Chapter 1 - Section 3
Video 1-6

Characterizing Algorithm

One criterion to evalute an algorithm is stability and unstability of it.
An algorithm is said to be stable, if for small changes in initial data produce correspondingly small changes in the final results. otherwise, it is unstable.
Some algorithms are stable only for certain choices of initial data, these algorithms are called conditionally stable.
The stability and unstability of an algorithm can be defined by magnitude of error too. Look at the following definition.

Def. 1.17. Suppose $E_0 > 0$ is an error produced at some stage of an algorithm and E_n is an error after n operations. Then

- If $E_n \simeq C n E_0$ for some constant C, then the growth of error is said to be linear.
- If $E_n \simeq C^n E_0$ for some $C > 1$, then the growth of error is called exponential.

The linear growth of error is unavoidable in an algorithm, but exponential growth of error should be avioded.

Example. 1 First, we can show that

$$P_n = c_1\left(\tfrac{1}{3}\right)^n + c_2\cdot 3^n \qquad (1)$$

is a solution to the recursive equation

$$P_n = \frac{10}{3}P_{n-1} - P_{n-2} \qquad (2)$$

for any constants c_1 and c_2 and $n = 2, 3, \dots$.

To show this, we have:

$$\frac{10}{3}P_{n-1} - P_{n-2} = \frac{10}{3}\left[c_1\left(\tfrac{1}{3}\right)^{n-1} + c_2\, 3^{n-1}\right]$$

$$- \left[c_1\left(\tfrac{1}{3}\right)^{n-2} + c_2\, 3^{n-2}\right]$$

$$= c_1\left(\tfrac{1}{3}\right)^{n-2}\left[\frac{10}{3}\cdot\frac{1}{3} - 1\right] + c_2\, 3^{n-2}\left[\frac{10}{3}\cdot 3 - 1\right]$$

$$= c_1\left(\tfrac{1}{3}\right)^{n-2}\left(\tfrac{1}{9}\right) + c_2\cdot 3^{n-2}\,(9)$$

$$= c_1\left(\tfrac{1}{3}\right)^n + c_2\, 3^n = P_n$$

So, (1) is the solution of (2).

Second, suppose $P_0 = 1$ and $P_1 = \frac{1}{3}$ are initial values. Then from (1),

$$\begin{aligned} P_0 &= c_1\left(\tfrac{1}{3}\right)^0 + c_2\cdot 3^0 \Rightarrow c_1 + c_2 = 1 \\ P_1 &= c_1\left(\tfrac{1}{3}\right)^1 + c_2\cdot 3^1 \Rightarrow \tfrac{1}{3}c_1 + 3c_2 = \tfrac{1}{3} \end{aligned} \Rightarrow \begin{cases} c_1 = 1 - c_2 \\ c_1 + 9c_2 = 1 \end{cases}$$

$\Rightarrow 1 - C_2 + 9C_2 = 1 \Rightarrow 8C_2 = 0 \Rightarrow C_2 = 0$ and $C_1 = 1 - 0 = 1$.

Therefore, the particular solution will be $P_n = (1/3)^n$, for all n, which a real solution.
Now, suppose, we use five-digit rounding arithmatic in computer, then

$\hat{P}_0 = 1.000 \ (= fl(P_0))$, $\hat{P}_1 = 0.33333$,

and with these values finding $\hat{c}_1$ and $\hat{c}_2$, we have
$\hat{c}_1 = 1.0000$ and $\hat{c}_2 = -0.12500 \times 10^{-5}$.

Having these floting numbers, we have,
$\hat{P}_n = 1.0000(1/3)^n - 0.12500 \times 10^{-5}(3)^n$;

which has absolute error

$$|P_n - \hat{P}_n| = 0.12500 \times 10^{-5}(3^n). \quad (3)$$

From error in (3) (E_n), we can see that the procedure to find P_n is unstable, since by small error in $E_0 = 0.12500 \times 10^{-5}$ the error in n iteration $E_n = 0.12500 \times 10^{-5}(3^n)$, which is growing exponentially with n.