1 Curvilinear Coordinates

$$\boxed{\begin{array}{l} \vec{r} = r\cos\phi\sin\theta\,\hat{x} + r\sin\phi\sin\theta\,\hat{y} + r\cos\theta\,\hat{z} \\ r\hat{r} = x\hat{x} + y\hat{y} + z\hat{z} \end{array}}$$

$$\hat{r} = \frac{\partial}{\partial r}\vec{r} = \boxed{\frac{\vec{r}}{r}} = \nabla r_{(x,y,z)}$$

$$\hat{\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\vec{r} = \boxed{\frac{\partial\hat{r}}{\partial\theta}} = r\nabla\theta_{(x,y,z)}$$

$$\hat{\phi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\vec{r} = \boxed{\frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi}} = r\sin\theta\nabla\phi_{(x,y,z)}$$

$$\hat{z} = \boxed{\frac{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}{\cos 2\theta}}$$

$$\begin{array}{l} \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} = \cos 2\theta\,\hat{z} \\ \sin\phi\,\hat{r} + \cos\phi\,\hat{\phi} = \sin\theta\,\hat{y} + \sin\phi\cos\theta\,\hat{z} \\ \hat{\phi} = -\sin\phi\sin\theta\,\hat{x} + \cos\phi\sin\theta\,\hat{y} \end{array} \Rightarrow$$

$$\hat{y} = \frac{\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}}{\sin\theta} - \frac{\sin\phi\cos\theta}{\sin\theta\cos 2\theta}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

$$= \boxed{-\frac{\sin\phi\sin\theta}{\cos 2\theta}\,\hat{r} + \frac{\sin\phi\cos\theta}{\cos 2\theta}\,\hat{\theta} + \frac{\cos\phi}{\sin\theta}\,\hat{\phi}}$$

$$\hat{x} = \boxed{\cot\phi\,\hat{y} - \frac{\hat{\phi}}{\sin\phi\sin\theta}}$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\left(\frac{d\theta}{dt}\frac{\partial\hat{r}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{r}}{\partial\phi}\right)$$

$$\boxed{\vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} + r\sin\theta\frac{d\phi}{dt}\hat{\phi}}$$

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\left(\frac{d\vec{r}}{dt} - \frac{dr}{dt}\hat{r}\right) = \boxed{\frac{v}{r}\left[\hat{v} - (\hat{r}\cdot\hat{v})\hat{r}\right]}$$

$$= \frac{d\theta}{dt}\frac{\partial\hat{r}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{r}}{\partial\phi} = \boxed{\frac{d\theta}{dt}\hat{\theta} + \sin\theta\frac{d\phi}{dt}\hat{\phi}}$$

$$\frac{d\hat{\theta}}{dt} = \frac{d\theta}{dt}\frac{\partial}{\partial\theta}\left(\frac{\partial\hat{r}}{\partial\theta}\right) + \frac{d\phi}{dt}\frac{\partial}{\partial\phi}\left(\frac{\partial\hat{r}}{\partial\theta}\right) = \boxed{-\frac{d\theta}{dt}\hat{r} + \cos\theta\frac{d\phi}{dt}\hat{\phi}}$$

$$\frac{d\hat{\phi}}{dt} = \frac{d\theta}{dt}\frac{\partial\hat{\phi}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi}\right) = -\frac{d\phi}{dt}\underbrace{\mathrm{Proj}_{\mathrm{xy}}\left(\frac{\hat{r}}{\sin\theta}\right)}_{\cos\phi\,\hat{x} + \sin\phi\,\hat{y}}$$

$$= -\frac{d\phi}{dt}\frac{\hat{r} - \cos\theta\,\hat{z}}{\sin\theta} = \boxed{\frac{d\phi}{dt}\frac{\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}}{\cos 2\theta}}$$

$$\frac{d\hat{\theta}}{dt} = \frac{d\theta}{dt}\hat{\theta} + \theta\left(-\frac{d\theta}{dt}\hat{r} + \cos\theta\frac{d\phi}{dt}\hat{\phi}\right)$$

$$\frac{d\hat{\phi}}{dt} = \frac{d\phi}{dt}\hat{\phi} + \phi\frac{d\phi}{dt}\left(\frac{\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}}{\cos 2\theta}\right)$$

$$\downarrow$$

$$\frac{dr}{dt} = \frac{d}{dt}(\vec{r}\cdot\vec{r})^{\frac{1}{2}} = \boxed{\hat{r}\cdot\vec{v} = v_{\parallel r}}$$

$$\frac{d\theta}{dt} = \nabla\theta\cdot\vec{v} = \boxed{\frac{\hat{\theta}\cdot\vec{v}}{r} = \frac{v_{\perp\theta}}{r} = \omega_\theta}$$

$$\frac{d\phi}{dt} = \nabla\phi\cdot\vec{v} = \boxed{\frac{\hat{\phi}\cdot\vec{v}}{r\sin\theta} = \frac{v_{\perp\phi}}{r\sin\theta} = \omega_\phi}$$

$$\vec{L} = \vec{r}\times\vec{p}$$

$$\boxed{m\vec{r}\times\vec{v}_\perp} = \boxed{m\vec{r}\times\vec{v}} = mr^2\left(\frac{d\theta}{dt}\hat{\phi} - \sin\theta\frac{d\phi}{dt}\hat{\theta}\right)$$

$$\underline{mr^2\frac{1}{r}\hat{r}\times\vec{v}} = \boxed{I\vec{\omega}} = \underline{mr^2}\left[\frac{v}{r}(\hat{\theta}\cdot\hat{v})\hat{\phi} - \frac{v}{r}(\hat{\phi}\cdot\hat{v})\hat{\theta}\right]$$

$$I\frac{1}{r}\hat{r}\times\vec{v}_\perp = I\vec{\underline{\omega}} = I\frac{v}{r}\left[(\hat{\theta}\cdot\hat{v})\hat{\phi} - (\hat{\phi}\cdot\hat{v})\hat{\theta}\right]$$

$$I\frac{v_\perp}{r}\hat{r}\times\widehat{v_\perp} = I\underline{\omega}\hat{\omega} = I\frac{v}{r}(\hat{\theta}\times\hat{\phi} = \hat{r})\times\hat{v}$$

- $\vec{\omega}\times\vec{r} = \frac{1}{r}(\hat{r}\times\vec{v}_\perp)\times\vec{r}$
  $$= (\hat{r}\cdot\hat{r})\vec{v}_\perp - (\vec{v}_\perp\cdot\hat{r})\hat{r}$$
  $$= \vec{v}_\perp$$

- $\vec{v}\times\vec{\omega} = \frac{v_\perp}{r}\left(v_\perp\hat{v}_\perp + v_\parallel\hat{r}\right)\times\hat{\omega}$
  $$= \frac{v_\perp}{r}\left(v_\perp\hat{r} - v_\parallel\widehat{v_\perp}\right)$$

$$\|\vec{v}\times\vec{\omega}\|^2 = v^2\frac{v_\perp^2}{r^2}$$

$$L_i = \sum_j I_{ij}\omega_j$$

$$E = \frac{\|\vec{L}\|^2}{2I} = \frac{1}{2}\vec{L}\cdot\vec{\omega}$$

$$\sum_m \frac{1}{2}m\|\vec{\omega}\times\vec{r}\|^2 = \frac{1}{2}\sum_{ij} I_{ij}\omega_j\omega_i$$

$$\overleftrightarrow{I} = \begin{bmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum myx & \sum m(x^2+z^2) & -\sum myz \\ -\sum mzx & -\sum mzy & \sum m(x^2+y^2) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} & I & \end{bmatrix}\begin{bmatrix} | \\ \omega \\ | \end{bmatrix}\cdot\begin{bmatrix} | \\ \omega \\ | \end{bmatrix}$$

$$\frac{d}{dt}\left(\vec{p}\times\vec{L}\right) = \frac{d\vec{p}}{dt}\times\vec{L} = f_{(r)}\hat{r}\times\left(\vec{r}\times m\frac{d\vec{r}}{dt}\right)$$
$$= mf_{(r)}\left[\vec{r}\left(\hat{r}\cdot\frac{d\vec{r}}{dt}\right) - \frac{d\vec{r}}{dt}\left(\hat{r}\cdot\vec{r}\right)\right]$$
$$= mf_{(r)}\left[\hat{r}\frac{1}{2}\frac{d}{dt}\left(\vec{r}\cdot\vec{r}\right) - \frac{1}{r}\frac{d\vec{r}}{dt}r^2\right]$$
$$= mf_{(r)}\left[\hat{r}r\frac{dr}{dt} - r\frac{d\vec{r}}{dt}\right]$$
$$= -\frac{mf(r)r}{I(r)}\left[-\frac{I(r)}{r}\frac{dr}{dt}\vec{r} + I_{(r)}\frac{d\vec{r}}{dt}\right]$$
$$= -\frac{mf(r)r}{I(r)}\frac{d}{dt}\left[I_{(r)}\vec{r}\right]$$
$$= -mf_{(r)}r^2\frac{d}{dt}\hat{r} = mk\frac{d}{dt}\hat{r}$$
$$\frac{d}{dt}\left(\frac{\vec{p}\times\vec{L}}{mk} - \hat{r}\right) = \frac{d}{dt}\vec{e}_{\text{ccen}} = 0$$

$$\vec{a} = \left[\ddot{r} - r\dot{\theta}^2 + r\dot{\phi}^2\frac{\sin^2\theta}{\cos 2\theta}\right]\hat{r}$$
$$+ \left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\frac{\tan 2\theta}{2}\right]\hat{\theta}$$
$$+ \left[2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta\right]\hat{\phi}$$

$$\vec{\tau} = \vec{r}\times\vec{F}$$
$$= (mr^2)\frac{\hat{r}\times\vec{a}}{r} = I\vec{\alpha}$$

$$\boxed{\frac{dv}{dt} = a(\hat{v}\cdot\hat{a}) = \hat{v}\cdot\vec{a}}$$

$$\tau = \begin{bmatrix}\tau_x \\ \tau_y \\ \tau_z\end{bmatrix}$$

$$=^* \begin{bmatrix} 0 & \tau_z & -\tau_y \\ -\tau_z & 0 & \tau_x \\ \tau_y & -\tau_z & 0 \end{bmatrix}$$

$$\|\vec{q}\times\vec{p}\|^2 = \begin{vmatrix} \|\vec{q}\|^2 & \vec{q}\cdot\vec{p} \\ \vec{p}\cdot\vec{q} & \|\vec{p}\|^2 \end{vmatrix}$$
$$= \vec{q}\cdot\vec{p}\times(\vec{q}\times\vec{p})$$

$$T = \hat{v} = \frac{\vec{v}}{v}$$
$$T' = \frac{(\vec{v}\cdot\vec{v})\vec{a} - (\vec{v}\cdot\vec{a})\vec{v}}{v^3} = \frac{\vec{v}\times(\vec{a}\times\vec{v})}{v^3}$$
$$\|T'\| = \frac{\sqrt{v^2a^2 - (\vec{v}\cdot\vec{a})^2}}{v^2} = \frac{\|\vec{a}\times\vec{v}\|}{v^2}$$
$$\hat{B} = \frac{\vec{a}\times\vec{v}}{\|\vec{a}\times\vec{v}\|} = \widehat{a\times v} \qquad (\hat{B}\cdot\vec{v}=0)$$
$$\hat{N} = \frac{T'}{\|T'\|} = \frac{\vec{v}\times(\vec{a}\times\vec{v})}{v\|\vec{a}\times\vec{v}\|} = \hat{v}\times\hat{B}$$

$$\boxed{\begin{aligned} \vec{a} &= a_T\hat{T} + a_N\hat{N} \\ a_T &= \vec{a}\cdot\hat{v} = \frac{dv}{dt} \\ a_N &= \frac{\|\vec{a}\times\vec{v}\|}{v} = \|\vec{a}\times\hat{v}\| \\ a^2 &= a_T^2 + a_N^2 = \|\frac{d\vec{v}}{dt}\|^2 \end{aligned}}$$

2   Lagrangian Equations

$$\boxed{\begin{array}{l} \mathcal{L} = T - U \ , \qquad p_i \equiv \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} \\[2mm] \rightarrow \ F_i \equiv \dfrac{dp_i}{dt} = \dfrac{\partial \mathcal{L}}{\partial q_i} \end{array}}$$

Newton's Laws:

$$\mathcal{L} = \tfrac{1}{2}m\dot{\mathbf{r}}^2 - U(\mathbf{r}) \ , \qquad \vec{p}_r = m\dot{\mathbf{r}}$$

$$\rightarrow \ \boxed{F = m\ddot{\mathbf{r}} = -\nabla U}$$

Angular:

$$\mathcal{L} = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\phi}^2 - U(r, \phi) \ , \qquad \begin{array}{l} p_r = m\dot{r} \\[2mm] p_\phi = mr^2\dot{\phi} = I\omega = I\dfrac{v_\perp}{r} \end{array} , \qquad \begin{array}{l} -\vec{F} = \nabla U = \dfrac{\partial U}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial U}{\partial \phi}\hat{\phi} \\[2mm] \vec{F} = m\ddot{\mathbf{r}} = (F\cdot\hat{r})\hat{r} + (F\cdot\hat{\phi})\hat{\phi} \end{array}$$

$$\rightarrow \quad \begin{array}{l} F_r = \boxed{-\dfrac{\partial U}{\partial r} + mr\dot{\phi}^2 = m\ddot{r}} \quad \text{(centripetal: } \tfrac{mv^2}{r} = mr\omega^2) \\[4mm] F_\phi = \boxed{\underbrace{-\dfrac{\partial U}{\partial \phi}}_{\tau} = \underbrace{mr^2\ddot{\phi}}_{I\alpha} + \underbrace{2mr\dot{r}\dot{\phi}}_{I\omega}} \quad \text{("coriolis": } 2m|\vec{\omega}\times\vec{v}| = 2m\dot{\phi}\dot{r}) \end{array}$$

$$\begin{array}{l} \dot{\phi}' = \dot{\phi} - \omega \\[1mm] \downarrow \\[1mm] m\ddot{\mathbf{r}}' = m\ddot{\mathbf{r}} - (mr\omega^2 - 2mr\dot{\phi}\omega)\hat{r} - 2m\dot{r}\omega\hat{\phi} \\[2mm] = m\ddot{\mathbf{r}} - \underbrace{mr\omega^2\hat{r}}_{\text{centrifugal force}} + \underbrace{2m\omega(r\dot{\phi}\hat{r} - \dot{r}\hat{\phi})}_{\text{coriolis force}} \end{array}$$

Note:

$$\boxed{\begin{array}{l} \dot{\hat{r}} = \dot{\phi}\hat{\phi} \\[2mm] \dot{\hat{\phi}} = -\dot{\phi}\hat{r} \end{array}} \rightarrow \begin{array}{l} \vec{r} = r\hat{r} \ = \ r\cos\phi\hat{x} + r\sin\phi\hat{y} \\[2mm] \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \\[2mm] \ddot{\vec{r}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\hat{r}} + r\ddot{\hat{r}} \ = \ (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} \end{array}$$

Electromagnetic:

$$\mathcal{L} = \tfrac{1}{2}m\dot{\mathbf{r}}^2 - q\left[V_{(t, \ \mathbf{r})} - \dot{\mathbf{r}}\cdot\vec{A}_{(t, \ \mathbf{r})}\right] \ , \qquad p_x = m\dot{x} + qA_x$$

$$\rightarrow \qquad m\ddot{x} + q\frac{dA_x}{dt} = -q\left[\frac{\partial V}{\partial x} - \dot{r}\cdot\frac{\partial\vec{A}}{\partial x}\right] \quad \rightarrow \quad m\ddot{x} = q\left(-\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} + \dot{r}\cdot\left[\frac{\partial\vec{A}}{\partial x} - \nabla A_x\right]\right)$$

$$m\ddot{x} + q\left[\frac{\partial A_x}{\partial t} + \dot{r}\cdot\nabla A_x\right] = q\left[-\frac{\partial V}{\partial x} + \dot{r}\cdot\frac{\partial\vec{A}}{\partial x}\right]$$

$$= q\left[-\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t}\right] + q\dot{y}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$+ q\dot{z}\left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right]$$

$$= qE_x + q\dot{y}B_z - q\dot{z}B_y$$

$$m\ddot{x} = qE_x + q\left[\dot{\mathbf{r}}\times\vec{B}\right]_x$$

$$\downarrow$$

$$\boxed{m\ddot{\mathbf{r}} = q\left(\vec{E} + \dot{\mathbf{r}}\times\vec{B}\right)}$$

3

Special Relativity:

$$\mathcal{L} = -\frac{1}{\gamma}mc^2 - U \ , \qquad\qquad\qquad \vec{p} = \gamma m \vec{v} \ \rightarrow \ \gamma m \dot{x} = \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$= \gamma m v^2 - \gamma m c^2 - U$$

$$= m \left(v^2 - c^2 \right) \left(1 - \frac{v^2}{c^2} \right)^{-1/2} - U$$

$$\approx \frac{1}{2} m v^2 - (U + mc^2) \qquad \text{(when } v \ll c\text{)}$$

Conservation of Energy:

$$\frac{d\mathcal{L}}{dt} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \right) + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \sum_i \left(\dot{p}_i \dot{q}_i + p_i \ddot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} \qquad\qquad \rightarrow \qquad \frac{\partial \mathcal{L}}{\partial t} = -\frac{d}{dt} \left(\sum_i p_i \dot{q}_i - \mathcal{L} \right)$$

$$= \frac{d}{dt} \left(\sum_i p_i \dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} \qquad\qquad\qquad = -\frac{d\mathcal{H}}{dt} \quad \text{If } \mathcal{L} \text{ is explicitly independent of time (implies coordinates are "natural"), then the Hamiltonian is conserved.}$$

$$\frac{1}{2}\sum_n m\dot{r}_n^2 = \frac{1}{2}\sum_n m \left(\sum_i \frac{\partial r_n}{\partial q_i}\dot{q}_i \right)^2 \qquad\qquad \mathcal{L} = \frac{1}{2}mv^2 - U \ = \ T_{(\dot{q}_i)} - U_{(q_i)} \ \rightarrow$$

$$= \frac{1}{2}\sum_{i,j} \left(m \sum_n \frac{\partial r_n}{\partial q_i}\frac{\partial r_n}{\partial q_j} \right) \dot{q}_i \dot{q}_j \qquad\qquad \mathcal{H} = \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}$$

$$= \frac{1}{2}\sum_i \sum_j A_{ij}\dot{q}_i \dot{q}_j \qquad\qquad \rightarrow \qquad = \sum_i \left(\sum_j A_{ij}\dot{q}_j \right)\dot{q}_i - \frac{1}{2}m\dot{r}^2 + U$$

$$\left(\text{for } \frac{\partial T}{\partial \dot{q}_i} \right) \quad = \frac{1}{2}\left(2\sum_{i\neq j} A_{ij}\dot{q}_i \dot{q}_j + A_{ii}\dot{q}_i^2 \right) + ... \qquad = \frac{1}{2}m\dot{r}^2 + U \quad \text{If } \mathcal{L} = \frac{1}{2}mv^2 - U \text{ and } U \text{ is independent of } v, \text{ then the Hamiltonian is the total energy.}$$

Lagrange Multipliers:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i}$$

$$\frac{dp}{dt} = -\nabla U + \lambda \nabla f$$

$$F_{\text{tot}} = F_{\text{ncnstr}} + F_{\text{cnstr}}$$

4

2.1   Examples

Atwood's Machine (Pulley):

Particle Confined to a Cylinder Surface:

Block Sliding on Wedge:

Bead on Spinning Wire Hoop:

Oscillations of Bead Near Equilibriuum:

3   Hamiltonian

$$\mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L} \ , \qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$\rightarrow \quad \bullet \ \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i}$$

$$\bullet \ \frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}$$

Newton Particle:

$$\mathcal{H} = \dot{x}(m\dot{x}) - \tfrac{1}{2}m\dot{x}^2 + U(x)$$
$$= \tfrac{1}{2}m\dot{x}^2 + U(x)$$
$$= T + U$$

Angular:

$$\mathcal{H} = m\dot{r}^2 + mr^2\dot{\theta}^2 - \left(\tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - U(r,\theta)\right) \ ,$$

$$= \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + U(r,\theta)$$

$$p_r = m\dot{r}$$
$$p_\theta = mr^2\dot{\theta} \ \equiv \ L = I\omega$$

Electromagnetic:

$$\mathcal{H} = \dot{\mathbf{r}} \cdot \vec{p}_r - \left(\tfrac{1}{2}m\dot{\mathbf{r}}^2 - q\phi_{(t,\ \mathbf{r})} + q\dot{\mathbf{r}} \cdot \vec{A}_{(t,\ \mathbf{r})}\right) \ ,$$

$$= m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \vec{A} - \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\phi - q\dot{\mathbf{r}} \cdot \vec{A}$$
$$= \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\phi$$

$$\vec{p}_r = m\dot{\mathbf{r}} + q\vec{A}$$

Special Relativity:

$$\mathcal{H} = \vec{v} \cdot (\gamma m\vec{v}) - \left(\gamma mv^2 - \gamma mc^2 - U\right) \ ,$$
$$= \gamma mc^2 + U$$
$$\approx \tfrac{1}{2}mv^2 + \left(U + mc^2\right) \qquad \text{(when } v \ll c\text{)}$$

$$\vec{p} = \gamma m\vec{v}$$

4 Hamilton-Jacobi Equations

$$K(Q,P,t) \equiv H(q,p,t) + \frac{\partial S_{(q,Q,t)}}{\partial t} = 0$$

$\dot{Q} = \frac{\partial K}{\partial P} = 0 \quad \Rightarrow \quad \boxed{Q = \alpha_Q = \frac{\partial S}{\partial P}}$ (constant)

$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \quad \Rightarrow \quad \boxed{P = \alpha_P = -\frac{\partial S}{\partial Q}}$ (constant)

$\dot{q} = \frac{\partial H}{\partial p} \;\Rightarrow\; \boxed{q = -\frac{\partial S}{\partial p}} \;,\; \dot{p} = -\frac{\partial H}{\partial q} \;\Rightarrow\; \boxed{p = \frac{\partial S}{\partial q}}$

<u>Solve for $S_{(q,\alpha_Q,t)}$</u>   ($n+1$ variables, nonlinear PDE)

$$H\left(q, \frac{\partial S(q,\alpha_Q,t)}{\partial q}, t\right) + \frac{\partial S(q,\alpha_Q,t)}{\partial t} = 0$$

$\frac{\partial H}{\partial t} = 0 \;\Rightarrow\; \boxed{S(q,Q,t) = W(q,Q) - Et}$

$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum p_i \dot{q}_i = \cancel{\frac{\partial S}{\partial t}} \cancel{+H} + \mathcal{L}$

$\Rightarrow \; \boxed{S = \int \mathcal{L}\, dt + \text{const.}}$

<u>Solve for $W_{(q,\alpha_Q)}$</u>   (n variables, nonlinear PDE)

$$H(q, \frac{\partial W}{\partial q}) = E \equiv \alpha_Q$$

<u>Harmonic Oscillator</u>

$-\frac{\partial S}{\partial t} = \frac{1}{2}p^2 - \frac{1}{2}\omega^2 q^2$

$\quad = \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{1}{2}\omega^2 q^2 \quad \left(S = s_1(q) + s_2(t)\right)$

$-\frac{\partial s_2(t)}{\partial t} = \frac{1}{2}\left(\frac{\partial s_1(q)}{\partial q}\right)^2 - \frac{1}{2}\omega^2 q^2 \equiv \alpha_Q$

$s_2(t) = -\alpha_Q t + \cancel{\text{const.}} \;,\; s_1(q) = \int \sqrt{2a_Q + \omega^2 q^2}\; dq$

$$Q \equiv \alpha_Q \;,\; P = -\int \frac{dq}{\sqrt{2a_Q + \omega^2 q^2}} + t$$

$$\alpha_P = -\frac{1}{\omega}\sin^{-1}\left[q\frac{\omega}{\sqrt{2\alpha_Q}}\right] + t$$

$$\boxed{q(t) = \frac{\sqrt{2\alpha_Q}}{\omega}\sin\left[\omega(t - \alpha_P)\right]}$$

5 Kinematics

Elastic Collisions:
$$m_0 v_0 = m_1 v_1 + m_2 v_2$$
$$\tfrac{1}{2} m_0 v_0^2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

$$\Rightarrow \boxed{\tfrac{1}{2} m_2 v_2^2 (m_1 + m_2) - \tfrac{1}{2} m_0 v_0^2 (m_1 - m_0) = (m_0 v_0)(m_2 v_2)}$$

- $$\boxed{\begin{array}{l} mv_0 = mv_1 + Mv_2 = mv_0 \left(1 - \frac{2M}{m+M}\right) + Mv_0 \left(\frac{2m}{m+M}\right) \\[4pt] \rightarrow \ M \in (\infty, \ m, \ 0] \ \Rightarrow \ v_1 \in (-v_0, \ 0, \ v_0] \end{array}}$$

Inelastic Collision: $\quad E_0 = \tfrac{1}{2} m v_0^2$

- $$\boxed{\begin{array}{l} mv_0 = (m+M)v_1 \\[4pt] \rightarrow \ E_1 = \left(\frac{m}{m+M}\right) E_0 \end{array}}$$

6 Orbits

Lagrangian : $\quad \mathcal{L} = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m r^2 \dot{\theta}^2 + \tfrac{1}{2} m r^2 \sin^2 \theta \dot{\phi}^2 - U(r)$

- $l = I\omega = mr^2 \dot{\theta}$

- $m\ddot{r} = -\dfrac{\partial}{\partial r} U_{\text{eff}} = -\dfrac{\partial}{\partial r} \left(\dfrac{l^2}{2mr^2} + U(r) \right)$

$$\left. \right\} \quad m \ \rightarrow \ \mu = \frac{mM}{m+M}$$

Hamiltonian: $\ E = \frac{p^2}{2m} + \frac{l^2}{2mr^2} + U(r)$

- Inf. Energy to get to $r = 0$ unless $l = 0$
- $U \sim 1/r$

Orbit Types:

$$\begin{aligned} E > 0 &: \ \text{Hyperbola} \\ E = 0 &: \ \text{Parabola} \\ E < 0 &: \ \text{Ellipse} \\ E = \text{Min}(U_{\text{eff}}) &: \ \text{Circle} \end{aligned}$$

Kepler's Laws:

1st Law : Elliptical Orbits $\quad$ (Sun [at/orbiting] focus)

2nd Law : Equal Area Sweep $\quad (r^2 d\theta = \frac{l}{m} dt)$

3rd Law : $T^2 = k^2 a^3 \quad$ T, Period
a, Semi-major axis
k, "constant" $\left(\frac{2\pi}{\sqrt{G[m_{\text{planet}} + M_{\text{sun}}]}} \right)$

7 Fluid Mechanics

$$\begin{aligned} \text{Bernoulli's Principle}: \quad \frac{\rho v^2}{2} + \rho g z + P_{\text{res}} &= \text{constant} \quad \text{[Energy Density]} \\ \text{Fluid Conservation}: \quad \rho A v &= \text{constant} \quad \text{[Mass Flow Rate]} \\ \text{Bouyant Force}: \quad F = \rho V g & \qquad (\rho, V, \text{ of displaced liquid}) \end{aligned}$$

Water Facts:

- $1 \text{ L} = 1 \text{ kg}$

8 Oscillators

8.1 Homogenous

$$(F = m\ddot{x}) = -kx - \overset{\text{(damp)}}{b\dot{x}}$$

$$\downarrow$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

$$z_{\text{tr}}(t) = \tilde{C}e^{rt} + [\tilde{D}_{\text{opt.}}\ te^{rt}]: \qquad \underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}}$$

$$(r^2 + 2\beta r + \omega_0^2)e^{rt} = 0$$

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

<u>Normal (Undamped)</u>: $\left(F = -kx\right) \Rightarrow$
$$\left(\ddot{x} = -\omega_0^2 x = -\tfrac{k}{m}x\right)$$

<u>Underdamped</u>: $\left(\beta < \omega_0\right)$

$$\boxed{z_{\text{tr}}(t) = \tilde{C}_1 e^{i\omega_0 t} + \tilde{C}_2 e^{-i\omega_0 t}}$$

$$\boxed{z_{\text{tr}}(t) = \left(\tilde{C}_1 e^{i\sqrt{\omega_0^2 - \beta^2}t} + \tilde{C}_2 e^{-i\sqrt{\omega_0^2 - \beta^2}t}\right)\underline{e^{-\beta t}}}$$

<u>Critically Damped</u>: $\left(\beta = \omega_0\right)$

<u>Overdamped</u>: $\left(\beta > \omega_0\right)$

$$\boxed{z_{\text{tr}}(t) = \left(\tilde{C}_1 + \tilde{C}_2 t\right)\underline{e^{-\beta t}}}$$
$$\text{Decay rate is maximized at } \beta = \omega_0$$

$$\boxed{z_{\text{tr}}(t) = \underline{\tilde{C}_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t}} + \tilde{C}_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}}$$
$$\text{(smaller, lasts longer)}$$

8.2 Inhomogenous (Driven)

$$\boxed{z(t) = z_{\text{st}}(t) + z_{\text{tr}}(t)}$$

$$m\ddot{x} = -kx - b\dot{x} + F_{\text{dr}}$$

$$\downarrow$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t$$

- $L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t)$

$$\boxed{z_{\text{st}}(t) = \tilde{C}e^{i\omega t} = Ae^{i(\omega t - \delta)}}: \qquad \underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}}$$

$$(-\omega^2 + 2i\beta\omega + \omega_0^2)\tilde{C}e^{i\omega t} = f_0 e^{i\omega t}$$

$$\tilde{C} = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = Ae^{-i\delta}$$

$$\boxed{A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \ , \quad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)}$$

Resonance (Max A^2) with fixed ω : $\boxed{\omega_0 = \omega}$

Resonance (Max A^2) with fixed ω_0 : $\boxed{\omega = \sqrt{\omega_0^2 - 2\beta^2}}$ $\qquad$ (usually $\beta \ll \omega$)

Full Width at Half Max, $A^2(\omega)$: FWHM $\approx 2\beta$

Quality Factor (Sharpness) : $Q = \frac{\omega_0}{2\beta} = \left(\pi\frac{1/\beta}{2\pi/\omega_0} = \pi\frac{\text{decay time}}{\text{period}}\right) = \left(2\pi\frac{\text{Energy stored}}{\text{Energy Dissipated}}\right)$

8.3 Parallel and Series

<u>Series, k_1+k_2+m:</u> $\dfrac{1}{K_{\text{eq}}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$

<u>Parallel, $k_1 k_2 + m$:</u> $K_{\text{eq}} = k_1 + k_2$

8.4 Normal Modes: 3 Springs + 2 Masses, $k_1+m_1+k_2+m_2+k_3$

1.) $m_1 \ddot{x}_1 = -k_1 x_1 - k_2 x_1 + k_2 x_2$

$\qquad = -(k_1+k_2)x_1 + k_2 x_2$

$\qquad m_2 \ddot{x}_2 = k_2 x_1 - k_2 x_2 - k_3 x_2$

$\qquad\qquad = k_2 x_1 - (k_2+k_3)x_2$

$\rightarrow$

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2.) $\mathbf{z}(t) = \mathbf{a}e^{i\omega t} = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix}e^{i\omega t}$

$\qquad = \begin{pmatrix} a_1 e^{-i\delta_1 t} \\ a_2 e^{-i\delta_2 t} \end{pmatrix}e^{i\omega t}$

$\rightarrow$

$$\mathbf{M}\ddot{\mathbf{z}} = -\mathbf{K}\mathbf{z}$$

$$-\omega^2 \mathbf{M}\mathbf{a}e^{i\omega t} = -\mathbf{K}\mathbf{a}e^{i\omega t}$$

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{a} = 0$$

$$\boxed{\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0}$$

<u>$x(t) = \text{Re}\big[z(t)\big]$ is the real solution.</u>

<u>Same m and k</u>

$$\begin{pmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 m \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \quad \rightarrow$$

$$\boxed{\omega = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}}$$

Smaller ω_1 is most symmetric motion
(both swing in phase)

Larger ω_2 swings out of phase

$$\boxed{z(t) = \tilde{A}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{i\omega_1 t} + \tilde{A}_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{i\omega_2 t}}$$

<u>Weak Coupling</u>

8.5 Single Pendulum (Use Lagrangian)

- $T = \frac{1}{2}mR^2\dot{\theta}^2$

- $U = mg(R - R\cos\theta)$

$\rightarrow$

$$
\begin{aligned}
mR^2\ddot{\theta} &= -mgR\sin\theta \\
&\approx -mgR\theta
\end{aligned}
$$

$\rightarrow$

$$
\boxed{\begin{aligned}
\ddot{\theta} &= -\left(\frac{g}{I/mR}\right)\theta = -\omega^2\theta \\
\theta(t) &= \mathrm{Re}\left[C_1 e^{i\omega t} + C_2 e^{-i\omega t}\right]
\end{aligned}}
$$

8.6 Double Pendulum (Use Lagrangian)

- $T = \frac{1}{2}m_1 L_1^2 \dot{\theta}_1{}^2 + \frac{1}{2}m_2(L_1\dot{\vec{\theta}}_1 + L_2\dot{\vec{\theta}}_2)^2$

  $= \frac{1}{2}(m_1 + m_2)L_1^2 \dot{\theta}_1{}^2 + \frac{1}{2}m_2 L_2^2 \dot{\theta}_2{}^2$
  $\quad + m_2 L_2 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$

- $U = m_1 g(L_1 - L_1\cos\theta_1)$
  $\quad + m_2 g(L_1 + L_2 - L_2\cos\theta_2 - L_1\cos\theta_1)$

$\rightarrow$
$$
\mathbf{M}\ddot{\theta} = -\mathbf{K}\theta \qquad \text{(small angle quadratic approx.)}
$$

$$
\begin{pmatrix} (m_1 + m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix}
\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}
= -
\begin{pmatrix} (m_1 + m_2)gL_1 + k_2 & 0 \\ 0 & m_2 g L_2 \end{pmatrix}
\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}
$$