# A simple model that explains why inequality is inevitable and ubiquitous

RENATO FABBRI, OSVALDO N. OLIVEIRA JR.
*São Carlos Institute of Physics, University of São Paulo, CP 369, 13560-970 São Carlos, SP, Brazil*
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**Abstract**

Inequality has always been a crucial issue for human kind, particularly concerning the highly unequal distribution of wealth, which is at the root of major problems facing humanity, including extreme poverty and wars. A quantitative observation of inequality has become commonplace in recent years with the discovery that many natural as well as man-made systems can be represented as scale-free networks, whose distribution of connectivity obeys a power law. These networks may be generated by the preferential attachment for the nodes, within the so-called rich-gets-richer paradigm. In this letter we introduce a simple model that explains the ubiquity of inequality, based on three simple assumptions applied to a generic system. The first assumption is that the amount of each resource input to the system is fixed, as in a conservation law. Second assumption is the diversity of the components. The third assumption is an uniform distribution of resources along component wealth. This implies that the more resources are allocated per component, less components with such cost the system presents, with the conservation of the amount of resources distributed through cost sweep. This can be geometrically described by the distribution of object sizes in a 3D space, where each dimension is assumed to be isotropic. Applying these assumptions to a generic system results in a power-law distribution, whose coefficient is the number of inputs that are independent from each other, i.e. the dimensionsionality of the allocated resources. Even though there is no restriction to the value of the coefficient, in practice we observe that existing systems normally exhibit a coefficient between 1.5 and 3.0. With our simple model it is not possible to determine whether this limitation in the coefficient values arises from a fundamental principle, but we indicate reasonable hypotheses. The assumptions in the model are analogous to the first and second laws of thermodynamics: conservation of resources and a time arrow pointing to inequality. Since these assumptions are easily justified based on established knowledge, the model proves unequivocally that inequality is ubiquitous. We also discuss ways to control this tendency to inequality, which is analogous to a decrease in entropy in a closed system induced with an external action.

*Keywords:* power laws, fundamental theory, complex systems, complex

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# 1 Introduction

Symmetry, and its symmetric twin, asymmetry, or inequality, is regarded as fundamental to cognition and the universal laws in nearly all fields, emanating from physics and philosophy. The ubiquity of symmetry is appreciated so profoundly that literature constantly recapitulates that the human mind present explicit and basic symmetry operations through thinking. Being the way our mind works, we model the world with symmetry regardless if the world present those symmetries. Models are useful, and the symmetry observed through scientific means are believed to reflect reality, but it is also foreseen that new knowledge should spout from a somewhat different paradigm.

In this letter, we present inequality as resulting from an uniform distribution of resources with respect to the resources already allocated to each component.

## 1.1 Power laws such as Zipf and Pareto

A power law is a functional relationship between two quantities $P(k)$ and $k$ in the form:

$$P(k) = Ck^{-\alpha} \tag{1}$$

where $k \in [k_L, k_R]$ and $C$ is constant. Assuming idealized phenomena:

$$C = \frac{1-\alpha}{k_R^{1-\alpha} - k_L^{1-\alpha}} \tag{2}$$

Often, $k_R \to \infty$ which implies $\alpha > 1$ as a condition for convergence of $P(k)$. In such cases, the power law has a well-defined mean only if $\alpha > 1$, a finite variance only when $\alpha > 2$. Well-defined skewness and kurtosis are restricted to the cases where $\alpha > 3$ and $\alpha > 4$ respectively.

In nearly all systems, power laws are observed through both theory and empirical data. Of special interest in the last decades are the scale-free complex networks, the basic characteristic of which is a power law distribution of connectivity (number of edges per node). Power laws also govern perception, as exposed by the Webner-Fechner and Stevens laws. As a rule of thumb, the distribution of resources among (often self-interested) components tends to follow a power law, which includes distribution of human wealth, interactions, friendships; connections among airports, synaptic count among neurons. Some advocate about a better fit and theoretical backbone for the superposition of a power law distribution and a Weibull distribution [1]. Most canonical examples in literature seem to be earthquake intensity and allometric relations of animal bodies, most canonical law examples seem to be Pareto and Zipf laws. Examples in basic physics are numerous, e.g. in a Newtonian context force is related to distance with $\alpha = 2$ and force is related to acceleration with $\alpha = -1$.

### 1.2 Related work

Power laws. See at least [2, 3].

## 2 Formalization

**Definition** A resource-based system is a complex system that has an underlying resource vital to the their components and their interdependent roles in the system, often expressed as a power law.

**Definition** Component wealth is the amount of resources $k$ allocated to the component.

We will use $p(k)$ to denote the fraction of components with component wealth $k$. Likewise, the same notation $p(k)$ will denote the probability of choosing a component with an amount $k$ of resources. The context should make it clear which is the appropriate meaning. This symmetry among probability, frequency and relative count is instrumental for the interpretative framework herein presented, which is conveniently present in the notation.

### 2.1 Propositions and corollaries

Inspired in the laws of thermodynamics, we derived four propositions. These principles can be though of as laws met in a very broad class. These propositions follow from the complexity of the system: there is so much involved, that the ignorance is assured and one needs to grasp hardly false assumptions. This leads to:

**Proposition 2.1** *There is diversity among the components of the resource-based system. (Zeroth law.)*

If there is a big set of components, there is hardly any way to avoid diversity. Be it location, size, age, the way someone regards them, etc., distinctions arise (as do symmetry). This can be regarded as both a statistical law, as a need for consideration: if a collection of objects is being considered by someone, they were put in some order or another system in which they differ.

**Proposition 2.2** *The amount of each resource input to the system is fixed. (First law.)*

This follows from the independence of each resource dimension, say $\lambda_1$ and $\lambda_2$, and the multiplicative relation they hold with the total resource $E = \lambda_1.\lambda_2$.

**Proposition 2.3** *In allocating resources, a resource-based system does not distinguish the resources already allocated per component. This is expressed as an uniform distribution $p_U(k)$ of resources k. (Second law.)*

That is, resources are allocated without distinction to component wealth, which has two major consequences: 1) diversity of component wealth is preserved; 2) the wealthier the components considered, the fewer they are.

**Proposition 2.4** *The ground state implied by the continuous presence of the second law is characterized by a power-law distribution $p(k)$ of components with component wealth k. Deviations from $p(k)$ tend to be transient or require effort, the expense of energy, such as work, or be imposed by hash conditions. (Third law.)*

This follows from second law. As the system continues to exist, and resources are continually allocated,

Proposition 2.1 is required for the attribution of different amount of resources for each component. If they are equal, by definition they have the same component wealth.

**Corollary 2.5** *The extension of allocation is* $[0, \infty]$

**Corollary 2.6** *The superior limit $k_2$ of the observed allocation of resources is an estimate of the amount of resources equally distributed along component wealth.*

**Corollary 2.7** *The superior limit $k_2$ of the observed allocation of resources is an estimate of the amount of resources equally distributed along component wealth.*

**Corollary 2.8** *$p(1)$ is another estimate of the amount of resources C equally distributed along component wealth.*

**Corollary 2.9** *The dimensionality of the allocated resources is the scaling factor $\alpha$.*

## 2.2 Phenomenological approach

Doubt propositions **??** and **??**. Suppose you have a power law observation in empirical data or driven from theory. Are the conceptualization presented above axiomatically still helpful? We provide here a general and mathematically grounded interpretation of any power law relationship, perfectly consonant with such assumptions. We understand that this short theoretical consideration suggests, sustains and deepen them.

Consider a constant speed $v$ for a wave propagation (suppose linear media with no dispersion), the number of oscillations $f$ per unit time is inversely proportional to the cycle length $\lambda$ (the period). In usual notation $f = \frac{v}{\lambda}$. The constant speed $v$ implies a power law between $f$ and $\lambda$ (with $\alpha = 1$ and $C = v$). Similarly, the general case of a power law can be interpreted as resulting from a constant amount of fundamental resources $C$ with dimensionality $\alpha$ being homogeneously distributed across "resources per component" $k$ resulting in $p(k)$ of such components.

Now let $f = \frac{v}{\lambda_1.\lambda_2}$, that is, the frequency of occurrence (or the probability of choosing such event at random) go with the inverse of two periods. If $\lambda_1 == \lambda_2 == \lambda$, then $f = \frac{v}{\lambda^2}$ In other words, the density given by constant $C$ resources in a supercube of edge $k$ in $\alpha$ dimensions gives the amount $p(k)$ of components of such size. Be $C_i$ the amount of resources allocated at specific resource costs $\lambda_{1,i}$ and $\lambda_{2,i}$ and assume linearity $\lambda_{1,i} = c.\lambda_{2,1}$, such that $p(k_i) = \frac{C}{C_i} = \frac{C}{\lambda_{1,i}.\lambda_{2,i}} = \frac{C}{c\lambda_{2,i}^2} = \frac{\widetilde{C}}{\lambda_{2,i}^2} \equiv \frac{v}{\lambda^2}$. Therefore one can consider that we only access $\lambda_1 == \lambda_2$ and a "normalized resource C" allocated by the environment democratically across resource per component (for the complex system being observed in which there is a power law).

Curiously enough, such interpretation seem to hold for both wave and probabilistic phenomena. This suggests that further mathematical parallels might be possible and useful in understanding complex systems through thermodynamics processes. One might glimpse the attribution of energy to complex systems cohesion and information embodiment and consecutive hypothesis for the nature of dark matter.

# 3 Paradigmatic examples

## 3.1 Workers in a factory

Let us consider a generic problem in which a System ($S$) provides an Output ($O$) depending on the Input ($I$) it receives. The following assumptions are established. 1) There may be several inputs, but for each input the amount of resources furnished to the System is fixed, as in a conservation law. 1) $S$ is made of a number of components that are not all equal to each other. That is to say, there is diversity in the nature of the components. 3) Distribution should be uniform with regard to the "size" of the component as in a geometric case where space is considered as isotropic.

There is no assumption for the Output (O), which is taken as to mean the performance (or richness) in terms of the components of S. Now assuming that there are N types of input, and for the sake of the argument, all of them have a time dependence (with 1/t), according to assumption 2) above. The Output is the product of the functions of these N inputs. O = (R1*R2*... RN)/tN since a given input can be written as Ri/t. The Output has therefore a power-law dependence on t with coefficient N. Now considering the values of N observed in practice (from many examples of power-law dependences), which is normally between 1.5 (2?) and 3, one infers that there are at least two types (?) of independent inputs and at maximum 3 independent inputs. Let us illustrate with a hypothetical case that may facilitate understanding the concepts. A piece of work is to be done in a company. What sort of resources can be established as inputs? We assume three inputs: number of workers, working hours and efficiency. We recall that all resources should be fixed and that there is diversity in the components. Then, first the total number of workers available are divided into groups of different sizes, Continuar exemplo ????

Falta resumir a literatura que mostra coeficiente entre 1.5 e 3. Mencionar casos em que é maior que 3. Incluir exemplos em que a distribuição uniforme se dá, mesmo que sejam empíricos. Lembro que você tinha isso para lista de e-mails, e acho que outros exemplos com maior apelo para a física precisariam ser incluídos.

### 3.2 Object sizes in your house or elsewhere

Que estah interessado na sua casa em objetos de dois tamanhos: um quase do tamanho de um cubo unitario (seja qual for a unidade q vc escolher); e outro quase do tamanho l (l inteiro e maior q um). Em cada cubo unitario ha rho de chance de ter um objeto com o tamanho almejado. Nos cubos de lado l tb. Dado um cubo de lado n (l divide n) ha, em media, rho . $(n/l)^3$ objetos de volume $l^3$ e $\rho.n^3$ objetos de volume $1^3$. Ou seja, a frequencia de objetos com volume X eh inversamente proporcional ao volume X.

## 4 Especial cases

The consequences of the interpretation of power laws presented in Section 2 are severe for understanding and dealing with phenomena. We expose selected cases in this section.

### 4.1 Scale free complex networks

We advocate that this is the similar case of that where edges reflect the resources allocated by individuals. If $f = v/resources_i = v/E_i = v/(N_i.(E_i/N_i)) = 2v/(N_i.k_i) \equiv v/(\lambda_1.\lambda_2)$. As $N_i$ and $k_i$ are both directly proportional to $E_i$, which is the primary resource, one can factor out another constant and consider the special case where $\lambda = \lambda_1 = \lambda_2$ and, consequently, $f = v/\lambda^2$. E.g. in a social network, the number of agents allocated and the time each of them put, are seen as the primary resource (individual . time).

Questions: *) the range of degree covered by scale-free networks is maximum, as do our perception, which also follows power laws. How far can we consider scale-free complex networks to be meta-sensors that captures and processes signals about the very reason of existance of the meta-sensor?

Theorem 1: every scale-free network with distribution of degree $p(k) = C/k^{\alpha}$ can be understood as having an equanimous distribution of resources in $\alpha$ dimensions. Corolary: if $p(E_i) = C/E_i$, with $\alpha = 1$. (might have to consider only edges with vertices of other connectivity, i.e. discard edges between vertices with same degree.)

### 4.2 Meta-sensors

Perception presents many psychophysical power-law relations between magnitude of the physical stimulus and the perceived (subjective) quantity [**?**, **?**]. This is usually attributed to the utility of perception capability, which is considerably enhanced upon broadening of the spectrum. Another explanation is on the physical phenomena itself. Consider a sound wave traveling with constant speed $v$. If the organism will consider wave lengths from $\lambda_1$ to $\lambda_2$, $f = \frac{v}{\lambda} \in [_]$ follows a power law with $\alpha = 1$.

In either case, the persistence of power laws in perception suggests a pertinence, and is regarded as such. This raises points a fit, in advance, for complex systems with power laws to be thought of as sensor (or meta-sensors) on signals of the domain of the resource $k$. For example, an interest group on hiking can be understood as a meta-sensor about hiking and involved community: current good places, equipment, people, proper behavior, etc. The power law, i.e. "the scale-free trace" to use the complex network jargon, sweeps the broadest diversity of engagement, which can be regarded as inversely proportional the diversity brought to the group by the participant [**?**]: as one allocates more resources (say time) in one system (or a set of them), it allocates less resources is the rest of the systems.

### 4.3 Equanimous inequality

Paradoxically, power laws, which is the current utmost inequality paradigm, follow from an equanimous consideration of resources and exhibit other equanimous aspects:

- $p(k) = C.k^{-\alpha} \Rightarrow p(k).k^{\alpha} = C$, with C constant. That is, the amount $k$ of resources per component times the amount of those components, which is the total "instantaneous" allocated resources, is constant $C$. The scaling factor $\alpha$ is herein interpreted as the number of dimensions in which such resources are being observed.
- Each component participates in numerous other complex systems, potentially infinite, and should present a broad, if not complete, sweep of resources allocated to itself. These resources are not necessarily of the same type. We assume that human systems, for example, present

power-law distributions of knowledge $p_k(k_k)$ and of wealth $p_w(k_w)$, with potentially different (relative) amount $k$ of resources. At the same time, within a fixed type of resource, resources allocated vary in different systems. For example, an individual tends to have many acquaintances (fixed resource) in its own family, work and neighborhood, a fewer knowns in such circles of distant family members, partners and friends.

- The distinction of each component particularities is often not of core importance to describe complex behavior. This reflects in symmetries among components. For example, Human individuals form complex social systems with power law distributions of relations. All the participants, by being humans, have the same amount of time available each day, resource, to engage in all the complex systems that are presented by the environment.

### 4.4 Wealth distribution

One manifestation of the power law which is most fundamental to daily experience in current society is the inequality of wealth distribution. There are continuous efforts to deal with this issue, usually advocating ways to minimize "social inequality". Considering the framework presented withing this letter:

- Such an inequality is a natural tendency that follows from presuppositions **??** and **??**, phenomenological mathematical backbone (Section **??**), more than fifty years and wide empirical evidence.
- This should make work to diminish it
- not any unequal outline, but a power law.

In particular, the (publicized) homogeneity of earnings in public institutions, and the (publicized) distribution of wealth in whole countries, reveal that there is indeed efforts to minimize the strong inequality imposed by power laws. Additionally, publicized data should be regarded with extra care and scepticism, as they do not present the expected power-law distributions.

In particular, power law like inequality seems inevitable, and a consequence of a distribution of wealth equanimous and insensitive along wealth allocated to each component. This implies the necessity of "work" for equalization. Also, we observe that the higher the $k_L$ of equation **??**, the higher all the probabilistic mass will be located, which implies greater wealth of the wealthier, "elites", hubs. In other words, the richer the least rich, the richer the more rich.

### 4.5 Naturalization of inequality

Power laws are very frequent in empirical data. This already grants its place among the study of natural phenomena. Many different explanations are given for the many cases where they are found, with most common denominators being fractals, chaos, networks. Cases in more traditional fields, such as Newtonian mechanics gravitational force relation to distance with $\alpha = 2$ (if masses are fixed), are usually not mentioned in specialized literature. In other words: there seems not to be an unifying theory of why power laws express such an ubiquitous spectrum of relationships.

If the framework in Section 2 is valid for all cases where power laws are found, the consideration of power laws as tied to natural phenomena *per se*, goes a step further. Phenomenologically, yes. That is: if there is a power law, the analysis developed in Section **??** holds. The power law relation can be regarded as an equanimous distribution of resources in $\alpha$ dimensions. If the acting of the "laws" given by presuppositions **??** and **??** is fundamentally what is taking place, that should depend on phenomena and standpoint. We advocate that this framework deepens the understanding of all power law incidences and is usually consonant with more explicit and intuitive relations of the system, its components and the context. The core meaning seems to emanate from the object sizes in the isotropic space. For the analysis, a reasonable geometric abstraction as such eases one to grasp the distribution of fundamental resources of a system of interest. To relate power laws to the environment is the most effective way we found to make explicit both the axiomatic and the phenomenological backbones of power law ubiquity.

## 5 Conclusions

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