A simple model that explains why inequality is ubiquitous

Renato Fabbri◑*, Osvaldo N. Oliveira Jr.‡,

São Carlos Institute of Physics, São Carlos, São Paulo, Brazil

◑This author contributed with initial concepts, writing, formalism and this final document.
‡This author contributed with writing, formalism, this final document and senior authorship.
* fabbri@usp.br

Abstract

Inequality has always been a crucial issue for human kind, particularly concerning the highly unequal distribution of wealth, which is at the root of major problems facing humanity, including extreme poverty and wars. A quantitative observation of inequality has become commonplace in recent years with the expanding recognition that many natural and man-made systems can be represented as scale-free networks, whose distribution of connectivity obeys a power law. In this paper we introduce a simple model that explains the ubiquity of inequality, based on two simple assumptions applied to a generic system with numerous parts. In the first assumption we consider the components as being diverse, while the second assumption is a uniform distribution of resources for the components. This implies that the more resources are allocated per component, the less numerous are such components. The second assumption is a conservation law: the amount of resources is conserved through component wealth. This can be geometrically depicted as the distribution of object sizes in an n-dimensional Euclidean space. Applying these assumptions to a generic system results in a power-law distribution, whose exponent is the number of inputs that are independent from each other, i.e. the dimensionality of the allocated resources. Even though there is no restriction to the value of the exponent, in practice we observe that existing systems normally exhibit an exponent between 1.5 and 3.0. We indicate reasonable hypotheses for this limitation. Since these assumptions are easily justified based on established knowledge, the model proves unequivocally that inequality is ubiquitous. We also discuss ways to control this tendency to inequality, which is analogous to induce a decrease in entropy in a closed system by an external action.

Introduction

Inequality has always been at the focus of social studies for the obvious importance for humanity and quality of life. In some respects, inequality can be regarded as asymmetry, thus being at the center stage of science since symmetry is fundamental to cognition and knowledge in nearly all fields [1, 2]. In fact, the ubiquity of symmetry is recurrent in the literature, according to which human thinking presents explicit, basic symmetry operations; therefore the world is modelled with symmetry regardless of its presence. In recent years, inequality in distributions has been addressed quantitatively with more emphasis, particularly in large systems represented as networks. Of special importance are the scale-free networks, whose distribution of connectivity (number of edges per node) obeys a power law and whose examples include human interactions, friendships, wealth, connections among airports and synaptic count between neurons [3]. Power laws also govern perception as expressed through the Stevens law and other phenomena such as earthquake intensity and allometric relations of animal bodies. Incidence in basic physics is commonplace, e.g. in a Newtonian gravity force F is related to distance d through $F = G\frac{m_1 m_2}{d^2}$. It is also known as the Pareto law and the most canonical example seems to be the Zipf law [4], although the name "Pareto law" is often associated to power laws in the context of income or wealth [5,6].

We strove to keep this article as short and simple as possible, and the outline is as follows. Next subsection is dedicated to relating previous work with the content here presented. Section 1 presents both an axiomatic and phenomenological description of the model. Paradigmatic observations of the model are discussed in Section 2, while implications for wealth distributions are considered separately in Section 3. A case in which inequality is desirable is presented in Section 4, which is followed by concluding remarks.

Related work

Models with power laws for dealing with broad classes of phenomena have been extensively discussed [2,7,8], but we could not find in the literature a truly unified framework for interpreting any power law, such as presented in this paper. In fact, one may regard Self-Organized Criticality (SOC) as such a unified model, but SOC models presuppose that the system is a dynamical system [2], as is the case for other models [9]. Furthermore, most models are very specific to wealth (e.g. [10]), meaning money and properties, while the simple model here presented is envisioned as way more general, and confluent with potentially any other model for power laws. This is true for economic models in general and instances where power-law distributions are negated or restricted to special cases (in favor e.g. of the Boltzmann distribution) or to a portion of the population (e.g. 5-10% richest) [11]. Also, it is worth keeping in mind that not all models of wealth distribution result in power laws [12], and that here the concept of wealth is wider than that of money and possessions. One may think of knowledge or health as a type of wealth in the following sections.

We argue that power laws, and their inherent extreme inequality, result potentially always from a uniform distribution of the resources, but even if they do not result from such uniform distribution, it is at least in agreement with it. This is closely related to other models and even hinted by them, but not directly expressed as in the next section. For example, many fractal-related models that result in power laws are akin to the example in Section 2.1.

1 Formalization

Definition 1 *A complex system is a system in which the whole is more than the sum of the individual parts.*

The definition of a **complex system** and its components can otherwise be left open, given the broad range of definitions currently in use [13]. For our purposes the system is generic and can have any type of component.

Definition 2 *A* **resource** *is anything that is used by a complex system to subsist and communicate.*

In practice, a complex system is commonly delimited and specified through its components and the resources that it keeps. The resources are not restricted to any type and can be e.g. geometrical volumes, wave periods or currency values as illustrated in the following sections.

Definition 3 *A* **resource-based system** *is a complex system that has an underlying resource vital to their components and their interdependent roles.*

Definition 4 *The* **component wealth** k *is the amount of resources (of any kind) allocated to a component.*

We will use $p(k)$ to denote the fraction of components with component wealth k, i.e. $p(k)$ will denote the probability of selecting at random a component with an amount k of resources. The correspondence among probability, frequency and relative count is instrumental for the interpretative framework in this article, which is conveniently embedded in the notation.

1.1 Propositions and corollaries

Proposition 0 *There is diversity among the components of the resource-based system.*

If there is a large set of components, there is hardly any way to avoid diversity. Be it location, size, age, the way someone regards them, etc., distinctions arise (as does symmetry). This can be regarded as a statistical law, or even as a deeper truth. The mere existence of two objects imposes diversity, otherwise they would be both the same. Proposition 0 is required for assigning different amounts of resources for each component: if they are equal, by definition they have the same wealth.

Proposition 1 *Resources allocation in a resource-based system is uniform across the incident component wealth. This is expressed as a uniform distribution $p_U(k) = C$ of resources with respect to component wealth k.*

That is, resources are allocated democratically with respect to component wealth values, which has two major consequences: 1) the interval of incident component wealth is maximized with a constant resource C distributed; 2) the wealthier the components considered, the fewer they are.

Corollary 1 *The different resource inputs are combined in the α dimensions of the resources.*

This follows from the distinction of each resource dimension, i.e. the incomparability of these amounts, say λ_1 and λ_2, and the multiplicative relation they hold with the total resource: $E = \widetilde{C_1}.\lambda_1.\lambda_2 \equiv \widetilde{C_2}\lambda^{\alpha=2}$ with $\widetilde{C_X}$ constant. Note that $E = \widetilde{C_1}.\lambda_1 + \widetilde{C_2}.\lambda_2 \equiv \widetilde{C_3}\lambda^{\alpha=1}$, that is: a resource resulting from the summation of other two unidimensional resources is a unidimensional resource. Therefore, one needs to use the product, not the sum. For example, if the resources are *workers* and *time*, a final resource $E = 5\,workers + 4h = 9$ holds little if any information, while $E = 5\,workers\,.\,4h = 20\,workers.hours$ is a reasonable measure of resources in a canonical metric.

Corollary 2 *The equilibrium state implied by the validity of Propositions 0 and 1 and Corollary 1 is characterized by a power-law distribution $p(k)$ of components with component wealth k.*

From the propositions, the same amount C of resources is allocated across all quantities k of resources allocated per component. Therefore, the fraction $p(k)$ of components with component wealth k is $p(k) = \frac{C}{k^\alpha}$, where α is the dimensionality of the resources compared to the dimension of k. As the system continues to exist, and resources are continually allocated, deviations from the equitable distribution $p_U(k)$ of resources along component wealth k tend to be transient. It is worth emphasizing that $p_U(k)$ is usually not made explicit in the literature, but only the power-law distribution $p(k)$ of components with component wealth k.

Corollary 3 *The extension of allocation is* $[k_L, k_R]$ *with* k_L *often being 0 or 1 while* $k_R \approx C^{1/\alpha}$.

This follows from Proposition 1. If the allocation of resources is insensitive to component wealth, it should sweep all possible values, and these usually have a lower boundary by being a positive quantity of resources. Systems are usually considered as a set of components or the components in which certain resources couple them, thus it is reasonable to assume $k_L = 0$ or $k_L = 1$.

The distributions $p(k)$ and $p_U(k)$ are also bounded above when the component wealth reaches $k_R \approx C^{1/\alpha}$, the amount of resources uniformly distributed along component wealth. In empirical data, k_R can vary considerably, due to nonlinearity of the resources scaling (most often $k_R < C^{1/\alpha}$) and to self-interested agents (most often $k_R > C^{1/\alpha}$). Power laws are often reported to strictly conduct empirical data only for a (broad) portion of component wealth range.

Corollary 4 *The upper limit* k_R *of the observed allocation of resources is an estimate of the amount of resources* C *equally distributed along component wealth* ($C^{1/\alpha} \approx k_R$).

This follows from Corollary 3.

Corollary 5 $N.p(1)$ *provides another estimate of the amount of resources* C *equally distributed along component wealth, where* N *is the number of components with* $k = 1$ ($C \approx N.p(1)$).

Corollary 6 *An estimate for* C *can also be found by* α, k_L *and* k_R *through Equation A2 in the Appendix.*

Corollary 7 *The dimensionality of the allocated resources is the scaling factor* α.

This last corollary follows from box counting or, most easily, through wave-like reasoning about power laws, discussed in the next section.

1.2 Phenomenological approach: power laws are consistent with the propositions

The propositions from the last subsection lead to power laws, and now we wish to verify whether phenomena governed by power laws are consistent with the propositions. In order to do that, we probe the definitions, propositions and corollaries from a more phenomenological standpoint, such as provided by empirical data. We take examples of 1D and 2D cases, and generalizations are straightforward.

(1D case) Consider a constant speed v for a wave propagation with no dispersion. Recall that the number of oscillations f per unit time is inversely proportional to the cycle length λ (the period). In usual notation $f = \frac{v}{\lambda}$. The constant speed v implies a power law between f and λ (with $\alpha = 1$ and $C = v$). This is the core insight: the general case of a power law can be interpreted as resulting from a constant amount C of fundamental resources with dimensionality α being homogeneously distributed across component wealth k and resulting in $p(k)$ of such components.

(2D case and beyond) Now let $f = \frac{v=C}{\lambda_1.\lambda_2}$, that is, the frequency of occurrence (or the probability of choosing such event at random) goes with the inverse of two periods while the speed is constant. If $\lambda_1 == \lambda_2 == \lambda$, then $f = \frac{v}{\lambda^2}$. In other words, the density is given by an amount v (or a quantity C of resources) in a hypercube of edge λ (or component wealth k) and $\alpha = 2$ dimensions. This reasoning yields the relative count, fraction or probability $p(k)$ of components with such component wealth k and corresponding hypercubes k^α. Let C_i be the amount of resources allocated at specific resource costs, and assume linearity $\lambda_{1,i} = c.\lambda_{2,i}$ such that $p(k_i) = \frac{C}{C_i} = \frac{C}{\lambda_{1,i}.\lambda_{2,i}} = \frac{C}{c\lambda_{2,i}^2} = \frac{\widetilde{C}}{\lambda_{2,i}^2} \equiv \frac{v}{\lambda^2}$. Therefore, we only access $\lambda_1 == \lambda_2$ and a "normalized resource C" allocated uniformly by the environment across component wealth: higher component wealth implies less numerous components.

The cases above were related to phenomena represented by power laws. For power-law distributions involving random variables, assume random variables $\lambda_{1,i}$ and $\lambda_{2,i}$ as possessing a uniform distribution of resources along component wealth themselves: $p_{\lambda_1}(\lambda_{1,i}) = \frac{C_i}{\lambda_{1,i}^{\alpha 1}}$ and $p_{\lambda_2}(\lambda_{2,i}) = \frac{C_i}{\lambda_{2,i}^{\alpha 2}}$. The product distribution of $\lambda_{3,i}^2 = \lambda_{1,i}.\lambda_{2,i}$ is then:

$$p_{\lambda_3^2}(\lambda_{3,i}^2) = \int_{-\infty}^{\infty} p_{\lambda_1}(\lambda_{1,i}) p_{\lambda_2}(\lambda_{3,i}^2/\lambda_{1,i}) \frac{1}{|\lambda_{1,i}|} d\lambda_1 \tag{1}$$

$$= \int_{-\infty}^{\infty} p_{\lambda_2}(\lambda_{2,i}) p_{\lambda_1}(\lambda_{3,i}^2/\lambda_{2,i}) \frac{1}{|\lambda_{2,i}|} d\lambda_2 \tag{2}$$

whose integrals have non-trivial limits. Note that $p_{\lambda_3^2}(\lambda_{3,i}^2) = \frac{N_i}{N} = p(\lambda_{x,i})$, where $p(\lambda_{x,i})$ is the probability that a component has component wealth $\lambda_{x,i}$ and N_i is the number of such components.

Together with the general model of Section 1.1, this analysis amounts to a quasi-"if and only if" mathematically grounded interpretation of any power-law relation.

1.3 Introduction to an estimate of α

Consider a generic problem in which a System (S) provides an Output (O) depending on the Input (I) it receives. Admit the following assumptions:

1. S is made of a number of components that are not all equal among themselves. That is to say, there is diversity in the components, in accordance with Proposition 0.

2. Distribution of resources is uniform with respect to the "size" of the components (component wealth) as in the geometric isotropic case of a house to be discussed in Section 2.1. This reflects Proposition 1

3. There may be several inputs, but for each input the amount of resources furnished to the System can be considered the same, in accordance with Corollary 1.

The Output (O) is assumed to be the performance (or richness) in terms of the components of S. If there are α types of independent inputs, i.e. resources in α dimensions, the Output is the product of these inputs and should be $O = C.k^{-\alpha}$, where k is a one-dimensional component wealth observation, as shown in Section 1.2.

Let us illustrate with a hypothetical case. Many tasks are to be done in a factory. Assume three inputs: number of workers N, working hours W and efficiency E. Sophisticated activities are developed by teams, so we assume that all concentrations $C.k^3 = n.w.e$ of these resources might be found in working groups. Assuming the tasks change all the time, all concentrations are important. By Proposition 1 they are equally important beforehand in the sense that resources will be spread uniformly over $[k_L, k_R]$ with $p_U(k) = \frac{1}{k_R - k_L}$. The number of such components, therefore, decreases with

$C.k^3 = n.w.e$, the component wealth with three dimensions. In other words, $p(k) = \frac{C}{k^3}$, that is, the fraction $p(k)$ of components (working groups) with component wealth k decreases with k^3 and distributes the same amount of resources $C = \frac{N.W.E}{k_R - k_L}$ along k.

The Output has therefore a power-law dependence on k with exponent α. Empirical values of α are normally between 1.5 and 3, from which one infers that there are fundamentally between two and three types of independent inputs to both natural and artificial systems. This can be envisioned, for a factory S, through resources of employees and time (the inputs I), with the fundamental resource being $person.hour$, the canonical resource for industrial production since the rise of Taylorism and taken to extreme limits in Fordism [14].

2 Paradigmatic observations of the propositions

The interpretation of power laws presented in Section 1 has an impact on the understanding of diverse systems. Figure 1 is a fundamental representation of the model presented, and should be kept in mind for reference while selected phenomena are scrutinized in this section.

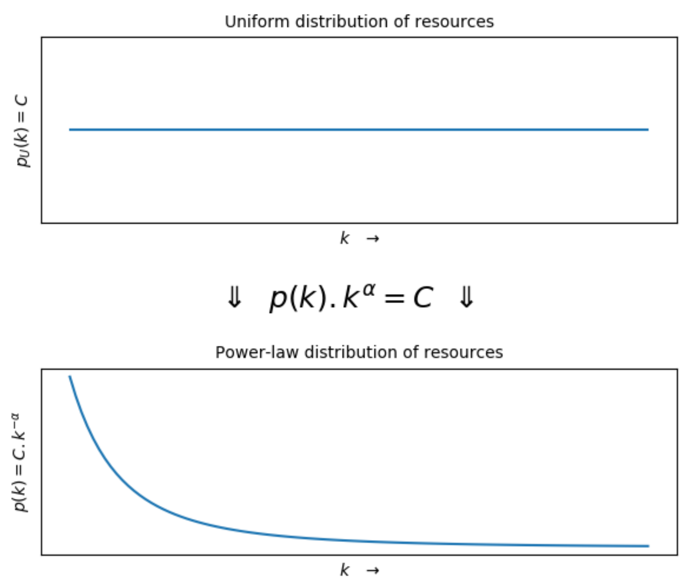

Fig 1. From uniform to power law. Let the system be insensitive to the amount of resources each component holds, i.e. let the amount of resources be conserved throughout the component wealth k. This assumption results in a power law if carried on in the terms described in Section 1.

2.1 Object sizes in a house

Pick the size of a person (or the size of his/her hand, or arm) as a measure unit l for length, pick $m \in (0, 1)$. In a house, there probably are more objects of volume $\approx (m.l)^3$ than those of volume $\approx l^3$. Furthermore, as we know nothing about the house, we may assume that the chance ρ of finding an object within an arbitrary cube of volume $\approx (m.l)^3 \pm \beta(ml)$ is the same as finding an object fitting an arbitrary cube of volume $\approx l^3 \pm \beta(l)$ for $\beta \in [0, 1]$. The house has a fixed volume so there are m^{-3} more of the smaller cubes and therefore m^{-3} more objects of such volume. The result is a power-law distribution of object volumes related to the length l with $\alpha = 3$: $p(l) = C.l^{-3}$. If the objects are mutually exclusive, α is probably lower, as the number of smaller cubes decreases considerably.

This example holds a geometrical interpretation of the formalism presented in the previous section. It also might be assumed true even if there is no isotropy and can be considered a "best guess" if total ignorance about the system is assumed. Extreme choices of l and m might exhibit spurious observations as the result of a bad fit of the scale for the analysis.

2.2 Scale-free complex networks

In a network, one has essentially E edges and N vertices. Assuming linearity of resources:

$$Resources = C_1 N + C_2 E \approx C_2 E = C_2 N \frac{E}{N} = C_2 \frac{N\overline{k}}{2} \tag{3}$$

where C_1 and C_2 are constants and $\overline{k}$ is the average degree, i.e. the mean of the number of edges attached to each vertex. If the nodes are not connected, they have minimized influence, also $N \ll E$ in most cases, thus the $C_1 N$ term is ignored. Then:

$$p_{E_i}(k_i) = \frac{C}{resources_i} = \frac{C}{C_2 E_i} = \frac{C/C_2}{N_i \frac{E_i}{N_i}} = \frac{2C/C_2}{N_i k_i} \equiv \frac{v}{\lambda_1 \lambda_2} \tag{4}$$

This can be regarded as the case of social networks where the number of agents allocated N_i and the time each of them put (related to k_i) are seen as the resource ($individuals.time$), and $\alpha \approx 2$ according to empirical evidence [3].

2.3 Natural laws

Power laws are very recurrent in empirical data. This already grants its place in the study of natural phenomena. Different explanations are given for the many cases where they are found, with most common denominators being fields such as fractals, chaos, networks and unifying models e.g. as given by the Theory of Self-Organized Criticality [2]. Cases in more traditional fields, such as Newtonian mechanics, where gravitational force relates to distance with $\alpha = 2$ (if masses are fixed), are usually not mentioned in specialized literature about power laws, but regarded as very special phenomena e.g. related to mediation by massless particles. In other words: there seems not to be a unifying theory of why power laws express such an ubiquitous spectrum of relationships.

If the framework in Section 1 is valid for all cases where power laws are found, the consideration of power laws as tied to natural phenomena *per se*, goes a step further. That is: if there is a power law, the analysis developed in Section 1.2 holds and the power law relation can be regarded as an equitable distribution of resources in α dimensions. If the acting of the "laws" given by Propositions 0 and 1 is fundamentally what is taking place, that should depend on phenomena and standpoint. We advocate that this framework deepens the understanding of potentially all power-law incidences and is usually consonant with more explicit and intuitive relations of the system, its components and the context. The core meaning seems to emanate, with the simplest formalism, from the object sizes in the isotropic space described in Section 2.1. That is a reasonable geometric abstraction for one to grasp the power-law inequality originated from a uniform distribution of fundamental resources through component wealth. Also, relating power laws to the environment is the most effective way we found to make explicit both the axiomatic and the phenomenological backbones of the power-law ubiquity described in Section 1. We hypothesize that $\alpha \approx 2$ is due to two basic resources input in any system: components and their time, energy or engagement. All other resources seem correlated to these two. We also hypothesize that deviations from $\alpha \approx 2$ are due to other resources less correlated to them or to any other nonlinearities in the relationship of resources.

Finally, this document addresses both physical laws (e.g. Newtonian gravitation and Coulomb's law) and statistical power-laws, and this issue deserves clarification. Let an ideal light source be located at a point and let it be omnidirectional. Imagine a spherical shell centered at such light source, with radius r. The power W of such light source is incident in the surface of the sphere with power $W/(4\pi r^2)$ per unit area. This same scheme may be applied to a number of cases and phenomena, making evident power-law distributions entailed by natural laws that are expressed as power-laws.

3 Implications for wealth distribution

One manifestation of inequality through power laws, which is most fundamental to daily experience in society, is found in the discrepant wealth distribution worldwide. There are continuing efforts to deal with this issue, usually advocating ways to minimize "social inequality". Considering the framework presented within this article:

- such inequality is a natural tendency that follows from Propositions 0 and 1 and is in accordance with the phenomenological discussion of Section 1.2.
- Deviations from a power-law distribution of wealth should require work. Occasional deviations are part of the statistical aspect of the phenomena involved, but the maintenance of a pattern different from the power law derived from the resources distribution are fated to require the expenditure of energy.
- Both deviations of power laws towards a more equitable or towards a more unequitable distribution should be ephemeral or require work.

In particular, the (publicized) homogeneity of earnings in public institutions, and the (publicized) distribution of wealth in whole countries, reveal that there is indeed efforts to minimize the strong inequality imposed by power laws. Additionally, publicized data should be regarded with extra care and skepticism, as they do not always present the expected power-law distributions.

In summary, power-law inequality seems to be an ubiquitous tendency and a consequence of a distribution of wealth equitable and insensitive along wealth allocated to each component. This implies the necessity of "work" for equalization. Also, we observe that the higher the k_L of equation A1, the higher all the probabilistic mass will be located, which implies greater wealth of the wealthier, the hubs or the "elite". In other words, the richer the least rich, the richer the more rich.

4 When inequality is desirable: the case of sensors and meta-sensors

Inequality is generally associated with negative implications, but unequal distributions may serve important, noble purposes, as in the example about perception we present here for the sake of the argument. Perception presents many psychophysical power-law relations between the magnitude of the physical stimulus and the perceived (subjective) quantity [7]. This can be attributed to the utility of perception, which is enhanced upon broadening of the spectrum of the perceived phenomenon. Another explanation is on the physical phenomena itself. Consider a sound wave traveling with constant speed v. If the organism is susceptible to wavelengths from λ_1 to λ_2, $f = \frac{v}{\lambda} \in [\frac{v}{\lambda_2}, \frac{v}{\lambda 1}]$ follows a power law with $\alpha = 1$. As made explicit in the discussion of Proposition 1, the power-law distribution maximizes the component-wealth incident domain and, therefore, the reception of signals. In other words, the power-law inequality maximizes versatility.

The existence of power laws in perception (and other sensing mechanisms) is advantageous, as acknowledged in the literature [15]. One may thus wonder whether complex systems with power-law behavior could be regarded as useful meta-sensors. By way of illustration, let us consider an interest group about hiking, which can be understood as a meta-sensor to find suitable places, equipment, people, proper behavior, etc. The power law, i.e. "the scale-free outline" to use the complex network jargon, sweeps a wide range of engagements (regarding the concentration of resources). The higher the component wealth k, the higher the engagement, but the lesser diversity is brought to the group by the participant [16]: as one allocates more resources (say time) in one system, it allocates less resources in outside systems.

5 Concluding Remarks

The model presented here is based on simple propositions, and does not require the system components to be "self-interested", unlike many other models leading to power laws (see a summary in [2, 4]. Therefore, the formalization sheds light into in the reasons why and the ways that self-interested (and not self-interested) agents organize themselves with extensive incidence of power laws. The propositions in Section 1.1 can be thought of as laws met in a very broad class of phenomena and follow from the system complexity. Because there are so many mechanisms into play, we had to adopt assumptions that are very unlikely to be false. The resulting axioms can be understood as statistical tendencies that hold with such universality that they can be seen as laws that govern phenomena ubiquitously. Furthermore, the assumptions in the model lead to a framework analogous to the laws of thermodynamics: conservation of resources and a time arrow pointing to inequality.

It is also worth emphasizing that physical laws expressed in the form of power laws and phenomena involving power-law distributions were treated on an equal footing. It remains to be seen whether such universality will be kept when specific problems are addressed. Still with regard to generality, if a complex system behaves according to the model proposed, then its energy will be related to the constant C of equation A1 for the distribution. Though we have not in any way treated the physical phenomena in complex systems in detail, one may speculate that if the Universe is composed of systems behaving as described in our model it might be possible to use the power-law concepts to seek for an explanation of dark matter or dark energy.

Perhaps the most important implication of our findings is that inequality - implicit in power-law behavior - is a natural tendency for diverse natural and human phenomena. Of course, there are many deviations from power laws caused by nonlinearities. But in the overwhelming number of systems obeying power laws, deviations tend to be transient or require effort. We hypothesize that inequality may be minimizable by the expenditure of energy and that the characteristic $\alpha \in [2, 3]$ is a consequence of the existence of two or three types of independent resources. Further work should link these findings to individual fields and contextualize the general framework.

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Appendix: Power laws

A power law is a functional relationship between two quantities $p(k)$ and k in the form:

$$p(k) = Ck^{-\alpha} \tag{A1}$$

where $k \in [k_L, k_R]$ and C is constant. There are four degrees of freedom in four characteristic variables: α, C, k_L and k_R. Suppose also $p(k)$ normalized so that $\int_{k_L}^{k_R} p(k) = 1$, i.e. the power law is fit to represent a probability density function. Assuming idealized phenomena:

$$\int_{k_L}^{k_R} p(k) = 1 \;\Rightarrow\; C = \frac{1-\alpha}{k_R^{1-\alpha} - k_L^{1-\alpha}} \tag{A2}$$

The cumulative distribution function $P(k)$, median m, mean μ and variance σ^2 are:

$$\begin{aligned}
P(k) &= \int_{k_L}^{k} p(\tilde{k})d\tilde{k} = \frac{C}{1-\alpha}(k^{1-\alpha} - k_L^{1-\alpha}) = \frac{k^{1-\alpha} - k_L^{1-\alpha}}{k_R^{1-\alpha} - k_L^{1-\alpha}} \\
&\left(m : \int_{k_L}^{m} p(k)dk = \int_{m}^{k_R} p(k)dk = \frac{1}{2} \right) \\
m &= \sqrt[1-\alpha]{k_L^{1-\alpha} + \frac{1-\alpha}{2C}} = \sqrt[1-\alpha]{k_R^{1-\alpha} - \frac{1-\alpha}{2C}} = \sqrt[1-\alpha]{\frac{k_R^{1-\alpha} - k_L^{1-\alpha}}{2}} \\
\mu &= \int_{k_L}^{k_R} kp(k)dk = C\,\frac{k_R^{2-\alpha} - k_L^{2-\alpha}}{2-\alpha} = \left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{k_R^{2-\alpha} - k_L^{2-\alpha}}{k_R^{1-\alpha} - k_L^{1-\alpha}}\right) \\
\sigma^2 &= \left[\int_{k_L}^{k_R} k^2 p(k)dk = C\,\frac{k_R^{3-\alpha} - k_L^{3-\alpha}}{3-\alpha} = \left(\frac{1-\alpha}{3-\alpha}\right)\left(\frac{k_R^{3-\alpha} - k_L^{3-\alpha}}{k_R^{1-\alpha} - k_L^{1-\alpha}}\right)\right] - \mu^2
\end{aligned} \tag{A3}$$

Often, $k_R \rightarrow \infty$, $\therefore$

$$\begin{aligned}
&if\ k_R \rightarrow \infty \Rightarrow \\
&\qquad if\ \alpha > 1 \;\Rightarrow \qquad C = \frac{\alpha-1}{k_L^{1-\alpha}} \\
&\qquad\qquad\qquad\qquad P(k) = 1 - \left(\frac{k_L}{k}\right)^{\alpha-1} \\
&\qquad\qquad\qquad\qquad m = \sqrt[1-\alpha]{\frac{\alpha-1}{2C}} \qquad = k_L . \sqrt[\alpha-1]{2} \\
&\qquad if\ \alpha > 2 \;\Rightarrow \qquad \mu = k_L^{2-\alpha}\frac{C}{\alpha-2} \qquad = k_L \frac{\alpha-1}{\alpha-2} \\
&\qquad if\ \alpha > 3 \;\Rightarrow \qquad \sigma^2 = k_L^{3-\alpha}\frac{C}{\alpha-3} - \mu^2 \qquad = k_L^2 \frac{\alpha-1}{(\alpha-3)(\alpha-2)^2}
\end{aligned} \tag{A4}$$

References

1. Deleuze G. Difference and repetition. Columbia University Press; 1994.

2. Marković D, Gros C. Power laws and self-organized criticality in theory and nature. Physics Reports. 2014;536(2):41–74.

3. Clauset A, Shalizi CR, Newman ME. Power-law distributions in empirical data. SIAM review. 2009;51(4):661–703.

4. Newman ME. Power laws, Pareto distributions and Zipf's law. Contemporary physics. 2005;46(5):323–351.

5. Adamic LA. Zipf, power-laws, and pareto-a ranking tutorial. Xerox Palo Alto Research Center, Palo Alto, CA, http://ginger hpl hp com/shl/papers/ranking/ranking html. 2000;.

6. Reed WJ. The Pareto, Zipf and other power laws. Economics Letters. 2001;74(1):15–19.

7. Schroeder MR. Fractals, chaos, power laws: Minutes from an infinite paradise. Courier Corporation; 2012.

8. Fabbri R, Pisani MM. Egalitarian aspects of scale-free networks Cereus UniRG; doi:10.18605/2175-7275/cereus.v10n4p98-106; 2018.

9. Slanina F. Inelastically scattering particles and wealth distribution in an open economy. Physical Review E. 2004 Apr 14;69(4):046102.

10. Bouchaud JP, Mézard M. Wealth condensation in a simple model of economy. Physica A: Statistical Mechanics and its Applications. 2000 Jul 15;282(3-4):536-45.

11. Sinha S, Chatterjee A, Chakraborti A, Chakrabarti BK. Econophysics: an introduction. John Wiley & Sons; 2010 Nov 22.

12. Dragulescu A, Yakovenko VM. Statistical mechanics of money. The European Physical Journal B-Condensed Matter and Complex Systems. 2000 Oct 1;17(4):723-9.

13. Holland JH. Complexity: A very short introduction. OUP Oxford; 2014.

14. Guillén MF. Models of management: Work, authority, and organization in a comparative perspective. University of Chicago Press; 1994.

15. Roederer JG. The physics and psychophysics of music: an introduction. Springer Science & Business Media; 2008.

16. Fabbri R, Fabbri R, Antunes DC, Pisani MM, de Oliveira Junior ON. Temporal stability in human interaction networks. Physica A: Statistical Mechanics and its Applications. 2017 Nov 15;486:92-105.