ISIMA lectures on celestial mechanics. 1

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The roots of solar system dynamics can be traced to two fundamental discoveries by Isaac Newton: first, that the acceleration of a body of mass m subjected to a force $\mathbf{F}$ is

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{\mathbf{F}}{m}, \tag{1}$$

and second, that the gravitational force exerted by a point mass m_2 at position $\mathbf{r}_2$ on a point mass m_1 at $\mathbf{r}_1$ is

$$\mathbf{F} = \frac{\mathfrak{G}\, m_1 m_2(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{2}$$

Newton's laws have now been superseded by the equations of general relativity, but remain accurate enough to describe all planetary system phenomena when they are supplemented by small relativistic corrections.

For general background on this material look at any mechanics textbook. The standard advanced reference is Murray and Dermott, *Solar System Dynamics*.

1. The two-body problem

The simplest significant problem in celestial mechanics, posed and solved by Newton but often known as the **two-body problem** (or **Kepler's problem**), is to determine the orbits of two particles of masses m_1 and m_2 under the influence of their mutual gravitational attraction.

From (1) and (2) the equations of motion are

$$\frac{d^2\mathbf{r}_1}{dt^2} = \frac{\mathfrak{G}\, m_2(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \qquad \frac{d^2\mathbf{r}_2}{dt^2} = \frac{\mathfrak{G}\, m_1(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \tag{3}$$

We change variables from $\mathbf{r}_1$ and $\mathbf{r}_2$ to

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \qquad \mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1; \tag{4}$$

$\mathbf{R}$ is the **center of mass** of the two particles and $\mathbf{r}$ is the **relative position**. These equations can be solved for $\mathbf{r}_1$ and $\mathbf{r}_2$ to yield

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{m_1 + m_2}\mathbf{r}, \qquad \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{5}$$

Taking the second time derivative of the first of equations (4) and using equations (3), we obtain

$$\frac{d^2\mathbf{R}}{dt^2} = 0, \tag{6}$$

which implies that the center of mass travels at uniform velocity, a result that is a consequence of the absence of any forces from external sources. Taking the second derivative of the second of equations (4) yields

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mathfrak{G} M}{r^3}\mathbf{r}, \tag{7}$$

where $r = |\mathbf{r}|$ and we have introduced $M \equiv m_1 + m_2$ to denote the total mass of the two particles. Equation (7) equally well describes the motion of a particle of negligible mass (a **test particle**) subject to the gravitational attraction of a fixed point mass M.

Equation (7) can also be written as

$$\frac{d^2\mathbf{r}}{dt^2} = -\boldsymbol{\nabla}\Phi_K, \tag{8}$$

where $\Phi_K(r) = -\mathfrak{G} M/r$ is the **Kepler potential**. The solution of equation (7) is known as the **Kepler orbit**.

The angular momentum per unit mass is

$$\mathbf{L} = \mathbf{r} \times \frac{d\mathbf{r}}{dt}. \tag{9}$$

Then

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = -\frac{\mathfrak{G} M}{r^2}\mathbf{r} \times \hat{\mathbf{r}} = 0. \tag{10}$$

Thus the angular momentum is conserved; moreover, since the angular momentum vector is normal to the plane containing the test particle's position and velocity vectors, the test particle must be confined to a plane (the **orbital plane**).

1.1. The shape of the Kepler orbit

We let (r, ψ) denote polar coordinates in the orbital plane, with ψ increasing in the direction of motion of the orbit; writing $\mathbf{r} = r\hat{\mathbf{r}}$ and using the relations $d\hat{\mathbf{r}}/dt = \dot{\psi}\hat{\boldsymbol{\psi}}$ and $d\hat{\boldsymbol{\psi}}/dt = -\dot{\psi}\hat{\mathbf{r}}$, the equation of motion (7) becomes

$$\ddot{r} - r\dot{\psi}^2 = -\frac{d\Phi_K(r)}{dr}, \qquad 2\dot{r}\dot{\psi} + r\ddot{\psi} = 0. \tag{11}$$

The second equation may be multiplied by r and integrated to yield

$$r^2\dot{\psi} = \text{constant} = L, \tag{12}$$

where $L = |\mathbf{L}|$. This is just a restatement of the conservation of angular momentum.

We may use equation (12) to eliminate $\dot{\psi}$ from the first of equations (11),

$$\ddot{r} - \frac{L^2}{r^3} = -\frac{d\Phi_K}{dr}. \tag{13}$$

Multiplying by $\dot{r}$ and integrating yields

$$\tfrac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} + \Phi_K(r) = E, \tag{14}$$

where E is a constant that is equal to the energy per unit mass of the test particle. We can also use equation (12) to write

$$\frac{d}{dt} = \dot{\psi}\frac{d}{d\psi} = \frac{L}{r^2}\frac{d}{d\psi}; \tag{15}$$

applying this relation in equation (14) yields

$$\frac{L^2}{2r^4}\left(\frac{dr}{d\psi}\right)^2 + \frac{L^2}{2r^2} + \Phi_K(r) = E. \tag{16}$$

Equation (14) implies that

$$\dot{r}^2 = 2E + \frac{2\mathfrak{G}M}{r} - \frac{L^2}{r^2}. \tag{17}$$

As $r \to 0$, the right side approaches $-L^2/r^2$ which is negative, while the left side is positive. Thus there must be a point of closest approach of the test particle to the central body, which is known as the **periapsis** or **pericenter**[1]. The periapsis distance q is given implicitly by the condition $\dot{r} = 0$, which yields the quadratic equation

$$2E + \frac{2\mathfrak{G}M}{q} - \frac{L^2}{q^2} = 0. \tag{18}$$

In the opposite limit, $r \to \infty$, the right side of equation (17) approaches $2E$. Since the left side is positive, when $E < 0$ there is a maximum distance Q that the particle can achieve, which is known as the **apoapsis** or **apocenter**, and is determined by solving equation (18)

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[1]For specific central bodies other names are used, such as perihelion (Sun), perigee (Earth), perijove (Jupiter), periastron (a star), etc. "Periapse" is incorrect—an apse is not an apsis.

with Q replacing q. Orbits with $E < 0$ are referred to as **bound** orbits since there is an upper limit to their distance from the central body. Particles on **unbound** orbits, with $E \geq 0$, have no apoapsis and eventually travel arbitrarily far from the central body.

To solve equation (13) we introduce the variable $u \equiv 1/r$, and change the independent variable from t to ψ using the relation (15). With these substitutions, $\ddot{r} = -L^2 u^2 d^2u/d\psi^2$, so equation (13) becomes

$$\frac{d^2u}{d\psi^2} + u = -\frac{1}{L^2}\frac{d\Phi_K}{du}. \tag{19}$$

Since $\Phi_K = -\mathfrak{G} M/r = -\mathfrak{G} Mu$, the right side is equal to a constant, $\mathfrak{G} M/L^2$, and the equation is easily solved to yield

$$u = \frac{1}{r} = \frac{\mathfrak{G} M}{L^2}[1 + e\cos(\psi - \varpi)], \tag{20}$$

where $e \geq 0$ and ϖ are constants of integration. We replace the angular momentum L by another constant of integration, a, defined by the relation[2]

$$L^2 = \mathfrak{G} Ma(1 - e^2), \tag{21}$$

so that the shape of the orbit is given by

$$r = \frac{a(1-e^2)}{1 + e\cos f}, \tag{22}$$

where $f = \psi - \varpi$ is known as the **true anomaly**[3].

The closest approach of the two bodies occurs at azimuth $\psi = \varpi$ and hence ϖ is known as the **azimuth of periapsis**. The periapsis distance is

$$q = a(1 - e). \tag{23}$$

Substituting this result and equation (21) into (18) reveals that the energy is simply related to the constant a:

$$E = -\frac{\mathfrak{G} M}{2a}. \tag{24}$$

Bound orbits have $E < 0$, so that $a > 0$ by equation (24) and hence $e < 1$ by equation (21). The apoapsis distance is obtained from equation (22) with $f = \pi$ and is given by

$$Q = a(1 + e). \tag{25}$$

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[2]The combination $a(1 - e^2)$ is sometimes called the **semilatus rectum**.

[3]The term "anomaly" is used to refer to any angular variable that is zero at periapsis and increases by 2π as the particle travels from periapsis to apoapsis and back.

The azimuth of the apocenter is $\psi = \varpi + \pi$; thus the periapsis and the apoapsis are joined by a straight line known as the **line of apsides**. Equation (22) describes an ellipse with one focus at the origin. Its major axis is the line of apsides and has length $q + Q = 2a$; thus the constant a is known as the **semi-major axis**. The **semi-minor axis** of the ellipse is easily determined as the maximum perpendicular distance of the orbit from the line of apsides, $b = \max_f[a(1-e^2)\sin f/(1+e\cos f)] = a(1-e^2)^{1/2}$. The **eccentricity** of the ellipse, $(1-b^2/a^2)^{1/2}$, is therefore equal to the constant e. The radial velocity is found by differentiating equation (22) and using the relation $\dot{f} = L/r^2$ (eq. 15):

$$\dot{r} = \frac{(\mathfrak{G} M)^{1/2} e \sin f}{[a(1-e^2)]^{1/2}}. \tag{26}$$

Unbound orbits have $E > 0$, $a < 0$ and $e > 1$. In this case equation (22) describes a hyperbola with focus at the origin and asymptotes at azimuth $\psi = \varpi \pm \cos^{-1}(-1/e)$. The constants a and e are still commonly referred to as semi-major axis and eccentricity even though these terms have no direct geometric interpretation. In the special case $E = 0$, a is infinite and $e = 1$, so equation (22) is undefined; however, in this case equation (18) implies that the periapsis distance $q = L^2/(2\mathfrak{G} M)$ so that equation (20) implies

$$r = \frac{2q}{1+\cos f}, \tag{27}$$

which describes a parabola.

1.2. Motion in the Kepler orbit

The **period** P of a bound orbit is the time taken to travel from periapsis to apoapsis and back. Since $d\psi/dt = L/r^2$, we have $\int_{t_1}^{t_2} dt = L^{-1}\int_{\psi_1}^{\psi_2} r^2 d\psi$; the integral on the right side is twice the area contained in the ellipse between azimuths ψ_1 and ψ_2, which leads to the simple result that $P = 2A/L$ where A is the area of the ellipse. Combining the geometrical result that the area is $A = \pi ab = \pi a^2(1-e^2)^{1/2}$ with equation (21) we obtain

$$P = 2\pi\left(\frac{a^3}{\mathfrak{G} M}\right)^{1/2}; \tag{28}$$

the period, like the energy, depends only on the semi-major axis. The **mean motion** or mean rate of change of azimuth, usually written n and equal to $2\pi/P$, thus satisfies

$$n^2 a^3 = \mathfrak{G} M, \tag{29}$$

which is **Kepler's third law**. If the particle passes through periapsis at $t = t_0$, the dimensionless variable

$$\ell = 2\pi \frac{t - t_0}{P} = n(t - t_0) \tag{30}$$

is referred to as the **mean anomaly**.

The position and velocity of a particle in the orbital plane at a given time t are specified by four **orbital elements**: two specify the size and shape of the orbit, which we can take to be e and a (or e and n, q and Q, L and E, etc.); one specifies the orientation of the orbit (ϖ); and one specifies the location or phase of the particle in its orbit (f, ℓ, or t_0).

The time-dependent trajectory $[r(t), \psi(t)]$ can be derived by solving the differential equations (17) for $r(t)$, then (12) for $\psi(t)$. However, there is a better approach. alternative.

First consider bound orbits. Since the radius of a bound orbit oscillates between $a(1-e)$ and $a(1+e)$ it is natural to introduce a variable $u(t)$, the **eccentric anomaly**, by the definition

$$r = a(1 - e\cos u). \tag{31}$$

Substituting equation (31) into the energy equation (17) and using equations (21) and (24) for L^2 and E, we obtain

$$\dot{r}^2 = a^2 e^2 \sin^2 u \dot{u}^2 = -\frac{\mathfrak{G} M}{a} + \frac{2\mathfrak{G} M}{a(1 - e\cos u)} - \frac{\mathfrak{G} M(1 - e^2)}{a(1 - e\cos u)^2}, \tag{32}$$

which simplifies to

$$(1 - e\cos u)^2 \dot{u}^2 = \frac{\mathfrak{G} M}{a^3} = n^2 = \dot{\ell}^2. \tag{33}$$

Defining $u(t)$ to increase with time, and recognizing that $u = 0$ when $\ell = 0$ (at pericenter), we may take the square root of this equation and integrate it to obtain **Kepler's equation**

$$\ell = u - e\sin u. \tag{34}$$

Kepler's equation cannot be solved analytically for u, but many efficient numerical solutions are available. The true, eccentric and mean anomalies f, u, and ℓ are all equal for circular orbits.

The relation between the true and eccentric anomalies is found by eliminating r from equations (22) and (31):

$$\cos f = \frac{\cos u - e}{1 - e\cos u}, \tag{35}$$

with the understanding that if f lies in the interval from $k\pi$ to $(k+1)\pi$ (k an integer) then u lies in the same interval. Alternative expressions are:

$$\sin f = \frac{(1 - e^2)^{1/2} \sin u}{1 - e\cos u}; \qquad \tan \tfrac{1}{2} f = \left(\frac{1 + e}{1 - e}\right)^{1/2} \tan \tfrac{1}{2} u. \tag{36}$$

Thus the position and velocity of a bound particle at any time t can be determined by the following procedure: Compute the mean anomaly from equation (30). Solve Kepler's equation to compute the eccentric anomaly. Then the radius r is given by equation (31); the true anomaly f is given by equations (35) or (36), and the azimuth $\psi = f + \varpi$. The radial velocity is

$$\dot{r} = n\frac{dr}{d\ell} = \frac{dr/du}{d\ell/du} = \frac{nae\sin u}{1 - e\cos u}, \tag{37}$$

and the azimuthal velocity is

$$r\dot{\psi} = \frac{L}{r} = \frac{(\mathfrak{G} M)^{1/2}(1-e^2)^{1/2}}{a^{1/2}(1-e\cos u)}, \tag{38}$$

where we have used equation (21).

The procedure for computing the position and velocity of an unbound particle is similar. The eccentric anomaly is given by expressions such as

$$\begin{aligned} r &= a(1 - e\cosh u), \\ \cos f &= \frac{e - \cosh u}{e\cosh u - 1}, \\ \ell &= u - e\sinh u, \end{aligned} \tag{39}$$

where the mean anomaly is defined by

$$\ell = n'(t - t_0), \qquad {n'}^2 a^3 = -\mathfrak{G} M. \tag{40}$$

The eccentric and mean anomalies increase from 0 to ∞ as the true anomaly increases from zero to $\cos^{-1}(-1/e)$.

For parabolic orbits there is no need to introduce the eccentric anomaly since the relation between time and true anomaly can be determined analytically. Since $\dot{f} = L/r^2$, we have

$$t - t_0 = \int_0^f \frac{r^2 df}{L} = 4\left(\frac{q^3}{2\mathfrak{G} M}\right)^{1/2} \int_0^f \frac{df}{(1+\cos f)^2}; \tag{41}$$

this can be integrated to yield

$$\left(\frac{2\mathfrak{G} M}{q^3}\right)^{1/2}(t - t_0) = 2\tan\tfrac{1}{2}f + \tfrac{2}{3}\tan^3\tfrac{1}{2}f, \tag{42}$$

which is a cubic equation that can be solved analytically for $\tan\frac{1}{2}f$.

Exercise: Let $\langle\cdot\rangle$ denote the time average of some quantity over one period of a Kepler orbit of semi-major axis a and eccentricity e. Show that

$$\langle r/a\rangle = 1-\tfrac{1}{2}e^2, \quad \langle (r/a)^2\rangle = 1+\tfrac{3}{2}e^2, \quad \langle (r/a)^2\cos^2 f\rangle = \tfrac{1}{2}+2e^2, \quad \langle (r/a)^2\sin^2 f\rangle = \tfrac{1}{2}(1-e^2),$$

$$\langle\langle a/r\rangle = 1, \quad \langle (a/r)^2\rangle = (1-e^2)^{-1/2}, \quad \langle (a/r)^3\rangle = (1-e^2)^{-3/2}.$$

$$\langle (a/r)^3\cos^2 f\rangle = \langle (a/r)^3\sin^2 f\rangle = \tfrac{1}{2}(1-e^2)^{-3/2}. \tag{43}$$

1.2.1. The eccentricity vector

There is an alternative derivation of the equation for the shape of a Kepler orbit. Take the cross-product of $\mathbf{L}$ with equation (7):

$$\mathbf{L}\times\ddot{\mathbf{r}} = -\frac{\mathfrak{G}M}{r^2}\mathbf{L}\times\hat{\mathbf{r}} = \frac{\mathfrak{G}M}{r^2}\hat{\mathbf{r}}\times(\mathbf{r}\times\dot{\mathbf{r}}). \tag{44}$$

Using the vector identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ we may show that $\hat{\mathbf{r}}\times(\mathbf{r}\times\dot{\mathbf{r}}) = \mathbf{r}(\hat{\mathbf{r}}\cdot\dot{\mathbf{r}})-\dot{\mathbf{r}}r$, which is simply equal to $-r^2 d\hat{\mathbf{r}}/dt$. Thus

$$\mathbf{L}\times\ddot{\mathbf{r}} = -\mathfrak{G}M\frac{d\hat{\mathbf{r}}}{dt}. \tag{45}$$

Integrating,

$$\mathbf{L}\times\dot{\mathbf{r}} = -\mathfrak{G}M(\hat{\mathbf{r}}+\mathbf{e}), \tag{46}$$

where $\mathbf{e}$ is a vector constant of the motion (the **eccentricity vector**). The eccentricity vector lies in the plane of the orbit, points towards periapsis, and its magnitude equals the eccentricity, $|\mathbf{e}| = e$. To derive equation (22) for the shape of the orbit, we simply take the dot product of equation (46) with $\hat{\mathbf{r}}$ and use the vector identity $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A}$ to show that $\hat{\mathbf{r}}\cdot\mathbf{L}\times\dot{\mathbf{r}} = -L^2/r$.

The components of the eccentricity vector also provide useful orbital elements. For example, if the orbital plane is the x-y plane, the components of the eccentricity vector are given by $\mathbf{e} = k\hat{\mathbf{x}}+h\hat{\mathbf{y}}$, where

$$k \equiv e\cos\varpi, \qquad h = e\sin\varpi. \tag{47}$$

The components k and h provide alternative orbital elements to e and ϖ; they are particularly useful for near-circular orbits, since ϖ is undefined when $e = 0$.

1.3. Motion in three dimensions

So far we have described the motion of a particle in its orbital plane. To specify the orbit completely we must also describe the spatial orientation of the orbital plane.

We work with the usual Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ϕ). We shall call the plane $z = 0$ (corresponding to $\theta = \frac{1}{2}\pi$) the **reference plane**. The **inclination** of the orbital plane to the reference plane is denoted I, with $0 \leq I \leq \pi$; thus $\cos I = \hat{\mathbf{z}} \cdot \hat{\mathbf{L}}$, where $\hat{\mathbf{z}}$ and $\hat{\mathbf{L}}$ are unit vectors in the direction of the z-axis and the angular momentum vector. Orbits with $0 \leq I \leq \frac{1}{2}\pi$ are called **direct** or **prograde**; orbits with $\frac{1}{2}\pi < I < \pi$ are **retrograde**.

Any bound Kepler orbit pierces the reference plane at two points known as the **nodes** of the orbit. The particle travels upwards ($\dot{z} > 0$) at the **ascending node** and downwards at the **descending node**. The azimuthal angle ϕ of the ascending node is denoted Ω and is called the **longitude of the ascending node**. The angle from ascending node to periapsis, measured in the direction of motion of the particle in the orbital plane, is denoted ω and is called the **argument of periapsis**.

An unfortunate feature of these elements is that neither ω nor Ω is defined for orbits in the reference plane ($I = 0$). Partly for this reason, the argument of periapsis is often replaced by a variable called the **longitude of periapsis** which is defined as $\varpi \equiv \Omega + \omega$. For orbits with zero inclination, the longitude of periapsis has a simple interpretation—it is the azimuthal angle between the x-axis and the periapsis, consistent with our earlier use of the same symbol for the azimuth of periapsis—but if the inclination is non-zero it is the sum of two angles measured in different planes (the reference plane and the orbital plane). Thus its traditional name is misleading: the longitude of periapsis is generally *not* equal to the azimuthal angle ϕ of the eccentricity vector, except for orbits of zero inclination. Despite this awkwardness, for most purposes the three elements (Ω, ϖ, I) provide the most convenient way to specify the orientation of a Kepler orbit.

The **mean longitude** is

$$\lambda \equiv \varpi + \ell, \tag{48}$$

where ℓ is the mean anomaly; like the longitude of perihelion, the mean longitude is the sum of angles measured in the reference plane (Ω) and the orbital plane ($\omega + \ell$).

The position and velocity of a particle in space at a given time t are specified by six orbital elements: two specify the size and shape of the orbit, which we can take to be e and a; three specify the orientation of the orbit (I, Ω, and ω), and one specifies the location of the particle in the orbit (f, u, ℓ, λ, or t_0). Thus, for example, the Cartesian coordinates

(x, y, z) are written in terms of the orbital elements as

$$\begin{aligned}\frac{x}{r} &= \cos\Omega\cos(f+\omega) - \sin\Omega\cos I\sin(f+\omega), \\ \frac{y}{r} &= \sin\Omega\cos(f+\omega) + \cos\Omega\cos I\sin(f+\omega), \\ \frac{z}{r} &= \sin I\sin(f+\omega);\end{aligned} \tag{49}$$

r is given in terms of the orbital elements by equation (22).

Exercise Show that

$$\left\langle \frac{z^2}{r^5} \right\rangle = \frac{e\sin^2 I}{2a^3(1-e^2)^{3/2}}, \tag{50}$$

where $\langle\cdot\rangle$ denotes a time average over the orbit.

When the eccentricity or inclination is small, the elements (e, ϖ, I, Ω) are sometimes replaced by

$$\begin{aligned} k &\equiv e\cos\varpi, \qquad & h &= e\sin\varpi, \\ q &\equiv \tan I\cos\Omega, \qquad & p &= \tan I\sin\Omega. \end{aligned} \tag{51}$$

The first line is the same as equation (47), although the substitution does not have a simple interpretation in terms of the eccentricity vector when the inclination is non-zero.

For some purposes the shape, size, and orientation of an orbit can be described most efficiently using the angular momentum and eccentricity vectors, $\mathbf{L}$ and $\mathbf{e}$. The two vectors describe five of the six orbital elements: the missing element is the one specifying the location of the particle in its orbit, f, u, ℓ, or t_0 (the six components of the two vectors determine only five elements because $\mathbf{e}$ is restricted to the plane normal to $\mathbf{L}$).

Note that ω and Ω are undefined for orbits with zero inclination, ω and ϖ are undefined for circular orbits, and ϖ, Ω, and I are undefined for radial orbits ($e \to 1$).

Exercise: Write a script to convert Cartesian positions and velocities to Keplerian orbital elements, and vice versa.

1.4. Gauss's f and g functions

A common problem is to determine the position $\mathbf{r}(t)$ of a particle in a Kepler orbit, given its position and velocity $(\mathbf{r}_0, \mathbf{v}_0)$ at time t_0. This can be done by evaluating the orbital elements

$a, e, I, \omega, \Omega, \ell$ from $\mathbf{r}_0, \mathbf{v}_0$, replacing ℓ by $\ell + n(t - t_0)$, and then determining the position from the new orbital elements. But there is a simpler method.

Since the particle is confined to the orbital plane, and $\mathbf{r}_0, \mathbf{v}_0$ are non-parallel vectors lying in this plane, we can write

$$\mathbf{r}(t) = f_G(t, t_0)\mathbf{r}_0 + g_G(t, t_0)\mathbf{v}_0, \tag{52}$$

which defines **Gauss's f and g functions**. This expression can also be used to derive the velocity of the particle:

$$\mathbf{v}(t) = \frac{\partial f_G(t, t_0)}{\partial t}\mathbf{r}_0 + \frac{\partial g_G(t, t_0)}{\partial t}\mathbf{v}_0. \tag{53}$$

To evaluate f_G and g_G for bound orbits we choose polar coordinates (r, ψ) and Cartesian coordinates (x, y) in the orbital plane, and assume that $\mathbf{r}_0$ lies along the $\hat{\mathbf{x}}$ axis ($\psi = 0$). Then the components of equation (52) along the x and y-axes are:

$$r\cos\psi = f_G r_0 + g_G \dot{r}_0, \qquad r\sin\psi = g_G r_0 \dot{\psi}_0. \tag{54}$$

We replace $r_0\dot{\psi}_0$ by $L/r_0 = [\mathfrak{G} Ma(1 - e^2)]^{1/2}/r_0$ (eqs. 12 and 21), use equation (26) to eliminate $\dot{r}_0$, and replace ψ by the difference in true anomalies, $f - f_0$. The resulting equations can be solved to yield:

$$f_G = \frac{\cos(f - f_0) + e\cos f}{1 + e\cos f}, \qquad g_G = \frac{(1 - e^2)^{3/2}}{n}\frac{\sin(f - f_0)}{(1 + e\cos f)(1 + e\cos f_0)}; \tag{55}$$

since these expressions contain only the orbital elements n, e, and f, they are valid in any coordinate system, not just the one we used for the derivation. For deriving velocities from equation (53) we need

$$\frac{\partial f_G}{\partial t} = n\frac{e\sin f_0 - e\sin f - \sin(f - f_0)}{(1 - e^2)^{3/2}}, \quad \frac{\partial g_G}{\partial t} = \frac{e\cos f_0 + \cos(f - f_0)}{1 + e\cos f_0}. \tag{56}$$

The f_G and g_G functions can also be expressed in terms of the eccentric anomaly, using equations (35), (36), (34), and (37):

$$f_G = \frac{\cos(u - u_0) - e\cos u_0}{1 - e\cos u_0}, \quad g_G = \frac{1}{n}[\sin(u - u_0) - e\sin u + e\sin u_0], \tag{57}$$

$$\frac{\partial f_G}{\partial t} = -\frac{\sin(u - u_0)}{(1 - e\cos u)(1 - e\cos u_0)}, \quad \frac{\partial g_G}{\partial t} = \frac{\cos(u - u_0) - e\cos u}{1 - e\cos u}. \tag{58}$$

1.5. Hamiltonian dynamics

The two-body equation of motion (7) can be derived from the Kepler Hamiltonian

$$H_K(\mathbf{q},\mathbf{p}) = \tfrac{1}{2}v^2 - \frac{\mathfrak{G}\, M}{|\mathbf{q}|} \tag{59}$$

and Hamilton's equations

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H_K}{\partial \mathbf{p}}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H_K}{\partial \mathbf{q}}, \tag{60}$$

where $\mathbf{q} = \mathbf{r}$ is the coordinate or position vector and $\mathbf{p} = \mathbf{v}$ is the momentum or velocity.

It can be shown that an alternative set of coordinates (the **Delaunay elements**) is

$$\ell, \quad \omega, \quad \Omega \tag{61}$$

with conjugate momenta

$$L \equiv (\mathfrak{G}\, Ma)^{1/2}, \quad \mathfrak{G} \equiv [\mathfrak{G}\, Ma(1-e^2)]^{1/2}, \quad H \equiv [\mathfrak{G}\, Ma(1-e^2)]^{1/2} \cos I. \tag{62}$$

The Kepler Hamiltonian is

$$H_K = -\frac{(\mathfrak{G}\, M)^2}{2L^2}. \tag{63}$$

The double usage of G and H is unfortunate but doesn't usually cause confusion. I will not give the derivation here, since it's relatively long, not very informative, and is given in many mechanics books or in *Galactic Dynamics*, Appendix E.

The Delaunay elements are not defined for circular, equatorial, or radial orbits. We can transform to new coordinates $(\theta_1, \theta_2, \theta_3)$ and momenta (J_1, J_2, J_3) by the generating function

$$S = J_1(\ell + \omega + \Omega) - J_2(\omega + \Omega) - J_3\Omega. \tag{64}$$

Then

$$\begin{gathered} L = \frac{\partial S}{\partial \ell} = J_1, \quad G = \frac{\partial S}{\partial \omega} = J_1 - J_2, \quad H = \frac{\partial S}{\partial \Omega} = J_2 - J_3, \\ \theta_1 = \frac{\partial S}{\partial J_1} = \omega + \Omega + \ell, \quad \theta_2 = \frac{\partial S}{\partial J_2} = -\omega - \Omega = -\varpi, \quad \theta_3 = \frac{\partial S}{\partial J_3} = -\Omega, \end{gathered} \tag{65}$$

which gives

$$\begin{gathered} J_1 = (\mathfrak{G}\, Ma)^{1/2}, \quad J_2 = (\mathfrak{G}\, Ma)^{1/2}[1 - (1-e^2)^{1/2}], \quad J_3 = (\mathfrak{G}\, Ma)^{1/2}(1-e^2)^{1/2}[1 - \cos I], \\ \theta_1 = \lambda, \quad \theta_2 = -\varpi, \quad \theta_3 = -\Omega; \end{gathered} \tag{66}$$

here $\lambda \equiv \varpi + \ell$ is called the **mean longitude**. These elements are well-defined for circular orbits but not for equatorial or radial orbits.

Next consider the generating function

$$S = J_a(\ell + \omega + \Omega) + \tfrac{1}{2}J_b^2 \cot(\omega + \Omega) + \tfrac{1}{2}J_c^2 \cot\Omega. \tag{67}$$

Then

$$L = \frac{\partial S}{\partial \ell} = J_a, \quad G = \frac{\partial S}{\partial \omega} = J_a - \tfrac{1}{2}J_b^2 \csc^2(\omega+\Omega), \quad H = \frac{\partial S}{\partial \Omega} = J_a - \tfrac{1}{2}J_b^2 \csc^2(\omega+\Omega) - \tfrac{1}{2}J_c^2 \csc^2\Omega,$$

$$\theta_a = \frac{\partial S}{\partial J_a} = \ell + \omega + \Omega, \quad \theta_b = \frac{\partial S}{\partial J_b} = J_b \cot(\omega + \Omega), \quad \theta_c = \frac{\partial S}{\partial J_c} = J_c \cot\Omega. \tag{68}$$

After some algebra, this yields

$$\begin{aligned} \theta_a &= \lambda, \quad J_a = \sqrt{\mathfrak{G}\, ma}, \\ \theta_b &= \sqrt{2\sqrt{\mathfrak{G}\, Ma}(1 - \sqrt{1-e^2})}\cos\varpi, \quad J_b = \sqrt{2\sqrt{\mathfrak{G}\, Ma}(1 - \sqrt{1-e^2})}\sin\varpi, \\ \theta_c &= \sqrt{2\sqrt{\mathfrak{G}\, Ma(1-e^2)}(1-\cos I)}\cos\Omega, \quad J_b = \sqrt{2\sqrt{\mathfrak{G}\, Ma(1-e^2)}(1-\cos I)}\sin\Omega. \end{aligned} \tag{69}$$

These are the Poincaré elements, which are well-defined for both circular and equatorial orbits (but not radial orbits). In the limit of small eccentricity and inclination these reduce to

$$\begin{aligned} \theta_a &= \lambda, \quad J_a = (\mathfrak{G}\, Ma)^{1/2}, \\ \theta_b &= (\mathfrak{G}\, Ma)^{1/4} e\cos\varpi, \quad J_b = (\mathfrak{G}\, Ma)^{1/4} e\sin\varpi, \\ \theta_c &= (\mathfrak{G}\, Ma)^{1/4} I\cos\Omega, \quad J_b = (\mathfrak{G}\, Ma)^{1/4} I\sin\Omega; \end{aligned} \tag{70}$$

apart from the constant of proportionality $(\mathfrak{G}\, Ma)^{1/4}$ these are just the Cartesian coordinates corresponding to the polar coordinate pairs e–ϖ and I–Ω.

2. Units for the solar system

Many formulae involve the combination $\mathfrak{G}\, M$, where $\mathfrak{G}$ is the gravitational constant and M is the mass of the Sun ($M_\odot$), the Earth ($M_\oplus$) or some other body. The constant $\mathfrak{G}$ is determined by laboratory experiments and is only known to a fractional accuracy of 1×10^{-4}. On the other hand, the combinations $\mathfrak{G}\, M_\odot$ (the **solar mass parameter**) $\mathfrak{G}\, M_\oplus$, etc. are known extremely well, since these can be determined from the relation between period and semi-major axis (Kepler's third law) for small bodies orbiting the Sun, Earth, etc., and both periods and semi-major axes can be measured with great accuracy. The value recommended by the International Astronomical Union is

$$\mathfrak{G}\, M_\odot = 1.3271243421 \times 10^{26}\,\mathrm{cm}^3\,\mathrm{s}^{-2} \tag{71}$$

with a fractional accuracy of about 10^{-10}.

The history of the determination of $\mathfrak{G} M_\odot$ is worth describing briefly. Until the mid-twentieth century virtually all of our information on the orbits of the planets came from tracking their positions on the sky, along with the position of the Sun. This information—angular positions of the planets and the Sun on the celestial sphere as a function of time—can be combined with the theory of Kepler orbits (plus small corrections arising from mutual interactions between the planets) to determine all of the orbital elements of the planets including the Earth, except for the overall scale of the system. Thus, for example, the ratio of semi-major axes of any two planets was known to high accuracy, but not the values of the semi-major axes (this indeterminacy is easy to understand from dimensions alone: measurements of angles, which are dimensionless, cannot be combined to find a quantity with dimensions of length). To reflect this uncertainty, astronomers introduced the **astronomical unit** (abbreviated AU), which was originally defined to be the semi-major axis of the Earth's orbit. Thus the semi-major axes of the planets were accurately known in astronomical units long before the value of the astronomical unit (in centimeters, say) was determined to comparable accuracy. Current practice is to *define* the astronomical unit to be

$$1\,\mathrm{AU} = 1.495978707 \times 10^{13}\,\mathrm{cm} \tag{72}$$

exactly; thus the AU is no longer precisely the same as the Earth's semi-major axis.

For (much) more detail see Urban and Seidelmann, eds., *Explanatory Supplement to the Astronomical Almanac*.

3. Nearly Circular Orbits

3.1. Expansions for small eccentricity

The determination of the trajectory $[r(t), \psi(t)]$ of a Kepler orbit generally requires the numerical solution of Kepler's equation. However, most planet and satellite orbits are nearly circular, so that expansions of the trajectory in powers of the eccentricity e can provide a powerful tool for analyzing orbital motion. Such techniques were extremely important in the days when all algebra was done by hand, and they remain useful for many problems in solar system dynamics. Our focus in this section will be on illustrating the derivations of these expansions; the formulae themselves will be given only to $\mathrm{O}(e^3)$ as higher order expansions can easily be derived by computer algebra.

(a) True anomaly in terms of eccentric anomaly We begin with the identity $\tan\frac{1}{2}x = -i[\exp(ix) - 1]/[\exp(ix) + 1]$. Using this identity in equation (36) and replacing e by a parameter β defined by

$$e = \frac{2\beta}{1+\beta^2}, \qquad \beta = \frac{1-(1-e^2)^{1/2}}{e}, \tag{73}$$

we find

$$\exp(if) = \exp(iu)\frac{1-\beta\exp(-iu)}{1-\beta\exp(iu)}; \tag{74}$$

taking the log,

$$f = u - i\ln[1-\beta\exp(-iu)] + i\ln[1-\beta\exp(iu)]. \tag{75}$$

Using the expansion $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \mathrm{O}(x^4)$, we obtain

$$f = u + 2\beta\sin u + \beta^2\sin 2u + \tfrac{2}{3}\beta^3\sin 3u + \mathrm{O}(\beta^4)], \tag{76}$$

or, in terms of the eccentricity,

$$f = u + e\sin u + \tfrac{1}{4}e^2\sin 2u + e^3(\tfrac{1}{4}\sin u + \tfrac{1}{12}\sin 3u) + \mathrm{O}(e^4). \tag{77}$$

(b) Eccentric anomaly in terms of true anomaly Equation (36) remains valid if u and f are interchanged and e is replaced by $-e$ (which means β is replaced by $-\beta$). Thus the analog of equation (77) is

$$u = f - e\sin f + \tfrac{1}{4}e^2\sin 2f - e^3(\tfrac{1}{4}\sin f + \tfrac{1}{12}\sin 3f) + \mathrm{O}(e^4). \tag{78}$$

(c) Mean anomaly in terms of true anomaly Combining Kepler's equation (34) with equation (78) and expanding as a Taylor series in e yields

$$\ell = f - 2e\sin f + \tfrac{3}{4}e^2\sin 2f - \tfrac{1}{3}e^3\sin 3f + \mathrm{O}(e^4). \tag{79}$$

The most important expansions are in terms of the mean anomaly, since time is the natural independent variable for a trajectory and mean anomaly is a linear function of time. To establish such expansions, we note that Kepler's equation (34) implies that the eccentric anomaly u changes by 2π when the mean anomaly ℓ changes by 2π. Thus any function $g(u)$ is a periodic function of the mean anomaly, which can be expanded in a Fourier series. In particular, setting $g(u) = \exp(iju)$ where j is an integer, we may write

$$\exp(iju) = \sum_{m=-\infty}^{\infty} c_m(j)\exp(im\ell), \text{ where } c_m(j) = \frac{1}{2\pi}\int_0^{2\pi} d\ell\,\exp[i(ju - m\ell)]. \tag{80}$$

Eliminating ℓ using Kepler's equation (34) we have

$$c_m(j) = \frac{1}{2\pi}\int_0^{2\pi} du(1 - e\cos u)\exp[i(j-m)u + ime\sin u]. \tag{81}$$

For $m \neq 0$ we use the identity

$$\exp(iz\sin u) = \sum_{k=-\infty}^{\infty} J_k(z)\exp(iku), \tag{82}$$

where $J_k(z)$ is a Bessel function. Setting $z = me$ we find that equation (81) simplifies to

$$c_m(j) = \frac{j}{m} J_{m-j}(me), \qquad m \neq 0. \tag{83}$$

For the case $m = 0$ it is easy to show that

$$c_0(j) = \delta_{j0} - \tfrac{1}{2}e\delta_{j1} - \tfrac{1}{2}e\delta_{j,-1}, \tag{84}$$

where δ_{mn} is 1 if $m = n$ and 0 if $m \neq n$.

Several useful expansions now follow easily:

(d) Eccentric anomaly as a function of mean anomaly Set $j = 1$ and take the imaginary part of equation (80). We obtain

$$\sin u = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{J_{m-1}(me)}{m}\sin m\ell; \tag{85}$$

Using the relations $J_{-n}(z) = (-1)^n J_n(z) = J_n(-z)$ and $J_{n-1}(z) + J_{n+1}(z) = 2nJ_n(z)/z$, this result simplifies to

$$\sin u = 2\sum_{m=1}^{\infty} \frac{J_m(me)}{me}\sin m\ell, \tag{86}$$

which is combined with Kepler's equation (34) to yield

$$u = \ell + 2\sum_{m=1}^{\infty} \frac{J_m(me)}{m}\sin m\ell. \tag{87}$$

Finally, the power series for Bessel functions, $J_n(z) = \sum_{k=0}^{\infty}(-1)^k(\frac{1}{2}z)^{n+2k}/[k!(n+k)!]$, can be used to convert equation (87) into a power series in eccentricity:

$$u = \ell + e\sin\ell + \tfrac{1}{2}e^2\sin 2\ell + \tfrac{1}{8}e^3(3\sin 3\ell - \sin\ell) + \mathrm{O}(e^4). \tag{88}$$

(e) True anomaly as a function of mean anomaly Inserting the series (88) into equation (77) we obtain

$$f = \ell + 2e \sin \ell + \tfrac{5}{4} e^2 \sin 2\ell + \tfrac{1}{12} e^3 (13 \sin 3\ell - 3 \sin \ell) + \mathrm{O}(e^4). \tag{89}$$

(f) Radius as a function of mean anomaly Take the real part of equation (80) with $j = 1$. We obtain

$$\cos u = -\tfrac{1}{2} e + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{J_{m-1}(me)}{m} \cos m\ell; \tag{90}$$

using the Bessel function identities given earlier in this section this result simplifies to

$$\cos u = -\tfrac{1}{2} e + 2 \sum_{m=1}^{\infty} \frac{J'_m(me)}{m} \cos m\ell. \tag{91}$$

Thus

$$\frac{r}{a} = 1 + \tfrac{1}{2} e^2 - 2e \sum_{m=1}^{\infty} \frac{J'_m(me)}{m} \cos m\ell. \tag{92}$$

Using the power series for Bessel functions we obtain a power series in eccentricity,

$$\frac{r}{a} = 1 + \tfrac{1}{2} e^2 - e \cos \ell - \tfrac{1}{2} e^2 \cos 2\ell + \tfrac{3}{8} e^3 (\cos \ell - \cos 3\ell) + \mathrm{O}(e^4). \tag{93}$$