Fourier Series

Introductory Examples

2025-03-16 19:30 GMT

Abstract

Introduce Fourier series approximation with a few periodic, piecewise linear functions.

Table of Contents

## Approximation Framework

Let denote the multiplicative group of complex numbers of modulus 1, with a topology whose basic open sets are the open arcs of the unit circle. Let denote Lebesgue measure on divided by so that . For any define as the following space of complex-valued functions

We then define the following norm on .

The definition of for can be extended to as follows. Given define as the following subset of non-negative real numbers .

Then we define

This norm is also called the *essential upper bound* of . If , is said to be *essentially bounded*. The normed space of essentially bounded functions is denoted .

Since we have for all

with equality only when is equal to some constant for almost all . Consequently

Moreover, if then

Therefore

Combining these subset relations we have

It turns out that for each , is a *complete* normed linear space i.e., a Banach space. That is, each Cauchy sequence in converges to a member of .

### -periodic functions

We will identify a -periodic function with a corresponding function if the following relation holds.

That is, we can define based on or, alternatively, we can define based on .

To facilitate construction of example functions we also define the following mapping from to .

For any we define the following inner product.

Then is a complete inner-product space, i.e., a Hilbert space.

### An orthonormal basis

For each integer define function as the following unit exponential function.

or equivalently

Then for any integers we have

where is the Kronecker delta, equal to 1 at and equal to zero for non-zero integers .

Therefore is an orthonormal set of functions in . It turns out that this set is moreover an orthonormal *basis* of .

## Fourier Coefficients

If and , the following formula defines to be the Fourier coefficient of of index .

Note that the magnitude of is bounded by for all .

We denote by the mapping

Thus maps to the normed linear space of bounded sequences . Moreover, from Mercer’s theorem it can be shown that

That is, maps to , the subspace of bounded sequences that converge to zero.

We now define as the following finite Fourier series approximation of of index .

More generally we’ll consider weighted approximations of the following form.

where is some specified weighting function, also known as a convergence factor.

## Example Functions

## Gibbs Phenomenon

## Kernel Functions

## DFT: the discrete Fourier transform

## Closing Remarks