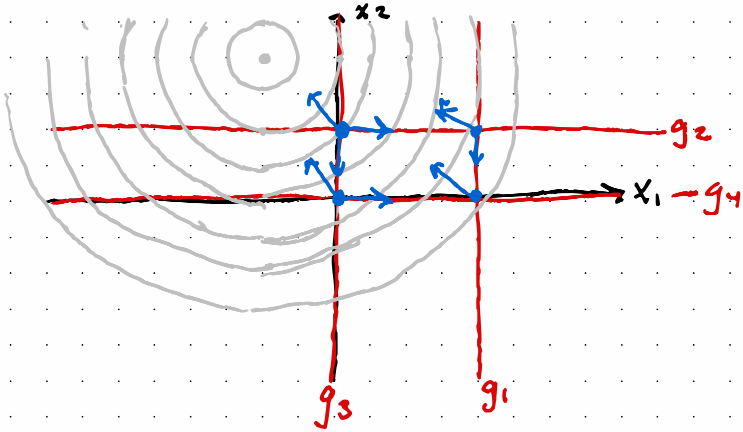
Homework 4

Problem 1 Solve graphically. Determine directions of feasible descent at corner points. Show gradient directions of f & g_is at these points. Verify graphical results analytically using KKT conditions

$$\min \quad f(x) = (x_1+1)^2 + (x_2-2)^2$$

s.t. $g_1 = x_1 - 2 \leq 0$ $\quad$ $g_3 = -x_1 \leq 0$

$g_2 = x_2 - 1 \leq 0$ $\quad$ $g_4 = -x_2 \leq 0$



Optimum solution is when g_2 & g_3 are active at $x_1 = 0$ $\quad x_2 = 1$

$$L = (x_1+1)^2 + (x_2-2)^2 + \mu_1(x_1-2) + \mu_2(x_2-1) + \mu_3(-x_1) + \mu_4(-x_2)$$

$$\nabla_x L = \begin{bmatrix} 2(x_1+1) + \mu_1 - \mu_3 \\ 2(x_2-2) + \mu_2 - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

at $x_1 = 0 \quad x_2 = 1$ $\qquad \mu_1 = 0 \quad \mu_2 > 0 \quad \mu_3 > 0 \quad \mu_4 = 0$

$2 - \mu_3 = 0$ $\quad$ $\mu_3 = 2$ ✓

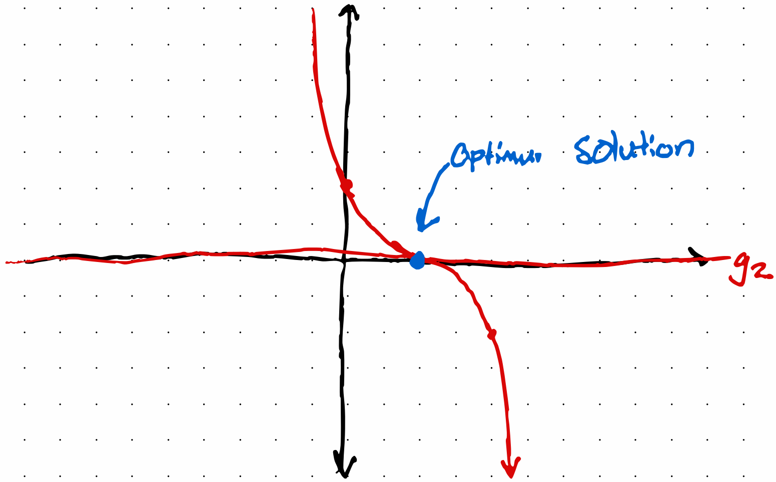
$-2 + \mu_2 = 0$ $\quad$ $\mu_2 = 2$ ✓ $\Big\}$ both μs are positive

$$L_{xx} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

P.d so satisfies KKT & $x_1 = 0, x_2 = 1$ is a solution

Problem 2 graph & find solution then apply optimality conditions can you find a solution based on the optimality conditions?

$$\min f = -x_1$$
$$\text{s.t. } g_1 = x_2 - (1-x_1)^3 \leq 0 \qquad x_2 \geq 0$$



$$L = -x_1 + \mu_1(x_2 - (1-x_1)^3) + \mu_2(-x_2)$$

$$\nabla_x L = \begin{bmatrix} -1 + \mu_1(3(1-x_1)^2) \\ \mu_1 - \mu_2 \end{bmatrix} = 0$$

if $\mu_1 = \mu_2 = 0$

$-1 = 0$ ✗ Not a solution
$0 = 0$ ✓

if $\mu_1 > 0 \quad x_2 = 0 \quad \mu_2 > 0 \quad x_1 = 1$

$-1 + \mu_1 \cdot 0 = -1$ ↳ says its not a solution

$\mu_1 = \mu_2$

KKT conditions will not be met because $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not a regular. This is because at this solution, both constraints are parallel and can't offset function gradient

Problem 3 Find a local solution to the problem

$$\text{Max } f = x_1x_2 + x_2x_3 + x_1x_3 = 0$$

$$\text{s.t } h = x_1 + x_2 + x_3 - 3 = 0$$

reformed

$$\min f = -x_1x_2 - x_2x_3 - x_1x_3 = 0$$

$$\text{s.t. } x_1 + x_2 + x_3 - 3 = 0$$

$$L = -x_1x_2 - x_2x_3 - x_1x_3 + \lambda(x_1 + x_2 + x_3 - 3)$$

$$\nabla_x L = \begin{bmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_3 + \lambda \\ -x_2 - x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\nabla_\lambda L = x_1 + x_2 + x_3 - 3 = 0$$

$$\begin{matrix} 0 & -1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 3 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{matrix}$$

$$\rightarrow x_1 = x_2 = x_3 = 1, \quad \lambda = 2$$

$$L_{xx} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \rightarrow \det(L_{xx}) = -2 \rightarrow \text{must check } 2^{nd} \text{ order conditions}$$

$$dx^T L_{xx} dx = \underset{1\times 3}{[dx_1 \ dx_2 \ dx_3]} \underset{3\times 3}{\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}}$$

Always positive for feasible direction

$$= [-dx_2 - dx_3 \quad -dx_1 - dx_3 \quad -dx_1 - dx_2] \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

$$= -dx_2dx_1 - dx_3dx_1 - dx_1dx_2 - dx_2dx_3 - dx_1dx_3 - dx_2dx_3$$

$$= -2dx_1dx_2 - 2dx_1dx_3 - 2dx_2dx_3$$

$$\frac{\partial h}{\partial x} dx = \left[\frac{\partial h}{\partial x_1} \ \frac{\partial h}{\partial x_2} \ \frac{\partial h}{\partial x_3}\right] \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = [1 \ 1 \ 1] \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = 0 \qquad dx_1 + dx_2 + dx_3 = 0$$

$$dx_1 = -dx_2 - dx_3$$

$$dx^T L_{xx} dx = -2\left[(-dx_2 - dx_3)dx_2 + (-dx_2 - dx_3)dx_3 + dx_2 dx_3\right]$$

$$= -2\left[-dx_2^2 - dx_3 dx_2 - dx_2 dx_3 - dx_3^2 + dx_2 dx_3\right]$$

$$= 2\left[dx_2^2 + dx_2 dx_3 + dx_3^2\right]$$ ← needs to be positive

$$\underbrace{\left(dx_2 + \tfrac{1}{2}dx_3\right)^2 + \tfrac{3}{4}dx_3^2}_{\text{always positive}} \leq 0$$

when $x_2 = x_3 = 0 \rightarrow x_1 = 0$ No movement

always positive

Since $dx^T L_{xx} dx > 0$ for all feasible disturbancess

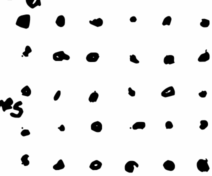
$$\boxed{x_1 = x_2 = x_3 = 1 \text{ is a KKT point and a local solution}}$$

Problem 5 Formulate garbage truck problem

N sites to visit. cost to move from node i to j is C_{ij} if an edge between nodes or ∞ if not.
site 0 is truck station where truck starts and returns

$$\min_{\{x_{ij}\}} \sum C_{ij} x_{ij} \qquad x_{ij} : \begin{cases} 0 & \text{don't go } i \to j \\ \mathbb{N} & \text{go } i \to j \text{ integer times} \end{cases}$$

0?

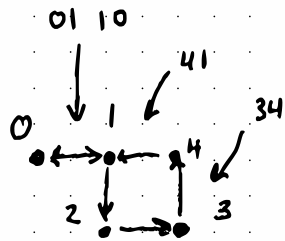


$$\text{s.t.} \quad \sum_i x_{ij} = \sum_i x_{ji} \quad \forall j$$

$$\sum_i x_{ij} \geq 1 \quad \forall j$$

times you enter node j = times exit node j

for every node j it must be entered at least once



- Enter node = Exit node
  +
  brings it back to start

since not building procedure this does not need to be explicitly programmed

- enter node must exit as next step
- every node needs to be entered once at least