Assume that everything is smooth unless the contrary is explicitly stated.

1. Let M be a connected manifold and let $f\colon M \to M$ be a map such that $f \circ f = f$. Show that $f(M)$ is an embedded submanifold of M.

2. Let M be a manifold and let $f\colon M \to M$ be a map such that $f \circ f = \mathrm{id}_M$. Show that the set of fixed points of f is an embedded submanifold of M.

3. Show that the diffeomorphism group of a connected manifold acts transitively.

4. Let X and Y be vector fields on a smooth manifold M, with flows denoted by φ_t and ψ_t, respectively. Fix a point p of M, and, whenever it makes sense, define

$$c(t) = (\psi_{-t} \circ \varphi_{-t} \circ \psi_t \circ \varphi_t)(p).$$

   Show that $c'(0) = 0$, and deduce from this that $f \mapsto (f \circ c)''(0)$ defines an element of T_pM. Prove that this element is precisely $2[X, Y]_p$.

5. Let D be a distribution on a manifold M, and let X be a (locally defined) vector field on M with flow φ_t. Show that the following are equivalent.

   (i) For all p in M and t for which it makes sense, $d(\varphi_t)_p(D_p) = D_{\varphi_t(p)}$.

   (ii) $\mathcal{L}_X$ takes local sections of D to local sections of D.

   (iii) $\mathcal{L}_X$ takes local annihilating forms for D to local annihilating forms for D.

   If any of these hold, then we say that X is an *infinitesimal symmetry* of D.

6. Let $\pi\colon M \to M'$ be a surjective submersion with connected fibres.

   (a) Show that $\ker d\pi$ is an involutive distribution on M, and describe the corresponding foliation.

   (b) Let D be a distribution on M containing $\ker d\pi$, and suppose that every local section of $\ker d\pi$ is also an infinitesimal symmetry of D. Show that there exists a unique distribution D' on M' such that $D'_{\pi(p)} = d\pi_p(D_p)$ for each p in M. (The next part of the problem outlines the proof of smoothness.)

   (c) Show that, for each p in M, it is possible to find an open neighbourhood U of p, local sections $Y_1, \ldots, Y_r$ of D on U, and a local *frame* $Y_1', \ldots, Y_r'$ of D' on $\pi(U)$ such that each Y_i is π-related to Y_i'. Deduce from this that D' is smooth, and involutive if D is.

7. Let $\pi\colon M \to M'$ be a surjective submersion with connected fibres, and let ω be a k-form on M. Show that ω is the pullback of a k-form on M' by π, if and only if $i_X\omega = 0$ and $\mathcal{L}_X\omega = 0$ for every section X of $\ker d\pi$. Interpret this geometrically and give an example to show that the conclusion is false if we drop the assumption that π has connected fibres.

8. Let $f\colon M \to N$ be a map which is transverse to the leaves of a foliation $\mathcal{F}$ on N. Show that the connected components of the pre-images of the leaves of $\mathcal{F}$ under f give rise to a foliation on M. Specialize to the intersection of two foliations.

9. Let M be a manifold with a foliation $\mathcal{F}$. The *leaf space* $M/\mathcal{F}$ is defined to be the set of leaves of $\mathcal{F}$ given the quotient topology. Show that the quotient map is open. Give an example to show that the quotient space need not be Hausdorff.

10. Consider the $(n-1)$-form ω on $\mathbb{R}^n \setminus \{0\}$ defined by

$$\omega = \frac{1}{|x|^n} \sum_{i=1}^{n} (-1)^{i-1} x^i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

   (a) Show that $\omega|_{S^{n-1}}$ is the standard volume form for S^{n-1}.
   (b) Show that ω is closed, but not exact.

11. Consider $\mathbb{R}^{2n}$ with the standard coordinates denoted by $x^1, y^1, \ldots, x^n, y^n$, and define ω by

$$\omega = \sum_{i=1}^{n} \left(-y^i \, dx^i + x^i \, dy^i \right).$$

   Show that ω restricts to a nowhere-vanishing 1-form on S^{2n-1}. Compare this with the easy direction of the hairy ball theorem.

12. Let ω be a volume form on a manifold M. Show that, for each vector field X on M, there exists a unique function f on M such that $\mathcal{L}_X \omega = f\omega$. Give the local coordinate expression for f in terms of X and ω. What happens if ω is the standard volume form on $\mathbb{R}^n$?

13. Let M be a compact manifold, and suppose that ω_0, ω_1 are two volume forms on M inducing the same orientation and volume. Show that there exists a diffeomorphism $f\colon M \to M$ such that $f^*\omega_1 = \omega_0$. (Hint: Consider $\omega_t = (1-t)\omega_0 + t\omega_1$.)

14. Given a vector bundle $E \to M$, we define the *Euler vector field* V^E on E to be the infinitesimal generator of the flow $(t, v) \mapsto e^t v$. Suppose that $E \to M$ and $F \to N$ are vector bundles, and that $\varphi\colon E \to F$ is a smooth map. Show that φ is a vector bundle morphism if and only if V^E is φ-related to V^F.

15. Let M and N be manifolds, and denote by λ^M and λ^N the tautological 1-forms on T^*M and T^*N, respectively. Use the previous problem to show that a diffeomorphism $F\colon T^*M \to T^*N$ is the cotangent lift of a diffeomorphism $f\colon M \to N$ if and only if $F^*\lambda^N = \lambda^M$.

16. (Milnor's exercise) For any manifold M, show that

$$M \to \mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R}), \qquad p \mapsto \mathrm{ev}_p$$

is a bijection. (Do it in the compact case first.)

17. With the previous problem in mind, construct a bijection between TM and $\mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R}[\varepsilon]/(\varepsilon^2))$. What if $\mathbb{R}[\varepsilon]/(\varepsilon^2)$ is replaced by $\mathbb{R}[\varepsilon]/(\varepsilon^{n+1})$? What if the domain and range of the set of algebra homomorphisms are swapped? (If you know about jets, then give your answer in terms of those. If not, then at least try to use the phrase "Taylor expansion" in your answer.)

18. Let G be a Lie group acting by isometries on a Riemannian manifold. Show that each component of its fixed point set is an embedded, totally geodesic submanifold.

19. Suppose that a Lie group G acts isometrically on a connected n-manifold M. Show that $\dim G \leq n + n(n-1)/2$, and give two examples where the bound is attained.

20. Let G be a Lie group admitting a bi-invariant metric g.

    (a) Show that the curvature tensor R acts on left-invariant vector fields by

    $$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]].$$

    (b) Show that

    $$\sec(X,Y) = \frac{1}{4}\left|[X,Y]\right|^2$$

    for every pair X, Y of orthonormal left-invariant vector fields, and conclude that, in the connected case, G is flat if and only if it is abelian.

21. Use the Gauss-Bonnet theorem and the previous problem to show that every compact, connected Lie group of dimension two is a torus.

22. Let M be a closed oriented surface in $\mathbb{R}^3$ with Gauss map ν. Show that

$$\nu^*(\mathrm{vol}_{S^2}) = K \mathrm{vol}_M,$$

and use this to prove that $\deg(\nu) = \chi(M)/2$.

23. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We define a $\mathfrak{g}$-valued 1-form θ on G by $\theta(v) = d(L_{g^{-1}})_g(v)$ for v in T_gG. (θ is known as the *Maurer-Cartan form.*) Show that

$$d\theta + \frac{1}{2}[\theta,\theta] = 0,$$

where the bracket on $\mathfrak{g}$-valued 1-forms is given by

$$[\omega,\eta](X,Y) = [\omega(X),\eta(Y)] - [\omega(Y),\eta(X)].$$

24. (Credit to Ibsen on Discord for this one.) Consider a manifold M with a connection ∇'. Let ∇ be the standard flat connection in a local coordinate patch U; going forward, we will assume that $M = U$. Define a map $A\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$A(X, Y) = A_X Y = \nabla'_X Y - \nabla_X Y.$$

Show that A is a $(1, 2)$-tensor. Now, by currying, we can identify A with an $\mathrm{End}(TM)$-valued 1-form on M given by $X \mapsto A_X$.

The flat connection ∇ induces a connection in $\mathrm{End}(TM)$, given by

$$\nabla_X(T)(Z) = \nabla_X(T(Z)) - T(\nabla_X Z).$$

This allows us to differentiate A to get an $\mathrm{End}(TM)$-valued 2-form dA on M given by

$$(dA)(X, Y) = \nabla_X A_Y - \nabla_Y A_X - A_{[X,Y]}.$$

dA is not quite the curvature, but we can add a "correction term" $A \wedge A$ to it to get the curvature tensor. (This is defined by the typical matrix multiplication expression, but with wedge products in the sum.) Prove this, and interpret the result geometrically using Stokes's theorem. Compare with the previous problem.