

1. The risk modeling

Given

$$y_n = \beta_n^T \mathcal{Z} + \sqrt{1 - \beta_n^T \beta_n} \epsilon_n$$

where $\mathcal{Z} \sim \mathcal{N}(0, I_S)$ and $\epsilon \sim \mathcal{N}(0, I_N)$, the resulting $y_n \sim \mathcal{N}(0, 1)$ as shown below

$$\mathbb{E} \{y_n\} = \sum_i \beta_{n,i} \mathbb{E} \{\mathcal{Z}_i\} + \sqrt{1 - \beta_n^T \beta_n} \mathbb{E} \{\epsilon_n\} = 0$$

$$\begin{aligned} \text{var} \{y_n\} &= \mathbb{E} \{(y_n - \mathbb{E} \{y_n\})^2\} \\ &= \mathbb{E} \left\{ (\beta_n^T \mathcal{Z} + \sqrt{1 - \beta_n^T \beta_n} \epsilon_n)^2 \right\} \\ &= \beta_n^T \beta_n \mathbb{E} \{(\mathcal{Z} - 0)^2\} + (1 - \beta_n^T \beta_n) \mathbb{E} \{(\epsilon_n - 0)^2\} \\ &= \beta_n^T \beta_n \text{var} \{\mathcal{Z}\} + (1 - \beta_n^T \beta_n) \text{var} \{\epsilon_n\} \\ &= 1 \end{aligned}$$

2. Motivation for threshold between different states $H_{c(n)}^c$

The motivation is to model discrete probability in the credit state matrix with a continuous distribution such as the gaussian. In this case, we want to set $H_{c(n)}^c$ such that

$$p(H_{c(n)}^{c-1} \leq y_n \leq H_{c(n)}^c) = p_{c(n)}^c$$

therefore we can write

$$p(y_n \leq H_{c(n)}^c) = \sum_{\gamma=1}^c p_{c(n)}^\gamma \quad y_n \xrightarrow{\mathcal{N}(0,1)} \quad \Phi(H_{c(n)}^c) = \sum_{\gamma=1}^c p_{c(n)}^\gamma$$

3. Confidence interval for monte carlo estimation

$$p(L_N(\mathcal{Z}, \epsilon) \geq l) \in p(L_N(\mathcal{Z}, \epsilon) \geq l) \pm CI$$

Idea is that for the two naive algorithms that are purported to be equivalent, the CI should be approximately the same

1. Likelihood Function for Two Level IS

Likelihood for the inner sampling conditioned on Z is given by

$$e^{\theta_x(Z)L + \psi(\theta_x(Z), Z)} \quad \text{where} \quad \psi(\theta) = \sum_{k=1}^m \log(1 + p_k(e^{\theta c_k} - 1))$$

The likelihood function for the outer sampling of Z consists of the following change of distribution

$$Z \sim \mathcal{N}(0, I) \quad \longrightarrow \quad Z \sim \mathcal{N}(\mu, I)$$

where μ is the twisting parameter for the outer importance sampling such that the resulting shifted normal distribution resembles the zero variance IS distribution, in other words,

$$\mu = \max_z P(L > x | Z = z) e^{\frac{-z^T z}{2}}$$

Then the likelihood for the outer IS is then

$$\begin{aligned} \frac{\mathcal{N}(0, I)}{\mathcal{N}(\mu, I)} &= \frac{\exp(-\frac{1}{2}z^T z)}{\exp(-\frac{1}{2}(z - \mu)^T(z - \mu))} \\ &= \exp\left(-\frac{1}{2}z^T z - \frac{1}{2}z^T z + z^T \mu - \frac{1}{2}\mu^T \mu\right) \\ &= e^{-\mu^T Z + \mu^T \mu/2} \end{aligned}$$

Therefore, the estimator for probability of tail event is given by

$$\mathbb{1}_{L > x} e^{\theta_x(Z)L + \psi(\theta_x(Z), Z)} e^{-\mu^T Z + \mu^T \mu/2}$$

Likelihood ratio expression

Show

$$\prod_{k=1}^m \left(\frac{p_k}{p_{k,\theta}} \right)^{Y_k} \left(\frac{1-p_k}{1-p_{k,\theta}} \right)^{1-Y_k} = e^{-\theta L + \phi(\theta)} \quad \text{where} \quad \phi(\theta) = \sum_{k=1}^m \log \left(1 + p_k (e^{\theta c_k} - 1) \right)$$

Proof.

$$\begin{aligned} lhs &= \exp \left\{ \sum_{k=1}^m Y_k \log \left(\frac{p_k}{p_{k,\theta}} \right) + (1 - Y_k) \log \left(\frac{1-p_k}{1-p_{k,\theta}} \right) \right\} \\ &= \exp \left\{ \sum_{k=1}^m Y_k \log \left(1 + p_k (e^{\theta c_k} - 1) \right) - Y_k \theta c_k + (1 - Y_k) \log \left(1 + p_k (e^{\theta c_k} - 1) \right) \right\} = rhs \end{aligned}$$

where

$$1 - p_{k,\theta} = \frac{1 - p_k}{1 + p_k (e^{\theta c_k} - 1)}$$

□

Glasserman&Li Algorithm

1. Outer Level IS for systematic risk Z

- (a) Find shifted parameter μ for outer IS for Z
- (b) Sample $Z \sim \mathcal{N}(\mu, I)$

2. Inner Level IS for each default indicators Y_k

- (a) Calculate conditional default probabilities $p_k(Z)$ for $k = 1, \dots, m$

$$p_k(Z) = P(Y_k = 1|Z) = p(X_k > x_k|Z) = P(a_k Z + b_k \epsilon_k > \Phi^{-1}(1 - p_k)|Z)$$

- (b) Compute the twisted parameters $\theta_x(Z)$

$$\theta_x(Z) = \begin{cases} \text{solution to } \frac{\partial}{\partial \theta} \psi_m(\theta, Z) = x & \psi'(0) = \mathbb{E}_p \{L|Z\} = \sum_{k=1}^m p_k(Z) c_k < x \\ 0 & \text{otherwise} \end{cases}$$

- (c) Compute default indicators (bernoulli) from twisted conditional default probabilities

$$p_{k, \theta_x(Z)}(Z) = \frac{p_k(Z) e^{\theta_x(Z) c_k}}{1 + p_k(Z) (e^{\theta_x(Z) c_k} - 1)} \quad k = 1, \dots, m$$

- (d) Compute Loss $L = c_1 Y_1 + \dots + c_m Y_m$ under twisted distribution

3. Return the estimator of tail probabilities

$$\mathbb{1}_{L > x} e^{\theta_x(Z) L + \psi(\theta_x(Z), Z)} e^{-\mu^T Z + \mu^T \mu / 2}$$

Therefore,

$$P(L > x) = \mathbb{E}_{Z \sim \mathcal{N}(\mu, I) \ Y_k \sim p_{k, \theta_x(Z)}} \left\{ \mathbb{1}_{L > x} e^{\theta_x(Z) L + \psi(\theta_x(Z), Z)} e^{-\mu^T Z + \mu^T \mu / 2} \right\}$$

Optimization

Want to solve for

$$\max_x \min_y f(x, y)$$

Same as

$$\max_x f(x, \hat{y}(x)) \quad \text{where} \quad \hat{y}(x) = \arg \min_y f(x, y)$$

Simply we write as a function of 1 variable

$$\max_x \hat{f}(x) \quad \text{where} \quad \hat{f}(x) = f(x, \hat{y}(x))$$

Want to compute the 1st order and 2nd order derivatives

$$\hat{f}'(x) = \frac{\partial f}{\partial x} f(x, \hat{y}(x)) + \frac{\partial f}{\partial y} f(x, \hat{y}(x)) \hat{y}'(x)$$

Want to compute $\hat{y}'(x)$ first find critical points

$$\frac{\partial f}{\partial y} f(x, y)|_{y=\hat{y}(x)} = 0$$

Solve for the function

$$f_{yx} = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x, y)|_{y=\hat{y}(x)}$$

Solve for $\hat{y}'(x)$

$$f_{yx}(x, \hat{y}(x)) + f_{yy}(x, \hat{y}(x)) \hat{y}'(x) = 0 \quad \rightarrow \quad \hat{y}'(x) = -\frac{f_{yx}(x, \hat{y}(x))}{f_{yy}(x, \hat{y}(x))}$$

Then compute second derivative, i.e. $\hat{f}''(x)$