1 Maxwell's Equations

Gauss's Law for Electricity (GLE)

$$\boxed{\oiint \vec{E} \cdot d\vec{a} = \frac{Q}{\epsilon_0}}$$

$$\iiint (\nabla \cdot \vec{E})\ dV = \frac{\iiint \rho\ dV}{\epsilon_0}$$

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$

Gauss's Law for Magnetism (GLM)

$$\boxed{\oiint \vec{B} \cdot d\vec{a} = 0}$$

$$\iiint (\nabla \cdot \vec{B})\ dV = 0$$

$$\boxed{\nabla \cdot \vec{B} = 0}$$

Faraday's Law of Induction (FLI)

$$\boxed{\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}}$$

$$\iint (\nabla \times \vec{E}) \cdot d\vec{a} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{a}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Maxwell-Ampere's Law (MAL)

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I + \epsilon_0 \frac{d\Phi_E}{dt} \right)}$$

$$\iint (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \iint \vec{J} \cdot d\vec{a} + \mu_0 \epsilon_0 \frac{d}{dt} \iint \vec{E} \cdot d\vec{a}$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)}$$

Lorentz Force Law (LFL)

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}}$$

Conservation of Charge (COC)

$$\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}}$$

1.1 Electrostatic/Magnetostatic Examples

Using GLE

1. Point Charges

$\vec{E} = E(r)\hat{r}$

$$\begin{aligned}\frac{Q}{\epsilon_0} &= \oiint E(r)\hat{r}\cdot d\vec{a}\\ &= E(r)\hat{r}\cdot \oiint r^2 \sin\phi\ |d\vec{\phi}\times d\vec{\theta}|\\ &= E(r)\hat{r}\cdot 4\pi r^2\hat{r}\end{aligned}$$

$$\vec{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \ \Rightarrow\ \frac{1}{4\pi\epsilon_0}\sum_i \frac{Q_i}{|\vec{\mathscr{r}}_i|^2}\hat{\mathscr{r}}_i$$

Coulomb's Law (CL):

$$\boxed{\vec{E}(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{dQ}{|\vec{\mathscr{r}}|^2}\hat{\mathscr{r}} \qquad \vec{\mathbf{r}}\in\mathbb{R}^3,\ \ \vec{\mathscr{r}} = \vec{\mathbf{r}} - \vec{l}'}$$

$$\boxed{\vec{F}(\vec{\mathbf{r}}) \ = \ q\vec{E}}$$

Using MAL

Biot-Savart Law (BSL):
(see potential, $\vec{A}$, for derivation)

$$\boxed{\begin{aligned}\vec{B}(\vec{\mathbf{r}}) &= k_\mu \int \frac{\vec{J}dV\times\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}, \qquad \vec{\mathbf{r}}\in\mathbb{R}^3,\quad \vec{\mathscr{r}} = \vec{\mathbf{r}} - \vec{l}'\\ &= k_\mu \int \frac{I(\vec{l}')\ d\vec{l}'\times\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}\end{aligned}}$$

$$\boxed{\vec{F} = q\hat{v}\times\vec{B}}$$

1. Infinite Line w/ Steady Current (SC)

$\vec{B} = B(r)\hat{\theta}$

Use MAL

$$\begin{aligned}\mu_o I &= \oint B(r)\hat{\theta}\cdot d\vec{L}\\ &= B(r)\hat{\theta}\cdot\oint r d\vec{\theta}\\ &= B(r)\hat{\theta}\cdot 2\pi r\hat{\theta}\end{aligned}$$

$$\vec{B}(r) = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$

cont.

2. Sphere w/ Constant Charge Density (CCD)

Let R be the radius.

$$\vec{E}(r) = \frac{Q(r)}{4\pi\epsilon_0 r^2}\hat{r}$$

$$Q(r) = \begin{cases} Q & (r > R) \\ Q_{r<R} = \iiint_0^r \frac{dQ}{dV}\ dV & (r < R) \end{cases}$$

- *Conductor*

$$Q_{r<R} = 0 \quad \Rightarrow \quad \vec{E}(r) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & (r > R) \\ 0 & (r < R) \end{cases}$$

- *Insulator w/ CCD and $\epsilon = \epsilon_0$*

$Q_{r<R} = \int \frac{Q}{V}\ dV = Q\frac{\int dV}{V} = Q\frac{r^3}{R^3}$

$$\Rightarrow \vec{E}(r) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & (r > R) \\ \\ \dfrac{Qr}{4\pi\epsilon_0 R^3}\hat{r} & (r < R) \end{cases}$$

Use BSL

$$\vec{B}(r) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{z} \times \hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2}$$

$$= \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{d\vec{z} \times \hat{\boldsymbol{\mathscr{r}}}}{r^2 + z^2}$$

$$= \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{d\vec{z} \times (r\hat{r} - z\hat{z})}{(r^2 + z^2)^{3/2}}$$

$$= \frac{\mu_0 I r}{4\pi}\int_{-\infty}^{\infty} \frac{dz}{(r^2 + z^2)^{3/2}}\ \hat{\theta}$$

$$= \frac{\mu_0 I r}{4\pi} \left.\frac{z}{r^2\sqrt{r^2 + z^2}}\right|_{-\infty}^{\infty} \hat{\theta}$$

$$= \frac{\mu_0 I}{2\pi r}\hat{\theta}$$

Constant Field Solutions (Capacitor/Solenoid)

3. Two Infinite Parallel Planes Capacitor w/ CCD ($+Q, -Q$)

$\vec{E} = E(z)\hat{z}$

$$\begin{aligned}\frac{Q}{\epsilon_0} &= \oiint E(z)\hat{z} \cdot d\vec{a} \\ &= E(z)\hat{z} \cdot \oiint xy\ |d\vec{x} \times d\vec{y}| \\ &= E(z)\hat{z} \cdot xy\hat{z}\end{aligned}$$

$$\vec{E}(z) = \frac{1}{\epsilon_0}\frac{dQ}{dA}\hat{z} = \frac{\sigma}{\epsilon_0}\hat{z}$$

4. One Infinite Plane w/ CCD

$\vec{E} = E(z)\hat{z}$

$$\begin{aligned}\frac{Q}{\epsilon_0} &= \oiint E(z)\hat{z} \cdot d\vec{a} \\ &= E(z)\hat{z} \cdot 2\oiint xy\ |d\vec{x} \times d\vec{y}| \\ &= E(z)\hat{z} \cdot 2xy\hat{z}\end{aligned}$$

$$\vec{E}(z) = \frac{1}{2\epsilon_0}\frac{dQ}{dA}\hat{z} = \frac{\sigma}{2\epsilon_0}\hat{z}$$

2. Infinite Long Solenoid Coil w/ SC

Let R be the coil radius. $\vec{B} = B(r)\hat{z} = B(r < R)\hat{z}$

$$\begin{aligned}\mu_o NI &= \oint B(r)\hat{z} \cdot d\vec{L} \\ &= (B_{r<R})\hat{z} \cdot \oint d\vec{L} \\ &= (B_{r<R})\hat{z} \cdot L\hat{z}\end{aligned}$$

$$\vec{B}(r) = \begin{cases} \frac{\mu_0 IN}{L}\hat{z} = \mu_0 I n_l\ \hat{z} & (0 < r < R) \\ 0 & (r > R) \end{cases}$$

3. Closed, Thin Solenoid Ring w/ SC

Let R_l be the ring radius and R_c be the coil radius.

$\vec{B} \ = \ B(r)\hat{\theta} \ = \ B(_{R_l - R_c < r < R_l + R_c})\hat{\theta} \ \approx \ B(R_l)\hat{\theta}$

$$\begin{aligned}\mu_o NI &= \oint B(r)\hat{\theta} \cdot d\vec{L} \\ &= B(r)\hat{\theta} \cdot 2\pi R_l\ \hat{\theta}\end{aligned}$$

$$\vec{B}(r) = \begin{cases} \dfrac{\mu_0 IN}{2\pi R_l}\hat{\theta} & (R_{l-c} < r < R_{l+c}) \\ 0 & \text{else} \end{cases}$$

Integrate w/ CFL and BSL

5. Ring w/ CCD centered at origin O

Let R be the ring radius, λ the charge density, and ϕ be $\angle OzR$.

$$E(0,0,z)\hat{z} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}^2}\vec{\mathscr{r}}$$

$$\vec{E}(0,0,z) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(Rd\theta)}{\mathscr{r}^2}\vec{\mathscr{r}}$$

$$= \frac{k\lambda}{z^2+R^2}\int_0^{2\pi} Rd\theta \;\cos\phi\;\hat{z}$$

$$= \boxed{\frac{k\lambda(2\pi R)z}{(z^2+R^2)^{3/2}}\;\hat{z}}$$

$$= \boxed{\frac{k\lambda}{D}\;\frac{2\pi R}{D}\cos\phi\;\hat{z}}$$

4. Ring w/ SC centered at origin O

Let R be the ring radius and ϕ be $\angle O_{\text{rigin}}zR$.

$$B(0,0,z)\hat{z} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}$$

$$\vec{B}(0,0,z) = \frac{\mu_0 I}{4\pi}\int \frac{Rd\vec{\theta}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{k_\mu I}{z^2+R^2}\int Rd\theta\;\hat{\theta}\times\frac{z\hat{z}-R\hat{r}}{\sqrt{z^2+R^2}} \qquad \text{or} \qquad \frac{k_\mu I}{z^2+R^2}\int_0^{2\pi} Rd\theta\;\sin\phi\;\hat{z}$$

$$= \frac{k_\mu IR}{(z^2+R^2)^{3/2}}\left[2\pi R\hat{z} + z\cancel{\int_0^{2\pi}\hat{r}d\theta}\right] \qquad \text{or} \qquad \frac{k_\mu I}{z^2+R^2}\int_0^{2\pi} Rd\theta\;\tfrac{R}{\sqrt{z^2+R^2}}\;\hat{z}$$

$$= \boxed{\frac{k_\mu I(2\pi R)R}{(z^2+R^2)^{3/2}}\;\hat{z}}$$

$$= \boxed{\frac{k_\mu I}{D}\;\frac{2\pi R}{D}\sin\phi\;\hat{z}}$$

6. Line Charge w/ CCD and one edge at the origin O

Let L be the line length, λ the charge density, and θ_0 be $\angle OyL$.

$$E_y(0,0,y)\hat{y} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}^2}\vec{\mathscr{r}}_y$$

$$\begin{aligned}
E_y(0,0,y) &= k\int \frac{\lambda dx}{\mathscr{r}^2}\vec{\mathscr{r}}_y \\
&= k\lambda \int \frac{dx}{y^2+x^2}\cos\theta \quad , \qquad x = y\tan\theta \\
&= \frac{k\lambda}{y^2}\int_0^{\theta_0} \frac{y\sec^2\theta}{1+\tan^2\theta}\cos\theta \\
&= \frac{k\lambda}{y}\int_0^{\theta_0}\cos\theta = \frac{k\lambda}{y}\sin\theta_0 \\
&= \boxed{\frac{k\lambda L}{y\sqrt{y^2+L^2}}} \quad = \quad \boxed{\frac{k\lambda}{y}\sin\theta_0}
\end{aligned}$$

Use this result to find *Finite Line Charge* and *Square Ring.*

5. Finite Wire w/ SC and one edge at the origin O

Let L be the wire length and θ_0 be $\angle OyL$.

$$B_z(0,0,y)\hat{z} = k_\mu I\int \frac{d\vec{l}\times\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}$$

$$\begin{aligned}
B_z(0,0,y)\hat{z} &= k_\mu I\int \frac{d\vec{x}\times\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2} \\
&= k_\mu I \int \frac{dx}{y^2+x^2}\sin(\theta+90) \quad , \qquad x = y\tan\theta \\
&= \frac{k_\mu I}{y^2}\int_0^{\theta_0} \frac{y\sec^2\theta}{1+\tan^2\theta}\cos\theta \\
&= \frac{k_\mu I}{y}\int_0^{\theta_0}\cos\theta = \frac{k_\mu I}{y}\sin\theta_0 \\
&= \boxed{\frac{k_\mu I L}{y\sqrt{y^2+L^2}}} = \boxed{\frac{k_\mu I}{y}\sin\theta_0}
\end{aligned}$$

Use this result to find *Finite Wire* and *Square Wire.*

1.2 Field Energies

The sum of the work to move a collection of charges considering the potential from each other charge comes out to be

$$\boldsymbol{W = \frac{1}{2}\sum_i q_i V(r_i)}$$

E-field Energy (electrostatic...)

$$E = \frac{1}{2}CV^2 = \frac{1}{2}VQ$$

$$\boxed{W_{\text{vol}} = \frac{1}{2}\iiint V\rho\, d\tau} = \frac{\epsilon_0}{2}\iiint V(\nabla\cdot\vec{E})\, d\tau$$

$$= \frac{\epsilon_0}{2}\iiint \vec{E}\cdot(\vec{E}) + \nabla\cdot(V\vec{E})\, d\tau$$

$$= \boxed{\frac{\epsilon_0}{2}\iiint \vec{E}^2\, d\tau + \frac{\epsilon_0}{2}\iint (V\vec{E})\cdot d\vec{a}}$$

$$\boxed{W_E = \frac{\epsilon_0}{2}\iiint \vec{E}^2\, d\tau} \quad (\text{if } \rho = 0 \text{ at } \infty)$$

B-field Energy

$$E = \frac{1}{2}LI^2 = \frac{1}{2}\Phi_B I = \frac{1}{2}\oint \vec{A}\cdot\vec{I}\, dl$$

$$\boxed{W_{\text{vol}} = \frac{1}{2}\iiint \vec{A}\cdot\vec{J}\, d\tau} = \frac{1}{2\mu_0}\iiint \vec{A}\cdot(\nabla\times\vec{B})\, d\tau$$

$$= \frac{1}{2\mu_0}\iiint \vec{B}\cdot(\vec{B}) - \nabla\cdot(\vec{A}\times\vec{B})\, d\tau$$

$$= \boxed{\frac{1}{2\mu_0}\iiint \vec{B}^2\, d\tau - \frac{1}{2\mu_0}\oint\!\!\!\!\!\!\iint (\vec{A}\times\vec{B})\cdot d\vec{a}}$$

$$\boxed{W_B = \frac{1}{2\mu_0}\iiint \vec{B}^2\, d\tau} \quad (\text{if } \vec{I} = 0 \text{ at } \infty)$$

1.3 Circuits/Ohm's Law

Ohm's Law

In Ohmic material, $\rightarrow$ $\boldsymbol{\sigma}$: Conductivity $J \approx \sigma(E + v \times B)$ $\boxed{J \approx \sigma E}$ $\Rightarrow$ $\boxed{V = IR}$ $\boxed{P = VI}$

Example: Wire w/ Two Plates

$$I = (\sigma E)A = \left(\frac{\sigma A}{L}\right)V \quad \Rightarrow \quad V = I\left(\frac{L}{\sigma A}\right) = IR$$

Resistor:	$V_R = IR$	$P = VI$		$R = \frac{\rho l}{A}$	$Z_R = R$
Capacitor:	$Q = CV_C$	$E_C = \frac{1}{2}CV_C^2$	$Q_\uparrow = Q_0(1 - e^{-t/\tau})$ $Q_\downarrow = Q_0 e^{-t/\tau}$	$C = \frac{\kappa\epsilon_0 A}{d} = \frac{\epsilon A}{d}$	$Z_C = \frac{1}{i\omega C}$
Inductor:	$\Phi_B = LI$ $V_L = -L\frac{dI}{dt}$	$E_L = \frac{1}{2}LI^2$	$I_\uparrow = I_0(1 - e^{-t/\tau})$ $I_\downarrow = I_0 e^{-t/\tau}$	$L = \frac{\mu_0 N^2 A}{l}$	$Z_L = i\omega L$

Open Circuit : $R = \infty$
Short Circuit : $R = 0$

Constants

$\tau_{RC} = RC$

$\tau_{RL} = L/R$

$\omega_{R,LC} = 1/\sqrt{LC}$

AC Filters

Low Pass (Non-Zero $\overline{\text{Probe}}$) :

- $R\overline{C}$
- $L\overline{R}$

High Pass (Non-Zero $\overline{\text{Probe}}$) :

- $C\overline{R}$
- $R\overline{L}$

Band Pass (Zero $\overline{\text{Probe}}$) :

- $R\overline{LC}$
- Bandwidth $= \left(\text{FWHM} = 2\beta = \frac{b}{m}\right) = \frac{R}{L}$

Other Components

$- > \vert -$	Diode	One Way Voltage (if > Bias Voltage)
$= \vert > -$	Op-Amp	$V_1 - V_2 \ \propto \ V_{\text{OA}}$ (Clipping If Too Large V_{OA})
$= \vert)-$	And	
$=) > -$	Or	

De Morgan's Law

- $\overline{A \cdot B} = \overline{A} + \overline{B}$
- $\overline{A + B} = \overline{A} \cdot \overline{B}$

1.4 Quasistatic (FLI)

Force on Wire in B-Field :

$$F = qv \times B$$

$$\boxed{F = LI \times B}$$

EMF :

$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt} = \oint \vec{E} \cdot d\vec{l}}$$

Mutual Inductance:

$$\boxed{\begin{array}{ll} \text{Flux Through } B: & \Phi_B = M \cdot I_A \\ \text{Flux Through } A: & \Phi_A = M \cdot I_B \end{array}}$$

Faraday's Law

1. Lorentz:

Square Circuit with $\vec{v}(t)$ leaving Constant B-Field (out)

I is out

2. Faraday:

Constant B-Field (out) with $-\vec{v}(t)$ leaving Square Circuit

I is out

3. Faraday:

Square Circuit in Increasing B-Field (out)

I is in

Examples:

B-Field Work:

$$\left(\vec{v} \cdot \vec{l} = 0\right) \rightarrow \quad W_B = \int \vec{F} \cdot d\vec{l}$$

$$= \int (q\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\boxed{W_B = 0} \quad \text{(magnetic fields do no work)}$$

Velocity of wire in (1.): $\quad I(t)R = V(t) = \left|\frac{d\Phi_B}{dt}\right| = Bhv(t)$

$$F_B(t) = -hI(t)B$$

$$= -\frac{B^2h^2v(t)}{R}$$

$$F = ma(t)$$

$$\boxed{m\frac{dv}{dt} = F_B + F_{\text{ext}} = F_{\text{ext}} - \frac{B^2h^2v(t)}{R}}$$

2 Potentials and Fields

2.1 Maxwell's Equations for Potentials

1. GLM for Potentials (GLMP)

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \;, \qquad \boldsymbol{\vec{A}' = \vec{A} + \nabla\lambda}$$

$$\boxed{\vec{B} = \nabla \times \vec{A} \;\Rightarrow\; \Phi_B = \oint \vec{A} \cdot d\vec{l}}$$

2. FLI for Potentials (FLIP)

$$\nabla \times \vec{E} = -\tfrac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{E} = \nabla \times \left(0 - \tfrac{\partial \vec{A}}{\partial t}\right)$$

$$\boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}} \;, \qquad \boldsymbol{V' = V + \tfrac{\partial \lambda}{\partial t}}$$

3. GLE for Potentials (GLEP)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$-\nabla^2 V - \frac{\partial(\nabla \cdot \vec{A})}{\partial t} = \frac{\rho}{\epsilon_0}$$

$$\boxed{\Box^2 V - \frac{\partial}{\partial t}(\partial_\mu A^\mu) = \frac{\rho}{\epsilon_0}}$$

4. MAL for Potentials (MALP)

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} + \nabla\left(\frac{1}{c^2}\frac{\partial V}{\partial t} + \nabla \cdot \vec{A}\right) = \mu_0 \vec{J}$$

$$\boxed{\Box^2 \vec{A} + \nabla(\partial_\mu A^\mu) = \mu_0 \vec{J}}$$

Field Energy (see Capacitor/Solenoid in Circuits):

$$\boxed{W_E = \frac{1}{2}\iiint V\rho \; d\tau} = \frac{\epsilon_0}{2}\int E^2 \; d\tau \quad (\text{if } \rho = 0 \text{ at } \infty)$$

$$\boxed{W_B = \frac{1}{2}\iiint \vec{A} \cdot \vec{J} \; d\tau} = \frac{1}{2\mu_0}\int B^2 \; d\tau \quad (\text{if } \vec{J} = 0 \text{ at } \infty)$$

2.2 Cases and Freedoms

GLMP and FLIP say that

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

In the electrostatic case,

$$\boxed{\textbf{Electrostatics:} \quad \begin{aligned} &(\boldsymbol{\nabla} \times \boldsymbol{E} \Leftrightarrow \boldsymbol{\partial_t A} \Leftrightarrow \boldsymbol{\partial_t B}) = \mathbf{0} \\ &\Rightarrow \ \boldsymbol{\partial_t \rho} = \mathbf{0} \end{aligned}}$$

GLEP and MALP say that

$$-\nabla^2 V - \frac{\partial(\nabla \cdot \vec{A})}{\partial t} = \frac{\rho}{\epsilon_0}$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} + \nabla\left(\frac{1}{c^2}\frac{\partial V}{\partial t} + \nabla \cdot \vec{A}\right) = \mu_0 \vec{J}$$

Freedom may be chosen to what $\nabla \cdot \vec{A}$ equals.

$$\boxed{\textbf{Coulomb Gauge:} \quad \boldsymbol{\nabla} \cdot \vec{\boldsymbol{A}} = \mathbf{0}}$$

- $\boxed{\textbf{Magnetostatics:} \quad \boldsymbol{\partial_t E} = \mathbf{0} \ \Rightarrow \ \boldsymbol{\partial_t \vec{J}} = \mathbf{0}}$

$$\boxed{\textbf{Lorenz Gauge:} \quad \boldsymbol{\nabla} \cdot \vec{\boldsymbol{A}} = -\frac{\mathbf{1}}{\boldsymbol{c^2}}\frac{\boldsymbol{\partial V}}{\boldsymbol{\partial t}} \ \Leftrightarrow \ \boldsymbol{\partial_\mu A^\mu} = \mathbf{0}}$$

In general, $\vec{A}$ and V can be [gauge] transformed while keeping $\vec{E}$ and $\vec{B}$ the same by

$$\boxed{V' = V - \frac{d\lambda}{dt}} \qquad (\lambda \text{ is a scalar function})$$

$$\boxed{\vec{A}' = \vec{A} + \nabla\lambda}$$

Electrostatic Potentials

Electrostatics: $\left(\boldsymbol{\partial_t \vec{A} = 0 \;\Rightarrow\; \partial_t \rho = 0}\right)$.

Using GLE to find E or using FLIP,

$$\nabla \times \vec{E} = 0 \;\Rightarrow \begin{cases} \oint \vec{E} \cdot d\vec{l} = 0 \\ \boxed{\vec{E} = -\nabla V} \end{cases}$$

$\Rightarrow$

$$\int_a^b \nabla \vec{V} \cdot d\vec{l} = \boxed{V(\vec{\mathbf{r}})\Big|_a^b = -\int_a^b \vec{E} \cdot d\vec{l} = W_E/q}$$

$$\boxed{V(\vec{\mathbf{r}}) = -\int \vec{E} \cdot d\vec{l} + V_0}$$

and from this (or GLEP)

$$\boxed{\begin{array}{ll} \textit{Poisson Equation:} & \boldsymbol{\nabla^2 V = -\dfrac{\rho(\vec{r'})}{\epsilon_0}} \\ \rho_\infty = 0 \;\Rightarrow\; \boldsymbol{\vec{V}}(\vec{\mathbf{r}}) \boldsymbol{= \dfrac{1}{4\pi\epsilon_0} \displaystyle\int \dfrac{\rho(\vec{r'})}{\mathscr{r}}\, d\tau'} & \end{array}}$$

Coulomb Gauge & Magnetostatic Potentials

Coulomb Guage: Choose $\left(\boldsymbol{\nabla \cdot \vec{A} = 0}\right)$

Using GLEP,

$$\boxed{\begin{array}{ll} \textit{Poisson Equation:} & \boldsymbol{\nabla^2 V = -\dfrac{\rho(\vec{r'}, t)}{\epsilon_0}} \\ \rho_\infty = 0 \;\Rightarrow\; \boldsymbol{V}(\vec{\mathbf{r}}) \boldsymbol{= \dfrac{1}{4\pi\epsilon_0} \displaystyle\int \dfrac{\rho(\vec{r'}, t)}{\mathscr{r}}\, d\tau'} & \end{array}}$$

If charges move, V updates immediately - not at light speed. Only $\vec{E}$ can be physically measured, and updates at light speed. $\vec{A}$ is difficult to find using MALP except for special cases like Magnetostatics.

As always, GLMP says

$$\boxed{\boldsymbol{\Phi_B = \int \vec{A} \cdot d\vec{l}}}$$

Magnetostatics: $\left(\boldsymbol{\partial_t \vec{E} = 0 \;\Rightarrow\; \partial_t \vec{J} = 0}\right)$

Using MALP,

$$\boxed{\begin{array}{ll} \textit{Poisson Equation:} & \boldsymbol{\nabla^2 \vec{A} = -\mu_0 \vec{J}(\vec{r'})} \\ \vec{J}_\infty = 0 \;\Rightarrow\; \boldsymbol{\vec{A}}(\vec{\mathbf{r}}) \boldsymbol{= k_\mu \displaystyle\int \dfrac{\vec{J}(\vec{r'})}{\mathscr{r}}\, d\tau'} & \end{array}}$$

2.2.1 Potential Examples

1. Point Charges

Reference Choice: $V(\infty) = 0$

$$V(r) = -\int_{\infty}^{r} \frac{kQ}{(r')^2} dr' = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

$$\boldsymbol{V(\vec{\mathbf{r}}) \;=\; \frac{1}{4\pi\epsilon_0}\sum_i \frac{Q_i}{\mathscr{r}_i}}$$

Coulomb Potential: $\boxed{V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|\vec{\mathscr{r}}_i|}}$

Work: $\boxed{W = \frac{1}{2}\sum q_i V(\vec{\mathbf{r}}_i)}$

1.

2.

3.

2. Sphere

Reference Choice: $V(\infty) = 0$

Let R be the radius.

$$V(r) = -\int_{\infty}^{r} \vec{E}(r') \cdot d\vec{r'}$$

- *Conductor*

$$E(r) = \begin{cases} \dfrac{kQ}{r^2} \\ 0 \end{cases} \quad \Rightarrow \quad V(r) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r} & (r > R) \\ \dfrac{Q}{4\pi\epsilon_0 R} & (r < R) \end{cases}$$

- *Insulator w/ CCD and $\epsilon = \epsilon_0$*

$$E(r) = \begin{cases} \dfrac{kQ}{r^2} \\ \dfrac{kQr}{R^3} \end{cases} \quad \Rightarrow \quad V(r) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r} & (r > R) \\ \dfrac{Q}{4\pi\epsilon_0 R} + \left.\dfrac{Qr'^2}{8\pi\epsilon_0 R^3}\right|_r^R & (r < R) \end{cases}$$

1.

2.

3.

Charges at ∞

3. (Infinite) Parallel Plate Capacitor

Reference Choice: $V(h) = 0$

Let the Capacitor Height be h

$$V(z) = -\int_h^z \frac{\sigma}{\epsilon_0}\hat{z} \cdot d\vec{z} = \frac{\sigma(h-z)}{\epsilon_0} \qquad (0 \le z \le h)$$

4. (Infinite) Single Plate w/ CCD

Reference Choice: $V(0) = 0$

$$V(z) = -\int_0^z \frac{\sigma}{2\epsilon_0}\hat{z} \cdot d\vec{z} = -\frac{\sigma z}{2\epsilon_0} \qquad (0 \le z < \infty)$$

Try $V(\infty) = 0$. (A charge distribution stretching to infinity DNE, so choose a diff. reference point.)

5. Infinite Line w/ CCD

Reference Choice: $V(1) = 0$

$$\begin{aligned} V(r) &= -\int_1^r \frac{\lambda}{2\pi r\epsilon_0}\hat{r} \cdot d\vec{r} \\ &= -\frac{\lambda}{2\pi\epsilon_0}\ln r \end{aligned}$$

Try $V(\infty) = 0$ (same problem above).

3.

4.

5.

2.3 Multipole Expansion

$$\begin{aligned}\mathscr{r}^2 &= r^2 + (r')^2 - 2(\vec{\mathbf{r}} \cdot \vec{r'}) \\ &= r^2 \left[1 + \frac{r'}{r} \left(\frac{r'}{r} - 2\frac{\vec{\mathbf{r}} \cdot \vec{r'}}{r'r} \right) \right] \\ &= r^2 (1 + \epsilon)\end{aligned} \qquad \Rightarrow \qquad \begin{aligned}\frac{1}{\mathscr{r}} &= \frac{1}{r}(1+\epsilon)^{-1/2} \\ &= \boldsymbol{\frac{1}{r} \sum_{n=0} \left(\frac{r'}{r} \right)^n P_n \left(\hat{r'} \cdot \hat{\mathbf{r}} \right)} \quad \text{(Legendre Polynomials)}\end{aligned}$$

$$\begin{aligned}V(\vec{\mathbf{r}}) &= \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} \rho(\vec{r'}) d\tau' \\ &= \boxed{\frac{1}{4\pi\epsilon_0} \sum_n \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha)\ \rho(\vec{r'}) d\tau'}\end{aligned}$$

$$= \frac{1}{4\pi\epsilon_0} \left[\begin{array}{l} \frac{1}{r} \int \rho(\vec{r'}) d\tau' + \frac{1}{r^2} \int \vec{r'} \cdot \hat{\mathbf{r}}\ \rho(\vec{r'}) d\tau' \\ + \frac{1}{r^3} \int (r')^2 P_2(\hat{r'} \cdot \hat{\mathbf{r}})\ \rho(\vec{r'}) d\tau' + \frac{1}{r^4} \int ... \end{array} \right]$$

$$\boxed{\begin{aligned}V_{\text{mon}} &= \frac{1}{4\pi\epsilon_0 r} \int \rho(\vec{r'}) d\tau' \\ V_{\text{dip}} &= \frac{1}{4\pi\epsilon_0 r^2} \left(\int \vec{r'}\ \rho(\vec{r'}) d\tau' \right) \cdot \hat{\mathbf{r}} = \frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}\end{aligned}}$$

V is the dipole term. **p** is the dipole moment.

$$\begin{aligned}\vec{A}(\vec{\mathbf{r}}) &= k_\mu \int \frac{1}{\mathscr{r}}\ \vec{J}(\vec{r'}) d\tau' \\ &= \boxed{k_\mu \sum_n \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha)\ \vec{J}(\vec{r'}) d\tau'}\end{aligned}$$

$$= k_\mu \left[\begin{array}{l} \frac{1}{r} \int \vec{J}(\vec{r'}) d\tau' + \frac{1}{r^2} \int \vec{r'} \cdot \hat{\mathbf{r}}\ \vec{J}(\vec{r'}) d\tau' \\ + \frac{1}{r^3} \int (r')^2 P_2(\hat{r'} \cdot \hat{\mathbf{r}})\ \vec{J}(\vec{r'}) d\tau' + \frac{1}{r^4} \int ... \end{array} \right]$$

Steady current:

$$\boxed{\begin{aligned}A_{\text{mon}} &= \frac{\mu_0 I}{4\pi r} \oint dl' = 0 \\ A_{\text{dip}} &= \frac{k_\mu}{r^2} I \int \vec{r'} \cdot \hat{\mathbf{r}}\ dl' = \frac{k_\mu}{r^2} I \int d\vec{a'} \times \hat{\mathbf{r}} \\ &= \frac{k_\mu}{r^2} (I\vec{a}) \times \hat{\mathbf{r}} = \frac{\mu_0 \vec{\mathbf{m}} \times \hat{\mathbf{r}}}{4\pi r^2}\end{aligned}}$$

A is the dipole term. **m** is the dipole moment.

Ideal Dipoles

Let dipole (2 charges) $\vec{\mathbf{p}} = p\hat{z} = 2dq\hat{z}$ and centered at the origin.

$\lim d \to 0,\ q \to \infty$:

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{d^n P_n(\cos\alpha)q + (-d)^n P_n(\cos\alpha)(-q)}{4\pi\epsilon_0\ r^{n+1}}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{P_n(\cos\alpha)qd^n}{r^{n+1}}[1+(-1)^n]$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{m=0}^{\infty}\frac{P_m(\cos\alpha)}{r^{2m+2}}(2qd)d^{2m}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{m=0}^{\infty}\frac{P_m(\cos\alpha)}{r^{2m+2}}pd^{2m}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{P_0(\cos\alpha)}{r^2}p + 0 + 0 + ...$$

$$\boxed{\text{Ideal Dip:}\qquad V_{\text{dip}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{\mathbf{p}}\cdot\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^{\,2}}}$$

Let dipole (ring) $\vec{\mathbf{m}} = m\hat{z} = Ia\hat{z}$ and centered at the origin.

$\lim a = \pi d^2 \to 0,\ I \to \infty$:

$$\vec{A}(\vec{\mathbf{r}}) = k_\mu \sum_{n=0}^{\infty}\frac{Id^n}{r^{n+1}}\int P_n(\cos\alpha)\ d\vec{l'}$$

$$= k_\mu \left(\begin{array}{c} \frac{I}{r}\int d\vec{l'} + \frac{I}{r^2}\int (d\hat{r'}\cdot\hat{\mathbf{r}})\ d\vec{l'} \\ + \frac{Id^2}{r^3}\int\left[\frac{3}{2}\left(1 + 2\frac{d\hat{r'}\cdot\hat{\mathbf{r}}}{d} + 1\right) - \frac{1}{2}\right] d\vec{l'} \\ + Id^2\sum_{n=3}^{\infty}\frac{d^{n-2}}{r^{n+1}}\int P_n(\hat{r'}\cdot\hat{\mathbf{r}})\ d\vec{l'} \end{array} \right)$$

$$= k_\mu\left(0 + \frac{I\pi d^2}{r^2}(\hat{z}\times\hat{\mathbf{r}}) + \frac{3I\pi d^3}{r^3}(\hat{z}\times\hat{\mathbf{r}}) + \frac{m}{\pi}(0+...)\right)$$

$$= k_\mu\left(0 + \frac{m}{r^2}(\hat{z}\times\hat{\mathbf{r}}) + 0 + 0 + ...\right)$$

$$\boxed{\text{Ideal Dip:}\qquad \vec{A}_{\text{dip}}(\vec{\mathbf{r}}) = k_\mu\frac{\vec{\mathbf{m}}\times\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^{\,2}}}$$

2.3.1 Multipole Examples

3.

4.

5.

3.

4.

5.

3 Electrodynamics in Matter

3.1 Ideal Dipoles

$$\boxed{\vec{\mathbf{p}} = \int r' \cdot \rho(r')\ d\tau'}$$

$$\vec{F}_{\text{dip}} = qE\Big|_{\vec{r}}^{\vec{r}+\vec{d}} = q\Delta\vec{E}$$

$$\approx \left[q\sum_i \left(\nabla E_i \cdot \vec{d}\right)\hat{i}\right]$$

$$\boxed{\vec{F}_{\text{dip}} = (\vec{\mathbf{p}}\cdot\nabla)\vec{E}}$$

$$U_{\text{ES dip}} = qV\Big|_{\vec{r}}^{\vec{r}+\vec{d}} = q\Delta V$$

$$= q\int_{\vec{r}}^{\vec{r}+\vec{d}} -\vec{E}\cdot d\vec{l}$$

$$\boxed{U_{\text{ES dip}} = -\vec{\mathbf{p}}\cdot\vec{E}}$$

$$\vec{N}_{\text{center}} = r\times F = \vec{d}\times q\vec{E}$$

$$\boxed{\vec{N}_{\text{dip}} = \vec{\mathbf{p}}\times\vec{E}}$$

$$\boxed{\textbf{Polarization:}\quad \vec{P} = \frac{d\vec{\mathbf{p}}}{d\tau}}$$

$$\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \nabla'\frac{1}{\mathscr{r}}\right)$$

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\int_\nu \frac{\vec{P}(\vec{r'})\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\ d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int_\nu \frac{-\nabla'\cdot\vec{P}(\vec{r'})}{\mathscr{r}}\ d\tau' + \frac{1}{4\pi\epsilon_0}\int_S \frac{\vec{P}(\vec{r'})\cdot\hat{n}}{\mathscr{r}}\ da'$$

$$\boxed{\rho_b = -\nabla\cdot\vec{P}} \qquad \boxed{\sigma_b = \vec{P}\cdot\hat{n}}$$

$$\boxed{\vec{\mathbf{m}} = \sum I\vec{a}}$$

$$\vec{F}_{\text{sqr. dip}} = q\vec{v}\times\vec{B}$$

$$= \pm IL\vec{x}\times B\hat{z}$$

$$= \pm\ ILB\ \hat{y}$$

$$\vec{N}_{\text{sqr. dip}} = 2\left[\frac{\pm\vec{W}}{2}\times\pm ILB\hat{y}\right]$$

$$= I(LW)\sin\theta B\ \hat{x}$$

$$\boxed{\vec{N}_{\text{dip}} = \vec{\mathbf{m}}\times\vec{B}}$$

$$\boxed{\vec{F}_{\text{dip}} = \nabla(\vec{\mathbf{m}}\cdot\vec{B})}\,???$$

$$\boxed{U_{\text{dip}} = -\vec{\mathbf{m}}\cdot\vec{B}}$$

$$\boxed{\textbf{Magnetization:}\quad \vec{M} = \frac{d\vec{\mathbf{m}}}{d\tau}}$$

$$\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \nabla'\frac{1}{\mathscr{r}}\right)$$

$$\vec{A}(\vec{\mathbf{r}}) = k_\mu\int_\nu \frac{\vec{M}(\vec{r'})\times\hat{\mathscr{r}}}{\mathscr{r}^2}\ d\tau'$$

$$= k_\mu\int_\nu \frac{\nabla'\times\vec{M}(\vec{r'})}{\mathscr{r}}\ d\tau' + k_\mu\int_S \frac{\vec{M}(\vec{r'})\times\hat{n}}{\mathscr{r}}\ da'$$

$$\boxed{\vec{J}_b = \nabla\times\vec{M}} \qquad \boxed{\vec{K}_b = \vec{M}\times\hat{n}}$$

3.2 Maxwell's Equations in Matter

GLE in Matter (GLEM)

$$\nabla \cdot \epsilon_0 \vec{E} = \rho = \rho_b + \rho_f$$
$$= -\nabla \cdot \vec{P} + \nabla \cdot D$$

$$\nabla \cdot \left(\epsilon_0 \vec{E} + \vec{P}\right) = \nabla \cdot D$$

$$\boxed{\begin{array}{ll} \multicolumn{2}{c}{\vec{D} = \epsilon_0 \vec{E} + \vec{P}} \\ & -\nabla \cdot \vec{P} = \rho_b \\ \nabla \cdot \vec{D} = \rho_f & \\ & \vec{P} \cdot \hat{n} = \sigma_b \end{array}}$$

COC in Matter (COCM)

$$\nabla \cdot \vec{J}_p = -\frac{\partial \rho_b}{\partial t}$$
$$= \frac{\partial}{\partial t}\left(\nabla \cdot \vec{\mathbf{P}}\right)$$

$$\boxed{\frac{\partial \vec{\mathbf{P}}}{\partial t} = \vec{J}_p}$$

MAL in Matter (MALM)

$$\nabla \times \frac{1}{\mu_0}\vec{B} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{J}_b + \vec{J}_f + \vec{J}_p + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$
$$= \nabla \times M + \vec{J}_f + \frac{\partial \vec{\mathbf{P}}}{\partial t} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \left(\frac{1}{\mu_0}\vec{B} - M\right) = \vec{J}_f + \frac{\partial}{\partial t}\left(\epsilon_0 \vec{E} + \vec{\mathbf{P}}\right)$$

$$\boxed{\begin{array}{ll} \multicolumn{2}{c}{\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}} \\ & \nabla \times \vec{M} = \vec{J}_b \\ \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} & \\ & \vec{M} \times \hat{n} = \vec{K}_b \end{array}}$$

Faraday's Law of Induction (FLI)

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$$\boxed{\text{Electrostatics:} \quad \nabla \times \vec{D} = \nabla \times \vec{P}}$$

Gauss's Law for Magnetism (GLM)

$$\boxed{\nabla \cdot \vec{B} = 0}$$

$$\boxed{\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}}$$

3.3 Linear Matter

Electric Susceptibility: χ_e

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

Susceptibility Tensor:

$$\vec{P} = \begin{pmatrix} \chi_{e_{xx}} & \chi_{e_{xy}} & \chi_{e_{xz}} \\ \chi_{e_{yx}} & \chi_{e_{yy}} & \chi_{e_{yz}} \\ \chi_{e_{zx}} & \chi_{e_{zy}} & \chi_{e_{zz}} \end{pmatrix} \epsilon_0 \vec{E}$$

Relative Permittivity: $\epsilon_r = 1 + \chi_e$

$$\begin{aligned} \vec{D} &= (1 + \chi_e)\epsilon_0 \vec{E} \\ &= \epsilon_r \epsilon_0 \vec{E} \\ &= \epsilon \vec{E} \end{aligned}$$

Magnetic Susceptibility: χ_m

$$\vec{M} = \chi_m \vec{H}$$

Susceptibility Tensor:

$$\vec{M} = \begin{pmatrix} \chi_{m_{xx}} & \chi_{m_{xy}} & \chi_{m_{xz}} \\ \chi_{m_{yx}} & \chi_{m_{yy}} & \chi_{m_{yz}} \\ \chi_{m_{zx}} & \chi_{m_{zy}} & \chi_{m_{zz}} \end{pmatrix} \vec{H}$$

Bound Current:

$$\begin{aligned} \vec{J}_b &= \nabla \times \left(\chi_m \vec{H}\right) \\ &= \chi_m \left(\vec{J}_f + \partial_t \vec{D}\right) \end{aligned}$$

Relative Permeability: $\mu_r = 1 + \chi_m$

$$\begin{aligned} \vec{B} &= (1 + \chi_m)\mu_0 \vec{H} \\ &= \mu_r \mu_0 \vec{H} \\ &= \mu \vec{H} \end{aligned}$$

4 Boundary Conditions

$$\boxed{\Delta\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

1. $\boxed{\Delta E_{\parallel} = 0}$

$$\oint \vec{E}\cdot d\vec{L} = -\oiint_{0-}^{0+} \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a}$$

$$(E_{\parallel}^{+} - E_{\parallel}^{-})L = 0$$

2. $\boxed{\Delta E_{\perp} = \frac{\sigma}{\epsilon_0}}$ $\boxed{\Delta D_{\perp} = \sigma_f}$

$$\oiint \vec{E}\cdot d\vec{a} = Q/\epsilon_0$$

$$(E_{\perp}^{+} - E_{\perp}^{-})a = \frac{\sigma a}{\epsilon_0}$$

Electrostatics: $\nabla\times\vec{E} = 0$

$\boxed{\Delta V = 0}$

$$V\Big|_{0-}^{0+} = -\int_{0-}^{0+}\vec{E}\cdot dL$$

$\boxed{\Delta\frac{\partial V}{\partial n} = -\frac{\sigma}{\epsilon_0}}$

$$\Delta(\nabla V)\cdot\hat{n}$$

$\boxed{\Delta\vec{D}_{\parallel} = \Delta\vec{P}_{\parallel}}$

$$\nabla\times\vec{D} = \nabla\times\vec{P}$$

$$\boxed{\Delta\vec{B} = \mu_0\vec{K}\times\hat{n}}$$

1. $\boxed{\Delta B_{\perp} = 0}$

$$\oiint \vec{B}\cdot d\vec{a} = 0$$

$$(B_{\perp}^{+} - B_{\perp}^{-})a = 0$$

2. $\boxed{\Delta\vec{B}_{\parallel} = \mu_0\vec{K}\times\hat{n}}$ $\boxed{\Delta\vec{H}_{\parallel} = \vec{K}_f\times\hat{n}}$

$$\oint \frac{\vec{B}}{\mu_0}\cdot dL = \oiint_{0-}^{0+}\left(\vec{J} + \frac{\epsilon_0\partial\vec{E}}{\partial t}\right)\cdot d\vec{a}$$

$$(B_{\parallel}^{+} - B_{\parallel}^{-})L = \mu_0 I_{enc}$$

$$\Delta B_{\parallel}L = \mu_0 KL = (\mu_0\vec{K}\times\hat{n})\cdot\vec{L}$$

$\boxed{\Delta A_{\parallel} = 0}$

$$\oint \vec{A}\cdot d\vec{l} = \Phi_B = 0$$

Magnetostatic: $\nabla\cdot\vec{A} = 0$

$\boxed{\Delta A_{\perp} = 0}$

$$\oiint_{0-}^{0+} \vec{A}\cdot d\vec{a} = 0$$

$\boxed{\Delta\frac{\partial\vec{A}}{\partial n} = -\mu_0\vec{K}}$

$$\Delta(\nabla\times\vec{A}) = \left(-\frac{\partial A_y^{+}}{\partial z} + \frac{\partial A_y^{-}}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x^{+}}{\partial z} - \frac{\partial A_x^{-}}{\partial z}\right)\hat{y}$$

$$= -\mu_0 K\hat{y}$$

5 Work-Energy, Radiation, and Momentum

The sum of the work to move a collection of charges considering the potential from each other charge comes out to be

$$\boldsymbol{W = \frac{1}{2}\sum_i q_i V(r_i)}$$

5.1 Field Energies

$$W_E = \frac{\epsilon_0}{2}\int \vec{E}^2 d\tau \qquad W_B = \frac{1}{2\mu_0}\int \vec{B}^2 d\tau$$

$$U_{EB} = \frac{1}{2}\int \epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2 \; d\tau = \frac{1}{2}\int u_{EB} \; d\tau$$

5.2 Energy Conservation

$$\boxed{\textit{Poynting Vector:} \quad \vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{2\mu_0}\mathrm{Re}(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*)}$$

$$\boxed{I = \langle P_{\text{ow}}/A\rangle = \langle S\rangle = \frac{1}{2}c\epsilon_0 E^2}$$

$$-P_{\text{ow}} = \frac{dW}{dT} + \frac{dU_{EB}}{dt} = -\int \vec{S}\cdot d\vec{a}$$

$$\frac{d}{dt}(u_{mech} + u_{EB}) = -\nabla\cdot\vec{S}$$

5.3 Radiation

<u>Accelerating Charge</u>

Larmor Formula $(v \ll c)$: $P_{\text{ow}} = \left(\frac{2k_\epsilon}{3c^3}\right) q^2 a^2$

<u>Electric Dipole Radiation</u>

Dipole Moment : $\vec{\mathbf{p}}(t) = p_0 \cos(\omega t)\hat{z}$

Intensity : $\langle S\rangle = \left(\frac{k_\epsilon}{8\pi c^3}\right) p_0^2\omega^4 \; \frac{\sin^2\theta}{r^2}$

Power : $\langle P\rangle_E = \left(\frac{k_\epsilon}{3c^3}\right) p_0^2\omega^4$

<u>Magnetic Dipole Radiation</u>: $\langle P\rangle_B = \left(\frac{k_\mu}{3c^3}\right) m_0^2\omega^4$

5.4 Momentum Conservation

$$\left(a \cdot \overleftrightarrow{T}\right)_i = \sum_n a_n T_{in}$$

$$T_{ij} = \epsilon_0 \left(E_i E_j - \tfrac{1}{2}\delta_{ij} E^2\right) + \tfrac{1}{\mu_0}\left(B_i B_j - \tfrac{1}{2}\delta_{ij} B^2\right)$$

$$\begin{aligned} F_i &= \int f_i d\tau \\ &= \oint_S \left(\overleftrightarrow{T} \cdot da\right)_i - \frac{1}{c^2}\frac{d}{dt}\int_V S d\tau \end{aligned}$$

$$\begin{aligned} \mathbf{f}_i &= \epsilon_0 E_i + (\vec{J} \times \vec{B})_i \\ &= \left(\nabla \cdot \overleftrightarrow{T}\right)_i - \frac{1}{c^2}\frac{\partial \vec{S}_i}{\partial t} \end{aligned}$$

$$\frac{dP_{mech}}{dt} = \int_V \left(\nabla \cdot \overleftrightarrow{T}\right)_i d\tau - \frac{dP_{EM}}{dt}$$

$$\frac{\partial}{\partial t}(p_{mech})_i = \left(\nabla \cdot \overleftrightarrow{T}\right)_i - \frac{\partial}{\partial t}(p_{EM})_i$$

$$\frac{d}{dt}(P_{mech} + P_{EM})_i = \int_V \left(\nabla \cdot \overleftrightarrow{T}\right)_i d\tau$$

$$\frac{\partial}{\partial t}(p_{mech} + p_{EM})_i = \left(\nabla \cdot \overleftrightarrow{T}\right)_i$$

$$L_{EM} = \vec{r} \times P_{EM}$$

$$l_{EM} = \vec{r} \times p_{EM}$$

6 Potentials in Lorenz Gauge (nonstatic sources)

See Potentials for Recap

If choose $\left(\boldsymbol{\nabla} \cdot \vec{\boldsymbol{A}} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \ \Leftrightarrow \ \partial_\mu A^\mu = 0\right)$

$$\boxed{\begin{aligned} \Box^2 V &= \frac{\rho}{\epsilon_0} \\ \Box^2 \vec{A} &= \mu_0 \vec{J} \end{aligned}}$$

Solutions satisfying these three equations (thus satisfying Maxwell's Eq.) are,

$$\boxed{\begin{aligned} V(\vec{\mathrm{r}}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'}, t_r)}{\mathscr{r}}\, d\tau' \\ \vec{A}(\vec{\mathrm{r}}, t) &= k_\mu \int \frac{\vec{J}(\vec{r'}, t_r)}{\mathscr{r}}\, d\tau' \end{aligned}}$$

where $t_r = t - \dfrac{\mathscr{r}}{c}$.

Notice that charges move, V and $\vec{A}$ update at the speed of light.
$t_r = t + \frac{\mathscr{r}}{c}$ is also a solution, though not physically real.

Using GLMP and FLIP to find the fields,

Jefimenko Equations:

$$\tilde{\mathbf{E}}(\tilde{\mathbf{r}}, \mathbf{t}) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho(\tilde{\mathbf{r}}, \mathbf{t_r})}{\mathscr{r}^2}\,\hat{\mathscr{r}} + \frac{\dot{\rho}(\tilde{\mathbf{r}}, \mathbf{t_r})}{\mathbf{c^2}}\,\hat{\mathscr{r}} + \frac{\tilde{\mathbf{J}}(\tilde{\mathbf{r}}, \mathbf{t_r})}{\mathbf{c^2}\,\mathscr{r}} \right] \mathbf{d}\tau'$$

$$\tilde{\mathbf{B}}(\tilde{\mathbf{r}}, \mathbf{t}) = \mathbf{k}_\mu \int \left[\frac{\tilde{\mathbf{J}}(\tilde{\mathbf{r}}, \mathbf{t_r})}{\mathscr{r}^2} + \frac{\dot{\tilde{\mathbf{J}}}(\tilde{\mathbf{r}}, \mathbf{t_r})}{\mathbf{c}\,\mathscr{r}} \right] \times \hat{\mathscr{r}}\,\mathbf{d}\tau'$$

It's usually easier solve for the potentials first instead of fields directly. In the electrostatic and magnetostatic limits, CL and BSL are recovered.

7 EM Waves

$$f(z,t) = \text{Re}[\tilde{f}(z,t)] = \text{Re}\left[Ae^{i(kz-wt+\delta)}\right]$$

ω is the same throughout! (?)

$\frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$

$\tilde{\mathbf{f}}(\mathbf{z},\mathbf{t};\delta=\mathbf{0}) : \tilde{A}_I e^{i(k_1 z - wt)} + \tilde{A}_R e^{i(-k_1 z - wt)} \Rightarrow \tilde{A}_T e^{i(k_2 z - wt)}$

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T; \quad k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T$$

$$\tilde{A}_R e^{i\delta_R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{A}_I e^{i\delta_I}; \quad \tilde{A}_T e^{i\delta_T} = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{A}_I e^{i\delta_I}$$

$$A_R = \left(\frac{|v_2 - v_1|}{v_2 + v_1}\right) A_I; \quad A_T = \left(\frac{2v_2}{v_2 + v_1}\right) A_I$$

7.1 Vacuum, $\vec{v}_{||}\vec{E}_{||}\hat{z}$

$$\tilde{B}_0 = \frac{k}{w}(\hat{z} \times \tilde{E}_0) = \frac{1}{c}(\hat{z} \times \tilde{E}_0)$$

$$\vec{S} = cu_{EM}\hat{z} = c\epsilon_0 E_0^2 \cos^2(kw, wt + \delta)\hat{z}$$

$$I_{nt} = \langle S \rangle = \frac{1}{2}c\epsilon_0 E_0^2$$

$$P_{res} = \frac{I_{nt}}{c}$$

7.2 Linear Media

$D = \epsilon E; \quad B = \mu H$

$\tilde{B}_0 = \frac{1}{v}(\hat{z} \times \tilde{E}_0)$

- $\boxed{n = \frac{c}{v}}$
- $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r}$

- $k_I v_1 = k_R v_1 = k_T v_2 = \omega$
- $k_I \sin\theta_I = \big(k_R \sin\theta_R = k_R \sin\theta_I\big) = k_T \sin\theta_T$
- Snell's Law : $\boxed{n_1 \sin\theta_I = n_2 \sin\theta_T}$

<u>Fresnel's Equations Oblique Incidence</u> $\left(\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-(n_1/n_2)^2 \sin{\theta_I}^2}}{\cos\theta_I}\ ,\ \ \beta = \frac{\mu_1 v_1}{\mu_2 v_2} \approx \frac{v_1}{v_2}\right)$

- P-Polarized ($E_\parallel$ to Plane of Incidence):

$$\tilde{E}_R = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_I\ ; \quad \tilde{E}_T = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_I$$

$$R = \frac{I_R}{I_I} = \left(\frac{E_R}{E_I}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_T}{E_I}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

Reflection Shift/Angles $\left(\alpha - \beta \stackrel{?}{=} 0\right)$: $\tan^2\theta_I \stackrel{?}{=} \left(\frac{n_2}{n_1}\right)^2 \frac{1-\beta^2}{1-(n_2/n_1)^2}$

In-Phase ($\delta = 0,\ \alpha > \beta$) : $\tan\theta_I > n_2/n_1$

Out-of-Phase ($\delta = \pi,\ \alpha < \beta$) : $\tan\theta_I < n_2/n_1$

Brewster's Angle ($R = 0$) : $\boxed{\tan\theta_{I=b} = n_2/n_1}$, $\boxed{\theta_R + \theta_T = 90}$

Critical Angle ($T = 0$) : $\boxed{\sin\theta_{I=c} = n_2/n_1}$, $\boxed{\theta_R = 90}$ $(n_1 > n_2)$ (evanescent if $> \theta_c$)

- S-Polarized ($E_\perp$ to Plane of Incidence):

$$\tilde{E}_R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)E_I\ ; \quad \tilde{E}_T = \left(\frac{2}{1+\alpha\beta}\right)E_I$$

$$R = \frac{I_R}{I_I} = \left(\frac{E_R}{E_I}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\alpha\left(\frac{E_T}{E_I}\right)^2 = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2$$

Reflection Shift/Angles $\left(1 - \alpha\beta \stackrel{?}{=} 0\right)$: $\alpha\beta \approx \frac{\sqrt{\beta^2 - \sin{\theta_I}^2}}{\cos\theta_I}$

In-Phase ($\delta = 0,\ 1 > \alpha\beta$) : $n_1 > n_2$

Out-of-Phase ($\delta = \pi,\ 1 < \alpha\beta$) : $n_2 > n_1$

Brewster Angle ($R = 0$) : $n_1 = n_2$ (None)

Critical Angle ($T = 0$) : $\boxed{\sin\theta_{I=c} = n_2/n_1}$, $\boxed{\theta_R = 90}$ $(n_1 > n_2)$ (evanescent if $> \theta_c$)

7.3 Diffraction and Interference

Double Slit Interference: $(d \ll L)$

Maxima : $d \sin\theta = m\lambda$

Minima : $d \sin\theta = (m + \frac{1}{2})\lambda$

Circular Aperture: (Diameter: $D \ll L$)

$\theta =$ Twice the normal, vertical angle

1st Minima : $D \sin\theta = 1.22\lambda$

Optical Path Length: $(n_1 \to n_2,\ \lambda \to \frac{\lambda}{n},\ v_n = f\frac{\lambda}{n})$

- $\delta = \frac{2\pi d}{\lambda/n} = k(nd)$
- $\Delta x_n = nd = nv\Delta t = c\Delta t$ (t, time through medium n) ($2dn$ for thin film reflec.)

Single Slit Diffraction: $(a \ll L,\ a \sim \lambda)$

Minima : $a \sin\theta = m\lambda, \quad m \neq 0$

Bragg [X-Ray] Diffraction: (Atom Distance : $d \sim \lambda$)

$\theta =$ Angle from Horizontal (not vertical/normal)

- Maxima : $(2d)\sin\theta = m\lambda$

Boundary Reflection: $(n_1 \to n_2)$

$n_2 < n_1 : \quad \delta \mathrel{+}= 0$

$n_2 > n_1 : \quad \delta \mathrel{+}= \pi$

7.4 Lenses and Mirrors $(\lambda \ll a)$

Draw Picture : 1. $\overline{f, y[s], L_{\text{ens}}} \to \overline{L_{\text{ens}}, y'[s'], \infty}$

2. $\overline{\infty, y[s], L_{\text{ens}}} \to \overline{f', L_{\text{ens}}, y'[s']}$

Imaging Eq. : $\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$

Thin Lens Eq. : $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ (Focal Length, $f = f'$)

Lensmaker Eq. : $\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ (R_2 is [-] for concave lens)

Lens Magnf. : $M_T \equiv \frac{y'}{y} = -\frac{s'}{s} = \frac{f}{f-s}$ Virtual: $f>s$ Real: $s<f$

Spherical Mirror : $f = R/2$

7.5 Other

Rayleigh Scattering $(\lambda \gg a)$: $I \propto I_0\left(\frac{a^6}{\lambda^4}\right)$ (Dipole Radiation, polarized)

[Sound] Doppler Effect $(v \ll c)$: $f_r = \left(\frac{v + v_r}{v - v_s}\right) f_s$ (frequency, f) (v_r, v_s are [+] if $\to\leftarrow$)

Standing Sound Wave

- Open Pipe : $L = n\left(\frac{\pi}{2}\right)$ (Ends are nodes/infl. pts. of 0 press.)
- Half Pipe : $L = (2n+1)\left(\frac{\pi}{4}\right)$ (Open End is a node, Closed is an antinode/maxi. press.)

Malus's Law : $I = I_0 \cos^2\theta$ (polarized)
$I = I_0/2$ (unpolarized)

7.6 Conductor; $J_{free} \neq 0$

$$J_{free} = \sigma E$$

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z - wt)}; \quad \tilde{B}(z,t) = \tilde{B}_0 e^{i(\tilde{k}z - wt)}$$

$$\tilde{k} = k + i\kappa; \quad \tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

$$k = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1}\,; \quad \kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1}$$

Skin depth: $d = \dfrac{1}{\kappa}$
Wave (phase) velocity: $v = \dfrac{\omega}{k}$
Group velocity (carries energy): $v_g = \frac{d\omega}{dk} < c$
Index Ref: $n = \dfrac{ck}{\omega}$

$$\frac{B_0}{E_0} = \frac{K}{\omega} = |\tilde{k}|/\omega = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}$$

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}$$

$$\tilde{E}_R = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_I; \quad \tilde{E}_T = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_I$$

7.7 Wave Guides

$$E^{||} = 0; \quad B^{\perp} = 0$$

TE Waves: $E_z = 0$; TM Waves: $B_z = 0$; TEM Waves: $E_z = B_z = 0$

$$E_x = \frac{i}{(w/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$
$$E_y = \frac{i}{(w/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$
$$B_x = \frac{i}{(w/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$
$$B_Y = \frac{i}{(w/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial Y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial X}\right)$$

Solving Rectangular Wave Guides:

$\text{TE}_{mn\neq 00}$: $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] B_z = 0$

$B_z = X(x)Y(y)$

$$\frac{\partial^2 X}{\partial x^2} = -k_x^2 X; \quad \frac{\partial^2 Y}{\partial y^2} = -k_y^2 Y$$
$$-k_x^2 - k_y^2 + (w/c)^2 - k^2 = 0$$

$B_z = B_0 \cos(m\pi x/a)\cos(n\pi y/b)$

$\omega < \omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}$

TM: $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] E_z = 0$

8 Del

$$\nabla F = \left\langle \frac{\partial}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial \theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right\rangle F$$

$$\nabla \cdot \vec{A} = \frac{1}{r^2\sin\theta} \left\langle \frac{\partial}{\partial r},\ \frac{\partial}{\partial \theta},\ \frac{\partial}{\partial \phi} \right\rangle \cdot\ r^2\sin\theta \left\langle A_r,\ \frac{1}{r}A_\theta,\ \frac{1}{r\sin\theta}A_\phi \right\rangle$$

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta} \left\| \begin{matrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{matrix} \right\|$$

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \vec{A} \cdot (B_i\vec{C}) - \vec{A} \cdot (\vec{B}C_i)$$

$$\begin{aligned} \vec{A} \times (\vec{B} \times \vec{C}) &= \vec{A} \cdot (\vec{B} \otimes \vec{C}) - \vec{A} \cdot (\vec{B}\vec{C}) \\ &= \vec{A} \cdot (\vec{B} \otimes \vec{C}) - \vec{A} \cdot (\vec{B} \otimes \vec{C})^T \\ &= (A^T(BC^T)^T)^T - (A^T(BC^T))^T \end{aligned}$$