.3) the value process dynamics as

$$dV_t = V_t \cdot \left\{ w^S \frac{dG_t}{S_t} + w^F \frac{dF_t}{F_t} \right\},$$

where the gain differential dG for the stock is given by

$$dG_t = dS_t + dD_t,$$

i.e.

$$dG_t = S_t(\mu + q)dt + \sigma S_t dW_t.$$

From the Itô formula we have the usual expression for the derivative dynamics

$$dF_t = \mu_F F_t dt + \sigma_F F_t dW_t,$$

where

$$\mu_F = \frac{1}{F}\left\{ \frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} \right\}, \qquad \sigma_F = \frac{1}{F} \cdot \sigma S \frac{\partial F}{\partial s}.$$

Collecting terms in the value equation gives us

$$dV_t = V_t \cdot \left\{ w^S(\mu + q) + w^F \mu_F \right\} dt + V_t \cdot \left\{ w^S \sigma + w^F \sigma_F \right\} dW_t,$$

and we now determine the portfolio weights in order to obtain a value process without a driving Wiener process, i.e. we define w^S and w^F as the solution to the system

$$\begin{aligned} w^S \sigma + w^F \sigma_F &= 0, \\ w^S + w^F &= 1. \end{aligned}$$

This system has the solution

$$w^S = \frac{\sigma_F}{\sigma_F - \sigma}, \qquad w^F = \frac{-\sigma}{\sigma_F - \sigma},$$

and leaves us with the value dynamics

$$dV_t = V_t \cdot \left\{ w^S(\mu + q) + w^F \mu_F \right\} dt.$$

Absence of arbitrage now implies that we must have the equation

$$w^S(\mu + q) + w^F \mu_F = r,$$

with probability 1, for all t, and, substituting the expressions for w^F, w^S, μ_F and σ_F into this equation, we get the equation

$$\frac{\partial F}{\partial t} + (r-q)S\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} - rF = 0.$$

The boundary value is obvious, so we have the following result.

Proposition 16.6 *The pricing function $F(t,s)$ of the claim $\Phi(S_T)$ solves the boundary value problem*

$$\begin{cases} \frac{\partial F}{\partial t}(t,s) + s(r-q)\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) = 0, \\ F(T,s) = \Phi(s). \end{cases} \tag{16.24}$$

Applying the Feynman–Kač representation theorem immediately gives us a risk neutral valuation formula.

Proposition 16.7 (Risk Neutral Valuation) *The pricing function has the representation*

$$F(t,s) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S_T)\right], \tag{16.25}$$

where the Q-dynamics of S are given by

$$dS_t = (r-q)S_t dt + \sigma(S_t) S_t dW_t^Q. \tag{16.26}$$

We recall that in the non-dividend case, the process S_t/B_t was a Q-martingale, so a natural question is to ask what kind of martingale properties we have in the present situation. The answer is given by the following result, which will be proved in much greater generality later in the chapter.

Proposition 16.8 *Under the martingale measure Q, the* **normalized gain process**

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s,$$

is a Q-martingale.

Proof Compute dG_t^Z and use the Q-dynamics of S. □

Note that this martingale property is quite reasonable from an economic point of view: in an arbitrage free economy today's stock price should be the arbitrage free value of all future discounted earnings which arise from holding the stock, plus the arbitrage free value of the stock price at the end of the time period under consideration. In other words, we expect that

$$S_0 = E^Q \left[\int_0^t e^{-rs} dD_s + e^{-rt} S_t \right], \tag{16.27}$$

and in the exercises the reader is invited to prove this "cost of carry" formula.

Since the pricing formulas (16.25)–(16.26) are so similar to the pricing formulas for the non-dividend case, one may perhaps guess that the corresponding pricing functions are closely related. This is indeed the case, and in order to show this let us denote the pricing function in (16.25) by F^q in order to highlight the dependence upon the parameter q. In particular this implies that the pricing function F^0 is the pricing function for the non-dividend case. If, for example, we are studying a European call, then F^0 is the Black–Scholes formula. It is now easy to prove the following result, which shows how to price a derivative for a dividend-paying stock in terms of the pricing function for the same derivative in the non-dividend case.

Proposition 16.9 *With notation as above, we have*

$$F^q(t, s) = F^0(t, se^{-q(T-t)}). \tag{16.28}$$

Proof We confine ourselves to prove the proposition when $t = 0$, i.e. we prove that $F^q(0, s) = F^0(0, se^{-qT})$. The case with an arbitrary t is then simply obtained by replacing T by $T - t$. Since the Q-dynamics of S are GBM we can write S_T as

$$S_T = se^{(r-q-\frac{1}{2}\sigma^2)T + \sigma W_T^Q},$$

and we can write this as

$$S_T = \left[se^{-qT}\right] e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T^Q}. \tag{16.29}$$

This is the value, at time T, of a dividend-paying stock with dividend yield q and initial stock price s. We see, however, by direct inspection of (16.29) that S_T has the same Q-distribution as the price, at time T, of a **non-dividend** stock with initial stock price se^{-qT}. Since the price is given by an expected value under Q, the result is thus proved. □

The moral of this result is thus that if, for example, you want to compute the price of a European call with continuous dividend yield, q, then you just replace s by $se^{-q(T-t)}$ in the standard Black–Scholes formula.

16.3 Continuous Dividends II: Martingale Methods

As we saw in Proposition 16.8, the natural martingale property of a price dividend pair (S, D) with continuous dividend process, is that the normalized gain process

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_u} dD_u$$

is a martingale under the risk neutral martingale measure with the bank account as numeraire. This result was derived within a simple Markovian framework using delta hedging and PDE techniques, but it should of course be possible to derive this result directly from the First Fundamental Theorem, and we would also like to understand the effect of using a numeraire other than the bank account.

16.3.1 *The Bank Account as Numeraire*

We start by using the bank account as numeraire, so we consider an arbitrage free market consisting of the bank account B and an asset with price process S and cumulative dividend process D. Our program is now as follows:

- Consider the self-financing portfolio where we hold exactly one unit of the asset S, and invest all dividends in the bank account. Denote the value process of this portfolio by V.
- The point is now that the portfolio V can be viewed as a **non-dividend-paying asset.**
- The market (V, B) is thus a standard market without dividends. The process V_t/B_t should consequently be a martingale under the risk neutral martingale measure Q with the numeraire B.

To carry out this program we have to derive the dynamics of the portfolio V, and to do this we denote by h_t the (so far unknown) number of units of the risk free asset B at time t, so the amount of money in the bank at time t is $h_t B_t$. Since we are holding exactly one unit of the underlying asset, the market value of the portfolio at time t will be given by

$$V_t = 1 \cdot S_t + h_t B_t. \tag{16.30}$$

We can now use the Itô formula to calculate the V dynamics as

$$dV_t = dS_t + h_t dB_t + B_t dh_t + dh_t dB_t = dS_t + h_t dB_t + B_t dh_t, \tag{16.31}$$

where we have used the fact that dB contains no dW term, so $dh_t dB_t = 0$. On the other hand, the self-financing condition says that

$$dV_t = 1 \cdot dS_t + 1 \cdot dD_t + h_t dB_t. \tag{16.32}$$

Comparing (16.31) with (16.32) we obtain

$$dS_t + h_t dB_t + B_t dh_t = dS_t + dD_t + h_t dB_t,$$

which gives us

$$dh_t = \frac{1}{B_t} dD_t,$$

or

$$h_t = \int_0^t \frac{1}{B_s} dD_s. \tag{16.33}$$

This result should not come as a big surprise. We can rewrite it as

$$h_t B_t = \int_0^t \frac{B_t}{B_s} dD_s.$$

and it simply says that the amount of money in the bank at time t is the sum of all the capitalized dividends payments over the interval $[0, t]$.

Since V can be viewed as a non-dividend-paying asset we know from general theory that V must be a Q martingale, so from (16.30) we conclude that the process

$$\frac{V_t}{B_t} = \frac{S_t + h_t B_t}{B_t} = \frac{S_t}{B_t} + h_t$$

is a Q martingale. Using (16.33) we have thus again derived the main result, but now in much greater generality.

Proposition 16.10 *Under the risk neutral martingale measure Q, the* **normalized gain process**

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_u} dD_u \tag{16.34}$$

is a Q-martingale. The arbitrage free price of a T-claim X is, as usual, given by

$$\Pi_t[X] = E^Q \left[e^{-\int_t^T r_s ds} X \middle| \mathcal{F}_t \right]. \tag{16.35}$$

16.3.2 *Continuous Dividend Yield Revisited*

In this section we will again analyze the continuous dividend yield model, but this time from a martingale perspective. We recall the model dynamics under P as

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{16.36}$$

$$dD_t = q S_t dt, \tag{16.37}$$

$$dB_t = r B_t dt. \tag{16.38}$$

Using the martingale machinery we always have the risk neutral valuation formula

$$\Pi_t[X] = e^{-r(T-t)} E^Q \left[\Phi(S_T) \middle| \mathcal{F}_t \right],$$

and in the present Markovian setting the pricing function is given by

$$F^q(t, s) = e^{-r(T-t)} E^Q_{t,s} \left[\Phi(S_T) \right],$$

so it only remains to find the martingale measure Q. Plugging the S and D dynamics into (16.34) gives us, after easy calculations,

$$dG_t^Z = Z_t (\mu - r + q) dt + Z_t \sigma dW_t$$

where $Z = S/B$.

As usual we now perform a Girsanov transformation form P to Q with likelihood dynamics

$$dL_t = L_t \varphi_t dW_t$$

which gives us

$$dW_t = \varphi_t dt + dW_t^Q.$$

Inserting this into dG^Z gives us the Q dynamics for G^Z as

$$dG_t^Z = Z_t \left(\mu - r + q + \sigma\varphi_t\right) dt + Z_t \sigma dW_t^Q.$$

From Proposition 16.10 we know that G^Z should be a Q-martingale, so we have the martingale condition

$$\mu - r + q + \sigma\varphi_t = 0,$$

with solution

$$\varphi = \frac{r - \mu - q}{\sigma}.$$

Plugging $dW_t = \varphi_t dt + dW_t^Q$ and the expression for φ into the P-dynamics for S, finally gives us the (expected) Q-dynamics of S.

Proposition 16.11 *For the model (16.36)–(16.38) with continuous dividend yield q, the Q-dynamics of S are given by*

$$dS_t = S_t \left(r - q\right) dt + S_t \sigma dW_t^Q. \tag{16.39}$$

16.3.3 *An Arbitrary Numeraire*

In this section we again consider a price dividend pair but instead of using the bank account B as numeraire we now want to use the price process A of another (non-dividend-paying asset) as numeraire. From general theory we know that, given absence of arbitrage, there will exist a martingale measure Q^A with the property that for every non-dividend-paying asset price S, the normalized price process S_t/A_t is a Q^A martingale. From this result, and by comparing with Proposition 16.10, one is easily led to conjecture that for the dividend-paying case, the process

$$\frac{S_t}{A_t} + \int_0^t \frac{1}{A_u} dD_u$$

should be a Q^A martingale but, as we will see below, this is **not** generally true.

We consider a price dividend pair (S, D) and the structure of the argument is exactly like in the previous section. We consider the portfolio where we hold exactly one unit of the asset S, and where all dividends are invested in the asset A. The value V of this portfolio can be viewed as the price process of a non-dividend-paying asset, so from general theory we conclude that the normalized process V/A should be a Q^A martingale.

For a general numeraire asset A the derivation of the V dynamics is more complicated than it was in the previous case when we used B as numeraire.

To facilitate the derivation we will need some new notation, which turns out to be very convenient.

Definition 16.12 *Let X and Y be Itô processes, with dynamics*

$$dX_t = \mu_X(t)dt + \sigma_X(t)dW_t,$$
$$dY_t = \mu_Y(t)dt + \sigma_Y(t)dW_t.$$

The process $\langle X, Y \rangle$ is defined by

$$\langle X, Y \rangle_t = \int_0^t \sigma_X(s)\sigma_Y(s)ds, \tag{16.40}$$

or equivalently by

$$d\langle X, Y \rangle_t = dX_t dY_t,$$

with the usual multiplication rules.

We now have some easy results concerning the angular bracket process.

Proposition 16.13 *For any real numbers μ and β and any Itô processes X, Y, and Z we have*

$$\langle X, Y \rangle_t = \langle Y, X \rangle_t,$$
$$\langle \mu X + \beta Y, Z \rangle_t = \mu \langle X, Z \rangle_t + \beta \langle Y, Z \rangle_t,$$
$$\langle X, \langle Y, Z \rangle \rangle \equiv 0.$$

Proof The proof is easy and left to the reader. □

A very important fact is that the angular bracket is also linear w.r.t stochastic integration. To formulate this we denote stochastic integration by $\bullet$ so that for two process h and X the integrated process $h \bullet X$ is defined by

$$(h \bullet X)_t = \int_0^t h_s dX_s.$$

Proposition 16.14 *With notation as above we have*

$$\langle h \bullet X, Y \rangle = h \bullet \langle X, Y \rangle,$$

or equivalently

$$d\langle h \bullet X, Y \rangle_t = h_t d\langle X, Y \rangle_t.$$

Proof Obvious from the definitions. □

We now go back to the construction of the self-financing portfolio. We denote the value process for the portfolio by V, and we denote by h_t the number of units of asset A in the portfolio at time t. Since, by definition, the portfolio consists of exactly one unit of the dividend-paying asset S, we have

$$V_t = S_t + h_t A_t. \tag{16.41}$$

Using Itô we obtain

$$dV_t = dS_t + h_t dA_t + A_t dh_t + d\langle h, A\rangle_t.$$

From the self-financing condition we have

$$dV_t = 1 \cdot dS_t + 1 \cdot dD_t + h_t dA_t,$$

and by comparing these expressions we obtain

$$dD_t = A_t dh_t + d\langle h, A\rangle_t. \tag{16.42}$$

In order to find an expression for dh we now multiply (16.42) by dA_t. This gives us

$$d\langle D, A\rangle_t = A_t d\langle h, A\rangle_t,$$

so

$$d\langle h, A\rangle_t = \frac{1}{A_t} d\langle D, A\rangle_t,$$

and from (16.42) we obtain

$$dh_t = \frac{1}{A_t} dD_t - \frac{1}{A_t^2} d\langle D, A\rangle_t,$$

or on integrated form

$$h_t = \int_0^t \frac{1}{A_s} dD_s - \int_0^t \frac{1}{A_s^2} d\langle D, A\rangle_s. \tag{16.43}$$

Applying the First Fundamental Theorem, we know that V/A should be a Q^A martingale, so from (16.41) and (16.43) we have the following result.

Proposition 16.15 *Assume that (S, D) is a price dividend pair and that A is the price process of a non-dividend-paying asset. Assuming absence of arbitrage we denote the martingale measure for the numeraire A by Q^A. Then the following hold:*

- *The normalized gain process G^Z defined by*

$$G_t^Z = \frac{S_t}{A_t} + \int_0^t \frac{1}{A_s} dD_s - \int_0^t \frac{1}{A_s^2} d\langle D, A\rangle_s \tag{16.44}$$

  is a Q^A martingale.
- *If the dividend process D has no driving Wiener component, or more generally if $d\langle D, A\rangle = 0$, then the gain process has the simpler form*

$$G_t^Z = \frac{S_t}{A_t} + \int_0^t \frac{1}{A_s} dD_s. \tag{16.45}$$

We see from this that the gain process G^Z above differs from the "naive" definition (16.45) by a term which is connected to the covariation $d\langle D, A\rangle$ between the dividend process D and the numeraire process A.

16.4 Exercises

Exercise 16.1 Prove Proposition 16.1. Assume that you are standing at T_n- and that the conclusion of the theorem does not hold. Show that by trading at T_n- and T_n you can then create an arbitrage. This is mathematically slightly imprecise, and the advanced reader is invited to provide a precise proof based on the martingale approach of Chapter 11.

Exercise 16.2 Prove the cost of carry formula (16.27).

Exercise 16.3 Prove Proposition 16.8.

Exercise 16.4 Consider the Black–Scholes model with a constant continuous dividend yield q. Prove the following put–call parity relation, where c_q (p_q) denotes the price of a European call (put):

$$p_q = c_q - se^{-q(T-t)} + Ke^{-r(T-t)}.$$

Exercise 16.5 Consider the Black–Scholes model with a constant continuous dividend yield q. The object of this exercise is to show that this model is complete. Consider therefore a contingent claim $\mathcal{X} = \Phi(S_T)$. Show that this claim can be replicated by a self-financing portfolio based on B and S, and derive the portfolio weights.

Hint: Copy the reasoning from Chapter 8, while using the self-financing dynamics given in Section 6.2.3.

Exercise 16.6 Consider the Black–Scholes model with a constant continuous dividend yield q. Use the result from the previous exercise in order to compute explicitly the replicating portfolio for the claim $\Phi(S_T) = S_T$.

17

FORWARD AND FUTURES CONTRACTS

Consider a financial market of the type presented in the previous chapters, with a (possibly stochastic) short rate r. We assume that the market is arbitrage free, and that pricing is done under some fixed risk neutral martingale measure Q (with, as usual, B as numeraire).

Let us now consider a fixed T-claim X, and assume that we are standing at time t. If we buy X and **pay today**, i.e. at time t, then we know that the arbitrage free price is given by

$$\Pi_t[X] = E^Q_{t,x}\left[X \cdot e^{-\int_t^T r(s)ds}\right] \tag{17.1}$$

and the payment streams are as follows:

1. At time t we pay $\Pi_t[X]$ to the underwriter of the contract.
2. At time T we receive X from the underwriter.

There are two extremely common variations of this type of contract, namely **forward** and **futures** contracts. Both these contracts have the same claim X as their underlying object, but they differ from our standard contract above by the way in which payments are made.

Before going into the contracts we recall from Proposition 11.21 the pricing formula for a zero coupon bond with face value 1 and maturity T as

$$p(t,T) = E^Q\left[e^{-\int_t^T r(s)ds} \,\middle|\, \mathcal{F}_t\right]. \tag{17.2}$$

We will also use the forward measure Q^T discussed in Section 15.5.

17.1 Forward Contracts

We will start with the conceptually easiest contract, which is the forward contract. This is an agreement between two parties to buy or sell a certain underlying claim at a fixed time T in the future. The difference between a forward contract and our standard claims studied so far is that for a forward contract all payments are made at time T. To be more precise we give a definition.

Definition 17.1 *Let X be a contingent T-claim. A* **forward contract** *on X,* **contracted at** *t, with* **time of delivery** *T, and with the* **forward price** *$f(t;T,X)$, is defined by the following payment scheme*:

- *The holder of the forward contract receives, at time T, the stochastic amount X from the underwriter.*

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

- *The holder of the contract pays, at time T, the amount $f(t;T,X)$ to the underwriter.*
- *The forward price $f(t;T,X)$ is determined at time t.*
- *The forward price $f(t;T,X)$ is determined in such a way that the price of the entire forward contract equals zero, at the time t when the contract is made.*

Forward markets are typically not standardized, so forward contracts are usually traded as OTC ("over the counter") instruments. Note that even if the value of a specific forward contract equals zero at the time t of writing the contract, it will typically have a non-zero market value which varies stochastically in the time interval $[t,T]$.

Our immediate project is to give a mathematical formalization of the definition above, and to derive a theoretical expression for the forward price process $f(t;T,X)$. This turns out to be quite simple, since the forward contract itself is a contingent T-claim ξ, defined by

$$\xi = X - f(t;T,X).$$

We are thus led to the following mathematical definition of the forward price process.

Definition 17.2 *Let X be a contingent T-claim as above. By the* **forward price process** *we mean a process $t \longmapsto f(t;T,X)$ with the property that*

$$\Pi_t[X - f(t;T,X)] = 0. \tag{17.3}$$

Writing the forward price as $f(t;T,X)$ formalizes the fact that the forward price is determined at t, given the information that is available at that time.

We now have the following basic formula for the forward price process.

Proposition 17.3 *The forward price process is given by any of the following expressions.*

$$f(t;T,X) = \frac{\Pi_t[X]}{p(t,T)}, \tag{17.4}$$

$$f(t;T,X) = \frac{1}{p(t,T)} E^Q\left[X \cdot e^{-\int_t^T r(s)ds} \,\middle|\, \mathcal{F}_t\right], \tag{17.5}$$

$$f(t;T,X) = E^T\left[X \middle| \mathcal{F}_t\right]. \tag{17.6}$$

where E^T denotes expectation under the T-forward measure Q^T.

Proof Using (17.3), the fact that $f(t;X,T) \in \mathcal{F}_t$, and risk neutral valuation, we immediately have the following identities, where we write suppress T and X and write $f(t)$ instead of $f(t;T,X)$

$$0 = \Pi_t[X - f(t)] = E^Q\left[\{X - f(t)\} \cdot e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= E^Q_{t,X_t}\left[X \cdot e^{-\int_t^T r_s ds}\right] - E^Q\left[f(t)\cdot e^{-\int_t^T r_s ds}\middle|\mathcal{F}_t\right] \\
&= \Pi_t[X] - f(t)E^Q\left[e^{-\int_t^T r_s ds}\middle|\mathcal{F}_t\right] \\
&= \Pi_t[X] - f(t)p(t,T).
\end{aligned}
$$

This immediately gives us (17.4)–(17.5). The relation (17.6) then follows from (17.5) and the general pricing formula (15.31). □

Note that when dealing with forward contracts there is some risk of conceptual confusion. If we fix t, T, and X, and let s be a fixed point in time with $t \le s \le T$, then there will be two different prices:

1. The forward price $f(s;T,X)$ which is paid to the underwriter at time T for a forward contract made at time s.
2. The (spot) price, at time s, of a fixed forward contract, entered at time t, and with time T of delivery. This spot price is easily seen to be equal to

$$\Pi_s[X] - p(s,T)f(t;T,X).$$

17.2 Futures Contracts

A futures contract is very much like a forward contract in the sense that it is an agreement between two parties to buy or sell a certain claim at a prespecified time T in the future. The principal difference between the two contracts lies in the way in which payments are being made. We first give a verbal definition of the futures contract.

Definition 17.4 *Let X be a contingent T-claim. A* **futures contract** *on X, with* **time of delivery** *T, is a financial asset with the following properties*:

(i) *At every point of time t with $0 \le t \le T$, there exists in the market a quoted object $F(t;T,X)$, known as the* **futures price** *for X at t, for delivery at T.*
(ii) *At the time T of delivery, the holder of the contract pays $F(T;T,X)$ and receives the claim X.*
(iii) *During an arbitrary time interval $(s,t]$ the holder of the contract receives the amount $F(t;T,X) - F(s;T,X)$.*
(iv) *The spot price, at any time t prior to delivery, of obtaining the futures contract, is by definition equal to zero.*

A rough way of thinking about a futures contract is to regard it as a forward contract where the payments are made continuously in time in the way described above, rather than all payments being made at time T. As the forward price increased, you would then get richer, and as the forward price decreased you would lose money. The reason that this way of looking at the futures contract is not entirely correct is the fact that if we start with a standard forward contract, with its associated forward price process f, and then introduce the above payment scheme over time, this will (through supply and demand) affect the original forward price process, so generically we will expect the futures price process F to be different from the forward price process f. The payment schedule above is known as "marking to market"; it is organized in such a way that the holder of a futures position, be it short or long, is required to keep a certain amount of money with the broker as a safety margin against default.

Futures contracts, as opposed to forward contracts, are traded in a standardized manner at an exchange. The volumes in which futures are traded over the world are astronomical, and one of the reasons for this is that on many markets it is difficult to trade (or hedge) directly in the underlying object. A typical example is the commodity market, where you actually have to deliver the traded object (tons of copper, timber, or ripening grapes), and thus are not allowed to go short. In these markets, the futures contract is a convenient financial instrument which does not force you to physically deliver the underlying object, while still making it possible for you to hedge (or speculate) against the underlying object.

We now note some properties of the futures contract:

- From (ii) and (iv) above it is clear that we must have

$$F(T;T,X) = X. \tag{17.7}$$

  Thus there is really no economic reason to actually deliver either the underlying claim or the payment at time T. This is also an empirical fact; the vast majority of all futures contracts are closed before the time of delivery.
- If you enter a futures contract at time t with a corresponding futures price $F(t;T,X)$, this does **not** mean that you are obliged to deliver X at time T at the price $F(t;T,X)$. The only contractual obligation is the payment stream defined above.
- The name futures **price** is therefore somewhat unfortunate from a linguistic point of view. If today's futures price is given by $F(t;T,X)$ (with $t < T$) this does not mean that anyone will ever pay the amount $F(t;T,X)$ in order to obtain some asset. It would perhaps be more clear to refer to $F(t;T,X)$ as the futures **quotation**.
- Since, by definition, the spot price of a futures contract equals zero, there is no cost or gain of entering or closing a futures contract.
- If the reader thinks that a futures contract conceptually is a somewhat complicated object, then the author is inclined to agree.

We now turn to the mathematical formalization of the futures contract, and is should by now be clear that the natural model for a futures contract is an asset with dividends. See Chapter 6 and Section 16.3 for more information on dividends.

Definition 17.5 *The* **futures contract** *on an underlying T-claim X is a price dividend pair (S, D) with a spot price process S_t and a cumulative dividend process D_t satisfying the following conditions*:

$$D_t = F(t; T, X). \tag{17.8}$$

$$F(T; T, X) = X. \tag{17.9}$$

$$S_t = 0, \quad \forall t \le T. \tag{17.10}$$

It now remains to investigate what the futures price process looks like. This turns out to be quite simple, and we can now prove the main result of the section.

Proposition 17.6 *Let X be a given contingent T-claim, and assume that market prices are obtained from the fixed risk neutral martingale measure Q. Then the following hold:*

- *The futures price process is a Q-martingale, and it is thus given by*

$$F(t; T, X) = E^Q[X | \mathcal{F}_t]. \tag{17.11}$$

- *If the short rate is deterministic, then the forward and the futures price processes coincide, and we have*

$$f(t; T, X) = F(t; T, X) = E^Q[X | \mathcal{F}_t]. \tag{17.12}$$

Proof From Proposition 16.3, it follows that the discounted gains process

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dF(s; T, X)$$

is a Q-martingale and thus, by the Martingale Representation Theorem, it can be written as

$$dG_t^Z = h_t dW_t^Q$$

for some adapted process h. In our case we furthermore have $S_t = 0$ for all t, so we obtain the representation

$$\frac{1}{B_t} dF(t; T, X) = h_t dW_t^Q.$$

Multiplying by B_t on both sides we get

$$dF(t; T, X) = B_t h_t dW_t^Q,$$

which (modulo a bit of integrability) implies that $F(t; T, X)$ is a Q-martingale. Using the martingale property and the fact that $F(T; T, X) = X$ we obtain

$$F(t; T, X) = E^Q[X | \mathcal{F}_t],$$

which proves the first part of the assertion. The second part follows from the fact that $Q^T = Q$ when the short rate is deterministic (see Lemma 15.13). □

Remark 17.2.1 *In the proof above we have used the fact that we have a Wiener-driven model, but the result is in fact true in much greater generality.*

17.3 Futures Options and Black-76

In this section we consider a futures contract on an underlying claim X with delivery date T_1, and denote the futures price process $F(t;T_1,X)$ by

$$F(t,T_1).$$

We sometimes also suppress T_1 and write F_t, i.e. $F_t = F(t,T_1)$, and we refer to this contract as a T_1-future.

17.3.1 *Generalities*

Today there is a huge volume of **futures options** being traded on the existing financial markets. The futures options market is in fact one of the most liquid markets in existence, so what then is a futures option?

A natural idea would perhaps be that a futures call option, on the T_1-futures contract above, would be a contract which gives the holder the right but not the obligation to buy the underlying T_1-future at a pre-specified price K, on the predetermined date T (where of course $T \le T_1$). This, however, is **completely wrong**. Since the spot price of a future is always equal to zero, no one would be willing to buy the T_1-future at the price K, since you could instead get the future for free on the futures market. the formal definition is instead as follows.

Definition 17.7 *A European futures call option, with strike price K and exercise date T, on a futures contract with delivery date T_1 will, if exercised at T, give to the holder:*

1. *The amount $F(T,T_1) - K$ in* **cash**.
2. *A long position in the underlying futures contract.*

We note that the long position can immediately be closed at zero cost, so Item 2 above has no financial value. For computational purposes we can thus use Item 1 above as the definition of the futures call option. A very natural question is now why futures options exist in the first place. Why would you buy a futures option instead of an option on the underlying market? There are in fact several good reasons for this:

- On many markets (such as commodity markets) the futures market is much more liquid than the underlying market.
- Futures options are typically settled in **cash**. This relieves you from handling the underlying (tons of copper, hundreds of pigs, etc.).
- The market place for futures and futures options is often the same. This facilitates hedging etc.

17.3.2 *The Black-76 Formula*

In this section we will derive the famous Black-76 formula for a European futures call option. As above, consider a futures contract with delivery date T_1 and use the notation $F_t = F(t,T_1)$.

Assumption 17.3.1 *We assume the following P-dynamics for F and B*

$$dF_t = \mu F_t dt + \sigma F_t dW_t, \tag{17.13}$$

$$dB_t = rB_t dt. \tag{17.14}$$

We will in fact treat a more general case than that of a call option. More precisely we want to price a derivative with exercise date $T < T_1$ with the T_1-futures price F as underlying, i.e. a claim of the form

$$\Phi(F_T),$$

and for a futures call with strike price K we would have the special case

$$\Phi(F) = \max[F - K, 0].$$

From standard risk neutral valuation we know that the price process $\Pi_t[\Phi]$ is of the form

$$\Pi_t[\Phi] = f(t, F_t)$$

where the pricing function f (not to be confused with a forward price) is given by

$$f(t, F) = e^{-r(T-t)} E^Q_{t,F}[\Phi(F_T)]$$

so it only remains to find the Q-dynamics for F.

We now recall that the futures price process F_t is a Q-martingale, so the Q-dynamics of F are given by

$$dF_t = \sigma F_t dW^Q_t.$$

We thus have

$$f(t, F) = e^{-r(T-t)} E^Q_{t,F}[\Phi(F_T)] \tag{17.15}$$

with Q-dynamics

$$dF_t = \sigma F_t dW^Q_t. \tag{17.16}$$

From Proposition 16.7 we recall that the pricing function f for an option on a stock with continuous dividend yield q is given by

$$f(t, s) = e^{-r(T-t)} E^Q_{t,s}[\Phi(S_T)], \tag{17.17}$$

with Q-dynamics

$$dS_t = (r - q)S_t + \sigma S_t dW^Q_t. \tag{17.18}$$

We now note that if we set $q = r$ in (17.18), the formulas (17.15)–(17.16) and (17.17)–(17.18) are (apart from the variable names) identical. This implies that we can draw on the results from Section 16.2.

Let thus $f^0(t, s)$ be the pricing function for the contract $\Phi(S_T)$ for the case when S is a Black–Scholes price stock with volatility σ and no dividends. Let $f(t, F)$ be the pricing formula for the futures claim $\Phi(F_T)$. We then have the following result.

Proposition 17.8 *With notation as above we have*

$$f(t,F) = f^0(t, Fe^{-r(T-t)}). \tag{17.19}$$

The moral of this is that in order to compute the price of the futures contract $\Phi(F_T)$, we simply reset today's futures price F to $Fe^{-r(T-t)}$ and use our old formulas for the contract $\Phi(S_T)$ where S is the stock price in a Black–Scholes model with volatility σ. In particular we can consider the case of a European call with strike price K. In this case the function f^0 is simply the Black–Scholes formula, and we obtain the following result.

Proposition 17.9 (Black-76) *The price of a futures option with exercise date T and exercise price K is given by*

$$\begin{aligned} c &= e^{-r(T-t)}\left\{FN[d_1] - KN[d_2]\right\}, \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2(T-t)\right\}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

17.4 Exercises

Exercise 17.1 Suppose that S is the price process of a non-dividend-paying asset. Show that the forward price $f(t;T,X)$ for the T-claim $\mathcal{Y} = S_T$ is given by

$$f(t;T,S_T) = \frac{S_t}{p(t,T)}.$$

Exercise 17.2 Suppose that S is the price process of a dividend-paying asset with dividend process D. Show that the forward price $f(t;T,S_T)$ is given by the cost of carry formula

$$f(t;T,S_T) = \frac{1}{p(t,T)}\left(S_t - E^Q\left[\int_t^T e^{-\int_t^s r(u)du}dD_s \middle| \mathcal{F}_t\right]\right).$$

Hint: Use the cost of carry formula for dividend-paying assets.

Exercise 17.3 Suppose that S is the price process of an asset in a standard Black–Scholes model, with r as the constant rate, and fix a contingent T-claim $\Phi(S_T)$. We know that this claim can be replicated by a portfolio based on the money account B, and on the underlying asset S. Show that it is also possible to find a replicating portfolio, based on the money account and on futures contracts for S_T.

17.5 Notes

For a wealth of information on forwards and futures, see Hull (2003) and Duffie (2001).

18

CURRENCY DERIVATIVES

In this chapter we will study a model which incorporates not only the usual domestic equity market, but also a market for the exchange rate between the domestic currency and a fixed foreign currency, as well as a foreign equity market. Financial derivatives defined in such situations are commonly known as **quanto products**.

We will start by studying derivatives written directly on the exchange rate X, and this will be done using the classical Black–Scholes methodology. We will then develop a much more general theory based on martingale techniques and, among other things, go on to study how to price (in the domestic currency) contracts written on foreign equity.

18.1 Pure Currency Contracts

Consider a situation where we have two currencies: the domestic currency (say euros), and the foreign currency (say US dollars). The spot exchange rate at time t is denoted by X_t, and by definition it is quoted as

$$\frac{\text{units of the domestic currency}}{\text{unit of the foreign currency}},$$

so in our example it is quoted as euros per US dollar. The point of quoting the exchange rate like this is that, from the domestic (European) perspective, the entity X_t is then a **domestic price** of a certain asset (the US dollar).

We will start our investigation by studying the simplest possible model—the Garman–Kohlhagen model. To this end we assume that the domestic short rate r^d, as well as the foreign short rate r^f, are deterministic constants, and we denote the corresponding riskless asset prices by B^d and B^f respectively. Furthermore we assume that the exchange rate is modeled by Geometric Brownian Motion. We can summarize this as follows.

Assumption 18.1.1 *We consider the following dynamics under the objective probability measure P:*

$$dX_t = X_t \alpha dt + X_t \sigma dW_t, \tag{18.1}$$

$$dB_t^d = r^d dB_t^d dt, \tag{18.2}$$

$$dB_t^f = r^f B_t^f dt, \tag{18.3}$$

where r^d, r^f, α, and σ are deterministic constants, and W is a scalar P-Wiener process.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

Our problem is that of pricing a currency derivative, i.e. a T-claim $\mathcal{Z}$ of the form

$$\mathcal{Z} = \Phi(X_T),$$

where Φ is some given deterministic function. To take a concrete and important example, we can consider the case when $\mathcal{Z} = \max[X_T - K, 0]$, i.e. we have a European call which gives the owner the option to buy one unit of the foreign currency at the price K (in the domestic currency).

At first glance it may perhaps seem that the problem of pricing the call option above is solved by use of the standard Black–Scholes formula, where we use domestic rate r^d as the short rate, and the stock price S is replaced by the exchange rate X. It is, however, important to understand that this line of argument is incorrect, and the reason is as follows. When we buy a stock (without dividends), this means that we buy a piece of paper, which we keep until we sell it. When we buy a foreign currency (say US dollars) we will, on the contrary, not just keep the physical dollar bills until we sell them again. Instead we will typically put the dollars into an account where they will grow with a certain interest rate. The obvious implication of this fact is that a foreign currency seems to play very much the same role as a **domestic stock with a continuous dividend yield**, and we will show below that this is indeed the case. First we formalize the institutional assumptions.

Assumption 18.1.2 *All markets are frictionless and liquid. All holdings of the foreign currency are invested in the foreign riskless asset, i.e. they will evolve according to the dynamics*

$$dB_t^f = r^f B_t^f \, dt.$$

Remark 18.1.1 *Interpreted literally this means that, for example, US dollars are invested in a US bank. In reality this does not have to be the case—US dollars bought in Europe will typically be placed in a European so-called* **Eurodollar account** *where they will command the Eurodollar interest rate which is very close to the US interest rate.*

Applying the standard theory of derivatives to the present **domestic** pricing problem have the usual risk neutral valuation formula

$$\Pi_t[\mathcal{Z}] = e^{-r^d(T-t)} E_{t,x}^Q [\Phi(X_T)],$$

and our only problem is to figure out what the martingale measure Q looks like. To do this we use the fact that Q is characterized by the property that every non-dividend-paying **domestic** asset has the short rate r^d as its local rate of return under Q. In order to use this characterization we have to translate the possibility of investing in the foreign riskless asset into domestic terms. Since B_t^f units of the foreign currency are worth $B_t^f \cdot X_t$ in the domestic currency we immediately have the following result.

Lemma 18.1 *The possibility of buying the foreign currency, and investing it at the foreign short rate, is equivalent to the possibility of investing in a* **non-dividend domestic asset** *with price process B^X, where*

$$B_t^X = B_t^f \cdot X_t. \tag{18.4}$$

The dynamics of B^X are given by

$$dB_t^X = B_t^X \left(\alpha + r^f\right) dt + B_t^X \sigma dW. \tag{18.5}$$

Summing up we see that our currency model is equivalent to a model of a domestic market consisting of the assets (B^d, B^X) where B^d is the domestic risk free asset and B^X is the domestic risky asset. It now follows directly from standard arbitrage theory that the martingale measure Q has the property that the Q-dynamics of B^X are given by

$$dB_t^X = r^d B_t^X dt + B_t^X \sigma dW_t^Q, \tag{18.6}$$

where W^Q is a Q-Wiener. Since by (18.4) we have

$$X_t = \frac{B_t^X}{B_t^f}, \tag{18.7}$$

we can use Itô's formula, (18.3) and (18.6) to obtain the Q-dynamics of X as

$$dX_t = X_t \left(r^d - r^f\right) dt + X_t \sigma dW_t^Q. \tag{18.8}$$

The basic pricing result follows immediately.

Proposition 18.2 (Pricing currency contracts) *The arbitrage free price $\Pi_t[\Phi]$ for the T-claim $\mathcal{Z} = \Phi(X_T)$ is given by $\Pi_t[\Phi] = F(t, X_t)$, where*

$$F(t,x) = e^{-r^d(T-t)} E_{t,x}^Q \left[\Phi(X_T)\right], \tag{18.9}$$

and where the Q-dynamics of X are given by

$$dX_t = X_t \left(r^d - r^f\right) dt + X_t \sigma dW_t^Q. \tag{18.10}$$

Alternatively $F(t,x)$ can be obtained as the solution to the boundary value problem

$$\begin{cases} \dfrac{\partial F}{\partial t} + x(r^d - r^f)\dfrac{\partial F}{\partial x} + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 F}{\partial x^2} - r^d F = 0, \\ \hfill F(T,x) = \Phi(x). \end{cases} \tag{18.11}$$

Proof The risk neutral valuation formula (18.9)–(18.10) follows from the standard risk neutral valuation formula and (18.8). The PDE result then follows via Feynman–Kač. □

Comparing (18.10) to (16.39) we see that our original guess was correct: a foreign currency is to be treated exactly as a (domestic) stock with a continuous dividend yield q with $q=r^f$. We may thus draw upon the results from Section 16.2 (see Proposition 16.9), which allows us to use pricing formulas for stock prices (without dividends) to price currency derivatives.

Proposition 18.3 (Currency Option Formula) *Let $F_0(t,x)$ be the pricing function for the claim $\mathcal{Z}=\Phi(X_T)$, in a model where we interpret X as the price of an ordinary stock without dividends. Let $F(t,x)$ be the pricing function of the same claim when X is interpreted as an exchange rate. Then the following relation holds*

$$F(t,x)=F_0(t,xe^{-r^f(T-t)}).$$

In particular, the price of the European call, where $\mathcal{Z}=max[X_T-K,0]$, on the foreign currency, is given by the modified Black–Scholes formula

$$F(t,x)=xe^{-r^f(T-t)}N\left[d_1\right]-e^{-r^d(T-t)}KN\left[d_2\right], \tag{18.12}$$

where

$$\begin{aligned} d_1(t,x) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{x}{K}\right)+\left(r^d-r^f+\frac{1}{2}\sigma_X^2\right)(T-t)\right\}, \\ d_2(t,x) &= d_1(t,x)-\sigma\sqrt{T-t}. \end{aligned}$$

18.2 The Martingale Approach

In this section we will derive the main results of foreign currency pricing theory using the martingale approach, and as can be expected this turns out to be both easier and more general than using the classical approach. We assume a filtered probability space $(\Omega,\mathcal{F},P,\mathbf{F})$ carrying a multidimensional Wiener process W, and we consider a domestic and a foreign market. The exchange rate process is denoted by X and the interpretation is again that

$$X_t=\frac{\text{units of the domestic currency}}{\text{unit of the foreign currency}}.$$

We allow the domestic short rate r^d, as well as the foreign short rate r^f, to be adapted random processes, and we denote the corresponding bank accounts by B^d and B^f respectively.

In this setting there will, at least potentially, exist several martingale measures and several associated likelihood processes.

The domestic and foreign martingale measures are denoted by Q^d and Q^f respectively, and we define the following likelihood processes:

$$L_t^d=\frac{dQ^d}{dP}, \qquad L_t^f=\frac{dQ^f}{dP}, \qquad L_t=\frac{dQ^f}{dQ^d}. \tag{18.13}$$

The corresponding Wiener processes under Q^d and Q^f are denoted by W^d and W^f. The Girsanov kernels corresponding to L^d and L^f and L are denoted by φ^d, φ^f, and φ so the likelihood processes have the dynamics

$$dL_t^d = L_t^d \varphi_t^d dW_t, \tag{18.14}$$

$$dL_t^f = L_t^f \varphi_t^f dW_t, \tag{18.15}$$

$$dL_t = L_t \varphi_t dW_t^d, \tag{18.16}$$

and the corresponding **market prices of risk** processes are defined as

$$\lambda_t^d = -\varphi_t^d, \quad \lambda_t^f = -\varphi_t^f, \quad \lambda_t = -\varphi_t. \tag{18.17}$$

The domestic and foreign stochastic discount factors are denoted by $\mathbf{M}^d$ and $\mathbf{M}^f$, so we have

$$\mathbf{M}_t^d = e^{-\int_0^t r_s^d ds} L_t^d, \tag{18.18}$$

$$\mathbf{M}_t^f = e^{-\int_0^t r_s^f ds} L_t^f. \tag{18.19}$$

Our task is now to find out how all these objects are related, and this is in fact quite easy. We fix an arbitrary exercise date T and consider an arbitrarily chosen foreign contingent claim Z^f. The claim denotes the amount of money in the **foreign** currency that the holder of the claim will receive at time T. If we interpret the foreign market as the US market and the domestic market as Euroland, then the claim Z^f will thus be denoted in dollars.

We will now compute the **domestic** price of the claim Z^f at time $t = 0$ in two different ways, and then we will compare the results.

We start by computing the arbitrage free price of Z^f at $t = 0$, expressed in the foreign currency. By (foreign) risk neutral valuation this is given by

$$\Pi_0^f [Z^f] = E^{Q^f} \left[e^{-\int_0^t r_s^f ds} Z^f \right].$$

By changing foreign currency to the domestic currency at time $t = 0$, the domestic arbitrage free price of Z^f is thus given by

$$\Pi_0^d [Z^f] = X_0 \Pi_0^f [Z^f] = X_0 E^{Q^f} \left[e^{-\int_0^t r_s^f ds} Z^f \right]. \tag{18.20}$$

On the other hand, by changing from the foreign to the domestic currency at $t = T$, we can express the foreign claim Z^f at time T as the **equivalent domestic claim** Z^d defined by

$$Z^d = X_T Z^f.$$

By (domestic) risk neutral valuation, the domestic arbitrage free price of Z^d is given by

$$\Pi_0^d [Z^f] = E^{Q^d} \left[e^{-\int_0^t r_s^d ds} Z^d \right] = E^{Q^d} \left[e^{-\int_0^t r_s^d ds} X_T Z^f \right]. \tag{18.21}$$

Given absence of arbitrage on the international market, the domestic prices at $t = 0$ of Z^f and Z^d must coincide, so from (18.20)and (18.21) we obtain

$$X_0 E^{Q^f} \left[e^{-\int_0^t r_s^f ds} Z^f \right] = E^{Q^d} \left[e^{-\int_0^t r_s^d ds} X_T Z^f \right].$$

We can now rewrite this, under the domestic martingale measure Q^d as

$$E^{Q^d}\left[X_0 L_T e^{-\int_0^t r_s^f ds} Z^f\right] = E^{Q^d}\left[e^{-\int_0^t r_s^d ds} X_T Z^f\right],$$

and since this holds for every T-claim $Z^f \in \mathcal{F}_T$ we conclude that

$$X_0 L_T e^{-\int_0^t r_s^f ds} = e^{-\int_0^t r_s^d ds} X_T, \quad P-a.s.$$

We may now state the first main result.

Proposition 18.4 *Assuming absence of arbitrage on a liquid international market, the exchange rate process X is given by any of the following expressions.*

$$X_t = X_0 L_t e^{\int_0^t (r_s^d - r_s^f) ds}, \tag{18.22}$$

$$X_t = X_0 \frac{\mathbf{M}_t^f}{\mathbf{M}_t^d}. \tag{18.23}$$

where $\mathbf{M}^d$ and $\mathbf{M}^f$ are the domestic and the foreign stochastic discount factors respectively.

Proof The relation (18.22) follows directly from the previous calculations, and (18.23) follows from (18.22), the definition of the stochastic discount factors, and the fact that on $\mathcal{F}_t$ we have

$$L_t = \frac{dQ^f}{dQ^d} = \frac{dQ^f/dP}{dQ^d/dP} = \frac{dL_t^f}{dL_t^d}.$$

□

We thus see that the exchange rate process is in fact (apart from the normalizing factor X_0) given by the ratio between the foreign and the domestic stochastic discount factors, and this can be used in order to obtain quite detailed information concerning the exchange rate dynamics. To fix notation, let us write the P dynamics of the exchange rate X as

$$dX_t = \alpha_t X_t dt + X_t \sigma_t dW_t, \tag{18.24}$$

where α and σ are adapted processes. We recall that the exchange rate volatility σ will be the same under equivalent measure transformations from P to Q^d and Q^f but that α will depend on the measure, through the Girsanov Theorem. We now have the following result.

Proposition 18.5 *With assumptions as above the following hold:*

1. *The dynamics of X under the domestic martingale measure Q^d are given by*
$$dX_t = X_t(r_t^d - r_t^f)dt + X_t \sigma_t dW_t^d. \tag{18.25}$$
2. *The exchange rate volatility process σ admits the representation*
$$\sigma_t = \varphi_t, \tag{18.26}$$

with φ as in (18.16).

3. *The foreign and domestic market prices of risk are related by*

$$\lambda_t^f = \lambda_t^d - \sigma_t, \tag{18.27}$$

or equivalently by

$$\varphi_t^f = \varphi_t^d + \sigma_t. \tag{18.28}$$

Proof The dt-term in (18.25) follows from (18.22) and (18.16), and the volatility term is the same under Q^d as under P. The relation (18.27)–(18.28) follow from (18.22), the fact that $L_t = L_t^f / L_t^d$, the dynamics in (18.14)–(18.15), the definition (18.17), and the Itô formula. The formula (18.26) follows in a similar way from (18.22) and (18.16). □

There is an important corollary to this result.

Corollary 18.6 *We have*

$$Q^d = Q^f \tag{18.29}$$

if and only if $\sigma = 0$, i.e. if and only if the exchange rate is deterministic.

Proof This follows immediately from (18.26). □

18.3 Domestic and Foreign Equity Markets

In this section we will model a market which, apart from the exchange rate and the domestic and foreign bank accounts, also includes a domestic equity with (domestic) price S^d, and a foreign equity with (foreign) price S^f. The restriction to a single domestic and foreign equity is made for notational convenience, and in most practical cases it is also sufficient.

Since we now have three risky assets we use a three-dimensional Wiener process in order to obtain a complete market.

Assumption 18.3.1 *The dynamic model of the entire economy, under the objective measure P, is as follows*:

$$dX_t = X_t\alpha_t dt + X_t\sigma_t dW_t, \tag{18.30}$$

$$dS_t^d = S_t^d\alpha_t^d dt + S_t^d\sigma_t^d ddW_t, \tag{18.31}$$

$$dS_t^f = S_t^f\alpha_t^f dt + S_t^f\sigma_t^f dW_t, \tag{18.32}$$

$$dB_t^d = r_t^d B_t^d dt, \tag{18.33}$$

$$dB_t^f = r_t^f B_t^f dt, \tag{18.34}$$

where the r-terms and the α-terms are adapted scalar processes, whereas the σ-terms are adapted three-dimensional row vector processes. The column vector process

$$W = \begin{bmatrix} W^1 \\ W^2 \\ W^3 \end{bmatrix}$$

is a three-dimensional Wiener process (as usual with independent components). Furthermore, the (3×3)*-dimensional volatility matrix*

$$\begin{bmatrix} -\sigma_t- \\ -\sigma_t^d- \\ -\sigma_t^f- \end{bmatrix}$$

is assumed to be invertible.

Remark 18.3.1 *The reason for the assumption about the volatility matrix is that this is the necessary and sufficient condition for completeness.*

Typical T-contracts which we may wish to price (in terms of the domestic currency) are given by the following list:

- A **foreign equity call, struck in foreign currency**, i.e. an option to buy one unit of the foreign equity at the strike price of K units of the foreign currency.
- A **foreign equity call, struck in domestic currency**, i.e. a European option to buy one unit of the foreign equity at time T, by paying K units of the domestic currency.
- An **exchange option** which gives us the right to exchange one unit of the domestic equity for one unit of the foreign equity.

More generally we will study pricing problems for T-claims $Z \in \mathcal{F}_T$ where Z is measured in the **domestic currency**, and we want to compute the **domestic** price. As a particular case we could for example have a simple claim of the form

$$Z = \Phi\left(X_T, S_T^d, S_T^f\right). \tag{18.35}$$

From the First Fundamental Theorem, applied to the domestic market, we know that the price is given by

$$\Pi_t[Z] = E^{Q^d}\left[e^{-\int_t^T r_u^d du} Z \middle| \mathcal{F}_t\right], \tag{18.36}$$

so in principle it only remains to find the Q^d-dynamics of S^d, S^f and X.

For S^d this is trivial. We know from general theory that the Q^d dynamics must be

$$dS_t^d = r_t^d S_t^d dt + S_t^d \sigma_t^d dW_t^d, \tag{18.37}$$

and from Proposition 18.5 we have the Q^d-dynamics of X as

$$dX_t = (r_t^d - r_t^f) X_t dt + X_t \sigma_t dW_t^d. \tag{18.38}$$

It thus remains to compute the Q^d-dynamics of S^f, but also this is easy. It follows directly from standard theory applied to the foreign market that we have the Q^f-dynamics

$$dS_t^f = r_t^f S_t^f dt + S_t^f \sigma_t^f dW_t^f, \tag{18.39}$$

so we now want to switch from Q^f to Q^d. We then recall from (18.13) that L is defined as

$$L_t = \frac{dQ^f}{dQ^d}, \quad \text{on } \mathcal{F}_t$$

with dynamics

$$dL_t = L_t \varphi_t dW_t^d,$$

and from (18.26) we recall that $\varphi_t = \sigma_t$. The Girsanov Theorem then implies that we have

$$dW_t^d = \sigma_t^* dt + dW_t^f, \tag{18.40}$$

where * denotes transpose. Plugging this into (18.39) finally gives us our result.

Proposition 18.7 *The Q^d-dynamics of S^d, S^f, and X are given by*

$$dS_t^d = r_t^d S_t^d dt + S_t^d \sigma_t^d dW_t^d, \tag{18.41}$$

$$dS_t^f = (r_t^f + \sigma_t^f \sigma^*) S_t^f dt + S_t^f \sigma_t^f dW_t^d, \tag{18.42}$$

$$dX_t = (r_t^d - r_t^f) X_t dt + X_t \sigma_t dW_t^d. \tag{18.43}$$

If we instead want to price in terms of the foreign currency we get the Q^f-dynamics by substituting (18.40) into (18.41)–(18.43).

18.4 An Extended Black–Scholes Model

In this section we specialize the theory developed above to the simplest possible case, i.e. the case when all coefficients in the model (18.30)–(18.34) are constant. We then have the model

$$dX_t = X_t \alpha dt + X_t \sigma dW_t, \tag{18.44}$$

$$dS_t^d = S_t^d \alpha_d dt + S_t^d \sigma_d dW_t, \tag{18.45}$$

$$dS_t^f = S_t^f \alpha_f dt + S_t^f \sigma_f dW_t, \tag{18.46}$$

$$dB_t^d = r^d B_t^d dt, \tag{18.47}$$

$$dB_t^f = r^f B_t^f dt. \tag{18.48}$$

Example 18.8 (Foreign call, struck in domestic currency) As an example we now compute the price of a foreign call, struck in domestic currency. This claim is a call option on the foreign asset S^f, with a strike price K, where K is quoted in the domestic currency. In order to express the claim in terms of the domestic currency we simply multiply the foreign stock price S^f by the exchange rate X. The claim, expressed in domestic currency, is thus given by

$$Z = \max\left[S_T^X - K, 0\right], \tag{18.49}$$

where

$$S_t^X = S_t^f \cdot X_t. \tag{18.50}$$

It is now obvious that we can interpret the process S^X as the price process of a domestically traded asset without dividends so we expect to have the Q^d dynamics

$$dS_t^X = r_t^d S_t^X dt + S_t^X(\sigma_f + \sigma)dW_t^d, \tag{18.51}$$

and this formula does indeed follow directly from (18.42)–(18.43). This implies that Z is a standard call option on the domestically traded asset S^X and, since S^X follows GBM with volatility $\|\sigma_f + \sigma\|$, we can use the standard Black–Scholes formula to obtain the following result.

Proposition 18.9 (Foreign call, struck in domestic currency) *The pricing function $F(t, s^x)$ where $s^x = S_t^f \cdot X_t$ is given by*

$$F(t, s^x) = s^x N[d_1] - e^{-r^d(T-t)} K N[d_2],$$

where

$$d_1(t, s^x) = \frac{1}{\|\sigma_f + \sigma\|\sqrt{T-t}} \left\{ \ln\left(\frac{s^x}{K}\right) + \left(r^d + \frac{1}{2}\|\sigma_f + \sigma\|^2\right)(T-t) \right\},$$

$$d_2(t, s^x) = d_1(t, s^x) - \|\sigma_f + \sigma\|\sqrt{T-t}.$$

18.5 The Siegel Paradox

This section constitutes a small digression in the sense that we will not derive any new pricing formulas. Instead we will take a closer look at a famous argument known as "The Siegel Paradox". This is really an entire family of related arguments, but one version goes as follows.

If we focus on the domestic market we say that the market is (on the aggregate) **risk neutral** if the following valuation formula holds, where S^d is the price process for any domestically traded asset and/or derivative:

$$S_t^d = e^{-r^d(T-t)} E^P\left[S_T^d \middle| \mathcal{F}_t \right]. \tag{18.52}$$

In many papers in economics and finance an assumption is made that the (domestic) market is in fact risk neutral, and this is of course a behavioral assumption. The risk neutral assumption is typically made because it facilitates computations, and because the object of the investigation is not concerned with risk aversion. In such a framework the risk neutrality assumption is not a big problem.

In an international setting it then seems natural to assume that both the domestic market **and** the foreign market are risk neutral, with the (quite plausible) argument that Americans are not very different from Europeans. In more formal terms this implies that, in addition to (18.52), the following formula also holds, where S^f is the foreign price of any asset traded on the foreign market:

$$S_t^f = e^{-r^f(T-t)} E^P\left[S_T^f \middle| \mathcal{F}_t \right]. \tag{18.53}$$

To move on to the Siegel Paradox, suppose that both the domestic and the foreign markets are risk neutral, i.e. that (18.52) and (18.53) hold. Suppose also, for the sake of notational simplicity, that the foreign and the domestic short rates are deterministic. We then have the following (perhaps surprising) argument.

Let us first consider a T claim of one dollar. The arbitrage free dollar value at $t=0$ is of course

$$e^{-r^f T},$$

so the euro value at $t=0$ is given by

$$X_0 e^{-r^f T}.$$

The 1-dollar claim is, however, identical to a T-claim of X_T euros. Given domestic risk neutrality, the euro value at $t=0$ is then given by

$$e^{-r^d T} E^P\left[X_T\right].$$

We thus have

$$X_0 e^{-r^f T} = e^{-r^d T} E^P\left[X_T\right]. \tag{18.54}$$

We now change perspective and consider a T-claim of one euro and compute the dollar value of this claim at $t=0$. The euro value at $t=0$ is of course

$$e^{-r^d T},$$

so the dollar value at $t=0$ is

$$\frac{1}{X_0} e^{-r^d T}.$$

The 1-euro claim is, however, identical to a T-claim of X_T^{-1} dollars, so by foreign risk neutrality we obtain the dollar value at $t=0$ as

$$e^{-r^f T} E^P\left[\frac{1}{X_T}\right],$$

which gives us

$$\frac{1}{X_0} e^{-r^d T} = e^{-r^f T} E^P\left[\frac{1}{X_T}\right]. \tag{18.55}$$

Combining (18.54) and (18.55) gives us the formula

$$E^P\left[\frac{1}{X_T}\right] = \frac{1}{E^P\left[X_T\right]},$$

which, by Jensen's inequality, is impossible unless X_T is deterministic. This is sometimes referred to as (one formulation of) "Siegel's Paradox".

We can also be more formal and use Proposition 18.5 to analyze the situation. The assumption of foreign and domestic risk neutrality is equivalent to the assumption that

$$Q^d = Q^f = P, \tag{18.56}$$

but formulas (18.27)–(18.28)now tells us that (18.56) **can never hold, unless** $\sigma = 0$, i.e. we have domestic and foreign risk neutrality at the same time if and only if the exchange rate is deterministic. It thus seems that Americans cannot be risk neutral at the same time as Europeans, so what is going on?

At a first glance the result above seems highly counter-intuitive, since the assumption about domestic and foreign risk neutrality seems to be quite innocent. This, however, is not the case. The paradox comes from the fact that the term "risk neutrality" is often interpreted as an assumption about the (aggregate) attitude towards risk, i.e. uncertainty, **as such**. However, from (18.52), which is an equation for objects measured in the domestic currency, and (18.53), which is an equation for objects measured in the foreign currency, it should be clear that risk neutrality is a property which holds only **relative to a specified numeraire**. To put it as a slogan, you may very well be risk neutral w.r.t. pounds sterling, and still be risk averse w.r.t. US dollars.

You can without contradictions assume that the entire world is risk neutral w.r.t euros or that the entire world is risk neutral w.r.t. dollars. What you cannot do, without assuming a deterministic exchange rate, is to assume that Americans are risk neutral w.r.t. dollars and Europeans are risk neutral w.r.t. euros. If you do that, and also assume (which we have done) liquid frictionless international markets, then you have in fact assumed that the entire world is risk neutral w.r.t. two different numeraires at the same time, and this is impossible unless the exchange rate is deterministic.

18.6 Exercises

Exercise 18.1 Consider the European call on the exchange rate described at the end of Section 18.1. Denote the price of the call by $c(t,x)$, and denote the price of the corresponding put option (with the same exercise price K and exercise date T) by $p(t,x)$. Show that the put–call parity relation between p and c is given by

$$p = c - xe^{-r^f(T-t)} + Ke^{-r^d(T-t)}.$$

Exercise 18.2 Compute the pricing function (in the domestic currency) for a **binary option** on the exchange rate. This option is a T-claim, $\mathcal{Z}$, of the form

$$\mathcal{Z} = 1_{[a,b]}(X_T),$$

i.e. if $a \le X_T \le b$ then you will obtain one unit of the domestic currency, otherwise you get nothing.

Exercise 18.3 Derive the dynamics of the domestic stock price S^d under the foreign martingale measure Q^f.

Exercise 18.4 Consider a model with the domestic short rate r^d and two foreign currencies, the exchange rates of which (from the domestic perspective) are denoted by X^1 and X^2 respectively. The foreign short rates are denoted by

r^1 and r^2 respectively. We assume that the exchange rates have P-dynamics given by

$$dX_t^i = X_t^i \alpha_i dt + X_t^i \sigma_i dW_t^i, \quad i = 1, 2,$$

where W_1, W_2 are P-Wiener processes with correlation ρ.

(a) Derive the pricing PDE for contracts, quoted in the domestic currency, of the form $\mathcal{Z} = \Phi(X_T^1, X_T^2)$.

(b) Derive the corresponding risk neutral valuation formula, and the Q^d-dynamics of X^1 and X^2.

(c) Compute the price, in domestic terms, of the "binary quanto contract" $\mathcal{Z}$, which gives, at time T, K units of foreign currency No. 1, if $a \leq X_T^2 \leq b$ (where a and b are given numbers), and zero otherwise. If you want to facilitate computations you may assume that $\rho = 0$.

18.7 Notes

The classic in this field is Garman and Kohlhagen (1983). See also Reiner (1992) and Amin and Jarrow (1991).

19

BONDS AND INTEREST RATES

19.1 Zero Coupon Bonds

In this chapter we will begin to study the particular problems which appear when we try to apply arbitrage theory to the bond market. The primary objects of investigation are **zero coupon bonds**, also known as **pure discount bonds**, of various maturities. All payments are assumed to be made in a fixed currency which, for convenience, we choose to be US dollars.

Definition 19.1 *A* **zero coupon bond** *with* **maturity date** T*, also called a* T*-bond, is a contract which guarantees the holder one dollar to be paid on the date* T*. The price at time* t *of a bond with maturity date* T *is denoted by* $p(t,T)$*.*

The convention that the payment at the time of maturity, known as the **principal value** or **face value**, equals one is made for computational convenience. **Coupon bonds**, which give the owner a payment stream during the interval $[0,T]$ are treated below. These instruments have the common property that they provide the owner with a deterministic cash flow, and for this reason they are also known as **fixed income** instruments.

We now make an assumption to guarantee the existence of a sufficiently rich and regular bond market.

Assumption 19.1.1 *We assume the following*:

- *There exists a (frictionless) market for* T*-bonds for every* $T > 0$*.*
- *The relation* $p(t,t) = 1$ *holds for all* t*.*
- *For each fixed* t*, the bond price* $p(t,T)$ *is differentiable w.r.t. time of maturity* T*.*

Note that the relation $p(t,t) = 1$ above is necessary in order to avoid arbitrage. The bond price $p(t,T)$ is thus a stochastic object with two variables, t and T, and for each outcome in the underlying sample space, the dependence upon these variables is very different:

- For a fixed value of t, $p(t,T)$ is a function of T. This function provides the prices, at the fixed time t, for bonds of all possible maturities. The graph of this function is called "the bond price curve at t", or "the term structure at t". Typically it will be a very smooth graph, i.e. for each t, $p(t,T)$ will be differentiable w.r.t. T. The smoothness property is in fact a part of our assumptions above, but this is mainly for convenience. All models to be considered below will automatically produce smooth bond price curves.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

- For a fixed maturity T, $p(t,T)$ (as a function of t) will be a scalar stochastic process. This process gives the prices, at different times, of the bond with fixed maturity T, and the trajectory will typically be very irregular (like a Wiener process).

We thus see that (our picture of) the bond market is different from any other market that we have considered so far, in the sense that the bond market contains an infinite number of assets (one bond type for each time of maturity). The basic goal in interest rate theory is roughly that of investigating the relations between all these different bonds. Somewhat more precisely we may pose the following general problems, to be studied below:

- What is a reasonable model for the bond market above?
- Which relations must hold between the price processes for bonds of different maturities, in order to guarantee an arbitrage free bond market?
- Is it possible to derive arbitrage free bond prices from a specification of the dynamics of the short rate of interest?
- Given a model for the bond market, how do you compute prices of interest rate derivatives, such as a European call option on an underlying bond?

19.2 Interest Rates

19.2.1 *Definitions*

Given the bond market above, we may now define a number of interest rates, and the basic construction is as follows. Suppose that we are standing at time t, and let us fix two other points in time, S and T, with $t < S < T$. The immediate project is to write a contract at time t which allows us to make an investment of one (dollar) at time S, and to have a **deterministic** rate of return, determined at the contract time t, over the interval $[S,T]$. This can easily be achieved as follows:

1. At time t we sell one S-bond. This will give us $p(t,S)$ dollars.
2. We use this income to buy exactly $p(t,S)/p(t,T)$ T-bonds. Thus our net investment at time t equals zero.
3. At time S the S-bond matures, so we are obliged to pay out one dollar.
4. At time T the T-bonds mature at one dollar a piece, so we will receive the amount $p(t,S)/p(t,T)$ dollars.
5. The net effect of all this is that, based on a contract at t, an investment of one dollar at time S has yielded $p(t,S)/p(t,T)$ dollars at time T.
6. Thus, at time t, we have made a contract guaranteeing a **riskless** rate of interest over the **future interval** $[S,T]$. Such an interest rate is called a **forward rate**.

We now go on to compute the relevant interest rates implied by the construction above. We will use two (out of many possible) ways of quoting forward rates, namely as continuously compounded rates or as simple rates.

The **simple** forward rate (or **LIBOR** rate) L, is the solution to the equation

$$1+(T-S)L=\frac{p(t,S)}{p(t,T)},$$

whereas the **continuously compounded** forward rate R is the solution to the equation

$$e^{R(T-S)}=\frac{p(t,S)}{p(t,T)}.$$

The simple rate notation is the one used in the market, whereas the continuously compounded notation is used in theoretical contexts. They are of course logically equivalent, and the formal definitions are as follows.

Definition 19.2

1. *The simple* **forward rate for** $[S,T]$ **contracted at** t*, henceforth referred to as the* **LIBOR** *forward rate, is defined as*

$$L(t;S,T)=-\frac{p(t,T)-p(t,S)}{(T-S)p(t,T)}.$$

2. *The simple* **spot rate for** $[S,T]$ *, henceforth referred to as the* **LIBOR** *spot rate, is defined as*

$$L(S,T)=-\frac{p(S,T)-1}{(T-S)p(S,T)}.$$

3. *The continuously compounded* **forward rate for** $[S,T]$ **contracted at** t *is defined as*

$$R(t;S,T)=-\frac{\log p(t,T)-\log p(t,S)}{T-S}.$$

4. *The* **continuously compounded spot rate***,* $R(S,T)$*, for the period* $[S,T]$ *is defined as*

$$R(S,T)=-\frac{\log p(S,T)}{T-S}.$$

5. *The* **instantaneous forward rate with maturity** T**, contracted at** t**,** *is defined by*

$$f(t,T)=-\frac{\partial \log p(t,T)}{\partial T}.$$

6. *The instantaneous* **short rate at time** t *is defined by*

$$r_t=f(t,t).$$

We note that spot rates are forward rates where the time of contracting coincides with the start of the interval over which the interest rate is effective, i.e. $t=S$. The instantaneous forward rate, which will be of great importance below, is the limit of the continuously compounded forward rate when $S\to T$.

It can thus be interpreted as the riskless rate of interest, contracted at t, over the infinitesimal interval $[T, T+dT]$.

We now go on to define the money account process B.

Definition 19.3 *The* **money account** *process is defined by*

$$B_t = e^{\int_0^t r_s ds},$$

i.e.

$$\begin{cases} dB_t = r - tB_t dt, \\ B_t = 1. \end{cases}$$

The interpretation of the money account is the same as before, i.e. you may think of it as describing a bank with a stochastic short rate. It can also be shown (see below) that investing in the money account is equivalent to investing in a self-financing "rolling over" trading strategy, which at each time t consists entirely of "just maturing" bonds, i.e. bonds which will mature at $t + dt$.

As an immediate consequence of the definitions we have the following useful formulas.

Lemma 19.4 *For $t \leq s \leq T$ we have*

$$p(t,T) = p(t,s) \cdot \exp\left\{-\int_s^T f(t,u)du\right\},$$

and in particular

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

If we wish to make a model for the bond market, it is obvious that this can be done in many different ways.

- We may specify the dynamics of the short rate (and then perhaps try to derive bond prices using arbitrage arguments).
- We may directly specify the dynamics of all possible bonds.
- We may specify the dynamics of all possible forward rates, and then use Lemma 19.4 in order to obtain bond prices.

All these approaches are of course related to each other, and we now go on to present a small "toolbox" of results to facilitate the analysis below. These results will not be used until Chapter 22, and the proofs are somewhat technical, so the next two subsections can be omitted at a first reading.

19.2.2 *Relations between $df(t,T)$, $dp(t,T)$, and $dr(t)$*

We will consider dynamics of the following form.

Short rate dynamics

$$dr_t = a(t)dt + b(t)dW_t. \tag{19.1}$$

Bond price dynamics

$$dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW(t). \tag{19.2}$$

Forward rate dynamics

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t). \tag{19.3}$$

The Wiener process W is allowed to be vector valued, in which case the volatilities $v(t,T)$ and $\sigma(t,T)$ are row vectors. The processes $a(t)$ and $b(t)$ are scalar adapted processes, whereas $m(t,T)$, $v(t,T)$, $\alpha(t,T)$, and $\sigma(t,T)$ are adapted processes parameterized by time of maturity T. The interpretation of the bond price equation (19.2) and the forward rate equation(19.3) is that these are scalar stochastic differential equations (in the t-variable) for each fixed time of maturity T. Thus (19.2) and (19.3) are both infinite dimensional systems of SDEs.

We will study the formal relations which must hold between bond prices and interest rates, and to this end we need a number of technical assumptions, which we collect below in an "operational" manner.

Assumption 19.2.1

1. *For each fixed ω, t all the objects $m(t,T)$, $v(t,T)$, $\alpha(t,T)$, and $\sigma(t,T)$ are assumed to be continuously differentiable in the T-variable. This partial T-derivative is sometimes denoted by $m_T(t,T)$ etc.*
2. *All processes are assumed to be regular enough to allow us to differentiate under the integral sign as well as to interchange the order of integration.*

The main result is as follows. Note that the results below hold, regardless of the measure under consideration, and in particular we do **not** assume that markets are free of arbitrage.

Proposition 19.5

1. *If $p(t,T)$ satisfies (19.2), then for the forward rate dynamics we have*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t,$$

where α and σ are given by

$$\begin{cases} \alpha(t,T) = v_T(t,T) \cdot v(t,T) - m_T(t,T), \\ \sigma(t,T) = -v_T(t,T). \end{cases} \tag{19.4}$$

2. *If $f(t,T)$ satisfies (19.3) then the short rate satisfies*

$$dr_t = a(t)dt + b(t)dW_t,$$

where

$$\begin{cases} a(t) = f_T(t,t) + \alpha(t,t), \\ b(t) = \sigma(t,t). \end{cases} \tag{19.5}$$

3. *If $f(t,T)$ satisfies (19.3) then $p(t,T)$ satisfies*

$$dp(t,T) = p(t,T)\left\{r_t + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right\}dt + p(t,T)S(t,T)dW_t,$$

where $\|\cdot\|$ denotes the Euclidean norm, and

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)ds, \\ S(t,T) = -\int_t^T \sigma(t,s)ds. \end{cases} \tag{19.6}$$

Proof The first part of the proposition is left to the reader (see the exercises).

For the second part we integrate the forward rate dynamics to get

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW_s. \tag{19.7}$$

Now we can write

$$\begin{aligned} \alpha(s,t) &= \alpha(s,s) + \int_s^t \alpha_T(s,u)du, \\ \sigma(s,t) &= \sigma(s,s) + \int_s^t \sigma_T(s,u)du, \end{aligned}$$

and, inserting this into (19.7), we have

$$\begin{aligned} r(t) &= f(0,t) + \int_0^t \alpha(s,s)ds + \int_0^t \int_s^t \alpha_T(s,u)duds \\ &\quad + \int_0^t \sigma(s,s)dW_s + \int_0^t \int_s^t \sigma_T(s,u)du\, dW_s. \end{aligned}$$

Changing the order of integration and identifying terms we obtain the result.

For the proof of the third part we give a slightly heuristic argument. The full formal proof, see Heath et al. (1992), is an integrated version of the proof given here, but the infinitesimal version below is (hopefully) easier to understand. Using the definition of the forward rates we may write

$$p(t,T) = e^{Y(t,T)}, \tag{19.8}$$

where Y is given by

$$Y(t,T) = -\int_t^T f(t,s)ds. \tag{19.9}$$

From the Itô formula we then obtain the bond dynamics as

$$dp(t,T) = p(t,T)dY(t,T) + \frac{1}{2}p(t,T)\left(dY(t,T)\right)^2, \tag{19.10}$$

and it remains to compute $dY(t,T)$. We have

$$dY(t,T) = -d\left(\int_t^T f(t,s)ds\right),$$

and the problem is that in the integral the t-variable occurs in two places: as the lower limit of integration, and in the integrand $f(t,s)$. This is a situation that is not covered by the standard Itô formula, but it is easy to guess the answer. The t appearing as the lower limit of integration should give rise to the term

$$\frac{\partial}{\partial t}\left(\int_t^T f(t,s)ds\right)dt.$$

Furthermore, since the stochastic differential is a linear operation, we should be allowed to move it inside the integral, thus providing us with the term

$$\left(\int_t^T df(t,s)ds\right).$$

We have therefore arrived at

$$dY(t,T) = -\frac{\partial}{\partial t}\left(\int_t^T f(t,s)ds\right)dt - \int_t^T df(t,s)ds,$$

which, using the fundamental theorem of integral calculus, as well as the forward rate dynamics, gives us

$$dY(t,T) = f(t,t)dt - \int_t^T \alpha(t,s)dtds - \int_t^T \sigma(t,s)dW_t ds.$$

We now exchange dt and dW_t with ds and recognize $f(t,t)$ as the short rate r_t, thus obtaining

$$dY(t,T) = r(t)dt + A(t,T)dt + S(t,T)dW_t,$$

with A and S as above. We therefore have

$$(dY(t,T))^2 = \|S(t,T)\|^2 dt,$$

and, substituting all this into (19.10), we obtain our desired result. $\square$

19.2.3 *An Expectation Hypothesis*

We now make a small digression to discuss the forward rate process $f(t,T)$. The economic interpretation of $f(t,T)$ is that this is the risk free rate of return which we may have on an investment over the infinitesimal interval $[T, T+dT]$ if the contract is made at t. On the other hand, the short rate r_t is the risk free rate of return over the infinitesimal interval $[T, T+dT]$, if the contract is made at T. Thus it is natural to view $f(t,T)$ (which can be observed at t) as an estimate of the future short rate r_t. More explicitly it is sometimes argued that if the market expects the short rate at T to be high, then it seems reasonable to assume that the forward rate $f(t,T)$ is also high, since otherwise there would be profits to be made on the bond market.

Our task now is to determine whether this reasoning is correct in a more precise sense, and to this end we study the most formalized version of the argument above, known as the "unbiased expectation hypothesis" for forward rates. This hypothesis then says that in an efficient market we must have

$$f(t,T) = E\left[r_T \mid \mathcal{F}_t\right], \tag{19.11}$$

i.e. the present forward rate is an unbiased estimator of the future spot rate. If we interpret the expression "an efficient market" as "an arbitrage free market" then we may use our general machinery to analyze the problem.

First we notice that there is no probability measure indicated in (19.11), so we have to make a choice.

Of course there is no reason at all to expect the hypothesis to be true under the objective measure P, but it is often claimed that it holds "in a risk neutral world". This more refined version of the hypothesis can then be formulated as

$$f(t,T) = E^Q\left[r_T \mid \mathcal{F}_t\right], \tag{19.12}$$

where Q is the usual risk neutral martingale measure. In fact, also this version of the expectation hypothesis is in general incorrect, which is shown by the following result, where we use the T-forward martingale measure Q^T, defined in Section 15.5.1.

Lemma 19.6 *Assume that, for all $T > 0$ we have $r_t/B_t \in L^1(Q)$. Then, for every fixed T, the T-forward rate process $f(t,T)$ is a Q^T-martingale, and in particular we have*

$$f(t,T) = E^T\left[r_T \mid \mathcal{F}_t\right]. \tag{19.13}$$

Proof Using Proposition 15.31 with $X = r_T$ we have

$$\Pi_t[X] = E^Q\left[r_T e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] = p(t,T) E^T\left[r_T \mid \mathcal{F}_t\right].$$

This gives us

$$\begin{aligned} E^T\left[r_T \mid \mathcal{F}_t\right] &= \frac{1}{p(t,T)} E^Q\left[r_T e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] \\ &= -\frac{1}{p(t,T)} E^Q\left[\frac{\partial}{\partial T} e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] = -\frac{1}{p(t,T)} \frac{\partial}{\partial T} E^Q\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] \\ &= -\frac{p_T(t,T)}{p(t,T)} = f(t,T). \end{aligned}$$

□

We thus see that the expectation hypothesis is false also under Q, but true under Q^T. Note that we have a different measure Q^T for different choices of the maturity date T.

19.2.4 *An Alternative View of the Money Account*

The object of this subsection is to show (heuristically) that the risk free asset B can in fact be replicated by a self-financing strategy, defined by "rolling over" just-maturing bonds. This is a "folklore" result, which is very easy to prove in discrete time, but surprisingly tricky in a continuous time framework.

Let us consider a self-financing portfolio which at each time t consists entirely of bonds maturing x units of time later (where we think of x as a small number). At time t the portfolio thus consists only of bonds with maturity $t+x$, so the value dynamics for this portfolio is given by

$$dV(t) = V(t) \cdot 1 \cdot \frac{dp(t,t+x)}{p(t,t+x)}, \tag{19.14}$$

where the constant 1 indicates that the weight of the $t+x$-bond in the portfolio equals one. We now want to study the behavior of this equation as x tends to zero, and to this end we use Proposition 19.5 to obtain

$$\frac{dp(t,t+x)}{p(t,t+x)} = \left\{ r(t) + A(t,t+x) + \frac{1}{2}\|S(t,t+x)\|^2 \right\} dt + S(t,t+x)dW(t).$$

Letting x tend to zero, (19.6) gives us

$$\lim_{x\to 0} A(t,t+x) = 0,$$
$$\lim_{x\to 0} S(t,t+x) = 0.$$

Furthermore we have

$$\lim_{x\to 0} p(t,t+x) = 1,$$

and, substituting all this into eqn (19.14), we obtain the value dynamics

$$dV_t = r_t V_t dt, \tag{19.15}$$

which we recognize as the dynamics of the money account.

The argument thus presented is of course only heuristical, and it requires some hard work to make it precise. Note, for example, that the rolling-over portfolio above does not fall into the general framework of self-financing portfolios, developed earlier. The problem is that, although at each time t the portfolio only consists of one particular bond (maturing at $t+x$), over an arbitrary short time interval, the portfolio will use an infinite number of different bonds. In order to handle such a situation, we need to extend the portfolio concept to include measure valued portfolios. This is done in Björk *et al.* (1995), and in Björk *et al.* (1997) the argument above is made precise.

19.3 Coupon Bonds, Swaps, and Yields

In most bond markets, there are only a relatively small number of zero coupon bonds traded actively. The maturities for these are generally short (typically between half a year and two years), whereas most bonds with a longer time to maturity are coupon bearing. Despite this empirical fact we will still assume the existence of a market for all possible pure discount bonds, and we now go on to introduce and price coupon bonds in terms of zero coupon bonds.

19.3.1 *Fixed Coupon Bonds*

The simplest coupon bond is the **fixed coupon bond**. This is a bond which, at some intermediary points in time, will provide predetermined payments (coupons) to the holder of the bond. The formal description is as follows:

- Fix a number of dates, i.e. points in time, $T_0, \dots, T_n$. Here T_0 is interpreted as the emission date of the bond, whereas $T_1, \dots, T_n$ are the coupon dates.
- At time $T_i, i = 1, \dots, n$, the owner of the bond receives the deterministic coupon c_i.
- At time T_n the owner receives the face value K.

We now go on to compute the price of this bond, and it is obvious that the coupon bond can be replicated by holding a portfolio of zero coupon bonds with maturities $T_i,\ i = 1, \dots, n$. More precisely we will hold c_i zero coupon bonds of maturity $T_i,\ i = 1, \dots, n-1$, and $K + c_n$ bonds with maturity T_n, so the price, $p(t)$, at a time $t < T_1$, of the coupon bond is given by

$$p(t) = K \cdot p(t, T_n) + \sum_{i=1}^{n} c_i \cdot p(t, T_i). \tag{19.16}$$

Very often the coupons are determined in terms of **return**, rather than in monetary (e.g. dollar) terms. The return for the ith coupon is typically quoted as a simple rate acting on the face value K, over the period $[T_{i-1}, T_i]$. Thus, if, for example, the ith coupon has a return equal to r_i, and the face value is K, this means that

$$c_i = r_i (T_i - T_{i-1}) K.$$

For a standardized coupon bond, the time intervals will be equally spaced, i.e.

$$T_i = T_0 + i\delta,$$

and the coupon rates $r_1, \dots, r_n$ will be equal to a common coupon rate r. The price $p(t)$ of such a bond will, for $t \le T_1$, be given by

$$p(t) = K \left(p(t, T_n) + r\delta \sum_{i=1}^{n} \cdot p(t, T_i) \right). \tag{19.17}$$

19.3.2 *Floating Rate Bonds*

There are various coupon bonds for which the value of the coupon is not fixed at the time the bond is issued, but rather reset for every coupon period. Most often the resetting is determined by some financial benchmark, like a market interest rate, but there are also bonds for which the coupon is benchmarked against a nonfinancial index.

As an example (to be used in the context of swaps below), we will confine ourselves to discussing one of the simplest floating rate bonds, where the coupon rate r_i is set to the spot LIBOR rate $L(T_{i-1}, T_i)$. Thus

$$c_i = (T_i - T_{i-1})L(T_{i-1}, T_i)K,$$

and we note that $L(T_{i-1}, T_i)$ is determined already at time T_{i-1}, but that c_i is not delivered until at time T_i. We now go on to compute the value of this bond at some time $t < T_0$, in the case when the coupon dates are equally spaced, with $T_i - T_{i-1} = \delta$, and to this end we study the individual coupon c_i. Without loss of generality we may assume that $K = 1$, and inserting the definition of the LIBOR rate (Definition 19.2) we have

$$c_i = \delta \frac{1 - p(T_{i-1}, T_i)}{\delta p(T_{i-1}, T_i)} = \frac{1}{p(T_{i-1}, T_i)} - 1.$$

The value at t, of the term -1 (paid out at T_i), is of course equal to

$$-p(t, T_i),$$

and it remains to compute the value of the term $\frac{1}{p(T_{i-1}, T_i)}$, which is paid out at T_i.

This is, however, easily done through the following argument:

- Buy, at time t, one T_{i-1}-bond. This will cost $p(t, T_{i-1})$.
- At time T_{i-1} you will receive the amount 1.
- Invest this unit amount in T_i-bonds. This will give you exactly $\frac{1}{p(T_{i-1}, T_i)}$ bonds.
- At T_i the bonds will mature, each at the face value 1. Thus, at time T_i, you will obtain the amount

$$\frac{1}{p(T_{i-1}, T_i)}.$$

This argument shows that it is possible to replicate the cash flow above, using a self-financing bond strategy, to the initial cost $p(t, T_{i-1})$. Thus the value at t, of obtaining $\frac{1}{p(T_{i-1}, T_i)}$ at T_i, is given by $p(t, T_{i-1})$, and the value at t of the coupon c_i is

$$p(t, T_{i-1}) - p(t, T_i).$$

Summing up all the terms we finally obtain the following valuation formula for the floating rate bond

$$p(t) = p(t, T_n) + \sum_{i=1}^{n} [p(t, T_{i-1}) - p(t, T_i)] = p(t, T_0). \tag{19.18}$$

In particular we see that if $t = T_0$, then $p(T_0) = 1$. The reason for this (perhaps surprisingly easy) formula is of course that the entire floating rate bond can be replicated through a self-financing portfolio (see the exercises).

19.3.3 *Interest Rate Swaps*

In this section we will discuss the simplest of all interest rate derivatives, the interest rate swap. This is basically a scheme where you exchange a payment stream at a fixed rate of interest, known as the **swap rate**, for a payment stream at a floating rate (typically a LIBOR rate).

There are many versions of interest rate swaps, and we will study the **forward swap settled in arrears**, which is defined as follows. We denote the principal by K, and the swap rate by R. By assumption we have a number of equally spaced dates $T_0, \dots, T_n$, and payment occurs at the dates $T_1, \dots, T_n$ (not at T_0). If you swap a fixed rate for a floating rate (in this case the LIBOR spot rate), then, at time T_i, you will receive the amount

$$K\delta L(T_{i-1}, T_i),$$

which is exactly Kc_i, where c_i is the ith coupon for the floating rate bond in the previous section. At T_i you will pay the amount

$$K\delta R.$$

The net cash flow at T_i is thus given by

$$K\delta\left[L(T_{i-1}, T_i) - R\right],$$

and using our results from the floating rate bond, we can compute the value at $t < T_0$ of this cash flow as

$$Kp(t, T_{i-1}) - K(1+\delta R)p(t, T_i).$$

The total value $\Pi(t)$, at t, of the swap is thus given by

$$\Pi(t) = K\sum_{i=1}^{n}\left[p(t, T_{i-1}) - (1+\delta R)p(t, T_i)\right],$$

and we can simplify this to obtain the following result.

Proposition 19.7 *The price, for $t < T_0$, of the swap above is given by*

$$\Pi(t) = Kp(t, T_0) - K\sum_{i=1}^{n} d_i p(t, T_i),$$

where

$$\begin{aligned} d_i &= R\delta, \quad i = 1, \dots, n-1, \\ d_n &= 1 + R\delta. \end{aligned}$$

The remaining question is how the swap rate R is determined. By definition it is chosen such that the value of the swap equals zero at the time when the contract is made. We have the following easy result.

Proposition 19.8 *If, by convention, we assume that the contract is written at $t = 0$, the swap rate is gven by*

$$R = \frac{p(0, T_0) - p(0, T_n)}{\delta\sum_1^n p(0, T_i)}.$$

In the case that $T_0 = 0$ *this formula reduces to*

$$R = \frac{1 - p(0,T_n)}{\delta \sum_1^n p(0,T_i)}.$$

19.3.4 *Yield and Duration*

Consider a zero coupon T-bond with market price $p(t,T)$. We now look for the bond's "internal rate of interest", i.e. the constant short rate of interest which will give the same value to this bond as the value given by the market. Denoting this value of the short rate by y, we thus want to solve the equation

$$p(t,T) = e^{-y \cdot (T-t)} \cdot 1,$$

where the factor 1 indicates the face value of the bond. We are thus led to the following definition.

Definition 19.9 *The continuously compounded* **zero coupon yield**, $y(t,T)$, *is given by*

$$y(t,T) = -\frac{\log p(t,T)}{T-t}.$$

For a fixed t, *the function* $T \longmapsto y(t,T)$ *is called the (zero coupon)* **yield curve**.

We note that the yield $y(t,T)$ is nothing more than the spot rate for the interval $[t,T]$. Now let us consider a fixed coupon bond of the form discussed in Section 19.3.1 where, for simplicity of notation, we include the face value in the coupon c_n. We denote its market value at t by $p(t)$. In the same spirit as above we now look for its internal rate of interest, i.e. the constant value of the short rate, which will give the market value of the coupon bond.

Definition 19.10 *The* **yield to maturity**, $y(t,T)$, *of a fixed coupon bond at time* t, *with market price* p, *and payments* c_i *at* T_i *for* $i = 1, \ldots, n$, *is defined as the value of* y *which solves the equation*

$$p(t) = \sum_{i=1}^{n} c_i e^{-y(T_i - t)}.$$

An important concept in bond portfolio management is the "Macaulay duration". Without loss of generality we may assume that $t = 0$.

Definition 19.11 *For the fixed coupon bond above, with price* p *at* $t = 0$, *and yield to maturity* y, *the* **duration**, D, *is defined as*

$$D = \frac{\sum_1^n T_i c_i e^{-yT_i}}{p}.$$

The duration is thus a weighted average of the coupon dates of the bond, where the discounted values of the coupon payments are used as weights, and it will in a sense provide you with the "mean time to coupon payment". As such it is an important concept, and it also acts as a measure of the sensitivity of the bond price w.r.t. changes in the yield. This is shown by the following obvious result.

Proposition 19.12 *With notation as above we have*

$$\frac{dp}{dy} = \frac{d}{dy}\left\{\sum_1^n c_i e^{-yT_i}\right\} = -D \cdot p. \tag{19.19}$$

Thus we see that duration is essentially for bonds (w.r.t. yield) what delta (see Section 10.2) is for derivatives (w.r.t. the underlying price). The bond equivalent of the gamma is **convexity**, which is defined as

$$C = \frac{\partial^2 p}{\partial y^2}.$$

19.4 Exercises

Exercise 19.1 A **forward rate agreement** (FRA) is a contract, by convention entered into at $t = 0$, where the parties (a lender and a borrower) agree to let a certain interest rate, $R^\star$, act on a prespecified principal, K, over some future period $[S,T]$. Assuming that the interest rate is continuously compounded, the cash flow to the lender is, by definition, given as follows:

At time S: $-K$.

At time T: $Ke^{R^\star(T-S)}$.

The cash flow to the borrower is of course the negative of that to the lender.

(a) Compute for any time $t < S$, the value, $\Pi(t)$, of the cash flow above in terms of zero coupon bond prices.
(b) Show that in order for the value of the FRA to equal zero at $t = 0$, the rate $R^\star$ has to equal the forward rate $R(0;S,T)$ (compare this result to the discussion leading to the definition of forward rates).

Exercise 19.2 Prove the first part of Proposition 19.5.

Hint: Apply the Itô formula to the process $\log p(t,T)$, write this in integrated form and differentiate with respect to T.

Exercise 19.3 Consider a coupon bond, starting at T_0, with face value K, coupon payments at $T_1,\ldots,T_n$ and a fixed coupon rate r. Determine the coupon rate r, such that the price of the bond, at T_0, equals its face value.

Exercise 19.4 Derive the pricing formula (19.18) directly, by constructing a self-financing portfolio which replicates the cash flow of the floating rate bond.

Exercise 19.5 Let $\{y(0,T);\ T \geq 0\}$ denote the zero coupon yield curve at $t = 0$. Assume that, apart from the zero coupon bonds, we also have exactly one fixed coupon bond for every maturity T. We make no particular assumptions about the coupon bonds, apart from the fact that all coupons are positive, and we denote the yield to maturity, again at time $t = 0$, for the coupon bond with maturity T, by $y_M(0,T)$. We now have three curves to consider: the forward rate curve $f(0,T)$, the zero coupon yield curve $y(0,T)$, and the coupon yield curve $y_M(0,T)$. The object of this exercise is to see how these curves are connected.

(a) Show that
$$f(0,T) = y(0,T) + T \cdot \frac{\partial y(0,T)}{\partial T}.$$

(b) Assume that the zero coupon yield curve is an increasing function of T. Show that this implies the inequalities
$$y_M(0,T) \leq y(0,T) \leq f(0,T), \quad \forall T,$$
(with the opposite inequalities holding if the zero coupon yield curve is decreasing). Give a verbal economic explanation of the inequalities.

Exercise 19.6 Prove Proposition 19.12.

Exercise 19.7 Consider a **consol bond**, i.e. a bond which will forever pay one unit of cash at $t = 1, 2, \ldots$. Suppose that the market yield y is constant for all maturities.

(a) Compute the price, at $t = 0$, of the consol.
(b) Derive a formula (in terms of an infinite series) for the duration of the consol.
(c) Use (a) and Proposition 19.12 in order to compute an analytical formula for the duration.
(d) Compute the convexity of the consol.

19.5 Notes

Fabozzi (2009) and Sundaresan (2009) are standard textbooks on bond markets.

20

SHORT RATE MODELS

20.1 Generalities

In this chapter we turn to the problem of how to model an arbitrage free family of zero coupon bond price processes $\{p(\cdot,T); T \geq 0\}$. We will start by applying the classical delta hedging approach of Black–Scholes, but at the end we also analyze the problem using martingale methods.

Since, at least intuitively, the price, $p(t,T)$, should in some sense depend upon the behavior of the short rate r over the interval $[t,T]$, a natural starting point is to give an a priori specification of the dynamics of the short rate. This has in fact been the "classical" approach to interest rate theory, so let us model the short rate, under the objective probability measure P, as the solution of an SDE of the form

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dW_t. \tag{20.1}$$

The short rate r is the only object given a priori, so the only exogenously given asset is the money account, with price process B defined by the dynamics

$$dB_t = r_t B_t dt. \tag{20.2}$$

As usual we interpret this as a model of a bank with the stochastic short interest rate r. The dynamics of B can then be interpreted as the dynamics of the value of a bank account. See Section 7.2 for a detailed discussion. To be quite clear let us formulate the above as a formalized assumption.

Assumption 20.1.1 *We assume the existence of one exogenously given (locally risk free) asset. The price, B, of this asset has dynamics given by eqn (20.2), where the dynamics of r, under the objective probability measure P, are given by eqn (20.1).*

As in Chapter 19, we make an assumption to guarantee the existence of a sufficiently rich bond market.

Assumption 20.1.2 *We assume that there exists a market for zero coupon T-bonds for every value of T.*

We thus assume that our market contains all possible bonds (plus, of course, the risk free asset above). Consequently it is a market containing an infinite number of assets—one bond price process $t \longmapsto p(t,T)$ for each T—but we again stress the fact that only the risk free asset is exogenously given. In other words:

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

- In this model the short rate r is considered as the underlying process.
- All bond price processes are regarded as derivatives of the underlying short rate r.

Our main goal is broadly to investigate the relationship which must hold in an arbitrage free market between the price processes of bonds with different maturities. As a second step we also want to obtain arbitrage free prices for other interest rate derivatives such as bond options and interest rate swaps.

Since we view bonds as interest rate derivatives it is natural to ask whether the bond prices are uniquely determined by the given r dynamics in (20.1) and the condition that the bond market shall be free of arbitrage. This question, and its answer, are fundamental.

Question:

Are bond prices uniquely determined

by the P-dynamics of the short rate r?

Answer:

No!

Let us start by viewing the bond market in the light of the meta-theorem 8.3.1.

1. The only exogenously given underlying traded asset is the risk free asset B.
2. The number N of exogenously given traded assets **excluding** the risk free asset is thus equal to zero.
3. The number R of random sources equals one (we have one driving Wiener process).
4. From the meta-theorem 8.3.1 we thus expect that the exogenously given market is arbitrage free but not complete.

The lack of completeness is thus quite clear: since the only exogenously given asset is the risk free one we have no possibility of forming interesting portfolios. The only thing we can do on the a priori given market is simply to invest our initial capital in the bank and then sit down and wait while the portfolio value evolves according to the dynamics (20.2). It is thus impossible to replicate an interesting derivative, even such a simple one as a T-bond. To sum up:

- The price of a particular bond will **not** be completely determined by the specification (20.1) of the r-dynamics and the requirement that the bond market is free of arbitrage.

- The reason for this fact is that arbitrage pricing is always a case of pricing a derivative **in terms of** the price of some underlying assets. In our market we do not have sufficiently many underlying assets.

We thus fail to determine a unique price of a particular bond. Fortunately this (perhaps disappointing) fact does not mean that bond prices can take any form whatsoever. On the contrary we have the following basic intuition.

Idea 20.1.1

- *Prices of bonds with* **different maturities** *will have to satisfy certain* **internal consistency relations** *in order to avoid arbitrage possibilities on the bond market.*
- *If we take the price of* **one** *particular "benchmark" bond as given, then the prices of* **all other** *bonds (with maturity prior to the benchmark) will be uniquely determined* **in terms of** *the price of the benchmark bond (and the r-dynamics).*

This fact is in complete agreement with the meta-theorem, since in the a priori given market consisting of one benchmark bond plus the risk free asset we will have $R = N = 1$ thus guaranteeing completeness.

20.2 The Term Structure Equation

To make the ideas presented in Section 20.1 more concrete we now begin our formal treatment.

Assumption 20.2.1 *We assume that there is a market for T-bonds for every choice of T and that the market is arbitrage free. We assume furthermore that, for every T, the price of a T-bond has the form*

$$p(t,T) = F(t,r_t;T), \tag{20.3}$$

where F is a smooth function of three real variables.

Conceptually it is perhaps easiest to think of F as a function of only two variables, namely r and t, whereas T is regarded as a parameter. Sometimes we will therefore write $F^T(t,r)$ instead of $F(t,r;T)$. The main problem now is to find out what F^T may look like on an arbitrage free market.

Just as in the case of stock derivatives we have a simple boundary condition. At the time of maturity a T-bond is of course worth exactly 1 pound, so we have the relation

$$F^T(T,r) = 1, \quad \text{for all } r. \tag{20.4}$$

Note that in the equation above the letter r denotes a real variable, while at the same time r is used as the name of the stochastic process for the short rate. To conform with our general notational principles we should really denote the stochastic process by a capital letter like R, and then denote an outcome of R by the letter r. Unfortunately the use of r as the name of the stochastic process seems to be so fixed that it cannot be changed. We will thus continue to use r

as a name both for the process and for a generic outcome of the process. This is somewhat sloppy, but we hope that the meaning will be clear from the context.

The strategy is now exactly the same as in Chapter 9:

1. We form a portfolio consisting of two bonds having different times of maturity, S and T. The corresponding pricing functions are $F^T(t,r)$ and $F^S(t,r)$ respectively.
2. We choose portfolio weights w^S and w^T on the two bonds such that the resulting portfolio V_t is locally risk free, i.e. with dynamics

$$dV_t = V_T k_t dt,$$

for some adapted process k.
3. Absence of arbitrage implies that $k_t = r_t$.
4. A closer inspection of this equality gives us a PDE.

The concrete calculations are now identical to those in Chapter 9, and the result is as follows.

Proposition 20.1 (Term structure equation) *In an arbitrage free bond market, there will exist a process $\lambda(t,r_t)$ such that, for every maturity T, the bond pricing function F^T will satisfy the term structure equation*

$$\begin{cases} F_t^T + \{\mu - \lambda\sigma v\} F_r^T + \dfrac{1}{2}\sigma^2(t,r)F_{rr}^T - rF^T = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad F^T(T,r) = 1. \end{cases} \tag{20.5}$$

The term structure equation is obviously closely related to the Black–Scholes equation, but it is a more complicated object due to the appearance of the market price of risk λ. The problem is that λ is not determined within the model. In order to be able to solve the term structure equation we must specify λ exogenously just as we have to specify μ and σ.

Despite this problem it is not hard to obtain a Feynman–Kač representation of F^T. This is done by fixing (t,r) and then using the process

$$e^{-\int_t^s r_u du} F^T(s, r_s). \tag{20.6}$$

If we apply the Itô formula to (20.6) and use the fact that F^T satisfies the term structure equation then, by using exactly the same technique as in Section 5.5, we obtain the following stochastic representation formula.

Proposition 20.2 (Risk neutral valuation) *Bond prices are given by the formula $p(t,T) = F(t,r_t;T)$ where*

$$F(t,r;T) = E_{t,r}^Q \left[e^{-\int_t^T r_s ds} \right]. \tag{20.7}$$

The Q-dynamics of r are given by

$$dr_t = \{\mu(t,r_t) - \lambda(t,r_t)\sigma(t,r_t)\} ds + \sigma(t,r_t) dW_t^Q, \tag{20.8}$$

and where W^Q is Q-Wiener.

The formula (20.7) has the usual natural economic interpretation, which is most easily seen if we write it as

$$F(t,r;T) = E^Q_{t,r}\left[e^{-\int_t^T r_s ds} \times 1\right]. \tag{20.9}$$

We see that the value of a T-bond at time t is given as the expected value of the final payoff of one pound, discounted to present value. The deflator used is the natural one, namely $\exp\left\{-\int_t^T r_s ds\right\}$, but we observe that the expectation is not to be taken using the underlying objective probability measure P. Instead we must, as usual, use the martingale measure Q and we see that we have different martingale measures for different choices of λ. See Chaper 9 for a more detailed discussion.

There remains one important and natural question, namely how we ought to choose λ in a concrete case. This question will be treated in some detail in Section 21.3, and the moral is that we must go to the actual market and, by using market data, infer the market's choice of λ.

The bonds treated above are of course contingent claims of a particularly simple type; they are deterministic. Let us close this section by looking at a more general type of contingent T-claim of the form

$$\mathcal{X} = \Phi(r_t), \tag{20.10}$$

where Φ is some real valued function. Using the same type of arguments as above it is easy to see that we have the following result.

Proposition 20.3 (General term structure equation) *Let $\mathcal{X}$ be a contingent T-claim of the form $\mathcal{X} = \Phi(r_t)$. In an arbitrage free market the price $\Pi_t[\Phi]$ will be given as*

$$\Pi_t[\Phi] = F(t, r_t), \tag{20.11}$$

where F solves the boundary value problem

$$\begin{cases} F_t + \{\mu - \lambda\sigma\} F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0, \\ \qquad\qquad\qquad\qquad F(T,r) = \Phi(r). \end{cases} \tag{20.12}$$

Furthermore F has the stochastic representation

$$F(t,r;T) = E^Q_{t,r}\left[e^{-\int_t^T r_s ds} \times \Phi(r_t)\right], \tag{20.13}$$

with Q-dynamics

$$dr_t = \{\mu(t,r_t) - \lambda(t,r_t)\sigma(t,r_t)\}\, ds + \sigma(t,r_t) dW_t^Q. \tag{20.14}$$

20.3 Martingale Analysis

We now apply the martingale machinery to our short rate model and, as could be expected, this is much easier and leads to more general results than the classical delta hedging approach. We recall our model as

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t, \tag{20.15}$$

$$dB_t = r_t B_t dt. \tag{20.16}$$

Referring to the First Fundamental Theorem we now look for a martingale measure Q, i.e. a measure $Q \sim P$ such that S_t/B_t is a Q-martingale for all underlying traded asset prices S. However, since B is the only traded asset we only need to find $Q \sim P$ such that

$$\frac{B_t}{B_t} = 1$$

is a Q-martingale. This gives us the following result.

Proposition 20.4 *In a short rate model,* **every** *Q such that $Q \sim P$ is a martingale measure.*

This result implies that every choice of Girsanov kernel φ in the likelihood dynamics

$$dL_t = L_t \varphi_t dW_t$$

where as usual $L_t = dQ/dP$ on $\mathcal{F}_t$, will generate a martingale measure. The Girsanov Theorem implies that

$$dW_t = \varphi_t dt + dW_t^Q,$$

where W^Q is Q-Wiener. Plugging this into the r dynamics gives us the Q-dynamics of r, and risk neutral valuation implies the following result.

Proposition 20.5 *With assumptions as above the following hold:*

1. *Every T-claim Z will be priced by the formula*

$$\Pi_t[Z] = E^Q\left[e^{-\int_t^T r_s ds} Z \middle| \mathcal{F}_t \right], \tag{20.17}$$

for some choice of φ, and the same φ is used for all claims.

2. *The Q-dynamics of r are given by*

$$dr_t = \{\mu(t, r_t) + \sigma(t, r_t)\varphi_t\} dt + \sigma(t, r_t) dW_t^Q. \tag{20.18}$$

3. *We have $\lambda_t = -\varphi_t$, i.e. the Girsanov kernel φ equals minus the market price of risk.*

We have an easy corollary of this in PDE terms.

Proposition 20.6 *If Z is of the form $Z = \Phi(r_T)$, and if λ is of the form $\lambda(t, r_t)$ then we have*

$$\Pi_t[\Phi] = F(t, r_t),$$

where the pricing function F solves the boundary problem

$$\begin{cases} F_t + \{\mu - \lambda\sigma\} F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0, \\ \qquad F(T, r) = \Phi(r). \end{cases} \tag{20.19}$$

Proof The Kolmogorov backward equation. □

20.4 Exercises

Exercise 20.1 We take as given an interest rate model with the following P-dynamics for the short rate:

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t.$$

Now consider a T-claim of the form $\mathcal{X} = \Phi(r_t)$ with corresponding price process $\Pi(t)$.

(a) Show that, under any martingale measure Q, the price process Π_t has a local rate of return equal to the short rate. In other words, show that the stochastic differential of Π_t is of the form

$$d\Pi_t = r_t \Pi_t dt + \sigma_\Pi \Pi_t dW_t.$$

(b) Show that the normalized price process

$$Z_t = \frac{\Pi_t}{B_t}$$

is a Q-martingale.

Exercise 20.2 The object of this exercise is to connect the forward rates defined in Chapter 19 to the framework above.

(a) Assuming that we are allowed to differentiate under the expectation sign, show that

$$f(t,T) = \frac{E^Q_{t,r_t}\left[r_t \exp\left\{-\int_t^T r_s ds\right\}\right]}{E^Q_{t,r_t}\left[\exp\left\{-\int_t^T r_s ds\right\}\right]}.$$

(b) Check that indeed $r_t = f(t,t)$.

Exercise 20.3 (Swap a fixed rate vs. a short rate) Consider the following version of an interest rate swap. The contract is made between two parties, A and B, and the payments are made as follows.

A (hypothetically) invests the principal amount K dollars at time 0 and lets it grow at a fixed rate R (to be determined below) over the time interval $[0, T]$.

At time T the principal will have grown to K_A dollars. A will then subtract the principal amount and pay the surplus $K - K_A$ to B (at time T).

B (hypothetically) invests the principal at the stochastic short rate over the interval $[0, T]$.

At time T the principal will have grown to K_B dollars. B will then subtract the principal amount and pay the surplus $K - K_B$ to A (at time T).

The **swap rate** for this contract is now defined as the value, R, of the fixed rate which gives this contract the value zero at $t = 0$. Your task is to compute the swap rate.

Exercise 20.4 (Forward contract) Consider a model with a stochastic rate. Fix a T-claim $\mathcal{X}$ of the form $\mathcal{X} = \Phi(r_t)$, and fix a point in time t, where $t < T$. From Proposition 20.3 we can in principle compute the arbitrage free price for $\mathcal{X}$ if we pay at time t. We may also consider a **forward contract** (see Section 7.7.1) on $\mathcal{X}$ **contracted at** t. This contract works as follows, where we assume that you are the buyer of the contract:

At time T you obtain the amount $\mathcal{X}$ dollars.

At time T you pay the amount K dollars.

The amount K is determined at t.

The **forward price for** $\mathcal{X}$ **contracted at** t is defined as the value of K which gives the entire contract the value zero at time t. Give a formula for the forward price.

20.5 Notes

The exposition in this chapter is standard. For further information, see the notes at the end of Chapter 21.

21

MARTINGALE MODELS FOR THE SHORT RATE

21.1 Q-Dynamics

Let us again study an interest rate model where the P-dynamics of the short rate are given by

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t. \tag{21.1}$$

As we saw in Chapter 20, the term structure (i.e. the family of bond price processes) will, together with all other derivatives, be completely determined by the general term structure equation

$$\begin{cases} F_t + \{\mu - \lambda\sigma\} F_r + \dfrac{1}{2}\sigma^2 F_{rr} - rF = 0, \\ \qquad\qquad\qquad\qquad\qquad F(T, r) = \Phi(r), \end{cases} \tag{21.2}$$

as soon as we have specified the following objects:

- The drift term μ
- The diffusion term σ
- The market price of risk λ.

Consider for a moment σ to be given a priori. Then it is clear from (21.2) that it is irrelevant exactly how we specify μ and λ *per se*. The object, apart from σ, that really determines the term structure (and all other derivatives) is the term $\mu - \lambda\sigma$ in eqn (21.2). Now, from Proposition 20.3 we recall that the term $\mu - \lambda\sigma$ is precisely the drift term of the short rate under the martingale measure Q. This fact is so important that we stress it again.

Result 21.1.1 *The term structure, as well as the prices of all other interest rate derivatives, are completely determined by specifying the* **r-dynamics under the martingale measure** Q.

Instead of specifying μ and λ under the objective probability measure P we will henceforth specify the dynamics of the short rate r directly under the martingale measure Q. This procedure is known as **martingale modeling**, and we formalize it as an assumption.

Assumption 21.1.1 *We assume that r under Q has dynamics given by*

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t. \tag{21.3}$$

For the rest if this chapter μ will always denote the drift term of the short rate under the martingale measure Q, and W will denote a Q-Wiener process.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

In the literature there are a large number of proposals on how to specify the Q-dynamics for r. We present a (far from complete) list of the most popular short rate models. If a parameter is time dependent this is written out explicitly. Otherwise all parameters are positive constants:

1. Vasiček
$$dr_t = (b - ar_t)\,dt + \sigma dW_t, \quad (a > 0), \tag{21.4}$$
2. Cox–Ingersoll–Ross (CIR)
$$dr_t = (b - ar_t)\,dt + \sigma\sqrt{r_t}dW_t, \tag{21.5}$$
3. Dothan
$$dr_t = ardt + \sigma r_t dW_t, \tag{21.6}$$
4. Black–Derman–Toy
$$dr_t = \theta(t) r_t dt + \sigma(t) r_t dW_t, \tag{21.7}$$
5. Ho–Lee
$$dr_t = \theta(t)dt + \sigma dW_t, \tag{21.8}$$
6. Hull–White (extended Vasiček)
$$dr_t = [\theta(t) - ar_t]\,dt + \sigma dW_t, \quad (a(t) > 0), \tag{21.9}$$
7. Hull–White (extended CIR)
$$dr_t = [\theta(t) - ar_t]\,dt + \sigma\sqrt{r_t}dW_t. \quad (a(t) > 0). \tag{21.10}$$

21.2 Properties of the Short Rate Models

In this section we will discuss some of the characteristics of the various short rate models above.

21.2.1 *Models with Linear Dynamics*

By definition, "linear dynamics" means that μ is linear in r and that σ is constant as a function of r. We see that in the list above, the Vasiček, the Ho–Lee, and the Hull–White extended Vasiček models have linear dynamics. These models are easy to handle analytically and the reason is the following.

Taking the Vasiček model as an example we can informally write it as

$$r_{t+dt} = (b - ar_t)\,dt + \sigma dW_t,$$

and we see that it is a continuous time version of an AR(1) model. This implies in particular that, conditional on r_t, the process value r_{t+dt} is normally distributed. With a constant or Gaussian initial value r_0 we thus expect the r-process to be Gaussian. This can also be seen from the solution to the SDE, which is

$$r_{=}e^{-at}r_0 + \int_0^t e^{a(t-s)}bds + \int_0^t e^{a(t-s)}\sigma dW_s$$

and it is easy to see that the stochastic integral is Gaussian.

Now, bond prices are given by expressions like

$$p(0,T) = E\left[e^{-\int_0^T r_s ds}\right], \tag{21.11}$$

and the normal property of r is inherited by the integral $\int_0^T r_s ds$ (an integral is just a sum). Thus we see that the computation of bond prices for a model with a normally distributed short rate boils down to the easy problem of computing the expected value of a log-normal random variable.

We finally note that for the linear models above, the short rate will, at any point in time, be negative with positive probability, and this is sometimes seen as a problem.

21.2.2 *Models with Mean Reversion*

Mean reversion means that the short rate has a tendency to revert to some long-term (possibly time-dependent) mean. This is the case for the Vasiček, and the CIR model, as well as the Hull–White extensions of these models. As an example we consider the Vasiček model, which we can write as

$$dr_t = a\left(\frac{b}{a} - r_t\right)dt + \sigma dW_t,$$

and we now see that if $r_t < \frac{b}{a}$ the drift is positive, so r has a tendency to increase. On the other hand, if $r_t > \frac{b}{a}$ then the drift is negative, so r has a tendency to decrease. The short rate r will thus have a tendency to revert to the value b/a, and one can in fact show (see the exercises) that r has a limiting Gaussian distribution with mean value b/a.

Mean reversion is in fact very important for short rate models, as opposed to stock price models. If a stock price increases from 100 to 150 dollars (say because of the introduction of a new technique), there is no compelling reason why the price should revert back to a lower value. If, on the other hand, the short rate goes high enough, say above 20 percent, then the government and/or the central bank will probably intervene to force the interest rate to go down. This is of course not a law of nature, but it is a well-established institutional fact.

Mean reversion is also important for forward rates (these will be treated in more detail in a later chapter). It is a well-documented empirical fact that the long end of the forward rate curve is much less volatile than the short end, and it can be shown that short rate mean reversion is necessary for a model to exhibit this volatility behavior.

21.2.3 *Lognormal Models*

The Dothan and the Black–Derman–Toy models are both GBM models, implying that the short rate is lognormal. Recalling the bond price formula

$$p(0,T) = E\left[e^{-\int_0^T r_s ds}\right], \tag{21.12}$$

we see that now we have a big computational problem. If we have a lognormal model then the exponent in the expectation above is a sum of lognormal random variables. We now recall that a sum of normal variables, and/or a product of lognormal variables are easy to handle. A sum of lognormal variables, on the other hand, is something of an analytical horror story. The lognormal models are thus hard to handle analytically.

It should be stressed that the analytical drawbacks for the lognormal models are tied to continuous time. A discrete time version of the Black–Derman–Toy model was for many years industry practice.

On the positive side, the lognormal models will produce strictly positive short rates with probability one.

21.2.4 *Square Root Models*

A square root model is a model with linear drift and square root diffusion term, so in our list the CIR model and the Hull–White extension of the CIR model are square root models, and we recall the CIR model as

$$dr_t = (b - ar_t)\,dt + \sigma\sqrt{r_t}dW_t. \tag{21.13}$$

Square root models are complicated to study in the sense that the standard theorems for existence and uniqueness of the SDE do not apply. In the standard results you need a Lipschitz condition on drift and diffusion, and the square root diffusion term $\sqrt{r}$ is obviously not Lipschitz.

Processes of this type can, subject to some modifications, be represented as squares of linear SDEs (see the exercises), so it is not surprising that the CIR model has a non-central χ^2-distribution. Under certain conditions on the parameters, the square root models are also positive. Given the connection with squares of linear models this is expected. A very loose argument for positivity of the CIR model goes as follows.

Assume that a, b and σ are positive, and assume that $r_t = 0$. Then the SDE has the form

$$dr_t = bdt$$

so the short rate r is kicked back to the positive half plane. This was just a very loose plausibility argument. One can in fact (but with considerable effort) prove the following result. See Jeanblanc et al. (2009).

Proposition 21.1 *Assume that $2b \geq \sigma^2$ and that $r_0 > 0$. Then $r_t > 0$ for all t.*

21.3 Inversion of the Yield Curve

Let us now address the question of how we will estimate the various model parameters in the martingale models above. To take a specific case, assume that we have decided to use the Vasiček model. Then we have to get values for a, b, and σ in some way, and a natural procedure would be to look in some textbook dealing with parameter estimation for SDEs. This procedure, however, is unfortunately completely nonsensical and the reason is as follows.

We have chosen to model our r-process by giving the Q-dynamics, which means that a, b, and σ are the parameters which hold under the martingale measure Q. When we make observations in the real world we are **not** observing r under the martingale measure Q, but under the objective measure P. This means that if we apply standard statistical procedures to our observed data we will not get our Q-parameters. What we get instead is pure nonsense.

This looks extremely disturbing but the situation is not hopeless. From Girsanov we know that the diffusion term is the same under P and under Q, so "in principle" it may be possible to estimate diffusion parameters using P-data.

When it comes to the estimation of parameters affecting the drift term of r we have to use completely different methods.

From Result 14.6.1 we recall the following moral.

Result 21.3.1 *In an incomplete market, the martingale measure is determined* **by the market**.

Thus, in order to obtain information about the Q-drift parameters we have to collect price information from the market, and the typical approach is that of **inverting the yield curve**, which works as follows:

- Choose a particular model involving one or several parameters. Let us denote the entire parameter vector by α. Thus we write the r-dynamics (under Q) as
$$dr_t = \mu(t, r_t; \alpha)dt + \sigma(t, r_t; \alpha)dW(t). \tag{21.14}$$
- Solve, for every conceivable time of maturity T, the term structure equation
$$\begin{cases} F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0, \\ F^T(T, r) = 1. \end{cases} \tag{21.15}$$
In this way we have computed the theoretical term structure as
$$p(t, T; \alpha) = F^T(t, r; \alpha).$$
Note that the form of the term structure will depend upon our choice of parameter vector. We have not made this choice yet.
- Collect price data from the bond market. In particular we may today (i.e. at $t = 0$) observe $p(0, T)$ for all values of T. Denote this **empirical term structure** by $\{p^\star(0, T);\ T \geq 0\}$.

- Now choose the parameter vector α in such a way that the theoretical curve $\{p(0,T;\alpha);\ T \geq 0\}$ fits the empirical curve $\{p^{\star}(0,T);\ T \geq 0\}$ as well as possible (according to some objective function). This gives us our estimated parameter vector $\alpha^{\star}$.
- Insert $\alpha^{\star}$ into μ and σ. Now we have pinned down exactly which martingale measure we are working with. Let us denote the result of inserting $\alpha^{\star}$ into μ and σ by $\mu^{\star}$ and $\sigma^{\star}$ respectively.
- We have now pinned down our martingale measure Q, and we can go on to compute prices of interest rate derivatives, like, say, $\mathcal{X} = \Gamma(r_T)$. The price process is then given by $\Pi_t[\Gamma] = G(t, r_t)$ where G solves the term structure equation

$$\begin{cases} G_t + \mu^{\star} G_r + \dfrac{1}{2}\left[\sigma^{\star}\right]^2 G_{rr} - rG = 0, \\ \qquad\qquad\qquad\qquad G(T, r) = \Gamma(r). \end{cases} \tag{21.16}$$

If the above program is to be carried out within reasonable time limits it is of course of great importance that the PDEs involved are easy to solve. It turns out that some of the models above are much easier to deal with analytically than others, and this leads us to the subject of so-called **affine term structures**.

21.4 Affine Term Structures

21.4.1 *Definition and Existence*

Definition 21.2 *If the term structure* $\{p(t,T);\ 0 \leq t \leq T,\ T > 0\}$ *has the form*

$$p(t,T) = F(t, r_t; T), \tag{21.17}$$

where F has the form

$$F(t, r; T) = e^{A(t,T) - B(t,T)r}, \tag{21.18}$$

and where A and B are deterministic functions, then the model is said to possess an **affine term structure** *(ATS).*

The functions A and B above are functions of the two real variables t and T, but conceptually it is easier to think of A and B as being functions of t, while T serves as a parameter. It turns out that the existence of an affine term structure is extremely pleasing from an analytical and a computational point of view, so it is of considerable interest to understand when such a structure appears. In particular we would like to answer the following question:

- For which choices of μ and σ in the Q-dynamics for r do we get an affine term structure?

We will try to give at least a partial answer to this question, and we start by investigating some of the implications of an affine term structure. Assume then that we have the Q-dynamics

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t, \tag{21.19}$$

and assume that this model actually possesses an ATS. In other words we assume that the bond prices have the form (21.18) above. Using (21.18) we may easily compute the various partial derivatives of F, and since F must solve the term structure equation (21.2), we thus obtain

$$A_t(t,T) - \{1 + B_t(t,T)\}\, r - \mu(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T) = 0. \tag{21.20}$$

The boundary value $F(T, r; T) \equiv 1$ implies

$$\begin{cases} A(T,T) = 0, \\ B(T,T) = 0. \end{cases} \tag{21.21}$$

Equation (21.20) gives us the relations which must hold between A, B, μ, and σ in order for an ATS to exist, and for a certain choice of μ and σ there may or may not exist functions A and B such that (21.20) is satisfied. Our immediate task is thus to give conditions on μ and σ which guarantee the existence of functions A and B solving (21.20). Generally speaking this is a fairly complex question, but we may give a very nice partial answer. We observe that if μ and σ^2 are both **affine** (i.e. linear plus a constant) functions of r, with possibly time-dependent coefficients, then eqn (21.20) becomes a separable differential equation for the unknown functions A and B.

Assume thus that μ and σ have the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t), \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases} \tag{21.22}$$

Then, after collecting terms, (21.20) transforms into

$$\begin{aligned} &A_t(t,T) - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) \\ &- \left\{ 1 + B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) \right\} r = 0. \end{aligned} \tag{21.23}$$

This equation holds for all t, T, and r, so let us consider it for a fixed choice of T and t. Since the equation holds for all values of r the coefficient of r must be equal to zero. Thus we have the equation

$$B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1. \tag{21.24}$$

Since the r-term in (21.23) is zero we see that the other term must also vanish, giving us the equation

$$A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T). \tag{21.25}$$

We may thus formulate our main result.

Proposition 21.3 (Affine term structure) *Assume that μ and σ are of the form*

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t), \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases} \tag{21.26}$$

Then the model admits an ATS of the form (21.18), where A and B satisfy the system

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1, \\ \qquad\qquad\qquad\qquad\qquad\qquad B(T,T) = 0. \end{cases} \tag{21.27}$$

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T), \\ A(T,T) = 0. \end{cases} \tag{21.28}$$

We note that eqn (21.27) is a Riccati equation for the determination of B which does not involve A. Having solved eqn (21.27) we may then insert the solution B into eqn (21.28) and simply integrate in order to obtain A.

An interesting question is if it is only for an affine choice of μ and σ^2 that we get an ATS. This is not generally the case, but it can fairly easily be shown that if we demand that μ and σ^2 are time independent, then a necessary condition for the existence of an ATS is that μ and σ^2 are affine. Looking at the list of models in Section 21.3 we see that all models except the Dothan and the Black–Derman–Toy models have an ATS.

We end this section with a comment on the procedure of calibrating the model to data described in Section 21.3. If we want a complete fit between the theoretical and the observed bond prices, this calibration procedure is formally that of solving the system of equations

$$p(0,T;\alpha) = p^{\star}(0,T) \quad \text{for all } T > 0. \tag{21.29}$$

We observe that this is an infinite dimensional system of equations (one equation for each T) with α as the unknown, so if we work with a model containing a finite parameter vector α (like the Vasiček model) there is no hope of obtaining a perfect fit. Now, one of the main goals of interest rate theory is to compute prices of various derivatives, like, for example, bond options, and it is well known that the price of a derivative can be very sensitive with respect to the price of the underlying asset. For bond options the underlying asset is a bond, and it is thus disturbing if we have a model for derivative pricing which is not even able to correctly price the underlying asset.

This leads to a natural demand for models which **can** be made to fit the observed bond data completely, and this is the reason why the Hull–White model has become so popular. In this model (and related ones) we introduce an infinite dimensional parameter vector α by letting some or all parameters be time dependent. Whether it is possible to actually solve the system (21.29) for a concrete model such as the Hull–White extension of the Vasiček model, and how this is to be done in detail, is of course not clear a priori but has to be dealt with in a deeper study. We carry out this study for the Hull–White model in Section 21.5

It should, however, be noted that the introduction of an infinite parameter, in order to fit the entire initial term structure, has its dangers in terms of numerical instability of the parameter estimates.

There is also a completely different approach to the problem of obtaining a perfect fit between today's theoretical bond prices and today's observed bond prices. This is the Heath–Jarrow–Morton approach, which roughly takes the observed term structure as an initial condition for the forward rate curve, thus automatically obtaining a perfect fit. This model will be studied in the next chapter.

21.5 Analytical Results for Some Standard Models

In this section we will apply the ATS theory above, in order to study the most common affine one-factor models.

21.5.1 *The Vasiček Model*

To illustrate the technique we now compute the term structure for the Vasiček model

$$dr_t = (b - ar_t)\,dt + \sigma dW_t. \tag{21.30}$$

Before starting the computations we note that this model has the property of being **mean reverting** (under Q) in the sense that it will tend to revert to the mean level b/a. The equations (21.27)–(21.28) become:

$$\begin{cases} B_t(t,T) - aB(t,T) = -1, \\ B(T,T) = 0. \end{cases} \tag{21.31}$$

$$\begin{cases} A_t(t,T) = bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) = 0. \end{cases} \tag{21.32}$$

Equation (21.31) is, for each fixed T, a simple linear ODE in the t-variable. It can easily be solved as

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}. \tag{21.33}$$

Integrating eqn (21.32) we obtain

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2(s,T)ds - b\int_t^T B(s,T)ds, \tag{21.34}$$

and, substituting the expression for B above, we obtain the following result.

Proposition 21.4 (The Vasiček term structure) *In the Vasiček model, bond prices are given by*

$$p(t,T) = e^{A(t,T)-B(t,T)r_t},$$

where

$$\begin{aligned} B(t,T) &= \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}, \\ A(t,T) &= \frac{\{B(t,T) - T + t\}\left(ab - \frac{1}{2}\sigma^2\right)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}. \end{aligned}$$

21.5.2 *The Ho–Lee Model*

For the Ho–Lee model the ATS equations become:

$$\begin{cases} B_t(t,T) = -1, \\ B(T,T) = 0. \end{cases}$$

$$\begin{cases} A_t(t,T) = \theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) = 0. \end{cases}$$

These are easily solved as

$$\begin{aligned} B(t,T) &= T - t, \\ A(t,T) &= \int_t^T \theta(s)(s-T)ds + \frac{\sigma^2}{2}\cdot\frac{(T-t)^3}{3}. \end{aligned}$$

It now remains to choose θ such that the theoretical bond prices, at $t = 0$, fit the observed initial term structure $\{p^\star(0,T); T \geq 0\}$. We thus want to find θ such that $p(0,T) = p^\star(0,T)$ for all $T \geq 0$. This is left as an exercise, and the solution is given by

$$\theta(t) = \frac{\partial f^\star(0,t)}{\partial T} + \sigma^2 t,$$

where $f^\star(0,t)$ denotes the observed forward rates. Plugging this expression into the ATS gives us the following bond prices.

Proposition 21.5 (The Ho–Lee term structure) *For the Ho–Lee model, the bond prices are given by*

$$p(t,T) = \frac{p^\star(0,T)}{p^\star(0,t)} \exp\left\{(T-t)f^\star(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r_t\right\}.$$

For completeness we also give the pricing formula for a European call on an underlying bond. We will not derive this result by solving the pricing PDE (this is in fact very hard), but instead we leave it to the reader to use Proposition 15.14. It is then an easy exercise to obtain the result below as a special case.

Proposition 21.6 (Bond options) *For the Ho–Lee model, the price at t, of a European call option with strike price K and exercise date T, on an underlying S-bond, we have the following pricing formula*:

$$c(t,T,K,S) = p(t,S)N(d) - p(t,T) \cdot K \cdot N(d - \sigma_p), \tag{21.35}$$

where

$$d = \frac{1}{\sigma_p} \log\left\{\frac{p(t,S)}{p(t,T)K}\right\} + \frac{1}{2}\sigma_p, \tag{21.36}$$

$$\sigma_p = \sigma(S-T)\sqrt{T}. \tag{21.37}$$

21.5.3 *The CIR Model*

The CIR model is much more difficult to handle than the Vasiček model, since we have to solve a Riccati equation. We cite the following result.

Proposition 21.7 (The CIR term structure) *The term structure for the CIR model is given by*

$$F^T(t,r) = A_0(T-t)e^{-B(T-t)r},$$

where

$$B(x) = \frac{2(e^{\gamma x}-1)}{(\gamma+a)(e^{\gamma x}-1)+2\gamma},$$

$$A_0(x) = \left[\frac{2\gamma e^{(a+\gamma)(x/2)}}{(\gamma+a)(e^{\gamma x}-1)+2\gamma}\right]^{2ab/\sigma^2},$$

and

$$\gamma = \sqrt{a^2+2\sigma^2}.$$

It is possible to obtain closed form expressions for European call options on zero coupon bonds within the CIR framework. Since these formulas are rather complicated, we refer the reader to Cox–Ingersoll–Ross (1985b).

21.5.4 *The Hull–White Model*

In this section we will make a fairly detailed study of a simplified version of the Hull–White extension of the Vasiček model. The Q-dynamics of the short rate are given by

$$dr_t = \{\theta(t) - ar_t\}\,dt + \sigma dW_t, \tag{21.38}$$

where a and σ are constants while θ is a deterministic function of time. In this model we typically choose a and σ in order to obtain a nice volatility structure whereas θ is chosen in order to fit the theoretical bond prices $\{p(0,T);\ T>0\}$ to the observed curve $\{p^\star(0,T);\ T>0\}$.

We have an affine structure so by Proposition 21.3 bond prices are given by

$$p(t,T) = e^{A(t,T)-B(t,T)r_t}, \tag{21.39}$$

where A and B solve

$$\begin{cases} B_t(t,T) = aB(t,T) - 1, \\ B(T,T) = 0. \end{cases} \tag{21.40}$$

$$\begin{cases} A_t(t,T) = \theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) = 0. \end{cases} \tag{21.41}$$

The solutions to these equations are given by

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}, \tag{21.42}$$

$$A(t,T) = \int_t^T \left\{\frac{1}{2}\sigma^2 B^2(s,T) - \theta(s)B(s,T)\right\} ds. \tag{21.43}$$

Now we want to fit the theoretical prices above to the observed prices and it is convenient to do this using the forward rates. Since there is a one-to-one correspondence (see Lemma 19.4) between forward rates and bond prices, we may just as well fit the theoretical forward rate curve $\{f(0,T);\ T > 0\}$ to the observed curve $\{f^\star(0,T);\ T > 0\}$, where of course $f^\star$ is defined by $f^\star(t,T) = -\frac{\partial \log p^\star(t,T)}{\partial T}$. In any affine model the forward rates are given by

$$f(0,T) = B_T(0,T)r_0 - A_T(0,T), \tag{21.44}$$

which, after inserting (21.42)–(21.43), becomes

$$f(0,T) = e^{-aT}r_0 + \int_0^T e^{-a(T-s)}\theta(s)ds - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2. \tag{21.45}$$

Given an observed forward rate structure $f^\star$ our problem is to find a function θ which solves the equation

$$f^\star(0,T) = e^{-aT}r_0 + \int_0^T e^{-a(T-s)}\theta(s)ds - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2, \quad \forall T > 0. \tag{21.46}$$

One way of solving (21.46) is to write it as

$$f^\star(0,T) = x(T) - g(T), \tag{21.47}$$

where x and g are defined by

$$\begin{cases} \dot{x} = -ax(t) + \theta(t), \\ x(0) = r(0), \end{cases}$$

$$g(t) = \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 = \frac{\sigma^2}{2}B^2(0,t). \tag{21.48}$$

We now have

$$\begin{aligned}\theta(T) &= \dot{x}(T) + ax(T) = f_T^\star(0,T) + \dot{g}(T) + ax(T) \\ &= f_T^\star(0,T) + \dot{g}(T) + a\{f^\star(0,T) + g(T)\},\end{aligned} \tag{21.49}$$

so we have in fact proved the following result.

Lemma 21.8 *Fix an arbitrary bond curve $\{p^\star(0,T);\ T>0\}$, subject only to the condition that $p^\star(0,T)$ is twice differentiable w.r.t. T. Choosing θ according to (21.49) will then produce a term structure $\{p(0,T);\ T>0\}$ such that $p(0,T) = p^\star(0,T)$ for all $T>0$.*

By choosing θ according to (21.49) we have, for a fixed choice of a and σ, determined our martingale measure. Now we would like to compute the theoretical bond prices under this martingale measure, and in order to do this we have to substitute our choice of θ into eqn (21.43). Then we perform the integration and substitute the result as well as eqn (21.42) into eqn (21.39). This leads to some exceedingly boring calculations which (of course) are left to the reader. The result is as follows.

Proposition 21.9 (The Hull–White term structure) *Consider the Hull–White model with a and σ fixed. Having inverted the yield curve by choosing θ according to (21.49) we obtain the bond prices as*

$$p(t,T) = \frac{p^\star(0,T)}{p^\star(0,t)} \exp\left\{B(t,T)f^\star(0,t) - \frac{\sigma^2}{4a}B^2(t,T)\left(1-e^{-2at}\right) - B(t,T)r_t\right\}, \tag{21.50}$$

where B is given by (21.42).

21.6 Bond Options in the Hull–White Model

As a concrete application of the Geman–El Karuoi–Rochet option pricing formula (15.15), we will now consider the case of interest rate options in the Hull–White model (extended Vasiček).

From Section 21.4 we recall that we have an affine term structure

$$p(t,T) = e^{A(t,T) - B(t,T)r_t}, \tag{21.51}$$

where A and B are deterministic functions.

The project is to price a European call option with exercise date T_1 and strike price K, on an underlying bond with date of maturity T_2, where $T_1 < T_2$. In the notation of the general theory above this means that $T = T_1$ and that $S_t = p(t,T_2)$. We start by checking Assumption 15.6.1, i.e. if the volatility, σ_z, of the process

$$Z_t = \frac{p(t,T_2)}{p(t,T_1)} \tag{21.52}$$

is deterministic. (In terms of the notation in Section 15.6 Z corresponds to $Z^{S,T}$ and σ_z corresponds to $\sigma^{S,T}$.)

Inserting (21.51) into (21.52) gives

$$Z_t = \exp\{A(t,T_2) - A(t,T_1) - [B(t,T_2) - B(t,T_1)]\, r_t\}.$$

Applying the Itô formula to this expression, and using (21.9), we get the Q-dynamics

$$dZ_t = Z_t\{\cdots\}\, dt + Z_t \cdot \sigma_t^z dW, \tag{21.53}$$

where

$$\sigma_t^z = -\sigma\,[B(t,T_2) - B(t,T_1)] = \frac{\sigma}{a} e^{at}\left[e^{-aT_2} - e^{-aT_1}\right]. \tag{21.54}$$

Thus σ_z is in fact deterministic, so we may apply Proposition 15.15. We obtain the following result, which also holds (why?) for the Vasiček model.

Proposition 21.10 (Hull–White bond option) *In the Hull–White short rate model (21.38) the price, at $t = 0$, of a European call with strike price K, and time of maturity T_1, on a bond maturing at T_2 is given by the formula*

$$c_0 = p(0,T_2)N[d_1] - K \cdot p(0,T_1)N[d_2], \tag{21.55}$$

where

$$d_2 = \frac{\ln\left(\frac{p(0,T_2)}{Kp(0,T_1)}\right) - \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}}, \tag{21.56}$$

$$d_1 = d_2 + \sqrt{\Sigma^2}, \tag{21.57}$$

$$\Sigma^2 = \frac{\sigma^2}{2a^3}\left\{1 - e^{-2aT_1}\right\}\left\{1 - e^{-a(T_2 - T_1)}\right\}^2. \tag{21.58}$$

21.7 Exercises

Exercise 21.1 Consider the Vasiček model, where we always assume that $a > 0$.

(a) Solve the Vasiček SDE explicitly, and determine the distribution of r_t. **Hint:** The distribution is Gaussian (why?), so it is enough to compute the expected value and the variance.

(b) As $t \to \infty$, the distribution of r_t tends to a limiting distribution. Show that this is the Gaussian distribution $N[b/a, \sigma/\sqrt{2a}]$. Thus we see that, in the limit, r will indeed oscillate around its mean reversion level b/a.

(c) Now assume that r_0 is a random variable, independent of the Wiener process W, having the Gaussian distribution obtained in (b). Show that this implies that r_t has the limit distribution in (b), for all values of t. Thus we have found the stationary distribution for the Vasiček model.

(d) Check that the density function of the limit distribution solves the time invariant Fokker–Planck equation, i.e. the Fokker–Planck equation with the $\frac{\partial}{\partial t}$-term is equal to zero.

Exercise 21.2 Show directly that the Vasiček model has an affine term structure without using the methodology of Proposition 21.3. Instead use the characterization of $p(t,T)$ as an expected value, insert the solution of the SDE for r, and look at the structure obtained.

Exercise 21.3 Try to carry out the program outlined above for the Dothan model and convince yourself that you will only get a mess.

Exercise 21.4 Consider the Ho–Lee model

$$dr_t = \theta(t)dt + \sigma dW_t.$$

Assume that the observed bond prices at $t=0$ are given by $\{p^\star(0,T);\ T \geq 0\}$. Assume furthermore that the constant σ is given. Show that this model can be fitted exactly to today's observed bond prices with θ as

$$\theta(t) = \frac{\partial f^\star}{\partial T}(0,t) + \sigma^2 t,$$

where $f^\star$ denotes the observed forward rates. (The observed bond price curve is assumed to be smooth.)

Hint: Use the affine term structure, and fit forward rates rather than bond prices (this is logically equivalent).

Exercise 21.5 Use the result of the previous exercise in order to derive the bond price formula in Proposition 21.5.

Exercise 21.6 It is often considered reasonable to demand that a forward rate curve always has an horizontal asymptote, i.e. that $\lim_{T\to\infty} f(t,T)$ exists for all t. (The limit will obviously depend upon t and r_t.) The object of this exercise is to show that the Ho–Lee model is not consistent with such a demand.

(a) Compute the explicit formula for the forward rate curve $f(t,T)$ for the Ho–Lee model (fitted to the initial term structure).

(b) Now assume that the initial term structure indeed has a horizontal asymptote, i.e. that $\lim_{T\to\infty} f^\star(0,T)$ exists. Show that this property is not respected by the Ho–Lee model, by fixing an arbitrary time t, and showing that $f(t,T)$ will be asymptotically linear in T.

Exercise 21.7 The object of this exercise is to indicate why the CIR model is connected to squares of linear diffusions. Let Y be given as the solution to the following SDE:

$$dY_t = \left(2aY_t + \sigma^2\right)dt + 2\sigma\sqrt{Y_t}dW_t, \quad Y(0) = y_0.$$

Define the process Z by $Z(t) = \sqrt{Y(t)}$. It turns out that Z satisfies a stochastic differential equation. Which?

Exercise 21.8 Use the general option formula (15.15) in order to prove the pricing formula of Proposition 21.6 for bond options in the Ho–Lee model.

21.8 Notes

Basic papers on short rate models are Vasiček (1977), Hull and White (1990), Ho and Lee (1986), Cox et al. (1985*b*), Dothan (1978), and Black et al. (1990). For a multivariate extension and notes on the affine term structure theory, see Duffie and Kan (1996). For a huge extension of the affine framework to general process theory see Duffie et al. (2003). An extensive analysis of the linear quadratic structure of the CIR model can be found in Magshoodi (1996) and Jeanblanc et al. (2009) The bond option formula for the Vasiček model was first derived by Jamshidian (1989). For examples of two-factor models see Brennan and Schwartz (1979) and Longstaff and Schwartz (1992).

22

FORWARD RATE MODELS

22.1 The Heath–Jarrow–Morton Framework

Up to this point we have studied interest models where the short rate r is the only explanatory variable. The main advantages with such models are as follows:

- Specifying r as the solution of an SDE allows us to use Markov process theory, so we may work within a PDE framework.
- In particular it is often possible to obtain analytical formulas for bond prices and derivatives.

The main drawbacks of short rate models are as follows:

- From an economic point of view it seems unreasonable to assume that the entire money market is governed by only one explanatory variable.
- It is hard to obtain a realistic volatility structure for the forward rates without introducing a very complicated short rate model.
- As the short rate model becomes more realistic, the inversion of the yield curve described above becomes increasingly more difficult.

These, and other considerations, have led various authors to propose models which use more than one state variable. One obvious idea would, for example, be to present an a priori model for the short rate as well as for some long rate, and one could of course also model one or several intermediary interest rates. The method proposed by Heath–Jarrow–Morton is at the far end of this spectrum—they choose the entire forward rate curve as their (infinite dimensional) state variable.

We now turn to the specification of the Heath–Jarrow–Morton (HJM) framework. We start by specifying everything under a given objective measure P.

Assumption 22.1.1 *We assume that, for every fixed $T > 0$, the forward rate $f(\cdot,T)$ has a stochastic differential which under the objective measure P is given by*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t^P, \tag{22.1}$$

$$f(0,T) = f^\star(0,T), \tag{22.2}$$

where W^P is a (d-dimensional) P-Wiener process whereas $\alpha(\cdot,T)$ and $\sigma(\cdot,T)$ are adapted processes.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

Note that conceptually eqn (22.1) is one stochastic differential in the t-variable for each fixed choice of T. The index T thus only serves as a "mark" or "parameter" in order to indicate which maturity we are looking at. Also note that we use the observed forward rated curve $\{f^\star(0,T);\ T \geq 0\}$ as the initial condition. This will automatically give us a perfect fit between observed and theoretical bond prices at $t=0$, thus relieving us of the task of inverting the yield curve.

Remark 22.1.1 *It is important to observe that the HJM approach to interest rates is not a proposal of a specific* **model***, like, for example, the Vasiček model. It is instead a* **framework** *to be used for analyzing interest rate models. Every short rate model can be equivalently formulated in forward rate terms, and for every forward rate model, the arbitrage free price of a contingent T-claim X will still be given by the pricing formula*

$$\Pi_0[X] = E^Q\left[e^{-\int_0^T r_s ds} \cdot X\right],$$

where the short rate as usual is given by $r(s) = f(s,s)$.

Suppose now that we have specified α, σ, and $\{f^\star(0,T);\ T \geq 0\}$. Then we have specified the entire forward rate structure and thus, by the relation

$$p(t,T) = e^{-\int_t^T f(t,s)ds}, \tag{22.3}$$

we have in fact specified the entire term structure $\{p(t,T);\ T > 0,\ \ 0 \leq t \leq T\}$. Since we have d sources of randomness (one for every Wiener process), and an infinite number of traded assets (one bond for each maturity T), we run a clear risk of having introduced arbitrage possibilities into the bond market. The first question we pose is thus very natural: How must the processes α and σ be related in order that the induced system of bond prices admits no arbitrage possibilities? The answer is given by the HJM drift condition below.

Theorem 22.1 (HJM drift condition) *Assume that the family of forward rates is given by (22.1) and that the induced bond market is arbitrage free. Then there exists a d-dimensional column-vector process*

$$\lambda(t) = [\lambda_1(t), \ldots, \lambda_d(t)]'$$

with the property that for all $T \geq 0$ and for all $t \leq T$, we have

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'ds - \sigma(t,T)\lambda(t). \tag{22.4}$$

In these formulas $'$ denotes transpose.

Proof From Proposition 19.5 we have the bond dynamics

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2\right\}dt + p(t,T)S(t,T)dW_t^P, \tag{22.5}$$

where

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)ds, \\ S(t,T) = -\int_t^T \sigma(t,s)ds. \end{cases} \tag{22.6}$$

The risk premium for the T-bond is thus given by

$$A(t,T) + \frac{1}{2}||S(t,T)||^2,$$

so from the First Fundamental Theorem, and the Girsanov Theorem, we conclude the existence of a d-dimensional market price of risk column-vector process λ such that

$$A(t,T) + \frac{1}{2}||S(t,T)||^2 = \sum_{i=1}^{d} S_i(t,T)\lambda_i(t).$$

Taking the T-derivative of this equation gives us eqn (22.4). □

22.2 Martingale Modeling

We now turn to the question of martingale modeling, and thus assume that the forward rates are specified directly under a martingale measure Q as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t), \tag{22.7}$$
$$f(0,T) = f^\star(0,T), \tag{22.8}$$

where W is a (d-dimensional) Q-Wiener process. Since a martingale measure automatically provides arbitrage free prices, we no longer have a problem of absence of arbitrage, but instead we have another problem. This is so because we now have the following two different formulas for bond prices:

$$p(t,T) = e^{-\int_t^T f(t,s)ds},$$
$$p(t,T) = E^Q\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right],$$

where the short rate r and the forward rates f are connected by $r(t) = f(t,t)$. In order for these formulas to hold simultaneously, we have to impose some sort of consistency relation between α and σ in the forward rate dynamics. The result is the famous Heath–Jarrow–Morton drift condition.

Proposition 22.2 (HJM drift condition) *Under the martingale measure Q, the processes α and σ must satisfy the following relation, for every t and every $T \geq t$:*

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)' ds. \tag{22.9}$$

Proof A short and brave argument is to observe that if we start by modeling directly under the martingale measure, then we may apply Proposition 22.1 with $\lambda = 0$. A more detailed argument is as follows.

From Proposition 19.5 we again have the bond price dynamics

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2\right\}dt + p(t,T)S(t,T)dW_t.$$

We also know that, under a martingale measure, the local rate of return has to equal the short rate r. Thus we have the equation

$$r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2 = r(t),$$

which gives us the result. □

The moral of Proposition 22.2 is that when we specify the forward rate dynamics (under Q) we may freely specify the volatility structure. The drift parameters are then uniquely determined. An "algorithm" for the use of an HJM model can be written schematically as follows:

1. Specify, by your own choice, the volatilities $\sigma(t,T)$.
2. The drift parameters of the forward rates are now given by

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)' ds. \tag{22.10}$$

3. Go to the market and observe today's forward rate structure

$$\{f^\star(0,T);\ T \geq 0\}.$$

4. Integrate in order to get the forward rates as

$$f(t,T) = f^\star(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s). \tag{22.11}$$

5. Compute bond prices using the formula

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}. \tag{22.12}$$

6. Use the results above in order to compute prices for derivatives.

To see at least how part of this machinery works we now study the simplest example conceivable, which occurs when the process σ is a deterministic constant. With a slight abuse of notation let us thus write $\sigma(t,T) \equiv \sigma$, where $\sigma > 0$. Equation (22.9) gives us the drift process as

$$\alpha(t,T) = \sigma\int_t^T \sigma ds = \sigma^2(T-t), \tag{22.13}$$

so eqn (22.11) becomes

$$f(t,T) = f^\star(0,T) + \int_0^t \sigma^2(T-s)ds + \int_0^t \sigma dW(s), \tag{22.14}$$

i.e.

$$f(t,T) = f^{\star}(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma W(t). \tag{22.15}$$

In particular we see that r is given as

$$r(t) = f(t,t) = f^{\star}(0,t) + \sigma^2 \frac{t^2}{2} + \sigma W(t), \tag{22.16}$$

so the short rate dynamics are

$$dr(t) = \left\{ f_T(0,t) + \sigma^2 t \right\} dt + \sigma dW(t), \tag{22.17}$$

which is exactly the Ho–Lee model, fitted to the initial term structure. Observe in particular the ease with which we obtained a perfect fit to the initial term structure.

22.3 The General Gaussian Model

In this section we will use the technique of change of numeraire, and in particular Proposition 15.15 in order to compute prices of bond options in a general Gaussian forward rate model. We specify the model (under Q) as

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)dW_t, \tag{22.18}$$

where W is a d-dimensional Q-Wiener process.

Assumption 22.3.1 *We assume that the volatility vector function*

$$\sigma(t,T) = [\sigma_1(t,T), \cdots, \sigma_d(t,T)]$$

is a **deterministic** *function of the variables t and T.*

Using Proposition 19.5 the bond price dynamics under Q are given by

$$dp(t,T) = p(t,T)r_t dt + p(t,T)v(t,T)dW_t, \tag{22.19}$$

where the volatility is given by

$$v(t,T) = -\int_t^T \sigma(t,s)ds. \tag{22.20}$$

We consider a European call option, with expiration date T_0 and exercise price K, on an underlying bond with maturity T_1 (where of course $T_0 < T_1$). In order to compute the price of the bond, we use Proposition 15.15, which means that we first have to find the volatility σ_{T_1,T_0} of the process

$$Z_t = \frac{p(t,T_1)}{p(t,T_0)}.$$

An easy calculation shows that in fact

$$\sigma_{T_1,T_0}(t) = v(t,T_1) - v(t,T_0) = -\int_{T_0}^{T_1} \sigma(t,s)ds. \tag{22.21}$$

This is clearly deterministic, so Assumption 15.6.1 is satisfied. We now have the following pricing formula.

Proposition 22.3 (Option prices for Gaussian forward rates) *The price, at $t=0$, of the option*

$$\mathcal{X} = max[p(T_0,T_1) - K, 0]$$

is given by

$$c_0 = p(0,T_1)N[d_1] - K \cdot p(0,T_0)N[d_2], \tag{22.22}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{p(0,T_1)}{Kp(0,T_0)}\right) + \frac{1}{2}\Sigma^2_{T_1,T_0}}{\sqrt{\Sigma^2_{T_1,T_0}}}, \\ d_2 &= d_1 - \sqrt{\Sigma^2_{T_1,T_0}}, \\ \Sigma^2_{T_1,T_0} &= \int_0^{T_0} \|\sigma_{T_1,T_0}(s)\|^2 ds, \end{aligned}$$

and σ_{T_1,T_0} is given by (22.21).

Proof Follows immediately from Proposition 15.15. □

22.4 The Musiela Parameterization

In many practical applications it is more natural to use time **to** maturity, rather than time **of** maturity, to parameterize bonds and forward rates. If we denote running time by t, time of maturity by T, and time to maturity by x, then we have $x = T - t$, and in terms of x the forward rates are defined as follows.

Definition 22.4 *For all $x \geq 0$ the forward rates $r(t,x)$ are defined by the relation*

$$r(t,x) = f(t, t+x). \tag{22.23}$$

Suppose now that we have the standard HJM-type model for the forward rates under a martingale measure Q

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t). \tag{22.24}$$

The question is to find the Q-dynamics for $r(t,x)$, and we have the following result, known as the Musiela equation.

Proposition 22.5 (The Musiela equation) *Assume that the forward rate dynamics under Q are given by (22.24). Then*

$$dr(t,x) = \{\mathbf{F}r(t,x) + D(t,x)\}\,dt + \sigma_0(t,x)dW(t), \tag{22.25}$$

where

$$\sigma_0(t,x) = \sigma(t,t+x),$$
$$D(t,x) = \sigma_0(t,x)\int_0^x \sigma_0(t,s)'ds,$$
$$\mathbf{F} = \frac{\partial}{\partial x}.$$

Proof Using a slight variation of the Itô formula we have

$$dr(t,x) = df(t,t+x) + \frac{\partial f}{\partial T}(t,t+x)dt,$$

where the differential in the term $df(t,t+x)$ only operates on the first t. We thus obtain

$$dr(t,x) = \alpha(t,t+x)dt + \sigma(t,t+x)dW(t) + \frac{\partial}{\partial x}r(t,x)dt,$$

and, using the HJM drift condition, we obtain our result. □

The point of the Musiela parameterization is that it highlights eqn (22.25) as an infinite dimensional SDE. It has become an indispensable tool of modern interest rate theory.

22.5 Exercises

Exercise 22.1 Show that for the Hull–White model

$$dr = (\Theta(t) - ar)dt + \sigma dW$$

the corresponding HJM formulation is given by

$$df(t,T) = \alpha(t,T)dt + \sigma e^{-a(T-t)}dW.$$

Exercise 22.2 (**Gaussian interest rates**) Take as given an HJM model (under the risk neutral measure Q) of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

where the volatility $\sigma(t,T)$ is a **deterministic** function of t and T.

(a) Show that all forward rates, as well as the short rate, are normally distributed.
(b) Show that bond prices are lognormally distributed.

Exercise 22.3 Consider the domestic and a foreign bond market, with bond prices being denoted by $p_d(t,T)$ and $p_f(t,T)$ respectively. Take as given a standard HJM model for the domestic forward rates $f_d(t,T)$, of the form

$$df_d(t,T) = \alpha_d(t,T)dt + \sigma_d(t,T)dW(t),$$

where W is a multidimensional Wiener process under the **domestic** martingale measure Q. The foreign forward rates are denoted by $f_f(t,T)$, and their dynamics, still under the domestic martingale measure Q, are assumed to be given by

$$df_f(t,T) = \alpha_f(t,T)dt + \sigma_f(t,T)dW(t).$$

Note that the same vector Wiener process is driving both the domestic and the foreign bond market. The exchange rate X (denoted in units of domestic currency per unit of foreign currency) has the Q dynamics

$$dX_t = \mu(t)X_t dt + X_t\sigma_X(t)dW(t).$$

Under a foreign martingale measure, the coefficient processes for the foreign forward rates will of course satisfy a standard HJM drift condition, but here we have given the dynamics of f_f under the domestic martingale measure Q. Show that under this measure the foreign forward rates satisfy the modified drift condition

$$\alpha_f(t,T) = \sigma_f(t,T)\left\{\int_t^T \sigma_f'(t,s)ds - \sigma_X'(t)\right\}.$$

Exercise 22.4 With notation as in the exercise above, we define the **yield spread** $g(t,T)$ by

$$g(t,T) = f_f(t,T) - f_d(t,T).$$

Assume that you are given the dynamics for the exchange rate and the domestic forward rates as above. You are also given the spread dynamics (again under the domestic measure Q) as

$$dg(t,T) = \alpha_g(t,T)dt + \sigma_g(t,T)dW(t).$$

Derive the appropriate drift condition for the coefficient process α_g in terms of σ_g, σ_d, and σ_X (but not involving σ_f).

Exercise 22.5 A **consol bond** is a bond which forever pays a constant continuous coupon. We normalize the coupon to unity, so over every interval with lenght dt the consol pays $1 \cdot dt$. No face value is ever paid. The price $C(t)$, at time t, of the consol is the value of this infinite stream of income, and it is obviously (why?) given by

$$C(t) = \int_t^\infty p(t,s)ds.$$

Now assume that bond price dynamics under a martingale measure Q are given by

$$dp(t,T) = p(t,T)r(t)dt + p(t,T)v(t,T)dW(t),$$

where W is a vector valued Q-Wiener process. Use the heuristic arguments given in the derivation of the HJM drift condition (see Section 19.2.2) in order to show that the consol dynamics are of the form

$$dC(t) = (C(t)r(t) - 1)\,dt + \sigma_C(t)dW(t),$$

where

$$\sigma_C(t) = \int_t^\infty p(t,s)v(t,s)ds.$$

Exercise 22.6 A Gaussian Interest Rate Model
Consider a HJM model (under the risk neutral measure Q) of the form

$$df(t,T) = \mu(t,T)dt + \sigma_1 \cdot (T-t)dW_t^1 + \sigma_2 e^{-a(T-t)}dW_t^2$$

where σ_1 and σ_2 are constants.

(a) Derive the bond price dynamics.
(b) Compute the pricing formula for a European call option on an underlying bond.

Exercise 22.7 Prove that a payment of $\frac{1}{p}(A-p)^+$ at time T^i is equivalent to a payment of $(A-p)^+$ at time T_{i-1}, where $p = p(T_{i-1}, T^i)$, and A is a deterministic constant.

Exercise 22.8 Prove Lemma 15.13.

22.6 Notes

The basic paper for this chapter is Heath et al. (1992). The Musiela parameterization was first systematically investigated in Musiela (1993), and developed further in Brace and Musiela (1994). Consistency problems for HJM models and families of forward rate curves were studied in Björk and Christensen (1999), Filipović (1999), and Filipović (2001). The question of when the short rate in a HJM model is in fact Markovian was first studied in Carverhill (1994) for the case of deterministic volatiliy, and for the case of a short rate depending volatility structure it was solved in Jeffrey (1995). The more general question when a given HJM model admits a realization in terms of a finite dimensional Markovian diffusion was, for various special cases, studied in Ritchken and Sankarasubramanian (1995), Cheyette (1996), Inui and Kijima (1998), Björk and Gombani (1999), and Chiarella and Kwon (2001). The necessary and sufficient conditions for the existence of finite dimensional Markovian realizations in the general case were first obtained, using methods from differential geometry, in Björk and Svensson (2001). This theory has then been developed further and extended considerably in Filipović and Teichmann (2003), Filipović and Teichmann (2004). A survey is given in Björk (2001). In Shirakawa (1991), Björk et al. (1995), Björk et al. (1997), and Jarrow and Madan (1995) the HJM theory has been extended to more general driving noise processes. There is an extensive literature on defaultable bonds. See Merton (1974), Duffie and Singleton (1999), Leland (1994), Jarrow and Madan (1995), Lando (2004), and Schönbucher (2003). Concerning practical estimation of the yield curve see Anderson et al. (1996).

23

LIBOR MARKET MODELS

In the previous chapters we have concentrated on studying interest rate models based on *infinitesimal* interest rates like the instantaneous short rate and the instantaneous forward rates. While these objects are nice to handle from a mathematical point of view, they have two main disadvantages:

- The instantaneous short and forward rates can never be observed in real life.
- If you would like to calibrate your model to cap- or swaption data, then this is typically very complicated from a numerical point of view if you use one of the "instantaneous" models.

A further fact from real life, which has been somewhat disturbing from a theoretical point of view, is the following:

- For a very long time, the market practice has been to value caps, floors, and swaptions by using a formal extension of the Black (1976) model. Such an extension is typically obtained by an approximation argument where the short rate at one point in the argument is assumed to be deterministic, while later on in the argument the LIBOR rate is assumed to be stochastic. This is of course logically inconsistent.
- Despite this, the market happily continues to use Black-76 for the pricing of caps, floors, and swaptions.

In a situation like this, where market practice seems to be at odds with academic work, there are two possible attitudes for the theorist: You can join them (the market) or you can try to beat them, and since the fixed income market does not seem to collapse because of the use of Black-76, the more realistic alternative seems to be to join them.

Thus there has appeared a natural demand for constructing logically consistent (and arbitrage free!) models having the property that the theoretical prices for caps, floors, and swaptions produced by the model are of the Black-76 form. This project has in fact been carried out very successfully, starting with Miltersen *et al.* (1997), Brace *et al.* (1997), and Jamshidian (1989). The basic structure of the models is as follows:

- Instead of modeling instantaneous interest rates, we model discrete **market rates** like LIBOR rates in the LIBOR market models, or forward swap rates in the swap market models.
- Under a suitable choice of numeraire(s), these market rates can in fact be modeled lognormally.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

- The market models will thus produce pricing formulas for caps and floors (the LIBOR models), and swaptions (the swap market models) which are of the Black-76 type and thus conforming with market practice.
- By construction the market models are thus very easy to calibrate to market data for caps/floors and swaptions respectively. They are then used to price more exotic products. For this later pricing part, however, we will typically have to resort to some numerical method, like Monte Carlo.

23.1 Caps: Definition and Market Practice

In this section we discuss LIBOR caps and the market practice for pricing and quoting these intruments. To this end we consider a fixed set of increasing maturities $T_0, T_1, \ldots, T_N$ and we define α_i, by

$$\alpha_i = T_i - T_{i-1}, \quad i = 1, \ldots N.$$

The number α_i is known as the **tenor**, and in a typical application we could for example have all α_i equal to a quarter of a year.

Definition 23.1 *We let $p_i(t)$ denote the zero coupon bond price $p(t, T_i)$ and let $L_i(t)$ denote the LIBOR forward rate (see Section 19.2), contracted at t, for the period $[T_{i-1}, T_i]$, i.e.*

$$L_i(t) = \frac{1}{\alpha_i} \cdot \frac{p_{i-1}(t) - p_i(t)}{p_i(t)}, \quad i = 1, \ldots, N. \tag{23.1}$$

We recall that a **cap** with **cap rate** R and **resettlement dates** $T_0, \ldots, T_N$ is a contract which at time T_i gives the holder of the cap the amount

$$X_i = \alpha_i \cdot \max\left[L_i(T_{i-1}) - R, 0\right], \tag{23.2}$$

for each $i = 1, \ldots, N$. The cap is thus a portfolio of the individual **caplets** $X_1, \ldots, X_N$. We note that the forward rate $L_i(T_{i-1})$ above is in fact the spot rate at time T_{i-1} for the period $[T_{i-1}, T_i]$, and determined already at time T_{i-1}. The amount X_i is thus determined at T_{i-1} but not paid out until at time T_i. We also note that, formally speaking, the caplet X_i is a call option on the underlying spot rate.

The market practice is to use the Black-76 formula for the pricing of caplets.

Definition 23.2 (Black's Formula for Caplets)
The Black-76 formula for the caplet

$$X_i = \alpha_i \cdot max[L(T_{i-1}, T_i) - R, 0] \tag{23.3}$$

is given by the expression

$$\mathbf{Capl_i^B}(t) = \alpha_i \cdot p_i(t) \left\{ L_i(t) N[d_1] - R N[d_2] \right\}, \quad i = 1, \ldots, N, \tag{23.4}$$

where

$$d_1 = \frac{1}{\sigma_i \sqrt{T_i - t}} \left[\ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\sigma_i^2 (T_i - t) \right], \tag{23.5}$$

$$d_2 = d_1 - \sigma_i \sqrt{T_i - t}. \tag{23.6}$$

The constant σ_i is known as the **Black volatility** *for caplet No. i. In order to make the dependence on the Black volatility σ_i explicit we will sometimes write the caplet price as $Capl_i^B(t;\sigma_i)$.*

It is implicit in the Black formula that the forward rates are lognormal (under some probability measure), but until recently there was no firm theoretical base for the use of the Black-76 formula for caplets. One of the main goals of this chapter is precisely that of investigating whether it is possible to build an arbitrage free model object which produces formulas of the Black type for caplet prices.

In the market, cap prices are not quoted in monetary terms but instead in terms of **implied Black volatilities**, and these volatilities can furthermore be quoted as **flat volatilities** or as **spot volatilities** (also known as **forward volatilities**). They are defined as follows.

Let us consider a fixed date t, the fixed set of dates $T_0, T_1, \ldots, T_N$ where $t \le T_0$, and a fixed cap rate R. We assume that, for each $i = 1, \ldots, N$, there is a traded cap with resettlement dates $T_0, T_1, \ldots, T_i$, and we denote the corresponding observed market price by $\mathbf{Cap_i^m}$. From this data we can easily compute the market prices for the corresponding caplets as

$$\mathbf{Capl_i^m}(t) = \mathbf{Cap_i^m}(t) - \mathbf{Cap_{i-1}^m}(t), \quad i = 1, \ldots, N \tag{23.7}$$

with the convention $\mathbf{Cap_0^m}(t) = 0$. Alternatively, given market data for caplets we can easily compute the corresponding market data for caps.

Definition 23.3 *Given market price data as above, the implied Black volatilities are defined as follows*:

- *The implied* **flat volatilities** $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ *are defined as the solutions of the equations*

$$\mathbf{Cap_i^m}(t) = \sum_{k=1}^{i} \mathbf{Capl_k^B}(t;\bar{\sigma}_i), i = 1, \ldots, N. \tag{23.8}$$

- *The implied* **forward** *or* **spot** *volatilities* $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ *are defined as solutions of the equations*

$$\mathbf{Capl_i^m}(t) = \mathbf{Capl_i^B}(t;\bar{\sigma}_i), i = 1, \ldots, N. \tag{23.9}$$

A sequence of implied volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ (flat or spot) is called a volatility **term structure**. *Note that we use the same notation $\bar{\sigma}_i$ for flat as well as for spot volatilities. In applications this will be made clear by the context.*

Summarizing the formal definition above, the flat volatility $\bar{\sigma}_i$ is volatility implied by the Black formula if you use *the same* volatility for each caplet in the cap with maturity T_i. The spot volatility σ_i is just the implied volatility from caplet No. i. The difference between flat and forward volatilities is thus similar to the difference between yields and forward rates. A typical shape of the volatility term structure (flat or spot) for caps with, say, a three-months tenor is that it has an upward hump for maturities around two–three years, but that the long end of the curve is downward sloping.

23.2 The LIBOR Market Model

We now turn from market practice to the construction of the so-called LIBOR market models. To motivate these models let us consider the theoretical arbitrage free pricing of caps. The price $c_i(t)$ of a caplet No. L_i is of course given by the standard risk neutral valuation formula

$$\mathbf{Capl_i}(t) = \alpha_i E^Q \left[e^{-\int_0^{T_i} r(s)ds} \cdot \max\left[L_i(T_{i-1}) - R, 0\right] \middle| \mathcal{F}_t \right], \quad i = 1, \ldots, N,$$

but it is much more natural to use the T_i forward measure to obtain

$$\mathbf{Capl_i}(t) = \alpha_i p_i(t) E^{T_i}\left[\max\left[L_i(T_{i-1}) - R, 0\right] \middle| \mathcal{F}_t\right], \quad i = 1, \ldots, N, \tag{23.10}$$

where E^{T_i} denotes expectation under the Q^{T_i}. In order to have a more compact notation we will from now on denote Q^{T_i} by Q^i.

The focal point of the LIBOR models is the following simple result.

Lemma 23.4 *For every $i = 1, \ldots, N$, the LIBOR process L_i is a martingale under the corresponding forward measure Q^{T_i}, on the interval $[0, T_{i-1}]$.*

Proof We have

$$\alpha_i \cdot L_i(t) = \frac{p_{i-1}(t)}{p_i(t)} - 1.$$

The process 1 is obviously a martingale under any measure. The process p_{i-1}/p_i is the price of the T_{i-1} bond normalized by the numeraire p_i. Since p_i is the numeraire for the martingale measure Q^{T_i}, the process p_{i-1}/p_i is thus trivially a martingale on the interval $[0, T_{i-1}]$. Thus $\alpha_i L_i$ is a martingale and hence L_i is also a martingale. □

The basic idea is now to define the LIBOR rates such that, for each i, $L_i(T)$ will be lognormal under "it's own" measure Q^i, since then all caplet prices in (23.10) will be given by a Black-type formula. In order to do this we consider the following objects as given a priori:

- A set of resettlement dates $T_0, \ldots, T_N$.
- An arbitrage free market bonds with maturities $T_0, \ldots, T_N$.
- A k-dimensional Q^N-Wiener process W^N.
- For each $i = 1, \ldots, N$ a *deterministic* function of time $\sigma_i(t)$.
- An initial non-negative forward rate term structure $L_1(0), \ldots, L_N(0)$.
- For each $i = 1, \ldots, N$, we define W^i as the k-dimensional Q^i-Wiener process generated by W^N under the Girsanov transformation $Q^N \to Q^i$.

Definition 23.5 *If the LIBOR forward rates have the dynamics*

$$dL_i(t) = L_i(t)\sigma_i(t)dW^i(t), \quad i = 1, \ldots, N, \tag{23.11}$$

where W^i is Q^i-Wiener as described above, then we say that we have a discrete tenor **LIBOR market model** *with volatilities $\sigma_1, \ldots, \sigma_N$.*

From the definition above it is not obvious that, given a specification of $\sigma_1, \ldots, \sigma_N$, there exists a corresponding LIBOR market model. In order to arrive at the basic pricing formulas as quickly as possible we will temporarily ignore the existence problem, but we will come back to it below and thus provide the missing link.

23.3 Pricing Caps in the LIBOR Model

Given a LIBOR market model, the pricing of a caplet, and hence also a cap, is trivial. Since L_i in (23.11) is just a GBM we obtain

$$L_i(T) = L_i(t) \cdot e^{\int_t^T \sigma_i(s)dW^i(s) - \frac{1}{2}\int_t^T \|\sigma_i(s)\|^2 ds}.$$

Since σ_i is assumed to be deterministic this implies that, conditional on $\mathcal{F}_t$, $L_i(T)$ is lognormal, i.e. we can write

$$L_i(T) = L_i(t)e^{Y_i(t,T)},$$

where $Y_i(t,T)$ is normally distributed with expected value

$$m_i(t,T) = -\frac{1}{2}\int_t^T \|\sigma_i(s)\|^2 ds, \tag{23.12}$$

and variance

$$\Sigma_i^2(t,T) = \int_t^T \|\sigma_i(s)\|^2 ds. \tag{23.13}$$

Using these results and (23.10), a simple calculation gives us the pricing formula for caps. Alternatively we see that the expectation E^i for the cap price in (23.10) is just the call price, within the Black–Scholes framework, in a world with zero short rate on an underlying traded asset with lognormal distribution as above.

Proposition 23.6 *In the LIBOR market model, the caplet prices are given by*

$$\mathbf{Capl_i}(t) = \alpha_i \cdot p_i(t)\left\{L_i(t)N[d_1] - RN[d_2]\right\}, \quad i = 1, \ldots, N, \tag{23.14}$$

where

$$d_1 = \frac{1}{\Sigma_i(t,T_{i-1})}\left[\ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\Sigma_i^2(t,T_{i-1})\right], \tag{23.15}$$

$$d_2 = d_1 - \Sigma_i(t,T_{i-1}), \tag{23.16}$$

with Σ_i defined by (23.13).

We thus see that each caplet price is given by a Black-type formula.

Remark 23.3.1 *Sometimes it is more convenient to work with a LIBOR model of the form*

$$dL_i(t) = L_i(t)\sigma_i(t)dW^i(t), \quad i = 1, \dots, N, \tag{23.17}$$

where $\sigma_i(t)$ is a scalar deterministic function, W^i is a scalar Q^i-Wiener process. Then the formulas above still hold if we replace $\|\sigma_i\|^2$ by σ_i^2. We can also allow for correlation between the various Wiener processes, but this will not affect the pricing of caps and floors. Such a correlation will however affect the pricing of more complicated products.

23.4 Terminal Measure Dynamics and Existence

We now turn to the question whether there always exists a LIBOR market model for any given specification of the deterministic volatilities $\sigma_1, \dots, \sigma_N$. In order to even get started we first have to specify all LIBOR rates $L_1, \dots, L_N$ under **one** common measure, and the canonical choice is the **terminal measure** Q^N.

Our problem is then basically that of carrying out a two-stage program:

- Specify all LIBOR rates under Q^N with dynamics of the form

$$dL_i(t) = L_i(t)\mu_i(t, L(t))\,dt + L_i(t)\sigma_i(t)dW^N(t), \quad i = 1, \dots, N \tag{23.18}$$

  where $L(t) = [L_1(t), \dots, L_N(t)]^\star$, and μ_i is some deterministic function.
- Show that, for some suitable choice of $\mu_1, \dots, \mu_N$, the Q^N dynamics in (23.18) will imply Q^i dynamics of the form (23.11).

In order to carry out this program we need to see how W^N is transformed into W^i as we change measure from Q^N to Q^i. We do this inductively by studying the effect of the Girsanov transformation from Q^i to Q^{i-1}.

Remark 23.4.1 *We have a small but irritating notational problem. LIBOR rates are typically denoted by the letter "L", but this is also a standard notation for a likelihood process. In order to avoid confusion we therefore introduce the notational convention that,* in this chapter only*, likelihood processes will be denoted by the letter η. In particular we introduce the notation*

$$\eta_i^j(t) = \frac{dQ^j}{dQ^i}, \quad \text{on } \mathcal{F}_t \text{ for } i, j = 1, \dots, N. \tag{23.19}$$

In order to get some idea of how we should choose the Q^N drifts of the LIBOR rates in (23.18) we will now perform some informal calculations. We thus (informally) assume that the LIBOR dynamics are of the form (23.18) under Q^N and that they are also of the form (23.11) under their own martingale measure. From Proposition 15.3 we know that the Radon–Nikodym derivative η_i^j is given by

$$\eta_i^j(t) = \frac{p_i(0)}{p_j(0)} \cdot \frac{p_j(t)}{p_i(t)}, \tag{23.20}$$

and in particular

$$\eta_i^{i-1}(t) = a_i \cdot \frac{p_{i-1}(t)}{p_i(t)} = a_i \left(1 + \alpha_i L_i(t)\right), \tag{23.21}$$

where $a_i = p_i(0)/p_{i-1}(0)$. From this formula we can now easily compute the η_i^{i-1} dynamics under Q^i as

$$d\eta_i^{i-1}(t) = a_i \alpha_i dL_i(t). \tag{23.22}$$

Assuming (still informally) that the L_i-dynamics are as in (23.11), and using (23.1) we then obtain

$$\begin{aligned} d\eta_i^{i-1}(t) &= a_i \alpha_i L_i(t) \sigma_i(t) dW^i(t) && (23.23)\\ &= a_i \alpha_i \frac{1}{\alpha_i} \left(\frac{p_{i-1}(t)}{p_i(t)} - 1 \right) \sigma_i(t) dW^i(t) && (23.24)\\ &= \eta_i^{i-1}(t) a_i \alpha_i \frac{1}{\eta_i^{i-1}(t)} \left(\frac{p_{i-1}(t)}{p_i(t)} - 1 \right) \sigma_i(t) dW^i(t) && (23.25) \end{aligned}$$

Using (23.21) we finally obtain

$$d\eta_i^{i-1}(t) = \eta_i^{i-1}(t) \frac{\alpha_i L_i(t)}{1 + \alpha_i L_i(t)} \sigma_i(t) dW^i(t). \tag{23.26}$$

Thus the Girsanov kernel for η_i^{i-1} is given by

$$\frac{\alpha_i L_i(t)}{1 + \alpha_i L_i(t)} \sigma_i^\star(t), \tag{23.27}$$

so from the Girsanov Theorem we have the relation

$$dW^i(t) = \frac{\alpha_i L_i(t)}{1 + \alpha_i L_i(t)} \sigma_i^\star(t) dt + dW^{i-1}(t). \tag{23.28}$$

Applying this inductively we obtain

$$dW^i(t) = -\sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1 + \alpha_k L_k(t)} \sigma_k^\star(t) dt + dW^N(t), \tag{23.29}$$

and plugging this into (23.11) we can finally obtain the Q^N dynamics of L_i (see (23.30) below).

All this was done under the informal assumption that there actually existed a LIBOR model satisfying both (23.11) and (23.18). We can however easily turn the argument around and we have the following existence result.

Proposition 23.7 *Consider a given volatility structure σ_1, σ_N, where each σ_i is assumed to be bounded, a probability measure Q^N and a standard Q^N-Wiener process W^N. Define the processes $L_1, \ldots, L_N$ by*

$$dL_i(t) = -L_i(t)\left(\sum_{k=i+1}^{N}\frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}\sigma_k(t)\sigma_i^\star(t)\right)dt + L_i(t)\sigma_i(t)dW^N(t), \tag{23.30}$$

for $i = 1, \ldots, N$ *where we use the convention* $\sum_N^N(\ldots) = 0$. *Then the* Q^i*-dynamics of* L_i *are given by (23.11). Thus there exists a LIBOR model with the given volatility structure.*

Proof Given that (23.30) has a solution for $i = 1, \ldots, N$, and that the Girsanov kernel in (23.27) satisfies the Novikov condition, the proof consists of exactly the calculations above. As for the existence of a solution of (23.30), this is trivial for $i = N$ since then the equation reads

$$dL_N(t) = L_i(t)\sigma_N(t)dW^N(t),$$

which is just GBM and since σ_N is bounded a solution does exist. Assume now that (23.30) admits a solution for $k = i+1, \ldots, N$. We can then write the $i:th$ component of (23.30) as

$$dL_i(t) = L_i(t)\mu_i\left[t, L_{i+1}(t), \ldots, L_N(t)\right]dt + L_i(T)\sigma_i(t)dW^N(t),$$

where the point is that μ_i does only depend on L_k for $k = i+1, \ldots, N$ and not on L_i. Denoting the vector $(L_{i+1}, \ldots, L_N)$ by L_{i+1}^N we thus have the explicit solution

$$\begin{aligned} L_i(t) &= L_i(0)\exp\left\{\int_0^t\left(\mu_i\left[s, L_{i+1}^N(s)\right] - \frac{1}{2}\|\sigma_i\|^2(s)\right)ds\right\} \\ &\quad\times\exp\left\{\int_0^t \sigma_i\left[s, L_{i+1}^N(s)\right]dW^N(s)\right\}, \end{aligned}$$

thus proving existence by induction. It also follows by induction that, given an initial positive LIBOR term structure, all LIBOR rate processes will be positive. From this we see that the Girsanov kernel in (23.27) is also bounded and thus it satisifies the Novikov condition. □

Remark 23.4.2 *Sometimes it is more convenient to work with a LIBOR model of the form*

$$dL_i(t) = L_i(t)\sigma_i(t)dW_i(t), \quad i = 1, \ldots, N, \tag{23.31}$$

where $\sigma_i(t)$ *is a scalar deterministic function,* W_i *is a scalar* Q^i*-Wiener process, and where we assume a given correlation structure* $dW_i(t)dW_j(t) = \rho_{ij}$. *This can easily be obtained by a small variation of the arguments above, and eqn (23.30) is then replaced by*

$$dL_i(t) = -L_i(t)\left(\sum_{k=i+1}^{N}\frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}\sigma_i(t)\sigma_k(t)\rho_{ik}\right)dt + L_i(t)\sigma_i(t)dW_i^N(t), \tag{23.32}$$

where W_i^N *is the* Q^N*-Wiener process generated by* W_i *under the Girsanov transformation* $Q^i \to Q^N$.

23.5 Calibration and Simulation

Suppose that we want to price some exotic (i.e. not a cap or a floor) interest rate derivative, like a Bermudan swaption. Performing this with a LIBOR model means that we typically carry out the following two steps:

- Use implied Black volatilities in order to calibrate the model parameters to market data.
- Use Monte Carlo (or some other numerical method) to price the exotic instrument.

In this section we mainly discuss the calibration part, and only comment briefly on the numerical aspects. For numerics and simulation see the Notes.

Let us thus assume that, for the resettlement dates $T_0, \ldots, T_N$, we are given an empirical term structure of implied forward volatilities, $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$, i.e., the implied Black volatilities for all caplets. For simplicity we assume that we are standing at time $t = 0$. Comparing the Black formula (23.4) with (23.14) we see that in order to calibrate the model we have to choose the deterministic LIBOR volatilities $\sigma_1(\cdot), \ldots, \sigma_N(\cdot)$, such that

$$\bar{\sigma}_i = \frac{1}{T_i} \int_0^{T_{i-1}} \|\sigma_i(s)\|^2 ds, \quad i = 1, \ldots, N. \tag{23.33}$$

Alternatively, if we use a scalar Wiener process for each LIBOR rate we must choose the scalar function $\sigma_i(\cdot)$ such that

$$\bar{\sigma}_i = \frac{1}{T_i} \int_0^{T_{i-1}} \sigma_i^2(s) ds, \quad i = 1, \ldots, N. \tag{23.34}$$

This is obviously a highly underdetermined system, so in applications it is common to make some structural assumption about the shape of the volatility functions. Below is a short and incomplete list of popular specifications. We use the formalism with a scalar Wiener process for each forward rate, and we recall that L_i lives on the time interval $0 \leq t \leq T_{i-1}$. We also introduce the temporary convention that $T_{-1} = 0$.

1. For each $i = 1, \ldots, N$, assume that the corresponding volatility is constant in time, i.e. that
$$\sigma_i(t) = \sigma_i$$
for $0 \leq t \leq T_{i-1}$.
2. For each $i = 1, \ldots, N$, assume that σ_i is piecewise constant, i.e. that
$$\sigma_i(t) = \sigma_{ij}, \quad \text{for } T_{j-1} < t \leq T_j,\ j = 0, \ldots, i-1.$$
3. As in item 2, but with the requirement that the volatility only depends on the number of resettlement dates left to maturity, i.e. that
$$\sigma_{ij} = \beta_{i-j}, \quad \text{for } T_{j-1} < t \leq T_j,\ j = 0, \ldots, i-1$$
where $\beta_1, \ldots, \beta_N$ are fixed numbers.

4. As in item 2, with the further specification that

$$\sigma_{ij} = \beta_i \gamma_j, \quad \text{for } T_{j-1} < t \le T_j, \ j = 0, \dots, i-1$$

where β_i and γ_j are fixed numbers.
5. Assume some simple functional parameterized form of the volatilities such as for example

$$\sigma_i(t) = q_i(T_{i-1} - t)e^{\beta_i(T_{i-1}-t)},$$

where $q_i(\cdot)$ is some polynomial and β_i is a real number.

Assuming that the model has been calibrated to market data, Monte Carlo simulation is the standard tool for computing prices of exotics. Since the SDEs (23.30) and (23.32) are too complicated to allow analytical solutions, we have to resort to simulation of discretized versions of the equations.

The simplest way to discretize (23.30) is to introduce a grid of length h and use the following recursive Euler scheme:

$$\begin{aligned} L_i((n+1)h) &= L_i(nh) - L_i(nh)\left(\sum_{k=i+1}^{N} \frac{\alpha_k L_k(nh)}{1+\alpha_k L_k(nh)} \sigma_i(nh)\sigma_k^\star(nh)\right) h \\ &\quad + L_i(nh)\sigma_i(nh)\left\{W^N((n+1)h) - W^N(nh)\right\}. \end{aligned} \tag{23.35}$$

However, from the point of view of numerical stability it is preferable to use a discretization of the SDE for $\ln(L_i)$. Using Itô we easily obtain

$$d\ln L_i(t) = -\left(\frac{1}{2}\sigma_i^2(t) + \sigma_i(t)\sigma_k^\star(t) \sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}\right) dt + \sigma_i(t) dW^N(t). \tag{23.36}$$

The point of this is that we now have a deterministic diffusion part, which leads to improved convergence of the corresponding discrete scheme

$$\begin{aligned} \ln L_i((n+1)h) &= \ln L_i(nh) \\ &\quad - h\left(\frac{1}{2}\sigma_i^2(nh) + \sum_{k=i+1}^{N} \frac{\alpha_k L_k(nh)}{1+\alpha_k L_k(nh)} \sigma_i(nh)\sigma_k^\star(nh)\right) \\ &\quad + \sigma_i(nh)\left\{W^N((n+1)h) - W^N(nh)\right\}. \end{aligned} \tag{23.37}$$

23.6 The Discrete Savings Account

In the LIBOR models discussed above there exists a forward neutral martingale measure $Q^i = Q^{T_i}$ for each $i = 1, \dots, N$, but so far we have not seen any risk neutral measure Q^B for the bank account B (also known as the *savings account*), and in fact we have not even seen a bank account process in the model. A natural question is therefore to investigate whether the LIBOR model will automatically imply a money account process B, as well as a corresponding risk neutral measure Q^B.

In this context it is, however, not quite clear what one would mean by a bank account. Since we are working in continuous time, one possibility is to look for a continuous bank account of the form we have seen earlier in the book, i.e. one with dynamics

$$dB(t) = r(t)B(t)dt,$$

where r is the continuously compounded short rate. However, since we have modeled discrete forward rates it would be unnatural to mix those with a continuously compounded short rate, so the natural choice would be to look for a bank account which is resettled at the points $T_0, \ldots, T_N$.

In order to construct the bank account we recall that the essential property is that it should be riskless on a local time scale, i.e. riskless between T_n and T_N for each n. The obvious way to achieve this is by forming the discretely rebalanced self-financing portfolio specified by constantly rolling over the bond of the shortest remaining maturity. More formally, suppose that we are standing at T_0 and consider the following portfolio strategy:

1. At T_0 invest one unit of money into the T_1 bond.
2. At T_1 sell the T_1 bond and invest everything in the T_2 bond.
3. Repeat this procedure recursively until T_N.

Denoting the value of this self-financing portfolio by B we immediately have

$$B(T_N) = \frac{B(T_n)}{p(T_n, T_N)}, \quad n = 0, \ldots, N-1, \tag{23.38}$$

and using the relation $p(T_n, T_N) = [1 + \alpha_N L(T_n, T_N)]^{-1}$ we obtain the discrete B dynamics as

$$B(T_0) = 1, \tag{23.39}$$

$$B(T_N) = (1 + \alpha_N L(T_n, T_N))\, B(T_n), \tag{23.40}$$

or, more explicitly

$$B(T_n) = \prod_{k=0}^{n-1} \frac{1}{p(T_k, T_{k+1})}, \tag{23.41}$$

alternatively

$$B(T_n) = \prod_{k=0}^{n-1} [1 + \alpha_{k+1} L(T_k, T_{k+1})]. \tag{23.42}$$

We note that B is indeed locally risk free in the sense that $B(T_N)$ is known already at time T_n, i.e. as a discrete time process B is *predictable* in the sense of Chapter C.

We can now easily determine the martingale measure corresponding to the discrete savings account.

Proposition 23.8 *The Radon–Nikodym derivative for the change from Q^N to Q^B is given by*

$$\frac{dQ^B}{dQ^N} = p(0, T_N)B(T_N), \quad on\ \mathcal{F}_{T_N}. \tag{23.43}$$

Proof From Proposition 15.3 we have (on $\mathcal{F}_{T_N}$)

$$\frac{dQ^B}{dQ^N} = \frac{B(T_N)}{p(T_N, T_N)} \cdot \frac{B(0)}{p(0, T_N)},$$

and since $P(T_N, T_N) = 1$ and $B(0) = 1$ we have the result. $\square$

23.7 Notes

The basic papers on the LIBOR and swap market models are Miltersen et al. (1997), Brace et al. (1997), and Jamshidian (1997). Since these basic papers were published there has appeared a huge literature on the subject. Very readable accounts can be found in Hunt and Kennedy (2000), Pelsser (2000), and the almost encyclopedic Brigo and Mercurio (2007).

24

POTENTIALS AND POSITIVE INTEREST

The purpose of this chapter is to present two approaches to interest rate theory which are based directly on stochastic discount factors, while also relating bond prices to the theory of probabilistic potential theory. A very appealing aspect of these approaches is that they both generate **positive term structures**, i.e. systems of bond prices for which the induced interest rates are all positive.

24.1 Generalities

As a general setup we consider a standard filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where P is the objective measure. We assume, as usual, that the market is pricing all assets, underlying and derivative, using a fixed martingale measure Q (with the money account as the numeraire).

We denote the likelihood process for the transition from the objective measure P to the martingale measure Q by L, i.e.

$$L_t = \frac{dQ_t}{dP_t},$$

where subindex t denotes the restriction of P and Q to $\mathcal{F}_t$. From Section 11.8 we recall that the **stochastic discount factor M** is defined by

$$\mathbf{M}_t = e^{-\int_0^t r_s ds} \cdot L_t, \tag{24.1}$$

and that the short rate can be recovered from the dynamics of $\mathbf{M}$ by the formula

$$d\mathbf{M}_t = -r_t \mathbf{M}_t + dM_t, \tag{24.2}$$

where M is a martingale defined by $M_t = B_t^{-1} L_t$.

We also recall that for any T-claim Y, the arbitrage free price process is given by

$$\Pi_t[Y] = \frac{E^P\left[\mathbf{M}_T Y \middle| \mathcal{F}_t\right]}{\mathbf{M}_t},$$

so, in particular, bond prices are given by the formula

$$p(t,T) = \frac{E^P\left[\mathbf{M}_T \middle| \mathcal{F}_t\right]}{\mathbf{M}_t}. \tag{24.3}$$

We now have the following important result.

Proposition 24.1 *Assume that the short rate is strictly positive and that the economically natural condition*

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

$$\lim_{T\to\infty} p(0,T) = 0 \tag{24.4}$$

is satisfied. Then the stochastic discount factor $\mathbf{M}$ *is a probabilistic* **potential**, *i.e.*

- $\mathbf{M}$ *is a non-negative supermartingale.*
- $E[\mathbf{M}_t] \to 0$ *as* $t \to \infty$.

Proof Assuming a positive short rate, we see from (24.1) that $\mathbf{M}$ is a non-negative martingale multiplied by a decreasing process, so $\mathbf{M}$ is clearly a supermartingale. The second condition follows directly from (24.4). □

Conversely one can show that any potential will serve as a stochastic discount factor. The moral is thus that modeling bond prices in a market with positive interest rates is equivalent to modeling a potential, and in the next sections we will describe two ways of doing this.

24.2 The Flesaker–Hughston Framework

Given a stochastic discount factor $\mathbf{M}$ and a positive short rate we may, for each fixed T, define the process $\{X(t,T);\ 0 \le t \le T\}$ by

$$X(t,T) = E^P[\mathbf{M}_T | \mathcal{F}_t], \tag{24.5}$$

and thus, according to (24.3) write bond prices as

$$p(t,T) = \frac{X(t,T)}{X(t,t)}. \tag{24.6}$$

We now have the following result.

Proposition 24.2 *For each fixed* t, *the mapping* $T \longmapsto X(t,T)$ *is smooth, and in fact*

$$\frac{\partial}{\partial T}X(t,T) = -E^P[r_T\mathbf{M}_T | \mathcal{F}_t]. \tag{24.7}$$

Furthermore, for each fixed T, *the process*

$$X_T(t,T) = \frac{\partial}{\partial T}X(t,T)$$

is a negative P*-martingale satisfying*

$$X_T(0,T) = p_T(0,T), \quad \textit{for all } T \ge 0.$$

Proof Using the definition of $\mathbf{M}$ and the Itô formula, we obtain

$$d\mathbf{M}_s = -r_s\mathbf{M}_s ds + B_s^{-1} dL_s,$$

so

$$\mathbf{M}_T = \mathbf{M}_t - \int_t^T r_s\mathbf{M}_s ds + \int_t^T B_s^{-1} dL_s.$$

Since L is a martingale, this gives us

$$E^P\left[\mathbf{M}_T\middle|\mathcal{F}_t\right]=\mathbf{M}_t-E^P\left[\int_t^T r_s\mathbf{M}_s ds\middle|\mathcal{F}_t\right].$$

Differentiating this expression w.r.t. T gives us (24.7). The martingale property follows directly from (24.7) and since $r>0$ it is clear that X_T is negative. The relation $X_T(0,T)=p_T(0,T)$ follows from (24.6). □

We can now state the basic result from Flesaker–Hughston, but first we need a formal definition.

Definition 24.3 *We say that* **the term structure is positive** *if the following conditions hold*:

- *For all t and T we have*
$$\frac{\partial}{\partial T}p(t,T)\leq 0.$$
- *For all t we have*
$$\lim_{T\to\infty}p(t,T)=0.$$

It is clear that in a positive term structure all forward rates are positive. We can now state the basic result from Flesaker–Hughston.

Theorem 24.4 *Assume that the term structure is positive. Then there exists a family of positive martingales $M(t,T)$ indexed by T and a positive deterministic function Φ such that*

$$p(t,T)=\frac{\int_T^\infty \Phi(s)M(t,s)ds}{\int_t^\infty \Phi(s)M(t,s)ds}. \tag{24.8}$$

The M family can, up to multiplicative scaling by the Φ function, be chosen as

$$M(t,T)=-X_T(t,T)=E^P\left[r_T\mathbf{M}_T\middle|\mathcal{F}_t\right], \tag{24.9}$$

where X is defined by (24.5). In particular, Φ can be chosen as

$$\Phi(s)=-p_T(0,s), \tag{24.10}$$

in which case the corresponding M is normalized to $M(0,s)=1$ for all $s\geq 0$.

Proof A positive term structure implies that $X(t,T)\to 0$ as $T\to\infty$, so we have

$$X(t,T)=-\int_T^\infty X_T(t,s)ds,$$

and thus we obtain from (24.6)

$$p(t,T)=\frac{\int_T^\infty X_T(t,s)ds}{\int_t^\infty X_T(t,s)ds}. \tag{24.11}$$

If we now define $M(t,T)$ by

$$M(t,T) = -X_T(t,T), \tag{24.12}$$

then (24.8) follows from (24.11) with $\Phi \equiv 1$. The function Φ is only a scale factor which can be chosen arbitrarily, and the choice in (24.10) is natural in order to normalize the M family. Since X_T is negative, M is positive and we are done. □

There is also a converse of the result above, due to Jin and Glasserman.

Proposition 24.5 *Consider a given family of positive martingales $M(t,T)$ indexed by T and a positive deterministic function Φ, such that the condition*

$$\int_0^\infty \Phi(s)M(t,s)ds < \infty$$

is satisfied $P - a.s.$ for all t. Then the specification

$$p(t,T) = \frac{\int_T^\infty \Phi(s)M(t,s)ds}{\int_t^\infty \Phi(s)M(t,s)ds} \tag{24.13}$$

will define an arbitrage free positive system of bond prices. Furthermore, the stochastic discount factor $\mathbf{M}$ generating the bond prices is given by

$$\mathbf{M}_t = \int_t^\infty \Phi(s)M(t,s)ds. \tag{24.14}$$

Proof Using the martingale property of the M family, as well as the positivity of M and Φ we obtain

$$E^P[\mathbf{M}_T | \mathcal{F}_t] = \int_T^\infty E^P[\Phi(s)M(T,s)|\mathcal{F}_t]\,ds = \int_T^\infty \Phi(s)M(t,s)ds$$
$$\leq \int_t^\infty \Phi(s)M(t,s)ds = \mathbf{M}_t.$$

We thus see that $\mathbf{M}$ is a supermartingale. From the formula above we have in particular that

$$\lim_{T\to\infty} E^P[\mathbf{M}_T] = \lim_{T\to\infty} \int_T^\infty \Phi(s)M(0,s)ds = 0,$$

so $\mathbf{M}$ is indeed a potential and can thus serve as a stochastic discount factor. The induced bond prices are thus given by

$$p(t,T) = \frac{E^P[\mathbf{M}_T|\mathcal{F}_t]}{\mathbf{M}_t},$$

and the calculation above shows the induced term structure given by (24.13). It is left to the reader to prove that the term structure is positive. □

We can also easily compute forward rates.

Proposition 24.6 *With bond prices given by (24.13), forward rates are given by*

$$f(t,T) = \frac{\Phi(T)M(t,T)}{\int_T^\infty \Phi(s)M(t,s)ds}, \tag{24.15}$$

and the short rate has the form

$$r_t = \frac{\Phi(t)M(t,t)}{\int_t^\infty \Phi(s)M(t,s)ds}. \tag{24.16}$$

Proof The proof is left as an exercise for the reader. □

The most-used instance of a Flesaker–Hughston model is the so-called **rational model**. In such a model we consider a given positive martingale K and two deterministic positive functions $\alpha(t)$ and $\beta(t)$. We then define the M family by

$$M(t,T) = \alpha(T) + \beta(T)K(t). \tag{24.17}$$

With this specification of M it is easily seen that bond prices will have the form

$$p(t,T) = \frac{A(T) + B(T)K(t)}{A(t) + B(t)K(t)} \tag{24.18}$$

where

$$A(t) = \int_t^\infty \Phi(s)\alpha(s)ds, \quad B(t) = \int_t^\infty \Phi(s)\beta(s)ds,$$

We can specialize this further by assuming K to be of the form

$$K(t) = e^{\int_0^t \gamma(s)dW_s - \frac{1}{2}\int_0^t \gamma^2(s)ds}$$

where γ is deterministic. Then K will be a lognormal martingale, the entire term structure will be analytically very tractable, and α and β can be chosen in order to fit the initial observed term structure.

24.3 Changing Base Measure

The arguments above do not at all depend upon the fact that P was assumed to be the objective probability measure. If instead we work with another base measure $P^0 \sim P$, we will of course have a Flesaker–Hughston representation of bond prices of the form

$$p(t,T) = \frac{\int_T^\infty \Phi(s)M^0(t,s)ds}{\int_t^\infty \Phi(s)M^0(t,s)ds}, \tag{24.19}$$

where $M^0(t,T)$ is a family of positive P^0 martingales, and the question is how M^0 relates to M.

Proposition 24.7 *With notation as above, we have*

$$M^0(t,T) = \frac{M(t,T)}{R_t^0}, \tag{24.20}$$

where

$$R_t^0 = \frac{dP^0}{dP} \quad on\ \mathcal{F}_t. \tag{24.21}$$

Proof From (24.9) we have, modulo scaling, the relation

$$M(t,T) = E^P\left[r_T B_T^{-1} L_T \middle| \mathcal{F}_t \right]$$

where $L_T = dQ/dP$ on $\mathcal{F}_T$. On the other hand we also have

$$M^0(t,T) = E^0\left[r_T B_T^{-1} L_T^0 \middle| \mathcal{F}_t \right],$$

where $L_T^0 = dQ/dP^0$ on $\mathcal{F}_T$. Using the Bayes formula we obtain

$$M^0(t,T) = E^0\left[r_T B_T^{-1} L_T^0 \middle| \mathcal{F}_t \right] = \frac{1}{R_t^0} E^P\left[r_T B_T^{-1} L_T^0 R_T^0 \middle| \mathcal{F}_t \right].$$

Using the fact that

$$L_T^0 R_T^0 = \frac{dQ}{dP^0} \cdot \frac{dP^0}{dP} = \frac{dQ}{dP} = L_T,$$

we thus have

$$M^0(t,T) = \frac{1}{R_t^0} E^P\left[r_T B_T^{-1} L_T \middle| \mathcal{F}_t \right] = \frac{M(t,T)}{R_t^0}.$$

□

24.4 Decomposition of a Potential

In Section 24.1 we saw that any SDF generating a positive term structure is a potential, so from a modeling point of view it is natural to ask how one can construct potentials from scratch. The main result used is the following.

Proposition 24.8 (Decomposition of a potential) *In a Wiener-driven setting, the following hold:*

1. *Let A be an increasing adapted process such that A_∞ is integrable, and define the martingale M by*

$$M_t = E^P\left[A_\infty \middle| \mathcal{F}_t \right]. \tag{24.22}$$

   Then the process Z, defined by

$$Z_t = -A_t + M_t, \tag{24.23}$$

   is a potential.
2. *Conversely, if Z is a potential then it admits a representation as*

$$Z_t = -A_t + M_t, \tag{24.24}$$

where A is an increasing adapted process with $A_\infty \in L^1$, and M is the martingale defined by

$$M_t = E^P\left[A_\infty \middle| \mathcal{F}_t\right]. \tag{24.25}$$

Remark 24.4.1 *The result above is valid also without the assumption of a Wiener-driven framework. In the general case A should be* **predictable** *rather than merely adapted. Loosely speaking, predictability means that we require that $A_t \in \mathcal{F}_{t-}$ for all t.*

Proof The proof of the first part is trivial and left to the reader as an exercise. The proof of the second part is much more difficult. It relies on the Doob–Meyer decomposition and the reader is referred to the literature. See Protter (2004). □

The point of this, for our purposes, is that by specifying an increasing adapted process A with $A_\infty < \infty$, we can generate a potential Z by the formula

$$Z_t = E^P\left[A_\infty \middle| \mathcal{F}_t\right] - A_t, \tag{24.26}$$

and then use Z as our SDF. In particular we may define A as

$$A_t = \int_0^t a_s ds \tag{24.27}$$

for some adapted non-negative process a. Then we easily obtain

$$Z_t = E^P\left[\int_0^\infty a_s ds \middle| \mathcal{F}_t\right] - \int_0^t a_s ds = \int_t^\infty E^P\left[a_s \middle| \mathcal{F}_t\right] ds. \tag{24.28}$$

We can now connect this to the Flesaker–Hughston framework. The family of processes $X(t,T)$ defined in (24.5) will, in the present framework, have the form

$$\begin{aligned} X(t,T) &= E^P\left[Z_T \middle| \mathcal{F}_t\right] = E^P\left[\int_T^\infty E^P\left[a_s \middle| \mathcal{F}_T\right] ds \middle| \mathcal{F}_t\right] \\ &= \int_T^\infty E^P\left[E^P\left[a_s \middle| \mathcal{F}_T\right] \middle| \mathcal{F}_t\right] ds = \int_T^\infty E^P\left[a_s \middle| \mathcal{F}_t\right] ds, \end{aligned} \tag{24.29}$$

so we have proved the following result.

Proposition 24.9 *Assume that a is a non-negative adapted process and define the potential Z by (24.26)–(24.27). Then, modulo scaling, the basic family of Flesaker–Hughston martingales is given by*

$$M(t,T) = -\frac{\partial}{\partial T} X(t,T) = E^P\left[a_T \middle| \mathcal{F}_t\right]. \tag{24.30}$$

24.5 The Markov Potential Approach of Rogers

As we have seen above, in order to generate an arbitrage free bond market model it is enough to construct a potential to act as stochastic discount factor (SDF), and in Section 24.4 we learned how to do this using the decomposition (24.22)–(24.23). In this section we will present a systematic way of constructing potentials along the lines above, in terms of Markov processes and their resolvents. The ideas are due to Rogers (1997), and we largely follow his presentation.

We consider a time-homogeneous Markov process X under the objective measure P, with infinitesimal generator $\mathcal{G}$. The reader unfamiliar with general Markov process theory can, without loss of good ideas, consider the case when X is a diffusion of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{24.31}$$

where μ and σ do not depend on running time t. In this case $\mathcal{G}$ is the usual Itô partial differential operator

$$\mathcal{G} = \mu(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}. \tag{24.32}$$

For any positive real valued, and sufficiently integrable, function g, and any positive real number α, we can now define the process A in the decomposition (24.23) as

$$A_t = \int_0^t e^{-\alpha s} g(X_s)ds, \tag{24.33}$$

where the exponential is introduced in order to allow at least all bounded functions g. In terms of the representation (24.27) we thus have

$$a_t = e^{-\alpha t} g(X_t), \tag{24.34}$$

and a potential Z is, according to (24.28), obtained as

$$Z_t = \int_t^\infty e^{-\alpha s} E^P\left[g(X_s) \middle| \mathcal{F}_t \right] ds = E^P\left[\int_t^\infty e^{-\alpha s} g(X_s)ds \middle| \mathcal{F}_t \right], \tag{24.35}$$

so we have

$$Z_t = -\int_0^t e^{-\alpha s} g(X_s)ds + M_t, \tag{24.36}$$

where M is the martingale

$$M_t = E^P\left[\int_0^\infty e^{-\alpha s} g(X_s)ds \middle| \mathcal{F}_t \right].$$

Using the Markov assumption we have in fact

$$Z_t = E^P\left[\int_t^\infty e^{-\alpha s} g(X_s)ds \middle| X_t \right], \tag{24.37}$$

and this expression leads to a well-known probabilistic object.

Definition 24.10 *For any non-negative α the* **resolvent** *R_α is an operator, mapping a bounded measurable real valued function g into the real valued function $[R_\alpha g]$ defined by the expression*

$$[R_\alpha g](x) = E_x^P\left[\int_0^\infty e^{-\alpha s} g(X_s)ds \right] \tag{24.38}$$

where subindex x denotes the conditioning $X_0 = x$.

We can now connect resolvents to potentials.

Proposition 24.11 *The potential Z defined by (24.37) can be expressed as*

$$Z_t = e^{-\alpha t} R_\alpha g(X_t). \tag{24.39}$$

Conversely, for any bounded non-negative g, the process

$$Z_t = e^{-\alpha t} R_\alpha g(X_t) \tag{24.40}$$

is a potential.

Proof Assume that Z is defined by (24.38). We then have, using the time invariance,

$$\begin{aligned} Z_t &= E^P\left[\int_0^\infty e^{-\alpha(t+s)} g(X_{t+s})ds \middle| X_t\right] = e^{-\alpha t} E^P\left[\int_0^\infty e^{-\alpha s} g(X_{t+s})ds \middle| X_t\right] \\ &= e^{-\alpha t} R_\alpha g(X_t). \end{aligned}$$

This proves the first part of the statement. The second part follows from the calculations above. □

If we define a potential Z as above, and use it as a stochastic discount factor, we can of course compute bond prices, and the short rate can easily be recovered.

Proposition 24.12 *If the stochastic discount factor Z is defined by (24.40) then bond prices are given by*

$$p(t,T) = e^{-\alpha(T-t)} \frac{E^P\left[R_\alpha g(X_T) | \mathcal{F}_t\right]}{R_\alpha g(X_t)} \tag{24.41}$$

and the short rate is given by

$$r_t = \frac{g(X_t)}{R_\alpha g(X_t)}. \tag{24.42}$$

Proof The formula (24.41) follows directly from the general formula (24.3). From (24.36) we have

$$dZ_t = -e^{-\alpha t} g(X_t)dt + dM_t,$$

and (24.42) now follows from (24.2) and (24.36). □

One problem with this scheme is that, for a concrete case, it may be very hard to compute the quotient in (24.42). To overcome this difficulty we will use the following standard result.

Proposition 24.13 *With notation as above we have essentially*

$$R_\alpha = (\alpha - \mathcal{G})^{-1}. \tag{24.43}$$

The phrase "essentially" above indicates that the result is "morally" correct, but that care has to be taken concerning the domain of the operators. We now provide a proof sketch for the diffusion case, i.e. when the X process satisfies an SDE of the form (24.31). The proof of the general case is similar, the only difference being that the Itô formula has to be replaced by the Dynkin formula. As usual we assume that all objects are well defined and "integrable enough".

Proof The proof is very straightforward. Suppressing the upper index P in E^P start by writing

$$R_\alpha g(x) = E_x\left[\int_0^\infty e^{-\alpha t} g(X_t)dt\right] = \int_0^\infty E_x\left[e^{-\alpha t} g(X_t)dt\right].$$

It is now natural to define the process Y by

$$Y_t = e^{-\alpha t} g(X_t),$$

and we easily obtain

$$dY_t = \left\{-\alpha Y_t + e^{-\alpha t}\mathcal{G}g(X_t)\right\}dt + e^{-\alpha t} g'(X_t)\sigma(X_t)dW_t.$$

Using the Itô formula one readily verifies (see the exercises) that Y can be written as

$$\begin{aligned}
Y_t &= e^{-\alpha t}Y_0 + \int_0^t e^{-\alpha(t-s)}e^{-\alpha s}\mathcal{G}g(X_s)ds + \int_0^t e^{-\alpha(t-s)}e^{-\alpha s}g'(X_s)\sigma(X_s)dW_s \\
&= e^{-\alpha t}g(x) + e^{-\alpha t}\int_0^t \mathcal{G}g(X_s)ds + e^{-\alpha t}\int_0^t g'(X_s)\sigma(X_s)dW_s.
\end{aligned}$$

Taking expectations we get

$$E_x\left[e^{-\alpha t}g(X_t)\right] = E_x[Y_t] = e^{-\alpha t}g(x) + e^{-\alpha t}\int_0^t E_x[\mathcal{G}g(X_s)]\,ds$$

and we now integrate this to obtain

$$\begin{aligned}
R_\alpha g(x) &= E_x\left[\int_0^\infty e^{-\alpha t}g(X_t)dt\right] \\
&= \int_0^\infty e^{-\alpha t}g(x)dt + \int_0^\infty e^{-\alpha t}\left(\int_0^t E_x[\mathcal{G}g(X_s)]\,ds\right)dt \\
&= \frac{g(x)}{\alpha} + \int_0^\infty E_x[\mathcal{G}g(X_s)]\left(\int_s^\infty e^{-\alpha t}dt\right)ds \\
&= \frac{g(x)}{\alpha} + \frac{1}{\alpha}E_x\left[\int_0^\infty e^{-\alpha s}\mathcal{G}g(X_s)ds\right].
\end{aligned}$$

We thus have

$$\alpha R_\alpha g = Ig + R_\alpha \mathcal{G}g,$$

where I is the identity operator. Using the fact the R_α is linear we can write this as

$$R_\alpha[\alpha - \mathcal{G}]\,g = Ig,$$

and since this holds for all g we have in fact shown that $R_\alpha = (\alpha - \mathcal{G})^{-1}$. □

We now go back to the short rate formula (24.42) and, using the identity $R_\alpha = (\alpha - \mathcal{G})^{-1}$, we see that with $f = R_\alpha g$ we have

$$\frac{g(X_t)}{R_\alpha g(X_t)} = \frac{(\alpha - \mathcal{G})f(X_t)}{f(X_t)},$$

where it usually is a trivial task to compute the last quotient.

This led Rogers to use the following scheme:

1. Choose a Markov process X, a real number α, and a function $f \geq 0$.
2. Define g by
$$g = (\alpha - \mathcal{G})\, f.$$
3. Choose α (and perhaps the parameters of f) such that g is non-negative.
4. Now we have $f = R_\alpha g$, and the short rate can be recaptured by
$$r(t) = \frac{(\alpha - \mathcal{G})f(X_t)}{f(X_t)}.$$

In this way Rogers produces a surprising variety of concrete analytically tractable non-negative interest rate models, and exchange rate models can also be treated within the same framework.

For illustration we consider the simplest possible example of a potential model, where the underlying Markov process is an n-dimensional Gaussian diffusion of the form

$$dX_t = -AX_t dt + dW_t. \tag{24.44}$$

In this case we have

$$\mathcal{G}f(x) = \frac{1}{2}\Delta f(x) - \nabla f(x)Ax \tag{24.45}$$

where Δ is the Laplacian and ∇f is the gradient viewed as a row vector. We now define f by

$$f(x) = e^{cx}$$

for some row vector $c \in R^n$. We immediately obtain

$$g(x) = (\alpha - \mathcal{G})\, f(x) = f(x)\left(\alpha - \frac{1}{2}\|c\|^2 + cAx\right).$$

The corresponding short rate is given by

$$r_t = \alpha - \frac{1}{2}\|c\|^2 + cAX_t, \tag{24.46}$$

so we have a Gaussian multi-factor model.

We end this section by connecting the Rogers theory to the Flesaker–Hughston framework, and this is quite straightforward. Comparing (24.30) to (24.34) we have

$$M(t,T) = e^{-\alpha T} E^P\left[g(X_T) | \mathcal{F}_t\right]. \tag{24.47}$$

24.6 Exercises

Exercise 24.1 Prove that the term structure constructed in Proposition 24.5 is positive.

Exercise 24.2 Prove proposition 24.6.

Exercise 24.3 Prove the first part of Proposition 24.8.

Exercise 24.4 Prove the first part of Proposition 24.8.

Exercise 24.5 Assume that the process Y has a stochastic differential of the form

$$dY_t = \{\alpha Y_t + \beta_t\} dt + \sigma_t dW_t,$$

where α is a real number whereas β and σ are adapted processes. Show that Y can be written as

$$Y_t = e^{\alpha t} Y_0 + \int_0^t e^{\alpha(t-s)} \beta_s ds + \int_0^t e^{\alpha(t-s)} \sigma_s dW_s.$$

Exercise 24.6 Without the normalizing function Φ, we can write the forward rates in a Flesaker–Hughston model as

$$f(t,T) = \frac{M(t,T)}{\int_T^\infty M(t,s)ds}.$$

A natural way of modeling the positive martingale family $M(t,T)$ is to write

$$dM(t,T) = M(t,T)\sigma(t,T)dW_t$$

for some chosen volatility structure σ, where σ and W are d-dimensional. Show that in this framework the forward rate dynamics are given by

$$df(t,T) = f(t,T)\{v(t,T) - \sigma(t,T)\} v^\star(t,T)dt + f(t,T)\{\sigma(t,T) - v(t,T)\} dW_t,$$

where * denotes transpose and

$$v(t,T) = \frac{\int_T^\infty M(t,s)\sigma(t,s)ds}{\int_T^\infty M(t,s)ds}.$$

Exercise 24.7 This exercise describes another way of producing a potential. Consider a fixed random variable $X_\infty \in L^2(P, \mathcal{F}_\infty)$. We can then define a martingale X by setting

$$X_t = E^P[X_\infty | \mathcal{F}_t].$$

Now define the process Z by

$$Z_t = E^P\left[(X_\infty - X_t)^2 \middle| \mathcal{F}_t \right].$$

(a) Show that

$$Z_t = E^P\left[X_\infty^2 \middle| \mathcal{F}_t\right] - X_t^2.$$

(b) Show that Z is a supermartingale and that Z is, in fact, a potential.

The point of this is that the potential Z, and thus the complete interest rate model generated by Z, is fully determined by a specification of the single random variable X_∞. This is called a "conditional variance model". See the Notes.

24.7 Notes

The Flesaker–Hughston fractional approach was developed in Flesaker and Hughston (1996) and Flesaker and Hughston (1997), but using completely different arguments from those above. In this chapter we basically follow Jin and Glasserman (2001). The Rogers potential theory approach was first presented in Rogers (1994). The general potential approach is developed further using conditional variance models and Wiener chaos in Brody and Hughston (2004), and in Hughston and Rafailidis (2005). A completely different, and more geometric, approach to modeling positive interest rates can be found in Brody and Hughston (2001) and Brody and Hughston (2002).

PART IV

OPTIMAL CONTROL AND INVESTMENT THEORY

25

STOCHASTIC OPTIMAL CONTROL

25.1 An Example

Let us consider an economic agent over a fixed time interval $[0,T]$. At time $t = 0$ the agent is endowed with initial wealth x_0 and his/her problem is how to allocate investments and consumption over the given time horizon. We assume that the agent's investment opportunities are the following:

- The agent can invest money in the bank at the deterministic short rate of interest r, i.e. he/she has access to the risk free asset B with

$$dB_t = r_t B_t dt. \tag{25.1}$$

- The agent can invest in a risky asset with price process S_t, where we assume that the S-dynamics are given by a standard Black–Scholes model

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{25.2}$$

We denote the agent's relative portfolio weights at time t by w_t^B (for the riskless asset), and w_t^S (for the risky asset) respectively. His/her consumption rate at time t is denoted by c_t.

We restrict the consumer's investment–consumption strategies to be self-financing, and as usual we assume that we live in a world where continuous trading and unlimited short selling is possible. If we denote the wealth of the consumer at time t by X_t, it now follows from Lemma 6.12 that (after a slight rearrangement of terms) the X-dynamics are given by

$$dX_t = X_t \left[w_t^B r + w_t^S \mu \right] dt - c_t dt + w_t^S \sigma X_t dW_t. \tag{25.3}$$

The object of the agent is to choose a portfolio–consumption strategy in such a way as to maximize his/her total utility over $[0,T]$, and we assume that this utility is given by

$$E\left[\int_0^T F(t, c_t)dt + \Phi(X_T) \right], \tag{25.4}$$

where F is the instantaneous utility function for consumption, whereas Φ is a "legacy" function which measures the utility of having some money left at the end of the period.

A natural constraint on consumption is the condition

$$c_t \geq 0, \ \forall t \geq 0, \tag{25.5}$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

and we also have of course the constraint

$$w_t^B + w_t^S = 1, \ \forall t \geq 0. \tag{25.6}$$

Depending upon the actual situation we may be forced to impose other constraints (it may, say, be natural to demand that the consumer's wealth never becomes negative), but we will not do this at the moment.

We may now formally state the consumer's utility maximization problem as follows:

$$\text{maximize}_{w^B,\ u^1,\ c} \quad E\left[\int_0^T F(t, c_t)dt + \Phi(X_T)\right] \tag{25.7}$$

$$dX_t = X_t \left[w_t^B r + w_t^S \mu\right] dt - c_t dt + w_t^S \sigma X_t dW_t, \tag{25.8}$$

$$X_0 = x_0, \tag{25.9}$$

$$c_t \geq 0, \ \forall t \geq 0, \tag{25.10}$$

$$w_t^B + w_t^S = 1, \ \forall t \geq 0. \tag{25.11}$$

A problem of this kind is known as a **stochastic optimal control problem**. In this context the process X is called the **state process** (or state variable), the processes w^B, w^S, c are called **control processes**, and we have a number of **control constraints**. In the next sections we will study a fairly general class of stochastic optimal control problems. The method used is that of **dynamic programming**, and at the end of the chapter we will solve a version of the problem above.

25.2 The Formal Problem

We now go on to study a fairly general class of optimal control problems. To this end, let $\mu(t, x, u)$ and $\sigma(t, x, u)$ be given functions of the form

$$\mu : R_+ \times R^n \times R^k \to R^n,$$
$$\sigma : R_+ \times R^n \times R^k \to R^{n \times d}.$$

For a given point $x_0 \in R^n$ we will consider the following **controlled** stochastic differential equation:

$$dX_t = \mu(t, X_t, u_t)\, dt + \sigma(t, X_t, u_t)\, dW_t, \tag{25.12}$$

$$X_0 = x_0. \tag{25.13}$$

We view the n-dimensional process X as a **state process**, which we are trying to "control" (or "steer"). We can (partly) control the state process X by choosing the k-dimensional **control process** u in a suitable way. W is a d-dimensional Wiener process, and we must now try to give a precise mathematical meaning to the formal expressions (25.12)–(25.13).

Remark 25.2.1 *In this chapter, where we will work under a fixed measure, all Wiener processes are denoted by the letter W.*

Our first modeling problem concerns the class of admissible control processes. In most concrete cases it is natural to require that the control process u is adapted to the X process. In other words, at time t the value u_t of the control process is only allowed to "depend" on past observed values of the state process X. One natural way to obtain an adapted control process is by choosing a deterministic function $g(t,x)$

$$g : R_+ \times R^n \to R^k,$$

and then defining the control process u by

$$u_t = g(t, X_t).$$

Such a function g is called a **feedback control law**, and in the sequel we will restrict ourselves to consider only feedback control laws. For mnemo-technical purposes we will often denote control laws by $\mathbf{u}(t,x)$, rather than $g(t,x)$, and write $u_t = \mathbf{u}(t, X_t)$. We use bold in order to indicate that $\mathbf{u}$ is a **function**. In contrast to this we use the notation u (italics) to denote the **value** of a control at a certain time. Thus $\mathbf{u}$ denotes a mapping, whereas u denotes a point in R^k.

Suppose now that we have chosen a fixed control law $\mathbf{u}(t,x)$. Then we can insert $\mathbf{u}$ into (25.12) to obtain the standard SDE

$$dX_t = \mu(t, X_t, \mathbf{u}(t, X_t))\, dt + \sigma(t, X_t, \mathbf{u}(t, X_t))\, dW_t. \tag{25.14}$$

In most concrete cases we also have to satisfy some **control constraints**, and we model this by taking as given a fixed subset $U \subseteq R^k$ and requiring that $u_t \in U$ for each t. We can now define the class of **admissible control laws**.

Definition 25.1 *A control law* $\mathbf{u}$ *is called* **admissible** *if*

- $\mathbf{u}(t,x) \in U$ *for all* $t \in R_+$ *and all* $x \in R^n$.
- *For any given initial point* (t,x) *the SDE*

$$\begin{aligned} dX_s &= \mu(s, X_s, \mathbf{u}(s, X_s))\, ds + \sigma(s, X_s, \mathbf{u}(s, X_s))\, dW_s, \\ X_t &= x \end{aligned}$$

  has a unique solution.

The class of admissible control laws is denoted by $\mathcal{U}$. For ease of notation we will often write $\mathbf{u}_t(x)$ instead of $\mathbf{u}(t,x)$.

For a given control law $\mathbf{u}$, the solution process X will of course depend on the initial value x, as well as on the chosen control law $\mathbf{u}$. To be precise we should therefore denote the process X by $X^{x,\mathbf{u}}$, but sometimes we will suppress x or $\mathbf{u}$. We note that eqn (25.14) looks rather messy, and since we will also have to deal with the Itô formula in connection with (25.14) we need some more streamlined notation.

Definition 25.2 *Consider eqn (25.14), and let $'$ denote matrix transpose.*

- *For any fixed vector $u \in R^k$, the functions μ^u, σ^u, and C^u are defined by*

$$\begin{aligned}\mu^u(t,x) &= \mu(t,x,u),\\ \sigma^u(t,x) &= \sigma(t,x,u),\\ C^u(t,x) &= \sigma(t,x,u)\sigma(t,x,u)'.\end{aligned}$$

- *For any control law $\mathbf{u}$, the functions $\mu^{\mathbf{u}}$, $\sigma^{\mathbf{u}}$, $C^{\mathbf{u}}(t,x)$, and $F^{\mathbf{u}}(t,x)$ are defined by*

$$\begin{aligned}\mu^{\mathbf{u}}(t,x) &= \mu(t,x,\mathbf{u}(t,x)),\\ \sigma^{\mathbf{u}}(t,x) &= \sigma(t,x,\mathbf{u}(t,x)),\\ C^{\mathbf{u}}(t,x) &= \sigma(t,x,\mathbf{u}(t,x))\sigma(t,x,\mathbf{u}(t,x))',\\ F^{\mathbf{u}}(t,x) &= F(t,x,\mathbf{u}(t,x)).\end{aligned}$$

- *For any fixed vector $u \in R^k$, the partial differential operator $\mathcal{A}^u$ is defined by*

$$\mathcal{A}^u = \sum_{i=1}^n \mu_i^u(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n C_{ij}^u(t,x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

- *For any control law $\mathbf{u}$, the partial differential operator $\mathcal{A}^{\mathbf{u}}$ is defined by*

$$\mathcal{A}^{\mathbf{u}} = \sum_{i=1}^n \mu_i^{\mathbf{u}}(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n C_{ij}^{\mathbf{u}}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Given a control law $\mathbf{u}$ we will sometimes write eqn (25.14) in a convenient shorthand notation as

$$dX_t^{\mathbf{u}} = \mu^{\mathbf{u}}dt + \sigma^{\mathbf{u}}dW_t. \tag{25.15}$$

For a given control law $\mathbf{u}$ with a corresponding controlled process $X^{\mathbf{u}}$ we will also often use the shorthand notation $\mathbf{u}_t$ instead of the clumsier expression $\mathbf{u}(t, X_t^{\mathbf{u}})$.

The reader should be aware of the fact that the existence assumption in the definition above is not at all an innocent one. In many cases it is natural to consider control laws which are "rapidly varying", i.e. feedback laws $\mathbf{u}(t,x)$ which are very irregular as functions of the state variable x. Inserting such an irregular control law into the state dynamics will easily give us a very irregular drift function $\mu(t,x,\mathbf{u}(t,x))$ (as a function of x), and we may find ourselves outside the nice Lipschitz situation in Proposition 5.1, thus leaving us with a highly nontrivial existence problem. The reader is referred to the literature for details.

We now go on to the objective function of the control problem, and therefore we consider as given a pair of functions

$$\begin{aligned}F &: R_+ \times R^n \times R^k \to R,\\ \Phi &: R^n \to R.\end{aligned}$$

Now we define the **value function** of our problem as the function

$$\mathcal{J}_0 : \mathcal{U} \to R,$$

defined by

$$\mathcal{J}_0(\mathbf{u}) = E\left[\int_0^T F(t, X_t^{\mathbf{u}}, \mathbf{u}_t)dt + \Phi\left(X_T^{\mathbf{u}}\right)\right],$$

where $X^{\mathbf{u}}$ is the solution to (25.14) with the given initial condition $X_0 = x_0$.

Our formal problem can thus be written as that of maximizing $\mathcal{J}_0(\mathbf{u})$ over all $\mathbf{u} \in \mathcal{U}$, and we define the **optimal value** $\hat{\mathcal{J}}_0$ by

$$\hat{\mathcal{J}}_0 = \sup_{\mathbf{u} \in \mathcal{U}} \mathcal{J}_0(\mathbf{u}).$$

If there exists an admissible control law $\hat{\mathbf{u}}$ with the property that

$$\mathcal{J}_0(\hat{\mathbf{u}}) = \hat{\mathcal{J}}_0,$$

then we say that $\hat{\mathbf{u}}$ is an **optimal control law** for the given problem. Note that, as for any optimization problem, the optimal law may not exist. For a given concrete control problem our main objective is of course to find the optimal control law (if it exists), or at least to learn something about the qualitative behavior of the optimal law.

25.3 Embedding the Problem

Given an optimal control problem we have two natural questions to answer:

(a) Does there exist an optimal control law?
(b) Given that an optimal control exists, how do we find it?

In this text we will mainly be concerned with problem (b) above, and the methodology used will be that of **dynamic programming**. The main idea is to embed our original problem into a much larger class of problems, and then to tie all these problems together with a partial differential equation (PDE) known as the Hamilton–Jacobi–Bellman equation. The control problem is then shown to be equivalent to the problem of finding a solution to the HJB equation.

We will now describe the embedding procedure, and for that purpose we choose a fixed point t in time, with $0 \le t \le T$. We also choose a fixed point x in the state space, i.e. $x \in R^n$. For this fixed pair (t, x) we now define the following control problem.

Definition 25.3 *The control problem $P(t, x)$ is defined as*

$$\underset{\mathbf{u}}{\text{maximize}} \quad E_{t,x}\left[\int_t^T F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + \Phi(X_T^{\mathbf{u}})\right], \tag{25.16}$$

given the dynamics

$$dX_s^{\mathbf{u}} = \mu\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s(X_s^{\mathbf{u}})\right) ds + \sigma\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s(X_s^{\mathbf{u}})\right) dW_s, \tag{25.17}$$

$$X_t = x, \tag{25.18}$$

and the constraints

$$\mathbf{u}(s,y) \in U, \quad \forall (s,y) \in [t,T] \times R^n. \tag{25.19}$$

Observe that we use the notation s and y above because the letters t and x are already used to denote the fixed chosen point (t,x).

We note that in terms of the definition above, our original problem is the problem $P(0,x_0)$. A somewhat drastic interpretation of the problem $P(t,x)$ is that you have fallen asleep at time zero. Suddenly you wake up, noticing that the time now is t and that your state process while you were asleep has moved to the point x. You now try to do as well as possible under the circumstances, so you want to maximize your utility over the remaining time, given the fact that you start at time t in the state x.

We now define the **value function** and the **optimal value function**.

Definition 25.4

- *The* **value function**

$$\mathcal{J} : R_+ \times R^n \times \mathcal{U} \to R$$

  is defined by

$$\mathcal{J}(t,x,\mathbf{u}) = E_{t,x}\left[\int_t^T F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + \Phi\left(X_T^{\mathbf{u}}\right)\right]$$

  given the dynamics (25.17)–(25.18).

- *The* **optimal value function**

$$V : R_+ \times R^n \to R$$

  is defined by

$$V(t,x) = \sup_{\mathbf{u} \in \mathcal{U}} \mathcal{J}(t,x,\mathbf{u}).$$

Thus $\mathcal{J}(t,x,\mathbf{u})$ is the expected utility of using the control law $\mathbf{u}$ over the time interval $[t,T]$, given the fact that you start in state x at time t. The optimal value function gives you the optimal expected utility over $[t,T]$ under the same initial conditions. For the benefit of the mathematical purist we note that for the problem $P(t,x)$ and a feedback law $\mathbf{u}$ we will in fact only use the mapping $(s,y) \longmapsto \mathbf{u}(s,y)$ with s restricted to the interval $[t,T]$.

The main object of interest for us is the optimal value function, and our goal is to derive a PDE for V. In order to do this we will rely on a very fundamental fact, referred to as the "Bellman optimality principle".

25.4 Time Consistency and the Bellman Principle

Let us now go back to the problem $P(t,x)$. We make the following simplifying assumption.

Assumption 25.4.1 *We assume that for every initial point (t,x) there exists an optimal control law for the problem $P(t,x)$. This control law is denoted by $\widehat{\mathbf{u}}^{tx}$.*

Note that the upper-case index (t,x) denotes the fixed initial point for problem $P(t,x)$. The object $\widehat{\mathbf{u}}^{tx}$ is thus a mapping $\widehat{\mathbf{u}}^{tx} : [t,T] \times R^n \to R^k$, so the control applied at some time $s \geq t$ will be given by the expression

$$\widehat{\mathbf{u}}_s^{tx}(X_s).$$

It is important to realize that a priori the optimal law for the problem $P(t,x)$ could very well depend on the choice of the starting point (n,x). It turns out, however, that the optimal law does **not** depend on the choice of the initial point. The formalization and proof of this statement is as follows.

Theorem 25.5 (The Bellman Optimality Principle) *Consider a fixed initial point (t,x) and consider the corresponding optimal law $\widehat{\mathbf{u}}^{tx}$. Then the law $\widehat{\mathbf{u}}^{tx}$ is also optimal for any subinterval of the form $[r,T]$ where $r \geq t$. In other words*

$$\widehat{\mathbf{u}}_s^{tx}(y) = \widehat{\mathbf{u}}_s^{r,X_r}(y)$$

for all $s \geq r$ and all $y \in R^n$. In particular, the optimal law for the initial point $t = 0$ will be optimal for all subintervals. This law will be denoted by $\widehat{\mathbf{u}}$.

In more pedestrian terms this means the following. Suppose that you optimize at time $t = 0$ and follow $\widehat{\mathbf{u}}$ up to time t, where you now have reached the state X_t. At time t you reconsider, and now decide to forget your original problem and instead solve problem $P(t,X_t)$. What the Bellman Principle tells you is that the law $\widehat{\mathbf{u}}$ (restricted to the time interval $[t,T]$ is optimal, not only for your original problem, but also for your new problem. In decision theoretic jargon we could say that our family of problems are **time consistent**, and in particular this implies that the expression "the optimal law" has a well-defined meaning—it does not depend on your choice of starting point.

Proof The proof is by contradiction. For simplicity and without loss of generality we only prove it for the case when $t = 0$, and we replace r above by t. Let us thus assume that for some $t > 0$ there exists a control law $\mathbf{u}^0$ such that $\mathbf{u}^0$ does better than $\widehat{\mathbf{u}}$ on $[t,T]$, i.e. that

$$E_{t,x}\left[\int_t^T F(s,X_s^{\mathbf{u}^0},\mathbf{u}_s^0)ds + \Phi(X_T^{\mathbf{u}^0})\right] \geq E_{t,x}\left[\int_t^T F(s,X_s^{\widehat{\mathbf{u}}},\widehat{\mathbf{u}}_s)ds + \Phi(X_T^{\widehat{\mathbf{u}}})\right]$$

for all $x \in R^n$ and strict inequality for some $x \in R^n$. We can then construct a new law $\mathbf{u}^\star$ on $[0,T]$ by the following formula:

$$\mathbf{u}_s^\star(y) = \begin{cases} \widehat{\mathbf{u}}_s(y) & \text{for} \quad 0 \le s < t \\ \mathbf{u}_s^0(y) & \text{for} \quad t \le s < T. \end{cases}$$

We then have

$$J_0(x_0, \mathbf{u}^\star) = E_{0,x_0}\left[\int_0^T F(s, X_s^{\mathbf{u}^*}, \mathbf{u}_s^*)ds + \Phi(X_T^{\mathbf{u}^*})\right]$$

$$= E_{0,x_0}\left[\int_0^t F(s, X_s^{\widehat{\mathbf{u}}}, \widehat{\mathbf{u}}_s)ds\right] + E_{0,x_0}\left[\int_t^T F(s, X_s^{\mathbf{u}^0}, \mathbf{u}_s^0)ds + \Phi(X_T^{\mathbf{u}^0})\right]$$

$$= E_{0,x_0}\left[\int_0^t F(s, X_s^{\widehat{\mathbf{u}}}, \widehat{\mathbf{u}}_s)ds\right] + E_{0,x_0}\left[E_{t,X_t}\left[\int_t^T F(s, X_s^{\mathbf{u}^0}, \mathbf{u}_s^0)ds + \Phi(X_T^{\mathbf{u}^0})\right]\right]$$

where we have used iterated expectations and the Markov property to obtain the last term. It now follows from the assumption concerning $\mathbf{u}^0$ that we have

$$E_{t,X_t}\left[\int_t^T F(s, X_s^{\mathbf{u}^0}, \mathbf{u}_s^0)ds + \Phi(X_T^{\mathbf{u}^0})\right] \ge E_{t,X_t}\left[\int_t^T F(s, X_s^{\widehat{\mathbf{u}}}, \widehat{\mathbf{u}}_s)ds + \Phi(X_T^{\widehat{\mathbf{u}}})\right]$$

with strict inequality with positive probability so we obtain

$$J_0(x_0, \mathbf{u}^\star) > E_{0,x_0}\left[\int_0^t F(s, X_s^{\widehat{\mathbf{u}}}, \widehat{\mathbf{u}}_s)ds\right] + E_{0,x_0}\left[\int_t^T F(s, X_s^{\widehat{\mathbf{u}}}, \widehat{\mathbf{u}}_s)ds + \Phi(X_T^{\widehat{\mathbf{u}}})\right]$$

$$= E_{0,x_0}\left[\int_0^T F(s, X_s^{\widehat{\mathbf{u}}}, \widehat{\mathbf{u}}_s)ds + \Phi(X_T^{\widehat{\mathbf{u}}})\right] = J_0(x_0, \widehat{\mathbf{u}}).$$

We have thus obtained the inequality

$$J_0(x_0, \mathbf{u}^\star) > J_0(x_0, \widehat{\mathbf{u}})$$

which contradicts the optimality of $\widehat{\mathbf{u}}$ on the interval $[0, T]$. □

25.5 Deriving the Hamilton–Jacobi–Bellman Equation

We now go on to derive a PDE for V. It should be noted that this derivation is largely heuristic. We make some rather strong regularity assumptions, and we disregard a number of technical problems. We will comment on these problems later, but to see exactly which problems we are ignoring we now make some basic assumptions.

Assumption 25.5.1 *We assume the following*:

1. *There exists an optimal control law* $\widehat{\mathbf{u}}$.
2. *The optimal value function* V *is regular in the sense that* $V \in C^{1,2}$.
3. *A number of limiting procedures in the following arguments can be justified.*

We now go on to derive the PDE, and to this end we fix $(t,x) \in (0,T) \times R^n$. Furthermore we choose a real number h (interpreted as a "small" time increment) such that $t+h < T$. We choose a fixed but arbitrary control law $\mathbf{u}$, and define the control law $\mathbf{u}^\star$ by

$$\mathbf{u}^\star(s,y) = \begin{cases} \mathbf{u}(s,y), & (s,y) \in [t,t+h] \times R^n \\ \widehat{\mathbf{u}}(s,y), & (s,y) \in (t+h,T] \times R^n. \end{cases}$$

In other words, if we use $\mathbf{u}^\star$ then we use the arbitrary control $\mathbf{u}$ during the time interval $[t,t+h]$, and then we switch to the optimal control law during the rest of the time period.

The whole idea of dynamic programming actually boils down to the following procedure:

- First, given the point (t,x) as above, we consider the following two strategies over the time interval $[t,T]$:

  Strategy I. Use the optimal law $\widehat{\mathbf{u}}$.
  Strategy II. Use the control law $\mathbf{u}^*$ defined above.

- We then compute the expected utilities obtained by the respective strategies.
- Finally, using the obvious fact that Strategy I by definition has to be at least as good as Strategy II, and letting h tend to zero, we obtain our fundamental PDE.

We now carry out this program.

Expected utility for Strategy I: This is trivial, since by definition the utility is the optimal one given by $\mathcal{J}(t,x,\widehat{\mathbf{u}}) = V(t,x)$.

Expected utility for Strategy II:
We divide the time interval $[t,T]$ into two parts, the intervals $[t,t+h]$ and $(t+h,T]$ respectively. The expected utility, using Strategy II, is given by

$$\mathcal{J}(t,x,\mathbf{u}^*) = E_{t,x}\left[\int_t^{t+h} F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds + \int_{t+h}^T F(s, X_s^{\widehat{\mathbf{u}}}, \widehat{\mathbf{u}}_s) ds + \Phi(X_T^{\widehat{\mathbf{u}}}\right]$$

$$= E_{t,x}\left[\int_t^{t+h} F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds + E_{t+h,X_{t+h}}\left[\int_{t+h}^T F(s, X_s^{\widehat{\mathbf{u}}}, \widehat{\mathbf{u}}_s) ds + \Phi(X_T^{\widehat{\mathbf{u}}}\right]\right]$$

$$= E_{t,x}\left[\int_t^{t+h} F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds + V(t+h, X_{t+h}^{\mathbf{u}})\right].$$

Comparing the strategies:
We now go on to compare the two strategies, and since by definition Strategy I is the optimal one, we must have the inequality

$$V(t,x) \geq E_{t,x}\left[\int_t^{t+h} F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds + V\left(t+h, X_{t+h}^{\mathbf{u}}\right)\right]. \tag{25.20}$$

We also note that the inequality sign is due to the fact that the arbitrarily chosen control law $\mathbf{u}$ which we use on the interval $[t, t+h]$ need not be the optimal one. In particular we have the following obvious fact.

Remark 25.5.1 *We have equality in (25.20) if and only if the control law* $\mathbf{u}$ *is an optimal law* $\widehat{\mathbf{u}}$. *(Note that the optimal law does not have to be unique.)*

Since, by assumption, V is smooth we now use the Itô formula to obtain (with obvious notation)

$$\begin{aligned} V\left(t+h, X_{t+h}^{\mathbf{u}}\right) = V(t,x) &+ \int_t^{t+h} \left\{\frac{\partial V}{\partial t}(s, X_s^{\mathbf{u}}) + \mathcal{A}^{\mathbf{u}} V(s, X_s^{\mathbf{u}})\right\} ds \\ &+ \int_t^{t+h} \nabla_x V(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}} dW_s. \end{aligned} \tag{25.21}$$

If we apply the expectation operator $E_{t,x}$ to this equation, and assume enough integrability, then the stochastic integral will vanish. We can then insert the resulting equation into the inequality (25.20). The term $V(t,x)$ will cancel, leaving us with the inequality

$$E_{t,x}\left[\int_t^{t+h} \left[F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) + \frac{\partial V}{\partial t}(s, X_s^{\mathbf{u}}) + \mathcal{A}^{\mathbf{u}} V(s, X_s^{\mathbf{u}})\right] ds\right] \leq 0. \tag{25.22}$$

Going to the limit: Now we divide by h, move h within the expectation and let h tend to zero. Assuming enough regularity to allow us to take the limit within the expectation, using the fundamental theorem of integral calculus, and recalling that $X_t = x$, we get

$$F(t,x,u) + \frac{\partial V}{\partial t}(t,x) + \mathcal{A}^u V(t,x) \leq 0, \tag{25.23}$$

where u denotes the value of the law $\mathbf{u}$ evaluated at (t,x), i.e. $u = \mathbf{u}(t,x)$.

Since the control law $\mathbf{u}$ was arbitrary, this inequality will hold for all choices of $u \in U$, and we will have equality if and only if $u = \widehat{\mathbf{u}}(t,x)$. We thus have the following equation:

$$\frac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \left\{F(t,x,u) + \mathcal{A}^u V(t,x)\right\} = 0.$$

During the discussion the point (t,x) was fixed, but since it was chosen as an arbitrary point we see that the equation holds in fact for all $(t,x) \in (0,T) \times R^n$. Thus we have a (non-standard type of) PDE, and we obviously need some boundary conditions. One such condition is easily obtained, since we obviously (why?) have $V(T,x) = \Phi(x)$ for all $x \in R^n$. We have now arrived at our goal, namely the Hamilton–Jacobi–Bellman equation (often referred to as the HJB equation).

Theorem 25.6 (Hamilton–Jacobi–Bellman equation) *Under Assumption 25.5.1, the following hold:*

1. *V satisfies the Hamilton–Jacobi–Bellman equation*

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u V(t,x)\} = 0, & \forall (t,x) \in (0,T) \times R^n \\ V(T,x) = \Phi(x), & \forall x \in R^n. \end{cases}$$

2. *For each $(t,x) \in [0,T] \times R^n$ the supremum in the HJB equation above is attained by $u = \widehat{\mathbf{u}}(t,x)$.*

Remark 25.5.2 *By going through the arguments above, it is easily seen that we may allow the constraint set U to be time- and state-dependent. If we thus have control constraints of the form*

$$u(t,x) \in U(t,x), \quad \forall t,x$$

then the HJB equation still holds with the obvious modification of the supremum part.

It is important to note that this theorem has the form of a **necessary** condition. It says that **if** V is the optimal value function, and **if** $\widehat{\mathbf{u}}$ is the optimal control, **then** V satisfies the HJB equation, and $\widehat{\mathbf{u}}(t,x)$ realizes the supremum in the equation. We also note that Assumption 25.5.1 is an ad hoc assumption. One would prefer to have conditions in terms of the initial data μ, σ, F, and Φ which would guarantee that Assumption 25.5.1 is satisfied. This can in fact be done, but at a fairly high price in terms of technical complexity. The reader is referred to the specialist literature.

A gratifying, and perhaps surprising, fact is that the HJB equation also acts as a **sufficient** condition for the optimal control problem. This result is known as the **Verification Theorem** for dynamic programming, and we will use it repeatedly below. Note that, as opposed to the necessary conditions above, the Verification Theorem is very easy to prove rigorously.

Theorem 25.7 (Verification Theorem) *Suppose that we have two functions $H(t,x)$ and $g(t,x)$, such that*

- *H is sufficiently integrable (see Remark 25.5.4 below), and solves the HJB equation*

$$\begin{cases} \dfrac{\partial H}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u H(t,x)\} = 0, & \forall (t,x) \in (0,T) \times R^n \\ H(T,x) = \Phi(x), & \forall x \in R^n. \end{cases}$$

- *The function g is an admissible control law.*
- *For each fixed (t,x), the supremum in the expression*

$$\sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u H(t,x)\}$$

  is attained by the choice $u = g(t,x)$.

Then the following hold:

1. *The optimal value function V to the control problem is given by*

$$V(t,x) = H(t,x).$$

2. *There exists an optimal control law $\widehat{\mathbf{u}}$, and in fact $\widehat{\mathbf{u}}(t,x) = g(t,x)$.*

Remark 25.5.3 *Note that we have used the letter H (instead of V) in the HJB equation above. This is because the letter V by definition denotes the optimal value function.*

Proof Assume that H and g are given as above. Now choose an arbitrary control law $\mathbf{u} \in \mathcal{U}$, and fix a point (t,x). We define the process $X^{\mathbf{u}}$ on the time interval $[t,T]$ as the solution to the equation

$$\begin{aligned} dX_s^{\mathbf{u}} &= \mu^{\mathbf{u}}(s, X_s^{\mathbf{u}})\, ds + \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}})\, dW_s, \\ X_t &= x. \end{aligned}$$

Inserting the process $X^{\mathbf{u}}$ into the function H and using the Itô formula we obtain

$$\begin{aligned} H(T, X_T^{\mathbf{u}}) = H(t,x) &+ \int_t^T \left\{ \frac{\partial H}{\partial t}(s, X_s^{\mathbf{u}}) + (\mathcal{A}^{\mathbf{u}} H)(s, X_s^{\mathbf{u}}) \right\} ds \\ &+ \int_t^T \nabla_x H(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}}) dW_s. \end{aligned}$$

Since H solves the HJB equation we see that

$$\frac{\partial H}{\partial t}(t,x) + F(t,x,u) + \mathcal{A}^u H(t,x) \leq 0$$

for all $u \in U$, and thus we have, for each s and P-a.s, the inequality

$$\frac{\partial H}{\partial t}(s, X_s^{\mathbf{u}}) + (\mathcal{A}^{\mathbf{u}} H)(s, X_s^{\mathbf{u}}) \leq -F^{\mathbf{u}}(s, X_s^{\mathbf{u}}).$$

From the boundary condition for the HJB equation we also have $H(T, X_T^{\mathbf{u}}) = \Phi(X_T^{\mathbf{u}})$, so we obtain the inequality

$$H(t,x) \geq \int_t^T F^{\mathbf{u}}(s, X_s^{\mathbf{u}}) ds + \Phi(X_T^{\mathbf{u}}) - \int_t^T \nabla_x H(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}} dW_s.$$

Taking expectations, and assuming enough integrability, we make the stochastic integral vanish, leaving us with the inequality

$$H(t,x) \geq E_{t,x}\left[\int_t^T F^{\mathbf{u}}(s, X_s^{\mathbf{u}}) ds + \Phi(X_T^{\mathbf{u}}) \right] = \mathcal{J}(t,x,\mathbf{u}).$$

Since the control law $\mathbf{u}$ was arbitrarily chosen this gives us

$$H(t,x) \geq \sup_{\mathbf{u} \in \mathcal{U}} \mathcal{J}(t,x,\mathbf{u}) = V(t,x). \tag{25.24}$$

To obtain the reverse inequality we choose the specific control law $\mathbf{u}(t,x) = \mathbf{g}(t,x)$. Going through the same calculations as above, and using the fact that by assumption we have

$$\frac{\partial H}{\partial t}(t,x) + F^{\mathbf{g}}(t,x) + \mathcal{A}^{\mathbf{g}} H(t,x) = 0,$$

we obtain the equality

$$H(t,x) = E_{t,x}\left[\int_t^T F^{\mathbf{g}}(s, X_s^{\mathbf{g}})ds + \Phi(X_T^{\mathbf{g}})\right] = \mathcal{J}(t,x,\mathbf{g}). \tag{25.25}$$

On the other hand we have the trivial inequality

$$V(t,x) \geq \mathcal{J}(t,x,\mathbf{g}), \tag{25.26}$$

so, using (25.24)–(25.26), we obtain

$$H(t,x) \geq V(t,x) \geq \mathcal{J}(t,x,\mathbf{g}) = H(t,x).$$

This shows that in fact

$$H(t,x) = V(t,x) = \mathcal{J}(t,x,\mathbf{g}),$$

which proves that $H = V$, and that $\mathbf{g}s$ is the optimal control law. □

Remark 25.5.4 *The assumption that H is "sufficiently integrable" in the theorem above is made in order for the stochastic integral in the proof to have expected value zero. This will be the case if, for example, H satisifes the condition*

$$\nabla_x H(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}}) \in \pounds^2$$

for all admissible control laws.

Remark 25.5.5 *Sometimes, instead of a maximization problem, we consider a minimization problem. Of course we now make the obvious definitions for the value function and the optimal value function. It is then easily seen that all the results above still hold if the expression*

$$\sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u V(t,x)\}$$

in the HJB equation is replaced by the expression

$$\inf_{u \in U} \{F(t,x,u) + \mathcal{A}^u V(t,x)\}.$$

Remark 25.5.6 *In the Verification Theorem we may allow the control constraint set U to be state and time dependent, i.e. of the form $U(t,x)$.*

25.6 Handling the HJB Equation

In this section we will describe the actual handling of the HJB equation, and in Section 25.7 we will study a classical example—the linear quadratic regulator.

We thus consider our standard optimal control problem with the corresponding HJB equation:

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u V(t,x)\} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad V(T,x) = \Phi(x). \end{cases} \tag{25.27}$$

Schematically we now proceed as follows:

1. Consider the HJB equation as a PDE for an unknown function V.
2. Fix an arbitrary point $(t,x) \in [0,T] \times R^n$ and solve, for this fixed choice of (t,x), the static optimization problem

   $$\max_{u \in U} \quad [F(t,x,u) + \mathcal{A}^u V(t,x)].$$

   Note that in this problem u is the only variable, whereas t and x are considered to be fixed parameters. The functions F, μ, σ, and V are considered as given.
3. The optimal choice of u, denoted by $\widehat{u}$, will of course depend on our choice of t and x, but it will also depend on the function V and its various partial derivatives (which are hiding under the sign $\mathcal{A}^u V$). To highlight these dependencies we write $\widehat{u}$ as

   $$\widehat{\mathbf{u}} = \widehat{\mathbf{u}}(t,x;V). \tag{25.28}$$

4. The function $\widehat{u}(t,x;V)$ is our candidate for the optimal control law, but since we do not know V this description is incomplete. Therefore we substitute the expression for $\widehat{u}$ in (25.28) into the PDE (25.27), giving us the PDE

   $$\frac{\partial V}{\partial t}(t,x) + F^{\widehat{\mathbf{u}}}(t,x) + \mathcal{A}^{\widehat{\mathbf{u}}} V(t,x) = 0, \tag{25.29}$$

   $$V(T,x) = \Phi(x). \tag{25.30}$$

5. Now we solve the PDE above! (See the remark below.) Then we put the solution V into expression (25.28). Using the Verification Theorem 25.7 we can now identify V as the optimal value function, and $\widehat{u}$ as the optimal control law.

Remark 25.6.1 *The hard work of dynamic programming consists in solving the highly nonlinear PDE in step* 5 *above. There are of course no general analytic methods available for this, so the number of known optimal control problems with an analytic solution is very small indeed. In an actual case one usually tries to* **guess** *a solution, i.e. we typically make an* **ansatz** *for V, parameterized by a finite number of parameters, and then we use the PDE in order to identify the parameters. The making of an ansatz is often helped by the intuitive observation that if there is an analytical solution to the problem, then it seems likely that V inherits some structural properties from the boundary function Φ as well as from the instantaneous utility function F.*

For a general problem there is thus very little hope of obtaining an analytic solution, and it is worth pointing out that many of the known solved control problems have, to some extent, been "rigged" in order to be analytically solvable.

25.7 The Linear Regulator

We now want to put the ideas from Section 25.6 into action, and for this purpose we study the most well known of all control problems, namely the linear quadratic regulator problem. In this classical engineering example we wish to **minimize**

$$E\left[\int_0^T \{X_t'QX_t + u_t'Ru_t\}\,dt + X_T'HX_T\right],$$

(where $'$ denotes transpose) given the dynamics

$$dX_t = \{AX_t + Bu_t\}\,dt + CdW_t.$$

One interpretation of this problem is that we want to control a vehicle in such a way that it stays close to the origin (the terms $x'Qx$ and $x'Hx$) while at the same time keeping the "energy" $u'Ru$ small.

As usual $X_t \in R^n$ and $\mathbf{u}_t \in R^k$, and we impose no control constraints on u. The matrices Q, R, H, A, B, and C are assumed to be known. Without loss of generality we may assume that Q, R, and H are symmetric, and we assume that R is positive definite (and thus invertible).

The HJB equation now becomes

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \inf\limits_{u \in R^k}\{x'Qx + u'Ru + [\nabla_x V](t,x)\,[Ax+Bu]\} \\ \qquad\qquad + \dfrac{1}{2}\sum\limits_{i,j}\dfrac{\partial^2 V}{\partial x_i \partial x_j}(t,x)\,[CC']_{i,j} = 0, \\ V(T,x) = x'Hx. \end{cases}$$

For each fixed choice of (t,x) we now have to solve the static unconstrained optimization problem to minimize

$$u'Ru + [\nabla_x V](t,x)\,[Ax+Bu].$$

Since, by assumption, $R > 0$ we get the solution by setting the gradient equal to zero, thus giving us the equation

$$2u'R = -(\nabla_x V)B,$$

which gives us the optimal u as

$$\widehat{u} = -\frac{1}{2}R^{-1}B'(\nabla_x V)'.$$

Here we see clearly (compare point 2 in the scheme above) that in order to use this formula we need to know V, and we thus try to make an educated guess about the structure of V. From the boundary value function $x'Hx$ and the quadratic term $x'Qx$ in the instantaneous cost function it seems reasonable to assume that V is a quadratic function. Consequently we make the following ansatz

$$V(t,x) = x'P(t)x + q(t),$$

where we assume that $P(t)$ is a deterministic symmetric matrix function of time, whereas $q(t)$ is a scalar deterministic function. It would of course also be natural to include a linear term of the form $L(t)x$, but it turns out that this is not necessary.

With this trial solution we have, suppressing the t-variable and denoting time derivatives by a dot,

$$\begin{aligned}
\frac{\partial V}{\partial t}(t,x) &= x'\dot{P}x + \dot{q}, \\
\nabla_x V(t,x) &= 2x'P, \\
\nabla_{xx} V(t,x) &= 2P \\
\widehat{u} &= -R^{-1}B'Px.
\end{aligned}$$

Inserting these expressions into the HJB equation we get

$$x'\dot{P}x + \dot{q} + x'Qx + x'PBR^{-1}RR^{-1}B'Px + 2x'PAx$$
$$-2x'PBR^{-1}B'Px + \sum_{i,j} P_{ij}[CC']_{ij} = 0.$$

We note that the last term above equals $tr[C'PC]$, where tr denotes the trace of a matrix, and furthermore we see that $2x'PAx = x'A'Px + x'PAx$ (this is just cosmetic). Collecting terms gives us

$$x'\left\{\dot{P} + Q - PBR^{-1}B'P + A'P + PA\right\}x + \dot{q} + tr[C'PC] = 0. \tag{25.31}$$

If this equation is to hold for all x and all t then firstly the bracket must vanish, leaving us with the matrix ODE

$$\dot{P} = PBR^{-1}B'P - A'P - PA - Q.$$

We are then left with the scalar equation

$$\dot{q} = -tr[C'PC].$$

We now need some boundary values for P and q, but these follow immediately from the boundary conditions of the HJB equation. We thus end up with the following pairs of equations:

$$\begin{cases} \dot{P} = PBR^{-1}B'P - A'P - PA - Q, \\ P(T) = H. \end{cases} \tag{25.32}$$

$$\begin{cases} \dot{q} = -tr[C'PC], \\ q(T) = 0. \end{cases} \tag{25.33}$$

The matrix equation (25.32) is known as a **Riccati equation**, and there are powerful algorithms available for solving it numerically. The equation for q can then be integrated directly.

Summing up we see that the optimal value function and the optimal control law are given by the following formulas. Note that the optimal control is linear in the state variable

$$V(t,x) = x'P(t)x + \int_t^T tr[C'P(s)C]ds, \tag{25.34}$$

$$\widehat{\mathbf{u}}(t,x) = -R^{-1}B'P(t)x. \tag{25.35}$$

25.8 Exercises

Exercise 25.1 Consider as before state process dynamics

$$dX_t = \mu(t,X_t,u_t)\,dt + \sigma(t,X_t,u_t)\,dW_t$$

and the usual restrictions for u. Our entire derivation of the HJB equation has so far been based on the fact that the objective function is of the form

$$\int_0^T F(t,X_t,u_t)dt + \Phi(X_T).$$

Sometimes it is natural to consider other criteria, like the expected **exponential utility** criterion

$$E\left[\exp\left\{\int_0^T F(t,X_t,u_t)dt + \Phi(X_T)\right\}\right].$$

For this case we define the optimal value function as the supremum of

$$E_{t,x}\left[\exp\left\{\int_t^T F(s,X_s,u_s)dt + \Phi(X_T)\right\}\right].$$

Follow the reasoning in Section 25.3 in order to show that the HJB equation for the expected exponential utility criterion is given by

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \sup_u \{V(t,x)F(t,x,u) + \mathcal{A}^u V(t,x)\} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V(T,x) = e^{\Phi(x)}. \end{cases}$$

Exercise 25.2 Solve the problem to minimize

$$E\left[\exp\left\{\int_0^T u_t^2 dt + X_T^2\right\}\right]$$

given the scalar dynamics

$$dX = (ax+u)dt + \sigma dW$$

where the control u is scalar and there are no control constraints.

Hint: Make the ansatz

$$V(t,x) = e^{A(t)x^2+B(t)}.$$

Exercise 25.3 Study the general linear–exponential–quadratic control problem of minimizing

$$E\left[\exp\left\{\int_0^T \{X_t'QX_t + u_t'Ru_t\}\,dt + X_T'HX_T\right\}\right]$$

given the dynamics

$$dX_t = \{AX_t + Bu_t\}\,dt + CdW_t.$$

Exercise 25.4 The object of this exercise is to connect optimal control to martingale theory. Consider therefore a general control problem of minimizing

$$E\left[\int_0^T F(t, X_t^{\mathbf{u}}, \mathbf{u}_t)dt + \Phi\left(X_T^{\mathbf{u}}\right)\right]$$

given the dynamics

$$dX_t = \mu\left(t, X_t, u_t\right)dt + \sigma\left(t, X_t, u_t\right)dW_t$$

and the constraints

$$\mathbf{u}(t,x) \in U.$$

Now, for any control law $\mathbf{u}$, define the **total cost process** $C(t;\mathbf{u})$ by

$$C(t;\mathbf{u}) = \int_0^t F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + E_{t,X_t^{\mathbf{u}}}\left[\int_t^T F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)dt + \Phi\left(X_T^{\mathbf{u}}\right)\right],$$

i.e.

$$C(t;\mathbf{u}) = \int_0^t F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + \mathcal{J}(t, X_t^{\mathbf{u}}, \mathbf{u}).$$

Use the HJB equation in order to prove the following claims:

(a) If $\mathbf{u}$ is an arbitrary control law, then C is a submartingale.
(b) If $\mathbf{u}$ is optimal, then C is a martingale.

25.9 Notes

Standard references on optimal control are Fleming and Rishel (1975) and Krylov (1980). A very clear exposition can be found in Øksendal (1998). For more recent work, using viscosity solutions, see Fleming and Soner (1993).

26

OPTIMAL CONSUMPTION AND INVESTMENT

26.1 A Generalization

In many concrete applications, in particular in economics, it is natural to consider an optimal control problem, where the state variable is constrained to stay within a prespecified domain. As an example it may be reasonable to demand that the wealth of an investor is never allowed to become negative. We will now generalize our class of optimal control problems to allow for such considerations.

Let us therefore consider the following controlled SDE

$$dX_t = \mu(t, X_t, u_t)\,dt + \sigma(t, X_t, u_t)\,dW_t, \tag{26.1}$$

$$X_0 = x_0, \tag{26.2}$$

where as before we impose the control constraint $u_t \in U$. We also consider as given a fixed time interval $[0,T]$, and a fixed domain $D \subseteq [0,T] \times R^n$, and the basic idea is that when the state process hits the boundary ∂D of D, then the activity is at an end. It is thus natural to define the **stopping time** τ by

$$\tau = \inf\{t \geq 0 \; |(t, X_t) \in \partial D\} \wedge T,$$

where $x \wedge y = \min[x,y]$. We consider as given an instantaneous utility function $F(t,x,u)$ and a "bequest function" $\Phi(t,x)$, i.e. a mapping $\Phi : \partial D \to R$. The control problem to be considered is that of maximizing

$$E\left[\int_0^\tau F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + \Phi(\tau, X_\tau^{\mathbf{u}})\right]. \tag{26.3}$$

In order for this problem to be interesting we have to demand that $X_0 \in D$, and the interpretation is that when we hit the boundary ∂D, the game is over and we obtain the bequest $\Phi(\tau, X_\tau)$. We see immediately that our earlier situation corresponds to the case when $D = [0,T] \times R^n$ and when Φ is constant in the t-variable.

In order to analyze our present problem we may proceed as in the previous sections, introducing the value function and the optimal value function exactly as before. The only new technical problem encountered is that of considering a stochastic integral with a stochastic limit of integration. Since this will take us outside the scope of the present text we will confine ourselves to giving the results. The proofs are (modulo the technicalities mentioned above) exactly as before.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

Theorem 26.1 (HJB equation) *Assume that the optimal value function V is in $C^{1,2}$, and that there exists an optimal law $\widehat{\mathbf{u}}$. Then the following hold:*

1. *V satisifies the HJB equation*

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u V(t,x)\} = 0, & \forall (t,x) \in D \\ V(t,x) = \Phi(t,x), & \forall (t,x) \in \partial D. \end{cases}$$

2. *For each $(t,x) \in D$ the supremum in the HJB equation above is attained by $u = \widehat{\mathbf{u}}(t,x)$.*

Theorem 26.2 (Verification Theorem) *Suppose that we have two functions $H(t,x)$ and $g(t,x)$, such that*

- *H is sufficiently integrable, and solves the HJB equation*

$$\begin{cases} \dfrac{\partial H}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u H(t,x)\} = 0, & \forall (t,x) \in D \\ H(t,x) = \Phi(t,x), & \forall (t,x) \in \partial D. \end{cases}$$

- *The function g is an admissible control law.*
- *For each fixed (t,x), the supremum in the expression*

$$\sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u H(t,x)\}$$

  is attained by the choice $u = g(t,x)$.

Then the following hold:

1. *The optimal value function V to the control problem is given by*

$$V(t,x) = H(t,x).$$

2. *There exists an optimal control law $\widehat{\mathbf{u}}$, and in fact $\widehat{\mathbf{u}}(t,x) = g(t,x)$.*

26.2 Optimal Consumption and Investment

In order to illustrate the technique we will now go back to the optimal consumption problem at the beginning of Chapter 25. We thus consider the problem of maximizing

$$E\left[\int_0^T U(t,c_t)dt + \Phi(X_T)\right], \tag{26.4}$$

given the wealth dynamics

$$dX_t = X_t\left[w_t^B r + w_t^S \mu\right]dt - c_t dt + w_t^S \sigma X_t dW_t. \tag{26.5}$$

As usual we impose the control constraints

$$c_t \geq 0, \ \forall t \geq 0,$$
$$w_t^B + w_t^S = 1, \ \forall t \geq 0.$$

In the formulation above, U is the local utility function for consumption whereas Φ is the utility function of remaining wealth. In a control problem of this kind it is important to be aware of the fact that one may quite easily formulate a nonsensical problem. To take a simple example, suppose that we have $\Phi = 0$, and suppose that F is increasing and unbounded in the c-variable. Then the problem above degenerates completely. It does not possess an optimal solution at all, and the reason is of course that the consumer can increase his/her utility to any given level by simply consuming an arbitrarily large amount at every t. The consequence of this hedonistic behavior is the fact that the wealth process will, with very high probability, become negative, but this is neither prohibited by the control constraints, nor punished by any bequest function.

An elegant way out of this dilemma is to choose the domain D of the preceding section as $D = [0,T] \times \{x \mid x > 0\}$. With τ defined as above this means, in concrete terms, that

$$\tau = \inf\{t > 0 : \; X_t = 0\} \wedge T.$$

A natural objective function in this case is thus given by

$$E\left[\int_0^\tau U(t, c_t)dt\right], \tag{26.6}$$

which automatically ensures that when the consumer has no wealth, then all activity is terminated.

We will now analyze this problem in some detail. Firstly we notice that we can get rid of the constraint $w_t^B + w_t^S = 1$ by defining a new control variable w_t as $w_t = w_t^S$, and then substituting $1 - w_t$ for w_t^B. This gives us the state dynamics

$$dX_t = w_t\,[\mu - r]\,X_t dt + (rX_t - c_t)\,dt + w_t \sigma X_t dW_t, \tag{26.7}$$

and the corresponding HJB equation is

$$\begin{cases} \dfrac{\partial V}{\partial t} + \sup\limits_{c \geq 0, w \in R}\left\{U(t,c) + wx(\mu - r)\dfrac{\partial V}{\partial x} + (rx - c)\dfrac{\partial V}{\partial x} + \dfrac{1}{2}x^2 w^2 \sigma^2 \dfrac{\partial^2 V}{\partial x^2}\right\} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V(T,x) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V(t,0) = 0. \end{cases}$$

The static optimization problem to be solved w.r.t. c and w is thus that of maximizing

$$U(t,c) + wx(\mu - r)\frac{\partial V}{\partial x} + (rx - c)\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2}$$

and, assuming an interior solution, the first order conditions are

$$U_c(t,c) = V_x, \tag{26.8}$$

$$w = \frac{-V_x}{x \cdot V_{xx}} \cdot \frac{\mu - r}{\sigma^2}, \tag{26.9}$$

where we have used subscripts to denote partial derivatives.

The first condition is well known from microeconomics. It says that, at the optimum, the marginal utility of consumption must equal the marginal (indirect) utility of wealth. In the second condition we recognize the inverse of the Arrow–Pratt relative risk aversion factor

$$-\frac{xV_{xx}}{V_x}.$$

We now specialize our example to the case when F is of the form

$$F(t,c) = e^{-\delta t}c^{\gamma},$$

where $0 < \gamma < 1$. The economic reasoning behind this is that we now have an infinite marginal utility at $c = 0$. This will force the optimal consumption plan to be positive throughout the planning period, a fact which will facilitate the analytical treatment of the problem. In terms of Remark 25.6.1 we are thus "rigging" the problem. The first order conditions are now

$$\gamma c^{\gamma-1} = e^{\delta t}V_x, \tag{26.10}$$

$$w = \frac{-V_x}{x \cdot V_{xx}} \cdot \frac{\mu - r}{\sigma^2}. \tag{26.11}$$

We again see that in order to implement the optimal consumption–investment plan (26.10)–(26.11) we need to know the optimal value function V. We therefore suggest a trial solution (see Remark 25.6.1), and in view of the shape of the instantaneous utility function it is natural to try a V-function of the form

$$V(t,x) = e^{-\delta t}h(t)x^{\gamma}, \tag{26.12}$$

where, because of the boundary conditions, we must demand that

$$h(T) = 0. \tag{26.13}$$

Given a V of this form we have (using $\cdot$ to denote the time derivative)

$$\frac{\partial V}{\partial t} = e^{-\delta t}\dot{h}x^{\gamma} - \delta e^{-\delta t}hx^{\gamma}, \tag{26.14}$$

$$\frac{\partial V}{\partial x} = \gamma e^{-\delta t}hx^{\gamma-1}, \tag{26.15}$$

$$\frac{\partial^2 V}{\partial x^2} = \gamma(\gamma-1)e^{-\delta t}hx^{\gamma-2}. \tag{26.16}$$

Inserting these expressions into (26.10)–(26.11) we get

$$\widehat{\mathbf{w}}(t,x) = \frac{\mu - r}{\sigma^2(1-\gamma)}, \tag{26.17}$$

$$\widehat{\mathbf{c}}(t,x) = xh(t)^{-1/(1-\gamma)}. \tag{26.18}$$

This looks very promising: we see that the candidate optimal portfolio is constant and that the candidate optimal consumption rule is linear in the wealth variable.

In order to use the Verification Theorem we now want to show that a V-function of the form (26.12) actually solves the HJB equation. We therefore substitute the expressions (26.14)–(26.18) into the HJB equation. This gives us the equation

$$x^{\gamma}\left\{\dot{h}(t)+Ah(t)+Bh(t)^{-\gamma/(1-\gamma)}\right\}=0,$$

where the constants A and B are given by

$$\begin{aligned} A &= \frac{\gamma(\mu-r)^2}{\sigma^2(1-\gamma)}+r\gamma-\frac{1}{2}\frac{\gamma(\mu-r)^2}{\sigma^2(1-\gamma)}-\delta \\ B &= 1-\gamma. \end{aligned}$$

If this equation is to hold for all x and all t, then we see that h must solve the ODE

$$\dot{h}(t)+Ah(t)+Bh(t)^{-\gamma/(1-\gamma)}=0, \tag{26.19}$$

$$h(T)=0. \tag{26.20}$$

An equation of this kind is known as a **Bernoulli equation**, and it can be solved explicitly (see the exercises).

Summing up, we have shown that if we define V as in (26.12) with h defined as the solution to (26.19)–(26.20), and if we define $\widehat{\mathbf{w}}$ and $\widehat{\mathbf{c}}$ by (26.17)–(26.18), then V satisfies the HJB equation, and $\widehat{\mathbf{w}}$, $\widehat{\mathbf{c}}$ attain the supremum in the equation. The Verification Theorem then tells us that we have indeed found the optimal solution.

26.3 The Mutual Fund Theorems

In this section we will briefly go through the "Merton mutual fund theorems", originally presented in Merton (1971).

26.3.1 *The Case with No Risk Free Asset*

We consider a financial market with n asset prices $S^1,\dots,S^n$. To start with we do **not** assume the existence of a risk free asset, and we assume that the price vector process S_t has the following dynamics under the objective measure P:

$$dS_t=D(S_t)\mu dt+D(S_t)\sigma dW_t. \tag{26.21}$$

Here W is a k-dimensional standard Wiener process, μ is an n-vector, σ is an $n\times k$ matrix, and $D(S)$ is the diagonal matrix

$$D(S)=diag[S^1,\dots,S^n].$$

In more pedestrian terms this means that

$$dS^i=S^i\mu_i dt+S^i\sigma_i dW, \quad i=1,\dots,n,$$

where σ_i is the ith row of the matrix σ.

We denote the investment strategy (relative portfolio) by w, and the consumption plan by c. If the pair (w,c) is self-financing, then it follows from the S-dynamics above, and from Lemma 6.12, that the dynamics of the wealth process X are given by

$$dX_t = X_t w_t' \mu dt - cdt + X_t w_t' \sigma dW_t. \tag{26.22}$$

We also take as given an instantaneous utility function $U(t,c)$, and we basically want to maximize

$$E\left[\int_0^T U(t,c_t)dt\right]$$

where T is some given time horizon. In order not to formulate a degenerate problem we also impose the condition that wealth is not allowed to become negative, and as before this is dealt with by introducing the stopping time

$$\tau = \inf\{t > 0 \mid X_t = 0\} \wedge T.$$

Our formal problem is then that of maximizing

$$E\left[\int_0^\tau U(t,c_t)dt\right]$$

given the dynamics (26.21)–(26.22), and subject to the control constraints

$$\sum_1^n w_{it} = 1, \tag{26.23}$$

$$c_t \geq 0. \tag{26.24}$$

Instead of (26.23) it is convenient to write

$$e'w_t = 1,$$

where e is the vector in R^n which has the number 1 in all components, i.e. $e' = (1,\ldots,1)$.

The HJB equation for this problem now becomes

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x,s) + \sup\limits_{e'w=1,\ c\geq 0} \{U(t,c) + \mathcal{A}^{c,w}V(t,x,s)\} = 0, \\ \hfill V(T,x,s) = 0, \\ \hfill V(t,0,s) = 0. \end{cases}$$

In the general case, when the parameters μ and σ are allowed to be functions of the price vector process S, the term $\mathcal{A}^{c,w}V(t,x,s)$ turns out to be rather forbidding (see Merton's original paper). It will in fact involve partial derivatives to the second order with respect to all the variables $x, s_1, \ldots, s_n$.

If, however, we assume that μ and σ are deterministic and constant over time, then we see by inspection that the wealth process X is a Markov process, and since the price processes do not appear, neither in the objective function nor in the definition of the stopping time, we draw the conclusion that in this case X itself will act as the state process, and we may forget about the underlying S-process completely.

Under these assumptions we may thus write the optimal value function as $V(t,x)$, with no s-dependence, and after some easy calculations the term $\mathcal{A}^{c,w}V$ turns out to be

$$\mathcal{A}^{c,w}V = xw'\mu\frac{\partial V}{\partial x} - c\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w'\Sigma w\frac{\partial^2 V}{\partial x^2},$$

where the matrix Σ is given by

$$\Sigma = \sigma\sigma'.$$

We now summarize our assumptions.

Assumption 26.3.1 *We assume that*

- *The vector μ is constant and deterministic.*
- *The matrix σ is constant and deterministic.*
- *The matrix σ has rank n, and in particular the matrix $\Sigma = \sigma\sigma'$ is positive definite and invertible.*

We note that, in terms of contingent claims analysis, the last assumption means that the market is complete. Denoting partial derivatives by subscripts we now have the following HJB equation

$$\begin{cases} V_t(t,x) + \sup_{w'e=1,\ c\geq 0}\left\{U(t,c) + (xw'\mu - c)V_x(t,x) + \frac{1}{2}x^2 w'\Sigma w V_{xx}(t,x)\right\} = 0, \\ V(T,x) = 0, \\ V(t,0) = 0. \end{cases}$$

If we relax the constraint $w'e = 1$, the Lagrange function for the static optimization problem is given by

$$L = U(t,c) + (xw'\mu - c)V_x(t,x) + \frac{1}{2}x^2 w'\Sigma w V_{xx}(t,x) + \lambda(1 - w'e).$$

Assuming the problem to be regular enough for an interior solution we see that the first order condition for c is

$$\frac{\partial F}{\partial c}(t,c) = V_x(t,x).$$

The first order condition for w is

$$x\mu' V_x + x^2 V_{xx} w'\Sigma = \lambda e',$$

so we can solve for w in order to obtain

$$\widehat{w} = \Sigma^{-1}\left[\frac{\lambda}{x^2 V_{xx}}e - \frac{xV_x}{x^2 V_{xx}}\mu\right]. \tag{26.25}$$

Using the relation $e'w = 1$ this gives λ as

$$\lambda = \frac{x^2 V_{xx} + x V_x e' \Sigma^{-1} \mu}{e' \Sigma^{-1} e},$$

and inserting this into (26.25) gives us, after some manipulation,

$$\widehat{w} = \frac{1}{e'\Sigma^{-1}e} \Sigma^{-1} e + \frac{V_x}{x V_{xx}} \Sigma^{-1} \left[\frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e} e - \mu \right]. \tag{26.26}$$

To see more clearly what is going on we can write this expression as

$$\widehat{\mathbf{w}}(t) = g + Y_t h, \tag{26.27}$$

where the fixed vectors g and h are given by

$$g = \frac{1}{e'\Sigma^{-1}e} \Sigma^{-1} e, \tag{26.28}$$

$$h = \Sigma^{-1} \left[\frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e} e - \mu \right], \tag{26.29}$$

whereas Y is given by

$$Y_t = \frac{V_x(t, X_t)}{X_t V_{xx}(t, X_t)}. \tag{26.30}$$

Thus we see that the optimal portfolio is moving stochastically along the one-dimensional "optimal portfolio line"

$$g + sh,$$

in the $(n-1)$-dimensional "portfolio hyperplane" Δ, where

$$\Delta = \{ w \in R^n \ | e'w = 1 \}.$$

We now make the obvious geometric observation that if we fix two points on the optimal portfolio line, say the points $w^a = g + ah$ and $w^b = g + bh$, then any point w on the line can be written as an affine combination of the basis points w^a and w^b. An easy calculation shows that if $w^s = g + sh$ then we can write

$$w^s = \mu w^a + (1 - \mu) w^b,$$

where

$$\mu = \frac{s - b}{a - b}.$$

The point of all this is that we now have an interesting economic interpretation of the optimality results above. Let us thus fix w^a and w^b as above on the optimal portfolio line. Since these points are in the portfolio plane Δ we can interpret them as the relative portfolios of two fixed mutual funds. We may then write (26.27) as

$$\widehat{\mathbf{w}}(t) = \mu(t) w^a + (1 - \mu(t)) w^b, \tag{26.31}$$

with

$$\mu(t) = \frac{Y_t - b}{a - b}.$$

Thus we see that the optimal portfolio $\widehat{\mathbf{w}}$ can be obtained as a"super portfolio" where we allocate resources between two fixed mutual funds.

Theorem 26.3 (Mutual fund theorem) *Assume that the problem is regular enough to allow for an interior solution. Then there exists a one-dimensional parameterized family of mutual funds, given by $w^s = g + sh$, where g and h are defined by (26.28)–(26.29), such that the following hold:*

1. *For each fixed s the relative portfolio w^s stays fixed over time.*
2. *For any fixed choice of $a \neq b$ the optimal portfolio $\widehat{\mathbf{w}}(t)$ is, for all values of t, obtained by allocating all resources between the fixed funds w^a and w^b, i.e.*

$$\widehat{\mathbf{w}}(t) = \mu^a(t)w^a + \mu^b(t)w^b,$$
$$\mu^a(t) + \mu^b(t) = 1.$$

3. *The relative proportions (μ^a, μ^b) of the portfolio wealth allocated to w^a and w^b respectively are given by*

$$\mu^a(t) = \frac{Y(t) - b}{a - b},$$
$$\mu^b(t) = \frac{a - Y(t)}{a - b},$$

where Y is given by (26.30).

26.3.2 *The Case with a Risk Free Asset*

Again we consider the model

$$dS_t = D(S_t)\mu dt + D(S_t)\sigma dW_t, \tag{26.32}$$

with the same assumptions as in the preceding section. We now also take as given the standard risk free asset B with dynamics

$$dB_t = rB_t dt.$$

Formally we can denote this as a new asset by superscript zero, i.e. $B = S^0$, and then we can consider relative portfolios of the form $w = (w_0, w_1, \ldots, w_n)'$ where of course $\sum_0^n w_i = 1$. Since B will play such a special role it will, however, be convenient to eliminate w_0 by the relation

$$w_0 = 1 - \sum_1^n w_i,$$

and then use the letter w to denote the portfolio weight vector for the risky assets only. Thus we use the notation

$$w = (w_1, \ldots, w_n)',$$

and we note that this truncated portfolio vector is allowed to take any value in R^n.

Given this notation it is easily seen that the dynamics of a self-financing portfolio are given by

$$dX_t = X_t \cdot \left\{ \sum_1^n w_{it}\mu_i + \left(1 - \sum_1^n w_{it}\right) r \right\} dt - cdt + X_t \cdot w_t' \sigma dW_t,$$

or more compactly

$$dX_t = X_t \cdot w_t'(\mu - re)dt + (rX_t - c)dt + X_t \cdot w_t' \sigma dW_t, \tag{26.33}$$

where as before $e \in R^n$ denotes the vector $(1, 1, \ldots, 1)'$.

The HJB equation now becomes

$$\begin{cases} V_t(t,x) + \sup_{c \geq 0, w \in R^n} \{U(t,c) + \mathcal{A}^{c,w} V(t,x)\} = 0, \\ V(T,x) = 0, \\ V(t,0) = 0, \end{cases}$$

where

$$\mathcal{A}^c V = xw'(\mu - re)V_x(t,x) + (rx - c)V_x(t,x) + \frac{1}{2}x^2 w' \Sigma w V_{xx}(t,x).$$

The first-order conditions for the static optimization problem are

$$\frac{\partial U}{\partial c}(t,c) = V_x(t,x),$$
$$\widehat{w} = -\frac{V_x}{xV_{xx}} \Sigma^{-1}(\mu - re),$$

and again we have a geometrically obvious economic interpretation.

Theorem 26.4 (Mutual fund theorem) *Given assumptions as above, the following hold*:

1. *The optimal portfolio consists of an allocation between two fixed mutual funds w^0 and w^f.*
2. *The fund w^0 consists only of the risk free asset.*
3. *The fund w^f consists only of the risky assets, and is given by*

$$w^f = \Sigma^{-1}(\mu - re).$$

4. *At each t the optimal relative allocation of wealth between the funds is given by*

$$\mu^f(t) = -\frac{V_x(t,X_t)}{X_t V_{xx}(t,X_t)},$$
$$\mu^B(t) = 1 - \mu^f(t).$$

Note that this result is not a corollary of the corresponding result from the previous section. Firstly it was an essential ingredient in the previous results that the volatility matrix of the price vector was invertible. In the case with a riskless asset the volatility matrix for the entire price vector $(B, S^1, \ldots, S^n)$ is of course degenerate, since its first row (having superscript zero) is identically equal to zero. Secondly, even if one assumes the results from the previous section, i.e. that the optimal portfolio is built up from two fixed portfolios, it is not at all obvious that one of these basis portfolios can be chosen so as to consist of the risk free asset alone.

26.4 Exercises

Exercise 26.1 Solve the problem of maximizing logarithmic utility

$$E\left[\int_0^T e^{-\delta t}\ln(c_t)dt + K\cdot\ln(X_T)\right],$$

given the usual wealth dynamics

$$dX_t = X_t\left[w_t^B r + w_t^S \mu\right]dt - c_t dt + w^S \sigma X_t dW_t,$$

and the usual control constraints

$$\begin{aligned} c_t &\geq 0, \;\forall t \geq 0, \\ w_t^B + w_t^S &= 1, \;\forall t \geq 0. \end{aligned}$$

Exercise 26.2 A **Bernoulli equation** is an ODE of the form

$$\dot{x}_t + A_t x_t + B_t x_t^\alpha = 0,$$

where A and B are deterministic functions of time and μ is a constant.

If $\alpha = 1$ this is a linear equation, and can thus easily be solved. Now consider the case $\alpha \neq 1$ and introduce the new variable y by

$$y_t = x_t^{1-\alpha}.$$

Show that y satisfies the **linear** equation

$$\dot{y}_t + (1-\alpha)A_t y_t + (1-\alpha)B_t = 0.$$

Exercise 26.3 Use the previous exercise in order to solve (26.19)–(26.20) explicitly.

Exercise 26.4 The following example is taken from Björk et al. (1987). We consider a consumption problem without risky investments, but with stochastic prices for various consumption goods:

$$\begin{aligned} N &= \text{the number of consumption goods,} \\ p_i(t) &= \text{price, at } t\text{, of good } i \text{ (measured as dollars per unit per unit time),} \\ p(t) &= [p_1(t), \ldots, p_N(t)]', \end{aligned}$$

$$
\begin{aligned}
c_i(t) &= \text{rate of consumption of good } i, \\
c(t) &= [c_1(t), \ldots, c_N(t)]', \\
X(t) &= \text{wealth process}, \\
r &= \text{short rate}, \\
T &= \text{time horizon}.
\end{aligned}
$$

We assume that the consumption price processes satisfy

$$dp_i = \mu_i(p)dt + \sqrt{2}\sigma_i(p)dW_i$$

where $W_1, \ldots, W_n$ are independent. The X-dynamics become

$$dX = rXdt - c'pdt,$$

and the objective is to maximize expected discounted utility, as measured by

$$E\left[\int_0^\tau U(t, c_t)dt\right]$$

where τ is the time of ruin, i.e.

$$\tau = \inf\{t \geq 0; X_t = 0\} \wedge T.$$

(a) Denote the optimal value function by $V(t,x,p)$ and write down the relevant HJB equation (including boundary conditions for $t = T$ and $x = 0$).

(b) Assume that U is of the form

$$U(t,c) = e^{-\delta t} \prod_{i=1}^{N} c_i^{\mu_i}$$

where $\delta > 0$, $0 < \mu_i < 1$ and $\mu = \sum_1^N \mu_i < 1$. Show that the optimal value function and the optimal control have the structure

$$
\begin{aligned}
V(t,x,p) &= e^{-\delta t} x^\mu \mu^{-\mu} G(t,p), \\
c_i(t,x,p) &= \frac{x}{p_i} \cdot \frac{\mu_i}{\mu} A(p)^\gamma G(t,p),
\end{aligned}
$$

where G solves the nonlinear equation

$$
\begin{cases}
\dfrac{\partial G}{\partial t} + (\mu r - \delta)G + (1-\mu)A^\gamma G^{-\mu\gamma} + \displaystyle\sum_i^N \mu_i \frac{\partial G}{\partial p_i} + \sum_i^N \sigma_i^2 \frac{\partial^2 G}{\partial p_i{}^2} = 0, \\
\hfill G(T,p) = 0, \quad p \in R^N.
\end{cases}
$$

If you find this too hard, then study the simpler case when $N = 1$.

(c) Now assume that the price dynamics are given by GBM, i.e.

$$dp_i = p_i\mu_i dt + \sqrt{2} p_i \sigma_i dW_i.$$

Try to solve the G-equation above by making the **ansatz**

$$G(t,p) = g(t)f(p).$$

Warning: This becomes somewhat messy.

26.5 Notes

The classic papers on optimal consumption are Merton (1969) and Merton (1971). For optimal trading under constraints and its relation to derivative pricing see Cvitanić (1997) and references therein. See also Korn (1997).

27

THE MARTINGALE APPROACH TO OPTIMAL INVESTMENT

In Chapter 25 we studied optimal investment and consumption problems, using dynamic programming. This approach transforms the original stochastic optimal control problem into the problem of solving a nonlinear deterministic PDE, namely the Hamilton–Jacobi–Bellman equation, so the probabilistic nature of the problem disappears as soon as we have formulated the HJB equation.

In this chapter we will present an alternative method of solving optimal investment problems. This method is commonly referred to as “the martingale approach” and it has the advantage that it is in some sense more direct and more probabilistic than dynamic programming, and we do not need to assume a Markovian structure. It should be noted however, that while dynamic programming can be applied to **any** Markovian stochastic optimal control problem, the martingale approach is only applicable to financial portfolio problems, and in order to get explicit results we also typically need to assume market completeness.

27.1 Generalities

We consider a financial market living on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$, where P is the objective probability measure. The basis carries an n-dimensional P-Wiener process W, and the filtration $\mathbf{F}$ is the one generated by the W process so $\mathbf{F} = \mathbf{F}^W$.

The financial market under consideration consists of n non-dividend-paying risky assets (“stocks”) with price processes $S^1, \ldots, S^n$, and a bank account with price process B. The formal assumptions concerning the price dynamics are as follows.

Assumption 27.1.1 *We assume the following:*

- *The risky asset prices have P-dynamics given by*

$$dS_t^i = \alpha_t^i S_t^i dt + S_t^i \sigma_t^i dW_t, \quad i = 1, \ldots, n. \tag{27.1}$$

  Here $\alpha^1, \ldots, \alpha^n$ are assumed to be $\mathbf{F}$-adapted scalar processes, and $\sigma^i, \ldots, \sigma^n$ are $\mathbf{F}^W$-adapted d-dimensional row vector processes.
- *The short rate r is allowed to be stochastic, i.e. the bank account has dynamics given by*

$$dB_t = r_t B_t dt.$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

Remark 27.1.1 *Note that we do not make any Markovian assumptions, so in particular the process α and σ are allowed to be arbitrary adapted path-dependent processes. Of particular interest is of course the Markovian case, i.e. when α and σ are deterministic functions of t and S_t so $\alpha_t = \alpha_t(S_t)$ and $\sigma_t = \sigma_t(S_t)$.*

Defining the stock vector process S, the rate of return vector process α, and the and the volatility matrix σ by

$$S = \begin{pmatrix} S^1 \\ \vdots \\ S^n \end{pmatrix}, \qquad \alpha = \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{pmatrix}, \qquad \sigma = \begin{pmatrix} -\sigma^1- \\ \vdots \\ -\sigma^n- \end{pmatrix},$$

we can write the stock price dynamics as

$$dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t, \tag{27.2}$$

where $D(S)$ denotes the diagonal matrix with $S^1, \dots, S^n$ on the diagonal.

We will need an important assumption concerning the volatility matrix.

Assumption 27.1.2 *We assume that with probability one the volatility matrix $\sigma(t)$ is non-singular for all t.*

The point of this assumption is the following result, the proof of which is obvious.

Proposition 27.1 *Under the assumptions above, the market model is complete.*

27.2 The Basic Idea

Let us consider an investor with initial capital x and a utility function U for terminal wealth. For any self-financing portfolio, we denote the corresponding portfolio value process by X and our problem is to maximize expected utility

$$E^P\left[U(X_T)\right],$$

over the class of self-financing adapted portfolios with the initial condition $X(0) = x$.

Note 27.2.1 *For problems including optimal consumption, see Section 27.8 below.*

In Chapter 25 we viewed this as a **dynamic** optimization problem and attacked it by using dynamic programming. A different way of formulating the problem is however as follows. Define $\mathcal{K}_T(x)$ as the set of contingent T-claims which can be replicated by a self-financing portfolio with initial capital x. Then our basic problem can be formulated as the **static** problem

$$\underset{X_T}{\text{maximize}} \quad E^P\left[U(X_T)\right]$$

subject to the **static** constraint

$$X_T \in \mathcal{K}_T(x).$$

In this formulation, the focus is not on the optimal portfolio strategy but instead on the terminal wealth X_T. We now have the following important observation, which follows immediately from market completeness.

Proposition 27.2 *With assumptions as above, the following conditions are equivalent for any random variable $X_T \in \mathcal{F}_T$:*

$$X_T \in \mathcal{K}_T(x). \tag{27.3}$$

$$E^Q\left[e^{-\int_0^T r_u du} \cdot X_T\right] = x. \tag{27.4}$$

The implication of this simple observation is that we can now decouple the problem of determining the optimal terminal wealth profile from the problem of determining the optimal portfolio. Schematically we proceed as follows:

- Instead of solving the dynamic control problem, we solve the **static** problem
$$\underset{X \in \mathcal{F}_T}{\text{maximize}} \quad E^P\left[U(X)\right] \tag{27.5}$$
subject to the budget constraint
$$E^Q\left[\frac{1}{B_T}X\right] = x, \tag{27.6}$$
where x is the initial wealth, and Q is the unique martingale measure.
- Given the optimal wealth profile $\hat{X}$, we can (in principle) compute the corresponding generating portfolio using martingale representation results.

27.3 The Optimal Terminal Wealth

The static problem (27.5) with the constraint (27.6) can easily be solved using Lagrange relaxation. We start by rewriting the budget constraint (27.6) as

$$E^P\left[\frac{1}{B_T}L_T X\right] = x,$$

where L is the likelihood process between Q and P. Equivalently we can write it as

$$E^P\left[\mathbf{M}_T X\right] = x,$$

where $\mathbf{M}$ is the stochastic discount factor, defined as usual by

$$M_t = \frac{1}{B_t}L_t.$$

We now relax the budget constraint to obtain the Lagrangian

$$\mathcal{L} = E^P\left[U(X)\right] - \lambda\left(E^P\left[\mathbf{M}_T X\right] - x\right),$$

so

$$\mathcal{L} = \int_{\Omega} \{U(X(\omega)) - \lambda \mathbf{M}_T(\omega) X(\omega)\} \, dP(\omega) + \lambda x.$$

It now remains to maximize the unconstrained Lagrangian over X, but this is trivial: Since we have no constraints we can maximize $\mathcal{L}$ for each ω. The optimality condition is

$$U'(X) = \lambda \mathbf{M}_T$$

so the optimal wealth profile is given by

$$\hat{X} = I\left(\lambda e^{-rT} L_T\right), \tag{27.7}$$

where I is the functional inverse of the utility function U, so

$$I = (U')^{-1}. \tag{27.8}$$

The likelihood dynamics are easily obtained from (27.2) and the Girsanov Theorem, so we can now collect our results in a proposition.

Proposition 27.3 *Under the assumptions above, the optimal terminal wealth profile $\hat{X}_T$ is given by*

$$\hat{X}_T = I\left(\lambda \mathbf{M}_T\right), \tag{27.9}$$

where the stochastic discount factor $\mathbf{M}$ is given by

$$\mathbf{M}_t = e^{-\int_0^t r_u du} L_t.$$

The likelihood dynamics are given by

$$dL_t = L_t \left\{\sigma_t^{-1}\left(\mathbf{r}_t - \alpha_t\right)\right\}' dW_t, \tag{27.10}$$

$$L_0 = 1, \tag{27.11}$$

and the Lagrange multiplier λ is determined by the budget constraint (27.6). The function I is defined by (27.8).

27.4 The Optimal Wealth Process

In Section 27.3 we saw that we could, in principle quite easily, derive a closed form expression for the optimal terminal wealth $\hat{X}_T$. The next step is to determine the optimal terminal wealth **process** $\widehat{X}_t$ for $0 \le t \le T$. In principle this is easily done. From risk neutral valuation we have the following result.

Proposition 27.4 *Under the assumptions above, the optimal wealth process $\hat{X}_t$ is given by*

$$\hat{X}_t = E^P\left[\left.\frac{\mathbf{M}_T}{\mathbf{M}_t} \cdot \widehat{X}_T \right| \mathcal{F}_t\right], \tag{27.12}$$

where $\widehat{X}_T$ is given by (27.9).

We note that in order to actually be able to compute $\widehat{X}_t$ we must in general add more structural assumptions. If, for example, we have a Markovian framework, then the expectation in (27.9) reduces to the solution of a Kolmogorov backward equation, which in a Wiener-driven model will be a parabolic boundary value problem.

27.5 The Optimal Portfolio

The next step is to determine the optimal portfolio strategy, i.e. the portfolio which generates $\hat{X}_T$. The general idea for how to do this is in fact quite simple, although it may be difficult to carry it out in a concrete case. It works roughly as follows.

If we denote the vector of relative portfolio weights on the n risky assets by $w_t = (w_t^1, \dots, w_t^n)$, then the dynamics of the induced wealth process X are given by

$$dX_t = X_t w_t \alpha_t dt + X_t(1 - w_t \mathbf{1}) r dt + X_t w_t \sigma_t dW_t, \tag{27.13}$$

where $\mathbf{1}$ denotes the column vector in R^n with 1 in every position.

On the other hand we have, in principle, determined the process $\widehat{X}_t$ by formulas (27.9) and (27.12). We then use the following strategy:

1. Use formula (27.12) to compute $d\widehat{X}_t$. This will give us an expression of the form
$$d\widehat{X}_t = (\dots)dt + \widehat{X}_t h_t dW_t \tag{27.14}$$
where we do not care at all about the dt-term, and where h is some row vector process.
2. Comparing (27.14) with the general portfolio dynamics formula (27.13) we deduce that
$$h_t = w_t \sigma_t.$$
By the complete market assumption, σ_t is invertible, and we can solve for w to obtain the optimal portfolio as
$$\widehat{w}_t = h_t \sigma_t^{-1}. \tag{27.15}$$

Proposition 27.5 *The vector process $\hat{w}$ of optimal portfolio weights on the risky assets is given by*

$$\widehat{w}_t = h_t \sigma_t^{-1}, \tag{27.16}$$

where h is given by (27.14), having used (27.9).

We see that we can "in principle" determine the optimal portfolio strategy $\hat{w}$. For a concrete model, the result of Proposition 27.5 does, however, not lead directly to a closed form expression for $\hat{u}$. The reason is that the formula (27.16) involves the process h which is not explicity given. To obtain closed form expressions, we therefore have to make some further model assumptions. In the next sections we will assume a particular form of the utility function U.

We will study the three most important cases, namely power, logarithmic, and exponential utility.

27.6 Log Utility

The simplest utility function (with a huge margin) is the case of log utility, i.e. we study the following problem.

Problem 27.6.1

$$\underset{X_T}{\text{maximize}}\, E^P\left[\ln(X_T)\right]$$

subject to

$$X_T \in \mathcal{K}_T(x).$$

27.6.1 *The Optimal Terminal Wealth*

With log utility we have

$$I(y) = \frac{1}{y},$$

so from Proposition 27.3 we obtain

$$\widehat{X}_T = I\left(\lambda \mathbf{M}_T\right) = \frac{1}{\lambda}\mathbf{M}_T^{-1}.$$

Finally λ is determined by

$$E^P\left[\mathbf{M}_T \widehat{X}_T\right] = x_0,$$

where x_0 denotes initial wealth. This gives us

$$E^P\left[\frac{1}{\lambda}\mathbf{M}_T\mathbf{M}_T^{-1}\right] = x_0,$$

so $\lambda = x_0^{-1}$ and we have proved the following result.

Proposition 27.6 *For log utility, the optimal terminal wealth is given by*

$$\widehat{X}_T = x_0 \mathbf{M}_T^{-1}. \tag{27.17}$$

Note that so far we have made no concrete assumptions about the underlying market, apart from the assumption of market completeness.

27.6.2 *The Optimal Wealth Process*

We have computed the optimal **terminal** wealth profile $\widehat{X}_T$ so our next problem is to determine what the optimal wealth **process** $\widehat{X}_t$ looks like for $0 \le t \le T$. From risk neutral valuation we have

$$\widehat{X}_t = E^P\left[\left.\frac{\mathbf{M}_T}{\mathbf{M}_t}\widehat{X}_T\right| \mathcal{F}_t\right]$$

so, plugging in (27.17), we obtain

$$\widehat{X}_t = E^P\left[\left.\frac{\mathbf{M}_T}{\mathbf{M}_t}x_0\mathbf{M}_T^{-1}\right|\mathcal{F}_t\right] = x_0\mathbf{M}_t^{-1}.$$

We thus have the following result.

Proposition 27.7 *For log utility, the optimal wealth process is given by*

$$\widehat{X}_t = x_0\mathbf{M}_t^{-1}. \tag{27.18}$$

This is in fact a rather remarkable result, since it says that the optimal wealth process X does **not** depend on the planning horizon T. For a general portfolio optimization problem, say with utility function U and time horizon T, the optimal solution will of course depend on all input parameters, such as U, T, and all other model parameters. We should thus write the optimal wealth process as

$$\widehat{X}_t = \widehat{X}_t^{U,T},$$

where the upper case indices U and T highlight the dependence on U and T. If we suppress the dependence on U we should still expect an expression of the form

$$\widehat{X}_t = \widehat{X}_t^T,$$

but for log utility there is no dependence on the time horizon T. In more concrete terms this implies that, for log utility, the portfolio which is optimal for a ten-year horizon will also be optimal for a five-year horizon. This peculiar property is often referred to as the "myopic" property of log utility, and it is because of this property that log utility is so easy to handle.

27.6.3 *The Optimal Portfolio*

We have computed the optimal wealth process, and our remaining problem is to compute the optimal portfolio. In order to do this we need some more specific assumptions on the structure of the underlying asset prices, and we confine ourselves to the simplest possible case.

Assumption 27.6.1 *Assume that we have a standard Black–Scholes model*

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

We now follow the strategy outlined in Section 27.5. From general portfolio theory we know that, with w denoting the weight on S, we have

$$d\widehat{X}_t = \widehat{X}_t\left\{(1-\hat{w}_t)\frac{dB_t}{B_t} + \hat{w}_t\frac{dS_t}{S_t}\right\}$$

which we write as

$$d\widehat{X}_t = \widehat{X}_t\{\dots\}dt + \widehat{X}_t\widehat{w}_t\sigma dW_t, \tag{27.19}$$

where we do not care about the dt-term. On the other hand, we have the formula (27.18)

$$\widehat{X}_t = x_0 \mathbf{M}_t^{-1},$$

and under Assumption 27.6.1 we have

$$\mathbf{M}_t = e^{-rt} L_t^{-1}$$

with the usual L dynamics

$$dL_t = L_t \varphi dW_t,$$

where

$$\varphi = \frac{r - \mu}{\sigma}.$$

This gives us the **M**-dynamics as

$$d\mathbf{M}_t = -r\mathbf{M}_t dt + \varphi \mathbf{M}_t dW_t,$$

so (27.18) and the Itô formula applied to $\mathbf{M}^{-1}$ gives us

$$d\widehat{X}_t = (\ldots)dt - \widehat{X}_t \varphi dW_t, \tag{27.20}$$

Comparing (27.20) with (27.19), we can finally identify the optimal portfolio weight as $\widehat{w}_t = -\varphi/\sigma$. We have thus proved the following result.

Proposition 27.8 *For log utility and the Black–Scholes model, the optimal weight on the risky asset is given by*

$$\widehat{w}_t = \frac{\mu - r}{\sigma^2}. \tag{27.21}$$

We note again that $\widehat{w}$ is a "myopic" portfolio in the sense that it does not depend on the time horizon T.

27.7 Other Utility Functions

The program that we followed above for log utility can also be carried out for power and exponential utility. For the Black–Scholes case, the computations are left to the reader as an exercise. The basic rule when solving problems involving utility functions is that you always start with log utility. If it turns out that you can handle log utility, then you go on to power and exponential utility. If you are unable to solve your problem for the log utility case then you have essentially three remaining possibilities: You give up the project, you resort to numerical methods, or you write a very abstract paper.

27.8 Optimal Consumption Problems

In the theory outlined above we have focused on problems where the objective was to maximize utility of terminal wealth. In most concrete applications, however, the goal is to optimize a mix of terminal wealth and intermediate consumption. The typical problem is then as follows.

Problem 27.8.1

$$\text{maximize} \quad E^P\left[\int_0^T U(t,c_t)dt + \phi(X_T)\right] \tag{27.22}$$

given initial wealth x, where c is the consumption rate (measured in dollars per unit time), U is the local utility function of consumption, and Φ is the utility of terminal wealth.

By far the most common special case is when we have exponential discounting, in which case the problem reads

Problem 27.8.2

$$\text{maximize} \quad E^P\left[\int_0^T e^{-\delta t}U(c_t)dt + e^{-\delta T}\Phi(X_T)\right] \tag{27.23}$$

where δ is the subjective discount factor.

We now make the following definition.

Definition 27.9 *Let c be a non-negative integrable random process and let $X_T \in \mathcal{F}_T$ be a random variable. We denote by $\mathcal{K}_T(x)$ the set of pairs (c, X_T) such that the consumption rate process c and the terminal wealth X_T can be replicated by a self-financing portfolio with initial wealth x.*

From market completeness and risk neutral valuation we now have the following result, which is parallel to Proposition 27.2.

Proposition 27.10 *The following statements are equivalent:*

$$(c, X_T) \in \mathcal{K}_T(x). \tag{27.24}$$

$$E^Q\left[\int_0^T \frac{1}{B_t}c_t dt + \frac{1}{B_T}X_T\right] = x. \tag{27.25}$$

We can write the budget constraint as

$$E^P\left[\int_0^T \mathbf{M}_t c_t dt + \mathbf{M}_T X_T\right] = x \tag{27.26}$$

where $\mathbf{M}$ is the stochastic discount factor, so we now have the following problem, where we restrict ourselves to a reformulation of Problem 27.8.2.

Problem 27.8.3

$$\underset{c, X_T}{\text{maximize}}\, E^P\left[\int_0^T e^{-\delta t}U(c_t)dt + e^{-\delta T}\Phi(X_T)\right] \tag{27.27}$$

subject to

$$E^P\left[\int_0^T \mathbf{M}_t c_t dt + \mathbf{M}_T X_T\right] = x.$$

The Lagrangian of this is

$$\mathcal{L} = E^P\left[\int_0^T \left\{e^{-\delta t}U(c_t) - \lambda \mathbf{M}_t c_t\right\} dt + e^{-\delta T}\Phi(X_T) - \lambda \mathbf{M}_T X_T\right] + \lambda x,$$

and we can now over maximize $c_t(\omega)$ for each (t,ω) and over $X_T(\omega)$ for each ω. The first-order conditions are

$$U'(\hat{c}_t) = \lambda e^{\delta t}\mathbf{M}_t, \qquad (27.28)$$

$$\Phi'(\widehat{X}_T) = \lambda e^{\delta T}\mathbf{M}_T, \qquad (27.29)$$

and we have our result.

Theorem 27.11 *The optimal consumption and terminal wealth are given by*

$$\widehat{c}_t = I(\lambda e^{\delta t}\mathbf{M}_t), \qquad (27.30)$$

$$\widehat{X}_T = I_\phi(\lambda e^{\delta T}\mathbf{M}_T). \qquad (27.31)$$

where

$$I = \left[U'\right]^{-1}, \qquad I_\Phi = \left[\Phi'\right]^{-1}. \qquad (27.32)$$

The Lagrange parameter λ is determined by the budget constraint (27.26).

Formula (27.28) is in fact a cornerstone in theoretical and empirical asset pricing theory, since it identifies the stochastic discount factor process as the marginal utility along the optimal consumption path (multiplied by a constant and an exponential function). This is so important that we formulate it as a separate result. We will later use this in connection with equilibrium theory.

Proposition 27.12 *For the problems 27.8.1 and 27.23, the stochastic discount factor admits the representations*

$$\mathbf{M}_t = \lambda^{-1}U_c(t,\hat{c}_t), \qquad (27.33)$$

$$\mathbf{M}_t = \lambda^{-1}e^{-\delta t}U'(\hat{c}_t). \qquad (27.34)$$

We recall from Proposition 11.23 that for any price process p_t, and for $s \le t$ we have the formula

$$p_s = E^P\left[\left.\frac{\mathbf{M}_t}{\mathbf{M}_s}p_t\right|\mathcal{F}_s\right],$$

so from (27.28) we obtain

$$p_s = E^P\left[\left.e^{-\delta(t-s)}\frac{U'(\widehat{c}_t)}{U'(\widehat{c}_s)}p_t\right|\mathcal{F}_s\right].$$

This implies that we can view the expression

$$\frac{\mathbf{M}_t}{\mathbf{M}_s} = e^{-\delta(t-s)} \frac{U'(\widehat{c}_t)}{U'(\widehat{c}_s)}$$

as "the stochastic discount factor from s to t". In discrete time the formula would be

$$p_n = E^P \left[\beta^{(m-n)} \frac{U'(\widehat{c}_m)}{U'(\widehat{c}_n)} p_m \middle| \mathcal{F}_n \right]$$

where β with $0 < \beta < 1$ is the subjective one-step discount factor. In particular we can then set $m = n+1$ and define the "one-step stochastic discount factor m" by

$$\mathbf{m}_{n+1} = \frac{\mathbf{M}_{n+1}}{\mathbf{M}_n}$$

and we would have

$$\mathbf{m}_{n+1} = \beta \frac{U'(\widehat{c}_{n+1})}{U'(\widehat{c}_n)}.$$

This is a standard result in discrete time asset pricing, identifying $\mathbf{m}$ as the marginal rate of substitution for consumption between n and $n+1$. In continuous time there is no immediate counterpart to the $\mathbf{m}$-process, since in continuous time there is no such thing as "the next point in time".

27.9 Exercises

Exercise 27.1 Consider the setup in the previous section and assume that $U(t,c) = \ln(c)$ and $\Phi(x) = a\ln(x)$, where a is a positive constant. Compute the optimal consumption plan, and the optimal terminal wealth profile.

Exercise 27.2 Consider the log-optimal portfolio e given by Proposition 27.7 as

$$X_t = x_0 B_t L_t^{-1}.$$

Show that this portfolio is the "P numeraire portfolio" in the sense that if Π is the arbitrage free price process for any asset in the economy, underlying or derivative, then the normalized asset price

$$\frac{\Pi_t}{X_t}$$

is a martingale under the objective probability measure P.

Hint: Use the abstract Bayes Theorem.

Exercise 27.3 Solve the problem of optimal terminal wealth, i.e. compute the optimal terminal wealth, the optimal wealth process, and the optimal portfolio for the case of a standard Black–Scholes model and power utility, i.e.

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \qquad \Phi(x) = A\frac{x^{1-\gamma}}{1-\gamma}.$$

for some constant $A \geq 0$.

Exercise 27.4 Solve the same problem as above, but with exponential utility, i.e.

$$U(x) = -\frac{1}{\gamma}e^{-\gamma x}, \quad \gamma > 0, \qquad \Phi(x) = -\frac{A}{\gamma}e^{-\gamma x}.$$

27.10 Notes

The basic papers for the martingale approach to optimal investment problems see Karatzas et al. (1987) and Cox and Huang (1989) for the complete market case. The theory for the (much harder) incomplete market case was developed in Karatzas et al. (1991) and Kramkov and Schachermayer (1999). A very readable overview of convex duality theory for the incomplete market case, containing an extensive bibliography, is given in Schachermayer (2002). See Chapter 32 for an outline of that theory.

28

OPTIMAL STOPPING THEORY AND AMERICAN OPTIONS

28.1 Introduction

The purpose of this chapter is to give an introduction to the theory of optimal stopping problems in discrete and continuous time. Since optimal stopping theory is rather technical, it is impossible to provide a rigorous derivation of the theory within the framework of this book, so the entire discussion is somewhat informal. We present the main ideas and tools of optimal stopping theory, such as the Snell Envelope Theorem, the system of variational inequalities, and the associated free boundary value problem. The reader is then referred to the specialist literature in the Notes for more precise information.

28.2 Generalities

Let $(\Omega, \mathcal{F}, P, \mathbf{F})$ be a filtered probability space in discrete or continuous time, where the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions. We recall the following definition.

Definition 28.1 *A non-negative random variable τ is called an (optional)* **stopping time** *w.r.t. the filtration* $\mathbf{F}$ *if it satisfies the condition*

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \textit{for all } t \geq 0. \tag{28.1}$$

Intuitively this means that the random time τ is non-anticipative, in the sense that at any time t we can actually determine whether the instant τ in time has occurred or not. It is an easy exercise to show that in discrete time the condition (28.1) can be replaced by

$$\{\tau = n\} \in \mathcal{F}_n, \quad \text{for all } n = 0, 1, 2, \ldots$$

Suppose now that we are given an integrable process Z and a fixed time horizon T (where it is allowed that $T = \infty$). We can then pose the problem of maximizing the expression

$$E\left[Z_\tau\right]$$

over the class of all stopping times τ, satisfying $0 \leq \tau \leq T$ with probability one.

An intuitive way of thinking about this is that we are playing a game where, at any time, we are allowed to push a red button which stops the game. If we push the button at time t then we will obtain the amount Z_t, and our objective is to play this game in such a way that, at time $t = 0$, the expected value of the

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

game is maximized. For this reason we will sometimes refer to the process Z as the "reward process".

We can thus formulate our problem as

$$\underset{0\leq\tau\leq T}{\text{maximize}} \qquad E\left[Z_\tau\right] \tag{28.2}$$

where it is understood that τ varies over the class of stopping times, and we say that a stopping time $\hat{\tau} \leq T$ is **optimal** if

$$E\left[Z_{\hat{\tau}}\right] = \sup_{0\leq\tau\leq T} E\left[Z_\tau\right].$$

For a given problem, an optimal stopping time does not necessarily exist. However, for every ϵ there will always exist an ϵ-optimal stopping time τ_ϵ with the property that

$$E\left[Z_{\tau_\epsilon}\right] \geq \sup_{0\leq\tau\leq T} E\left[Z_\tau\right] - \epsilon.$$

We now have a number of natural questions:

- Under what conditions does there exist an optimal stopping time $\hat{\tau}$?
- If $\hat{\tau}$ exists, how do you find it?
- Is $\hat{\tau}$ unique?
- How do you compute the optimal value $\sup_{0\leq\tau\leq T} E\left[Z_\tau\right]$?

Below we will study these questions in discrete as well as in continuous time. In both cases we will use the same approach, namely that of dynamic programming. For references to the literature, see the Notes at the end of the text.

28.3 Some Simple Results

We start by noticing that in some (very rare) cases, an optimal stopping problem admits a trivial solution.

Proposition 28.2 *The following hold:*

1. *If Z is a submartingale, then late stopping is optimal, i.e. $\hat{\tau} = T$.*
2. *If Z is a supermartingale, then it is optimal to stop immediately, i.e. $\hat{\tau} = 0$.*
3. *If Z is a martingale, then all stopping times τ with $0 \leq \tau \leq T$ are optimal.*

Proof Obvious. □

It is of course very seldom that one comes across one of these cases, one of the few exceptions being the American call on an underlying stock without dividends in Section 28.6.1. It is nevertheless useful to be able to recognize when these simple cases appear. The following result is obvious.

Proposition 28.3 *Assume that the process Z has dynamics*

$$dZ_t = \mu_t dt + \sigma_t dW_t, \tag{28.3}$$

where μ and σ are adapted processes and σ is square integrable. Then the following hold:

- *If $\mu_t \geq 0$, $P - a.s$ for all t, then Z is a submartingale.*
- *If $\mu_t \leq 0$, $P - a.s$ for all t, then Z is a supermartingale.*
- *If $\mu_t = 0$, $P - a.s$ for all t, then Z is a martingale.*

There are also a number of results connecting martingale theory to the theory of convex (or harmonic) functions. The reader will probably recall the following basic results for real valued functions.

Proposition 28.4 *The following hold:*

- *If f is linear and g is convex, then $g(f(x))$ is convex.*
- *If f is convex and g is convex and increasing, then $g(f(x))$ is convex.*
- *If f is concave and g is concave and increasing, then $g(f(x))$ is concave.*

The connection to martingale theory is as follows:

$$\begin{aligned} \text{martingale} &\sim \text{linear function}, \\ \text{submartingale} &\sim \text{convex function}, \\ \text{supermartingale} &\sim \text{concave function}. \end{aligned}$$

The connections are in fact much deeper, but this will do nicely for our purposes. We now have the following result, which is parallel to Proposition 28.4.

Proposition 28.5 *Given enough integrability, the following hold:*

1. *If Z is a martingale and g is convex, then $g(Z_t)$ is a submartingale.*
2. *If Z is a submartingale and g is convex and increasing, then $g(Z_t)$ is a submartingale.*
3. *If Z is a supermartingale and g is concave and increasing, then $g(Z_t)$ is a supermartingale.*

28.4 Discrete Time

Our method of attacking the problem (28.2) is to use Dynamic Programming. The idea is to embed the original problem (28.2) into a large class of problems, indexed by time, and then to connect these problems by means of a simple recursion. We will start by analyzing a fairly general problem, and later on we specialize to a Markovian framework.

28.4.1 *The General Case*

In order to emphasize that we are now working in discrete time we denote a typical point in time by n or k, rather than by s or t.

Definition 28.6 *Consider the optimal stopping problem (28.2) above.*

- *For any point in time n and a given stopping time τ with $n \leq \tau \leq T$ we define the* **value process** $J(\tau)$ *by*

$$J_n(\tau) = E\left[Z_\tau \middle| \mathcal{F}_n\right]. \tag{28.4}$$

- *The* **optimal value process** V *is defined by*

$$V_n = \sup_{n \leq \tau \leq T} E\left[Z_\tau \middle| \mathcal{F}_n\right], \tag{28.5}$$

where, for brevity of notation, we use "sup" to denote the essential supremum.
- *A stopping time which realizes the (essential) supremum in (28.5) is said to be* **optimal at** n*, and it will be denoted by* $\hat{\tau}_n$*.*

We now try to understand the nature of the optimal value process V, and to do this we consider a fixed time n. We then compare three different stopping strategies:

Strategy 1: We use the optimal stopping strategy $\hat{\tau}_n$.
Strategy 2: We stop immediately.
Strategy 3: We do not stop at time n. Instead we wait until time $n+1$, and from time $n+1$ we behave optimally, i.e. we use the stopping time $\hat{\tau}_{n+1}$.

Let us now compute the values of these strategies, and compare them:

- The value of Strategy 1 (henceforth S1) is obviously, and by definition, given by V_n.
- The value of S2 is equally obviously given by Z_n.
- For S3 we do not stop at time n. Instead we wait one step and we find ourselves at time $n+1$. By definition we are assumed to behave optimally from $n+1$ and onwards, so at time $n+1$ the value of our strategy is given by V_{n+1}. The value of this, from the point of view of time n, is then (at least intuitively) given by the conditional expectation $E\left[V_{n+1} \middle| \mathcal{F}_n\right]$.

More formally we argue as follows. The value, at time n, of using $\hat{\tau}_{n+1}$ is by definition given by $E\left[Z_{\hat{\tau}_{n+1}} \middle| \mathcal{F}_n\right]$. Using iterated expectations we then obtain

$$E\left[Z_{\hat{\tau}_{n+1}} \middle| \mathcal{F}_n\right] = E\left[E\left[Z_{\hat{\tau}_{n+1}} \middle| \mathcal{F}_{n+1}\right] \middle| \mathcal{F}_n\right] = E\left[V_{n+1} \middle| \mathcal{F}_n\right].$$

We now compare S1 with S2 and S3. Since S1 by definition is the optimal one, the value of this strategy is as least as high as the value of S2 and S3. We thus have the trivial inequalities

$$V_n \geq Z_n, \tag{28.6}$$

$$V_n \geq E\left[V_{n+1} \middle| \mathcal{F}_n\right]. \tag{28.7}$$

We now make the simple but important observation that at time n we have only two possibilities for the optimal stopping time $\hat{\tau}_n$: It is either optimal to stop immediately, in which case we have $\hat{\tau}_n = n$ and $V_n = Z_n$, or else it is optimal not to stop at time n, in which case $\hat{\tau}_n = \hat{\tau}_{n+1}$ and $V_n = E[V_{n+1}|\mathcal{F}_n]$. We thus have the equation

$$V_n = \max\{Z_n, E[V_{n+1}|\mathcal{F}_n]\}.$$

We have thus more or less proved our first result.

Proposition 28.7 *The optimal value process V is the solution of the following backward recursion:*

$$V_n = max\{Z_n, E[V_{n+1}|\mathcal{F}_n]\}, \tag{28.8}$$

$$V_T = Z_T. \tag{28.9}$$

Furthermore, it is optimal to stop at time n if and only if $V_n = Z_n$. If stopping at n is not optimal, then $V_n > Z_n$ and $V_n = E[V_{n+1}|\mathcal{F}_n]$.

Proof This is obvious from the arguments given above. □

We note that the recursion above implicitly defines the optimal stopping strategy. We know that $V_n = Z_n$ if and only if it is optimal to stop at n, and that $V_n > Z_n$ if and only if it is optimal to continue. Thus the optimal stopping policy is to stop the first time the optimal value process equals the payoff process.

Proposition 28.8 *The following hold:*

- *An optimal stopping rule $\hat{\tau}$ at time $t = 0$ is given by*

$$\hat{\tau} = min\{n \geq 0: \ V_n = Z_n\}. \tag{28.10}$$

- *For a fixed n an optimal stopping time $\hat{\tau}_n$ is given by*

$$\hat{\tau}_n = min\{k \geq n: \ V_k = Z_k\}. \tag{28.11}$$

- *For any n we have*

$$V_{\hat{\tau}_n} = Z_{\hat{\tau}_n}. \tag{28.12}$$

Remark 28.4.1 *Note that an optimal stopping time will always exist in discrete time when $T < \infty$. It does, however, not need to be unique. The stopping time in (28.10) is in fact the smallest optimal stopping time.*

We will now study the optimal value process in some more detail. To do this we need some new definitions.

Definition 28.9 *Consider a fixed process Y.*

- *We say that a process X* **dominates** *the process Y if $X_n \geq Y_n$ $P-a.s.$ for all n.*
- *Assuming that $E[Y_n] < \infty$ for all $n \leq T$, the* **Snell envelope** *S, of the process Y is defined as the smallest supermartingale dominating Y. More precisely: S is a supermartingale dominating Y, and if D is another supermartingale dominating Y, then $S_n \leq D_n$ $P-a.s.$ for all n.*

It is not obvious that the Snell envelope exists, but existence does in fact follow from the following result, the proof of which is left to the reader.

Proposition 28.10 *Let $\{D^\alpha\}_{\alpha \in A}$ be a family of supermartingales, indexed by α. Then the process X, defined by*

$$X_n = \inf_{\alpha \in A} D_n^\alpha$$

is a supermartingale.

Using this result we can easily prove the existence of a Snell envelope.

Proposition 28.11 *Consider a process Y with $E[Y_n] < \infty$ for all $n \leq T$. Then there exists a Snell envelope for Y.*

Proof Define $\mathcal{D}$ as the family of supermartingales dominating Y. Now define S by

$$S_n = \inf_{D \in \mathcal{D}} D_n.$$

By the previous proposition, S will be a supermartingale. It will obviously dominate Y and by construction it will be minimal. □

Going back to the optimal value process V, we see from (28.6) that V dominates Z. Furthermore we see from (28.7) that V is a supermartingale, and in fact we have the following result.

Theorem 28.12 (The Snell Envelope Theorem) *The optimal value process V is the Snell envelope of the reward process Z.*

Proof Since V is a supermartingale dominating Z, in order to show that $V = S$ we only have to prove the minimality of V. Let us thus assume that X is a supermartingale dominating Z. We then have to show that $V_n \leq X_n$ for all n. For $n = T$ we have $V_T = Z_T$, and since X dominates Z we have $X_T \geq Z_T$, so obviously $X_T \geq V_T$. We then proceed by induction and thus assume that $X_{n+1} \geq V_{n+1}$. Since X is a supermartingale we have $X_n \geq E[X_{n+1} | \mathcal{F}_n]$, so by the induction assumption we have $X_n \geq E[V_{n+1} | \mathcal{F}_n]$. Furthermore we know that X dominates Z so we have $X_n \geq Z_n$, and we thus obtain the inequality

$$X_n \geq \max\{Z_n, E[V_{n+1} | \mathcal{F}_n]\}.$$

Using (28.8) we conclude that $X_n \geq V_n$. □

Since V is a supermartingale we have $E[V_{n+1} | \mathcal{F}_n] \leq V_n$, so V is decreasing in conditional average, and this has a rather clear economic interpretation in terms of missed opportunities. Indeed, if you are standing at n, then V_n is by definition the optimal expected value of the game. The expectation $E[V_{n+1} | \mathcal{F}_n]$ is the expected value of waiting until tomorrow (and then using an optimal stopping strategy). If it is de facto optimal to stop today, then obviously you lose something by waiting until tomorrow, and this loss is measured by the gap in the supermartingale inequality.

From this you also are led to expect that if it is **not** optimal to stop at time n, then $E[V_{n+1}|\mathcal{F}_n] = V_n$, i.e. the V process should be a martingale on the interval $[n, n+1]$. This intuition is formalized by the result below that says that V is indeed a martingale until the optimal stopping time, but first we need another definition.

Definition 28.13 *For any process X, and any stopping time τ, the* **stopped process** *X^τ is defined by*

$$X_n^\tau = X_{n\wedge\tau}, \quad n = 0, 1, \ldots, T,$$

where we have used the notation $a \wedge b = min\{a, b\}$.

We also need two small standard results, the proofs of which are left to the reader.

Proposition 28.14 *Assume the a process X is a supermartingale on a finite time interval $[0, T]$. Then the following hold:*

- *If τ is a stopping time, then the stopped process X^τ is a supermartingale.*
- *If $E[X_T] = X_0$, then X is in fact a martingale.*

The basic result is now the following.

Proposition 28.15 *Consider a fixed n. Then the stopped process $V^{\hat{\tau}_n}$ is a martingale on the interval $[n, T]$.*

Proof WLOG (without loss of generality) we may assume that $n = 0$, and we denote the optimal stopping time $\hat{\tau}_0$ by $\hat{\tau}$. Since V is a supermartingale it follows that also $V^{\hat{\tau}}$ is a supermartingale, and it is enough to show that $V_0 = E\left[V_T^{\hat{\tau}}\right]$. This follows from the following equalities

$$V_0 = E[Z_{\hat{\tau}}] = E[V_{\hat{\tau}}] = E[V_{\hat{\tau}\wedge T}] = E\left[V_T^{\hat{\tau}}\right].$$

□

28.4.2 *Markovian Models*

In this section we will restrict ourselves to Markovian models. This will lead to greater analytical tractability, and we consider the simplest possible model. More precisely we consider a model driven by a finite state Markov chain X on the state space $\mathcal{X} = \{1, \ldots, N\}$ and we assume that the optimal stopping problem is of the form

$$\max_{0\le\tau\le T} E[\alpha^\tau g(X_\tau)], \tag{28.13}$$

where the discount factor α is a real number with $\alpha < 1$, and g is a mapping $g : \mathcal{X} \to R$.

We assume furthermore that X is time homogeneous with transition matrix $\mathbf{A}$, i.e.

$$\mathbf{A}_{i,j} = P(X_{n+1} = j \,|X_n = i). \tag{28.14}$$

In this setting, the optimal value process V_n at time n will be a deterministic function of X_n and with a slight abuse of notation we denote this function also by V_n so we have $V_n = V_n(X_n)$.

We now need to introduce some useful notational conventions. We start by noting that any real valued function f defined on $\mathcal{X}$, i.e. $f : \mathcal{X} \to R$, is completely specified by its values $f(1), \ldots, f(N)$ on $\mathcal{X}$. We can thus view f as a vector in R^N, and we will regard it as the column vector

$$f = \begin{bmatrix} f(1) \\ f(2) \\ \vdots \\ f(N) \end{bmatrix}.$$

We can also view X as a process living on the set of unit vectors in R^N, so that instead of writing $X_n = i$ we write $X_n = e_i$ where e_i is the $i : th$ unit column vector in R^N. This implies that, for a fixed n, we can write the random variable $f(X_n)$ as

$$f(X_n) = f^\star X_n,$$

where * denotes transpose. In particular, we can write a conditional expectation of the form $E\left[f(X_{n+1}) \middle| X_n \right]$ as

$$E\left[f(X_{n+1}) \middle| X_n \right] = g(X_n)$$

where the function g on vector form is given by

$$g = \mathbf{A} f.$$

With this notation, the recursion from Proposition 28.7 reads as follows.

Proposition 28.16 *The optimal value functions $V_1, \ldots, V_T$ are determined by the recursion*

$$\begin{aligned} V_n &= max[\alpha^n g, \mathbf{A} V_{n+1}], \\ V_T &= \alpha^T g, \end{aligned}$$

where the maximum is interpreted component wise.

In a model with discounted payoffs like the present one it is natural to slightly redefine the optimal value function. We thus define an alternative optimal value function W by

$$W_n = \alpha^{-n} V_n.$$

This simply means that the W function is the optimal value for the problem

$$\underset{n \le \tau \le T}{\text{maximize}} \qquad E\left[\alpha^{\tau - n} g(X_\tau) \middle| X_n \right]$$

and we have the recursion

$$W_n = \max\left[g, \alpha \mathbf{A} W_{n+1} \right], \tag{28.15}$$

$$W_T = g. \tag{28.16}$$

28.4.3 *Infinite Horizon*

We now relax the assumption of having a finite time horizon and consider the infinite horizon problem

$$\underset{\tau \geq 0}{\text{maximize}} \qquad E\left[\alpha^\tau g(X_\tau) \middle| X_0\right]. \tag{28.17}$$

With some extra effort it can be shown that most of the results of the preceding sections still hold, and we summarize as follows.

Proposition 28.17 *Define the optimal value function by*

$$V_n = \sup_{\tau \geq n} E\left[\alpha^\tau g(X_\tau) \middle| X_n\right].$$

Then the following hold:

- *The optimal value process $V_n(X_n)$ is the smallest supermartingale dominating the process $\alpha^n g(X_n)$.*
- *V satisfies the recursion*

$$V_n = max[\alpha^n g, \mathbf{A}V_{n+1}]. \tag{28.18}$$

- *The optimal stopping time $\hat{\tau}_n$ is given by*

$$\hat{\tau}_n = \inf\left\{k \geq n;\ V_k(X_k) = \alpha^k g(X_k)\right\}. \tag{28.19}$$

We note that as opposed to the finite horizon case, we no longer have a boundary value at T, so it is not clear how to carry out the infinite recursion (28.18). More about this later.

In the present setting we may express the proposition above in pure function-theoretic language. To do this we need a definition.

Definition 28.18 *A function sequence $\{f_0, f_1, f_2, \ldots\}$ where $f_n : \mathcal{X} \to R$, is called* **excessive** *if it satisfies the relation*

$$f_n \geq \mathbf{A}f_{n+1}, \quad n = 0, 1, \ldots \tag{28.20}$$

*The sequence is called α-***excessive** *if*

$$f_n \geq \alpha\mathbf{A}f_{n+1}, \quad n = 0, 1, \ldots \tag{28.21}$$

We can now reformulate Proposition 28.17 as follows.

Proposition 28.19 *With notation as above, The optimal value function sequence V_n is the smallest excessive function sequence dominating the sequence $\alpha^n g$.*

In the infinite horizon setting, the alternative optimal value process W, defined in the previous section as

$$W_n = \sup_{\tau \geq n} E\left[\alpha^{\tau - n} g(X_\tau) \middle| X_n\right] \tag{28.22}$$

is much nicer to handle than the function sequence V. The reason for this is that because of the assumed time homogeneity of the process X and the infinite horizon, the function $W_n(i)$ will in fact not depend on running time n. Thus we can write $W_n(i) = W(i)$ where W is the **optimal value function**, and from (28.15) we obtain the recursion

$$W = \max[g, \alpha \mathbf{A} W].$$

It is also clear from the time invariance, that the state space $\mathcal{X}$ will be divided into two regions, the **continuation region** C and the **stopping region** S such that whenever $X_n \in C$ it is optimal to continue the game, whereas we stop immediately if $X_n \in S$. We now have the following result.

Proposition 28.20 *For the infinite horizon case the following hold:*

- *The optimal value function sequence W_n is time invariant, so $W_n = W$ for $n = 1, 2, \ldots$, and in particular we have*

$$W(i) = \sup_{\tau \geq 0} E\left[\alpha^{\tau} g(X_\tau) \middle| X_0 = i\right], \quad i = 1, \ldots, N. \tag{28.23}$$

- *W satisfies the recursion*

$$W = max[g, \alpha \mathbf{A} W]. \tag{28.24}$$

- *The function W is the smallest α-excessive function dominating the function g.*
- *The continuation region $C \in \mathcal{X}$ is given by*

$$C = \{i \in \mathcal{X};\ W(i) > g(i)\}. \tag{28.25}$$

- *The stopping region S is given by*

$$S = \{i \in \mathcal{X};\ W(i) = g(i)\}. \tag{28.26}$$

- *An optimal stopping time is given by*

$$\hat{\tau} = \inf\{k \geq 0;\ W(X_k) = g(X_k)\}. \tag{28.27}$$

As we noted above, it is not immediately clear how to compute the optimal value function W. We know that it satisfies the recursion (28.24) but it is not obvious how to solve this equation. The good news is that the W function can in fact be computed numerically with a very fast algorithm. To see this we recall that W is the smallest α-excessive function dominating g. W will therefore be the optimal solution of the following finite dimensional optimization problem:

$$\underset{W}{\text{minimize}} \qquad \sum_{i=1}^{N} W(i) \tag{28.28}$$

subject to the constraints

$$W \geq g, \tag{28.29}$$

$$W - \alpha \mathbf{A} W \geq 0. \tag{28.30}$$

Here we can obviously replace the sum $\sum_{i=1}^{n} W(i)$ by any sum of the form $\sum_{i=1}^{n} c(i)W(i)$ where $c(1),\ldots,c(N)$ are positive.

The point is that the optimization problem above is a **Linear Programming Problem**. Using standard software this problem can be solved in a fraction of a second even for very large values of N.

28.5 Continuous Time

We now turn to the continuous time theory. As can be expected, this is technically more complicated than the discrete time theory, and we will only present some of the main ideas. We will often sweep technical problems under the carpet by assuming "enough regularity", and we refer the reader to the specialist literature for technical details and precise formulations.

28.5.1 *General Theory*

The setup in continuous time is that we consider a given semimartingale Z, on a finite time interval $[0,T]$, satisfying the integrability condition

$$\sup_{0\le\tau\le T} E\left[|Z_\tau|\right] < \infty. \tag{28.31}$$

The problem to be solved is again the following:

$$\underset{0\le\tau\le T}{\text{maximize}} \qquad E\left[Z_\tau\right]. \tag{28.32}$$

As before, we embed this in a wider class of problems by defining the optimal value process V by

$$V_t = \sup_{t\le\tau\le T} E\left[\,Z_\tau|\,\mathcal{F}_t\right]. \tag{28.33}$$

and we denote an optimal stopping time (which does not necessarily exist) for this problem by $\hat{\tau}_t$. The main results from discrete time now carry over to continuous time. The arguments are basically the same as in Section 28.4.1 but it should be noted that in continuous time there are a number of quite hard technical problems to handle. We do not go into these problems here, but refer the reader to the extremely well written Appendix D on optimal stopping in Karatzas and Shreve (1998).

We recall the definition of the Snell envelope.

Definition 28.21 *Consider a fixed process Y.*

- *We say that a process X* **dominates** *the process Y if $X_t \ge Y_t$ $P-a.s.$ for all $t\ge 0$.*
- *Assuming that $E\left[Y_t\right] < \infty$ the* **Snell envelope** *S, of the process Y is defined as the smallest supermartingale dominating Y. More precisely: S is a supermartingale dominating Y, and if D is another supermartingale dominating Y, then $S_t \le D_t$, $P-a.s.$ for all $t \ge 0$.*

As in discrete time we do have existence of a Snell envelope.

Proposition 28.22 *For any integrable semimartingale Y, the Snell envelope exists.*

The proof of this result is more complicated than in discrete time since we can no longer rely on the simple recursion results from Section 28.4.1. We thus have to use a different technique, and the reader is again referred to Karatzas and Shreve (1998). In continuous time there are also some other nontrivial technical complications. For example, since the optimal value process V is defined by $V_t = \sup_{t \le \tau \le T} E[Z_\tau | \mathcal{F}_t]$, where the *sup* denotes the essential supremum, this implies that for each fixed t, the process value V_t is only defined P-a.s. It is now a rather hard problem to show that we can choose a version of the V process which is cadlag, i.e. right continuous and with left limits.

The basic optimal stopping theorem mirrors the one in discrete time.

Theorem 28.23 (The Snell Envelope Theorem) *Given the integrability condition (28.31) the following hold:*

- *The optimal value process V is the Snell envelope of the payoff process Z.*
- *If there exists a (not necessarily unique) optimal stopping time, then the smallest one is given by*

$$\hat{\tau}_t = \inf\{s \ge t;\ V_s = Z_s\}. \tag{28.34}$$

- *For any fixed t, the stopped process $V^{\hat{\tau}_t}$ is a martingale on the interval $[t,T]$.*

28.5.2 *Diffusion Models*

We now specialize the general model of the previous section to a diffusion setting by considering the SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{28.35}$$

For simplicity we assume that X is scalar, but the theory extends in an obvious way to vector valued SDEs. We also consider a given payoff function $\Phi(t,x)$, i.e. $\Phi : [0,T] \times R \to R$, and we study the optimal stopping problem

$$\underset{0 \le \tau \le T}{\text{maximize}}\, E[\Phi(\tau, X_\tau)]. \tag{28.36}$$

We attack this problem by dynamic programming, i.e. we embed the original problem within a large class of problems indexed by t and x. We then connect these problems by a PDE, which in our case turns out to be a so-called **free boundary value problem**. It should again be emphasized that we are sweeping a number of technical problems (mostly concerning regularity) under the carpet.

Definition 28.24 *For a fixed $(t,x) \in [0,T] \times R$, and each stopping time τ with $\tau \ge t$, the* **value function** *J is defined by*

$$J(t,x;\tau) = E_{t,x}\left[\Phi(\tau, X_\tau)\right]. \tag{28.37}$$

The **optimal value function** $V(t,x)$ *is defined by*

$$V(t,x) = \sup_{t \le \tau \le T} E_{t,x}\left[\Phi(\tau, X_\tau)\right] = \sup_{t \le \tau \le T} J(t,x;\tau). \tag{28.38}$$

Here, as always, the lower index (t,x) indicates that the expected value is obtained by integrating over the measure induced by the SDE

$$\begin{aligned} dX_s &= \mu(s, X_s)ds + \sigma(s, X_s)dW_s, \\ X_t &= x. \end{aligned}$$

A stopping time which realizes the supremum for V above is called **optimal** *and will be denoted by $\hat{\tau}_{tx}$. For brevity we will often suppress the x and denote it by $\hat{\tau}_t$.*

We now go to the dynamic programming argument, and to this end we fix (t,x) and a "small" time increment h, where later on $h \to 0$. We now consider three strategies:

Strategy 1: We use the optimal stopping strategy $\hat{\tau}_t$.
Strategy 2: We stop immediately.
Strategy 3: We do not stop at time t. Instead we wait until time $t+h$, and from time $t+h$ we behave optimally, i.e. we use the stopping time $\hat{\tau}_{t+h}$.

We will now compare these strategies, but before doing this, we need some ad hoc assumptions.

Assumption 28.5.1 *We assume the following:*

- *There exists an optimal stopping time $\hat{\tau}_{t,x}$ for each (t,x).*
- *The optimal value function V is "regular enough". More precisely we assume that $V \in C^{1,2}$.*
- *All processes below are "integrable enough", in the sense that expected values exist, stochastic integrals are true (instead of being merely local) martingales, etc.*

Comparing the three strategies S1, S2, and S3 above, we have the following results:

- The value of S1 is obviously given by $V(t,x)$.
- The value of S2 is obviously given by $\Phi(t,x)$.
- The value of S3 is given (why?) by $E_{t,x}\left[V(t+h, X_{t+h})\right]$.

Since, by definition, S1 is better than (or equal to) S2 and S3 we have the inequalities

$$V(t,x) \ge \Phi(t,x), \tag{28.39}$$

$$V(t,x) \ge E_{t,x}\left[V(t+h, X_{t+h})\right]. \tag{28.40}$$

Remark 28.5.1 *If h is "small enough" then it seems to be clear, at least intuitively, that one of S2 and S3 is always optimal. In general it could of course be the case that it is optimal to stop in the interior of the interval $[t, t+h]$, but as $h \to 0$, such a strategy will be indistinguishable from S2. In other words:* **In the limit, one of the inequalities (28.39)–(28.40) will be an equality**.

From the Itô formula we have

$$V(t+h, X_{t+h}) = V(t,x) + \int_t^{t+h} \left(\frac{\partial}{\partial s} + \mathbf{A}\right) V(s, X_s) ds \tag{28.41}$$

$$+ \int_t^{t+h} \sigma(s, X_s) \frac{\partial V}{\partial x}(s, X_s) dW_s, \tag{28.42}$$

where $\mathbf{A}$ is the usual Itô operator defined by

$$\mathbf{A}f(t,x) = \mu(t,x)\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}(t,x). \tag{28.43}$$

Plugging (28.42) into (28.40) we obtain

$$V(t,x) \geq V(t,x) + E_{t,x}\left[\int_t^{t+h} \left(\frac{\partial}{\partial s} + \mathbf{A}\right) V(s, X_s) ds\right]$$
$$+ E_{t,x}\left[\int_t^{t+h} \sigma(s, X_s)\frac{\partial V}{\partial x}(s, X_s) dW_s\right],$$

and, using the fact that the expected value of the stochastic integral will vanish, we have

$$E_{t,x}\left[\int_t^{t+h} \left(\frac{\partial}{\partial s} + \mathbf{A}\right) V(s, X_s) ds\right] \leq 0.$$

We may now divide by h and let $h \to 0$ to obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{A}\right) V(t,x) \leq 0.$$

We thus see that the optimal value function has the two properties

$$V(t,x) \geq \Phi(t,x), \tag{28.44}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{A}\right) V(t,x) \leq 0. \tag{28.45}$$

The arguments above do, however, provide more information than this. From Remark 28.5.1 we draw the following conclusion:

- It is optimal to stop at (t,x) if and only if

  $$V(t,x) = \Phi(t,x),$$

  in which case

  $$\left(\frac{\partial}{\partial t} + \mathbf{A}\right) V(t,x) < 0.$$

- It is optimal to continue if and only if

$$V(t,x) > \Phi(t,x),$$

in which case

$$\left(\frac{\partial}{\partial t} + \mathbf{A}\right) V(t,x) = 0.$$

If we thus define the **continuation region** C by

$$C = \{(t,x);\ V(t,x) > \Phi(t,x)\}, \tag{28.46}$$

we see that the structure of the optimal stopping rule is that it is optimal to stop at (t,x) if and only if $(t,x) \in C^c$, where ^{c} denotes the complement.

If, on the other hand, $(t,x) \in C$, then it is optimal to continue, which implies that the inequality (28.45) will be an equality, thus giving us the relation

$$\left(\frac{\partial}{\partial t} + \mathbf{A}\right) V(t,x) = 0, \quad (t,x) \in C.$$

If we now collect our heuristic arguments we have the following (somewhat imprecise) result.

Proposition 28.25 *If the optimal value function V is regular enough, the following hold:*

$$V(T,x) = \Phi(T,x), \tag{28.47}$$

$$V(t,x) \geq \Phi(t,x), \quad \forall (t,x), \tag{28.48}$$

$$V(t,x) = \Phi(t,x), \quad (t,x) \in C^c, \tag{28.49}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{A}\right) V(t,x) \leq 0, \quad \forall (t,x), \tag{28.50}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{A}\right) V(t,x) = 0, \quad (t,x) \in C. \tag{28.51}$$

Here C^c denotes the complement of C. Furthermore, given (t,x), the optimal stopping time $\hat{\tau}_{t,x}$ is given by

$$\hat{\tau}_{t,x} = \inf\{s \geq t;\ V(s,X_s) = \Phi(s,X_s)\}. \tag{28.52}$$

The big problem with our heuristic result above is that it is very hard to give reasonable conditions on μ, σ, and Φ which guarantee that V is regular enough. It may easily happen that, even for seemingly very natural choices of μ, σ, and Φ, V is not $C^{1,2}$, and in many cases V is not even C^1 in the x variable across the boundary ∂C. The moral of this is that if we want to study optimal stopping problems rigorously, we need a much more complicated technical machinery, and the reader is referred to the specialist literature.

We can in fact reformulate Proposition 28.25 in the following way.

Proposition 28.26 *Given enough regularity, the optimal value function is characterized by the following relations:*

$$V(T,x) = \Phi(T,x), \tag{28.53}$$

$$V(t,x) \geq \Phi(t,x), \quad \forall(t,x), \tag{28.54}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{A}\right) V(t,x) \leq 0, \quad \forall(t,x), \tag{28.55}$$

$$max\left\{\Phi(t,x) - V(t,x), \left(\frac{\partial}{\partial t} + \mathbf{A}\right) V(t,x)\right\} = 0, \quad \forall(t,x) \tag{28.56}$$

The point of this reformulation as a set of **variational inequalities** is that the continuation region C does not appear.

We note that from Proposition 28.25 above, it follows that on C the optimal value function solves a boundary value problem.

Proposition 28.27 *Assuming enough regularity, the optimal value function satisfies the following parabolic equation:*

$$\frac{\partial V}{\partial t}(t,x) + \mu(t,x)\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 V}{\partial x^2}(t,x) = 0, \quad (t,x) \in C,$$
$$V(t,x) = \Phi(t,x), \quad (t,x) \in \partial C.$$

The PDE above is known as a **free boundary value problem**, because of the fact that the domain C and its boundary ∂C are not given a priori but have to be determined as a part of the solution. Generally speaking, there is little hope of having an analytical solution of a free boundary value problem, so typically one has to resort to numerical schemes.

Remark 28.5.2 *When trying to solve the free boundary value problem above, it is common to add the condition that V should be smooth, not only in the interior of C, but also that it should be C^1 at the boundary of C. This is called a* **smooth fit condition**, *and it is a largely heuristic condition, which does not necessarily hold. See Peskir and Shiryaev (2006) for details.*

Although it is generally very hard to determine the continuation region C, there is an easily applicable partial result.

Proposition 28.28 *It is never optimal to stop at a point where*

$$\frac{\partial \Phi}{\partial t}(t,x) + \mu(t,x)\frac{\partial \Phi}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 \Phi}{\partial x^2}(t,x) > 0. \tag{28.57}$$

Expressed otherwise, we have

$$\left\{(t,x); \ \left(\frac{\partial}{\partial t} + \mathbf{A}\right)\Phi(t,x) > 0\right\} \subseteq C. \tag{28.58}$$

Proof If the condition (28.57) is satisfied, then the process $\Phi(t, X_t)$ is a submartingale close to (t,x) and it is therefore optimal to continue. □

We note that the inclusion in (28.58) is generally a strict one, i.e. it can be optimal to continue also at points outside the set in the left-hand side of (28.58).

28.5.3 *Connections to the General Theory*

The connections between the free boundary value formulation in the previous section and the general Snell theory are more or less obvious. The inequality (28.48) says that the optimal value process $V(t, X_t)$ dominates the reward process $\Phi(t, X_t)$, and (28.50) says that $V(t, X_t)$ is a supermartingale. The relation (28.51) or, alternatively, the free boundary value formulation, is a differential statement of the fact that the stopped optimal value process is a martingale.

28.6 American Options

We now specialize the theory to study the pricing of American options.

28.6.1 *The American Call without Dividends*

We recall from Section 7.9 that for an American call on an underlying stock without dividends, early exercise is never optimal, so the American call price will in fact coincide with the European call price. In Section 7.9 we provided an elementary proof of this, but we can also use our formal theory to derive the same result. The optimal stopping problem is

$$\underset{0 \leq \tau \leq T}{\text{maximize}} \qquad E^Q\left[e^{-r\tau}\max\{S_\tau - K, 0\}\right] \tag{28.59}$$

where the stock price dynamics under the risk neutral measure Q are given by

$$dS_t = rS_t dt + \sigma_t S_t dW_t. \tag{28.60}$$

Here r is the short rate (assumed to be positive), and σ is an arbitrary adapted process. In terms of the general theory, this means that the reward process Z is given by

$$Z_t = e^{-rt}\max\{S_t - K, 0\} = \max\left\{e^{-rt}S_t - e^{-rt}K, 0\right\}.$$

We now note that, from arbitrage theory, the process $e^{-rt}S_t$ is a Q martingale, whereas the "process" $e^{-rt}K$ is a deterministic decreasing function of time and hence a supermartingale. The process $e^{-rt}S_t - e^{-rt}K$ is thus a Q submartingale. Since the mapping $x \longmapsto \max\{x, 0\}$ is convex and increasing we see that the reward process Z is a convex and increasing function of a submartingale and thus, according to Proposition 28.5, Z is itself a submartingale. Hence the optimal stopping problem is trivial, with optimal stopping time given by $\hat{\tau} = T$.

28.6.2 *The American Put Option*

If we specialize the arguments from the previous sections to the case of an American put option with last exercise day T and strike price K, within a Black–Scholes model, this implies that we are considering an optimal stopping problem of the form

$$\underset{0 \le \tau \le T}{\text{maximize}} \qquad E^Q\left[e^{-r\tau}\max\{K - S_\tau, 0\}\right]. \tag{28.61}$$

Here Q denotes the risk neutral measure, and the Q dynamics of S are again given by

$$dS_t = rS_t dt + S_t \sigma dW_t. \tag{28.62}$$

Because of the discounting factor it is convenient to define a slightly modified optimal value function V by

$$V(t,x) = \sup_{t \le \tau \le T} E^Q_{t,x}\left[e^{-r(\tau - t)}\max\{K - S_\tau, 0\}\right].$$

With this notation we have the following basic result for the American put.

Proposition 28.29 *Assume that a sufficiently regular function $V(t,s)$, and an open set $C \subseteq R_+ \times R_+$, satisfies the following conditions:*

1. *C has a continously differentiable boundary b_t, i.e. $b \in C^1$ and $(t, b_t) \in \partial C$.*
2. *V satisfies the PDE*

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 V}{\partial s^2} - rV = 0, \quad (t,s) \in C. \tag{28.63}$$

3. *V satisfies the final time boundary condition*

$$V(T,s) = max[K - s, 0], \quad s \in R_+. \tag{28.64}$$

4. *V satisfies the inequality*

$$V(t,s) > max[K - s, 0], \quad (t,s) \in C. \tag{28.65}$$

5. *V satisfies*

$$V(t,s) = max[K - s, 0], \quad (t,s) \in C^c. \tag{28.66}$$

6. *V satisfies the* **smooth fit condition**

$$\lim_{s \downarrow b(t)} \frac{\partial V}{\partial s}(t,s) = -1, \quad 0 \le t < T. \tag{28.67}$$

Then the following hold:

- *V is the optimal value function.*
- *C is the continuation region.*
- *The optimal stopping time is given by*

$$\hat{\tau} = \inf\{t \ge 0;\ S_t = b_t\}. \tag{28.68}$$

There is of course also an alternative characterization of the optimal value function in terms of a set of variational inequalities.

There is a large literature on the American put, but there are no analytical formulas for the pricing function or the optimal boundary. For practical use, the following alternatives are available:

- Solve the free boundary value problem numerically.
- Solve the variational inequalities numerically.
- Approximate the Black–Scholes model by a binomial model and compute the exact binomial American put price.

28.6.3 *The Perpetual American Put*

As we have seen, optimal stopping problems are notoriously difficult to solve analytically. One happy exception is (within the Black–Scholes model) the perpetual American put, i.e. an American put with infinite time horizon. This option is fairly simple to analyze, since the infinite horizon and the time invariance of the stock price dynamics implies that the option price as well as the optimal boundary are constant as functions of running time.

From economic arguments it is reasonable to expect that there exists a (constant) critical price b such that we exercise the option whenever $S_t \le b$. Since the optimal value function (i.e. the pricing function) $V(t,s)$ in this case will be independent of running time t and thus of the form $V(t,s) = V(s)$ we see that the free boundary value problem for V reduces to the ODE

$$rs\frac{\partial V}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 V}{\partial s^2} - rV = 0, \quad s > b. \tag{28.69}$$

We now want to find a function V and a real number b such that V and b satisfy (28.69). It is not hard to see that the general solution of the ODE is of the form

$$V(s) = As + Bs^{-\gamma}, \tag{28.70}$$

where γ is given by

$$\gamma = \frac{2r}{\sigma^2}. \tag{28.71}$$

Since V must be bounded as $s \to \infty$ (why?) it follows directly that $A = 0$. We then use the boundary condition and the smooth fit condition:

$$\begin{aligned} V(b) &= K - b, \\ \frac{\partial V}{\partial s}(b+) &= -1. \end{aligned}$$

From this we easily deduce the following result.

Proposition 28.30 *For a perpetual American put with strike K, the pricing function V and the critical price b are given by*

$$b = \frac{\gamma K}{1+\gamma} \tag{28.72}$$

$$V(s) = \frac{K}{1+\gamma}\left(\frac{b}{s}\right)^{\gamma}, \quad s > b. \tag{28.73}$$

28.7 Exercises

Exercise 28.1 Prove that, in discrete time, random time τ is a stopping time if and only if $\{\tau = n\} \in \mathcal{F}_n$ for all n.

Exercise 28.2 Construct a (trivial) example in continuous time, of an optimal stopping problem for which there is no optimal stopping time.

Exercise 28.3 Prove Proposition 28.2.

Exercise 28.4 Prove Proposition 28.3.

Exercise 28.5 Prove Proposition 28.4.

Exercise 28.6 Prove Proposition 28.5.

Exercise 28.7 Let f and g be concave functions. Show that h defined by $h(x) = \min\{f(x), g(x)\}$ is concave.

Exercise 28.8 Let X and Y be supermartingales. Show that Z defined by

$$Z_n(\omega) = \min\{X_n(\omega), Y_n(\omega)\}$$

is a supermartingale. Compare with the previous exercise and with the relations between martingale theory and convex theory given in Section 28.3.

Exercise 28.9 Prove Proposition 28.14.

Exercise 28.10 Assume a standard Black–Scholes model for the stock price and assume that $r = 0$. In this (highly unrealistic) case, one can easily solve the American put problem on a finite time horizon $[0,T]$. Do this.

Exercise 28.11 Consider an ODE of the form

$$f(s) + asf'(s) + bs^2 f''(s) = 0.$$

Introduce a new variable x by $x = \ln(s)$. Show that the ODE by this change of variable will be transformed into a linear ODE with constant coefficients. More precisely, find the ODE satisfied by the function F, defined by $F(x) = f(e^x)$, i.e. $f(s) = F(\ln s)$.

Exercise 28.12 Consider the ODE (28.69). Use the transformation in the previous exercise to show that the ODE has a general solution of the form (28.70).

Exercise 28.13 Prove Proposition 28.30.

28.8 Notes

The basic paper on the Snell envelope is Snell (1952). A standard reference is Shiryaev (2008), and the monograph Peskir and Shiryaev (2006) is an almost encyclopedic text on optimal stopping theory with finance applications. In Karatzas and Shreve (1998) there is a detailed discussion of American options, and Appendix D contains an extremely well written account of the Snell theory for continuous time processes. A precise and very readable introduction to optimal stopping problems can be found in Øksendal (1998).

PART V

INCOMPLETE MARKETS

29

INCOMPLETE MARKETS

29.1 Introduction

In the following chapters we will investigate some aspects of derivative pricing in incomplete markets. A market can however be incomplete in many different ways, and below is a short list:

1. There are more random sources than there are risky underlying assets.
2. There are constraints on admissible portfolios, like, for example, a short selling constraint.
3. The underlying object is not traded, like in the case of weather derivatives.
4. The underlying is traded but the market is not a liquid one, as is the case for many commodity markets.
5. The underlying is traded but portfolios can not easily and/or without large costs be carried forward in time. This is, for example, the case for electricity derivatives.

In the following chapters we will focus on pricing rather than hedging, the main reason being that the pricing theory is a bit more accessible than the hedging theory, and there seems to be more concrete results for pricing than for hedging. See the notes for some references to hedging theory for incomplete markets.

We will mostly focus on item 1 above, i.e. the case when we have more random sources than traded assets. We will work within the general framework of Chapter 14 so our most general model is given by (14.3)–(14.4) which we recall as

$$dS_t = D(S_t)\,\mu_t dt + D(S_t)\,\sigma_t dW_t, \tag{29.1}$$

$$dB_t = r_t B_t dt, \tag{29.2}$$

where S is n-dimensional whereas W is N-dimensional. As usual we look for a martingale measure Q so we perform a Girsanov transformation with likelihood dynamics

$$dL_t = L_t \varphi_t^* dW_t.$$

We then define Q by $dQ = L_t dP$ on $\mathcal{F}_t$, and the Girsanov Theorem then says that we can write

$$dW_t = \varphi_t dt + dW_t^Q,$$

where W^Q is Q-Wiener. The Q dynamics of S are thus given by

$$dS_t = S_t \left\{\mu_t + \sigma_t \varphi_t\right\} dt + \sigma_t dW_t^Q$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

The condition for Q to be a martingale measure is then that the "martingale equation"

$$\mu_t + \sigma_t \varphi_t = \mathbf{r}_t \tag{29.3}$$

has a solution φ, and in order to guarantee this we make the following assumption.

Assumption 29.1.1 *We assume that the model is generically arbitrage free, i.e. that σ_t, viewed as a linear mapping $\sigma_t : R^N \to R^n$, is surjective P-a.s. for all t. We also assume that all relevant likelihood processes L are true martingales (as opposed to being merely local martingales).*

The assumption that σ_t is surjective implies that $n \leq N$. In this chapter, the interesting case is of course when $n < N$. This implies that the martingale equation (29.3) will have infinitely many solutions, implying that the model is incomplete.

In order to obtain more concrete results, we will now introduce two commonly used factor models, which we sometimes will use to illustrate the theory.

29.2 A Markov Factor Model

A commonly used model is the so-called Markovian factor model. In order to construct it we need the following assumptions and notation:

- There exists an underlying non-financial k-dimensional factor process Y, with dynamics
$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t$$
where μ_Y and σ_Y are vector and matrix functions of appropriate dimensions.
- The S-dynamics are of the form
$$dS_t = D(S_t)\mu(Y_t)dt + D(S_t)\sigma(Y_t)dW_t,$$
where (with a slight abuse of earlier notation) μ and σ are vector and matrix functions of appropriate dimensions.
- The $n \times N$ volatility matrix $\sigma(y)$ is surjective for all $y \in R^k$.

Definition 29.1 *With assumptions as above, the* **Markov factor model** *is defined as*

$$dS_t = D(S_t)\mu(Y_t)dt + D(S_t)\sigma(Y_t)dW_t, \tag{29.4}$$

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t, \tag{29.5}$$

$$dB_t = rB_t dt, \tag{29.6}$$

where, for simplicity, we assume that r is constant.

The process Y represents some underlying non-financial factors, and we see that Y is allowed to modulate the drift and diffusion of the price process S. Given our assumptions, the process Y is an autonomous Markov process, and in a model of this sort it is important that we assume that the vector process Y does **not** contain any **price process**. Any price component of Y should instead be included in S.

29.3 The Independent Factor Markov Model

The independent factor Markov model is a special case of the Markov factor model above, but it has more structure. We keep the assumptions and notation from Section 29.2, but we also need some more assumptions.

Assumption 29.3.1 *We assume the following:*

- *We can split the Wiener process W as*

$$W = \begin{bmatrix} W^S \\ W^Y \end{bmatrix}$$

where W^S is n-dimensional, and W^Y is m-dimensional, so $n + m = N$.

- *The S-dynamics are of the form*

$$dS_t = D(S_t)\mu_S(Y_t)dt + D(S_t)\sigma_S(Y_t)dW_t^S,$$

where μ and σ_S are vector and matrix functions of appropriate dimensions.

- *The Y-dynamics are of the form*

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t^Y,$$

where μ_Y and σ_Y are vector and matrix functions of appropriate dimensions.

- *The $n \times n$ volatility matrix $\sigma_S(y)$ is invertible for all $y \in R^k$.*

We can now define our model.

Definition 29.2 *With assumptions as above, the* **independent factor model** *is defined as*

$$dS_t = D(S_t)\mu(Y_t)dt + D(S_t)\sigma_S(Y_t)dW_t^S, \tag{29.7}$$

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t^Y, \tag{29.8}$$

$$dB_t = rB_t dt, \tag{29.9}$$

where, for simplicity, we assume that r is constant.

In order to connect to the notation of the Markov factor model (29.4)–(29.6) we have the following relations, with the Markov factor notation on the left-hand side of the equality sign:

$$\mu(y) = \mu(y), \quad \sigma(y) = [\sigma_S(y), 0], \quad \mu_Y(y) = \mu_Y(y), \quad \sigma_Y(y) = [0, \sigma_Y(y)].$$

Under our assumptions we see that the factor process Y is an autonomous Markov process which is independent of the driving Wiener process W^S.

The assumption that σ^S is invertible roughly says that the model is "complete, conditional on Y", i.e. that the incompleteness in the model stems from the factor process Y. Without Y, the model would be complete.

Our independent factor model above is purely Wiener driven, but in a more general setting Y could be an arbitrary Markov process (typically independent of W^S). A frequently studied example is when Y is a finite state continuous time Markov chain.

29.3.1 *Absence of Arbitrage*

The Markov factor model (29.4)–(29.6) and the independent factor model (29.7)–(29.9) are both special cases of the general model in Chapter 14. Thus, since we have assumed that $\sigma(y)$ is surjective and that $\sigma_S(y)$ is invertible for all $y \in R^k$, we see that both models are free of arbitrage. Given a solution φ to the martingale equation, the Q-dynamics will have the form

$$dS_t = D(S_t)\mathbf{r}dt + D(S_t)\sigma(Y_t)dW_t^Q, \tag{29.10}$$

$$dY_t = \{\mu_Y(Y_t) + \sigma(Y_t)\varphi_t\}\, dt + \sigma_Y(Y_t)dW_t^Q. \tag{29.11}$$

Note, however, that there is no guarantee that the Girsanov kernel φ will be Markovian, i.e. of the form $\varphi_t = \varphi(t, S_t, Y_t)$, so the pair (S, Y) is not necessarily Markovian under Q.

29.3.2 *Incompleteness*

Starting with the Markov factor model, the martingale equation (29.3) has the form

$$\sigma(Y_t)\varphi_t = \mathbf{r} - \mu(Y_t). \tag{29.12}$$

Since $\sigma(y)$ is surjective for all $y \in R^k$ the equation will possess a solution, implying absence of arbitrage, but in the generic case when $n < N$ the solution is not unique, implying that the model is incomplete.

In order to have a better understanding of the incompleteness we now move to the independent factor model. The assumed splitting of W into W^S and W^Y implies that we can split φ as

$$\varphi = \begin{bmatrix} \varphi^S \\ \varphi^Y \end{bmatrix}$$

and this implies that the L dynamics are of the form

$$dL_t = L_t\varphi_t^S dW_t^S + L_t\varphi_t^Y dW_t^Y.$$

The Girsanov Theorem then allows us to write

$$d\begin{bmatrix} W_t^S \\ W_t^Y \end{bmatrix} = \begin{bmatrix} \varphi_t^S \\ \varphi_t^Y \end{bmatrix} dt + d\begin{bmatrix} W_t^{SQ} \\ W_t^{YQ} \end{bmatrix}$$

where W^{SQ} and W^{YQ} are Q-Wiener, and this implies that the martingale equation (29.12) takes the form

$$\mu(Y_t) + \sigma_S(Y_t)\varphi_t^S = \mathbf{r}, \tag{29.13}$$

involving only the process φ^S. Since σ^S is invertible this implies that φ^S is uniquely determined as

$$\varphi_t^S = \sigma^{-1}(Y_t)\left[\mathbf{r} - \mu(Y_t)\right]. \tag{29.14}$$

Recall, however, that the full Girsanov kernel φ is given by

$$\varphi = \begin{bmatrix} \varphi^S \\ \varphi^Y \end{bmatrix},$$

so we have in fact only determined the component φ^S. The Girsanov kernel component φ^Y is, on the other hand, without constraints, so we have proved the following result.

Proposition 29.3 *For the independent factor model above, there is a unique φ^S given by (29.14) generating a martingale measure. The process φ^Y is, however, completely undetermined. In other words,* **every** *choice of φ^Y will generate a martingale measure. There will thus exist an infinite number of martingale measures generated by Girsanov kernels of the form $\left[\varphi^S, \varphi^Y\right]$, implying that the market is (seriously) incomplete.*

29.4 Methods to Handle Market Incompleteness

A we saw above, our market is generically incomplete, so the problem is what to do in this situation. Luckily for us, there are in fact a number of techniques and arguments available which allows us to price derivatives and manage portfolios in an incomplete market. Below is a partial list:

1. Instead of looking at all possible Girsanov transformations, we restrict ourselves to a smaller family of Girsanov transformations and hope that this will give us a unique Q. This is done in Section 30.1 in the context of generalized Esscher transformations.
2. We introduce a (generalized) “distance” $f(P,Q)$ between probability measures, and then choose the equivalent martingale measure Q which minimizes $f(P,Q)$. This is done, to some extent in Section 30.2, and in Chapter 31 the topic is investigated in more detail in terms of the concept of f-divergence.
3. In Chapter 32 we outline a duality theory for portfolio optimization in incomplete markets.
4. Inspired by microeconomic theory we can try to price derivatives using utility indifference techniques. This is done in Chapter 33.
5. We may also completely give up the project of finding a unique martingale pricing measure. Instead we try to find “reasonable” (in some sense)

arbitrage free *bounds* for derivative asset prices. This program is outlined in Chapter 34.

6. We can embed the market model within a full-fledged dynamic equilibrium model. The market equilibrium will then produce a unique martingale measure. See Chapters 35–38.

29.5 Notes

For derivatives on weather, commodities, and energy, see, for example, Geman (1999), Geman (2005), Clewlow and Strickland (2000). For superreplication of contingent claims in incomplete markets, and optimal trading and hedging under portfolio constraints see Karatzas and Shreve (1998), and also Cvitanić (1997).

30

THE ESSCHER TRANSFORM AND THE MINIMAL MARTINGALE MEASURE

In this chapter and Chapter 31 we will present some of the methods that have been used in the literature in order to find a unique risk neutral martingale measure in an incomplete market model.

We will mainly use the general model defined by (29.1)–(29.2), but we will also specialize to the factor Markov model (29.4)–(29.6), as well as to the independent factor Markov model (29.7)–(29.9).

30.1 The Esscher Transform

In this section we will try to determine a unique equivalent martingale measure by restricting the admissible Girsanov transformation to a subclass of transformations. We will do this by using a generalization of the so-called *Esscher transform*, which is a particular type of absolutely continuous measure transformation.

30.1.1 *The Standard Esscher Transform*

We start by describing the standard Esscher transform. This transform has a long history in insurance mathematics and, in the present context, the general picture is that we consider a filtered space $(\Omega, \mathcal{F}, P, \mathbf{F})$ carrying a random process X.

Assumption 30.1.1 *We assume that $X_0 = 0$, and that X has independent stationary increments. We also assume that X has exponential moments of all orders.*

Given this assumption we define the mapping $h : R_+ \times R \to R$ by

$$h(t,a) = E\left[e^{aX_t}\right]. \tag{30.1}$$

From the independent stationary increment assumption it follows easily that

$$h(t+s,a) = h(t,a) \cdot h(s,a),$$

so we have in fact

$$h(t,a) = h^t(a) \tag{30.2}$$

where $h(a) = h(1,a)$. From this we see (why?) that if we define the process L^a by

$$L_t^a = \frac{e^{aX_t}}{h^t(a)} \tag{30.3}$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

then L^a is a positive $(P, \mathbf{F})$-martingale with $L_0^a = 1$. This allows us to define the Esscher transform as follows.

Definition 30.1 *For any real number a, the probability measure Q^a is defined by*

$$L_t^a = \frac{dQ^a}{dP}, \quad \text{on } \mathcal{F}_t. \tag{30.4}$$

The measure change from P to Q^a is referred to as an **Esscher transform**.

The main use of the standard Esscher transform in financial economics has been in the context of asset prices driven by a Levy process. Since Levy processes are outside the scope of this book, we exemplify by a brief study of the Esscher transform in the (rather trivial) context of a standard Black–Scholes model of the form

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

We already know how to handle this model, but let us forget about that and look for an Esscher transform. The obvious candidate for the X process is the return process R defined by $dR_t = S_t^{-1} dS_t$, giving us X as

$$dX_t = \mu dt + \sigma dW_t.$$

This is obviously a process with stationary independent increments, and a simple calculation gives us

$$h(t, a) = e^{(a\mu + \frac{1}{2}a^2\sigma^2)t}.$$

We thus have

$$L_t^a = \frac{e^{a\mu t + a\sigma W_t}}{e^{(a\mu + \frac{1}{2}a^2\sigma^2)t}} = e^{a\sigma W_t - \frac{a^2\sigma^2}{2}t},$$

giving us the L^a dynamics

$$dL_t^a = L_t^a a\sigma dW_t. \tag{30.5}$$

We recognize this as a standard Girsanov transformation with the slight reparameterization $\varphi = a\sigma$ of the Girsanov kernel, and we can then look for a martingale measure. We will of course only recapture the usual risk neutral measure Q by setting $a = (r - \mu)/\sigma^2$, so in this simple case the Esscher transform does not add anything of interest.

30.1.2 *The Generalized Esscher Transform*

We now go back to our general model from (29.1)–(29.2) which we recall as

$$dS_t = D(S_t)\mu_t dt + D(S_t)\sigma_t dW_t, \tag{30.6}$$

$$dB_t = r_t B_t dt, \tag{30.7}$$

where S is n-dimensional whereas W is N-dimensional, and $n \leq N$. We also recall that, by assumption, the matrix σ_t is surjective, and we now try to apply something like an Esscher transform to this model. It is then clear that, apart from the process W, there is no obvious candidate for the X process of the standard Esscher transform. We may of course once again identify the X-process by the (multidimensional) return process R, defined by

$$R_t = D^{-1}(S_t)dS_t$$

giving us

$$dX_t = \mu_t dt + \sigma_t dW_t, \tag{30.8}$$

but this process does not have independent increments so there is no obvious way to generalize (30.3). We can, however, quite easily generalize formula (30.5), and this leads us to the following definition.

Definition 30.2 *For the process X defined by (30.8), and for any adapted n-dimensional column vector process ψ, we define the* **generalized Esscher transform** *by the likelihood dynamics*

$$dL_t^\psi = L_t^\psi \psi_t^* \sigma_t dW_t. \tag{30.9}$$

We will refer to the process ψ as the **Esscher kernel***, and henceforth refer to the generalized Esscher transform as "the Esscher transform".*

We see that the Esscher transform above corresponds to a reparameterized Girsanov transform with a Girsanov kernel φ of the form

$$\varphi = \sigma_t^* \psi_t. \tag{30.10}$$

Since the general Girsanov kernel φ is N-dimensional, whereas the Esscher kernel ψ is n-dimensional, and we are interested only in the case $n < N$, we see that the Esscher transformation amounts to a substantial restriction of the class of Girsanov transformations. The Girsanov Theorem now gives us

$$dW_t = \sigma_t^* \psi_t dt + dW_t^Q$$

and, in terms of the Esscher transform, the martingale equation

$$\sigma_t \varphi_t = \mathbf{r}_t - \mu_t$$

will take the form

$$\sigma_t \sigma_t^* \psi_t = \mathbf{r}_t - \mu_t. \tag{30.11}$$

At this point we recall the following easy result from linear algebra.

Proposition 30.3 *Assume that the $n \times N$ matrix A is surjective as a mapping $A : R^N \to R^n$. Then A^* is injective and the $n \times n$ matrix AA^* is invertible.*

Proof The proof is left to the reader. It can be found in most textbooks on linear algebra. $\square$

Applying this result to the volatility matrix σ_t gives us the following result.

Theorem 30.4 *There is a unique solution ψ^E to the martingale equation (30.11), given by*

$$\psi_t^E = \left[\sigma_t \sigma_t^*\right]^{-1} (\mathbf{r}_t - \mu_t). \tag{30.12}$$

and the corresponding Girsanov kernel is

$$\varphi_t^E = \sigma_t^* \left[\sigma_t \sigma_t^*\right]^{-1} (\mathbf{r}_t - \mu_t). \tag{30.13}$$

Thus there exists a unique martingale measure generated by an Esscher transform. This unique martingale measure is referred to as the **Esscher measure***, and we denote it by Q^E.*

In the next sections we will study the effect of an Esscher transform on the factor Markov models from Chapter 29.

30.1.3 *The Markov Factor Model*

For the Markov factor model (29.4)–(29.6), which we recall as

$$\begin{aligned}
dS_t &= D(S_t)\mu(Y_t)dt + D(S_t)\sigma(Y_t)dW_t, \\
dY_t &= \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t, \\
dB_t &= rB_t dt,
\end{aligned}$$

Theorem 30.4 translates as follows.

Proposition 30.5 *There is a unique solution ψ^E to the martingale equation (30.11), given by*

$$\psi_t^E = \left[\sigma(Y_t)\sigma^*(Y_t)\right]^{-1} (\mathbf{r} - \mu(Y_t)) \tag{30.14}$$

and the corresponding Girsanov kernel is

$$\varphi_t^E = \sigma^*(Y_t) \left[\sigma(Y_t)\sigma^*(Y_t)\right]^{-1} (\mathbf{r} - \mu(Y_t)). \tag{30.15}$$

Since φ is of the form $\varphi_t = \varphi(Y_t)$ we have the following important implication.

Corollary 30.6 *The process (S, Y) remains a Markov process under the Esscher measure Q^E.*

30.1.4 *The Independent Factor Markov Model*

We now specialize to the model (29.7)–(29.9):

$$\begin{aligned}
dS_t &= D(S_t)\mu(Y_t)dt + D(S_t)\sigma_S(Y_t)dW_t^S, \\
dY_t &= \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t^Y, \\
dB_t &= rB_t dt,
\end{aligned}$$

and we recall the relations

$$W = \begin{bmatrix} W^S \\ W^Y \end{bmatrix}, \quad \mu_t = \mu(Y_t), \quad \sigma_t = [\sigma_S(Y_t), 0].$$

We then have

$$\sigma_t^* = \begin{bmatrix} \sigma_S^*(Y_t) \\ 0 \end{bmatrix}.$$

Using Theorem 30.4 we obtain $\psi_t^E = [\sigma_S(Y_t)\sigma_S^*(Y_t)]^{-1}\{\mathbf{r} - \mu(Y_t)\}$. Recalling that $\sigma_S(y)$ is assumed to be invertible, we get the following result.

Proposition 30.7 *For the independent factor Markov model, the Esscher kernel generating Q^E is given by*

$$\psi_t^E = [\sigma_S^*(Y_t)]^{-1}\sigma_S^{-1}(Y_t)\{\mathbf{r} - \mu(Y_t)\}, \tag{30.16}$$

and the corresponding Girsanov kernel is

$$\varphi_t^E = \begin{bmatrix} \sigma_S^{-1}(Y_t)\{\mathbf{r} - \mu(Y_t)\} \\ 0 \end{bmatrix}. \tag{30.17}$$

Applying the Girsanov Theorem to this result gives us the following.

Proposition 30.8 *For the independent factor Markov model we have*

$$d\begin{bmatrix} W_t^S \\ W_t^Y \end{bmatrix} = \begin{bmatrix} \sigma_S^{-1}(Y_t)\{\mathbf{r} - \mu(Y_t)\}(\mathbf{r} - \mu(Y_t)) \\ 0 \end{bmatrix} dt + d\begin{bmatrix} W_t^{SE} \\ W_t^{YE} \end{bmatrix} \tag{30.18}$$

where W^{SE} and W^{YE} are Q^E-Wiener. In particular, this implies that we have $W^{YE} = W^Y$.

We see that we have changed the measure for the W^S part of the Wiener process, whereas the distribution of W^Y, and thus also the distribution of Y, remains unchanged. This indicates that the Esscher transform, at least for this case, has some minimality property, and we will pursue this topic in Section 30.2.

30.2 The Minimal Martingale Measure

We now go back to the general model (30.6)–(30.7) and, as always, we assume that we have generic absence of arbitrage, i.e. that σ_t is surjective, so the martingale measure equation

$$\sigma_t \varphi_t = \mathbf{r}_t - \mu_t \tag{30.19}$$

always possesses a (typically not unique) solution. In the context of an incomplete market we then again face the problem of choosing **one** particular measure among the infinitely many martingale measures. The following (loose) idea then seems natural.

Idea 30.2.1 *Find the martingale measure Q which, in some sense, is closest to the objective measure P.*

30.2.1 *Definition and Existence*

The problem is now how to define what we mean by the expression "Q is close to P". This problem will be studied systematically in Chapter 31 but a rather natural way is to study the Euclidian norm of the Girsanov kernel taking us from P to Q. As a first attempt, it thus seems reasonable to look for a solution φ to (30.19) which minimizes the Euclidian norm $\|\varphi_t\|$ at every t. We formalize this into a definition.

Definition 30.9 *Consider the following optimization problem for every fixed t:*

$$\begin{aligned} &\underset{\varphi}{\text{minimize}} \quad \|\varphi_t\|^2 \\ &s.t. \\ &\sigma_t\varphi_t = \mathbf{r}_t - \mu_t. \end{aligned}$$

The measure Q^M generated by the Girsanov kernel φ^M is referred to as the **minimal martingale measure**. *The corresponding Q^M-Wiener process is denoted by W^M, and is defined as usual by*

$$dW_t = \varphi_t^M dt + dW_t^M.$$

It is perhaps not clear that the minimal martingale measure exists, but we recall the following result from elementary linear algebra.

Proposition 30.10 *Assume that the $n \times k$ matrix A is surjective as a mapping $A : R^N \to R^n$. For any $y \in R^n$, consider the optimization problem*

$$\begin{aligned} &\min_{x \in R^k} \quad \|x\|^2, \\ &s.t. \quad Ax = y. \end{aligned}$$

Then the following hold:

- *$A^\star$ is injective and $AA^\star$ is invertible.*
- *The unique optimal solution of the minimum norm problem is given by*

$$\hat{x} = A^\star \left(AA^\star\right)^{-1} y. \tag{30.20}$$

Proof Left to the reader as an exercise. □

From this we have the following obvious result.

Proposition 30.11 *Assume again that the volatility matrix σ_t is surjective P-a.s. for all t. Then the minimal martingale measure Q^M exists, and it is generated by the Girsanov kernel φ^M defined by*

$$\varphi^M = \sigma_t^* \left[\sigma_t\sigma_t^*\right]^{-1} \left[\mathbf{r} - \mu_t\right]. \tag{30.21}$$

The corresponding market price of risk is denoted by λ_t^M so we have $\lambda_t^M = -\varphi_t^M$.

30.2.2 *Basic Properties of Q^M*

Comparing Proposition 30.11 to Theorem 30.4 we have the following result.

Proposition 30.12 *For the model (30.6)–(30.7), the minimal martingale measure Q^M coincides with the Esscher martingale measure Q^E.*

We commented earlier, in connection with Proposition 30.1.4, that the Esscher measure had some sort of minimality property, at least in the case of the independent factor model. The minimality property of Q^M, and hence of Q^E, does in fact go much deeper then just minimizing the norm of φ. To see this we need a small definition.

Definition 30.13 *Two scalar martingales X and Y are said to be* **strongly orthogonal** *if the process $M_t = X_t \cdot Y_t$ is a martingale.*

We now have the following result.

Theorem 30.14 *Consider the model (30.6)–(30.7), and define the n-dimensional vector P-martingale X by*

$$dX_t = \sigma_t dW_t.$$

Define furthermore the Q^M-martingale X^M by

$$dX_t^M = \sigma_t dW_t^M.$$

Now assume that Y is a scalar P-martingale which is strongly orthogonal to every component of X under P. Then the following hold:

- *The process Y will still be a martingale under Q^M.*
- *Under the measure Q^M the process Y will be strongly orthogonal to every component of X^M.*

Proof The proof is left to the reader as an exercise. □

This result, which holds in much more general situations, roughly says that the minimal martingale measure turns all normalized asset prices into martingales, while keeping as much as possible of the geometrical martingale structure intact.

The minimal martingale measure is in fact a very important object in financial economics. It turns up as a natural object of study in many applications and it has been the subject of intense research. See the Notes for more information.

30.2.3 *Economic Interpretation of Q^M*

As we have seen above, the minimal martingale measure Q^M has some very nice mathematical properties, so Q^M is a rather natural choice for pricing derivatives in an incomplete market. However, if we choose one particular martingale measure then we have indirectly made some sort of assumption concerning the aggregate risk aversion in the market. We thus need to clarify what the choice of Q^M amounts to in economic terms.

This question is resolved by recalling the Hansen–Jagannathan bounds from Proposition 14.9: We have, for every admissible Girsanov kernel, the inequality

$$|SR| = \frac{|\mu_t^p - r_t|}{\|\sigma_t^p\|} \leq \|\varphi_t\|, \tag{30.22}$$

which says that the Sharpe ratio of every asset, underlying or derivative, is bounded by the norm of φ. The obvious economic interpretation of the Sharpe ratio is that it is a measure of the aggregate risk aversion in the market, so we have the following result.

Proposition 30.15 *The price system defined by the minimal martingale measure Q^M has the lowest possible uniform Sharpe ratio that is consistent with the exogenously specified underlying assets.*

30.3 Notes

The Esscher transformation was introduced in Esscher (1932), and further developed in Gerber and Shiu (1994), Kallsen and Shiryayev (2002), and in many other papers. The minimal martingale measure turns up as a natural object in many areas of finance, such as quadratic hedging theory and the theory of local risk minimization. It is a canonical object of study in mathematical finance and it has been the object of intensive research. See Föllmer and Sondermann (1986), Schweizer (1991), and Schweizer (2001).

31

MINIMIZING f-DIVERGENCE

We now return to the problem of defining a concept of "closeness" between the two measures P and Q. This will be done using the concept of f-divergence.

31.1 Definition and Basic Properties

One possibility to define how far apart two measures are, would of course be to define a metric on the convex space of probability measures, but it turns out that another construction is very useful in statistics, finance, physics, and in many other applications.

Definition 31.1 *Let $f : R_+ \to R$ be a strictly convex function with $f(1) = 0$. Let furthermore P and Q be two measures such that $Q << P$ on $(\Omega, \mathcal{F})$, and let L denote the Radon–Nikodym derivative*

$$L = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}.$$

*The f-**divergence** $f(Q,P)$ between P and Q is defined by*

$$f(Q,P) = E^P[f(L)] \tag{31.1}$$

if the expectation above is well defined, and $f(Q,P) = +\infty$ otherwise.

We start by stating two simple properties of the f-divergence.

Proposition 31.2 *The following hold.*

1. *We have $f(Q,P) \geq 0$ for all $Q << P$.*
2. *$f(Q,P) = 0$ if and only if $Q = P$.*

Proof By the Jensen inequality we have

$$f(Q,P) = E^P[f(L)] \geq f\left(E^P[L]\right) = f(1) = 0,$$

which proves 1. Since f is strictly convex, the Jensen inequality is an equality if and only if L is deterministic, and this happens if and only if $L = 1$, which is equivalent to saying that $Q = P$. □

It follows from this result that it is natural to interpret $f(Q,P)$ as a measure of the "distance" between Q and P. Note, however, that in general $f(Q,P)$ is not a metric.

Definition 31.3 *Let $\mathcal{K}$ be a convex set of probability measures, dominated by P, and consider the problem of minimizing $f(Q,P)$, over $Q \in \mathcal{K}$. If there exists an optimal Q^*, i.e.*

$$f(Q^*,P) = \inf_{Q \in \mathcal{K}} F(Q,P),$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

then we say that Q^* is the **f-projection** *of P onto $\mathcal{K}$, and we denote it by $f(\mathcal{K}, P)$.*

The interpretation is of course that Q^* is the measure $Q \in \mathcal{K}$ which is closest to P in the sense of f-divergence.

Below is a list of some of the most commonly used f-divergences and their trade names.

$f(x) = x\ln(x)$	Relative entropy
$f(x) = -\ln(x)$	Reverse relative entropy
$f(x) = x^2 - 1$	Variance
$f(x) = \frac{1}{2}\lvert x-1\rvert$	Total variation
$f(x) = (\sqrt{x}-1)^2$	Hellinger distance.

The way the f-divergence is typically used in finance is as follows, where we are given a model of an incomplete market.

1. Set $\mathcal{K} = \mathcal{M}$, where $\mathcal{M}$ is the class of equivalent martingale measures.
2. Choose a concrete f-divergence from the list above.
3. Minimize the divergence $f(Q, P)$ over $Q \in \mathcal{M}$, thus projecting P on $\mathcal{M}$.
4. Use the f-projection Q^* to price all derivatives in the market.

Depending on the choice of f we then obtain such objects as the minimal entropy measure, the minimal reverse entropy measure, the minimal variance measure, etc.

It is not an easy task to determine the f-projection for a concrete model, the only exception being the reverse entropy $f(x) = -\ln(x)$, which is treated in Section 31.2 below. In the case when the return process of the underlying stock is a Levy process, it is often possible to obtain explicit formulas for various minimal measures, but apart from such cases it is typically very hard to give an explicit characterization. See the Section 31.3 below for an example. These problems have been intensively studied in the literature (see the Notes), and the most frequently used divergence in finance seems to be the relative entropy.

As we will see below, the purely mathematical theory of f-divergences is very interesting, but one may of course, for good reasons, ask whether there is also some economic reason to study divergence. The answer is yes: There is a deep duality between minimizing f-divergence and maximizing derivatives and portfolio value. See Section 31.4 and Chapter 32.

31.2 Minimal Reverse Entropy

In this section we consider the general model

$$dS_t = D(S_t)\mu_t dt + D(S_t)\sigma_t dW_t, \tag{31.2}$$

$$dB_t = r_t B_t dt, \tag{31.3}$$

from Chapter 14, and our job is to determine the minimal reverse entropy measure, i.e. to determine the f projection of P onto $\mathcal{M}$ for the case $f(x) = -\ln(x)$. The mathematical problem is then as follows.

$$\underset{\varphi}{\text{maximize}} \quad E^P[\ln(L_T)] \tag{31.4}$$

subject to the usual martingale measure constraint

$$\mu_t + \sigma_t \varphi_t = r_t, \quad 0 \le t \le T. \tag{31.5}$$

Since

$$L_T = e^{\int_0^T \varphi_t dW_t - \frac{1}{2}\int_0^T \|\varphi_t\|^2 dt}$$

we have

$$E^P[\ln(L_T)] = E^P\left[-\frac{1}{2}\int_0^T \|\varphi_t\|^2 dt\right]$$

so we want to minimize

$$E^P\left[\int_0^T \|\varphi_t\|^2 dt\right] \tag{31.6}$$

subject to the martingale constraint (31.5) above.

This, however, is a trivial problem. We simply minimize $\|\varphi_t\|^2$ for each separate (t,ω), subject to the martingale constraint. The resulting measure is immediately recognized as the minimal martingale measure from Definition 30.9.

Proposition 31.4 *The minimal reverse entropy measure Q^{RE} coincides with the minimal martingale measure Q^M.*

31.3 Minimal Entropy in a Factor Model

In order to illustrate the problems of analytically determining the f-projection for a non-trivial f, we will now try to determine the minimal entropy measure, i.e. $f(x) = x\ln(x)$, for a concrete model. The model is a Markovian factor model defined as follows, where S and Y are scalar, and where W^1 and W^2 are independent scalar Wiener processes.

$$dS_t = \mu(Y_t)S_t dt + S_t\sigma(Y_t)dW_t^1, \tag{31.7}$$

$$dY_t = \alpha(Y_t)dt + \beta(Y_t)dW_t^1 + \delta(Y_t)dW_t^2. \tag{31.8}$$

For notational simplicity we assume that the short rate is equal to zero (otherwise we normalize). Denoting the likelihood process by X, the problem is to

$$\underset{\varphi_1,\varphi_2}{\text{minimize}} \quad E^p[X_T \ln(X_T)] \tag{31.9}$$

given the state dynamics

$$dX_t = X_t\left\{\varphi_{1t}dW_t^1 + \varphi_{2t}dW_t^2\right\}, \tag{31.10}$$

$$dY_t = \alpha(Y_t)dt + \beta(Y_t)dW_t^1 + \delta(Y_t)dW_t^2, \tag{31.11}$$

and the martingale measure constraint

$$\mu(Y_t) + \varphi_{1t}\sigma(Y_t) = 0. \tag{31.12}$$

Remark 31.3.1 *The reason why we denote the likelihood process by X rather than by the usual L is purely aesthetic. The likelihood X acts as a state variable, and the author feels more comfortable with x as a variable than he does with l.*

We note that price process S does not turn up directly in the equations (31.7)–(31.12) above – it only appears indirectly through the martingale condition (31.12). The implication of this is that S does not act as a state variable in the optimization problem.

This minimal entropy problem was first analyzed in Benth and Karlsen (2005), where the authors used duality methods involving utility indifference pricing in order to derive a number of results, including a semi-analytical solution to the minimal entropy problem. We will use dynamic programming directly on the original problem to re-derive the semi-analytical solution of Benth and Karlsen (2005).

The HJB equation takes the following form where lower case index denotes partial derivative, where we denote φ_2 by u, and where we have solved (31.12) as $\varphi_1 = -\mu/\sigma$.

$$\begin{cases} V_t(t,x,y) + \inf_u \mathbf{A}^u V(t,x,y) &= 0 \\ V(T,x,y) &= x\ln(x). \end{cases} \tag{31.13}$$

The infinitesimal operator $\mathbf{A}^u$ is given by

$$\mathbf{A}^u V = \alpha(y)V_y + \frac{1}{2}x^2\left\{\frac{\mu^2(y)}{\sigma^2(y)} + u^2\right\}V_{xx} + \frac{1}{2}\left\{\beta^2(y)+\delta^2(y)\right\}V_{yy} + x\left\{-\beta(y)\frac{\mu(y)}{\sigma(y)} + u\delta(y)\right\}V_{xy}.$$

Assuming provisionally that $V_{xx} > 0$, the optimal u is easily obtained as

$$\hat{u} = -\frac{\delta(y)V_{xy}}{xV_{xx}}. \tag{31.14}$$

At this point we need an Ansatz, and a natural (and perhaps somewhat optimistic) guess is that the optimal u does not depend on x. Given this guess it follows easily from (31.10) that, given $X_t = x$, we have

$$X_T = xe^{\int_t^T A(s,Y_s)ds + \int_t^T B(s,Y_s)dW_s^1 + \int_t^T C(s,Y_s)dW_s^2}$$

where the functions A, B, and C do not depend on x. From this it follows directly that we can write

$$E^P_{t,x,y}\left[X_T\ln(X_T)\right] = x\ln(x) + xg(t,y)$$

where g is some function to be determined. We thus make the Ansatz

$$V(t,x,y) = x\ln(x) + xg(t,y), \tag{31.15}$$

which gives us

$$V_t = xg_t, \quad V_x = 1 + \ln(x) + g, \quad V_{xx} = \frac{1}{x}, \quad V_y = xg_y, \quad V_{yy} = xg_{yy}, \quad V_{xy} = g_y.$$

Plugging this into (31.14) gives us

$$\widehat{u}(t,y) = -\delta(y)g_y(t,y),$$

and we note (with some satisfaction) that $\hat{u}$ does not depend on x. Plugging everything into the HJB equation gives us a PDE for g (after variable separation) and, denoting the likelihood process by the more standard L, we have the following result.

Proposition 31.5 *The minimal entropy measure Q^{ME} is generated by the Girsanov transformation*

$$dL_t = L_t \left\{ \varphi_{1t} dW_t^1 + \varphi_{2t} dW_t^2 \right\}$$

where

$$\varphi_{1t} = -\frac{\mu(Y_t)}{\sigma(Y_t)}, \quad \varphi_{2t} = -\delta(Y_t)g_y(t,Y_t), \tag{31.16}$$

where g solves the boundary value problem

$$\begin{cases} g_t(t,y) + F(y,g_y) + \dfrac{1}{2}\left\{\beta^2(y) + \delta^2(y)\right\} g_{yy}(t,y) + \dfrac{1}{2}\dfrac{\mu^2(y)}{\sigma^2(y)} = 0, \\ g(T,y) = 0, \end{cases} \tag{31.17}$$

and where

$$F(y,g_y) = \left\{ \alpha(y) - \beta(y)\frac{\mu(y)}{\sigma(y)} - \delta^2(y) \right\} g_y(t,y) + \frac{1}{2}\delta^2(y)g_y^2(t,y).$$

Remark 31.3.2 *In the proposition above we have made the ad hoc assumption that the PDE (31.17) admits a classical solution g such that the Ansatz (31.15) satisfies the conditions of the Verification Theorem. This has to be checked in each particular case.*

There is an interesting and easy corollary of this result.

Corollary 31.6 *Consider the following special case of the model (31.7)–(31.8)*

$$\begin{aligned} dS_t &= \mu S_t dt + S_t \sigma dW_t^1, \\ dY_t &= \alpha(Y_t)dt + \beta(Y_t)dW_t^1 + \delta(Y_t)dW_t^2, \end{aligned}$$

where μ and σ are constant. For this case the minimal entropy measure coincides with the minimal martingale measure, so we have

$$Q^{ME} = Q^M.$$

Proof It is easy to check that the solution of (31.17) has the form $g(t,y) = g(t)$, implying that $g_y = 0$. This implies, by (31.16) that $\varphi_2 = 0$, which identifies Q^{ME} as the minimal martingale measure. □

31.4 Duality

In this section we will briefly discuss the duality that exists between the problems of, on the one hand, minimizing f-divergence and, on the other hand, maximizing utility of financial derivatives. This duality will then be investigated in greater depth in Chapter 32.

To this end we consider an arbitrage free (typically incomplete) financial market living on the time interval $[0,T]$. We make no particular assumption about the structure of the market, but we assume the following.

Assumption 31.4.1

1. *We assume that the risk free short rate is zero, so $r_t = 0$. This is for simplicity of notation.*
2. *We consider a financial agent with zero subjective discount factor and a utility function U.*

We also need some assumptions concerning the utility function U. These will ensure that we will later have a well-behaved optimization problem without corner solutions.

Assumption 31.4.2 *We assume that the utility function U is strictly increasing, differentiable, strictly concave. We also assume that it satisfies*

$$\lim_{x \to +\infty} U(x) = 0.$$

Defining the effective domain $dom(U)$ by $dom(U) = \{x : U(x) > -\infty\}$ we consider two cases.

Case 1 *This is the case when $dom(U) = (0, \infty)$, and a typical example would be power utility. We then assume that*

$$\lim_{x \downarrow 0} U'(x) = +\infty.$$

Case 2 *This is the case when $dom(U) = (-\infty, \infty)$, and a typical example would be exponential utility. We then assume that*

$$\lim_{x \downarrow -\infty} U'(x) = +\infty.$$

Assumption 31.4.1 above implies that for a T-claim X, the utility at $t = 0$, for the agent is given by

$$E^P [U(X)].$$

Now, given an equivalent martingale measure Q, this measure will generate a price system Π^Q for all contingent claims (whether they can be replicated or not). Recalling that $r = 0$ the price system is of course given by the usual risk neutral formula

$$\Pi_t^Q[X] = E^Q [X | \mathcal{F}_t], \quad X \in \mathcal{F}_T.$$

We will now go on to study utility maximization, and in connection with this we will need some results concerning convex duality. These can be found in Appendix D.

31.4.1 *Utility Maximization of Financial Derivatives*

We now assume that the agent has initial wealth x, and we pose the following problem.

Problem 31.4.1 *Determine the T-claim X which maximizes utility, while still being affordable under the price system Q.*

Using risk neutral valuation, we can reformulate this into the following mathematical problem.

Problem 31.4.2

$$\underset{X \in \mathcal{F}_T}{\text{maximize}} \quad E^P[U(X)],$$

subject to the constraint

$$E^Q[X] \le x. \tag{31.18}$$

Defining the likelihood process L^Q by

$$L_T^Q = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_T, \tag{31.19}$$

and replacing the constraint (31.18) by

$$E^P\left[L_T^Q X\right] \le x, \tag{31.20}$$

we define the Lagrangian $L(X, y, Q)$ by

$$L(x, y, Q) = E^P[U(X)] - y\left(E^P\left[L_T^Q X\right] - x\right)$$

where y is the Lagrange multiplier. We can write this as

$$L(x, y, Q) = E^P\left[U(X) - yL_T^Q X\right] + yx \tag{31.21}$$

and this leads us to the following definitions.

Definition 31.7 *For every fixed martingale measure Q, we define the functions $G(y, Q)$ and $u(x, Q)$ as follows:*

1. *The function $G(y, Q)$ is defined by*

$$G(y, Q) = \sup_{X \in \mathcal{F}_T} L(X, y, Q). \tag{31.22}$$

2. *The indirect utility function $u(x, Q)$ is defined by*

$$u(x, Q) = \sup_{X \in \mathcal{F}_T} E^P[U(X)] \tag{31.23}$$

subject to

$$E^P\left[L_T^Q X\right] \leq x. \tag{31.24}$$

We can easily now prove the following result.

Proposition 31.8 *Denote by V the conjugate of U, i.e. $V = U^*$, so*

$$V(y) = \sup_{x \in R}\{U(x) - xy\}, \quad y > 0. \tag{31.25}$$

Then the following hold:

1. *We have the duality formula*

$$G(y,Q) = E^P\left[V(yL_T^Q)\right] + yx. \tag{31.26}$$

2. *The indirect utility function admits the representation*

$$u(x,Q) = \inf_{y>0} G(y,Q) \tag{31.27}$$

or, equivalently,

$$u(x,Q) = \inf_{y>0}\left\{E^P\left[V(yL_T^Q)\right] + yx\right\}. \tag{31.28}$$

3. *The optimal X in Problem 31.4.2 is given by*

$$\widehat{X}(x) = I(y_x^Q L_T^Q), \tag{31.29}$$

where y_x^Q is determined by

$$E^P\left[L_T^Q I(y_x^Q L_T^Q)\right] = x. \tag{31.30}$$

4. *We also have*

$$u(x,Q) = E^P\left[U(I(y_x^Q L_T^Q))\right]. \tag{31.31}$$

Proof We form the Lagrangian (31.21), i.e.

$$L(x,y,Q) = E^P\left[U(X) - yL_T^Q X\right] + yx.$$

We then recall the standard arguments from Chapter 27 to obtain

$$\begin{aligned} G(y,Q) &= \sup_X\left\{E^P\left[U(X) - yL_T^Q X\right]\right\} + yx \\ &= E^P\left[\sup_X\left\{U(X) - yL_T^Q X\right\}\right] + yx \\ &= E^P\left[V(yL_T^Q)\right] + yx, \end{aligned}$$

which proves (31.26). The optimality condition is

$$U'(X) = yL_T^Q,$$

giving us

$$\widehat{X}(x) = I\left(y_x^Q L_T^Q\right)$$

where the optimal Lagrange multiplier y_x^Q is determined by

$$E^P\left[L_T^Q I\left(y_x^Q L_T^Q\right)\right] = x.$$

This proves (31.29)–(31.31), and it remains to prove (31.27)–(31.28).

For every $X \in \mathcal{F}_T$ satisfying the budget constraint $E^P\left[L_T^Q X\right] \le x$, and for all $y > 0$, we have

$$\begin{aligned} E^P\left[U(X)\right] &\le E^P\left[U(X)\right] + y(x - E^P\left[L_T^Q X\right]) \\ &= E^P\left[U(X) - yL_T^Q X\right] + yx \\ &\le E^P\left[V(yL_T^Q)\right] + yx. \end{aligned} \tag{31.32}$$

We thus have

$$E^P\left[U(X)\right] \le u(x,Q) \le \inf_{y>0}\left\{E^P\left[V(yL_T^Q)\right] + yx\right\} \tag{31.33}$$

for every X with $E^P\left[L_T^Q X\right] \le x$, and for all $y > 0$. Choosing $X = \widehat{X} = I(yL_T^Q)$ will make the second inequality in (31.32) hold with equality, and choosing $y = y_x^Q$ such that $E^P\left[L_T^Q I(y_x^Q L_T^Q)\right] = x$ will make the first inequality in (31.32) hold with equality. We thus have

$$\begin{aligned} E^P\left[U(\widehat{X})\right] &\le u(x,Q) \le \inf_{y>0}\left\{E^P\left[V(yL_T^Q)\right] + yx\right\} \\ &\le \left\{E^P\left[V(y_x^Q L_T^Q)\right] + y_x^Q x\right\} \\ &= E^P\left[U(\widehat{X})\right]. \end{aligned}$$

□

31.4.2 *Minimax Measures*

In the previous section we have discussed utility maximization given a fixed martingale measure Q. If the market is complete, then Q is unique so $u(x,Q)$ is the maximal utility obtainable given initial wealth x. If, on the other hand, the market is incomplete, then we have many (typically infinitely many) martingale measures. It is then natural to study the worst possible case, so we make the following definition.

Definition 31.9 *Define the function $u(x)$ by*

$$u(x) = \inf_{Q \in \mathcal{M}} u(x,Q). \tag{31.34}$$

A martingale measure $\widehat{Q} = \widehat{Q}(x)$ is called a **minimax measure**, *given the initial wealth x, if it satisfies the relation*

$$u(x, \widehat{Q}) = u(x). \tag{31.35}$$

Note that $\widehat{Q}(x)$ will typically depend on x.

We can now connect minimax measures to f-divergences.

Proposition 31.10 *With $V = U^*$ as above, define the function V_y by*

$$V_y(z) = V(yz). \tag{31.36}$$

We then have

$$u(x) = \inf_{y>0} \{V_y(\mathcal{M}, P) + yx\}, \tag{31.37}$$

where $V_y(\mathcal{M}, P)$ is the V_y-projection of P onto $\mathcal{M}$.

Proof From (31.28) we obtain

$$\begin{aligned} u(x) &= \inf_{Q \in \mathcal{M}} \inf_{y>0} \left\{ E^P \left[V(yL_T^Q) \right] + yx \right\} = \inf_{y>0} \inf_{Q \in \mathcal{M}} \left\{ E^P \left[V(yL_T^Q) \right] + yx \right\} \\ &= \inf_{y>0} \left\{ \inf_{Q \in \mathcal{M}} E^P \left[V(yL_T^Q) \right] + yx \right\} = \inf_{y>0} \{V_y(\mathcal{M}, P) + yx\}. \end{aligned}$$

□

This duality can in fact be pushed further, and we have the following result.

Proposition 31.11 *Define the function $v(y)$ by*

$$v(y) = \inf_{Q \in \mathcal{M}} E^P \left[V(yL_T^Q) \right]. \tag{31.38}$$

We then have the duality

$$u = v^*, \qquad v = u^*, \tag{31.39}$$

i.e.

$$u(x) = \inf_{y>0} \{v(y) + yx\}. \tag{31.40}$$

Proof The proof follows immediately from the formulas in the proof of Proposition 31.10. □

The moral of all this, so far, is that the problem of maximizing utility in an incomplete market is intimately related to the problem of minimizing the f-divergence for $f = V_y = U_y^*$.

According to our definitions, $u(x)$ is the maximal utility obtained from affordable T-claims, given a worst case pricing scenario in terms of the minimal measure Q^*. A very natural question is then if the measure Q^* is also in some way connected to the problem of maximizing expected utility of (self-financing) portfolio value in an incomplete market. The answer to this question is essentially affirmative. Apart from some (very nontrivial) technical modifications, this will be demonstrated in Chapter 32, where we will outline a duality theory between optimal portfolios and dually optimal (minimal) measures.

31.4.3 *Log Utility*

We will now illustrate the ideas above with the simplest possible example—the case of log utility, i.e. when we have

$$U(x) = \ln(x).$$

We want to compute

$$V_y(\mathcal{M}, P) = \inf_{Q \in \mathcal{M}} E^P\left[u^*(yL_T^Q)\right]$$

and an easy computation shows that

$$V(y) = -\ln(y).$$

We thus want to maximize

$$E^P\left[\ln(yL_T^Q)\right] = \ln(y) + E^P\left[\ln(L_T^Q)\right]$$

over $Q \in \mathcal{M}$, so we simply want to maximize $E^P\left[\ln(L_T^Q)\right]$. This, however, is exactly the problem of computing the f-projection for reverse entropy. We have already done that in Section 31.2, so we can copy the result.

Proposition 31.12 *For the log utility function*

$$U(x) = \ln(x),$$

the minimax measure $\widehat{Q}(x)$ is independent of x, and we have

$$\widehat{Q}(x) = Q^M$$

where Q^M is the minimal martingale measure from Definition 30.9.

31.4.4 *Exponential Utility*

We now move to the second simplest example—the case of exponential utility, i.e. when we have

$$U(x) = -\frac{1}{\alpha}e^{-\alpha x}, \quad \alpha > 0.$$

This utility function is a very popular choice in applications. It has constant absolute risk aversion (CARA), which is rather unrealistic from an economic point of view. The CARA property is, however, also the main reason for its popularity—computations become very simple.

We want to compute

$$U_y^*(\mathcal{M}, P) = \inf_{Q \in \mathcal{M}} E^P\left[V(yL_T^Q)\right]$$

and an easy computation shows that

$$V(y) = \frac{1}{\alpha}\left\{y\ln(y) - y\right\}.$$

Recalling that $E^P\left[L_T^Q\right] = 1$, we obtain

$$\begin{aligned} E^P\left[V(yL_T^Q)\right] &= \frac{1}{\alpha}E^P\left[yL_T^Q\ln(yL_T^Q) - yL_T^Q\right] \\ &= \frac{1}{\alpha}E^P\left[yL_T^Q\ln(y) + yL_T^Q\ln(L_T^Q) - yL_T^Q\right] \\ &= \frac{1}{\alpha}\left\{y\ln(y) + yE^P\left[L_T^Q\ln(L_T^Q)\right] - y\right\}, \end{aligned}$$

so we have the optimization problem

$$\underset{Q\in\mathcal{M}}{\text{minimize}} \quad E^P\left[L_T^Q\ln(L_T^Q)\right].$$

This is exactly the problem of finding the minimal entropy measure, and we note with some relief that the initial wealth x does not enter into the optimization problem. This is due to the CARA property of exponential utility. Denoting the entropy divergence by H, i.e. $H(x) = x\ln(x)$ we thus have

$$U_y^*(\mathcal{M}, P) = \frac{1}{\alpha}\left\{y\ln(y) + yH(\mathcal{M}, P) - y\right\}$$

and we can now use (31.37) to compute $u(x)$ as

$$u(x) = \inf_{y>0}\left[\frac{1}{\alpha}\left\{y\ln(y) + yH(\mathcal{M}, P) - y\right\} + yx\right].$$

This is very easy and we obtain

$$u(x) = -\frac{1}{\alpha}e^{-H(\mathcal{M},P)-\alpha x}.$$

We summarize our main result.

Proposition 31.13 *For the exponential utility function*

$$U(x) = -\frac{1}{\alpha}e^{-\alpha x},$$

the minimax measure $\widehat{Q}(x)$ is independent of x and α, and we have

$$\widehat{Q}(x) = Q^{ME}$$

where Q^{ME} is the minimal entropy measure.

The above program can be carried out for a large number of utility functions. Below is a list of some frequently used utility functions u, as well as the corresponding divergence functions f:

$u(x) = -e^{-\alpha x}$	$f(x) = x\ln(x)$	Relative entropy
$u(x) = \ln(x)$	$f(x) = -\ln(x)$	Reverse relative entropy
$u(x) = x^2$	$f(x) = x^2 - 1$	Variance
$u(x) = -x^{-1}$	$f(x) = (\sqrt{x} - 1)^2$	Hellinger distance
$u(x) = \frac{1}{p}x^p$	$f(x) = -\frac{1}{q}x^q$	q-moment measure ($\frac{1}{p} + \frac{1}{q} = 1, p < 1, p \neq 0$).

31.5 Notes

Goll and Rüschendorf (2001) is a basic reference for the general theory of f-divergence and the connections to finance. Minimal entropy was introduced by Miyahara (1976) and developed by, among others, Frittelli in several papers such as Frittelli (2000). For a detailed study of entropy in Levy-driven markets, including an extensive bibliography, see Miyahara (2011).

32

PORTFOLIO OPTIMIZATION IN INCOMPLETE MARKETS

In this chapter we will present an outline of a duality theory for portfolio optimization in incomplete markets. The final results of this theory basically says that the dual problem to the portfolio maximization problem is a minimization problem over the class of martingale measures. We will follow Schachermayer (2002) very closely, but we will go very lightly on a number of technical issues, including in particular some very hard technical problems from functional analysis. For (much) more details the reader is thus referred to Schachermayer (2002) and Kramkov and Schachermayer (1999).

32.1 Setup

We consider an incomplete financial market living on a probability space $(\Omega, \mathcal{F}, P)$ on a finite time interval $[0, T]$. For simplicity we assume that $r = 0$, where r is the short rate. We consider an agent with zero subjective discount factor, and we use the following notation:

$$
\begin{aligned}
\mathcal{M} &= \text{the set of equivalent martingale measures.} \\
K_T(x) &= \text{the set of reachable portfolio values } X_T, \text{ at time } T, \text{ given } X_0 = x. \\
U(x) &= \text{the utility function of the agent.}
\end{aligned}
$$

Expected utility of a T-claim Y is thus given by the usual formula $E^P[U(Y)]$. We now need some assumptions. The first ensures absence of arbitrage.

Assumption 32.1.1 *We assume that the market is free of arbitrage in the sense that* $\mathcal{M} \neq \emptyset$.

We also need some assumptions concerning the utility function, and we recall these from the previous chapter.

Assumption 32.1.2 *We assume that the utility function U is strictly increasing, differentiable, strictly concave. We also assume that it satisfies*

$$\lim_{x \to +\infty} U(x) = 0.$$

Defining the effective domain dom(U) by dom$(U) = \{x : U(x) > -\infty\}$ we consider two cases.

Case 1 *This is the case when dom$(U) = (0, \infty)$. We then assume that*

$$\lim_{x \downarrow 0} U'(x) = +\infty.$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

Case 2 *This is the case when* $dom(U) = (-\infty, \infty)$. *We then assume that*

$$\lim_{x \downarrow -\infty} U'(x) = +\infty.$$

Given this setup we may formulate our main problem.

Problem 32.1.1 *The problem is*

$$\underset{X \in K(x)}{\text{maximize}} \quad E^P\left[U(X)\right]. \tag{32.1}$$

In the case of a complete market we know from Chapter 27 how to solve this problem by using the martingale approach, but in the present chapter we are interested in the incomplete market case. As we will see, the portfolio optimization problem, both for the complete and the incomplete market case, is closely related to convex duality, and the reader will find the necessary results quoted in Appendix D.

As it turns out, the general duality theory for an incomplete market is technically quite complicated and thus unfortunately outside the scope of this book. However, in the special case of a finite sample space Ω, it is possible to present a rather detailed and complete analysis. The strategy of the present chapter is thus as follows:

1. In Section 32.2 we present the duality theory for the case of a complete market.
2. In Section 32.3 we extend the duality theory to the case of an incomplete market on a finite Ω.
3. In Section 32.4 we outline very briefly how the duality theory extends to a general Ω.

32.2 The Complete Market Case

In this section we assume that the market is complete so there is a unique martingale measure. By the usual arguments from Chapter 27, Problem 32.1.1 is then equivalent to the following problem.

Problem 32.2.1

$$\underset{X \in \mathcal{F}_T}{\text{maximize}} \quad E^P\left[U(X)\right], \tag{32.2}$$

subject to constraint

$$E^Q\left[X\right] \leq x. \tag{32.3}$$

This, however, is just a special case of Problem 31.4.2, for the case of a complete market with the unique Q above. The relevant Lagrangian is now given by

$$L(x,y) = E^P\left[U(X) - yL_T X\right] + yx \tag{32.4}$$

where $L = dQ/dP$. We can now copy the definitions and results from Sections 31.4.1–31.4.2, for the special case of a unique Q. We then have the following main result for the complete market case.

Theorem 32.1 (The complete market case) *Consider the market defined above, and define the functions u and v by*

$$u(x) = \sup_{X \in K_T(x)} E^P\left[U(X)\right], \tag{32.5}$$

$$v(y) = E^P\left[V(yL_T)\right], \quad y > 0, \tag{32.6}$$

where $V = U^*$. Then the following hold:

1. *The value functions u and v are conjugate, so*

$$v(y) = u^*(y). \tag{32.7}$$

2. *The optimal X in (32.5) is unique and satisfies the equivalent formulas*

$$U'(\widehat{X}(x)) = \hat{y}L_T, \qquad \widehat{X}(x) = I\left(\hat{y}L_T\right), \tag{32.8}$$

   where $\hat{y} = \hat{y}(x)$ is determined by any of the equivalent formulas

$$u'(x) = y, \quad v'(y) = -x. \tag{32.9}$$

3. *The following relations hold*

$$u'(x) = E^p\left[U'(\widehat{X}(x))\right], \qquad v'(y) = E^Q\left[V'(yL_T)\right],$$

$$xu'(x) = E^p\left[\widehat{X}(x)U'(\widehat{X}(x))\right], \qquad yv'(y) = E^Q\left[yV'(yL_T)\right].$$

Proof Definition (32.6) is a copy of (31.38) for the trivial case when Q is unique. The relation (32.7) is a restatement of (31.39) for the special case of a unique Q. The formulas (32.8)–(32.9) are the standard optimality conditions in the Lagrangian. Moving on to Item 3, we take the y derivative in (32.6) to obtain

$$v'(y) = E^P\left[L_T V'(yL_T)\right] = E^Q\left[V'(yL_T)\right],$$

which proves the formula for $v'(y)$, and the formula for $yv'(y)$ is a trivial consequence. From duality we have $u'(x) = \hat{y}$, and from (32.8) it follows that $E^P\left[U'(\widehat{X}(x))\right] = \hat{y}$, which proves the formula for $u'(x)$. From duality we also have $xu'(x) = -yv'(y) = E^P\left[yL_T V'(yL_T)\right]$. From duality we have $V'(yL_T) = -\widehat{X}$ and we have $yL_T = U'(\widehat{X})$, which proves the formula for $xu'(x)$. □

We thus see that there exists a very nice duality theory for the complete market case there, and the question is now whether there exists a similar duality theory also for the incomplete market case.

32.3 Incomplete Market, Finite Ω

In this section we keep the assumptions of no arbitrage, but we no longer assume that the market is complete. It turns out, not surprisingly, that the incomplete market case is much more difficult to analyze then the complete market case, so in this section we therefore assume that the sample space is finite so $\Omega = \{\omega_1, \ldots, \omega_K\}$. We also assume that the market is living in discrete time. The case of a general Ω and continuous time is discussed in Section 32.4.

Recalling that $K_T(x)$ is the set of random variables $Z \in \mathcal{F}_T$ such that $Z = X_T$ for some self-financing portfolio with $X_0 = x$, our main problem is as follows.

Problem 32.3.1

$$\text{maximize}_{X \in \mathcal{F}_T} \; E^p [U(X)]$$

subject to the constraint

$$X \in K_T(x).$$

As it is stated, this problem is a bit hard to study, the reason being that the constraint $X \in K_T(x)$ is rather implicit. Our strategy is to replace it with a simpler and more tractable, but equivalent, constraint. To that end we we will use the following small but nice result, where we recall that $\mathcal{M}$ is the set of equivalent martingale measures.

Proposition 32.2 *A random variable $Z \in \mathcal{F}_T$ is dominated by some element in $K_T(x)$ if and only if Z satisfies the budget constraint*

$$E^Q [Z] \leq x, \quad \textit{for all } Q \in \mathcal{M}. \tag{32.10}$$

Proof In one direction the proof is trivial: If $Z \leq X_T$ with probability one for $X_T \in K_T(x)$, then we know from risk neutral valuation (i.e. the martingale property of the wealth process X_n) that $E^Q [X_T] = x$ for every martingale measure $Q \in \mathcal{M}$. Since $Z \leq X$, the inequality (32.10) now follows. In the other direction, the proof is a little bit harder, but it can be proved using elementary linear algebra, so it is left to the reader. □

It follows directly from this result that Problem 32.3.1 can be replaced by the following problem.

Problem 32.3.2

$$\underset{X \in \mathcal{F}_T}{\text{maximize}} \; E^p [U(X)]$$

subject to the constraints

$$E^Q [X] \leq x, \quad \textit{for all } Q \in \mathcal{M}.$$

We now have a maximization problem subject to a number of constraints, so we are close to having a standard constrained optimization problem which can be treated by Lagrangian techniques. Note, however, that we have infinitely many constraints (one for each $Q \in \mathcal{M}$), so the problem is still a bit too hard for us.

To overcome this obstacle we note that a measure Q on Ω is completely determined by the point masses $(Q(\omega_1), \ldots, Q(\omega_K))$ so we can view every measure on Ω as a point in R^K. It is then easy to prove the following result.

Proposition 32.3 *The set $\mathcal{M}$ is a closed bounded convex polytope in R^K. In particular, $\mathcal{M}$ is the convex hull of the extremal points of $\mathcal{M}$. We denote these extremal points by $Q^1, \ldots, Q^M$.*

Proof The proof is left to the reader. □

Using this result we can give the final formulation of the optimal portfolio problem.

Problem 32.3.3

$$\underset{X \in \mathcal{F}_T}{\text{maximize}} \ E^p [U(X)]$$

subject to the constraints

$$E^{Q^i}[X] \leq x, \quad i = 1, \ldots, K.$$

The point of this is that we now have a concave maximization problem with a finite number of (linear) inequality constraints. We can thus attack the problem with standard Lagrangian techniques. As in the complete market case we define the function u.

Definition 32.4 *The function $u(x)$ is defined by*

$$u(x) = \sup \left\{ E^P [U(X)] : \ E^{Q^i}[X] \leq x, \quad i = 1, \ldots, K \right\}. \tag{32.11}$$

By the arguments above it follows that u is indeed the indirect utility function (i.e. the optimal value function) for the original Problem 32.3.1.

The Lagrangian for Problem 32.3.3 is given by

$$L(X, \mu) = E^P [U(X)] - \sum_{i=1}^{K} \mu_i \left(E^{Q^i}[X] - x \right), \quad \mu_i > 0, \ i = 1, \ldots, K, \tag{32.12}$$

where $\mu = (\mu_1, \ldots, \mu_K)$. We can write this as

$$L(X, \mu) = E^P [U(X)] - \sum_{i=1}^{K} \mu_i \left(E^P \left[L_T^i X \right] - x \right), \tag{32.13}$$

where

$$L_T^i = \frac{dQ^i}{dP}, \quad \text{on } \mathcal{F}_T, \quad i = 1, \ldots K.$$

Recalling the Saddle Point Theorem D.4 we can now state the following, where $\mu > 0$ is shorthand for $\mu_i > 0 \ i = 1, \ldots, K$.

Lemma 32.5 *Defining $G(\mu)$ by*

$$G(\mu) = \sup_X \ L(X, \mu) \tag{32.14}$$

we have

$$u(x) = \inf_{\mu > 0} \ G(\mu). \tag{32.15}$$

We now need a small observation concerning convex combinations, but first we recall the following definition.

Definition 32.6 *The* **unit simplex** *in* R^n *is denoted by* Δ^{n-1} *and defined by*

$$\Delta^{n-1} = \left\{ (\alpha_1, \ldots, \alpha_n) \in R^n : \sum_{i=1}^{n} \alpha_i = 1, \quad \alpha_i \geq 0, \ i = 1, \ldots, n \right\}.$$

Lemma 32.7 *Assume that* $\alpha \in \Delta^{K-1}$*. Define the measure* Q^α *by*

$$Q^\alpha = \sum_{i=1}^{K} \alpha_i Q^i \tag{32.16}$$

and use the notation

$$L_T^\alpha = \frac{dQ^\alpha}{dP}, \quad \text{on } \mathcal{F}_T.$$

Then we have

$$L_T^\alpha = \sum_{i=1}^{K} \alpha_i L_T^i. \tag{32.17}$$

Proof The easy proof is left to the reader. □

Let us now define $y \in R_+$ and $\alpha = (\alpha_1, \ldots, \alpha_K) \in \Delta^{K-1}$ by

$$y = \sum_{i=1}^{K} \mu_i, \qquad \alpha_i = \frac{\mu_i}{y}, \quad i = 1, \ldots, K. \tag{32.18}$$

We then write the Lagrangian as

$$L(X, \mu) = E^P \left[U(X) \right] - \left(E^P \left[\sum_{i=1}^{K} \mu_i L^i X \right] - \sum_{i=1}^{K} \mu_i x \right), \tag{32.19}$$

which, using Lemma 32.7 and the notation above, can be written as

$$L(X, y, \alpha) = E^P \left[U(X) \right] - y \left(E^{Q^\alpha} [X] - x \right). \tag{32.20}$$

In this expression y is an arbitrary positive real number and α is an arbitrary point in Δ^{K-1}. As α varies over the unit simplex Δ^{K-1}, the measure Q^α will vary over all elements of $\mathcal{M}$ so we can write the Lagrangian as

$$L(X, y, Q) = E^P \left[U(X) - y L^Q X \right] + yx, \quad y > 0, \quad Q \in \mathcal{M}. \tag{32.21}$$

It now follows from Lemma 32.5 and the arguments above that we have the following result.

Lemma 32.8 *Defining the function* $G(y, Q)$ *by*

$$\mathbf{G}(y, Q) = \sup_X L(X, y, Q), \tag{32.22}$$

we have

$$u(x) = \inf_{y>0} \inf_{Q \in \mathcal{M}} G(y, Q). \tag{32.23}$$

Defining, as before, V by $V = U^*$ it follows from (31.26) that

$$G(y,Q) = E^P\left[V\left(yL^Q\right)\right] + yx, \quad y > 0, \quad Q \in \mathcal{M}. \tag{32.24}$$

We thus have

$$u(x) = \inf_{y>0} \inf_{Q \in \mathcal{M}} \left\{E^P\left[V\left(yL^Q\right)\right] + yx\right\}, \tag{32.25}$$

and we can now follow the arguments in Section 31.4.2. We thus define $v(y)$ by

$$v(y) = \inf_{Q \in \mathcal{M}} E^P\left[V\left(yL^Q\right)\right] \tag{32.26}$$

and obtain

$$u(x) = \inf_{y>0} \left\{v(y) + yx\right\}, \tag{32.27}$$

so we deduce that

$$u = v^*, \qquad v = u^*.$$

We can now state the results for the incomplete market case.

Theorem 32.9 (Incomplete market, finite Ω) *Assume that Ω is finite, time is discrete, and that the market is free of arbitrage, i.e $\mathcal{M} \neq \emptyset$. Define V as the convex conjugate of U so that*

$$V(y) = \sup_x \left\{U(x) - xy\right\}, \quad y > 0. \tag{32.28}$$

Define furthermore the functions $u(x)$ and $v(y)$ by

$$u(x) = \sup_{X \in K_T(x)} \quad E^P\left[U(X)\right] \tag{32.29}$$

$$v(x) = \inf_{Q \in \mathcal{M}} \quad E^P\left[V(yL_T^Q)\right]. \tag{32.30}$$

Then the following hold:

1. *The functions u and v are convex conjugates, i.e.*

$$v(y) = \sup_x \left\{u(x) - xy\right\}, \quad y > 0, \tag{32.31}$$

$$u(x) = \inf_{y>0} \left\{v(y) + xy\right\}, \quad x \in dom(u). \tag{32.32}$$

2. *The optimization problems in (32.29) and (32.30) have unique optimal solutions $\widehat{X}(x)$ and $\widehat{Q}(y)$, and satisfy*

$$\widehat{X}(x) = I\left(y\frac{d\widehat{Q}(y)}{dP}\right), \tag{32.33}$$

$$y\frac{d\widehat{Q}(y)}{dP} = U'(\widehat{X}(x)), \tag{32.34}$$

where x and y are related by the (equivalent) relations

$$u'(x) = y, \quad v'(y) = -x. \tag{32.35}$$

3. *The following formulas hold:*

$$u'(x) = E^p\left[U'(\widehat{X}(x))\right], \qquad v'(y) = E^Q\left[V'(y\frac{d\widehat{Q}(y)}{dP}\right], \tag{32.36}$$

$$xu'(x) = E^p\left[\widehat{X}(x)U'(\widehat{X}(x))\right], \quad yv'(y) = E^Q\left[yV'(y\frac{d\widehat{Q}(y)}{dP})\right]. \tag{32.37}$$

32.4 Incomplete Market, General Ω

The case of an incomplete market, continuous time, and a general Ω is, perhaps not surprisingly, much harder to analyze. In particular there are hard topological problems to be handled. It is thus outside the scope of this book to present the full theory and the interested reader is instead referred to Schachermayer (2002) and references therein for full proofs and precise statements.

The moral of the general theory is, however, very positive: Given an extra assumption on the utility function and given some modifications of the constraints $X \in K_T(x)$ and $Q \in \mathcal{M}$ in (32.29)–(32.30), the finite dimensional results carry over to the general case. We now proceed to give a flavor of the general results, and we start with an important extra assumption on U.

Assumption 32.4.1 *We assume that the utility function U exhibits* **reasonable asymptotic elasticity**, *i.e. that it satisfies*

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1. \tag{32.38}$$

In Case 2 of Assumption 32.1.2 we also demand

$$\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1. \tag{32.39}$$

To get some intuition about this, we note that (32.38) implies that

$$\liminf_{x\to\infty} \frac{xU''(x)}{U'(x)} > 0,$$

which says that the relative risk aversion is asymptotically strictly positive.

Before stating the first central result we need a definition.

Definition 32.10

$C_T(x) = \{Y \in \mathcal{F}_T : Y \geq 0,$ *and Y is dominated by some element in* $K_T(x)\}$

$\mathcal{M}^a =$ *the class of local martingale measures*

$$D^0 = \left\{L_T : L_T = \frac{dQ}{dP},\ Q \in \mathcal{M}^a\right\}$$

$D =$ *the closure of* D^0 *in the topology of convergence in probability.*

We can now state our first main result which covers the case when U is defined on the entire real line (like exponential utility).

Theorem 32.11 (The case $dom(U) = (0, \infty)$**)** *Assume that the utility function U satisfies Assumption 32.1.2 (Case 1), and that U has reasonable asymptotic elasticity. Let V be the conjugate of U and define the functions u and v by*

$$u(x) = \sup_{X_T \in C_T(x)} E^P \left[U(X_T)\right], \quad v(y) = \inf_{L_T \in D} E^P \left[V(yL_T)\right]. \tag{32.40}$$

Then the following hold:

1. *The functions u and v are conjugate.*
2. *The optimal $\widehat{X}_T$ and $\widehat{L}_T$ exist. They are unique and they satisfy*

$$\widehat{X}_T(x) = I(y\widehat{L}_T), \quad y\widehat{L}_T = U'(\widehat{X}_T(x)),$$

where x and y are related by

$$u'(x) = y, \quad v'(y) = -x.$$

3. *We have the relations*

$$u'(x) = E^P\left[U'(\widehat{X}_T(x))\right], \quad v'(y) = E^{\widehat{Q}}\left[V'\left(\frac{d\widehat{Q}(y)}{dP}\right)\right],$$
$$xu'(x) = E^P\left[\widehat{X}_T(x)U'(\widehat{X}_T(x))\right], \quad yv'(y) = E^{\widehat{Q}}\left[V'(y\widehat{L}_T)\right].$$

We now move to the case when U is defined on $(0, \infty)$. In this case, the sets $C_T(x)$ and D has to be modified. For D this turns out to be easy: we replace D by $\mathcal{M}^a$. For $C_T(x)$ the situation is more complicated. Define the set $C_U^b(x)$ by

$$C_U^b(x) = \left\{Y \in \mathcal{F}_T : Y \le X \text{ for some } X \in K_T(x) \text{ and } E^P\left[|U(Y)|\right] < \infty\right\},$$

and define the set $C_U(x)$ as the set of $X \in \mathcal{F}_T$ such that $U(X)$ is in the L^1 closure of $\left\{U(Y) : \ Y \in C_U^b(x)\right\}$. We then have the following result.

Theorem 32.12 (The case $dom(U) = (-\infty, \infty)$**)** *Assume that the utility function U satisfies Assumption 32.1.2 (Case 2), and that U has reasonable asymptotic elasticity. Let V be the conjugate of U and define the functions u and v by*

$$u(x) = \sup_{X_T \in C_U(x)} E^P \left[U(X_T)\right], \quad v(y) = \inf_{Q \in \mathcal{M}^a} E^P \left[V(yL_T^Q)\right]. \tag{32.41}$$

Then the following hold:

1. *The functions u and v are conjugate.*
2. *The optimal $\widehat{X}_T(x)$ and $\widehat{Q}(y)$ exist. They are unique and they satisfy*

$$\widehat{X}_T(x) = I(y\frac{d\widehat{Q}(y)}{dP}), \quad y\frac{d\widehat{Q}(y)}{dP} = U'(\widehat{X}_T(x)),$$

where x and y are related by

$$u'(x) = y, \quad v'(y) = -x.$$

3. *We have the relations*

$$u'(x) = E^P\left[U'(\widehat{X}_T(x))\right], \quad v'(y) = E^{\widehat{Q}}\left[V'\left(\frac{d\widehat{Q}(y)}{dP}\right)\right],$$

$$xu'(x) = E^P\left[\widehat{X}_T(x)U'(\widehat{X}_T(x))\right], \quad yv'(y) = E^{\widehat{Q}}\left[V'\left(y\frac{d\widehat{Q}(y)}{dP}\right)\right].$$

4. *If $\widehat{Q}(y) \in \mathcal{M}$ and $x = -v'(y)$, then $\widehat{X}_T(x) \in K_T(x)$ so $\widehat{X}_T(x)$ is in fact the terminal value of a self financing portfolio with initial value x.*

32.5 Notes

Karatzas et al. (1991) and Kramkov and Schachermayer (1999) are basic references for optimal investment in incomplete markets. Schachermayer (2002), parts of which we mainly followed above, provides a very readable overview of convex duality theory for the incomplete market case, and it also contains an extensive bibliography. An interesting alternative dual theory, related to the Pontryagin maximum principle, is presented in Rogers (2002).

33

UTILITY INDIFFERENCE PRICING AND OTHER TOPICS

Let us again consider the general (incomplete) model (14.3)–(14.4) which we recall as

$$dS_t = D(S_t)\mu_t dt + D(S_t)\sigma_t dW_t, \tag{33.1}$$

$$dB_t = r_t B_t dt, \tag{33.2}$$

where S is n-dimensional whereas W is N-dimensional, the matrix σ_t is full rank and $n \le N$. In the incomplete case when $n < N$ we have more random sources than underlying assets so the martingale equation

$$r_t = \mu_t + \sigma_t \varphi_t$$

does not admit a unique solution φ. Defining, as before, the "market price of risk" vector λ by $\lambda_t = -\varphi_t$, the common (although perhaps somewhat loose) economic interpretation is that the component λ_t^i reflects the aggregate risk aversion of the market relative to the risk factor W_t^i. In a complete market the risk aversion for every risk factor is thus uniquely determined by the underlying S-dynamics, but in an incomplete market the underlying S-dynamics do not contain enough information for the determination of a unique market price of risk vector.

We then recall Result 14.6.1.

Result 33.0.1 *In an incomplete market, the martingale measure and the market price of risk is determined by* **the market**.

The implication of this is that if we want to price in an incomplete market setting we must get information about the market attitude towards risk, and one way of doing this is of course to specify a utility function. We have already seen this idea in action in the previous chapters, and it will be pursued in great detail in Chapters 35–38, where we study dynamic general equilibrium theory.

33.1 Global Indifference Pricing

In the present section we will briefly discuss the connection between utility functions and derivative pricing in a more restricted setting than that of dynamic equilibrium. We assume that there exist an economic agent with utility function U, and a subjective discount factor $\delta = 0$. The utility, at time $t = 0$, of a T-claim Y is then given by

$$E^P[U(Y)]. \tag{33.3}$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

The wealth process of the agent is denoted by X, and the initial wealth of the agent is denoted by x. We now discuss how to price a fixed T-claim Y, and to this end we define the function $V(x,z) = V(x,z;Y)$ by

$$V(x,z) = \sup_{X_T \in \mathcal{K}(x)} E^P \left[U(X_T + zY) \right] \tag{33.4}$$

where $\mathcal{K}(x)$ denotes the wealth profiles X_T that can be reached at time T, by trading in the underlying assets, given initial wealth x. The entity $V(x,z)$ represents the maximal utility you can achieve by trading on the market, if your initial wealth equals x, and you get z units of the claim Y at time T. The question is now how much you are willing to pay, at time $t=0$, for getting z units of Y at time T, and one very reasonable answer is given by the **utility indifference buy price** $p(z)$, defined as follows.

Definition 33.1 *The utility indifference buy price $p(z)$ is defined as the solution to the equation*

$$V(x - p(z), z) = V(x, 0). \tag{33.5}$$

In other words, you are indifferent between the following alternatives:

1. Paying $p(z)$ at time $t=0$ and getting z units of Y at time T.
2. Paying zero at time $t=0$ and getting zero units of the claim Y at time T.

There is a substantial literature on this topic but since the problems are hard and there are very few concrete analytical results, we will confine ourselves to a few comments:

- In order to compute $V(x,z)$ above we need to extend the duality theory for optimal portfolios to a case which also includes the exogenously given claim Y. This can be done, but the theory is not easy.
- In a Markovian setting we may attack the problem with dynamic programming, and this sometimes leads to concrete results.
- The price $p(z)$ is typically not linear in z.
- We cannot in general interpret the “global” utility indifference price above as a **market** price. It should rather be seen as a “personal valuation” or “reservation price” of, for example, an individual or an insurance company.
- In the general case, the price $p(z)$ is also a function of initial wealth x. In order to obtain a price which is independent of x, it is therefore very common to assume exponential utility.

To get some small feeling for indifference pricing, let us consider the simplest possible case, which occurs when we have no underlying risky assets, the short rate is zero, and we assume exponential utility

$$U(x) = -\frac{1}{\gamma} e^{-\gamma x}.$$

In this case the pricing equation (33.5) takes the form

$$E^P \left[e^{-\gamma(x - p(z) + zY)} \right] = E^P \left[e^{-\gamma x} \right].$$

We can easily solve for $p(z)$ to obtain

$$p(z) = -\frac{1}{\gamma} \ln \left(E^P \left[e^{-\gamma(zY)} \right] \right)$$

and we note that in this case the price is indeed independent of x.

33.2 Marginal Indifference Pricing

The reader familiar with microeconomics and equilibrium theory will perhaps find the global utility indifference pricing ideas above a bit strange. In microeconomics, the typical objects of interest are not "global" objects. Instead the focus is typically on "local" concepts, like *marginal* utilities and *marginal* rates of substitution, the reason being that these are the objects that turn up naturally in equilibrium theory. It therefore seems natural to develop a "marginal" theory of indifference pricing and, with some small changes, we now follow Davis (1997).

Consider the incomplete market and the claim Y of the previous section, and define the functions $V(x)$ and $V(x,p,\delta,\eta)$ by

$$V(x) = \sup_{X_T \in \mathcal{K}_T(x)} E^P \left[U\left(X_T \right) \right] \tag{33.6}$$

$$V(x,p,\delta) = \sup_{X_T \in \mathcal{K}_T(x)} E^P \left[U \left(X_T + \frac{\delta}{p} Y \right) \right] \tag{33.7}$$

so we have in fact $V(x) = V(x,p,0)$ for all (x,p). The function V is the usual indirect utility function of wealth, and the interpretation of $V(x,p,\delta)$ is as follows:

- The price (at $t = 0$) per unit of the derivative Y is given by p.
- At time $t = 0$ you get (for free) δ dollars to invest in the derivative Y. This will allow you to buy δ/p units of Y.
- Your initial wealth to invest in the stock market is still equal to x.
- At time T you will have your portfolio value X_T plus the value of δ/p units of Y.
- The value $V(x,p,\delta)$ is your utility after investing optimally in the stock market.

We can now define the marginal indifference price.

Definition 33.2 *Assume that $V(x,p,\delta)$ is continuously differentiable, and that there is a unique solution $p(x)$ to the equation*

$$\frac{\partial V}{\partial \delta}(x,p,0) = \frac{\partial V}{\partial x}(x,p,0). \tag{33.8}$$

Then we refer to $p(x)$ as the **marginal indifference price**.

The intuition of this is that, at the indifference price, you are indifferent between investing a small amount in the stock market or investing the same amount in the derivative Y. From this argument it seems that the marginal indifference pricing is, in some sense, closely connected to equilibrium pricing, and we will see that this is indeed the case.

It follows from an Envelope Theorem that

$$\frac{\partial V}{\partial \delta}(x,p,0) = E^P\left[U'\left(\widehat{X}_T\right)\frac{Y}{p}\right]$$

and we have the usual result (like in Theorem 32.11)

$$\frac{\partial V}{\partial x}(x,p,0) = E^P\left[U'\left(X_T\right)\right].$$

We thus have the equation

$$E^P\left[U'\left(\widehat{X}_T\right)\frac{Y}{p}\right] = E^P\left[U'\left(\widehat{X}_T\right)\right],$$

which gives us our main result.

Proposition 33.3 *Suppose that $V(x)$ is differentiable and that $V'(x) > 0$ for all x. The marginal indifference price is given by*

$$p(x) = \frac{E^P\left[U'(\widehat{X}_T)Y\right]}{V'(x)} \tag{33.9}$$

or, equivalently,

$$p(x) = \frac{E^P\left[U'(\widehat{X}_T)Y\right]}{E^P\left[U'\left(\widehat{X}_T\right)\right]}.$$

From a microeconomic point of view, this is a very pleasing result. We can interpret the object

$$\frac{U'(\widehat{X}_T)}{E^P\left[U'\left(\widehat{X}_T\right)\right]} \tag{33.10}$$

as an equilibrium stochastic discount factor, and formulas of this type will turn up again in the context of dynamic equilibrium theory in Chapters 35–38. See for example Propositions 36.6 and 38.1.

33.3 Hedging

As we have seen, in an incomplete market there is generically not a unique price for a given claim, and it is impossible to hedge perfectly. A natural idea is then to try to find an approximate hedge, and the obvious first choice is to use a quadratic loss function. This line of ideas was first investigated by Föllmer and Sondermann (1986) and then generalized in Schweizer (1991) and many other papers. The quadratic hedging approach has subsequently been the object of intensive research and has led to a large and deep literature. In quadratic hedging we keep the portfolios self-financing and we search for the best hedge.

An alternative is to go for a perfect hedge but to relax the idea of a self-financing portfolio. This line of ideas leads to the theory of “local risk miminization”. For an overview with an extensive bibliography see Schweizer (2001).

33.4 Notes

For a collection of papers on indifference pricing see Carmona (2009). For explicit results concerning indifference pricing see Zariphopoulo (2001), Benth and Karlsen (2005), and the overview in Henderson and Hobson (2009). The basic reference on marginal indifference pricing is Davis (1997). For textbook treatments of hedging in incomplete markets, see Bingham and Kiesel (2004) and Dana and Jeanblanc (2003).

34

GOOD DEAL BOUNDS

In Chapters 30–33 we discussed incomplete markets and in particular we considered several ways of choosing one "canonical" martingale measure from the (infinite) class of equivalent martingale measures, thus obtaining a unique arbitrage free price. The approach of the present chapter is different in the sense that we do **not** look for a **unique** price for a given derivative. Instead we try to derive reasonable (in some sense) **pricing bounds** for financial derivatives.

34.1 General Ideas

In order to get some intuition let us consider a T-claim X on some incomplete market. One possible way to obtain pricing bounds would be to compute the upper and lower no arbitrage bounds, but the drawback with this approach is that the bounds obtained in this way are typically extremely wide, and completely useless from a practical point of view. We thus need tighter bounds, and then the following idea turns out to be useful.

Idea 34.1.1

1. With no arbitrage pricing we (only) rule out the pricing rules which allow for arbitrage possibilities.
2. The underlying reason for ruling out arbitrage is that we interpret an arbitrage possibility as an astronomical mispricing in the market. Alternatively we view it as an astronomically "good deal".
3. There are, however, pricing rules that have the following properties:
   - The pricing rule allows for business opportunities which are far too good to be realistic on any market (with a realistic risk aversion).
   - Nevertheless the pricing rule is **not** an arbitrage opportunity.
4. We should then perhaps rule out not only prices which allow arbitrage but also prices which allow for deals which are "too good".

As an example of an unreasonable pricing rule consider the following lottery:

- The lottery will take place one and only one time in history (so it is not repeated) and only one lottery ticket is sold.
- The outcome of the lottery is decided by the flip of a fair coin
  - ∗ If the result is "heads", then the holder of the ticket receives one million dollars.
  - ∗ If the result is "tails", then the holder of the ticket receives nothing.
- The price of the lottery ticket is one dollar.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

It is clear that, from a common sense point of view, setting the price to one dollar, as we did above, amounts to an almost astronomical mispricing, and you would of course never expect a price system like this on any "reasonable" market. Note, however, that the pricing of the ticket **is** arbitrage free. This is a typical example of a deal which is "too good", and we would like to rule out such deals.

We then have to formalize the idea of a "good deal" and this is exactly what was done in Cochrane and Saá Requejo (2000). The authors defined a "good deal" as an asset price process with a high Sharpe ratio (where the term "high" obviously has to be defined). To see more concretely what is gong on, we recall the definition of the Sharpe ratio SR of an asset as

$$SR_t = \frac{\mu_t - r_t}{v_t}$$

where μ is the local mean rate of return, r is the short rate, and v is the volatility. The way we think about this is that the Sharpe ratio is the risk premium per unit of volatility, so the Sharpe ratio tells us something about the aggregate risk aversion on the market. To be more concrete, let us assume that the underlying asset is a common stock. Given empirical data, a reasonable volatility would then be anything in the interval 20–60 percent per annum, so let us assume that $v = 40$ percent per annum. In percentage terms we would then have

$$\mu - r = SR \cdot 40.$$

This implies that a Sharpe ratio of $SR = 5$ implies that the asset would have an excess rate of return of 200 percent per annum, which of course is extremely high. Empirical studies do in fact tell us that typical values of the Sharpe ratio is somewhere in the interval $[0.5, 1]$.

In a first attempt, a mathematical formalization of the pricing problem would then be to find the maximum (minimum) arbitrage free price process for the derivative, subject to an upper bound on the Sharpe ratio. However, this way of formalizing the problem turns out have two major drawbacks:

1. The optimization problem turns out to be mathematically intractable.
2. A much more serious problem is the following: Suppose that we have found upper and lower pricing bounds on a derivative, subject to a bound on the Sharpe ratio of the derivative. Then it may in principle still be possible to form a self-financing portfolio, based on the underlying assets and the newly introduced derivative, such that the portfolio has a very high Sharpe ratio.

What we need is thus a formalization of the pricing problem which gives us a mathematically tractable problem, and which at the same time allows us to have complete control over the Sharpe ratios of *all portfolios* based on the underlying assets and the derivative.

This is precisely where the Hansen–Jagannathan bounds from Proposition 14.9 comes in useful, and the idea is now that instead of putting a bound on the Sharpe ratio of the derivative under study, we put a bound on the right-hand side of the Hansen–Jagannathan inequality (i.e. the norm of the market price of risk vector). In the final formulation, the pricing problem is thus that of finding the maximum (minimum) arbitrage free price process, given a bound on the right-hand side of the Hansen–Jagannathan inequality.

In the original paper Cochrane and Saá Requejo (2000) this was done under the objective measure P, using the formalism of stochastic discount factors, but we will instead follow Björk and Slinko (2006) where the analysis is done under Q using the formalism of martingale measures rather than SDFs, the reason being that the analysis becomes much more streamlined in this way. In particular it turns out that in the Q-formalism the pricing problem has the form of a standard stochastic control problem which can be attacked by dynamic programming.

34.2 The Model

We now move on to a concrete model, and we recall the factor model from Definition 29.1:

$$dS_t = D(S_t)\mu(Y_t)dt + D(S_t)\sigma(Y_t)dW_t, \tag{34.1}$$
$$dY_t = a(Y_t)dt + b(Y_t)dW_t, \tag{34.2}$$
$$dB_t = r(Y_t)B_t dt, \tag{34.3}$$

where S, W, and Y are of dimensions n, N, and k respectively.

34.3 The Good Deal Bounds

Given the setup above, our task is to price a T-claim X of the form

$$X = \Phi(S_T, Y_T).$$

We start by searching for a martingale measure Q so, as usual, we introduce a likelihood process $L = dQ/dP$ by

$$\begin{cases} dL_t = L_t \varphi_t^* dW_t, \\ L_0 = 1. \end{cases} \tag{34.4}$$

The Girsanov Theorem implies that $dW_t = \varphi_t dt + dW_t^Q$ and, using this, the condition for Q to be a martingale measure is, as usual,

$$\mu(y) + \sigma(y)\varphi_t = \mathbf{r}(y), \tag{34.5}$$

where we have used the notation $\mathbf{r}(y) = [r(y), \dots, r(y)]^*$.

From Proposition 14.9 we recall that, given a pricing measure Q, generated by a Girsanov kernel φ satisfying (34.5), the Hansen–Jagannathan bounds

$$(SR_t)^2 \le \|\varphi_t\|^2$$

will hold for all underlying assets, for all derivatives, and for all portfolios. We are now ready to define the upper and lower good deal pricing bounds.

Definition 34.1 *Given a T-claim of the form $\Phi(S_T, Y_T)$, and constant $B > 0$, the* **upper good deal pricing bound** *process is defined as the optimal value process for the following optimal control problem,*

$$\underset{\varphi}{\text{maximize}} \; E^Q \left[e^{-\int_0^T r(Y_u)du} \Phi\left(S_T, Y_T\right) \right], \tag{34.6}$$

where the state processes (S, Y) have Q dynamics

$$dS_t = D(S_t)\mathbf{r}(Y_t)dt + D(S_t)\sigma(Y_t)dW_t^Q \tag{34.7}$$

$$dY_t = \{a(Y_t) + b(Y_t)\varphi_t\}\,dt + b(Y_t)dW_t^Q \tag{34.8}$$

and where the process W^Q is Q-Wiener. The N-dimensional control process φ_t is subject to the constraints

$$\mu(Y_t) + \sigma(Y_t)\varphi_t = \mathbf{r}(Y_t), \tag{34.9}$$

$$\|\varphi_t\|^2 \leq B^2. \tag{34.10}$$

We note the following:

- The expected value in (34.6) is the standard risk neutral valuation formula for contingent claims.
- The condition (34.9) guarantees that the induced measure Q is indeed a martingale measure for S.
- The induced Q dynamics of S and Y are given in (34.7)–(34.8).
- The constraint (34.10) is the constraint to rule out "good deals", not only for the underlying asset price S, but also for all derivatives and all portfolios based on underlying and derivatives.
- In order to obtain the lower pricing bound, we solve the corresponding minimum problem.
- The constant B is chosen at will, and reflects your own opinion about what constitutes a good deal. A typical choice would be $B = 0.5$.

Since we are in a standard setting for dynamic programming (DynP), we know from general DynP-theory that the optimal value function $V(t, s, y)$ will satisfy the Hamilton–Jacobi–Bellman equation on the time interval $[0, T]$. After some standard calculations we thus have the following result.

Theorem 34.2 *The upper good deal bound function is the solution V to the following boundary value problem:*

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,s,y) + \sup\limits_{\varphi} \mathbf{A}^{\varphi} V(t,s,y) - r(y)V(t,s,y) & = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad V(T,s,y) & = \Phi(s,y) \end{cases} \tag{34.11}$$

subject to the constraints

$$\mu(y) + \sigma(y)\varphi(t,s,y) = r(y), \tag{34.12}$$

$$\|\varphi(t,s,y)\|^2 \leq B^2. \tag{34.13}$$

The infinitesimal operator $\mathbf{A}^{\varphi}$ above is given by

$$\mathbf{A}^{\varphi} V(t,s,y) = V_s(t,s,y)D(s)\mathbf{r}(y) + V_y(t,s,y)\{a(y) + b(y)\varphi(t,s,y)\}$$
$$+ \frac{1}{2}\mathbf{t}r\left\{[\sigma^*(y), b^*(y)]\, H \left[\begin{array}{c} \sigma(y) \\ b(y) \end{array}\right]\right\}, \tag{34.14}$$

where V_s and V_y denotes the gradients w.r.t. s and y (on row vector form), whereas H is the Hessian block matrix defined in the obvious way as

$$H = \left[\begin{array}{c} H_{ss}, H_{sy} \\ H_{ys}, H_{yy} \end{array}\right].$$

34.4 The Embedded Optimization Problem

In the HJB equation we always have an embedded static optimization problem, and for the good deal bound HJB equation (34.11) this turns out to be very simple. By inspection of the infinitesimal operator in (34.14) we note the following.

Lemma 34.3 *Given* (t,s,y) *the embedded optimization problem in the HJB equation (34.11) is:*

$$\underset{\varphi}{\text{maximize}}\ V_y(t,s,y)b(y)\varphi \tag{34.15}$$

subject to the constraints

$$\mu(y) + \sigma(y)\varphi = r(y), \tag{34.16}$$
$$\|\varphi\|^2 \leq B^2. \tag{34.17}$$

As far as optimization problems are concerned this is an easy one. We have an N-dimensional control variable φ, a linear objective function with n linear constraints, and one quadratic constraint. From this it is immediately clear that the norm constraint (34.17) will be binding.

34.5 Relations to the Minimal Martingale Measure

Let us now recall the minimal martingale measure Q^M and the corresponding Girsanov kernel φ^M from Section 30.2. Denoting the optimal value function corresponding to the upper and lower good deal bounds by V^u and V^l, and denoting the price under Q^M by V^M we note the following:

- The only formal restriction on B is that we must have

$$\sup_t \|\varphi_t^M\| \leq B.$$

- We always have the inequality

$$V^l \leq V^M \leq V^u.$$

34.6 An Option with Basis Risk

As a first example we consider the following model:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t^1, \\
dY_t &= aY_t dt + Y_t \left\{ b_1 dW_t^1 + b_2 dW_t^2 \right\}, \\
dB_t &= rB_t dt,
\end{aligned}
$$

where r, μ, σ, a, b_1, and b_2 are positive constants, and where W^1 and W^2 are independent Wiener processes. We also assume that $Y_0 > 0$, implying that Y_t is strictly positive for all t. The price process S is a standard Black–Scholes model, whereas the Y process is an underlying non-traded index.

Our task is to compute good deal bounds for a European Call option on the Y process, i.e. a T-claim X of the form

$$
X = \max\left[Y_T - K, 0\right].
$$

We note that if Y were a traded asset there would be a unique price for the call option, given by the Black–Scholes formula with volatility $\sqrt{b_1^2 + b_2^2}$. In the present setting Y is not the price of a traded asset, and since we have one traded asset and two independent Wiener processes, the market is not complete so we cannot replicate X. We do, however, have a nontrivial correlation between the driving noises of S a W, so the intuition is that we can partially hedge (and thus partially price) the claim X. We also note that if $b_2 = 0$ then the market is complete so then there should be a unique price for X.

For this model the Girsanov transformation will be of the form

$$
dL_t = L_t \varphi_{1t} dW_t^1 + L_t \varphi_{2t} dW_t^2
$$

with Girsanov kernel $\varphi = (\varphi_1, \varphi_2)$. This implies that the good deal pricing HJB equation for $V(s,t,y)$ takes the form

$$
\begin{cases}
V_t + \sup\limits_{\varphi} \mathbf{A}^{\varphi} V(t,s,y) - rV(t,s,y) = 0, \\
\qquad\qquad\qquad\qquad V(T,s,y) = \max\left[y - K, 0\right],
\end{cases}
\tag{34.18}
$$

for $s > 0$ and $y > 0$, where

$$
\begin{aligned}
\mathbf{A}^{\varphi} V &= \mu s V_s + y\left(a + b_1\varphi_1 + b_2\varphi_2\right) V_y \\
&\quad + \frac{1}{2}\sigma^2 s^2 V_{ss} + \frac{1}{2}\left(b_1^2 + b_2^2\right) y^2 V_{yy} + ys\sigma b_1 V_{sy}.
\end{aligned}
$$

The constraints are

$$
\mu + \sigma\varphi_1 = r, \tag{34.19}
$$

$$
\varphi_1^2 + \varphi_2^2 \leq B^2. \tag{34.20}
$$

The embedded optimization problem is to maximize

$$
y\left(b_1\varphi_1 + b_2\varphi_2\right) V_y
$$

given the constraints. As always, the quadratic constraint (34.20) will be binding. We can thus solve the system

$$\mu + \sigma\varphi_1 = r, \quad \varphi_1^2 + \varphi_2^2 = B^2,$$

to obtain

$$\varphi_1 = -\frac{R}{\sigma}, \quad \varphi_2 = \pm\sqrt{B^2 - \frac{R^2}{\sigma^2}}$$

where $R = \mu - r$. From the shape of the problem, it is obvious (why?) that $V_y > 0$, and since $y > 0$ the problem reduces to that of maximizing

$$b_1\varphi_1 + b_2\varphi_2.$$

Since b_2 is positive, the optimal φ_2 must be given by the positive root above, so we have

$$\widehat{\varphi}_1 = -\frac{R}{\sigma},$$

$$\widehat{\varphi}_2 = \sqrt{B^2 - \frac{R^2}{\sigma^2}}.$$

We note in passing that, for obvious reasons (exactly why?), the minimal martingale measure Q^M is generated by

$$\varphi_1^M = -\frac{R}{\sigma}, \quad \varphi_2^M = 0.$$

Thus

$$\|\varphi_t^M\|^2 = \frac{R^2}{\sigma^2}, \quad t \geq 0,$$

which highlights why we must choose B such that $B^2 \geq \frac{R^2}{\sigma^2}$ in the formula for $\widehat{\varphi}_2$ above.

In order to compute the upper good deal price bound we note that we can write the Q-dynamics of Y as

$$dY_t = \alpha Y_t dt + \beta dW_t^Q$$

where W^Q is a scalar Q-Wiener process and

$$\alpha = a - b_1\frac{R}{\sigma} + b_2\sqrt{B^2 - \frac{R^2}{\sigma^2}},$$

$$\beta = \sqrt{b_1^2 + b_2^2}.$$

By risk neutral valuation we have the upper good deal bound pricing formula $c_Y^u(t,y)$ as

$$c_Y^u(t,y) = e^{-r(T-t)}E_{t,y}^Q\left[\max\left[Y_T - K, 0\right]\right],$$

and, given $Y_t = y$, we have

$$Y_T = ye^{(\alpha - \frac{1}{2}\beta^2)(T-t) + \beta\left[W_T^Q - W_t^Q\right]},$$

which we can write as

$$Y_T = \left\{ ye^{(\alpha - r)(T-t)} \right\} e^{(r - \frac{1}{2}\beta^2)(T-t) + \beta \left[W_T^Q - W_t^Q \right]}.$$

This, however, corresponds exactly to the Q-formula for the value at T of a **traded asset** in a standard Black–Scholes model, with volatility β, and initial value at t given by $ye^{(\alpha - r)(T-t)}$. Denoting the standard Black–Scholes formula by $c(t, s, \sigma)$ we thus have the following result.

Proposition 34.4 *The upper good deal bound pricing formula $c_Y^u(t, y)$ for the call option on Y is given by*

$$c_Y^u(t, y) = c(t, ye^{(\alpha - r)(T-t)}, \beta), \tag{34.21}$$

with α and β as above. In order to obtain the lower price bound we replace α by

$$a - b_1 \frac{R}{\sigma} - b_2 \sqrt{B^2 - \frac{R^2}{\sigma^2}}.$$

We note that in this case the optimal value function $V(t, s, y) = c_Y^u(t, y)$ does not depend on s.

34.7 Notes

The good deal theory presented above is first investigated in Cochrane and Saá Requejo (2000) where other examples are studied. The theory was extended and streamlined in Björk and Slinko (2006) where point process applications are also studied. For an interesting but slightly different view of good deal bounds, see Cerny (2003) and Cerny and Hodges (2001). A related approach to obtain asset price bounds, based on gains–loss ratios, is presented in Bernardo and Ledoit (2000). See Rodriguez (2000) for an interesting connection of Bernardo and Ledoit (2000) to linear programming.

PART VI

DYNAMIC EQUILIBRIUM THEORY

35

EQUILIBRIUM THEORY: A SIMPLE PRODUCTION MODEL

When we have studied consumption and investment problems in the previous chapters, all underlying price processes have been exogenously given. We have also typically assumed that the underlying market is complete, implying that there has been a unique exogenously given stochastic discount factor $\mathbf{M}$ as well as unique martingale measure Q. We are now moving on to **equilibrium theory** where the perspective is slightly different.

- In the previous lecture the short rate r process was **exogenously** given.
- We now move to an **equilibrium model** where the short rate process r_t will be determined **endogenously** within the model.
- In later lectures we will also discuss how other asset price processes (apart from r) are determined by equilibrium.

In the literature, there are two main classes of equilibrium models: production models and endowment models. We will study both types and we start with production models. For the simplest case of a production model the setup is roughly as follows:

- We assume the existence of one or several economic **agents** with given utility functions for consumption.
- The agents are exogenously given a **production technology**.
- The agents make decisions about
  * Investment in the production technology
  * Consumption
  * Investment in the risk free asset B.
- The agents act so as to maximize expected utility.
- The short rate process r, and hence the risk free asset price process B, is then determined by the equilibrium condition that supply equals demand on the market for B.

We now go on to analyze the simplest possible equilibrium model in a production economy. The model is a Kindergarten version of the Cox–Ingersoll–Ross model Cox et al. (1985a), but it is instructive to start with this simple model, and in Chapter 36 we will extend it to a more interesting model. We will use dynamic programming to analyze the model. The model can also be analyzed using the martingale approach but we leave this to the reader.

35.1 The Model

We start with a formal assumption concerning production technology. The assumption is a simplified version of the one presented in Cox et al. (1985a).

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

Assumption 35.1.1 *We assume that there exists a single consumption good with unit price. We also assume that there exists a constant returns-to-scale physical production technology process S (for production of the consumption good) with return dynamics*

$$dS_t = \mu S_t dt + S_t \sigma dW_t. \tag{35.1}$$

The economic agents can invest unlimited positive amounts in this technology, but since it is a matter of physical investment, short positions are not allowed.

The consumption good could for example be "apples", but since the price by assumption is one dollar per apple we may as well use "dollars" as the numeraire. For simplicity, we will thus refer to the consumption good as "dollars", but always with the provision that these "dollars" can be invested as well as consumed.

The exact meaning of the production process S is perhaps less clear to the reader. The precise interpretation is as follows:

- At any time t you are allowed to invest dollars in the production process.
- If you, at time t_0, invest q dollars, and wait until time t_1 then you will receive the amount of
$$q \cdot \frac{S_{t_1}}{S_{t_0}}$$
in dollars. In particular we see that the return on the investment is linear in q, hence the term "constant returns to scale".
- Since this is a matter of physical investment, short selling is not allowed.

A moment of reflection shows that, from a purely formal point of view, investment in the technology S is in fact **equivalent to the possibility of investing in a risky asset** with price process S, but again with the constraint that short selling is not allowed. An alternative way of interpreting the model would be to say that S is the exogenously given price process of a financial asset, with short selling allowed, and that we are thus studying a partial equilibrium model.

We also need a risk free asset, and this is provided by the next assumption.

Assumption 35.1.2 *We assume that there exists a risk free asset in* **zero net supply** *with dynamics*

$$dB_t = r_t B_t dt,$$

where r is the short rate process, which will be determined endogenously. The risk free rate r is assumed to be of the form

$$r_t = r(t, X_t)$$

where X denotes portfolio value.

We now need to define the term "zero net supply" above.

Definition 35.1 *To say that an asset is in* **zero net supply** *means that in equilibrium the net amount sold of the asset must equal the net amount bought. In other words, if someone wants to sell an asset, someone else has to buy it.*

Interpreting the production technology S as above, the wealth dynamics will be given by the usual formula

$$dX_t = X_t w_t(\mu - r)dt + (r_t X_t - c_t)dt + X_t w_t \sigma dW_t, \tag{35.2}$$

where w is the portfolio weight on S, $1-w$ is the weight on B, and c is the consumption rate. Finally we need an economic agent.

Assumption 35.1.3 *We assume that there exists a representative agent who wishes to maximize the usual expected utility*

$$E^P\left[\int_0^T U(t, c_t)dt + \Phi(X_T)\right].$$

One would obviously like to have more than one agent, but we note the following:

- Assuming a representative agent facilitates the computations enormously.
- It can be shown that the general case with a finite number of different agents can be reduced to the case of a representative agent.
- We may thus WLOG assume the existence of a representative agent.

The main reason why the assumption of a representative agent simplifies the computation is the following rather trivial observation which, because of its importance, we state as a proposition.

Proposition 35.2 *If an asset is in zero net supply and the economy has a representative agent, then there is no trade in the asset in equilibrium.*

Proof If the agent sells some amount of the asset then, because of zero net supply and market clearing, someone else has to buy the same amount. But the agent is the only agent in the market, implying that in equilibrium there is no trade at all. □

35.2 Equilibrium

We now go on to study equilibrium in our model. Intuitively this is a situation where the agent is optimal and the market clears for the risk free asset. In view of Proposition 35.2 above, this implies that the optimal weight on B is zero. Thus we have $1 - w = 0$, implying that $w = 1$.

Definition 35.3 *An equilibrium of the model is a triple* $\{\hat{c}(t,x), \hat{w}(t,x), r(t,x)\}$ *of real valued functions such that the following hold:*

1. *Given the risk free rate process* $r(t, X_t)$*, the optimal consumption and investment are given by* $\hat{c}$ *and* $\hat{w}$ *respectively.*

2. *The market clears for the risk free asset, i.e.*

$$\widehat{w}(t,x) \equiv 1.$$

In order to determine the equilibrium risk free rate, we now go on to study the optimal consumption/investment problem for the representative agent. Note that the agent is a price taker in the sense that he/she considers the short rate process r, and hence also the risk free asset price process B, as exogenously given. The agent optimizes the portfolio given r, and our problem is thus to specify the short rate process in such a way that the optimal choice of the agent is $\widehat{w} \equiv 1$. At first sight this may sound like an impossibly hard problem, but it is in fact surprisingly easy to solve.

The HJB equation for the agent reads as

$$V_t + U(t,\hat{c}) + wx(\mu - r)V_x + (rx - \hat{c})V_x + \frac{1}{2}w^2x^2\sigma^2V_{xx} = 0, \tag{35.3}$$

where now r is shorthand for $r(t,x)$. The optimal consumption $\hat{c}$ and portfolio weight $\hat{w}$ are determined by the usual first order conditions:

$$U_c(t,\hat{c}) = V_x(t,x), \tag{35.4}$$

$$\hat{u}(t,x) = -\frac{(\mu - r)}{\sigma^2} \cdot \frac{V_x(t,x)}{xV_{xx}(t,x)}. \tag{35.5}$$

In equilibrium there will exist an equilibrium short rate $r(t,x)$, a martingale measure Q, a Girsanov kernel $\varphi(t,x)$, and a stochastic discount factor $\mathbf{M}(t,x)$. In order to obtain these objects we now simply plug the equilibrium condition $\widehat{w}_t \equiv 1$ into (35.5) to obtain our main result.

Proposition 35.4 *The following will hold:*

- *The equilibrium short rate is given by* $r(t,\widehat{X}_t)$ *where*

$$r(t,x) = \mu + \sigma^2 \frac{xV_{xx}(t,x)}{V_x(t,x)}. \tag{35.6}$$

- *The dynamics of the equilibrium wealth process are*

$$d\widehat{X}_t = \left(\mu\widehat{X}_t - \hat{c}_t\right)dt + \widehat{X}_t\sigma dW_t. \tag{35.7}$$

- *The Girsanov kernel has the form* $\varphi(t,\widehat{X}_t)$ *where*

$$\varphi(t,x) = \sigma \frac{xV_{xx}(t,x)}{V_x(t,x)}. \tag{35.8}$$

- *The equilibrium stochastic discount factor* $\mathbf{M}$ *is given by any of the formulas*

$$\mathbf{M}_t = \lambda U_c(t,\hat{c}_t), \tag{35.9}$$

$$\mathbf{M}_t = \lambda V_x(t,\widehat{X}_t), \tag{35.10}$$

where $\lambda = U_c(0,\hat{c}_0) = V_x(0,\widehat{X}_0)$.

- *The optimal value function V is determined by the* **equilibrium** *HJB equation*

$$\begin{cases} V_t + U(t,\hat{c}) + (\mu x - \hat{c})V_x + \dfrac{1}{2}\sigma^2 x^2 V_{xx} = 0, \\ V(T,x) = \Phi(x), \end{cases} \tag{35.11}$$

where $\hat{c}$ is determined by (35.4).

Proof Formulas (35.6), (35.7), and (35.11) follow directly when we plug the equilibrium condition $\widehat{u}_t \equiv 1$, into (35.5), into the state dynamics (35.2), and into the HJB equation (35.3). The representation (35.8) follows from (35.6) and the usual martingale condition $r = \mu + \sigma\varphi$. From (27.33) we have $\mathbf{M}_t = \lambda^{-1} U_c(t, \hat{c}_t)$, and from (35.4) we have $U_c(t,\hat{c}) = V_x(t,x)$, giving us (35.9)–(35.10). The expression for λ follows from the fact that $\mathbf{M}_0 = 1$. □

Note 35.2.1 *If, for a moment, we forget about equilibrium, view the control problem of the agent as a standard optimal control problem, with $r(t,x)$ as exogenously given, and plug in the first-order condition (35.5) into the HJB equation (35.3) we would get*

$$V_t + U(t,\hat{c}) + (rx - \hat{c})V_x - \frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2} \cdot \frac{V_x^2}{V_{xx}} = 0. \tag{35.12}$$

This is the **non-equilibrium** *HJB equation and we note that it is highly nonlinear in V_x and V_{xx}. The* **equilibrium** *HJB equation (35.11), on the other hand, is much simpler and we see that, apart from the $\hat{c}$ terms, it is in fact linear in V_x and V_{xx}. As we will see in Section 35.3, this is (of course) not a coincidence.*

35.3 Introducing a Central Planner

So far we have assumed that the economic setting is that of a **representative agent** investing and consuming in a market, and we have studied the equilibrium for that market.

An alternative to this setup is when, instead of a representative agent, we consider a **central planner**. The difference between these two concepts is that the central planner does not have access to a financial market, and in particular he/she does not have access to the risk free asset B.

The optimization problem for the central planner is simply that of maximizing expected utility when everything that is not consumed is invested in the production process. This obviously sounds very much like the problem of a representative agent who, in equilibrium, does not invest anything in the risk free asset, so a very natural conjecture is that the equilibrium consumption of the representative agent coincides with the optimal consumption of the central planner. We will see.

The formal problem of the central planner is to maximize

$$E^P\left[\int_0^T U(t, c_t)dt + \Phi(X_T)\right],$$

given the wealth dynamics

$$dX_t = (\mu X_t - c_t)dt + \sigma X_t dW_t.$$

The HJB equation for this problem is

$$\begin{cases} V_t + \sup_c \left\{ U(t,c) + (\mu x - c)V_x + \frac{1}{2}\sigma^2 x^2 V_{xx} \right\} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V(T,x) = \Phi(x) \end{cases}$$

with the usual first-order condition

$$U_c(t,c) = V_x(t,x).$$

Substituting the optimal c we thus obtain the PDE

$$\begin{cases} V_t + U(t,\hat{c}) + (\mu x - \hat{c})V_x + \frac{1}{2}\sigma^2 x^2 V_{xx} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad V(T,x) = \Phi(x) \end{cases}$$

and we see that this is identical to (35.11). We have thus proved the following result.

Proposition 35.5 *Given assumptions as above, the following hold:*

- *The optimal consumption for the central planner coincides with the equilibrium consumption of the representative agent.*
- *The optimal wealth process for the central planner is identical with the equilibrium wealth process for the representative agent.*

This result implies in particular that the following scheme is valid:

- Solve the (fairly simple) problem for the central planner, thus providing us with the optimal value function V.
- Define the "shadow interest rate" r by (35.6).
- Now forget about the central planner and consider the optimal consumption/investment problem of a representative agent with access to the production technology S and a risk free asset B with dynamics

$$dB_t = r(t, X_t)B_t dt$$

  where r is defined as above.
- The economy will then be in equilibrium, so $\widehat{w} = 1$, and we will recover the optimal consumption and wealth processes of the central planner.

35.4 Exercises

Exercise 35.1 Consider the case when $\Phi(x) = 0$, and

$$U(t,c) = e^{-\delta t} \ln(c).$$

(a) Analyze this case using DynP and the *Ansatz*

$$V(t,x) = f(t) \ln(x) + g(t).$$

(b) Analyze, as far as possible, the same problem using the martingale approach.

Exercise 35.2 Analyze the case of power utility, i.e. when

$$U(t,c) = e^{-\delta t} \frac{c^{1-\beta}}{1-\beta}$$

for $\beta > 0$.

35.5 Notes

The model in this section is a special case of the general production model of Cox et al. (1985a). For other textbook treatments of equilibrium theory see Back (2017), Dana and Jeanblanc (2003), Duffie (2001), Karatzas and Shreve (1998).

36

THE COX–INGERSOLL–ROSS FACTOR MODEL

We now go on to extend the model of the previous chapter to a factor model, which is a special case of the CIR factor model in Cox et al. (1985*a*).

36.1 The Model

In the model some objects are assumed to be given exogenously whereas other objects are determined by equilibrium, and we also have economic agents.

36.1.1 *Exogenous Objects*

We start with the exogenous objects.

Assumption 36.1.1 *The following objects are considered as given a priori*:

1. *A 2-dimensional Wiener process* W.
2. *A scalar factor process* Y *of the form*
$$dY_t = a(Y_t)dt + b(Y_t)dW_t \tag{36.1}$$
where a *is a scalar real valued function and* b *is a two-dimensional row vector function.*
3. *A constant returns-to-scale production technology process* S *with dynamics*
$$dS_t = \mu(Y_t)S_t dt + S_t\sigma(Y_t)dW_t. \tag{36.2}$$

The interpretation of this is that Y is a process which in some way influences the economy. It could for example describe the weather. The interpretation of the production technology is as in Chapter 35 and we have again a short selling constraint.

36.1.2 *Endogenous Objects*

In this model we also have some processes that are to be determined endogenously in equilibrium. They are as follows, where we use the notation

$$X_t = \text{the portfolio value at time } t,$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

to be more precisely defined below:

1. A risk free asset B, in zero net supply, with dynamics
$$dB_t = r_t B_t dt$$
where the risk free rate r is assumed to be of the form
$$r_t = r(t, X_t, Y_t).$$
2. A financial derivative in zero net supply, with price process $F(t, X_t, Y_t)$, representing the price process of a T-claim $H(X_T, Y_T)$. F will have dynamics of the form
$$dF = \mu_F F dt + F \sigma_F dW_t$$
where μ_F and σ_F are of the form
$$\mu_F = \mu_F(t, X_t, Y_t), \quad \sigma_F = \sigma_F(t, X_t, Y_t),$$
and will be determined in equilibrium.

We also need an important assumption.

Assumption 36.1.2 *We assume that the* 2×2 *diffusion matrix*
$$\begin{pmatrix} -\sigma- \\ -\sigma_F- \end{pmatrix}$$
is invertible P*-a.s. for all* t*.*

The implication of this assumption is that, apart from the short selling constraint for S, the market consisting of S, F, and B is complete.

36.1.3 *Economic Agents*

The basic assumption in Cox et al. (1985a) is that there are a finite number of agents with identical initial capital, identical beliefs about the world, and identical preferences. In the present complete market setting this implies that we may as well consider a single representative agent. The object of the agent is (loosely) to maximize expected utility of the form
$$E^P\left[\int_0^T U(t, c_t, Y_t) dt\right],$$
where c is the consumption rate (measured in dollars per time unit) and U is the utility function. We see that the underlying non-financial factor Y is allowed to affect the utility function.

36.2 The Portfolio Problem

In this section we discuss the relevant portfolio theory, formulate the agent's optimal control problem and derive the relevant HJB equation.

36.2.1 *Portfolio Dynamics*

The agent can invest in S, F, and B. We will use the following notation:

$$\begin{aligned} X &= \text{portfolio market value}, \\ w^S &= \text{portfolio weight on } S, \\ w^F &= \text{portfolio weight on } F, \\ 1 - w^S - w^F &= \text{portfolio weight on } B. \end{aligned}$$

Using standard theory we see that the portfolio dynamics are given by

$$dX_t = w_t^S X_t \frac{dS_t}{S_t} + w_t^F X_t \frac{dF_t}{F_t} + (1 - w_t^S - w_t^F) X_t \frac{dB_t}{B_t} - c_t dt$$

where, for simplicity of notation, lower-case index t always indicates running time, but where other variables are suppressed. This gives us the portfolio dynamics as

$$dX_t = X_t \left\{ w^S(\mu - r) + w^F(\mu_F - r) \right\} dt + (rX_t - c)\, dt + X_t \left\{ w^S \sigma + w^F \sigma_F \right\} dW_t,$$

and we write this more compactly as

$$dX_t = X_t \mu_X(t, X_t, Y_t, u_t) dt - c_t dt + X_t \sigma_X(t, X_t, Y_t, u_t) dW_t, \tag{36.3}$$

where we use the shorthand notation

$$u = (w^S, w^F),$$

and where μ_X and σ_X are defined by

$$\mu_X = w^S [\mu - r] + w^F [\mu_F - r] + r, \tag{36.4}$$

$$\sigma_X = w^S \sigma + w^F \sigma_F. \tag{36.5}$$

36.2.2 *The Control Problem and the HJB Equation*

The control problem for the agent is, given r and F, to maximize

$$E^P \left[\int_0^\tau U(t, c_t, Y_t) dt \right],$$

where

$$\tau = \inf \{ t \geq 0 : \ X_t = 0 \} \wedge T$$

subject to the portfolio dynamics

$$dX_t = X_t \mu_X(t, X_t, Y_t, u_t) dt - c_t dt + X_t \sigma_X(t, X_t, Y_t, u_t) dW_t,$$

and the control constraints

$$c \geq 0, \quad w^S \geq 0.$$

The HJB equation for this is straightforward and reads as

$$\begin{cases} V_t + \sup_{c,u} \{U + \mathbf{A}^u V\} = 0, \\ V(T,x) = 0, \\ V(t,0) = 0, \end{cases} \tag{36.6}$$

where the infinitesimal operator $\mathbf{A}^u$ is defined by

$$\mathbf{A}^u V = (x\mu_X - c)V_x + aV_y + \frac{1}{2}x^2\sigma_X^2 V_{xx} + \frac{1}{2}b^2 V_{yy} + x\sigma_X bV_{xy}.$$

Here, for vectors h and g in R^2, we have used the notation

$$\begin{aligned} hg &= (h,g), \\ g^2 &= \|g\|^2, \\ h^2 &= \|h\|^2, \end{aligned}$$

where (b,g) denotes inner product.

36.3 Equilibrium

Since B and F are in zero net supply, we have the following definition of equilibrium.

Definition 36.1 *An equilibrium is a list of processes*

$$\{r, \mu_F, \sigma_F, w^S, w^F, c, V\}$$

such that (V, w^S, w^F, c) *solves the HJB equation given* (r, μ_F, σ_F)*, and the market clearing conditions*

$$w_t^S = 1, \quad w_t^F = 0$$

are satisfied.

We will now study the implications of the equilibrium conditions on the short rate r and the dynamics of F. We do this by studying the first-order conditions for optimality in the HJB equations, with the equilibrium conditions in force.

The first-order conditions (FOC), with the equilibrium conditions $w^S = 1$ and $w^F = 0$ inserted, are easily seen to be as follows:

$$(w^S) \qquad x(\mu - r)V_x + x^2\sigma^2 V_{xx} + x\sigma bV_{xy} = 0, \tag{36.7}$$

$$(w^F) \qquad x(\mu_F - r)V_x + x^2\sigma\sigma_F V_{xx} + xb\sigma_F V_{xy} = 0, \tag{36.8}$$

$$(c) \qquad U_c = V_x, \tag{36.9}$$

where (w^S) indicates that it is the FOC for w^S etc.

Substituting the equilibrium conditions $w^S = 1$ and $w^F = 0$ into the HJB equation and the portfolio dynamics, gives us the following result.

Proposition 36.2 (The Equilibrium HJB Equation) *In equilibrium, the following hold*:

- *The HJB equations takes the form*

$$\begin{cases} V_t + U(t,\hat{c}) + (\mu x - \hat{c})V_x + aV_y + \frac{1}{2}x^2\sigma^2 V_{xx} + \frac{1}{2}b^2 V_{yy} + xb\sigma V_{xy} = 0 \\ V(T,x,y) = 0 \\ V(t,0,y) = 0 \end{cases} \tag{36.10}$$

  where $\hat{c}$ is determined by (36.9).
- *The equilibrium portfolio dynamics are given by*

$$d\hat{X}_t = (\mu\hat{X}_t - \hat{c}_t)dt + \hat{X}_t\sigma dW_t. \tag{36.11}$$

Remark 36.3.1 *We will see below that "everything" in the model, like the risk free rate, the Girsanov kernel, the stochastic discount factor, etc, is determined by the equilibrium optimal value function V. It is then important, and perhaps surprising, to note that the equilibrium HJB equation (36.10) is completely determined by* **exogenous data**, *i.e. by the Y and S dynamics. In other words, the equilibrium short rate, the stochastic discount factor, etc, do* **not** *depend on the particular choice of derivative F that we use in order to complete the market.*

36.4 The Short Rate and the Risk Premium for F

From the FOC (36.7) for w^S we immediately obtain our first main result.

Proposition 36.3 *The equilibrium short rate $r(t,x,y)$ is given by*

$$r = \mu + \sigma^2 \frac{xV_{xx}}{V_x} + \sigma b \frac{V_{xy}}{V_x} \tag{36.12}$$

From the equilibrium optimality condition (36.8) for w^F we have the following result.

Proposition 36.4 *The risk premium for F in equilibrium is given by*

$$\mu_F - r = -\left[\frac{xV_{xx}}{V_x}\sigma\sigma_F + \frac{V_{xy}}{V_x}b\sigma_F\right]. \tag{36.13}$$

36.5 The Equilibrium Stochastic Discount Factor

Since every equilibrium must be arbitrage free, we can in fact push the analysis further. We denote by φ the Girsanov kernel for the likelihood process $L = \frac{dQ}{dP}$, so L has dynamics

$$dL_t = L_t\varphi_t dW_t.$$

We know from arbitrage theory that the martingale conditions for S and F are

$$r = \mu + \sigma\varphi,$$
$$r = \mu_F + \sigma_F\varphi.$$

On the other hand we have, from (36.12) and (36.13),

$$r = \mu + \sigma\left\{\frac{xV_{xx}}{V_x}\sigma + \frac{V_{xy}}{V_x}b\right\},$$
$$r = \mu_F + \sigma_F\left\{\frac{xV_{xx}}{V_x}\sigma + \frac{V_{xy}}{V_x}b\right\}.$$

Using Assumption 36.1.2 we can thus solve for the vector φ to obtain the following important result.

Proposition 36.5 *The Girsanov kernel φ is given by*

$$\varphi = \frac{xV_{xx}}{V_x}\sigma + \frac{V_{xy}}{V_x}b. \tag{36.14}$$

Moving on to the stochastic discount factor, we recall from (27.33) the formula

$$U_c(t, \widehat{c}_t) = \lambda\mathbf{M}_t,$$

and we also recall the first-order condition (36.9) which says that $U_c = V_x$ holds along the optimal path. We thus have the following result.

Proposition 36.6 *The equilibrium stochastic discount factor is given by any of the formulas*

$$\mathbf{M}_t = \frac{U_c(t, \hat{c}_t)}{U_c(0, \hat{c}_0)}, \tag{36.15}$$

$$\mathbf{M}_t = \frac{V_x(t, \widehat{X}_t, Y_t)}{V_x(0, x_0, y_0)}. \tag{36.16}$$

The more adventurous reader can also derive (36.15)–(36.16) using a bare hands approach by first recalling from Proposition 14.8 that the dynamics of $Z_t = \lambda\mathbf{M}_t$ are given by

$$dZ_t = -rZ_t dt + Z_t\varphi dW_t.$$

We can then use the Itô formula on $V_x(t, \widehat{X}_t, Y_t)$ and the Envelope Theorem D.3 on the HJB equation in equilibrium to compute dV_x. This is quite messy, but after lengthy calculations one obtains

$$dV_x = -rV_x dt + V_x\varphi dW_t,$$

with r and φ as in (36.12) and (36.14). Comparing this with the Z dynamics above implies that $V_x = \mathbf{M}$ up to a multiplicative constant thus proving (36.16), and the optimality condition $U_c = V_x$ gives us (36.15).

36.6 Risk Neutral Valuation

From general arbitrage theory we then immediately have the standard risk neutral valuation formula

$$F(t,x,y) = E^{Q}_{t,x,y}\left[e^{-\int_t^T r_s ds} H(X_T, Y_T)\right], \tag{36.17}$$

where H is the contract function for F. We have already determined the Girsanov kernel φ by (36.14) so the equilibrium Q-dynamics of X and Y are given by

$$\begin{aligned} d\hat{X}_t &= \hat{X}_t\left[\mu + \sigma\varphi\right]dt - \hat{c}_t dt + \hat{X}_t \sigma dW_t^Q, \\ dY_t &= \left[a + b\varphi\right]dt + bdW_t^Q. \end{aligned}$$

We thus deduce that the pricing function F is the solution of the PDE

$$\begin{cases} F_t + F_x x(\mu + \sigma\varphi) - cF_x + \dfrac{1}{2}x^2\sigma^2 F_{xx} \\ +F_y(a + b\varphi) + \dfrac{1}{2}F_{yy}b^2 + xF_{xy}b\sigma - rF = 0 \\ \qquad\qquad F(T,x,y) = H(x,y), \end{cases} \tag{36.18}$$

which is Kolmogorov backward equation for the expectation above.

36.7 Introducing a Central Planner

As in Section 35.3 we now introduce a central planner into the economy. This means that there is no market for B and F, so the central planner only chooses the consumption rate, invests everything into S, and the problem is thus to maximize

$$E^P\left[\int_0^\tau U(t, c_t, Y_t)dt + \Phi(X_T)\right]$$

subject to the dynamics

$$\begin{aligned} dX_t &= (\mu X_t - c)dt + X_t \sigma dt, \\ dY_t &= a(Y_t)dt + b(Y_t)dW_t \end{aligned}$$

and the constraint $c \geq 0$.

The Bellman equation for this problem is

$$\begin{cases} V_t + \sup\limits_c \left\{ U + (\mu x - c)V_x + aV_y + \dfrac{1}{2}\sigma^2 x^2 V_{xx} + \dfrac{1}{2}b^2 V_{yy} + xb\sigma V_{xy} \right\} = 0 \\ \qquad\qquad V(T,x) = \Phi(x) \\ \qquad\qquad V(t,0) = 0. \end{cases}$$

We now see that this is exactly the equilibrium Bellman equation (36.10) in the CIR model. We thus have the following result.

Proposition 36.7 *Given assumptions as above, the following hold:*

- *The optimal consumption for the central planner coincides with the equilibrium consumption of the representative agent.*
- *The optimal wealth process for the central planner is identical with the equilibrium wealth process for the representative agent.*

This result implies in particular that the following scheme is valid:

- Solve the (fairly simple) problem for the central planner, thus providing us with the optimal value function V.
- Define the "shadow interest rate" r by (36.12), and the Girsanov kernel φ by (36.14).
- For a derivative with contract function H, define F by (36.17) and (36.18).
- Define and σ_F and μ_F by applying Itô to $F(t, X_t, Y_t)$.
- The F dynamics will now be
$$dF = \mu_F F dt + F h dW_t.$$
- Now forget about the central planner and consider the optimal consumption/investment problem of a representative agent with access to the production technology S, the derivative F, and the risk free asset B with dynamics
$$dB_t = r(t, X_t) B_t dt$$
where r is defined as above.
- The economy will then be in equilibrium, so $w^S = 1$, $w^F = 0$ and we will recover the optimal consumption and wealth processes of the central planner.

36.8 Exercises

Exercise 36.1 Assume that the utility function U does not depend on y and has the form
$$U(t,c) = e^{-\delta t} \frac{1}{1-\beta} c^{1-\beta}$$
where β and δ are positive real numbers. We also assume that $\Phi = 0$.

(a) Prove that the equilibrium optimal value function $V(t,x,y)$ has the form
$$V(t,x,y) = U(t,x) f(t,y)$$
with U as above, where f satisfies the PDE
$$\begin{cases} f_t + [a + b\sigma(1-\beta)] f_y + \dfrac{b^2}{2} f_{yy} + \beta f^{1-\frac{1}{\beta}} + \left[\mu(1-\beta) - \delta - \dfrac{1}{2}\sigma^2\beta(1-\beta)\right] f = 0, \\ \hfill f(T,y) = 0. \end{cases}$$

(b) Compute (in terms of f) the equilibrium interest rate r and the Girsanov kernel φ. Note that they do not depend on the x-variable.

Exercise 36.2 With $\Phi = 0$, assume and that we have log utility, i.e.

$$U(t, c, y) = e^{-\delta t} \ln(c).$$

(a) Show that the HJB equation has a solution of the form

$$V(t, x, y) = e^{-\delta t} f(t, y) \ln(x) + e^{-\delta t} g(t, y)$$

and derive the relevant PDEs for f and g.

(b) Solve the PDE for f and derive explicit expressions for r and φ.
Hint: Feynman–Kač.

36.9 Notes

This model studied above is a special case of the model presented in Cox et al. (1985a), where the authors also allow for several production processes, but where only PDE methods are used. A very general multi-agent equilibrium model, allowing for several production processes, as well as endowments, is studied in detail in Karatzas et al. (1990).

37

THE COX–INGERSOLL–ROSS INTEREST RATE MODEL

We now specialize to the model in Cox et al. (1985*b*). In this model the authors study power utility, but all concrete formulas are actually derived under the assumption of log utility, i.e.

$$U(t,c,y) = e^{-\delta t}\ln(c), \tag{37.1}$$

so we restrict ourselves to this particular case. We still assume an underlying factor model of the form

$$dS_t = \mu(Y_t)S_t dt + S_t\sigma(Y_t)dW_t, \tag{37.2}$$

$$dY_t = a(Y_t)dt + b(Y_t)dW_t. \tag{37.3}$$

Given the assumption of log utility, it is easy to see that the equilibrium HJB equation has a solution of the form

$$V(t,x,y) = e^{-\delta t}f(t,y)\ln(x) + e^{-\delta t}g(t,y)$$

and we obtain the following PDE for f

$$\begin{cases} f_t + af_y + \dfrac{1}{2}b^2 f_{yy} - \delta f + 1 = 0, \\ \hfill F(T,y) = 0. \end{cases}$$

Using Feynman–Kač it follows that f is given by the formula

$$f(t,y) = \frac{1}{\delta}\left[1 - e^{-\delta(T-t)}\right]$$

so we have

$$\frac{xV_{xx}}{V_x} = -1, \quad \frac{V_{xy}}{V_x} = 0,$$

and plugging this into the formula (36.12) gives us the short rate as

$$r(t,y) = \mu(y) - \sigma^2(y). \tag{37.4}$$

In view of this formula it is now natural to specialize further by assuming that

$$\mu(y) = \mu \cdot y,$$
$$\sigma(y) = \sigma \cdot \sqrt{y},$$

which means that the S dynamics are of the form

$$dS_t = \mu S_t Y_t dt + \sigma S_t \sqrt{Y_t} dW_t.$$

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

This gives us

$$r(t, y) = (\mu - \sigma^2)y.$$

Now, in order to have a positive Y process, which is necessary for $\sqrt{Y_t}$ to make sense, we introduce the assumption that the Y dynamics are of the form

$$dY_t = \{AY_t + B\}\, dt + b\sqrt{Y_t} dW_t \tag{37.5}$$

where A, B and b are positive constants (see Section 21.2.4) so in the earlier notation we have

$$a(y) = Ay + B,$$
$$b(y) = b\sqrt{y}$$

and, using (36.14) for the Girsanov kernel, we have proved the following result.

Proposition 37.1 *For the CIR model, with log utility and (S, Y) dynamics of the form*

$$dS_t = \mu S_t Y_t dt + S_t \sqrt{Y_t} \sigma dW_t, \tag{37.6}$$
$$dY_t = \{AY_t + B\}\, dt + \sqrt{Y_t} b dW_t, \tag{37.7}$$

where σ and b are two-dimensional row vectors, and where $2A \geq \|b\|^2$ the following hold:

- *The short rate is given by*

$$r(t, Y_t) = (\mu - \sigma^2)Y_t.$$

- *The short rate dynamics under P are*

$$dr_t = [A + B_0]\, dt + b_0 \sqrt{r_t} dW_t,$$

  where

$$B_0 = (\mu - \sigma^2)B,$$
$$b_0 = \sqrt{\mu - \sigma^2} b.$$

- *The Girsanov kernel is given by*

$$\varphi(t, y) = -\sigma\sqrt{y}.$$

- *The Q dynamics of r are*

$$dr_t = [A_0 r_t + B_0]\, dt + b_0 \sqrt{r_t} dW_t^Q$$

  where

$$A_0 = A - \sigma b\sqrt{\mu - \sigma^2}.$$

Remark 37.0.1 *The condition guaranteeing that the Y equation has a positive solution is*

$$2A \geq \|b\|^2.$$

This will also guarantee that the SDE for the short rate has a positive solution. In order to have a positive short rate we obviously also need to assume that

$$\mu \geq \|\sigma\|^2.$$

37.1 Exercises

Exercise 37.1 Compute the stochastic discount factor and the Girsanov kernel for the model (37.2)–(37.3).

Exercise 37.2 Compute the stochastic discount factor for the model (37.6)–(37.7).

Exercise 37.3 This exercise shows that you can basically generate an arbitrarily chosen process as the short rate in a CIR model with log utility.

Consider the CIR setting above with log utility, but with production dynamics of the form

$$dS_t = \mu S_t Y_t dt + \sigma S_t dW_t,$$

where σ is a constant 2-dimensional row vector and the Y-dynamics are arbitrary. Compute the short rate.

37.2 Notes

The model in this section is the one presented in Cox et al. (1985*b*).

38

ENDOWMENT EQUILIBRIUM: UNIT NET SUPPLY

In the previous chapters we have studied equilibrium models in economies with a production technology. An alternative to that setup is to model an economy where each agent is exogenously endowed with a stream of income/consumption. This can be done in several ways, and we start with the simplest one, characterized by unit net supply of risky assets. There is also a corresponding zero net supply version of the theory, but since the results turn out to be identical we confine ourselves to the unit net supply theory.

38.1 The Model

In the model some objects are assumed to be given exogenously whereas other objects are determined by equilibrium, and we also have economic agents.

38.1.1 *Exogenous Objects*

We start with the exogenous objects.

Assumption 38.1.1 *The following objects are considered as given a priori:*

1. *A 1-dimensional Wiener process W.*
2. *A scalar and strictly positive process e of the form*
$$de_t = a(e_t)dt + b(e_t)dW_t \tag{38.1}$$
where a and b is a scalar real valued functions.

The interpretation of this is that e is an **endowment process** which provides the owner with a consumption stream at the rate e_t units of the consumption good per unit time, so during the time interval $[t, t+dt]$ the owner will obtain $e_t dt$ units of the consumption good. A more concrete interpretation is that you have an orchard (like the cherry orchard in the Chekhov play). You do not invest anything in the orchard and you do not work: The cherries simply fall off the trees for you to eat. Based on Lucas (1978), the endowment process is in fact often referred to as a "Lucas tree" or a "Lucas orchard".

38.1.2 *Endogenous Objects*

The endogenous objects in the model are as follows:

1. A risk free asset B, in zero net supply, with dynamics
$$dB_t = r_t B_t dt$$
where the risk free rate r is determined in equilibrium.

Arbitrage Theory in Continuous Time. Tomas Björk, Oxford University Press (2020).
© Tomas Björk. DOI: 10.1093/oso/9780198851615.001.0001

2. A price dividend pair (S, D) in unit net supply, where by assumption

$$dD_t = e_t dt.$$

In other words: Holding the asset S provides the owner with the dividend process e over the time interval $[0, T]$. Since S is defined in terms of e we can write the dynamics of S as

$$dS_t = \alpha_t S_t dt + \gamma_t S_t dW_t$$

where α and γ will be determined in equilibrium.

3. We stress the fact that, apart for providing the owner with the dividend process e over $[0, T]$, the asset S gives no further benefits to the owner. In equilibrium we will thus have

$$S_t = \frac{1}{\mathbf{M}_t} E^P \left[\int_t^T \mathbf{M}_s e_s ds \middle| \mathcal{F}_t \right],$$

where $\mathbf{M}$ is the equilibrium stochastic discount factor. In particular we will have

$$S_T = 0.$$

38.1.3 *Economic Agents*

We consider a single representative agent who wants to maximize expected utility of the form

$$E^P \left[\int_0^T U(t, c_t) dt \right]$$

where c is the consumption rate (measured in dollars per time unit) and U is the utility function.

Assumption 38.1.2 *We assume that the utility function U is smooth, increasing in c, strictly concave, and satisfying $U_c(t, 0) = +\infty$. We also assume that the agent has initial wealth $X_0 = S_0$. In other words: The agent has enough money to buy the right to the dividend process e. Another interpretation is that the agent is the initial owner of the single existing unit of S (unit net supply). At $t = 0$ she sells this unit at the price S_0 and then the game is on.*

We will use the notation

$$\begin{aligned} w_t &= \text{portfolio weight on the risky asset}, \\ 1 - w_t &= \text{portfolio weight on the risk free asset}, \\ c_t &= \text{rate of consumption}. \end{aligned}$$

38.1.4 *Equilibrium Conditions*

The natural equilibrium conditions are that the agent will, at all times, hold the risky asset and that she will consume all dividends. Formally this reads as follows:

$$\begin{aligned} w_t &= 1, \quad \text{(the asset } (S,D) \text{ in unit net supply)},\\ 1 - w_t &= 0, \quad (B \text{ in zero net supply}),\\ c_t &= e_t, \quad \text{(market clearing for consumption)}. \end{aligned}$$

Note that the market clearing condition $c_t = e_t$ for consumption is by no means self-evident. In particular it implies that you cannot save or invest the endowment $e_t dt$ that you get at time t in order to consume more at a later date. One possible interpretation is that the consumption good is instantly perishable: The cherries you harvest at time t are rotten already at time $t + dt$.

38.2 The Martingale Approach

We now go on to study the model above using the martingale approach. It is also possible to analyze the model using dynamic programming, but the martingale approach is much easier.

38.2.1 *The Control Problem*

We assume again that the initial wealth of the agent is given by $X_0 = S_0$. Since we have a complete market we can rely on the usual martingale approach arguments to formulate the agent's control problem as follows.

Problem 38.2.1

$$\underset{c}{\text{maximize}} \; E^P \left[\int_0^T U(t, c_t) dt \right]$$

subject to the constraints

$$c_t \geq 0, \qquad E^P \left[\int_0^T \mathbf{M}_t c_t dt \right] \leq S_0.$$

As usual, $\mathbf{M}$ denotes the stochastic discount factor. The first constraint is obvious and the second one is the budget constraint.

Since the asset S provides the owner with the income stream defined by e and nothing else we can apply arbitrage theory and risk neutral valuation to deduce that

$$S_0 = E^P \left[\int_0^T \mathbf{M}_t e_t dt \right].$$

We can thus rewrite the budget constraint as

$$E^P\left[\int_0^T \mathbf{M}_t c_t dt\right] \le E^P\left[\int_0^T \mathbf{M}_t e_t dt\right],$$

so our problem can be formulated as follows.

Problem 38.2.2

$$\text{maximize } E^P\left[\int_0^T U(t, c_t) dt\right]$$

subject to the constraints

$$c_t \ge 0, \qquad E^P\left[\int_0^T \mathbf{M}_t c_t dt\right] \le E^P\left[\int_0^T \mathbf{M}_t e_t dt\right].$$

It follows from the assumptions on U that the optimal consumption will always be strictly positive. The relevant Lagrangian is thus given by

$$E^P\left[\int_0^T \{U(t, c_t) - \lambda \mathbf{M}_t c_t dt\}\right] + \lambda E^P\left[\int_0^T \mathbf{M}_t e_t dt\right],$$

where λ is the Lagrange multiplier. The optimality condition for c is thus

$$U_c(t, c_t) = Z_t, \tag{38.2}$$

where

$$Z_t = \lambda \mathbf{M}_t. \tag{38.3}$$

38.2.2 *Equilibrium*

It is now surprisingly easy to derive formulas for the equilibrium short rate, stochastic discount factor, and Girsanov kernel. The clearing condition $\hat{c}_t = e_t$ and the optimality condition (38.2) gives us

$$Z_t = U_c(t, e_t),$$

so we can use the Itô formula to obtain

$$dZ_t = \left\{U_{ct}(t, e_t) + a(e_t) U_{cc}(t, e_t) + \frac{1}{2} b^2(e_t) U_{ccc}(t, e_t)\right\} dt + b(e_t) U_{cc}(t, e_t) dW_t.$$

As usual we refer to (14.39) to obtain

$$dZ_t = -r_t Z_t dt + Z_t \varphi_t dW_t,$$

and we can now directly identify the equilibrium rate and the Girsanov kernel as follows.

Proposition 38.1 *The equilibrium stochastic discount factor is given by*

$$\mathbf{M}_t = \frac{U_c(t,e_t)}{U_c(0,e_0)}. \tag{38.4}$$

The equilibrium short rate is given by

$$r(t,e) = -\frac{U_{ct}(t,e) + a(e)U_{cc}(t,e) + \frac{1}{2}b^2(e)U_{ccc}(t,e)}{U_c(t,e)}. \tag{38.5}$$

The equilibrium Girsanov kernel is given by

$$\varphi(t,e) = \frac{U_{cc}(t,e)}{U_c(t,e)} \cdot b(e). \tag{38.6}$$

Remark 38.2.1 *In the proof above it seems that we have only used the consumption market clearing condition*

$$c_t \equiv e_t,$$

and not at all the clearing conditions for the risky and the risk free assets

$$1 - u_t \equiv 0,$$
$$u_t \equiv 1.$$

It is, however, easy to see that the clearing condition $c = e$ actually implies the other two. The equilibrium consumption stream $c = e$ discussed above can certainly be replicated by holding exactly one unit of (S,D) and putting zero weight on the risk free asset. It now follows from market completeness and the Martingale Representation Theorem that this is the unique portfolio providing the holder with the consumption stream e.

38.2.3 *Log Utility*

To exemplify we now specialize to the log utility case when the local utility function is of the form

$$U(t,c) = e^{-\delta t}\ln(c).$$

In this case we have

$$U_c = \frac{1}{c}e^{-\delta t}, \quad U_{tc} = -\frac{\delta}{c}e^{-\delta t}, \quad U_{cc} = -\frac{1}{c^2}e^{-\delta t}, \quad U_{ccc} = \frac{2}{c^3}e^{-\delta t}.$$

Plugging this into the formula (38.5) gives us the short rate as

$$r(t,e) = \delta + \frac{a(e)}{e} - \frac{b^2(e)}{e^2}.$$

Given this expression it is natural to specialize further to the case when the e dynamics are of the form

$$de_t = ae_t dt + be_t dW_t,$$

where (with a slight abuse of notation) a and b in the right-hand side are real constants, so that

$$a(e) = a \cdot e, \quad b(e) = b \cdot e.$$

We then obtain a constant short rate of the form

$$r = \delta + a - b^2.$$

38.3 Extending the Model

In the previous sections we have assumed that the endowment process e satisfies an SDE of the form

$$de_t = a(e_t)dt + b(e_t)dW_t.$$

A natural extension of this setup would of course be to consider a factor model of the form

$$\begin{aligned} de_t &= a(e_t, Y_t)dt + b(e_t, Y_t)dW_t, \\ dY_t &= \mu(Y_t)dt + \sigma(Y_t)dW_t \end{aligned}$$

where Y is an underlying factor process, and W is a two-dimensional Wiener process. In this section we will extend our endowment theory to include a fairly general model for the endowment process, and as a special case we will consider the factor model above.

38.3.1 *The General Scalar Case*

We extend the earlier model by simply assuming that the (scalar) endowment process has the structure

$$de_t = a_t dt + b_t dW_t, \tag{38.7}$$

where W is a k-dimensional Wiener process, and where the scalar process a and the k-dimensional row vector process b are adapted to some given filtration $\mathbf{F}$ (which, apart from W, may include many other driving processes). This setup will obviously include various factor models, and will also include non-Markovian models. We assume that we have $N+1$ random sources in the model.

In order to build the model we introduce as usual the asset-dividend pair (S, D) where

$$dD_t = e_t dt,$$

and we assume, as before, that S is in unit net supply. The interpretation is again that S provides the holder with the endowment e (and nothing else). We then introduce a risk free asset B and a number of derivatives $F_1, \ldots, F_N$ which are defined in terms of the random sources, so that the market consisting of $S, B, F_1, \ldots, F_N$ is complete.

We can now apply the usual martingale approach, and a moment of reflection will convince you that the argument in Section 38.2.2 goes through without any essential change. We thus conclude that for this extended model we have the following result.

Proposition 38.2 *If the endowment process e has dynamics according to (38.7), then the following hold:*

- *The equilibrium stochastic discount factor is given by*

$$\mathbf{M}_t = \frac{U_c(t, e_t)}{U_c(0, e_0)}. \tag{38.8}$$

- *The equilibrium short rate process is given by*

$$r_t = -\frac{U_{ct}(t,e) + a_t U_{cc}(t,e_t) + \frac{1}{2}\|b_t\|^2 U_{ccc}(t,e_t)}{U_c(t,e_t)}. \tag{38.9}$$

- *The Girsanov kernel is given by*

$$\varphi_t = \frac{U_{cc}(t,e_t)}{U_c(t,e_t)} \cdot b_t. \tag{38.10}$$

Remark 38.3.1 *We note that the measure transformation from P to Q only affects the Wiener process W driving the endowment process e. The distribution of other driving processes will thus* **not** *be changed.*

38.3.2 *A Factor Model*

We exemplify the theory of the Section 38.3.1 by considering a factor model of the form

$$de_t = a(e_t, Y_t)dt + b(e_t, Y_t)dW_t, \tag{38.11}$$

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t. \tag{38.12}$$

where W is a standard two-dimensional Wiener process. For simplicity we assume log utility, so

$$U(t,c) = e^{-\delta t}\ln(c).$$

In this case the equilibrium rate and the Girsanov kernel will be of the form $r_t = r(e_t, Y_t)$, $\varphi_t = \varphi(e_t, Y_t)$ and after some easy calculations we obtain

$$r(e,y) = \delta + \frac{a(e,y)}{e} - \frac{\|b(e,y)\|^2}{e^2},$$

$$\varphi(e,y) = -\frac{b(e,y)}{e}.$$

Given these expressions it is natural to make the further assumption that a and b are of the form

$$a(e,y) = e \cdot a(y),$$

$$b(e,y) = e \cdot b(y),$$

which implies

$$
\begin{aligned}
r(y) &= \delta + a(y) - \|b(y)\|^2 \\
\varphi(y) &= -b(y).
\end{aligned}
$$

The obvious idea is now to specialize further to the case when

$$
\begin{aligned}
a(y) &= a \cdot y, \\
b(y) &= \sqrt{y} \cdot b,
\end{aligned}
$$

and in order to guarantee positivity of Y we assume

$$
\begin{aligned}
\mu(y) &= \beta + \mu \cdot y, \\
\sigma(y) &= \sigma \cdot \sqrt{y}
\end{aligned}
$$

where $2\beta \geq \|\sigma\|^2$. We then have the following result.

Proposition 38.3 *Assume that the model has the structure*

$$
\begin{aligned}
de_t &= ae_t Y_t dt + e_t b \sqrt{Y_t} dW_t, \\
dY_t &= \{\beta + \mu Y_t\} dt + \sigma \sqrt{Y_t} dW_t.
\end{aligned}
$$

Then the equilibrium short rate and the Girsanov kernel are given by

$$
\begin{aligned}
r_t &= \delta + \left(a - \|b\|^2\right) Y_t, \\
\varphi_t &= \sqrt{Y_t} \cdot b.
\end{aligned}
$$

We thus see that we have essentially re-derived the Cox–Ingersoll–Ross short rate model, but now within an endowment framework.

We finish this section with a remark on the structure of the Girsanov transformation. Let us assume that, for a general utility function $U(t, c)$, the processes e and Y are driven by independent Wiener processes, so the model has the form

$$
\begin{aligned}
de_t &= a(e_t, Y_t)dt + b(e_t, Y_t)dW_t^e, \\
dY_t &= \mu(Y_t)dt + \sigma(Y_t)dW_t^Y,
\end{aligned}
$$

where W^e and W^Y are independent and where b and σ are scalar. Then the Girsanov kernel has the vector form

$$
\varphi_t = \frac{U_{cc}(t, e_t)}{U_c(t, e_t)} \cdot [b(e_t, Y_t), 0]
$$

so the likelihood dynamics are

$$
dL_t = L_t \frac{U_{cc}(t, e_t)}{U_c(t, e_t)} \cdot b(e_t, Y_t) dW_t^e,
$$

implying that the Girsanov transformation will only affect W^e and not W^Y.

38.4 Several Endowment Processes

A natural extension of the model in Section 38.3.1 would be to consider, not only one scalar endowment process, but a finite number of endowment processes $e_1, \dots e_d$, with dynamics

$$de_{it} = a_{it}dt + b_{it}dW_t, \quad i = 1, \dots, d, \tag{38.13}$$

where W is k-dimensional Wiener (with $k \geq d$) and we have $N+1$ random sources in the filtration. We then introduce the price dividend pairs $(S_1, D_1), \dots (S_d, D_d)$ where

$$dD_{it} = e_{it}dt.$$

As usual we assume that the risky $S_1, \dots, S_d$ assets are in unit net supply and that asset i gives the owner the right to the dividend process D_i. We also assume the existence of a risk free asset B, and we assume the existence of a sufficient number of derivative assets in zero net supply, such that the market is complete.

This model looks, prima facie, more general than the model of Section 38.3.1, but this is in fact not the case.

Assuming a representative agent with utility $U(t, c)$ and denoting the aggregate endowment by η so

$$\eta_t = \sum_{i=1}^{d} e_{it} \tag{38.14}$$

we see that the optimization problem of the representative agent is to maximize expected utility

$$E^P\left[\int_0^T U(t, c_t)dt\right]$$

subject to the (aggregate) budget constraint

$$E^P\left[\int_0^T \mathbf{M}_t c_t dt\right] \leq E^P\left[\int_0^T \mathbf{M}_t \eta_t dt\right].$$

The equilibrium market clearing condition is of course

$$\hat{c}_t = \eta_t.$$

From a formal point of view this is exactly the same problem that we studied above apart from the fact that e is replaced by η. We may thus copy the result from Proposition 38.2 to state the following result.

Proposition 38.4 *Write the aggregate endowment process η dynamics as*

$$d\eta_t = a_t dt + b_t dW_t,$$

where

$$a_t = \sum_{i=1}^{d} a_{it}, \quad b_t = \sum_{i=1}^{d} b_{it},$$

then the following hold:

- *The equilibrium stochastic discount factor is given by*

$$\mathbf{M}_t = \frac{U_c(t,\eta_t)}{U_c(0,\eta_0)}. \tag{38.15}$$

- *The equilibrium short rate process is given by*

$$r_t = -\frac{U_{ct}(t,\eta_t) + a_t U_{cc}(t,\eta_t) + \frac{1}{2}\|b_t\|^2 U_{ccc}(t,\eta_t)}{U_c(t,\eta_t)}. \tag{38.16}$$

- *The Girsanov kernel is given by*

$$\varphi_t = \frac{U_{cc}(t,\eta_t)}{U_c(t,\eta_t)} \cdot b_t. \tag{38.17}$$

We finish this section by noting that although the case of a multidimensional endowment process will formally reduce to the case of a scalar endowment, the multidimensional case may still lead to new computational and structural problems.

Suppose, for example, that we model each e_i as a Markov process of the form

$$de_{it} = a_i(e_{it})dt + b_i(e_{it})dW_t, \quad i = 1, \dots, d,$$

where a_i is a deterministic real valued function, and b_i is a deterministic k-dimensional row vector function. Then the aggregate endowment will have dynamics

$$d\eta_t = \left\{ \sum_{i=1}^{d} a_i(e_{it}) \right\} dt + \left\{ \sum_{i=1}^{d} b_i(e_{it}) \right\} dW_t,$$

so η is **not** Markov. In particular, if each e_i is GBM, this does **not** imply that η is GBM.

38.5 Exercises

Exercise 38.1 Consider a model with log utility

$$U(t,c) = e^{-\delta t} \ln(c)$$

and scalar endowment following GBM

$$de_t = ae_t dt + be_t W_t,$$

where a and b are real numbers and W is P-Wiener.

(a) Compute the P and Q dynamics of S.

(b) Use the result of (a) and let $T \to \infty$. What do you get?

Exercise 38.2 Consider a model with power utility

$$U(t,c) = e^{-\delta t} \frac{1}{\gamma} c^{\gamma},$$

where $\gamma < 1$, and scalar GBM endowment as in the previous exercise. Compute the short rate and the Girsanov kernel. Compute the S-dynamics when $T \to \infty$.

Exercise 38.3 Consider a model with exponential utility

$$U(t,c) = e^{-\delta t}\frac{1}{\gamma}e^{-\gamma c}$$

and scalar endowment with dynamics

$$de_t = (b - ae_t)dt + \sigma\sqrt{e_t}dW_t$$

where a, b, and σ are positive real numbers such that $2b > \sigma^2$ (so that e stays positive). Compute the short rate, the Girsanov kernel, and the S-dynamics.

38.6 Notes

Basic references for endowment models are Huang (1987) and Karatzas et al. (1990). For textbook treatments see Back (2017), Duffie (2001), Dana and Jeanblanc (2003), and Karatzas and Shreve (1998).

A problem with additive utility is that it ties time preferences to risk aversion in a restrictive way. The theory of *recursive preferences* allows you separate time preferences from risk aversion while retaining the Bellman optimality principle. This approach is now more or less standard, and some basic references are Kreps and Porteus (1978) and Epstein and Zin (1989) in discrete time, and Duffie and Epstein (1992) in continuous time.

Another problem with time additive utility is that it does not allow consumption at different points in time to be substitutes or complements. One way of dealing with this problem is to allow for *habit formation* models. In *internal* models you let the utility of consumption depend on a smoothed average of previous consumption (if you bought a new car yesterday your utility of a new one today is small). See Detemple and Zapatero (1991).

In *external* models you allow the utility of consumption depend on some external factor, like for example average consumption in the economy ("catching up with the Joneses"). See Abel (1990).

Appendix A

MEASURE AND INTEGRATION

The purpose of this appendix is to give a brief introduction to measure theory and to the associated integration theory on general measure spaces.

A.1 Sets and Mappings

Let X be an arbitrary set. We then say that X is **finite** if it contains only finitely many elements. If X is not finite, we say that it is **infinite**. We will use the notation $f : X \to Y$ to denote a function (or "mapping") f which takes values in Y and which has domain X. If we apply the f to an element $x \in X$, we denote the function value by $f(x)$.

Definition A.1 *Let X and Y be sets and let*

$$f : X \to Y$$

be a given mapping.

1. *The mapping f is* **injective** *if, for all x and z in X, it holds that*
$$x \neq z \quad \Rightarrow \quad f(x) \neq f(z).$$
2. *The mapping f is* **surjective** *if, for all $y \in Y$, there exists an $x \in X$ such that*
$$y = f(x).$$
3. *The mapping f is* **bijective** *if it is both injective and surjective.*
4. *The* **image** *of X under f is denoted by $Im(f)$ and defined by*
$$Im(f) = \{f(x); \quad x \in X\}.$$
5. *For any set $B \subseteq Y$, the* **inverse image** *or* **pullback** *of B under f is denoted by $f^{-1}(B)$, and defined by*
$$f^{-1}(B) = \{x; \quad f(x) \in B\}.$$
We also say that B is **lifted** *to $f^{-1}(B)$.*
6. *In particular, for every $y \in Y$ we write*
$$f^{-1}(y) = \{x; \quad f(x) = y\}.$$
7. *For any set $A \subseteq X$ the* **direct image** *of A under f is denoted by $f(A)$ and defined by*
$$f(A) = \{f(x); \quad x \in A\}.$$

Note that the inverse image of a set always exists (although it could be the empty set) even if the function f does not have an inverse. For our purposes the inverse image concept is much more important than the direct image. The following very useful result shows that the set algebraic operations are preserved under the inverse image.

Proposition A.2 *The following relations always hold*:

1. *For any $A, B \subseteq Y$ we have*
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$
2. *For any $A, B \subseteq Y$ we have*
$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$
3. *For any $B \subseteq Y$ we have*
$$f^{-1}(B^c) = \left[f^{-1}(B)\right]^c$$
where ^{c} denotes the complement.
4. *For any indexed collection $\{B_\gamma\}_{\gamma \in \Gamma}$ of sets in Y we have*
$$f^{-1}\left(\bigcap_{\gamma \in \Gamma} B_\gamma\right) = \bigcap_{\gamma \in \Gamma} f^{-1}(B_\gamma).$$
5. *For any indexed collection $\{B_\gamma\}_{\gamma \in \Gamma}$ of sets in Y we have*
$$f^{-1}\left(\bigcup_{\gamma \in \Gamma} B_\gamma\right) = \bigcup_{\gamma \in \Gamma} f^{-1}(B_\gamma).$$

Proof We leave the proof to the reader. □

We define the set of **natural numbers** as the set $N = \{1, 2, \ldots\}$, and the set of integers Z as $Z = \{0, \pm 1, \pm 2, \ldots\}$. Using N we can give a name to the "smallest" type of infinite set.

Definition A.3 *An infinite set X is* **countable** *if there exists a bijection*
$$f : N \to X.$$
If X is infinite but not countable, it is said to be **uncountable**.

Note that in our definition a countable set is always infinite. The intuitive interpretation of this is that a countable set has "as many" elements as N, whereas an uncountable set has "more" elements than N. If X is countable we may, for each $n \in N$ define $x_n \in X$ by $x_n = f(n)$. Since f is a bijection we can thus write X as
$$X = \{x_1, x_2, x_3, \ldots\},$$

so we see that in this sense we can really count off the elements in X, one by one.

A countable set is thus an infinite set, but in a sense it is "almost finite", and thus very easy and nice to handle. In mathematics in general and in particular in probability theory, the difference between countable and uncountable sets is crucial, and therefore it is important to be able to tell if a set is countable or not. The following result is a good start in that direction.

Proposition A.4 *The set Q of rational numbers is countable.*

Proof It is enough to show that the set Q_+ of positive rational numbers is countable, and we do this by first representing each rational number p/q by the integer lattice point $(p, q) \in R^2$. We now prove that this set of integer lattice points is countable by simply presenting a scheme for counting them.

The scheme begins with $(1,1), (2,1), (1,2), (1,3), (2,2), (3,1), (4,1), (3,2)$, and if the reader draws a two-dimensional graph of this scheme, she will easily see how to continue. □

This simple idea can be extended considerably.

Proposition A.5 *Assume that we are give a countable family $\{X_n\}_{i=1}^{\infty}$ of sets, where each set X_n is countable. Then the union $\bigcup_{i=1}^{\infty} X_n$ is countable.*

Proof Because of the countability assumption, the set X_i can be written as $\{x_{i1}, x_{i2}, x_{i3}, \ldots\}$, so the set X_i can be bijectively mapped into the integer lattice points of the form $(i,1), (i,2), (i,3), \ldots$. Thus the union $\bigcup_{i=1}^{\infty} X_n$ can be mapped bijectively onto the entire set of positive integer lattice points in R^2 and we have already proved that this set is countable. □

The most important example of an uncountable set is the set of real numbers.

Proposition A.6 *The set R of real numbers in uncountable.*

Proof Omitted. □

A.2 Measures and Sigma-Algebras

Let X be a set and let us denote the class of all subsets of X by 2^X. More formally we thus see that 2^X, commonly known as **the power set of** X, is a set, the elements of which are subsets of X.

We now want to formalize the idea of a mass distribution on X, and the reader may think of a large plate (the set X) with mashed potatoes on it. For every subset $A \subseteq X$ we would now like to define the non-negative real number $\mu(A)$ as "the amount of mashed potatoes which lies on the set A" or "the measure of A". However, when one tries to formalize this intuitively simple notion, one encounters technical problems, and the main problem is the fact that in the generic situation, there exists subsets $A \subseteq X$ which are so "nasty" that it is mathematically impossible to define $\mu(A)$. Typically we are therefore forced to define the measure $\mu(A)$ only for certain "nice" subsets $A \subseteq X$. These "nice" sets are called "measurable sets", and the technical concept needed is that of a **sigma-algebra** (or σ-algebra). In order to define this concept, let $\mathcal{F}$ be a family of subsets of X, i.e. $\mathcal{F} \subseteq 2^X$.

Definition A.7 *A family $\mathcal{F}$ of subsets of X is a σ-algebra if the following hold:*

1. $\emptyset \in \mathcal{F}$.
2. $A \in \mathcal{F} \quad \Rightarrow \quad A^c \in \mathcal{F}$.
3. $A_n \in \mathcal{F}, \quad \text{for } n = 1, 2, \ldots \quad \Rightarrow \quad \bigcup_{i=1}^{\infty} A_n \in \mathcal{F}$.
4. $A_n \in \mathcal{F}, \quad \text{for } n = 1, 2, \ldots \quad \Rightarrow \quad \bigcap_{i=1}^{\infty} A_n \in \mathcal{F}$.

Thus a σ-algebra contains the empty set, and is closed under complement, countable unions and countable intersections. In fact, it is only necessary to require that $\mathcal{F}$ is closed under complements and countable unions (see the exercises). Note that the conditions (3) and (4) only concern **countable** unions.

Trivial examples of σ-algebras are

$$\mathcal{F} = 2^X, \quad \mathcal{F} = \{\emptyset, X\}.$$

Definition A.8 *A pair $(X, \mathcal{F})$ where X is a set and $\mathcal{F}$ is a σ-algebra on X is called a* **measurable space**. *The subsets of X which are in $\mathcal{F}$ are called* $\mathcal{F}$**-measurable sets**.

We are now in a position to define the concept of a (non-negative) measure.

Definition A.9 *A finite measure μ on a measurable space (X, μ) is a mapping*

$$\mu : \mathcal{F} \to R_+,$$

such that:

1. $\mu(A) \geq 0, \quad \forall A \in \mathcal{F}$.
2. $\mu(\emptyset) = 0$.
3. *If $A_n \in \mathcal{F}$ $\forall n = 1, 2, \ldots$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The intuitive interpretation of (2) is obvious; there is no mass on the empty set. If A and B are disjoint sets, it is also obvious that the mass on $A \cup B$ equals the sums of the masses on A and B. The condition (3) above is an extension of this property to the case of an infinite collection of sets, and is known as the **sigma-additivity** of the measure.

Generally speaking, it is a hard problem to construct nontrivial measures, and we will come back to this below. In our applications, we will typically be given a measure which is defined a priori.

Definition A.10 *A* **measure space** *is a triple* $(X,\mathcal{F},\mu)$*, where* μ *is a measure on the measurable space* $(X,\mathcal{F})$*.*

A.3 Integration

Let $(X,\mathcal{F},\mu)$ be a measure space, and let $f: X \to R$ be a given function. The object of the present section is to give a reasonable definition of the formal expression

$$\int_X f(x) d\mu(x), \tag{A.1}$$

and we do this in a couple of simple steps.

Definition A.11 *For an arbitrary* $A \subseteq X$ *the* **indicator function** I_A *is defined by*

$$I_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in A^c. \end{cases}$$

If $f = c \cdot I_A$ where c is a real number and A is measurable, then there is a very natural definition of (A.1), namely

$$\int_X f(x) d\mu(x) = \int_X c \cdot I_A(x) d\mu(x) = c \cdot \mu(A), \tag{A.2}$$

i.e. the "area under the graph of f" $=$ "the base" $\cdot$ "the height" $= \mu(A) \cdot c$.

Observe that we must demand that A is $\mathcal{F}$-measurable, since otherwise the right-hand side of (A.2) is not defined. This also gives us a natural definition of linear combinations of indicator functions.

Definition A.12 *A mapping* $f: X \to R$ *is* **simple** *if it can be written as*

$$f(x) = \sum_{i=1}^{n} c_i \cdot I_{A_i}(x), \tag{A.3}$$

where $A_1, \ldots, A_n$ *are measurable and* $c_1, \ldots, c_n$ *are real numbers.*

Definition A.13 *For a simple function, as in (A.3), the integral is defined by*

$$\int_X f(x) d\mu(x) = \sum_{i=1}^{n} c_i \cdot \mu(A_i).$$

We now want to extend this integral concept to functions which are not simple. Let us therefore consider an arbitrary non-negative function. The intuitive idea is now to carry out the following program:

- Approximate f from below by simple functions, i.e find simple f_n, $n = 1, 2, \ldots$, such that $f_n(x) \uparrow f(x)$ for all x.

- Define the integral of f as the limit of the integrals of the approximating simple functions, i.e.

$$\int_X f(x)d\mu(x) = \lim_n \int_X f_n(x)d\mu(x).$$

The problem with this natural idea is the fact that not all functions can be well approximated by simple functions, so we cannot define the integral for an arbitrary function.

Instead we have to be contented with defining the integral concept (A.1) for those functions f which **can** be approximated by simple functions. This is the class of **measurable** functions, and the formal definition is as follows.

Definition A.14 *A function $f : X \rightarrow R$ is $\mathcal{F}$-**measurable** if, for every interval $I \subseteq R$ it holds that $f^{-1}(I) \in \mathcal{F}$, i.e. if it holds that*

$$\{x \in X; f(x) \in I\} \in \mathcal{F}$$

for all intervals I. We will often write this as $f \in \mathcal{F}$.

When testing if a given function is measurable, the following result is often of great use.

Proposition A.15 *The following properties are equivalent:*

1. *f is $\mathcal{F}$-measurable.*
2. $\{x \in X; f(x) < \alpha\} \in \mathcal{F}, \quad \forall \alpha \in R.$
3. $\{x \in X; f(x) \leq \alpha\} \in \mathcal{F}, \quad \forall \alpha \in R.$
4. $\{x \in X; f(x) > \alpha\} \in \mathcal{F}, \quad \forall \alpha \in R.$
5. $\{x \in X; f(x) \geq \alpha\} \in \mathcal{F}, \quad \forall \alpha \in R.$

Proof Use Proposition A.2. □

The following important result shows that measurability is preserved under the most common operations.

Proposition A.16 *Assume that f and g are measurable on a measurable space $(X, \mathcal{F})$. Then the following hold:*

1. *For all real numbers α and β the functions*

$$\alpha f + \beta g, \quad f \cdot g$$

   are measurable.

2. *If $g(x) \neq 0$ for all x, then*

$$\frac{f}{g}$$

   is measurable.

3. *If* $\{f_n\}_{n=1}^{\infty}$ *is a (countable) sequence of measurable functions, then the functions*

$$\sup_n f_n, \quad \inf_n f_n, \quad \limsup_n f_n, \quad \liminf_n f_n$$

are measurable.

Proof The proof is omitted. □

We can now go on to define the integral of a non-negative function on a measure space.

Definition A.17 *Let* $f : X \to R$ *be non-negative and measurable on the measure space* $(X, \mathcal{F}, \mu)$*. The integral of* f *w.r.t* μ *over* X *is then defined by*

$$\int_X f(x)d\mu(x) = \sup_{\varphi} \int_X \varphi(x)d\mu(x), \tag{A.4}$$

where the supremum is over the class of simple functions φ *such that* $0 \le \varphi \le f$*.*

We now want to extend this definition to functions which are not necessarily non-negative. Let therefore f be an arbitrary measurable function. It then follows from Proposition A.16 that also $|f|$ is measurable, since we can write

$$f = f^+ - f^-,$$

where

$$f^+ = \max[f, 0], \quad f^- = \max[-f, 0].$$

Definition A.18 *A measurable function* f *is* **integrable***, which we will write as* $f \in L^1(X, \mathcal{F}, \mu)$*, if*

$$\int_X |f(x)|d\mu(x) < \infty.$$

For an integrable function f*, the integral over* X *is defined by*

$$\int_X f(x)d\mu(x) = \int_X f^+(x)d\mu(x) - \int_X f^-(x)d\mu(x).$$

If A *is any measurable set, the integral of* f *over* A *is defined by*

$$\int_A f(x)d\mu(x) = \int_X I_A(x)f(x)d\mu(x).$$

We will often write $\int_X f(x)d\mu(x)$ *as* $\int_X f(x)\mu(dx)$ *or as* $\int_X f d\mu$*. When the underlying measure space is unambiguous, we will write* L^1 *as shorthand for* $L^1(X, \mathcal{F}, \mu)$*.*

Example A.19 We now give a simple but important example of a measure space. Let X be the set of natural numbers and let $\mathcal{F}$ be the power set. On this space we now define the **counting measure** ν by

$$\nu(A) = \text{the number of points in } A. \tag{A.5}$$

In other words, the counting measure puts unit mass on every single natural number. We immediately see that on $(N, 2^N, \nu)$, *every* real valued function will be measurable, and a function f is integrable if and only if

$$\sum_{n=1}^{\infty} |f(n)| < \infty.$$

The integral of any $f \in L^1$ is easily seen to be given by

$$\int_X f(x) d\mu(x) = \sum_{n=1}^{\infty} f(n).$$

Thus we see that our integration theory also allows us to treat ordinary sums as integrals.

We now have some very natural properties of the integral. The proof is not trivial and is omitted.

Proposition A.20 *The following relations hold:*

1. *For any $f, g \in L^1(X, \mathcal{F}, \mu)$ and any real numbers α and β it holds that*
$$\int_X (\alpha f(x) + \beta g(x))\, d\mu(x) = \alpha \int_X f(x) d\mu(x) + \beta \int_X g(x) d\mu(x).$$
2. *If $f(x) \le g(x)$ for all x, then*
$$\int_X f(x) d\mu(x) \le \int_X g(x) d\mu(x).$$
3. *For any function in L^1 it holds that*
$$\left| \int_X f(x) d\mu(x) \right| \le \int_X |f(x)|\, d\mu(x).$$

One of the most striking properties of the integral concept defined above is the surprising ease with which it handles convergence problems. The three basic results are as follows.

Theorem A.21 (The Fatou Lemma) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that*

$$f_n \ge 0, \quad n = 1, 2, \ldots$$

and

$$\lim_{n \to \infty} f_n(x) = f(x), \quad \forall x \in X,$$

for some limit function f. Then

$$\int_X f(x)d\mu(x) \le \liminf_n \int_X f_n(x)d\mu(x).$$

Theorem A.22 (Monotone Convergence) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that:*

1. *The sequence is non-negative, i.e.*

$$f_n \ge 0, \quad n = 1, 2, \dots$$

2. *The sequence is increasing, i.e. for all x we have*

$$f_1(x) \le f_2(x) \le \dots \le f_n(x) \le f_{n+1}(x) \le \dots$$

Then, defining the function f by

$$f(x) = \lim_n f_n(x),$$

it holds that

$$\int_X f(x)d\mu(x) = \lim_n \int_X f_n(x)d\mu(x).$$

Theorem A.23 (The Lebesgue Dominated Convergence Theorem) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that*

$$f_n(x) \to f(x)$$

for some limit function f. Suppose that there exists a non-negative function $g \in L^1$ such that

$$|f_n(x)| \le g(x), \quad \forall n, \ \forall x \in X.$$

Then

$$\int_X f(x)d\mu(x) = \lim_n \int_X f_n(x)d\mu(x).$$

A.4 Sigma-Algebras and Partitions

The purpose of this section is to give some more intuition for the measurability concept for functions. We will do this by considering the simplest case of a sigma-algebra, namely when it is generated by a *partition*.

Definition A.24 *A* **partition** $\mathcal{P}$ *of the space X is a finite collection of sets $\{A_1, A_2, \dots, A_n\}$ such that:*

1. *The sets cover X, i.e.*

$$\bigcup_{i=1}^{n} A_i = X,$$

2. *The sets are disjoint, i.e.*

$$i \neq j \quad \Rightarrow \quad A_i \cap A_j = \emptyset.$$

The sets $A_1, \ldots A_n$ are called the **components** *of $\mathcal{P}$, and the sigma-algebra consisting of all possible unions (including the empty set) of the components in $\mathcal{P}$ is denoted by $\sigma(\mathcal{P})$. This sigma-algebra is called* **the sigma-algebra generated by** *$\mathcal{P}$.*

We now have a result which shows (in this restricted setting) what measurability for a function "really means".

Proposition A.25 *Let $\mathcal{P}$ be a given partition of X. A function $f : X \to R$ is $\sigma(\mathcal{P})$-measurable if and only if f is constant on each component of $\mathcal{P}$.*

Proof First assume that f is measurable, and consider a fixed but arbitrarily chosen real number y. Since f is measurable we know that $f^{-1}(y)$ is in $\sigma(\mathcal{P})$ so it is a union of some of the components of $\mathcal{P}$. If the union is non-empty this means precisely that $f = y$ on that union, so in particular it takes the constant value y on all components in the union. The converse is trivial. $\square$

The sigma-algebras that we are going to consider later on in the text are in general **not** generated by partitions. They are typically much more complicated, so the proposition above is of little "practical" interest. The point of the discussion concerning partitions is instead that when you **informally think about** sigma-algebras, it is very helpful to have this simple case at the back of your head.

As an example, we see directly from Proposition A.25 why we must restrict ourselves to integrating measurable functions only. The problem with a non-measurable function is that the function is varying wildly compared with the fine structure of the sigma-algebra. In particular this implies that a non-measurable function cannot be well approximated by simple functions.

It is also very instructive to see exactly what goes wrong when we try to integrate non-measurable functions. Even for a non-measurable function f we can of course in principle define the integral by (A.4), but this integral will not have the nice properties in Proposition A.20. See the exercises for concrete examples.

A.5 Sets of Measure Zero

Consider again a measure space $(X, \mathcal{F}, \mu)$. If $N \in \mathcal{F}$ and $\mu(N) = 0$, we say that N is a **null set**. If a certain property holds for all $x \in X$ except for on a null set, then we say that the property hold **almost everywhere** (w.r.t. μ), and in shorthand we write "μ-a.e.". For example, if we write

$$f \geq 0, \quad \mu - a.e.$$

this means that there exists a null set N such that $f(x) \geq 0$ for all $x \in N^c$. It is easy to see that if f and g are integrable, and $f = g$, almost everywhere, then for every $A \in \mathcal{F}$ we have

$$\int_A f d\mu = \int_A g d\mu.$$

In fact, there is an important converse of this statement, which shows that you can test whether two functions are equal almost everywhere or not, by testing their integrals. We omit the proof.

Proposition A.26

- *Assume that f and g are integrable and that*

$$\int_A f d\mu = \int_A g d\mu,$$

  for every $A \in \mathcal{F}$ Then $f = g$, μ-a.e.
- *Assume that f and g are integrable and that*

$$\int_A f d\mu \geq \int_A g d\mu,$$

  for every $A \in \mathcal{F}$ Then $f \geq g$, μ-a.e.

A.6 The L^p Spaces

Let p be a real number with $1 \leq p < \infty$. We define the function class $L^p(X, \mathcal{F}, \mu)$ as the class of measurable functions f such that

$$\int_X |f(x)|^p d\mu(x) < \infty,$$

and for $f \in L^p$ we define the **L^p-norm** $\|f\|_p$by

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p}$$

For $p = \infty$ the norm is defined by

$$\|f\|_\infty = \text{ess sup } |f| = \inf\{M \in R;\ |f| \leq M,\ a.e.\}$$

The two main inequalities for L^p spaces are the Minkowski and the Hölder inequalities.

Proposition A.27 *The following hold for $1 \leq p \leq \infty$.*

1. *The Minkowski inequality:*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

2. *The Hölder inequality:*

$$\int_X |f(x)g(x)|d\mu(x) \leq \|f\|_p \cdot \|g\|_q$$

where p and q are **conjugate** *i.e.* $1/p+1/q=1$.

From the Minkowski inequality it follows that if we identify functions which are equal almost everywhere, then L^p is a normed vector space. If $\{f_n\}$ is a sequence of functions in L^p and f is a function in L^p such that

$$\|f_n - f\| \to 0, \quad \text{as } n \to \infty,$$

then we say that f_n converges to f in L^p and we write

$$f_n \xrightarrow{L^p} f.$$

Definition A.28 *A sequence $\{f_n\}_n$ of functions in L^p is a* **Cauchy sequence** *if, for all $\epsilon < 0$ there exists an integer N such that*

$$\|f_n - f_m\| < \epsilon,$$

for all $n,m \geq N$.

It is easy to see (prove this) that if f_n converges to some f in L^p then $\{f_n\}_n$ is Cauchy. The converse is not necessarily true for generally normed spaces, but it is in fact true for the L^p spaces.

Proposition A.29 *Every L^p space, for $1 \leq p \leq \infty$, is* **complete** *in the sense that every Cauchy sequence converges to some limit point. In other words, if $\{f_n\}$ is a Cauchy sequence in L^p, then there exists a (unique) element $f \in L^p$ such that $f_n \to f$ in L^p.*

We will mainly be dealing with L^1 and L^2, and for L^2 there is furthermore an inner product, which is the natural generalization of the scalar product on R^n.

Definition A.30 *For any two elements f and g in L^2 we define the* **inner product** (f,g) *by*

$$(f,g) = \int_X f(x) \cdot g(x)d\mu(x).$$

It is easy to see the inner product is bilinear (i.e. linear in each variable) and that the inner product is related to the L^2 norm by

$$\|f\|_2 = \sqrt{(f,f)}.$$

The vector space structure, the inner product, and the completeness of L^2, ensures that L^2 is a *Hilbert space.*

A.7 Hilbert Spaces

Hilbert spaces are infinite dimensional vector spaces which generalize the finite dimensional Euclidian spaces R^n.

Definition A.31 *Consider a real vector space* $\mathbf{H}$. *A mapping* $(\ ,\ ) : \mathbf{H} \times \mathbf{H} \to R$, *is called an* **inner product** *on* $\mathbf{H}$ *if it has the following properties:*

- *It is bilinear, i.e. for any* $\alpha, \beta \in R$ *and* $f, g, h \in \mathbf{H}$

$$(\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h)$$

and

$$(h, \alpha f + \beta g) = \alpha(h, f) + \beta(h, g).$$

- *It is symmetric, i.e.*

$$(f, g) = (g, f), \quad \forall f, g \in \mathbf{H}.$$

- *It is positive definite, i.e.*

$$(f, f) \geq 0, \quad \textit{for all } f \in \mathbf{H} \textit{ with equality if and only if } f = 0.$$

The inner product generalizes the standard scalar product on R^n and in particular it induces a *norm* and the concept of *orthogonality*.

Definition A.32

- *For any* $f \in \mathbf{H}$ *the* **norm** *of* f *is denoted by* $\|f\|$ *and defined by*

$$\|f\| = \sqrt{(f, f)}.$$

- *Two vectors* $f, g \in \mathbf{H}$ *are said to be* **orthogonal** *if* $(f, g) = 0$. *We write this as* $f \perp g$.
- *For any linear subspace* $M \in \mathbf{H}$ *we define its* **orthogonal complement** $M^{\perp}$ *as*

$$M^{\perp} = \{f \in \mathbf{H};\ f \perp M\}.$$

We interpret $\|f\|$ as "the length of f", and the following result shows that the norm concept really is a norm in the technical sense of satisfying the triangle inequality.

Proposition A.33

- *For all* $f, g \in \mathbf{H}$ *the Cauchy–Schwartz inequality holds, i.e.*

$$\|(f, g)| \leq \|f\| \cdot \|g\|.$$

- *The norm* $\|\cdot\|$ *satisfies the* **triangle inequality**, *i.e. for any* $f, g \in \mathbf{H}$ *we have*

$$\|f + g\| \leq \|f\| + \|g\|.$$

Proof To prove the Cauchy–Schwartz inequality we note that for all $f, g \in \mathbf{H}$ and $s \in R$ we have $(f - sg, f - sg) = \|f - sg\|^2 \geq 0$. We thus have $\|f\|^2 + \|g\|^2 - 2s(f, g) \geq 0$. Minimizing this over all real numbers s and plugging the optimal s into the inequality gives us the Cauchy inequality. To prove the triangle inequality, write $\|f + g\|^2$ as $(f + g, f + g)$ and expand using the bilinearity and Cauchy–Schwartz. $\square$

A vector space with an inner product is called an *inner product space*, and on such a space we may use the induced norm to define the concept of a Cauchy sequence and of completeness (as for the L^p spaces above).

Definition A.34 *A* **Hilbert space** *is an inner product space which is complete under the induced norm* $\|\cdot\|$.

We note that $L^2(X, \mathcal{F}, \mu)$ above is a Hilbert space and it is in fact the most important example of a Hilbert space. The single most important result for Hilbert spaces is probably the projection theorem. The proof is omitted.

Theorem A.35 *Assume that* $\mathbf{M}$ *is a closed linear subspace of a Hilbert space* $\mathbf{H}$.

- *Assume that* f *is a fixed vector in* $\mathbf{H}$. *Consider the optimization problem*

$$\min_{g \in \mathbf{M}} \|f - g\|.$$

Then there exists a unique solution $\hat{g}$ *and it is characterized by the condition that*

$$f - \hat{g} \perp \mathbf{M}.$$

- *We have the decomposition of* $\mathbf{H}$ *as*

$$\mathbf{H} = M \oplus M^{\perp}.$$

The direct sum sign $\oplus$ *means that* $\mathbf{H} = M + M^{\perp}$ *and that* $M \cap M^{\perp} = \{0\}$.

The optimal $\hat{g}$ above is called the **orthogonal projection** of f onto $\mathbf{M}$. We see that the optimal "error vector" $f - \hat{g}$ is perpendicular to the subspace $\mathbf{M}$, exactly like the situation in R^n.

Definition A.36 *A linear mapping* $F : \mathbf{H} \to R$ *is called a* **linear functional**. *For any* $f \in \mathbf{H}$ *we sometimes write* Ff *rather than* $F(f)$. *A linear functional is said to be* **bounded** *if there exists a constant* K *such that*

$$|F(f)| \leq K\|f\|, \quad \forall f \in \mathbf{H}.$$

It is relatively easy to see that a linear functional is bounded if and only if it is continuous. It is also easy to see that if we choose a fixed $f \in \mathbf{H}$ and define the mapping $F : \mathbf{H} \to R$ by

$$Fg = (f, g), \quad \forall f \in \mathbf{H}, \tag{A.6}$$

then F is a bounded linear functional. The next result shows that **all** bounded linear functionals on a Hilbert space are of this form.

Theorem A.37 (Riesz Representation Theorem)
Assume that

$$F : \mathbf{H} \to R$$

is a bounded linear functional. Then there exists a unique $g \in \mathbf{H}$ *such that*

$$Ff = (f, g), \quad \forall g \in \mathbf{H}.$$

Proof Define M by $M = ker[F] = \{f \in \mathbf{H};\ Ff = 0\}$. The M is a closed subspace and we can decompose $\mathbf{H}$ as $\mathbf{H} = M + M^{\perp}$. From the definition of M it is clear that F restricted to $M^{\perp}$ is linear with trivial kernel. It is thus a vector space isomorphism between $M^{\perp}$ and R, so $M^{\perp}$ has to be one-dimensional and we can write $M^{\perp} = Rg_0$ for some $g_0 \in M^{\perp}$. Now define g by

$$g = \frac{Fg_0}{\|g_0\|^2} g_0.$$

Then $(g_0, g) = Fg_0$ so by linearity we have $Ff = (f, g)$ for all $f \in M^{\perp}$ and hence (exactly why?) also for all $f \in \mathbf{H}$. □

Note that we have to assume that the linear functional F is *bounded*, i.e. continuous. On a finite dimensional Euclidian space R^n, *all* linear functionals are continuous, but on a Hilbert space there may exists linear functionals which are not continuous. It is only for the continuous ones that we have the Riesz representation above.

A.8 Sigma-Algebras and Generators

As usual we consider some basic space X. Let $\mathcal{S}$ be some a priori given class of subsets of X, i.e. $\mathcal{S} \subseteq 2^X$, where $\mathcal{S}$ is not assumed to be a sigma-algebra. The question is whether there is some natural way of extending $\mathcal{S}$ to a sigma-algebra. We can of course always extend $\mathcal{S}$ to the power algebra 2^X, but in most applications this is going too far, and instead we would like to extend $\mathcal{S}$ in some *minimal* way to a sigma-algebra. This minimal extension can in fact always be achieved, and intuitively one is easily led to something like the following argument:

- Assume for example that $\mathcal{S}$ is not closed under complement formation. Then we extend $\mathcal{S}$ to $\mathcal{S}_1$ by adjoining to $\mathcal{S}$ all sets which can be written as complements of sets in $\mathcal{S}$. Thus $\mathcal{S}_1$ is closed under complements.
- If $\mathcal{S}$ is a sigma-algebra then we are finished. If it is not, then assume for example that it is not closed under countable unions. We then extend $\mathcal{S}_1$ to $\mathcal{S}_2$ by adjoining to $\mathcal{S}_1$ all countable unions of sets in $\mathcal{S}_1$. Thus $\mathcal{S}_2$ is closed under countable unions.
- It can now very well happen that $\mathcal{S}_2$ is not closed under complement formation (or under countable intersections). In that case we extend $\mathcal{S}_2$ to $\mathcal{S}_3$ by adjoining to $\mathcal{S}_2$ all complements of sets in $\mathcal{S}_2$.
- And thus we go on ...

In this way it perhaps seems likely that "at last" we will obtain the (unique?) minimal extension of $\mathcal{S}$ to a sigma-algebra. Unfortunately the method is not constructive (unless of course X is finite) so we need a more indirect method.

Proposition A.38 *Let $\{\mathcal{F}_\alpha;\ \alpha \in A\}$ be an indexed family of sigma-algebras on some basic set X, where A is some index set, i.e. for each $\alpha \in A$ $\mathcal{F}_\alpha$ is a sigma-algebra. Define $\mathcal{F}$ by*

$$\mathcal{F} = \bigcap_{\alpha \in A} \mathcal{F}_\alpha.$$

Then $\mathcal{F}$ is a sigma-algebra.

Proof The proof is left to the reader as an exercise. □

We can now go back to our family $\mathcal{S}$ above.

Proposition A.39 *Let $\mathcal{S}$ be an arbitrary family of subsets of X. Then there exists a unique minimal extension of $\mathcal{S}$ to a sigma-algebra. More precisely, there exists a $\mathcal{G} \subseteq 2^X$ such that:*

- *$\mathcal{G}$ extends $\mathcal{S}$, i.e. $\mathcal{S} \subseteq \mathcal{G}$.*
- *$\mathcal{G}$ is a sigma-algebra on X.*
- *$\mathcal{G}$ is minimal, i.e. if $\mathcal{F}$ is any sigma-algebra on X such that $\mathcal{S} \subseteq \mathcal{F}$, then $\mathcal{G} \subseteq \mathcal{F}$.*

Proof Define $\mathcal{G}$ by

$$\mathcal{G} = \bigcap \mathcal{F}$$

where the intersection is taken over all sigma-algebras $\mathcal{F}$ such that $\mathcal{S} \subseteq \mathcal{F}$. It follows from Proposition A.38 that $\mathcal{G}$ is a sigma-algebra, it obviously extends $\mathcal{S}$ and from the construction we easily see (why?) that it is minimal. □

Definition A.40

- *The sigma-algebra $\mathcal{G}$ in the previous proposition is called* **the sigma-algebra generated by** *$\mathcal{S}$, and we write*
$$\mathcal{G} = \sigma\{\mathcal{S}\}.$$
- *The family $\mathcal{S}$ is called a* **generator system** *for $\mathcal{G}$.*
- *If $\{\mathcal{F}_\gamma;\ \gamma \in \Gamma\}$ is an indexed family of sigma-algebras on X, we denote by*
$$\bigvee_{\gamma \in \Gamma} \mathcal{F}_\gamma$$
the smallest sigma-algebra which contains each $\mathcal{F}_\gamma$.
- *If $\{f_\gamma;\ \gamma \in \Gamma\}$ is an indexed family of real valued functions on X, we denote by*
$$\mathcal{G} = \sigma\{f_\gamma;\ \gamma \in \Gamma\},$$
the smallest sigma-algebra $\mathcal{G}$ such that f_γ is measurable for each $\gamma \in \Gamma$.

It is important to understand that even if the elements in a generator system $\mathcal{S}$ are “simple” (in some sense), the sigma-algebra generated by $\mathcal{S}$ can be very complicated.

We first give some rather trivial examples:

1. If X is the interval $[0,1]$ and $\mathcal{S} = \{[0,1/2]\}$ it is easy to see that $\sigma\{\mathcal{S}\} = \{X, \emptyset, [0,1/2], (1/2,1]\}$.
2. If $X = N$ and $\mathcal{S}$ is the class of all singleton sets of N, i.e. $\mathcal{S} = \{\{n\};\ n \in N\}$, then $\sigma\{\mathcal{S}\} = 2^N$.

We now come to the single most important sigma-algebra.

Definition A.41 *If the set X is given by $X = R^n$ then we define the* **Borel algebra** $\mathcal{B}(R^n)$ *as the sigma-algebra which is generated by the class of open sets on R^n. The elements on the Borel algebra are called Borel sets.*

The Borel algebra is an extremely complicated object and the reader should be aware of the following facts:

- There is no "constructive" definition of the Borel algebra. In other words, it is not possible to give anything like a concrete description of what "the typical Borel set" looks like.
- The Borel algebra is strictly included in the power algebra. Thus there exist subsets of R^n which are not Borel sets.
- However, all subsets of R^n which ever turn up "in practice" are Borel sets. Reformulating this, one can say that it is enormously hard to construct a set which is not a Borel set. The pedestrian can therefore, and without danger, informally regard a Borel set as "an arbitrary subset" of R^n.

There are a large number of alternative ways of generating the Borel algebra. By recalling that a set is open if and only if its complement is closed it is easily seen that the Borel algebra is also generated by the class of all closed sets. Below is a list of some of the most common generator systems for the Borel algebra on R. The extensions to R^n are obvious.

Proposition A.42 *The Borel algebra $\mathcal{B}(R)$ can be defined in any of the following ways:*

$$
\begin{aligned}
\mathcal{B}(R) &= \sigma\{\textit{open sets}\}, \\
\mathcal{B}(R) &= \sigma\{\textit{closed sets}\}, \\
\mathcal{B}(R) &= \sigma\{\textit{intervals of the type } (a,b]\}, \\
\mathcal{B}(R) &= \sigma\{\textit{intervals of the type } (a,b)\}, \\
\mathcal{B}(R) &= \sigma\{\textit{intervals of the type } [a,b]\}, \\
\mathcal{B}(R) &= \sigma\{\textit{intervals of the type } [a,b)\}, \\
\mathcal{B}(R) &= \sigma\{\textit{intervals of the type } (-\infty,b)\}, \\
\mathcal{B}(R) &= \sigma\{\textit{intervals of the type } (-\infty,b]\}, \\
\mathcal{B}(R) &= \sigma\{\textit{intervals of the type } [a,\infty)\}, \\
\mathcal{B}(R) &= \sigma\{\textit{intervals of the type } (a,\infty)\}, \\
\mathcal{B}(R) &= \sigma\{\textit{all intervals}\}.
\end{aligned}
$$

At first glance it seems impossible to prove any results at all about the Borel algebra, since we do not have a constructive definition of it. The reason that we have any control over the class of Borel sets is the fact that it is the *minimal* extension of the class of all intervals, and this can be seen in the following useful alternative characterization of the class of measurable functions.

Proposition A.43 *Let $(X, \mathcal{F})$ be a measurable space and let $f : X \to R$ be a given function. Then f is $\mathcal{F}$-measurable if and only if $f^{-1}(B) \in \mathcal{F}$ for every Borel set $B \subseteq R$.*

Proof If $f^{-1}(B) \in \mathcal{F}$ for every Borel set $B \subseteq R$, then f is measurable since in particular it lifts intervals back to measurable sets. Assume now that f is measurable. We then have to show that f lifts arbitrary Borel sets to $\mathcal{F}$-measurable sets. To do this, define the class $\mathcal{G}$ of subsets of R by

$$\mathcal{G} = \left\{ B \subseteq R;\ f^{-1}(B) \in \mathcal{F} \right\}.$$

The class $\mathcal{G}$ is thus the class of "good" subsets of R which do lift back to measurable sets on X. We now want to prove that every Borel set is a good set, i.e. that $\mathcal{B} \subseteq \mathcal{G}$. Now, using Proposition A.2 it is not hard to prove (do this!) that $\mathcal{G}$ is a sigma-algebra. Since f was assumed to be measurable, it lifts intervals back to measurable sets, so $\mathcal{G}$ does in fact contain all intervals Thus $\mathcal{G}$ is a sigma-algebra containing all intervals, but since the Borel algebra is the smallest sigma-algebra containing all intervals it must hold that $\mathcal{B} \subseteq \mathcal{G}$ which was what we wanted to prove. □

A particularly important example of a measurable space is $(R, \mathcal{B}(R))$, and a real valued function $f : R \to R$ which is measurable w.r.t $\mathcal{B}(R)$ is called **Borel measurable** or a **Borel function**. The following result shows that there are plenty of Borel functions.

Proposition A.44 *Every continuous function $f : R^n \to R$ is Borel measurable.*

Proof This follows immediately from the fact that a function is continuous if and only if it lifts open sets to open sets. □

From Proposition A.16 it follows that every function that we can construct, by starting with continuous functions and then using the standard algebraic operations and limiting procedures, will be a Borel function. It does in fact require a considerable amount of creativity to construct a function which is not a Borel function, so the reader can with very little danger interpret "a Borel function" as "an arbitrary function".

As an important consequence of Proposition A.43 we see that measurability is preserved under composition with Borel functions.

Proposition A.45 *Assume that $f : X \to R$ is a $\mathcal{F}$-measurable mapping, and that $g : R \to R$ is a Borel function. Then the composite mapping $h = g \circ f$ defined by $h(x) = g(f(x))$ is $\mathcal{F}$-measurable.*

Proof Easy exercise for the reader. □

A.9 Product Measures

Let $(X, \mathcal{F}, \mu)$ and let $(Y, \mathcal{G}, \nu)$ be two measure spaces. We now want to construct a measure on the product space $X \times Y$ along the same lines as when we construct the area measure on R^2 from the length measure on R.

Definition A.46 *A* **measurable rectangle** *is a set $Z \subseteq X \times Y$ of the form*

$$Z = A \times B,$$

where $A \in \mathcal{F}$ and $B \in \mathcal{G}$. The **product sigma-algebra** *$\mathcal{F} \otimes \mathcal{G}$ is defined by*

$$\mathcal{F} \otimes \mathcal{G} = \sigma \{\textit{measurable rectangles}\}.$$

There is now a natural definition of the measure of a measurable rectangle, namely $\lambda(A \times B)$ = "the base times the height" $= \mu(A) \cdot \nu(B)$. Thus we have defined a product measure on the class of measurable rectangles, and the following result shows that this measure can in fact be extended to the entire product sigma-algebra.

Proposition A.47 *There exists a unique measure λ on $\{X \times Y, \mathcal{F} \otimes \mathcal{G}\}$ such that*

$$\lambda(A \times B) = \mu(A) \cdot \nu(B),$$

for every measurable rectangle $A \times B$. This measure λ is called the **product measure** *and denoted by $\lambda = \mu \times \nu$.*

We end this section by formulating a very useful result which shows that instead of integrating w.r.t. the product measure we can perform iterated integrals.

Theorem A.48 (The Fubini Theorem) *Consider the product measure space $\{X \times Y, \mathcal{F} \otimes \mathcal{G}, \mu \times \nu\}$ and let $f : X \times Y \to R$ be a measurable mapping. Assume* **either** *that f is integrable over $X \times Y$* **or** *that f is non-negative. Then we have*

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \left[\int_Y f(x,y) d\nu(y) \right] d\mu(x) = \int_Y \left[\int_X f(x,y) d\mu(x) \right] d\nu(y).$$

Note that included in the Fubini Theorem is the statement that the function

$$x \longmapsto \int_Y f(x,y) d\nu(y)$$

is $\mathcal{F}$-measurable, and correspondingly for the other integral.

The Fubini Theorem may of course be extended to any finite product space.

A.10 The Lebesgue Integral

On the class of intervals on R we have a natural length measure m defined by

$$m([a,b]) = b - a.$$

The obvious question is whether this length measure can be extended to a proper measure on the Borel algebra. That this is indeed the case is shown by the following highly nontrivial result.

Proposition A.49 *On the measurable space $(R, \mathcal{B})$ there exists a unique measure m with the property that for any interval $[a,b]$*

$$m([a,b]) = b - a.$$

This measure is called the (scalar) **Lebesgue measure**, and by taking products we can easily form the n-dimensional Lebesgue measure on R^n.

Equipped with the Lebesgue measure we can now start integrating real valued functions defined on the real line. We know that all continuous functions are Borel measurable, and at this point we could encounter a problem, since for continuous functions we also have the Riemann integral. If a function f is continuous and if A is a finite interval we can form two integrals, namely

$$\int_A f(x)dm(x), \quad \text{(Lebesgue)},$$

and

$$\int_A f(x)dx, \quad \text{(Riemann)},$$

and if we are unlucky these integral concepts could differ. Happily enough, it can be proved that whenever the Riemann integral is well defined, it will coincide with the Lebesgue integral. The advantage of using the Lebesgue integral instead of the Riemann integral is that the Lebesgue theory allows us to integrate all Borel functions, whereas the Riemann integral only allows us to integrate Riemann integrable functions (which is a much smaller class). Furthermore, a pointwise convergent sequence of non-negative Riemann integrable functions may converge to a limit function for which the Riemann integral is not even defined, whereas the class of Borel functions is closed under pointwise convergence.

A.11 The Radon–Nikodym Theorem

One of the big breakthroughs in arbitrage pricing came from the realization that absence of arbitrage is very closely connected to the existence of certain absolutely continuous measure transformations. The basic mathematical tool is the Radon–Nikodym Theorem which we will prove below, and although our prime application will be in the context of probability theory, we present the theory for arbitrary finite measures.

Definition A.50 *Consider a measurable space $(X, \mathcal{F})$ on which there are defined two separate measures μ and ν:*

- *If, for all $A \in \mathcal{F}$, it holds that*
$$\mu(A) = 0 \quad \Rightarrow \quad \nu(A) = 0, \tag{A.7}$$
then ν is said to be **absolutely continuous** *with respect to μ on $\mathcal{F}$ and we write this as $\nu << \mu$.*
- *If we have both $\mu << \nu$ and $\nu << \mu$, then μ and ν are said to be* **equivalent** *and we write $\mu \sim \nu$.*
- *If there exists two events, A and B such that:*
  * $A \cup B = X$,
  * $A \cap B = \emptyset$,
  * $\mu(B) = 0, \quad$ *and* $\quad \nu(A) = 0$,

  then μ and ν are said to be mutually **singular**, *and we write $\mu \perp \nu$.*

We now give some simple examples of these concepts.

Example A.51

- The simplest example of absolute continuity occurs when X is finite or at most countable, say $X = N$, and $\mathcal{F} = 2^X = 2^N$. Every measure μ on N is of course determined by its point masses $\mu(n)$, $n \in N$, and the relation $\nu << \mu$ simply means that
$$\mu(n) = 0 \quad \Rightarrow \quad \nu(n) = 0.$$
- Let μ be "Poisson(c)-measure", defined by its point masses on the natural numbers as
$$\mu(n) = e^{-c}\frac{c^n}{n!}, \quad n \in N,$$
and let ν be Lebesgue measure on the positive real line. Then, viewed as measures on $(R, \mathcal{B}(R))$, we have $\mu \perp \nu$ since μ puts all its mass on N whereas ν puts all its mass on $R \cap N^c$.

Consider a fixed measure space $(X, \mathcal{F}, \mu)$, and let $f : X \to R$ be a non-negative measurable mapping in $L^1(X, \mathcal{F}, \mu)$. We can then define a new measure ν on $(X, \mathcal{F})$ by setting
$$\nu(A) \stackrel{def}{=} \int_A f(x) d\mu(x), \quad A \in \mathcal{F}. \tag{A.8}$$

It now follows fairly easily that ν is a measure on $(X, \mathcal{F}, \mu)$ and from the definition it also follows directly that $\nu << \mu$ on $\mathcal{F}$. Thus (A.8) provides us with a way of constructing measures which are absolutely continuous w.r.t. the base measure μ, and a natural question is whether *all* measures which are absolutely continuous w.r.t. μ are obtained in this way. The affirmative answer to this question is given by the following central result.

Theorem A.52 (The Radon–Nikodym Theorem) *Consider the measure space $(X, \mathcal{F}, \mu)$, where we assume that μ is finite, i.e. that $\mu(X) < \infty$. Let ν be* *a measure on* $(X, \mathcal{F})$ *such that* $\nu << \mu$ *on* $\mathcal{F}$. *Then there exists a non-negative function* $f : X \to R$ *such that:*

$$f \text{ is } \mathcal{F}\text{-measurable,} \tag{A.9}$$

$$\int_X f(x)d\mu(x) < \infty, \tag{A.10}$$

$$\nu(A) = \int_A f(x)d\mu(x), \quad \text{for all } A \in \mathcal{F}. \tag{A.11}$$

The function f *is called the* **Radon–Nikodym derivative** *of* ν *w.r.t* μ. *It is uniquely determined* μ*-a.e. and we write*

$$f(x) = \frac{d\nu(x)}{d\mu(x)}, \tag{A.12}$$

or alternatively

$$d\nu(x) = f(x)d\mu(x). \tag{A.13}$$

Proof We sketch a proof which is due to von Neumann. Define a new measure λ by setting $\lambda(A) = \mu(A) + \nu(A)$, for all $A \in \mathcal{F}$. For any $g \in L^2(\lambda)$ we can define the linear mapping $\Phi : X \to R$ by

$$\Phi(g) = \int_X g(x)d\nu(x),$$

and by the triangle and Cauchy–Schwartz inequalities we have

$$|\Phi(g)| = \left| \int_X g d\nu \right| \le \int_X |g| d\nu \le \int_X |g| d\lambda \le \sqrt{\lambda(X)} \cdot \|g\|_{L^2(\lambda)}.$$

Thus, from the Riesz Representation Theorem there exists an $f \in L^2(\lambda)$ such that $\Phi(g) = (g, f)$ for all $g \in L^2(\lambda)$, i.e.

$$\int_X g d\nu = \int_X g f d\lambda, \quad \forall g \in L^2(\lambda). \tag{A.14}$$

By choosing $g = I_A$ for arbitrary $A \in \mathcal{F}$ and using the fact that $0 \le \nu(A) \le \lambda(A)$ we see that $0 \le f \le 1$. We now write (A.14) as

$$\int_X g d\nu = \int_X g f d\nu + \int_X g f d\mu,$$

i.e.

$$\int_X g(1 - f)d\nu = \int_X g f d\mu, \quad \forall g \in L^2(\lambda). \tag{A.15}$$

Since this holds for all $g \in L^2(\lambda)$ and in particular for all indicator functions, we can write this on "differential form" (see the exercises for a justification of this) as

$$(1 - f)d\nu = f d\mu.$$

It is now tempting to multiply through by $(1-f)^{-1}$ to obtain

$$d\nu = \frac{f}{(1-f)} d\mu,$$

and thus to define the Radon–Nikodym derivative by $f/(1-f)$, but the problem is of course what happens when $f = 1$. Define therefore A by $A = \{x \in X;\ f(x) = 1\}$, and set $g = I_A$. From (A.15) we obtain

$$\int_A f d\mu = \int_A (1-f) d\nu = 0,$$

and (since $f \geq 0$) this implies that $\mu(A) = 0$. We now use the assumption that $\nu << \mu$ to deduce that also $\nu(A) = 0$, so we can safely write

$$d\nu = \frac{f}{(1-f)} d\mu,$$

and we see that the Radon–Nikodym derivative is in fact given by

$$\frac{d\nu}{d\mu} = \frac{f}{(1-f)}.$$

□

Example A.53 Going back to Example A.51 we consider the case when $X = N$, and $\mathcal{F} = 2^N$, and we recall that the relation $\nu << \mu$ means that

$$\mu(n) = 0 \quad \Rightarrow \quad \nu(n) = 0.$$

Thus, given μ and ν with $\nu << \mu$, the problem of finding a Radon–Nikodym derivative f will in this case boil down to the problem of finding an f such that

$$\nu(n) = f(n)\mu(n), \quad \forall n \in N. \tag{A.16}$$

We see that for those n where $\mu(n) \neq 0$ we can solve (A.16) by defining $f(n)$ as

$$f(n) = \frac{\nu(n)}{\mu(n)},$$

so the only problem occurs when $\mu(n) = 0$. However, from the absolute continuity it follows that $\nu(n) = 0$ whenever $\mu(n) = 0$, so for those n equations (A.16) becomes

$$0 = f(n) \cdot 0$$

and we see that for those n we can define f arbitrarily, say by putting $f(n) = 17$. Consequently f is not uniquely defined, but we see that the set where it is not uniquely defined (i.e. for those n where $\mu(n) = 0$) has μ measure zero.

It is important to realize that the concept of absolute continuity is defined relative to the given sigma-algebra. If, for example, $\mathcal{G} \subseteq \mathcal{F}$ then it could well happen that $\mu << \nu$ on $\mathcal{G}$ while it does not hold that $\mu << \nu$ on $\mathcal{F}$. A trivial example is given by setting $X = \{1,2,3\}$, and defining

$$\mathcal{F} = 2^X \quad \mathcal{G} = \{X, \emptyset, \{1\}, \{2,3\}\}$$

and

$$\begin{array}{lll} \mu(1) = 2, & \mu(2) = 0, & \mu(3) = 2, \\ \nu(1) = 8, & \nu(2) = 5, & \nu(3) = 13. \end{array}$$

Here we obviously do **not** have $\nu << \mu$ on $\mathcal{F}$, since $\mu(2) = 0$ while $\nu(2) \neq 0$. We do however, have $\nu << \mu$ on $\mathcal{G}$ with () Radon–Nikodym derivative given by

$$f(n) = \begin{cases} 4 & \text{for } n = 1 \\ 9 & \text{for } n = 2 \\ 9 & \text{for } n = 3 \end{cases}$$

(Note in particular that f is $\mathcal{G}$-measurable.)

We thus see that if we enlarge the sigma-algebra we may lose absolute continuity, but if $\nu << \mu$ on some measurable space $(X, \mathcal{F})$ and $\mathcal{G} \subseteq \mathcal{F}$, then $\nu << \mu$ also on $\mathcal{G}$.

A.12 Exercises

Exercise A.1 Prove Proposition A.2.

Exercise A.2 Which of the properties stated in Proposition A.2 are still valid (and which are not necessarily valid) if we replace f^{-1} with f in expressions like $f^{-1}(a \cup B)$ etc.?

Exercise A.3 Show that in the definition of a sigma-algebra, the closedness property under countable intersections in fact follows from the other defining properties.

Exercise A.4 Let μ be a measure. Show formally, using the axioms of a measure, that the following relations hold for all measurable sets A and B:

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B),$$
$$B \subseteq A \Rightarrow \mu(A \cap B^c) = \mu(A) - \mu(B).$$

Exercise A.5 Let X be a finite set $X = \{x_1, \ldots, x_K\}$ and define $\mathcal{F}$ as the power algebra 2^X. Let furthermore μ be a measure on $(X, \mathcal{F})$ and define p_n by $p_n = \mu(\{x_n\})$, i.e. p_n denotes the mass on the point x_n. Let f be any real valued non-negative function, and show in detail, by using Definition A.17 that

$$\int_X f(x) d\mu(x) = \sum_{i=1}^{K} f(x_n) p_n.$$

Exercise A.6 Define X by $X = [0, 2]$ and define $\mathcal{F}$ by $\mathcal{F} = \{\emptyset, \Omega, [0, 1), [1, 2]\}$. Define a measure μ by setting

$$\mu([0, 1)) = 1, \quad \mu([1, 2]) = 1, \mu([0, 2]) = 2,$$

and define the functions $f, g : X \to R$ by $f(x) = x$ and $g(x) = 2 - x$.

(a) Show that f and g are not measurable.
(b) Despite the fact that neither f nor g is measurable we now define

$$\int_X f(x)d\mu(x) \stackrel{def}{=} \sup \int_X \varphi(x)d\mu(x),$$

where the supremum is taken over all non-negative measurable simple functions φ such that $\varphi \leq g$. We make the corresponding definition for g. Now compute and compare $\int_X f d\mu$, $\int_X g d\mu$, and $\int_X (f+g)d\mu$.

Exercise A.7 The object of this exercise is to show that a measurable function can be well approximated by simple functions. Let therefore $f: X \to R$ be a non-negative measurable function on some measurable space $(X, \mathcal{F})$, and also assume that there exists some constant M such $0 \leq f(x) \leq M$ for all $x \in X$. Show that for every n there exists a simple function f_n such that $f(x) \leq f_n(x) \leq f(x) + 1/n$ for all $x \in X$.

Hint: Consider sets of the form $\{x \in X;\ k/n \leq f(x) \leq (k+1)/n\}$.

Exercise A.8 Continuing the exercise above, show that there exists an increasing sequence of simple functions f_n such that $f_n(x) \uparrow f(x)$ for all $x \in X$.

Exercise A.9 Fill in the details in the proof of Proposition A.33.

Exercise A.10 Prove proposition A.38.

Exercise A.11 Describe the sigma-algebra on R which is generated by the class of all singleton sets, i.e. of all sets of the form $\{x\}$, where $x \in R$.

Exercise A.12 Prove Proposition A.45 by using Proposition A.43.

Exercise A.13 Let $\{\mathcal{F}_n;\ n = 1, 2, \ldots\}$ be a sequence of sigma-algebras on some common space X. Does it always hold that

$$\mathcal{G} \stackrel{def}{=} \bigcup_{i=1}^{\infty} \mathcal{F}_n$$

is a sigma-algebra? What if the sequence is increasing?

Exercise A.14 Consider two measures μ and ν with $\nu << \mu$. For any functions $g \in L^1(\nu)$ and $f \in L^1(\mu)$ we define the "differential equality"

$$g d\nu = f d\mu \tag{A.17}$$

as being shorthand for

$$\int_A g(x)d\nu(x) = \int_A f(x)d\mu(x), \quad \forall A \in \mathcal{F}.$$

(a) Prove that

$$f = \frac{d\nu}{d\mu} \quad \Leftrightarrow \quad d\nu = f d\mu.$$

(b) Show that for any $h \in L^1(gd\nu)$ it holds that

$$gd\nu = fd\mu \quad \Rightarrow \quad hgd\nu = hfd\mu.$$

(c) Assume that $\lambda << \nu << \mu$ and prove the "chain rule"

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu}.$$

(d) Assuming $\nu \sim \mu$, prove that

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu}\right)^{-1}.$$

A.13 Notes

Royden (1988) gives a very clear and readable presentation of measure theory, and also treats point set topology and basic functional analysis.

Appendix B

PROBABILITY THEORY

A probability space is simply a measure space $(\Omega, \mathcal{F}, P)$ where the measure P has the property that it has total mass equal to unity, i.e.

$$P(\Omega) = 1.$$

The underlying space Ω often is referred to as the **sample space**, and the elements of the sigma-algebra $\mathcal{F}$ are called **events**.

B.1 Random Variables and Processes

In this section we will discuss random variables and random processes.

Definition B.1 *A* **random variable** X *is a mapping*

$$X : \Omega \to R$$

such that X is $\mathcal{F}$-measurable.

Remark B.1.1 *As the reader probably has observed, the letter X, which in Appendix A was used to designate a measure space, is now used as the name of a random variable. This is an unfortunate clash of notation, but since both uses of the letter X is standard within its respective area of application we will simply accept this. There is, however, no risk of confusion: from now on the measure space will always be a probability space, and thus denoted by Ω. The only use of X will be as a name for a random variable or a random process.*

The interpretation of a random variable is as follows:

- Somewhere, hidden from us, a point $\omega \in \Omega$ in the sample space is "randomly" chosen by, say, the God of Chance.
- We are not allowed to observe ω directly, but we are allowed to observe measurements on the sample space, i.e. we can observe the real number $X(\omega)$, which gives us partial information about ω.

Definition B.2 *The* **distribution measure** μ_X *for a random variable X is a measure on $(R, \mathcal{B})$ defined by*

$$\mu_X(B) = P(\{\omega \in \Omega;\ X(\omega) \in B\}), \quad B \in \mathcal{B},$$

i.e.

$$\mu_X(B) = P(X^{-1}(B)).$$

The (cumulative) **distribution function** *of X is denoted by F_X and defined by*

$$F_X(x) = P(\{\omega \in \Omega;\ X(\omega) \le x\}).$$

Note that since X is assumed to be measurable, the event $\{\omega \in \Omega;\ X(\omega) \in B\}$ is in $\mathcal{F}$ so its P measure is well defined. We will often write this event as $\{X \in B\}$ and then the definition of the distribution measure becomes

$$\mu_X(B) = P(X \in B),$$

and the distribution function can be written as

$$F_X(x) = P(X \le x), \quad x \in R.$$

We now go on to introduce the measure theoretic definition of an expected value.

Definition B.3 *For any $X \in L^1(\Omega, \mathcal{F}, P)$ its* **expected value***, denoted by $E[X]$, is defined by*

$$E[X] = \int_\Omega X(\omega) dP(\omega).$$

For $X \in L^2$ the **variance** *is defined by*

$$Var[X] = E\left[(X - E[X])^2\right].$$

We note that the definition above gives the expected value and the variance as integrals over the (abstract) sample space Ω. The following result connects these formulas to the standard elementary formulas where expectations and variances are computed as integrals over the real line.

Proposition B.4 *Let $g : R \to R$ be a Borel function such that the composite random variable $g(X)$ is integrable. The we have*

$$E[g(X)] = \int_\Omega g(X(\omega))\, dP(\omega) = \int_R g(x) d\mu_X(x).$$

Proof We leave the proof as an exercise. See the exercises for hints. □

We note that the first equality above holds by definition. The point of the result is thus the second equality. The careful reader notes (with satisfaction) that, by Proposition A.45, the measurability of $g(X)$ is guaranteed.

In order to convince the reader that the framework above really is of some value we will now prove a useful result which shows that in some cases an expected value can be computed in terms of an ordinary Lebesgue integral.

Proposition B.5 *Let X be a non-negative random variable. Then it holds that*

$$E[X] = \int_0^\infty P(X \ge t)\, dt.$$

Proof By using the Fubini Theorem we have

$$\begin{aligned}E[X] &= \int_\Omega X(\omega)dP(\omega) = \int_\Omega \left[\int_0^{X(\omega)} 1 \cdot dt\right] dP(\omega) \\ &= \int_\Omega \left[\int_0^\infty I\{t \le X(\omega)\} \cdot dt\right] dP(\omega) = \int_0^\infty \left[\int_\Omega I\{t \le X(\omega)\} dP(\omega)\right] dt \\ &= \int_0^\infty \left[\int_{A_t} dP(\omega)\right] dt,\end{aligned}$$

where the event $A_t = \{X \ge t\}$. We thus have

$$E[X] = \int_0^\infty P(A_t)dt = \int_0^\infty P(X \ge t)\, dt. \qquad \square$$

We now go on to the concept of a random process.

Definition B.6 *A random process on the probability space* $(\Omega, \mathcal{F}, P)$ *is a mapping*

$$X : R_+ \times \Omega \to R,$$

such that for each $t \in R_+$ *the mapping*

$$X(t, \cdot) : \Omega \to R$$

is $\mathcal{F}$*-measurable.*

We interpret $X(t,\omega)$ as "the value at time t given the outcome ω", and we will also use the alternative notation $X_t(\omega)$.

Note the following:

- For each fixed t the mapping

$$\omega \longmapsto X(t,\omega),$$

  which we also denote by X_t, is a **random variable**.
- For each $\omega \in \Omega$ the mapping

$$t \longmapsto X(t,\omega)$$

  is a **deterministic function of time**. This function, which we may draw as a graph, is called the **realization** or **trajectory** of X for the outcome ω. When we observe a random process over time (like the evolution of a stock price) we thus see the trajectory of a single ω.

In the definition above we have defined processes only on the time interval $0 \le t < \infty$, but we can of course consider processes defined on just a subinterval or on the integers (a "discrete time process").

B.2 Partitions and Information

Consider a sample space Ω and a given partition $\mathcal{P} = \{A_1, \ldots, A_K\}$ of Ω. We can now give an intuitive interpretation of $\mathcal{P}$ in *information terms* along the following lines:

- Someone (the God of Chance?) chooses a point ω in the sample space. We do not know exactly which point has been chosen.
- What we do get information about is exactly which component of $\mathcal{P}$ that ω belongs to. More formally one can think of this as an experiment where we are allowed to observe the random variable Y defined by

$$Y(\omega) = \sum_{i=1}^{K} n \cdot I_{A_n}(\omega).$$

If we observe $Y(\omega) = n$ then we know with certainty that ω lies in A_n.

We thus see that having access to a certain partition can be interpreted as having access to a certain amount of information. We note that the trivial partition $\mathcal{P} = \{\Omega\}$ corresponds to "no information at all". The other extreme case occurs when Ω is finite, say $\Omega = \{\omega_1, \ldots, \omega_N\}$, and the partition is given by $\mathcal{F} = \{\{\omega_1\}, \{\omega_2\}, \ldots, \{\omega_N\}\}$. This case corresponds to "full information".

In some cases we may even compare the informational content in two separate partitions. Consider, as an example, the space $\Omega = [0, 1]$ with two partitions

$$\mathcal{P}_1 = \{A_1, A_2, A_3, A_4\},$$

where

$$A_1 = \left[0, \frac{1}{3}\right), \quad A_2 = \left[\frac{1}{3}, \frac{1}{2}\right), \quad A_3 = \left[\frac{1}{2}, \frac{3}{4}\right), \quad A_4 = \left[\frac{3}{4}, 1\right]$$

and

$$\mathcal{P}_2 = \{B_1, B_2, B_3\},$$

where

$$B_1 = \left[0, \frac{1}{3}\right), \quad B_2 = \left[\frac{1}{3}, \frac{3}{4}\right), \quad B_3 = \left[\frac{3}{4}, 1\right].$$

It is now natural to say that $\mathcal{P}_1$ *contains more information* than $\mathcal{P}_2$ since the partition $\mathcal{P}_1$ has been obtained by subdividing some of the components of $\mathcal{P}_2$ into smaller pieces. There is thus a natural partial order relation between different partitions of a given sample space, and in the example above we say that $\mathcal{P}_1$ is *finer* than $\mathcal{P}_2$.

Definition B.7 *For a given sample space Ω, a partition $\mathcal{S}$ is said to be* **finer** *than a partition $\mathcal{P}$ if every component in $\mathcal{P}$ is a union of components in $\mathcal{S}$.*

The interpretation of this is of course that "$\mathcal{S}$ contains more information than $\mathcal{P}$".

Consider again the sample space Ω and some given but otherwise arbitrary mapping $f : \Omega \to R$. For simplicity we assume that f only takes finitely many values, and we denote these values by $x_1, x_2, \ldots, x_K$. The interpretation is that f is a measurement on Ω and that we gain knowledge about the (unknown) sample point ω by observing the measurement $f(\omega)$. We now note that f generates a natural partition $\mathcal{P}(f)$ defined by

$$\mathcal{P}(f) = \{A_1, \ldots, A_K\},$$

where

$$A_n = \{\omega \in \Omega;\ f(\omega) = x_n\}, \quad n = 1, 2, \ldots, K,$$

i.e.

$$A_n = f^{-1}(x_n), \quad n = 1, 2, \ldots, K.$$

It is then natural to interpret the partition $\mathcal{P}(f)$ as "the information generated by f", since by observing f we can exactly tell in which component of $\mathcal{P}(f)$ that ω lies.

We also see that given the information in $\mathcal{P}(f)$, i.e. given information about in which components ω lies, we can exactly determine the value $f(\omega)$. The reason for this is of course that f is constant on each component on $\mathcal{P}(f)$ and we can easily generalize this observation to the following definition and lemma.

Definition B.8 *A given mapping $f : \Omega \to R$ is called measurable w.r.t a partition $\mathcal{P}$ if and only if it is constant on the components of $\mathcal{P}$.*

Lemma B.9 *Take as given a sample space Ω, a finite valued mapping $f : \Omega \to R$ and a partition $\mathcal{P}$. If f is $\mathcal{P}$-measurable then the value of f is completely determined by the information in $\mathcal{P}$ in the sense that if we know in which component $\mathcal{P}$ is located, then we know the function value $f(\omega)$.*

Consider again the sample space Ω and a finite valued mapping $f : \Omega \to R$. If we also are given another mapping $g : R \to R$ and define $h : \Omega \to R$ by $h(\omega) = g(f(\omega))$, then it is obvious that h generates less information than f, i.e. that $\mathcal{P}(f)$ is finer than $\mathcal{P}(h)$, which also can be expressed by saying that h is constant on the components in $\mathcal{P}(f)$, i.e. h is $\mathcal{P}(f)$-measurable. There is also a converse of this result which will have important generalizations later on in the text.

Proposition B.10 *Consider a fixed Ω and two finite valued mappings $f : \Omega \to R$ and $h : \Omega \to R$. Assume that h is $\mathcal{P}(f)$-measurable. Then there exists a function $g : R \to R$ such that $h = g \circ f$.*

Proof Exercise for the reader. □

B.3 Sigma-Algebras and Information

Let us again consider the sample space Ω and a given partition $\mathcal{P}$. We note the following facts:

- The partition $\mathcal{P}$ generates a natural sigma-algebra, namely $\sigma\{\mathcal{P}\}$.
- From $\sigma\{\mathcal{P}\}$ we can easily reconstruct the original partition $\mathcal{P}$, since the components of $\mathcal{P}$ are precisely the *atoms* in $\sigma\{\mathcal{P}\}$, i.e. the sets in $\sigma\{\mathcal{P}\}$ which have no proper subsets (apart from $\emptyset$) in $\sigma\{\mathcal{P}\}$.
- If $\mathcal{F}$ and $\mathcal{S}$ are two partitions, then

$$\mathcal{S} \quad \text{is finer than} \quad \mathcal{P}$$
$$\text{if and only if}$$
$$\mathcal{P} \subseteq \mathcal{S}.$$

- For any mapping $f : \Omega \to R$ it holds that

$$f \text{ is } \mathcal{P}\text{-measurable}$$
$$\text{if and only if}$$
$$f \text{ is } \sigma\{\mathcal{P}\}\text{-measurable.}$$

As long as we are working with finite partitions it is thus logically equivalent if we work with partitions or if we work with the corresponding sigma-algebras. From a technical point of view, however, the sigma-algebra formalism is superior to the partition formalism, since a sigma-algebra is closed under the usual set of theoretic operations. Furthermore, our development of measure theory demands that we have a sigma-algebra as the basic object. Thus, even if the intuitive information concept is perhaps most natural to formulate within the partition framework, it turns out that the sigma-algebra formalism is vastly superior in the long run. It should also be emphasized that the equivalence between partitions and sigma-algebras only holds when the partition is finite. In the general case, there is simply no alternative to the sigma-algebra formalism.

We will therefore henceforth formalize the intuitive information concept in terms of sigma-algebras, and in particular we will interpret the relation

$$\mathcal{G} \subseteq \mathcal{F}$$

between two sigma-algebras $\mathcal{G}$ and $\mathcal{F}$ as

" $\mathcal{G}$ contains less information than $\mathcal{F}$."

Let us again take a sample space Ω as given, and consider a mapping $\Omega\colon XR$. We recall an earlier definition:

Definition B.11 *The sigma-algebra $\sigma\{X\}$ is defined as the smallest sigma-algebra $\mathcal{F}$ such that X is $\mathcal{F}$-measurable.*

We will refer to $\sigma\{X\}$ as "the sigma-algebra generated by X". Technically speaking it is the intersection of all sigma-algebras $\mathcal{G}$ such that X is $\mathcal{G}$-measurable, but we can in fact give a more explicit representation.

Proposition B.12 *We have the representation*

$$\sigma\{X\} = \left\{X^{-1}(B);\ B \in \mathcal{B}(R)\right\}.$$

Proof Exercise for the reader. □

Definition B.13 *Let $\mathcal{K}$ be an arbitrary family of mappings from Ω to R. Then $\sigma\{\mathcal{K}\}$ is defined as the smallest sigma-algebra $\mathcal{G}$ such that X is $\mathcal{G}$-measurable for all $X \in \mathcal{K}$.*

We now have a general result for sigma-algebras, which is parallel to Proposition B.10 for partitions. The proof is not easy and therefore omitted.

Proposition B.14 *Let $X_1, \dots, X_N$ be given mappings $X_n : \Omega \to R$, and assume that a mapping $Z : \Omega \to R$ is $\sigma\{X_1, \dots, X_N\}$-measurable. Then there exists a Borel function $f : R^N \to R$ such that for all $\omega \in \Omega$ we have*

$$X(\omega) = f(X_1(\omega), \dots, X_N(\omega)).$$

This proposition thus formalizes the idea that if a random variable X is measurable w.r.t. a certain sigma-algebra then "the value of the variable is completely determined by the information contained in the sigma-algebra".

We now pass on to random processes, and note that every random process generates an entire family of interesting sigma-algebras.

Definition B.15 *Let $\{X_t;\ t \geq 0\}$ be a random process, defined on the probability space $(\Omega, \mathcal{F}, P)$. We then define the* **sigma-algebra generated by X over the interval $[\mathbf{0}, \mathbf{t}]$** *by*

$$\mathcal{F}_t^X = \sigma\{X_s;\ s \leq t\}.$$

The intuitive interpretation is that "$\mathcal{F}_t^X$ is the information generated by observing X over the time interval $[0,t]$". There is in general no very explicit description of $\mathcal{F}_t^X$, but it is not hard to show that $\mathcal{F}_t^X$ is generated by all events of the form $\{X_s \in B\}$ for all $s \leq t$ and all Borel sets B.

If Z is a random variable then, based on the discussions above, we interpret the statement

"Z is $\mathcal{F}_t^X$-measurable"

as

"Z is a function of the entire X-trajectory over the interval $[0,t]$."

From the definition it is immediately clear that

$$s \leq t \quad \Rightarrow \quad \mathcal{F}_s^X \subseteq \mathcal{F}_t^X,$$

so every random process X will in this way generate an increasing family of sigma-algebras. We now generalize this concept.

Definition B.16 *A* **filtration** $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$, *sometimes written as* $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$, *on the probability space $(\Omega, \mathcal{F}, P)$ is an indexed family of sigma-algebras on Ω such that*:

- $\mathcal{F}_t \subseteq \mathcal{F}, \quad \forall t \geq 0,$
- $s \leq t \quad \Rightarrow \quad \mathcal{F}_s^X \subseteq \mathcal{F}_t^X.$

Given a filtration $\mathbf{F}$ *as above, the sigma-algebra* $\mathcal{F}_\infty$ *is defined as*

$$\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t.$$

A filtration thus formalizes the idea of an non-decreasing information flow over time. We now introduce one of the most basic concepts for stochastic processes.

Definition B.17 *Consider a given filtration* $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ *on some probability space, and a random process* X *on the same space. We say that the process* X *is* **adapted** *to the filtration* $\mathbf{F}$ *if*

$$X_t \in \mathcal{F}_t, \quad \forall t \geq 0.$$

The interpretation of this definition is that "For every fixed t, the process value X_t is completely determined by the information $\mathcal{F}_t$ that we have access to at time t." Alternatively we can say that "an adapted process does not look into the future". W note in passing that the process X is always adapted to the **internal filtration** $\mathcal{F}_t^X$ generated by X.

Example B.18 Let Z be any random process with continuous trajectories, and define the filtration $\mathbf{F}$ as the internal filtration $\mathcal{F}_t = \mathcal{F}_t^Z$. The following processes are adapted:

$$\begin{aligned}
X_t &= sup_{s \leq t} |Z_s|, \\
X_t &= Z_{t/2}, \\
X_t &= \int_0^t Z_s ds.
\end{aligned}$$

The processes

$$\begin{aligned}
X_t &= Z_{t+1}, \\
X_t &= \int_0^{t+2} Z_s ds
\end{aligned}$$

are **not** adapted.

In a typical financial application, the filtration $\mathbf{F}$ is generated by the observed asset prices. A natural requirement for a portfolio strategy is that the portfolio decision that is taken at time t, is only allowed to depend upon the public information that we have access to at time t (by observing asset prices). The formalization of this idea is to demand that the portfolio strategy should be adapted.

B.4 Independence

We consider again a given probability space $(\Omega, \mathcal{F}, P)$, and recall the standard definition of independent events.

Definition B.19 *Two events $A, B \in \mathcal{F}$ are* **independent** *if*

$$P(A \cap B) = P(A) \cdot P(B).$$

We now generalize this definition to sigma-algebras, random variables, and processes.

Definition B.20

- *Two sigma-algebras $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ are independent if*

$$P(G \cap H) = P(G) \cdot P(H),$$

for all $G \in \mathcal{G}$ and all $H \in \mathcal{H}$.
- *Two random variables X and Y are independent if the sigma-algebras $\sigma\{X\}$ and $\sigma\{Y\}$ are independent.*
- *Two stochastic processes X and Y are independent if the sigma-algebras $\sigma\{X_t;\ t \geq 0\}$ and $\sigma\{Y_t;\ t \geq 0\}$ are independent.*
- *An indexed family $\{\mathcal{G}_\gamma;\ \gamma \in \Gamma\}$ of sigma algebras, where $\mathcal{G}_\gamma \in \mathcal{F}$ for each $\gamma \in \Gamma$ are mutually independent if*

$$P\left(\bigcap_{i=1}^{n} G_n\right) = \prod_{i=1}^{n} P(G_n),$$

for every finite sub collection $G_1, \ldots, G_n$ where $G_i \in \mathcal{G}_{\gamma_i}$ and where $\gamma_i \neq \gamma_j$ for $i \neq j$. The extension to random variables and processes is the obvious one.

We note that two random variables X and Y are independent if and only if

$$P(X \in B_1 \ \&\ Y \in B_2) = P(X \in B_1) \cdot P(Y \in B_2),$$

for all Borel sets B_1 and B_2.

We now formulate and sketch the proof of a very useful result.

Proposition B.21 *Suppose that the random variables X and Y are independent. Assume furthermore that X, Y, and XY are in L^1. Then we have*

$$E[X \cdot Y] = E[X] \cdot E[Y]. \tag{B.1}$$

Proof We do the proof in several steps:

1. Choose arbitrary $A \in \sigma\{X\}$ and $B \in \sigma\{Y\}$. Then we have

$$\begin{aligned} E[I_A \cdot I_B] = E[I_{A \cap B}] = \int_{A \cap B} dP(\omega) = P(A \cap B) \\ = P(A) \cdot P(B) = E[I_A] \cdot E[I_B]. \end{aligned}$$

Thus the proposition holds for indicator functions.
2. From the previous item and from the linearity of the integral it follows that (B.1) holds for all simple functions (check this in detail).

3. In the general case we can WLOG (without loss of generality) assume that X and Y are non-negative. In that case there exist (see the exercises) sequences $\{X_n\}$ and $\{Y_n\}$ of simple random variables such that

$$X_n \uparrow X, \quad X_n \in \sigma\{X\},$$
$$Y_n \uparrow Y, \quad Y_n \in \sigma\{Y\}.$$

From item 2 above we have

$$E[X_n \cdot Y_n] = E[X_n] \cdot E[Y_n].$$

Now we let $n \to \infty$ and use the Monotone Convergence Theorem. □

We also have the following simple but useful corollary.

Corollary B.22 *If X and y are independent random variables, and if f and g are Borel functions, then $f(X)$ and $g(Y)$ are independent. In particular, if $f(X)$, $g(Y)$, and $f(X)g(Y)$ are in L^1, then*

$$E[f(X) \cdot g(Y)] = E[f(X)] \cdot E[g(Y)].$$

Proof Exercise for the reader. □

B.5 Conditional Expectations

Apart from the concept of independence, the most important concept in probability theory is that of conditional expectation. We will need to treat this concept in its most general (but also most useful) version, namely that of a conditional expectation given a sigma-algebra. We start, however, with a more elementary discussion in order to motivate the more abstract arguments later on.

Consider a fixed probability space $(\Omega, \mathcal{F}, P)$, and suppose that A and B are events in $\mathcal{F}$ with $P(B) \neq 0$. We recall the elementary definition of conditional probability.

Definition B.23 *The probability of A, conditional on B is defined by*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \tag{B.2}$$

The intuition behind this definition is as follows:

- The probability for any event A is the fraction of the total mass which is is located on A, so
$$P(A) = \frac{P(A)}{P(\Omega)}.$$
- When we condition on B, we **know** that B has happened. Thus the effective sample space is now B rather than Ω. This explains the normalizing factor in the denominator of (B.2).
- The only part of A that can occur if we know that B has occurred is precisely $A \cap B$.

What we are looking for is now a sensible definition of the object

$$E[X|\mathcal{G}],$$

where X is a random variable and $\mathcal{G}$ is a sigma-algebra included in $\mathcal{F}$. The interpretation should be that $E[X|\mathcal{G}]$ is "the expectation of X given that we have access to the information in $\mathcal{G}$". It is not trivial to formalize this rather vague notion, so we start with some heuristics.

We therefore recall that the unconditional expected value is given by

$$E[X] = \int_\Omega X(\omega)P(d\omega),$$

i.e. $E[X]$ is a weighted average of the values of X, where we have used the "probabilities" $P(d\omega)$ as weights.

Suppose now that we have obtained information about the outcome of the random experiment, in the sense that we know that sample point ω is in the set B. The natural definition of the expected value of X given B is then obtained by taking the weighted average of X over the new effective sample space B. We must of course normalize the probability measure so that we have total mass equal to unity on the new space B. Thus we normalize the probabilities as

$$\frac{P(d\omega)}{P(B)}$$

and we may thus define the object $E[X|B]$.

Definition B.24 *Suppose $B \in \mathcal{F}$ with $P(B) > 0$, and that $X \in L^1(\Omega, \mathcal{F}, P)$. Then "the conditional expectation of X given B" is defined by*

$$E[X|B] = \frac{1}{P(B)} \int_B X(\omega)dP(\omega).$$

We now consider a slightly more general case, where we are given a finite partition $\mathcal{P} = \{A_1, \ldots, A_K\}$ with $A_n \in \mathcal{F}$ for $n = 1, \ldots, K$. Having access to the information contained in $\mathcal{P}$ is, according to our earlier discussion, equivalent to knowing exactly in which of the components $A_1, \ldots, A_K$ that the outcome ω lies. Now consider the following schedule:

- Someone (the God of Chance?) chooses "randomly" a point ω in the sample space. We do not know exactly which point has been chosen.
- We are informed about in exactly which component of the partition that ω lies.
- As soon as we know in which component ω lies, say for example in A_n, then we can compute the conditional expectation of X given A_n according to the formula above.

From this we see that exactly which conditional expectation we will compute will depend on in which component that ω lies. We may therefore define a mapping from Ω to the real line by

$$\omega \longmapsto E[X|A_n], \quad \text{if } \omega \in A_n,\ n = 1, \dots, K.$$

This leads us to the following definition.

Definition B.25 *With assumptions as above, and also assuming that $P(A_n) > 0$ for all n, we define $E[X|\mathcal{P}]$, "the conditional expectation of X given the information in $\mathcal{P}$", by*

$$E[X|\mathcal{P}](\omega) = \sum_{n=1}^{K} I_{A_n}(\omega) E[X|A_n], \tag{B.3}$$

i.e.

$$E[X|\mathcal{P}](\omega) = \frac{1}{P(A_n)} \int_{A_n} X dP, \quad \textit{when } \omega \in A_n. \tag{B.4}$$

We note that the object $E[X|\mathcal{P}]$ is not a real number but a mapping from Ω to R, i.e. it is a **random variable**. We also note that, by definition, $E[X|\mathcal{P}]$ is constant on each component of $\mathcal{P}$, i.e. it is $\sigma\{\mathcal{P}\}$-measurable.

We would now like to extend the definition above to the case when we condition on a general sigma-algebra, and not only on a finite partition. This is, however, not entirely straightforward, and a major problem with the definition above is that we had to assume that $P(A_n) > 0$ for all n, since otherwise we divide by zero in (B.4). We therefore have to take a more indirect approach, and start by listing some important properties of the conditional expectation above.

Proposition B.26 *Assume that $(\Omega, \mathcal{F}, P)$, X, and $\mathcal{P}$ are as above. Define the sigma-algebra $\mathcal{G} \subseteq \mathcal{F}$ by $\mathcal{G} = \sigma\{\mathcal{P}\}$. Then the conditional expectation $E[X|\mathcal{P}]$ is characterized as the unique random variable Z on $(\Omega, \mathcal{F}, P)$ with the following properties*:

(i) *Z is $\mathcal{G}$ measurable.*
(ii) *For every $G \in \mathcal{G}$ it holds that*

$$\int_G Z(\omega) dP(\omega) = \int_G X(\omega) dP(\omega).$$

Proof Exercise for the reader. □

The point of this result is that it characterizes the conditional expectation in a way which does not require the components of $\mathcal{P}$ to have strictly positive probabilities. In fact, the conditions (i)–(ii) above can be formulated for *any* sigma-algebra $\mathcal{G}$ even if $\mathcal{G}$ is *not* generated by a finite partition. This is the starting point for our final definition of conditional expectations.

Definition B.27 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable in $L^1(\Omega, \mathcal{F}, P)$. Let furthermore $\mathcal{G}$ be a sigma-algebra such that $\mathcal{G} \subseteq \mathcal{F}$. If Z is a random variable with the properties that*:

(i) *Z is $\mathcal{G}$-measurable.*
(ii) *For every $G \in \mathcal{G}$ it holds that*

$$\int_G Z(\omega)dP(\omega) = \int_G X(\omega)dP(\omega). \tag{B.5}$$

Then we say that Z is the **conditional expectation of X given the sigma-algebra $\mathcal{G}$**. *In that case we denote Z by the symbol*

$$E[X|\mathcal{G}].$$

The price that we have to pay for this very general definition of conditional expectation is that we have a nontrivial existence problem, since it is not immediately clear that in the general case there will always exist a random variable Z as above. We note that X itself will obviously always satisfy (ii), but in the general case it will not satisfy (i). We do however have an existence result, and the proof is a nice application of the Radon–Nikodym Theorem.

Theorem B.28 *Let $(\Omega, \mathcal{F}, P)$, X, and $\mathcal{G}$ be as in Definition B.27. Then the following hold*:

- *There will always exist a random variable Z satisfying conditions (i)–(ii) above.*
- *The variable Z is unique, i.e. if both Y and Z satisfy (i)–(ii) then $Y = Z$, P – a.s.*

Proof Define the measure ν on $(\Omega, \mathcal{G})$ by

$$\nu(G) \stackrel{def}{=} \int_G X(\omega)dP(\omega).$$

Trivially we then have $\nu << P$ and we see directly by inspection that if we define Z by

$$Z = \frac{d\nu}{dP}, \quad \text{on } \mathcal{G},$$

then Z will be $\mathcal{G}$-measurable and it will have the property that

$$\nu(G) = \int_G ZdP,$$

i.e

$$\int_G ZdP = \int_G XdP$$

for all, $G \in \mathcal{G}$. □

In passing we note that if $\mathcal{G}$ is the trivial sigma-algebra $\mathcal{G} = \{\Omega, \emptyset\}$ then it follows directly from the definition above (prove this!) that

$$E[X|\mathcal{G}] = E[X].$$

We now have some natural and simple rules for calculating conditional expectations.

Proposition B.29 *The following hold*:

$$X \leq Y \quad \Rightarrow \quad E[X|\mathcal{G}] \leq E[Y|\mathcal{G}], \quad P-a.s. \tag{B.6}$$

$$E[\alpha X + \beta Y|\mathcal{G}] = \alpha E[X|\mathcal{G}] + \beta E[Y|\mathcal{G}], \quad \forall \alpha, \beta \in R. \tag{B.7}$$

Proof The relation (B.6) follows more or less directly from Proposition A.26. In order to prove (B.7) we *define* Z by $Z = \alpha E[X|\mathcal{G}] + \beta E[Y|\mathcal{G}]$. Then Z is obviously $\mathcal{G}$-measurable and we only have to show that for every $G \in \mathcal{G}$ we have

$$\int_G (\alpha E[X|\mathcal{G}] + \beta E[Y|\mathcal{G}])\, dP = \int_G (\alpha X + \beta Y)\, dP.$$

Using the definition of the conditional expectation and linearity, the right-hand side of the above becomes

$$\begin{aligned}&\int_G (\alpha E[X|\mathcal{G}] + \beta E[Y|\mathcal{G}])\, dP = \alpha \int_G E[X|\mathcal{G}]\, dP + \beta \int_G E[Y|\mathcal{G}]\, dP \\ &= \alpha \int_G X dP + \beta \int_G Y dP = \int_G (\alpha X + \beta Y)\, dP.\end{aligned}$$

□

One of the most important and frequently used properties of conditional expectation is the rule of iterated expectations.

Proposition B.30 *Assume the setting above and also assume that the sigma-algebra $\mathcal{H}$ satisfies $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$. Then the following hold*:

$$E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}], \tag{B.8}$$

$$E[X] = E[E[X|\mathcal{G}]]. \tag{B.9}$$

Proof We start by noting that (B.9) is a special case of (B.8) since $E[X] = E[X|\mathcal{H}]$ where $\mathcal{H}$ is the trivial sigma-algebra. In order to prove (B.8) we define Z by $Z = E[X|\mathcal{H}]$. We now have to show that Z is $\mathcal{H}$-measurable and that for all events $H \in \mathcal{H}$ we have

$$\int_H Z dP = \int_H E[X|\mathcal{G}]\, dP. \tag{B.10}$$

The measurability is immediately clear (why?). As for (B.10) we note that since $\mathcal{H} \subseteq \mathcal{G}$ we have $H \in \mathcal{H} \Rightarrow H \in \mathcal{G}$ so

$$\int_H E[X|\mathcal{G}]\, dP = \int_H X dP = \int_H E[X|\mathcal{H}]\, dP = \int_H Z dP.$$

□

Suppose now that X is $\mathcal{G}$-measurable. We have earlier said that the intuitive interpretation of this is that X is uniquely determined by the information contained in $\mathcal{G}$. When we condition on $\mathcal{G}$ this should imply that we know X and thus can treat it as deterministic (conditionally on $\mathcal{G}$). This intuition is formalized by the following result, where we leave the proof as an exercise.

Proposition B.31 *I X is $\mathcal{G}$-measurable and if X,Y and XY are in L^1, then*

$$E[X|\mathcal{G}] = X, \qquad P-a.s. \tag{B.11}$$
$$E[XY|\mathcal{G}] = X \cdot E[Y|\mathcal{G}], \quad P-a.s. \tag{B.12}$$

There is a Jensen inequality also for conditional expectations.

Proposition B.32 *Assume that $f : R \to R$ is convex and that X and $f(X)$ are integrable. Then*

$$f(E[X|\mathcal{G}]) \leq E[f(X)|\mathcal{G}], \quad P-a.s.$$

Assume that X and Y are defined on the same space $(\Omega, \mathcal{F}, P)$. The we can define the conditional expectation of Y, given X.

Definition B.33 *For any integrable Y and for any X we define*

$$E[Y|X] \stackrel{def}{=} E[Y|\sigma\{X\}].$$

Since $E[Y|X]$ by this definition automatically is $\sigma\{X\}$-measurable, Proposition B.14 guarantees that there exists a Borel function g such that

$$E[Y|X] = g(X), \quad P-a.s. \tag{B.13}$$

Using this g we may now define conditional expectations on the distribution side instead of on the Ω side.

Definition B.34 *We define the object $E[Y|X = x]$ by*

$$E[Y|X = x] = g(x), \quad x \in R,$$

where g is given by (B.13).

From the law of iterated expectations and Proposition B.4 we obtain the following result, which should be well known from elementary probability theory.

Proposition B.35 *If μ_X denotes the distribution measure for X then, for any random variable Y:*

$$E[Y] = \int_R E[Y|X = x]\, d\mu_X(x).$$

If X and $\mathcal{G}$ are independent, i.e. if $\sigma\{X\}$ and $\mathcal{G}$ are independent sigma-algebras, then it seems reasonable to expect that $\mathcal{G}$ does not contain any information about X. The technical formulation of this intuition is as follows.

Proposition B.36 *Assume that X is integrable, and that X and $\mathcal{G}$ are independent. Then*

$$E[Y|X] = E[Y].$$

Proof Left as an exercise. □

It is well known that $E[X]$ is the optimal mean square deterministic predictor of X. The corresponding result for conditional expectations is as follows.

Proposition B.37 *Let $(\Omega, \mathcal{F}, P)$ be a given probability space, let $\mathcal{G}$ be a sub-sigma-algebra of $\mathcal{F}$, and let X be a square integrable random variable.*

Consider the problem of minimizing

$$E\left[(X-Z)^2\right]$$

where Z is allowed to vary over the class of all square integrable $\mathcal{G}$ measurable random variables. The optimal solution $\hat{Z}$ is then given by

$$\hat{Z} = E\left[X \mid \mathcal{G}\right].$$

Proof Left to the reader. See the exercises for a hint. □

In geometrical terms this means that $E\left[X \mid \mathcal{G}\right]$ is the orthogonal projection (in $L^2(\Omega, \mathcal{F}, P)$ of X onto the closed subspace $L^2(\Omega, \mathcal{G}, P)$. For square integrable random variables one may in fact use this as the definition of the conditional expectation. This definition can then be extended from L^2 to L^1 by continuity, since L^2 is dense in L^1.

B.6 Equivalent Probability Measures

In this section we discuss absolute continuity and equivalence for the particular case of probability measures. The results in this section will be heavily used in Chapter 11.

Let therefore P and Q be probability measures on $(\Omega, \mathcal{F})$. We immediately have the following simple result.

Lemma B.38 *For two probability measures P and Q, the relation $P \sim Q$ on $\mathcal{F}$ holds if and only if*

$$P(A) = 1 \quad \Leftrightarrow \quad Q(A) = 1, \quad \textit{for all } A \in \mathcal{F}. \tag{B.14}$$

Proof Exercise for the reader. □

In the context of probability measures we thus notice that although two equivalent measures P and Q may assign completely different probabilities to a fixed event A, but all events which are impossible under P (i.e. $P(A) = 0$) are also impossible under Q. Equivalently: all events which are certain under P (i.e. $P(A = 1)$) are also certain under Q. It also follows directly (prove this!) from the definition that if an event A has strictly positive P-probability, then it also has strictly positive Q-probability (and vice versa).

From the Radon–Nikodym Theorem we know that $Q << P$ on the probability space $(\Omega, \mathcal{F})$ if and only if there exists $\mathcal{F}$-measurable mapping $L : \Omega \to R_+$ such that

$$\int_A dQ(\omega) = \int_A L(\omega) dP(\omega) \tag{B.15}$$

for all $A \in \mathcal{F}$. Since Q is a probability measure L must also have the property that

$$\int_{\Omega} L dP = 1,$$

i.e

$$E^P[L] = 1.$$

In other words, the Radon–Nikodym derivative L is a non-negative random variable with $E^P[L] = 1$, and it is often referred to as the **likelihood ratio** between Q and P. Written in terms of expected values, it follows from (B.15) that, for any random variable $X \in L^1(Q)$ we have

$$E^Q[X] = E^P[L \cdot X]. \tag{B.16}$$

Suppose now that $Q << P$ on $\mathcal{F}$ and that we also have a smaller sigma-algebra $\mathcal{G} \subseteq \mathcal{F}$. We then have two Radon–Nikodym derivatives: $L^{\mathcal{F}}$ on $\mathcal{F}$, and $L^{\mathcal{G}}$ on $\mathcal{G}$, and these are typically not equal since $L^{\mathcal{F}}$ will generically not be $\mathcal{G}$-measurable. The following result shows how they are related.

Proposition B.39 *Assume that $Q << P$ on $\mathcal{F}$ and that $\mathcal{G} \subseteq \mathcal{F}$. Then the Radon–Nikodym derivatives $L^{\mathcal{F}}$ and $L^{\mathcal{G}}$ are related by*

$$L^{\mathcal{G}} = E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right]. \tag{B.17}$$

Proof We have to show that $E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right]$ is $\mathcal{G}$-measurable (which is obvious) and that, for any $G \in \mathcal{G}$

$$\int_G dQ = \int_G E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right] dP.$$

This, however, follows immediately from the trivial calculation

$$\int_G dQ = \int_G L^{\mathcal{F}} dP = \int_G E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right] dP,$$

where we have used the fact that $G \in \mathcal{G} \subseteq \mathcal{F}$. □

Example B.40 To see an example of the result above let $\Omega = \{1, 2, 3\}$ and define

$$\mathcal{F} = 2^{\Omega} \quad \mathcal{G} = \{\Omega, \emptyset, \{1\}, \{2, 3\}\}$$

and

$$\begin{aligned} P(1) &= 1/4, \quad P(2) = 1/2, \quad P(3) = 1/4, \\ Q(1) &= 1/3, \quad Q(2) = 1/3, \quad Q(3) = 1/3. \end{aligned}$$

We see directly that

$$L^{\mathcal{F}}(1) = 4/3, \quad L^{\mathcal{F}}(2) = 2/3, \quad L^{\mathcal{F}}(3) = 4/3$$

and it is obvious that $L^{\mathcal{F}}$ is not $\mathcal{G}$-measurable. Since $P(\{2, 3\}) = 3/4$, and $Q(\{2, 3\}) = 2/3$, the local scale factor on $\{2, 3\}$ is $8/9$, so $L^{\mathcal{G}}$ is given by

$$L^{\mathcal{G}}(1) = 4/3, \quad L^{\mathcal{G}}(2) = 8/9, \quad L^{\mathcal{G}}(3) = 8/9,$$

and we also have the simple calculation

$$E^P\left[L^{\mathcal{F}}\middle|\{2,3\}\right] = \frac{P(2)L^{\mathcal{F}}(2)+P(3)L^{\mathcal{F}}(3)}{P(2)+P(3)} = \frac{8}{9}.$$

The formula (B.16) gives us expectations under Q in terms of expectations under P, and a natural question is how conditional expected values under Q are related to conditional expectations under P. The following very useful result, known as the "Abstract Bayes' Formula" solves this problem.

Proposition B.41 (Bayes' Theorem) *Assume that X is a random variable on $(\Omega, \mathcal{F}, P)$, and let Q be another probability measure on $(\Omega, \mathcal{F})$ with Radon–Nikodym derivative*

$$L = \frac{dQ}{dP} \quad on\ \mathcal{F}.$$

Assume that $X \in L^1(\Omega, \mathcal{F}, Q)$ and that $\mathcal{G}$ is a sigma-algebra with $\mathcal{G} \subseteq \mathcal{F}$. Then

$$E^Q[X|\mathcal{G}] = \frac{E^P[L \cdot X|\mathcal{G}]}{E^P[L|\mathcal{G}]}, \quad Q-a.s. \tag{B.18}$$

Proof We start by proving that

$$E^Q[X|\mathcal{G}] \cdot E^P[L|\mathcal{G}] = E^P[L \cdot X|\mathcal{G}], \quad P-a.s. \tag{B.19}$$

We show this by proving that for an arbitrary $G \in \mathcal{G}$ the P-integral of both sides coincide. The left-hand side becomes

$$\int_G E^Q[X|\mathcal{G}] \cdot E^P[L|\mathcal{G}]\,dP = \int_G E^P\left[L \cdot E^Q[X|\mathcal{G}]\middle|\mathcal{G}\right]dP$$
$$= \int_G L \cdot E^Q[X|\mathcal{G}]\,dP = \int_G E^Q[X|\mathcal{G}]\,dQ = \int_G X dQ.$$

Integrating the right-hand side we obtain

$$\int_G E^P[L \cdot X|\mathcal{G}]\,dP = \int_G L \cdot X dP = \int_G X dQ.$$

Thus (B.19) holds P-a.s. and since $Q << P$ also Q-a.s. It remains to show that $E^P[L|\mathcal{G}] \neq 0$ Q-a.s. but this follows from the calculation

$$Q\left(E^P[L|\mathcal{G}] = 0\right) = \int_{\{E^P[L|\mathcal{G}]=0\}} dQ = \int_{\{E^P[L|\mathcal{G}]=0\}} L dP$$
$$= \int_{\{E^P[L|\mathcal{G}]=0\}} E^P[L|\mathcal{G}]\,dP = 0.$$

□

B.7 Exercises

Exercise B.1 Prove Proposition B.4 by carrying out the following steps:

(a) Prove the proposition in the case when $g = I_A$ where A is an arbitrary Borel set.

(b) Prove that the proposition holds when g is a simple function.
(c) You can WLOG assume that g is non-negative (why?), so now approximate g by simple functions.

Exercise B.2 Prove Proposition B.10.

Exercise B.3 Prove Proposition B.12.

Exercise B.4 Prove Corollary B.22.

Exercise B.5 Prove Proposition B.26.

Exercise B.6 Prove (B.6) by using Proposition A.26.

Exercise B.7 Prove Proposition B.31 by first proving it when X is an indicator function, then extend by linearity to simple functions and at last by approximating X with a sequence of simple functions.

Exercise B.8 Prove Proposition B.31 by the following steps:

(i) Choose a fixed X.
(ii) Show that for any $A \in \sigma\{X\}$ you have $E[I_A|\mathcal{G}] = E[I_A]$.
(iii) Extend by linearity to simple $\sigma\{X\}$-measurable functions and at last by approximating X with a sequence of simple functions.

Exercise B.9 Let $h : R \to R$ be a function such that $h \geq 0$, $h' \geq 0$, and $h(0) = 0$. Assume that X is a non-negative random variable. Prove that

$$E[h(X)] = \int_0^\infty h'(t)P(X \geq t)dt.$$

Exercise B.10 Prove Proposition B.36 by starting with the case when $X = I_A$ and then do the usual steps.

Exercise B.11 Prove Proposition B.37 by going along the following lines:

(a) Prove that the "estimation error" $X - E[X|\mathcal{G}]$ is orthogonal to $L^2(\Omega, \mathcal{G}, P)$ in the sense that for any $Z \in L^2(\Omega, \mathcal{G}, P)$ we have

$$E[Z \cdot (X - E[X|\mathcal{G}])] = 0.$$

(b) Now prove the proposition by writing

$$X - Z = (X - E[X|\mathcal{G}]) - (E[X|\mathcal{G}] - Z)$$

and use the result just proved.

B.8 Notes

For the mathematician, Durrett (1996) is a very good standard reference on probability theory. For the economist (and also for many mathematicians) the text by Jacod and Protter (2000) is the perfect, and amazingly far-reaching, reference.

Appendix C

MARTINGALES AND STOPPING TIMES

C.1 Martingales

Let $(\Omega, \mathcal{F}, P, \mathbf{F})$ be a filtered probability space, and let X be a random process in continuous or discrete time.

Definition C.1 *The process X is an* **F-martingale** *if*

1. *X is $\mathbf{F}$-adapted.*
2. *$X_t \in L^1$ for each t.*
3. *For every s and t with $0 \le s \le t$ it holds that*

$$X_s = E\left[X_t \middle| \mathcal{F}_s\right], \quad P-a.s.$$

If the equality sign is replaced by $\le$ ($\ge$) then x is said to be a **submartingale** (**supermartingale**).

Note that the martingale property is always with respect to some given filtration. In all honesty it should be mentioned that while martingale theory in discrete time is a fairly straightforward activity, martingale theory in continuous time is sometimes rather complicated and there are lots of highly nontrivial technical problems. In order for the theory to work well in continuous time we typically want our processes to have right continuous trajectories with left limits, and we also need to assume that the filtration $\mathbf{F}$ has some regularity properties. However, in almost all concrete situations these technical problems can be taken care of, so with almost no danger the reader can safely forget about the technicalities. For the rest of this book we simply ignore these problems. Below we only give proofs for the discrete time case. The proofs for the continuous time results are typically obtained by sampling the continuous time processes at discrete points in time and then performing a (often nontrivial) limiting argument.

It follows immediately from the definition that a martingale is characterized by the property that the conditional expectations of a forward increment equals zero, i.e. that

$$E\left[X_t - X_s \middle| \mathcal{F}_s\right] = 0, \quad \text{for all } s \le t.$$

For martingales in discrete time, it is in fact enough to demand that the martingale property holds for one single time step.

Proposition C.2 *An adapted integrable discrete time process $\{X_n;\ n = 0, 1, \ldots\}$ is a martingale w.r.t the filtration $\{\mathcal{F}_n;\ n = 0, 1, \ldots\}$ if and only if*

$$E\left[X_{n+1} \middle| \mathcal{F}_n\right] = X_n, \quad n = 0, 1, 2, \ldots$$

Proof Easy exercise. □

Two of the most common types of martingales are the following.

Example C.3 Let Y be any integrable random variable on the filtered space $(\Omega, \mathcal{F}, P, \mathbf{F})$, and define the process X by

$$X_t = E\left[Y \middle| \mathcal{F}_t\right], \quad t \geq 0. \tag{C.1}$$

Then it is an easy exercise to see that X is an $\mathcal{F}_t$-martingale. In particular, this implies that **on a compact interval** $[0, T]$ any given martingale M is always generated by its final value M_T by the formula

$$M_t = E\left[M_T \middle| \mathcal{F}_t\right], \quad 0 \leq t \leq T. \tag{C.2}$$

Note that this only holds on a finite closed interval. The more complicated case of an infinite or open interval will be discussed below.

Example C.4 If X is a process with independent increments on $(\Omega, \mathcal{F}, P, \mathbf{F})$, and if also $E[X_t - X_s] = 0$, for all s, t, then X is a martingale.

Example C.5 Let $\{Z_n;\ n = 1, 2, \ldots\}$ be a family of independent integrable random variables, and define the discrete time process X by

$$X_n = \sum_{i=1}^{n} Z_i, \tag{C.3}$$

the X is a martingale w.r.t. the filtration $\mathcal{F}^X$.

There is a close connection between martingale theory, the theory of convex functions, and the theory of harmonic functions. The correspondence is as follows

Martingale theory	**Convex theory**	**Harmonic theory**
martingale	linear function	harmonic function
submartingale	convex function	subharmonic function
supermartingale	concave function	superharmonic function

We will not go deeper into this, but from convexity theory we recognize directly the structure of the following result.

Proposition C.6 *Let X be a process on $(\Omega, \mathcal{F}, P, \mathbf{F})$.*

- *If X is a martingale and if $f : R \to R$ is a convex (concave)function such that $f(X_t)$ is integrable for all t, then the process Y defined by*

  $$Y_t = f(X_t),$$

  is a submartingale (supermartingale).
- *If X is a submartingale and if $f : R \to R$ is a convex non-decreasing function such that $f(X_t)$ is integrable for all t, then the process Y defined by*

  $$Y_t = f(X_t)$$

  is a submartingale.

Proof Jensen's inequality for conditional expectations. □

On every **finite** interval $[0,T]$, every martingale X is of the form

$$X_t = E\left[X_T | \mathcal{F}_t\right],$$

and a natural question is if **every** martingale X also on the infinite interval $[0,\infty]$ has a representation of this form, i.e. if there always exists some random variable X_∞ such that

$$X_t = E\left[X_\infty | \mathcal{F}_t\right], \tag{C.4}$$

In general the answer to this question is no, and a symmetric random walk on the integers is a typical counter example. In order to have a representation of the form (C.4) we need some further integrability of X. We will not prove the most general (and hard) version of the results but for completeness sake we will cite the most general convergence theorem without proof.

Theorem C.7 *Suppose that X is a submartingale satisfying the condition*

$$\sup_{t\geq 0} E\left[X_t^+\right] < \infty.$$

Then there exists a random variable Y such that $X_t \to Y$, P-a.s.

We now move to the more manageable quadratic case.

Definition C.8 *A martingale X is called* **square integrable** *if there exists a constant M such that*

$$E\left[X_t^2\right] \leq M, \quad \textit{for all } t \in [0,\infty).$$

We now have the following nice result.

Proposition C.9 (Martingale Convergence) *Assume that x is a square integrable martingale. Then there exists a random variable, which we denote by X_∞, such that $X_t \to X_\infty$ in L^2 and P-a.s. as $t \to \infty$. Furthermore we have the representation*

$$X_t = E\left[X_\infty | \mathcal{F}_t\right], \quad \textit{for all } t \geq 0. \tag{C.5}$$

Proof Since $x \longmapsto x^2$ is convex, the process X_t^2 is a submartingale, which implies that the mapping $m_t = E\left[X_t^2\right]$ is nondecreasing. The assumption that X is square integrable is thus equivalent to the existence of a real number $c < \infty$ such that $m_t \uparrow c$. We will now prove L^2-convergence by showing that X_t is Cauchy in L^2. We have

$$\begin{aligned} E\left[(X_t - X_s)^2\right] &= E\left[X_t^2 - 2X_sX_t + X_s^2\right] = E\left[E\left[X_t^2 - 2X_sX_t + X_s^2 \middle| \mathcal{F}_s\right]\right] \\ &= E\left[X_t^2\right] - 2E\left[X_s E\left[X_t | \mathcal{F}_s\right]\right] + E\left[X_s^2\right] = E\left[X_t^2\right] - E\left[X_s^2\right] = m_t - m_s. \end{aligned}$$

Since $m_t \to c$ it follows that m_t is Cauchy and thus that X_t is Cauchy in L^2. Since L^2 is complete this implies the existence of a random variable $Y \in L^2$ such that $X_t \to Y$ in L^2. The almost sure convergence then follows from Theorem C.7

In order to prove (C.5) it is enough (why?) to show that for every s and every $A \in \mathcal{F}_s$ we have

$$\int_A X_s dP = \int_A Y dP,$$

and this follows easily from the fact that for every $t > s$ the martingale property implies that

$$\int_A X_s dP = \int_A X_t dP.$$

If now $t \to \infty$, it follows (how?) from the L^2-convergence that, as $t \to \infty$, we have

$$\int_A X_t dP \to \int_A Y dP.$$

□

C.2 Discrete Stochastic Integrals

In this section we discuss briefly the simplest type of stochastic integration, namely integration of discrete time processes. This will thus serve as an introduction to the more complicated Wiener case later on, and it is also important in its own right. The cental concept here is that of a predictable process.

Definition C.10 *Consider a filtered space $(\Omega, \mathcal{F}, P, \mathbf{F})$ in discrete time, i.e. $n = 0, 1, 2, \ldots$.*

- *A random process X is* **F-predictable** *if, for each n, X_n is $\mathcal{F}_{n-1}$ measurable. Here we use the convention $\mathcal{F}_{-1} = \mathcal{F}_0$.*
- *For any random process X, the* **increment process** *ΔX is defined by*

$$(\Delta X)_n = X_n - X_{n-1}, \tag{C.6}$$

  with the convention $X_{-1} = 0$.
- *For any two processes X and Y, the* **discrete stochastic integral** *process $X \star Y$ is defined by*

$$(X \star Y)_n = \sum_{k=0}^{n} X_k (\Delta Y)_k. \tag{C.7}$$

  Instead of $(X \star Y)_n$ we will sometimes write $\int_0^n X_s dY_s$.

Note that a predictable process is "known one step ahead in time". The reason why we define ΔX by "backward increments" is that in this way ΔX is adapted whenever X is.

The main result for stochastic integrals is that when you integrate a predictable process X w.r.t. a martingale M, the result is a new martingale.

Proposition C.11 *Assume that the space $(\Omega, \mathcal{F}, P, \mathbf{F})$ carries the processes X and M where X is predictable, M is a martingale, and $X_n(\Delta M)_n$ is integrable for each n. Then the stochastic integral $X \star M$ is a martingale.*

Proof Left as an exercise to the reader. □

C.3 Likelihood Processes

Martingale theory is closely connected with absolutely continuous measure transformations and arbitrage theory. This is discussed in detail in Chapter 11 and here we will only state some basic facts.

We consider a filtered probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$ on a compact interval $[0, T]$. Suppose now that L_T is some non-negative integrable random variable in $\mathcal{F}_T$. We can then define a new measure Q on $\mathcal{F}_T$ by setting

$$dQ = L_T dP, \quad \text{on } \mathcal{F}_T,$$

and if

$$E^P[L_T] = 1,$$

the new measure will also be a probability measure.

From its definition, L_T will be the Radon–Nikodym derivative of Q w.r.t. P on $\mathcal{F}_T$ so $Q << P$ on $\mathcal{F}_T$. Hence we will also have $Q << P$ on $\mathcal{F}_t$ for all $t \leq T$ and thus, by the Radon–Nikodym Theorem, there will exist a random process $\{L_t;\ 0 \leq t \leq T\}$ defined by

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t. \tag{C.8}$$

The L process is known as the **likelihood process** for the measure transformation from P to Q and it has the following fundamental property, which will be used frequently.

Proposition C.12 *With assumptions as above, the likelihood process L, defined by (C.8), is a $(P, \mathbf{F})$-martingale.*

Proof The statement follows directly from Proposition B.39. □

Using the likelihood process, we can also characterize a Q-martingale in terms of the P measure.

Proposition C.13 *A process M is a Q-martingale if and only if the process $L \cdot M$ is a P-martingale.*

Proof Exercise for the reader. □

C.4 Stopping Times

Consider again a filtered space $(\Omega, \mathcal{F}, P, \mathbf{F})$ and a martingale X on the space. A natural question, which we will encounter in connection with American options, is whether the martingale property also holds when the deterministic times are replaced by stochastic times, i.e. whether we always have the equality

$$E[X_T | \mathcal{F}_S] = X_S, \tag{C.9}$$

where S and T are random times with $S \leq T$. It is rather clear that we cannot expect a strong theory unless we restrict the study to those random times which in some sense are adapted to the information flow given by the filtration. These are the so-called *stopping times*.

Definition C.14 *A* **stopping time** *w.r.t. the filtration* $\mathbf{F}$ *is a non-negative random variable* T *such that*

$$\{T \leq t\} \in \mathcal{F}_t, \quad \textit{for every } t \geq 0. \tag{C.10}$$

A stopping time is thus characterized by the fact that at any time t we can, based upon the information available at t, decide whether T has occurred or not. This definition may seem a bit abstract, but in most concrete situations it is very easy to see whether a random time is a stopping time or not. A typical example of a stopping time is obtained if X is an adapted discrete time process and we define T as a *hitting time*, i.e. we define T by

$$T \stackrel{def}{=} \inf\{n \geq 0;\ X_n \in A\},$$

where $A \subseteq R$ is some Borel set. T is thus the first time when X enters into the set A, and intuitively it is obvious that we can decide whether the event $\{T \leq n\}$ has occurred, based upon observations of X at the times $0, 1, 2, \ldots, n$. Thus T is a stopping time, and we obtain a formal proof by choosing a fixed n and noting that

$$\{T(\omega) \leq n\} = \{X_t(\omega) \in A, \text{ for some } t \leq n\} = \bigcup_{t=0}^{n} \{X_t \in A\}.$$

Since X is adapted we have $\{X_t \in A\} \in \mathcal{F}_t \subseteq \mathcal{F}_n$, so $\{T \leq n\} \in \mathcal{F}_n$.

A typical example of a random time which is **not** a stopping time is given by

$$T(\omega) = \sup\{n \geq 0;\ X_n \in A\}.$$

In this definition, T is thus the *last* time that X visits A, and it is again intuitively obvious that in the generic case we cannot decide whether T has occurred or not based upon the basis of observations upon $X_0, X_1, \ldots X_n$ since this would imply that at time n we already know if X will visit A or not at some time in the future.

In order to be able to even formulate the equality (C.9) we must define what we mean by the expression $\mathcal{F}_T$ for a stopping time T. Intuitively the interpretation is of course that $\mathcal{F}_T$ = "the information generated by the flow $\mathbf{F}$ up to the random time T", but it is not obvious how to formalize this in mathematical terms. The generally accepted definition is the following.

Definition C.15 *Let* T *be a* $\mathbf{F}$ *stopping time. The sigma-algebra* $\mathcal{F}_T$ *is defined as the class of events satisfying*

$$A \in \mathcal{F}_\infty, \tag{C.11}$$

$$A \cap \{T \leq t\} \in \mathcal{F}_t, \quad \textit{for all } t \geq 0. \tag{C.12}$$

We now have some natural results.

Proposition C.16 *Let S and t be stopping times on the filtered space $(\Omega, \mathcal{F}, P, \mathbf{F})$, and let X be an adapted process, which in the continuous time case is assumed to have trajectories which are either left- or right-continuous. Define $\vee$ and $\wedge$ by $x \vee y = max[x,y]$ and $x \wedge y = min[x,y]$ for any real number, and define $S \vee T$ by $(S \vee T)(\omega) = S(\omega) \vee T(\omega)$. Then the following hold:*

- *If $S \leq T$, P-a.s. then $\mathcal{F}_S \subseteq \mathcal{F}_T$.*
- *$S \vee T$ and $X \wedge T$ are stopping times.*
- *If T is P-a.s. finite or if X_∞ is well defined in $\mathcal{F}_\infty$, then X_T is $\mathcal{F}_T$-measurable.*

Proof The first two items are left as easy exercises, and we prove the third item only for the discrete time case. To show that X_T is $\mathcal{F}_T$-measurable we have to show that $\{X_T \in B\} \in \mathcal{F}_T$ for every Borel set B, so we thus have to show that for every n we have $\{X_T \in B\} \cap \{T \leq n\} \in \mathcal{F}_n$. We obtain

$$
\begin{aligned}
\{X_T \in B\} \cap \{T \leq n\} &= \{X_T \in B\} \cap \bigcup_{k=0}^{N} \{T = k\} \\
&= \bigcup_{k=0}^{N} (\{X_k \in B\} \cap \{T = k\})
\end{aligned}
\tag{C.13}
$$

Since X is adapted and T is a stopping time, $\{X_k \in B\}$ and $\{T = k\}$ are in $\mathcal{F}_k$ which is included in $\mathcal{F}_n$. □

We now prove that the martingale property is stable under stopping.

Proposition C.17 *Let X be a martingale and let T be a stopping time. Then the* **stopped process** *X^T, defined by*

$$
X_t^T = X_{T \wedge t},
\tag{C.14}
$$

is a martingale.

Proof We only give the proof for the discrete time case. For this we define the process h by $h_n = I\{n \leq T\}$, $n = 0, 1, 2\ldots$, where I denotes the indicator of the event within the bracket. Now, $\{n \leq T\} = \{T < n\}^c = \{T \leq n-1\}^c$. Since T is a stopping time we thus see that $h_n \in \mathcal{F}_{n-1}$ so h is predictable. Furthermore, we have the obvious equality

$$
X_n^T = \sum_{k=0}^{n} h_k (\Delta X)_k,
$$

so from Proposition C.11 we see that X^T is a martingale. □

We finish by stating a fairly general version of the "Optional Sampling Theorem" which shows that the martingale property is preserved under random sampling, but we only give the proof for a simple special case.

Theorem C.18 (The Optional Sampling Theorem) *Assume that X is a martingale satisfying*

$$\sup_{t \geq 0} E\left[X_t^2\right] \leq \infty.$$

Let S and T be stopping times such that $S \leq T$. Then

$$E\left[X_T \mid \mathcal{F}_S\right] = X_S, \quad P - a.s. \tag{C.15}$$

If X is a submartingale satisfying the same integrability condition then (C.15) holds with $=$ replaced by $\geq$.

Proof We will be content with proving the result in discrete time and for the case when X is a martingale, the submartingale case being a bit harder. From Proposition C.9 it follows that there exists an integrable random variable Y such that

$$X_n = E\left[Y \mid \mathcal{F}_n\right], \quad n = 0, 1, \ldots \tag{C.16}$$

It is thus enough (why?) to show that for any stopping time T we have

$$E\left[Y \mid \mathcal{F}_T\right] = X_T,$$

i.e. we have to show that for every $A \in \mathcal{F}_T$ we have

$$\int_A Y dP = \int_A X_T dP.$$

By writing A as $A = \bigcup_n (A \cap \{T = n\})$, noting that $A \cap \{T = n\} \in \mathcal{F}_n$, and using (C.16) we obtain

$$\int_A Y dP = \sum_{n=0}^{\infty} \int_{A \cap \{T=n\}} Y dP = \sum_{n=0}^{\infty} \int_{A \cap \{T=n\}} X_n dP = \int_A X_T dP. \qquad \square$$

C.5 Exercises

Exercise C.1 Show that for any integrable random variable Y on a filtered space $(\Omega, \mathcal{F}, P, \mathbf{F})$, the process X defined by

$$X_t = E\left[Y \mid \mathcal{F}_t\right], \quad t \geq 0$$

is a martingale.

Exercise C.2 Let $\{Z_n\}$ be a sequence of i.i.d. (independent identically distributed) random variables with finite exponential moments of all orders. Define the function $\varphi : R \to R$ by

$$\varphi(\lambda) = E\left[e^{\lambda Z_n}\right],$$

and define the process X by

$$X_n = \frac{e^{\lambda S_n}}{[\varphi(\lambda)]^n}, \quad \text{where} \quad S_n = \sum_{k=1}^{n} Z_k.$$

Prove that X is an $\mathcal{F}_n$-martingale, where $\mathcal{F}_n = \sigma\{Z_i;\ i = 1, \ldots, n\}$.

Exercise C.3 Prove that, for any stopping time T, $\mathcal{F}_T$, defined by (C.10), is indeed a sigma-algebra.

Exercise C.4 Prove Proposition C.11.

Exercise C.5 Prove Proposition C.6.

Exercise C.6 Show that in discrete time, the defining property $\{T \leq t\} \in \mathcal{F}_t$ for a stopping time, can be replaced by the weaker condition

$$\{T = n\} \in \mathcal{F}_n, \quad \text{for all } n.$$

Exercise C.7 Prove the first two items in Proposition C.16.

Exercise C.8 A Wiener process W is a continuous time process with $W_0 = 0$, continuous trajectories, and Gaussian increments such that for $s < t$ the increment $W_t - W_s$ is normally distributed with mean zero and variance $t - s$. Furthermore, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$, where the filtration is the internal one generated by W.

(a) Show that W is a martingale.
(b) Show that $W_t^2 - t$ is a martingale.
(c) Show that for any real number λ

$$e^{\lambda W_t - \frac{1}{2}\lambda t}$$

is a martingale.
(d) For $b < 0 < a$ we define the stopping time T as the first time that W hits one of the "barriers" a or b, i.e.

$$T = \inf\{n \geq 0;\ X_n = a, \text{ or } X_n = b\}.$$

Define p_a and p_b as

$$p_a = P(W \text{ hits the } a \text{ barrier before hitting the } b \text{ barrier}),$$
$$p_b = P(W \text{ hits the } b \text{ barrier before hitting the } a \text{ barrier}),$$

so $p_a = P(W_T = a)$ and $p_b = P(W_T = b)$. Use the fact that every stopped martingale is a martingale to infer that $E[W_T] = 0$, and show that

$$p_a = \frac{-b}{a-b}, \quad p_b = \frac{a}{a+b}.$$

You may without proof use the fact that $P(T < \infty) = 1$.
(e) Use the technique above to show that

$$E[T] = |ab|.$$

(f) Let T be as above and let $b = -a$. Use the Optional Sampling Theorem, Proposition C.17 and item (iii) above to show that the Laplace transform $\varphi(\lambda)$ of the distribution of T is given by

$$\varphi(\lambda) = E\left[e^{-\lambda T}\right] = e^{-a\sqrt{2\lambda}}, \quad \lambda \geq 0.$$

Exercise C.9 Prove Proposition C.13.

Hint: Use the Bayes' Formula (B.18).

Appendix D

CONVEX DUALITY

In this section we give a brief recapitulation of some standard duality results from convexity theory. See Rockafeller (1970) for the full theory.

D.1 Conjugate Functions

Definition D.1 *Assume that $U : R \to R$ a strictly increasing, strictly concave function with effective domain (a, ∞), satisfying*

$$\lim_{x \downarrow a} U'(x) = +\infty.$$

We then define the **conjugate function** $f^* : R_+ \to R$ *by*

$$U^*(y) = \sup_{x \in R} \{U(x) - yx\}, \quad y > 0. \tag{D.1}$$

We then have the following result.

Proposition D.2 *The conjugate function U^* is convex and we have the inversion formula*

$$U(x) = \inf_{y > 0} \{U^*(x) + yx\}. \tag{D.2}$$

Furthermore we have

$$\frac{dU^*}{dy}(y) = -\hat{x}(y), \tag{D.3}$$

$$\frac{dU}{dx}(x) = \hat{y}(x), \tag{D.4}$$

where $\widehat{y}(x)$ and $\widehat{x}(y)$ are the optimal x and y in (D.1) and (D.2).

D.2 Lagrange Functions and Saddle Points

Consider a concave function $U : R^n \to R$ and the convex functions $h_i : R^n \to R$, $i = 1, \ldots, K$. We study the following optimization problem.

Problem D.2.1 (Primal problem)

$$\text{maximize}_{x \in R^n} \; U(x) \tag{D.5}$$

subject to

$$h_i(x) \leq 0, \quad i = 1, \ldots, K. \tag{D.6}$$

We define the vector function h as $h = (h_1, \dots h_K)$ and we now define some useful objects, where R_+^K denotes the vectors in R^K with strictly positive components. We use the notational convention $y > 0$ for $y \in R_+^K$.

Definition D.3 *For $y \in R^K$ we define the following functions:*

1. *The* **Lagrangian** *function $L : R^n \times R_+^K \to R$ is defined by*
$$L(x, y) = f(x) - yh(x), \quad y > 0, \tag{D.7}$$
where $yh(x)$ denotes the inner product $\sum_i y_i h_i(x)$.
2. *The functions F and G are defined by*
$$F(x) = \inf_{y>0} L(x, y), \tag{D.8}$$
$$G(y) = \sup_{x \in R^n} L(x, y). \tag{D.9}$$

We then have the following fundamental result.

Theorem D.4 (Saddle Point Theorem) *Assume that the objective function U is concave and that $h_1, \dots, h_K$ are convex. Denote the optimal solution to Problem D.2.1 by $\hat{x}$. Assume furthermore that there exists some $x \in R^n$ satisfying $h_i(x) < 0$ for all i. Then the following hold:*

1. *There exists a Lagrange multiplier $\hat{y} \geq 0$ such that L has a saddle point at $(\hat{x}, \hat{y})$, i.e.*
$$L(x, \hat{y}) \leq L(\hat{x}, \hat{y}) \leq L(\hat{x}, y), \quad x \in R^n, \ y \geq 0. \tag{D.10}$$
2. *We have*
$$\sup_{x \in R^n} F(x) = U(\hat{x}) = \inf_{y \geq 0} G(y) \tag{D.11}$$
or equivalently
$$\sup_{x \in R^n} \inf_{y \geq 0} L(x, y) = U(\hat{x}) = \inf_{y \geq 0} \sup_{x \in R^n} L(x, y). \tag{D.12}$$

Given this result it is natural to define the **dual problem** as follows.

Problem D.2.2 (Dual problem)

$$\underset{y>0}{\text{minimize}} \ G(y). \tag{D.13}$$

The Saddle Point Theorem above shows that solving the dual problem is in fact equivalent to solving the original primal problem.

D.3 An Envelope Theorem

Consider a mapping $U : R^n \times R \to R$ and a subset $E \subseteq R^n$. We can then formulate the following problem.

Problem D.3.1

$$\underset{x}{\text{maximize}}\ U(x,a), \quad a \in R,$$

subject to

$$x \in E.$$

In this problem x is the control variable, E is the constraint, and a acts as a fixed parameter, so we have one problem for every fixed choice of a. Note that we assume that E does not depend on a. Let us assume that there is a unique optimal point $\widehat{x}(a)$ for each fixed a, and define u as the indirect utility function, i.e.

$$u(a) = U(\widehat{x}(a), a).$$

We then have the following result, which is a part of an entire family of "envelope theorems".

Theorem D.5 (Envelope Theorem) *Assume that U, $\widehat{x}$, and u are differentiable. Then we have*

$$u'(a) = \frac{\partial U}{\partial a}(\widehat{x}(a), a). \tag{D.14}$$

The point of this result is that by definition we have

$$u'(a) = \frac{dU}{da}(\widehat{x}(a), a) = \frac{\partial U}{\partial x}(\widehat{x}(a), a) \cdot \frac{d\widehat{x}}{da}(a) + \frac{\partial U}{\partial a}(\widehat{x}(a), a),$$

so the Envelope Theorem says that the first term vanishes. For a proof see Milgrom and Segal (2002).

REFERENCES

Abel, A.B. (1990). Asset prices under habit formation and catching up with the Joneses. *American Economic Review*, **80**(2), 38–42.

Amin, K. and Jarrow, R. (1991). Pricing foreign currency options under stochastic interest rates. *Journal of International Money and Finance*, **10**(3), 310–329.

Anderson, N., Breedon, F., Deacon, M., Derry, A., and Murphy, G. (1996). *Estimating and Interpreting the Yield Curve*. Wiley, Chichester.

Back, K. (2017). *Asset Pricing and Portfolio Choice Theory* (2nd edn). Oxford University Press, Oxford.

Barone-Adesi, G. and Elliott, R. (1991). Approximations for the values of American options. *Stochastic Analysis and Applications*, **9**(2), 115–131.

Benninga, S., Björk, T., and Wiener, Z. (2002). On the use of numeraires in option pricing. *Journal of Derivatives*, **10**(2), 43–58.

Benth, F.E. and Karlsen, K.H. (2005). A pde representation of the density of the minimal entropy martingale measure in stochastic volatility markets. *Stochastics*, **77**(2), 109–137.

Bernardo, A.E. and Ledoit, O. (2000). Gain, loss, and asset pricing. *Journal of Political Economy*, **108**(1), 144–172.

Bingham, N.H. and Kiesel, R. (2004). *Risk Neutral Valuation* (2nd edn). Springer, Berlin.

Björk, Tomas (2001). A geometric view of interest rate theory. In *Option Pricing, Interest Rates and Risk Management* (eds E. Jouini, J. Cvitanic, and M. Musiela). Cambridge University Press, Cambridge.

Björk, T. and Christensen, B. (1999). Interest rate dynamics and consistent forward rate curves. *Mathematical Finance*, **9**(4), 323–348.

Björk, T., Di Masi, G., Kabanov, Y., and Runggaldier, W. (1997). Towards a general theory of bond markets. *Finance and Stochastics*, **1**, 141–174.

Björk, Tomas and Gombani, Andrea (1999). Minimal realizations of interest rate models. *Finance and Stochastics*, **3**(4), 413–432.

Björk, T., Kabanov, Y., and Runggaldier, W. (1995). Bond market structure in the presence of a marked point process. *Mathematical Finance*, **7**(2), 211–239.

Björk, Tomas and Slinko, I. (2006). Towards a general theory of good deal bounds. *Review of Finance*, **10**(2), 221–260.

Björk, Tomas and Svensson, Lars (2001). On the existence of finite dimensional realizations for nonlinear forward rate models. *Mathematical Finance*, **11**(2), 205–243.

Black, F. (1976). The pricing of commodity contracts. *Journal of Financial Economics*, **3**(1–2), 167–179.

Black, F., Derman, E., and Toy, W. (1990). A one-factor model of interest rates and its application to treasury bond options. *Financial Analysts Journal*, **46**(1), 33–39.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, **81**(3), 637–654.

Brace, A., Gatarek, D., and Musiela, M. (1997). The market model of interest rate dynamics. *Mathematical Finance*, **7**(2), 127–154.

Brace, A. and Musiela, M. (1994). A multifactor Gauss Markov implementation of Heath, Jarrow, and Morton. *Mathematical Finance*, **4**(3), 259–283.

Brennan, M.J. and Schwartz, E.S. (1979). A continuous time approach to pricing bonds. *Journal of Banking and Finance*, **3**(2), 133–155.

Brigo, D. and Mercurio, F. (2007). *Interest Rate Models* (2nd edn). Springer, Berlin, Heidelberg.

Brody, D.C. and Hughston, L.P. (2001). Interest rates and information geometry. *Proceedings of the Royal Society A*, **457**, 1343–1363.

Brody, D.C. and Hughston, L.P. (2002). Entropy and information in the interest rate term structure. *Quantitative Finance*, **2**(1), 70–80.

Brody, D.C. and Hughston, L.P. (2004). Chaos and coherence: A new framework for interest rate modelling. *Proceedings of the Royal Society A*, **460**, 85–110.

Carmona, R. (ed.) (2009). *Indifference Pricing: Theory and Applications.* Princeton University Press, Princeton N.J.

Carverhill, A. (1994). When is the short rate Markovian? *Mathematical Finance*, **4**(4), 305–312.

Cerny, A. (2003). Generalised sharpe ratios and asset pricing in incomplete markets. *European Finance Review*, **7**, 191–233.

Cerny, A. and Hodges, S. (2001). The theory of good deal pricing in financial markets. In H. Geman et al. (eds). Mathematical Finance. Bachelier Congress 200. Springer-Verlag. Berlin, Heidelberg 2002.

Cheyette, O. (1996). Markov representation of the Heath–Jarrow–Morton model. Preprint. BARRA Inc.

Chiarella, Carl and Kwon, Oh Kang (2001). Forward rate dependent Markovian transformations of the Heath–Jarrow–Morton term structure model. *Finance and Stochastics*, **5**(2), 237–257.

Clewlow, L. and Strickland, C. (2000). *Energy Derivatives.* Lacima Group, London.

Cochrane, J. (2001). *Asset Pricing.* Princeton University Press, Princeton, N.J.

Cochrane, J. and Saá Requejo, J. (2000). Beyond arbitrage: Good-deal asset price bounds in incomplete markets. *Journal of Political Economy*, **108**(1), 79–119.

Cox, J.C. and Huang, C.F. (1989). Optimal consumption and portfolio choice when asset prices follow a diffusion process. *Journal of Economic Theory*, **49**(1), 33–83.

Cox, J., Ingersoll, J., and Ross, S. (1985*a*). An intertemporal general equilibrium model of asset prices. *Econometrica*, **53**(2), 363–384.

Cox, J., Ingersoll, J., and Ross, S. (1985*b*). A theory of the term structure of interest rates. *Econometrica*, **53**(2), 385–408.

Cox, J., Ross, S., and Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*, **7**(3), 229–264.

Cox, J. and Rubinstein, M. (1992). *Options Markets.* Prentice Hall, Englewood Cliffs, N.J.

Cvitanić, J. (1997). Optimal trading under constraints. In *Financial Mathematics*, Springer Lecture Notes in Mathematics, Vol 1656 (ed. W. Runggaldier). Springer Verlag, Berlin, Heidelberg, New York.

Dana, R. and Jeanblanc, M. (2003). *Financial Markets in Continuous Time.* Springer Verlag, Berlin, Heidelberg, New York.

Davis, M.H.A. (1997). Option pricing in incomplete markets. In *Mathematics of Derivative Securities* (eds M. Demptster and S. Pliska), pp. 216–266. Cambridge University Press, Cambridge.

Delbaen, F. (1992). Representing martingale measures when asset prices are continuous and bounded. *Mathematical Finance*, **2**(2), 107–130.

Delbaen, F. and Schachermayer, W. (1994). A general version of the fundamental theorem of asset pricing. *Matematische Annalen*, **300**, 463–520.

Delbaen, F. and Schachermayer, W. (1998). The fundamental theorem for unbounded processes. *Matematische Annalen*, **312**, 215–250.

Dellacherie, C. and Meyer, P.A. (1972). *Probabilités et Potentiel.* Hermann, Paris.

Detemple, J. and Zapatero, F. (1991). Asset prices in an exchange economy with habit formation. *Econometrica*, **59**(6), 1633–1657.

Doob, J.L. (1984). *Classical Potential Theory and its Probabilistic Counterpart.* Springer Verlag, New York.

Dothan, M. (1978). On the term structure of interest rates. *Journal of Financial Economics*, **6**(1), 59–69.

Duffie, D. (1989). *Futures Markets.* Prentice Hall, Englewood Cliffs, NJ.

Duffie, D. (2001). *Dynamic Asset Pricing Theory* (3rd edn). Princeton University Press, Princeton, N.J.

Duffie, D. and Epstein, L. (1992). Stochastic differential utility. *Econometrica*, **60**(2), 353–394.

Duffie, D., Filipovic, D., and Schachermayer, W. (2003). Affine processes and applications in finance. *Annals of Applied Probability*, **13**(3), 984–1053.

Duffie, D. and Huang, C. (1986). Multiperiod securities markets with differential information. *Journal of Mathematical Economics*, **15**(3), 283–303.

Duffie, D. and Kan, R. (1996). A yield factor model of interest rates. *Mathematical Finance*, **6**(4), 379–406.

Duffie, D. and Singleton, K. (1999). Modeling term structures of defaultable bonds. *Review of Financial Studies*, **12**(4), 687–720.

Durrett, R. (1996). *Probability.* Duxbury Press, Belmont. C.A.

Elliott, R.J. (1982). *Stochastic Calculus and Applications.* Springer Verlag, New York.

Epstein, L. and Zin, S.E. (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns A theoretical framework. *Econometrica*, **57**(4), 937–969.

Esscher, F. (1932). On the probability function in the collective theory of risk. *Skandinavisk Aktuarietidskrift*, **15**, 175–195.

Fabozzi, F. (2009). *Bond Markets, Analysis, and Strategies* (7th edn). Prentice Hall, Englewood Cliffs, N.J.

Filipović, D. (1999). A note on the Nelson-Siegel family. *Mathematical Finance*, **9**(4), 349–359.

Filipović, D. (2001). *Consistency Problems for Heath–Jarrow–Morton Interest Rate Models.* Springer Lecture Notes in Mathematics, Vol. 1760. Springer Verlag, Berlin, Heidelberg.

Filipović, D. and Teichmann, J. (2003). Existence of invariant manifolds for stochastic equations in infinite dimension. *Journal of Functional Analysis*, **197**(2), 398–432.

Filipović, D. and Teichmann, J. (2004). On the geometry of the term structure of interest rates. *Proceedings of the Royal Society*, **460**, 129–167.

Fleming, W. and Rishel, R. (eds) (1975). *Deterministic and Stochastic Optimal Control.* Springer Verlag, Berlin, Heidelberg, New York.

Fleming, W. and Soner, M. (eds) (1993). *Controlled Markov Processes and Viscosity Solutions.* Springer Verlag, New York.

Flesaker, B. and Hughston, L. (1996). Positive interest. *RISK Magazine*, **9**, 46–9.

Flesaker, B. and Hughston, L. (1997). International models for interest rates and foreign exchange. *Net Exposure*, **3**, 55–79.

Föllmer, H. and Sondermann, D. (1986). Hedging of non-redundant contingent claims under incomplete information. In *Contributions to Mathematical Economics* (ed. W. a. M.-C. Hildenbrand) pp. 205–223. North-Holland, Amsterdam.

Frittelli, M. (2000). The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance*, **10**(1), 215–225.

Garman, M. and Kohlhagen, S. (1983). Foreign currency option values. *Journal of International Money and Finance*, **2**(3), 231–237.

Geman, H. (1989). The importance of the forward neutral probability in a stochastic approach of interest rates. Working paper, ESSEC.

Geman, H. (ed.) (1999). *Insurance and Weather Derivatives.* Risk Books, London.

Geman, H. (ed.) (2005). *Commodities and Commodity Derivatives.* Wiley, New York.

Geman, H., El Karoui, N., and Rochet, J.-C. (1995). Changes of numéraire, changes of probability measure and option pricing. *Journal of Applied Probability*, **32**(2), 443–458.

Gerber, H. and Shiu, E. (1994). Option pricing by Esscher transforms. *Transactions of the Society of Actuaries*, **46**, 51–92.

Geske, R. and Johnson, H. (1984). The American put option valued analytically. *Journal of Finance*, **39**(5), 1511–1524.

Goll, T. and Rüschendorf, L. (2001). Minimax and minimal distance martingale measures and their relationship to portfolio optimization. *Finance and Stochastics*, **5**(4), 557–581.

Hansen, L. and Jagannathan, R. (1991). Implications of security market data for models of dynamic economies. *Journal of Political Economy*, **99**(2), 225–262.

Harrison, J. and Kreps, J. (1979). Martingales and arbitrage in multiperiod markets. *Journal of Economic Theory*, **20**(3), 381–408.

Harrison, J. and Pliska, S. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes & Applications*, **11**(3), 215–260.

Heath, D., Jarrow, R., and Morton, A. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, **60**(1), 77–105.

Henderson, V. and Hobson, D. (2009). Utility indifference pricing – an overview. In *Indifference Pricing: Theory and Applications* (ed. R. Carmona), pp. 44–74. Princeton University Press, Princeton, N.J.

Ho, T. and Lee, S. (1986). Term structure movements and pricing interest rate contingent claims. *Journal of Finance*, **41**(5), 1011–1029.

Huang, C.F. (1987). An intertemporal general equilibrium asset pricing model: The case of diffusion information. *Econometrica*, **55**(1), 117–142.

Hughston, L.P. and Rafailidis, A. (2005). A chaotic approach to interest rate modelling. *Finance and Stochastics*, **9**, 43–65.

Hull, J. (2003). *Options, Futures, and Other Derivatives* (5th edn). Prentice Hall, Englewood Cliffs, N.J.

Hull, J. and White, A. (1990). Pricing interest-rate-derivative securities. *Review of Financial Studies*, **3**(4), 573–592.

Hunt, P. and Kennedy, J. (2000). *Financial Derivatives in Theory and Practice.* Wiley, Chichester.

Inui, Koji and Kijima, Masaaki (1998). A Markovian framework in multi-factor Heath–Jarrow–Morton models. *Journal of Financial and Quantitative Analysis*, **33**(3), 423–440.

Jacod, J. and Protter, P. (2000). *Probability Essentials.* Springer Verlag, Berlin Heidelberg.

Jacod, J. and Shiryaev, A.N. (1987). *Limit Theorems for Stochastic Processes.* Springer Verlag, Berlin.

Jamshidian, F. (1989). An exact bond option formula. *Journal of Finance*, **44**(1), 205–209.

Jamshidian, F. (1997). LIBOR and swap market models and measures. *Finance and Stochastics*, **1**(4), 293–330.

Jarrow, B. and Madan, D. (1995). Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns. *Mathematical Finance*, **5**(4), 311–336.

Jarrow, R., Lando, D., and Turnbull, S. (1997). A Markov model for the term structure of credit risk spreads. *Review of Financial Studies*, **10**(2), 481–523.

Jeanblanc, M., Yor, M., and Chesney, M. (2009). *Mathematical Methods for Financial Markets.* Springer Verlag, New York, Heidelberg.

Jeffrey, Andrew (1995). Single factor Heath–Jarrow–Morton term structure models based on Markov spot interest rate dynamics. *Journal of Financial and Quantitative Analysis*, **30**(4), 619–642.

Jin, Y. and Glasserman, P. (2001). Equilibrium positive interest rates: A unified view. *Review of Financial Studies*, **14**(1), 187–214.

John, F. (1982). *Partial Differential Equations.* Springer Verlag, Berlin.

Joshi, M. (2008). *The Concepts and Practice of Mathematical Finance* (2nd edn). Cambridge University Press, Cambridge.

Kallsen, J. and Shiryayev, A.N. (2002). The cumulant process and Esscher's change of measure. *Finance and Stochastics*, **6**(2), 313–338.

Karatzas, I., Lehoczky, J., and Shreve, S. (1987). Optimal portfolio and consumption decision for a "small investor" on a finite horizon. *SIAM Journal of Control and Optimization*, **25**(6), 1157–1186.

Karatzas, I., Lehoczky, J., and Shreve, S. (1990). Existence and uniqueness of multi-agent equilibrium in a stochastic dynamic consumption/investment model. *Mathematics of Operations Research*, **15**(1), 90–128.

Karatzas, I., Lehoczky, J., Shreve, S., and Xu, G.L. (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal of Control and Optimization*, **29**(3), 702–730.

Karatzas, Ioannis and Shreve, Steven (1998). *Methods of Mathematical Finance.* Springer, New York, Heidelberg, Berlin.

Karatzas, Ioannis and Shreve, Steven (2008). *Brownian Motion and Stochastic Calculus* (2nd edn). Springer, New York, Heidelberg, Berlin.

Korn, R. (1997). *Optimal Portfolios.* World Scientific, Singapore.

Kramkov, D. and Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investments in incomplete markets. *Annals of Applied Probability*, **9**(3), 904–950.

Kreps, D. (1981). Arbitrage and equilibrium in economies with infinitely many commodities. *Journal of Mathematical Economics*, **8**(1), 15–35.

Kreps, D. and Porteus, E. (1978). Temporal resolution of uncertainty and dynamic choice theory. *Econometrica*, **46**(1), 185–200.

Krylov, R. (1980). *Controlled Diffusion Processes.* Springer Verlag, New York.

Lando, D. (2004). *Credit Risk Modeling.* Princeton University Press, Princeton, N.J.

Leland, H. (1994). Corporate debt value, bond covenants and optimal capital structure. *Journal of Finance*, **49**(4), 1213–1252.

Long, J.B. (1990). The numeraire portfolio. *Journal of Financial Economics*, **26**(1), 29–69.

Longstaff, F.A. and Schwartz, E.S. (1992). Interest rate volatility and the term structure: A two-factor general equilibrium model. *Journal of Finance*, **47**(4), 1259–1282.

Lucas, R. (1978). Asset prices in an exchange economy. *Econometrica*, **46**(6), 1429–1445.

Magshoodi, Y. (1996). Solution of the extended CIR term structure and bond option valuation. *Mathematical Finance*, **6**(1), 89–109.

Margrabe, W. (1978). The value of an option to exchange one asset for another. *Journal of Finance*, **33**(1), 177–186.

Merton, R. (1969). Lifetime portfolio selection under uncertainty: The continuous time case. *Review of Economics and Statistics*, **51**(3), 247–257.

Merton, R. (1971). Optimum consumption and portfolio rules in a continuous time model. *Journal of Economic Theory*, **3**(4), 373–413.

Merton, R. (1973). The theory of rational option pricing. *Bell Journal of Economics and Management Science*, **4**(1), 141–183.

Merton, R. (1974). On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, **29**(2), 449–479.

Meyer, P.A. (1976). Un cours sur les integrales stochastiques. In *Seminaire de Probabilités X*, pp. 174–329. Springer Verlag, Berlin, Heidelberg.

Milgrom, P. and Segal, Y. (2002). Envelope theorems for arbitrary choice sets. *Econometrica*, **70**(2), 583–601.

Miltersen, K., Sandmann, K., and Sondermann, D. (1997). Closed form solutions for term structure derivatives with log-normal interest rates. *Journal of Finance*, **52**(1), 409–430.

Miyahara, Y. (1976). Canonical martingale measures of incomplete assets markets. In *Proceedings of the Seventh Japan–Russia Symposium* (ed. S. Watanabe). World Scientific, Singapore.

Miyahara, Y. (2011). *Option Pricing in Incomplete Markets.* Imperial College Press, London.

Musiela, M. (1993). Stochastic PDEs and term structure models. Preprint.

Musiela, M. and Rutkowski, M. (1997). *Martingale Methods in Financial Modelling.* Springer-Verlag, Berlin.

Øksendal, B. (1998). *Stochastic Differential Equations.* Springer-Verlag, Berlin.

Pelsser, A. (2000). *Efficient Methods of Valuing Interest Rate Derivatives.* Springer-Verlag, Berlin.

Peskir, G. and Shiryaev, A.N. (2006). *Optimal Stopping and Free Boundary Value Problems.* Birkhäuser, Basel.

Protter, P. (2004). *Stochastic Integration and Differential Equations* (2nd edn). Springer-Verlag, Berlin.

Reiner, E. (1992). Quanto mechanics. *RISK*, **5**, 59–63.

Rendleman, R. and Bartter, B. (1979). Two state option pricing. *Journal of Finance*, **34**(5), 1092–1110.

Revuz, D. and Yor, M. (1991). *Continuous Martingales and Brownian Motion.* Springer-Verlag, Berlin, Heidelberg.

Ritchken, P. and Sankarasubramanian, L. (1995). Volatility structures of forward rates and the dynamics of the term structure. *Mathematical Finance*, **5**(1), 55–72.

Rockafeller, T. (1970). *Convex analysis.* Princeton University Press, Princeton, N.J.

Rodriguez, I. (2000). A simple linear programming approach to gain, loss and asset pricing. Topics in Theoretical Economics. **2**(1). ISSN (Online) 1534-598X. DOI: https://doi.org/10.2202/1534-598X.1064.

Rogers, L.C.G. (1994). The potential approach to the term structure of interest rates and foreign exchange rates. *Mathematical Finance*, **7**(2), 157–176.

Rogers, L.C.G. (2002). Duality in constrained optimal investment and consumption problems: A synthesis. In *Paris-Princeton Lectures on Mathematical Finance 2002* (ed. R. e. a. Carmonq). Springer-Verlag, Berlin, Heidelberg, New York.

Royden, H.L. (1988). *Real Analysis* (3rd edn). Prentice Hall, New York.

Rudin, W. (1991). *Functional Analysis.* McGraw–Hill, New York.

Schachermayer, W. (1994). Martingale measures for discrete time processes with infinite horizon. *Mathematical Finance*, **4**(1), 25–56.

Schachermayer, W. (2002). Optimal investment in incomplete financial markets. In *Mathematical Finance–Bachelier Congress 2000* (ed. H. Geman), pp. 427–462. Springer-Verlag, Berlin, Heidelberg, New York.

Schönbucher, P. (2003). *Credit Derivatives Pricing Models.* Wiley, Chichester.

Schweizer, M. (1991). Option hedging for semimartingales. *Stochastic Processes and their Applications*, **37**(2), 339–363.

Schweizer, M. (2001). A guided tour through quadratic hedging approaches. In *Option Pricing, Interest Rates and Risk Management* (ed. E. Jouini), pp. 538–574. Cambridge University Press, Cambridge.

Shirakawa, H. (1991). Interest rate option pricing with Poisson–Gaussian forward rate curve processes. *Mathematical Finance*, **1**(4), 77–94.

Shiryaev, A.N. (2008). *Optimal stopping rules* (2nd edn). Springer Verlag, Berlin, Heidelberg.

Snell, J.L. (1952). Application of martingale system theorems. *Transactions of the American Mathematical Society*, **73**(2), 293–312.

Steele, J.M. (2001). *Stochastic Calculus and Financial Applications.* Springer Verlag, New York Berlin Heidelberg.

Sundaresan, S. (2009). *Fixed Income Markets and their Derivatives* (3rd edn). Academic Press, Cambridge, M.A.

Vasiček, O. (1977). An equilibrium characterization of the term stucture. *Journal of Financial Economics*, **5**(2), 177–188.

Zariphopoulo, T. (2001). A solution approach to valuation with unhedgeable risk. *Finance and Stochastics*, **5**(1), 61–82.

INDEX

Note: Figures are indicated by an italic "f'', following the page number.