1 Solving System of Linear Equations $Ax = b$

1.1 p-Norm and Condition Number

Vector p-Norm: $\boxed{\|\vec{x}\|_p = \sqrt[p]{\sum_i |x_i|^p}}$

$$\text{1-Norm}: \quad \|\vec{x}\|_1 = \sum_i |x_i|$$

$$\infty\text{-Norm}: \quad \|\vec{x}\|_\infty = \max |x_i|$$

- $\|x\|_1 \geq \|x\|_2 \geq \|x\|_\infty$
- $\|x\|_1 \leq \sqrt{n}\,\|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$

Function/Vector Condition Number:

$$\text{cond}\Big(f(x)\Big) = \left|\frac{[f(\hat{x}) - f(x)]/f(x)}{[\hat{x} - x]/x}\right| = \left|\frac{\Delta y/y}{\Delta x/x}\right| = \left|\frac{y' \cdot \Delta x/y}{\Delta x/x}\right| = \left|\frac{xf'(x)}{f(x)}\right|$$

Matrix p-Norm: $\boxed{\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}}$

$$\text{1-Norm}: \quad \|A\|_1 = \max_j \sum_i |a_{ij}|$$

$$\infty\text{-Norm}: \quad \|A\|_\infty = \max_i \sum_j |a_{ij}|$$

$$\left.\begin{array}{l} \bullet\ \|AB\| \leq \|A\| \cdot \|B\| \\ \bullet\ \|Ax\| \leq \|A\| \cdot \|x\| \end{array}\right\} \text{For } p\text{-norms (not necessarily in general)}$$

Matrix Condition Number:

$$\boxed{\text{cond}_p(A) = \|A\|_p \cdot \|A^{-1}\|_p} \qquad (\infty \text{ if singular})$$

$$= \frac{\max_{x \neq 0} \|Ax\|_p / \|x\|_p}{\min_{x \neq 0} \|Ax\|_p / \|x\|_p} = \text{cond}_p(\gamma A) \geq 1$$

- Diagonal, D: $\text{cond}(D) = \frac{\max |d_i|}{\min |d_i|}$
- $\|z\| = \|A^{-1}y\| \leq \|A^{-1}\| \cdot \|y\|$

  $\rightarrow \frac{\|z\|}{\|y\|} \leq \max \frac{\|z\|}{\|y\|} \stackrel{?}{=} \|A^{-1}\|$ (optimize)

1.2 Error Bounds and Residuals

Error Bound: $\boxed{\dfrac{\|\hat{x}-x\|}{\|x\|} \lessapprox \operatorname{cond}(A)\ \epsilon_{\text{mach}}}$ $\rightarrow$ A computed solution is expected to lose about $\log_{10}(\operatorname{cond}(A))$ digits, so the input data must be more accurate to these digits and the working precision must carry more than these digits.

$A\hat{x} = b + \Delta b = Ax + A\Delta x$

- $\|b\| \leq \|A\| \cdot \|x\|$
- $\|\Delta x\| \leq \|A^{-1}\| \cdot \|\Delta b\|$

$\rightarrow \boxed{\frac{\|\Delta x\|}{\|x\|} \leq \operatorname{cond}(A)\ \frac{\|\Delta b\|}{\|b\|}}$

$A\hat{x} + r = b$

- $\|\Delta x\| = \|A^{-1}(A\hat{x} - b)\| = \|-A^{-1}r\|$
  $\leq \|A^{-1}\| \cdot \|r\|$

$\rightarrow \boxed{\frac{\|\Delta x\|}{\|\hat{x}\|} \leq \operatorname{cond}(A)\ \frac{\|r\|}{\|A\|\cdot\|\hat{x}\|}}$

$(A + \Delta A)\hat{x} = b$

- $\|\Delta x\| = \|-A^{-1}(\Delta A)\hat{x}\|$
  $\leq \|A^{-1}\| \cdot \|\Delta A\| \cdot \|\hat{x}\|$

$\rightarrow \boxed{\frac{\|\Delta x\|}{\|x\|} \leq \operatorname{cond}(A)\ \frac{\|\Delta A\|}{\|A\|}}$

$(A + \Delta A)\hat{x} = b$

- $\|r\| = \|b - A\hat{x}\| = \|\Delta A \cdot \hat{x}\|$
  $\leq \|\Delta A\| \cdot \|\hat{x}\|$

$\rightarrow \boxed{\frac{\|r\|}{\|A\|\cdot\|\hat{x}\|} \leq \frac{\|\Delta A\|}{\|A\|}}\ ,\quad \frac{\|\Delta x\|}{\|x\|} \leq \frac{\|A^{-1}\|\cdot\|r\|}{\|\hat{x}\|} \leq \operatorname{cond}(A)\ \frac{\|\Delta A\|}{\|A\|}$

$$\Big[A(t)x(t) \ = \ b(t)\Big] = \Big[\big(A_0 + \Delta A \cdot t\big)x(t) \ = \ b_0 + \Delta b \cdot t\Big]$$

- $x'(t) = \frac{b'(t) - A'(t)x(t)}{A(t)} = A^{-1}(t)\Big[\Delta b - \Delta A \cdot x(t)\Big]$
- $x(t) = x_0 + x'(0)t + \mathcal{O}(t^2)$

$\rightarrow \boxed{\frac{\|x(t)-x_0\|}{\|x_0\|} \leq \operatorname{cond}(A)\left(\frac{\|\Delta b\|}{\|b\|} + \frac{\|\Delta A\|}{\|A\|}\right)|t| + \mathcal{O}(t^2)}$

1.3 Gaussian Elimination with LU/PLU/PLDUQ Decomposition

Elementary Elimination Matrices, L_k

$$\begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & \frac{-a_{k+1}}{a_k} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \frac{-a_n}{a_k} & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- a_k is the "pivot"
- is lower triangular
- $\forall i \neq j \quad (L_k^{-1})_{ij} = -(L_k)_{ij}$

Ex :

$$\begin{pmatrix} 1 & 0 & \dots \\ -a_1/a_2 & 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \end{pmatrix}$$

LU/PLU Factorization (w/ partial pivoting)

$$\boxed{\begin{array}{ll} A = LU & \begin{array}{l}(L \text{ is gen. triang.})\\ (U \text{ is upp. triang.})\end{array} \\ L = (\dots L_2 P_2 L_1 P_1)^{-1} & \end{array}}$$

$$\{\dots\} b = (\dots L_2 P_2 L_1 P_1) A x$$

$$L^{-1} b = (P_1^T L_1^{-1} P_2^T L_2^{-1} \dots)^{-1} A x$$

$$= L^{-1}(LU)x = y$$

$$\boxed{\underset{\text{(forw.-sub.)}}{b = Ly} \ , \ \underset{\text{(back.-sub.)}}{y = Ux}}$$

- Permutation matrix, P_i, rowswaps s.t. $a_k \neq 0$
- P_i rowswaps s.t. a_k is largest s.t. $a_{k+i}/a_k \leq 1$ for numerical stability/ minimize errors
- Pivoting isn't needed if A is diag. dom. ($a_{jj} > \sum_{i, i \neq j} a_{ij}$)
- A can be singular

$$\boxed{\begin{array}{ll} A = PLU & \begin{array}{l}(P \text{ is rowswap permu.})\\ (L \text{ is unit low. triang.})\\ (U \text{ is upp. triang.})\end{array} \\ P = (\dots P_2 P_1)^{-1} & \end{array}}$$

$$\{\dots\} b = (\dots P_2 P_1) A x$$

$$P^T b = (P_1^T P_2^T \dots)^{-1} A x$$

$$= P^T (PLU) x = Ly$$

$$\boxed{P^T b = Ly \ , \ \ y = Ux}$$

$$\boxed{P^T A = LDU \qquad \text{(D is diag.)}}$$

- LDU is unique up to D
- LDU is unique if L/U are unit low./upp. diag., resp.

$$\boxed{P^T A Q^T = LDU \qquad \begin{array}{l}\text{(P is permu. for rows)}\\ \text{(Q is permu. for cols.)}\end{array}}$$

- "Complete pivoting" search for largest a_k
- Would be most numerically stable
- Expensive, so not really used

Error Bound: $\frac{\|r\|}{\|A\|\|x\|} \leq \frac{\|\Delta A\|}{\|A\|} \leq \underset{\text{(Wilkinson)}}{\rho\, n^2 \epsilon_{\text{mach}}} \sim \underset{\text{(usually)}}{n \epsilon_{\text{mach}}}$

(growth factor, ρ, is the largest entry at any point during factorization - usually at U - divided by the largest entry of A)

1.4 Gaussian-Jordan with MD Decomposition

Elementary Elimination Matrices, M_k

$$\begin{pmatrix} 1 & \dots & \frac{-a_1}{a_k} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & \frac{-a_{k+1}}{a_k} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \frac{-a_n}{a_k} & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{k-1} \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- a_k is the "pivot"
- $\forall i \neq j \quad (M_k^{-1})_{ij} = -(M_k)_{ij}$

MD Factorization (w/ partial pivoting)

$$\boxed{A = MD \quad \begin{matrix}(M \text{ is elem. elim.}) \\ (D \text{ is diag.})\end{matrix} \qquad M = (\dots M_2 P_2 M_1 P_1)^{-1}}$$

$$\{\dots\}b = (\dots M_2 P_2 M_1 P_1)Ax$$

$$M^{-1}b = (P_1^T M_1^{-1} P_2^T M_2^{-1} \dots)^{-1} Ax$$

$$= M^{-1}(MD)x = y$$

$$\boxed{M^{-1}b = y \ , \quad y = Dx \ \text{(division)}}$$

- Permutation matrix, P_i, rowswaps s.t. $a_k \neq 0$
- P_i rowswaps cannot ensure numerical stability (≤ 1)
- Division is $\mathcal{O}(n)$, so may be useful for parallel comps.
- Can also find A^{-1}

Finding A^{-1}

$$D^{-1}M^{-1}(A|I) = (I|A^{-1})$$

$$= D^{-1}M^{-1}\left[\begin{array}{cc|cc} a_{11} & \dots & 1 & 0 \\ \vdots & a_{nn} & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & a'_{11} & \dots \\ 0 & 1 & \vdots & a'_{nn} \end{array}\right]$$

1.5 Symmetric Matrices

Positive Definite: $\boxed{x^T A x \geq 0}$

Cholesky Factorization for Sym., Pos. Def.: $\boxed{A = LL^T = LDL^T}$

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots \\ a_{21} & a_{22} & a_{32} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 & \dots \\ l_{21} & l_{22} & 0 & \dots \\ l_{31} & l_{32} & l_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} & \dots \\ 0 & l_{22} & l_{32} & \dots \\ 0 & 0 & l_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} l_{11}^2 & \dots & \dots & \dots \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & \dots & \dots \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Pivoting not needed
- Well defined (always works)
- Only lower triangle needed for storage
- $A = LDL^T$ is sometimes useful, where D is diag.

Symmetric Indefinite Matrices

- Pivoting Needed : $\boxed{PAP^T = LDL^T}$
- Ideally, D is diag., but if not possible, then D is tridiag. (Aasen) or 1x1/2x2 block diag. (Bunch, Parlett, Kaufmann, etc.)

1.6 Banded Matrices

- Similar to normal Gaussian Elim., but less work since more zeroes
- Pivoting means bandwidth will expand no more than double
- Only $\mathcal{O}(\beta n)$ storage needed

1.7 Rank-1 Update with Sherman-Morrison

$$\tilde{A}\tilde{x} = b = (A - uv^T)\tilde{x}$$

$$\rightarrow \ \tilde{x} = \tilde{A}^{-1}b$$

$$\boxed{\begin{aligned} \tilde{A}^{-1} &= (A-uv^T)^{-1} = A^{-1} + \tfrac{A^{-1}u}{1-v^T(A^{-1}u)}\, v^T A^{-1} \\ \tilde{A}^{-1}b &= \tilde{x} = (A^{-1}b) + \tfrac{A^{-1}u}{1-v^T(A^{-1}u)}\, v^T(A^{-1}b) \\ &\qquad x + \tfrac{y}{1-v^Ty}\, v^T x \end{aligned}}$$

General Woodbury Formula: $\boxed{(A-UV^T)^{-1} = A^{-1} + (A^{-1}U)(I - V^TA^{-1}U)^{-1}\ v^TA^{-1}}$

- U and V are general $n \times k$ matrices
- No guarantee of numerical stability, so caution is needed

1.8 Complexity

$$\begin{aligned}
\text{Explicit Inversion}: &\quad \begin{aligned} LUA^{-1} &= I \\ D^{-1}M^{-1}I &= A^{-1}\end{aligned} \rightarrow \mathcal{O}(n^3) \quad , \quad A^{-1}b = x \rightarrow \mathcal{O}(n^2) \\
\text{Gaussian Elimination}: &\quad A = LU \rightarrow \mathcal{O}(n^3/3) \quad , \quad LUx = b \rightarrow \mathcal{O}(n^2) \\
\text{Gaussian-Jordan}: &\quad A = MD \rightarrow \mathcal{O}(n^3/2) \quad , \quad MDx = b \rightarrow \mathcal{O}(n) \\
\text{Symmetric}: &\quad \begin{aligned} A &= LL^T \\ PAP^T &= LDL^T\end{aligned} \rightarrow \mathcal{O}(n^3/6) \quad , \quad LL^Tx = b \rightarrow \mathcal{O}(n^2) \\
\text{Banded}: &\quad A_\beta = LU \rightarrow \mathcal{O}(\beta^2 n) \quad , \quad LUx = b \rightarrow \mathcal{O}(\beta n) \\
\text{Sherman-Woodbury}: &\quad \tilde{A} = A - uv^T \rightarrow \mathcal{O}(n^2) \quad , \quad \tilde{x} = \tilde{A}b \rightarrow \mathcal{O}(n^2)
\end{aligned}$$

1.9 Diagonal Scaling

Ill-conditioned

$$\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \epsilon \end{pmatrix}$$

Well-conditioned

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/\epsilon \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\epsilon \end{pmatrix}\begin{pmatrix} 1 \\ \epsilon \end{pmatrix}$$

- No general way to correct poor scaling

1.10 Iterative Refinement

$$\begin{aligned} r_0 &= b - Ax_0 = A\Delta x_0 \\ r_1 &= b - A(x_0 + \Delta x_0) = b - Ax_1 = A\Delta x_1 \\ r_2 &= b - A(x_1 + \Delta x_1) = b - Ax_2 = A\Delta x_2 \end{aligned}$$

$$\boxed{x = x_0 + \lim_{n=0}^{\infty} \Delta x_n}$$ (terminate when r_n is small enough)

- Double storage needed to hold original matrix
- r_n usually must be computed with higher precision than x_n
- Useful for badly scaled systmes, or making unstable systems stable
- If x_n is not accurate, r_n might not need better accuracy

2 Least $\|r\|$ Linear Regression/Fit for $Ax + r = b$

- $A = A_{m\times n}$ $\boxed{(m > n,\ \text{underdetermined})}$
- $\|r(y = Ax)\|$ is cont. & coer. $\rightarrow \exists \|r(y)\|_{\min}$
- $r(y)$ is strictly convex $\rightarrow$ $y = Ax$ is unique
- $\boxed{\begin{array}{c}\text{rank}(A) = n \\ \text{(full column rank)}\end{array}}$ $\Rightarrow$ $\begin{aligned} A(x_1 - x_2) &= 0 \quad \text{(unique } x\text{)} \\ (x_1 - x_2) &= 0 \rightarrow x_1 = x_2 \end{aligned}$

Example - Vandermonde Matrix, A:

$$Ax = \begin{pmatrix} -\vec{f}(t_1)- \\ \vdots \\ -\vec{f}(t_m)- \end{pmatrix} \begin{pmatrix} | \\ \vec{x} \\ | \end{pmatrix} = \begin{pmatrix} y(t_1) \\ \vdots \\ y(t_m) \end{pmatrix} = \begin{pmatrix} | \\ \vec{y} \\ | \end{pmatrix} = (x^T A^T)^T \quad , \quad y(t) = \textstyle\sum_{i=1}^n x_i f_i(t) = \vec{x} \cdot \vec{f}$$

Decompose b :

$$\begin{aligned} b &= Ax + r \\ &= y + r \\ &= Pb + P_\perp b \end{aligned}$$

Projector of A, P

$$\text{Projector}: \quad \underset{\text{(Idempotent)}}{P^2 = P} \rightarrow \underset{\text{(Projector of A)}}{PA = A}$$

$$\text{Orthogonal Projector}: \quad P^T = P \rightarrow P_\perp A = (I - P)A = 0$$

Minimize residual, r:

$$\begin{aligned} \nabla \|r\|_2^2 = 0 \qquad &\left(\tfrac{\partial r^2}{\partial x_i} = 0\right) \\ &= \nabla \left[(b - Ax)^T (b - Ax)\right] \\ &= \nabla \left(b^T b - 2x^T A^T b + x^T A^T Ax\right) \\ 0 &= 2A^T Ax - 2A^T b \\ &\downarrow \\ A^T Ax &= A^T b \quad \text{(Solvable with Cholesky)} \end{aligned}$$

$$\begin{aligned} \|r\|_2^2 &= \|Pr + P_\perp r\|_2^2 = \|b - Ax\|^2 \\ &= \|Pr\|^2 + \|P_\perp r\|^2 \\ &= \cancel{\|Pb - Ax\|_2^2} + \|P_\perp b\|_2^2 \\ &\downarrow \\ Ax &= Pb \\ A^T Ax &= A^T Pb = (P^T A)^T b \\ A^T Ax &= A^T b \quad \text{(System of Normal Equations)} \end{aligned}$$

Cross-Product Matrix of A: $\boxed{A^T A}$

$$\begin{aligned} \text{Symmetric}: \quad & (A^T A)^T = A^T A \\ \text{Pos. Def.}: \quad & \text{rank}(A) = n \\ & \rightarrow \langle x | A^T Ax \rangle = x^T A^T Ax \\ & \qquad = (Ax)^T (Ax) \\ & \qquad = \|Ax\|^2 \geq 0 \\ \text{Nonsingular}: \quad & A^T Ax = 0 \\ & \rightarrow \|Ax\|^2 = 0 = Ax \\ & \rightarrow (x = 0) \end{aligned}$$

System of Normal Equations: $\boxed{A^T Ax = A^T b}$

Pseudoinverse, A^+

$$\boxed{\begin{aligned} x &= (A^T A)^{-1} A^T b \\ &\equiv A^+ b \end{aligned}} \rightarrow \boxed{\begin{aligned} A^+ &\equiv (A^T A)^{-1} A^T \\ A^+ A &= I \end{aligned}}$$

Ortho. Proj., P

$$\boxed{\begin{aligned} Ax &= A(A^T A)^{-1} A^T b \\ &= Pb \end{aligned}} \rightarrow \boxed{\begin{aligned} P &= A(A^T A)^{-1} A^T \\ &= AA^+ \end{aligned}}$$

<u>System of Normal Equations Issues:</u>

- Info can be lost forming A^TA, e.g, $A = \begin{pmatrix} 1 & 0 \\ \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \;\rightarrow\; A^TA = \begin{pmatrix} 1+\epsilon^2 & 1 \\ 1 & 1+\epsilon^2 \end{pmatrix} \;\approx\; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (singular)

- System of Normal Equations: $\boxed{\text{cond}(A^TA) = [\text{cond}(A)]^2}$

2.1 Error Bounds and Residuals

<u>Error Bound</u>: $\boxed{\dfrac{\|\Delta x\|}{\|x\|} \lessapprox \text{cond}(A)\ \epsilon_{\text{mach}}}$ $\rightarrow$ A computed solution is expected to lose about $\log_{10}(\text{cond}(A))$ digits, so the input data must be more accurate to these digits and the working precision must carry more than these digits.

<u>Norm and Conditioning:</u>

$$\boxed{\|A\| = \max_{x\neq 0}\left(\frac{\|Ax\|}{\|x\|} = \frac{\|AA^+b\|}{\|A^+b\|}\right)}$$

$$\boxed{\text{cond}(A) = \begin{cases} \|A\|_2 \cdot \|A^+\|_2 & \text{rank}(A) = n \\ \infty & \text{rank}(A) < n \end{cases}}$$

$$A^TA(x+\Delta x) = A^TA(b+\Delta b)$$

- $\|\Delta x\| \;\leq\; \|A^+\| \cdot \|\Delta b\|$

$$\rightarrow \boxed{\begin{aligned} \frac{\|\Delta x\|}{\|\hat{x}\|} &\leq \left(\text{cond}(A)\frac{\|b\|}{\|Ax\|}\right)\frac{\|\Delta b\|}{\|b\|} \\ &= \left(\text{cond}(A)\frac{1}{\cos\theta}\right)\frac{\|\Delta b\|}{\|b\|} \end{aligned}}$$

- Cond. number is a func. of $\text{cond}(A)$ and b
- $Pb \approx 0$ or $\theta \approx 90^\circ$ is highly sensitive

$$(A+\Delta A)^T(A+\Delta A)(x+\Delta x) = (A+\Delta A)^Tb$$

- $\cancel{A^TAx} + A^T\Delta Ax + (\Delta A)^TAx + \cancel{(\Delta A)^T\Delta Ax} \quad = \quad \cancel{A^Tb} + (\Delta A)^Tb$
  $+ A^TA\Delta x + \cancel{A^T\Delta A\Delta x} + \cancel{(\Delta A)^TA\Delta x} + \cancel{(\Delta A)^T\Delta A\Delta x}$

- $\|\Delta x\| = \|(A^TA)^{-1}(\Delta A)^Tr - A^+\Delta Ax\|$
  $\leq \|(A^TA)^{-1}\| \cdot \|\Delta A\| \cdot \|r\| + \|A^+\| \cdot \|\Delta A\| \cdot \|x\|$

$$\rightarrow \boxed{\begin{aligned} \frac{\|\Delta x\|}{\|\hat{x}\|} &\leq \left([\text{cond}(A)]^2\frac{\|r\|}{\|Ax\|} + \text{cond}(A)\right)\frac{\|\Delta A\|}{\|A\|} \\ &= \left([\text{cond}(A)]^2\tan\theta + \text{cond}(A)\right)\frac{\|\Delta A\|}{\|A\|} \end{aligned}}$$

2.2 Solving $A^TAx = A^Tb$ with an Augmented Matrix

$$\begin{aligned} r + Ax &= b \\ A^Tr \quad &= 0 \end{aligned} \quad\Rightarrow\quad \begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix}\begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \quad\Rightarrow\quad \begin{pmatrix} \alpha I & A \\ A^T & 0 \end{pmatrix}\begin{pmatrix} r/\alpha \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

- Solvable with LU Decomp or Symm. Pos. Def. Methods
- α "controls the relative weights of the two subsystems in chooing pivots from either"
- $\alpha = \max a_{ij}/1000$ (rule of thumb)
- MATLAB uses it for large, sparse systems

2.3 QR Decomposition

Motivation: $Q^T A = \begin{pmatrix} R \\ 0 \end{pmatrix}$ (R is upp. triag.) $\rightarrow$ $Q^T Ax + Q^T r = Q^T b$, $\begin{pmatrix} Rx \\ 0 \end{pmatrix} + \begin{pmatrix} r'_1 \\ r'_2 \end{pmatrix} = \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix}$ $\rightarrow$ $\|r'\|^2 = \cancel{\|b'_1 - Rx\|^2} + \|b'_2\|^2$

$\downarrow$

$\boxed{Rx = b'_1}$, $r' = \begin{pmatrix} 0 \\ b'_2 \end{pmatrix}$ (solve with back-sub)

Orthogonal Matrix, Q

$$\boxed{Q^T Q = QQ^T = I}$$

QR Factorization

$$\boxed{A = Q \begin{pmatrix} R \\ 0 \end{pmatrix}}$$

Reduced QR Factorization

$$\boxed{A = Q \begin{pmatrix} R \\ 0 \end{pmatrix} = \begin{pmatrix} Q_\parallel & Q_\perp \end{pmatrix} \begin{pmatrix} R \\ 0 \end{pmatrix} = Q_\parallel R}$$

1. Q^T is a span(A) Plane Rotation through $\mathbb{R}^m$ to span$\big([R \;\; 0]^T\big)$

2-norm Preserved (Q is a rotation/reflection)

- $\|Qv\|^2 = \langle v | Q^T Q v \rangle = \|v\|^2$
  $\|Q^T v\|^2 = \langle v | QQ^T v \rangle = \|v\|^2$

- $Q^T = H_n \dots H_1$
- $H_i^T H_i = H_i H_i^T = I$
- $A = [a_1 \;\; \dots \;\; a_n]$
- $I_n = [e_1 \;\; \dots \;\; e_n]$

$$\begin{aligned} H_1 a_1 &= \alpha_1 e_1 \qquad (\|a_1\| = |\alpha_1|) \\ H_i \dots H_1 a_i &= \textstyle\sum_j^i c_j e_j \;=\; H_n \dots H_1 a_i \\ &\qquad (\|a_i\|^2 = |\alpha_1|^2 = \textstyle\sum_j^i c_j^2) \\ \langle r | a_i \rangle &= 0 \qquad (1 \le i \le n) \\ \langle H_i \dots H_1 r | e_j \rangle &= 0 \qquad (1 \le j \le i) \end{aligned}$$

$\rightarrow$ $Q^T A$ rotates A until the column vectors are aligned with certain axes described above

2. A is a Lin. Sum of $Q_\parallel$'s Orthogonal Column Vectors Given by R

$$\left\{ Q_\parallel = Q_{m\times n} \;\middle|\; \text{span}(Q_\parallel) = \text{span}(A) \right\}$$

$\rightarrow$ $Q^+ = (Q^T Q)^{-1} Q^T = Q^T$

$\rightarrow$ $P = Q_\parallel Q_\parallel^T$

$\rightarrow$ $Q_\parallel^T Ax = Q_\parallel^T Pb = \cancel{Q_\parallel^T Q_\parallel} Q_\parallel^T b$

$= Q_\parallel^T b$ (System of Orthogonal Equations?)

$$A = Q_\parallel R = \begin{pmatrix} | & | & | \\ \vec{q_1} & \dots & \vec{q_n} \\ | & | & | \end{pmatrix} \begin{pmatrix} r_{11} & \dots & r_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & r_{nn} \end{pmatrix} = \begin{pmatrix} | & | & | \\ \vec{a_1} & \dots & \vec{a_n} \\ | & | & | \end{pmatrix}$$

- $\vec{a_j} = \sum_i^j r_{ij} \cdot \vec{q_i}$

$\rightarrow$ R transforms the $Q_\parallel$ column vectors about span(A), an $\mathbb{R}^n$ plane, until they equal the column vectors of A

2.3.1 Householder Transformation/Elementary Reflector, H

$$\begin{aligned} H\vec{a_1} &= \alpha_1 \vec{e_1} \qquad \overset{\|a_1\| = |\alpha_1|}{\text{(rotation)}} \\ &= \underset{[v_\perp \text{ bisects } \theta(a_1, e_1)]}{\boxed{\vec{a_1} - 2\hat{v}(\hat{v}\cdot\vec{a_1})}} \end{aligned} \quad \rightarrow \quad \boxed{H = I - 2vv^T = I - \frac{2vv^T}{v^Tv}}$$

- $H = H^T = H^{-1}$ (symmetric and orthogonal)

- $\alpha_1 e_1 = a_1 - (2v_1)\dfrac{v_1 \cdot a_1}{v_1 \cdot v_1} \quad \Rightarrow \quad v_1 = (a_1 - \alpha e_1)\dfrac{v_1 \cdot \cancel{v_1}}{\cancel{2v_1} \cdot a_1}$ (magnitude doesn't matter)

$$\rightarrow \quad \boxed{v_1 = (a_1 - \alpha e_1)}$$

$$\alpha_1 = \pm\|a_1\| \quad \rightarrow \quad \boxed{\alpha_i = -\text{sign}(a_i)\|a_i\|}$$

(avoid "cancellation" in finite-calc. of v above)

$$H_j \dots H_1 a_i = a_i^j \quad \rightarrow \quad \boxed{v_{j+1} = \begin{pmatrix} 0 \\ \vdots \\ (a_i^j)_i \\ \vdots \\ (a_i^j)_m \end{pmatrix} - \alpha_i e_i}$$

- Store v_i and R into A and an extra n-vector.
- Q and H can be computed if needed.
- When column i is completed, row i is too.

2.3.2 Givens Rotation, G

$$\boxed{G = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}} \rightarrow Gx = G\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \pm\begin{pmatrix} \|a\| \\ 0 \end{pmatrix}$$

- creates 0's one at a time
- useful for spare matrices
- When column i is completed, row i is too.

$$\rightarrow \boxed{c = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}\ , \quad s = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}}$$

Avoid squaring any number $\gg 1$ to prevent overflow/underflow

- $t = \frac{a_2}{a_1} < 1 \quad \rightarrow \quad c = \frac{1}{\sqrt{1+t^2}} \quad , \quad s = c \cdot t$
- $\tau = \frac{a_1}{a_2} < 1 \quad \rightarrow \quad s = \frac{1}{\sqrt{1+\tau^2}} \quad , \quad c = s \cdot \tau$

2.3.3 Gram-Schmidt Orthogonalization

$$Q_\parallel^T = \begin{pmatrix} \widehat{q_1} : q_1 = a_1 \\ \widehat{q_2} : q_2 = a_2 - \widehat{q_1}(\widehat{q_1}\cdot a_2) \\ \widehat{q_3} : q_3 = a_3 - \widehat{q_1}(\widehat{q_1}\cdot a_3) - \widehat{q_2}(\widehat{q_2}\cdot a_3) \\ \vdots \\ \widehat{q_n} : q_n = a_n - \sum_j^n \widehat{q_j}(\widehat{q_j}\cdot a_n) \end{pmatrix}, \quad R = \begin{pmatrix} \|a_1\| & \widehat{q_1}\cdot a_2 & \widehat{q_1}\cdot a_3 & \dots & \widehat{q_1}\cdot a_n \\ 0 & \|a_2\| & \widehat{q_2}\cdot a_3 & \dots & \widehat{q_2}\cdot a_n \\ 0 & 0 & \|a_3\| & \dots & \widehat{q_3}\cdot a_n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \|a_n\| \end{pmatrix}$$

Classical, Column Oriented: Find $\widehat{q_i}$, then solve for $\widehat{q_{i+1}}$, and continue up to $\widehat{q_n}$.

- For a program, obviously a_k can be replaced by q_k, so less storage is needed.
- Cancellation that causes loss of orthogonality occurs more when ill-conditioned.
- As a result, performing $Q_{\|}^T b = b_1'$ isn't always best.
- Can't column-pivot, since that depends on rows being completed first.

Modified, Row Oriented: Let $q_i^{[k]} = a_i - \sum_j^k \widehat{q_j}(\widehat{q_j} \cdot a_i)$. For all $1 \le i \le n$, solve for $q_i^{[k]}$ starting first at $k = 1$, then continue until $k = n$.

- Allows for column pivoting since rows are completed first.
- Cancellation, though still present, is less severe.

Augmented Matrix:

$$\big(A \mid b\big) = \big(Q_{\|} \mid q_{n+1}\big)\begin{pmatrix} R & b_1' \\ 0 & \rho \end{pmatrix}$$

$$\begin{pmatrix} | & | & | & | \\ a_1 & .. & a_n & b \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ \widehat{q_1} & \dots & \widehat{q_n} & q_{n+1} \\ | & | & | & | \end{pmatrix}\begin{pmatrix} r_{11} & \dots & r_{1n} & | \\ 0 & \ddots & \vdots & b_1' \\ \vdots & \ddots & r_{nn} & | \\ 0 & \dots & 0 & \rho \end{pmatrix}$$

- Use Gram-Schmidt QR on this, then solve $Rx = b_1'$
- This method is prefered numerically to reduce cancelling effects
- Text didn't recommend what q_{+1} or ρ should be.
- ρ or $(q_{n+1})_i$ looks like it should be 0.
- Idk, not much explained.

Reorthogonalizing: Repeating procedure to straighten vectors (usually not needed)

2.3.4 Factorization with Column-Pivoting

- Column with largest norm is pivoted to the current column i to be reduced, and current row i is completed too.
- Choose the next pivoting column based on norms of the smaller columns from remaining uncompleted submatrix.
- Repeat until the end (rank might be n) or if the max norm is smaller than some tolerance (rank might be $k < n$)
- Pivoting avoids working with 0's on the diag.

2.3.5 Rank Deficiency (or Other) Case

If $\text{rank}(A) = k < n$:

$$\boxed{\begin{array}{ccc} (Q^T A P)(P^T x) & = & Q^T b \\ \begin{pmatrix} R & S \\ 0 & 0' \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} & = & \begin{pmatrix} b_1' \\ b_2' \end{pmatrix} \end{array}}$$

- $0'$ is approx. 0 since the remaining norms are too small.
- $R = R_{k \times k}$
- S is the remaining columns after R is completed.
- There are multiple solutions for x.

- For a quick solution, $\boxed{Rz = b_1' \ , \quad x = P\begin{pmatrix} z \\ 0 \end{pmatrix}}$
- For the minimized-norm solution with the smallest $\|x\|$, S must be annihilated.
- For another method or if underdetermined ($m < n$), something like SVD Decomp. can be used.

2.4 Singular Value Decomposition (SVD)

$$\boldsymbol{A} = \boxed{\boldsymbol{U\Sigma V^T}} = \begin{pmatrix} | & & | & & | \\ \boldsymbol{u_1} & \dots & \boldsymbol{u_k} & .. & \boldsymbol{u_m} \\ | & & | & & | \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma_1} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\sigma_2} & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \ddots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{\sigma_n} \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \text{---} \boldsymbol{v_1} \text{---} \\ \vdots \\ \text{---} \boldsymbol{v_k} \text{---} \\ \vdots \\ \text{---} \boldsymbol{v_n} \text{---} \end{pmatrix}$$

$$= \boxed{\boldsymbol{U_\parallel \Sigma_1 V^T}} = (\boldsymbol{U_\parallel}\ \ \boldsymbol{U_\perp}) \begin{pmatrix} \boldsymbol{\Sigma_1} \\ \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{V_{0\perp}^T} \\ \boldsymbol{V_{0\parallel}^T} \end{pmatrix} = \boxed{\sum_{\boldsymbol{i}}^{\boldsymbol{n}} \boldsymbol{\sigma_i \cdot u_i v_i^T}}$$

$$\boldsymbol{AA^T = U\mathcal{D}U^T}$$
$$\boldsymbol{A^TA = VDV^T}$$
$$\boldsymbol{\mathcal{D} = \Sigma\Sigma^T}$$
$$\boldsymbol{D = \Sigma^T\Sigma}$$
$$\boldsymbol{\sigma_i = \sqrt{d_i}}$$

- Underdetermined, $m < n$ is possible too.
- Analagous to Gaussian-Jordan Diagonalization method.
- U and V are orthogonal; u_i and v_i are the respective "left" and "right" singular vectors.
- Usually, the singular values are ordered such that $\sigma_1 \geq \sigma_2 \geq \dots$
- $\forall(k < i),\ \sigma_i = 0 \quad \Rightarrow \quad \text{rank}(A) = k < n$
- $U_\parallel = U_{m\times k}: \quad \text{span}(U_\parallel) = \text{span}(A)\ ,\quad \text{span}(U_\perp) = \text{span}(A)^\perp$
- $V_{0\perp} = V_{n\times k}: \quad \text{span}(V_{0\parallel}) = \text{null}(A)\ ,\quad \text{span}(V_{0\perp}) = \text{null}(A)^\perp$ $\qquad \text{null}(A) = \{x : Ax = 0\}$

Pseudoinverse:
$$\boxed{\begin{aligned} A^+ &\equiv V\Sigma^+U^T \\ \Sigma^+ &\equiv \left[\Sigma^T \ \text{ and } \ \sigma_i \to 1/\sigma_i \quad \forall(\sigma_i \neq 0)\right] \end{aligned}}$$

- $Ax + r = b \ \rightarrow$

  $\boxed{x = A^+b = \left(V\Sigma^+U^T\right)b}$
- $\boxed{x_{\text{min}} = \sum_{\sigma_i \neq 0} \frac{u_i \cdot b}{\sigma_i}\ v_i}$ useful for ill-conditioned or rank deficient since small σ can be dropped.

2.4.1 Other uses

Euclidean 2-norm : $\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_{\text{max}}$

Euclid. Cond. Num. : $\text{cond}_2(A) = \sigma_{\text{max}}/\sigma_{\text{min}}$

Lower Rank Approx. : $A \ \approx\ A_k = \sum_i^k \sigma_i \left(u_i v_i^T\right)$
- Closest rank$= k$ matrix to A in the Frobinius norm.
- Frobinius Norm = Euclid. Norm for a "vector" in $\mathbb{R}^{mn}$.

Total Least Squares : $[A \mid y]_{m\times(n+1)} = U\Sigma V^T_{(n+1)\times(n+1)}$

-$\widehat{A}$ is an A with uncertainty, like how y normally is.

$\text{rank}([\widehat{A} \mid y]) \leq n \ \rightarrow\ \sigma_{n+1} = 0 \ \rightarrow\ \widehat{A} \cdot v_{n+1} = 0$

$\left[\widehat{A} \mid y\right] \cdot \begin{bmatrix} x \\ -1 \end{bmatrix} = 0 \ \rightarrow\ \begin{bmatrix} x \\ -1 \end{bmatrix} \propto v_{n+1} = \begin{bmatrix} \vec{\nu_n} \\ \nu_{n+1} \end{bmatrix} \ \rightarrow\ \boxed{x = \frac{\vec{\nu_n}}{-\nu_{n+1}}}$

2.5 Complexity

Normal, Cholesky

- $A^T A = A'$ costs $\frac{mn^2}{2}$
- $A' = LL^T$ costs $\frac{n^3}{6}$
- Rel. Err. $\propto$ $[\text{cond}(A)]^2$
- Bad if $\text{cond}(A) \approx 1/\sqrt{\epsilon_{\text{mach}}}$

Householder

- $Q^T A = R$ costs $mn^2 - \frac{n^3}{3}$
- Rel. Err. $\propto$ $[\text{cond}(A)]^2 \|r\|_2 + \text{cond}(A)$
- Bad if $\text{cond}(A) \approx 1/\epsilon_{\text{mach}}$
- More accurate than Cholesky and broadly applicable
- Usable for rank deficient or nearly rank-deficient

Givens

- The normal implementation needs 50% more work than Householder.
- A more complex implementation makes it comparable to Householder.
- Useful if matrix is sparse or zeros need to be maintained.

SVD

- Most expensive cost at $\propto$ $mn^2 + n^3$, perhaps 4-10 times or more.
- Robust and reliable.

3 Matrix Information

$$\begin{aligned} \text{Symmetric}: &\quad S = S^T \\ \text{Hermitian}: &\quad H = H^\dagger \\ \text{Orthogonal}: &\quad QQ^T = Q^T Q = I \\ \text{Unitary}: &\quad UU^\dagger = U^\dagger U = I \\ \text{Normal}: &\quad AA^\dagger = A^\dagger A \end{aligned}$$

D

$H = UDU^{-1}$

- D is real

$U = e^{iH}$

- $U = e^{iH} = U_H e^{iD} (U_H)^{-1}$

3.1 Error Bound and Conditioning

$A + \Delta A = Q(D + \Delta D)Q^{-1}$

- $v = (\Delta\lambda I - D)^{-1}(\Delta D)v$
- $\|(\Delta\lambda I - D)^{-1}\|_2^{-1} \leq \|\Delta D\|_2$

$$|\Delta\lambda - \lambda_i| \leq \|Q(\Delta A)Q^{-1}\|_2$$

$\rightarrow \boxed{|\Delta\lambda - \lambda_i| \leq \text{cond}(Q)\ \|\Delta A\|_2}$

$(A + \Delta A)(x + \Delta x) = (\lambda + \Delta\lambda)(x + \Delta x)$

- $Ax = \lambda x\ , \quad y^H A = \lambda y^H$
- $\underline{\lambda \text{ is simple}} \ \Rightarrow \ y^H x \neq 0 \ \ (?)$
- $y^H \cancel{Ax} + \cancel{y^H A \Delta x} + y^H (\Delta A)x + y^H \cancel{(\Delta A)\Delta x}$
  $\approx y^H \cancel{\lambda x} + \cancel{y^H \lambda \Delta x} + y^H (\Delta\lambda)x + y^H \cancel{(\Delta\lambda)\Delta x}$

$$\rightarrow \boxed{|\Delta\lambda| \ \lessapprox \ \frac{\|y\|_2 \cdot \|x\|_2}{|y^H x|} \ \|\Delta A\|_2 \ = \ \frac{1}{\cos\theta} \ \|\Delta A\|_2}$$

- $AA^\dagger = A^\dagger A \ \rightarrow \ \text{cond}(A) = 1$
- Non-simple (multiple) eigenvalue is complicated:
- allows $y^H x = 0$, depends on eigenvalue spacings, vector angles, etc.
- Balancing can improve conditioning - diagonal rescaling