Chapter 4

Stochastic volatility – introduction

Chapter 2 was devoted to local volatility, a very constrained form of stochastic volatility: implied volatilities have 100% correlation among themselves and with S, and their volatilities are determined by the market smile used for calibration of the model. Volatilities of implied volatilities – as generated by the model – will likely bear no resemblance to conservative levels based on historically observed volatilities of implied volatilities, or to implied volatilities of volatilities, whenever they are observable.

Can we design models that let us freely specify the dynamics of implied volatilities? We tackle this issue, first starting from a general point of view, then specializing to a modeling framework that can be used practically, based on forward variances of specific European payoffs.

4.1 Modeling vanilla option prices

It is natural to try and model the dynamics of the prices of vanilla options directly. Let C_{KT} be the price of a call option with strike K, maturity T, and let us assume without loss of generality zero interest rates and repos. Because an option is an asset that can be bought or sold, its pricing drift is r – which we have taken to be zero. The dynamics of C_{KT}, along with S can be written as:

$$\begin{cases} dS &= \overline{\sigma} S dW^S \\ dC_{KT} &= \Lambda_{KT}\, dW^{KT} \end{cases} \tag{4.1}$$

with the initial conditions:

$$\begin{cases} S_{t=0} &= S^{\text{market}}_{t=0} \\ C_{KT}\,(t=0) &= C^{\text{market}}_{KT}\,(t=0) \end{cases}$$

and the terminal condition:

$$C_{KT}\,(t=T) \;=\; (S_T - K)^+ \tag{4.2}$$

where W^S, W^{KT} are Brownian motions, Λ_{KT} is the volatility of C_{KT} and $\overline{\sigma}$ is the instantaneous volatility of S. We will omit the t subscripts in the processes whenever possible.

4.1.1 Modeling implied volatilities

Let us try to work with implied volatilities $\widehat{\sigma}_{KT}$. In the pricing equation, $\widehat{\sigma}_{KT}$ acquires a drift that represents the cost of financing a delta position on $\widehat{\sigma}_{KT}$. Taking a directional position on $\widehat{\sigma}_{KT}$ entails trading the corresponding delta-hedged vanilla option. Expressing that the pricing drift of C_{KT} is the interest rate – here zero – will then determine the pricing – or "risk-neutral" – drift of $\widehat{\sigma}_{KT}$. The joint SDEs of S and $\widehat{\sigma}_{KT}$ read:

$$\begin{cases} dS &= \overline{\sigma} S dW^S \\ d\widehat{\sigma}_{KT} &= \mu_{KT} dt \,+\, \lambda_{KT}\, dW^{KT} \end{cases} \tag{4.3}$$

where W^{KT} is a Brownian motion. $\widehat{\sigma}_{KT}$ is related to C_{KT} through: $C_{KT} = P_{BS}(t, S, \widehat{\sigma}_{KT}, T)$. The terminal condition (4.2) then becomes:

$$\lim_{t\to T}\, (T-t)\, \widehat{\sigma}^2_{KT,\, t} \;=\; 0 \tag{4.4}$$

and the initial condition is simply:

$$\widehat{\sigma}_{KT,\, t=0} \;=\; \widehat{\sigma}^{\text{market}}_{KT,\, t=0}$$

The SDE for C_{KT} reads:

$$\begin{aligned} dC_{KT} \;=\; & \left(\frac{dP_{BS}}{dt} + \frac{\overline{\sigma}^2}{2} S^2 \frac{d^2 P_{BS}}{dS^2} + \frac{\lambda^2_{KT}}{2} \frac{d^2 P_{BS}}{d\widehat{\sigma}^2_{KT}} + \rho \lambda_{KT} \overline{\sigma} S \frac{d^2 P_{BS}}{dS d\widehat{\sigma}_{KT}} \right. \\ & \left. + \frac{dP_{BS}}{d\widehat{\sigma}_{KT}} \mu_{KT} \right) dt \,+\, S \frac{dP_{BS}}{dS} \overline{\sigma} dW^S \,+\, \frac{dP_{BS}}{d\widehat{\sigma}_{KT}} \lambda_{KT}\, dW^{KT} \end{aligned}$$

where ρ is the correlation between W^S and W^{KT}. Expressing that the drift of C^{KT} vanishes determines μ_{KT}. The derivatives of P_{BS} are all available analytically – the resulting expression for μ^{KT} is:

$$\mu_{KT} \;=\; \frac{1}{\widehat{\sigma}_{KT}} \left(\frac{\widehat{\sigma}_{KT} - \overline{\sigma}^2}{2\,(T-t)} - \frac{1}{2} d_1 d_2 \lambda^2_{KT} + \frac{d_2}{\sqrt{T-t}} \rho \overline{\sigma} \lambda_{KT} \right)$$

which we directly quote from [80], where d_1, d_2 are standard expressions appearing in Black-Scholes formulas:

$$d_1 \;=\; \frac{1}{\widehat{\sigma}_{KT}\sqrt{T-t}} \ln \frac{F_T(S_t)}{K} + \frac{\widehat{\sigma}_{KT}\sqrt{T-t}}{2}, \quad d_2 \;=\; d_1 - \widehat{\sigma}_{KT}\sqrt{T-t}$$

The joint dynamics of $(S, \widehat{\sigma}_{KT})$ in (4.3) would now be completely specified if the process $\overline{\sigma}$ was known. The processes for $\overline{\sigma}$ and $\widehat{\sigma}_{KT}$ cannot be chosen arbitrarily, for a solution to (4.3) to exist. For example it is possible to prove that, for short maturities, the at-the-money implied volatility should tend to the instantaneous volatility – see [43] for a proof:[1]

$$\lim_{T\to t} \widehat{\sigma}_{ST,t} \;=\; \overline{\sigma}_t \tag{4.5}$$

[1] Surprisingly, the proof of this simple and natural result is rather technical. Also note that the non-explosion condition imposed on μ_{KT} in [80] is not needed: as long as (4.4) holds, how fast or how slowly the left-hand side tends to zero does not matter.

One may think that once some technical conditions on processes $\overline{\sigma}$ and $\widehat{\sigma}_{KT}$ are satisfied, we should be able to explicitly construct – at least numerically – a solution to (4.3) that complies with (4.4). The trouble is that once we have a solution to (4.3) over $[0, T]$, the ensuing dynamics for S determines prices of European options for maturities up to T, hence the implied volatilities $\widehat{\sigma}_{k\tau}$ of all vanilla options with $\tau < T$.

This is problematic, as our intention in modeling the $\widehat{\sigma}_{KT}$ was to be able to choose their initial values freely. What this means is that information about the initial smile has to be embedded in the process for $\overline{\sigma}$.[2] Moreover, such embedding is presumably convoluted: an arbitrary configuration of the $\widehat{\sigma}_{KT}$ is consistent with a dynamics of S – hence a process $\overline{\sigma}$ may exist – only if the no-arbitrage conditions highlighted in Section 2.2.2 are satisfied. Whenever these conditions are violated, $\overline{\sigma}$ does not exist and this has to manifest itself in the structural impossibility of constructing a process for $\overline{\sigma}$.

The upshot is that direct modeling of implied volatilities of vanilla options is impractical.

4.2 Modeling the dynamics of the local volatility function

We try here a different line of approach: given a non-arbitrageable configuration of implied volatilities $\widehat{\sigma}_{KT}$ a local volatility function $\sigma(t, S)$ exists, given by the Dupire formula (2.3). There is a one-to-one mapping between the volatility surface and the local volatility function. We are not using a local volatility *model*; we are using the local volatility *function* to represent at any time t the full set of implied volatilities.

We can then generate a dynamics for implied volatilities by generating a dynamics for this local volatility function which we denote by σ_t: $\sigma_t(\tau, \mathcal{S})$ is the local volatility function associated with the volatility surface at time t. For a fixed couple $(\tau, \mathcal{S})$, $\sigma_t(\tau, \mathcal{S})$ is a *process* that exists for $t \leq \tau$.

No-arbitrage restrictions on $\widehat{\sigma}_{KT}$ translate into the simple condition that $\sigma_t^2(\tau, \mathcal{S})$ be positive for all $\tau, \mathcal{S}$. In what follows we work with variances σ^2. In the local volatility model, the local volatility function is fixed and the instantaneous volatility $\overline{\sigma}_t$ of S_t is given by:

$$\overline{\sigma}_t = \sigma_{t_0}(t, S_t)$$

where the t_0 subscript indicates that $\sigma_{t_0}(t, S_t)$ was obtained from implied volatilities observed at time t_0. In a model where the local volatility function itself is dynamic, $\overline{\sigma}$ is given by:

$$\overline{\sigma}_t = \sigma_t(t, S_t) \tag{4.6}$$

[2]This is precisely what occurs in the local volatility model, the simplest of all market models.

This follows from the definition of σ_t. From equation (2.6) the local volatility is given by:

$$\sigma_t^2(\tau, \mathcal{S}) = \frac{E_t[\sigma_\tau^2 \delta(S_\tau - \mathcal{S})]}{E_t[\delta(S_\tau - \mathcal{S})]} = E_t[\sigma_\tau^2 \mid S_\tau = \mathcal{S}]$$

Setting $\mathcal{S} = S_t$ and taking the limit $\tau \to t$ removes the conditionality in the expectation and yields (4.6). In contrast to the local volatility model, the dynamics of S_t is generated by the short end of the volatility function σ_t only. Local volatilities $\sigma_t(\tau, \mathcal{S})$ for $\tau > t$ do not appear explicitly in the SDE for S_t. Their role is to encode information on implied volatilities $\widehat{\sigma}_{KT}$: prices of vanilla options for maturities $\tau > t$ are derived from $\sigma_t(\tau, \mathcal{S})$ by solving the forward equation (2.7). In our notation:

$$\frac{dC_t^{\mathcal{S}\tau}}{d\tau} + (r-q)\,\mathcal{S}\frac{dC_t^{\mathcal{S}\tau}}{d\mathcal{S}} - \frac{\sigma_t^2(\tau, \mathcal{S})}{2}\mathcal{S}^2\frac{d^2C_t^{\mathcal{S}\tau}}{d\mathcal{S}^2} = -qC_t^{\mathcal{S}\tau}$$

with the initial condition $C_t^{\mathcal{S}t} = (S_t - \mathcal{S})^+$, where $C_t^{\mathcal{S}\tau}$ is the value at time t of a call option of maturity τ, strike $\mathcal{S}$.

We now use notation $\sigma_{\tau\mathcal{S}}$ for $\sigma_t(\tau, \mathcal{S})$ and omit the t-subscript in processes whenever possible. The pricing SDEs for S, $\sigma_{\tau\mathcal{S}}^2$ are given by:

$$\begin{cases} dS = (r-q)\,Sdt + \sigma_{tS}SdW^S \\ d\sigma_{\tau\mathcal{S}}^2 = \mu_{\tau\mathcal{S}}dt + \lambda_{\tau\mathcal{S}}dW^{\tau\mathcal{S}} \end{cases} \tag{4.7}$$

where $W^{\tau\mathcal{S}}$ is a Brownian motion. Just as in the previous section, we use zero interest rates and repos without loss of generality and determine $\mu_{\tau\mathcal{S}}$ by imposing that option prices have vanishing drift. $\sigma_{\tau\mathcal{S}}^2$ is given by

$$\sigma_{\tau\mathcal{S}}^2 = 2\left.\frac{\frac{dC_{KT}}{dT}}{K^2\frac{d^2C_{KT}}{dK^2}}\right|_{\substack{K=\mathcal{S}\\T=\tau}} \tag{4.8}$$

Both numerator and denominator are prices of linear combinations of vanilla options – a calendar spread for the numerator, a butterfly spread for the denominator – hence their drifts vanish. Remembering from equation (2.5), page 27, that

$$\frac{d^2C_{\mathcal{S}\tau}}{d\mathcal{S}^2} = e^{-r(\tau-t)}E_t[\delta(S_\tau - \mathcal{S})] = e^{-r(\tau-t)}\rho_{\tau\mathcal{S}}$$

where $\rho_{\tau\mathcal{S}}$ is the probability density that $S_\tau = \mathcal{S}$, yields:

$$\mathcal{S}^2\,\sigma_{\tau\mathcal{S}}^2\rho_{\tau\mathcal{S}} = 2e^{r(\tau-t)}\left.\frac{dC_{KT}}{dT}\right|_{\substack{K=\mathcal{S}\\T=\tau}} \tag{4.9}$$

Since the right-hand side of (4.9) has zero drift, so must the left-hand side: $\sigma_{\tau\mathcal{S}}^2\rho_{\tau\mathcal{S}}$ has vanishing drift.[3]

[3]In the left-hand side of (4.9) $\mathcal{S}$ is not a process – only $\sigma_{\tau\mathcal{S}}$ and $\rho_{\tau\mathcal{S}}$ are processes.

$\rho_{\mathcal{TS}}$ is a process: it is the (undiscounted) price of a sharp butterfly spread: this implies that its drift vanishes as well. $\rho_{\mathcal{TS}}$ is a *function* of S and t and a *functional* of the local volatility function:

$$\rho_{\mathcal{TS}} \equiv \rho_{\mathcal{TS}}\left(t, S,\ \sigma^2\right)$$

Its SDE then simply reads:

$$d\rho_{\mathcal{TS}} = \frac{d\rho_{\mathcal{TS}}}{dS}dS + \iint \frac{\delta\rho_{\mathcal{TS}}}{\delta\sigma^2_{ux}}d\sigma^2_{ux}$$

where the double integral stands for $\int_t^{\mathcal{T}} du \int_0^{\infty} dx$.

Using now expression (4.7) for $d\sigma^2_{\mathcal{TS}}$ and $d\sigma^2_{ux}$, we get the drift of $\sigma^2_{\mathcal{TS}}\rho_{\mathcal{TS}}$:

$$\rho_{\mathcal{TS}}\mu_{\mathcal{TS}}dt + \lambda_{\mathcal{TS}}\left(S\frac{d\rho_{\mathcal{TS}}}{dS}\sigma_{tS}\left\langle dW^S dW^{\mathcal{TS}}\right\rangle + \iint \frac{\delta\rho_{\mathcal{TS}}}{\delta\sigma^2_{ux}}\lambda_{ux}\left\langle dW^{ux}dW^{\mathcal{TS}}\right\rangle\right)$$

Expressing that it vanishes yields:

$$\mu_{\mathcal{TS}} = -\frac{\lambda_{\mathcal{TS}}}{\rho_{\mathcal{TS}}}\left(S\frac{d\rho_{\mathcal{TS}}}{dS}\sigma_{tS}\frac{\left\langle dW^S dW^{\mathcal{TS}}\right\rangle}{dt} + \iint \frac{\delta\rho_{\mathcal{TS}}}{\delta\sigma^2_{ux}}\lambda_{ux}\frac{\left\langle dW^{ux}dW^{\mathcal{TS}}\right\rangle}{dt}\right) \tag{4.10}$$

We now derive a more explicit expression for $\frac{\delta\rho_{\mathcal{TS}}}{\delta\sigma^2_{ux}}$, which appears in the right-hand side of (4.10). $\rho_{\mathcal{TS}}\left(t, S,\ \sigma^2\right)$ is an expectation: $\rho_{\mathcal{TS}} = E_t[\delta\left(S_{\mathcal{T}} - \mathcal{S}\right)]$. It solves the usual backward equation:

$$\frac{d\rho_{\mathcal{TS}}}{dt} + \frac{\sigma^2_{tS}}{2}S^2\frac{d^2\rho_{\mathcal{TS}}}{dS^2} = 0 \tag{4.11}$$

with the terminal condition $\rho\left(t = \mathcal{T}\ S,\ \mathcal{TS}\right) = \delta\left(S - \mathcal{S}\right)$. By definition the functional derivative $\frac{\delta\rho_{\mathcal{TS}}}{\delta\sigma^2_{ux}}$ is such that, at order one in a perturbation $\delta\sigma^2$ of the local volatility function,

$$\delta\rho_{\mathcal{TS}} = \iint \frac{\delta\rho_{\mathcal{TS}}}{\delta\sigma^2_{ux}}\,\delta\sigma^2_{ux}$$

Consider a perturbation $\delta\sigma^2$ of σ^2. At first order in $\delta\sigma^2$, the perturbation $\delta\rho_{\mathcal{TS}}$ solves the following equation:

$$\frac{d\delta\rho_{\mathcal{TS}}}{dt} + \frac{\sigma^2_{tS}}{2}S^2\frac{d^2\delta\rho_{\mathcal{TS}}}{dS^2} = -\frac{\delta\sigma^2_{tS}}{2}S^2\frac{d^2\rho_{\mathcal{TS}}}{dS^2} \tag{4.12}$$

with the terminal condition at $t = \mathcal{T}$: $\delta\rho_{\mathcal{TS}} = 0$. The solution to (4.12) is given by the Feynman-Kac theorem:

$$\delta\rho_{\mathcal{TS}} = E\left[\int_t^{\mathcal{T}} \frac{\delta\sigma^2_{uS_u}}{2}S_u^2\left.\frac{d^2\rho_{\mathcal{TS}}}{dS^2}\right|_{S=S_u} du\right]$$

where the expectation is taken with respect to the dynamics generated by the local volatility function σ_{ux}^2. Let us rewrite this expectation using the probability density for S at time u – which is ρ_{ux}:

$$\delta\rho_{\tau\mathcal{S}} = \frac{1}{2}\int_t^\tau du \int_0^\infty dx\ \rho_{ux}x^2\ \frac{d^2\rho_{\tau\mathcal{S}}\left(ux,\sigma^2\right)}{dx^2}\ \delta\sigma_{ux}^2$$

which yields:

$$\frac{\delta\rho_{\tau\mathcal{S}}}{\delta\sigma_{ux}^2} = \frac{1}{2}\rho_{ux}x^2\frac{d^2\rho_{\tau\mathcal{S}}\left(ux,\sigma^2\right)}{dx^2} \tag{4.13}$$

The sagacious reader will have noticed that we have already derived this result in Section 2.9, in the context of the vega hedge in the local volatility model. $\rho_{\tau\mathcal{S}}$ is the price of a European option of maturity τ: its sensitivity to σ_{ux}^2 is then given by equation (2.117) which involves the product of the probability density and the dollar gamma: (4.13) is identical to (2.117), page 82. We have our final expression for $\mu_{\tau\mathcal{S}}$:

$$\begin{aligned}\mu_{\tau\mathcal{S}} = -\frac{\lambda_{\tau\mathcal{S}}}{\rho_{\tau\mathcal{S}}}&\left(S\frac{d\rho_{\tau\mathcal{S}}\left(tS,\sigma^2\right)}{dS}\sigma_{tS}\frac{\left\langle dW^S dW^{\tau\mathcal{S}}\right\rangle}{dt}\right.\\&\left.+\frac{1}{2}\int_t^\tau du\int_0^\infty dx\ \rho_{ux}x^2\frac{d^2\rho_{\tau\mathcal{S}}\left(ux,\sigma^2\right)}{dx^2}\ \lambda_{ux}\frac{\left\langle dW^{ux}dW^{\tau\mathcal{S}}\right\rangle}{dt}\right)\end{aligned} \tag{4.14}$$

This expression for $\mu_{\tau\mathcal{S}}$ was first published by Iraj Kani and Emanuel Derman in [39] – albeit without the first piece in the right-hand side – and more recently rederived by René Carmona and Sergey Nadtochiy in [21], who also prove that the instantaneous volatility $\overline{\sigma}_t$ of S_t is indeed $\sigma_t\left(t,S_t\right)$: this establishes the connection between the SDEs for $\sigma_{\tau\mathcal{S}}^2$ and S in (4.7).

Drift $\mu_{\tau\mathcal{S}}$ in (4.14) is computationally expensive: not only is it non-local in the sense that it depends on the whole local volatility function, but it involves *forward* transition densities $\rho_{\tau\mathcal{S}}\left(ux,\sigma^2\right)$ for all u in $[t,\tau]$. It seems difficult to come up with explicit non-trivial solutions to (4.7) based on the direct modeling of local volatilities.

Inspection of expression (4.14) suggests however two simple solutions for which $\mu_{\tau\mathcal{S}}$ vanishes:

- $\lambda_{\tau\mathcal{S}} \equiv 0$: this implies $\mu_{\tau\mathcal{S}} \equiv 0$. The local volatility function is frozen: this recovers the local volatility model.

- $\left\langle dSdW^{\tau\mathcal{S}}\right\rangle \equiv 0$ and $\left\langle dW^{ux}dW^{\tau\mathcal{S}}\right\rangle \equiv 0$: local volatilities have zero correlation among themselves and with S. All points of the local volatility function have their own uncorrelated dynamics: such a model amounts to randomly drawing the instantaneous volatility of S as time advances in such a way that the expectation of it square matches the square of the local volatility function calibrated on the initial smile: $E[\overline{\sigma}_t^2] = \sigma_{t_0}^2(t, S_t)$. While this appears to generate a non-trivial dynamics for implied volatilities it recovers in fact the local volatility model.[4]

Indeed, imagine sitting at time t with a spot value S and let us compute prices of a vanilla option of maturity T using expression (2.30), derived in Section 2.4.1, where we use as base model the local volatility model with local volatility function σ_{t_0}:

$$P_{\overline{\sigma}}(t) \;=\; P_{\sigma_0}(t, S) + E_{\overline{\sigma}}\left[\int_t^T \frac{1}{2} e^{-ru} S_u^2 \frac{d^2 P_{\sigma_0}}{dS^2}\left(\overline{\sigma}_u^2 - \sigma_0(u, S_u)^2\right) du\right]$$

As, by construction, $E_{\overline{\sigma}}[\overline{\sigma}_u^2] = \sigma_0(u, S_u)^2$ we get $P_{\overline{\sigma}}(t) = P_{\sigma_0}(t, S)$. This can also be established by noting that, in our model, because each point of the local volatility function is driven by an independent random variable, $E[\overline{\sigma}_u^2 \mid S_u = S] = \sigma_0^2(u, S)$: the general Dupire equation (2.6) then shows that prices in both models are identical. Over any finite time interval the randomness of σ_t averages out and the model behaves as though the local volatility function was fixed, equal to $\sigma_0(t, S)$.

Johannes Wissel proposes in [86] an approach that consists in modeling a discrete set of vanilla options prices: he introduces "local implied volatilities", whose relationship to vanilla option prices is more direct than that of local volatilities. The drifts of these local implied volatilities are non-local as well, except we now have a large but finite number of local implied volatilities. Still, an explicit non-trivial example of a model has not been available yet.

4.2.1 Conclusion

Again, we hit a snag in our attempt to model the dynamics of implied volatilities. Implied volatilities of vanilla options are unwieldy objects, as are their associated local volatilities: because they are related indirectly to option prices, their drifts are complex. In addition, we are handling the dynamics of a two-dimensional set of processes – reducing the dimensionality may be the price to pay to gain some tractability.

Observe that implied volatilities can be defined for any European payoff, as long as it is convex or concave. Are there particular payoffs whose implied volatilities are easier to handle?

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[4] I thank Bruno Dupire for pointing this out to me.

4.3 Modeling implied volatilities of power payoffs

Consider payoff S_T^p, which we call a *power payoff*, following the terminology of Schweizer and Wissel in [81], and denote its price by Q^{pT}.[5]

In the absence of cash-amount dividends, which is the assumption we make throughout this section, in the Black-Scholes model with implied volatility $\widehat{\sigma}$, Q^{pT} is given by:

$$Q^{pT} = e^{-r(T-t)} F_T^p e^{\frac{p(p-1)}{2}(T-t)\widehat{\sigma}^2} \tag{4.15}$$

where F_T is the forward for maturity T.

Given the market price Q^{pT} of a power payoff, its implied volatility $\widehat{\sigma}_{pT}$ is obtained by inverting (4.15). Power payoffs are concave for $p \in]0,1[$ and convex otherwise: $\widehat{\sigma}_{pT}$ is well-defined as long as $p \neq 0$ and $p \neq 1$.

4.3.1 Implied volatilities of power payoffs

A power payoff is a European option: its market price can be calculated through replication on vanilla options, using expression (3.7), page 107. However, given a market smile, existence of arbitrary moments of S_T is not guaranteed.

For $p \in [0,1]$ power payoffs have finite prices, as S^p is bounded above by an affine function of S. For values of $p > 1$ or values of $p < 0$, market prices of power payoffs may not be finite.

In [67] Roger Lee relates the existence of moments of S_T to the asymptotic behavior of implied volatilities for large and small strikes. Specifically, he shows that:

- When $K \to 0$ or $K \to \infty$, $T\widehat{\sigma}_{KT}^2$ grows at most linearly in $\ln K$, with a slope that cannot be higher than 2 or lower than -2.[6] Moreover, the asymptotic slope of $T\widehat{\sigma}_{KT}^2$ as a function of $\ln K$ is related to the largest/lowest index for which moments of S_T are finite:

- Let $p_+ = \sup\{p\colon E[S_T^{1+p}] < \infty\}$. Then
$$\limsup_{K\to\infty} \frac{T\widehat{\sigma}_{KT}^2}{\ln K} = 2 - 4\left(\sqrt{p_+^2 + p_+} - p_+\right)$$

- Let $p_- = \sup\{p\colon E[S_T^{-p}] < \infty\}$. Then
$$\limsup_{K\to 0} \frac{T\widehat{\sigma}_{KT}^2}{\ln K} = -2 + 4\left(\sqrt{p_-^2 + p_-} - p_-\right)$$

[5] This section is based on joint work with Pierre Henry-Labordère.

[6] These bounds on the slope of the integrated variance f as a function of log-moneyness can be derived by imposing that the denominator in the Dupire formula (2.19), page 33, is positive. Assuming that, asymptotically, $f = ay + b$, positivity of the denominator (a butterfly spread) requires that $|a| < 2$.

In practice, even for indexes, implied volatilities exist only in a limited range of strikes: the asymptotic behavior of the smile is then set by the extrapolation chosen by the trader.

Which power payoffs have finite prices then depends on non-observable implied volatility data. Typically $T\widehat{\sigma}_{KT}^2$ is parametrized as a function of $\ln\left(K/F_T\right)$ as the Dupire equation acquires a simple form (see expression (2.19), page 33), and no-arbitrage conditions are more easily handled. Choosing an affine extrapolation in these units amounts – through its slope – to deciding which power payoffs have finite prices.

Calculating implied volatilities of power payoffs

Implied volatilities $\widehat{\sigma}_{pT}$ are obtained from market prices Q^{pT} by inverting (4.15). For $p \in]0,1[$ prices of power payoffs are always finite; there exists a direct formula of $\widehat{\sigma}_{pT}$ as a weighted average of vanilla implied volatilities, which we now derive.

Undiscounted prices of power payoffs are related to the characteristic function $L(p)$ of $x = \ln \frac{S_T}{F_T}$:

$$e^{r(T-t)}\frac{Q^{pT}}{F_T^p} \;=\; E\left[\left(\frac{S_T}{F_T}\right)^p\right] \;=\; E\left[e^{px}\right] \;=\; L(p) \tag{4.16}$$

$L(p)$ is obtained from the market smile through:

$$L(p) \;=\; \int_0^{\infty} e^{r(T-t)}\frac{d^2C_{KT}}{dK^2}e^{p\ln\frac{K}{F_T}}dK$$

where we have used the fact that the probability density of S_T is given by: $\rho(S_T) = e^{r(T-t)}\left.\frac{d^2C_{KT}}{dK^2}\right|_{K=S_T}$

Andrew Matytsin – see [73] – introduces a measure of moneyness z defined by:

$$z(K) \;=\; \frac{\ln\left(\frac{F_T}{K}\right)}{\widehat{\sigma}_{KT}\sqrt{T}} - \frac{\widehat{\sigma}_{KT}\sqrt{T}}{2} \tag{4.17}$$

Replacing C_{KT} with its expression as a function of $\widehat{\sigma}_{KT}$, he gives the following formula for $L(p)$:

$$L(p) \;=\; \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}e^{-p\left(\frac{\omega^2}{2}+z\omega\right)}\left(1+p\frac{d\omega}{dz}\right)dz \tag{4.18}$$

where $\omega(z) \;=\; \widehat{\sigma}_{K(z)T}\sqrt{T}$.

Let us introduce a p-dependent measure of moneyness, y, defined by $y = z + p\omega(z)$:

$$y(K) \;=\; \frac{\ln\left(\frac{F_T}{K}\right)}{\widehat{\sigma}_{KT}\sqrt{T}} + \left(p - \frac{1}{2}\right)\widehat{\sigma}_{KT}\sqrt{T} \tag{4.19}$$

For $p = 0$, $y\,(K)$ is the Black-Scholes d_2, while for $p = 1$, it is d_1. Mapping $K \to y$ maps $[0, +\infty]$ into $[-\infty, +\infty]$ and, most importantly, is monotonic.[7] Thus, $K\,(y, p)$ is well-defined.

From (4.18), performing a change of variable from z to y yields:

$$L(p) \;=\; \int_{-\infty}^{+\infty} dy \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} e^{\frac{p(p-1)}{2}\widehat{\sigma}^2_{K(y,p)T}}$$

where $\widehat{\sigma}_{K(y,p)T}$ is the implied volatility for "moneyness" y, that is for strike K such that y and K are related through (4.19).

Using now (4.16) and (4.15) we get the following direct relationship between vanilla and power-payoff implied volatilities:

$$e^{\frac{p(p-1)}{2}\widehat{\sigma}^2_p T} \;=\; \int_{-\infty}^{+\infty} dy \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} e^{\frac{p(p-1)}{2}\widehat{\sigma}^2_{K(y,p)T}} \tag{4.20}$$

which holds for all $p \in [0, 1]$. $\widehat{\sigma}_{K(y,p)T}$ is easily obtained numerically.[8]

Taking the limits $p \to 0$ and $p \to 1$ yields the implied volatilities of two practically important payoffs.

Implied volatility of the log contract

Let us take the limit $p \to 0$. For small p, $S^p = 1 + p \ln S$, thus $\lim_{p\to 0} \widehat{\sigma}_p$ is the implied volatility of the log contract, which is replicated with a density of vanilla options proportional to $\frac{1}{K^2}$ – see Section 3.1.2. It has the property that its Black-Scholes dollar gamma and vega are independent of S. It is closely related to the variance swap, extensively studied in Chapter 5.

Expanding each side of (4.20) at order one in p yields:

$$1 - \frac{p}{2}\widehat{\sigma}^2_0 T \;=\; \int_{-\infty}^{+\infty} dy \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \left(1 - \frac{p}{2}\widehat{\sigma}^2_{K(y,0)T} T\right)$$

which supplies the following formula for the log-contract implied volatility:

$$\widehat{\sigma}^2_{\ln S} \;=\; \int_{-\infty}^{+\infty} dy\, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \widehat{\sigma}^2_{K(y)T} \tag{4.21a}$$

$$y(K) \;=\; \frac{\ln\left(\frac{F_T}{K}\right)}{\widehat{\sigma}_{KT}\sqrt{T}} - \frac{\widehat{\sigma}_{KT}\sqrt{T}}{2} \tag{4.21b}$$

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[7] From Roger Lee's work we know that for $K \to 0$ and $K \to \infty$, $\widehat{\sigma}^2_{KT} T$ grows at most linearly in $\ln K$: $\widehat{\sigma}^2_{KT} T \propto a \ln K$ with $|a| < 2$. Using these two properties one easily shows that $\lim_{K\to 0} z(K) = +\infty$ and $\lim_{K\to\infty} z(K) = -\infty$.
In [46], Masaaki Fukasawa shows that the $K \to y$ mapping is monotonically decreasing for both $p = 0$ and $p = 1$. Because $y(K)$ is affine in p this implies that $y(K)$ is monotonic for all $p \in [0, 1]$ – I am indebted to Ling Ling Cao for this observation.

[8] Choose a set of strikes K_i. For each K_i, calculate the corresponding value y_i of y using (4.19) and record the couple $(y_i,\ \widehat{\sigma}_{K_i T})$. Then build an interpolation, for example a spline, of these couples to generate the function $\widehat{\sigma}_{K(y,p)T}$.

This expression was first published by Neil Chriss and William Morokoff in [32]. In comparison with formula (3.7), which expresses the price of the power payoff as a weighted integral of vanilla option prices, (4.21a) is less sensitive to numerical discretization of the integral.

For example, if $\widehat{\sigma}_{KT}$ is constant, equal to $\widehat{\sigma}_0$, a Gauss-Hermite quadrature yields $\widehat{\sigma}_{\ln S} = \widehat{\sigma}_0$ no matter how few points we use. In practice using about 10 points provides good accuracy.

Implied volatility of the S ln S contract

Now set $p = 1 - \varepsilon$ and take the limit $\varepsilon \to 0$. For small ε, $S^p = S - \varepsilon S \ln S$, thus $\lim_{p\to 1} \widehat{\sigma}_p$ is the implied volatility of the $S \ln S$ contract, which is replicated with a density of vanilla options proportional to $\frac{1}{K}$ – see Section 3.1.9.1.

This payoff has the property that its dollar gamma – hence its vega – is proportional to S. Proceeding as above, we get:

$$\widehat{\sigma}^2_{S\ln S} = \int_{-\infty}^{+\infty} dy \, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \widehat{\sigma}^2_{K(y)T} \tag{4.22a}$$

$$y(K) = \frac{\ln\left(\frac{F_T}{K}\right)}{\widehat{\sigma}_{KT}\sqrt{T}} + \frac{\widehat{\sigma}_{KT}\sqrt{T}}{2} \tag{4.22b}$$

4.3.2 Forward variances of power payoffs

Power payoffs are of the type $h\left(\frac{S_T}{F_T}\right)$ with h a convex (resp. concave) function for $p < 0$ or $p > 1$ (resp. $p \in]0, 1[$). They fall in the class considered in Section 2.2.2.2 for which (a) an implied volatility can be defined, (b) the convex order condition for implied volatilities (2.17) holds.

We then have:

$$(T_2 - t)\, \widehat{\sigma}^2_{pT_2} \geq (T_1 - t)\, \widehat{\sigma}^2_{pT_1} \tag{4.23}$$

This allows us to define *positive* forward variances ξ, either discrete, or continuous.

- Discrete forward variances $\xi^{pT_1T_2}$ are defined by:

$$\xi^{pT_1T_2} = \frac{(T_2 - t)\, \widehat{\sigma}^2_{pT_2} - (T_1 - t)\, \widehat{\sigma}^2_{pT_1}}{T_2 - T_1} \tag{4.24}$$

- Continuous forward variances ξ^{pT} are defined by:

$$\xi^{pT} = \frac{d}{dT}\left((T - t)\, \widehat{\sigma}^2_{pT}\right) \tag{4.25}$$

The set of ξ^{pT} for all T is called the variance curve for index p; these are the state variables whose dynamics we will model.

We could have defined as well forward variances for implied volatilities of vanilla options, for a given moneyness, in Section 4.1.1, but their unwieldiness makes it a poitnless exercise.[9]

4.3.3 The dynamics of forward variances

In what follows, we work with continuous forward variances – discrete forward variances will reappear in Chapter 7.8. Again, we work with zero interest rates without loss of generality, and omit the t subscripts for processes.

$$d\xi^{pT} = \lambda^{pT} dW^{pT} + \bullet\, dt$$

We determine the drift of ξ^{pT} so that the drift of Q^{pT} vanishes. The SDEs for ξ^{pT} and S are:

$$\begin{cases} dS = \overline{\sigma} S dW^S \\ d\xi^{pT} = \mu^{pT} dt + \lambda^{pT} dW^{pT} \end{cases} \tag{4.26}$$

with initial conditions $S_{t=0} = S_{t=0}^{\text{market}}$ and $\xi_{t=0}^{pT} = \xi_{t=0}^{pT\text{ market}}$. With zero interest rates, repos, Q^{pT} is given by:

$$Q^{pT} = S^p e^{\frac{p(p-1)}{2}\int_t^T \xi^{p\tau} d\tau} \tag{4.27}$$

In what follows we deal with a single variance curve, for a given index p – we omit it in the notation. The SDE for Q^T is given by:

$$\begin{aligned} \frac{dQ^T}{Q^T} = & \left[\frac{p(p-1)}{2}\left(\overline{\sigma}^2 - \xi^t\right) + \frac{p(p-1)}{2}\int_t^T \mu^\tau d\tau \right. \\ & \left. + \frac{p^2(p-1)}{2}\overline{\sigma}\int_t^T \lambda^\tau \rho^{S\tau} d\tau + \frac{p^2(p-1)^2}{8}\int_t^T\int_t^T dudv\rho^{uv}\lambda^u\lambda^v \right] dt \\ & + \left[p\overline{\sigma} dW^S + \frac{p(p-1)}{2}\int_t^T \lambda^\tau dW^\tau \, d\tau \right] \end{aligned} \tag{4.28}$$

where

$$\rho^{S\tau} = \frac{\langle dW^S dW^\tau \rangle}{dt}, \rho^{uv} = \frac{\langle dW^u dW^v \rangle}{dt}$$

[9]Strike K cannot be kept constant as the maturity is varied. Provided K scales linearly with the forward F_T: $K_T = \alpha F_T$, the convex order condition for option prices (2.9) holds and translates into the condition (2.15) on implied volatilities:

$$(T_2 - t)\,\widehat{\sigma}^2_{\alpha F_{T_2}, T_2} \geq (T_1 - t)\,\widehat{\sigma}^2_{\alpha F_{T_1}, T_1}$$

The counterpart of equation (4.25) for vanilla option implied volatilities would thus have read:

$$\xi_t^{\alpha, T} = \frac{d}{dT}\left((T-t)\,\widehat{\sigma}^2_{\alpha F_T, T}\right)$$

The first two lines in equation (4.28) are the drift of Q^T, which has to vanish for all T. Taking the derivative of this drift with respect to T yields:

$$\frac{p\,(p-1)}{2}\mu^T + \frac{p^2\,(p-1)}{2}\overline{\sigma}\lambda^T\rho^{ST} + \frac{p^2\,(p-1)^2}{4}\int_t^T \rho^{Tu}\lambda^T\lambda^u du \;=\; 0$$

from which we get the expression of μ^T, already given in [81]:

$$\mu^T \;=\; -\left(p\overline{\sigma}\lambda^T\rho^{ST} \;+\frac{p\,(p-1)}{2}\int_t^T \rho^{Tu}\lambda^T\lambda^u du\right) \tag{4.29}$$

Taking $T \to t$ in (4.28) leaves only the first term in the drift of Q^T: $\frac{p(p-1)}{2}\left(\overline{\sigma}^2 - \xi^t\right)$. We then get the extra condition:

$$\xi_t^t \;=\; \overline{\sigma}_t^2 \tag{4.30}$$

Thus, the short end of the variance curve is equal to the instantaneous variance of S, *for all values of* p.

Using now the expression of μ^T in (4.29) yields the final SDEs for S and ξ^T – reinstating the p and t indices:

$$dS_t \;=\; \sqrt{\xi_t^t}S_t dW_t^S \tag{4.31a}$$

$$d\xi_t^{pT} \;=\; -\left(p\sqrt{\xi_t^t}\lambda_t^T\rho^{ST} \;+\frac{p\,(p-1)}{2}\int_t^T \rho^{Tu}\lambda_t^T\lambda_t^u du\right)dt \;+\; \lambda_t^T dW_t^T \tag{4.31b}$$

4.3.4 Markov representation of the variance curve

Generally, the solution of (4.31b) requires evolving each forward variance ξ^T individually in a Monte-Carlo simulation – this is impractical. It may be that, for a well chosen covariance structure of forward variances, each ξ^T can be written as a function of a finite number of state variables, i.e. possesses a Markov representation. Then one only needs to evolve a finite set of state variables to generate the full variance curve at time t.

The question of building Markov representations has been especially addressed in the context of yield curve modeling. Formula (4.27) for Q^{pT} is similar to the expression of a zero-coupon bond, up to the factor S^p, with ξ^T playing the role of the forward rate for date T. It is then tempting, following the work of Oren Cheyette in [30] in the context of the HJM framework for the yield curve, to try and derive a class of solutions to (4.31b) that have a Markov representation.

We carry out the typical derivation one would go through in a yield curve context – for the sake of it, since we must warn the reader that the outcome is fruitless in the case of power payoffs.

While we do not carry the p index, we will now carry t subscripts. Let us assume, following [30], that the covariance structure of the ξ_t^T is such that SDE (4.31b) for

ξ^T has the following particular form:

$$dS_t = \sqrt{\xi_t^t} S_t dW_t^S \tag{4.32a}$$

$$d\xi_t^T = \mu_t^T dt + \sum_{i=0}^{n} \alpha_i (T) \beta_{it} dW_t^i \tag{4.32b}$$

where W_i , $i = 0 \ldots n$ are n correlated Brownian motions, β_{it} are processes, and $\alpha_i (T)$ are functions of T. The correlations of W_t^i and W_t^j is ρ_{ij} and the correlation of W_t^i and W_t^S is ρ_{iS}.

The volatility structure in the above equations is not inapt. For example, taking $\beta_{it} = e^{k_i t}$, a function, and $\alpha_i (T) = \alpha_i e^{-k_i T}$ yields:

$$d\xi_t^T = \mu^T dt + \sum_{i=0}^{n} \alpha_i e^{-k_i (T-t)} dW_t^i$$

The dynamics of the ξ^T is time-homogeneous – volatilities and correlations of $\xi^T, \xi^{T'}$ are only a function of $T - t$ and $T - t, T' - t$, respectively – the "volatilities" of ξ^T being expressed as a linear combination of exponentials.

Mirroring the derivation in [30], we integrate (4.32b) and try to express ξ_t^T as a function of as few processes as possible. We have:

$$\begin{aligned}
\xi_t^T &= \xi_0^T - p \Sigma_i \int_0^t \alpha_i(T) \beta_{i\tau} \rho_{iS} \sqrt{\xi_\tau^\tau} d\tau \\
&\quad - \frac{p\,(p-1)}{2} \int_0^t \left(\int_\tau^T \Sigma_{ij} \rho_{ij} \alpha_i(T) \beta_{i\tau} \alpha_j(u) \beta_{j\tau} du \right) d\tau + \Sigma_i \int_0^t \alpha_i(T) \beta_{i\tau} dW_\tau^i \\
&= \xi_0^T + \Sigma_i \alpha_i(T) \Bigg[-p \rho_{iS} \int_0^t \beta_{i\tau} \sqrt{\xi_\tau^\tau} d\tau \\
&\qquad - \frac{p\,(p-1)}{2} \Sigma_j \int_0^t \rho_{ij} \beta_{i\tau} \beta_{j\tau} \left(A_j(T) - A_j(\tau) \right) d\tau + \int_0^t \beta_{i\tau} dW_\tau^i \Bigg]
\end{aligned}$$

where we introduce $A_j(\tau) = \int_0^\tau \alpha_j(u) du$.

We now define process $B_{ij,t} = \rho_{ij} \int_0^t \beta_{it} \beta_{jt}$. A little manipulation yields:

$$\begin{aligned}
\xi_t^T &= \xi_0^T + \Sigma_i \alpha_i(T) \left[-\frac{p\,(p-1)}{2} \Sigma_j B_{ij,t} \left(A_j^T - A_j^t \right) \right. \\
&\quad \left. + \int_0^t \left(-p \rho_{iS} \beta_{i\tau} \sqrt{\xi_\tau^\tau} d\tau - \frac{p\,(p-1)}{2} \Sigma_j B_{ij,t} \alpha_j(\tau) d\tau + \beta_{i\tau} dW_\tau^i \right) \right]
\end{aligned}$$

On top of processes $B_{ij,t}$ we thus need to define n processes x_{it}:

$$\begin{aligned}
x_{i0} &= 0 \\
dx_{it} &= -p \rho_{iS} \beta_{it} \sqrt{\xi_t^t} dt - \frac{p\,(p-1)}{2} \Sigma_j B_{ij,t} \alpha_j(t) dt + \beta_{it} dW_t^i
\end{aligned}$$

The variance curve is given at time t by:

$$\xi_t^T = \xi_0^T + \Sigma_i \alpha_i(T) \left[-\frac{p(p-1)}{2} \Sigma_j B_{ij,t} \left(A_j^T - A_j^t \right) + x_{it} \right] \tag{4.33}$$

Setting $T = t$ gives the expression of the short end of the curve – the instantaneous variance of S_t:

$$\xi_t^t = \xi_0^t + \Sigma_i \alpha_i(t) x_{it} \tag{4.34}$$

Thus, in a Monte-Carlo simulation of our model, we only need to evolve processes x_{it}, $V_{ij,t}$ and S_t, according to the following SDEs:

$$\begin{aligned} dx_{it} &= -p\rho_{iS}\beta_{it}\sqrt{\xi_0^t + \Sigma_i \alpha_i(t) x_{it}}\,dt - \frac{p(p-1)}{2}\Sigma_j B_{ij,t}\alpha_j(t)dt \qquad (4.35a)\\ &\quad + \beta_{it} dW_t^i \\ dB_{ij,t} &= \rho_{ij}\beta_{it}\beta_{jt}\, dt \qquad (4.35b)\\ dS_t &= \sqrt{\xi_0^t + \Sigma_i \alpha_i(t) x_{it}}\; S_t\, dW_t^S \qquad (4.35c) \end{aligned}$$

The variance curve at time t is given by (4.33). Inspection of equations (4.35) shows that processes β_{it} can depend arbitrarily on the x_{it} and $V_{ij,t}$, thus β_{it} can, for example, be an arbitrary function of the variance curve ξ_t at time t. This would generate the equivalent of a “local volatility” model for the ξ_t^T.

Unlike forward rates though, the instantaneous variance cannot be negative. Looking at the SDE for processes x_i above it is not clear that it is possible to define processes β_i and functions α_i that ensure that $\xi_t^t = \xi_0^t + \Sigma_i \alpha_i(t) x_{it} \geq 0$.[10]

The conclusion is that, unfortunately the ansatz (4.32b) used in [30] cannot be transposed to the framework of forward variances. This does not mean there are no low-dimensional Markov representations of the variance curve.

Indeed, any stochastic volatility model written on the instantaneous variance $V_t = \xi_t^t$ does provide a Markov representation of the variance curve for any value of p, though it may not be explicit. Think for example of the Heston model.

We refer the reader to Chapter 7 for examples of models with low-dimensional Markov representations for forward variances associated to $p \to 0$.

4.3.5 Dynamics for multiple variance curves

Given an initial variance curve $\xi_{t=0}$ for a particular value p^*, SDEs (4.31) generate the joint dynamics of $(S_t, \xi_t^{p^*})$. For a given market smile, we will generally have a

[10] Why not assume a lognormal dynamics for ξ_t^T: $d\xi_t^T = \xi_t^T \left(\mu_t^T dt + \Sigma_i \alpha_i(T) \beta_{it} dW_t^i \right)$ and look for a Markov representation of $\ln \xi_t^T$? We encourage the reader to try for herself; there does not seem to be a solution unless $p = 0$ or $p = 1$.

set of values of p for which prices of power payoffs are finite, hence the $\xi_{t=0}^{pT}$ are well-defined. Is it possible to generate a joint dynamics for S_t and a set of ξ_t^p?

A solution to SDEs (4.31) for a given p^* provides the full dynamics of S_t, hence prices for power payoffs and a dynamics for variance curves ξ_t^p with $p \neq p^*$. If our objective is to be able to independently set the initial values of variances curves for multiple values of p, this implies that information about initial curves $\xi_{t=0}^p$ has to be embedded in the SDE (4.31b) for $\xi_t^{p^*}$. It is not clear how this can be done practically.

The joint dynamics of multiple variance curves has to comply with condition (4.30) which expresses that the short ends of variance curves collapse one onto another at all times, as ξ_t^{pt} is the instantaneous variance of S_t. The author does not know of an example of direct modeling of the joint dynamics for multiple curves that is able to calibrate to market prices of power payoffs. Obviously the local volatility model provides a solution – albeit not explicit and not very exciting as it is driven by a single Brownian motion.

Even though we are not able to handle the dynamics of multiple curves, we are able to explicitly construct the joint dynamics of S_t and the variance curve, for a particular value p^*. We generate a dynamics for the full volatility surface by modeling the dynamics of one particular variance curve $\xi_t^{p^*}$. This produces a dynamics for other variance curves ξ_t^p with $p \neq p^*$ that obeys (4.31b), except we cannot set their initial condition. We will generally have $\xi_{t=0}^p \neq \xi_{t=0}^{p\text{market}}$: our models will not provide exact calibration to the vanilla smile.[11]

4.3.6 The log contract, again

Consider the payoff $\frac{S^p-1}{p}$: it has the same implied volatility as the power payoff of index p. Taking the limit $p \to 0$:

$$\lim_{p\to 0} \frac{S^p - 1}{p} \;=\; \ln S$$

We have already made this observation in Section 4.3.1 and have derived an expression for the implied volatility of the log contract, in case there are no cash-amount dividends.

We first encountered the log contract in the discussion of cliquet hedges in Chapter 3. Using the same notation as in Section 3.1.4, we simply denote by $\widehat{\sigma}_T$ the implied volatility of the log contract of maturity T and by ζ_t the associated variance curve:

$$\zeta_t^T \;\equiv\; \xi^{p=0,T} \;=\; \frac{d}{dT}\left((T-t)\,\widehat{\sigma}_T^2\,(t)\right)$$

Log contracts have finite prices: they do not require anything beyond the non-abitrageability of the smile, thus $\widehat{\sigma}_T$ is always well-defined. Power payoffs satisfy the

[11] At least we are able to calibrate exactly the term structure of implied volatilities $\widehat{\sigma}_{p^*T}$. In the case of implied volatilities of vanilla options, we were not even able to handle the dynamics of *one* $\widehat{\sigma}_{KT}$, let alone a term structure $\widehat{\sigma}_{K_T T}$.

convex order condition, thus ζ_t^T is positive. Taking the limit $p \to 0$ in (4.31) yields the following joint dynamics for (S_t, ξ_t):

$$\begin{cases} dS_t &= \sqrt{\zeta_t^t} S_t dW_t^S \\ d\zeta_t^T &= \lambda_t^T dW_t^T \end{cases} \tag{4.36}$$

Thus, forward variances associated to log contracts have no drift.

In practice, log contracts themselves are not traded, as vanilla options are not traded over a sufficiently wide range of strikes to allow for exact replication. Moreover, our analysis does not carry over to the case of dividends with fixed cash amounts – which cannot be represented by a proportional yield q – as the property that $E[S_{T_2}|S_{T_1}] = \frac{F_{T_2}}{F_{T_1}}$ needed to prove the convex order condition (4.23) no longer holds.

Luckily, closely related instruments known as variance swaps are traded; (4.36) will still hold, except the ζ_t^T will be replaced by variance swap forward variances – they are the basic building blocks in the models of Chapter 7.

Before we do this, we pause to study variance swaps in detail.

Chapter's digest

▶ Hoping to construct a market model for vanilla options by modeling implied volatilities of vanilla options directly is a dead end.

▶ There is a one-to-one mapping between a non-arbitrageable vanilla smile and its corresponding local volatility function. Morover, no-arbitrage conditions simply translate in the requirement that local volatilities be real. It is then tempting to specify a dynamics for the local volatility function. It turns out, however, that the drift of local volatilities is non-local and involves forward transition densities, thus is computationally too expensive. Again, we reach an impasse – rather than trying to model the dynamics of implied volatilities of vanilla payoffs – or their associated local volatilities – are there other types of convex payoffs whose implied volatilities are less unwieldy objects?

▶ One good candidate is the family of power payoffs $\left(\frac{S_T}{F_T}\right)^p$. For each value of p such that the market price of the corresponding power payoff is finite for all T, a term structure of forward variances ξ^{pT} can be defined. Setting the volatilities of the ξ^{pT} determines their drifts. For general values of p, it is not clear that there exist particular forms of the volatilities of the ξ^{pT} that give rise to a Markov representation of the variance curve. For $p \to 0$, however, the drift of ξ^{pT} vanishes. $\xi^{p=0T}$ are forward variances associated to the term structure of log contracts, which are closely related to variance swaps.