A simple model that explains why inequality is ubiquitous

RENATO FABBRI AND OSVALDO N. OLIVEIRA JR.
São Carlos Institute of Physics, University of São Paulo, CP 369, 13560-970 São Carlos, SP, Brazil

October 31, 2016

Abstract

Inequality has always been a crucial issue for human kind, particularly concerning the highly unequal distribution of wealth, which is at the root of major problems facing humanity, including extreme poverty and wars. A quantitative observation of inequality has become commonplace in recent years with expanding recognition that many natural and man-made systems can be represented as scale-free networks, whose distribution of connectivity obeys a power law. These networks may be generated by the preferential attachment for the nodes, within the so-called rich-gets-richer paradigm. In this letter we introduce a simple model that explains the ubiquity of inequality, based on three simple assumptions applied to a generic system with numerous parts. The first assumption is the diversity of the components. The second assumption is a uniform distribution of resources with respect to the resources in individual components. The third assumption is that the amount of each resource input to the system is fixed. This implies that the more resources are allocated per component, the less numbered are such components, with the conservation of the amount of resources distributed through $component\ wealth = \frac{resources}{component}$ sweep. The second and third assumptions are conservations laws: energy is conserved through each resource and through component wealth. This can be geometrically described by the distribution of object sizes in a 3D space, where each dimension is assumed to be isotropic. Applying these assumptions to a generic system results in a power-law distribution, whose coefficient is the number of inputs that are independent from each other, i.e. the dimensionsionality of the allocated resources. Even though there is no restriction to the value of the coefficient, in practice we observe that existing systems normally exhibit a coefficient between 1.5 and 3.0. With our simple model it is not possible to determine whether this limitation in the coefficient values arises from a fundamental principle, but we indicate reasonable hypotheses. The assumptions in the model lead to a framework analogous to the laws of thermodynamics: conservation of resources and a time arrow pointing to inequality. Since these assumptions are easily justified based on established knowledge, the model proves unequivocally that inequality is ubiquitous. We also discuss ways to control this tendency to inequality, which is analogous to a decrease in entropy in a closed system induced by an external action.

Keywords: power laws, fundamental theory, complex systems, complex networks, anthropological physics

1 Introduction

Inequality has always been at the focus of social studies for the obvious importance for humanity and quality of life. In some respects, inequality can be regarded as asymmetry, thus being at the center stage of science since symmetry is regarded as fundamental to cognition and knowledge in nearly all fields [**?**, **?**]. In fact, the ubiquity of symmetry is reflected in the recurrent mentioning in the literature that human thinking presents explicit, basic symmetry operations; therefore we model the world with symmetry regardless of its presence. In recent years, inequality in distributions has been addressed quantitatively, particularly in large systems represented as networks [**?**]. Of special importance are the scale-free networks [?], whose distribution of connectivity (number of edges per node) obeys a power law [?]. Examples include distribution of human wealth, interactions, friendships, connections among airports, synaptic count among neurons, earthquake intensity and allometric relations of animal bodies [?]. Power laws also govern perception, as in Webner-Fechner and Stevens laws. Some advocate that data are better fitted with superposition of a power law distribution and a Weibull distribution [**?**]. The most canonical law examples seem to be Pareto and Zipf laws. Examples in basic physics are numerous, e.g. in a Newtonian context force F is related to distance d through gravity ($F = G\frac{m_1\, m_2}{d^2}$) with $\alpha = 2$ and force is related to acceleration ($F = m\, a^1$) with $\alpha = -1$.

Models with power laws for dealing with broad classes of phenomena have been extensively discussed [**?**, **?**], but a unified framework for interpreting any power law, such as discussed in this letter, was not found in the literature. We demonstrate that the extreme inequality of power laws results from a uniform distribution with respect to the resources allocated to each component.

Section 2 presents both an axiomatic and phenomenological description of the model. Section 3 exposes separately a geometric and a productive system paradigms. Especial cases are considered in Section 4, which is followed by concluding remarks. The most elementary mathematical description of power-law distributions is outlined in the Appendix.

2 Formalization

Definition 1 *In a complex system the whole is more than the sum of the individual parts.*

The definition of a **complex system** and its components can otherwise be left open, given the broad range of definitions currently in use. For our purposes the system is generic and can have any type of component.

Definition 2 *A* **resource** *is anything that is used by a complex system to subsist and communicate.*

In practice, a complex system is commonly delimited, specified through its components and the resources that keep it. The resources are not restricted to any type and can be, for example, geometrical volumes, wave cycles or currency values.

Definition 3 *A* **resource-based system** *is a complex system that has an underlying resource vital to their components and their interdependent roles.*

Definition 4 *The* **component wealth** *k is the amount of resources (of a certain kind) allocated to a component.*

We will use $p(k)$ to denote the fraction of components with component wealth k. Likewise, the same notation $p(k)$ will denote the probability of choosing a component with an amount k of resources. The context should make it clear which is the appropriate meaning. This correspondence among probability, frequency and relative count is instrumental for the interpretative framework in this article, which is conveniently embedded in the notation. It is also useful to take k as a wave period λ. Then, $p(k)$ is immediately understood as the frequency f. Note that k is a measure of the resources, not total resources (or total component wealth), which might be correlated to k and have k as one of the resources.

2.1 Propositions and corollaries

Proposition 0 *There is diversity among the components of the resource-based system.*

If there is a large set of components, there is hardly any way to avoid diversity. Be it location, size, age, the way someone regards them, etc., distinctions arise (as does symmetry). This can be regarded as a statistical law, or even as a deeper truth. The mere existence of two objects imposes diversity, otherwise they would be both the same.

Proposition 0 is required for assigning different amounts of resources for each component: if they are equal, by definition they have the same component wealth.

Proposition 1 *Resources allocation in a resource-based system is uniform across the incident component wealth. This is expressed as a uniform distribution $p_U(k)$ of resources with respect to component wealth k.*

That is, resources are allocated democratically with respect to component wealth values, which has two major consequences: 1) the interval of incident component wealth is maximized with respect to a constant resource C distributed; 2) the wealthier the components considered, the fewer they are. Indeed, the resources are evidence for the persistence of the complex system under consideration, i.e. given a set of components the system is often delimited by the allocated resources and components without resources allocated are discarded.

Corollary 1 *The different resource inputs are combined in the α dimensions of the resources.*

This follows from the distinction of each resource dimension, i.e. the incomparability of these amounts, say λ_1 and λ_2, and the multiplicative relation they hold with the total resource: $E = \widetilde{C_1}.\lambda_1.\lambda_2 \equiv \widetilde{C_2}\lambda^{\alpha=2}$ with $\widetilde{C_X}$ constant. Note that $E = \widetilde{C_1}.\lambda_1 + \widetilde{C_2}.\lambda_2 \equiv \widetilde{C_3}\lambda^{\alpha=1}$, that is: a resource resulting from the summation of other two unidimensional resources is a unidimensional resource and these resources are not of distinct dimensions. For example, if the resources are *components* and *time*, a final resource $E = 5\,components + 4h = 9$ holds little if any information, while $E = 5\,components\,.\,4h = 20$ *components.hours* is a reasonable measure of resources in a canonical metric.

Corollary 2 *The equilibrium state implied by the validity of Propositions 0, 1 and 1 (propositions 0 and 1?) is characterized by a power-law distribution $p(k)$ of components with component wealth k.*

From the propositions, the same amount C of resources is allocated across all quantities k of resources allocated per component. Therefore, the fraction $p(k)$ of components with component wealth k is $p(k) = \frac{C}{k^\alpha}$, where α is the dimensionality of the resources compared to the dimension of k. As the system continues to exist, and resources are continually allocated, deviations from the equanimous distribution $p_U(k)$ of resources along component wealth k tend to be transient. It is worth emphasizing that $p_U(k)$ is usually not made explicit in the literature, but only the power-law distribution $p(k)$ of components with component wealth k.

Corollary 3 *The extension of allocation is $[k_L, k_R]$ with R_L often 0 or 1 and $k_R \approx C$.*

This follows from Proposition 1. If the allocation of resources is insensitive to component wealth, it should sweep all possible values, and these usually have a lower boundary by being a positive quantity of resources. Systems are usually considered as a set of components or the components in which certain resources couple them, thus it is reasonable to assume $k_L = 0$ and $k_R = 1$.

The distributions $p(k)$ and $p_U(k)$ are also bounded above when the component wealth reaches $k_R \approx C$, the amount of resources uniformly distributed along component wealth. In empirical data, k_R can vary considerably, due to nonlinearity of the resources scaling (most often $k_R < C$) and to self-interested agents (most often $k_R > C$). Power laws are reported to conduct empirical data sovereignly and often are strict only for a (broad) portion of component wealth range. (não entendi "sovereignly")

Corollary 4 *The upper limit k_R of the observed allocation of resources is an estimate of the amount of resources C equally distributed along component wealth ($C \approx k_2$).*

This follows from Corollary 3.

Corollary 5 $p(1)$ *provides another estimate of the amount of resources C equally distributed along component wealth ($C \approx N.p(1)$, where N is the number of components).*

Corollary 6 *An estimate for C can also be found by α, k_L and k_R through Equation 6. (as equações não foram numeradas?)*

Corollary 7 *The dimensionality of the allocated resources is the scaling factor α.*

This last corollary follows from box counting or, most easily, through wave-like reasoning about power laws, discussed in the next section.

2.2 Phenomenological approach: power laws are consistent with the assumptions

We probe the definitions, propositions and corollaries above from a more phenomenological standpoint, such as provided via data and descriptive models. Suppose there is a power law relation in empirical data or driven from specific knowledge. Is the conceptualization from the last section still helpful? We provide here a general, mathematically grounded interpretation of any power law relationship, consistent with such assumptions. Both approaches pose a quasi-"if and only if" interpretation of power laws.

(1D case) Consider a constant speed v for a wave propagation (suppose linear media with no dispersion). Recall that the number of oscillations f per unit time is inversely proportional to the cycle length λ (the period). In usual notation $f = \frac{v}{\lambda}$. The constant speed v implies a power law between f and λ (with $\alpha = 1$ and $C = v$). This is the core insight: the general case of a power law can be interpreted as resulting from a constant amount C of fundamental resources with dimensionality α being homogeneously distributed across component wealth k and resulting in $p(k)$ of such components.

(2D case and beyond) Now let $f = \frac{v=C}{\lambda_1.\lambda_2}$, that is, the frequency of occurrence (or the probability of choosing such event at random) go with the inverse of two periods while the speed is constant. If $\lambda_1 == \lambda_2 == \lambda$, then $f = \frac{v}{\lambda^2}$. In other words, the density given by an amount v (or a quantity C of resources) in a hypercube of edge λ (or component wealth k) and $\alpha = 2$ dimensions. This reasoning yields the relative count, fraction or probability $p(k)$ of components with such component wealth k and corresponding hypercubes k^α. Let C_i be the amount of resources allocated at specific resource costs, say $\lambda_{1,i}$ and $\lambda_{2,i}$ to continue with the wave argumentation, and let us assume linearity $\lambda_{1,i} = c.\lambda_{2,i}$ such that $p(k_i) = \frac{C}{C_i} = \frac{C}{\lambda_{1,i}.\lambda_{2,i}} = \frac{C}{c\lambda_{2,i}^2} = \frac{\widetilde{C}}{\lambda_{2,i}^2} \equiv \frac{v}{\lambda^2}$. Therefore, we only access $\lambda_1 == \lambda_2$ and a "normalized resource C" allocated by the environment uniformly across component wealth: higher component wealth implies less numerous components.

The cases above were related to phenomena represented by power laws. For power-law distributions involving random variables, we assume random variables $\lambda_{1,i}$ and $\lambda_{2,i}$ as possessing a uniform distribution of resources along component wealth themselves: $p_{\lambda_1}(\lambda_{1,i}) = \frac{C_i}{\lambda_{1,i}^{\alpha_1}}$ and $p_{\lambda_2}(\lambda_{2,i}) = \frac{C_i}{\lambda_{2,i}^{\alpha_2}}$. Consider the product distribution of the $\lambda_{3,i}^2 = \lambda_{1,i}.\lambda_{2,i}$ variable:

$$p_{\lambda_3^2}(\lambda_{3,i}^2) = C_1.C_2.\lambda_3^{-2\alpha_2}\frac{(R_{1,R} - R_{1,L})^{\alpha_2-\alpha_1}}{\alpha_2 - \alpha_1} = C_1.C_2.\lambda_3^{-2\alpha_1}\frac{(R_{2,R} - R_{2,L})^{\alpha_1-\alpha_2}}{\alpha_1 - \alpha_2} \tag{1}$$

where R_R and R_L are the upper and lower limits of the corresponding resources in the system. Note that $p_{\lambda_3^2}(\lambda_{3,i}^2) = \frac{N_i}{N} = p(\lambda_{0,i})$, where $p(\lambda_{0,i}$ is the probability that a component has a measured component wealth $\lambda_{0,i}$ correlated to $\lambda_{3,i}^2$. The number N_i of components with component wealth $\lambda_{3,i}^2$ and the total number of components N are conserved for $\lambda_{0,i}$. If one assumes $\lambda_{0,i} \propto \lambda_{3,i}$, then $p(k_{0,i}) \propto k_{0,i}^2$.

3 Paradigmatic observations of the propositions

3.1 Object sizes in your house or elsewhere

Pick your size (or the size of your hand, or your arm) as a measure unit l for length, pick $m \in (0, 1)$. In your house, there probably are more objects of volume $\approx (m.l)^3$ than those of volume $\approx l^3$. Furthermore, as we know nothing about your house, we can assume that the chance ρ of finding an object fitting an arbitrary cube of volume $\approx (m.l)^3$ is the same as finding an object fitting an arbitrary cube of volume of volume $\approx l^3$. Your house has a fixed volume so there are m^{-3} more of the smaller cubes and therefore m^{-3} more objects of such volume. The result is a power-law distribution of object volumes related to the length l with $\alpha = 3$: $p(l) = C.l^{-3}$. If the objects are mutually exclusive, α is probably lower, as the number of smaller cubes decreases considerably.

This example holds a geometrical interpretation of the formalism presented in the previous section. It also might be assumed true even if there is no isotropy and can be considered a "best guess" if total ignorance about the system is assumed. Extreme choices of l and m (and of dl and $dm = mdl$ for tolerance in size) will often exhibit spurious observations as the result of a bad fit of the scale for the analysis.

3.2 Workers in a factory

Consider a generic problem in which a System (S) provides an Output (O) depending on the Input (I) it receives. The following assumptions are established:

1. S is made of a number of components that are not all equal among themselves. That is to say, there is diversity in the components, in accordance with Proposition 0.
2. Distribution of resources is uniform with regard to the "size" of the components (component wealth) as in the geometric isotropic case of your house. This reflects Proposition 1
3. There may be several inputs, but for each input the amount of resources furnished to the System can be considered the same in accordance with Proposition 1.

The Output (O) is assumed to be the performance (or richness) in terms of the components of S. If there are α types of independent inputs, i.e. resources in α dimensions, the Output is the product of these functions should be $O = C.k^{-\alpha}$, where k is a one dimensional component wealth observation, as shown in Section 2.2.

The Output has therefore a power-law dependence on k with coefficient α. Empirical values of α are normally between 1.5 and 3, from which one infers that there are fundamentally between two and three types of independent inputs to both natural and artificial systems. This can be envisioned, for a factory S, through resources of employees and time, with the fundamental resource being $person.hour$, the canonical resource for industrial production since Taylorism and taken to extreme limits in Fordism.

Let us illustrate with a hypothetical case. Many, many tasks are to be done in a company. Assume three inputs: number of workers N, working hours W and efficiency E. Sophisticated activities are developed by teams, so we assume that all concentrations $C.k^3 = n.w.e$ of these resources might be found in working groups. Assuming the tasks change all the time, all concentrations are important. By Proposition 1 they are equally important beforehand in the sense that resources will be spread uniformly over $[k_L, k_R]$

with $p_U(k) = \frac{1}{k_R - k_L}$. The number of such components, therefore, decreases with $C.k^3 = n.w.e$, the component wealth with three dimensions. In other words, $p(k) = \frac{C}{k^3}$, that is, the fraction $p(k)$ of components (working groups) with component wealth k decrease with k^3 and distributes the same amount of resources $C = \frac{N.W.E}{k_R - k_L}$ along k.

4 Implications

The consequences of the interpretation of power laws presented in Section 2 are severe for understanding and dealing with diverse (if not almost all) phenomena. Selected cases are scrutinized in this section.

4.1 Scale-free complex networks

In a network, one has essentially E edges and N vertices. Assuming linearity of resources:

$$Resources = C_1 N + C_2 E \approx C_2 E = C_2 N \frac{E}{N} = C_2 \frac{N\bar{k}}{2} \tag{2}$$

where C_1 and C_2 are constants and $\bar{k}$ is the average degree, i.e. the mean of the number of edges attached to each vertex. As in the literature, nodes can be disregarded, since if the nodes are not connected, they have minimized influence. Therefore:

$$p_{E_i}(k_i) = \frac{C}{resources_i} = \frac{C}{C_2 E_i} = \frac{C/C_2}{N_i \frac{E_i}{N_i}} = \frac{2C/C_2}{N_i k_i} \equiv \frac{v}{\lambda_1 \lambda_2} \tag{3}$$

As N_i and k_i are both directly proportional to E_i, which is the fundamental resource, one can factor out another constant and consider the special case where $\lambda = \lambda_1 = \lambda_2$ and, consequently, $f = v/\lambda^2$. This is the case for a social network, the number of agents allocated N_i and the time each of them put (related to k_i) are seen as the primary resource (*individuals.time*), and $\alpha \approx 2$ as empirical evidenced [**?**].

Also:

$$p_{E_i}(k_i) = \frac{N_i}{N} = \frac{2C/C_2}{N_i k_i} \Rightarrow \frac{N_i^2}{N^2} = 2\frac{C}{N.C_2} k_i^{-1} \Rightarrow p_{k_i}(k_i) \propto k_i^{-\frac{1}{2}} \tag{4}$$

so that if $p(k_i)$ is observed only with respect to the degree, $\alpha = 0.5$, which is far from empirical evidence because it only captures the distribution of one of the resources with respect to k_i.

4.2 Meta-sensors

Perception presents many psychophysical power-law relations between the magnitude of the physical stimulus and the perceived (subjective) quantity [?]. This is usually attributed to the utility of perception capability, which is enhanced upon broadening of the spectrum. Another explanation is on the physical phenomena itself. Consider a sound wave traveling with constant speed v. If the organism will consider wave lengths from λ_1 to λ_2, $f = \frac{v}{\lambda} \in [\frac{v}{\lambda_2}, \frac{v}{\lambda 1}]$ follows a power law with $\alpha = 1$. In any case, as made explicit by the discussion of Proposition 1, the power-law distribution maximizes component-wealth incident domain and, therefore, the reception of signals related to the existence of the system. In other words, power-law inequality maximized versatility.

In either case, the persistence of power laws in perception suggests a pertinence, and is regarded as such by literature. This raises a fit, in advance, (não entendi o significado de “fit”?) for complex systems with power laws to be analysed as sensors (or meta-sensors). For example, an interest group on hiking can be understood as a meta-sensor about hiking and involved community to find current good places, equipment, people, proper behavior, etc. The power law, i.e. “the scale-free trace” to use the complex network jargon, sweeps a wide range of engagement (regarding the concentration of resources). The higher the component wealth k the higher the engagement, but the lesser diversity is brought to the group by the participant [?]: as one allocates more resources (say time) in one system, it allocates less resources is outside systems.

4.3 Equanimous inequality

Paradoxically, power laws, which are the current utmost inequality paradigm, follow from an equanimous consideration of resource inputs and exhibit other equanimous aspects:

- $p(k) = C.k^{-\alpha} \Rightarrow p(k).k^{\alpha} = C$, with C constant. That is, the amount k^{α} of resources per component times the amount of those components, which is the total “instantaneous” allocated resources, is constant C.
- Each component participates in numerous other complex systems, potentially infinite, and should present a broad, if not complete, sweep of resources allocated to itself. These resources are not necessarily of the same type. We assume that human systems, for example, present power-law distributions of knowledge $p_k(k_k)$ and of wealth $p_w(k_w)$, with potentially different (relative) amount k of resources. At the same time, within a fixed type of resource, resources allocated vary in different systems. For example, an individual tends to have many acquaintances (resource) in its own family, work and neighborhood,

and fewer knowns in such circles of distant family members, partners and friends.

- The total resource available to each component is potentially the same, but spread differently across systems. For example, human individuals form complex social systems with power-law distributions of relations. All the participants, by being humans, have the same amount of time (resource) available each day to engage in all the complex systems that are presented by the environment. One can even say assume that each individual creates the same amount of relationships with the world each day, be they with other people, ideas, things, etc.

4.4 Wealth distribution

One manifestation of inequality through power laws, which is most fundamental to daily experience in society, is found in the discrepant wealth distributions worldwide. There are continuous efforts to deal with this issue, usually advocating ways to minimize "social inequality". Considering the framework presented within this letter:

- Such inequality is a natural tendency that follows from Propositions 0, 1 and 1 in accordance with the phenomenological analysis of Section 2.2.
- Deviations from a power-law distribution of wealth should require work. Occasional deviations are part of the statistical aspect of the phenomena involved, but the maintenance of a pattern different from the power law derived from the resources distribution are fated to require the expenditure of energy.
- Both deviations of power laws towards a more equanimous or towards a more inequanimous distribution should be ephemeral or require work.

In particular, the (publicized) homogeneity of earnings in public institutions, and the (publicized) distribution of wealth in whole countries, reveal that there is indeed efforts to minimize the strong inequality imposed by power laws. Additionally, publicized data should be regarded with extra care and skepticism, as they do not present the expected power-law distributions.

In particular, power-law inequality seems inevitable, and a consequence of a distribution of wealth equanimous and insensitive along wealth allocated to each component. This implies the necessity of "work" for equalization. Also, we observe that the higher the k_L of equation 5, the higher all the probabilistic mass will be located, which implies greater wealth of the wealthier, the hubs or the "elites". In other words, the richer the least rich, the richer the more rich.

4.5 Naturalization of inequality

Power laws are very frequent in empirical data. This already grants its place among the study of natural phenomena. Many different explanations are given for the many cases where they are found, with most common denominators being fields such as fractals, chaos, networks and unifying models e.g. as given by the Theory of Self-Organized Criticality [**?**]. Cases in more traditional fields, such as Newtonian mechanics gravitational force relation to distance with $\alpha = 2$ (if masses are fixed), are usually not mentioned in specialized literature. In other words: there seems not to be a unifying theory of why power laws express such an ubiquitous spectrum of relationships.

If the framework in Section 2 is valid for all cases where power laws are found, the consideration of power laws as tied to natural phenomena *per se*, goes a step further. Phenomenologically we have a fit (?) of the analysis. That is: if there is a power law, the analysis developed in Section 2.2 holds and the power law relation can be regarded as an equanimous distribution of resources in α dimensions. If the acting of the "laws" given by Propositions 0, 1 and 1 is fundamentally what is taking place, that should depend on phenomena and standpoint. We advocate that this framework deepens the understanding of potentially all power law incidences and is usually consonant with more explicit and intuitive relations of the system, its components and the context. The core meaning seems to emanate, with the simplest formalism, from the object sizes in the isotropic space shown in Section 3.1. That is a reasonable geometric abstraction for one to grasp the power law inequality originated from a uniform distribution of fundamental resources through component wealth. Also, relating power laws to the environment is the most effective way we found to make explicit both the axiomatic and the phenomenological backbones of power law ubiquity described in Section 2. We hypothesize that $\alpha \approx 2$ is due to two basic resources input in any system: components and their time, energy or engagement. All other resources seem correlated to these two. We also hypothesize that deviations from $\alpha \approx 2$ are due to other resources less correlated to them or to any other nonlinearities in the relationship of resources.

5 Discussion

Our model does not require the components to be "self-interested". This is often required in this context, and we understand that the formalization presented here shed light into in the reasons why and the ways that self-interested agents organize themselves with extensive incidence of power laws.

These principles can be thought of as laws met in a very broad class of phenomena. These propositions follow from the complexity of the system: there is so much involved, that the ignorance is assured and one needs to grasp hardly false assumptions. (não entendi o argumento?) The resulting axioms can be viewed as statistical tendencies that hold with such universality that they can be seen as laws that govern phenomena ubiquitously.

Deviations from $p(k)$ tend to be transient or require effort, the expenditure of energy (as work needed to reduce entropy), or be imposed by harsh conditions (such as an apple that does not fall if stuck in the ceiling).

All sorts of nonlinearities should account for many kinds of deviations in empirical power laws. Also, both wave and probabilistic interpretations suggest that further mathematical parallels might be useful in understanding complex systems, be it through thermodynamic, wave or quantum theories. One might glimpse diverse severe hypothesis, such as a lower bound for the energy involved in the integration of the components into a complex system and if such energy can account for part of the phenomena currently explained through dark matter assumptions. (não estou seguro se essa extensão será compreensível para o leitor. Discutir?)

6 Conclusions

The presented framework most importantly laid inequality ubiquitous through power laws which follow from simple assumptions. These assumptions can also be observed in any power-law incidence. Therefore, the interpretation of diverse natural and human phenomena are impacted and inequality is posed as a natural tendency. Immediate consequences were therefore scrutinized case-by-case. We hypothesize that inequality should be minimizable by the expenditure of energy and that the characteristic $\alpha = 2$ is a consequence of the existence of two types of independent resources. Further work should link these findings to individual fields and contextualize the general framework.

Acknowledgments

Financial support was obtained from the United Nations Development Program (UNDP, contract 2013/00056, project BRA/12/018); the Brazilian National Counsel of Technological and Scientific Development (CNPq, process 140860/2013-4, project 870336/1997-5, advisor: Osvaldo N. Oliveira Jr.) and FAPESP. Authors thank the Brazilian General Secretariat of the Republic Presidency (SG-PR) and all labMacambira.sf.net members for pursuing this and other developments.

Appendix: Power laws

A power law is a functional relationship between two quantities $p(k)$ and k in the form:

$$p(k) = Ck^{-\alpha} \tag{5}$$

where $k \in [k_L, k_R]$ and C is constant. There are four degrees of freedom in four characteristic variables: α, C, k_L and k_R. Suppose also $p(k)$ normalized so that $\int_{k_L}^{k_R} p(k) = 1$, i.e. the power law is fit to represent a probability density function. Assuming idealized phenomena:

$$\int_{k_L}^{k_R} p(k) = 1 \;\Rightarrow\; C = \frac{1-\alpha}{k_R^{1-\alpha} - k_L^{1-\alpha}} \tag{6}$$

The cumulative distribution function $P(k)$, the and median m, mean μ and variance σ^2 are:

$$\begin{aligned}
P(k) &= \int_{k_L}^{k} p(\tilde{k})d\tilde{k} = \frac{C}{1-\alpha}(k^{1-\alpha} - k_L^{1-\alpha}) = \frac{k^{1-\alpha} - k_L^{1-\alpha}}{k_R^{1-\alpha} - k_L^{1-\alpha}} \\
&\left(m : \int_{k_L}^{m} p(k)dk = \int_{m}^{k_R} p(k)dk = \frac{1}{2} \right) \\
m &= \sqrt[1-\alpha]{k_L^{1-\alpha} + \frac{1-\alpha}{2C}} = \sqrt[1-\alpha]{k_R^{1-\alpha} - \frac{1-\alpha}{2C}} = \sqrt[1-\alpha]{\frac{k_R^{1-\alpha} - k_L^{1-\alpha}}{2}} \\
\mu &= \int_{k_L}^{k_R} kp(k)dk = C\,\frac{k_R^{2-\alpha} - k_L^{2-\alpha}}{2-\alpha} = \left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{k_R^{2-\alpha} - k_L^{2-\alpha}}{k_R^{1-\alpha} - k_L^{1-\alpha}}\right) \\
\sigma^2 &= \left[\int_{k_L}^{k_R} k^2 p(k)dk = C\,\frac{k_R^{3-\alpha} - k_L^{3-\alpha}}{3-\alpha} = \left(\frac{1-\alpha}{3-\alpha}\right)\left(\frac{k_R^{3-\alpha} - k_L^{3-\alpha}}{k_R^{1-\alpha} - k_L^{1-\alpha}}\right) \right] - \mu^2
\end{aligned} \tag{7}$$

Often, $k_R \to \infty$ which implies: $\alpha > 1$ as a condition for convergence of $p(k)$ and for a finite median m. Also, the mean μ is finite only if $\alpha > 2$, and the variance σ^2 is finite only when $\alpha > 3$:

$$
\begin{aligned}
&if\ k_R \to \infty \Rightarrow \\
&\qquad if\ \alpha > 1\ \Rightarrow \qquad C = \frac{\alpha - 1}{k_L^{1-\alpha}} \\
&\qquad\qquad\qquad P(k) = 1 - \left(\frac{k_L}{k}\right)^{\alpha - 1} \\
&\qquad\qquad\qquad m = \sqrt[1-\alpha]{\frac{\alpha - 1}{2C}} \qquad = k_L.\sqrt[\alpha-1]{2} \qquad (8) \\
&\qquad if\ \alpha > 2\ \Rightarrow \qquad \mu = k_L^{2-\alpha}\frac{C}{\alpha - 2} \qquad = k_L\frac{\alpha - 1}{\alpha - 2} \\
&\qquad if\ \alpha > 3\ \Rightarrow \qquad \sigma^2 = k_L^{3-\alpha}\frac{C}{\alpha - 3} - \mu^2 \quad = k_L^2\frac{\alpha - 1}{(\alpha - 3)(\alpha - 2)^2}
\end{aligned}
$$