Assume that everything is smooth unless the contrary is explicitly stated.

1. Let M be a connected manifold and let $f\colon M \to M$ be a map such that $f \circ f = f$. Show that $f(M)$ is an embedded submanifold of M.

2. Let M be a manifold and let $f\colon M \to M$ be a map such that $f \circ f = \mathrm{id}_M$. Show that $f(M)$ is an embedded submanifold of M.

3. Show that the action of the diffeomorphism group of a connected manifold is transitive. (Hint: Show that, if p and q are in the open unit ball of $\mathbb{R}^n$, then there exists a compactly-supported vector field whose flow takes p to q.)

4. Let X and Y be vector fields on a smooth manifold M, with flows denoted by φ_t and ψ_t, respectively. Fix a point p of M, and, whenever it makes sense, define

$$c(t) = (\psi_t \circ \varphi_t \circ \psi_{-t} \circ \varphi_{-t})(p).$$

   Show that $c'(0) = 0$, and deduce from this that $f \mapsto (f \circ c)''(0)$ defines an element of T_pM. Prove that this element is precisely $2[X, Y]_p$.

5. Let D be a distribution on a manifold M, and let X be a (locally defined) vector field on M with flow φ_t. Show that the following are equivalent.

   (i) For all p in M and t for which it makes sense, $d(\varphi_t)_p(D_p) = D_{\varphi_t(p)}$.

   (ii) $\mathcal{L}_X$ takes local sections of D to local sections of D.

   (iii) $\mathcal{L}_X$ takes local annihilating forms for D to local annihilating forms for D.

   If any of these hold, then we say that X is an *infinitesimal symmetry* of D.

6. Let $\pi\colon M \to M'$ be a surjective submersion with connected fibres.

   (a) Show that $\ker d\pi$ is an involutive distribution on M, and describe the corresponding foliation.

   (b) Let D be a distribution on M containing $\ker d\pi$, and suppose that every local section of $\ker d\pi$ is also an infinitesimal symmetry of D. Show that there exists a unique distribution D' on M' such that $D'_{\pi(p)} = d\pi_p(D_p)$ for each p in M. (The next part of the problem outlines the proof of smoothness.)

   (c) Show that, for each p in M, it is possible to find an open neighbourhood U of p, local sections $Y_1, \ldots, Y_r$ of D on U, and a local *frame* $Y'_1, \ldots, Y'_r$ of D' on $\pi(U)$ such that each Y_i is π-related to Y'_i. Deduce from this that D' is smooth, and involutive if D is.

7. Let $\pi\colon M \to M'$ be a surjective submersion with connected fibres, and let ω be a k-form on M. Show that ω is the pullback of a k-form on M' by π, if and only if $i_X\omega = 0$ and $\mathcal{L}_X\omega = 0$ for every section X of $\ker d\pi$. Interpret this geometrically and give an example to show that the conclusion is false if we drop the assumption that π has connected fibres.

8. Let $f\colon M \to N$ be a map which is transverse to the leaves of a foliation $\mathcal{F}$ on N. Show that the connected components of the pre-images of the leaves of $\mathcal{F}$ under f give rise to a foliation on M. (To get a flat chart, inspect the proof of the transversality theorem.) Specialize to the intersection of two foliations.

9. Let M be a manifold with a foliation $\mathcal{F}$. The *leaf space* $M/\mathcal{F}$ is defined to be the set of leaves of $\mathcal{F}$ given the quotient topology. Show that the quotient map is open. Give an example to show that the quotient space need not be Hausdorff.

10. Consider the $(n-1)$-form ω on $\mathbb{R}^n \setminus \{0\}$ defined by

$$\omega = \frac{1}{|x|^n} \sum_{i=1}^{n} (-1)^{i-1} x^i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

    (a) Show that $\omega|_{S^{n-1}}$ is the standard volume form for S^{n-1}.
    (b) Show that ω is closed, but not exact.

11. Consider $\mathbb{R}^{2n}$ with the standard coordinates denoted by $x^1, y^1, \ldots, x^n, y^n$, and define ω by

$$\omega = \sum_{i=1}^{n} \left(-y^i \, dx^i + x^i \, dy^i \right).$$

Show that ω restricts to a nowhere-vanishing 1-form on S^{2n-1}. Compare this with the easy direction of the hairy ball theorem.

12. Let ω be a volume form on a manifold M. Show that, for each vector field X on M, there exists a unique function f on M such that $\mathcal{L}_X\omega = f\omega$. Give the local coordinate expression for f in terms of X and ω. What happens if ω is the standard volume form on $\mathbb{R}^n$?

13. Let M be a compact manifold, and suppose that ω_0, ω_1 are two volume forms on M inducing the same orientation and volume. Show that there exists a diffeomorphism $f\colon M \to M$ such that $f^*\omega_1 = \omega_0$. (Hint: Consider $\omega_t = (1-t)\omega_0 + t\omega_1$.)

14. Let E be a vector bundle over a manifold M, and let X be the Euler vector field on E. (The flow of X is given by multiplication by e^t.) Let $F\colon E \to E$ be a smooth mapping for which X is F-related to itself. Prove that F is actually a bundle map.

15. Use the previous problem to show that a diffeomorphism of a cotangent bundle which preserves the tautological 1-form is actually a cotangent lift.

16. (Milnor's exercise) For any manifold M, show that

$$M \to \mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R}), \qquad p \mapsto \mathrm{ev}_p$$

is a bijection. (Do it in the compact case first.)

17. With the previous problem in mind, construct a bijection between TM and $\mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R}[\varepsilon]/(\varepsilon^2))$. What if $\mathbb{R}[\varepsilon]/(\varepsilon^2)$ is replaced with $\mathbb{R}[\varepsilon]/(\varepsilon^{n+1})$? What if the domain and range of the set of algebra homomorphisms are swapped?

18. Let G be a Lie group acting by isometries on a Riemannian manifold. Show that each component of its fixed point set is an embedded, totally geodesic submanifold.

19. Let G be a Lie group admitting a bi-invariant metric g.

    (a) Show that the curvature tensor R acts on left-invariant vector fields by

    $$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]].$$

    (b) Show that

    $$\sec(X,Y) = \frac{1}{4}\,|[X,Y]|^2$$

    for every pair X, Y of orthonormal left-invariant vector fields, and conclude that, in the connected case, G is flat if and only if it is abelian.

20. Use the Gauss-Bonnet theorem and the previous problem to show that every compact, connected Lie group of dimension two is a torus.

21. (Credit to Ibsen on Discord for this one.) Consider a manifold M with a connection ∇'. Let ∇ be the standard flat connection in a local coordinate patch U; going forward, we will assume that $M = U$. Define a map $A\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

    $$A(X,Y) = A_X Y = \nabla'_X Y - \nabla_X Y.$$

    Show that A is a $(1,2)$-tensor. Now, by curring, we can identify A with an $\mathrm{End}(TM)$-valued 1-form on M given by $X \mapsto A_X$.

    The flat connection ∇ induces a connection in $\mathrm{End}(TM)$, given by

    $$\nabla_X(T)(Z) = \nabla_X(T(Z)) - T(\nabla_X Z).$$

This allows us to differentiate A to get an $\mathrm{End}(TM)$-valued 2-form dA on M given by

$$(dA)(X,Y) = \nabla_X A_Y - \nabla_Y A_X - A_{[X,Y]}.$$

dA is not quite the curvature, but we can add a "correction term" $A \wedge A$ to it to get the curvature tensor. (This is defined by the typical matrix multiplication expression, but with wedge products in the sum.) Prove this, and interpret the result geometrically using Stokes's theorem.