1 Curvilinear Coordinates

$$\boxed{\begin{aligned}\vec{r} &= r\cos\phi\sin\theta\,\hat{x} + r\sin\phi\sin\theta\,\hat{y} + r\cos\theta\,\hat{z}\\ &= r\hat{r}\end{aligned}}$$

$$\begin{aligned}
\hat{r} &= \frac{\partial}{\partial r}\vec{r} = \boxed{\frac{\vec{r}}{r}} = \nabla r\\
\hat{\theta} &= \frac{1}{r}\frac{\partial}{\partial\theta}\vec{r} = \boxed{\frac{\partial\hat{r}}{\partial\theta}} = r\nabla\theta\\
\hat{\phi} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\vec{r} = \boxed{\frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi}} = r\sin\theta\nabla\phi
\end{aligned}$$

$$\begin{aligned}
\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} &= \cos 2\theta\,\hat{z}\\
\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi} &= \sin\theta\,\hat{y} + \sin\phi\cos\theta\,\hat{z}\\
\hat{\phi} &= -\sin\phi\sin\theta\,\hat{x} + \cos\phi\sin\theta\,\hat{y}
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
z &= \boxed{\frac{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}{\cos 2\theta}}\\
y &= \frac{\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}}{\sin\theta} - \frac{\sin\phi\cos\theta}{\sin\theta\cos 2\theta}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]\\
&= \boxed{-\frac{\sin\phi\sin\theta}{\cos 2\theta}\,\hat{r} + \frac{\sin\phi\cos\theta}{\cos 2\theta}\,\hat{\theta} + \frac{\cos\phi}{\sin\theta}\,\hat{\phi}}\\
x &= \boxed{\cot\phi\,\hat{y} - \frac{\hat{\phi}}{\sin\phi\sin\theta}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat{r}}{dt} &= \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\left(\frac{d\vec{r}}{dt} - \frac{dr}{dt}\hat{r}\right) = \boxed{\frac{v}{r}\left[\hat{v} - (\hat{r}\cdot\hat{v})\,\hat{r}\right]}\\
&= \frac{d\theta}{dt}\frac{\partial\hat{r}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{r}}{\partial\phi} = \boxed{\frac{d\theta}{dt}\,\hat{\theta} + \sin\theta\,\frac{d\phi}{dt}\,\hat{\phi}}\\
\frac{d\hat{\theta}}{dt} &= \frac{d\theta}{dt}\frac{\partial}{\partial\theta}\left(\frac{\partial\hat{r}}{\partial\theta}\right) + \frac{d\phi}{dt}\frac{\partial}{\partial\phi}\left(\frac{\partial\hat{r}}{\partial\theta}\right) = \boxed{-\frac{d\theta}{dt}\,\hat{r} + \cos\theta\,\frac{d\phi}{dt}\,\hat{\phi}}\\
\frac{d\hat{\phi}}{dt} &= \frac{d\theta}{dt}\cancel{\frac{\partial\hat{\phi}}{\partial\theta}} + \frac{d\phi}{dt}\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi}\right) = -\frac{d\phi}{dt}\underbrace{\mathrm{Proj}_{\mathrm{xy}}\left(\frac{\hat{r}}{\sin\theta}\right)}_{\cos\phi\hat{x} + \sin\phi\hat{y}}\\
&= -\frac{d\phi}{dt}\frac{\hat{r} - \cos\theta\,\hat{z}}{\sin\theta} = \boxed{\frac{d\phi}{dt}\frac{\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}}{\cos 2\theta}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \frac{dr}{dt}\hat{r} + r\left(\frac{d\theta}{dt}\frac{\partial\hat{r}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{r}}{\partial\phi}\right)\\
&\boxed{\vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} + r\sin\theta\frac{d\phi}{dt}\hat{\phi}}\\
\frac{d\vec{\theta}}{dt} &= \frac{d\theta}{dt}\hat{\theta} + \theta\left(-\frac{d\theta}{dt}\,\hat{r} + \cos\theta\,\frac{d\phi}{dt}\,\hat{\phi}\right)\\
\frac{d\vec{\phi}}{dt} &= \frac{d\phi}{dt}\hat{\phi} + \phi\frac{d\phi}{dt}\left(\frac{\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}}{\cos 2\theta}\right)\\
&\downarrow\\
\frac{dr}{dt} &= \frac{d}{dt}(\vec{r}\cdot\vec{r})^{\frac{1}{2}} = \boxed{\hat{r}\cdot\vec{v} = \omega\vec{r}\cdot\hat{v}}\\
\frac{d\theta}{dt} &= \nabla\theta\cdot\vec{v} = \boxed{\frac{\hat{\theta}\cdot\vec{v}}{r} = \omega\hat{\theta}\cdot\hat{v}}\\
\frac{d\phi}{dt} &= \nabla\phi\cdot\vec{v} = \boxed{\frac{\hat{\phi}\cdot\vec{v}}{r\sin\theta} = \frac{\omega\,\hat{\phi}\cdot\hat{v}}{\sin\theta}}
\end{aligned}$$

$$\begin{aligned}
\vec{L} &= \vec{r}\times\vec{p}\\
(mrv)\hat{r}\times\hat{v} &= m\vec{r}\times\vec{v} = mr^2\left(\frac{d\theta}{dt}\hat{\phi} - \sin\theta\frac{d\phi}{dt}\hat{\theta}\right)\\
&= I\vec{\omega} = mr^2\left[\frac{v}{r}(\hat{\theta}\cdot\hat{v})\hat{\phi} - \frac{v}{r}(\hat{\phi}\cdot\hat{v})\hat{\theta}\right]\\
\boxed{(mrv)\hat{r}\times\hat{v}} &= \boxed{(I\omega)\hat{\omega}} = mr^2\omega\left[(\hat{\theta}\cdot\hat{v})\hat{\phi} - (\hat{\phi}\cdot\hat{v})\hat{\theta}\right]\\
\underline{mrv}(\hat{\theta}\times\hat{\phi})\times\hat{v} &= \underline{I\omega}\hat{\omega} = \underline{I\omega}\left[(\hat{\theta}\cdot\hat{v})\hat{\phi} - (\hat{\phi}\cdot\hat{v})\hat{\theta}\right]
\end{aligned}$$

$$\begin{aligned}
\vec{a} = &\left[\ddot{r} - r\dot{\theta}^2 + r\dot{\phi}^2\frac{\sin^2\theta}{\cos 2\theta}\right]\hat{r}\\
&+ \left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\frac{\tan 2\theta}{2}\right]\hat{\theta}\\
&+ \left[2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta\right]\hat{\phi}
\end{aligned}$$

$$\begin{aligned}
\vec{\tau} &= \vec{r}\times\vec{F}\\
&= (mr^2)\frac{\hat{r}\times\vec{a}}{r} = I\vec{\alpha}
\end{aligned}$$

$$\begin{aligned}
T &= \hat{v} = \frac{\vec{v}}{v}\\
T' &= \frac{(\vec{v}\cdot\vec{v})\,\vec{a} - (\vec{v}\cdot\vec{a})\,\vec{v}}{v^3} = \frac{\vec{v}\times(\vec{a}\times\vec{v})}{v^3}\\
\|T'\| &= \frac{\sqrt{v^2a^2 - (\vec{v}\cdot\vec{a})^2}}{v^2} = \frac{\|\vec{v}\times\vec{a}\|}{v^2}\\
N &= \frac{T'}{\|T'\|} = \frac{\vec{v}\times(\vec{a}\times\vec{v})}{v\|\vec{v}\times\vec{a}\|} = \hat{v}\times(\widehat{a\times v})\\
B &= \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|}\\
\|\vec{q}\times\vec{p}\|^2 &= q^2p^2 - (\vec{q}\cdot\vec{p})^2
\end{aligned}$$

$$\begin{aligned}
\vec{a} &= a_T\vec{T} + a_N\vec{N}\\
a_T &= \vec{a}\cdot\vec{T}\\
a^2 &= a_T^2 + a_N^2
\end{aligned}$$

$$
\begin{aligned}
\tfrac{d}{dt}\left(\vec{p} \times \vec{L}\right) &= \tfrac{d\vec{p}}{dt} \times \vec{L} = f_{(r)}\hat{r} \times \left(\vec{r} \times m\tfrac{d\vec{r}}{dt}\right) \\
&= mf_{(r)}\left[\vec{r}\left(\hat{r} \cdot \tfrac{d\vec{r}}{dt}\right) - \tfrac{d\vec{r}}{dt}\left(\hat{r} \cdot \vec{r}\right)\right] \\
&= mf_{(r)}\left[\hat{r}\,\tfrac{1}{2}\tfrac{d}{dt}\left(\vec{r} \cdot \vec{r}\right) - \tfrac{1}{r}\tfrac{d\vec{r}}{dt}r^2\right] \\
&= mf_{(r)}\left[\hat{r}r\tfrac{dr}{dt} - r\tfrac{d\vec{r}}{dt}\right] \\
&= -\tfrac{mf(r)r}{I(r)}\left[-\tfrac{I(r)}{r}\tfrac{dr}{dt}\vec{r} + I_{(r)}\tfrac{d\vec{r}}{dt}\right] \\
&= -\tfrac{mf(r)r}{I(r)}\tfrac{d}{dt}\left[I_{(r)}\vec{r}\right] \\
&= -m\cancel{f_{(r)}r^2}\tfrac{d}{dt}\hat{r} = mk\tfrac{d}{dt}\hat{r} \\
\tfrac{d}{dt}\left(\tfrac{\vec{p}\times\vec{L}}{mk} - \hat{r}\right) &= \tfrac{d}{dt}\vec{e} = 0
\end{aligned}
$$

2 Lagrangian Equations

$$\boxed{\begin{aligned} &\mathcal{L} = T - U \ , \qquad p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \\ &\rightarrow \ F_i \equiv \frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i} \end{aligned}}$$

<u>Newton's Laws</u>:

$$\mathcal{L} = \tfrac{1}{2} m \dot{\mathbf{r}}^2 - U(\mathbf{r}) \ , \qquad \vec{p}_r = m\dot{\mathbf{r}}$$

$$\rightarrow \ \boxed{F = m\ddot{\mathbf{r}} = -\nabla U}$$

<u>Angular</u>:

$$\mathcal{L} = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\phi}^2 - U(r,\phi) \ , \qquad \begin{aligned} p_r &= m\dot{r} \\ p_\phi &= m r^2 \dot{\phi} = I\omega \, \overset{(1 \text{ if } \dot{r}=0)}{\frac{\hat{\phi}\cdot\hat{v}}{\cancel{\sin\theta}}} \end{aligned} \ , \qquad \begin{aligned} -\vec{F} &= \nabla U = \tfrac{\partial U}{\partial r}\hat{r} + \tfrac{1}{r}\tfrac{\partial U}{\partial \phi}\hat{\phi} \\ \vec{F} &= m\ddot{\mathbf{r}} = (F\cdot\hat{r})\hat{r} + (F\cdot\hat{\phi})\hat{\phi} \end{aligned}$$

$$F_r = \boxed{-\tfrac{\partial U}{\partial r} + m r \dot{\phi}^2 = m\ddot{r}} \quad (\text{centripetal: } \tfrac{mv^2}{r} = m r \omega^2)$$

$$\rightarrow \quad F_\phi = \boxed{\underbrace{-\tfrac{\partial U}{\partial \phi}}_{\tau} = \underbrace{m r^2 \ddot{\phi}}_{I\alpha} + \underbrace{2 m r \dot{r} \dot{\phi}}_{\dot{I}\omega}} \quad (\text{“coriolis”}: \ 2m|\vec{\omega}\times\vec{v}| = 2m\dot{\phi}\dot{r})$$

$$\dot{\phi}' = \dot{\phi} - \omega$$

$$\downarrow$$

$$\begin{aligned} m\ddot{\mathbf{r}}' &= m\ddot{\mathbf{r}} - (m r \omega^2 - 2 m r \dot{\phi}\omega)\hat{r} - 2 m \dot{r} \omega \hat{\phi} \\ &= m\ddot{\mathbf{r}} - \underbrace{m r \omega^2 \hat{r}}_{\text{centrifugal force}} + \underbrace{2 m \omega (r \dot{\phi} \hat{r} - \dot{r} \hat{\phi})}_{\text{coriolis force}} \end{aligned}$$

Note: $\boxed{\begin{aligned} \dot{\hat{r}} &= \dot{\phi}\hat{\phi} \\ \dot{\hat{\phi}} &= -\dot{\phi}\hat{r} \end{aligned}}$ $\rightarrow$ $$\begin{aligned} \vec{r} &= r\hat{r} \ = \ r\cos\phi\hat{x} + r\sin\phi\hat{y} \\ \dot{\vec{r}} &= \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \\ \ddot{\vec{r}} &= \ddot{r}\hat{r} + 2\dot{r}\dot{\hat{r}} + r\ddot{\hat{r}} \ = \ (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} \end{aligned}$$

<u>Electromagnetic</u>:

$$\mathcal{L} = \tfrac{1}{2} m \dot{\mathbf{r}}^2 - q\Big[V(t,\ \mathbf{r}) - \dot{\mathbf{r}} \cdot \vec{A}(t,\ \mathbf{r})\Big] \ , \qquad p_x = m\dot{x} + qA_x$$

$$\rightarrow \qquad m\ddot{x} + q\tfrac{dA_x}{dt} = -q\left[\tfrac{\partial V}{\partial x} - \dot{r}\cdot\tfrac{\partial \vec{A}}{\partial x}\right]$$

$$m\ddot{x} + q\left[\tfrac{\partial A_x}{\partial t} + \dot{r}\cdot\nabla A_x\right] = q\left[-\tfrac{\partial V}{\partial x} + \dot{r}\cdot\tfrac{\partial \vec{A}}{\partial x}\right]$$

$$\begin{aligned} \rightarrow \quad m\ddot{x} &= q\left(-\tfrac{\partial V}{\partial x} - \tfrac{\partial A_x}{\partial t} + \dot{r}\cdot\left[\tfrac{\partial \vec{A}}{\partial x} - \nabla A_x\right]\right) \\ &= q\left[-\tfrac{\partial V}{\partial x} - \tfrac{\partial A_x}{\partial t}\right] + q\dot{y}\left[\tfrac{\partial A_y}{\partial x} - \tfrac{\partial A_x}{\partial y}\right] \\ &\quad + q\dot{z}\left[\tfrac{\partial A_z}{\partial x} - \tfrac{\partial A_x}{\partial z}\right] \\ &= qE_x + q\dot{y}B_z - q\dot{z}B_y \\ m\ddot{x} &= qE_x + q\left[\dot{\mathbf{r}}\times\vec{B}\right]_x \end{aligned}$$

$$\downarrow$$

$$\boxed{m\ddot{\mathbf{r}} = q\left(\vec{E} + \dot{\mathbf{r}}\times\vec{B}\right)}$$

Special Relativity:

$$\mathcal{L} = -\tfrac{1}{\gamma}mc^2 - U\ , \qquad\qquad \vec{p} = \gamma m\vec{v} \ \rightarrow\ \gamma m\dot{x} = \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$= \gamma m v^2 - \gamma m c^2 - U$$

$$= m\left(v^2 - c^2\right)\left(1 - \tfrac{v^2}{c^2}\right)^{-1/2} - U$$

$$\approx \tfrac{1}{2}mv^2 - (U + mc^2) \qquad \text{(when } v \ll c\text{)}$$

Conservation of Energy:

$$\frac{d\mathcal{L}}{dt} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right) + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \sum_i \left(\dot{p}_i\dot{q}_i + p_i\ddot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t} \qquad \rightarrow \qquad \frac{\partial \mathcal{L}}{\partial t} = -\frac{d}{dt}\left(\sum_i p_i\dot{q}_i - \mathcal{L}\right)$$

$$= \frac{d}{dt}\left(\sum_i p_i\dot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t} \qquad\qquad = -\frac{d\mathcal{H}}{dt}$$

If $\mathcal{L}$ is explicitly independent of time (implies coordinates are "natural"), then the Hamiltonian is conserved.

$$\tfrac{1}{2}\sum_n m\dot{r}_n^2 = \tfrac{1}{2}\sum_n m\left(\sum_i \frac{\partial r_n}{\partial q_i}\dot{q}_i\right)^2$$

$$= \tfrac{1}{2}\sum_{i,j}\left(m\sum_n \frac{\partial r_n}{\partial q_i}\frac{\partial r_n}{\partial q_j}\right)\dot{q}_i\dot{q}_j$$

$$= \tfrac{1}{2}\sum_i\sum_j A_{ij}\dot{q}_i\dot{q}_j$$

$$\left(\text{for } \tfrac{\partial T}{\partial \dot{q}_i}\right) \qquad = \tfrac{1}{2}\left(2\sum_{i\neq j} A_{ij}\dot{q}_i\dot{q}_j + A_{ii}\dot{q}_i^2\right) + \ldots$$

$$\rightarrow$$

$$\mathcal{L} = \tfrac{1}{2}mv^2 - U \ = \ T(\dot{q}_i) - U(q_i) \ \rightarrow$$

$$\mathcal{H} = \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}$$

$$= \sum_i\left(\sum_j A_{ij}\dot{q}_j\right)\dot{q}_i - \tfrac{1}{2}m\dot{\mathbf{r}}^2 + U$$

$$= \tfrac{1}{2}m\dot{\mathbf{r}}^2 + U$$

If $\mathcal{L} = \frac{1}{2}mv^2 - U$ and U is independent of v, then the Hamiltonian is the total energy.

Lagrange Multipliers:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} + \lambda\frac{\partial f}{\partial q_i}$$

$$\frac{dp}{dt} = -\nabla U + \lambda\nabla f$$

$$F_{\text{tot}} = F_{\text{ncnstr}} + F_{\text{cnstr}}$$

2.1 Examples

Atwood's Machine (Pulley):

Particle Confined to a Cylinder Surface:

Block Sliding on Wedge:

Bead on Spinning Wire Hoop:

Oscillations of Bead Near Equilibriuum:

3 Hamiltonian

$$\boxed{\begin{array}{l} \mathcal{H} = \displaystyle\sum_i \dot{q}_i p_i - \mathcal{L}\ , \qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}} \\ \to \begin{array}{l} \bullet\ \dfrac{dp_i}{dt} = -\dfrac{\partial \mathcal{H}}{\partial q_i} \\ \bullet\ \dfrac{dq_i}{dt} = \dfrac{\partial \mathcal{H}}{\partial p_i} \end{array} \end{array}}$$

Newton Particle:

$$\begin{aligned} \mathcal{H} &= \dot{x}(m\dot{x}) - \tfrac{1}{2}m\dot{x}^2 + U(x) \\ &= \tfrac{1}{2}m\dot{x}^2 + U(x) \\ &= T + U \end{aligned}$$

Angular:

$$\mathcal{H} = m\dot{r}^2 + mr^2\dot{\theta}^2 - \left(\tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - U(r,\theta)\right)\ , \qquad \begin{aligned} p_r &= m\dot{r} \\ p_\theta &= mr^2\dot{\theta}\ \equiv\ L = I\omega \end{aligned}$$

$$= \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + U(r,\theta)$$

Electromagnetic:

$$\mathcal{H} = \dot{\mathbf{r}} \cdot \vec{p}_r - \left(\tfrac{1}{2}m\dot{\mathbf{r}}^2 - q\phi_{(t,\ \mathbf{r})} + q\dot{\mathbf{r}} \cdot \vec{A}_{(t,\ \mathbf{r})}\right)\ , \qquad \vec{p}_r = m\dot{\mathbf{r}} + q\vec{A}$$

$$\begin{aligned} &= m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \vec{A} - \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\phi - q\dot{\mathbf{r}} \cdot \vec{A} \\ &= \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\phi \end{aligned}$$

Special Relativity:

$$\mathcal{H} = \vec{v} \cdot (\gamma m\vec{v}) - \left(\gamma mv^2 - \gamma mc^2 - U\right)\ , \qquad \vec{p} = \gamma m\vec{v}$$

$$\begin{aligned} &= \gamma mc^2 + U \\ &\approx \tfrac{1}{2}mv^2 + \left(U + mc^2\right) \qquad \text{(when } v \ll c\text{)} \end{aligned}$$

4 Hamilton-Jacobi Equations

$$\boxed{K(Q,P,t) \equiv H(q,p,t) + \frac{\partial S_{(q,Q,t)}}{\partial t} = 0}$$

$$\dot{Q} = \frac{\partial K}{\partial P} = 0 \quad \Rightarrow \quad \boxed{Q = \alpha_Q = \frac{\partial S}{\partial P}} \quad \text{(constant)}$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \quad \Rightarrow \quad \boxed{P = \alpha_P = -\frac{\partial S}{\partial Q}} \quad \text{(constant)}$$

$$\dot{q} = \frac{\partial H}{\partial p} \quad \Rightarrow \quad \boxed{q = -\frac{\partial S}{\partial p}} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q} \quad \Rightarrow \quad \boxed{p = \frac{\partial S}{\partial q}}$$

Solve for $S_{(q,\alpha_Q,t)}$ ($n+1$ variables, nonlinear PDE)

$$H\left(q, \frac{\partial S_{(q,\alpha_Q,t)}}{\partial q}, t\right) + \frac{\partial S_{(q,\alpha_Q,t)}}{\partial t} = 0$$

$$\underline{\frac{\partial H}{\partial t} = 0} \quad \Rightarrow \quad \boxed{S(q,Q,t) = W(q,Q) - Et}$$

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum p_i \dot{q}_i = \cancel{\frac{\partial S}{\partial t} + H} + \mathcal{L}$$

$$\Rightarrow \quad \boxed{S = \int \mathcal{L}\ dt + \text{const.}}$$

Solve for $W_{(q,\alpha_Q)}$ (n variables, nonlinear PDE)

$$H(q, \tfrac{\partial W}{\partial q}) = E \equiv \alpha_Q$$

Harmonic Oscillator

$$-\frac{\partial S}{\partial t} = \frac{1}{2}p^2 - \frac{1}{2}\omega^2 q^2$$

$$= \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{1}{2}\omega^2 q^2 \qquad \big(S = s_1(q) + s_2(t)\big)$$

$$-\frac{\partial s_2(t)}{\partial t} = \frac{1}{2}\left(\frac{\partial s_1(q)}{\partial q}\right)^2 - \frac{1}{2}\omega^2 q^2 \equiv \alpha_Q$$

$$s_2(t) = -\alpha_Q t + \cancel{\text{const.}} \quad , \quad s_1(q) = \int \sqrt{2a_Q + \omega^2 q^2}\ dq$$

$$Q \equiv \alpha_Q \quad , \quad P = -\int \frac{dq}{\sqrt{2a_Q + \omega^2 q^2}} + t$$

$$\alpha_P = -\frac{1}{\omega}\sin^{-1}\left[q\frac{\omega}{\sqrt{2\alpha_Q}}\right] + t$$

$$\boxed{q(t) = \frac{\sqrt{2\alpha_Q}}{\omega}\sin\left[\omega(t - \alpha_P)\right]}$$

5 Kinematics

Elastic Collisions:
$$m_0 v_0 = m_1 v_1 + m_2 v_2$$
$$\tfrac{1}{2} m_0 v_0^2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

$$\Rightarrow \boxed{\tfrac{1}{2} m_2 v_2^2 (m_1 + m_2) - \tfrac{1}{2} m_0 v_0^2 (m_1 - m_0) = (m_0 v_0)(m_2 v_2)}$$

- $$\boxed{\begin{array}{l} mv_0 \;=\; mv_1 + Mv_2 \;=\; mv_0\left(1 - \tfrac{2M}{m+M}\right) + Mv_0\left(\tfrac{2m}{m+M}\right) \\ \rightarrow\; M \in (\infty,\ m,\ 0] \;\Rightarrow\; v_1 \in (-v_0,\ 0,\ v_0] \end{array}}$$

Inelastic Collision: $E_0 = \frac{1}{2} m v_0^2$

- $$\boxed{\begin{array}{l} mv_0 = (m+M)v_1 \\ \rightarrow\; E_1 = \left(\tfrac{m}{m+M}\right) E_0 \end{array}}$$

6 Orbits

Lagrangian: $\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m r^2 \sin^2\theta \dot{\phi}^2 - U(r)$

- $l = I\omega = mr^2\dot{\theta}$
- $m\ddot{r} = -\dfrac{\partial}{\partial r} U_{\text{eff}} = -\dfrac{\partial}{\partial r}\left(\dfrac{l^2}{2mr^2} + U(r)\right)$

$\Big\}\quad m \;\rightarrow\; \mu = \frac{mM}{m+M}$

Hamiltonian: $E = \frac{p^2}{2m} + \frac{l^2}{2mr^2} + U(r)$

- Inf. Energy to get to $r = 0$ unless $l = 0$
- $U \sim 1/r$

Orbit Types:

$E > 0$:	Hyperbola
$E = 0$:	Parabola
$E < 0$:	Ellipse
$E = \text{Min}(U_{\text{eff}})$:	Circle

Kepler's Laws:

1st Law : Elliptical Orbits (Sun [at/orbiting] focus)

2nd Law : Equal Area Sweep ($r^2 d\theta \;=\; \frac{l}{m} dt$)

3rd Law : $T^2 = k^2 a^3$
T, Period
a, Semi-major axis
k, "constant" $\left(\frac{2\pi}{\sqrt{G[m_{\text{planet}} + M_{\text{sun}}]}}\right)$

7 Fluid Mechanics

Bernoulli's Principle :	$\frac{\rho v^2}{2} + \rho g z + P_{\text{res}}$	$=$ constant	[Energy Density]
Fluid Conservation :	$\rho A v$	$=$ constant	[Mass Flow Rate]
Bouyant Force :	$F = \rho V g$		(ρ, V, of displaced liquid)

Water Facts:

- 1 L = 1 kg

8 Oscillators

8.1 Homogenous

$$(F = m\ddot{x}) = -kx - \overset{\text{(damp)}}{b\dot{x}}$$

$$\downarrow$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

$z_{\text{tr}}(t) = \tilde{C}e^{rt} + [\tilde{D}_{\text{opt.}}\ te^{rt}]$: $\underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}}$

$$(r^2 + 2\beta r + \omega_0^2)e^{rt} = 0$$

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Normal (Undamped): $\big(F = -kx\big) \;\Rightarrow$
$\big(\ddot{x} = -\omega_0^2 x = -\frac{k}{m}x\big)$

$$\boxed{z_{\text{tr}}(t) = \tilde{C}_1 e^{i\omega_0 t} + \tilde{C}_2 e^{-i\omega_0 t}}$$

Underdamped: $\big(\beta < \omega_0\big)$

$$\boxed{z_{\text{tr}}(t) = \left(\tilde{C}_1 e^{i\sqrt{\omega_0^2-\beta^2}t} + \tilde{C}_2 e^{-i\sqrt{\omega_0^2-\beta^2}t}\right)\underline{e^{-\beta t}}}$$

Critically Damped: $\big(\beta = \omega_0\big)$

$$\boxed{z_{\text{tr}}(t) = \big(\tilde{C}_1 + \tilde{C}_2 t\big)\underline{e^{-\beta t}}}$$

Decay rate is maximized at $\beta = \omega_0$

Overdamped: $\big(\beta > \omega_0\big)$

$$\boxed{z_{\text{tr}}(t) = \underbrace{\tilde{C}_1 e^{-\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t}}_{\text{(smaller, lasts longer)}} + \tilde{C}_2 e^{-\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t}}$$

8.2 Inhomogenous (Driven)

$$\boxed{z(t) = z_{\text{st}}(t) + z_{\text{tr}}(t)}$$

$$m\ddot{x} = -kx - b\dot{x} + F_{\text{dr}}$$

$$\downarrow$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t$$

- $L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t)$

$\boxed{z_{\text{st}}(t) = \tilde{C}e^{i\omega t} = Ae^{i(\omega t-\delta)}}$: $\underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}}$

$$(-\omega^2 + 2i\beta\omega + \omega_0^2)\tilde{C}e^{i\omega t} = f_0 e^{i\omega t}$$

$$\tilde{C} = \frac{f_0}{\omega_0^2-\omega^2+2i\beta\omega} = Ae^{-i\delta}$$

$$\boxed{A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}\ ,\quad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)}$$

Resonance (Max A^2) with fixed ω : $\boxed{\omega_0 = \omega}$

Resonance (Max A^2) with fixed ω_0 : $\boxed{\omega = \sqrt{\omega_0^2 - 2\beta^2}}$ (usually $\beta \ll \omega$)

Full Width at Half Max, $A^2(\omega)$: FWHM $\approx 2\beta$

Quality Factor (Sharpness) : $Q = \frac{\omega_0}{2\beta} = \left(\pi\frac{1/\beta}{2\pi/\omega_0} = \pi\frac{\text{decay time}}{\text{period}}\right) = \left(2\pi\frac{\text{Energy stored}}{\text{Energy Dissipated}}\right)$

8.3 Parallel and Series

Series, k_1+k_2+m: $\dfrac{1}{K_{\text{eq}}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$

Parallel, k_1k_2+m: $K_{\text{eq}} = k_1 + k_2$

8.4 Normal Modes: 3 Springs + 2 Masses, k_1+m_1+k_2+m_2+k_3

1.)
$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2x_1 + k_2x_2 \\ &= -(k_1+k_2)x_1 + k_2x_2 \\[1em] m_2\ddot{x}_2 &= k_2x_1 - k_2x_2 - k_3x_2 \\ &= k_2x_1 - (k_2+k_3)x_2 \end{aligned} \quad \rightarrow \quad \begin{gathered} \mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x} \\ \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{gathered}$$

2.)
$$\begin{aligned} \mathbf{z}(t) = \mathbf{a}e^{i\omega t} &= \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} e^{i\omega t} \\ &= \begin{pmatrix} a_1e^{-i\delta_1 t} \\ a_2e^{-i\delta_2 t} \end{pmatrix} e^{i\omega t} \end{aligned} \quad \rightarrow \quad \begin{aligned} \mathbf{M}\ddot{\mathbf{z}} &= -\mathbf{K}\mathbf{z} \\ -\omega^2\mathbf{M}\mathbf{a}e^{i\omega t} &= -\mathbf{K}\mathbf{a}e^{i\omega t} \\ (\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} &= 0 \\ \boxed{\det(\mathbf{K} - \omega^2\mathbf{M}) = 0} \end{aligned}$$

$x(t) = \text{Re}[z(t)]$ is the real solution.

Same m and k

$$\begin{pmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 m \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \quad \rightarrow \quad \boxed{\omega = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}}$$

Smaller ω_1 is most symmetric motion (both swing in phase)

Larger ω_2 swings out of phase

$$\boxed{z(t) = \tilde{A}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_1 t} + \tilde{A}_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_2 t}}$$

Weak Coupling

8.5 Single Pendulum (Use Lagrangian)

- $T = \frac{1}{2} m R^2 \dot{\theta}^2$
- $U = mg(R - R\cos\theta)$

$$\rightarrow \quad \begin{aligned} mR^2\ddot{\theta} &= -mgR\sin\theta \\ &\approx -mgR\theta \end{aligned} \quad \rightarrow \quad \boxed{\begin{gathered} \ddot{\theta} = -\left(\frac{g}{I/mR}\right)\theta = -\omega^2\theta \\ \theta(t) = \mathrm{Re}\big[C_1 e^{i\omega t} + C_2 e^{-i\omega t}\big] \end{gathered}}$$

8.6 Double Pendulum (Use Lagrangian)

- $T = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^{\,2} + \frac{1}{2} m_2 (L_1 \dot{\vec{\theta}}_1 + L_2 \dot{\vec{\theta}}_2)^2$
  $= \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\theta}_1^{\,2} + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^{\,2}$
  $+ m_2 L_2 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$
- $U = m_1 g (L_1 - L_1\cos\theta_1)$
  $+ m_2 g (L_1 + L_2 - L_2\cos\theta_2 - L_1\cos\theta_1)$

$\rightarrow$

$$\mathbf{M}\ddot{\theta} = -\mathbf{K}\theta \qquad \text{(small angle quadratic approx.)}$$

$$\begin{pmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = - \begin{pmatrix} (m_1+m_2) g L_1 + k_2 & 0 \\ 0 & m_2 g L_2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$