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# LARGE DEVIATIONS IN MULTIFACTOR PORTFOLIO CREDIT RISK

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The measurement of portfolio credit risk focuses on rare but significant large-loss events. This paper investigates rare event asymptotics for the loss distribution in the widely used Gaussian copula model of portfolio credit risk. We establish logarithmic limits for the tail of the loss distribution in two limiting regimes. The first limit examines the tail of the loss distribution at increasingly high loss thresholds; the second limiting regime is based on letting the individual loss probabilities decrease toward zero. Both limits are also based on letting the size of the portfolio increase. Our analysis reveals a qualitative distinction between the two cases: in the rare-default regime, the tail of the loss distribution decreases exponentially, but in the large-threshold regime the decay is consistent with a power law. This indicates that the dependence between defaults imposed by the Gaussian copula is qualitatively different for portfolios of high-quality and lower-quality credits.

KEY WORDS: large deviations, portfolio credit risk, multifactor model, Gaussian copula

## 1. INTRODUCTION

The measurement of portfolio credit risk focuses on rare but significant large-loss events, with banks and other financial institutions seeking to measure the 99.9th or 99.99th percentile of their credit loss distributions over, e.g., a one-year horizon. Credit portfolios are often large, including exposure to thousands or even tens of thousands of obligors; and the default probabilities of high-quality credits are extremely small over short or intermediate time horizons. These features of the credit risk context lead us to consider rare-event asymptotics of loss distributions for large credit portfolios.

An essential feature of a portfolio view of credit risk is a mechanism for capturing dependence between the defaults of multiple obligors. Because historical data on defaults are limited, theory and practice have looked to the equity market for information on "correlations" between obligors. The most widely used mechanism for mapping correlations

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(such as correlations between asset returns) to dependence between defaults is the Gaussian copula framework (as in Gupton et al. [1997] and Li [2000]), which is loosely based on foundational work of Merton (1974), in which default occurs when firm value crosses a boundary. The mixed Poisson model of CreditRisk+ (Wilde [1997]) is in some respects closer to the intensity-based models of Duffie and Singleton (1999) and Jarrow and Turnbull (1995). A key feature of both the Gaussian copula and the mixed Poisson model is that defaults become independent conditional on a set of underlying factors. Our analysis of the Gaussian copula takes advantage of this structure.

We prove logarithmic limits for the loss distribution in the Gaussian copula framework under two different limiting regimes. The first limit examines the tail of the loss distribution at increasingly high loss thresholds; the second limiting regime is based on letting the individual loss probabilities decrease toward zero. Both limits are also based on letting the size of the portfolio increase.

The two limits we consider correspond to the two sources of rarity in the problem of measuring portfolio credit risk—the rarity of large losses resulting from multiple defaults and the rarity of individual defaults of high-quality credits. Our analysis reveals an interesting and surprising qualitative distinction between the cases: in the rare-default regime, the tail of the loss distribution decreases exponentially, but in the large-threshold regime the rate of decrease follows a power law. slows down as P gets smaller

In both cases, the rate of decrease is determined primarily by the correlation structure underlying the Gaussian copula, and a good deal of our analysis is devoted to identifying this structure. We consider models with a finite number of types of obligors. Each type is characterized by a vector of factor loadings (coefficients) on the factors in the Gaussian copula. Each loading vector determines a half-space of directions along which moving the factors increases the conditional default probabilities for obligors of that type. The default probabilities determine how far the factors must move for defaults of that type to become "likely"; and the exposures determine which combinations of types must default to reach a loss threshold. This structure determines a region of factor outcomes in which a large loss (exceeding a specified threshold) becomes "likely." The points in this region closest to the origin determine the rate of decrease of the tail of the loss distribution.

Identifying the rate of decrease of the tail of the loss distribution—and understanding whether the tail is exponential or polynomial—is useful in developing approximations. Our investigation is also useful for the design and analysis of efficient Monte Carlo methods for rare event simulation. Glasserman and Li (2005) analyze an importance sampling method in the setting of single-factor models. The formulation considered here suggests a different algorithm that explicitly takes account of the added complexity of the multifactor case and the presence of multiple types of obligors. We use the tools developed here in Glasserman et al. (2005) to establish the asymptotic optimality of this method.

Several authors (Martin et al. 2001; Gordy 2002; Dembo et al. 2004) have developed saddlepoint and other tail approximations for loss distributions when defaults are independent. These approximations apply as well to conditional loss distributions when defaults are conditionally independent, but then extending them to the unconditional loss distribution requires, in principle, integrating out the conditioning variables. (An exception is Gordy [2002], who works in the more tractable mixed Poisson setting.) Our analysis applies directly to the unconditional loss distribution; indeed, we find that the decay rate of the tail of the loss distribution is dominated by the effect of the conditioning variables (the factors in the Gaussian copula). Thus, our results are qualitatively different from those derived from independent defaults. The decay rate we find in the large threshold limit is consistent with that in Lucas et al. (2001), but their limit applies to an approximation to the loss distribution whereas we deal with the distribution itself.

The rest of this paper is organized as follows. Section 2 introduces notation and states our two main results: large-deviations limits for the loss distribution in the rare-default and large-threshold limits. Section 3 proves the first result and Section 4 proves the second. Each of these proofs is divided in two parts—the proof of an upper bound and the proof of a lower bound.

## 2. NOTATION AND MAIN RESULTS

We consider the distribution of losses from default in a fixed portfolio over a fixed horizon. We use the following notation:

$$
\begin{aligned}
m &= \text{ the number of obligors to which the portfolio is exposed;}\\
Y_k &= \text{ default indicator } (= 1 \text{ for default}, = 0 \text{ otherwise}) \text{ for the } k\text{-th obligor;}\\
p_k &= \text{ marginal probability that the } k\text{-th obligor defaults;}\\
c_k &= \text{ loss resulting from default of the } k\text{-th obligor;}\\
L_m &= c_1 Y_1 + \cdots + c_m Y_m = \text{ total loss from defaults.}
\end{aligned}
$$

The loss $c_k$ will in some cases be assumed deterministic and in others allowed to be a random variable. Dependence among the default indicators $Y_k$ is given by a *multifactor Gaussian copula model with a finite number of types.* We set $Y_k = \mathbf{1}\{X_k > \Phi^{-1}(1 - p_k)\}$, with $\Phi$ the cumulative normal distribution and $X_1, X_2, \ldots$ correlated standard normal random variables so that $\mathsf{P}(Y_k = 1) = p_k$. Correlations between these latent variables determine the dependence among the default indicators. In practice, these correlations are often derived from correlations in asset values or equity returns. The assumption of a finite number of types is specified as follows:

**M1** There are $d$ factors and $t$ types of obligors. $\{\mathcal{I}_1^{(m)}, \ldots, \mathcal{I}_t^{(m)}\}$ is a partition of the set of obligors $\{1, \ldots, m\}$ into types. If $k \in \mathcal{I}_j^{(m)}$, then the $k$-th obligor is of type $j$ and its latent variable is given by

$$X_k = \mathbf{a}_j^\top \mathbf{Z} + b_j \varepsilon_k$$

where $\mathbf{a}_j \in \mathbb{R}^d$ with $0 < \|\mathbf{a}_j\| < 1$, $\mathbf{Z}$ is a $d$ dimensional standard normal random vector, $b_j = \sqrt{1 - \mathbf{a}_j^\top \mathbf{a}_j}$ and $\varepsilon_k$ are independent standard normal random variables. Let $n_j^{(m)} = |\mathcal{I}_j^{(m)}|$ denote the number of obligors of type $j$. We assume that for each $j = 1, \ldots, t$, $r_j = \lim_{m\to\infty} \frac{1}{m} n_j^{(m)} > 0$.

means: asymptotically, number of groups is tractable

We consider asymptotics of $\mathsf{P}(L_m > x_m)$ for large $m$, allowing the loss threshold $x_m$ to increase with the size of the portfolio. We consider two limiting regimes:

**Small default probability limit:** small $p_k$'s and moderate $x_m$;
**Large loss threshold limit:** large $x_m$ and moderate $p_k$'s.

As mentioned in the Introduction, these two regimes exhibit qualitatively different behavior, with the large-threshold limit reflecting a heavier tail. This distinction results from the greater impact of dependence (through the Gaussian copula) in the second case.

We close this section with some notational conventions:

- $[\mathbf{v}]_i$ denotes the $i$-th component of the vector $\mathbf{v}$;
- $\|\mathbf{v}\| = \sqrt{\mathbf{v}^\top \mathbf{v}}$ for a vector $\mathbf{v}$;
- $\mathbf{e} = (1, \ldots, 1)^\top \in \mathbb{R}^d$;
- $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$;
- $\phi(\cdot)$ and $\Phi(\cdot)$ are a density function and a cdf of standard normal random variable, respectively.

## 2.1. Small Default Probability Regime

We specify the parameters of the small default probability regime by imposing the assumptions in **M2** in addition to those in **M1**:

**M2** 1. The default loss, $c_k$, is deterministic and $0 < c_k \leq \bar{c} < \infty$ for $k = 1, \ldots, m$.

2. If the $k$-th obligor is of type $j$ then its default probability is given by $p_k = p_j^{(m)} \triangleq \Phi(-s_j\sqrt{m})$ where $s_j > 0$. The conditional default probability (given the factors $\mathbf{Z}$) of the same obligor is given by

$$p_k(\mathbf{Z}) = p_j^{(m)}(\mathbf{Z}) = \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{Z} + \Phi^{-1}(p_j^{(m)})}{b_j}\right) = \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{Z} - s_j\sqrt{m}}{b_j}\right). \tag{2.1}$$

3. For each type $j = 1, \ldots, t$,

means: asymptotically, weights in a group j cannot be too large

$$C_j \triangleq \lim_{m\to\infty} \frac{1}{m} \sum_{k\in\mathcal{I}_j^{(m)}} c_k < \infty \quad \text{and}$$

$$C \triangleq \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} c_k = \sum_{j=1}^{t} \lim_{m\to\infty} \frac{1}{m} \sum_{k\in\mathcal{I}_j^{(m)}} c_k = \sum_{j=1}^{t} C_j.$$

4. The total loss from defaults and the portfolio default threshold are

$$L_m = \sum_{k=1}^{m} c_k Y_k^{(m)} \quad \text{and} \quad x_m = q \sum_{k=1}^{m} c_k$$

where $Y_k^{(m)} = \mathbf{1}\{X_k > \Phi^{-1}(1 - p_j^{(m)})\}$ and $0 < q < 1$. We impose a mild restriction on the possible values of $q$; $q$ is not a value in the finite set, $\{\frac{1}{C}\sum_{j\in\mathcal{J}} C_j : \mathcal{J} \subset \{1, \ldots, t\}\}$.

Using the fact that $\Phi(-t) \sim \phi(t)/t$ as $t \to \infty$, it follows that the default probabilities in **M2**-2 satisfy $p_j^{(m)} = \exp(-s_j^2 m/2 + o(m))$. So, this assumption specifies that the default probabilities decrease exponentially in $m$ and the last assumption implies that the loss threshold $x_m$ is $O(m)$. The particular parametrization in **M2**-2 is convenient because of the simplification it provides in (2.1).

Now we introduce a central concept in analysis of the small default probability model that we call the *q-minimal index set*. These are sets of obligor types. We say that $\mathcal{J}$ is a $q$-minimal index set if $\mathcal{J} \subset \{1, \ldots, t\}$ and

$$\max_{\mathcal{J}' \subsetneq \mathcal{J}} \sum_{j\in\mathcal{J}'} C_j < qC \leq \sum_{j\in\mathcal{J}} C_j. \tag{2.2}$$

Thus, if all obligors of types in $\mathcal{J}$ default, the loss exceeds $qC$ (which is the limit of $x_m/m$), but the default of all obligors of any subset of types does not suffice to reach the threshold. Note that the inequality $qC \le \sum_{j\in\mathcal{J}} C_j$ is in fact strict because of the restriction on $q$. We define $\mathcal{M}_q$ to be the family of all $q$-minimal index sets. Note the following two simplest and most extreme cases:

just plug into above condition...

- $q < \frac{\min_j C_j}{C}$ : $\mathcal{M}_q = \{\{1\}, \{2\}, \ldots, \{t\}\}$ and the default of any type suffices;
- $q > 1 - \frac{\min_j C_j}{C}$ : $\mathcal{M}_q = \{\{1, 2, \ldots, t\}\}$ and defaults of all types are required.

For each type $j \in \{1, \ldots, t\}$, we define the halfspace

$$G_j \triangleq \left\{\mathbf{z} \in \mathbb{R}^d : \mathbf{a}_j^\top \mathbf{z} \ge s_j\right\}$$

and set

$$G_{\mathcal{J}} \triangleq \bigcap_{j\in\mathcal{J}} G_j = \left\{\mathbf{z} \in \mathbb{R}^d : \mathbf{a}_j^\top \mathbf{z} \ge s_j \quad \text{for all } j \in \mathcal{J}\right\}, \quad \mathcal{J} \in \mathcal{M}_q,$$

and

$$G_{\mathcal{M}_q} \triangleq \bigcup_{\mathcal{J}\in\mathcal{M}_q} G_{\mathcal{J}}.$$

Our analysis relies heavily on such sets, so some explanation of their role may be helpful. Factor outcomes $\mathbf{z}$ for which $\mathbf{a}_j^\top \mathbf{z}$ is large make the type $j$ conditional default probability $p_j^{(m)}(\mathbf{z})$ large; see (2.1). The set $G_j$ should therefore be interpreted as the set of factor outcomes that make defaults of type $j$ "likely." The factor outcomes in $G_{\mathcal{J}}$ make defaults "likely" for all obligor types in the set $\mathcal{J}$, and $G_{\mathcal{M}_q}$ is the set of factor outcomes that make it "likely" that the portfolio loss will exceed $qC$ through the default of *some* collection of types.

EXAMPLE 2.1. To illustrate, we consider a simple example. There are four obligor types with $C_1 = 2$, $C_2 = 2$, $C_3 = 3$, and $C_4 = 3$, so $C = 10$. Set $q = 0.45$. Then $\mathcal{M}_q = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. Figure 2.1 shows the coefficient vector for each type (indicated by arrows) and the resulting sets $G_j, j = 1, 2, 3, 4$. $G_{\{1,3\}}$ is indicated by slanted lines and $G_{\{2,4\}}$ is indicated by crossed lines. $G_{\mathcal{M}_q} = G_{\{1,3\}} \cup G_{\{2,4\}}$ since $G_{\{2,3\}} \subset G_{\{1,3\}}$, $G_{\{1,4\}} \subset G_{\{2,4\}}$, and $G_{\{3,4\}} = \emptyset$.

Under a standard multivariate normal distribution, the "most likely" point in a set is the point closest to the origin. (These are points $\mathsf{P}_{\{1,3\}}$, $\mathsf{P}_{\{2,3\}}$, $\mathsf{P}_{\{1,4\}}$, and $\mathsf{P}_{\{2,4\}}$ in Figure 2.1 for the associated sets.) With this in mind, we define $\gamma_{\mathcal{J}}$ as the *unique* solution of the following *linearly constrained* problem:

$$\gamma_{\mathcal{J}} \triangleq \begin{cases} \operatorname{argmin}\{\|\mathbf{z}\| : \mathbf{z} \in G_{\mathcal{J}}\} & \text{if } G_{\mathcal{J}} \ne \emptyset \\ (\infty, \ldots, \infty)^\top & \text{if } G_{\mathcal{J}} = \emptyset. \end{cases} \tag{2.3}$$

Define

$$\gamma_* \triangleq \operatorname{argmin}\{\|\gamma_{\mathcal{J}}\| : \mathcal{J} \in \mathcal{M}_q\} \tag{2.4}$$

where, if a tie occurs, we choose the arbitrary minimal one. (This is the point $\mathsf{P}_{(1,3)}$ in Figure 2.1.) Note that $\gamma_* = (\infty, \ldots, \infty)^\top$ and $\|\gamma_*\| = \infty$ if $G_{\mathcal{J}} = \emptyset$ for all $\mathcal{J} \in \mathcal{M}_q$, by definition. Now we are ready to state the large deviations result for the small default probability regime. It says, roughly, that the rate of decrease of the tail of the loss distribution

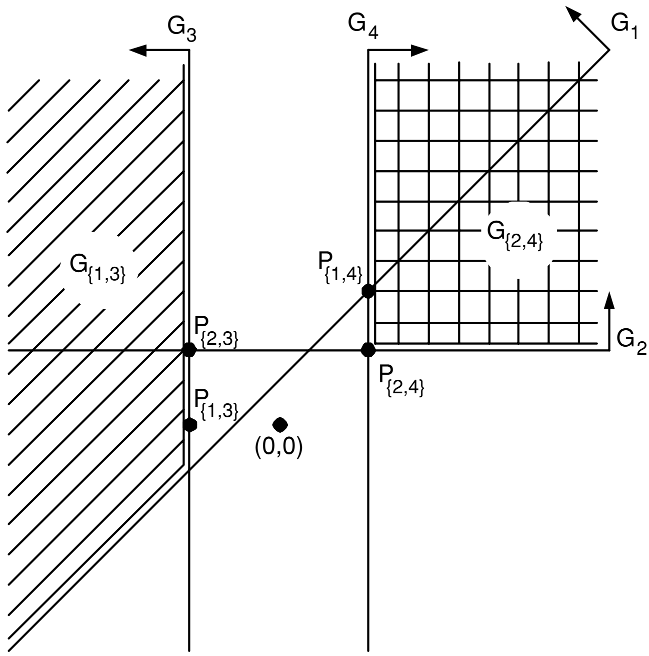


FIGURE 2.1. Illustration of the halfspaces associated with obligor types

is determined by the "most likely" factor outcome leading to large losses. The proof is given in Section 3.3.

THEOREM 2.1. *If the assumptions* **M1** *and* **M2** *are satisfied, then*

$$\lim_{m\to\infty} \frac{1}{m} \log \mathsf{P}(L_m > x_m) = -\frac{1}{2}\|\gamma_*\|^2.$$

## 2.2. Large Loss Threshold Regime

We now consider asymptotics as the loss threshold $x_m$ grows at a faster rate while the default probabilities remain fixed. We also allow random recovery, so the loss resulting from the default of an obligor is random. We add **M3** to **M1** to specify the parameters:

**M3** 1. The marginal default probabilities satisfy $0 < \underline{p} \le p_k \le \bar{p} < 1$ for $k = 1, \dots, m$.

2. If the $k$-th obligor is of type $j$ then its conditional default probability is given by

no assumption of groups …

$$p_k(\mathbf{Z}) = \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{Z} + \Phi^{-1}(p_k)}{b_j}\right).$$

3. The maximum loss for obligor $k$ is $l_k$ and $0 < \underline{l} \le l_k \le \bar{l} < \infty$ for $k = 1, \dots, m$. The actual loss upon the default of obligor $k$ is $l_k U_k$, with $U_k$ a $[\underline{u}, 1]$-valued random variable and $\underline{u}$ a constant satisfying $0 < \underline{u} \le 1$. For each obligor type $j$, $\{U_k\}_{k\in\mathcal{I}_j^{(m)}}$ is an iid sequence from a distribution with

mean $u_j^*$. These loss random variables are independent of $\mathbf{Z}$ and $\{\varepsilon_k\}$. We assume that $\frac{1}{m}\sum_{k\in\mathcal{I}_j^{(m)}} l_k$ converges for all $j$. We use $u_k$ to denote the mean of $U_k$; this is $u_j^*$ if the $k$-th obligor is of type $j$.

4. The total loss from defaults and the portfolio default threshold are

$$L_m = \sum_{k=1}^{m} l_k U_k Y_k \quad \text{and} \quad x_m = q_m \sum_{k=1}^{m} l_k u_k$$

where $Y_k = \mathbf{1}\{X_k > \Phi^{-1}(1-p_k)\}$ and $q_m = \Phi(s\sqrt{\log m})$ for some $0 < s < 1$.

In the last of these conditions, we have $q_m \uparrow 1$ and the rate of increase is such that $1 - q_m$ is roughly $m^{-s^2/2}$; see the discussion following **M2**. The specific parametrization in **M3**-4 leads to some simplification in the analysis, but the key feature is the order of magnitude of $1 - q_m$. For any rate of decrease of $1 - q_m$ that is faster than $1/m$ (e.g., $1 - q_m \sim m^{-3/2}$ or $1 - q_m \sim e^{-m}$), it is shown in Kang (2005) that a result similar to Theorem 2.2 holds under additional conditions on the model parameters.

Let

$$G \triangleq \bigcap_{j=1}^{t} \{\mathbf{z} \in \mathbb{R}^d : \mathbf{a}_j^\top \mathbf{z} \geq s b_j\}$$

and let $\gamma$ be the *unique* solution of the following linearly constrained problem

$$\gamma \triangleq \begin{cases} \operatorname{argmin}\{\|\mathbf{z}\| : \mathbf{z} \in G\} & \text{if } G \neq \emptyset \\ (\infty, \dots, \infty)^\top & \text{if } G = \emptyset. \end{cases} \tag{2.5}$$

Now we can state the large deviations result for large loss threshold regime. The proof will be given in Section 4.3.

THEOREM 2.2. *If the assumptions* **M1** *and* **M3** *are satisfied, then*

$$\lim_{m\to\infty} \frac{1}{\log m} \log \mathsf{P}(L_m > x_m) = -\frac{1}{2}\|\gamma\|^2.$$

Observe that in Theorem 2.2 we normalize by log $m$, indicating that the probability decays like $m^{-\|\gamma\|^2/2}$, whereas in Theorem 2.1 we normalize by $m$, indicating that the probability decays like $\exp(-m\|\gamma_*\|^2/2)$.

## 3. ANALYSIS OF THE SMALL DEFAULT PROBABILITY REGIME

In this section, we assume the conditions **M1** and **M2**. For each obligor type $j = 1, \dots, t$, define

$$\begin{aligned} G_j^{(m,\epsilon)} &\triangleq \{\mathbf{z} \in \mathbb{R}^d : \mathbf{a}_j^\top \mathbf{z} \geq (1-\epsilon_m) s_j \sqrt{m}\} \\ &= \left\{\mathbf{z} \in \mathbb{R}^d : p_j^{(m)}(\mathbf{z}) = \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{z} - s_j\sqrt{m}}{b_j}\right) \geq \Phi\left(-\frac{s_j}{b_j}\epsilon_m\sqrt{m}\right)\right\}, \end{aligned}$$

and then

$$G_{\mathcal{J}}^{(m,\epsilon)} \triangleq \bigcap_{j\in\mathcal{J}} G_j^{(m,\epsilon)} \quad \text{for } \mathcal{J} \in \mathcal{M}_q \quad \text{and} \quad G^{(m,\epsilon)} \triangleq \bigcup_{\mathcal{J}\in\mathcal{M}_q} G_{\mathcal{J}}^{(m,\epsilon)}$$

where $\epsilon_m > 0$ satisfies $\epsilon_m \to 0$ and $\epsilon_m\sqrt{m} \to \infty$. One possible choice is $\epsilon_m = m^{-1/3}$.

The assumption that $\|\mathbf{a}_j\| > 0$ implies $\mathbf{a}_j \neq \mathbf{0}$. If all $\mathbf{a}_j \geq \mathbf{0}$, then $G_{\mathcal{J}}^{(m,\epsilon)} \neq \emptyset$ for all $\mathcal{J} \subset \{1,\ldots,t\}$. However, if some of the coefficient vectors $\mathbf{a}_j$ have negative components, then these sets may be empty. Hence we define a subfamily of $\mathcal{M}_q$,

$$\mathcal{S}_q \triangleq \left\{\mathcal{J} \in \mathcal{M}_q : G_{\mathcal{J}}^{(m,\epsilon)} \neq \emptyset \text{ for all } m\right\}.$$

EXAMPLE 3.1. (Continued.) We can re-interpret Figure 2.1 as depicting the sets $G_j^{(m,\epsilon)}, j = 1,2,3,4$. As before, $\mathcal{M}_q = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$, so taking into consideration the sets with nonempty intersection we find that $\mathcal{S}_q = \{\{1,3\}$, $\{1,4\},\{2,3\},\{2,4\}\}$. (Shortly, it can be easily seen that $G_{\{2,3\}}^{(m,\epsilon)}$ and $G_{\{1,4\}}^{(m,\epsilon)}$ are contained in $G_{\{1,3\}}^{(m,\epsilon)}$ and $G_{\{2,4\}}^{(m,\epsilon)}$, respectively, and our treatments on index sets $\{1, 3\}$ and $\{2, 4\}$ have

{3,4} is emptyset

also effects on any index sets whose $G_{\mathcal{J}}^{(m,\epsilon)}$ is included in $G_{\{1,3\}}^{(m,\epsilon)} \cup G_{\{2,4\}}^{(m,\epsilon)}$. So it is sufficient to consider only the two dominating index sets. However, we will not investigate this any more in this paper because the asymptotics is insensitive to the choice of $\mathcal{S}_q$.)

The next lemma and corollary give conditions under which $G_{\mathcal{J}}^{(m,\epsilon)}$ and $G_{\mathcal{J}}$ are nonempty.

LEMMA 3.1. *Suppose that* $f(\cdot)$ *is a function such that* $\lim_{m\to\infty} f(m) = \infty$, $g_j \in \mathbb{R}$ *and a real sequence* $\{\eta_m\}$ *satisfies* $\lim_{m\to\infty}\eta_m = 0$. *For any* $\mathcal{J} \subset \{1,\ldots,t\}$,

$$G_{\mathcal{J}} = \left\{\mathbf{z} : \mathbf{a}_j^\top \mathbf{z} \geq s_j \quad \text{for all } j \in \mathcal{J}\right\} \neq \emptyset$$

*if and only if*

$$\left\{\mathbf{z} : \mathbf{a}_j^\top \mathbf{z} \geq g_j + (1-\eta_m)s_j f(m) \quad \text{for all } j \in \mathcal{J}\right\} \neq \emptyset \quad \text{for all } m.$$

*Proof*. Consider the "only if" direction. Suppose $\mathbf{z}^* \in G_{\mathcal{J}} \neq \emptyset$ and fix $m$. Because $\mathbf{a}_j^\top \mathbf{z}^* \geq s_j > 0$ for each $j \in \mathcal{J}$, we can choose $\lambda_m > 0$ sufficiently large that $\mathbf{a}_j^\top(\lambda_m \mathbf{z}^*) \geq g_j + (1-\eta_m)s_j f(m)$ for all $j \in \mathcal{J}$. For the "if" direction, noting that $\lim_{m\to\infty}(1-\eta_m)s_j f(m) = \infty$ for all $j \in \mathcal{J}$ is sufficient. □

COROLLARY 3.1. $G_{\mathcal{J}}^{(m,\epsilon)} \neq \emptyset$ *for all* $m$ *if and only if* $G_{\mathcal{J}} \neq \emptyset$.

*Proof*. By taking $f(m) = \sqrt{m}$, $g_j = 0$, and $\eta_m = \epsilon_m$, Lemma 3.1 gives the result. □

Now for each $\mathcal{J} \in \mathcal{S}_q$, we define $\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}$ as the *unique* solution of the following linearly constrained problem:

$$\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)} \triangleq \operatorname{argmin}\left\{\|\mathbf{z}\| : \mathbf{z} \in G_{\mathcal{J}}^{(m,\epsilon)}\right\}. \tag{3.1}$$

We consider a useful convergence property of $\frac{1}{\sqrt{m}}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}$.

LEMMA 3.2. *Assume that* $\mathbf{a}_j \neq \mathbf{0}$, $h_j > 0$, and $v_j^{(m)} \to 0$ *as* $m \to \infty$ *for all* $j = 1,\ldots,t$. *For any* $\mathcal{J} \subset \{1,\ldots,t\}$ *such that* $\{\mathbf{z} : \mathbf{a}_j^\top \mathbf{z} \geq h_j + v_j^{(m)}, j \in \mathcal{J}\} \neq \emptyset$ *for all* $m$,

$$\left\{\mathbf{z} : \mathbf{a}_j^\top \mathbf{z} \geq h_j \quad \text{for all } j \in \mathcal{J}\right\} \neq \emptyset$$

*and* asymptotically v_j term vanishes

$$\mathbf{z}_{\mathcal{J}}^{(m)} = \operatorname{argmin}\left\{\|\mathbf{z}\| : \mathbf{a}_j^\top \mathbf{z} \geq h_j + v_j^{(m)} \quad \textit{for all } j \in \mathcal{J}\right\}$$

*converges to*

$$\mathbf{z}_{\mathcal{J}}^{\infty} = \operatorname{argmin}\left\{\|\mathbf{z}\| : \mathbf{a}_j^\top \mathbf{z} \geq h_j \quad \textit{for all } j \in \mathcal{J}\right\}$$

*as* $m \to \infty$ . *Furthermore,* $\mathbf{z}_{\mathcal{J}}^{\infty} \neq \mathbf{0}$.

*Proof*. Fix arbitrary $\mathcal{J}$. Define, for $\mathbf{h} = (h_1, \ldots, h_l)^\top$,

$$\mathcal{P}(\mathbf{h}) = \left\{\mathbf{z} : \mathbf{a}_j^\top \mathbf{z} \geq h_j \quad \text{for all } j \in \mathcal{J}\right\}.$$

Let $\mathbf{v}^{(m)}$ be the vector with $j$-th component $v_j^{(m)}$. Since $\mathbf{v}^{(m)} \to \mathbf{0}$, there exists an $m$ such that $\mathbf{h} + \mathbf{v}^{(m)} > 0$. So $\mathbf{z}_{\mathcal{J}}^{(m)}$ multiplied by a large positive scalar lies in $\mathcal{P}(\mathbf{h})$, which proves the first claim. Again since $\mathbf{v}^{(m)} \to \mathbf{0}$, there exists an $M$ such that for $m > M$,

$$\mathcal{P}(\mathbf{h} + \mathbf{e}) \subset \mathcal{P}\big(\mathbf{h} + \mathbf{v}^{(m)}\big).$$

Set $\mathbf{z}^0 = \operatorname{argmin}\{\|\mathbf{z}\| : \mathbf{z} \in \mathcal{P}(\mathbf{h} + \mathbf{e})\}$ and $\mathcal{B} \triangleq \{\mathbf{z} : \|\mathbf{z}\| \leq \|\mathbf{z}^0\|\}$. Now for $m > M$, $\mathbf{z}^0 \in \mathcal{P}(\mathbf{h} + \mathbf{v}^{(m)})$ implies $\|\mathbf{z}_{\mathcal{J}}^{(m)}\| \leq \|\mathbf{z}^0\|$ and hence $\mathbf{z}_{\mathcal{J}}^{(m)} \in \mathcal{P}(\mathbf{h} + \mathbf{v}^{(m)}) \cap \mathcal{B}$. If we extend the parameter $m$ to a continuous one on $[0, 1]$ (e.g., by linear interpolation of $\mathbf{v}'(1/m) = \mathbf{v}^{(m)}$ and $\mathbf{v}'(0) = \mathbf{0}$), then we can apply the *Maximum Theorem* (e.g., Theorem 9.17 of Sundaram [1996]) since $\|\cdot\|$ is strictly convex and $\mathcal{P}(\mathbf{h} + \mathbf{v}^{(m)}) \cap \mathcal{B}$ is a compact and convex-valued continuous correspondence. Hence we get $\mathbf{z}_{\mathcal{J}}^{(m)} \to \mathbf{z}_{\mathcal{J}}^{\infty}$. Finally $\mathbf{z}_{\mathcal{J}}^{\infty}$ is nonzero since $\mathbf{a}_j^\top \mathbf{0} = 0 < h_j$. $\square$

To connect Lemma 3.2 to the definitions of $\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}$, first note that for any $h_j > 0$ and $\mathcal{J}$,

$$\begin{aligned} \frac{1}{f(m)} &\operatorname{argmin}\left\{\|\mathbf{z}\| : \mathbf{a}_j^\top \mathbf{z} \geq h_j f(m) \quad \text{for all } j \in \mathcal{J}\right\} \\ &= \operatorname{argmin}\left\{\|\mathbf{z}\| : \mathbf{a}_j^\top \mathbf{z} \geq h_j \quad \text{for all } j \in \mathcal{J}\right\} \end{aligned} \tag{3.2}$$

where $f(\cdot)$ is any positive-valued function. Consider a $\mathcal{J}$ satisfying $G_{\mathcal{J}} \neq \emptyset$. For this $\mathcal{J}$, $\min\{\|\mathbf{z}\| : \mathbf{a}_j^\top \mathbf{z} \geq s_j \text{ for all } j \in \mathcal{J}\} < \infty$ and by Corollary 3.1, $G_{\mathcal{J}}^{(m,\epsilon)}$ is nonempty and $\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}$ is well-defined for any $m$. Furthermore,

$$\begin{aligned} \gamma_{\mathcal{J}} &= \operatorname{argmin}\left\{\|\mathbf{z}\| : \mathbf{a}_j^\top \mathbf{z} \geq s_j \text{ for all } j \in \mathcal{J}\right\} && \text{by (2.3)} \\ &= \lim_{m\to\infty} \frac{1}{\sqrt{m}} \boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)} && \text{by (3.2) with } f(\cdot) = \sqrt{\cdot} \text{ and Lemma 3.2.} \end{aligned} \tag{3.3}$$

For future reference we record an implication of (3.3):

$$\|\gamma_{\mathcal{J}}\| = \lim_{m\to\infty} \frac{1}{\sqrt{m}} \left\|\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\right\| < \infty \quad \text{for all } \mathcal{J} \text{ satisfying } G_{\mathcal{J}} \neq \emptyset. \tag{3.4}$$

## 3.1. Upper Bound Computation

LEMMA 3.3. *If $\mathcal{S}_q \neq \emptyset$, then for each $\mathcal{J} \in \mathcal{S}_q$,*

$$\min\left\{\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)\top} \mathbf{z} : \mathbf{z} \in G_{\mathcal{J}}^{(m,\epsilon)}\right\} = \left\|\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\right\|^2 \tag{3.5}$$

*Proof*. $G_{\mathcal{J}}^{(m,\epsilon)} = \bigcap_{j\in\mathcal{J}} G_j^{(m,\epsilon)}$ is a convex polyhedron and the definition of $\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}$ implies that

$$\{\mathbf{z} \in \mathbb{R}^d : \|\mathbf{z}\| \le \|\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\|\} \cap G_{\mathcal{J}}^{(m,\epsilon)} = \{\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\}.$$

Then there is a separating plane and for this case it is uniquely given by

$$\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)^\top}\mathbf{z} = \boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)^\top}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}.$$

See Theorem 11.3 in Rockafellar (1997). So

$$G_{\mathcal{J}}^{(m,\epsilon)} \subset \{\mathbf{z} : \boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)^\top}\mathbf{z} \ge \boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)^\top}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\}$$

which concludes the proof. □

As is often the case for large deviations results, we introduce a change of measure as in the proof. We select a measure under which the rare event is no longer rare, so that the rate of decrease of the original probability is given by the rate of decrease of the likelihood ratio relating the original to the new probability. The particular change of measure we introduce is intended to approximate the "the most likely" way (or ways) for the rare event to occur. It involves shifting the factor mean to increase the default probabilities of sets of types and then increasing the conditional default probabilities given the factors. (See Glasserman and Li [2005] for an application of these ideas in the simpler setting of a single-factor homogeneous model.)

Define a sequence of changes of measure using two likelihood ratios. The first one is the likelihood ratio relating the $d$-dimensional standard normal distribution to a mixture of $N(\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}, \mathbf{I}_d)$, $\mathcal{J} \in \mathcal{S}_q$ with weights $\lambda_{\mathcal{J}}$'s, for each $m$. The weights can be any $\lambda_{\mathcal{J}} > 0$ for all $\mathcal{J} \in \mathcal{S}_q$ with $\sum_{\mathcal{J}\in\mathcal{S}_q} \lambda_{\mathcal{J}} = 1$. This likelihood ratio is given explicitly by $(M_{\mathbf{Z}}^{(m,\epsilon)})^{-1}$, which is defined by

pretty easy to derive in papers/meetingsmeeting.pdf

$$M_{\mathbf{Z}}^{(m,\epsilon)} \triangleq \begin{cases} \displaystyle\sum_{\mathcal{J}\in\mathcal{S}_q} \lambda_{\mathcal{J}} \exp\left(\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)^\top}\mathbf{Z} - \frac{1}{2}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)^\top}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\right) & \text{if } \mathcal{S}_q \ne \emptyset \\ 1 & \text{if } \mathcal{S}_q = \emptyset. \end{cases}$$

Note that the sum is over the subfamily $\mathcal{S}_q$.

EXAMPLE 3.2. (Continued.) As before, with the interpretation of Figure 2.1 as depicting the sets $G_j^{(m,\epsilon)}$, $j = 1, 2, 3, 4$, $\mathcal{S}_q = \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}$. The minimal points for each index sets in $\mathcal{S}_q$ are $\mathsf{P}_{\{1,3\}}$, $\mathsf{P}_{\{1,4\}}$, $\mathsf{P}_{\{2,3\}}$, and $\mathsf{P}_{\{2,4\}}$. These four points are used as $\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}$'s to define the mixture of shifted normal distributions.

The second part of the change of measure is defined by a conditional likelihood given $\mathbf{Z}$. First we define a conditional cumulant generating function divided by $m$,

$$\psi_m(\theta, \mathbf{z}) \triangleq \frac{1}{m}\log \mathsf{E}[e^{\theta L_m} \mid \mathbf{Z} = \mathbf{z}] \tag{3.6}$$
$$= \frac{1}{m}\sum_{j=1}^{t}\sum_{k\in\mathcal{I}_j^{(m)}} \log\left(1 + p_j^{(m)}(\mathbf{z})(e^{\theta c_k} - 1)\right).$$

Then, given $\mathbf{Z}$, we set up the conditional likelihood ratio associated with the change of default probabilities by

$$e^{-\theta_m(\mathbf{Z})L_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}$$

where

$$\theta_m(\mathbf{z}) \triangleq \operatorname*{argmin}_{\theta \geq 0}\{-\theta x_m + m\psi_m(\theta, \mathbf{z})\}. \tag{3.7}$$

The new conditional default probability is an exponentially tilted one from $Y_k$ by the amount $\theta_m(\mathbf{Z})$.

The likelihood ratio for the combined change of measure is given by

$$\frac{d\mathsf{P}}{d\mathsf{P}_m} = e^{-\theta_m(\mathbf{Z})L_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}.$$

We write $\mathsf{E}_m$ for expectation under the probability measure $\mathsf{P}_m$, under which $\mathbf{Z}$ is distributed by the mixture defined above and $Y_k$ is a default indicator with the exponentially tilted conditional default probability.

The approach we used for the upper bound computation is to express the default probability by

only necessary for proving asymptotics of tail distribution

$$\begin{aligned}
\mathsf{P}(L_m > x_m) &= \mathsf{E}_m\big[\mathbf{1}\{L_m > x_m\}e^{-\theta_m(\mathbf{Z})L_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\big] \quad \text{by the definition of } \mathsf{P}_m\\
&\leq \mathsf{E}_m\big[e^{-\theta_m(\mathbf{Z})x_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\big] \quad \text{dropping the indicator}\\
&= \mathsf{E}_m\big[e^{-\theta_m(\mathbf{Z})x_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\mathbf{1}_{(G^{(m,\epsilon)})^c}(\mathbf{Z})\big]\\
&\quad + \mathsf{E}_m\big[e^{-\theta_m(\mathbf{Z})x_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\big]\\
&\leq \mathsf{E}_m\big[e^{-\theta x_m + m\psi_m(\theta,\mathbf{Z})}\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\mathbf{1}_{(G^{(m,\epsilon)})^c}(\mathbf{Z})\big] \quad \text{by definition (3.7)} &(3.8)\\
&\quad + \mathsf{E}_m\big[\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\big] \quad \text{by definition (3.7), because } \theta_m(\mathbf{Z}) \geq 0 &(3.9)
\end{aligned}$$

where $\theta$ is an arbitrary positive real number. We then find the upper bounds on both the terms in (3.8) and (3.9).

For ease of reference we state the following lemma whose proof can be found on page 7 in Dembo and Zeitouni (1998);

LEMMA 3.4. *Let $N$ be a fixed integer. Then, for every $a_\epsilon^i \geq 0$,*

$$\limsup_{\epsilon \to 0} \epsilon \log\left(\sum_{i=1}^{N} a_\epsilon^i\right) = \max_{1 \leq i \leq N} \limsup_{\epsilon \to 0} \epsilon \log a_\epsilon^i.$$

Using Lemma 3.3, we prove the following lemma.

LEMMA 3.5. *If $\mathcal{S}_q \neq \emptyset$, then*

$$\limsup_{m \to \infty} \frac{1}{m} \log \mathsf{E}_m\big[\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\big] \leq -\frac{1}{2}\|\gamma_*\|^2.$$

*Proof*.

$$
\begin{aligned}
&\mathsf{E}_m\big[\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\big] \\
&\quad= \mathsf{E}_m\Big[\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\mathbf{1}_{\bigcup_{\mathcal{J}\in\mathcal{S}_q} G_{\mathcal{J}}^{(m,\epsilon)}}(\mathbf{Z})\Big] \\
&\quad\le \mathsf{E}_m\Big[\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\sum_{\mathcal{J}\in\mathcal{S}_q}\mathbf{1}_{G_{\mathcal{J}}^{(m,\epsilon)}}(\mathbf{Z})\Big] \\
&\quad= \sum_{\mathcal{J}\in\mathcal{S}_q}\mathsf{E}_m\left[\left(\sum_{\mathcal{J}'\in\mathcal{S}_q}\lambda_{\mathcal{J}'}\exp\left(\boldsymbol{\mu}_{\mathcal{J}'}^{(m,\epsilon)\top}\mathbf{Z}-\frac{1}{2}\boldsymbol{\mu}_{\mathcal{J}'}^{(m,\epsilon)\top}\boldsymbol{\mu}_{\mathcal{J}'}^{(m,\epsilon)}\right)\right)^{-1}\mathbf{1}_{G_{\mathcal{J}}^{(m,\epsilon)}}(\mathbf{Z})\right] \\
&\quad\le \sum_{\mathcal{J}\in\mathcal{S}_q}\mathsf{E}_m\left[\frac{1}{\lambda_{\mathcal{J}}}\exp\left(-\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)\top}\mathbf{Z}+\frac{1}{2}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)\top}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\right)\mathbf{1}_{G_{\mathcal{J}}^{(m,\epsilon)}}(\mathbf{Z})\right].
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}_m\big[\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\big] \\
&\le \limsup_{m\to\infty}\frac{1}{m}\log\left(\sum_{\mathcal{J}\in\mathcal{S}_q}\mathsf{E}_m\left[\frac{1}{\lambda_{\mathcal{J}}}\exp\left(-\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)\top}\mathbf{Z}+\frac{1}{2}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)\top}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\right)\mathbf{1}_{G_{\mathcal{J}}^{(m,\epsilon)}}(\mathbf{Z})\right]\right) \\
&= \max_{\mathcal{J}\in\mathcal{S}_q}\left\{\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}_m\left[\exp\left(-\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)\top}\mathbf{Z}+\frac{1}{2}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)\top}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\right)\mathbf{1}_{G_{\mathcal{J}}^{(m,\epsilon)}}(\mathbf{Z})\right]\right\} \\
&\hspace{25em}\text{by Lemma3.4.}
\end{aligned}
$$

For each $\mathcal{J}\in\mathcal{S}_q$,

$$
\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}_m\big[e^{-\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)\top}\mathbf{Z}+\frac{1}{2}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)\top}\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}}\mathbf{1}_{G_{\mathcal{J}}^{(m,\epsilon)}}(\mathbf{Z})\big] \\
&\le \lim_{m\to\infty}\frac{1}{2m}\big\|\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\big\|^2+\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}_m\Big[e^{-\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)\top}\mathbf{Z}}\mathbf{1}_{G_{\mathcal{J}}^{(m,\epsilon)}}(\mathbf{Z})\Big] \\
&\le \frac{1}{2}\|\boldsymbol{\gamma}_{\mathcal{J}}\|^2+\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}_m\Big[e^{-\|\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\|^2}\mathbf{1}_{G_{\mathcal{J}}^{(m,\epsilon)}}(\mathbf{Z})\Big] \quad \text{by (3.4) and Lemma 3.3} \\
&\le \frac{1}{2}\|\boldsymbol{\gamma}_{\mathcal{J}}\|^2+\limsup_{m\to\infty}\frac{1}{m}\log e^{-\|\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\|^2} \\
&= \frac{1}{2}\|\boldsymbol{\gamma}_{\mathcal{J}}\|^2-\lim_{m\to\infty}\frac{1}{m}\big\|\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}\big\|^2 \\
&= -\frac{1}{2}\|\boldsymbol{\gamma}_{\mathcal{J}}\|^2 \quad \text{by (3.4).}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{E}_m\big[\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\big] &\le \max_{\mathcal{J}\in\mathcal{S}_q}\left\{-\frac{1}{2}\|\boldsymbol{\gamma}_{\mathcal{J}}\|^2\right\} \\
&= -\frac{1}{2}\|\boldsymbol{\gamma}_*\|^2, \quad \text{by (2.4).}
\end{aligned}
$$

$\square$

Define $\mathcal{K} \subset \{1, \dots, t\}$ to be a *cutset* of $\mathcal{M}_q$ if $\mathcal{K} \cap \mathcal{J} \neq \emptyset$ for all $\mathcal{J} \in \mathcal{M}_q$ and $\mathcal{K}$ is minimal—that is, for any $\mathcal{K}' \subsetneqq \mathcal{K}$, there exists a $\mathcal{J} \in \mathcal{M}_q$ such that $\mathcal{K}' \cap \mathcal{J} = \emptyset$. We denote the family of all cutsets by $\mathcal{C}_q$.

Now we construct a covering of $(G^{(m,\epsilon)})^c$.

LEMMA 3.6.

$$\left(G^{(m,\epsilon)}\right)^c \subset \bigcup_{\mathcal{K} \in \mathcal{C}_q} \bigcap_{j \in \mathcal{K}} \left(G_j^{(m,\epsilon)}\right)^c.$$

*Proof*. Suppose

$$\mathbf{z} \in \left(G^{(m,\epsilon)}\right)^c = \bigcap_{\mathcal{J} \in \mathcal{M}_q} \bigcup_{j \in \mathcal{J}} \left(G_j^{(m,\epsilon)}\right)^c.$$

Then for each $\mathcal{J} \in \mathcal{M}_q$, there exists a $j_{\mathcal{J}} \in \mathcal{J}$ such that $\mathbf{z} \in (G_{j_{\mathcal{J}}}^{(m,\epsilon)})^c$. Now $\{j_{\mathcal{J}} : \mathcal{J} \in \mathcal{M}_q\}$ is a *cut* of $\mathcal{M}_q$, (that is, $j_{\mathcal{J}} \in \{j_{\mathcal{J}'} : \mathcal{J}' \in \mathcal{M}_q\} \cap \mathcal{J} \neq \emptyset$ for all $\mathcal{J} \in \mathcal{M}_q$) and it contains a cutset $\mathcal{K}$. For this cutset $\mathcal{K}$,

$$\mathbf{z} \in \bigcap_{\mathcal{J} \in \mathcal{M}_q} \left(G_{j_{\mathcal{J}}}^{(m,\epsilon)}\right)^c \subset \bigcap_{j \in \mathcal{K}} \left(G_j^{(m,\epsilon)}\right)^c$$

and this completes the proof. □

By Lemma 3.6 it is clear that

$$\mathbf{1}_{(G^{(m,\epsilon)})^c} \leq \sum_{\mathcal{K} \in \mathcal{C}_q} \mathbf{1}_{\bigcap_{j \in \mathcal{K}} (G_j^{(m,\epsilon)})^c}. \tag{3.10}$$

Now observe that for each $\mathcal{K} \in \mathcal{C}_q$, $\mathbf{z} \in \bigcap_{l \in \mathcal{K}} (G_l^{(m,\epsilon)})^c$ implies

$$p_j^{(m)}(\mathbf{z})\left(e^{\theta c_k} - 1\right) \leq \begin{cases} \Phi\left(-\frac{s_j}{b_j}\epsilon_m\sqrt{m}\right)\left(e^{\theta \bar{c}} - 1\right) & \text{if } j \in \mathcal{K}, \text{i.e. } \mathbf{z} \in \left(G_j^{(m,\epsilon)}\right)^c \\ e^{\theta c_k} - 1 & \text{if } j \notin \mathcal{K} \end{cases} \tag{3.11}$$

and using this, we also have for $\mathbf{z} \in \bigcap_{l \in \mathcal{K}} (G_l^{(m,\epsilon)})^c$,

$$\begin{aligned} e^{m\psi_m(\theta,\mathbf{z})} &= \prod_{j=1}^{t} \prod_{k \in \mathcal{I}_j^{(m)}} \left(1 + p_j^{(m)}(\mathbf{z})\left(e^{\theta c_k} - 1\right)\right) \\ &\leq \prod_{j \in \mathcal{K}} \prod_{k \in \mathcal{I}_j^{(m)}} \left(1 + \Phi\left(-\frac{s_j}{b_j}\epsilon_m\sqrt{m}\right)\left(e^{\theta \bar{c}} - 1\right)\right) \times \prod_{j \notin \mathcal{K}} \prod_{k \in \mathcal{I}_j^{(m)}} e^{\theta c_k}. \end{aligned} \tag{3.12}$$

Then this deterministic upper bound (3.12) gives

$$\begin{aligned} &\frac{1}{m} \log \mathsf{E}'_m\left[e^{m\psi_m(\theta,\mathbf{Z})} \mathbf{1}_{\bigcap_{j \in \mathcal{K}} (G_j^{(m,\epsilon)})^c}(\mathbf{Z})\right] \\ &\quad \leq \frac{1}{m} \sum_{j \in \mathcal{K}} n_j^{(m)} \log\left(1 + \Phi\left(-\frac{s_j}{b_j}\epsilon_m\sqrt{m}\right)\left(e^{\theta \bar{c}} - 1\right)\right) + \theta \sum_{j \notin \mathcal{K}} \frac{1}{m} \sum_{k \in \mathcal{I}_j^{(m)}} c_k \\ &\quad \to \theta \sum_{j \notin \mathcal{K}} C_j \qquad \text{as } m \to \infty \end{aligned} \tag{3.13}$$

where $\{\mathsf{E}'_m\}_{m=1}^{\infty}$ are expectations under $\{\mathsf{P}'_m\}_{m=1}^{\infty}$, an arbitrary sequence of probability measures. This implies

LEMMA 3.7.

$$\limsup_{m\to\infty}\frac{1}{m}\log \mathsf{E}_m\left[e^{-\theta x_m+m\psi_m(\theta,\mathbf{Z})}\left(M_{\mathbf{Z}}^{(m,\epsilon)}\right)^{-1}\mathbf{1}_{(G^{(m,\epsilon)})^c}(\mathbf{Z})\right]\le\theta\left(\max_{\mathcal{K}\in\mathcal{C}_q}\left\{\sum_{j\notin\mathcal{K}}C_j\right\}-qC\right).$$

*Proof*. First introduce an intermediate change of measure:

$$\frac{d\mathsf{P}}{d\mathsf{P}'_m}=e^{-\theta_m(\mathbf{Z})L_m+m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}.$$

Note that

$$\left(M_{\mathbf{Z}}^{(m,\epsilon)}\right)^{-1}\frac{d\mathsf{P}}{d\mathsf{P}'_m}=\frac{d\mathsf{P}}{d\mathsf{P}_m}\quad\text{and hence}$$

$$\mathsf{E}_m\left[\left(M_{\mathbf{Z}}^{(m,\epsilon)}\right)^{-1}Y\right]=\mathsf{E}'_m[Y]\quad\text{for any random variable } Y. \tag{3.14}$$

$$\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{m}\log \mathsf{E}_m\left[e^{-\theta x_m+m\psi_m(\theta,\mathbf{Z})}\left(M_{\mathbf{Z}}^{(m,\epsilon)}\right)^{-1}\mathbf{1}_{(G^{(m,\epsilon)})^c}(\mathbf{Z})\right]\\
&\le\limsup_{m\to\infty}\frac{1}{m}\log\left(\sum_{\mathcal{K}\in\mathcal{C}_q}\mathsf{E}_m\left[e^{-\theta x_m+m\psi_m(\theta,\mathbf{Z})}\left(M_{\mathbf{Z}}^{(m,\epsilon)}\right)^{-1}\mathbf{1}_{\bigcap_{j\in\mathcal{K}}(G_j^{(m,\epsilon)})^c}(\mathbf{Z})\right]\right)\qquad\text{by (3.10)}\\
&=\max_{\mathcal{K}\in\mathcal{C}_q}\left\{\limsup_{m\to\infty}\frac{1}{m}\log \mathsf{E}_m\left[e^{-\theta x_m+m\psi_m(\theta,\mathbf{Z})}\left(M_{\mathbf{Z}}^{(m,\epsilon)}\right)^{-1}\mathbf{1}_{\bigcap_{j\in\mathcal{K}}(G_j^{(m,\epsilon)})^c}(\mathbf{Z})\right]\right\}\qquad\text{by Lemma 3.4}\\
&\le\max_{\mathcal{K}\in\mathcal{C}_q}\left\{-\theta qC+\limsup_{m\to\infty}\frac{1}{m}\log \mathsf{E}_m\left[e^{m\psi_m(\theta,\mathbf{Z})}\left(M_{\mathbf{Z}}^{(m,\epsilon)}\right)^{-1}\mathbf{1}_{\bigcap_{j\in\mathcal{K}}(G_j^{(m,\epsilon)})^c}(\mathbf{Z})\right]\right\}\\
&=\max_{\mathcal{K}\in\mathcal{C}_q}\left\{-\theta qC+\limsup_{m\to\infty}\frac{1}{m}\log \mathsf{E}'_m\left[e^{m\psi_m(\theta,\mathbf{Z})}\mathbf{1}_{\bigcap_{j\in\mathcal{K}}(G_j^{(m,\epsilon)})^c}(\mathbf{Z})\right]\right\}\qquad\text{by (3.14)}\\
&\le\theta\left(\max_{\mathcal{K}\in\mathcal{C}_q}\left\{\sum_{j\notin\mathcal{K}}C_j\right\}-qC\right)\qquad\text{by (3.13).}
\end{aligned}$$

Note that Lemma 3.7 holds even if $\mathcal{S}_q=\emptyset$ (and hence $G^{(m,\epsilon)}=\emptyset$). □

The sign of this upper bound is determined by the next lemma:

LEMMA 3.8.

$$\max_{\mathcal{K}\in\mathcal{C}_q}\left\{\sum_{j\notin\mathcal{K}}C_j\right\}<qC.$$

*Proof*. Suppose that for some $\mathcal{K}\in\mathcal{C}_q$, $\sum_{j\notin\mathcal{K}}C_j=\sum_{j\in\mathcal{K}^c}C_j\ge qC$ where $\mathcal{K}^c=\{1,\dots,t\}\backslash\mathcal{K}$. Then by the definition of a $q$-minimal index set, there exists a $\mathcal{J}\in\mathcal{M}_q$ such that $\mathcal{J}\subset\mathcal{K}^c$. Furthermore $\mathcal{K}\cap\mathcal{J}\subset\mathcal{K}\cap\mathcal{K}^c=\emptyset$, which contradicts the assumption that $\mathcal{K}\in\mathcal{C}_q$. □

Now we get the following conclusion:

THEOREM 3.1. *Suppose the assumptions* **M1** *and* **M2**. *Then*

$$\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{P}(L_m>x_m)\le-\frac{1}{2}\|\gamma_*\|^2.$$

*Proof*. First assume that $\mathcal{S}_q \neq \emptyset$. Then $\|\gamma_*\| < \infty$ and for any $\theta > 0$,

$$\mathsf{P}(L_m > x_m) \leq \mathsf{E}_m\left[e^{-\theta x_m + m\psi_m(\theta,\mathbf{Z})}\left(M_{\mathbf{Z}}^{(m,\epsilon)}\right)^{-1}\mathbf{1}_{(G^{(m,\epsilon)})^c}(\mathbf{Z})\right] + \mathsf{E}_m\left[\left(M_{\mathbf{Z}}^{(m,\epsilon)}\right)^{-1}\mathbf{1}_{G^{(m,\epsilon)}}\right]$$

by (3.8) and (3.9). Considering sufficiently large $\theta$, Lemmas 3.4, 3.5, 3.7, and 3.8 together imply

$$\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{P}(L_m > x_m) \\
&\leq \max\left\{\theta\left(\max_{\mathcal{K}\in\mathcal{C}_q}\left\{\sum_{j\notin\mathcal{K}} C_j\right\} - qC\right), -\frac{1}{2}\|\gamma_*\|^2\right\} \\
&= -\frac{1}{2}\|\gamma_*\|^2.
\end{aligned}$$

If $\mathcal{S}_q = \emptyset$, then similarly

$$\begin{aligned}
&\mathsf{P}(L_m > x_m) \\
&= \mathsf{E}_m\left[\mathbf{1}\{L_m > x_m\}e^{-\theta_m(\mathbf{Z})L_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}\right] \\
&\leq \mathsf{E}_m\left[e^{-\theta_m(\mathbf{Z})x_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}\right] \\
&\leq \mathsf{E}_m\left[e^{-\theta x_m + m\psi_m(\theta,\mathbf{Z})}\right].
\end{aligned}$$

If $\mathcal{S}_q = \emptyset$ then $(M_{\mathbf{Z}}^{(m,\epsilon)})^{-1} = 1$, $G^{(m,\epsilon)} = \emptyset$ and $\|\gamma_*\| = \infty$. Lemma 3.7 then implies

$$\begin{aligned}
\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{P}(L_m > x_m) &\leq \theta\left(\max_{\mathcal{K}\in\mathcal{C}_q}\left\{\sum_{j\notin\mathcal{K}} C_j\right\} - qC\right) \\
&\to -\infty = -\frac{1}{2}\|\gamma_*\|^2 \quad \text{as } \theta\to\infty.
\end{aligned}$$

By combining these two cases, we complete the proof. □

### 3.2. Lower Bound Computations

*3.2.1. Lower Bounds for Partial Portfolios.* To develop a lower bound on the tail of the loss distribution for the full portfolio, we consider losses in subsets of the full portfolio. We use $\rho$ for the threshold ratio for a partial portfolio instead of $q$, the threshold ratio for the full portfolio. We take $0 < \rho < 1$. Consider any set of types $\mathcal{J}$ satisfying $G_{\mathcal{J}} \neq \emptyset$ and consider the partial portfolio restricted to types in $\mathcal{J}$. The size of the partial portfolio is

$$n_{\mathcal{J}}^{(m)} \triangleq \sum_{j\in\mathcal{J}} n_j^{(m)} = \sum_{j\in\mathcal{J}} |\mathcal{I}_j^{(m)}|,$$

the corresponding loss for the partial portfolio is

$$L_{\mathcal{J}}^{(m)} \triangleq \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{I}_j^{(m)}} c_k Y_k^{(m)}$$

and we set the *partial threshold* at

$$x_{\mathcal{J}}^{(m)} \triangleq \rho\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{I}_j^{(m)}} c_k.$$

Recall that if obligor $k$ is of type $j$, then its conditional default probability is given by $p_j^{(m)}(\mathbf{z})$, as defined in (2.1). For any $\epsilon > 0$ with $(1+\epsilon)\rho < 1$, set

$$H_j^{(m)}(\epsilon) \triangleq \{\mathbf{z} \in \mathbb{R}^d : \mathbf{a}_j^\top \mathbf{z} \geq b_j \Phi^{-1}((1+\epsilon)\rho) + s_j\sqrt{m}\}, \quad j \in \mathcal{J}.$$

Note that by Lemma 3.1, $H_j^{(m)}(\epsilon) \neq \emptyset$ for all $m$. For notational convenience, we set

$$H_{\mathcal{J}}^{(m)}(\epsilon) \triangleq \bigcap_{j\in\mathcal{J}} H_j^{(m)}(\epsilon).$$

For each $j \in \mathcal{J}$, $\mathbf{z} \in H_j^{(m)}(\epsilon)$ implies

$$\sum_{k\in\mathcal{I}_j^{(m)}} c_k \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{z} - s_j\sqrt{m}}{b_j}\right) \geq (1+\epsilon)\rho \sum_{k\in\mathcal{I}_j^{(m)}} c_k.$$

Hence for $\mathbf{z} \in H_{\mathcal{J}}^{(m)}(\epsilon)$

$$\begin{aligned}\mathsf{E}\big[L_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z} = \mathbf{z}\big] &= \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{I}_j^{(m)}} c_k \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{z} - s_j\sqrt{m}}{b_j}\right) \geq (1+\epsilon)\rho \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{I}_j^{(m)}} c_k \\ &\geq (1+\epsilon) x_{\mathcal{J}}^{(m)}. \end{aligned} \tag{3.15}$$

We define $\boldsymbol{\nu}_{\mathcal{J}}^{(m)}(\epsilon)$ to be the unique solution of

$$\boldsymbol{\nu}_{\mathcal{J}}^{(m)}(\epsilon) \triangleq \operatorname{argmin}\{\|\mathbf{z}\| : \mathbf{z} \in H_{\mathcal{J}}^{(m)}(\epsilon)\}.$$

LEMMA 3.9. *For each* $\epsilon > 0$ *with* $(1+2\epsilon)\rho < 1$ *and any set of types* $\mathcal{J}$ *satisfying* $G_{\mathcal{J}} \neq \emptyset$, *there exists a* $\delta > 0$ *such that*

$$B_{\mathcal{J}}^{(m)}(\epsilon,\delta) \triangleq \{\mathbf{z} \in \mathbb{R}^d : \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) \leq \mathbf{z} \leq \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) + \delta\mathbf{e}\} \subset H_{\mathcal{J}}^{(m)}(\epsilon) \quad \textit{for all } m.$$

*Proof*. Because $\Phi(\cdot)$ has compact range, it is uniformly continuous, so there exists a $\delta' > 0$ such that $|x-y| \leq \delta'$ implies $|\Phi(x) - \Phi(y)| \leq \epsilon\rho$. Let $\bar{a} = \max\{|[\mathbf{a}_j]_i| : 1 \leq j \leq t, 1 \leq i \leq d\}$ and $\underline{b} = \min\{b_1, \ldots, b_t\}$. The assumption $0 < \|\mathbf{a}_j\| < 1$ implies $\bar{a} > 0$ and $\underline{b} > 0$. Set

$$\delta = \frac{\delta' \underline{b}}{d\bar{a}}.$$

Then for any $j \in \mathcal{J}$ and any $\mathbf{z} \in B_{\mathcal{J}}^{(m)}(\epsilon,\delta)$,

$$\begin{aligned}&\left|\frac{\mathbf{a}_j^\top \mathbf{z} - s_j\sqrt{m}}{b_j} - \frac{\mathbf{a}_j^\top \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) - s_j\sqrt{m}}{b_j}\right| \\ &\quad = \left|\frac{\mathbf{a}_j^\top(\mathbf{z} - \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon))}{b_j}\right| \leq \sum_{i=1}^d \left|\frac{[\mathbf{a}_j]_i \cdot [\mathbf{z} - \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon)]_i}{b_j}\right| \leq \frac{d\bar{a}\delta}{\underline{b}} = \delta'\end{aligned}$$

and hence

$$\left|\Phi\left(\frac{\mathbf{a}_j^\top \mathbf{z} - s_j\sqrt{m}}{b_j}\right) - \Phi\left(\frac{\mathbf{a}_j^\top \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) - s_j\sqrt{m}}{b_j}\right)\right| \leq \epsilon\rho.$$

So

$$\Phi\left(\frac{\mathbf{a}_j^\top \mathbf{z} - s_j\sqrt{m}}{b_j}\right) \geq \Phi\left(\frac{\mathbf{a}_j^\top \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) - s_j\sqrt{m}}{b_j}\right) - \epsilon\rho$$
$$\geq (1+2\epsilon)\rho - \epsilon\rho = (1+\epsilon)\rho,$$

which completes the proof. □

We will also need the following convergence property:

LEMMA 3.10. *A sequence of events* $\{A_m\}_{m=1}^{\infty}$ *and a sequence of positive integers* $n_m$ *with* $\lim_{m\to\infty} n_m = \infty$ are given. *Suppose that, given* $A_m$, *the* $X_k^{(m)}, k = 1, \ldots, n_m$, *are conditionally independent random variables of conditional mean* 0 *for which*

$$\limsup_{m\to\infty} \frac{1}{(n_m)^2} \sum_{k=1}^{n_m} \mathsf{Var}\big(X_k^{(m)} \mid A_m\big) = 0.$$

*Set*

$$S_m = \frac{1}{n_m} \sum_{k=1}^{n_m} X_k^{(m)}.$$

*Then for any* $\epsilon > 0$,

$$\lim_{m\to\infty} \mathsf{P}(|S_m| > \epsilon | A_m) = 0.$$

*Note that* $X_k^{(m)}$ *and* $X_k^{(l)}(m \neq l)$ *may have different distributions.*

*Proof*. By the conditional independence of $X_k^{(m)}$ and the fact that $\mathsf{E}[S_m \mid A_m] \equiv 0$,

$$\mathsf{Var}(S_m \mid A_m) = \frac{1}{(n_m)^2} \mathsf{Var}\left(\sum_{k=1}^{n_m} X_k^{(m)} \middle| A_m\right)$$
$$= \frac{1}{(n_m)^2} \sum_{k=1}^{n_m} \mathsf{Var}\big(X_k^{(m)} \mid A_m\big) \to 0 \quad \text{as } m \to \infty.$$

Then using Chebyshev's inequality, for any $\epsilon > 0$,

$$\mathsf{P}(|S_m| > \epsilon \mid A_m) \leq \frac{\mathsf{E}\big[S_m^2 \mid A_m\big]}{\epsilon^2} = \frac{\mathsf{Var}(S_m \mid A_m)}{\epsilon^2} \to 0 \quad \text{as } m \to \infty. \qquad □$$

To apply this lemma to our setting, we center the loss from obligor $k$ by setting

$$X_k^{(m)} \triangleq c_k\big(Y_k^{(m)} - p_k^{(m)}(\mathbf{Z})\big)$$

where, as before, $p_k^{(m)}(\mathbf{z}) = \mathsf{E}[Y_k^{(m)} \mid \mathbf{Z} = \mathbf{z}]$. Let $\{\mathbf{z}^{(m)}\}$ be an arbitrary sequence in $\mathbb{R}^d$. For each $m$, the random variables $X_k^{(m)}, k \in \bigcup_{j\in\mathcal{J}} \mathcal{I}_j^{(m)}$, satisfy the conditions of Lemma 3.10 with $A_m = \{\mathbf{Z} = \mathbf{z}^{(m)}\}$ and $n_m = n_{\mathcal{J}}^{(m)}$ because

- the $Y_k^{(m)}$'s are conditionally independent (given $\mathbf{Z}$);
- $\mathsf{E}[X_k^{(m)} \mid \mathbf{Z} = \mathbf{z}^{(m)}] = 0$,
- and

$$\limsup_{m\to\infty} \frac{1}{\big(n_{\mathcal{J}}^{(m)}\big)^2} \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{I}_j^{(m)}} \mathsf{Var}\big(X_k^{(m)} \mid \mathbf{Z} = \mathbf{z}^{(m)}\big) \leq \limsup_{m\to\infty} \frac{1}{n_{\mathcal{J}}^{(m)}} \bar{c}^2 = 0$$

since

$$\mathsf{Var}\big(X_k^{(m)} \,\big|\, \mathbf{Z} = \mathbf{z}^{(m)}\big) \le \frac{1}{4} c_k^2 \le \bar{c}^2.$$

If we define

$$\begin{aligned} S_{\mathcal{J}}^{(m)} &\triangleq \frac{1}{n_{\mathcal{J}}^{(m)}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{I}_j^{(m)}} X_k^{(m)} \\ &= \frac{1}{n_{\mathcal{J}}^{(m)}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{I}_j^{(m)}} c_k \big(Y_k^{(m)} - p_k^{(m)}(\mathbf{Z})\big), \end{aligned}$$

then Lemma 3.10 implies that for any $\epsilon > 0$,

$$\lim_{m\to\infty} \mathsf{P}\big(\big|S_{\mathcal{J}}^{(m)}\big| \le \epsilon \,\big|\, \mathbf{Z} = \mathbf{z}^{(m)}\big) = 1. \tag{3.16}$$

Now fix an arbitrary $\mathbf{b}_B \in [\mathbf{0}, \delta\mathbf{e}]$. The set $B_{\mathcal{J}}^{(m)}(\epsilon, \delta)$ varies according to $m$ but writing $B_{\mathcal{J}}^{(m)}(\epsilon, \delta) = \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) + [\mathbf{0}, \delta\mathbf{e}]$ enables us to fix elements relatively within $B_{\mathcal{J}}^{(m)}(\epsilon, \delta)$. Set $\mathbf{z}_B^{(m)} = \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) + \mathbf{b}_B$. Then

$$\begin{aligned} &\mathsf{P}\left( S_{\mathcal{J}}^{(m)} > -\frac{\epsilon}{n_{\mathcal{J}}^{(m)}} x_{\mathcal{J}}^{(m)} \,\middle|\, \mathbf{Z} = \mathbf{z}_B^{(m)} \right) \\ &\quad = \mathsf{P}\left( L_{\mathcal{J}}^{(m)} - \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{I}_j^{(m)}} c_k p_k^{(m)}(\mathbf{Z}) > -\epsilon x_{\mathcal{J}}^{(m)} \,\middle|\, \mathbf{Z} = \mathbf{z}_B^{(m)} \right) \\ &\quad = \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > \mathsf{E}\big[L_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z}\big] - \epsilon x_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z} = \mathbf{z}_B^{(m)}\big) \\ &\quad \le \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z} = \mathbf{z}_B^{(m)}\big) \end{aligned}$$

because (3.15) and Lemma 3.9 imply that

$$\mathsf{E}\big[L_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z}\big] - \epsilon x_{\mathcal{J}}^{(m)} \ge x_{\mathcal{J}}^{(m)} \quad \text{on } \big\{\mathbf{Z} \in B_{\mathcal{J}}^{(m)}(\epsilon, \delta)\big\}.$$

So

$$\begin{aligned} &\mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z} = \mathbf{z}_B^{(m)}\big) \\ &\quad \ge \mathsf{P}\left( S_{\mathcal{J}}^{(m)} > -\frac{\epsilon}{n_{\mathcal{J}}^{(m)}} x_{\mathcal{J}}^{(m)} \,\middle|\, \mathbf{Z} = \mathbf{z}_B^{(m)} \right) \\ &\quad \ge \mathsf{P}\left( S_{\mathcal{J}}^{(m)} > -\frac{1}{2}\epsilon C^* \,\middle|\, \mathbf{Z} = \mathbf{z}_B^{(m)} \right) \qquad \text{for all sufficiently large } m \\ &\quad \ge \mathsf{P}\left( \big|S_{\mathcal{J}}^{(m)}\big| < \frac{1}{2}\epsilon C^* \,\middle|\, \mathbf{Z} = \mathbf{z}_B^{(m)} \right) \\ &\quad \to 1 \quad \text{by (3.16)} \end{aligned} \tag{3.17}$$

where

$$C^* = \lim_{m\to\infty} \frac{1}{n_{\mathcal{J}}^{(m)}} x_{\mathcal{J}}^{(m)} = \frac{\rho \sum_{j \in \mathcal{J}} C_j}{\sum_{j \in \mathcal{J}} r_j} > 0.$$

By the dominated convergence theorem,

$$
\begin{aligned}
&\lim_{m\to\infty} \int_{\mathbf{b}\in[\mathbf{0},\delta\mathbf{e}]} \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z} = \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) + \mathbf{b}\big)\, d\mathbf{b} \\
&\quad = \int_{\mathbf{b}\in[\mathbf{0},\delta\mathbf{e}]} \lim_{m\to\infty} \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z} = \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) + \mathbf{b}\big)\, d\mathbf{b} \\
&\quad = \int_{\mathbf{b}\in[\mathbf{0},\delta\mathbf{e}]} 1\, d\mathbf{b} \quad \text{by (3.17)} \\
&\quad = \delta^d. \qquad (3.18)
\end{aligned}
$$

For $1 \le i \le d$, set

$$
\nu_i^{(m)} = \max\big\{\big|\big[\boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon)\big]_i\big|, \big|\big[\boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon)\big]_i + \delta\big|\big\}.
$$

The standard normal probability density function $\phi$ has the property that for any constants $-\infty < \alpha < \beta < \infty$,

$$
\min_{\alpha\le x\le\beta} \phi(x) = \phi(\max\{|\alpha|, |\beta|\}). \tag{3.19}
$$

Thus,

$$
\begin{aligned}
&\mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)}\big) \\
&\quad \ge \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)}, \mathbf{Z} \in B_{\mathcal{J}}^{(m)}(\epsilon, \delta)\big) \\
&\quad = \int_{\mathbf{b}\in[\mathbf{0},\delta\mathbf{e}]} \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z} = \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) + \mathbf{b}\big) \prod_{i=1}^{d} \phi\big(\big[\boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon)\big]_i + b_i\big)\, d\mathbf{b} \\
&\quad \ge \int_{\mathbf{b}\in[\mathbf{0},\delta\mathbf{e}]} \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z} = \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) + \mathbf{b}\big)\, d\mathbf{b} \times \prod_{i=1}^{d} \phi\big(\nu_i^{(m)}\big), \qquad (3.20)
\end{aligned}
$$

using (3.19) for the last inequality.

To analyze the last factor in (3.20), we recall the definition of $H_j^{(m)}(\epsilon)$ and find that

$$
\begin{aligned}
&\lim_{m\to\infty} \frac{1}{\sqrt{m}} \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) \\
&\quad = \lim_{m\to\infty} \operatorname{argmin}\left\{\|\mathbf{z}\| : \mathbf{a}_j^\top \mathbf{z} \ge \frac{b_j \Phi^{-1}((1+2\epsilon)\rho)}{\sqrt{m}} + s_j \text{ for all } j \in \mathcal{J}\right\} \quad \text{by (3.2)} \\
&\quad = \lim_{m\to\infty} \operatorname{argmin}\big\{\|\mathbf{z}\| : \mathbf{a}_j^\top \mathbf{z} \ge s_j \text{ for all } j \in \mathcal{J}\big\} \qquad \text{by Lemma 3.2} \\
&\quad = \boldsymbol{\gamma}_{\mathcal{J}}. \quad \text{by (2.3)}
\end{aligned}
$$

This implies

$$
\begin{aligned}
\lim_{m\to\infty} \frac{1}{\sqrt{m}} \nu_i^{(m)} &= \lim_{m\to\infty} \max\left\{\frac{1}{\sqrt{m}}\big|\big[\boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon)\big]_i\big|, \frac{1}{\sqrt{m}}\big|\big[\boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon)\big]_i + \delta\big|\right\} \\
&= \lim_{m\to\infty} \frac{1}{\sqrt{m}}\big|\big[\boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon)\big]_i\big| = |[\boldsymbol{\gamma}_{\mathcal{J}}]_i|,
\end{aligned}
$$

which in turn implies

$$
\lim_{m\to\infty} \frac{1}{m} \log\left(\prod_{i=1}^{d} \phi\big(\nu_i^{(m)}\big)\right) = -\frac{1}{2}\|\boldsymbol{\gamma}_{\mathcal{J}}\|^2. \tag{3.21}
$$

So,

$$
\begin{aligned}
&\liminf_{m\to\infty} \frac{1}{m} \log \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)}\big) \\
&\quad \ge \liminf_{m\to\infty} \frac{1}{m} \log\left( \int_{\mathbf{b}\in[\mathbf{0},\delta\mathbf{e}]} \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z} = \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) + \mathbf{b}\big)\, d\mathbf{b} \times \prod_{i=1}^{d} \phi\big(\nu_i^{(m)}\big)\right) \quad \text{by (3.20)} \\
&\quad \ge \liminf_{m\to\infty} \frac{1}{m} \log\left( \int_{\mathbf{b}\in[\mathbf{0},\delta\mathbf{e}]} \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)} \,\big|\, \mathbf{Z} = \boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon) + \mathbf{b}\big) d\mathbf{b}\right) \\
&\qquad + \liminf_{m\to\infty} \frac{1}{m} \log\left( \prod_{i=1}^{d} \phi\big(\nu_i^{(m)}\big)\right) \\
&\quad = \lim_{m\to\infty} \frac{1}{m} \log(\delta^d) - \frac{1}{2}\|\gamma_{\mathcal{J}}\|^2 \quad \text{by (3.18) and (3.21)} \\
&\quad = -\frac{1}{2}\|\gamma_{\mathcal{J}}\|^2.
\end{aligned}
$$

We summarize the result as the following theorem.

THEOREM 3.2. *If the assumptions* **M1** *and* **M2** *are satisfied and* $0 < \rho < 1$, *then for any* $\mathcal{J}$ *satisfying* $G_{\mathcal{J}} \neq \emptyset$,

$$\liminf_{m\to\infty} \frac{1}{m} \log \mathsf{P}\big(L_{\mathcal{J}}^{(m)} > x_{\mathcal{J}}^{(m)}\big) \ge -\frac{1}{2}\|\gamma_{\mathcal{J}}\|^2.$$

Note that $\gamma_{\mathcal{J}}$ does not depend on $\rho$.

*3.2.2. A Lower Bound for the Full Portfolio.* We now build on the lower bound for the partial portfolio to get a lower bound on the full portfolio. Consider $\mathcal{J}_* \in \mathcal{S}_q$ satisfying

$$\gamma_{\mathcal{J}_*} = \gamma_*, \tag{3.22}$$

with $\gamma_*$ as defined in (2.4). Because the loss in the full portfolio is at least as large as that in any partial portfolio, we have

$$\mathsf{P}(L_m > x_m) \ge \mathsf{P}\left( \sum_{j\in\mathcal{J}_*} \sum_{k\in\mathcal{I}_j^{(m)}} c_k Y_k^{(m)} > x_m \right) \tag{3.23}$$

with, as before, $x_m = q\sum_{k=1}^{m} c_k$.

By the restriction on $q$, $qC < \sum_{j\in\mathcal{J}_*} C_j$ and there exists a $\delta > 0$ such that $(1+\delta)qC < \sum_{j\in\mathcal{J}_*} C_j$. Define

$$\rho = \frac{(1+\delta)qC}{\displaystyle\sum_{j\in\mathcal{J}_*} C_j}$$

which is less than 1. If we use this new threshold ratio, then the partial portfolio loss and partial threshold for $\mathcal{J}_*$ are

$$L_{\mathcal{J}_*}^{(m)} = \sum_{j\in\mathcal{J}_*} \sum_{k\in\mathcal{I}_j^{(m)}} c_k Y_k^{(m)} \quad \text{and} \quad x_{\mathcal{J}_*}^{(m)} = \rho \sum_{j\in\mathcal{J}_*} \sum_{k\in\mathcal{I}_j^{(m)}} c_k.$$

LEMMA 3.11. *There exists an* $M$ *such that for all* $m > M$,

$$\big\{L_{\mathcal{J}_*}^{(m)} > x_{\mathcal{J}_*}^{(m)}\big\} \subset \big\{L_{\mathcal{J}_*}^{(m)} > x_m\big\}.$$

*Proof*. Because

$$\lim_{m\to\infty}\frac{1}{m}x_{\mathcal{J}_*}^{(m)}=(1+\delta)qC=(1+\delta)\lim_{m\to\infty}\frac{1}{m}x_m,$$

if we set $x^*=(1+\frac{\delta}{2})qC$, then there exist $M_1$ and $M_2$ such that

$$\frac{1}{m}x_{\mathcal{J}_*}^{(m)}>x^*\quad\text{for all } m>M_1 \text{ since } x^*<\lim_{m\to\infty}\frac{1}{m}x_{\mathcal{J}_*}^{(m)}$$

and

$$\frac{1}{m}x_m<x^*\quad\text{for all } m>M_2 \text{ since } x^*>\lim_{m\to\infty}\frac{1}{m}x_m.$$

Take $M=\max\{M_1,M_2\}$. Then $m>M$ implies $x_{\mathcal{J}_*}^{(m)}>x_m$ and hence

$$\{L_{\mathcal{J}_*}^{(m)}>x_{\mathcal{J}_*}^{(m)}\}\subset\{L_{\mathcal{J}_*}^{(m)}>x_m\}.\qquad\square$$

Now $0<\rho<1$ and $G_{\mathcal{J}_*}\neq\emptyset$ from $\mathcal{J}_*\in\mathcal{S}_q$ satisfies the conditions of Theorem 3.2 and we get

$$\begin{aligned}\liminf_{m\to\infty}\frac{1}{m}\log\mathsf{P}\big(L_{\mathcal{J}_*}^{(m)}>x_m\big)&\geq\liminf_{m\to\infty}\frac{1}{m}\log\mathsf{P}\big(L_{\mathcal{J}_*}^{(m)}>x_{\mathcal{J}_*}^{(m)}\big)\quad\text{by Lemma 3.11}\\&\geq-\frac{1}{2}\|\gamma_{\mathcal{J}_*}\|^2\quad\text{by Theorem 3.2}\\&=-\frac{1}{2}\|\gamma_*\|^2\quad\text{by (3.22).}\end{aligned}$$

Combining this with (3.23), we finally get the asymptotic lower bound for the full portfolio.

THEOREM 3.3. *If the assumptions* **M1** *and* **M2** *are satisfied and* $\mathcal{S}_q\neq\emptyset$, *then*

$$\liminf_{m\to\infty}\frac{1}{m}\log\mathsf{P}(L_m>x_m)\geq-\frac{1}{2}\|\gamma_*\|^2.$$

## 3.3. Proof of Theorem 2.1

We can conclude the proof of Theorem 2.1.

*Proof*. If $\mathcal{S}_q\neq\emptyset$, then combining Theorems 3.1 and 3.3 gives

$$\begin{aligned}-\frac{1}{2}\|\gamma_*\|^2&\leq\liminf_{m\to\infty}\frac{1}{m}\log\mathsf{P}(L_m>x_m),\quad\text{by Theorem 3.3}\\&\leq\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{P}(L_m>x_m)\\&\leq-\frac{1}{2}\|\gamma_*\|^2,\quad\text{by Theorem 3.1.}\end{aligned}$$

If $\mathcal{S}_q=\emptyset$, then by Theorem 3.1,

$$\begin{aligned}\lim_{m\to\infty}\frac{1}{m}\log\mathsf{P}(L_m>x_m)&\leq\limsup_{m\to\infty}\frac{1}{m}\log\mathsf{P}(L_m>x_m)\\&\leq-\frac{1}{2}\|\gamma_*\|^2=-\infty.\qquad\square\end{aligned}$$

## 4. ANALYSIS OF LARGE LOSS THRESHOLD REGIME

Recall that, in this regime, we are assuming the conditions **M1** and **M3** in Section 2. The analysis in this section is quite parallel to *small default probability regime* and we omit some of it which can be deduced easily from the counterparts in Section 3.

Let us define

$$G_j^{(m,\epsilon)} \triangleq \left\{\mathbf{z} \in \mathbb{R}^d : \mathbf{a}_j^\top \mathbf{z} \geq sb_j(1-\epsilon_m)\sqrt{\log m} - \Phi^{-1}(\bar{p})\right\} \quad \text{for } j = 1, \ldots, t$$
$$= \left\{\mathbf{z} \in \mathbb{R}^d : \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{z} + \Phi^{-1}(\bar{p})}{b_j}\right) \geq \Phi((1-\epsilon_m)s\sqrt{\log m})\right\}$$

where $\epsilon_m = \frac{1}{\sqrt{\log m}}$. We also set

$$G^{(m,\epsilon)} \triangleq \bigcap_{j=1}^{t} G_j^{(m,\epsilon)}.$$

Note that as we mentioned in Section 2.1, these sets may be empty because of the negative factor-loading components. The Corollary 3.1 shows that $G \neq \emptyset$ is equivalent to $G^{(m,\epsilon)} \neq \emptyset$ for all $m$. Now we define $\boldsymbol{\mu}^{(m,\epsilon)}$ be the *unique* solution of the following *linearly constrained* problem:

$$\boldsymbol{\mu}^{(m,\epsilon)} \triangleq \operatorname{argmin}\left\{\|\mathbf{z}\| : \mathbf{z} \in G^{(m,\epsilon)}\right\}. \tag{4.1}$$

Furthermore, with $\sqrt{\log m}$ and $\boldsymbol{\mu}^{(m,\epsilon)}$ instead of $\sqrt{m}$ and $\boldsymbol{\mu}_{\mathcal{J}}^{(m,\epsilon)}$, the same arguments as those used for (3.3) also show that the limit of $\frac{1}{\sqrt{\log m}}\boldsymbol{\mu}^{(m,\epsilon)}$ exists and coincide by (3.2) and Lemma 3.2. Hence we have

$$\gamma = \lim_{m\to\infty} \frac{1}{\sqrt{\log m}}\boldsymbol{\mu}^{(m,\epsilon)}. \tag{4.2}$$

### 4.1. Upper Bound Computation

Each $U_k$ is $[\underline{u}, 1]$-valued. Denote the cumulant generating function of $U_k$ of type $j$ (whose mean $u_k = u_j^*$) by

$$\Lambda_j(\lambda) \triangleq \log \mathsf{E}[e^{\lambda U_k}]. \tag{4.3}$$

$\Lambda_j(\cdot)$ is twice continuously differentiable on $\mathbb{R}$ because $U_k$ is bounded. (See e.g., pp. 72–73 of Durrett (1996). Actually $\Lambda_j(\cdot)$ is an analytic function.) $\Lambda_j(0) = 0$, $\Lambda_j'(0) = \mathsf{E}[U_k] = u_k = u_j^* > 0$.

Let us define a sequence of changes of measure using two likelihood ratios. The first one is a likelihood ratio of $d$-dimensional standard normal distribution and $N(\boldsymbol{\mu}^{(m,\epsilon)}, \mathbf{I}_d)$. This ratio is given explicitly by $(M_{\mathbf{Z}}^{(m,\epsilon)})^{-1}$ where

$$M_{\mathbf{Z}}^{(m,\epsilon)} \triangleq \begin{cases} e^{\boldsymbol{\mu}^{(m,\epsilon)\top}\mathbf{Z} - \frac{1}{2}\boldsymbol{\mu}^{(m,\epsilon)\top}\boldsymbol{\mu}^{(m,\epsilon)}} & \text{if } G \neq \emptyset \\ 1 & \text{if } G = \emptyset. \end{cases}$$

The second one is given by

$$e^{-\theta_m(\mathbf{Z})L_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}$$

where $\theta_m(\mathbf{Z})$ is given by (3.7) and the conditional cumulant generating function divided by $m$ is given by

$$
\begin{aligned}
\psi_m(\theta, \mathbf{z}) &= \frac{1}{m} \log \mathsf{E}\big[e^{\theta L_m} \,\big|\, \mathbf{Z} = \mathbf{z}\big] \\
&= \frac{1}{m} \sum_{k=1}^{m} \log \mathsf{E}\big[e^{\theta l_k U_k Y_k} \,\big|\, \mathbf{Z} = \mathbf{z}\big] \quad \text{by conditional independence} \\
&= \frac{1}{m} \sum_{k=1}^{m} \log \mathsf{E}\big[\mathsf{E}\big[e^{\theta l_k U_k Y_k} \,\big|\, \mathbf{U}_k, \mathbf{Z} = \mathbf{z}\big] \,\big|\, \mathbf{Z} = \mathbf{z}\big] \\
&= \frac{1}{m} \sum_{k=1}^{m} \log \mathsf{E}\big[1 + p_k(\mathbf{z})(e^{\theta l_k U_k} - 1) \,\big|\, \mathbf{Z} = \mathbf{z}\big] \\
&= \frac{1}{m} \sum_{k=1}^{m} \log \big(1 + p_k(\mathbf{z})\big(\mathsf{E}\big[e^{\theta l_k U_k} \,\big|\, \mathbf{Z} = \mathbf{z}\big] - 1\big)\big) \\
&= \frac{1}{m} \sum_{k=1}^{m} \log \big(1 + p_k(\mathbf{z})\big(\mathsf{E}\big[e^{\theta l_k U_k}\big] - 1\big)\big) \quad \text{by independence of } U_k \text{ and } \mathbf{Z} \\
&= \frac{1}{m} \sum_{j=1}^{t} \sum_{k \in \mathcal{I}_j^{(m)}} \log \big(1 + p_k(\mathbf{z})\big(e^{\Lambda_j(\theta l_k)} - 1\big)\big). \qquad (4.4)
\end{aligned}
$$

That is, given $\mathbf{Z}$, we tilt $L_m$ exponentially. We can decompose this change of measure into the one for $Y_k$ given $\mathbf{Z}$ and another one for $U_k$ given $\mathbf{Z}$ and $Y_k$. But the aggregated representation is sufficient for this paper. The final combined likelihood ratios are given by

$$\frac{d\mathsf{P}}{d\mathsf{P}_m} = e^{-\theta_m(\mathbf{Z})L_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})} \big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}.$$

From now on, $\mathsf{E}_m$ denotes the expectation under the probability measure $\mathsf{P}_m$, under which $\mathbf{Z}$ is distributed by $N(\boldsymbol{\mu}^{(m,\epsilon)}, \mathbf{I}_d)$, $Y_k$ is a Bernoulli random variable with the exponentially tilted conditional default probability, and $U_k$ is also twisted exponentially.

LEMMA 4.1. *If $G \neq \emptyset$, then*

$$\min\big\{\boldsymbol{\mu}^{(m,\epsilon)^\top} \mathbf{z} : \mathbf{z} \in G^{(m,\epsilon)}\big\} = \|\boldsymbol{\mu}^{(m,\epsilon)}\|^2 \qquad (4.5)$$

*Proof*. The proof steps are exactly parallel to those of Lemma 3.3. □

Using Lemma 4.1, we prove the following lemma.

LEMMA 4.2. *If $G \neq \emptyset$, then*

$$\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}_m\big[\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1} \mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\big] \leq -\frac{1}{2}\|\gamma\|^2.$$

*Proof*.

$$
\begin{aligned}
&\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}_m\Big[e^{-\boldsymbol{\mu}^{(m,\epsilon)\top}\mathbf{Z}+\frac{1}{2}\boldsymbol{\mu}^{(m,\epsilon)\top}\boldsymbol{\mu}^{(m,\epsilon)}}\mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\Big] \\
&\quad \le \lim_{m\to\infty} \frac{1}{2\log m}\|\boldsymbol{\mu}^{(m,\epsilon)}\|^2 + \limsup_{m\to\infty}\frac{1}{\log m}\log \mathsf{E}_m\Big[e^{-\boldsymbol{\mu}^{(m,\epsilon)\top}\mathbf{Z}}\mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\Big] \\
&\quad \le \frac{1}{2}\|\gamma\|^2 + \limsup_{m\to\infty}\frac{1}{\log m}\log \mathsf{E}_m\Big[e^{-\|\boldsymbol{\mu}^{(m,\epsilon)}\|^2}\mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\Big] \quad \text{by Lemma 4.1 and (4.2)} \\
&\quad = \frac{1}{2}\|\gamma\|^2 + \limsup_{m\to\infty}\frac{1}{\log m}\log e^{-\|\boldsymbol{\mu}^{(m,\epsilon)}\|^2}\mathsf{P}_m\big(G^{(m,\epsilon)}\big) \\
&\quad = \frac{1}{2}\|\gamma\|^2 - \lim_{m\to\infty}\frac{1}{\log m}\big\|\boldsymbol{\mu}^{(m,\epsilon)}\big\|^2 + \limsup_{m\to\infty}\frac{1}{\log m}\log \mathsf{P}_m\big(G^{(m,\epsilon)}\big) \\
&\quad \le -\frac{1}{2}\|\gamma\|^2. \qquad \square
\end{aligned}
$$

LEMMA 4.3. *If* $\Lambda_j(\cdot)$ *is given by (4.3), then there exists a positive constant* $D$ *such that*

$$\log\big(1+\alpha\big(e^{\Lambda_j(\theta)}-1\big)\big) \le \alpha u_k\theta + D\theta^2$$

*for all* $\theta \in [0,1]$ *and* $\alpha \in [0,1]$.

*Proof*. Set $f(\theta;\alpha) = \log(1+\alpha(e^{\Lambda_j(\theta)}-1)) - \alpha u_k\theta$. $f(0;\alpha) = 0$. We compute Taylor's expansion of $\theta$ at 0.

$$f'(\theta;\alpha) = \frac{\alpha\Lambda_j'(\theta)e^{\Lambda_j(\theta)}}{1+\alpha\big(e^{\Lambda_j(\theta)}-1\big)} - \alpha u_k$$

and $f'(0;\alpha) = 0$. For $\theta \in [0,1]$,

$$
\begin{aligned}
f(\theta;\alpha) &= f(0;\alpha) + f'(0;\alpha)\theta + \frac{1}{2}f^{(2)}(t_\theta;\alpha)\theta^2 \\
&= \frac{1}{2}f^{(2)}(t_\theta;\alpha)\theta^2 \\
&\le D\theta^2
\end{aligned}
$$

where $t_\theta \in [0,\theta]$ and

$$D \triangleq \frac{1}{2}\sup\{|f^{(2)}(t;\alpha)| : t\in[0,1], \alpha\in[0,1]\}.$$

$D < \infty$ since $f^{(2)}(t;\alpha)$ is continuous on $[0,1]\times[0,1]$. $f(\theta;\alpha) \le D\theta^2$ shows the inequality. $\square$

LEMMA 4.4. *For sufficiently large* $m$,

$$q_m - \Phi((1-\epsilon_m)s\sqrt{\log m}) \ge \frac{m^{-(1-\epsilon_m)^2 s^2/2}}{\sqrt{2\pi}s\sqrt{\log m}}\left(\frac{1}{2} - m^{-s^2\epsilon_m(2-\epsilon_m)/2}\right).$$

*Proof*. Using the inequalities

$$\frac{x}{x^2+1}\frac{e^{-x^2/2}}{\sqrt{2\pi}} \le 1-\Phi(x) \le \frac{1}{x}\frac{e^{-x^2/2}}{\sqrt{2\pi}}, \tag{4.6}$$

$$
\begin{aligned}
q_m &- \Phi((1-\epsilon_m)s\sqrt{\log m}) \\
&= \Phi(s\sqrt{\log m}) - \Phi((1-\epsilon_m)s\sqrt{\log m}) \\
&\geq \frac{1}{\sqrt{2\pi}}\left(\frac{(1-\epsilon_m)s\sqrt{\log m}}{((1-\epsilon_m)s\sqrt{\log m})^2+1}e^{-((1-\epsilon_m)s\sqrt{\log m})^2/2} - \frac{1}{s\sqrt{\log m}}e^{-(s\sqrt{\log m})^2/2}\right) \\
&\geq \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2s\sqrt{\log m}}e^{-(1-\epsilon_m)^2 s^2\log m/2} - \frac{1}{s\sqrt{\log m}}e^{-s^2\log m/2}\right) \\
&= \frac{1}{\sqrt{2\pi}s\sqrt{\log m}}\left(\frac{1}{2}m^{-(1-\epsilon_m)^2 s^2/2} - m^{-s^2/2}\right) \\
&= \frac{m^{-(1-\epsilon_m)^2 s^2/2}}{\sqrt{2\pi}s\sqrt{\log m}}\left(\frac{1}{2} - m^{-s^2\epsilon_m(2-\epsilon_m)/2}\right). \qquad \square
\end{aligned}
$$

Now we consider a covering $\bigcup_{j=1}^t (G_j^{(m,\epsilon)})^c$ of $(G^{(m,\epsilon)})^c$. It is clear that

$$
\mathbf{1}_{(G^{(m,\epsilon)})^c} \leq \sum_{j=1}^t \mathbf{1}_{(G_j^{(m,\epsilon)})^c}. \tag{4.7}
$$

We set

$$
\theta_m \triangleq m^{-1+\frac{s^2}{2}}.
$$

Note that $\lim_{m\to\infty}\theta_m = 0$. From now on, $\{\mathsf{E}'_m\}_{m=1}^\infty$ are expectations under an arbitrary sequence of probability measures $\{\mathsf{P}'_m\}_{m=1}^\infty$ unless they are specified otherwise.

Now observe that for $\mathbf{z} \in (G_j^{(m,\epsilon)})^c$, with $q_m^\epsilon \triangleq \Phi((1-\epsilon_m)s\sqrt{\log m})$,

$$
\begin{aligned}
-\theta_m q_m &\sum_{k\in\mathcal{I}_j^{(m)}} l_k u_k + \sum_{k\in\mathcal{I}_j^{(m)}} \log\bigl(1 + p_k(\mathbf{z})\bigl(e^{\Lambda_j(\theta_m l_k)} - 1\bigr)\bigr) \\
&\leq \sum_{k\in\mathcal{I}_j^{(m)}}\left(-\theta_m q_m l_k u_k + \log\left(1 + \Phi\left(\frac{\mathbf{a}_j^\top\mathbf{z} + \Phi^{-1}(\bar p)}{b_j}\right)\bigl(e^{\Lambda_j(\theta_m l_k)} - 1\bigr)\right)\right) \text{ by } p_k \leq \bar p \\
&\leq \sum_{k\in\mathcal{I}_j^{(m)}}\bigl(-\theta_m q_m l_k u_k + \log\bigl(1 + q_m^\epsilon\bigl(e^{\Lambda_j(\theta_m l_k)} - 1\bigr)\bigr)\bigr) \text{ by } \mathbf{z} \in \bigl(G_j^{(m,\epsilon)}\bigr)^c \\
&\leq \sum_{k\in\mathcal{I}_j^{(m)}}\bigl(-\theta_m q_m l_k u_k + q_m^\epsilon u_k l_k \theta_m + D\bar l^2\theta_m^2\bigr) \text{ for large } m \text{ by Lemma 4.3} \\
&= -\theta_m n_j^{(m)}\left(\bigl(q_m - q_m^\epsilon\bigr)\frac{1}{n_j^{(m)}}\sum_{k\in\mathcal{I}_j^{(m)}} l_k u_k - D\bar l^2\theta_m\right) \\
&\leq -\theta_m n_j^{(m)}\bigl(\bigl(q_m - q_m^\epsilon\bigr)\underline{l}\cdot\underline{u} - D\bar l^2\theta_m\bigr)
\end{aligned} \tag{4.8}
$$

where the last inequality follows because $q_m - q_m^\epsilon > 0$, $l_k \geq \underline{l}$, and $U_k \geq \underline{u}$. Then for any sequence of nonnegative random variables $\{Y_m\}_{m=1}^\infty$ with

$$
y_m \triangleq \frac{1}{\log m}\log \mathsf{E}'_m[Y_m] \text{ satisfying } \limsup_{m\to\infty} y_m < \infty, \tag{4.9}
$$

we have, with $\beta_m^\epsilon \triangleq \frac{1}{2} - m^{-s^2\epsilon_m(2-\epsilon_m)/2}$,

$$
\begin{aligned}
&\frac{1}{\log m}\log \mathsf{E}'_m\Big[Y_m \cdot e^{-\theta_m q_m \sum_{k\in\mathcal{I}_j^{(m)}} l_k u_k + \sum_{k\in\mathcal{I}_j^{(m)}} \log(1+p_k(\mathbf{Z})(e^{\Lambda_j(\theta_m l_k)}-1))} \mathbf{1}_{(G_j^{(m,\epsilon)})^c}(\mathbf{Z})\Big] \\
&\le \frac{1}{\log m}\log \mathsf{E}'_m\Big[Y_m \cdot e^{-\theta_m n_j^{(m)}((q_m-q_m^\epsilon)\underline{l}\cdot\underline{u} - D\bar{l}^2\theta_m)} \mathbf{1}_{(G_j^{(m,\epsilon)})^c}(\mathbf{Z})\Big] \quad \text{by (4.8)} \\
&\le y_m + \frac{1}{\log m}\log\big(e^{-\theta_m n_j^{(m)}((q_m-q_m^\epsilon)\underline{l}\cdot\underline{u} - D\bar{l}^2\theta_m)}\big) \\
&= y_m - \frac{\theta_m n_j^{(m)}}{\log m}\big((q_m-q_m^\epsilon)\underline{l}\cdot\underline{u} - D\bar{l}^2\theta_m\big) \\
&\le y_m - \frac{\theta_m n_j^{(m)}}{\log m}\left(\frac{m^{-(1-\epsilon_m)^2 s^2/2}}{\sqrt{2\pi}s\sqrt{\log m}}\beta_m^\epsilon \cdot \underline{l}\cdot\underline{u} - D\bar{l}^2\theta_m\right) \quad \text{by Lemma 4.4} \\
&= y_m - \left(\frac{n_j^{(m)}}{m}\right)\frac{\theta_m m^{1-(1-\epsilon_m)^2 s^2/2}}{\log m\sqrt{2\pi}s\sqrt{\log m}}\left(\beta_m^\epsilon\cdot\underline{l}\cdot\underline{u} - D\bar{l}^2\theta_m\frac{\sqrt{2\pi}s\sqrt{\log m}}{m^{-(1-\epsilon_m)^2 s^2/2}}\right) \\
&= y_m - \left(\frac{n_j^{(m)}}{m}\right)\frac{m^{\epsilon_m(2-\epsilon_m)s^2/2}}{\sqrt{2\pi}s(\log m)^{3/2}}\left(\beta_m^\epsilon\cdot\underline{l}\cdot\underline{u} - D\bar{l}^2\frac{\sqrt{2\pi}s\sqrt{\log m}}{m^{1-s^2+\epsilon_m(2-\epsilon_m)s^2/2}}\right)
\end{aligned}
$$

$$
\tag{4.10} \to \limsup_{m\to\infty} y_m - r_j\frac{\infty}{\sqrt{2\pi}s}\left(\left(\frac{1}{2}-0\right)\underline{l}\cdot\underline{u} - D\bar{l}^2\sqrt{2\pi}s\cdot 0\right) = -\infty \quad \text{as } m\to\infty
$$

where the last convergence is understood as a lim sup. Note that

$$
\begin{aligned}
\log\left(\frac{m^{\epsilon_m(2-\epsilon_m)s^2/2}}{(\log m)^{3/2}}\right) &= \frac{s^2}{2}\frac{1}{\sqrt{\log m}}\left(2-\frac{1}{\sqrt{\log m}}\right)\log m - \frac{3}{2}\log(\log m) \\
&= \frac{s^2}{2}\left(2-\frac{1}{\sqrt{\log m}}\right)\sqrt{\log m} - \frac{3}{2}\log(\log m) \\
&\sim s^2\sqrt{\log m} - \frac{3}{2}\log(\log m) \to \infty, \\
\log\big(m^{-s^2\epsilon_m(2-\epsilon_m)/2}\big) &= -\frac{s^2}{2}\frac{1}{\sqrt{\log m}}\left(2-\frac{1}{\sqrt{\log m}}\right)\log m \\
&= -\frac{s^2}{2}\left(2-\frac{1}{\sqrt{\log m}}\right)\sqrt{\log m} \to -\infty,
\end{aligned}
$$

and

$$
\frac{\sqrt{\log m}}{m^{1-s^2+\epsilon_m(2-\epsilon_m)s^2/2}} \le \frac{\sqrt{\log m}}{m^{(1-s^2)/2}} \to 0.
$$

For each type $j$ and any $\mathbf{z} \in \mathbb{R}^d$,

$$
\begin{aligned}
&-\theta_m q_m \sum_{k\in\mathcal{I}_j^{(m)}} l_k u_k + \sum_{k\in\mathcal{I}_j^{(m)}} \log\big(1 + p_k(\mathbf{z})\big(e^{\Lambda_j(\theta_m l_k)} - 1\big)\big) \\
&\quad\le \sum_{k\in\mathcal{I}_j^{(m)}} \big(-\theta_m q_m l_k u_k + u_k \theta_m l_k + D l_k^2 \theta_m^2\big) \quad \text{by Lemma 4.3} \\
&\quad\le \theta_m(1-q_m)\left(\sum_{k\in\mathcal{I}_j^{(m)}} l_k u_k\right) + n_j^{(m)} D\bar{l}^2 \theta_m^2 \\
&\quad\le \theta_m(1-q_m) n_j^{(m)} \bar{l} + n_j^{(m)} D\bar{l}^2 \theta_m^2. \qquad (4.11)
\end{aligned}
$$

Define a sequence of nonnegative random variables $\{Y_m\}_{m=1}^{\infty}$ for type $j$,

$$
Y_m \triangleq \prod_{j'\neq j} e^{-\theta_m q_m \sum_{k\in\mathcal{I}_{j'}^{(m)}} l_k u_k + \sum_{k\in\mathcal{I}_{j'}^{(m)}} \log(1+p_k(\mathbf{Z})(e^{\Lambda_{j'}(\theta_m l_k)}-1))}.
$$

Then

$$
\begin{aligned}
&\frac{1}{\log m} \log \mathsf{E}'_m[Y_m] \\
&\quad\le \frac{1}{\log m} \log \mathsf{E}'_m\left[\prod_{j'\neq j} e^{\theta_m(1-q_m) n_{j'}^{(m)} \bar{l} + n_{j'}^{(m)} D\bar{l}^2 \theta_m^2}\right] \quad \text{by (4.11)} \\
&\quad= \sum_{j'\neq j} \frac{n_{j'}^{(m)} \bar{l}}{\log m} \theta_m((1-q_m) + D\bar{l}\theta_m) \\
&\quad\le \frac{m\bar{l}}{\log m} \theta_m((1-q_m) + D\bar{l}\theta_m) \quad \text{since } \sum_{j'\neq j} n_{j'}^{(m)} \le m \\
&\quad\le \bar{l} \frac{m\theta_m}{\log m} \frac{1}{\sqrt{2\pi} s \sqrt{\log m}} e^{(-s^2 \log m)/2} + \bar{l}^2 D \frac{m\theta_m^2}{\log m} \quad \text{by (4.6)} \\
&\quad\le \bar{l} \frac{\theta_m m^{1-s^2/2}}{\sqrt{2\pi} s (\log m)^{3/2}} + \bar{l}^2 D \frac{m\theta_m^2}{\log m} \\
&\quad= \bar{l} \frac{1}{\sqrt{2\pi} s (\log m)^{3/2}} + \bar{l}^2 D \frac{m^{-1+s^2}}{\log m} \triangleq y_m \to 0. \qquad (4.12)
\end{aligned}
$$

From (4.12), observe that $Y_m$ with $y_m$ satisfies the condition (4.9) for (4.10) to hold. Hence for each $j$,

$$
\begin{aligned}
&e^{-\theta_m x_m + m\psi_m(\theta_m, \mathbf{Z})} \mathbf{1}_{(G_j^{(m,\epsilon)})^c}(\mathbf{Z}) \\
&\quad= \prod_{j'=1}^{t} e^{-\theta_m q_m \sum_{k\in\mathcal{I}_{j'}^{(m)}} l_k u_k + \sum_{k\in\mathcal{I}_{j'}^{(m)}} \log(1+p_k(\mathbf{Z})(e^{\Lambda_{j'}(\theta_m l_k)}-1))} \mathbf{1}_{(G_j^{(m,\epsilon)})^c}(\mathbf{Z}) \quad \text{by (4.4)} \\
&\quad= Y_m \cdot e^{-\theta_m q_m \sum_{k\in\mathcal{I}_j^{(m)}} l_k u_k + \sum_{k\in\mathcal{I}_j^{(m)}} \log(1+p_k(\mathbf{Z})(e^{\Lambda_j(\theta_m l_k)}-1))} \mathbf{1}_{(G_j^{(m,\epsilon)})^c}(\mathbf{Z}).
\end{aligned}
$$

This representation and (4.10) imply

$$\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}'_m\big[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \mathbf{1}_{(G_j^{(m,\epsilon)})^c}(\mathbf{Z})\big] = -\infty. \tag{4.13}$$

Then we have the following lemma.

LEMMA 4.5.

$$\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}_m\big[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1} \mathbf{1}_{(G^{(m,\epsilon)})^c}(\mathbf{Z})\big] = -\infty.$$

*Proof*. As in the proof of Lemma 3.7, let us first introduce an intermediate change of measure:

$$\frac{d\mathsf{P}}{d\mathsf{P}'_m} = e^{-\theta_m(\mathbf{Z}) L_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})}$$

such that

$$\mathsf{E}_m\big[\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1} Y\big] = \mathsf{E}'_m[Y] \quad \text{for any random variable } Y. \tag{4.14}$$

$$\begin{aligned}
&\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}_m\big[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1} \mathbf{1}_{(G^{(m,\epsilon)})^c}(\mathbf{Z})\big] \\
&\quad \le \limsup_{m\to\infty} \frac{1}{\log m} \log \left( \sum_{j=1}^{t} \mathsf{E}_m[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1} \mathbf{1}_{(G_j^{(m,\epsilon)})^c}(\mathbf{Z})] \right) \quad \text{by (4.7)} \\
&\quad = \max_{1\le j\le t} \left\{ \limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}_m\big[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1} \mathbf{1}_{(G_j^{(m,\epsilon)})^c}(\mathbf{Z})\big] \right\} \quad \text{by Lemma 3.4} \\
&\quad = \max_{1\le j\le t} \left\{ \limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{E}'_m\big[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \mathbf{1}_{(G_j^{(m,\epsilon)})^c}(\mathbf{Z})\big] \right\} \quad \text{by (4.14)} \\
&\quad \le \max_{1\le j\le t} \{-\infty\} \quad \text{by (4.13)} \\
&\quad = -\infty.
\end{aligned}$$

□

Note that Lemma 4.5 is still true even if $G = \emptyset$. Now we get the following conclusion:

THEOREM 4.1. *Suppose the assumptions* **M1** *and* **M3** *hold. Then*

$$\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{P}(L_m > x_m) \le -\frac{1}{2}\|\gamma\|^2.$$

*Proof*. First consider the case of $G \ne \emptyset$. Using the definition of $\theta_m$,

$$\begin{aligned}
&\mathsf{P}(L_m > x_m) \\
&\quad = \mathsf{E}_m\big[\mathbf{1}\{L_m > x_m\} e^{-\theta_m(\mathbf{Z}) L_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})} \big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\big] \quad \text{by the definition of } \mathsf{P}_m \\
&\quad \le \mathsf{E}_m\big[e^{-\theta_m(\mathbf{Z}) x_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})} \big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1}\big] \quad \text{dropping the indicator} \\
&\quad = \mathsf{E}_m\big[e^{-\theta_m(\mathbf{Z}) x_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})} \big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1} \mathbf{1}_{(G^{(m,\epsilon)})^c}(\mathbf{Z})\big] \\
&\qquad + \mathsf{E}_m\big[e^{-\theta_m(\mathbf{Z}) x_m + m\psi_m(\theta_m(\mathbf{Z}),\mathbf{Z})} \big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1} \mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\big] \\
&\quad \le \mathsf{E}_m\big[e^{-\theta_m x_m + m\psi_m(\theta_m,\mathbf{Z})} \big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1} \mathbf{1}_{(G^{(m,\epsilon)})^c}(\mathbf{Z})\big] \quad \text{by definition (3.7)} \\
&\qquad + \mathsf{E}_m\big[\big(M_{\mathbf{Z}}^{(m,\epsilon)}\big)^{-1} \mathbf{1}_{G^{(m,\epsilon)}}(\mathbf{Z})\big] \quad \text{by definition (3.7), because } \theta_m(\mathbf{Z}) \ge 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\limsup_{m\to\infty} \frac{1}{\log m} \log \mathsf{P}(L_m > x_m) \\
&\le \max\left\{-\infty, -\frac{1}{2}\|\gamma\|^2\right\} \quad \text{by Lemma 4.2 and Lemma 4.5} \\
&= -\frac{1}{2}\|\gamma\|^2.
\end{aligned}$$

If $G = \emptyset$, then $M_{\mathbf{Z}}^{(m,\epsilon)} = 1$ and Lemma 4.5 completes the proof. □

## 4.2. Lower Bound Computation

We start with the homogenization of the default probabilities by replacing them by lower bound $\underline{p}$. For any conditions on $\mathbf{a}_j$ and $c_k$,

$$\{L_m > x_m\} \supset \{\underline{L}_m > x_m\}$$

where $\underline{L}_m$ corresponds to the portfolio with uniform default probabilities, $\underline{p}$. So any lower bounds on the event $\{\underline{L}_m > x_m\}$ is also valid for $\{L_m > x_m\}$. Hence from now on we denote the total default loss corresponding to $\underline{p}$ by $L_m$. Then we compute the lower bound, which coincides with the upper bound of the previous section. The underlying idea is very similar to the proof of the small default probability model.

First note that the conditional default probability of the $k$-th obligor of type $j$ is

$$p_k(\mathbf{z}) = \mathsf{P}\big(X_k > \Phi^{-1}(1-\underline{p}) \,\big|\, \mathbf{Z} = \mathbf{z}\big) = \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{z} + \Phi^{-1}(\underline{p})}{b_j}\right).$$

Define for $\xi > 0$,

$$\begin{aligned}
(4.15)\qquad H_j^{(m)}(\xi) &\triangleq \left\{\mathbf{z} \in \mathbb{R}^d : \mathbf{a}_j^\top \mathbf{z} \ge b_j s\sqrt{\log m} - \Phi^{-1}(\underline{p}) + \xi s b_j\right\} \quad \text{for } j = 1, \ldots, t \\
&= \left\{\mathbf{z} \in \mathbb{R}^d : \mathbf{a}_j^\top \mathbf{z} \ge b_j(1 + \xi\epsilon_m)s\sqrt{\log m} - \Phi^{-1}(\underline{p})\right\} \\
&= \left\{\mathbf{z} \in \mathbb{R}^d : \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{z} + \Phi^{-1}(\underline{p})}{b_j}\right) \ge \Phi((1 + \xi\epsilon_m)s\sqrt{\log m})\right\}
\end{aligned}$$

where, as before, $\epsilon_m = \frac{1}{\sqrt{\log m}}$. We also define

$$H^{(m)}(\xi) \triangleq \bigcap_{j=1}^{t} H_j^{(m)}(\xi).$$

Note that $G \ne \emptyset$ implies the nonemptiness of $H^{(m)}(\xi)$. We define $\boldsymbol{\nu}^{(m)}(\xi)$ as the unique solution of

$$\boldsymbol{\nu}^{(m)}(\xi) \triangleq \operatorname{argmin}\left\{\|\mathbf{z}\| : \mathbf{z} \in H^{(m)}(\xi)\right\}.$$

Now we get a similar result to Lemma 3.9.

LEMMA 4.6. *For each $\xi > 0$, there exists a $\delta > 0$ such that*

$$B^{(m)}(\xi, \delta) \triangleq \left\{\mathbf{z} \in \mathbb{R}^d : \boldsymbol{\nu}^{(m)}(2\xi) \le \mathbf{z} \le \boldsymbol{\nu}^{(m)}(2\xi) + \delta\mathbf{e}\right\} \subset H^{(m)}(\xi) \quad \textit{for all } m.$$

*Proof*. Let $\bar{a} = \max\{|[\mathbf{a}_j]_i| : 1 \le j \le t, 1 \le i \le d\}$ and $\underline{b} = \min\{b_1, \dots, b_t\}$. Then the assumption $0 < \|\mathbf{a}_j\| < 1$ implies $\bar{a} > 0$ and $\underline{b} > 0$. Set

$$\delta = \frac{\xi \underline{b} s}{d\bar{a}}.$$

Then for any $\mathbf{z} \in B^{(m)}(\xi, \delta)$,

$$\begin{aligned}
\mathbf{a}_j^\top \mathbf{z} &= \mathbf{a}_j^\top \mathbf{z} - \mathbf{a}_j^\top \boldsymbol{\nu}^{(m)}(2\xi) + \mathbf{a}_j^\top \boldsymbol{\nu}^{(m)}(2\xi) \\
&\ge -\left|\mathbf{a}_j^\top \mathbf{z} - \mathbf{a}_j^\top \boldsymbol{\nu}^{(m)}(2\xi)\right| + b_j s\sqrt{\log m} - \Phi^{-1}(\underline{p}) + 2\xi s b_j \\
&\ge -\sum_{i=1}^{d} \left|[\mathbf{a}_j]_i \cdot [\mathbf{z} - \boldsymbol{\nu}^{(m)}(2\xi)]_i\right| + b_j s\sqrt{\log m} - \Phi^{-1}(\underline{p}) + 2\xi s b_j \\
&\ge -d\bar{a}\delta + b_j s\sqrt{\log m} - \Phi^{-1}(\underline{p}) + 2\xi s b_j \\
&\ge b_j s\sqrt{\log m} - \Phi^{-1}(\underline{p}) + \xi s b_j + \xi s(b_j - \underline{b}) \\
&\ge b_j s\sqrt{\log m} - \Phi^{-1}(\underline{p}) + \xi s b_j.
\end{aligned}$$

So $\mathbf{z} \in H^{(m)}(\xi)$. □

By Lemma 4.6, there exists a $\delta$ such that $B^{(m)}(1, \delta) \subset H^{(m)}(1)$. Then $\mathbf{z} \in B^{(m)}(1, \delta)$ implies

$$\begin{aligned}
\mathsf{E}[L_m \mid \mathbf{Z} = \mathbf{z}] &= \sum_{j=1}^{m} l_k u_k \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{z} + \Phi^{-1}(\underline{p})}{b_j}\right) \\
&\ge \Phi((1 + \epsilon_m)s\sqrt{\log m}) \sum_{k=1}^{m} l_k u_k.
\end{aligned} \tag{4.16}$$

Now define

$$X_k^{(m)} \triangleq m^{s^2/2}\sqrt{\log m}\, l_k(U_k Y_k - u_k p_k(\mathbf{Z})) \tag{4.17}$$

where, as before,

$$p_k(\mathbf{z}) = \mathsf{E}[Y_k \,|\, \mathbf{Z} = \mathbf{z}] = \Phi\left(\frac{\mathbf{a}_j^\top \mathbf{z} + \Phi^{-1}(\underline{p})}{b_j}\right).$$

Let $\{\mathbf{z}^{(m)}\}$ be an arbitrary sequence in $\mathbb{R}^d$. For each $m$, by setting $A_m = \{\mathbf{Z} = \mathbf{z}^{(m)}\}$ and $n_m = m$, $X_k^{(m)} (k = 1, \dots, m)$ defined by (4.17) satisfies the conditions of Lemma 3.10 because

- the $Y_k$'s are conditionally independent (given $\mathbf{Z}$);
- $\mathsf{E}[X_k^{(m)} \mid \mathbf{Z} = \mathbf{z}^{(m)}] = 0$ since $U_k$ is independent of $\mathbf{Z}$ and $\varepsilon_k$,
- and

$$\limsup_{m\to\infty} \frac{1}{m^2} \sum_{k=1}^{m} \mathsf{Var}\big(X_k^{(m)} \,\big|\, \mathbf{Z} = \mathbf{z}^{(m)}\big) \le \limsup_{m\to\infty} \frac{5\bar{l}^2}{4} \frac{\log m}{m^{1-s^2}} = 0$$

since

$$\mathsf{Var}\big(X_k^{(m)} \,\big|\, \mathbf{Z} = \mathbf{z}^{(m)}\big) \le \frac{5}{4} l_k^2 m^{s^2} \log m.$$

Define

$$
\begin{aligned}
S_m &\triangleq \frac{1}{m}\sum_{k=1}^{m} X_k^{(m)} \\
&= \frac{m^{s^2/2}\sqrt{\log m}}{m}\sum_{k=1}^{m} l_k(U_k Y_k - u_k p_k(\mathbf{Z})) \\
&= \frac{m^{s^2/2}\sqrt{\log m}}{m}(L_m - \mathsf{E}[L_m \mid \mathbf{Z}]). \qquad (4.18)
\end{aligned}
$$

Lemma 3.10 implies for any $\epsilon > 0$,

$$
\lim_{m\to\infty} \mathsf{P}\big(|S_m| > \epsilon \,\big|\, \mathbf{Z} = \mathbf{z}^{(m)}\big) = 0. \qquad (4.19)
$$

LEMMA 4.7. *For sufficiently large $m$,*

$$
\Phi((1+\epsilon_m)s\sqrt{\log m}) - q_m \geq \frac{1}{2\sqrt{2\pi}s\sqrt{\log m}\, m^{s^2/2}}.
$$

*Proof*. Using the inequalities (4.6),

$$
\begin{aligned}
&\Phi((1+\epsilon_m)s\sqrt{\log m}) - q_m \\
&\quad = \Phi((1+\epsilon_m)s\sqrt{\log m}) - \Phi(s\sqrt{\log m}) \\
&\quad \geq \frac{1}{\sqrt{2\pi}}\left(\frac{s\sqrt{\log m}}{(s\sqrt{\log m})^2+1}e^{-(s\sqrt{\log m})^2/2} - \frac{1}{(1+\epsilon_m)s\sqrt{\log m}}e^{-((1+\epsilon_m)s\sqrt{\log m})^2/2}\right) \\
&\quad \geq \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2s\sqrt{\log m}}m^{-s^2/2} - \frac{1}{s\sqrt{\log m}}m^{-(1+\epsilon_m)^2 s^2/2}\right) \\
&\quad = \frac{m^{-s^2/2}}{\sqrt{2\pi}s\sqrt{\log m}}\left(\frac{1}{2} - m^{-\epsilon_m(2+\epsilon_m)s^2/2}\right) \\
&\quad \sim \frac{m^{-s^2/2}}{\sqrt{2\pi}s\sqrt{\log m}}\left(\frac{1}{2}\right)
\end{aligned}
$$

since

$$
\log m^{-\epsilon_m(2+\epsilon_m)s^2/2} = -\frac{s^2}{\sqrt{\log m}}(1+\epsilon_m/2)\log m \sim -s^2\sqrt{\log m} \to -\infty. \qquad \square
$$

As before, we fix an arbitrary $\mathbf{b}_B \in [\mathbf{0}, \delta\mathbf{e}]$. The set $B^{(m)}(1,\delta)$ varies according to $m$ but writing $B^{(m)}(1,\delta) = \boldsymbol{\nu}^{(m)}(2) + [\mathbf{0}, \delta\mathbf{e}]$ enables us to fix elements relatively within $B^{(m)}(1,\delta)$. Set $\mathbf{z}_B^{(m)} = \boldsymbol{\nu}^{(m)}(2) + \mathbf{b}_B$. Then on $\{\mathbf{Z} = \mathbf{z}_B^{(m)}\}$ for sufficiently large $m$,

$$
\begin{aligned}
&\frac{m^{s^2/2}\sqrt{\log m}}{m}(x_m - \mathsf{E}[L_m \mid \mathbf{Z}]) \\
&\leq \frac{m^{s^2/2}\sqrt{\log m}}{m}\left(x_m - \Phi((1+\epsilon_m)s\sqrt{\log m})\sum_{k=1}^{m} l_k u_k\right) \quad \text{by (4.16)} \\
&= m^{s^2/2}\sqrt{\log m}(q_m - \Phi((1+\epsilon_m)s\sqrt{\log m}))\frac{1}{m}\sum_{k=1}^{m} l_k u_k \\
&\leq -m^{s^2/2}\sqrt{\log m}\left(\frac{1}{2\sqrt{2\pi}\, s\sqrt{\log m}\, m^{s^2/2}}\right)\frac{1}{m}\sum_{k=1}^{m} l_k u_k \quad \text{by Lemma 4.7}
\end{aligned}
$$

$$
= -\left(\frac{1}{2\sqrt{2\pi}\, s}\right)\frac{1}{m}\sum_{k=1}^{m} l_k u_k \tag{4.20}
$$

$$
\to \frac{-1}{2\sqrt{2\pi}\, s} C > 0 \quad \text{as } m \to \infty \tag{4.21}
$$

where $C \triangleq \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} l_k u_k = \lim_{m\to\infty} \sum_{j=1}^{t} u_j^* \cdot \frac{1}{m}\sum_{k\in\mathcal{I}_j^{(m)}} l_k$ and the assumption **M3**-3 assures us of its existence. Note that for $\epsilon$ satisfying $0 < \epsilon < \frac{1}{2\sqrt{2\pi}s}C$, there exists an $M$ such that $m > M$ implies

$$
\left(\frac{1}{2\sqrt{2\pi}\, s}\right)\frac{1}{m}\sum_{k=1}^{m} l_k u_k > \epsilon \quad \text{by (4.21).} \tag{4.22}
$$

Now using these results, for large $m$ and any $\epsilon$ satisfying $0 < \epsilon < \frac{1}{2\sqrt{2\pi}s}C$, we get

$$
\begin{aligned}
&\mathsf{P}\big(L_m > x_m \,\big|\, \mathbf{Z} = \mathbf{z}_B^{(m)}\big) \\
&= \mathsf{P}\big(L_m - \mathsf{E}[L_m \mid \mathbf{Z}] > x_m - \mathsf{E}[L_m \mid \mathbf{Z}] \,\big|\, \mathbf{Z} = \mathbf{z}_B^{(m)}\big) \\
&= \mathsf{P}\left(S_m > \frac{m^{s^2/2}\sqrt{\log m}}{m}(x_m - \mathsf{E}[L_m \mid \mathbf{Z}]) \,\middle|\, \mathbf{Z} = \mathbf{z}_B^{(m)}\right) \quad \text{by (4.18)} \\
&\geq \mathsf{P}\left(S_m > -\left(\frac{1}{2\sqrt{2\pi}\, s}\right)\frac{1}{m}\sum_{k=1}^{m} l_k u_k \,\middle|\, \mathbf{Z} = \mathbf{z}_B^{(m)}\right) \quad \text{by (4.20)} \\
&\geq \mathsf{P}\left(|S_m| \leq \epsilon, \left(\frac{1}{2\sqrt{2\pi}\, s}\right)\frac{1}{m}\sum_{k=1}^{m} l_k u_k > \epsilon \,\middle|\, \mathbf{Z} = \mathbf{z}_B^{(m)}\right) \\
&= \mathsf{P}\big(|S_m| \leq \epsilon \,\big|\, \mathbf{Z} = \mathbf{z}_B^{(m)}\big) \quad \text{by (4.22)} \\
&\to 1 \quad \text{as } m \to \infty \text{ by (4.19).}
\end{aligned}
$$

So we get

$$
\lim_{m\to\infty} \mathsf{P}\big(L_m > x_m \,\big|\, \mathbf{Z} = \mathbf{z}_B^{(m)}\big) = 1 \tag{4.23}
$$

and by the dominating convergence theorem and (4.23) as in (3.18),

$$\lim_{m\to\infty} \int_{\mathbf{b}\in[\mathbf{0},\delta\mathbf{e}]} \mathsf{P}\big(L_m > x_m \,\big|\, \mathbf{Z} = \boldsymbol{\nu}^{(m)}(2) + \mathbf{b}\big)\, d\mathbf{b} = \delta^d. \tag{4.24}$$

For $1 \le i \le d$, set

$$\nu_i^{(m)} = \max\big\{\big|\big[\boldsymbol{\nu}^{(m)}(2)\big]_i\big|, \big|\big[\boldsymbol{\nu}^{(m)}(2)\big]_i + \delta\big|\big\}.$$

Then, replacing $L_{\mathcal{J}}^{(m)}$, $x_{\mathcal{J}}^{(m)}$, $\boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon)$, and $B_{\mathcal{J}}^{(m)}(\epsilon,\delta)$ by $L_m$, $x_m$, $\boldsymbol{\nu}^{(m)}(2)$, and $B^{(m)}$ $(1, \delta)$, respectively, and using (3.19) as in (3.20), we get

$$\mathsf{P}(L_m > x_m) \ge \int_{\mathbf{b}\in[\mathbf{0},\delta\mathbf{e}]} \mathsf{P}(\, L_m > x_m |\, \mathbf{Z} = \boldsymbol{\nu}^{(m)}(2)) + \mathbf{b})\, d\mathbf{b} \times \prod_{i=1}^{d} \phi\big(\nu_i^{(m)}\big). \tag{4.25}$$

Recalling the definition (4.15) of $H_j^{(m)}(\xi)$ and modifying steps for (3.21), we get

$$\lim_{m\to\infty} \frac{1}{\log m} \log\left(\prod_{i=1}^{d} \phi\big(\nu_i^{(m)}\big)\right) = -\frac{1}{2}\|\gamma\|^2. \tag{4.26}$$

Now, in the derivation steps of Theorem 3.2 in Section 3.2.1, we use (4.24), (4.25), and (4.26) by replacing $\frac{1}{m}$, $L_{\mathcal{J}}^{(m)}$, $x_{\mathcal{J}}^{(m)}$, and $\boldsymbol{\nu}_{\mathcal{J}}^{(m)}(2\epsilon)$ by $\frac{1}{\log m}$, $L_m$, $x_m$, and $\boldsymbol{\nu}^{(m)}(2)$, respectively. Then, we get the following asymptotic lower bound.

THEOREM 4.2. *If the assumptions* **M1** *and* **M3** *are satisfied and* $G \neq \emptyset$, *then*

$$\liminf_{m\to\infty} \frac{1}{\log m} \mathsf{P}(L_m > x_m) \ge -\frac{1}{2}\|\gamma\|^2.$$

### 4.3. Proof of Theorem 2.2

From Theorems 4.1 and 4.2, the proof is almost identical to that of Theorem 2.1 in Section 3.3.

## 5. CONCLUDING REMARKS

We have proved logarithmic limits for the loss distribution in the widely used Gaussian copula model of portfolio credit risk. The two main results of the paper consider two limiting regimes, one based on increasingly high loss thresholds (which is relevant to value-at-risk measured at high confidence levels), and one based on small loss probabilities. We find an interesting qualitative distinction between the two cases in that the rare-default limit yields an exponential decrease in the tail of the loss distribution whereas the large-threshold regime yields a power law decay.

We conclude with some observations on this distinction and possible extensions. A property of the Gaussian copula is that it produces zero *extreme tail dependence* in the absence of perfect correlation (see, e.g., Embrechts et al. 2001). This means that random variables linked through a Gaussian copula become independent in the extremes. In the setting of small default probabilities, defaults require extreme values of the underlying factors, so the absence of extreme tail dependence in the Gaussian copula suggests that defaults become nearly independent as default probabilities decrease; and when obligors are independent, the tail of the loss distribution decays exponentially. In contrast, in the large-threshold limit with moderate default probabilities the dependence introduced by low-quality credits -> default probability smaller

the Gaussian copula has a dominant effect and produces a heavier tail. This suggests a qualitative difference in the dependence induced by the Gaussian copula for portfolios of high-quality and low-quality credits. These observations further suggest that we should not expect to see the same distinction—exponential versus power law decay—in a model with, e.g., a $t$-copula, because the multivariate $t$ distribution has positive extreme tail dependence.

The market prices of synthetic CDOs tied to indices are often quoted through a Gaussian copula model using "implied correlations." These are skewed in the sense that more senior tranches—which are sensitive to defaults of the most highly rated underlying credits—have higher implied correlations. Higher implied correlations may, in part, offset the diminished dependence in the Gaussian copula at low default probabilities.

A question not addressed in this paper is the decay rate for the case $\|\gamma\| = \infty$. It may be possible to introduce a faster-growing normalizing function of $m$ to achieve a finite limit, but this remains an open problem. As with any asymptotic result, there is also the question of whether the asymptotics are visible in practice. In Glasserman et al. (2005), we apply the results developed here to design efficient rare-event simulation techniques.

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