1 Curvilinear Coordinates

$$\boxed{\begin{aligned}\vec{r} &= r\cos\phi\sin\theta\,\hat{x} + r\sin\phi\sin\theta\,\hat{y} + r\cos\theta\,\hat{z}\\ r\hat{r} &= x\hat{x} + y\hat{y} + z\hat{z}\end{aligned}}$$

$$\begin{aligned}
\hat{r} &= \frac{\partial}{\partial r}\vec{r} = \boxed{\frac{\vec{r}}{r}} = \nabla r = \frac{\nabla r}{\|\nabla r\|}\\
\hat{\theta} &= \frac{1}{r}\frac{\partial}{\partial\theta}\vec{r} = \boxed{\frac{\partial\hat{r}}{\partial\theta}} = r\nabla\theta = \frac{\nabla\theta}{\|\nabla\theta\|}\\
\hat{\phi} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\vec{r} = \boxed{\frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi}} = r\sin\theta\nabla\phi = \frac{\nabla\phi}{\|\nabla\phi\|}
\end{aligned}$$

$$\begin{aligned}
\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} &= \cos 2\theta\,\hat{z}\\
\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi} &= \sin\theta\,\hat{y} + \sin\phi\cos\theta\,\hat{z}\\
\hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}
\end{aligned} \quad\Rightarrow$$

$$1.)\ \hat{z} = \boxed{\frac{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}{\cos 2\theta}}$$

$$2.)\ \hat{y} = \frac{\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}}{\sin\theta} - \frac{\sin\phi\cos\theta}{\sin\theta\cos 2\theta}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right] = \boxed{-\frac{\sin\phi\sin\theta}{\cos 2\theta}\,\hat{r} + \frac{\sin\phi\cos\theta}{\cos 2\theta}\,\hat{\theta} + \frac{\cos\phi}{\sin\theta}\,\hat{\phi}}$$

$$3.)\ \hat{x} = \boxed{\cot\phi\,\hat{y} - \frac{\hat{\phi}}{\sin\phi}}$$

$$\begin{aligned}
\frac{d\hat{r}}{dt} &= \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\left(\frac{d\vec{r}}{dt} - \frac{dr}{dt}\hat{r}\right) = \boxed{\frac{v}{r}\left[\hat{v} - (\hat{r}\cdot\hat{v})\,\hat{r}\right]}\\
&= \frac{d\theta}{dt}\frac{\partial\hat{r}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{r}}{\partial\phi} = \boxed{\frac{d\theta}{dt}\,\hat{\theta} + \sin\theta\,\frac{d\phi}{dt}\,\hat{\phi}}\\
\frac{d\hat{\theta}}{dt} &= \frac{d\theta}{dt}\frac{\partial}{\partial\theta}\left(\frac{\partial\hat{r}}{\partial\theta}\right) + \frac{d\phi}{dt}\frac{\partial}{\partial\phi}\left(\frac{\partial\hat{r}}{\partial\theta}\right) = \boxed{-\frac{d\theta}{dt}\,\hat{r} + \cos\theta\,\frac{d\phi}{dt}\,\hat{\phi}}\\
\frac{d\hat{\phi}}{dt} &= \cancel{\frac{d\theta}{dt}\frac{\partial\hat{\phi}}{\partial\theta}} + \frac{d\phi}{dt}\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi}\right) = -\frac{d\phi}{dt}\underbrace{\mathrm{Proj}_{\mathrm{xy}}\left(\frac{\hat{r}}{\sin\theta}\right)}_{\cos\phi\hat{x} + \sin\phi\hat{y}}\\
&= -\frac{d\phi}{dt}\frac{\hat{r} - \cos\theta\,\hat{z}}{\sin\theta} = \boxed{\frac{d\phi}{dt}\frac{\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}}{\cos 2\theta}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \frac{dr}{dt}\hat{r} + r\left(\frac{d\theta}{dt}\frac{\partial\hat{r}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{r}}{\partial\phi}\right)\\
&\boxed{\vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} + r\sin\theta\frac{d\phi}{dt}\hat{\phi}}\\
\frac{d\vec{\theta}}{dt} &= \frac{d\theta}{dt}\hat{\theta} + \theta\left(-\frac{d\theta}{dt}\,\hat{r} + \cos\theta\,\frac{d\phi}{dt}\,\hat{\phi}\right)\\
\frac{d\vec{\phi}}{dt} &= \frac{d\phi}{dt}\hat{\phi} + \phi\frac{d\phi}{dt}\left(\frac{\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}}{\cos 2\theta}\right)
\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}
\frac{dr}{dt} &= \frac{d}{dt}(\vec{r}\cdot\vec{r})^{\frac{1}{2}} = \boxed{\hat{r}\cdot\vec{v} = v_{\parallel r}}\\
\frac{d\theta}{dt} &= \nabla\theta\cdot\vec{v} = \boxed{\frac{\hat{\theta}\cdot\vec{v}}{r} = \frac{v_{\perp\theta}}{r} = \omega_\theta}\\
\frac{d\phi}{dt} &= \nabla\phi\cdot\vec{v} = \boxed{\frac{\hat{\phi}\cdot\vec{v}}{r\sin\theta} = \frac{v_{\perp\phi}}{r\sin\theta} = \omega_\phi}
\end{aligned}$$

$$\begin{aligned}
\vec{L} &= \vec{r}\times\vec{p}\\
\boxed{m\vec{r}\times\vec{v}_\perp} &= \boxed{m\vec{r}\times\vec{v}} = mr^2\left(\frac{d\theta}{dt}\hat{\phi} - \sin\theta\frac{d\phi}{dt}\hat{\theta}\right)\\
\boxed{m\vec{r}\times\vec{v}_\perp} &= m\vec{r}\times(\vec{\omega}\times\vec{r}) = m\left[\vec{\omega}\|\vec{r}\|^2 - \vec{r}(\vec{\omega}\cdot\vec{r})\right]\\
&\underline{\overleftrightarrow{I}}\,\vec{\omega} = \underline{m\left[\|\vec{r}\|^2\mathbb{1}_3 - \vec{r}\,\vec{r}^T\right]}\,\vec{\omega}
\end{aligned}$$

- $\vec{\omega}\times\vec{r} \equiv \vec{v}_\perp$
- $\vec{r}_\perp\times\vec{v} = r_\perp^2\vec{\omega}$

$$\sum\overleftrightarrow{I} = \begin{bmatrix}\sum m(y^2+z^2) & -\sum mxy & -\sum mxz\\ -\sum myx & \sum m(x^2+z^2) & -\sum myz\\ -\sum mzx & -\sum mzy & \sum m(x^2+y^2)\end{bmatrix}$$

$$E = \sum_m \frac{\|\vec{L}\|^2}{2I} = \frac{1}{2}\vec{L}\cdot\vec{\omega} = \frac{1}{2}\sum_{ij} I_{ij}\,\omega^j\,\omega^i$$

$$L_i = \sum_j I_{ij}\,\omega^j$$

$$\sum_m \frac{1}{2}m\|\vec{\omega}\times\vec{r}\|^2 = \frac{1}{2}\begin{bmatrix} & I & \end{bmatrix}\begin{bmatrix}|\\ \omega\\ |\end{bmatrix}\cdot\begin{bmatrix}|\\ \omega\\ |\end{bmatrix}$$

$$
\begin{aligned}
\tfrac{d}{dt}\big(\vec{p}\times\vec{L}\big) &= \tfrac{d\vec{p}}{dt}\times\vec{L} = f_{(r)}\hat{r}\times\big(\vec{r}\times m\tfrac{d\vec{r}}{dt}\big) \\
&= mf_{(r)}\left[\vec{r}\big(\hat{r}\cdot\tfrac{d\vec{r}}{dt}\big) - \tfrac{d\vec{r}}{dt}\big(\hat{r}\cdot\vec{r}\big)\right] \\
&= mf_{(r)}\left[\hat{r}\tfrac{1}{2}\tfrac{d}{dt}\big(\vec{r}\cdot\vec{r}\big) - \tfrac{1}{r}\tfrac{d\vec{r}}{dt}r^2\right] \\
&= mf_{(r)}\left[\hat{r}r\tfrac{dr}{dt} - r\tfrac{d\vec{r}}{dt}\right] \\
&= -\tfrac{mf(r)r}{I(r)}\left[-\tfrac{I(r)}{r}\tfrac{dr}{dt}\vec{r} + I_{(r)}\tfrac{d\vec{r}}{dt}\right] \\
&= -\tfrac{mf(r)r}{I(r)}\tfrac{d}{dt}\left[I_{(r)}\vec{r}\right] \\
&= -m\cancel{f_{(r)}r^2}\tfrac{d}{dt}\hat{r} = mk\tfrac{d}{dt}\hat{r}
\end{aligned}
$$

$$
\tfrac{d}{dt}\Big(\tfrac{\vec{p}\times\vec{L}}{mk} - \hat{r}\Big) = \tfrac{d}{dt}\vec{e}_{\text{ccen}} = 0
$$

$$
\begin{aligned}
\vec{a} = &\left[\ddot{r} - r\dot{\theta}^2 + r\dot{\phi}^2\tfrac{\sin^2\theta}{\cos 2\theta}\right]\hat{r} \\
&+ \left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\tfrac{\tan 2\theta}{2}\right]\hat{\theta} \\
&+ \left[2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta\right]\hat{\phi}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\tau} &= \vec{r}\times\vec{F} \\
&= (mr^2)\tfrac{\hat{r}\times\vec{a}}{r} = I\vec{\alpha}
\end{aligned}
\qquad
\tau = \begin{bmatrix}\tau_x\\ \tau_y\\ \tau_z\end{bmatrix}
=^{*} \begin{bmatrix} 0 & \tau_z & -\tau_y \\ -\tau_z & 0 & \tau_x \\ \tau_y & -\tau_z & 0 \end{bmatrix}
$$

$$
\boxed{\tfrac{dv}{dt} = a(\hat{v}\cdot\hat{a}) = \hat{v}\cdot\vec{a} = \tfrac{d}{dt}\|\vec{v}\|}
$$

$$
\begin{aligned}
\|\vec{q}\times\vec{p}\|^2 &= \begin{vmatrix}\vec{q}\cdot\vec{q} & \vec{q}\cdot\vec{p} \\ \vec{p}\cdot\vec{q} & \vec{p}\cdot\vec{p}\end{vmatrix} \\
&= \vec{q}\cdot\vec{p}\times(\vec{q}\times\vec{p})
\end{aligned}
$$

$$
\boxed{\tfrac{dt}{ds} = \tfrac{1}{v}}
$$

$$
\begin{aligned}
T &= \hat{v} = \tfrac{\vec{v}}{v} \\
\tfrac{dT}{dt} &= \tfrac{(\vec{v}\cdot\vec{v})\vec{a} - (\vec{v}\cdot\vec{a})\vec{v}}{v^3} = \tfrac{\vec{v}\times(\vec{a}\times\vec{v})}{v^3} = \tfrac{(\vec{v}\times\vec{a})\times\vec{v}}{v^3} \\
\|\tfrac{dT}{dt}\| &= \tfrac{\sqrt{v^2a^2 - (\vec{v}\cdot\vec{a})^2}}{v^2} = \tfrac{\|\vec{a}\times\vec{v}\|}{v^2}\ , \quad \tfrac{dT}{ds} = k\hat{N} \\
\hat{N} &= \tfrac{T'}{\|T'\|} = \tfrac{(\vec{v}\times\vec{a})\times\vec{v}}{\|\vec{v}\times\vec{a}\|v} = \hat{B}\times\hat{v} \\
\hat{B} &= \tfrac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} = \widehat{v\times a} = \hat{v}\times\hat{N} \qquad \underline{(\hat{B}\cdot\vec{v} = 0)} \\
\tfrac{d\hat{B}}{dt} &= \tfrac{\vec{v}\times\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|} - \left[\tfrac{\vec{v}\times\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|}\cdot\hat{B}\right]\hat{B}\ , \quad \tau = \hat{N}\cdot\tfrac{d\hat{B}}{ds}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\vec{a} &= a_T\hat{T} + a_N\hat{N} \\
a_T &= \vec{a}\cdot\hat{v} = \tfrac{dv}{dt} \\
a_N &= \tfrac{\|\vec{a}\times\vec{v}\|}{v} = \|\vec{a}\times\hat{v}\| \\
a^2 &= a_T^2 + a_N^2 = \|\tfrac{d\vec{v}}{dt}\|^2
\end{aligned}
}
$$

<u>Frenet Trihedron</u>

Differentiable (in this book) : C^∞

No singular pts. Order 0 (Regular) : $\vec{v}(t) \neq 0$

- $\|\vec{v}(t)\| = c \to 1 \;\Rightarrow\; \int_s \|\vec{v}(t)\|\,dt = t = \Delta s$

  $\to s:\ \vec{x}(t) = \vec{x}(s)$

- $\frac{1}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \boxed{\vec{v}\cdot\vec{a} = 0}$

No singular pts. Order 1 : $\vec{a}(t) \neq 0$

- Curvature, $k \neq 0$ (see right) • Vertex, $k' = 0$

$1 = \|\vec{t}\| = \|\vec{n}\| = \|\vec{b}\|\ ,\quad 0 = \vec{t}\cdot\vec{n} = \vec{n}\cdot\vec{b} = \vec{b}\cdot\vec{t}$

- $\vec{v}(s) = \vec{t}(s)$ $\quad\boxed{(t = n\times b)}$
- $\vec{a}(s) = \boxed{\vec{t}'(s) = k_{(s)}\vec{n}_{(s)}}$, $k(s) \geq 0$ (can be L or R-handed) (can be neg. if in $\mathbb{R}^2$)

$*$ $k(s) > 0$ for well defined curve with $\hat{n}$

- $\boxed{\vec{b} = \vec{t}\times\vec{n}}$, $\frac{d}{dt}(\vec{b}\cdot\vec{b}) = \vec{b}\cdot\vec{b}' = 0$, $*$ $\boxed{\vec{b}'(s) = \tau_{(s)}\vec{n}_{(s)}}$
- $\boxed{\vec{n} = \vec{b}\times\vec{t}}$, $*$ $\boxed{\vec{n}'(s) = -k\vec{t} - \tau\vec{b}}$, $*$ t-n pl. = osculating pl.

- $t''(s) = k'n - k^2t - k\tau b$ • $b''(s) = \tau'n - \tau kt - \tau^2 b$ • $n''(s) = -k't - \tau'b - (k^2+\tau^2)n$
- $|\tau| = \|b'\|$ • $\tau = -\frac{(t\times t')\cdot t''}{k^2} = -\frac{t\cdot(t'\times t'')}{\|t'\|^2}$ • $k = \|t'\| = \frac{(b\times b')\cdot b''}{\tau^2} = \frac{b\cdot(b'\times b'')}{\|b'\|^2}$
- $n \Rightarrow k, \tau :$ $\quad *\ \|n'\|^2 = k^2 + \tau^2$ $\quad *\ \frac{(n\times n')\cdot n''}{\|n'\|^2} = \frac{k'\tau - k\tau'}{k^2+\tau^2} = \frac{\frac{d}{ds}(k/\tau)}{(k/\tau)^2+1} = \frac{d}{ds}\arctan(k/\tau)$

2 Lagrangian Equations

$$\boxed{\begin{aligned} &\mathcal{L} = T - U \ , \qquad p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \\ &\rightarrow \ F_i \equiv \frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i} \end{aligned}}$$

Newton's Laws:

$$\mathcal{L} = \tfrac{1}{2} m \dot{\mathbf{r}}^2 - U(\mathbf{r}) \ , \qquad \vec{p}_r = m\dot{\mathbf{r}}$$

$$\rightarrow \ \boxed{F = m\ddot{\mathbf{r}} = -\nabla U}$$

Angular:

$$\mathcal{L} = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\phi}^2 - U(r,\phi) \ , \qquad \begin{aligned} p_r &= m\dot{r} \\ p_\phi &= mr^2\dot{\phi} = I\omega = I\tfrac{v_\perp}{r} \end{aligned} \ , \qquad \begin{aligned} -\vec{F} \ &= \ \nabla U \ = \ \tfrac{\partial U}{\partial r}\hat{r} + \tfrac{1}{r}\tfrac{\partial U}{\partial \phi}\hat{\phi} \\ \vec{F} \ &= \ m\ddot{\mathbf{r}} \ = (F\cdot\hat{r})\hat{r} + (F\cdot\hat{\phi})\hat{\phi} \end{aligned}$$

$$F_r = \boxed{-\tfrac{\partial U}{\partial r} + mr\dot{\phi}^2 = m\ddot{r}} \quad (\text{centripetal: } \tfrac{mv^2}{r} = mr\omega^2)$$

$$\rightarrow \quad F_\phi = \boxed{\underbrace{-\tfrac{\partial U}{\partial \phi}}_{\tau} = \underbrace{mr^2\ddot{\phi}}_{I\alpha} + \underbrace{2mr\dot{r}\dot{\phi}}_{\dot{I}\omega}} \quad \left(\begin{matrix} \text{“coriolis”:} \\ 2m|\vec{\omega}\times\vec{v}| = 2m\dot{\phi}\dot{r} \end{matrix}\right)$$

$$\dot{\phi}' = \dot{\phi} - \omega \ , \quad r' = r \qquad \left(\hat{r} = R(\omega)\hat{r}' \ , \quad \hat{\phi} = R(\omega)\hat{\phi}'\right)$$

$$\downarrow$$

$$m\ddot{\mathbf{r}} = m\ddot{\mathbf{r}}' - (mr\omega^2 + 2mr\dot{\phi}\omega)\hat{r} + (2m\dot{r}\omega + mr\dot{\omega})\,\hat{\phi}$$

$$m\ddot{\mathbf{r}}' = m\ddot{\mathbf{r}} + \underbrace{mr\omega^2\,\hat{r}}_{\text{centrifugal force}} + \underbrace{2m\omega(r\dot{\phi}\,\hat{r} - \dot{r}\,\hat{\phi})}_{\text{coriolis force}} - \underbrace{mr\dot{\omega}\,\hat{\phi}}_{\text{Euler force}}$$

Note: $\boxed{\begin{aligned} \dot{\hat{r}} &= \dot{\phi}\hat{\phi} \\ \dot{\hat{\phi}} &= -\dot{\phi}\hat{r} \end{aligned}}$ $\rightarrow$

$$\begin{aligned} \vec{r} &= r\hat{r} \ = \ r\cos\phi\hat{x} + r\sin\phi\hat{y} \\ \dot{\vec{r}} &= \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \\ \ddot{\vec{r}} &= \ddot{r}\hat{r} + 2\dot{r}\dot{\hat{r}} + r\ddot{\hat{r}} \ = \ (\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi} \end{aligned}$$

Electromagnetic:

$$\mathcal{L} = \tfrac{1}{2} m\dot{\mathbf{r}}^2 - q\Big[V(t,\ \mathbf{r}) - \dot{\mathbf{r}}\cdot\vec{A}(t,\ \mathbf{r})\Big] \ , \qquad p_x = m\dot{x} + qA_x$$

$$\rightarrow \qquad m\ddot{x} + q\tfrac{dA_x}{dt} = -q\left[\tfrac{\partial V}{\partial x} - \dot{r}\cdot\tfrac{\partial \vec{A}}{\partial x}\right] \quad \rightarrow \quad m\ddot{x} = q\left(-\tfrac{\partial V}{\partial x} - \tfrac{\partial A_x}{\partial t} + \dot{r}\cdot\left[\tfrac{\partial \vec{A}}{\partial x} - \nabla A_x\right]\right)$$

$$m\ddot{x} + q\left[\tfrac{\partial A_x}{\partial t} + \dot{r}\cdot\nabla A_x\right] = q\left[-\tfrac{\partial V}{\partial x} + \dot{r}\cdot\tfrac{\partial \vec{A}}{\partial x}\right]$$

$$\begin{aligned} &= q\left[-\tfrac{\partial V}{\partial x} - \tfrac{\partial A_x}{\partial t}\right] + q\dot{y}\left[\tfrac{\partial A_y}{\partial x} - \tfrac{\partial A_x}{\partial y}\right] \\ &\quad + q\dot{z}\left[\tfrac{\partial A_z}{\partial x} - \tfrac{\partial A_x}{\partial z}\right] \\ &= qE_x + q\dot{y}B_z - q\dot{z}B_y \end{aligned}$$

$$m\ddot{x} = qE_x + q\left[\dot{\mathbf{r}}\times\vec{B}\right]_x$$

$$\downarrow$$

$$\boxed{m\ddot{\mathbf{r}} = q\left(\vec{E} + \dot{\mathbf{r}}\times\vec{B}\right)}$$

Special Relativity:

$$\mathcal{L} = -\tfrac{1}{\gamma}mc^2 - U \ , \qquad \vec{p} = \gamma m\vec{v} \ \rightarrow \ \gamma m\dot{x} = \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$= \gamma m v^2 - \gamma m c^2 - U$$

$$= m\left(v^2 - c^2\right)\left(1 - \tfrac{v^2}{c^2}\right)^{-1/2} - U$$

$$\approx \tfrac{1}{2}mv^2 - (U + mc^2) \qquad \text{(when } v \ll c\text{)}$$

Conservation of Energy:

$$\frac{d\mathcal{L}}{dt} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right) + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \sum_i \left(\dot{p}_i\dot{q}_i + p_i\ddot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t} \quad \rightarrow \quad \frac{\partial \mathcal{L}}{\partial t} = -\frac{d}{dt}\left(\sum_i p_i\dot{q}_i - \mathcal{L}\right)$$

$$= \frac{d}{dt}\left(\sum_i p_i\dot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t} \qquad = -\frac{d\mathcal{H}}{dt}$$

If $\mathcal{L}$ is explicitly independent of time (implies coordinates are "na/tural"), then the Hamiltonian is conserved.

$$\tfrac{1}{2}\sum_n m\dot{r}_n^2 = \tfrac{1}{2}\sum_n m\left(\sum_i \frac{\partial r_n}{\partial q_i}\dot{q}_i\right)^2$$

$$= \tfrac{1}{2}\sum_{i,j}\left(m\sum_n \frac{\partial r_n}{\partial q_i}\frac{\partial r_n}{\partial q_j}\right)\dot{q}_i\dot{q}_j$$

$$= \tfrac{1}{2}\sum_i\sum_j A_{ij}\dot{q}_i\dot{q}_j$$

$$T = \tfrac{1}{2}\left(2\sum_{i\neq j} A_{ij}\dot{q}_i\dot{q}_j + \sum_i A_{ii}\dot{q}_i^2\right) + \ldots$$

$$\rightarrow$$

$$\mathcal{L} = \tfrac{1}{2}mv^2 - U \ = \ T(\dot{q}_i) - U(q_i) \ \rightarrow$$

$$\mathcal{H} = \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}$$

$$= \sum_i\left(\sum_j A_{ij}\dot{q}_j\right)\dot{q}_i - \tfrac{1}{2}m\dot{\mathbf{r}}^2 + U$$

$$= \tfrac{1}{2}m\dot{\mathbf{r}}^2 + U$$

If $\mathcal{L} = \frac{1}{2}mv^2 - U$ and U is independent of v, then the Hamiltonian is the total energy.

Lagrange $(x^i, v^i) \leftrightarrow$ Hamiltonian (q^i, p_i):

$$v^i(q^i,p_i) = \frac{\partial q^i}{\partial t} = \frac{\partial H(q^i,p_i)}{\partial p_i} \qquad \text{(also for Newton. } \leftarrow \text{ Hamil.)}$$

- $\exists p_i(q^i \rightarrow x^i, v^i) \ \Leftarrow \ \boxed{\left|\frac{\partial^2 H}{\partial p_i p_j}\right| \neq 0}$ (invert. + diff.) (not req. for uniq. H)
- $\left|\frac{\partial^2 H}{\partial p_i p_j}\right| = \left|\frac{\partial^2 L}{\partial v_i v_j}\right|^{-1} \neq 0$ (uniq. cond. for L)

Lagrange Multipliers:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_i}$$

$$\frac{dp}{dt} = -\nabla U + \lambda \nabla f$$

$$F_{\text{tot}} = F_{\text{ncnstr}} + F_{\text{cnstr}}$$

2.1 Examples

Atwood's Machine (Pulley):

Particle Confined to a Cylinder Surface:

Block Sliding on Wedge:

Bead on Spinning Wire Hoop:

Oscillations of Bead Near Equilibriuum:

3 Hamiltonian

$$\boxed{\begin{aligned} &\mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L}\ , \qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}} \\ &\rightarrow \quad \bullet\ \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i} \\ & \quad \bullet\ \frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \end{aligned}}$$

Newton Particle:

$$\begin{aligned} \mathcal{H} &= \dot{x}(m\dot{x}) - \tfrac{1}{2}m\dot{x}^2 + U(x) \\ &= \tfrac{1}{2}m\dot{x}^2 + U(x) \\ &= T + U \end{aligned}$$

Angular:

$$\begin{aligned} \mathcal{H} &= m\dot{r}^2 + mr^2\dot{\theta}^2 - \left(\tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - U(r,\theta)\right)\ , \\ &= \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + U(r,\theta) \end{aligned}$$

$$p_r = m\dot{r}$$
$$p_\theta = mr^2\dot{\theta} \ \equiv\ L = I\omega$$

Electromagnetic:

$$\begin{aligned} \mathcal{H} &= \dot{\mathbf{r}} \cdot \vec{p}_r - \left(\tfrac{1}{2}m\dot{\mathbf{r}}^2 - q\phi_{(t,\ \mathbf{r})} + q\dot{\mathbf{r}} \cdot \vec{A}_{(t,\ \mathbf{r})}\right)\ , \\ &= m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \vec{A} - \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\phi - q\dot{\mathbf{r}} \cdot \vec{A} \\ &= \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\phi \end{aligned}$$

$$\vec{p}_r = m\dot{\mathbf{r}} + q\vec{A}$$

Special Relativity:

$$\begin{aligned} \mathcal{H} &= \vec{v} \cdot (\gamma m\vec{v}) - \left(\gamma m v^2 - \gamma m c^2 - U\right)\ , \\ &= \gamma m c^2 + U \\ &\approx \tfrac{1}{2}mv^2 + \left(U + mc^2\right) \qquad \text{(when } v \ll c\text{)} \end{aligned}$$

$$\vec{p} = \gamma m \vec{v}$$

Poisson Brackets

$$\{f,g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$$

- $\{q_i, g_{(q,p,t)}\} = \frac{\partial g}{\partial p_i}\ ,\ \{p_i, g_{(q,p,t)}\} = -\frac{\partial g}{\partial q_i}$

$$\Rightarrow \quad \bullet\ \{f,g\} = \sum_i -\{p_i, f\}\{q_i, g\} + \{p_i, g\}\{q_i, f\}$$
$$= \sum_i \{q_i, f\}\{p_i, g\} - \{q_i, g\}\{p_i, f\}$$

Hamilton Eq. : $\dot{q}_i = \{q_i, H\},\ \dot{p}_i = \{p_i, H\} \ \Rightarrow\ \{f_{(q,p,t)}, H\} = \sum_i \frac{\partial f}{\partial q_i}\dot{q} + \dot{p}\frac{\partial f}{\partial p_i} = \dot{f} - \frac{\partial f}{\partial t}$

Canonical Transforms

$$\begin{matrix} q \rightarrow \bar{q}_{(q,p)} \\ p \rightarrow \bar{p}_{(q,p)} \end{matrix} \ \text{s.t.}\ \begin{matrix} \{\bar{q}_i, \bar{q}_j\} = 0 = \{\bar{p}_i, \bar{p}_j\} \\ \{\bar{q}_i, \bar{p}_j\} = \delta_{ij} \end{matrix} \ \left(\begin{matrix} \text{Point Transforms} \\ \bar{q}(q) \text{ are canonical.} \end{matrix}\right) \Rightarrow \begin{matrix} \dot{\bar{q}} = \frac{\partial H}{\partial \bar{p}} \\ \dot{\bar{p}} = -\frac{\partial H}{\partial \bar{q}} \end{matrix}\ ,\quad \{f,g\}_{q,p} = \{f,g\}_{\bar{q},\bar{p}}$$

Generator of Transformation

$$\{f,g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$$

$$\boxed{\frac{df}{d\lambda_g} - \frac{\partial f}{\partial t}\frac{\partial t}{\partial \lambda_g}} \equiv \sum_i \frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial \lambda_g} + \frac{\partial p_i}{\partial \lambda_g}\frac{\partial f}{\partial p_i}$$

$$\boxed{\frac{\partial g}{\partial t}\frac{\partial t}{\partial \lambda_f} - \frac{dg}{d\lambda_f}} \equiv \sum_i -\frac{\partial p_i}{\partial \lambda_f}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial q_i}{\partial \lambda_f}$$

$$\{g,H\} = \sum_i \boxed{\frac{\partial g}{\partial q_i}}\frac{\partial H}{\partial p_i} - \boxed{\frac{\partial g}{\partial p_i}}\frac{\partial H}{\partial q_i}$$

$$\equiv \sum_i \boxed{-\frac{\partial p_i}{\partial \lambda}}\frac{\partial H}{\partial p_i} - \boxed{\frac{\partial q_i}{\partial \lambda}}\frac{\partial H}{\partial q_i}$$

$$\boxed{\dot{g} - \frac{\partial g}{\partial t} = -\frac{dH}{d\lambda} + \frac{\partial H}{\partial t}\frac{\partial t}{\partial \lambda}}$$

1. $\delta H = 0$ $\qquad \delta H = \epsilon_\lambda \{H, g\}$

2. $\bar{q}_i = q_i + \delta q_i \quad , \quad \bar{p}_i = p_i + \delta p_i \quad \Rightarrow \quad \boxed{\frac{\partial H}{\partial \lambda} = 0 = -\frac{dg}{dt}}$

$$\equiv q_i + \epsilon_\lambda \frac{\partial g}{\partial p_i} \qquad \equiv p_i - \epsilon_\lambda \frac{\partial g}{\partial q_i}$$

$$= q_i + \epsilon_\lambda \{q_i, g\} \qquad = p_i + \epsilon_\lambda \{p_i, g\}$$

(e.g. $g = p$ or $g = l_z$)

3. $\Rightarrow \; \delta f = \epsilon_\lambda \{f, g\} \; \rightarrow \; \frac{df}{d\lambda_g} - \frac{\partial f}{\partial t}\frac{\partial t}{\partial \lambda_g} = \{f, g\}$

- $g = l_z \;\Rightarrow\; \begin{array}{l} \delta x = -\epsilon y = -(\delta\theta) y \\ \delta y = \epsilon x = (\delta\theta) x \end{array} \;\Rightarrow\; \boxed{\begin{array}{l} \frac{\partial x}{\partial \theta} = -y \\ \frac{\partial y}{\partial \theta} = x \end{array}}$

4 Hamilton-Jacobi Equations

$$\boxed{K(Q,P,t) \equiv H(q,p,t) + \frac{\partial S_{(q,Q,t)}}{\partial t} = 0}$$

$$\dot{q} = \frac{\partial H}{\partial p} \;\Rightarrow\; \boxed{q = -\frac{\partial S}{\partial p}} \;,\; \dot{p} = -\frac{\partial H}{\partial q} \;\Rightarrow\; \boxed{p = \frac{\partial S}{\partial q}}$$

$$\dot{Q} = \frac{\partial K}{\partial P} = 0 \quad \Rightarrow \quad \boxed{Q = \frac{\partial S}{\partial P} \equiv \alpha_Q \;\; \text{(constant)}}$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \;\Rightarrow\; \boxed{P = -\frac{\partial S}{\partial Q} \equiv \alpha_P \;\; \text{(constant)}}$$

Solve for $S_{(q,\alpha_Q,t)}$ ($n+1$ variables, nonlinear PDE)

$$H\left(q, \frac{\partial S_{(q,\alpha_Q,t)}}{\partial q}, t\right) + \frac{\partial S_{(q,\alpha_Q,t)}}{\partial t} = 0$$

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum p_i \dot{q}_i + 0 = \cancel{\frac{\partial S}{\partial t} + H} + \mathcal{L}$$

$$\Rightarrow \; \boxed{S = \int \mathcal{L}\; dt + \text{const.}}$$

$$\frac{\partial H}{\partial t} = 0 \;\Rightarrow\; \boxed{S(q,Q,t) = W(q,Q) - Et}$$

Solve for $W_{(q,\alpha_Q)}$ (n variables, nonlinear PDE)

$$H(q, \tfrac{\partial W}{\partial q}) = E \equiv \alpha_Q$$

Harmonic Oscillator

$$-\frac{\partial S}{\partial t} = \frac{1}{2}p^2 - \frac{1}{2}\omega^2 q^2$$

$$= \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{1}{2}\omega^2 q^2 \qquad \big(S = s_1(q) + s_2(t)\big)$$

$$-\frac{\partial s_2(t)}{\partial t} = \frac{1}{2}\left(\frac{\partial s_1(q)}{\partial q}\right)^2 - \frac{1}{2}\omega^2 q^2 \equiv \alpha_Q$$

$$s_2(t) = -\alpha_Q t + \cancel{\text{const.}} \quad , \quad s_1(q) = \int \sqrt{2a_Q + \omega^2 q^2}\; dq$$

$$Q \equiv \alpha_Q \quad , \quad P = -\int \frac{dq}{\sqrt{2a_Q + \omega^2 q^2}} + t$$

$$\alpha_P = -\frac{1}{\omega}\sin^{-1}\left[q\frac{\omega}{\sqrt{2\alpha_Q}}\right] + t$$

$$\boxed{q(t) = \frac{\sqrt{2\alpha_Q}}{\omega}\sin\left[\omega(t - \alpha_P)\right]}$$

5 Kinematics

<u>Elastic Collisions</u>: $\begin{array}{c} p_0 = p_1 + p_2 \\ \frac{p_0^2}{2m_0} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \end{array}$

$\Rightarrow$ $\boxed{\frac{p_0^2}{2m_0}(m_0 - m_{12}) + \frac{p_{21}^2}{2m_{21}}(m_1 + m_2) = p_0 \cdot p_{21}}$

- $\boxed{\begin{array}{l} mv_0 \;=\; mv_1 + Mv_2 \;=\; mv_0\left(1 - \frac{2M}{m+M}\right) + Mv_0\left(\frac{2m}{m+M}\right) \\ \to\; M \in (\infty,\; m,\; 0] \;\Rightarrow\; v_1 \in (-v_0,\; 0,\; v_0] \end{array}}$

<u>Inelastic Collision</u>: $E_0 = \frac{1}{2}mv_0^2$

- $\boxed{\begin{array}{l} mv_0 = (m+M)v_1 \\ \to\; E_1 = \left(\frac{m}{m+M}\right)E_0 \end{array}}$

6 Orbits

<u>Lagrangian</u> : $\mathcal{L} = \sum\limits_{1,2} \frac{1}{2}m_i\|\dot{r_i}\|^2 - U_{(\|r_1 - r_2\|)}$

$(MR = \sum m_i r_i)$ $= \frac{1}{2}M\|R\|^2 + \frac{1}{2}\mu\|\dot{r}\|^2 - U_{(\|r\|)}$

$= \frac{1}{2}MR^2 - U_{(\|r\|)} + \begin{array}{l} \frac{1}{2}m_1\|\dot{\overline{r_1}}\|^2 + \frac{1}{2}m_2\|\dot{\overline{r_2}}\|^2 \\ \sum \frac{1}{2}m_i\dot{\overline{r_i}}^2 + \frac{1}{2}m_i\overline{r_i}^2\left[\dot{\theta_i}^2 + \sin^2\theta_i\dot{\phi_i}^2\right] \\ \sum \frac{1}{2}m_i\dot{\overline{r_i}}^2 + \frac{1}{2}m_i\overline{r_i}^2\|w_i\|^2 \text{ (coplanar } w_i) \end{array}$

- $\begin{array}{l} \overline{r_1} = R + \frac{m_2}{M}r \\ \overline{r_2} = R - \frac{m_1}{M}r \end{array}$

- $l = I\omega_r = \underline{\mu r^2}\dot{\theta}_r = \underline{\frac{m_1 m_2}{M = m_1 + m_2}\|r_1 - r_2\|^2\dot{\theta}_r}$
- $m\ddot{r} = -\frac{\partial}{\partial r}U_{\text{eff}} = -\frac{\partial}{\partial r}\left[\frac{l^2}{2\mu r^2} + U_{(\|r\|)}\right]$
- $L_z = \sum m_i\overline{r_i}^2\sin^2\theta_i\dot{\phi_i} \;\Rightarrow\; \underline{L_x, L_y}^*$

* angles about the 3 axes can't be treated simultaneously as gen. coord., since not independent; 2 angles per point suffice to determine position. Fully describing a rigid body needs 3 trans. DOF and also 3 rot. DOF. But these can't be defined as rotations about Cartesian axes (see Euler angles).

<u>Hamiltonian</u>: $E = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + U(r)$

- Inf. Energy to get to $r = 0$ unless $l = 0$
- $U \sim 1/r$

<u>Orbit Types</u>:

$E > 0$:	Hyperbola
$E = 0$:	Parabola
$E < 0$:	Ellipse
$E = \text{Min}(U_{\text{eff}})$:	Circle

<u>Kepler's Laws</u>:

1st Law :	Elliptical Orbits	(Sun [at/orbiting] focus)
2nd Law :	Equal Area Sweep	$(r^2 d\theta \;=\; \frac{l}{m}dt)$
3rd Law :	$T^2 = k^2 a^3$	T, Period a, Semi-major axis k, "constant" $\left(\frac{2\pi}{\sqrt{G[m_{\text{planet}} + M_{\text{sun}}]}}\right)$

7 Fluid Mechanics

Bernoulli's Principle :	$\frac{\rho v^2}{2} + \rho g z + P_{\text{res}}$	$=$ constant	[Energy Density]
Fluid Conservation :	$\rho A v$	$=$ constant	[Mass Flow Rate]
Bouyant Force :	$F = \rho V g$		(ρ, V, of displaced liquid)

<u>Water Facts</u> :

- 1 L = 1 kg

8 Oscillators

8.1 Homogenous

$$(F = m\ddot{x}) = -kx - \overset{\text{(damp)}}{b\dot{x}}$$

$$\downarrow$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

$z_{\text{tr}}(t) = \tilde{C}e^{rt} + [\tilde{D}_{\text{opt.}}\ te^{rt}]$: $\underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}}$

$$(r^2 + 2\beta r + \omega_0^2)e^{rt} = 0$$

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

<u>Normal (Undamped)</u>: $\big(F = -kx\big) \Rightarrow \big(\ddot{x} = -\omega_0^2 x = -\frac{k}{m}x\big)$

$$\boxed{z_{\text{tr}}(t) = \tilde{C}_1 e^{i\omega_0 t} + \tilde{C}_2 e^{-i\omega_0 t}}$$

<u>Underdamped</u>: $\big(\beta < \omega_0\big)$

$$\boxed{z_{\text{tr}}(t) = \left(\tilde{C}_1 e^{i\sqrt{\omega_0^2 - \beta^2}t} + \tilde{C}_2 e^{-i\sqrt{\omega_0^2-\beta^2}t}\right)\underline{e^{-\beta t}}}$$

<u>Critically Damped</u>: $\big(\beta = \omega_0\big)$

$$\boxed{z_{\text{tr}}(t) = \big(\tilde{C}_1 + \tilde{C}_2 t\big)\underline{e^{-\beta t}}}$$

Decay rate is maximized at $\beta = \omega_0$

<u>Overdamped</u>: $\big(\beta > \omega_0\big)$

$$\boxed{z_{\text{tr}}(t) = \underbrace{\tilde{C}_1 e^{-\left(\beta - \sqrt{\beta^2-\omega_0^2}\right)t}}_{\text{(smaller, lasts longer)}} + \tilde{C}_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}}$$

8.2 Inhomogenous (Driven)

$$\boxed{z(t) = z_{\text{st}}(t) + z_{\text{tr}}(t)}$$

$$m\ddot{x} = -kx - b\dot{x} + F_{\text{dr}}$$

$$\downarrow$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t$$

- $L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t)$

$\boxed{z_{\text{st}}(t) = \tilde{C}e^{i\omega t} = Ae^{i(\omega t - \delta)}}$: $\underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}}$

$$(-\omega^2 + 2i\beta\omega + \omega_0^2)\tilde{C}e^{i\omega t} = f_0 e^{i\omega t}$$

$$\tilde{C} = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = Ae^{-i\delta}$$

$$\boxed{A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\ ,\quad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)}$$

Resonance (Max A^2) with fixed ω : $\boxed{\omega_0 = \omega}$

Resonance (Max A^2) with fixed ω_0 : $\boxed{\omega = \sqrt{\omega_0^2 - 2\beta^2}}$ (usually $\beta \ll \omega$)

Full Width at Half Max, $A^2(\omega)$: FWHM $\approx 2\beta$

Quality Factor (Sharpness) : $Q = \frac{\omega_0}{2\beta} = \left(\pi\frac{1/\beta}{2\pi/\omega_0} = \pi\frac{\text{decay time}}{\text{period}}\right) = \left(2\pi\frac{\text{Energy stored}}{\text{Energy Dissipated}}\right)$

8.3 Parallel and Series

Series, k_1+k_2+m: $\dfrac{1}{K_{\text{eq}}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$

Parallel, k_1k_2+m: $K_{\text{eq}} = k_1 + k_2$

8.4 Normal Modes: 3 Springs + 2 Masses, k_1+m_1+k_2+m_2+k_3

1.)
$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2x_1 + k_2x_2 \\ &= -(k_1+k_2)x_1 + k_2x_2 \\[1em] m_2\ddot{x}_2 &= k_2x_1 - k_2x_2 - k_3x_2 \\ &= k_2x_1 - (k_2+k_3)x_2 \end{aligned} \quad \rightarrow \quad \begin{gathered} \mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x} \\[1em] \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{gathered}$$

2.)
$$\begin{aligned} \mathbf{z}(t) = \mathbf{a}e^{i\omega t} &= \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} e^{i\omega t} \\ &= \begin{pmatrix} a_1e^{-i\delta_1 t} \\ a_2e^{-i\delta_2 t} \end{pmatrix} e^{i\omega t} \end{aligned} \quad \rightarrow \quad \begin{aligned} \mathbf{M}\ddot{\mathbf{z}} &= -\mathbf{K}\mathbf{z} \\ -\omega^2\mathbf{M}\mathbf{a}e^{i\omega t} &= -\mathbf{K}\mathbf{a}e^{i\omega t} \\[1em] (\mathbf{K}-\omega^2\mathbf{M})\mathbf{a} &= 0 \\ \boxed{\det(\mathbf{K}-\omega^2\mathbf{M}) = 0} \end{aligned}$$

$x(t) = \text{Re}\big[z(t)\big]$ is the real solution.

Same m and k

$$\begin{pmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 m \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \quad \rightarrow \quad \begin{gathered} \boxed{\omega = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}} \\[1em] \boxed{z(t) = \tilde{A}_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_1 t} + \tilde{A}_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_2 t}} \end{gathered}$$

Smaller ω_1 is most symmetric motion
(both swing in phase)

Larger ω_2 swings out of phase

Weak Coupling

8.5 Single Pendulum (Use Lagrangian)

- $T = \frac{1}{2} m R^2 \dot{\theta}^2$
- $U = mg(R - R\cos\theta)$

$$\rightarrow \quad \begin{aligned} mR^2\ddot{\theta} &= -mgR\sin\theta \\ &\approx -mgR\theta \end{aligned} \quad \rightarrow \quad \boxed{\begin{aligned} \ddot{\theta} = -\left(\tfrac{g}{I/mR}\right)\theta = -\omega^2\theta \\ \theta(t) = \mathrm{Re}\big[C_1 e^{i\omega t} + C_2 e^{-i\omega t}\big] \end{aligned}}$$

8.6 Double Pendulum (Use Lagrangian)

- $T = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^{\,2} + \frac{1}{2} m_2 (L_1 \dot{\vec{\theta}}_1 + L_2 \dot{\vec{\theta}}_2)^2$
  $= \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\theta}_1^{\,2} + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^{\,2}$
  $+ m_2 L_2 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$
- $U = m_1 g (L_1 - L_1 \cos\theta_1)$
  $+ m_2 g (L_1 + L_2 - L_2\cos\theta_2 - L_1\cos\theta_1)$

$\rightarrow$
$$\mathbf{M}\ddot{\theta} = -\mathbf{K}\theta \qquad \text{(small angle quadratic approx.)}$$

$$\begin{pmatrix} (m_1 + m_2) L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\begin{pmatrix} (m_1 + m_2) g L_1 + k_2 & 0 \\ 0 & m_2 g L_2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

- $\begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t - t \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 - x_2 \end{pmatrix}$
- $m_1 \begin{pmatrix} 1 \\ v_1 \end{pmatrix} + m_2 \begin{pmatrix} 1 \\ v_2 \end{pmatrix} = m_1' \begin{pmatrix} 1 \\ v_1' \end{pmatrix} + m_2' \begin{pmatrix} 1 \\ v_2' \end{pmatrix}$