

Simulating Deformable Objects

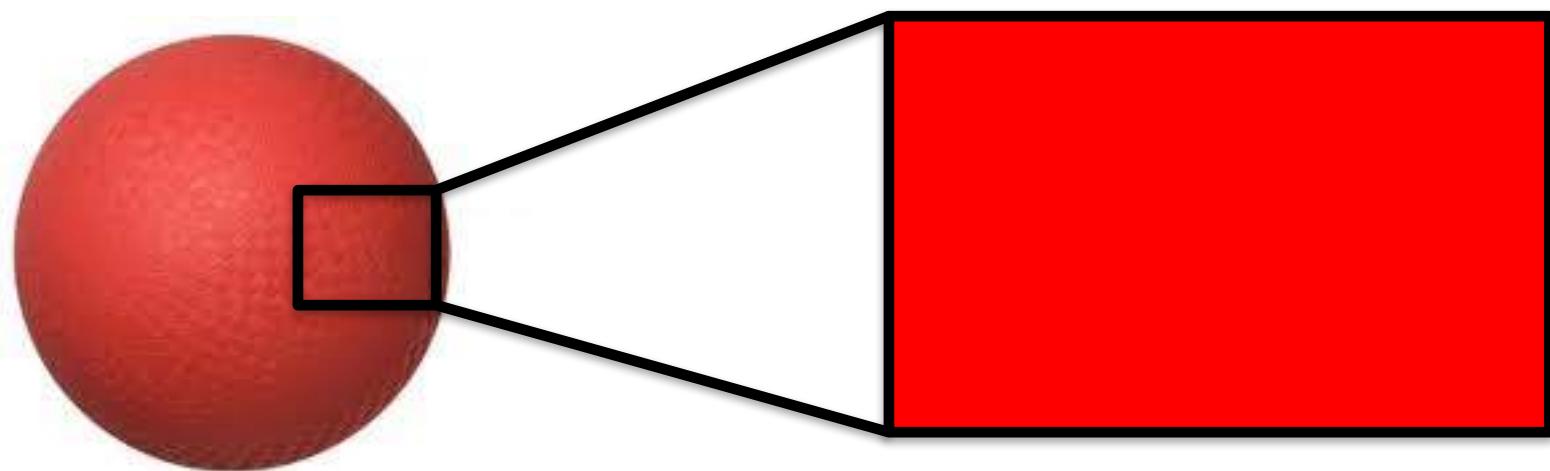
David Levin

Department of Computer Science

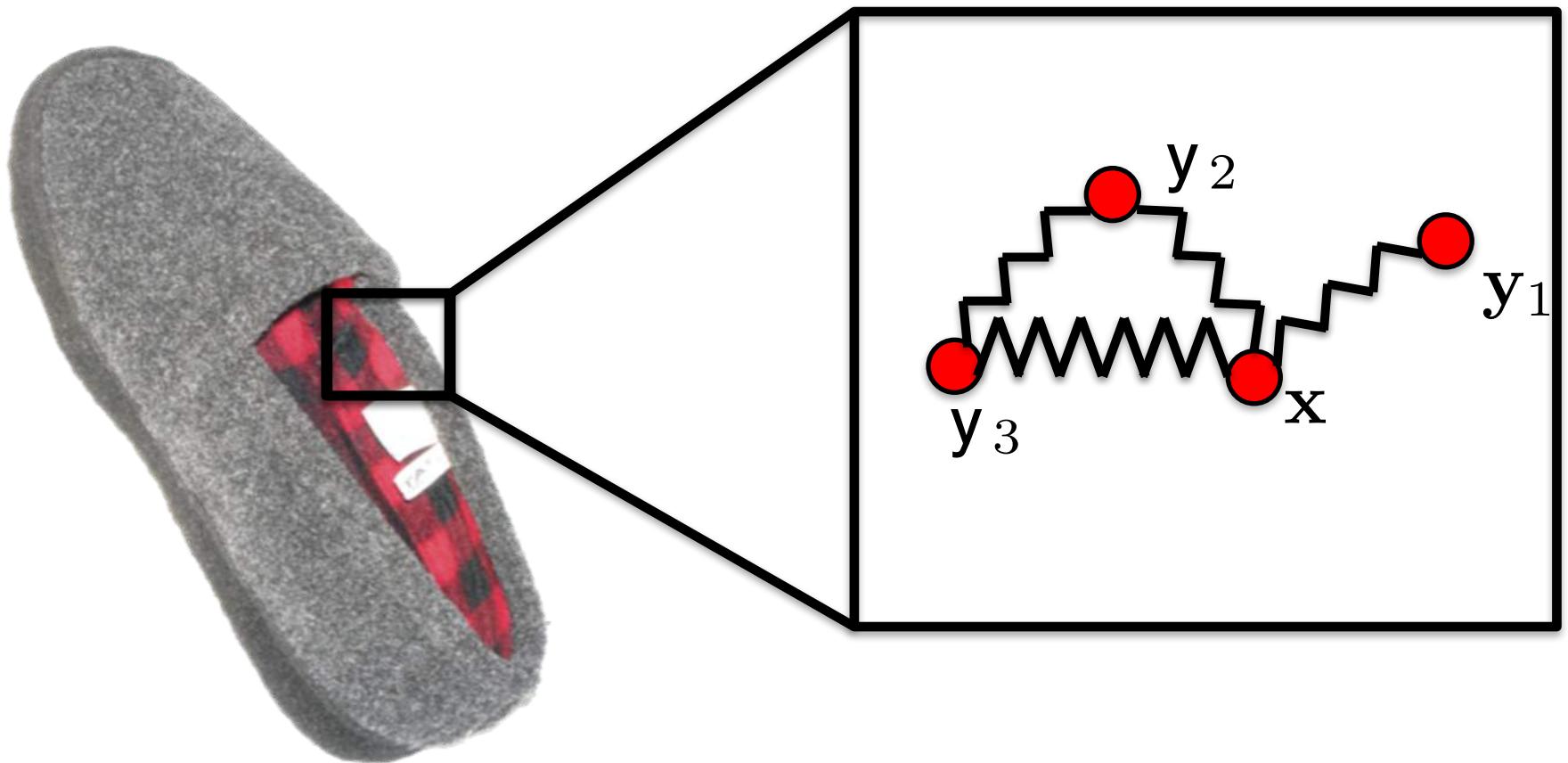
Class Schedule for the Next Few Weeks

- For undergraduates: Graduate courses don't have reading week
- We will have a lecture November 7th.
- No lecture October 24th, I'm travelling.

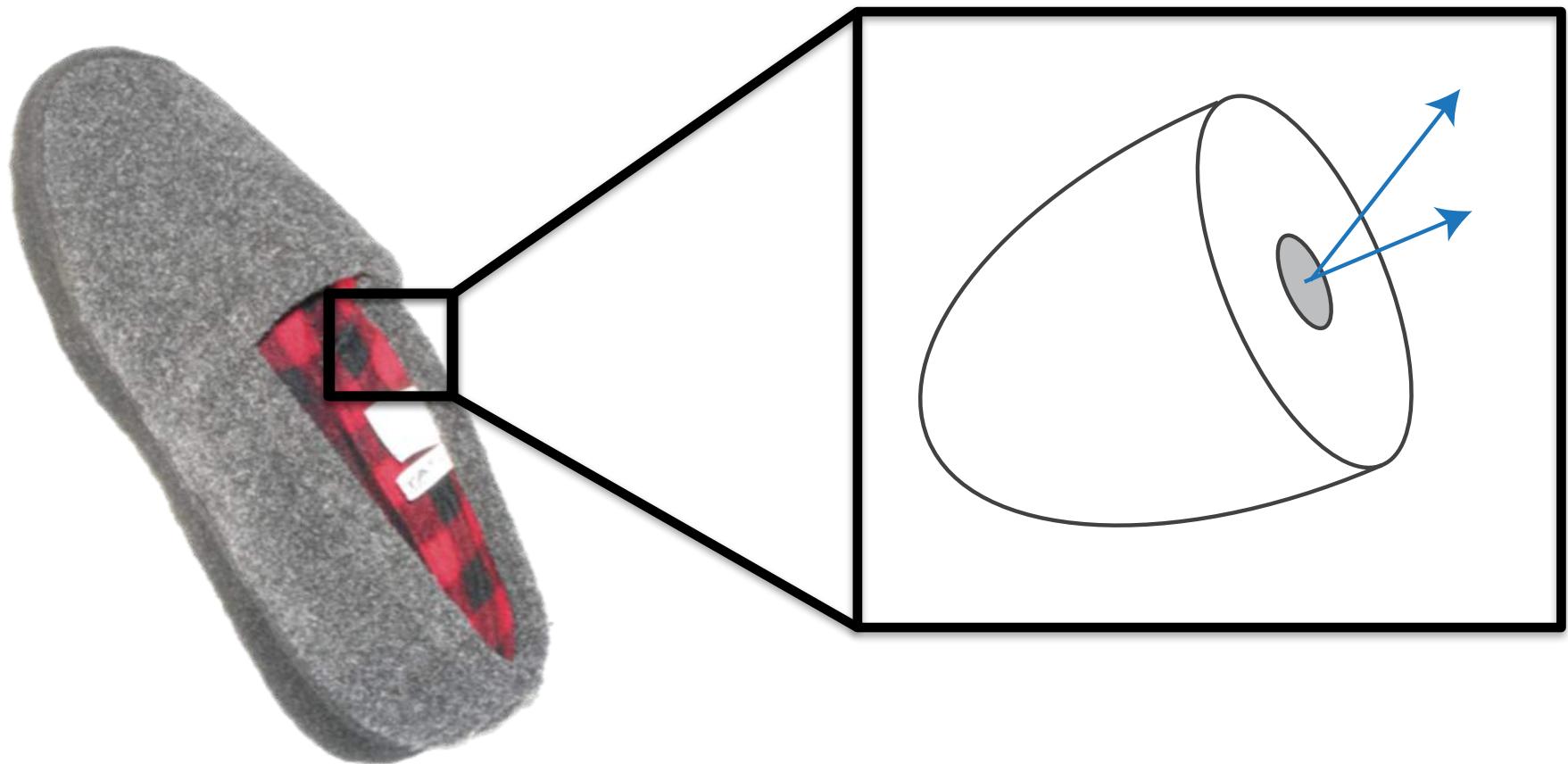
An Introduction to Continuum Mechanics



An Example



An Example

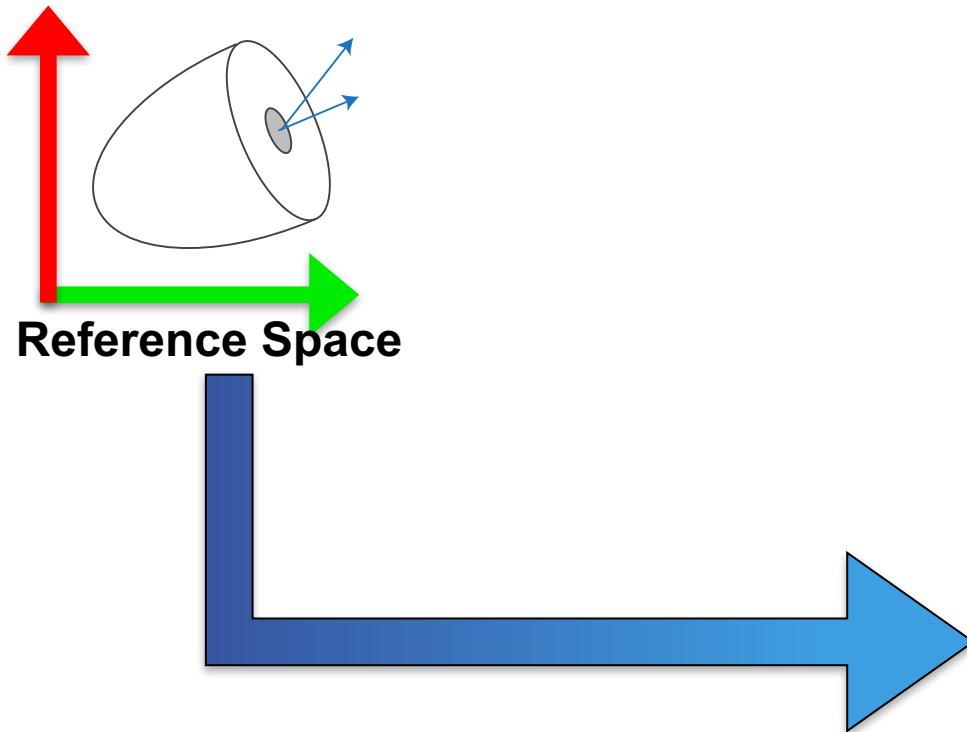


Requirements for Continuum Mechanics

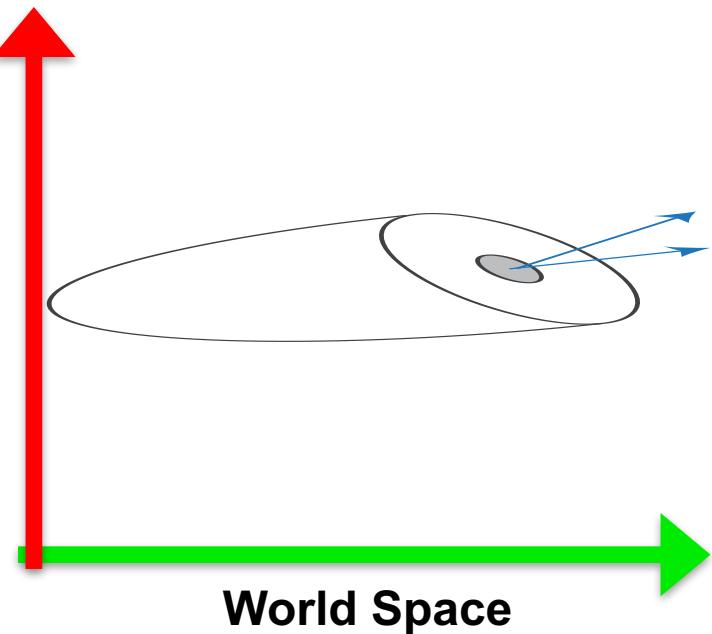
1. Material Model
2. *Measure of Deformation*

Continuum Mechanics: Deformation

- Defining a measure of deformation:

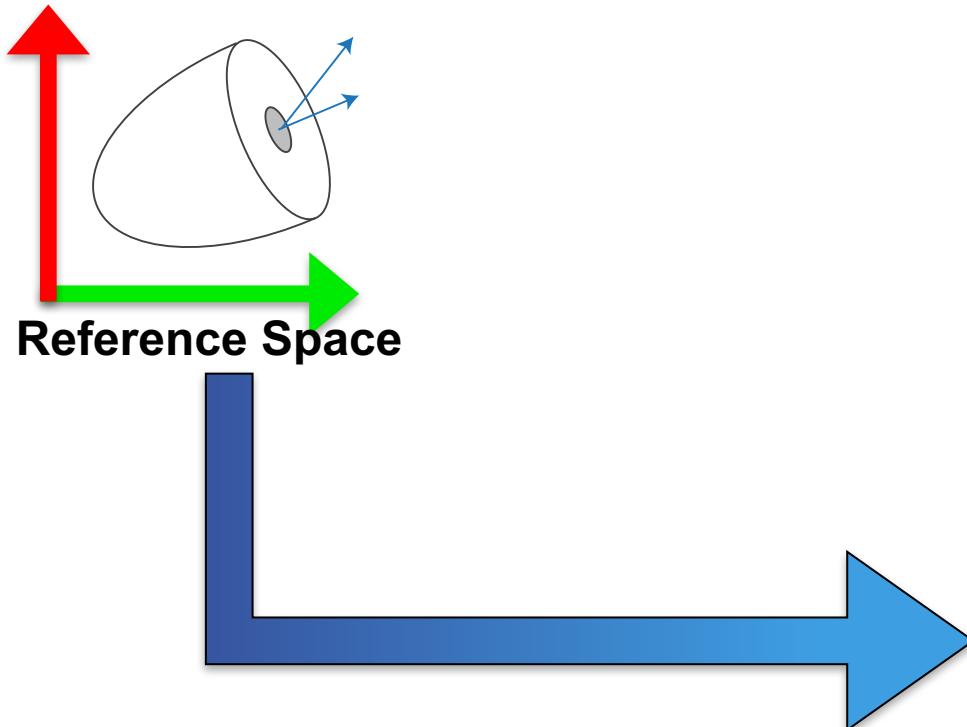


$$\text{World}_{\text{Ref}} \mathbf{x} = \text{World}_{\text{Ref}} \phi(^{\text{Ref}} \mathbf{x})$$

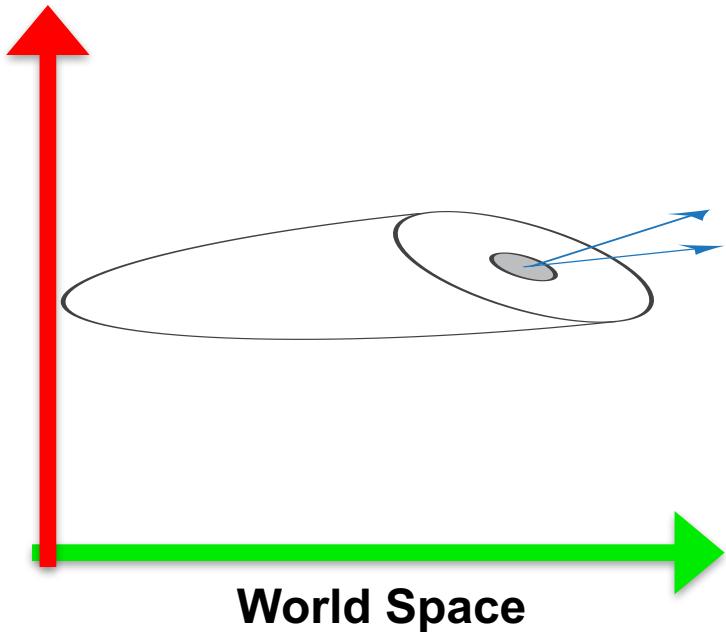


Continuum Mechanics

- We will leave our mapping undefined (for now)

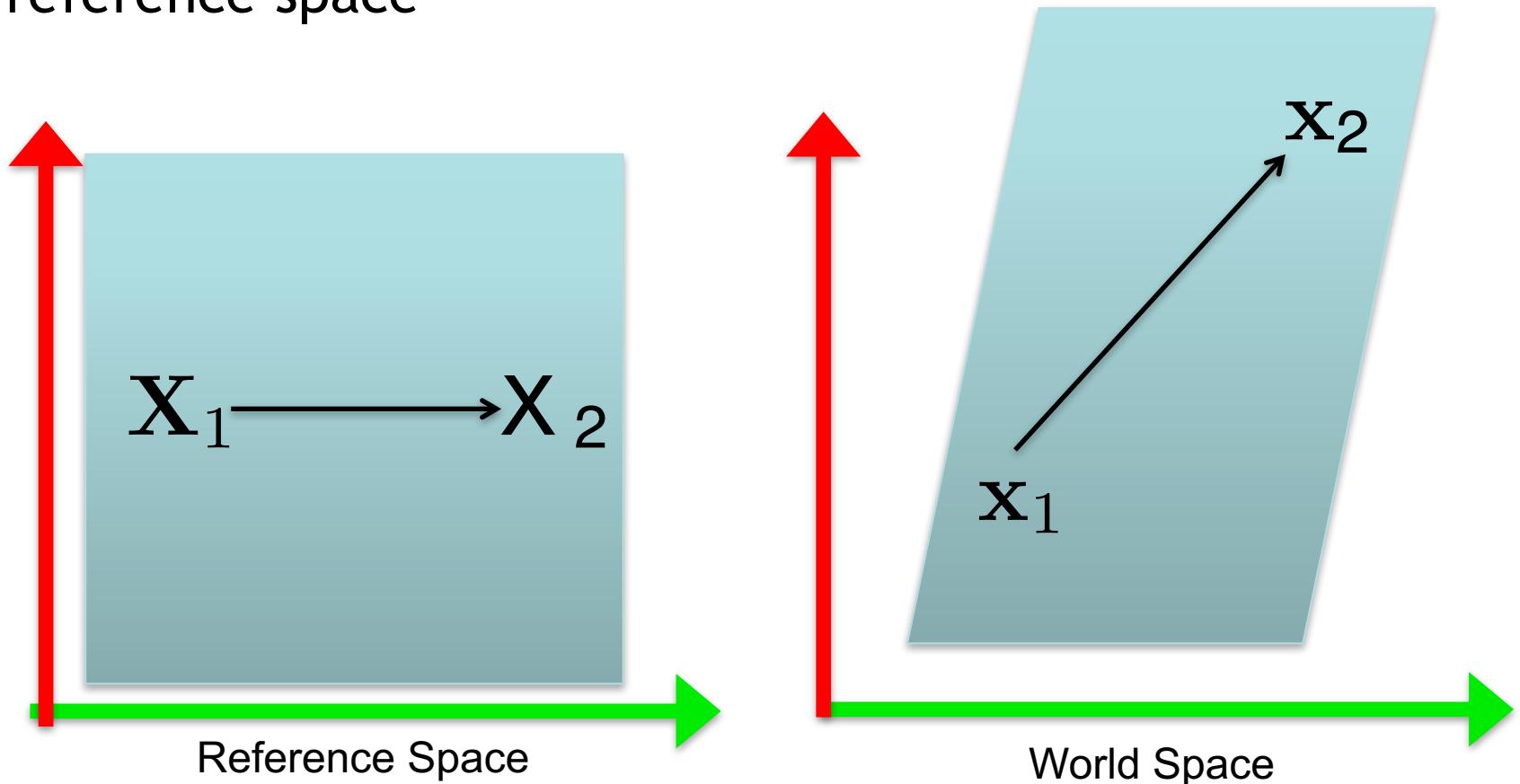


$$\text{World}_{\text{Ref}} \mathbf{x} = \text{World}_{\text{Ref}} \phi(^{\text{Ref}} \mathbf{x})$$



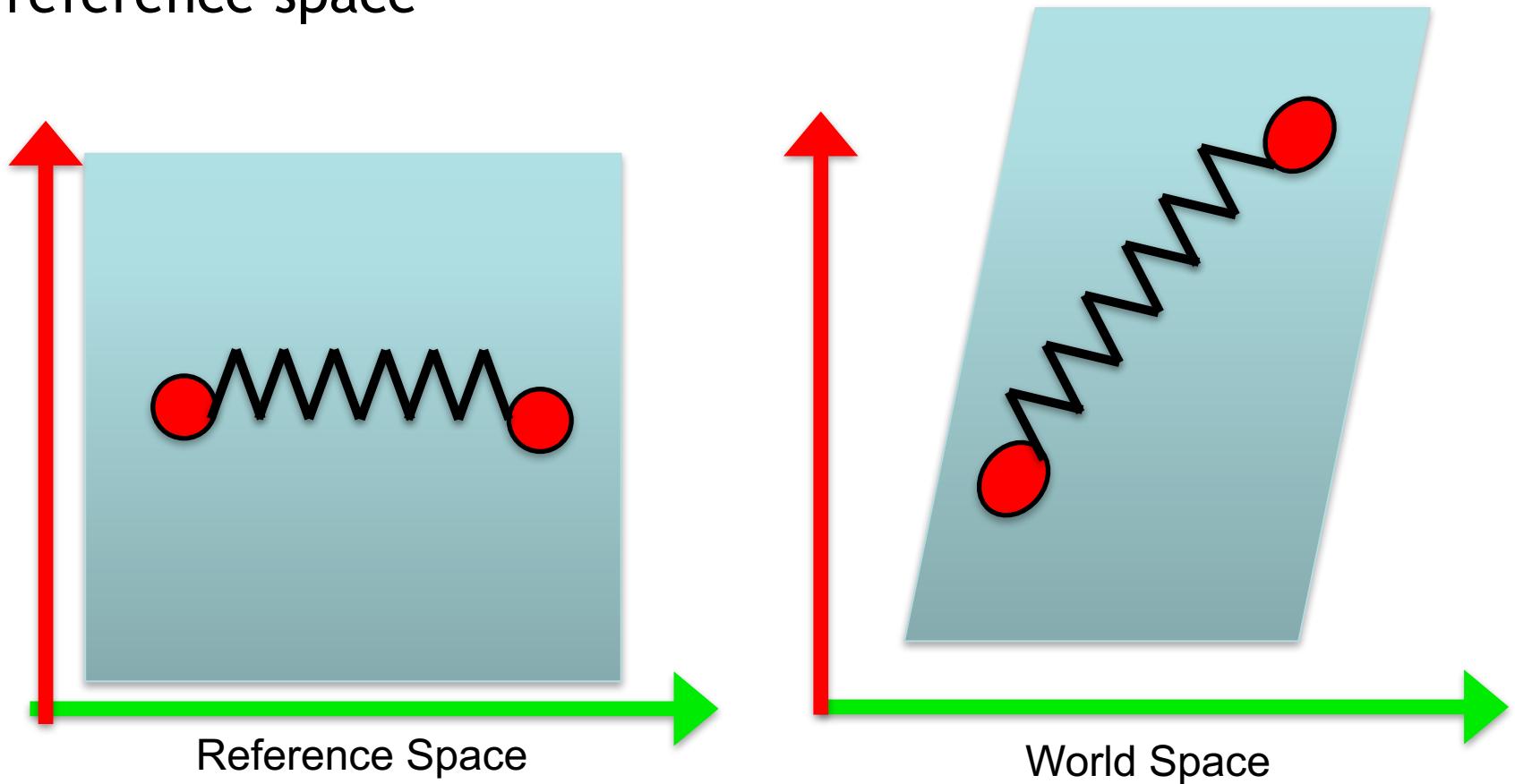
Continuum Mechanics: Deformation

- Consider the effect of $\frac{World}{Ref} \phi$ on a single vector in our reference space



Continuum Mechanics: Deformation

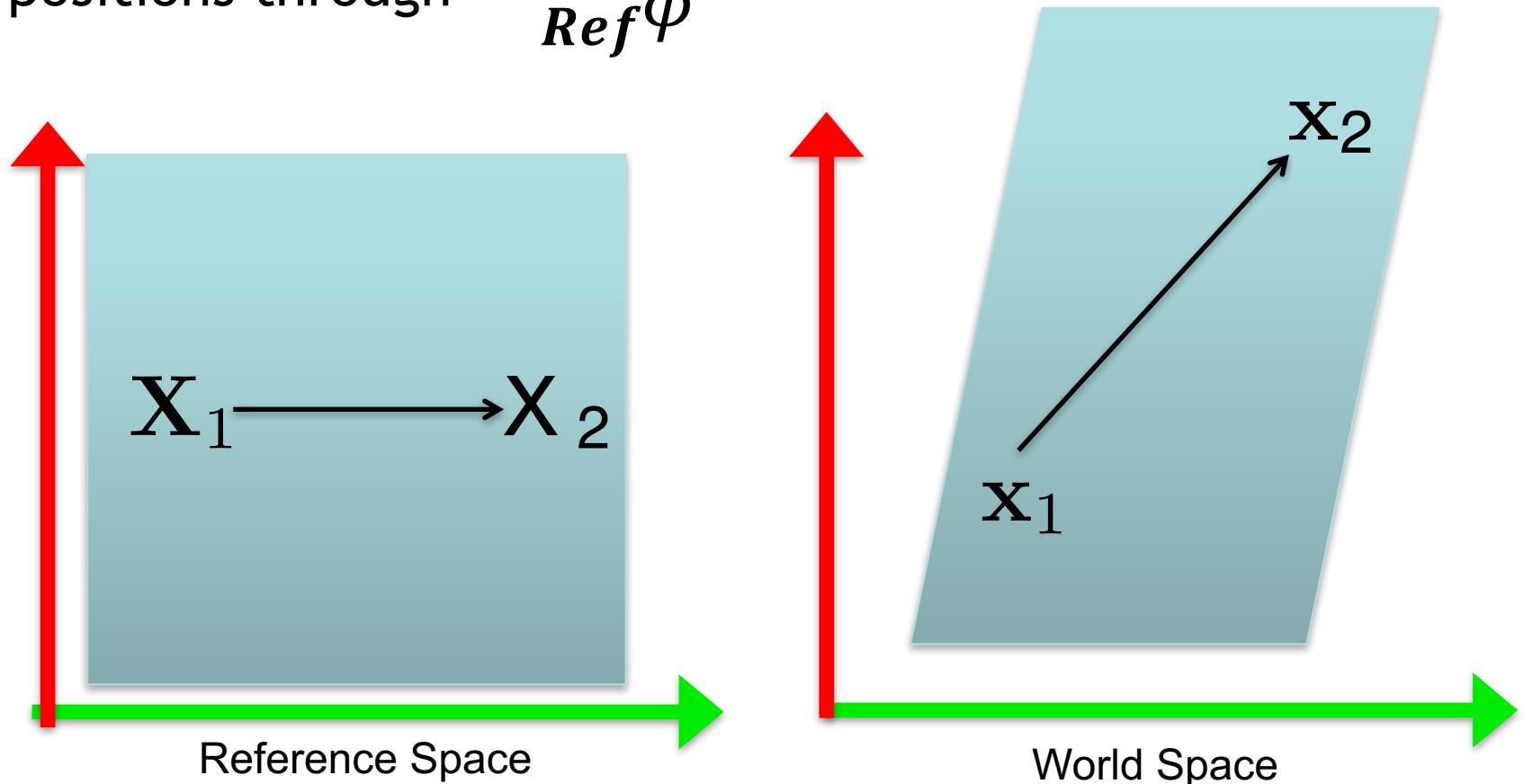
- Consider the effect of $\frac{World}{Ref} \phi$ on a single vector in our reference space



We are pretending we have springs EVERYWHERE in our object!!!

Continuum Mechanics: Deformation

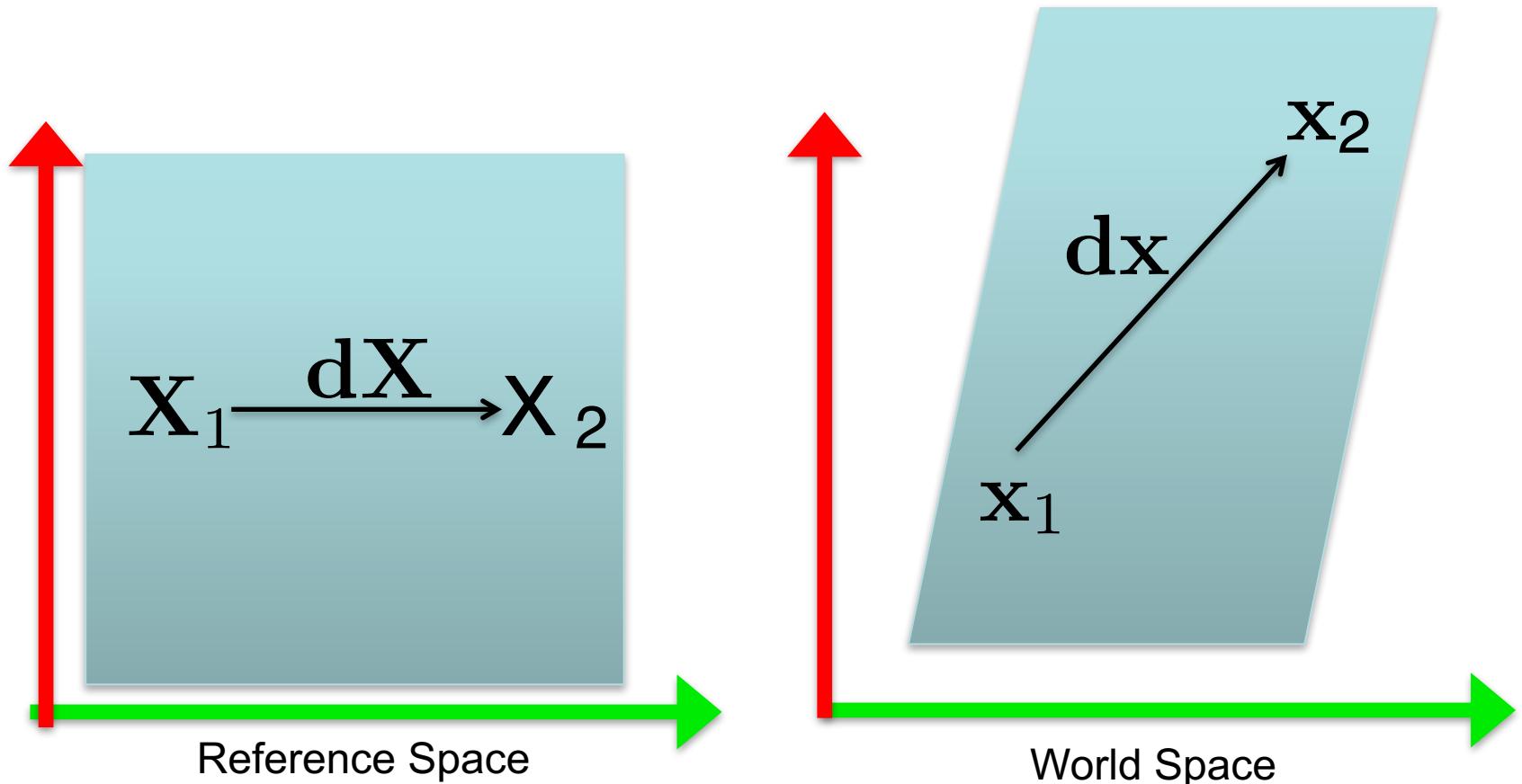
- New positions are given by passing the reference positions through ϕ



$$\mathbf{x}_2 = \phi(\mathbf{X}_2)$$

Continuum Mechanics: Deformation

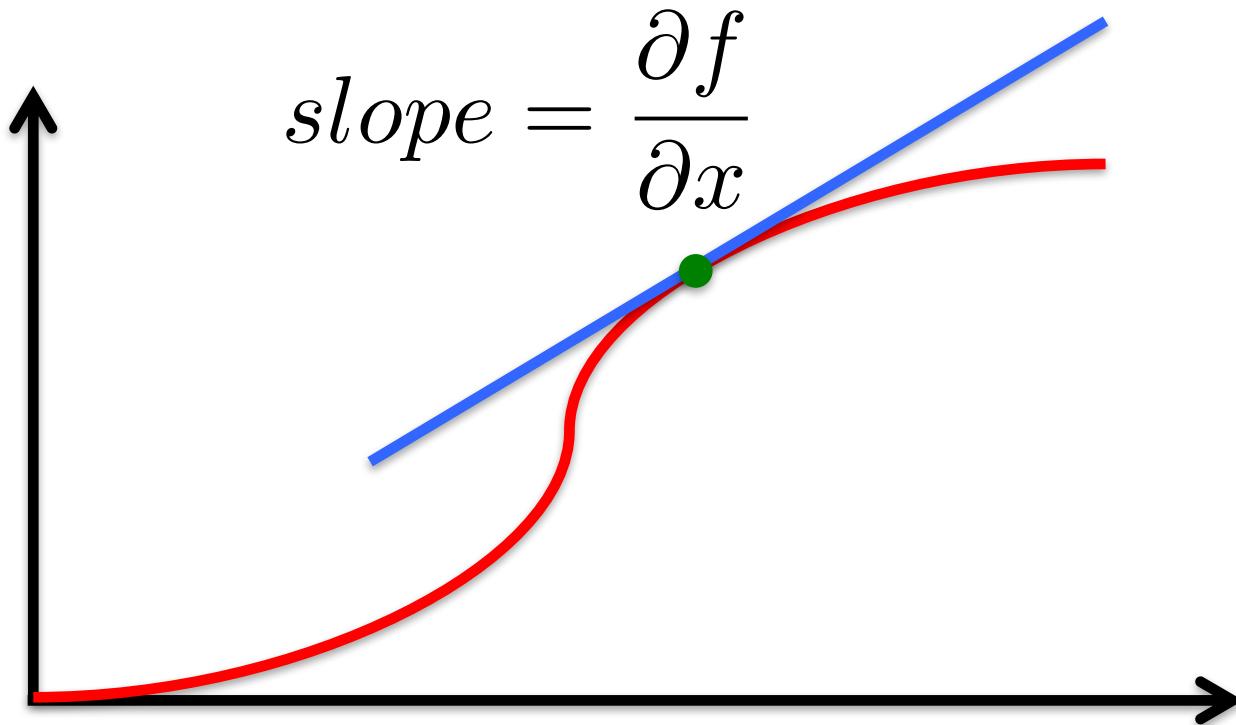
- Just rephrasing so we can see the spring vector



$$\mathbf{x}_1 + d\mathbf{x} = \phi(\mathbf{X}_1 + d\mathbf{X})$$

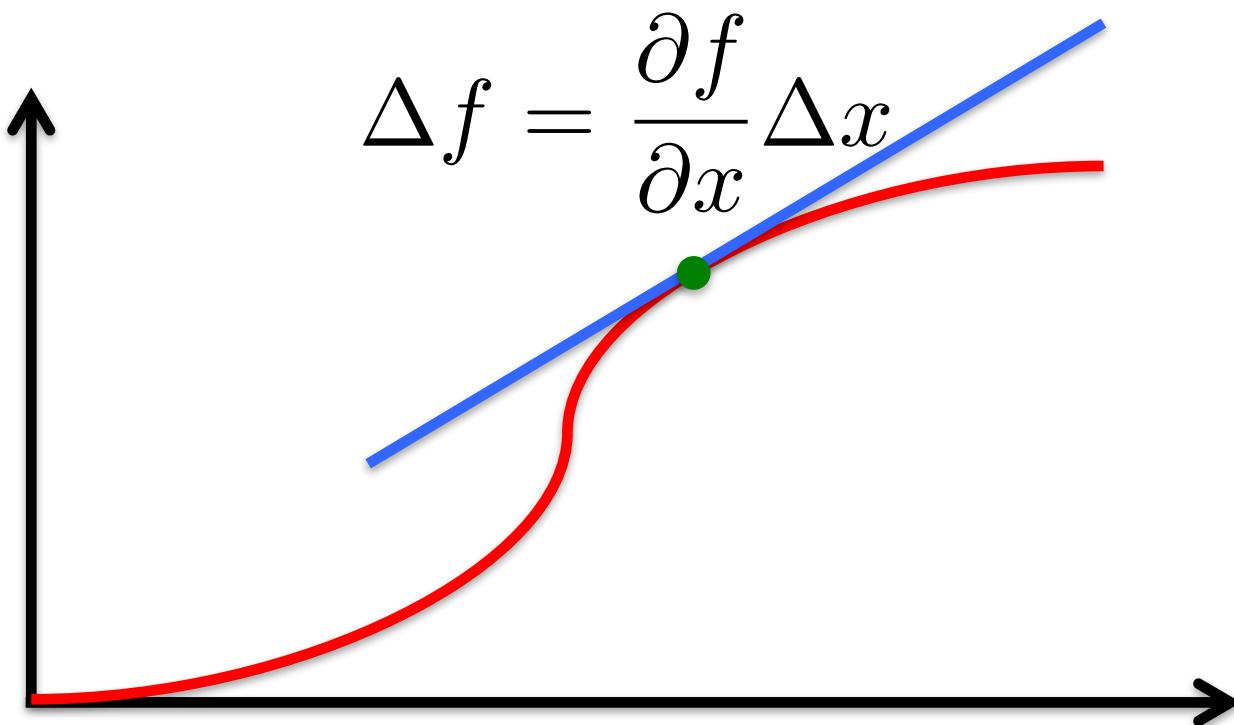
An Aside: Taylor Expansion

- Approximate small change in a non-linear function



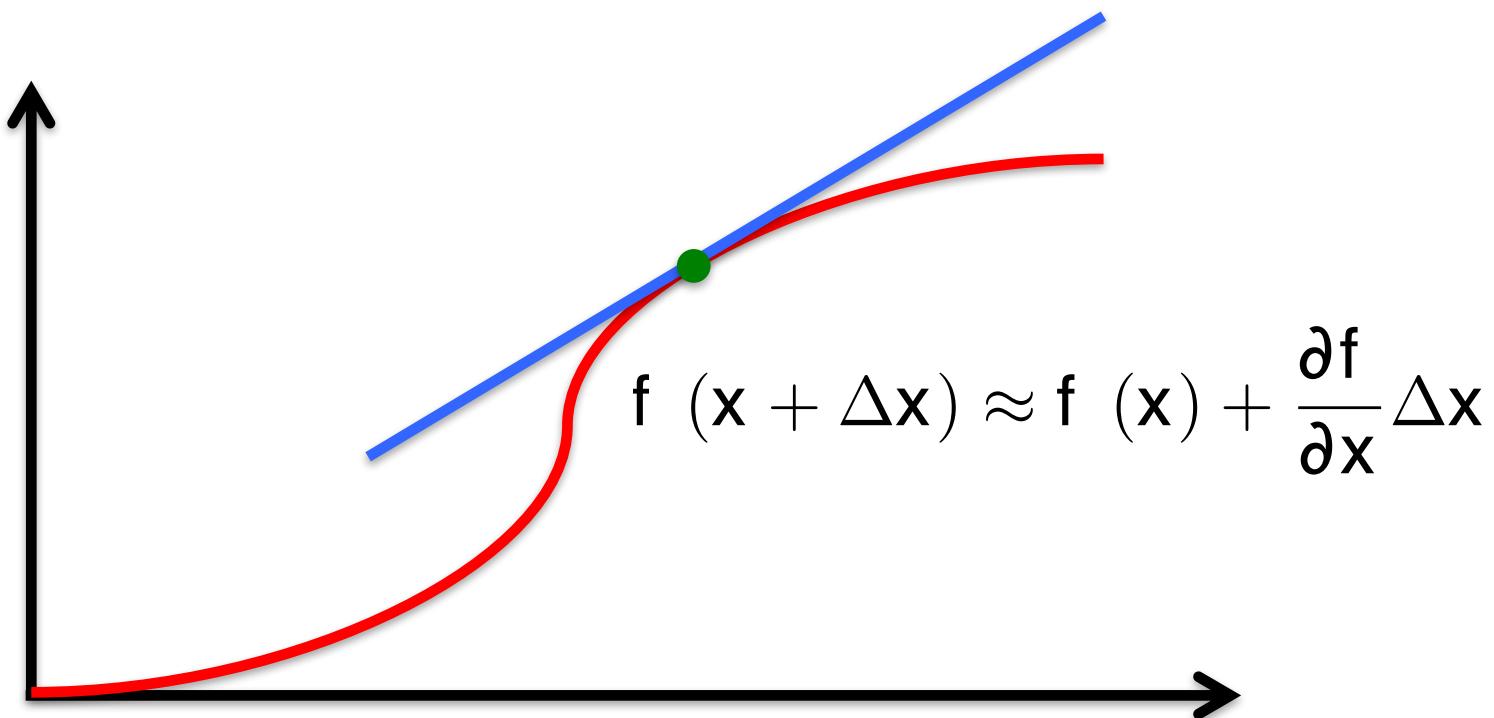
An Aside: Taylor Expansion

- Approximate small change in a non-linear function



An Aside: Taylor Expansion

- Approximate small change in a non-linear function



Taylor Expansion: Multi-dimensional Functions

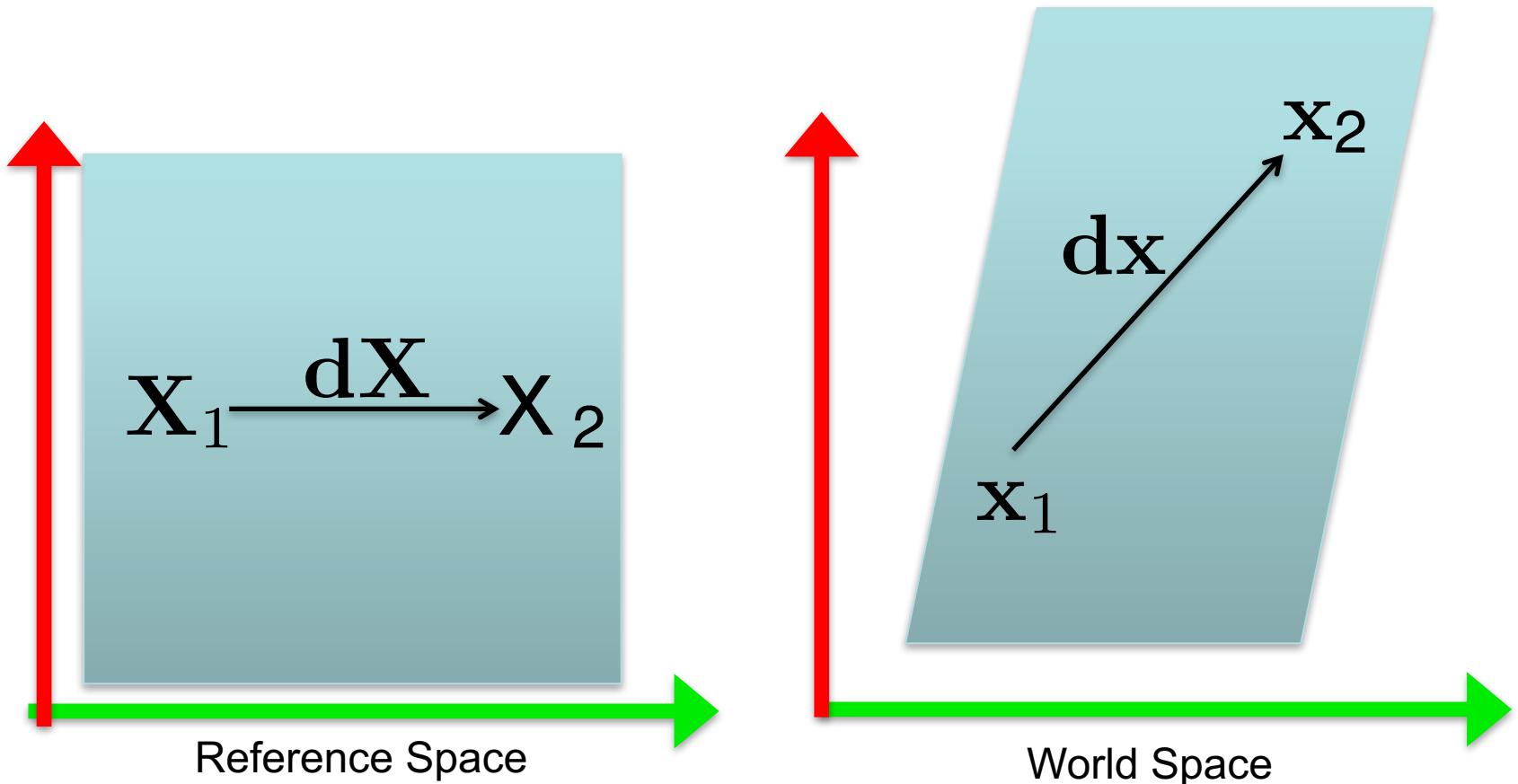
- Almost exactly the same!

$$f(x + \Delta x) \approx f(x) + \boxed{\frac{\partial f}{\partial x} \Delta x}$$

Gradient Matrix or “Jacobian”

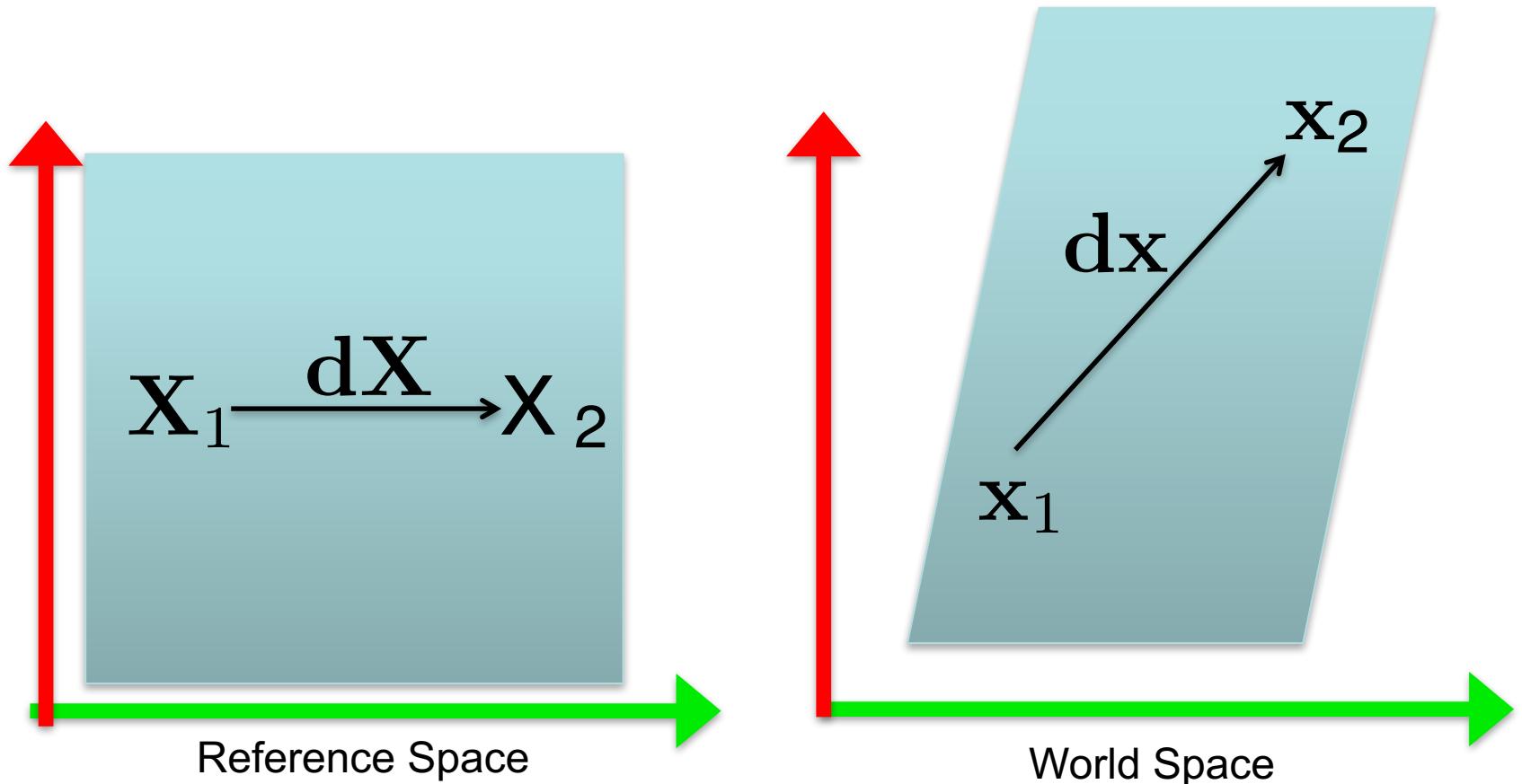
Continuum Mechanics: Deformation

- Apply Taylor Expansion



$$\mathbf{x}_1 + d\mathbf{x} = \phi(\mathbf{X}_1 + d\mathbf{X})$$

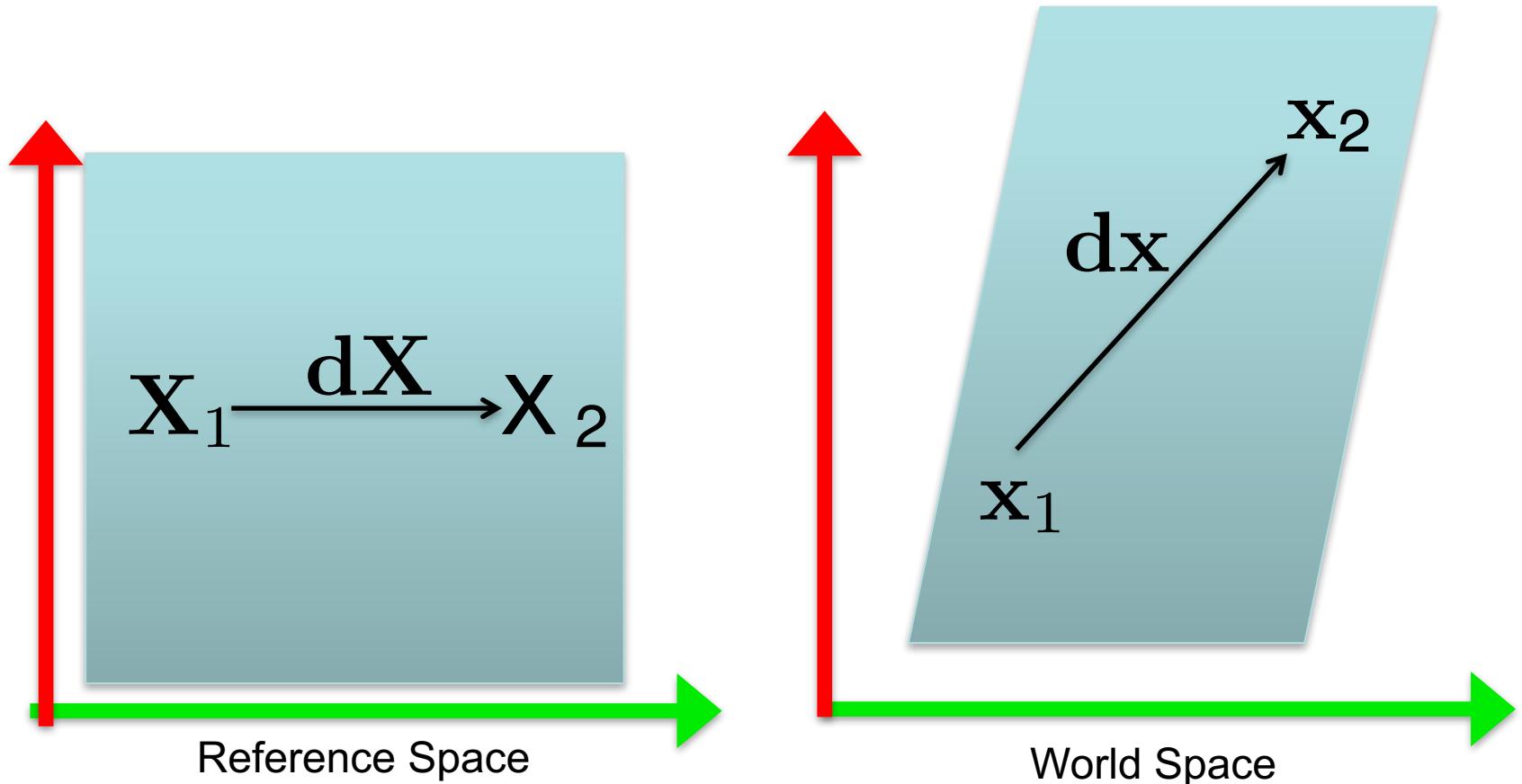
Continuum Mechanics: Deformation



$$\mathbf{x}_1 + \mathbf{d}\mathbf{x} \approx \phi(\mathbf{X}_1) + \frac{\partial \phi}{\partial \mathbf{X}} \mathbf{d}\mathbf{X}$$

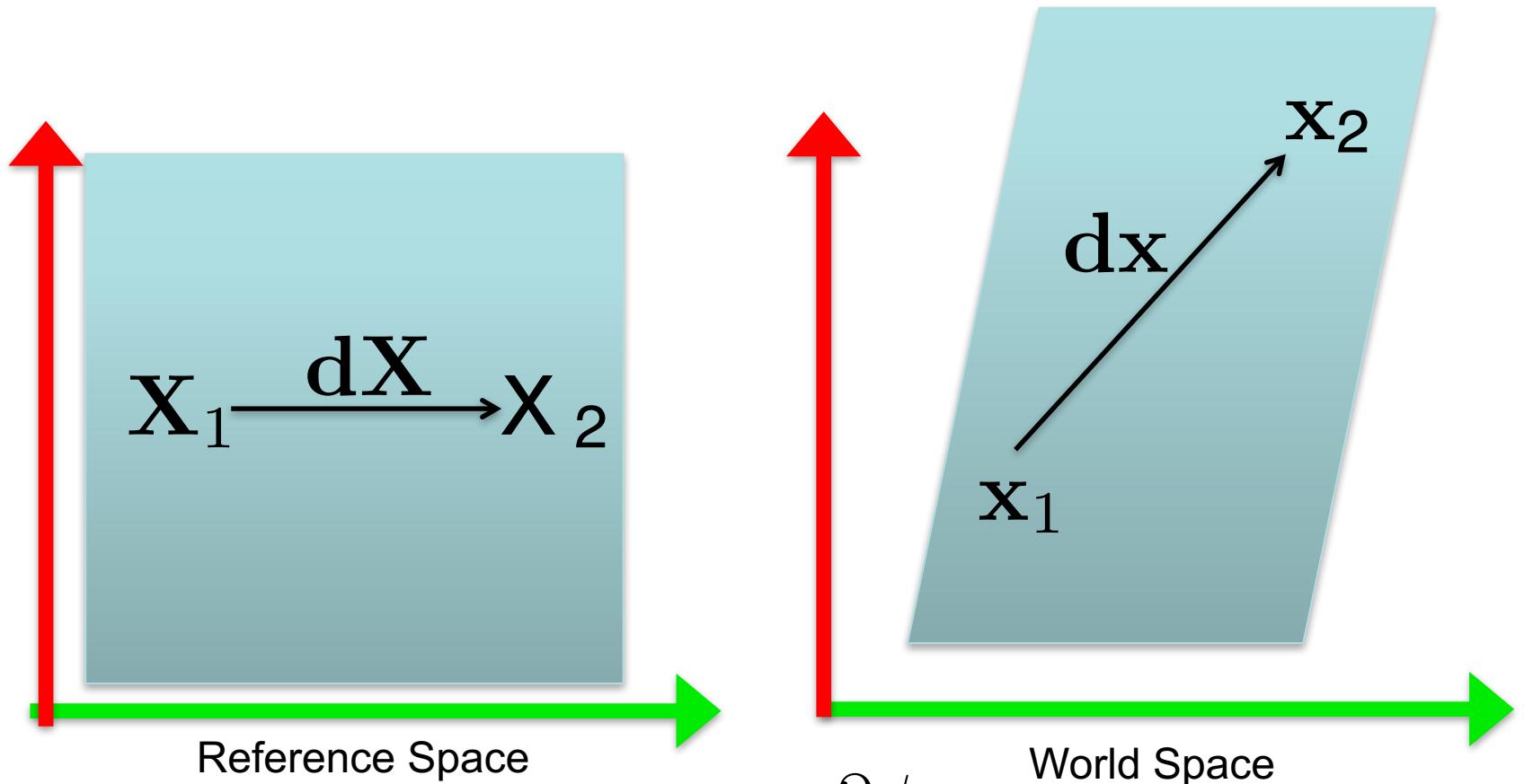
Continuum Mechanics: Deformation

- \mathbf{X}_1 and $\phi(\mathbf{X}_1)$ are the same so we are left with ...



$$\mathbf{x}_1 + d\mathbf{x} \approx \phi(\mathbf{X}_1) + \frac{\partial \phi}{\partial \mathbf{X}} d\mathbf{X}$$

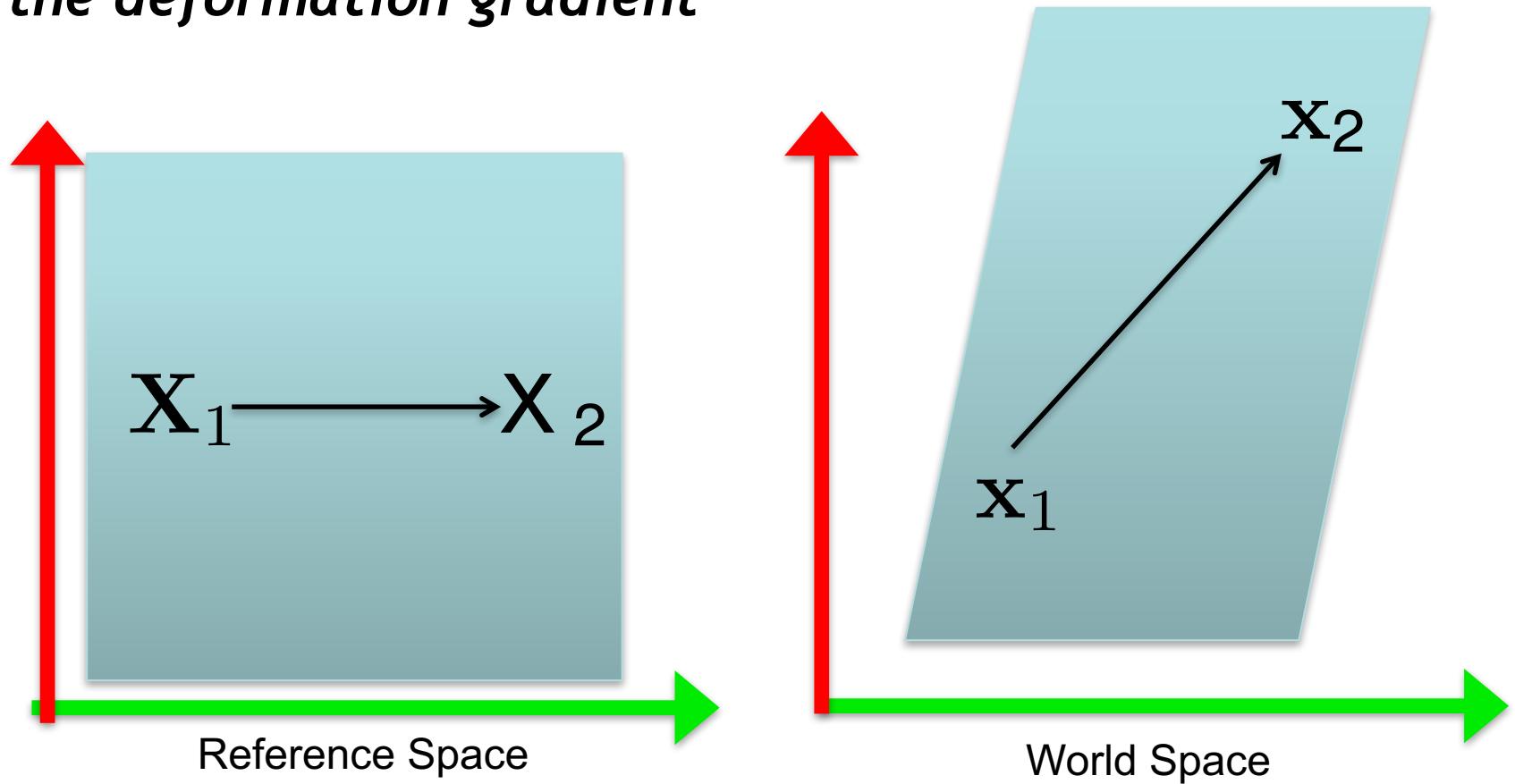
Continuum Mechanics: Deformation



$$d\mathbf{x} \approx \frac{\partial \phi}{\partial \mathbf{X}} d\mathbf{X}$$

Continuum Mechanics: Deformation

- \mathbf{F} is our deformation measure called *the deformation gradient*



$$d\mathbf{x} \approx \mathbf{F} d\mathbf{X}$$

$d\mathbf{x}$ is second term in taylor expansion ...

Continuous Deformation vs. Mass Spring

- Spring Force:

$$-k \left(\left(\frac{l}{l_0} - 1 \right) \frac{\mathbf{x} - \mathbf{y}_i}{|\mathbf{x} - \mathbf{y}_i|} \right)$$

- Deformation:

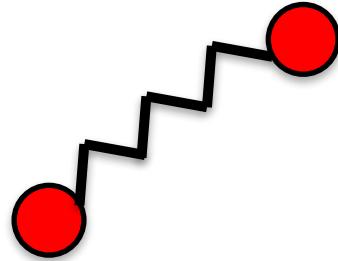
$$\left(\frac{l}{l_0} - 1 \right) \frac{\mathbf{x} - \mathbf{y}_i}{|\mathbf{x} - \mathbf{y}_i|}$$

Equivalent to \mathbf{F} deformation gradient

Continuous Deformation vs. Mass Spring

- Undefomed Spring:

$$\frac{l}{l_0} = ?$$

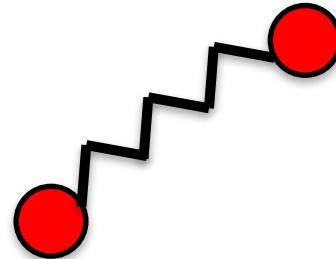


- Undefomed Continuum:

Continuous Deformation vs. Mass Spring

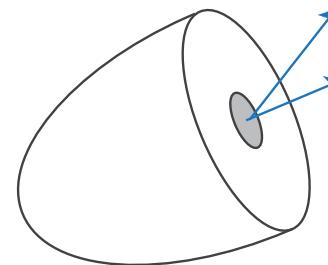
- Undefomed Spring:

$$\frac{l}{l_0} = 1$$



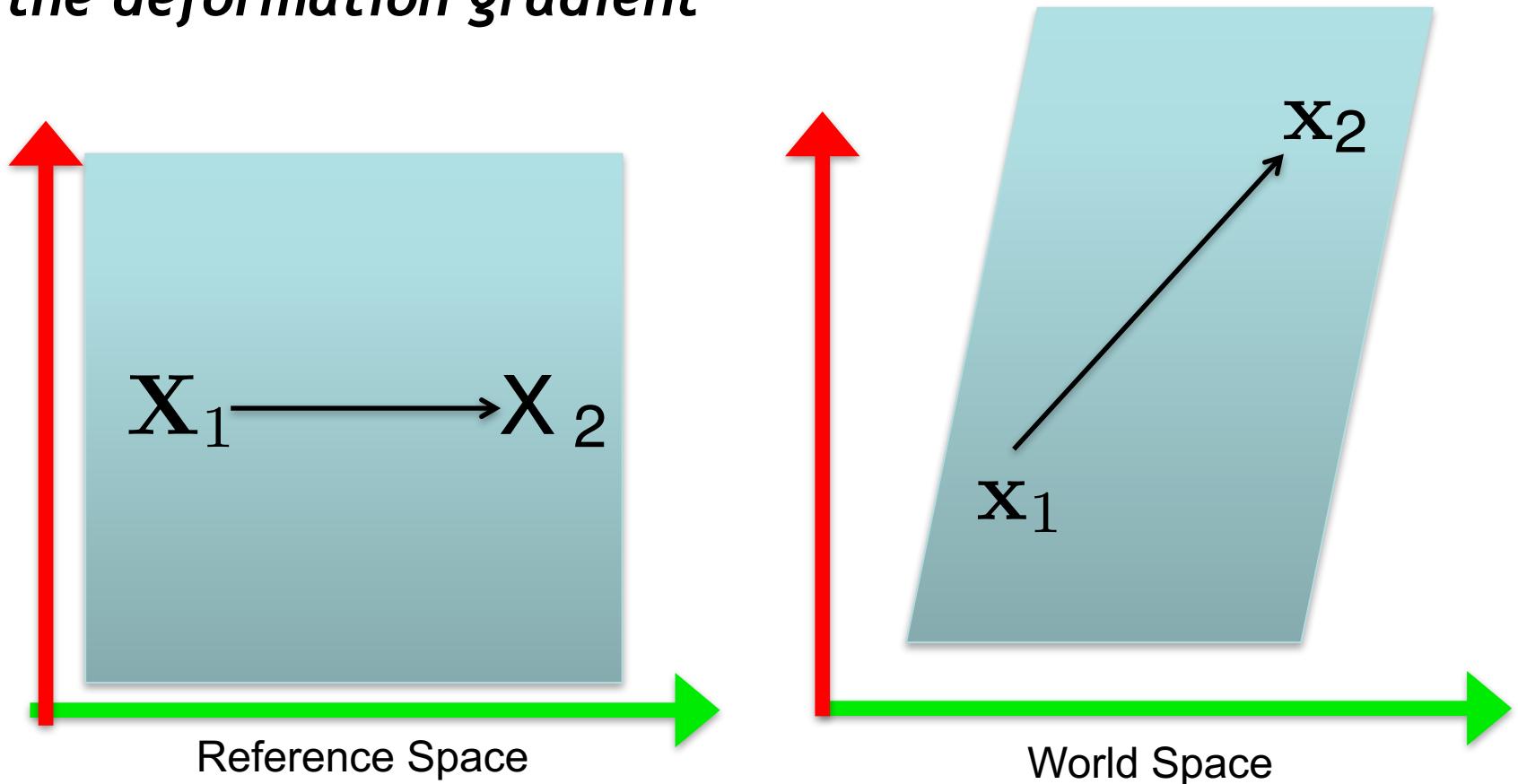
- Undefomed Continuum:

$$F = ?$$



A Few Slides Ago...

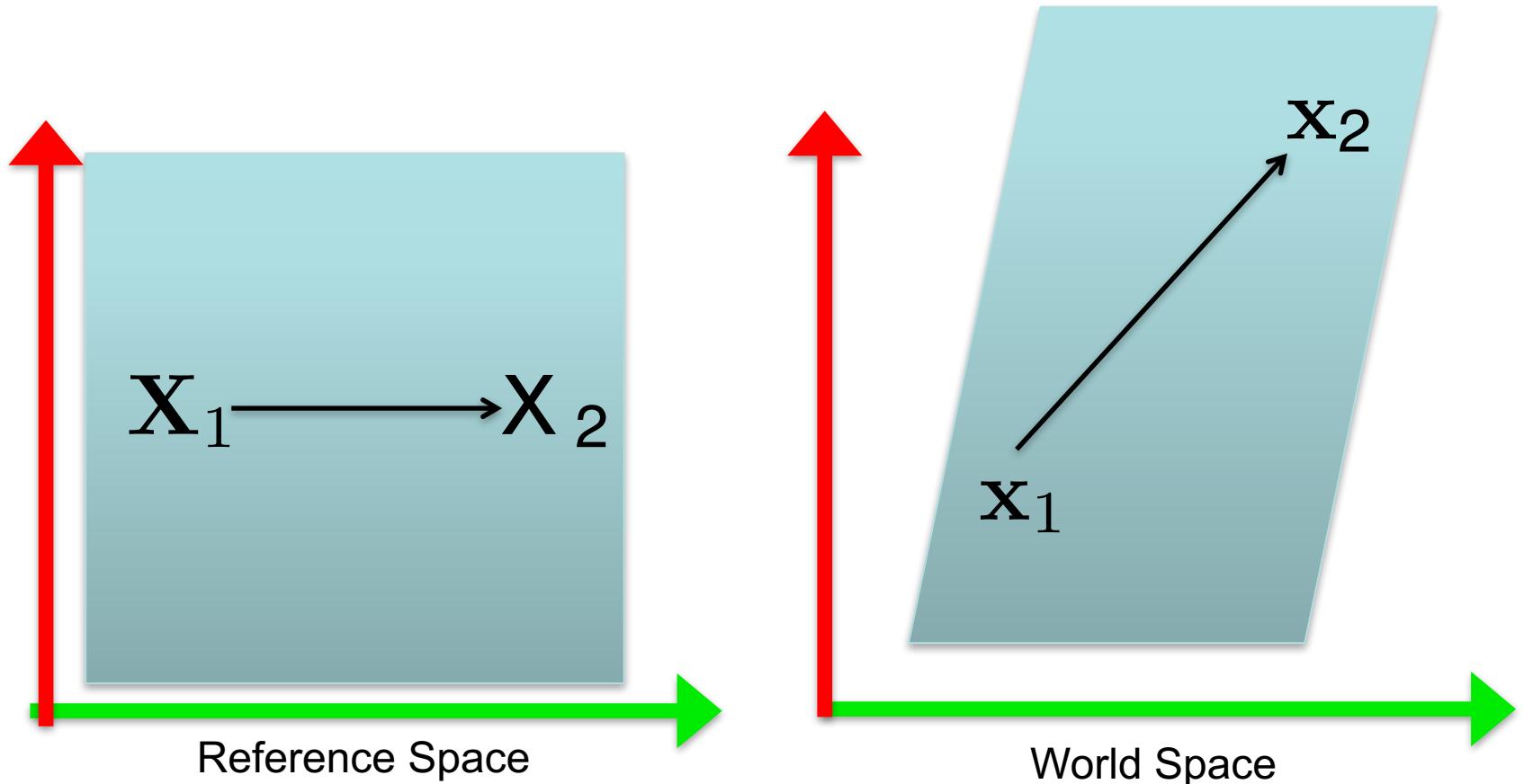
- F is our deformation measure called *the deformation gradient*



$$dx \approx FdX$$

A Few Slides Ago...

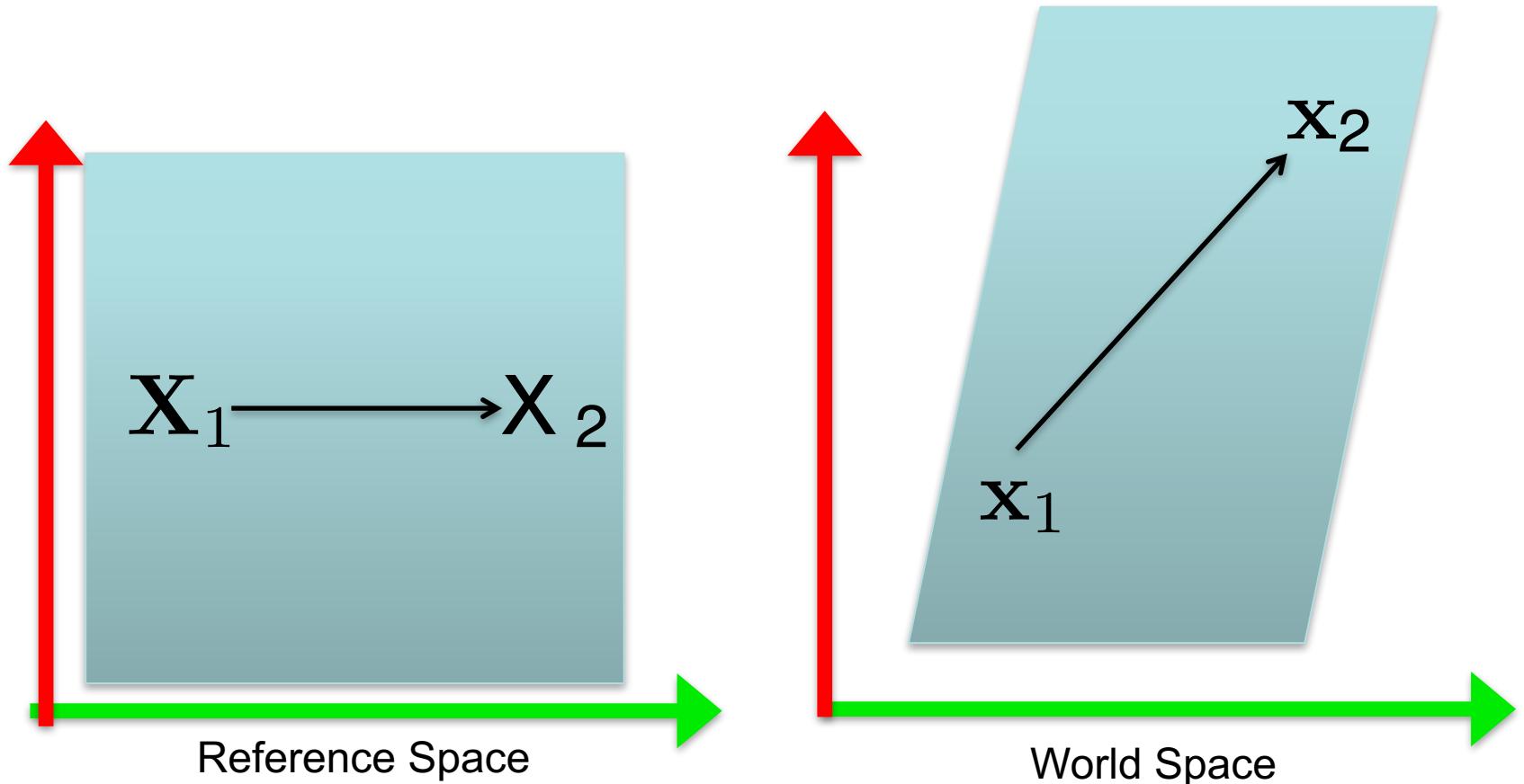
- If there is no deformation $dx = dX$



$$dx \approx FdX$$

A Few Slides Ago...

- If there is no deformation $dx = dX$

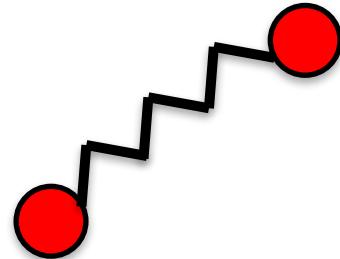


$$dX \approx FdX$$

Continuous Deformation vs. Mass Spring

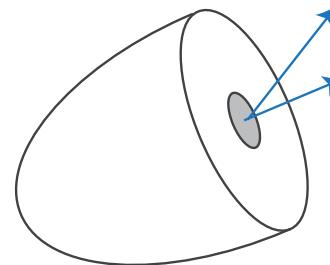
- Undefomed Spring:

$$\frac{l}{l_0} = 1$$



- Undefomed Continuum:

$$\mathbf{F} = \mathbf{I}$$



Continuous Deformation vs. Mass Spring

- Spring Force:

$$-k \left(\left(\frac{l}{l_0} - 1 \right) \frac{\mathbf{x} - \mathbf{y}_i}{|\mathbf{x} - \mathbf{y}_i|} \right)$$

- Deformation:

$$\boxed{\left(\frac{l}{l_0} - 1 \right)} \frac{\mathbf{x} - \mathbf{y}_i}{|\mathbf{x} - \mathbf{y}_i|}$$

This is called **strain**

Properties of a Strain

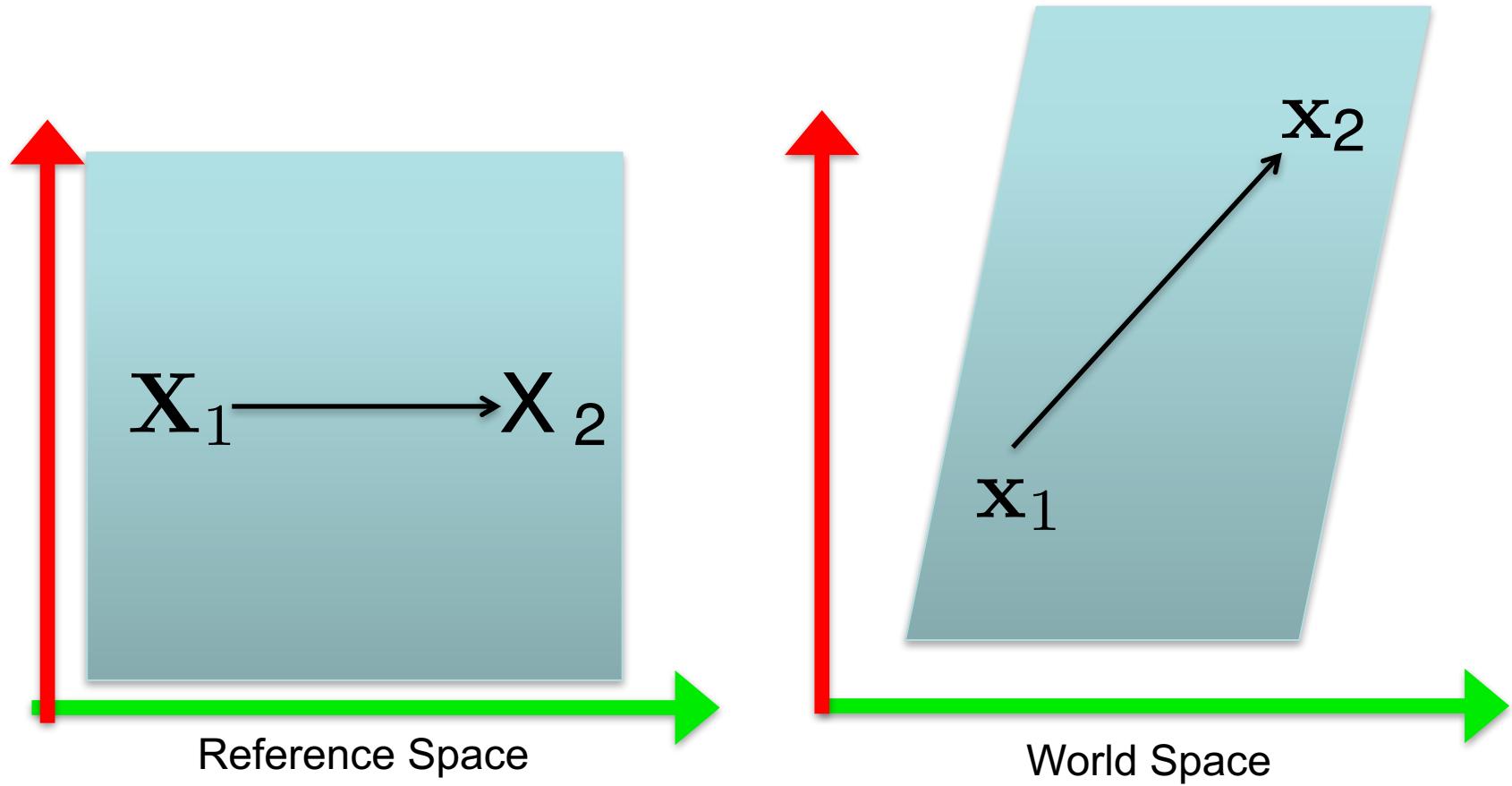
- Spring strain

$$\left(\frac{l}{l_0} - 1 \right)$$

- Property 1: 0 if spring is undeformed
- Property 2: invariant to rigid motion
- Can we find a similar measure that would work for an arbitrary volume ?
- Any guesses ?

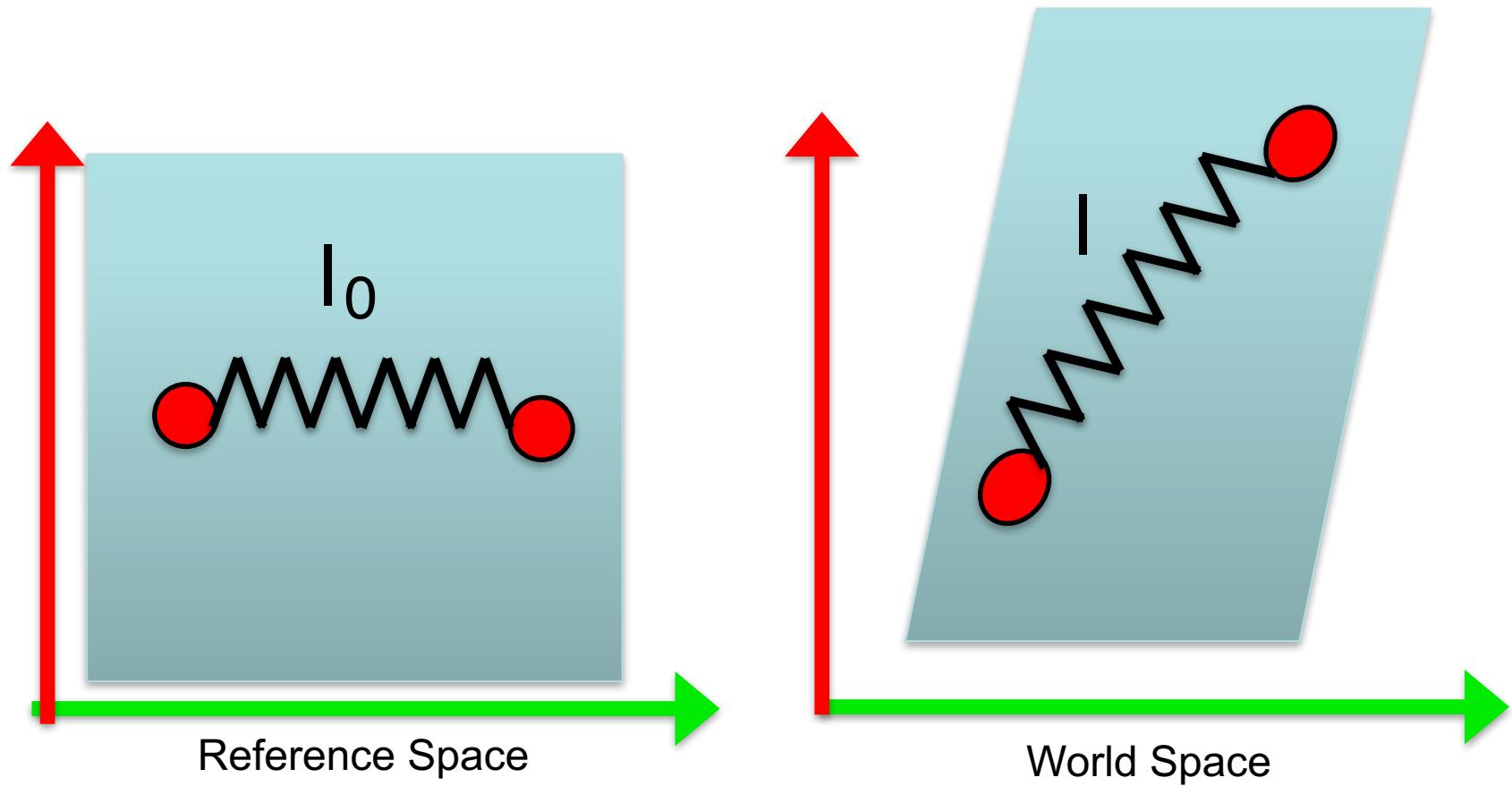
Proposed Strain Measure

- Let's try and use the difference between the lengths of our deformed vector and its undeformed counterpart



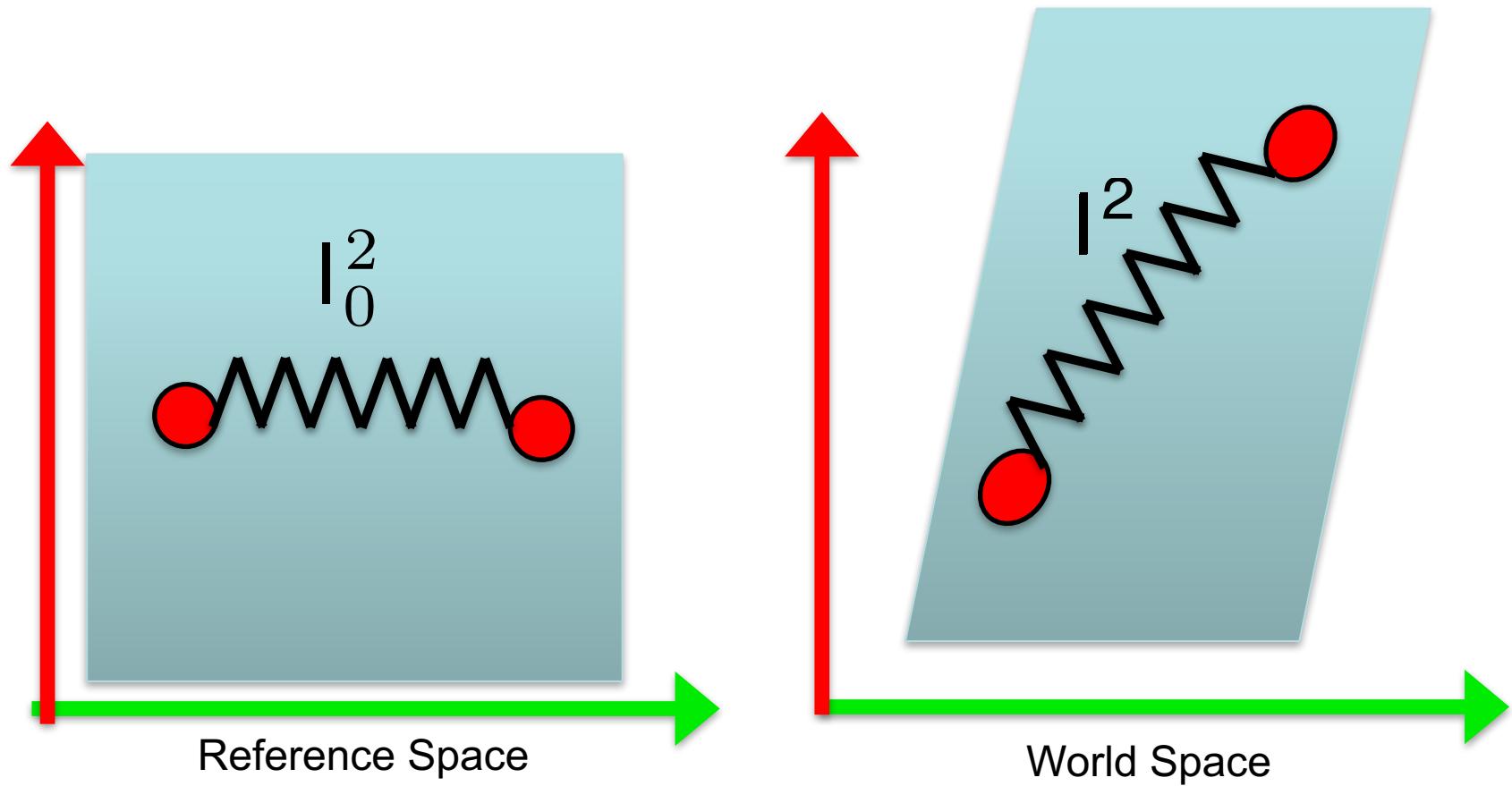
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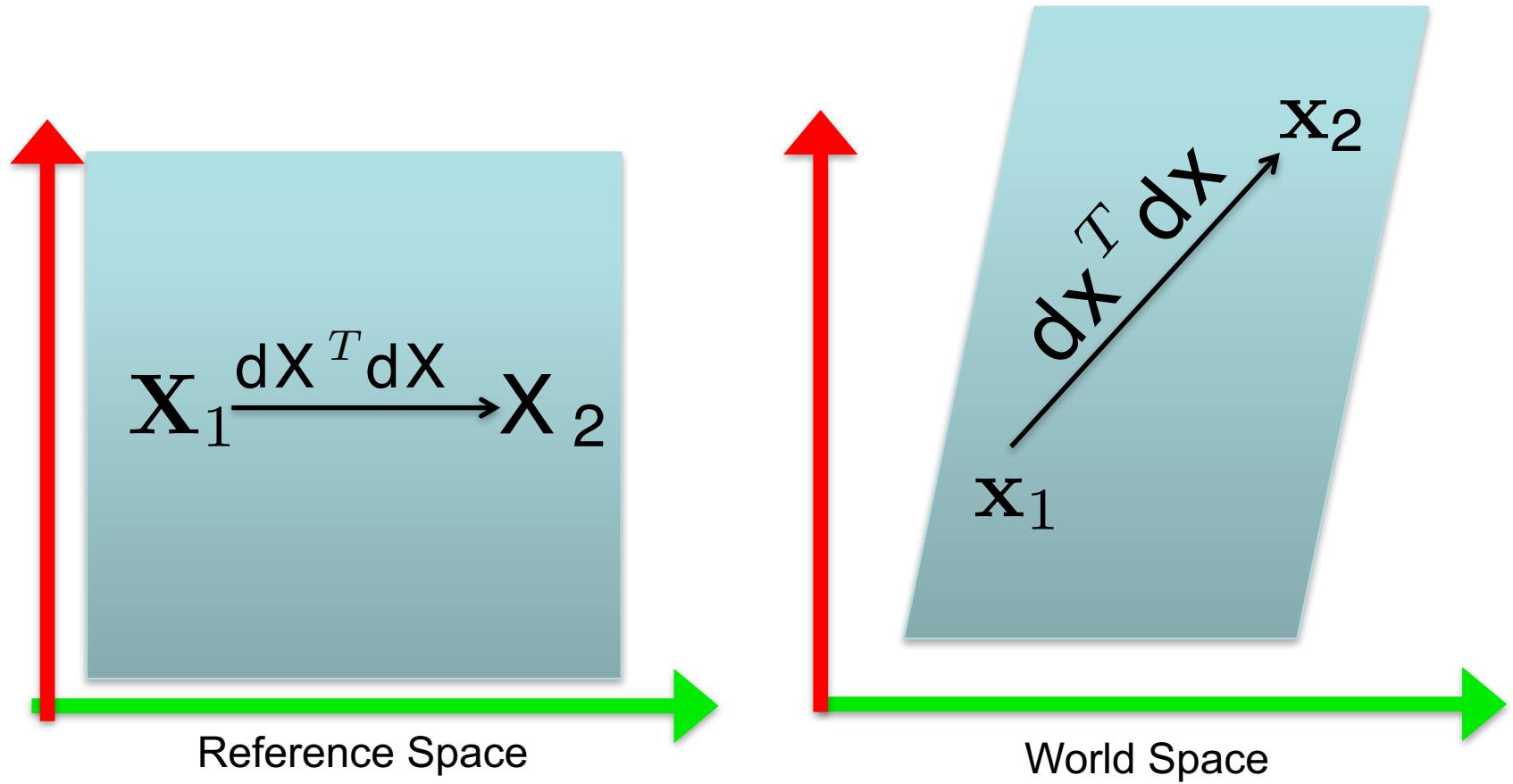
Proposed Strain Measure

- Let's try and use the difference between the lengths of our deformed vector and its undeformed counterpart



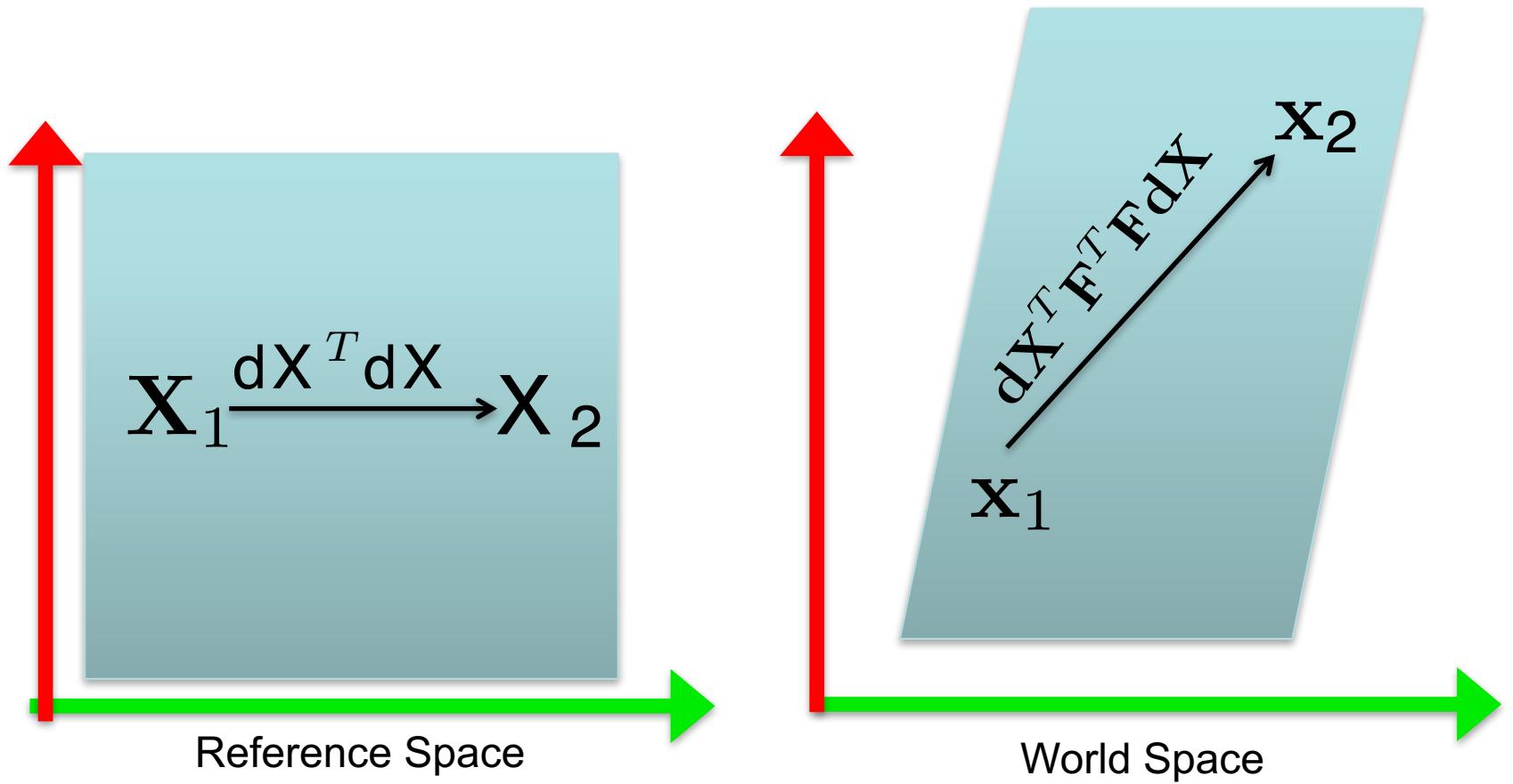
Proposed Strain Measure: Distance Between Points

- Substitute in formula for length squared



Proposed Strain Measure: Distance Between Points

- Use $\mathbf{d}\mathbf{x} \approx \mathbf{F}\mathbf{d}\mathbf{X}$



Proposed Strain Measures: Distance Between Points

- We want to quantify change in shape so we can take the difference of the original and deformed lengths

$$l^2 - l_0^2$$

$$dX^T F^T F dX - dX^T dX$$

$$dX^T (F^T F - I) dX$$

Green Lagrange Strain:

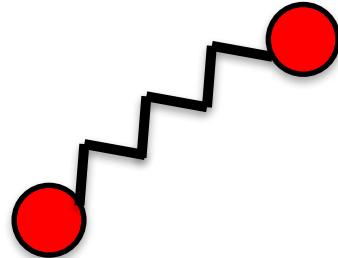
$$\boxed{\frac{1}{2} \square F^T F - I^L}$$

where 1/2 come from

Continuous Deformation vs. Mass Spring

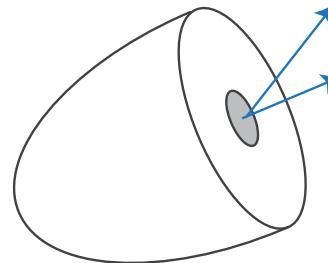
- Undefomed Spring:

$$\frac{l}{l_0} - 1 = ?$$



- Undefomed Continuum:

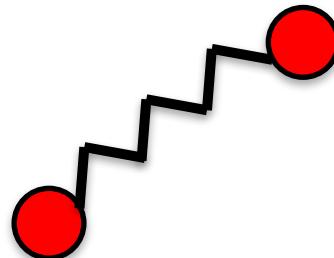
$$\frac{1}{2} \mathbb{F}^T \mathbb{F} - \mathbb{I} = ?$$



Recall...

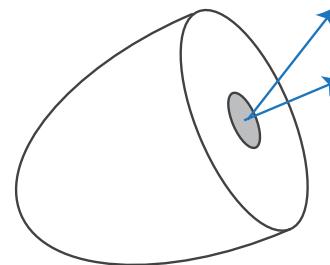
- Undefomed Spring:

$$\frac{l}{l_0} - 1 = 0$$



- Undefomed Continuum:

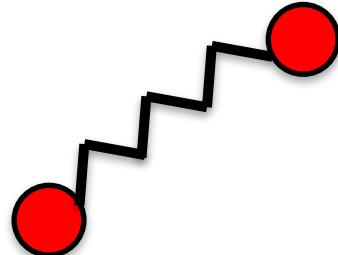
$$\frac{1}{2} \mathbf{F}^T \mathbf{F} - \mathbf{I} = ?$$



Continuous Deformation vs. Mass Spring

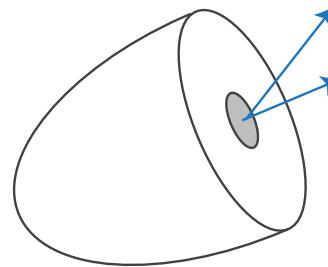
- Undefomed Spring:

$$\frac{l}{l_0} - 1 = 0$$



- Undefomed Continuum:

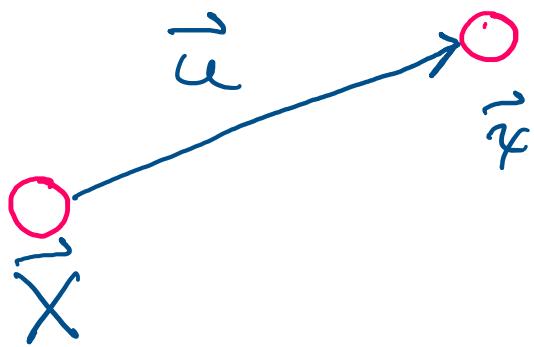
$$\frac{1}{2} \mathbf{F}^T \mathbf{F} - \mathbf{I} = 0$$



For Small Deformations

Position $\vec{x}(\vec{X}) = \vec{X} + \vec{u}(\vec{X})$

where $\vec{u} \in \mathbb{R}^3$ is the displacement



We can rewrite F using \vec{u}

For Small Deformations

$$F = \frac{d\vec{x}}{d\vec{X}} = \frac{d(\vec{X} + \vec{u})}{d\vec{X}} = I + \frac{d\vec{u}}{d\vec{X}}$$

Strain: $\frac{1}{2} \left[I - \left(I + \frac{d\vec{u}}{d\vec{X}} \right)^T \left(I + \frac{d\vec{u}}{d\vec{X}} \right) \right]$

$$\Rightarrow \frac{1}{2} \left[I - I + \frac{d\vec{u}^T}{d\vec{X}} + \frac{d\vec{u}}{d\vec{X}} + \frac{d\vec{u}^T}{d\vec{X}} + \frac{d\vec{u}}{d\vec{X}} \right]$$

For Small Deformations

$$F = \frac{d\vec{x}}{d\vec{X}} = \frac{d(\vec{X} + \vec{u})}{d\vec{X}} = I + \frac{d\vec{u}}{d\vec{X}}$$

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$\Rightarrow \frac{1}{2} \left[I - \cancel{I} + \underbrace{\frac{d\vec{u}^T}{d\vec{X}} \frac{d\vec{u}}{d\vec{X}}^T}_{\text{if } u \text{ is very small...}} + \frac{d\vec{u}^T}{d\vec{X}} + \frac{d\vec{u}}{d\vec{X}} \right]$

if u is very small ...

For Small Deformations

$$F = \frac{d\vec{x}}{d\vec{X}} = \frac{d(\vec{X} + \vec{u})}{d\vec{X}} = I + \frac{d\vec{u}}{d\vec{X}}$$

Strain: $\frac{1}{2} \left[I - \left(I + \frac{d\vec{u}}{d\vec{X}} \right)^T \left(I + \frac{d\vec{u}}{d\vec{X}} \right) \right]$

$$\Rightarrow \frac{1}{2} \left[I - I + \underbrace{\frac{d\vec{u}^T}{d\vec{X}} \frac{d\vec{u}}{d\vec{X}}^T}_{\text{this is very small} \Rightarrow \text{Ignore!}} + \frac{d\vec{u}^T}{d\vec{X}} + \frac{d\vec{u}}{d\vec{X}} \right]$$

this is very small \Rightarrow Ignore!

For Small Deformations

$$F = \frac{d\vec{x}}{d\vec{X}} = \frac{d(\vec{X} + \vec{u})}{d\vec{X}} = I + \frac{d\vec{u}}{d\vec{X}}$$

Strain: $\frac{1}{2} \left[I - \left(I + \frac{d\vec{u}}{d\vec{X}} \right)^T \left(I + \frac{d\vec{u}}{d\vec{X}} \right) \right]$

$\Rightarrow \frac{1}{2} \left[I - I + \cancel{\frac{d\vec{u}^T}{d\vec{X}}} + \cancel{\frac{d\vec{u}}{d\vec{X}}^T} + \frac{d\vec{u}^T}{d\vec{X}} + \frac{d\vec{u}}{d\vec{X}} \right]$

this is very small \Rightarrow Ignore!

For Small Deformations

$$F = \frac{d\vec{x}}{d\vec{X}} = \frac{d(\vec{X} + \vec{u})}{d\vec{X}} = I + \frac{d\vec{u}}{d\vec{X}}$$

Strain: $\frac{1}{2} \left[I - \left(I + \frac{d\vec{u}}{d\vec{X}} \right)^T \left(I + \frac{d\vec{u}}{d\vec{X}} \right) \right]$

$\Rightarrow \frac{1}{2} \left[I - I + \cancel{\frac{d\vec{u}^T}{d\vec{X}}} + \cancel{\frac{d\vec{u}}{d\vec{X}}^T} + \frac{d\vec{u}^T}{d\vec{X}} + \frac{d\vec{u}}{d\vec{X}} \right]$

this is very small \Rightarrow Ignore!

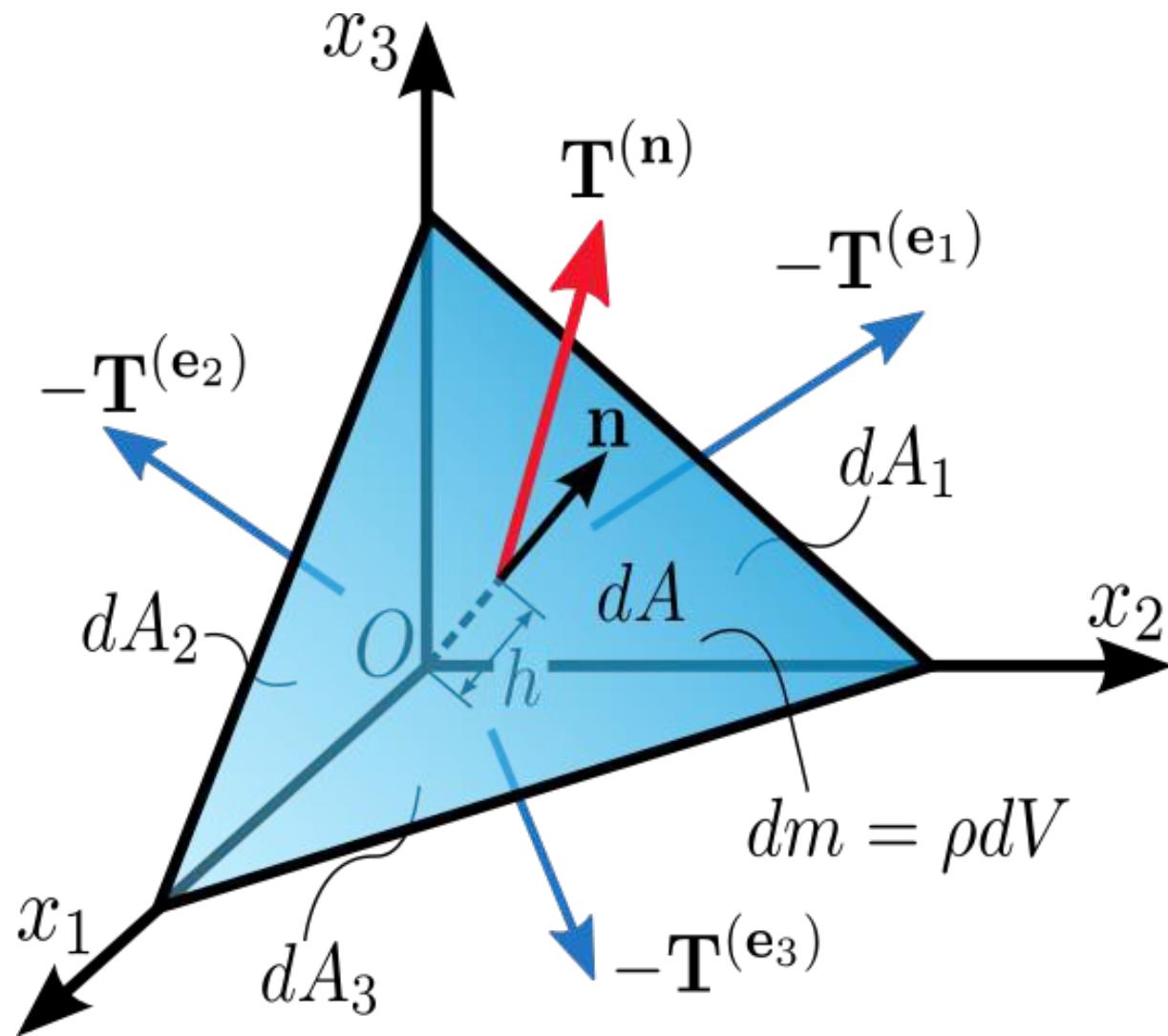
Continuum Mechanics: The Required Stuff

1. Material Model
2. Measure of Deformation

$$\frac{1}{2} \left(\frac{\partial \vec{\mathbf{u}}^T}{\partial \vec{\mathbf{X}}} + \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}} \right)$$

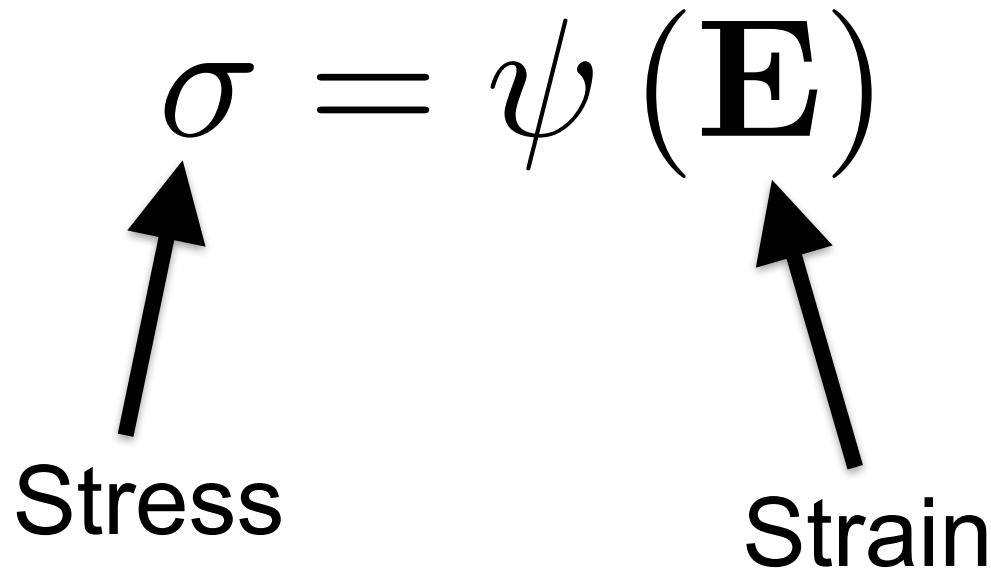
- We just need the material model now

What is Stress ?



Material Models in Continuum Mechanics

- Materials models in continuum mechanics convert strain into a force per unit area called a stress

$$\sigma = \psi(E)$$


The diagram illustrates the relationship between stress and strain. At the top center is the equation $\sigma = \psi(E)$. Two arrows point upwards from the words "Stress" and "Strain" towards the variable E in the equation, indicating that both stress and strain are inputs to the function ψ .

Material Models in Continuum Mechanics

- Materials models in continuum mechanics convert strain into a force per unit area called a stress

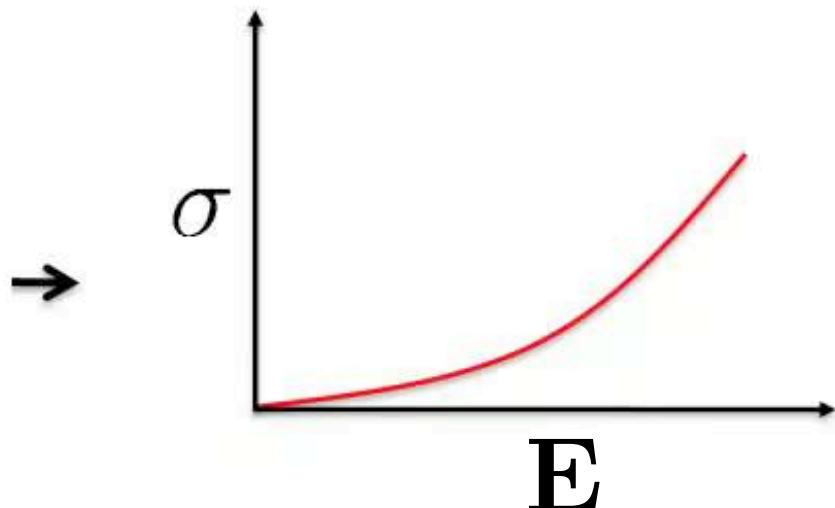
$$\sigma = \psi(E)$$

The diagram illustrates the relationship between Stress, Strain, and a Material Model. At the top center is the equation $\sigma = \psi(E)$. Three arrows point upwards from below to the components of the equation: a long arrow points to σ , a medium arrow points to E , and a short arrow points to ψ . Below the equation, the word "Material Model" is written in large, bold, black capital letters. To the left of "Material Model" is the word "Stress" and to the right is the word "Strain", both in large, bold, black capital letters.

Material Representation

Stress-Strain relationship 1D

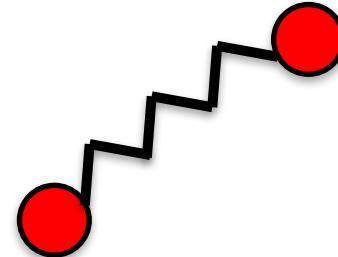
- Strain $E = l/l_0 - 1$
- Stress $\sigma = f/A$



Continuous Deformation vs. Mass Spring

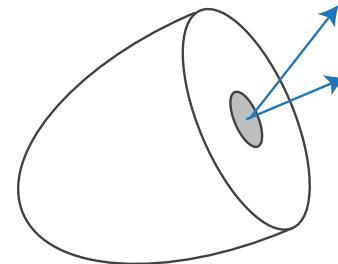
- Mass Spring:

$$\mathbf{f} = -\boxed{k} \left(\left(\frac{l}{l_0} - 1 \right) \frac{\mathbf{x} - \mathbf{y}_i}{|\mathbf{x} - \mathbf{y}_i|} \right)$$



- Continuum Mechanics:

$$\sigma = \psi(\mathbf{E})$$

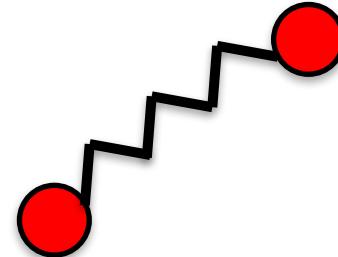


What is ψ ?

Continuous Deformation vs. Mass Spring

- Mass Spring:

$$\mathbf{f} = -\boxed{k} \left(\left(\frac{l}{l_0} - 1 \right) \frac{\mathbf{x} - \mathbf{y}_i}{|\mathbf{x} - \mathbf{y}_i|} \right)$$

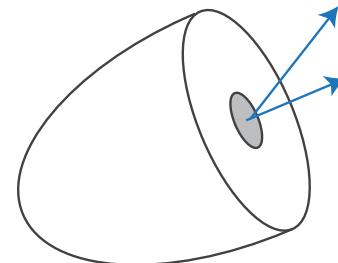


- Continuum Mechanics:

$$\sigma = \mathbf{K} \mathbf{E}$$



tensor



Continuum Mechanics: The Required Stuff

- Material Model

$$\sigma = KE$$

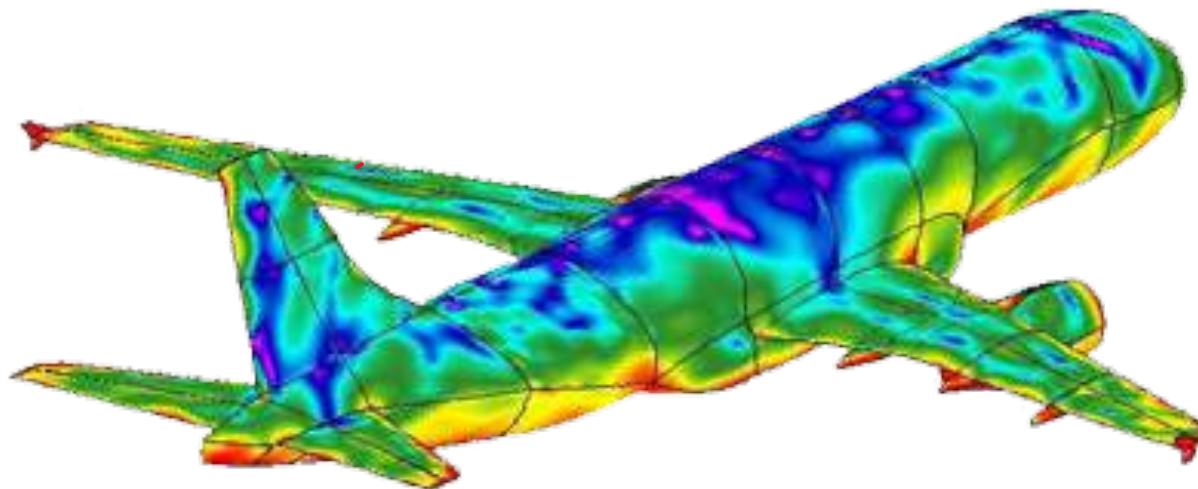
- Measure of Deformation

$$\frac{1}{2} \left(\frac{\partial \vec{u}}{\partial \vec{X}}^T + \frac{\partial \vec{u}}{\partial \vec{X}} \right)$$

- We have all the pieces now

Static Structural Analysis

- Solve for the rest shape of the object under some external forces



Static Structural Analysis

$$\underline{f_{\text{INT}}(u)} + \underbrace{f_{\text{EXT}}}_{\text{External Forces like gravity}} = 0 \quad \text{Force Balance}$$

Internal Forces From deformation

Hmm... something that equals 0...

Static Structural Analysis

$$f_{\text{INT}}(u) + f_{\text{EXT}} = 0 \quad \text{Optimize!}$$



$$u^* = \underset{\mathcal{U}}{\operatorname{argmin}} \int \frac{1}{2} E^T K E - \underbrace{U^T F_{\text{EXT}}}_{\substack{\text{Work done} \\ \text{by external force}}} d\mathcal{N}$$

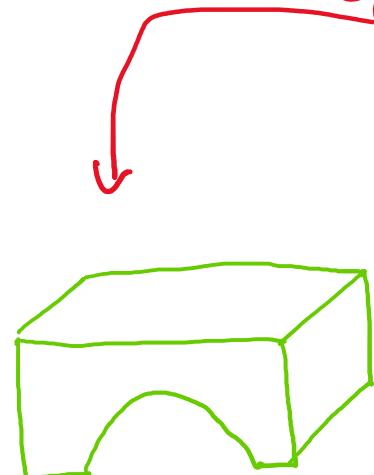
$$E = \begin{bmatrix} E_{xx} \\ E_{xy} \\ E_{zz} \\ E_{yz} \\ E_{xz} \\ E_{xy} \end{bmatrix} \in \mathbb{R}^{6 \times 1}$$

Internal Potential Energy

Static Structural Analysis

$$U^* = \operatorname{argmin} \int \frac{1}{2} E^T K E - U^T F_{EXT} d\mathcal{N}$$

Volume
Object



\int
 \cdot

Hooke's Law

$$\begin{bmatrix} (1-b) & b & b & 0 & 0 & 0 \\ b & (1-b) & b & 0 & 0 & 0 \\ b & b & (1-b) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2b) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2b) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2b) \end{bmatrix}$$

Static Structural Analysis

$$U^* = \operatorname{argmin}_U \int \frac{1}{2} E^T K E - U^T F_{EXT} d\Delta$$

\int \downarrow Hooke's Law

$$\left[\begin{array}{ccccccc} (1-b) & b & b & 0 & 0 & 0 \\ b & (1-b) & b & 0 & 0 & 0 \\ b & b & (1-b) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2b) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2b) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2b) \end{array} \right]$$

$\frac{a}{(1+b)(1-2b)}$

a = Young's Modulus

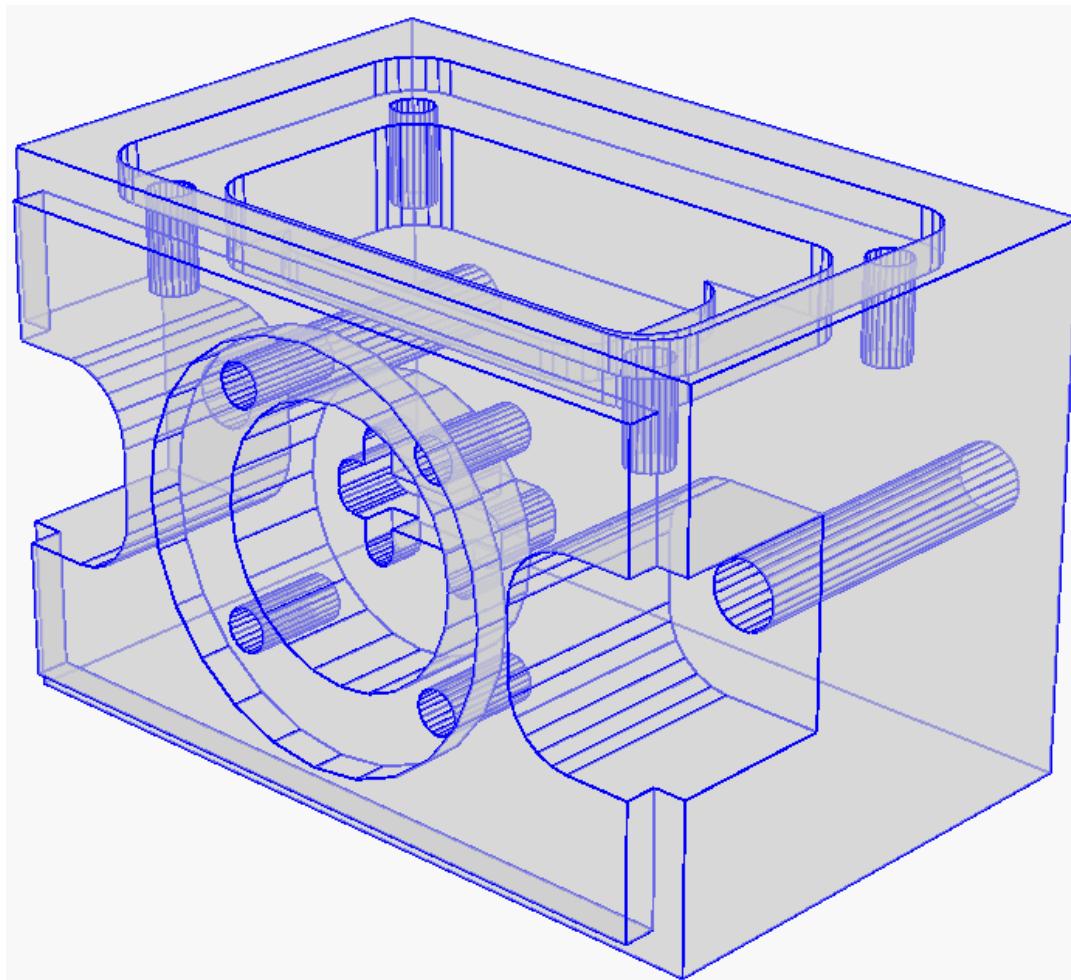
(10^6 is a good value)

b = Poisson's Ratio

(0.45 is good)

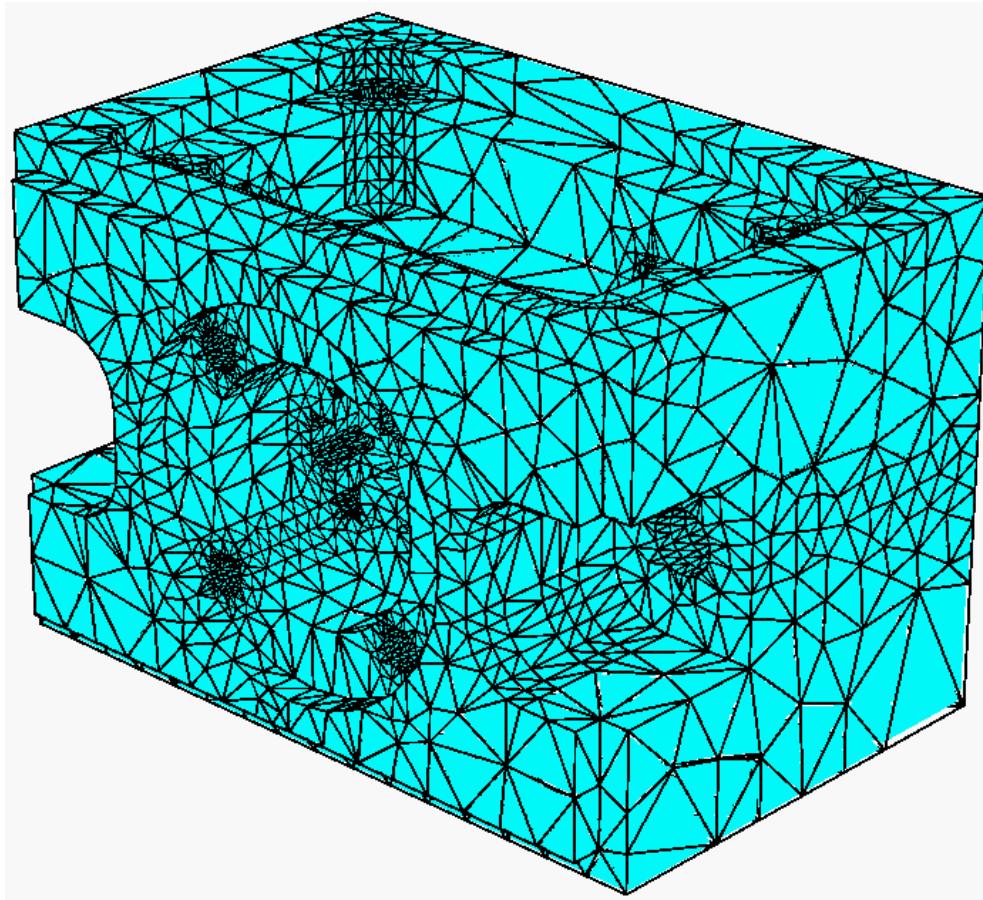
Discretization

1. Input Object



Discretization

2. Divide object into “elements”, small simple volumetric shapes



Back to the Math!

$$U^* = \operatorname{argmin} \int \frac{1}{2} E^T K E - U^T F_{EXT} d\mathcal{N}$$

Divide into tetrahedra

$$U^* = \operatorname{argmin} \sum_{i=1}^N \int \frac{1}{2} E^T K E - U^T F_{EXT} d\mathcal{N}_i$$

number of tetrahedra

$\sum_{i=1}^N$

\mathcal{N}_i

Back to the Math!

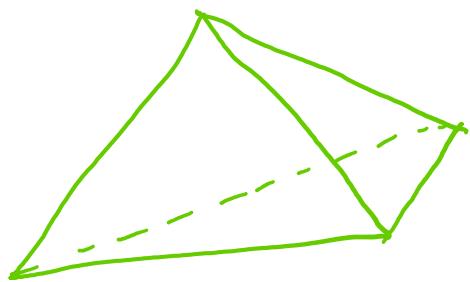
$$u^* = \operatorname{argmin} \sum_i^N \int_{\Omega_i} \frac{1}{2} E^T |E - U^T F_{EXT}| d\Omega_i$$

$\Omega_i \rightarrow$ What about E ?

$$E = \begin{bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \\ E_{yz} \\ E_{xz} \\ E_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x_x} \\ \frac{\partial u_y}{\partial x_y} \\ \frac{\partial u_z}{\partial x_z} \\ \frac{1}{2} \left(\frac{\partial u_y}{\partial x_z} + \frac{\partial u_z}{\partial x_y} \right) \\ \frac{1}{2} \left(\frac{\partial u_x}{\partial x_z} + \frac{\partial u_z}{\partial x_x} \right) \\ \frac{1}{2} \left(\frac{\partial u_x}{\partial x_y} + \frac{\partial u_y}{\partial x_x} \right) \end{bmatrix} = \text{What Next?}$$

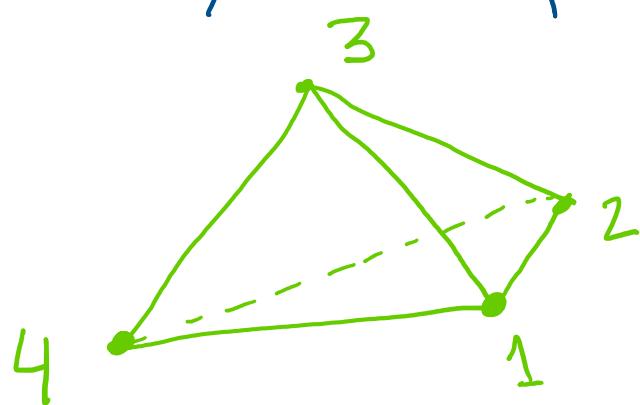
Back to the Math!

We need a way to represent $u(x)$ inside



Back to the Math!

We need a way to represent $u(x)$ inside

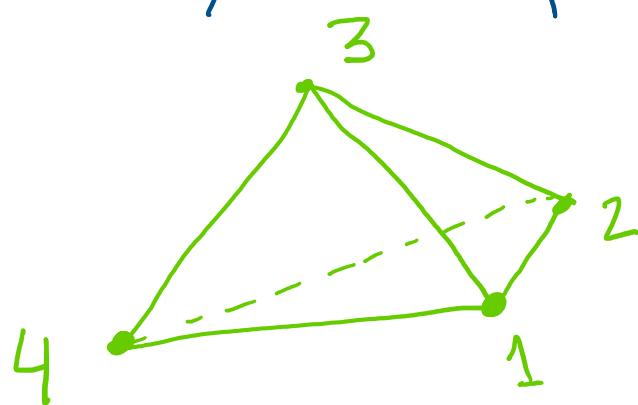


We'll use Barycentric Coordinates

1. One coordinate ϕ_j for each (j^{th}) vertex
2. $\phi_j(x) = 1$ if $x =$ position of vertex j
 $= 0$ else

Back to the Math!

We need a way to represent $u(x)$ inside

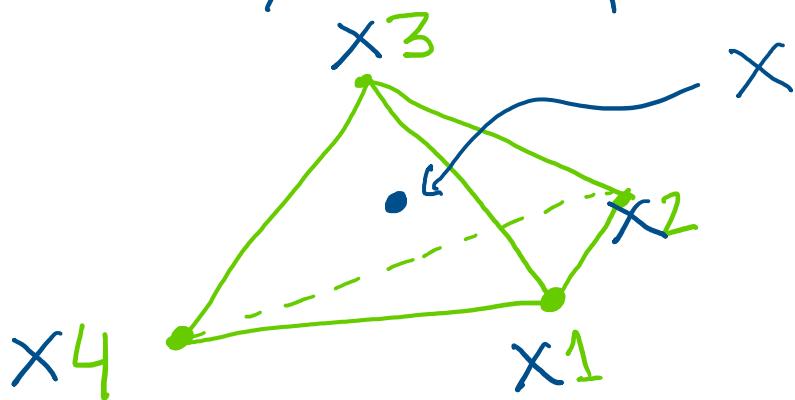


We'll use Barycentric Coordinates

1. One coordinate ϕ_j for each (j^{th}) vertex
2. $\phi_j(x) = 1$ if $x =$ position of vertex j
3. $\sum_{j=1}^4 \phi_j(x) = 1$

Back to the Math!

We need a way to represent $u(x)$ inside



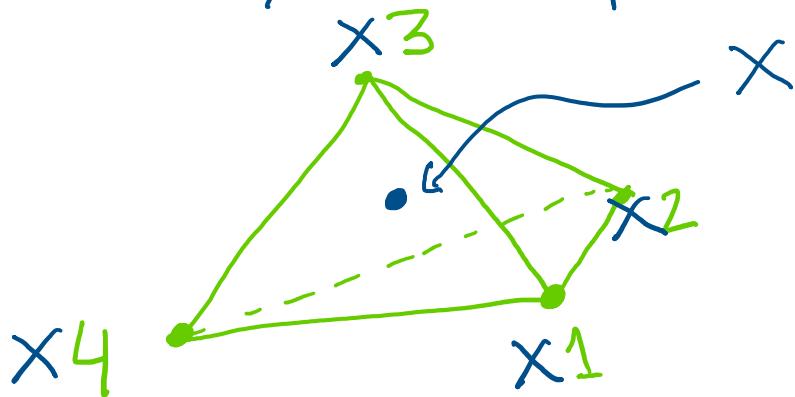
We'll use Barycentric Coordinates

$$x = x_1 \phi_1 + x_2 \phi_2 + x_3 \phi_3 + x_4 \phi_4$$

\Rightarrow 3 equations, 4 unknowns

Back to the Math!

We need a way to represent $u(x)$ inside



We'll use Barycentric Coordinates

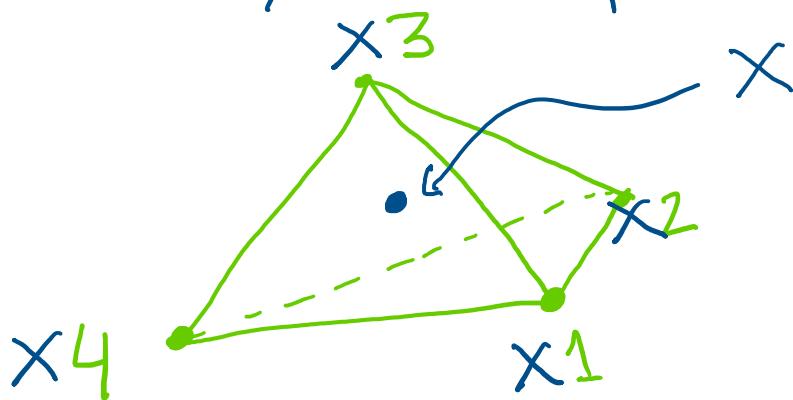
$$x = x_1 \phi_1 + x_2 \phi_2 + x_3 \phi_3 + x_4 \phi_4$$

\Rightarrow 3 equations, 4 unknowns

* Use $\sum \phi_j = 1$ OR $\phi_4 = 1 - \phi_1 - \phi_2 - \phi_3$

Back to the Math!

We need a way to represent $u(x)$ inside



We'll use Barycentric Coordinates

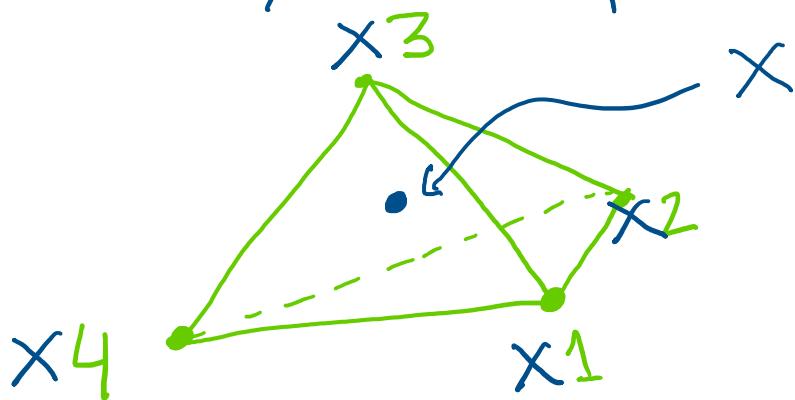
$$x = x_0 \phi_0 + x_1 \phi_1 + x_2 \phi_2 + x_3 \phi_3 + x_4 \phi_4$$

$$(1) \quad x = x_0 \phi_0 + x_1 \phi_1 + x_2 \phi_2 + x_3 \phi_3 + x_4 (1 - \phi_0 - \phi_1 - \phi_2 - \phi_3)$$

$$(2) \quad x - x_4 = (x_1 - x_4) \phi_1 + (x_2 - x_4) \phi_2 + (x_3 - x_4) \phi_3$$

Back to the Math!

We need a way to represent $u(x)$ inside



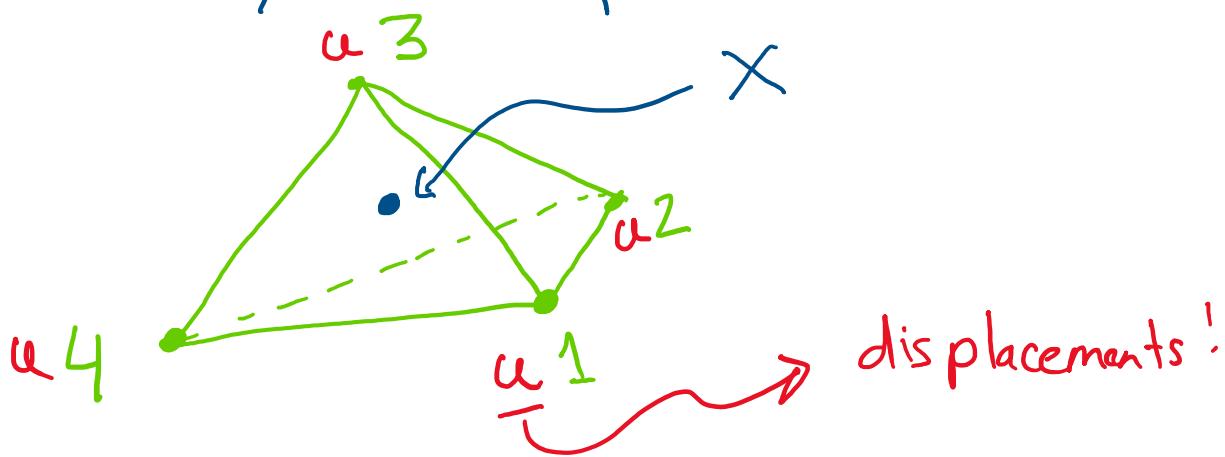
We'll use Barycentric Coordinates

$$x - x_4 = (x_1 - x_4)\phi_1 + (x_2 - x_4)\phi_2 + (x_3 - x_4)\phi_3$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} x - x_1 \\ (x - x_2) \\ (x - x_3) \end{bmatrix} \begin{bmatrix} x - x_4 \end{bmatrix}, \quad \phi_4(x) = 1 - \phi_1 - \phi_2 - \phi_3$$

Back to the Math!

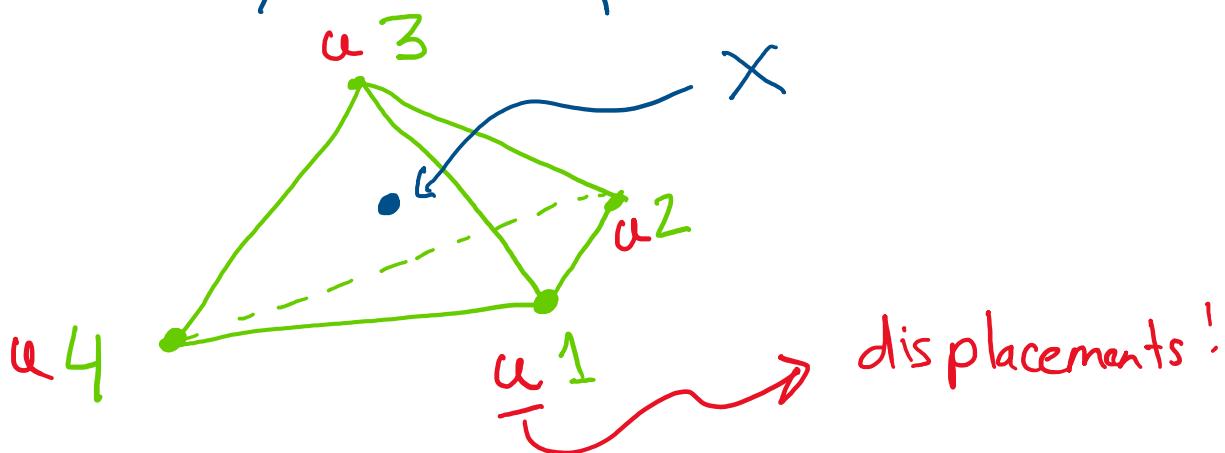
We need a way to represent $u(x)$ inside



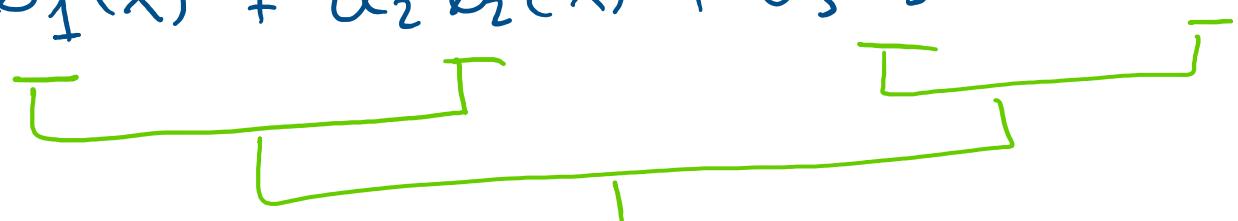
$$u(x) = u_1 \phi_1(x) + u_2 \phi_2(x) + u_3 \phi_3(x) + u_4 \phi_4(x)$$

Back to the Math!

We need a way to represent $u(x)$ inside



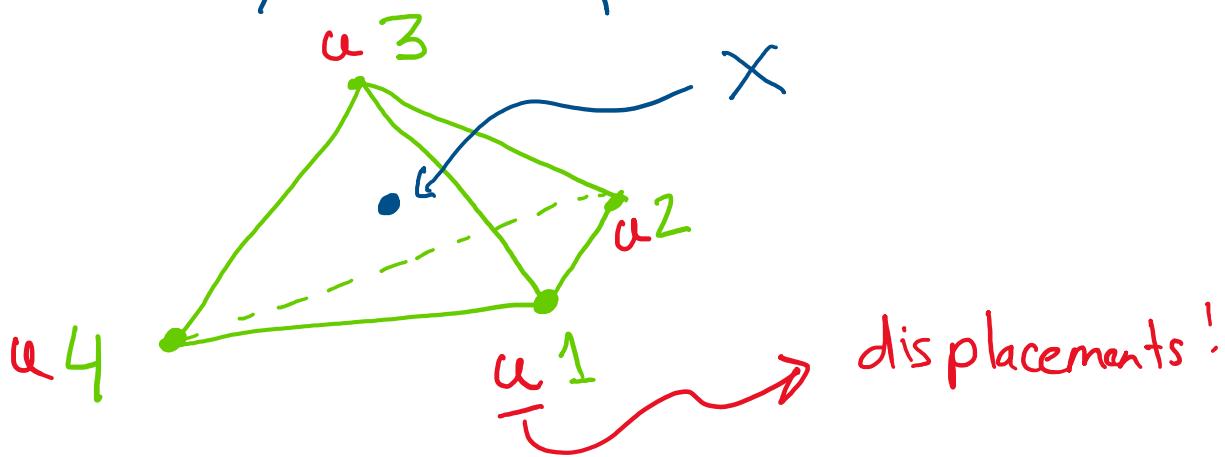
$$u(x) = u_1 \phi_1(x) + u_2 \phi_2(x) + u_3 \phi_3(x) + u_4 \phi_4(x)$$



Shape Functions

Back to the Math!

We need a way to represent $u(x)$ inside



$$u(x) = u_1 \phi_1(x) + u_2 \phi_2(x) + u_3 \phi_3(x) + u_4 \phi_4(x)$$

We can take the derivative of this!

Back to the Math!

$$u^* = \operatorname{argmin} \sum_i^N \int_{\Omega_i} \frac{1}{2} E^T |K| E - U^T F_{EXT} d\Omega_i$$

$\Omega_i \rightarrow$ What about E ?

$$E = \begin{bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \\ E_{yz} \\ E_{xz} \\ E_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x_x} \\ \frac{\partial u_y}{\partial x_y} \\ \frac{\partial u_z}{\partial x_z} \\ \frac{1}{2} \left(\frac{\partial u_y}{\partial x_z} + \frac{\partial u_z}{\partial x_y} \right) \\ \frac{1}{2} \left(\frac{\partial u_x}{\partial x_z} + \frac{\partial u_z}{\partial x_x} \right) \\ \frac{1}{2} \left(\frac{\partial u_x}{\partial x_y} + \frac{\partial u_y}{\partial x_x} \right) \end{bmatrix} = \text{What Next?}$$

Use Shape Funcs!

Back to the Math!

$$u(x) = u_1 \phi_1(x) + u_2 \phi_2(x) + u_3 \phi_3(x) + u_4 \phi_4(x)$$

$$\frac{\partial u_i}{\partial x_j} = u_{1i} \frac{\partial \phi_1}{\partial x_j} + u_{2i} \frac{\partial \phi_2}{\partial x_j} + u_{3i} \frac{\partial \phi_3}{\partial x_j} + u_{4i} \frac{\partial \phi_4}{\partial x_j}$$

Now we can build E !

Back to the Math!

$$u^* = \operatorname{argmin} \sum_i^N \left\| \frac{1}{2} E^T \mathbf{1} - U^T F_{EXT} \right\|_2^2$$

$E = \begin{bmatrix} \frac{\partial u_x}{\partial x_x} \\ \frac{\partial u_y}{\partial x_y} \\ \frac{\partial u_z}{\partial x_z} \\ \frac{1}{2} \left(\frac{\partial u_y}{\partial x_z} + \frac{\partial u_z}{\partial x_y} \right) \\ \frac{1}{2} \left(\frac{\partial u_x}{\partial x_z} + \frac{\partial u_z}{\partial x_x} \right) \\ \frac{1}{2} \left(\frac{\partial u_x}{\partial x_y} + \frac{\partial u_y}{\partial x_x} \right) \end{bmatrix} = \begin{bmatrix} \text{Matrix } B \\ \vdots \end{bmatrix}$

$B \in 6 \times 12$

$$U = \begin{bmatrix} U_{1x} & U_{1y} & U_{1z} \\ U_{2x} & U_{2y} & U_{2z} \\ U_{3x} & U_{3y} & U_{3z} \\ U_{4x} & U_{4y} & U_{4z} \end{bmatrix} \quad \text{Dimensions: } 12 \times 1$$

Back to the Math!

$$u^* = \operatorname{argmin}_i \sum_{\Omega_i}^N \int \frac{1}{2} E^T K E - u^T F_{EXT} d\Omega_i$$

Use $E = Bu$

$$u^* = \operatorname{argmin}_i \sum_{\Omega_i}^N \int \frac{1}{2} u_i^T B(x)^T K B(x) u_i - u_i^T F_{EXT} d\Omega_i$$

Back to the Math!

$$u^* = \operatorname{argmin}_i \sum_{j=1}^N \left\{ \frac{1}{2} u_i^T B(x)^T K B(x) u_i - u_i^T F_{\text{ext}} d \eta_i \right\}$$



$$u^* = \operatorname{argmin} \frac{1}{2} \sum_{i=1}^N u_i^T \left\{ B(x)^T K B(x) d \eta_i u_i - \dots \right\}$$

η_i

K_i

Back to the Math!

$$u^* = \operatorname{argmin} \frac{1}{2} \sum_{i=1}^N u_i^T K_i u_i - u_i^T f_{\text{ext}}$$

Write as big quadratic function

$$u^* = \operatorname{argmin} \frac{1}{2} u^T K u - u^T f$$

How do you find optimal point?

Solution Procedure

$$\nabla_u \left(\frac{1}{2} u^\top K u - u^\top f \right) = 0$$

$$Ku = f$$

Step 3: Build Per Element Equations

- In all our Elements

$$\mathbf{K}_i \mathbf{y} = \mathbf{f}$$

Step 5: Assembly

- We have a collection of Element “Stiffness” matrices (our \mathbf{K}_i 's)
- We will “assemble” these Element matrices into a global matrix for our problem

Step 5: Assembly

- For a particular element we have

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y_p \\ y_{p+1} \end{pmatrix}$$

Contribution of y_p to “force” at y_{p+1}

Step 5: Assembly

- Now we know how to add our element matrix into the global matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + \begin{bmatrix} & & \\ & & \\ & \xrightarrow{\text{Row } p+1, \text{ Column } p} & \\ & & \end{bmatrix} = K$$

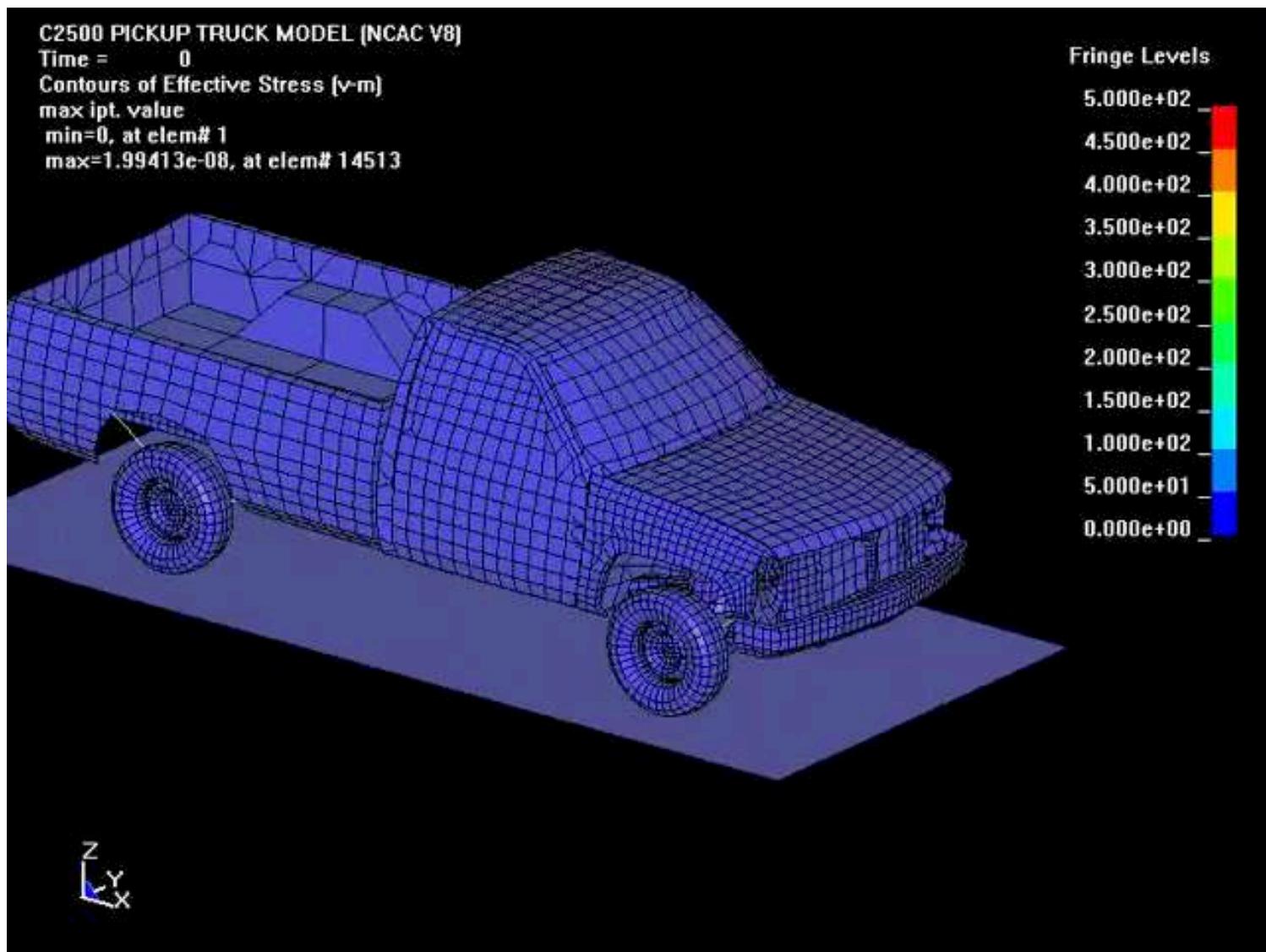
Step 6: Solve

- Now we have a large Linear System

$$K y = f$$

- Which we can solve anyway we like (though some ways are better than others)

So now you can do this:



Lots of Recent Fabrication Work Uses These Methods

- We'll talk more about “microstructures” later in the

Elastic Textures for Additive Fabrication

Julian Panetta*, Qingnan Zhou*, Luigi Malomo,
Nico Pietroni, Paolo Cignoni, Denis Zorin

(* Joint First Authors)

Still Lots of Work to Do

- Handling uncertainty (Shameless Self-Promotion)

Stochastic Structural Analysis for Context-Aware Design and Fabrication

Timothy Langlois*†|| Ariel Shamir*‡ Daniel Dror*‡ Wojciech Matusik*§ David I.W. Levin*¶

*Disney Research †Cornell University ‡Adobe Research §The Interdisciplinary Center ¶University of Toronto



Next Time

- We'll talk about where the Psi in this equation comes from

$$\sigma = \psi(\mathbf{E})$$