# 1 Maxwell's Equations

### Gauss's Law for Electricity (GLE)

$$\boxed{\oiint \vec{E} \cdot d\vec{a} = \frac{Q}{\epsilon_0}}$$

$$\iiint (\nabla \cdot \vec{E})\, dV = \frac{\iiint \rho\, dV}{\epsilon_0}$$

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$

### Gauss's Law for Magnetism (GLM)

$$\boxed{\oiint \vec{B} \cdot d\vec{a} = 0}$$

$$\iiint (\nabla \cdot \vec{B})\, dV = 0$$

$$\boxed{\nabla \cdot \vec{B} = 0}$$

### Faraday's Law of Induction (FLI)

$$\boxed{\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}}$$

$$\iint (\nabla \times \vec{E}) \cdot d\vec{a} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{a}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

### Maxwell-Ampere's Law (MAL)

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 \left( I + \epsilon_0 \frac{d\Phi_E}{dt} \right)}$$

$$\iint (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \iint \vec{J} \cdot d\vec{a} + \mu_0 \epsilon_0 \frac{d}{dt} \iint \vec{E} \cdot d\vec{a}$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)}$$

### Lorentz Force Law (LFL)

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}}$$

$$\boxed{\vec{f} \equiv \frac{\partial \vec{F}}{\partial V} = \rho\vec{E} + \vec{J} \times \vec{B}}$$

Jefimenko Equations:

$$\vec{E}(\vec{\mathbf{r}}, t) = k_\epsilon \int \left[ \frac{\rho(\vec{\mathbf{r}}, t_r)}{\mathscr{r}^2}\, \hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{\mathbf{r}}, t_r)}{c\, \mathscr{r}}\, \hat{\mathscr{r}} - \frac{\dot{\vec{J}}(\vec{\mathbf{r}}, t_r)}{c^2\, \mathscr{r}} \right] d\tau'$$

$$\vec{B}(\vec{\mathbf{r}}, t) = k_\mu \int \left[ \frac{\vec{J}(\vec{\mathbf{r}}, t_r)}{\mathscr{r}^2} + \frac{\dot{\vec{J}}(\vec{\mathbf{r}}, t_r)}{c\, \mathscr{r}} \right] \times \hat{\mathscr{r}}\, d\tau'$$

## Conservation of Charge (COC)

$$\boxed{\nabla \cdot J = -\frac{\partial \rho}{\partial t}}$$

$$\left[\frac{\partial}{\partial t}\right] \left(\nabla \cdot E = \frac{\rho}{\epsilon_0}\right)$$

$$\leftarrow \left(\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}\right)$$

$$\boxed{\nabla \cdot \frac{\partial E}{\partial t} = \frac{\partial \rho}{\partial t}}$$

$$\left[\nabla \cdot\right] \left(\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}\right) \leftarrow \left(\nabla \cdot E = \frac{\rho}{\epsilon_0}\right)$$

$$\boxed{0 = 0}$$

$$\left[\frac{\partial}{\partial t}\right] \left(\nabla \cdot B = 0\right) \qquad \leftarrow \left(\nabla \times E = -\frac{\partial B}{\partial t}\right)$$

$$\left[\nabla \cdot\right] \left(\nabla \times E = -\frac{\partial B}{\partial t}\right) \leftarrow \left(\nabla \cdot B = 0\right)$$

## Laplacian

$$\boxed{\nabla^2 E = \frac{\nabla \rho}{\epsilon_0} + \nabla \times \frac{\partial B}{\partial t}}$$

$$\left[\nabla\right]\left(\nabla \cdot E = \frac{\rho}{\epsilon_0}\right) \leftarrow \left(\nabla \times E = -\frac{\partial B}{\partial t}\right)$$

$$\boxed{\nabla^2 B = -\mu_0(\nabla \times J) - \nabla \times \frac{\partial E}{\partial t}}$$

$$\left[\nabla\right]\left(\nabla \cdot B = 0\right) \leftarrow \left(\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}\right)$$

## D'alambertian

$$\boxed{\Box^2 B = \nabla^2 B - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = -\mu_0(\nabla \times J)}$$

$$\left[\nabla \times\right] \left(\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}\right) \leftarrow \begin{pmatrix} \nabla \cdot B = 0 \\ \nabla \times E = -\frac{\partial B}{\partial t} \end{pmatrix}$$

$$\left[\frac{\partial}{\partial t}\right] \left(\nabla \times E = -\frac{\partial B}{\partial t}\right)$$

$$\leftarrow \left(\frac{\partial E}{\partial t} = c^2(\nabla \times B - \mu_0 J)\right)$$

$$\boxed{\nabla \times \frac{\partial E}{\partial t} + \frac{\partial^2 B}{\partial t^2} = 0}$$

$$\boxed{\Box^2 E = \nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial J}{\partial t} + \frac{\nabla \rho}{\epsilon_0}}$$

$$\left[\nabla \times\right] \left(\nabla \times E = -\frac{\partial B}{\partial t}\right) \qquad \leftarrow \begin{pmatrix} \nabla \cdot E = \frac{\rho}{\epsilon_0} \\ \nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t} \end{pmatrix}$$

$$\left[\frac{\partial}{\partial t}\right] \left(\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}\right)$$

$$\leftarrow \left(\frac{\partial B}{\partial t} = -\nabla \times E\right)$$

$$\boxed{\nabla \times \frac{\partial B}{\partial t} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial J}{\partial t}}$$

## 1.1 Special Cases

Zero Fields :

$$\partial_t B = 0 = \nabla \times E \quad \Rightarrow \quad \left[\nabla^2 E = \frac{\nabla\rho}{\epsilon_0}\right] \;,\; \left[\nabla^2 B = -\mu_0(\nabla \times J)\right] \;,\; \left[\epsilon_0 \frac{\partial^2 E}{\partial t^2} = -\frac{\partial J}{\partial t}\right]$$

$$\partial_t E = 0 = \nabla \times B - \mu_0 J \quad \Rightarrow \quad \left[\frac{\partial \rho}{\partial t} = 0\right] \;,\; \left[\nabla^2 B = -\mu_0(\nabla \times J)\right] \;;\; \left[\frac{\partial^2 B}{\partial t^2} = 0\right] \;,\; \left[\nabla^2 E = \mu_0 \frac{\partial J}{\partial t} + \frac{\nabla\rho}{\epsilon_0}\right]$$

- $\partial_t B \;,\; \partial_t E = 0 \;\Rightarrow\; \frac{\partial J}{\partial t} \;,\; \frac{\partial \rho}{\partial t} = 0$
- $\boxed{\rho \;,\; J = 0 \;\Rightarrow\; \Box^2 E \;,\; \Box^2 B = 0}$

From Jefimenko :

$$\frac{\partial J}{\partial t} = 0 \;\Rightarrow\; \left[\partial_t B = 0\right] \;\Rightarrow\; \begin{pmatrix} \frac{\partial^2 E}{\partial t^2} = 0 \\ \nabla^2 E = \frac{\nabla \rho}{\epsilon_0} \end{pmatrix}$$

$$\frac{\partial \rho}{\partial t} = 0 \;,\; \frac{\partial^2 J}{\partial t^2} = 0 \;\Rightarrow\; \left[\partial_t E = 0\right]$$

- $\frac{\partial \rho}{\partial t} \;,\; \frac{\partial J}{\partial t} = 0 \;\Rightarrow\; \partial_t B \;,\; \partial_t E = 0$

Slight rationalization:

If $\boldsymbol{\frac{\partial \rho}{\partial t} = 0}$ : $-\frac{\partial Q}{\partial t} = \oiint J \cdot da = I|_a = 0 = \frac{\partial I}{\partial a}$

If $\boldsymbol{\frac{\partial J}{\partial t} = 0}$ : $\iint \frac{\partial J}{\partial t} \cdot da = \frac{\partial I}{\partial t} = 0$

- $I_{(a,t)} = I_0$
- $\partial_t\big(E = k\int \frac{\rho \hat{\boldsymbol{\imath}} dV}{\|\boldsymbol{\imath}\|^2}\big) = 0$

  $\rightarrow \nabla \times B = \mu_0 J \rightarrow B = k_\mu \int \frac{I d\vec{l} \times \hat{\boldsymbol{\imath}}}{\|\boldsymbol{\imath}\|^2}$
- $\partial_t B = 0$

$$** \boxed{\boldsymbol{\frac{\partial E}{\partial t} \;,\; \frac{\partial B}{\partial t} = 0 \;\Leftrightarrow\; \frac{\partial \rho}{\partial t} \;,\; \frac{\partial J}{\partial t} = 0}} **$$

Single charges do not constitute a current. Free charges (unaffected by external forces holding them) will always, by self-interaction through their E-field, create an unconstant current and thus unconstant B-field. Under static conditions, a free space must have a $\rho = 0$. See below for more info.

Electrostatic Metal Conductors (when $J = 0 = B$)

— Suppose there is a finite set of positive point charges $\{\delta_i \mid i \leq n,\ v_i = 0,\ E(x_i) = 0\}$, such as a charge at every integer coordinate in $\mathbb{R}^3$. If $\delta_i$ is a free charge only to move by electrical forces, by Gauss's Law the only way $x_i$ is a stable critical point is if there is a negative charge at $x_i$, which there isn't. $\forall n > 1$, any free charge in $\mathbb{R}^3$ cannot be in stable equilibrium. See below for neutral equilibrium.

— On a surface/boundary where movement is restricted to lower dimensions (perhaps by mechanical forces), Gauss's Law won't apply, so these lesser free charges can be in stable equilibrium, such as charges evenly distributed over a sphere. Note that E = 0 at $x_i$ and symmetric points in between $x_i$. In the $\lim n \to \infty$ then $E_{\|} = 0$ for the entire boundary.

— If a surface dist. creates a neutral equi. inside the surface (see right) and a point charge were inside the boundary at $x_i$, the surface charges will polarize and make $E(x_i)$ unstable.

- $\rho(\vec{r}_m) = 0 \;\Rightarrow\; E = E_{!m}$

$$\left[\underline{\vec{E}_{!m\|}(\vec{b}) = 0} \;\Rightarrow\; \oint_{l \in B} \vec{E} \cdot d\vec{l} = 0\right] \Leftrightarrow \underline{\left[\phi_{!m}(\vec{b}) = \phi_0\right]}$$

$$\wedge \;\; \vec{E}_{!m}(\vec{r}_m) \neq 0 \;\Rightarrow\; \oint_{l \in B, V_m} \vec{E} \cdot d\vec{l} \neq 0 \;\; \boxtimes$$

$$\Rightarrow \;\; \underline{\vec{E}_{!m}(\vec{r}_m) = 0} \;,\; \phi_{!m}(\vec{r}_m) = \phi_0 \qquad \text{(neutr. equi.)}$$

$$\vec{E}_{!m\|}(\vec{b}) \neq 0 \;\wedge\; \vec{E}_{!m}(\vec{r}_m) \neq 0$$

$$\wedge \;\; J = J_0 \neq 0 \qquad \big(\text{circuit wire}\big)$$

## 1.2 Electrostatic/Magnetostatic Examples

**Using GLE**

### 1. Point Charges

$\vec{E} = E(r)\hat{r}$

$$\frac{Q}{\epsilon_0} = \oiint E(r)\hat{r} \cdot d\vec{a}$$
$$= E(r)\hat{r} \cdot \oiint r^2 \sin\phi \; |d\vec{\phi} \times d\vec{\theta}|$$
$$= E(r)\hat{r} \cdot 4\pi r^2 \hat{r}$$

$$\vec{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \;\Rightarrow\; \frac{1}{4\pi\epsilon_0}\sum_i \frac{Q_i}{|\vec{\mathscr{r}}_i|^2}\hat{\mathscr{r}}_i$$

**Coulomb's Law (CL):**

$$\boxed{\vec{E}(\mathbf{\vec{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{dQ}{|\vec{\mathscr{r}}|^2}\hat{\mathscr{r}} \qquad \mathbf{\vec{r}} \in \mathbb{R}^3, \;\; \vec{\mathscr{r}} = \mathbf{\vec{r}} - \vec{l'}}$$

$$\boxed{\vec{F}(\mathbf{\vec{r}}) \;=\; q\vec{E}}$$

**Using MAL**

**Biot-Savart Law (BSL):**
(see potential, $\vec{A}$, for derivation)

$$\boxed{\begin{aligned} \vec{B}(\mathbf{\vec{r}}) &= k_\mu \int \frac{\vec{J}dV \times \hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}, \qquad \mathbf{\vec{r}} \in \mathbb{R}^3, \quad \vec{\mathscr{r}} = \mathbf{\vec{r}} - \vec{l'} \\ &= k_\mu \int \frac{I(\vec{l'})\; d\vec{l'} \times \hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2} \end{aligned}}$$

$$\boxed{\vec{F} = q\hat{v} \times \vec{B}}$$

### 1. Infinite Line w/ Steady Current (SC)

$\vec{B} = B(r)\hat{\theta}$

Use MAL

$$\mu_o I = \oint B(r)\hat{\theta} \cdot d\vec{L}$$
$$= B(r)\hat{\theta} \cdot \oint r d\vec{\theta}$$
$$= B(r)\hat{\theta} \cdot 2\pi r\hat{\theta}$$

$$\vec{B}(r) = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$

cont.

## 2. Sphere w/ Constant Charge Density (CCD)

Let $R$ be the radius.

$$\vec{E}(r) = \frac{Q(r)}{4\pi\epsilon_0 r^2}\hat{r}$$

$$Q(r) = \begin{cases} Q & (r > R) \\ Q_{r<R} = \iiint_0^r \frac{dQ}{dV}\ dV & (r < R) \end{cases}$$

- *Conductor*

$$Q_{r<R} = 0 \quad \Rightarrow \quad \vec{E}(r) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & (r > R) \\ 0 & (r < R) \end{cases}$$

- *Insulator w/ CCD and $\epsilon = \epsilon_0$*

$$Q_{r<R} = \int \tfrac{Q}{V}\ dV = Q\tfrac{\int dV}{V} = Q\tfrac{r^3}{R^3}$$

$$\Rightarrow \ \vec{E}(r) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & (r > R) \\ \\ \dfrac{Qr}{4\pi\epsilon_0 R^3}\hat{r} & (r < R) \end{cases}$$

Use BSL

$$\begin{aligned} \vec{B}(r) &= \frac{\mu_0 I}{4\pi}\int \frac{d\vec{z} \times \hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} \\ &= \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{d\vec{z} \times \hat{\boldsymbol{\mathscr{r}}}}{r^2 + z^2} \\ &= \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{d\vec{z} \times (r\hat{r} - z\hat{z})}{(r^2 + z^2)^{3/2}} \\ &= \frac{\mu_0 I r}{4\pi}\int_{-\infty}^{\infty} \frac{dz}{(r^2 + z^2)^{3/2}}\ \hat{\theta} \\ &= \frac{\mu_0 I r}{4\pi} \frac{z}{r^2\sqrt{r^2 + z^2}}\Bigg|_{-\infty}^{\infty}\ \hat{\theta} \\ &= \frac{\mu_0 I}{2\pi r}\hat{\theta} \end{aligned}$$

**Constant Field Solutions (Capacitor/Solenoid)**

### 3. Two Infinite Parallel Planes Capacitor w/ CCD ($+Q, -Q$)

$\vec{E} = E(z)\hat{z}$

$$\begin{aligned}\frac{Q}{\epsilon_0} &= \oiint E(z)\hat{z} \cdot d\vec{a} \\ &= E(z)\hat{z} \cdot \oiint xy\ |d\vec{x} \times d\vec{y}| \\ &= E(z)\hat{z} \cdot xy\hat{z}\end{aligned}$$

$$\vec{E}(z) = \frac{1}{\epsilon_0}\frac{dQ}{dA}\hat{z} = \frac{\sigma}{\epsilon_0}\hat{z}$$

### 4. One Infinite Plane w/ CCD

$\vec{E} = E(z)\hat{z}$

$$\begin{aligned}\frac{Q}{\epsilon_0} &= \oiint E(z)\hat{z} \cdot d\vec{a} \\ &= E(z)\hat{z} \cdot 2\oiint xy\ |d\vec{x} \times d\vec{y}| \\ &= E(z)\hat{z} \cdot 2xy\hat{z}\end{aligned}$$

$$\vec{E}(z) = \frac{1}{2\epsilon_0}\frac{dQ}{dA}\hat{z} = \frac{\sigma}{2\epsilon_0}\hat{z}$$

### 2. Infinite Long Solenoid Coil w/ SC

Let $R$ be the coil radius. $\vec{B} = B(r)\hat{z} = B(r < R)\hat{z}$

$$\begin{aligned}\mu_o NI &= \oint B(r)\hat{z} \cdot d\vec{L} \\ &= (B_{r<R})\hat{z} \cdot \oint d\vec{L} \\ &= (B_{r<R})\hat{z} \cdot L\hat{z}\end{aligned}$$

$$\vec{B}(r) = \begin{cases} \frac{\mu_0 IN}{L}\hat{z} = \mu_0 In_l\ \hat{z} & (0 < r < R) \\ 0 & (r > R) \end{cases}$$

### 3. Closed, Thin Solenoid Ring w/ SC

Let $R_l$ be the ring radius and $R_c$ be the coil radius.

$\vec{B}\ =\ B(r)\hat{\theta}\ =\ B(R_l - R_c < r < R_l + R_c)\hat{\theta}\ \approx\ B(R_l)\hat{\theta}$

$$\begin{aligned}\mu_o NI &= \oint B(r)\hat{\theta} \cdot d\vec{L} \\ &= B(r)\hat{\theta} \cdot 2\pi R_l\ \hat{\theta}\end{aligned}$$

$$\vec{B}(r) = \begin{cases} \dfrac{\mu_0 IN}{2\pi R_l}\hat{\theta} & (R_{l-c} < r < R_{l+c}) \\ 0 & \text{else} \end{cases}$$

**Integrate w/ CFL and BSL**

## 5. Ring w/ CCD centered at origin $O$

Let $R$ be the ring radius, $\lambda$ the charge density, and $\phi$ be $\angle OzR$.

$$E(0,0,z)\hat{z} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}^2}\,\hat{\mathscr{r}}$$

$$\vec{E}(0,0,z) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(Rd\theta)}{\mathscr{r}^2}\,\hat{\mathscr{r}}_z$$

$$= \frac{k\lambda}{z^2+R^2}\int_0^{2\pi} Rd\theta \;\; \cos\phi \;\hat{z}$$

$$= \boxed{\frac{k\lambda(2\pi R)z}{(z^2+R^2)^{3/2}}\;\hat{z}}$$

$$= \boxed{\frac{2\pi R}{D}\;\frac{k\lambda}{D}\cos\phi\;\hat{z}}$$

## 4. Ring w/ SC centered at origin $O$

Let $R$ be the ring radius and $\phi$ be $\angle OzR$.

$$B(0,0,z)\hat{z} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}$$

$$\vec{B}(0,0,z) = \frac{\mu_0 I}{4\pi}\int \frac{Rd\vec{\theta}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{k_\mu I}{z^2+R^2}\int Rd\theta\;\hat{\theta}\times\frac{z\hat{z}-R\hat{r}}{\sqrt{z^2+R^2}} \qquad \text{or} \qquad \frac{k_\mu I}{z^2+R^2}\int_0^{2\pi} Rd\theta\;\;\sin\phi\;\hat{z}$$

$$= \frac{k_\mu IR}{(z^2+R^2)^{3/2}}\left[2\pi R\hat{z} + z\cancel{\int_0^{2\pi}\hat{r}d\theta}\right] \qquad \text{or} \qquad \frac{k_\mu I}{z^2+R^2}\int_0^{2\pi} Rd\theta\;\tfrac{R}{\sqrt{z^2+R^2}}\;\hat{z}$$

$$= \boxed{\frac{k_\mu I(2\pi R)R}{(z^2+R^2)^{3/2}}\;\hat{z}}$$

$$= \boxed{\frac{2\pi R}{D}\;\frac{k_\mu I}{D}\sin\phi\;\hat{z}}$$

### 6. Line Charge w/ CCD and one edge at the origin $O$

Let $L$ be the line length, $\lambda$ the charge density, and $\theta_0$ be $\angle OyL$.

$$E_y(0,0,y)\hat{y} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\boldsymbol{\mathscr{r}}^2}\,\hat{\boldsymbol{\mathscr{r}}}_y$$

$$\begin{aligned}
E_y(0,0,y) &= k\int \frac{\lambda dx}{\boldsymbol{\mathscr{r}}^2}\,\hat{\boldsymbol{\mathscr{r}}}_y \\
&= k\lambda\int \frac{dx}{y^2+x^2}\cos\theta \quad , \qquad x = y\tan\theta \\
&= \frac{k\lambda}{y^2}\int_0^{\theta_0} \frac{y\sec^2\theta\, d\theta}{1+\tan^2\theta}\cos\theta \\
&= \frac{k\lambda}{y}\int_0^{\theta_0} \cos\theta\, d\theta = \frac{k\lambda}{y}\sin\theta_0 \\
&= \boxed{\frac{k\lambda L}{y\sqrt{y^2+L^2}}} \quad = \quad \boxed{\frac{k\lambda}{y}\sin\theta_0}
\end{aligned}$$

Use this result to find *Finite Line*, *Square Ring*, and *Infinite Line*.

### 5. Finite Wire w/ SC and one edge at the origin $O$

Let $L$ be the wire length and $\theta_0$ be $\angle OyL$.

$$B_z(0,0,y)\hat{z} = k_\mu I\int \frac{d\vec{l}\times\hat{\boldsymbol{\mathscr{r}}}}{|\vec{\boldsymbol{\mathscr{r}}}|^2}$$

$$\begin{aligned}
B_z(0,0,y)\,\hat{z} &= k_\mu I\int \frac{d\vec{x}\times\hat{\boldsymbol{\mathscr{r}}}}{|\vec{\boldsymbol{\mathscr{r}}}|^2} \\
&= k_\mu I\int \frac{dx\,\hat{z}}{y^2+x^2}\sin(\theta+90) \quad , \qquad x = y\tan\theta \\
&= \frac{k_\mu I}{y^2}\int_0^{\theta_0} \frac{y\sec^2\theta\, d\theta}{1+\tan^2\theta}\cos\theta\,\hat{z} \\
&= \frac{k_\mu I}{y}\int_0^{\theta_0} \cos\theta\, d\theta\,\hat{z} = \frac{k_\mu I}{y}\sin\theta_0\,\hat{z} \\
&= \boxed{\frac{k_\mu I L}{y\sqrt{y^2+L^2}}\,\hat{z}} = \quad \boxed{\frac{k_\mu I}{y}\sin\theta_0\,\hat{z}}
\end{aligned}$$

Use this result to find *Finite Wire*, *Square Wire*, and *Infinite Wire*.

## 1.3 Field Energies

The sum of the work to move a collection of charges considering the potential from each other charge comes out to be

$$\boldsymbol{W = \frac{1}{2}\sum_i q_i V(r_i)}$$

E-field Energy (electrostatic...)

$$E = \frac{1}{2}CV^2 = \frac{1}{2}VQ$$

$$\boxed{W_{\text{vol}} = \frac{1}{2}\iiint V\rho\ d\tau} = \frac{\epsilon_0}{2}\iiint V(\nabla\cdot\vec{E})\ d\tau$$

$$= \frac{\epsilon_0}{2}\iiint \vec{E}\cdot(\vec{E}) + \nabla\cdot(V\vec{E})\ d\tau$$

$$= \boxed{\frac{\epsilon_0}{2}\iiint \vec{E}^2\ d\tau + \frac{\epsilon_0}{2}\iint (V\vec{E})\cdot d\vec{a}}$$

$$\boxed{W_E = \frac{\epsilon_0}{2}\iiint \vec{E}^2\ d\tau} \quad (\text{if } \rho = 0 \text{ at } \infty)$$

B-field Energy

$$E = \frac{1}{2}LI^2 = \frac{1}{2}\Phi_B I = \frac{1}{2}\oint I\vec{A}\cdot d\vec{l}$$

$$\boxed{W_{\text{vol}} = \frac{1}{2}\iiint \vec{A}\cdot\vec{J}\ d\tau} = \frac{1}{2\mu_0}\iiint \vec{A}\cdot(\nabla\times\vec{B})\ d\tau$$

$$= \frac{1}{2\mu_0}\iiint \vec{B}\cdot(\vec{B}) - \nabla\cdot(\vec{A}\times\vec{B})\ d\tau$$

$$= \boxed{\frac{1}{2\mu_0}\iiint \vec{B}^2\ d\tau - \frac{1}{2\mu_0}\oiint(\vec{A}\times\vec{B})\cdot d\vec{a}}$$

$$\boxed{W_B = \frac{1}{2\mu_0}\iiint \vec{B}^2\ d\tau} \quad (\text{if } \vec{I} = 0 \text{ at } \infty)$$

$$V(\vec{x}) = V\big|_0 + \vec{x}\cdot\vec{\nabla}V\big|_0 + \tfrac{1}{2}\sum_{i,j} x_i x_j \tfrac{\partial}{\partial x_i}\tfrac{\partial}{\partial x_j}V\big|_0 + \ldots = V\big|_0 - \vec{x}\cdot\vec{E}\big|_0 - \tfrac{3}{6}\sum_{i,j} x_i x_j \tfrac{\partial}{\partial x_i}E_j\big|_0 + \underbrace{\left[\tfrac{1}{6}r^2\vec{\nabla}\cdot\vec{E}\big|_0\right]}_{0} + \ldots$$

$$W_{\text{ext.}} = \int \rho(x')V_{\text{ext.}(x')}d\tau' = \boxed{qV\big|_0 - \vec{p}\cdot\vec{E}\big|_0 - \tfrac{1}{6}\sum_{i,j} Q_{ij}\tfrac{\partial}{\partial x_i}E_j\big|_0 + \ldots}$$

## 1.4 Circuits/Ohm's Law

Ohm's Law

In Ohmic material, $\rightarrow$ $\boldsymbol{\sigma}$: Conductivity $J \approx \sigma(E + v \times B) \Rightarrow \boxed{J \approx \sigma E}$ $\boxed{V = IR}$ $\boxed{P = VI}$

Example: Wire w/ Two Plates

$$I = (\sigma E)A = \left(\frac{\sigma A}{L}\right) V \quad \Rightarrow \quad V = I\left(\frac{L}{\sigma A}\right) = IR$$

| | | | | | |
|---|---|---|---|---|---|
| Resistor: | $V_R = IR$ | $P = VI$ | | $R = \dfrac{\rho l}{A}$ | $Z_R = R$ |
| Capacitor: | $Q = CV_C$ | $E_C = \dfrac{1}{2}CV_C^2$ | $Q_\uparrow = Q_0(1 - e^{-t/\tau})$<br>$Q_\downarrow = Q_0 e^{-t/\tau}$ | $C = \dfrac{\kappa \epsilon_0 A}{d} = \dfrac{\epsilon A}{d}$ | $Z_C = \dfrac{1}{i\omega C}$ |
| Inductor: | $\Phi_B = LI$<br>$V_L = -L\dfrac{dI}{dt}$ | $E_L = \dfrac{1}{2}LI^2$ | $I_\uparrow = I_0(1 - e^{-t/\tau})$<br>$I_\downarrow = I_0 e^{-t/\tau}$ | $L = \dfrac{\mu_0 N^2 A}{l}$ | $Z_L = i\omega L$ |

Open Circuit : $R = \infty$
Short Circuit : $R = 0$

Constants

$\tau_{RC} = RC$

$\tau_{RL} = L/R$

$\omega_{R,LC} = 1/\sqrt{LC}$

AC Filters

Low Pass (Non-Zero $\overline{\text{Probe}}$) :

- $R\overline{C}$
- $L\overline{R}$

High Pass (Non-Zero $\overline{\text{Probe}}$) :

- $C\overline{R}$
- $R\overline{L}$

Band Pass (Zero $\overline{\text{Probe}}$) :

- $R\overline{LC}$
- Bandwidth $= \left(\text{FWHM} = 2\beta = \dfrac{b}{m}\right) = \dfrac{R}{L}$

Other Components

| | | |
|---|---|---|
| $- > \|-$ | Diode | One Way Voltage (if > Bias Voltage) |
| $= \| > -$ | Op-Amp | $V_1 - V_2 \propto V_{\text{OA}}$ (Clipping If Too Large $V_{\text{OA}}$) |
| $= \|)-$ | And | |
| $=) > -$ | Or | |

De Morgan's Law

- $\overline{A \cdot B} = \overline{A} + \overline{B}$
- $\overline{A + B} = \overline{A} \cdot \overline{B}$

## 1.5 Quasistatic (FLI)

Force on Wire in B-Field :

$F = qv \times B$

$\boxed{F = LI \times B}$

EMF :

$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt} = \oint \vec{E} \cdot d\vec{l}}$$

Mutual Inductance (See Inductor):

$$\boxed{\begin{array}{ll} \text{Flux Through } B: & \Phi_B = MI_A;\ V_A = -M\frac{dI_A}{dt} \\ \text{Flux Through } A: & \Phi_A = MI_B;\ V_B = -M\frac{dI_B}{dt} \end{array}}$$

Faraday's Law

1. Lorentz:

   Square Circuit with $\vec{v}(t)$ leaving Constant B-Field (out)

   $I$ is out

2. Faraday:

   Constant B-Field (out) with $-\vec{v}(t)$ leaving Square Circuit

   $I$ is out

3. Faraday:

   Square Circuit in Increasing B-Field (out)

   $I$ is in

**Examples:**

$B$-Field Work:

$$\left(\vec{v} \cdot \vec{l} = 0\right) \;\rightarrow\; W_B = \int \vec{F} \cdot d\vec{l}$$

$$= \int (q\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\boxed{W_B = 0} \quad \text{(magnetic fields do no work)}$$

Velocity of wire in (1.): $I(t)R = V(t) = \left|\frac{d\Phi_B}{dt}\right| = Bhv(t)$

$$F_B(t) = -hI(t)B$$

$$= -\frac{B^2h^2v(t)}{R}$$

$$F = ma(t)$$

$$\boxed{m\frac{dv}{dt} = F_B + F_{\text{ext}} = F_{\text{ext}} - \frac{B^2h^2v(t)}{R}}$$

# 2 Potentials and Fields

## 2.1 Maxwell's Equations for Potentials

**1. GLM for Potentials (GLMP)**

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \ , \qquad \boldsymbol{\vec{A}' = \vec{A} + \nabla\lambda}$$

$$\boxed{\vec{B} = \nabla \times \vec{A}} \ \Rightarrow \ \boxed{\Phi_B = \oint \vec{A} \cdot d\vec{l}}$$

**2. FLI for Potentials (FLIP)**

$$\nabla \times \vec{E} = -\tfrac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{E} = \nabla \times \left(0 - \tfrac{\partial \vec{A}}{\partial t}\right) \ , \qquad \boldsymbol{V' = V + \tfrac{\partial \lambda}{\partial t}}$$

$$\boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}} \ \Rightarrow \ \boxed{V\Big|_a^b = -\int_a^b \vec{E} \cdot d\vec{l} - \frac{\partial}{\partial t}\int_a^b \vec{A} \cdot d\vec{l}}$$

**3. GLE for Potentials (GLEP)**

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$-\nabla^2 V - \frac{\partial(\nabla \cdot \vec{A})}{\partial t} = \frac{\rho}{\epsilon_0}$$

$$\boxed{-\Box^2 V - \frac{\partial}{\partial t}(\partial_\mu A^\mu) = \frac{\rho}{\epsilon_0}}$$

**4. MAL for Potentials (MALP)**

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} + \nabla\left(\frac{1}{c^2}\frac{\partial V}{\partial t} + \nabla \cdot \vec{A}\right) = \mu_0 \vec{J}$$

$$\boxed{-\Box^2 \vec{A} + \nabla(\partial_\mu A^\mu) = \mu_0 \vec{J}}$$

**Field Energy (see Capacitor/Solenoid in Circuits)**:

$$\boxed{W_E = \frac{1}{2}\iiint V\rho \, d\tau} \ = \frac{\epsilon_0}{2}\int E^2 \, d\tau \quad (\text{if } \rho = 0 \text{ at } \infty)$$

$$\boxed{W_B = \frac{1}{2}\iiint \vec{A} \cdot \vec{J} \, d\tau} \ = \frac{1}{2\mu_0}\int B^2 \, d\tau \quad (\text{if } \vec{J} = 0 \text{ at } \infty)$$

## 2.2 Cases and Freedoms

GLMP and FLIP say that

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

In the electrostatic case, $\boxed{\textbf{Electrostatics:} \quad \boldsymbol{\nabla \times E = \partial_t B = 0}}$

GLEP and MALP say that

$$-\nabla^2 V - \frac{\partial(\nabla \cdot \vec{A})}{\partial t} = \frac{\rho}{\epsilon_0}$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} + \nabla\left(\frac{1}{c^2}\frac{\partial V}{\partial t} + \nabla \cdot \vec{A}\right) = \mu_0 \vec{J}$$

Freedom may be chosen to what $\nabla \cdot \vec{A}$ equals:

**Coulomb Gauge:** $\boldsymbol{\nabla \cdot \vec{A} = 0}$

- **Magnetostatics:** $\boldsymbol{\partial_t E = 0 \;\Leftrightarrow\; \nabla \times B = \mu_0 J}$

**Lorenz Gauge:** $\boldsymbol{\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \;\Leftrightarrow\; \partial_\mu A^\mu = 0}$

In general, $\vec{A}$ and $V$ can be [gauge] transformed while keeping $\vec{E}$ and $\vec{B}$ the same by

$$\boxed{V' = V - \frac{d\lambda}{dt}} \qquad (\lambda \text{ is a scalar function})$$

$$\boxed{\vec{A}' = \vec{A} + \nabla\lambda}$$

**Electrostatic Potentials**

Electrostatics: $\boldsymbol{\partial_t \vec{B} = 0}$.

$$\nabla \times \vec{E} = 0 \;\Rightarrow\; \oint \vec{E} \cdot d\vec{l} = 0$$

Choose $\frac{\partial A}{\partial t} = 0 \;\Rightarrow\; \boxed{E = -\nabla V}$

$$\int_a^b \nabla \vec{V} \cdot d\vec{l} = \boxed{V(\vec{\mathbf{r}})\Big|_a^b = -\int_a^b \vec{E} \cdot d\vec{l} = W_E/q}$$

$$\boxed{V(\vec{\mathbf{r}}) = -\int \vec{E} \cdot d\vec{l} + V_0}$$

and from this (or GLEP)

*Poisson Equation*: $\boldsymbol{\nabla^2 V = -\frac{\rho(\vec{r'})}{\epsilon_0}}$

$\rho_\infty = 0 \;\Rightarrow\; \boldsymbol{\vec{V}(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'})}{\mathscr{r}}\, d\tau'}$

*PE Uniqueness Theorem*: Given $\rho$ and $V$ at the boundary, $V$ is unique in the space containing $\rho$ enclosed by the surface.

**Coulomb Gauge & Magnetostatic Potentials**

Coulomb Guage: Choose $\boldsymbol{\left(\nabla \cdot \vec{A} = 0\right)}$

Using GLEP,

*Poisson Equation*: $\boldsymbol{\nabla^2 V = -\frac{\rho(\vec{r'}, t)}{\epsilon_0}}$

$\rho_\infty = 0 \;\Rightarrow\; \boldsymbol{V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'}, t)}{\mathscr{r}}\, d\tau'}$

If charges move, $V$ updates immediately - not at light speed. Only $\vec{E}$ can be physically measured, and updates at light speed. $\vec{A}$ is difficult to find using MALP except for special cases like Magnetostatics.

As always, GLMP says

$$\boxed{\boldsymbol{\Phi_B = \oint \vec{A} \cdot d\vec{l}}}$$

Magnetostatics: $\boldsymbol{\partial_t \vec{E} = 0}$

Using MALP,

*Poisson Equation*: $\boldsymbol{\nabla^2 \vec{A} = -\mu_0 \vec{J}(\vec{r'})}$

$\vec{J}_\infty = 0 \;\Rightarrow\; \boldsymbol{\vec{A}(\vec{\mathbf{r}}) = k_\mu \int \frac{\vec{J}(\vec{r'})}{\mathscr{r}}\, d\tau'}$

### 2.2.1 Electrostatic Potential Examples

#### 1. Point Charges

Reference Choice: $V(\infty) = 0$

$$V(r) = -\int_{\infty}^{r} \frac{kQ}{(r')^2} dr' = \frac{Q}{4\pi\epsilon_0} \frac{1}{r}$$

$$\boldsymbol{V(\vec{\mathbf{r}}) \;=\; \frac{1}{4\pi\epsilon_0} \sum_i \frac{Q_i}{\mathscr{r}_i}}$$

**Coulomb Potential**: $\boxed{V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{|\vec{\mathscr{r}}_i|}}$

**Work**: $\boxed{W = \frac{1}{2} \sum q_i V(\vec{\mathbf{r}}_i)}$

#### 2. Sphere

Reference Choice: $V(\infty) = 0$

Let $R$ be the radius.

$$V(r) = -\int_{\infty}^{r} \vec{E}(r') \cdot d\vec{r'}$$

- *Conductor*

$$E(r) = \begin{cases} \dfrac{kQ}{r^2} \\ \\ 0 \end{cases} \quad \Rightarrow \quad V(r) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r} & (r > R) \\ \\ \dfrac{Q}{4\pi\epsilon_0 R} & (r < R) \end{cases}$$

- *Insulator w/ CCD and* $\epsilon = \epsilon_0$

$$E(r) = \begin{cases} \dfrac{kQ}{r^2} \\ \\ \dfrac{kQr}{R^3} \end{cases} \quad \Rightarrow \quad V(r) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r} & (r > R) \\ \\ \dfrac{Q}{4\pi\epsilon_0 R} + \left.\dfrac{Qr'^2}{8\pi\epsilon_0 R^3}\right|_r^R & (r < R) \end{cases}$$

**Charges at $\infty$**

### 3. (Infinite) Parallel Plate Capacitor

Reference Choice: $V(h) = 0$

Let the Capacitor Height be $h$

$$V(z) = -\int_h^z \frac{\sigma}{\epsilon_0}\hat{z} \cdot d\vec{z} = \frac{\sigma(h-z)}{\epsilon_0} \qquad (0 \le z \le h)$$

### 4. (Infinite) Single Plate w/ CCD

Reference Choice: $V(0) = 0$

$$V(z) = -\int_0^z \frac{\sigma}{2\epsilon_0}\hat{z} \cdot d\vec{z} = -\frac{\sigma z}{2\epsilon_0} \qquad (0 \le z < \infty)$$

Try $V(\infty) = 0$. (A charge distribution stretching to infinity DNE, so choose a diff. reference point.)

### 5. Infinite Line w/ CCD

Reference Choice: $V(1) = 0$

$$\begin{aligned} V(r) &= -\int_1^r \frac{\lambda}{2\pi r\epsilon_0}\hat{r} \cdot d\vec{r} \\ &= -\frac{\lambda}{2\pi\epsilon_0}\ln r \end{aligned}$$

Try $V(\infty) = 0$ (same problem above).

**Method of Images (Uniqueness Theorem)**

**6.** $\begin{bmatrix} \bullet\ \delta_q \textbf{ at } x_0 \\ \bullet\ V(0,y,z)=0 \end{bmatrix} == \begin{bmatrix} \bullet\ \delta_q \textbf{ at } x_0 \\ \bullet\ \delta_{-q} \textbf{ at } -x_0 \end{bmatrix}$

**7.** $\begin{bmatrix} \bullet\ \delta_q \textbf{ at } x_0 \\ \bullet\ \textbf{Conductive sphere at origin O, radius } R < x_0 \\ \quad \textbf{and charge } Q \Leftrightarrow \textbf{potential } V = \frac{k(Q+q\lambda)}{R^2}\textbf{.} \end{bmatrix}$

$$== \begin{bmatrix} \bullet\ \delta_q \text{ at } x_0 \\ \bullet\ \delta_{-q\lambda} \text{ at } \left\{x_1 \middle| \frac{R}{x_0} = \frac{x_1}{R} = \lambda\right\} \\ \bullet\ \delta_{q\lambda+Q} \text{ at } \mathbf{O} \quad (V=0 \text{ grounded sphere} \to\ Q = -q\lambda) \end{bmatrix}$$

**8.** $\begin{bmatrix} \bullet\ \delta_q \textbf{ at } x_0 \\ \bullet\ \textbf{Conductive sphere at origin O, radius } R > x_0 \\ \quad \textbf{and charge } Q \Leftrightarrow \textbf{potential } V = \frac{k(Q+q)}{R^2}\textbf{.} \end{bmatrix}$

$$== \begin{bmatrix} \bullet\ \delta_q \text{ at } x_0 \\ \bullet\ \delta_{-q/\lambda} \text{ at } \left\{x_1 \middle| \frac{R}{x_0} = \frac{x_1}{R} = \lambda\right\} \\ \bullet\ \delta_{q+Q} \text{ at } \mathbf{O} \quad (V=0 \text{ grounded sphere} \to\ Q = -q) \end{bmatrix}$$

**9.** $\begin{bmatrix} \bullet\ \delta_{\pm q} \textbf{ at } \left\{ \mp z_0 \middle| z_0 \gg 0 \right\} \\ \bullet\ \vec{E}_{\pm q}(r \ll z_0) \approx \frac{2kqq}{r}\hat{z} = E_0\hat{z} \quad (\textbf{Const. E-field}) \\ \bullet\ \textbf{Conductive sphere at origin O, radius } R \ll z_0 \end{bmatrix}$

$$== \begin{bmatrix} \bullet\ \delta_{\pm q} \text{ at } \left\{ \mp z_0 \middle| z_0 \gg 0 \right\} \\ \bullet\ \delta_{\mp q\lambda} \text{ at } \left\{ \mp z_1 \middle| \frac{R}{z_0} = \frac{z_1}{R} = \lambda \right\} \quad (\text{Dipole}) \\ \Rightarrow\ V(r{\ge}R, \theta) \approx -E_0(\underbrace{r - \tfrac{R^3}{r^2}}_{\mp q,\ \pm q\lambda})\cos\theta \end{bmatrix}$$

## 2.3 Green's Functions

Green's 1st Identity : $$\oiint [f\vec{\nabla}g]\cdot\hat{n}\,dS = \iiint \vec{\nabla}\cdot[f\vec{\nabla}g]d\tau = \iiint f\nabla^2 g + \vec{\nabla}f\cdot\vec{\nabla}g\,d\tau$$

Green's 2nd Identity : $$\oiint \left[f\vec{\nabla}g - g\vec{\nabla}f\right]\cdot\hat{n}\,dS = \iiint \left[f\nabla^2 g - g\nabla^2 f\right]d\tau$$

Green's 3rd Identity : $$\iiint \left[-f\nabla^2 g\right]d\tau = \iiint \left[-g\nabla^2 f\right]d\tau + \oiint \left[g\vec{\nabla}f - f\vec{\nabla}g\right]\cdot\hat{n}\,dS$$

$$\iiint \left[-V\nabla^2 G\right]d\tau = \iiint \left[-G\nabla^2 V\right]d\tau + \oiint \left[G\vec{\nabla}V\right]\cdot\hat{n}\,dS - \oiint \left[V\vec{\nabla}G\right]\cdot\hat{n}\,dS$$

$$\boxed{V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\iiint G(\vec{x},\vec{x}')\rho(\vec{x})\,d\tau' - \frac{1}{4\pi}\oiint G(\vec{x},\vec{x}')\vec{E}(\vec{x})\cdot\hat{n}\,dS' - \frac{1}{4\pi}\oiint V(\vec{x})\frac{\partial G}{\partial n'}\,dS'}$$

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$$

$$\nabla^2\frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\delta(\vec{x}-\vec{x}')$$

$$\nabla^2 F = 0$$

$$\int \nabla^2 G\,d\tau = \oint \frac{\partial G}{\partial n}dS = -4\pi$$

<u>Dirichlet Boundary Conditions</u>: $$V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\iiint G(\vec{x},\vec{x}')\rho(\vec{x})\,d\tau' - \frac{1}{4\pi}\oiint V(\vec{x})\frac{\partial G}{\partial n'}\,dS' \qquad \left(G = 0 \text{ on } S \to \text{find } \frac{\partial G}{\partial n'}\right)$$

<u>Neumann Boundary Conditions</u>: $$V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\iiint G(\vec{x},\vec{x}')\rho(\vec{x})\,d\tau' + \frac{1}{4\pi}\oiint G(\vec{x},\vec{x}')\frac{\partial V}{\partial n'}\,dS' + \langle V\rangle \qquad \left(\frac{dG}{dn} = -\frac{4\pi}{S} \text{ on } S \to \text{find } G\right)$$

<u>Dirichlet Laplace Eq. Solution on Sphere</u>:

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \frac{-R/x'}{\left|\vec{x}-\frac{R^2}{x'}\frac{\vec{x}'}{x'}\right|} = \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\theta_{xx'}}} - \frac{1}{\sqrt{x^2x'^2/R^2+R^2-2xx'\cos\theta_{xx'}}}$$

$$\left.\frac{\partial G}{\partial x'}\right|_{x'=R} = -\frac{2x'-\cancel{2x\cos\theta}}{2}\left[x^2+x'^2-2xx'\cos\theta\right]^{-3/2} + \frac{2x^2x'/R^2-\cancel{2x\cos\theta}}{2}\left[x^2\cancel{x'^2/R^2}+R^2-2xx'\cos\theta\right]^{-3/2}$$

$$= \boxed{(x^2/x'-x')\left[x^2+x'^2-2xx'\cos\theta\right]^{-3/2}}$$

$$\left.\frac{\partial G}{\partial n'}\right|_{x'=R} = -\left.\frac{\partial G}{\partial x'}\right|_{x'=R} \quad (|\vec{x}|>R) \qquad \Rightarrow \qquad \boxed{V(\vec{x}) = -\oiint V(R,\vec{x}')\frac{\partial G}{\partial n}\,dS}$$

$$= \left.\frac{\partial G}{\partial x'}\right|_{x'=R} \quad (|\vec{x}|<R)$$

$$\nabla_x^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') = -\tfrac{4\pi}{r^2}\delta(r-r')\,\delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta') = -\tfrac{4\pi}{r}\delta(\rho-\rho')\,\delta(\phi-\phi')\,\delta(z-z')$$

$$= -\tfrac{4\pi}{r^2}\delta(r-r')\sum_{l,m} Y_{lm}^* Y_{lm} \qquad = -\tfrac{4\pi}{r}\delta(\rho-\rho')\sum_m \tfrac{1}{2\pi}e^{im(\phi-\phi')}\int_{-\infty}^{\infty}\tfrac{1}{2\pi}e^{ik(z-z')}\,dk$$

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[g_l(r,r')Y_{lm}^*(\theta,\phi)\right]Y_{lm}(\theta,\phi) \qquad \Leftarrow \qquad \left[\tfrac{1}{r}\tfrac{\partial^2}{\partial r^2}r + \tfrac{l(l+1)}{r^2}\right]g_l = -\tfrac{4\pi}{r^2}\delta(r-r')$$

$$= \int_0^{\infty}\sum_{m=-\infty}^{\infty} g_m(k,\rho,\rho')\left[\tfrac{1}{2\pi}e^{im(\phi-\phi')}\right]\tfrac{1}{\pi}\cos[k(z-z')]\,dk \qquad \Leftarrow \qquad \left[\tfrac{1}{\rho}\tfrac{\partial}{\partial\rho}\left(\rho\tfrac{\partial}{\partial\rho}\right) - \left(k^2+\tfrac{m^2}{\rho^2}\right)\right]g_m = -\tfrac{4\pi}{\rho}\delta(\rho-\rho')$$

---

$$-\lambda_n\Psi_n = \left[\nabla_x^2 + f(\vec{x})\right]\Psi_n \;\Rightarrow\; \begin{array}{lcl} \left[\nabla_x^2 + f(\vec{x}) + \lambda_n\right]\Psi_n & = & 0 \\ \left[\nabla_x^2 + f(\vec{x}) + \lambda\right]G(\vec{x}-\vec{x}') & = & -4\pi\delta(\vec{x}-\vec{x}') \end{array}$$

$$G_\lambda(\vec{x}-\vec{x}') = \sum_n a_n(\vec{x}')\Psi_n(\vec{x}') = \sum_n \tfrac{4\pi}{\lambda-\lambda_n}\Psi_n^*(\vec{x}')\Psi_n(\vec{x}') \quad\Rightarrow\quad \begin{array}{l} f(\vec{x})=0 \\ \lambda_n = k^2 \end{array} :\; G_0 = \boxed{\tfrac{1}{|\vec{x}-\vec{x}'|} = \tfrac{1}{2\pi^2}\iiint \tfrac{1}{k^2}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,d^3k}$$

$$P_l(\cos\gamma) = \sum_{m=-l}^{l}|Y_{lm}(\theta,\phi)\rangle\langle Y_{lm}(\theta,\phi)|P_l(\cos\gamma) = \sum_{m=-l}^{l}\tfrac{4\pi}{2l+1}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$\tfrac{1}{|\vec{x}-\vec{x}'|} = \tfrac{1}{r}\sum_{l=0}\underbrace{\left(\tfrac{r'}{r}\right)^l}_{<1}P_l(\cos\gamma) \quad\text{or}\quad \tfrac{1}{r'}\sum_{l=0}\underbrace{\left(\tfrac{r}{r'}\right)^l}_{<1}P_l(\cos\gamma) \;=\; \tfrac{1}{r_<}\sum_{l=0}\left(\tfrac{r_<}{r_>}\right)^l\sum_{m=-l}^{l}\tfrac{4\pi}{2l+1}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

## 2.4 Multipole Expansion

$$\begin{aligned} \mathscr{r}^2 &= r^2 + (r')^2 - 2(\vec{\mathbf{r}}\cdot\vec{r'}) \\ &= r^2\left[1 + \frac{r'}{r}\left(\frac{r'}{r} - 2\frac{\vec{\mathbf{r}}\cdot\vec{r'}}{r'r}\right)\right] \\ &= r^2(1+\epsilon) \quad \text{or} \quad (r')^2(1+\epsilon') \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{1}{\mathscr{r}} &= \frac{1}{r}(1+\epsilon)^{-1/2} \\ &= \boldsymbol{\frac{1}{r}\sum_{n=0}\left(\frac{r'}{r}\right)^n P_n\left(\hat{r'}\cdot\hat{\mathbf{r}}\right)} \quad \text{(Legendre Polynomials)} \end{aligned}$$

---

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\,\rho(\vec{r'})d\tau'$$

$$= \boxed{\frac{1}{4\pi\epsilon_0}\sum_n \frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\ \rho(\vec{r'})d\tau'}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\begin{array}{c} \frac{1}{r}\int \rho(\vec{r'})d\tau' + \frac{1}{r^2}\int \vec{r'}\cdot\hat{\mathbf{r}}\ \rho(\vec{r'})d\tau' \\ + \frac{1}{r^3}\int (r')^2 P_2(\hat{r'}\cdot\hat{\mathbf{r}})\ \rho(\vec{r'})d\tau' + \frac{1}{r^4}\int ... \end{array}\right]$$

$$\boxed{\begin{aligned} V_{\text{mon}} &= \frac{1}{4\pi\epsilon_0 r}\int \rho(\vec{r'})d\tau' \\ V_{\text{dip}} &= \frac{1}{4\pi\epsilon_0 r^2}\left(\int \vec{r'}\ \rho(\vec{r'})d\tau'\right)\cdot\hat{\mathbf{r}} = \frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} \end{aligned}}$$

$V$ is the dipole term. $\mathbf{p}$ is the dipole moment.

$Q_{\text{quad. tens.}}$ : $\boxed{Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\rho(x')\,d\tau'}$

$$\vec{A}(\vec{\mathbf{r}}) = k_\mu \int \frac{1}{\mathscr{r}}\ \vec{J}(\vec{r'})d\tau'$$

$$= \boxed{k_\mu \sum_n \frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\ \vec{J}(\vec{r'})d\tau'}$$

$$= k_\mu\left[\begin{array}{c} \frac{1}{r}\int \vec{J}(\vec{r'})d\tau' + \frac{1}{r^2}\int \vec{r'}\cdot\hat{\mathbf{r}}\ \vec{J}(\vec{r'})d\tau' \\ + \frac{1}{r^3}\int (r')^2 P_2(\hat{r'}\cdot\hat{\mathbf{r}})\ \vec{J}(\vec{r'})d\tau' + \frac{1}{r^4}\int ... \end{array}\right]$$

Steady current: $\boxed{\begin{aligned} A_{\text{mon}} &= \frac{\mu_0 I}{4\pi r}\oint dl' = 0 \\ A_{\text{dip}} &= \frac{k_\mu}{r^2}I\int \vec{r'}\cdot\hat{\mathbf{r}}\ dl' = \frac{k_\mu}{r^2}I\int d\vec{a'}\times\hat{\mathbf{r}} \\ &= \frac{k_\mu}{r^2}(I\vec{a})\times\hat{\mathbf{r}} = \frac{\mu_0\vec{\mathbf{m}}\times\hat{\mathbf{r}}}{4\pi r^2} \end{aligned}}$

$A$ is the dipole term. $\mathbf{m}$ is the dipole moment.

**Ideal Dipoles**

Let dipole (2 charges) $\vec{\mathbf{p}} = p\hat{z} = 2dq\hat{z}$ and centered at the origin.

$\lim d \to 0,\ q \to \infty\ :$

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{d^n P_n(\cos\alpha)q + (-d)^n P_n(\cos\alpha)(-q)}{4\pi\epsilon_0\ r^{n+1}}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{P_n(\cos\alpha)qd^n}{r^{n+1}}[1+(-1)^n]$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{m=0}^{\infty}\frac{P_m(\cos\alpha)}{r^{2m+2}}(2qd)d^{2m}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{m=0}^{\infty}\frac{P_m(\cos\alpha)}{r^{2m+2}}pd^{2m}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{P_0(\cos\alpha)}{r^2}p + 0 + 0 + ...$$

$$\boxed{\text{Ideal Dip:} \qquad V_{\text{dip}}(\vec{\mathbf{r}}) = k_\epsilon\frac{\vec{\mathbf{p}}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}}$$

Let dipole (ring) $\vec{\mathbf{m}} = m\hat{z} = Ia\hat{z}$ and centered at the origin.

$\lim a = \pi d^2 \to 0,\ I \to \infty\ :$

$$\vec{A}(\vec{\mathbf{r}}) = k_\mu\sum_{n=0}^{\infty}\frac{Id^n}{r^{n+1}}\int P_n(\cos\alpha)\ d\vec{l'}$$

$$= k_\mu\left(\begin{array}{c} \frac{I}{r}\int d\vec{l'} + \frac{I}{r^2}\int (d\hat{r'}\cdot\hat{\mathbf{r}})\ d\vec{l'} \\ + \frac{Id^2}{r^3}\int\left[\frac{3}{2}\left(1+2\frac{d\hat{r'}\cdot\hat{\mathbf{r}}}{d}+1\right)-\frac{1}{2}\right]d\vec{l'} \\ + Id^2\sum_{n=3}^{\infty}\frac{d^{n-2}}{r^{n+1}}\int P_n(\hat{r'}\cdot\hat{\mathbf{r}})\ d\vec{l'} \end{array}\right)$$

$$= k_\mu\left(0 + \frac{I\pi d^2}{r^2}(\hat{z}\times\hat{\mathbf{r}}) + \frac{3I\pi d^3}{r^3}(\hat{z}\times\hat{\mathbf{r}}) + \frac{m}{\pi}(0+...)\right)$$

$$= k_\mu\left(0 + \frac{m}{r^2}(\hat{z}\times\hat{\mathbf{r}}) + 0 + 0 + ...\right)$$

$$\boxed{\text{Ideal Dip:} \qquad \vec{A}_{\text{dip}}(\vec{\mathbf{r}}) = k_\mu\frac{\vec{\mathbf{m}}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}}$$

### 2.4.1 Multipole Examples

$$\underline{\underline{\text{Vertical Origin}}}\ \underline{\text{Dipole } E\text{-Field}}: \quad \vec{E}(\vec{x}) = -\vec{\nabla}V = -k\vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right) = -k\left[\frac{-3\hat{r}(\vec{p}\cdot\vec{r})}{r^4} + \frac{1}{r^3}\vec{\nabla}\left(\vec{p}\cdot\vec{r}\right)\right] = k\frac{3\hat{r}(\vec{p}\cdot\hat{r})-p\hat{z}}{r^3} = \boxed{k\frac{3\hat{r}(\vec{p}\cdot\hat{r})-\vec{p}}{r^3}}$$

$$\hat{n} = \langle n_x, n_y, n_z\rangle = \langle\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta\rangle = \sqrt{\frac{4\pi}{3}}\left\langle -\frac{Y_{11}-Y_{1-1}}{\sqrt{2}}, \frac{iY_{11}+iY_{1-1}}{\sqrt{2}}, Y_{10}\right\rangle(\theta,\phi) \iff A_1Y_1(\theta,\phi) = \left\langle -\frac{n_x+in_y}{\sqrt{2}}, \frac{n_x-in_y}{\sqrt{2}}, n_z\right\rangle$$

$$\begin{aligned}
\iiint_{r<R}\vec{E}\,d\tau &= -k_\epsilon \oiint \iiint \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d\tau'\,\hat{n}(\vec{x})\,dS = -k_\epsilon R^2\iiint\rho(\vec{x}')\underline{\oiint \frac{\hat{n}(\vec{x})}{|\vec{x}-\vec{x}'|}\,d\Omega}\ d\tau' \\
\text{(sphere)}\quad &= -k_\epsilon R^2\iiint\rho(\vec{x}')\underline{\oiint \frac{\hat{n}(\vec{x}')\cos\gamma\sin\gamma}{|\vec{x}-\vec{x}'|}\,d\gamma\,d\beta}\ d\tau' \\
&= -k_\epsilon R^2\iiint\rho(\vec{x}')\int_{-1}^{1}\frac{2\pi x\hat{n}(\vec{x}')}{\sqrt{r^2+r'^2-2rr'x}}\,dx\ d\tau' \\
&= -\frac{4\pi k_\epsilon}{3}R^2\iiint\frac{r_<}{r_>^2}\rho(\vec{x}')\hat{n}(\vec{x}')\ d\tau' = \boxed{-\frac{4\pi k_\epsilon\vec{p}}{3}\ \text{or}\ \frac{4\pi R^3}{3}E_{\rho,r>R}(0)}
\end{aligned}$$

$$\begin{aligned}
&= -k_\epsilon R^2\iiint\rho(\vec{x}')\sum_{m=-1}^{1}A_{1m}Y^*_{1m}(\theta',\phi')\oiint\hat{n}(\vec{x})Y_{1m}(\theta,\phi)\,d\Omega\ d\tau' \\
&= -k_\epsilon R^2\iiint\rho(\vec{x}')\underline{\oiint\hat{n}(\vec{x})\frac{r_<}{r_>^2}P_1(\cos\gamma)\,d\Omega}\ d\tau' \\
&= -k_\epsilon R^2\iiint\frac{r_<}{r_>^2}\rho(\vec{x}')\underline{\oiint\hat{n}(\vec{x}')\cos\theta\,d\Omega}\ d\tau' \\
&= -\frac{4\pi k_\epsilon}{3}R^2\iiint\frac{r_<}{r_>^2}\rho(\vec{x}')\hat{n}(\vec{x}')\ d\tau'
\end{aligned}$$

$$\boxed{E_{\text{id. dip.}}(x) = \underbrace{k_\epsilon\frac{3\hat{n}(\vec{p}\cdot\hat{n})-\vec{p}}{|\vec{x}-\vec{x}'|^3}}_{\int\equiv 0\text{ by conv.}}\underbrace{-\frac{4\pi k_\epsilon\vec{p}}{3}\delta(\vec{x}-\vec{x})}_{\text{for real dip.}}}$$

## 2.5 Multipole Moments

$$V_{\text{ext.}}(\vec{x}) = k\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d\tau' = k\int \rho(\vec{x}')\sum_{l,m}\frac{r_<^l}{r_>^{l+1}}\frac{4\pi}{2l+1}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\,d\tau'$$

$$= 4\pi k\sum_{l,m}\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\underline{\int \rho(\vec{x}')r'^lY^*_{lm}(\theta',\phi')\,d\tau'} \equiv \boxed{4\pi k\sum_{l,m}\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\underline{q_{lm}}}$$

# 3 Electrodynamics in Matter

## 3.1 Ideal Dipoles

$$\boxed{\vec{\mathbf{p}} = \int r' \cdot \rho(r')\ d\tau'}$$

$$\vec{F}_{\text{dip}} = qE\Big|_{\vec{r}}^{\vec{r}+\vec{d}} = q\Delta\vec{E}$$

$$\approx \left[q\sum_i \left(\nabla E_i \cdot \vec{d}\right)\hat{i}\right]$$

$$\boxed{\vec{F}_{\text{dip}} = (\vec{\mathbf{p}} \cdot \nabla)\vec{E}}$$

$$U_{\text{ES dip}} = qV\Big|_{\vec{r}}^{\vec{r}+\vec{d}} = q\Delta V$$

$$= q\int_{\vec{r}}^{\vec{r}+\vec{d}} -\vec{E} \cdot d\vec{l}$$

$$\boxed{U_{\text{ES dip}} = -\vec{\mathbf{p}} \cdot \vec{E}}$$

$$\vec{N}_{\text{center}} = r \times F = \vec{d} \times q\vec{E}$$

$$\boxed{\vec{N}_{\text{dip}} = \vec{\mathbf{p}} \times \vec{E}}$$

$$\boxed{\textbf{Polarization}:\quad \vec{P} = \frac{d\vec{\mathbf{p}}}{d\tau}} \qquad \left(\frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2} = \nabla'\frac{1}{\boldsymbol{\imath}}\right)$$

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\int_\nu \frac{\vec{P}(\vec{r'}) \cdot \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\ d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int_\nu \frac{-\nabla' \cdot \vec{P}(\vec{r'})}{\boldsymbol{\imath}}\ d\tau' + \frac{1}{4\pi\epsilon_0}\int_S \frac{\vec{P}(\vec{r'}) \cdot \hat{n}}{\boldsymbol{\imath}}\ da'$$

$$\boxed{\rho_b = -\nabla \cdot \vec{P}} \qquad \boxed{\sigma_b = \vec{P} \cdot \hat{n}}$$

$$\boxed{\vec{\mathbf{m}} = \sum I\vec{a}}$$

$$\vec{F}_{\text{sqr. dip}} = q\vec{v} \times \vec{B}$$

$$= \pm IL\vec{x} \times B\hat{z}$$

$$= \pm\ ILB\ \hat{y}$$

$$\vec{N}_{\text{sqr. dip}} = 2\left[\frac{\pm\vec{W}}{2} \times \pm ILB\hat{y}\right]$$

$$= I(LW)\sin\theta B\ \hat{x}$$

$$\boxed{\vec{N}_{\text{dip}} = \vec{\mathbf{m}} \times \vec{B}}$$

$$\boxed{\vec{F}_{\text{dip}} = \nabla(\vec{\mathbf{m}} \cdot \vec{B})}\ ???$$

$$\boxed{U_{\text{dip}} = -\vec{\mathbf{m}} \cdot \vec{B}}$$

$$\boxed{\textbf{Magnetization}:\quad \vec{M} = \frac{d\vec{\mathbf{m}}}{d\tau}} \qquad \left(\frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2} = \nabla'\frac{1}{\boldsymbol{\imath}}\right)$$

$$\vec{A}(\vec{\mathbf{r}}) = k_\mu\int_\nu \frac{\vec{M}(\vec{r'}) \times \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\ d\tau'$$

$$= k_\mu\int_\nu \frac{\nabla' \times \vec{M}(\vec{r'})}{\boldsymbol{\imath}}\ d\tau' + k_\mu\int_S \frac{\vec{M}(\vec{r'}) \times \hat{n}}{\boldsymbol{\imath}}\ da'$$

$$\boxed{\vec{J}_b = \nabla \times \vec{M}} \qquad \boxed{\vec{K}_b = \vec{M} \times \hat{n}}$$

## 3.2 Maxwell's Equations in Matter

**GLE in Matter (GLEM)**

$$\nabla \cdot \epsilon_0 \vec{E} = \rho = \rho_b + \rho_f$$

$$= -\nabla \cdot \vec{P} + \nabla \cdot D$$

$$\nabla \cdot \left(\epsilon_0 \vec{E} + \vec{P}\right) = \nabla \cdot D$$

$$\boxed{\begin{array}{ll} \multicolumn{2}{c}{\vec{D} = \epsilon_0 \vec{E} + \vec{P}} \\ & -\nabla \cdot \vec{P} = \rho_b \\ \nabla \cdot \vec{D} = \rho_f & \\ & \vec{P} \cdot \hat{n} = \sigma_b \end{array}}$$

**COC in Matter (COCM)**

$$\nabla \cdot \vec{J}_p = -\frac{\partial \rho_b}{\partial t}$$

$$= \frac{\partial}{\partial t}\left(\nabla \cdot \vec{\mathbf{P}}\right)$$

$$\boxed{\frac{\partial \vec{\mathbf{P}}}{\partial t} = \vec{J}_p}$$

**MAL in Matter (MALM)**

$$\nabla \times \frac{1}{\mu_0}\vec{B} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{J}_b + \vec{J}_f + \vec{J}_p + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$= \nabla \times M + \vec{J}_f + \frac{\partial \vec{\mathbf{P}}}{\partial t} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \left(\frac{1}{\mu_0}\vec{B} - M\right) = \vec{J}_f + \frac{\partial}{\partial t}\left(\epsilon_0 \vec{E} + \vec{\mathbf{P}}\right)$$

$$\boxed{\begin{array}{ll} \multicolumn{2}{c}{\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}} \\ & \nabla \times \vec{M} = \vec{J}_b \\ \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} & \\ & \vec{M} \times \hat{n} = \vec{K}_b \end{array}}$$

**Faraday's Law of Induction (FLI)**

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$$\boxed{\text{Electrostatics:} \quad \nabla \times \vec{D} = \nabla \times \vec{P}}$$

**Gauss's Law for Magnetism (GLM)**

$$\boxed{\nabla \cdot \vec{B} = 0}$$

$$\boxed{\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}}$$

## 3.3 Linear Matter

Electric Susceptibility: $\chi_e$

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

Susceptibility Tensor:

$$\vec{P} = \begin{pmatrix} \chi_{e_{xx}} & \chi_{e_{xy}} & \chi_{e_{xz}} \\ \chi_{e_{yx}} & \chi_{e_{yy}} & \chi_{e_{yz}} \\ \chi_{e_{zx}} & \chi_{e_{zy}} & \chi_{e_{zz}} \end{pmatrix} \epsilon_0 \vec{E}$$

Relative Permittivity: $\epsilon_r = 1 + \chi_e$

$$\begin{aligned} \vec{D} &= (1 + \chi_e)\epsilon_0 \vec{E} \\ &= \epsilon_r \epsilon_0 \vec{E} \\ &= \epsilon \vec{E} \end{aligned}$$

Energy Density : $u_p = \dfrac{1}{2}\vec{E} \cdot \vec{P} = \dfrac{1}{2}\sum_{ij} \chi_{ij} E_i E_j$

(to make polarization)

$$= \tfrac{1}{2} \begin{bmatrix} & & \\ & \chi & \\ & & \end{bmatrix} \begin{bmatrix} | \\ E \\ | \end{bmatrix} \cdot \begin{bmatrix} | \\ E \\ | \end{bmatrix}$$

Magnetic Susceptibility: $\chi_m$

$$\vec{M} = \chi_m \vec{H}$$

Susceptibility Tensor:

$$\vec{M} = \begin{pmatrix} \chi_{m_{xx}} & \chi_{m_{xy}} & \chi_{m_{xz}} \\ \chi_{m_{yx}} & \chi_{m_{yy}} & \chi_{m_{yz}} \\ \chi_{m_{zx}} & \chi_{m_{zy}} & \chi_{m_{zz}} \end{pmatrix} \vec{H}$$

Bound Current:

$$\begin{aligned} \vec{J}_b &= \nabla \times \left(\chi_m \vec{H}\right) \\ &= \chi_m \left(\vec{J}_f + \partial_t \vec{D}\right) \end{aligned}$$

Relative Permeability: $\mu_r = 1 + \chi_m$

$$\begin{aligned} \vec{B} &= (1 + \chi_m)\mu_0 \vec{H} \\ &= \mu_r \mu_0 \vec{H} \\ &= \mu \vec{H} \end{aligned}$$

# 4 Boundary Conditions

$$\boxed{\Delta\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

1.) $\boxed{\Delta E_\parallel = 0}$

$$\oint \vec{E}\cdot d\vec{L} = -\oiint_{0-}^{0+} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$(E_\parallel^+ - E_\parallel^-)L = 0$$

2.) $\boxed{\Delta E_\perp = \frac{\sigma}{\epsilon_0}}$ $\boxed{\Delta D_\perp = \sigma_f}$

$$\oiint \vec{E}\cdot d\vec{a} = Q/\epsilon_0$$

$$(E_\perp^+ - E_\perp^-)a = \frac{\sigma a}{\epsilon_0}$$

Electrostatics: $\nabla\times\vec{E} = 0$

$\boxed{\Delta V = 0}$

$$V\Big|_{0-}^{0+} = -\int_{0-}^{0+} \vec{E}\cdot dL$$

$\boxed{\Delta\frac{\partial V}{\partial n} = -\frac{\sigma}{\epsilon_0}}$

$$\Delta(\nabla V)\cdot\hat{n}$$

$\boxed{\Delta\vec{D}_\parallel = \Delta\vec{P}_\parallel}$

$$\nabla\times\vec{D} = \nabla\times\vec{P}$$

$\boxed{\Delta\vec{B} = \mu_0\vec{K}\times\hat{n}}$ ; $\boxed{\Delta A_\parallel = 0}$

$$\oint \vec{A}\cdot d\vec{l} = \Phi_B = 0$$

1.) $\boxed{\Delta B_\perp = 0}$

$$\oiint \vec{B}\cdot d\vec{a} = 0$$

$$(B_\perp^+ - B_\perp^-)a = 0$$

2.) $\boxed{\Delta\vec{B}_\parallel = \mu_0\vec{K}\times\hat{n}}$ $\boxed{\Delta\vec{H}_\parallel = \vec{K}_f\times\hat{n}}$

$$\oint \frac{\vec{B}}{\mu_0}\cdot dL = \oiint_{0-}^{0+}\left(\vec{J} + \frac{\epsilon_0\partial\vec{E}}{\partial t}\right)\cdot d\vec{a}$$

$$(B_\parallel^+ - B_\parallel^-)L = \mu_0 I_{enc}$$

$$\Delta B_\parallel L = \mu_0 K L = (\mu_0\vec{K}\times\hat{n})\cdot\vec{L}$$

Magnetostatic: $\nabla\cdot\vec{A} = 0$

$\boxed{\Delta A_\perp = 0}$

$$\oiint_{0-}^{0+} \vec{A}\cdot d\vec{a} = 0$$

$\boxed{\Delta\frac{\partial\vec{A}}{\partial n} = -\mu_0\vec{K}}$

$$\Delta(\nabla\times\vec{A}) = \left(-\frac{\partial A_y^+}{\partial z} + \frac{\partial A_y^-}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x^+}{\partial z} - \frac{\partial A_x^-}{\partial z}\right)\hat{y}$$

$$= -\mu_0 K\hat{y}$$

# 5 Energy, Radiation, Momentum, and Angular Momentum

## 5.1 Energy Conservation

$$\frac{dW_{mech}}{dt} = \vec{F}\cdot\vec{v} = q\vec{E}\cdot\vec{v} + \cancel{q\vec{v}\times\vec{B}\cdot\vec{v}}$$
$$= \int \vec{E}\cdot\vec{J}\, d\tau \qquad (J = \rho_+ v_+ + \rho_- v_-)$$
$$= \int -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \tfrac{1}{\mu_0}B^2\right) - \nabla\cdot\left(\tfrac{1}{\mu_0}\vec{E}\times\vec{B}\right) d\tau$$

$$\boxed{\begin{aligned}\frac{dW_{mech}}{dt} &= -\frac{d}{dt}\int u_{em}\, d\tau - \int \nabla\cdot\vec{S}\, d\tau \\ &= -\int \frac{\partial u_{em}}{\partial t}\, d\tau - \oint \vec{S}\cdot d\vec{a}\end{aligned}}$$

$$\boxed{\begin{aligned}&\textit{Field Energy Density}: \quad u_{em} = \tfrac{dW_e}{d\tau} + \tfrac{dW_m}{d\tau} = \tfrac{\epsilon_0}{2}E^2 + \tfrac{1}{2\mu_0}B^2 \\ &\textit{Poynting Vector}\ \text{(Energy Flux Density)}: \quad \vec{S} = \tfrac{1}{\mu_0}(\vec{E}\times\vec{B}) = \tfrac{1}{2\mu_0}\mathrm{Re}(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*)\end{aligned}}$$

$$\boxed{\begin{aligned}\int \vec{S}\cdot d\vec{a} &= -\left(\frac{dW_{mech}}{dT} + \frac{dU_{em}}{dt}\right) = P_{\text{ow}} \\ \nabla\cdot\vec{S} &= -\frac{\partial}{\partial t}(u_{mech} + u_{em})\end{aligned}}$$

$$\boxed{I = \langle P_{\text{ow}}/A\rangle = \langle S\rangle = \frac{1}{2}c\epsilon_0 E^2}$$

### 5.1.1 Radiation

Accelerating Charge

Larmor Formula $(v \ll c)$ : $P_{\text{ow}} = \left(\dfrac{2k_\epsilon}{3c^3}\right) q^2 a^2$

Electric Dipole Radiation

Dipole Moment : $\vec{\mathbf{p}}(t) = p_0 \cos(\omega t)\hat{z}$

Intensity : $\langle S\rangle = \left(\dfrac{k_\epsilon}{8\pi c^3}\right) p_0^2\omega^4 \, \dfrac{\sin^2\theta}{r^2}$

Power : $\langle P\rangle_E = \left(\dfrac{k_\epsilon}{3c^3}\right) p_0^2\omega^4$

Magnetic Dipole Radiation: $\langle P\rangle_B = \left(\dfrac{k_\mu}{3c^3}\right) m_0^2\omega^4$

$$\begin{aligned}(E_1+E_2)\times(B_1+B_2) &= E_1\times B_1 + (E_1\times B_2 + E_2\times B_1) + E_2\times B_2 \\ (E_1+E_2)\cdot(E_1+E_2) &= |E_1|^2 + (2E_1\cdot E_2) + |E_2|^2 \\ (B_1+B_2)\cdot(B_1+B_2) &= |B_1|^2 + (2B_1\cdot B_2) + |B_2|^2 \\ &\downarrow \\ \text{E\&M Energy} &= \tfrac{1}{2}m_1 v_{I1}^2 + \left(\begin{smallmatrix}\text{Crossterms;}\\ \text{radiation,...}\end{smallmatrix}\right) + \tfrac{1}{2}m_2 v_{I2}^2 \\ &= \tfrac{1}{2}L_1 I_1^2 + M I_1 I_2 + \tfrac{1}{2}L_2 I_2^2 \\ \text{(heat dissipation)}\quad & \quad +R_1 I_1^2 \qquad\qquad +R_2 I_2^2 \\ \text{Grav. Energy} &= \tfrac{1}{2}m_1 v_1^2 + -m\tfrac{GM}{r} + \tfrac{1}{2}m_2 v_2^2\end{aligned}$$

## 5.2 Momentum/Angular Momentum Conservation

$$\frac{\partial \vec{F}}{\partial \tau} = \vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$$

$$= \epsilon_0\left[(\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E}\right] + \tfrac{1}{\mu_0}\left[(\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B}\right]$$

$$-\frac{1}{2}\nabla\left(\epsilon_0 E^2 + \tfrac{1}{\mu_0}B^2\right) - \tfrac{\partial}{\partial t}\left(\epsilon_0\vec{E}\times\vec{B}\right)$$

$$= \nabla\cdot\overleftrightarrow{T} - \frac{\partial}{\partial t}\left(\epsilon_0\mu_0\vec{S}\right) = \nabla\cdot\overleftrightarrow{T} - \frac{\partial\vec{g}_{em}}{\partial t}$$

$$\boxed{\begin{aligned}\frac{dp_{mech}}{dt} &= \int\nabla\cdot\overleftrightarrow{T}\,d\tau - \frac{d}{dt}\int\vec{g}_{em}\,d\tau \\ &= \oint\overleftrightarrow{T}\cdot d\vec{a} - \int\frac{\partial\vec{g}_{em}}{\partial t}\,d\tau = \vec{F}\end{aligned}}$$

$$\boxed{\oint\overleftrightarrow{T}\cdot d\vec{a} = \frac{d\vec{p}_{mech}}{dt} + \frac{d\vec{p}_{em}}{dt}}$$

*Field Momentum Density* : $\vec{g}_{em} = \epsilon_0\mu_0\vec{S} = \epsilon_0(\vec{E}\times\vec{B})$

*Maxwell Stress Tensor* (Force per Area on Surface) : $\overleftrightarrow{T}$ ,

$$\begin{aligned}T_{ij} &= \epsilon_0\left(E_iE_j - \tfrac{1}{2}\delta_{ij}E^2\right) + \tfrac{1}{\mu_0}\left(B_iB_j - \tfrac{1}{2}\delta_{ij}B^2\right) \\ &= \epsilon_0E_iE_j + \tfrac{1}{\mu_0}B_iB_j - \delta_{ij}u_{em}\end{aligned}$$

$$T_{xx} = \epsilon_0E_x^2 + \tfrac{1}{\mu_0}B_x^2 - u_{em} \quad , \quad T_{xy} = \epsilon_0E_xE_y + \tfrac{1}{\mu_0}B_xB_y$$

$$\boxed{\nabla\cdot\overleftrightarrow{T} = \vec{f} + \frac{\partial\vec{g}_{em}}{\partial t} = \frac{\partial}{\partial t}\left(\vec{g}_{mech} + \vec{g}_{em}\right)}$$

*Field Angular Momentum Density* : $\vec{l}_{em} = \vec{r}\times\vec{g}_{em} = \epsilon_0\vec{r}\times\left(\vec{E}\times\vec{B}\right)$

### 5.2.1 Examples (including hidden momentum)

• A circuit has momentum $\int\epsilon_0(\vec{E}\times\vec{B})\,d\tau$, as it provides energy (center of energy) from the battery to the resistor.

• Hidden momentum is always relativistic.

• A magnetic dipole alone has no hidden momentum; in an electric field, the hidden momentum is $-\int\epsilon_0(\vec{E}\times\vec{B})\,d\tau$.

• A spinning shell of charge / a long solenoid alone has no hidden momentum; an electric dipole in the middle will give the moving charges hidden momentum of $-\int\epsilon_0(\vec{E}\times\vec{B})\,d\tau$.

• When one of the fields is removed/changes (discharge, etc.), the hidden momentum goes into the B-field creating charges, along with any mechanical EM response impulse; the impulsed applied on removal doesn't really depend on the initial field momentum or hidden momentum.

• When a field changes, any angular momentum stored in the fields is transfered to the other objects in the predictable way (as opposed to above, since it has no center of energy equivalent). If there are no other objects then there wouldn't be angular momentum anyway.
• Hidden momentum can exist when center of energy is moving (no examples given).

# 6 Potentials in Lorenz Gauge (nonstatic sources)

See Potentials for Recap

If choose $\left(\boldsymbol{\nabla} \cdot \vec{\boldsymbol{A}} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \;\Leftrightarrow\; \partial_\mu A^\mu = 0\right)$

$$\boxed{\begin{aligned} -\Box^2 V &= \frac{\rho}{\epsilon_0} \\ -\Box^2 \vec{A} &= \mu_0 \vec{J} \end{aligned}}$$

Solutions satisfying these three equations (thus satisfying Maxwell's Eq.) are,

$$\boxed{\begin{aligned} V(\vec{r}, t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'}, t_r)}{\scriptstyle\mathcal{R}}\, d\tau' \\ \vec{A}(\vec{r}, t) &= k_\mu \int \frac{\vec{J}(\vec{r'}, t_r)}{\scriptstyle\mathcal{R}}\, d\tau' \end{aligned}}$$

where $t_r = t - \dfrac{\scriptstyle\mathcal{R}}{c}$.

Notice that charges move, $V$ and $\vec{A}$ update at the speed of light. $t_r = t + \frac{\scriptstyle\mathcal{R}}{c}$ is also a solution, though not physically real.

Using GLMP and FLIP to find the fields,

*Jefimenko Equations*:

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\int \left[\frac{\rho(\vec{r}, t_r)}{\scriptstyle\mathcal{R}^2}\hat{\scriptstyle\mathcal{R}} + \frac{\dot{\rho}(\vec{r}, t_r)}{c\scriptstyle\mathcal{R}}\hat{\scriptstyle\mathcal{R}} - \frac{\dot{\vec{J}}(\vec{r}, t_r)}{c^2\scriptstyle\mathcal{R}}\right] d\tau'$$

$$\vec{B}(\vec{r}, t) = k_\mu \int \left[\frac{\vec{J}(\vec{r}, t_r)}{\scriptstyle\mathcal{R}^2} + \frac{\dot{\vec{J}}(\vec{r}, t_r)}{c\scriptstyle\mathcal{R}}\right] \times \hat{\scriptstyle\mathcal{R}}\, d\tau'$$

It's usually easier solve for the potentials first instead of fields directly. In the electrostatic and magnetostatic limits, CL and BSL are recovered.

# 7 EM Waves

$$f(z,t) = \mathrm{Re}[\tilde{f}(z,t)] = \mathrm{Re}\left[Ae^{i(kz-wt+\delta)}\right]$$

$\omega$ is the same throughout! (?)

$\dfrac{\lambda_1}{\lambda_2} = \dfrac{k_2}{k_1} = \dfrac{v_1}{v_2} = \dfrac{n_2}{n_1}$

$\tilde{\mathbf{f}}(\mathbf{z}, \mathbf{t}; \delta = \mathbf{0}) : \tilde{A}_I e^{i(k_1 z - wt)} + \tilde{A}_R e^{i(-k_1 z - wt)} \Rightarrow \tilde{A}_T e^{i(k_2 z - wt)}$

$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T; \quad k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T$

$\tilde{A}_R e^{i\delta_R} = \left(\dfrac{v_2 - v_1}{v_2 + v_1}\right) \tilde{A}_I e^{i\delta_I}; \quad \tilde{A}_T e^{i\delta_T} = \left(\dfrac{2v_2}{v_2 + v_1}\right) \tilde{A}_I e^{i\delta_I}$

$A_R = \left(\dfrac{|v_2 - v_1|}{v_2 + v_1}\right) A_I; \quad A_T = \left(\dfrac{2v_2}{v_2 + v_1}\right) A_I$

## 7.1 Vacuum, $\vec{v}_{||}\vec{E}_{||}\hat{z}$

$$\tilde{B}_0 = \frac{k}{w}(\hat{z} \times \tilde{E}_0) = \frac{1}{c}(\hat{z} \times \tilde{E}_0)$$

$$\vec{S} = cu_{EM}\hat{z} = c\epsilon_0 E_0^2 \cos^2(kw, wt + \delta)\hat{z}$$

$$I_{nt} = \langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2$$

$$P_{res} = \frac{I_{nt}}{c}$$

## 7.2 Linear Media

$D = \epsilon E; \quad B = \mu H$

$\tilde{B}_0 = \frac{1}{v}(\hat{z} \times \tilde{E}_0)$

- $\boxed{n = \frac{c}{v}}$
- $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r}$

- $k_I v_1 = k_R v_1 = k_T v_2 = \omega$
- $k_I \sin\theta_I = \big(k_R \sin\theta_R = k_R \sin\theta_I\big) = k_T \sin\theta_T$
- Snell's Law : $\boxed{n_1 \sin\theta_I = n_2 \sin\theta_T}$

Fresnel's Equations Oblique Incidence $\left(\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-(n_1/n_2)^2 \sin{\theta_I}^2}}{\cos\theta_I}\ , \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2} \approx \frac{v_1}{v_2}\right)$

- P-Polarized ($E_\parallel$ to Plane of Incidence):

$$\tilde{E}_R = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_I\ ; \quad \tilde{E}_T = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_I$$

$$R = \frac{I_R}{I_I} = \left(\frac{E_R}{E_I}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_T}{E_I}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

Reflection Shift/Angles $\left(\alpha - \beta \stackrel{?}{=} 0\right)$ : $\tan^2\theta_I \stackrel{?}{=} \left(\frac{n_2}{n_1}\right)^2 \frac{1-\beta^2}{1-(n_2/n_1)^2}$

In-Phase ($\delta = 0,\ \alpha > \beta$) : $\tan\theta_I > n_2/n_1$

Out-of-Phase ($\delta = \pi,\ \alpha < \beta$) : $\tan\theta_I < n_2/n_1$

Brewster's Angle ($R = 0$) : $\boxed{\tan\theta_{I=b} = n_2/n_1}$ , $\boxed{\theta_R + \theta_T = 90}$

Critical Angle ($T = 0$) : $\boxed{\sin\theta_{I=c} = n_2/n_1}$ , $\boxed{\theta_R = 90}$ ($n_1 > n_2$) (evanescent if $> \theta_c$)

- S-Polarized ($E_\perp$ to Plane of Incidence):

$$\tilde{E}_R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)E_I\ ; \quad \tilde{E}_T = \left(\frac{2}{1+\alpha\beta}\right)E_I$$

$$R = \frac{I_R}{I_I} = \left(\frac{E_R}{E_I}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\alpha\left(\frac{E_T}{E_I}\right)^2 = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2$$

Reflection Shift/Angles $\left(1 - \alpha\beta \stackrel{?}{=} 0\right)$ : $\alpha\beta \ \approx \ \frac{\sqrt{\beta^2 - \sin{\theta_I}^2}}{\cos\theta_I}$

In-Phase ($\delta = 0,\ 1 > \alpha\beta$) : $n_1 > n_2$

Out-of-Phase ($\delta = \pi,\ 1 < \alpha\beta$) : $n_2 > n_1$

Brewster Angle ($R = 0$) : $n_1 = n_2$ (None)

Critical Angle ($T = 0$) : $\boxed{\sin\theta_{I=c} = n_2/n_1}$ , $\boxed{\theta_R = 90}$ ($n_1 > n_2$) (evanescent if $> \theta_c$)

## 7.3 Diffraction and Interference

Double Slit Interference: $(d \ll L)$

Maxima : $d \sin\theta = m\lambda$

Minima : $d \sin\theta = (m + \frac{1}{2})\lambda$

Circular Aperture: (Diameter: $D \ll L$)

$\theta =$ Twice the normal, vertical angle

1st Minima : $D \sin\theta = 1.22\lambda$

Optical Path Length: $(n_1 \to n_2,\ \lambda \to \frac{\lambda}{n},\ v_n = f\frac{\lambda}{n})$

- $\delta = \frac{2\pi d}{\lambda/n} = k(nd)$
- $\Delta x_n = nd = nv\Delta t = c\Delta t$ ($t$, time through medium $n$) ($2dn$ for thin film reflec.)

Single Slit Diffraction: $(a \ll L,\ a \sim \lambda)$

Minima : $a \sin\theta = m\lambda, \quad m \neq 0$

Bragg [X-Ray] Diffraction: (Atom Distance : $d \sim \lambda$)

$\theta =$ Angle from Horizontal (not vertical/normal)

- Maxima : $(2d) \sin\theta = m\lambda$

Boundary Reflection: $(n_1 \to n_2)$

$n_2 < n_1 : \quad \delta \mathrel{+}= 0$

$n_2 > n_1 : \quad \delta \mathrel{+}= \pi$

## 7.4 Lenses and Mirrors $(\lambda \ll a)$

Draw Picture : 1. $\overline{f, y[s], L_{\text{ens}}} \to \overline{L_{\text{ens}}, y'[s'], \infty}$

2. $\overline{\infty, y[s], L_{\text{ens}}} \to \overline{f', L_{\text{ens}}, y'[s']}$

Imaging Eq. : $\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$

---

Thin Lens Eq. : $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ (Focal Length, $f = f'$)

Lensmaker Eq. : $\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ ($R_2$ is [-] for concave lens)

Lens Magnf. : $M_T \equiv \frac{y'}{y} = -\frac{s'}{s} = \frac{f}{f-s}$ Virtual: $f>s$ Real: $s<f$

Spherical Mirror : $f = R/2$

## 7.5 Other

Rayleigh Scattering $(\lambda \gg a)$ : $I \propto I_0 \left(\frac{a^6}{\lambda^4}\right)$ (Dipole Radiation, polarized)

[Sound] Doppler Effect $(v \ll c)$ : $f_r = \left(\frac{v + v_r}{v - v_s}\right) f_s$ (frequency, $f$) ($v_r,\ v_s$ are [+] if $\to\leftarrow$)

Standing Sound Wave

- Open Pipe : $L = n\left(\frac{\pi}{2}\right)$ (Ends are nodes/infl. pts. of 0 press.)
- Half Pipe : $L = (2n+1)\left(\frac{\pi}{4}\right)$ (Open End is a node, Closed is an antinode/maxi. press.)

Malus's Law : $I = I_0 \cos^2\theta$ (polarized)
$I = I_0/2$ (unpolarized)

### 7.6 Conductor; $J_{free} \neq 0$

$$J_{free} = \sigma E$$

$\tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z - wt)}$; $\quad \tilde{B}(z,t) = \tilde{B}_0 e^{i(\tilde{k}z - wt)}$

$\tilde{k} = k + i\kappa$; $\quad \tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$

$k = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1}$ ; $\quad \kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1}$

Skin depth: $d = \frac{1}{\kappa}$
Wave (phase) velocity: $v = \frac{\omega}{k}$
Group velocity (carries energy): $v_g = \frac{d\omega}{dk} < c$
Index Ref: $n = \frac{ck}{\omega}$

$$\frac{B_0}{E_0} = \frac{K}{\omega} = |\tilde{k}|/\omega = \sqrt{\epsilon\mu\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}}$$

$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}$

$\tilde{E}_R = \left(\frac{1 - \tilde{\beta}}{1 + \tilde{\beta}}\right)\tilde{E}_I$; $\quad \tilde{E}_T = \left(\frac{2}{1 + \tilde{\beta}}\right)\tilde{E}_I$

### 7.7 Wave Guides

$$E^{||} = 0; \quad B^{\perp} = 0$$

TE Waves: $E_z = 0$;  TM Waves: $B_z = 0$;  TEM Waves: $E_z = B_z = 0$

$$
\begin{aligned}
E_x &= \frac{i}{(w/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right) \\
E_y &= \frac{i}{(w/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right) \\
B_x &= \frac{i}{(w/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right) \\
B_Y &= \frac{i}{(w/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial Y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial X}\right)
\end{aligned}
$$

Solving Rectangular Wave Guides:

$\mathrm{TE}_{mn\neq 00}$: $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] B_z = 0$

$B_z = X(x)Y(y)$

$$\frac{\partial^2 X}{\partial x^2} = -k_x^2 X; \quad \frac{\partial^2 Y}{\partial y^2} = -k_y^2 Y$$

$$-k_x^2 - k_y^2 + (w/c)^2 - k^2 = 0$$

$B_z = B_0 \cos(m\pi x/a)\cos(n\pi y/b)$

$\omega < \omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}$

TM: $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] E_z = 0$

# 8 Del

$$\nabla F = \left\langle \frac{\partial}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial \theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \right\rangle_{r,\theta,\phi} F = \left(\hat{r},\ \hat{\theta},\ \hat{\phi}\right)\cdot\left(\frac{\partial}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial \theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) F$$

$$= \begin{pmatrix} \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z} \\ \cos\phi\cos\theta\,\hat{x} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z} \\ -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{pmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \end{bmatrix} F$$

$$= \left\langle \begin{matrix} \cos\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \end{matrix} \right\rangle_{\hat{x},\hat{y},\hat{z}} F = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F$$

$$\nabla\cdot\vec{A} = \frac{1}{r}\frac{1}{r\sin\theta}\left\langle \frac{\partial}{\partial r},\ \frac{\partial}{\partial\theta},\ \frac{\partial}{\partial\phi}\right\rangle \cdot [r\cdot r\sin\theta]\left\langle A_r,\ \frac{1}{r}A_\theta,\ \frac{1}{r\sin\theta}A_\phi\right\rangle$$

$$\nabla\times\vec{A} = \frac{1}{r}\frac{1}{r\sin\theta}\left\| \begin{matrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{matrix} \right\|$$

$$[\vec{A}\times(\vec{B}\times\vec{C})]_i = \vec{A}\cdot(B_i\vec{C}) - \vec{A}\cdot(\vec{B}C_i)$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \boxed{(\vec{A}\cdot(\vec{B}\otimes\vec{C})^T)^T - (\vec{A}\cdot\vec{B}\otimes\vec{C})^T} = (\vec{A}\otimes\vec{B})\cdot\vec{C} - (\vec{A}\cdot\vec{B}\otimes\vec{C})^T$$
$$= (A^T(BC^T)^T)^T - (A^TBC^T)^T = (AB^T)^TC - (A^TBC^T)^T$$

$$d\vec{r} = d\hat{x} + d\hat{y} + d\hat{z}$$
$$= \partial\hat{r} + r\,\partial\hat{\theta} + r\sin\theta\,\partial\hat{\phi}$$

$$\left[\hat{r},\ \hat{\theta},\ \hat{\phi}\right] = \left[\frac{\partial}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\vec{r}$$
$$= [\nabla_{sp}\otimes\vec{r}\,]^T$$

$$\theta = \theta(x,y,z) \quad \left(x^2+y^2 = z^2\tan^2\theta\right)$$
$$\phi = \phi(x,y,z) \quad \left(y = x\tan\phi\right)$$

$$d\theta = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy + \frac{\partial\theta}{\partial z}dz$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}$$

$$\begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix} = \begin{bmatrix} -\nabla r- \\ -\nabla\theta- \\ -\nabla\phi- \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} | & | & | \\ \nabla r & \nabla\theta & \nabla\phi \\ | & | & | \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\phi} \end{bmatrix}$$

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} -\nabla_{sp}x- \\ -\nabla_{sp}y- \\ -\nabla_{sp}z- \end{bmatrix} \begin{bmatrix} \partial r \\ r\,\partial\theta \\ r\sin\theta\,\partial\phi \end{bmatrix}$$