1 Analytic/Holomorphic Functions

Differentiable : $\exists f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{z}$ $\quad \begin{matrix} f = u+iv \\ h = \sigma + i\tau \end{matrix}$ $\Leftrightarrow$ $\frac{\text{Cauchy-Riemann Eq.}}{\boxed{\partial f/\partial\bar z = 0}}$: $\boxed{\begin{matrix} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, & \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \\ r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, & r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta} \end{matrix}}$

- $\begin{matrix} \exists\partial_{x,y} f\big|_z \in C^1, \text{CR Eq.}\big|_z \\ f\big|_z \in C^1, \exists\partial_{x,y} f\big|_z, \text{CR Eq.}\big|_z \end{matrix} \Rightarrow \text{Holo}\big|_z$
- $\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y}$, $\begin{matrix} x = \frac{1}{2}(z+\bar z) \\ y = \frac{1}{2i}(z-\bar z) \end{matrix}$
- $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$
- $\Delta u = \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0$
- $u(z) = c$ OR $u^2 + v^2 = c \Rightarrow f(z) = c$

Holomorphic at z_0 : $\begin{matrix} \exists V_{z_0}, \forall z \in V_{z_0}, \exists f'(z) \\ \text{Holo}\big|_{z_0} \Rightarrow \exists\partial_{x,y} f\big|_{z_0}, \text{CR}\big|_{z_0} \end{matrix}$ Smooth : $f(z) \in C^\infty$ Entire : Holomorphic over all $\mathbb{C}$

Analytic : $f(z_0) \in C^\omega \subset C^\infty$: $\begin{matrix} \exists V_{z_0}, \forall z \in V_{z_0} \\ \exists\delta > 0, \forall |z| < \delta \end{matrix}$, $f(z_0 + z) = \sum a_n z^n \rightarrow \boxed{f(z) = \sum a_n (z-z_0)^n}$

- $\sum a_n (z_1 - z_0)^n \Rightarrow \sum \left| a_n (z-z_0)^n \right|$: $\boxed{|z - z_0| < |z_1 - z_0|}$
- $\boxed{a_n = \frac{f^n(z_0)}{n!}}$
- Root Test : $\lim \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R}$
- Ratio Test : $\lim \sqrt[n]{a_n} = \frac{1}{R}$
- $\frac{1}{R} = \limsup \sqrt[n]{a_n}$

Sum : $\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$

Green's Theorem : $\oint u\,dx + v\,dy = \iint \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy$

$$\oint \overline{f(z)}\,dz = \int \left[u\,dx + v\,dy\right] + i\int \left[u\,dy - v\,dx\right]$$

$$\underline{\text{CR for } \bar f \ \rightarrow} = \iint \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dy\,dx + i\iint \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] dy\,dx$$

$$\underline{\text{sourceless}}\ \underline{\text{irrotational}} = \oint \vec f \cdot d\vec l \ \ \text{(curl)} \quad + i\oint \left(\vec f \cdot \hat n\right) dl \ \ \text{(flux)}$$

Cauchy's Theorem : $\boxed{\oint_\gamma f(z)dz = 0} = i\iint \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\,dx\,dy$ $\quad\left(\begin{matrix} f' \text{ must be cont. to use Green's Theorem} \\ \text{Goursat proves w/o cont. w/ triangles} \end{matrix}\right)$

1. simple closed 2. closed squares/triangles+$\square$ 3. $\exists F$ ($F' = f$, well-defined, path ind.) 4. $\square$ $\left(\oint f dz = \oint F' dz = 0\right)$

- D is simp.-con. $\Rightarrow \exists F$ (cont. $F' = f$, holo.)
- no zero $\Rightarrow \boxed{f(z) = e^{g(z)}}$, $\underline{g(z) = \text{Log } f(z_0) + \int_{z_0}^{z} \frac{f'}{f}\,dw}$
- Morera's Theorem : f is cont., $\forall\gamma_{\in C^1} \in D$, $\oint_\gamma f(z)dz = 0 \Rightarrow f$ is holo. in D

Cauchy's Formula : $\boxed{f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0} dz} \Rightarrow f(z) = \sum a_n (z - z_0)^n$ $\boxed{\text{(Holo} \rightarrow \text{Analytic)}}$

1. $f(z_0) = \lim_{r\to 0} \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta = \underbrace{\boxed{\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta}}_{\text{(Mean Value Theorem)}} \le \max\left|f(z_0 + re^{i\theta})\right| \rightarrow 0$ (if f is cont.)

- Louisville's Theorem : f is entire, $\exists M > 0$, $\forall |f(z)| \le M \Rightarrow f(z) = c$
- Analytic : $\exists F$(holo., cont. F') $\Rightarrow F = \sum b_n (z-z_0)^n \Rightarrow$ cont. $f = \sum a_n (z-z_0)^n \Rightarrow$ $\underline{\text{cont. } f'}$
- $\boxed{a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{k+1}}\,dz} = \frac{f^n(z_0)}{n!}$
- $\exists z_0, \forall k, f^{(k)}(z_0) = 0 \Rightarrow f(z) = 0$

Zero/Singularity/Pole of Order m, z_0 :

Zero : $f(z) = \sum_{m} a_n (z-z_0)^n \quad (m \ge 1) \quad = \quad g(z)(z-z_0)^m$

Removable Singularity : $f(z) = \sum_{0} a_n (z-z_0)^n \quad (m = 0) \quad = \quad a_0 + \dots$

Pole : $f(z) = \sum_{-m} a_n (z-z_0)^n \quad (m \ge 1) \quad = \quad \dfrac{H(z) = \frac{1}{h(z)}}{(z-z_0)^m} = \dfrac{1}{g(z)}$

Essential Singularity : $f(z) = \sum_{-\infty} a_n (z-z_0)^n \quad (m = \infty)$

Residue : $\mathrm{Res}(f; z_0) = \frac{1}{2\pi i} \oint f(\zeta)\, d\zeta$

- $\boxed{\oint_\gamma f(\zeta)\, d\zeta = 2\pi i \sum_{\text{sing.}} \mathrm{Res}(f; z_0)}$
- $\underline{g(z) = (z-z_0)^n} \Rightarrow \boxed{\mathrm{Res}(g; z_0) = \begin{cases} 0 & n \neq -1 \\ 1 & n = -1 \end{cases}}$
- Pole $\to \boxed{\mathrm{Res}(f; z_0) = a_{-1} = \dfrac{H^{(m-1)}(z_0)}{(m-1)!}}$

$*$ $f(z) = \dfrac{H(z)}{z-z_0} \to \mathrm{Res}(f; z_0) = H(z_0)$

- $G'(z_0) \neq 0 \to \mathrm{Res}\left(\frac{H}{G}; z_0\right) = \dfrac{H(z_0)}{G'(z_0)}$

Laurent Series : $f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n = \sum_{0}^{\infty} a_n (z-z_0)^n + \underline{\sum_{1}^{\infty} b_n (z-z_0)^{-n}}$ $\leftarrow$ principal part

- $\boxed{a_n = \dfrac{1}{2\pi i} \oint_{|\zeta - z_0| \text{ analytic}} \dfrac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta}$
- $\boxed{b_n = \dfrac{1}{2\pi i} \oint_{|\zeta - z_0| \text{ analytic}} f(\zeta)(\zeta-z_0)^{n-1}\, d\zeta}$

$*$ Exam. : $f(z) = P(z) + \dfrac{Q(z)}{R(z)} = P(z) + \dfrac{a}{z-3} + \dfrac{b}{z-5} = P(z) + \sum \begin{cases} \frac{-a}{3}\left(\frac{z}{3}\right)^n - \frac{b}{5}\left(\frac{z}{5}\right)^n & |z| < 1 \\ \frac{a}{z}\left(\frac{3}{z}\right)^n - \frac{b}{5}\left(\frac{z}{5}\right)^n & 1 < |z| < 5 \\ \frac{a}{z}\left(\frac{3}{z}\right)^n + \frac{b}{z}\left(\frac{5}{z}\right)^n & 5 < |z| \end{cases}$

- $f(z) = \sum_{i} P\left(\frac{1}{z-z_i}\right) + \underset{\text{(analytic)}}{P(z)}$

$*$ $\dfrac{1}{(1-z)^2} = (1 + z + z^2 ...)^2 = 1 + 2z + 3z^2 ...$ $\quad *$ $\boxed{\dfrac{1}{(1-z)^k} = \sum \dfrac{(n+k-1)!}{n!(k-1)!} z^n}$ (n stars, $k-1$ bars) [last bar can't move]

$*$ $f(z) = \dfrac{H(z)}{R(z)} = \dfrac{H(z)}{(z-z_1)(z-z_2)^3} = \boxed{\dfrac{H_1(z)}{z-z_1} = \dfrac{H_2(z)}{(z-z_2)^3}} = \boxed{\dfrac{H_1(z_1)}{z-z_1} + \dfrac{H_2''(z_2)/2!}{z-z_2} + \dfrac{H_2'(z_2)/1!}{(z-z_2)^2} + \dfrac{H_2(z_2)}{(z-z_2)^3} + P(z)}$

$= \boxed{\dfrac{\mathrm{Res}(f; z_1)}{z-z_1} + \dfrac{\mathrm{Res}(f; z_2)}{z-z_2} + \dfrac{\mathrm{Res}((z-z_2)f; z_2)}{(z-z_2)^2} + \dfrac{\mathrm{Res}((z-z_2)^2 f; z_2)}{(z-z_2)^3} + P(z)}$

$H_i(z) = Q_i(z) + (z-z_i)^{n_i} P(z)$

$n < n_i$, $\underline{H_i^{(n)}(z_i) = Q_i^{(n)}(z_i)}$

(or $\frac{Q}{R}(\infty) = 0 \Rightarrow P'(z) = 0$)

$= \dfrac{Q(z) + P(z)\,R(z)}{R(z)} = \boxed{\dfrac{Q_1(z_1)}{z-z_1} + \dfrac{Q_2''(z_2)/2!}{z-z_2} + \dfrac{Q_2'(z_2)/1!}{(z-z_2)^2} + \dfrac{Q_2(z_2)}{(z-z_2)^3} + P(z)}$

$= \dfrac{1}{z}\dfrac{Q_1(z_1)}{1-\frac{z_1}{z}} + \dfrac{1}{z}\dfrac{Q_2''(z_2)/2!}{1-\frac{z_2}{z}} + \dfrac{1}{z^2}\dfrac{Q_2'(z_2)/1!}{(1-\frac{z_2}{z})^2} + \dfrac{1}{z^3}\dfrac{Q_2(z_2)}{(1-\frac{z_2}{z})^3} + P(z)$

$= \boxed{\frac{Q_1(z_1)}{0!} Z_1\left\{\binom{n}{n} z_1^n\right\} + \frac{Q_2''(z_2)}{2!} Z_1\left\{\binom{n}{n} z_2^n\right\} + \frac{Q_2'(z_2)}{1!} Z_2\left\{\binom{n+1}{n} z_2^n\right\} + \frac{Q_2(z_2)}{0!} Z_3\left\{\binom{n+2}{n} z_2^n\right\} + P(z)}$

$*$ $Z\{a_n = -b a_{n-1} - c a_{n-2} + g_{n-2}\} = \dfrac{\vec{R}(z) \cdot (a_0 + \frac{a_1}{z}, a_0, 0)}{R(z) = z^2 + bz + c} + \dfrac{Z\{g_n\}}{R(z)} = \underset{Z\{\text{homogeneous}\}}{a_0 + \dfrac{Q(z)}{R(z)}} + \underset{Z\{\text{particular}\}}{\dfrac{Z\{g_n\}}{R(z)}}$

$\underline{\text{Green's/Stokes' Theorem}}$:

$$\oint \begin{bmatrix} u \\ v \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = \iint \begin{vmatrix} \nabla_x & \nabla_y \\ -(-u) & v \end{vmatrix} dx\, dy$$

$$\oint \begin{bmatrix} v \\ -u \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} = \iint \vec{\nabla} \cdot \begin{bmatrix} v \\ -u \end{bmatrix} dx\, dy$$

$\underline{\text{2D Div. Theorem}}$:

$$\oint \begin{bmatrix} v \\ -u \end{bmatrix} \cdot \hat{n}\ dl = \iint \vec{\nabla} \cdot \begin{bmatrix} v \\ -u \end{bmatrix} dA$$

$\underline{\text{Green's 2D 1st Identity}}$:

$$\oint \begin{bmatrix} f\nabla_x g \\ f\nabla_y g \end{bmatrix} \cdot \hat{n}\ dl = \iint \vec{\nabla} \cdot \left[f\,\vec{\nabla} g\right] dA$$

$$\boxed{\oint \left[f\,\vec{\nabla} g\right] \cdot \hat{n}\ dl = \iint \left[f\,\vec{\nabla}^2 g + \vec{\nabla} g \cdot \vec{\nabla} f\right] dA} \Rightarrow \underline{\oint \left[f\,\vec{\nabla} f\right] \cdot \hat{n}\ dl = \iint \left[f\,\vec{\nabla}^2 f + \|\vec{\nabla} f\|^2\right] dA}$$

$\underline{\text{Green's 2D 2nd Identity}}$:

$$\boxed{\oint \left[f\,\vec{\nabla} g - g\,\vec{\nabla} f\right] \cdot \hat{n}\ dl = \iint \left[f\,\vec{\nabla}^2 g - g\,\vec{\nabla}^2 f\right] dA}$$

$\underline{\text{Green's 2D 3rd Identity}}$:

$$\boxed{\vec{\nabla}^2 G = \delta^2(z - z_0)} \Rightarrow \boxed{f_{(z_0)} = \oint \left[f\,\vec{\nabla} G - G\,\vec{\nabla} f\right] \cdot \hat{n}\ dl + \iint \left[G\,\vec{\nabla}^2 f\right] dA}$$

$$f_{(z_0)} = \oint f_{(z)} \left[\vec{\nabla} G \cdot \hat{n}\right] dz$$

- f is harmonic
- G is 0 on the boundary

2 Conformal Mapping

- $e^z = e^x e^{iy} = e^x(\cos y + i\sin y)$: $\begin{cases} x \in (-\infty,0],[0,\infty) \\ y\in[0,\pi],[\pi,2\pi]+\theta_0 \end{cases} \to \begin{matrix} R\in(0,1],[1,\infty) \\ \theta\in[0,\pi],[\pi,2\pi] \end{matrix}$

- $\log z = \ln R_0 + i\arg(z)$: $\begin{cases} R_0\in(0,1],[1,\infty) \\ \theta_0\in[-\pi,0],[0,\pi] \end{cases} \to \begin{matrix} u\in(-\infty,0],[0,\infty) \\ v\in[-\pi,0],[0,\pi]+2\pi k \end{matrix}$

- $\cos z = \cos x\cosh y - i\sin x\sinh y$: $\begin{cases} x\in[0,\pm\pi/2) \\ y\in[0,\pm\infty) \end{cases} \to \begin{matrix} u\in[0,\infty) \\ v\in[0,\pm_x\pm_y\infty) \end{matrix}$

- $\sin z = \sin x\cosh y + i\cos x\sinh y$: $\begin{cases} x\in[0,\pm\pi/2) \\ y\in[0,\pm\infty) \end{cases} \to \begin{matrix} u\in[0,\pm_x\infty) \\ v\in[0,\pm_y\infty) \end{matrix}$

- $Az+B$ $\begin{matrix} * \; Cz = Re^{i\theta}z \quad \text{(rotation+scale)} \\ * \; z + C = z + a + bi \quad \text{(translation)} \end{matrix}$: $\begin{cases} \text{rotation+scale+translation} \\ \text{for lines and circles} \end{cases}$

- $\dfrac{1}{z} = \dfrac{x-iy}{x^2+y^2} = u+iv \;\Rightarrow\; z = x+iy = \dfrac{u-iv}{u^2+v^2}$

$$: \begin{cases} \underline{\alpha(x^2+y^2)+\beta x+\gamma y=\delta} = \frac{\alpha+\beta u-\gamma v}{u^2+v^2} \;\Rightarrow\; \delta(u^2+v^2)-\beta u+\gamma v=\alpha \\ \beta^2+\gamma^2+4\alpha\gamma>0 \,,\; \boxed{\alpha\neq 0 \,,\; \begin{matrix} \delta\neq 0 \;\text{(circle w/o 0)} \\ \delta=0 \;\text{(circle w/ 0)} \end{matrix}} \to \beta^2+\gamma^2+4\alpha\gamma>0 \,,\; \boxed{\begin{matrix}\text{(circle w/o 0)} \\ \text{(line w/o 0)}\end{matrix}} \\ \boxed{\alpha=0 \,,\; \begin{matrix} \delta\neq 0 \;\text{(line w/o 0)} \\ \delta=0 \;(\text{line } y=\frac{-\beta}{\gamma}x) \end{matrix}} \to \beta^2+\gamma^2>0 \,,\; \boxed{\begin{matrix}\text{(circle w/ 0)} \\ (\text{line } v=\frac{\beta}{\gamma}u)\end{matrix}} \end{cases}$$

- $T(z) = \dfrac{az+b}{cz+d} = \frac{\alpha z+\beta}{\gamma z+1} = \underline{\frac{1}{c}\left[(bc-ad)\frac{1}{cz+d}+a\right]}$ $\quad \begin{matrix} * \; ad-bc=0 \;\Rightarrow\; T'=0 \\ * \; T(\zeta)=\zeta \;\Rightarrow\; 0=c\zeta^2+(a-d)\zeta-b \\ * \; T^{-1}(z)=\frac{-dz+b}{cz-a} \end{matrix}$

  : $\{$lines+circles $\to$ lines+circles

- $\underline{T(\pm x_0)=\pm x_0=\epsilon} \;\Rightarrow\; \underline{\epsilon=\pm\sqrt{\frac{b}{c\neq 0}}\geq 0}$, $\underline{(a=d)}$ $\Rightarrow$ $\boxed{T(z)=\dfrac{az+c\epsilon^2}{cz+a} \;\to\; T_\epsilon(z)=\dfrac{az+\epsilon^2}{z+a}}$

  : $\{$Circle of Appolonius through $x=\pm\epsilon$ $\to$ Itself $\Rightarrow$ $\underline{\lim_{\epsilon\to 0} T_\epsilon : \text{Dipole Circles} \;\to\; \text{Itself}}$

$$T^n_\epsilon(z) = \frac{\alpha_n z+\epsilon^2\gamma_n}{\gamma_n z+\alpha_n} \;\Rightarrow\; T^{n-1}\begin{bmatrix}\alpha_1\\ \gamma_1\end{bmatrix} = \begin{bmatrix}\alpha_1 & \epsilon^2\gamma_1 \\ \gamma_1 & \alpha_1\end{bmatrix}^{n-1}\left[\begin{matrix}\alpha_1\\ \gamma_1\end{matrix}\middle|\begin{matrix}\cancel{\epsilon^2\gamma_1}\\ \cancel{\alpha_1}\end{matrix}\right] = \left[\begin{matrix}\alpha_n\\ \gamma_n\end{matrix}\middle|\begin{matrix}\cancel{\epsilon^2\gamma_n}\\ \cancel{\alpha_n}\end{matrix}\right] \overset{(\alpha_1\neq\beta_1)}{=} \underline{\frac{1}{2\epsilon}}\left[\begin{matrix}\epsilon(\alpha_1+\epsilon\gamma_1)^n+\epsilon(\alpha_1-\epsilon\gamma_1)^n \\ (\alpha_1+\epsilon\gamma_1)^n-(\alpha_1-\epsilon\gamma_1)^n\end{matrix}\middle|\begin{matrix}\cancel{\epsilon^2\gamma_n}\\ \cancel{\alpha_n}\end{matrix}\right]$$

(induction)

$$\begin{matrix} 2\epsilon\alpha_n=\epsilon(a+\epsilon)^n+\epsilon(a-\epsilon)^n \\ 2\epsilon\gamma_n=(a+\epsilon)^n-(a-\epsilon)^n \end{matrix} \;,\; \underline{\epsilon=1} \;\left|\; \begin{matrix} \lim_{n\to\infty}\frac{\alpha_n}{\gamma_n} = \epsilon\frac{1+x^n}{1-x^n} \\ (x=\frac{a-\epsilon}{a+\epsilon}) \;=\; \frac{2\epsilon}{1-x^n}-\epsilon \end{matrix} \right. \;\to\; \begin{matrix} 0<|x|<1 \;(\Leftrightarrow \underline{a>0}) \\ 1<|x| \;(\Leftrightarrow \underline{a<0}) \end{matrix} \;\Rightarrow\; \lim_{n\to\infty} T^n_1(z) = \begin{matrix}\epsilon=1 \\ -\epsilon=-1\end{matrix}$$

$$\boxed{\text{Source}=\mp 1 \;\to\; \text{Sink}=\pm 1}$$

$$x=\frac{Re^{i\theta}-\epsilon}{Re^{i\theta}+\epsilon} = \frac{R^2-\epsilon^2+i2R\epsilon\sin\theta}{R^2+\epsilon^2+2R\epsilon\cos\theta} = Ce^{i\phi} \;\Rightarrow\; \frac{\alpha_n}{\gamma_n} = \frac{2\epsilon}{1-C^n e^{in\phi}}-\epsilon = \frac{(2\epsilon-2\epsilon C^n\cos n\phi)+i2\epsilon C^n\sin n\phi}{C^{2n}+1-2C^n\cos n\phi}-\epsilon$$

$$\begin{matrix} -1\leq X=\frac{R^2-\epsilon^2}{R^2+\epsilon^2}\leq 1 \\ 0\leq Y=\frac{2R\epsilon}{R^2+\epsilon^2}\leq 1 \end{matrix} \;,\; \begin{matrix} \tan\phi=\frac{Y}{X}\sin\theta \\ X^2+Y^2=1 \end{matrix} \;,\; \begin{matrix} 0\leq C=\frac{\sqrt{X^2+Y^2\sin^2\theta}}{1+Y\cos\theta}\overset{?}{=}1 \\ C\overset{?}{=}1 \to 0\overset{?}{=}Y(Y\cos\theta+1)\underline{\cos\theta} \end{matrix} \;:\; \lim_{n\to\infty}\frac{\alpha_n}{\gamma_n} \begin{matrix} =\frac{2\epsilon}{1-e^{in\phi}}-\epsilon \;\; (C=1:\mathrm{Re}(a)=0) \\ =-\epsilon \;\; (C>1:\underline{\mathrm{Re}(a)<0}, \text{ also } a=-\epsilon) \\ =\epsilon \;\; (C<1:\underline{\mathrm{Re}(a)>0}) \end{matrix}$$

$$\underline{\text{General Circles of Appolonius}}: \quad \begin{array}{l} \zeta_{\pm} = \pm A\epsilon + B \\ B = \frac{\zeta_+ + \zeta_-}{2} \\ A = \frac{\zeta_+ - \zeta_-}{2} \end{array} \quad , \quad \begin{array}{rl} T_{\zeta(z)} = & \frac{(d+2cB)z + c(A^2\epsilon^2 - B^2)}{cz+d} \\ = & \boxed{\dfrac{(D + \zeta_+ + \zeta_-)z - \zeta_+\zeta_-}{z + D}} \end{array}$$

$$\begin{array}{r} T(\zeta) = \frac{1}{A}\zeta - \frac{B}{A} = z \\ F(z) = \frac{az+\epsilon^2}{z+a} \\ T^{-1}(z) = Az + B = \zeta \end{array} \ : \ T^{-1} \circ F \circ T(\zeta) = A\left[\frac{a\zeta - aB + A\epsilon^2}{\zeta - B + Aa}\right] + B = \boxed{\dfrac{(Aa+B)\zeta - \zeta_+\zeta_-}{\zeta + (Aa - B)}}$$

3 Harmonic Functions

4 Transforms

$$\begin{aligned} f_{(z)}g_{(z)} &= \left(a_0 + a_1 z + a_2 z^2 + \dots\right)\left(b_0 + b_1 z + b_2 z^2 + \dots\right) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)\, z + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, z^2 + \dots \\ &= \sum_n c_n z^n \;\Rightarrow\; \boxed{c_n = \sum_k^n a_n b_{n-k}} \end{aligned}$$