Simulating Deformable Objects

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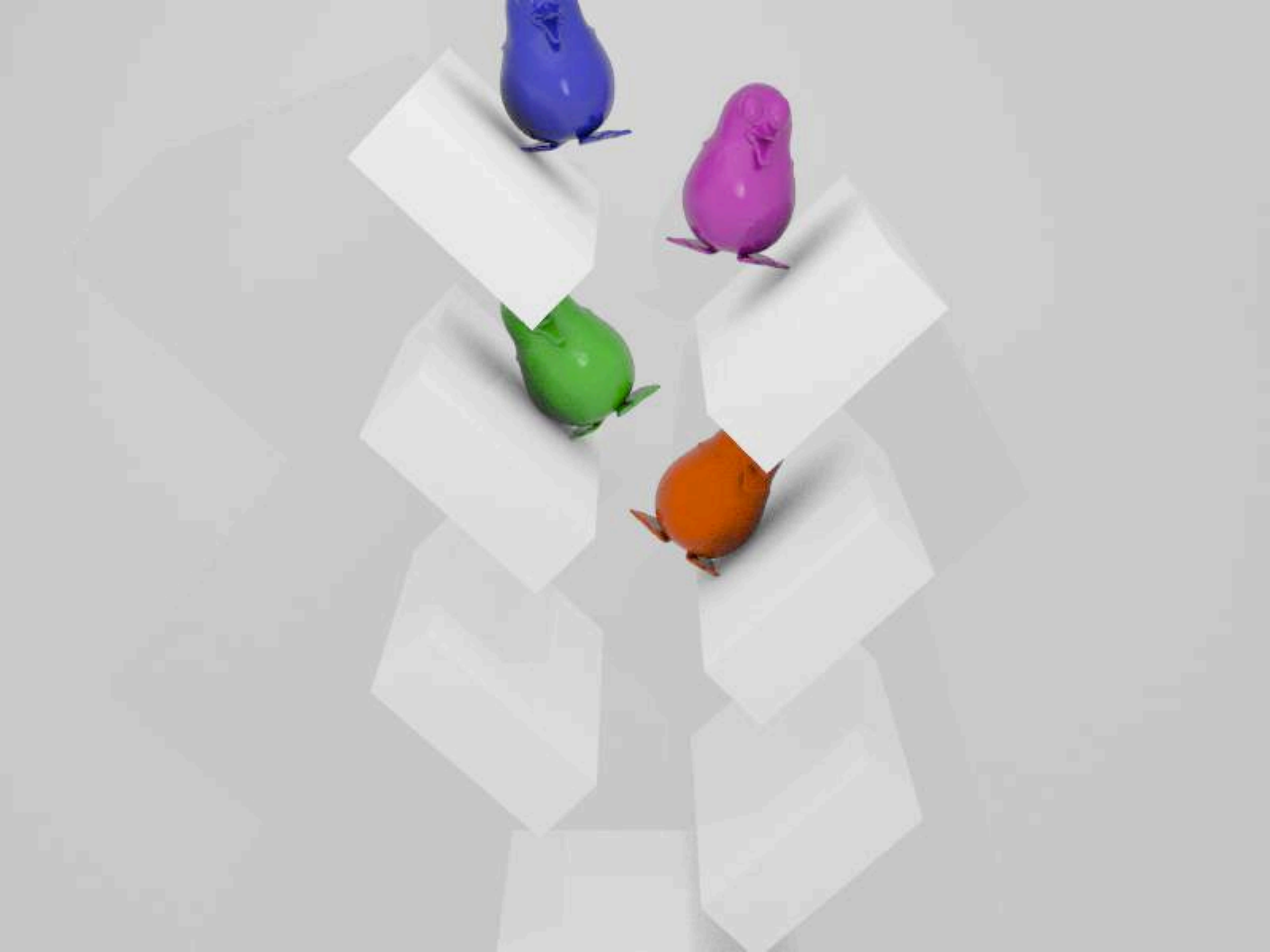
Questions

Plan for Today

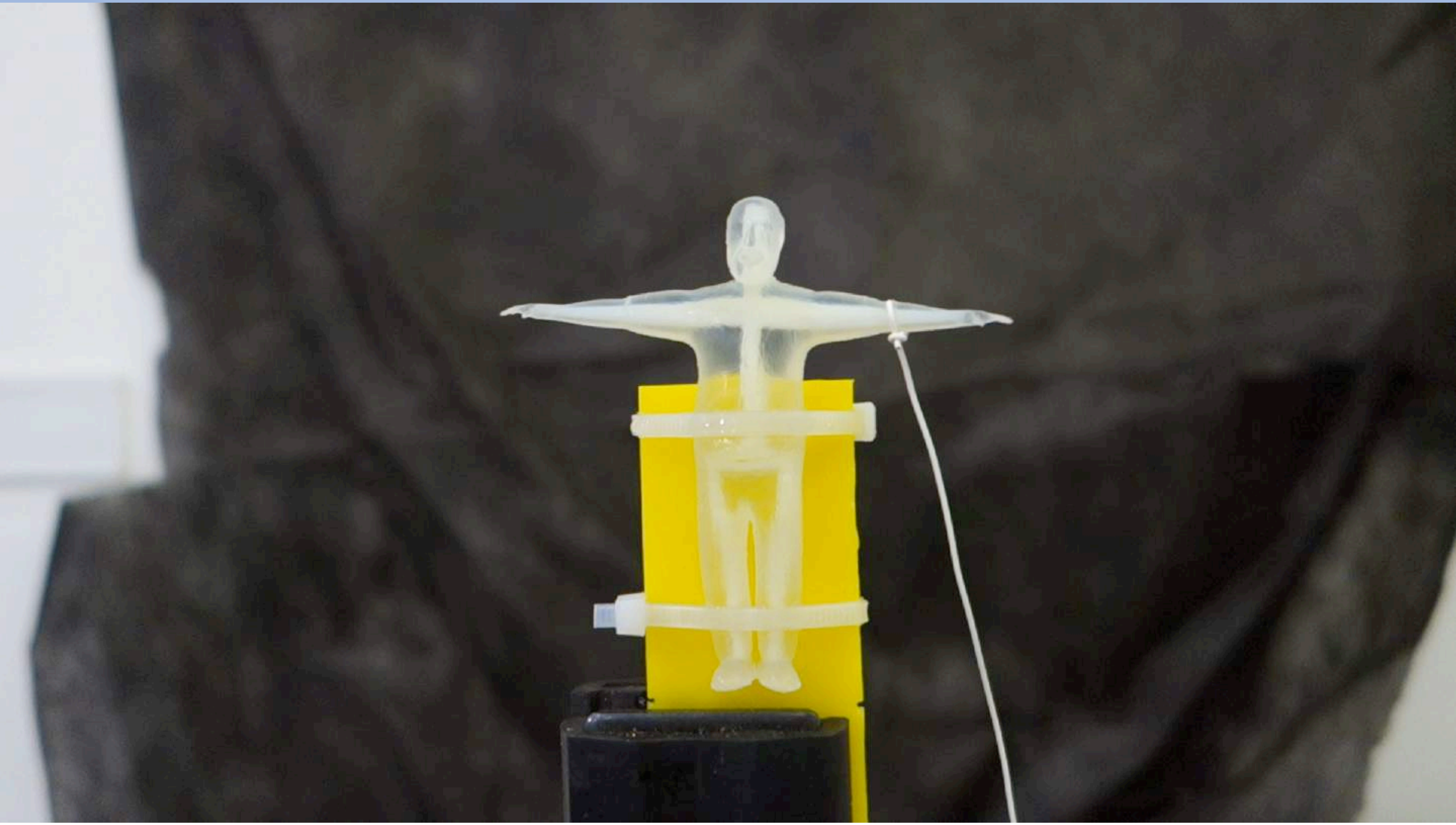
- Introduction to Physics Simulation
- Mass-Spring Simulation



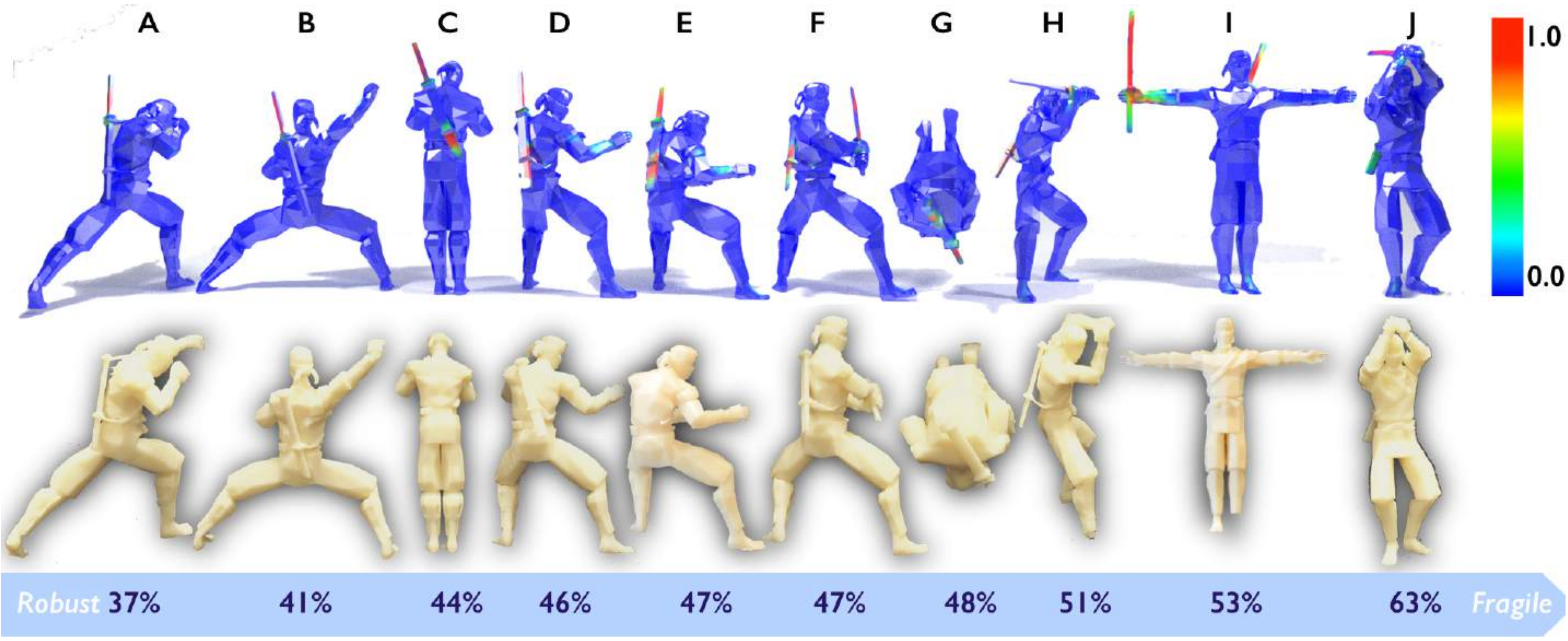
Simulation is a big word

- Definition: Imitation of a situation or process
- We are going to focus on simulation of deformable solid bodies like what we’ve seen in the preceding videos
- This is very useful for fabrication because …

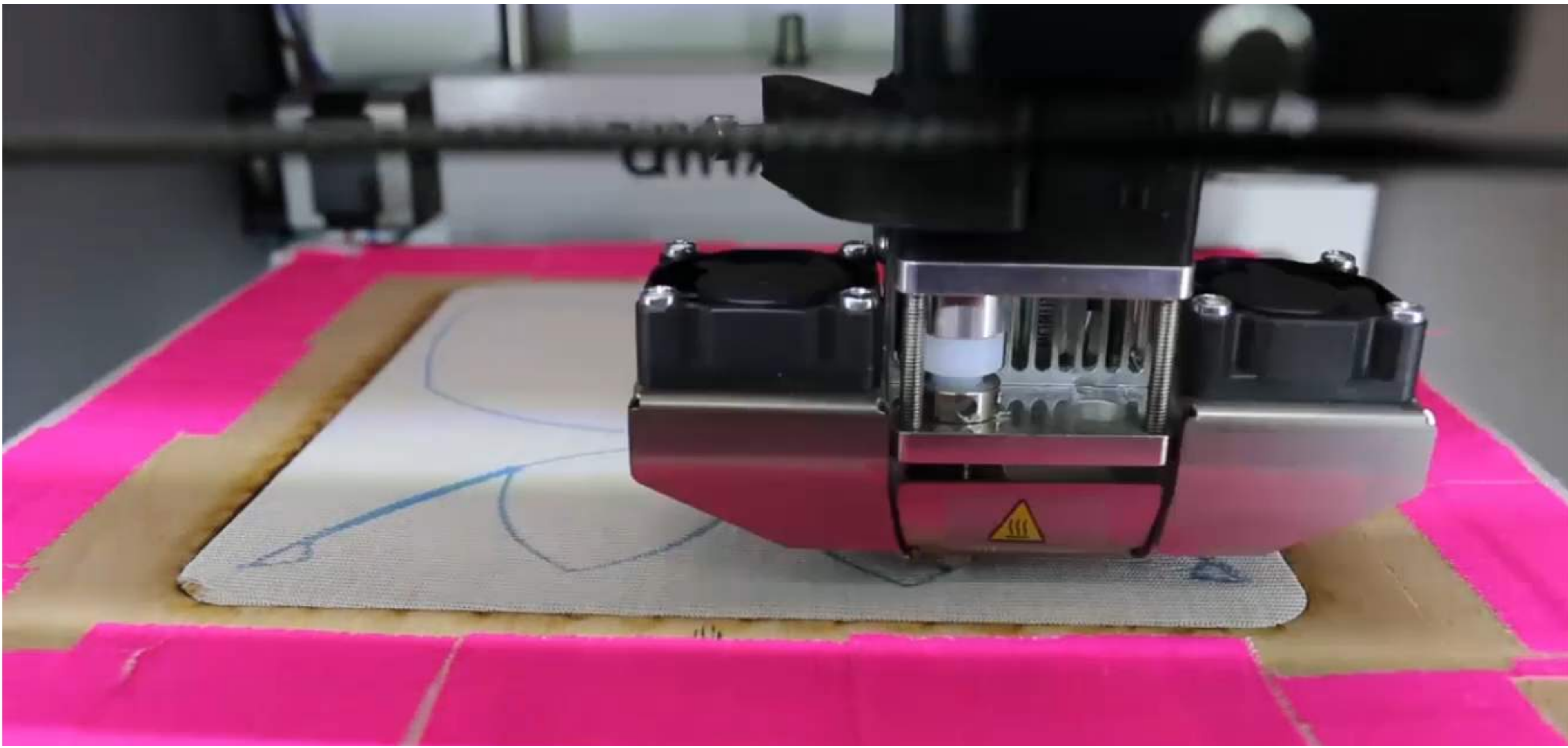
We might want to know how things bend



We might want to know when things break



We might want to know how things will change shape



Newton's Second Law

In 1D

mass (kg)

$$f = ma$$

force (N)

acceleration (m/s)

Statics vs. Dynamics

Newton's law describes time varying motion of an object.

Recall: $a = \ddot{x}(t) = \frac{d^2u}{dt^2}$

$\therefore$ Newton's law $\rightarrow m\ddot{x}(t) = F$

Integrate to get $x(t)$.

Statics vs. Dynamics

We call this time varying motion "Dynamics"

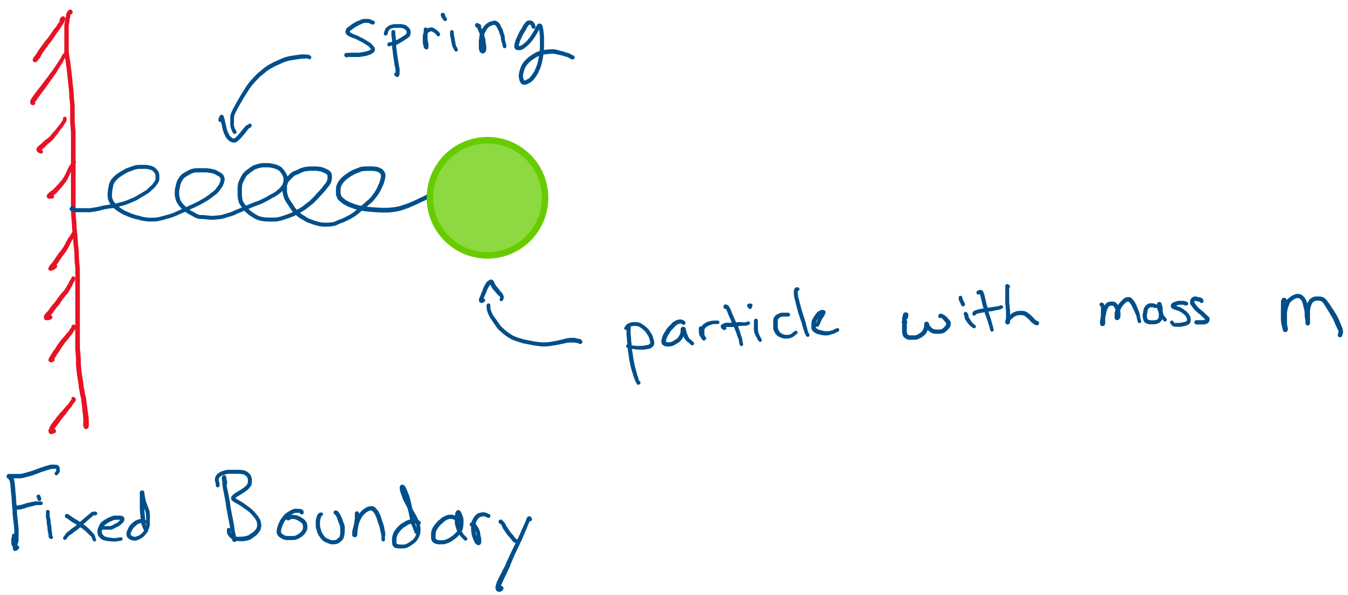
"Statics" is concerned with the case $\ddot{u}(t) = 0$, or $F = 0$

We'll talk more about statics next lecture.

Today: Crash Course in Dynamics

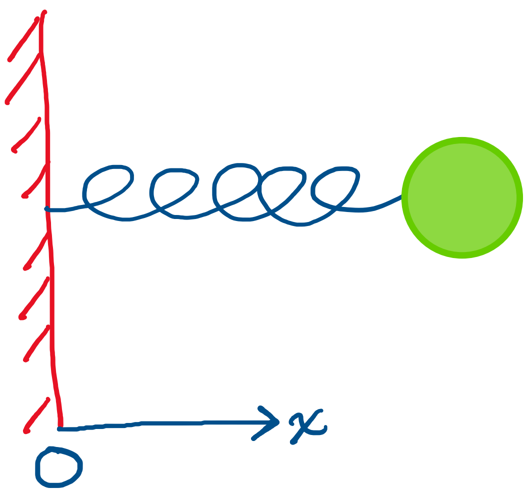
1. 1D Example Mass Spring System
2. Generalizing Newton's Law
    1. Kinetic and Potential Energy
3. 3D Mass Spring System
4. Numerical Time Integration
    1. Explicit Integration
    2. Implicit Integration
5. Implicit Integration as Optimization (If there's time)

1D Mass Spring System



Hooke's Law: Force exerted by a spring is linearly proportional to it's stretch

1D Mass Spring System



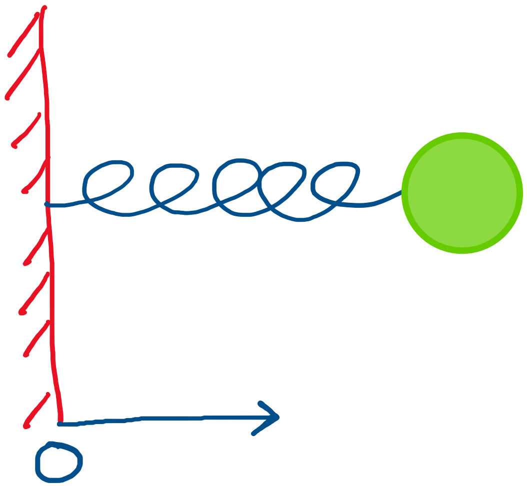
Assume spring has ∅ rest length.

i.e $F = 0$ when position of mass is $x = 0$

Let $x(t)$ be position of mass at time t

Let $F_{spring} = -\underbrace{k}_{\text{spring constant}} x(t)$

1D Mass Spring System



Newton's Law:

$$ma = F$$

$$m\ddot{x}(t) = -kx$$

Step 1: Rewrite as 1st order system

$$\begin{aligned} m\dot{v} &= -kx \\ \dot{x} &= v \end{aligned} \Rightarrow \begin{bmatrix} \dot{x} \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -\frac{k}{m} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \end{bmatrix}$$

Step 2: Eigen problem

$$\begin{bmatrix} x \\ v \end{bmatrix}(t) = \exp(At) \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

1D Mass Spring System

Generalizing Newton's Law

Re-interpret Newton's law:

$$m\ddot{x} = F \longrightarrow m\ddot{x} - F = 0$$

We want this to equal zero.

What question does this raise?

Generalizing Newton's Law

Re-interpret Newton's law:

$m\ddot{x} = f \longrightarrow m\ddot{x} - f = 0$

We want this to equal zero.

What question does this raise?

Can we solve this as an optimization?

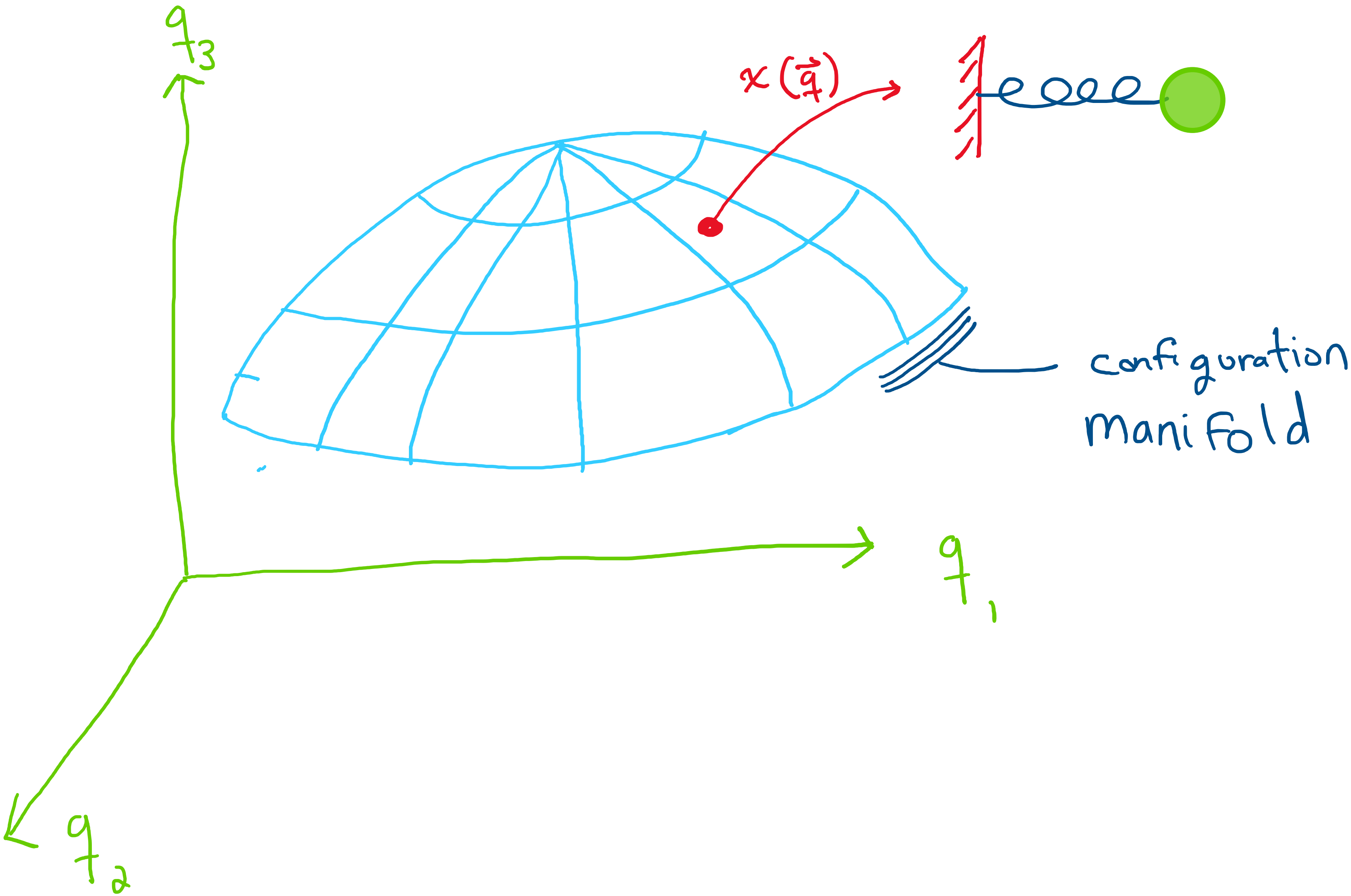
Kinetic and Potential Energy

We can but we need some new concepts.

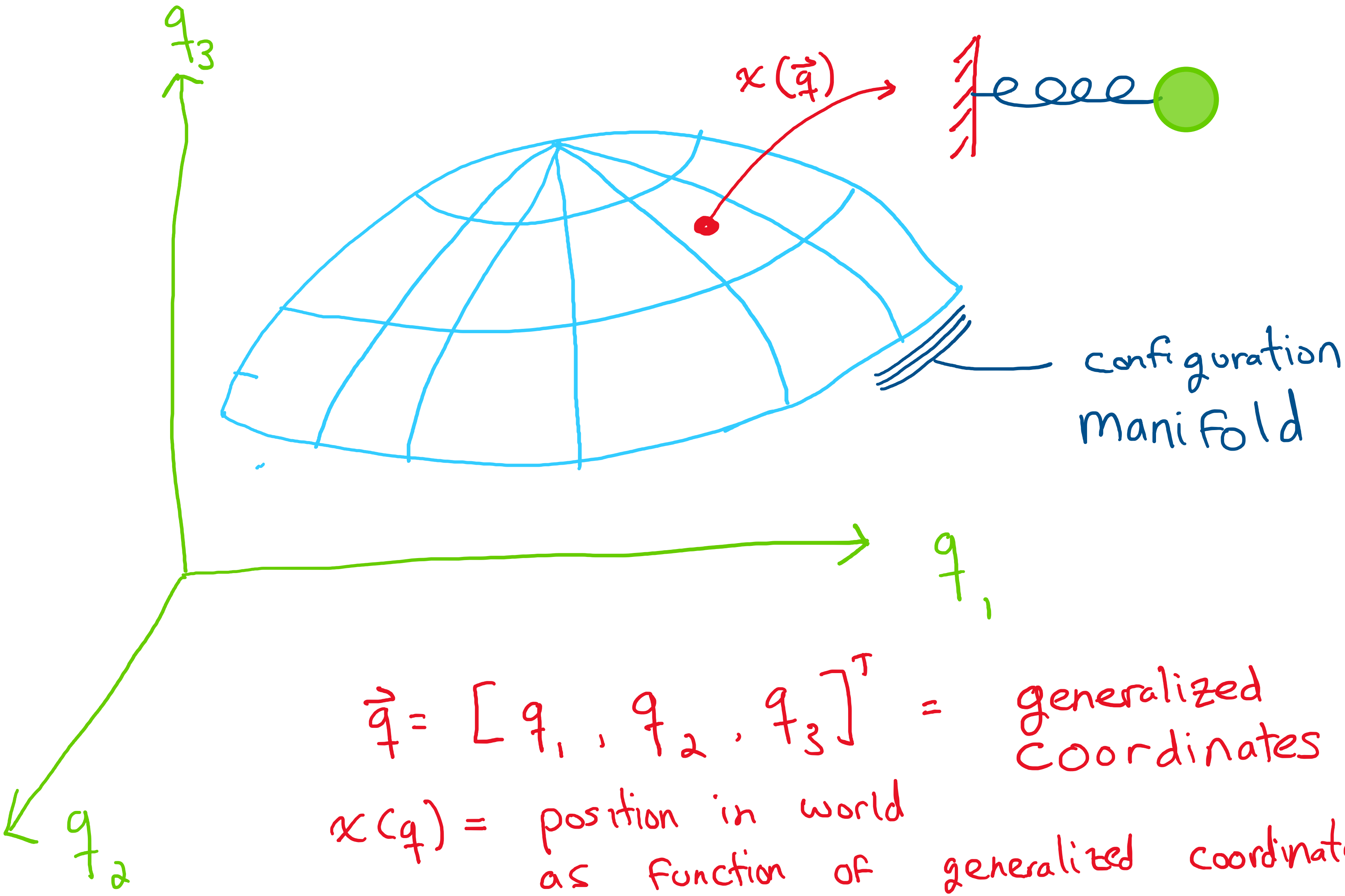
Kinetic Energy: Energy due to motion

Potential Energy: Energy due to position

Generalized Coordinates and Configuration Space



Generalized Coordinates and Configuration Space



$$\vec{q} = [q_1, q_2, q_3]^T = \text{generalized coordinates}$$

$x(q)$ = position in world as function of generalized coordinates

Generalizing Newton's Law

We are going to build a function called the Lagrangian:

$$L(\vec{q}) = T(\dot{q}) - V(q)$$

Potential Energy

Kinetic Energy

Generalizing Newton's Law

Hamilton's Principle of Least Action:

Define $S(q, \dot{q}) = \int_0^t T(\dot{q}(t)) - V(q(t))\, dt$

$\dot{q} = \frac{dq}{dt}$

Any physical trajectory $q(t)$, "extremizes" S.

Q: How do we extremize S?

Lagrangian Mechanics

This method of finding q is named after its inventor Lagrange.

Let's go through this for a single spring and mass in 3D

Start by choosing $\vec{q}(t) = \vec{x}(t)$

$\therefore \quad \vec{x}(q) = \vec{q}, \qquad \vec{q} \in \mathbb{R}^3$

Lagrangian Mechanics

Step 1: Define Kinetic Energy (T)

Standard Definition for a particle is

$$T = \frac{1}{2} m v(t)^T v(t)$$

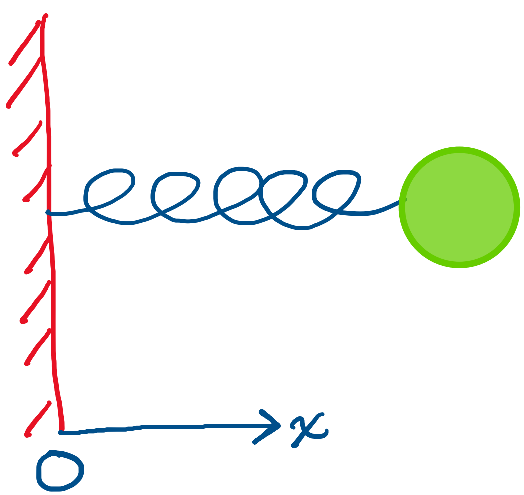
$$\hookrightarrow \frac{dx}{dt} = \frac{dq}{dt} = \dot{q}$$

$$\Rightarrow T(\dot{q}) = \frac{1}{2} m \dot{q}^T \dot{q}$$

Lagrangian Mechanics

Step 2: Define Potential Energy

We want an energy that is Ø when the spring is in it's rest configuration



when $x = q = 0$, potential energy is minimized.

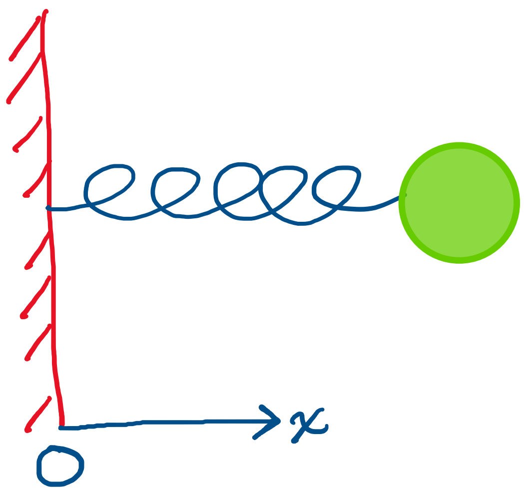
Q: What simple formula could we use ?

Lagrangian Mechanics

Step 2: Define Potential Energy

$$\text{Potential Energy} = \frac{1}{2}k\|x\|_2^2 = \frac{1}{2}k\,x^T x$$

$$= \frac{1}{2}k\,q^T q = V(q)$$

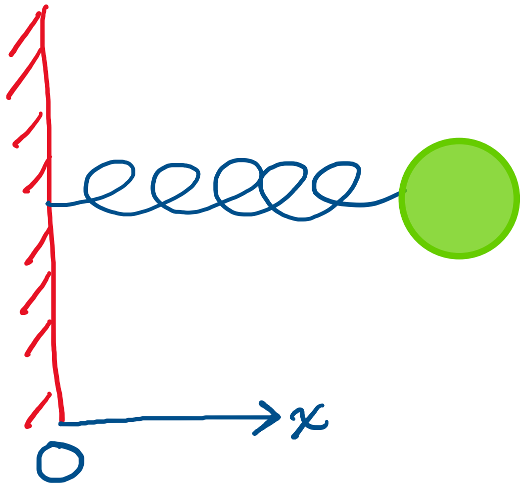


Lagrangian Mechanics

Step 3: Form Lagrangian & S

$$L = V - T = \frac{1}{2} m \dot{q}^T \dot{q} - \frac{1}{2} k q^T q$$

$$S(q, \dot{q}) = \int_t \frac{1}{2} m \dot{q}^T \dot{q} - \frac{1}{2} k q^T q \, dt$$

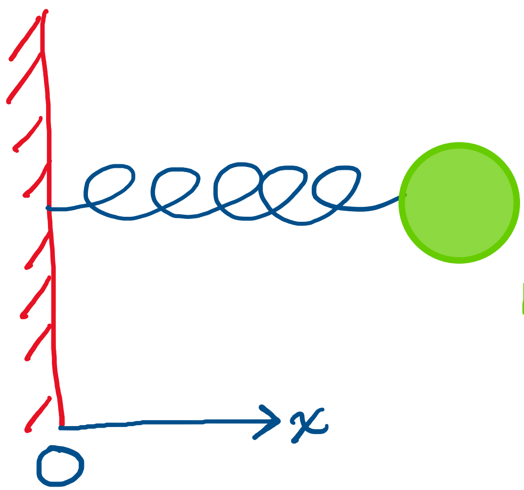


Lagrangian Mechanics

Step 4: Extremize!

Like optimization but with functions.

Given a function $f(y, \dot{y} \ldots)$, find function $y(t)$ that minimizes F



F - called a Functional

4a.) assume $y(t)$ extremizes F

4b.) perturb $y(t)$, make sure it doesn't change F

Lagrangian Mechanics

1.) $$S(q, \dot{q}) = \int_t \frac{1}{2} m \dot{q}^T \dot{q} - \frac{1}{2} k q^T q \, dt$$

2.) Assume $q, \dot{q}$ minimize S

3.) Perturb!

small perterbation

$$S(q + \delta q, \dot{q} + \delta \dot{q}) = ?$$

Hint: We saw this last lecture

Lagrangian Mechanics

3.) Taylor Expansion:

$$S(q+\delta q, \dot{q}+\delta\dot{q}) \approx S(q,\dot{q}) + \frac{\partial S}{\partial q}\delta q + \frac{\partial S}{\partial \dot{q}}\delta\dot{q}$$

$$\Rightarrow \int_t m\dot{q}\,\delta\dot{q} - kq\,dt$$

4.) Integration by parts

Lagrangian Mechanics

3.) Taylor Expansion:

$$S(q+\delta q, \dot{q}+\delta\dot{q}) \approx S(q,\dot{q}) + \frac{\partial S}{\partial q}\delta q + \frac{\partial S}{\partial \dot{q}}\delta\dot{q}$$

$$\Rightarrow \int_t m\dot{q}\,\delta\dot{q} - kq\,dt$$

4.) Integration by parts

$$m\dot{q}\,\delta q\Big|_0^t - \int_t m\ddot{q} + kq\,dt$$

Lagrangian Mechanics

Remember we want $S(q+\delta q, \dot{q}+\delta\dot{q}) = 0$

$$\Rightarrow \frac{\partial S}{\partial q}\delta q + \frac{\partial S}{\partial \dot{q}}\delta\dot{q} = 0$$

$$\Rightarrow m\dot{q}\,\delta q\Big|_{0}^{t} - \int_{t}(m\ddot{q} + kq)\,\delta q\,dt = 0$$

Lagrangian Mechanics

Remember we want $S(q+\delta q, \dot{q}+\delta\dot{q}) = 0$

$$\Rightarrow \frac{\partial S}{\partial q}\delta q + \frac{\partial S}{\partial \dot{q}}\delta\dot{q} = 0$$

$$\Rightarrow m\dot{q}\,\delta q\Big|_0^t - \int_t (m\ddot{q} + kq)\,\delta q\,dt = 0$$

$\delta q(t) = \delta q(0) = 0$, so just this has to be zero.

$m\ddot{q} = -kq \Rightarrow$ Newton's law! But it works for 3D points.

Lagrangian Mechanics

Important:

Force $= -\nabla_q$ Potential Energy

$$-kx = -kq = -\nabla_q \left(\frac{1}{2} k q^T q \right)$$

Lagrangian Mechanics

What about

Spring has rest length l_0

$\vec{q} = ?$

$T = ?$

$V = ?$

Lagrangian Mechanics

What about 

Spring has rest length l_0

$$\vec{q} = \left[\vec{x}_1^T \quad \vec{x}_2^T\right]^T$$

$$T = m\,\vec{x}_1^T\vec{x}_1 + m\,\vec{x}_2^T\vec{x}_2 = \vec{q}^T \overbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}^{M} \vec{q}$$

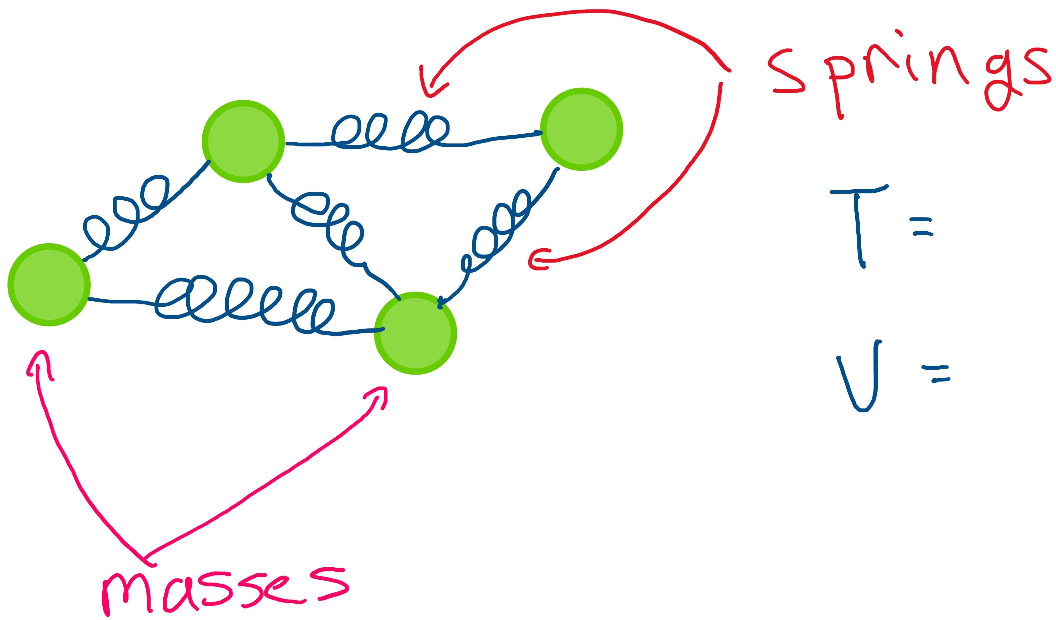
$$V = k\left(1 - \frac{|x_2 - x_1|}{l_0}\right)^2$$ ← measures stretch of spring

Can plug into S to get equation of the form

$$M\ddot{q} = F$$

3D Mass Spring System

Upgrade to more springs

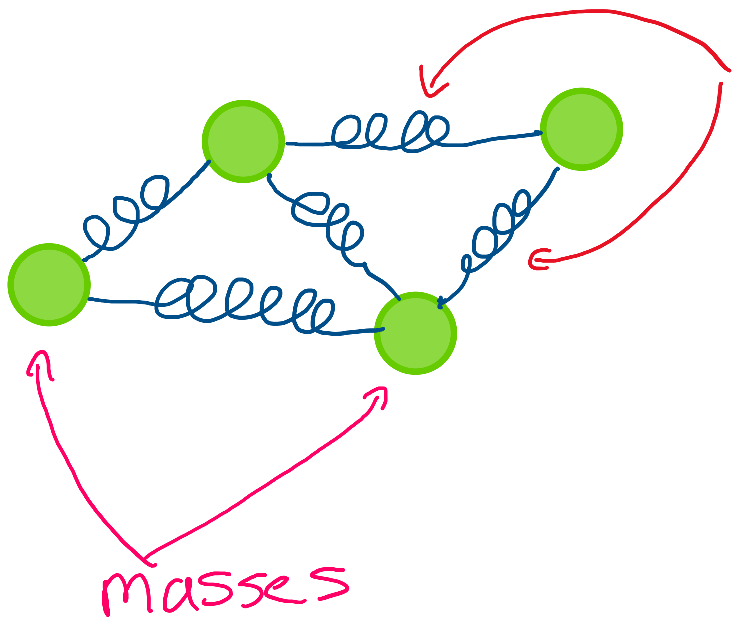


T = ?

V = ?

3D Mass Spring System

Upgrade to more springs

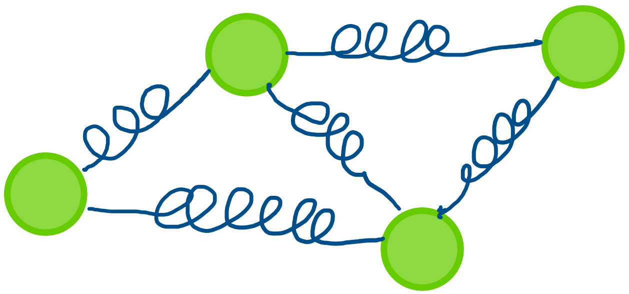


$$T = \;?$$

$$V = \sum_{i=1}^{\text{Num Springs}} k \left\| 1 - \frac{\overbrace{\left| x_1^i - x_2^i \right|}^{\text{END POINTS OF } i^{th} \text{ SPRING}}}{l_0} \right\|_2^2$$

Plug into $S \longrightarrow M\ddot{q} = f(q)$

3D Mass Spring System

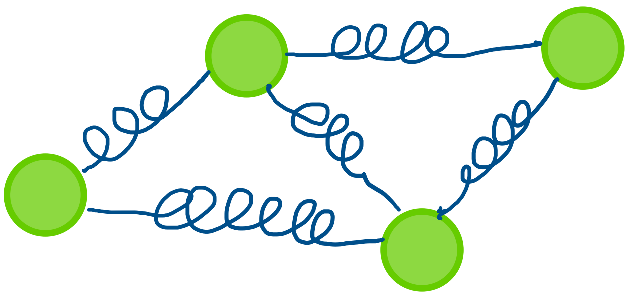


$$M\ddot{q} = f(q)$$

How do I solve this for q?

Numerical Integration!

Integration

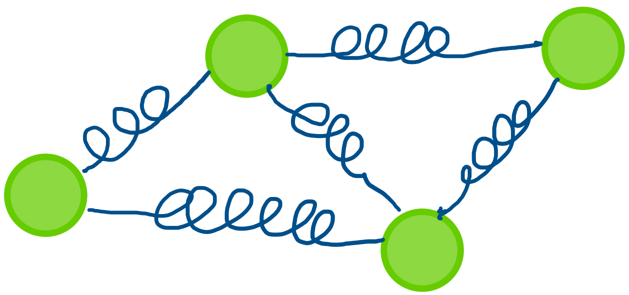


$$M\ddot{q} = f(q)$$

Bring back $v = \dot{q}$

$$\left.\begin{aligned} M\dot{v} &= f(q) \\ \dot{q} &= v \end{aligned}\right\}$$ What to do about derivatives?

Integration



$M\ddot{q} = f(q)$

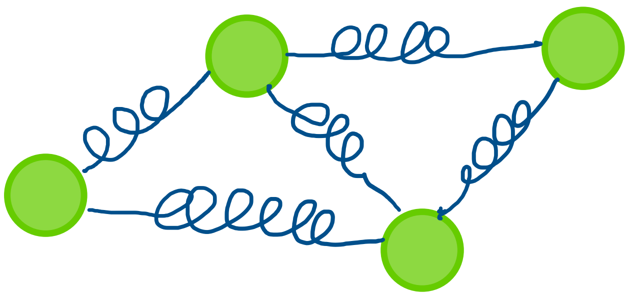
Bring back $v = \dot{q}$

$$\left.\begin{aligned} M\dot{v} &= f(q) \\ \dot{q} &= v \end{aligned}\right\}$$

What to do about derivatives?

A: Finite Difference

Integration



$$M\ddot{q} = f(q)$$

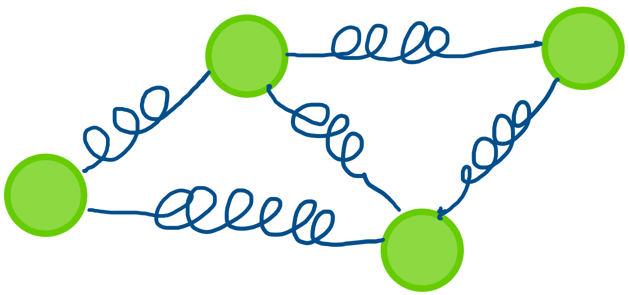
Bring back $V = \dot{q}$

$$\left.\begin{aligned} M\dot{v} &= f(q) \\ \dot{q} &= V \end{aligned}\right\} \text{ What to do about derivatives?}$$

A: Finite Difference

$$\dot{q} = \frac{q^{t+1} - q^{t}}{h}, \quad \dot{V} = \frac{V^{t+1} - v^{t}}{h}$$

Explicit Integration



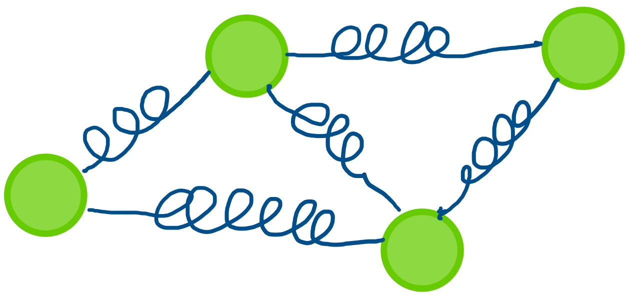
$$M \ddot{q} = f(q)$$

Bring back $v = \dot{q}$

explicit choice

$$M v^{t+1} = M v^t + h f(q^t)$$
$$q^{t+1} = q^t + h v^{t}$$

Explicit Integration



$$M\ddot{q} = f(q)$$

Bring back $v = \dot{q}$

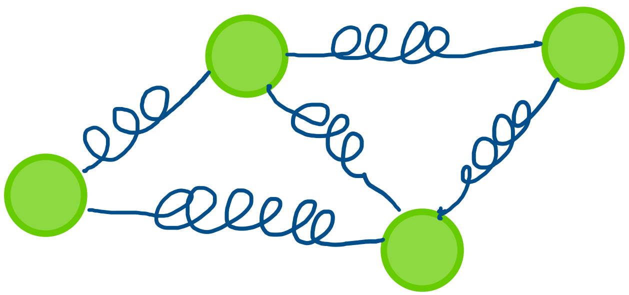
explicit choice

$$Mv^{t+1} = Mv^{t} + h\,f(q^{t})$$

$$q^{t+1} = q^{t} + h\,v^{t}$$

Explicit Euler $\rightarrow$ UNSTABLE FOR SPRINGS ☹

Implicit Integration



$$M\ddot{q} = f(q)$$

Bring back $v = \dot{q}$

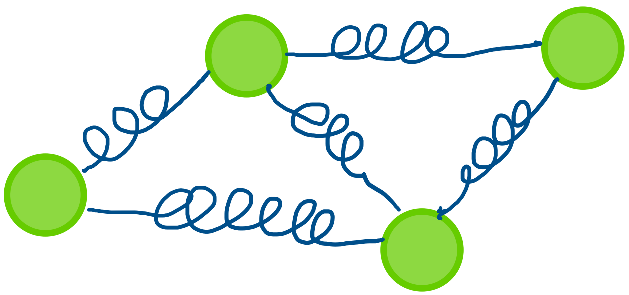
Implicit Choice

$$Mv^{t+1} = Mv^{t} + h\,f(q^{t+1})$$

$$q^{t+1} = q^{t} + h\,v^{t+1}$$

How do we solve this ?

Implicit Integration



$$M\ddot{q} = f(q)$$

Bring back $v = \dot{q}$

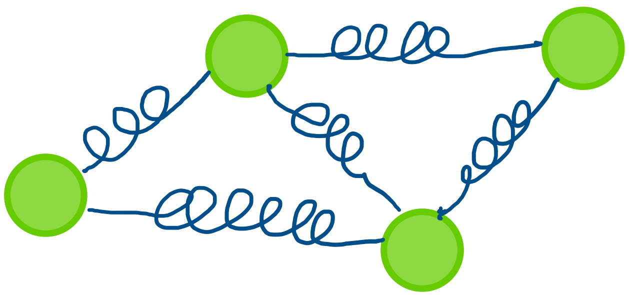
Implicit Choice

$$M v^{t+1} = M v^{t} + h f(q^{t} + h v^{t+1})$$

$$q^{t+1} = q^{t} + h v^{t+1}$$

How do we solve this ?

Semi-Implicit Integration



$$M\ddot{q} = f(q)$$

Bring back $v = \dot{q}$

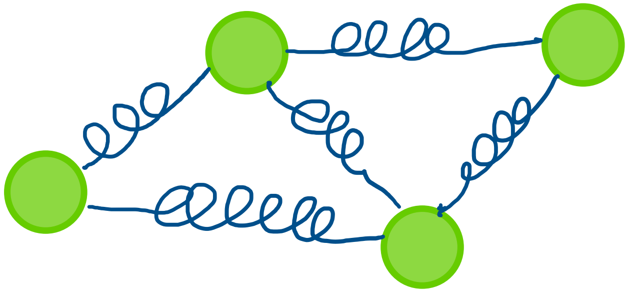
Semi-Implicit Choice

$$Mv^{t+1} = Mv^{t} + h^{2}\left.\frac{\partial f}{\partial q}\right|_{t} v^{t+1} + hf(q^{t})$$

$$q^{t+1} = q^{t} + hv^{t+1}$$

How do we solve this?

Semi-Implicit Integration



$$M\ddot{q} = f(q)$$

Bring back $v = \dot{q}$

$$(M - h^2 K)\, v^{t+1} = M v^t + h f(q^t)$$

$$q^{t+1} = q^t + h v^{t+1}$$

How do we solve this ?

We can solve this

Stiffness Matrix

Important: $K = - \dfrac{\partial^2 \text{Potential Energy}}{\partial q^2}$

For assignment 2 you need to derive and implement K for a single spring!

Implicit Integration as Optimization

Let's define an energy

$$E(v^{t+1}) = \frac{1}{2}(v^{t+1} - v^t)^T M (v^{t+1} - v^t) + \Psi(q^{t+1})$$

Then

$$v^{t+1^*} = \operatorname{argmin} E(v^{t+1}) \iff M v^{t+1^*} = F(q^{t+1^*})$$

How do we prove this?

Implicit Integration as Optimization

Let's define an energy

$$E(v^{t+1}) = \frac{1}{2}(v^{t+1} - v^t)^T M (v^{t+1} - v^t) + \Psi(q^{t+1})$$

Then

$$v^{t+1^*} = \operatorname{argmin} E(v^{t+1}) \iff Mv^{t+1^*} = hF(q^{t+1^*})$$

How do we prove this?

Show $\nabla_{v^{t+1}} E(v^{t+1^*}) = 0 \iff Mv^{t+1^*} = hf(q^{t+1^*})$

In Action

Fast Simulation of Mass Spring Systems, Liu et al. SIGGRAPH Asia 2013