

1. The risk modeling

Given

$$y_n = \beta_n^T \mathcal{Z} + \sqrt{1 - \beta_n^T \beta_n} \epsilon_n$$

where $\mathcal{Z} \sim \mathcal{N}(0, I_S)$ and $\epsilon \sim \mathcal{N}(0, I_N)$, the resulting $y_n \sim \mathcal{N}(0, 1)$ as shown below

$$\mathbb{E}\{y_n\} = \sum_i \beta_{n,i} \mathbb{E}\{\mathcal{Z}_i\} + \sqrt{1 - \beta_n^T \beta_n} \mathbb{E}\{\epsilon_n\} = 0$$

$$\begin{aligned} \text{var}\{y_n\} &= \mathbb{E}\{(y_n - \mathbb{E}\{y_n\})^2\} \\ &= \mathbb{E}\left\{(\beta_n^T \mathcal{Z} + \sqrt{1 - \beta_n^T \beta_n} \epsilon_n)^2\right\} \\ &= \beta_n^T \beta_n \mathbb{E}\{(\mathcal{Z} - 0)^2\} + (1 - \beta_n^T \beta_n) \mathbb{E}\{(\epsilon_n - 0)^2\} \\ &= \beta_n^T \beta_n \text{var}\{\mathcal{Z}\} + (1 - \beta_n^T \beta_n) \text{var}\{\epsilon_n\} \\ &= 1 \end{aligned}$$

2. Motivation for threshold between different states $H_{c(n)}^c$

The motivation is to model discrete probability in the credit state matrix with a continuous distribution such as the gaussian. In this case, we want to set $H_{c(n)}^c$ such that

$$p(H_{c(n)}^{c-1} \leq y_n \leq H_{c(n)}^c) = p_{c(n)}^c$$

therefore we can write

$$p(y_n \leq H_{c(n)}^c) = \sum_{\gamma=1}^c p_{c(n)}^\gamma \quad y_n \xrightarrow{\mathcal{N}(0,1)} \quad \Phi(H_{c(n)}^c) = \sum_{\gamma=1}^c p_{c(n)}^\gamma$$

3. Confidence interval for monte carlo estimation

$$p(L_N(\mathcal{Z}, \epsilon) \geq l) \in p(L_N(\mathcal{Z}, \epsilon) \geq l) \pm CI$$

Idea is that for the two naive algorithms that are purported to be equivalent, the CI should be approximately the same

GL 2005: Inner Level Optimization

Inner level optimization involves

$$\min_{\theta \geq 0} \{-\theta l + \psi(\theta)\} \quad \text{where} \quad \psi(\theta) = \sum_{k=1}^m \log(1 + p_k(e^{\theta c_k} - 1))$$

ψ is strictly convex and passes through the origin. The above optimization is equivalent to

$$\theta^* = \begin{cases} \text{solution to } \frac{\partial \psi}{\partial \theta} = l & l > \frac{\partial \psi}{\partial \theta} \big|_{\theta=0} \\ 0 & \text{otherwise} \end{cases}$$

where

$$\frac{\partial \psi}{\partial \theta} \big|_{\theta=0} = \sum_{k=1}^m \frac{p_k e^{\theta c_k} c_k}{1 + p_k(e^{\theta c_k} - 1)} \big|_{\theta=0} = \sum_{k=1}^m p_k c_k$$

GL 2005: Likelihood Function for Two Level IS

Likelihood for the inner sampling conditioned on Z is given by

$$e^{\theta_x(Z)L + \psi(\theta_x(Z), Z)} \quad \text{where} \quad \psi(\theta) = \sum_{k=1}^m \log(1 + p_k(e^{\theta c_k} - 1))$$

The likelihood function for the outer sampling of Z consists of the following change of distribution

$$Z \sim \mathcal{N}(0, I) \quad \longrightarrow \quad Z \sim \mathcal{N}(\mu, I)$$

where μ is the twisting parameter for the outer importance sampling such that the resulting shifted normal distribution resembles the zero variance IS distribution, in other words,

$$\mu = \max_z P(L > x | Z = z) e^{\frac{-z^T z}{2}}$$

Then the likelihood for the outer IS is then

$$\begin{aligned} \frac{\mathcal{N}(0, I)}{\mathcal{N}(\mu, I)} &= \frac{\exp(-\frac{1}{2}z^T z)}{\exp(-\frac{1}{2}(z - \mu)^T(z - \mu))} \\ &= \exp\left(-\frac{1}{2}z^T z - \frac{1}{2}z^T z + z^T \mu - \frac{1}{2}\mu^T \mu\right) \\ &= e^{-\mu^T Z + \mu^T \mu/2} \end{aligned}$$

Therefore, the estimator for probability of tail event is given by

$$\mathbb{1}_{L > x} e^{\theta_x(Z)L + \psi(\theta_x(Z), Z)} e^{-\mu^T Z + \mu^T \mu/2}$$

GL 2005: Likelihood Expression for Exponential Twisting

Show

$$\prod_{k=1}^m \left(\frac{p_k}{p_{k,\theta}} \right)^{Y_k} \left(\frac{1-p_k}{1-p_{k,\theta}} \right)^{1-Y_k} = e^{-\theta L + \phi(\theta)}$$

where

$$\begin{aligned}\phi(\theta) &= \sum_{k=1}^m \log \left(1 + p_k (e^{\theta c_k} - 1) \right) \\ L &= \sum_{k=1}^m c_k Y_k \\ p_{k,\theta} &= \frac{p_k e^{\theta c_k}}{1 + p_k (e^{\theta c_k} - 1)}\end{aligned}$$

Proof.

$$\begin{aligned}lhs &= \exp \left\{ \sum_{k=1}^m Y_k \log \left(\frac{p_k}{p_{k,\theta}} \right) + (1 - Y_k) \log \left(\frac{1-p_k}{1-p_{k,\theta}} \right) \right\} \\ &= \exp \left\{ \sum_{k=1}^m Y_k \log \left(1 + p_k (e^{\theta c_k} - 1) \right) - Y_k \theta c_k + (1 - Y_k) \log \left(1 + p_k (e^{\theta c_k} - 1) \right) \right\} = rhs\end{aligned}$$

where

$$1 - p_{k,\theta} = \frac{1 - p_k}{1 + p_k (e^{\theta c_k} - 1)}$$

□

GL 2005: Algorithm

1. Outer Level IS for systematic risk Z

- (a) Find shifted parameter μ for outer IS for Z
- (b) Sample $Z \sim \mathcal{N}(\mu, I)$

2. Inner Level IS for each default indicators Y_k

- (a) Calculate conditional default probabilities $p_k(Z)$ for $k = 1, \dots, m$

$$p_k(Z) = P(Y_k = 1|Z) = p(X_k > x_k|Z) = P(a_k Z + b_k \epsilon_k > \Phi^{-1}(1 - p_k)|Z)$$

- (b) Compute the twisted parameters $\theta_x(Z)$

$$\theta_x(Z) = \begin{cases} \text{solution to } \frac{\partial}{\partial \theta} \psi_m(\theta, Z) = x & \psi'(0) = \mathbb{E}_p \{L|Z\} = \sum_{k=1}^m p_k(Z) c_k < x \\ 0 & \text{otherwise} \end{cases}$$

- (c) Compute default indicators (bernoulli) from twisted conditional default probabilities

$$p_{k, \theta_x(Z)}(Z) = \frac{p_k(Z) e^{\theta_x(Z) c_k}}{1 + p_k(Z) (e^{\theta_x(Z) c_k} - 1)} \quad k = 1, \dots, m$$

- (d) Compute Loss $L = c_1 Y_1 + \dots + c_m Y_m$ under twisted distribution

3. Return the estimator of tail probabilities

$$\mathbb{1}_{L > x} e^{\theta_x(Z) L + \psi(\theta_x(Z), Z)} e^{-\mu^T Z + \mu^T \mu / 2}$$

Therefore,

$$P(L > x) = \mathbb{E}_{Z \sim \mathcal{N}(\mu, I) \mid Y_k \sim p_{k, \theta_x(Z)}} \left\{ \mathbb{1}_{L > x} e^{\theta_x(Z) L + \psi(\theta_x(Z), Z)} e^{-\mu^T Z + \mu^T \mu / 2} \right\}$$

Optimization

Want to solve for

$$\max_x \min_y f(x, y)$$

Same as

$$\max_x f(x, \hat{y}(x)) \quad \text{where} \quad \hat{y}(x) = \arg \min_y f(x, y)$$

Simply we write as a function of 1 variable

$$\max_x \hat{f}(x) \quad \text{where} \quad \hat{f}(x) = f(x, \hat{y}(x))$$

Want to compute the 1st order and 2nd order derivatives

$$\hat{f}'(x) = \frac{\partial f}{\partial x} f(x, \hat{y}(x)) + \frac{\partial f}{\partial y} f(x, \hat{y}(x)) \hat{y}'(x)$$

Want to compute $\hat{y}'(x)$ first find critical points

$$\frac{\partial f}{\partial y} f(x, y)|_{y=\hat{y}(x)} = 0$$

Solve for the function

$$f_{yx} = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x, y)|_{y=\hat{y}(x)}$$

Solve for $\hat{y}'(x)$

$$f_{yx}(x, \hat{y}(x)) + f_{yy}(x, \hat{y}(x)) \hat{y}'(x) = 0 \quad \rightarrow \quad \hat{y}'(x) = -\frac{f_{yx}(x, \hat{y}(x))}{f_{yy}(x, \hat{y}(x))}$$

Then compute second derivative, i.e. $\hat{f}''(x)$