$$\boxed{\vec{\nabla} = \left[\vec{\nabla}\left(r,\theta,\phi\right)\right]\bar{\partial}_\circ}$$

$$d = [dx\ dy\ dz]\ \vec{\nabla} = d\vec{l}^{\,T}\vec{\nabla}$$

$$d(r,\theta,\phi) = [dx\ dy\ dz]\ \vec{\nabla}(r,\theta,\phi)$$

$$\boxed{\partial\bar{l}_\circ^{\,T} = d\vec{l}^{\,T}\vec{\nabla}(r,\theta,\phi)}$$

$$\partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ = d\vec{l}^{\,T}\left[\vec{\nabla}(r,\theta,\phi)\right]\bar{\partial}_\circ$$

$$\boxed{d = \partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ = d\vec{l}^{\,T}\vec{\nabla}}$$

$$\underline{\vec{\nabla}} = \begin{bmatrix}\frac{\partial}{\partial x}\\[2pt]\frac{\partial}{\partial y}\\[2pt]\frac{\partial}{\partial z}\end{bmatrix} = \begin{bmatrix}| & | & |\\ \nabla r & \nabla\theta & \nabla\phi\\ | & | & |\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r}\\[2pt]\frac{\partial}{\partial\theta}\\[2pt]\frac{\partial}{\partial\phi}\end{bmatrix}$$

$$\underline{\partial\bar{l}_\circ} = \begin{bmatrix}dr\\ d\theta\\ d\phi\end{bmatrix} = \left[\vec{\nabla}(r,\theta,\phi)\right]^T d\vec{l}$$

$$= \begin{bmatrix}-\nabla r-\\ -\nabla\theta-\\ -\nabla\phi-\end{bmatrix}\begin{bmatrix}dx\\ dy\\ dz\end{bmatrix}$$

$$\theta = \theta_{(x,y,z)}\quad \left(x^2+y^2 = z^2\tan^2\theta\right)$$

$$\phi = \phi_{(x,y,z)}\quad \left(y = x\tan\phi\right)$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z}$$

$$d\theta = dx\frac{\partial\theta}{\partial x} + dy\frac{\partial\theta}{\partial y} + dz\frac{\partial\theta}{\partial z}$$

$$dy_{\vec{r}_\circ}(\vec{r}_\circ')\big|_{t=0} = \left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi\right)\frac{1}{dt}\big|_{t=0}$$

$$\boxed{\vec{b}^{\,i}\cdot\vec{b}_i = \delta_{ij}}:\quad \begin{array}{l}\vec{\nabla}\phi\cdot\frac{\partial\vec{r}}{\partial\phi} = 1\\[4pt]\vec{\nabla}\phi\cdot\frac{\partial\vec{r}}{\partial\theta} = 0\end{array}\ \Rightarrow\ \begin{bmatrix}-\vec{\nabla}r-\\ -\vec{\nabla}\theta-\\ -\vec{\nabla}\phi-\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\end{bmatrix}\vec{r} = \mathbb{1}_3\ \Rightarrow\ \boxed{\frac{\partial y}{\partial\phi} = [0\ 1\ 0]\begin{bmatrix}-\vec{\nabla}r-\\ -\vec{\nabla}\theta-\\ -\vec{\nabla}\phi-\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{\partial y}{\partial\phi}^{T}}$$

$$d = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} = d\vec{l}\cdot\vec{\nabla} \qquad = \left[\frac{dr}{\|\nabla r\|}, \frac{d\theta}{\|\nabla\theta\|}, \frac{d\phi}{\|\nabla\phi\|}\right]\left[\|\nabla r\|\frac{\partial}{\partial r}, \|\nabla\theta\|\frac{\partial}{\partial\theta}, \|\nabla\phi\|\frac{\partial}{\partial\phi}\right]^T$$

$$= dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial\theta} + d\phi\frac{\partial}{\partial\phi} = \partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ \qquad = [dr, rd\theta, r\sin\theta d\phi]\left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right]^T$$

$$= \frac{dr}{\|\nabla r\|}\|\nabla r\|\frac{\partial}{\partial r} + \frac{d\theta}{\|\nabla\theta\|}\|\nabla\theta\|\frac{\partial}{\partial\theta} + \frac{d\phi}{\|\nabla\phi\|}\|\nabla\phi\|\frac{\partial}{\partial\phi} \quad \cdots = d\bar{l}_\circ^{\,T}\bar{\nabla}_\circ = d\vec{l}_\circ^{\,T}\vec{\nabla}_\circ$$

$$d\vec{l} = d\vec{r} = [dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial\theta} + d\phi\frac{\partial}{\partial\phi}](x,y,z)^T$$

$$d(x,y,z) = \left[\frac{dr}{\|\nabla r\|}\|\nabla r\|\frac{\partial}{\partial r} + \frac{d\theta}{\|\nabla\theta\|}\|\nabla\theta\|\frac{\partial}{\partial\theta} + \frac{d\phi}{\|\nabla\phi\|}\|\nabla\phi\|\frac{\partial}{\partial\phi}\right](x,y,z)$$

$$(dx,dy,dz) = dr\,\hat{r}^T + r\,d\theta\,\hat{\theta}^T + r\sin\theta\,d\phi\,\hat{\phi}^T$$

$$(\hat{r},\hat{\theta},\hat{\phi}) \equiv \left(\|\nabla r\|\frac{\partial\vec{r}}{\partial r}, \|\nabla\theta\|\frac{\partial\vec{r}}{\partial\theta}, \|\nabla\phi\|\frac{\partial\vec{r}}{\partial\phi}\right)$$

$$= \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\otimes(x,y,z)^T$$

$$= \bar{\nabla}_\circ^T\otimes\vec{r}$$

$$\boxed{d\vec{l} = (dx,dy,dz)\cdot(\hat{x},\hat{y},\hat{z})} = \boxed{(dr, rd\theta, r\sin\theta d\theta)\cdot\left(\hat{r},\hat{\theta},\hat{\phi}\right) = d\bar{l}_\circ} = d\bar{l}_\circ^{\,T}\cdot\left(\hat{r},\hat{\theta},\hat{\phi}\right)$$

$$\boxed{\vec{\nabla} = (\hat{x},\hat{y},\hat{z})\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)} = \boxed{\left(\hat{r},\hat{\theta},\hat{\phi}\right)\cdot\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) = \vec{\nabla}_\circ} = \left(\frac{\partial\vec{r}}{\partial r}, \frac{1}{r}\frac{\partial\vec{r}}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\vec{r}}{\partial\phi}\right)\bar{\nabla}_\circ^T$$

$$\vec{\nabla} = [\bar{\nabla}_\circ^T\otimes\vec{r}]\bar{\nabla}_\circ = \left[\bar{\nabla}_\circ^T\otimes(x,y,z)^T\right]\begin{bmatrix}\frac{\partial}{\partial r}\\[2pt]\frac{1}{r}\frac{\partial}{\partial\theta}\\[2pt]\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\end{bmatrix} \quad\Rightarrow\ \frac{\partial}{\partial x} = \frac{\partial x}{\partial r}\frac{\partial}{\partial r} + \underline{\|\nabla\theta\|^2\frac{\partial x}{\partial\theta}}\frac{\partial}{\partial\theta} + \underline{\|\nabla\phi\|^2\frac{\partial x}{\partial\phi}}\frac{\partial}{\partial\phi}$$

$$= [\vec{\nabla}(r,\theta,\phi)]\bar{\partial}_\circ = [\vec{\nabla}(r,\theta,\phi)]\begin{bmatrix}\frac{\partial}{\partial r}\\[2pt]\frac{\partial}{\partial\theta}\\[2pt]\frac{\partial}{\partial\phi}\end{bmatrix} \quad\Rightarrow\ \frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \underline{\frac{\partial\theta}{\partial x}}\frac{\partial}{\partial\theta} + \underline{\frac{\partial\phi}{\partial x}}\frac{\partial}{\partial\phi}\ \Rightarrow\ \boxed{\frac{\partial\phi}{\partial y} = \frac{\partial y}{\partial\phi}\|\nabla\phi\|^2}$$

	contravariant $_i$		(equal since orthog.)		covariant ^{i}	
$\hat{r} = (\hat{r}_x, \hat{r}_y, \hat{r}_z) =$	$\frac{\vec{r}}{r} =$	$\frac{\partial}{\partial r}\vec{r} = \frac{\partial\vec{r}}{\partial r}\|\frac{\partial\vec{r}}{\partial r}\|^{-1}$	$\overset{\rightarrow}{=}$	$\|\nabla r\|\frac{\partial\vec{r}}{\partial r}$	$\overset{\leftarrow}{=}\ \frac{\nabla r}{\|\nabla r\|}\ =$	∇r
$\hat{\theta} = (\hat{\theta}_x, \hat{\theta}_y, \hat{\theta}_z) =$	$\frac{\partial\hat{r}}{\partial\theta} =$	$\frac{1}{r}\frac{\partial}{\partial\theta}\vec{r} = \frac{\partial\vec{r}}{\partial\theta}\|\frac{\partial\vec{r}}{\partial\theta}\|^{-1}$	$\overset{\rightarrow}{=}$	$\|\nabla\theta\|\frac{\partial\vec{r}}{\partial\theta}$	$\overset{\leftarrow}{=}\ \frac{\nabla\theta}{\|\nabla\theta\|}\ =$	$r\nabla\theta$
$\hat{\phi} = (\hat{\phi}_x, \hat{\phi}_y, \hat{\phi}_z) = \frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi} =$	$\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\vec{r} =$	$\frac{\partial\vec{r}}{\partial\phi}\|\frac{\partial\vec{r}}{\partial\phi}\|^{-1}$	$\overset{\rightarrow}{=}$	$\|\nabla\phi\|\frac{\partial\vec{r}}{\partial\phi}$	$\overset{\leftarrow}{=}\ \frac{\nabla\phi}{\|\nabla\phi\|}\ =$	$r\sin\theta\nabla\phi$

1

1 Frenet Equations

$$(a \times b) \cdot (c \times d) = a \cdot b \times (c \times d)$$

$$= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

$$\boxed{\frac{dt}{ds} = \frac{1}{v}}$$

$$T = \hat{v} = \frac{\vec{v}}{v}$$

$$\frac{dT}{dt} = \frac{(\vec{v} \cdot \vec{v})\vec{a} - (\vec{v} \cdot \vec{a})\vec{v}}{v^3} = \frac{\vec{v} \times (\vec{a} \times \vec{v})}{v^3} = \frac{(\vec{v} \times \vec{a}) \times \vec{v}}{v^3}$$

$$\left\| \frac{dT}{dt} \right\| = \frac{\sqrt{v^2 a^2 - (\vec{v} \cdot \vec{a})^2}}{v^2} = \frac{\|\vec{a} \times \vec{v}\|}{v^2} \;, \quad \frac{dT}{ds} = k\hat{N}$$

$$\hat{N} = \frac{T'}{\|T'\|} = \frac{(\vec{v} \times \vec{a}) \times \vec{v}}{\|\vec{v} \times \vec{a}\| v} = \hat{B} \times \hat{v}$$

$$\hat{B} = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|} = \widehat{v \times a} = \hat{v} \times \hat{N} \quad (\hat{B} \cdot \vec{v} = 0)$$

$$\frac{d\hat{B}}{dt} = \frac{\vec{v} \times \dot{\vec{a}}}{\|\vec{v} \times \vec{a}\|} - \left[\frac{\vec{v} \times \dot{\vec{a}}}{\|\vec{v} \times \vec{a}\|} \cdot \hat{B} \right] \hat{B} \;, \quad \tau = \hat{N} \cdot \frac{d\hat{B}}{ds}$$

$$\boxed{\begin{aligned} \vec{a} &= a_T \hat{T} + a_N \hat{N} \\ a_T &= \vec{a} \cdot \hat{v} = \frac{dv}{dt} \\ a_N &= \frac{\|\vec{a} \times \vec{v}\|}{v} = \|\vec{a} \times \hat{v}\| \\ a^2 &= a_T^2 + a_N^2 = \left\| \frac{d\vec{v}}{dt} \right\|^2 \end{aligned}}$$

Frenet Trihedron

Differentiable (in this book) : C^∞

No singular pts. Order 0 (Regular) : $\vec{v}(t) \neq 0$

- $\|\vec{v}(t)\| = c \to 1 \Rightarrow \int_s \|\vec{v}(t)\| dt = t = \Delta s$

  $\to s: \; \vec{x}(t) = \vec{x}(s)$

- $\frac{1}{2} \frac{d}{dt}(\vec{v} \cdot \vec{v}) = \boxed{\vec{v} \cdot \vec{a} = 0}$

No singular pts. Order 1 : $\vec{a}(t) \neq 0$

- Curvature, $k \neq 0$ (see right)   • Vertex, $k' = 0$

$1 = \|\vec{t}\| = \|\vec{n}\| = \|\vec{b}\| \;, \quad 0 = \vec{t} \cdot \vec{n} = \vec{n} \cdot \vec{b} = \vec{b} \cdot \vec{t}$

- $\vec{v}(s) = \vec{t}(s)$ $\boxed{(t = n \times b)}$

- $\vec{a}(s) = \boxed{\vec{t'}(s) = k(s)\vec{n}(s)}$, $k(s) \geq 0$ (can be L or R-handed) (can be neg. if in $\mathbb{R}^2$)

- $*$ $k(s) > 0$ for well defined curve with $\hat{n}$

- $\boxed{\vec{b} = \vec{t} \times \vec{n}}$, $\frac{d}{dt}(\vec{b} \cdot \vec{b}) = \vec{b} \cdot \vec{b'} = 0$, $*$ $\boxed{\vec{b'}(s) = \tau_{(s)}\vec{n}(s)}$

- $\boxed{\vec{n} = \vec{b} \times \vec{t}}$, $*$ $\boxed{\vec{n'}(s) = -k\vec{t} - \tau\vec{b}}$, $*$ t-n pl. = osculating pl.

- $t''(s) = k'n - k^2 t - k\tau b$ • $b''(s) = \tau'n - \tau k t - \tau^2 b$ • $n''(s) = -k't - \tau'b - (k^2 + \tau^2)n$

- $|\tau| = \|b'\|$ • $\tau = -\frac{(t \times t') \cdot t''}{k^2} = -\frac{t \cdot (t' \times t'')}{\|t'\|^2}$ • $k = \|t'\| = \frac{(b \times b') \cdot b''}{\tau^2} = \frac{b \cdot (b' \times b'')}{\|b'\|^2}$

- $n \Rightarrow k, \tau :$ $*$ $\|n'\|^2 = k^2 + \tau^2$ $*$ $\frac{(n \times n') \cdot n''}{\|n'\|^2} = \frac{k'\tau - k\tau'}{k^2 + \tau^2} = \frac{\frac{d}{ds}(k/\tau)}{(k/\tau)^2 + 1} = \frac{d}{ds} \arctan(k/\tau)$

Indicatrix [of Tangents] :

- $\vec{t}(\theta_{(s)}) = (\cos\theta, \sin\theta) = (x'_{(s)}, y'_{(s)})$
- $\vec{t'}(\theta) = \underline{\theta'_{(s)}}(-\sin\theta, \cos\theta) = \underline{k(s)\vec{n}}$
- $\theta_{(s)} = \arctan(y'/x')$
- $\int_0^l k(s) \, ds = \theta(s) \Big|_0^l = 2\pi I_{\text{rot. index}}$

Local Canonical Form at $t = 0$:

- $(\hat{t}, \hat{n}, \hat{b}) = (\hat{x}, \hat{y}, \hat{z})$
- $\vec{r}(s) - \vec{r}(0) \approx (s - \frac{k^2 s^3}{6}, \frac{k}{2}s^2 + \frac{k' s^3}{6}, \frac{-k\tau}{6}s^3)$
- $\tau < 0 \Rightarrow \frac{dz}{ds} > 0$

Isoperimetric Inequality : $0 \leq l^2 - 4\pi A$

Four-Vertex Theorem : A simple closed curve has ≥ 4 vertices

Cauchy-Crofton Formula (measure of number of times lines intersect a curve) :

- Tangent line at (ρ, θ) : $x\cos\theta + y\sin\theta = \rho$ • Curve c : $y = 0$, $x \in (-l/2, l/2)$, $C = \sum c_i$
- $\int \text{Lines that cross } c = \int_0^{2\pi} \int_0^{|\cos\theta|l/2} d\rho \, d\theta = 2l \Rightarrow \int_0^{2\pi} \int_0^\infty n_C \, d\rho \, d\theta = 2l$

2 Jacobian/Differential, $dF_{\alpha(0)} : \mathbb{R}^n \to \mathbb{R}^m$

- $\boxed{\alpha(0) = \beta(0)}$ $\Rightarrow$ $\underline{F(t=0)} = F \circ \alpha\big|_{t=0} = F \circ \beta\big|_{t=0}$

- $\boxed{\alpha'(0) = \beta'(0)}$ $\Rightarrow$ $\frac{\partial x}{\partial \alpha_i}\big|_{t=0} = \frac{\partial x}{\partial \beta_i}\big|_{t=0} \cdot \frac{\cancel{d\beta_i/dt}}{\cancel{d\alpha_i/dt}}\big|_{\cancel{t=0}}$ $\Rightarrow$ $\underline{dF_{\alpha(0)}(\alpha'(0)) = dF_{\beta(0)}(\beta'(0))}$

- $*\ F = (x_0, x_1, \ldots, x_m)$ $\Rightarrow$ $\underline{dF_{\alpha(0)}(\alpha'(0))} \equiv \frac{d}{dt}(F \circ \alpha)\big|_{t=0} = \begin{bmatrix} \frac{\partial x_1}{\partial \alpha_0} & \frac{\partial x_1}{\partial \alpha_1} & \cdots \\ \frac{\partial x_2}{\partial \alpha_0} & \frac{\partial x_2}{\partial \alpha_1} & \cdots \\ \vdots & & \end{bmatrix}_{t=0} \begin{bmatrix} \frac{d\alpha_0}{dt} \\ \frac{d\alpha_1}{dt} \\ \vdots \end{bmatrix}_{t=0} = \boxed{J_{F(0)} \cdot \alpha'(0)}$

- $d(G \circ F)_p = dG_{F(p)} \circ dF_p$

3 Del

$$\nabla F = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) F$$

$$= \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{bmatrix} F = \begin{bmatrix} \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z} \\ \cos\phi\cos\theta\,\hat{x} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z} \\ -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{bmatrix} F$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F = \begin{bmatrix} \cos\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \end{bmatrix} F = \begin{bmatrix} \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} \\ \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi} \\ \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi} \end{bmatrix} F = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F$$

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{1}{r\sin\theta}\left\langle \frac{\partial}{\partial r},\ \frac{\partial}{\partial\theta},\ \frac{\partial}{\partial\phi} \right\rangle \cdot [r \cdot r\sin\theta]\left\langle A_r,\ \frac{1}{r}A_\theta,\ \frac{1}{r\sin\theta}A_\phi \right\rangle$$

$$\nabla \times \vec{A} = \frac{1}{r}\frac{1}{r\sin\theta}\left\| \begin{matrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{matrix} \right\| = \left\| \begin{matrix} \frac{\partial\vec{r}}{\partial r} & \frac{\partial\vec{r}}{\partial\theta} & \frac{\partial\vec{r}}{\partial\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{matrix} \right\|$$

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \vec{A} \cdot (B_i\vec{C}) - \vec{A} \cdot (\vec{B}C_i)$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \boxed{(\vec{A} \cdot (\vec{B} \otimes \vec{C})^T)^T - (\vec{A} \cdot \vec{B} \otimes \vec{C})^T} = (\vec{A} \otimes \vec{B}) \cdot \vec{C} - (\vec{A} \cdot \vec{B} \otimes \vec{C})^T$$

$$= (A^T(BC^T)^T)^T - (A^T BC^T)^T = (AB^T)^T C - (A^T BC^T)^T$$