1 Wave Function

$$\Psi_p = e^{i(2\pi x/\lambda - 2\pi t/T)}$$
$$= e^{i(kx-\omega t)}$$
$$= e^{\frac{i}{\hbar}(px-Et)}$$

$$\breve{p}\Psi_p = p\Psi_p = \hbar k \Psi_p \qquad\qquad \breve{E}\Psi_p = E\Psi_p = \hbar\omega\Psi_p$$

$$\boxed{\breve{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}} \qquad\qquad \boxed{\breve{E} = -\frac{\hbar}{i}\frac{\partial}{\partial t}}$$

- $|\boldsymbol{f}\rangle \equiv \int f_{(x)}|x\rangle\,dx$

- $\langle\boldsymbol{x}|\boldsymbol{x'}\rangle \equiv \delta_{(x-x')}$

- $\langle\boldsymbol{x}|\boldsymbol{\hat{x}}|\boldsymbol{x'}\rangle \equiv x\langle x|x'\rangle$

1. $\langle x|\hat{x}|f\rangle = xf(x)$

$$\boxed{\breve{x}\langle x|f\rangle \equiv x\langle x|f\rangle}$$

2. $\langle\boldsymbol{x}|\boldsymbol{\hat{p}}|\boldsymbol{x'}\rangle \equiv \frac{\hbar}{i}\delta'_{(x-x')}$
$$= \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|x'\rangle$$

3. $\langle x|\hat{p}|f\rangle = \int f_{(x')}\delta'(x-x')\,dx'$

$$\boxed{\breve{p}\langle x|f\rangle \equiv \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|f\rangle}$$

1.1 Schrodinger Ψ

$$\boxed{\breve{E}|\Psi\rangle = \widehat{H}|\Psi\rangle} = (\widehat{T}+\widehat{V})|\Psi\rangle \qquad\qquad \boxed{\breve{E}\langle x|\Psi\rangle = \breve{H}\Psi} = (\breve{T}+\breve{V})\Psi = \left[\frac{\breve{p}^2}{2m}+V(\vec{r},t)\right]\Psi$$

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \left[\frac{\hat{p}^2}{2m}+V(\hat{x},t)\right]|\Psi\rangle \qquad\qquad \boxed{i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \left[\frac{-\hbar^2}{2m}\nabla^2+V(\breve{r},t)\right]\Psi(\vec{r},t)}$$

$$\boxed{-\breve{E}\langle\Psi| = \langle\Psi|\hat{H}} \qquad\qquad \boxed{-\breve{E}\langle\Psi|x\rangle = \breve{H}\Psi^*}$$

<u>If $V = V(x)$</u>

$\Psi(x,t) = \psi(x)\phi(t) \;\Rightarrow$

- $E_n\phi_n(t) = i\hbar\dfrac{\partial}{\partial t}\phi_n(t) \;\Rightarrow\; \boxed{\phi_n(t) = e^{-\frac{i}{\hbar}E_n t}}$

- $\boxed{E_n\psi_n(x) = \left(\dfrac{-\hbar^2}{2m}\partial_x^2 + V(x)\right)\psi_n(x)}$

  $-\ \psi$ can be lin. sum of real or complex, so choose real ψ

- Linear : $\boxed{\begin{aligned}\Psi_{(x,t)} &= \sum_n \psi_{n(x)}e^{-\frac{i}{\hbar}E_n t}c_n \\ &= \sum_n \langle x|n\rangle e^{-\frac{i}{\hbar}E_n t}\langle n|\Psi\rangle \\ &= \int_{x'}\langle x|\left[\sum_n|n\rangle e^{-\frac{i}{\hbar}E_n t}\langle n|\right]|x'\rangle\Psi_{(x')}\,dx' \\ &= \int_{x'}U_{(x,t;\,x',0)}\Psi_{(x')}\,dx'\end{aligned}}$

- $\sigma_H^2 = \langle H^2\rangle - \langle H\rangle^2 = 0 \;\Rightarrow\;$ measuring stationary state, Ψ_n, returns one E_n (determinate state)

1.2 Usage

- $\langle f|g\rangle = \int_{-\infty}^{\infty} f(x)^* g(x)\ dx$
- $\langle f|g\rangle_{ab} = \int_{a}^{b} f(x)^* g(x)\ dx$

- $|f\rangle = \int f(x')|x'\rangle dx' \sim f(x) \equiv \langle x|f\rangle$
- $\langle f| = \int f(x)^*[...]\ dx$

- $\langle f|f\rangle = \int_{a}^{b} |f|^2 dx < \infty \ \Rightarrow\ f \in L_{2(a,b)}$
- $\left(\int_{a}^{b} |f|^p\, dx < \infty \ \Rightarrow\ f \in L_{p(a,b)}\right)$

$$\langle x|\Psi\rangle = \Psi = \begin{cases} \sum_n c_n f_n \\ \int_n c_n f_n\ dn \end{cases}, \quad \langle f_m|f_n\rangle = \begin{cases} \delta_{mn} \\ \delta_{(m-n)} \end{cases}, \quad \begin{array}{c}\text{(see Born int.)}\\ |c_n|^2 = \begin{cases} P(n) \\ \text{PDF}_{(n)} \end{cases}\end{array}$$

$$\Rightarrow \boxed{c_n = \langle f_n|\Psi\rangle}$$

$\forall\{f_n\} \in L_2:$

$$|\Psi\rangle = \begin{cases} \sum_n c_n|f_n\rangle = \sum_n \langle f_n|\Psi\rangle\,|f_n\rangle = \left(\sum_n |f_n\rangle\langle f_n|\right)|\Psi\rangle = |\Psi\rangle \\[2ex] \int_n c_n|f_n\rangle\ dn = \int_n \langle f_n|\Psi\rangle|f_n\rangle\ dn = \left(\int_n |f_n\rangle\langle f_n|\ dn\right)|\Psi\rangle = |\Psi\rangle \end{cases}$$

$\breve{x}\Psi_y = x\Psi_y = y\Psi_y$

$\Rightarrow \boxed{\Psi_y = \delta_{(x-y)} = \langle x|y\rangle}$

$\langle x|\hat{p}|p\rangle = \int \langle x|\hat{p}|x'\rangle\langle x'|p\rangle\,dx'$

$p\langle x|p\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|p\rangle$

$\langle x|\hat{p}|p\rangle = \breve{p}\Psi_p = p\Psi_p$

$\Rightarrow \boxed{\Psi_p = Ae^{\frac{i}{\hbar}px} = \langle x|p\rangle}$

$\langle x|\hat{H}|n\rangle = E_n\langle x|n\rangle$

$\breve{H}\Psi_n = E_n\Psi_n$

(See Potential Examples)

$\Psi_{(x,t)} = \int_{-\infty}^{\infty} \Psi_y c_{y(t)}\ dy$

$= \int_{-\infty}^{\infty} \delta_{(x-y)}\Psi_{(y,t)}\ dy$

$\Psi_{(x,t)} = \int_{-\infty}^{\infty} \Psi_p \phi_{(E_p,t)} c_p\ dp$

$= \int_{-\infty}^{\infty} \frac{e^{\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}\frac{p^2}{2m}t}\Phi_{(p,0)}\ dp$

$\Psi_{(x,t)} = \int_{-\infty}^{\infty} \Psi_n \phi_{(E_n,t)} c_n\ dn$

$= \int_{-\infty}^{\infty} \Psi_n e^{\frac{-i}{\hbar}E_n t} c_n\ dn$

$c_x(t) = \langle \Psi_x|\Psi_{(x,t)}\rangle = \langle x|\Psi\rangle$

$\Psi_{(x,t)} = \int_{-\infty}^{\infty} \delta_{(x-y)}\Psi_{(y,t)}\ dy$

$c_p(t) = \langle \Psi_p|\Psi_{(x,t)}\rangle = \langle p|\Psi\rangle$

$\boxed{\Phi_{(p,t)} = \int_{-\infty}^{\infty} \frac{e^{\frac{-i}{\hbar}px}}{\sqrt{2\pi\hbar}}\Psi_{(x,t)}\ dx}$

$c_n(t) = \langle \Psi_n|\Psi_{(x,t)}\rangle = \langle n|\Psi\rangle$

$\Psi_{(n,t)} = \int_{-\infty}^{\infty} \Psi_n^* \Psi_{(x,t)}\ dx$

Born Interpretation: $\text{PDF}(x) = |\Psi(x)|^2 = \Psi^* \Psi$

$P_{(a<x<b)} = \int_a^b |\Psi|^2 dx \equiv \langle \Psi | \Psi \rangle_{ab}$

Boundary Conditions:

- $\boxed{\langle \Psi | \Psi \rangle = 1}$ (physical, bound states only)

- $\Psi(x)$ isn't always cont. (see extra)

- $\Psi(\pm\infty) = 0$

- $\frac{\partial \Psi(x)}{\partial x}$ is cont. except at $V = \infty$

- $\text{Min}(V) \leq E_\Psi \in \mathbb{R}$

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} E\Psi dx = \int_{-\epsilon}^{\epsilon} \widehat{H}\Psi dx \Rightarrow$$

- $\langle \Psi_n | \Psi_n \rangle \to \infty \Rightarrow \Psi_n$ not PHYSICAL

  sol. but $\Psi = \int c_n \Psi_n$ can if $\langle \Psi | \Psi \rangle = 1$

$$\lim_{\epsilon \to 0} \frac{\hbar^2}{2m}\Delta(\frac{d\Psi}{dx}) = \int_{-\epsilon}^{\epsilon} V\Psi dx$$

- $E[f_{(x)}] = \int_{-\infty}^{\infty} f_{(x)} \, \text{PDF}(x) \, dx = \int_{-\infty}^{\infty} f_{(x)} \, |\Psi(x)|^2 \, dx = \int_{-\infty}^{\infty} \Psi(x)^* f_{(x)} \Psi(x) \, dx = \boxed{\langle \Psi | f\Psi \rangle \equiv \langle f_{(x)} \rangle}$

- $\int_x \Psi^* \Psi \, dx = \int_x \left(\int_n c_n^*(t) \Psi_n^*(x) \, dn \right) \left(\int_{n'} c_{n'}(t) \Psi_{n'}(x) \, dn' \right) \, dx$

$$= \int_n c_n^*(t) \int_{n'} c_{n'}(t) \, \delta(n-n') \, dn' \, dn = \int_n |c_n(t)|^2 \, dn \Rightarrow \boxed{\text{PDF}(n) = |c_n|^2 = c_n^* c_n}$$

Adjoint (herm. adj./herm. conj.): $\boxed{\{A^\dagger : \langle f | Af \rangle = \langle A^\dagger f | f \rangle\}} \Rightarrow \langle h | \hat{A} g \rangle = \langle \hat{A}^\dagger h | g \rangle$ (let $f=h+g$, $f=h+ig$)

Hermitian Operator: $\{A : \hat{A}^\dagger = \hat{A}\}$

- $\boxed{\exists \{\Psi_n\} : \hat{A}\Psi_{n(x)} = a_n \Psi_{n(x)}}$ (spectral theorem)

- $\langle a \rangle = a \in \mathbb{R} \Rightarrow \hat{A}$ can be an observable

- $\boxed{\langle \Psi_m | \Psi_n \rangle \in \{\delta_{mn}, \, \delta_{(m-n)}\}}$

- $\boxed{\text{Axiom: } \{\Psi_n\} \text{ for } \hat{A} \text{ are complete}}$

  Non-degenerate: $(m \neq n), (a_m \neq a_n) \Rightarrow \langle \Psi_m | \Psi_n \rangle \in \{\delta_{mn}, \, \delta_{(m-n)}\}$

  Degenerate: $(m \neq n), (a_m = a_n), (\Psi_m \neq \Psi_n), \langle \Psi_m | \Psi_n \rangle \neq 0 \Rightarrow$ Use Gram-Schmidt

  to find orthogonal $\langle \Psi_m' | \Psi_n' \rangle = \langle a\Psi_m + b\Psi_n | c\Psi_m + d\Psi_n \rangle = 0$

Expectation: $E[\hat{A}_{(x,p)}]$

- $\int_{-\infty}^{\infty} \hat{A}_{(x,p)}^* \, \Psi^* \Psi \, dx = \langle \hat{A}\Psi | \Psi \rangle = \boxed{\langle \Psi | \hat{A}\Psi \rangle \equiv \langle \hat{A}_{(x,p)} \rangle}$ (won't work if $\int A \, |\Psi|^2 \, dx$)

$$\langle \Psi | \hat{A}\Psi \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{A}\Psi \, dx = \int_{-\infty}^{\infty} \left(\int_n c_n^* \Psi_n^* \, dn \right) \left(\int_{n'} c_{n'} \hat{A}\Psi_{n'} \, dn' \right) \, dx$$

$$= \int_n a_n |c_n|^2 \, dn = E[a] \equiv \langle a \rangle \qquad c_n = \text{PDF}(n) \text{ (see above and Momentum Space)}$$

$$\boxed{\langle a \rangle = \langle \Psi | \hat{A}\Psi \rangle = \langle \Psi | \hat{A} | \Psi \rangle = \langle A \rangle}$$

- $\boxed{\langle \sigma_a^2 \rangle = \langle a^2 \rangle - \langle a \rangle^2} \Rightarrow \sigma_A^2 = 0$ for Ψ_n (determinate state)

<u>Matrix Operators:</u>

Given complete $\{e_n\}: \ \langle e_m | e_n \rangle = \delta_{mn}$

1.) $\boxed{Q_{mn}^{(e)} \equiv \langle e_m | \widehat{Q}_{(x,p)} | e_n \rangle}$

$$|\beta\rangle = \widehat{Q}|\alpha\rangle = \sum_m |e_m\rangle \left[\begin{array}{c} \langle e_m|\beta\rangle = \langle e_m|\widehat{Q}|\alpha\rangle \\[4pt] \sum_n b_n \langle e_m|e_n\rangle = \sum_n a_n \boxed{\langle e_m|\widehat{Q}|e_n\rangle} \\[4pt] \boxed{b_m = \sum_n \left(Q_m^{(e)}\right)_n a_n} \end{array} \right] = $$

$$\sum_m b_m |e_m\rangle = \sum_{n,m} \langle e_n|\alpha\rangle \ Q_{mn}^{(e)} |e_m\rangle$$
$$= \sum_{n,m} Q_{mn}^{(e)} |e_m\rangle\langle e_n|\alpha\rangle$$
$$\Rightarrow \boxed{\widehat{Q} = \sum_{m,n} Q_{mn}^{(e)} |e_m\rangle\langle e_n|}$$

2.) Find $\widehat{Q}$ as a matrix

$$\begin{array}{c} |f\rangle = \sum_n c_n^{(e)}{}_{[f]} \ |e_n\rangle \\ \downarrow \\ f_{(x)} = \sum_n c_n^{(e)}{}_{[f]} \ e_{n(x)} \end{array} \quad = \quad \begin{pmatrix} \vdots \\ c_{n[f]} \\ \vdots \end{pmatrix}^{(e)} \cdot \begin{pmatrix} \vdots \\ e_{n(x)} \\ \vdots \end{pmatrix} \quad \equiv \quad \boxed{\begin{array}{c} \vec{c}^{\ (e)}{}_{[f]} \cdot \vec{e}_{(x)} \\[4pt] \int_n c^{(e)}{}_{[f](n)} \cdot e_{(n,x)} \ dn \end{array}} \quad , \quad \boxed{c_n^{(e)}{}_{[f]} = \langle e_n|f\rangle}$$

$\widehat{Q}|f\rangle$
$= \left(\sum_{m,n'} Q_{mn'}^{(e)} |e_m\rangle\langle e_{n'}|\right) \sum_n c_n^{(e)} |e_n\rangle$
$= \sum_{m,n} \left(\sum_{n'} Q_{mn'}^{(e)} c_n^{(e)} \langle e_{n'}|e_n\rangle\right) |e_m\rangle$
$= \sum_m \left(\sum_n \left(Q_m^{(e)}\right)_n c_n^{(e)}\right) |e_m\rangle$

$$\boxed{\begin{array}{c} \widehat{Q} \left[\begin{pmatrix} | \\ c \\ | \end{pmatrix}^{(e)} \cdot \begin{pmatrix} | \\ e \\ | \end{pmatrix} \right] = \left[\begin{pmatrix} & \vdots & \\ - & Q_m & - \\ & \vdots & \end{pmatrix}^{(e)} \begin{pmatrix} | \\ c \\ | \end{pmatrix} \right] \cdot \begin{pmatrix} | \\ e \\ | \end{pmatrix} \\[16pt] \widehat{Q} \ |f\rangle = \boxed{\widehat{Q} \left[\vec{c}^{\ (e)}{}_{[f]} \cdot \vec{e} \ \right] = \left[\overline{Q}^{(e)} \ \vec{c}^{\ (e)}{}_{[f]} \right] \cdot \vec{e}} \\[12pt] \begin{array}{l} \langle x|\widehat{Q}|f\rangle = \int_m \left[\overline{Q}^{(\delta)} f\right]_{(m)} \cdot \delta_{(x-m)} \ dm \\[4pt] \text{e.g.} \\[4pt] \qquad = \int_m \left[\int_n Q_m^{(\delta)}{}_{(n)} \cdot f_{(n)} \ dn\right] \delta_{(x-m)} \ dm = \widehat{Q}f_{(x)} \end{array} \end{array}}$$

3.) Terms

Diagonalizable: $\quad A \equiv PDP^{-1}$

Conj. Transpose, $\dagger$: $\quad A^\dagger \equiv A^{T*} = A^{*T}$

Hermitian, H: $\quad H = H^\dagger$
$\qquad\qquad\qquad H = UDU^{-1} = UDU^\dagger \quad$ (spectral theorem)

Unitary, U: $\quad U: \ UU^\dagger = U^\dagger U = 1$
$\qquad\qquad\qquad \exists H: \ U = e^{iH} = (U')e^{iD}(U')^\dagger$

<u>Hermitian Operator $\sim$ Hermitian Matrix</u>

$\langle Qx|y\rangle = \langle x|Qy\rangle \qquad$ (if inf. size then must be in Hilbert Space)

(draw it out)

$\rightarrow (\overline{Q}x)^{*T} \cdot y_m|e_m\rangle = y_m x^{*T} \cdot (\overline{Q}_m^*)$

$\qquad\qquad = x^{*T} \cdot \overline{Q}^{*T} y_m|e_m\rangle$

$\qquad\qquad = x^{*T} \cdot \overline{Q} y_m|e_m\rangle$

$\rightarrow \overline{Q}^\dagger \equiv \overline{Q}^{*T} = \overline{Q} \qquad \boxtimes$

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4.) Eigenvalue Equation

General Case:

$$\widehat{Q}|q_i\rangle = q_i|q_i\rangle$$

$$|q_i\rangle = \sum c_n^{(e)}{}_{[q_i]}|e_n\rangle$$

$$\boxed{\begin{aligned}\overline{Q}^{(e)} &= \ UDU^{\dagger} \qquad \text{(Spectral Theorem)}\\[4pt]
&= \begin{pmatrix} | & | \\ \vec{c}_{[q_0]} & \vec{c}_{[q_1]} & \cdots \\ | & | \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & \cdots \\ 0 & q_1 & \cdots \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} - & \vec{c}^{\,*}{}_{[q_0]} & - \\ - & \vec{c}^{\,*}{}_{[q_1]} & - \\ & \vdots \end{pmatrix}^{(e)}\\[4pt]
&\text{where } \langle\vec{c}_m|\vec{c}_n\rangle = \delta_{mn} \ \text{ since } \ Q^{\dagger}Q = QQ^{\dagger} \quad \text{(normal)}\end{aligned}}$$

$$q_i|q_i\rangle \ = \ \widehat{Q}|q_i\rangle$$

$$\left[q_i\ \vec{c}^{\,(e)}{}_{[q_i]}\right]\cdot\vec{e}_{(x)} \ = \ \left[\ \overline{Q}^{(e)}\ \vec{c}^{\,(e)}{}_{[q_i]}\right]\cdot\vec{e}_{(x)}$$

$$*\forall n \qquad\quad (qc)_n\ |e_{n\,(x)}\rangle = (Qc)_n\ |e_{n\,(x)}\rangle$$

$$\Downarrow^* \qquad\qquad \langle e_{n(x)}|(qc)_n|e_{n(x)}\rangle = \langle e_{n(x)}|(Qc)_n|e_{n(x)}\rangle$$

$$q_i\ \vec{c}^{\,(e)}{}_{[q_i]} \ = \ \overline{Q}^{(e)}\ \vec{c}^{\,(e)}{}_{[q_i]} \qquad\qquad (qc)_n = (Qc)_n$$

$$q_i\begin{pmatrix}|\\ \vec{c}_{[q_i]}\\ |\end{pmatrix}^{(e)} = \begin{pmatrix} | & | \\ \vec{c}_{[q_0]} & \vec{c}_{[q_1]} & \cdots \\ | & | \end{pmatrix}^{(e)}\begin{pmatrix} q_0 & 0 & \cdots \\ 0 & q_1 & \cdots \\ \vdots & \vdots \end{pmatrix}\begin{pmatrix} - & \vec{c}^{\,*}{}_{[q_0]} & - \\ - & \vec{c}^{\,*}{}_{[q_1]} & - \\ & \vdots \end{pmatrix}^{(e)}\begin{pmatrix}|\\ \vec{c}_{[q_i]}\\ |\end{pmatrix}^{(e)} \qquad \boxed{\begin{aligned}&q_i\ ,\ \vec{c}^{\,(e)}{}_{[q_i]}:\\[4pt] &\det\left(\overline{Q}^{(e)} - Iq_i\right) = 0\end{aligned}}$$

Special Case:

$$\begin{aligned}|q_n\rangle &= |e_n\rangle\\ \widehat{Q}|e_n\rangle &= q_n|e_n\rangle\end{aligned} \quad\Bigg|\quad \begin{aligned}\widehat{Q}|a\rangle &= \sum_n \widehat{Q}|e_n\rangle\langle e_n|a\rangle\\ &= \Big(\sum_n q_n|e_n\rangle\langle e_n|\Big)|a\rangle\end{aligned} \quad\Rightarrow\quad \boxed{\widehat{Q} = \sum_n q_n|e_n\rangle\langle e_n|}\quad\Rightarrow\quad \boxed{\overline{Q}^{(e)} = \begin{pmatrix} q_0 & 0 & \cdots \\ 0 & q_1 & \cdots \\ \vdots & \vdots & q_i \end{pmatrix}}$$

$$Q_{mn}^{(e)} = q_n\delta_{mn}$$

$$\overline{Q}^{(e)} = \begin{pmatrix} q_0 & 0 & \cdots \\ 0 & q_1 & \cdots \\ \vdots & \vdots \end{pmatrix}^{(e)} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots \end{pmatrix}^{(e)}\begin{pmatrix} q_0 & 0 & \cdots \\ 0 & q_1 & \cdots \\ \vdots & \vdots \end{pmatrix}\begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots \end{pmatrix}^{(e)} \qquad \boxed{\vec{c}^{\,(e)}{}_{[q_i]} = \left(\cdots\ 0\ 0\ 1_{(i)}\ 0\ 0\ \cdots\right)^T}$$

5. Unitary Transformation and Trace

- $\boxed{|b_i\rangle = U|a_i\rangle \ \Leftrightarrow\ U = \sum|b_n\rangle\langle a_n|}$

- $\boxed{\overline{U}_{ij} = \langle a_i|U|a_j\rangle = \langle a_i|b_j\rangle}$

- $\boxed{\langle b_i|U\widehat{Q}U^{\dagger}|b_j\rangle = \langle a_i|\widehat{Q}|a_j\rangle}$

- $\boxed{\begin{aligned}(A)|a_i\rangle &= a_i|a_i\rangle\\ (UAU^{\dagger})|b_i\rangle &= a_i|b_i\rangle\end{aligned}}$

- $\text{Tr}(Q) = \sum\langle a_i|Q|a_i\rangle = \sum\langle b_i|Q|b_i\rangle$

- $\text{Tr}(QP) = \text{Tr}(PQ)$

- $\text{Tr}(U^{\dagger}QU) = \text{Tr}(Q)$

- $\text{Tr}(|a_i\rangle\langle a_j|) = \delta_{ij}$

- $\text{Tr}(|b_i\rangle\langle a_i|) = \langle a_i|b_i\rangle$

$\Phi(p,t)$ - Momentum Space (generalizable Born Interpretation):

$$\int_x \Psi^*\Psi dx = \int_x \int_p c_p^*(t)\Psi_p^*(x)dp \int_{p'} c_{p'}(t)\Psi_{p'}(x)dp' \ dx$$

$$= \int_p c_p^*(t) \int_{p'} c_{p'}(t) \int_x \Psi_p^*(x)\Psi_{p'}(x)dx \ dp'dp$$

$$= \int_p \Phi^* \int_{p'} \Phi' \ \delta(p-p')dp'dp$$

$$= \int_p \Phi^*\Phi \ dp \ \Rightarrow \ \boxed{\text{PDF}(p) = |\Phi|^2 = \Phi^*\Phi}$$

$$\boxed{\langle\Psi|\Psi\rangle = \langle\Phi|\Phi\rangle}$$

Anything in x-space can be done in p-space
(or generalize to any transform, c_n)

$$x\Phi = xe^{-\frac{i}{\hbar}px} = -\frac{\hbar}{i}\frac{\partial}{\partial p}\Phi = \frac{\hbar}{i}\frac{\partial}{\partial(-p)}\Phi$$

- $\langle p|\hat{p}|p'\rangle \equiv p\langle p|p'\rangle \equiv p\delta(p-p')$
1. $\langle p|\hat{p}|f\rangle = pf(p) = \boxed{p\langle p|f\rangle \equiv \breve{p}\langle p|f\rangle}$
2. $\langle p|\hat{x}|p'\rangle = \iint \langle p|x\rangle\langle x|\hat{x}|x'\rangle\langle x'|p'\rangle dxdx'$

$$= \frac{1}{2\pi\hbar}\int xe^{\frac{i}{\hbar}x(p'-p)}dx$$

$$= -\frac{\hbar}{i}\delta'(p-p') = -\frac{\hbar}{i}\frac{\partial}{\partial p}\langle p|p'\rangle$$

3. $\langle p|\hat{x}|f\rangle = \int\langle p|\hat{x}|p'\rangle\langle p'|f\rangle \ dp'$

$$= \boxed{-\frac{\hbar}{i}\frac{\partial}{\partial p}\langle p|f\rangle \equiv \breve{x}\langle p|f\rangle}$$

$\Rightarrow \ A(x,\hat{p}_x) \to A(\hat{x}_p, p)$

$\Rightarrow \boxed{\langle a\rangle = \langle\Phi|A(\hat{x}_p,p)|\Phi\rangle}$

Heisenberg Uncertainty Proof:

$$\langle f|g\rangle \equiv \left\langle (\widehat{A}-\langle a\rangle)\Psi \middle| (\widehat{B}-\langle b\rangle)\Psi\right\rangle$$

$$= \left\langle \Psi\middle|(\widehat{A}-\langle a\rangle)\middle|(\widehat{B}-\langle b\rangle)\Psi\right\rangle$$

$$= \langle\Psi|\widehat{A}(\widehat{B}\Psi)\rangle - \langle a\rangle\langle b\rangle = \langle\widehat{A}\widehat{B}\rangle - \langle a\rangle\langle b\rangle$$

$$\sigma_A^2\sigma_B^2 = \|(\widehat{A}-\langle a\rangle)\Psi\|^2 \ \|(\widehat{B}-\langle b\rangle)\Psi\|^2$$

$$\equiv \langle f|f\rangle\langle g|g\rangle \geq \|\langle f|g\rangle\|^2 \quad \text{(see Schwarz Ineq.)}$$

$$\geq \ [\text{Im}(\langle f|g\rangle)]^2 \ = \ \left(\frac{1}{2i}[\langle f|g\rangle - \langle f|g\rangle^*]\right)^2$$

$$= \left(\frac{1}{2i}\langle\widehat{A}\widehat{B}-\widehat{B}\widehat{A}\rangle\right)^2 \equiv \boxed{\left(\frac{1}{2i}\left\langle[\widehat{A},\widehat{B}]\right\rangle\right)^2}$$

Commutator

- $[\widehat{A},\widehat{B}]f \equiv \widehat{A}(\widehat{B}f) - \widehat{B}(\widehat{A}f)$
- $[A,BC] = [A,B]C + B[A,C]$
- $[AB,C] = A[B,C] + [A,C]B$
- $[x,\hat{p}] = i\hbar$
- $\boxed{\sigma_A\sigma_B \ \geq \ \left\|\frac{1}{2i}\left\langle[\widehat{A},\widehat{B}]\right\rangle\right\|}$

$$\Rightarrow \boxed{\Delta x\Delta p \ \geq \ \hbar/2}$$

- $\boxed{[\hat{p},f_{(x)}] = \hat{p}f_{(x)} = \frac{\hbar}{i}\frac{\partial f}{\partial x}}$
$\boxed{[\hat{x},g_{(p)}] = \hat{x}g_{(p)} = -\frac{\hbar}{i}\frac{\partial g}{\partial p}}$

Commutator of Hermitian $\widehat{A},\widehat{B}$

- $[A,B]^\dagger = -[A,B]$
- $\exists\Psi_n$ s.t. $\left(\widehat{A}\Psi_n = a\Psi_n\right)$, $\left(\widehat{B}\Psi_n = b\Psi_n\right)$

$\Leftrightarrow [\widehat{A},\widehat{B}] = 0$

$\Rightarrow \boxed{\sigma_A\sigma_B \ \geq \ 0} \ \text{\small(Both can be measured concurrently)}$

$\boxed{AB = BA}$

Anti-Hermitian Operators: $A^\dagger = -A$

- $\langle A\rangle = ai, \quad a\in\mathbb{R}$
- $[A,B]^\dagger = -[A,B]$

Operator Evolution (Heisenberg Equation)

$$\frac{d}{dt}\left\langle \Psi(x,t)\Big|Q\Big|\Psi(x,t)\right\rangle = \left\langle \frac{\partial \Psi}{\partial t}\Big|Q\Big|\Psi\right\rangle + \left\langle \Psi\Big|\frac{\partial Q}{\partial t}\Big|\Psi\right\rangle + \left\langle \Psi\Big|Q\Big|\frac{\partial \Psi}{\partial t}\right\rangle$$

$$\boxed{\begin{aligned}\frac{d}{dt}\langle Q\rangle &= \frac{1}{i\hbar}\left\langle \left[\widehat{Q},\widehat{H}\right]\right\rangle + \left\langle \frac{\partial \widehat{Q}}{\partial t}\right\rangle \\ i\hbar\frac{d}{dt}\langle Q\rangle &= \left\langle \left[\widehat{Q},\widehat{H}\right]\right\rangle + i\hbar\left\langle \frac{\partial \widehat{Q}}{\partial t}\right\rangle\end{aligned}}$$ (Q is conserved when this equals 0)

- Conservations: $\dfrac{d\langle \Psi|\Psi\rangle}{dt} = 0, \ \dfrac{d\langle H\rangle}{dt} = 0$

- Ehrenfest's Theorem: $m\dfrac{d\langle x\rangle}{dt} = \langle p\rangle, \ \dfrac{d\langle p\rangle}{dt} = -\left\langle \dfrac{\partial V}{\partial x}\right\rangle \ \Rightarrow$ other classical eq.

- Virial Theorem: $\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\left\langle [H,x]p + x[H,p]\right\rangle = \left\langle \left[\frac{p^2}{2m},x\right]p + x[V,p]\right\rangle$

$$= \frac{i}{\hbar}\left\langle \frac{1}{2m}p[p,x]p - \frac{1}{2m}[p,x]p^2 - x[p,V]\right\rangle$$

$$\boxed{\frac{d\langle xp\rangle}{dt} = 2\langle T\rangle - \left\langle x\frac{\partial V}{\partial x}\right\rangle} \to 0 = \frac{d}{dt}\left\langle \Psi_n(x)\Big|Q_{(x,p)}\Big|\Psi_n(x)\right\rangle$ (for stationary states)

- Energy-Time Uncertainty: $(Q = Q(x,\hat{p}) \neq Q(x,\hat{p},t)) \ \Rightarrow \ \sigma_H\sigma_Q \geq \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right|$

$$\Rightarrow \boxed{\begin{aligned}\sigma_Q &\equiv \frac{d\langle Q\rangle}{dt}\Delta t \ \approx \ \Delta\langle Q\rangle \\ \sigma_H\left(\frac{\sigma_Q}{|d\langle Q\rangle/dt|}\right) &\geq \frac{\hbar}{2} \\ \Delta E\Delta t &\geq \frac{\hbar}{2}\end{aligned}}$$ Δt is the amount of time it would take $\langle Q\rangle$ to change "appreciably", or one std. dev. at the constant rate $\frac{d}{dt}\langle Q\rangle$

  Mass Lifetime:

$\Delta(mc^2)\Delta t \geq \frac{\hbar}{2}$ ☑

  Orthogonal Time Example:

$\Psi(x,\tau) = \frac{\sqrt{2}}{2}\left(\Psi_1 e^{-\frac{i}{\hbar}E_1\tau} + \Psi_2 e^{-\frac{i}{\hbar}E_2\tau}\right)$

$\left\langle \Psi(x,0)\Big|\Psi(x,\tau)\right\rangle = 0 = \frac{1}{2}\left(e^{-\frac{i}{\hbar}E_1\tau} + e^{-\frac{i}{\hbar}E_2\tau}\right)$

$\Rightarrow \ \tau\frac{E_2-E_1}{2} = \frac{\pi}{2}\,\hbar\left(\frac{1}{2}+n\right) \geq \frac{\hbar}{2}$ ☑

Translation Operator

$$f(x + \Delta x) \approx f(x) + \frac{df}{dx}\Delta x$$

$$= f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + ... = \left\{\begin{array}{l} f(x') = \sum_n \frac{f^{(n)}(a)}{n!}(x'-a)^n \\ (x' = x + \Delta x), \ (a = x) \end{array}\right\}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(\Delta x)^n = \sum_{n=0}^{\infty} \frac{(\Delta x \nabla)^n}{n!} f(x)$$

$$\boxed{f(x + \Delta x) = e^{\frac{i}{\hbar}(\Delta x)\breve{p}} \, f(x)} \quad \Leftrightarrow \quad \boxed{f(x) = e^{\frac{i}{\hbar}x\breve{p}} \, f(0)} \quad * \ \langle x | e^{\frac{i}{\hbar}x\hat{p}} | x'\rangle = e^{\frac{i}{\hbar}x\breve{p}} \langle x | x'\rangle \ *$$

Time Translation :

$$f(t + \Delta t) = \underline{f(t) + f'(t)\Delta t + ...} = \sum_n \frac{(\Delta t)^n}{n!}\left(\frac{\partial}{\partial t}\right)^n f(t)$$

$$\langle x_N | \hat{U}(t) | x_0 \rangle$$
$$= \breve{U}_{(x_N, t; \, x_0, 0)}$$

$$\langle x | \hat{U}(t) | \Psi \rangle$$
$$= \Psi_{(x,t)}$$

$$i\hbar\frac{\partial}{\partial t}\Psi = \breve{H}_{(x,p,t)}\Psi$$

$$\frac{\partial f}{\partial t} = \left[\frac{-i\breve{H}}{\hbar}\right] f \quad \Rightarrow$$

$$\frac{\partial^n f}{\partial t^n} \neq \left[\frac{-i\breve{H}(t)}{\hbar}\right]^n f$$

$$\left\{\begin{array}{l} f(t_0 + \Delta t) \approx e^{\frac{-i\Delta t}{\hbar}\breve{H}(t_0)} f(t_0) \quad \text{(1st order)} \\[2mm] \boxed{f(0+t) = \lim_{N\to\infty} \prod_{n=0}^{N-1} e^{\frac{-i}{\hbar}\breve{H}(n\Delta t)\Delta t} f(0)} \\[2mm] \sim \neq e^{\frac{-i}{\hbar}\int \breve{H}(t)\, dt} f(0) \quad \left(\begin{array}{l}\text{since } 0 \neq \\ [H(t_0), H(t_1)]\end{array}\right) \end{array}\right.$$

Pictures: $\langle Q \rangle_{(t)} = \langle \Psi_{(x,t)} | \, Q_{(x,p,t)} \, | \Psi_{(x,t)}\rangle$

- $\boxed{\text{Schrodinger Picture: } \langle Q \rangle_{(t)} = \left\langle e^{\frac{-i}{\hbar}t\hat{H}}\Psi_{(t=0)} \middle| Q_{(x,p,t)} \middle| e^{\frac{-i}{\hbar}t\hat{H}}\Psi_{(t=0)} \right\rangle}$

$$Q = Q(x,p) \Rightarrow \langle Q \rangle_{(t)} = \left\langle \sum e^{\frac{-i}{\hbar}E_n t}c_n\Psi_n(x) \middle| Q \middle| \sum e^{\frac{-i}{\hbar}E_n t}c_n\Psi_n(x)\right\rangle \quad \text{(nice for stationary states)}$$

- $\boxed{\text{Heisenberg Picture: } \langle Q \rangle_{(t)} = \left\langle \Psi_{(t=0)} \middle| e^{\frac{i}{\hbar}t\hat{H}}Q e^{\frac{-i}{\hbar}t\hat{H}} \middle| \Psi_{(t=0)}\right\rangle}$

- $\boxed{\text{Dirac Picture: } \langle Q \rangle_{(t)} = \left\langle e^{\frac{-i}{\hbar}\int \hat{H}_1(t)dt}\Psi_{(t=0)} \middle| e^{\frac{i}{\hbar}t\hat{H}_0}Q e^{\frac{-i}{\hbar}t\hat{H}_0} \middle| e^{\frac{-i}{\hbar}\int \hat{H}_1(t)dt}\Psi_{(t=0)}\right\rangle}$

$$\langle Q \rangle_{(t+\Delta t)} = \langle Q \rangle_{(t)} + \frac{d\langle Q\rangle}{dt}\Delta t + ... \quad \Rightarrow$$

A 1st order approximation of $\langle Q \rangle_{(t+\Delta t)}$

should yield $\frac{d\langle Q\rangle}{dt} = \frac{1}{i\hbar}\left\langle\left[Q, H\right]\right\rangle + \frac{\partial Q}{\partial t}$

Schrodinger Picture

1.) $i\hbar \frac{\partial}{\partial t}\langle Q_S \rangle = \langle [Q_S, H_S]\rangle$

4.) $|\Psi_{S(t)}\rangle = U_{S(t,t_0)}|\Psi_{S(t_0)}\rangle$

$\Rightarrow i\hbar \frac{\partial}{\partial t}|\Psi_S\rangle = H_S\Psi_S = [H_S^0 + H_S^1(t)]|\Psi_S\rangle$

$= \sum E_n |n_S^0\rangle e^{-\frac{i}{\hbar}E_n t}\langle n_{S(t)}^1|\Psi_{(0)}\rangle$

$+ \sum |n_S^0\rangle e^{-\frac{i}{\hbar}E_n t} \cdot i\hbar\frac{\partial}{\partial t}\langle n_{S(t)}^1|\Psi_{(0)}\rangle$

$\Rightarrow i\hbar\frac{\partial}{\partial t}U_{S(t,t_0)}^0 = H_S^0 U_{S(t,t_0)}^0$

$\Rightarrow U_{S(t,t_0)}^0 = e^{-\frac{i}{\hbar}H_S^0(t-t_0)}$

Heisenberg Picture

1.) $Q_{H(t)} \equiv U_S^\dagger Q_S U_S \qquad H_S \neq H_S(t)$

$\Rightarrow i\hbar\frac{\partial}{\partial t}Q_H = [Q_H, H_H] \qquad \Downarrow$
$\qquad\qquad\qquad\qquad\qquad\qquad H_H = H_S$

2.) $U_H \equiv U_{S(t,t_0)}^\dagger U_{S(t,t_0)} = \mathbb{I}$

3.) $|q_{H(t)}\rangle \equiv U_{S(t,t_0)}^\dagger |q_S\rangle$

$\Rightarrow Q_H|q_{H(t)}\rangle = q|q_{H(t)}\rangle$

$\Rightarrow i\hbar\frac{\partial}{\partial t}|q_{H(t)}\rangle = -H_S|q_{H(t)}\rangle$

4.) $|\Psi_H\rangle \equiv U_{S(t,t_0)}^\dagger |\Psi_{S(t)}\rangle = |\Psi_{S(t_0)}\rangle$

$= U_{H(t,t_0)}|\Psi_{H(t_0)}\rangle$

$\Rightarrow i\hbar\frac{\partial}{\partial t}|\Psi_H\rangle = 0$

Dirac/Interaction Picture (see transition amplitude)

1.) $Q_{I(t)} \equiv U_S^{0\dagger} Q_S U_S^0$

$\Rightarrow i\hbar\frac{\partial}{\partial t}Q_I = [Q_I, H_I^0] \qquad (H_S^0 = H_I^0 \text{ see Heis. pic.})$

2.) $U_{I(t,t_0)} \equiv U_S^{0\dagger}(t,t_0) U_{S(t,t_0)}$

3.) $|q_{I(t)}\rangle \equiv U_S^{0\dagger}(t,t_0)|q_S\rangle$

$\Rightarrow Q_I|q_{I(t)}\rangle = q|q_{I(t)}\rangle$

$\Rightarrow i\hbar\frac{\partial}{\partial t}|q_{I(t)}\rangle = -H_S^0|q_{I(t)}\rangle$

4.) $|\Psi_{I(t)}\rangle \equiv U_S^{0\dagger}(t,t_0)|\Psi_{S(t)}\rangle$

$= U_{I(t,t_0)}|\Psi_{I(t_0)}\rangle \qquad (\text{since } |\Psi_{I(t_0)}\rangle = |\Psi_{S(t_0)}\rangle)$

$\Rightarrow i\hbar\frac{\partial}{\partial t}|\Psi_I\rangle = U_S^{0\dagger}H_S^1(t)U_S^0|\Psi_I\rangle = H_I^1|\Psi_I\rangle$

$\Rightarrow i\hbar\frac{\partial}{\partial t}U_{I(t,t_0)} = H_I^1(t)U_{I(t,t_0)}$

$\Rightarrow U_{I(t,t_0)} = \mathbb{I} + \frac{1}{i\hbar}\int_{t_0}^t H_I^1(t')U_I(t',t_0)\,dt'$

- $U_{I(t,t_0)} = \mathbb{I} + \frac{1}{i\hbar}\int_{t_0}^t H_I^1(t')U_I(t',t_0)\,dt'$

$= \mathbb{I} + \mathcal{O}(H_I^1)$

$= \mathbb{I} + \frac{1}{i\hbar}\int_{t_0}^t H_I^1(t')\,dt' + \mathcal{O}([H_I^1]^2)$

$= \mathbb{I} + \frac{1}{i\hbar}\int_{t_0}^t H_I^1(t')\,dt'$

$+ \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} H_I^1(t')H_I^1(t'')\,dt''dt' + \dots$

- $U_{S(t,t_0)} = U_S^0 + \frac{1}{i\hbar}\int_{t_0}^t U_S^0 H_I^1(t')\,dt' + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} U_S^0 H_I^1(t')H_I^1(t'')\,dt''dt' + \dots$

$= U^0(t,t_0)$

$+ \frac{1}{i\hbar}\int_{t_0}^t U^0(t,t_0)U^{0\dagger}(t',t_0)H^1 U^0(t',t_0)\,dt'$

$+ \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} U^0(t,t_0)U^{0\dagger}(t',t_0)H^1 U^0(t',t_0)U^{0\dagger}(t'',t_0)H^1 U^0(t'',t_0)\,dt''dt' + \dots$

Infinitismal t Path Integral

$$S[x(t)] = \int_0^t \mathcal{L}(x, \dot{x})\, dt \;\to\; \mathcal{L}\,\delta t = \left[\tfrac{1}{2}m\left(\tfrac{x_1 - x_0}{\delta t}\right)^2 - V\left(\tfrac{x_1 + x_0}{2}, t_0 + \tfrac{\delta t}{2}\right)\right]\delta t$$

$$\langle x|\hat{U}_{(\epsilon)}|\Psi\rangle = \int \langle x|\hat{U}_{(\epsilon)}|x'\rangle\langle x'|\Psi_{(x,t)}\rangle\, dx' = \int \breve{U}_{(x,\,t+\epsilon;\,x',\,t)}\Psi_{(x',t)}\, dx' = \Psi_{(x,\,t+\epsilon)}$$

- $\breve{U}_{(x_1,\,\epsilon;\,x_0,\,0)} = A e^{\frac{i}{\hbar}S} = A e^{\frac{i}{\hbar}\mathcal{L}\epsilon}$

$$= A\exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2}m\tfrac{(x_1-x_0)^2}{\epsilon} - \epsilon V\left(\tfrac{x_1+x_0}{2}, 0+\tfrac{\epsilon}{2}\right)\right]\right\}$$

$$(\eta = x_0 - x_1) \quad = A\exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2}m\tfrac{\eta^2}{\epsilon}\right]\right\}\exp\left\{-\tfrac{i}{\hbar}\epsilon V\left(x_1 + \tfrac{\eta}{2}, 0+\tfrac{\epsilon}{2}\right)\right\}$$

$$\approx A\exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2}m\tfrac{\eta^2}{\epsilon}\right]\right\}\exp\left\{-\tfrac{i}{\hbar}\epsilon V_{(x_1,0)}\right\}$$

$$\approx A\exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2}m\tfrac{\eta^2}{\epsilon}\right]\right\}\left[1 - \tfrac{i}{\hbar}\epsilon V_{(x_1,0)}\right]$$

> $\frac{\eta^2}{\epsilon} \lesssim \pi$ Explanation
>
> The integral involving $\breve{U}$ is over all η. The phase of the complex exponential will vary/oscillate too wildly and destructively interfere if η^2/ϵ were to grow too big, so $\eta^2 \sim \epsilon$ is all that matters. This means the integral is over $\sqrt{\epsilon}$, not ϵ. Because of this (somehow, Bibl. given), using a finite difference formula for derivatives is legitimate in this case, though not in general.

$$\Psi_{(x,\epsilon)} = \int_{-\infty}^{\infty} \breve{U}_{(x,\epsilon;\,x',0)}\Psi_{(x',0)}\, dx'$$

$$= A\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}S}\Psi_{(x',0)}\, dx'$$

$$= A\int_{-\infty}^{\infty} \exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2}m\tfrac{(x-x')^2}{\epsilon} - \epsilon V\left(\tfrac{x+x'}{2}, 0+\tfrac{\epsilon}{2}\right)\right]\right\}\Psi_{(x',0)}\, dx'$$

$$= A\int_{-\infty}^{\infty} \exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2}m\tfrac{\eta^2}{\epsilon} - \epsilon V\left(\tfrac{x+\eta/2}{2}, 0+\tfrac{\epsilon}{2}\right)\right]\right\}\Psi_{(x+\eta,0)}\, d\eta$$

$$\approx A\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{\eta^2}{\epsilon}\right]}\left[1 - \tfrac{i}{\hbar}\epsilon V_{(x,0)}\right]\left[\Psi_{(x,0)} + \eta\Psi'_{(x,0)} + \tfrac{\eta^2}{2}\Psi''_{(x,0)}\right]d\eta$$

$$\approx A\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{\eta^2}{\epsilon}\right]}\left[\left(1 - \tfrac{i}{\hbar}\epsilon V_{(x,0)}\right)\Psi_{(x,0)} + \cancel{\eta\Psi'_{(x,0)}} + \tfrac{\eta^2}{2}\Psi''_{(x,0)}\right]d\eta$$

$$= \cancel{A}\sqrt{\tfrac{2\hbar\epsilon\pi}{-im}}\left[\left(1 - \tfrac{i}{\hbar}\epsilon V_{(x,0)}\right) + \tfrac{1}{2}\cdot\tfrac{2\hbar\epsilon}{-im}\cdot\tfrac{1}{2}\tfrac{\partial^2}{\partial x^2}\right]\Psi_{(x,0)}$$

$$= \Psi_{(x,0)} - \tfrac{i}{\hbar}\epsilon\breve{H}\Psi_{(x,0)}$$

$$\boxed{i\hbar\,\frac{\Psi_{(x,\epsilon)} - \Psi_{(x,0)}}{\epsilon - 0} = \breve{H}\Psi_{(x,0)}}$$

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Finite t, Free Particle Propagator

$$S[x(t)] = \int_0^t \mathcal{L}(x, \dot{x})\, dt = \lim_{N\to\infty} \sum_{n=0}^{N-1} \left[\tfrac{1}{2}m\left(\tfrac{x_{n+1}-x_n}{\delta t}\right)^2\right]\delta t$$

$$\langle x_N|\hat{U}(t)|\Psi(0)\rangle = \int \langle x_N|\hat{U}(t)|x_0\rangle\Psi(x_0,0)\, dx_0 = \int \breve{U}(x_N, t; x_0, 0)\Psi(x_0,0)\, dx_0 = \Psi(x_N, t)$$

$$\Rightarrow \quad \langle x_N|\hat{U}(t)|x_0\rangle = \langle x_N|e^{-\frac{i}{\hbar}Ht}e^{\frac{i}{\hbar}Ht_0}|x_0\rangle = \langle x_N, t_N\,|\,x_0, t_0\rangle$$

$$\langle x_N|\hat{U}(t)|x_0\rangle = \lim_{N\to\infty}\langle x_N|\hat{U}^N(\epsilon)|x_0\rangle = \lim_{N\to\infty}\langle x_N|\hat{U}(\epsilon)\ldots\hat{U}(\epsilon)\hat{U}(\epsilon)|x_0\rangle$$

$$= \lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle x_N|\hat{U}(\epsilon)|x_{N-1}\rangle\ldots\langle x_2|\hat{U}(\epsilon)|x_1\rangle\langle x_1|\hat{U}(\epsilon)|x_0\rangle\, dx_1\, dx_2\ldots dx_{N-1}$$

$$\breve{U}(x_N, t; x_0, 0) = \int_{x_0}^{x_N} A e^{\frac{i}{\hbar}S}\mathcal{D}[x(t)]$$

$$= \lim_{N\to\infty} A \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{(x_1-x_0)^2}{\epsilon}+\frac{1}{2}m\frac{(x_2-x_1)^2}{\epsilon}+\frac{1}{2}m\frac{(x_3-x_2)^2}{\epsilon}+\ldots\right]}dx_1\, dx_2\ldots dx_{N-1}$$

$$= \lim_{N\to\infty} A\sqrt{\frac{2\hbar\epsilon}{m}}^{N-1}\int\cdots\int_{-\infty}^{\infty} e^{-\frac{(y_1-y_0)^2}{i}-\frac{(y_2-y_1)^2}{i}}dy_1\, e^{\left[-\frac{(y_3-y_2)^2}{i}\ldots\right]}dy_2\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\frac{2\hbar\epsilon}{m}}^{N-1}\int\cdots\int_{-\infty}^{\infty} e^{-\frac{y_1^2+y_1^2-2y_1(y_2+y_0)+y_2^2+y_0^2}{i}}dy_1\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\frac{2\hbar\epsilon}{m}}^{N-1}\int\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{i}\left(2\left[y_1^2-y_1(y_0+y_2)+\frac{(y_0+y_2)^2}{4}\right]-\frac{(y_2+y_0)^2}{2}+y_2^2+y_0^2\right)}dy_1\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\frac{2\hbar\epsilon}{m}}^{N-1}\int\cdots\int_{-\infty}^{\infty} e^{-\frac{2}{i}\left[y_1-\frac{y_2+y_0}{2}\right]^2}dy_1\, e^{-\frac{(y_2-y_0)^2}{2i}}e^{-\frac{(y_3-y_2)^2}{i}\ldots}dy_2\, e^{[\ldots]}\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\frac{2\hbar\epsilon}{m}}^{N-1}\int\cdots\sqrt{\frac{\pi i}{2}}\int_{-\infty}^{\infty} e^{-\frac{(y_2-y_0)^2}{2i}}e^{-\frac{(y_3-y_2)^2}{i}\ldots}dy_2\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\frac{2\hbar\epsilon}{m}}^{N-1}\int\cdots\sqrt{\frac{(\pi i)^2}{3}}\int_{-\infty}^{\infty} e^{-\frac{(y_3-y_0)^2}{3i}}e^{-\frac{(y_4-y_3)^2}{i}\ldots}dy_3\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\frac{2\hbar\epsilon}{m}}^{N-1}\sqrt{\frac{(\pi i)^{N-1}}{N}}\, e^{-\frac{(y_N-y_0)^2}{Ni}} \quad = \quad \lim_{N\to\infty} A\sqrt{\frac{2\hbar\epsilon\pi}{-im}}^N\sqrt{\frac{-im}{2\hbar\epsilon N\pi}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x_N-x_0)^2}{N\epsilon}}$$

$$\boxed{\text{Free Particle}: \quad \breve{U}(x_N, t; x_0, 0) = \sqrt{\frac{-im}{2\hbar t\pi}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x_N-x_0)^2}{t}}}$$

$$\boxed{\int_{x_0}^{x_N}\mathcal{D}[x(t)] = \lim_{N\to\infty}\sqrt{\frac{-im}{2\hbar\epsilon\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{-im}{2\hbar\epsilon\pi}}\, dx_1\cdots\int_{-\infty}^{\infty}\sqrt{\frac{-im}{2\hbar\epsilon\pi}}\, dx_{N-1}}$$

All Paths Explanation

$$S[x] = S[x_{cl}] + \cancel{S'[x_{cl}]\eta} + \mathcal{O}(\eta^2)$$

1st order variation of S from x_{cl} equals 0. This means propagator integrand for paths near x_{cl} will have about the same phase, and will add constructively. Paths very different from x_{cl} (like those with faster than light motion) will vary in action, and because $\hbar$ is so small their phases will vary wildly, meaning the sum will destructively interfere. The result is that only paths near the classical path will be important, with $S[x]/\hbar \lesssim \pi$.

Action-Energy Relationship

$$S_{(x_{cl} + \Delta x_{cl}, \dot{x} + \Delta \dot{x}_{cl}, \tau + \Delta\tau)} = S_{cl} + \Delta S_{cl}$$

$$= \int_0^{\tau + \Delta\tau} \mathcal{L}(x_{cl} + \Delta x_{cl}, \dot{x}_{cl} + \Delta \dot{x}_{cl}, t)\, dt$$

$$dS = \frac{\partial S}{\partial \tau} d\tau + \left[\frac{\partial S}{\partial x} dx + \frac{\partial S}{\partial \dot{x}} d\dot{x} = \int (dL) dt \right] \quad (\eta = dx,\ \eta(0) = 0)$$

$$\Delta S_{cl} = \mathcal{L}_{(\tau)} \Delta\tau + \int_0^{\tau} \left. \frac{\partial \mathcal{L}}{\partial x}\right|_{cl} \eta + \left.\frac{\partial \mathcal{L}}{\partial \dot{x}}\right|_{cl} \dot{\eta}\, dt$$

$$= \mathcal{L}_{(\tau)} \Delta\tau + \int_0^{\tau} \left[\cancel{\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}}\right]_{cl} \eta + \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_{cl}} \eta \right] dt$$

$$= \left[\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{x} \right]_{cl,\tau} \Delta\tau$$

$$\boxed{\Delta S_{cl} = -H_{(t_f)} \Delta t_f}$$

(Time-Independent) Hamiltonian-Lagrangian Propagator Relationship / Finite Path Integral

$$\breve{U}_{(x_N, t;\, x_0, 0)} = \langle x_N | e^{-\frac{i}{\hbar} H t} | x_0 \rangle = \langle x_N | [e^{-\frac{i}{\hbar} H \frac{t}{N}}]^N | x_0 \rangle = \lim_{N\to\infty} \langle x_N | [e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} \epsilon} e^{-\frac{i}{\hbar} V(\hat{x}) \epsilon}]^N | x_0 \rangle \quad \text{(not trivial)}$$

$$= \lim_{N\to\infty} \int_{x\ldots} \langle x_N | e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} \epsilon} e^{-\frac{i}{\hbar} V(\hat{x}) \epsilon} | x_{N-1} \rangle \ldots \langle x_1 | e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} \epsilon} e^{-\frac{i}{\hbar} V(\hat{x}) \epsilon} | x_0 \rangle\, dx \ldots$$

$$= \lim_{N\to\infty} \int_{x\ldots} \ldots \left[\underbrace{\int \langle x_1 | e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} \epsilon} | p \rangle \langle p | x_0 \rangle\, dp}_{\text{free part. prop.}}\ e^{-\frac{i}{\hbar} V(x_0) \epsilon} \right] dx \ldots$$

1. $$= \lim_{N\to\infty} \int_{x\ldots} \ldots \left[\int \frac{e^{\frac{i}{\hbar} p(x_1 - x_0)}}{2\pi\hbar} e^{-\frac{i}{\hbar}\frac{p^2}{2m}\epsilon} e^{-\frac{i}{\hbar} V(x_0)\epsilon}\, dp \right] dx \ldots = \boxed{\int e^{\frac{i}{\hbar} \int p\dot{x} - H(x,p)\, dt} [\mathcal{D}x \mathcal{D}p]} \quad \text{(Phase Space)}$$

2. $$= \lim_{N\to\infty} \int_{x\ldots} \ldots \left[\sqrt{\frac{-im}{2\pi\hbar\epsilon}}\ e^{\frac{im(x_1 - x_0)^2}{\epsilon}} e^{-\frac{i}{\hbar} V(x_0)\epsilon} \right] dx \ldots = \lim_{N\to\infty} \int_{x\ldots} \ldots \sqrt{\frac{-im}{2\pi\hbar\epsilon}}\ e^{\frac{i}{\hbar}\mathcal{L}\epsilon}\, dx_1 \ldots$$

$$= \boxed{\int_{x_0}^{x_N} e^{\frac{i}{\hbar} S} [\mathcal{D}x] = \int_{x_0}^{x_N} e^{\frac{i}{\hbar} \int \mathcal{L}\, dt} [\mathcal{D}x]} \quad \text{(Configuration Space)} \qquad \text{(above is only integrable if } p \text{ is quadratic in } H\text{)}$$

Trace of Propagator

$$G_{(t)} = \int \langle x | e^{-\frac{i}{\hbar} H t} | x \rangle\, d^3 x$$

$$= \sum_n \int \langle x | n \rangle e^{-\frac{i}{\hbar} E_n t} \langle n | x \rangle\, dx$$

$$G_{(t)} = \sum_n e^{-\frac{i}{\hbar} E_n t} \sim \sum_n e^{-\beta E_n} = Z_{(\beta)}$$

1.3 Extra

$L_2 \subset$ Hilbert Space = complete inner product space

$$\rho(x,t) \equiv \|\Psi\|^2 \quad,\quad P_a^b(t) = \int_a^b \rho\,dx \quad,\quad P(t) = P_{-\infty}^\infty(t) \quad,\quad \Psi = \sqrt{\rho}\,e^{\frac{i}{\hbar}S} \quad \text{e.g.,}\quad e^{\frac{i}{\hbar}(p\cdot x - Et)}$$

- $$\breve{E}\rho = \breve{E}(\Psi^*\Psi) = \Psi^*(\breve{E}\Psi) + \Psi(\breve{E}\Psi^*)$$
  $$= \Psi^*(\breve{H}\Psi) - \Psi(\breve{H}\Psi^*)$$
  $$= \Psi^*(\tfrac{p^2}{2m} + V)\Psi - \Psi(\tfrac{p^2}{2m} + V)\Psi^*$$
  $$-\tfrac{\hbar}{i}\tfrac{\partial\rho}{\partial t} = \tfrac{\hbar}{i}\nabla\cdot\left(\Psi^*\tfrac{p}{2m}\Psi - \Psi\tfrac{p}{2m}\Psi^*\right)$$

  (Probability Current)
  $$\tfrac{\partial\rho}{\partial t} = -\nabla\cdot J = -\nabla\cdot\left(\Psi^*\tfrac{p}{2m}\Psi - \Psi\tfrac{p}{2m}\Psi^*\right)$$
  $$= -\nabla\cdot\tfrac{\rho\nabla S}{m} \quad (\text{e.g., } \nabla S = p)$$
  $$\left[\tfrac{d}{dt}P_a^b = J_{(a,t)} - J_{(b,t)}\right], \quad \left[\int J\,dV = \langle\Psi|\tfrac{p}{m}|\Psi\rangle = \tfrac{\langle p\rangle}{m}\right]$$

- $(V \in \mathbb{R}) \qquad \Rightarrow \quad \tfrac{d}{dt}P = 0 \qquad \Rightarrow \quad P(t) \equiv 1$

  $(V = V_0 - i\Gamma) \quad \Rightarrow \quad \tfrac{d}{dt}P = \tfrac{-2\Gamma}{\hbar}P \quad \Rightarrow \quad P(t) = e^{-2(\Gamma/\hbar)t}$

- $\langle\Psi_n|\Psi_n\rangle \,,\, \langle\Psi_m|\Psi_m\rangle = 1 \;\Rightarrow\; \tfrac{d}{dt}\langle\Psi_n|\Psi_m\rangle = 0$

Schwarz Inequality:
$$\left\|\int_a^b f^*g\,dx\right\|^2 \le \left\|\int_a^b f^*f\,dx\right\|\left\|\int_a^b g^*g\,dx\right\|$$
$$\|\langle f|g\rangle_{ab}\|^2 \le \|\langle f|f\rangle_{ab}\|\,\|\langle g|g\rangle_{ab}\|$$

$$\Big[V(x) = V(-x)\Big] \;\Rightarrow\; \Big[\Psi(x) \Rightarrow \Psi(-x)\Big] \;\Rightarrow\; \Big[\Psi(-x) = \Psi(x)\Big] \,\cup\, \Big[\Psi(-x) = -\Psi(x)\Big]$$

Discontinuity in Ψ means the possiblity of $\sigma_p \to \infty$

Prob 3.29: $\Psi(x,0) = \begin{cases} \frac{1}{\sqrt{2n\lambda}}e^{2\pi i x/\lambda}, & -n\lambda < x < n\lambda \\ 0 & \text{else} \end{cases}$

$\sigma_p \to \infty$ because the integral of $\delta^2(x)$ is infinite

$$\int_{-\infty}^\infty f(x)D_1(x)dx = \int_{-\infty}^\infty f(x)D_2(x)dx \;\Rightarrow\; \delta(cx) = \frac{1}{|c|}\delta(x)$$

$$\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ik(x-x')}dx' \;\Rightarrow\; F[\delta(x)] = \frac{1}{2\pi}$$

$$\delta'(x - x') = \frac{1}{2\pi}\int_{-\infty}^\infty ik e^{ik(x-x')}dx' \;\Rightarrow\; \int\delta'(x-x')f(x')dx' = f'(x)$$

Poisson Brackets

$$\{f,g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$$

$$\{\omega(q,p,t), H\} = \sum_i \frac{\partial \omega}{\partial q_i}\dot{q} + \dot{p}\frac{\partial \omega}{\partial p_i} = \dot{\omega} - \frac{\partial \omega}{\partial t}$$

Hamilton Eq. : $\dot{q} = \{q, H\}, \ \dot{p} = \{p, H\}$

Canonical Transforms

$$\begin{array}{l} q \to \bar{q}(q,p) \\ p \to \bar{p}(q,p) \end{array} \quad \text{s.t.} \quad \begin{array}{l} \{\bar{q}_i, \bar{q}_j\} = 0 = \{\bar{p}_i, \bar{p}_j\} \\ \{\bar{q}_i, \bar{p}_j\} = \delta_{ij} \end{array} \quad \left(\begin{array}{l} \text{Point Transforms} \\ \bar{q}(q) \text{ are canonical.} \end{array} \right)$$

$$\Rightarrow \quad \begin{array}{l} \dot{\bar{q}} = \frac{\partial H}{\partial \bar{p}} \\ \dot{\bar{p}} = -\frac{\partial H}{\partial \bar{q}} \end{array} \quad , \quad \{f,g\}_{q,p} = \{f,g\}_{\bar{q},\bar{p}}$$

Generator of Transformation

1. $\delta H = 0$

$$\delta H = \epsilon_\lambda \{H, g\}$$

$$g = l_z$$

2. $\bar{q}_i = q_i + \delta q_i \qquad , \quad \bar{p}_i = p_i + \delta p_i$

$\qquad \equiv q_i + \epsilon_\lambda \frac{\partial g}{\partial p_i} \qquad \equiv p_i - \epsilon_\lambda \frac{\partial g}{\partial q_i}$

$\qquad = q_i + \epsilon_\lambda \{q_i, g\} \qquad = p_i + \epsilon_\lambda \{p_i, g\}$

$$\Rightarrow \quad \boxed{\frac{\partial H}{\partial \lambda} = 0 = \frac{dg}{dt}}$$

(e.g. $g = p$ or $g = l_z$)

$$\delta x = -\epsilon y = -(\delta\theta)y$$
$$\delta y = \epsilon x = (\delta\theta)x$$

$$\Rightarrow \quad \boxed{\begin{array}{l} \frac{\partial x}{\partial \theta} = -y \\ \frac{\partial y}{\partial \theta} = x \end{array}}$$

3. $\Rightarrow \ \delta f = \epsilon_\lambda \{f, g\} \ \to \ \frac{\partial f}{\partial \lambda} = \{f, g\}$

Tensors and Tensor Operators

rank-2 Tensor : $|t^{(2)}\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3} t_{ij}|i\rangle|j\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3}|ij\rangle\langle ij|t^{(2)}\rangle$

rank-2 Carte. Tens. Oper., T_{ij} : Set of $3^{n=2}$ Operators

rank-k Spher. Tens. Oper., T_k^q : Set of $2k+1$ Operators s.t. $U[R]T_k^q U^\dagger[R] = \sum_{q'=-k}^{k} D_{q'q}^k T_k^{q'}$

$$\Rightarrow U T_k^q U^\dagger U|jm\rangle = \sum_{q'}\sum_{m'} D_{q'q}^k D_{m'm}^j T_{q'}^k |jm'\rangle$$

$$\sim \ U|kq\rangle|jm\rangle = \sum_{q'}\sum_{m'} D_{q'q}^k D_{m'm}^j |kq'\rangle|jm'\rangle$$

$\bullet \ T_1^{\pm 1} = \mp \frac{V_x \pm iV_y}{\sqrt{2}}$

$\quad T_1^0 \ = V_z$

(CG coeff.)

Wigner-Eckhart : $\langle \alpha_2 j_2 m_2 | T_k^q | \alpha_1 j_1 m_1 \rangle = \langle \alpha_2 j_2 \| T_k \| \alpha_1 j_1 \rangle \cdot \langle j_2 m_2 | kq, j_1 m_1 \rangle$

$$0 = \left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + l(l+1) - \frac{m^2}{\sin^2\theta}\right]P_l^m = \left[\frac{d}{dx}\left([1-x^2]\frac{d}{dx}\right) + l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m \; {\scriptstyle(x=\cos\theta)}$$

$$= \left[\frac{d}{d\xi}\left(\xi[1-\xi]\frac{d}{d\xi}\right) + l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m \; {\scriptstyle(\xi=\frac{1}{2}[1-x])}$$

Legendre Polynomial	Associated Legendre Function	Spherical Harmonics
$(m=0 \leftrightarrow$ Azimuthal Symmetry$)$	(not a polynomial if m is odd)	

$$Y_{lm}(\theta,\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi} \times$$

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-l)^l \qquad P_l^m(x) = \sqrt{1-x^2}^{\,|m|}\left(\frac{d}{dx}\right)^{|m|}P_l(x) \qquad (-1)^m Y_{l,-m}^* \quad \sqrt{\frac{(l-m)!}{(l+m)!}\frac{2l+1}{2}}P_l^m(\cos\theta)$$

$$\delta_{l'l} = \frac{2l+1}{2}\int_{-1}^1 P_l(x)P_{l'}(x)\,dx \qquad \delta_{l'l} = \frac{(l-m)!}{(l+m)!}\frac{2l+1}{2}\int_{-1}^1 P_l^m(x)P_{l'}^m(x)\,dx$$

$$\delta_{l'l}\delta_{m'm} = \iint Y_{m'l'}^* Y_{ml}\,d\Omega$$

- $\boxed{P_l(\cos\theta) = P_l^0(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}Y_{l0}(\theta,\phi)}$ • $f(0,\phi) = \sum_l \sum_m \langle x|Y_{l0}\rangle\langle Y_{l0}^*|f\rangle = \sum_{l=0}^\infty \sqrt{\frac{2l+1}{4\pi}}P_l(1)\int\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)f(\theta,\phi)\,d\Omega$

- $\boxed{0 = \left[\nabla^2 + \frac{l(l+1)}{r^2} - \frac{m^2}{r^2\sin^2\theta}\right]P_l^m(\cos\theta)} = \left[\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{l(l+1)}{r^2} - \frac{m^2}{r^2\sin^2\theta}\right]P_l^m(\cos\theta)$

$$= \left[\nabla^2 + \frac{l(l+1)}{r^2} + \frac{\nabla^2 e^{im\phi}}{e^{im\phi}}\right]P_l^m(\cos\theta) = \frac{1}{e^{im\phi}}\left[\nabla^2 + \frac{l(l+1)}{r^2}\right]\left[e^{im\phi}P_l^m(\cos\theta)\right] = \boxed{\begin{matrix}\left[\nabla^2 + \frac{l(l+1)}{r^2}\right] & Y_{lm}(\theta,\phi),\, P_l(\cos\theta) \\ (\nabla^2 = \nabla'^2) & Y_{lm'}(\gamma,\beta),\, P_l(\cos\gamma)\end{matrix}}$$

$$\to \; g(0,\beta) = Y_{lm}^*[\theta'+0,\phi(0,\beta)] = \sum_{l'=0}^\infty \sum_{m'=-l'}^{l'} Y_{l'0}(\gamma,\beta)A_{l'0}[m] \quad (m'=0)\Big|_{\gamma=0} = \sum_l \sqrt{\frac{2l+1}{4\pi}}\int Y_{l0}^*(\gamma,\beta)Y_{lm}^*(\theta'+\gamma,\phi)\,d\Omega$$

$$Y_{lm}^*(\theta',\phi') = \left[Y_{lm}^*(\theta,\phi) = \sum_{l'=0}^\infty \sum_{m'=-l'}^{l'} Y_{l'm'}(\gamma,\beta)A_{lm'}[m] \quad (l'=l)\right]_{\gamma=0} \qquad \frac{2l+1}{4\pi}\int P_l(\cos\gamma)Y_{lm}^*(\theta,\phi)\,d\Omega$$

$$\Rightarrow \; P_l(\cos\gamma) = \sum_{m=-l}^l Y_{lm}(\theta,\phi)\left[A_{lm}(\theta',\phi') = \int P_l(\cos\gamma)Y_{lm}^*(\theta,\phi)\,d\Omega\right] = \boxed{\frac{4\pi}{2l+1}\sum_{m=-l}^l Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi')}$$

$$0 = \left[\frac{1}{\rho^{2-1}}\frac{d}{d\rho}\left(\rho^{2-1}\frac{d}{d\rho}\right) + k^2 - \frac{m^2}{\rho^2}\right]R_{(\rho)} = \left[\frac{d^2}{d\rho^2} + \frac{2-1}{\rho}\frac{d}{d\rho} + k^2 - \frac{m^2}{\rho^2}\right]R_{(\rho)}$$

$$= \left[\frac{d^2}{dx^2} + \frac{2-1}{x}\frac{d}{dx} + 1 - \frac{m^2}{x^2}\right]R_{(x=k\rho)} \qquad (a = \text{radius of cylinder})$$

Bessel/Neumann Function $(m \in \mathbb{R})$

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{j=0}^\infty \left(\frac{x}{2}\right)^{2j}\frac{(-1)^j}{j!}\frac{1}{\Gamma(j+m+1)}$$

$$N_m(x) = \frac{\cos m\pi \cdot J_m(x) - J_{-m}}{\sin m\pi} \qquad \left(N(0) \to \infty\right)$$

$$R_m(\rho) = \sum_{n=1}^\infty A_n J_m\left(\frac{x_{mn}}{a}\rho = k_{mn}\rho\right) \qquad \left(J_m(x_{mn}) = 0\right)$$

$$= \sum_{n=1}^\infty B_n J_m\left(\frac{y_{mn}}{a}\rho = k_{mn}\rho\right) \qquad \left(J_m'(y_{mn}) = 0\right)$$

$$\delta_{n'n} = \frac{2}{a^2}\frac{1}{J_{m+1}^2(x_{mn}\frac{\rho}{a})}\int_0^a \rho J_m(x_{mn'}\tfrac{\rho}{a})J_m(x_{mn}\tfrac{\rho}{a})\,d\rho$$

3rd Kind (Hankel)

$$H_m^{(1)}(x) = J_m + iN_m$$

$$H_m^{(2)}(x) = J_m - iN_m$$

Modified Bessel $(k^2 \to -k^2)$

$$I_m(x) = i^{-m}J_m(ix)$$

$$K_m(x) = \frac{\pi}{2}i^{m+1}H_m^{(1)}(ix)$$

Spherical Bessel : $j_l = \sqrt{\frac{2\pi}{z}}J_{l+1/2}$

$$\delta(k-k') = \frac{2k^2}{\pi}\int_0^\infty r^2 j_l(k'r)j_l(kr)\,dr$$

"Cylindrical Series" (?) $0 \le \rho \le a \qquad a \to \infty, \int dk$

$$\Psi_{(\vec r)} = \sum_{m,\,n}\frac{1}{\sqrt{2\pi}}e^{im\phi} \times \begin{matrix}[A_{mn}e^{k_{mn}z} + B_{mn}e^{-k_{mn}z}] \\ \times [C_{mn}J_m(k_{mn}\rho) + D_{mn}N_m(k_{mn}\rho)]\end{matrix}$$

Associated Laguerre Polynomials: $L \equiv$

Laguerre Polynomials: $L_{\equiv}$

2 Simple 1D Potentials

2.1 Infinite Square Well (1-D)

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{otherwise} \end{cases}$$

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin k_n x$$

$$k_n = \frac{2\pi}{\lambda} = \frac{2\pi}{2a/n} = \frac{n\pi}{a} \qquad \forall n = 1, 2, 3, \ldots \qquad \boxed{!! \ \hat{p}\Psi_n \neq p\Psi_n \ !!} \qquad \text{wave isn't infinite}$$

$$E_n = \frac{p^2}{2m} = \frac{\hbar^2 k_n^2}{2m}$$

2.1.1 3-D Rectangular Box

$$\Psi_{n_x n_y n_z}(x, y, z) = \Psi_{n_x}(x)\Psi_{n_y}(y)\Psi_{n_z}(z) = \sqrt{\frac{8}{a_x a_y a_z}}(\sin k_{n_x} x)(\sin k_{n_y} y)(\sin k_{n_z} z)$$

$$k_{n_i} = \frac{n_i \pi}{a_i} \qquad \forall n_x, n_y, n_z = 1, 2, 3, \ldots$$

$$E_{n_x n_y n_z} = \frac{\hbar^2}{2m}(k_{n_x}^2 + k_{n_y}^2 + k_{n_z}^2)$$

2.2 Harmonic Oscillator (1-D): $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$

$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)$
$\qquad\qquad = \frac{1}{2m}\left(-ip + m\omega x\right)\left(ip + m\omega x\right) \sim E \sim \hbar\omega$ $\qquad \Rightarrow \qquad \boxed{a = a_- = \frac{1}{\sqrt{2m}}\frac{1}{\sqrt{\hbar\omega}}\left(i\hat{p} + m\omega x\right)}$

$\boxed{\begin{array}{c} aa^\dagger = \frac{H}{\hbar\omega} + \frac{1}{2} \\ aa^\dagger|n\rangle = \left(\frac{E_n}{\hbar\omega} + \frac{1}{2}\right)|n\rangle \end{array}}$, $\boxed{\begin{array}{c} a^\dagger a = \frac{H}{\hbar\omega} - \frac{1}{2} \\ a^\dagger a|n\rangle = \left(\frac{E_n}{\hbar\omega} - \frac{1}{2}\right)|n\rangle \end{array}}$ $\rightarrow$ $\boxed{\left[a, a^\dagger\right] = 1}$ $\leftarrow$ $\boxed{\begin{array}{c} H = \hbar\omega(a^\dagger a + \frac{1}{2}) \\ = \hbar\omega(aa^\dagger - \frac{1}{2}) \end{array}}$

$\qquad\qquad\qquad$ or $[H, a_\pm] = (\pm\hbar\omega)a_\pm$

$(aa^\dagger)a\Psi_n = a(a^\dagger a)\Psi_n$ $\qquad\qquad\qquad\qquad (a^\dagger a)a^\dagger|n\rangle = a^\dagger(aa^\dagger)|n\rangle$

$\left(\frac{H}{\hbar\omega} + \frac{1}{2}\right)a\Psi_n = a\left(\frac{H}{\hbar\omega} - \frac{1}{2}\right)\Psi_n$ $\qquad\qquad \left(\frac{H}{\hbar\omega} - \frac{1}{2}\right)a^\dagger|n\rangle = a^\dagger\left(\frac{H}{\hbar\omega} + \frac{1}{2}\right)|n\rangle$

$\left(\frac{E_{an}}{\hbar\omega} + \frac{1}{2}\right)a\Psi_n = \left(\frac{E_n}{\hbar\omega} - \frac{1}{2}\right)a\Psi_n$ $\qquad\qquad \left(\frac{E_{a^\dagger n}}{\hbar\omega} - \frac{1}{2}\right)\cancel{a}|a^\dagger n\rangle = \left(\frac{E_n}{\hbar\omega} + \frac{1}{2}\right)\cancel{a}|a^\dagger(n)\rangle$

$E_{an}(a\Psi_n) = (E_n - \hbar\omega)(a\Psi_n)$ $\qquad\qquad\qquad E_{a^\dagger n}|a^\dagger n\rangle = (E_n + \hbar\omega)|a^\dagger(n)\rangle$

$\qquad\qquad\qquad \Downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Downarrow$

$E_{n-1}|n-1\rangle = (E_n - \hbar\omega)|n-1\rangle$ $\qquad\qquad E_{n+1}|n+1\rangle = (E_n + \hbar\omega)|n+1\rangle$

(Why ladders):

$\qquad\qquad$ (use induction)

$\boxed{\left[\frac{H}{\hbar\omega}, a_\pm\right] = (\pm 1)a_\pm} \Rightarrow \left[\frac{H}{\hbar\omega}, a_\pm^m\right] = (\pm 1)ma_\pm$

$\boxed{\frac{H}{\hbar\omega}|n\rangle = \frac{E_n}{\hbar\omega}|n\rangle = c_n|n\rangle} \Rightarrow \frac{H}{\hbar\omega}a_\pm^m|n\rangle = (\pm 1\cdot m + c_n)|n\rangle$

$\left.\begin{array}{c} Ha_\pm^m|n\rangle = \hbar\omega(c_n \pm m)a_\pm^m|n\rangle \\ * \ Ha_\pm^n|0\rangle = \hbar\omega(c_0 \pm n)a_\pm^n|0\rangle \ * \\ = E_{n0}\cdot a_\pm^n|0\rangle \end{array}\right.$

$E_n \geq \text{Min}(V) \Rightarrow a\Psi_0 = 0$ $\quad$ (else is un-normalizable) $\qquad\qquad aa^\dagger|a_+^n(0)\rangle = \left(\frac{\hbar\omega(n+1/2)}{\hbar\omega} + \frac{1}{2}\right)|a_+^n(0)\rangle$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \boxed{aa^\dagger|a_+^n(0)\rangle = (n+1)|a_+^n(0)\rangle}$

$0 = (ip + m\omega x)\Psi_0$ $\quad \boxed{\begin{array}{c} \Psi_0 = Ae^{-\frac{m\omega}{\hbar}\frac{x^2}{2}} \\ A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\ \frac{1}{\sigma^2} = \frac{m\omega}{\hbar} \end{array}}$ $\qquad\qquad$ • $\langle a_+^n(0)|aa^\dagger|a_+^n(0)\rangle = n+1$

$\hbar\frac{d}{dx}\Psi_0 = -m\omega x\Psi_0$ $\qquad\qquad\qquad\qquad\qquad\qquad$ • $a^\dagger|a_+^n(0)\rangle = \sqrt{n+1}\ |a_+^{n+1}(0)\rangle$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ • $a|a_+^{n+1}(0)\rangle = \sqrt{n+1}|a_+^n(0)\rangle \ ** $

$a^\dagger a|0\rangle = \left(\frac{E_0}{\hbar\omega} - \frac{1}{2}\right)|0\rangle = 0$ $\qquad\qquad a^\dagger a|n,0\rangle = \left(\frac{\hbar\omega(n+1/2)}{\hbar\omega} - \frac{1}{2}\right)|n,0\rangle$

$E_0|0\rangle = \frac{1}{2}\hbar\omega|0\rangle \qquad (c_0 = \frac{1}{2})$ $\qquad\qquad \boxed{a^\dagger a|a_+^n(0)\rangle = n|a_+^n(0)\rangle}$

$\boxed{E_n = \hbar\omega(\frac{1}{2} + n)}$

$\frac{H}{\hbar\omega}|a_+^n(0)\rangle = \left(\frac{1}{2} + n\right)|a_+^n(0)\rangle = \underline{c\circ a_+^n(0)}|a_+^n(0)\rangle$ $\qquad c_n = \frac{1}{2} + n$ $\qquad \boxed{a|n\rangle = \sqrt{n}|n-1\rangle}$

$\frac{H}{\hbar\omega}a_\pm^m|a_+^n(0)\rangle = \left(\frac{1}{2} + n \pm m\right)a_\pm^m|a_+^n(0)\rangle$ $\quad \rightarrow \quad a_+^n(0) = n$ $\quad \rightarrow \quad \boxed{a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle}$

$\frac{H}{\hbar\omega}|a_+^{n\pm m}(0)\rangle = \underline{(c\circ a_+^{n\pm m}(0))}|a_+^{n\pm m}(0)\rangle$ $\qquad a_\pm(n) = n \pm 1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \boxed{\Psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\Psi_0}$

2.2.1 Position/Momentum Operators

$$\boxed{x = \frac{1}{2}\frac{\sqrt{2m}\sqrt{\hbar\omega}}{m\omega}(a + a^\dagger)} \qquad \boxed{\hat{p} = \frac{1}{2}\frac{\sqrt{2m}\sqrt{\hbar\omega}}{i}(a - a^\dagger)}$$

Show Virial Theorem Works

$$2\langle T \rangle = N\langle V \rangle$$

$$E_n = 2\langle V \rangle_n$$
$$= 2\langle \Psi_n | V | \Psi_n \rangle$$
$$= 2\left\langle \Psi_n \left| \frac{1}{2}mw^2 \frac{2m\hbar\omega}{(2m\omega)^2}(a + a^\dagger)^2 \right| \Psi_n \right\rangle$$
$$= \frac{2m^2\hbar\omega^3}{(2m\omega)^2}\left(0 + \left\langle \Psi_n \left| (aa^\dagger + a^\dagger a) \right| \Psi_n \right\rangle + 0\right)$$

$$E_n = (n + 1/2)\hbar\omega \quad \boxed{\checkmark}$$

Heisenberg Picture

$$\frac{da_\pm}{dt} = \mp i\omega a_\pm$$
$$\Rightarrow \qquad a_\pm(t) = a_\pm(0)e^{\mp i\omega t}$$
$$x(t) \pm \frac{ip(t)}{m\omega} = x(0)e^{\mp i\omega t} \pm \frac{ip(0)}{m\omega}e^{\mp i\omega t}$$

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$$
$$\frac{p(t)}{m\omega} = -x(0)\sin\omega t + \frac{p(0)}{m\omega}\cos\omega t$$

Test the Uncertainty Principle

$$\sigma_x \sigma_p \geq \tfrac{1}{2}\left|\left\langle [x, p] \right\rangle\right|$$

$$xp - px = \frac{2m\hbar\omega}{4m\omega i}\begin{pmatrix} a^2 - aa^\dagger + a^\dagger a - (a^\dagger)^2 \\ -a^2 + a^\dagger a - aa^\dagger + (a^\dagger)^2 \end{pmatrix}$$

$$= \frac{\hbar}{i}(a^\dagger a - aa^\dagger) = i\hbar(n + 1 - n)$$

$$\Rightarrow \quad \sigma_x \sigma_p \geq \frac{\hbar}{2} \quad \boxed{\checkmark}$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \qquad\qquad \sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2$$

$$= \frac{2m\hbar\omega}{4m^2\omega^2}\begin{bmatrix} \langle (a + a^\dagger)^2 \rangle \\ -\langle a + a^\dagger \rangle^2 \end{bmatrix} \qquad = \frac{2m\hbar\omega}{-4}\begin{bmatrix} \langle (a - a^\dagger)^2 \rangle \\ -\langle a - a^\dagger \rangle^2 \end{bmatrix}$$

$$= \frac{\hbar}{2m\omega}\langle aa^\dagger + a^\dagger a \rangle \qquad = \frac{\hbar m\omega}{2}\langle aa^\dagger + a^\dagger a \rangle$$

$$= \frac{\hbar}{m\omega}\left(n + \tfrac{1}{2}\right) \qquad\qquad = \hbar m\omega\left(n + \tfrac{1}{2}\right)$$

$$\Rightarrow \quad \sigma_x \sigma_p = \hbar\left(n + \tfrac{1}{2}\right) \geq \tfrac{\hbar}{2} \quad \boxed{\checkmark}$$

2.2.2 Analytic Method

$$\Psi_n = A\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}$$

$$A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}x$$

$$H_n(x) = (-1)^n\, e^{-x^2}\left(\frac{d}{dx}\right)^n e^{x^2}$$

Hermite Polynomials:

$$e^{-z^2 + 2zx} = \sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x)$$

2.2.3 Coherent States

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$\boxed{\sigma_x \sigma_p = \frac{\hbar}{2}}$$

$$\langle\alpha|\alpha\rangle = \langle\alpha| \left(\begin{array}{l} \sum_{n=0}^{\infty} \langle\Psi_n|\alpha\rangle \, |\Psi_n\rangle = \\[1em] \sum_{n=0}^{\infty} \left\langle \frac{(a^{\dagger})^n}{\sqrt{n!}} \Psi_0 \middle| \alpha \right\rangle |\Psi_n\rangle = \\[1em] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle\Psi_0|\alpha\rangle \, |\Psi_n\rangle \end{array} \right)$$

$$\Rightarrow \quad \boxed{|\alpha\rangle = e^{-\alpha^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\Psi_n\rangle} \quad \rightarrow \quad |\alpha_{=0}\rangle = |\Psi_0\rangle$$

$$a|\alpha_{(x,t)}\rangle = e^{-\frac{\alpha^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, e^{\frac{-i}{\hbar} E_n t} a|\Psi_n\rangle$$

$$= \langle\Psi_0|\alpha\rangle^2 \sum_{n=0}^{\infty} \frac{(\alpha^2)^n}{n!} \, \langle\Psi_n|\Psi_n\rangle$$

$$= e^{-\frac{\alpha^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, e^{\frac{-i}{\hbar}\hbar\omega(\frac{1}{2}+n)t} \sqrt{n}|\Psi_{n-1}\rangle$$

$$= \langle\Psi_0|\alpha\rangle^2 \, e^{\alpha^2} = 1$$

$$= \left(\alpha e^{\frac{-i}{\hbar}\hbar\omega t}\right) e^{-\frac{\alpha^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, e^{\frac{-i}{\hbar}\hbar\omega(\frac{1}{2}+n)t}|\Psi_n\rangle$$

$$\boxed{a|\alpha_{(x,t)}\rangle = \left(\alpha e^{-i\omega t}\right)|\alpha_{(x,t)}\rangle}$$

$|\alpha\rangle$ are obviously not orthogonal. They are an overcomplete basis.

2.2.4 3-D Harmonic Potential

$$V(r) = \frac{1}{2}kr^2$$

$$\boxed{\begin{array}{c} \text{(Isotropic)} \\ E_{n_x n_y n_z} = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right) = \hbar\omega\left(n + \frac{3}{2}\right) \quad l = n - 2k \in \{n, n-2, ..., 0\} \end{array}}$$

2.3 Free Particle (1-D)

$$V(x) = 0$$

$$\Psi_{(x,t)} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi_{(x,0)} e^{\frac{i}{\hbar}[px - E(p)t]} \, dp$$

$$= \int \langle x|p \rangle e^{-\frac{i}{\hbar}E(p)t} \langle p|\Psi \rangle \, dp$$

$$\Phi_{(x,t)} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi_{(x,0)} e^{\frac{-i}{\hbar}[px + E(p)t]} \, dx$$

$$\langle x|U_{(t)}|\Psi \rangle = \iint \langle x|p \rangle e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} \langle p|x' \rangle \, dp \, \langle x'|\Psi \rangle \, dx'$$

$$= \iint_{-\infty}^{\infty} \frac{1}{2\pi\hbar} e^{-\frac{i}{\hbar}\left[\frac{p^2 t}{2m} - p(x - x')\right]} \, dp \, \Psi_{(x',0)} \, dx'$$

$$= \int \sqrt{\frac{-im}{2\pi\hbar t}} e^{\frac{im(x-x')^2}{2\hbar t}} \Psi_{(x',0)} \, dx'$$

($E < 0 \to \Psi = e^{\pm kx}$ is possible and also not normalizable, but solution above is already a complete set)

- $E(p) = \frac{p^2}{2m}$

- $v_{\text{wave}} = \boxed{v_{\text{phase}} = \frac{\omega(k)}{k}} = \frac{E}{p} = \frac{v_{\text{classical}}}{2}$

- $v_{\text{particle}} \approx \boxed{v_{\text{group}} = \frac{d\omega(k)}{dk}} = 2v_{\text{wave}}$ $\begin{pmatrix} \text{dispersion} \\ \text{relation} \end{pmatrix}$

Heisenberg Pic. Free Particle

$$x_H(t) = x_H(0) + \frac{p_H(0)}{m}t$$

$$[x_H(0),\ x_H(t)] = \left[x_H(0), \frac{p_H(0)}{m}t\right] = \frac{i\hbar t}{m}$$

$$\boxed{\sigma_{x_t}\sigma_{x_0} \geq \frac{\hbar t}{2m}}$$

2.4 Delta Potential (1-D)

Potential Well: $\qquad V(x) = -\alpha\delta(x)$ $\qquad$ ($\alpha \to -\alpha$ for potential wall)

Bound State ($E < 0$) [only for Well]:

$$\Psi = \sqrt{k}e^{k|x|} = \begin{cases} \sqrt{k}e^{kx} & x \leq 0 \\ \sqrt{k}e^{-kx} & x \geq 0 \end{cases}$$

$$k = \frac{m\alpha}{\hbar^2}$$

$$E = -\frac{(\hbar k)^2}{2m}$$

Scattering State ($E > 0$) [for both]:

$$\Psi = \begin{cases} Ae^{iKx} + Be^{-iKx} & x < 0 \\ Fe^{iKx} & x > 0 \end{cases}$$

$$E = \frac{(\hbar K)^2}{2m}\ , \qquad\qquad \beta \equiv \frac{k}{K} = \frac{m\alpha/\hbar^2}{K}$$

$$B = \frac{i\beta}{1 - i\beta}A\ , \qquad\qquad F = \frac{1}{1 - i\beta}A$$

$$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1 + \beta^2}\ , \qquad T = \frac{|F|^2}{|A|^2} = \frac{1}{1 + \beta^2}$$

Can't normalize. All free particles have ranges of p and thus E, so R and T are approx. in the vicinity of E.

2.5   Finite Square Potential (1-D)

2.5.1   Potential Well $\qquad V(x) = \begin{cases} -V_0 & -a < x < a \\ 0 & \text{otherwise} \end{cases}$

($V_0 \to -V_0$ **for wall and do cases for** $E > V_0, E = V_0, E < V_0,$ **and change to** $\sinh, \cosh$ **if needed**)

$$k \; ; K : \qquad E = \frac{-(\hbar k)^2}{2m} = \frac{(\hbar K)^2}{2m}$$

$$l : \quad E + V_0 = \frac{(\hbar l)^2}{2m}$$

$$v : \qquad V_0 = \frac{\hbar^2 v^2}{2m} = \frac{\hbar^2(l^2 + k^2)}{2m} = \frac{\hbar^2(l^2 - K^2)}{2m}$$

$$\frac{k_a}{l_a} \equiv \sqrt{\frac{(ka)^2}{(la)^2}} = \sqrt{\frac{(la)^2 + (ka)^2}{(la)^2} - 1}$$

$$\boxed{\frac{k_a}{l_a} \equiv \sqrt{\left(\frac{v_a}{l_a}\right)^2 - 1}} \; , \quad v_a^2 = \begin{cases} l_a^2 + k_a^2 \\ l_a^2 - K_a^2 \end{cases}$$

Bound State $(E_n < 0)$ [only for well]:

$$\Psi_{\text{even}}(x) = \begin{cases} \Psi(-x) & x < 0 \\ D\cos(lx) & 0 < x < a \\ Fe^{-kx} & a < x \end{cases} \qquad \Psi_{\text{odd}}(x) = \begin{cases} -\Psi(-x) & x < 0 \\ C\sin(lx) & 0 < x < a \\ Fe^{-kx} & a < x \end{cases}$$

- $F = D\cos(la)e^{ka}$

- $\frac{-(\partial_x \Psi)(a)}{\Psi(a)} = k = l\tan(la) \Rightarrow$

  $\tan(l_a) = \sqrt{(v_a/l_a)^2 - 1}$

  big $v_a \to l \approx_< \frac{n\pi}{2a} \to E_n + V_0 = \frac{\hbar^2 l^2}{2m}$; $\underline{n \text{ odd}}$

- $\boxed{n_{\max} = \left\lfloor \frac{v_a}{\pi} \right\rfloor + 1}$

- $F = D\sin(la)e^{ka}$

- $\frac{-(\partial_x \Psi)(a)}{\Psi(a)} = k = -l\cot(la) \Rightarrow$

  $-\cot(l_a) = \sqrt{(v_a/l_a)^2 - 1}$

  big $v_a \to l \approx_< \frac{n\pi}{2a} \to E_n + V_0 = \frac{\hbar^2 l^2}{2m}$; $\underline{n \text{ even}}$

- $\boxed{n_{\max} = \left\lfloor \frac{v_a + \frac{\pi}{2}}{\pi} \right\rfloor}$

Scattering State $(E > 0)$ [for both]:

$$\Psi = \begin{cases} Ae^{iKx} + Be^{-iKx} & x < -a \\ C\sin lx + D\cos lx & -a < x < a \\ Fe^{iKx} & a < x \end{cases} \qquad \frac{d\Psi}{dx} = \begin{cases} iKAe^{iKx} - iKBe^{-iKx} & x < -a \\ lC\cos lx - lD\sin lx & -a < x < a \\ iKFe^{iKx} & a < x \end{cases}$$

$$B = i\sin(2l_a)\left(\frac{l_a^2 - K_a^2}{2K_a l_a}\right) F$$

$$F = \frac{e^{-2iK_a}}{\cos(2l_a) - i\left(\frac{l_a^2 + K_a^2}{2K_a l_a}\right)\sin(2l_a)} A$$

(Can't normalize. See delta potential.)

$$T^{-1} = 1 + \left(\frac{l_a^2 - K_a^2}{2K_a l_a}\right)^2 \sin^2(2l_a)$$

$$= 1 + \frac{V_0^2}{4E(E + V_0)} \sin^2\left(2a\sqrt{\frac{E + V_0}{\hbar^2/2m}}\right)$$

(full transmission at inf. sqr. well $E_n + V_0 = \frac{\hbar^2 l^2}{2m}$; $l = \frac{n\pi}{2a}$)

3   2D and 3D Schrodinger Equation

General dimensions, D

$$\frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left(r^{D-1} \frac{\partial}{\partial r} \right) R(r) = \left[\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} \right] R(r)$$

$$= \left[\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} \right] r^n u(r)$$

$$= \left[\frac{\partial^2}{\partial r^2} + \frac{D-1+2n}{r} \frac{\partial}{\partial r} + \frac{2n(2D-4+2n)}{4r^2} \right] u$$

$$= \left[\frac{\partial^2}{\partial r^2} - \frac{(D-1)(D-3)}{4r^2} \right] u \quad (n = \frac{1-D}{2}, 0, 2-D)$$

$$R(r) = u(r)/\sqrt{r}^{D-1} \ \sim \ e^{\frac{i}{\hbar} p_r r}/\sqrt{r}^{D-1}$$

$$L^2 \sim \hbar^2 \ , \quad \hat{p}_r = \frac{\hbar}{i} \left(\frac{\partial}{\partial r} + \frac{D-1}{2r} \right) \ , \quad \hat{p}'_r = \frac{\hbar}{i} \frac{\partial}{\partial r}$$

$$ER(r) = \left[\frac{\hat{p}_r^2}{2M} + V(r) + \frac{L^2 - \hbar^2(D-1)^2/4}{2(Mr^2)} \right] R(r)$$

$$Eu(r) = \left[\frac{\hat{p}'^2_r}{2M} + V(r) + \frac{L^2 - \hbar^2(D-1)(D-3)/4}{2(Mr^2)} \right] u(r)$$

3.1   2D Schrodinger

If $V = V(\rho)$

$\Psi(\vec{\mathbf{r}}) = R_m(\rho) \Phi_m(\phi) \ \Rightarrow$

$$ER = \left[\frac{-\hbar^2}{2M} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + V(\rho) + \frac{\hbar^2 m^2}{2M\rho^2} \right] R$$

$$\boxed{ Eu = \frac{-\hbar^2}{2M} \frac{\partial^2 u}{\partial \rho^2} + \left[V(\rho) + \frac{\hbar^2(m^2 + 1/4)}{2M\rho^2} \right] u }$$

$$Eu\Phi = \left[\frac{\hat{p}'^2_\rho}{2M} + V(\rho) + \frac{\hat{L}_z^2 + \hbar^2/4}{2(M\rho^2)} \right] u\Phi$$

- $R_m(\rho) = u_m(\rho)/\sqrt{\rho} \quad \left(\int \Psi r \, dr d\phi = 1 \right)$

- $\Phi_m(\phi) = e^{im\phi}$

- $L_z = (\vec{r} \times \vec{p})_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$

3.2   3D Schrodinger

<u>If $V = V(r)$</u>

$$\Psi(\vec{r}) = R_l(r)Y_l^m(\theta,\phi) = R_l(r)\Theta_l^m(\theta)\Phi_m(\phi) \;\Rightarrow$$

$$ER = \left[\frac{-\hbar^2}{2M}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r) + \frac{\hbar^2 l(l+1)}{2Mr^2}\right]R$$

$$\boxed{Eu = \frac{-\hbar^2}{2M}\frac{\partial^2 u}{\partial r^2} + \left[V(r) + \frac{\hbar^2 l(l+1)}{2Mr^2}\right]u}$$

$$Eu\Theta = \left[\frac{\hat{p'}_r^2}{2M} + V(r) + \frac{\hat{L}^2}{2(Mr^2)}\right]u\Theta$$

- $R_l(r) = u_l(r)/r \qquad \left(\int \Psi r^2 \sin\theta\, dr d\theta d\phi = 1\right)$
- $\Phi_m(\phi) = e^{im\phi}$
- $\Theta_l^m(\theta) = A P_l^m(\cos\theta)$
  - $A = \epsilon\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}, \quad \epsilon = \begin{cases}(-1)^m & (m\geq 0)\\ 1 & (m\leq 0)\end{cases}$
  - $P_l^m(x) =$ Assoc. Legendre Func. (see extra)
- $l \in \mathbb{N}_0, \; m \in \{-l,...,-1,0,1,...,l\}$
- $\widehat{L}_i = (\vec{r}\times\vec{p})_i$

$$\langle u|\nabla_r^2 u\rangle = \langle\nabla_r^2 u|u\rangle \;\Rightarrow\; \left[u^*\frac{\partial u}{\partial r} - u\frac{\partial u^*}{\partial r}\right]_0^\infty \begin{smallmatrix}\leftarrow 1.)\\ \leftarrow 2.)\end{smallmatrix} = 0$$

1.) $\int_0^\infty u^2\, dr = 1 \;\Rightarrow\; \boxed{u(\infty) = 0 \text{ or } e^{ir}}$

2.) $\boxed{u(0) = c = 0}$ $\left\{c\neq 0 \to \begin{smallmatrix}\Psi_{l=0}(r)\sim\frac{c}{r}\\ \nabla^2(\frac{1}{r})\sim\delta^3(r)\end{smallmatrix} \to \begin{smallmatrix}H\Psi\neq E\Psi\\ \text{if } V(r)\neq\delta^3(r)\end{smallmatrix}\right\}$

$V \sim r^{-2<a}, \; l\neq 0: \; \lim\limits_{r\to 0} u'' \sim \frac{l(l+1)}{r^2}u, \; u\sim r^{l+1}$

$V \sim r^{-2<a<-1}, \; E>0: \; \lim\limits_{r\to\infty} p_r^2 u \sim Eu, \; u\sim e^{\pm ikr}$

$V \sim r^{-1\leq a}, \; E>0: \; u\gtrsim re^{\pm ikr}$

$\quad$ Sim for $E<0: \; u\sim e^{\pm kr} \text{ or } re^{\pm kr} \text{ etc.}$

3.2.1   3D Free Particle, $V=0$

$$\frac{\hbar^2}{2M}\left[-\frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2}\right]u = \frac{\hbar^2 k^2}{2M}u \;\Rightarrow\; \left[-\frac{\partial^2}{\partial\rho^2} + \frac{l(l+1)}{\rho^2}\right]|l\rangle = |l\rangle = \begin{cases}A\rho^{l+1}\\ B\rho^{-l}\end{cases} \text{or(?) ...}$$

$$a_l \equiv \frac{\partial}{\partial\rho} + \frac{l+1}{\rho} \qquad a_l^\dagger = -\frac{\partial}{\partial\rho} + \frac{l+1}{\rho}$$

$$a_l a_l^\dagger|l\rangle = |l\rangle \qquad a_l^\dagger a_l|l\rangle = a_{l+1}a_{l+1}^\dagger|l\rangle$$
$$a_l^\dagger|l\rangle = e^{i\theta_l}|a_{l(l)}^\dagger\rangle$$

$$a_l^\dagger\left(a_l a_l^\dagger\right)|l\rangle = \underline{a_l^\dagger|l\rangle}$$
$$\left(a_l^\dagger a_l\right)\underline{a_l^\dagger|l\rangle} = \left(a_{l+1}a_{l+1}^\dagger\right)\underline{a_l^\dagger|l\rangle}$$
$$\underline{a_l^\dagger|l\rangle} = e^{i\theta_l}|l+1\rangle$$

Spherical Bessel :	$rR_0^B = u_0^B \sim \sin(\rho) = \sin(kr)$
Spherical ~~Neumann~~ :	$rR_0^N = u_0^N \sim -\cos(\rho)$

$$\cancel{e^{i\theta_l}}\frac{\rho}{k}R_{l+1} = a_l^\dagger\left(\frac{\rho}{k}R_l\right) = \left(-\frac{\partial}{\partial\rho} + \frac{l+1}{\rho}\right)\left(\frac{\rho}{k}R_l\right)$$

$$R_{l+1} = \left(-\frac{\partial}{\partial\rho} + \frac{l}{\rho}\right)R_l = -\rho^l\frac{\partial}{\partial\rho}\left(\rho^{-l}R_l\right)$$

$$\frac{R_l}{\rho^l} = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\frac{R_{l-1}}{\rho^{l-1}} = \left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^l R_0$$

$$\boxed{R_l = C_l(-\rho)^l\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^l R_0}$$

<u>Infinite Spherical Well:</u> $\quad V(r) = \begin{cases}0 & r\leq a\\ \infty & r>a\end{cases}, \quad E_n = \frac{\hbar^2 k_n^2}{2m}$

<u>Bessel:</u> $\begin{aligned}R_l^B(\rho) &= C_l(-\rho)^l\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^l R_0^B(\rho)\\ R_0^B(\rho) &\sim k_n\sin(\rho)/\rho = \sin(k_n r)/r\end{aligned} \quad\Rightarrow\quad \begin{aligned}\beta_l^n &\equiv k_n a: \; R_l^B(\beta_l^n) = 0\\ \beta_0^n &= \frac{n\pi}{a}\cdot a\end{aligned}$

3.2.2 Hydrogen Atom, $V = -\frac{ke^2}{r}$

$$Eu = \left(\frac{\hat{p}_r^2}{2m} + V(r) + \frac{\hat{L}^2}{2(mr^2)} \right) u \qquad u(r) = rR(r)$$

$$\boxed{Eu = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial r^2} u + \left[-\frac{ke^2}{r} + \frac{\hbar^2 l(l+1)}{2mr^2} \right] u}$$

$$\boxed{\Psi_{nlm}(\vec{r}) = R_{nl}(r)\, Y_l^m(\theta, \phi) = R_{nl}(r)\, \Theta_l^m(\theta)\, \Phi_m(\phi)}$$

- $\Phi_m(\phi) = e^{im\phi}$

- $\Theta_l^m(\theta) = A P_l^m(\cos\theta)$

  - $A = \epsilon \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}}$, $\quad \epsilon = \begin{cases} (-1)^m & (m \geq 0) \\ 1 & (m \leq 0) \end{cases}$

  - $P_l^m(x)$ Assoc. Legendre Func. (see extra)

- $R_{nl}(r) = \frac{B}{r} \rho^{l+1} e^{-\rho} \nu(\rho)$

  - $\rho = k_n r$, $\quad k_n = \frac{1}{a_0 n}$ (fine structure below)

  - $\nu(\rho) = L_{n-l-1}^{2l+1}(2\rho)$ Assoc. Laguerre Pol. (see extra)

  - $B = \sqrt{2k_n \frac{(n-l-1)!}{2n[(n+l)!]^3}}\, 2^{l+1}$

$$\boxed{\alpha \equiv \frac{kqq}{\hbar c} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c} \approx \frac{1}{137}}$$

$$\boxed{a_0 \equiv \frac{\hbar^2}{m(kqq)} = \frac{4\pi\epsilon_0 \hbar^2}{me^2}}$$

$$\boxed{E_n = -\frac{\hbar^2 k_n^2}{2m} = -\frac{\hbar^2}{2ma_0^2} \frac{1}{n^2} = -\frac{1}{2}\alpha^2 (mc^2) \frac{1}{n^2} \approx -13.6 \frac{1}{n^2} \,[\text{eV}]}$$

$$\boxed{\frac{1}{\lambda} = \frac{\alpha^2 (mc^2)}{2hc} \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right), \quad R = 1.097\ \text{E7}\ [\text{m}^{-1}]}$$

Quantum Numbers - n, l, m:

- $\left(n \in \{1, 2, 3, ...\} \right)$, $\left(l \in \{0, 1, 2, ..., n-1\} \right)$, $\left(m \in \{-l, ..., -1, 0, 1, ..., l\} \right)$

  - Degeneracy is n^2

(outdated) Bohr Model:

- $L = (\bar{r})(\bar{p}) = (a_0 n^2)(\hbar k_n) = n\hbar$ (not correct!!)
- Electrons don't radiate about the nucleus
- Energy diff. follows Rydberg formula

4   Spin and L

4.1   Hydrogen Atom

Angular Momentum :

$$\boxed{\widehat{L}_i \equiv (\vec{r} \times \vec{p})_i} \;,\; \boxed{L_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}}$$

$$\boxed{\widehat{L}_\pm \equiv \widehat{L}_x \pm i\widehat{L}_y}$$

$$\boxed{\widehat{L}^2 \equiv L_x^2 + L_y^2 + L_z^2}$$

$$\boxed{L_\pm L_\mp = \widehat{L}^2 - L_z^2 \pm \hbar L_z}$$

Commutation Relations:

$$\boxed{[\hat{x}, L_y] = i\hbar\hat{z}} \;,\; \boxed{[p_x, L_y] = i\hbar p_z} \;,\; \boxed{[L_x, L_y] = i\hbar L_z}$$

$$\boxed{[L^2, L_i] = [H, L_i] = [H, L^2] = 0} \quad \text{(can measure concurrently)}$$

$$\rightarrow \quad \boxed{\begin{array}{c} L_z Y_{m'} = \hbar m' Y_{m'} \\ \hline L^2 Y_{m'} = \hbar^2 \lambda_{m'} Y_{m'} \end{array}} \quad \Rightarrow \quad \begin{array}{l} \langle L^2 - L_z^2 \rangle = \langle L_x^2 + L_y^2 \rangle \geq 0 \\ \bullet \;\; \sqrt{\lambda_{m'}} \geq m' \geq -\sqrt{\lambda_{m'}} \end{array}$$

Let $\;(L_\pm)^n Y_\mu \equiv |m\rangle$ $\qquad$ (see harm. osc. for why ladders)

$$\boxed{\left[\frac{L_z}{\hbar}, L_\pm\right] = (\pm 1)L_\pm} \quad \Rightarrow \quad \left[\frac{L_z}{\hbar}, (L_\pm)^n\right] = \pm n(L_\pm)^n \quad \Rightarrow \quad \begin{array}{l} L_z\big[(L_\pm)^n Y_\mu\big] = (\mu \pm n)\hbar\big[(L_\pm)^n Y_\mu\big] \\ \bullet \;\; L_z|m\rangle = (\mu \pm n)\hbar|m\rangle \end{array}$$

$$\boxed{[L^2, L_\pm] = 0} \quad \Rightarrow \quad \left[L^2, (L_\pm)^n\right] = 0 \quad \Rightarrow \quad \begin{array}{l} L^2\big[(L_\pm)^n Y_\mu\big] = \lambda_\mu\big[(L_\pm)^n Y_\mu\big] \\ \bullet \;\; L^2|m\rangle = \lambda_\mu|m\rangle \end{array}$$

Then $\;\left(\sqrt{\lambda_\mu} \geq (\mu \pm n) \geq -\sqrt{\lambda_\mu}\right) \;\Rightarrow\;$ **Let** $\qquad$ (else un-normalizable solution)

$$\underline{L_+|m_t\rangle = 0} \;,\quad L_z|m_t\rangle = \hbar l \;,$$
$$L^2|m_t\rangle = \lambda\hbar^2 \;,\quad L^2 = L_-L_+ + L_z^2 + \hbar L_z$$
$$\bullet \;\; L^2|m_t\rangle = \hbar^2 l(l+1)|m_t\rangle = \lambda\hbar^2|m_t\rangle$$

$$\underline{L_-|m_b\rangle = 0} \;,\quad L_z|m_b\rangle = \hbar l' \;,$$
$$L^2|m_b\rangle = \lambda\hbar^2 \;,\quad L^2 = L_+L_- + L_z^2 - \hbar L_z$$
$$\bullet \;\; L^2|m_b\rangle = \hbar^2 l'(l'-1)|m_b\rangle = \lambda\hbar^2|m_b\rangle$$

$$\left[\lambda = l'(l'-1) = l(l+1)\right] \;\Rightarrow\; \left[l' = -l\right] \;\Rightarrow\; \begin{bmatrix} L_z|m_t\rangle = \hbar l|m_t\rangle \\ L_z|m_b\rangle = -\hbar l|m_b\rangle \end{bmatrix} \quad \begin{array}{l}\text{(Spherical Harmonics do} \\ \text{not allow half-integer } l)\end{array}$$

Schrodinger Y_l^m: $\qquad \boxed{\begin{array}{c|l} \begin{array}{c} l \in \{0, 1, 2, ...\} \\[4pt] m \in \{-l, -l+1, ..., l-1, l\} \end{array} & \begin{array}{l} L_z\big|Y_l^m\big\rangle = \hbar m\big|Y_l^m\big\rangle = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\big|Y_l^m\big\rangle \\[4pt] L^2\big|Y_l^m\big\rangle = \hbar^2 l(l+1)\big|Y_l^m\big\rangle \\[4pt] L_\pm\big|Y_l^m\big\rangle = \hbar\sqrt{l(l+1)-m(m\pm1)}\,\big|Y_l^m\big\rangle \end{array} \end{array}}$

4.2 Generalized

Angular Momentum:

$$\widehat{J}_i \equiv \text{???} \qquad \boxed{J^2 \equiv J_x^2 + J_y^2 + J_z^2}$$

$$\boxed{J_\pm \equiv J_x \pm iJ_y} \quad \boxed{J_\pm J_\mp = J^2 - J_z^2 \pm \hbar J_z}$$

Commutation Relations:

$$\boxed{\left[J_i, J_j\right] = i\hbar J_k \,\epsilon_{ij}} \;\Leftrightarrow\; \boxed{J \times J = i\hbar J}$$

$$\boxed{\left[J^2, J_z\right] = 0} = \left[H, J_z\right] \quad \text{(if spher. symm.)}$$

General:

$$\boxed{\begin{array}{c} j \in \left\{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, ...\right\} \\[2mm] m \in \{-j, -j+1, ..., j-1, j\} \end{array} \; \begin{array}{l} J_z|jm\rangle = \hbar m|jm\rangle \\[2mm] J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle \\[2mm] J_\pm|jm\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\,|jm\rangle \\[2mm] J_x|jm\rangle = \frac{J_+ + J_-}{2}|jm\rangle \end{array}}$$

Generator of Rotations:

$$U[R_{(\theta)}] = e^{-\frac{i}{\hbar}\theta\hat{\theta}\cdot L} = \lim_{N\to\infty}\left[\mathbb{1} - \frac{i}{\hbar}\frac{\theta}{N}\hat{\theta}\cdot L\right]^N \;\Leftrightarrow\; \boxed{U[R_{(\epsilon_z\hat{z})}] = \mathbb{1} - \frac{i}{\hbar}\epsilon_z L_z}$$

1.) $\;U[R_{(\epsilon_z\hat{z})}]|x,y\rangle \equiv |x - \epsilon_z y, \epsilon_z x + y\rangle$

$\quad\Rightarrow\; \langle x,y|U[R_{(\epsilon_z\hat{z})}]|\Psi\rangle = \Psi(x + \epsilon_z y, \epsilon_z x - y)$

$\quad\Rightarrow\; \underline{\langle x,y|L_z|\Psi\rangle = (XP_y - YP_x)\Psi_{(x,y)}}$

or 2.) $\quad U^\dagger XU \equiv X - \epsilon_z Y \;\Rightarrow\; [X, L_z] = -i\hbar Y$

$\qquad U^\dagger P_y U \equiv \epsilon_z P_x + P_y \;\Rightarrow\; [P_y, L_z] = i\hbar P_x$

$\qquad U^\dagger YU,\; U^\dagger P_x U,\; \Rightarrow\; \ldots$

$$\Rightarrow\; \underline{L_z = XP_y - YP_x}$$

Consistency Check: $\; U[R_{(-\epsilon_z\hat{z})}]T_{(-\epsilon_x\hat{x} - \epsilon_y\hat{y})}U[R_{(\epsilon_z\hat{z})}]T_{(\epsilon_x\hat{x} + \epsilon_y\hat{y})} = T_{(-\epsilon_y\epsilon_z\hat{x} + \epsilon_x\epsilon_z\hat{y})}$

$\qquad\qquad ?U[R_{(-\epsilon_y\hat{y})}]U[R_{(-\epsilon_x\hat{x})}]U[R_{(\epsilon_y\hat{y})}]U[R_{(\epsilon_x\hat{x})}] = \mathbb{1} + \frac{i}{\hbar}\epsilon_x\epsilon_y L_z = U[R_{(-\epsilon_x\epsilon_y\hat{z})}]$

$\qquad\qquad\qquad\qquad (L_z = xp_y - yp_x)$

Tensors and Tensor Operators

rank-2 Tensor :

$$|t^{(2)}\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3} t_{ij}|i\rangle|j\rangle$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} |ij\rangle\langle ij|t^{(2)}\rangle$$

rank-2 Cartesian Tens. Oper., T_{ij} :

Set of $3^{n=2}$ Operators

rank-k Spherical Tensor Operator, T_k^q :

Set of $2k + 1$ Operators s.t.

$$U[R]T_k^q U^\dagger[R] = \sum_{q'=-k}^{k} D_{q'q}^k T_k^{q'}$$

$$\downarrow$$

$$UT_k^q U^\dagger U|jm\rangle = \sum_{q'}\sum_{m'} D_{q'q}^k D_{m'm}^j T_{q'}^k|jm'\rangle$$

$$\sim\; U|kq\rangle|jm\rangle = \sum_{q'}\sum_{m'} D_{q'q}^k D_{m'm}^j |kq'\rangle|jm'\rangle$$

• $T_1^{\pm1} = \mp\frac{V_x \pm iV_y}{\sqrt{2}}$

$\;\; T_1^0 = V_z$

Wigner-Eckhart : $\;\langle\alpha_2 j_2 m_2|T_k^q|\alpha_1 j_1 m_1\rangle = \langle\alpha_2 j_2\|T_k\|\alpha_1 j_1\rangle \cdot \langle j_2 m_2|kq; j_1 m_1\rangle$

$\qquad\qquad\qquad\qquad\qquad$ (CG coeff.)

4.3 1 Particle w/ Spin, $s = \frac{1}{2}$

*Find the Eigenvectors, e_i , of S_z and S^2 in the form of $|\chi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{i\phi_1} \\ \sin\frac{\theta}{2} e^{i\phi_2} \end{pmatrix} = e^{i\gamma}\begin{pmatrix} \cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix}$ $\quad \gamma = \frac{\phi_1+\phi_2}{2}$ $\quad \phi = \phi_2 - \phi_1$

$$\boxed{* \quad e_i \in \left\{ \quad |\tfrac{1}{2}\,\tfrac{1}{2}\rangle \equiv |\uparrow\,\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix} = \tfrac{1}{2}(1+\sigma_z) \quad,\quad |\tfrac{1}{2}\,\tfrac{-1}{2}\rangle \equiv |\downarrow\,\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix} = \tfrac{\sigma_x}{2}(1+\sigma_z) \quad \right\}}$$

$$\left. \begin{array}{l} S^2|\uparrow\,\rangle = \frac{3\hbar^2}{4}|\uparrow\,\rangle \\[2mm] S^2|\downarrow\,\rangle = \frac{3\hbar^2}{4}|\downarrow\,\rangle \end{array} \right\} \Rightarrow$$

$$S^2 = \frac{3\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \frac{3\hbar^2}{4}\sigma_0 = *\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{3\hbar^2}{4}&0\\0&\frac{3\hbar^2}{4}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}^{T*}$$

$$= \underline{\text{Casimir Op:}}\ -(g_{yz})^2 - (g_{zx})^2 - (g_{xy})^2 = \frac{3\hbar^2}{4}\mathbb{1} \qquad \left(\text{only for } s=1/2 \text{ systems}\right)$$

$$\left. \begin{array}{l} S_-|\uparrow\,\rangle = \hbar|\downarrow\,\rangle \\[2mm] S_+|\downarrow\,\rangle = \hbar|\uparrow\,\rangle \\[2mm] S_+|\uparrow\,\rangle = S_-|\downarrow\,\rangle = 0 \end{array} \right\} \Rightarrow$$

$$S_+ = \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad S_- = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix} \qquad \left(\begin{smallmatrix}\text{can't}\\\text{measure}\end{smallmatrix}\right)$$

$$= \hbar|\uparrow\,\rangle\langle\downarrow| \qquad = \hbar|\downarrow\,\rangle\langle\uparrow|$$

Lad. Op. (see harm.)

$[S_+, S_-] = (2\hbar)S_z$

$[S_z, S_\pm] = (\pm\hbar)S_+$

$$\left. \begin{array}{l} S_z|\uparrow\,\rangle = \frac{\hbar}{2}|\uparrow\,\rangle \\[2mm] S_z|\downarrow\,\rangle = -\frac{\hbar}{2}|\downarrow\,\rangle \end{array} \right\} \Rightarrow$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\hbar}{2}\sigma_z = *\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{\hbar}{2}&0\\0&-\frac{\hbar}{2}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}^{T*}$$

$$= \tfrac{\hbar}{2}|\uparrow\,\rangle\langle\uparrow| - \tfrac{\hbar}{2}|\downarrow\,\rangle\langle\downarrow|$$

$$\left. \begin{array}{l} S_x = \frac{1}{2}(S_+ + S_-) \\[2mm] S_y = \frac{1}{2i}(S_+ - S_-) \end{array} \right\} \Rightarrow$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

$$\boxed{\{S_i, S_j\} = \tfrac{\hbar^2}{2}\delta_{ij}} \qquad \left(\text{only for } s=1/2 \text{ systems}\right)$$

Properties of Pauli Matrices, σ_i

- $\sigma_i = \sigma_i^\dagger = \sigma_i^{-1}$

* $\sigma_i^2 = 1 = (\hat{n}\cdot\sigma)^2 \leftrightarrow \begin{array}{l}(\hat{n}\cdot\sigma+1)(\hat{n}\cdot\sigma-1)=0 \\ (\hat{n}\cdot S + \frac{\hbar}{2})(\hat{n}\cdot S - \frac{\hbar}{2}) = (S_z + \frac{\hbar}{2})(S_z - \frac{\hbar}{2})\end{array}$

- $\sigma_i\sigma_j = -\sigma_j\sigma_i = i\sigma_k = (\sigma_j\sigma_i)^\dagger \Rightarrow \left[\frac{\sigma_i}{2i}, \frac{\sigma_j}{2i}\right] = \frac{\sigma_k}{2i}$

* $(A\cdot\sigma)(B\cdot\sigma) = A\cdot B + i(A\times B)\cdot\sigma \qquad \left(\text{if } [A_i,\sigma_i] = 0 = [B_i,\sigma_i]\right)$

* $(\sigma_i\sigma_j)^2 = -1$

* $(\sigma_i\sigma_j\sigma_k)^2 = -1 = \mathbb{i}^2$

- $\text{Tr}\,\sigma_i = 0 \Rightarrow \text{Tr}(\sigma_\alpha\sigma_\beta) = 2\delta_{\alpha\beta} \qquad \alpha \in (0, x, y, z)$

* $\sum c_\alpha \sigma_\alpha = 0 \to c_\alpha = 0 \Rightarrow M_{2\times 2} = \sum \frac{1}{2}\text{Tr}(M\sigma_\alpha)\sigma_\alpha$

- $\det(\sigma_i) = -1$

Gamma Matrices, γ_α

- $\gamma_i = \begin{bmatrix}0&\sigma_i\\-\sigma_i&0\end{bmatrix},\ \gamma_t = \begin{bmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{bmatrix} \text{ or } \begin{bmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{bmatrix}$

* $\gamma_t^2 = 1:\ e^{j\phi} = \cosh\phi + \gamma_t\sinh\phi$

* $\gamma_x^2 = \gamma_y^2 = \gamma_z^2 = -1$

- $-\gamma_i = \gamma_i^\dagger = \gamma_i^{-1}\ ,\ \ \gamma_t = \gamma_t^\dagger = \gamma_t^{-1}$

- $\gamma_\alpha\gamma_\beta = -\gamma_\beta\gamma_\alpha$

* $(\gamma_i\gamma_j)^\dagger = \gamma_j\gamma_i\ ,\ (\gamma_t\gamma_i)^\dagger = -\gamma_i\gamma_t$

* $(\gamma_i\gamma_j)^2 = -1\ ,\ (\gamma_t\gamma_i)^2 = 1$

* $(\gamma_i\gamma_j\gamma_k)^2 = 1\ ,\ (\gamma_t\gamma_i\gamma_k)^2 = -1$

* $(\gamma_t\gamma_x\gamma_y\gamma_z)^2 = -1 = \mathbb{i}^2$

General Direction, $\hat{n}$, on Bloch Sphere $(1, \theta, \phi) \Leftrightarrow$ Pauli Vector, V (see Properties of σ_i)

$$\hat{n} \cdot \vec{S} = \cos\phi \sin\theta\, S_x + \sin\phi \sin\theta\, S_y + \cos\theta\, S_z = \boxed{\frac{\hbar}{2}\begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}} = V_{\hat{n}}$$

- $V_{\vec{n}}^2 = \|\vec{n}\|^2 I$ • $\det(V_{\vec{n}}) = -\|\vec{n}\|^2$ • $V = V^\dagger$ • $\mathrm{Tr}(V) = 0$ • $\underset{\text{over } R_{\hat{n}} \perp}{\text{Reflect } V} : -RVR^{\searrow}$

- $\underset{xy\text{-plane by }\psi}{\text{Rotate } V \text{ in}} : -\left[\cos\frac{\psi}{2}\sigma_x + \sin\frac{\psi}{2}\sigma_y\right]\left[-\sigma_x V \sigma_x\right][..] = \underline{\left[\cos\frac{\psi}{2}\mathbb{1} - \sin\frac{\psi}{2}\sigma_x\sigma_y\right]V[..]^{-1}} = \boxed{(\pm)UV(\pm)U^\dagger}$ $(U \in SU(2))$

- $i\vec{n}\cdot\sigma = in_x\sigma_x + in_y\sigma_y + in_z\sigma_z = n_x\sigma_y\sigma_z + n_y\sigma_z\sigma_x + n_z\sigma_x\sigma_y$

* $\hat{n}\cdot\vec{S}|\chi_\pm\rangle = \pm\frac{\hbar}{2}|\chi_\pm\rangle \Rightarrow \boxed{|\chi_+\rangle = e^{i\gamma}\begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} = \begin{bmatrix} 1 \\ \tan\frac{\theta}{2}e^{i\phi} \end{bmatrix} , \; |\chi_-\rangle = e^{i\gamma}\begin{bmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\cot\frac{\theta}{2}e^{i\phi} \end{bmatrix}}$

*   <u>Riemann Sphere $\to \mathbb{C}P^1$</u>: $|\chi\rangle = \begin{bmatrix} 1 \\ f(\theta,\phi) \end{bmatrix} \to \left.\begin{array}{l} f(\theta,\phi) \in (-\infty, \infty): \;\; 0 \leq \theta < \pi \\ f(\theta,\phi) \in (-\infty, \infty): \;\; \pi \leq \theta < 2\pi \end{array}\right\}$ (double cover)

<u>Projectors and Nilpotents:</u>

$P_U^\pm = \frac{1}{2}(1 \pm U)$
    $(U^2 = 1)$

$P_z^\pm = \frac{\sigma_z}{2}(1 \pm \sigma_z)\sigma_z$
    $= |\pm\rangle = |z^\pm\rangle$

$P_{tz}^\pm = \frac{\gamma_t\gamma_z}{2}(1 \pm \gamma_t\gamma_z)\gamma_t\gamma_z$

$|\xi\rangle = \begin{bmatrix} a+bi & 0 \\ c+di & 0 \end{bmatrix} = \boxed{(aP_z^+ + b\sigma_x\sigma_y P_z^+) + (c\sigma_x P_z^+ + d\sigma_y P_z^+)}$

$|\xi\rangle\langle\chi| = \begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix} = \xi\xi^T\epsilon = \begin{bmatrix} \xi^1 & 0 \\ \xi^2 & 0 \end{bmatrix}\begin{bmatrix} -\xi^2 & \xi^1 \\ 0 & 0 \end{bmatrix}$ $\det(|\xi\rangle\langle\chi|) = 0 \, , \; \epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$= x\sigma_x + y\sigma_y + z\sigma_z = \left(\xi^1|z^+\rangle + \xi^2\sigma_x|z^+\rangle\right)\left(-\xi^2\langle z^+|^* + \xi^1\sigma_x\langle z^+|^*\right)$

$\{\alpha_\pm, \alpha_\pm\} = 0$

$\{\alpha_-, \alpha_+\} = 1$

$\alpha_-\alpha_+ + \alpha_+\alpha_- = P_+ + P_-$

$P_\pm P_\mp = 0 = P_\mp P_\pm$

$\alpha_{UV}^\pm = \frac{1}{2}(U + V)$ $(U^2 = 1 \, , \; V^2 = -1)$

$Cl(1,3,\mathbb{C}): \begin{array}{l} a_{tz}^\pm = \frac{1}{2}(\gamma_t + \gamma_z) \;\; \Rightarrow \;\; P_{tz}^\pm = a_{tz}^\mp a_{tz}^\pm = \frac{1}{2}(1 \pm \gamma_t\gamma_z) \\ b_{ixy}^\pm = \frac{1}{2}(i\gamma_x + \gamma_y) \;\; \Rightarrow \;\; P_{ixy}^\pm = b_{ixy}^\mp b_{ixy}^\pm = \frac{1}{2}(1 \pm i\gamma_x\gamma_y) \end{array}$

$\{a_\pm, b_\pm\} = 0 = \{a_\pm, b_\mp\}$
$[P_{tz}^\pm, P_{ixy}^\pm] = 0 = [P_{tz}^\pm, P_{ixy}^\mp]$ $\to$ $\boxed{|\epsilon\rangle = \epsilon^1 P_{tz}^+ P_{ixy}^+ + \epsilon^2 \alpha_{tzixy}^+ P_{tz}^+ P_{ixy}^+}$

<u>$R \in SO(3) \sim \mathbb{R}P^3$</u> : $\begin{array}{l} R(\theta) = e^{\theta g} \\ R(0) = \mathbb{1} \end{array} \Rightarrow \left.\frac{dR}{d\theta}\right|_{\theta=0} = \boxed{g[R] \in \begin{array}{l} \mathfrak{so}(3) \\ \mathfrak{su}(2) \end{array}}$

- $\det(R = e^{\theta g}) = 1 = e^{\theta\,\mathrm{Tr}(g)} \Rightarrow \boxed{\mathrm{Tr}(g) = 0}$

- $\boxed{\begin{array}{c} v^T[R^T R]v = v^T\mathbb{1}v \\ R^T = R^{-1} \end{array}} \Rightarrow g^T + g = 0 \leftrightarrow \boxed{g^T = -g}$

$[g_{yz}, g_{zx}] = g_{xy} \equiv g_z = [g_x, g_y]$

$[g_{zx}, g_{xy}] = g_{yz} \equiv g_x = [g_y, g_z]$

$[g_{xy}, g_{yz}] = g_{zx} \equiv g_y = [g_z, g_x]$

$$U \in SU(2) = 2 \times SO(3) = Spin(3)$$
$$= \underline{\frac{2n \text{ grade,}}{\|\pm U_i\|^2 = \mathbb{1}}} \in Cl(3,0) \sim S^3_{phere}$$

$:\quad \boxed{\begin{array}{l} \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} = e^{-i\frac{\psi}{2}\hat{n}\cdot\sigma} = \cos\frac{\psi}{2}\mathbb{1} - \sin\frac{\psi}{2}\,\mathfrak{i}\,\vec{n}\cdot\sigma \\ \sim\ e^{-\frac{\psi}{2}\sigma_i\sigma_j} = \cos\frac{\psi}{2}\mathbb{1} - \sin\frac{\psi}{2}\sigma_i\sigma_j \end{array}} \Rightarrow \boxed{g[U] \in \begin{array}{l}\mathfrak{su}(2)\\ \mathfrak{so}(3)\end{array}}$

- $\begin{array}{c}\text{Rotate } \hat{k} \text{ to } \hat{k}' \text{ by angle } \theta \\ \hat{\theta} = (-\sin\phi, \cos\phi, 0) = \hat{k} \times \hat{k}'\end{array}$ $\quad \underline{U[R(\theta)] = \cos\frac{\theta}{2}\mathbb{1} - \sin\frac{\theta}{2}i(\hat{\theta}\cdot\sigma)} = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}e^{i\phi} & \cos\frac{\theta}{2} \end{bmatrix}$

- $\begin{array}{c}\text{Rotate about } \hat{\psi} \text{ by angle } \psi \\ \hat{\psi} = \hat{n} = (n^x, n^y, n^z)\end{array}$ $\quad \underline{U[R(\psi)] = \cos\frac{\psi}{2}\mathbb{1} - \sin\frac{\psi}{2}i(\hat{\psi}\cdot\sigma)} = \begin{bmatrix} \cos\frac{\psi}{2} - in^z\sin\frac{\psi}{2} & (-in^x - n^y)\sin\frac{\psi}{2} \\ (-in^x + n^y)\sin\frac{\psi}{2} & \cos\frac{\psi}{2} + in^z\sin\frac{\psi}{2} \end{bmatrix}$

- $\det(U) = \alpha^*\alpha + \beta^*\beta = 1 \Rightarrow \boxed{\mathrm{Tr}(g) = 0}$

- $\boxed{\begin{array}{c}\xi^\dagger[U^\dagger U]\chi = \xi^\dagger\mathbb{1}\chi \\ U^\dagger = U^{-1}\end{array}} \Rightarrow g^\dagger + g = 0 \leftrightarrow \boxed{g^\dagger = -g}$

- $\boxed{V' = (\pm)UV(\pm)U^\dagger = U[vv^T\epsilon]U^{-1}}$, $\begin{array}{c}\epsilon = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\\ (\det vv^T = 0)\end{array}$

$-\frac{1}{2}\sigma_x\sigma_y = g_{xy} = [g_{yz}, g_{zx}] = -\frac{\mathfrak{i}}{2}\sigma_z = \frac{1}{2}\hat{i}$
$-\frac{1}{2}\sigma_y\sigma_z = g_{yz} = [g_{zx}, g_{xy}] = -\frac{\mathfrak{i}}{2}\sigma_x = \frac{1}{2}\hat{j}$
$-\frac{1}{2}\sigma_z\sigma_x = g_{zx} = [g_{xy}, g_{yz}] = -\frac{\mathfrak{i}}{2}\sigma_y = \frac{1}{2}\hat{k}$

$\underline{\Lambda \in SO^+(1,3)}:\ \begin{array}{l}\Lambda_{(\theta)} = e^{\theta g}\\ \Lambda_{(0)} = \mathbb{1}\end{array} \Rightarrow \frac{d\Lambda}{d\theta}\big|_{\theta=0} = \boxed{g[\Lambda] \in \begin{array}{l}\mathfrak{so}^+(1,3)\\ \mathfrak{sl}(2,\mathbb{C})\end{array}}$

- $\boxed{v^T[\eta]v = v^T[\Lambda^T\eta\Lambda]v = v^T\eta\Lambda^{-1}\cdot\Lambda v} \Rightarrow \boxed{g^T = -\eta g \eta}$, $\eta = (+,-,-,-)$

- Rotation, J_{ij}: $\boxed{\Lambda^T_{ij} = \Lambda^{-1}_{ij}} \Rightarrow \boxed{\mathrm{Tr}(J) = 0,\ J^T = -J}$

- Boost, K_{ti}: $\boxed{\Lambda^T_{ti} = \Lambda_{ti}} \Rightarrow \boxed{\mathrm{Tr}(K) = 0,\ K^T = K}$

$\left.\begin{array}{l}[J_{yz}, J_{zx}] = J_{xy}\\[4pt] [K_{tx}, K_{ty}] = -J_{xy}\\[4pt] [J_{yz}, K_{ty}] = K_{tz}\\[4pt] [J_{yz}, K_{tz}] = -K_{ty}\end{array}\right\} \times 3$

$\underline{L \in SL(2,\mathbb{C}) = 2 \times SO^+(1,3) =}$
$\underline{Spin(1,3) = \subset Cl(3,0) = \subset Cl(1,3)}$

$:\quad \begin{bmatrix}\alpha & \beta\\ \gamma & \alpha^*\end{bmatrix} \sim \begin{array}{l} e^{\theta J_{ij}} = \begin{array}{l}\cos\frac{\psi}{2}\mathbb{1} - \sin\frac{\psi}{2}\sigma_i\sigma_j\\ \cos\frac{\psi}{2}\mathbb{1} + \sin\frac{\psi}{2}\gamma_i\gamma_j\end{array}\\[10pt] e^{\theta K_{ti}} = \begin{array}{l}\cosh\frac{\psi}{2}\mathbb{1} - \sinh\frac{\psi}{2}\sigma_i\\ \cosh\frac{\psi}{2}\mathbb{1} + \sinh\frac{\psi}{2}\gamma_i\end{array}\end{array} \Rightarrow \boxed{\begin{array}{c}\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{so}^+(1,3)\\ g[L] \in \mathfrak{su}(2) \oplus (\mp^l_r i)\mathfrak{su}(2)\\ \mathfrak{su}(2)_\mathbb{C}\end{array}}$

- $\boxed{\begin{array}{c}\psi^T[\epsilon]\phi = \psi^T[L^T\epsilon L]\phi = \psi^T\epsilon L^{-1}\cdot L\phi\\ = -\phi^T\epsilon\psi = 0 \text{ if } \psi = \phi\end{array}} \Rightarrow \boxed{g^T\epsilon = -\epsilon g}$, $\epsilon = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

- Rotation, J_{ij}: $\boxed{L^\dagger_{ij} = L^{-1}_{ij}} \Rightarrow \boxed{\mathrm{Tr}(J) = 0,\ J^T = -J}$

- Boost, K_{ti}: $\boxed{L^\dagger_{ti} = L_{ti}} \Rightarrow \boxed{\mathrm{Tr}(K) = 0,\ K^T = K}$

$[J_{yz}, J_{zx}] = J_{xy} = -\frac{1}{2}\sigma_x\sigma_y \to \frac{1}{2}\gamma_x\gamma_y$
$[K_{tx}, K_{ty}] = -J_{xy}$
$[J_{yz}, K_{ty}] = K_{tz} = \mp^l_r\frac{1}{2}\sigma_z \to \pm^l_r\frac{1}{2}\gamma_t\gamma_z$
$[J_{yz}, K_{tz}] = -K_{ty}$

- Left/Left Dual: $\begin{array}{l}w_{ld}\cdot w_l : \psi^T\epsilon\cdot\phi = \psi^T\epsilon L^{-1}\cdot L\phi\\ w^T_{ld}:\ -\epsilon\psi\ \to [L^T]^{-1}[-\epsilon\psi]\end{array}$

- Right Dual/Right: $\begin{array}{l}w_r\cdot w_{rd} : \psi^\dagger\epsilon\cdot\phi^* = \psi^\dagger\epsilon[L^*]^{-1}\cdot L^*\phi^*\\ w^T_r:\ -\epsilon\psi^*\ \to\ [L^\dagger]^{-1}[-\epsilon\psi^*]\end{array}$

- $\boxed{W' = (\pm)LW(\pm)L^\dagger = L[w_l w_l^\dagger][(L^\dagger)^{-1}]^{-1} = L_{ij}[w_l w_{rd}]L^{-1}_{ij} \text{ or } L_{ti}[w_l w_{rd}][L^{-1}_{ti}]^{-1}}$ $\quad (\det w_l w_l^\dagger = 0)$

- $\underline{(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})}:\ \begin{bmatrix}L & 0\\ 0 & [L^\dagger]^{-1}\end{bmatrix}\begin{bmatrix}w_l\\ w_r^T\end{bmatrix}$

- $\underline{(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})}:\ \begin{bmatrix}c\tilde{t}\\ \tilde{r}\end{bmatrix} = [A^{-1}L \otimes [L^\dagger]^{-1}A]\begin{bmatrix}ct\\ \vec{r}\end{bmatrix} = [SO(3)]\begin{bmatrix}ct\\ \vec{r}\end{bmatrix}$

- $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C} = \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C}$:

$$A^\pm_i = \frac{1}{2}(J_{jk} \pm K_{ti})$$
$$[A^\pm_i, A^\pm_j] = A^\pm_k\ ,\ \ [A^+, A^-] = 0$$

4.4   2 Objects w/ Spin   Objects could be orbital momentum, another particle spin, etc.

4.4.1   2 Objects w/ Spin $\frac{1}{2}$:   **Dim** : $2 \otimes 2 = 3 \oplus 1$ $\Rightarrow$ $(2s_1 + 1)(2s_2 + 1) = \sum\limits_{s=|s_1-s_2|}^{s_1+s_2} 2s + 1$
Spin : $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

*Find Eigenvectors, e_i , of $(S^{(1,2)})_z$ and $(S^{(1,2)})^2$ in the form of $|\chi_i \chi_j\rangle$ (using $(S^{(1,2)})_\pm$)

$$\boxed{\chi_i \chi_j \rightarrow |\chi_i \chi_j\rangle \equiv |\chi_i\rangle |\chi_j\rangle \equiv |\chi_i\rangle \otimes |\chi_j\rangle}$$

Choose $|\chi_i\rangle \equiv S_z$-Eigenvector w/ Spin $\frac{1}{2}$ (e.g, $|\frac{1}{2}\frac{-1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, as opposed to $\begin{pmatrix} .6 \\ .8 \end{pmatrix}$)

$$S^{(i)} \equiv \begin{pmatrix} S_x^{(i)} \\ S_y^{(i)} \\ S_z^{(i)} \end{pmatrix} \qquad\qquad S^{(1,2)} \equiv \left(S^{(1)} + S^{(2)}\right) \equiv \begin{pmatrix} S_x^{(1)} + S_x^{(2)} \\ S_y^{(1)} + S_y^{(2)} \\ S_z^{(1)} + S_z^{(2)} \end{pmatrix}$$

- $S_z^{(2)} S_x^{(1)} \left(|\chi_1\rangle \otimes |\chi_2\rangle\right) = \left(S_x^{(1)}|\chi_1\rangle\right) \otimes \left(S_z^{(2)}|\chi_2\rangle\right)$

- $\left(S^{(1,2)}\right)^2 = \left(S^{(1)} + S^{(2)}\right) \cdot \left(S^{(1)} + S^{(2)}\right)$

- $S^{(i)} \cdot S^{(j)} \equiv \underline{S_x^{(i)} S_x^{(j)} + S_y^{(i)} S_y^{(j)}} + S_z^{(i)} S_z^{(j)}$

  $= S_z^{(i)} S_z^{(j)} + \underline{\frac{1}{2} S_+^{(i)} S_-^{(j)} + \frac{1}{2} S_-^{(i)} S_+^{(j)}}$

  $(S^{(i)})^2 \equiv S^{(i)} \cdot S^{(i)}$

1. $\left(S^{(1,2)}\right)_z$

$$\left(S^{(1,2)}\right)_z \chi_1\chi_2 = \left(S_z^{(1)} + S_z^{(2)}\right)|\chi_1\rangle|\chi_2\rangle$$

$$= S_z^{(1)}|\chi_1\rangle \otimes |\chi_2\rangle + |\chi_1\rangle \otimes S_z^{(2)}|\chi_2\rangle$$

$$\left(S^{(1,2)}\right)_z |\chi_1\chi_2\rangle = \hbar(m_1 + m_2)\,|\chi_1\chi_2\rangle$$

$$\Rightarrow \quad \underline{e_i = a_i|\uparrow\uparrow\,\rangle + b_i|\uparrow\downarrow\,\rangle + c_i|\downarrow\uparrow\,\rangle + d_i|\downarrow\downarrow\,\rangle}$$

$$|\uparrow\uparrow\,\rangle \;=\; |\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle$$
$$|\uparrow\downarrow\,\rangle \;=\; |\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\tfrac{-1}{2}\rangle$$
$$|\downarrow\uparrow\,\rangle \;=\; |\tfrac{1}{2}\tfrac{-1}{2}\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle$$
$$|\downarrow\downarrow\,\rangle \;=\; |\tfrac{1}{2}\tfrac{-1}{2}\rangle \otimes |\tfrac{1}{2}\tfrac{-1}{2}\rangle$$

2. Use $\left(S^{(1,2)}\right)_\pm$ on $|\uparrow\,\rangle \otimes |\uparrow\,\rangle$ to GUESS e_i from "nice" bevahior

$$S_- |\uparrow\uparrow\,\rangle = \tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$
$$S_- \left[\tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\right] = |\downarrow\downarrow\,\rangle$$
$$S_- |\downarrow\downarrow\,\rangle = 0$$

S_+ works too

If $\tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$ then maybe $\tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ works (try $S_\pm$ on it).

Guess for $\{e_i\}$:

$$|1\ 1\rangle \equiv |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle = |\uparrow\uparrow\,\rangle$$

$$|1\ 0\rangle \equiv \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle + |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) = \tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|1\ \text{-}1\rangle \equiv |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle = |\downarrow\downarrow\,\rangle$$

$$|0\ 0\rangle \equiv \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle - |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) = \tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

3. Check if the guesses are eigenvectors of $\left(S^{(1,2)}\right)^2$ [and do $\left(S^{(1,2)}\right)_z$ to see eigenvalues]

(work has been skipped, do it yourself, check answer below)

$$S^2|1\ 1\rangle = \hbar^2(1)(1+1)|1\ 1\rangle \quad (s=1) \qquad S_z|1\ 1\rangle = \hbar(1)|1\ 1\rangle \quad (m=1)$$

$$S^2|1\ 0\rangle = \hbar^2(1)(1+1)|1\ 0\rangle \quad (s=1) \qquad S_z|1\ 0\rangle = \hbar(0)|1\ 0\rangle \quad (m=0)$$

$$S^2|1\ \text{-}1\rangle = \hbar^2(1)(1+1)|1\ \text{-}1\rangle \quad (s=1) \qquad S_z|1\ \text{-}1\rangle = \hbar(-1)|1\ \text{-}1\rangle \quad (m=-1)$$

$$S^2|0\ 0\rangle = \hbar^2(0)(0+1)|0\ 0\rangle \quad (s=0) \qquad S_z|0\ 0\rangle = \hbar(0)|0\ 0\rangle \quad (m=0) \qquad ☑$$

* $e_i \in$
$$\begin{cases}
|1\ 1\rangle = {}_{1\leftarrow}\cancel{e^{i\phi}}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle = |\uparrow\uparrow\,\rangle \\[2mm]
|1\ 0\rangle = \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle + |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) = \tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\[2mm]
|1\ \text{-}1\rangle = |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle = |\downarrow\downarrow\,\rangle \\[4mm]
|0\ 0\rangle = \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle - |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) = \tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)
\end{cases}$$

Triplet: $s=1$ (for the first three)

Singlet: $s=0$ (for $|0\ 0\rangle$)

4.4.2 2 Objects w/ Any Spin:

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus |j_1 - j_2| \ \Rightarrow\ (2j_1 + 1)(2j_2 + 1) = \sum_{j=|j_1-j_2|}^{j_1+j_2} 2j + 1$$

- $|\chi_1\rangle$ has spin, j_1; and $|\chi_2\rangle$ has spin, j_2

- $j_{\max} = j_2 + j_1$ and $j_{\min} = |j_2 - j_1|$

- Possible total $|j\ m\rangle$ must satisfy

  1.) $j_{\min} \le j \le j_{\max}$,   **2.)** $-j \le m \le j$,

  3.) have integer differences

If j_1 and j_2 are known from the start,

$$|j\ m\rangle = \sum_{m_1,m_2} |j_1\ m_1\rangle \otimes |j_2\ m_2\rangle\ \langle j_1\ m_1| \otimes \langle j_2\ m_2||j\ m\rangle$$
$$= \sum_{m_1,m_2} |j_1\ m_1\rangle \otimes |j_2\ m_2\rangle\ C^{j_1 j_2 j}_{m_1 m_2 m}$$

where the sum is over all poss. int. diff. values that satisfy

$$m_1 + m_2 = m, \quad -j_1 \le m_1 \le j_1, \quad -j_2 \le m_2 \le j_2,$$

and C are the corresponding Clebsh-Gordan coefficients, whose squared value is the probability of measuring the $\chi_1 \otimes \chi_2$ state represented by that term.

Possible Combined $|j\ m\rangle$

$$(2j_{\max}+1) \begin{cases} \left|j_{\max}\ j_{\max}\right\rangle = 1 \leftarrow \cancel{e^{i\phi}} \cdots \\ \left|j_{\max}\ j_{\max}-1\right\rangle \\ \vdots \\ \left|j_{\max}\ -j_{\max}\right\rangle \end{cases}$$

$$(2j_{\max}-1) \begin{cases} \left|j_{\max}-1\ j_{\max}-1\right\rangle = 1 \cdots \\ \left|j_{\max}-1\ j_{\max}-2\right\rangle \\ \vdots \end{cases}$$

$$\vdots$$

$$(2j_{\min}+1) \begin{cases} \left|j_{\min}\ j_{\min}\right\rangle = 1 \cdots \\ J_- \left|j_{\min}\ j_{\min}\right\rangle \sim \cdots \\ \vdots \\ \left|j_{\min}\ -j_{\min}\right\rangle \end{cases}$$

If the top state in a j-set (see box above) is known, applying the J_- lowering operator (and normalizing) provides the coefficients for the rest of the set of varying m. The coefficients for each top state of a set are (by convention) positive, real, and normalized to 1. This makes all of the coefficients real. For the top state of the initial $j_{\max}$-set, $|j_{\max}\ j_{\max}\rangle$, there is only one product-ket in the sum; its coefficient is thus set to 1. For an arbitrary set below the first, the top state has product-ket coefficients such that the state is orthogonal to all other previously determined states that have the same m. To reduce some work to solve for them, use

$$C^{j_1 j_2 j}_{m_1 m_2 m} = (-1)^{j_1 + j_2 - j} \cdot C^{j_1 j_2 j}_{-m_1 -m_2 -m}$$

$$\langle j_1\ m_1|\langle j_2\ m_2|j\ m\rangle = (-1)^{j_1 + j_2 - j} \cdot \langle j_1\ -m_1|\langle j_2\ -m_2|j\ -m\rangle$$

If m_1 and m_2 are also known from the start, then $m = m_1 + m_2$, and

$$|j_1\ m_1\rangle \otimes |j_2\ m_2\rangle = \sum_j C^{j_1 j_2 j}_{m_1 m_2 m}\ |j\ (m_1 + m_2)\rangle$$

where the sum is only over all possible s as satisfied above - **1.), 2.) and 3.)**. In this case, the total z-component, m, is known. The only unknown is the total spin, s, whose probability to be measured is C^2.

4.5   Electron in Magnetic Field

$\mu_{\text{clas.}} = IA = \frac{q}{2\pi r}v(\pi r^2) = \frac{q}{2\pi r}\frac{L}{mr}(\pi r^2) = \left(\frac{q}{2m}\right)L \;\rightarrow\; \frac{e\hbar}{2m}\cdot n$ (Bohr magneton)

$\mu_{\text{quan.}} = \left(\frac{g_e q}{2m}\right)S = \left(\frac{q}{m}\right)S = \gamma S$

$$\tau_\mu = \mu \times B \qquad\qquad H = -\mu \cdot B$$
$$F_\mu = \nabla(\mu \cdot B) \qquad,\qquad = -\gamma S \cdot B$$

Larmor Precession

$$\chi(t) = \cos(\alpha/2)\begin{pmatrix}1\\0\end{pmatrix}e^{-\frac{i}{\hbar}E_1 t} + \sin(\alpha/2)\begin{pmatrix}0\\1\end{pmatrix}e^{-\frac{i}{\hbar}E_2 t}$$

$B = B_0\hat{k}$

$H = -\gamma B_0 S_z$

$\qquad = -\gamma B_0\begin{pmatrix}\frac{\hbar}{2} & 0\\ 0 & -\frac{\hbar}{2}\end{pmatrix}$ $\Rightarrow$

$$= \begin{pmatrix}\cos(\alpha/2)e^{-\frac{i}{\hbar}E_1 t}\\ \sin(\alpha/2)e^{-\frac{i}{\hbar}E_2 t}\end{pmatrix}$$

$$\begin{pmatrix}\langle S_x\rangle\\ \langle S_y\rangle\\ \langle S_z\rangle\end{pmatrix} = \begin{pmatrix}\frac{\hbar}{2}\sin(\alpha)\cos(\gamma B_0 t)\\ -\frac{\hbar}{2}\sin(\alpha)\sin(\gamma B_0 t)\\ \frac{\hbar}{2}\cos(\alpha)\end{pmatrix}$$ (torque from B with S leads to precession)

Stern-Gerlach

5 Bosons and Fermions

Distinguishable Particles: $\boxed{\psi(r_1, r_2) \equiv \psi_a(r_1)\psi_b(r_2)}$

Indistinguishable Particles:

$$\boxed{P_x f(x_1, x_2;\ y_1, y_2;\ ...)\ =\ \pm\ f(x_2, x_1;\ y_1, y_2;\ ...)} \quad , \quad \boxed{\iint |\Psi_{(x_1, x_2)}|^2 dx_1 dx_2 = \iint \text{Pr}_{(x_1, x_2)} \tfrac{dx_1 dx_2}{2}}$$

Boson:
$(s \in \{0, 1, 2, ...\})$

$$\psi_+(r_1, r_2) \equiv \frac{1}{\sqrt{2}}\Big[\psi_a(r_1)\psi_b(r_2) + \psi_a(r_2)\psi_b(r_1)\Big]$$

$$\boxed{\psi(r_1, r_2) = \psi(r_2, r_1)} \qquad \rightarrow \qquad \boxed{P_i \Psi = \Psi} \quad \text{(symmetric)}$$

Fermion:
$(s \in \{\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, ...\})$

$$\psi_-(r_1, r_2) \equiv \frac{1}{\sqrt{2}}\Big[\psi_a(r_1)\psi_b(r_2) - \psi_a(r_2)\psi_b(r_1)\Big]$$

$$\boxed{\psi(r_1, r_2) = -\psi(r_2, r_1)} \qquad \rightarrow \qquad \boxed{P_i \Psi = -\Psi} \quad \text{(antisymmetric)}$$

5.1 Exchange Forces: $\quad \Big\langle (x_1 - x_2)^2 \Big\rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$

$$\boxed{\begin{aligned}
&\text{Dist. Part.} \quad : \quad \big\langle (\Delta x)^2 \big\rangle = \big\langle (\Delta x)^2 \big\rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b \\[2mm]
&\text{Symmetric:} \quad \big\langle (\Delta x)^2 \big\rangle = \big\langle (\Delta x)^2 \big\rangle_d - 2\,\big\| \langle \psi_b | x | \psi_a \rangle \big\|^2 \quad \text{(attractive if overlap)} \\[2mm]
&\text{Antisymmetric:} \quad \big\langle (\Delta x)^2 \big\rangle = \big\langle (\Delta x)^2 \big\rangle_d + 2\,\big\| \langle \psi_b | x | \psi_a \rangle \big\|^2 \quad \text{(repulsive if overlap)}
\end{aligned}}$$

- $\langle x_1 x_2 \rangle = \dfrac{1}{2} \displaystyle\int \Big[\psi_a(r_1)^*\psi_b(r_2)^* \pm \psi_b(r_1)^*\psi_a(r_2)^*\Big] x_1 x_2 \Big[\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)\Big] dx_1 dx_2$

$$= \frac{1}{2}\langle x \rangle_a \langle x \rangle_b + \frac{1}{2}\langle x \rangle_b \langle x \rangle_a$$

$$\pm \frac{1}{2}\Big\langle \psi_b(r_1)\Big| x_1 \Big|\psi_a(r_1)\Big\rangle \Big\langle \psi_a(r_2)\Big| x_2 \Big|\psi_b(r_2)\Big\rangle \pm \frac{1}{2}\Big\langle \psi_a(r_1)\Big| x_1 \Big|\psi_b(r_1)\Big\rangle \Big\langle \psi_b(r_2)\Big| x_2 \Big|\psi_a(r_2)\Big\rangle$$

$$= \langle x \rangle_a \langle x \rangle_b \pm \big\| \langle \psi_b | x | \psi_a \rangle \big\|^2$$

Two Electrons:

$$\psi(r_1, r_2)\chi(m_1, m_2) = \begin{cases} \quad \text{(singlet)} \\ -\psi(r_1, r_2)\chi(m_2, m_1) \\[4mm] \quad \text{(triplet)} \\ -\psi(r_2, r_1)\chi(m_1, m_2) \end{cases} \begin{array}{l} \Rightarrow \quad \begin{array}{l} \chi \ \text{is antisymmetric so} \\ \psi \ \text{is symmetric} \end{array} \quad \Rightarrow \quad \text{Attractive (ground state)} \\[6mm] \Rightarrow \quad \begin{array}{l} \chi \ \text{is symmetric so} \\ \psi \ \text{is antisymmetric} \end{array} \quad \Rightarrow \quad \text{Repulsive} \end{array}$$

5.2 Statistics

Sterling's Approx:
$$\log(z!) \approx z \log(z) - z \qquad z \gg 1 \ \text{ or } \ z = 0$$
$$\frac{d}{dz} \log(z!) \approx \log(z)$$

Lagrange Multiplier:
$$G(X, \alpha, \beta) = \log(Q(X)) + \alpha f_1(X) + \beta f_2(X)$$
$$\frac{\partial G}{\partial \alpha}[Q_{\max}] = 0, \qquad \frac{\partial G}{\partial \beta}[Q_{\max}] = 0, \qquad \frac{\partial G}{\partial N_n}[Q_{\max}] = 0$$

$$\sum_n N_n = N \qquad\qquad\qquad \sum_n N_n E_n = E$$

$$f_1(X) = N - \sum_n N_n = 0 \qquad\qquad f_2(X) = E - \sum_n N_n E_n = 0$$

Let there be N_n particles in the E_n energy level having d_n degeneracies, and $Q(N_1, N_2, ...)$ be the number of possible configurations for such a state given $X = (N_1, N_2, ..., N_n)$.

Dist.

$$\begin{cases}
\textbf{1.)} \ Q(X) = \prod_n \binom{N - N_1 - ... - N_{n-1}}{N_n} d_n^{N_n} \\[4pt]
\qquad\quad = N! \prod_n \frac{d_n^{N_n}}{N_n!} \\[12pt]
\textbf{2.)} \ \log(Q) = \log(N!) + \sum_n N_n \log(d_n) \\[4pt]
\qquad\qquad\qquad\qquad - \log(N_n!)
\end{cases}$$

3.) $\dfrac{\partial G}{\partial N_n} \approx \dfrac{\log(d_n) - \log(N_n)}{-\alpha - \beta E_n} = 0$

4.) $N_n = \dfrac{d_n}{e^{\beta E_n + \alpha}}$

Fermion

$$\begin{cases}
\textbf{1.)} \ Q(X) = \prod_n \binom{d_n}{N_n} \\[12pt]
\textbf{2.)} \ \log(Q) = \sum_n \log(d_n!) - \log(N_n!) \\[4pt]
\qquad\qquad\qquad\qquad - \log[(d_n - N_n)!]
\end{cases}$$

3.) $\dfrac{\partial G}{\partial N_n} \approx \dfrac{-\log(N_n) + \log(d_n - N_n)}{-\alpha - \beta E_n} = 0$

4.) $N_n = \dfrac{d_n}{e^{\beta E_n + \alpha} + 1}$

Boson

$$\begin{cases}
\textbf{1.)} \ Q(X) = \prod_n \binom{N_n + d_n - 1}{N_n} \\[12pt]
\textbf{2.)} \ \log(Q) = \sum_n \log[(N_n + d_n - 1)!] \\[4pt]
\qquad\qquad\qquad - \log(N_n!) \\[4pt]
\qquad\qquad\qquad - \log[(d_n - 1)!]
\end{cases}$$

3.) $\dfrac{\partial G}{\partial N_n} \approx \dfrac{\log(N_n + d_n - 1) - \log(N_n)}{-\alpha - \beta E_n} = 0$

4.) $N_n = \dfrac{d_n - 1}{e^{\beta E_n + \alpha} - 1} \approx \dfrac{d_n}{e^{\beta E_n + \alpha} - 1}$

Given some substance in thermal equilibrium,

$$\beta = \frac{1}{k_b T} \qquad \mu(T) \equiv -\frac{\alpha}{k_b T}$$

where μ depends on the situation.

$$\frac{N_n}{d_n} : \qquad n(\epsilon) \quad = \quad \begin{cases} \frac{1}{e^{(\epsilon-\mu)/k_b T}} & \text{Maxwell-Boltzmann} \\[2ex] \frac{1}{e^{(\epsilon-\mu)/k_b T}+1} & \text{Fermi-Dirac} \\[2ex] \frac{1}{e^{(\epsilon-\mu)/k_b T}-1} & \text{Bose-Einstein} \end{cases}$$

6 Perturbation Theory

$$H^{(0)}\psi_n = E_n\psi_n$$
$$\downarrow$$
$$H\psi_n' = E_n'\psi_n'$$

$$\left[H^{(0)} + \lambda H^{(1)}\right]\left[\psi_n + \lambda\psi_n^{(1)} + \lambda^2\psi_n^{(2)} + ...\right] = \left[E_n + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + ...\right]\left[\psi_n + \lambda\psi_n^{(1)} + \lambda^2\psi_n^{(2)} + ...\right]$$

$$\cancel{\lambda^0 H^{(0)}\psi_n} \qquad\qquad \cancel{\lambda^0 E_n\psi_n}$$

$$+\ \lambda^1(H^{(0)}\psi_n^{(1)} + H^{(1)}\psi_n) \qquad +\ \lambda^1(E_n\psi_n^{(1)} + E_n^{(1)}\psi_n)$$
$$+\ \lambda^2(H^{(0)}\psi_n^{(2)} + H^{(1)}\psi_n^{(1)}) \quad = \quad +\ \lambda^2(E_n\psi_n^{(2)} + E_n^{(1)}\psi_n^{(1)} + E_n^{(2)}\psi_n) \qquad (\lambda=1)$$
$$+\ ... \qquad\qquad\qquad +\ ...$$

6.1 Non-Degenerate Theory

$\underline{E_n^{(1)},\ \psi_n^{(1)}}:$
$$E_n\psi_n^{(1)} + E_n^{(1)}\psi_n = H^{(0)}\psi_n^{(1)} + H^{(1)}\psi_n$$

$$\langle\psi_m|\quad(-H^{(1)} + E_n^{(1)})|\psi_n\rangle = \langle\psi_m|\quad(H^{(0)} - E_n)|\psi_n^{(1)}\rangle = \sum c_i^{(1)}(E_i - E_n)\langle\psi_m|\psi_i\rangle$$

$$-\langle\psi_m|H^{(1)}|\psi_n\rangle + E_n^{(1)}\langle\psi_m|\psi_n\rangle = c_m^{(1)}(E_m - E_n)$$

$$\boxed{E_n^{(1)} = \langle\psi_n|H^{(1)}|\psi_n\rangle} \qquad \boxed{\psi_n^{(1)} = \sum_{m\neq n}\frac{\langle\psi_m|H^{(1)}|\psi_n\rangle}{E_n - E_m}\psi_m + (0)\psi_n}$$

$\underline{E_n^{(2)},\ |n^{(2)}\rangle}:$
$$-\langle m^{(0)}|H^{(1)}|n^{(1)}\rangle + E_n^{(1)}\langle m^{(0)}|n^{(1)}\rangle$$
$$+E_n^{(2)}\langle m^{(0)}|n^{(0)}\rangle$$
$$=\quad \langle m^{(0)}|H^{(0)} - E_n^{(0)}|n^{(2)}\rangle$$
$$=\quad c_m^{(2)}(E_m - E_n)$$

$$\boxed{E_n^{(2)} = \sum_{m\neq n}\frac{|\langle m|H^{(1)}|n\rangle|^2}{E_n - E_m}} = \langle n|H^{(1)}|n^{(1)}\rangle \quad,\quad \boxed{|n^{(2)}\rangle = \sum_{m\neq n}\frac{\langle m|H^{(1)} - E_n^{(1)}|n^{(1)}\rangle}{E_n - E_m}\cdot|m\rangle}$$

$\underline{E_n^{(i+1)},\ |n^{(i+1)}\rangle}:$
$$E_n^{(i+1)} = \langle n|H^{(1)}|n^{(i)}\rangle$$

$$|n^{(i+1)}\rangle = \sum_{m\neq n}\frac{\langle m|H^{(1)}|n^{(i)}\rangle - \sum_{j=0}^{i}E_n^{(j+1)}\langle m|n^{(i-j)}\rangle}{E_n - E_m}\cdot|m\rangle$$

6.2 Degenerate Perturbation Theory (see Matrix Operators)

$$\Psi = \sum_i \left(c_i^{(\psi)}[\Psi] \right) \psi_i$$

$$\equiv \sum_i c_i^{(\psi)} \psi_i$$

$$= c_0^{(\psi)} \psi_0 + c_1^{(\psi)} \psi_1 + ...$$

$\forall \psi_i :$

- $H^{(0)} \psi_i = E_n \psi_i$ (ψ_n are degenerate eigenfunctions of $H^{(0)}$)
- $\langle \psi_i | \psi_j \rangle = \delta_{ij}$
- $\langle \psi_i | \hat{Q} | \psi_j \rangle \equiv Q_{ij}$

$$E_n \Psi^{(1)} + E^{(1)} \Psi = H^{(0)} \Psi^{(1)} + H^{(1)} \Psi \qquad \text{(first order)}$$

$$\cancel{E_n \langle \psi_i | \Psi^{(1)} \rangle} + E^{(1)} \langle \psi_i | \Psi \rangle = \cancel{\langle H^{(0)} \psi_i | \Psi^{(1)} \rangle} + \langle \psi_i | H^{(1)} | \Psi \rangle$$

$$= \langle \psi_i | H^{(1)} | c_0 \psi_0 + c_1 \psi_1 + ... \rangle$$

$$c_i E^{(1)} = c_0 \langle \psi_i | H^{(1)} | \psi_0 \rangle + c_1 \langle \psi_i | H^{(1)} | \psi_1 \rangle + ...$$

$$E^{(1)} \begin{pmatrix} c_0[\Psi] \\ c_1[\Psi] \\ \vdots \end{pmatrix}^{(\psi)} = \begin{pmatrix} H_{00}^{(1)} & H_{01}^{(1)} & ... \\ H_{10}^{(1)} & H_{11}^{(1)} & ... \\ \vdots & \vdots & \end{pmatrix}^{(\psi)} \begin{pmatrix} c_0[\Psi] \\ c_1[\Psi] \\ \vdots \end{pmatrix}^{(\psi)} \Rightarrow \boxed{ \begin{array}{c} \text{(solve for } E^{(1)}, \vec{c}^{\,(\psi)}[\Psi]) \\ \left\| \begin{array}{ccc} H_{aa}^{(1)} - E^{(1)} & H_{ab}^{(1)} & ... \\ H_{ba}^{(1)} & H_{bb}^{(1)} - E^{(1)} & ... \\ \vdots & \vdots & \end{array} \right\| = 0 \end{array} }$$

In general,

$$E_i^{(1)} \vec{c}^{\,(\psi)}[\Psi_i] = \overline{H^{(1)}}^{(\psi)} \vec{c}^{\,(\psi)}[\Psi_i] \qquad \text{(ith eigen-)}$$

$$E_i^{(1)} \begin{pmatrix} | \\ \vec{c}[\Psi_i] \\ | \end{pmatrix}^{(\psi)} = \begin{pmatrix} | & | & \\ \vec{c}[\Psi_i] & \vec{c}[\Psi_i] & ... \\ | & | & \end{pmatrix}^{(\psi)} \begin{pmatrix} E_0^{(1)} & 0 & ... \\ 0 & E_1^{(1)} & ... \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} - & \vec{c}^{\,*}[\Psi_i] & - \\ - & \vec{c}^{\,*}[\Psi_i] & - \\ & \vdots & \end{pmatrix}^{(\psi)} \begin{pmatrix} | \\ \vec{c}[\Psi_i] \\ | \end{pmatrix}^{(\psi)}$$

Instead of solving the characteristic polynomial, it would be wise to choose a basis $\{\psi\}$ such that $\vec{c}^{\,(\psi)}[\Psi_i] = (...0\ 0\ 1_{(i)}\ 0\ 0\ ...)^T \Leftrightarrow \Psi_i = \psi_i$, making $\overline{H^{(1)}}^{(\psi)}$ diagonal with eigenvalue entries. These are the energy eigenvalues, $E_i^{(1)} = \left(H^{(1)} \right)_{ii}^{(\psi)} = \langle \psi_i | H^{(1)} | \psi_i \rangle$, which is just like first-order non-Perturbation energy. This also means $|\psi_i\rangle$ are eigenfunctions of $H^{(1)}$ (see Matrix Operators).

It is best to find a hermitian operator, $\hat{A}$, that commutes with $H^{(0)}$ and $H^{(1)}$, whose eigenvalues within the degenerate basis are unique. The corresponding eigenfunctions will be a basis that makes $H^{(1)}$ diagonal. This will also make them eigenfunctions of $H^{(1)}$.

1. $A = A^\dagger$

2. $[A, H^{(0)}] = 0 \rightarrow \left\{ \exists \{\Psi\} \,\middle|\, (A\Psi_n = a_n \Psi_n),\ (H^{(0)} \Psi_n = E_n \Psi_n) \right\}$

3. $\{\psi\} \subset \{\Psi\}$ s.t $\forall \psi_i : \begin{cases} \left(H^{(0)} \psi_i = E_n \psi_i \right), & \leftarrow \text{degenerate} \\ \left(A \psi_i = a_i \psi_i | \right), & \left(\forall_{(i \neq j)}\ a_i \neq a_j \right) \end{cases}$

4. $[A, H^{(1)}] = 0 \Rightarrow 0 = \langle A\psi_i | H^{(1)} | \psi_j \rangle - \langle \psi_i | H^{(1)} | A\psi_j \rangle$

$$0 = (a_i - a_j) H_{ij}^{(1)}$$

$$0 = H_{ij}^{(1)} \quad \boxtimes$$

6.3 Hydrogen Energy Corrections

6.3.1 Fine Structure - $\alpha^4 mc^2$

The Dirac Equation can derive the total fine structure correction with a α^4 order approx.

1. Relativistic, $\hat{p}^4$

$$T = \sqrt{p^2 c^2 + m^2 c^4} - mc^2 = mc^2 \sqrt{1 + \frac{p^2}{m^2 c^2}} - mc^2$$

$$= mc^2 \left[\frac{\left(\frac{1}{2}\right)}{1!} \left(\frac{p^2}{m^2 c^2} \right) + \frac{\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right)}{2!} \left(\frac{p^2}{m^2 c^2} \right)^2 + \dots \right]$$

$$= \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} + \dots$$

$$\downarrow$$

$$H_r^{(1)} = -\frac{p^4}{8m^3 c^2} \qquad \text{(For some reason } \hat{p}^4 \text{ needs to be hermitian to use perturbation theory.}$$
$$\text{It only isn't when } l = 0, \text{ while } \hat{p}^2 \text{ always is hermitian. See Prob. 6.15)}$$

L^2 and L_z should commute with p^4 because the perturbation is spherically symmetric, meaning l and m_l should be conserved (see Operator Evolution). Their eigenvalues are also distinct (taking the eigenfunctions of nlm_l together) within each set of n^2 degeneracies, so their eigenvectors and eigenvalues can be used. n, l and m_l the "good" numbers.

$$\langle r^{-1} \rangle = \frac{1}{n^2 a_0^1}$$

$$\langle r^{-2} \rangle = \frac{1}{(l+1/2)n^3 a_0^2}$$

$$\left\langle \psi_{nlm_l} | H_r^{(1)} | \psi_{nlm_l} \right\rangle = \frac{-1}{8m^3 c^2} \langle \psi_{nlm_l} | p^4 | \psi_{nlm_l} \rangle$$

$$= \frac{-1}{8m^3 c^2} \langle p^2 \psi_{nlm_l} | p^2 | \psi_{nlm_l} \rangle$$

$$= \frac{-1}{8m^3 c^2} \left\langle \left[2m(E_n - V) \right]^2 \right\rangle$$

$$= \frac{-4m^2}{8m^3 c^2} \langle E_n^2 - 2E_n V + V^2 \rangle$$

$$= -\frac{E_n^2}{2mc^2} \left[\frac{4n}{l + 1/2} - 3 \right]$$

2. Spin-Orbit Coupling, $\boldsymbol{S_e \cdot L_e}$

In the electron's frame of reference, the proton is spinning around it, creating a B-field affecting its magnetic dipole moment. The non-inertial reference frame requires multiplying by the Thomas procession correction, which in this case is $C_T = g_e - 1 = 1/2$. In the lab frame, the moving electron's magnetic dipole moment creates an electric dipole moment, which is affected by the proton charge. The latter is much harder to calculate.

$$H_{so}^{(1)} = -C_T \ \mu_e \cdot B(L_e) \qquad \text{(See Electron in Magnetic Field)}$$

$$= \tfrac{1}{2}\tfrac{qS}{m} \cdot \tfrac{k_\mu}{r^3} \int I d\vec{l} \times \vec{r} \quad \left(\sim \ \tfrac{1}{2}\tfrac{qS}{m} \cdot \tfrac{k_\mu}{r^3} \int \tfrac{mqd\vec{v}\times\vec{r}}{m} \right)$$

$$= \tfrac{1}{2}\tfrac{qS}{m} \cdot \tfrac{k_\epsilon}{c^2}\tfrac{2\pi}{r} I = \tfrac{1}{2}\tfrac{qS}{m} \cdot \tfrac{k_\epsilon}{c^2}\tfrac{2\pi}{r}\tfrac{q(L/mr)}{2\pi r}$$

$$= \frac{kqq}{2m}\frac{1}{mc^2}\frac{S \cdot L}{r^3} = \frac{e^2}{8\pi\epsilon_0 m^2 c^2}\frac{S \cdot L}{r^3}$$

$S \cdot L$ does not commute with L or S (meaning m_l and m_s are bad), but $[S \cdot L, S^2] = [S \cdot L, L^2] = 0$. The sum of the two, $J \equiv L+S$, and J^2 also commute with the perturbation. They are all conserved, and their unique eigenvalues per set of degeneracies - $l, s=1/2, j, m_j$ - are the "good" numbers (along with n).

$$S \cdot L = \frac{1}{2}\left(J^2 - L^2 - S^2\right)$$

$$\langle r^{-3} \rangle = \frac{1}{l(l+1/2)(l+1)n^3 a_0^3}$$

(note: divergent at $l = 0$)

$$\langle nljm_j|H_{so}^{(1)}|nljm_j\rangle = \frac{kqq}{2m}\frac{1}{mc^2}\frac{\hbar^2[j(j+1)-l(l+1)-s(s+1)]}{2l(l+1/2)(l+1)n^3 a_0^3}$$

$$= \frac{kqq}{4mn^4}\frac{\hbar^2\alpha^3 m^3 c^3}{\hbar^3 mc^2}\frac{n[j(j+1)-l(l+1)-s(s+1)]}{l(l+1/2)(l+1)}$$

$$= \frac{kqq}{4\hbar cn^4}\frac{\alpha^3 m^2 c^4}{mc^2}\frac{n[j(j+1)-l(l+1)-s(s+1)]}{l(l+1/2)(l+1)}$$

$$= \frac{E_n^2}{mc^2}\left\{\frac{n\left[j(j+1)-l(l+1)-3/4\right]}{l(l+1/2)(l+1)}\right\}$$

3. Darwin Term (correction for $H_{so}^{(1)}$ when $l = 0$) skipped

4. Total Correction

$$E_{fs}^{(1)} = E_r^{(1)} + E_{so}^{(1)}$$

$$= -\frac{E_n^2}{2mc^2}\left[\frac{4n}{l+\frac{1}{2}} - 3\right] + \frac{E_n^2}{mc^2}\left\{\frac{n\left[j(j+1)-l(l+1)-3/4\right]}{l(l+1/2)(l+1)}\right\}$$

$$= \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right) \qquad (j=l\pm1/2)$$

$\downarrow$

$$E_{nj} = E_n + E_{fs}^{(1)}$$

$$= E_n\left[1 - \frac{E_n}{2mc^2}\left(\frac{4n}{j+1/2} - 3\right)\right]$$

$$= -\frac{\alpha^2 mc^2}{2n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - 3/4\right)\right]$$

Fine structure splits the l energy degeneracies. However, since $j = l \pm 1/2$, there are still two j degeneracies if $n > 2$. Overall, the good numbers to use for stationary state solutions to the hydrogen atom w/ fine structure correction are $n, l, s=1/2, j, m_j$. Note, J^2, L^2, and S^2 always commute(?)

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6.3.2 Zeeman Effect (Ext. B-Field)

$$H_B^{(1)} = -(\mu_s + \mu_l) \cdot B_{\text{ext}} \qquad \text{(see Electron in Magnetic Field)}$$

$$= -\left(\frac{g_e q}{2m} S + \frac{q}{2m} L\right) \cdot B_{\text{ext}}$$

$$= \frac{e}{2m}(2S + L) \cdot B_{\text{ext}}$$

Weak Zeeman $(B_{\text{ext}} \ll B_{\text{int}})$

$$H_{WZ}^{(1)} = \frac{e}{2m} B_{\text{ext}} \cdot (2S + L)$$

$$= \frac{e}{2m} B_{\text{ext}} \cdot (J + S)$$

Fine structure perturbation dominate the Zeeman perturbation, so the fine structure numbers are the good ones: $n, l, s_{=1/2}, j$, and m_j. m_l and m_s can't be used for $\langle L \rangle$ or $\langle S \rangle$, so instead use the fact that the "vector" $J = L + S$ is conserved, so a **time-averaged** S-component to the J "vector" can be defined as $S_{\text{ave}} = \frac{S \cdot J}{J^2} J$, where $S \cdot J = \frac{1}{2}(J^2 + S^2 - L^2)$.

$$E_{\text{WZ}}^{(1)} = \frac{e}{2m} B_{\text{ext}} \cdot \langle nljm_j | J + S_{\text{ave}} | nljm_j \rangle$$

$$= \frac{e}{2m} B_{\text{ext}} \cdot \left\langle J \left(1 + \frac{S \cdot J}{J^2}\right) \right\rangle$$

$$= \frac{e}{2m} B_{\text{ext}} \cdot \langle J \rangle \left(1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}\right)$$

$$= \frac{e\hbar}{2m} B_{\text{ext}} m_j \left(1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}\right) \qquad \text{(let } B_{\text{ext}} \text{ be parallel to the z-axis)}$$

$$= \mu_B B_{\text{ext}} m_j g_j \qquad \begin{array}{l} \mu_B = \text{Bohr magneton} = 5.788 \times 10^{-5} \text{ ev/T} \\ g_j = \text{Lande g-factor} \end{array}$$

Strong Zeeman $(B_{\text{ext}} \gg B_{\text{int}})$

For a strong magnetic field parallel to the z-axis, m_l and m_s are stuck in the same place, making them and l conserved. The external torque, however, means that the total angular momentums, j and m_j are not. Though unneeded, obviously $s_{=1/2}$.

$$E_{SZ}^{(1)} = \frac{e}{2m} B_{\text{ext}} \langle 2S_z + L_z \rangle$$

$$= \mu_B B_{\text{ext}} (2m_s + m_l)$$

The spin-orbit correction must be changed with respect to the new good numbers, m_l and m_s. The relativistic correction uses the same numbers, so it stays the same.

$$E_{so}^{(1)} = \frac{e^2}{8\pi\epsilon_0 m^2 c^2} \left\langle \frac{S_x L_x + S_y L_z + S_z L_z}{r^3} \right\rangle \qquad E_{fs}^{(1)} = E_{so}^{(1)} + E_r^{(1)}$$

$$= \frac{e^2}{8\pi\epsilon_0 m^2 c^2} \frac{0 + 0 + \hbar^2 m_s m_l}{l(l+1/2)(l+1)n^3 a_0^3} \qquad\qquad = \frac{E_n^2}{2mc^2}\frac{4nm_s m_l}{l(l+1/2)(l+1)} + \frac{E_n^2}{2mc^2}\left[3 - \frac{4n}{l+1/2}\right]$$

$$\rightarrow$$

$$= \frac{4nE_n^2}{2mc^2}\left[\frac{m_s m_l}{l(l+1/2)(l+1)} + \frac{3}{4n} - \frac{1}{l+1/2}\right]$$

$$= \frac{kqq}{2m^2 c^2}\frac{\hbar^2}{(\hbar/\alpha mc)^3 n^3}\frac{m_s m_l}{l(l+1/2)(l+1)}$$

$$\downarrow$$

$$= \frac{kqq}{2\hbar c}\frac{\alpha^3 m^2 c^4}{4mc^2 n^4}\frac{4nm_s m_l}{l(l+1/2)(l+1)} \qquad\qquad E_{nlm_l m_s} = E_n + E_{SZ}^{(1)} + E_{fs}^{(1)}$$

$$= \frac{E_n^2}{2mc^2}\frac{4nm_s m_l}{l(l+1/2)(l+1)}$$

Intermediate Zeeman $(B_{\text{ext}} \sim B_{\text{int}})$

There are no good numbers here (see Degenerate Perturbation Theory). The basis is chosen to be $|j\ m_j\rangle = \sum_i C_i |l\ m_l\rangle \otimes |s\ m_s\rangle$ (see 2 Objects w/ Any Spin), as it makes $\overline{H^{(1)}}^{(e)}$ easier (instead of using l, m_l, m_s).

1.) $\psi_i = |j\ m_j\rangle_i$

2.) $\left(\langle l\ m_l|\langle s\ m_s|\right)_x \left(|l\ m_l\rangle|s\ m_s\rangle\right)_y = \delta_{xy}$

3.) $Q_{rc}^{(\psi)} = \langle\psi_r|\hat{Q}|\psi_c\rangle$

4.) ψ_i s.t. $\begin{cases} 0 \le l < n \\ j_{(l\pm)} = l \pm 1/2, \\ 2l^2 < i \le 2(l+1)^2 \end{cases}$

$$\langle jm_j|H_{fs}^{(1)}|jm_j\rangle = \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right)$$

$$\equiv \gamma_n\left(3 - \frac{4n}{j+1/2}\right) \qquad\qquad \overline{H^{(1)}}^{(jm_j)} = \overline{H_{fs}^{(1)}}^{(jm_j)} + \overline{H_{IZ}^{(1)}}^{(jm_j)}$$

See Griffith Prob. 6.25 for example with $n = 2$

$$\langle jm_j|H_{IZ}^{(1)}|jm_j\rangle = \langle jm_j|H_{IZ}^{(1)}\left(C_i|lm_l\rangle \otimes |sm_s\rangle\right)$$

$$= \mu_B B_{\text{ext}}(2m_s + m_l)C_i^2$$

$$\equiv \beta(2m_s + m_l)C_i^2$$

6.3.3 Stark Effect (Small Ext. E-Field)

- $H^{(1)} = -p \cdot E = eE \cdot r$ (small r)

- $n = 1 \;\rightarrow\; H^{(1)} = 0$

- $n = 2 \;\rightarrow\; \begin{cases} H^{(1)} = 0 & m = \pm 1 \\[2mm] H^{(1)} = ke|E|a_0 & m = 0 \quad\text{(k is some constant)} \end{cases}$

6.3.4 Lamb Shift (quantitized E-field) - $\alpha^5 mc^2$ (skipped)

6.3.5 Hyperfine (Spin-Spin), $S_p \cdot S_e$ - $m/m_p \; \alpha^4 mc^2$

(Coupling between the electron magnetic moment and the magnetic field from the proton magnetic moment)

$$\mu_e = -\frac{g_e e}{2m_e} S_e = -\frac{e}{m_e} S_e, \qquad \mu_p = \frac{g_p e}{2m_p} S_p$$

$$B(\mu_p) = \frac{\mu_0}{4\pi r^3}[3(\vec{\mu_p} \cdot \hat{r})\hat{r} - \vec{\mu_p}] + \frac{2\mu_0}{3}\vec{\mu_p}\delta^3(r)$$

$$H_{hf}^{(1)} = -\mu_e \cdot B(\mu_p)$$
$$= \dots$$
$$\downarrow$$
$$E_{hf}^{(1)} = \left(\frac{e}{m_e}\right)\left(\frac{2\mu_0}{3}\frac{g_p e}{2m_p}\right)\langle S_e \cdot S_p\rangle \left|\psi_{nlm}(0)\right|^2$$

In the ground state, $\left|\psi_{100}(0)\right|^2 = 1/(\pi a_0^3)$. S_e^2, S_p^2, and the sum $S = S_e + S_p$ commute with $S_e \cdot S_p$, so s_e, s_p, m_s, s^2 are the good numbers. S_e and S_p do not, so m_{se} and m_{sp} are not good numbers.

$$E_{hf}^{(1)} = \left(\frac{e}{m_e}\right)\left(\frac{2}{3\epsilon_0 c^2}\frac{g_p e}{2m_p}\right)\frac{1}{2\pi a_0^3}\langle S^2 - S_e^2 - S_p^2\rangle$$

$$= \frac{g_p e^2}{4\pi\epsilon_0 c^2 m_p m_e}\frac{4\alpha^3 m_e^3 c^3 \hbar^2}{3\hbar^3}\left[\frac{s(s+1)}{2} - 3/4\right]$$

$$= \frac{4}{3}g_p\frac{m_e}{m_p}\alpha^4 m_e c^2\left[\frac{s(s+1)}{2} - 3/4\right]$$

$$= \frac{4}{3}g_p\frac{m_e}{m_p}\alpha^4 m_e c^2 \cdot \begin{cases} \frac{1}{4} & s = 1 \;\text{(triplet)} \\[2mm] \frac{-3}{4} & s = 0 \;\text{(singlet)} \end{cases} \;\rightarrow\; \begin{array}{l} \Delta E = 5.88 \times 10^{-6}\text{ eV} \\[2mm] \lambda = 21\text{ cm}, \quad \nu = 1420\text{ MHz} \end{array}$$

6.4 Transition Amplitude (See Pictures)

$H_{(t)} = H^0 + H^1_{(t)}$

$\downarrow$

$U_{(t)}|i^0\rangle = \sum_n |n^0\rangle e^{-\frac{i}{\hbar}E_n^0 t}\langle n^0|U_I|i^0\rangle$

$|\Psi_{(t)}\rangle = \boxed{\sum_n |n^0\rangle e^{-\frac{i}{\hbar}E_n^0 t} d_{n(t)}}$

$\downarrow$

$0 = \langle f^0|i\hbar\frac{\partial}{\partial t} - H^0 - H^1_{(t)}|\Psi_{(t)}\rangle$

$= \sum_n \langle f^0|\left[i\hbar\dot{d}_n - H^1_{(t)}d_n\right]|n^0\rangle e^{-\frac{i}{\hbar}E_n^0 t}$

$\downarrow$

$\dot{d}_{f(t)} = \sum_n \frac{1}{i\hbar}\langle f^0|H^1_{(t)}|n^0\rangle e^{\frac{i}{\hbar}(E_f^0 - E_n^0)t} d_{n(t)}$

$= \sum_n \frac{1}{i\hbar}\langle f^0|H^1_{(t)}|n^0\rangle e^{i\omega_{fn}t} d_{n(t)}$

$d_{n(t)} = d_{n(0)} + \int_0^t \dot{d}_n\,dt'$:

- $d_{n(0)} = \delta_{ni}$ (if $|d_{n\neq i}(t)| \ll 1$)   (0$^{\text{th}}$ order)

- $\dot{d}_{f(t)} \approx \frac{1}{i\hbar}\langle f^0|H^1_{(t)}|i^0\rangle e^{i\omega_{fi}t}$

- $\boxed{d_{n(t)} \approx \delta_{ni} + \frac{1}{i\hbar}\int_0^t \langle n^0|H^1_{(t')}|i^0\rangle e^{i\omega_{ni}t'}\,dt'}$ $\left(\substack{1^{\text{st}}\\ \text{order}}\right)$

- $\dot{d}_{f(t)} \approx \frac{1}{i\hbar}\overline{H^1_{fi}}_{(t)}e^{i\omega_{fi}t}$

  $\quad + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \sum_n \overline{H^1_{fn}}_{(t)} e^{i\omega_{fn}t} \overline{H^1_{ni}}_{(t')} e^{i\omega_{ni}t'}\,dt'$

- $\dots$

Interactive Picture Method:

$U_{I(t,t_0)} = \mathbb{I} + \frac{1}{i\hbar}\int_{t_0}^t H^1_{I(t')}\,dt' + \left(\frac{1}{i\hbar}\right)^2\int_{t_0}^t\int_{t_0}^{t'} H^1_{I(t')}H^1_{I(t'')}\,dt''dt' + \dots$

- $\langle f^0|U_{I(t,t_0)}|i^0\rangle = \langle f^0|e^{\frac{i}{\hbar}E_f^0(t-t_0)}U_{(t,t_0)}|i^0\rangle$

  $\equiv d_{f(t)} = \boxed{\delta_{fi} + \frac{1}{i\hbar}\int_{t_0}^t\langle f^0|H^1_{(t')}|i^0\rangle e^{i\omega_{fi}(t'-t_0)}\,dt'}$ (1$^{\text{st}}$ order)

  $\quad + \left(\frac{1}{i\hbar}\right)^2\int_{t_0}^t\int_{t_0}^{t'}\sum_n\langle f^0|H^1_{(t')}|n^0\rangle e^{i\omega_{fn}(t'-t_0)}\langle n^0|H^1_{(t'')}|i^0\rangle e^{i\omega_{ni}(t''-t_0)}\,dt''dt' + \dots$

Normal Schrodinger Propagator:

$U_{S(t,t_0)} = U^0_{(t,t_0)} + \frac{1}{i\hbar}\int_{t_0}^t U^0_{(t,t_0)}U^{0\dagger}_{(t',t_0)}H^1_{(t')}U^0_{(t',t_0)}\,dt'$

$\quad + \left(\frac{1}{i\hbar}\right)^2\int_{t_0}^t\int_{t_0}^{t'} U^0_{(t,t_0)}U^{0\dagger}_{(t',t_0)}H^1_{(t')}U^0_{(t',t_0)}U^{0\dagger}_{(t'',t_0)}H^1_{(t'')}U^0_{(t'',t_0)}\,dt''dt' + \dots$

- $\langle f^0|U_{(t,t_0)}|i^0\rangle = \boxed{\delta_{fi}e^{-\frac{i}{\hbar}E_f^0(t-t_0)} + \frac{1}{i\hbar}\int_{t_0}^t e^{-\frac{i}{\hbar}E_f^0(t-t')}\langle f^0|H^1_{(t')}|i^0\rangle e^{-\frac{i}{\hbar}E_i^0(t'-t_0)}\,dt'}$ (1$^{\text{st}}$ order)

  $\quad + \left(\frac{1}{i\hbar}\right)^2\int_{t_0}^t\int_{t_0}^{t'}\sum_n e^{-\frac{i}{\hbar}E_f^0(t-t')}\langle f^0|H^1_{(t')}|n^0\rangle e^{-\frac{i}{\hbar}E_n^0(t'-t'')}\langle n^0|H^1_{(t'')}|i^0\rangle e^{-\frac{i}{\hbar}E_i^0(t''-t_0)}\,dt''dt' + \dots$

6.5 Variation Principle - Approx. Ground State Energy

$$\psi = \sum c_n \psi_n \;\rightarrow\; E(\psi) > E_0 = E(\psi_0)$$
$$\psi \equiv f(b,x), \qquad \langle H \rangle = \langle T \rangle + \langle V \rangle$$

$$\Rightarrow$$

$$b_{\min} : \;\; \frac{d}{db}\langle H \rangle = 0$$
$$E_0 \approx \left\langle f(b_{\min},x) \Big| H \Big| f(b_{\min},x) \right\rangle$$

6.6 Selection Rules - Orbital Transitions

Electric Dipole Approximation ONLY: $\lambda_\gamma \gg$ atom length $\rightarrow$ E, B feels homogenously oscillating to the atom

$\psi_{nlm} \rightarrow \psi_{n'l'm'}$:

- $\Delta m \in \{-1, \cancel{0}^?, 1\}$

$$s(\gamma) = 1 \;\rightarrow\; m_s(\gamma) \in \{-\hbar, \cancel{0}^?, \hbar\}$$
$$E = E\hat{z} \;\rightarrow\; \Delta m = 0$$

- $\Delta l = \pm 1$

  $1s \leftrightarrow 2p$

  Exception:$(2s \rightarrow 1s)$ through two-photon emission

- $\Delta j \in \{-1, 0, 1\}$

  Exception:$(j = 0 \rightarrow j = 0)$ not allowed

7 Blackbody Radiation

- Power Spectrum : $\;\; I'(\omega) = \dfrac{\hbar^3 \omega^3}{h^2 c^2} \dfrac{1}{e^{\hbar\omega/k_b T} - 1} \;\; \left[\frac{I}{\Omega \cdot f}\right]$ ($\mu = 0$ for photons since photon number isnt conserved)

- Stefan-Boltzmann Law : $\;\; I = \dfrac{dP}{dA} \propto T^4$!! important !!

- Wien's Displacement Law : $\;\; \lambda_{\max} = \dfrac{2.9 \times 10^{-3}}{T} \; [\text{m}]$ (mode of spectrum)

8 Adiabatic Theorem - Slow Changing of Potential

$$t = 0 \; \rightarrow \quad \begin{array}{l} H_{(t=0)} = H^{(0)} \\ H_{(0)}|n\rangle = E_n|n\rangle \end{array} \qquad t = t \; \rightarrow \quad \begin{array}{l} H = H^{(0)}{}_{(t)} \\ H_{(t)}|n(t)\rangle = E_{n(t)}|n(t)\rangle \end{array}$$

Dynamic Phase : $\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')\,dt'$

$$|\Psi_{m(t)}\rangle \equiv \sum_n |n(t)\rangle e^{i\theta_n(t)} \langle n(t)|m(0)\rangle$$

$$\approx |m(t)\rangle e^{i\theta_m(t)} e^{i\gamma_m(t)}$$

$$= |m(t)\rangle e^{i\theta_m(t)} e^{\frac{i}{\hbar}\int A^m \cdot dR}$$

$$\sum_n \cancel{H|n\rangle e^{i\theta_n}} c_n = i\hbar \sum_n |\dot{n}\rangle e^{i\theta_n} c_n + \cancel{|n\rangle i\dot{\theta}_n e^{i\theta_n} c_n} + |n\rangle e^{i\theta_n}\dot{c}_n$$

$$\langle m|\dot{H}|n\rangle + \langle m|H|\dot{n}\rangle = \cancel{\langle m|\dot{E}_n|n\rangle} + \langle m|E_n|\dot{n}\rangle$$

$$\Downarrow$$

$$\dot{c}_m = \frac{d}{dt}\langle m(t)|m(0)\rangle = -\langle m|\dot{m}\rangle c_m - \sum_{n \neq m} \frac{\langle m|\dot{H}|n\rangle}{E_n - E_m} e^{i(\theta_n - \theta_m)}\langle n|\Psi\rangle$$

(not trivial)

$$\approx -\langle m(t)|\dot{m}(t)\rangle c_m \quad \Rightarrow \quad c_m(t) \approx c_m(0) e^{\frac{i}{\hbar} i\hbar \int \langle m|\dot{m}\rangle dt'}$$

$$c_n(t) \approx \delta_{nm} e^{i\gamma_m(t)}$$

Berry Phase : $\gamma_m(t) = i\int_0^t \langle m(t')|\dot{m}(t')\rangle dt' \in \mathbb{R}$

Berry/Geometric Phase

$$\gamma_m(t) = i\int_0^t \langle m(t')|\dot{m}(t')\rangle dt' = \boxed{\frac{1}{\hbar}\int_{R_i}^{R_f} i\hbar\langle m|\nabla_R m\rangle \cdot dR}$$

$$\Rightarrow \frac{1}{\hbar}\oint i\hbar\langle m|\nabla_R m\rangle \cdot dR = \frac{1}{\hbar}\iint \nabla_R \times i\hbar\langle m|\nabla_R m\rangle \cdot da$$

$$\sim \quad \boxed{\frac{1}{\hbar}\oint A^m \cdot dR} \quad = \frac{1}{\hbar}\iint \nabla_R \times A^m \cdot da = \frac{1}{\hbar}\Phi_B^m$$

Aharanov-Bohm Effect:

$$i\hbar\frac{\partial \Psi}{\partial t} = \left[\frac{(p - qA)^2}{2m} + V + \cancel{q\phi}\right]\Psi$$

$$\Rightarrow \quad \Psi = e^{\frac{i}{\hbar}\int_{\mathcal{O}}^r qA\cdot dr'}\psi \;, \quad \breve{E}\psi = \breve{H}\psi$$

$$= \boxed{e^{ig}\psi}$$

$$\vec{A} = \frac{\Phi_B}{2\pi r}\hat{\phi} \quad \Rightarrow \quad \Delta g_{\text{closed}} = \frac{q}{\hbar}\oint \frac{\Phi_B}{2\pi r}\hat{\phi} \cdot r\,d\hat{\phi}$$

$$= \frac{q\Phi_B}{\hbar} = \gamma_m$$

Infinitismal Path Integral

R : Slow degree of freedom (nucleus)

r : Fast degree of freedom (electron)

$$\mathbb{I} = \int dR \sum_n |R, n(R)\rangle\langle R, n(R)|$$

$$\approx \int dR |R, n(R)\rangle\langle R, n(R)|$$

$$\langle \chi(\epsilon)|e^{-\frac{i}{\hbar}H\epsilon}|\chi(0)\rangle = \langle R(\epsilon)|e^{-\frac{i}{\hbar}H_f\epsilon}|R(0)\rangle\langle n(R(\epsilon))|e^{-\frac{i}{\hbar}H_s\epsilon}|n(R(0))\rangle$$

$$\breve{U}(R_1, \epsilon; R_0, 0) = \sqrt{\frac{-im}{2\pi\hbar\epsilon}} e^{\frac{i}{\hbar}\mathcal{L}_s\epsilon} e^{-\frac{i}{\hbar}E_n(R_0)\epsilon}\langle n(R_1)|n(R_0)\rangle$$

$$\Psi(R_1, \epsilon) = \langle \chi(\epsilon)|\hat{U}(\epsilon)|\Psi(R_0, 0)\rangle \approx \sqrt{\frac{-im}{2\pi\hbar\epsilon}}\int e^{\frac{i}{\hbar}\mathcal{L}_s\epsilon} e^{-\frac{i}{\hbar}E_n(R_1+\eta)\epsilon}\langle n(R_1)|n(R_1+\eta)\rangle\Psi(R_1+\eta, 0)\,d\eta \qquad (\eta = R_0 - R_1)$$

$$\approx \sqrt{\frac{-im}{2\pi\hbar\epsilon}}\int e^{\frac{i}{\hbar}\frac{m}{2}\frac{\eta^2}{\epsilon}}\left[1 - \frac{i}{\hbar}\epsilon(V_s + E_n)\right]\langle n(R_1)|\left[|n(R_1)\rangle + \eta|\partial n(R_1)\rangle + \frac{\eta^2}{2}|\partial^2 n(R_1)\rangle\right]\left[1 + \eta\frac{d}{dR} + \frac{\eta^2}{2}\frac{d^2}{dR^2}\right]\Psi(R_1, 0)\,d\eta$$

$$\approx \sqrt{\frac{-im}{2\pi\hbar\epsilon}}\int e^{\frac{i}{\hbar}\frac{m}{2}\frac{\eta^2}{\epsilon}}\left[1 - \frac{i}{\hbar}\epsilon V(R_1, 0) + \cancel{\eta(\ldots)} + \frac{\eta^2}{2}\frac{d^2}{dR^2} + \eta^2\langle n|\partial n\rangle\frac{d}{dR} + \frac{\eta^2}{2}\langle n|\partial^2 n\rangle\right]\Psi(R_1, 0)\,d\eta$$

$$\breve{E}|\Psi\rangle = \hat{H}|\Psi\rangle : \hat{H} = \frac{P_s^2}{2m} + V_s + \hat{H}_f$$

$$\breve{E}\Psi = \breve{H}\Psi \quad : \breve{H} = \frac{(P_s - A^n)^2}{2m} + V + \Phi^n$$

$$\boxed{A^n = i\hbar\langle n|\partial n\rangle}$$

$$\boxed{\Phi^n = \frac{\hbar^2}{2m}\left[\langle \partial n|\partial n\rangle - \langle \partial n|n\rangle\langle n|\partial n\rangle\right]}$$

$$\left(\begin{array}{l} \langle n|\partial n\rangle + \langle \partial n|n\rangle = 0 \\ A^n \text{ is added/subtracted in} \end{array}\right)$$

9 Integral Form

$$\psi(r) = \psi_0(r) + \int g(r - r_0)V(r_0)\psi(r_0)\ d^3r \qquad g(r) = -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}$$

$$= \psi_0 + \int gV\psi_{(r_0)}$$

$$= \psi_0 + \int gV\psi_0 + \int\int gVgV\psi_{(r_0)}$$

$$= \psi_0 + \int gV\psi_0 + \int\int gVgV\psi_0 + \int\int gVgVgV\psi_0 + ...$$

10 Klein-Gordon Equation (Spinless Free Particle)

$$\mathcal{L} = \mathcal{T} - \mathcal{U} = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}\kappa^2\phi^2$$

$$(p^2c^2 + m^2c^4)\psi = E^2\psi$$

$$\boxed{= -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\kappa^2\phi^2}$$

$$(-E^2 + p^2c^2 + m^2c^4)\psi = 0$$

$$\left[-(E/c)^2 + p^2 + (mc)^2\right]\psi = 0$$

$$d\mathcal{L} = \frac{1}{c^2}\frac{\partial\phi}{\partial t}\frac{\partial\epsilon}{\partial t} - \frac{\partial\phi}{\partial x}\frac{\partial\epsilon}{\partial x} - \kappa^2\phi\epsilon \qquad (\epsilon = d\phi)$$

$$\frac{\left[-(E/c)^2 + p^2 + (mc)^2\right]}{\hbar^2}\psi = 0$$

$$S_{[\phi]} = \int dt\int dx\,\mathcal{L}\left(\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}\right)$$

$$\left[\frac{1}{c^2}\frac{\partial}{\partial t}^2 - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0$$

$$dS_{[\phi]} = 0 = \int dt\int dx\,d\mathcal{L}$$

$$\boxed{(-\Box^2 + \mu^2)\psi = 0}$$

$$= \int dt\int dx\left[\frac{1}{c^2}\frac{\partial\phi}{\partial t}\frac{\partial\epsilon}{\partial t} - \frac{\partial\phi}{\partial x}\frac{\partial\epsilon}{\partial x} - \kappa^2\phi\epsilon\right]$$

$$= \int dt\int dx\left[-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial x^2} - \kappa^2\phi\right]\epsilon + 0 + \ldots$$

$$\boxed{\Box^2\phi = \kappa^2\phi}$$

11 Dirac Equation

$$\mu^2 = \square^2 \qquad\qquad\qquad E^2 = p^2c^2 + m^2c^4 = H^2$$

$$
\begin{aligned}
m &= \sqrt{\nabla^2 - \partial_t^2} \\
&= A\partial_x + B\partial_y + C\partial_z + iD\partial_t \\
&= i\gamma^\mu \partial_\mu
\end{aligned}
\qquad\qquad
\begin{aligned}
\sqrt{p^2 + m^2} &= \alpha \cdot p + \beta m \\
&= \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m
\end{aligned}
$$

$$
\begin{aligned}
\partial_x^2 + \partial_y^2 + \partial_z^2 - \tfrac{\partial}{\partial t}^2 &= (A\partial_x + B\partial_y + C\partial_z + iD\partial_t)^2 \\
&= A^2\partial_x^2 + B^2\partial_y^2 + C^2\partial_z^2 - D^2\partial_t^2 \\
&\quad + [AB + BA]\partial_x\partial_y + [AC + CA]\partial_x\partial_z + [BC + CB]\partial_y\partial_z \\
&\quad + [AD + DA]i\partial_x\partial_t + [BD + DB]i\partial_y\partial_t + [CD + DC]i\partial_z\partial_t
\end{aligned}
$$

$$D = \gamma^0, \qquad A = i\gamma^1 = i\beta\alpha_1, \qquad B = i\gamma^2 = i\beta\alpha_2, \qquad C = i\gamma^3 = i\beta\alpha_3$$

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\gamma^\mu = \left[\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} \right] \qquad \gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} (i\hbar\gamma^\mu\partial_\mu - mc)\psi &= 0 \\ (i\slashed{\partial} - m)\psi &= 0 \qquad \text{(natural units)} \end{aligned}}
\qquad
\boxed{\begin{aligned} i\hbar\tfrac{\partial}{\partial t}\psi &= (c\alpha \cdot p + \beta mc^2)\psi \\ i\hbar\tfrac{\partial}{\partial t}\psi &= (c\alpha \cdot (p - qA) + \beta mc^2 + q\phi)\psi \end{aligned}}$$

$$
\begin{aligned}
i\hbar\tfrac{\partial}{\partial t}\psi &= (c\alpha \cdot (p - qA) + \beta mc^2 + q\phi)\psi \\
&= (c\alpha \cdot \pi + \beta mc^2 + q\phi)\psi
\end{aligned}
\qquad\qquad
i\hbar\tfrac{\partial}{\partial t}\psi = (c\alpha \cdot p + \beta mc^2)\psi
$$

$$
\begin{aligned}
\psi(t) &= \psi(p)e^{i(p\cdot r - Et)} \\
\phi &= 0
\end{aligned}
\;\Rightarrow\; E\psi = (\alpha \cdot \pi + \beta m)\psi
\qquad\qquad
\psi(t) = \psi(p)e^{i(p\cdot r - Et)} \;\Rightarrow\; E\psi = (\alpha \cdot p + \beta m)\psi
$$

$$
\begin{bmatrix} E - m & -\sigma\cdot\pi \\ -\sigma\cdot\pi & E + m \end{bmatrix}\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = 0 \Leftrightarrow
\begin{aligned} (E-m)\psi_+ &= (\sigma\cdot\pi)\psi_- \\ (E+m)\psi_- &= (\sigma\cdot\pi)\psi_+ \end{aligned}
\qquad
\begin{bmatrix} E - m & -\sigma\cdot p \\ -\sigma\cdot p & E + m \end{bmatrix}\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = 0 \Leftrightarrow
\begin{aligned} (E-m)\psi_+ &= (\sigma\cdot p)\psi_- \\ (E+m)\psi_- &= (\sigma\cdot p)\psi_+ \end{aligned}
$$

$$(E - m)\psi_\pm = \frac{(\sigma\cdot\pi)(\sigma\cdot\pi)}{E+m}\psi_\pm$$

$$E_s\psi_\pm \approx \frac{(\sigma\cdot\pi)^2}{2m}\psi_\pm \qquad \begin{array}{l}\text{(Pauli's Eq.)} \\ (\sim \text{to Schrodinger})\end{array}$$

$$\boxed{\begin{array}{c}\sigma \cdot A\sigma \cdot B = \\ A\cdot B + i\sigma\cdot(A\times B)\end{array}} = \frac{\sigma\cdot\pi\sigma\cdot\pi}{2m}\psi_\pm = \frac{\pi\cdot\pi + i\sigma\cdot(\pi\times\pi)}{2m}\psi_\pm$$

$$= \left[\frac{\pi^2}{2m} - \frac{q\hbar}{2m}\sigma\cdot B\right]\psi_\pm$$

$$\boxed{(g_e = 2)} \quad = \left[\frac{\pi^2}{2m} - \frac{g_e q}{2m}S\cdot B\right]\psi_\pm$$

$$\boxed{p = 0 \;\Rightarrow\; \begin{cases} \psi = \begin{bmatrix} \psi_+ \\ 0 \end{bmatrix}, & E = m \\[2ex] \psi = \begin{bmatrix} 0 \\ \psi_- \end{bmatrix}, & E = -m \end{cases}}$$

$$\psi_\pm = \frac{(\sigma\cdot p)^2}{E^2 - m^2}\psi_\pm = \frac{p^2}{E^2 - m^2}\psi_\pm \;\Rightarrow\; \boxed{E_\pm = \pm\sqrt{p^2 + m^2}}$$

$$\boxed{\int \|\psi_+\|^2 + \|\psi_-\|^2\, d^3r = 1}$$

Hydrogen Fine Structure

$$E\psi = H\psi$$
$$E\psi = (\alpha \cdot p + \beta m + q\phi)\psi \quad \Rightarrow \quad \begin{aligned}(E - m - V)\psi_+ &= (\sigma \cdot p)\psi_- \\ (E + m - V)\psi_- &= (\sigma \cdot p)\psi_+\end{aligned}$$

Left column:

$$(E - m - V)\psi_+ = (\sigma \cdot p)\left(\tfrac{1}{E + m - V}\right)(\sigma \cdot p)\psi_+$$

$$(E_s - V)\psi_+ = \tfrac{1}{2m}(\sigma \cdot p)\left(1 + \tfrac{E_S - V}{2m}\right)^{-1}(\sigma \cdot p)\psi_+$$

$$\approx \tfrac{p^2}{2m}\psi_+ \qquad \text{(1st order, } v^2)$$

$$\approx \tfrac{p^2}{2m}\psi_+ - \tfrac{\sigma \cdot p}{(2m)^2}(E_S - V)(\sigma \cdot p)\psi_+ \qquad \text{(2nd order, } v^4)$$

$$= \tfrac{p^2}{2m}\psi_+ - \tfrac{\sigma \cdot p}{(2m)^2}\left[(\sigma \cdot p)\overbrace{(E_S - V)\psi_+}^{\text{1st order}} + [E_S - V, \sigma \cdot p]\,\psi_+\right]$$

$$\approx \left[\tfrac{p^2}{2m} - \tfrac{p^2}{(2m)^2}\tfrac{p^2}{2m} - \tfrac{(\sigma \cdot p)(\sigma \cdot [p,V])}{(2m)^2}\right]\psi_+$$

$$E_S\psi_+ = \left[\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} - \tfrac{i\sigma \cdot (p \times [p,V])}{4m^2} - \underbrace{\tfrac{p[p,V]}{4m^2}}\right]\psi_+ = H\psi_+$$
$$\text{(isn't Hermitian)}$$

Right column:

$$1 = \int \|\psi_+\|^2 + \|\psi_-\|^2 \, d^3 r$$

$$= \int \|\psi_+\|^2 + \left\|\tfrac{\sigma \cdot p}{E + m - V}\psi_+\right\|^2 d^3 r$$

$$\approx \int \|\psi_+\|^2 + \left\|\tfrac{\sigma \cdot p}{2m}\psi_+\right\|^2 d^3 r$$

$$= \int \psi_+^\dagger \left(1 + \tfrac{p^2}{4m^2}\right)\psi_+ \, d^3 r$$

$$\approx \left\langle \left(1 + \tfrac{p^2}{8m^2}\right)\psi_+ \,\middle|\, \left(1 + \tfrac{p^2}{8m^2}\right)\psi_+ \right\rangle$$

$$\equiv \langle \psi_S | \psi_S \rangle$$

Lower section:

$$E_S\left(1 + \tfrac{p^2}{8m^2}\right)^{-1}\psi_S = H\left(1 + \tfrac{p^2}{8m^2}\right)^{-1}\psi_S$$

$$E_S\psi_S = \left(1 + \tfrac{p^2}{8m^2}\right)H\left(1 + \tfrac{p^2}{8m^2}\right)^{-1}\psi_S$$

$$= \left(H + \tfrac{p^2 H}{8m^2}\right)\left(1 - \tfrac{p^2}{8m^2} + \mathcal{O}_{(p^4)}\right)\psi_S$$

$$\approx \left(H + \left[\tfrac{p^2}{8m^2}, H\right]\right)\psi_S \approx \left(H + \left[\tfrac{p^2}{8m^2}, V\right]\right)\psi_S \qquad \text{(2nd order, } v^4)$$

$$E_S\psi_S = \left(\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} - \tfrac{i\sigma \cdot (p \times [p,V])}{4m^2} - \tfrac{p[p,V]}{4m^2} + \tfrac{[p,V]p + p[p,V]}{8m^2}\right)\psi_S$$

$$= \left(\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} - \tfrac{i\sigma \cdot (p \times [p,V])}{4m^2} - \tfrac{[p,[p,V]]}{8m^2}\right)\psi_S$$

$$= \boxed{(H_S + H_{\text{rel.}} + H_{\text{so}} + H_{\text{darwin}})\psi_S}$$

$$= \left(\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} - \tfrac{1}{4m^2}\sigma \cdot (p \times \nabla V) + \overbrace{\tfrac{1}{8m^2}\nabla^2 V}^{\text{Darwin} \Rightarrow}\right)\psi_S$$

$$= \left(\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} - \tfrac{1}{2m^2}S \cdot \left[\vec{p} \times \tfrac{qq\vec{r}}{4\pi r^3}\right] + \tfrac{1}{8m^2}[qq\delta^3{}_{(r)}]\right)\psi_S$$

$$= \left(\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} + \underbrace{\tfrac{e^2}{8\pi m^2}\tfrac{S \cdot L}{r^3}}_{l \neq 0} + \underbrace{\tfrac{e^2}{8m^2}\delta^3{}_{(r)}}_{l = 0}\right)\psi_S$$

Right side of lower section:

$$\overline{V(r)} = V_{(r)} + \sum_i \overline{\tfrac{\partial V}{\partial r_i}\delta r_i}$$

$$+ \tfrac{1}{2!}\sum_{ij}\overline{\tfrac{\partial^2 V}{\partial r_i \partial r_j}\delta r_i \delta r_j}$$

$$+ \mathcal{O}(\delta r^3)$$

$$= V_{(r)} + \tfrac{1}{2}(\delta r)^2 \nabla^2 V + ..$$

$$(\delta r \sim \tfrac{\hbar}{mc})$$

Exact Energy Eigenvalues :

$$E_{nj} = mc^2\left[1 + \left(\frac{\alpha}{n - (j + 1/2) + \sqrt{(j + 1/2)^2 - \alpha^2}}\right)^2\right]^{-1/2}$$