1 Solving Nonlinear Equations

1.1 One Dimension/Equation **skipped a lot**

Root Multiplicity, m: $0 = f(\bar{x}) = f'(\bar{x}) = ... = f^{(m-1)}(\bar{x})$ (Simple Root: $m = 1$)

k-th Iteration Error: $\boxed{e_k = x_k - \bar{x}}$ Convergence Rate, r: $\boxed{\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C}$ ($0 < C < 1$ if $r = 1$)

Iterval Bisection (Finding $y = 0$): $\boxed{[f(a) < 0]\ ,\ [f(b) > 0]\ ,\ [f \text{ is cont.}] \ \Rightarrow\ \exists m \text{ s.t. } f(m) = 0}$

Fixed-Point Iteration (Finding $y = x$): $\boxed{\text{cont. } f(x) = 0 \ \Rightarrow\ \text{Find } g(x) = x} \rightarrow \boxed{x_{k+1} = g(x_k)}$

$\sim$ Banach-Fixed Point Theorem (there are many FP theorems)

- g is Contractive (over a domain): $\text{dist}(g(x), g(y)) \leq q \cdot \text{dist}(x, y)$ $q \in [0, 1)$
- $e_{k+1} = [x_{k+1} - \bar{x}] = [g(x_k) - g(\bar{x})] = g'(\xi_k)(x_k - \bar{x}) = g'(\xi_k)e_k$
- $\forall |g'(\xi_k)| < G < 1 \ \Rightarrow\ \Big(|e_{k+1}| \leq G|e_k| \leq ... \leq G^k|e_0|\Big) \ \Rightarrow\ \lim_{k\to\infty} e_k = 0$ ($G = \max g'$ over domain)
- $\lim_{k\to\infty} |g'(\xi_k)| = \boxed{\underset{\text{(one contractive condition)}}{\Big(0 < |g'(\bar{x})| < 1\Big)} = C}$ ($r = 1$)
- $\boxed{g'(\bar{x}) = 0} \ \Rightarrow\ \big[g(x_k) - g(\bar{x})\big] = \frac{g''(\xi_k)}{2}(x_k - \bar{x})^2 \ \Rightarrow\ \boxed{\left|\frac{g''(\bar{x})}{2}\right| = C}$ ($r = 2$ if $\bar{x}$ is an $m = 2$ root of g)

Newton's Method (Finding $y = 0$):

$f(\bar{x}) = 0 = f(x_k + h_k) \ \approx\ f(x_k) + f'(x_k)h_k \ \Rightarrow\ \boxed{x_{k+1} = x_k + h_k = x_k - \frac{f(x_k)}{f'(x_k)}}$

- $\boxed{g(x) \equiv x - \frac{f(x)}{f'(x)}} \ \Rightarrow\ g(\bar{x}) = \bar{x}\ ,\ \boxed{g'(\bar{x}) = \frac{f(\bar{x})f''(\bar{x})}{f'(\bar{x})^2} = 0}\ ,\ \boxed{r = 2}$ (if $\bar{x}$ is a simple root of f)
- $\bar{x}$ is an $m > 1$ root of f $\Rightarrow$ $\boxed{r = 1\ ,\ C = 1 - 1/m}$ (proof not given)

Secant Method (Finding $y = 0$):

$f'(x_k) \ \approx\ \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$ Approx. $f'(x_k)$ with a secant line's slope $\Rightarrow$ $\boxed{x_{k+1} = x_k + h_k = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)}$

- $\boxed{r = r_+ \approx 1.618}$: $r_+^2 - r_+ - 1 = 0$ (proof hard)
- Lower cost of iter. offsets the larger number of iter. compared to Newton's Method with derivatives

1.2 m Dimensions/System of Equations **stuff skipped**

Newton's Method (Solving $\vec{y} = 0$):

$$\boxed{\{J_f(\vec{x})\}_{ij} = \frac{\partial f_i(\vec{x})}{\partial x_j}} : \quad \boxed{J_f(\vec{x}_k)\vec{h}_k = -\vec{f}(x_k)} \quad \Rightarrow \quad \boxed{\vec{x}_{k+1} = \vec{x}_k + \vec{h}_k = \vec{x}_k - J_f(\vec{x}_k)^{-1}\vec{f}(\vec{x}_k)}$$

- $\boxed{\vec{g}(\vec{x}) \equiv \vec{x} - J_f(\vec{x})^{-1}\vec{f}(\vec{x})} \Rightarrow$ $$J_g(\bar{x}) = \underbrace{\cancel{I - J_f(\bar{x})^{-1} J_f(\bar{x})}}_{\text{(if } J_f(\bar{x}) \text{ is nonsingular)}} + \sum_{i=1}^{n} f_i(\bar{x}) H_i(\bar{x})$$ H_i = component matrix of the tensor, $D_x J_f(\bar{x})$

  $$= \mathcal{O} \quad \Rightarrow \quad \boxed{r = 2}$$ (uh.....)

- LU fact. of the Jacobian costs $\mathcal{O}(n^3)$

Broyden's [Secant Updating] Method (Solving $\vec{y} = 0$):

$$\boxed{B_k\vec{h}_k = -\vec{f}(x_k)} \quad \Rightarrow \quad \boxed{\vec{x}_{k+1} = \vec{x}_k + \vec{h}_k} \,, \quad \boxed{B_{k+1} = B_k + \frac{f(x_{k+1})h_k^T}{h_k^T h_k}}$$ (cost is $\mathcal{O}(n^3)$)

- $B_{k+1}(\vec{x}_{k+1} - \vec{x}_k) \; = \; B_{k+1}\vec{h}_k \; = \; f(\vec{x}_{k+1}) - f(\vec{x}_k)$
- B_k factorization is updated to factorization of B_{k+1} at cost $\mathcal{O}(n^2)$ instead of directly from the above eq.
- Lower cost of iter. offsets the larger number of iter. compared to Newton's Method with derivatives