A simple model that explains why inequality is ubiquitous

RENATO FABBRI, OSVALDO N. OLIVEIRA JR.
São Carlos Institute of Physics, University of São Paulo, CP 369, 13560-970 São Carlos, SP, Brazil

October 20, 2015

Abstract

Inequality has always been a crucial issue for human kind, particularly concerning the highly unequal distribution of wealth, which is at the root of major problems facing humanity, including extreme poverty and wars. A quantitative observation of inequality has become commonplace in recent years with increasing recognition that many natural and man-made systems can be represented as scale-free networks, whose distribution of connectivity obeys a power law. These networks may be generated by the preferential attachment for the nodes, within the so-called rich-gets-richer paradigm. In this letter we introduce a simple model that explains the ubiquity of inequality, based on three simple assumptions applied to a generic system with a large number of parts. The first assumption is the diversity of the components, while the second is related to a uniform distribution of resources for individual components. In the third assumption, we keep the amount of each resource input to the system fixed. This implies that the more resources are allocated per component, the smaller is the number of such components in the system, thus conserving the amount of resources distributed. The second and third assumptions are conservations laws, with the resources being conserved in each input and across the component wealth. This can be geometrically described by the distribution of object sizes in a 3D space, where each dimension is assumed to be isotropic. Applying these assumptions to a generic system results in a power-law distribution, whose coefficient is the number of inputs that are independent from each other, i.e. the dimensionsionality of the allocated resources. Even though there is no restriction to the value of the coefficient, in practice we observe that existing systems normally exhibit a coefficient between 1.5 and 3.0. With our simple model it is not possible to determine whether this limitation in the coefficient values arises from a fundamental principle, but we indicate reasonable hypotheses. The assumptions in the model lead to a framework analogous to the laws of thermodynamics: conservation of resources and a time arrow pointing to inequality. Since these assumptions are easily justified based on established knowledge, the model proves unequivocally that inequality is ubiquitous. We also discuss ways to control this tendency to inequality, which is analogous to a decrease in entropy in a closed system induced with an external action.

Keywords: power laws, fundamental theory, complex systems, complex networks, anthropological physics

1 Introduction

Inequality has always been at the focus of social studies for the obvious importance for humanity and quality of life. In some respects, inequality can be regarded as asymmetry, thus being at the center stage of science since symmetry is regarded as fundamental to cognition and knowledge in nearly all fields [1, 2]. In fact, the ubiquity of symmetry is appreciated so profoundly that human thinking is believed to present explicit, basic symmetry operations, and therefore the world is modelled with symmetry, regardless of its presence. In recent years, inequality in distributions has been addressed quantitatively, particularly in large systems represented as networks. Of particular importance are the scale-free networks [?], whose distribution of connectivity (number of edges per node) obeys a power law [?]. Significantly, a wide range of natural and man-made systems have been represented as networks with power-law distributions [3] [?]. As a rule of thumb, the distribution of resources among (often self-interested) components tends to follow a power law, which includes distribution of human wealth, interactions, friendships; connections among airports and synaptic count among neurons. Other examples of power-law distributions are associated with earthquake intensity and allometric relations of animal bodies, while the most canonical law examples seem to be Pareto and Zipf laws [ref.]. Some advocate that a more precise fitting can be obtained with superposition of a power law distribution and a Weibull distribution [4], but inequality remains.

Power laws are also prevailing in other phenomena and areas, which is not necessarily related to distribution of resources. For example, power laws govern perception, as indicated by the Webner-Fechner and Stevens laws [ref. ?]. In basic physics there are various power laws, e.g. in a Newtonian context force F is related to distance d through gravity ($F = G\frac{m_1\, m_2}{d^2}$) with $\alpha = 2$ and force is related to acceleration ($F = m\, a^1$) with $\alpha = -1$.

Not surprisingly, there has been extensive body of research to explain power law behavior, with models for generating scale-free networks [?] and for dealing with broad classes of phenomena [2, 5]. But to the best of our knowledge, a unified framework for interpreting any power law is lacking. This is what we do in this Letter, with the axiomatic and phenomenological description a simple model in Section 2. ???? Section 3 exposes separately a geometric and a productive system paradigms. Special cases are considered in Section 4, which is followed by concluding remarks. The most elementary mathematical description of power-law distributions is outlined in the Appendix.

2 Formalization

Definition 1 *A complex system is taken as one in which the whole is more than the sum of the individual parts.*

Other definitions currently in use in the literature could also apply, since for our purposes the system is generic and can have any type of component.

Definition 2 *A* **resource** *is anything that is used by a complex system to subsist and communicate.*

In practice, a complex system is commonly delimited and specified through its components and the resources to keep it. The resources are not restricted to any type and can be, for example, geometrical volumes, wave cycles or currency values.

Definition 3 *A* **resource-based system** *is a complex system that has an underlying resource vital to the their components and their interdependent roles in the system.*

Definition 4 *The* **component wealth** k *is the amount of resources (of a certain kind) allocated to a component.*

We will use $p(k)$ to denote the fraction of components with component wealth k. Likewise, the same notation $p(k)$ will denote the probability of choosing a component with an amount k of resources. The context should make it clear which is the appropriate meaning. This correspondence among probability, frequency and relative count is instrumental for the interpretative framework in this article, which is conveniently embedded in the notation. It is also useful to observe k as a wave period λ. Then, $p(k)$ is more immediately understood as the frequency f. Notice that k is a measure of the resources, not total resources (or total component wealth), which might be correlated to k and have k as one of the resources.

2.1 Propositions and corollaries

Proposition 0 *There is diversity among the components of the resource-based system.*

For a large set of components, there is hardly any way to avoid diversity. This could occur with location, size, age, etc. This can be regarded as both a statistical law, or even as a deeper truth. The mere existence of two objects imposes diversity, otherwise they would be both the same.

Proposition 0 is important for the distribution of resources among components: if the latter are all equal, by definition they have the same component wealth.

Proposition 1 *The allocation of resources in a resource-based system is uniform across the components. This is expressed as a uniform distribution $p_U(k)$ of resources with respect to component wealth k.*

That is, resources are allocated democratically with respect to component wealth values, which has two major consequences: 1) there is a maximum value for the wealth of a component; 2) the wealthier the components considered, the fewer they are. In the analysis, components without resources allocated are discarded.

Corollary 1 *The different resource inputs are combined in the α dimensions of the resources.*

This is important to consider resources with distinct dimensions, as the amounts of different resources may be incomparable. Let λ_1 and λ_2 be distinct resources, leading to a total resource: $E = \widetilde{C_1}.\lambda_1.\lambda_2 \equiv \widetilde{C_2}\lambda^{\alpha=2}$ with $\widetilde{C_X}$ constant. Note that $E = \widetilde{C_1}.\lambda_1 + \widetilde{C_2}.\lambda_2 \equiv \widetilde{C_3}\lambda^{\alpha=1}$, that is: a resource resulting from the summation of other two unidimensional resources is an unidimensional resource and these resources are not of distinct dimensions. For example, if the resources are *components* and *time*, a final resource $E = 5\,components + 4h = 9$ holds little if any information, while $E = 5\,components\,.\,4h = 20$ *components.hours* is a reasonable measure of resources in a canonical metric. (nao entendi?)

Corollary 2 *The equilibrium state implied by Propositions 0, 1 and 1 is characterized by a power-law distribution $p(k)$ of components with component wealth k.*

From the propositions, the same amount C of resources is allocated across all quantities k of resources allocated per component. Therefore, the fraction $p(k)$ of components with component wealth k is $p(k) = \frac{C}{k^\alpha}$, where α is the dimensionality of the resources compared to the dimension of k. As the system continues to exist, and resources are continually allocated, deviations from the equanimous distribution $p_U(k)$ of resources along component wealth k tend to be transient. It is worthnoting that $p_U(k)$ is usually not made explicit in the literature, but only the power-law distribution $p(k)$ of components with component wealth k.

Corollary 3 *The extension of allocation is $[k_L, k_R]$ with R_L often 0 or 1 and $k_R \approx C$.*

This follows from Proposition 1. If the allocation of resources is insensitive to component wealth, it should sweep all possible values, which have a lower limit by being a positive quantity of resources. Systems are normally considered as a set of components Coupled by the resources, thus $k_L = 0$ and $k_R = 1$ are reasonable, respectively. (não entendi?)

The distributions $p(k)$ and $p_U(k)$ also have an upper limit as the component wealth reaches $k_R \approx C$, the total amount of resources uniformly distributed. In empirical data, k_R can vary considerably, due to nonlinearity of the resources scaling (most often $k_R < C$) and to self-interested agents (most often $k_R > C$). Power laws are reported to conduct empirical data sovereignly and often are strict only for a (broad) portion of component wealth range. (nao entendi?)

Corollary 4 *The upper limit k_R for the allocation of resources is an estimate of the amount of resources C equally distributed along component wealth ($C \approx k_2$).*

This follows from Corollary 3. (acho que isso já foi mencionado no corolário anterior, e então fica repetitivo)

Corollary 5 *$p(1)$ provides another estimate of the amount of resources C equally distributed along component wealth ($C \approx N.p(1)$, where N is the number of components).*

Corollary 6 *An estimate for C can also be found by α, k_L and k_R through Equation 6.*

Corollary 7 *The dimensionality of the allocated resources is the scaling factor α.*

This last corollary follows from box counting or, most easily, through wave-like reasoning about power laws, as explained in the next section.

2.2 Phenomenological approach: power laws are consistent with the assumptions

One important step for checking the consistency of our model is to verify whether the assumptions above lead to any type of power law. The simple argument presented below taking wave propagation as example corresponds to a quasi-"if and only if"interpretation of power laws.

(1D case) Consider a wave propagation with constant speed v (suppose linear media with no dispersion). The number of oscillations f per unit time is inversely proportional to the cycle length λ (the period). In usual notation $f = \frac{v}{\lambda}$. The constant speed v implies a power law between f and λ (with $\alpha = 1$ and $C = v$). This is the core insight, the general case that a power law can be interpreted as resulting from a constant amount C of fundamental resources with dimensionality α being homogeneously distributed across component wealth k and resulting in $p(k)$ of such components.

(2D case and beyond) Now let $f = \frac{v=C}{\lambda_1.\lambda_2}$, that is, the frequency of occurrence (or the probability of choosing such event at random) go with the inverse of two periods while the speed is constant. If $\lambda_1 == \lambda_2 == \lambda$, then $f = \frac{v}{\lambda^2}$. In other words, the density given by an amount v (or a quantity C of resources) in a hypercube of edge λ (or component wealth k) and $\alpha = 2$ dimensions. This same reasoning yields the relative count, The fraction or probability $p(k)$ of components with such component wealth k and corresponding hypercubes k^α. Let C_i be the amount of resources allocated at specific resource costs, say $\lambda_{1,i}$ and $\lambda_{2,i}$ to strengthen the wave argumentation, and assume linearity $\lambda_{1,i} = c.\lambda_{2,i}$ such that $p(k_i) = \frac{C}{C_i} = \frac{C}{\lambda_{1,i}.\lambda_{2,i}} = \frac{C}{c\lambda_{2,i}^2} = \frac{\widetilde{C}}{\lambda_{2,i}^2} \equiv \frac{v}{\lambda^2}$. Therefore one can consider that we only access $\lambda_1 == \lambda_2$ and a "normalized resource C" allocated by the environment uniformly across component wealth: higher component wealth implies less numerous components. (não entendi?)

Na parte debaixo não entendi a argumentação. Como o que está em verde fecha a seção em que se procurava mostrar que as hipóteses levam a power laws.

The cases above were related to phenomena represented by power laws. For power-law distributions involving random variables, we assume random variables $\lambda_{1,i}$ and $\lambda_{2,i}$ as presenting an uniform distribution of resources along component wealth themselves: $p_{\lambda_1}(\lambda_{1,i}) = \frac{C_i}{\lambda_{1,i}^{\alpha_1}}$ and $p_{\lambda_2}(\lambda_{2,i}) = \frac{C_i}{\lambda_{2,i}^{\alpha_2}}$. Consider de product distribution of the $\lambda_{3,i}^2 = \lambda_{1,i}.\lambda_{2,i}$ variable:

$$p_{\lambda_3^2}(\lambda_{3,i}^2) = C_1.C_2.\lambda_3^{-2\alpha_2}\frac{(R_{1,R} - R_{1,L})^{\alpha_2-\alpha_1}}{\alpha_2 - \alpha_1} = C_1.C_2.\lambda_3^{-2\alpha_1}\frac{(R_{2,R} - R_{2,L})^{\alpha_1-\alpha_2}}{\alpha_1 - \alpha_2} \tag{1}$$

where R_R and R_L are the upper and lower limits of the respective resources in the system considered. Notice that $p_{\lambda_3^2}(\lambda_{3,i}^2) = \frac{N_i}{N} = p(\lambda_{0,i})$, where $p(\lambda_{0,i}$ is the probability that a component has a measured component wealth $\lambda_{0,i}$ correlated to $\lambda_{3,i}^2$. The number N_i of components with component wealth $\lambda_{3,i}^2$ and the total number of components N are conserved for $\lambda_{0,i}$. If assumed $\lambda_{0,i} \propto \lambda_{3,i}$, than $p(k_{0,i}) \propto k_{0,i}^2$.

3 Paradigmatic observations of the propositions

In the following we discuss a few paradigmatic cases where the propositions can be applied to specific systems.

3.1 Object sizes in a house

Take a measure unit l for length and consider $m \in (0,1)$. In a house, there are probably more objects of volume $\approx (m.l)^3$ than those of volume $\approx l^3$. We may assume that the chance ρ of finding an object fitting an arbitrary cube of volume $\approx (m.l)^3$ is the same as that for an object with of volume $\approx l^3$.

Since the house has a fixed volume, m^{-3} more of the smaller cubes can be fitted and therefore there should be m^{-3} more objects of such volume. The result is a power-law distribution of object volumes related to the length l with $\alpha = 3$: $p(l) = C.l^{-3}$. Note that if the objects are mutually exclusive, α is probably lower, as the number of smaller cubes will decrease considerably.

This example gives a geometrical interpretation of the formalism presented in section ??. It also might be assumed true even if there is no isotropy and can be considered a "best guess" if total ignorance about the system is assumed. Extreme choices of l and m (and of dl and $dm = mdl$ for tolerance in size) will often exhibit spurious observations as the result of a bad fit of the scale for the analysis (nao entendi?).

3.2 Workers in a factory

In a factory where many tasks need to be performed, we may consider three types of inputs: number of workers N, working hours W and efficiency E. This is similar to the canonical resource *person.hour* since Taylorism and taken to the extreme in Fordism [ref.?].

We assume that all concentrations $C.k^3 = n.w.e$ of these resources might be found in working groups. Assuming the tasks changes all the time, all concentrations are important. By Proposition 1 they are equally important beforehand in the sense that resources will be spread uniformly over $[k_L, k_R]$ with $p_U(k) = \frac{1}{k_R - k_L}$. The number of such components, therefore, decreases with $C.k^3 = n.w.e$, the component wealth with three dimensions. In other words, $p(k) = \frac{C}{k^3}$, that is, the fraction $p(k)$ of components (working groups) with component wealth k decrease with k^3 and distributes the same amount of resources $C = \frac{N.W.E}{k_R - k_L}$ along k. (não entendi as contas)

4 Implications

The consequences of the interpretation of power laws presented in Section 2 are severe for understanding and dealing with diverse (if not almost all) phenomena. Selected cases are discussed in this section.

4.1 Scale-free complex networks

In a network, one has essentially E edges and N vertices. Assuming linearity of resources:

$$Resources = C_1 N + C_2 E \approx C_2 E = C_2 N \frac{E}{N} = C_2 \frac{N\bar{k}}{2} \tag{2}$$

where C_1 and C_2 are constants and $\bar{k}$ is the average degree, i.e. the mean of the number of edges attached to each vertex. As in the literature, nodes can be disregarded, since if they are not connected. Therefore:

$$p_{E_i}(k_i) = \frac{C}{resources_i} = \frac{C}{C_2 E_i} = \frac{C/C_2}{N_i \frac{E_i}{N_i}} = \frac{2C/C_2}{N_i k_i} \equiv \frac{v}{\lambda_1 \lambda_2} \tag{3}$$

Since N_i and k_i are both directly proportional to E_i, which is the fundamental resource, one can factor out another constant and consider the special case where $\lambda = \lambda_1 = \lambda_2$ and, consequently, $f = v/\lambda^2$. This is the case for a social network, the number of agents allocated N_i and the time each of them put (related to k_i), are seen as the primary resource (*individuals.time*), and $\alpha \approx 2$ as shown by empirical evidence [3].

Also:

$$p_{E_i}(k_i) = \frac{N_i}{N} = \frac{2C/C_2}{N_i k_i} \Rightarrow \frac{N_i^2}{N^2} = 2\frac{C}{N.C_2} k_i^{-1} \Rightarrow p_{k_i}(k_i) \propto k_i^{-\frac{1}{2}} \tag{4}$$

so that if $p(k_i)$ is observed only with respect to the degree, $\alpha = 0.5$, which is far from empirical evidence because it only captures the distribution of one of the resources with respect to k_i.

4.2 Equanimous inequality

Paradoxically, power laws, which are the current utmost inequality paradigm, follow from an equanimous consideration of resource inputs and exhibit other equanimous aspects:

- $p(k) = C.k^{-\alpha} \Rightarrow p(k).k^{\alpha} = C$, with C constant. That is, the amount k^{α} of resources per component times the amount of those components, which is the total "instantaneous" allocated resources, is constant C.
- Each component participates in numerous other complex systems, potentially infinite, and should present a broad, if not complete, sweep of resources allocated to itself. These resources are not necessarily of the same type. We assume that human systems, for example, present power-law distributions of knowledge $p_k(k_k)$ and of wealth $p_w(k_w)$, with potentially different (relative) amount k of resources. At the same time, within a fixed type of resource, resources allocated vary in different systems. For instance, an individual tends to have many acquaintances (resource) in its own family, work and neighborhood, but fewer in circles of distant family members, partners and friends.
- The total resource available to each component is potentially the same, but spread differently across systems. For example, human individuals form complex social systems with power-law distributions of relations. All the participants, by being humans, have the same amount of time (resource) available each day to engage in all the complex systems that are presented by the environment. One can even assume that each

individual creates the same amount of relationships with the world each day, be them with other people, ideas, things, etc.

4.3 Wealth distribution

One manifestation of inequality through power laws most fundamental in society is the discrepant wealth distributions worldwide. There are continuous efforts to deal with this issue, usually advocating ways to minimize “social inequality”. Considering the framework presented within this letter:

- Such an inequality is a natural tendency that follows from Propositions 0, 1 and 1 in accordance with the phenomenological analysis of Section 2.2.
- Deviations from a power-law distribution of wealth should require work. Occasional deviations are part of the statistical aspect of the phenomena involved, but the maintenance of a pattern different from the power law derived from the resources distribution should require expenditure of energy.
- Both deviations of power laws towards a more equanimous or towards a more inequanimous distribution should be ephemeral or require work.

In summary, power-law inequality seems inevitable, and a consequence of a distribution of wealth equanimous and insensitive along wealth allocated to each component. This implies the necessity of “work” for equalization. Also, we observe that the higher the k_L of equation 5, the higher all the probabilistic mass will be located, which implies greater wealth of the wealthier, the hubs or the “elites”. In other words, the richer the least rich, the richer the more rich. (não entendi?)

4.4 When inequality is good

Inequality is normally treated as a bad consequence, especially in social studies. There are cases, however, where inequality in distributions is highly beneficial. One such example is given by perception, which presents many psychophysical power-law relations between the magnitude of the physical stimulus and the perceived (subjective) quantity [5]. This enhances the utility of perception capability by broadening the spectrum. The explanation is in the physical phenomenon itself. Consider a sound wave traveling with constant speed v. If the organism can capture wavelengths from λ_1 to λ_2, $f = \frac{v}{\lambda} \in [\frac{v}{\lambda_2}, \frac{v}{\lambda 1}]$ follows a power law with $\alpha = 1$. As discussed with regard to Proposition 1, the power-law distribution maximizes the component-wealth domain and, in the present case, the reception of signals. In other words, power-law inequality maximized versatility.

The existence of power laws in perception suggests a pertinence, and is regarded as such by literature. This raises a fit, in advance, for complex systems with power laws to be analysed as sensors (or meta-sensors). (nao entendi os significados de pertinence e fit na sentença). For example, an group interested in hiking can be understood as a meta-sensor about hiking, and involved a community to find current good places, equipment, people, proper behavior, etc. The power law, i.e. "the scale-free trace" to use the complex network jargon, sweeps a wide range of types of engagement (regarding the concentration of resources). The higher the component wealth k the higher the engagement, but the lesser diversity is brought to the group by the participant [6]: as one allocates more resources (say time) in one system, it allocates less resources in outside systems. (não entendi a argumentação).

5 Discussion

A discussão poderia começar com algo mais genérico, e podemos rearranjar isso depois.

Our model does not require the components to be "self-interested". This is often required in this context (por quem?), and we understand that the formalization presented here shed insight in the reasons why and the ways that self-interested agents organize themselves with extensive incidence of power laws. These principles can be thought of as laws met in a broad class of phenomena. The resulting axioms can be viewed as statistical tendencies that hold with such universality that they can be seen as laws that govern phenomena ubiquitously. Deviations from $p(k)$ tend to be transient or require effort, the expenditure of energy (as work in needed to reduce entropy), or be imposed by harsh conditions.

Power laws are very frequent in empirical data, which already grants its place among the study of natural phenomena. Different explanations are given for the many cases where they are found, with most common denominators being fields such as fractals, chaos, networks and unifying models e.g. as given by the Theory of Self-Organized Criticality [2]. Cases in more traditional fields, such as in Newtonian mechanics with the gravitational force relation to distance with $\alpha = 2$ (if masses are fixed), are usually not mentioned in the specialized literature. In other words: there seems not to be a unifying theory of why power laws express such ubiquitous spectrum of relationships.

If the framework in Section 2 is valid for all cases where power laws are found, the consideration of power laws as tied to natural phenomena *per se* goes a step further. Phenomenologically we have a fit of the analysis. (o que quer dizer?) That is: if there is a power law, the analysis developed in Section 2.2 holds and the power law relation can be regarded as an equanimous distribution of resources in α dimensions. If the acting of the "laws" given by Propositions 0, 1 and 1 is fundamentally what is taking place. We advocate that this framework deepens the understanding of potentially all power law incidences. The core meaning seems to emanate, with the simplest formalism, from the object sizes in the isotropic space shown in Section 3.1. That is a reasonable geometric abstraction for one to grasp the power law inequality originated from a uniform distribution of fundamental resources through component wealth. Also, relating power laws to the environment is the most effective way we found to make explicit both the axiomatic and the phenomenological backbones of power law ubiquity described in Section 2.

We hypothesize that $\alpha \approx 2$ is due to two basic resources input in any system: components and their time, energy or engagement. All other resources end correlated to these two. We also hypothesize that deviations from $\alpha \approx 2$ are due to other resources less correlated to them or to any other nonlinearities in the relationship of resources.

6 Conclusions

The presented framework most importantly laid inequality ubiquitous through power laws which follow from simple assumptions. These assumptions can also be observed in any power-law incidence. Therefore, the interpretation of diverse natural and human phenomena are impacted and inequality is posed as a natural tendency. Immediate consequences were therefore scrutinized case-by-case. We hypothesize that inequality should be minimizable by the expenditure of energy and that the characteristic $\alpha = 2$ is a consequence of the existence of two types of independent resources. Further work should link these findings to individual fields and contextualize the general framework.

Acknowledgments

Financial support was obtained from the United Nations Development Program (UNDP, contract 2013/00056, project BRA/12/018); the Brazilian National Counsel of Technological and Scientific Development (CNPq, process 140860/2013-4, project 870336/1997-5, advisor: Osvaldo N. Oliveira Jr.) and FAPESP. Authors thank the Brazilian General Secretariat of the Republic Presidency (SG-PR) and all labMacambira.sf.net members for pursuing this and other developments.

Appendix: Power laws

A power law is a functional relationship between two quantities $p(k)$ and k in the form:

$$p(k) = Ck^{-\alpha} \tag{5}$$

where $k \in [k_L, k_R]$ and C is constant. There are four degrees of freedom in four characteristic variables: α, C, k_L and k_R. Suppose also $p(k)$ normalized so that $\int_{k_L}^{k_R} p(k) = 1$, i.e. the power law is fit to represent a probability density function. Assuming idealized phenomena:

$$\int_{k_L}^{k_R} p(k) = 1 \;\Rightarrow\; C = \frac{1-\alpha}{k_R^{1-\alpha} - k_L^{1-\alpha}} \tag{6}$$

The cumulative distribution function $P(k)$, the and median m, mean μ and variance σ^2 are:

$$\begin{aligned}
P(k) &= \int_{k_L}^{k} p(\tilde{k})d\tilde{k} = \frac{C}{1-\alpha}(k^{1-\alpha} - k_L^{1-\alpha}) = \frac{k^{1-\alpha} - k_L^{1-\alpha}}{k_R^{1-\alpha} - k_L^{1-\alpha}} \\
&\left(m : \int_{k_L}^{m} p(k)dk = \int_{m}^{k_R} p(k)dk = \frac{1}{2} \right) \\
m &= \sqrt[1-\alpha]{k_L^{1-\alpha} + \frac{1-\alpha}{2C}} = \sqrt[1-\alpha]{k_R^{1-\alpha} - \frac{1-\alpha}{2C}} = \sqrt[1-\alpha]{\frac{k_R^{1-\alpha} - k_L^{1-\alpha}}{2}} \\
\mu &= \int_{k_L}^{k_R} kp(k)dk = C\,\frac{k_R^{2-\alpha} - k_L^{2-\alpha}}{2-\alpha} = \left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{k_R^{2-\alpha} - k_L^{2-\alpha}}{k_R^{1-\alpha} - k_L^{1-\alpha}}\right) \\
\sigma^2 &= \left[\int_{k_L}^{k_R} k^2 p(k)dk = C\,\frac{k_R^{3-\alpha} - k_L^{3-\alpha}}{3-\alpha} = \left(\frac{1-\alpha}{3-\alpha}\right)\left(\frac{k_R^{3-\alpha} - k_L^{3-\alpha}}{k_R^{1-\alpha} - k_L^{1-\alpha}}\right) \right] - \mu^2
\end{aligned} \tag{7}$$

Often, $k_R \to \infty$ which implies: $\alpha > 1$ as a condition for convergence of $p(k)$ and for a finite median m. Also, the mean μ is finite only if $\alpha > 2$, and the variance σ^2 is finite only when $\alpha > 3$:

$$
\begin{aligned}
&if\ k_R \to \infty \Rightarrow \\
&\qquad if\ \alpha > 1 \ \Rightarrow \qquad C = \frac{\alpha - 1}{k_L^{1-\alpha}} \\
&\qquad\qquad\qquad P(k) = 1 - \left(\frac{k_L}{k}\right)^{\alpha-1} \\
&\qquad\qquad\qquad m = \sqrt[1-\alpha]{\frac{\alpha-1}{2C}} \qquad = k_L.\sqrt[\alpha-1]{2} \qquad (8) \\
&\qquad if\ \alpha > 2 \ \Rightarrow \qquad \mu = k_L^{2-\alpha}\frac{C}{\alpha-2} \qquad = k_L\frac{\alpha-1}{\alpha-2} \\
&\qquad if\ \alpha > 3 \ \Rightarrow \qquad \sigma^2 = k_L^{3-\alpha}\frac{C}{\alpha-3} - \mu^2 \quad = k_L^2\frac{\alpha-1}{(\alpha-3)(\alpha-2)^2}
\end{aligned}
$$

References

1. G. Deleuze, *Difference and repetition*. Columbia University Press, 1994.
2. D. Marković and C. Gros, "Power laws and self-organized criticality in theory and nature," *Physics Reports*, vol. 536, no. 2, pp. 41–74, 2014.
3. A. Clauset, C. R. Shalizi, and M. E. Newman, "Power-law distributions in empirical data," *SIAM review*, vol. 51, no. 4, pp. 661–703, 2009.
4. K. Xu, L. Liu, and X. Liang, "Complex networks: A mixture of power-law and weibull distributions," *arXiv preprint arXiv:0908.0588*, 2009.
5. M. R. Schroeder, *Fractals, chaos, power laws: Minutes from an infinite paradise*. Courier Corporation, 2012.
6. R. e. a. Fabbri, "Temporal stability in human interaction networks: sector sizes, topological prominence and time activity," *arXiv preprint arXiv:1310.7769*, 2013.