Project 13 - Two-sided Jacobi algorithm for SVD

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Since my student number ends with 35, I have been tasked with project $13 = 35$ mod 22. My repository, wherein the examination project lies, can be found at `https://github.com/ttlindkvist/ppnm/`. I have done testing to ensure correctness and timed the algorithm to visualize the $O(n^3)$ operational cost. Since the algorithm has been successfully implemented and the $O(n^3)$ cost has been confirmed, I would self-evaluate this to a score of 10/10.

1 Elementary Jacobi diagonalization

As we know we can use Jacobi rotations to diagonalize a real symmetric matrix A into

$$A = VDV^T \tag{1}$$

This can be done by sweeping across the upper triangle of the matrix cyclically, eliminating off-diagonal elements pair-by-pair, by a well-tuned Jacobi rotation $J(p, q)$ eliminating the pair A_{pq}, A_{qp}.

$$A \rightarrow J(p, q)^T A J(p, q) \tag{2}$$

But what if we want to find the singular values of a non-symmetric, square matrix? Can we reuse some of what we know from this method? It turns out the answer is yes! But first what is SVD?

2 SVD

The singular value decomposition (SVD) of a matrix A is a factorization of the matrix on the form

$$A = UDV^T, \tag{3}$$

where U and V are orthogonal matrices and D is a diagonal matrix containing the singular values of A. These *singular values* are the square roots of the eigenvalues of the real symmetric matrix $A^T A$.

3 Two-sided Jacobi method

Many parts of the Jacobi algorithm for the symmetric case can be reused!

3.1 Square matrices

In fact the only modification necessary is a Givens rotation symmetrizing the two off-diagonal elements, that are to be eliminated by the following Jacobi transformation

$$A \rightarrow J(p,q)^T \left(G^T(p,q)A \right) J(p,q) \tag{4}$$

The matrices U and V are updated in accordance with transformations on the left and right side of A.

$$U \rightarrow UGJ \tag{5}$$
$$V \rightarrow VJ \tag{6}$$

With U and V initialized to identity. This process is, just like the Jacobi diagonalization of a symmetric matrix, repeated over all non-diagonal elements of the matrix, called a sweep, until two consecutive sweeps leave the diagonal elements unchanged, and we are left with the SVD for $A = UDV^T$.

The angle of the Givens rotation can be found by first looking at the 2x2 case:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7}$$

$$G^T(\theta)A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\cos\theta - c\sin\theta & b\cos\theta - d\sin\theta \\ a\sin\theta + c\cos\theta & b\sin\theta + d\cos\theta \end{pmatrix} \tag{8}$$

Require off-diagonals to be equal

$$b\cos\theta - d\sin\theta = a\sin\theta + c\cos\theta \tag{9}$$
$$\sin\theta(a+d) = \cos\theta(b-c) \tag{10}$$
$$\tan\theta = \frac{b-c}{a+d} \tag{11}$$

Which generalizes to the case of symmetrizing two off-diagonal elements, A_{pq} and A_{qp}, in a arbitrarily sized square matrix A.

$$\tan\theta = \frac{A_{pq} - A_{qp}}{A_{pp} + A_{qq}} \tag{12}$$

3.2 Tall matrices

For tall matrices A of size $N \times M$, $N \geq M$, a small adjustment has to be made. First a QR-factorization is necessary, and is done using my own Gram-Schmidt implementation from a homework

$$A = QR. \tag{13}$$

Since R now is a square matrix, we can use the above method for generating a SVD for R

$$R = U'DV^T. \tag{14}$$

So we can write A as

$$A = Q(U'DV^T), \tag{15}$$

which is almost the SVD. But since the product of two orthogonal matrices is also! an orthogonal matrix, the SVD for A is simply

$$A = UDV^T, \quad U = QU'. \tag{16}$$

4   Testing

How do we prove that the method works? As seen from the output (see github), it works for small matrices by visual inspection, but we could just be lucky.

Testing is done by generating a bunch of large, tall matrices, and checking if the resulting U and V are orthogonal (by checking U^TU and V^TV is identity), and by checking that D is diagonal. Then, by definition, the algorithm must be producing the correct output.

5   Computational cost

Just as the algorithm for symmetric matrices, the cyclic method for SVD is an $O(n^3)$ operation as seen on figure 1. This stems from the fact that a loop over the upper triangle of a matrix is an $O(n^2)$ operation and the following update is an $O(n)$ operation. A fit to the time taken to SVD different sized random square matrices, shows the $O(n^3)$ cost. What is also seen is a comparison with the GSL implementation of a one-sided Jacobi algorithm - which is around 8-11 times faster in these use-cases.
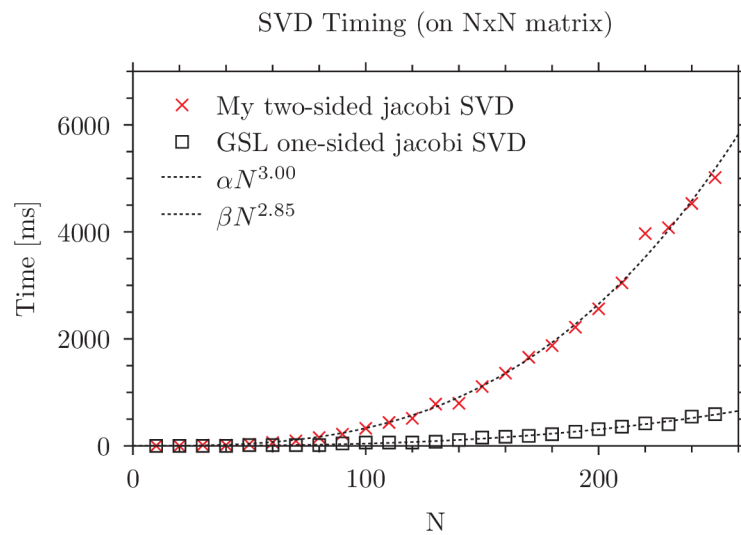
Figure 1: Timing of my implementation of the two sided Jacobi SVD versus the GSL implementation of the one-sided Jacobi SVD on random N by N matrices.