1 Analytic/Holomorphic Functions

Differentiable : $\exists f'(z) = \lim\limits_{h\to 0} \frac{f(z+h)-f(z)}{z}$ $\quad \begin{matrix} f = u+iv \\ h = \sigma + i\tau \end{matrix} \Rightarrow$ $\frac{\text{Cauchy-Riemann}}{\text{Equations}}$: $\boxed{\begin{matrix} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, & \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \\ r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, & r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta} \end{matrix}}$

- $\left(\text{C-R Eq.}\big|_z\right), \left(f(z) \in C^1\right) \Rightarrow \exists f'(z)$
- $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ • $\Delta u = \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0$ • $u(z) = c$ OR $u^2 + v^2 = c \Rightarrow f(z) = c$

Holomorphic : $\forall z \in D,\ \exists f'(z)$ Smooth : $f(z) \in C^\infty$ Entire : Analytic everywhere

Analytic : $f(z_0) \in C^\omega \subset C^\infty$: $\exists \delta > 0,\ \forall |z| < \delta,\ f(z_0+z) = \sum a_n z^n \ \to\ \boxed{f(z) = \sum a_n (z-z_0)^n}$

- $\sum a_n (z_1 - z_0)^n \Rightarrow \sum \left|a_n (z-z_0)^n\right|$: $\boxed{|z - z_0| < |z_1 - z_0|}$ • $\boxed{a_n = \frac{f^n(z_0)}{n!}}$
- Root Test : $\lim \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R}$ • Ratio Test : $\lim \sqrt[n]{a_n} = \frac{1}{R}$ • $\frac{1}{R} = \limsup \sqrt[n]{a_n}$

Sum : $\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$

Green's Theorem : $\oint u\,dx + v\,dy = \iint \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy$

$\oint \overline{f(z)}\,dz = \int \left[u\,dx + v\,dy\right] + i\int \left[u\,dy - v\,dx\right]$

CR for $\bar{f}$ $\to = \iint \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dy\,dx + i\iint \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] dy\,dx$

sourceless irrotational $= \oint \vec{f} \cdot d\vec{l}$ (curl) $+ i\oint \left(\vec{f} \cdot \hat{n}\right) dl$ (flux)

Cauchy's Theorem : $\boxed{\oint_\gamma f(z)\,dz = 0} = i\iint \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\,dx\,dy$ (f' must be cont. to use Green's Theorem / Goursat proves w/o cont. w/ triangles)

1. simple closed 2. closed squares/triangles+□ 3. $\exists F$ ($F' = f$, well-defined, path ind.) 4. □ $\left(\oint f\,dz = \oint F'\,dz = 0\right)$

- D is simp.-con. $\Rightarrow \exists F$ (cont. $F' = f$, holo.) • no zero $\Rightarrow \boxed{f(z) = e^{g(z)}}$, $g(z) = \text{Log } f(z_0) + \int_{z_0}^{z} \frac{f'}{f}\,dw$
- Morera's Theorem : f is cont., $\forall \gamma_{\in C^1} \in D$, $\oint_\gamma f(z)\,dz = 0 \Rightarrow f$ is holo. in D

Cauchy's Formula : $\boxed{f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0}\,dz} \Rightarrow f(z) = \sum a_n (z-z_0)^n$ (Power series/analytic)

1. $f(z_0) = \lim\limits_{r\to 0} \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta = \boxed{\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta} \le \max\left|f(z_0 + re^{i\theta})\right| \to 0$ (if f is cont.)

(Mean Value Theorem)

- Louisville's Theorem : f is entire, $\exists M > 0,\ f(z) \le M \Rightarrow f(z) = c$
- Analytic : $\exists F$(holo., cont. F') $\Rightarrow F = \sum b_n (z-z_0)^n \Rightarrow$ cont. $f = \sum a_n (z-z_0)^n \Rightarrow$ cont. f'
- $\boxed{a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{k+1}}\,dz} = \frac{f^n(z_0)}{n!}$ • $\exists z_0,\ \forall k,\ f^{(k)}(z_0) = 0 \Rightarrow f(z) = 0$

Zero/Singularity/Pole of Order m, z_0 :

$$\text{Zero}: \; f(z) = \sum_{m} a_n (z-z_0)^n \quad (m \geq 1) \;=\; g(z)(z-z_0)^m$$

$$\text{Removable Singularity}: \; f(z) = \sum_{0} a_n (z-z_0)^n \quad (m = 0) \;=\; a_0 + \dots$$

$$\text{Pole}: \; f(z) = \sum_{-m} a_n (z-z_0)^n \quad (m \geq 1) \;=\; \frac{1}{g(z)} = \frac{H(z)=\frac{1}{h(z)}}{(z-z_0)^m}$$

$$\text{Essential Singularity}: \; f(z) = \sum_{-\infty} a_n (z-z_0)^n \quad (m = \infty)$$

Residue : $\text{Res}(f; z_0) = \frac{1}{2\pi i} \oint f(\zeta)\, d\zeta$

- $\boxed{\oint_\gamma f(\zeta)\, d\zeta = 2\pi i \sum_{\text{sing.}} \text{Res}(f; z_0)}$
- $g(z) = (z-z_0)^j \;\Rightarrow\; \text{Res}(g; z_0) = \begin{cases} 0 & j \neq -1 \\ 1 & j = -1 \end{cases}$
- Pole $\rightarrow$ $\boxed{\text{Res}(f; z_0) = a_{-1} = \frac{H^{(m-1)}(z_0)}{(m-1)!}}$

$*$ $f(z) = \frac{H(z)}{z-z_0} \rightarrow \text{Res}(f; z_0) = H(z_0)$

- $G'(z_0) \neq 0 \rightarrow \text{Res}\left(\frac{H}{G}; z_0\right) = \frac{H(z_0)}{G'(z_0)}$

Laurent Series : $f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n = \sum_{0}^{\infty} a_n (z-z_0)^n + \underline{\sum_{1}^{\infty} b_n (z-z_0)^{-n}} \;\leftarrow$ principal part

- $$f(z) = P(z) + \frac{Q(z)}{R(z)} = P(z) + \frac{a}{z-3} + \frac{b}{z-5} = P(z) + \sum \begin{cases} \frac{-a}{3}\left(\frac{z}{3}\right)^n - \frac{b}{5}\left(\frac{z}{5}\right)^n & |z| < 1 \\ \frac{a}{z}\left(\frac{3}{z}\right)^n - \frac{b}{5}\left(\frac{z}{5}\right)^n & 1 < |z| < 5 \\ \frac{a}{z}\left(\frac{3}{z}\right)^n + \frac{b}{z}\left(\frac{5}{z}\right)^n & 5 < |z| \end{cases}$$

Green's/Stokes' Theorem :

$$\oint \begin{bmatrix} u \\ v \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = \iint \begin{vmatrix} \nabla_x & \nabla_y \\ -(-u) & v \end{vmatrix} dx\, dy$$

$$\oint \begin{bmatrix} v \\ -u \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} = \iint \vec{\nabla} \cdot \begin{bmatrix} v \\ -u \end{bmatrix} dx\, dy$$

2D Div. Theorem :

$$\oint \begin{bmatrix} v \\ -u \end{bmatrix} \cdot \hat{n}\; dl = \iint \vec{\nabla} \cdot \begin{bmatrix} v \\ -u \end{bmatrix} dA$$

Green's 2D 1st Identity :

$$\oint \begin{bmatrix} f\nabla_x g \\ f\nabla_y g \end{bmatrix} \cdot \hat{n}\; dl = \iint \vec{\nabla} \cdot \left[f\vec{\nabla} g\right] dA$$

$$\boxed{\oint \left[f\vec{\nabla} g\right] \cdot \hat{n}\; dl = \iint \left[f\vec{\nabla}^2 g + \vec{\nabla} g \cdot \vec{\nabla} f\right] dA}$$

$$\Rightarrow \underline{\oint \left[f\vec{\nabla} f\right] \cdot \hat{n}\; dl = \iint \left[f\vec{\nabla}^2 f + \|\vec{\nabla} f\|^2\right] dA}$$

Green's 2D 2nd Identity :

$$\boxed{\oint \left[f\vec{\nabla} g - g\vec{\nabla} f\right] \cdot \hat{n}\; dl = \iint \left[f\vec{\nabla}^2 g - g\vec{\nabla}^2 f\right] dA}$$

Green's 2D 3rd Identity :

$$\boxed{\vec{\nabla}^2 G = \delta^2(z-z_0)} \Rightarrow \boxed{f(z_0) = \oint \left[f\vec{\nabla} G - G\vec{\nabla} f\right] \cdot \hat{n}\; dl + \iint \left[G\vec{\nabla}^2 f\right] dA}$$

$$\boxed{f(z_0) = \oint f(z)\left[\vec{\nabla} G \cdot \hat{n}\right] dz}$$

- f is harmonic
- G is 0 on the boundary

2 Conformal Mapping

- $e^z = e^x e^{iy} = e^x(\cos y + i\sin y):\ \begin{cases} x \in (-\infty, 0], [0, \infty) \\ y \in [0, \pi], [\pi, 2\pi] + \theta_0 \end{cases} \rightarrow \begin{matrix} R \in (0, 1], [1, \infty) \\ \theta \in [0, \pi], [\pi, 2\pi] \end{matrix}$
- $\log\ z = \ln R_0 + i\arg(z):\ \begin{cases} R_0 \in (0, 1], [1, \infty) \\ \theta_0 \in [-\pi, 0], [0, \pi] \end{cases} \rightarrow \begin{matrix} u \in (-\infty, 0], [0, \infty) \\ v \in [-\pi, 0], [0, \pi] + 2\pi k \end{matrix}$
- $\cos z = \cos x \cosh y - i \sin x \sinh y:\ \begin{cases} x \in [0, \pm\pi/2) \\ y \in [0, \pm\infty) \end{cases} \rightarrow \begin{matrix} u \in [0, \infty) \\ v \in [0, \pm_x \pm_y \infty) \end{matrix}$
- $\sin z = \sin x \cosh y + i \cos x \sinh y:\ \begin{cases} x \in [0, \pm\pi/2) \\ y \in [0, \pm\infty) \end{cases} \rightarrow \begin{matrix} u \in [0, \pm_x\infty) \\ v \in [0, \pm_y\infty) \end{matrix}$

3 Harmonic Functions

4 Transforms

$$
\begin{aligned}
f(z)g(z) &= \left(a_0 + a_1 z + a_2 z^2 + \dots\right)\left(b_0 + b_1 z + b_2 z^2 + \dots\right) \\
&= a_0 b_0 + (a_0 b_1 + a_1 b_0)\, z + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, z^2 + \dots \\
&= \sum_n c_n z^n \;\Rightarrow\; \boxed{c_n = \sum_k^n a_n b_{n-k}}
\end{aligned}
$$