1 An Easy First Order Differential (& Transport Equation)

$$\boldsymbol{A(x)\ u_x + B(x,y)\ u_y = f(u)}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{B(x,y)}{A(x)} \quad \Rightarrow \quad \int \frac{dy}{\beta(y)} = \int \frac{dx}{\alpha(x)} \quad \Rightarrow \quad Y(y) = X(x) + C$$

$$\begin{aligned} C &= Y(y) - X(x) \\ \epsilon &= x \end{aligned}$$

$$u_x = u_C \frac{dC}{dx} + u_\epsilon \frac{d\epsilon}{dx} = \frac{-1}{\alpha(x)} u_C + u_\epsilon$$

$$u_y = u_C \frac{dC}{dy} + u_\epsilon \frac{d\epsilon}{dy} = \frac{1}{\beta(y)} u_C$$

$$\begin{aligned} f(u) &= A(x)u_x + B(x,y)u_y \\ &= A(x)\left[\frac{-1}{\alpha(x)} u_C + u_\epsilon\right] + \frac{B(x,y)}{\beta(y)} u_C \\ &= A(x)u_\epsilon + \left[\frac{B(x,y)}{\beta(y)} - \frac{A(x)}{\alpha(x)}\right] u_C \\ &= A(\epsilon)\frac{\partial u}{\partial \epsilon} + [0, \text{ hopefully}] \end{aligned}$$

f(u) ≠ 0 : $\displaystyle \int \frac{du}{f(u)} = \int \frac{d\epsilon}{A(\epsilon)}$

$\gamma(u) = h(\epsilon) + g(C)$

$u(x,y) = \gamma^{-1}\{\ h(x) + g[\ Y(y) - X(x)\ ]\ \}$

f(u) = 0 : $\displaystyle \frac{\partial u}{\partial \epsilon} = 0$

$u(x,y) = g(C) = g[\ Y(y) - X(x)\ ]$

2 Quadratic-esque Equations (pre-Wave Equation)

$$\begin{aligned} \mathbf{0} &= \boldsymbol{\alpha u_{xx} + \beta u_{xt} + \gamma u_{tt}} \\ &= \boldsymbol{[a\partial_x + b\partial_t][c\partial_x + d\partial_t]u} \end{aligned}$$

$0 \;=\; [a\partial_x + b\partial_t]v \;=\; [c\partial_x + d\partial_t]w$

$u = v\left(t - \frac{b}{a}x\right) + w\left(t - \frac{d}{c}x\right)$

3 Diffusion Equation

$$\boldsymbol{u_t - ku_{xx} = f(x,t)}$$

3.1 $\boldsymbol{x \in (-\infty, \infty)}$

1. $\boldsymbol{u(x,0) = \phi(x)}$

$$u = \int_{-\infty}^{\infty} S(x-y)\phi(y)dy \;+\; \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s)f(y,s)dyds$$

   (a) $\boldsymbol{f(x,t) = 0:}$ $u = \int_{-\infty}^{\infty} S(x-y)\phi(y)dy$

3.2 $\boldsymbol{x \in (0, \infty)}$

1. $\boldsymbol{u(x,0) = \phi_u(x)}$

   (a) **Dirichlet:** $\boldsymbol{u(0,t) = h(t)}$

$$\phi(x) = \begin{cases} \phi_u(x) & x > 0 \\ -\phi_u(-x) & x < 0 \end{cases}$$

$v(x,t) = u - h(t)$

- $v(x,0) = \phi_v = \phi_u(x) - h(0)$
- $v(0,t) = 0$
- $v_t - kv_{xx} = f(x,t) - h'(t)$

$u = h(t) + v(x,t)$

i. $\boldsymbol{u(0,t) = h(t) = 0}$ **:**

$$\begin{aligned} u &= \int_{-\infty}^{\infty} S(x-y)\phi(y)dy \ + \ \int_\Delta S(x-y,t-s)f(y,s)d\Delta \\ &= \int_0^{\infty} S(x-y)\phi_u(y)dy \ - \ \int_{-\infty}^{0} S(x-y)\phi_u(-y)dy \\ &\quad + \int_\Delta S(x-y,t-s)f(y,s)d\Delta \\ &= \int_0^{\infty} [S(x-y) - S(x+y)]\phi_u(y)dy \ + \ \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds \end{aligned}$$

(b) **Neumann:** $\boldsymbol{u_x(0,t) = h(t)}$

$$\phi(x) = \begin{cases} \phi_u(x) & x > 0 \\ \phi_u(-x) & x < 0 \end{cases}$$

$v(x,t) = u - xh(t)$

- $v(x,0) = \phi_v = \phi_u - xh(0)$
- $v(0,t) = 0$
- $v_t - kv_{xx} = f(x,t) - xh'(t)$

$u = xh(t) + v(x,t)$

i. $\boldsymbol{u_x(0,t) = h(t) = 0}$ **:**

$$\begin{aligned} u &= \int_{-\infty}^{\infty} S(x-y)\phi(y)dy \ + \ \int_\Delta S(x-y,t-s)f(y,s)d\Delta \\ &= \int_0^{\infty} S(x-y)\phi_u(y)dy + \int_{-\infty}^{0} S(x-y)\phi_u(-y)dy \\ &\quad + \int_\Delta S(x-y,t-s)f(y,s)d\Delta \\ &= \int_0^{\infty} [S(x-y) + S(x-y)]\phi_u(y)dy \ + \ \int_\Delta S(x-y,t-s)f(y,s)d\Delta \end{aligned}$$

3.3 $\boldsymbol{x \in (0, L)}$

1. $\boldsymbol{u(x,0) = \phi(x), \quad f(x,t) = 0}$

$u = \sum X(x)\ T(t)$

- $\dfrac{T'(t)}{kT(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$

  (a) $\boldsymbol{\lambda > 0}$ **:**
     - $X = A_x \cos(\sqrt{\lambda}x) + B_x \sin(\sqrt{\lambda}x)$
     - $T = A_t e^{-\lambda kt}$

  (b) $\boldsymbol{\lambda = 0}$ **:**
     - $X = A_x x + B_x$
     - $T = A_t$

  (c) $\boldsymbol{\lambda < 0}$ **:**
     - $X = A_x \cosh(\sqrt{\lambda}x) + B_x \sinh(\sqrt{\lambda}x)$
     - $T = A_t e^{\lambda kt}$

(a) **Dirichlet:** $\boldsymbol{u(0,t) = u(L,t) = 0}$

$$\sqrt{\lambda} = \frac{n\pi}{L},\ n \in \mathbb{N}$$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x)$$

$$u = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 kt} \sin(\frac{n\pi}{L}x)$$

(b) **Neumann:** $\boldsymbol{u_x(0,t) = u_x(L,t) = 0}$

$$\sqrt{\lambda} = \frac{n\pi}{L},\ n \in \mathbb{N}_0$$

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi}{L}x)$$

$$u = \frac{1}{2}A_0 \ + \ \sum_{n=1}^{\infty} A_n \ e^{-(\frac{n\pi}{L})^2 kt} \cos(\frac{n\pi}{L}x)$$

(c) **Robin:** $\boldsymbol{u_x(0,t) - a_0 u(0,t) = u_x(L,t) + a_L u(L,t) = 0}$

4 Wave Equation

$$\boldsymbol{u_{tt} - c^2 u_{xx} = f(x,t)}$$

4.1 $x \in (-\infty, \infty)$

1. $u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$

$$u = \frac{1}{2}[\phi(x-ct)+\phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

- $f(x,t) = 0:$ $\quad u = \frac{1}{2}[\phi(x-ct)+\phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy$

4.2 $x \in (0, \infty)$

1. $u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$

   (a) **Direchlet:** $u(0,t) = h(t)$

   $x \in (ct, \infty):$

$$u = \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

   $x \in (0, ct):$

$$^*u = \frac{1}{2}[\phi(ct+x)-\phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x}\psi(y)dy + \frac{1}{2c}\int_0^t\int_{c(t-s)-x}^{c(t-s)+x} f(y,s)dyds + h(t-\frac{x}{c})$$

   - $f(x,t) = h(t) = 0:$ $\quad u = \frac{1}{2}[\phi(x-ct)+\phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy$

4.3 $x \in (0, L)$

1. $u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad f(x,t) = 0$

$$u = \sum X(x)\ T(t)$$

- $\dfrac{T''(t)}{c^2T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$

  (a) $\lambda > 0:$
  - $X = A_x\cos(\sqrt{\lambda}x) + B_x\sin(\sqrt{\lambda}x)$
  - $T = A_t\cos(\sqrt{\lambda}ct) + B_t\sin(\sqrt{\lambda}ct)$

  (b) $\lambda = 0:$

- $X = A_x x + B_x$
- $T = A_t t + B_t$

(c) $\boldsymbol{\lambda < 0}$ **:**

- $X = A_x \cosh(\sqrt{\lambda}x) + B_x \sinh(\sqrt{\lambda}x)$
- $T = A_t \cosh(\sqrt{\lambda}ct) + B_t \sinh(\sqrt{\lambda}ct)$

(a) **Direchlet:** $\boldsymbol{u(0,t) = u(L,t) = 0}$

$$\sqrt{\lambda} = \frac{n\pi}{L},\ n \in \mathbb{N}_0$$

$$\phi(x) = \sum_{n=0}^{\infty} A_n \sin(\frac{n\pi x}{L}), \quad \psi(x) = \sum_{n=0}^{\infty} \frac{n\pi c}{L} B_n \sin(\frac{n\pi x}{L})$$

$$u = \sum_{n=0}^{\infty} \left(A_n \cos(\frac{n\pi}{L}ct) + B_n \sin(\frac{n\pi}{L}ct)\right) \sin(\frac{n\pi}{L}x)$$

(b) **Neumann:** $\boldsymbol{u_x(0,t) = u_x(L,t) = 0}$

$$\sqrt{\lambda} = \frac{n\pi}{L},\ n \in \mathbb{N}_0$$

$$\phi(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\tfrac{n\pi}{L}x), \quad \psi(x) = \tfrac{1}{2}B_0 + \sum_{n=1}^{\infty} \tfrac{n\pi c}{L} B_n \cos(\tfrac{n\pi}{L}x)$$

$$u = \frac{1}{2}A_0 \ + \ \frac{1}{2}B_0 t \ + \ \sum_{n=1}^{\infty} \left(A_n \cos(\frac{n\pi ct}{L}) + B_n \sin(\frac{n\pi ct}{L})\right) \cos(\frac{n\pi}{L}x)$$

5 Error function

$S(x-y,t) = \frac{1}{\sqrt{4kt\pi}} e^{-\frac{(x-y)^2}{4kt}}$

$\int_a^b e^{-p^2} dp = \dfrac{\sqrt{\pi}}{2}[erf(b) - erf(a)]$

- $erf(\infty) = 1 \quad | \quad erf(-\infty) = -1 \quad | \quad erf(0) = 0$
- $\int_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi} \quad | \quad \int_{-\infty}^{0} e^{-p^2} dp = \dfrac{\sqrt{\pi}}{2}$

6 Fourier Series

6.1 Sine

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} S_n \sin\left(\frac{g(n)}{L}\pi x\right) \\
\int_0^L f(x)\sin\left(\frac{h(m)}{L}\pi x\right)dx &= \sum_{n=0}^{\infty}\int_0^L S_n \sin\left(\frac{g(n)}{L}\pi x\right)\sin\left(\frac{h(m)}{L}\pi x\right)dx \\
&= S_{h(m)}\frac{L}{2} \\
S_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{g(n)}{L}\pi x\right)dx
\end{aligned}
$$

6.2 Cosine

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} S_n \cos\left(\frac{g(n)}{L}\pi x\right) \\
\int_0^L f(x)\cos\left(\frac{h(m)}{L}\pi x\right)dx &= \sum_{n=0}^{\infty}\int_0^L S_n \cos\left(\frac{g(n)}{L}\pi x\right)\cos\left(\frac{h(m)}{L}\pi x\right)dx \\
&= S_{h(m)}\frac{L}{2} \\
S_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{g(n)}{L}\pi x\right)dx
\end{aligned}
$$

6.3 Full

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{g(n)}{L}\pi x\right) + B_n \sin\left(\frac{g(n)}{L}\pi x\right) \\
A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{g(n)}{L}\pi x\right)dx \\
B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{g(n)}{L}\pi x\right)dx
\end{aligned}
$$