

# 1 Wave Function

$$\begin{aligned}\Psi_p &= e^{i(2\pi x/\lambda - 2\pi t/T)} \\ &= e^{i(kx - \omega t)} \\ &= e^{\frac{i}{\hbar}(px - Et)}\end{aligned}$$

$$\check{p}\Psi_p = p\Psi_p = \hbar k\Psi_p$$

$$\check{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

$$\check{E}\Psi_p = E\Psi_p = \hbar\omega\Psi_p$$

$$\check{E} = -\frac{\hbar}{i} \frac{\partial}{\partial t}$$

$$\bullet |f\rangle \equiv \int f(x) |x\rangle dx$$

$$\bullet \langle x|x'\rangle \equiv \delta(x - x')$$

$$\bullet \langle x|\hat{x}|x'\rangle \equiv x\langle x|x'\rangle$$

$$1. \langle x|\hat{x}|f\rangle = xf(x)$$

$$\check{x}\langle x|f\rangle \equiv x\langle x|f\rangle$$

$$\begin{aligned}2. \langle x|\hat{p}|x'\rangle &\equiv \frac{\hbar}{i}\delta'(x - x') \\ &= \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|x'\rangle\end{aligned}$$

$$3. \langle x|\hat{p}|f\rangle = \int f(x')\delta'(x - x')dx'$$

$$\check{p}\langle x|f\rangle \equiv \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|f\rangle$$

## 1.1 Schrodinger $\Psi$

$$\check{E}|\Psi\rangle = \hat{H}|\Psi\rangle = (\hat{T} + \hat{V})|\Psi\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \left[\frac{\hat{p}^2}{2m} + V(\hat{x}, t)\right]|\Psi\rangle$$

$$-\check{E}\langle\Psi| = \langle\Psi|\hat{H}$$

$$\check{E}\langle x|\Psi\rangle = \check{H}\Psi = (\check{T} + \check{V})\Psi = \left[\frac{\check{p}^2}{2m} + V(\check{\mathbf{r}}, t)\right]\Psi$$

$$i\hbar\frac{\partial}{\partial t}\Psi(\check{\mathbf{r}}, t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\check{\mathbf{r}}, t)\right]\Psi(\check{\mathbf{r}}, t)$$

$$-\check{E}\langle\Psi|x\rangle = \check{H}\Psi^*$$

If  $V = V(x)$

$$\Psi(x, t) = \psi(x)\phi(t) \Rightarrow$$

$$\bullet E_n\phi_n(t) = i\hbar\frac{\partial}{\partial t}\phi_n(t) \Rightarrow \phi_n(t) = e^{-\frac{i}{\hbar}E_nt}$$

$$\bullet E_n\psi_n(x) = \left(\frac{-\hbar^2}{2m}\partial_x^2 + V(x)\right)\psi_n(x)$$

–  $\psi$  can be lin. sum of real or complex,  
so choose real  $\psi$

$$\begin{aligned}\bullet \text{Linear : } \Psi(x, t) &= \sum_n \psi_n(x) e^{-\frac{i}{\hbar}E_nt} c_n \\ &= \sum_n \langle x|n\rangle e^{-\frac{i}{\hbar}E_nt} \langle n|\Psi\rangle \\ &= \int_{x'} \langle x| \left[ \sum_n |n\rangle e^{-\frac{i}{\hbar}E_nt} \langle n| \right] |x'\rangle \Psi(x') dx' \\ &= \int_{x'} U(x, t; x', 0) \Psi(x') dx'\end{aligned}$$

$$\bullet \sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2 = 0 \Rightarrow \text{measuring stationary state, } \Psi_n, \text{ returns one } E_n \text{ (determinate state)}$$

## 1.2 Usage

- $\langle f|g\rangle = \int_{-\infty}^{\infty} f(x)^* g(x) dx$
  - $\langle f|g\rangle_{ab} = \int_a^b f(x)^* g(x) dx$
  - $|f\rangle = \int f(x')|x'\rangle dx' \sim f(x) \equiv \langle x|f\rangle$
  - $\langle f| = \int f(x)^* [\dots] dx$
  - $\langle f|f\rangle = \int_a^b |f|^2 dx < \infty \Rightarrow f \in L_2(a,b)$
  - $\left(\int_a^b |f|^p dx < \infty \Rightarrow f \in L_p(a,b)\right)$
- 

$$\langle x|\Psi\rangle = \Psi = \begin{cases} \sum_n c_n f_n \\ \int_n c_n f_n dn \end{cases}, \quad \langle f_m|f_n\rangle = \begin{cases} \delta_{mn} \\ \delta_{(m-n)} \end{cases}, \quad \begin{matrix} \text{(see Born int.)} \\ |c_n|^2 = \begin{cases} P(n) \\ \text{PDF}(n) \end{cases} \end{matrix}$$

$$\Rightarrow \boxed{c_n = \langle f_n|\Psi\rangle}$$

$\forall \{f_n\} \in L_2$ :

$$|\Psi\rangle = \begin{cases} \sum_n c_n |f_n\rangle = \sum_n \langle f_n|\Psi\rangle |f_n\rangle = \left(\sum_n |f_n\rangle \langle f_n|\right) |\Psi\rangle = |\Psi\rangle \\ \int_n c_n |f_n\rangle dn = \int_n \langle f_n|\Psi\rangle |f_n\rangle dn = \left(\int_n |f_n\rangle \langle f_n| dn\right) |\Psi\rangle = |\Psi\rangle \end{cases}$$

$\check{x}\Psi_y = x\Psi_y = y\Psi_y$ $\Rightarrow \boxed{\Psi_y = \delta_{(x-y)} = \langle x y\rangle}$	$\langle x \hat{p} p\rangle = \int \langle x \hat{p} x'\rangle \langle x' p\rangle dx'$ $p\langle x p\rangle = \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x p\rangle$ $\langle x \hat{p} p\rangle = \check{p}\Psi_p = p\Psi_p$ $\Rightarrow \boxed{\Psi_p = A e^{\frac{i}{\hbar} p x} = \langle x p\rangle}$	$\langle x \hat{H} n\rangle = E_n \langle x n\rangle$ $\check{H}\Psi_n = E_n \Psi_n$ (See Potential Examples)
$\Psi_{(x,t)} = \int_{-\infty}^{\infty} \Psi_y c_y(t) dy$ $= \int_{-\infty}^{\infty} \delta_{(x-y)} \Psi_{(y,t)} dy$	$\Psi_{(x,t)} = \int_{-\infty}^{\infty} \Psi_p \phi_{(E_p,t)} c_p dp$ $= \int_{-\infty}^{\infty} \frac{e^{\frac{i}{\hbar} p x}}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} \frac{p^2}{2m} t} \Phi_{(p,0)} dp$	$\Psi_{(x,t)} = \int_{-\infty}^{\infty} \Psi_n \phi_{(E_n,t)} c_n dn$ $= \int_{-\infty}^{\infty} \Psi_n e^{\frac{-i}{\hbar} E_n t} c_n dn$
$c_x(t) = \langle \Psi_x \Psi_{(x,t)}\rangle = \langle x \Psi\rangle$ $\Psi_{(x,t)} = \int_{-\infty}^{\infty} \delta_{(x-y)} \Psi_{(y,t)} dy$	$c_p(t) = \langle \Psi_p \Psi_{(x,t)}\rangle = \langle p \Psi\rangle$ $\boxed{\Phi_{(p,t)} = \int_{-\infty}^{\infty} \frac{e^{\frac{-i}{\hbar} p x}}{\sqrt{2\pi\hbar}} \Psi_{(x,t)} dx}$	$c_n(t) = \langle \Psi_n \Psi_{(x,t)}\rangle = \langle n \Psi\rangle$ $\Psi_{(n,t)} = \int_{-\infty}^{\infty} \Psi_n^* \Psi_{(x,t)} dx$

Born Interpretation:  $\text{PDF}(x) = |\Psi(x)|^2 = \Psi^* \Psi$

$$P(a < x < b) = \int_a^b |\Psi|^2 dx \equiv \langle \Psi | \Psi \rangle_{ab}$$

$$- \quad \boxed{\langle \Psi | \Psi \rangle = 1} \quad (\text{physical, bound states only})$$

- $\Psi(\pm\infty) = 0$
- $\text{Min}(V) \leq E_\Psi \in \mathbb{R}$
- $\langle \Psi_n | \Psi_n \rangle \rightarrow \infty \Rightarrow \Psi_n$  not PHYSICAL  
sol. but  $\Psi = \int c_n \Psi_n$  can if  $\langle \Psi | \Psi \rangle = 1$

$$\bullet \quad E[f(x)] = \int_{-\infty}^{\infty} f(x) \text{PDF}(x) dx = \int_{-\infty}^{\infty} f(x) |\Psi(x)|^2 dx = \int_{-\infty}^{\infty} \Psi(x)^* f(x) \Psi(x) dx = \boxed{\langle \Psi | f \Psi \rangle \equiv \langle f(x) \rangle}$$

$$\begin{aligned} \bullet \quad \int_x \Psi^* \Psi dx &= \int_x \left( \int_n c_n^*(t) \Psi_n^*(x) dn \right) \left( \int_{n'} c_{n'}(t) \Psi_{n'}(x) dn' \right) dx \\ &= \int_n c_n^*(t) \int_{n'} c_{n'}(t) \delta(n-n') dn' dn = \int_n |c_n(t)|^2 dn \Rightarrow \boxed{\text{PDF}(n) = |c_n|^2 = c_n^* c_n} \end{aligned}$$

$$\text{Adjoint (herm. adj./herm. conj.): } \{A^\dagger : \langle f | A f \rangle = \langle A^\dagger f | f \rangle\} \Rightarrow \langle h | \hat{A} g \rangle = \langle \hat{A}^\dagger h | g \rangle \quad (\text{let } f=h+g, f=h+ig)$$

Hermitian Operator:  $\{A : \hat{A}^\dagger = \hat{A}\}$

- $\boxed{\exists \{\Psi_n\} : \hat{A} \Psi_n(x) = a_n \Psi_n(x)}$  (spectral theorem)
- $\langle a \rangle = a \in \mathbb{R} \Rightarrow \hat{A}$  can be an observable
- $\boxed{\langle \Psi_m | \Psi_n \rangle \in \{\delta_{mn}, \delta_{(m-n)}\}}$
- $\boxed{\text{Axiom: } \{\Psi_n\} \text{ for } \hat{A} \text{ are complete}}$

$$\text{Non-degenerate: } (m \neq n), (a_m \neq a_n) \Rightarrow \langle \Psi_m | \Psi_n \rangle \in \{\delta_{mn}, \delta_{(m-n)}\}$$

$$\begin{aligned} \text{Degenerate: } (m \neq n), (a_m = a_n), (\Psi_m \neq \Psi_n), \langle \Psi_m | \Psi_n \rangle \neq 0 &\Rightarrow \text{Use Gram-Schmidt} \\ \text{to find orthogonal } \langle \Psi'_m | \Psi'_n \rangle &= \langle a \Psi_m + b \Psi_n | c \Psi_m + d \Psi_n \rangle = 0 \end{aligned}$$

Expectation:  $E[\hat{A}(x,p)]$

$$\bullet \quad \int_{-\infty}^{\infty} \hat{A}(x,p)^* \Psi^* \Psi dx = \langle \hat{A} \Psi | \Psi \rangle = \boxed{\langle \Psi | \hat{A} \Psi \rangle \equiv \langle \hat{A}(x,p) \rangle} \quad (\text{won't work if } \int A |\Psi|^2 dx)$$

$$\begin{aligned} \langle \Psi | \hat{A} \Psi \rangle &= \int_{-\infty}^{\infty} \Psi^* \hat{A} \Psi dx = \int_{-\infty}^{\infty} \left( \int_n c_n^* \Psi_n^* dn \right) \left( \int_{n'} c_{n'} \hat{A} \Psi_{n'} dn' \right) dx \\ &= \int_n a_n |c_n|^2 dn = E[a] \equiv \langle a \rangle \quad c_n = \text{PDF}(n) \quad (\text{see above and Momentum Space}) \end{aligned}$$

$$\boxed{\langle a \rangle = \langle \Psi | \hat{A} \Psi \rangle = \langle \Psi | \hat{A} | \Psi \rangle = \langle A \rangle}$$

$$\bullet \quad \boxed{\langle \sigma_a^2 \rangle = \langle a^2 \rangle - \langle a \rangle^2} \Rightarrow \sigma_A^2 = 0 \quad \text{for } \Psi_n \quad (\text{determinate state})$$

## Matrix Operators:

Given complete  $\{e_n\} : \langle e_m | e_n \rangle = \delta_{mn}$

1.)  $\boxed{Q_{mn}^{(e)} \equiv \langle e_m | \hat{Q}_{(x,p)} | e_n \rangle}$

$$| \beta \rangle = \hat{Q} | \alpha \rangle = \sum_m | e_m \rangle \left[ \begin{array}{l} \langle e_m | \beta \rangle = \langle e_m | \hat{Q} | \alpha \rangle \\ \sum_n b_n \langle e_m | e_n \rangle = \sum_n a_n \boxed{\langle e_m | \hat{Q} | e_n \rangle} \\ \boxed{b_m = \sum_n \left( Q_m^{(e)} \right)_n a_n} \end{array} \right] = \begin{array}{l} \sum_m b_m | e_m \rangle = \sum_{n,m} \langle e_n | \alpha \rangle Q_{mn}^{(e)} | e_m \rangle \\ = \sum_{n,m} Q_{mn}^{(e)} | e_m \rangle \langle e_n | \alpha \rangle \\ \Rightarrow \boxed{\hat{Q} = \sum_{m,n} Q_{mn}^{(e)} | e_m \rangle \langle e_n |} \end{array}$$

2.) Find  $\hat{Q}$  as a matrix

$$\begin{array}{l} | f \rangle = \sum_n c_n^{(e)[f]} | e_n \rangle \\ \downarrow \\ f(x) = \sum_n c_n^{(e)[f]} e_n(x) \end{array} = \begin{pmatrix} \vdots \\ c_n[f] \\ \vdots \end{pmatrix}^{(e)} \cdot \begin{pmatrix} \vdots \\ e_n(x) \\ \vdots \end{pmatrix} \equiv \boxed{\vec{c}^{(e)[f]} \cdot \vec{e}(x)} = \boxed{\int_n c^{(e)[f](n)} \cdot e(n,x) dn}, \quad \boxed{c_n^{(e)[f]} = \langle e_n | f \rangle}$$

$$\hat{Q} | f \rangle$$

$$\begin{aligned} &= \left( \sum_{m,n'} Q_{mn'}^{(e)} | e_m \rangle \langle e_{n'} | \right) \sum_n c_n^{(e)} | e_n \rangle \\ &= \sum_{m,n} \left( \sum_{n'} Q_{mn'}^{(e)} c_n^{(e)} \langle e_{n'} | e_n \rangle \right) | e_m \rangle \\ &= \sum_m \left( \sum_n (Q_m^{(e)})_n c_n^{(e)} \right) | e_m \rangle \end{aligned}$$

$$\begin{aligned} \hat{Q} \left[ \begin{pmatrix} | \\ c \\ | \end{pmatrix}^{(e)} \cdot \begin{pmatrix} | \\ e \\ | \end{pmatrix} \right] &= \left[ \begin{pmatrix} - & \vdots & - \\ & Q_m & \\ & \vdots & \end{pmatrix}^{(e)} \begin{pmatrix} | \\ c \\ | \end{pmatrix}^{(e)} \right] \cdot \begin{pmatrix} | \\ e \\ | \end{pmatrix} \\ \hat{Q} | f \rangle &= \boxed{\hat{Q} [ \vec{c}^{(e)[f]} \cdot \vec{e} ]} = \boxed{[ \bar{Q}^{(e)} \vec{c}^{(e)[f]} ] \cdot \vec{e}} \\ \langle x | \hat{Q} | f \rangle &= \int_m [ \bar{Q}^{(\delta)} f ]_{(m)} \cdot \delta_{(x-m)} dm \\ \text{e.g.} \quad &= \int_m \left[ \int_n Q_m^{(\delta)}(n) \cdot f(n) dn \right] \delta_{(x-m)} dm = \hat{Q} f(x) \end{aligned}$$

## 3.) Terms

Diagonalizable:  $A \equiv P D P^{-1}$

Conj. Transpose,  $\dagger$ :  $A^\dagger \equiv A^{T*} = A^{*T}$

Hermitian,  $H$ :  $H = H^\dagger$

$$H = U D U^{-1} = U D U^\dagger \quad (\text{spectral theorem})$$

Unitary,  $U$ :  $U : U U^\dagger = U^\dagger U = 1$

$$\exists H : U = e^{iH} = (U') e^{iD} (U')^\dagger$$

Hermitian Operator  $\sim$  Hermitian Matrix

$$\langle Qx | y \rangle = \langle x | Qy \rangle \quad (\text{if inf. size then must be in Hilbert Space})$$

(draw it out)

$$\begin{aligned} \rightarrow (\bar{Q}x)^{*T} \cdot y_m | e_m \rangle &= y_m x^{*T} \cdot (\bar{Q}_m^*) \\ &= x^{*T} \cdot \bar{Q}^{*T} y_m | e_m \rangle \\ &= x^{*T} \cdot \bar{Q} y_m | e_m \rangle \end{aligned}$$

$$\rightarrow \bar{Q}^\dagger \equiv \bar{Q}^{*T} = \bar{Q} \quad \square$$

#### 4.) Eigenvalue Equation

General Case:

$$\begin{aligned}\hat{Q}|q_i\rangle &= q_i|q_i\rangle \\ |q_i\rangle &= \sum c_n^{(e)}[q_i]|e_n\rangle\end{aligned}$$

$$\begin{aligned}\overline{Q}^{(e)} &= UDU^\dagger \quad (\text{Spectral Theorem}) \\ &= \begin{pmatrix} | & | & | \\ \vec{c}_{[q_0]} & \vec{c}_{[q_1]} & \dots \\ | & | & | \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & \dots \\ 0 & q_1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} -\vec{c}_{[q_0]}^* & - \\ -\vec{c}_{[q_1]}^* & - \\ \vdots & \vdots \end{pmatrix}^{(e)} \\ &\text{where } \langle \vec{c}_m | \vec{c}_n \rangle = \delta_{mn} \text{ since } Q^\dagger Q = Q Q^\dagger \quad (\text{normal})\end{aligned}$$

$$q_i|q_i\rangle = \hat{Q}|q_i\rangle$$

$$[q_i \ \vec{c}^{(e)}[q_i]] \cdot \vec{e}(x) = [\overline{Q}^{(e)} \ \vec{c}^{(e)}[q_i]] \cdot \vec{e}(x)$$

$\Downarrow^*$

$$q_i \vec{c}^{(e)}[q_i] = \overline{Q}^{(e)} \vec{c}^{(e)}[q_i]$$

$$q_i \begin{pmatrix} | & | & | \\ \vec{c}_{[q_i]} & & \\ | & | & | \end{pmatrix}^{(e)} = \begin{pmatrix} | & | & | \\ \vec{c}_{[q_0]} & \vec{c}_{[q_1]} & \dots \\ | & | & | \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & \dots \\ 0 & q_1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} -\vec{c}_{[q_0]}^* & - \\ -\vec{c}_{[q_1]}^* & - \\ \vdots & \vdots \end{pmatrix}^{(e)} \begin{pmatrix} | & | & | \\ \vec{c}_{[q_i]} & & \\ | & | & | \end{pmatrix}^{(e)}$$

$$\begin{aligned}&\underline{q_i, \vec{c}^{(e)}[q_i]} : \\ &\det(\overline{Q}^{(e)} - I_{q_i}) = 0\end{aligned}$$

Special Case:

$$\begin{aligned}&\left. \begin{aligned}|q_n\rangle &= |e_n\rangle \\ \hat{Q}|e_n\rangle &= q_n|e_n\rangle\end{aligned} \right| \begin{aligned}\hat{Q}|a\rangle &= \sum_n \hat{Q}|e_n\rangle \langle e_n|a\rangle \\ &= \left( \sum_n q_n |e_n\rangle \langle e_n| \right) |a\rangle\end{aligned} \Rightarrow \begin{aligned}\hat{Q} &= \sum_n q_n |e_n\rangle \langle e_n| \\ Q_{mn}^{(e)} &= q_n \delta_{mn}\end{aligned} \Rightarrow \overline{Q}^{(e)} = \begin{pmatrix} q_0 & 0 & \dots \\ 0 & q_1 & \dots \\ \vdots & \vdots & q_i\end{pmatrix}\end{aligned}$$

$$\overline{Q}^{(e)} = \begin{pmatrix} q_0 & 0 & \dots \\ 0 & q_1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}^{(e)} = \begin{pmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & \dots \\ 0 & q_1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}^{(e)} \quad \vec{c}^{(e)}[q_i] = (\dots 0 0 1_{(i)} 0 0 \dots)^T$$

#### 5. Unitary Transformation and Trace

- $|b_i\rangle = U|a_i\rangle \Leftrightarrow U = \sum |b_n\rangle \langle a_n|$
- $\overline{U}_{ij} = \langle a_i | U | a_j \rangle = \langle a_i | b_j \rangle$
- $\langle b_i | U \hat{Q} U^\dagger | b_j \rangle = \langle a_i | \hat{Q} | a_j \rangle$
- $\begin{aligned}(A)|a_i\rangle &= a_i|a_i\rangle \\ (UAU^\dagger)|b_i\rangle &= a_i|b_i\rangle\end{aligned}$
- $\text{Tr}(Q) = \sum \langle a_i | Q | a_i \rangle = \sum \langle b_i | Q | b_i \rangle$
- $\text{Tr}(QP) = \text{Tr}(PQ)$
- $\text{Tr}(U^\dagger Q U) = \text{Tr}(Q)$
- $\text{Tr}(|a_i\rangle \langle a_j|) = \delta_{ij}$
- $\text{Tr}(|b_i\rangle \langle a_i|) = \langle a_i | b_i \rangle$

$\Phi(p, t)$  - Momentum Space (generalizable Born Interpretation):

$$\begin{aligned}
 \int_x \Psi^* \Psi dx &= \int_x \int_p c_p^*(t) \Psi_p^*(x) dp \int_{p'} c_{p'}(t) \Psi_{p'}(x) dp' dx \\
 &= \int_p c_p^*(t) \int_{p'} c_{p'}(t) \int_x \Psi_p^*(x) \Psi_{p'}(x) dx dp' dp \\
 &= \int_p \Phi^* \int_{p'} \Phi' \delta(p - p') dp' dp \\
 &= \int_p \Phi^* \Phi dp \Rightarrow \boxed{\text{PDF}(p) = |\Phi|^2 = \Phi^* \Phi}
 \end{aligned}$$

$$\boxed{\langle \Psi | \Psi \rangle = \langle \Phi | \Phi \rangle}$$

Anything in  $x$ -space can be done in  $p$ -space  
(or generalize to any transform,  $c_n$ )

Heisenberg Uncertainty Proof:

$$\begin{aligned}
 \langle f | g \rangle &\equiv \langle (\hat{A} - \langle a \rangle) \Psi | (\hat{B} - \langle b \rangle) \Psi \rangle \\
 &= \langle \Psi | (\hat{A} - \langle a \rangle) | (\hat{B} - \langle b \rangle) \Psi \rangle \\
 &= \langle \Psi | \hat{A} \hat{B} \Psi \rangle - \langle a \rangle \langle b \rangle = \langle \hat{A} \hat{B} \rangle - \langle a \rangle \langle b \rangle \\
 \sigma_A^2 \sigma_B^2 &= \|(\hat{A} - \langle a \rangle) \Psi\|^2 \|(\hat{B} - \langle b \rangle) \Psi\|^2 \\
 &\equiv \langle f | f \rangle \langle g | g \rangle \geq \|\langle f | g \rangle\|^2 \quad (\text{see Schwarz Ineq.}) \\
 &\geq [\text{Im}(\langle f | g \rangle)]^2 = \left( \frac{1}{2i} [\langle f | g \rangle - \langle f | g \rangle^*] \right)^2 \\
 &= \left( \frac{1}{2i} \langle \hat{A} \hat{B} - \hat{B} \hat{A} \rangle \right)^2 \equiv \boxed{\left( \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2}
 \end{aligned}$$

Commutator of Hermitian  $\hat{A}, \hat{B}$

- $[A, B]^\dagger = -[A, B]$
- $\exists \Psi_n$  s.t.  $(\hat{A} \Psi_n = a \Psi_n)$  ,  $(\hat{B} \Psi_n = b \Psi_n)$   
 $\Leftrightarrow [\hat{A}, \hat{B}] = 0$   
 $\Rightarrow \boxed{\sigma_A \sigma_B \geq 0} \quad (\text{Both can be measured concurrently})$   
 $\boxed{AB = BA}$

$$x\Phi = x e^{-\frac{i}{\hbar} p x} = -\frac{\hbar}{i} \frac{\partial}{\partial p} \Phi = \frac{\hbar}{i} \frac{\partial}{\partial(-p)} \Phi$$

- $\langle p | \hat{p} | p' \rangle \equiv p \langle p | p' \rangle \equiv p \delta(p - p')$
- 1.  $\langle p | \hat{p} | f \rangle = p f(p) = \boxed{p \langle p | f \rangle \equiv \check{p} \langle p | f \rangle}$
- 2.  $\langle p | \hat{x} | p' \rangle = \iint \langle p | x \rangle \langle x | \hat{x} | x' \rangle \langle x' | p' \rangle dx dx'$   
 $= \frac{1}{2\pi\hbar} \int x e^{\frac{i}{\hbar} x(p' - p)} dx$   
 $= -\frac{\hbar}{i} \delta'(p - p') = -\frac{\hbar}{i} \frac{\partial}{\partial p} \langle p | p' \rangle$
- 3.  $\langle p | \hat{x} | f \rangle = \int \langle p | \hat{x} | p' \rangle \langle p' | f \rangle dp'$   
 $= \boxed{-\frac{\hbar}{i} \frac{\partial}{\partial p} \langle p | f \rangle \equiv \check{x} \langle p | f \rangle}$

$$\begin{aligned}
 \Rightarrow A(x, \hat{p}_x) &\rightarrow A(\hat{x}_p, p) \\
 \Rightarrow \boxed{\langle a \rangle} &= \langle \Phi | A(\hat{x}_p, p) | \Phi \rangle
 \end{aligned}$$

Commutator

- $[\hat{A}, \hat{B}] f \equiv \hat{A}(\hat{B} f) - \hat{B}(\hat{A} f)$
- $[A, BC] = [A, B]C + B[A, C]$
- $[AB, C] = A[B, C] + [A, C]B$
- $[x, \hat{p}] = i\hbar$
- $\sigma_A \sigma_B \geq \left\| \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right\|$   
 $\Rightarrow \boxed{\Delta x \Delta p \geq \hbar/2}$
- $[\hat{p}, f(x)] = \hat{p} f(x) - \frac{\hbar}{i} \frac{\partial f}{\partial x}$   
 $[\hat{x}, g(p)] = \hat{x} g(p) - \frac{\hbar}{i} \frac{\partial g}{\partial p}$

Anti-Hermitian Operators:  $A^\dagger = -A$

- $\langle A \rangle = ai, \quad a \in \mathbb{R}$
- $[A, B]^\dagger = -[A, B]$

## Operator Evolution (Heisenberg Equation)

$$\frac{d}{dt} \left\langle \Psi(x, t) \left| Q \right| \Psi(x, t) \right\rangle = \left\langle \frac{\partial \Psi}{\partial t} \left| Q \right| \Psi \right\rangle + \left\langle \Psi \left| \frac{\partial Q}{\partial t} \right| \Psi \right\rangle + \left\langle \Psi \left| Q \right| \frac{\partial \Psi}{\partial t} \right\rangle$$

$$\boxed{\begin{aligned} \frac{d}{dt} \langle Q \rangle &= \frac{1}{i\hbar} \left\langle [\hat{Q}, \hat{H}] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle \\ i\hbar \frac{d}{dt} \langle Q \rangle &= \left\langle [\hat{Q}, \hat{H}] \right\rangle + i\hbar \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle \end{aligned}} \quad (Q \text{ is conserved when this equals } 0)$$

- Conservations:  $\frac{d\langle \Psi | \Psi \rangle}{dt} = 0, \quad \frac{d\langle H \rangle}{dt} = 0$
- Ehrenfest's Theorem:  $m \frac{d\langle x \rangle}{dt} = \langle p \rangle, \quad \frac{d\langle p \rangle}{dt} = - \left\langle \frac{\partial V}{\partial x} \right\rangle \Rightarrow \text{other classical eq.}$
- Virial Theorem:  $\begin{aligned} \frac{d}{dt} \langle xp \rangle &= \frac{i}{\hbar} \langle [H, x]p + x[H, p] \rangle = \left\langle \left[ \frac{p^2}{2m}, x \right] p + x[V, p] \right\rangle \\ &= \frac{i}{\hbar} \langle \frac{1}{2m} p[p, x]p - \frac{1}{2m} [p, x]p^2 - x[p, V] \rangle \end{aligned}$   

$$\boxed{\frac{d\langle xp \rangle}{dt} = 2\langle T \rangle - \left\langle x \frac{\partial V}{\partial x} \right\rangle} \rightarrow 0 = \frac{d}{dt} \left\langle \Psi_n(x) \left| Q_{(x,p)} \right| \Psi_n(x) \right\rangle \quad (\text{for stationary states})$$
- Energy-Time Uncertainty:  $(Q = Q(x, \hat{p}) \neq Q(x, \hat{p}, t)) \Rightarrow \sigma_H \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|$

$$\Rightarrow \boxed{\begin{aligned} \sigma_Q &\equiv \frac{d\langle Q \rangle}{dt} \Delta t \approx \Delta \langle Q \rangle \\ \sigma_H \left( \frac{\sigma_Q}{|d\langle Q \rangle/dt|} \right) &\geq \frac{\hbar}{2} \\ \Delta E \Delta t &\geq \frac{\hbar}{2} \end{aligned}} \quad \begin{array}{l} \Delta t \text{ is the amount of time it would} \\ \text{take } \langle Q \rangle \text{ to change "appreciably",} \\ \text{or one std. dev. at the constant} \\ \text{rate } \frac{d}{dt} \langle Q \rangle \end{array}$$

Mass Lifetime:

$$\Delta(mc^2) \Delta t \geq \frac{\hbar}{2} \quad \checkmark$$

Orthogonal Time Example:

$$\Psi(x, \tau) = \frac{\sqrt{2}}{2} (\Psi_1 e^{-\frac{i}{\hbar} E_1 \tau} + \Psi_2 e^{-\frac{i}{\hbar} E_2 \tau})$$

$$\left\langle \Psi(x, 0) \left| \Psi(x, \tau) \right\rangle = 0 = \frac{1}{2} (e^{-\frac{i}{\hbar} E_1 \tau} + e^{-\frac{i}{\hbar} E_2 \tau})$$

$$\Rightarrow \tau \frac{E_2 - E_1}{2} = \frac{\pi}{2} \hbar \left( \frac{1}{2} + n \right) \geq \frac{\hbar}{2} \quad \checkmark$$

## Translation Operator

$$\begin{aligned}
 f(x + \Delta x) &\approx f(x) + \frac{df}{dx} \Delta x \\
 &= f(x) + f'(x) \Delta x + \frac{f''(x)}{2!} (\Delta x)^2 + \dots = \left\{ f(x') = \sum_n \frac{f^{(n)}(a)}{n!} (x' - a)^n \right\} \\
 &\quad \left( x' = x + \Delta x, (a = x) \right) \\
 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (\Delta x)^n = \sum_{n=0}^{\infty} \frac{(\Delta x \nabla)^n}{n!} f(x)
 \end{aligned}$$

$$\boxed{f(x + \Delta x) = e^{\frac{i}{\hbar}(\Delta x)\hat{p}} f(x)} \Leftrightarrow \boxed{f(x) = e^{\frac{i}{\hbar}x\hat{p}} f(0)} * \langle x | e^{\frac{i}{\hbar}x\hat{p}} | x' \rangle = e^{\frac{i}{\hbar}x\hat{p}} \langle x | x' \rangle *$$

$$\begin{array}{l}
 \text{Time Translation :} \\
 \langle x_N | \hat{U}(t) | x_0 \rangle \\
 = \check{U}(x_N, t; x_0, 0) \\
 \langle x | \hat{U}(t) | \Psi \rangle \\
 = \Psi(x, t)
 \end{array}
 \left| \begin{array}{l}
 f(t + \Delta t) = \underline{f(t) + f'(t)\Delta t + \dots} = \sum_n \frac{(\Delta t)^n}{n!} \left( \frac{\partial}{\partial t} \right)^n f(t) \\
 i\hbar \frac{\partial}{\partial t} \Psi = \check{H}(x, p, t) \Psi \\
 \frac{\partial f}{\partial t} = \left[ \frac{-i\check{H}}{\hbar} \right] f \Rightarrow \left\{ \begin{array}{l} f(t_0 + \Delta t) \approx e^{\frac{-i\Delta t}{\hbar} \check{H}(t_0)} f(t_0) \quad (1\text{st order}) \\
 \boxed{f(0 + t) = \lim_{N \rightarrow \infty} \prod_{n=0}^{N-1} e^{\frac{-i}{\hbar} \check{H}(n\Delta t) \Delta t} f(0)} \\
 \sim \neq e^{\frac{-i}{\hbar} \int \check{H}(t) dt} f(0) \quad \left( \begin{array}{l} \text{since } 0 \neq \\ [H(t_0), H(t_1)] \end{array} \right) \end{array} \right. \\
 \frac{\partial^n f}{\partial t^n} \neq \left[ \frac{-i\check{H}(t)}{\hbar} \right]^n f
 \end{array}
 \right.$$

Pictures:  $\langle Q \rangle_{(t)} = \langle \Psi(x, t) | Q(x, p, t) | \Psi(x, t) \rangle$

- Schrodinger Picture:  $\langle Q \rangle_{(t)} = \left\langle e^{\frac{-i}{\hbar}t\hat{H}} \Psi_{(t=0)} \left| Q(x, p, t) \right| e^{\frac{-i}{\hbar}t\hat{H}} \Psi_{(t=0)} \right\rangle$
- $Q = Q(x, p) \Rightarrow \langle Q \rangle_{(t)} = \left\langle \sum e^{\frac{-i}{\hbar}E_n t} c_n \Psi_{n(x)} \left| Q \right| \sum e^{\frac{-i}{\hbar}E_n t} c_n \Psi_{n(x)} \right\rangle$  (nice for stationary states)
- Heisenberg Picture:  $\langle Q \rangle_{(t)} = \left\langle \Psi_{(t=0)} \left| e^{\frac{i}{\hbar}t\hat{H}} Q e^{\frac{-i}{\hbar}t\hat{H}} \right| \Psi_{(t=0)} \right\rangle$
- Dirac Picture:  $\langle Q \rangle_{(t)} = \left\langle e^{\frac{-i}{\hbar} \int \hat{H}_1(t) dt} \Psi_{(t=0)} \left| e^{\frac{i}{\hbar}t\hat{H}_0} Q e^{\frac{-i}{\hbar}t\hat{H}_0} \right| e^{\frac{-i}{\hbar} \int \hat{H}_1(t) dt} \Psi_{(t=0)} \right\rangle$

$$\begin{array}{l}
 \langle Q \rangle_{(t+\Delta t)} = \langle Q \rangle_{(t)} + \frac{d\langle Q \rangle}{dt} \Delta t + \dots \Rightarrow \text{A 1st order approximation of } \langle Q \rangle_{(t+\Delta t)} \\
 \text{should yield } \frac{d\langle Q \rangle}{dt} = \frac{1}{i\hbar} \langle [Q, H] \rangle + \frac{\partial Q}{\partial t}
 \end{array}$$



### Schrodinger Picture

$$\begin{aligned}
1.) \quad i\hbar \frac{\partial}{\partial t} \langle Q_S \rangle &= \langle [Q_S, H_S] \rangle \\
4.) \quad |\Psi_S(t)\rangle &= U_{S(t,t_0)} |\Psi_S(t_0)\rangle \\
&\Rightarrow i\hbar \frac{\partial}{\partial t} |\Psi_S\rangle = H_S \Psi_S = [H_S^0 + H_S^1(t)] |\Psi_S\rangle \\
&= \sum E_n |n_S^0\rangle e^{-\frac{i}{\hbar} E_n t} \langle n_S^1(t) | \Psi(0) \rangle \\
&\quad + \sum |n_S^0\rangle e^{-\frac{i}{\hbar} E_n t} \cdot i\hbar \frac{\partial}{\partial t} \langle n_S^1(t) | \Psi(0) \rangle \\
&\Rightarrow i\hbar \frac{\partial}{\partial t} U_{S(t,t_0)}^0 = H_S^0 U_{S(t,t_0)}^0 \\
&\Rightarrow U_{S(t,t_0)}^0 = e^{-\frac{i}{\hbar} H_S^0 (t-t_0)}
\end{aligned}$$

### Heisenberg Picture

$$\begin{aligned}
1.) \quad Q_{H(t)} &\equiv U_S^\dagger Q_S U_S & H_S \neq H_S(t) \\
&\Rightarrow i\hbar \frac{\partial}{\partial t} Q_H = [Q_H, H_H] & \downarrow \\
& & H_H = H_S \\
2.) \quad U_H &\equiv U_{S(t,t_0)}^\dagger U_{S(t,t_0)} = \mathbb{I} \\
3.) \quad |q_H(t)\rangle &\equiv U_{S(t,t_0)}^\dagger |q_S\rangle \\
&\Rightarrow Q_H |q_H(t)\rangle = q |q_H(t)\rangle \\
&\Rightarrow i\hbar \frac{\partial}{\partial t} |q_H(t)\rangle = -H_S |q_H(t)\rangle \\
4.) \quad |\Psi_H\rangle &\equiv U_{S(t,t_0)}^\dagger |\Psi_S(t)\rangle = |\Psi_S(t_0)\rangle \\
&= U_{H(t,t_0)} |\Psi_H(t_0)\rangle \\
&\Rightarrow i\hbar \frac{\partial}{\partial t} |\Psi_H\rangle = 0
\end{aligned}$$

### Dirac/Interaction Picture (see transition amplitude)

$$\begin{aligned}
1.) \quad Q_I(t) &\equiv U_S^{0\dagger} Q_S U_S^0 \\
&\Rightarrow i\hbar \frac{\partial}{\partial t} Q_I = [Q_I, H_I^0] \quad (H_S^0 = H_I^0 \text{ see Heis. pic.}) \\
2.) \quad U_I(t, t_0) &\equiv U_S^{0\dagger}(t, t_0) U_S(t, t_0) \\
3.) \quad |q_I(t)\rangle &\equiv U_S^{0\dagger}(t, t_0) |q_S\rangle \\
&\Rightarrow Q_I |q_I(t)\rangle = q |q_I(t)\rangle \\
&\Rightarrow i\hbar \frac{\partial}{\partial t} |q_I(t)\rangle = -H_S^0 |q_I(t)\rangle \\
4.) \quad |\Psi_I(t)\rangle &\equiv U_S^{0\dagger}(t, t_0) |\Psi_S(t)\rangle \\
&= U_I(t, t_0) |\Psi_I(t_0)\rangle \quad (\text{since } |\Psi_I(t_0)\rangle = |\Psi_S(t_0)\rangle) \\
&\Rightarrow i\hbar \frac{\partial}{\partial t} |\Psi_I\rangle = U_S^{0\dagger} H_S^1(t) U_S^0 |\Psi_I\rangle = H_I^1 |\Psi_I\rangle \\
&\Rightarrow i\hbar \frac{\partial}{\partial t} U_I(t, t_0) = H_I^1(t) U_I(t, t_0) \\
&\Rightarrow U_I(t, t_0) = \mathbb{I} + \frac{1}{i\hbar} \int_{t_0}^t H_I^1(t') U_I(t', t_0) dt'
\end{aligned}$$

$$\begin{aligned}
\bullet \quad U_I(t, t_0) &= \mathbb{I} + \frac{1}{i\hbar} \int_{t_0}^t H_I^1(t') U_I(t', t_0) dt' \\
&= \mathbb{I} + \mathcal{O}(H_I^1) \\
&= \mathbb{I} + \frac{1}{i\hbar} \int_{t_0}^t H_I^1(t') dt' + \mathcal{O}([H_I^1]^2) \\
&= \mathbb{I} + \frac{1}{i\hbar} \int_{t_0}^t H_I^1(t') dt' \\
&\quad + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} H_I^1(t') H_I^1(t'') dt'' dt' + \dots
\end{aligned}$$

$$\begin{aligned}
\bullet \quad U_{S(t,t_0)} &= U_S^0 + \frac{1}{i\hbar} \int_{t_0}^t U_S^0 H_I^1(t') dt' + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} U_S^0 H_I^1(t') H_I^1(t'') dt'' dt' + \dots \\
&= U^0(t, t_0) \\
&\quad + \frac{1}{i\hbar} \int_{t_0}^t U^0(t, t_0) U^{0\dagger}(t', t_0) H^1 U^0(t', t_0) dt' \\
&\quad + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} U^0(t, t_0) U^{0\dagger}(t', t_0) H^1 U^0(t', t_0) U^{0\dagger}(t'', t_0) H^1 U^0(t'', t_0) dt'' dt' + \dots
\end{aligned}$$

## Infinitesimal $t$ Path Integral

$$S[x(t)] = \int_0^t \mathcal{L}(x, \dot{x}) dt \rightarrow \mathcal{L} \delta t = \left[ \frac{1}{2} m \left( \frac{x_1 - x_0}{\delta t} \right)^2 - V \left( \frac{x_1 + x_0}{2}, t_0 + \frac{\delta t}{2} \right) \right] \delta t$$

$$\langle x | \hat{U}(\epsilon) | \Psi \rangle = \int \langle x | \hat{U}(\epsilon) | x' \rangle \langle x' | \Psi(x, t) \rangle dx' = \int \check{U}(x, t + \epsilon; x', t) \Psi(x', t) dx' = \Psi(x, t + \epsilon)$$

- $\check{U}(x_1, \epsilon; x_0, 0) = A e^{\frac{i}{\hbar} S} = A e^{\frac{i}{\hbar} \mathcal{L} \epsilon}$ 

$$= A \exp \left\{ \frac{i}{\hbar} \left[ \frac{1}{2} m \frac{(x_1 - x_0)^2}{\epsilon} - \epsilon V \left( \frac{x_1 + x_0}{2}, 0 + \frac{\epsilon}{2} \right) \right] \right\}$$

$$(\eta = x_0 - x_1) = A \exp \left\{ \frac{i}{\hbar} \left[ \frac{1}{2} m \frac{\eta^2}{\epsilon} \right] \right\} \exp \left\{ -\frac{i}{\hbar} \epsilon V \left( x_1 + \frac{\eta}{2}, 0 + \frac{\epsilon}{2} \right) \right\}$$

$$\approx A \exp \left\{ \frac{i}{\hbar} \left[ \frac{1}{2} m \frac{\eta^2}{\epsilon} \right] \right\} \exp \left\{ -\frac{i}{\hbar} \epsilon V(x_1, 0) \right\}$$

$$\approx A \exp \left\{ \frac{i}{\hbar} \left[ \frac{1}{2} m \frac{\eta^2}{\epsilon} \right] \right\} \left[ 1 - \frac{i}{\hbar} \epsilon V(x_1, 0) \right]$$

$$\frac{\eta^2}{\epsilon} \lesssim \pi \quad \text{Explanation}$$

The integral involving  $\check{U}$  is over all  $\eta$ . The phase of the complex exponential will vary/oscillate too wildly and destructively interfere if  $\eta^2/\epsilon$  were to grow too big, so  $\eta^2 \sim \epsilon$  is all that matters. This means the integral is over  $\sqrt{\epsilon}$ , not  $\epsilon$ . Because of this (somehow, Bibl. given), using a finite difference formula for derivatives is legitimate in this case, though not in general.

$$\begin{aligned} \Psi(x, \epsilon) &= \int_{-\infty}^{\infty} \check{U}(x, \epsilon; x', 0) \Psi(x', 0) dx' \\ &= A \int_{-\infty}^{\infty} e^{\frac{i}{\hbar} S} \Psi(x', 0) dx' \\ &= A \int_{-\infty}^{\infty} \exp \left\{ \frac{i}{\hbar} \left[ \frac{1}{2} m \frac{(x - x')^2}{\epsilon} - \epsilon V \left( \frac{x + x'}{2}, 0 + \frac{\epsilon}{2} \right) \right] \right\} \Psi(x', 0) dx' \\ &= A \int_{-\infty}^{\infty} \exp \left\{ \frac{i}{\hbar} \left[ \frac{1}{2} m \frac{\eta^2}{\epsilon} - \epsilon V \left( \frac{x + \eta/2}{2}, 0 + \frac{\epsilon}{2} \right) \right] \right\} \Psi(x + \eta, 0) d\eta \\ &\approx A \int_{-\infty}^{\infty} e^{\frac{i}{\hbar} \left[ \frac{1}{2} m \frac{\eta^2}{\epsilon} \right]} \left[ 1 - \frac{i}{\hbar} \epsilon V(x, 0) \right] \left[ \Psi(x, 0) + \eta \Psi'(x, 0) + \frac{\eta^2}{2} \Psi''(x, 0) \right] d\eta \\ &\approx A \int_{-\infty}^{\infty} e^{\frac{i}{\hbar} \left[ \frac{1}{2} m \frac{\eta^2}{\epsilon} \right]} \left[ \left( 1 - \frac{i}{\hbar} \epsilon V(x, 0) \right) \Psi(x, 0) + \cancel{\eta \Psi'(x, 0)} + \frac{\eta^2}{2} \Psi''(x, 0) \right] d\eta \\ &= A \sqrt{\frac{2\hbar\epsilon\pi}{im}} \left[ \left( 1 - \frac{i}{\hbar} \epsilon V(x, 0) \right) + \frac{1}{2} \cdot \frac{2\hbar\epsilon}{-im} \cdot \frac{1}{2} \frac{\partial^2}{\partial x^2} \right] \Psi(x, 0) \\ &= \Psi(x, 0) - \frac{i}{\hbar} \epsilon \check{H} \Psi(x, 0) \end{aligned}$$

$$\boxed{i\hbar \frac{\Psi(x, \epsilon) - \Psi(x, 0)}{\epsilon - 0} = \check{H} \Psi(x, 0)}$$

## Finite $t$ , Free Particle Propagator

$$S[x(t)] = \int_0^t \mathcal{L}(x, \dot{x}) dt = \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \left[ \frac{1}{2} m \left( \frac{x_{n+1} - x_n}{\delta t} \right)^2 \right] \delta t$$

$$\langle x_N | \hat{U}(t) | \Psi(0) \rangle = \int \langle x_N | \hat{U}(t) | x_0 \rangle \Psi(x_0, 0) dx_0 = \int \check{U}(x_N, t; x_0, 0) \Psi(x_0, 0) dx_0 = \Psi(x_N, t)$$

$$\Rightarrow \langle x_N | \hat{U}(t) | x_0 \rangle = \langle x_N | e^{-\frac{i}{\hbar} H t} e^{\frac{i}{\hbar} H t_0} | x_0 \rangle = \langle x_N, t_N | x_0, t_0 \rangle$$

$$\begin{aligned} \langle x_N | \hat{U}(t) | x_0 \rangle &= \lim_{N \rightarrow \infty} \langle x_N | \hat{U}^N(\epsilon) | x_0 \rangle = \lim_{N \rightarrow \infty} \langle x_N | \hat{U}(\epsilon) \dots \hat{U}(\epsilon) \hat{U}(\epsilon) | x_0 \rangle \\ &= \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle x_N | \hat{U}(\epsilon) | x_{N-1} \rangle \dots \langle x_2 | \hat{U}(\epsilon) | x_1 \rangle \langle x_1 | \hat{U}(\epsilon) | x_0 \rangle dx_1 dx_2 \dots dx_{N-1} \end{aligned}$$

$$\begin{aligned} \check{U}(x_N, t; x_0, 0) &= \int_{x_0}^{x_N} A e^{\frac{i}{\hbar} S} \mathcal{D}[x(t)] \\ &= \lim_{N \rightarrow \infty} A \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{\frac{i}{\hbar} \left[ \frac{1}{2} m \frac{(x_1 - x_0)^2}{\epsilon} + \frac{1}{2} m \frac{(x_2 - x_1)^2}{\epsilon} + \frac{1}{2} m \frac{(x_3 - x_2)^2}{\epsilon} + \dots \right]} dx_1 dx_2 \dots dx_{N-1} \\ &= \lim_{N \rightarrow \infty} A \sqrt{\frac{2\hbar\epsilon}{m}}^{N-1} \int \dots \int_{-\infty}^{\infty} e^{-\frac{(y_1 - y_0)^2}{i} - \frac{(y_2 - y_1)^2}{i}} dy_1 e^{\left[ -\frac{(y_3 - y_2)^2}{i} \dots \right]} dy_2 \dots \\ &= \lim_{N \rightarrow \infty} A \sqrt{\frac{2\hbar\epsilon}{m}}^{N-1} \int \dots \int_{-\infty}^{\infty} e^{-\frac{y_1^2 + y_1^2 - 2y_1(y_2 + y_0) + y_2^2 + y_0^2}{i}} dy_1 \dots \\ &= \lim_{N \rightarrow \infty} A \sqrt{\frac{2\hbar\epsilon}{m}}^{N-1} \int \dots \int_{-\infty}^{\infty} e^{-\frac{1}{i} \left( 2 \left[ y_1^2 - y_1(y_0 + y_2) + \frac{(y_0 + y_2)^2}{4} \right] - \frac{(y_2 + y_0)^2}{2} + y_2^2 + y_0^2 \right)} dy_1 \dots \\ &= \lim_{N \rightarrow \infty} A \sqrt{\frac{2\hbar\epsilon}{m}}^{N-1} \int \dots \int_{-\infty}^{\infty} e^{-\frac{2}{i} \left[ y_1 - \frac{y_2 + y_0}{2} \right]^2} dy_1 e^{-\frac{(y_2 - y_0)^2}{2i}} e^{-\frac{(y_3 - y_2)^2}{i}} \dots dy_2 e^{[\dots]} \dots \\ &= \lim_{N \rightarrow \infty} A \sqrt{\frac{2\hbar\epsilon}{m}}^{N-1} \int \dots \sqrt{\frac{\pi i}{2}} \int_{-\infty}^{\infty} e^{-\frac{(y_2 - y_0)^2}{2i}} e^{-\frac{(y_3 - y_2)^2}{i}} \dots dy_2 \dots \\ &= \lim_{N \rightarrow \infty} A \sqrt{\frac{2\hbar\epsilon}{m}}^{N-1} \int \dots \sqrt{\frac{(\pi i)^2}{3}} \int_{-\infty}^{\infty} e^{-\frac{(y_3 - y_0)^2}{3i}} e^{-\frac{(y_4 - y_3)^2}{i}} \dots dy_3 \dots \\ &= \lim_{N \rightarrow \infty} A \sqrt{\frac{2\hbar\epsilon}{m}}^{N-1} \sqrt{\frac{(\pi i)^{N-1}}{N}} e^{-\frac{(y_N - y_0)^2}{Ni}} = \lim_{N \rightarrow \infty} A \sqrt{\frac{2\hbar\epsilon\pi}{-im}}^N \sqrt{\frac{-im}{2\hbar\epsilon N\pi}} e^{\frac{i}{\hbar} \frac{m}{2} \frac{(x_N - x_0)^2}{N\epsilon}} \end{aligned}$$

Free Particle :  $\check{U}(x_N, t; x_0, 0) = \sqrt{\frac{-im}{2\hbar t\pi}} e^{\frac{i}{\hbar} \frac{m}{2} \frac{(x_N - x_0)^2}{t}}$

$\int_{x_0}^{x_N} \mathcal{D}[x(t)] = \lim_{N \rightarrow \infty} \sqrt{\frac{-im}{2\hbar\epsilon\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{-im}{2\hbar\epsilon\pi}} dx_1 \dots \int_{-\infty}^{\infty} \sqrt{\frac{-im}{2\hbar\epsilon\pi}} dx_{N-1}$

### All Paths Explanation

$$S[x] = S[x_{cl}] + S'[x_{cl}]\eta + \mathcal{O}(\eta^2)$$

1<sup>st</sup> order variation of  $S$  from  $x_{cl}$  equals 0. This means propagator integrand for paths near  $x_{cl}$  will have about the same phase, and will add constructively. Paths very different from  $x_{cl}$  (like those with faster than light motion) will vary in action, and because  $\hbar$  is so small their phases will vary wildly, meaning the sum will destructively interfere. The result is that only paths near the classical path will be important, with  $S[x]/\hbar \lesssim \pi$ .

### Action-Energy Relationship

$$S(x_{cl} + \Delta x_{cl}, \dot{x} + \Delta \dot{x}_{cl}, \tau + \Delta \tau) = S_{cl} + \Delta S_{cl}$$

$$= \int_0^{\tau + \Delta \tau} \mathcal{L}(x_{cl} + \Delta x_{cl}, \dot{x}_{cl} + \Delta \dot{x}_{cl}, t) dt$$

$$dS = \frac{\partial S}{\partial \tau} d\tau + \left[ \frac{\partial S}{\partial x} dx + \frac{\partial S}{\partial \dot{x}} d\dot{x} = \int (dL) dt \right] \quad (\eta = dx, \eta(0) = 0)$$

$$\begin{aligned} \Delta S_{cl} &= \mathcal{L}(\tau) \Delta \tau + \int_0^\tau \frac{\partial \mathcal{L}}{\partial x} \Big|_{cl} \eta + \frac{\partial \mathcal{L}}{\partial \dot{x}} \Big|_{cl} \dot{\eta} dt \\ &= \mathcal{L}(\tau) \Delta \tau + \int_0^\tau \left[ \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right]_{cl} \eta + \frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{x}} \eta \right] dt \\ &= \left[ \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{x} \right]_{cl, \tau} \Delta \tau \end{aligned}$$

$$\Delta S_{cl} = -H(t_f) \Delta t_f$$

### (Time-Independent) Hamiltonian-Lagrangian Propagator Relationship / Finite Path Integral

$$\check{U}(x_N, t; x_0, 0) = \langle x_N | e^{-\frac{i}{\hbar} H t} | x_0 \rangle = \langle x_N | [e^{-\frac{i}{\hbar} H \frac{t}{N}}]^N | x_0 \rangle = \lim_{N \rightarrow \infty} \langle x_N | [e^{-\frac{i}{\hbar} \frac{p^2}{2m} \epsilon} e^{-\frac{i}{\hbar} V(\hat{x}) \epsilon}]^N | x_0 \rangle \quad (\text{not trivial})$$

$$= \lim_{N \rightarrow \infty} \int_{x \dots} \langle x_N | e^{-\frac{i}{\hbar} \frac{p^2}{2m} \epsilon} e^{-\frac{i}{\hbar} V(\hat{x}) \epsilon} | x_{N-1} \rangle \dots \langle x_1 | e^{-\frac{i}{\hbar} \frac{p^2}{2m} \epsilon} e^{-\frac{i}{\hbar} V(\hat{x}) \epsilon} | x_0 \rangle dx \dots$$

$$= \lim_{N \rightarrow \infty} \int_{x \dots} \dots \left[ \underbrace{\int \langle x_1 | e^{-\frac{i}{\hbar} \frac{p^2}{2m} \epsilon} | p \rangle \langle p | x_0 \rangle dp}_{\text{free part. prop.}} e^{-\frac{i}{\hbar} V(x_0) \epsilon} \right] dx \dots$$

$$1. = \lim_{N \rightarrow \infty} \int_{x \dots} \dots \left[ \int \frac{e^{\frac{i}{\hbar} p(x_1 - x_0)}}{2\pi\hbar} e^{-\frac{i}{\hbar} \frac{p^2}{2m} \epsilon} e^{-\frac{i}{\hbar} V(x_0) \epsilon} dp \right] dx \dots = \boxed{\int e^{\frac{i}{\hbar} \int p \dot{x} - H(x, p) dt} [\mathcal{D}x \mathcal{D}p]} \quad (\text{Phase Space})$$

$$2. = \lim_{N \rightarrow \infty} \int_{x \dots} \dots \left[ \sqrt{\frac{-im}{2\pi\hbar\epsilon}} e^{\frac{im(x_1 - x_0)^2}{\epsilon}} e^{-\frac{i}{\hbar} V(x_0) \epsilon} \right] dx \dots = \lim_{N \rightarrow \infty} \int_{x \dots} \dots \sqrt{\frac{-im}{2\pi\hbar\epsilon}} e^{\frac{i}{\hbar} \mathcal{L} \epsilon} dx_1 \dots$$

$$= \boxed{\int_{x_0}^{x_N} e^{\frac{i}{\hbar} S} [\mathcal{D}x] = \int_{x_0}^{x_N} e^{\frac{i}{\hbar} \int \mathcal{L} dt} [\mathcal{D}x]} \quad (\text{Configuration Space}) \quad (\text{above is only integrable if } p \text{ is quadratic in } H)$$

### Trace of Propagator

$$G(t) = \int \langle x | e^{-\frac{i}{\hbar} H t} | x \rangle d^3 x$$

$$= \sum_n \int \langle x | n \rangle e^{-\frac{i}{\hbar} E_n t} \langle n | x \rangle dx$$

$$G(t) = \sum_n e^{-\frac{i}{\hbar} E_n t} \sim \sum_n e^{-\beta E_n} = Z(\beta)$$

### 1.3 Extra

$L_2 \subset$  Hilbert Space = complete inner product space

$$\rho(x, t) \equiv \|\Psi\|^2, \quad P_a^b(t) = \int_a^b \rho dx, \quad P(t) = P_{-\infty}^\infty(t), \quad \Psi = \sqrt{\rho} e^{\frac{i}{\hbar} S} \quad \text{e.g., } e^{\frac{i}{\hbar}(p \cdot x - Et)}$$

$$\begin{aligned} \bullet \quad & \begin{aligned} \check{E}\rho &= \check{E}(\Psi^*\Psi) = \Psi^*(\check{E}\Psi) + \Psi(\check{E}\Psi^*) \\ &= \Psi^*(\check{H}\Psi) - \Psi(\check{H}\Psi^*) \\ &= \Psi^*\left(\frac{p^2}{2m} + V\right)\Psi - \Psi\left(\frac{p^2}{2m} + V\right)\Psi^* \\ -\frac{\hbar}{i}\frac{\partial\rho}{\partial t} &= \frac{\hbar}{i}\nabla \cdot \left(\Psi^*\frac{p}{2m}\Psi - \Psi\frac{p}{2m}\Psi^*\right) \end{aligned} & \left. \begin{aligned} & \text{(Probability Current)} \\ \frac{\partial\rho}{\partial t} &= -\nabla \cdot J = -\nabla \cdot \left(\Psi^*\frac{p}{2m}\Psi - \Psi\frac{p}{2m}\Psi^*\right) \\ &= -\nabla \cdot \frac{\rho\nabla S}{m} \quad (\text{e.g., } \nabla S = p) \\ \left[\frac{d}{dt}P_a^b = J_{(a,t)} - J_{(b,t)}\right], & \quad \left[\int J dV = \langle\Psi|\frac{p}{m}|\Psi\rangle = \frac{\langle p\rangle}{m}\right] \end{aligned} \right\} \\ \bullet \quad & (V \in \mathbb{R}) \quad \Rightarrow \quad \frac{d}{dt}P = 0 \quad \Rightarrow \quad P(t) \equiv 1 \\ & (V = V_0 - i\Gamma) \quad \Rightarrow \quad \frac{d}{dt}P = \frac{-2\Gamma}{\hbar}P \quad \Rightarrow \quad P(t) = e^{-2(\Gamma/\hbar)t} \\ \bullet \quad & \langle\Psi_n|\Psi_n\rangle, \quad \langle\Psi_m|\Psi_m\rangle = 1 \quad \Rightarrow \quad \frac{d}{dt}\langle\Psi_n|\Psi_m\rangle = 0 \end{aligned}$$

Schwarz Inequality:

$$\left\| \int_a^b f^* g dx \right\|^2 \leq \left\| \int_a^b f^* f dx \right\| \left\| \int_a^b g^* g dx \right\|$$

$$\|\langle f|g\rangle_{ab}\|^2 \leq \|\langle f|f\rangle_{ab}\| \|\langle g|g\rangle_{ab}\|$$

$$\left[ V(x) = V(-x) \right] \Rightarrow \left[ \Psi(x) \Rightarrow \Psi(-x) \right] \Rightarrow \left[ \Psi(-x) = \Psi(x) \right] \cup \left[ \Psi(-x) = -\Psi(x) \right]$$

Discontinuity in  $\Psi$  means the possibility of  $\sigma_p \rightarrow \infty$

Prob 3.29:  $\Psi(x, 0) = \begin{cases} \frac{1}{\sqrt{2n\lambda}} e^{2\pi i x/\lambda}, & -n\lambda < x < n\lambda \\ 0 & \text{else} \end{cases}$

$\sigma_p \rightarrow \infty$  because the integral of  $\delta^2(x)$  is infinite

$$\int_{-\infty}^{\infty} f(x) D_1(x) dx = \int_{-\infty}^{\infty} f(x) D_2(x) dx \Rightarrow \delta(cx) = \frac{1}{|c|} \delta(x)$$

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dx' \Rightarrow F[\delta(x)] = \frac{1}{2\pi}$$

$$\delta'(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} i k e^{ik(x-x')} dx' \Rightarrow \int \delta'(x - x') f(x') dx' = f'(x)$$

## Poisson Brackets

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i}$$

$$\{\omega(q, p, t), H\} = \sum_i \frac{\partial \omega}{\partial q_i} \dot{q}_i + \dot{p}_i \frac{\partial \omega}{\partial p_i} = \dot{\omega} - \frac{\partial \omega}{\partial t}$$

$$\text{Hamilton Eq. : } \dot{q} = \{q, H\}, \quad \dot{p} = \{p, H\}$$

## Canonical Transforms

$$\begin{array}{lcl} q \rightarrow \bar{q}(q, p) & \text{s.t.} & \{\bar{q}_i, \bar{q}_j\} = 0 = \{\bar{p}_i, \bar{p}_j\} \\ p \rightarrow \bar{p}(q, p) & & \{\bar{q}_i, \bar{p}_j\} = \delta_{ij} \end{array} \quad \left( \begin{array}{l} \text{Point Transforms} \\ \bar{q}(q) \text{ are canonical.} \end{array} \right)$$

$$\Rightarrow \begin{array}{l} \dot{\bar{q}} = \frac{\partial H}{\partial \bar{p}} \\ \dot{\bar{p}} = -\frac{\partial H}{\partial \bar{q}} \end{array}, \quad \{f, g\}_{q,p} = \{f, g\}_{\bar{q}, \bar{p}}$$

## Generator of Transformation

$$1. \delta H = 0$$

$$\delta H = \epsilon_\lambda \{H, g\}$$

$$g = l_z$$

$$\begin{aligned} 2. \quad \bar{q}_i &= q_i + \delta q_i, \quad \bar{p}_i = p_i + \delta p_i \\ &\equiv q_i + \epsilon_\lambda \frac{\partial g}{\partial p_i} \quad \equiv p_i - \epsilon_\lambda \frac{\partial g}{\partial q_i} \\ &= q_i + \epsilon_\lambda \{q_i, g\} \quad = p_i + \epsilon_\lambda \{p_i, g\} \end{aligned}$$

$$\Rightarrow \boxed{\frac{\partial H}{\partial \lambda} = 0 = \frac{dg}{dt}} \quad (\text{e.g. } g = p \text{ or } g = l_z)$$

$$\Rightarrow \begin{aligned} \delta x &= -\epsilon y = -(\delta \theta) y \\ \delta y &= \epsilon x = (\delta \theta) x \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} \frac{\partial x}{\partial \theta} &= -y \\ \frac{\partial y}{\partial \theta} &= x \end{aligned}}$$

$$3. \Rightarrow \delta f = \epsilon_\lambda \{f, g\} \rightarrow \frac{\partial f}{\partial \lambda} = \{f, g\}$$

## Tensors and Tensor Operators

$$\text{rank-2 Tensor : } |t^{(2)}\rangle = \sum_{i=1}^3 \sum_{j=1}^3 t_{ij} |i\rangle |j\rangle = \sum_{i=1}^3 \sum_{j=1}^3 |ij\rangle \langle ij| t^{(2)}\rangle$$

$$\text{rank-2 Carte. Tens. Oper., } T_{ij} : \text{ Set of } 3^{n=2} \text{ Operators}$$

$$\text{rank-}k \text{ Spher. Tens. Oper., } T_k^q : \text{ Set of } 2k+1 \text{ Operators s.t. } U[R] T_k^q U^\dagger[R] = \sum_{q'=-k}^k D_{q'q}^k T_k^{q'}$$

$$\begin{aligned} \Rightarrow U T_k^q U^\dagger U |jm\rangle &= \sum_{q'} \sum_{m'} D_{q'q}^k D_{m'm}^j T_{q'}^k |jm'\rangle \\ &\sim U |kq\rangle |jm\rangle = \sum_{q'} \sum_{m'} D_{q'q}^k D_{m'm}^j |kq'\rangle |jm'\rangle \end{aligned}$$

(CG coeff.)

$$\text{Wigner-Eckhart : } \langle \alpha_2 j_2 m_2 | T_k^q | \alpha_1 j_1 m_1 \rangle = \langle \alpha_2 j_2 | T_k | \alpha_1 j_1 \rangle \cdot \langle j_2 m_2 | kq, j_1 m_1 \rangle$$

## Functions

$$\text{Associated Legendre Functions: } P_l^m \equiv \sqrt{1-x^2}^{|m|} \left( \frac{d}{dx} \right)^{|m|} P_l(x) \quad (\text{not a polynomial if odd})$$

$$\text{Legendre Polynomials: } P_l(x) \equiv \frac{1}{2^l l!} \left( \frac{d}{dx} \right)^l (x^2 - 1)^l$$

$$\text{Associated Laguerre Polynomials: } L \equiv$$

$$\text{Laguerre Polynomials: } L \equiv$$

## 2 Simple 1D Potentials

### 2.1 Infinite Square Well (1-D)

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{otherwise} \end{cases}$$

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin k_n x$$

$$k_n = \frac{2\pi}{\lambda} = \frac{2\pi}{2a/n} = \frac{n\pi}{a} \quad \forall n = 1, 2, 3, \dots \quad \boxed{!! \hat{p}\Psi_n \neq p\Psi_n !!} \quad \text{wave isn't infinite}$$

$$E_n = \frac{p^2}{2m} = \frac{\hbar^2 k_n^2}{2m}$$

#### 2.1.1 3-D Rectangular Box

$$\Psi_{n_x n_y n_z}(x, y, z) = \Psi_{n_x}(x) \Psi_{n_y}(y) \Psi_{n_z}(z) = \sqrt{\frac{8}{a_x a_y a_z}} (\sin k_{n_x} x) (\sin k_{n_y} y) (\sin k_{n_z} z)$$

$$k_{n_i} = \frac{n_i \pi}{a_i} \quad \forall n_x, n_y, n_z = 1, 2, 3, \dots$$

$$E_{n_x n_y n_z} = \frac{\hbar^2}{2m} (k_{n_x}^2 + k_{n_y}^2 + k_{n_z}^2)$$

## 2.2 Harmonic Oscillator (1-D): $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \frac{1}{2m}(p^2 + m^2\omega^2x^2) \Rightarrow \boxed{a = a_- = \frac{1}{\sqrt{2m}}\frac{1}{\sqrt{\hbar\omega}}(i\hat{p} + m\omega x)}$$

$$= \frac{1}{2m}(-ip + m\omega x)(ip + m\omega x) \sim E \sim \hbar\omega$$

$$\boxed{aa^\dagger = \frac{H}{\hbar\omega} + \frac{1}{2}} \quad \boxed{aa^\dagger|n\rangle = (\frac{E_n}{\hbar\omega} + \frac{1}{2})|n\rangle} \quad , \quad \boxed{a^\dagger a = \frac{H}{\hbar\omega} - \frac{1}{2}} \quad \boxed{a^\dagger a|n\rangle = (\frac{E_n}{\hbar\omega} - \frac{1}{2})|n\rangle} \quad \rightarrow \quad \boxed{[a, a^\dagger] = 1} \quad \leftarrow \quad \boxed{H = \hbar\omega(a^\dagger a + \frac{1}{2}) = \hbar\omega(aa^\dagger - \frac{1}{2})}$$

or  $[H, a_\pm] = (\pm\hbar\omega)a_\pm$

$$\begin{aligned} (aa^\dagger)a\Psi_n &= a(a^\dagger a)\Psi_n & (a^\dagger a)a^\dagger|n\rangle &= a^\dagger(aa^\dagger)|n\rangle \\ \left(\frac{H}{\hbar\omega} + \frac{1}{2}\right)a\Psi_n &= a\left(\frac{H}{\hbar\omega} - \frac{1}{2}\right)\Psi_n & \left(\frac{H}{\hbar\omega} - \frac{1}{2}\right)a^\dagger|n\rangle &= a^\dagger\left(\frac{H}{\hbar\omega} + \frac{1}{2}\right)|n\rangle \\ \left(\frac{E_{an}}{\hbar\omega} + \frac{1}{2}\right)a\Psi_n &= \left(\frac{E_n}{\hbar\omega} - \frac{1}{2}\right)a\Psi_n & \left(\frac{E_{a^\dagger n}}{\hbar\omega} - \frac{1}{2}\right)a^\dagger|n\rangle &= \left(\frac{E_n}{\hbar\omega} + \frac{1}{2}\right)a^\dagger|n\rangle \\ E_{an}(a\Psi_n) &= (E_n - \hbar\omega)(a\Psi_n) & E_{a^\dagger n}(a^\dagger|n\rangle) &= (E_n + \hbar\omega)(a^\dagger|n\rangle) \\ \downarrow & & \downarrow & \\ E_{n-1}|n-1\rangle &= (E_n - \hbar\omega)|n-1\rangle & E_{n+1}|n+1\rangle &= (E_n + \hbar\omega)|n+1\rangle \end{aligned}$$

(Why ladders):

$$\begin{aligned} & \boxed{[\frac{H}{\hbar\omega}, a_\pm] = (\pm 1)a_\pm} \Rightarrow \boxed{[\frac{H}{\hbar\omega}, a_\pm^m] = (\pm 1)ma_\pm} \quad \text{(use induction)} \\ & \boxed{\frac{H}{\hbar\omega}|n\rangle = \frac{E_n}{\hbar\omega}|n\rangle = c_n|n\rangle} \Rightarrow \boxed{\frac{H}{\hbar\omega}a_\pm^m|n\rangle = (\pm 1 \cdot m + c_n)|n\rangle} \end{aligned} \quad \left| \begin{aligned} Ha_\pm^m|n\rangle &= \hbar\omega(c_n \pm m)a_\pm^m|n\rangle \\ * Ha_\pm^n|0\rangle &= \hbar\omega(c_0 \pm n)a_\pm^n|0\rangle * \\ &= E_{n0} \cdot a_\pm^n|0\rangle \end{aligned} \right.$$

$$E_n \geq \text{Min}(V) \Rightarrow a\Psi_0 = 0 \quad (\text{else is un-normalizable})$$

$$\begin{aligned} 0 &= (ip + m\omega x)\Psi_0 \\ \hbar\frac{d}{dx}\Psi_0 &= -m\omega x\Psi_0 \end{aligned} \quad \left| \begin{aligned} \Psi_0 &= Ae^{-\frac{m\omega}{\hbar}\frac{x^2}{2}} \\ A &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\ \frac{1}{\sigma^2} &= \frac{m\omega}{\hbar} \end{aligned} \right.$$

$$\begin{aligned} a^\dagger a|0\rangle &= \left(\frac{E_0}{\hbar\omega} - \frac{1}{2}\right)|0\rangle = 0 \\ E_0|0\rangle &= \frac{1}{2}\hbar\omega|0\rangle \quad (c_0 = \frac{1}{2}) \end{aligned}$$

$$aa^\dagger|a_+^n(0)\rangle = \left(\frac{\hbar\omega(n+1/2)}{\hbar\omega} + \frac{1}{2}\right)|a_+^n(0)\rangle$$

$$\boxed{aa^\dagger|a_+^n(0)\rangle = (n+1)|a_+^n(0)\rangle}$$

- $\langle a_+^n(0)|aa^\dagger|a_+^n(0)\rangle = n+1$
- $a^\dagger|a_+^n(0)\rangle = \sqrt{n+1}|a_+^{n+1}(0)\rangle$
- $a|a_+^{n+1}(0)\rangle = \sqrt{n+1}|a_+^n(0)\rangle * *$

$$a^\dagger a|n, 0\rangle = \left(\frac{\hbar\omega(n+1/2)}{\hbar\omega} - \frac{1}{2}\right)|n, 0\rangle$$

$$\boxed{a^\dagger a|a_+^n(0)\rangle = n|a_+^n(0)\rangle}$$

$$\frac{H}{\hbar\omega}|a_+^n(0)\rangle = \left(\frac{1}{2} + n\right)|a_+^n(0)\rangle = \underline{c \circ a_+^n(0)}|a_+^n(0)\rangle$$

$$\frac{H}{\hbar\omega}a_\pm^m|a_+^n(0)\rangle = \left(\frac{1}{2} + n \pm m\right)a_\pm^m|a_+^n(0)\rangle$$

$$\frac{H}{\hbar\omega}|a_+^{n\pm m}(0)\rangle = \underline{(c \circ a_+^{n\pm m}(0))}|a_+^{n\pm m}(0)\rangle$$

$$\begin{aligned} c_n &= \frac{1}{2} + n \\ a_+^n(0) &= n \\ a_\pm(n) &= n \pm 1 \end{aligned} \quad \rightarrow$$

$$\boxed{E_n = \hbar\omega(\frac{1}{2} + n)}$$

$$\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle \\ a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \end{aligned}$$

$$\boxed{\Psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\Psi_0}$$



### 2.2.1 Position/Momentum Operators

$$x = \frac{1}{2} \frac{\sqrt{2m}\sqrt{\hbar\omega}}{m\omega} (a + a^\dagger)$$

$$\hat{p} = \frac{1}{2} \frac{\sqrt{2m}\sqrt{\hbar\omega}}{i} (a - a^\dagger)$$

Show Virial Theorem Works

$$2\langle T \rangle = N\langle V \rangle$$

$$\begin{aligned} E_n &= 2\langle V \rangle_n \\ &= 2\langle \Psi_n | V | \Psi_n \rangle \\ &= 2 \left\langle \Psi_n \left| \frac{1}{2} m \omega^2 \frac{2m\hbar\omega}{(2m\omega)^2} (a + a^\dagger)^2 \right| \Psi_n \right\rangle \\ &= \frac{2m^2\hbar\omega^3}{(2m\omega)^2} (0 + \langle \Psi_n | (aa^\dagger + a^\dagger a) | \Psi_n \rangle + 0) \end{aligned}$$

$$E_n = (n + 1/2)\hbar\omega \quad \checkmark$$

Heisenberg Picture

$$\begin{aligned} \frac{da_\pm}{dt} &= \mp i\omega a_\pm \\ \Rightarrow a_\pm(t) &= a_\pm(0)e^{\mp i\omega t} \\ x(t) \pm \frac{ip(t)}{m\omega} &= x(0)e^{\mp i\omega t} \pm \frac{ip(0)}{m\omega}e^{\mp i\omega t} \end{aligned}$$

$$\begin{aligned} x(t) &= x(0) \cos \omega t + \frac{p(0)}{m\omega} \sin \omega t \\ \frac{p(t)}{m\omega} &= -x(0) \sin \omega t + \frac{p(0)}{m\omega} \cos \omega t \end{aligned}$$

Test the Uncertainty Principle

$$\sigma_x \sigma_p \geq \frac{1}{2} \left| \left\langle [x, p] \right\rangle \right|$$

$$\begin{aligned} xp - px &= \frac{2m\hbar\omega}{4m\omega i} \begin{pmatrix} a^2 - aa^\dagger + a^\dagger a - (a^\dagger)^2 \\ -a^2 + a^\dagger a - aa^\dagger + (a^\dagger)^2 \end{pmatrix} \\ &= \frac{\hbar}{i} (a^\dagger a - aa^\dagger) = i\hbar(n + 1 - n) \\ \Rightarrow \sigma_x \sigma_p &\geq \frac{\hbar}{2} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \sigma_x^2 &= \langle x^2 \rangle - \langle x \rangle^2 & \sigma_p^2 &= \langle p^2 \rangle - \langle p \rangle^2 \\ &= \frac{2m\hbar\omega}{4m^2\omega^2} \begin{bmatrix} \langle (a + a^\dagger)^2 \rangle \\ -\langle a + a^\dagger \rangle^2 \end{bmatrix} &= \frac{2m\hbar\omega}{-4} \begin{bmatrix} \langle (a - a^\dagger)^2 \rangle \\ -\langle a - a^\dagger \rangle^2 \end{bmatrix} \\ &= \frac{\hbar}{2m\omega} \langle aa^\dagger + a^\dagger a \rangle &= \frac{\hbar m\omega}{2} \langle aa^\dagger + a^\dagger a \rangle \\ &= \frac{\hbar}{m\omega} (n + \frac{1}{2}) &= \hbar m\omega (n + \frac{1}{2}) \\ \Rightarrow \sigma_x \sigma_p &= \hbar (n + \frac{1}{2}) \geq \frac{\hbar}{2} \quad \checkmark \end{aligned}$$

### 2.2.2 Analytic Method

$$\Psi_n = A \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

$$A = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x$$

$$H_n(x) = (-1)^n e^{-x^2} \left( \frac{d}{dx} \right)^n e^{x^2}$$

Hermite Polynomials:

$$e^{-z^2+2zx} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x)$$

### 2.2.3 Coherent States

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$\sigma_x \sigma_p = \frac{\hbar}{2}$$

$$\begin{aligned} \langle \alpha | \alpha \rangle &= \langle \alpha | \left( \sum_{n=0}^{\infty} \langle \Psi_n | \alpha \rangle | \Psi_n \rangle = \sum_{n=0}^{\infty} \left\langle \frac{(a^\dagger)^n}{\sqrt{n!}} \Psi_0 \middle| \alpha \right\rangle | \Psi_n \rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle \Psi_0 | \alpha \rangle | \Psi_n \rangle \right) \\ &= \langle \Psi_0 | \alpha \rangle^2 \sum_{n=0}^{\infty} \frac{(\alpha^2)^n}{n!} \langle \Psi_n | \Psi_n \rangle \\ &= \langle \Psi_0 | \alpha \rangle^2 e^{\alpha^2} = 1 \end{aligned} \Rightarrow \begin{aligned} &\boxed{|\alpha\rangle = e^{-\alpha^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\Psi_n\rangle} \rightarrow |\alpha=0\rangle = |\Psi_0\rangle \\ &a|\alpha(x,t)\rangle = e^{-\frac{\alpha^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{\frac{-i}{\hbar} E_n t} a |\Psi_n\rangle \\ &= e^{-\frac{\alpha^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{\frac{-i}{\hbar} \hbar \omega (\frac{1}{2}+n)t} \sqrt{n} |\Psi_{n-1}\rangle \\ &= \left( \alpha e^{\frac{-i}{\hbar} \hbar \omega t} \right) e^{-\frac{\alpha^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{\frac{-i}{\hbar} \hbar \omega (\frac{1}{2}+n)t} |\Psi_n\rangle \\ &\boxed{a|\alpha(x,t)\rangle = \left( \alpha e^{-i\omega t} \right) |\alpha(x,t)\rangle} \end{aligned}$$

$|\alpha\rangle$  are obviously not orthogonal. They are an overcomplete basis.

### 2.2.4 3-D Harmonic Potential

$$V(r) = \frac{1}{2} k r^2$$

$$\begin{aligned} &\text{(Isotropic)} \\ &\boxed{E_{n_x n_y n_z} = \hbar \omega \left( n_x + n_y + n_z + \frac{3}{2} \right) = \hbar \omega \left( n + \frac{3}{2} \right) \quad l = n - 2k \in \{n, n-2, \dots, 0\}} \end{aligned}$$

## 2.3 Free Particle (1-D)

$$V(x) = 0$$

$$\begin{aligned}\Psi_{(x,t)} &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi_{(x,0)} e^{\frac{i}{\hbar}[px - E(p)t]} dp & \langle x|U(t)|\Psi\rangle &= \iint \langle x|p\rangle e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} \langle p|x'\rangle dp \langle x'|\Psi\rangle dx' \\ &= \int \langle x|p\rangle e^{-\frac{i}{\hbar}E(p)t} \langle p|\Psi\rangle dp & &= \iint_{-\infty}^{\infty} \frac{1}{2\pi\hbar} e^{-\frac{i}{\hbar}\left[\frac{p^2 t}{2m} - p(x-x')\right]} dp \Psi_{(x',0)} dx' \\ \Phi_{(x,t)} &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi_{(x,0)} e^{\frac{-i}{\hbar}[px + E(p)t]} dx & &= \int \sqrt{\frac{-im}{2\pi\hbar t}} e^{\frac{im(x-x')^2}{2\hbar t}} \Psi_{(x',0)} dx'\end{aligned}$$

( $E < 0 \rightarrow \Psi = e^{\pm kx}$  is possible and also not normalizable, but solution above is already a complete set)

<ul style="list-style-type: none"> <li>• <math>E(p) = \frac{p^2}{2m}</math></li> <li>• <math>v_{\text{wave}} = \boxed{v_{\text{phase}} = \frac{\omega(k)}{k}} = \frac{E}{p} = \frac{v_{\text{classical}}}{2}</math></li> <li>• <math>v_{\text{particle}} \approx \boxed{v_{\text{group}} = \frac{d\omega(k)}{dk}} = 2v_{\text{wave}} \quad \left( \begin{smallmatrix} \text{dispersion} \\ \text{relation} \end{smallmatrix} \right)</math></li> </ul>	<u>Heisenberg Pic. Free Particle</u> $x_H(t) = x_H(0) + \frac{p_H(0)}{m}t$ $[x_H(0), x_H(t)] = \left[ x_H(0), \frac{p_H(0)}{m}t \right] = \frac{i\hbar t}{m}$ <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> <math>\sigma_{x_t} \sigma_{x_0} \geq \frac{\hbar t}{2m}</math> </div>
--	--

## 2.4 Delta Potential (1-D)

**Potential Well:**

$$V(x) = -\alpha\delta(x)$$

( $\alpha \rightarrow -\alpha$  for potential wall)

Bound State ( $E < 0$ ) [only for Well]:

$$\Psi = \sqrt{k}e^{k|x|} = \begin{cases} \sqrt{k}e^{kx} & x \leq 0 \\ \sqrt{k}e^{-kx} & x \geq 0 \end{cases}$$

$$k = \frac{m\alpha}{\hbar^2}$$

$$E = -\frac{(\hbar k)^2}{2m}$$

Scattering State ( $E > 0$ ) [for both]:

$$\Psi = \begin{cases} Ae^{iKx} + Be^{-iKx} & x < 0 \\ Fe^{iKx} & x > 0 \end{cases}$$

$$E = \frac{(\hbar K)^2}{2m}, \quad \beta \equiv \frac{k}{K} = \frac{m\alpha/\hbar^2}{K}$$

$$B = \frac{i\beta}{1-i\beta}A, \quad F = \frac{1}{1-i\beta}A$$

$$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}, \quad T = \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}$$

Can't normalize. All free particles have ranges of  $p$  and thus  $E$ , so  $R$  and  $T$  are approx. in the vicinity of  $E$ .

## 2.5 Finite Square Potential (1-D)

**2.5.1 Potential Well**  $V(x) = \begin{cases} -V_0 & -a < x < a \\ 0 & \text{otherwise} \end{cases}$  ( $V_0 \rightarrow -V_0$  for wall and do cases for  $E > V_0, E = V_0, E < V_0$ , and change to sinh, cosh if needed)

$$\begin{aligned} k; K : \quad E &= \frac{-(\hbar k)^2}{2m} = \frac{(\hbar K)^2}{2m} \\ l : \quad E + V_0 &= \frac{(\hbar l)^2}{2m} \\ v : \quad V_0 &= \frac{\hbar^2 v^2}{2m} = \frac{\hbar^2 (l^2 + k^2)}{2m} = \frac{\hbar^2 (l^2 - K^2)}{2m} \end{aligned} \quad \left| \quad \begin{aligned} \frac{k_a}{l_a} &\equiv \sqrt{\frac{(ka)^2}{(la)^2}} = \sqrt{\frac{(la)^2 + (ka)^2}{(la)^2}} - 1 \\ \frac{k_a}{l_a} &\equiv \sqrt{\left(\frac{v_a}{l_a}\right)^2 - 1}, \quad v_a^2 = \begin{cases} l_a^2 + k_a^2 \\ l_a^2 - K_a^2 \end{cases} \end{aligned} \right.$$

Bound State ( $E_n < 0$ ) [only for well]:

$$\Psi_{\text{even}}(x) = \begin{cases} \Psi(-x) & x < 0 \\ D \cos(lx) & 0 < x < a \\ F e^{-kx} & a < x \end{cases}$$

- $F = D \cos(la) e^{ka}$
- $\frac{-(\partial_x \Psi)(a)}{\Psi(a)} = k = l \tan(la) \Rightarrow$   
 $\tan(la) = \sqrt{(v_a/l_a)^2 - 1}$   
 $\text{big } v_a \rightarrow l \approx \frac{n\pi}{2a} \rightarrow E_n + V_0 = \frac{\hbar^2 l^2}{2m} ; \underline{n \text{ odd}}$
- $n_{\text{max}} = \left\lfloor \frac{v_a}{\pi} \right\rfloor + 1$

$$\Psi_{\text{odd}}(x) = \begin{cases} -\Psi(-x) & x < 0 \\ C \sin(lx) & 0 < x < a \\ F e^{-kx} & a < x \end{cases}$$

- $F = D \sin(la) e^{ka}$
- $\frac{-(\partial_x \Psi)(a)}{\Psi(a)} = k = -l \cot(la) \Rightarrow$   
 $-\cot(la) = \sqrt{(v_a/l_a)^2 - 1}$   
 $\text{big } v_a \rightarrow l \approx \frac{n\pi}{2a} \rightarrow E_n + V_0 = \frac{\hbar^2 l^2}{2m} ; \underline{n \text{ even}}$
- $n_{\text{max}} = \left\lfloor \frac{v_a + \frac{\pi}{2}}{\pi} \right\rfloor$

Scattering State ( $E > 0$ ) [for both]:

$$\Psi = \begin{cases} A e^{iKx} + B e^{-iKx} & x < -a \\ C \sin lx + D \cos lx & -a < x < a \\ F e^{iKx} & a < x \end{cases}$$

$$\frac{d\Psi}{dx} = \begin{cases} iK A e^{iKx} - iK B e^{-iKx} & x < -a \\ lC \cos lx - lD \sin lx & -a < x < a \\ iK F e^{iKx} & a < x \end{cases}$$

$$B = i \sin(2la) \left( \frac{l_a^2 - K_a^2}{2K_a l_a} \right) F$$

$$T^{-1} = 1 + \left( \frac{l_a^2 - K_a^2}{2K_a l_a} \right)^2 \sin^2(2la)$$

$$F = \frac{e^{-2iKa}}{\cos(2la) - i \left( \frac{l_a^2 + K_a^2}{2K_a l_a} \right) \sin(2la)} A$$

$$= 1 + \frac{V_0^2}{4E(E + V_0)} \sin^2 \left( 2a \sqrt{\frac{E + V_0}{\hbar^2/2m}} \right)$$

(Can't normalize. See delta potential.)

(full transmission at inf. sq. well  $E_n + V_0 = \frac{\hbar^2 l^2}{2m} ; l = \frac{n\pi}{2a}$ )

### 3 2D and 3D Schrodinger Equation

General dimensions,  $D$

$$\begin{aligned}
 \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) R(r) &= \left[ \frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} \right] R(r) \\
 &= \left[ \frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} \right] r^n u(r) \\
 &= \left[ \frac{\partial^2}{\partial r^2} + \frac{D-1+2n}{r} \frac{\partial}{\partial r} + \frac{2n(2D-4+2n)}{4r^2} \right] u \\
 &= \left[ \frac{\partial^2}{\partial r^2} - \frac{(D-1)(D-3)}{4r^2} \right] u \quad (n = \frac{1-D}{2}, 0, 2, \dots, D)
 \end{aligned}$$

$$R(r) = u(r)/\sqrt{r}^{D-1} \sim e^{\frac{i}{\hbar} p_r r} / \sqrt{r}^{D-1}$$

$$L^2 \sim \hbar^2, \quad \hat{p}_r = \frac{\hbar}{i} \left( \frac{\partial}{\partial r} + \frac{D-1}{2r} \right), \quad \hat{p}'_r = \frac{\hbar}{i} \frac{\partial}{\partial r}$$

$$ER(r) = \left[ \frac{\hat{p}_r^2}{2M} + V(r) + \frac{L^2 - \hbar^2(D-1)^2/4}{2(Mr^2)} \right] R(r)$$

$$Eu(r) = \left[ \frac{\hat{p}'_r^2}{2M} + V(r) + \frac{L^2 - \hbar^2(D-1)(D-3)/4}{2(Mr^2)} \right] u(r)$$

#### 3.1 2D Schrodinger

If  $V = V(\rho)$

$$\Psi(\vec{r}) = R_m(\rho) \Phi_m(\phi) \Rightarrow$$

$$ER = \left[ \frac{-\hbar^2}{2M} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + V(\rho) + \frac{\hbar^2 m^2}{2M \rho^2} \right] R$$

$$Eu = \frac{-\hbar^2}{2M} \frac{\partial^2 u}{\partial \rho^2} + \left[ V(\rho) + \frac{\hbar^2(m^2 + 1/4)}{2M \rho^2} \right] u$$

$$Eu\Phi = \left[ \frac{\hat{p}'_\rho{}^2}{2M} + V(\rho) + \frac{\hat{L}_z^2 + \hbar^2/4}{2(M\rho^2)} \right] u\Phi$$

$$\bullet R_m(\rho) = u_m(\rho)/\sqrt{\rho} \quad \left( \int \Psi r dr d\phi = 1 \right)$$

$$\bullet \Phi_m(\phi) = e^{im\phi}$$

$$\bullet L_z = (\vec{r} \times \vec{p})_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

### 3.2 3D Schrodinger

If  $V = V(r)$

$$\Psi(\vec{r}) = R_l(r)Y_l^m(\theta, \phi) = R_l(r)\Theta_l^m(\theta)\Phi_m(\phi) \Rightarrow$$

$$ER = \left[ \frac{-\hbar^2}{2M} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + V(r) + \frac{\hbar^2 l(l+1)}{2Mr^2} \right] R$$

$$Eu = \frac{-\hbar^2}{2M} \frac{\partial^2 u}{\partial r^2} + \left[ V(r) + \frac{\hbar^2 l(l+1)}{2Mr^2} \right] u$$

$$Eu\Theta = \left[ \frac{\hat{p}_r^2}{2M} + V(r) + \frac{\hat{L}^2}{2(Mr^2)} \right] u\Theta$$

$$\langle u | \nabla_r^2 u \rangle = \langle \nabla_r^2 u | u \rangle \Rightarrow \left[ u^* \frac{\partial u}{\partial r} - u \frac{\partial u^*}{\partial r} \right]_0^\infty \stackrel{\leftarrow 1.}{\stackrel{\leftarrow 2.}{=}} 0$$

$$1.) \int_0^\infty u^2 dr = 1 \Rightarrow \boxed{u(\infty) = 0 \text{ or } e^{ir}}$$

$$2.) \boxed{u(0) = c = 0} \left\{ c \neq 0 \rightarrow \begin{matrix} \Psi_{l=0}(r) \sim \frac{c}{r} \\ \nabla^2(\frac{1}{r}) \sim \delta^3(r) \end{matrix} \rightarrow \begin{matrix} H\Psi \neq E\Psi \\ \text{if } V(r) \neq \delta^3(r) \end{matrix} \right\}$$

$$\bullet R_l(r) = u_l(r)/r \quad \left( \int \Psi r^2 \sin \theta dr d\theta d\phi = 1 \right)$$

$$\bullet \Phi_m(\phi) = e^{im\phi}$$

$$\bullet \Theta_l^m(\theta) = AP_l^m(\cos \theta)$$

$$- A = \epsilon \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}}, \quad \epsilon = \begin{cases} (-1)^m & (m \geq 0) \\ 1 & (m \leq 0) \end{cases}$$

$$- P_l^m(x) = \text{Assoc. Legendre Func. (see extra)}$$

$$\bullet l \in \mathbb{N}_0, \quad m \in \{-l, \dots, -1, 0, 1, \dots, l\}$$

$$\bullet \hat{L}_i = (\vec{r} \times \vec{p})_i$$

$$V \sim r^{-2 < a}, \quad l \neq 0 : \lim_{r \rightarrow 0} u'' \sim \frac{l(l+1)}{r^2} u, \quad u \sim r^{l+1}$$

$$V \sim r^{-2 < a < -1}, \quad E > 0 : \lim_{r \rightarrow \infty} p_r^2 u \sim Eu, \quad u \sim e^{\pm ikr}$$

$$V \sim r^{-1 \leq a}, \quad E > 0 : u \gtrsim r e^{\pm ikr}$$

$$\text{Sim for } E < 0 : u \sim e^{\pm kr} \text{ or } r e^{\pm kr} \text{ etc.}$$

#### 3.2.1 3D Free Particle, $V = 0$

$$-\frac{\hbar^2}{2M} \left[ \frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2} \right] u = \frac{\hbar^2 k^2}{2M} u \Rightarrow \left[ \frac{\partial^2}{\partial \rho^2} + \frac{l(l+1)}{\rho^2} \right] |l\rangle = |l\rangle$$

$$a_l \equiv \frac{\partial}{\partial \rho} + \frac{l+1}{\rho} \quad a_l^\dagger = -\frac{\partial}{\partial \rho} + \frac{l+1}{\rho}$$

$$a_l a_l^\dagger |l\rangle = |l\rangle \quad a_l^\dagger a_l |l\rangle = a_{l+1} a_{l+1}^\dagger |l\rangle$$

$$a_l^\dagger |l\rangle = e^{i\theta_l} |a_l^\dagger(l)\rangle$$

$$a_l^\dagger (a_l a_l^\dagger) |l\rangle = a_l^\dagger |l\rangle$$

$$(a_l^\dagger a_l) a_l^\dagger |l\rangle = (a_{l+1} a_{l+1}^\dagger) a_l^\dagger |l\rangle$$

$$\underline{a_l^\dagger |l\rangle} = \cancel{e^{i\theta_l}} |l+1\rangle$$

$$\text{Bessel : } r \underline{R_0^B} = u_0^B \sim \sin(\rho) = \sin(kr)$$

$$\text{Neumann : } r \underline{R_0^N} = u_0^N \sim -\cos(\rho)$$

$$\cancel{e^{i\theta_l}} \frac{\rho}{k} R_{l+1} = a_l^\dagger \left( \frac{\rho}{k} R_l \right) = \left( -\frac{\partial}{\partial \rho} + \frac{l+1}{\rho} \right) \left( \frac{\rho}{k} R_l \right)$$

$$R_{l+1} = \left( -\frac{\partial}{\partial \rho} + \frac{l}{\rho} \right) R_l = -\rho^l \frac{\partial}{\partial \rho} (\rho^{-l} R_l)$$

$$\frac{R_l}{\rho^l} = -\frac{1}{\rho} \frac{\partial}{\partial \rho} \frac{R_{l-1}}{\rho^{l-1}} = \left( -\frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^l R_0$$

$$\boxed{R_l = C_l (-\rho)^l \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^l R_0}$$

Infinite Spherical Well:  $V(r) = \begin{cases} 0 & r \leq a \\ \infty & r > a \end{cases}, \quad E_n = \frac{\hbar^2 k_n^2}{2m}$

$$\begin{aligned} \text{Bessel: } R_l^B(\rho) &= C_l (-\rho)^l \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^l R_0^B(\rho) & \Rightarrow & \beta_l^n \equiv k_n a : R_l^B(\beta_l^n) = 0 \\ R_0^B(\rho) &\sim k_n \sin(\rho)/\rho = \sin(k_n r)/r & & \beta_0^n = \frac{n\pi}{a} \cdot a \end{aligned}$$

### 3.2.2 Hydrogen Atom, $V = -\frac{k\epsilon^2}{r}$

$$Eu = \left( \frac{\hat{p}_r^2}{2m} + V(r) + \frac{\hat{L}^2}{2(mr^2)} \right) u \quad u(r) = rR(r)$$

$$Eu = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial r^2} u + \left[ -\frac{ke^2}{r} + \frac{\hbar^2 l(l+1)}{2mr^2} \right] u$$

$$\Psi_{nlm}(\vec{r}) = R_{nl}(r) Y_l^m(\theta, \phi) = R_{nl}(r) \Theta_l^m(\theta) \Phi_m(\phi)$$

- $\Phi_m(\phi) = e^{im\phi}$
- $\Theta_l^m(\theta) = AP_l^m(\cos \theta)$
- $A = \epsilon \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}}$ ,  $\epsilon = \begin{cases} (-1)^m & (m \geq 0) \\ 1 & (m \leq 0) \end{cases}$
- $P_l^m(x)$  Assoc. Legendre Func. (see extra)
- $R_{nl}(r) = \frac{B}{r} \rho^{l+1} e^{-\rho} \nu(\rho)$
- $\rho = k_n r$ ,  $k_n = \frac{1}{a_0 n}$  (fine structure below)
- $\nu(\rho) = L_{n-l-1}^{2l+1}(2\rho)$  Assoc. Laguerre Pol. (see extra)
- $B = \sqrt{2k_n \frac{(n-l-1)!}{2n[(n+l)!]^3}} 2^{l+1}$

$$\alpha \equiv \frac{kq q}{\hbar c} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

$$a_0 \equiv \frac{\hbar^2}{m(kq q)} = \frac{4\pi\epsilon_0 \hbar^2}{me^2}$$

$$E_n = -\frac{\hbar^2 k_n^2}{2m} = -\frac{\hbar^2}{2ma_0^2} \frac{1}{n^2} = -\frac{1}{2} \alpha^2 (mc^2) \frac{1}{n^2} \approx -13.6 \frac{1}{n^2} [\text{eV}]$$

$$\frac{1}{\lambda} = \frac{\alpha^2 (mc^2)}{2\hbar c} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) = R \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right), \quad R = 1.097 \text{ E7 } [\text{m}^{-1}]$$

Quantum Numbers -  $n, l, m$ :

- $(n \in \{1, 2, 3, \dots\}), (l \in \{0, 1, 2, \dots, n-1\}), (m \in \{-l, \dots, -1, 0, 1, \dots, l\})$
- Degeneracy is  $n^2$

(outdated) Bohr Model:

- $L = (\vec{r})(\vec{p}) = (a_0 n^2)(\hbar k_n) = n\hbar$  (not correct!!)
- Electrons don't radiate about the nucleus
- Energy diff. follows Rydberg formula

## 4 Spin and $L$

### 4.1 Hydrogen Atom

Angular Momentum :

$$\widehat{L}_i \equiv (\vec{r} \times \vec{p})_i, \quad L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

$$\widehat{L}_{\pm} \equiv \widehat{L}_x \pm i\widehat{L}_y$$

$$\widehat{L}^2 \equiv L_x^2 + L_y^2 + L_z^2$$

$$L_{\pm}L_{\mp} = \widehat{L}^2 - L_z^2 \pm \hbar L_z$$

Commutation Relations:

$$[\hat{x}, L_y] = i\hbar \hat{z}, \quad [p_x, L_y] = i\hbar p_z, \quad [L_x, L_y] = i\hbar L_z$$

$$[L^2, L_i] = [H, L_i] = [H, L^2] = 0 \quad (\text{can measure concurrently})$$

$$\begin{aligned} & \begin{aligned} & L_z Y_{m'} = \hbar m' Y_{m'} \\ & L^2 Y_{m'} = \hbar^2 \lambda_{m'} Y_{m'} \end{aligned} \Rightarrow \begin{aligned} & \langle L^2 - L_z^2 \rangle = \langle L_x^2 + L_y^2 \rangle \geq 0 \\ & \bullet \sqrt{\lambda_{m'}} \geq m' \geq -\sqrt{\lambda_{m'}} \end{aligned} \end{aligned}$$

**Let**  $(L_{\pm})^n Y_{\mu} \equiv |m\rangle$  (see harm. osc. for why ladders)

$$\begin{aligned} \left[ \frac{L_z}{\hbar}, L_{\pm} \right] = (\pm 1) L_{\pm} & \Rightarrow \left[ \frac{L_z}{\hbar}, (L_{\pm})^n \right] = \pm n (L_{\pm})^n \Rightarrow \begin{aligned} & L_z [(L_{\pm})^n Y_{\mu}] = (\mu \pm n) \hbar [(L_{\pm})^n Y_{\mu}] \\ & \bullet L_z |m\rangle = (\mu \pm n) \hbar |m\rangle \end{aligned} \\ \left[ L^2, L_{\pm} \right] = 0 & \Rightarrow [L^2, (L_{\pm})^n] = 0 \Rightarrow \begin{aligned} & L^2 [(L_{\pm})^n Y_{\mu}] = \lambda_{\mu} [(L_{\pm})^n Y_{\mu}] \\ & \bullet L^2 |m\rangle = \lambda_{\mu} |m\rangle \end{aligned} \end{aligned}$$

**Then**  $\left( \sqrt{\lambda_{\mu}} \geq (\mu \pm n) \geq -\sqrt{\lambda_{\mu}} \right) \Rightarrow$  **Let** (else un-normalizable solution)

$$\left. \begin{aligned} & \underline{L_+ |m_t\rangle = 0}, \quad L_z |m_t\rangle = \hbar l, \\ & L^2 |m_t\rangle = \lambda \hbar^2, \quad L^2 = L_- L_+ + L_z^2 + \hbar L_z \\ & \bullet L^2 |m_t\rangle = \hbar^2 l(l+1) |m_t\rangle = \lambda \hbar^2 |m_t\rangle \end{aligned} \right| \begin{aligned} & \underline{L_- |m_b\rangle = 0}, \quad L_z |m_b\rangle = \hbar l', \\ & L^2 |m_b\rangle = \lambda \hbar^2, \quad L^2 = L_+ L_- + L_z^2 - \hbar L_z \\ & \bullet L^2 |m_b\rangle = \hbar^2 l'(l'-1) |m_b\rangle = \lambda \hbar^2 |m_b\rangle \end{aligned}$$

$$\left[ \lambda = l'(l'-1) = l(l+1) \right] \Rightarrow [l' = -l] \Rightarrow \left[ \begin{aligned} & L_z |m_t\rangle = \hbar l |m_t\rangle \\ & L_z |m_b\rangle = -\hbar l |m_b\rangle \end{aligned} \right] \quad (\text{Spherical Harmonics do not allow half-integer } l)$$

Schrodinger  $Y_l^m$ :

$$\left. \begin{aligned} & l \in \{0, 1, 2, \dots\} \\ & m \in \{-l, -l+1, \dots, l-1, l\} \end{aligned} \right| \begin{aligned} & L_z |Y_l^m\rangle = \hbar m |Y_l^m\rangle = \frac{\hbar}{i} \frac{\partial}{\partial \phi} |Y_l^m\rangle \\ & L^2 |Y_l^m\rangle = \hbar^2 l(l+1) |Y_l^m\rangle \\ & L_{\pm} |Y_l^m\rangle = \hbar \sqrt{l(l+1) - m(m \pm 1)} |Y_l^{m \pm 1}\rangle \end{aligned}$$



## 4.2 Generalized

Angular Momentum:

$$\hat{J}_i \equiv ???$$

$$J^2 \equiv J_x^2 + J_y^2 + J_z^2$$

$$J_{\pm} \equiv J_x \pm iJ_y$$

$$J_{\pm}J_{\mp} = J^2 - J_z^2 \pm \hbar J_z$$

Commutation Relations:

$$[J_i, J_j] = i\hbar J_k \epsilon_{ij} \Leftrightarrow J \times J = i\hbar J$$

$$[J^2, J_z] = 0 = [H, J_z] \quad (\text{if spher. symm.})$$

General:

$j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$ $m \in \{-j, -j+1, \dots, j-1, j\}$	$J_z jm\rangle = \hbar m jm\rangle$ $J^2 jm\rangle = \hbar^2 j(j+1) jm\rangle$ $J_{\pm} jm\rangle = \hbar \sqrt{j(j+1)-m(m\pm 1)} jm\rangle$ $J_x jm\rangle = \frac{J_+ + J_-}{2} jm\rangle$
---	--

Generator of Rotations:

$$U[R(\theta)] = e^{-\frac{i}{\hbar} \theta \hat{\theta} \cdot L} = \lim_{N \rightarrow \infty} \left[ I - \frac{i}{\hbar} \frac{\theta}{N} \hat{\theta} \cdot L \right]^N \Leftrightarrow U[R(\epsilon_z \hat{z})] = I - \frac{i}{\hbar} \epsilon_z L_z$$

$$1.) \quad U[R(\epsilon_z \hat{z})]|x, y\rangle \equiv |x - \epsilon_z y, \epsilon_z x + y\rangle$$

$$\Rightarrow \langle x, y | U[R(\epsilon_z \hat{z})] | \Psi \rangle = \Psi(x + \epsilon_z y, \epsilon_z x - y)$$

$$\Rightarrow \langle x, y | L_z | \Psi \rangle = (XP_y - YP_x) \Psi(x, y)$$

$$\text{or } 2.) \quad U^\dagger X U \equiv X - \epsilon_z Y \Rightarrow [X, L_z] = -i\hbar Y$$

$$U^\dagger P_y U \equiv \epsilon_z P_x + P_y \Rightarrow [P_y, L_z] = i\hbar P_x$$

$$U^\dagger Y U, U^\dagger P_x U, \Rightarrow \dots$$

$$\Rightarrow \underline{L_z = XP_y - YP_x}$$

$$\text{Consistency Check: } U[R(-\epsilon_z \hat{z})] T(-\epsilon_x \hat{x} - \epsilon_y \hat{y}) U[R(\epsilon_z \hat{z})] T(\epsilon_x \hat{x} + \epsilon_y \hat{y}) = T(-\epsilon_y \epsilon_z \hat{x} + \epsilon_x \epsilon_z \hat{y})$$

$$? U[R(-\epsilon_y \hat{y})] U[R(-\epsilon_x \hat{x})] U[R(\epsilon_y \hat{y})] U[R(\epsilon_x \hat{x})] = I + \frac{i}{\hbar} \epsilon_x \epsilon_y L_z = U[R(-\epsilon_x \epsilon_y \hat{z})]$$

$$(L_z = xp_y - yp_x)$$

Tensors and Tensor Operators

rank-2 Tensor :

$$|t^{(2)}\rangle = \sum_{i=1}^3 \sum_{j=1}^3 t_{ij} |i\rangle |j\rangle$$

$$= \sum_{i=1}^3 \sum_{j=1}^3 |ij\rangle \langle ij | t^{(2)} \rangle$$

rank-2 Cartesian Tens. Oper.,  $T_{ij}$  :

Set of  $3^{n=2}$  Operators

rank-k Spherical Tensor Operator,  $T_k^q$  :

Set of  $2k+1$  Operators s.t.

$$U[R] T_k^q U^\dagger[R] = \sum_{q'=-k}^k D_{q'q}^k T_k^{q'} \quad \downarrow$$

$$U T_k^q U^\dagger U |jm\rangle = \sum_{q'} \sum_{m'} D_{q'q}^k D_{m'm}^j T_{q'}^k |jm'\rangle$$

$$\sim U |kq\rangle |jm\rangle = \sum_{q'} \sum_{m'} D_{q'q}^k D_{m'm}^j |kq'\rangle |jm'\rangle$$

$$\bullet \quad T_1^{\pm 1} = \mp \frac{V_x \pm iV_y}{\sqrt{2}} \\ T_1^0 = V_z$$

$$\text{Wigner-Eckhart: } \langle \alpha_2 j_2 m_2 | T_k^q | \alpha_1 j_1 m_1 \rangle = \langle \alpha_2 j_2 | T_k | \alpha_1 j_1 \rangle \cdot \langle j_2 m_2 | kq; j_1 m_1 \rangle$$

(CG coeff.)

### 4.3 1 Particle w/ Spin, $s = \frac{1}{2}$

\*Find the Eigenvectors,  $e_i$ , of  $S_z$  and  $S^2$  in the form of  $|\chi\rangle = \begin{pmatrix} \cos \frac{\theta}{2} e^{i\phi_1} \\ \sin \frac{\theta}{2} e^{i\phi_2} \end{pmatrix} = e^{i\gamma} \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ \sin \frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix}$   $\gamma = \frac{\phi_1 + \phi_2}{2}$   
 $\phi = \phi_2 - \phi_1$

$$* \quad e_i \in \left\{ \begin{array}{l} |\frac{1}{2} \frac{1}{2}\rangle \equiv |\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |\frac{1}{2} \frac{-1}{2}\rangle \equiv |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array} \right\}$$

$$\left. \begin{array}{l} S^2 |\uparrow\rangle = \frac{3\hbar^2}{4} |\uparrow\rangle \\ S^2 |\downarrow\rangle = \frac{3\hbar^2}{4} |\downarrow\rangle \end{array} \right\} \Rightarrow S^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3\hbar^2}{4} \sigma_0 = * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3\hbar^2}{4} & 0 \\ 0 & \frac{3\hbar^2}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{T*}$$

$$= \frac{3\hbar^2}{4} \mathbb{I} \quad (\text{only for } s = 1/2 \text{ systems})$$

$$\left. \begin{array}{l} S_- |\uparrow\rangle = \hbar |\downarrow\rangle \\ S_+ |\downarrow\rangle = \hbar |\uparrow\rangle \\ S_+ |\uparrow\rangle = S_- |\downarrow\rangle = 0 \end{array} \right\} \Rightarrow \begin{array}{ll} S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ = \hbar |\uparrow\rangle\langle\downarrow| & = \hbar |\downarrow\rangle\langle\uparrow| \end{array} \quad (\text{can't measure})$$

$$\left. \begin{array}{l} S_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle \\ S_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle \end{array} \right\} \Rightarrow S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2} \sigma_z = * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{T*}$$

$$= \frac{\hbar}{2} |\uparrow\rangle\langle\uparrow| - \frac{\hbar}{2} |\downarrow\rangle\langle\downarrow|$$

$$\left. \begin{array}{l} S_x = \frac{1}{2}(S_+ + S_-) \\ S_y = \frac{1}{2i}(S_+ - S_-) \end{array} \right\} \Rightarrow \begin{array}{l} S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_x \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_y \\ \boxed{\{S_i, S_j\} = \frac{\hbar^2}{2} \delta_{ij}} \quad (\text{only for } s = 1/2 \text{ systems}) \end{array}$$

#### Properties of Pauli Matrices, $\sigma_i$

- $\sigma_i = \sigma_i^\dagger = \sigma_i^{-1}$
- \*  $\sigma_i^2 = I = (\hat{n} \cdot \sigma)^2 \Leftarrow \begin{array}{l} (\hat{n} \cdot \sigma)^2 - I^2 = (\hat{n} \cdot \sigma + I)(\hat{n} \cdot \sigma - I) = \\ 0 = (\hat{n} \cdot S + \frac{\hbar}{2})(\hat{n} \cdot S - \frac{\hbar}{2}) = (S_z + \frac{\hbar}{2})(S_z - \frac{\hbar}{2}) \end{array}$
- $\sigma_i \sigma_j = -\sigma_j \sigma_i = i\sigma_k = (\sigma_j \sigma_i)^\dagger \Rightarrow \left[ \frac{\sigma_i}{2}, \frac{\sigma_j}{2} \right] = i\frac{\sigma_k}{2} \quad (i \neq j)$
- \*  $(A \cdot \sigma)(B \cdot \sigma) = A \cdot B + i(A \times B) \cdot \sigma \quad (\text{if } [A_i, \sigma_i] = 0 = [B_i, \sigma_i])$
- \*  $(\sigma_i \sigma_j)^2 = -1 \quad (\text{even if } \sigma_i^2 = \pm 1, \text{ but } (\gamma_t \gamma_i)^2 = 1 \text{ [see right]})$
- $\text{Tr } \sigma_i = 0 \Rightarrow \text{Tr}(\sigma_\alpha \sigma_\beta) = 2\delta_{\alpha\beta} \quad \alpha \in (0, x, y, z)$
- \*  $\sum c_\alpha \sigma_\alpha = 0 \rightarrow c_\alpha = 0 \Rightarrow M_{2 \times 2} = \sum \frac{1}{2} \text{Tr}(M \sigma_\alpha) \sigma_\alpha$
- $\det(\sigma_i) = -1$

#### Misc.

- $U[R(\theta)] = e^{\theta g_i} \Leftarrow \begin{array}{l} * U_{\theta=0} = I \\ * g_i = \frac{dU}{d\theta} \Big|_{\theta=0} \end{array}$
- $\det U = 1 = \det(e^{\theta g_i}) = e^{\theta \text{tr}(g_i)=0}$
- $\left[ \frac{-i\sigma_x}{2}, \frac{-i\sigma_y}{2} \right] = \frac{-i\sigma_z}{2} = \underline{g_z} = -g_z^\dagger \quad (\mathfrak{su}(2))$
- Rotation = Unitary, Boost = Hermitian
- $g_z^\dagger = g_i \quad (\text{for boosts } \in \mathfrak{sl}(2, \mathbb{C}))$
- $g_i^T = -g \quad (\text{for } \mathfrak{so}(3))$
- $g_i^T = -\eta g \eta \quad (\text{for } \mathfrak{so}^+(1, 3))$
- $j^2 = 1 : e^{j\phi} = \cosh \phi + j \sinh \phi$

General Direction,  $\hat{n}$ , on Bloch Sphere  $(1, \theta, \phi) \Leftrightarrow$  Pauli Vector,  $V$  (see Properties of  $\sigma_i$ )

$$\hat{n} \cdot \vec{S} = \cos \phi \sin \theta S_x + \sin \phi \sin \theta S_y + \cos \theta S_z$$

$$V_{\hat{n}} = \frac{\hbar}{2} \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{bmatrix}$$

- $V_{\vec{n}}^2 = \|\vec{n}\|^2 I$  •  $\det(V_{\vec{n}}) = -\|\vec{n}\|^2$
- $V = V^\dagger$  •  $\text{Tr}(V) = 0$  • Reflect  $V$  over  $R_{\hat{n}} \perp$  :  $-RV R^{-1}$
- Rotate  $V$  in  $xy$ -plane by  $\psi$  :  $-\left[\cos \frac{\psi}{2} \sigma_x + \sin \frac{\psi}{2} \sigma_y\right] [-\sigma_x V \sigma_x][\dots]$   
 $= \left[\cos \frac{\psi}{2} I - \sin \frac{\psi}{2} \sigma_x \sigma_y\right] V [\dots]^{-1} = \boxed{UVU^\dagger}$

$$\hat{n} \cdot \vec{S} |\chi_\pm\rangle = \pm \frac{\hbar}{2} |\chi_\pm\rangle \Rightarrow |\chi_+\rangle = e^{i\gamma} \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} = \begin{bmatrix} 1 \\ \tan \frac{\theta}{2} e^{i\phi} \end{bmatrix}, |\chi_-\rangle = e^{i\gamma} \begin{bmatrix} -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \cos \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\cot \frac{\theta}{2} e^{i\phi} \end{bmatrix}$$

Riemann Sphere  $\rightarrow \mathbb{CP}^1$ :  $|\chi\rangle = \begin{bmatrix} 1 \\ f(\theta, \phi) \end{bmatrix} \rightarrow \left. \begin{array}{l} f(\theta, \phi) \in (-\infty, \infty) : 0 \leq \theta < \pi \\ f(\theta, \phi) \in (-\infty, \infty) : \pi \leq \theta < 2\pi \end{array} \right\}$  (double cover)

$SU(2)$  Matrices:  $\begin{bmatrix} \cos \frac{\psi}{2} I - \sin \frac{\psi}{2} i(n^x \sigma_x + n^y \sigma_y + n^z \sigma_z) \\ \dots - \sin \frac{\psi}{2} (n^x \sigma_y \sigma_z + n^y \sigma_z \sigma_x + n^z \sigma_x \sigma_y) \end{bmatrix} = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \Leftrightarrow U^\dagger = U^{-1}, \det(U) = \alpha^* \alpha + \beta^* \beta = 1$

- Rotate  $\hat{k}$  to  $\hat{k}'$  by angle  $\theta$   
 $\hat{\theta} = (-\sin \phi, \cos \phi, 0) = \hat{k} \times \hat{k}'$   $U[R(\theta)] = e^{-i\frac{\theta}{2}\hat{\theta} \cdot \sigma} = \cos \frac{\theta}{2} I - \sin \frac{\theta}{2} i(\hat{\theta} \cdot \sigma) = \begin{bmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} e^{-i\phi} \\ \sin \frac{\theta}{2} e^{i\phi} & \cos \frac{\theta}{2} \end{bmatrix}$
- Rotate about  $\hat{\psi}$  by angle  $\psi$   
 $\hat{\psi} = \hat{n} = (n^x, n^y, n^z)$   $U[R(\psi)] = \cos \frac{\psi}{2} I - \sin \frac{\psi}{2} i(\hat{\psi} \cdot \sigma) = \begin{bmatrix} \cos \frac{\psi}{2} - in^z \sin \frac{\psi}{2} & (-in^x - n^y) \sin \frac{\psi}{2} \\ (-in^x + n^y) \sin \frac{\psi}{2} & \cos \frac{\psi}{2} + in^z \sin \frac{\psi}{2} \end{bmatrix}$

## 4.4 2 Objects w/ Spin Objects could be orbital momentum, another particle spin, etc.

4.4.1 2 Objects w/ Spin  $\frac{1}{2}$ :  $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0 \Rightarrow (2s_1 + 1)(2s_2 + 1) = \sum_{s=|s_1-s_2|}^{s_1+s_2} 2s + 1$

\*Find Eigenvectors,  $e_i$ , of  $(S^{(1,2)})_z$  and  $(S^{(1,2)})^2$  in the form of  $|\chi_i \chi_j\rangle$  ( using  $(S^{(1,2)})_\pm$  )

$$\chi_i \chi_j \rightarrow |\chi_i \chi_j\rangle \equiv |\chi_i\rangle |\chi_j\rangle \equiv |\chi_i\rangle \otimes |\chi_j\rangle$$

Choose  $|\chi_i\rangle \equiv S_z$ -Eigenvector w/ Spin  $\frac{1}{2}$  (e.g,  $|\frac{1}{2} \frac{-1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , as opposed to  $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ )

$$S^{(i)} \equiv \begin{pmatrix} S_x^{(i)} \\ S_y^{(i)} \\ S_z^{(i)} \end{pmatrix} \quad \left| \quad S^{(1,2)} \equiv (S^{(1)} + S^{(2)}) \equiv \begin{pmatrix} S_x^{(1)} + S_x^{(2)} \\ S_y^{(1)} + S_y^{(2)} \\ S_z^{(1)} + S_z^{(2)} \end{pmatrix} \right.$$

- $S_z^{(2)} S_x^{(1)} (|\chi_1\rangle \otimes |\chi_2\rangle) = (S_x^{(1)} |\chi_1\rangle) \otimes (S_z^{(2)} |\chi_2\rangle)$
- $(S^{(1,2)})^2 = (S^{(1)} + S^{(2)}) \cdot (S^{(1)} + S^{(2)})$
- $S^{(i)} \cdot S^{(j)} \equiv \underline{S_x^{(i)} S_x^{(j)} + S_y^{(i)} S_y^{(j)} + S_z^{(i)} S_z^{(j)}}$   
 $= \underline{S_z^{(i)} S_z^{(j)} + \frac{1}{2} S_+^{(i)} S_-^{(j)} + \frac{1}{2} S_-^{(i)} S_+^{(j)}}$   
 $(S^{(i)})^2 \equiv S^{(i)} \cdot S^{(i)}$

1.  $(S^{(1,2)})_z$

$$\begin{aligned}
 (S^{(1,2)})_z |\chi_1 \chi_2\rangle &= (S_z^{(1)} + S_z^{(2)}) |\chi_1\rangle |\chi_2\rangle \\
 &= S_z^{(1)} |\chi_1\rangle \otimes |\chi_2\rangle + |\chi_1\rangle \otimes S_z^{(2)} |\chi_2\rangle \\
 (S^{(1,2)})_z |\chi_1 \chi_2\rangle &= \hbar(m_1 + m_2) |\chi_1 \chi_2\rangle \\
 \Rightarrow \underline{e_i = a_i |\uparrow\uparrow\rangle + b_i |\uparrow\downarrow\rangle + c_i |\downarrow\uparrow\rangle + d_i |\downarrow\downarrow\rangle}
 \end{aligned}
 \quad \left| \begin{array}{l} |\uparrow\uparrow\rangle = |\frac{1}{2}\frac{1}{2}\rangle \otimes |\frac{1}{2}\frac{1}{2}\rangle \\ |\uparrow\downarrow\rangle = |\frac{1}{2}\frac{1}{2}\rangle \otimes |\frac{1}{2}\frac{-1}{2}\rangle \\ |\downarrow\uparrow\rangle = |\frac{1}{2}\frac{-1}{2}\rangle \otimes |\frac{1}{2}\frac{1}{2}\rangle \\ |\downarrow\downarrow\rangle = |\frac{1}{2}\frac{-1}{2}\rangle \otimes |\frac{1}{2}\frac{-1}{2}\rangle \end{array} \right.$$

2. Use  $(S^{(1,2)})_\pm$  on  $|\uparrow\rangle \otimes |\uparrow\rangle$  to GUESS  $e_i$  from “nice” behavior

$$\begin{aligned}
 S_- |\uparrow\uparrow\rangle &= \frac{\sqrt{2}}{2} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
 S_- \left[ \frac{\sqrt{2}}{2} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \right] &= |\downarrow\downarrow\rangle \\
 S_- |\downarrow\downarrow\rangle &= 0
 \end{aligned}
 \quad \left| \begin{array}{l} \text{Guess for } \{e_i\}: \\ |1\ 1\rangle \equiv |\frac{1}{2}\frac{1}{2}\rangle |\frac{1}{2}\frac{1}{2}\rangle = |\uparrow\uparrow\rangle \\ |1\ 0\rangle \equiv \frac{1}{\sqrt{2}} \left( |\frac{1}{2}\frac{1}{2}\rangle |\frac{1}{2}\frac{-1}{2}\rangle + |\frac{1}{2}\frac{-1}{2}\rangle |\frac{1}{2}\frac{1}{2}\rangle \right) = \frac{\sqrt{2}}{2} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1\ -1\rangle \equiv |\frac{1}{2}\frac{-1}{2}\rangle |\frac{1}{2}\frac{-1}{2}\rangle = |\downarrow\downarrow\rangle \\ |0\ 0\rangle \equiv \frac{1}{\sqrt{2}} \left( |\frac{1}{2}\frac{1}{2}\rangle |\frac{1}{2}\frac{-1}{2}\rangle - |\frac{1}{2}\frac{-1}{2}\rangle |\frac{1}{2}\frac{1}{2}\rangle \right) = \frac{\sqrt{2}}{2} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \end{array} \right.$$

$S_+$  works too

If  $\frac{\sqrt{2}}{2} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$  then maybe  $\frac{\sqrt{2}}{2} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$  works (try  $S_\pm$  on it).

3. Check if the guesses are eigenvectors of  $(S^{(1,2)})^2$  [and do  $(S^{(1,2)})_z$  to see eigenvalues]

(work has been skipped, do it yourself, check answer below)

$$\begin{aligned}
 S^2 |1\ 1\rangle &= \hbar^2(1)(1+1) |1\ 1\rangle & (s=1) & \quad S_z |1\ 1\rangle = \hbar(1) |1\ 1\rangle & (m=1) \\
 S^2 |1\ 0\rangle &= \hbar^2(1)(1+1) |1\ 0\rangle & (s=1) & \quad S_z |1\ 0\rangle = \hbar(0) |1\ 0\rangle & (m=0) \\
 S^2 |1\ -1\rangle &= \hbar^2(1)(1+1) |1\ -1\rangle & (s=1) & \quad S_z |1\ -1\rangle = \hbar(-1) |1\ -1\rangle & (m=-1) \\
 S^2 |0\ 0\rangle &= \hbar^2(0)(0+1) |0\ 0\rangle & (s=0) & \quad S_z |0\ 0\rangle = \hbar(0) |0\ 0\rangle & (m=0) \quad \checkmark
 \end{aligned}$$

$$* \quad e_i \in \left\{ \begin{array}{l} |1\ 1\rangle = \cancel{1} \leftarrow e^{i\phi} |\frac{1}{2}\frac{1}{2}\rangle |\frac{1}{2}\frac{1}{2}\rangle = |\uparrow\uparrow\rangle \\ |1\ 0\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}\frac{1}{2}\rangle |\frac{1}{2}\frac{-1}{2}\rangle + |\frac{1}{2}\frac{-1}{2}\rangle |\frac{1}{2}\frac{1}{2}\rangle \right) = \frac{\sqrt{2}}{2} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1\ -1\rangle = |\frac{1}{2}\frac{-1}{2}\rangle |\frac{1}{2}\frac{-1}{2}\rangle = |\downarrow\downarrow\rangle \end{array} \right\} \quad \text{Triplet : } s=1$$

$$\left\{ \begin{array}{l} |0\ 0\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}\frac{1}{2}\rangle |\frac{1}{2}\frac{-1}{2}\rangle - |\frac{1}{2}\frac{-1}{2}\rangle |\frac{1}{2}\frac{1}{2}\rangle \right) = \frac{\sqrt{2}}{2} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \end{array} \right\} \quad \text{Singlet : } s=0$$

#### 4.4.2 2 Objects w/ Any Spin:

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus |j_1 - j_2| \Rightarrow (2j_1 + 1)(2j_2 + 1) = \sum_{j=|j_1-j_2|}^{j_1+j_2} 2j + 1$$

- $|\chi_1\rangle$  has spin,  $j_1$ ; and  $|\chi_2\rangle$  has spin,  $j_2$
- $j_{\max} = j_2 + j_1$  and  $j_{\min} = |j_2 - j_1|$
- Possible total  $|j \ m\rangle$  must satisfy

- 1.)  $j_{\min} \leq j \leq j_{\max}$ ,
- 2.)  $-j \leq m \leq j$ ,
- 3.) have integer differences

If  $j_1$  and  $j_2$  are known from the start,

$$\begin{aligned} |j \ m\rangle &= \sum_{m_1, m_2} |j_1 \ m_1\rangle \otimes |j_2 \ m_2\rangle \langle j_1 \ m_1| \otimes \langle j_2 \ m_2| |j \ m\rangle \\ &= \sum_{m_1, m_2} |j_1 \ m_1\rangle \otimes |j_2 \ m_2\rangle C_{m_1 m_2 m}^{j_1 j_2 j} \end{aligned}$$

where the sum is over all poss. int. diff. values that satisfy

$$m_1 + m_2 = m, \quad -j_1 \leq m_1 \leq j_1, \quad -j_2 \leq m_2 \leq j_2,$$

and  $C$  are the corresponding Clebsh-Gordan coefficients, whose squared value is the probability of measuring the  $\chi_1 \otimes \chi_2$  state represented by that term.

If the top state in a  $j$ -set (see box above) is known, applying the  $J_-$  lowering operator (and normalizing) provides the coefficients for the rest of the set of varying  $m$ . The coefficients for each top state of a set are (by convention) positive, real, and normalized to 1. This makes all of the coefficients real. For the top state of the initial  $j_{\max}$ -set,  $|j_{\max} \ j_{\max}\rangle$ , there is only one product-ket in the sum; its coefficient is thus set to 1. For an arbitrary set below the first, the top state has product-ket coefficients such that the state is orthogonal to all other previously determined states that have the same  $m$ . To reduce some work to solve for them, use

$$\begin{aligned} C_{m_1 m_2 m}^{j_1 j_2 j} &= (-1)^{j_1+j_2-j} \cdot C_{-m_1 -m_2 -m}^{j_1 j_2 j} \\ \langle j_1 \ m_1 | \langle j_2 \ m_2 | j \ m\rangle &= (-1)^{j_1+j_2-j} \cdot \langle j_1 \ -m_1 | \langle j_2 \ -m_2 | j \ -m\rangle \end{aligned}$$

If  $m_1$  and  $m_2$  are also known from the start, then  $m = m_1 + m_2$ , and

$$|j_1 \ m_1\rangle \otimes |j_2 \ m_2\rangle = \sum_j C_{m_1 m_2 m}^{j_1 j_2 j} |j \ (m_1+m_2)\rangle$$

where the sum is only over all possible  $s$  as satisfied above - **1.), 2.) and 3.)**. In this case, the total z-component,  $m$ , is known. The only unknown is the total spin,  $s$ , whose probability to be measured is  $C^2$ .

Possible Combined  $|j \ m\rangle$

$$\begin{aligned} (2j_{\max} + 1) &\left\{ \begin{array}{l} |j_{\max} \ j_{\max}\rangle = 1 \leftarrow \text{top} \dots \\ |j_{\max} \ j_{\max}-1\rangle \\ \vdots \\ |j_{\max} \ -j_{\max}\rangle \end{array} \right. \\ (2j_{\max} - 1) &\left\{ \begin{array}{l} |j_{\max}-1 \ j_{\max}-1\rangle = 1 \dots \\ |j_{\max}-1 \ j_{\max}-2\rangle \\ \vdots \end{array} \right. \\ \vdots & \\ (2j_{\min} + 1) &\left\{ \begin{array}{l} |j_{\min} \ j_{\min}\rangle = 1 \dots \\ J_- |j_{\min} \ j_{\min}\rangle \sim \dots \\ \vdots \\ |j_{\min} \ -j_{\min}\rangle \end{array} \right. \end{aligned}$$

## 4.5 Electron in Magnetic Field

$$\mu_{\text{clas.}} = IA = \frac{q}{2\pi r} v(\pi r^2) = \frac{q}{2\pi r} \frac{L}{mr} (\pi r^2) = \left(\frac{q}{2m}\right) L \rightarrow \frac{e\hbar}{2m} \cdot n \quad (\text{Bohr magneton})$$

$$\mu_{\text{quan.}} = \left(\frac{geq}{2m}\right) S = \left(\frac{q}{m}\right) S = \gamma S$$

$$\begin{aligned} \tau_\mu &= \mu \times B & H &= -\mu \cdot B \\ F_\mu &= \nabla(\mu \cdot B) & &= -\gamma S \cdot B \end{aligned},$$

### Larmor Precession

$$\begin{aligned} \chi(t) &= \cos(\alpha/2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar} E_1 t} + \sin(\alpha/2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\frac{i}{\hbar} E_2 t} \\ B &= B_0 \hat{k} \\ H &= -\gamma B_0 S_z \\ &= -\gamma B_0 \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} \end{aligned} \Rightarrow \begin{aligned} &= \begin{pmatrix} \cos(\alpha/2) e^{-\frac{i}{\hbar} E_1 t} \\ \sin(\alpha/2) e^{-\frac{i}{\hbar} E_2 t} \end{pmatrix} \\ \begin{pmatrix} \langle S_x \rangle \\ \langle S_y \rangle \\ \langle S_z \rangle \end{pmatrix} &= \begin{pmatrix} \frac{\hbar}{2} \sin(\alpha) \cos(\gamma B_0 t) \\ -\frac{\hbar}{2} \sin(\alpha) \sin(\gamma B_0 t) \\ \frac{\hbar}{2} \cos(\alpha) \end{pmatrix} \end{aligned} \quad (\text{torque from } B \text{ with } S \text{ leads to precession})$$

### Stern-Gerlach

## 5 Bosons and Fermions

Distinguishable Particles:  $\boxed{\psi(r_1, r_2) \equiv \psi_a(r_1)\psi_b(r_2)}$

Indistinguishable Particles:

$$\boxed{P_x f(x_1, x_2; y_1, y_2; \dots) = \pm f(x_2, x_1; y_1, y_2; \dots)} \quad , \quad \boxed{\iint |\Psi(x_1, x_2)|^2 dx_1 dx_2 = \iint \text{Pr}(x_1, x_2) \frac{dx_1 dx_2}{2}}$$

Boson:

$$(s \in \{0, 1, 2, \dots\}) \quad \psi_+(r_1, r_2) \equiv \frac{1}{\sqrt{2}} \left[ \psi_a(r_1)\psi_b(r_2) + \psi_a(r_2)\psi_b(r_1) \right]$$

$$\boxed{\psi(r_1, r_2) = \psi(r_2, r_1)} \quad \rightarrow \quad \boxed{P_i \Psi = \Psi} \quad (\text{symmetric})$$

Fermion:

$$(s \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}) \quad \psi_-(r_1, r_2) \equiv \frac{1}{\sqrt{2}} \left[ \psi_a(r_1)\psi_b(r_2) - \psi_a(r_2)\psi_b(r_1) \right]$$

$$\boxed{\psi(r_1, r_2) = -\psi(r_2, r_1)} \quad \rightarrow \quad \boxed{P_i \Psi = -\Psi} \quad (\text{antisymmetric})$$

### 5.1 Exchange Forces: $\left\langle (x_1 - x_2)^2 \right\rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$

Dist. Part. :	$\langle (\Delta x)^2 \rangle = \langle (\Delta x)^2 \rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b$
Symmetric:	$\langle (\Delta x)^2 \rangle = \langle (\Delta x)^2 \rangle_d - 2 \left\  \langle \psi_b   x   \psi_a \rangle \right\ ^2$ (attractive if overlap)
Antisymmetric:	$\langle (\Delta x)^2 \rangle = \langle (\Delta x)^2 \rangle_d + 2 \left\  \langle \psi_b   x   \psi_a \rangle \right\ ^2$ (repulsive if overlap)

$$\begin{aligned}
 \bullet \langle x_1 x_2 \rangle &= \frac{1}{2} \int \left[ \psi_a(r_1)^* \psi_b(r_2)^* \pm \psi_b(r_1)^* \psi_a(r_2)^* \right] x_1 x_2 \left[ \psi_a(r_1) \psi_b(r_2) \pm \psi_b(r_1) \psi_a(r_2) \right] dx_1 dx_2 \\
 &= \frac{1}{2} \langle x \rangle_a \langle x \rangle_b + \frac{1}{2} \langle x \rangle_b \langle x \rangle_a \\
 &\quad \pm \frac{1}{2} \left\langle \psi_b(r_1) \left| x_1 \right| \psi_a(r_1) \right\rangle \left\langle \psi_a(r_2) \left| x_2 \right| \psi_b(r_2) \right\rangle \pm \frac{1}{2} \left\langle \psi_a(r_1) \left| x_1 \right| \psi_b(r_1) \right\rangle \left\langle \psi_b(r_2) \left| x_2 \right| \psi_a(r_2) \right\rangle \\
 &= \langle x \rangle_a \langle x \rangle_b \pm \left\| \langle \psi_b | x | \psi_a \rangle \right\|^2
 \end{aligned}$$

Two Electrons:

$$\psi(r_1, r_2) \chi(m_1, m_2) = \begin{cases} \text{(singlet)} & \Rightarrow \begin{array}{l} \chi \text{ is antisymmetric so} \\ -\psi(r_1, r_2) \chi(m_2, m_1) \end{array} & \Rightarrow \text{Attractive (ground state)} \\ \text{(triplet)} & \Rightarrow \begin{array}{l} \chi \text{ is symmetric so} \\ -\psi(r_2, r_1) \chi(m_1, m_2) \end{array} & \Rightarrow \text{Repulsive} \end{cases}$$

## 5.2 Statistics

Sterling's Approx:  $\log(z!) \approx z \log(z) - z \quad z \gg 1 \text{ or } z = 0$

$$\frac{d}{dz} \log(z!) \approx \log(z)$$

Lagrange Multiplier:  $G(X, \alpha, \beta) = \log(Q(X)) + \alpha f_1(X) + \beta f_2(X)$

$$\frac{\partial G}{\partial \alpha}[Q_{\max}] = 0, \quad \frac{\partial G}{\partial \beta}[Q_{\max}] = 0, \quad \frac{\partial G}{\partial N_n}[Q_{\max}] = 0$$


---

$$\sum_n N_n = N \quad \sum_n N_n E_n = E$$

$$f_1(X) = N - \sum_n N_n = 0 \quad f_2(X) = E - \sum_n N_n E_n = 0$$

Let there be  $N_n$  particles in the  $E_n$  energy level having  $d_n$  degeneracies, and  $Q(N_1, N_2, \dots)$  be the number of possible configurations for such a state given  $X = (N_1, N_2, \dots, N_n)$ .

Dist.  $\left\{ \begin{array}{ll} \text{1.) } Q(X) = \prod_n \binom{N - N_1 - \dots - N_{n-1}}{N_n} d_n^{N_n} \\ \quad = N! \prod_n \frac{d_n^{N_n}}{N_n!} & \text{3.) } \frac{\partial G}{\partial N_n} \approx \frac{\log(d_n) - \log(N_n)}{-\alpha - \beta E_n} = 0 \\ \text{2.) } \log(Q) = \log(N!) + \sum_n N_n \log(d_n) & \text{4.) } N_n = \frac{d_n}{e^{\beta E_n + \alpha}} \\ \quad - \log(N_n!) \end{array} \right.$

Fermion  $\left\{ \begin{array}{ll} \text{1.) } Q(X) = \prod_n \binom{d_n}{N_n} & \text{3.) } \frac{\partial G}{\partial N_n} \approx \frac{-\log(N_n) + \log(d_n - N_n)}{-\alpha - \beta E_n} = 0 \\ \text{2.) } \log(Q) = \sum_n \log(d_n!) - \log(N_n!) & \text{4.) } N_n = \frac{d_n}{e^{\beta E_n + \alpha} + 1} \\ \quad - \log[(d_n - N_n)!] \end{array} \right.$

Boson  $\left\{ \begin{array}{ll} \text{1.) } Q(X) = \prod_n \binom{N_n + d_n - 1}{N_n} & \text{3.) } \frac{\partial G}{\partial N_n} \approx \frac{\log(N_n + d_n - 1) - \log(N_n)}{-\alpha - \beta E_n} = 0 \\ \text{2.) } \log(Q) = \sum_n \log[(N_n + d_n - 1)!] & \text{4.) } N_n = \frac{d_n - 1}{e^{\beta E_n + \alpha} - 1} \approx \frac{d_n}{e^{\beta E_n + \alpha} - 1} \\ \quad - \log(N_n!) & \\ \quad - \log[(d_n - 1)!] \end{array} \right.$



Given some substance in thermal equilibrium,

$$\beta = \frac{1}{k_b T} \quad \mu(T) \equiv -\frac{\alpha}{k_b T}$$

where  $\mu$  depends on the situation.

$$\frac{N_n}{d_n} : \quad n(\epsilon) = \begin{cases} \frac{1}{e^{(\epsilon-\mu)/k_b T}} & \text{Maxwell-Boltzmann} \\ \frac{1}{e^{(\epsilon-\mu)/k_b T} + 1} & \text{Fermi-Dirac} \\ \frac{1}{e^{(\epsilon-\mu)/k_b T} - 1} & \text{Bose-Einstein} \end{cases}$$

## 6 Perturbation Theory

$$H^{(0)}\psi_n = E_n\psi_n$$

$$\downarrow$$

$$H\psi'_n = E'_n\psi'_n$$

$$\left[ H^{(0)} + \lambda H^{(1)} \right] \left[ \psi_n + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \dots \right] = \left[ E_n + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots \right] \left[ \psi_n + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \dots \right]$$

$$\begin{array}{ccc} \cancel{\lambda^0 H^{(0)} \psi_n} & & \cancel{\lambda^0 E_n \psi_n} \\ + \lambda^1 (H^{(0)} \psi_n^{(1)} + H^{(1)} \psi_n) & = & + \lambda^1 (E_n \psi_n^{(1)} + E_n^{(1)} \psi_n) \\ + \lambda^2 (H^{(0)} \psi_n^{(2)} + H^{(1)} \psi_n^{(1)}) & & + \lambda^2 (E_n \psi_n^{(2)} + E_n^{(1)} \psi_n^{(1)} + E_n^{(2)} \psi_n) \\ + \dots & & + \dots \end{array} \quad (\lambda=1)$$

### 6.1 Non-Degenerate Theory

$$\underline{E_n^{(1)}, \psi_n^{(1)}} : \quad E_n \psi_n^{(1)} + E_n^{(1)} \psi_n = H^{(0)} \psi_n^{(1)} + H^{(1)} \psi_n$$

$$\begin{aligned} \langle \psi_m | (-H^{(1)} + E_n^{(1)}) | \psi_n \rangle &= \langle \psi_m | (H^{(0)} - E_n) | \psi_n^{(1)} \rangle = \sum c_i^{(1)} (E_i - E_n) \langle \psi_m | \psi_i \rangle \\ - \langle \psi_m | H^{(1)} | \psi_n \rangle + E_n^{(1)} \langle \psi_m | \psi_n \rangle &= c_m^{(1)} (E_m - E_n) \end{aligned}$$

$$\boxed{E_n^{(1)} = \langle \psi_n | H^{(1)} | \psi_n \rangle}$$

$$\boxed{\psi_n^{(1)} = \sum_{m \neq n} \frac{\langle \psi_m | H^{(1)} | \psi_n \rangle}{E_n - E_m} \psi_m + (0) \psi_n}$$

$$\begin{aligned} \underline{E_n^{(2)}, |n^{(2)}\rangle} : \quad & - \langle m^{(0)} | H^{(1)} | n^{(1)} \rangle + E_n^{(1)} \langle m^{(0)} | n^{(1)} \rangle \\ & + E_n^{(2)} \langle m^{(0)} | n^{(0)} \rangle \end{aligned} \quad \begin{aligned} &= \langle m^{(0)} | H^{(0)} - E_n^{(0)} | n^{(2)} \rangle \\ &= c_m^{(2)} (E_m - E_n) \end{aligned}$$

$$\boxed{E_n^{(2)} = \sum_{m \neq n} \frac{|\langle m | H^{(1)} | n \rangle|^2}{E_n - E_m}} = \langle n | H^{(1)} | n^{(1)} \rangle \quad ,$$

$$\boxed{|n^{(2)}\rangle = \sum_{m \neq n} \frac{\langle m | H^{(1)} - E_n^{(1)} | n^{(1)} \rangle}{E_n - E_m} \cdot |m\rangle}$$

$$\underline{E_n^{(i+1)}, |n^{(i+1)}\rangle} : \quad E_n^{(i+1)} = \langle n | H^{(1)} | n^{(i)} \rangle$$

$$|n^{(i+1)}\rangle = \sum_{m \neq n} \frac{\langle m | H^{(1)} | n^{(i)} \rangle - \sum_{j=0}^i E_n^{(j+1)} \langle m | n^{(i-j)} \rangle}{E_n - E_m} \cdot |m\rangle$$

## 6.2 Degenerate Perturbation Theory (see Matrix Operators)

$$\begin{aligned}
 \Psi &= \sum_i \left( c_i^{(\psi)} [\Psi] \right) \psi_i & \bullet \quad H^{(0)} \psi_i &= E_n \psi_i & \underline{(\psi_n \text{ are degenerate eigenfunctions of } H^{(0)})} \\
 &\equiv \sum_i c_i^{(\psi)} \psi_i & , \quad \forall \psi_i : & \bullet \quad \langle \psi_i | \psi_j \rangle &= \delta_{ij} \\
 &= c_0^{(\psi)} \psi_0 + c_1^{(\psi)} \psi_1 + \dots & \bullet \quad \langle \psi_i | \hat{Q} | \psi_j \rangle &\equiv Q_{ij}
 \end{aligned}$$

$$E_n \Psi^{(1)} + E^{(1)} \Psi = H^{(0)} \Psi^{(1)} + H^{(1)} \Psi \quad (\text{first order})$$

$$\begin{aligned}
 E_n \overline{\langle \psi_i | \Psi^{(1)} \rangle} + E^{(1)} \langle \psi_i | \Psi \rangle &= \overline{\langle H^{(0)} \psi_i | \Psi^{(1)} \rangle} + \langle \psi_i | H^{(1)} | \Psi \rangle \\
 &= \langle \psi_i | H^{(1)} | c_0 \psi_0 + c_1 \psi_1 + \dots \rangle \\
 c_i E^{(1)} &= c_0 \langle \psi_i | H^{(1)} | \psi_0 \rangle + c_1 \langle \psi_i | H^{(1)} | \psi_1 \rangle + \dots
 \end{aligned}$$

$$E^{(1)} \begin{pmatrix} c_0[\Psi] \\ c_1[\Psi] \\ \vdots \end{pmatrix}^{(\psi)} = \begin{pmatrix} H_{00}^{(1)} & H_{01}^{(1)} & \dots \\ H_{10}^{(1)} & H_{11}^{(1)} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}^{(\psi)} \begin{pmatrix} c_0[\Psi] \\ c_1[\Psi] \\ \vdots \end{pmatrix}^{(\psi)} \Rightarrow \boxed{\begin{array}{c} \text{(solve for } E^{(1)}, \vec{c}^{(\psi)}[\Psi]) \\ \left\| \begin{pmatrix} H_{aa}^{(1)} - E^{(1)} & H_{ab}^{(1)} & \dots \\ H_{ba}^{(1)} & H_{bb}^{(1)} - E^{(1)} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \right\| = 0 \end{array}}$$

In general,

$$E_i^{(1)} \vec{c}^{(\psi)}[\Psi_i] = \overline{H^{(1)}}^{(\psi)} \vec{c}^{(\psi)}[\Psi_i] \quad (i\text{th eigen-})$$

$$E_i^{(1)} \begin{pmatrix} \vec{c} | \\ \vec{c}[\Psi_i] | \\ | \end{pmatrix}^{(\psi)} = \begin{pmatrix} \vec{c} | & \vec{c} | & \dots \\ \vec{c}[\Psi_i] | & \vec{c}[\Psi_i] | & \dots \\ | & | & \ddots \end{pmatrix}^{(\psi)} \begin{pmatrix} E_0^{(1)} & 0 & \dots \\ 0 & E_1^{(1)} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} - & \vec{c}^*[\Psi_i] & - \\ - & \vec{c}^*[\Psi_i] & - \\ \vdots & \vdots & \ddots \end{pmatrix}^{(\psi)} \begin{pmatrix} \vec{c} | \\ \vec{c}[\Psi_i] | \\ | \end{pmatrix}^{(\psi)}$$

Instead of solving the characteristic polynomial, it would be wise to choose a basis  $\{\psi\}$  such that  $\vec{c}^{(\psi)}[\Psi_i] = (\dots 0 \ 0 \ 1_{(i)} \ 0 \ 0 \ \dots)^T \Leftrightarrow \Psi_i = \psi_i$ , making  $\overline{H^{(1)}}^{(\psi)}$  diagonal with eigenvalue entries. These are the energy eigenvalues,  $E_i^{(1)} = (H^{(1)})_{ii}^{(\psi)} = \langle \psi_i | H^{(1)} | \psi_i \rangle$ , which is just like first-order non-Perturbation energy. This also means  $|\psi_i\rangle$  are eigenfunctions of  $H^{(1)}$  (see Matrix Operators).

It is best to find a hermitian operator,  $\hat{A}$ , that commutes with  $H^{(0)}$  and  $H^{(1)}$ , whose eigenvalues within the degenerate basis are unique. The corresponding eigenfunctions will be a basis that makes  $H^{(1)}$  diagonal. This will also make them eigenfunctions of  $H^{(1)}$ .

1.  $A = A^\dagger$
2.  $[A, H^{(0)}] = 0 \rightarrow \left\{ \exists \{\Psi\} \mid (A\Psi_n = a_n\Psi_n), (H^{(0)}\Psi_n = E_n\Psi_n) \right\}$
3.  $\{\psi\} \subset \{\Psi\}$  s.t.  $\forall \psi_i : \begin{cases} (H^{(0)}\psi_i = E_n\psi_i), & \leftarrow \text{degenerate} \\ (A\psi_i = a_i\psi_i), & (\forall (i \neq j) \ a_i \neq a_j) \end{cases}$
4.  $[A, H^{(1)}] = 0 \Rightarrow \begin{aligned} 0 &= \langle A\psi_i | H^{(1)} | \psi_j \rangle - \langle \psi_i | H^{(1)} | A\psi_j \rangle \\ 0 &= (a_i - a_j) H_{ij}^{(1)} \\ 0 &= H_{ij}^{(1)} \quad \checkmark \end{aligned}$

## 6.3 Hydrogen Energy Corrections

### 6.3.1 Fine Structure - $\alpha^4 mc^2$

The Dirac Equation can derive the total fine structure correction with a  $\alpha^4$  order approx.

#### 1. Relativistic, $\hat{p}^4$

$$\begin{aligned}
 T &= \sqrt{p^2 c^2 + m^2 c^4} - mc^2 = mc^2 \sqrt{1 + \frac{p^2}{m^2 c^2}} - mc^2 \\
 &= mc^2 \left[ \frac{(\frac{1}{2})}{1!} \left( \frac{p^2}{m^2 c^2} \right) + \frac{(\frac{1}{2})(1 - \frac{1}{2})}{2!} \left( \frac{p^2}{m^2 c^2} \right)^2 + \dots \right] \\
 &= \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} + \dots \\
 &\downarrow \\
 H_r^{(1)} &= -\frac{p^4}{8m^3 c^2} \quad \text{(For some reason } \hat{p}^4 \text{ needs to be hermitian to use perturbation theory. It only isn't when } l = 0, \text{ while } \hat{p}^2 \text{ always is hermitian. See Prob. 6.15)}
 \end{aligned}$$

$L^2$  and  $L_z$  should commute with  $p^4$  because the perturbation is spherically symmetric, meaning  $l$  and  $m_l$  should be conserved (see Operator Evolution). Their eigenvalues are also distinct (taking the eigenfunctions of  $nlm_l$  together) within each set of  $n^2$  degeneracies, so their eigenvectors and eigenvalues can be used.  $n, l$  and  $m_l$  the “good” numbers.

$$\left. \begin{aligned}
 \langle r^{-1} \rangle &= \frac{1}{n^2 a_0} \\
 \langle r^{-2} \rangle &= \frac{1}{(l+1/2)n^3 a_0^2}
 \end{aligned} \right| \begin{aligned}
 \langle \psi_{nlm_l} | H_r^{(1)} | \psi_{nlm_l} \rangle &= \frac{-1}{8m^3 c^2} \langle \psi_{nlm_l} | p^4 | \psi_{nlm_l} \rangle \\
 &= \frac{-1}{8m^3 c^2} \langle p^2 \psi_{nlm_l} | p^2 | \psi_{nlm_l} \rangle \\
 &= \frac{-1}{8m^3 c^2} \langle [2m(E_n - V)]^2 \rangle \\
 &= \frac{-4m^2}{8m^3 c^2} \langle E_n^2 - 2E_n V + V^2 \rangle \\
 &= -\frac{E_n^2}{2mc^2} \left[ \frac{4n}{l+1/2} - 3 \right]
 \end{aligned}$$

#### 2. Spin-Orbit Coupling, $\mathbf{S}_e \cdot \mathbf{L}_e$

In the electron's frame of reference, the proton is spinning around it, creating a  $B$ -field affecting its magnetic dipole moment. The non-inertial reference frame requires multiplying by the Thomas precession correction, which in this case is  $C_T = g_e - 1 = 1/2$ . In the lab frame, the moving electron's magnetic dipole moment creates an electric dipole moment, which is affected by the proton charge. The latter is much harder to calculate.

$$\begin{aligned}
H_{so}^{(1)} &= -C_T \mu_e \cdot B(L_e) \quad (\text{See Electron in Magnetic Field}) \\
&= \frac{1}{2} \frac{qS}{m} \cdot \frac{k_\mu}{r^3} \int I d\vec{l} \times \vec{r} \quad \left( \sim \frac{1}{2} \frac{qS}{m} \cdot \frac{k_\mu}{r^3} \int \frac{mqd\vec{v} \times \vec{r}}{m} \right) \\
&= \frac{1}{2} \frac{qS}{m} \cdot \frac{k_e}{c^2} \frac{2\pi}{r} I = \frac{1}{2} \frac{qS}{m} \cdot \frac{k_e}{c^2} \frac{2\pi}{r} \frac{q(L/mr)}{2\pi r} \\
&= \frac{kqq}{2m} \frac{1}{mc^2} \frac{S \cdot L}{r^3} = \frac{e^2}{8\pi\epsilon_0 m^2 c^2} \frac{S \cdot L}{r^3}
\end{aligned}$$

$S \cdot L$  does not commute with  $L$  or  $S$  (meaning  $m_l$  and  $m_s$  are bad), but  $[S \cdot L, S^2] = [S \cdot L, L^2] = 0$ . The sum of the two,  $J \equiv L + S$ , and  $J^2$  also commute with the perturbation. They are all conserved, and their unique eigenvalues per set of degeneracies -  $l, s=1/2, j, m_j$  - are the “good” numbers (along with  $n$ ).

$$\begin{array}{l}
S \cdot L = \frac{1}{2} (J^2 - L^2 - S^2) \\
\langle r^{-3} \rangle = \frac{1}{l(l+1/2)(l+1)n^3 a_0^3} \\
\text{(note: divergent at } l=0)
\end{array}
\left| \begin{array}{l}
\langle n l j m_j | H_{so}^{(1)} | n l j m_j \rangle = \frac{kqq}{2m} \frac{1}{mc^2} \frac{\hbar^2 [j(j+1) - l(l+1) - s(s+1)]}{2l(l+1/2)(l+1)n^3 a_0^3} \\
= \frac{kqq}{4mn^4} \frac{\hbar^2 \alpha^3 m^3 c^3}{\hbar^3 m c^2} \frac{n [j(j+1) - l(l+1) - s(s+1)]}{l(l+1/2)(l+1)} \\
= \frac{kqq}{4\hbar c n^4} \frac{\alpha^3 m^2 c^4}{mc^2} \frac{n [j(j+1) - l(l+1) - s(s+1)]}{l(l+1/2)(l+1)} \\
= \frac{E_n^2}{mc^2} \left\{ \frac{n [j(j+1) - l(l+1) - 3/4]}{l(l+1/2)(l+1)} \right\}
\end{array} \right.$$

3. Darwin Term (correction for  $H_{so}^{(1)}$  when  $l=0$ ) skipped

#### 4. Total Correction

$$\begin{aligned}
E_{fs}^{(1)} &= E_r^{(1)} + E_{so}^{(1)} \\
&= -\frac{E_n^2}{2mc^2} \left[ \frac{4n}{l + \frac{1}{2}} - 3 \right] + \frac{E_n^2}{mc^2} \left\{ \frac{n [j(j+1) - l(l+1) - 3/4]}{l(l+1/2)(l+1)} \right\} \\
&= \frac{E_n^2}{2mc^2} \left( 3 - \frac{4n}{j + 1/2} \right) \quad (j=l\pm 1/2) \\
&\downarrow \\
E_{nj} &= E_n + E_{fs}^{(1)} \\
&= E_n \left[ 1 - \frac{E_n}{2mc^2} \left( \frac{4n}{j + 1/2} - 3 \right) \right] \\
&= -\frac{\alpha^2 m c^2}{2n^2} \left[ 1 + \frac{\alpha^2}{n^2} \left( \frac{n}{j + 1/2} - 3/4 \right) \right]
\end{aligned}$$

Fine structure splits the  $l$  energy degeneracies. However, since  $j = l \pm 1/2$ , there are still two  $j$  degeneracies if  $n > 2$ . Overall, the good numbers to use for stationary state solutions to the hydrogen atom w/ fine structure correction are  $n, l, s=1/2, j, m_j$ . Note,  $J^2, L^2$ , and  $S^2$  always commute(?)

### 6.3.2 Zeeman Effect (Ext. $B$ -Field)

$$\begin{aligned}
H_B^{(1)} &= -(\mu_s + \mu_l) \cdot B_{\text{ext}} && \text{(see Electron in Magnetic Field)} \\
&= -\left(\frac{g_e q}{2m} S + \frac{q}{2m} L\right) \cdot B_{\text{ext}} \\
&= \frac{e}{2m} (2S + L) \cdot B_{\text{ext}}
\end{aligned}$$

Weak Zeeman ( $B_{\text{ext}} \ll B_{\text{int}}$ )

$$\begin{aligned}
H_{WZ}^{(1)} &= \frac{e}{2m} B_{\text{ext}} \cdot (2S + L) \\
&= \frac{e}{2m} B_{\text{ext}} \cdot (J + S)
\end{aligned}$$

Fine structure perturbation dominate the Zeeman perturbation, so the fine structure numbers are the good ones:  $n, l, s=1/2, j$ , and  $m_j$ .  $m_l$  and  $m_s$  can't be used for  $\langle L \rangle$  or  $\langle S \rangle$ , so instead use the fact that the “vector”  $J = L + S$  is conserved, so a **time-averaged**  $S$ -component to the  $J$  “vector” can be defined as  $S_{\text{ave}} = \frac{S \cdot J}{J^2} J$ , where  $S \cdot J = \frac{1}{2} (J^2 + S^2 - L^2)$ .

$$\begin{aligned}
E_{WZ}^{(1)} &= \frac{e}{2m} B_{\text{ext}} \cdot \langle nljm_j | J + S_{\text{ave}} | nljm_j \rangle \\
&= \frac{e}{2m} B_{\text{ext}} \cdot \left\langle J \left( 1 + \frac{S \cdot J}{J^2} \right) \right\rangle \\
&= \frac{e}{2m} B_{\text{ext}} \cdot \langle J \rangle \left( 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \right) \\
&= \frac{e\hbar}{2m} B_{\text{ext}} m_j \left( 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \right) && \text{(let } B_{\text{ext}} \text{ be parallel to the z-axis)} \\
&= \mu_B B_{\text{ext}} m_j g_j && \begin{array}{l} \mu_B = \text{Bohr magneton} = 5.788 \times 10^{-5} \text{ eV/T} \\ g_j = \text{Lande g-factor} \end{array}
\end{aligned}$$

Strong Zeeman ( $B_{\text{ext}} \gg B_{\text{int}}$ )

For a strong magnetic field parallel to the z-axis,  $m_l$  and  $m_s$  are stuck in the same place, making them and  $l$  conserved. The external torque, however, means that the total angular momentums,  $j$  and  $m_j$  are not. Though unneeded, obviously  $s=1/2$ .

$$\begin{aligned}
E_{SZ}^{(1)} &= \frac{e}{2m} B_{\text{ext}} \langle 2S_z + L_z \rangle \\
&= \mu_B B_{\text{ext}} (2m_s + m_l)
\end{aligned}$$

The spin-orbit correction must be changed with respect to the new good numbers,  $m_l$  and  $m_s$ . The relativistic correction uses the same numbers, so it stays the same.

$$\begin{aligned}
E_{\text{so}}^{(1)} &= \frac{e^2}{8\pi\epsilon_0 m^2 c^2} \left\langle \frac{S_x L_x + S_y L_y + S_z L_z}{r^3} \right\rangle & E_{\text{fs}}^{(1)} &= E_{\text{so}}^{(1)} + E_{\text{r}}^{(1)} \\
&= \frac{e^2}{8\pi\epsilon_0 m^2 c^2} \frac{0 + 0 + \hbar^2 m_s m_l}{l(l+1/2)(l+1)n^3 a_0^3} & \rightarrow &= \frac{E_n^2}{2mc^2} \frac{4nm_s m_l}{l(l+1/2)(l+1)} + \frac{E_n^2}{2mc^2} \left[ 3 - \frac{4n}{l+1/2} \right] \\
&= \frac{kqq}{2m^2 c^2} \frac{\hbar^2}{(\hbar/\alpha mc)^3 n^3} \frac{m_s m_l}{l(l+1/2)(l+1)} & &= \frac{4nE_n^2}{2mc^2} \left[ \frac{m_s m_l}{l(l+1/2)(l+1)} + \frac{3}{4n} - \frac{1}{l+1/2} \right] \\
&= \frac{kqq}{2\hbar c} \frac{\alpha^3 m^2 c^4}{4mc^2 n^4} \frac{4nm_s m_l}{l(l+1/2)(l+1)} & \downarrow & \\
&= \frac{E_n^2}{2mc^2} \frac{4nm_s m_l}{l(l+1/2)(l+1)} & E_{nlm_l m_s} &= E_n + E_{\text{SZ}}^{(1)} + E_{\text{fs}}^{(1)}
\end{aligned}$$

### Intermediate Zeeman ( $B_{\text{ext}} \sim B_{\text{int}}$ )

There are no good numbers here (see Degenerate Perturbation Theory). The basis is chosen to be  $|j m_j\rangle = \sum_i C_i |l m_l\rangle \otimes |s m_s\rangle$  (see 2 Objects w/ Any Spin), as it makes  $\overline{H^{(1)}}^{(e)}$  easier (instead of using  $l, m_l, m_s$ ).

$$\begin{aligned}
1.) \psi_i &= |j m_j\rangle_i & 2.) \left( \langle l m_l | \langle s m_s | \right)_x \left( |l m_l\rangle |s m_s\rangle \right)_y &= \delta_{xy} \\
3.) Q_{rc}^{(\psi)} &= \langle \psi_r | \hat{Q} | \psi_c \rangle & 4.) \psi_i \text{ s.t. } &\begin{cases} 0 \leq l < n \\ j_{(l\pm)} = l \pm 1/2, \\ 2l^2 < i \leq 2(l+1)^2 \end{cases}
\end{aligned}$$

$$\begin{aligned}
\langle j m_j | H_{fs}^{(1)} | j m_j \rangle &= \frac{E_n^2}{2mc^2} \left( 3 - \frac{4n}{j+1/2} \right) \\
&\equiv \gamma_n \left( 3 - \frac{4n}{j+1/2} \right) & \overline{H^{(1)}}^{(jm_j)} &= \overline{H_{fs}^{(1)}}^{(jm_j)} + \overline{H_{IZ}^{(1)}}^{(jm_j)} \\
\langle j m_j | H_{IZ}^{(1)} | j m_j \rangle &= \langle j m_j | H_{IZ}^{(1)} \left( C_i |l m_l\rangle \otimes |s m_s\rangle \right) & \text{See Griffith Prob. 6.25 for example with } n=2 \\
&= \mu_B B_{\text{ext}} (2m_s + m_l) C_i^2 \\
&\equiv \beta (2m_s + m_l) C_i^2
\end{aligned}$$

### 6.3.3 Stark Effect (Small Ext. $E$ -Field)

- $H^{(1)} = -p \cdot E = eE \cdot r$  (small  $r$ )
- $n = 1 \rightarrow H^{(1)} = 0$
- $n = 2 \rightarrow \begin{cases} H^{(1)} = 0 & m = \pm 1 \\ H^{(1)} = ke|E|a_0 & m = 0 \end{cases}$  ( $k$  is some constant)

### 6.3.4 Lamb Shift (quantitized $E$ -field) - $\alpha^5 mc^2$ (skipped)

### 6.3.5 Hyperfine (Spin-Spin), $S_p \cdot S_e$ - $m/m_p \alpha^4 mc^2$

(Coupling between the electron magnetic moment and the magnetic field from the proton magnetic moment)

$$\left. \begin{aligned} \mu_e &= -\frac{g_e e}{2m_e} S_e = -\frac{e}{m_e} S_e, & \mu_p &= \frac{g_p e}{2m_p} S_p \\ B(\mu_p) &= \frac{\mu_0}{4\pi r^3} [3(\vec{\mu}_p \cdot \hat{r})\hat{r} - \vec{\mu}_p] + \frac{2\mu_0}{3} \vec{\mu}_p \delta^3(r) \end{aligned} \right| \begin{aligned} H_{hf}^{(1)} &= -\mu_e \cdot B(\mu_p) \\ &= \dots \\ &\downarrow \\ E_{hf}^{(1)} &= \left(\frac{e}{m_e}\right) \left(\frac{2\mu_0}{3} \frac{g_p e}{2m_p}\right) \langle S_e \cdot S_p \rangle |\psi_{nlm}(0)|^2 \end{aligned}$$

In the ground state,  $|\psi_{100}(0)|^2 = 1/(\pi a_0^3)$ .  $S_e^2, S_p^2$ , and the sum  $S = S_e + S_p$  commute with  $S_e \cdot S_p$ , so  $s_e, s_p, m_s, s^2$  are the good numbers.  $S_e$  and  $S_p$  do not, so  $m_{se}$  and  $m_{sp}$  are not good numbers.

$$\begin{aligned} E_{hf}^{(1)} &= \left(\frac{e}{m_e}\right) \left(\frac{2}{3\epsilon_0 c^2} \frac{g_p e}{2m_p}\right) \frac{1}{2\pi a_0^3} \langle S^2 - S_e^2 - S_p^2 \rangle \\ &= \frac{g_p e^2}{4\pi\epsilon_0 c^2 m_p m_e} \frac{4\alpha^3 m_e^3 c^3 \hbar^2}{3\hbar^3} \left[ \frac{s(s+1)}{2} - 3/4 \right] \\ &= \frac{4}{3} g_p \frac{m_e}{m_p} \alpha^4 m_e c^2 \left[ \frac{s(s+1)}{2} - 3/4 \right] \\ &= \frac{4}{3} g_p \frac{m_e}{m_p} \alpha^4 m_e c^2 \cdot \begin{cases} \frac{1}{4} & s = 1 \text{ (triplet)} \\ \frac{-3}{4} & s = 0 \text{ (singlet)} \end{cases} \rightarrow \begin{aligned} \Delta E &= 5.88 \times 10^{-6} \text{ eV} \\ \lambda &= 21 \text{ cm}, \quad \nu = 1420 \text{ MHz} \end{aligned} \end{aligned}$$



## 6.4 Transition Amplitude (See Pictures)

$  \begin{aligned}  H(t) &= H^0 + H^1(t) \\  &\downarrow \\  U(t) i^0\rangle &= \sum_n  n^0\rangle e^{-\frac{i}{\hbar} E_n^0 t} \langle n^0   U_I   i^0 \rangle \\   \Psi(t)\rangle &= \boxed{\sum_n  n^0\rangle e^{-\frac{i}{\hbar} E_n^0 t} d_n(t)} \\  &\downarrow \\  0 &= \langle f^0   i\hbar \frac{\partial}{\partial t} - H^0 - H^1(t)   \Psi(t) \rangle \\  &= \sum_n \langle f^0   \left[ i\hbar \dot{d}_n - H^1(t) d_n \right]   n^0 \rangle e^{-\frac{i}{\hbar} E_n^0 t} \\  &\downarrow \\  \dot{d}_f(t) &= \sum_n \frac{1}{i\hbar} \langle f^0   H^1(t)   n^0 \rangle e^{\frac{i}{\hbar} (E_f^0 - E_n^0) t} d_n(t) \\  &= \sum_n \frac{1}{i\hbar} \langle f^0   H^1(t)   n^0 \rangle e^{i\omega_{fn} t} d_n(t)  \end{aligned}  $	$  \begin{aligned}  d_n(t) &= d_n(0) + \int_0^t \dot{d}_n dt' : \\  \bullet \quad d_n(0) &= \delta_{ni} \quad (\text{if }  d_{n \neq i}(t)  \ll 1) \quad (0^{\text{th}} \text{ order}) \\  \bullet \quad \dot{d}_f(t) &\approx \frac{1}{i\hbar} \langle f^0   H^1(t)   i^0 \rangle e^{i\omega_{fi} t} \\  \bullet \quad \boxed{d_n(t) \approx \delta_{ni} + \frac{1}{i\hbar} \int_0^t \langle n^0   H^1(t')   i^0 \rangle e^{i\omega_{ni} t'} dt'} &\quad (1^{\text{st}} \text{ order}) \\  \bullet \quad \dot{d}_f(t) &\approx \frac{1}{i\hbar} \overline{H_{fi}^1}(t) e^{i\omega_{fi} t} \\  &\quad + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \sum_n \overline{H_{fn}^1}(t) e^{i\omega_{fn} t} \overline{H_{ni}^1}(t') e^{i\omega_{ni} t'} dt' \\  \bullet \quad \dots  \end{aligned}  $
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Interactive Picture Method:

$$U_I(t, t_0) = \mathbb{I} + \frac{1}{i\hbar} \int_{t_0}^t H_I^1(t') dt' + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} H_I^1(t') H_I^1(t'') dt'' dt' + \dots$$

$$\begin{aligned}
 \bullet \quad \langle f^0 | U_I(t, t_0) | i^0 \rangle &= \langle f^0 | e^{\frac{i}{\hbar} E_f^0 (t-t_0)} U(t, t_0) | i^0 \rangle \\
 &\equiv d_f(t) = \boxed{\delta_{fi} + \frac{1}{i\hbar} \int_{t_0}^t \langle f^0 | H^1(t') | i^0 \rangle e^{i\omega_{fi}(t'-t_0)} dt'} \quad (1^{\text{st}} \text{ order}) \\
 &\quad + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} \sum_n \langle f^0 | H^1(t') | n^0 \rangle e^{i\omega_{fn}(t'-t_0)} \langle n^0 | H^1(t'') | i^0 \rangle e^{i\omega_{ni}(t''-t_0)} dt'' dt' + \dots
 \end{aligned}$$

Normal Schrodinger Propagator:

$$\begin{aligned}
 U_S(t, t_0) &= U^0(t, t_0) + \frac{1}{i\hbar} \int_{t_0}^t U^0(t, t_0) U^{0\dagger}(t', t_0) H^1(t') U^0(t', t_0) dt' \\
 &\quad + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} U^0(t, t_0) U^{0\dagger}(t', t_0) H^1(t') U^0(t', t_0) U^{0\dagger}(t'', t_0) H^1(t'') U^0(t'', t_0) dt'' dt' + \dots \\
 \bullet \quad \langle f^0 | U(t, t_0) | i^0 \rangle &= \boxed{\delta_{fi} e^{-\frac{i}{\hbar} E_f^0 (t-t_0)} + \frac{1}{i\hbar} \int_{t_0}^t e^{-\frac{i}{\hbar} E_f^0 (t-t')} \langle f^0 | H^1(t') | i^0 \rangle e^{-\frac{i}{\hbar} E_i^0 (t'-t_0)} dt'} \quad (1^{\text{st}} \text{ order}) \\
 &\quad + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} \sum_n e^{-\frac{i}{\hbar} E_f^0 (t-t')} \langle f^0 | H^1(t') | n^0 \rangle e^{-\frac{i}{\hbar} E_n^0 (t'-t'')} \langle n^0 | H^1(t'') | i^0 \rangle e^{-\frac{i}{\hbar} E_i^0 (t''-t_0)} dt'' dt' + \dots
 \end{aligned}$$

## 6.5 Variation Principle - Approx. Ground State Energy

$$\begin{aligned} \psi = \sum c_n \psi_n \rightarrow E(\psi) > E_0 = E(\psi_0) & \Rightarrow b_{\min} : \frac{d}{db} \langle H \rangle = 0 \\ \psi \equiv f(b, x), \quad \langle H \rangle = \langle T \rangle + \langle V \rangle & E_0 \approx \left\langle f(b_{\min}, x) \left| H \right| f(b_{\min}, x) \right\rangle \end{aligned}$$

## 6.6 Selection Rules - Orbital Transitions

Electric Dipole Approximation ONLY:  $\lambda_\gamma \gg$  atom length  $\rightarrow E, B$  feels homogenously oscillating to the atom

$$\psi_{nlm} \rightarrow \psi_{n'l'm'}:$$

- $\Delta m \in \{-1, \overset{?}{0}, 1\}$

$$s(\gamma) = 1 \rightarrow m_s(\gamma) \in \{-\hbar, \overset{?}{0}, \hbar\}$$

$$E = E\hat{z} \rightarrow \Delta m = 0$$

- $\Delta l = \pm 1$

$$1s \leftrightarrow 2p$$

Exception:  $(2s \rightarrow 1s)$  through two-photon emission

- $\Delta j \in \{-1, 0, 1\}$

Exception:  $(j = 0 \rightarrow j = 0)$  not allowed

## 7 Blackbody Radiation

- Power Spectrum :  $I'(\omega) = \frac{\hbar^3 \omega^3}{h^2 c^2} \frac{1}{e^{\hbar\omega/k_b T} - 1} \left[ \frac{I}{\Omega \cdot f} \right] \quad (\mu = 0 \text{ for photons since photon number isn't conserved})$

- Stefan-Boltzmann Law :  $I = \frac{dP}{dA} \propto T^4 \quad \text{!! important !!}$

- Wien's Displacement Law :  $\lambda_{\max} = \frac{2.9 \times 10^{-3}}{T} [\text{m}] \quad (\text{mode of spectrum})$

## 8 Adiabatic Theorem - Slow Changing of Potential

$$\begin{array}{ll}
 t = 0 \rightarrow & H_{(t=0)} = H^{(0)} \\
 & H_{(0)}|n\rangle = E_n|n\rangle \\
 t = t \rightarrow & H = H^{(0)}(t) \\
 & H(t)|n(t)\rangle = E_n(t)|n(t)\rangle
 \end{array}$$

Dynamic Phase :  $\theta_n(t) = -\frac{1}{\hbar} \int_0^t E_n(t') dt'$

$$\begin{aligned}
 |\Psi_m(t)\rangle &\equiv \sum_n |n(t)\rangle e^{i\theta_n(t)} \langle n(t)|m(0)\rangle \\
 &\approx |m(t)\rangle e^{i\theta_m(t)} e^{i\gamma_m(t)} \\
 &= |m(t)\rangle e^{i\theta_m(t)} e^{\frac{i}{\hbar} \int A^m \cdot dR}
 \end{aligned}$$

$$\begin{aligned}
 \sum_n \cancel{H|n\rangle} e^{i\theta_n} c_n &= i\hbar \sum_n |\dot{n}\rangle e^{i\theta_n} c_n + \cancel{|n\rangle i\dot{\theta}_n e^{i\theta_n} c_n} + |n\rangle e^{i\theta_n} \dot{c}_n \\
 \langle m|\dot{H}|n\rangle + \langle m|H|\dot{n}\rangle &= \cancel{\langle m|\dot{E}_n|n\rangle} + \langle m|E_n|\dot{n}\rangle \\
 &\Downarrow
 \end{aligned}$$

$$\dot{c}_m = \frac{d}{dt} \langle m(t)|m(0)\rangle = -\langle m|\dot{m}\rangle c_m - \sum_{n \neq m} \frac{\langle m|\dot{H}|n\rangle}{E_n - E_m} e^{i(\theta_n - \theta_m)} \langle n|\Psi\rangle$$

$$\begin{aligned}
 &\text{(not trivial)} \\
 &\approx -\langle m(t)|\dot{m}(t)\rangle c_m \Rightarrow c_m(t) \approx c_m(0) e^{\frac{i}{\hbar} \int \langle m|\dot{m}\rangle dt'}
 \end{aligned}$$

$$c_n(t) \approx \delta_{nm} e^{i\gamma_m(t)}$$

Berry Phase :  $\gamma_m(t) = i \int_0^t \langle m(t')|\dot{m}(t')\rangle dt' \in \mathbb{R}$

### Berry/Geometric Phase

$$\gamma_m(t) = i \int_0^t \langle m(t')|\dot{m}(t')\rangle dt' = \frac{1}{\hbar} \int_{R_i}^{R_f} i\hbar \langle m|\nabla_R m\rangle \cdot dR$$

$$\begin{aligned}
 \Rightarrow \frac{1}{\hbar} \oint i\hbar \langle m|\nabla_R m\rangle \cdot dR &= \frac{1}{\hbar} \iint \nabla_R \times i\hbar \langle m|\nabla_R m\rangle \cdot da \\
 \sim \boxed{\frac{1}{\hbar} \oint A^m \cdot dR} &= \frac{1}{\hbar} \iint \nabla_R \times A^m \cdot da = \frac{1}{\hbar} \Phi_B^m
 \end{aligned}$$

### Aharanov-Bohm Effect:

$$\begin{aligned}
 i\hbar \frac{\partial \Psi}{\partial t} &= \left[ \frac{(p - qA)^2}{2m} + V + \cancel{\phi} \right] \Psi \\
 \Rightarrow \Psi &= e^{\frac{i}{\hbar} \int_{\mathcal{O}}^r qA \cdot dr'} \psi, \quad \check{E}\psi = \check{H}\psi \\
 &= \boxed{e^{ig}\psi}
 \end{aligned}$$

$$\begin{aligned}
 \vec{A} &= \frac{\Phi_B}{2\pi r} \hat{\phi} \Rightarrow \Delta g_{\text{closed}} = \frac{q}{\hbar} \oint \frac{\Phi_B}{2\pi r} \hat{\phi} \cdot r d\hat{\phi} \\
 &= \frac{q\Phi_B}{\hbar} = \gamma_m
 \end{aligned}$$

### Infinitesimal Path Integral

$R$  : Slow degree of freedom (nucleus)

$r$  : Fast degree of freedom (electron)

$$\begin{aligned}
 \mathbb{I} &= \int dR \sum_n |R, n(R)\rangle \langle R, n(R)| \\
 &\approx \int dR |R, n(R)\rangle \langle R, n(R)|
 \end{aligned}$$

$$\langle \chi(\epsilon) | e^{-\frac{i}{\hbar} H \epsilon} | \chi(0) \rangle = \langle R(\epsilon) | e^{-\frac{i}{\hbar} H_f \epsilon} | R(0) \rangle \langle n(R(\epsilon)) | e^{-\frac{i}{\hbar} H_s \epsilon} | n(R(0)) \rangle$$

$$\check{U}(R_1, \epsilon; R_0, 0) = \sqrt{\frac{-im}{2\pi\hbar\epsilon}} e^{\frac{i}{\hbar} \mathcal{L}_s \epsilon} e^{-\frac{i}{\hbar} E_n(R_0) \epsilon} \langle n(R_1) | n(R_0) \rangle$$

$$\Psi(R_1, \epsilon) = \langle \chi(\epsilon) | \hat{U}(\epsilon) | \Psi(R_0, 0) \rangle \approx \sqrt{\frac{-im}{2\pi\hbar\epsilon}} \int e^{\frac{i}{\hbar} \mathcal{L}_s \epsilon} e^{-\frac{i}{\hbar} E_n(R_1 + \eta) \epsilon} \langle n(R_1) | n(R_1 + \eta) \rangle \Psi(R_1 + \eta, 0) d\eta \quad (\eta = R_0 - R_1)$$

$$\approx \sqrt{\frac{-im}{2\pi\hbar\epsilon}} \int e^{\frac{i}{\hbar} \frac{m}{2} \frac{\eta^2}{\epsilon}} \left[ 1 - \frac{i}{\hbar} \epsilon (V_s + E_n) \right] \langle n(R_1) | \left[ |n(R_1)\rangle + \eta |\partial n(R_1)\rangle + \frac{\eta^2}{2} |\partial^2 n(R_1)\rangle \right] \left[ 1 + \eta \frac{d}{dR} + \frac{\eta^2}{2} \frac{d^2}{dR^2} \right] \Psi(R_1, 0) d\eta$$

$$\approx \sqrt{\frac{-im}{2\pi\hbar\epsilon}} \int e^{\frac{i}{\hbar} \frac{m}{2} \frac{\eta^2}{\epsilon}} \left[ 1 - \frac{i}{\hbar} \epsilon V(R_1, 0) + \cancel{\eta \dots} + \frac{\eta^2}{2} \frac{d^2}{dR^2} + \eta^2 \langle n | \partial n \rangle \frac{d}{dR} + \frac{\eta^2}{2} \langle n | \partial^2 n \rangle \right] \Psi(R_1, 0) d\eta$$

$$\begin{array}{l}
 \check{E}|\Psi\rangle = \hat{H}|\Psi\rangle : \hat{H} = \frac{P_s^2}{2m} + V_s + \hat{H}_f \\
 \check{E}\Psi = \check{H}\Psi : \check{H} = \frac{(P_s - A^n)^2}{2m} + V + \Phi^n
 \end{array}
 \left| \begin{array}{l}
 A^n = i\hbar \langle n | \partial n \rangle \\
 \Phi^n = \frac{\hbar^2}{2m} [\langle \partial n | \partial n \rangle - \langle \partial n | n \rangle \langle n | \partial n \rangle]
 \end{array} \right.
 \left( \begin{array}{l}
 \langle n | \partial n \rangle + \langle \partial n | n \rangle = 0 \\
 A^n \text{ is added/subtracted in}
 \end{array} \right)$$

## 9 Integral Form

$$\begin{aligned}
\psi(r) &= \psi_0(r) + \int g(r-r_0)V(r_0)\psi(r_0) d^3r & g(r) &= -\frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \\
&= \psi_0 + \int gV\psi(r_0) \\
&= \psi_0 + \int gV\psi_0 + \int \int gVgV\psi(r_0) \\
&= \psi_0 + \int gV\psi_0 + \int \int gVgV\psi_0 + \int \int gVgVgV\psi_0 + \dots
\end{aligned}$$

## 10 Klein-Gordon Equation (Spinless Free Particle)

$$\begin{aligned}
(p^2c^2 + m^2c^4)\psi &= E^2\psi \\
(-E^2 + p^2c^2 + m^2c^4)\psi &= 0 \\
[-(E/c)^2 + p^2 + (mc)^2]\psi &= 0 \\
\frac{[-(E/c)^2 + p^2 + (mc)^2]}{\hbar^2}\psi &= 0 \\
\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\psi &= 0 \\
\boxed{(-\square^2 + \mu^2)\psi} &= 0
\end{aligned}$$

$$\begin{aligned}
\mathcal{L} = \mathcal{T} - \mathcal{U} &= \frac{1}{2c^2} \left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2} \left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}\kappa^2\phi^2 \\
&= \boxed{-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\kappa^2\phi^2}
\end{aligned}$$

$$d\mathcal{L} = \frac{1}{c^2} \frac{\partial\phi}{\partial t} \frac{\partial\epsilon}{\partial t} - \frac{\partial\phi}{\partial x} \frac{\partial\epsilon}{\partial x} - \kappa^2\phi\epsilon \quad (\epsilon = d\phi)$$

$$S[\phi] = \int dt \int dx \mathcal{L}(\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t})$$

$$\begin{aligned}
dS[\phi] &= 0 = \int dt \int dx d\mathcal{L} \\
&= \int dt \int dx \left[ \frac{1}{c^2} \frac{\partial\phi}{\partial t} \frac{\partial\epsilon}{\partial t} - \frac{\partial\phi}{\partial x} \frac{\partial\epsilon}{\partial x} - \kappa^2\phi\epsilon \right] \\
&= \int dt \int dx \left[ \underbrace{-\frac{1}{c^2} \frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial x^2} - \kappa^2\phi}_{\square^2\phi} \right] \epsilon + 0 + \dots
\end{aligned}$$

$$\boxed{\square^2\phi = \kappa^2\phi}$$

## 11 Dirac Equation

$$\mu^2 = \square^2$$

$$E^2 = p^2 c^2 + m^2 c^4 = H^2$$

$$\begin{aligned} m &= \sqrt{\nabla^2 - \partial_t^2} \\ &= A\partial_x + B\partial_y + C\partial_z + iD\partial_t \\ &= i\gamma^\mu \partial_\mu \end{aligned}$$

$$\begin{aligned} \sqrt{p^2 + m^2} &= \alpha \cdot p + \beta m \\ &= \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m \end{aligned}$$

$$\begin{aligned} \partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{\partial^2}{\partial t^2} &= (A\partial_x + B\partial_y + C\partial_z + iD\partial_t)^2 \\ &= A^2\partial_x^2 + B^2\partial_y^2 + C^2\partial_z^2 - D^2\partial_t^2 \\ &\quad + [AB + BA]\partial_x\partial_y + [AC + CA]\partial_x\partial_z + [BC + CB]\partial_y\partial_z \\ &\quad + [AD + DA]i\partial_x\partial_t + [BD + DB]i\partial_y\partial_t + [CD + DC]i\partial_z\partial_t \end{aligned}$$

$$D = \gamma^0, \quad A = i\gamma^1 = i\beta\alpha_1, \quad B = i\gamma^2 = i\beta\alpha_2, \quad C = i\gamma^3 = i\beta\alpha_3$$

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\gamma^\mu = \left[ \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} \right] \quad \gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

$$\begin{aligned} (i\hbar\gamma^\mu\partial_\mu - mc)\psi &= 0 \\ (i\not{\partial} - m)\psi &= 0 \quad (\text{natural units}) \end{aligned}$$

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi &= (c\alpha \cdot p + \beta mc^2)\psi \\ i\hbar\frac{\partial}{\partial t}\psi &= (c\alpha \cdot (p - qA) + \beta mc^2 + q\phi)\psi \end{aligned}$$

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi &= (c\alpha \cdot (p - qA) + \beta mc^2 + q\phi)\psi \\ &= (c\alpha \cdot \pi + \beta mc^2 + q\phi)\psi \end{aligned}$$

$$\begin{aligned} \psi(t) &= \psi(p)e^{i(p \cdot r - Et)} \\ \phi &= 0 \end{aligned} \Rightarrow E\psi = (\alpha \cdot \pi + \beta m)\psi$$

$$\begin{bmatrix} E - m & -\sigma \cdot \pi \\ -\sigma \cdot \pi & E + m \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = 0 \Leftrightarrow \begin{aligned} (E - m)\psi_+ &= (\sigma \cdot \pi)\psi_- \\ (E + m)\psi_- &= (\sigma \cdot \pi)\psi_+ \end{aligned}$$

$$(E - m)\psi_\pm = \frac{(\sigma \cdot \pi)(\sigma \cdot \pi)}{E + m}\psi_\pm$$

$$E_s\psi_\pm \approx \frac{(\sigma \cdot \pi)^2}{2m}\psi_\pm \quad \begin{array}{l} \text{(Pauli's Eq.)} \\ (\sim \text{to Schrodinger}) \end{array}$$

$$\begin{aligned} \frac{\sigma \cdot A \sigma \cdot B}{A \cdot B + i\sigma \cdot (A \times B)} &= \frac{\sigma \cdot \pi \sigma \cdot \pi}{2m}\psi_\pm = \frac{\pi \cdot \pi + i\sigma \cdot (\pi \times \pi)}{2m}\psi_\pm \end{aligned}$$

$$= \left[ \frac{\pi^2}{2m} - \frac{q\hbar}{2m}\sigma \cdot B \right] \psi_\pm$$

$$(g_e = 2) = \left[ \frac{\pi^2}{2m} - \frac{g_e q}{2m}S \cdot B \right] \psi_\pm$$

$$i\hbar\frac{\partial}{\partial t}\psi = (c\alpha \cdot p + \beta mc^2)\psi$$

$$\psi(t) = \psi(p)e^{i(p \cdot r - Et)} \Rightarrow E\psi = (\alpha \cdot p + \beta m)\psi$$

$$\begin{bmatrix} E - m & -\sigma \cdot p \\ -\sigma \cdot p & E + m \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = 0 \Leftrightarrow \begin{aligned} (E - m)\psi_+ &= (\sigma \cdot p)\psi_- \\ (E + m)\psi_- &= (\sigma \cdot p)\psi_+ \end{aligned}$$

$$p = 0 \Rightarrow \begin{cases} \psi = \begin{bmatrix} \psi_+ \\ 0 \end{bmatrix}, & E = m \\ \psi = \begin{bmatrix} 0 \\ \psi_- \end{bmatrix}, & E = -m \end{cases}$$

$$\psi_\pm = \frac{(\sigma \cdot p)^2}{E^2 - m^2}\psi_\pm = \frac{p^2}{E^2 - m^2}\psi_\pm \Rightarrow E_\pm = \pm\sqrt{p^2 + m^2}$$

$$\int \|\psi_+\|^2 + \|\psi_-\|^2 d^3r = 1$$

## Hydrogen Fine Structure

$$\begin{aligned} E\psi &= H\psi \\ E\psi &= (\alpha \cdot p + \beta m + q\phi)\psi \Rightarrow \begin{aligned} (E - m - V)\psi_+ &= (\sigma \cdot p)\psi_- \\ (E + m - V)\psi_- &= (\sigma \cdot p)\psi_+ \end{aligned} \end{aligned}$$

$$\begin{aligned} (E - m - V)\psi_+ &= (\sigma \cdot p) \left( \frac{1}{E+m-V} \right) (\sigma \cdot p)\psi_+ \\ (E_s - V)\psi_+ &= \frac{1}{2m} (\sigma \cdot p) \left( 1 + \frac{E_s - V}{2m} \right)^{-1} (\sigma \cdot p)\psi_+ \\ &\approx \frac{p^2}{2m} \psi_+ \quad (1^{\text{st}} \text{ order}, v^2) \\ &\approx \frac{p^2}{2m} \psi_+ - \frac{\sigma \cdot p}{(2m)^2} (E_s - V) (\sigma \cdot p)\psi_+ \quad (2^{\text{nd}} \text{ order}, v^4) \\ &= \frac{p^2}{2m} \psi_+ - \frac{\sigma \cdot p}{(2m)^2} \left[ (\sigma \cdot p) \overbrace{(E_s - V)\psi_+}^{1^{\text{st}} \text{ order}} + [E_s - V, \sigma \cdot p] \psi_+ \right] \\ &\approx \left[ \frac{p^2}{2m} - \frac{p^2}{(2m)^2} \frac{p^2}{2m} - \frac{(\sigma \cdot p)(\sigma \cdot [p, V])}{(2m)^2} \right] \psi_+ \\ E_S \psi_+ &= \left[ \frac{p^2}{2m} + V - \frac{p^4}{8m^3} - \frac{i\sigma \cdot (p \times [p, V])}{4m^2} - \underbrace{\frac{p[p, V]}{4m^2}}_{\text{(isn't Hermitian)}} \right] \psi_+ = H\psi_+ \end{aligned}$$

$$\begin{aligned} 1 &= \int \|\psi_+\|^2 + \|\psi_-\|^2 d^3r \\ &= \int \|\psi_+\|^2 + \left\| \frac{\sigma \cdot p}{E+m-V} \psi_+ \right\|^2 d^3r \\ &\approx \int \|\psi_+\|^2 + \left\| \frac{\sigma \cdot p}{2m} \psi_+ \right\|^2 d^3r \\ &= \int \psi_+^\dagger \left( 1 + \frac{p^2}{4m^2} \right) \psi_+ d^3r \\ &\approx \left\langle \left( 1 + \frac{p^2}{8m^2} \right) \psi_+ \left| \left( 1 + \frac{p^2}{8m^2} \right) \psi_+ \right. \right\rangle \\ &\equiv \langle \psi_S | \psi_S \rangle \end{aligned}$$

$$\begin{aligned} E_S \left( 1 + \frac{p^2}{8m^2} \right)^{-1} \psi_S &= H \left( 1 + \frac{p^2}{8m^2} \right)^{-1} \psi_S \\ E_S \psi_S &= \left( 1 + \frac{p^2}{8m^2} \right) H \left( 1 + \frac{p^2}{8m^2} \right)^{-1} \psi_S \\ &= \left( H + \frac{p^2 H}{8m^2} \right) \left( 1 - \frac{p^2}{8m^2} + \mathcal{O}(p^4) \right) \psi_S \\ &\approx \left( H + \left[ \frac{p^2}{8m^2}, H \right] \right) \psi_S \approx \left( H + \left[ \frac{p^2}{8m^2}, V \right] \right) \psi_S \quad (2^{\text{nd}} \text{ order}, v^4) \\ E_S \psi_S &= \left( \frac{p^2}{2m} + V - \frac{p^4}{8m^3} - \frac{i\sigma \cdot (p \times [p, V])}{4m^2} - \frac{p[p, V]}{4m^2} + \frac{[p, V]p + p[p, V]}{8m^2} \right) \psi_S \\ &= \left( \frac{p^2}{2m} + V - \frac{p^4}{8m^3} - \frac{i\sigma \cdot (p \times [p, V])}{4m^2} - \frac{[p, [p, V]]}{8m^2} \right) \psi_S \\ &= \boxed{(H_S + H_{\text{rel.}} + H_{\text{so}} + H_{\text{darwin}}) \psi_S} \\ &= \left( \frac{p^2}{2m} + V - \frac{p^4}{8m^3} - \frac{1}{4m^2} \sigma \cdot (p \times \nabla V) + \overbrace{\frac{1}{8m^2} \nabla^2 V}^{\text{Darwin} \Rightarrow} \right) \psi_S \\ &= \left( \frac{p^2}{2m} + V - \frac{p^4}{8m^3} - \frac{1}{2m^2} S \cdot [\vec{p} \times \frac{q\vec{q}\vec{r}}{4\pi r^3}] + \frac{1}{8m^2} [qq\delta^3(r)] \right) \psi_S \\ &= \left( \frac{p^2}{2m} + V - \frac{p^4}{8m^3} + \underbrace{\frac{e^2}{8\pi m^2} \frac{S \cdot L}{r^3}}_{l \neq 0} + \underbrace{\frac{e^2}{8m^2} \delta^3(r)}_{l=0} \right) \psi_S \end{aligned}$$

$$\begin{aligned} \overline{V(r)} &= V(r) + \sum_i \overline{\frac{\partial V}{\partial r_i} \delta \vec{r}_i} \\ &\quad + \frac{1}{2!} \sum_{ij} \overline{\frac{\partial^2 V}{\partial r_i \partial r_j} \delta r_i \delta r_j} \\ &\quad + \mathcal{O}(\delta r^3) \\ &= V(r) + \frac{1}{2} (\delta r)^2 \nabla^2 V + .. \\ &\quad (\delta r \sim \frac{\hbar}{mc}) \end{aligned}$$

$$\begin{array}{l} \text{Exact Energy} \\ \text{Eigenvalues} \end{array} : E_{nj} = mc^2 \left[ 1 + \left( \frac{\alpha}{n - (j + 1/2) + \sqrt{(j + 1/2)^2 - \alpha^2}} \right)^2 \right]^{-1/2}$$