1 Differential Topology/Geometry Problems

1. This problem outlines a proof of the Frobenius theorem.

   (a) Let $X_1, \ldots, X_k$ be smooth vector fields on a manifold M which are linearly independent at every point. Show that, if $X_1, \ldots, X_k$ are commuting, then, for every point $p \in M$, there is a coordinate chart $(U, x^1, \ldots, x^n)$ near p such that, on U,
   $$X_i = \frac{\partial}{\partial x^i}, \qquad i = 1, \ldots, k.$$

   (b) Let $D \subseteq TM$ be an involutive distribution of rank k. Show that, for every $p \in M$, there exists a local frame $Y_1, \ldots, Y_k$ for D in a neighborhood of p, such that $[Y_i, Y_j] = 0$. Conclude that, near every point of M, there exists a flat chart for D, thus proving the Frobenius theorem.

2. Let $X, Y \in \mathfrak{X}(M)$ have flows φ_t, ψ_t, respectively, and fix a point $m \in M$. Let
   $$c(t) := (\psi_t \circ \varphi_t \circ \psi_{-t} \circ \varphi_{-t})(m).$$
   Prove that $c'(0) = 0 \in T_m M$, and deduce that $c''(0) \colon f \mapsto (f \circ c)''(0)$ defines a tangent vector to M at m. Prove that this tangent vector is precisely $2[X, Y]_m$.

3. It is well known that, if M is a smooth manifold and $X, Y \in \mathfrak{X}(M)$ have flows F_t, G_s, respectively, then $\mathcal{L}_X Y = 0$ if and only if $F_t \circ G_s = G_s \circ F_t$, whenever both sides are defined. If M is a submanifold of $\mathbb{R}^n$, then we can give a more quantitative substance to the notion that $\mathcal{L}_X Y$ measures how much the flows of X and Y commute. Fix $p \in M$, and define
   $$A(s, t) := G_s \circ F_t(p) - F_t \circ G_s(p)$$
   for (s, t) small enough. Show that
   $$A(0,0) = \frac{\partial A}{\partial s}(0,0) = \frac{\partial A}{\partial t}(0,0) = \frac{\partial^2 A}{\partial s^2}(0,0) = \frac{\partial^2 A}{\partial t^2}(0,0) = 0,$$
   $$\frac{\partial^2 A}{\partial s \partial t}(0,0) = \mathcal{L}_X Y|_p.$$
   (Bonus: Compute the second mixed partials of A at an arbitrary (s, t) in terms of the Euclidean connection $\nabla_X Y = X(Y^i)\partial_i$.)

4. This problem explores equivalent formulations of involutivity of distributions in terms of differential forms. If D is a smooth distribution on a manifold M, then we say that $\omega \in \Omega^k(M)$ *annihilates* D, if $\omega(X_1, \ldots, X_k) = 0$ whenever $X_1, \ldots, X_k$ are local sections of D.

(a) Show that a codimension-k distribution $D \subset TM$ is smooth, if and only if near every point, there exist *local defining forms* for D, i.e., locally defined 1-forms $\omega^1, \ldots, \omega^k$, which are linearly independent at every point, such that $D = \ker(\omega^1) \cap \cdots \cap \ker(\omega^k)$.

(b) Suppose that D is a smooth distribution in TM. Prove that D is involutive, if and only if for every 1-form ω annihilating D on an open set U, $d\omega$ also annihilates D on U.

(c) With some terminology, we can obtain an elegant algebraic characterization of involutivity. Let $\mathcal{I}^k(D)$ denote the space of all k-forms on M which annihilate D, and let

$$\mathcal{I}(D) := \bigoplus_{k=0}^{n} \mathcal{I}^k(D).$$

Show that $\mathcal{I}(D)$ is an ideal in the algebra $\Omega^*(M)$.

(d) Show that D is involutive, if and only if $d(\mathcal{I}(D)) \subseteq \mathcal{I}(D)$; if this holds, we say $\mathcal{I}(D)$ is a *differential ideal* in $\Omega^*(M)$.

(e) For computations, it is useful to have an involutivity criterion in terms of local defining forms. Let D be smooth and of codimension k. Let $\omega^1, \ldots, \omega^k$ be local defining forms for D on an open set U. Show that a p-form η ($p \geq 1$) on U annihilates D, if and only if

$$\eta = \sum_{i=1}^{k} \omega^i \wedge \beta^i$$

for some $\beta^1, \ldots, \beta^k \in \Omega^{p-1}(U)$. Note that this is equivalent to $\eta \wedge \omega^1 \wedge \cdots \wedge \omega^k = 0$. Now, use these facts to deduce the following criteria for involutivity of D on U:

(i) D is involutive on U;

(ii) $d\omega^1, \ldots, d\omega^k$ annihilate D on U;

(iii) There exist $\alpha^i_j \in \Omega^1(U)$ such that

$$d\omega^i = \sum_{j=1}^{k} \omega^i \wedge \alpha^i_j, \qquad i = 1, \ldots, k;$$

(iv) $d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^k = 0$ for each $i = 1, \ldots, k$.

5. Let $\omega \in \Omega^1(M)$ be a smooth 1-form. A smooth positive function μ defined on an open subset $U \subseteq M$ is said to be a *integrating factor for* ω if $\mu\omega$ is exact on U. Show that, if ω is nowhere-vanishing on M, then ω admits integrating factors in a neighborhood of every point if and only if $d\omega \wedge \omega \equiv 0$. In particular, every nowhere-vanishing smooth 1-form on a 2-manifold admits local integrating factors.

6. Let $\pi\colon M \to M'$ be a surjective submersion with connected fibres.

   (a) Show that $\ker d\pi = \cup_{p\in M} \ker d\pi_p$ is a smooth, involutive distribution on M. Describe the foliation of M determined by $\ker d\pi$.

   (b) Let D be a smooth distribution on M. For a (local) vector field X with flow φ_t, show that the following are equivalent:

      (i) For all $p \in M$ and all t for which it makes sense, $d(\varphi_t)_p(D_p) = D_{\varphi_t(p)}$.

      (ii) For all local sections Z of D, $\mathcal{L}_X Z$ is a local section of D.

      (iii) For all local annihilating forms ω of D, $\mathcal{L}_X\omega$ is a local annihilating form of D.

   If any of these hold, we say that X is an *infinitesimal symmetry* of the distribution D.

   (c) Let D be a smooth distribution on M containing $\ker d\pi$, such that every (local) vertical vector field is an infinitesimal symmetry of D. Show that there exists a unique distribution D' on M' such that $D'_{\pi(p)} = d\pi_p(D_p)$ for all $p \in M$.

   (d) Show that for every $p \in M$, one can find local sections $Y_1, \dots, Y_r$ of D defined in a neighbourhood U of p, and a local *frame* $Y_1', \dots, Y_r'$ for D' defined on $\pi(U)$. Thus, D' is a smooth distribution. Deduce from this that if D is involutive, then so is D'.

7. Let $\pi\colon M \to M'$ be a surjective submersion.

   (a) Suppose that π has connected fibres. Show that a k-form ω on M is the pullback by π of a unique k-form on M', if and only if, for all vertical vector fields X, $\iota_X\omega = 0$ and $\mathcal{L}_X\omega = 0$.

   (b) Give an example where part (a) fails if the fibres of π are not connected.

   (c) (I don't know if this is possible, yet.) Use this to prove part (c) of the previous problem.

8. Let M be a smooth manifold. Define the *tautological* 1-*form* λ on T^*M by

$$\lambda_{(x,\varphi)}(V) := \varphi(d\pi_{(x,\varphi)}(V)), \qquad (x,\varphi) \in T^*M, V \in T_{(x,\varphi)}(T^*M).$$

   (a) Show that λ is a smooth 1-form, and that $\omega := -d\lambda$ is a symplectic form on T^*M (i.e., a non-degenerate closed 2-form).

   (b) Show that for every 1-form β on M, if β is regarded as a mapping $\beta\colon M \to T^*M$, then $\beta^*\lambda = \beta$.

   (c) Show that if $f\colon M \to M$ is a diffeomorphism of smooth manifolds, then its cotangent lift $f^*\colon T^*M \to T^*M$, defined by $f^*(x,\varphi) := (f(x), ((df_x)^{-1})^*\varphi)$, satisfies $(f^*)^*\lambda = \lambda$.

(d) Using the following steps, show that the converse to part (c) is true, i.e., that every diffeomorphism of T^*M with itself which preserves the tautological 1-form is a cotangent lift. Let $F\colon T^*M \to T^*M$ be a diffeomorphism satisfying $F^*\lambda = \lambda$. Prove the following:

(i) If X denotes the *Euler vector field*, defined by $\iota_X \omega = \lambda$, then $F_* X = X$.

(ii) The flow φ_t of X commutes with F for all t, and $\varphi_t(x, \alpha) = (x, e^t \alpha)$.

(iii) F maps fibres of T^*M to fibres of T^*M, and so there exists a map $f\colon M \to M$ such that $\pi \circ F = f \circ \pi$. Moreover, f is a diffeomorphism. By part (b), $F^{-1} \circ f^*$ is the identity map on T^*M, i.e., F is the cotangent lift of f.

9. Consider $\omega \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ defined by

$$\omega := \frac{1}{|x|^n} \sum_{k=1}^{n} (-1)^{k-1} x^k \, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n.$$

(a) Show that $\omega|_{S^{n-1}}$ is the standard volume form for S^{n-1}.

(b) Show that ω is closed, but not exact.

10. Let $\mathbb{R}^{2n}$ have coordinates $(x^1, y^1, \ldots, x^n, y^n)$, and define $\omega \in \Omega^1(\mathbb{R}^{2n})$ by

$$\omega := \sum_{j=1}^{n} (-y^i \, dx^i + x^i \, dy^i).$$

Show that $\omega|_{S^{2n-1}}$ is nowhere-vanishing.

11. Let ω be a nowhere-vanishing top-degree form on M, and fix $X \in \mathfrak{X}(M)$. Show that there is a unique $f \in C^\infty(M)$ such that $\mathcal{L}_X \omega = f\omega$, and write f in local coordinates. If $M = \mathbb{R}^n$ and $\omega = dx^1 \wedge \cdots \wedge dx^n$, what is f?

12. Let $f, g\colon M \to N$ be homotopic smooth maps, where M is a compact, oriented, boundary-less m-manifold. Show that, for every m-form ω on N,

$$\int_M f^*\omega = \int_M g^*\omega.$$

13. Let M be a connected smooth manifold and let $f\colon M \to M$ be a smooth map such that $f \circ f = f$. Show that $f(M)$ is an embedded submanifold of M.

14. Show that every closed subset of $\mathbb{R}^n$ is the fixed-point set of some smooth $\mathbb{R}$-action. Is the same true if $\mathbb{R}^n$ is replaced with a manifold which admits a nowhere-vanishing smooth vector field?

15. Let M be a connected smooth manifold. Show that the diffeomorphism group of M acts transitively on M. (Hint: First show that, if p, q are in the open unit ball B^n in $\mathbb{R}^n$, then there exists a smooth compactly-supported vector field X on $\mathbb{R}^n$ whose flow's time-one map takes p to q.)

16. Let M be a compact manifold. If ω_0, ω_1 are two volume forms on M defining the same orientation, with
$$\int_M \omega_0 = \int_M \omega_1,$$
prove that there exists a diffeomorphism $f\colon M \to M$ with $f^*\omega_1 = \omega_0$. (Hint: Moser's argument.)

17. Prove that every closed smooth manifold admits a Morse function which takes on different values at its critical points. (Hint: Starting from an arbitrary Morse function on the manifold, perturb it near its critical points in such a way that the new function has different values at its critical points and remains Morse.)

18. (a) Let M be a non-orientable manifold. Show that, no matter what N is, $M \times N$ is non-orientable.
    (b) If V is a vector space, show that both orientations on V define the same orientation on $V \times V$.
    (c) Show that there is a neighborhood in $M \times M$ of the diagonal of M which has a natural orientation, regardless of the orientability of M.

19. Let M be a closed, oriented manifold, and let X be a smooth vector field on M such that

    - each component of the set of zeroes of X is a submanifold F of M;
    - for every p in F, the range of the induced linear operator on T_pM is a complement to T_pF, and hence induces an invertible linear operator A_p on T_pM/T_pF.

    It is easy to see that $\operatorname{sign}(\det A_p)$ depends only on F. Denote this number by $\varepsilon_F = \pm 1$.

    Assuming for simplicity that every F is orientable, prove that
    $$\chi(M) = \sum_F \varepsilon_F \chi(F).$$
    (Hint: Construct a vector field Y on M which is tangent to each F and whose restriction thereof has only nondegenerate zeroes, and consider the zeroes of $X + tY$ for small enough t. Also, use Poincare-Hopf for M and for the F's.)

20. This exercise outlines a proof that de Rham cohomology is a homotopy invariant.

   (a) Let $\varphi, \psi \colon M \to N$ be smoothly-homotopic smooth maps. Show that φ, ψ induce the same maps in de Rham cohomology, i.e., $\varphi^*\omega - \psi^*\omega$ is exact, for all closed ω. (Hint: First, prove it for $i_0, i_1 \colon M \to M \times [0,1]$ by using the fundamental theorem of calculus to rewrite $i_1^*\omega - i_0^*\omega$, where $\omega \in \Omega^p(M \times [0,1])$ is closed.)

   (b) Use (but do not prove, unless you want to) the following two facts about homotopy to prove that, if M and N are (continuously) homotopy-equivalent manifolds, then $H^k_{dR}(M) \cong H^k_{dR}(N)$ for each k:

      (i) (Whitney Approximation Theorem) If $F \colon M \to N$ is a continuous map, then F is continuously homotopic to a smooth map.

      (ii) If $F, G \colon M \to N$ are smooth maps which are continuously homotopic, then they are also smoothly homotopic.

   (c) As an application of the homotopy invariance of de Rham cohomology, prove that any contractible open subset U of $\mathbb{R}^n$ has each $H^k_{dR}(U) = 0$, and conclude the *Poincare lemma*: every closed form on a smooth manifold is locally exact.

21. Let M be a smooth manifold. Denote by $\pi_{TM} \colon TM \to M$ the natural projection of the tangent bundle of M, and by $\pi_{TTM} \colon TTM \to TM$ the natural projection of the tangent bundle to TM. By differentiating π_{TM}, we obtain another map $d\pi_{TM} \colon TTM \to TM$.

   (a) Show that $(TTM, d\pi_{TM}, TM)$ is a vector bundle over TM. (More generally, if $\pi \colon E \to M$ is any smooth vector bundle, then $(TE, d\pi, TM)$ is a smooth vector bundle over TM; this is called the *secondary vector bundle structure*.)

   (b) It turns out that (TTM, π_{TTM}, TM) and $(TTM, d\pi_{TM}, TM)$ are naturally isomorphic as vector bundles. Define the *canonical swap* $s_M \colon TTM \to TTM$ as follows: if $x = (x^i)$ are local coordinates for M, inducing local coordinates $(x, v) = (x^i, v^i)$ for TM, which themselves induce local coordinates $(x, v; \xi, \eta) = (x^i, v^i; \xi^i, \eta^i)$ for TTM, define

$$s_M(x, v; \xi, \eta) := (x, \xi; v, \eta).$$

   Show that s_M is well-defined, independent of the choice of local coordinates.

   (c) Show that s_M is an involutive isomorphism of $(TTM, d\pi_{TM}, TM)$ with the bundle (TTM, π_{TTM}, TM). That is, $s_M = s_M^{-1}$, and the following diagram commutes:

$$\begin{tikzcd} TTM \arrow[rr, "s_M"] \arrow[dr, "d\pi_{TM}"'] & & TTM \arrow[dl, "\pi_{TTM}"] \\ & TM & \end{tikzcd}$$

(d) Show that if $f\colon \mathbb{R}^2 \to M$, $(s,t) \mapsto f(s,t)$, is C^2, then $\partial_t\partial_s f = s_M \partial_s\partial_t f$. This property can be used to give a coordinate-free definition of the canonical swap. Let $J^2_{(0,0)}(\mathbb{R}^2, M)$ denote the space of 2-jets at the origin $(0,0) \in \mathbb{R}^2$ of smooth maps $\mathbb{R}^2 \to M$. Define two maps

$$\begin{aligned} p\colon J^2_{(0,0)}(\mathbb{R}^2, M) &\to TTM, & p([f]) &:= \partial_s\partial_t f(0,0),\\ J\colon J^2_{(0,0)}(\mathbb{R}^2, M) &\to J^2_{(0,0)}(\mathbb{R}^2, M), & J([f]) &:= [f \circ \alpha], \end{aligned}$$

where $\alpha\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the mapping that swaps the coordinates. Show that p is a smooth submersion, and that J is smooth. Show that there exists a unique smooth map $j\colon TTM \to TTM$ such that the following diagram commutes:

$$\begin{tikzcd} J^2_{(0,0)}(\mathbb{R}^2, M) \arrow[r, "J"] \arrow[d, "p"'] & J^2_{(0,0)}(\mathbb{R}^2, M) \arrow[d, "p"] \\ TTM \arrow[r, dashed, "j"'] & TTM \end{tikzcd}$$

Show that $j = s_M$.

22. This exercise explores some "canonical constructions" on vector bundles. Let $\pi_E\colon E \to M$ be a rank-k vector bundle over the n-manifold M.

(a) Given $\xi \in E$ lying in the fibre over $x \in M$, define the *vertical lift* $\mathrm{vl}_\xi\colon E_x \to T_\xi(E_x)$ by

$$\mathrm{vl}_\xi(\eta) := \left.\frac{d}{dt}\right|_{t=0} \xi + t\eta.$$

Show that vl_ξ is a linear isomorphism of vector spaces. Show that the vertical lift determines a vector bundle isomorphism of the pullback bundle $\pi_E^*(E)$ with the vertical bundle VE:

$$\begin{tikzcd} \pi_E^*(E) \arrow[rr, "\mathrm{vl}"] \arrow[dr, "\mathrm{proj}_1"'] & & VE \arrow[dl, "\pi_{TE}|_{VE}"] \\ & E & \end{tikzcd}$$

(b) The vertical lift allows us to define the *canonical vector field* V on E, by $V_\xi := \mathrm{vl}_\xi \xi$. Find the flow of V, and express V in terms of local bundle coordinates.

(c) Specializing to the tangent bundle (TM, π_{TM}, M) of M, the vertical lift and canonical vector field allow for the definition of the *canonical endomorphism*, or the *tangent structure* of M. It is a $(1,1)$-tensor field J on the total space

TM defined by $J_\xi(X) := \mathrm{vl}_\xi(d\pi_{TM}(X))$, $\xi \in TM, X \in T_\xi TM$. Prove that J is indeed a $(1,1)$-tensor field, and give its local coordinate expression in terms of local bundle coordinates.

(d) Prove the following properties of J:

  (i) $\mathrm{image}(J) = \ker(J) = VTM$;

  (ii) $\mathcal{L}_V J = -J$, where V is the canonical vector field on TM;

  (iii) $J[X,Y] = J[JX,Y] + J[X,JY]$.

In fact, these three properties characterize TM, in the following sense, explaining the terminology "tangent structure." If (E, π_E, M) is a vector bundle over M whose total space admits a $(1,1)$-tensor field J satisfying (i)-(iii) (with VTM replaced by VE), then there is a bundle isomorphism $(E, \pi_E, M) \cong (TM, \pi_{TM}, M)$ taking J to the tangent structure on TM.

23. Let M be a smooth manifold and let $\mathfrak{g}$ be a Lie algebra. If ω and η are $\mathfrak{g}$-valued k-forms and l-forms on M, respectively, we define their wedge product to be the $\mathfrak{g}$-valued $(k+l)$-form $[\omega \wedge \eta]$ on M given by

$$[\omega \wedge \eta]_p(v_1, \dots, v_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)[\omega_p(v_{\sigma(1)}, \dots, v_{\sigma(k)}), \eta_p(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})].$$

(a) Verify that, if ω and η are $\mathfrak{g}$-valued 1-forms, then for every $X, Y \in \mathfrak{X}(M)$,

$$[\omega \wedge \eta](X,Y) = [\omega(X), \eta(Y)] - [\omega(Y), \eta(X)].$$

(b) Prove that

$$[\omega \wedge \eta] = (-1)^{|\omega||\eta|+1}[\eta \wedge \omega].$$

Now let G be a Lie group, with Lie algebra $\mathfrak{g}$. We can define a canonical $\mathfrak{g}$-valued 1-form, the *Maurer-Cartan* form, by $\theta_g(v) := d(L_{g^{-1}})_g(v)$. Using the wedge product on Lie algebra valued 1-forms just developed, we study the Maurer-Cartan form.

(c) Prove that θ is a left-invariant $\mathfrak{g}$-valued 1-form on G.

(d) Prove the *Maurer-Cartan* equation:

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0.$$

(Hint: It suffices to check the equation on left-invariant vector fields.)

With the language of principal bundles, the Maurer-Cartan equation has a more geometric interpretation. Let $P \to M$ be a right principal G-bundle over a manifold M. A *principal G-connection* is a $\mathfrak{g}$-valued 1-form ω on P such that

(i) (Equivariance) For all $g \in G$, $R_g^*\omega = \mathrm{Ad}_{g^{-1}}\omega$, where R_g is the action of g on P.

(ii) (Reproduces Lie algebra generators) For all $\xi \in \mathfrak{g}$, if ξ_P is the vector field on P whose flow is $(t, p) \mapsto p \cdot \exp t\xi$, then $\omega(\xi_P) \equiv \xi$ on P.

If ω is a principal G-connection, then we define its *curvature* to be the $\mathfrak{g}$-valued 2-form Ω on P defined by

$$\Omega := d\omega + \frac{1}{2}[\omega \wedge \omega].$$

Consider the Lie group G as a right principal G-bundle over a singleton $\{*\}$, with the action of G being right-multiplication.

(e) Prove that the Maurer-Cartan form θ is a principal G-connection on the G-bundle $G \to \{*\}$ just described. Deduce that its curvature vanishes identically.

Therefore, the Maurer-Cartan equation simply states that the curvature form associated to the connection form θ on $G \mapsto \{*\}$ vanishes.

24. Let $E \to M$ be a vector bundle with a linear connection ∇. Analogously to the exterior derivative of a differential form on M, we can define, using ∇, the "covariant exterior derivative" of an E-valued differential form on M: if $\omega \in \Omega^k(M; E)$, then we define

$$\begin{aligned}(d^\nabla\omega)(X_0, \ldots, X_k) &= \sum_{i=0}^{k}(-1)^i \nabla_{X_i}(\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k)) \\ &\quad + \sum_{0 \le i < j \le k}(-1)^{i+j}\omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k).\end{aligned}$$

It is clear that $d^\nabla \colon \Omega^k(M; E) \to \Omega^{k+1}(M; E)$ for all $k \ge 0$. Note that, if E is the trivial line bundle $M \times \mathbb{R}$ and ∇ is the standard flat connection $s \mapsto Xs$, then d^∇ reduces to the usual exterior derivative. Prove the following:

(a) If $s \in \Gamma(E) = \Omega^0(M; E)$, then $d^\nabla s = \nabla s$ and $d^\nabla(d^\nabla s) = Rs$. (Observe that, unlike the usual exterior derivative, we may not have $d^\nabla \circ d^\nabla = 0$.)

(b) If $s \in \Gamma(E)$, then

$$((d^\nabla)^3 s)(X, Y, Z) = R(X, Y)\nabla_Z s + R(Y, Z)\nabla_X s + R(Z, X)\nabla_Y s.$$

(c) If E is the tangent bundle of M, and if τ denotes the torsion tensor of ∇, then

$$\begin{aligned}(d^\nabla \mathrm{id})(X, Y) &= \tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \\ (d^\nabla(d^\nabla \mathrm{id}))(X, Y, Z) &= R(X, Y)Z + R(Y, Z)X + R(Z, X)Y.\end{aligned}$$

Therefore, Bianchi's first identity for torsion-free connections simply states that the derivative of zero is zero.

25. Let (M, ω) be a closed symplectic manifold. Show that $[\omega] \neq 0 \in H^2_{dR}(M)$, and conclude that the only sphere S^n which admits a symplectic structure is S^2.

26. Let $E \to M$ be a rank-r vector bundle equipped with an affine connection ∇. For a local frame $e_1, \ldots, e_r$ for $E \to M$, we define the connection 1-forms ω^i_j by $\nabla_X e_j = \omega^i_j(X) e_i$, and the curvature 2-forms by $R(X, Y)e_j = \Omega^i_j(X, Y)e_i$.

    (a) Prove the *second structural equation*:

    $$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j.$$

    (b) Prove the *second Bianchi identity*:

    $$d\Omega^i_j = \Omega^i_k \wedge \omega^k_j - \omega^i_k \wedge \Omega^k_j.$$

    Now we specialize to the tangent bundle $E = TM$, in which the torsion tensor is well-defined: $\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. We define the torsion 1-forms τ^i by $\tau(X, Y) = \tau^i(X, Y)e_i$, and the dual 1-forms θ^i by $\theta^i(e_j) = \delta^i_j$.

    (c) Prove the *first structural equation*:

    $$\tau^i = d\theta^i + \omega^i_k \wedge \theta^k.$$

    (d) Prove the *first Bianchi identity*:

    $$d\tau^i = \Omega^i_k \wedge \theta^k - \omega^i_k \wedge \tau^k.$$

    The four equations in this exercise may be recast more elegantly using the following matrix notation. Write ω and Ω for the matrices whose (i, j)-th entries are ω^i_j and Ω^i_j, respectively. Interpret τ, θ as column vectors. The exterior derivative dA of a matrix A of differential forms is equal to the matrix whose components are the exterior derivatives of those of A. Finally, if $A = [A^i_j]$ and $B = [B^i_j]$ are matrices of differential forms, where A has as many columns as B does rows, then $A \wedge B$ is the matrix whose (i, j)-th entry is $A^i_k \wedge B^k_j$. We thus have:

    $$\begin{aligned} \Omega &= d\omega + \omega \wedge \omega, \\ d\Omega &= \Omega \wedge \omega - \omega \wedge \Omega, \\ \tau &= d\theta + \omega \wedge \theta, \\ d\tau &= \Omega \wedge \theta - \omega \wedge \tau. \end{aligned}$$

27. This problem uses the notation and results of the previous problem.

(a) Suppose that $e_1, \ldots, e_r$ and $\bar{e}_1, \ldots, \bar{e}_r$ are frames for $E \to M$ over an open set U in M. Then, there exists a $GL(r, \mathbb{R})$-valued function $a = [a^i_j]$ on U such that $\bar{e}_j = e_i a^i_j$. If we consider the row vectors $e = [e_1 \cdots e_r]$, $\bar{e} = [\bar{e}_1 \cdots \bar{e}_r]$, then $\bar{e} = ea$. Show that, if ∇ is a connection in $E \to M$, then, on U,

$$\overline{\omega} = a^{-1}\omega a + a^{-1} da,$$
$$\overline{\Omega} = a^{-1}\Omega a.$$

Now, if $P(X)$ is a degree-k polynomial in $X = [x^i_j]$, an $r \times r$ matrix of indeterminates, such that $P(A^{-1}XA) = P(X)$ for all $A \in GL(r, \mathbb{R})$, then it follows that $P(\Omega)$ is a *globally-defined* differential $2k$-form on M. (We shall call such a polynomial *invariant*.) In the remainder of the problem, you will prove that $P(\Omega)$ is closed, and that its cohomology class is independent of the choice of connection. An algebraic lemma helps us simplify the problem significantly.

Lemma 1.1. *The set of all invariant polynomials in $X = [x^i_j]$ is equal to the set $\mathbb{R}[\Sigma_1, \ldots, \Sigma_r]$, where Σ_k is the kth* trace polynomial, $\Sigma_k(X) := \mathrm{tr}(X^k) \in \mathbb{Z}[x^i_j]$.

It is therefore enough to consider $\Sigma_k(\Omega)$. (Why?)

(b) Prove the *generalized second Bianchi identity*:

$$d(\Omega^k) = \Omega^k \wedge \omega - \omega \wedge \Omega^k,$$

and use this to prove that $d(\mathrm{tr}(\Omega^k)) = 0$.

(c) Let ∇^t be a family of connections in $E \to M$ whose connection and curvature matrices ω_t and Ω_t on a framed open set U vary smoothly in t. Prove that

$$\frac{d}{dt}\mathrm{tr}(\Omega_t^k) = k \cdot d\,\mathrm{tr}\left(\Omega_t^{k-1} \wedge \frac{d}{dt}\omega_t\right).$$

(d) Prove that

$$\mathrm{tr}\left(\Omega_t^{k-1} \wedge \frac{d}{dt}\omega_t\right)$$

is independent of the frame, and therefore defines a global form on M.

(e) Combine the previous exercises, as follows, to prove that the cohomology class $[\mathrm{tr}(\Omega^k)]$ is independent of the connection. Let ∇^0, ∇^1 be connections in $E \to M$. Then, $\nabla^t := (1-t)\nabla^0 + t\nabla^1$ is a connection in $E \to M$. Let U be a framed open set, and let ω_t, Ω_t be the connection and curvature matrices of ∇^t over U, respectively. Show that

$$\mathrm{tr}(\Omega_1^k) - \mathrm{tr}(\Omega_0^k)$$

is an exact form on M, and conclude that the cohomology class $[\mathrm{tr}(\Omega^k)]$ is independent of the connection. (Hint: Rewrite the above expression using the fundamental theorem of calculus, and apply the previous parts of the problem.)

28. Let G be a Lie group.

    (a) For $X, Y \in \mathfrak{g}$, show that $[X, Y] = 0$ if and only if

    $$\exp tX \exp sY = \exp sY \exp tX, \qquad s, t \in \mathbb{R}.$$

    (b) Show that, if G is abelian, then so is $\mathfrak{g}$, and that the converse holds if G is connected. Give an example to show that the converse is not necessarily true if G is not connected.

    (c) Show that $\exp\colon \mathfrak{g} \to G$ is a Lie group homomorphism, if and only if the identity component of G is abelian. (For the "only if" direction, consider $t \mapsto \exp tX \exp tY$.)

29. Let G, H be Lie groups.

    (a) Show that any continuous homomorphism $\gamma\colon \mathbb{R} \to H$ is smooth. (Hint: Let $V \subseteq \mathfrak{h}$ be a neighborhood of 0, star-shaped with respect to 0, such that exp is a diffeomorphism from $2V$ onto $\exp(2V)$. Choose $t_0 > 0$ small enough so that $\gamma(t) \in \exp(V)$ whenever $|t| \leq t_0$, and consider $X_0 \in V$ given by $\exp X_0 = \gamma(t_0)$. Show that for every dyadic rational q, $\gamma(qt_0) = \exp(qX_0)$, and then use continuity to conclude that γ is smooth.)

    (b) Show that any continuous homomorphism $F\colon G \to H$ is smooth. (Hint: Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$, and consider the map $\alpha\colon \mathbb{R}^n \to G$ given by

    $$\alpha(t_1, \ldots, t_n) := (\exp t_1 X_1) \cdots (\exp t_n X_n).)$$

    (c) Show that, if G is a second-countable, Hausdorff, locally Euclidean topological group, then there is at most one smooth structure on G making it into a Lie group.

2 Riemannian Geometry Problems

30. Let (M, g) be a Riemannian manifold. We will develop the *Sasaki metric* $\widehat{g}$, a Riemannian metric on TM induced by g with various special properties.

(a) Let $(p, v) \in TM$, and choose $V, W \in T_{(p,v)}(TM)$. Define $\widehat{g}_{(p,v)}(V, W)$ as follows: choose curves $\alpha(t) = (p(t), v(t))$ and $\beta(t) = (q(t), w(t))$ in TM with $\alpha(0) = \beta(0) = (p, v)$ and $\alpha'(0) = V, \beta'(0) = W$, and set

$$\widehat{g}_{(p,v)}(V, W) := g_p(d\pi(V), d\pi(W)) + g_p\left(\frac{Dv}{dt}(0), \frac{Dw}{dt}(0)\right).$$

Show that $\widehat{g}_{(p,v)}(V, W)$ is well-defined, and that $\widehat{g}$ is a Riemannian metric on TM.

(b) Denote by M_0 the zero section in TM, i.e.,

$$M_0 = \{(p, 0) \in T_pM : p \in M\}.$$

Let $i_0 \colon M_0 \hookrightarrow TM$ be the inclusion map, and let $\varphi \colon M \to M_0$ be the diffeomorphism $p \mapsto (p, 0)$. Prove that $(i \circ \varphi)^*\widehat{g} = g$. Similarly, if $i_p \colon T_pM \hookrightarrow TM$ is the inclusion of the fiber $\pi^{-1}(\{p\})$, then show that $i_p^*\widehat{g}$ is the Riemannian metric on T_pM induced by the inner product g_p.

(c) We call $\xi \in T_{(p,v)}(TM)$ *horizontal* if it is orthogonal (with respect to $\widehat{g}$) to the submanifold $\pi^{-1}(\{p\})$ of TM, and we call a curve in TM *horizontal* if all of its velocity vectors are horizontal. Show that a curve $t \mapsto (p(t), v(t))$ in TM is horizontal if and only if $v(t)$ is parallel along $p(t)$ in M. Deduce that the geodesic vector field G on TM is horizontal.

(d) Show that the trajectories of G are geodesics of $\widehat{g}$. (Hint: Use the fact that geodesics are locally length-minimizing.)

(e) (Liouville's theorem) Prove that $\operatorname{div}(G) = 0$. (Hint: Use normal coordinates on M.)

31. Let X be a unit vector field on (M, g) such that $\nabla_X X \equiv 0$. Show that X is locally a gradient, if and only if the orthogonal distribution determined by X is integrable. Give an example to show that, even if this condition holds, X need not be a gradient globally.

32. Let (M, g) be a Riemannian manifold. We say that $X \in \mathfrak{X}(M)$ is a *Killing vector field* if $\mathcal{L}_X g = 0$. Prove that:

(a) X is a Killing vector field, if and only if

$$\langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 0, \qquad Y, Z \in \mathfrak{X}(M).$$

(b) If X is a Killing vector field which is orthogonal to a geodesic γ at one point $\gamma(t_0)$, then X is orthogonal to γ everywhere.

(c) If X is a Killing vector field such that $X_p = 0$, then X is tangent to all of the geodesic spheres centered at p. Deduce that, if M is odd-dimensional, then the zeroes of a Killing vector field cannot be isolated.

(d) Killing vector fields restrict to Jacobi fields along geodesics. Deduce that if M is connected, and X is a Killing vector field such that $X_p = 0$ and $\nabla_v X = 0$ for all $v \in T_pM$, then $X \equiv 0$.

33. Answer the following questions and explain: Is it possible to equip

(a) T^n with a metric of negative sectional curvature? Constant positive sectional curvature?

(b) S^n with a metric of non-positive sectional curvature?

(c) $S^1 \times \mathbb{RP}^2$ with a metric of negative sectional curvature? Positive sectional curvature? Non-negative sectional curvature?

(d) $S^2 \times S^2$ with a metric of non-positive sectional curvature? Non-negative sectional curvature?

34. Let (M, g) be a Riemannian manifold. Prove *Bochner's formula*: for $u \in C^\infty(M)$,

$$\frac{1}{2}\Delta(|\operatorname{grad} u|^2) = |\nabla^2 u|^2 + \langle \operatorname{grad} \Delta u, \operatorname{grad} u\rangle + \operatorname{Ric}(\operatorname{grad} u, \operatorname{grad} u).$$

(Hint: To prove it at a point $p \in M$, there are two ways to proceed. One is to choose a local orthonormal frame (E_i) with $\nabla E_i(p) = 0$ for each i, and to compute. Another is to use normal coordinates centered at p.)

35. Let (M, g) be a Riemannian manifold. Given $X \in \mathfrak{X}(M)$ and an oriented open set $U \subseteq M$, we define the *divergence* $\operatorname{div} X$ on U by the equation $(\operatorname{div} X)\, dV_g = \mathcal{L}_X(dV_g)$, where dV_g is the Riemannian volume form of U with the induced metric and chosen orientation. If the orientation is flipped, both sides change sign, implying that we may extend $\operatorname{div} X$ to a well-defined smooth function on all of M.

(a) Prove the *divergence theorem*: if M is compact (with or without boundary), then

$$\int_M (\operatorname{div} X)\, dV_g = \int_{\partial M} \langle X, N\rangle\, dV_{\widehat{g}},$$

where N is the outward unit normal on ∂M and $\widehat{g}$ is the induced metric on ∂M. (For the non-orientable case, pass to the orientation double covering.)

(b) Show that, for $u \in C^\infty(M)$,

$$\operatorname{div}(uX) = u \operatorname{div} X + \langle \operatorname{grad} u, X\rangle,$$

and deduce the integration by parts formula:

$$\int_M \langle \operatorname{grad} u, X \rangle \, dV_g = \int_{\partial M} u \langle X, N \rangle \, dV_g - \int_M u \operatorname{div} X \, dV_g.$$

(c) Prove *Green's identities*:

$$\int_M u \, \Delta v \, dV_g = \int_{\partial M} u N v \, dV_{\widehat{g}} - \int_M \langle \operatorname{grad} u, \operatorname{grad} v \rangle \, dV_g,$$
$$\int_M (u \Delta v - v \Delta u) \, dV_g = \int_{\partial M} (u N v - v N u) \, dV_{\widehat{g}}.$$

(d) Now suppose that $\partial M = \emptyset$. Let $u \in C^\infty(M)$ satisfy $\Delta u = -\lambda u$ for some $\lambda \in \mathbb{R}$. Show that

$$\lambda \int_M |\operatorname{grad} u|^2 \, dV_g \leq n \int_M |\nabla^2 u|^2 \, dV_g,$$

where $n = \dim M$. (Hint: Consider the norm of the 2-tensor $\nabla^2 u - (\Delta u) g/n$ and use one of Green's identities. It may help to note that $\Delta u = \operatorname{tr}_g(\nabla^2 u)$.)

(e) Combine the results of this exercise and Bochner's formula (previous problem) to deduce the following theorem of Lichnerowicz: if (M, g) is a compact Riemannian n-manifold without boundary, and there is a constant $\kappa > 0$ such that $\operatorname{Ric}(v, v) \geq \kappa |v|^2$ for all $v \in TM$, then, for all positive $\lambda > 0$ such that $\Delta u = -\lambda u$ for some $u \in C^\infty(M)$ not identically zero, then $\lambda \geq n\kappa/(n-1)$.

36. Show that $\{z = x^2 + y^2\}$ has no conjugate point relative to a geodesic γ with $\gamma(0) = (0, 0, 0)$.

37. Let G be a Lie group acting smoothly and isometrically on a Riemannian manifold M. Show that each component of the fixed-point set of the action is a smoothly embedded totally geodesic submanifold of M. (Different components may have different dimensions.)

38. Let G be a Lie group with a bi-invariant Riemannian metric g.

(a) Show that, for all left-invariant vector fields X, Y, Z on G,

$$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]],$$

where $R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

(b) Show that, for orthonormal $X, Y \in \operatorname{Lie}(G)$, $\sec(X, Y) = |[X, Y]|^2/4$. Conclude that, if G is connected, then G is flat if and only if it is abelian.

39. Use the Gauss-Bonnet theorem to prove that every compact, connected Lie group of dimension 2 is isomorphic to the torus $S^1 \times S^1$. (Use the previous exercise.)

40. Let (M, g) be a Riemannian manifold.

    (a) Let $(x^1, \ldots, x^n)$ be Riemannian normal coordinates centered at $p \in M$. Show that
    $$g_{ij}(x) = \delta_{ij} - \sum_{k,l} R_{iklj}(p) x^k x^l + O(|x|^3).$$
    (Hint: Let $\gamma(t) = (tv^1, \ldots, tv^n)$ be a radial geodesic starting at p, and let $J(t) = tw^i \partial_i|_{\gamma(t)}$ be a Jacobi field along γ. Compute the derivatives of $|J|^2$ in two different ways.)

    (b) Suppose that $\dim M = 2$. For $r > 0$ small enough, let $A(r)$ be the area of the geodesic disk of radius r centered at p. Prove that
    $$\sec(p) = \frac{12}{\pi} \lim_{r \to 0^+} \frac{\pi r^2 - A(r)}{r^4}.$$
    Derive a similar formula for the sectional curvature at p, but in terms of the cirumference of geodesic circles at p.

41. Let (M, g) be a Riemannian manifold whose sectional curvatures are non-positive. Show that no point of M has conjugate points along any geodesic.

42. Let (M, g) be a complete, connected Riemannian manifold with positive sectional curvature. Prove that, if M_1, M_2 are compact, totally geodesic submanifolds such that $\dim M_1 + \dim M_2 \geq \dim M$, then $M_1 \cap M_2 \neq \emptyset$. (Assume for the sake of contradiction that the intersection is empty, and show that there exists a shortest geodesic segment γ connecting M_1 and M_2 and a parallel vector field along γ tangent to M_1 and M_2 at the endpoints. Derive a contradiction using the second variation formula.)

43. Let M be a smooth manifold. Suppose that $g(t)$, $t \in \mathbb{R}$, is a smoothly varying family of Riemannian metrics on M, subject to the evolution rule
    $$\frac{d}{dt} g(t) = h(g(t)),$$
    for some $h \colon \mathrm{Sym}_2(TM) \to \mathrm{Sym}_2(TM)$. Prove the following formulas that describe how various geometric quantities evolve with time as the metric tensor does:

    (a) Metric inverse:
    $$\frac{d}{dt} g^{ij} = -h^{ij} = -g^{ik} g^{jl} h_{kl}.$$

(b) Christoffel symbols:

$$\frac{d}{dt}\Gamma_{ij}^{k} = \frac{1}{2}g^{kl}\left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\right).$$

(c) Curvature tensor:

$$\frac{d}{dt}R_{ijk}{}^{l} = \frac{1}{2}g^{kl}\Big(\nabla_i\nabla_j h_{kp} + \nabla_i\nabla_k h_{jp} - \nabla_i\nabla_p h_{jk} \\ - \nabla_j\nabla_i h_{kp} - \nabla_j\nabla_k h_{ip} + \nabla_j\nabla_p h_{ik}\Big).$$

(d) Ricci curvature:

$$\frac{d}{dt}R_{ij} = \frac{1}{2}g^{kl}\left(\nabla_q\nabla_i h_{jp} + \nabla_q\nabla_j h_{ip} - \nabla_q\nabla_p h_{ij} - \nabla_i\nabla_j h_{qp}\right).$$

(e) Scalar curvature:

$$\frac{d}{dt}R = -\Delta H + \nabla^p\nabla^q h_{pq} - h^{pq}R_{pq} = -\Delta H + \mathrm{tr}_g^2\left(\nabla(\nabla h) - h\otimes \mathrm{Ric}\right),$$

where $H = \mathrm{tr}_g(h) = g^{ij}h_{ij}$.

(f) Volume element:

$$\frac{d}{dt}d\mu = \frac{1}{2}H\,d\mu.$$

(g) Volume:

$$\frac{d}{dt}\int_M d\mu = \int_M \frac{1}{2}H\,d\mu,$$

assuming M is oriented and compact.

(h) Total scalar curvature:

$$\frac{d}{dt}\int_M R\,d\mu = \int_M\left(\frac{1}{2}RH - h^{ij}R_{ij}\right)d\mu = \int_M\left(\frac{1}{2}RH - \mathrm{tr}_g^2(h\otimes\mathrm{Ric})\right)d\mu$$

assuming M is oriented and closed.

The case when $h = -2\cdot\mathrm{Ric}$ is of particular interest in the study of Hamilton's Ricci flow.