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Appendix to "Fast Simulation of Multifactor Portfolio Credit Risk"

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A Appendix: Proofs

A.1 Small Default Probability Regime: Proof of Theorem 2

To compute the upper bound of the second moment of the IS estimator, we decompose it into a sum of two terms. Recall that $\mathsf{E}_m[\cdot]$ denotes expectation under the IS distribution. For any $\theta > 0$,

$$
\begin{aligned}
&M_2(x_m, \theta_m(\mathbf{Z})) \\
&= \mathsf{E}_m\left[\mathbf{1}\{L_m > x_m\} e^{-2\theta_m(\mathbf{Z})L_m + 2m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})} \left(\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right)^2\right] \\
&= \mathsf{E}\left[\mathbf{1}\{L_m > x_m\} e^{-\theta_m(\mathbf{Z})L_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right] \quad \text{by recovering original measure} \\
&\le \mathsf{E}\left[e^{-\theta_m(\mathbf{Z})x_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right] \quad \text{dropping the indicator} \\
&= \mathsf{E}\left[e^{-\theta_m(\mathbf{Z})x_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \mathbf{1}_{\left(G^{(m)}\right)^c}(\mathbf{Z})\right] \\
&\quad + \mathsf{E}\left[e^{-\theta_m(\mathbf{Z})x_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \mathbf{1}_{G^{(m)}}(\mathbf{Z})\right] \\
&\le \mathsf{E}\left[e^{-\theta x_m + m\psi_m(\theta,\mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \mathbf{1}_{\left(G^{(m)}\right)^c}(\mathbf{Z})\right] \quad \text{by definition (7)} \qquad (27) \\
&\quad + \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \mathbf{1}_{G^{(m)}}(\mathbf{Z})\right] \quad \text{by definition (7) and replacing } \theta_m(\mathbf{Z}) \text{ by } 0\ . \qquad (28)
\end{aligned}
$$

Then we find the upper bounds of the last two terms in (27) and (28).

We start our proof from the existence of some limits being used in the statements of theorems. The following corollary guarantees the boundedness of $\boldsymbol{\mu}_{\mathcal{J}}^{(m)}$ and $\boldsymbol{\mu}^{(m)}$ if Assumptions 1 and 2 are satisfied, respectively.

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Corollary 1 *$G_{\mathcal{J}}^{(m)} \neq \emptyset$ for all m if and only if $G_{\mathcal{J}} \neq \emptyset$.*

Proof: See Corollary 3.1 in GKS [3]. ■

Consider $\mathcal{J}$ satisfying $G_{\mathcal{J}} \neq \emptyset$. This implies $\left\|\boldsymbol{\gamma}_{\mathcal{J}}\right\| < \infty$. Then, by Corollary 1, $G_{\mathcal{J}}^{(m)}$ are nonempty and $\boldsymbol{\mu}_{\mathcal{J}}^{(m)}$ are well defined. Furthermore, the following result is shown in GKS [3]:

$$\boldsymbol{\gamma}_{\mathcal{J}} = \lim_{m\to\infty} \frac{1}{\sqrt{m}} \boldsymbol{\mu}_{\mathcal{J}}^{(m)}. \tag{29}$$

Lemma 3 *Assume that $\{\mathbf{v}_m\}_{m=1}^{\infty} \subset \mathbb{R}^d$ and a function $f(\cdot)$ satisfy*

$$\lim_{m\to\infty} f(m) = \infty \quad \text{and} \quad 0 < \lim_{m\to\infty} \frac{1}{f(m)} \|\mathbf{v}_m\|^2 < \infty.$$

Then for d-dimensional standard normal vector $\mathbf{Z}$,

$$\limsup_{m\to\infty} \frac{1}{f(m)} \log \mathsf{P}\left(\|\mathbf{Z}\| \geq \| \mathbf{v}_m\|\right) \leq - \lim_{m\to\infty} \frac{1}{2f(m)} \| \mathbf{v}_m\|^2$$

Proof: $\|\mathbf{Z}\|^2$ is a random variable of χ_d^2-distribution and has density function $c \cdot y^{\frac{d}{2}-1} e^{-y/2} \mathbf{1}_{[0,\infty)}$ where $c = \frac{1}{\Gamma(d/2)2^{d/2}}$ is a constant. Choose $0 < \epsilon < 1$. There exists an M such that for all $y \geq M$, $y^{\frac{d}{2}-1} \leq e^{\epsilon y/2}$. Then for m satisfying $\| \mathbf{v}_m\|^2 \geq M$,

$$\begin{aligned}
\mathsf{P}\left(\|\mathbf{Z}\| \geq \| \mathbf{v}_m\|\right) &= \mathsf{P}\left(\|\mathbf{Z}\|^2 \geq \| \mathbf{v}_m\|^2\right) \\
&= \int_{\| \mathbf{v}_m\|^2}^{\infty} c \cdot y^{\frac{d}{2}-1} e^{-y/2} dy \\
&\leq \int_{\| \mathbf{v}_m\|^2}^{\infty} c \cdot e^{-(1-\epsilon)y/2} dy \\
&= \frac{2c}{1-\epsilon} e^{-\frac{1-\epsilon}{2}\| \mathbf{v}_m\|^2}
\end{aligned} \tag{30}$$

From the conditions, $\lim_{m\to\infty} \|\mathbf{v}_m\| = \infty$ holds and hence (30) is true for all large m.

$$\begin{aligned}
\limsup_{m\to\infty} \frac{1}{f(m)} \log \mathsf{P}\left(\|\mathbf{Z}\| \geq \| \mathbf{v}_m\|\right) &\leq \limsup_{m\to\infty} \frac{1}{f(m)} \log\left(\frac{2c}{1-\epsilon}\right) - \liminf_{m\to\infty} \frac{1-\epsilon}{2f(m)} \| \mathbf{v}_m\|^2 \\
&= -(1-\epsilon) \lim_{m\to\infty} \frac{1}{2f(m)} \| \mathbf{v}_m\|^2 .
\end{aligned}$$

ϵ was arbitrary and we complete the proof. ■

Set $f(m) = m$ and $\mathbf{v}_m = \boldsymbol{\mu}_{\mathcal{J}}^{(m)}$. By (29), this setting satisfies the conditions of Lemma 3 and we get the following corollary.

Corollary 2 *For each $\mathcal{J} \in \mathcal{S}_q$,*

$$\limsup_{m\to\infty} \frac{1}{m} \log \mathsf{P}\left(\|\mathbf{Z}\| \geq \left\|\boldsymbol{\mu}_{\mathcal{J}}^{(m)}\right\|\right) \leq -\frac{1}{2} \left\|\boldsymbol{\gamma}_{\mathcal{J}}\right\|^2 .$$

From the definition (17), it can be shown that

Lemma 4 *For each $\mathcal{J} \in \mathcal{S}_q$,*

$$\min \left\{ {\boldsymbol{\mu}_{\mathcal{J}}^{(m)}}^{\top} \mathbf{z} \ : \ \mathbf{z} \in G_{\mathcal{J}}^{(m)} \right\} = \left\| \boldsymbol{\mu}_{\mathcal{J}}^{(m)} \right\|^2 \ . \tag{31}$$

Proof: See the proof of Lemma 3.3 in GKS [3]. ■

For easy reference we state the following lemma whose proof can be found on page 7 in Dembo and Zeitouni [2];

Lemma 5 *Let N be a fixed integer. Then, for every $a_\epsilon^i \geq 0$,*

$$\limsup_{\epsilon \to 0} \ \epsilon \log \left(\sum_{i=1}^{N} a_\epsilon^i \right) = \max_{1 \leq i \leq N} \limsup_{\epsilon \to 0} \ \epsilon \log a_\epsilon^i \ .$$

Using Corollary 2 and Lemma 4, we prove the following lemma.

Lemma 6 *If $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I}_d)$ and $\mathcal{S}_q \neq \emptyset$, then*

$$\limsup_{m \to \infty} \frac{1}{m} \log \mathsf{E} \left[\left(M_{\mathbf{Z}}^{(m)} \right)^{-1} \ \mathbf{1}_{G^{(m)}}(\mathbf{Z}) \right] \leq - \left\| \boldsymbol{\gamma}_* \right\|^2 \ .$$

Proof:

$$\begin{aligned}
&\mathsf{E} \left[\left(M_{\mathbf{Z}}^{(m)} \right)^{-1} \ \mathbf{1}_{G^{(m)}}(\mathbf{Z}) \right] \\
&\quad = \ \mathsf{E} \left[\left(M_{\mathbf{Z}}^{(m)} \right)^{-1} \ \mathbf{1}_{\bigcup_{\mathcal{J} \in \mathcal{S}_q} G_{\mathcal{J}}^{(m)}}(\mathbf{Z}) \right] \\
&\quad \leq \ \mathsf{E} \left[\left(M_{\mathbf{Z}}^{(m)} \right)^{-1} \ \sum_{\mathcal{J} \in \mathcal{S}_q} \mathbf{1}_{G_{\mathcal{J}}^{(m)}}(\mathbf{Z}) \right] \\
&\quad = \ \sum_{\mathcal{J} \in \mathcal{S}_q} \mathsf{E} \left[\left(\sum_{\mathcal{J}' \in \mathcal{S}_q} \lambda_{\mathcal{J}'} \exp \left({\boldsymbol{\mu}_{\mathcal{J}'}^{(m)}}^{\top} \mathbf{Z} - \frac{1}{2} {\boldsymbol{\mu}_{\mathcal{J}'}^{(m)}}^{\top} \boldsymbol{\mu}_{\mathcal{J}'}^{(m)} \right) \right)^{-1} \ \mathbf{1}_{G_{\mathcal{J}}^{(m)}}(\mathbf{Z}) \right] \\
&\quad \leq \ \sum_{\mathcal{J} \in \mathcal{S}_q} \mathsf{E} \left[\frac{1}{\lambda_{\mathcal{J}}} \exp \left(- {\boldsymbol{\mu}_{\mathcal{J}}^{(m)}}^{\top} \mathbf{Z} + \frac{1}{2} {\boldsymbol{\mu}_{\mathcal{J}}^{(m)}}^{\top} \boldsymbol{\mu}_{\mathcal{J}}^{(m)} \right) \ \mathbf{1}_{G_{\mathcal{J}}^{(m)}}(\mathbf{Z}) \right] \ .
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\limsup_{m \to \infty} \frac{1}{m} \log \mathsf{E} \left[\left(M_{\mathbf{Z}}^{(m)} \right)^{-1} \ \mathbf{1}_{G^{(m)}}(\mathbf{Z}) \right] \\
&\quad \leq \ \limsup_{m \to \infty} \frac{1}{m} \log \left(\sum_{\mathcal{J} \in \mathcal{S}_q} \mathsf{E} \left[\frac{1}{\lambda_{\mathcal{J}}} \exp \left(- {\boldsymbol{\mu}_{\mathcal{J}}^{(m)}}^{\top} \mathbf{Z} + \frac{1}{2} {\boldsymbol{\mu}_{\mathcal{J}}^{(m)}}^{\top} \boldsymbol{\mu}_{\mathcal{J}}^{(m)} \right) \ \mathbf{1}_{G_{\mathcal{J}}^{(m)}}(\mathbf{Z}) \right] \right) \\
&\quad = \ \max_{\mathcal{J} \in \mathcal{S}_q} \left\{ \limsup_{m \to \infty} \frac{1}{m} \log \mathsf{E} \left[\exp \left(- {\boldsymbol{\mu}_{\mathcal{J}}^{(m)}}^{\top} \mathbf{Z} + \frac{1}{2} {\boldsymbol{\mu}_{\mathcal{J}}^{(m)}}^{\top} \boldsymbol{\mu}_{\mathcal{J}}^{(m)} \right) \ \mathbf{1}_{G_{\mathcal{J}}^{(m)}}(\mathbf{Z}) \right] \right\} \quad \text{by Lemma 5.}
\end{aligned}$$

For each $\mathcal{J} \in \mathcal{S}_q$,

$$\limsup_{m \to \infty} \frac{1}{m} \log \mathsf{E} \left[e^{- {\boldsymbol{\mu}_{\mathcal{J}}^{(m)}}^{\top} \mathbf{Z} + \frac{1}{2} {\boldsymbol{\mu}_{\mathcal{J}}^{(m)}}^{\top} \boldsymbol{\mu}_{\mathcal{J}}^{(m)}} \mathbf{1}_{G_{\mathcal{J}}^{(m)}}(\mathbf{Z}) \right]$$

$$
\begin{aligned}
&\le \lim_{m\to\infty} \frac{1}{2m}\left\|\boldsymbol{\mu}_{\mathcal{J}}^{(m)}\right\|^2 + \limsup_{m\to\infty}\frac{1}{m}\log \mathsf{E}\left[e^{-\boldsymbol{\mu}_{\mathcal{J}}^{(m)\top}\mathbf{z}}\mathbf{1}_{G_{\mathcal{J}}^{(m)}}(\mathbf{Z})\right] \\
&\le \frac{1}{2}\left\|\boldsymbol{\gamma}_{\mathcal{J}}\right\|^2 + \limsup_{m\to\infty}\frac{1}{m}\log \mathsf{E}\left[e^{-\left\|\boldsymbol{\mu}_{\mathcal{J}}^{(m)}\right\|^2}\mathbf{1}_{G_{\mathcal{J}}^{(m)}}(\mathbf{Z})\right] \qquad \text{by (29) and Lemma 4} \\
&= \frac{1}{2}\left\|\boldsymbol{\gamma}_{\mathcal{J}}\right\|^2 + \limsup_{m\to\infty}\frac{1}{m}\log e^{-\left\|\boldsymbol{\mu}_{\mathcal{J}}^{(m)}\right\|^2}\mathsf{P}\left(G_{\mathcal{J}}^{(m)}\right) \\
&= \frac{1}{2}\left\|\boldsymbol{\gamma}_{\mathcal{J}}\right\|^2 - \lim_{m\to\infty}\frac{1}{m}\left\|\boldsymbol{\mu}_{\mathcal{J}}^{(m)}\right\|^2 + \limsup_{m\to\infty}\frac{1}{m}\log \mathsf{P}\left(G_{\mathcal{J}}^{(m)}\right) .
\end{aligned}
$$

The definition of $\boldsymbol{\mu}_{\mathcal{J}}^{(m)}$ imply

$$
G_{\mathcal{J}}^{(m)} \subset \left\{\mathbf{z} \;:\; \|\mathbf{z}\| \ge \left\|\boldsymbol{\mu}_{\mathcal{J}}^{(m)}\right\|\right\} .
$$

Using this and Corollary 2, we have

$$
\limsup_{m\to\infty}\frac{1}{m}\log \mathsf{P}\left(G_{\mathcal{J}}^{(m)}\right) \le -\frac{1}{2}\left\|\boldsymbol{\gamma}_{\mathcal{J}}\right\|^2 .
$$

So, again by (29),

$$
\limsup_{m\to\infty}\frac{1}{m}\log \mathsf{E}\left[e^{-\boldsymbol{\mu}_{\mathcal{J}}^{(m)\top}\mathbf{Z}+\frac{1}{2}\boldsymbol{\mu}_{\mathcal{J}}^{(m)\top}\boldsymbol{\mu}_{\mathcal{J}}^{(m)}}\mathbf{1}_{G_{\mathcal{J}}^{(m)}}(\mathbf{Z})\right] \quad \le \quad -\left\|\boldsymbol{\gamma}_{\mathcal{J}}\right\|^2
$$

and

$$
\begin{aligned}
\limsup_{m\to\infty}\frac{1}{m}\log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\mathbf{1}_{G^{(m)}}(\mathbf{Z})\right] &\le \max_{\mathcal{J}\in\mathcal{S}_q}\left\{-\left\|\boldsymbol{\gamma}_{\mathcal{J}}\right\|^2\right\} \\
&= -\left\|\boldsymbol{\gamma}_*\right\|^2 \quad \text{by (19).}
\end{aligned}
$$

■

Define $\mathcal{K} \subset \{1,\cdots,t\}$ to be a *cutset* of $\mathcal{M}_q$ if $\mathcal{K}\cap\mathcal{J} \ne \emptyset$ for all $\mathcal{J}\in\mathcal{M}_q$ and $\mathcal{K}$ is minimal in this property: that is, for any $\mathcal{K}' \subsetneqq \mathcal{K}$, there exists a $\mathcal{J}\in\mathcal{M}_q$ such that $\mathcal{K}'\cap\mathcal{J}=\emptyset$. We denote the family of all cutsets by $\mathcal{C}_q$. Using the cutsets, we construct a covering of $\left(G^{(m)}\right)^c$.

Lemma 7

$$
\left(G^{(m)}\right)^c \subset \bigcup_{\mathcal{K}\in\mathcal{C}_q}\bigcap_{j\in\mathcal{K}}\left(G_j^{(m)}\right)^c .
$$

Proof: See Lemma 3.6 in GKS [3]. ■

By Lemma 7, it is clear that

$$
\mathbf{1}_{\left(G^{(m)}\right)^c} \le \sum_{\mathcal{K}\in\mathcal{C}_q}\mathbf{1}_{\bigcap_{j\in\mathcal{K}}\left(G_j^{(m)}\right)^c} . \tag{32}
$$

The following deterministic upper bound of $e^{m\psi_m(\theta,\mathbf{z})}$ on $\bigcap_{j\in\mathcal{K}}\left(G_j^{(m)}\right)^c$ can be derived as in GKS [3]: For $\mathbf{z}\in\bigcap_{l\in\mathcal{K}}\left(G_l^{(m)}\right)^c$,

$$
e^{m\psi_m(\theta,\mathbf{z})}
$$

$$
\begin{aligned}
&\le \prod_{j\in\mathcal{K}}\prod_{k\in\mathcal{I}_j^{(m)}}\left(1+\Phi\left(-\frac{s_j}{b_j}\epsilon_m\sqrt{m}+\alpha_2^{(m)}\Phi^{-1}(q)\right)\left(e^{\theta\bar{c}}-1\right)\right)\times\prod_{j\notin\mathcal{K}}\prod_{k\in\mathcal{I}_j^{(m)}} e^{\theta c_k}\ , \\
&\le \prod_{j\in\mathcal{K}}\prod_{k\in\mathcal{I}_j^{(m)}}\left(1+\Phi\left(-\frac{s_j}{b_j}\epsilon_m\sqrt{m}+\left|\Phi^{-1}(q)\right|\right)\left(e^{\theta\bar{c}}-1\right)\right)\times\prod_{j\notin\mathcal{K}}\prod_{k\in\mathcal{I}_j^{(m)}} e^{\theta c_k}\ , \qquad (33)
\end{aligned}
$$

which plays a crucial role in establishing the tight upper bound of second moment. Using the upper bound (33), we get for any sequence of non-negative random variables $\{Y_m\}_{m=1}^{\infty}$ satisfying $\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}\left[Y_m\right]<\infty$,

$$
\begin{aligned}
&\frac{1}{m}\log\mathsf{E}\left[e^{m\psi_m(\theta,\mathbf{Z})}\cdot Y_m\cdot\mathbf{1}_{\bigcap_{j\in\mathcal{K}}\left(G_j^{(m)}\right)^c}(\mathbf{Z})\right] \\
&\quad\le \frac{1}{m}\sum_{j\in\mathcal{K}}\left|\mathcal{I}_j^{(m)}\right|\log\left(1+\Phi\left(-\frac{s_j}{b_j}\epsilon_m\sqrt{m}+\left|\Phi^{-1}(q)\right|\right)\left(e^{\theta\bar{c}}-1\right)\right)+\theta\sum_{j\notin\mathcal{K}}\frac{1}{m}\sum_{k\in\mathcal{I}_j^{(m)}}c_k \\
&\qquad +\frac{1}{m}\log\mathsf{E}\left[Y_m\right]\quad\text{by (33) and non-negativity of } Y_m \\
&\quad\to \theta\sum_{j\notin\mathcal{K}}C_j+\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}\left[Y_m\right]\qquad\text{as } m\to\infty \qquad (34)
\end{aligned}
$$

where the last convergence is limsup-sense.

Lemma 8 *If $\mathbf{Z}\sim N(\mathbf{0},\mathbf{I}_d)$ and $\mathcal{S}_q\neq\emptyset$, then*

$$
\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}\left[e^{-\theta x_m+m\psi_m(\theta,\mathbf{Z})}\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\ \mathbf{1}_{\left(G^{(m)}\right)^c}(\mathbf{Z})\right] \\
&\quad\le\ \theta\left(\max_{\mathcal{K}\in\mathcal{C}_q}\left\{\sum_{j\notin\mathcal{K}}C_j\right\}-qC\right)+\|\boldsymbol{\gamma}_*\|^2\ .
\end{aligned}
$$

Proof: First observe the following limits:

$$
\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right] \\
&\quad=\ \limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}\left[\left(\sum_{\mathcal{J}\in\mathcal{S}_q}\lambda_{\mathcal{J}}\exp\left(\boldsymbol{\mu}_{\mathcal{J}}^{(m)\top}\mathbf{Z}-\frac{1}{2}\boldsymbol{\mu}_{\mathcal{J}}^{(m)\top}\boldsymbol{\mu}_{\mathcal{J}}^{(m)}\right)\right)^{-1}\right] \\
&\quad\le\ \limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}\left[\frac{1}{\lambda_{\mathcal{J}_*}}\exp\left(-\boldsymbol{\mu}_{\mathcal{J}_*}^{(m)\top}\mathbf{Z}+\frac{1}{2}\boldsymbol{\mu}_{\mathcal{J}_*}^{(m)\top}\boldsymbol{\mu}_{\mathcal{J}_*}^{(m)}\right)\right] \\
&\quad=\ \lim_{m\to\infty}\frac{1}{m}\left\|\boldsymbol{\mu}_{\mathcal{J}_*}^{(m)}\right\|^2 \\
&\quad=\ \|\boldsymbol{\gamma}_*\|^2\quad\text{by (19) and (29)}
\end{aligned}
$$

where the inequality is obtained by choosing one term among the positive summands in the denominator.

Taking $Y_m=\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}$ with an upper bound, $\|\boldsymbol{\gamma}_*\|^2$, of $\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}\left[Y_m\right]$ in (34) gives

$$
\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}\left[e^{-\theta x_m+m\psi_m(\theta,\mathbf{Z})}\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\ \mathbf{1}_{\left(G^{(m)}\right)^c}(\mathbf{Z})\right]
$$

$$
\begin{aligned}
&\leq \limsup_{m\to\infty} \frac{1}{m} \log \left(\sum_{\mathcal{K}\in\mathcal{C}_q} \mathsf{E}\left[e^{-\theta x_m + m\psi_m(\theta,\mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \ \mathbf{1}_{\bigcap_{j\in\mathcal{K}}\left(G_j^{(m)}\right)^c}(\mathbf{Z}) \right] \right) \quad \text{by (32)} \\
&= \max_{\mathcal{K}\in\mathcal{C}_q} \left\{ \limsup_{m\to\infty} \frac{1}{m} \log \mathsf{E}\left[e^{-\theta x_m + m\psi_m(\theta,\mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \ \mathbf{1}_{\bigcap_{j\in\mathcal{K}}\left(G_j^{(m)}\right)^c}(\mathbf{Z}) \right] \right\} \quad \text{by Lemma 5} \\
&\leq \max_{\mathcal{K}\in\mathcal{C}_q} \left\{ -\theta q C + \limsup_{m\to\infty} \frac{1}{m} \log \mathsf{E}\left[e^{m\psi_m(\theta,\mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \ \mathbf{1}_{\bigcap_{j\in\mathcal{K}}\left(G_j^{(m)}\right)^c}(\mathbf{Z}) \right] \right\} \\
&\leq \max_{\mathcal{K}\in\mathcal{C}_q} \left\{ -\theta q C + \theta \sum_{j\notin\mathcal{K}} C_j + \|\boldsymbol{\gamma}_*\|^2 \right\} \quad \text{by (34)} \\
&= \theta \left(\max_{\mathcal{K}\in\mathcal{C}_q} \left\{ \sum_{j\notin\mathcal{K}} C_j \right\} - qC \right) + \|\boldsymbol{\gamma}_*\|^2 \ .
\end{aligned}
$$

■

The following lemma on cutsets has an important implication for the sign of the upper bound of the second moment. For a sufficiently large θ, the sign of the upper bound in Lemma 8 is determined by Lemma 9.

Lemma 9

$$
\max_{\mathcal{K}\in\mathcal{C}_q} \left\{ \sum_{j\notin\mathcal{K}} C_j \right\} < qC \ .
$$

Proof: See Lemma 3.8 in GKS [3]. ■

Now we are ready to prove the main theorem.

Proof: (Theorem 2) For any $\theta > 0$,

$$
\begin{aligned}
&M_2(x_m, \theta_m(\mathbf{Z})) \\
&\quad \leq \ \mathsf{E}\left[e^{-\theta x_m + m\psi_m(\theta,\mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \ \mathbf{1}_{\left(G^{(m)}\right)^c}(\mathbf{Z}) \right] + \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \ \mathbf{1}_{G^{(m)}}(\mathbf{Z}) \right]
\end{aligned}
$$

by (27) and (28). Considering sufficiently large θ, Lemma 5, Lemma 6, Lemma 8, and Lemma 9 imply

$$
\begin{aligned}
&\limsup_{m\to\infty} \frac{1}{m} \log M_2(x_m, \theta_m(\mathbf{Z})) \\
&\quad \leq \ \max\left\{ \theta \left(\max_{\mathcal{K}\in\mathcal{C}_q} \left\{ \sum_{j\notin\mathcal{K}} C_j \right\} - qC \right) + \|\boldsymbol{\gamma}_*\|^2 \ , \ -\|\boldsymbol{\gamma}_*\|^2 \right\} \\
&\quad = \ -\|\boldsymbol{\gamma}_*\|^2 \ .
\end{aligned}
$$

Now, by combining this upper bound and Theorem 1with Jensen's inequality, we have

$$
-\|\boldsymbol{\gamma}_*\|^2 \ = \ 2 \liminf_{m\to\infty} \frac{1}{m} \log \mathsf{P}\left(L_m > x_m\right) \quad \text{by Theorem 1}
$$

$$
\begin{aligned}
&\leq \limsup_{m\to\infty} \frac{1}{m} \log M_2(x_m, \theta_m(\mathbf{Z})) \quad \text{by Jensen's inequality} \\
&\leq -\|\boldsymbol{\gamma}_*\|^2 .
\end{aligned}
$$

■

A.2 Large Loss Threshold Regime: Proof of Theorem 4

The approach we use for this regime is exactly same as the one used in Section A.1. Since we set

$$
M_{\mathbf{Z}}^{(m)} = e^{\boldsymbol{\mu}^{(m)\top}\mathbf{Z} - \frac{1}{2}\boldsymbol{\mu}^{(m)\top}\boldsymbol{\mu}^{(m)}},
$$

if we replace $G^{(m)}$ by $G^{(m)}_{\{1,\ldots,t\}}$, all the lines to reach the terms (27) and (28) also hold for the IS estimator under the current parameter regime. That is, we have

$$
\begin{aligned}
&M_2(x_m, \theta_m(\mathbf{Z})) \\
&\quad \leq \mathsf{E}\left[e^{-\theta x_m + m\psi_m(\theta, \mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)} \right)^{-1} \mathbf{1}_{\left(G^{(m)}_{\{1,\ldots,t\}}\right)^c}(\mathbf{Z}) \right] \qquad (35) \\
&\qquad + \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)} \right)^{-1} \mathbf{1}_{G^{(m)}_{\{1,\ldots,t\}}}(\mathbf{Z}) \right] \qquad (36)
\end{aligned}
$$

for any $\theta > 0$.

The following limit can be shown in a similar way to (29):

$$
\boldsymbol{\gamma} = \lim_{m\to\infty} \frac{1}{\sqrt{\log m}} \boldsymbol{\mu}^{(m)}. \qquad (37)
$$

Furthermore, using the definition (22), we can show the following lemma which is similar to Lemma 4.

Lemma 10 *If $G \neq \emptyset$, then*

$$
\min\left\{ \boldsymbol{\mu}^{(m)\top}\mathbf{z} \ : \ \mathbf{z} \in G^{(m)}_{\{1,\ldots,t\}} \right\} = \left\| \boldsymbol{\mu}^{(m)} \right\|^2 . \qquad (38)
$$

With $f(\cdot) = \log m$, $\mathbf{v}_m = \boldsymbol{\mu}^{(m)}$ and (37), we can deduce the following corollary from Lemma 3.

Corollary 3 *If $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I}_d)$, then*

$$
\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{P}\left(\|\mathbf{Z}\| \geq \left\| \boldsymbol{\mu}^{(m)} \right\| \right) \leq -\frac{1}{2} \|\boldsymbol{\gamma}\|^2 .
$$

Using Corollary 3 and Lemma 10, we prove the following lemma.

Lemma 11 *If $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I}_d)$ and $G \neq \emptyset$, then*

$$
\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)} \right)^{-1} \mathbf{1}_{G^{(m)}_{\{1,\ldots,t\}}}(\mathbf{Z}) \right] \leq -\|\boldsymbol{\gamma}\|^2 .
$$

Proof:

$$\begin{aligned}
&\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \mathbf{1}_{G^{(m)}_{\{1,\ldots,t\}}}(\mathbf{Z})\right] \\
&= \limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}\left[e^{-\boldsymbol{\mu}^{(m)\top}\mathbf{Z}+\frac{1}{2}\boldsymbol{\mu}^{(m)\top}\boldsymbol{\mu}^{(m)}} \mathbf{1}_{G^{(m)}_{\{1,\ldots,t\}}}(\mathbf{Z})\right] \\
&\leq \lim_{m\to\infty} \frac{1}{2\log m}\|\boldsymbol{\mu}^{(m)}\|^2 + \limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}\left[e^{-\boldsymbol{\mu}^{(m)\top}\mathbf{Z}} \mathbf{1}_{G^{(m)}_{\{1,\ldots,t\}}}(\mathbf{Z})\right] \\
&\leq \frac{1}{2}\|\boldsymbol{\gamma}\|^2 + \limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}\left[e^{-\|\boldsymbol{\mu}^{(m)}\|^2} \mathbf{1}_{G^{(m)}_{\{1,\ldots,t\}}}(\mathbf{Z})\right] \qquad \text{by Lemma 10 and (37)} \\
&= \frac{1}{2}\|\boldsymbol{\gamma}\|^2 + \limsup_{m\to\infty} \frac{1}{\log m} \log e^{-\|\boldsymbol{\mu}^{(m)}\|^2} \mathsf{P}\left(G^{(m)}_{\{1,\ldots,t\}}\right) \\
&= \frac{1}{2}\|\boldsymbol{\gamma}\|^2 - \lim_{m\to\infty} \frac{1}{\log m}\|\boldsymbol{\mu}^{(m)}\|^2 + \limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{P}\left(G^{(m)}_{\{1,\ldots,t\}}\right) \\
&= -\frac{1}{2}\|\boldsymbol{\gamma}\|^2 + \limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{P}\left(G^{(m)}_{\{1,\ldots,t\}}\right) \ .
\end{aligned}$$

The definition of $\boldsymbol{\mu}^{(m)}$ implies

$$G^{(m)}_{\{1,\ldots,t\}} \subset \left\{\mathbf{z} \ : \ \|\mathbf{z}\| \geq \|\boldsymbol{\mu}^{(m)}\|\right\} \ .$$

Using this and Corollary 3, we have

$$\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{P}\left(G^{(m)}_{\{1,\ldots,t\}}\right) \leq -\frac{1}{2}\|\boldsymbol{\gamma}\|^2 \ ,$$

which completes the proof. ∎

Now we consider a covering $\bigcup_{j=1}^t \left(G_j^{(m)}\right)^c$ of $\left(G^{(m)}_{\{1,\ldots,t\}}\right)^c$ and its direct implication,

$$\mathbf{1}_{\left(G^{(m)}_{\{1,\ldots,t\}}\right)^c} \leq \sum_{j=1}^t \mathbf{1}_{\left(G_j^{(m)}\right)^c} \ . \tag{39}$$

We set

$$\theta_m \triangleq m^{-1+\frac{s^2}{2}} \ .$$

Note that $\lim_{m\to\infty} \theta_m = 0$.

The following convergence can be shown as in GKS [3]. (The proof is almost identical to the original one given there by replacing $\Phi^{-1}(1-\overline{p}) + b_j(1-\frac{1}{\sqrt{\log m}})\Phi^{-1}(q)$ by $\alpha_1^{(m)}\Phi^{-1}(1-\overline{p}_j) + b_j(1-\frac{1}{\sqrt{\log m}})\Phi^{-1}(q)$): For any sequence of non-negative random variables $\{Y_m\}_{m=1}^\infty$ with $\frac{1}{\log m}\log \mathsf{E}'_m\left[Y_m\right] = y_m$ satisfying $\limsup_{m\to\infty} y_m < \infty$, we have

$$\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}'_m\left[Y_m \cdot e^{-\theta_m q_m \sum_{k\in\mathcal{I}_j^{(m)}} l_k u_k + \sum_{k\in\mathcal{I}_j^{(m)}} \log\left(1+p_k(\mathbf{Z})\left(e^{\Lambda_j(\theta_m l_k)}-1\right)\right)} \mathbf{1}_{\left(G_j^{(m)}\right)^c}(\mathbf{Z})\right] = -\infty \tag{40}$$

where $\{\mathsf{E}'_m\}_{m=1}^\infty$ are expectations under an arbitrary sequence of probability measures $\{\mathsf{P}'_m\}_{m=1}^\infty$. Using the following upper bound (with appropriate constant D)

$$-\theta_m q_m \sum_{k\in\mathcal{I}_j^{(m)}} l_k u_k + \sum_{k\in\mathcal{I}_j^{(m)}} \log\left(1+p_k(\mathbf{z})\left(e^{\Lambda_j(\theta_m l_k)}-1\right)\right) \leq \theta_m(1-q_m)n_j^{(m)}\bar{l} + n_j^{(m)} D\bar{l}^2\theta_m^2 \tag{41}$$

and defining a sequence of non-negative random variables by $\{Y_m\}_{m=1}^{\infty}$,

$$Y_m \triangleq \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \cdot \prod_{j' \neq j} e^{-\theta_m q_m \sum_{k\in\mathcal{I}_{j'}^{(m)}} l_k u_k + \sum_{k\in\mathcal{I}_{j'}^{(m)}} \log\left(1+p_k(\mathbf{Z})\left(e^{\Lambda_{j'}(\theta_m l_k)}-1\right)\right)} , \tag{42}$$

we have

$$\begin{aligned}
&\frac{1}{\log m} \log \mathsf{E}\left[Y_m\right] \\
&\leq \frac{1}{\log m} \log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \cdot \prod_{j'\neq j} e^{\theta_m(1-q_m) n_{j'}^{(m)} \bar{l} + n_{j'}^{(m)} D \bar{l}^2 \theta_m^2}\right] \quad \text{by (41)} \\
&= \sum_{j'\neq j} \frac{n_{j'}^{(m)}\bar{l}}{\log m} \theta_m \left((1-q_m) + D\bar{l}\theta_m\right) + \frac{1}{\log m} \log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right] \\
&\leq \frac{m\bar{l}}{\log m} \theta_m \left((1-q_m) + D\bar{l}\theta_m\right) + \frac{1}{\log m} \log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right] \quad \text{since } \sum_{j'\neq j} n_{j'}^{(m)} \leq m \\
&\leq \bar{l} \frac{m\theta_m}{\log m} \frac{1}{\sqrt{2\pi} s \sqrt{\log m}} e^{(-s^2 \log m)/2} + \bar{l}^2 D \frac{m\theta_m^2}{\log m} + \frac{1}{\log m} \log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right] \\
&= \bar{l} \frac{\theta_m m^{1-s^2/2}}{\sqrt{2\pi} s (\log m)^{3/2}} + \bar{l}^2 D \frac{m\theta_m^2}{\log m} + \frac{1}{\log m} \log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right] \\
&= \bar{l} \frac{1}{\sqrt{2\pi} s (\log m)^{3/2}} + \bar{l}^2 D \frac{m^{-1+s^2}}{\log m} + \frac{1}{\log m} \log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right] \triangleq y_m
\end{aligned}$$

where the inequality in the fifth line comes from the following well-known inequality,

$$1 - \Phi(x) \leq \frac{1}{x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} .$$

Observe that

$$\log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right] = \log \mathsf{E}\left[e^{-\boldsymbol{\mu}^{(m)\top}\mathbf{Z} + \frac{1}{2}\|\boldsymbol{\mu}^{(m)}\|^2}\right] = \|\boldsymbol{\mu}^{(m)}\|^2$$

and hence

$$\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1}\right] = \limsup_{m\to\infty} \frac{1}{\log m} \|\boldsymbol{\mu}^{(m)}\|^2 = \|\boldsymbol{\gamma}\|^2 \quad \text{by (37)} .$$

This implies $\limsup_{m\to\infty} y_m = \|\boldsymbol{\gamma}\|^2 < \infty$. So $\{Y_m\}_{m=1}^{\infty}$ given by (42) with $\{y_m\}_{m=1}^{\infty}$ satisfies the condition of (40) by taking $\mathsf{P}'_m = \mathsf{P}$. Combining all these results, we have, for each j,

$$\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}\left[e^{-\theta_m x_m + m\psi_m(\theta_m, \mathbf{Z})} \cdot \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \cdot \mathbf{1}_{\left(G_j^{(m)}\right)^c}(\mathbf{Z})\right] = -\infty. \tag{43}$$

Then we have the following lemma.

Lemma 12

$$\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}\left[e^{-\theta_m x_m + m\psi_m(\theta_m, \mathbf{Z})} \cdot \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \cdot \mathbf{1}_{\left(G_{\{1,\ldots,t\}}^{(m)}\right)^c}(\mathbf{Z})\right] = -\infty .$$

Proof:

$$
\begin{aligned}
&\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}\left[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \cdot \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \cdot \mathbf{1}_{\left(G^{(m)}_{\{1,\ldots,t\}}\right)^c}(\mathbf{Z})\right] \\
&\le \limsup_{m\to\infty} \frac{1}{\log m} \log \left(\sum_{j=1}^{t} \mathsf{E}\left[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \cdot \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \cdot \mathbf{1}_{\left(G^{(m)}_{j}\right)^c}(\mathbf{Z})\right]\right) \quad \text{by (39)} \\
&= \max_{1\le j\le t} \left\{ \limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}\left[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \cdot \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \cdot \mathbf{1}_{\left(G^{(m)}_{j}\right)^c}(\mathbf{Z})\right]\right\} \quad \text{by Lemma 5} \\
&\le \max_{1\le j\le t} \{-\infty\} \quad \text{by (43)} \\
&= -\infty\ .
\end{aligned}
$$

■

Now we prove the main theorem.

Proof: (Theorem 4) Substituting θ_m for θ in (35), we get

$$
\begin{aligned}
&M_2(x_m, \theta_m(\mathbf{Z})) \\
&\quad \le \mathsf{E}\left[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \mathbf{1}_{\left(G^{(m)}_{\{1,\ldots,t\}}\right)^c}(\mathbf{Z})\right] + \mathsf{E}\left[\left(M_{\mathbf{Z}}^{(m)}\right)^{-1} \mathbf{1}_{G^{(m)}_{\{1,\ldots,t\}}}(\mathbf{Z})\right]
\end{aligned}
$$

by (35) and (36). Hence,

$$
\begin{aligned}
&\limsup_{m\to\infty} \frac{1}{\log m} \log M_2(x_m, \theta_m(\mathbf{Z})) \\
&\quad \le \max\left\{-\infty\ ,\ -\|\boldsymbol{\gamma}\|^2\right\} \quad \text{by Lemma 11 and Lemma 12} \\
&\quad = -\|\boldsymbol{\gamma}\|^2\ .
\end{aligned}
$$

The final step for asymptotic optimality is exactly same as that in the proof of Theorem 2. ■

A.3 Proof of Lemma 1

Proof: (Lemma 1) Recall that the definition (17) implies that $\boldsymbol{\mu}_{\mathcal{J}}^{(m)}$ is the minimum norm point in the polyhedron $G_{\mathcal{J}}^{(m)}$ defined by an intersection of $|\mathcal{J}|$ halfspaces in $\mathbb{R}^d$. Assume that $|\mathcal{J}| > d$. Denote $\boldsymbol{\mu}_{\mathcal{J}}^{(m)}$ by $\mathbf{v}$ to simplify notation. Consider the following primal linear programming (LP) problem (P) and its dual problem (D):

$$
\textbf{(P)}\quad \begin{array}{ll} \min & \mathbf{v}^\top \mathbf{z} \\ \text{s.t.} & \mathbf{a}_j^\top \mathbf{z} \ge d_j \quad \text{for } j \in \mathcal{J} \end{array}
\qquad \text{and} \qquad
\textbf{(D)}\quad \begin{array}{ll} \max & \sum_{j\in\mathcal{J}} d_j \pi_j \\ \text{s.t.} & \sum_{j\in\mathcal{J}} \mathbf{a}_j \pi_j = \mathbf{v} \\ & \pi_j \ge 0 \quad \text{for } j \in \mathcal{J} \end{array}
$$

where (P) is the same LP as the one in Lemma 4. Hence $\mathbf{v}$ is the finite optimal solution of (P) and LP duality implies the dual problem (D) also has a finite optimal solution. Furthermore, (D) is a standard form LP and hence there exists an optimal basic feasible solution (see, e.g., Bertsimas and Tsitsiklis [1]). Because the number of constraints in (D) is d, the optimal basic feasible solution has at most d non-zero components. Denote these by π_j^* and define $\mathcal{P} = \{j : \pi_j^* > 0\}$.

Then $|\mathcal{P}| \leq d$. Note the following complementary slackness condition between the optimal pair $\mathbf{v}$ and π_j^*'s:

$$\pi_j^*(\mathbf{a}_j^\top \mathbf{v} - d_j) = 0 \quad \text{for all } j \in \mathcal{P}. \tag{44}$$

Now consider the following *linearly constrained strictly convex* optimization problem (CP) and its KKT optimality condition:

$$\textbf{(CP)} \quad \begin{array}{lll} \min & \|\mathbf{z}\| & \\ \text{s.t.} & \mathbf{a}_j^\top \mathbf{z} \geq d_j & \text{for } j \in \mathcal{P} \end{array} \qquad \text{and} \qquad \textbf{(KKT)} \quad \begin{array}{ll} \mathbf{a}_j^\top \mathbf{z} \geq d_j & \text{for } j \in \mathcal{P} \\ \pi_j \geq 0 & \text{for } j \in \mathcal{P} \\ \pi_j(\mathbf{a}_j^\top \mathbf{z} - d_j) = 0 & \text{for } j \in \mathcal{P} \\ \frac{\mathbf{z}}{\|\mathbf{z}\|} = \sum_{j \in \mathcal{P}} \mathbf{a}_j \pi_j & \end{array}.$$

Observe that $\mathbf{v}$ (as a primal feasible solution) and $\frac{\pi_j^*}{\|\mathbf{v}\|}$ for $j \in \mathcal{P}$ (as corresponding Lagrange multipliers) satisfy the (KKT) conditions: specifically, (44) implies the complementarity condition in (KKT), and the dual feasibility of π_j^*'s in (D) implies the last condition of (KKT). Because (CP) is a strictly convex optimization problem, satisfaction of (KKT) implies that $\mathbf{v}$ is an optimal solution for the original problem which has $|\mathcal{P}|$ ($\leq d$) constraints (see, e.g., Nocedal and Wright [4]). Taking $\mathcal{J}' = \mathcal{P}$ completes the proof. ■

A.4 Derivation of the Subset Sum Problem

To identify $\{\boldsymbol{\mu}_{\mathcal{J}}^{(m)} : \mathcal{J} \in \mathcal{S}_q\}$, we need to check whether, for each $\mathbf{v} \in \mathcal{V}$, there is a $\mathcal{J} \in \mathcal{S}_q$ such that $\mathbf{v} = \boldsymbol{\mu}_{\mathcal{J}}^{(m)}$ using the information $\mathcal{H}(\mathbf{v})$ and $\mathcal{F}(\mathbf{v})$. We will show that this problem can be formulated as a *minimal cover* problem (MCP), and then show that the MCP can be transformed into a knapsack problem.

MCP: An index set A is given, and $\{C_i\}_{i \in A}$ with $C_i > 0$ and a subset $B \subset A$ ($B \neq \emptyset$) are given. For a given positive number b, is there a subset $J \subset A \setminus B$ such that

$$\sum_{j \in J \cup B} C_j \geq b \qquad \text{and} \qquad \sum_{j \in J \cup B \setminus \{k\}} C_j < b \text{ for all } k \in J \cup B \text{ ?}$$

Then we have the following lemma:

Lemma 13 *The answer to MCP is YES if and only if there exists a $J \subset A \setminus B$ such that*

$$\begin{array}{ll} \text{i)} & \displaystyle\sum_{j \in J \cup B} C_j \geq b, \\ \text{ii)} & \displaystyle\sum_{j \in J \cup B \setminus \{k\}} C_j < b \textit{ for all } k \in J, \textit{ and} \\ \text{iii)} & \displaystyle\sum_{j \in J \cup B} C_j - \min_{i \in B} C_i < b. \end{array}$$

Proof: If we notice the relation $\sum_{j \in J \cup B \setminus \{k\}} C_j < b$ for all $k \in B \Leftrightarrow \sum_{j \in J \cup B} C_j - \min_{i \in B} C_i < b$, then the proof is complete. ■

Set $b' = b - \sum_{j\in B} C_j$. Using Lemma 13, we can rewrite the MCP as

MCP′: $\{C_i\}_{i\in A}$ with $C_i > 0$ and a subset $B \subset A$ are given. For a given positive number b, is there a subset $J \subset A \setminus B$ such that

$$\begin{aligned}
&\text{i)} \quad \sum_{j\in J} C_j \geq b', \\
&\text{ii)} \quad \sum_{j\in J\setminus\{k\}} C_j < b' \text{ for all } k \in J, \text{ and} \\
&\text{iii)} \quad \sum_{j\in J} C_j < b' + \min_{i\in B} C_i \ ?
\end{aligned}$$

Consider the following 0-1 knapsack problem (KP):

$$\textbf{(KP)} \qquad \begin{aligned}
f^* = \min \quad & \sum_{j\in A\setminus B} C_j x_j \\
\text{s.t.} \quad & \sum_{j\in A\setminus B} C_j x_j \geq b' \\
& x_j \in \{0,1\} \text{ for all } j \in A \setminus B \ .
\end{aligned}$$

Then any set $G \subset A \setminus B$ corresponding to an optimal solution of (KP) satisfies condition i) of MCP′ by feasibility. If $\sum_{j\in G\setminus\{k\}} C_j \geq b'$ for some $k \in G$, then $G \setminus \{k\}$ is another feasible set with strictly smaller value than the optimal one, and this contradicts the optimality of G. Hence G satisfies ii) of MCP′. Therefore, we conclude that $f^* < b' + \min_{i\in B} C_i$ if and only if the answer to MCP is YES. Now set $A = \mathcal{H}(\mathbf{v})$ and take a B from $\mathcal{F}(\mathbf{v})$. Then by setting $b = qC$, the answer to MCP determines whether there is a $\mathcal{J}$ such that $B \subset \mathcal{J} \subset \mathcal{H}(\mathbf{v})$, $\mathcal{J} \in \mathcal{S}_q$, and $\mathbf{v} = \boldsymbol{\mu}_{\mathcal{J}}^{(m)}$. Hence by checking this question for all $B \in \mathcal{F}(\mathbf{v})$, we can decide whether $\mathbf{v} \in \{\boldsymbol{\mu}_{\mathcal{J}}^{(m)} : \mathcal{J} \in \mathcal{S}_q\}$. The final transformation to the maximization form using the minimal index set notations results in (SSP).

With $\min_{i\in B} C_i = 1$, MCP′ is equivalent to knapsack feasibility problem and hence MCP is NP-complete.

A.5 The Single Factor Case

We give a rather complete description of how to find $\{\boldsymbol{\mu}_{\mathcal{J}}^{(m)} : \mathcal{J} \in \mathcal{S}_q\}$ instead of $\mathcal{S}_q$ in the single factor case. Notice that $\mathcal{S}_q$ depends only on the C_j's and q. Hence it is independent of the number of factors. Assume that we are given t factor loading coefficients and $d_j > 0$ for all j. After scaling, we can assume that each type is associated with a hyperplane defining $G_j^{(m)}$, $z \geq v_j$ for positive v_j or $z \leq v_j$ for negative v_j where $v_j \triangleq d_j/a_j$. If there are both positive and negative v_j's for j's in $\mathcal{J}$, then $\{z : z \geq v_j \text{ if } v_j > 0 \text{ and } z \leq v_j \text{ if } v_j < 0 \text{ for } j \in \mathcal{J}\} = \emptyset$. Hence it is sufficient to consider the positive v_j's and negative v_j's separately. First consider the positive v_j's and assume that their indices of factor types are $1,\ldots,p$ and $v_i < v_j$ if $i < j$. Then $\min\{\|\mathbf{z}\| : \mathbf{a}_j^\top \mathbf{z} \geq d_j, j \in \{1,\ldots,p\}\} = \max_{j\in\{1,\ldots,p\}} v_j$. If we use the notation in Section 5.1 , then $\mathcal{V} = \{v_1,\ldots,v_p\}$, $\mathcal{H}(v_j) = \{1,\ldots,j\}$, and $\mathcal{F}(v_j) = \{\{j\}\}$ for $j = 1,\ldots,p$.

Now for a given q value, we decide which elements in $\mathcal{V}$ are in $\{\boldsymbol{\mu}_{\mathcal{J}}^{(m)} : \mathcal{J} \in \mathcal{S}_q\}$. For this single factor case, the associated subset sum problem is, for $j = 2, \ldots, p$,

$$\begin{aligned} f_j^* = \max \quad & \sum_{1 \le i \le j-1} C_i x_i \\ \text{s.t.} \quad & \sum_{1 \le i \le j-1} C_i x_i \le \sum_{1 \le i \le j} C_i - qC \\ & x_i \in \{0, 1\} \text{ for } i = 1, \ldots, j-1. \end{aligned}$$

Then $v_j \in \{\boldsymbol{\mu}_{\mathcal{J}}^{(m)} : \mathcal{J} \in \mathcal{S}_q\}$ if and only if $f_j^* > \sum_{1 \le i \le j} C_i - qC$. In this case, we solve $p-1$ subset sum problems since we just need to check whether $\sum_{1 \le i \le p} C_i \ge qC$ for v_1. (The last inequality does not always hold because there may be some negative v_j's in the original problem.) If we follows the same steps for negative v_j's, then by combining the two results we completely characterize the desired set $\{\boldsymbol{\mu}_{\mathcal{J}}^{(m)} : \mathcal{J} \in \mathcal{S}_q\}$ for the single factor case.

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