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Jan Sokolowski
Jean-Paul Zolesio

Introduction to Shape Optimization

Shape Sensitivity Analysis

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Jan Sokolowski
Systems Research Institute
Polish Academy of Sciences
ul. Newelska 6
01-447 Warszawa
Poland

Jean-Paul Zolesio
Centre National
de la Recherche Scientifique
et Institut non Lineaire de Nice
B.P. 71 Faculté des Sciences
06108 Nice Cedex 2
France

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Notation

D	domain in $\mathbb{R}^N$ with piecewise smooth boundary ∂D
Ω	measurable set in $\mathbb{R}^N$ or in D, or domain of class C^k
$\Gamma = \partial\Omega$	boundary of Ω
n	unit normal vector field on Γ, outward to Ω
$\mathcal{N}_0$	unitary extension of n to an open neighbourhood of Γ in $\mathbb{R}^N$
$y(\Omega)$	function in $W^{s,p}(\Omega)$
χ_Ω or χ	characteristic function of Ω
Ω^c	$= D \setminus \overline{\Omega}$ (or $\mathbb{R}^N \setminus \overline{\Omega}$)
χ_{Ω^c} or χ^c	characteristic function of Ω^c
$\lvert\Omega\rvert$	N-dimensional measure of Ω
κ	mean curvature of Γ
$\mathrm{Char}(D)$	$= \{\chi \in L^2(D)$ such that $(1-\chi)\chi = 0$ a.e. on $D\}$
B	unit ball in $\mathbb{R}^N$
B_0	unit ball in $\mathbb{R}^{N-1}$, $B_0 \subset B$
$J(\Omega)$	domain functional (or cost functional)
$P_D(\Omega)$	perimeter of Ω in D
$A(x), B(x), C(x)$	continuous matrix functions on $\overline{D}$
${}^*A(x)$	transpose of $A(x)$
T_t	transformation of $\mathbb{R}^N$ or of $\overline{D}$ into $\mathbb{R}^N$
$V(t)(\cdot)$	$=V(t,x)$ speed vector field
DT_t	Jacobian of T_t
$\gamma_t = \gamma(t)$	$= \det(DT_t)$
$M(T_t)$	$M(T_t) = \gamma(t)^* DT_t^{-1}$
$\omega_t = \omega(t)$	$= \lVert M(T_t).n \rVert_{\mathbb{R}^N}$ on Γ
$\epsilon(V)$	$= \frac{1}{2}({}^*DV + DV)$
$dJ(\Omega; V)$	Eulerian derivative
$G(\Omega)$	shape gradient
G	density of the shape gradient
$g(\Gamma)$	the density gradient
g	$\in L^1(\mathbb{R}^N)$ a (non unique) distributed representation of $g(\Gamma)$
γ_Γ	trace operator on Γ, e.g. $\gamma_\Gamma \in \mathcal{L}(H^1(\Omega); H^{\frac{1}{2}}(\Gamma))$
ψ	obstacle function

∇_Γ	tangential gradient on Γ
$\frac{\partial}{\partial n}$	normal derivative on Γ
$\frac{\partial}{\partial n_A}$	conormal derivative on Γ associated to the matrix A
div_Γ	tangential divergence on Γ
Δ_Γ	Laplace-Beltrami operator on Γ
$\dot{y}(\Omega; V)$	material derivative of $y(\Omega)$ at Ω in direction of the speed field
$y'(\Omega; V)$	shape (domain) derivative of $y(\Omega)$ at Ω in direction of the speed field V
$y * T_t$	transported distribution
$y'(\Gamma; V)$	boundary shape derivative of $y(\Gamma)$ at Γ in direction of the speed field V
$C_0^\infty(\Omega)$, $C_0^\infty(\Omega; \mathbb{R}^N)$	space of smooth functions (or of vector smooth functions) with compact supports in Ω

1. Introduction to shape optimization

1.1. Preface

This book is motivated largely by a desire to solve shape optimization problems that arise in applications, particularly in structural mechanics and in the optimal control of distributed parameter systems. Many such problems can be formulated as the minimization of functionals defined over a class of admissible domains.

Shape optimization is quite indispensable in the design and construction of industrial structures. For example, aircraft and spacecraft have to satisfy, at the same time, very strict criteria on mechanical performance while weighing as little as possible. The shape optimization problem for such a structure consists in finding a geometry of the structure which minimizes a given functional (e.g. such as the weight of the structure) and yet simultaneously satisfies specific constraints (like thickness, strain energy, or displacement bounds).

The geometry of the structure can be considered as a given domain in the three–dimensional Euclidean space. The domain is an open, bounded set whose topology is given, e.g. it may be simply or doubly connected. The boundary is smooth or piecewise smooth, so boundary value problems that are defined in the domain and associated with the classical partial differential equations of mathematical physics are well posed. In general the cost functional takes the form of an integral over the domain or its boundary where the integrand depends smoothly on the solution of a boundary value problem. The shape optimization problem consists in the minimization of such a functional with respect to the geometrical domain which must belong to the admissible family.

Much of the book is concerned with the shape sensitivity analysis for unilateral problems describing such physical phenomena as contact problems in elasticity, elasto–plastic torsion problems, and the obstacle problem.

An elastic membrane in the plane which is fixed along its boundary may serve as a simple example. If a transversal force is applied, then the transversal displacement of the membrane in the state of static equilibrium is a scalar function defined in the domain occupied by the membrane. The specific functional that has been intensively studied for the purpose of structural optimization is the compliance, i.e. the work an external force in that process. That functional proves to be the integral of the product of force and displacement over the geometrical domain.

If arbitrarily small variations of the boundary of the geometrical domain are taken into account, both the functional and the force's displacement of the membrane are perturbed.

To solve any shape optimization problem would ideally mean to find the minimum – whenever it exists – of a specific cost functional over a set of admissible domains. However, it turns out that very few adequate existence results are available. In general, the existence results for such problems are obtained, provided that some unrealistic constraints are imposed on the family of admissible domains. These constraints are often not satisfactory from the physical point of view; for example, one cannot perform the modelling of a large ocean wave by a graph without excluding the interesting phenomenon of a breaking wave. The constraints under discussion also give no realistic framework for the development of optimization algorithms; for example, one is unable to project onto a tangent cone to the family of domains having a pointwise bounded curvature of the boundaries.

In practice, engineers are interested in increasing the stiffness of a plate, improving the drag of aircraft's wing, decreasing the weight of a radiator, etc. Many studies in the field of structural optimization have been undertaken for the last thirty years. The results on mechanical formulation of the problems, their functional analysis and on control theory have recently been combined. For a review of such results the reader is referred to (Haug et al. 1981).

Let us recall that a structure to be optimized (considered in the structural optimization) is generally an assemblage of different parts like beams, plates, shells, and three–dimensional bodies. In this book we restrict our attention to the continuous formulation of such optimization problems for general elastic structures.

In such problems the sensitivity analysis plays a central role and was intensively studied in the 1980s by the authors. The first results concerning the differentiability with respect to perturbations of a geometrical domain were obtained by Hadamard for the first eigenvalue of a membrane; assuming that the boundary of a domain was smooth, he used perturbations of the boundary along the normal direction. This technique cannot be extended to more general situations, in particular, to domains with piecewise smooth boundaries. A straightforward approach to perturbations of geometrical domain considers hypographs in the Cartesian coordinates. In this case linear function spaces can be used to parametrize the domains. Making use of such a parametrization one can obtain the existence results for the shape optimization problems in a standard way. A more general setting of perturbations of geometrical domains is to consider a one–parameter family of smooth transformations of the N–dimensional Euclidean space. That general setting fails to have any linear structure for transformations, however the associated vector field (whose flow is the transformation) does possess the linear structure of an appropriate linear space. Usual constraints on the geometrical domain can be easily taken into account by the choice of a linear subspace of the vector fields. In this technique, fluid mechanics developments have been of great importance for deriving the shape analysis used in this book. For example, the prescribed volume of a geometrical domain is preserved for the perturbed domains provided that the vector field is divergence–free. The difference from fluid mechanics is that the material derivative of a solution of a boundary value problem is not sufficient for the sensitivity analysis of shape cost functionals.

In the book we introduce the shape derivative of solutions to linear and unilateral boundary value problems. The shape derivative depends only on the normal component of the vector field on the boundary of the geometrical domain. Then the boundary value problem whose solutions give the shape derivative is characterized. In particular, for the linear case the existence of the so–called shape gradient is proved and its form is derived.

Let us begin with a review of some concepts used in the book, such as the material derivative, the shape derivative of solutions to PDE's, the first and second order derivatives of domain functionals.

For the sake of simplicity we restrict our considerations to the domain of integration of ordinary differential equations. Let us consider a simple example: $\Omega = (0, a)$ is a domain in $\mathbb{R}$ and y denotes the solution to the following boundary value problem

$$-\frac{d^2y}{dx^2}(x) + y(x) = \sin x \quad x \in (0, a), \quad y(0) = y(a) = 0.$$

It is assumed that perturbations of a are in the form $a + tv$, $t > 0$, and the mapping T_t from Ω onto $\Omega_t = (0, a + tv)$ is defined by $T_t(x) = \frac{a+tv}{a}x$. The solution y_t on Ω_t is given by

$$y_t(x) = -\frac{1}{2}\frac{\sin(a + tv)}{\operatorname{sh}(a + tv)}\operatorname{sh}x + \frac{1}{2}\sin x \quad t \geqq 0.$$

Let y^t be the solution transported to the fixed domain Ω:

$$y^t = y_t \circ T_t = -\frac{1}{2}\frac{\sin(a + tv)}{\operatorname{sh}(a + tv)}\operatorname{sh}(\frac{a + tv}{a}x) + \frac{1}{2}\sin(\frac{a + tv}{a}x).$$

Then the shape derivative takes the form

$$y'(x) = \frac{\partial y_t}{\partial t}(x)_{|t=0} = -\frac{v}{2}\frac{\operatorname{sh}a \cos a - \sin a \operatorname{ch}a}{(\operatorname{sh}a)^2}\operatorname{sh}x$$

while the material derivative is given by

$$\dot{y}(x) = \frac{dy^t}{dt}(x)_{|t=0} = y'(x) + \frac{vx}{2a}(\cos x - \frac{\sin a}{\operatorname{sh}a}\operatorname{ch}x).$$

It can be verified that y' satisfies

$$-\frac{d^2y'}{dx^2}+y'=0,\quad y'(0)=0,\quad y'(a)=-\frac{v}{2}\frac{\text{sh}a\ \cos a-\sin a\ \text{ch}a}{\text{sh}a}$$

while $\dot{y}$ is the solution to the following equation

$$-\frac{d^2\dot{y}}{dx^2}+\dot{y}=-\frac{v}{a}\cos x,\quad \dot{y}(0)=\dot{y}(a)=0.$$

Both $\dot{y}$ and y' depend linearly on v, but y' depends only through the boundary value.

Let us consider the variational inequality associated with minimization of the following quadratic functional

$$I(\phi)=\int_0^{1+tv}\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2-f\phi\right]dx$$

over the convex subset $K_t=\{\phi\in\mathcal{H}_t|\phi(0)\geqq 0\}$ of the Hilbert space $\mathcal{H}_t=\{\phi\in H^1(0,1+tv)|\phi(1+tv)=0\}$.

The minimum of $I(.)$ over K_t is achieved at the unique element z_t in K_t characterized by the following variational inequality

$$z_t\in K_t:\quad \int_0^{1+tv}\frac{dz_t}{dx}\left(\frac{d\phi}{dx}-\frac{dz_t}{dx}\right)dx\geqq\int_0^{1+tv}f(\phi-z_t)dx\quad\forall\phi\in K_t\ .$$

The minimizer z_t can be evaluated explicitly using the solution u_t to the linear problem associated with minimization of $I(.)$ over $\mathcal{H}_t$:

$$-\frac{d^2u_t}{dx^2}(x)=f(x)\quad x\in\Omega_t=(0,1+tv)$$

subject to the boundary conditions

$$\frac{du_t}{dx}(0)=0,\ u_t(1+tv)=0\ .$$

It can be verified that $\theta(x)=1-\frac{x}{1+tv}$, an element of $\mathcal{H}_t$, is orthogonal to the subspace $H_0^1(\Omega_t)$; $\mathcal{H}_t$ is equipped with the scalar product

$$\int_0^{1+tv}\frac{d\phi}{dx}\frac{d\psi}{dx}dx=(\phi,\psi)_{\mathcal{H}_t}\ .$$

Each element $\phi\in\mathcal{H}_t$ takes the form $\phi=\phi(0)(1-\frac{x}{1+tv})+\psi$, for some $\psi\in H_0^1(\Omega_t)$. The metric projection P_t in $\mathcal{H}_t$ onto K_t satisfies $P_t(\psi)=\psi$. Thus

$$P_t(\phi)=\phi-\min\{0,\phi(0)\}\left(1-\frac{x}{1+tv}\right)=\phi-[\phi(0)]^-\left(1-\frac{x}{1+tv}\right)$$

and we have $z_t = P_t(u_t)$. Hence

$$z_t(x) = u_t(x) - [u_t(0)]^- \left(1 - \frac{x}{1+tv}\right) .$$

Therefore, there exists the shape derivative

$$z'(x;v) = \lim_{t\downarrow 0} \frac{1}{t}(z_t(x) - z_0(x))$$

as well as the material derivative

$$\dot{z}(x;v) = \lim_{t\downarrow 0} \frac{1}{t}((z_t \circ T_t)(x) - z_0(x)) .$$

It is assumed that $z_t(x) = 0$ for $x \geqq 1 + tv$.

To make our considerations more specific we suppose that $u_0(0) = 0$. If u' denotes the shape derivative for the linear problem, then u' is linear with respect to v where $v = 1$, or $v = -1$. It follows that

$$\begin{aligned} z'(x;v) &= \lim_{t\downarrow 0} \frac{1}{t}(z_t(x) - z_0(x)) \\ &= \lim_{t\downarrow 0} \frac{1}{t}((P_t u_t)(x) - (P_0 u_0)(x)) \\ &= \lim_{t\downarrow 0} \frac{1}{t}\left(u_t(x) - [u_t(0)]^- \left(1 - \frac{x}{1+tv}\right) - u_0(x) + [u_0(0)]^-(1-x)\right) \\ &= u'(x) - [u'(0)]^-(1-x) . \end{aligned}$$

provided that we assume $u_t(x) = 0$, for $x \geqq 1 + tv$.

Since the term $[u'(0)]^- = \min\{0, -u'(0)\}$ is not linear with respect to v, it follows that the shape derivative $z'(x;v)$ fails to be linear with respect to v. The same argument applies to the material derivative $\dot{z}(x;v)$. On the other hand we obtain

$$\begin{aligned} &z'(x;v) = u'(x) \quad \text{for } u_0(0) < 0 \\ &z'(x;v) = u'(x) - u'(0)(1-x) \quad \text{for } u_0(0) > 0 . \end{aligned}$$

Let us consider a cost functional to be minimized with respect to the domain Ω. The simplest example is as follows:

$$J(t) = \frac{1}{2}\int_0^{1+tv} (u_t - y_d)^2 dx \quad t > 0 .$$

Then the semi–derivative is given by

$$J'(0) = \lim_{t\downarrow 0} \frac{1}{t}(J(t) - J(0))$$

$$= (u(1) - y_d(1))^2 v + \int_0^1 (u - y_d) u' dx \ .$$

It is obvious that $J'(0)$ is linear with respect to v. Let us introduce the adjoint state

$$-\frac{d^2p}{dx^2}(x) = u(x) - y_d(x), \quad \frac{dp}{dx}(0) = p(1) = 0 \ .$$

Hence

$$J'(0) = [(u(1) - y_d(1))^2 + \int_0^1 \frac{du}{dx}(x)\frac{dp}{dx}(x)dx]v \ .$$

The second derivative is given by

$$\begin{aligned}\frac{d^2J}{dt^2}(0) &= \lim_{t\downarrow 0}\frac{1}{t}(J'(t) - J'(0)) \\ &= v[-2y_d(1)u'(1) + v\frac{du}{dx}(1)\frac{dp}{dx}(1) + \int_0^1 \left(\frac{du'}{dx}\frac{dp}{dx} + \frac{du}{dx}\frac{dp'}{dx}\right)dx] \ .\end{aligned}$$

Here p' denotes the shape derivative of the solution to the adjoint equation. In the case of the unilateral problem with the functional

$$J(t) = \int_0^{1+tv} (z_t - y_d)^2 dx,$$

we have

$$J'(0) = v(z(1) - y_d(1))^2 + \int_0^1 (z(x) - y_d(x))z'(x;v)dx \ .$$

If the mapping $v \to z'(.,v)$ fails to be linear, then the functional $J(.)$ is not differentiable at $t = 0$, and the adjoint state cannot be introduced.

Let us consider the general setting of the shape optimization problem. Such a problem can be formulated as the following minimization problem:

$$\Omega^* \in \mathcal{U}_{ad} \ : \quad J(\Omega^*) = \inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega) \tag{1.1}$$

where the set $\mathcal{U}_{ad}$ of admissible domains in $\mathbb{R}^N$, $N = 1, 2, 3, \ldots$, includes the classes of all admissible geometries for the problem under consideration. Usually the cost functional $J(.)$ takes the form $J(\Omega) = h(\Omega, y(\Omega))$, where the element $y(\Omega)$ is given as the solution of a boundary value problem well posed in the domain Ω. In general, the element $y(\Omega)$ belongs to a functional space $W(\Omega)$, usually a Sobolev space of functions defined in the domain Ω. The existence of a solution to the problem (1.1) is ensured, provided the set $\mathcal{U}_{ad}$ is endowed with a topology (convergence of domains) such that

(i) the mapping $\Omega \to J(\Omega)$ is lower semicontinuous, and
(ii) the set $\mathcal{U}_{ad}$ is compact.

Roughly speaking, the compactness of the set of admissible domains $\mathcal{U}_{ad}$ follows, if for example pointwise constraints on the absolute value of the curvature of the boundary $\Gamma = \partial\Omega$ are prescribed for any admissible domain $\Omega \in \mathcal{U}_{ad}$. In order to ensure the existence of an optimal solution to the problem (1.1), the standard approach of control theory is proposed, i.e. a regularizing term can be introduced. Therefore the cost functional takes the form

$$J_\alpha(\Omega) = J(\Omega) + \alpha E(\Omega) \quad \alpha > 0,$$

where the regularizing term $E(\Omega)$ enjoys the following property: the closure of the set

$$\{\Omega | E(\Omega) \leqq M\}$$

is compact; $M > 0$ is a constant.

The main part of this book is concerned with the shape sensitivity analysis of the mapping $\Omega \to J(\Omega)$. It is supposed that the boundary Γ of the domain Ω is piecewise smooth. The infinitesimal perturbation Ω_t of the domain Ω is defined as follows: $\Omega_t = T_t(\Omega)$, where T_t is a smooth one–to–one mapping defined in a neighbourhood of $\overline{\Omega}$. The mapping T_t can be considered as the flow of the vector field $V(t) = \frac{\partial T_t}{\partial t} \circ T_t^{-1}, t \geqq 0$; this point of view has been introduced in (Zolesio 1976). The Eulerian derivative of $J(.)$ in direction $V(.,.)$ at $t = 0$ defined by

$$\lim_{t \downarrow 0} \frac{1}{t}(J(\Omega_t) - J(\Omega))$$

depends on the vector field $V(0)$. All the transformations T_t considered in Chapt. 2, can be constructed using vector fields in $C^0([0,\varepsilon); V^k(D))$ (see Sect. 2.10). For a differentiable domain functional the structure theorem (Theorem 2.5), which defines the general form of the shape gradient, is given (following (Zolesio 1979)) by

$$G(\Omega) = \gamma_\Gamma^*(g_n) \ .$$

The gradient is an element of the space of distributions $\mathcal{D}'(\mathbb{R}^N; \mathbb{R}^N)$, supported on the boundary $\Gamma = \partial\Omega$. In order to obtain the explicit form of $G(\Omega)$ one has to define: the material derivative $\dot{y}(\Omega; V)$, the shape derivative $y'(\Omega; V)$, and the boundary shape derivative $y'(\Gamma; V)$. The forms of those derivatives are derived for elliptic, parabolic, and hyperbolic problems. The material and the shape derivatives are given as the unique solutions to the associated partial differential equations. In particular, these equations involve tangential operators on the boundary, e.g. the Laplace–Beltrami operator.

In Chapt. 3 the necessary optimality conditions for problem (1.1) are stated. Moreover, related results on the shape sensitivity analysis for linear problems including systems of equations of linear elasticity, the Kirchhoff plate, multiple eigenvalue problems, heat transfer equations, and wave equations are presented.

Chapt. 4 is concerned with the shape sensitivity analysis of variational inequalities. We present related results on the differential stability of the metric projection in Hilbert spaces due to Haraux (1977), Mignot (1976), Sokołowski (1981c; 1985a,b,c; 1986b; 1987b; 1988b,c,d), Sokołowski and Zolesio (1985a,b; 1987a), and Zolesio (1985b).

For other results on the shape optimization the reader is referred to the monographs by Banichuk (1983), Haslinger et al. (1988), Haug et al. (1981), Haug et al. (1986), Pironneau (1984), Troicki et al. (1982) and Zolesio (1988).

We also provide a list of references at the end of this book.

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2. Preliminaries and the material derivative method

This chapter is concerned with mathematical methods used in the shape sensitivity analysis. In particular in Sect. 2.9 the so-called material derivative and the speed method are introduced. The latter is applied in Chaps. 3 and 4 for the shape sensitivity analysis of the boundary value problems of elliptic type as well as for the initial – boundary value problems of parabolic and hyperbolic types. In Chap. 4 the speed method is used for the shape sensitivity analysis of nonlinear problems of elliptic type. In Sect. 3.3 of Chap. 3, the necessary optimality conditions for a model shape optimization problem are derived using the speed method. In this chapter we describe mathematical tools that can be used to prove the existence of solutions to related shape optimization problems, e.g. the notion of the perimeter of a bounded domain in $\mathbb{R}^N$ (see (DeGiorgi et al. 1972)) is discussed. In Sect. 2.7 we introduce, following (Micheletti 1972), the notion of the convergence of domains that ensures e.g. the convergence of normal vector fields on the boundaries, as well as the curvatures of the boundaries, etc. In Sect. 2.1 the domain $\Omega \subset \mathbb{R}^N$ with the boundary $\Gamma = \partial\Omega$ is defined. The notion of an integral on the manifold $\Gamma = \partial\Omega$ is discussed in Sect. 2.2. Functional spaces used in the book are examined in Sect. 2.3, in particular the Sobolev spaces (see e.g. (Adams 1975; Lions et al. 1968)) are introduced. In Sect. 2.4 the notion of weak solutions to elliptic boundary value problems is investigated using Stampacchia's theorem, and the Lax–Milgram lemma in the symmetric case. Several examples of the second order and the fourth order elliptic problems related to applications, e.g. in structural mechanics, are given. In Sect. 2.5 the notion of a functional depending on the domain $\Omega \subset \mathbb{R}^N$ is introduced. In Sect. 2.6 a shape functional for an elliptic boundary value problem is defined. The notion of the perimeter is used to define a regularizing term occurring in the expression of the shape functional (see (2.46)) for the transmission problem. The convergence of minimizing sequences for a model shape optimization problem is studied. Sect. 2.7 is concerned with the analysis of the convergence of domains in $\mathbb{R}^N$; for this purpose the method introduced by (Micheletti 1972) is applied. Furthermore, under some specific assumptions, the explicit form of one–to–one transformations of domains in $\mathbb{R}^N$ is derived. In Sect. 2.8 families of perturbations of a given domain in $\mathbb{R}^N$ are defined. Such a family can be defined in a number of ways, in particular that proposed by

Hadamard (1908) is presented. In Sect. 2.9 a general method of defining such a family is described. Using the speed method, the shape gradient of a given shape functional is defined in Sect. 2.11. In Sect. 2.12 the case of multiple eigenvalues is studied with the use of non–smooth optimization technique. Differential properties of the mapping T_t associated with the speed method are obtained in Sect. 2.13. Sect. 2.14 deals with the differentiability, with respect to the space variables, of the composed functions $f \circ T_t$ for given mappings $f : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ or given distributions. In Sect. 2.15 additional properties of the mapping T_t are obtained. In Sect. 2.16 the speed method is used to define the derivatives of domain integrals in the directions of vector fields. In Sect. 2.18 the derivatives of boundary integrals are derived. Tangential differential operators on Γ are defined in Sects. 2.19 and 2.20. Sect. 2.29 is concerned with elliptic problems on the manifold $\Gamma = \partial\Omega$. Sect. 2.22 deals with the transformation of differential operators, accomplished by means of the mapping T_t associated with the speed method. Formulae useful in the shape sensitivity analysis of partial differential equations are given. The notion of the material derivatives of functions defined:

(i) on the domain Ω,
(ii) on the boundary Γ of Ω

is introduced in Sects. 2.26 and 2.26, respectively.

The notion of the shape derivative is presented in Sect. 2.30. Finally, in Sect. 2.31 the notion of the shape derivatives of functions defined on the manifold $\Gamma = \partial\Omega$ is introduced. In Chaps. 3 and 4 the shape derivatives of solutions to specific boundary value problems will be considered. It will be shown that these shape derivatives actually depend on the normal component of the speed vector field on $\Gamma = \partial\Omega$. This property of the shape derivatives is crucial for the shape optimization.

2.1. Domains in $\mathbb{R}^N$ of class C^k

We denote by Ω an open set in $\mathbb{R}^N$ which is generally assumed to be bounded; hence $\overline{\Omega}$ is compact. Γ denotes the boundary of $\Omega : \Gamma = \overline{\Omega} \setminus \Omega$. Moreover it is assumed that Ω is a smooth domain of class C^k:

Γ is a C^k manifold and Ω is located on one side of Γ; local coordinates are defined as follows: there exists a family $\mathcal{O}_1, \dots, \mathcal{O}_m$, of open sets in $\mathbb{R}^N$ and mappings c_i from $\mathcal{O}_i$ onto

$$B = \{\xi = (\xi_1, \dots, \xi_{N-1}, \xi_N) \in \mathbb{R}^N \text{ such that } \|\xi\|_{\mathbb{R}^N} \leqq 1\},$$

c_i is a one–to–one mapping,

$$c_i \in C^k(\mathcal{O}_i; \mathbb{R}^N) \text{ with } c_i^{-1} \in C^k(B; \mathbb{R}^N)$$

and

$$c_i(\mathcal{O}_i \cap \Omega) = B_+ \equiv \{\xi \in B | \xi_N \geqq 0\},$$
$$c_i(\mathcal{O}_i \cap \Gamma) = B_0 \equiv \{\xi \in B | \xi_N = 0\} \ .$$

The family $\mathcal{O}_i$, $i = 1, 2, \ldots, m$, covers $\overline{\Omega}$, it means that

$$\overline{\Omega} \subset \bigcup_{i=1,\ldots,m} \mathcal{O}_i \ .$$

It is also supposed that there is given a partition of unity $r_i \in C_0^\infty(\mathcal{O}_i)$ such that $0 \leqq r_i \leqq 1$ and $\sum_{i=1}^m r_i = 1$ on Γ. The local coordinates on Γ are defined by $h_i = c_i^{-1}$. Hence for any x belonging to $\Gamma \cap \mathcal{O}_i$ we can write $x = h_i(\xi)$ where $\xi = c_i(x)$. The tangent linear space $\mathcal{T}_x\Gamma$ to Γ at x is spanned by the $N-1$ vectors

$$Dh(\xi) \cdot e_i, \quad i = 1, 2, \ldots, N-1,$$

where e_i, $i = 1, 2, \ldots, N$, is the canonical basis of $\mathbb{R}^N$, $e_i = (\overbrace{0, \ldots, 0, 1}^{i}, 0, \ldots, 0)$. It can be easily verified that the vector ${}^*Dh(\xi)^{-1} \cdot e_N$ is orthogonal to the tangent space $\mathcal{T}_x\Gamma$:

$$\langle Dh(\xi) \cdot e_i, {}^*Dh(\xi)^{-1} \cdot e_N \rangle_{\mathbb{R}^N} = \langle e_i, e_N \rangle_{\mathbb{R}^N} = 0 \ .$$

It might be well to point out that the $N-1$ vectors $\tau_i = Dh(\xi) \cdot e_i$, $i = 1, 2, \ldots, N-1$, which form a basis of the tangent space $\mathcal{T}_x\Gamma$, are not orthogonal:

$$\langle \tau_i, \tau_j \rangle_{\mathbb{R}^N} = \langle {}^*Dh(\xi) \cdot Dh(\xi) \cdot e_i, e_j \rangle_{\mathbb{R}^N} \ .$$

The normal vector field m on Γ can be defined as follows

$$m(x) = \sum_{i=1}^{m} r_i {}^*Dh(c_i(x))^{-1} \cdot e_n$$

and the unitary normal field on Γ is given by

$$n(x) = \|m(x)\|_{\mathbb{R}^N}^{-1} m(x) \ .$$

Furthermore we assume that the vector field $n(x)$, $x \in \Gamma$, is outward pointing on Γ. It is important to observe that the vector fields m and n are not only defined on the boundary Γ, but m is also defined in $U = \cup_{i=1,\ldots,m}\mathcal{O}_i$, U is an open neighbourhood of Γ. We denote by $\mathcal{N}_0$ an unitary extension of the normal field n. $\mathcal{N}_0$ is defined in a neighbourhood of $\overline{\Omega}$ in $\mathbb{R}^N$. In fact, $n(x)$ is only defined in the neighbourhood U' of Γ in $\mathbb{R}^N$,

$$U' = \{x \in U \text{ such that } m(x) \neq 0\}.$$

Let us consider $r_0 \in C^\infty(\mathbb{R}^N)$, $0 \leqq r_0 \leqq 1$, $r_0 \equiv 0$ on $\mathbb{R}^N \setminus U'$ and $r_0 \equiv 1$ on U'', U'' is an open set, $U' \supset U'' \supset \Gamma$; let a vector $e \in \mathbb{R}^N$ be such that $\|e\|_{\mathbb{R}^N} = 1$. Then the unitary vector field $\mathcal{N}_0$ is defined as follows

$$\mathcal{N}_0(x) = (1 - r_0(x))e + r_0(x)n(x) \ .$$

A vector field V defined on Γ is said to be in $C^\ell(\Gamma)$, $0 \leqq \ell \leqq k$, if and only if $(r_i V) \circ h_i$ is in $C^\ell(B_0)$ for all i.

If Ω is a smooth open set with C^k boundary, $k \geqq 1$, (in such a case $c_i \in C^k(\mathcal{O}_i)$ and $h_i \in C^k(B)$), then the fields m and n defined on Γ are only elements of $C^{k-1}(\Gamma; \mathbb{R}^N)$. This loss of regularity of the normal field on Γ is due to the contribution of the term Dh in the expressions for m and n. The same conclusion is valid for the unitary extension $\mathcal{N}_0$ of the normal field n:

$$\mathcal{N}_0 \text{ is in } C^{k-1}(\mathbb{R}^N) \ .$$

Since $c \circ h$ is the identity, using the chain rule it can be shown that $(Dh)^{-1} = (Dc) \circ h$. Therefore we can also write that

$$m(x) = \sum_{i=1}^{m} r_i(x)^* Dc_i(x) \cdot e_n \ .$$

Mean curvature κ of the boundary $\Gamma = \partial\Omega$.

Let $\Omega \subset \mathbb{R}^3$ be a domain of class C^2 in $\mathbb{R}^3$; Ω is located on one side of $\Gamma = \partial\Omega$ and Γ is a manifold of class C^2. With each point x_0 of Γ the second fundamental form of Γ with the eigenvalues (k_1, k_2) is associated. The mean curvature of Γ at x_0 is defined as follows (DaCarmo 1976, p.146)

$$\kappa(x_0) = \frac{1}{2}(k_1 + k_2) \ .$$

The eigenvalues k_1 and k_2 are associated with the eigenvectors $\tau_1(x_0)$ and $\tau_2(x_0)$, the so–called principal curvature directions, which are obtained in the following way.

Let $x_0 \in \Gamma$ be a given point and denote by n_0 the unit normal field at x_0, it is assumed that n_0 is outward pointing on Γ. For any fixed unit tangent vector τ, $\tau \in \mathcal{T}_{x_0}\Gamma$, one has to consider the two–dimensional linear manifold spanned by the vectors n_0, τ (this manifold is of the form $E(\tau) = x_0 + (\mathbb{R} \cdot n \oplus \mathbb{R} \cdot \tau)$), and the plane curve $\eta(\tau) = \Gamma \cap E(\tau)$, i.e. a curve in $E(\tau)$ such that $x_0 \in \eta(\tau)$. In the local coordinates (τ, n_0) , $\eta(\tau)$ is the graph of a function $f(\cdot): (-\varepsilon, \varepsilon) \to \mathbb{R}$, $\varepsilon > 0$, such that $f(0) = 0$ (i.e. $x_0 \in \eta(\tau)$). Moreover the curvature

$$-f''(\alpha)(1 + f'(\alpha)^2)^{-\frac{3}{2}} \quad \alpha \in (-\varepsilon, \varepsilon)$$

is defined in a neighbourhood of x_0 on $\eta(\tau)$. The sign of this expression is chosen in such a way that the curvature is positive, provided that the osculating circle to $\eta(\tau)$ in the plane $E(\tau)$ is located in Ω, otherwise the curvature is negative. The curvature of the plane curve $\eta(\tau)$ at the point x_0 is denoted by $k(\tau) = -f''(0)(1+f'(0)^2)^{-\frac{3}{2}}$; we have also (Da Carmo 1976) that $k(\tau) = R^{-1}(\tau)$, where $R(\tau)$ is the radius of curvature of $\eta(\tau)$ at x_0, i.e. $|R(\tau)|$ is the radius of the osculating circle at x_0 to $\eta(\tau)$ in $E(\tau)$.

It can be shown (Da Carmo 1976) that there exist two tangential directions τ_1, τ_2 at x_0 such that τ_1 minimizes (τ_2 maximizes) $|k(\tau)|$. They are called the principal curvature directions of Γ at x_0, and the mean curvature takes the form $\kappa = \frac{1}{2}(k(\tau_1) + k(\tau_2))$.

2.2. Surface measures on Γ

The surface measure on Γ can be defined with the use of the cofactor matrix notation. For a mapping $h: \mathbb{R}^N \to \mathbb{R}^N$, we denote by $M(h)$ the matrix of cofactors of the matrix Dh, i.e.

$$M(h) = \det(Dh)^*(Dh)^{-1},$$

where $^*Dh$ denotes the transpose of Dh. It is known that $(^*Dh)^{-1} = {}^*((Dh)^{-1})$, hence the form $^*(Dh)^{-1}$ is well defined.

For any continuous function f defined on Γ with compact support on $\Gamma_i = \Gamma \cap \mathcal{O}_i$ the following relation holds

$$\int_\Gamma f d\Gamma = \int_{B_0} f \circ h_i \|M(h_i) \cdot e_n\|_{\mathbb{R}^N} d\xi',$$

where $\xi' = (\xi_1, \xi_2, \ldots, \xi_{N-1})$. The term $\|M(h_i) \cdot e_n\|_{\mathbb{R}^N}$ is continuous with respect to $x \in \Gamma$ (since $(Dh)^{-1} = (Dc) \circ h$, h and c are C^1) and bounded on $\overline{B}_{0,i}$, $1 \leqq i \leqq m$, where the sets $B_{0,i} = c(\operatorname{spt}(r_i \cap \Gamma))$ are included in B_0. Therefore $f \in L^1(\Gamma)$ if and only if $f \circ h \in L^1_{\mathrm{loc}}(B_0)$ for all integers i, $1 \leqq i \leqq m$. Hence

$$\int_\Gamma f d\Gamma = \sum_{i=1}^m \int_{B_0} (r_i f) \circ h_i \|M(h_i) \cdot e_n\|_{\mathbb{R}^N} d\xi' \ .$$

2.3. Functional spaces

Let Ω be an open set in $\mathbb{R}^N$. $\mathcal{D}(\Omega)$ is the linear space of infinitely many times differentiable functions with compact supports in Ω; a sequence ϕ_k converges to ϕ in $\mathcal{D}(\Omega)$ if and only if there exists a compact set $\overline{\mathcal{O}}$, $\overline{\mathcal{O}} \subset \Omega$, such that for all $k = 1, 2, \dots,$ $\operatorname{spt} \phi_k \subset \overline{\mathcal{O}}$, and all derivatives $(\frac{\partial}{\partial x})^\alpha \phi_k$ converge to $(\frac{\partial}{\partial x})^\alpha \phi$ uniformly on $\overline{\mathcal{O}}$ as $k \to \infty$.

The following notation is introduced

$$\left(\frac{\partial}{\partial x}\right)^\alpha \phi = \frac{\partial^{\alpha_1} \dots \partial^{\alpha_N}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}} \phi,$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ and α_i, $1 \leqq i \leqq N$, are integers.

The dual space $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. We denote by $\langle \cdot, \cdot \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)}$ the duality bilinear form between spaces $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$ (for the sake of simplicity we shall write $\langle \cdot, \cdot \rangle$ whenever possible).

For any element $\mathcal{F}$ in $\mathcal{D}'(\Omega)$ one can define, following (Schwartz 1966), the derivative $(\frac{\partial}{\partial n})^\alpha \mathcal{F}$ as an element of $\mathcal{D}'(\Omega)$ such that

$$\forall \phi \in \mathcal{D}(\Omega): \quad \left\langle \left(\frac{\partial}{\partial n}\right)^\alpha \mathcal{F}, \phi \right\rangle = (-1)^{|\alpha|} \langle \mathcal{F}, \left(\frac{\partial}{\partial n}\right)^\alpha \phi \rangle,$$

where $|\alpha| = \alpha_1 + \dots + \alpha_N$.

It should be emphasize that the space $L^1_{\text{loc}}(\Omega)$ can be identified as a subspace of $\mathcal{D}'(\Omega)$ in the following way:

For any element $f \in L^1_{\text{loc}}(\Omega)$ we define the distribution $\mathcal{F}_f$ of the form

$$\langle \mathcal{F}_f, \phi \rangle = \int_\Omega f \phi dx \quad \forall \phi \in \mathcal{D}(\Omega) \ .$$

We shall identify the distribution $\mathcal{F}_f$ with the element f.

For any measurable set Ω in $\mathbb{R}^N$, the characteristic function χ_Ω (for the sake of simplicity we shall write χ whenever possible) is defined by

$$\chi_\Omega(x) = \begin{cases} 1 & \text{for } x \in \Omega \\ 0 & \text{for } x \in \Omega^c = \mathbb{R}^N \setminus \Omega \ . \end{cases}$$

Ω^c is the complement of Ω in $\mathbb{R}^N$. If the Lebesgue measure of $\Omega \subset \mathbb{R}^N$ is finite, then χ_Ω is in $L^1(\mathbb{R}^N)$. In general χ_Ω is in $L^1_{\text{loc}}(\mathbb{R}^N)$, therefore χ_Ω can be identified with an element of the space of distributions $\mathcal{D}'(\mathbb{R}^N)$. Let us consider the gradient

$$\nabla \chi_\Omega = \left(\frac{\partial}{\partial x_1} \chi_\Omega, \dots, \frac{\partial}{\partial x_N} \chi_\Omega\right)$$

which is an element of $\mathcal{D}'(\mathbb{R}^N, \mathbb{R}^N)$ defined as follows

$$\langle \nabla \chi_\Omega, \phi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = -\int_\Omega \operatorname{div} \phi dx$$

$$\forall \phi = (\phi_1, \dots, \phi_N) \in \mathcal{D}(\mathbb{R}^N; \mathbb{R}^N) \ .$$

Let Ω be a domain of class C^k, $k \geqq 1$, using the well-known Stokes' formula

$$\int_\Omega \operatorname{div}\phi dx = \int_\Gamma \phi \cdot n d\Gamma$$

we get

$$\langle \nabla \chi_\Omega, \phi \rangle = -\int_\Gamma \phi \cdot n d\Gamma,$$

where n is the outward unit normal vector on $\Gamma = \partial\Omega$. Making use of the trace operator $\gamma_\Gamma \in \mathcal{L}(\mathcal{D}^{k-1}(\overline{\Omega}; \mathbb{R}^N); \mathcal{D}^{k-1}(\Gamma; \mathbb{R}^N))$ introduced by Schwarz (1966), where $\mathcal{D}^k(\Gamma)$ denotes the set of functions ϕ defined on Γ such that $\sum_{i=1}^m (r_i\phi) \circ h_i$ is in $C^k(B_0)$, we have

$$\langle \nabla \chi_\Omega, \phi \rangle = -\langle {}^\star\gamma_\Gamma \cdot n, \phi \rangle$$

that is the gradient of χ_Ω takes the form

$$\nabla \chi_\Omega = -{}^\star\gamma_\Gamma \cdot n \ .$$

$\nabla \chi_\Omega$ is an element of $\mathcal{D}^{1-k}(\mathbb{R}^N; \mathbb{R}^N) = (\mathcal{D}^{k-1}(\mathbb{R}^N; \mathbb{R}^N))'$, where ${}^\star\gamma_\Gamma$ denotes the transposed operator, and we have

$${}^\star\gamma_\Gamma \in \mathcal{L}(\mathcal{D}^{k-1}(\mathbb{R}^N; \mathbb{R}^N); \mathcal{D}^{1-k}(\mathbb{R}^N; \mathbb{R}^N)) \ .$$

Let us consider a domain Ω of class C^k, $k \geqq 1$.

For any p, $1 \leqq p < +\infty$, and for $s \geqq 0$ the Sobolev space $W^{s,p}(\Omega)$ (Adams 1975) is defined as the closure of the space $C^\infty(\overline{\Omega})$ in the following norms:

if s is an integer

$$\|\phi\|^p_{W^{s,p}(\Omega)} = \sum_{|\alpha| \leqq s} \int_\Omega \left| \left(\frac{\partial}{\partial x}\right)^\alpha \phi(x) \right|^p dx$$

and for arbitrary s, $s \geqq 0$,

$$\|\phi\|^p_{W^{s,p}(\Omega)} = \|\phi\|^p_{W^{[s],p}(\Omega)} + \int\int_{\Omega\times\Omega} \frac{|\phi(x) - \phi(y)|^p}{\|x-y\|^{N+(s-[s])p}_{\mathbb{R}^N}} dx dy,$$

where $[s]$ is the integer such that

$$[s] \leqq s < [s] + 1 \ .$$

For $p = 2$, $W^{s,2}(\Omega)$ is a Hilbert space; it is denoted by $H^s(\Omega)$.

Let $\Omega \subset \mathbb{R}^N$ be a measurable set. Let us assume that $0 \leqq s < \frac{1}{2}$, and consider the $W^{s,p}(\mathbb{R}^N)$ norm of the characteristic function χ_Ω given by

$$\|\chi_\Omega\|_{W^{s,p}(\mathbb{R}^N)} = 2 \int_\Omega \int_{\Omega^c} \|x-y\|^{-(N+sp)}_{\mathbb{R}^N} dx dy \ .$$

The following result holds true (Baiocchi et al. 1984).

Proposition 2.1 *If Ω is a domain of class C^1 in $\mathbb{R}^N$, then the characteristic function χ_Ω belongs to $W^{s,p}(\mathbb{R}^N)$ for $s < 1/p$, $1 \leqq p < +\infty$.*

2.4. Linear elliptic boundary value problems

Let us consider an element $y(\Omega)$ of $W^{s,p}(\Omega)$ which depends on the domain $\Omega \subset \mathbb{R}^N$. In general this element is given in the form of a solution to a boundary value problem defined in Ω. The sensitivity analysis of the mapping $\Omega \to y(\Omega)$ will be carried on in an abstract way. However, we shall start with a simple example of the linear elliptic boundary value problem that can serve as a model. Two basic linear problems, to be formulated later on, are used as mathematical models for the small displacements of a membrane and for the Kirchhoff plate in the state of static equilibrium. First we consider the Laplace equation in Ω with two different boundary conditions: the Dirichlet boundary conditions on a part Γ_0 of the boundary $\Gamma = \partial\Omega$, where the displacement of the membrane is prescribed, and the Neumann boundary conditions on $\Gamma_1 = \Gamma \backslash \Gamma_0$. The Kirchhoff model of the plate leads to the fourth order biharmonic equation with boundary conditions of different types. Such examples of mathematical boundary value problems are used only as models. The use of membrane as a model of the Laplace equation is not of crucial importance in our considerations; many other examples, arising in the shape optimization, can be related to the same boundary value problem. For example, we can consider the steady state heat transfer equation, or the steady irrotational flow of an incompressible perfect fluid giving rise to the similar mathematical model.

Let $A(x) = a_{ij}(x)$, $x \in \overline{\Omega}$, $1 \leqq i,j \leqq N$, be a $N \times N$ matrix function such that

$$A(\cdot) \in C(\overline{\Omega}; \mathbb{R}^{N^2})$$

and

$$\langle A(x) \cdot \xi, \xi \rangle_{\mathbb{R}^N} \geqq \alpha \|\xi\|^2_{\mathbb{R}^N} \quad \forall \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$$

for some $\alpha > 0$ and all $x \in \overline{\Omega}$.

Let $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\Gamma)$ be given, and denote by $y \in H^1(\Omega)$ a solution to the following problem

$$-\mathrm{div}(A \cdot \nabla y) + y = f \quad \text{in } \mathcal{D}'(\Omega), \tag{2.1}$$

$$y = g \quad \text{in } H^{\frac{1}{2}}(\Gamma) \ . \tag{2.2}$$

We introduce the bilinear form on $H^1(\Omega) \times H^1(\Omega)$

$$a(\phi, z) = \int_\Omega \langle A(x) \cdot \nabla\phi(x), \nabla z(x)\rangle_{\mathbb{R}^N} dx + \int_\Omega \phi z dx$$

and the closed convex subset K of $H^1(\Omega)$ associated with the Dirichlet data g

$$K = \{\phi \in H^1(\Omega) | \gamma_\Gamma \phi = g\} \ .$$

To prove the existence and uniqueness of a solution to (2.1), (2.2) the use is made of Stampacchia's theorem.

Theorem 2.2 *Let H be a Hilbert space, K a closed convex set in H, $a(\cdot,\cdot)$ a continuous bilinear form on $H \times H$ such that*

$$\exists \alpha > 0: \quad a(\phi, \phi) \geqq \alpha \|\phi\|_H^2 \quad \forall \phi \in H \ .$$

Then for any continuous linear form $L(\cdot)$ on H there exists the unique element $y \in K$ such that

$$a(y, \phi - y) \geqq L(\phi - y) \quad \forall \phi \in K \ .$$

Remark. In general it is not assumed that the bilinear form $a(\cdot,\cdot)$ is symmetric. If it is the case, then y is also the unique solution in K of the minimization problem $\mathcal{J}(y) \leqq \mathcal{J}(\phi)\ \forall \phi \in K$, where $\mathcal{J}(\cdot)$ is the quadratic energy functional

$$\mathcal{J}(\phi) = \frac{1}{2} a(\phi, \phi) - L(\phi) \ .$$

For the linear form on $H^1(\Omega)$ defined by $L(\phi) = \int_\Omega f\phi dx$, where f is given in $L^2(\Omega)$, we can prove using Theorem 2.2 the existence and uniqueness of an element $y \in H^1(\Omega)$ with $\gamma_\Gamma y = g$ and

$$\int_\Omega \langle A \cdot \nabla y, \nabla(\phi - y)\rangle_{\mathbb{R}^N} dx + \int_\Omega y(\phi - y) dx \geqq \int_\Omega f(\phi - y) dx$$

for all ϕ in $H^1(\Omega)$ with $\phi = g$ on Γ.

Since an element z of the space $H_0^1(\Omega)$ can be considered as the difference $z = \phi - y$, where $\phi = y + z \in K$, we have

$$\int_\Omega \langle A \cdot \nabla y, \nabla z\rangle_{\mathbb{R}^N} dx + \int_\Omega yz dx \geqq \int_\Omega fz dx \quad \forall z \in H_0^1(\Omega) \ .$$

$H_0^1(\Omega)$ is a linear space, therefore z can be replaced by $-z$ and we obtain

$$y \in K: \quad \int_\Omega \langle A \cdot \nabla y, \nabla z\rangle_{\mathbb{R}^N} dx + \int_\Omega yz dx = \int_\Omega fz dx \quad \forall z \in H_0^1(\Omega) \ . \quad (2.3)$$

Making use of the classical "variational interpretation" of (2.3) y can be identified as a weak solution in $H^1(\Omega)$ to the problem (2.1) and (2.2). To this end we select $z \in \mathcal{D}(\Omega)$ and then (2.1) follows from (2.3).

Another approach to the solution of the boundary value problem (2.1) and (2.2) can be formulated as follows, first the homogenous Dirichlet boundary value problem is considered (i.e. with $g = 0$) using the well-known Lax–Milgram lemma.

Theorem 2.3 *Let H, $a(\cdot,\cdot)$ and $L(\cdot)$ be such as in Theorem 2.2. There exists the unique element $y \in H$ which satisfies the boundary value problem*

$$a(y,\phi) = L(\phi) \quad \forall \phi \in H \ .$$

Proof. It is sufficient to apply Theorem 2.2 with $K = H$ and for $\phi = y \pm z$ (with $z \in H$) in the variational inequality of Theorem 2.3. □

Let us suppose now that g is given in the space $H^1(\Omega)$, and consider the following linear form defined on the space $H_0^1(\Omega)$

$$L(z) = \int_\Omega [fz - \langle A \cdot \nabla g, \nabla z\rangle_{\mathbb{R}^N} - gz]dx \ .$$

From Theorem 2.3 it follows that there exists the unique element $u \in H_0^1(\Omega)$ such that

$$-\text{div}(A \cdot \nabla u) + u = f + \text{div}(A \cdot \nabla g) - g \quad \text{in } \mathcal{D}'(\Omega) \tag{2.4}$$

$$u = 0 \quad \text{on } \Gamma \ . \tag{2.5}$$

Assume that $y = u + g$, hence y is an element of K and satisfies (2.1) and (2.2).

Non–smooth right–hand sides of elliptic equations

It is clear that it need not be assumed in (2.1) and (2.4) that the element f is in $L^2(\Omega)$. For an element $f \in H^{-1}(\Omega)$, the existence of a solution to (2.1) and (2.2) can be proved using the same argument as above for $f \in L^2(\Omega)$. For example, if f is given in $L^2(\Omega)$, then we can prove the existence of the unique solution $y \in H^1(\Omega)$ to the problem

$$-\text{div}(A \cdot \nabla y) + y = \frac{\partial f}{\partial x_1} \quad \text{in } \mathcal{D}'(\Omega) \tag{2.6}$$

$$y = g \quad \text{in } H^{\frac{1}{2}}(\Gamma) \ . \tag{2.7}$$

To eliminate the lower order term appearing on the left–hand sides of equations (2.1), (2.4) and (2.6), one has to make use of the Poincaré inequality. It might be well to point out that the first (positive) eigenvalue of the second order elliptic operator in $H_0^1(\Omega)$ is defined as follows

$$\lambda_1 = \min\{\int_\Omega \langle A \cdot \nabla\phi, \nabla\phi\rangle_{\mathbb{R}^N} dx |\ \phi \in H_0^1(\Omega), \quad \|\phi\|_{H^1(\Omega)} = 1\} \ . \tag{2.8}$$

Hence by the definition of λ_1 we have the following inequality for any ϕ in $H_0^1(\Omega)$, $\phi \neq 0$,

$$\lambda_1 \leqq \left(\int_\Omega \langle A \cdot \nabla\phi, \nabla\phi\rangle_{\mathbb{R}^N} dx\right) / \|\phi\|^2_{L^2(\Omega)}$$

whence it follows that

$$\|\phi\|^2_{L^2(\Omega)} \leqq \frac{1}{\lambda_1} \int_\Omega \langle A \cdot \nabla\phi, \nabla\phi\rangle_{\mathbb{R}^N} dx \ . \tag{2.9}$$

As a consequence of (2.9), without loss of the generality, the lower order term u on the left–hand side of the equation (2.4) can be neglected. Hence the following problem is to be considered:

Find $u \in H_0^1(\Omega)$ such that

$$-\operatorname{div}(A \cdot \nabla u) = \frac{\partial f}{\partial x_1} + \operatorname{div}(A \cdot \nabla g) - g \quad \text{in } \mathcal{D}'(\Omega), \tag{2.10}$$

$$u = 0 \quad \text{in } H^{\frac{1}{2}}(\Gamma) \ . \tag{2.11}$$

From the inequality (2.9) it follows that the bilinear form

$$a_0(\phi, z) = \int_\Omega \langle A \cdot \nabla\phi, \nabla z\rangle_{\mathbb{R}^N} dx$$

is coercive on $H_0^1(\Omega)$ equipped with the following norm

$$\|z\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla z|^2 dx$$

for we have

$$\begin{aligned} a_0(\phi, \phi) &\geqq \frac{\lambda_1}{2}\|\phi\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_\Omega \nabla\phi \cdot \nabla\phi dx \\ &\geqq \min\left(\frac{\lambda_1}{2}, \frac{\alpha}{2}\right) \|\phi\|^2_{H_0^1(\Omega)} \quad \forall \phi \in H_0^1(\Omega) \ . \end{aligned}$$

Therefore Theorem 2.3 ensures the existence and uniqueness of the solution u to (2.10) and (2.11) in the space $H_0^1(\Omega)$.

Let $y = u + g$; by direct calculation with the use of (2.10) we obtain that $y \in K$ is the solution to the following problem:

$$-\operatorname{div}(A \cdot \nabla y) = \frac{\partial f}{\partial x_1} \quad \text{in } \mathcal{D}'(\Omega), \tag{2.12}$$

$$y = g \quad \text{in } H^{\frac{1}{2}}(\Gamma) \ . \tag{2.13}$$

Finally, let us consider the Neumann boundary conditions for the problem (2.12):

$$\frac{\partial y}{\partial n} = 0 \quad \text{i.e. the homogenous Neumann condition on } \Gamma.$$

As far as applications in structural mechanics are concerned, the condition $y = 0$ on Γ means that the displacement is prescribed on the boundary $\Gamma = \partial\Omega$, e.g. the membrane is clamped.

We introduce the vector field $n_A = A \cdot n$ on Γ; n_A is a C^{k-1} transversal field on Γ (here we assume that the mapping $x \to A_{ij}(x)$ is $C^k(\Gamma)$), the transversality of n_A is derived from the positive definiteness of A

$$\langle n_A, n\rangle_{\mathbb{R}^N} = \langle A \cdot n, n\rangle_{\mathbb{R}^N} \geqq \alpha\|n\|^2_{\mathbb{R}^N} = \alpha > 0 \ .$$

In particular, if n is outward pointing on Γ, then n_A is also outward pointing on Γ.

The condition $\frac{\partial y}{\partial n_A} = 0$ on Γ means that the displacement y is not prescribed on the boundary Γ. In order to ensure the existence of a solution to the boundary value problem under consideration, it is obvious that some additional conditions on the right–hand side term of the equation (the source term) should be imposed.

Let $f \in L^2(\Omega)$ be an element such that $\int_\Omega f(x)dx = 0$, then the mapping $\phi \to \int_\Omega f\phi dx$ is a linear continuous form defined on the quotient space $H^1(\Omega)/\mathbb{R}$ (in other words, one has to identify in the space $H^1(\Omega)/\mathbb{R}$ elements ϕ and z such that there exists a constant c with $\phi = z + c$) equipped with the norm $\|\phi\| \equiv (a_0(\phi,\phi))^{\frac{1}{2}}$. Hence it follows from Theorem 2.3 that there exists the unique solution $y \in H^1(\Omega)/\mathbb{R}$ to the problem

$$-\text{div}(A \cdot \nabla y) = f \quad \text{in } \Omega, \tag{2.14}$$

$$\frac{\partial y}{\partial n_A} = 0 \quad \text{on } \Gamma \ . \tag{2.15}$$

In the case of non–homogenous Neumann boundary conditions, i.e. $\frac{\partial y}{\partial n_A} = g$, it is supposed that $g \in H^1(\Omega)$ is an element with

$$\int_\Gamma g d\Gamma = 0 \ .$$

Moreover we introduce the linear form

$$L(\phi) = \int_\Omega f\phi dx + \int_\Gamma g\phi d\Gamma$$

which is continuous on the space $H^1(\Omega)/\mathbb{R}$. Using Theorem 2.3 we can prove the existence and uniqueness of a solution to the elliptic boundary value problem in the space $H^1(\Omega)/\mathbb{R}$

$$-\text{div}(A \cdot \nabla y) = f \quad \text{in } \Omega, \tag{2.16}$$

$$\frac{\partial y}{\partial n_A} = 0 \quad \text{on } \Gamma \ . \tag{2.17}$$

The source term of the equation (2.16) can be treated in the same way as it was done for the Dirichlet problem, e.g. one can make use of a distribution $\frac{\partial f}{\partial x_1} \in \mathcal{D}'(\Omega)$, where $f \in L^2(\Omega)$. Let us introduce the linear form

$$L(\phi) = -\int_\Omega f \frac{\partial \phi}{\partial x_1} dx + \int_\Gamma (g + f n_1)\phi d\Gamma$$

and assume that the condition $L(1) = 0$ is satisfied, hence

$$\int_\Gamma (g + f n_1) d\Gamma = 0 . \tag{2.18}$$

The equation (2.18) is well defined provided that $f \in H^s(\Omega)$ for some $s > \frac{1}{2}$. This requirement results from the condition that the trace of f on Γ has to be in $L^2(\Gamma)$ (see (Adams 1975)).

The compatibility conditions $\int_\Omega f dx = 0$, $\int_\Gamma g d\Gamma = 0$ for the problem (2.16) and (2.17) can be replaced with the equivalent condition

$$0 = L(1) = \int_\Omega f dx + \int_\Gamma g d\Gamma . \tag{2.19}$$

In the next sections we shall consider the boundary value problems defined in the perturbed domain Ω_t. It will be assumed that for the Neumann problem defined on the family of domains, the functions $f \in L^2(\mathbb{R}^N)$ and $g \in H^1(\mathbb{R}^N)$ are given. In general it is not supposed that the condition (2.19) is satisfied for any domain Ω. However for any given domain Ω with the boundary $\Gamma = \partial\Omega$ we define the elements

$$f(\Omega) = f|_\Omega - \frac{1}{\text{meas}(\Omega)} \int_\Omega f dx \tag{2.20}$$

$$g(\Gamma) = g|_\Gamma - \frac{1}{\text{meas}(\Gamma)} \int_\Gamma g d\Gamma . \tag{2.21}$$

Under these assumptions, the functions $f(\Omega)$ and $g(\Gamma)$ satisfy the conditions

$$\int_\Omega f(\Omega) dx = \int_\Gamma g(\Omega) d\Gamma = 0$$

and the problem (2.16) and (2.17) with the data $f(\Omega)$ and $g(\Gamma)$ is well posed in any domain Ω.

Transmission problems

Let us consider the transmission conditions related to the problem (2.16). In this case, the matrix $A(x)$ is not supposed to be continuous with respect to x, but it can possess the discontinuity lines or interfaces.

Let $D \subset \mathbb{R}^2$ be a fixed domain, $B(x)$ and $C(x)$ two continuous matrix functions defined on $\overline{D}$. We define on $\overline{D}$ the following matrix function

$$A(\Omega) = \chi_{\Omega^c} B + \chi_\Omega C$$

for a given measurable set Ω, $\Omega \subset D$. It should be recalled that $\Omega^c = D \setminus \Omega$ is the complement of Ω in D, χ_{Ω^c} and χ_Ω denote the characteristic functions of sets Ω^c and Ω, respectively.

Let us consider the following problem:

$$\begin{aligned} &\text{Find } y \text{ in } H_0^1(D) \text{ such that} \\ &-\operatorname{div}(A(\Omega)\cdot\nabla y) = \frac{\partial f}{\partial x_1} \quad \text{in } H^{-1}(D) \end{aligned} \tag{2.22}$$

and with the prescribed transmission conditions on the interface $\Gamma = \partial\Omega$ (it is assumed that Γ is smooth):

$$\langle (C - B)\cdot n, \nabla y\rangle_{\mathbb{R}^2} = g \quad \text{on } \Gamma, \tag{2.23}$$

where $g \in H^1(D)$ is a given element. In addition, since $y \in H_0^1(\Omega)$, we have prescribed the homogenuous Dirichlet boundary condition on ∂D

$$y = 0 \quad \text{on } \partial D\ . \tag{2.24}$$

For the problem formulated we introduce the bilinear form

$$a(\Omega;\phi,z) = \int_D \langle A(\Omega)\cdot\nabla\phi, \nabla z\rangle_{\mathbb{R}^2} dx$$

defined on $H_0^1(D;\mathbb{R}^2)$ and the linear form

$$L(\phi) = -\int_D f\frac{\partial f}{\partial x_1}\phi dx + \int_\Gamma g\phi d\Gamma\ .$$

It is assumed that the matrices $B(x)$ and $C(x)$ are uniformly positive definite, $B(x)$, $C(x) \geqq \alpha\mathcal{I}$ for some $\alpha > 0$, then $A(x) \equiv A(\Omega)(x) \geqq \alpha\mathcal{I}$, $x \in \overline{D}$. Making use of Theorem 2.3 and Green's formula, one can prove the existence and uniqueness of the solution $y \in H_0^1(D)$ to the problem (2.22) and (2.23). Finally let us consider the boundary conditions on Γ involving the tangential derivatives of an unknown solution. For any element $\phi \in C^1(\overline{\Omega})$, $\Omega \subset \mathbb{R}^N$, we denote by $\nabla_\Gamma\phi$ the tangential gradient of ϕ on Γ, i.e. $\nabla_\Gamma\phi$ is the tangential component of the gradient $\nabla\phi$ on the boundary Γ

$$\nabla_\Gamma\phi = \nabla\phi - \frac{\partial\phi}{\partial n}n \quad \text{on } \Gamma\ .$$

Let us consider the boundary value problem:

$$\begin{aligned} &\text{Find } y \in \mathcal{D}'(\Omega) \text{ such that} \\ &-\operatorname{div}(A(\Omega)\cdot\nabla y) = f \quad \text{in } \mathcal{D}'(\Omega), && (2.25)\\ &\frac{\partial}{\partial n_A}y - \Delta_\Gamma y = g \quad \text{on } \Gamma, && (2.26) \end{aligned}$$

where $A(\cdot)$ is a continuous matrix function on Γ, Δ_Γ is the so-called Laplace–Beltrami operator on Γ (defined in Sect. 2.20). In particular for any ϕ in $C^1(\Gamma)$ the following identity holds

$$-\int_\Gamma \Delta_\Gamma y \phi d\Gamma = \int_\Gamma \nabla_\Gamma y \cdot \nabla_\Gamma \phi d\Gamma \ . \tag{2.27}$$

It is assumed that y is such that $\Delta_\Gamma y \in L^2(\Gamma)$ and $\nabla_\Gamma y \in L^2(\Gamma; \mathbb{R}^N)$. Let us define the energy space $W(\Omega)$ for the problem (2.25) and (2.26) as the closure of the space $C^1(\overline{\Omega})/\mathbb{R}$ in the norm

$$\|\phi\|_W^2 = \int_\Omega \langle A \cdot \nabla\phi, \nabla\phi\rangle_{\mathbb{R}^N} dx + \int_\Gamma \langle \nabla_\Gamma \phi, \nabla_\Gamma \phi\rangle_{\mathbb{R}^N} d\Gamma \ .$$

The linear form $L(\phi) = \int_\Omega f\phi dx + \int_\Gamma g\phi d\Gamma$, with the compatibility conditions satisfied by the data $\int_\Omega f dx + \int_\Gamma g d\Gamma = 0$, is continuous on the space $W(\Omega)$. Therefore the existence and uniqueness of the solution $y \in W(\Omega)$ to the problem (2.25) and (2.26) results from Theorem 2.3.

Fourth order elliptic problems

Let $f \in L^2(\Omega)$ be a given element, let us consider the following elliptic boundary value problem:

$$\Delta(h\Delta y) = f \quad \text{in } \mathcal{D}'(\Omega), \tag{2.28}$$

$$y = 0 \quad \text{on } \Gamma, \tag{2.29}$$

$$\frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma, \tag{2.30}$$

where

$$h \in C(\overline{\Omega}), \quad h(x) \geqq \alpha_0 > 0 \quad \forall x \in \overline{\Omega} \ . \tag{2.31}$$

A weak solution to the system (2.28)–(2.30) is an element of the Sobolev space $H_0^2(\Omega)$. It is assumed that the domain Ω is of class C^k, $k \geqq 1$. The space $H_0^2(\Omega)$ is the closed subspace of $H^2(\Omega)$ defined by two conditions (2.29) and (2.30), see e.g. (Adams 1975) for the details.

Let us introduce the bilinear form

$$a(\phi, z) = \int_\Gamma h\Delta\phi\Delta z dx \quad \forall \phi, z \in H_0^2(\Omega) \ .$$

Making use of the elliptic regularity for the Laplace equation, one can show that the norm $a(\phi,\phi)^{\frac{1}{2}}$ is equivalent on the space $H_0^2(\Omega)$ to the $H^2(\Omega)$ norm.

Hence using the linear form $L(\phi) = \int_\Omega f\phi dx$, we can apply Theorem 2.3 and prove the existence of the unique solution in $H_0^2(\Omega)$ to the problem (2.28)–(2.30). For $h(x)$ =const, the problem (2.28)–(2.30) can be considered as a model of the Kirchhoff plate of the constant thickness, clamped on $\Gamma = \partial\Omega$, see e.g. (Washizu 1982).

Now, let us consider the boundary conditions for the Kirchhoff plate which is free on the boundary $\Gamma = \partial\Omega$. The problem is to find a weak solution in $H^2(\Omega)$ to the elliptic equation:

$$\Delta(h\Delta y) = f \quad \text{in } \mathcal{D}'(\Omega) \tag{2.32}$$

with the following boundary conditions:

$$\Delta y = 0 \quad \text{on } \Gamma, \tag{2.33}$$

$$\frac{\partial}{\partial n}\Delta y = 0 \quad \text{on } \Gamma\ . \tag{2.34}$$

In order to assure the uniqueness of solutions to (2.32)–(2.34) the Hilbert space $H = H^2(\Omega)/\mathcal{M}$ is taken into consideration, where $\mathcal{M}$ is the closed subspace of $H^2(\Omega)$ defined by

$$\mathcal{M} = \{\phi \in H^2(\Omega) | \Delta\phi = 0 \quad \text{in } \Omega\}$$

i.e. the subspace of the harmonic functions in $H^2(\Omega)$. It is obvious that for the space H with the quotient norm, the bilinear form $a(\phi, z)$ is coercive on H. Let us characterize linear forms on H. The linear form $L(\phi) = \int_\Omega f\phi dx$ is defined on $L^2(\Omega)$; this form is defined on H if and only if the element f satisfies the orthogonality condition

$$\int_\Omega f\phi dx = 0 \quad \forall\phi \in \mathcal{M}\ .$$

In order to have the orthogonality condition in an explicit form, let us consider an element $F \in H^2(\Omega) \cap H_0^1(\Omega)$, depending on $f \in L^2(\Omega)$, such that the following equation holds

$$\Delta F = f \quad \text{in } \Omega, \quad F = 0 \quad \text{on } \Gamma\ .$$

The orthogonality condition can be written as follows

$$\int_\Omega \Delta F\phi dx = 0\ .$$

Applying Green's formula to this condition we can show that for $\phi \in \mathcal{M}$

$$\int_\Gamma \left(\frac{\partial}{\partial n}F\phi - \frac{\partial}{\partial n}\phi F\right) d\Gamma = 0\ .$$

Thus we have the auxiliary condition for F, $\frac{\partial}{\partial n}F = 0$ on Γ. Therefore f satisfies the orthogonality condition if and only if

$$f = \Delta F \quad \text{with} \quad F \in H_0^2(\Omega)\ . \tag{2.35}$$

Finally we can conclude that for any $f \in L^2(\Omega)$, f satisfying (2.35), the problem (2.32), (2.33) and (2.34) is well posed on the quotient space $H = H^2(\Omega)/\mathcal{M}$.

For the fourth order boundary value problems under consideration, it is not necessary to assume that the right–hand side of (2.28) or (2.32) is in the space $L^2(\Omega)$.

For the Dirichlet problem (2.28)–(2.30) one has to choose a linear form $L(\cdot)$ which is continuous on the space $H_0^2(\Omega)$. Hence we can use, e.g. the linear forms

$$L_f(\phi) = \int_\Omega f \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx \quad f \in L^2(\Omega),$$

since $\phi \in H_0^2(\Omega)$, i.e. ϕ satisfies the conditions (2.29) and (2.30) then

$$L_f(\phi) = \left\langle \frac{\partial^2 f}{\partial x_i \partial x_j}, \phi \right\rangle .$$

$\langle \cdot, \cdot \rangle$ denotes the duality pairing between the Sobolev space $H_0^2(\Omega)$ and its dual $H^{-2}(\Omega)$ which is a closed subspace of $\mathcal{D}'(\Omega)$.

But this means that one can prove the existence and uniqueness of the solution $y \in H_0^2(\Omega)$ to the following problem

$$\Delta(h\Delta y) = \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{in } \mathcal{D}'(\Omega), \tag{2.36}$$

$$y = \frac{\partial}{\partial n} y = 0 \quad \text{on } \Gamma . \tag{2.37}$$

for any f given in $L^2(\Omega)$.

2.5. Shape functionals

Throughout this book any shape functional is denoted by $J(\Omega)$, $J(\cdot) \colon \Omega \to J(\Omega) \in \mathbb{R}$, where Ω is a domain of class C^k for $k \geqq 1$. Some examples with non–smooth domains Ω will be also considered.

Very simple examples of the domain functionals are:

$$J_1(\Omega) = \text{meas}\,(\Omega) = \int_\Omega dx, \tag{2.38}$$

$$J_2(\Omega) = \text{meas}\,(\Gamma) = \int_\Gamma d\Gamma, \tag{2.39}$$

$$J_3(\Omega) = \text{total curvature of } \Gamma = \int_\Gamma \kappa^2 d\Gamma . \tag{2.40}$$

Let us observe that the functional $J_1(\Omega)$ can be defined in terms of χ_Ω, the characteristic function of Ω,

$$J_1(\Omega) = \int_{\mathbb{R}^N} \chi_\Omega dx \ .$$

In general for functionals having this property, i.e. those which can be defined in terms of χ_Ω, the following equality holds $J(\Omega) = j(\chi_\Omega)$. The functional $j(\cdot)$ can be extended to the larger class of functions $\omega(\cdot) \in L^\infty(\mathbb{R}^N)$ such that $0 \leqq \omega(x) \leqq 1$ for almost every (a.e.) $x \in \mathbb{R}^N$. Thus the extended functional also denoted $j(\cdot)$ is defined on $L^\infty(\mathbb{R}^N)$, and we can use the weak – (*) topology on $L^\infty(\mathbb{R}^N)$ to define the continuity (or semi-continuity) of the functional $\Omega \to j(\chi_\Omega) \equiv J(\Omega)$.

We can also make use of the Banach structure of the space $L^\infty(\mathbb{R})$ to define, in an appropriate way, the derivatives of $J(\cdot)$ with respect to the domain Ω.

The following result characterizes the convergence of characteristic functions.

Lemma 2.4 *Let χ_k be a sequence of characteristic functions such that $\chi_k \to \chi$ weak – (*) in $L^\infty(\mathbb{R}^N)$ as $k \to \infty$, i.e.*

$$\int_{\mathbb{R}^N} \chi_k \phi dx \to \int_{\mathbb{R}^N} \chi \phi dx \quad \forall \in L^1(\mathbb{R}^N) \ .$$

If χ is a characteristic function, then $\chi_k \to \chi$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

Proof. Let $\psi \in \mathcal{D}(\mathbb{R}^N)$ be a fixed element, then for any ϕ in $L^2(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (\chi_k \psi) \phi dx \to \int_{\mathbb{R}^N} (\chi \psi) \phi dx.$$

Hence $\chi_k \psi$ converges to $\chi \psi$ weakly in $L^2(\mathbb{R}^N)$ as $k \to \infty$, but

$$\|\chi_k \psi\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \psi^2 \chi_k dx \ .$$

This expression converges to $\int_{\mathbb{R}^N} \chi \psi^2 dx = \|\chi \psi\|^2_{L^2(\mathbb{R}^N)}$, since $\psi \in \mathcal{D}(\mathbb{R}^N)$ is arbitrary, that completes the proof. □

Domain functionals such as $J_1(\Omega)$, depending on the characteristic function χ_Ω of the domain Ω, are only particular cases of functionals under consideration. In general, we cannot expect that a domain functional $J(\Omega)$ enjoys this property. Clearly, the functional $J_3(\Omega)$ depends on the boundary Γ of the domain Ω and cannot be extended to the set of functions $\omega(\cdot) \in L^\infty(\mathbb{R}^N)$ such that $0 \leqq \omega(x) \leqq 1$ for a.e. x. Moreover let us observe that the functional $J_2(\Omega)$ can be extended to the following class of non-smooth measurable domains: a measurable set E in $\mathbb{R}^N$ is said to have the finite perimeter $\mathcal{P}(E)$ provided that meas$(E) < \infty$ and

$$\mathcal{P}(E) = \sup\{\int_E \mathrm{div}\phi dx \mid \phi \in \mathcal{D}^1(\mathbb{R}^N, \mathbb{R}^N)\ , \quad \max_{x\in\mathbb{R}^N} \|\phi(x)\|_{\mathbb{R}^N} \leqq 1\} < \infty,$$

where $\mathcal{D}^1(\mathbb{R}^N)$ denotes the space of functions continuously differentiable with compact supports in $\mathbb{R}^N$.

The class of measurable sets E with finite perimeters has been introduced by Caccioppoli (see e.g. (E. De Giorgi et al. 1972)).

If E is a C^2 domain, then the supremum appearing in the definition of $\mathcal{P}(E)$ is attained for any extension $\mathcal{N}_0$ of the normal field n on ∂E, such that $\mathcal{N}_0 \in C^1(\mathbb{R}^N, \mathbb{R}^N)$. Hence we have

$$\mathcal{P}(E) = \int_E \mathrm{div}(\mathcal{N}_0) d\Gamma = \int_{\partial E} n \cdot n d\Gamma \tag{2.41}$$

and $\mathcal{P}(E) = J_2(\partial E)$. This proves that the functional $J_2(\Omega)$ can be extended to the class of measurable sets with finite perimeters as well as to the larger class of functions $\omega \in L^\infty(\mathbb{R}^N)$, $0 \leqq \omega(x) \leqq 1$ for a.e. x, such that

$$\sup\{\int_{\mathbb{R}^N} \omega \mathrm{div}(\phi) dx | \phi \in \mathcal{D}^1(\mathbb{R}^N, \mathbb{R}^N), \quad \max_x \|\phi(x)\|_{\mathbb{R}^N} \leqq 1\}$$

is finite.

2.6. Shape functionals for problems governed by linear elliptic boundary value problems

In many shape optimization problems the following situation occurs: a shape functional $J(\Omega)$ depends on the domain Ω via the solution $y(\Omega)$ to a boundary value problem defined in Ω. For the second order elliptic problems the functional $J(\Omega)$ takes the form

$$J(\Omega) = \int_\Omega F_1(x, y(x), \nabla y(x)) dx + \int_\Gamma F_0(x, y(x), \nabla y(x)) d\Gamma(x) + \alpha \mathcal{E}(\Omega), \tag{2.42}$$

where $\mathcal{E}(\Omega)$ denotes the regularizing term that ensures the existence of the optimal domain minimizing the functional $J(\Omega)$ over the appropriate class of domains, $\alpha > 0$ is a constant.

If $F_0 \equiv 0$, then it is said that J is a distributed cost functional; if $F_1 \equiv 0$, then J is referred to as a boundary cost functional. Examples of F_1 and F_0 are

$$F_1(x, y(x), \nabla y(x)) = \frac{1}{2}(y(x) - y_g(x))^2, \tag{2.43}$$

where y_g is a given function in $L^2(\mathbb{R}^N)$,

$$F_0(x, y(x), \nabla y(x)) = \frac{1}{2}(y(x) - z_g(x))^2,$$

here z_g is a given element in $H^s(\mathbb{R}^N)$, $s > \frac{1}{2}$.

One can assume that

$$F_0(x, y(x), \nabla y(x)) = \frac{1}{2}\left(\frac{\partial y}{\partial n}(x) - z_g(x)\right)^2$$

provided that $y = y(\Omega)$ is smooth enough, i.e. the normal derivative $\partial y/\partial n$ is in $L^2(\Gamma)$. The perimeter of the domain Ω can be taken as the regularizing term $\mathcal{E}(\Omega)$ in the domain functional (2.42). Such assumption is adopted in the following example.

2.6.1. Shape functionals for transmission problems

Let us assume that $y(\Omega)$ is a solution to the problem (2.22). Making use of the variational form of the equation one can show that the following integral identity holds for the element $y = y(\Omega)$

$$y \in H_0^1(D) \; : \quad \int_D \langle A(\Omega) \cdot \nabla y, \nabla \phi \rangle_{\mathbb{R}^N} dx = -\int_D f \frac{\partial \phi}{\partial x_1} dx \qquad \forall \phi \in H_0^1(D), \tag{2.44}$$

where

$$A(\Omega) = (1 - \chi_\Omega)B + \chi_\Omega C \; .$$

Moreover we introduce the set of characteristic functions

$$\text{Char}(D) = \{\chi \in L^2(D) | \; \chi(1 - \chi) = 0 \quad \text{a.e. in} \quad D\}$$

equipped with the $L^2(D)$ topology.

Proposition 2.5 *The mapping $\chi_\Omega \to y(\Omega)$, where $y(\Omega)$ denotes a solution to the problem (2.44), is continuous from the set Char(D) into $H_0^1(D)$.*

Proof. Let Ω_k, $k = 1, 2, \ldots$, and Ω_0 be measurable subsets of $D \subset \mathbb{R}^N$ and let $y_k = y(\Omega_k)$, $k = 1, 2, \ldots$, be the solution to (2.44) with the domain Ω replaced by Ω_k in the definition of $A(\Omega)$. We have, with $y_0 = y(\Omega_0)$,

$$\int_D \langle A(\Omega_k) \cdot \nabla (y_k - y_0), \nabla \phi \rangle_{\mathbb{R}^N} dx = \tag{2.45}$$
$$\int_D \langle (A(\Omega_k) - A(\Omega_0)) \cdot \nabla y(\Omega_0), \nabla \phi \rangle_{\mathbb{R}^N} dx \quad \forall \phi \in H_0^1(\Omega) \; .$$

The matrix A is uniformly positive definite, i.e. we assume that there exists $\alpha_0 > 0$ such that

$$\langle B\cdot\xi,\xi\rangle_{\mathbb{R}^N} \geqq \alpha_0\|\xi\|^2_{\mathbb{R}^N} \quad \text{and} \quad \langle C\cdot\xi,\xi\rangle \geqq \alpha_0\|\xi\|^2_{\mathbb{R}^N}$$

for all the vectors ξ in $\mathbb{R}^N$.

By the Poincaré inequality for $\phi = y_k - y_0$ we get

$$\alpha_1\|y_k - y_0\|_{H^1_0(D)} \leqq \left(\int_D \|[(\chi_{\Omega_0} - \chi_{\Omega_k})B + (\chi_{\Omega_k} - \chi_{\Omega_0})C]\cdot\nabla y(\Omega_0)\|^2_{\mathbb{R}^N}dx\right)^{\frac{1}{2}}$$

The family of domains Ω_k, $k = 1,2,\ldots$, is said to converge in measure to the domain Ω_0 if and only if the sequence of characteristic functions $\{\chi_{\Omega_k}\}$ converges to χ_{Ω_0} in the set Char(D) i.e. in $L^2(D)$ norm or equivalently in $L^p(D)$ norm for any p, $1 \leqq p < +\infty$. Then the sequence of functions

$$f_k = (\chi_{\Omega_0} - \chi_{\Omega_k})B + (\chi_{\Omega_k} - \chi_{\Omega_0})C$$

converges to zero in $L^p(D;\mathbb{R}^{N^2})$ as $k \to \infty$. We have also $\|f_k(x)\|_{\mathbb{R}^N} \leqq$const $\forall x \in \overline{D}$, because B and C are continuous matrix functions. By the Lebesgue theorem there exists a subsequence f_{k_i} such that $f_{k_i}(x) \to 0$ as $i \to \infty$ for almost every x in $\overline{D}$. Then the function on the right–hand side of (2.45) belongs to $L^1(D)$, and converges to zero for almost every x in $\overline{D}$. By the dominated convergence theorem the right–hand side of (2.45) converges to zero as $i \to \infty$. Making use of (2.45), one can show that the norm $\|y_k - y_0\|_{H^1(D)}$ is bounded and converges to zero, the limit $y_0 \in H^1_0(D)$ is uniquely determined. Therefore the sequence $\{y_k\}$ converges $y_k \to y_0$ in the space $H^1_0(D)$ as $k \to \infty$. Let Ω be such that $\chi_\Omega \in$ Char(D), and define the shape functional

$$J(\Omega) = \int_D |\nabla y(\Omega)|^2 dx + \alpha\mathcal{P}_D(\Omega), \tag{2.46}$$

where

$$\mathcal{P}_D(\Omega) = \sup\{\int_\Omega \mathrm{div}\phi dx \mid \phi \in \mathcal{D}^1(D,\mathbb{R}^N)\ ,\quad \max_{x\in D}\|\phi(x)\|_{\mathbb{R}^N} \leqq 1\}$$

is the perimeter of Ω in D, $\alpha > 0$ is a constant.

Since the mapping $H \ni y \to \|y\|^2_H \in \mathbb{R}$ is weakly lower semi–continuous in any Hilbert space H, then the first term on the right–hand side of (2.46) is lower semi–continuous on the set Char$(D) \subset L^2(D)$. The same remains valid for the second term. □

Lemma 2.6 *The functional $\chi_\Omega \to \mathcal{P}_D(\Omega)$ is lower semi–continuous on the set Char$(D) \subset L^2(D)$.*

Proof. For any $\phi \in \mathcal{D}^1(D; \mathbb{R}^N)$, the mapping

$$\chi_\Omega \to \int_\Omega \operatorname{div}\phi dx = \int_D \chi_\Omega \operatorname{div}\phi dx$$

is continuous on the set $\operatorname{Char}(D) \subset L^2(D)$. Therefore the supremum with respect to ϕ, appearing in the definition of the perimeter $\mathcal{P}_D(\Omega)$, is lower semi–continuous.

□

Finally we have the following result:

Proposition 2.7 *The shape functional defined by (2.46) is lower semi–continuous on the set* $Char(D) \subset L^2(D)$.

□

Let us consider the shape optimization problem related to the transmission boundary value problem (2.44).

In order to minimize the shape functional (or cost) $J(\Omega)$ over the set $\operatorname{Char}(D) \subset L^2(D)$ we have to apply, in view of Proposition 2.7, the compactness result given by E. De Giorgi et al. (1972).

Proposition 2.8 *Let D be a bounded domain in $\mathbb{R}^N$. For any $M > 0$, the set*

$$Char(D, M) = \{\chi_\Omega \in \operatorname{Char}(D) | \mathcal{P}_D(\Omega) \leqq M\}$$

is compact in $L^2(D)$.

□

This yields the following existence result for the shape optimization problem under consideration.

Theorem 2.9 *There exists a measurable set Ω_0 in D such that*

$$J(\Omega_0) \leqq J(\Omega)$$

for all measurable sets $\Omega \subset D$.

Proof. If for a given set Ω the supremum in the definition of perimeter $\mathcal{P}_D(\Omega)$ is not finite, then we set

$$\mathcal{P}_D(\Omega) = +\infty \quad \text{and} \quad J(\Omega) = +\infty \ .$$

Let $j_0 = J(\emptyset)$, (where $\emptyset$ is the empty set, so $\chi_\emptyset = 0$ on D), and $j_0 > 0$. It is easy to show that the minimization of $J(\Omega)$ over the set $\operatorname{Char}(D)$ is equivalent to the minimization of this functional over the set $\operatorname{Char}(D, j_0)$. Since the set $\operatorname{Char}(D, j_0)$ is compact and $J(\cdot)$ is lower semi–continuous on the set $\operatorname{Char}(D, j_0)$, then by the Weierstrass theorem there exists a solution Ω_0 to the minimization problem. □

The considered interface optimization in the transmission boundary value problem is a particular example of shape optimization problems. It exemplifies the "two fluids" problem where the state equation of the optimization problem is defined in the domain Ω as well as in its complement $\Omega^c = D \setminus \Omega$ in D. The energy space for the state equation under consideration is the Sobolev space $H_0^1(D)$, where $D = \Omega \cup \Gamma \cup \Omega^c$; hence it is a function space independent of Ω. For more general shape optimization problems, i.e. such that the shape functional $J(\Omega)$ depends, e.g. on a weak solution to the Dirichlet or Neumann problem for the second order elliptic equation defined in Ω, the following principal difficulty is encountered: the energy space for the state equation in the form of an elliptic problem, e.g. the space $H_0^1(\Omega)$ for the Dirichlet problem, or the space $H^1(\Omega)/\mathbb{R}$ for the Neumann problem, depends on the variable domain Ω.

In this case the notion of continuity of the mapping $\Omega \to y(\Omega)$ (the element $y(\Omega)$ is defined by the state equation) should be defined in an appropriate way. A possible way of overcoming this difficulty in the case of the Dirichlet problem is the penalization technique which enables us to construct an approximation of the Dirichlet problem by means of a family of transmission problems depending on a parameter.

2.6.2. Approximation of the homogeneous Dirichlet problem

Let $\Omega \subset \mathbb{R}^N$ be a domain of class C^1, $\overline{\Omega} \subset D$, where D is a given sufficiently smooth domain in $\mathbb{R}^N$. Denote by $y = y(\Omega) \in H_0^1(\Omega)$ the solution to the Dirichlet problem

$$-\operatorname{div}(A \cdot \nabla y) = f \quad \text{in } L^2(\Omega), \tag{2.47}$$

$$y = 0 \quad \text{on } \Gamma, \tag{2.48}$$

where $f \in L^2(\Omega)$ is a given element and A is a continuous matrix function, $A \in C(\overline{D}; \mathbb{R}^{N^2})$, $A(\cdot)$ is uniformly positive definite on $\overline{D}$. Let $\varepsilon > 0$ be a real parameter. Let us consider the element $y_\varepsilon(\Omega) \in H_0^1(\Omega)$, given as the unique solution to the following variational problem

$$\begin{aligned} & y_\varepsilon = y_\varepsilon(\Omega) \in H_0^1(\Omega): \\ & \frac{1}{\varepsilon}\int_{\Omega^c} \langle A \cdot \nabla y_\varepsilon, \nabla \phi \rangle_{\mathbb{R}^N} dx + \int_\Omega \langle A \cdot \nabla y_\varepsilon, \nabla \phi \rangle_{\mathbb{R}^N} dx \\ & = \int_D f \phi dx \quad \forall \phi \in H_0^1(\Omega) \ . \end{aligned} \tag{2.49}$$

It will be shown that under appropriate assumptions on the domain Ω the sequence $\{y_\varepsilon(\Omega)\}$ converges in $H_0^1(\Omega)$ to the element $y^0 = y(\Omega)^0$ as $\varepsilon \downarrow 0$, where

$$y(\Omega)^0 = \begin{cases} y(\Omega) & \text{in } \overline{\Omega} \\ 0 & \text{in } \Omega^c \end{cases} .$$

Substituting $\phi = y_\varepsilon$ in (2.49) we get

$$\alpha \int_{\Omega^c} \|\nabla y_\varepsilon\|^2_{\mathbb{R}^N} dx \leqq \varepsilon \int_D f y_\varepsilon dx - \varepsilon \int_\Omega \langle A \cdot \nabla y_\varepsilon, \nabla y_\varepsilon \rangle_{\mathbb{R}^N} dx . \tag{2.50}$$

Hence due to (2.49) y_ε is bounded:

$$\|y_\varepsilon\|^2_{H^1_0(D)} \leqq \frac{1}{\alpha} \|f\|_{L^2(D)} \quad \varepsilon > 0 . \tag{2.51}$$

Making use of (2.51) and applying the Cauchy–Schwartz inequality, in view of (2.50), it follows that

$$\int_{\Omega^c} \|\nabla y_\varepsilon\|^2 dx \leqq \varepsilon \left[\frac{1}{\alpha} + \frac{C_1}{\alpha^2} \right] \|f\|^2_{L^2(D)}, \tag{2.52}$$

where

$$C_1 = \|A\|_{L^\infty(D;\mathbb{R}^{N^2})} .$$

A weak solution to (2.47) – (2.48) satisfies the following integral identity

$$y \in H^1_0(\Omega): \quad \int_\Omega \langle A \cdot \nabla y, \nabla \phi \rangle_{\mathbb{R}^N} dx = \int_\Omega f \phi dx \quad \forall \phi \in H^1_0(\Omega) . \tag{2.53}$$

Assuming that $\phi \in H^1_0(\Omega)$ and subtracting (2.49) from (2.53) we get

$$\int_\Omega \langle A \cdot \nabla (y_\varepsilon - y), \nabla \phi \rangle_{\mathbb{R}^N} dx = 0 , \tag{2.54}$$

that is for the scalar product

$$a(\phi, z) = \int_D \langle A \cdot \nabla \phi, \nabla z \rangle_{\mathbb{R}^N} dx$$

the element $y_\varepsilon - y$ is orthogonal to the following closed subspace of $H^1_0(D)$:

$$H^1(D; \Omega^c) = \{\phi \in H^1(D) | \phi = 0 \text{ a.e. in } \Omega^c\} .$$

The element $y(\Omega)$ is the a–projection in $H^1_0(D)$ of $y_\varepsilon(\Omega)$ on the subspace $H^1(D;\Omega^c)$. It is seen from (2.51) that the element y_ε belongs to a bounded subset of the space $H^1_0(D)$.Therefore, there exists a weakly convergent subsequence $\{y_{\varepsilon_k}\}$, $y_{\varepsilon_k} \rightharpoonup z$ weakly in $H^1_0(D)$ as $k \to \infty$. From (2.54) it follows that $z \in H^1(D;\Omega^c)$. On the other hand, from (2.51) it is inferred that $y_{\varepsilon_k}|_{\Omega^c}$ converges weakly in the space $H^1(\Omega^c, \partial D) = \{\phi \in H^1(\Omega^c) | \phi = 0 \quad \text{on } \partial D\}$, $y_{\varepsilon_k}|_{\Omega^c} \rightharpoonup w$, where $w \in H^1(\Omega^c, \partial D)$. Making use of the compact embedding of the Sobolev space H^1 into L^2 for a bounded domain, one can show that y_{ε_k} and

$y_{\varepsilon_k}|_{\Omega^c}$ converge in $L^2(D)$ and $L^2(\Omega^c)$, respectively. Thus $w(x) = z(x)$ for a.e. $x \in \Omega^c$.

Since the mapping

$$w \to \int_{\Omega^c} \langle A \cdot \nabla w, \nabla w \rangle_{\mathbb{R}^N} dx$$

is weakly lower semi–continuous on $H^1(\Omega^c, \partial D)$, then passing to the limit, we we have that

$$\int_{\Omega^c} \langle A \cdot \nabla z, \nabla z \rangle_{\mathbb{R}^N} dx = 0 \quad z \in H^1(D; \Omega^c) \ .$$

The a–projection is a linear mapping, therefore it is continuous from $H_0^1(D)$–weak into $H^1(D; \Omega^c)$–weak. Hence $z = y(\Omega)$ and the sequence $\{y_\varepsilon(\Omega)\}$ converges weakly to $y(\Omega)^0$ as $\varepsilon \downarrow 0$. This proves the following result.

Proposition 2.10 *For any $f \in L^2(D)$ and any domain Ω of class C^1 in D, the sequence $\{y_\varepsilon(\Omega)\}_{\varepsilon>0}$ of solutions to the transmission problem (2.49) converges weakly in $H_0^1(D)$:*

$$y_\varepsilon(\Omega) \rightharpoonup y(\Omega)^0$$

as $\varepsilon \downarrow 0$, where $y(\Omega)^0$ denotes a weak solution to the homogeneous Dirichlet problem (2.47) and (2.48).

□

Using an extension operator $P \equiv P_{\Omega^c}$: $H^1(\Omega^c) \to H^1(D)$, we can improve the convergence result given by Proposition 2.10. Let $P \in \mathcal{L}(H^1(\Omega^c); H^1(D))$ be a linear mapping such that $(P\phi)|_{\Omega^c} = \phi$ for all $\phi \in H^1(\Omega^c)$. Assuming that Ω is smooth enough, so the mapping P exists, one can select the test function $\phi = P(y_\varepsilon|_{\Omega^c})$ in (2.49). It follows that

$$\frac{1}{\varepsilon}\int_{\Omega^c} \langle A \cdot \nabla y_\varepsilon, \nabla y_\varepsilon \rangle_{\mathbb{R}^N} dx = \int_D f P(y_\varepsilon|_{\Omega^c}) dx - \int_\Omega \langle A \cdot \nabla y_\varepsilon, \nabla P(y_\varepsilon|_{\Omega^c}) \rangle_{\mathbb{R}^N} dx \ .$$

Using

$$\|\nabla P(y_\varepsilon|_{\Omega^c})\|_{L^2(\Omega^c)} \leqq \|P(y_\varepsilon|_{\Omega^c})\|_{H_0^1(D)} \leqq C_1 \|P\| \|y_\varepsilon|_{\Omega^c}\|_{H^1(\Omega^c)}$$

we obtain

$$\frac{\alpha}{\varepsilon}\|\nabla y_\varepsilon\|_{L^2(\Omega^c)} \leqq (\|f\|_{L^2(D)} + C_2\|P\|)\|\nabla y_\varepsilon\|_{L^2(\Omega^c)},$$

where

$$C_2 = C_1 \|A\|_{L^\infty(D;\mathbb{R}^{N^2})}$$

hence

$$\|\nabla y_\varepsilon\|_{L^2(\Omega^c)} \leqq \varepsilon(\|f\|_{L^2(D)} + C_2\|P\|) \ .$$

From the foregoing it is evident that $\frac{1}{\varepsilon}\nabla y_\varepsilon$ belongs to a bounded subset of the space $L^2(\Omega^c;\mathbb{R}^N)$ and we can suppose that the sequence $\{\frac{1}{\varepsilon_k}\nabla y_{\varepsilon_k}\}$ converges weakly to $\nabla z \in L^2(\Omega^c;\mathbb{R}^N)$ with $z \in H^1(\Omega^c, \partial D)$ as $k \to \infty$. Passing to the limit in (2.49) we obtain

$$\int_{\Omega^c} \langle A \cdot \nabla z, \nabla\phi\rangle_{\mathbb{R}^N} dx + \int_\Omega \langle A \cdot \nabla y(\Omega), \nabla\phi\rangle_{\mathbb{R}^N} dx = \int_D f\phi dx \quad \forall\phi \in H_0^1(D) \ .$$

Applying Green's formula to the integral identity it follows that

$$\begin{aligned} -\mathrm{div}(A \cdot \nabla z) &= f \quad \text{in } \Omega^c, \\ \frac{\partial z}{\partial n_A} = \frac{\partial y}{\partial n_A} & \quad \text{on } \partial\Omega = \Gamma, \end{aligned}$$

where $n_A = A \cdot n$, and $z = 0$ on ∂D.

Assuming that $\phi = y_\varepsilon - y^0$ we obtain

$$\int_\Omega \langle A \cdot \nabla(y_{\varepsilon_k} - y^0), \nabla(y_{\varepsilon_k} - y^0)\rangle_{\mathbb{R}^N} dx =$$
$$\int_D f(y_{\varepsilon_k} - y^0)dx - \frac{1}{\varepsilon_k}\int_{\Omega^c} \langle A \cdot \nabla y_{\varepsilon_k}, \nabla y_{\varepsilon_k}\rangle_{\mathbb{R}^N} dx \ .$$

The right–hand side converges to zero as $k \to \infty$.

Therefore

$$\int_D \langle A \cdot \nabla(y_\varepsilon - y^0), \nabla(y_\varepsilon - y^0)\rangle_{\mathbb{R}^N} dx \leqq \int_D f(y_\varepsilon - y^0)dx + \varepsilon C_1 \|P\|^2 \ .$$

Proposition 2.10 implies that $y_\varepsilon \rightharpoonup y^0$ weakly in $H_0^1(D)$, therefore $y_\varepsilon \to y^0$ strongly in $L^2(D)$. This yields the proof of the following result.

Proposition 2.11 *Assuming that there exists a continuous extension operator $P \in \mathcal{L}(H^1(\Omega^c); H^1(D))$ we have that $y_\varepsilon \to y^0(\Omega)$ strongly in $H_0^1(D)$ as $\varepsilon \downarrow 0$.*

□

In Proposition 2.11 we assumed the existence of a continuous extension operator to show that the elements $\frac{1}{\varepsilon} y_\varepsilon|_{\Omega^c}$ are bounded in $H^1(\Omega^c; D)$ uniformly with respect to $\varepsilon > 0$. This assumption requires an additional smoothness of Γ and can be avoided by the application of the Banach–Steinhaus theorem since for any test function $\phi \in H_0^1(\Omega)$ from (2.49) it follows that the term

$$\frac{1}{\varepsilon}\int_{\Omega^c} \langle A \cdot \nabla y_\varepsilon, \nabla\phi\rangle_{\mathbb{R}^N} dx$$

is bounded uniformly with respect to $\varepsilon > 0$. The space $H^1(\Omega^c; \partial D)$ is equipped with the norm $a(\phi,\phi)^{\frac{1}{2}}$ associated with the matrix function A, and $\frac{1}{\varepsilon} y_\varepsilon$ is weakly bounded uniformly with respect to $\varepsilon > 0$ hence by the Banach–Steinhaus theorem, the term $\frac{1}{\varepsilon} y_\varepsilon$ is bounded in the norm of the space $H^1(\Omega^c; \partial D)$ uniformly with respect to $\varepsilon > 0$, and we have:

Proposition 2.12 *Assume that Ω is smooth enough, i.e. it can be supposed that y^0 belongs to $H_0^1(D)$, then $y_\varepsilon \to y(\Omega)^0$ strongly in $H_0^1(D)$ as $\varepsilon \downarrow 0$.*

□

In order to ensure the existence of an optimal shape for the related shape optimization problems an appropriate regularization can be used. First let us consider the perimeter $\mathcal{P}_D(\Omega)$ as the regularization term.

We shall consider a simple cost functional of the form

$$J(\Omega) = \frac{1}{2} \int_\Omega (y(\Omega) - y_g)^2 dx + \alpha \mathcal{P}_D(\Omega) \tag{2.55}$$

however, the method of finding an optimal solution, worked–out for this functional, is quite general and can be used for any cost functional of the form (2.42).

Let $y(\Omega)$ denote a weak solution to (2.47) and (2.48), where Ω is a sufficiently smooth domain, e.g. it can be assumed that Ω is a bounded domain of class C^1. Our aim is to determine the solution $y(\Omega)$ to the homogeneous Dirichlet boundary value problem for any measurable set Ω in D. For this purpose the following subspace of $H_0^1(D)$ is taken into consideration:

for any measurable set Ω in D we denote
$H^1(D; D \setminus \Omega) = \{\phi \in H_0^1(D) | \nabla \phi = 0 \text{ a.e. on } D \setminus \Omega\}$,

$H^1(D; D \setminus \Omega)$ is a closed linear subspace of the space $H_0^1(D)$. Furthermore, if Ω is a domain of class C^1 such that $\overline{\Omega} \subset D$, then $H^1(D; D \setminus \Omega) = H^1(\Omega)$; here each element $y \in H_0^1(\Omega)$ is identified with its extension y^0 in $H_0^1(D)$, where y^0 denotes the extension of y by zero on $D \setminus \Omega$.

The following cost functional

$$J_\varepsilon(\Omega) = \frac{1}{2} \int_D (y_\varepsilon(\Omega) - y_g)^2 dx + \alpha \mathcal{P}_D(\Omega)$$

is considered, where $y_\varepsilon = y_\varepsilon(\Omega)$ denotes a weak solution to (2.49) for $\varepsilon > 0$. Applying Theorem 2.3 to the penalized problem (2.49), one can show that there exists a set Ω_ε with the finite perimeter $\mathcal{P}_D(\Omega_\varepsilon)$ in D such that $J_\varepsilon(\Omega_\varepsilon) \leqq J_\varepsilon(\Omega)$ for all measurable sets Ω in D.

From Proposition 2.12 it follows that

$$J_\varepsilon(\Omega) \to J(\Omega) \quad \text{as } \varepsilon \downarrow 0$$

for any smooth domain Ω.

Unfortunately we cannot claim that there exists a cluster point of the family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ which is a smooth domain. On the other hand there exists a set $\Omega^* \subset D$ with the finite perimeter in D and a sequence $\{\Omega_{\varepsilon_k}\}_{k=1}^\infty$, $\varepsilon_k \downarrow 0$ as $k \to \infty$, which converges to the set Ω^*. However in general the corresponding sequence $\{y_k\}_{k=1}^\infty$, $y_k = y_{\varepsilon_k}(\Omega_{\varepsilon_k})$, need not to converge to the element $y(\Omega^*)$ as $\varepsilon_k \downarrow 0$.

Therefore we shall consider the continuity of the mapping $\Omega \to y(\Omega)$ to obtain an existence result for the shape optimization problem with the cost functional depending on a weak solution $y(\Omega)$ to the Dirichlet problem (2.47) and (2.48). To ensure the continuity of this mapping the family of admissible subdomains of D should be equipped with a suitable topology. As we have seen in the case of the transmission problem the suitable topology is the $L^2(D)$ topology for characteristic functions. In the case of the Dirichlet problem this topology leads in particular to the notion of the Sobolev space $H_0^1(\Omega)$ where Ω is only a measurable subset of D and cannot be used for our purposes.

We may expect that the Hausdorff topology would ensure the existence of an optimal domain. We shall briefly explain why the Hausdorff topology is not appropriate for the existence problem.

The distance between two closed subsets $A, B \subset D$ is given by

$$d(A,B) = \sup(\varrho(A,B), \varrho(B,A)),$$

where

$$\varrho(A,B) = \max_{x\in A} \min_{y\in B} \|x-y\|_{\mathbb{R}^N} \ .$$

For a given sequence of open sets

$$\Omega_k = D \setminus A_k, \quad \text{with } \overline{A}_k = A_k \ , \ k = 1,2,...,$$

Ω is a limit, which we denote $\Omega_k \xrightarrow{H} \Omega$, provided that

$$\Omega = D \setminus A, \ \overline{A} = A$$
$$\lim_{k\to\infty} d(A, A_k) = 0 \ .$$

For a given open set Ω, the Sobolev space $H_0^1(\Omega)$ is the closure in the norm of $H^1(\Omega)$ of the space $C_0^\infty(\Omega)$. The space $H_0^1(\Omega)$ is a closed subspace of the Sobolev space $H_0^1(D)$, $\Omega \subset D$.

Let $f \in L^2(D)$ be a given element, $\{\Omega_k\}$ a family of domains such that $\Omega_k \xrightarrow{H} \Omega$ as $k \to \infty$. Let us consider the Dirichlet problem defined in the domain Ω_k

$$y_k \in H_0^1(\Omega_k): \quad \int_{\Omega_k} \nabla y_k \cdot \nabla\phi dx = \int_{\Omega_k} f\phi dx \quad \forall \phi \in H_0^1(\Omega_k) \ .$$

It is easy to show that there exists an element $\overline{y} \in H_0^1(D)$ such that for a subsequence also denoted $\{y_k\}$

$$y_k \rightharpoonup \overline{y} \quad \text{weakly in} \quad H_0^1(D)$$

as $k \to \infty$, furthermore

$$\int_\Omega \nabla\overline{y} \cdot \nabla\phi dx = \int_\Omega f\phi dx \quad \forall\phi \in H_0^1(\Omega) \ .$$

Unfortunately we cannot claim that $\overline{y} \in H_0^1(\Omega)$ i.e. $\overline{y}$ solves the Dirichlet problem in Ω. Let us observe that for $\overline{y} \in H_0^1(D)$, the set

$$Z = \{x \in \overline{D} | \ \overline{y}(x) = 0\}$$

is not in general closed. Finally, let us describe the regularization method which ensures the existence of an optimal domain for the Dirichlet problem. The regularization method leads to the stronger topology, compared to the Hausdorff topology, on the family of subdomains Ω of D.

Let $(x, x_{N+1}) \in \mathbb{R}^{N+1}$, and denote

$$\overline{D} = \overline{Q} \times [0, L] \quad L > 0,$$

where $Q \subset \mathbb{R}^N$ is a sufficiently smooth bounded domain and L is a positive constant. Any admissible domain Ω takes the form of a hypograph in D

$$\Omega \equiv \Omega_f = \{(x, x_{N+1}) \in \mathbb{R}^{N+1} | \ x \in Q \quad 0 < x_{N+1} < f(x) \leqq L\},$$

where f is an element of the convex set

$$K_m = \{f \in H^m(Q) | \ 0 < \alpha \leqq f(x) \leqq L \quad x \in Q\} \ .$$

The regularizing term is defined as follows

$$\mathcal{E}_m(\Omega) = \|f\|^2_{H^m(Q)} \ .$$

We assume for simplicity that $m > N$, therefore by the Sobolev embedding theorem $H^m(Q) \hookrightarrow C^1(\overline{\Omega})$. We have the following lemmae, the proofs are omitted here.

Lemma 2.13 *If* $\{f_k\} \subset K_m$ *and*

$$f_k \rightharpoonup f \quad \textit{weakly in} \quad H^m(Q)$$

then

$$\Omega_{f_k} \xrightarrow{H} \Omega_f \ .$$

□

Lemma 2.14 *Let us assume*

(i) $\{f_k\} \subset K_m$, $f_k \rightharpoonup f$ *weakly in* $H^m(Q)$
(ii) $y_k \in H_0^1(\Omega_{f_k})$, $\|y_k\|_{H_0^1(D)} \leqq C$

then there exists a subsequence also denoted $\{y_k\}$ *and an element* $y \in H_0^1(\Omega_f)$ *such that*

$$y_k \rightharpoonup y \quad \textit{weakly in} \quad H_0^1(D) \ .$$

□

Using the above lemmae we obtain that the mapping $f \to y(\Omega_f)$ is continuous from $H^m(Q)$–weak into $H_0^1(D)$–weak and therefore there exists an element $f^\star \in K_m$ such that the following domain functional

$$J(\Omega_f) = \frac{1}{2}\int_{\Omega_f} (y(\Omega_f) - y_g)^2 dx + \alpha \mathcal{E}_m(\Omega_f)$$

attains its minimum $J(\Omega_{f^\star})$ on the set of admissible domains

$$\mathcal{U}_{\text{ad}} \equiv \{\Omega | \Omega = \Omega_f \text{ for some } f \in K_m\} \ .$$

2.7. Convergence of domains

In the previous section the set Char(D) equipped with the $L^2(D)$ topology was introduced and the compact sets Char(D, M) depending on a constant M were defined. Unfortunately the convergence of characteristic functions does not preserve the regularity of the domains. Stronger topologies on families of domains are to be defined in order to ensure the convergence of normal fields, curvatures, etc.

The principal idea used here for constructions of such topologies was introduced by Micheletti (1972). Let us consider two domains Ω_1 and Ω_2 of class C^k in $D \subset \mathbb{R}^N$, we have to assume that there exists a transformation T defined on D, which maps D onto D, such that T and T^{-1} are in $C^k(\overline{D})$ and $T(\Omega_1) = \Omega_2$. Such a transformation is not unique, therefore we shall restrict our consideration to the transformation T with the minimal norm

$$\|T - \mathcal{I}\| + \|T^{-1} - \mathcal{I}\| \tag{2.57}$$

among all T such that $T(\Omega_1) = \Omega_2$. If the minimal value of the norm is zero, then $T = \mathcal{I}$ (the identity mapping) and $\Omega_2 = \Omega_1$. If this value is small, then Ω_2 is close to Ω_1 in the sense of the topology defined here.

Therefore one has to construct the mapping T such that $T(\Omega_1) = \Omega_2$ for a given simply connected domains Ω_1 and Ω_2.

Let Ω_1 and Ω_2 be two bounded and simply connected domains in D (or in $\mathbb{R}^N$). The following two situations are to be distinguished:

(i) $\overline{\Omega}_1 \subset \Omega_2$, (2.58)

(ii) Ω_1 is starshaped with respect to a given point x_0. (2.59)

Let us assume that $\overline{\Omega}_1 \subset \Omega_2$; in order to construct the mapping T the following boundary value problem is considered:

Find $z \in H^1(\Omega_2 \setminus \overline{\Omega}_1)$ such that

$$\Delta z = 0 \quad \text{in } \Omega_2 \setminus \overline{\Omega}_1, \tag{2.60}$$

$$z = 0 \quad \text{on } \Gamma_1 = \partial\Omega_1, \tag{2.61}$$

$$z = 1 \quad \text{on } \Gamma_2 = \partial\Omega_2 \ . \tag{2.62}$$

The domain $\Omega_2 \setminus \overline{\Omega}_1$ is not simply connected. Two parts Γ_1 and Γ_2 of the boundary $\partial\Omega$ of the domain $\Omega_2 \setminus \overline{\Omega}_1$ can be considered as the level curves of the solution z to the problem (2.60)–(2.62). Let us examine the family of level curves

$$z^{-1}(t) = \{x \in \Omega_2 \setminus \overline{\Omega}_1 | z(x) = t\} \quad 0 \leqq t \leqq 1 \ .$$

It follows that

$$\Gamma_1 = z^{-1}(0) \quad \text{and} \quad \Gamma_2 = z^{-1}(1) \ .$$

For a fixed t, $0 \leqq t \leqq 1$, the open set $\mathcal{O}_t = \{x \in \Omega_2 \setminus \overline{\Omega}_1 | 0 < z(x) < t\}$ is well defined. It is assumed that this set has the following properties:

$$\mathcal{O}_0 = \emptyset,$$

$$\mathcal{O}_1 = \Omega_2 \setminus \overline{\Omega}_1 \ ,$$

and $t \to \mathcal{O}_t$ is a monotone increasing family of sets.

From (2.60)–(2.62) it follows that z is a harmonic function in $\mathcal{O}_t$ which attains the maximal value on the boundary $z^{-1}(t) \subset \partial\mathcal{O}_t$, i.e. the maximal value of z in $\mathcal{O}_t$ is obtained at any point B of the set $z^{-1}(t)$. From the maximum principle, see e.g. (Protter et al. 1984), it can be inferred that at any boundary point B, where the maximum of z is obtained, we have $\frac{\partial z}{\partial n}(B) > 0$. Furthermore, since $z^{-1}(t)$ is the level set of z, we get

$$\frac{\partial z}{\partial n}(B) = \|\nabla z(B)\|_{\mathbb{R}^N} \quad \text{on } z^{-1}(t) \ .$$

Thus for $0 < t \leqq 1$ we have

$$\min_{B \in z^{-1}(t)} \|\nabla z(B)\|_{\mathbb{R}^N} > 0 \ .$$

z attains its minimum on $\Gamma_1 = z^{-1}(0)$, hence by the same argument as above applied to $-z$, one can show that

$$\min_{B\in\Gamma_1} \|\nabla z(B)\|_{\mathbb{R}^N} > 0 \ .$$

Whence for the gradient ∇z on $\overline{\mathcal{O}}_1$ we have

$$\min_{B\in\overline{\mathcal{O}}_1} \|\nabla z(B)\|_{\mathbb{R}^N} > 0 \ . \tag{2.63}$$

Following (Zolesio 1976), we introduce the autonomeous field

$$V(x) = \|\nabla z(x)\|_{\mathbb{R}^N}^{-2} \nabla z(x)$$

and the flow associated with this field:

$$T_t(X) = e^{tV}(X) \ . \tag{2.64}$$

In other words, $T_t(X) = x(t,X)$, where $x(\cdot,\cdot)$ denotes a solution to the system of ordinary differential equations

$$\frac{d}{dt}x(t,X) = V(x(t,X)), \tag{2.65}$$

where $X \in \Gamma_1$, and the initial condition is given by

$$x(0,X) = X \ . \tag{2.66}$$

Proposition 2.15 *The transformations T_t, $t \in [0,1]$, have the following properties:*

$$T_0 = \mathcal{I} \text{ (the identity mapping on } \Gamma_1)$$

T_t mapps Γ_1 onto $z^{-1}(t)$, that is

$$T_t(\Gamma_1) = z^{-1}(t) \text{ for } 0 < t \leqq 1$$

and in particular for $t = 1$ we get

$$T_t(\Gamma_1) = \Gamma_2 \ .$$

Proof. Let X be a point in Γ_1 and consider the solution $x = x(t,X)$ defined by the system (2.65) and (2.66), we have to prove that $z(x(t,X)) = t$, in view of (2.60)–(2.62) the initial condition becomes $z(X) = 0$, then it is enough to show that the derivative of the mapping $t \longrightarrow z(x(t,X))$ is equal to the identity. But

$$\frac{d}{dt}z(x(t,X)) = \nabla z(x(t,X)) \cdot \frac{d}{dt}x(t,X)$$

and

$$\frac{d}{dt}x(t,X) = V(x(t,X)) = \|\nabla z(x)\|_{\mathbb{R}^N}^{-2} \nabla z(x)$$

hence

$$\frac{d}{dt}z(x(t,X)) = \mathcal{I} \ .$$

From the classical results on the regularity of solutions to the Laplace equation (2.60)–(2.62) (see e.g. (Necas 1967)) it can be inferred that if Γ_1 and Γ_2 are C^{k+1}, then z is in $C^k(\overline{U})$ (with $U = \Omega_2 \setminus \overline{\Omega}_1$) and the field V is in $C^{k-1}(\overline{U};\mathbb{R}^N)$. By the classical results of the theory of ordinary differential equations, the mapping $X \to T_t(X) = x(t,X)$ and its inverse are in $C^{k-1}(\mathbb{R}^N;\mathbb{R}^N)$ (it is assumed here that V is extended to $\mathbb{R}^N$ as an element of $C^{k-1}(\mathbb{R}^N;\mathbb{R}^N)$).

In particular, if Γ_1 is C^{k+1}, then $z^{-1}(t) = T_t(\Gamma_1)$ is C^{k-1}. In the case of the system of differential equations (2.65) and (2.66) one can assume that $k \geqq 2$ under the condition that the vector field V is in $C^{k-1}(\mathbb{R}^N;\mathbb{R}^N)$.

If, e.g. Γ_1 is a C^3 manifold, then using the method of level sets of the solution to (2.60)–(2.62), the resulting level sets $z^{-1}(t)$, $0 < t \leqq 1$, are a priori only C^1 manifolds. However, the level set $z^{-1}(0) = \Gamma_1$ is given as a C^3 manifold.

In fact for any t, $0 < t < 1$, the level set $z^{-1}(t)$ is included in the interior of the domain $\Omega_2 \setminus \overline{\Omega}_1$, thus by the standard elliptic regularity result for the solutions to (2.60)–(2.62) it follows that $z(\cdot)$ is C^∞ in a neighbourhood of $z^{-1}(t)$.

□

Assuming that the field V is extended to $\mathbb{R}^N$ and this extension is in the space $C^1(\mathbb{R}^N;\mathbb{R}^N)$, one can show that the transformation T_t maps Ω_1 onto Ω_2.

In the second case, i.e. when $\overline{\Omega}_1$ is not included in the domain Ω_2, one has to apply an appropriate transformation making it possible to reduce the case under consideration to the first one. Ω_2 is assumed to be the starshaped domain with respect to the point x_0. For simplicity it is assumed that $x_0 = 0$ and the domain Ω_2^r is defined by $\Omega_2^r = r\Omega_2$ with $r > 0$, r large enough, such that $\overline{\Omega}_1 \subset \Omega_2^r$ (such r exists, since $\overline{\Omega}_1 \subset \mathbb{R}^N$ is compact and Ω_2 is an open set with $0 \in \Omega_2$).

2.8. Transformations T_t of domains

To carry out the sensitivity analysis of the shape functionals $J(\Omega)$ one needs to introduce a family of perturbations $\{\Omega_t\}$ of a given domain $\Omega \subset \mathbb{R}^N$ for $0 \leqq t < \varepsilon$. It is assumed that the domains $\Omega \equiv \Omega_0$ and Ω_t for $0 < t < \varepsilon$ have the same topological properties and possess the same regularity, e.g. Ω, and Ω_t for $0 < t < \varepsilon$ are simply connected domains of class C^k, where $k \geqq 1$. Hence one can construct a family of transformations $T_t\colon \mathbb{R}^N \to \mathbb{R}^N$ for $0 \leqq t < \varepsilon$ which are one–to–one, and T_t maps Ω onto Ω_t.

It is supposed that for this family of transformations the following conditions are satisfied.

(A1) $T_t(\cdot)$ and $T_t^{-1}(\cdot)$ belong to $C^k(\mathbb{R}^N;\mathbb{R}^N)$ for all $t \in [0,\varepsilon)$.

(A2) The mappings $t \to T_t(x)$ and $t \to T_t^{-1}(x)$ belong to $C^1([0,\varepsilon))$ for all $x \in \mathbb{R}^N$.

Equivalently, it can be assumed that a domain $\Omega \subset \mathbb{R}^N$ and the family of transformations $T_t: \mathbb{R}^N \to \mathbb{R}^N$ are given. The family of domains $\{\Omega_t\}$ is then defined by $\Omega_t = T_t(\Omega)$. It is obvious that for a given family of domains $\{\Omega_t\}$, the family T_t of transformations is not uniquely determined. Since only small deformations (or perturbations) of Ω are considered, we can make use in fact of the transformations which are defined as the perturbations of the identity mapping $\mathcal{I}$ in $\mathbb{R}^N$. An example of such transformations can be as follows

$$T_t = \mathcal{I} + t\Theta,$$

where Θ is a smooth vector field defined on $\mathbb{R}^N$

$$\Theta \in W^{k,\infty}(\mathbb{R}^N, \mathbb{R}^N) \quad \text{or} \quad \Theta \in C^k(\mathbb{R}^N, \mathbb{R}^N).$$

The above parameterization of domains was studied, e.g. by Murat et al. (1976) and Pironneau (1984).

A classical tool making it possible to construct the mapping T_t is to consider domains which are hypographs, but this approach is rather restrictive. We denote $x = (x', x_N) \in \mathbb{R}^N$, where $x' = (x_1, \ldots, x_{N-1})$. Let $Q \subset \mathbb{R}^{N-1}$ be a bounded domain and suppose that $f \in C^k(\overline{Q})$,

$$\min\{f(x) | x \in \overline{Q}\} > 0 \ .$$

The domain $\Omega_f \equiv \Omega(f) \subset \mathbb{R}^N$ associated with f is defined as follows:

$$\Omega(f) = \{x \in \mathbb{R}^N | x' \in Q \quad 0 < x_N < f(x')\} \ .$$

The boundary $\partial\Omega(f)$ of the domain $\Omega(f)$ is by definition only piecewise smooth. The graph $\Gamma_f = \{(x', f(x')) \in \mathbb{R}^N | x' \in \overline{Q}\}$ is the part of the boundary $\partial\Omega(f)$ which depends on f. For any element g given in $C^k(\overline{Q})$ there exists $\varepsilon > 0$ such that

$$f + tg \quad \text{is admissible for} \quad |t| < \varepsilon$$

in the sense that

$$\min\{(f + tg)(x) | x \in \overline{Q}\} > 0 \ .$$

The following notation is introduced

$$\Omega_t = \Omega(f + tg) \ .$$

Thus the transformation T_t can be constructed,

$$T_t(x', x_N) = \left(x', x_N \frac{f(x') + tg(x')}{f(x')}\right) \tag{2.67}$$

with the inverse

$$T_t^{-1}(x', y_N) = \left(x', y_N \frac{f(x')}{f(x') + tg(x')}\right) \tag{2.68}$$

and it turns out that T_t is a one–to–one mapping from $\overline{Q}\times[0,\infty)$ onto $\overline{Q}\times[0,\infty)$.

A particular case of the parameterization of domains relies on the use of the polar coordinates.

Let f_0 and f be two periodic functions, $f_0,\ f\in C^k([0,2\Pi])$, the domain $\Omega(f)$ is defined by

$$\Omega(f)=\{(\rho,\theta)\in \mathbb{R}^2|0\leqq\theta<2\Pi,\ f_0(\theta)<\rho<f(\theta)\}\ .$$

It is supposed that

$$\min\{f_0(\theta)|\theta\in[0,2\Pi]\}>0$$

and

$$\min\{(f-f_0)(\theta)|\theta\in[0,2\Pi]\}>0\ .$$

For any admissible element $g\in C^k([0,2\Pi])$ with $g(0)=g(2\Pi)$ there exists $\varepsilon>0$,

$$\min\{(f+tg-f_0)(\theta)|\theta\in[0,2\Pi]\}>0$$

for t, $|t|<\varepsilon$. We denote

$$\Omega_t=\Omega(f+tg)\ .$$

The transformation T_t is as follows

$$T_t(\rho,\theta)=\left(f_0(\theta)+(\rho-f_0(\theta))\frac{(f+tg)(\theta)-f_0(\theta)}{f(\theta)-f_0(\theta)},\theta\right)\ .$$

Numerous examples of parameterizations of domains can be described as follows. The function f, used to define the domain $\Omega(f)\subset\mathbb{R}^N$, depends on a vector parameter a, i.e. $f=f(a)$, ($a\in\mathbb{R}^N$ or a is an element of an open set A in a Banach space). Hence $\Omega(a)=\Omega(f(a))$. Moreover let us consider an admissible direction b of the parameter a and assume that

$$\Omega_t=\Omega(t)=\Omega(a+tb)\ .$$

The associated transformation T_t is defined in the same way as before.

For the particular case of domains of class C^∞ (the boundary Γ of the domain Ω is a smooth manifold, Ω is located on one side of Γ) we can apply the parameterization of domains proposed by Hadamard (1908). This approach has been used by several authors, e.g. Dems, Mróz, Murat, Pironeau, Rousselet and Simon.

To present this method we describe briefly the Hadamard parameterization for a smooth domain Ω in $\mathbb{R}^N$. The normal field n on Γ is in $C^\infty(\Gamma;\mathbb{R}^N)$, let $g\in C^\infty(\Gamma)$ be a given element; since Γ is assumed to be compact, then there exists $\varepsilon>0$ such that for any t, $|t|<\varepsilon$,

$$\Gamma_t=\Gamma+tgn=\{y|y=x+tg(x)n(x)\quad\text{for } x\in\Gamma\}$$

is the boundary of the domain Ω_t of class C^∞. Making use of an extension $\mathcal{N}_0$ to $\mathbb{R}^N$ of the normal vector field n defined on Γ, $\mathcal{N}_0 \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$, we can define the transformation $T_t(x) = x + tg_0(x)\mathcal{N}_0(x)$, where g_0 denotes an extension of $g \in C^\infty(\Gamma)$ to $\mathbb{R}^N$, $g_0 \in C^\infty(\mathbb{R}^N)$.

Thus the transformation constructed is a particular case of that considered in the foregoing example. Let us assume that $\Phi = g_0\mathcal{N}_0$, that is Φ is proportional to the normal field n on Γ.

Observe that for any $\Psi \in W^{k,\infty}(\mathbb{R}^N; \mathbb{R}^N)$, there exists $\varepsilon > 0$ such that for t, $|t| < \varepsilon$, the transformation $T_t = \mathcal{I} + t\Psi$ is one–to–one. If Ψ is a linear mapping then the inverse T_t^{-1} is defined by

$$T_t^{-1} = \sum_{k=0}^{\infty}(-1)^k t^k \Psi^k, \tag{2.69}$$

where Ψ^k denotes the composition $\Psi \circ \Psi \circ \ldots \Psi$ of the mapping $\Psi\colon \mathbb{R}^N \to \mathbb{R}^N$, k times. It is worth noting that the structure of the mapping T_t^{-1} is not the same as that of the mapping T_t and, in particular, T_t^{-1} is not associated with the element $-\Psi$.

Equivalently for the linear mapping $\Phi \in W^{k,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ proportional to the normal field n on Γ,

$$x_t = x + tg(x)n(x) \in \Gamma_t \ ,$$

where $x \in \Gamma$.

However, if n_t is the normal field on Γ, then the vector $y = x_t - tg(x)n_t(x_t) \in \mathbb{R}^N$ is different from $x \in \mathbb{R}^N$ and in general the element y does not belong to the boundary Γ_t of the domain Ω_t.

If the domain Ω is of class C^k, then the normal field n on $\Gamma = \partial\Omega$ is only of the class C^{k-1}. Hence the Hadamard parameterization results in the perturbed domain Ω_t of class C^{k-1}. Therefore the Hadamard parameterization preserves the class of domains only for C^∞ domains.

The parameterization of domains can be also associated with the level curves of a given function. In fact, there are two possible cases that are of some interest for our purposes. Let z be a smooth function

$$z \in C^1(D) \quad \text{with} \quad \|\nabla z(x)\|_{\mathbb{R}^N} > 0$$

for all $x \in D \setminus \{x_z\}$, where x_z is a given element and D is a smooth bounded domain in $\mathbb{R}^N$.

Suppose that z attains its minimum on $\overline{D}$ at the point x_z and assume that

$$\Gamma_t = z^{-1}(t) = \{x \in D | z(x) = t\} \tag{2.70}$$

$$\Omega_t = \{x \in D | z(x) < t\}. \tag{2.71}$$

In the second case we assume that a family of functions $\{z_t\}$ depending on the parameter t is given. The function z_t is defined on the fixed domain $\overline{D}$ for each $t \in [0, \delta)$, in this case $\Gamma_t = z_t^{-1}(1)$, and

$$\Omega_t = \{x \in D | z(x) < 1\}.$$

For these two important situations we shall derive in the next section the explicit expressions for the related transformations $T_t: \overline{D} \to D$ which map the domain Ω onto domains Ω_t.

The domains defined by the level curves of a given function or a given family of functions occur in problems, where the notion of the monotone rearrangement is involved as well as in problems, where the controlability of the free boundary is considered, e.g. the controlability of the boundaries of coincidence sets of the solutions to obstacle problems (see Chap. 4). We have already introduced in Sect. 2.7 the set $z^{-1}(t)$ to construct the transformation T_t which maps any bounded simply connected domain Ω_1 onto the given domain Ω_2 using an autonomous vector field V selected in an appropriate way. T_t is obtained as the exponential mapping $T_t = e^{tV}$. This mapping has the following property: the inverse T_t^{-1} is associated with the vector field $-V$. Let us observe that for the vector field V which belongs to $W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ the differential equation

$$\frac{d}{dt}x(t) = V(x(t)) \tag{2.72}$$

is locally well–posed. In the next section a generalization of this method is presented. It relies on the use of time dependent vector fields (speed fields) V (non autonomous vector fields V).

2.9. The speed method

Let us consider the general case of constructing the transformation T_t. Let D be a domain in $\mathbb{R}^N$ (here D is not supposed to be bounded) with the boundary ∂D piecewise C^k for a given integer $k \geqq 0$.

Let T_t be a one–to–one mapping from $\overline{D}$ onto $\overline{D}$ such that

$$T_t \quad \text{and} \quad T_t^{-1} \quad \text{belong to } C^k(\overline{D}; \mathbb{R}^N) \tag{2.73}$$

and

$$t \to T_t(x), T_t^{-1}(x) \in C([0, \varepsilon)) \quad \forall x \in \overline{D} \tag{2.74}$$

thus $(t, x) \to T_t(x) \in C([0, \varepsilon); C^k(\overline{D}; \mathbb{R}^N)) \equiv C(0, \varepsilon; C^k(\overline{D}; \mathbb{R}^N))$.

For any $X \in \overline{D}$ and $t > 0$ the point $x(t) = T_t(X)$ moves along the trajectory $x(\cdot)$ with the velocity

$$\left\| \frac{d}{dt} x(t) \right\|_{\mathbb{R}^N} = \left\| \frac{\partial}{\partial t} T_t(X) \right\|_{\mathbb{R}^N} .$$

The point X may be thought of as the Lagrangian (or material) coordinate while x is the Eulerian (or actual) coordinate. The speed vector field $V(t, x(t))$ at the point $x(t)$ is to be defined in the Eulerian coordinates; therefore it is assumed that $V(t, x)$ has the form as follows:

$$V(t, x) = \left(\frac{\partial}{\partial t} T_t \right) \circ T_t^{-1}(x) . \tag{2.75}$$

It is obvious that from (2.73) and (2.74) it can be inferred that the vector field $V(t)$, defined as $V(t)(x) = V(t, x)$, satisfies the relation

$$V \in C(0, \varepsilon; C^k(\overline{D}; \mathbb{R}^N)) . \tag{2.76}$$

If V is a vector field such that (2.76) holds, then the transformation T_t depending on V, and such that conditions (2.73) and (2.74) are satisfied, is defined below.

Let $x = x(t, X)$ denote the solution to the system of ordinary differential equations

$$\frac{d}{dt} x(t, X) = V(t, x(t, X)), \tag{2.77}$$

$$x(0, X) = X . \tag{2.78}$$

Using the classical results, see (Dieudonne 1970), one can show the local existence of a solution to the system (2.77) and (2.78) in the following way.

For any $z \in D$ there exists a neighbourhood U_z of z in D, an interval $I(z)$ in $\mathbb{R}_+$, $0 \in I(z)$, and the mapping $x: I(z) \times U_z \ni (t, X) \to x(t, X) \in D$ such that for any $X \in U_z$ the mapping $t \to x(t, X)$ is the unique solution to (2.77) and (2.78) defined for $t \in I(z)$.

If the domain Ω is bounded, then $\overline{\Omega}$ can be covered by a finite family of open sets $U_i = U_{z_i}$. Let $U = \cup_i U_i$ and $I = \cap_i I(z_i)$, hence there exists the unique mapping $x: I \times U \to D$ such that for any X in U, $t \to x(t, X)$ is the solution to (2.77) and (2.78), where U is a neighbourhood of $\overline{\Omega}$ in D. The classical regularity result, see e.g. (Dieudonne 1970), implies the existence of the partial derivatives

$$\left(\frac{\partial}{\partial X} \right)^{\alpha} x(t, X), \quad |\alpha| = \alpha_1 + \ldots + \alpha_N \leqq k,$$

with

$$\left(\frac{\partial}{\partial X} \right)^{\alpha} x(t, X) \in C(I; C^{k-|\alpha|}(U; \mathbb{R}^N)).$$

If $D \subset \mathbb{R}^N$ is a bounded domain, then we set $U = D$ and use the same argument. In order to ensure that the mapping $X \to x(t, X)$ maps $\overline{D}$ onto $\overline{D}$, we need the following assumption.

Let the boundary ∂D of the domain D be piecewise smooth, suppose that the normal field $n = n(x)$ exists for a.e. $x \in \partial D$ and the vector field $V(t) = V(t, \cdot)$ satisfies the condition

$$V(t,x) \cdot n(x) = 0 \quad \text{for a.e.} \quad x \in \partial D \ . \tag{2.79}$$

If $n = n(x)$ is not defined at a singular point $x \in \partial D$ (i.e. ∂D is non–smooth at x) we set

$$V(t,x) = 0 \ . \tag{2.80}$$

It is clear that if T_t maps $\overline{D}$ onto $\overline{D}$, and for T_t holds (2.73) and (2.74), then the vector field V defined by (2.75) satisfies (2.77), (2.78) (with $x(t) = T_t(X)$), and (2.79), (2.80). Thus we can state the following theorem.

Theorem 2.16 *Let D be a bounded domain in $\mathbb{R}^N$ with the piecewise smooth boundary ∂D, and $V \in C(0,\varepsilon; C^k(\overline{D};\mathbb{R}^N))$ be a given vector field which satisfies (2.79) and (2.80).*

Then there exists an interval I, $0 \in I$, and the one–to–one transformation $T_t(V)\colon \mathbb{R}^N \to \mathbb{R}^N$ such that $T_t(V)$ maps $\overline{D}$ onto $\overline{D}$. Furthermore $T_t(V)$ satisfies conditions (2.73), (2.74) and (2.77), (2.78). In particular the vector field V can be written in the form

$$V = \partial_t T_t(V) \circ T_t(V)^{-1} \equiv \frac{\partial T_t}{\partial t}(V) \circ T_t(V)^{-1} \ .$$

On the other hand, if T_t is a transformation of $\overline{D}$, T_t satisfies (2.73) and (2.74) and V is defined by the formula $V = \partial_t T_t \circ T_t^{-1}$, then (2.79) and (2.80) hold for V. Furthermore $V \in C(0,\varepsilon; C^k(\overline{D};\mathbb{R}^N))$ and the transformation $T_t(X) = x(t,X)$ is defined as the local solution to the system of ordinary differential equations (2.77) and (2.78), that is $T_t = T_t(V)$.

Proof. We give only the proof of the first part of Theorem 2.16. Therefore one has to show that the inverse mapping T_t^{-1} exists and has the same properties as T_t.

Let V_t be given,

$$V_t(s) = V(t-s) \quad t > 0 \ .$$

It can be easily verified that the transformation T_t^{-1} is associated (via the problem (2.77) and (2.78)) with the vector field $-V_t$, i.e. $T_t(V)^{-1} = T_t(-V_t)$; hence $T_t(V)^{-1}$ belongs to $C^k(\overline{D};\mathbb{R}^N)$. In order to prove the continuity of T_t^{-1} with respect to t, i.e. to show that $T_t(V)^{-1}$ belongs to $C(I; C^k(\overline{D};\mathbb{R}^N))$, we make use of the continuity of the solution to (2.77) and (2.78) with respect to the initial data X and the vector field $V \in \mathcal{C}^{0,k}$, since the mapping $t \to V_t$ is continuous from I into $C(I; C^k(\overline{D};\mathbb{R}^N))$. To prove the regularity of the derivative $D_x X$, where $X = X(t,x) = T_t(-V_t)(x)$, with respect to X, we apply the above continuity of the mapping $(t \to V_t)$ to the system of linear differential equations whose solution is the partial derivative $(\frac{\partial}{\partial x})^\alpha X$. In particular for the first order derivatives we have

$$\frac{d}{ds} D_x X(s) = -DV_t(s, X(s,x)) \cdot D_x X(s) \tag{2.81}$$

$$D_x X(0) = \mathcal{I} \ . \tag{2.82}$$

□

For the transformation T_t investigated in Sect. 2.8, we shall derive the formulae for the associated speed vector fields. First, let us consider the transformation $T_t = \mathcal{I} + t\Phi$; hence

$$\partial_t T_t = \Phi \ .$$

Therefore

$$V(t,x) = \Phi \circ (\mathcal{I} + t\Phi)^{-1}(x) \ .$$

Furthermore, from (2.70) it can be inferred that for the domains $\Omega_t = \Omega(f + tg)$ defined in Sect. 2.8, the following relation holds

$$\partial_t T_t(x', x_N) = (0, x_N g(x')/f(x')) \ .$$

Thus

$$V(t,x) = \left(0, x_N \frac{g(x')}{f(x') + tg(x')}\right) \ .$$

Finally, if the domains Ω_t are defined by the level curves $z^{-1}(t)$, then as it was already shown the speed vector field is defined by the formula

$$V(t,x) = \|\nabla z\|^{-2}_{\mathbb{R}^N}(x) \nabla z(x),$$

here $V(t,x) = V(x)$ is an autonomous vector field (independent of t). On the other hand, if the domain Ω_t is defined by the level curves $z_t^{-1}(1)$, then it can be shown (see Chap. 4) that the speed vector field takes the form

$$V(t,x) = -z_t'(x) \|\nabla z_t\|^{-2}_{\mathbb{R}^N}(x) \nabla z_t(x),$$

where $z_t'(x)$ denotes the partial derivative $\frac{\partial}{\partial t} z_t(x)$.

In the next section we shall consider transformations defined by smooth speed vector fields with compact supports

$$V \in C(0, \varepsilon; \mathcal{D}(D; \mathbb{R}^N)) \ .$$

For such a choice of the vector field V, no additional assumption on the open set D (here D is not supposed to be bounded, its boundary ∂D is not assumed to be smooth) is needed to ensure the existence of the family

$$T_t(V) \quad t \in [0, \delta] \quad \text{for some} \quad \delta > 0.$$

2.10. Admissible speed vector fields $V^k(D)$

From Theorem 2.16, under assumption that the domain $D \subset \mathbb{R}^N$ is bounded, it follows that the one–to–one transformation $T_t(V)$ of $\overline{D}$ exists for any vector field $V \in C(0,\varepsilon; C^k(\overline{D};\mathbb{R}^N))$ which satisfy the conditions (2.79) and (2.80).

The shape sensitivity analysis carried on in the next sections will not be restricted to the domains Ω included in a given bounded domain $D \subset \mathbb{R}^N$. In particular, our analysis will include the case of $D = \mathbb{R}^N$.

Theorem 2.16, and conditions (2.79) and (2.80) make it possible to introduce the notion of a set of admissible speed fields associated with the domain D in $\mathbb{R}^N$ and corresponding to the required smoothness C^k of the constructed family of domains $\{\Omega_t\}$, $t \in [0,\varepsilon)$.

Definition 2.17 Let D be a domain in $\mathbb{R}^N$ whose boundary ∂D is piecewise C^k, $k \geqq 1$. It is supposed that the outward unit normal field n exists a.e. on ∂D, i.e. except for singular points $\overline{x}$ of ∂D. The following notation is used

$$\begin{aligned} V^k(D) = \{ & V \in \mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N) | \\ & \langle V, n\rangle_{\mathbb{R}^N} = 0 \quad \text{on } \partial D \text{ except for the singular points } \overline{x} \text{ of } \partial D, \\ & V(\overline{x}) = 0 \text{ for all singular points } \overline{x} \} \ . \end{aligned}$$

$V^k(D)$ is equipped with the topology induced by $\mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N)$.

If $V \in C(0,\varepsilon; V^k(D))$, then there exists a compact set $\overline{\mathcal{O}}$ in $\mathbb{R}^N$ such that the support of $V(t)$ is included in $\overline{\mathcal{O}}$ for all $0 \leqq t \leqq \varepsilon$. In such a case, one can make use of Theorem 2.16 and define the transformation $T_t(V)$ in the form of a one–to–one mapping on $\overline{\mathcal{O}}$. If the following conditions are satisfied: $\langle V, n\rangle_{\mathbb{R}^N} = 0$ on ∂D and $V(\overline{x}) = 0$, then $T_t(V)$ maps $\overline{\mathcal{O}} \cap \overline{D}$ onto $\overline{\mathcal{O}} \cap \overline{D}$ and $(\mathbb{R}^N \setminus \overline{\mathcal{O}}) \cap \overline{D}$ onto itself. Finally, the restriction of the mapping $T_t(V)$ to $\overline{D}$, $T_t(V)|_{\overline{D}}$ is a one–to–one transformation of $\overline{D}$ possessing all properties required for the mapping $T_t(V)$.

Theorem 2.18 *Let D be a bounded domain in $\mathbb{R}^N$ with the piecewise smooth boundary ∂D, and $V \in C(0,\varepsilon; V^k(D))$ be a vector field. Then there exists an interval $I = [0,\delta)$, $0 < \delta \leqq \varepsilon$, and a one-to-one transformation $T_t(V)$ for each $t \in I$ which maps $\overline{D}$ onto $\overline{D}$ and satisfies all properties listed in Theorem 2.16.*

2.11. Eulerian derivatives of shape functionals

Given the following data: an open set D in $\mathbb{R}^N$, a measurable subset Ω of D, an integer $k \geqq 0$, a vector field $V \in C(0,\varepsilon;V^k(D))$, and the associated transformation $T_t(V)$ from $\overline{D}$ onto $\overline{D}$.

First we suppose that the functional $J(\Omega)$ is well defined for any measurable set Ω in D. Examples of such functionals have been given in Sections 2.5 and 2.6, e.g. the functionals J_1 and J_2 discussed in Sect. 2.5, or any quadratic functional connected with the transmission problem examined in Sect. 2.6.2. Let $\Omega_t = T_t(V)(\Omega)$, $t \in [0,\delta)$, be a given family of deformations of Ω, hence Ω_t is the measurable subset of D for any $t \in [0,\delta)$. The Eulerian derivative of the domain functional $J(\Omega)$ can be defined as follows.

Definition 2.19 For any vector field $V \in C(0,\varepsilon;V^k(D))$, the Eulerian derivative of the domain functional $J(\Omega)$ at Ω in the direction of a vector field V is defined as the limit

$$dJ(\Omega;V) = \lim_{t\downarrow 0}(J(\Omega_t) - J(\Omega))/t, \tag{2.83}$$

where

$$\Omega_t = T_t(V)(\Omega) \ .$$

Definition 2.20 The functional $J(\Omega)$ is shape differentiable (or for simplicity differentiable) at Ω if

(i) there exists the Eulerian derivative $dJ(\Omega;V)$ for all directions V
(ii) the mapping $V \to dJ(\Omega;V)$ is linear and continuous from $C(0,\varepsilon;C^k(D;\mathbb{R}^N))$ into $\mathbb{R}$.

Example. Consider the functional $J(\Omega) =$ measure of Ω, and assume that

$$J(\Omega_t) = \int_{\Omega_t} dx \ ,$$

by the change of variables $x = T_t(V)(X)$ we get

$$J(\Omega_t) = \int_{\Omega} \gamma(t)(x)dx,$$

where $\gamma(t) = \det(DT_t)$.
In the next section it will be shown that for $t > 0$, t small enough,

$$\min_{x\in\overline{D}} \gamma(t) > 0$$

and $t \to \gamma(t)$ is differentiable in $\mathcal{D}^k(D)$, i.e.

$$\frac{1}{t}(\gamma(t) - 1) \to \operatorname{div} V(0) \quad \text{as} \quad t \to 0 \ .$$

The convergence takes place in the space $\mathcal{D}^{k-1}(D)$, and in fact in $C^k(\overline{\mathcal{O}})$, where $\overline{\mathcal{O}}$ denotes the support of the vector field V for $V \in \mathcal{D}^k(D; \mathbb{R}^N)$. Finally, we obtain the Eulerian derivative

$$dJ(\Omega; V) = \int_\Omega \operatorname{div} V(0) dx \ ,$$

that is

$$\begin{aligned} dJ(\Omega; V) &= \langle \chi_\Omega, \operatorname{div} V(0) \rangle_{\mathcal{D}^{-k}(\Omega) \times \mathcal{D}^k(\Omega)} \\ &= \langle -\nabla \chi_\Omega, V(0) \rangle_{\mathcal{D}^{-k}(D; \mathbb{R}^N) \times \mathcal{D}^k(D; \mathbb{R}^N)} \ . \end{aligned} \tag{2.84}$$

Similar results can be expected for all shape differentiable functionals.

Proposition 2.21 *Let D be an open set in $\mathbb{R}^N$ and suppose that the functional $J(\Omega)$ defined on the family of sets*

$$\{\textit{measurable set } \Omega | \Omega \subset D \subset \mathbb{R}^N\}$$

is shape differentiable. Then there exists the distribution $G(\Omega) \in \mathcal{D}^{-k}(\Omega) = (\mathcal{D}^k(\Omega))'$ such that

$$\begin{aligned} dJ(\Omega; V) = \langle G(\Omega), V(0) \rangle_{\mathcal{D}^{-k}(D; \mathbb{R}^N) \times \mathcal{D}^k(D; \mathbb{R}^N)} & \\ \forall V \in C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N)) \ . & \end{aligned} \tag{2.85}$$

Proof. We shall show that

$$dJ(\Omega; V) = dJ(\Omega; V(0)) \ .$$

The continuity of the mapping $V(0) \to dJ(\Omega; V(0))$ on $\mathcal{D}^k(D; \mathbb{R}^N)$ yields (2.85). For any $V \in C(0, \varepsilon; V^k(D))$, $m \in \mathbb{N}$, the following notation is used

$$V_m(t) = \begin{cases} V(t) & \text{if } 0 \leqq t \leqq \frac{1}{m} \\ V(\frac{1}{m}) & \text{if } t > \frac{1}{m} \ . \end{cases}$$

It is clear that for any $m \in \mathbb{N}$ we have $\Omega_t(V_m) = \Omega_t(V)$ for t, $0 \leqq t \leqq \frac{1}{m}$, where $\Omega_t(V_m) = T_t(V_m)(\Omega)$; hence $dJ(\Omega; V) = dJ(\Omega; V_m)$ for any $m > 0$.

For $0 \leqq t \leqq \frac{1}{m}$ we have $\operatorname{spt} V(t) \subset \overline{\mathcal{O}}$, where $\overline{\mathcal{O}}$ is a compact subset of D. Hence $\operatorname{spt} V_m(t) \subset \overline{\mathcal{O}}$ for all t. Therefore for all $\alpha \in \mathbb{N}^N$ and for s, $0 < s < \varepsilon$, $s > 1/m$,

$$\sup_{\substack{0\leq t\leq s\\ x\in\overline{\mathcal{O}}}} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} V_m(t,x) - \left(\frac{\partial}{\partial x}\right)^{\alpha} V(0,x)\right\|_{\mathbb{R}^N} =$$

$$\sup_{\substack{0\leq t\leq \frac{1}{m}\\ x\in\overline{\mathcal{O}}}} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} (V(t,x) - V(0,x))\right\|_{\mathbb{R}^N} .$$

The last term converges to zero as $m \to \infty$, because $V \in C(0,\varepsilon;\mathcal{D}^k(D;\mathbb{R}^N))$ and the derivatives

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} V(t,x)$$

are uniformly continuous on $[0,s] \times \overline{\mathcal{O}}$ for all α, $|\alpha| \leqq k$.

□

We denote by $G(\Omega)$ the gradient of the domain functional $J(\Omega)$ (or the shape gradient of $J(\Omega)$, to dispel doubts if any). By definition $G(\Omega) \in \mathcal{D}^{-k}(D;\mathbb{R}^N)$. For $J(\Omega) = \int_\Omega dx$ we have $G(\Omega) = -\nabla\chi_\Omega$. One can see from this simple example that the distribution $G(\Omega) \in \mathcal{D}'(D;\mathbb{R}^N)$ is more regular since the linear form

$$\langle G(\Omega),\phi\rangle = \int_\Omega \mathrm{div}\phi dx$$

is defined for any $\phi \in \mathcal{D}^1(D;\mathbb{R}^N)$, thus $G(\Omega) \in \mathcal{D}^{-1}(D;\mathbb{R}^N) = (\mathcal{D}^1(D;\mathbb{R}^N))'$.

If Ω is an arbitrary measurable subset of D, meas $(\Omega) < +\infty$, then

$$\chi_\Omega \in L^2(D) \quad \text{hence} \quad G(\Omega) \in H^{-1}(D;\mathbb{R}^N) .$$

Furthermore, if Ω is a smooth domain, then

$$\chi_\Omega \in H^s(D) \quad s < \frac{1}{2} .$$

This result implies that $G(\Omega) \in H^s(D)$ for any s, $s < -\frac{1}{2}$.

If the boundary Γ of the domain Ω is smooth, then we can use Stokes' theorem. Hence for

$$J(\Omega) = \int_\Omega dx$$

we obtain

$$dJ(\Omega;V) = \int_\Omega \mathrm{div}V(0)dx = \int_\Gamma V(0)\cdot n d\Gamma .$$

Let $\gamma_\Gamma\colon H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ denotes the (restriction) trace mapping on Γ, $\gamma_\Gamma u = u|_\Gamma$ (see e.g. (Lions et al. 1968)). In general we have

$$\gamma_\Gamma \in \mathcal{L}(C^k(\mathbb{R}^N);C^k(\Gamma)),$$

for all integers $k \geqq 0$. Furthermore

$$dJ(\Omega; V) = \langle n, \gamma_\Gamma \cdot V(0) \rangle_{\mathcal{D}^{-k+1}(\Gamma;\mathbb{R}^N) \times \mathcal{D}^{k-1}(\Gamma;\mathbb{R}^N)} = \langle {}^*\gamma_\Gamma \cdot n, V(0) \rangle_{\mathcal{D}^{-k+1}(\mathbb{R}^N;\mathbb{R}^N) \times \mathcal{D}^{k-1}(\mathbb{R}^N;\mathbb{R}^N)},$$

where ${}^*\gamma_\Gamma$ denotes the transpose of γ_Γ.

It can be shown that the distribution

$$G = {}^*\gamma_\Gamma \cdot n \quad \text{is supported on} \quad \Gamma = \partial\Omega$$

and linearly depends on the normal vector field n on $\partial\Omega$. We shall prove that in general shape gradients of domain functionals defined on sufficiently smooth domains have the same property.

Proposition 2.22 *Let us suppose that the domain functional $J(\cdot)$ is defined on the class of measurable subsets of D and is shape differentiable at Ω. Then* $\operatorname{spt}(G(\Omega)) \subset \overline{\Omega}$, *where $G(\Omega)$ is the shape gradient of $J(\cdot)$ at Ω.*

Proof. Let $V \in \mathcal{D}(D; \mathbb{R}^N)$ be a vector field such that $(\operatorname{spt} V) \cap \overline{\Omega} = \emptyset$. Hence

$$T_t(V)(\Omega) = \Omega_t(V) = \Omega \ ,$$

since $T_t(V)|_{\overline{\Omega}} \equiv \mathcal{I}$ is the identity mapping. Thus $dJ(\Omega; V) = 0$. Therefore it can be shown that the distribution $G(\Omega)$ restricted to $D \setminus \overline{\Omega}$ vanishes and, as a result, the support of the distribution $G(\Omega)$ is included in $\overline{\Omega}$. □

Proposition 2.23 *Let the domain functional $J(\cdot)$ be defined for any $\Omega \subset \mathbb{R}^N$ with the boundary $\partial\Omega$ of class C^k. Then the distribution $G(\Omega)$ is supported on Γ.*

Proof. If $V \in \mathcal{D}(\Omega; \mathbb{R}^N)$, then $\Omega_t(V) = \Omega$ and $dJ(\Omega; V) = 0$. Hence the restriction of the distribution $G(\Omega)$ to the open set Ω is equal to zero, i.e. $\operatorname{spt}(G(\Omega)) \subset \overline{\Omega}^c$. From Proposition 2.22 it follows that $\operatorname{spt}(G(\Omega)) \subset \overline{\Omega}$, hence $\operatorname{spt}(G(\Omega)) \subset \Gamma = \overline{\Omega} \cap \overline{\Omega}^c$ as was to be shown. □

Proposition 2.24 *Let Ω be a domain with the boundary $\partial\Omega$ of class C^k and $V \in \mathcal{D}(D; \mathbb{R}^N)$ be an element such that*

$$\langle V, n \rangle_{\mathbb{R}^N} = 0 \quad on \ \Gamma.$$

Then

$$\langle G(\Omega), V \rangle_{\mathcal{D}'(D;\mathbb{R}^N) \times \mathcal{D}(D;\mathbb{R}^N)} = 0 \ .$$

Proof. Suppose that $\langle V, n \rangle_{\mathbb{R}^N} = 0$ on $\Gamma = \partial\Omega$. For any $X \in \Gamma$, $x(t, X) = T_t(V)(X) \in \Gamma$. The speed vector field $V(x)$ belongs to the tangent space $\mathcal{T}_x\Gamma$, hence it follows that Γ is globally invariant with respect to the transformation $T_t(V)$. From this it is inferred that $\Omega_t = \Omega$ and $dJ(\Omega; V) = 0$ □

Consider the domain functional $J(\cdot)$ which is shape differentiable at any domain of class C^k. Let Ω be a given domain of class C^{k-1}. The normal vector field n on $\partial\Omega$ belongs to $C^k(\Gamma; \mathbb{R}^N)$. Furthermore let us denote by $\mathcal{N}_0$ an extension of the field n such that

$$\mathcal{N}_0 \in C^k(\overline{D}; \mathbb{R}^N)$$

and by $V_{\mathcal{N}}$ the following vector field

$$V_{\mathcal{N}} = \langle V, \mathcal{N}_0 \rangle_{\mathbb{R}^N} \mathcal{N}_0 \in C^k(\overline{D}; \mathbb{R}^N) \quad \forall V \in C^k(\overline{D}; \mathbb{R}^N) \ .$$

If

$$F(\Omega) = \{V \in C^k(\overline{D}; \mathbb{R}^N) | \langle V, n \rangle_{\mathbb{R}^N} = 0 \quad \text{on } \Gamma\},$$

then $F(\Omega)$ is a closed subspace of the space $C^k(\overline{D}; \mathbb{R}^N)$.

Proposition 2.25 *The mapping* $\{V\} \to v = \langle V, n \rangle_{\mathbb{R}^N}$

$$\textit{from} \quad C^k(\overline{D}; \mathbb{R}^N)/F(\Omega) \quad \textit{into} \quad C^k(\Gamma)$$

is an isomorphism.

The space $C^k(\overline{D}; \mathbb{R}^N)/F(\Omega)$ *is equipped with the quotient topology of a Frechet space (or a Banach space when* $\overline{D}$ *is compact).*

Proof. The linear mapping $\{V\} \to \langle V, n \rangle_{\mathbb{R}^N}$ is well defined because from $V_1 - V_2 \in F(\Omega)$ it follows that $\langle V_1 - V_2, n \rangle_{\mathbb{R}^N} = 0$. To proceed further one has to construct the inverse mapping defined as follows

$$v = \sum_{i=1}^{m} v_i \quad \text{for any} \quad v \in C^k(\Gamma),$$

where $v_i = v r_i$ (see Sect. 2.1), $r_i \in \mathcal{D}(U_i)$, $i = 1, 2, \ldots, m$, is a partition of the unity on $\Gamma = \partial\Omega$. Let us introduce the following notation

$$\omega_i(\xi', 0) = v_i \circ h_i(\xi'),$$

where $v_i \circ h_i \in \mathcal{D}^k(B_0)$. B_0 is the unit ball in $\mathbb{R}^{N-1}$, and $\omega_i(\xi', 0)$ can be extended to B, the unit ball of $\mathbb{R}^N$, in the following way

$$\tilde{\omega}_i(\xi', \xi_N) = \omega_i(\xi', 0) \quad \xi = (\xi', \xi_N) \in \mathbb{R}^N \ .$$

We define

$$Pv = \sum_{i=1}^{m} r_i \tilde{\omega}_i \circ c_i$$

$Pv \in C^k(\overline{D})$ is an extension to $\overline{D}$ of the element v defined on Γ. The mapping $v \to \{Pv\ \mathcal{N}_0\}$ is the required inverse mapping, as was to be shown. □

In the following the space $C^k(\Gamma)$ is identified with the quotient space $C^k(\overline{D}; \mathbb{R}^N)/F(\Omega)$ using the isomorphism $\{V\} \to \langle V, n\rangle_{\mathbb{R}^N}$. From Proposition 2.23 it follows that the closed subspace $F(\Omega)$ is included in the kernel of the continuous linear mapping

$$(V \to dJ(\Omega; V)) \in \mathcal{L}(C^k(\overline{D}; \mathbb{R}^N), \mathbb{R}) \ .$$

Proposition 2.26 *There exists a linear continuous mapping*

$$dJ(\Gamma; \cdot): C^k(\Gamma) \to \mathbb{R}$$

such that for all vector fields $V \in C^k(\overline{D}; \mathbb{R}^N)$

$$dJ(\Omega; V) = dJ(\Gamma; \langle V, n\rangle_{\mathbb{R}^N}) \ .$$

Proof. The canonical mapping $\imath: V \to \{V\}$, from $C^k(\overline{D}; \mathbb{R}^N)$ onto $C^k(\overline{D}; \mathbb{R}^N)/F(\Omega) = C^k(\Gamma)$ is surjective. Furthermore

$$\imath(V) = \langle V|_\Gamma, n\rangle_{\mathbb{R}^N} \ .$$

The closed set $F(\Omega)$ is included in the kernel of the linear mapping $dJ(\Omega, \cdot)$. Hence we have the following factorization:

$$dJ(\Gamma; \cdot) = dJ(\Omega; \cdot) \circ \imath \ ,$$

equivalent to $dJ(\Omega; V) = dJ(\Gamma; \langle V, n\rangle_{\mathbb{R}^N})$ □

Let us introduce the following notation

$$dJ(\Gamma; v_n), \quad v_n(x) = \langle V(x), n(x)\rangle_{\mathbb{R}^N} \quad x \in \Gamma \ .$$

Finally the structure of the gradient is determined.

Theorem 2.27 (the Hadamard Formula) *Let $J(\cdot)$ be a shape functional which is shape differentiable at every domain Ω of class C^k, $\Omega \subset D$. Furthermore let us assume that $\Omega \subset D$ is a domain with boundary of class C^{k-1}. There exists the scalar distribution*

$$g(\Gamma) \in \mathcal{D}^{-k}(\Gamma)$$

such that the gradient $G(\Omega) \in \mathcal{D}^{-k}(\Omega; \mathbb{R}^N)$ of the functional $J(\cdot)$ at Ω , with $\mathrm{spt} G(\Omega) \in \Gamma$*, is given by*

$$G(\Omega) = {}^*\gamma_\Gamma(g \cdot n), \tag{2.86}$$

where $\gamma_\Gamma \in \mathcal{L}(\mathcal{D}(\overline{D}; \mathbb{R}^N), \mathcal{D}(\Gamma; \mathbb{R}^N))$ is the trace operator and ${}^*\gamma_\Gamma$ denotes the transpose of γ_Γ.

From (2.86) it follows that

$$dJ(\Omega;V) = dJ(\Gamma;v_n) = \langle g, v_n\rangle_{\mathcal{D}^{-k}(\Gamma)\times\mathcal{D}^k(\Gamma)} \ .$$

In general $g = g(\Omega) \in \mathcal{D}^{-k}(\Gamma)$, however for a specific class of shape functionals one can assume that $g(\Omega)$ is an integrable function on Γ. If this is the case, then

$$dJ(\Omega;V) = \int_\Gamma g(x)\langle V(0,x), n(x)\rangle_{\mathbb{R}^N} d\Gamma \ . \tag{2.87}$$

In the particular case of the functional $J(\Omega) = \int_\Omega dx$, we proved that $g \equiv 1$ on Γ. In general, if $g \in L^1(\Gamma)$, then g is obtained in the form of the trace on Γ of an element $G \in W^{1,1}(\Omega)$. The element G is not uniquely determined while the element g, the density of the gradient, is unique. We have

$$\begin{aligned}
&\langle G(\Omega), V(0)\rangle_{\mathcal{D}^{-k}(D;\mathbb{R}^N)\times\mathcal{D}^k(D;\mathbb{R}^N)} = \\
&\int_\Gamma G\langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma = \\
&\int_\Omega \operatorname{div}(GV(0))dx = \langle \chi_\Omega, \operatorname{div}(GV(0))\rangle = \\
&-\langle G\nabla\chi_\Omega, V(0)\rangle_{\mathcal{D}^{-k}(D;\mathbb{R}^N)\times\mathcal{D}^k(D;\mathbb{R}^N)} \ .
\end{aligned}$$

Hence the gradient $G(\Omega)$ satisfies

$$G(\Omega) = {}^*\gamma_\Gamma(gn) = -G\nabla\chi_\Omega \ . \tag{2.88}$$

2.12. Non–differentiable shape functionals

In Chaps. 3 and 4 we shall consider the domain functionals $J\colon \Omega \to J(\Omega)$ such that the Eulerian derivative $dJ(\Omega;V)$ exists at Ω in any direction V, $V \in C(0,\varepsilon;C^k(\overline{D};\mathbb{R}^N))$. However, some domain functionals are not shape differentiable, because the mapping $V \to dJ(\Omega;V)$ is nonlinear in such cases. An example of non–differentiable domain functional is the multiple eigenvalue $\lambda(\Omega)$ of an elliptic eigenvalue problem (see Chap. 3 for details), let us recall that an eigenvalue problem can be stated as follows:

$$a_\Omega(y,\phi) = \lambda(\Omega)b_\Omega(y,\phi) \quad \forall\phi,$$

where y denotes an eigenfunction, $a_\Omega(\cdot,\cdot)$, $b_\Omega(\cdot,\cdot)$ are bilinear forms. It can be shown (see e.g. (Zolesio 1979a; 1981)) that the mapping $V \to d\lambda(\Omega;V)$ is concave. In Chap. 4 the unilateral problems will be examined. In general, the mapping $V \to dJ(\Omega;V)$ associated with such problems is neither linear nor concave. Nevertheless the unilateral problems have the same properties as those derived for the shape differentiable functionals. In particular, if the mapping $V \to dJ(\Omega; V)$ is continuous in the appropriate topology, then $dJ(\Omega; V) = dJ(\Omega; V(0))$. Furthermore, if $\langle V(0), n\rangle_{\mathbb{R}^N} = 0$ on Γ, and Γ is sufficiently smooth, then $\Omega_t = \Omega$, and we have $dJ(\Omega; V) = dJ(\Omega; V(0)) = 0$. In the particular case of the multiple eigenvalues, the shape derivative of the domain functional $J(\Omega) \equiv \lambda(\Omega)$ enjoys the following property

$$-dJ(\Omega; V) = \sup_{G \in \mathcal{M}} \langle G, V(0)\rangle_{H^{-s}(\mathbb{R}^N;\mathbb{R}^N) \times H^s(\mathbb{R}^N;\mathbb{R}^N)}$$

for some $s > 0$, where $\mathcal{M}$ is a given set of distributions with supports in $\overline{\Omega}$. Functions of the form are well known, they are used in the convex analysis (see e.g. (Ekeland et al. 1976)). Let us recall that the support function of a convex set $\mathcal{M}$ (here $\mathcal{M}$ is a convex set in the Sobolev space $H^{-s} \equiv H^{-s}(\mathbb{R}^N; \mathbb{R}^N)$) is defined (Ekeland et al. 1976) as follows:

$$S_{\mathcal{M}}(G) = \begin{cases} 0 & \text{if } G \text{ belongs to } \mathcal{M} \\ +\infty & \text{otherwise} \end{cases} .$$

The conjugate function $S^*_{\mathcal{M}}: H^s(\mathbb{R}^N; \mathbb{R}^N) \to \mathbb{R}$ is defined in the following way:

$$S^*_{\mathcal{M}}(V) = \sup_{G \in H^{-s}} [\langle G, V\rangle - S_{\mathcal{M}}(G)] .$$

Hence

$$S^*_{\mathcal{M}}(V) = -dJ(\Omega; V) .$$

If the set $\mathcal{M}$ is not convex, then (Ekeland et al. 1976, p.18)

$$S_{\mathcal{M}} \equiv S_{\overline{\mathrm{co}}(\mathcal{M})},$$

where $\overline{\mathrm{co}}(\mathcal{M})$ denotes the closure in the space $H^{-s}(\mathbb{R}^N; \mathbb{R}^N)$ of the convex hull of set $\mathcal{M}$. In order to obtain the necessary optimality conditions for the minimization problem

$$\min_{\Omega} J(\Omega)$$

we should characterize the closed convex cone $\partial J(\Omega) \subset H^{-s}(\mathbb{R}^N; \mathbb{R}^N)$. This cone is defined as follows

$$\partial J(\Omega) = \{G \in H^{-s}(D; \mathbb{R}^N) | -dJ(\Omega; V) \geqq \langle G, V\rangle \quad \forall V \in H^s(D; \mathbb{R}^N)\}$$

In fact (Ekeland et al. 1976) the cone $\partial J(\Omega)$ can be defined equivalently by

$$\partial J(\Omega) = \partial S^*_{\mathcal{M}}(0) .$$

In this definition the use is made of the notion of subdifferential $\partial S^*_{\mathcal{M}}(0)$, which is the convex cone with the elements $G \in H^{-s}(D; \mathbb{R}^N)$. For this cone the following condition is satisfied

$$S^*_{\mathcal{M}}(V) \geqq S^*_{\mathcal{M}}(0) + \langle G, V\rangle \quad \forall V \in H^s(D; \mathbb{R}^N)$$

In particular $S^*_{\mathcal{M}}(0) = 0$.

It can be inferred that (Ekeland et al. 1976)

$$G \in \partial S^*_{\mathcal{M}}(0) \quad \text{if and only if} \quad 0 \in \partial S^{**}_{\mathcal{M}}(G) \ . \tag{2.89}$$

Furthermore it can be shown that $S_{\overline{\text{co}}(\mathcal{M})}$ is lower semi–continuous, thus $S_{\mathcal{M}} = S_{\overline{\text{co}}(\mathcal{M})}$ is lower semi–continuous and $S^{**}_{\mathcal{M}} = S_{\mathcal{M}}$ (it is evident that $f^{**} = f$ for any convex semi–continuous function f).

Finally from (2.89) it follows that $G \in \partial J(\Omega)$ if and only if

$$S_{\overline{\text{co}}(\mathcal{M})}(G) \leqq S_{\overline{\text{co}}(\mathcal{M})}(G') \quad \forall G' \in H^{-s}(D; \mathbb{R}^N) \ .$$

By definition, $S_{\mathcal{M}}$ attains the value 0 or $+\infty$. Therefore the above inequality is equivalent to the condition $G \in \overline{\text{co}}(\mathcal{M})$. The foregoing result may be stated as the following proposition:

Proposition 2.28 *Let $J(\Omega)$ be a domain functional such that*

$$dJ(\Omega; V) = \inf_{G \in \mathcal{M}} \langle G, V(0) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the spaces $H^{-s}(\mathbb{R}^N; \mathbb{R}^N)$ and $H^s(\mathbb{R}^N; \mathbb{R}^N)$ for some $s > 0$, and $\mathcal{M}$ is a given subset of the space $H^{-s}(\mathbb{R}^N; \mathbb{R}^N)$.

If

$$\partial J(\Omega) = \{G \in H^{-s}(\mathbb{R}^N; \mathbb{R}^N) | dJ(\Omega; V) \leqq \langle G, V(0) \rangle \quad \forall V \in H^s(\mathbb{R}^N; \mathbb{R}^N)\},$$

then

$$\partial J(\Omega) = \overline{\text{co}}(\mathcal{M}),$$

where $\overline{\text{co}}(\mathcal{M})$ denotes the closure in $H^{-s}(\mathbb{R}^N; \mathbb{R}^N)$ of the convex hull of the set $\mathcal{M} \subset H^{-s}(\mathbb{R}^N; \mathbb{R}^N)$.

□

2.13. Properties of T_t transformations

Given a vector field $V \in C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N))$, where D is an open set in $\mathbb{R}^N$, ∂D is piecewise C^k. D is not supposed to be bounded. Let us consider the transformations

$$T_t = T_t(V) \in C^k(\overline{D}; \mathbb{R}^N)$$

for fixed t.

It is assumed that $\langle V(t,x), n(x) \rangle_{\mathbb{R}^N} = 0$ a.e. on ∂D, and $V(t,x) = 0$ at any singular point x of the boundary ∂D. It should be remarked that a normal vector $n(x)$ does not exist at the singular point $x \in \partial D$. T_t is a one-to-one transformation of $\overline{D}$ into $\mathbb{R}^N$, $T_t(\partial D) = \partial D$.

Let us denote by $DT_t(X)$ the Jacobian matrix of T_t evaluated at X, ${}^*DT_t(X)$ is the transpose of $DT_t(X)$. It is evident that

$$({}^*DT_t)^{-1} = {}^*((DT_t)^{-1})$$

so, to simplify the notation, we shall write ${}^*DT_t^{-1}$. The application of the chain rule yields the following useful results

Proposition 2.29 *We have*

(i)

$$(\nabla\phi) \circ T_t = {}^*DT_t^{-1} \cdot \nabla(\phi \circ T_t) \quad \forall \phi \in C^1(\mathbb{R}^N),$$

(ii)

$$D(T \circ S) = \{(DT) \circ S\} \cdot DS \quad \forall (T,S) \in C^1(\mathbb{R}^N; \mathbb{R}^N) \ .$$

Proof. Let DT be the Jacobian matrix function of the transformation T, hence

$$(DT)_{ij} = \frac{\partial}{\partial x_j} T_i \ ,$$

$$\frac{\partial}{\partial x_i} T_j = ({}^*DT)_{ij} \ .$$

First we shall prove the property (ii). It follows that

$$\frac{\partial}{\partial x_j}(T_i \circ S) = \sum_{k=1}^{N} \{\frac{\partial}{\partial x_k} T_i \circ S\} \frac{\partial}{\partial x_j} S_k \ ,$$

whence

$$D(T \circ S)_{ij} = \sum_{k=1}^{N} (DT)_{ij} \circ S (DS)_{kj} \ .$$

Finally we obtain

$$\frac{\partial}{\partial x_i}(\phi \circ T) = \sum_{k=1}^{N} \frac{\partial \phi}{\partial x_k} \circ T \frac{\partial}{\partial x_i} T_k = \sum_{k=1}^{N} {}^*(DT)_{ik}((\nabla\phi) \circ T)_k \ ,$$

thus

$$\nabla(\phi \circ T) = {}^*DT \cdot (\nabla\phi) \circ T \ .$$

This proves the property (i). □

Lemma 2.30 *We have*

$$D(T_t^{-1}) = (DT_t)^{-1} \circ T_t^{-1} \ . \tag{2.90}$$

□

Let

$$\begin{aligned} &\gamma(t) = \det(DT_t) \\ &\det D(T_t^{-1}) = \gamma(t)^{-1} \circ T_t^{-1} \ . \end{aligned} \tag{2.91}$$

Furthermore we shall prove that transformations $T_t(\cdot)$ have the following properties (see Sect. 2.1 for applications).

Lemma 2.31 *The mapping $t \to DT_t(V)$ ($t \to \gamma(t)$) is differentiable in $C^{k-1}(\mathbb{R}^N;\mathbb{R}^N)$ (in $C^{k-1}(\mathbb{R})$). The derivative at $t=0$ is given by*

$$\left(\frac{\partial}{\partial t}DT_t(V)\right)_{|t=0} = DV(0), \tag{2.92}$$

$$(\gamma'(0) = \mathrm{div}V(0)) \ . \tag{2.93}$$

Therefore, for any compact set $\overline{\mathcal{O}}$, all integers l, $0 \leqq l \leqq k-1$, and indices $\alpha \in \mathbb{N}^N$,

$$|\alpha| = \alpha_1 + \cdots + \alpha_N \leqq 1,$$

we obtain as $t \to 0$

i)

$$\max_{x\in\overline{\mathcal{O}}}\left|\left(\frac{\partial}{\partial x}\right)^\alpha \left[\left(\frac{\partial}{\partial x_i}(T_t)_j(x) - \delta_{i,j}\right)/t - \frac{\partial}{\partial x_i}V_j(0,x)\right]\right| \to 0 \ , \tag{2.94}$$

ii)

$$\max_{x\in\overline{\mathcal{O}}}\left|\left(\frac{\partial}{\partial x}\right)^\alpha \left[\frac{\gamma(t)(x)-1}{t} - \mathrm{div}V(0,x)\right]\right| \to 0 \ . \tag{2.95}$$

Proof. First we shall prove the property (i).
The matrix DT_t is determined by the solution to the following linear differential system

$$DT_t(X) - \mathcal{I} = \int_0^t DV(s, T_s(X)) \cdot DT_s(X)ds \ . \tag{2.96}$$

But $T_s \to T_0 = \mathcal{I}$ in $C^k(\mathbb{R}^N;\mathbb{R}^N)$ as $s \to 0$, hence

$$\frac{1}{t}(DT_t(x) - \mathcal{I}) = \frac{1}{t}\int_0^t DV(s, T_s(X)) \cdot DT_s(X)ds \ .$$

The last term of the above expression converges in $C^{k-1}(\mathbb{R}^N;\mathbb{R}^N)$ to the limit $DV(0,X)\cdot DT_0(X)$ as $t \to 0$, hence because of $T_0 = \mathcal{I}$, we obtain (2.92) and (2.94).

The property (ii) can be proved by induction with respect to N. First, assume that $N = 2$ and denote $x = x_1$, $y = x_2$, $V = \mathrm{col}(V_x, V_y$ and $T = \mathrm{col}(T_x, T_y)$ then

$$\begin{aligned}\gamma'(0) = &\frac{\partial}{\partial x}V_x(0)\frac{\partial}{\partial y}(T_y)_0 + \frac{\partial}{\partial y}V_y(0)\frac{\partial}{\partial x}(T_x)_0\\ &-\frac{\partial}{\partial x}V_y(0)\frac{\partial}{\partial y}(T_x)_0 - \frac{\partial}{\partial x}(T_y)_0\frac{\partial}{\partial y}V_x(0)\ .\end{aligned}$$

The two last terms on the right–hand side of this equation vanish, because T_t is equal to the identity for $t = 0$. Hence

$$\frac{\partial}{\partial x}(T_y)_0 = \frac{\partial}{\partial y}(T_x)_0 = 0.$$

Finally we consider the space $\mathbb{R}^{N+1}$; let $j = 1, 2, \ldots, N+1$, and assume that

$$x_{N+1} \quad \text{is fixed,} \quad x = (x', x_{N+1}).$$

We define a mapping $\mathbb{R}^N \to \mathbb{R}^N$ in the form:

$$T_t^j : x' \to [(T_t)_1(x), \ldots, (T_t)_{j-1}(x), (T_t)_{j+1}(x), \ldots, (T_t)_{N+1}(x)]\ .$$

Then

$$\det[D_x T_t(x)] = \sum_{j=1}^{N+1}(-1)^{N+1+j}\det[D_{x'}T_t^j((x', x_{N+1}))]\frac{\partial}{\partial x_j}(T_t)_{N+1}(x)$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\gamma(0,x) &= \sum_{j=1}^{N+1}(-1)^{N+1+j}\mathrm{div}_{x'}V^j(0,x)\delta_{j,N+1} + \frac{\partial}{\partial x_{N+1}}V_{N+1}(0,x)\\ &= \mathrm{div}_{x'}V_{N+1}(0,(x', x_{N+1})) + \frac{\partial}{\partial x_{N+1}}V_{N+1}(0,x)\ .\end{aligned}$$

□

2.14. Differentiability of transformed functions

Now we shall investigate the properties of functions of the form $(f \circ T_t)(\cdot)$ where T_t is a given transformation.

Proposition 2.32 *If $f \in W^{1,1}(\mathbb{R}^N)$ and $V \in \mathcal{D}^k(\mathbb{R}^N; \mathbb{R}^N)$, then $t \to f \circ T_t$ is differentiable in $L^1(\mathbb{R}^N)$ and the derivative is given by*

$$\left(\frac{\partial}{\partial t}(f \circ T_t)\right)\bigg|_{t=0} = \langle \nabla f, V(0) \rangle_{\mathbb{R}^N} \ . \tag{2.97}$$

Proof. First, we suppose that $f \in C^\infty(\mathbb{R}^N)$ and make use of the following expansion:

$$f(T_t(x)) - f(x) = \int_0^1 \nabla f[x + s(T_t(x) - x)] \cdot (T_t(x) - x) ds \ .$$

Hence

$$\begin{aligned}
&\left[\frac{1}{t}(f \circ T_t - f) - \nabla f \cdot V(0)\right](x) = \\
&\left[\int_0^1 (\nabla f[x + s(T_t(x) - x)]) ds - \nabla f(x)\right] \cdot \frac{T_t(x) - x}{t} + \\
&\nabla f(x) \cdot \left(\frac{T_t(x) - x}{t} - V(0, x)\right) \ .
\end{aligned}$$

Let

$$z = x + s(T_t(x) - x) = \Lambda(s,t)(x)$$

then

$$dx = \det(D([(1-s)\mathcal{I} + sT_t]^{-1}))dz \ .$$

One has to show that

$$g(t) = \int_\Omega \left\| \int_0^1 (\nabla f[x + s(T_t(x) - x)] - \nabla f(x)) ds \right\|_{\mathbb{R}^N} dx \to 0 \quad \text{as} \quad t \to 0$$

and that

$$g(t) \leqq \int_0^1 ds \left(\int_\Omega \|\nabla f[x + s(T_t(x) - x)] - \nabla f(x)\|_{\mathbb{R}^N} \right) dx \ .$$

Taking into account that for any s the mapping $t \to \nabla f \circ \Lambda(s,t)$ is continuous in $L^1(\mathbb{R}^N)$, we can prove that for any s

$$h(s,t) = \int_\Omega \|\nabla f[x + s(T_t(x) - x)] - \nabla f(x)\|_{\mathbb{R}^N} dx \to 0 \quad \text{as} \quad t \to 0.$$

It should be remarked that $h(s,t) \leqq h(1,t)$ for all s. Hence it indicates that $g(t) \leq h(1,t) \to 0$ as was to be shown. □

Proposition 2.33 *Assume that*

$$(t \to f(t)) \in C(0, \varepsilon; W^{1,1}(\mathbb{R}^N)) \cap C^1(0, \varepsilon; L^1(\mathbb{R}^N)) \ ,$$

and

$$V \in C(0, \varepsilon; \mathcal{D}^k(\mathbb{R}^N; \mathbb{R}^N)), \quad k \geqq 1 .$$

Then the mapping

$$t \to f(t) \circ T_t = f(t, T_t(\cdot))$$

is differentiable in $L^1(\mathbb{R}^N)$*, the derivative is given by*

$$\left(\frac{\partial}{\partial t}(f(t) \circ T_t)\right)\Bigg|_{t=0} = f'(0) + \nabla_x f(0) \cdot V(0) .$$

Proof. By our assumptions there exists an element $f'(0) \in L^1(\mathbb{R}^N)$,

$$\left\| \frac{1}{t}(f(t) - f(0)) - f'(0) \right\|_{L^1(\mathbb{R}^N)} \to 0 \quad \text{as} \quad t \to 0 .$$

Hence

$$\frac{1}{t}[f(t) \circ T_t - f(0)] = \frac{1}{t}[f(t) - f(0)] \circ T_t + \frac{1}{t}[f(0) \circ T_t - f(0)] =$$

$$f'(\xi t) \circ T_t + \frac{1}{t}[f(0) \circ T_t - f(0)] \to f'(0) + \nabla_x f(0) \cdot V(0) \quad \text{as} \quad t \to 0 .$$

□

Remark. From the proof of Proposition 2.33 it can be inferred that the following convergence

$$\left\| \frac{1}{t}(f(t) \circ T_t - f) - \nabla f \cdot V(0) \right\|_{L^1(\mathbb{R}^N)} \to 0$$

takes place as $t \to 0$. In specific cases an estimate for the speed of this convergence is necessary. For this purpose one has to consider an element f, e.g. $f \in W^{2,1}(\mathbb{R}^N)$ and use the same reasoning as before but applying the second order expansion of f. Such an approach is much more technical but yields Proposition 2.36.

Let us introduce the following notation

$$g(s) = f(x + s(T_t(x) - x)) .$$

Lemma 2.34 *We have*

$$g(s) - g(0) - sg'(0) = \int_0^s (s - \rho) g''(\rho) d\rho . \tag{2.98}$$

Proof. Let

$$h(s) = g(s) - g(0) - sg'(0)$$

therefore we have

$$h(0) = 0 \quad \text{and} \quad h'(s) = g'(s) - g'(0) \ .$$

Furthermore the following relation holds

$$g'(\rho) = g'(0) + \int_0^\rho g''(\tau)d\tau \ .$$

Hence it can be shown that

$$h(s) = \int_0^s \int_0^\rho g''(\tau)d\tau = \int_0^s (s-\rho)g''(\rho)d\rho$$

This proves our lemma. □

It is evident that

$$\begin{aligned}
g'(s) &= \nabla f(x + s(T_t(x) - x)) \cdot (T_t(x) - x) \\
g'(0) &= \nabla f(x) \cdot (T_t(x) - x) \\
g''(s) &= \langle D^2 f(x + s(T_t(x) - x)) \cdot (T_t(x) - x), T_t(x) - x\rangle_{\mathbb{R}^N} \ .
\end{aligned}$$

Here $D^2 f(\cdot)$ denotes the Hessian matrix function of f, $D^2 f(x)$ is a symmetric matrix for all x. For $s = 1$ we have

$$\begin{aligned}
&[f(T_t(x)) - f(x)] - \nabla f(x) \cdot (T_t(x) - x) = \\
&\left\langle \left[\int_0^1 (1-s)D^2 f(x + s(T_t(x) - x))ds\right] \cdot (T_t(x) - x), T_t(x) - x \right\rangle_{\mathbb{R}^N}
\end{aligned} \tag{2.99}$$

and

$$\begin{aligned}
&\frac{1}{t}\|(f \circ T_t - f - \nabla f \cdot (T_t - \mathcal{I}))\|_{L^1(\mathbb{R}^N)} \leqq \\
&\frac{1}{t}\|T_t - \mathcal{I}\|^2_{L^\infty} \max_{0 \leqq s \leqq 1} \int_{\mathbb{R}^N} \|D^2 f(x + s(T_t(x) - x))\|_{\mathbb{R}^N} dx,
\end{aligned} \tag{2.100}$$

here we denote $L^\infty = L^\infty(\mathbb{R}^N; \mathbb{R}^N)$.
In this section an estimate of the left–hand side of (2.99) will be derived. The right–hand side of the expression (2.100) involves the term sought

$$\|T_t - \mathcal{I}\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^N)} \ .$$

We have to derive a more explicit expression for the integral term on the right–hand side of (2.100). For this purpose the following change of variables is to be applied.

For a given s, we introduce the variable z, $z = x + s(T_t(x) - x))$. Then $x = [(1-s)\mathcal{I} + sT_t]^{-1}(z)$ and it follows that

$$dx = \det(D[(1-s)\mathcal{I} + sT_t]^{-1})(z)dz = b(s,t)(z)dz \ .$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^N} \|D^2 f(x+s(T_t(x)-x))\|_{\mathbb{R}^N} dx = \\ \int_{\mathbb{R}^N} \|D^2 f(z)\|^2_{\mathcal{L}(\mathbb{R}^N;\mathbb{R}^N)} b(s,t)(z) dz \ .\end{aligned} \tag{2.101}$$

It can be easily verified that for a given vector field

$$V \in C(0,\varepsilon;\mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N))$$

there exists a constant $C_1 > 0$, $C_1 = C_1(V)$, such that

$$\max_{\substack{0\leqq s\leqq 1\\ 0\leqq t\leqq\varepsilon\\ z\in\mathbb{R}^N}} |b(s,t)(z)| \leqq C_1 \ .$$

Hence making use of (2.101) and (2.100) we have

$$\begin{aligned}\frac{1}{t}\|(f\circ T_t - T_t - \nabla f\cdot(T_t-\mathcal{I}))\|_{L^1(\mathbb{R}^N)} \leqq \\ \frac{1}{t}C_1(V)\|T_t-\mathcal{I}\|^2_{L^\infty}\|D^2 f\|_{L^1(\mathbb{R}^N,\mathbb{R}^{N^2})} \ .\end{aligned} \tag{2.102}$$

The term $\frac{1}{t}(f\circ T_t - T_t)$ is to be compared with $\nabla f\cdot V(0)$. The inequality (2.102) can be rewritten in the following way:

$$\begin{aligned}\left\|\frac{1}{t}(f\circ T_t - T_t) - \nabla f\cdot V(0)\right\|_{L^1(\mathbb{R}^N)} \leqq \\ \left\|\frac{1}{t}(T_t-\mathcal{I}) - V(0)\right\|_{L^\infty}\|\nabla f\|_{L^1(\mathbb{R}^N)} + \\ \frac{1}{t}C_1(V)\|T_t-\mathcal{I}\|^2_{L^\infty}\|D^2 f\|_{L^2(\mathbb{R}^N,\mathcal{L}(\mathbb{R}^N;\mathbb{R}^N))}\end{aligned} \tag{2.103}$$

Before proceeding further with a reasoning, we present the following result.

Proposition 2.35 *For any vector field $V(\cdot,\cdot)$,*

$$V \in C(0,\varepsilon;\mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N)), \quad k \geqq 1,$$

there exists the function $\mu(V,t) \geqq 0$ such that $\mu(V,t) \to 0$ as $t \to 0$ and

$$\left\|\frac{1}{t}(T_t(V)-\mathcal{I}) - V(0)\right\|_{L^\infty(\mathbb{R}^N)} \leqq \mu(V,t) \ . \tag{2.104}$$

Proof. For any ε_0, $0 < \varepsilon_0 < \varepsilon$, the vector field V is uniformly continuous on $[0,\varepsilon_0]\times\overline{\mathcal{O}}$, where $[0,\varepsilon_0]\times\overline{\mathcal{O}} \supset \mathrm{spt}V$ (the support $\overline{\mathcal{O}}(t)$ of $V(t)$ is not fixed but for $0 \leqq t \leqq \varepsilon_0$ there exists a set $\overline{\mathcal{O}}$ such that $\overline{\mathcal{O}}(t) \subset \overline{\mathcal{O}}$, and $\overline{\mathcal{O}}$ is a compact set in $\mathbb{R}^N$).

The modulus of continuity $\omega(\cdot)$ of the vector field V is used,

$$\omega(r) = \max\{\|V(t,x) - V(s,y)\|_{\mathbb{R}^N} | (s,t) \in [0,\varepsilon_0] \times [0,\varepsilon_0],$$
$$(x,y) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}},\ |s-t| + \|x-y\|_{\mathbb{R}^N} \leqq r\} \ .$$

It follows that $\omega(r) \to 0$ as $r \to 0$, $\omega(\cdot)$ is continuous (i.e. $V(\cdot,\cdot)$ is uniformly continuous). Furthermore

$$\left\|\frac{1}{t}(T_t(x) - x) - V(0,x)\right\|_{\mathbb{R}^N} \leqq$$
$$\frac{1}{t}\int_0^t \|V(s,T_s(x)) - V(0,x)\|_{\mathbb{R}^N} ds \leqq$$
$$\frac{1}{t}\int_0^t \omega(r(s))ds \equiv \mu(t) \ ,$$

where the following inequality is used

$$|s| + \|T_s(x) - x\|_{\mathbb{R}^N} \leqq |s| + \int_0^s \|V(\rho, t_\rho(x))\|_{\mathbb{R}^N} d\rho \leqq |r(s)|,$$

here we denote

$$r(s) = s(1 + \|V\|_{L^\infty((0,\varepsilon_0)\times\mathbb{R}^N)}) \ .$$

Since $\omega(\cdot)$ is continuous, $\omega(0) = 0$, hence $\mu(t) \to 0$ as $t \to 0$ as was to be shown. □

Combining Lemma 2.34 and the estimate (2.103) we obtain the following inequality:

$$\left\|\frac{1}{t}(f \circ T_t - T_t) - \nabla f \cdot V(0)\right\|_{L^1(\mathbb{R}^N)} \leqq \|\nabla f\|_{L^1(\mathbb{R}^N)}\mu(V,t) +$$
$$C_1(V)\|D^2 f\|_{L^1(\mathbb{R}^N,\mathcal{L}(\mathbb{R}^N;\mathbb{R}^N))}(\|V(0)\|_{L^1} + \mu(V,t))\|T_t - \mathcal{I}\|_{L^\infty} \ ,$$

here we denote $L^1 = L^1(\mathbb{R}^N;\mathbb{R}^N)$.
Thus we get

Proposition 2.36 *For any vector field $V \in C(0,\varepsilon_0;\mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N))$, $k \geqq 1$, there exists a function $\mu(V,\cdot) \geqq 0$, $\mu(V,t) \to 0$ as $t \to 0$, and a constant $N(V)$ such that for any element $f \in W^{2,1}(\mathbb{R}^N)$*

$$\left\|\frac{1}{t}(f \circ T_t - f) - \nabla f \cdot V(0)\right\|_{L^1(\mathbb{R}^N)} \leqq$$
$$\|\nabla f\|_{W^{1,1}}[\mu(V,t) + N(V)\|T_t - \mathcal{I}\|_{L^\infty}] \ . \tag{2.105}$$

Proof. For any element $f \in \mathcal{D}(\mathbb{R}^N)$ the inequality (2.105) directly follows from our calculations. Thus for $f \in W^{2,1}(\mathbb{R}^N)$ one can use the standard density argument, because the space $\mathcal{D}(\mathbb{R}^N)$ is dense in the space $W^{2,1}(\mathbb{R}^N)$. □

Proposition 2.36 gives the derivative of the mapping $t \to f \circ T_t$ in $L^1(\mathbb{R}^N)$. From the point of view of examples considered in this book it is advantageous to show the differentiability of the mapping in $W^{1,1}(\mathbb{R}^N)$. Assume that the element f is given in $W^{2,1}(\mathbb{R}^N)$. We shall examine, in the same way as in the proof of Proposition 2.36, the following convergence

$$\frac{1}{t}[\nabla(f \circ T_t) - \nabla f] \to \nabla(\nabla f \cdot V(0))$$

in the norm of the space $L^1 = L^1(\mathbb{R}^N; \mathbb{R}^N)$ as $t \to 0$. From

$$\nabla(f \circ T_t) = {}^*DT_t \cdot (\nabla f) \circ T_t$$

it follows that

$$\frac{1}{t}[\nabla(f \circ T_t) - \nabla f] = \frac{1}{t}{}^*DT_t \cdot [(\nabla f) \circ T_t - \nabla f] + \frac{1}{t}[{}^*DT_t - \mathcal{I}] \cdot \nabla f \ .$$

For the second term on the right–hand side of this expression we have

$$\frac{1}{t}\left[\left(\frac{\partial f}{\partial x_i}\right) \circ T_t - \frac{\partial f}{\partial x_i}\right] - \nabla\left(\frac{\partial f}{\partial x_i}\right) \cdot V(0) \to 0 \quad \text{in } L^1(\mathbb{R}^N) \text{ as } t \to 0.$$

On the other hand

$${}^*DT_t \to \mathcal{I} \text{ in } \mathcal{D}^{k-1}(\mathbb{R}^N; \mathbb{R}^{N^2}) \text{ as } t \to 0.$$

The foregoing results may be stated as the following proposition

Proposition 2.37 *Let there be given a vector field $V \in C(0,\varepsilon; \mathcal{D}^k(\mathbb{R}^N; \mathbb{R}^N))$, $k \geqq 1$, and an element $f \in W^{2,1}(\mathbb{R}^N)$.*

Then

$$\frac{1}{t}[f \circ T_t - f] \to \nabla f \cdot V(0) \quad \textit{strongly in } W^{1,1}(\mathbb{R}^N)$$

as $t \to 0$.

□

Finally the form of partial derivative of the mapping $t \to f(t) \circ T_t$ with respect to t will be determined.

Proposition 2.38 *Let f be a given element in*

$$C(0,\varepsilon; W^{2,1}(\mathbb{R}^N)) \cap C^1(0,\varepsilon; W^{1,1}(\mathbb{R}^N))$$

and let V be a vector field, $V \in C(0,\varepsilon; \mathcal{D}^k(\mathbb{R}^N; \mathbb{R}^N))$, where $k \geqq 1$ is an integer. Then the mapping $t \to f(t) \circ T_t$ is differentiable in $W^{1,1}(\mathbb{R}^N)$, the derivative at $t = 0$ is given by

$$\left[\frac{\partial}{\partial t}(f(t) \circ T_t)\right]_{t=0} = f'(0) + \langle \nabla f(0), V(0) \rangle_{\mathbb{R}^N} \ . \tag{2.106}$$

Proof. We have

$$f(t) \circ T_t - f(0) = [f(t) \circ T_t - f(t)] + [f(t) - f(0)] \ .$$

By our assumptions the second term on the right–hand side of this equation converges to $f'(0)$ in the norm of space $W^{1,1}(\mathbb{R}^N)$. Hence one has to consider the first term. From the inequality (2.105) appearing in Proposition 2.36 we get

$$\left\| \frac{1}{t}(f(t) \circ T_t - f(t)) - \nabla f(t) \cdot V(0) \right\|_{L^1(\mathbb{R}^N)} \leqq$$
$$\|\nabla f(t)\|_{W^{1,1}(\mathbb{R}^N)} [\mu(V,t) + N(V)\|T_t - \mathcal{I}\|_{L^\infty} \ .$$

Making use of our assumptions on f, we can show that the mapping

$$t \to \|\nabla f(t)\|_{W^{1,1}(\mathbb{R}^N)}$$

is continuous. □

In reference to Proposition 2.38 the following question can be formulated: let D be a given domain, $D \subset \mathbb{R}^N$ and let f be in $L^2(D)$, is the mapping $t \to f \circ T_t$ differentiable in $H^{-1}(D)$?

The answer to this question is positive provided that the weak topology of the space $H^{-1}(D)$ is considered.

Proposition 2.39 *Let $f \in L^2(D)$, $V \in C(0,\varepsilon; V^k(D))$ be given, $k \geqq 1$, then the mapping $t \to f \circ T_t$ is weakly differentiable in the space $H^{-1}(D)$.*

Proof. Let $\phi \in H_0^1(D)$ be given and let us introduce the following notation

$$S_t = T_t^{-1} \ ,$$
$$\lambda(t) = \gamma(t)^{-1} \circ T_t^{-1} = \gamma(t)^{-1} \circ S_t \ .$$

We have

$$\frac{1}{t}\int_D (f \circ T_t - f)\phi dx = \frac{1}{t}\int_D f(\lambda(t)\phi \circ S_t - \phi) dx \ .$$

Furthermore

$$\frac{1}{t}(\lambda(t)\phi \circ S_t - \phi) = \lambda(t)\frac{1}{t}(\phi \circ S_t - \phi) + \frac{1}{t}(\lambda(t) - 1)\phi,$$

the right–hand side of this equality converges to $-\nabla\phi \cdot V(0) + \lambda'(0)\phi$ strongly in $L^2(D)$ as $t \to 0$. Moreover it is evident that

$$\lambda'(0) = -\mathrm{div} V(0) \ .$$

It should be remarked that $S_t = T_t^{-1}$ is the transformation associated with the speed vector field $-V_t$. Therefore

$$\int_D \frac{1}{t}(f \circ T_t - f)\phi dx \to -\int_D f \text{div}(\phi V(0)) dx = \langle f \cdot V(0), \phi \rangle_{H^{-1}(D)\times H_0^1(D)}$$

as $t \to 0$; this proves Proposition 2.39. □

Remark. We present the following counterexample showing that one cannot expect that the mapping $t \to f \circ T_t$ is strongly differentiable in $H^{-1}(D)$ for any $f \in L^2(D)$.

Let $D = (-1,1)$, $V(x) = 1-x^2$ and $T_t = e^{tV}$, i.e. $T_t x = x + \int_0^t [1-(T_s x)^2] ds$. The transformation T_t maps $\overline{D}$ onto $\overline{D}$, because $V = 0$ on ∂D.

We have $V(0) = 1$ and for $t > 0$, t small enough, $T_t(0) = t\mathcal{I} + o(t)$, where $o(t)/t \to 0$ in $\mathbb{R}$ as $t \to 0$. Let us assume that

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \ . \end{cases}$$

Hence $f \in L^2(D)$. For $\phi \in H_0^1(D) \subset C(D)$ it follows that

$$\frac{1}{t}\int_D (f \circ T_t - f)\phi dx = \frac{1}{t}\int_{-t}^0 \phi(x) dx = \frac{1}{t}\int_{-t}^0 \phi(0) dx + \frac{1}{t}\int_{-t}^0 (\phi(x) - \phi(0)) dx \ .$$

Obviously, the first term on the right–hand side of this equality converges to $\phi(0) = \langle \delta_0, \phi \rangle$ and the second one converges to zero as $t \to 0$. However, this convergence is not uniform with respect to ϕ on the unit ball of the space $H_0^1(D)$. Let

$$\phi_t(x) = \begin{cases} t + \frac{x}{2t} & \text{if } \quad -2t^2 \leqq x \leqq -t^2 \\ -\frac{x}{2t} & \text{if } \quad -t^2 \leqq x \leqq 0 \\ 0 & \text{elsewhere} \ . \end{cases}$$

We have

$$\int_D |\nabla \phi_t|^2 dx = 2\int_0^{t^2} \frac{1}{2t^2} dx = 1$$

hence

$$\|\phi_t\|_{H_0^1(D)} = 1$$

whence it follows that the element ϕ_t belongs to the unit ball of the space $H_0^1(D)$. For the element ϕ_t,

$$\phi_t(0) = 0$$

and

$$\frac{1}{t}\int_{-t}^0 \phi_t(x) dx = \frac{1}{t}\int_{-t}^0 -\frac{x}{2t} dx = \frac{1}{t}\cdot\frac{1}{2t}[x^2]_{-t}^0 = \frac{1}{2} \ .$$

Therefore

$$\frac{1}{t}\int_D (f \circ T_t - f)\phi_t dx = \frac{1}{2} \quad \forall t \ .$$

Thus

$$\frac{1}{t}\|[f \circ T_t - f]\|_{H^{-1}(D)} = \sup_{\substack{\phi \in H_0^1(D) \\ \|\phi\|_{H_0^1(D)} \leqq 1}} \left| \int_D \frac{1}{t}[f \circ T_t - f]\phi dx \right| \geqq \frac{1}{2}$$

and we cannot show that

$$\lim_{t \to 0} \left\| \frac{1}{t}[f \circ T_t - f] \right\|_{H^{-1}(D)} = 0 \ .$$

This is why the conclusion of Proposition 2.39 is optimal for $f \in L^2(D)$ and one is not in a position to improve it.

Now we shall consider the differentiability of the mapping $t \to f \circ T_t$, where f is a measure or a distribution and f cannot be represented in a form of an integrable function. Let $D \subset \mathbb{R}^N$ be an open set with the piecewise smooth boundary ∂D. We make use of the following notation:

$$\begin{aligned} H^{-1}(D) &= (H_0^1(D))' \\ &= \{h \in \mathcal{D}'(D) | h = f + \operatorname{div} g, \ \text{where } f \in L^2(D) \text{ and } g \in L^2(D; \mathbb{R}^N)\} \ . \end{aligned}$$

Let $h \in H^{-1}(D)$ be a distribution, the transported distribution $f * T_t$ is defined as follows:

$$\begin{aligned} &\langle h * T_t, \phi \rangle_{\mathcal{D}'(D) \times \mathcal{D}(D)} = \langle h, (\gamma(t)^{-1}\phi) \circ T_t^{-1} \rangle_{\mathcal{D}'(D) \times \mathcal{D}(D)} = \\ &\int_D (f \circ T_t \phi - \sum_{i=1}^N g_i \frac{\partial}{\partial x_i}([\gamma(t)^{-1}\phi] \circ T_t^{-1}) dx \quad \forall \phi \in \mathcal{D}(D) \ . \end{aligned}$$

Applying the change of variables $x = T_t(X)$ we have

$$h * T_t = f \circ T_t + \gamma(t)^{-1} \operatorname{div}(DT_t^{-1} \cdot g \circ T_t) \ .$$

Let $f \in L^2(D)$, $g \in L^2(D; \mathbb{R}^N)$ be given, then from Proposition 2.39 it follows that the mappings $t \to f \circ T_t$, $t \to g \circ T_t$ are weakly differentiable in $H^{-1}(D)$ and $H^{-1}(D; \mathbb{R}^N)$, respectively. For any distribution $h \in H^{-1}(D)$ we can prove the following result on the differentiability of the mapping $t \to h \circ T_t$

Proposition 2.40 *Given an element $h \in H^{-1}(D)$ of the form*

$$h = f + \operatorname{div} g, \quad \textit{where} \quad (f, g) \in L^2(D; \mathbb{R}^{N+1}) \ .$$

The distribution transformed to the fixed domain Ω, i.e. the mapping

$$t \to h * T_t = f \circ T_t + \gamma(t)^{-1} \operatorname{div}(DT_t^{-1} \cdot g \circ T_t)$$

is weakly differentiable in $H^{-1}(D)$.

□

The strong material derivative of an element $f \in L^2(\mathbb{R}^N)$ exists in the Sobolev space $H^{-2}(\mathbb{R}^N)$.

Proposition 2.41 *Let $f \in L^2(\mathbb{R}^N)$, $V \in C(0,\varepsilon);\mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N))$ be given, then the mapping $t \to \frac{1}{t}[f \circ T_t(V) - f]$ is strongly differentiable in the space $H^{-2}(\mathbb{R}^N)$.*

Proof. Using the same reasoning as in the proof of Proposition 2.39 and applying Proposition 2.37 one can show that

$$\frac{1}{t}[\gamma(t)\phi \circ T_t - \phi] \to \operatorname{div}(\phi V(0)) \quad \text{strongly in } H^1(\mathbb{R}^N)$$

as $t \to 0$ for all $\phi \in H^2(\mathbb{R}^N)$. □

2.15. Derivatives for $t > 0$

In the above section the derivatives of different terms with respect to t were determined at $t = 0$. Using the formulae and Lemma 2.43 one can construct the form of these derivatives for $t > 0$.

First, the following lemma is to be proved.

Lemma 2.42 *We have*

$$T_{t+\varepsilon}(V) - T_t(V) = (T_\varepsilon(V_t) - \mathcal{I}) \circ T_t(V), \tag{2.107}$$

where V_t stands for the translated speed vector field

$$V_t(s,x) = V(t+s,x) \ . \tag{2.108}$$

Proof. It can be shown that

$$T_{t+\varepsilon}(V)(X) = X + \int_0^{t+\varepsilon} V(s, T_s(V)(X))ds =$$

$$[X + \int_0^t V(s, T_s(V)(X))ds] + \int_t^{t+\varepsilon} V(s, T_s(V)(X))ds =$$

$$T_t(V)(X) + \int_0^\varepsilon V_t(\rho, T_{t+\rho}(V)(X))d\rho \ .$$

Thus

$$T_{t+\varepsilon}(V)(X) = T_\varepsilon(V_t)(T_t(V)(X)) \ . \tag{2.109}$$

This proves Lemma 2.42. □

Making use of Lemma 2.42 we are in a position to construct the form of the derivatives of transformations $T_t(V)$ with respect to t for $t > 0$.

Lemma 2.43
For $t > 0$

$$\frac{\partial}{\partial t}T_t(V)(X) = \left(\frac{\partial}{\partial \varepsilon}T_\varepsilon(V_t)\right)\bigg|_{\varepsilon=0}(T_t(V)(X)) = V(t, T_t(V)(X)) \ . \tag{2.110}$$

For the sake of simplicity, we write $\frac{\partial}{\partial t}T_t = V(t)\circ T_t$. Using the same approach one can determine the derivatives with respect to t of the Jacobian matrix DT_t and the determinant $\gamma(t) = \det(DT_t)$.

Proposition 2.44 *Let $D \subset \mathbb{R}^N$ be a bounded domain, and $V \in C(0, \varepsilon; C^k(\overline{D}; \mathbb{R}^N))$, $k \geqq 1$, be a vector field, $\langle V, n\rangle_{\mathbb{R}^N} = 0$ a.e on ∂D. Furthermore it is assumed that $V = 0$ at any singular point of ∂D, i.e. at any point where the normal vector field n is not defined. Then the mappings*

$$t \to DT_t(V) \quad \textit{and} \quad t \to \gamma(t)$$

are strongly differentiable in $C^{k-1}(\overline{D}; \mathbb{R}^N)$ and $C^{k-1}(D)$, respectively. The derivatives are as follows

$$\frac{\partial}{\partial t}(DT_t(V))(x) = D_x V(t, T_t(V)(x)) \cdot DT_t(V)(x) \tag{2.111}$$

and

$$\frac{\partial}{\partial t}\gamma(t) = \operatorname{div} V(t, T_t(V))\gamma(t) \ . \tag{2.112}$$

For the sake of simplicity we shall write

$$\frac{\partial}{\partial t}DT_t = DV(t) \circ T_t \cdot DT_t \ ,$$

and

$$\frac{\partial}{\partial t}\gamma(t) = \gamma(t)\operatorname{div} V(t) \circ T_t \ .$$

Proof. It is sufficient to show that

$$\begin{aligned} DT_{t+\varepsilon}(V) - DT_t(V) &= D[T_\varepsilon(V) \circ T_t(V)] - DT_t(V) \\ &= (DT_\varepsilon(V_t) - \mathcal{I}) \circ T_t(V) \cdot DT_t(V) \ , \end{aligned}$$

and

$$\gamma(t + \varepsilon) = \det DT_\varepsilon(V_t) \circ T_t(V)\gamma(t) \ .$$

□

2.16. Derivatives of domain integrals

Let $V \in \mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N)$, $k \geqq 1$, $Y \in W^{1,1}(\mathbb{R}^N)$, and let $\Omega \subset \mathbb{R}^N$ be a measurable subset. Moreover the standard notation is used,

$$\Omega_t = T_t(V)(\Omega) \ .$$

We shall consider the following domain functional

$$J(\Omega) = \int_\Omega Y dx \ .$$

Proposition 2.45 *The domain functional $J(\Omega)$ is shape differentiable,*

$$dJ(\Omega;V) = \left(\frac{\partial}{\partial t}\int_{\Omega_t} Y dx\right)\bigg|_{t=0} = \int_\Omega \mathrm{div}(YV(0))dx \ . \tag{2.113}$$

Proposition 2.46 *If Ω is a domain with the boundary Γ of class C^k, $k \geqq 1$, then*

$$dJ(\Omega;V) = \int_\Gamma Y\langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma \ . \tag{2.114}$$

Proof. First using the change of variables $x = T_t(V)(X)$ the integral defined on Ω_t is transformed to the domain Ω, hence

$$J(\Omega_t) = \int_{\Omega_t} Y dx = \int_\Omega Y \circ T_t(V)\gamma(t)dx,$$

where $\gamma(t) = \det(DT_t)$, DT_t is the Jacobian matrix of the transformation $T_t(V)$. From Proposition 2.44 it follows that $\gamma'(0) = \mathrm{div}V(0)$, thus

$$\frac{d}{dt}J(\Omega_t)\bigg|_{t=0} = \int_\Omega (\nabla y \cdot V(0) + y\mathrm{div}V(0))dx$$

because of

$$\nabla y \cdot V + y\mathrm{div}V = \mathrm{div}(yV).$$

If Γ is a C^k manifold, then Stokes' formula yields (2.114). □

2.17. Change of variables in boundary integrals

Let Γ be a C^k manifold, and let $V \in C(0,\varepsilon;\mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N))$, $f \in L^1(\Gamma)$, where $\Gamma_t = T_t(V)(\Gamma)$. The manifold Γ_t is covered by the family of open sets $U_i^t =$

$T_t(U_i)$. We define the functions $c_i^t = c_i \circ T_t^{-1}$ from U^t into B, $i = 1, \dots, m$, and $h_i^t = (c_i^t)^{-1} = T_t \circ c_i^{-1}$. By the same reasoning as in Sect. 2.1 it follows that

$$\int_{\Gamma_t} f d\Gamma_t = \sum_{i=1}^{m} \int_{B_0} (r_i^t f) \circ h_i^t \|M(h_i^t) \cdot e_N\|_{\mathbb{R}^N} d\xi',$$

where $r_i^t = r_i \circ T_t^{-1}$, $i = 1, 2, \dots, m$, are such that $\sum_{i=1}^m r_i^t \equiv 1$ in an open neighbourhood of Γ_t in $\mathbb{R}^N$; we use the notation $(r_i^t f) \circ h_i^t = [r_i(f \circ T_t)] \circ c_i^{-1}$. It can be verified that the following chain rule formula holds:

$$M(g \circ h) = M(g) \circ h \cdot M(h),$$

where M is the cofactor matrix defined in Sect. 2.1. Hence

$$M(h_i^t) = M(T_t) \circ h_i \cdot M(h_i)$$

and

$$\int_{\Gamma_t} f d\Gamma_t = \sum_{i=1}^{m} \int_{B_0} [r_i(f \circ T_t)] \circ h_i \|M(T_t) \circ h_i \cdot M(h_i) \cdot e_N\|_{\mathbb{R}^N} d\xi' \ .$$

The normal field on $\Gamma_i = \Gamma \cap U_i$ is given by

$$n = \|M(h_i) \cdot e_N\|_{\mathbb{R}^N}^{-1} M(h_i) \cdot e_N \ .$$

Furthermore

$$\|M(T_t) \circ h_i \cdot M(h_i) \cdot e_N\|_{\mathbb{R}^N} = \|M(T_t) \cdot n\|_{\mathbb{R}^N} \circ h_i \|M(h_i) \cdot e_N\|_{\mathbb{R}^N} \quad (2.115)$$

and

$$\int_{\Gamma_t} f d\Gamma_t = \sum_{i=1}^{m} \int_{\Gamma} r_i f \circ T_t \|M(T_t) \cdot n\|_{\mathbb{R}^N} d\Gamma \ .$$

The foregoing will enable us to derive, using the transformation T_t, the formula for the transformation of boundary integrals:

Proposition 2.47 *For any $f \in L^1(\Gamma_t)$,*

$$\int_{\Gamma_t} f d\Gamma_t = \int_{\Gamma} f \circ T_t \|M(T_t) \cdot n\|_{\mathbb{R}^N} d\Gamma, \quad (2.116)$$

where $M(T_t) = \det(DT_t)^* DT^{-1}$ is the cofactor matrix of the Jacobian matrix DT_t.

Let

$$\omega(t) = \|M(DT_t) \cdot n\|_{\mathbb{R}^N} \quad (2.117)$$

or equivalently

$$\omega(t) = \gamma(t)\|{}^*(DT_t)^{-1} \cdot n\|_{\mathbb{R}^N} , \tag{2.118}$$

where $\gamma(t) = \det(DT_t)$, $\gamma(t) > 0$ on Γ.

The unit normal field $\mathbf{n_t}$ on Γ_t

The normal vector field m_t is defined locally on Γ_t

$$m_t(x) = {}^*D(T_t \circ h)^{-1}(\xi) \cdot e_N,$$

where

$$\xi = (T_t \circ h)^{-1}(x) \quad \text{for} \quad x \in \Gamma_t \cap T_t(U_i).$$

Hence the outward unit normal field on Γ_t is given by

$$n_t(x) = \|M(T_t \circ h) \cdot e_N\|^{-1}_{\mathbb{R}^N} M(T_t \circ h) \cdot e_N \ .$$

On the other hand it can be shown that

$$M(T \circ h) = M(T) \circ h \cdot M(h)$$

whence

$$n_t(x) = (\|M(T_t) \circ h \cdot M(h) \cdot e_N\|^{-1}_{\mathbb{R}^N} M(T_t) \circ h \cdot M(h) \cdot e_N)(\xi) \ .$$

Since we have

$$n(T_t^{-1}(x)) = (\|M(h) \cdot e_N\|^{-1}_{\mathbb{R}^N} M(h) \cdot e_N)(\xi),$$

then the form of the unit normal vector field on Γ_t can be derived from (2.116).

Proposition 2.48 *The unit normal vector field on Γ_t is given by*

$$n_t(T_t(X)) = (\|{}^*DT_t^{-1} \cdot n\|^{-1}_{\mathbb{R}^N} {}^*(DT_t)^{-1} \cdot n)(X)$$

for $X \in \Gamma$.

2.18. Derivatives of boundary integrals

Let $f \in W^{2,1}(\mathbb{R}^N)$, $V \in C(0,\varepsilon;\mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N))$, $k \geqq 1$, and let $\Omega \subset \mathbb{R}^N$ be a domain with the boundary Γ of class C^k. We shall consider the following surface integral on Γ_t:

$$J(\Omega_t) = \int_{\Gamma_t} f d\Gamma_t = \int_\Gamma f \circ T_t \omega(t) d\Gamma \ .$$

Making use of Proposition 2.45 we can state the following lemma

Lemma 2.49 *The mapping $t \to \omega(t)$ is differentiable from $[0,\delta)$ into $C^{k-1}(\mathbb{R}^N)$, $\delta > 0$, the derivative $\omega'(t)$ at $t = 0$ is given by*

$$\omega'(0) = \mathrm{div}V(0) - \langle DV(0) \cdot n, n\rangle_{\mathbb{R}^N} \ . \tag{2.119}$$

Furthermore for any compact $\overline{\mathcal{O}} \subset \Omega$ and all multi-indices $\alpha \in \mathbb{N}^N$, $|\alpha| = \alpha_1 + \cdots + \alpha_N \leqq 1$,

$$\lim_{t\to 0}\max_{x\in\overline{\mathcal{O}}}\left\| \left(\frac{\partial}{\partial x}\right)^\alpha \left[\frac{\omega(t)-1}{t} - \omega'(0)\right]\right\|_{\mathbb{R}^N} = 0 \ .$$

Proof. From (2.118) it follows that

$$\omega(t)^2 = \langle A(t)\cdot n, n\rangle_{\mathbb{R}^N},$$

where $A(t)$ is the $N \times N$ matrix,

$$A(t) = \gamma^2(t)(DT_t)^{-1} \cdot {}^*(DT)^{-1} \ .$$

Using the differentiability properties of $\gamma(t)$ and DT_t^{-1} with respect to t, at $t = 0$, one can prove the proposition as stated. □

Proposition 2.50 *Let $f \in W^{2,1}(\mathbb{R}^N)$ and $V \in C(0,\varepsilon;\mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N))$, $k \geqq 1$ be given. The functional*

$$J(\Omega) = \int_\Gamma f d\Gamma$$

is shape differentiable. The Eulerian derivative is given by

$$\begin{aligned} dJ(\Omega; V) &= \left(\frac{d}{dt}\int_{\Gamma_t} f d\Gamma_t\right)\Big|_{t=0} \\ &= \int_\Gamma \{\nabla f \cdot V(0) + f[\mathrm{div}V(0) - \langle DV(0)\cdot n, n\rangle_{\mathbb{R}^N}]\} d\Gamma \ . \end{aligned} \tag{2.120}$$

Proof. Applying the change of variables, the same as used in (2.116), we obtain

$$\frac{1}{t}(J(\Omega_t) - J(\Omega)) = \int_\Gamma \frac{1}{t}(f\circ T_t - f)\omega(t) d\Gamma + \int_\Gamma f\left(\frac{\omega(t)-1}{t}\right) d\Gamma \ .$$

From Proposition 2.37 it follows that

$$\frac{1}{t}(f \circ T_t - f) \to \nabla f \cdot V(0) \quad \text{strongly in } W^{1,1}(\mathbb{R}^N)$$

as $t \to 0$. For the trace operator on Γ we have $\gamma_\Gamma[\frac{1}{t}(f\circ T_t - f)] \to \gamma_\Gamma \cdot (\nabla f \cdot V(0))$ strongly in $L^1(\Gamma)$ as $t \to 0$. It can be verified that $\omega(t) \to 1$ strongly in $L^\infty(\Gamma)$ as $t \to 0$, hence

$$\frac{1}{t}\int_\Gamma (f\circ T_t - f)\omega(t)d\Gamma \to \int_\Gamma \nabla f\cdot V d\Gamma$$

as $t\to 0$. Since $f\in L^1(\Gamma)$ and $(\omega(t)-1)/t \to \omega'(0)$ strongly in $L^\infty(\Gamma)$ as $t\to 0$ thus

$$\int_\Gamma \frac{1}{t}(\omega(t)-1)f d\Gamma \to \int_\Gamma \omega'(0) f d\Gamma$$

as was to be shown. □

2.19. The tangential divergence of the field V on Γ

Let D be an open set in $\mathbb{R}^N$ with the boundary ∂D; it is assumed that ∂D is piecewise C^1. For any bounded domain $\Omega\subset D$, with the boundary $\Gamma=\partial D$ of class C^1, the domain functional

$$\Omega\to J(\Omega)=\int_\Gamma f d\Gamma$$

is shape differentiable for any element $f\in W^{2,1}(D)$. In other words for any vector field $V\in C(0,\varepsilon;C^1(D;\mathbb{R}^N))$,

$$\langle V(t,x),n(t)\rangle_{\mathbb{R}^N}=0 \quad \text{for} \quad x\in\partial D,$$

(except for the singular points $\overline{x}$ of the boundary ∂D, at which it is supposed that $V(t,\overline{x})=0$), the mapping

$$t\to\int_{T_t(V)(\Gamma)} f d\Gamma_t$$

is differentiable. (2.120) implies that the Eulerian derivative

$$\frac{d}{dt}\left(\int_{T_t(V)(\Gamma)} f d\Gamma\right)\Bigg|_{t=0} = dJ(\Omega;V)$$

is linear and continuous with respect to the vector field $V\in C(0,\varepsilon;C^1(\overline{D};\mathbb{R}^N))$. By the reasoning presented in the previous section it follows that the shape functional $J(\cdot)$ is shape differentiable for $\Gamma=\partial\Omega$ of class C^1 and for $V\in C(0,\varepsilon;C^1(\overline{D};\mathbb{R}^N))$.

Now let us suppose that Ω is a given domain with the boundary Γ of class C^{k+1}. In particular, if $k=1$, then Γ is of class C^2. From Theorem 2.27. it follows that the shape gradient G can be given in the following form $G={}^*\gamma_\Gamma(gn)$, where $g\in\mathcal{D}^{-1}(\Gamma)$ is a scalar distribution supported on Γ. The outward normal field n on Γ is chosen as an orientation on Γ. In particular, the transverse order of the distribution G is zero and we have

$$\langle G, V(0)\rangle = \int_\Gamma g\langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma \ .$$

It now remains to identify the element g, referred to as the density of the shape gradient G.

Proposition 2.51 *Let $\Omega \subset D$ be a given domain with the boundary Γ of class C^2. Suppose that $V_1, V_2 \in C^1(\overline{D}; \mathbb{R}^N)$, and*
(i) $\langle V_i, n\rangle_{\mathbb{R}^N} = 0$ on ∂D except for the singular points of ∂D at which $V_i = 0$, $i = 1, 2$.
(ii) $V_1|_\Gamma = V_2|_\Gamma$, where $V_i|_\Gamma \in C^1(\Gamma)$, $i = 1, 2$.
Then for all $x \in \Gamma$ the following identity holds

$$\begin{aligned} &\mathrm{div} V_1(x) - \langle DV_2(x)\cdot n(x), n(x)\rangle_{\mathbb{R}^N} = \\ &\mathrm{div} V_2(x) - \langle DV_2(x)\cdot n(x), n(x)\rangle_{\mathbb{R}^N} \ . \end{aligned} \tag{2.121}$$

Proof. Let V stand for $V_2 - V_1$; hence $V \equiv 0$ on Γ. In particular $\langle V, n\rangle_{\mathbb{R}^N} = 0$ on Γ and

$$\langle G, V\rangle = \int_\Gamma g\langle V, n\rangle_{\mathbb{R}^N} d\Gamma = 0 \ .$$

From (2.120) it follows that

$$\begin{aligned} &\int_\Gamma f[\mathrm{div} V - \langle DV\cdot n, n\rangle_{\mathbb{R}^N}] d\Gamma = \\ &\langle g, (\gamma_\Gamma V)\cdot n\rangle_{\mathcal{D}^{-1}(\Gamma)\times\mathcal{D}^1(\Gamma)} - \int_\Gamma \nabla f\cdot V d\Gamma = 0 \ . \end{aligned}$$

The element $f \in H^1(\Gamma)$ is arbitrary, hence

$$\mathrm{div} V - \langle DV\cdot n, n\rangle_{\mathbb{R}^N} = 0 \quad \text{on } \Gamma \ .$$

□

From the foregoing it is evident that we are in a position to define the tangential divergence of V on Γ.

Definition 2.52
(i) Let Ω be a given domain with the boundary Γ of class C^2, and $V \in C^1(U; \mathbb{R}^N)$ be a vector field; U is an open neighbourhood of the manifold $\Gamma \subset \mathbb{R}^N$. Then the following notation is used to define the tangential divergence

$$\mathrm{div}_\Gamma V = (\mathrm{div} V - \langle DV\cdot n, n\rangle_{\mathbb{R}^N})\,|_\Gamma \in C(U) \ .$$

(ii) Let Ω be a bounded domain with the boundary of class C^2, and let $V \in C^1(\Gamma; \mathbb{R}^N)$ be a given vector field on Γ. The tangential divergence of V on Γ is given by

$$\mathrm{div}_\Gamma V = (\mathrm{div}\tilde{V} - \langle D\tilde{V} \cdot n, n\rangle_{\mathbb{R}^N})|_\Gamma \in C(\Gamma),$$

where $\tilde{V}$ is any C^1 extension of V to an open neighbourhood of $\Gamma \subset \mathbb{R}^N$.

It should be remarked that the notion of tangential divergence introduced above is known in the differential geometry and in fact for any vector field $V \in C^1(\Gamma; \mathbb{R}^N)$ we have

$$\mathrm{div}_\Gamma V \equiv V^\alpha|_\alpha,$$

where $u|_\alpha$ is the so–called covariant derivative on the manifold Γ (see e.g. (Kosinski 1986)).

The notion of tangential gradient ∇_Γ on Γ

$$\nabla_\Gamma : C^2(\Gamma) \to C^1(\Gamma; \mathbb{R}^N)$$

is also introduced.

Definition 2.53 Let an element $h \in C^2(\Gamma)$ be given and let $\tilde{h}$ be an extension of h, $\tilde{h} \in C^2(U)$ and $\tilde{h}|_\Gamma = h$ on Γ; U is an open neighbourhood of Γ in $\mathbb{R}^N$. The following notation is used:

$$\nabla_\Gamma h = \nabla\tilde{h}|_\Gamma - \frac{\partial \tilde{h}}{\partial n} n \ . \tag{2.122}$$

It can be shown that such an extension $\tilde{h}$ exists: for this purpose we shall use the notation of Sect. 2.1. Let $x \in U$, $\xi = c(x)$ and $\xi_0 = (\xi', 0) \in B_0$, then $\tilde{h}(x) = h(c^{-1}(\xi_0))$. The functions c and c^{-1} are in $C^2(U)$ thus $\tilde{h} \in C^2$.

Finally we can prove that the term $\nabla\tilde{h}|_\Gamma - \frac{\partial \tilde{h}}{\partial n} n$ on the right–hand side of (2.122) is well defined and independent of the choice of the C^2 extension $\tilde{h}$ of h. For this purpose the functional

$$J(\Omega) = \int_\Gamma \tilde{h} d\Gamma$$

is to be considered, where $\tilde{h}$ is an extension of h; for $V \in \mathcal{D}^1(D; \mathbb{R}^N)$ we have

$$dJ(\Omega; V) = \int_\Gamma (\nabla\tilde{h} \cdot V + h\mathrm{div}_\Gamma V) d\Gamma =$$

$$\int_\Gamma \left(\nabla\tilde{h} - \frac{\partial \tilde{h}}{\partial n} n\right) \cdot V d\Gamma + \int_\Gamma \left(\frac{\partial \tilde{h}}{\partial n} V \cdot n + h\mathrm{div}_\Gamma V\right) d\Gamma \ .$$

From Proposition 2.26 it follows that

$$dJ(\Omega; V) = \langle g(h), V \cdot n \rangle_{\mathcal{D}^{-1}(\Gamma)\times\mathcal{D}^{1}(\Gamma)} \ .$$

The vector field V is chosen such that

$$V \cdot n \equiv \langle V, n \rangle_{\mathbb{R}^N} = 0 \quad \text{on } \Gamma.$$

Hence

$$\int_\Gamma \left(\nabla \tilde{h} - \frac{\partial \tilde{h}}{\partial n} n \right) \cdot V d\Gamma = - \int_\Gamma h \operatorname{div}_\Gamma V d\Gamma \ . \tag{2.123}$$

Now let us suppose that $h = 0$ on Γ, in view of (2.123),

$$\int_\Gamma \left(\nabla \tilde{h} - \frac{\partial \tilde{h}}{\partial n} n \right) \cdot V d\Gamma = 0 \ .$$

Therefore $(\nabla \tilde{h} - \frac{\partial \tilde{h}}{\partial n} n)(x)$ is an element of the tangent space $\mathcal{T}_x\Gamma$ for any $x \in \Gamma$. Since it is supposed that $\tilde{h} \in C^2(U)$, then one can choose V so that $V = \nabla \tilde{h} - \frac{\partial \tilde{h}}{\partial n} n$ on Γ. Hence

$$\int_\Gamma \left\| \nabla \tilde{h} - \frac{\partial \tilde{h}}{\partial n} \right\|^2_{\mathbb{R}^N} d\Gamma = 0$$

and

$$\left(\nabla \tilde{h} - \frac{\partial \tilde{h}}{\partial n} n \right)(x) = 0$$

for all $x \in \Gamma$. This indicates that the following proposition can be stated:

Proposition 2.54 *Let $\tilde{h} \in C^2(U)$ be a given element. The restriction to Γ of the vector field $\nabla \tilde{h} - \frac{\partial \tilde{h}}{\partial n} n$ depends on the restriction $h = \tilde{h}|_\Gamma$, i.e. $h = 0$ implies that $\nabla \tilde{h} - \frac{\partial \tilde{h}}{\partial n} n = 0$ on Γ.*

□

Remark. $\operatorname{div}_\Gamma V$ and $\nabla_\Gamma \tilde{h}$ are defined on Γ for fields $V \in C^2(U; \mathbb{R}^N)$ and $\tilde{h} \in C^2(U)$. These definitions can be easily extended to elements $\tilde{V}$ and $\tilde{h}$ in the spaces $H^{s+\frac{1}{2}}(U; \mathbb{R}^N)$ and $H^s(U)$ for any $s > 1$, respectively. It can be accomplished by means of the usual density argument of the space $\mathcal{D}(\overline{U})$ in the Sobolev space $H^{s+\frac{1}{2}}(U)$. Let a sequence $\{V_k\} \subset \mathcal{D}(U; \mathbb{R}^N)$ be given such that $V_k \to \tilde{V}$ in the space $H^s(U; \mathbb{R}^N)$ as $k \to \infty$. In other words, if $\operatorname{div}\tilde{V}_k - \langle D\tilde{V}_k \cdot \mathcal{N}_0, \mathcal{N}_0 \rangle_{\mathbb{R}^N}$ converges to $\operatorname{div} V - \langle DV \cdot \mathcal{N}_0, \mathcal{N}_0 \rangle_{\mathbb{R}^N}$ (where $\mathcal{N}_0$ is a C^{k-1} extension of n to U) in the norm of Sobolev space $H^{s-\frac{1}{2}}(U)$, then $\operatorname{div}_\Gamma V_k$ converges in the norm of Sobolev space $H^{s-\frac{1}{2}}(\Gamma)$ to $\operatorname{div}_\Gamma V$, where V is defined as follows

$$V = \tilde{V}|_\Gamma \in H^s(\Gamma;\mathbb{R}^N)$$

Furthermore let a sequence $\{\tilde{h}_k\} \subset D(U)$ be given such that $\tilde{h}_k \to \tilde{h}$ in the space $H^{s+\frac{1}{2}}(U)$ as $k \to \infty$. Then $\nabla\tilde{h}_k - \frac{\partial}{\partial n}\tilde{h}_k n$ converges to $\nabla\tilde{h} - \frac{\partial}{\partial n}\tilde{h}n$ in the space $H^{s-1}(\Gamma)$; the limit element is denoted by $\nabla_\Gamma h$, where $h = \tilde{h}|_\Gamma \in H^s(\Gamma)$.

Proposition 2.55 *For all $V \in H^s(\Gamma;\mathbb{R}^N)$ and $h \in H^s(\Gamma)$ with $s > 1$, $\mathrm{div}_\Gamma V$ and $\nabla_\Gamma h$ are well defined elements of the spaces $H^{s-1}(\Gamma;\mathbb{R}^N)$ and $H^{s-1}(\Gamma)$, respectively.*

□

2.20. Tangential gradients and Laplace–Beltrami operators on Γ

First, an equivalent definition of the Sobolev space $H^1(\Gamma)$ on the manifold Γ is given. We introduce the scalar product

$$(\phi,\psi)_{H^1(\Gamma)} = \int_\Gamma (\phi\psi + \nabla_\Gamma\phi\cdot\nabla_\Gamma\psi)d\Gamma$$

which is well defined for all $\phi,\psi \in C^k(\Gamma)$, Γ is of class C^k, $k \geqq 1$. The Sobolev space $H^1(\Gamma)$ is the closure of the space $C^k(\Gamma)$ with respect to the norm induced by the scalar product. Therefore the space $C^k(\Gamma)$ is dense by definition in the space $H^1(\Gamma)$, and the formula (2.113) can be extended to the space $H^1(\Gamma)$ in the following way.

Definition 2.56 For any element $h \in H^1(\Gamma)$, $\nabla_\Gamma h$ is by definition an element of the space $L^2(\Gamma;\mathbb{R}^N)$ such that for all vector fields

$$V \in C^1(\Gamma;\mathbb{R}^N),\ \langle V,n\rangle_{\mathbb{R}^N} = 0 \quad \text{on } \Gamma$$

we have

$$\int_\Gamma \nabla_\Gamma h\cdot V d\Gamma = -\int_\Gamma h\,\mathrm{div}_\Gamma V d\Gamma\ . \tag{2.124}$$

Remark. By means of (2.124) we can define the tangential divergence $\mathrm{div}_\Gamma V$ of any vector field $V \in H^1_\Gamma(\Gamma;\mathbb{R}^N)$, where

$$H^1_\Gamma(\Gamma;\mathbb{R}^N) = \{V \in H^1(\Gamma;\mathbb{R}^N) | \langle V,n\rangle_{\mathbb{R}^N} = 0 \ \text{ a.e. on } \Gamma\}\ .$$

$\mathrm{div}_\Gamma V \in L^2(\Gamma)$ is defined as follows

$$\int_\Gamma \mathrm{div}_\Gamma V h d\Gamma = -\int_\Gamma \nabla_\Gamma h\cdot V d\Gamma,$$

for all $h \in H^1(\Gamma)$.

Finally the foregoing result may be stated as the following proposition:

Proposition 2.57 *Let $\Omega \subset \mathbb{R}^N$ be a domain of class C^2. For all elements $(h, V) \in H^1(\Gamma) \times H^1_\Gamma(\Gamma; \mathbb{R}^N)$ we have*

$$\int_\Gamma \nabla_\Gamma h \cdot V d\Gamma = -\int_\Gamma h \mathrm{div}_\Gamma V d\Gamma \ . \tag{2.125}$$

If for $V \in H^1(\Gamma; \mathbb{R}^N)$ the tangential component of V is defined in the following way

$$V_\tau = V - \langle V, n \rangle_{\mathbb{R}^N} n,$$

then

$$\mathrm{div}_\Gamma(V) = \mathrm{div}_\Gamma(V_\tau) + \kappa \langle V, n \rangle_{\mathbb{R}^N}$$

here κ stands for the mean curvature of Γ.

□

Proposition 2.58 *Let Ω be a domain of class C^k for $k \geqq 2$, $N = 3$, i.e $\Omega \subset \mathbb{R}^3$. If $S \subset \Gamma$ is a smooth manifold with the boundary ∂S, then (2.125) represents the well known Stokes' formula*

$$\int_S \mathrm{div}_\Gamma V d\Gamma = \int_{\partial S} V \cdot \nu d\ell, \tag{2.126}$$

where ν is the unit tangent vector to the manifold Γ, normal to the boundary ∂S of S, ν is outward pointing on ∂S.

□

The identities (2.125) and (2.126) provide formulae for integration by part on the boundary: the former for the manifold Γ without boundary; and the latter for a part S of the manifold Γ; S has the boundary ∂S which is a $(N-2)$ – dimensional manifold.

In the next section the formula (2.125) will be extended to any vector field $V \in H^1(\Gamma; \mathbb{R}^N)$ which in general is not tangent to Γ.

Definition 2.59 Let $h \in H^2(\Gamma)$, then we have $\nabla_\Gamma h \in H^1_\Gamma(\Gamma; \mathbb{R}^N)$. The Laplace–Beltrami operator Δ_Γ on Γ is defined as follows

$$\Delta_\Gamma h = \mathrm{div}_\Gamma(\nabla_\Gamma h) \quad \forall h \in H^2(\Gamma) \ .$$

Hence $\Delta_\Gamma h \in L^2(\Gamma)$, and from (2.125) it follows that the element $\Delta_\Gamma h \in L^2(\Gamma)$ is uniquely determined by the integral identity

$$\int_\Gamma \Delta_\Gamma h \psi d\Gamma = -\int_\Gamma \nabla_\Gamma h \cdot \nabla \psi d\Gamma \quad \forall \psi \in H^1(\Gamma). \tag{2.127}$$

It can be inferred from (2.127) that for any fixed element $h \in H^1(\Gamma)$ the mapping

$$\psi \to \int_\Gamma \Delta_\Gamma h \psi d\Gamma$$

is well defined as a linear continuous form on the Sobolev space $H^1(\Gamma)$. Thus for any element $h \in H^1(\Gamma)$ we have

$$\Delta_\Gamma h \in (H^1(\Gamma))' = H^{-1}(\Gamma) \ .$$

The linear subspace $H^1(\Gamma)/\mathbb{R}$ of $H^1(\Gamma)$ can be defined in the standard way. The scalar product on $H^1(\Gamma)/\mathbb{R}$ is defined as follows

$$(\phi, \psi)_\Gamma = \int_\Gamma \nabla_\Gamma \phi \nabla_\Gamma \psi d\Gamma \ .$$

If $(\phi, \psi)_\Gamma = 0$ for a given element $\phi \in H^1(\Gamma)$, then $\nabla_\Gamma \phi = 0$ a.e. on Γ and $\phi =$ constant. Since $\phi \in H^1(\Gamma)$, then there exists (Lions et al. 1968) an element $\Phi \in H^{3/2}(\Omega)$, the extension of ϕ, and

$$\Phi|_\Gamma = \phi, \quad \text{furthermore} \quad \frac{\partial \Phi}{\partial n} = 0 \quad \text{on } \Gamma \ .$$

Therefore $\nabla \Phi = \nabla_\Gamma \phi$ on Γ, and ϕ is a constant, that is $\phi = 0$ as an element of the quotient space $H^1(\Gamma)/\mathbb{R}$.

Remark. It should be noted that on the right–hand side of (2.127) there is the scalar product of vector fields $\nabla_\Gamma h$ and $\nabla_\Gamma \psi$ tangent to the manifold $\Gamma = \partial\Omega$. On the other hand, if ψ is a smooth function defined in an open neighbourhood of Γ in $\mathbb{R}^N$, then

$$\nabla_\Gamma h \cdot (\nabla \psi|_\Gamma) = \nabla_\Gamma h \cdot \nabla_\Gamma \psi$$

because of

$$\frac{\partial \psi}{\partial n} n \cdot \nabla_\Gamma h = 0.$$

Hence, if ψ is the restriction to Γ of a given function ψ defined in $\mathbb{R}^N$, then

$$\int_\Gamma \Delta_\Gamma h \psi d\Gamma = - \int_\Gamma \nabla_\Gamma h \cdot \nabla \psi d\Gamma \quad \forall \psi \in H^2(\mathbb{R}^N) \ .$$

2.21. Variational problems on Γ

Let us introduce the following bilinear form

$$a_\Gamma(\phi, \psi) = \int_\Gamma \langle A \cdot \nabla_\Gamma \phi, \nabla_\Gamma \psi \rangle_{\mathbb{R}^N} d\Gamma, \tag{2.128}$$

where $A(x)$ is a $N \times N$ matrix defined for $x \in \Gamma$, $A \in L^\infty(\Gamma; \mathbb{R}^{N^2})$ with

$$\alpha \|\xi\|^2_{\mathbb{R}^N} \leqq \langle A(x) \cdot \xi, \xi \rangle_{\mathbb{R}^N} \leqq \beta \|\xi\|^2_{\mathbb{R}^N}$$

for all $\xi \in \mathbb{R}^N$, where $0 < \alpha < \beta$; α, β are given constants.

Let

$$L(\cdot) \in (H^1(\Gamma)/\mathbb{R})' \tag{2.129}$$

be a given linear form. By direct application of Theorem 2.3 one can prove the existence and uniqueness of the solution to the variational problem associated with the bilinear form (2.128).

Proposition 2.60 *There exists the unique element $y \in H^1(\Gamma)/\mathbb{R}$ such that*

$$\int_\Gamma \langle A \cdot \nabla_\Gamma y, \nabla_\Gamma \phi \rangle_{\mathbb{R}^N} d\Gamma = L(\phi) \quad \forall \phi \in H^1(\Gamma)/\mathbb{R} \ . \tag{2.130}$$

□

A simple example of the linear form $L(\cdot) \in (H^1(\Gamma)/\mathbb{R})'$ is as follows. Let the following condition holds

$$\int_\Gamma f d\Gamma = 0 \tag{2.131}$$

for a given $f \in L^2(\Gamma)$.

We define the linear form

$$L(\phi) = \int_\Gamma f \phi d\Gamma \quad \forall \phi \in H^1(\Gamma)/\mathbb{R} \ .$$

For such a choice of $L(\cdot)$ the corresponding solution $y = y(\Gamma)$ to the problem (2.130) satisfies the Laplace–Beltrami equation on the manifold Γ:

$$-\mathrm{div}_\Gamma(A \cdot \nabla_\Gamma y) = f \quad \text{in } L^2(\Gamma) \ . \tag{2.132}$$

Here it is supposed that Γ is a sufficiently smooth manifold without boundary. In Chap. 3 variational problems will be considered for Γ only piecewise C^k and boundary conditions defined on the singular part of Γ.

Finally the fourth order problem on the manifold Γ is discussed. The scalar product $((\phi, \psi))$ is given by

$$((\phi, \psi)) = \int_\Gamma (\Delta_\Gamma \phi \Delta_\Gamma \psi + \nabla_\Gamma \phi \cdot \nabla \psi + \phi \psi) d\Gamma \tag{2.133}$$

for ϕ, $\psi \in C^2(\Gamma; \mathbb{R}^N)$

The Sobolev space $H^2(\Gamma)$ is the closure of the space $C^2(\Gamma)$, Γ is C^k for $k \geqq 2$, with respect to the norm induced by the scalar product (2.133).

For a given matrix function $A(\cdot)$ satisfying the same conditions as in (2.128), and an element $h(\cdot) \in L^\infty(\Gamma)$,

$$0 < \alpha_0 \leqq h(x) \leqq \beta_0 \quad \text{for a.e. } x \in \Gamma,$$

where $0 < \alpha_0 \leqq \beta_0$, the following bilinear form is constructed

$$a(\phi,\psi) = \int_\Gamma \{h\Delta_\Gamma\phi\Delta_\Gamma\psi + \langle A\cdot\nabla_\Gamma\phi, \nabla_\Gamma\psi\rangle_{\mathbb{R}^N} + \phi\psi\}d\Gamma \qquad \forall\phi,\psi \in H^2(\Gamma)\ . \tag{2.134}$$

Using Theorem 2.3 one can prove the existence and uniqueness of the solution to the variational problem associated with the bilinear form (2.134).

Proposition 2.61 *Let $f \in L^2(\Gamma)$ be given. There exists the unique element $y(\Gamma) \in H^2(\Gamma)$ such that*

$$\Delta_\Gamma(h\Delta_\Gamma y) - \mathrm{div}_\Gamma(A\cdot\nabla_\Gamma y) + y = f \quad in\ L^2(\Gamma)\ . \tag{2.135}$$

□

The quotient space $H^2(\Gamma)/\mathbb{R}$ equipped with the scalar product

$$(\phi,\psi) = \int_\Gamma (\Delta_\Gamma\phi\Delta_\Gamma\psi + \nabla_\Gamma\phi\cdot\nabla_\Gamma\psi)d\Gamma$$

can be examined in the same way as in the case of the second order problems on Γ. For any element $f \in L^2(\Gamma)$ satisfying the condition (2.131), one can prove the existence and uniqueness of a weak solution to the problem

$$y \in H^2(\Gamma)/\mathbb{R}: \quad \Delta_\Gamma(h\Delta_\Gamma y) - \Delta_\Gamma y = f \quad \text{in } L^2(\Gamma)\ . \tag{2.136}$$

2.22. The transport of differential operators

We have

$$(\nabla\phi)\circ T_t = {}^*DT_t^{-1}\cdot\nabla(\phi\circ T_t)\ .$$

Making use of the notation

$$A(t) = \gamma(t)\cdot(DT_t)^{-1}\cdot{}^*DT_t)^{-1}, \quad \gamma(t) = \det(DT_t)$$

the following formulae can be derived.

Lemma 2.62 *Let $\phi \in H^2(\mathbb{R}^N)$ and $\varphi \in H^4(\mathbb{R}^N)$, then*

$$(\Delta\phi)\circ T_t = \gamma(t)^{-1}\mathrm{div}(A(t)\cdot\nabla(\phi\circ T_t))$$
$$(\Delta^2\varphi)\circ T_t = \gamma(t)^{-1}\mathrm{div}(A(t)\cdot\Delta(\gamma(t)^{-1}\mathrm{div}(A\cdot\nabla(\varphi\circ T_t)))) \ .$$

□

Next the transport of tangential operators on the manifold Γ is to be considered. Let $\tilde{h}\in C^2(\mathbb{R}^N)$ be given and let Γ be a manifold of class C^2; for a given vector field $V\in C(0,\varepsilon;\mathcal{D}^1(\mathbb{R}^N;\mathbb{R}^N))$ the transformed boundary is denoted by $\Gamma_t = T_t(V)(\Gamma)$. Let $h_t = \tilde{h}|_{\Gamma_t}\in C^2(\Gamma_t)$; the tangential gradient $\nabla_{\Gamma_t}h_t$ is defined as follows

$$\nabla_{\Gamma_t}h_t = \nabla\tilde{h} - \langle n_t,\nabla\tilde{h}\rangle_{\mathbb{R}^N}n_t \quad \text{on } \Gamma_t \ .$$

Hence

$$(\nabla\tilde{h})\circ T_t = {}^*(DT_t)^{-1}\nabla(\tilde{h}\circ T_t)$$

and we have

$$n_t\circ T_t = \|{}^*(DT_t)^{-1}\cdot n\|_{\mathbb{R}^N}^{-1}\,{}^*(DT_t)^{-1}\cdot n \quad \text{on } \Gamma \ .$$

Moreover

$$(\nabla_{\Gamma_t}h_t)\circ T_t = {}^*(DT_t)^{-1}\cdot[\nabla(\tilde{h}\circ T_t) - \langle B(t)\cdot n,\nabla(\tilde{h}\circ T_t)\rangle_{\mathbb{R}^N}n], \qquad (2.137)$$

where

$$B(t) = \|{}^*(DT_t)^{-1}\cdot n\|_{\mathbb{R}^N}^{-2}(DT_t)^{-1}\cdot{}^*(DT_t)^{-1} \ .$$

Furthermore

$$\langle B(t)\cdot n, n\rangle_{\mathbb{R}^N} = 1, \quad {}^*B(t) = B(t) \quad \text{and} \quad B(0) = \mathcal{I} \ .$$

Thus (2.137) can be rewritten as follows

$$\begin{aligned}&(\nabla_{\Gamma_t}h_t)\circ T_t = \\ &{}^*(DT_t)^{-1}\cdot\nabla_{\Gamma}(h\circ T_t) + \langle(B(t)-\mathcal{I})\cdot n,\nabla(\tilde{h}\circ T_t)\rangle_{\mathbb{R}^N}{}^*(DT_t)^{-1}\cdot n\end{aligned} \qquad (2.138)$$

By this means we get the formula for the transport of the tangential gradient on the manifold Γ.

2.23. Integration by parts on Γ

For vector fields $V\in C^1(\Gamma;\mathbb{R}^N)$ the formula similar to (2.123) will be derived. In general the condition $\langle V,n\rangle_{\mathbb{R}^N} = 0$ on Γ is not satisfied for such vector fields.

Let us consider the domain $\Omega\subset\mathbb{R}^N$ with the boundary $\Gamma = \partial\Omega$ of class C^2, hence the unit normal vector field n on Γ is C^1. For $V\in\mathcal{D}^1(\mathbb{R}^N;\mathbb{R}^N)$ the field $\langle V,\mathcal{N}_0\rangle_{\mathbb{R}^N}\mathcal{N}_0$ is defined in $\mathbb{R}^N$, where $\mathcal{N}_0$ is a smooth unitary extension to $\mathbb{R}^N$ of the normal vector field n on Γ. Let $f\in H^2(\mathbb{R}^N)$ be given. We shall consider the following integral

$$J(\Omega) = \int_\Gamma f d\Gamma \ .$$

For the Eulerian derivative of $J(\Omega)$ we have

$$dJ(\Gamma; v_n) = dJ(\Omega; \langle V, \mathcal{N}_0\rangle_{\mathbb{R}^N}\mathcal{N}_0)$$

that is

$$\int_\Gamma \left(\frac{\partial f}{\partial n} v_n + f\mathrm{div}_\Gamma(v_n n)\right) d\Gamma = \int_\Gamma (\nabla f \cdot V + f\mathrm{div}_\Gamma V) d\Gamma, \tag{2.139}$$

where $v_n = \langle V, n\rangle_{\mathbb{R}^N}$ stands for the normal component of V on Γ. Thus $v_n \in C^1(\Gamma)$. Using Definition 2.52 of the tangential divergence on Γ the following formulae are derived.

Lemma 2.63 *Let $\phi \in H^1(\Gamma)$ and $V \in C^1(\Gamma; \mathbb{R}^N)$, then*

$$\mathrm{div}_\Gamma(\phi V) = \langle \nabla_\Gamma \phi, V\rangle_{\mathbb{R}^N} + \phi \mathrm{div}_\Gamma V, \tag{2.140}$$

$$\mathrm{div}_\Gamma(v_n n) = v_n \mathrm{div}_\Gamma(n) \ . \tag{2.141}$$

Proof. We show that (2.141) holds. From (2.140) we get

$$\mathrm{div}_\Gamma(v_n n) = \langle \nabla_\Gamma v_n, n\rangle_{\mathbb{R}^N} + v_n \mathrm{div}_\Gamma n \ .$$

Since $\nabla_\Gamma v_n$ is a tangent vector on Γ, it can be inferred that $\langle \nabla_\Gamma v_n, n\rangle_{\mathbb{R}^N} = 0$, which proves (2.141). □

To determine the form of $\mathrm{div}_\Gamma n$ on Γ, one has to make use of the following simple lemma:

Lemma 2.64 *Let $\mathcal{N}_0 \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ be an unitary extension of the vector field n on Γ, then*

$$\langle D(\mathcal{N}_0) \cdot n, n\rangle_{\mathbb{R}^N} = 0 \quad \text{on } \Gamma \ . \tag{2.142}$$

Proof. We have

$$\langle \mathcal{N}_0, \mathcal{N}_0\rangle_{\mathbb{R}^N} = 1 \text{ in an open neighbourhood of } \Gamma.$$

Therefore

$$\nabla \langle \mathcal{N}_0, \mathcal{N}_0\rangle_{\mathbb{R}^N} = 2 {}^*D\mathcal{N}_0 \cdot \mathcal{N}_0 = 0,$$

whence

$$\langle D\mathcal{N}_0 \cdot n, n\rangle_{\mathbb{R}^N} = \langle n, {}^*D\mathcal{N}_0 \cdot n\rangle_{\mathbb{R}^N} = 0 \ .$$

□

Lemma 2.14 *For any unitary extension $\mathcal{N}_0$ of the normal vector field n on Γ we have*

$$\mathrm{div}_\Gamma n = \mathrm{div}\mathcal{N}_0 \quad on\ \Gamma\ . \tag{2.143}$$

□

From (2.139), (2.141) and (2.143) the following identity is obtained

$$\int_\Gamma (\nabla f \cdot V + f \mathrm{div}_\Gamma V) d\Gamma = \int_\Gamma \left(\frac{\partial f}{\partial n} v_n + f v_n \mathrm{div}\mathcal{N}_0 \right) d\Gamma\ . \tag{2.144}$$

This is the formula for integration by parts on the C^2 manifold Γ, where Γ is the boundary of the domain Ω.

When (2.144) is considered the element $\mathrm{div}\mathcal{N}_0 = \mathrm{div}_\Gamma n \in C(\Gamma)$ does not depend on v_n and f. We shall show that $\mathrm{div}_\Gamma n$ is the mean curvature κ of the manifold Γ.

From Lemma 2.65 it follows that the restriction to Γ of $\mathrm{div}\mathcal{N}_0$ is independent of the choice of an unitary extension $\mathcal{N}_0 \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ of the normal field n to an open neighbourhood of the boundary Γ.

We shall construct, in a very simple way, an extension $\mathcal{N}_0$ such that we shall be able to evaluate the term $\mathrm{div}_\Gamma \mathcal{N}_0$. First, let us consider $\Omega \subset \mathbb{R}^2$, then $\Gamma = \partial\Omega$ is a one–dimensional curve on the plane. Thus, Γ is locally the graph of a C^2 function ρ. For any point $x_0 \in \Gamma$ we define the unit orthogonal vectors τ_0 and n_0 tangent and normal to Γ at x_0, respectively. Then there exist:

(i) a neighbourhood U of x_0 in $\mathbb{R}^2$ such that $\Omega \cap U$ is a hypergraph: $x = \alpha\tau_0 + \beta n_0$ for any point $x \in U$,
(ii) a mapping $\alpha \to \rho(\alpha)$, $\rho(\cdot) \in C^2(-\varepsilon, \varepsilon)$, $\rho(0) = 0$,

$$U \cap \Omega = \{x = \alpha\tau_0 + \beta n_0 \quad \text{for} \quad \beta < \rho(\alpha)\}$$

and

$$\Gamma \cap U = \{x = \alpha\tau_0 + \beta n_0 \quad \text{for} \quad \beta = \rho(\alpha)\}\ .$$

The normal field is of the form

$$n(\alpha, \rho(\alpha)) = (1 + [\rho'(\alpha)]^2)^{-\frac{1}{2}}(-\rho'(\alpha) + 1)\ .$$

We define

$$\mathcal{N}_0(\alpha, \beta) = n(\alpha, \rho(\alpha))\ .$$

The direct calculations yield

$$\mathrm{div}(\mathcal{N}_0)(\alpha, \rho(\alpha)) = -(1 + \rho'(\alpha)^2)^{-\frac{3}{2}}\rho''(\alpha) = \kappa(x)\ .$$

The particular case of $N = 3$ is discussed in (Zolesio 1979a).

Let us now direct our attention again to the formula for integration by parts on Γ. Taking into account that $\mathrm{div}_\Gamma n = \kappa$, where κ is the mean curvature of the surface Γ and making use of (2.141), we get the following formula for integration by parts on Γ:

$$\int_{\Gamma}(\nabla f \cdot V + f \operatorname{div}_{\Gamma} V) d\Gamma = \int_{\Gamma}\left(\frac{\partial f}{\partial n} + \kappa f\right) v_n d\Gamma, \tag{2.145}$$

where $v_n = \langle V, n\rangle_{\mathbb{R}^N}$ on Γ.

If $V = \phi e_i = \phi(\underbrace{0,\dots,}_{i-1}1,0,\dots,0)$ in a neighbourhood of Γ, then

$$\int_{\Gamma}\left[\frac{\partial f}{\partial x_i}\phi + f \operatorname{div}_{\Gamma}(\phi e_i)\right] d\Gamma = \int_{\Gamma}\left(\frac{\partial f}{\partial n} + \kappa f\right)\phi n_i d\Gamma \ .$$

It is known that $\operatorname{div}_{\Gamma}(\phi e_i) = \operatorname{div}(\phi e_i) - \langle D(\phi e_i)\cdot n, n\rangle_{\mathbb{R}^N}$ on Γ. However

$$\operatorname{div}(\phi e_i) = \nabla \phi e_i = \frac{\partial \phi}{\partial x_i},$$

$$D(\phi e_i)_{kl} = \frac{\partial}{\partial x_i}(\phi e_i)_k = \frac{\partial}{\partial x_i}(\phi \delta_{i,k}) = \frac{\partial \phi}{\partial x_i}\delta_{i,k},$$

where $\delta_{i,k}$ is the Kronecker delta. Then

$$\langle D(\phi e_i)\cdot n, n\rangle_{\mathbb{R}^N} = \sum_{k,l=1}^{N} \frac{\partial \phi}{\partial x_i}\delta_{i,k} n_k n_l = \frac{\partial \phi}{\partial n} n_i \ .$$

Finally we have

$$\operatorname{div}_{\Gamma}(\phi e_i) = \frac{\partial \phi}{\partial x_i} - \frac{\partial \phi}{\partial n} n_i \ .$$

From the foregoing it is evident that the following proposition can be stated.

Proposition 2.66 (Integration by parts on Γ) *Let Ω be a domain of class C^2 with the boundary Γ, $f, \phi \in H^2(\Omega)$. Then*

$$\int_{\Gamma}\frac{\partial f}{\partial x_i}\phi d\Gamma = -\int_{\Gamma} f \frac{\partial \phi}{\partial x_i} d\Gamma + \int_{\Gamma}\left(\frac{\partial}{\partial n}(f\phi) + \kappa f \phi\right) n_i d\Gamma \ .$$

□

Proposition 2.67 *Let $f \in H^2(\Omega)$ and $\phi \in H^3(\Omega)$. Then*

$$\int_{\Gamma}\nabla f \cdot \nabla \phi d\Gamma = -\int_{\Gamma} f \Delta \phi d\Gamma +$$
$$\int_{\Gamma}\left[\frac{\partial f}{\partial n}\frac{\partial \phi}{\partial n} + f\langle D^2\phi \cdot n, n\rangle_{\mathbb{R}^N} + \kappa f \frac{\partial \phi}{\partial n}\right] d\Gamma \ .$$

Proof. We have

$$\int_\Gamma \frac{\partial f}{\partial x_i}\frac{\partial \phi}{\partial x_i} d\Gamma = -\int_\Gamma f \frac{\partial^2 \phi}{\partial x_i^2} d\Gamma + \int_\Gamma \left(\frac{\partial}{\partial n}\left(f \frac{\partial \phi}{\partial x_i}\right) + \kappa f \frac{\partial \phi}{\partial x_i}\right) n_i d\Gamma \ .$$

Furthermore

$$\sum_{i=1}^N \frac{\partial}{\partial n}\left(f\frac{\partial \phi}{\partial x_i}\right) n_i = \frac{\partial f}{\partial n}\frac{\partial \phi}{\partial n} + f \langle D^2\phi \cdot n, n\rangle_{\mathbb{R}^N} \ .$$

Since

$$\frac{\partial}{\partial n}\left(\frac{\partial \phi}{\partial x_i}\right) = \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_i}\right) n_j$$

then

$$\sum_{i=1}^N \frac{\partial}{\partial n}\left(\frac{\partial \phi}{\partial x_i}\right) n_i = \sum_{i,j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} n_i n_j \ .$$

□

In the particular case of $\frac{\partial}{\partial n} f = 0$ on Γ we obtain

$$\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma \phi d\Gamma = -\int_\Gamma f \Delta \phi d\Gamma + \int_\Gamma f \left[\langle D^2\phi \cdot n, n\rangle_{\mathbb{R}^N} + \kappa \frac{\partial \phi}{\partial n}\right] d\Gamma \ .$$

Moreover

$$\frac{\partial^2 \phi}{\partial n^2} = \langle D^2 \phi \cdot n, n\rangle_{\mathbb{R}^N} \quad \text{on } \Gamma$$

and

$$\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma \phi d\Gamma = -\int_\Gamma f \Delta \phi d\Gamma$$

from which the following representation of the Laplacian on Γ can be derived.

Proposition 2.68 *Let Γ be of class C^2 and let $\phi \in H^3(\Omega)$, then*

$$\Delta \phi = \Delta_\Gamma \phi + \kappa \frac{\partial \phi}{\partial n} + \frac{\partial^2 \phi}{\partial n^2} \quad \textit{on } \Gamma \ .$$

□

Proposition 2.69 *Let us suppose that $y \in H^s(\Omega)$ is the solution to the Dirichlet problem on Ω:*

$$\begin{aligned} -\Delta y &= f \quad \textit{in } \Omega, \\ y &= g \quad \textit{on } \Gamma, \end{aligned}$$

where $f \in H^{s-2}(\Omega)$ and $g \in H^{s-\frac{1}{2}}(\Gamma)$ are given elements. Then we have

$$\kappa \frac{\partial y}{\partial n} + \frac{\partial^2 y}{\partial n^2} = -f - \Delta_\Gamma g \quad on\ \Gamma$$

provided that s is sufficiently large, say $s > \frac{5}{2}$.

Proposition 2.70 *Let us suppose that* $y \in H^s(\Omega)$ *is the solution to the Neumann problem on* Ω*:*

$$-\Delta y = f \quad in\ \Omega,$$
$$\frac{\partial y}{\partial n} = g \quad on\ \Gamma,$$

where $f \in H^{s-2}(\Omega)$ *and* $g \in H^{s-\frac{3}{2}}(\Gamma)$ *are given elements. Then we have*

$$\Delta_\Gamma y + \frac{\partial^2 y}{\partial n^2} = -f - \kappa g \quad on\ \Gamma$$

provided that s is sufficiently large, say $s > \frac{5}{2}$.

□

2.24. The transport of Laplace–Beltrami operators

Let $h \in H^{5/2}(\mathbb{R}^N)$, then

$$h|_{\Gamma_t} \in H^2(\Gamma_t) \quad \text{and} \quad \Delta_{\Gamma_t} h \in L^2(\Gamma_t),$$

where Δ_{Γ_t} is defined as follows

$$\int_{\Gamma_t} \Delta_{\Gamma_t} h \psi d\Gamma_t = -\int_{\Gamma_t} \nabla_{\Gamma_t} h \cdot \nabla \psi d\Gamma_t$$

for all ψ in $\mathcal{D}(\mathbb{R}^N)$. To derive the formula one has to make use of the identity

$$\nabla_{\Gamma_t} h \cdot \nabla \psi = \nabla_{\Gamma_t} h \cdot \nabla_{\Gamma_t} \psi \ .$$

Using the change of variables $x = T_t(V)(X)$,

$$\int_\Gamma (\Delta_{\Gamma_t} h) \circ T_t \phi \omega(t) d\Gamma = -\int_\Gamma \langle (\nabla_{\Gamma_t} h) \circ T_t, {}^*DT_t^{-1} \cdot \nabla \phi \rangle \omega(t) d\Gamma =$$

for all $\phi \in \mathcal{D}(\mathbb{R}^N)$; from (2.132) and (2.137),

$$= -\int_\Gamma \langle C(t) \cdot [\nabla(h \cdot T_t) - \langle B(t) \cdot n, \nabla(h \cdot T_t) \rangle_{\mathbb{R}^N} n], \nabla \phi \rangle_{\mathbb{R}^N} d\Gamma,$$

where

$$\begin{aligned}C(t) &= \omega(t) DT_t^{-1} \cdot {}^*DT_t^{-1}, \\ B(t) &= \|{}^*DT_t^{-1} \cdot n\|_{\mathbb{R}^N}^{-2} DT_t^{-1} \cdot {}^*DT_t^{-1}, \\ \omega(t) &= \gamma(t) \|{}^*DT_t^{-1} \cdot n\|_{\mathbb{R}^N}, \\ \gamma(t) &= \det(DT_t) \ .\end{aligned}$$

Finally integration by parts on Γ yields

$$\begin{aligned}&\int_\Gamma (\Delta_{\Gamma_t} h) \circ T_t \phi \omega(t) d\Gamma = \\ &- \int_\Gamma \mathrm{div}_\Gamma (C(t) \cdot [\nabla(h \circ T_t) - \langle B(t) \cdot n, \nabla(h \circ T_t)\rangle_{\mathbb{R}^N} n]) \phi d\Gamma \\ &+ \int_\Gamma \left(\frac{\partial \phi}{\partial n} + \kappa \phi\right) \langle C(t) \cdot [\nabla(h \circ T_t) - \langle B(t) \cdot n, \nabla(h \circ T_t)\rangle_{\mathbb{R}^N} n], n\rangle_{\mathbb{R}^N} d\Gamma \ .\end{aligned}$$

Let ϕ be an element such that $\frac{\partial \phi}{\partial n} = 0$ on Γ, hence the Laplace–Beltrami operator transformed to the domain Ω using the standard change of variables is of the form:

$$\begin{aligned}(\Delta_{\Gamma_t} h) \circ T_t = \omega(t)^{-1} \{ &-\mathrm{div}_\Gamma (C(t) \cdot [\nabla(h \circ T_t) - \langle B(t) \cdot n, \nabla(h \circ T_t)\rangle_{\mathbb{R}^N} n]) \\ &+ \kappa C(t) \cdot [\nabla(h \circ T_t) - \langle B(t) \cdot n, \nabla(h \circ T_t)\rangle_{\mathbb{R}^N} n] \}\end{aligned} \tag{2.146}$$

This expression may be of advantage in calculations however the transformed Laplace–Beltrami operator is not in the divergence form. The equivalent expression in the divergence form can be derived in the following way.

Let $\phi_t \in H^{3/2}(\mathbb{R}^N)$ denote for any t an extension of $\phi \in H^1(\Gamma)$,

$$\phi_t|_\Gamma = \phi \quad \text{and} \quad \frac{\partial}{\partial (C(t) \cdot n)} \phi_t = 0 \quad \text{on } \Gamma \ .$$

It is evident that the vector field $C(t) \cdot n$ is transverse on Γ, i.e.

$$\langle C(t) \cdot n, n\rangle_{\mathbb{R}^N} = \omega(t) \|{}^*DT_t^{-1} \cdot n\|_{\mathbb{R}^N}^2 > 0 \ .$$

Hence the extension ϕ_t exists. We define $\psi_t \in H^{3/2}(\mathbb{R}^N)$ by

$$\psi_t = \phi_t \circ T_t(V)^{-1} \quad \text{for} \quad V \in \mathcal{D}^1(\mathbb{R}^N; \mathbb{R}^N) \ .$$

Let $h \in H^2(\mathbb{R}^N)$, then

$$\int_\Gamma (\Delta_{\Gamma_t} h) \circ T_t \phi \omega(t) d\Gamma = \int_{\Gamma_t} \Delta_{\Gamma_t} h \psi_t d\Gamma_t = - \int_{\Gamma_t} \nabla_{\Gamma_t} h \cdot \nabla_{\Gamma_t} \psi d\Gamma_t \ .$$

This follows from the fact that the gradient $\nabla_\Gamma \psi_t$ is a tangent vector field on Γ. Accordingly we have

$$\langle n_t, \nabla\psi_t\rangle_{\mathbb{R}^N} \circ T_t =$$
$$\langle {}^*(DT_t)^{-1} \cdot n, {}^*(DT_t)^{-1} \cdot \phi_t\rangle_{\mathbb{R}^N}\omega(t)^{-1} =$$
$$\omega(t)^{-2}\langle C(t) \cdot n, \nabla\phi_t\rangle_{\mathbb{R}^N} = 0 \ .$$

Making use of the assumption $\partial\phi_t/\partial(C(t) \cdot n) = 0$ on Γ which implies that $C(t) \cdot \nabla\phi_t$ is a tangent vector field on Γ, we get

$$-\int_{\Gamma_t} \nabla_{\Gamma_t} h \cdot \nabla_{\Gamma_t}\psi_t d\Gamma_t =$$
$$-\int_{\Gamma_t} \nabla h \cdot \nabla\psi_t d\Gamma_t =$$
$$-\int_{\Gamma} \langle C(t) \cdot \nabla\phi_t, \nabla(h \circ T_t)\rangle_{\mathbb{R}^N} d\Gamma =$$

$C(t) \cdot \nabla\phi_t$ is also a tangent vector field on Γ,

$$= -\int_{\Gamma} \langle C(t) \cdot \nabla\phi_t, \nabla_{\Gamma}(h \circ T_t)\rangle_{\mathbb{R}^N} d\Gamma =$$
$$-\int_{\Gamma} \langle \nabla\phi_t, C(t) \cdot \nabla_{\Gamma}(h \circ T_t)\rangle_{\mathbb{R}^N} d\Gamma =$$

Integration by parts on Γ yields

$$= \int_{\Gamma} (\Delta_{\Gamma_t} h) \circ T_t \phi\omega(t) d\Gamma =$$
$$\int_{\Gamma} \phi \mathrm{div}_{\Gamma}(C(t) \cdot \nabla_{\Gamma}(h \circ T_t)) d\Gamma \ .$$

On the other hand, $\phi \equiv \phi_t$ on Γ; therefore

$$0 = \langle C \cdot n, \nabla\phi\rangle_{\mathbb{R}^N} = \langle C \cdot n, \nabla_{\Gamma}\phi\rangle_{\mathbb{R}^N} + \langle C \cdot n, n\rangle_{\mathbb{R}^N} \frac{\partial\phi}{\partial n}$$

whence

$$\frac{\partial\phi}{\partial n} = -\langle C \cdot n, n\rangle^{-1}_{\mathbb{R}^N} \langle C \cdot n, \nabla_{\Gamma}\phi\rangle_{\mathbb{R}^N} \ .$$

Thus

$$-\int_{\Gamma} \frac{\partial}{\partial n}\phi_t \langle C \cdot n, \nabla_{\Gamma}(h \circ T_t)\rangle_{\mathbb{R}^N} d\Gamma =$$
$$\int_{\Gamma} E \cdot \nabla_{\Gamma}\phi d\Gamma = \int_{\Gamma} (-\mathrm{div}_{\Gamma} E\phi + \kappa\phi E \cdot n) d\Gamma,$$

where

$$E = \langle C \cdot n, n\rangle^{-1}_{\mathbb{R}^N} \langle C \cdot n, \nabla_{\Gamma}(h \circ T_t)\rangle_{\mathbb{R}^N} C \cdot n \ .$$

In particular

$$E \cdot n = \langle C \cdot n, \nabla_\Gamma (h \circ T_t)\rangle_{\mathbb{R}^N} \ .$$

Therefore two terms involving the mean curvature κ cancel each other. As a result we obtain

$$\int_\Gamma (\Delta_{\Gamma_t} h) \circ T_t \phi \omega(t) d\Gamma = \int_\Gamma \phi \operatorname{div}_\Gamma (C(t) \cdot \nabla_\Gamma (h \circ T_t)) d\Gamma$$

$$- \int_\Gamma \phi \operatorname{div}_\Gamma (\langle C \cdot n, n\rangle_{\mathbb{R}^N}^{-1} \langle C \cdot n, \nabla_\Gamma (h \circ T_t)\rangle_{\mathbb{R}^N} C \cdot n) d\Gamma \ .$$

This leads to the transported Laplace–Beltrami operator in the divergence form:

$$\begin{aligned}(\Delta_{\Gamma_t} h) \circ T_t =& \omega(t)^{-1} \operatorname{div}_\Gamma [C(t) \cdot (\nabla_\Gamma [h \circ T_t] - \\ & \langle C(t) \cdot n, n\rangle_{\mathbb{R}^N}^{-1} \langle C(t) \cdot n, \nabla_\Gamma (h \circ T_t)\rangle_{\mathbb{R}^N} n)],\end{aligned} \tag{2.147}$$

where

$$C(t) = \omega(t) DT_t^{-1} \cdot {}^* DT_t^{-1} \ .$$

2.25. Material derivatives

Let $\Omega \subset D$ be a bounded domain, where ∂D is piecewise smooth, $\partial\Omega = \Gamma$ is C^k (i.e. Ω is a domain of class C^k, see Sect. 2.1) and let V be a vector field such that $\langle V, n\rangle_{\mathbb{R}^N} = 0$ on ∂D, except for the singular points of ∂D where we suppose that $V = 0$. Moreover we assume that an element $y(\Omega) \in W^{s,p}(\Omega)$, $s \in [0, k]$, $1 \leqq p < +\infty$, is given.

From Theorem 2.3 it follows that

$$y(\Omega_t) \circ T_t(V) \in W^{s,p}(\Omega) \quad \text{for} \quad 0 \leqq t < \varepsilon \ . \tag{2.148}$$

Definition 2.71 The element $\dot{y}(\Omega; V) \in W^{s,p}(\Omega)$ is the material derivative of $y(\Omega) \in W^{s,p}(\Omega)$ in the direction of a vector field $V \in C(0, \varepsilon; V^k(D))$ if there exists the limit

$$\dot{y}(\Omega; V) = \lim_{t \to 0} \frac{1}{t}(y(\Omega_t) \circ T_t(V) - y(\Omega)).$$

As far as this limit is considered we can take into account the strong or weak convergence in $W^{s,p}(\Omega)$. In the first case $\dot{y}(\Omega; V)$ is called the strong (s,p) derivative, in the second – the weak one.

Remark. In general, we shall be concerned with an element $y(\Omega) \in \mathcal{W}(\Omega)$, where $\mathcal{W}(\Omega)$ is a Banach space. It is said that $\dot{y}(\Omega; V)$ is the weak (strong) material derivative of y in $\mathcal{W}$ if $\frac{1}{t}(y(\Omega_t) \circ T_t - y(\Omega))$ is weakly (strongly) convergent to $\dot{y}(\Omega; V)$ in $\mathcal{W}(\Omega)$ as $t \downarrow 0$.

A simple but useful example of the material derivative is as follows:

Let $Y \in W^{m,p}(D)$ for some $m \in \mathbb{N}$, $p \geqq 1$, and let $y(\Omega) = Y|_\Omega$ denotes the restriction of Y to Ω; hence $y(\Omega) \in W^{m,p}(\Omega)$ and

$$y(\Omega_t) = Y|_{\Omega_t} \quad \text{thus} \quad y(\Omega_t) \circ T_t = (Y|_{\Omega_t}) \circ T_t$$

that is

$$y(\Omega_t) \circ T_t = (Y \circ T_t)|_\Omega \in W^{m,p}(\Omega) \ .$$

Using the same arguments as in the proofs of Proposition 2.32 and 2.39, one can show that the mapping $t \to y \circ T_t$ is strongly differentiable in $W^{m-1,p}(D)$ at $t = 0$ for $m \geqq 1$, and weakly differentiable in $W^{m-1,p}(D)$ for $m \leqq 0$. Hence the regularity of the mapping $t \to (y \circ T_t)|_\Omega$ can be described as follows:

Proposition 2.72 *Let $Y \in W^{m,p}(D)$, $m \leqq k$, $m \in \mathbb{N}$, where k determines the regularity of the vector field V. Then*
(i) For $m \geqq 1$ the mapping $\Omega \to Y|_\Omega$ is differentiable in the sense that there exits the strong (m,p) material derivative of the form:

$$\dot{y}(\Omega_t; V) = \nabla Y|_\Omega \cdot V(0) \ .$$

(ii) For $m \leqq 0$ the mapping $\Omega \to Y|_\Omega$ is weakly differentiable in the sense that there exits the weak (m,p) material derivative of the form:

$$\dot{y}(\Omega_t; V) = \nabla Y|_\Omega \cdot V(0) \ .$$

Here it is assumed that $V(0) \in C^k(\overline{D})$, $k \geqq 1$.

Proof. It is evident that for $m \geqq 1$ we have

$$\left\| \frac{1}{t}((Y \circ T_t)|_\Omega - Y|_\Omega) - \nabla Y|_\Omega \cdot V(0) \right\|_{W^{m-1,p}(\Omega)} \leqq$$
$$\left\| \frac{1}{t}(Y \circ T_t - Y) - \nabla Y \cdot V(0) \right\|_{W^{m-1,p}(D)} \ .$$

From the assumptions adopted it follows that the last term on the right–hand side of this inequality converges to zero as $t \to 0$.

ii) For $m \leqq 0$ by our assumptions we have

$$\frac{1}{t}(Y \circ T_t - Y) \rightharpoonup \nabla Y \cdot V(0) \quad \text{weakly in } W^{m-1,p}(D)$$

as $t \to 0$. Since for $t > 0$, $\frac{1}{t}(Y \circ T_t - Y)$ remains bounded in $W^{m-1,p}(D)$, then for some $C > 0$, C independent of t,

$$\frac{1}{t} \left\| (Y \circ T_t)|_\Omega - Y|_\Omega \right\|_{W^{m-1,p}(\Omega)} \leqq C$$

whence it follows that there exists a subsequence $t_n \downarrow 0$ such that

$$\frac{1}{t_n}[(Y \circ T_{t_n} - Y)|_{\Omega}] \rightharpoonup z \quad \text{weakly in} \quad W^{m-1,p}(\Omega) \ .$$

Since

$$\langle z, \phi \rangle_{\mathcal{D}'(D)\times\mathcal{D}(D)} = \langle \nabla Y \cdot V(0), \phi \rangle_{\mathcal{D}'(D)\times\mathcal{D}(D)}$$

for all $\phi \in W^{-(m-1),p}(\Omega)$, the dual space of $W^{(m-1),p}(\Omega)$, here we recall that $m-1 \leqq 0$, it follows that $z = \nabla Y|_{\Omega} \cdot V(0)$. □

2.26. Material derivatives on Γ

Let us define the material derivative of an element $y(\Gamma) \in W^{r,p}(\Gamma)$, it is assumed that this derivative is well defined for all the boundaries Γ of domains Ω of class C^k. Hence for a given domain Ω with the boundary Γ and for

$$\Gamma_t = T_t(V)(\Gamma) \quad \text{with } V \in C(0,\varepsilon; \mathcal{D}^k(\mathbb{R}^N; \mathbb{R}^N))$$

$y(\Gamma_t)$ is well defined as an element of the space $W^{r,p}(\Gamma_t)$

Definition 2.74 We say that

$$\dot{y} = \dot{y}(\Gamma; V) \in W^{r,p}(\Gamma)$$

is the weak (strong) (r,p) material derivative of an element $y(\Gamma)$ at Γ, in the direction of a field V, if there exists the limit

$$\frac{1}{t}(y(\Gamma_t) \circ T_t(V) - y(\Gamma)) \to \dot{y}(\Gamma; V),$$

as $t \to 0$ for weak (strong) convergence in the space $W^{r,p}(\Gamma)$.

Proposition 2.75 *Let $\dot{y}(\Omega; V)$ be the weak (strong) (s,p) material derivative of an element $y(\Omega)$ at Ω, in the direction of a field V. Then for $s > \frac{1}{p}$ there exists the weak (strong) $(s-\frac{1}{p}, p)$ material derivative $\dot{y}(\Gamma; V)$ of the element $y(\Gamma) = y(\Omega)|_{\Gamma}$*

$$\dot{y}(\Gamma; V) = \dot{y}(\Omega; V)|_{\Gamma} \in W^{s-\frac{1}{p},p}(\Gamma) \ .$$

Proof. Since

$$z_t = \frac{1}{t}(y(\Omega_t) \circ T_t - y(\Omega)) \to \dot{y}(\Omega; V)$$

weakly (strongly) in the space $W^{s,p}(\Omega)$ as $t \downarrow 0$, then we can make use of the continuity of the trace operator on Γ which implies that

$$z_t \to \dot{y}(\Omega; V)|_\Gamma \text{ weakly (strongly) in } W^{s-\frac{1}{p},p}(\Gamma)$$

as $t \downarrow 0$. □

Using this result we can determine the form of the Eulerian derivative of a given domain functional $J(\Omega)$ depending on the element $y(\Gamma)$.

For a given vector field V tangent to Γ one can derive the explicit form of the material derivative.

Proposition 2.76 *Let $V \in \mathcal{D}^k(D; \mathbb{R}^N)$ be a given vector field such that $\langle V, n\rangle_{\mathbb{R}^N} = 0$ on $\Gamma = \partial\Omega$ and let $\dot{y}(\Omega; V)$ be the weak (s,p) material derivative of $y(\Omega)$. If it is supposed that the derivative $\dot{y}(\Omega; \mathcal{V})$ exists for all admissible directions $\mathcal{V}$, then for the vector field V we have*

$$\dot{y}(\Omega; V) = \nabla y(\Omega) \cdot V \in W^{s,p}(\Omega)$$

Proof. The condition $\langle V, n\rangle_{\mathbb{R}^N} = 0$ on Γ implies that $T_t(V)(\Omega) = \Omega_t = \Omega$ and $y(\Omega_t) = y(\Omega)$, hence

$$\frac{1}{t}(y(\Omega_t) \circ T_t - y(\Omega)) = \frac{1}{t}(y(\Omega) \circ T_t - y(\Omega)) \ .$$

The weak limit in $W^{s,p}(\Omega)$ of the left-hand side of this equality is equal to $\dot{y}(\Omega; V)$ by our assumption, the weak limit of the right-hand side is equal to $\nabla y(\Omega) \cdot V$, this concludes the proof. □

The same result remains valid for the boundary material derivative.

Proposition 2.77 *Let $\dot{y}(\Gamma; V)$ be the weak (r,p) material derivative of $y(\Gamma)$, and suppose that the material derivative $\dot{y}(\Gamma; V)$ exists for any admissible direction*

$$V \in C(0, \varepsilon; \mathcal{D}(D; \mathbb{R}^N))$$

such that $\langle V, n\rangle_{\mathbb{R}^N} = 0$ on $\Gamma = \partial\Omega$, then

$$\dot{y}(\Gamma; V) = \nabla_\Gamma y(\Gamma) \cdot V \in W^{r,p}(\Omega) \ .$$

2.27. The material derivative of a solution to the Laplace equation with Dirichlet boundary conditions

Let us consider the homogeneous Dirichlet problem

$$-\Delta y(\Omega_t) = h \quad \text{in } H^{-1}(\Omega_t),$$
$$y(\Omega_t) = 0 \quad \text{on } \Gamma_t \ .$$

This section is concerned with the particular case of the Dirichlet problem for $h \in L^2(\Omega_t)$. For this case the form of the material derivative $\dot{y}(\Omega; V)$ of the solution $y(\Omega)$ can be derived in a simple way.

Assume that $V \in C(0, \varepsilon; C^2(\mathbb{R}^N; \mathbb{R}^N))$, i.e. $k = 2$. We shall determine the form of the weak $(0, 2)$ material derivative $\dot{y}(\Omega; V)$ of the solution $y(\Omega)$ to the Dirichlet problem.

First the Dirichlet problem is transformed to the fixed domain Ω using the change of variables $T_t(V)$, in other words the form of boundary value problems for elements $y^t \equiv y(\Omega_t) \circ T_t$, $t \in [0, \varepsilon)$, is derived.

The right hand–side of the Laplace equation is transformed to the fixed domain Ω. Hence one has to consider two elements:

$$h|_{\Omega_t} \in H^{-1}(\Omega_t)$$

and

$$(h|_{\Omega_t}) \star T_t \in H^{-1}(\Omega).$$

We shall derive sufficient conditions for the mapping

$$t \to (h|_{\Omega_t}) \star T_t$$

to be weakly differentiable in the Sobolev space $H^{-1}(D)$.

Let $h \in H^{-1}(D)$ be a given element, the transformed distribution $h \star T_t \in H^{-1}(D)$ is defined as follows

$$\langle h \star T_t, \phi \rangle_{H^{-1}(D) \times H_0^1(D)} = \langle h, (\gamma(t)^{-1} \phi) \circ T_t^{-1} \rangle_{H^{-1}(D) \times H_0^1(D)} \quad \forall \phi \in H_0^1(D) \ .$$

The restriction

$$h|_{\Omega} \in \mathcal{D}'(\Omega)$$

of the distribution $h \in H^{-1}(D)$ is given by

$$\langle h|_{\Omega}, \phi \rangle_{\mathcal{D}'(D) \times \mathcal{D}(D)} = \langle h, \phi^0 \rangle_{\mathcal{D}'(D) \times \mathcal{D}(D)} \quad \forall \phi \in \mathcal{D}(\Omega),$$

where ϕ^0 denotes the extension of $\phi \in \mathcal{D}(\Omega)$ to $\overline{D}$, $\phi^0(x) = 0$ on $\overline{D} \setminus \Omega$. Since $h|_{\Omega} \in H^{-1}(\Omega)$, then

$$\|h|_{\Omega}\|_{H^{-1}(\Omega)} = \sup_{\|\phi\|_{H_0^1(\Omega)} \leqq 1} |\langle h, \phi \rangle| \leqq \sup_{\|\phi\|_{H_0^1(D)} \leqq 1} |\langle h, \phi \rangle| = \|h\|_{H^{-1}(D)} \ .$$

Let $\Omega_t = T_t(V)(\Omega)$ and

$$h|_{\Omega_t} = \chi_{\Omega_t} h \in H^{-1}(\Omega_t) \ .$$

The transformed distribution is defined by

$$(h|_{\Omega_t}) \star T_t = (\chi_{\Omega_t} h) \star T_t \in H^{-1}(\Omega) \ .$$

Proposition 2.78 *Let $\Omega_t = T_t(V)(\Omega)$ then*

$$(h|_{\Omega_t}) \star T_t = (h \star T_t)|_{\Omega} \ . \tag{2.149}$$

Proof. For any $\phi \in H_0^1(\Omega)$ we have

$$\langle h|_{\Omega_t} \star T_t, \phi \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \langle h|_{\Omega_t}, (\gamma(t)^{-1}\phi) \circ T_t^{-1} \rangle_{H^{-1}(\Omega_t)\times H_0^1(\Omega_t)}$$

Making use of the extension ϕ^0 of ϕ one can show that

$$\begin{aligned} &= \langle h, (\gamma(t)^{-1}\phi^0) \circ T_t^{-1} \rangle_{H^{-1}(D)\times H_0^1(D)} \\ &= \langle h \star T_t, \phi^0 \rangle_{H^{-1}(D)\times H_0^1(D)} = \end{aligned}$$

and since $\phi^0 = 0$ on $\overline{D} \setminus \Omega$

$$= \langle (h \star T_t)|_{\Omega}, \phi \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \ .$$

For the element $y(\Omega_t) = y_t \in H_0^1(\Omega_t)$ the following integral identity holds:

$$\int_{\Omega_t} \nabla y_t \cdot \nabla \phi dx = \langle h, \phi^0 \rangle_{H^{-1}(D)\times H_0^1(D)} \quad \forall \phi \in H_0^1(\Omega_t) \ , \tag{2.150}$$

or equivalently

$$\langle -\mathrm{div}(\chi_{\Omega_t} \nabla y_t), \phi \rangle_{H^{-1}(\Omega_t)\times H_0^1(\Omega_t)} = \langle h|_{\Omega_t}, \phi \rangle_{H^{-1}(\Omega_t)\times H_0^1(\Omega_t)} \ .$$

Applying the standard change of variables, i.e. replacing x by $T_t(x)$ in the left–hand side of this equality,

$$\int_{\Omega} \langle A(t) \cdot \nabla(y_t \circ T_t), \nabla(\phi \circ T_t) \rangle_{\mathbb{R}^N} dx = \langle h|_{\Omega_t}, \phi \rangle,$$

where $A(t) = \gamma(t) DT_t^{-1} \cdot {}^*DT_t^{-1}$. Thus

$$\begin{aligned} &\langle -\mathrm{div}(\chi_{\Omega} A(t) \cdot \nabla(y_t \circ T_t)), \psi \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \\ &\langle h|_{\Omega_t}, \psi \circ T_t \rangle_{H^{-1}(\Omega_t)\times H_0^1(\Omega_t)} \quad \forall \psi = \phi \circ T_t^{-1} \in H_0^1(\Omega) \ . \end{aligned}$$

Let us observe that for any element $\Lambda \in \mathcal{D}'(D)$, the element $\Lambda \star T_t^{-1}$ is defined as follows:

$$\langle \Lambda \star T_t^{-1}, \psi \rangle = \langle \Lambda, (\gamma(t)\psi) \circ T_t \rangle \quad \forall \psi \in \mathcal{D}(D) \ .$$

Therefore

$$\begin{aligned} &\langle -\gamma(t)^{-1} \mathrm{div}(\chi_{\Omega} A(t) \cdot \nabla(y_t \circ T_t)), \gamma(t)\psi \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \\ &\langle h|_{\Omega_t}, (\gamma(t)\psi) \circ T_t \rangle_{H^{-1}(\Omega_t)\times H_0^1(\Omega_t)} \ . \end{aligned}$$

Whence it follows that

$$[-\gamma(t)^{-1}\mathrm{div}(\chi_\Omega A(t)\cdot\nabla(y_t\circ T_t))]\star T_t^{-1}=h|_{\Omega_t}$$

or

$$-\gamma(t)^{-1}\mathrm{div}(\chi_\Omega A(t)\cdot\nabla(y_t\circ T_t))=(h|_{\Omega_t})\star T_t=(h\star T_t)|_\Omega\ .$$

□

Let $y^t=y(\Omega_t)\circ T_t\in H_0^1(\Omega)$ be a solution to the following problem

$$\int_\Omega\langle A(t)\cdot\nabla y^t,\nabla\phi\rangle_{\mathbb{R}^N}dx=\langle\gamma(t)(h\star T_t)|_\Omega,\phi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\quad\forall\phi\in H_0^1(\Omega)\ .$$

It has been already shown that the mapping $t\to h\star T_t$ is weakly differentiable in the space $H^{-2}(D)$, however the mapping $t\to(h\star T_t)|_\Omega$ fails to have this property in the space $H^{-2}(D)$. In order to obtain the required differentiability of the mapping $t\to y^t$, it is necessary to introduce additional assumptions on the distribution h, the domain Ω and the speed field V.

It is assumed that the support of the singular part of h is included in Ω and in Ω_t for $t>0$, t small enough. For any element $h\in H^{-1}(D)$ one can find elements $f\in L^2(D)$ and $g\in L^2(D;\mathbb{R}^N)$ such that $h=f-\mathrm{div}g$. Therefore we assume that there exists a compact set $\overline{\mathcal{O}}$, $\overline{\mathcal{O}}\subset\Omega\cup\Omega^c$,

$$g(x)=0\quad\text{for a.e. } x\in\overline{D}\setminus\overline{\mathcal{O}}\ . \tag{2.151}$$

Lemma 2.79 *Let us suppose that the condition (2.151) is satisfied, then the mapping $t\to(h\star T_t)|_\Omega$ is weakly differentiable in $(H_0^1(\Omega)\cap H^2(\Omega))'$, the dual space of the space $H_0^1(\Omega)\cap H^2(\Omega)$.*

Proof. Let $\phi\in H_0^1(\Omega)\cap H^2(\Omega)$ be a given element, then on the set $\overline{\Omega}\setminus\overline{\mathcal{O}}$ the element ϕ can be modified, i.e. there exists an element $\tilde{\phi}\in H^2(\Omega)$ such that

$$\phi-\tilde{\phi}\in H^2(D\setminus\overline{\mathcal{O}})\ .$$

From (2.150) it follows that

$$\langle\nabla g,\phi\rangle_{H^{-1}(D)\times H_0^1(D)}=\langle\nabla g,\tilde{\phi}\rangle_{H^{-1}(D)\times H_0^1(D)}$$

thus

$$\langle(h\star T_t)|_\Omega,\phi\rangle=\langle(h\star T_t)|_\Omega,\tilde{\phi}^0\rangle_{H^{-2}(D)\times H_0^2(D)}\ .$$

□

Proposition 2.80 *Let $V\in C(D;\mathbb{R}^N)$, and $h=f+\mathrm{div}g$, where $f\in L^2(D)$ and $g\in L^2(D;\mathbb{R}^N)$ are given elements. Moreover it is assumed that for a given compact set $\overline{\mathcal{O}}$, $\overline{\mathcal{O}}\subset\Omega\cup\Omega^c$, g satisfies the assumption (2.151). Then the mapping $t\to y^t=y(\Omega_t)\circ T_t\in H^1(\Omega)$ is weakly differentiable in $L^2(\Omega)$, its derivative is given by*

$$\forall \psi \in L^2(\Omega) \ : \quad \int_\Omega \dot{y}(\Omega; V)\psi dx =$$

$$-\int_\Omega \langle (\operatorname{div} V(0)\mathcal{I} - 2\epsilon(V(0))) \cdot \nabla y(\Omega), \nabla((-\Delta)^{-1}\psi)\rangle_{\mathbb{R}^N} dx +$$

$$\int_\Omega \operatorname{div}(fV(0))((-\Delta)^{-1}\psi)dx + \int_\Omega \nabla(\nabla((-\Delta)^{-1}\psi)\cdot V(0))\cdot g dx \ .$$

Proof. For a given element $\phi \in H_0^1(\Omega) \cap H^2(\Omega)$ we have

$$\int_\Omega \langle A(t) \cdot \nabla\phi, \nabla y^t\rangle_{\mathbb{R}^N} dx = \int_\Omega f \circ T_t \gamma(t)\phi dx - \int_D \nabla((\tilde{\phi}^0) \circ T_t) \cdot g dx,$$

where $\tilde{\phi} \in H_0^2(\Omega)$ is an element such that $\tilde{\phi} = \phi$ in an open neighborhood in $\mathbb{R}^N$ of the compact set $\overline{\mathcal{O}}$. Let $\tilde{\phi}^0$ be an extension of $\tilde{\phi}$ to $\overline{D}$. Using Green's formula one can show that

$$-\int_\Omega y^t \Delta\phi dx = \int_\Omega y^t \operatorname{div}([A(t) - \mathcal{I}] \cdot \nabla\phi)dx - \int_D \nabla[(\tilde{\phi}^0) \circ T_t] \cdot g dx \ . \quad (2.152)$$

Since the inverse $(-\Delta)^{-1}$ of the Laplace operator with the homogeneous Dirichlet boundary condition is an isomorphism from $L^2(\Omega)$ onto $H^2(\Omega) \cap H_0^1(\Omega)$, the right–hand side of (2.152) is differentiable with respect to t at $t = 0$. The derivative of the right–hand side of this equation is given by

$$\int_\Omega y \operatorname{div}(A'(0) \cdot \nabla\phi)dx + \int_D \nabla(\nabla\tilde{\phi}^0 \cdot V(0)) \cdot g dx \ .$$

On the other hand

$$\nabla(\nabla\tilde{\phi}^0 \cdot V(0)) = \nabla(\nabla\phi \cdot V(0)) \quad \text{on } \overline{\mathcal{O}} \ .$$

Since $g \equiv 0$ on $\Omega \setminus \overline{\mathcal{O}}$, then it can be assumed that $g \neq 0$ on $\overline{\mathcal{O}}$. Therefore the integral on D can be written in the form, $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\int_D \nabla(\nabla\tilde{\phi}^0 \cdot V(0)) \cdot g dx = \int_\Omega \nabla(\nabla\phi \cdot V(0)) \cdot g dx \ .$$

Let $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$, $-\Delta\phi = \psi$, or equivalently $\phi = (-\Delta)^{-1}\psi$, where $\psi \in L^2(\Omega)$. Therefore

$$\int_\Omega \dot{y}(\Omega; V)\psi dx = \int_\Omega y \operatorname{div}(A'(0) \cdot \nabla\phi)dx + \int_\Omega \operatorname{div}(fV(0))\phi dx$$
$$+ \int_\Omega \nabla(\nabla\phi \cdot V(0)) \cdot g dx,$$

where

$$A'(0) = \left(\frac{d}{dt}A(t)\right)_{t=0} = \mathrm{div}V(0)\mathcal{I} - 2\epsilon(V(0)),$$
$$\epsilon(V(0)) \text{ is the symmetrized part of } DV(0),$$
$$\epsilon(V(0)) = \frac{1}{2}({}^*DV(0) + DV(0)),$$

i.e $\epsilon(V(0))$ is the strain tensor associated with the speed $V(0)$. Applying Green's formula we obtain

$$\int_\Omega \dot{y}(\Omega; V)\psi dx = \int_\Omega -\langle A'(0) \cdot \nabla y, \nabla\phi\rangle_{\mathbb{R}^N} dx$$
$$+ \int_\Omega \mathrm{div}(fV(0))\phi dx + \int_\Omega \nabla(\nabla\phi \cdot V(0)) \cdot g dx,$$

as was to be shown. □

For the particular case of the homogeneous Dirichlet boundary value problem with the right–hand side $f \in L^2(\Omega)$, the following corollary can be stated.

Corollary 2.81 *Let $y(\Omega) \in H_0^1(\Omega_t)$ be the solution to the problem:*

$$-\Delta y(\Omega_t) = f \quad in\ \Omega_t,$$
$$y(\Omega_t) = 0 \quad on\ \Gamma_t,$$

where $f \in L^2(D)$ is a given element. Moreover, let $V \in C^1([0,\varepsilon); C^2(\overline{D}; \mathbb{R}^N))$ be a given vector field. Then the mapping $t \to y(\Omega_t) \circ T_t$ is weakly differentiable in $L^2(\Omega)$ and the weak (0,2) material derivative is given by the following formula

$$\int_\Omega \dot{y}(\Omega; V)\psi dx =$$
$$- \int_\Omega \langle(\mathrm{div}V(0)\mathcal{I} - 2\epsilon(V(0))) \cdot \nabla y(\Omega), \nabla((-\Delta)^{-1} \cdot \psi)\rangle_{\mathbb{R}^N} dx \tag{2.153}$$
$$+ \int_\Omega \mathrm{div}(fV(0)((-\Delta)^{-1}\psi) dx \quad \forall\psi \in L^2(\Omega)\ .$$

The right–hand side of (2.153) is linear and continuous with respect to $\psi \in H^{-1}(\Omega)$, because the inverse operator $(-\Delta)^{-1}$ is an isomorphism from $H^{-1}(\Omega)$ onto $H_0^1(\Omega)$. Thus the material derivative $\dot{y}(\Omega; V)$ is defined for any vector field $V \in C^1([0,\varepsilon); C^2(\overline{D}; \mathbb{R}^N))$. In fact for $f \in H^1(D)$ it is easy to show that the strong material derivative exists.

2.28. Strong material derivatives for Dirichlet problems

It is assumed that $f \in H^1(D)$ is a given element, Ω is a given domain of class C^k in D, $k \geqq 1$, and $V \in C(0,\varepsilon; V^k(D))$ is a given vector field; the transformed domain is denoted by $\Omega_t = T_t(V)(\Omega)$. Let us consider the element $y(\Omega_t) \in H_0^1(\Omega_t)$: a weak solution to the homogeneous Dirichlet boundary value problem in Ω_t,

$$\begin{aligned} -\Delta y(\Omega_t) &= f \quad \text{in } \Omega_t, \\ y(\Omega_t) &= 0 \quad \text{on } \Gamma_t = \partial\Omega_t \ . \end{aligned} \tag{2.154}$$

The element $y_t = y(\Omega_t)$ satisfies the integral identity

$$\int_{\Omega_t} \nabla y_t \cdot \nabla \phi_t dx = \int_{\Omega_t} f \phi_t dx$$

for all $\phi_t \in H_0^1(\Omega_t)$.

Using the change of variables $x = T_t(V)(X)$ one can show that for the element transformed to the domain Ω,

$$y^t = y(\Omega_t) \circ T_t \in H_0^1(\Omega),$$

the following integral identity holds

$$\int_\Omega \langle A(t) \cdot \nabla y_t, \nabla \psi \rangle_{\mathbb{R}^N} dx = \int_\Omega \gamma(t) g \circ T_t \psi dx \quad \forall \psi = \phi_t \circ T_t \in H_0^1(\Omega) \ . \tag{2.155}$$

Let us assume that

$$z^t = \frac{1}{t}(y^t - y) \in H_0^1(\Omega)$$

then

$$\begin{aligned} \int_\Omega \nabla z^t \cdot \nabla \psi dx = &-\frac{1}{t} \int_\Omega \langle (A(t) - \mathcal{I}) \cdot \nabla y^t, \nabla \psi \rangle_{\mathbb{R}^N} dx \\ &+\frac{1}{t} \int_\Omega (\gamma(t) f \circ T_t - f) \psi dx \ . \end{aligned} \tag{2.156}$$

From (2.155) it follows that

$$\|y^t\|_{H_0^1(\Omega)} \leqq C \ .$$

Moreover, using (2.156) we can show that y^t converges strongly to $y(\Omega)$ in $H_0^1(D)$ as $t \to 0$. Applying this convergence result to the right–hand side of (2.156) we have

$$\frac{1}{t}(A(t) - \mathcal{I}) \to A'(0) \quad \text{strongly in } L^\infty(D; \mathbb{R}^N)$$

and

$$\frac{1}{t}(\gamma(t)f \circ T_t - f) \to \operatorname{div}(fV(0)) \quad \text{strongly in } L^2(D),$$

it is assumed here that $k \geqq 1$. From the foregoing it can be inferred that z^t is bounded, i.e.

$$\|z^t\|_{H_0^1(\Omega)} \leqq C \ .$$

We can suppose that $z^k = z^{t_k} \rightharpoonup z$ weakly in $H_0^1(\Omega)$ (for a sequence $\{t_k\}$, $t_k \to 0$ as $k \to \infty$); for the weak limit z the following integral identity holds

$$\int_\Omega \nabla z \cdot \nabla \phi dx = - \int_\Omega \langle A'(0) \cdot \nabla y, \nabla \phi \rangle_{\mathbb{R}^N} dx + \int_\Omega \operatorname{div}(fV(0))\phi dx \quad \forall \phi \in H_0^1(\Omega) \ . \tag{2.157}$$

Let us assume that in (2.156) ϕ is taken as $z^k = z^{t_k}$. It is known that the sequence $\{\nabla y^{t_k}\}$ converges strongly to ∇y in $L^2(\Omega; \mathbb{R}^N)$ as $k \to +\infty$; furthermore

$$\frac{1}{t_k}(A(t_k) - \mathcal{I}) \cdot \nabla y^{t_k} \to A'(0) \cdot \nabla y \quad \text{strongly in } L^2(\Omega; \mathbb{R}^N),$$
$$\nabla z^k \rightharpoonup \nabla z \quad \text{weakly in } L^2(\Omega; \mathbb{R}^N) \ .$$

We can pass to the limit in (2.156) and obtain

$$\|z^k\|^2_{H_0^1(\Omega)} \to \|z\|^2_{H_0^1(\Omega)} \quad \text{as } k \to +\infty \ .$$

As it has been already shown, the convergence of z^k to z in $H_0^1(\Omega)$ assures the strong convergence; the element z is uniquely determined, hence from (2.157) it can be inferred by usual reasoning that z^t converges to z strongly in $H_0^1(D)$.

Proposition 2.82 *Let Ω be a given domain of class C^k in D, $k \geqq 1$, and let $V \in C(0, \varepsilon; V^k(D))$ be a given vector field. Suppose that $f \in H^1(D)$ is given, then the solution $y(\Omega_t)$ of the homogeneous Dirichlet boundary value problem (2.153) has the strong (1,2) material derivative z in the direction V, that is*

$$\frac{1}{t}(y(\Omega_t) \circ T_t(V) - y(\Omega)) \to z \quad \textit{strongly in } \ H_0^1(\Omega)$$

as $t \to 0$; the element z is uniquely determined as the unique solution to (2.157).

We now turn to the case $k \geqq 2$. Taking into account the classical implicit function theorem one has to consider the mapping

$$\Phi : [0, \varepsilon) \times (H^2(\Omega) \cap H_0^1(\Omega)) \to L^2(\Omega)$$

which is given by

$$\Phi(t, y) = -\operatorname{div}(a(t)\nabla y) - \gamma(t) f \circ T_t \ .$$

By standard regularity results applied to the elliptic problem defined in the domain Ω with the boundary Γ of class C^k (see e.g. (Nečas 1967)), it follows that for any t, $0 \leqq t < \varepsilon$, $y \to \Phi(t,y)$ is an isomorphism from $H^2(\Omega) \cap H_0^1(\Omega)$ onto $L^2(\Omega)$.

On the other hand, from our assumption that $f \in H^1(D)$ it follows that the mapping $t \to \Phi(t,y)$ is differentiable in the $L^2(\Omega)$ norm. By the implicit function theorem it follows that the mapping $t \to y^t$, where the element y^t is given as the unique solution to (2.154) and satisfies $\Phi(t,y^t) = 0$, is differentiable in $H^2(\Omega) \cap H_0^1(\Omega)$. The derivative at $t = 0$ is of the form

$$z = -D_y\Phi(0,y)^{-1} \cdot \frac{\partial_t\Phi}{\partial t}(0,y) .$$

This proves the following result:

Proposition 2.83 *Let Ω be a domain of class C^k in D, $k \geqq 2$, let $V \in C(0,\varepsilon;V^k(D))$ and $f \in H^1(D))$ be given. Then the solution $y(\Omega_t)$ to the homogeneous Dirichlet boundary value problem (2.153) has the strong (2,2) material derivative z in the direction V, that is*

$$\frac{1}{t}(y(\Omega_t) \circ T_t(V) - y(\Omega)) \to z \quad \textit{strongly in} \quad H^2(\Omega) \cap H_0^1(\Omega)$$

as $t \to 0$; the element z is given as the unique solution to (2.157).

2.29. The material derivative of a solution to the Laplace equation with Neumann boundary conditions

Let Ω be a domain in D with the boundary Γ of class C^k, $k \geqq 1$, let $V \in C(0,\varepsilon;V^k(D))$ and let $f \in L^2(D)$ be given elements. It is assumed that $y(\Omega_t) \in H^1(\Omega)/\mathbb{R}$ is the solution to the following Neumann boundary value problem

$$-\Delta y(\Omega_t) = f - \frac{1}{|\Omega_t|}\int_{\Omega_t} f dx \quad \text{in } \Omega_t , \tag{2.158}$$

$$\frac{\partial y}{\partial n_t}(\Omega_t) = 0 \quad \text{on } \Gamma_t , \tag{2.159}$$

where $|\Omega| = \text{meas}(\Omega)$.
For all elements ϕ_t in $H^1(\Omega_t)/\mathbb{R}$ we have

$$\int_{\Omega_t} \nabla y_t \cdot \nabla\phi_t dx = \int_{\Omega_t}\Big(f - \frac{1}{|\Omega_t|}\int_{\Omega_t} f dx\Big)\phi_t dx .$$

Applying the usual change of variables to the above integral identity,

$$x = T_t(V)(X),\ y^t = y_t \circ T_t,\ \phi = \phi_t \circ T_t \in H^1(\Omega),$$

we obtain

$$\int_\Omega \langle A(t) \cdot \nabla y^t, \nabla \phi \rangle_{\mathbb{R}^N} dx = \int_\Omega \gamma(t) \Big(f \circ T_t - \int_{\Omega_t} f dx \Big) \phi dx \ . \tag{2.160}$$

The element

$$F(t) = \gamma(t) f \circ T_t - \frac{1}{|\Omega_t|} \int_\Omega \gamma(t) f \circ T_t dx \tag{2.161}$$

satisfies

$$\int_\Omega F(t,x) dx = \int_{\Omega_t} f dx - \frac{1}{|\Omega_t|} \int_{\Omega_t} f dx \int_\Omega \gamma(t) dx = 0 \ .$$

This result is due to the fact that

$$\int_\Omega \gamma(t) f \circ T_t dx = \int_{\Omega_t} f dx \quad \text{and} \quad \int_\Omega \gamma(t) dx = \int_{\Omega_t} dx = |\Omega_t| \ .$$

Assuming that in (2.160) ϕ is taken as y^t we have

$$\|y^t\|_{H^1(\Omega)/\mathbb{R}} \leqq C$$

because $k \geqq 1$ and

$$\|A(t)\|_{W^{1,\infty}(D;\mathbb{R}^{N^2})} \leqq C \quad \text{for } t \in [0, \varepsilon) \ .$$

For $f \in L^2(D)$ we have that $F(t) \to F(0)$ strongly in $L^2(D)$ as $t \to 0$, where

$$F(0) = f - \frac{1}{|\Omega|} \int_\Omega f dx \ .$$

From (2.160) it follows that $y^t \to y$ strongly in $H^1(\Omega)/\mathbb{R}$.
First assuming that $\phi = y^t$ in (2.160) we get

$$\|y^t\|^2_{H^1(\Omega)/\mathbb{R}} \leqq C \ .$$

Let us consider a subsequence

$$y^k = y^{t_k},\ t_k \to 0 \text{ as } k \to \infty,$$

then

$$y^k \rightharpoonup y = y(\Omega)$$

weakly in $H^1(\Omega)/\mathbb{R}$ as $t \to 0$, hence $y^t \rightharpoonup y(\Omega)$ weakly in $H^1(\Omega)/\mathbb{R}$ as $t \to 0$.
For $z^t = \frac{1}{t}(y^t - y)$ one obtains

$$\int_\Omega \nabla z^t \cdot \nabla \phi dx = - \int_\Omega \langle \frac{1}{t}(A(t) - \mathcal{I}) \cdot \nabla y^t, \nabla \phi \rangle_{\mathbb{R}^N} dx +$$
$$\int_\Omega \frac{1}{t}(F(t) - F(0)) \phi dx \quad \forall \phi \in H^1(\Omega)/\mathbb{R} \ .$$

Furthermore $\frac{1}{t}(A(t) - \mathcal{I}) \cdot \nabla y^t$ converges weakly in $L^2(\Omega; \mathbb{R}^N)$ to the element $A'(0) \cdot \nabla y$. For $f \in H^1(D)$ we have

$$\frac{1}{t}(F(t) - F(0)) \to \operatorname{div}(fV(0)) + \operatorname{div}V(0)\frac{1}{|\Omega|^2}\int_\Omega f dx$$
$$-\frac{1}{|\Omega|}\int_\Omega \operatorname{div}(fV(0))d\Gamma = F'(0) \in L^2\Omega) \text{ strongly in } L^2(\Omega) \text{ as } t \to 0 \ .$$

Thus

$$\int_\Omega F'(0)dx = \int_\Gamma f\langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma + \frac{1}{|\Omega|^2}\int_\Gamma \langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma \int_\Omega f dx$$
$$-\int_\Gamma f\langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma = \frac{1}{|\Omega|^2}\int_\Gamma \langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma \int_\Omega f dx \ .$$

From the foregoing we obtain

Proposition 2.84 *Let Ω be a given domain of class C^k in D, $k \geqq 1$, and let $V \in C(0, \varepsilon; \mathcal{D}(D; \mathbb{R}^N))$ be a given vector field. Then for a given element $f \in H^1(D)$ and the field V,*

$$\int_\Gamma \langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma \int_\Omega f dx = 0,$$

the solution $y(\Omega)$ to the Neumann boundary value problem (2.158) and (2.159) has the weak material derivative in $H^1(\Omega)/\mathbb{R}$ in the direction of the speed vector field V.

2.30. Shape derivatives

We denote by D a given domain in $\mathbb{R}^N$. Let Ω be a domain of class C^k in D, and let $y(\Omega)$ be an element of the specific Sobolev space denoted by $W(\Omega)$. It is assumed that the following conditions are satisfied:

The weak material derivative $\dot{y}(\Omega; V)$ exists in $W(\Omega)$ and $\nabla y(\Omega) \cdot V(0) \in W(\Omega)$ for all vector fields $V \in C(0, \varepsilon; V^k(D))$, $k \geqq 1$.

Definition 2.85 The shape derivative of $y(\Omega)$ in the direction V is the element $y'(\Omega; V) \in W(\Omega)$ defined by

$$y'(\Omega; V) = \dot{y}(\Omega; V) - \nabla y(\Omega) \cdot V(0) \ . \tag{2.163}$$

Proposition 2.86 *If the mapping $V \to \dot{y}(\Omega; V)$ is continuous from $C(0, \varepsilon; V^k(D))$ into $W(\Omega)$, then*

$$y'(\Omega; V) = y'(\Omega; V(0)) \ . \tag{2.164}$$

Proof. Let us consider the domain functional

$$J(\Omega) = \int_\Omega y(\Omega)\phi dx,$$

where ϕ is an element given in $W(D)$. Applying the change of variables $x = T_t(V)(X)$ to the above integral we have

$$J(\Omega_t) = \int_\Omega y(\Omega_t) \circ T_t \phi \circ T_t \gamma(t) dx \ .$$

The Eulerian derivative of this functional is of the form

$$dJ(\Omega; V) = \int_\Omega \dot{y}(\Omega; V)\phi dx + \int_\Omega y(\Omega)\mathrm{div}(\phi V(0)) dx \ .$$

By Proposition 2.21 it follows that $dJ(\Omega; V) = dJ(\Omega; V(0))$. Therefore $\dot{y}(\Omega; V) = \dot{y}(\Omega; V(0))$ and we obtain (2.164). □

Proposition 2.87 *Let us assume that the mapping $V \to \dot{y}(\Omega; V)$ is linear and continuous from $C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N))$ into $W(\Omega)$. If V_1 and V_2 are two vector fields in $C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N))$,*

$$\langle V_1(0), n\rangle_{\mathbb{R}^N} = \langle V_2(0), n\rangle_{\mathbb{R}^N} \qquad on\ \Gamma = \partial\Omega,$$

then

$$y'(\Omega; V_1) = y'(\Omega; V_2) \ .$$

Proof.

$$y'(\Omega; V_1) - y'(\Omega; V_2) = y'(\Omega; V_1 - V_2) = y'(\Omega; V_1(0) - V_2(0)) \ .$$

We have $\langle (V_1(0) - V_2(0)), n\rangle_{\mathbb{R}^N} = 0$ on Γ, then from Proposition 2.25 it follows that $\dot{y}(\Omega; V_1(0)) - V_2(0)) = \nabla y(\Omega) \cdot (V_1(0) - V_2(0))$; hence as a result we get (2.165). □

2.31. Derivatives of domain integrals (II)

We shall extend the results obtained in Sect. 2.16 to the case of $y(\Omega)$ having the weak material derivative in $L^1(\Omega)$.

Let D be a given domain in $\mathbb{R}^N$ and let $y(\Omega) \in L^1(\Omega)$ be a given element such that there exists the weak material derivative $\dot{y}(\Omega; V)$ in $L^1(\Omega)$ as well as the shape derivative $y'(\Omega; V)$ in $L^1(\Omega)$ for any vector field $C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N))$. In other words

$$\int_\Omega \frac{1}{t}(y(\Omega) \circ T_t(V) - y(\Omega))\phi dx \to \int_\Omega \dot{y}(\Omega; V)\phi dx \quad \forall \phi \in L^\infty(\Omega)$$

as $t \to 0$. Furthermore it is assumed that $\nabla y(\Omega) \in L^1(\Omega; \mathbb{R}^N)$. Let us consider the domain functional

$$J(\Omega) = \int_\Omega y(\Omega) dx \ .$$

Applying the change of variables $x = T_t(V)(X)$ to the integral we obtain

$$J(\Omega_t) = \int_\Omega \gamma(t) y(\Omega_t) \circ T_t dx \ .$$

It is assumed here that the mapping $t \to \gamma(t)$ is differentiable in the norm of the space $L^\infty(D)$ (for $k \geqq 1$). The Eulerian derivative of $J(\Omega)$ has the form

$$dJ(\Omega; V) = \int_\Omega \dot{y}(\Omega; V) dx + \int_\Omega y(\Omega) \mathrm{div} V(0) dx \ . \tag{2.166}$$

From (2.163) it follows that

$$dJ(\Omega; V) = \int_\Omega y'(\Omega; V) dx + \int_\Omega \mathrm{div}(y(\Omega) V(0)) dx \ . \tag{2.167}$$

In other words, (2.167) constitutes a generalization of (2.113).

Finally, if Ω is a C^k domain, $k \geqq 1$, then by Stokes' formula it follows

$$dJ(\Omega; V) = \int_\Omega y'(\Omega; V) dx + \int_\Gamma y(\Omega) \langle V(0), n \rangle_{\mathbb{R}^N} d\Gamma \ . \tag{2.168}$$

2.32. Shape derivatives on Γ

We denote by D a given domain in $\mathbb{R}^N$; for any domain Ω in D with the boundary Γ of class C^k, $k \geqq 2$, an element $z(\Gamma)$ of the specific Sobolev space denoted $W(\Gamma)$ is considered. It is assumed that the following conditions are satisfied:

(i) The weak material derivative $\dot{y}(\Gamma; V)$ exists in $W(\Gamma)$.
(ii) $\nabla_\Gamma z(\Gamma) \cdot V(0)$ belongs to the space $W(\Gamma)$ for all vector fields $V \in C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N))$,

where $k \geqq 3$ is a fixed integer.

Definition 2.88 The shape derivative of z in the direction V is the element of $W(\Gamma)$ defined by

$$z'(\Gamma; V) = \dot{z}(\Gamma; V) - \nabla_\Gamma z(\Gamma) \cdot V(0) \ . \tag{2.169}$$

Proposition 2.89 *Assume that the mapping $V \to \dot{y}(\Omega; V)$ is continuous from $C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N))$ into $W(\Gamma)$. Then*

$$z'(\Gamma; V) = z'(\Gamma; V(0)) \ . \tag{2.170}$$

Proof. Let us consider the domain functional

$$J(\Omega) = \int_\Gamma z(\Gamma)\phi d\Gamma \ ,$$

where ϕ is an element given in $\mathcal{D}(D)$.
Using the change of variables $x = T_t(V)(X)$ we have

$$J(\Omega_t) = \int_\Gamma z(\Gamma_t) \circ T_t \omega(t) d\Gamma.$$

Hence the Eulerian derivative is given by

$$dJ(\Omega; V) = \int_\Gamma \dot{z}(\Gamma; V)\phi d\Gamma + \int_\Gamma z\nabla\phi \cdot V(0) d\Gamma \ . \tag{2.171}$$

According to our assumptions $C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N)) \ni V \to dJ(\Omega; V) \in \mathbb{R}^N$ is a continuous mapping, thus from Proposition 2.26 it follows that $dJ(\Omega; V) = dJ(\Omega; V(0))$. Moreover by (2.171) it follows that $\dot{z}(\Gamma; V) = \dot{z}(\Gamma; V(0))$ as was to be shown. □

Proposition 2.90 *Let us assume that the mapping $V \to \dot{z}(\Gamma; V)$ is linear and continuous from $C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N))$ into $W(\Gamma)$. If V_1 and V_2 are two vector fields in $C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N))$,*

$$\langle V_1(0), n\rangle_{\mathbb{R}^N} = \langle V_2(0), n\rangle_{\mathbb{R}^N} \quad \text{on } \Gamma,$$

then we have

$$z'(\Gamma; V_1) = z'(\Gamma; V_2) \ .$$

Proof. Denote by V the difference $V_2 - V_1$, then $\langle V(0), n\rangle_{\mathbb{R}^N} = 0$ on Γ and $z'(\Gamma; V) = z'(\Gamma; V(0))$. It can be inferred that $T_t(V(0))(\Omega) = \Omega$. Hence from Proposition 2.87 it follows that

$$\dot{z}(\Gamma; V(0)) = \nabla_\Gamma z(\Gamma) \cdot V(0) \ .$$

Making use of (2.169) we can conclude the proof. □

2.33. Derivatives of boundary integrals (II)

We extend the results obtained in Sect. 2.18 to the case of $z(\Gamma)$ having the weak material derivative in $L^1(\Gamma)$.

Let D be a given domain in $\mathbb{R}^N$ and let $z(\Gamma) \in L^1(\Gamma)$ be an element such that there exist the weak material derivative $\dot{z}(\Gamma; V)$ in $L^1(\Gamma)$ and the shape derivative $z'(\Gamma; V) \in L^1(\Gamma)$ for any vector field $V \in C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N))$, where $k \geqq 2$ is an integer. Therefore

$$\int_\Gamma \frac{1}{t}(z(\Gamma_t) \cdot T_t(V) - z(\Gamma))\phi d\Gamma \to \int_\Gamma \dot{z}(\Gamma; V)\phi d\Gamma \quad \forall \phi \in L^\infty(\Gamma)$$

as $t \to 0$. Furthermore

$$\nabla_\Gamma z(\Gamma) \in L^1(\Gamma; \mathbb{R}^N) \ .$$

Let us consider the domain functional

$$J(\Omega_t) = \int_{\Gamma_t} z(\Gamma_t) d\Gamma \ .$$

The change of variables $x = T_t(V)(X)$ yields

$$J(\Omega) = \int_\Gamma z(\Gamma_t) \circ T_t \omega(t) d\Gamma,$$

where $t \to \omega(t)$ is differentiable in the norm of the space $L^\infty(\Gamma)$ for $k \geqq 2$. Therefore the Eulerian derivative of J is of the form

$$dJ(\Omega; V) = \int_\Gamma \dot{z}(\Gamma; V) d\Gamma + \int_\Gamma z(\Gamma) \mathrm{div}_\Gamma(V(0)) d\Gamma \ . \tag{2.172}$$

From (2.163) it follows that

$$dJ(\Omega; V) = \int_\Gamma z'(\Gamma; V) d\Gamma + \int_\Gamma [\nabla_\Gamma z(\Gamma) \cdot V(0) + z(\Gamma)\mathrm{div}_\Gamma(V(0))] d\Gamma,$$

that is

$$dJ(\Omega; V) = \int_\Gamma z'(\Gamma; V) d\Gamma + \int_\Gamma \mathrm{div}_\Gamma(z(\Gamma)V(0)) d\Gamma \ .$$

On the other hand, in view of (2.141), we have

$$\int_\Gamma \mathrm{div}_\Gamma(z(\Gamma)V(0)) d\Gamma = \int_\Gamma z(\Gamma)\kappa \langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma,$$

where κ is the mean curvature on the manifold Γ. Finally we obtain

$$dJ(\Omega;V) = \int_\Gamma [z'(\Gamma;V) + \kappa z(\Gamma)\langle V(0), n\rangle_{\mathbb{R}^N}] d\Gamma \ . \tag{2.173}$$

In the particular case of $z(\Gamma) = y(\Omega)|_\Gamma$ we obtain

$$z'(\Gamma;V) = y'(\Omega;V)|_\Gamma + \frac{\partial y}{\partial n}(\Omega)\langle V(0), n\rangle_{\mathbb{R}^N}$$

and (2.173) can be rewritten in the form

$$dJ(\Omega;V) = \int_\Gamma y'(\Omega;V)|_\Gamma d\Gamma + \int_\Gamma \left(\frac{\partial}{\partial n} y(\Omega) + \kappa y(\Omega)\right) \langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma \ . \tag{2.174}$$

3. Shape Derivatives for Linear Problems

In this chapter the form of the material derivatives and the shape derivatives for linear boundary value problems as well as initial boundary value problems is derived.

In Sect. 3.1 the form of the shape derivative for a second order elliptic boundary value problem with non–homogeneous Dirichlet boundary conditions is determined. In Sect. 3.2 the same problem but with non–homogeneous Neumann boundary conditions is considered.

In Sect. 3.3 the necessary optimality conditions for the general shape optimization problem are established, and two specific examples are presented. In particular, the form of the second order shape derivative of the integral domain functional discussed in the first example is obtained. In the second example, the same as in Sect. 2.6.1, the domain functional involves the perimeter of a given domain. Such a term is essential for the applications, in particular, to free boundary problems. Parabolic equations are considered in Sect. 3.4.

The shape derivative of the solution to the system of equations of linear elasticity is determined in Sect. 3.5. In Sect. 3.6 the multiple eigenvalue problem is investigated by means of non–smooth analysis. In Sect. 3.7 the shape derivative of the solution to the Kirchhoff plate problem is derived. Domains with corners in the plane, and in $\mathbb{R}^3$ are studied in Sect. 3.8. Sect. 3.9 presents results on the shape sensitivity analysis of elliptic boundary value problems with singularities. Finally in Sect. 3.10 an initial boundary value problem of hyperbolic type is considered.

The results discussed in this chapter can be applied, in particular to shape optimization problems for linear partial differential equations or systems of equations of elliptic, parabolic and hyperbolic types.

3.1. The shape derivative for the Dirichlet boundary value problem

Let D be a given domain in $\mathbb{R}^N$. It is assumed that for any domain Ω of class C^k in D there are given three elements $h(\Omega), z(\Omega), y(\Omega)$ such that $h(\Omega) \in \mathrm{L}^2(\Omega), z(\Gamma) \in H^{\frac{1}{2}}(\Gamma)$, and $y(\Omega) \in H^1(\Omega)$ is a solution to the Dirichlet boundary value problem

$$-\Delta y(\Omega) = h(\Omega) \quad \text{in } L^2(\Omega), \tag{3.1}$$

$$y(\Omega) = z(\Gamma) \quad \text{on } \Gamma \ . \tag{3.2}$$

It is assumed that for any vector field $V \in C(0, \varepsilon; \mathcal{D}^k(D; \mathbb{R}^N))$ and for the given elements $h(\Omega)$, $z(\Gamma)$, $y(\Omega)$, there exist the shape derivatives $h'(\Omega)$, $z'(\Gamma)$, $y'(\Omega)$ in $L^2(\Omega)$, $H^{\frac{1}{2}}(\Gamma)$, $H^1(\Omega)$, respectively. In particular we have

$$\frac{1}{t}(y(\Omega_t) \circ T_t(V) - y(\Omega)) \rightharpoonup \dot{y}(\Omega; V) \quad \text{weakly in } H^1(\Omega) \tag{3.3}$$

as $t \to 0$, and

$$\nabla y(\Omega) \cdot V(0) \in H^1(\Omega) \ . \tag{3.4}$$

Let us consider the weak form of the equations (3.1) and (3.2) written as the integral identity

$$\int_\Omega y(\Omega)\Delta\phi dx = \int_\Gamma z(\Gamma)\frac{\partial \phi}{\partial n} d\Gamma \quad \forall \phi \in H^2(\Omega) \cap H_0^1(\Omega) \ . \tag{3.5}$$

Let $\phi \in \mathcal{D}(\mathbb{R}^N)$, then from (3.2) it follows that

$$\int_{\Gamma_t} y(\Omega_t)\phi d\Gamma_t = \int_{\Gamma_t} z(\Omega_t)\phi d\Gamma_t \ .$$

Taking the derivative with respect to t at $t = 0$ of both sides of this identity, we obtain

$$\int_\Gamma y'(\Omega; V)|_\Gamma \phi d\Gamma + \int_\Gamma \left[\frac{\partial}{\partial n}(y(\Omega)\phi) + \kappa y(\Omega)\phi\right] \langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma =$$
$$\int_\Gamma (z(\Gamma)\phi)'(\Gamma; V) d\Gamma + \int_\Gamma \kappa z(\Gamma)\phi \langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma \ .$$

For a given element $\phi \in \mathcal{D}(\mathbb{R}^N)$ we have

$$\phi'(\Gamma; V) = \frac{\partial \phi}{\partial n}\langle V(0), n\rangle_{\mathbb{R}^N} \ .$$

If it is assumed that $\frac{\partial \phi}{\partial n} = 0$ on Γ, then

$$\int_\Gamma y'(\Omega;V)|_\Gamma \phi d\Gamma + \int_\Gamma \Big(\frac{\partial}{\partial n} y(\Omega) + \kappa y(\Omega)\Big)\phi\langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma =$$
$$\int_\Gamma z'(\Gamma;V)\phi d\Gamma + \int_\Gamma \kappa z(\Gamma)\phi\langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma \ .$$

From (3.2) it follows that

$$y'(\Omega;V)|_\Gamma = -\frac{\partial}{\partial n} y(\Omega)\langle V(0), n\rangle_{\mathbb{R}^N} + z'(\Gamma;V) \quad \text{on } \Gamma \ . \tag{3.6}$$

On the other hand, since $\phi \in \mathcal{D}(\Omega)$, then $\phi \in \mathcal{D}(\Omega_t)$. Thus for $t > 0$, t small enough, we obtain

$$\int_{\Omega_t} \nabla y(\Omega_t) \cdot \nabla \phi dx = \int_{\Omega_t} h(\Omega_t)\phi dx \ . \tag{3.7}$$

Taking the derivative with respect to t at $t = 0$ of both sides of this identity we have

$$\int_\Omega \nabla y'(\Omega;V) \cdot \nabla \phi dx = \int_{\Omega_t} h'(\Omega;V)\phi dx \ ,$$

that is

$$-\Delta y'(\Omega;V) = h'(\Omega;V) \quad \text{in } \mathcal{D}'(\Omega) \ . \tag{3.8}$$

Proposition 3.1 *Let $(h(\Omega), z(\Gamma)) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma)$ be given elements such that there exist the shape derivatives $(h'(\Omega), z'(\Gamma))$ in $L^2(\Omega) \times H^{1/2}(\Gamma)$. Then the solution $y(\Omega)$ to the Dirichlet boundary value problem (3.1) and (3.2) has the shape derivative $y'(\Omega;V)$ in $H^1(\Omega)$ determined as the unique solution to the Dirichlet boundary value problem (3.6) and (3.8).*

3.2. The shape derivative for the Neumann boundary value problem

Let D be a given domain in $\mathbb{R}^N$. It is assumed that for any domain Ω of class C^k in D there are given three elements $h(\Omega), z(\Gamma) and y(\Omega)$ such that $h(\Omega) \in L^2(\Omega)$ and $z(\Gamma) \in H^{\frac{1}{2}}(\Omega)/\mathbb{R}$,

$$\int_\Omega h(\Omega)dx + \int_\Gamma z(\Gamma)d\Gamma = 0 \ . \tag{3.9}$$

In this section $y(\Omega)$ denotes a solution to the Neumann boundary value problem

$$-\Delta y(\Omega) = h(\Omega) \quad \text{in } L^2(\Omega),$$
$$\frac{\partial}{\partial n} y(\Omega) = z(\Omega) \quad \text{in } H^{\frac{1}{2}}(\Gamma) \ .$$

Let us consider the following integral identity

$$\int_{\Omega_t} \nabla y(\Omega_t) \cdot \nabla \phi dx = \int_{\Omega_t} h(\Omega_t)\phi dx + \int_{\Gamma_t} z(\Gamma)\phi d\Gamma \ , \tag{3.10}$$

where $\phi \in \mathcal{D}(\mathbb{R}^N)$ is a given element and $y(\Omega_t) \in H^1(\Omega_t)/\mathbb{R}$.
Taking the derivative of (3.10) with respect to t at $t = 0$ we obtain

$$\int_\Omega \nabla y'(\Omega; V) \cdot \nabla \phi dx + \int_\Gamma \nabla y \cdot \nabla \phi \langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma =$$
$$\int_\Omega h'(\Omega; V)\phi dx + \int_\Gamma h(\Omega)\phi \langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma +$$
$$\int_\Gamma \left[z'(\Gamma; V)\phi + z(\Gamma)\frac{\partial \phi}{\partial n} + \kappa z(\Gamma)\phi\langle V(0), n\rangle_{\mathbb{R}^N} \right] d\Gamma \ .$$

Assuming that ϕ is in $\mathcal{D}(\Omega)$ we get

$$-\Delta y'(\Omega; Y) = h'(\Omega; V) \quad \text{in } \Omega \ . \tag{3.11}$$

If the test function ϕ is such that $\frac{\partial \phi}{\partial n} = 0$ on Γ, then Green's formula yields

$$\int_\Gamma \frac{\partial}{\partial n} y'(\Omega; V)\phi d\Gamma - \int_\Gamma \mathrm{div}_\Gamma(\langle V(0), n\rangle_{\mathbb{R}^N} \nabla_\Gamma y)\phi d\Gamma =$$
$$\int_\Gamma \left[h(\Omega)\langle V(0), n\rangle_{\mathbb{R}^N} + z'(\Gamma; V) + \kappa z(\Gamma)\langle V(0), n\rangle_{\mathbb{R}^N}\right] \phi d\Gamma \ .$$

If $v_n = \langle V(0), n\rangle_{\mathbb{R}^N}$ on Γ, then the following Neumann boundary conditions can be set out for $y'(\Omega; V)$

$$\frac{\partial}{\partial n}\left[y'(\Omega; V)\right] = \mathrm{div}_\Gamma(v_n \nabla_\Gamma y(\Omega)) + \tag{3.12}$$
$$\left[h(\Omega) + \kappa z(\Gamma)\right] v_n + z'(\Gamma; V) \quad \text{on } \Gamma \ .$$

We shall show that the compatibility condition (3.9) is satisfied in an appropriate way for the problem (3.11) and (3.12).

Proposition 3.2 *For the terms on the right-hand side of formulae (3.11) and (3.12) the compatibility condition (3.9) holds, i.e.*

$$\int_\Omega h'(\Omega; V)dx - \int_\Gamma \mathrm{div}_\Gamma(v_n \nabla_\Gamma y(\Omega))d\Gamma + \tag{3.13}$$
$$\int_\Gamma \left[y(\Omega) + \kappa z(\Gamma)\right] v_n d\Gamma + \int_\Gamma z'(\Gamma; V)d\Gamma = 0,$$

hence there exists the unique solution $y'(\Omega;V) \in H^1(\Omega)/\mathbb{R}$ to the problem (3.11) and (3.12).

Proof. From (3.9) it follows that in (3.10) ϕ can be replaced with $\phi + c$, where c is any constant. Differentiation with respect to t does not change this property. Therefore ϕ can be replaced with $\phi + c$ in the integral identity obtained by differentiation of (3.13) with respect to t. This yields

$$\int_\Omega h' dx + \int_\Gamma h v_n d\Gamma + \int_\Gamma z' d\Gamma + \int_\Gamma \kappa z v_n d\Gamma = 0 \ .$$

On the other hand we have

$$\int_\Gamma \operatorname{div}_\Gamma (v_n \nabla_\Gamma y) d\Gamma = - \int_\Gamma v_n \nabla_\Gamma y \cdot \nabla_\Gamma 1 d\Gamma = 0$$

as was to be shown. □

Proposition 3.3 *Let $(h(\Omega), z(\Gamma)) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma)$ be given elements such that there exist the shape derivatives $(h'(\Omega;V), z'(\Gamma;V))$ in $L^2(\Omega) \times H^{1/2}(\Gamma)$. Then the solution $y(\Omega)$ to the Neumann boundary value problem has the shape derivative $y'(\Omega;V)$ in $H^1(\Omega)/\mathbb{R}$. This derivative is given as the unique solution to the Neumann boundary value problem (3.11) and (3.12).*

3.3. Necessary optimality conditions

Let us consider the domain functional $J(\Omega)$ defined for any domain Ω of class C^k in D and depending on an element $y(\Omega) \in W^{s,p}(\Omega)$

$$\begin{aligned} J(\Omega) = & \int_\Omega F_1(x, y(\Omega)(x), \nabla y(\Omega)(x)) dx + \\ & \int_\Gamma F_0(x, y(\Omega)(x), \nabla y(\Omega)(x)) d\Gamma \ . \end{aligned} \tag{3.14}$$

We assume that for any vector field $V \in C(0,\varepsilon; V^k(D))$, where $s > 3/2$, $1 \leqq p < \infty$, there exists the shape derivative $y'(\Omega;V)$ in $W^{s,p}(\Omega)$ of $y(\Omega) \in W^{s,p}(\Omega)$.

Furthermore, it is assumed that the elements $F_0(x,y,q)$ and $F_1(x,y,q)$ are sufficiently smooth with respect to $y = y(\Omega)(x), q = \nabla y(\Omega)(x)$, respectively. The Eulerian derivative of $J(\Omega)$ in the direction of a vector field $V \in C(0,\varepsilon; V^k(D))$ has the form

$$dJ(\Omega;V) = \int_\Omega \frac{\partial F_1}{\partial y}(x, y(\Omega)(x), \nabla y(\Omega)(x))y'(\Omega;V)(x)dx+ \tag{3.15}$$
$$\int_\Omega \nabla_q F_1(x, y(\Omega)(x), \nabla y(\Omega)(x)) \cdot \nabla(y'(\Omega;V))dx+$$
$$\int_\Gamma F_1(x, y(\Omega)(x), \nabla y(\Omega)(x))\langle V(0,x), n\rangle_{\mathbb{R}^N} d\Gamma+$$
$$\int_\Gamma \frac{\partial F_0}{\partial y}(x, y(\Omega)(x), \nabla y(\Omega)(x))z'(\Gamma;V)(x)d\Gamma+$$
$$\int_\Gamma \nabla_q F_0(x, y(\Omega)(x), \nabla y(\Omega)(x)) \cdot \nabla y'(\Omega;V)d\Gamma+$$
$$\int_\Gamma \langle \nabla_q F_0(x, y(\Omega)(x), \nabla y(\Omega)(x)), D^2 y(\Omega) \cdot n\rangle_{\mathbb{R}^N} \langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma+$$
$$\int_\Gamma \kappa(x) F_0(x, y(\Omega)(x), \nabla y(\Omega)(x))\langle V(0,x), n(x)\rangle_{\mathbb{R}^N} d\Gamma\ ,$$

where $z(\Gamma) = y(\Omega)|_\Gamma$. Moreover for $x \in \Gamma$ we have

$$z'(\Gamma;V)(x) = y'(\Omega;V) + \frac{\partial y}{\partial n}(\Omega)(x)\langle V(0;x), n(x)\rangle_{\mathbb{R}^N}\ . \tag{3.16}$$

Let us assume that the shape derivative $y'(\Omega;V) \in W^{s,p}(\Omega) = W$ is determined as the unique solution to the following linear problem

$$\langle Ay'(\Omega;V), \phi\rangle_{W'\times W} = L(V,\phi) \quad \forall \phi \in W\ , \tag{3.17}$$

where $A \in \mathcal{L}(W;W')$ and $L(V,\cdot) \in W'$ are given elements.
Let $p(\Omega) \in W'$ be the adjoint state defined as a weak solution to the following adjoint problem

$$\langle \psi, A^* p\rangle_{W\times W'} = \int_\Omega \frac{\partial F_1}{\partial y}(x, y(\Omega)(x), \nabla y(\Omega)(x))\psi dx+ \tag{3.18}$$
$$\int_\Omega \nabla_q F_1(x, y(\Omega)(x), \nabla y(\Omega)(x)) \cdot \nabla \psi dx+$$
$$\int_\Gamma \frac{\partial F_0}{\partial y}(x, y(\Omega)(x), \nabla y(\Omega)(x))\psi d\Gamma+$$
$$\int_\Gamma \nabla_q F_0(x, y(\Omega)(x), \nabla y(\Omega)(x)) \cdot \nabla \psi d\Gamma \quad \forall \psi \in W\ .$$

To obtain the Eulerian derivative $dJ(\Omega;V)$ of the domain functional $J(\Omega)$ we have to make use of the identity

$$\langle y'(\Omega;V), A^* p\rangle = \langle Ay'(\Omega;V), p\rangle = L(V,p)\ .$$

Hence

$$
\begin{aligned}
dJ(\Omega;V) = L(V,p) + \int_\Gamma F_1(x,y(\Omega)(x))\langle V(0),n\rangle_{\mathbb{R}^N} d\Gamma + \qquad (3.19)\\
\int_\Gamma \frac{\partial F_0}{\partial y}(x,y(\Omega)(x)\frac{\partial y}{\partial n}\langle V(0),n\rangle_{\mathbb{R}^N} d\Gamma + \\
\int_\Gamma \langle \nabla_q F_0(x,y(\Omega)(x), D^2 y\cdot n\rangle_{\mathbb{R}^N}\langle V(0),n\rangle_{\mathbb{R}^N} d\Gamma + \\
\int_\Gamma \kappa(x) F_0(x,y(\Omega)(x)\langle V(0),n\rangle_{\mathbb{R}^N} d\Gamma \ .
\end{aligned}
$$

In general, the mapping $V \to L(V,p)$ is linear and continuous on $C(0,\varepsilon;V^k(D))$ and depends only on $v_n = \langle V(0),n\rangle_{\mathbb{R}^N}$. As far as the second order elliptic boundary value problems are concerned, see Sects. 3.1 and 3.2 for details. Let us consider the following example of the shape functional (3.14):

$$J(\Omega) = \frac{1}{2}\int_\Omega (y(\Omega) - z_g)^2 dx, \tag{3.20}$$

where z_g is a given element in $H^1(\Omega)$, i.e. we have

$$F_0(x,p,q) = \frac{1}{2}(y(\Omega) - z_g(x))^2 \tag{3.21}$$

and

$$F_1 \equiv 0 \ .$$

Here $y(\Omega)$ denotes the solution to the following Dirichlet boundary value problem

$$-\Delta y(\Omega) = h(\Omega) \quad \text{in } \Omega, \tag{3.22}$$
$$y(\Omega) = 0 \quad \text{on } \Gamma \ . \tag{3.23}$$

Using the results of Sect. 3.1 one can show that the domain derivative $y'(\Omega;V)$ is given as the unique solution to the following problem

$$-\Delta y'(\Omega;V) = h'(\Omega;V) \quad \text{in } \Omega, \tag{3.24}$$
$$y'(\Omega;V) = -\frac{\partial y}{\partial n}(\Omega;V)\langle V(0),n\rangle_{\mathbb{R}^N} \quad \text{on } \Gamma \ . \tag{3.25}$$

In particular, if the element $h(\Omega)$ is defined as the restriction to Ω of an element $f \in H^1(D)$, then $h'(\Omega;V) = 0$ and the adjoint problem is of the form

$$-\Delta p(\Omega) = y(\Omega) - z_g \quad \text{in } \Omega, \tag{3.26}$$
$$p(\Omega) = 0 \quad \text{on } \Gamma \ . \tag{3.27}$$

Therefore we have

$$dJ(\Omega;V)=\int_\Omega (y(\Omega)-z_g)y'(\Omega;V)dx \\ +\frac{1}{2}\int_\Gamma (y(\Omega)-z_g)^2\langle V(0),n\rangle_{\mathbb{R}^N}d\Gamma \tag{3.28}$$

and from (3.26) it follows that

$$dJ(\Omega;V)=-\int_\Omega \Delta p(\Omega)y'(\Omega;V)dx \\ +\frac{1}{2}\int_\Gamma (y(\Omega)-z_g)^2\langle V(0),n\rangle_{\mathbb{R}^N}d\Gamma\ . \tag{3.29}$$

Using Green's formula we obtain

$$\int_\Omega \Delta p(\Omega)y'(\Omega;V)dx=\int_\Omega \nabla p(\Omega)\cdot\nabla y'(\Omega;V)dx \\ \int_\Gamma \frac{\partial}{\partial n}p(\Omega)y'(\Omega;V)d\Gamma\ . \tag{3.30}$$

Making use of (3.25) it is possible to show that

$$\int_\Gamma \frac{\partial p}{\partial n}y'(\Omega;V)d\Gamma=-\int_\Gamma \frac{\partial p}{\partial n}\frac{\partial y}{\partial n}\langle V(0),n\rangle_{\mathbb{R}^N}d\Gamma\ , \tag{3.31}$$

and

$$\int_\Omega \nabla p\cdot\nabla y' dx=-\int_\Omega \Delta y' p dx+\int_\Gamma \frac{\partial y'}{\partial n}pd\Gamma=0\ . \tag{3.32}$$

Finally we get

$$dJ(\Omega;V)=\int_\Gamma g\langle V(0),n\rangle_{\mathbb{R}^N}d\Gamma,$$

where

$$g(x)=-\nabla y(\Omega)(x)\cdot\nabla p(\Omega)(x)+\frac{1}{2}(y(\Omega)(x)-z_g(x))^2 \quad \text{on } \Gamma, \tag{3.33}$$

here $y(\Omega)$ and $p(\Omega)$ are given as the unique solutions to (2.183) and (2.185), respectively.
The results of Sect. 2.13 enables us to show that the derivative with respect to t at $t\neq 0$ of the mapping $t\to J(\Omega_t)$ is of the form

$$\frac{d}{dt}J(\Omega_t)=dJ(\Omega_t,V_t)\ ,$$

with $V_t(s)\equiv V(t+s)$; hence from (3.33) it follows that

$$\frac{d}{dt}J(\Omega_t)=\int_{\Gamma_t}\langle V(t),n_t\rangle_{\mathbb{R}^N}d\Gamma_t\ , \tag{3.34}$$

where

$$V \in C(0, \varepsilon; V^k(D)), \quad \Omega_t = T_t(V)(\Omega), \quad \Gamma_t = T_t(V)(\Gamma), \quad \Gamma = \partial\Omega,$$

$y(\Omega_t)$ and $p(\Omega_t)$ denote solutions to (2.183) and (2.185), respectively. These solutions are associated with the domain Ω_t of class C^k.
Let us consider the particular case of an autonomous vector field V (i.e. V is independent of t). From (3.34) it can be inferred that the second order derivative of $J(\Omega_t)$ with respect to t is defined as follows

$$\frac{d^2}{dt^2} J(\Omega_t)_{|t=0} = \frac{d}{d\varepsilon}\Big(\int_\Gamma [-\langle B(\varepsilon)\cdot\nabla y^\varepsilon, \nabla p^\varepsilon\rangle_{\mathbb{R}^N} + \frac{1}{2}(y^\varepsilon - z_g \circ T_\varepsilon)^2\omega(\varepsilon)]\,\langle V\circ T_\varepsilon, n^\varepsilon\rangle_{\mathbb{R}^N} d\Gamma\Big)_{|\varepsilon=0}, \tag{3.35}$$

where

$$\begin{aligned} y^\varepsilon &= y(\Omega_\epsilon)\circ T_\varepsilon\ , \\ p^\varepsilon &= p(\Omega_\epsilon)\circ T_\varepsilon\ , \\ n^\varepsilon &= n_\varepsilon \circ T_\varepsilon\ , \\ B(\varepsilon) &= \omega(\varepsilon) DT_\varepsilon^{-1}\cdot {}^*(DT_\varepsilon)^{-1}\ , \\ \omega(\varepsilon) &= \|M(T_\varepsilon)\cdot n\|_{\mathbb{R}^N}\ , \\ M(T_\varepsilon) &= \det(DT_\varepsilon)\ {}^*DT_\varepsilon^{-1}\ . \end{aligned}$$

It should be remarked that

$$\omega'(0) = \mathrm{div}_\Gamma V, \quad B'(0) = \mathrm{div}_\Gamma V\mathcal{I} - 2\epsilon(V)\ , \tag{3.36}$$

where $\epsilon(V)$ is the symmetrized part of DV

$$\epsilon(V) = \frac{1}{2}(DV + {}^*DV)\ .$$

Furthermore we have

$$n_\varepsilon \circ T_\varepsilon = \|M(T_\varepsilon)\cdot n\|^{-1}_{\mathbb{R}^N} M(T_\varepsilon)\cdot n \quad \text{on } \Gamma\ .$$

Therefore

$$\begin{aligned} \frac{dn^\varepsilon}{d\varepsilon}_{|\varepsilon=0} &= (-\mathrm{div}_\Gamma V\mathcal{I} + \mathrm{div} V\mathcal{I} - {}^*DV)\cdot n \\ &= \langle DV\cdot n, n\rangle_{\mathbb{R}^N} n - {}^*DV\cdot n\ . \end{aligned} \tag{3.37}$$

Lemma 3.4 *We have*

$$\dot{n}(\Gamma; V) = \frac{d}{d\varepsilon} n^\varepsilon_{|\varepsilon=0} = -({}^*DV\cdot n)_\tau\ , \tag{3.38}$$

where $({}^*DV\cdot n)_\tau$ *is the tangential component of the vector field* ${}^*DV\cdot n$ *on* Γ.

Finally we shall derive the form of the second order derivative of the domain functional $J(\Omega_t)$ with respect to the parameter t:

$$\frac{d^2}{dt^2}J(\Omega_t) = -\int_\Gamma [\nabla\dot{y}\cdot\nabla p + \nabla y\cdot\nabla\dot{p} + \langle B'(0)\cdot\nabla p, \nabla y\rangle_{\mathbb{R}^N}]\,\langle V, n\rangle_{\mathbb{R}^N} d\Gamma$$
$$+\int_\Gamma \left[(y-z_g)(\dot{y}-\nabla z_g\cdot V) + \frac{1}{2}(y-z_g)^2 \mathrm{div}_\Gamma V\right]\langle V, n\rangle_{\mathbb{R}^N} d\Gamma \tag{3.39}$$
$$+\int_\Gamma \left[\nabla y\cdot\nabla p + \frac{1}{2}(y-z_g)^2\right]\left(\langle DV\cdot V, n\rangle_{\mathbb{R}^N} - \langle V, ({}^*DV\cdot n)_\tau\rangle_{\mathbb{R}^N}\right) d\Gamma\ ,$$

where $\dot{y} = \dot{y}(\Omega; V)$ and $\dot{p} = \dot{p}(\Omega; V)$ denote the material derivatives of $y(\Omega)$ and $p(\Omega)$, respectively.

In order to the derive necessary optimality conditions for the minimization problems formulated in Sect. 2.6.1 with the domain functional (2.46), we have to determine the shape derivative of the perimeter $\mathcal{P}_D(\Omega_t)$. Assuming that Ω is a domain of class C^k we have

$$\mathcal{P}_D(\Omega_t) = \int_{\Gamma_t} d\Gamma_t\ , \tag{3.40}$$

and

$$\frac{d}{dt}\mathcal{P}_D(\Omega_t) = \int_{\Gamma_t} \kappa_t\langle V, n_t\rangle_{\mathbb{R}^N} d\Gamma_t\ , \tag{3.41}$$

where κ_t is the mean curvature of the manifold Γ_t. Let us recall that

$$\kappa_t = \mathrm{div}_\Gamma n_t \tag{3.42}$$

that is

$$= \mathrm{div}(\mathcal{N}_t) - \langle D\mathcal{N}_t\cdot n_t, n_t\rangle_{\mathbb{R}^N}\ , \tag{3.43}$$

where $\mathcal{N}_t$ is any smooth C^{k-1} extension of the normal field n_t to a neighbourhood of Γ_t. The following identity will be used below.

Lemma 3.5 *For any extension $\mathcal{N}$ of the normal vector field n on Γ we have*

$$\mathrm{div}\mathcal{N}\circ T = \mathrm{Tr}\left[D(\mathcal{N}\circ T)\cdot\ {}^*DT^{-1}\right]\ . \tag{3.44}$$

In a similar way one can show that

$$\mathrm{div}(\mathcal{N}_t)\circ T_t = \mathrm{Tr}\left[D(\mathcal{N}_t\circ T_t)\cdot\ {}^*DT_t^{-1}\right]\ . \tag{3.45}$$

Remark. Making use of (3.43) and (3.45), an explicit form of the material derivative $\dot{\kappa}(\Gamma; V)$ on Γ can be determined.

Necessary optimality conditions. We derive necessary optimality conditions for the shape optimization problem with the cost functional (2.46). Let us suppose that Ω_0 is a smooth domain included in D,

$$\int_{\Omega_0} dx = |\Omega_0|, \tag{3.46}$$

and

$$J(\Omega_0) + \alpha \mathcal{P}_D(\Omega_0) \leqq J(\Omega) + \alpha \mathcal{P}_D(\Omega) \tag{3.47}$$

for all domains Ω in D, where $\alpha > 0$ is given.
Then for all vector fields $V \in V^k(D)$ with

$$\int_\Gamma \langle V, n \rangle_{\mathbb{R}^N} d\Gamma = 0 \ , \tag{3.48}$$

the following necessary optimality conditions hold

$$dJ(\Omega_0; V) + \alpha \int_\Gamma \kappa \langle V, n \rangle_{\mathbb{R}^N} d\Gamma = 0 \ , \tag{3.49}$$

$$\begin{aligned} &\left(\frac{d^2}{dt^2} J(\Omega_t)\right)_{|t=0} + \alpha \int_\Gamma \kappa \left[\, (\mathrm{div}_\Gamma V) \langle V, n \rangle_{\mathbb{R}^N} + \langle DV \cdot V, n \rangle_{\mathbb{R}^N} \right. \\ &\left. - \langle V, ({}^*DV \cdot n) \rangle_{\mathbb{R}^N} \, \right] d\Gamma + \alpha \int_\Gamma \dot{\kappa}(\Gamma; V) \langle V, n \rangle_{\mathbb{R}^N} d\Gamma \geqq 0 \ . \end{aligned} \tag{3.50}$$

3.4. Parabolic equations

In this section the standard notation of (Lions et al. 1968) is used. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the sufficiently smooth boundary $\Gamma = \partial\Omega$.
Let $H \subset H^1(\Omega)$ be a closed subspace such that

$$H_0^1(\Omega) \subset H \subset H^1(\Omega) \ .$$

Moreover the following notation is introduced

$$\begin{aligned} &I = (t_0, t_1) \\ &W(I; H) = \{\phi \in L^2(I; H) \mid \frac{\partial \phi}{\partial t} \in L^2(I; H')\}, \end{aligned} \tag{3.51}$$

where H' denotes the dual space, $t_0 < t_1$ are given.
Furthermore the following Cartesian products are defined

$$Q = \Omega \times I, \quad \Sigma = \Gamma \times I \ .$$

Using these products the space $H^{2,1}(Q)$ is introduced

$$H^{2,1}(Q) = \{\phi \in L^2(Q) \,|\, \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x_i}, \frac{\partial^2 \phi}{\partial x_i \partial x_j} \in L^2(Q) \quad i,j = 1,\dots,N\} \;. \tag{3.52}$$

3.4.1. Neumann boundary conditions

Under the assumption that $H = H^1(\Omega)$ the following parabolic initial–boundary value problem is formulated

$$\frac{\partial y}{\partial t} - \Delta y = f \quad \text{in } Q, \tag{3.53}$$

$$\frac{\partial y}{\partial n} = 0 \quad \text{on } \Sigma, \tag{3.54}$$

$$y(t_0, x) = z(x) \quad \text{in } \Omega, \tag{3.55}$$

where $f \in L^2(Q)$ and $z \in L^2(Q)$ are given elements.
The weak form of (3.53)–(3.55) is as follows

$$y(t) \in H^1(\Omega) : \int_\Omega \Big(\frac{\partial y}{\partial t}(t)\phi + \nabla y(t) \cdot \nabla \phi\Big) dx = \int_\Omega f(t)\phi dx \tag{3.56}$$

for a.e. $t \in I = (t_0, t_1)$ and for all $\phi \in H^1(\Omega)$,

$$y(t_0) = y_0 \quad \text{in } \Omega \;.$$

It can be shown, see e.g. (Lions et al. 1968), that any weak solution $y \in W(I; H)$ to (3.56) has the property

$$y(\cdot) \in C(I; L^2(\Omega)) \;. \tag{3.57}$$

Let us suppose that $f \in L^2(\mathbb{R}^{N+1})$ and $z \in H^1(\mathbb{R}^N)$; under this assumption parabolic problems, defined in the cylinders $Q_s = \Omega_s \times I$ with $\Omega_s = T_s(V)(\Omega)$, are formulated, here $V(\cdot,\cdot) \in C(0,\varepsilon; C^1(\mathbb{R}^N; \mathbb{R}^N))$ is given and $s \in [0,\varepsilon)$ is a parameter. For a given parameter s the parabolic problem has the following form:

Find an element $y_s \in W(I; H^1(\Omega_s))$ such that

$$y_s(t) \in H^1(\Omega_s) : \int_\Omega \Big(\frac{\partial y_s}{\partial t}(t)\phi + \nabla y_s(t) \cdot \nabla \phi\Big) dx = \int_\Omega f_s(t)\phi dx \tag{3.58}$$

for a.e. $t \in I = (t_0, t_1)$ and for all $\phi \in H^1(\Omega_s)$,

$$y_s(t_0) = z_s \quad \text{in } \Omega,$$

where

$$z_s = z|_{\Omega_s}, \quad f_s = f|_{\Omega_s} \ . \tag{3.59}$$

In order to determine the form of the material derivative $\dot{y} = y(\Omega; V)$ of the solution to (3.53)–(3.55) in the direction of a vector field $V(\cdot,\cdot)$, it should be remarked that for the element $y_s \in W(I; H^1(\Omega_s))$ the following integral identity holds

$$\int_{t_0}^{t_1}\int_{\Omega_s} \{-y_s(t,x)\frac{\partial\phi}{\partial t}(t,x) + \nabla y_s(t,x)\cdot\nabla\phi(t,x)\}dxdt = \tag{3.60}$$
$$\int_{t_0}^{t_1}\int_{\Omega_s} f_s(t,x)\phi(t,x)dxdt + \int_{\Omega_s} z_s(x)\phi(t_0,x)dx \quad \forall\phi\in H^{2,1}(Q_s)$$

with $\phi(t_1,\cdot)=0$.

Let us assume that the integral identity (3.60) is transformed to the fixed cylinder $Q = \Omega \times I$ using the change of variables $x = T_s(V)(X)$; this yields the element $y^s \equiv y_s \circ T_s \in W(I; H^1(\Omega))$. It can be shown that the following integral identity is satisfied:

$$\int_{t_0}^{t_1}\int_{\Omega} \{-\gamma(s)(x) y^s(x,t)\frac{\partial\phi}{\partial t}(x,t) + \tag{3.61}$$
$$\langle A(s)(x)\cdot\nabla y^s(x,t), \nabla\phi(x,t)\rangle_{\mathbb{R}^N}\}dxdt =$$
$$\int_{t_0}^{t_1}\int_{\Omega} f^s(x,t)\phi(t,x)dxdt + \int_{\Omega} z^s(x)\phi(t_0,x)dx$$
$$\text{for all } \phi\in H^{1,2}(Q) \text{ with } \phi(t_1,x)=0 \quad \text{in } \Omega,$$

where

$$f^s = \gamma(s) f\circ T_s, \ z^s = \gamma(s) z\circ T_s \ .$$

Lemma 3.6 *The strong material derivative $\dot{y} = y(\Omega; V)$ in $W(I; H^1(\Omega))$ of the solution to (3.53)–(3.55) is given as the unique solution to the following system*

$$\frac{\partial\dot{y}}{\partial t} - \Delta y = -\gamma'(0)\frac{\partial y}{\partial t} + \operatorname{div}(A'(0)\cdot\nabla y) + \operatorname{div}(fV) \quad \textit{in } Q, \tag{3.62}$$

$$\frac{\partial\dot{y}}{\partial t} + \langle A'(0)\cdot\nabla y, n\rangle_{\mathbb{R}^N} = 0 \quad \textit{on } \Sigma, \tag{3.63}$$

$$\dot{y}(t_0) = \operatorname{div}(zV) \quad \textit{in } \Omega \ . \tag{3.64}$$

Proof. Let $s\in[0,\delta)$ be given. From (3.61) it follows that the element $y_s \in W(I; H^1(\Omega))$ satisfies the following system of equations

$$\gamma(s)\frac{\partial y^s}{\partial t} - \operatorname{div}(A(s)\cdot\nabla y^s) = f^s \quad \text{in } Q, \tag{3.65}$$

$$\langle A(s)\cdot\nabla y^s, n\rangle_{\mathbb{R}^N} = 0 \quad \text{on } \Sigma, \tag{3.66}$$

$$y^s(t_0) = z^s \quad \text{in } \Omega \ . \tag{3.67}$$

Multiplying (3.65) by y^s and integrating over $\Omega \times (0,t)$ we obtain

$$\int_{t_0}^{t}\int_{\Omega}\gamma(s)\frac{\partial y^s}{\partial t}y^s dxdt + \int_{t_0}^{t}\int_{\Omega}\langle A(s)\cdot\nabla y^s, \nabla y^s\rangle_{\mathbb{R}^N} dxdt = \int_{t_0}^{t}\int_{\Omega} f^s y^s dxdt \tag{3.68}$$

for all $t \in (t_0, t_1]$. From this identity it can be inferred that

$$\|y^s(t,\cdot)\|_{L^2(\Omega)} + \|y^s\|_{L^2(I;H^1(\Omega))} \leqq C\{\|f^s\|_{L^2(I;(H^1(\Omega))')} + \|z^s\|_{L^2(\Omega)}\} . \tag{3.69}$$

This inequality is due to the fact that for $s > 0$, s small enough, we have

$$\gamma(s) = 1 + s\gamma'(0) + o(s) \quad \text{in } L^\infty(\Omega) . \tag{3.70}$$

From (3.65), (3,69) and (3.70) it follows that

$$\left\|\frac{\partial y}{\partial t}\right\|_{L^2(I;(H^1(\Omega))')} \leqq C\{\|f^s\|_{L^2(I;(H^1(\Omega))')} + \|z^s\|_{L^2(\Omega)}\} . \tag{3.71}$$

Therefore

$$\|y^s\|_{W(I;H^1(\Omega))} + \|y^s(t)\|_{L^2(\Omega))} \leqq C\{\|f^s\|_{L^2(I;(H^1(\Omega))')} + \|z^s\|_{L^2(\Omega))}\} . \tag{3.72}$$

Let us consider the following system of equations

$$\gamma(s)\frac{\partial}{\partial t}(y^s - y) - \operatorname{div}(A(s)\cdot\nabla(y^s - y)) = f^s - f + (1-\gamma(s))\frac{\partial y}{\partial t} + \operatorname{div}(A(s) - \mathcal{I})\cdot\nabla y) \quad \text{in } Q, \tag{3.73}$$

$$y^s - y = z^s - z \quad \text{in } \Omega . \tag{3.74}$$

By the same reasoning as above it follows that

$$\|y^s - y\|_{W(I;H^1(\Omega))} \leqq \{\|f^s - f\|_{L^2(I;(H^1(\Omega))')} + \|1-\gamma(s)\|_{L^\infty(\Omega)} + \|A(s) - \mathcal{I}\|_{L^\infty(\Omega:\mathbb{R}^{N^2})} + \|z^s - z\|_{L^2(\Omega)}\} . \tag{3.75}$$

Let w^s stands for

$$w^s \equiv \frac{1}{s}(y^s - y) - \dot{y}, \tag{3.76}$$

then the element $w^s \in W(I;H^1(\Omega))$ is a weak solution to the following linear parabolic problem

$$\int_{\Omega} \frac{\partial w^s}{\partial t} \phi dx + \int_{\Omega} \nabla w^s \cdot \nabla \phi dx = L_s(\phi) \tag{3.77}$$

for a.e. $t \in I = (t_0, t_1)$ and for all $\phi \in H^1(\Omega)$,

$$w^s(t_0) = \frac{1}{s}(z_s - z) - \operatorname{div}(zV) \quad \text{in } \Omega, \tag{3.78}$$

where

$$L_s(\phi) = \int_{\Omega} \{\frac{1}{s}(1 - \gamma(s))\frac{\partial y^s}{\partial t}\phi + \gamma' \frac{\partial y}{\partial t}\phi + \frac{1}{s}\langle (A(s) - \mathcal{I})\nabla y^s, \nabla \phi \rangle_{\mathbb{R}^N} \tag{3.79}$$
$$- \langle A' \cdot \nabla y, \nabla \phi \rangle_{\mathbb{R}^N} + \frac{1}{s}(f^s - f)\phi - \operatorname{div}(fV)\phi\} dx \ .$$

From (3.79) and (3.75) it follows that

$$\begin{aligned} | L_s(\phi) | \leqq C\{ & \|\frac{1}{s}(1 - \gamma(s)) - \gamma'(0)\|_{L^\infty(\Omega)} + \\ & \|\frac{1}{s}(A(s) - \mathcal{I}) - A'\|_{L^\infty(\Omega;\mathbb{R}^{N^2})} + \\ & \|\frac{1}{s}(f^s - f) - \operatorname{div}(fV)\|_{L^2(I;(H^1(\Omega))')} + \\ & \|y^s - y\|_{W(I;H^1(\Omega))}\} \|\phi\|_{L^2(I;H^1(\Omega))} \to 0 \end{aligned} \tag{3.80}$$

as $s \to 0$.
Furthermore

$$\|w^s(t_0)\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad s \to 0 \tag{3.81}$$

hence

$$\|w^s\|_{W(I;H^1(\Omega))} \leqq C\{\|L_s\|_{L^2(I;(H^1(\Omega))')} + \|w^s(t_0)\|_{L^2(\Omega)}\} \to 0 \tag{3.82}$$

as $s \to 0$, which concludes the proof of Lemma 3.6.

□

Let us consider the shape derivative $y' = y'(\Omega; V)$ of the solution to the system (3.53)–(3.55).
It should be emphasized that the shape derivative exists

$$y' = \dot{y} - \nabla y \cdot V \in W(I; H^1(\Omega)) \tag{3.83}$$

provided that ∇y belongs to $W(I; H^1(\Omega; \mathbb{R}^N))$. To satisfy this requirement some regularity assumptions are to be imposed upon the solution to (3.53)–(3.55). In order to determine the form of y', we make use of the following integral identity

$$\int_{t_0}^{t_1} \int_{\Omega_s} \{-y_s \frac{\partial \phi}{\partial t} + \nabla y_s \cdot \nabla \phi\} dx dt = \tag{3.84}$$
$$\int_{t_0}^{t_1} \int_{\Omega_s} f \phi dx dt + \int_{\Omega_s} z(x)\phi(t_0, x) dx \ .$$

Differentiation with respect to s at $s = 0$ yields

$$\begin{aligned}&\int_{t_0}^{t_1}\int_\Omega\{-y'\frac{\partial\phi}{\partial t}+\nabla y'\cdot\nabla\phi\}dxdt+ \\ &\int_{t_0}^{t_1}\int_\Gamma\{-y\frac{\partial\phi}{\partial t}+\nabla y\cdot\nabla\phi\}\langle V,n\rangle_{\mathbb{R}^N}d\Gamma dt= \\ &\int_{t_0}^{t_1}\int_\Gamma f\phi\langle V,n\rangle_{\mathbb{R}^N}d\Gamma dt+\int_\Gamma z(x)\phi(t_0,x)\langle V(x),n(x)\rangle_{\mathbb{R}^N}d\Gamma\end{aligned} \tag{3.85}$$

$$\text{for all } \phi\in H^{2,1}(Q) \text{ with } \phi(t_1,\cdot)=0 \quad \text{in } \Omega\ .$$

Considering the boundary integrals we have to note first that the Neumann boundary condition $\frac{\partial y}{\partial n} = 0$ on Σ implies $\nabla y = \nabla_\Gamma y$ on Σ, hence

$$\begin{aligned}&\int_\Gamma\nabla y\cdot\nabla\phi\langle V,n\rangle_{\mathbb{R}^N}d\Gamma= \\ &\int_\Gamma\nabla y\cdot\nabla_\Gamma\phi\langle V,n\rangle_{\mathbb{R}^N}d\Gamma= \\ &\int_\Gamma \operatorname{div}_\Gamma(\langle V,n\rangle_{\mathbb{R}^N}\nabla_\Gamma y d\Gamma+\int_\Gamma\kappa\phi y\langle V,n\rangle_{\mathbb{R}^N}d\Gamma\ .\end{aligned} \tag{3.86}$$

The first equality results from the assumption $\frac{\partial\phi}{\partial n} = 0$, and the second one is obtained by integration by parts.
Thus we have the following result

Lemma 3.7 *The shape derivative of the solution to the system (3.53)-(3.55) is determined as the solution to the following parabolic problem*

$$\frac{\partial y'}{\partial t}-\Delta y'=0 \quad \text{in } Q, \tag{3.87}$$

$$\frac{\partial y'}{\partial n}=(-\frac{\partial y}{\partial t}-\kappa y+f)\langle V,n\rangle_{\mathbb{R}^N}+\operatorname{div}_\Gamma(\langle V,n\rangle_{\mathbb{R}^N}) \quad \text{on } \Sigma, \tag{3.88}$$

$$y'(t_0)=0 \quad \text{in } \Omega\ . \tag{3.89}$$

Proof. Integration by parts of (3.85) with $\phi \in \mathcal{D}(Q)$ yields (3.87). The initial condition (3.89) follows from (3.55) because of (3.83). Finally the boundary condition (3.88) is obtained by integrations by parts of (3.85), in view of (3.87) and (3.86). □

3.4.2. Dirichlet boundary conditions

Let $U \in H^{2,1}(\mathbb{R}^{N+1})$, $U = U(t,x)$, be given; we shall use the following notation

$$u = U|_\Sigma \tag{3.90}$$

and suppose that $\frac{\partial U}{\partial n} = 0$ on Σ.
Let us consider the following parabolic problem

$$\frac{\partial w}{\partial t} - \Delta w = f \quad \text{in } Q, \tag{3.91}$$

$$w = u \quad \text{on } \Sigma, \tag{3.92}$$

$$w(t_0, x) = z(x) \quad \text{in } \Omega, \tag{3.93}$$

where $f \in L^2(\mathbb{R}^{N+1})$ and $z \in H^1(\mathbb{R}^N)$ are given elements.

Lemma 3.8 *The strong material derivative $\dot{w} \in W(I; H^1(\Omega))$ of the solution $w \in H^{2,1}(Q)$ to (3.91)-(3.93) is determined as the unique solution to the following parabolic problem*

$$\frac{\partial \dot{w}}{\partial t} - \Delta \dot{w} = -\gamma(0)\frac{\partial w}{\partial t} + \operatorname{div}(A'(0) \cdot \nabla w) + \operatorname{div}(fV) \quad \text{in } Q, \tag{3.94}$$

$$\dot{w} = \nabla_\Gamma u \cdot V_\tau \quad \text{on } \Sigma, \tag{3.95}$$

$$\dot{w}(t_0, x) = \operatorname{div}(z(x)V(0, x)) \quad \text{in } \Omega, \tag{3.96}$$

here V_τ stands for the tangential component of V on Γ,

$$V_\tau = V - \langle V, n \rangle_{\mathbb{R}^N} n \quad \text{on } \Gamma\ . \tag{3.97}$$

Proof. Let $\Omega_s = T_s(V)(\Omega)$, and let w_s be a solution to the parabolic equation (3.91)–(3.93) defined in the cylinder Q_s for a given $s \in [0, \delta)$; if it is assumed that

$$\xi_s \equiv w_s - U|_{Q_s} \tag{3.98}$$

then $\xi_s \in W(I; H_0^1(\Omega_s))$. Furthermore

$$\int_{t_0}^{t_1} \int_{\Omega_s} \{-(\xi_s + U)\frac{\partial \phi}{\partial t} + \nabla(\xi_s + U) \cdot \nabla\phi\} dx dt \tag{3.99}$$

$$= \int_{t_0}^{t_1} \int_{\Omega_s} f\phi dx dt + \int_{\Omega_s} z(x)\gamma(s)\phi(t_0, x) dx$$

for all $\phi \in H^{2,1}(Q)$ with $\phi|_\Sigma = 0$ and $\phi(t_1, x) = 0$ in Ω_s .

Applying the change of variables $x = T_s(V)(X)$ to the integral identity (3.99) we have the following integral identity, $\xi^s = \xi_s \circ T_s \in W(I; H_0^1(\Omega_s))$,

$$\int_{t_0}^{t_1} \int_\Omega \{-(\xi^s + U^s)\frac{\partial \phi}{\partial t}\gamma(s) + \langle A(s) \cdot \nabla(\xi^s + U^s), \nabla\phi\rangle_{\mathbb{R}^N}\} dx dt \tag{3.100}$$

$$= \int_{t_0}^{t_1} \int_\Omega f^s \phi dx dt + \int_\Omega z^s(x)\gamma(s)\phi(t_0, x) dx$$

for all $\phi \in H^{2,1}(Q)$ with $\phi(t_1, x) = 0$ in Ω,

where $U^s = U \circ T_s$.
By the same reasoning as in the proof of Lemma 3.6 it follows that for the strong material derivative $\dot{\xi} \in W(I; H_0^1(\Omega))$ the following integral identity holds

$$\int_{t_0}^{t_1} \int_\Omega \{-(\dot{\xi} + \dot{U})\frac{\partial \phi}{\partial t} + \langle \nabla(\dot{\xi} + \dot{U}), \nabla \phi \rangle_{\mathbb{R}^N}\} dxdt + \tag{3.101}$$
$$\int_{t_0}^{t_1} \int_\Omega \{-\gamma'(0) w \frac{\partial \phi}{\partial t} + \langle A' \cdot \nabla w, \nabla \phi \rangle_{\mathbb{R}^N}\} dxdt =$$
$$\int_{t_0}^{t_1} \int_\Omega \mathrm{div}(fV) \phi dxdt + \int_\Omega \mathrm{div}(zV)\phi(x,0)dx$$

$$\text{for all } \phi \in H^{2,1}(Q) \text{ with } \phi(t_1, x) = 0 \quad \text{in } \Omega,$$

where $\dot{U} = \nabla U \cdot V$ in Q. Moreover

$$\dot{U} = \nabla U \cdot V_\tau + \frac{\partial U}{\partial n}\langle V, n\rangle_{\mathbb{R}^N} = \nabla_\Gamma u \cdot V_\tau \quad \text{on } \Gamma \ . \tag{3.102}$$

We have

$$w^s = \xi^s + U^s \ . \tag{3.103}$$

Hence from the existence of the strong material derivative $\dot{U} \in W(I; H^1(\Omega))$, it can be inferred that the strong material derivative $\dot{w} = \dot{\xi} + \dot{U} \in W(I; H^1(\Omega))$ exists. Furthermore, from (3.101) it follows that (3.94) and (3.96) are met. Finally using (3.102), (3.95) is obtained.
The shape derivative $w' \in W(I; H^1(\Omega))$ of the solution $w \in H^{2,1}(Q)$ to (3.91)–(3.93) exists and is of the form

$$w' = \dot{w} - \nabla w \cdot V \tag{3.104}$$

provided that $\nabla w \in W(I; H^1(\Omega; \mathbb{R}^N))$. Therefore to derive the form of w' we shall use the integral identity (3.99).
First let us note that (3.96) and (3.104) yields

$$w'(t_0, x) = 0 \quad \text{in } \Omega \ . \tag{3.105}$$

Furthermore by means of (3.95) we have

$$w' = -\frac{\partial w}{\partial n}\langle V, n\rangle_{\mathbb{R}^N} \quad \text{on } \Sigma \ . \tag{3.106}$$

Finally let $\phi \in \mathcal{D}(\mathbb{R}^{N+1})$, then $\mathrm{spt}\phi \subset Q_s$ for $s > 0$, s small enough.
Differentiation of (3.99) with respect to s at $s = 0$ results in the following integral identity

$$\int_{t_0}^{t_1}\int_{\Omega}\{-w'\frac{\partial\phi}{\partial t}+\nabla w'\cdot\nabla\phi\}dxdt+ \tag{3.107}$$
$$\int_{t_0}^{t_1}\int_{\Gamma}\{-w\frac{\partial\phi}{\partial t}+\nabla w\cdot\nabla\phi\}\langle V,n\rangle_{\mathbb{R}^N}d\Gamma dt =$$
$$\int_{t_0}^{t_1}\int_{\Gamma}f\phi\langle V,n\rangle_{\mathbb{R}^N}d\Gamma dt \quad \forall\phi\in\mathcal{D}(Q)\ .$$

Thus

$$\frac{\partial w'}{\partial t}-\Delta w'=0 \quad \text{in } Q\ . \tag{3.108}$$

Therefore we have the following result.

Lemma 3.9 *The shape derivative $w'\in W(I;H^1(\Omega))$ for the solution to (3.91)-(3.93) is given as the unique solution to (3.105),(3.106) and (3.108).*

□

3.5. Shape sensitivity in elasticity

This section is concerned with the elliptic boundary value problems in elasticity. The standard notation is used (e.g. (Washizu 1982)):

$$\phi_{,i}=\partial\phi/\partial x_i, \quad i=1,2,3,$$
$$\nabla\phi=\mathrm{col}(\phi_{,1},\phi_{,2},\phi_{,3})\ .$$

It is assumed that $\phi\in H^1(\Omega)$. If $\Gamma=\partial\Omega$ is a smooth manifold, then

$$\nabla\phi=\nabla_\Gamma\phi+\frac{\partial\phi}{\partial n}n,$$

*material derivative

where

$$\frac{\partial\phi}{\partial n}=\nabla\phi\cdot n=\phi_{,i}n_i=\Sigma_{i=1}^3 n_i\frac{\partial\phi}{\partial x_i}\ ,$$

and

$\nabla_\Gamma\phi$ is a tangent vector field on Γ.

In these equations we have made use of the summation convention over repeated indices $i,j,k,l=1,2,3$. For any vector function $\phi\in H^1(\mathbb{R}^3,\mathbb{R}^3)$ the following notation is used

$$(D\phi)_{ij}=\phi_{i,j}\ ,\quad ({}^*D\phi)_{ij}=\phi_{j,i}\quad i,j=1,2,3\ .$$

The linearized strain tensor $\epsilon(\phi)$ is of the form

$$\epsilon(\phi)=\frac{1}{2}(D\phi+{}^*D\phi),\quad \epsilon_{ij}=\frac{1}{2}(\phi_{i,j}+\phi_{j,i})\ .$$

Let us consider the fourth order tensor function $\mathcal{C}(\cdot) = \{c_{ijkl}(\cdot)\}$, $i,j,k,l = 1,2,3$, such that the following symmetry conditions are satisfied

$$c_{ijkl}(x) = c_{jikl}(x) = c_{klij}(x) \quad x \in \mathbb{R}^3, \qquad i,j,k,l = 1,2,3. \tag{3.109}$$

Furthermore $c_{ijkl}(\cdot) \in L^\infty_{loc}(\mathbb{R}^N)$, $i,j,k,l = 1,2,3$, and there exists $\alpha_0 > 0$ such that

$$\xi : \mathcal{C} : \xi = c_{ijkl}(x)\xi_{ij}\xi_{kl} \geqq \alpha_0 \xi_{ij}\xi_{kl} = \alpha_0 \xi : \xi \tag{3.110}$$

for all $x \in \mathbb{R}^3$ and for all second order symmetric tensors ξ. The stress tensor $\sigma = \sigma(\phi)$ is defined by

$$\sigma = \mathcal{C} : \epsilon(\phi), \tag{3.111}$$

$$\text{i.e.} \quad \sigma_{ij} = c_{ijkl}\epsilon_{kl} = c_{ijkl}\phi_{i,j} \quad i,j,k,l = 1,2,3.$$

The normal component σ_n of the stress tensor on the boundary Γ,

$$\sigma_n = n \cdot \sigma \cdot n = \sigma_{ij}n_i n_j \tag{3.112}$$

and the tangential component σ_τ,

$$\sigma_\tau = \sigma \cdot n - \sigma_n n \tag{3.113}$$

are well defined, e.g. for $\phi \in H^2(\mathbb{R}^3; \mathbb{R}^3)$. Let us assume that

$$a(z,\phi) = \int_\Omega \epsilon(z) : \mathcal{C} : \epsilon(\phi) dx \quad \forall z, \phi \in H^1(\Omega; \mathbb{R}^3), \tag{3.114}$$

$$\langle F, \phi \rangle = \int_\Omega f \cdot \phi dx + \int_{\Gamma_2} P \cdot \phi d\Gamma, \tag{3.115}$$

here $\Omega \subset \mathbb{R}^3$ is a given domain with the smooth boundary $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, meas$\Gamma_0 > 0$; $f \in L^2(\Omega; \mathbb{R}^3)$ and $P \in H^1(\Gamma_2; \mathbb{R}^3)$ are given elements.

Under these assumptions there exists a weak solution to the variational equation

$$u \in H : \quad a(u,\phi) = \langle F, \phi \rangle \quad \forall \phi \in H, \tag{3.116}$$

where

$$H = \{\phi \in H^1(\Omega; \mathbb{R}^3) | \phi = 0 \quad \text{on } \Gamma_0 \ , \quad \phi_n = \phi \cdot n = 0 \quad \text{on } \Gamma_2\} \ . \tag{3.117}$$

It can be shown (Fichera 1972) that for the weak solution u, the following system of equations is satisfied

$$-\mathrm{div}\sigma = f \quad \text{in } \Omega, \tag{3.118}$$

$$u = 0 \quad \text{on } \Gamma_0 \ , \tag{3.119}$$

$$\sigma \cdot n = P \quad \text{on } \Gamma_1 \ , \tag{3.120}$$

$$u \cdot n = 0 \quad \text{on } \Gamma_2 \ , \tag{3.121}$$

$$\sigma_\tau = 0 \quad \text{on } \Gamma_2 \ , \tag{3.122}$$

in the weak sense.
It has been known (Fichera 1972) that for meas$\Gamma_0 > 0$ the bilinear form $a(\cdot,\cdot)$ is coercive, i.e. there exists a constant $\alpha > 0$ such that

$$\int_\Omega \sigma(\phi) : C : \sigma(\phi) dx \geqq \alpha \|\phi\|^2_{H^1(\Omega;\mathbb{R}^3)} \quad \forall \phi \in H \ . \tag{3.123}$$

This implies the existence and uniqueness of the weak solution to (3.118)–(3.122) defined by (3.116). The equivalent form of (3.118)–(3.122) expressed in terms of the displacement u is for i=1,2,3, as follows

$$-(c_{ijkl}(x)u_{k,l}(x))_{,j} = f_i(x) \quad \text{in } \Omega, \tag{3.124}$$

$$u_i = 0 \quad \text{on } \Gamma_0 \ , \tag{3.125}$$

$$c_{ijkl}(x)u_{k,l}(x)n_i(x)n_j(x) = P_i(x) \quad \text{on } \Gamma_1 \ , \tag{3.126}$$

$$u_i(x)n_i(x) = 0 \quad \text{on } \Gamma_2 \ , \tag{3.127}$$

$$c_{ijkl}(x)u_{k,l}n_j(x) = \sigma_n(x)n_i(x) \quad \text{on } \Gamma_2 \ . \tag{3.128}$$

Frictionless contact problems and contact problems with given friction for linear elastic solids are formulated in Sects. 4.6 and 4.7, respectively. The contact problems take the form of variational inequalities.

Let us examine the linear model (3.124)–(3.128). The principal aim of our consideration is the shape sensitivity analysis of the system (3.124)–(3.128), therefore it is assumed that data are smooth enough, e.g.

$$P(\cdot), f(\cdot) \in C^1(\mathbb{R}^3;\mathbb{R}^3), \tag{3.129}$$

$$c_{ijkl}(\cdot) \in C^1(\mathbb{R}^3) \quad i,j,k,l = 1,2,3 \ . \tag{3.130}$$

The system (3.124)–(3.128) is to be defined in the domain $\Omega_t \in \mathbb{R}^3$, with the boundary $\Gamma_t = \overline{\Gamma}_0^t \cup \overline{\Gamma}_1^t \cup \overline{\Gamma}_2^t$ for $t \in [0,\delta)$. For this purpose the following notation is introduced

$$a_t(z,\phi) = \int_{\Omega_t} Dz : C : D\phi dx \quad \forall z, \phi \in H_t \ , \tag{3.131}$$

$$H_t = \{\phi \in H^1(\Omega_t;\mathbb{R}^3) | \phi = 0 \quad \text{on } \Gamma_0^t \ , \quad \phi \cdot n_t = 0 \quad \text{on } \Gamma_2^t\}, \tag{3.132}$$

$$\langle F_t, \phi \rangle_t = \int_{\Omega_t} f \cdot \phi dx + \int_{\Gamma_1^t} P \cdot \phi d\Gamma \quad \forall \phi \in H_t \ . \tag{3.133}$$

The following variational equation holds

$$u_t \in H_t : \quad a_t(u_t,\phi) = \langle F_t, \phi \rangle_t \quad \forall \phi \in H_t \tag{3.134}$$

for a weak solution to (3.124)–(3.128).
For the transported solution

$$u^t = u_t \circ T_t \tag{3.135}$$

the following equation is satisfied

$$u^t \in H: \quad a^t(u^t,\phi) = \langle F^t,\phi\rangle \quad \forall\phi \in H, \tag{3.136}$$

where

$$a^t(z,\phi) = a_t(z\circ T_t^{-1},\phi\circ T_t^{-1})$$
$$= \int_\Omega \gamma(t)\epsilon^t(z):C^t:\epsilon^t(\phi)dx \qquad \forall z,\phi\in H, \tag{3.137}$$
$$\langle F^t,\phi\rangle = \langle F_t,\phi\circ T_t^{-1}\rangle_t$$
$$= \int_\Omega f^t\cdot\phi dx + \int_{\Gamma_1} P^t\cdot\phi d\Gamma \quad \forall\phi\in H\ . \tag{3.138}$$

In these equations the following notation has been used

$$\epsilon^t(\phi) = \frac{1}{2}\{D\phi\cdot DT_t^{-1} + {}^*(DT_t)^{-1}\cdot{}^*D\phi\}, \tag{3.139}$$
$$f^t = \gamma(t)f\circ T_t, \quad P^t = \omega(t)P\circ T_t\ .$$

It can be shown that the derivatives $a'(\cdot,\cdot),\langle F',.\rangle$ of the bilinear form $a^t(\cdot,\cdot)$ and the linear form $\langle F^t,.\rangle$ with respect to t at $t=0$ are given, respectively by

$$a'(z,\phi) = \int_\Omega \{\epsilon'(z):\sigma(\phi) + \epsilon'(\phi):\sigma(z) + \epsilon(z):C':\epsilon(\phi)\}dx \tag{3.140}$$
$$\forall z,\phi\in H,$$

$$\langle F',\phi\rangle = \int_\Omega \{\mathrm{div}Vf\cdot\phi + \phi\cdot Df\cdot V\}dx + \tag{3.141}$$
$$\int_{\Gamma_1}\{\mathrm{div}_\Gamma VP\cdot\phi + \phi_i\nabla_\Gamma P_i\cdot V_\tau\}d\Gamma \quad \forall\phi\in H^1(\Omega;\mathbb{R}^3),$$

where

$$\epsilon'(\phi) = -\frac{1}{2}\{D\phi\cdot DV + {}^*DV\cdot{}^*D\phi\}, \tag{3.142}$$
$$C' = \{c'_{ijkl}\}, \quad c'_{ijkl} = \mathrm{div}Vc_{ijkl} + \nabla c_{ijkl}\cdot V\ . \tag{3.143}$$

Theorem 3.10 *The following variational equation*

$$\dot u\in H^1(\Omega;\mathbb{R}^3)\ :\quad a(\dot u,\phi) = \langle F',\phi\rangle - a'(u,\phi) \quad \forall\phi\in H \tag{3.144}$$
$$\textit{with } \dot u = 0 \quad \textit{on } \Gamma_0\ , \quad \dot u\cdot n = n\cdot DV\cdot u_\tau \quad \textit{on } \Gamma_2 \tag{3.145}$$

holds for the strong material derivative $\dot u\in H^1(\Omega;\mathbb{R}^3)$ of the solution $u(\Omega)$ to (3.124)-(3.128) in the direction of a vector field $V(\cdot,\cdot)$.

Proof. From our assumptions it follows that the mappings

$$[0,\delta) \ni t \to f \circ T_t \in L^2(\Omega;\mathbb{R}^3), \tag{3.146}$$

$$[0,\delta) \ni t \to P \circ T_t \in L^2(\Gamma_1;\mathbb{R}^3) \tag{3.147}$$

are strongly differentiable.

Moreover

$$(Df \cdot V)_i = \nabla f_i \cdot V,$$

$$(D_\Gamma P \cdot V_\tau)_i = \nabla_\Gamma P_i \cdot V_\tau \ , \quad V_\tau = V - n\langle V, n\rangle_{\mathbb{R}^3} \ .$$

Therefore we can differentiate the integral identity (3.136) with respect to t at $t = 0$. This yields the integral identity (3.145). Since $u^t = 0$ on Γ_0, then $\dot{u} = 0$ on Γ_0. On the other hand, $u_t \cdot n_t = 0$ on Γ_2^t, thus

$$u^t \cdot n^t = 0 \quad \text{on } \Gamma_2 \ ,$$

where

$$n^t = (n_t \circ T_t) = \|{}^*DT_t^{-1} \cdot n\|_{\mathbb{R}^3}^{-1} {}^*DT_t^{-1} \cdot n \ .$$

Hence

$$\frac{d}{dt}(u^t \cdot n^t)_{|t=0} = 0 \quad \text{on } \Gamma_2 \ , \tag{3.148}$$

or equivalently

$$\dot{u} \cdot n - u \cdot {}^*DV \cdot n = 0 \quad \text{on } \Gamma_2 \ . \tag{3.149}$$

Taking into account (3.127), one can show that (3.144) holds. □

Finally the form of the shape derivative $u' = u'(\Omega; V)$ of the solution $u(\Omega)$ to (3.124)–(3.128) is derived. It is assumed that the following regularity assumption is satisfied

$$Du \cdot V \in H^1(\Omega;\mathbb{R}^3) \ . \tag{3.150}$$

Therefore the shape derivative exists, is defined by

$$u' = \dot{u} - Du \cdot V \in H^1(\Omega;\mathbb{R}^3) \tag{3.151}$$

and satisfies the following variational equation

$$a(u', \phi) = \langle F', \phi\rangle - a'(u,\phi) - a(Du \cdot V, \phi) \ . \tag{3.152}$$

Let us suppose that $V = 0$ on $\overline{\Gamma}_0 \cap \overline{\Gamma}_1, \overline{\Gamma}_1 \cap \overline{\Gamma}_2, \overline{\Gamma}_0 \cap \overline{\Gamma}_2, \langle V, n\rangle_{\mathbb{R}^3} = 0$ on Γ, then $\Omega = \Omega_t$. Thus $u' = 0$ and

$$\langle f', \phi\rangle - a'(u,\phi) - a(Du \cdot V, \phi) = 0 \ . \tag{3.153}$$

Let us recall, that for $u_t \in H^1(\Omega_t;\mathbb{R}^3)$ the following integral identity holds

$$\int_\Omega \sigma_t \colon \epsilon(\phi) dx = \int_{\Omega_t} f \cdot \phi dx + \int_{\Gamma_1^t} P \cdot \phi d\Gamma \ . \tag{3.154}$$

Differentiation of (3.154) with respect to t at $t = 0$ yields

$$\int_\Omega \sigma' : \epsilon(\phi)dx + \int_\Gamma v_n \sigma : \epsilon(\phi)d\Gamma = \int_\Gamma v_n f \cdot \phi d\Gamma + \int_{\Gamma_1} v_n \kappa P \cdot \phi d\Gamma \tag{3.155}$$

for all $\phi \in H^2(\Omega; \mathbb{R}^3)$.
Let us assume that $\partial\phi/\partial n = 0$ on Γ, and let $\phi_\tau = \phi - \phi \cdot n$, then

$$\begin{aligned} &\int_\Gamma v_n \sigma : \epsilon(\phi)d\Gamma = \int_\Gamma v_n \sigma : D\phi d\Gamma = \int_\Gamma v_n \sigma : D_\Gamma \phi d\Gamma = \\ &\int_\Gamma \{-\mathrm{div}_\Gamma(v_n\sigma)\phi + v_n \kappa n \cdot \sigma \cdot \phi\} d\Gamma = \\ &\int_\Gamma \{-\phi_n \mathrm{div}_\Gamma(v_n\sigma)n - \phi_\tau \mathrm{div}_\Gamma(v_n\sigma) + v_n\kappa n \cdot \sigma \cdot \phi\} d\Gamma \ . \end{aligned} \tag{3.156}$$

Hence

$$\begin{aligned} &\int_\Omega \sigma' : \epsilon(\phi)dx = -\int_\Omega \mathrm{div}\sigma' \cdot \phi + \int_\Gamma n \cdot \sigma' \cdot \phi d\Gamma = \\ &\int_\Gamma \{-\phi \cdot \mathrm{div}_\Gamma(v_n\sigma) + v_n\kappa n \cdot \sigma \cdot \phi + v_n f \cdot \phi + v_n \kappa \cdot P \cdot \phi\} d\Gamma = \end{aligned} \tag{3.157}$$

In this equation the use has been made of the conditions: $\phi = 0$ on Γ_0 and $\phi \cdot n = 0$ on Γ_2. Hence $\phi = \phi_\tau$ on Γ_2, $\sigma \cdot n = P$ on Γ_1 and $\sigma \cdot n = \sigma_n n$ on Γ_2 because of $\sigma_\tau = 0$ on Γ_2 . Thus using (2.125) we have

$$= \int_{\Gamma_1} \phi \cdot [v_n f + v_n \kappa P - \mathrm{div}_\Gamma(v_n \sigma_\tau)] d\Gamma + \int_\Gamma v_n \phi_\tau f_\tau d\Gamma \ . \tag{3.158}$$

□

Theorem 3.11 *If for the vector field $V(\cdot,\cdot)$ the following conditions are satisfied*

$$V = 0 \quad on\ \overline{\Gamma}_0 \cap \overline{\Gamma}_1\ , \quad \overline{\Gamma}_1 \cap \overline{\Gamma}_2\ , \quad \overline{\Gamma}_0 \cap \overline{\Gamma}_2\ , \tag{3.159}$$

then the shape derivative $u' = u'(\Omega; V) \in H^1(\Omega; \mathbb{R}^3)$ of the solution $u(\Omega)$ to (3.124)-(3.128) satisfies the following system of equations (representing a boundary value problem)

$$\mathrm{div}\sigma' = 0 \quad in\ \Omega, \tag{3.160}$$

$$u' = -v_n \frac{\partial u}{\partial n} \quad on\ \Gamma_0\ , \quad v_n = \langle V(0), n\rangle_{\mathbb{R}^3}, \tag{3.161}$$

$$\sigma' \cdot n = v_n f + v_n \kappa P - \mathrm{div}_\Gamma(v_n \sigma_\tau) \quad on\ \Gamma_1\ , \tag{3.162}$$

$$u' \cdot n = u_\tau \cdot {}^*DV \cdot n \quad on\ \Gamma_2\ , \tag{3.163}$$

$$\sigma_\tau = v_n f_\tau \quad on\ \Gamma_2\ . \tag{3.164}$$

Proof. The equation (3.160) follows from (3.157) for the test functions $\phi \in \mathcal{D}(\Omega; \mathbb{R}^3)$. We derive the boundary conditions (3.161)–(3.164). We have

$$u' = \dot{u} - Du \cdot V = -v_n \frac{\partial u}{\partial n} - V_\tau \cdot D_\Gamma u = -v_n \frac{\partial u}{\partial n} \quad \text{on } \Gamma_0 \ . \tag{3.165}$$

Hence (3.161) is obtained. The condition (3.162) follows from (3.157). In order to derive the condition (3.163), the following equation

$$u_t \cdot n_t = 0 \quad \text{on } \Gamma_2^t = T_t(\Gamma_2) \tag{3.166}$$

is to be used; from (3.166) it follows that

$$u' \cdot n = -u \cdot n' \quad \text{on } \Gamma_2 \ . \tag{3.167}$$

On the other hand

$$n' = \dot{n} - \nabla_\Gamma n \cdot V_\tau = -{}^*DV \cdot n \ . \tag{3.168}$$

Since $V_\tau = 0$, then

$$u' \cdot n = u \cdot {}^*DV \cdot n \ . \tag{3.169}$$

Taking into account that (3.166) holds on Γ_2 we get (3.163). Finally (3.164) follows from (3.157), which concludes the proof. □

3.6. Shape sensitivity analysis of the smallest eigenvalue

Let us consider the following eigenvalue problem :

$$\text{Find} \quad (\lambda, z) \in \mathbb{R}^+ \times H \quad \text{such that}$$

$$a(z, \phi) = \lambda\rho \int_\Omega h(x) z(x) \phi(x) dx \quad \forall \phi \in H(\Omega) \ . \tag{3.170}$$

In this formulation we use the notation of the previous section; it is assumed that $\rho > 0$ is a given number, and $h(.) \in C(\mathbb{R}^2)$ is a given function such that $0 < h_{min} \leqq h(x) \leqq h_{max}$ for all $x \in \mathbb{R}^3$.

Let $\lambda(\Omega)$ be the smallest eigenvalue and $M(\Omega) \subset H(\Omega)$ the set of eigenfunctions corresponding to $\lambda(\Omega)$. It is assumed that for the elements of the set $M(\Omega)$ the following identity holds

$$\rho \int_\Omega h(x) \phi(x) \phi(x) dx = 1 \quad \forall \phi \in M(\Omega) \ . \tag{3.171}$$

It is well known that the smallest eigenvalue can be determined as follows

$$\lambda(\Omega) = \min\{a(\phi,\phi)/(\rho\int_\Omega h|\phi|^2dx) | \phi \in H, \phi \neq 0\} \tag{3.172}$$
$$= \min\{a(\phi,\phi) | \rho\int_\Omega h|\phi|^2dx = 1,\ \phi \in H\}\ .$$

Since $a(\phi,\phi) \geqq \alpha\|\phi\|_{H^1(\Omega;\mathbb{R}^3)}$ for all $\phi \in H$, then from (3.172) it follows that $\lambda(\Omega) > 0$. Let the family $\{\Omega_t\} \subset \mathbb{R}^3$, $t \in [0,\delta)$, be given and let λ_t be defined as follows

$$\lambda_t = \lambda(\Omega_t) = \min\{a_t(\phi,\phi) | \phi \in H_t |\ \rho\int_{\Omega_t} h|\phi|^2dx = 1\}\ . \tag{3.173}$$

We shall derive the form of the directional derivative

$$\dot{\lambda} = d\lambda(\Omega;V) = \lim_{t\downarrow 0}(\lambda(\Omega_t) - \lambda(\Omega))/t\ . \tag{3.174}$$

Since the bilinear form $a_t(.,.)$ is coercive and continuous uniformly with respect to $t \in [0,\delta)$ we can assume that there exists a constant C such that $\|\phi_t\|_{H^1(\Omega_t;\mathbb{R}^3)} \leqq C$ for all $t \in [0,\delta)$.

Lemma 3.7 *We have*

$$\dot{\lambda} = \inf\{a'(\phi,\phi) - \rho\int_\Omega \mathrm{div}(hV)|\phi|^2dx | \phi \in M(\Omega)\}\ . \tag{3.175}$$

Proof.
Since

$$\lambda_t = \inf\{a_t(\phi,\phi)/(\rho\int_{\Omega_t} h|\phi|^2dx | \phi \in H_t,\ \phi \neq 0\}, \tag{3.176}$$

then the transformation of the integrals defined on Ω_t to the fixed domain Ω yields

$$\lambda_t = \inf\{F(t,\phi) | \|\phi\|_{H^1(\Omega;\mathbb{R}^3)} \leqq C\}, \tag{3.177}$$

where

$$F(t,\phi) = a^t(\phi,\phi)/(\rho\int_\Omega \gamma(t)h^t|\phi|^2dx)\ .$$

Let us assume that

$$M_t = \{\phi \in H | \lambda_t = F(t,\phi)\}, \tag{3.178}$$

then

$$\phi \in M_t \quad \text{if and only if} \quad \phi \circ T_t^{-1} \in M(\Omega_t) \quad t \in [0,\delta), \tag{3.179}$$

and by (3.177) it follows that

$$\|\phi\|_{H^1(\Omega;\mathbb{R}^3)} \leqq C \quad \text{for all } \phi \in M_t \text{ and for all } t \in [0,\delta)\ . \tag{3.180}$$

Let $\{t_k\}_{k=1}^{\infty}$ be a given sequence such that $t_k \to 0$ as $k \to \infty$. For any sequence $\{z_k\} \subset M_{t_k} \subset H$, there exists a subsequence also denoted by $\{z_k\}$ such that

$$z_k \rightharpoonup z^\star \quad \text{weakly in} \quad H^1(\Omega; \mathbb{R}^3), \text{ where } z^\star \in M_0 = M(\Omega) \ . \tag{3.181}$$

$F(., \phi)$ is right-differentiable at t=0 for any fixed element $\phi \in H^1(\Omega; \mathbb{R}^3)$,

$$\begin{aligned}\frac{\partial F}{\partial t}(0, \phi) &= \lim_{s \downarrow 0}(F(s, \phi) - F(0, \phi))/s \\ &= a'(\phi, \phi) - \rho\lambda \int_\Omega \operatorname{div}(hV)|\phi|^2 dx \ .\end{aligned} \tag{3.182}$$

The right-derivative of λ_t at $t = 0$ is determined in two steps.
Step 1 :

$$\begin{aligned}\lambda_t - \lambda_0 =& F(t, \phi_t) - F(0, \phi) \\ \leqq& F(t, \phi) - F(0, \phi) \quad \forall \phi \in M(\Omega),\end{aligned} \tag{3.183}$$

thus

$$\begin{aligned}\limsup_{t \downarrow 0}(\lambda_t - \lambda_0)/t \leqq& \lim_{t \downarrow 0}(F(t, \phi) - F(0, \phi))/t \\ =& \frac{\partial F}{\partial t}(0, \phi) \quad \forall \phi \in M(\Omega) \ .\end{aligned} \tag{3.184}$$

Step 2 :

$$\begin{aligned}\lambda_t - \lambda_0 = F(t, \phi_t) - F(0, \phi) \geqq F(t, \phi_t) - F(0, \phi_t) \\ \forall \phi_t \in M_t \text{ and } \quad \forall \phi \in M(\Omega) \ .\end{aligned} \tag{3.185}$$

There exists s, $0 \leqq s \leqq t$,

$$\frac{F(t, \phi_t) - F(0, \phi_t)}{t} = \frac{\partial F}{\partial t}(s, \phi_t) \ . \tag{3.186}$$

For $t \downarrow 0$ we have that $s \downarrow 0$ and there exists an element $\phi^\star \in M(\Omega)$ such that

$$\phi_t \rightharpoonup \phi^\star \quad \text{weakly in} \quad H^1(\Omega; \mathbb{R}^3) \ . \tag{3.187}$$

Hence

$$\liminf_{t \downarrow 0} \frac{\partial F}{\partial t}(s, \phi_t) \geqq \frac{\partial F}{\partial t}(0, \phi^\star) \tag{3.188}$$

and

$$\liminf_{t \downarrow 0}(\lambda_t - \lambda_0)/t \geqq \frac{\partial F}{\partial t}(0, \phi^\star) \ . \tag{3.189}$$

Finally from Step 1,

$$\begin{aligned}\frac{\partial F}{\partial t}(0, \phi^\star) \leqq& \liminf_{t \to 0}\Big(\frac{\lambda_t - \lambda_0}{t}\Big) \leqq \limsup_{t \to 0}\Big(\frac{\lambda_t - \lambda_0}{t}\Big) \\ \leqq& \frac{\partial F}{\partial t}(0, \phi) \quad \phi^\star \in M(\Omega) \text{ and } \forall \phi \in M(\Omega) \ .\end{aligned} \tag{3.190}$$

Therefore

$$\dot{\lambda} = \lim_{t \to 0}(\lambda_t - \lambda_0)/t = \inf\{\frac{\partial F}{\partial t}(0, \phi) | \phi \in M(\Omega)\} \tag{3.191}$$

which concludes the proof of Lemma 3.12. □

Let us observe that from (3.191) it follows that

$$\dot{\lambda} = d\lambda(\Omega; V) = \min\left\{\frac{\partial F}{\partial t}(0, \phi) | \phi \in \overline{\text{co}} M(\Omega)\right\} \tag{3.192}$$

(see Sect. 2.12 in Chap. 2 for details), where

$$H \supset \overline{\text{co}} M(\Omega) = \text{weak closure of } \text{co}(M(\Omega)) \ . \tag{3.193}$$

In (3.193) the following notation is used

$$\text{co}(M(\Omega)) = \{\phi \in H | \phi = \alpha_i \phi_i, \ \alpha_i \geqq 0, \ \Sigma_i \alpha_i = 1, \ \phi_i \in M(\Omega)\} \ . \tag{3.194}$$

Since the set $M(\Omega)$ is bounded, then the weak closure of $\text{co}(M(\Omega))$ can be determined in the following way

$$\phi \in \overline{\text{co}} M(\Omega) \quad \text{if and only if} \quad \exists \{\phi_k\} \subset \text{co}(M(\Omega)) \quad \text{such that}$$

$$\phi_k \rightharpoonup \phi \quad \text{weakly in} \quad H^1(\Omega; \mathbb{R}^3) \text{ as } k \to \infty \ .$$

It may be useful to characterize the subdifferential of λ as

$$-\lambda(\Omega) = \max\{a(\phi, \phi) | \phi \in H \text{ with } \rho \int_\Omega h |\phi|^2 dx = 1\} \ . \tag{3.195}$$

The form of this subdifferential can be used to derive the necessary optimality conditions for related shape optimization problems.

From (3.183),(3.192), and Propositions 2.36 and 2.38 in Sect. 2.14 it follows that

$$\dot{\lambda} = \min\{\langle G(\phi), V(0)\rangle_{\mathcal{D}'(\mathbb{R}^N) \times \mathcal{D}(\mathbb{R}^N)} | \phi \in \overline{\text{co}} M(\Omega)\}, \tag{3.196}$$

where for any $\phi \in H$ the distribution $G(\phi) \in \mathcal{D}'(\mathbb{R}^N)$ which is supported on Γ, $\text{spt} G(\phi) \subset \Gamma = \partial\Omega$, is defined by the identity

$$\begin{aligned} \langle G(\phi), V(0)\rangle_{\mathcal{D}'(\mathbb{R}^N) \times \mathcal{D}(\mathbb{R}^N)} &= a'(\phi, \phi) - \\ &\rho\lambda \int_\Omega \text{div}(hV(0)) |\phi|^2 dx \quad \forall \phi \in H \ . \end{aligned} \tag{3.197}$$

From (3.196) using the classical result of non–smooth analysis, see e.g. (Ekeland et al. 1976) we obtain:

Lemma 3.13 *The subdifferential of $\lambda(\Omega)$ has the form*

$$\partial(\lambda)(\Omega) = \{-G(\phi) \in \mathcal{D}'(\mathbb{R}^N) | \phi \in \overline{\text{co}} M(\Omega)\} \ . \tag{3.198}$$

For ϕ sufficiently smooth the form of the distribution $G(\phi)$ can be identified as follows.

Let us suppose that $V = 0$ on $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ and $\bar{\Gamma}_0 \cap \bar{\Gamma}_2$, furthermore let us assume that $V_\tau = 0$ on $\partial\Omega$. We have

$$F(t,\phi) = a_t(\phi,\phi)/(\rho \int_{\Omega_t} h|\phi|^2 dx), \tag{3.199}$$

where the bilinear form $a_t(.,.)$ is defined by (3.114).

Let ϕ be a sufficiently smooth function defined in $\mathbb{R}^3$, $\phi = 0$ on Γ_0^t, $\phi \cdot n_t = 0$ on Γ_2^t. Then

$$\frac{\partial F}{\partial t}(0,\phi) = \int_\Gamma v\sigma(\phi) \colon \epsilon(\phi) d\Gamma - \lambda \int_\Gamma v_n \rho h |\phi|^2 d\Gamma \ . \tag{3.200}$$

For the first term on the right–hand side of this equation we have

$$\int_\Gamma v_n \sigma \colon \epsilon d\Gamma = \int_\Gamma v_n \sigma \colon D\phi d\Gamma = \int_\Gamma v_n n \cdot \sigma \cdot \frac{\partial \phi}{\partial n} d\Gamma + \int_\Gamma v_n \sigma \colon D_\Gamma \phi d\Gamma = \tag{3.201}$$

Integration by parts on Γ yields

$$= \int_\Gamma v_n n \cdot \sigma \cdot \frac{\partial \phi}{\partial n} d\Gamma - \int_\Gamma \operatorname{div}_\Gamma(v_n \sigma) \cdot \phi d\Gamma + \int_\Gamma v_n \kappa n \cdot \sigma \cdot \phi d\Gamma = \tag{3.202}$$

Taking into account (2.125) and the boundary conditions: $\phi = 0$ on Γ_0, $\sigma \cdot n = 0$ on Γ_1, $\phi \cdot n = 0$, $\sigma_\tau = 0$ on Γ_2, we get

$$\begin{aligned} &= \int_{\Gamma_0} v_n n \cdot \sigma \cdot \frac{\partial \phi}{\partial n} d\Gamma - \int_{\Gamma_1} \operatorname{div}_\Gamma(v_n \sigma_\tau) \cdot \phi d\Gamma + \\ &\int_{\Gamma_2} \{ v_n \sigma_n n \cdot \frac{\partial \phi}{\partial n} - v_n \kappa n \cdot \sigma \cdot \phi_\tau \} d\Gamma \ . \end{aligned} \tag{3.203}$$

Therefore the following representation of $G(\phi) \in \mathcal{D}'(\mathbb{R}^N)$ can be obtained for ϕ sufficiently regular:

$$\begin{aligned} &\langle G(\phi), V(0) \rangle_{\mathcal{D}'(\mathbb{R}^N) \times \mathcal{D}(\mathbb{R}^N)} = \langle g(\phi), v_n \rangle_{\mathcal{D}'(\Gamma) \times \mathcal{D}(\Gamma)} = \\ &= \int_{\Gamma_0} v_n n \cdot \sigma \cdot \frac{\partial \phi}{\partial n} d\Gamma - \int_{\Gamma_1} \{ \operatorname{div}_\Gamma(v_n \sigma_\tau) + \lambda v_n \rho h \phi \} \cdot \phi d\Gamma \\ &+ \int_{\Gamma_2} \{ v_n \sigma_n n \cdot \frac{\partial \phi}{\partial n} - v_n \kappa n \cdot \sigma \cdot \phi_\tau - \lambda v_n \rho h |\phi_\tau|^2 \} d\Gamma \ . \end{aligned} \tag{3.204}$$

Remark. If $\partial\Omega = \Gamma_0$, then $\phi = 0$ on Γ_0 and

$$\dot{\lambda} = \max \{ \int_\Gamma v_n n \cdot \sigma \cdot \frac{\partial \phi}{\partial n} d\Gamma | \phi \in \overline{\text{co}} M(\Omega) \} \ . \tag{3.205}$$

3.7. Shape sensitivity analysis of the Kirchhoff plate

In this section the linear model of thin, solid elastic plate is considered. The static response of the plate $w = w(\Omega)(x)$, $x \in \Omega \subset \mathbb{R}^2$, can be determined using the Kirchhoff plate equation

$$(b_{ijkl}(x)w_{,kl}(x))_{,ij} = f(x) \quad \text{in } \Omega \tag{3.206}$$

with the following boundary conditions given on $\Gamma = \partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$

$$w = 0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_0 \ , \tag{3.207}$$

$$w = 0, \quad M_n = 0 \quad \text{on } \Gamma_1 \ , \tag{3.208}$$

$$M_n = 0, \quad Q = 0 \quad \text{on } \Gamma_2 \ , \tag{3.209}$$

i.e. the plate is clamped, simply supported and free on the portions Γ_0, Γ_1 and Γ_2 of the boundary $\partial\Omega$. M_n denotes the moment and Q is the effective shear force given by

$$M_n = M_{ij}n_in_j \ , \tag{3.210}$$

$$Q = -M_{kl,l}n_k - \frac{\partial}{\partial\tau}(M_{n\tau}), \tag{3.211}$$

where

$$M_{ij} = b_{ijkl}w_{,lk}, \quad M_{n\tau} = M_{kl}n_l\tau_k \ .$$

For the tensor function $\mathcal{B}(.) = \{b_{ijkl}(.)\} \in C^2(\mathbb{R}^2; \mathbb{R}^{16})$, $i,j,k,l = 1,2$, the usual symmetry conditions hold

$$b_{ijkl}(x) = b_{jikl}(x) = b_{klij}(x) \quad x \in \Omega, \quad i,j,k,l = 1,2 \ . \tag{3.212}$$

Let us assume that there exists a constant $\nu > 0$ such that

$$b_{ijkl}(x)\xi_{ij}\xi_{kl} \geqq \nu\xi_{ij}\xi_{ij} \quad \text{for all } x \in \mathbb{R}^2 \tag{3.213}$$

and for all the symmetric second order tensors ξ. For a weak solution $w \in H^2(\Omega)$ to (3.206)–(3.209) the following integral identity is satisfied

$$w \in H(\Omega): \quad a(w,\phi) = (f,\phi) \quad \forall\phi \in H(\Omega), \tag{3.214}$$

where

$$H(\Omega) = \{\phi \in H^2(\Omega) | \phi = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1 \ , \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma_0\}, \quad (3.215)$$

$$a(z, \phi) = \int_\Omega b_{ijkl}(x) z_{,ij}(x) \phi_{,kl}(x) dx \quad \forall \phi, z \in H^2(\Omega), \quad (3.216)$$

$$(f, \phi) = \int_\Omega f(x)\phi(x) dx \quad \forall \phi \in H(\Omega) \ . \quad (3.217)$$

The following Green's formula holds for all $z \in H^4(\Omega)$ and $\phi \in H^2(\Omega)$:

$$a(z, \phi) = \int_\Omega (b_{ijkl}(x) z_{,lk}(x))_{,ij} \phi(x) dx + \int_\Gamma \{Q\phi + M_n \frac{\partial \phi}{\partial n}\} d\Gamma \ . \quad (3.218)$$

Using the formula (3.218) it is possible to define the weak form of the equation (3.206), along with the nonhomogeneous boundary conditions prescribed on Γ_0, Γ_1 and Γ_2, respectively. However for the sake of simplicity we restrict our considerations to the system (3.206)–(3.209) with the homogeneous boundary conditions.

Let $w \in H(\Omega_t)$ denote a solution to the system (3.206)–(3.209) defined in the domain $\Omega_t \subset \mathbb{R}^2$, $\Omega_t = T_t(V)(\Omega)$, $t \in [0, \delta)$,

$$w_t \in H_t : \quad a_t(w_t, \phi) = (f, \phi)_t \quad \forall \phi \in H_t \ , \quad (3.219)$$

where

$$H_t = \{\phi \in H^2(\Omega_t) | \phi = 0 \quad \text{on } \Gamma_0^t \cup \Gamma_1^t \ , \quad \frac{\partial \phi}{\partial n_t} = 0 \quad \text{on } \Gamma_0^t\}, \quad (3.220)$$

$$a_t(z, \phi) = \int_{\Omega_t} b_{ijkl}(x) z_{,ij}(x) \phi_{,kl}(x) dx \quad \forall z, \phi \in H^2(\Omega_t), \quad (3.221)$$

$$(f, \phi)_t = \int_{\Omega_t} f(x)\phi(x) dx \quad \forall \phi \in H_t \ . \quad (3.222)$$

To proceed further with a discussion we have to derive the form of the material derivative

$$\dot{w} = \lim_{t \to 0} (w_t \circ T_t - w)/t \ . \quad (3.223)$$

From (3.219) it follows that for $w^t = w_t \circ T_t \in H^2(\Omega)$ the following variational equation is met

$$w^t \in H(\Omega) : \quad a^t(w^t, \phi) = (f^t, \phi) \quad \forall \phi \in H(\Omega), \quad (3.224)$$

where

$$a^t(z, \phi) = \int_\Omega \xi^t(z) : \mathcal{B}^t : \xi^t(\phi) dx \quad \forall z, \phi \in H^2(\Omega), \quad (3.225)$$

$$\xi^t(z) = D({}^*DT_t^{-1} \cdot \nabla z) \cdot DT_t^{-1} \ , \quad (3.226)$$

$$\mathcal{B}^t = \{b_{ijkl}^t\}, \quad b_{ijkl} = \gamma(t)(b_{ijkl} \circ T_t), \quad (3.227)$$

$$f^t = \gamma(t)(f \circ T_t) \ . \quad (3.228)$$

Lemma 3.14 *For the strong material derivative $\dot{w}$ the following variational equation is satisfied*

$$\dot{w} \in H(\Omega): \quad a(\dot{w}, \phi) = (f', \phi) - a'(w, \phi) \quad \forall \phi \in H(\Omega), \tag{3.229}$$

where

$$f' = \operatorname{div}(fV), \tag{3.230}$$

$$a'(z, \phi) = \int_\Omega \{\xi'(z): \mathcal{B}: \xi(\phi) + \xi(z): \mathcal{B}': \xi(\phi) + \xi(z): \mathcal{B}: \xi'(\phi)\} dx \quad \forall z, \phi \in H^2(\Omega),$$

$$\xi'(z) = -D({}^*DV \cdot \nabla z) - D(\nabla z) \cdot DV, \tag{3.231}$$

$$\mathcal{B}' = \{b'_{ijkl}\}, \quad b'_{ijkl} = \operatorname{div}(b_{ijkl} V), \tag{3.232}$$

$$\xi(\phi) = D(\nabla \phi) = \{\phi_{,ij}\} \ . \tag{3.233}$$

Proof. Since the mapping

$$[0, \varepsilon) \ni t \to f \circ T_t \in (H(\Omega))' \tag{3.234}$$

is strongly differentiable (see Proposition 2.41), then we can prove this lemma using Theorem 4.30 of Chap. 4. □

Finally the form of the shape derivative w' is to be determined

$$w' = \dot{w} - \nabla w \cdot V \ . \tag{3.235}$$

It is supposed that

$$\nabla w \cdot V \in H^2(\Omega), \tag{3.236}$$

therefore $w' \in H^2(\Omega)$.

Differentiating (3.219) with respect to t at t=0 and assuming that $\phi \in H^2(\mathbb{R}^2)$ is a given function, $\phi \in H^2(\Omega_t)$ for $t > 0$, t small enough, we have

$$\frac{d}{dt} a_t(w_t, \phi)_{|t=0} = \int_\Omega b_{ijkl} w'_{,kl} \phi_{,ij} dx + \int_\Gamma v_n b_{ijkl} w_{,kl} \phi_{,ij} d\Gamma, \tag{3.237}$$

$$\frac{d}{dt}(f, \phi)_{t_{|t=0}} = \int_\Gamma v_n f \phi d\Gamma, \tag{3.238}$$

here it is assumed that the trace $\gamma_\Gamma f \in L^2(\Gamma)$ is well defined.

Integration by parts of the first integral on the right–hand side of (3.237), yields

$$M'_{ij} = b_{ijkl} w'_{,kl} \ . \tag{3.239}$$

Hence

$$\int_{\Omega} M'_{ij}\phi_{,ij}dx = \int_{\Omega} M'_{ij,ji}\phi dx + \int_{\Gamma}\{M'_{ij}n_j\phi_{,i} - M'_{ij,j}n_i\phi\}d\Gamma \ . \tag{3.240}$$

It should be remarked that on $\Gamma = \partial\Omega$ we have

$$\phi_{,i} = n_i\frac{\partial\phi}{\partial n} + (\nabla_\Gamma\phi)_i \ . \tag{3.241}$$

Therefore

$$\int_{\Gamma} M'_{ij}n_j\phi_{,i}d\Gamma = \int_{\Gamma} M'_n\frac{\partial\phi}{\partial n}d\Gamma + \int_{\Gamma} n\cdot M'\cdot\nabla_\Gamma\phi d\Gamma =$$

and integrating by parts on Γ we get

$$\begin{aligned} &= \int_{\Gamma}\{M'_n\frac{\partial\phi}{\partial n} - \mathrm{div}_\Gamma(M'\cdot n)\phi + \kappa M'_n\phi\}d\Gamma \\ &= \int_{\Gamma}\{M'_n\frac{\partial\phi}{\partial n} - \mathrm{div}_\Gamma(M'_{n\tau})\phi\}d\Gamma, \end{aligned} \tag{3.242}$$

where

$$M'_n = M'_{ij}n_in_j \tag{3.243}$$

$$M'_{n\tau} = M'\cdot n - M'_n n \ . \tag{3.244}$$

Integration by parts on Γ accomplished for the second term on the right–hand side of (3.237) leads to

$$\int_{\Gamma} v_nM_{ij}\phi_{,ij}d\Gamma = \tag{3.245}$$

accompanied by the appropriate extension $\mathcal{N}$ of the normal vector field on Γ with $\partial\mathcal{N}/\partial n = 0$, and for ϕ such that $\partial^2\phi/\partial n^2 = 0$ on Γ, yields

$$= \int_{\Gamma} v_nM_{ij}[n_j\nabla_\Gamma(\frac{\partial\phi}{\partial n})_i + \nabla_\Gamma(n_i\frac{\partial\phi}{\partial n} + (\nabla_\Gamma\phi)_i)_j]d\Gamma = \tag{3.246}$$

Integrating by parts on Γ we have

$$\begin{aligned} &= -\int_{\Gamma}\{\mathrm{div}_\Gamma(v_nM_{n\tau})\frac{\partial\phi}{\partial n} + \mathrm{div}_\Gamma(v_nM_\tau)\cdot n\frac{\partial\phi}{\partial n} + \mathrm{div}_\Gamma(v_nM_\tau)\cdot\nabla_\Gamma\phi\}d\Gamma \\ &= -\int_{\Gamma}\{\frac{\partial\phi}{\partial n}(\mathrm{div}_\Gamma(v_nM_\tau) - \mathrm{div}_\Gamma(v_nM_{n\tau}) \\ &\qquad + \mathrm{div}_\Gamma(v_nM_\tau)\cdot n) + \phi\mathrm{div}_\Gamma(\mathrm{div}_\Gamma(v_nM_\tau)_\tau)\}d\Gamma \ . \end{aligned}$$

From (3.237) and (3.238) it follows that

$$\int_\Omega M'_{ij,ji}\phi dx + \int_\Gamma (M'_n \frac{\partial \phi}{\partial n} - \phi(n \cdot \mathrm{div} M' + \mathrm{div}_\Gamma(M'_{n\tau}))d\Gamma +$$
$$\int_\Gamma \{-\frac{\partial \phi}{\partial n}(\mathrm{div}_\Gamma(v_n M_{n\tau}) + \mathrm{div}_\Gamma(v_n M_\tau)\cdot n) \tag{3.247}$$
$$+ \phi \mathrm{div}_\Gamma(\mathrm{div}_\Gamma(v_n M_\tau)_\tau)\} d\Gamma = \int_\Gamma v_n f \phi d\Gamma \ .$$

Hence the shape derivative w' is a solution to the following boundary value problem

$$(b_{ijkl}(x) w'(x)_{,kl})_{,ij} = 0 \quad \text{in } \Omega, \tag{3.248}$$
$$w' = 0, \quad \frac{\partial w'}{\partial n} = -v_n \frac{\partial^2 w}{\partial n^2} \quad \text{on } \Gamma_0 \ , \tag{3.249}$$
$$w' = -v_n \frac{\partial w}{\partial n}, \quad M'_n = \mathrm{div}_\Gamma(v_n M_{n\tau}) + \mathrm{div}_\Gamma(v_n M_\tau)\cdot n \quad \text{on } \Gamma_1 \ , \tag{3.250}$$
$$\begin{aligned} M'_n &= \mathrm{div}_\Gamma(v_n M_{n\tau} + \mathrm{div}(v_n M_\tau)\cdot n \quad \text{on } \Gamma_2 \ , \\ Q' &= -n \cdot \mathrm{div} M' - \mathrm{div}_\Gamma(M'_{n\tau}) \\ &= -\mathrm{div}_\Gamma(\mathrm{div}_\Gamma(v_n M_\tau)_\tau) + v_n f \quad \text{on } \Gamma_2 \ . \end{aligned} \tag{3.251}$$

The boundary conditions for w' and $\partial w'/\partial n$ in (3.249) and (3.250) are derived from (3.235).

3.8. Shape derivatives of boundary integrals: the non–smooth case

Let us consider a non–smooth domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$. First a non–smooth domain in the plane will be examined. Let $\Omega \subset \mathbb{R}^2$ be a given domain with the piecewise smooth boundary Γ, i.e. it is supposed that there exist $a_i \in \Gamma$, $1 \leqq i \leqq m$, m an integer, such that $\Gamma = \Gamma \setminus \{a_1, \ldots, a_m\}$ is of class $C^k, k \geqq 1$. Moreover it is assumed that Γ has corners located at points a_i, $i = 1, \ldots, m$; therefore the unit tangent vector τ on Γ is not continuous at these points. The tangent and normal vectors at a_i are defined as follows

$$\tau_i = \tau_i^- - \tau_i^+, \quad n_i = n_i^- - n_i^+ \ , \tag{3.252}$$

where, e.g. n_i^+ (n_i^-) denotes the right (left) limit at $a_i \in \Gamma$ of the outward unit normal vector field n on Γ. It should be remarked that the existence of limits $\tau^\pm$ at a_i is equivalent to the existence of limits $n^\pm$ of the normal field n at a_i. Here it is assumed that the unit tangent vector τ on Γ corresponds to the natural orientation on Γ.

Let $V \in C(0,\varepsilon;\mathcal{D}^k(\mathbb{R}^2,\mathbb{R}^2))$ be given and let $T_t(V)$ be the associated transformation of $\mathbb{R}^2$. We shall examine the domain $\Omega_t = T_t(V)(\Omega)$ with the boundary $\Gamma_t = T(V)(\Gamma)$ and corners located at the points $a_i^t = T_t(V)(a_i), 1 \leqq i \leqq m$.

Let $f \in H^{3/2}(\mathbb{R}^2)$ and let us consider the shape functional $J(\Omega_t) = \int_{\Gamma_t} f d\Gamma_t$. Applying the change of variables $x = T_t(V)(X)$ to the shape functional we obtain

$$J(\Omega_t) = \int_\Gamma f \circ T_t \omega(t) d\Gamma$$

(see Sect. 2.17 of Chap. 2).

Let $\Gamma_i = (a_i a_{i+1}), i = 1,\dots m-1$, be the part of Γ which joins the points a_i, a_{i+1}, in the positive direction of Γ. For $i = m$ we set $\Gamma_m = (a_m, a_1)$. Finally by the application of Stokes' formula on Γ_i we obtain (see Chap. 2, Sect. 2.34) for $1 \leqq i \leqq m$,

$$\begin{aligned}\int_{\Gamma_i} \mathrm{div}_\Gamma(fV(0))d\Gamma =& f(a_{i+1})\langle V(0,a_{i+1}), \tau^-(a_{i+1})\rangle_{\mathbb{R}^2} \\ &- f(a_i)\langle V(0,a_{i+1}), \tau^+(a_i)\rangle_{\mathbb{R}^2},\end{aligned}$$

where

$$\tau^-(a_{i+1}) = \lim_{\substack{x \to a_{i+1} \\ x \in \Gamma_i = (a_i, a_{i+1})}} \tau(x),$$

$$\tau^+(a_i) = \lim_{\substack{x \to a_i \\ x \in \Gamma_i = (a_i, a_{i+1})}} \tau(x) \ .$$

For $\tau^+(a_m)$ and $\tau^-(a_1)$ the same formulae on $\Gamma_m = (a_m, a_1)$ can be used. Here $\omega(t) = \|M(T_t) \cdot n\|_{\mathbb{R}^2}$. Furthermore the mapping $t \to \omega(t)$ is differentiable in $C^{k-1}(\mathbb{R}^N)$, and the derivative at $t = 0$ is of the form

$$\omega'(0) = \mathrm{div}V(0) - \langle DV(0) \cdot n, n\rangle_{\mathbb{R}^2} \quad \text{on } \Gamma \ .$$

Hence

$$dJ(\Omega;V) = \int_\Gamma (\nabla f \cdot V(0) + f \mathrm{div}_\Gamma V(0)) d\Gamma,$$

where $\mathrm{div}V(0)$ is defined everywhere in $\mathbb{R}^2$, thus almost everywhere on Γ. Using the identity

$$\mathrm{div}_\Gamma(fV) = \nabla_\Gamma \cdot V + f \mathrm{div}_\Gamma V$$

we obtain

$$dJ(\Omega;V) = \int_\Gamma \mathrm{div}_\Gamma(fV(0))d\Gamma + \int_\Gamma \frac{\partial f}{\partial n}\langle V(0) \cdot n\rangle_{\mathbb{R}^2} d\Gamma \ .$$

Therefore the following proposition can be formulated

Proposition 3.15 *Let us suppose that* $f \in H^{3/2}(\mathbb{R}^N)$, *then*

$$\Big(\frac{d}{dt}\int_{\Gamma_t} f d\Gamma_t\Big)|_{t=0} = \int_\Gamma \Big(\frac{\partial f}{\partial n} + \kappa f\Big)\langle V(0), n\rangle_{\mathbb{R}^2} d\Gamma + \sum_{i=1}^{m} \langle V(0, a_i), \tau^-(a_i) - \tau^+(a_i)\rangle_{\mathbb{R}^2} \ . \tag{3.253}$$

Finally let us consider an example of a non–smooth domain in $\mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be a given bounded domain, and let $\Gamma = \partial\Omega$ be piecewise C^k. Furthermore it is assumed that $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup S$, where $S = \overline{\Gamma}_1 \cap \overline{\Gamma}_2$ is a one–dimensional manifold without boundary of class C^k and $\overline{\Gamma}_i$, $i = 1, 2$, are of class C^k. For any point $x \in \overline{\Gamma}_i$, $\mathcal{T}_x\overline{\Gamma}_i$ stands for the tangent space to $\overline{\Gamma}_i$ at $x \in \overline{\Gamma}_i$, $i = 1, 2$. Let $\tau \in C^{k-1}(S)$ be a vector field,

$$\tau(x) \in \mathcal{T}_x\overline{\Gamma}_1 \cap \mathcal{T}_x\overline{\Gamma}_2$$

for any $x \in S$; τ is a unit tangent vector on S oriented along S according to the positive orientation. Let us assign to each $x \in S$ a unit vector $\mu_i(x) \in \mathcal{T}_x\overline{\Gamma}_i$ such that

$$\langle \tau(x), \mu_i(x)\rangle_{\mathbb{R}^3} = 0 \ .$$

We assume that $\mu_i(x)$ is outward pointing on Γ_i. From Stokes' formula it follows that

$$\int_{\Gamma_i} \operatorname{div}(fV) d\Gamma = \int_S fV \cdot \mu_i d\ell \ .$$

Thus

$$\int_{\partial\Gamma} \operatorname{div}(fV) d\Gamma = \int_S f\langle V, \mu_1 + \mu_2\rangle_{\mathbb{R}^3} d\ell \ .$$

This result makes it possible to determine the form of the shape derivative of the surface integral on $\partial\Omega$

Proposition 3.16 *Let us suppose that $f \in H^{3/2}(\mathbb{R}^3)$, then*

$$\Big(\frac{d}{dt}\int_{\partial\Omega_t} f d\Gamma_t\Big)\Big|_{t=0} = \int_{\partial\Omega} \Big(\frac{\partial f}{\partial n} + \kappa f\Big)\langle V(0), n\rangle_{\mathbb{R}^3} d\Gamma + \int_S f\langle V(0), \mu_1 + \mu_2\rangle_{\mathbb{R}^3} d\ell \ . \tag{3.254}$$

If the boundary $\partial\Omega$ of the domain Ω is piecewise C^k, $k \geqq 1$, then the form of the shape derivatives of surface integrals on $\partial\Omega$ can be derived in the same way.

3.9. Shape sensitivity analysis of boundary value problems with singularities

Let $\Omega \subset \mathbb{R}^2$ be a given domain with the sufficiently smooth boundary $\Gamma = \partial\Omega$.

Let $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \{A\} \cup \{B\}$, with $\overline{\Gamma}_0 = \Gamma_0 \cup \{A\} \cup \{B\}$ and $\overline{\Gamma}_1 = \Gamma_1 \cup \{A\} \cup \{B\}$ (see Fig.3.17). We shall make use of the following notation

$$H^1_{\Gamma_0}(\Omega) = \{\phi \in H^1(\Omega) | \phi = 0 \quad \text{on } \Gamma_0\} \ .$$

It is well known, see e.g. (Temam 1985), that there exists $\alpha > 0$ such that

$$\int_\Omega |\nabla\phi|^2 dx \geqq \alpha \|\phi\|^2_{H^1(\Omega)} \quad \forall \phi \in H^1_{\Gamma_0}(\Omega)$$

provided that $\int_{\Gamma_0} d\ell > 0$.

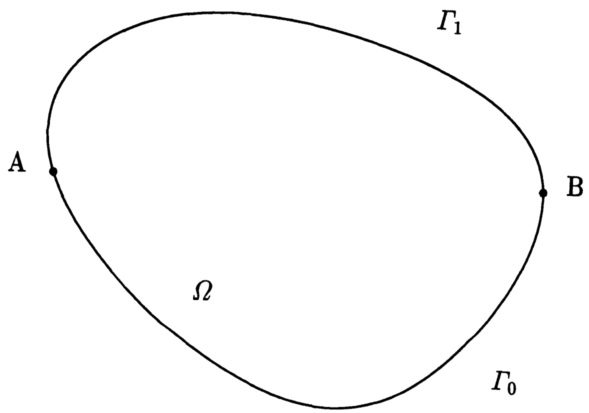

Fig. 3.17. Domain $\Omega \in \mathbb{R}^2$

The following boundary value problem is considered

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma_0 \ ,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_1 \ ,$$

where $f \in L^2(\mathbb{R}^2)$ is a given element.

For a weak solution to this problem the following integral identity holds

$$u \in H^1_{\Gamma_0}(\Omega) \ : \quad \int_\Omega \nabla u \cdot \nabla \phi dx = \int_\Omega f \phi dx \quad \forall \phi \in H^1_{\Gamma_0}(\Omega) \ .$$

Lemma 3.18 *The weak material derivative*

$$\dot{u} = \dot{u}(\Omega; V) \in H^1_{\Gamma_0}(\Omega)$$

of the solution $u \in H^1_{\Gamma_0}(\Omega)$ *is given as the unique solution to the following integral equation*

$$\dot{u} \in H^1_{\Gamma_0}(\Omega) \ :$$

$$\int_\Omega \nabla \dot{u} \cdot \nabla \phi dx = \int_\Omega div(fV) \phi dx - \int_\Omega \langle A' \cdot \nabla u, \nabla \phi \rangle_{\mathbb{R}^2} dx$$

$$\forall \phi \in H^1_{\Gamma_0}(\Omega) \ .$$

Proof. Let

$$u_s \in H^1_{\Gamma_0^s}(\Omega_s) : \quad \int_{\Omega_s} \nabla u_s \cdot \nabla \phi dx = \int_{\Omega_s} f \phi dx \quad \forall \phi \in H^1_{\Gamma_0^s}(\Omega_s),$$

where $\Omega_s = T_s(\Omega)$ and $\Gamma_0^s = T_s(\Gamma_0)$. Let u^s stand for

$$u^s = u_s \circ T_s \in H^1_{\Gamma_0}(\Omega) \ .$$

Then

$$u^s \in H^1_{\Gamma_0}(\Omega) : \quad \int_\Omega \langle A_s \cdot \nabla u^s, \nabla \phi \rangle_{\mathbb{R}^2} dx = \int_\Omega f^s \phi dx \tag{3.255}$$

$$\forall \phi \in H^1_{\Gamma_0}(\Omega),$$

where $f^s = \gamma(s) f \circ T_s$, $s \in [0, \delta)$.

The mapping

$$[0, \delta) \ni s \to f^s \in (H^1_{\Gamma_0}(\Omega))'$$

is weakly differentiable at $s = 0$, its derivative is given by

$$f' = \mathrm{div}(fV) \ .$$

Therefore we can differentiate (3.255) with respect to s at $s = 0$, as was to be shown. □

Remark. In order to ensure the existence of the strong material derivative $\dot{u}(\Omega; V) \in H^1_{\Gamma_0}(\Omega)$ the following assumption is necessary :

The mapping
$$[0,\delta) \ni s \to f \circ T_s \in (H^1_{\Gamma_0}(\Omega))'$$
is strongly differentiable. If $f \in H^r(\mathbb{R}^2)$, $r > 2$, then the assumption is satisfied.

Let us consider the shape derivative $u' = u'(\Omega; V)$ of the solution $u \in H^1_{\Gamma_0}(\Omega)$ to the above elliptic equation. Since $\nabla u \cdot V \in L^2(\Omega)$ for all admissible vector fields V thus
$$u' = \dot{u} - \nabla u \cdot V \in L^2(\Omega) \ .$$

Let ϕ be a given smooth function, then for $u^s \in H^1_{\Gamma_0}(\Omega)$ we have
$$\int_\Omega \nabla u^s \cdot \nabla \phi dx = -\int_\Omega u^s \Delta\phi dx + \int_\Gamma u^s \frac{\partial \phi}{\partial n} d\Gamma \ .$$
If it is assumed that
$$\Delta\phi \in L^2(\Omega), \quad \phi = 0 \quad \text{on } \Gamma_0 \ , \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma_1 \ ,$$
whence
$$\int_\Omega \nabla u^s \cdot \nabla \phi dx = -\int_\Omega u^s \Delta \phi dx$$
because $u^s = 0$ on Γ_0. From the foregoing it can be inferred that
$$\int \nabla \dot{u} \cdot \nabla \phi dx = -\int_\Omega \dot{u} \Delta \phi dx$$
hence
$$-\int_\Omega \dot{u}\Delta\phi dx = \int_\Omega \operatorname{div}(fV)\phi dx - \int_\Omega \langle A' \cdot \nabla u, \nabla \phi \rangle_{\mathbb{R}^2} dx \ .$$
It should be noted here that we use the same symbol for the scalar product in $L^2(\Omega)$ and the duality pairing between $(H^1_{\Gamma_0}(\Omega))'$ and $H^1_{\Gamma_0}(\Omega)$.

Therefore the following integral identity
$$\int_\Omega u' \Delta \phi dx = L(u, V, \phi) = -\int_\Omega \nabla u \cdot V \Delta\phi dx$$
$$-\int_\Omega \operatorname{div}(fV)\phi dx + \int_\Omega \langle A' \cdot \nabla u, \nabla \phi \rangle_{\mathbb{R}^2} dx$$
holds for the shape derivative $u' \in L^2(\Omega)$. For any sufficiently smooth vector field V with
$$V(A) = V(B) = 0, \quad v_n = \langle V, n \rangle_{\mathbb{R}^2} = 0 \quad \text{on } \Gamma$$
we have that $u' = 0$. Thus there exist:
$$\text{distributions } l_A(\phi),\ l_B(\phi) \in \mathcal{D}^{-1}(\Gamma; \mathbb{R}^2),$$
$$g_0(\phi) \in \mathcal{D}^{-1}(\Gamma_0),\ g_1(\phi) \in \mathcal{D}^{-1}(\Gamma_1)$$

such that

$$\text{spt } l_A(\phi) = \{A\},$$
$$\text{spt } l_B(\phi) = \{B\} \ .$$

Furthermore

$$\begin{aligned}\int_\Omega u'\Delta\phi dx =& \langle \ell_A(\phi), V(A)\rangle_{\mathbb{R}^2} + \langle \ell_B(\phi), V(B)\rangle_{\mathbb{R}^2} + \\ & \langle g_0(\phi), v_n\rangle_{\mathcal{D}^{-1}(\Gamma_0)\times\mathcal{D}^1(\Gamma_0)} + \langle g_1(\phi), v_n\rangle_{\mathcal{D}^{-1}(\Gamma_1)\times\mathcal{D}^1(\Gamma_1)} \\ & \text{for all } \phi \in H^1_{\Gamma_0}(\Omega) \text{ with } \Delta\phi \in L^2(\Omega),\end{aligned}$$

where we denote $\langle \ell_A(\phi), V(A)\rangle_{\mathbb{R}^2} = \langle l_A(\phi), V\rangle$. If it is assumed that $\phi \in \mathcal{D}(\Omega)$, then

$$\int_\Omega u'\Delta\phi dx = \int_\Omega \Delta u'\phi dx = 0 \ .$$

Hence

$$\Delta u' = 0 \quad \text{in } \mathcal{D}'(\Omega) \ .$$

The boundary conditions for u' are non–smooth, because we have

$$u' = \dot{u} - v_n\frac{\partial u}{\partial n} \in H^{-1/2}(\Gamma) \quad \text{on } \Gamma \ .$$

Thus

$$u' = -v_n\frac{\partial u}{\partial n} \quad \text{on } \Gamma_0 \ ,$$
$$u' = \dot{u} \quad \text{on } \Gamma_1 \ .$$

In order to determine the form of distributions l_A, l_B, g_0, g_1, the form of the singularity of u is to be obtained first, i.e.

$$u = u_1 + u_2$$

where $u - u_1 \in H^2(\Omega)$, u_2 is the singular term, see (Grisvard 1985) for the details.

The form of distributions l_A, l_B can be determined making use of the term $L(u, V, \phi)$ provided that the explicit form of the singular part u_2 of the solution u is known.

3.10. Hyperbolic initial boundary value problems

Let $\Omega \subset \mathbb{R}^N$ be a given domain of class C^k, k integer, $k \geqq 1$; the following notation is used: $I = (t_0, t_1)$, $t_1 > t_0$, $Q = I \times \Omega \subset \mathbb{R}^{N+1}$. For a given vector field $V \in C(0, \varepsilon; \mathcal{D}^k(\mathbb{R}^N, \mathbb{R}^N))$, $T_s(V)$ denotes the associated transformation.

In this section t denotes the time variable, $t \in I$, while s is the transformation parameter, $s \in [0, \delta)$, $0 < \delta < \varepsilon$.

Let $\Omega_s = T_s(V)(\Omega)$, $s \in [0, \delta)$, and let $Q_s = I \times \Omega_s \subset \mathbb{R}^{N+1}$.

For any s, $0 \leqq s < \delta$, we have to consider a partial differential equation of the hyperbolic type, i.e. the wave equation defined in the cylinder Q_s.

Let $f \in L^1(I; L^2(\mathbb{R}^N))$ be given, and denote by $\Box$ the wave operator :

$$\Box\, \phi = -\Delta\phi + \frac{\partial^2 \phi}{\partial t^2} \quad \forall \phi \in C^2(\mathbb{R}^{N+1}) \ . \tag{3.256}$$

Let $y = y_s$, $s \in [0, \delta)$, be a solution to the following mixed hyperbolic problem

$$\Box\, y = f \quad \text{in } Q_s \ , \tag{3.257}$$

$$y(t_0) = \frac{\partial y}{\partial t}(t_0) = 0 \quad \text{in } \Omega_s \ , \tag{3.258}$$

$$y = 0 \quad \text{on } \Sigma_s \ , \tag{3.259}$$

where $\Sigma_s = I \times \Gamma_s$ is the lateral boundary of Q_s, and $\Gamma_s = \partial\Omega_s$.
We derive the standard a priori estimates for solutions to (3.257)–(3.259).

Proposition 3.19 *For any ϕ in $C^\infty(\overline{Q}_s)$ such that $\phi|_{\Sigma_s} = 0$ we have*

$$\|\phi\|_{L^\infty(I; H_0^1(\Omega_s))} \leqq \|\Box\, \phi\|_{L^1(I; L^2(\Omega_s))} \tag{3.260}$$

and

$$\left\|\frac{\partial \phi}{\partial t}\right\|_{L^\infty(I; L^2(\Omega_s))} \leqq C \|\Box\, \phi\|_{L^1(I; L^2(\Omega_s))} \ . \tag{3.261}$$

Proof. Let us assume that the quadratic energy functional is defined by

$$E(t) = \frac{1}{2}\int_{\Omega_s} \left(|\nabla\phi(t,x)|^2 + \left(\frac{\partial \phi}{\partial t}(t,x)\right)^2\right) dx \ .$$

Since $\phi = 0$ on Γ_s, then using Green's formula we get

$$E'(t) = \int_{\Omega_s} \Box\, \phi \frac{\partial \phi}{\partial t} dx \leqq \|\Box\, \phi\|_{L^2(\Omega_s)} \left\|\frac{\partial \phi}{\partial t}\right\|_{L^2(\Omega_s)} \ .$$

From $E(t_0) = 0$, it follows that for any $t \in I$

$$\begin{aligned} E(t) &\leqq \int_{t_0}^{t} \|\Box\, \phi\|_{L^2(\Omega_s)} \left\|\frac{\partial \phi}{\partial t}\right\|_{L^2(\Omega_s)} dt \\ &\leqq \int_{t_0}^{t_1} \|\Box\, \phi\|_{L^2(\Omega_s)} \left\|\frac{\partial \phi}{\partial t}\right\|_{L^2(\Omega_s))} dt \\ &\leqq \|\Box\, \phi\|_{L^1(I; L^2(\Omega_s))} \left\|\frac{\partial \phi}{\partial t}\right\|_{L^\infty(I; L^2(\Omega_s))} \ . \end{aligned}$$

Hence for any $t \in I$ we obtain

$$\frac{1}{2}\int_{\Omega_s}\left(\frac{\partial y}{\partial t}\right)^2 dx \leqq E(t) \leqq \|\Box\,\phi\|_{L^1(I;L^2(\Omega_s))}\left\|\frac{\partial \phi}{\partial t}\right\|_{L^\infty(I;L^2(\Omega_s))}$$

which leads to (3.261).
For any $t \in I$ we also have

$$\frac{1}{2}\int_{\Omega_s}|\nabla y|^2 dx \leqq \left\|\frac{\partial \phi}{\partial t}\right\|_{L^\infty(I;L^2(\Omega_s))}\|\Box\,\phi\|_{L^1(I;L^2(\Omega_s))}\ .$$

This proves (3.260). □

Let the problem (3.257)–(3.259) be defined in the cylinder Q_s. We denote by

$$y_s \in L^\infty(I; H_0^1(\Omega_s))$$

a weak solution to the problem (3.257)–(3.259).

Proposition 3.20 *Let $f \in L^1(I; L^2_{loc}(\mathbb{R}^N))$. For any bounded domain of class C^k, $k \geqq 1$, and for any $s \in [0,\delta)$, there exists the unique solution y_s to (3.257)–(3.259) such that*

$$y_s \in L^\infty(I; H_0^1(\Omega_s)) \tag{3.262}$$

$$\frac{\partial y_s}{\partial t} \in L^\infty(I; L^2(\Omega_s))\ . \tag{3.263}$$

Proof. Applying Galerkin's method and making use of a priori estimates (3.260)–(3.261), one can show that (3.262) and (3.263) hold. □

Proposition 3.21 *Let y_s be the solution to (3.257)–(3.259), then the following inequality is met*

$$\|y_s\|_{L^\infty(I;H_0^1(\Omega_s))} + \left\|\frac{\partial y_s}{\partial t}\right\|_{L^\infty(I;L^2(\Omega_s))} \leqq 2\|\Box\,y_s\|_{L^1(I;L^2(\Omega_s))}\ . \tag{3.264}$$

Let $y^s = y_s \circ T_s$ denote the element transported to the fixed domain Ω, i.e. defined by

$$y^s(t,x) = y_s(t, T_s(x))\ .$$

It can be shown that for all s, $0 \leqq s < \delta$, we have

$$y^s \in L^\infty(I; H_0^1(\Omega)) \tag{3.265}$$

$$\frac{\partial y^s}{\partial t} \in L^\infty(I; L^2(\Omega))\ .$$

The transported wave operator $\Box_s$ is defined in Q,

$$(\Box\,\phi)\circ T_s = \Box_s(\phi\circ T_s)\ . \tag{3.266}$$

Since the variables t and s are independent, then

$$\left(\frac{\partial^2\phi}{\partial t^2}\circ T_s\right) = \frac{\partial^2}{\partial t^2}(\phi\circ T_s)$$

It should be remarked that

$$(\Delta\phi)\circ T_s = \gamma(s)^{-1}\mathrm{div}(A(s)\cdot\nabla(\phi\circ T_s)),$$

where

$$A_s = \gamma(s)DT_s^{-1}\cdot {}^*DT_s^{-1}\ .$$

Thus

$$\Box_s\phi = -\gamma(s)^{-1}\mathrm{div}(A(s)\cdot\nabla\phi) + \frac{\partial^2\phi}{\partial t^2}\ . \tag{3.267}$$

Using the change of variables $x = T_s(X)$ in the estimate (3.264) we obtain the following estimate of the transported solution y^s.

Lemma 3.22 *There exists a constant $\alpha > 0$ such that for any s, $0 \leqq s < \delta$,*

$$\alpha\|y^s\|_{L^\infty(I;H_0^1(\Omega))} + \alpha\left\|\frac{\partial y^s}{\partial t}\right\|_{L^\infty(I;L^2(\Omega))} \leqq \tag{3.268}$$

$$2\int_{t_0}^{t_1}\left(\int_\Omega\left[-\left(\sqrt{\gamma(s)}\right)^{-1}\mathrm{div}(A(s)\cdot\nabla y^s) + \sqrt{\gamma(s)}\frac{\partial^2 y^s}{\partial t^2}\right]^2 dx\right)^{\frac{1}{2}} dt =$$

$$2\|\sqrt{\gamma(s)}\Box_s y^s\|_{L^1(I;L^2(\Omega))}$$

Proof. Applying the change of variables $x = T_s(V)(X)$ in (3.264), it follows that (3.268) holds. By the assumptions adopted, $\alpha > 0$ is a given constant,

$$A_s(x) = A(s,x) \geqq \alpha\mathcal{I}$$

$$\gamma(s)(x) = \gamma(s,x) \geqq \alpha$$

for all $s\in[0,\delta)$ and $x\in\overline{\Omega}$. □

Let us assume that $z^s = y^s - y$. Hence, from (3.268) it follows that

$$\alpha\|z^s\|_{L^\infty(I;H_0^1(\Omega))} + \alpha\left\|\frac{\partial z^s}{\partial t}\right\|_{L^\infty(I;L^2(\Omega))} \leqq$$

$$2\|\gamma(s)\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|\Box_s z^s\|_{L^1(I;L^2(\Omega))}\ .$$

On the other hand

$$\begin{aligned}\Box_s z^s &= \Box_s y^s - \Box_s y \\ &= (\Box\, y_s) \circ T_s - \Box_s y \\ &= (f \circ T_s - f) + (\Box - \Box_s) y \ .\end{aligned} \tag{3.269}$$

In order to determine an estimate of the norm of the element $\frac{1}{s} z^s$ (as s approaches zero) for $s > 0$, it is sufficient to examine the last two terms.

Proposition 3.23 *Let $\phi \in L^1(I; H^2(\Omega))$ be given such that $\frac{\partial \phi}{\partial t} \in L^1(I; L^2(\Omega))$. Then the mapping $s \to \Box_s \cdot \phi$ is differentiable in the norm of the space $L^1(I; L^2(\Omega))$, and the derivative at $s = 0$ is given by*

$$\left(\frac{d}{ds} \Box_s \cdot \phi\right)\Big|_{s=0} = \Box' \phi = \mathrm{div} V(0) \Delta \phi - \mathrm{div}(A'(0) \cdot \nabla \phi), \tag{3.270}$$

where

$$\begin{aligned} A'(0) &= \mathrm{div} V(0) \mathcal{I} - 2\epsilon(V(0)) \\ \epsilon(V(0)) &= \frac{1}{2}(DV(0) + {}^*DV(0)) \ . \end{aligned}$$

Proof. Under our assumptions the mappings $s \to \gamma(s, \cdot)$ and $s \to A_s(\cdot)$ are differentiable in $C^{k-1}(\overline{\Omega})$ and $C^{k-1}(\overline{\Omega}; \mathbb{R}^N)$, respectively. Then from (3.267) it follows that (3.270) holds. □

Proposition 3.24 *Let us suppose that $f \in L^1(I; H^1_{loc}(\mathbb{R}^N))$, then the mapping $s \to f(t, T_s(\cdot))$ is differentiable in the norm of the space $L^1(I; L^2(\Omega))$, the derivative at $s = 0$ is given by*

$$\frac{d}{ds}(f \circ T_s)_{|_{s=0}}(t, x) = \nabla_x f(t, x) \cdot V(0, x) \ . \tag{3.271}$$

The proof of Proposition 3.24 is similar to that of Proposition 2.32 given in Chap. 2, Sect. 2.14. Therefore it is omitted here. □

Lemma 3.25 *Let us assume that $\Box' y \in L^1(I; L^2(\Omega))$. The material derivative $\dot{y} \in L^\infty(I; H^1_0(\Omega))$, with $\partial \dot{y} / \partial t \in L^\infty(I; L^2(\Omega))$, of the solution y to (3.257)–(3.259) satisfies the equation*

$$\begin{aligned} &\Box\, \dot{y} = \mathrm{div}(f V) - \Box' y \quad \textit{in } Q, \\ &\dot{y} = 0 \quad \textit{on } \Sigma, \\ &\dot{y}(t_0) = 0, \quad \frac{\partial \dot{y}}{\partial t}(t_0) = 0 \quad \textit{in } \Omega \ . \end{aligned}$$

Proof. We shall show that

$$\frac{1}{s}(y^s - y) - \dot{y} \to 0 \quad \text{strongly in} \quad L^\infty(I; H_0^1(\Omega))$$
$$\frac{1}{s}\left(\frac{\partial y^s}{\partial t} - \frac{\partial y}{\partial t}\right) - \frac{\partial \dot{y}}{\partial t} \to 0 \quad \text{strongly in} \quad L^\infty(I; L^2(\Omega)) \quad \text{as } s \to 0 \ .$$

Let us assume that

$$z^s = \frac{1}{s}(y^s - y) - \dot{y},$$

then

$$\begin{aligned}
&\Box_s z^s = F_s \quad \text{in Q},\\
&z^s = 0 \quad \text{on } \Sigma,\\
&z^s(t_0) = 0 \ , \quad \frac{\partial z^s}{\partial t}(t_0) = 0 \quad \text{in } \Omega,
\end{aligned}$$

where

$$F_s = \frac{1}{s}(\gamma(s) f \circ T_s - f) - \operatorname{div}(fV) + \frac{1}{s}(\Box - \Box_s)y - \Box' y \ .$$

From Proposition 3.22 and 3.23 it follows that

$$F_s \to 0 \quad \text{strongly in} \quad L^1(I; L^2(\Omega)) \quad \text{as } s \to 0 \ .$$

Using a priori estimates we obtain

$$\|z^s\|_{L^\infty(I;H_0^1(\Omega))} + \left\|\frac{\partial z^s}{\partial t}\right\|_{L^\infty(I;L^2(\Omega))} \leqq C\|\Box_s z^s\|_{L^1(I;L^2(\Omega))} =$$
$$C\|F_s\|_{L^1(I;L^2(\Omega))} \to 0 \ .$$

This concludes the proof of Lemma 3.25. □

Now we are able to determine the form of the shape derivative $y' = y'(\Omega; V)$.

Lemma 3.26 *Let us suppose that*

$$\Box(\nabla y \cdot V) \in L^1(I; L^2(\Omega)),$$

then the shape derivative y' is given as the unique solution to the following hyperbolic problem

$$\begin{aligned}
&\Box y' = 0 \quad \text{in } Q,\\
&y' = -\frac{\partial y}{\partial n}\langle V, n\rangle_{\mathbb{R}^N} \quad \text{on } \Sigma,\\
&y'(t_0) = 0, \quad \frac{\partial y}{\partial t}(t_0) = 0 \quad \text{in } \Omega \ .
\end{aligned}$$

Proof. Since $y' = \dot{y} - \nabla y \cdot V$, then

$$\begin{aligned}
&\Box\, y' = \Box\, \dot{y} - \Box(\nabla y \cdot V) \in L^1(I; L^2(\Omega)) \\
&y'(t_0) = \frac{\partial y'}{\partial t}(t_0) = 0 \quad \text{in } \Omega, \\
&y' = \dot{y} - \nabla y \cdot V = -\frac{\partial y}{\partial n}\langle V, n\rangle_{\mathbb{R}^N} \quad \text{ on } \Sigma\ .
\end{aligned}$$

Thus

$$\begin{aligned}
&y' \in L^\infty(I; H^1(\Omega)) \\
&\frac{\partial y'}{\partial t} \in L^\infty(I; L^2(\Omega))\ .
\end{aligned}$$

Finally in order to show that $\Box\, y' = 0$ in Q, we have to consider under assumption that $s \in [0, \delta)$ the following integral identity

$$\int_{t_0}^{t_1} \int_{\Omega_s} \{y_s \frac{\partial \phi}{\partial t^2} + \nabla y_s \cdot \nabla \phi\} dx dt = \int_{t_0}^{t_1} \int_{\Omega_s} f \phi dx dt$$

$$\text{for all } \phi \in \mathcal{D}(\mathbb{R}^{N+1}) \text{ such that } \phi|_{Q_s} \in \mathcal{D}(Q_s)\ .$$

Differentiating with respect to s at $s = 0$ we obtain the integral identity

$$\int_{t_0}^{t_1} \int_{\Omega} \{y' \frac{\partial^2 \phi}{\partial t^2} + \nabla y' \cdot \nabla \phi\} dx dt = 0 \quad \forall \phi \in \mathcal{D}(Q),$$

as was to be shown.

$\square$

4. Shape Sensitivity Analysis of Variational Inequalities

This chapter is concerned with the shape sensitivity analysis of variational inequalities. First we shall examine a simple example.

Example 4.1 Let y denote the deflection of an elastic membrane of the reference configuration $\Omega \subset \mathbb{R}^2$ subjected to the pressure f. It is assumed that the deflection of the membrane is constrained by an obstacle ψ, therefore the following unilateral condition

$$y \geqq \psi \tag{4.1}$$

must be considered.
The deflection y is determined by minimizing the energy functional

$$I(\phi) = \frac{1}{2}\int_\Omega |\nabla\phi|^2 dx - \int_\Omega f\phi dx \tag{4.2}$$

subject to (4.1). This problem has a unique solution provided that, e.g. $y = 0$ on $\partial\Omega$. Therefore y minimizes the functional (4.2) over the convex closed set

$$K = \{\phi \in H_0^1(\Omega) | \phi(x) \geqq \psi(x) \quad \text{in } \Omega\} \ .$$

This set is non–empty for $\psi(\cdot) \in L^1_{loc}(\Omega)$, $\psi \leqq 0$ on $\partial\Omega$. The necessary and sufficient optimality conditions for the minimization problem under discussion have the form of a variational inequality. The problem considered can be formulated as follows:

Find an element $y = y(\Omega) \in K = K(\Omega)$ such that

$$\int_\Omega \nabla y \cdot \nabla(\phi - y)dx \geqq \int_\Omega f(\phi - y)dx \quad \forall\phi \in K;$$

here it is assumed that $f \in L^2(\Omega)$,

or in an equivalent way:

Find a solution to the following complementary problem

$$y - \psi \geqq 0\ , \quad -\Delta y - f \geqq 0 \quad \text{in } \Omega,$$
$$(y - \psi)(\Delta y + f) = 0 \quad \text{in } \Omega\ .$$

From the regularity results obtained by Brezis et al. (1968) (see also (Kinderlehrer et al. 1980)) it follows that

$$y \in H^2(\Omega) \cap H_0^1(\Omega) \ .$$

Let us denote by $\mathcal{Z} \subset \Omega$ the so-called coincidence set

$$\mathcal{Z} = \{x \in \Omega | y(x) = \psi(x)\} \ .$$

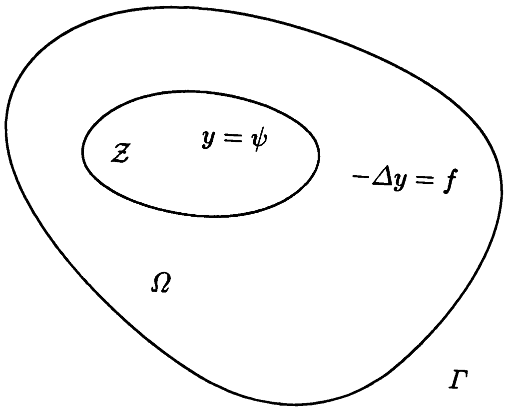

Fig. 4.1. Domain $\Omega \subset \mathbb{R}^2$

It is evident that $\phi + y \in K$ for any $\phi \in H_0^1(\Omega)$, $\phi \geqq 0$, therefore

$$\int_\Omega \nabla y \cdot \nabla \phi dx - \int_\Omega f \phi dx \geqq 0 \quad \forall \phi \geqq 0 \ .$$

Hence there exists a non-negative Radon measure μ given by

$$\begin{aligned} \int \phi d\mu &= \int_\Omega (-\Delta y - f)\phi dx \\ &= \int_\Omega (\nabla y \cdot \nabla \phi - f\phi) dx \quad \forall \phi \in C_0^\infty(\Omega) \end{aligned}$$

with the property that for $\mathcal{Z}$ compact

$$\mu(\Omega \setminus \mathcal{Z}) = 0 \ .$$

It should be noted that in general the set $\mathcal{Z}$ is not closed.

Let $\Pi : H^{-1}(\Omega) \ni f \to y \in H_0^1(\Omega)$ be the non–linear mapping associated with the unilateral problem under consideration.
It can be shown (Mignot 1976; Haraux 1977) that the mapping Π is directionally differentiable and the differential of Π in a direction $h \in H^{-1}(\Omega)$, denoted by $\Pi'(h)$, minimizes the quadratic functional

$$J(\phi) = \frac{1}{2}\int_\Omega |\nabla \phi|^2 dx - \int_\Omega h\phi dx$$

over the convex cone

$$S = \{\phi \in H_0^1(\Omega) | \phi \geqq 0 \quad \text{q.e. on } \mathcal{Z}, \ \int_\Omega \phi d\mu = 0\} \ .$$

Here q.e. means quasi–everywhere, i.e. everywhere possibly except for a set of capacity zero. The definition of capacity is given in Sect. 4.3 by the formula (4.54).

Making use of the approach relying on the material derivative method it can be shown (Sokołowski et al. 1985a; 1987a), that the domain derivative y' of the solution y to the unilateral problem is the unique minimizer of the functional

$$j(\phi) = \frac{1}{2}\int_\Omega |\nabla \phi|^2 dx$$

over the cone

$$S_v(\Omega) = \{\phi \in H^1(\Omega) | \phi = -v_n \frac{\partial y}{\partial n} \text{ on } \Gamma, \ \phi \geqq 0 \text{ q.e. on } \mathcal{Z}, \int \phi d\mu = 0\}$$

provided that the obstacle ψ is sufficiently smooth (see Sect. 4.3 for the details).
Finally let us observe that the solution y to the unilateral problem can be characterized as the metric projection in the Sobolev space $H_0^1(\Omega)$ of an element $F \in H_0^1(\Omega)$ onto the convex set $K \subset H_0^1(\Omega)$

$$y = P_K F : \quad \int_\Omega |\nabla(y - F)|^2 dx = \min_{u \in K} \int_\Omega |\nabla(u - F)|^2 dx,$$

where $F = \Delta^{-1} f$, i.e.

$$F \in H_0^1(\Omega) : \quad \int_\Omega \nabla F \cdot \nabla \phi dx = \int_\Omega f \phi dx \quad \forall \phi \in H_0^1(\Omega) \ .$$

It should be noted that we use the same symbol for the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ and for the scalar product in $L^2(\Omega)$.
We shall briefly outline the main results on the differential stability of the metric projection in a Hilbert space onto a closed and convex subset. They will be used in the shape sensitivity analysis of variational inequalities. First we shall discuss some examples.

Example 4.2 Let us consider the projection mapping in $\mathbb{R}$ onto the set $K = [0,1]$. We have

$$\forall x \in \mathbb{R}: \quad P_K(x) = \begin{cases} 1 & x > 1 \\ x & 0 \leqq x \leqq 1 \\ 0, & x < 0 \ . \end{cases}$$

It is evident that the mapping $x \to P_K(x)$ is differentiable everywhere except at $x = 0$ and $x = 1$. For $h = \pm 1$ and for $\varepsilon > 0$, ε small enough, it can be shown that at the point $y = 0$ we have

$$P_K(y + \varepsilon h) = P_K y + \varepsilon h^+ \ ,$$

where

$$h^+ = \begin{cases} h, & h \geqq 0 \\ 0, & h < 0 \ . \end{cases}$$

Thus for $\varepsilon > 0$, ε small enough, the following equality holds

$$[P_K(y + \varepsilon h) - P_K(y)]/\varepsilon = h^+ = \lim_{\varepsilon \downarrow 0} [P_K(y + \varepsilon h) - P_K(y)]/\varepsilon \ .$$

Hence at $y = 0$ we have

$$P_K(y + \varepsilon h) = P_K(y) + \varepsilon Q(h) + o(\varepsilon),$$

where the mapping $Q(\cdot) : \mathbb{R} \to \mathbb{R}$ is defined by $Q(h) = h^+$ for all $h \in \mathbb{R}$.
The mapping $Q(\cdot)$ is called the conical differential of the projection $P_K(\cdot)$ at $y = 0$, this name will be used throughout.
Let us recall a way in which a variational inequality can be used to characterize the mapping $P_K(\cdot)$. Since for a given $x \in \mathbb{R}$ we have

$$(P_K(x) - x)^2 \leqq (v - x)^2 \quad \forall v \in K,$$

then by the standard reasoning it follows that the element $P_K(x)$ is given as the unique solution to the following variational inequality:

$$K \ni P_K(x) : \quad (P_K(x) - x)(v - P_K(x)) \geqq 0 \quad \forall v \in K \ .$$

Example 4.3 Let $K \subset \mathbb{R}^N$ be a compact, convex set with non-empty interior and ∂K of class C^2. It is assumed that a convex function $\psi \in C^2(\mathbb{R}^N)$ is given such that $\psi(\overline{x}) \leqq 0$ for some $\overline{x} \in \mathbb{R}^N$ and

$$\partial K = \{x \in \mathbb{R}^N | \psi(x) = 0\} \ .$$

It can be shown, e.g. using the results obtained by Malanowski (1985), that the projection in $\mathbb{R}^N$ onto the set K is directionally differentiable, i.e. for given elements $f, h \in \mathbb{R}^N$ and for $\varepsilon > 0$, ε small enough, we have

$$P_K(f + \varepsilon h) = P_K(f) + \varepsilon Q(h) + o(\varepsilon) \ ,$$

where the element $Q = Q(h) \in \mathbb{R}^N$ is given as the unique solution to the following variational inequality

$$\begin{aligned} &Q \in S_K(f) \ : \\ &(AQ, x - Q)_{\mathbb{R}^N} \geqq (h, x - Q)_{\mathbb{R}^N} \quad \forall x \in S_K(f) \ . \end{aligned}$$

In this inequality the following notation is used:

$$\begin{aligned} &A = I + \lambda D^2\psi(u) \ , \\ &u = P_K(f) \ , \\ &\lambda = \begin{cases} \|f - u\|_{\mathbb{R}^N} / \|D\psi(u)\|_{\mathbb{R}^N} & f \notin K; \\ 0 & \text{otherwise} \end{cases} \\ &S_K(f) = \begin{cases} x \in \mathbb{R}^N : \ D\psi(u).x \leqq 0 \text{ and } \lambda D\psi(u).x = 0 & \text{if } f \notin \text{int}K \\ \mathbb{R}^N & \text{if } f \in \text{int}K \ . \end{cases} \end{aligned}$$

In the case under consideration $Q(\cdot) : \mathbb{R}^N \to \mathbb{R}^N$ is the metric projection onto $S_K(f)$ with respect to a scalar product in $\mathbb{R}^N$ depending on the Lagrange multiplier λ.

4.1. Differential stability of the metric projection in Hilbert spaces

We shall briefly outline the main results on the directional differentiability of the projection in a Hilbert space onto a convex, closed subset.
Let H be a separable Hilbert space, and let $K \subset H$ a convex, closed subset. Moreover, it is assumed that there is given a bilinear form

$$a(\cdot, \cdot) : H \times H \to \mathbb{R}$$

which is coercive and continuous, i.e.

$$\begin{aligned} &a(v, v) \geq \alpha \|v\|_H^2 \quad \forall v \in H \\ &|a(v, z)| \leq M \|v\|_H \|z\|_H \quad \forall v, z \in H, \end{aligned} \tag{4.3}$$

where $\alpha > 0$ and M are given constants.

For the sake of simplicity, it is assumed that the bilinear form is symmetric

$$a(v,z) = a(z,v) \quad \forall v,z \in H \ . \tag{4.4}$$

Let $P_K(f)$ stand for the a–projection in H of an element $f \in H$ onto the convex set K. The element $y = P_K(f)$ minimizes the quadratic functional

$$I(v) = \frac{1}{2}a(v-f, v-f) \tag{4.5}$$

over the set K. Therefore to characterize y the following variational problem can be stated:

$$\begin{aligned} &\text{Find } y \in K \text{ such that} \\ &a(y-f, v-f) \geqq 0 \quad \forall v \in K \ . \end{aligned} \tag{4.6}$$

It can be shown that the mapping $P_K(\cdot) : H \to K \subset H$ is Lipschitz continuous

$$\|P_K(f_1) - P_k(f_2)\|_H \leqq \frac{M}{\alpha}\|f_1 - f_2\|_H \quad \forall f_1, f_2 \in H \ . \tag{4.7}$$

Therefore by a generalization of Rademacher's theorem (Mignot 1976), there exists a dense subset $\Upsilon \subset H$ on which $P_K(\cdot)$ is Gateaux differentiable, i.e. for any $f \in \Upsilon$ we can find a linear continuous mapping $P'_K(\cdot) = P'_K(f,\cdot) : H \to H$ such that

$$\forall h \in H : P_K(f + \varepsilon h) = P_K(f) + \varepsilon P'_K(h) + o(\varepsilon) \ , \tag{4.8}$$

where $\|o(\varepsilon)\|_H/\varepsilon \to 0$ as $\varepsilon \downarrow 0$.
Below we shall use the concept of the conical differential of the projection $P_K(\cdot)$.

Definition 4.4 The mapping $P_K(\cdot)$ is conically differentiable at $f \in H$, if there exists a continuous mapping

$$\begin{aligned} &Q(\cdot) : H \to H, \\ &Q(\alpha h) = \alpha Q(h) \quad \text{for all } \alpha > 0 \text{ and for all } h \in H \end{aligned} \tag{4.9}$$

such that for $\varepsilon > 0$, ε small enough,

$$\forall h \in H : P_K(f + \varepsilon h) = P_K(f) + \varepsilon Q(h) + o(\varepsilon), \tag{4.10}$$

where $\|o(\varepsilon)\|_H/\varepsilon \to 0$ as $\varepsilon \to 0$, uniformly with respect to h on compact subsets of H.

In order to determine the form of the mapping $Q(\cdot)$ defined by (4.10) for a class of sets K we need the following notation.
For a given element $y \in K$, $C_K(y)$ denotes the radial cone

$$C_K(y) = \{\phi \in H | \exists \varepsilon > 0 \text{ such that } y + \varepsilon\phi \in K\} \ . \tag{4.11}$$

In general the convex cone $C_K(y)$ is not closed, we denote by $T_K(y) = \text{cl}(C_K(y))$ its closure in H, $T_K(y)$ is the tangent cone. $N_K(y)$ denotes the normal cone to K at $y \in K$ of the form

$$N_K(y) = \{\phi \in H | a(\phi, z - y) \leqq 0 \quad \forall z \in K\} \ . \tag{4.12}$$

The normal cone is convex and closed. Finally we denote by $S_K(f) \subset H$ the convex and closed cone of the form

$$S_K(f) = \{v \in T_K(y) | a(f - y, v) = 0\} \ , \tag{4.13}$$

where $y = P_K(f)$ with $f \in H$. Let us assume that there is given a continuous mapping

$$f(\cdot) : [0, \delta) \to H$$

which is right differentiable at 0, i.e. there exists an element $f'(0) \in H$ such that

$$\lim_{s \downarrow 0} \|(f(s) - f(0))/s - f'(0)\|_H = 0 \ .$$

The following notation is used

$$\begin{aligned} y(s) &= P_K(f(s)) \quad s \in [0, \delta), \\ \varrho(s) &= (y(s) - y(0))/s \ . \end{aligned}$$

It evidently follows from (4.7) that

$$\|\varrho(s))\|_H \leqq M/\alpha \quad \forall s \in (0, \delta) \ .$$

It can be shown, the reader is referred to the proof of Theorem 4.6, that for every weak limit point ϱ of the function $s \to \varrho(s)$ at $s = 0^+$ the following condition is satisfied

$$\varrho \in S_K(f(0)) \ .$$

The orthogonal subspace in H to the element $f - y \in H$ is denoted by

$$[f - y]^\perp = \{\phi \in H | a(y - f, \phi) = 0\} \ .$$

We denote $(\varphi, v) = a(\varphi, v)$ for all $\varphi, v \in H$, $\|\varphi\|_H = a(\varphi, \varphi)^{\frac{1}{2}}$.

Proposition 4.5 *Let $K \subset H$ be a closed, convex subset of the Hilbert space H. Then for any $f \in H$, and any element $w \in C_K(u) \cap [f - u]^\perp$, $u = P_K(f)$ we have*

$$P_K(f + tw) = P_K(f) + tw \ . \tag{4.14}$$

Therefore

$$P_K(f + tw) = P_K(f) + tw + o(t) \quad \forall w \in \text{cl}(C_K(u) \cap [f - u]^\perp) \ . \tag{4.15}$$

Proof. Since the projection $P_K(\cdot)$ is Lipschitz continuous, then by means of the density argument one can see from (4.14) that (4.15) holds. Before proceeding further with the proof we have to show that (4.14) is met.
Since $w \in C_K(u)$, then

$$u + tw \in K$$

for $t > 0$, t small enough.
On the other hand for all $v \in K$

$$\begin{aligned}&(f + tw - (u + tw), v - (f + tw)) = \\ &(f - u, v - u) - t(f - u, w) = \\ &(f - u, v - u) \leqq 0 \ .\end{aligned}$$

Hence for $t > 0$, t small enough,

$$P_K(f + tw) = u + tw = P_K f + tw \quad \forall w \in C_K(u) \cap [f - u]^{\perp} \ .$$

□

Theorem 4.6 *Let $f \in H$ be a given element, and let $u = P_K f$.*
If for any element $w \in S_K(u)$,

$$P_K(f + tw) = P_K(f) + tw + o(t) \ ,$$

then for $t > 0$, t small enough,

$$\forall h \in H : P_K(f + th) = P_K f + tP_S h + o(t), \tag{4.16}$$

where $P_S(\cdot)$ is the metric projection in H onto $S_K(u)$.

Proof. (Haraux 1977) Let $z \in H$ be given, then

$$\|P_K(f + tz) - P_K f\|_H \leqq t\|z\|_H \ .$$

By the definition of $C_K(u)$, if

$$\frac{1}{t}(P_K(f + tz) - P_K f) \rightharpoonup \xi \quad \text{weakly in } H \text{ as } t \downarrow 0,$$

then $\xi \in T_K(u) = \text{cl}(C_K(u))$. Furthermore, we shall show that $\xi \in [f - u]^{\perp}$. It should be remarked that for any element $g \in H$ we have

$$(g - P_K g, v - P_K g) \leqq 0 \quad \forall v \in K \ .$$

Let us assume that

$$g = f + tz \ , \quad v = P_K f = u \ ,$$

then

$$P_K g = P_K f + t\xi(t) \ .$$

Hence

$$(f + tz - (P_K f + t\xi(t)), u - P_K f + t\xi(t))) \leqq 0$$

and

$$t^2(\xi(t), \xi(t) - z) \leqq t(f - P_K f, \xi(t)) = (f - P_K f, P_K(f + tz) - P_K f) \leqq 0 \ .$$

As a result, the following inequalities are obtained:

$$0 \geq \limsup_{t \downarrow 0} (\xi(t), \xi(t) - z) \geq \liminf_{t \downarrow 0} (\xi(t), \xi(t) - z) \geq (\xi, \xi - z) \ .$$

Moreover

$$t(\xi(t), \xi(t) - z) \leqq (f - u, \xi(t)) \leqq 0 \ .$$

Let $\xi(t) \rightharpoonup \xi \in S$ weakly in H as $t \downarrow 0$, then $\xi \in T_K(u)$ and

$$(f - u, \xi) = 0 \ ,$$

i.e. $\xi \in S = S_K(u)$, the latter remains valid for any weak limit of the function $t \to \xi(t)$ at $t = 0^+$. Let $f \in H$ be fixed, $u = P_K f$, $S = S_K(u)$. Then for any element $w_1 \in S$,

$$\limsup_{t \downarrow 0} (\xi(t), w_1) \leqq 0 \ .$$

For a given element $z \in H$, $S^\star = \{v \in H | (v, \varphi) \leqq 0 \quad \forall \varphi \in S\}$,

$$z = w + w_1 \quad \text{with } w \in S \text{ and } w_1 \in S^\star \ .$$

For the projection $P_K(\cdot)$ the following inequality holds

$$(P_K h - P_K g, h - g) \geq \|P_K h - P_K g\|_H^2 \quad \forall h, g \in H \ .$$

Therefore

$$\begin{aligned} &\|P_K(f + tz) - P_K(f + tw)\|_H^2 \leqq t(w_1, P_K(f + tz) - P_K(f + tw)) \\ &= t^2(w, w_1) + t^2(w_1, \xi(t)) = t^2(w_1, \xi(t)) \ . \end{aligned}$$

This result makes it possible to show that

$$\begin{aligned} &\limsup_{t \downarrow 0} \left\| \frac{P_K(f + tz) - P_K(f + tw)}{t} \right\|_H = \\ &\limsup_{t \downarrow 0} (w_1, \xi(t)) \leqq 0 \ . \end{aligned}$$

Hence

$$P_K(f + tz) = P_K(f + tw) + o(t) =$$

Proposition 4.5 implies

$$\begin{aligned} &= P_K f + tP_S z + o(t) \\ &= P_K f + tw + o(t) \ . \end{aligned}$$

□

For some specific convex sets K, the explicit form of the tangent cone $T_K(u) = \mathrm{cl}(C_K(u))$ will be determined.

Example 4.7 (Rao et al. 1990) Let us consider the space $H = H^2(\Omega) \cap H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^N, N \leqq 3$, with the scalar product

$$(y, z)_H = \int_\Omega \Delta y \Delta z dx \quad \forall y, z \in H \ .$$

Let us denote by K the following convex cone

$$K = \{ \varphi \in H | \varphi \geq \psi \quad \text{in } \Omega \},$$

where $\psi \in H$ is given.

Proposition 4.8 *Let $u \in K$ be and $\Xi = \{ x \in \Omega | u(x) = \psi(x) \}$ be compact. Then*

$$T_K(u) = \{ \varphi \in H | \varphi \geqq 0 \quad on\ \Xi \} \ .$$

Proof. For $N \leqq 3$, by the Sobolev embedding theorem, it follows that $H_0^2(\Omega) \hookrightarrow C(\overline{\Omega})$.
It is evident that

$$T_K(u) \subset \{ \varphi \in H_0^2(\Omega) \mid \varphi(x) \geqq 0 \quad \text{on } \Xi \} \ .$$

To proceed further with our proof, it is therefore sufficient to show that any element $\phi \in H_0^2(\Omega)$, $\phi(\cdot) \geqq 0$ on Ξ, belongs to $T_K(u)$.
Since Ξ is compact, then there exists $0 \leq \eta \in C_0^\infty(\Omega)$, $\eta \equiv 1$ on Ξ. From the Sobolev embedding theorem it follows that $u, \psi, \phi \in C(\overline{\Omega})$. Hence for any $\varepsilon > 0$ there exists $t > 0$ such that

$$t(\phi + \varepsilon\eta) + u - \psi \geqq 0 \quad \text{in } \Omega \ .$$

Thus

$$\phi + \varepsilon\eta \in C_K(u) \quad \varepsilon > 0$$

and

$$\phi + \varepsilon\eta \to \phi \quad \text{in } H_0^2(\Omega) \text{ strongly as } \varepsilon \downarrow 0 \ ;$$

hence $\phi \in \mathrm{cl}(C_K(u)) = T_K(u)$. □

Example 4.9 Let us introduce the following notation

$$H = L^2(\Omega), \quad \Omega \subset \mathbb{R}^N \text{ is a given domain,}$$
$$K = \{ \phi \in L^2(\Omega) | \phi(x) \geqq 0 \text{ for a. e. } x \in \Omega \} \ .$$

Let

$$f \in L^2(\Omega) ,$$

then

$$u = P_K f = f^+ = \max\{f, 0\}$$

and

$$\Xi = \{x \in \Omega | f(x) = 0\} .$$

Therefore

$$T_K(u) = \{\phi \in L^2(\Omega) | \phi(x) \geqq 0 \quad \text{for a.e. } x \in \Xi\} .$$

Example 4.10 (Mignot 1976)
Let

$$H = H_0^1(\Omega) ,$$
$$K = \{\phi \in H_0^1(\Omega) | \phi(x) \geqq 0 \text{ for a.e. } x \in \Omega\} .$$

For any element $f \in H_0^1(\Omega)$ we have

$$u = P_K f \in K : \quad \int_\Omega \nabla(u - f) \cdot \nabla(\phi - u) dx \geqq 0 \quad \forall \phi \in K .$$

Then

$$T_K(u) = \{\phi \in H_0^1(\Omega) | \phi(x) \geqq 0 \text{ for q.e. } \ x \in \Xi\} , \tag{4.27}$$

where $\Xi = \{x \in \Omega | u(x) = 0\}$.

Definition 4.11 A convex and closed set $K \subset H$ is called polyhedric, if for all $f \in H$ the following condition is satisfied

$$S_K(f) = \mathrm{cl}(\{v \in C_K(y) | a(f - y, v) = 0\}), \tag{4.28}$$

where $y = P_K f$, and the cone $S_K(f)$ is defined by (4.13).

It should be noted that the inclusion

$$\mathrm{cl}(\{v \in C_K(y) | a(f - y, v) = 0\}) \subset S_K(f)$$

holds for any element $f \in H$.
The sets K in Examples 4.2, 4.9 and 4.10 are polyhedric, however in general the sets K in Examples 4.3, 4.7 fail to be polyhedric.
For any polyhedric set $K \subset H$ the form of the conical differential of the metric projection onto K has been derived by Mignot (1976), see also (Haraux 1977).

Corollary 4.12 *Let $f(\cdot) : [0, \delta) \to H$ be right–differentiable in the norm of H at $s = 0$.*

It is supposed that for the convex and closed set $K \subset H$ the following condition is satisfied

$$T_K(f) \cap [f-g]^\perp = \mathrm{cl}(C_K(f) \cap [f-g]^\perp),$$

where $f = f(0)$, $g = P_K(f(0))$.
Then for $s > 0$, s small enough,

$$P_K(f(s)) = P_K(f(0)) + sP_{S_K(f(0))}(f'(0)) + o(s), \tag{4.29}$$

where $\|o(s)\|_H/s \to 0$ as $s \to 0$. □

In particular, it follows from (4.16) that the projection $P_K(\cdot)$ is conically differentiable at $f = f(0) \in H$. Moreover we have

$$Q(h) = P_{S_K(f)}(h) \quad \forall h \in H \ . \tag{4.30}$$

It should be noted that in general

$$Q(h) \neq -Q(-h) \ .$$

Theorem 4.6 remains valid for a non–symmetric, coercive bilinear form $a(\cdot,\cdot)$ provided that the Hilbert space H is the so–called Dirichlet space (Mignot 1976). It should be emphasized that from Stampacchia's theorem, see e.g. (Kinderlehrer et al. 1980), it follows that in the non–symmetric case there exists the unique solution to the variational inequality (4.6) – the reader is referred to Chap. 2 for applications of the theorem to elliptic boundary value problems.
In this chapter we present an example of the set $K \subset L^2(\Omega, \mathbb{R}^N)$ that is not polyhedric, nevertheless it is possible to derive the form of the conical differential of the metric projection onto K (Sokołowski 1985a).

Example 4.13 Let us consider the metric projection in the space $H = L^2(\Omega; \mathbb{R}^N)$ onto the set

$$K = \{\ v \in L^2(\Omega; \mathbb{R}^N) | \frac{1}{2}\sum_{i=1}^N a_i v_i^2(\xi) \leqq 1 \text{ for a.e. } \xi \in \Omega\}, \tag{4.31}$$

where $a_i > 0$, $i = 1, \dots, N$, are given constants. Let

$$\psi(x) = \frac{1}{2}\sum_{i=1}^N a_i x_i^2 - 1 \quad x \in \mathbb{R}^N$$

and

$$U = \{x \in \mathbb{R}^N | \psi(x) \leqq 0\} \ .$$

Let $P_U(\cdot) : \mathbb{R}^N \to \mathbb{R}^N$ denote the metric projection in $\mathbb{R}^N$ onto U, $f(\cdot) \in L^\infty(\Omega; \mathbb{R}^N)$ be a given element, and $u(\xi)$, $\xi \in \Omega$, be the projection of $f(\xi) \in \mathbb{R}^N$ onto U. We have

$$u(\xi) = P_U(f(\xi)) \ .$$

Let $\lambda(\xi)$ be the associated Lagrange multiplier

$$\lambda(\xi) = \begin{cases} \|f(\xi) - u(\xi)\|_{\mathbb{R}^N} / \|D\psi(u(\xi))\|_{\mathbb{R}^N} & \text{for } f(\xi) \notin U \\ 0 & \text{for } f(\xi) \in U \end{cases}$$

and introduce the symmetric matrix

$$A(\xi) = [(1 + \lambda(\xi)a_i)\delta_{ij}]_{N \times N},$$

where $\xi \in \Omega$, $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$.
In the particular case of the set $U \subset \mathbb{R}^N$, the cone (4.23) is of the form

$$S_U(f(\xi)) = \{x \in \mathbb{R}^N | D\psi(u(\xi)) \cdot x \leqq 0 \text{ and } \lambda(\xi) D\psi(u(\xi)) \cdot x = 0\} \ .$$

It can be easily shown, in the case under consideration, that the condition (4.28) is not satisfied. It is evident that the projection $P_U(\cdot)$ is differentiable at $f(\xi)$ provided that the associated Lagrange multiplier $\lambda(\xi) \neq 0$. In general the right–derivative $q = q(\xi) \in \mathbb{R}^N$ of $P_U(\cdot)$ at $f(\xi) \in \mathbb{R}^N$, in any direction $h \in \mathbb{R}^N$, is given as the unique solution to the following variational inequality

$$\begin{aligned} & q \in S_U(f(\xi)) : \\ & \langle A(\xi) \cdot q - h, v - q \rangle_{\mathbb{R}^N} \geqq 0 \quad \forall v \in S_U(f(\xi)) \ . \end{aligned}$$

Therefore the projection $P_K(\cdot)$ in $L^2(\Omega; \mathbb{R}^N)$ onto the set (4.31) is right–differentiable at f in any direction $h \in L^2(\Omega; \mathbb{R}^N)$.
The right–derivative $q(\cdot) \in L^2(\Omega; \mathbb{R}^N)$ is obtained as the unique solution to the variational inequality

$$\begin{aligned} q(\cdot) \in S_K(f) = \{v(\cdot) \in L^2(\Omega; \mathbb{R}^N) | D\psi(u(\xi)) \cdot v(\xi) \leqq 0 \text{ and} \\ \lambda(\xi) D\psi(u(\xi)) \cdot v(\xi) = 0 \quad \text{a.e. in } \Omega\} \end{aligned}$$

$$\int_\Omega (A(\xi) \cdot q(\xi) - h(\xi), v(\xi) - q(\xi))_{\mathbb{R}^N} d\xi \geqq 0 \quad \forall v(\cdot) \in S_K(f) \ .$$

□

One can easily check that $q \neq P_{S_K(f)}(h)$, hence the set K in Example 4.13 is not polyhedric. This indicates that Theorem 4.6 in its present form cannot be extended to convex sets for which (4.28) is not satisfied.

In the following sections of this chapter we shall examine the boundary–values problems which can be formulated as variational inequalities, with convex sets defined by unilateral conditions prescribed in a domain $\Omega \subset \mathbb{R}^N$ or on the boundary $\Gamma = \partial\Omega$. Sect. 4.3 deals with obstacle problems, defined in the Sobolev space $H^1(\Omega)$, with non–symmetric bilinear forms. The domain derivatives of solutions to the obstacle problems are obtained. Furthermore, we provide a result on the shape differentiability of the capacity, and we consider the shape

sensitivity analysis of the free boundaries (i.e. the boundaries of the coincidence sets) associated with the obstacle problems under consideration. In Sect. 4.4 the Signorini problems with unilateral conditions on the boundaries are investigated. In Sect. 4.5 we introduce the variational inequalities of the second kind corresponding to the minimization problems with non–differentiable integral functionals. Using the classical approach of Cea et al. 1971, we introduce the saddle–point formulations of the variational inequalities. The differential stability of solutions for such formulations is investigated following (Sokołowski 1988d). The form of the shape and material derivatives of the solutions to the variational inequalities is obtained.
In Sect. 4.6 and 4.7 the unilateral problems in elasticity are considered in two cases:

(i) frictionless contact problems
(ii) contact problems with given friction.

We establish the abstract results on the differential stability of solutions to a class of variational inequalities. In particular, in the case (i) we shall show, using the abstract results, that the following convex set is polyhedric

$$K = \{ \ \phi \in H^1(\Omega;\mathbb{R}^3) | \phi|_{\Gamma_0} = 0, \quad \phi \cdot n \geqq 0 \quad \text{on } \Gamma_2 \} \ .$$

In the case (ii) we use the saddle point formulations of the variational inequalities of the second kind for the purposes of the sensitivity analysis.
In Sect. 4.8 elasto–plastic torsion problems are considered. Finally in Sect. 4.9 we provide the results on the shape sensitivity analysis of elasto–visco–plastic problems.
We suppose that $\Omega \subset \mathbb{R}^N$, $N = 2$ or 3, is a bounded domain with the sufficiently smooth boundary $\Gamma = \partial\Omega$. For a given vector field $V(\cdot,\cdot) \in C(0,\varepsilon;\mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N))$, $k \geqq 1$, the family $\{\Omega_t\} \subset \mathbb{R}^N$, $t \in [0,\delta)$, $0 < \delta < \varepsilon$, is defined by $\Omega_t = T_t(V)(\Omega)$.

It should be remarked that for a solution $u_t \in K(\Omega_t)$ of a specific unilateral problem, e.g. the Signorini problem in elasticity, defined in the domain $\Omega_t = T_t(V)(\Omega)$, $t \in (0,\delta)$, in general

$$u^t \equiv u_t \circ T_t \notin K(\Omega),$$

therefore we cannot obtain a unilateral problem for $u_t \circ T_t$, applying the change of variables $T_t(V)(\cdot)$.

We can circumvent this difficulty by the use of an appropriate transformation $\mathcal{F}(\cdot,\cdot)$ with the following property

$$z^t \equiv \mathcal{F}(t, u_t \circ T_t) \in K(\Omega) \text{ if and only if } u_t \in K(\Omega_t) \ .$$

The element $z^t \in K(\Omega)$ is determined as the solution to an auxiliary variational inequality defined in Ω. We shall prove that the right–derivative

$$\dot{z} = \lim_{t \downarrow 0}(z^t - z^0)/t$$

exists in the appropriate Sobolev space; in general this is equivalent to the existence of the strong material derivative

$$\dot{u} = \lim_{t \downarrow 0}(u^t - u^0)/t \ .$$

Finally in the scalar case, making use of the relation

$$u' = \dot{u} - \nabla u \cdot V,$$

it is possible to show that the shape derivative exists.

In the case of a system of elliptic equations, we use the relation

$$u' = \dot{u} - Du \cdot V \ ,$$

to derive the form of the shape derivative $u' = u'(\Omega; V)$ for the variational inequality under consideration.

First we present some results to be used below for the differential stability analysis of solutions to variational inequalities. They constitute an extension of the implicit function theorem to the case of variational inequalities.

4.2. Sensitivity analysis of variational inequalities in Hilbert spaces

Let $K \subset H$ be a convex and closed subset of a Hilbert space H, and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between H' and H, where H' denotes the dual of H.

We shall consider the following family of variational inequalities depending on a parameter $t \in [0, \delta)$, $\delta > 0$,

$$y_t \in K : \qquad a_t(y_t, \varphi - y_t) \geqq \langle f_t, \varphi - y_t \rangle \qquad \forall \varphi \in K \ . \tag{4.32}$$

Moreover, let $y_t = \mathcal{P}_t(f_t)$ be a solution to (4.32).

Theorem 4.14 *Let us assume that*

(i) the bilinear form $a_t(\cdot, \cdot) : H \times H \to \mathbb{R}$ is coercive and continuous uniformly with respect to $t \in [0, \delta)$. Let $\mathcal{A}_t \in \mathcal{L}(H; H')$ be the linear operator defined as follows $a_t(\phi, \varphi) = \langle \mathcal{A}_t \phi, \varphi \rangle$ $\forall \phi, \varphi \in H$; it is supposed that there exists $\mathcal{A}' \in \mathcal{L}(H; H')$ such that

$$\mathcal{A}_t = \mathcal{A}_0 + t\mathcal{A}' + o(t) \quad \text{in } \mathcal{L}(H; H') \ . \tag{4.33}$$

(ii) for $t > 0$, t small enough, the following equality hols

$$f_t = f_0 + tf' + o(t) \quad in\ H', \tag{4.34}$$

where $f_t, f_0, f' \in H'$

(iii) $K \subset H$ *is convex and closed, and for the solutions to the variational inequality*

$$\Pi f = \mathcal{P}_0(f) \in K : \quad a_0(\Pi f, \varphi - \Pi f) \geqq \langle f, \varphi - \Pi f\rangle \quad \forall \varphi \in K \tag{4.35}$$

the following differential stability result holds

$$\forall h \in H : \qquad \Pi(f_0 + \varepsilon h) = \Pi f_0 + \varepsilon \Pi' h + o(\varepsilon) \quad in\ H \tag{4.36}$$

for $\varepsilon > 0$, ε *small enough, where the mapping* $\Pi' : H' \to H$ *is continuous and positively homogeneous.*

Then the solutions to the variational inequality (4.32) are right-differentiable with respect to t *at* $t = 0$, *i.e. for* $t > 0$, t *small enough,*

$$y_t = y_0 + ty' + o(t) \quad in\ H,$$

where

$$y' = \Pi'(f' - \mathcal{A}'y_0) \ .$$

Proof. The usual argument leads to

$$\begin{aligned} a_0(y_0 - y_t, y_0 - y_t) \leqq &\langle f_t - f_0, y_0 - y_t\rangle + \\ & a_0(y_t, y_t - y_0) - a_t(y_t, y_t - y_0) \ . \end{aligned} \tag{4.38}$$

Using (4.12) we obtain

$$\begin{aligned} \alpha\|y_0 - y_t\|_H^2 \leqq &\|f_t - f_0\|_{H'}\|y_0 - y_t\|_H + \\ & |a_0(y_t, y_t - y_0) - a_t(y_t, y_t - y_0)| \ . \end{aligned} \tag{4.39}$$

From the assumption (i) it follows that there exists a constant C such that for $t > 0$, t small enough,

$$|a_0(y_t, y_t - y_0) - a_t(y_t, y_t - y_0)| \leqq Ct\|y_t\|_H\|y_t - y_0\|_H \ . \tag{4.40}$$

Therefore from (4.39) and (4.40) we have that

$$\|y_t - y_0\|_H \leqq Ct \quad t \in [0, \delta) \ . \tag{4.41}$$

With simple calculations it is possible to show that for the element $y_t \in H$ the following variational inequality holds

$$\begin{aligned} & y_t \in K : \\ & a_0(y_t, \varphi - y_t) \geq \langle f_0 + t(f' - \mathcal{A}'y_0), \varphi - y_t\rangle + \langle \varrho(t), \varphi - y_t\rangle \quad \forall \varphi \in K, \end{aligned} \tag{4.42}$$

where

$$\langle \varrho(t), \varphi \rangle = \sum_{i=1}^{3} \langle \varrho_i(t), \varphi \rangle \quad \forall \varphi \in H,$$

$$\varrho_1(t) = f_t - f_0 - tf', \tag{4.43}$$

$$\langle \varrho_2(t), \varphi \rangle = a_0(y_o, \varphi) - a_t(y_0, \varphi) + t\langle \mathcal{A}' y_0, \varphi \rangle \quad \forall \varphi \in H, \tag{4.44}$$

$$\langle \varrho_3(t), \varphi \rangle = a_0(y_t - y_o, \varphi) - a_t(y_t - y_0, \varphi) \quad \forall \varphi \in H \ . \tag{4.45}$$

Assumptions (i) and (ii) imply $\|\varrho_i(t)\|_{H'}/t \to 0$ as $t \downarrow 0$, $i = 1, 2$. Furthermore, from (4.33) it follows that for $t > 0$, t small enough,

$$|\langle \varrho_3(t), \varphi \rangle| \leqq r(t) \|y_t - y_0\|_H \|\varphi\|_H \quad \forall \varphi \in H, \tag{4.46}$$

where $r(t) \downarrow 0$ as $t \downarrow 0$. Therefore taking into account (4.41) we obtain

$$\|\varrho_3(t)\|_{H'}/t \to 0 \quad \text{as } t \downarrow 0 \ . \tag{4.47}$$

Thus

$$\|\varrho(t)\|_{H'}/t \to 0 \quad \text{as } t \downarrow 0 \tag{4.48}$$

and

$$\begin{aligned} y_t &= \mathcal{P}_0(f_0 + t(f' - \mathcal{A}' y_0) + \varrho(t)) \\ &= \mathcal{P}_0(f_0 + t(f' - \mathcal{A}' y_0)) + o(t) \\ &= \mathcal{P}_0(f_0) + t\Pi'(f' - \mathcal{A}' y_0) + o(t), \end{aligned} \tag{4.49}$$

where $\|o(t)\|_H/t \to 0$ as $t \downarrow 0$. □

We shall present several examples of convex sets in the Sobolev spaces for which the assumption (iii) of Theorem 4.14 is satisfied.

4.3. The obstacle problem in $H^1(\Omega)$

We assume that there are given elements $g \in H^{1/2}(\Gamma)$ and $\psi \in L^1_{\text{loc}}(\Omega)$ such that the convex and closed set of the form

$$K = \{\phi \in H^1(\Omega) \mid \phi = g \quad \text{on } \Gamma, \quad \phi(x) \geqq \psi(x) \text{ a.e. in } \Omega\} \tag{4.50}$$

is non–empty.

In order to define an obstacle problem we introduce the bilinear form

$$\begin{aligned} a(z, \phi) = \int_\Omega \{ &\langle A(x) \cdot \nabla z(x), \nabla \phi(x) \rangle_{\mathbb{R}^N} + \sum_{i=1}^{N} a_i(x) \frac{\partial z}{\partial x_i}(x) \phi(x) + \\ &a_0(x) z(x) \phi(x) \} dx \quad \forall z, \phi \in H^1(\Omega) \end{aligned} \tag{4.51}$$

and the linear form

$$\langle f, \phi \rangle = \int_\Omega \{f_0(x)\phi(x) + \sum_{i=1}^{N} f_i(x)\frac{\partial \phi}{\partial x_i}(x)\}dx \quad \forall \phi \in H^1(\Omega), \tag{4.52}$$

where $A(\cdot) = [a_{ij}(\cdot)]_{N\times N}$, $a_0(\cdot), a_1(\cdot), ..., a_N(\cdot)$, $f_0(\cdot), f_1(\cdot), ..., f_N(\cdot)$ are continuous functions in $\mathbb{R}^N$, $a_{ij}(x) = a_{ji}(x)$ for all $x \in \mathbb{R}^N$ and $i, j = 1, ..., N$. It is assumed that the condition (4.12) is satisfied.
Let us consider the following problem:

$$\begin{aligned} &\text{Find an element } z \in K \text{ such that} \\ &a(z, \phi - z) \geqq \langle f, \phi - z \rangle \quad \forall \phi \in K . \end{aligned} \tag{4.53}$$

It should be remarked that the bilinear form (4.51) is not symmetric. Therefore we cannot apply Theorem 4.6, which was stated only for symmetric bilinear forms,. to the variational inequality (4.53).

In order to define the cone (4.23) for the convex set (4.50) and the bilinear form (4.51) we have to direct our attention to the notion of capacity of a set in $\mathbb{R}^N$ (Ziemer 1989).

Let $A \subset \Omega$ be a set such that there exists $\phi \in H^1(\Omega)$ with $\phi(x) \geqq \chi_A(x)$. The following notation is introduced

$$\mathrm{cap}_a(A) = \inf\{a(\phi, \phi) \mid \phi \geqq \chi_A \quad \phi \in C_0^\infty(\Omega)\} . \tag{4.54}$$

It is said that a given condition is satisfied quasi–everywhere on Ω (q.e. on Ω) if this condition holds everywhere on Ω except for a set of the capacity zero.
We denote by $z = \Pi(f)$ the unique solution to the variational inequality (4.53) with the non–symmetric bilinear form (4.51). One can show that the mapping $f \to \Pi(f)$ is conically differentiable.

Theorem 4.15 (Mignot). *For $\varepsilon > 0$, ε small enough, we have*

$$\Pi(f + \varepsilon h) = \Pi(f) + \varepsilon \Pi'(h) + o(\varepsilon), \tag{4.55}$$

where $\|(o(\varepsilon)\|_{H^1(\Omega)}/\varepsilon \to 0$ as $\varepsilon \to 0$. The element $Q = \Pi'(h) \in H^1(\Omega)$ is given as the unique solution to the following variational inequality

$$\begin{aligned} &Q \in S_K(f) : \\ &a(Q, \phi - Q) \geqq \langle h, \phi - Q \rangle \quad \forall \phi \in S_K(f) . \end{aligned} \tag{4.56}$$

Here

$$\begin{aligned} &\langle h, \phi \rangle = \int_\Omega \{h_0(x)\phi(x) + \sum_{i=1}^{N} h_i(x)\frac{\partial \phi}{\partial x_i}(x)\}dx \\ &\qquad h_0(\cdot), h_1(\cdot), ..., h_N(\cdot) \in L^2(\Omega) \end{aligned} \tag{4.57}$$

are given elements, and

$$\begin{aligned} S_K(f) = \{\phi \in H^1(\Omega) \mid \phi = 0 \ on\ \Gamma, \\ \phi(x) \geqq 0 \ q.e.\ on\ \mathcal{Z}, \quad a(\phi, y) = \langle f, \phi \rangle\}, \end{aligned} \tag{4.58}$$

where $\mathcal{Z} = \{x \in \Omega \mid y(x) = \psi(x)\}$ *for* $\psi \in H^1(\Omega)$.

□

Proof of Theorem 4.15 was given by Mignot (1976). Theorem 4.15 combined with Theorem 4.14 will be used below to derive the form of the material derivatives and the shape derivatives of solutions to the variational inequality (4.53).

Let $\{\Omega_t\} \in \mathbb{R}^N$ be the family of domains given by $\Omega_t = T_t(V)(\Omega)$, $t \in [0, \delta)$, depending on the vector field $V(\cdot,\cdot) \in C(0, \varepsilon; \mathcal{D}^k(\mathbb{R}^N, \mathbb{R}^N)), k \geqq 1$. Let $K(\Omega_t)$ stand for:

$$K(\Omega_t) = \{\phi \in H^1(\Omega_t) \mid \phi = g \text{ on } \partial\Omega_t \ , \ \phi(x) \geqq \psi(x) \text{ a.e. in } \Omega_t\} \ .$$

The following variational inequality parameterized by $t \in [0, \delta)$ is considered

$$\begin{aligned} & z_t \in K(\Omega_t) : \\ & a_t(z_t, \phi - z_t) \geqq \langle f_t, \phi - z_t \rangle_t \quad \forall \phi \in K(\Omega_t), \end{aligned} \tag{4.59}$$

where it is assumed that $g(\cdot), \psi(\cdot) \in C^1(\mathbb{R}^N)$; $\langle \cdot,\cdot \rangle_t$ is the duality pairing between $(H^1(\Omega_t))'$ and $H^1(\Omega_t)$. The bilinear form $a_t(\cdot,\cdot)$ and the element $f_t \in (H^1(\Omega_t))'$ are defined, respectively by

$$\begin{aligned} a_t(y, \phi) = \int_{\Omega_t} \{ \langle A(x) \cdot \nabla z(x), \nabla \phi(x) \rangle_{\mathbb{R}^N} + \sum_{i=1}^{N} a_i(x) \frac{\partial z}{\partial x_i}(x) \phi(x) \qquad (4.60) \\ + a_0(x) z(x) \phi(x) \} dx \quad \forall \phi \in H^1(\Omega_t), \end{aligned}$$

$$\langle f_t, \phi \rangle_t = \int_{\Omega_t} \{ f_0(x) \phi(x) + \sum_{i=1}^{N} f_i(x) \frac{\partial \phi}{\partial x_i}(x) \} dx \quad \forall \phi \in H^1(\Omega_t) \ . \tag{4.61}$$

We shall determine the form of the shape derivative $z' \in H^1(\Omega)$ of the solution z to the variational inequality (4.53). In the particular case of the variational inequality (4.53) with non–homogeneous Dirichlet boundary conditions, the shape derivative z' can be defined as follows

$$z'(x) = \frac{\partial z}{\partial t}(0^+, x) \quad x \in \Omega, \tag{4.62}$$

where

$$z(t, x) = \begin{cases} z_t(x) & x \in \Omega, \quad t \in [0, \delta) \\ g(x) & x \in \mathbb{R}^N / \Omega, \quad t \in [0, \delta) \ . \end{cases} \tag{4.63}$$

Theorem 4.16 *Let us assume that*

(i) $K(\Omega_t)$ *is a non-empty, closed and convex subset of the space* $H^1(\Omega_t)$ *for all* $t \in [0, \delta)$,

(ii) $a_{ij}(\cdot), a_i(\cdot), f_i(\cdot) \in C^1(\mathbb{R}^N)$ *for* $i = 0, 1, ..., N$ *and* $j = 1, ..., N$,

(iii) there exists a constant $\alpha > 0$ *such that for any* $t \in [0, \delta)$

$$
\begin{aligned}
& a_t(\phi, \phi) \geqq \alpha \|\phi\|^2_{H^1(\Omega_t)} \\
& \textit{for all } \phi \in H^1(\Omega_t) \textit{ with } \phi = g \textit{ on } \partial\Omega_t,
\end{aligned}
\tag{4.64}
$$

(iv) $-\beta(x) = g(x) - \psi(x) > 0$ *for all* $x \in \mathbb{R}^N$, $\beta(\cdot)$, $1/\beta(\cdot) \in C^1(\mathbb{R}^N)$, $\psi \in C^2(\mathbb{R}^N)$.

Then the domain derivative $z' \in H^1(\Omega)$ *of the solution* z *to the variational inequality (4.53) is given as the unique solution to the following variational inequality*

$$
z' \in S_v(\Omega): \quad a(z', \phi - z') \geqq 0 \quad \forall \phi \in S_v(\Omega), \tag{4.65}
$$

where the convex closed cone $S_v(\Omega) \subset H^1(\Omega)$ *is defined by*

$$
\begin{aligned}
S_v(\Omega) = \{\phi \in H^1(\Omega) \mid \phi = -v_n \frac{\partial}{\partial n}(z - g) \textit{ on } \Gamma, \quad \phi \geqq 0 \textit{ q.e. on } \mathcal{Z}, \\
a(z, \phi) = \langle f, \phi \rangle\} \ .
\end{aligned}
$$

Here

$$
\begin{aligned}
& \mathcal{Z} = \{x \in \Omega \mid z(x) = \psi(x)\} \\
& v_n(x) = \langle V(0, x), n(x) \rangle_{\mathbb{R}^N}, x \in \Gamma = \partial\Omega \ .
\end{aligned}
$$

□

Theorem 4.16 was given by Sokołowski et al. (1987).
The proof of Theorem 4.16 will be given for the particular case of the following variational inequality

$$
\begin{aligned}
& y \in K = \{\phi \in H_0^1(\Omega) \mid \phi(x) \geqq \psi(x) \quad \text{in } \Omega\} \\
& \int_\Omega \langle \nabla y(x), \nabla\phi(x) - \nabla y(x) \rangle_{\mathbb{R}^N} dx \geqq \\
& \int_\Omega f(x)(\phi(x) - y(x)) dx \quad \forall \phi \in K,
\end{aligned}
\tag{4.67}
$$

where $\psi(\cdot) \in C^1(\mathbb{R}^N)$ is a given element, $\psi(x) < 0$ on $\Gamma = \partial\Omega$.
We denote by $\tilde{y}(x, t)$ the extension of $y_t \in H_0^1(\Omega_t)$ to $\mathbb{R}^N$

$$
\tilde{y}(x, t) = \begin{cases} y_t(x) & x \in \Omega_t \ , \quad t \in [0, \delta) \\ 0 & x \in \mathbb{R}^N / \Omega_t \ , \quad t \in [0, \delta) \end{cases} \tag{4.68}
$$

then

$$
y'(x) = \frac{\partial \tilde{y}}{\partial t}(0^+, x) \quad x \in \Omega \ . \tag{4.69}
$$

From Theorem 4.16 the following corollary can be inferred.

Corollary 4.17 *For $t > 0, t$ small enough, we have*

$$\tilde{y}|_\Omega = y + ty' + o(t) \quad in\ H^1(\Omega), \tag{4.70}$$

where $\|o(t)\|_{H^1(\Omega)}/t \to 0$ as $t \downarrow 0$ and the element $y' \in H^1(\Omega)$ is given as the unique solution to the following variational inequality

$$y' \in S_v(\Omega): \quad \int_\Omega \langle \nabla y', \nabla(\phi - y') \rangle_{\mathbb{R}^N} dx \geqq 0 \quad \forall \phi \in S_v(\Omega), \tag{4.71}$$

where the cone $S_v(\Omega) \subset H^1(\Omega)$ is of the form

$$\begin{aligned} S_v(\Omega) = \{\phi \in H^1(\Omega) \mid \phi = -v_n \frac{\partial y}{\partial n} \ on\ \Gamma,\ \phi \geqq 0\ q.e.\ on\ \mathcal{Z}, \\ \int_\Omega \nabla y \cdot \nabla \phi dx = \int_\Omega f \phi dx\} \end{aligned} \tag{4.72}$$

and

$$\mathcal{Z} = \{x \in \Omega \mid y_0(x) = \psi(x)\} \ . \tag{4.73}$$

Proof. Let $y_t \in H_0^1(\Omega_t)$, $t \in [0, \delta)$, denote a solution to the variational inequality (4.67) defined in the domain Ω_t

$$\begin{aligned} &y_t \in K(\Omega_t) = \{\phi \in H_0^1(\Omega_t) \mid \phi(x) \geqq \psi(x) \quad \text{in } \Omega\} \\ &\int_\Omega \langle \nabla y_t, \nabla(\phi - y_t) \rangle_{\mathbb{R}^N} \geqq \int_\Omega f(\phi - y_t) dx \quad \forall \phi \in K(\Omega_t) \ . \end{aligned} \tag{4.74}$$

First, the form of the material derivative $\dot{y} \in H_0^1(\Omega)$ will be derived. Let

$$y^t = y_t \circ T_t \in H_0^1(\Omega) \quad t \in [0, \delta) \ . \tag{4.75}$$

Taking into account (4.74) and (4.75), one can show that for $y^t \in K$ the following variational inequality holds

$$\int_\Omega \langle A_t \cdot \nabla y^t, \nabla(\phi - y^t) \rangle_{\mathbb{R}^N} dx \geqq \int_\Omega f^t(\phi - y^t) dx \quad \forall \phi \in K \ . \tag{4.76}$$

From Theorems 4.14 and 4.15 it follows that

$$\begin{aligned} &\dot{y} \in \mathcal{S}: \\ &\int_\Omega \langle \nabla \dot{y}, \nabla(\phi - \dot{y}) \rangle_{\mathbb{R}^N} dx \geqq \int_\Omega f'(\phi - \dot{y}) dx \\ &- \int_\Omega \langle A' \cdot \nabla y^0, \nabla(\phi - \dot{y}) \rangle dx \quad \forall \phi \in \mathcal{S}, \end{aligned} \tag{4.77}$$

where

$$\mathcal{S} = \{\phi \in H_0^1(\Omega) \mid \phi \geqq 0 \text{ q.e. on } \mathcal{Z}, \int_\Omega \langle \nabla y^0, \nabla \phi \rangle_{\mathbb{R}^N} dx = \int_\Omega f \phi dx\} . \tag{4.78}$$

Using the formula

$$y' = \dot{y} - \nabla y^0 \cdot V \tag{4.79}$$

for any vector field V with the compact support in a sufficiently small neighbourhood of $\Gamma = \partial\Omega$, we obtain

$$\begin{aligned} & y' \in S_v(\Omega) : \\ & \int_\Omega \langle \nabla y', \nabla(\phi - y') \rangle_{\mathbb{R}^N} dx \geqq \int_\Omega f'(\phi - y') dx \\ & - \int_\Omega \langle A' \cdot \nabla y^0, \nabla(\phi - y') \rangle_{\mathbb{R}^N} dx \\ & - \int_\Omega \langle \nabla(\nabla y^0 \cdot V), \nabla(\phi - y') \rangle_{\mathbb{R}^N} dx \quad \forall \phi \in S_v(\Omega) . \end{aligned} \tag{4.80}$$

The regularity result $y = y^0 \in H^2(\Omega)$ derived by Brezis et al. (1968) for the solutions to the variational inequality (4.67) was used in (4.80).
Therefore, taking into account (4.22), one can show that $y' \in H^1(\Omega)$.
The following notation is introduced

$$\begin{aligned} G(y^0, V; \phi) = \int_\Omega \{ & f'\phi - \langle A' \cdot \nabla y^0, \nabla \phi \rangle_{\mathbb{R}^N} \\ & - \langle \nabla(\nabla y^0 \cdot V), \nabla \phi \rangle_{\mathbb{R}^N} dx \quad \forall \phi \in H_0^1(\Omega) . \end{aligned} \tag{4.81}$$

For any vector field $V(\cdot,\cdot)$ with $v_n(x) = \langle V(0,x), n(x) \rangle_{\mathbb{R}^N} = 0$ for all $x \in \Gamma$ we have that $y' = 0$. Hence from (4.80),

$$0 \geqq G(y^0, V; \phi) \quad \forall \phi \in \{S_v(\Omega) - S_v(\Omega)\} . \tag{4.82}$$

So taking $\pm V$ in (4.82), we obtain $G(y^0, V, \phi) = 0$. Therefore there exists the distribution $g_n(\phi) \in \mathcal{D}^{-1}(\Gamma)$ such that for an arbitrary vector field V the following equality is satisfied (see Chapt. 2, Sect. 2.11)

$$G(y^0, V; \phi) = \langle g_n(\phi), v_n \rangle_{\mathcal{D}^{-1}(\Gamma) \times \mathcal{D}^1(\Gamma)} . \tag{4.83}$$

Applying Green's formula to (4.81) for test functions ϕ sufficiently smooth, e.g. $\phi \in H^2(\Omega)$, we have that $g_n(\phi) = 0$.
This concludes the proof of Corollary 4.17. □

4.3.1. Differentiability of the Newtonian capacity

We shall consider the following particular case of the obstacle problem. Let $D \subset \mathbb{R}^N$ be a given domain with the smooth boundary ∂D and let $\overline{\Omega} \subset D$. We denote by $K(\Omega)$ the convex set

$$K(\Omega) = \{\phi \in H_0^1(D) \mid \phi(x) \geqq \chi_\Omega(x) \quad \text{in } \Omega\} \ .$$

Let $y(\Omega)$ be the unique solution to the following variational inequality

$$y = y(\Omega) \in K(\Omega): \quad \int_D \nabla y \cdot \nabla(\phi - y)dx \geqq 0 \quad \forall \phi \in K(\Omega),$$

here it is assumed that $K(\Omega) \neq \emptyset$.
Let $V(\cdot,\cdot)$ be a given vector field with the compact support $\operatorname{spt} V \subset D$ such that $\Omega \subset \operatorname{spt} V$, and let $\{\Omega_t\}$, $t \in [0,\delta)$, denote the associated family of domains. The following variational inequality is introduced

$$y_t = y(\Omega_t) \in K(\Omega_t) = \{\phi \in H_0^1(D) \mid \phi(x) \geqq \chi_{\Omega_t}(x) \text{ in } \Omega\}$$

$$\int_D \nabla y_t \cdot \nabla(\phi - y_t)dx \geqq 0 \quad \forall \phi \in K(\Omega_t) \ .$$

It is assumed here that $K(\Omega_t) \neq \emptyset$ for $t > 0$, t small enough.

Lemma 4.18 *There exists the strong limit*

$$\dot{y} = \lim_{t \downarrow 0}(y_t - y_0)/t \quad \textit{in } H_0^1(D)$$

given as the unique solution to the following variational inequality

$$\dot{y} \in S: \quad \int_\Omega \nabla \dot{y} \cdot \nabla(\phi - \dot{y})dx \geqq$$
$$\int_\Omega \langle A' \cdot \nabla y, \nabla(\phi - y)\rangle_{\mathbb{R}^N} dx \quad \forall \phi \in S,$$

where

$$S = \{\phi \in H_0^1(D) \mid \phi(x) \geqq 0 \ \textit{q.e. on } \mathcal{Z}, \quad \int_\Omega \nabla y \cdot \nabla \phi dx = 0\}$$
$$\mathcal{Z} = \{x \in \Omega \mid y(x) = \chi_\Omega(x)\} \ .$$

4.3.2. The shape controlability of the free boundary

Let $\Omega \in \mathbb{R}^2$ be a given domain, $\Omega_t = T_t(V)(\Omega)$, $t \in [0,\delta)$.
We shall consider the variational inequality (4.67) defined in the domain Ω_t,

$$y_t \in K(\Omega_t) = \{\phi \in H_0^1(\Omega_t) \mid \phi(x) \geqq \psi(x) \quad \text{in } \Omega_t\}$$

$$\int_{\Omega_t} \langle \nabla y_t, \nabla(\phi - y_t)\rangle_{\mathbb{R}^2} dx \geqq \int_{\Omega_t} f(\phi - y_t)dx \quad \forall \phi \in K(\Omega_t) \ .$$

It is assumed that the elements f, $\psi \in C(\mathbb{R}^2)$ satisfy the following conditions: $f(x) < 0$ in an open neighbourhood of Ω, $\max_{x\in\Omega}\psi(x) > 0$ and $\psi(x) < 0$ on $\Gamma = \partial\Omega$. Furthermore it is supposed that f and ψ are smooth enough. So using the result of Brezis et al. (1968) we get

$$y_t \in W^{2,\infty}(\Omega_t) \quad \text{for all } t \in [0,\delta) \ .$$

From the Sobolev embedding theorem it follows that in this particular case $y_t \in C(\overline{\Omega}_t)$. Therefore the coincidence set $\mathcal{Z}_t \subset \Omega_t$ is closed

$$\mathcal{Z}_t = \{x \in \Omega_t \mid y_t(x) = \psi(x)\}$$

and under assumptions made we have $\text{meas}(\mathcal{Z}_t) > 0$ for all $t \in [0,\delta)$. The Radon measure μ_t is defined by

$$\int \phi d\mu_t = \int_{\Omega_t} \{\nabla y_t \cdot \nabla\phi - f\phi\}dx = \int_{\Omega_t} (-\Delta y_t - f)\phi dx = \\ - \int_{\mathcal{Z}_t} (\Delta\psi + f)\phi dx \quad \forall\phi \in C_0(\Omega_t) \cap H_0^1(\Omega_t) \ .$$

We denote by Σ_t the boundary of the coincidence set $\mathcal{Z}_t$. If Σ_t is a C^1 manifold, then y_t is a solution to the following system

$$\begin{aligned} -\Delta y_t &= f \quad \text{in } \Omega_t \setminus \mathcal{Z}_t \ , \\ y_t &= 0 \quad \text{on } \Gamma_t \ , \\ y_t &= \psi \quad \frac{\partial y_t}{\partial n} = \frac{\partial \psi}{\partial n} \quad \text{on } \Sigma_t \ , \\ y_t &= \psi \quad \text{on } \mathcal{Z}_t \ . \end{aligned}$$

Corollary 4.17 implies that the shape derivative $y_t' \in H^1(\Omega)$, $t \in [0,\delta)$, of the solution $y_t \in H^1(\Omega)$ to the obstacle problem under study, in the direction of a vector field $V(\cdot,\cdot)$, is given as the unique solution to the following variational inequality

$$y_t' \in S(\Omega_t): \quad \int_{\Omega_t} \langle\nabla y_t', \nabla(\phi - y_t')\rangle_{\mathbb{R}^2} dx \geqq 0 \quad \forall\phi \in S(\Omega_t),$$

where the cone $S(\Omega_t)$ is of the form

$$\begin{aligned} S(\Omega_t) = \{\phi \in H^1(\Omega_t) \mid \phi(x) = -\frac{\partial y_t}{\partial n}(x)\langle V(t,x), n_t(x)\rangle_{\mathbb{R}^2} \text{ for } x \in \partial\Omega_t, \\ \phi(x) = 0 \text{ for } y_t(x) = \psi(x) \text{ and } \Delta\psi(x) + f(x) < 0, \\ \phi(x) \geqq 0 \text{ for } y_t(x) = \psi(x) \text{ and } \Delta\psi(x) + f(x) = 0\} \ . \end{aligned}$$

Let us observe that the set $\mathcal{Z}_t$ is defined by the condition $y_t(x) - \psi(x) = 0$, so on $\Sigma_t = \partial\mathcal{Z}_t$ we have $y_t(x) - \psi(x) = 0$ for all $t \in [0,\delta)$. We shall construct the vector field $W(\cdot,\cdot)$, see Theorem 4.20, with the following property

$$T_t(W)(\Sigma_0) = \Sigma_t$$

for all $t \in [0, \delta)$, i.e. the field $W(\cdot, \cdot)$ defines the evolution of the free boundary Σ_t.

Let us consider the family of functions $z_t(.) \in C^2(\Omega_t)$, $t \in [0, \delta)$, and suppose that for some $s > 0$ the level sets

$$\Sigma_t^s = \{x \in \Omega_t \mid z_t(x) = s\} = z_t^{-1}(s)$$

are C^1 manifolds for all $t \in [0, \delta)$. We shall construct the vector field $W_s(\cdot, \cdot)$ which defines the evolution with respect to $t \in [0, \delta)$ of the manifold Σ_t^s.

Lemma 4.19 *Let us assume that*

(i) $z_t(.) \in C^2(\Omega_t)$ *for all* $t \in [0, \delta)$, *where* $\Omega_t = T_t(V)(\Omega)$,

(ii) $\|\nabla z_t\|_{\mathbb{R}^2} > 0$ *on* $\Sigma_t^s = z_t^{-1}(s)$ *for some* $s > 0$,

(iii) there exist the material and shape derivatives

$$\dot{z}_t = \dot{z}_t(V) \in H^1(\Omega_t),$$
$$z_t' = z_t'(V) \in C^1(\Omega_t).$$

Then

$$\Sigma_t^s = T_t(W_s)(\Sigma_0^s) \quad \textit{for all } t \in [0, \delta),$$

where $\Sigma_0^s = z_0^{-1}(s)$.

The vector field $W_s(\cdot, \cdot)$ *is given by*

$$W_s(t, x) = -z_t'(V)(x) \|\nabla z_t(x)\|_{\mathbb{R}^2}^{-2} \nabla z_t(x)$$

for all $t \in [0, \delta)$ *and all* x *in an open neighbourhood of* $\Sigma_t^s \subset \mathbb{R}^2$.

Proof. For simplicity s is omitted in superscripts and subscripts, e.g. we shall write $W(\cdot, \cdot)$, Σ_t and $\hat{\Sigma}_t$ for $W_s(\cdot, \cdot)$, Σ_t^s and $\hat{\Sigma}_t^s$, respectively.

It suffices to show that for

$$\hat{\Sigma}_t = T_t(W)(\Sigma_0) \quad t \in [0, \delta)$$

we have

$$z_t(x) = s \quad x \in \hat{\Sigma}_t \ .$$

Let U be the vector field defined as follows:

$$U = \eta_1 V + \eta_2 W, \quad \eta_i = \eta_i(t, x), \quad i = 1, 2,$$

where $\eta_1 \equiv 1$ in an open neighbourhood of $\Gamma_t = \partial \Omega_t$, $\eta_2 \equiv 1$ in an open neighbourhood of Σ_t, $1 \geqq \eta_i(t, x) \geqq 0$ for $i = 1, 2$, $\eta_1 + \eta_2 \equiv 1$.

We have

$$\int_{\Sigma_0} z_0(x)\, g(x)\, d\ell = s \int_{\Sigma_0} g(x)\, d\ell \quad \forall g \in C^\infty(\mathbb{R}^2),$$

because of $z_0(x) = s$ on Σ_0. We define the function

$$f(t) = \int_{\hat{\Sigma}_t} z_t(x) \left(g \circ T_t^{-1}(U)\right)(x) \left(\omega_t^{-1}(T_t(U)) \circ T_t^{-1}(U)\right)(x)\, d\ell \ .$$

It is evident that

$$f(0) = s \int_{\Sigma_0} g(x)\, d\ell \ .$$

Furthermore

$$f(t) = \int_{\Sigma_0} (z_t \circ T_t(U))(x)\, g(x)\, d\ell \ .$$

Evaluating the derivative of $f(\cdot)$ at t we have

$$f(t+\varepsilon) = \int_{\Sigma_0} (z_{t+\varepsilon} \circ T_{t+\varepsilon}(U))(x)\, g(x)\, d\ell \ .$$

Therefore

$$\lim_{\varepsilon \downarrow 0} \left(f(t+\varepsilon) - f(t)\right)/\varepsilon = \int_{\Sigma_0} \dot{z}_t(U)\, g(x)\, d\ell,$$

where

$$\dot{z}_t(U) = z_t'(U) + \nabla z_t \cdot U$$

and for $t > 0$, t small enough,

$$U = W = -z_t'(V)\, |\nabla z_t|^{-2} \nabla z_t \ .$$

in an open neighbourhood of Σ_0.
Moreover

$$z_t'(U) = z_t'(V)$$

since z_t' depends on the restriction $U|_{\Gamma_t} \equiv V$. Thus

$$\dot{z}_t(U) = 0$$

and for $t > 0$, t small enough,

$$f(t) = \text{const} \ .$$

Finally

$$\begin{aligned} f(t) = f(0) &= s \int_{\Sigma_0} g(x)\, d\ell \\ &= s \int_{\hat{\Sigma}_t} \left(g \circ T_t^{-1}(U)\right)(x)\, \omega_t\left(T_t^{-1}(U)\right) d\ell, \end{aligned}$$

where

$$\omega_t\left(T_t^{-1}(U)\right) = \omega_t^{-1}\left(T_t(U)\right) \circ T_t^{-1}(U) \ .$$

Hence by the definition of $f(t)$ we get

$$s\int_{\hat{\Sigma}_t} g_t d\ell = \int_{\hat{\Sigma}_t} z_t(x)\, g_t d\ell$$

for all g_t given by

$$g_t = g \circ T_t^{-1}(U)\, \omega_t\left(T_t^{-1}(U)\right) \ .$$

Therefore

$$z_t(x) = s \quad \text{on } \hat{\Sigma}_t$$

as was to be shown. □

On the free boundary $\Sigma_t \equiv \partial \mathcal{Z}_t$, that is the zero level set $z_t^{-1}(0)$ for the family

$$z_t = y_t - \psi \quad t \in [0, \delta)$$

we have

$$z_t(x) = 0 \text{ and } \nabla z_t(x) = 0 \text{ for } x \in \Sigma_t \ .$$

Therefore Lemma 4.19 cannot be directly applied to the obstacle problem.
Let us suppose that for the cone $S(\Omega_t) \subset H^1(\Omega_t)$ the following condition is satisfied

(A3) $\quad \{S(\Omega_t) - S(\Omega_t)\}$ is a linear subspace of $H_0^1(\Omega_t)$.

Thus the shape derivative y_t' is linear with respect to the normal component $\langle V(t,x), n_t(x)\rangle_{\mathbb{R}^2}$ of the vector field $V(t,\cdot)$ on $\partial\Omega_t$. The condition (A3) holds provided that, e.g.

$$\Delta\psi(x) + f(x) < 0 \quad \text{for all } x \ .$$

In this case the shape derivative y_t' is given as the solution to the following boundary–value problem in $\Omega_t \setminus \mathcal{Z}_t$

$$\begin{aligned} \Delta y_t' &= 0 \quad \text{in } \Omega_t \setminus \mathcal{Z}_t \ , \\ y_t' &= -\frac{\partial y_t}{\partial n}\langle V(t), n_t\rangle_{\mathbb{R}^2} \quad \text{on } \Gamma_t \ , \\ y_t' &= 0 \quad \text{on } \Sigma_t \ . \end{aligned}$$

Furthermore $y_t = \psi$ on $\mathcal{Z}_t$.
Let $s > 0$ be small enough, and let us consider the level curve Σ_t^s of $z_t = y_t - \psi$

$$\Sigma_t^s = \{x \in \Omega_t | z_t(x) = s\} \ .$$

Due to the assumptions imposed on $z_t \in C^2(\Omega_t)$, this curve is a C^1 manifold located in a small neighbourhood of the free boundary $\Sigma_t \subset \mathbb{R}^2$. For any point $x \in \Sigma_t^s$ there exists the unique point $\xi \in \Sigma_t$ such that for $r > 0$, r sufficiently small,

$$x = \xi + rn_t(\xi) + o(r),$$

where n_t is the unit normal vector on the C^1 manifold Σ_t, n_t is outward pointing on Σ_t.

Expanding in the Taylor series on Σ_t^s we have

$$y_t'(x) = y_t'(\xi) + s\nabla y_t'(\xi)\cdot n_t(\xi) + o(s),$$
$$\nabla z_t(x) = \nabla z_t(\xi) + sD^2 z_t(\xi)\cdot n_t(\xi) + o(s) \quad .$$

Taking into account that on the free boundary Σ_t the following conditions are satisfied

$$z_t = y_t - \psi = 0,$$
$$\nabla z_t = 0,$$
$$y_t' = z_t' = 0 \quad \text{for } \psi' = 0$$

one can show that

$$y_t'(x) = s\nabla y_t'(\xi)\cdot n_t(\xi) + o(s) \quad \text{for all } \xi \in \Sigma_t \ ,$$
$$\nabla z_t(x) = sD^2 z_t(\xi)\cdot n_t(\xi) + o(s) \quad \text{for all } \xi \in \Sigma_t \ .$$

Applying Lemma 4.19 to the level set Σ_t^s of $z_t(\cdot)$ it is possible to prove that the evolution of the manifold Σ_t^s with respect to $t \in [0,\delta)$ is defined by the vector field

$$\begin{aligned} W_s(t,x) &= -\, z_t'(x)\frac{\nabla z_t(x)}{\|\nabla z_t(x)\|^2_{\mathbb{R}^2}} = -y_t'(x)\frac{\nabla z_t(x)}{\|\nabla z_t(x)\|^2_{\mathbb{R}^2}} \\ &= -\left(s\frac{\partial y_t'}{\partial n_t}(\xi) + o(s)\right)\frac{\left(sD^2 z_t(\xi)\cdot n_t(\xi) + o(s)\right)}{\|sD^2 z_t(\xi)\cdot n_t(\xi) + o(s)\|^2_{\mathbb{R}^2}} \end{aligned}$$

for all $x \in \Sigma_t^s$ and for all $\xi \in \Sigma_t$.
We obtain the limit as $s \downarrow 0$

$$W_s(t,x) \to -\frac{\partial y_t'}{\partial n_t}(\xi)\frac{D^2 z_t(\xi)\cdot n_t(\xi)}{\|D^2 z_t(\xi)\cdot n_t(\xi)\|^2_{\mathbb{R}^2}} \quad \xi \in \Sigma_t \ .$$

Therefore we can define the vector field $W(t,\xi)$ on Σ_t

$$W(t,\xi) = -\frac{\partial y_t'}{\partial n_t}(\xi)\frac{D^2 z_t(\xi)\cdot n_t(\xi)}{\|D^2 z_t(\xi)\cdot n_t(\xi)\|^2_{\mathbb{R}^2}} \quad \text{for all } \xi \in \Sigma_t$$

using the above limit.
This form of the vector field $W(t,\xi)$, $\xi \in \Sigma_t$, $t \in [0,\delta)$, can be further simplified. Let us recall that on the free boundary Σ_t we have

$$\nabla\left(\frac{\partial}{\partial x_i} z_t\right) = \frac{\partial}{\partial n}\left(\frac{\partial z_t}{\partial x_i}\right) n_t \quad \text{for } \nabla z_t = 0 \ .$$

Hence the following equation holds on Σ_t

$$\frac{\partial^2 z_t}{\partial x_i \partial x_j} = \frac{\partial}{\partial n}\left(\frac{\partial z_t}{\partial x_i}\right) n_j, \quad \text{where } n_j \text{ is the } j-\text{th component of } n_t \ .$$

Thus

$$\left(D^2 z_t \cdot n_t\right)_j = \frac{\partial^2 z_t}{\partial x_i \partial x_j} n_i = \frac{\partial}{\partial n}\left(\frac{\partial z_t}{\partial x_i}\right) n_i n_j \ .$$

From the foregoing it follows that

$$D^2 z_t \cdot n_t = \langle D^2 z_t \cdot n_t, n_t \rangle_{\mathbb{R}^2} n_t \quad \text{on } \Sigma_t \ .$$

On the other hand, Σ_t is the level set of z_t, then from the general formula (see Proposition 2.66) it follows that on Σ_t we have

$$\Delta z_t = \Delta_{\Sigma_t} z_t + \kappa_t \frac{\partial z_t}{\partial n_t} + \langle D^2 z_t \cdot n_t, n_t \rangle_{\mathbb{R}^2},$$

where Δ_{Σ_t} is the Laplace–Beltrami operator on Σ_t (see Sect. 2.20), κ_t is the mean curvature of the free boundary Σ_t. Let us observe that on Σ_t

$$z_t = 0 \quad \text{hence} \quad \Delta_{\Sigma_t} z_t = 0$$

and

$$\nabla z_t = 0 \quad \text{thus} \quad \partial z_t / \partial n_t = 0 \ .$$

Finally we get

$$\Delta z_t = \langle D^2 z_t \cdot n_t, n_t \rangle_{\mathbb{R}^2} \quad \text{on } \Sigma_t \ .$$

On the other hand

$$\Delta z_t = \Delta (y_t - \psi) = -(f + \Delta\psi) \quad \text{on } \Sigma_t \ ,$$

thus

$$\begin{aligned} W(t,\xi) &= -\nabla y_t' |\langle D^2 z_t \cdot n_t, n_t \rangle_{\mathbb{R}^2}|^{-2} \langle D^2 z_t \cdot n_t, n_t \rangle_{\mathbb{R}^2} = \\ &- \nabla y_t' |\Delta z_t|^{-2} \Delta z_t = (f(\xi) + \Delta\psi(\xi))^{-1} \nabla y_t'(\xi) \quad \xi \in \Sigma_t \ . \end{aligned}$$

This result makes it possible to state the following theorem:

Theorem 4.20 *Let $y_t \in C^2(\Omega_t)$ be a solution to the obstacle problem and let the shape derivative y_t' be linear with respect to the normal component of the vector field $V(t,\cdot)$ on Γ_t.*
The evolution, with respect to t, of the free boundary $\Sigma_t = \partial Z_t$ is defined by the vector field

$$W(t,x) = (f(x) + \Delta\psi(x))^{-1} \nabla y_t'(x) \quad x \in \Sigma_t \ ,$$

where the shape derivative y'_t is given as the unique solution to the following linear boundary–value problem

$$\begin{aligned} \Delta y'_t &= 0 \quad in\ \Omega_t \setminus \mathcal{Z}_t\ , \\ y'_t &= -\frac{\partial y_t}{\partial n} \langle V(t), n_t \rangle_{\mathbb{R}^2} \quad on\ \Gamma_t\ , \\ y'_t &= 0 \quad on\ \Sigma_t\ . \end{aligned}$$

□

This theorem can be used to determine the form of the shape derivative $dJ(\Omega; V)$ of the following shape functional

$$J(\Omega) = \int_{\mathcal{Z}} dx = \int_{\{y=\psi\}} dx\ . \tag{4.85}$$

Corollary 4.21 *The shape functional (4.85) is shape differentiable*

$$dJ(\Omega; V) = \int_\Sigma W(0,x) \cdot n(x)\, d\Sigma = \int_{\partial \mathcal{Z}} [f(x) + \Delta\psi]^{-1} \frac{\partial y'}{\partial n} d\Sigma,$$

where y' is given as the unique solution to the following boundary value problem

$$\begin{aligned} \Delta y' &= 0 \quad in\ \Omega \setminus \mathcal{Z}, \\ y' &= -\frac{\partial y}{\partial n} \langle V, n \rangle_{\mathbb{R}^2} \quad on\ \Gamma = \partial\Omega, \\ y' &= 0 \quad on\ \Sigma = \partial \mathcal{Z}\ . \end{aligned}$$

□

Let us observe that the form of the shape derivative $dJ(\Omega; V)$ can be simplified provided that an appropriate adjoint state equation is introduced.

If p is the unique solution to the following boundary value problem

$$\begin{aligned} -\Delta p &= 0 \quad \text{in } \Omega \setminus \mathcal{Z} \\ p &= 0 \quad \text{on } \Gamma = \partial\Omega \\ p &= [f + \Delta\psi]^{-1} \quad \text{on } \Sigma = \partial\mathcal{Z}, \end{aligned}$$

then

$$dJ(\Omega; V) = \int_\Sigma p \frac{\partial y'}{\partial n} d\Sigma = -\int_\Gamma \frac{\partial p}{\partial n} \frac{\partial y}{\partial n} \langle V, n \rangle_{\mathbb{R}^2} d\Gamma\ .$$

Remark. Here we have the particular situation for free boundary problems, where the shape derivative $V \to dJ(\Omega; V)$ is a linear mapping of the form $dJ(\Omega; V) = \langle G, V \rangle_{\mathcal{D}' \times \mathcal{D}}$. It should be remarked that for the distribution $G \in \mathcal{D}'(\mathbb{R}^2; \mathbb{R}^2)$ given by

$$G = {}^*\gamma_\Gamma(g_n)$$

the element g_n is as follows

$$g_n = -\frac{\partial p}{\partial n}\frac{\partial y}{\partial n} \ .$$

On the other hand, one cannot expect for an arbitrary shape functional $J(\Omega)$ depending on solutions to variational inequalities to obtain the shape derivative $V \to dJ(\Omega; V)$ as a linear mapping, we refer the reader to the preface of this book in Chap. 1 for an elementary example. Furthermore it can be seen, e.g. from (4.81) that in general the shape derivative $y'(\Omega; V)$ for the obstacle problems fails to be linear with respect to the normal component of the vector field $V(0, \cdot)$ on Γ.

4.4. The Signorini problem

Let us assume that an obstacle is located on the part Γ_1 of the boundary Γ of the sufficiently smooth domain Ω. Moreover, it is supposed that

$$K = \{\phi \in H^1(\Omega) | \phi = 0 \quad \text{on } \Gamma_0 = \Gamma \setminus \Gamma_1 \ , \quad \phi \geqq 0 \quad \text{on } \Gamma_1\} \tag{4.86}$$

$$a(z, \phi) = \int_\Omega \langle \nabla z(x), \nabla \phi(x) \rangle_{\mathbb{R}^N} dx \quad \forall z, \phi \in H^1(\Omega), \tag{4.87}$$

$$\langle f, \phi \rangle = \int_\Omega f(x)\phi(x)\, dx \quad \forall \phi \in H^1(\Omega), \tag{4.88}$$

where $f \in L^2(\Omega)$.
In this case the Hilbert space H is given by

$$H^1_{\Gamma_0} = \{\phi \in H^1(\Omega) | \phi = 0 \quad \text{on } \Gamma_0\} \ .$$

We denote by $z = \Pi(f)$ the unique solution to the following variational inequality

$$z \in K\colon \quad a(z, \phi - z) \geqq \langle f, \phi - z \rangle \quad \forall \phi \in K \ , \tag{4.89}$$

by $\mathcal{Z} \subset \partial\Omega$ the coincidence set

$$\mathcal{Z} = \{x \in \Gamma_1 | z(x) = 0\}, \tag{4.90}$$

and by μ the Radon measure supported on Γ, and defined as follows

$$\int \phi d\mu = a(z, \phi) - \langle f, \phi \rangle \quad 0 \leqq \phi \in C^1(\overline{\Omega}) \ . \tag{4.91}$$

It can be shown that $\mu \geqq 0$ and if $\mathcal{Z}$ is closed, then

$$\mu(\mathcal{Z}^c) = 0, \tag{4.9}$$

where $\mathcal{Z}^c = \Gamma \setminus \mathcal{Z}$.

By the results of Mignot (1976) it follows that the set (4.86) is polyhedr and we have

$$S_K(f) = \{\phi \in H^1_{\Gamma_0}(\Omega) | \phi \geqq 0 \quad \text{q.e. on } \mathcal{Z}, \quad \int \phi d\mu = 0\} \ . \tag{4.9:}$$

Therefore using Theorem 4.6 one can show that for any element $h \in (H^1_{\Gamma_0}(\Omega))'$ and for $\varepsilon > 0$, ε small enough,

$$\Pi(f + \varepsilon h) = \Pi(f) + \varepsilon \Pi'(h) + o(\varepsilon), \tag{4.94}$$

where $\|o(\varepsilon)\|_{H^1(\Omega)}/\varepsilon \to 0$ as $\varepsilon \downarrow 0$. The element $Q = \Pi'(h)$ is given as th(unique solution to the following variational inequality

$$\begin{aligned} &Q \in S_K(f) \ : \\ &a(Q, \phi - Q) \geqq \langle h, \phi - Q \rangle \quad \forall \phi \in S_K(f), \end{aligned} \tag{4.95}$$

where the cone $S_K(f) \subset H^1_{\Gamma_0}(\Omega)$ is defined by (4.93).

From (4.94) and (4.95) it follows that

$$\Pi'(h) = P_{S_K(f)}(\mathcal{G}h) \quad \forall h \in \left(H^1_{\Gamma_0}(\Omega)\right)', \tag{4.96}$$

where

$$\begin{aligned} &\varphi = \mathcal{G}h \in H^1_{\Gamma_0}(\Omega), \\ &a(\varphi, \phi) = \langle h, \phi \rangle \quad \forall \phi \in H^1_{\Gamma_0}(\Omega) \ . \end{aligned} \tag{4.97}$$

We now turn to the shape sensitivity analysis of the variational inequality (4.89). It is assumed that the condition

$$\nabla z \cdot V \in H^1(\Omega) \tag{4.98}$$

is satisfied for a given vector field $V(\cdot, \cdot)$.

We denote by $z_t \in H_t = \{\phi \in H^1(\Omega_t) | \phi = 0 \quad \text{on } \Gamma^t_0\}$ a weak solution to the variational inequality (4.89) defined in the domain $\Omega_t = T_t(V)(\Omega)$, $t \in [0, \delta)$,

$$z_t \in K(\Omega_t) = \{\phi \in H^1(\Omega_t) | \phi = 0 \quad \text{on } \Gamma^t_0, \quad \phi \geqq 0 \text{ on } \Gamma^t_1\} \tag{4.99}$$

$$\int_{\Omega_t} \langle \nabla z_t, \nabla(\phi - z_t) \rangle_{\mathbb{R}^N} dx \geqq \int_{\Omega_t} f(\phi - z_t) dx \quad \forall \phi \in K(\Omega_t) \ . \tag{4.100}$$

Let $z^t = z_t \circ T_t \in H^1_{\Gamma_0}(\Omega)$; the standard change of variables yields

$$\begin{aligned} &z^t \in K(\Omega) \ : \\ &\int_\Omega \langle A_t \cdot \nabla z^t, \nabla(\phi - z^t) \rangle_{\mathbb{R}^N} dx \geqq \int_\Omega f^t (\phi - z^t) dx \quad \forall \phi \in K \ . \end{aligned} \tag{4.101}$$

Using Theorem 4.14 one can determine the form of the material derivative $\dot{z} \in H^1_{\Gamma_0}(\Omega)$.

Proposition 4.22 *The material derivative $\dot{z} \in H^1_{\Gamma_0}(\Omega)$ of the solution $z \in K$ to the variational inequality (4.89) in the direction of a vector field $V(\cdot,\cdot)$ is given as the unique solution to the following variational inequality*

$$\begin{aligned} &\dot{z} \in S_K(f) \ : \\ &\int_\Omega \langle \nabla \dot{z}, \nabla(\phi - \dot{z})\rangle_{\mathbb{R}^N} dx \geqq \int_\Omega f'(\phi - \dot{z})\, dx \\ &- \int_\Omega \langle A' \cdot \nabla z, \nabla(\phi - \dot{z})\rangle_{\mathbb{R}^N} dx \quad \forall \phi \in S_K(f) \ . \end{aligned} \tag{4.102}$$

In (4.102) it is assumed that $z = z^0 = z_0$.

In order to derive the form of the shape derivative $z' \in H^1(\Omega)$ we shall use the following definition, see Chap. 2, Sect. 2.30,

$$z' = \dot{z} - \nabla z \cdot V \ . \tag{4.103}$$

Taking into account the assumption (4.98), one can determine $z' \in H^1(\Omega)$.

Using the same argument as for the obstacle problem we obtain

Proposition 4.23 *The shape derivative $z' \in H^1(\Omega)$ of the solution to the variational inequality (4.89), in the direction of a vector field $V(\cdot,\cdot)$ for which (4.98) holds, is given as the unique solution to the following variational inequality:*

$$\begin{aligned} z' \in S_v(\Gamma_1) = \{\phi \in H^1(\Omega) | \phi &= -v_n \frac{\partial z}{\partial n} \quad on\ \Gamma_0, \\ \phi &\geqq -v_n \frac{\partial z}{\partial n} \quad on\ \mathcal{Z}, \\ \int \left(\phi + v_n \frac{\partial z}{\partial n}\right) d\mu &= 0\} \end{aligned} \tag{4.104}$$

$$\int_\Omega \langle \nabla z', \nabla(\phi - z')\rangle_{\mathbb{R}^N} dx \geqq 0 \quad \forall \phi \in S_v(\Gamma_1) \tag{4.105}$$

□

The reader is referred, e.g. to the paper of Neittaanmäki et al. (1988) for the related numerical results.

Remark. Let us observe that the existence of the material derivative $\dot{z} \in H^1(\Omega)$ implies the interior regularity of the solution z to the variational inequality (4.89). Let $V(\cdot,\cdot) \in C(0,\varepsilon; \mathcal{D}^k(\mathbb{R}^N;\mathbb{R}^N))$ be given with $V(0,\cdot) \in \mathcal{D}(\Omega;\mathbb{R}^N)$, then

$$z' = 0 \quad \text{thus} \quad \dot{z} = \nabla z \cdot V \in H_0^1(\Omega),$$

but $V(0,\cdot) \in \mathcal{D}(\Omega;\mathbb{R}^N)$ is arbitrary, therefore

$$\nabla z \in H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \ .$$

The same argument can be used to establish the interior regularity of solutions to all variational inequalities considered in Chap. 4.

4.5. Variational inequalities of the second kind

This section is concerned with the sensitivity analysis of variational inequalities of the second kind.

Let us consider the following problem:

Problem (P): Find an element $u \in H^1(\Omega)$ that minimizes the functional

$$\begin{aligned} J(\phi) = \frac{1}{2}a(\phi,\phi) - (f,\phi) + j(\phi) = \frac{1}{2}\int_\Omega (|\nabla\phi(x)|^2 + |\phi(x)|^2)dx \\ - \int_\Omega f(x)\phi(x)\,dx + \int_{\partial\Omega} |\phi(x)|d\Gamma \end{aligned} \tag{4.106}$$

over the space $H^1(\Omega)$.

It should be noted that the convex, non–smooth functional

$$j(\phi) = \int_{\partial\Omega} |\phi(x)|d\Gamma \quad \phi \in L^2(\partial\Omega) \tag{4.107}$$

can be defined as follows

$$j(\phi) = \max\{\int_{\partial\Omega} \mu(x)\phi(x)\,d\Gamma| - 1 \leqq \mu(x) \leqq 1 \text{ for a.e. } x \in \partial\Omega\} \ . \tag{4.108}$$

Let us introduce the following notation

$$\Lambda = \{\mu \in L^\infty(\partial\Omega)\,| - 1 \leqq \mu(x) \leqq 1 \quad \text{for a.e. } x \in \partial\Omega\} \ . \tag{4.109}$$

Hence

$$j(\phi) = \max\{\langle\mu,\phi\rangle | \phi \in \Lambda\}, \tag{4.110}$$

where

$$\langle\mu,\phi\rangle = \int_{\partial\Omega} \mu(x)\phi(x)\,d\Gamma \quad \forall \mu,\phi \in L^2(\partial\Omega) \ . \tag{4.111}$$

It is evident that the functional

$$J(\phi) = \frac{1}{2}a(\phi,\phi) - (f,\phi) + \max\{\langle\mu,\phi\rangle | \mu \in \Lambda\} \tag{4.112}$$

is non–differentiable on the space $H^1(\Omega)$. For the unique solution $u \in H^1(\Omega)$ to the problem (P) the following variational inequality of the second kind is satisfied

$$a(u,\phi - u) + (f,\phi - u) + j(\phi) - j(u) \geqq 0 \quad \forall\phi \in H^1(\Omega) \ . \tag{4.113}$$

We shall show that the solution $u(\Omega) \in H^1(\Omega)$ to the variational inequality (4.113) is directionally differentiable with respect to $f \in (H^1(\Omega))'$, and also that the shape derivative $u'(\Omega; V)$ exists.

First, let us observe that due to (4.110), the element $u \in H^1(\Omega)$ can be obtained by solving the following problem

Problem (PL): Find $(u,\lambda) \in H^1(\Omega) \times \Lambda$ such that

$$\mathcal{L}(u,\mu) \leqq \mathcal{L}(u,\lambda) \leqq \mathcal{L}(\phi,\lambda) \quad \text{for all } \mu \in \Lambda \text{ and for all } \phi \in H^1(\Omega), \tag{4.114}$$

where

$$\mathcal{L}(\phi,\mu) = \frac{1}{2}a(\phi,\phi) - (f,\phi) + \langle\mu,\gamma_\Gamma\phi\rangle \tag{4.115}$$

for all $\phi \in H^1(\Omega)$ and for all $\mu \in H^{-\frac{1}{2}}(\partial\Omega)$,

here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$; $H^{-\frac{1}{2}}(\partial\Omega) = (H^{\frac{1}{2}}(\partial\Omega))'$ stands for the dual space.

The second inequality of (4.114) is equivalent to the variational equation

$$u \in H^1(\Omega: \quad a(u,\phi) = (f,\phi) - \langle\lambda,\gamma_\Gamma\phi\rangle \quad \forall\phi \in H^1(\Omega) \ . \tag{4.116}$$

Therefore

$$u = -z + w, \tag{4.117}$$

where the elements $z = z(\lambda)$, $w = w(f) \in H^1(\Omega)$ are given as the unique solutions to the following equations

$$a(z(\lambda),\phi) = \langle\lambda,\gamma_\Gamma\phi\rangle \quad \forall\phi \in H^1(\Omega), \tag{4.118}$$

$$a(w(f),\phi) = (f,\phi) \quad \forall\phi \in H^1(\Omega), \tag{4.119}$$

respectively.

On the other hand, the first inequality of (4.114) can be simplified to obtain

$$\lambda \in \Lambda: \quad \langle\mu,\gamma_\Gamma u\rangle \leqq \langle\lambda,\gamma_\Gamma u\rangle \quad \forall\mu \in \Lambda \ . \tag{4.120}$$

Taking into account (4.117), we have the following variational inequality

$$\lambda \in \Lambda : \quad \langle \mu - \lambda, \gamma_\Gamma z(\lambda) \rangle \geqq \langle \mu - \lambda, \gamma_\Gamma w(f) \rangle \quad \forall \mu \in \Lambda \ . \tag{4.121}$$

We denote by $b(\cdot,\cdot) : H^{-\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \mathbb{R}$ the symmetric bilinear form

$$b(\mu, \eta) = \langle \eta, \gamma_\Gamma z(\mu) \rangle \quad \mu, \eta \in H^{-\frac{1}{2}}(\partial\Omega) \ . \tag{4.122}$$

The bilinear form (4.122) is coercive, i.e. there exists $\alpha > 0$ such that

$$b(\mu, \mu) \geqq \alpha \|\mu\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} \quad \forall \mu \in H^{-\frac{1}{2}}(\partial\Omega), \tag{4.123}$$

due to the fact that $\langle \mu, \gamma_\Gamma z(\mu) \rangle = a(z(\mu), z(\mu)) \geqq \alpha_0 \|z(\mu)\|^2_{H^1(\Omega)}$ for all $\mu \in H^{-\frac{1}{2}}(\partial\Omega)$, where $\alpha_0 > 0$.

We now turn to the analysis of the differential stability of solutions to the variational inequality (4.121) written in the form

$$\lambda \in \Lambda \subset H^{-\frac{1}{2}}(\partial\Omega) \ : \quad b(\lambda, \mu - \lambda) \geqq \langle \mu - \lambda, \gamma_\Gamma w \rangle \quad \forall \mu \in \Lambda \ . \tag{4.124}$$

It can be shown that the set (4.109) is a closed and convex subset of the Sobolev space $H^{-\frac{1}{2}}(\partial\Omega)$. We use the following notation.

$$\Xi^\pm = \{x \in \partial\Omega | \lambda(x) = \pm 1\}, \tag{4.125}$$

$$\Xi_0 = \{x \in \Xi^+ \cup \Xi^- | u(x) = 0\} \ . \tag{4.126}$$

Moreover, it is assumed that

$$\text{meas}\left(\Xi^\pm \setminus \text{int}\Xi^\pm\right) = 0 \tag{4.127}$$

$$\text{meas}\left(\Xi_0 \setminus \text{int}\Xi_0\right) = 0, \tag{4.128}$$

where meas(Ξ) is the one–dimensional Lebesgue measure of the set $\Xi \subset \partial\Omega$ and int$E$ for a set $E \subset \partial\Omega$ means the relative interior.
We need the assumptions (4.127) and (4.128) to obtain the closure in $H^{-\frac{1}{2}}(\partial\Omega)$ of the following sets:

$$\begin{aligned} K_1 = \{\phi \in L^2(\partial\Omega) \,|\, \phi(x) \geqq 0 \quad &\text{a.e. on } \Xi^+, \\ \phi(x) \leqq 0 \quad &\text{a.e. on } \Xi^-\} \end{aligned} \tag{4.129}$$

$$K_2 = \{\phi \in K_1 \,|\, \phi(x) = 0 \quad \text{a.e. on } \Xi_0\} \tag{4.130}$$

in the form

$$\begin{aligned} \text{cl}K_1 = \{\phi \in H^{-\frac{1}{2}}(\partial\Omega | \langle \phi, \eta \rangle \geqq 0 \text{ for all } \eta \in C_0(\partial\Omega) \text{ such that} \\ \text{spt}\eta \subset \Xi^+ \text{ and } \eta(x) \geqq 0 \quad \text{on } \Xi^+, \\ \text{or spt}\eta \subset \Xi^- \text{ and } \eta(x) \leqq 0 \quad \text{on } \Xi^-\}, \end{aligned} \tag{4.131}$$

$$\text{cl}K_2 = \{\phi \in \text{cl}K_1 | \langle \phi, \eta \rangle = 0 \text{ for all } \eta \in C_0(\partial\Omega) \text{ with spt}\eta \subset \Xi_0\} \tag{4.132}$$

We denote by $C_\Lambda(\lambda) \subset H^{-\frac{1}{2}}(\partial\Omega)$ the radial cone

$$C_\Lambda(\lambda) = \{\mu \in H^{-\frac{1}{2}}(\partial\Omega) | \exists \varepsilon > 0 \text{ such that } \lambda + \varepsilon\mu \in \Lambda\} \ . \tag{4.133}$$

It is obvious, because of (4.109), that

$$C_\Lambda(\lambda) \subset L^\infty(\partial\Omega) \ . \tag{4.134}$$

Finally we introduce the notation

$$M = \{\mu \in H^{-\frac{1}{2}}(\partial\Omega) | \langle \mu, \gamma_\Gamma u\rangle = 0\} \ , \tag{4.135}$$

M is a linear and closed subspace of the space $H^{-\frac{1}{2}}(\partial\Omega)$.

The following result was proved by Sokołowski (1988d).

Lemma 4.24 *Let us assume that the sets Ξ^+, Ξ^- and Ξ_0 are sufficiently regular. Then*

$$\begin{aligned} S =& \mathrm{cl}(C_\lambda(\Lambda)) \cap M = \mathrm{cl}(C_\lambda(\Lambda) \cap M) \\ =& \{\mu \in H^{-\frac{1}{2}}(\partial\Omega) | \langle \mu, \eta\rangle \geqq 0 \textit{ for all } \eta \in C_0(\partial\Omega) \\ & \textit{such that } \mathrm{spt}\eta \subset \Xi^+ \textit{ and } \eta \geqq 0 \textit{ or } \mathrm{spt}\eta \subset \Xi^- \textit{ and } \eta \leqq 0, \\ & \langle \mu, \varphi\rangle = 0 \textit{ for all } \varphi \in C_0(\partial\Omega) \textit{ with } \mathrm{spt}\varphi \subset \Xi_0\} \ . \end{aligned}$$

□

Lemma 4.24 implies that the set $\Lambda \subset H^{-\frac{1}{2}}(\partial\Omega)$ is polyhedric. Therefore, according to Theorem 4.6, the metric projection in the space $H^{-\frac{1}{2}}(\partial\Omega)$ with respect to the norm $\|\phi\|_{-\frac{1}{2},\Gamma} = (b(\phi,\phi))^{\frac{1}{2}}$ onto the set $\Lambda \subset H^{-\frac{1}{2}}(\partial\Omega)$ is conically differentiable.

The following theorem was proved by Sokołowski (1988d).

Theorem 4.25 *Let*

$$f_\varepsilon = f + \varepsilon f' + o(\varepsilon) \quad \textit{in } \left(H^1(\Omega)\right)' \tag{4.137}$$

and let $\lambda_\varepsilon \in \Lambda$ denote the solution to the variational inequality

$$\begin{aligned} &\lambda_\varepsilon \in \Lambda \ : \\ &b(\lambda_\varepsilon, \mu - \lambda_\varepsilon) \geqq \langle \mu - \lambda_\varepsilon, \gamma_\Gamma w(f_\varepsilon)\rangle \quad \forall \mu \in \Lambda \ . \end{aligned}$$

Then for $\varepsilon > 0$, ε small enough,

$$\lambda_\varepsilon = \lambda + \varepsilon\lambda' + o(\varepsilon) \quad \textit{in } H^{-\frac{1}{2}}(\partial\Omega), \tag{4.138}$$

where $\|o(\varepsilon)\|_{H^{-\frac{1}{2}}(\partial\Omega)}/\varepsilon \to 0$ as $\varepsilon \downarrow 0$.

The element $\lambda' \in H^{-\frac{1}{2}}(\partial\Omega)$ is given as the unique solution to the following variational inequality

$$\begin{aligned}&\lambda' \in S \ : \\ &b(\lambda', \mu - \lambda') \geqq \langle \mu - \lambda', \gamma_\Gamma w(f') \rangle \quad \forall \mu \in S \ .\end{aligned} \tag{4.139}$$

□

Finally we turn to the shape sensitivity analysis of the problem (P). For this purpose the problems (P_t) in the domains Ω_t, $t \in [0, \delta)$, are defined.

Problem (P_t): Find an element $u_t \in H^1(\Omega_t)$ that minimizes the functional

$$\begin{aligned}J_t(\phi) = &\frac{1}{2} \int_{\Omega_t} (|\nabla \phi(x)|^2 + |\phi(x)|^2) dx \\ &- \int_{\Omega_t} f(x)\phi(x)\, dx + \int_{\partial\Omega_t} |\phi(x)|\, d\Gamma\end{aligned} \tag{4.140}$$

over the space $H^1(\Omega_t)$.

It can be shown that there exists the unique solution $u_t \in H^1(\Omega_t)$ to the problem (P_t) for any $t \in [0, \delta)$; for $t = 0$ the problem (P_0) becomes the problem (P).

In order to determine the form of the strong material derivative $\dot{u} = \dot{u}(\Omega; V) \in H^1(\Omega)$ of the solution u to the problem (P) in the direction of a vector field $V(\cdot, \cdot)$ we introduce the notation

$$u^t = u_t \circ T_t \in H^1(\Omega) \quad t \in [0, \delta) \ . \tag{4.141}$$

The element $u^t \in H^1(\Omega)$, $t \in [0, \delta)$, is given as the unique solution to the auxiliary problem (P^t). The form of this problem will be derived.
To this end, by the change of variables $x = T_t(X), X \in \Omega, x \in \Omega_t$, in (4.140), we obtain the integral functional defined in Ω

$$J^t(\phi) = J_t(\phi \circ T_t^{-1}) \quad \forall \phi \in H^1(\Omega) \tag{4.142}$$

and given by

$$\begin{aligned}J^t(\phi) =& \frac{1}{2} a^t(\phi, \phi) - (f^t, \phi) + j_t(\phi) \\ =& \frac{1}{2} \int_\Omega \{\langle A_t(x) \cdot \nabla \phi(x), \nabla \phi(x) \rangle_{\mathbb{R}^N} + |\phi(x)|^2 \gamma(t)(x)\}\, dx \\ &- \int_\Omega f^t(x)\phi(x)\, dx + \int_{\partial\Omega} |\phi(x)| \omega(t)(x)\, d\Gamma \quad \forall \phi \in H^1(\Omega) \ .\end{aligned} \tag{4.143}$$

In (4.143) the following notation has been used

$$A_t(x) = \det(DT_t(x)) DT_t^{-1}(x) \cdot {}^*DT_t^{-1}(x) \quad x \in \Omega, \tag{4.144}$$

$$\gamma(t)(x) = \det(DT_t(x)) \quad x \in \Omega, \tag{4.145}$$

$$\omega(t)(x) = \| \det(DT_t(x))\, {}^*DT_t^{-1}(x) \cdot n(x) \|_{\mathbb{R}^N} \quad x \in \partial\Omega, \tag{4.146}$$

$$f^t(x) = (f \circ T_t)(x) \gamma(t)(x) \quad x \in \Omega \ . \tag{4.147}$$

The auxiliary problem, defined in the fixed domain Ω, can be stated as follows.

Problem (P^t): Find an element $u^t \in H^1(\Omega)$ that minimizes the functional $J^t(\phi)$ over the space $H^1(\Omega)$.

To obtain the form of the material derivative $\dot{u}(\Omega; V)$ we shall use the same reasoning as in the proof of directional differentiability of solutions to the problem (P) with respect to $f \in (H^1(\Omega))'$ (see Theorem 4.25).
It is assumed that

$$u^t = -z^t\left(\lambda^t\right) + w^t\left(f^t\right) = -z^t + w^t, \tag{4.148}$$

where

$$w^t \in H^1(\Omega)\ : a^t\left(w^t, \phi\right) = \left(f^t, \phi\right) \quad \forall \phi \in H^1(\Omega), \tag{4.149}$$
$$z^t = z^t\left(\lambda^t\right) \in H^1(\Omega)\ : a^t\left(z^t, \phi\right) = \langle \lambda^t, \gamma_\Gamma \phi \rangle \quad \forall \phi \in H^1(\Omega)\ . \tag{4.150}$$

The element $\lambda^t \in L^\infty(\partial\Omega) \subset H^{-\frac{1}{2}}(\partial\Omega)$ is given as the unique solution to the following variational inequality

$$\begin{aligned} &\lambda^t \in \Lambda\ : \\ &b_t\left(\lambda^t, \mu - \lambda^t\right) \geqq \langle \mu - \lambda^t, \omega(t)\gamma_\Gamma w^t \rangle \quad \forall \mu \in \Lambda, \end{aligned} \tag{4.151}$$

where the bilinear form

$$b^t\left(\cdot,\cdot\right) : H^{-\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \mathbb{R} \tag{4.152}$$

is defined as follows

$$b^t\left(\xi, \mu\right) = \langle \mu, \gamma_\Gamma z^t\left(\xi\right) \rangle \quad \forall \mu, \xi \in H^{-\frac{1}{2}}(\partial\Omega)\ . \tag{4.153}$$

For any $\xi \in H^{-\frac{1}{2}}(\partial\Omega)$ the element $z^t(\xi) \in H^1(\Omega)$ introduced in (4.153) is a solution to the equation

$$a^t\left(z^t\left(\xi\right), \phi\right) = \langle \xi, \gamma_\Gamma \phi \rangle \quad \forall \phi \in H^1(\Omega)\ . \tag{4.154}$$

The use of (4.146) implies that the material derivative $\dot{u}$ is given by

$$\dot{u} = -z'\left(\lambda\right) - z\left(\dot{\lambda}\right) + \dot{w}\ . \tag{4.155}$$

The elements $z'\left(\lambda\right)$ and $\dot{w} \in H^1(\Omega)$ can be determined, according to (4.149), (4.150) and (4.154), by solving the following equations:

$$z' = z'(\lambda) \in H^1(\Omega) :$$
$$a(z', \phi) = \int_{\partial\Omega} \lambda(x)\phi(x)\,d\Gamma - a'(z, \phi) \quad \forall\phi \in H^1(\Omega), \tag{4.156}$$
$$\dot{w} \in H^1(\Omega) :$$
$$a(\dot{w}, \phi) = \int_{\Omega} f'\phi dx - a'(w, \phi) \quad \forall\phi \in H^1(\Omega), \tag{4.157}$$

where it is assumed that $\lambda = \lambda^0$, $z = z^0$, $w = w^0$,

$$a'(y, \phi) = \int_{\Omega} \{A'(x) \cdot \nabla y(x), \nabla\phi(x)\rangle_{\mathbb{R}^N} + \tag{4.158}$$
$$\gamma'(x)y(x)\phi(x)\}dx \quad \forall y, \phi \in H^1(\Omega),$$
$$\gamma'(x) = \operatorname{div}(V(0,x)), \tag{4.159}$$
$$A'(x) = \operatorname{div}V(0,x)\mathcal{I} - DV(0,x) - {}^*DV(0,x) \ . \tag{4.160}$$

Finally we shall characterize the material derivative $\dot{\lambda} \in H^{-\frac{1}{2}}(\partial\Omega)$ of the solution $\lambda(\Gamma)$ to the problem (4.124) in the direction of a vector field $V(\cdot,\cdot)$. The form of this derivative was obtained by Sokołowski (1988d).

Lemma 4.26 *For $t > 0$, t small enough,*

$$\lambda^t = \lambda + t\dot{\lambda} + o(t) \quad in\ H^{-\frac{1}{2}}(\partial\Omega), \tag{4.161}$$

where $\|o(t)\|_{H^{-\frac{1}{2}}(\partial\Omega)}/t \to 0$ as $t \downarrow 0$.

The element $\dot{\lambda} \in H^{-\frac{1}{2}}(\partial\Omega)$ is given as the unique solution to the following variational inequality

$$\dot{\lambda} \in S_\Lambda(\lambda) :$$
$$b\left(\dot{\lambda}, \mu - \dot{\lambda}\right) \geqq \langle \mu - \dot{\lambda}, \gamma_\Gamma\dot{w} + \omega'\gamma_\Gamma w\rangle - b'\left(\lambda, \mu - \dot{\lambda}\right) \quad \forall\mu \in S_\Lambda(\lambda) \tag{4.162}$$

In (4.162) the notation is used

$$b'(\lambda, \mu) = \langle \mu, \gamma_\Gamma z'(\lambda) + \omega'\gamma_\Gamma z(\lambda)\rangle \quad \forall\lambda, \mu \in H^{-\frac{1}{2}}(\partial\Omega), \tag{4.163}$$
$$\omega'(x) = \operatorname{div}V(0,x) - \langle DV(0,x) \cdot n(x), n(x)\rangle_{\mathbb{R}^N} \ . \tag{4.164}$$

The proof of Lemma 4.26 relies on the application of Theorem 4.14 combined with Theorem 4.20.

Let us observe that from (4.116) and (4.148) it follows that the element $u^t \in H^1(\Omega)$ is a solution to the equation

$$a^t(u^t, \phi) = (f^t, \phi) - \langle \omega(t)\lambda^t, \gamma_\Gamma\phi\rangle \quad \forall\phi \in H^1(\Omega) \ . \tag{4.165}$$

Therefore

$$a(\dot{u},\phi)+a'(u,\phi)=(f,\phi)-\langle\dot{\lambda}+\omega'\lambda,\gamma_\Gamma\phi\rangle\quad\forall\phi\in H^1(\Omega)\ . \tag{4.166}$$

In (4.166) it is assumed that $u=u^0$ and $\lambda=\lambda^0$.
Moreover we have

$$f'(x)=\operatorname{div}(f(x)V(0,x))\quad x\in\Omega\ . \tag{4.167}$$

The material derivative $\dot{u}\in H^1(\Omega)$ minimizes the quadratic functional, in view of (4.165),

$$I(\phi)=\frac{1}{2}a(\phi,\phi)+a'(u,\phi)-(f',\phi)+\langle\dot{\lambda}+\omega'\lambda,\gamma_\Gamma\phi\rangle \tag{4.168}$$

over the space $H^1(\Omega)$.

On the other hand, (4.162) implies

$$\begin{aligned}&\langle\dot{\lambda},\gamma_\Gamma\dot{w}+\omega'\gamma_\Gamma w\rangle-\left(\dot{\lambda},\dot{\lambda}\right)-b'\left(\lambda,\dot{\lambda}\right)\\&\geqq\langle\mu,\gamma_\Gamma\dot{w}+\omega'\gamma_\Gamma w\rangle-\left(\dot{\lambda},\mu\right)-b'(\lambda,\mu)\quad\forall\mu\in\Lambda\ .\end{aligned} \tag{4.169}$$

Hence on the basis of (4.155) to (4.157) and (4.163) we obtain

$$\langle\dot{\lambda},\gamma_\Gamma\dot{u}+\omega'\gamma_\Gamma u\rangle\geqq\langle\mu,\gamma_\Gamma\dot{u}+\omega'\gamma_\Gamma u\rangle\ . \tag{4.170}$$

Thus we can replace the term $\langle\dot{\lambda},\gamma_\Gamma\phi\rangle$ in (4.168) with the following one

$$\max\{\langle\mu,\gamma_\Gamma\dot{u}+\omega'\gamma_\Gamma u\rangle\mid\mu\in S_\Lambda(\lambda)\}\ . \tag{4.171}$$

This leads to the following result obtained by Sokołowski (1988d).

Theorem 4.27 *For $t>0$, t small enough,*

$$u^t=u+t\dot{u}+o(t)\quad in\quad H^1(\Omega), \tag{4.172}$$

where $\|o(t)\|_{H^1(\Omega)}/t\to0$ as $t\downarrow0$.

The element $\dot{u}\in H^1(\Omega)$ minimizes the functional

$$\begin{aligned}I(\phi)=&\frac{1}{2}a(\phi,\phi)+a'(u,\phi)+(f',\phi)+\langle\omega'\lambda,\gamma_\Gamma\phi\rangle+\\&\max\{\langle\mu,\gamma_\Gamma\phi+\omega'\gamma_\Gamma u\rangle\mid\mu\in S_\Lambda(\lambda)\}\end{aligned} \tag{4.173}$$

over the space $H^1(\Omega)$.

We shall determine the form of the domain (shape) derivative $u'=u'(\Omega;V)\in H^1(\Omega)$ of the solution $u(\Omega)$ to the problem (P) in the direction a of vector field $V(\cdot,\cdot)$. We have

$$u' = \dot{u} - \nabla u \cdot V(0) \ . \tag{4.174}$$

Therefore the shape derivative $u' \in H^1(\Omega)$ is well defined provided that

$$\nabla u \cdot V(0) \in H^1(\Omega) \ . \tag{4.175}$$

For any vector field $V(\cdot,\cdot) \in C(0,\varepsilon; C^1(\mathbb{R}^N;\mathbb{R}^N))$ with

$$v_n(x) = \langle V(0,x), n(x)\rangle_{\mathbb{R}^N} = 0 \quad x \in \partial\Omega \tag{4.176}$$

it follows that

$$u' = 0, \text{ i.e. } \dot{u} = \nabla u \cdot V(0) \ . \tag{4.177}$$

Furthermore $\dot{\lambda} = V_\tau \cdot \nabla_\Gamma \lambda = \langle V, \nabla_\Gamma \lambda\rangle_{\mathbb{R}^N}$, where $\nabla_\Gamma \lambda$ is the tangential gradient of $\lambda(\Gamma)$ on $\Gamma = \partial\Omega$; V_τ denotes the tangential component of $V(0)$ on Γ.
If the condition (4.176) is satisfied, then from (4.166) and (4.177) we obtain

$$\begin{aligned} &a(\nabla u \cdot V(0), \phi) + a'(u,\phi) = \\ &(f,\phi) - \langle V_\tau \cdot \nabla_\Gamma \lambda + \omega'\lambda, \gamma_\Gamma \phi\rangle \quad \forall \phi \in H^1(\Omega) \ . \end{aligned} \tag{4.178}$$

Thus

$$\begin{aligned} &u' \in H^1(\Omega) \ : \\ &a(u',\phi) = \langle g_n(\phi), v_n\rangle_{\mathcal{D}^{-1}(\partial\Omega)\times\mathcal{D}^1(\partial\Omega)} \\ &\qquad + \langle \omega'\lambda, \gamma_\Gamma\phi\rangle + \langle \lambda', \gamma_\Gamma \phi\rangle \quad \forall\phi \in H^1(\Omega) \ . \end{aligned} \tag{4.179}$$

This equation results from (4.166), (4.174) and (4.178) for any vector field $V(\cdot,\cdot) \in C(0,\varepsilon;\ C^1(\mathbb{R}^N;\mathbb{R}^N))$. The linear mapping

$$H^1(\Omega) \ni \phi \to g_n(\phi) \in \mathcal{D}^{-1}(\partial\Omega)$$

is continuous.
Here $\lambda' = \lambda'(\Gamma; V)$ denotes the boundary shape derivative of the element $\lambda(\Gamma)$ in the direction of a vector field $V(\cdot,\cdot)$ given by

$$\begin{gathered} \lambda' = \dot{\lambda} - V_\tau \cdot \nabla_\Gamma \lambda, \\ \text{where } V_\tau(x) = V(0,x) - v_n(x)\, n(x) \quad x \in \partial\Omega \ . \end{gathered}$$

The element $\lambda' \in H^{-\frac{1}{2}}(\partial\Omega)$ is given as the unique solution to the following variational inequality

$$\begin{aligned} &\lambda' \in S_\Lambda(\lambda) \ : \\ &b(\lambda', \mu - \lambda') \geqq \langle \mu - \lambda', v_n \kappa \gamma_\Gamma u + v_n \gamma_\Gamma \frac{\partial u}{\partial n} + \gamma_\Gamma w' - \gamma_\Gamma z'(\lambda)\rangle \quad \forall \mu \in S_\Lambda(\lambda), \end{aligned}$$

where κ denotes the mean curvature of the boundary $\partial\Omega$.

Therefore the shape derivative $u' = u'(\Omega; V) \in H^1(\Omega)$ can be determined as the unique solution to the variational problem derived by Sokołowski (1988d).

Theorem 4.28 *Let us assume that (4.175) is satisfied. Then the shape derivative $u' \in H^1(\Omega)$ minimizes the functional*

$$I(\phi) = \frac{1}{2} a(\phi, \phi) - \langle g_n(\phi), v_n \rangle_{\mathcal{D}^{-1}(\partial\Omega)\times\mathcal{D}^1(\partial\Omega)}$$
$$+ \max\{\langle \mu, \gamma_\Gamma \phi + v_n \kappa \gamma_\Gamma u + v_n \gamma_\Gamma \frac{\partial u}{\partial n} \rangle | \mu \in S_\Lambda(\lambda)\}$$

over the space $H^1(\Omega)$.

The distribution $g_n(\phi) \in \mathcal{D}^{-1}(\partial\Omega)$ has the following representation

$$\langle g_n(\phi), v_n \rangle_{\mathcal{D}^{-1}(\partial\Omega)\times\mathcal{D}^1(\partial\Omega)} = \int_{\partial\Omega} v_n [\nabla_\Gamma u \cdot \nabla_\Gamma \phi - f\phi + \kappa\lambda\phi] d\Gamma$$

for any $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\partial\phi/\partial n = 0$ on $\partial\Omega$.

□

4.6. Sensitivity analysis of the Signorini problem in elasticity

This section is concerned with the sensitivity analysis of solutions to the system of equations describing the deformations of plane elastic solids. First, the mathematical model will be presented. For simplicity, it is assumed that N=2, however the same results can be obtained for N=3.

Let us consider the deformations of a plane elastic body of reference configuration $\overline{\Omega} \subset \mathbb{R}^2$. It is assumed that the body is subjected to body forces $f = (f_1, f_2)$ and that surface tractions $P = (P_1, P_2)$ are applied to a portion Γ_1 of the boundary $\Gamma = \partial\Omega$ of the body. Moreover, it is assumed that the body is fixed along a portion Γ_0 of the boundary, and that frictionless contact conditions are prescribed on a portion Γ_2 of the boundary $\partial\Omega$.

Let $u = (u_1, u_2)$ and $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, denote arbitrary displacement and stress fields in the body. We consider Hookean elastic materials

$$\sigma_{ij}(x) = c_{ijkl}(x) u_{k,l}(x) \quad x \in \Omega,$$

where $\{c_{ijkl}(x)\}$, $i, j, k, l = 1, 2$, denote the components of Hooke's tensor $\mathcal{C}$ at $x \in \Omega$, $u_{k,l} = \partial u_k / \partial x_l$; the summation convention over repeated indices $i, j, k, l = 1, 2$ is used.

It is assumed that

$$c_{ijkl}(x) = c_{jikl}(x) = c_{klij}(x) \quad x \in \Omega,$$
$$c_{ijkl}(\cdot) \in L^{\infty}(\Omega) \quad \text{for all } i,j,k,l = 1,2 \ .$$

Furthermore it is supposed that there exists a positive constant $\alpha_0 > 0$ such that

$$c_{ijkl}(x)\, e_{ij} e_{kl} \geqq \alpha_0 e_{ij} e_{ij} \quad \text{for all } x \in \Omega$$
$$\text{and for all symmetric matrices } [e_{ij}]_{2\times 2} \ .$$

It is said that a stress field $\sigma = \sigma(x)$ is in equilibrium at a point x in the interior of Ω if

$$(\operatorname{div}\sigma(x))_i = -\sigma_{ij}(x)_{,j} = f_i(x) \quad x \in \Omega, \ i = 1,2,$$

where

$$\sigma_{ij,j} = \sum_{j=1}^{2} \frac{\partial \sigma_{ij}}{\partial x_j}, \quad i = 1,2 \ .$$

It is said that a displacement field $u = u(x)$ satisfies the kinematic boundary conditions on Γ_0 if

$$u_i(x) = 0 \quad x \in \Gamma_0 \ , \quad i = 1,2 \ .$$

If P is the traction applied on Γ_1, then for stress produced the following relation

$$(\sigma(x) \cdot n(x))_i = \sigma_{ij}(x) n_j(x) = P_i(x) \qquad x \in \Gamma_1, \quad i = 1,2,$$

must hold.

If the body is unilaterally supported by a frictionless rigid foundation and the portion Γ_2 of the boundary $\partial\Omega$ is a candidate for the contact region, i.e. the contact occurs at a portion $\mathcal{Z} \subset \Gamma_2$ which is not known a priori, then the unilateral boundary conditions on Γ_2 are given by:

$$\begin{cases} u \cdot n \leqq 0, \quad \sigma_n \leqq 0, \quad \sigma_n u \cdot n = 0, \\ \sigma_\tau = 0, \end{cases}$$

where σ_n and σ_τ denote the normal and tangential components of the stress $\sigma \cdot n$ on Γ, respectively.

Let u denote a specific displacement field of the body, corresponding to the equilibrium state of the body determined for given data: body forces f and tractions P . The displacement field u is given as the unique weak solution to the following nonlinear system:

$$-(c_{ijkl}(x) u_{k,l}(x))_{,j} = f_i(x) \quad \text{in } \Omega, \quad i = 1,2,$$
$$u_i = 0 \quad \text{on } \Gamma_0, \ i = 1,2,$$
$$c_{ijkl} u_{k,l} n_j = P_i \quad \text{on } \Gamma_1 \ , \ i = 1,2,$$
$$u_i n_i \leqq 0, \ \sigma_n = c_{ijkl} u_{k,l} n_j n_i \leqq 0, \ \sigma_n u_i n_i = 0,$$
$$c_{ijkl} u_{k,l} n_j = \sigma_n n_i \quad \text{on } \Gamma_2 \ .$$

Let us recall that for a weak solution to the system under consideration the following variational inequality is satisfied

$$
\begin{aligned}
&u \in K \ : \\
&a(u, \phi - u) \geqq \langle F, \phi - u \rangle \quad \forall \phi \in K,
\end{aligned}
\tag{4.180}
$$

where the bilinear form $a(\cdot,\cdot)$, the element $F \in (H^1(\Omega;\mathbb{R}^2))'$, and the convex and closed set $K \subset H^1(\Omega;\mathbb{R}^2)$ are defined as follows:

$$
a(z,\phi) = \int_\Omega Dz{:}\mathcal{C}{:}\,D\phi dx = \int_{\partial\Omega} c_{ijkl}(x)\, z_{i,j}(x)\, \phi_{k,l}(x)\, dx
$$

$$
\forall z, \phi \in H^1(\Omega : \mathbb{R}^2),
$$

$$
\langle F, \phi \rangle = \int_\Omega f \cdot \phi dx + \int_{\Gamma_1} P \cdot \phi d\Gamma \quad \forall \phi \in H^1(\Omega;\mathbb{R}^2),
$$

$$
K = \{\phi \in H^1(\Omega;\mathbb{R}^2) | \phi = 0 \quad \text{on } \Gamma_0\ , \quad \phi \cdot n \leqq 0 \quad \text{on } \Gamma_2\}\ .
$$

It is assumed that $f \in L^2(\Omega;\mathbb{R}^2)$, $P \in L^2(\Gamma_1;\mathbb{R}^2)$ are given, and that $\text{meas}(\Gamma_0) > 0$; therefore there exists (Fichera 1972) the unique weak solution to the variational inequality (4.180).

In order to determine the form of the material and shape derivatives of solutions to the variational inequality (4.180) we have to direct our attention to the following abstract result.

4.6.1. Differential stability of solutions to variational inequalities in Hilbert spaces

Let $H, \mathcal{H}$ be Hilbert spaces, and let $\mathcal{H}', H'$ denote the dual spaces. Moreover it is assumed that a linear and continuous mapping $\mathcal{R} \in \mathcal{L}(H;\mathcal{H})$ is given. We denote by K the closed and convex subset of the space H defined as follows:

$$
K = \{\phi \in W | \mathcal{R}\phi \in \mathcal{K} \subset \mathcal{H}\}\ . \tag{4.181}
$$

Let the bilinear form $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ be coercive, continuous and symmetric

$$
a(\phi,\phi) \geqq \alpha \|\phi\|_H^2 \quad \forall \phi \in H, \tag{4.182}
$$

$$
|a(\phi,\psi)| \leqq \beta \|\phi\|_H \|\psi\|_H \quad \forall \phi,\psi \in H, \tag{4.183}
$$

$$
a(\phi,\psi) = a(\psi,\phi) \quad \forall \phi,\psi \in H,
$$

where $\alpha > 0$ and β are constants.
We shall study the differentiability of the mapping

$$
H' \ni f \longrightarrow \Pi(f) \in K \subset H, \tag{4.184}
$$

where for each $f \in H'$, the element $\Pi(f) \in K$ is determined as the unique solution to the following variational inequality:

$$\begin{aligned} &\Pi(f) \in K \ : \\ &a(\Pi(f), \phi - \Pi(f)) \geqq \langle f, \phi - \Pi(f) \rangle \quad \forall \phi \in K \ . \end{aligned} \tag{4.185}$$

In the case of the Signorini problem (4.180) we have

$$H = H^1_{\Gamma_0}(\Omega; \mathbb{R}^2), \ K = \{\phi \in W | \mathcal{R}\phi \in \mathcal{K}\},$$

$$\mathcal{H} = \{\phi_n = \mathcal{R}\phi \in H^{\frac{1}{2}}(\Gamma_2) | \phi \in H^1(\Omega; \mathbb{R}^2), \ \phi = 0 \ \text{ on } \Gamma_0\}, \tag{4.186}$$

$$\mathcal{K} \subset H^{\frac{1}{2}}(\Gamma_2), \ \mathcal{K} = \{h \in \mathcal{H} | h(x) \leqq 0 \ \text{ on } \Gamma_2\}, \tag{4.187}$$

$$(\mathcal{R}\phi)(x) \ \stackrel{def}{=} \ \phi_n(x) = \sum_{i=1}^{2} \phi_i(x) n_i(x) \quad x \in \Gamma_2 \tag{4.188}$$

$$\forall \phi = (\phi_1, \phi_2) \in H^1(\Omega; \mathbb{R}^2) \ .$$

We shall prove that the conical differentiability of the mapping (4.184) is equivalent to the conical differentiability of the projection $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K} \subset \mathcal{H}$.

Since the operator $\mathcal{R}$ maps H onto $\mathcal{H}$ and $0 \in \mathcal{K} \subset \mathcal{H}$, therefore

$$\ker \mathcal{R} \cap K = \ker \mathcal{R} \ . \tag{4.189}$$

Introducing the notation

$$H_1 = \ker \mathcal{R}, \ H_2 = H_1^{\perp} \tag{4.190}$$

we have

$$H = H_1 \oplus H_2 \ . \tag{4.191}$$

It can be shown that there exists the inverse operator $\mathcal{R}^{-1} \in \mathcal{L}(\mathcal{H}; H_2)$. The scalar product $((\cdot,\cdot))_{\mathcal{H}}$ is defined in the following way:

$$((h_1, h_2))_{\mathcal{H}} = a\left(\mathcal{R}^{-1} h_1, \mathcal{R}^{-1} h_2\right) \quad \forall h_1, h_2 \in \mathcal{H} \ . \tag{4.192}$$

We denote by $P_{\mathcal{K}}$ the orthogonal projection

$$P_{\mathcal{K}} : \mathcal{H} \longrightarrow \mathcal{K} \subset \mathcal{H}, \tag{4.193}$$

i.e. for a given element $\xi \in \mathcal{H}$, the element $p = P_{\mathcal{K}}(\xi)$ is determined as the unique solution to the variational inequality:

$$\begin{cases} p = P_{\mathcal{K}}(\xi) \in \mathcal{K} \\ ((p - \xi, h - p)) \geqq 0 \quad \forall h \in \mathcal{K} \ . \end{cases} \tag{4.194}$$

Let $f \in H'$ be a given element, and let $\Phi(f) \in \mathcal{H}$ be the unique solution to the variational equation:

$$((\Phi(f), h)) = \langle f, \mathcal{R}^{-1}h\rangle \quad \forall h \in \mathcal{H}, \tag{4.195}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between H' and H.

It should be remarked that the linear mapping

$$H' \ni f \longrightarrow \in \Phi(f) \in \mathcal{H} \tag{4.196}$$

is continuous.

The foregoing results allow to decompose the variational inequality (4.185) in the following way: the solution $y = \Pi(f)$ to (4.185) can be written in the form:

$$\Pi(f) = y_1 + y_2 \quad y_i \in H_i,\ i = 1,2, \tag{4.197}$$

where $y_1 \in H_1$ is given as the unique solution to the variational equation:

$$y_1 \in H_1 \ :\ a(y_1, \eta) = \langle f, \eta\rangle \quad \forall \eta \in H_1 \ . \tag{4.198}$$

The element $y_2 \in H_2$ is defined as follows:

$$y_2 = \mathcal{R}^{-1} P_{\mathcal{K}}(\Phi(f)) \ . \tag{4.199}$$

The results obtained may be stated as the following lemma:

Lemma 4.29 *The mapping (4.184) is conically differentiable if and only if the metric projection (4.193) is conically differentiable.*

□

Finally let us consider the variational inequality:

$$\begin{aligned} &y_\varepsilon \in K \ : \\ &a_\varepsilon(y_\varepsilon, \phi - y_\varepsilon) \geqq \langle f_\varepsilon, \phi - y_\varepsilon\rangle \quad \forall \phi \in K, \end{aligned} \tag{4.200}$$

where $\varepsilon \in [0,\delta)$ is a parameter, $\delta > 0$, and $a_\varepsilon(\cdot,\cdot) : H \times H \to \mathbb{R}$ denotes the family of bilinear forms such that (4.182) and (4.183) are satisfied uniformly with respect to the parameter $\varepsilon \in [0,\delta)$. We denote by $\mathcal{A}_\varepsilon \in \mathcal{L}(H;H')$ the linear operator defined by

$$\langle \mathcal{A}_\varepsilon z, \phi\rangle = a_\varepsilon(z,\phi) \quad \forall z,\phi \in H \ . \tag{4.201}$$

Furthermore we denote by $((\cdot,\cdot))_{\mathcal{H}}$ the scalar product of the form: $((h,\eta))_{\mathcal{H}} \overset{def}{=} a_0(\mathcal{R}^{-1}h, \mathcal{R}^{-1}\eta) \quad \forall h,\eta \in \mathcal{H}$.

It is assumed that the mapping (4.193) is conically differentiable, i.e. for $\varepsilon > 0$, ε small enough,

$$\forall h \in \mathcal{H} : P_{\mathcal{K}}(\xi + \varepsilon h) = P_{\mathcal{K}}(\xi) + \varepsilon Q(h) + o(\varepsilon), \tag{4.202}$$

where $\|o(\varepsilon)\|_{\mathcal{H}}/\varepsilon \to 0$ as $\varepsilon \downarrow 0$.

Theorem 4.30 *Let us assume that*
(i) there exists an operator $\mathcal{A}' \in \mathcal{L}(H; H')$ *such that*

$$\lim_{\varepsilon \downarrow 0} \| (\mathcal{A}_\varepsilon - \mathcal{A}_0) / \varepsilon - \mathcal{A}' \|_{\mathcal{L}(H;H')} = 0 \tag{4.203}$$

(ii) there exists an element $f' \in H'$ *such that*

$$\lim_{\varepsilon \downarrow 0} \| (f_\varepsilon - f_0) / \varepsilon - f' \|_{H'} = 0 \tag{4.204}$$

(iii) the relation (4.202) holds.

Then for $\varepsilon > 0$, ε *small enough, the solution to (4.200) is of the form*

$$y_\varepsilon = y_0 + \varepsilon y' + o(\varepsilon) \quad in\ H, \tag{4.205}$$

where $\|o(\varepsilon)\|_H / \varepsilon \to 0$ *as* $\varepsilon \downarrow 0$.

The element $y' \in H$ *is given by:*

$$y' = \Pi_1 (f' - \mathcal{A}' y_0) + \mathcal{R}^{-1} \mathcal{Q} (\Phi (f' - \mathcal{A}' y_0)), \tag{4.206}$$

where for any $\theta \in H'$, *the elements* $\Pi_1(\theta), \Phi(\theta)$ *are determined as the unique solutions to the following variational equations:*

$$\begin{cases} \Pi_1 = \Pi_1 (\theta) \in H_1 \\ a_0 (\Pi_1, \eta) = \langle \theta, \eta \rangle \quad \forall \eta \in H_1 \end{cases} \tag{4.207}$$

and

$$\begin{cases} \Phi = \Phi (\theta) \in \mathcal{H} \\ a_0 (\mathcal{R}^{-1} \Phi, \mathcal{R}^{-1} h) = \langle \theta, \mathcal{R}^{-1} h \rangle \quad \forall h \in \mathcal{H}, \end{cases} \tag{4.208}$$

respectively.

□

To prove Theorem 4.30 one can use the same reasoning as that of the proof of Theorem 4.14 provided that the condition (4.202) is satisfied.

In order to show that (4.202) holds, it suffices to prove that the set (4.180) is polyhedric.

From the results obtained by Mignot (1976) we have that the set

$$\mathcal{K} = \{h \in \mathcal{H}(\Gamma_2) | h \leqq 0\}$$

is polyhedric for $\mathcal{H}(\Gamma_2) \subset L^2(\Gamma_2)$, the linear subspace with the scalar product $((\cdot,\cdot))$, if conditions (A4)–(A6) are satisfied:

(A4) $\eta^+, \eta^- \in \mathcal{H}(\Gamma_2) \quad \forall \eta \in \mathcal{H}(\Gamma_2)$,

(A5) $((\eta^+, \eta^-)) \leqq 0 \quad \forall \eta \in \mathcal{H}(\Gamma_2)$,

(A6) $\mathcal{H}(\Gamma_2) \cap C_0(\Gamma_2)$ is dense in $C_0(\Gamma_2)$,

where $\eta^+ = \max\{0, \eta\}$, $\eta^- = \max\{0, -\eta\}$.

If this is the case, then $\{\mathcal{H}(\Gamma_2),\ ((\cdot,\cdot))\}$ is the so-called Dirichlet space.

Lemma 4.31 *The assumptions (A4)–(A6) hold for the space $\mathcal{H}$ defined by (4.186) with the scalar product (4.192).*

Proof. It is evident that the assumption (A4) is satisfied, because the space $\mathcal{H}(\Gamma_2)$ is a closed linear subspace of the Sobolev space $H^{\frac{1}{2}}(\Gamma_2)$. The general properties of the Sobolev spaces $H^{\frac{1}{2}}$ given by Lions et al. (1968) imply that the assumption (A6) is also satisfied.

Hence we have to prove that (A5) holds. First, we suppose that the outward unit normal vector on Γ_2 is of the form $n = (1,0)$, whence

$$(\mathcal{R}\zeta)(\cdot) = \zeta_1(\cdot) \in \mathcal{H}(\Gamma_2) \quad \forall \zeta = (\zeta_1, \zeta_2) \in H \ . \tag{4.209}$$

Let $\eta \in \mathcal{H}(\Gamma_2)$ be a given element, and let η^* be defined as follows

$$\zeta^* = \mathcal{R}^{-1}\eta, \quad \zeta^* = (\zeta_1^*, \zeta_2^*) \ . \tag{4.210}$$

Making use of the definition of the scalar product in $\mathcal{H}$ we have

$$((\eta,\eta)) = \inf\{a(\zeta,\zeta) : \zeta \in H, \mathcal{R}\zeta = \eta\} = a(\zeta^*,\zeta^*) \ . \tag{4.211}$$

On the other hand

$$a(\zeta,\zeta) = a_1(\zeta_1,\zeta_1) + a_2(\zeta_1,\zeta_2) + a_3(\zeta_2,\zeta_1) + a_4(\zeta_2,\zeta_2), \tag{4.212}$$

where $a_i(\cdot,\cdot)$, $i = 1,\ldots,4$, are the appropriate bilinear forms. Taking into account the necessary and sufficient optimality conditions for the minimization problem (4.211), we obtain

$$a(\zeta^*,\zeta^*) = a_1(\zeta_1^*,\zeta_1^*) - a_4(\zeta_2^*,\zeta_2^*) \ . \tag{4.213}$$

Let

$$\psi^* = \mathcal{R}^{-1}|\eta|, \tag{4.214}$$

where $|\eta| = \eta^+ - \eta^-$ for $\eta \in \mathcal{H}(\Gamma_2)$.
Furthermore, since $\zeta_i^* \in H^1(\Omega)$, $i = 1,2$, then $|\zeta_i^*| \in H^1(\Omega)$, $i = 1,2$, where

$$|\zeta^*| = (|\zeta_1^*|, |\zeta_2^*|) \in H \ . \tag{4.215}$$

It should be noted that

$$|\zeta^*| - \psi^* \in H_1 = \ker \mathcal{R}, \tag{4.216}$$

$$\psi \in H_2 = (\ker \mathcal{R})^{\perp} \ . \tag{4.217}$$

Thus

$$a(\psi^*, |\zeta^*| - \psi^*) = 0 \tag{4.218}$$

and

$$\begin{aligned}
&a(\psi^*,\psi^*) - a(|\zeta^*|,|\zeta^*|) = \\
&2a(\psi^*,\psi^* - |\zeta^*|) - a(\psi^* - |\zeta^*|,\psi^* - |\zeta^*|) = \\
&a(\psi^* - |\zeta^*|,\psi^* - |\zeta^*|) \leqq 0 \ .
\end{aligned} \tag{4.219}$$

Hence from (4.219) it follows that

$$\begin{aligned}
((|\eta|,|\eta|)) = a(\psi^*,\psi^*) \leqq a(|\zeta^*|,|\zeta^*|) = a_1(|\zeta_1^*|,|\zeta_1^*|) \\
- a_4(|\zeta_2^*|,|\zeta_2^*|) = a_1(\zeta_1^*,\zeta_1^*) - a_4(\zeta_2^*,\zeta_2^*) = ((\eta,\eta)) \ .
\end{aligned} \tag{4.220}$$

The condition $((\eta^+,\eta^-)) \leqq 0$ for all $\eta \in \mathcal{H}(\Gamma_2)$ results from (4.220).

If the condition (4.209) is not satisfied, then one can use the following transformation:

$$\begin{aligned}
\psi_1 &= \mathcal{N}_1\phi_1 + \mathcal{N}_2\phi_2 \ , \\
\psi_2 &= -\mathcal{N}_2\phi_1 + \mathcal{N}_1\phi_2 \ ,
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{N} = (\mathcal{N}_1,\mathcal{N}_2) \quad \mathcal{N}_1,\mathcal{N}_2 \in W^{1,\infty}(\Omega) , \\
&\mathcal{N}(x) = n(x) \ \text{ for a.e. } x \in \Gamma_2 \ , \\
&\mathcal{N}_1^2(x) + \mathcal{N}_2^2(x) \geqq c > 0 \text{ a.e. on } \Omega \ .
\end{aligned}$$

Hence

$$\phi \cdot n = \psi_1 \quad \text{on } \Gamma_2 \ .$$

This concludes the proof of Lemma 4.31. □

4.6.2. Shape sensitivity analysis

The foregoing results make it possible to derive the form of the functional sensitivity coefficient for solutions to the variational inequality (4.177). The following notation is used

$$H = \{\phi \in H^1(\Omega;\mathbb{R}^2) \,|\, \phi = 0 \quad \text{on } \Gamma_0\} \ = H^1_{\Gamma_0}(\Omega;\mathbb{R}^2) \ .$$

Let $F \in H'$ be a given element of the form (4.179), and let

$$u = \Pi(F)$$

be the solution to the variational inequality (4.177).

Theorem 4.32 *For $t > 0$, t small enough,*

$$\forall h \in H' : \Pi(F + th) = \Pi(F) + t\Pi'(h) + o(t) , \tag{4.221}$$

where $\|o(t)\|_{H^1(\Omega;\mathbb{R}^2)}/t \to 0$ as $t \downarrow 0$.

The element $Q = \Pi'(h)$ is given as the unique solution to the following variational inequality

$$\begin{aligned}&\mathcal{Q} \in S \ : \\ &a(\mathcal{Q}, \phi - \mathcal{Q}) \geqq \langle h, \phi - \mathcal{Q}\rangle \quad \forall \phi \in S,\end{aligned} \tag{4.222}$$

where

$$\begin{aligned} S =& \{\phi \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^2) | \phi \cdot n \geqq 0 \quad on\ \mathcal{Z}, \quad a(u,\phi) = \langle F, \phi\rangle\} \\ =& \{\phi \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^2) | \phi \cdot n \geqq 0 \quad on\ \mathcal{Z}^0, \quad \phi\cdot n = 0 \quad on\ \mathcal{Z}^+\}\end{aligned} \tag{4.223}$$

and

$$\mathcal{Z} = \{x \in \Gamma_2 | u(x)\cdot n(x) = 0\}, \tag{4.224}$$

$$\mathcal{Z}^+ = \ \text{fine spt}\mu, \tag{4.225}$$

$$\mathcal{Z}^0 = \mathcal{Z} \setminus \mathcal{Z}^+ \ . \tag{4.226}$$

The Radon measure $\mu \geqq 0$ concentrated on Γ_2 is defined as follows

$$\int \phi \cdot n d\mu = a(u,\phi) - \langle F, \phi\rangle \quad 0 \leqq \phi \in C^1(\overline{\Omega};\mathbb{R}^2) \ . \tag{4.227}$$

□

Theorem 4.32 results from Theorem 4.30, because from Lemma 4.31 it follows that the set (4.181) is polyhedric. This proves that the assumption (4.202) holds for the Signorini problem.

Finally let us consider the shape sensitivity analysis of the problem (4.177). We denote by u_t a solution to the variational inequality (4.177) defined in the domain $\Omega_t \subset \mathbb{R}^2$,

$$\begin{aligned}&u_t \in K(\Omega_t) \ : \\ &a_t(u_t, \phi - u_t) \geqq \langle F_t, \phi - u_t\rangle_t \quad \forall \phi \in K(\Omega_t),\end{aligned} \tag{4.228}$$

where

$$K(\Omega_t) = \{\phi \in H^1(\Omega_t;\mathbb{R}^2) | \phi = 0 \text{ on } \Gamma_0^t,\ \phi\cdot n_t \geqq 0 \text{ on } \Gamma_2^t\}, \tag{4.229}$$

$$\langle F_t, \phi\rangle_t = \int_{\Omega_t} f\cdot\phi dx + \int_{\Gamma_1^t} P\cdot\phi d\Gamma \quad \forall \phi \in H^1(\Omega_t;\mathbb{R}^2) \tag{4.230}$$

$$\text{with } f \in L^2(\mathbb{R}^2;\mathbb{R}^2) \text{ and } P \in H^1(\mathbb{R}^2;\mathbb{R}^2),$$

$$a_t(z,\phi) = \int_{\Omega_t} Dz{:}C{:}D\phi dx = \int_{\Omega_t} c_{ijkl}(x)\, z_{i,j}(x)\, \phi_{k,l}(x)\, dx \tag{4.231}$$

$$\forall z, \phi \in H^1(\Omega_t;\mathbb{R}^2) \text{ with } c_{ijkl} \in L^\infty_{\text{loc}}(\mathbb{R}^2) \quad \forall i,j,k,l = 1,2 \ .$$

Theorem 4.33 *The mapping*

$$[0,\delta) \ni t \longrightarrow u_t \circ T_t \in H \tag{4.232}$$

is differentiable at $t = 0^+$, *i.e. for* $t > 0$, t *small enough,*

$$u^t = u_t \circ T_t = u + t\dot{u} + o(t), \tag{4.233}$$

where $\|o(t)\|_{H^1(\Omega;\mathbb{R}^2)}/t \to 0 \quad as \quad t \downarrow 0.$

The strong material derivative $\dot{u} \in H^1(\Omega;\mathbb{R}^2)$ *is the unique solution to the following variational inequality*

$$\begin{aligned} &\dot{u} \in S(\Omega) \ : \\ &a(\dot{u}, \phi - \dot{u}) \geqq \langle F' - \mathcal{B}u, \phi - \dot{u}\rangle + a(DV \cdot u, \phi - \dot{u}) \quad \forall \phi \in S(\Omega), \end{aligned} \tag{4.234}$$

where

$$S(\Omega) = \{\phi \in W | \phi \cdot n \leqq n \cdot DV \cdot u \quad on \quad \mathcal{Z}, \\ a(u,\phi) - \langle F, \phi\rangle = a(DV \cdot u, \phi)\}, \tag{4.235}$$

$$\langle \mathcal{B}z, \phi\rangle = \int_\Omega \{\epsilon(z) : \mathcal{C}' : \epsilon(\phi) + \epsilon'(z) : \mathcal{C} : \epsilon(\phi) + \\ \epsilon(z) : \mathcal{C} : \epsilon'(\phi)\} dx \quad \forall z, \phi \in H^1(\Omega;\mathbb{R}^2), \tag{4.236}$$

$$\epsilon(\phi) = \frac{1}{2}(D\phi + {}^*D\phi), \tag{4.237}$$

$$\epsilon'(\phi) = \frac{1}{2}\{D(DV \cdot \phi) + {}^*(D(DV \cdot \phi)) - D\phi \cdot DV - {}^*DV \cdot {}^*D\phi\}, \tag{4.238}$$

$$\mathcal{C}' = \{c'_{ijkl}\}, \ c'_{ijkl} = \mathrm{div} V c_{ijkl} + \ \langle \nabla c_{ijkl}, V\rangle_{\mathbb{R}^2}\ , \tag{4.239}$$

$$\langle F', \phi\rangle = \int_\Omega \{\sum_{i=1}^2 (\mathrm{div}(f_i V)\phi_i) + f \cdot DV \cdot \phi\} dx + \\ \int_{\Gamma_1} \{\sum_{i=1}^2 (\mathrm{div}(P_i V)\phi_i) - (n \cdot DV \cdot n) P \cdot \phi + P \cdot DV \cdot \phi\} d\Gamma \\ \forall \phi \in H^1(\Omega;\mathbb{R}^2)\ . \tag{4.240}$$

Proof. We have

$$\phi_t \in K(\Omega_t) \text{ if and only if } \phi = DT_t^{-1} \cdot (\phi_t \circ T_t) \in K(\Omega)\ . \tag{4.241}$$

The following notation is used

$$z^t = DT_t^{-1} \cdot (u_t \circ T_t)\ . \tag{4.242}$$

It can be shown that the element $z^t \in H$ is the solution to the variational inequality

$$\begin{aligned} & z^t \in K \ : \\ & a^t\left(z^t, \phi - z^t\right) \geqq \langle F(t), \phi - z^t \rangle \quad \forall \phi \in K, \end{aligned} \tag{4.243}$$

where the bilinear form $a^t(\cdot,\cdot)$ and the linear form $F(t) \in H'$ are defined as follows:

$$a^t(z,\phi) = \int_\Omega \epsilon^t(z) : \mathcal{C}^t : \epsilon^t(\phi)\, dx \quad \forall z, \phi \in H, \tag{4.244}$$

$$\epsilon^t(\phi) = \frac{1}{2}\{D(DT_t \cdot \phi) \cdot DT_t^{-1} + {}^*DT_t^{-1} \cdot {}^*(D(DT_t \cdot \phi))\}, \tag{4.245}$$

$$\mathcal{C}^t = \{c^t_{ijkl}\} \quad \text{with } c^t_{ijkl} = \det(DT_t)\, c_{ijkl} \circ T_t\ , \tag{4.246}$$

$$\langle F(t), \phi \rangle = \int_\Omega f^t \cdot \phi dx + \int_{\Gamma_1} P^t \cdot \phi d\Gamma \quad \forall \phi \in H, \tag{4.247}$$

$$f^t = \det(DT_t)\, {}^*DT_t \cdot (f \circ T_t)\,, \tag{4.248}$$

$$P^t = \|M(DT_t) \cdot n\|_{\mathbb{R}^2}\, {}^*DT_t \cdot (P \circ T_t)\,, \tag{4.249}$$

$$M(DT_t) = \det(DT_t)\, {}^*DT_t^{-1}\ . \tag{4.250}$$

Applying Theorem 4.30 to (4.243) we get that the mapping

$$[0, \delta) \ni t \longrightarrow z^t \in H \tag{4.251}$$

is differentiable at $t = 0^+$, i.e. for $t > 0$, t small enough,

$$z^t = z^0 + t\dot{z} + o(t)\,, \tag{4.252}$$

where $\|o(t)\|_{H^1(\Omega;\mathbb{R}^2)}/t \ \to 0$ as $t \downarrow 0$.
$\dot{z}$ is given as the unique solution to the variational inequality

$$\begin{aligned} & \dot{z} \in S \ : \\ & a(\dot{z}, \phi - \dot{z}) \geqq \langle F' - \mathcal{B}u, \phi - \dot{z} \rangle \quad \forall \phi \in S\ . \end{aligned} \tag{4.253}$$

In the foregoing the substitution $z^0 = u$ is used.

On the other hand, since

$$u^t = u_t \circ T_t = DT_t \cdot z^t \tag{4.254}$$

then

$$\dot{u} = \dot{z} - DV \cdot z^0 = \dot{z} - DV \cdot u\ . \tag{4.255}$$

Therefore from (4.253), in view of (4.255), it follows (4.234). This proves Theorem 4.33. □

To derive the form of shape derivative u' we use the relation

$$u' = \dot{u} - Du \cdot V\ . \tag{4.256}$$

It is evident that $u' \in H^1(\Omega; \mathbb{R}^2)$ provided that

$$Du \cdot V \in H^1(\Omega; \mathbb{R}^2) \ .$$

It is assumed that the vector field $V = 0$ on $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_2$, and this vector field is of the form

$$V = v_n \mathcal{N}_0 \ , \text{ where } \mathcal{N}_0 = n \text{ on } \Gamma \ . \tag{4.257}$$

Theorem 4.34 *For the shape derivative u' of the solution u to the Signorini variational inequality (4.177), in the direction of the vector field $v_n\mathcal{N}_0$, the following system holds:*

$$\sigma_{ij}(u')_{,j} = 0 \quad \text{in } \Omega, \quad i = 1, 2, \tag{4.258}$$

along with the boundary conditions

on Γ_0:

$$u' = -v_n \frac{\partial u}{\partial n} \ . \tag{4.259}$$

on Γ_1:

$$\sigma(u') \cdot n = v_n (f + \kappa P) - \operatorname{div}_\Gamma (v_n P_\tau) \ . \tag{4.260}$$

Furthermore, if Γ_2 is divided into three subsets:

$$\Gamma_2 \setminus \mathcal{Z}, \quad \mathcal{Z}^+, \quad \mathcal{Z}^0 = \mathcal{Z} \setminus \mathcal{Z}^+ \tag{4.261}$$

defined by (4.224), (4.225) and (4.226), respectively, then there exist the following sets of the boundary conditions on Γ_2:

(i) on $\mathcal{Z}^0$:

$$u' \cdot n \leqq u \cdot \nabla_\Gamma v_n - v_n n \cdot \frac{\partial u}{\partial n} \ , \tag{4.262}$$

$$\sigma_n(u') \leqq 0, \tag{4.263}$$

$$\sigma_n(u') \left[u' \cdot n - u \cdot \nabla_\Gamma v_n - v_n n \cdot \frac{\partial u}{\partial n} \right] = 0, \tag{4.264}$$

$$\sigma_\tau(u') = v_n f_\tau \ . \tag{4.265}$$

(ii) on $\mathcal{Z}^+$:

$$u' \cdot n = u \cdot \nabla_\Gamma v_n - v_n n \cdot \frac{\partial u}{\partial n}, \tag{4.266}$$

$$\sigma_\tau(u') = v_n f_\tau + \sigma_n(u) \nabla_\Gamma v_n \ . \tag{4.267}$$

(iii) on $\Gamma_2 \setminus \mathcal{Z}$:

$$\sigma(u') \cdot n = v_n f \ . \tag{4.268}$$

□

4.7. The Signorini problem with given friction

In this section we shall consider, following Sokołowski (1988d), the contact problem with given friction on the part Γ_2 of the boundary $\Gamma = \partial\Omega$ of the domain $\Omega \subset \mathbb{R}^2$.

We denote by $u \in H^1(\Omega;\mathbb{R}^2)$ a solution to the following system

$$-(c_{ijkl}(x)u_{k,l}(x))_{,j} = f_i(x) \quad \text{in } \Omega, \quad i = 1,2, \tag{4.269}$$

$$u_i = 0 \quad \text{on } \Gamma_0, \quad i = 1,2, \tag{4.270}$$

$$c_{ijkl}u_{k,l}n_j = P_i \quad \text{on } \Gamma_1, \quad i = 1,2, \tag{4.271}$$

$$u_i n_i \geqq 0, \quad \sigma_n \leqq 0, \quad \sigma_n u_i n_i = 0 \quad \text{on } \Gamma_2 \ , \tag{4.272}$$

$$\sigma_\tau \cdot u + |u \cdot \tau| = 0, \quad -1 \leqq \sigma_\tau \leqq 1 \quad \text{on } \Gamma_2 \ , \tag{4.273}$$

i.e. the stick–sleep condition (4.273) is prescribed on Γ_2. A weak solution to (4.269)–(4.273) minimizes the functional

$$J(\phi) = \frac{1}{2}a(\phi,\phi) - \langle F,\phi\rangle + \int_{\Gamma_2} |\phi \cdot \tau| d\Gamma, \tag{4.274}$$

over the convex and closed cone

$$K = \{\phi \in H^1(\Omega;\mathbb{R}^2) | \phi = 0 \quad \text{on } \Gamma_0, \quad \phi \cdot n \leqq 0 \quad \text{on } \Gamma_2\} \ . \tag{4.275}$$

It should be emphasized that u is also the unique solution to the following variational problem :

Find an element $u \in K$ such that

$$a(u,\phi - u) + \int_{\Gamma_2} \{|\phi \cdot \tau| - |u \cdot \tau|\} d\Gamma \geqq \langle F, \phi - u\rangle \quad \forall \phi \in K \ . \tag{4.276}$$

It is assumed that the boundary $\partial\Omega$ of Ω is sufficiently smooth and $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2$.

Let $F, h \in H'$ be given, and let F^ε stand for

$$F^\varepsilon = F + \varepsilon h \quad \varepsilon \in [0,\delta) \ . \tag{4.277}$$

We shall consider the following variational problem:

Find an element $u^\varepsilon \in K$ such that

$$a(u^\varepsilon,\phi - u^\varepsilon) + \int_{\Gamma_2} \{|\phi \cdot \tau| - |u^\varepsilon \cdot \tau|\} d\Gamma \geqq \langle F^\varepsilon, \phi - u^\varepsilon\rangle \quad \forall \phi \in K \tag{4.278}$$

The saddle–point formulation of the variational inequality (4.276) will be introduced. To this end, we need the following notation

$$\mathcal{U} = \{\varphi \in H^{\frac{1}{2}}(\Gamma_2) | \exists z \in H \text{ such that } \varphi = z \cdot \tau \quad \text{on } \Gamma_2\}, \tag{4.279}$$

$\mathcal{U}'$ denotes the dual space with the dual norm $\|.\|_{\mathcal{U}'}$. The symbol

$$\int_{\Gamma_2} \mu(x)\varphi(x)\,d\Gamma \quad \forall \mu \in \mathcal{U}' \quad \forall \varphi \in \mathcal{U}$$

is used for the duality paring between spaces $\mathcal{U}'$ and $\mathcal{U}$, the latter is obtained by the extension to $\mathcal{U}', \mathcal{U}$ of the scalar product in $L^2(\Gamma_2)$. Furthermore we introduce the notation

$$\begin{aligned}(\mathcal{U}')^+ = \{\mu \in \mathcal{U}' | \int_{\Gamma_2} \mu(x)\varphi(x)\,d\Gamma \geqq 0 \quad \forall \varphi \in \mathcal{U},\\ \varphi(x) \geqq 0 \quad \text{for a.e. } x \in \Gamma_2\},\end{aligned} \tag{4.280}$$

$$(\mathcal{U}')^* = (\mathcal{U}')^+ - (\mathcal{U}')^+ \ . \tag{4.281}$$

In this section we denote by $\Lambda \subset \mathcal{U}'$ the convex, closed set of the form

$$\begin{aligned}\Lambda =& \{\mu \in (\mathcal{U}')^* \,|\, -1 \leqq \mu \leqq 1 \quad \text{on } \Gamma_2\}\\ =& \text{ the unit ball of } L^\infty(\Gamma_2) \ .\end{aligned} \tag{4.282}$$

We define the functional

$$\begin{aligned}\mathcal{L}(F;\phi,\mu) = \frac{1}{2}a(\phi,\phi) - \langle F,\phi\rangle + \int_{\Gamma_2} \mu(x)(\phi\cdot\tau)(x)\,d\Gamma\\ \forall F \in H' \quad \forall \phi \in H \quad \forall \mu \in \mathcal{U}'\end{aligned} \tag{4.283}$$

Let $(u^\varepsilon, \lambda^\varepsilon) \in K \times \Lambda$ be the unique element for which the saddle–point conditions are satisfied,

$$\mathcal{L}(F^\varepsilon;u^\varepsilon,\mu) \leqq \mathcal{L}(F^\varepsilon;u^\varepsilon,\lambda^\varepsilon) \leqq \mathcal{L}(F^\varepsilon;\phi,\lambda^\varepsilon) \quad \forall \mu \in \Lambda \quad \forall \phi \in K \ . \tag{4.284}$$

In order to determine the form of the right–derivative of u^ε with respect to ε at $\varepsilon = 0$, the following cones are defined:

$$\begin{aligned}S_K = S_K(u^0) = \{\phi \in H | \phi(x)\cdot n(x) = 0 \quad \text{on } \mathcal{Z}\setminus\mathcal{Z}_0 \ ,\\ \phi(x)\cdot n(x) \geqq 0 \quad \text{on } \mathcal{Z}_0\},\end{aligned} \tag{4.285}$$

$$\begin{aligned}S_\Lambda = S_\Lambda(\lambda^0) = \{\mu \in \mathcal{U}' | \mu(x) \geqq 0 \quad \text{on } \Xi_1^0,\\ \mu(x) \leqq 0 \quad \text{on } \Xi_2^0,\\ \mu(x) = 0 \quad \text{on } \Xi_1^+ \cup \Xi_2^+\},\end{aligned} \tag{4.286}$$

where

$$\Xi_1 = \{x \in \Gamma_2 | \lambda^0(x) = -1\},$$
$$\Xi_2 = \{x \in \Gamma_2 | \lambda^0(x) = +1\},$$
$$\Xi_i^0 = \{x \in \Xi_i | u^0(x) = 0\}, \quad i = 1, 2,$$
$$\Xi_i^+ = \Xi_i \setminus \Xi_i^0, \quad i = 1, 2 \ .$$

Theorem 4.35 *For $\varepsilon > 0$, ε small enough,*

$$u^\varepsilon = u^0 + \varepsilon q + o(\varepsilon) \quad \textit{in } H^1(\Omega; \mathbb{R}^2), \tag{4.287}$$

where $\|o(\varepsilon)\|_{H^1(\Omega;\mathbb{R}^2)}/\varepsilon \to 0$ as $\varepsilon \downarrow 0$.
The element $q \in H^1(\Omega; \mathbb{R}^2)$ minimizes the functional

$$I(\phi) = \frac{1}{2} a(\phi, \phi) - \langle h, \phi \rangle + \max\{\int_{\Gamma_2} \mu(x)(\phi \cdot \tau)(x)\, d\Gamma | \mu \in S_\Lambda(\lambda^0)\} \tag{4.288}$$

over the cone $S_K(u^0)$.

Proof.
Taking advantage of (4.278), (4.277), (4.279) and using the standard arguments one can show that

$$\|u^\varepsilon - u^0\|_{H^1(\Omega;\mathbb{R}^2)} \leqq C\varepsilon \quad \varepsilon \in [0, \delta) \ . \tag{4.289}$$

Let

$$(\mathcal{R}_1\phi)(x) = \phi(x) \cdot n(x) = \phi_n(x), \tag{4.290}$$
$$(\mathcal{R}_2\phi)(x) = (\phi \cdot \tau)(x) \tag{4.291}$$

for $x \in \Gamma_2$ and for all $\phi \in H$.
For simplicity, it is assumed here that the boundary $\partial\Omega$ of the domain Ω is smooth enough and we have

$$\phi_n, \phi \cdot \tau \in H^{\frac{1}{2}}(\Gamma_2) \quad \forall \phi \in H \ . \tag{4.292}$$

However, the condition (4.292) can be weakened. We define the linear and closed subspaces $H_i \subset H \quad i = 1, 2, 3$ such that

$$H = \bigoplus_{i=1}^{3} H_i \tag{4.293}$$

and

$$H_1 = \{\phi \in H | a(\phi, \eta) = 0 \quad \forall \eta \in H_3 \ , \ \mathcal{R}_2\phi = 0\}, \tag{4.294}$$
$$H_2 = \{\phi \in H | a(\phi, \eta) = 0 \quad \forall \eta \in H_3 \ , \ \mathcal{R}_1\phi = 0\}, \tag{4.295}$$
$$H_3 = \{\phi \in H | \mathcal{R}_1\phi = 0, \ \mathcal{R}_2\phi = 0\} \ . \tag{4.296}$$

It should be noted that (4.295) and (4.296) imply that

$$K \cap H_i = H_i \quad i = 1,2 \ . \tag{4.297}$$

Therefore the symbol K will stand for the convex and closed set:

$$K = \{\phi \in H_1 | (\mathcal{R}_1\phi)(x) \geqq 0 \text{ for a.e. } x \in \Gamma_2\} \ . \tag{4.298}$$

Furthermore we use the notation

$$S = \{\phi \in H_1 | \phi_n(x) = 0 \quad \text{for a.e. } x \in \mathcal{Z} \setminus \mathcal{Z}_0 \ , \qquad \phi_n(x) \geqq 0 \quad \text{for a.e. } x \in \mathcal{Z}_0\}, \tag{4.299}$$

where a.e. means almost everywhere with respect to the one–dimensional Lebesgue measure on Γ_2.

The functional (4.275) can be written in the form

$$J(\phi) = \sum_{i=1}^{3} \frac{1}{2} a(\phi^i, \phi^i) - \langle F, \phi^i \rangle + \int_{\Gamma_2} |(\phi \cdot \tau)(x)| d\Gamma \tag{4.300}$$
$$\forall \phi = \phi^1 + \phi^2 + \phi^3 \in H \ .$$

Hence the saddle–point conditions (4.284) can be formulated:

Find $(u_\varepsilon^1, u_\varepsilon^2, u_\varepsilon^3, \lambda^\varepsilon) \in K \times H_2 \times H_3 \times \Lambda$ such that

$$a(u_\varepsilon^1, \phi - u_\varepsilon^1) - \langle F^\varepsilon, \phi - u_\varepsilon^1 \rangle \geqq 0 \quad \forall \phi \in K \subset H_1 \ , \tag{4.301}$$
$$a(u_\varepsilon^2, \phi) - \langle F^\varepsilon, \phi \rangle + \int_{\Gamma_2} \lambda^\varepsilon(x)(\phi \cdot \tau)(x) \, d\Gamma = 0 \quad \forall \phi \in H_2 \ , \tag{4.302}$$
$$\int_{\Gamma_2} \mu(x) (u_\varepsilon^2 \cdot \tau)(x) \, d\Gamma \geqq \int_{\Gamma_2} \lambda^\varepsilon(x) (u_\varepsilon^2 \cdot \tau)(x) \, d\Gamma \quad \forall \mu \in \Lambda, \tag{4.303}$$
$$a(u_\varepsilon^3, \phi) - \langle F^\varepsilon, \phi \rangle = 0 \quad \forall \phi \in H_3 \ . \tag{4.304}$$

In order to prove (4.287), it suffices to show that for $\varepsilon > 0$, ε small enough,

$$u_\varepsilon^i = u_0^i + \varepsilon q_i + o(\varepsilon) \quad \text{in } H_i \ , \quad i = 1,2,3, \tag{4.305}$$

where $\|o(\varepsilon)\|_{H^1(\Omega;\mathbb{R}^2)}/\varepsilon \to 0$ as $\varepsilon \downarrow 0$, and $q_i \in H_i$, $i = 1,2,3$, are uniquely determined.

For each case under study, i.e. for $i = 1,2$ and 3, the specific proof will be given.

Case $i = 1$.

Let us consider the variational inequality (4.301). We can apply Theorem 4.10 to (4.301) and obtain (4.305) for $i = 1$. Therefore the element $q_1 \in H_1$ is given as the unique solution to the following variational inequality

$$\begin{aligned} & q_1 \in S \subset H_1 \\ & a(q_1, \phi - q_1) - \langle h, \phi - q_1 \rangle \geqq 0 \quad \forall \phi \in S \subset H_1 \ . \end{aligned} \tag{4.306}$$

Case $i = 2$.

Let an element $z(\mu) \in H_2$, $\mu \in \mathcal{U}'$, be a solution to the following variational equation

$$a(z(\mu), \phi) + \int_{\Gamma_2} \mu(x)(\phi \cdot \tau)(x)\, d\Gamma = 0 \quad \forall \phi \in H_2 \ . \tag{4.307}$$

If

$$w^\varepsilon = u_\varepsilon^1 - z(\lambda^\varepsilon),$$

then

$$w^\varepsilon \in H_2 : \quad a(w^\varepsilon, \phi) = \langle F^\varepsilon, \phi \rangle \quad \forall \phi \in H_2 \tag{4.308}$$

and for $\varepsilon > 0$, ε small enough,

$$w^\varepsilon = w^0 + \varepsilon w' + o(\varepsilon) \quad \text{in } H_2 \ , \tag{4.309}$$

where $\|o(\varepsilon)\|_{H_2}/\varepsilon \to 0$ as $\varepsilon \downarrow 0$. The element $w' \in H_2$ is given as the unique solution to the following variational equation

$$w' \in H_2 : \quad a(w', \phi) = \langle h, \phi \rangle \quad \forall \phi \in H_2 \ . \tag{4.310}$$

The condition (4.303) is equivalent to the variational inequality

$$\begin{aligned} & \lambda^\varepsilon \in \Lambda : \\ & b(\lambda^\varepsilon, \mu - \lambda^\varepsilon) + \int_{\Gamma_2} (\mu(x) - \lambda^\varepsilon(x))(w^\varepsilon \cdot \tau)(x)\, d\Gamma \geqq 0 \quad \forall \mu \in \Lambda, \end{aligned} \tag{4.311}$$

where the bilinear form $b(\cdot,\cdot) : \mathcal{U}' \times \mathcal{U}' \to \mathbb{R}$ is given by

$$b(\mu, \lambda) = \int_{\Gamma_2} \lambda(x)(z(\mu; x) \cdot \tau(x))\, d\Gamma \quad \forall \mu, \lambda \in \mathcal{U}' \ . \tag{4.312}$$

Under the same assumptions as in Sect. 4.5, one can prove, making use of the results obtained by Sokołowski (1988d), that for the cone (4.286) associated to the set Λ of the form (4.282) the following condition is satisfied

$$S_\Lambda(\lambda^0) = \mathrm{cl} C_\Lambda(\lambda^0) \cap M, \tag{4.313}$$

where $C_\Lambda(\lambda^0)$ is the radial cone to Λ at λ^0, and the set M is defined by

$$M = \{\mu \in \mathcal{U}' | \int_{\Gamma_2} \mu(x)\left(u^0 \cdot \tau\right)(x)\, d\Gamma = 0\} \ . \tag{4.314}$$

Applying Theorem 4.6 to the variational inequality (4.311) one can show that for $\varepsilon > 0$, ε small enough,

$$\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda' + o(\varepsilon) \quad \text{in } \mathcal{U}', \tag{4.315}$$

where $||o(\varepsilon)||_{\mathcal{U}'}/\varepsilon \to 0$ as $\varepsilon \downarrow 0$.

The element $\lambda' \in \mathcal{U}'$ is given as the unique solution to the following variational inequality

$$\begin{aligned} &\lambda' \in S_\Lambda\left(\lambda^0\right) \ : \\ &b(\lambda', \mu - \lambda') + \int_{\Gamma_2} (\mu(x) - \lambda'(x))(w' \cdot \tau)(x)\, d\Gamma \geqq 0 \\ &\qquad\qquad\qquad\qquad \forall \mu \in S_\Lambda\left(\lambda^0\right) \ . \end{aligned} \tag{4.316}$$

From (4.302) it can be inferred, taking into account (4.277) and (4.315), that for $\varepsilon > 0$, ε small enough, (4.305) holds for $i = 2$, where q_2 is given as the unique solution to the following variational equation

$$\begin{aligned} &q_2 \in H_2 : \\ &a(q_2, \phi) - \langle h, \phi\rangle + \int_{\Gamma_2} \lambda'(x)(\phi \cdot \tau)(x)\, d\Gamma = 0 \quad \forall \phi \in H_2 \ . \end{aligned} \tag{4.317}$$

Case $i = 3$.
It is easy to show that for $i = 3$ the equations (4.277) and (4.304) yield (4.305), where $q_3 \in H_3$ is given as the unique solution to the following variational equation

$$q_3 \in H_3 : \quad a\left(q^3, \phi\right) - \langle h, \phi\rangle = 0 \quad \forall \phi \in H_3 \ . \tag{4.318}$$

From (4.305), (4.306), (4.316), (4.317) and (4.318) it follows that (4.287) holds. This concludes the proof of Theorem 4.13. □

4.7.1. Shape sensitivity analysis

Let us consider the shape differentiability of the solutions to the Signorini problem with given friction. Let

$$\Omega_t = T_t(V)(\Omega),$$

where $V \in C(0,\varepsilon; C^1(\mathbb{R}^2;\mathbb{R}^2))$ is a given vector field. The notation $\Gamma_i^t = T_t(V)(\Gamma_i)$, $i = 0,1,2$, is used, and $\partial\Omega_t = \overline{\Gamma}_0^t \cup \overline{\Gamma}_1^t \cup \overline{\Gamma}_2^t$.

We shall examine the contact problem in the domain Ω_t for $t \in [0, \delta)$.

Problem (P_t): Find an element $u_t \in K(\Omega_t)$ that minimizes the functional

$$J_t(\phi) = \frac{1}{2} a_t(\phi, \phi) - \langle F_t, \phi \rangle + j_t(\phi) \tag{4.319}$$

over the set $K(\Omega_t) \subset H_t$, where

$$H_t = \{\phi \in H^1(\Omega_t; \mathbb{R}^2) | \phi = 0 \text{ on } \Gamma_0^t\}, \tag{4.320}$$

$$a_t(z, \phi) = \int_{\Omega_t} c_{ijkl}(x) z_{i,j}(x) \phi_{k,l}(x) \, dx \quad \forall z, \phi \in H_t, \tag{4.321}$$

$$\langle F_t, \phi \rangle = \int_{\Omega_t} f_i(x) \phi_i(x) \, dx + \int_{\Gamma_1^t} P_i(x) \phi_i(x) \, d\Gamma \quad \forall \phi \in H_t \ . \tag{4.322}$$

In (4.321) and (4.322) the restrictions to Ω_t of the elements

$$c_{ijkl}(\cdot), f_i(\cdot), P_i(\cdot) \in H^1(\mathbb{R}^2) \quad i, j, k, l = 1, 2$$

are also denoted by c_{ijkl}, f_i, P_i, respectively.
Finally let

$$j_t(\phi) = \int_{\Gamma_2^t} |(\phi \cdot \tau_t)(x)| d\Gamma, \tag{4.323}$$

where $\tau_t(x)$, $x \in \Gamma_2^t$, denotes the unit tangent vector on Γ_2^t; $n_t(x), x \in \partial\Omega_t$, is the outward unit normal vector on $\partial\Omega_t$.

It can be shown that there exists the unique solution to the problem (P_t), $t \in [0, \delta)$; for $t = 0$ the problem (P_0) becomes the problem (P). Therefore we denote its solution by $u_0 = u$. Let $\tilde{u}_t$ be an extension of the element $u_t \in H^1(\Omega_t; \mathbb{R}^2)$ to an open neighborhood of $\Omega_t \subset \mathbb{R}^2$ such that the restriction $\tilde{u}_t|_\Omega \in H^1(\Omega; \mathbb{R}^2)$ is well defined for $t > 0$, t small enough.

Theorem 4.36 *Let us assume that $Du \cdot V(0) \in H^1(\Omega; \mathbb{R}^2)$, and the sets Ξ^0, Ξ_i^+, Ξ_i^- are sufficiently regular.*
Then for $t > 0$, t small enough,

$$\tilde{u}_t|_\Omega = u + tu' + o(t) \quad \text{in } H^1(\Omega; \mathbb{R}^2), \tag{4.324}$$

where $\|o(t)\|_{H^1(\Omega;\mathbb{R}^2)}/t \to 0$ as $t \downarrow 0$.
The strong shape derivative $u' \in H^1(\Omega; \mathbb{R}^2)$ is given as the unique solution to the following problem.

Problem (P'): *Find an element $u' \in S_v(\Omega)$ that minimizes the functional*

$$I(\phi) = \frac{1}{2} a(\phi, \phi) + g(V; \phi) + j'(V; \phi) \tag{4.325}$$

over the cone $S_v(\Omega) \subset H^1(\Omega;\mathbb{R}^2)$.

We use the following notation:

$$\begin{aligned} S_v(\Omega) = \{\phi \in H^1(\Omega;\mathbb{R}^2) | \phi = -v_n Du \cdot n - v_\tau Du \cdot \tau \quad &on\ \Gamma_0, \\ \phi \cdot n \geqq u_\tau n \cdot DV \cdot \tau - v_n n \cdot Du \cdot n - v_\tau n \cdot Du \cdot \tau \quad &on\ \Xi^+, \\ \phi \cdot n = u_\tau n \cdot DV \cdot \tau - v_n n \cdot Du \cdot n - v_\tau n \cdot Du \cdot \tau \quad &on\ \Xi^0\}, \end{aligned} \tag{4.326}$$

$$\begin{aligned} j'(V;\phi) = \max\{\int_{\Gamma_2} \xi[\phi_\tau + u_\tau(\mathrm{div}V - n \cdot DV \cdot n) + v_n \tau \cdot Du \cdot n \\ v_\tau \tau \cdot Du \cdot \tau] d\Gamma \mid \xi \in S_\lambda(\Lambda)\} \ . \end{aligned} \tag{4.327}$$

A representation of the linear functional $g(V;\cdot)$ *for* $\phi|_{\partial\Omega} \in H^2(\Omega;\mathbb{R}^2)$ *can be given in the form*

$$\begin{aligned} &g(V;\phi) = \\ &\int_{\partial\Omega} \{v_n \sigma(u) : \epsilon(\phi) + v_n(f \cdot \phi) + \phi_n(f \cdot V) - (\sigma(u) \cdot n) \cdot (DV \cdot \phi)\} d\Gamma + \\ &\int_{\Gamma_1} \{\mathrm{div}(P_i V)\phi_i - (n \cdot DV \cdot n) P \cdot \phi + P \cdot DV \cdot \phi\} d\Gamma \ . \end{aligned} \tag{4.328}$$

Proof (Sokołowski et al. 1987a). Let

$$u^t = u_t \circ T_t \ , \tag{4.329}$$
$$z^t = DT_t^{-1} \cdot u^t \ , \tag{4.330}$$

where DT_t^{-1} is the inverse of the Jacobian matrix of the mapping $T_t(V)$. From (4.329) and (4.330) it follows that

$$\dot{u} = \lim_{t \downarrow 0} \frac{1}{t}(u^t - u) = \dot{z} + DV \cdot u \tag{4.331}$$

provided that there exists the limit

$$\dot{z} = \lim_{t \downarrow 0} \frac{1}{t}(z^t - u) \quad \text{in } H^1(\Omega;\mathbb{R}^2) \ . \tag{4.332}$$

Therefore we have to show that there exists the limit $\dot{z} \in H^1(\Omega;\mathbb{R}^2)$ defined by (4.332). Finally the shape derivative u' is determined from the equation

$$u' = \dot{u} - Du \cdot V = \dot{z} + DV \cdot u - Du \cdot V \ . \tag{4.333}$$

Since

$$\phi \in K(\Omega_t) \text{ if and only if } DT_t^{-1} \cdot (\phi \circ T_t) \in K(\Omega) \quad \forall t \in [0,\delta), \tag{4.334}$$

then for the element $z^t \in H^1(\Omega;\mathbb{R}^2)$ defined by (4.330) we have that $z^t \in K(\Omega)$ for all $t \in [0,\delta)$; however, in general $u^t \notin K(\Omega)$.

Applying the change of variables in (4.319), and making use of (4.330), one can show that the element z^t is given as the unique solution to the following problem

Problem (P^t)**:** Find an element $z^t \in K(\Omega)$ that minimizes the functional

$$J^t(\phi) = \frac{1}{2}a^t(\phi,\phi) - \langle F^t,\phi\rangle + j^t(\phi) \tag{4.335}$$

over the set $K(\Omega)$.

We use the notation:

$$a^t(z,\phi) = a_t(DT_t\cdot(z\circ T_t^{-1}), DT_t\cdot(\phi\circ T_t^{-1})), \tag{4.336}$$

$$\langle F^t,\phi\rangle = \langle F_t, DT_t\cdot(\phi\circ T_t^{-1})\rangle \quad \forall z,\phi\in H^1(\Omega;\mathbb{R}^2), \tag{4.337}$$

$$\begin{aligned} j^t(\phi) &= \int_{\Gamma_2^t} |(\phi\circ T_t^{-1})\cdot\tau(x)|d\Gamma \qquad\qquad (4.338)\\ &= \int_{\Gamma_2}\left\{\left\| DT_t\cdot\phi - \phi_n\gamma(t)\frac{{}^*DT_t^{-1}\cdot n}{\|{}^*DT_t^{-1}\cdot n\|_{\mathbb{R}^2}}\right\|_{\mathbb{R}^2}\right\}d\Gamma \\ &\qquad\qquad \forall\phi\in H^1(\Omega;\mathbb{R}^2)\ . \end{aligned}$$

The explicit form of $a^t(\cdot,\cdot)$ and $\langle F^t,\cdot\rangle$ is given by (4.244)–(4.250). Let $\dot a(z,\phi)$ stand for the following limit

$$\begin{aligned} \dot a(z,\phi) &= \lim_{t\downarrow 0}\frac{1}{t}(a^t(z,\phi) - a(z,\phi)) \qquad\qquad (4.339)\\ &= a'(z,\phi) + a(DV\cdot z,\phi) + a(z, DV\cdot\phi) \quad \forall z,\phi\in H^1(\Omega;\mathbb{R}^2)\ . \end{aligned}$$

In (4.339) the notation is the same as that used in Sect. 4.7, e.g.

$$\begin{aligned} a'(z,\phi) = \int_\Omega \{&\operatorname{div}(c_{ijkl}V)z_{i,j}\phi_{k,l} - c_{ijkl}[\epsilon'_{ij}(z)\epsilon_{kl}(\phi) \qquad (4.340)\\ &+ \epsilon'_{ij}(z)\epsilon_{kl}(\phi)]\}dx\ . \end{aligned}$$

Furthermore

$$\begin{aligned} \langle F',\phi\rangle = &\int_\Omega \{\operatorname{div}(f_iV)\phi_i + f\cdot DV\cdot\phi\}dx + \\ &\int_{\Gamma_1}\{\operatorname{div}(P_iV)\phi - (n\cdot DV\cdot n)(P\cdot\phi) + P\cdot DV\cdot\phi\}d\Gamma \\ &\qquad\qquad \forall\phi\in H^1(\Omega;\mathbb{R}^2)\ . \end{aligned}$$

It can be shown that

$$\langle F', \phi\rangle = \lim_{t\downarrow 0} \frac{1}{t}\{\langle F^t, DT_t \cdot \phi\rangle - \langle F, \phi\rangle\} \ .$$

By standard reasoning it follows that the solution z^t to the problem (P^t) is Lipschitz continuous with respect to $t \in [0, \delta)$.

Lemma 4.37 *For $t > 0$, t small enough, we have*

$$\|z^t - u\|_{H^1(\Omega;\mathbb{R}^2)} \leqq Ct \ . \tag{4.341}$$

Therefore, from (4.329) and (4.330) it follows that

$$\|u^t - u\|_{H^1(\Omega;\mathbb{R}^2)} \leqq Ct, \tag{4.342}$$

where the constant C is independent of $t \in [0, \delta)$.

Finally, let us examine the non–smooth term (4.338). It is assumed that for $t > 0$, t small enough, the following condition

$$DT_t \cdot n - \gamma(t) \frac{{}^*DT_t^{-1} \cdot n}{\|{}^*DT_t^{-1} \cdot n\|_{\mathbb{R}^2}} = 0 \tag{4.343}$$

is satisfied by the vector field $V(\cdot, \cdot)$; for $t = 0$ the condition (4.343) reduces to the equality $n - n = 0$.

We can simplify (4.338); from (4.343) it can be inferred that

$$j^t(\phi) = \int_{\Gamma_2} (\phi \cdot \tau)(x) r_t(x) d\Gamma, \tag{4.344}$$

where

$$r_t = \|DT_t \cdot \tau\|_{\mathbb{R}^2} \|\gamma(t)^* DT_t^{-1} \cdot n\|_{\mathbb{R}^2} \ . \tag{4.345}$$

For $t > 0$, t small enough, we have that

$$r_t = 1 + t\dot{r} + o(t) \quad \text{in } C(\Gamma_2), \tag{4.346}$$

where

$$\begin{aligned} \dot{r} =& \left(\|DT_t \cdot \tau\|_{\mathbb{R}^2}\right)^{\cdot} + \left(\|\det(DT_t)^* DT_t^{-1} \cdot n\|_{\mathbb{R}^2}\right)^{\cdot} \\ =& \tau \cdot DV \cdot \tau + \operatorname{div} V - n \cdot DV \cdot n \ . \end{aligned} \tag{4.347}$$

The functional (4.344) can be written as follows

$$j^t(\phi) = \max\{\int_{\Gamma_2} \xi(x)(\phi \cdot \tau)(x) r_t(x) d\Gamma | \quad \xi \in \Lambda\} \ . \tag{4.348}$$

Furthermore it is possible to show that there exists the unique pair (z^t, λ^t), $t \in [0, \delta)$, such that

$$\mathcal{L}^t(z^t, \xi) \leqq \mathcal{L}^t(z^t, \lambda^t) \leqq \mathcal{L}^t(\phi, \lambda^t) \quad \text{for all } \xi \in \Lambda \text{ and for all } \phi \in K(\Omega), \tag{4.349}$$

where

$$\mathcal{L}^t(\phi, \xi) = \frac{1}{2}a^t(\phi, \phi) - \langle F^t, \phi\rangle + \int_{\Gamma_2} \xi(x)(\phi \cdot \tau)(x) r_t(x) d\Gamma \tag{4.350}$$
$$\text{for all } \phi \in H^1(\Omega; \mathbb{R}^2) \text{ and for all } \xi \in \Lambda\ .$$

The element $\lambda^t \in L^\infty(\Gamma_2)$, $t \in [0, \delta)$, is given as the unique solution to the following variational inequality

$$\begin{aligned} &\lambda^t \in \Lambda\ : \\ &b^t(\lambda^t, \xi - \lambda^t) \geqq \int_{\Gamma_2} (\xi - \lambda^t) w_\tau^t r_t d\Gamma \quad \forall \xi \in \Lambda, \end{aligned} \tag{4.351}$$

where

$$b^t(\lambda, \xi) = \int_{\Gamma_2} \lambda(x) \psi_\tau^t(\xi)(x) r_t(x) d\Gamma, \tag{4.352}$$

$$\psi_\tau^t(\xi)(x) = \psi^t(x) \cdot \tau(x) \quad x \in \Gamma_2\ , \tag{4.353}$$

$$\psi^t \in H: \quad a^t(\psi^t, \phi) = \int_{\Gamma_2} \xi(x)(\phi \cdot \tau)(x) r_t(x) d\Gamma \quad \forall \phi \in H, \tag{4.354}$$

and

$$\begin{aligned} &w_\tau^t = w^t \cdot \tau, \\ &w^t \in H: \quad a(w^t, \phi) = \langle F^t, \phi\rangle \quad \forall \phi \in H\ . \end{aligned} \tag{4.355}$$

The solution λ^t to (4.351) is right–differentiable with respect to the parameter t at $t = 0$.

Lemma 4.38 *For $t > 0$, t small enough,*

$$\lambda^t = \lambda + t\dot{\lambda} + o(t) \quad \textit{in } (H^{\frac{1}{2}}(\Gamma_2))', \tag{4.356}$$

where $\|o(t)\|_{(H^{\frac{1}{2}}(\Gamma))'}/t \to 0$ as $t \downarrow 0$.

The element $\dot{\lambda}$ is given as the unique solution to the following variational inequality

$$\begin{aligned} &\dot{\lambda} \in S_\Lambda(\lambda)\ : \\ &b(\dot{\lambda}, \xi - \dot{\lambda}) + \dot{b}(\lambda, \xi - \dot{\lambda}) \geqq \int_{\Gamma_2} (\xi - \dot{\lambda})(\dot{w}_\tau + \dot{r} w_\tau) d\Gamma \quad \forall \xi \in S_\Lambda(\lambda), \end{aligned} \tag{4.357}$$

where the cone $S_\Lambda(\lambda) \subset (H^{\frac{1}{2}}(\Gamma_2))'$ *is defined by (4.286)*

$$\dot{b}(\lambda, \xi) = \int_{\Gamma_2} \{\dot{\psi}_\tau(\lambda) + u_\tau \dot{r}\} \xi d\Gamma, \tag{4.358}$$

$$\dot{\psi} \in H : a(\dot{\psi}, \phi) + \dot{a}(\psi, \phi) = \int_{\Gamma_2} \lambda \psi_\tau \dot{r} d\Gamma \quad \forall \phi \in H, \tag{4.359}$$

$$\dot{w} \in H : a(\dot{\psi}, \phi) + \dot{a}(w, \phi) + \dot{a}(w, \psi) = \langle F, \phi \rangle \quad \forall \phi \in H \ . \tag{4.360}$$

To prove Lemma 4.38 one can use the same reasoning as that of the proof of Theorem 4.35. Therefore, the proof of this lemma is left to the reader.

Let us observe that the element $z^t \in K(\Omega)$ minimizes the functional

$$I^t(\phi) = \frac{1}{2} a^t(\phi, \phi) - \langle F^t, \phi \rangle + \int_{\Gamma_2} \lambda^t \phi \cdot \tau r_t d\Gamma \tag{4.361}$$

over the set $K(\Omega)$. Furthermore, from Lemma 4.38 it follows that

$$\left(\int_{\Gamma_2} \lambda^t \phi \cdot \tau r_t d\Gamma \right)^{\cdot} = \int_{\Gamma_2} \{\dot{\lambda} \phi \cdot \tau + \lambda \dot{r} \phi_\tau\} d\Gamma \ . \tag{4.362}$$

Therefore, taking into account (4.342), and using the same argument as in the proof of Theorem 4.35, we obtain the following result.

Lemma 4.39 *For* $t > 0$, *t small enough,*

$$z^t = u + t\dot{z} + o(t) \quad \text{in } H^1(\Omega; \mathbb{R}^2), \tag{4.363}$$

where $\|o(t)\|_{H^1(\Omega;\mathbb{R}^2)}/t \to 0$ *as* $t \downarrow 0$.

The element $\dot{z} \in H^1(\Omega; \mathbb{R}^2)$ *is given as the unique solution to the following problem.*

Problem $(\dot{P})$: *Find an element* $\dot{z} \in S_K(u)$ *that minimizes the functional*

$$I(\phi) = \frac{1}{2} a(\phi, \phi) + \dot{a}(u, \phi) - \langle \dot{F}, \phi \rangle + \tag{4.364}$$

$$\max\{\int_{\Gamma_2} \xi(\phi \cdot \tau + \dot{r} u_\tau) d\Gamma | \xi \in S_\Lambda(\lambda)\} + \int_{\Gamma_2} \lambda \phi \cdot \tau \dot{r} d\Gamma$$

over the cone $S_K(u)$.

Proof. Using the reasoning of the proof given for the shape sensitivity analysis presented in Sect. 4.6, one can show that the element $\dot{z} \in S_K(u)$ minimizes the functional

$$I_1(\phi) = \frac{1}{2}a(\phi,\phi) + \dot{a}(u,\phi) - \langle \dot{F},\phi\rangle + \int_{\Gamma_2} (\dot{\lambda} + \lambda\dot{r})\phi \cdot \tau d\Gamma$$

over the cone $S_K(u)$ of the form (4.285). From (4.357) it follows that

$$\int_{\Gamma_2} \dot{\lambda}(\dot{z}_\tau + \dot{r}z_\tau)d\Gamma \geqq \int_{\Gamma_2} \xi(\dot{z}_\tau + \dot{r}z_\tau)d\Gamma \quad \forall \xi \in S_\Lambda(\lambda) \ . \tag{4.365}$$

Making use of (4.365) and taking into account that $z_\tau = u_\tau$, one can show that the functional $I_1(\cdot)$ can be replaced with (4.364). This proves Lemma 4.39.

In order to conclude the proof of Theorem 4.36, it should be noted that from (4.331) and (4.333) it follows that $u' \in \{S_K(u) - DV \cdot u + Du \cdot V\}$. On the other hand, if we select $V(\cdot,\cdot)$ with compact support in Ω, then $u' = 0$ and $\dot{u} = Du \cdot V$. Hence the following equation holds

$$a(Du \cdot V, \phi) + a(DV \cdot u, \phi) - \langle \dot{F}, \phi\rangle = 0, \tag{4.366}$$

for any $V = V(0)$ with compact support in Ω.

Using (4.366), the linear form (4.328) can be determined. This proves Theorem 4.36.

4.8. Elasto–Plastic torsion problems

Let $\Omega \subset \mathbb{R}^2$ be a given bounded domain with the smooth boundary $\Gamma = \partial\Omega$. We shall examine the following variational inequality :

Find $u \in \mathcal{K}(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla(\phi(x) - u(x))dx \geqq \mu \int_\Omega (\phi(x) - u(x))dx \quad \forall \phi \in \mathcal{K}(\Omega), \tag{4.367}$$

where $\mu > 0$ is a given constant and $\mathcal{K}(\Omega)$ is a closed and convex subset of the Sobolev space $H_0^1(\Omega)$,

$$\mathcal{K}(\Omega) = \{\phi \in H_0^1(\Omega) | |\nabla\phi(x)| \leqq k \text{ for a.e. } x \in \Omega\} \ . \tag{4.368}$$

Here $k > 0$ is a given constant, it is assumed that $k = 1$ in (4.368).

It can be shown (Friedman 1982) that there exists the unique solution to (4.367). Let us denote by $P \subset \Omega$ the so–called plastic region:

$$P = \{ \ x \in \Omega \ | \ |\nabla u(x)| = k\}, \tag{4.369}$$

then $E = \Omega \setminus P$ is the so–called elastic region and we have

$$-\Delta u(x) = \mu \quad \text{in } E \ . \tag{4.370}$$

The elastic region E and the plastic region P are not known a priori and should be determined; therefore the problem (4.367) is a free boundary problem. Let us recall that for the solutions to the variational inequality (4.367) the following regularity result

$$u \in H^2(\Omega) \cap H_0^1(\Omega) \tag{4.371}$$

is obtained (Brezis et al. 1968; Friedman 1982).

On the other hand, it can be shown (Brezis et al. 1968) that the solution to (4.367) is also the unique solution to the variational inequality

$$\begin{aligned} & u \in K_\rho(\Omega) \ : \\ & \int_\Omega \nabla u(x) \cdot \nabla(\phi(x) - u(x))dx \geqq \mu \int_\Omega (\phi(x) - u(x))dx \\ & \qquad\qquad\qquad\qquad\qquad \forall \phi \in K_\rho(\Omega), \end{aligned} \tag{4.372}$$

where

$$K_\rho(\Omega) = \{ \ \phi \in H_0^1(\Omega) \mid \phi(x) \leqq \rho(x) \text{ for a.e. } x \in \Omega\}, \tag{4.373}$$

$$\rho(x) = \min_{\xi \in \partial\Omega} \|x - \xi\|_{\mathbb{R}^2} \quad x \in \overline{\Omega} \ . \tag{4.374}$$

Material derivative $\dot{u}(\Omega)$

We denote by $\rho_t(\cdot)$ the distance function:

$$\rho_t(x) = \min_{\xi \in \partial\Omega_t} \|x - \xi\|_{\mathbb{R}^2} \quad x \in \Omega_t \ , \tag{4.375}$$

where $\Omega_t = T_t(V)(\Omega)$, $V(\cdot,\cdot)$ is a given vector field.

It is assumed that the following condition is satisfied

$$\rho_t(\cdot) \in H_0^1(\Omega_t) \quad \text{for } t > 0, \ t \text{ small enough},$$

and there exists an element $\dot{\rho}(\cdot) \in H_0^1(\Omega)$ such that

$$\lim_{t \to 0} \|(\rho_t \circ T_t - \rho_0)/t - \dot{\rho}\|_{H_0^1(\Omega)} = 0 \ . \tag{4.376}$$

Here $\dot{\rho}$ denotes the material derivative of the distance function ρ in the direction of the vector field $V(\cdot,\cdot)$.

Let us consider the following variational inequality defined in the domain Ω_t

$$\begin{aligned} & u_t \in K_{\rho_t}(\Omega_t) \ : \\ & \int_{\Omega_t} \nabla u_t(x) \cdot \nabla(\phi(x) - u_t(x))dx \geqq \mu \int_{\Omega_t} (\phi(x) - u_t(x))dx \\ & \qquad\qquad\qquad\qquad\qquad \forall \phi \in K_{\rho_t}(\Omega_t) \ . \end{aligned}$$

Theorem 4.40 *For $t > 0$, t small enough, we have*

$$u_t \circ T_t = u_0 + t\dot{u} + o(t), \tag{4.378}$$

where $\|o(t)\|_{H_0^1(\Omega)}/t \to 0$ as $t \to 0$.

The strong material derivative $\dot{u} \in H_0^1(\Omega)$ is given as the unique solution to the following variational inequality

$$\begin{aligned}
&\dot{u} \in S_\rho(\Omega) : \\
&\int_\Omega \nabla\dot{u}(x) \cdot \nabla(\phi(x) - \dot{u}(x))dx \geqq \\
&\int_\Omega \{F'(x)(\phi(x) - \dot{u}(x))dx - \\
&\int_\Omega \langle A'(x) \cdot \nabla(\phi(x) - \dot{u}(x))\rangle_{\mathbb{R}^2} dx \quad \forall\phi \in S_\rho(\Omega),
\end{aligned} \tag{4.379}$$

where

$$S_\rho(\Omega) = \{\phi \in H_0^1(\Omega) | \quad \phi(x) \leqq \dot{\rho}(x) \text{ q.e. on } P, \\ \int_P \nabla u_0 \cdot \nabla\phi(x)dx = \mu \int_P \phi(x)dx\}, \tag{4.380}$$

$$F'(x) = \mu \mathrm{div} V(0,x) \quad x \in \Omega, \tag{4.381}$$

$$A'(x) = \mathrm{div} V(0,x)\mathcal{I} - DV(0,x) - {}^*DV(0,x) \ . \tag{4.382}$$

Proof. Let $u^t = u_t \circ T_t \in H_0^1(\Omega)$, $t \in [0, \delta)$. The element u^t is the unique solution to the following variational inequality

$$\begin{aligned}
&u^t \in K_{\rho^t}(\Omega) \ : \\
&\int_\Omega \langle A_t(x) \cdot \nabla u^t(x), \nabla(\phi(x) - u^t(x))\rangle_{\mathbb{R}^2} dx \geqq \\
&\int_\Omega F_t(x)(\phi(x) - u^t(x))dx \quad \forall\phi \in K_{\rho^t}(\Omega),
\end{aligned} \tag{4.383}$$

where

$$K_{\rho^t}(\Omega) = \{\phi \in H_0^1(\Omega) | \quad \phi(x) \leqq \rho^t(x) \quad \text{for a.e. } x \in \Omega\}, \tag{4.384}$$

$$\rho^t(x) = (\rho_t \circ T_t)(x) \quad x \in \Omega, \tag{4.385}$$

$$A_t(x) = \det(DT_t(x))DT_t^{-1}(x) \cdot {}^*DT_t^{-1}(x), \tag{4.386}$$

$$F_t(x) = \mu\det(DT_t(x)) \ . \tag{4.387}$$

We shall prove that there exists an element $\dot{u} \in H_0^1(\Omega)$ such that

$$\lim_{t \downarrow 0} \|(u^t - u^0)/t - \dot{u}\|_{H_0^1(\Omega)} = 0 \ . \tag{4.388}$$

In order to apply the results on the differential stability of solutions to the obstacle problems, we introduce the inequality

$$
\begin{aligned}
& w^t \in K(\Omega) \ : \\
& \int_\Omega \langle A_t(x) \cdot \nabla w^t(x), \nabla(\phi(x) - w^t(x)) \rangle_{\mathbb{R}^2} dx \geqq \int_\Omega F_t(x)(\phi(x) - w^t(x)) dx \\
& - \int_\Omega \langle A_t(x) \cdot \nabla \rho^t(x), \nabla(\phi(x) - w^t(x) \rangle_{\mathbb{R}^2} dx \quad \forall \phi \in K(\Omega),
\end{aligned} \tag{4.389}
$$

where

$$
K(\Omega) = \{\phi \in H_0^1(\Omega) | \quad \phi(x) \leqq 0 \text{ a.e. in } \Omega\} \ . \tag{4.390}
$$

Hence for $t > 0$, t small enough, we have

$$
w^t = w^0 + t\dot{w} + o(t) \quad \text{in } H_0^1(\Omega), \tag{4.391}
$$

where the element $\dot{w}$ is given as the unique solution to the following variational inequality

$$
\begin{aligned}
& \dot{w} \in S_0(\Omega) \ : \\
& \int_\Omega \nabla \dot{w}(x) \cdot \nabla(\phi(x) - \dot{w}(x)) dx \geqq \int_\Omega F'(x)(\dot{w}(x)) dx - \\
& \int_\Omega \langle A'(x) \cdot \nabla \rho(x) + \dot{\rho}(x), \nabla(\phi(x) - \dot{w}(x)) \rangle_{\mathbb{R}^2} dx \quad \forall \phi \in S_0(\Omega) \ .
\end{aligned} \tag{4.392}
$$

In (4.392) the use has been made of the notation

$$
\begin{aligned}
S_0(\Omega) = \{\phi \in H_0^1(\Omega) | \phi(x) \leqq 0 \text{ q.e. on } \Xi, & \int_\Xi \nabla w^0(x) \cdot \nabla \phi(x) dx \\
&= \mu \int_\Xi \phi(x) dx - \int_\Xi \nabla \rho(x) \cdot \nabla \phi(x) dx\},
\end{aligned} \tag{4.393}
$$

$$
\Xi = \{x \in \Omega | w^0(x) = 0\} \ . \tag{4.394}
$$

Since

$$
u^t = w^t + \rho_t \tag{4.395}
$$

then (4.377) and (4.391) yield (4.378). This concludes the proof of Theorem 4.40. □

Remark. In order to use the material derivative $\dot{u}$ in the shape optimization, the form of the material derivative $\dot{\rho}(x)$, $x \in P$, should be determined. We have

$$
\dot{\rho}(x) = -\langle n(z(x)), V(0,x) - V(0, z(x)) \rangle_{\mathbb{R}^2} \quad x \in P, \tag{4.396}
$$

where

$$
z(x) = \arg\min\{\|\xi - x\|_{\mathbb{R}^2} | \xi \in \partial\Omega\} \quad x \in P \tag{4.397}
$$

and $n(\xi)$, $\xi \in \partial\Omega$, denotes the outward unit normal vector on $\partial\Omega$.

Domain derivative $u'(\Omega)$
Let $\tilde{u}_t(x)$, $x \in \Omega_t$, $t \in [0,\delta)$, denote an extension of $u_t \in H_0^1(\Omega_t)$ to $\mathbb{R}^2$ defined by

$$\tilde{u}_t(x) = \begin{cases} u_t(x) & x \in \Omega_t, t \in [0,\delta); \\ 0 & x \in \mathbb{R}^2 \setminus \Omega_t, t \in [0,\delta) \ . \end{cases} \tag{4.398}$$

For $t > 0$, t small enough, we have

$$\tilde{u}_{t|\Omega} = u_0 + tu' + o(t) \quad \text{in } H^1(\Omega), \tag{4.399}$$

where $\|o(t)\|_{H_0^1(\Omega)}/t \to 0$ as $t \to 0$.

We shall derive the form of the domain derivative u'. It is assumed that the set

$$\{\mathcal{S}_v(\Omega) - \mathcal{S}_v(\Omega)\} \cap H^2(\Omega)$$

is dense in the set $\{\mathcal{S}_v(\Omega) - \mathcal{S}_v(\Omega)\} \subset H^1(\Omega)$.

Theorem 4.41 *The shape derivative $u' = u'(\Omega, V)$ of the solution $u = u(\Omega)$ to the variational inequality (4.367) in the direction of a vector field $V(\cdot,\cdot)$ is given as the unique solution to the variational inequality*

$$\begin{aligned} & u' \in \mathcal{S}_v(\Omega) \ : \\ & \int_\Omega \nabla u'(x) \cdot \nabla(\phi(x) - u'(x))dx \geqq 0 \quad \forall \phi \in \mathcal{S}_v(\Omega), \end{aligned} \tag{4.400}$$

where

$$\begin{aligned} \mathcal{S}_v(\Omega) = \{\phi \in H^1(\Omega) | \phi(x) = v_n(x) \ on \ \partial\Omega, \\ \phi(x) \geqq \rho'(x) \ q.e. \ on \ P, \\ \int_P (\Delta\rho(x) + \mu)(\phi(x) - \rho'(x))dx = 0\}, \end{aligned} \tag{4.401}$$

$$v_n(x) = \langle V(0,x), n(x)\rangle_{\mathbb{R}^2} \quad x \in \partial\Omega, \tag{4.402}$$

$$\rho'(x) = v_n(z(x)) \quad x \in P, \tag{4.403}$$

$$\Delta\rho(x) = 1/(\rho(x) - R(z(x))) \quad x \in P, \tag{4.404}$$

$R(\cdot)$ is the radius of curvature of $\partial\Omega$.

The proof of Theorem 4.41 is omitted here. To prove this theorem one can use the same reasoning as that of the proof of, e.g. Theorem 4.16.

4.9. Elasto–Visco–Plastic problems

Let $\Omega \subset \mathbb{R}^3$ be a given domain with the sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. We denote by $\mathcal{E}$ the set of symmetric tensors of the second order. For a given element $\sigma \in \mathcal{E}$ we use the notation

$$\operatorname{tr} \sigma = \sum_{i=1}^{3} \sigma_{ii} \ , \tag{4.405}$$

$$\sigma^D = \sigma - \frac{1}{3} \operatorname{tr}(\sigma)\mathcal{I} \ . \tag{4.406}$$

We shall examine the following problem:

Problem (P)**:** Find an element $\sigma = \sigma(\Omega) \in L^2(\Omega;\mathcal{E})$ that minimizes the functional

$$I(\sigma) \equiv \frac{1}{2}\mathcal{A}(\sigma,\sigma) - \int_\Gamma (\sigma \cdot n) \cdot u_0 d\Gamma \tag{4.407}$$

subject to

$$\operatorname{div}\sigma + f = 0 \quad \text{in } \Omega, \tag{4.408}$$

$$\sigma \cdot n = g \quad \text{on } \Gamma_1 \ , \tag{4.409}$$

$$\sigma^D(x) \in K^D \quad \text{for a.e. } x \in \Omega, \tag{4.410}$$

where $f, u_0, g \in H^1(\mathbb{R}^3;\mathbb{R}^3)$ are given elements, $K^D \subset \mathcal{E}^D$, $\mathcal{E} = \mathcal{E}^D \oplus \mathbb{R}\mathcal{I}$, K^D is a convex subset such that $0 \in K^D$. For simplicity, it is assumed that

$$K^D = \{\sigma \in \mathcal{E}^D |\ \mathcal{F}(\sigma^D) \leqq 0\}, \tag{4.411}$$

where

$$\mathcal{F}(\sigma^D) = \frac{1}{2}|\sigma^D|^2 - k^2 \tag{4.412}$$

and $k > 0$ is a given constant, i.e. we consider the Huber–Von Mises yield condition. The bilinear form $\mathcal{A}(\cdot,\cdot)$ is defined as follows

$$\mathcal{A}(\sigma,\zeta) = \frac{1}{9k}\int_\Omega \operatorname{tr}(\sigma)\cdot\operatorname{tr}(\zeta)dx + \frac{1}{2\mu}\int_\Omega \sigma^D : \zeta^D dx \qquad \forall \sigma,\zeta \in L^2(\Omega;\mathcal{E}), \tag{4.413}$$

where $k, \mu > 0$ are given constants.

The solution σ of the elasto–plastic (i.e. elastic perfectly plastic) problem (P) can be considered as the stress field in the body of reference configuration Ω with prescribed displacement field u_0 on Γ_0 and prescribed tractions g on Γ_1; f denotes the vector of body forces. We replace (4.410) by a penalization term in the functional (4.407) and therefore we obtain the so–called Perzyna model.

Let P_K denote the metric projection in $\mathcal{E} = \mathcal{E}^D \oplus \mathbb{R}\mathcal{I}$ onto $K = K^D \oplus \mathbb{R}\mathcal{I}$ and let $\alpha > 0$ be given. We shall examine the following elasto–visco–plastic problem:

Problem (P_α)**:** Find an element $\sigma \in L^2(\Omega; \mathcal{E})$ that minimizes the functional

$$\begin{aligned} I_\alpha(\sigma) = \frac{1}{2}\mathcal{A}(\sigma, \sigma) - \int_{\Gamma_0} (\sigma \cdot n) \cdot u_0 d\Gamma + \\ \frac{1}{4\alpha} \int_\Omega |\sigma - P_K(\sigma)|^2_{\mathcal{E}} dx \end{aligned} \tag{4.414}$$

subject to

$$\begin{aligned} &\mathrm{div}\sigma + f = 0 \quad \text{in } \Omega, \\ &\sigma \cdot n = g \quad \text{in } \Gamma_1 \ . \end{aligned}$$

For any $\alpha > 0$, there exists the unique solution to the problem (P_α). It is assumed that $\alpha > 0$ is fixed. We denote by σ the solution to (P_α).

In order to determine the form of the material derivative $\dot{\sigma}$ of the solution to (P_α), we define the elasto–visco–plastic problem (P_α^t) in the domain Ω_t depending on the vector field $V(\cdot, \cdot)$.

Problem (P_α^t)**:** Find an element $\sigma_t \in L^2(\Omega_t; \mathcal{E})$ that minimizes the functional

$$\begin{aligned} I_t(\sigma) =& \frac{1}{2}\mathcal{A}_t(\sigma, \sigma) - \int_{\Gamma_0^t} (\sigma \cdot n_t) \cdot u_0 d\Gamma \\ &+ \frac{1}{4\alpha} \int_{\Omega_t} |\sigma - P_K(\sigma)|^2_{\mathcal{E}} dx \end{aligned} \tag{4.415}$$

subject to

$$\mathrm{div}\sigma + f = 0 \quad \text{in } \Omega_t \ , \tag{4.416}$$

$$\sigma \cdot n_t = g \quad \text{in } \Gamma_1^t \ . \tag{4.417}$$

We denote by f, u_0, g the restrictions to Ω_t, Γ_0^t, Γ_1^t of elements $f \in H^1(\mathbb{R}^3; \mathbb{R}^3)$, $u_0, g \in H^1(\mathbb{R}^3; \mathbb{R}^3)$, respectively. Furthermore, it is assumed that there exists an element $\sigma^* \in H^1(\mathbb{R}^3; \mathbb{R}^3)$ such that

$$\mathrm{div}\sigma^* + f = 0 \quad \text{in } \Omega_t \quad t \in [0, \delta], \tag{4.418}$$

$$\sigma^* \cdot n_t = g \quad \text{on } \Gamma_1^t \quad t \in [0, \delta] \ . \tag{4.419}$$

It can be easily shown that for fixed $\alpha > 0$ and under the assumptions introduced, the solutions to the problems (P_α^t), $t \in [0, \delta)$, are Lipschitz continuous with respect to t, i.e.

$$\|\sigma_t \circ T_t - \sigma_0\|_{L^2(\Omega;\mathcal{E})} \leqq Ct \ . \tag{4.420}$$

However, in order to obtain the form of the material derivative $\dot{\sigma}$ one has to assume that for a given field $V(\cdot,\cdot)$ the following condition holds

(A7) There exists the strong limit

$$\lim_{t\downarrow 0}(\sigma_t \circ T_t - \sigma)/t = \dot{\sigma} \quad \text{in } L^2(\Omega;\mathcal{E}) \ . \tag{4.421}$$

We shall show below that the element $\dot{\sigma}$ is uniquely determined as the solution to the auxiliary elasto–visco–plastic problem.

Remark. Let us observe that (4.421) implies the interior regularity of the stress field σ. In particular, for any vector field $V(\cdot,\cdot)$ with compact support in Ω, we have

$$\dot{\sigma} = \nabla\sigma \cdot V \ . \tag{4.422}$$

Therefore, $\sigma \in H^1(\Omega;\mathcal{E})$ is the necessary condition for (4.421) to be satisfied for any vector field $V(\cdot,\cdot)$.

Remark. The condition (4.420) is not sufficient for the existence of the material derivative $\dot{\sigma}$.

Theorem 4.42 *The material derivative $\dot{\sigma}$ of the solution σ to the problem (P_α), in the direction of a vector field $V(\cdot,\cdot)$ such that the condition (A7) is satisfied, solves the following problem:*

Problem $(\dot{P})$: *Find an element $\dot{\sigma} \in L^2(\Omega;\mathcal{E})$ that minimizes the functional*

$$\begin{aligned}\dot{I}_\alpha(\zeta) =& \frac{1}{2}\mathcal{A}(\zeta,\zeta) + \frac{1}{2\mu}\int_\Omega \gamma'\sigma^D : \zeta^D dx + \\ & \frac{1}{4\alpha}\int_\Omega |\zeta - \beta - P_S(\zeta-\beta)|^2 dx + \frac{1}{9k}\int_\Omega \gamma' \operatorname{tr}(\sigma)\operatorname{tr}(\zeta)dx + \\ & \int_{\Gamma_0}\{\gamma'(\zeta\cdot n)u_0 - (\zeta\cdot({}^*DV\cdot n))\cdot u_0 + (\zeta\cdot n)\cdot(\nabla u_0\cdot V)\}d\Gamma\end{aligned}$$

subject to

$$\int_\Omega \zeta : \epsilon(\phi)dx = \int_\Omega (\zeta : B')\cdot\epsilon(\phi)dx + \int_\Omega (\nabla\sigma_* \cdot V)\cdot\epsilon(\phi)dx$$

$$\forall\phi \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3) \ .$$

In the formulation of the Problem $(\dot{P})$ the following notation is used: $\beta = \gamma'(P_K\sigma - \sigma)$, P_S is metric projection in $\mathcal{E}^D$ onto the cone S_K defined by (4.437), $\gamma' = \operatorname{div}V$, $B' = -\operatorname{div}V\mathcal{I} + {}^*DV$.

Proof. Let

$$\eta_t = \sigma_t - \sigma^*, \tag{4.423}$$

where σ_t is the solution to (P_t) and σ^* denotes the restriction to Ω_t of the element $\sigma^* \in H^1(\mathbb{R}^3; \mathcal{E})$. Therefore, for the element η_t the following relations

$$\operatorname{div}\eta_t = 0 \quad \text{in } \Omega_t , \tag{4.424}$$

$$\eta_t \cdot n_t = 0 \quad \text{on } \Gamma_1^t \tag{4.425}$$

are satisfied.

Hence

$$\int_{\Omega_t} \eta_t : \epsilon(\phi) dx = 0 \quad \forall \phi \in H^1_{\Gamma_0}(\Omega_t; \mathbb{R}^3) \ . \tag{4.426}$$

Applying to (4.426) the usual change of variables, the following system in the fixed domain Ω is obtained

$$\int_{\Omega_t} \eta_t : \epsilon(\phi \circ T_t^{-1}) dx = \frac{1}{2} \int_\Omega \gamma(t) \eta^t : \{D\phi \cdot DT_t^{-1} + {}^*DT_t^{-1} \cdot {}^*D\phi\} dx =$$

$$\int_\Omega \gamma(t)(\eta^t : {}^*DT_t^{-1}) : \epsilon(\phi) dx = \int_\Omega \sigma^t : \epsilon(\phi) dx \quad \forall \phi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^3), \tag{4.427}$$

where

$$\sigma^t \equiv \gamma(t)\eta^t : {}^*DT_t^{-1} = \gamma(t)(\sigma \circ T_t) : {}^*DT_t^{-1} - \gamma(t)(\sigma^* \circ T_t) : {}^*DT_t^{-1} \ . \tag{4.428}$$

Thus

$$\sigma_t \circ T_t = \frac{1}{\gamma(t)} \sigma^t \cdot {}^*DT_t + \sigma^* \circ T_t = \sigma^t \cdot B_t + \sigma_*^t \tag{4.429}$$

and

$$\sigma_*^t = \sigma^* \circ T_t \ , \tag{4.430}$$

$$B_t = \frac{1}{\gamma(t)} {}^*DT_t \ . \tag{4.431}$$

pplying again the change of variables to the functional (4.412) to Ω, the resulting functional $I^t(\cdot)$ defined on Ω is derived,

$$I^t(\sigma^t) = \frac{1}{18k} \int_\Omega \gamma(t)[\operatorname{tr}(\sigma^t : B_t) + \operatorname{tr}\sigma_*^t]^2 dx + \frac{1}{4\mu} \int_\Omega \gamma(t) \|(\sigma^t : B_t)^D \tag{4.432}$$

$$+ (\sigma_*^t)^D\|^2_{\mathcal{E}^D} dx + \frac{1}{4\alpha} \int_\Omega \gamma(t) \|(\sigma^t : B_t)^D + (\sigma_*^t)^D - \phi_t\|^2_{\mathcal{E}^D} dx$$

$$- \int_{\Gamma_0} u_0^t \cdot \sigma^t \cdot n d\Gamma + \text{const},$$

where the element ϕ_t is determined as the unique solution to the following variational inequality

$$\begin{aligned}
&\phi_t^D \in K \ : \\
&\int_\Omega \gamma(t)(\phi_t^D - (\sigma^t\!: B_t)^D - (\sigma_*^t)^D, \zeta^D - \phi_t^D)_{\mathcal{E}^D}\, dx \geqq 0 \qquad (4.433) \\
&\qquad\qquad\qquad\qquad \forall \zeta^D(x) \in K, \\
&\phi_t = \phi_t^D + \mathrm{tr}(\sigma^t)\mathcal{I} \ . \qquad (4.434)
\end{aligned}$$

Making use of Example 4.2, and taking into account the assumption (A7), it is possible to show that for $t > 0$, t small enough, we have

$$\begin{aligned}
&\phi_t = \phi_0 + o(t) \quad \text{in } L^2(\Omega; \mathcal{E}), \qquad (4.435) \\
&\dot\phi = \dot\phi^D + \mathrm{tr}(\dot\phi)\mathcal{I},
\end{aligned}$$

where the element $\dot\phi^D \in L^2(\Omega; \mathcal{E}^D)$ is given as the unique solution to the following variational inequality

$$\begin{aligned}
&\dot\phi^D \in S_K \subset L^2(\Omega; \mathcal{E}^D) \ : \\
&\int_\Omega ((1+\lambda)(\dot\phi^D - \gamma'\sigma^D - \gamma'(\sigma^*)^D - \dot\sigma^D - (\sigma\!: \dot B)^D \qquad (4.436) \\
&- (\nabla\sigma^*\!: V)^D), \zeta^D - \dot\phi^D)_{\mathcal{E}^D}\, dx \geqq 0 \quad \forall \zeta^D \in S_K \ .
\end{aligned}$$

Here

$$\begin{aligned}
S_K = \{\eta \in L^2(\Omega; \mathcal{E}^D) |\ & \lambda(x)\xi^D(x) : \eta^D(x) = 0, \qquad (4.437) \\
& \xi^D(x) : \eta^D(x) \leqq 0 \quad \text{in } \Omega\},
\end{aligned}$$

$$\xi^D(x) = P_K(\sigma^D(x) + \sigma^{*D}(x)), \qquad (4.438)$$

$$\lambda(x) = \begin{cases} \frac{1}{2\sqrt{k}} |\sigma^D(x) + \sigma^{*D}(x)|_{\mathcal{E}^D} & \text{for } \sigma^D(x) + \sigma^{*D}(x) \notin K \\ 0 & \text{for } \sigma^D(x) + \sigma^{*D}(x) \in K \ . \end{cases} \qquad (4.439)$$

The functional (4.433) can be written in the form

$$I^t(\sigma^t) = \mathcal{A}^t(\sigma^t, \sigma^t) - \langle F^t, \sigma^t \rangle \qquad (4.440)$$

with the appropriate bilinear form $\mathcal{A}^t(\cdot,\cdot)$ and linear form $\langle F^t, \cdot \rangle$, respectively; the element ϕ_t is considered as the data. The necessary and sufficient optimality conditions for the problem (P_α^t) transported to the fixed domain Ω are of the form

$$\mathcal{A}^t(\sigma^t, \zeta) = \langle F^t, \zeta \rangle \quad \forall \zeta \in L^2(\Omega; \mathcal{E}), \qquad (4.441)$$

$$\int_\Omega \sigma^t\!: \epsilon(\phi)dx = 0 \quad \forall \phi \in H^1_{\Gamma_0}(\Omega; \mathrm{IR}^3) \ . \qquad (4.442)$$

Differentiating (4.442), (4.443) with respect to t yields

$$\mathcal{A}(\sigma', \zeta) + \mathcal{A}'(\sigma, \zeta) = \langle F', \zeta \rangle \quad \forall \zeta \in L^2(\Omega; \mathcal{E}),$$
$$\int_\Omega \sigma' : \epsilon(\phi) dx = 0 \quad \forall \phi \in H^1_{\Gamma_0}(\Omega; \mathrm{I\!R}^3)$$

as a result the problem $(\dot{P})$ is obtained. □

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