$$\boxed{\vec{\nabla} = \left[\vec{\nabla}\,(r,\theta,\phi)\right]\bar{\partial}_\circ}$$

$$d = [dx\ dy\ dz]\ \vec{\nabla} = d\vec{l}^{\,T}\,\vec{\nabla}$$

$$d(r,\theta,\phi) = [dx\ dy\ dz]\ \vec{\nabla}(r,\theta,\phi)$$

$$\boxed{\partial\bar{l}_\circ^{\,T} = d\vec{l}^{\,T}\,\vec{\nabla}(r,\theta,\phi)}$$

$$\partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ = d\vec{l}^{\,T}\left[\vec{\nabla}(r,\theta,\phi)\right]\bar{\partial}_\circ$$

$$\boxed{d = \partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ = d\vec{l}^{\,T}\,\vec{\nabla}}$$

$$\underline{\vec{\nabla}} = \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix} = \begin{bmatrix}| & | & |\\ \nabla r & \nabla\theta & \nabla\phi\\ | & | & |\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\\ \frac{\partial}{\partial \phi}\end{bmatrix}$$

$$\underline{\partial\bar{l}_\circ} = \begin{bmatrix}dr\\ d\theta\\ d\phi\end{bmatrix} \begin{aligned} &= [\vec{\nabla}(r,\theta,\phi)]^T d\vec{l}\\ &= \begin{bmatrix}-\nabla r-\\ -\nabla\theta-\\ -\nabla\phi-\end{bmatrix}\begin{bmatrix}dx\\ dy\\ dz\end{bmatrix}\end{aligned}$$

$$\theta = \theta(x,y,z) \quad \left(x^2+y^2 = z^2\tan^2\theta\right)$$

$$\phi = \phi(x,y,z) \quad \left(y = x\tan\phi\right)$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z}$$

$$d\theta = dx\frac{\partial\theta}{\partial x} + dy\frac{\partial\theta}{\partial y} + dz\frac{\partial\theta}{\partial z}$$

$$dy_{\vec{r}_\circ}(\vec{r}_\circ{}')\big|_{t=0} = \left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \phi}d\phi\right)\frac{1}{dt}\Big|_{t=0}$$

$$\boxed{\vec{b}^{\,i}\cdot\vec{b}_i = \delta_{ij}} : \quad \begin{aligned}\vec{\nabla}\phi\cdot\frac{\partial\vec{r}}{\partial\phi} &= 1\\ \vec{\nabla}\phi\cdot\frac{\partial\vec{r}}{\partial\theta} &= 0\end{aligned} \quad \Rightarrow \begin{bmatrix}-\vec{\nabla}r-\\ -\vec{\nabla}\theta-\\ -\vec{\nabla}\phi-\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi}\end{bmatrix}\vec{r} = \mathbb{1}_3 \quad \Rightarrow \quad \boxed{\frac{\partial y}{\partial\phi} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}-\vec{\nabla}r-\\ -\vec{\nabla}\theta-\\ -\vec{\nabla}\phi-\end{bmatrix}^{-1}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} = \frac{\partial y}{\partial\phi}^T}$$

$$\begin{aligned}d &= dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} = d\vec{l}\cdot\vec{\nabla} & &= \left[\frac{dr}{\|\nabla r\|},\frac{d\theta}{\|\nabla \theta\|},\frac{d\phi}{\|\nabla \phi\|}\right]\left[\|\nabla r\|\frac{\partial}{\partial r},\ \|\nabla\theta\|\frac{\partial}{\partial \theta},\ \|\nabla\phi\|\frac{\partial}{\partial \phi}\right]^T\\ &= dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial \theta} + d\phi\frac{\partial}{\partial \phi} = \partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ & &= [dr, rd\theta, r\sin\theta d\phi]\left[\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial \theta},\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right]^T\\ &= \frac{dr}{\|\nabla r\|}\|\nabla r\|\frac{\partial}{\partial r} + \frac{d\theta}{\|\nabla \theta\|}\|\nabla \theta\|\frac{\partial}{\partial \theta} + \frac{d\phi}{\|\nabla \phi\|}\|\nabla \phi\|\frac{\partial}{\partial \phi} & &\cdots = d\bar{l}_\circ^{\,T}\bar{\nabla}_\circ = d\vec{l}_\circ^{\,T}\vec{\nabla}_\circ\end{aligned}$$

$$d\vec{l} = d\vec{r} = [dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial \theta} + d\phi\frac{\partial}{\partial \phi}](x,y,z)^T$$

$$d(x,y,z) = \left[\frac{dr}{\|\nabla r\|}\|\nabla r\|\frac{\partial}{\partial r} + \frac{d\theta}{\|\nabla \theta\|}\|\nabla \theta\|\frac{\partial}{\partial \theta} + \frac{d\phi}{\|\nabla \phi\|}\|\nabla \phi\|\frac{\partial}{\partial \phi}\right](x,y,z)$$

$$(dx,dy,dz) = dr\,\hat{r}^T + r\,d\theta\,\hat{\theta}^T + r\sin\theta\,d\phi\,\hat{\phi}^T$$

$$\begin{aligned}(\hat{r},\hat{\theta},\hat{\phi}) &\equiv \underline{\left(\|\nabla r\|\frac{\partial\vec{r}}{\partial r},\ \|\nabla \theta\|\frac{\partial\vec{r}}{\partial \theta},\ \|\nabla \phi\|\frac{\partial\vec{r}}{\partial \phi}\right)}\\ &= \left(\frac{\partial}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial \theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\otimes(x,y,z)^T\\ &= \bar{\nabla}_\circ^T\otimes\vec{r}\end{aligned}$$

$$\boxed{d\vec{l} = (dx,dy,dz)\cdot(\hat{x},\hat{y},\hat{z})} = \boxed{(dr, rd\theta, r\sin\theta d\theta)\cdot(\hat{r},\hat{\theta},\hat{\phi}) = d\vec{l}_\circ} = d\bar{l}_\circ^{\,T}\cdot(\hat{r},\hat{\theta},\hat{\phi})$$

$$\boxed{\vec{\nabla} = (\hat{x},\hat{y},\hat{z})\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)} = \boxed{(\hat{r},\hat{\theta},\hat{\phi})\cdot\left(\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial \theta},\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) = \vec{\nabla}_\circ} = \left(\frac{\partial\vec{r}}{\partial r},\frac{1}{r}\frac{\partial\vec{r}}{\partial \theta},\frac{1}{r\sin\theta}\frac{\partial\vec{r}}{\partial \phi}\right)\bar{\nabla}_\circ^T$$

$$\vec{\nabla} = [\bar{\nabla}_\circ^T\otimes\vec{r}]\bar{\nabla}_\circ = \left[\bar{\nabla}_\circ^T\otimes(x,y,z)^T\right]\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial \theta}\\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\end{bmatrix} \quad \Rightarrow \quad \frac{\partial}{\partial x} = \underline{\frac{\partial x}{\partial r}}\frac{\partial}{\partial r} + \underline{\|\nabla\theta\|^2\frac{\partial x}{\partial \theta}}\frac{\partial}{\partial \theta} + \underline{\|\nabla\phi\|^2\frac{\partial x}{\partial \phi}}\frac{\partial}{\partial \phi}$$

$$= [\vec{\nabla}(r,\theta,\phi)]\bar{\partial}_\circ = [\vec{\nabla}(r,\theta,\phi)]\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\\ \frac{\partial}{\partial \phi}\end{bmatrix} \quad \Rightarrow \quad \frac{\partial}{\partial x} = \underline{\frac{\partial r}{\partial x}}\frac{\partial}{\partial r} + \underline{\frac{\partial \theta}{\partial x}}\frac{\partial}{\partial \theta} + \underline{\frac{\partial \phi}{\partial x}}\frac{\partial}{\partial \phi} \quad \Rightarrow \quad \boxed{\frac{\partial\phi}{\partial y} = \frac{\partial y}{\partial \phi}\|\nabla\phi\|^2}$$

$$\begin{array}{ccccccccccccc} & & & & & & \underline{\text{contravariant}}_i & & \text{(equal since orthog.)} & & \underline{\text{covariant}}^i & & \\ \hat{r} & = & (\hat{r}_x,\hat{r}_y,\hat{r}_z) & = & \frac{\vec{r}}{r} & = \frac{\partial}{\partial r}\vec{r} & = \frac{\partial\vec{r}}{\partial r}\left\|\frac{\partial\vec{r}}{\partial r}\right\|^{-1} & \overset{\rightarrow}{=} & \|\nabla r\|\frac{\partial\vec{r}}{\partial r} & \overset{\leftarrow}{=} & \frac{\nabla r}{\|\nabla r\|} & = & \nabla r\\ \hat{\theta} & = & (\hat{\theta}_x,\hat{\theta}_y,\hat{\theta}_z) & = & \frac{\partial\hat{r}}{\partial\theta} & = \frac{1}{r}\frac{\partial}{\partial \theta}\vec{r} & = \frac{\partial\vec{r}}{\partial \theta}\left\|\frac{\partial\vec{r}}{\partial \theta}\right\|^{-1} & \overset{\rightarrow}{=} & \|\nabla \theta\|\frac{\partial\vec{r}}{\partial \theta} & \overset{\leftarrow}{=} & \frac{\nabla \theta}{\|\nabla \theta\|} & = & r\nabla\theta\\ \hat{\phi} & = & (\hat{\phi}_x,\hat{\phi}_y,\hat{\phi}_z) & = & \frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi} & = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\vec{r} & = \frac{\partial\vec{r}}{\partial \phi}\left\|\frac{\partial\vec{r}}{\partial \phi}\right\|^{-1} & \overset{\rightarrow}{=} & \|\nabla \phi\|\frac{\partial\vec{r}}{\partial \phi} & \overset{\leftarrow}{=} & \frac{\nabla \phi}{\|\nabla \phi\|} & = & r\sin\theta\nabla\phi \end{array}$$

Fundamental Theorem:

$$\begin{aligned}
f\circ r(t_f) - f\circ r(t_i) &= \sum \int_{t_i}^{t_f} \frac{dr^i}{dt}\frac{\partial f}{\partial r^i}\,dt &&= \underline{\int_{t_i}^{t_f} \frac{dx}{dt}\frac{\partial f}{\partial x}\,dt + \int_{t_i}^{t_f}\frac{dy}{dt}\frac{\partial f}{\partial y}\,dt} \\
\boxed{\int_\gamma \vec{\nabla} f\cdot d\vec{r}} &= \sum \int_{r^i_i}^{r^i_f} dr^i\,\frac{\partial f}{\partial r^i} &&= \underline{\int_{x_i}^{x_f} dx\,\frac{\partial f}{\partial x} + \int_{y_i}^{y_f} dy\,\frac{\partial f}{\partial y}} \\
f(x_f,y_f) - f(x_i,y_i) &= \underline{\sum \Delta_{r^i}\frac{\partial f}{\partial r^i}\big|_{c_i}} &&= (x_f-x_i)\int_{x_i}^{x_f}\frac{dx}{x_f-x_i}\frac{\partial f}{\partial x} + (y_f-y_i)\int_{y_i}^{y_f}\frac{dy}{y_f-y_i}\frac{\partial f}{\partial y} \\
\begin{pmatrix} x(t,x_f,x_i) = x_i + t(x_f-x_i) \\ y(t,x_f,x_i) = y_i + t(y_f-y_i)\end{pmatrix} & &&= \underline{(x_f-x_i)\int_0^1 dt\,\frac{\partial f}{\partial x}\big|_{r(t,x_f,x_i)} + (y_f-y_i)\int_0^1 dt\,\frac{\partial f}{\partial y}\big|_{r(t,x_f,x_i)}}
\end{aligned}$$

$$\text{Partial}:\ \underline{\frac{\partial f}{\partial x}(x_i,y_i)} = 0 + \underline{\int_0^1 dt\,\frac{\partial f}{\partial x}\big|_{r_i}} + 0\int_0^1 dt\,\frac{\partial^2 f}{\partial^2 x}\big|_{r_i} + 0\int_0^1 dt\,\frac{\partial^2 f}{\partial y\partial x}\big|_{r_i}$$

$$\begin{aligned}
Y^i{}_j &= X^i{}_k X^k{}_l X^l{}_j \\
\frac{\partial Y^i{}_j}{\partial X^a{}_b} &= \delta^i{}_a e^b X X e_j + e^i X e_a e^b X e_j + e^i X X e_a \delta^b{}_j \\
\frac{\partial Y}{\partial X^a{}_b} &= \mathbb{1} e_a e^b X X + X e_a e^b X + X X e_a e^b \mathbb{1} \\
&= \left[(XX)^T(\mathbb{1} e_a e^b)^T + X^T(X e_a e^b)^T + \mathbb{1}^T(XX e_a e^b)^T\right]^T \\
dY &= \left[\mathbb{1} e_a e^b XX + X e_a e^b X + XX e_a e^b \mathbb{1}\right] dX^a{}_b \\
dY &= \mathbb{1}(dX)XX + X(dX)X + XX(dX)\mathbb{1}
\end{aligned}$$

$$\begin{aligned}
\mathrm{vec}\left(\frac{\partial Y}{\partial X^a{}_b}\right) &= \left[(XX)^T\otimes \mathbb{1} + X^T\otimes X + \mathbb{1}^T\otimes (XX)\right]\mathrm{vec}(e_a e^b) \\
\mathrm{vec}(dY) &= \left[(XX)^T\otimes \mathbb{1} + X^T\otimes X + \mathbb{1}^T\otimes (XX)\right]\mathrm{vec}(dX) = d\mathrm{vec}(Y)
\end{aligned}$$

$$\frac{\partial Y}{\partial X} \equiv \frac{\partial \mathrm{vec(Y)}}{\partial \mathrm{vec(X)}} = (XX)^T\otimes \mathbb{1} + X^T\otimes X + \mathbb{1}^T\otimes (XX)$$

$$\begin{aligned}
\mathrm{vec}(D) &= \mathrm{vec}(ABC) = (C^T\otimes A)\mathrm{vec}(B) \\
ABC_j &= (C^k{}_j A)B_k = (C_j^T\otimes A)\mathrm{vec}(B) = D_j \\
e^i ABC_j &= (C^k{}_j A^i)B_k = (C_j^T\otimes A^i)\mathrm{vec}(B) = D^i{}_j
\end{aligned}$$

$$\underline{\text{Dual Space}}:\quad V:\ B = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix},\quad \underline{\mathrm{Hom}(V,\mathbb{R}) = V^*}:\ B^* = \begin{bmatrix} -v^1- \\ -v^2- \end{bmatrix} \ \to\ B^*B = \mathbb{1}_2$$

- $\underline{\text{Adjoint of Linear Maps}},\ F^* \ \Big|\ \begin{matrix} F: V\to W \\ Fv = w\end{matrix},\quad \begin{matrix} F^*: W^*\to V^* \\ F^*(a)\in V^*\end{matrix},\quad \begin{aligned} \underline{F^*(a)\cdot v} &\equiv a\cdot w \\ &= a^T F v \\ \underline{(F^T a)\cdot v} &= (F^T a)^T v\end{aligned} \quad \Rightarrow\ \boxed{F^* = F^T}$

- $\dfrac{\underline{f: V\otimes W^* \to\ \mathrm{Hom}(V,W)}}{\text{(related to } (V^*)^* = \mathrm{Hom}(V^*,\mathbb{R}))} \ \Big|\ \begin{matrix} B: W\to V \\ a\in W^* \\ B = f(v\otimes a)\end{matrix},\quad \begin{aligned} f(v\otimes a)w &= (va^T)w \\ &= v(a\cdot w) \\ Bw &= (a\cdot w)v\end{aligned} \qquad \begin{matrix} *\ \ W = V\ \Rightarrow\ Bv = (a\cdot v)v \\ \Rightarrow\ \boxed{f: V\otimes V^*\to\mathbb{R}}\end{matrix}$

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$$\nabla F = \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{bmatrix} F = \begin{bmatrix} \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z} \\ \cos\phi\cos\theta\,\hat{x} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z} \\ -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{bmatrix} F$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F = \begin{bmatrix} \cos\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \end{bmatrix} F \;=\; \begin{bmatrix} \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} \\ \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi} \\ \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi} \end{bmatrix} F \;=\; \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F$$

$$\nabla \cdot \vec{A} \;=\; \frac{1}{r}\frac{1}{r\sin\theta}\left\langle \frac{\partial}{\partial r},\ \frac{\partial}{\partial\theta},\ \frac{\partial}{\partial\phi} \right\rangle \;\cdot\; [r \cdot r\sin\theta]\left\langle A_r,\ \frac{1}{r}A_\theta,\ \frac{1}{r\sin\theta}A_\phi \right\rangle$$

$$\nabla \times \vec{A} \;=\; \frac{1}{r}\frac{1}{r\sin\theta}\begin{Vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{Vmatrix} = \begin{Vmatrix} \frac{\partial\vec{r}}{\partial r} & \frac{\partial\vec{r}}{\partial\theta} & \frac{\partial\vec{r}}{\partial\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{Vmatrix}$$

$$\nabla f = df$$

$$A \cdot B = {}^*\left[(A_i\,dx^i) \wedge {}^*(B_j\,dx^j)\right]$$

$$\nabla \cdot A = {}^*d^*(A_i\,dx^i)$$

$$\nabla \times A = 2^*d(A_i\,dx^i)$$

$$\begin{aligned}
[\vec{A}\times(\vec{B}\times\vec{C})]_i &= \vec{A}\cdot B_i\vec{C} - \vec{A}\cdot\vec{B}C_i \\
[\vec{A}\times(\vec{B}\times\vec{C})]^T &= \vec{A}_r * \vec{B}_r\vec{C}_c - \vec{A}_r * \vec{B}_c\vec{C}_r \\
\vec{A}\times(\vec{B}\times\vec{C}) &= \boxed{(A_rB_c)C_c - (A_r * B_c)C_c} \\
&= \underline{(A\odot B)C - (A\cdot B)C} \\
\text{(A, B commute)} &= B(A\cdot C) - (A\cdot B)C \\
\vec{\nabla}\times(\vec{\nabla}\times\vec{C}) &= \nabla(\nabla\cdot C) - (\nabla\cdot\nabla)C \\
[\vec{A}\times(\vec{\nabla}\times\vec{C})]^T &= \vec{A}_r(\vec{\nabla}_r\vec{C}_c) - (\vec{A}\cdot\vec{\nabla})\vec{C}^T \\
[\vec{\nabla}\times(\vec{B}\times\vec{C})]_i &= \vec{\nabla}\cdot B_i\vec{C} - \vec{\nabla}\cdot\vec{B}C_i &&= \vec{C}\cdot\vec{\nabla}B_i + \vec{\nabla}\cdot\vec{C}B_i - \vec{\nabla}\cdot\vec{B}C_i - \vec{B}\cdot\vec{\nabla}C_i \\
[\vec{\nabla}\times(\vec{B}\times\vec{C})]^T &= \vec{\nabla}_r * (\vec{B}_r\vec{C}_c) - \vec{\nabla}_r * (\vec{B}_c\vec{C}_r) &&= \vec{C}_r * \vec{\nabla}_c\vec{B}_r + \vec{\nabla}_r * \vec{C}_c\vec{B}_r - \vec{\nabla}_r * \vec{B}_c\vec{C}_r - \vec{B}_r * \vec{\nabla}_c\vec{C}_r
\end{aligned}$$

$$\begin{aligned}
\vec{A}\times(\vec{B}\times\vec{C}) &= \begin{bmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \\
&= \begin{bmatrix} A_1B_1 & A_2B_1 & A_3B_1 \\ A_1B_2 & A_2B_2 & A_3B_2 \\ A_1B_3 & A_2B_3 & A_3B_3 \end{bmatrix}\vec{C} - (A\cdot B)\mathbb{1}_3\vec{C} \\
&= (A_rB_c)C_c - (A_r * B_c)C_c \\
&= \underline{(A^T{}_r{\otimes}_cB)C - (A^TB)C} = \underline{(A\odot B)C - (A\cdot B)C}
\end{aligned}$$

$$\begin{aligned}
[(\vec{A}\times\vec{B})\times\vec{C}\,]_i &= \vec{A}B_i\cdot\vec{C} - A_i\vec{B}\cdot\vec{C} \\
(\vec{A}\times\vec{B})\times\vec{C} &= \boxed{(\vec{A}_r\vec{B}_c)\vec{C}_c - \vec{A}_c\vec{B}_r * \vec{C}_c} \\
&= \underline{(A^T{}_r{\otimes}_cB)C - (AB^T)C} \\
&= \underline{(A\odot B)C - A(B\cdot C)} \\
\text{(B, C commute)} &= (A\cdot C)B - A(B\cdot C) \\
(\vec{A}\times\vec{\nabla})\times\vec{C} &= (\vec{A}_r\vec{\nabla}_c)\vec{C}_c - \vec{A}_c(\vec{\nabla}_r\cdot\vec{C}_c) \\
(\vec{\nabla}\times\vec{B})\times\vec{C} &= (\vec{\nabla}_r\vec{B}_c)\vec{C} - (\vec{\nabla}_c\vec{B}_r)\vec{C}
\end{aligned}$$

$$\begin{aligned}
[(\vec{A}\times\vec{B})\times\vec{C}\,]^T &= [A_1\ A_2\ A_3]\begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -C_3 & C_2 \\ C_3 & 0 & -C_1 \\ -C_2 & C_1 & 0 \end{bmatrix} \\
&= A^T\begin{bmatrix} B_1C_1 & B_2C_1 & B_3C_1 \\ B_1C_2 & B_2C_2 & B_3C_2 \\ B_1C_3 & B_2C_3 & B_3C_3 \end{bmatrix} - A^T\mathbb{1}_3(B\cdot C) \\
&= A_r(B_rC_c) - A_r(B_r * C_c) \\
&= A^T(B\odot C) - A^T(B\cdot C)
\end{aligned}$$

Orthogonal Coord. Change:

$$\vec{v} = v^i \frac{\partial}{\partial x^i} = v'^i \frac{\partial}{\partial x'^i} \qquad e^n = e^n{}_i dx^i \qquad \vec{v} = v^i \frac{\partial}{\partial x^i} = v^i \delta_i^k \frac{\partial}{\partial x^k} = \left(v^i \frac{\partial x'^j}{\partial x^i} \right) \left(\frac{\partial x^k}{\partial x'^j} \frac{\partial}{\partial x^k} \right) \equiv v'^j e'_j$$

$$v^i e_i = v'^i e'_i \qquad e_n = e_n{}^i \tfrac{\partial}{\partial x^i}$$

- $e'_j = \dfrac{\partial x^i}{\partial x'^j} e_i$
- $v'^j = \dfrac{\partial x'^j}{\partial x^i} v^i$, $dx'^j = \dfrac{\partial x'^j}{\partial x^i} dx^i$

$$\delta_i^k = g_{ij} g^{jk} = \delta_k^i = g^{ij} g_{jk}$$

$$g'_{ij} = \frac{\partial x^m}{\partial y_i} \frac{\partial x^n}{\partial y_j} g_{mn} \qquad \tfrac{\partial x^k}{\partial y^i} \tfrac{\partial y^i}{\partial x^i} = \tfrac{\partial x^m}{\partial y_i} \tfrac{\partial x^n}{\partial y_j} \eta_{mn} \cdot \tfrac{\partial y^j}{\partial x^p} \tfrac{\partial y^k}{\partial x^q} \eta^{pq}$$

$$g'^{ij} = \frac{\partial y_i}{\partial x^m} \frac{\partial y_j}{\partial x^n} g^{mn} \qquad = \tfrac{\partial x^m}{\partial y_i} \eta_{mn} \cdot \tfrac{\partial y^k}{\partial x^q} \eta^{qn}$$

$$= \tfrac{\partial x^m}{\partial y_i} \eta_{mn} \cdot \tfrac{\partial y^k}{\partial x^m} \eta^{mn}$$

2 Frenet Equations

$$a\cdot(b\times c) = (a\times b)\cdot c$$

$$a\times(b\times c) = (c\cdot a)b - (b\cdot a)c$$

$$(a\times b)\times c = b(c\cdot a) - a(c\cdot b)$$

$$(a\times b)\cdot(c\times d) = a\cdot b\times(c\times d)$$

$$= \left|\begin{bmatrix} a\cdot \\ b\cdot \end{bmatrix}[c\ d]\right| = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

$$\boxed{\frac{dt}{ds} = \frac{1}{v}}$$

$$T = \hat{v} = \frac{\vec{v}}{v}$$

$$\frac{dT}{dt} = \frac{(\vec{v}\cdot\vec{v})\vec{a} - (\vec{v}\cdot\vec{a})\vec{v}}{v^3} = \frac{\vec{v}\times(\vec{a}\times\vec{v})}{v^3} = \frac{(\vec{v}\times\vec{a})\times\vec{v}}{v^3}$$

$$\left\|\frac{dT}{dt}\right\| = \frac{\sqrt{v^2a^2-(\vec{v}\cdot\vec{a})^2}}{v^2} = \frac{\|\vec{a}\times\vec{v}\|}{v^2}\ , \quad \frac{dT}{ds} = k\hat{N}$$

$$\hat{N} = \frac{T'}{\|T'\|} = \frac{(\vec{v}\times\vec{a})\times\vec{v}}{\|\vec{v}\times\vec{a}\|v} = \hat{B}\times\hat{v}$$

$$\hat{B} = \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} = \widehat{v\times a} = \hat{v}\times\hat{N} \qquad \underline{(\hat{B}\cdot\vec{v}=0)}$$

$$\frac{d\hat{B}}{dt} = \frac{\vec{v}\times\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|} - \left[\frac{\vec{v}\times\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|}\cdot\hat{B}\right]\hat{B}\ , \quad \frac{dB}{ds} = \tau\hat{N}$$

$$\tau = \hat{N}\cdot\frac{d\hat{B}}{ds} = \frac{\hat{B}\cdot\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|} = \frac{(\vec{v}\times\vec{a})\cdot\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|^2}$$

$$\boxed{\begin{aligned} \vec{a} &= a_T\hat{T} + a_N\hat{N} \\ a_T &= \vec{a}\cdot\hat{v} = \frac{dv}{dt} \\ a_N &= \frac{\|\vec{a}\times\vec{v}\|}{v} = \|\vec{a}\times\hat{v}\| \\ a^2 &= a_T^2 + a_N^2 = \left\|\frac{d\vec{v}}{dt}\right\|^2 \end{aligned}}$$

Frenet Trihedron for Regular *Parametrized* Curves

Differentiable (in this book) : C^∞

No singular pts. Order 0 (Regular) : $\vec{v}(t) \neq 0$

- $\|\vec{v}(t)\| = c \to 1 \ \Rightarrow \int_s \|\vec{v}(t)\|\,dt = t = \Delta s$

  $\to s:\ \vec{x}(t) = \vec{x}(s)$
- $\frac{1}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \boxed{\vec{v}\cdot\vec{a} = 0}$

No singular pts. Order 1 : $\vec{a}(t) \neq 0$

- Curvature, $k\neq 0$ (see right) • Vertex, $k' = 0$

$1 = \|\vec{t}\| = \|\vec{n}\| = \|\vec{b}\|\ , \quad 0 = \vec{t}\cdot\vec{n} = \vec{n}\cdot\vec{b} = \vec{b}\cdot\vec{t}$

- $\vec{v}(s) = \vec{t}(s)$ $\quad\boxed{(t = n\times b)}$
- $\vec{a}(s) = \boxed{\vec{t}'(s) = k(s)\vec{n}(s)}$, $k(s)\geq 0$ (can be L or R-handed) (can be neg. if in $\mathbb{R}^2$)

$*$ $k(s) > 0$ for well defined curve with $\hat{n}$

- $\boxed{\vec{b} = \vec{t}\times\vec{n}}$, $\frac{d}{dt}(\vec{b}\cdot\vec{b}) = \vec{b}\cdot\vec{b}' = 0$, $*$ $\boxed{\vec{b}'(s) = \tau(s)\vec{n}(s)}$
- $\boxed{\vec{n} = \vec{b}\times\vec{t}}$, $*$ $\boxed{\vec{n}'(s) = -k\vec{t} - \tau\vec{b}}$, $*$ t-n pl. = osculating pl.

- $t''(s) = k'n - k^2t - k\tau b$ • $b''(s) = \tau' n - \tau k t - \tau^2 b$ • $n''(s) = -k't - \tau' b - (k^2+\tau^2)n$
- $|\tau| = \|b'\|$ • $\tau = -\frac{(t\times t')\cdot t''}{k^2} = -\frac{t\cdot(t'\times t'')}{\|t'\|^2}$ • $k = \|t'\| = \frac{(b\times b')\cdot b''}{\tau^2} = \frac{b\cdot(b'\times b'')}{\|b'\|^2}$
- $n \Rightarrow k,\tau:$ $\quad *\ \|n'\|^2 = k^2+\tau^2$ $\quad *\ \frac{(n\times n')\cdot n''}{\|n'\|^2} = \frac{k'\tau - k\tau'}{k^2+\tau^2} = \frac{\frac{d}{ds}(k/\tau)}{(k/\tau)^2+1} = \frac{d}{ds}\arctan(k/\tau)$

Indicatrix of Tangents, $\vec{t}(\theta(s))$:

- $\vec{t}(\theta(s)) = (\cos\theta, \sin\theta) = (x'(s), y'(s))$
- $\vec{t}'(\theta) = \underline{\theta'(s)}(-\sin\theta, \cos\theta) = \underline{k(s)}\vec{n}$
- $\theta(s) = \arctan(y'/x')$
- $\int_0^l k(s)\,ds = \theta(s)\Big|_0^l = 2\pi I_{\text{rot. index}}$
- $k(s) = \lim_{s\to 0}\frac{r\theta(s)}{s}\Big|_{r=1}$ (See Gaussian K)

Local Canonical Form at $t = 0$:

- $(\hat{t},\hat{n},\hat{b}) = (\hat{x},\hat{y},\hat{z})$
- $\vec{r}(s) - \vec{r}(0) \approx (s - \frac{k^2s^3}{6},\ \frac{k}{2}s^2 + \frac{k's^3}{6},\ \frac{-k\tau}{6}s^3)$
- $\tau < 0 \ \Rightarrow\ \frac{dz}{ds} > 0$

Isoperimetric Inequality : $0 \leq l^2 - 4\pi A$

Four-Vertex Theorem : A simple closed curve has ≥ 4 vertices

Cauchy-Crofton Formula (measure of number of times lines intersect a curve) :

- Tangent line at (ρ,θ) : $x\cos\theta + y\sin\theta = \rho$ • Curve c : $y = 0,\ x\in(-l/2, l/2)$, $C = \sum c_i$
- $\int$ Lines that cross $c = \int_0^{2\pi}\int_0^{|\cos\theta|l/2} d\rho\, d\theta = 2l \ \Rightarrow\ \int_0^{2\pi}\int_0^{\infty} n_C\ d\rho\, d\theta = 2l$

3 Jacobian/Differential, $dF_{\alpha(0)} : \underline{\mathbb{R}^n \to \mathbb{R}^m}$

- $\boxed{\alpha(0) = \beta(0)} \;\Rightarrow\; \underline{F(t=0)} = F \circ \alpha\big|_{t=0} = F \circ \beta\big|_{t=0}$
- $\boxed{\alpha'(0) = \beta'(0)} \;\Rightarrow\; \frac{\partial x}{\partial \alpha_i}\Big|_{t=0} = \frac{\partial x}{\partial \beta_i}\Big|_{t=0} \cdot \cancel{\frac{d\beta_i/dt}{d\alpha_i/dt}\Big|_{t=0}} \;\Rightarrow\; \boxed{dF_{\alpha(0)}(\alpha'(0)) = dF_{\beta(0)}(\beta'(0))}$ (doesn't depend on α)

$*\; F = (f_0, f_1, \ldots, f_m) \;\Rightarrow\; \underline{dF_{\alpha(0)}(\alpha'(0))} \equiv \frac{d}{dt}(F \circ \alpha)\big|_{t=0} = \begin{bmatrix} \frac{\partial f_0}{\partial \alpha_0} & | & \cdots \\ \frac{\partial f_1}{\partial \alpha_0} & F_{\alpha_1} & \cdots \\ \vdots & | & \end{bmatrix}_{t=0} \begin{bmatrix} \frac{d\alpha_0}{dt} \\ \frac{d\alpha_1}{dt} \\ \vdots \end{bmatrix}_{t=0} = \boxed{J_{F(0)} \cdot \alpha'(0)}$

$*$ Surface Tangent (see below): $q = \gamma(t=0) = \big(u(0), v(0)\big) = X^{-1} \circ \alpha(0)$
$X(q) = X \circ \gamma(0) = \alpha(0) \in S \;\Rightarrow\; dX_q(\gamma'(0)) = \alpha'(0)$

- $d(G \circ F)_p = dG_{F(p)} \circ dF_p$
- $\underline{\text{Regular } \textbf{Value}, F(p)}$: onto $dF_{\forall p}$/Full Rank
  $*\; F : R^n \to R \;\Rightarrow\; dF_p \neq 0$
- $\underline{\text{Critical } \textbf{Point}, p}$: !onto dF_p

F is a $\underline{\text{Homeomorphism}}$ onto image $F(X)$:
- F is bijective between X & $F(X)$
- F is cont. • F^{-1} is cont.

F is a $\underline{\text{Diffeomorphism}}$ onto image $F(X)$:
- $F \in C^\infty$ (cont. part. deri. of all orders)
- $F^{-1} \in C^\infty$ • F is a bijection

$\Rightarrow F^{-1} \circ F = 1 \Rightarrow dF^{-1}dF = 1$ ($\exists$ left-inv)
$\Rightarrow F \circ F^{-1} = 1 \Rightarrow dF\,dF^{-1} = 1$ ($\exists$ right-inv)

$\underline{\text{Inverse Function}}$ $\underline{\text{Theorem}}$ (IFT) :
- $F : \underline{\mathbb{R}^n \to \mathbb{R}^n}$, $F \in C^\infty$
- $\exists dF_p^{-1}$ (sq. matrix dF_p is an isomorphism/non-zero det.)

$\Rightarrow \exists F^{-1} \in C^\infty$ (locally at F(p))

4 Surfaces, S : $X(q) = X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big) = p \in S \subset \mathbb{R}^3$

Regular Parametrized Surface

- $\forall p \in S$, $\underline{\exists X \in C^\infty}$, $X : V_q$ (neighborhood of q) $\to V_p \cap S$ (diff. parametrizations are possible, btw)
- dX_q is one-to-one = (maybe non sq.) matrix col. are lin. ind. = any 2x2 $|\text{sub-}J_X| \neq 0 \;\Rightarrow\; \exists$(tangent at all points)

Regular Surface (is reg. param. surface)

- X is a homeo. in V_q (or X is one-to-one) $\to$ $\underline{X^{-1} \in C^0}$ (is cont.) $\forall p \in S,\ X^{-1}(V_p) = V_q$ $\Rightarrow$ $\exists$ no self-intersections; cont. = doesn't depend on parametrization (see coor. change below)

- Coordinate Change, h, between Two Param. $\underline{\text{is a Diffeomorphism (need for diff. func. on } S)}$:

$*\; X^{-1}$ is a homeomorphism $\to$ $\underline{h = X^{-1} \circ Y \text{ is is a homeomorphism from } Y \text{ to } X}$ $\Rightarrow\Rightarrow$ $\underline{h^{-1} \text{ is a homomorphism}}$

$*\; p \in S$, $p = Y(\epsilon, \eta) = X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$, $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ (can change axes to make this true)

$F(u,v,t) = \big(x(u,v), y(u,v), z(u,v) + t\big)$: $F(u,v,t)$, $X(u,v) \in C^\infty$, $\exists dF^{-1} \overset{(IFT)}{\Rightarrow} F^{-1} \in C^\infty$

$(u,v) = X^{-1} \circ Y(\epsilon, \eta) = h(\epsilon, \eta) \overset{\simeq}{=} \big(F^{-1} \circ Y\big)(\epsilon, \eta) \;\Rightarrow\; \underline{h \in C^\infty} \;\Rightarrow\Rightarrow\; \underline{h^{-1} \in C^\infty}$ (same for $Y^{-1} \circ X$)

$*$ Needed that $X^{-1} \in C^0$ **on a [3D] neigh. for every point** $[\forall p \in S,\ X^{-1}(V_p) = V_q \overset{\simeq}{=} F^{-1}(V_p)]$, to avoid $(t \neq 0,\ F^{-1} \circ Y \neq h)$

$*$ Ex: $\gamma(t) = (\cos t, \sin 2t)$, $\gamma(\mathbb{R}) = \alpha(I_1) = \beta(I_2)$, (∞ - graph $\underline{\text{not reg.}}$)

$I_1 = (-\frac{\pi}{2}, \frac{3\pi}{2}) = (-\frac{\pi}{2}, \frac{\pi}{2}) \cup \frac{\pi}{2} \cup (\frac{\pi}{2}, \frac{3\pi}{2})$
$I_2 = (\frac{\pi}{2}, \frac{5\pi}{2}) = \underline{(\frac{3\pi}{2}, \frac{5\pi}{2}) \cup \frac{3\pi}{2} \cup (\frac{\pi}{2}, \frac{3\pi}{2})}$

$\Rightarrow$ β^{-1} is 1:1 but not cont. for any **[2D] neigh.** of $(0,0)$
$F^{-1}(x,y) = (t', u) \neq \beta^{-1}(x,y) \overset{\simeq}{=} (t, 0)$ near $(0,0)$
$\underline{\beta^{-1} \circ \alpha(I_1)}$ is 1:1 but not cont., so not diffeo.

- $\underline{f \in C^\infty} \;\Rightarrow\; \boxed{\big(\vec{x}, f(\vec{x})\big) \text{ is a reg. surf.}}$

- $\begin{array}{l} f : \mathbb{R}^n \to \mathbb{R} \\ f(X) = c \\ \underline{\text{is a reg. val.}} \end{array}$, $\begin{array}{l} f \in C^\infty \\ F(X) = \big(x_1, \dots, x_{n-1}, f(X)\big) \\ \exists dF_p^{-1} \end{array} \overset{\text{(IFT)}}{\Rightarrow} \begin{array}{l} \exists F^{-1} \in C^\infty \\ F^{-1}\big(f_1, \dots, f_{n-1}, f(\vec{x})\big) = X \end{array}$, $\begin{array}{l} x_n = f_n^{-1} : \mathbb{R}^n \to \mathbb{R} \\ \underline{x_n = f_n^{-1} \in C^\infty} \end{array}$

$\to \begin{array}{l} x_n = f_n^{-1}\big(x_1, \dots, x_{n-1}, f(\vec{x}) = c\big) \\ \quad = \underline{f'^{-1}_n\big(x_1, \dots, x_{n-1}\big)} \end{array} \Rightarrow \begin{array}{l} S = \underline{\big(x_1, \dots, x_{n-1}, f'^{-1}_n\big)} \text{ where } f(\vec{x}) = c \\ S == \text{ Surface } f^{-1}(c) \end{array} \Rightarrow \boxed{\begin{array}{c} \underline{\text{Regular Value Theorem}} \\ \text{Surface } f^{-1}(c) \text{ is reg.} \end{array}}$

- $\underline{\frac{\partial(x,y)}{\partial(u,v)} \neq 0} \;\Rightarrow\; \pi_{\text{proj.}} \circ X(u,v) \equiv \big(x(u,v), y(u,v), \cancel{z(x,y)}\big) \overset{(IFT)}{\Rightarrow} \big(\pi \circ X\big)^{-1}(x,y) = \big(u(x,y), v(x,y)\big)$

$*\; X(u,v) = \big(x(u,v), y(u,v), \underline{z(u,v)}\big) \;\Rightarrow\; z\big(u(x,y), v(x,y)\big) = z \circ \big(\pi \circ X\big)^{-1}(x,y) = \boxed{\begin{array}{c} \underline{\text{Implicit Func. Theor.}} \\ \text{(locally orientable)} \\ f(x,y) = z \in C^\infty \end{array}}$

$*\; \dfrac{\text{Know } S \text{ is reg. sur.}}{X \text{ is param?}}, \begin{array}{l} X \in C^\infty \\ dX_q \text{ is 1:1} \end{array}, \underline{X \text{ is 1:1}} \;\Rightarrow\; \underline{\big(\pi \circ X\big)^{-1} \circ \pi} \circ X(u,v) = \underline{X^{-1}} \circ X(u,v) \;\Rightarrow\; \boxed{X^{-1} \in C^0}$

- $\begin{array}{l} \underline{\text{Surface}} \\ \underline{\text{Tangent}} \end{array}$: $\begin{array}{l} q = \gamma(t=0) = \big(u(0), v(0)\big) = X^{-1} \circ \alpha(0) \\ X(q) = X \circ \gamma(0) = \alpha(0) \in S \;\Rightarrow\; dX_q\big(\gamma'(0)\big) = \alpha'(0) = \frac{\partial X}{\partial u}(q)\, u'(0) + \frac{\partial X}{\partial v}(q)\, v'(0) \end{array}$

1st Fund. Form : $\langle \alpha'(0), \alpha'(0) \rangle = [u' \; v'] \begin{bmatrix} X_u \\ X_v \end{bmatrix} [X_u \; X_v] \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{array}{c} \|X_u\|^2 (u')^2 + 2\langle X_u, X_v \rangle u'v' + \|X_v\|^2 (v')^2 \\ \boxed{E(u')^2 + 2Fu'v' + G(v')^2} \end{array}$

$\boxed{\begin{array}{l} \bullet\; X_u = \alpha'\binom{u=t}{v=v_0} \\ \bullet\; X_v = \alpha'\binom{u=u_0}{v=t} \end{array}}$ • $\begin{array}{c} \underline{\text{Line}} \\ \underline{\text{Element}} \end{array}$: $\begin{array}{l} ds = \|\alpha'(t)\| dt \\ ds^2 = c_i c_j g_{ij} dx^i dx^j \end{array}$ • $\begin{array}{c} \underline{\text{Area}} \\ \underline{\text{Element}} \end{array}$: $\begin{array}{l} dA = \|X_u \times X_v\| du\, dv \\ = \underset{(\sqrt{1-\cos^2})}{\sqrt{EG - F^2}}\, du\, dv = \underset{\underline{\text{(Volume too!!!)}}}{\sqrt{\det(g_{ij})}\, dx^1 \dots dx^n} \end{array}$

$*$ Regular Curves, $C \in R^3$ (instead of Regular Parametrized Curves)

- $\forall p \in C,\; \exists \alpha \in C^\infty,\; \alpha : I_t$ (neighborhood of t) $\subset R \to V_p \cap C$ (neighborhood of p)
- $\forall t \in I$, $d\alpha_t$ is 1:1 • α is a homeo. in I_t

  $*$ Change of param. are homeomorphisms $\Rightarrow$ Properties like arc length, curvature, torsion, etc. aren't param. dependent

$*$ Coordinate Curves : $\alpha(t) = X \circ \gamma(t) \;\Big|\; \gamma \in \big\{\big(u(t), v_0\big),\; \big(u_0, v(t)\big)\big\}$ (maps of parallels and meridians)

Function, $f : S \subset \mathbb{R}^n \to \mathbb{R}$

- $\big(\forall p \in S,\; \underline{f(p) \neq 0}\big) \;\Rightarrow\; \big(\forall p \in S,\; \underline{f(p) > 0}\big)$ or $\big(\forall p \in S,\; \underline{f(p) < 0}\big)$
- *Differentiable on S* : $f \circ X \in C^\infty$ (doesn't depend on param./coord. change)
- E.g., $X^{-1}(p),\; \vec{v} \cdot p,\; |p - p_0|^2 \;\Rightarrow\; \boxed{X^{-1} \in C^\infty}$, $\boxed{U \text{ is diffeo. to } X(U)}$

Function, $\phi : S_1 \to S_2$ is a Diffeomorphism from S_1 to S_2 • $\boxed{d\phi_p : T_p(S_1) \to T_{\phi p}(S_2)}$

- *Differentiable* : $X_2^{-1} \circ (\phi \circ X_1) \in C^\infty$ (doesn't depend on param./coord. change)
- Differential Map : $\beta'(0) = d\phi_p(w) = d\phi_p \alpha'(0) = d\phi_p dX_q(u'(0), v'(0))^T$ (p.85???)
- Inverse Function Theorem : $\phi \in C^\infty,\; \exists d\phi_p^{-1} \;\Rightarrow\; \phi^{-1} \in C^\infty$ (Diffeomorphism from $S_1 \to S_2$??????)

5 Gauss Map (Normals), $N(p) = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{X_u \times X_v}{EG-F^2} : S \to S^2$

$N'(p) = \underline{dN_p \alpha'(0)} = \begin{bmatrix} (dN_p) \\ N_x\ N_y\ N_z \end{bmatrix} \begin{bmatrix} (dX_q) \\ X_u\ X_v \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \equiv \begin{bmatrix} N_u\ N_v \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \overset{\text{(see below)}}{=} \begin{bmatrix} b_1 & b_2 = aX_u + bX_v \end{bmatrix} \begin{bmatrix} (dN_p) \end{bmatrix} \begin{bmatrix} c_1 = au' + bu' \\ c_2 \end{bmatrix}$

- $\boxed{S = f^{-1}(c) \ \Leftrightarrow \ Orientated}$ = $\underline{\text{normals } N(p) \text{ are in same dir. } (\pm 1)}$ = $\boxed{\exists \frac{\partial(\hat{u},\hat{v})}{\partial(u,v)} > 0 \text{ over all } S}$

2nd Fund. [Quadratic] Form : $\langle -dN_p(\alpha'(0)), \alpha'(0) \rangle = \underline{\langle \alpha'(0), -dN_p(\alpha'(0)) \rangle}$ (self-adjoint=orthog. eig)

$*$ $\langle N(s), \alpha'(s) \rangle = 0 \ \Rightarrow \ \boxed{\langle N(s=0), \alpha''(0) \rangle} = \underset{\text{(depends on } \alpha'(0))}{\underline{-\langle dN_p \alpha'(0), \alpha'(0) \rangle}} = \underset{\text{(Normal Curvature of } \alpha \text{ at } p)}{\boxed{\langle N, kn \rangle (p) \equiv k_n(p)}} =$ k of α from a normal (cross) section of S

$= -\langle N'(0), \alpha'(0) \rangle$

$*$ $\underline{\langle -dN_p \alpha', \alpha' \rangle} = -[N_u\ N_v] \begin{bmatrix} u' \\ v' \end{bmatrix} [u'\ v'] \begin{bmatrix} X_u \\ X_v \end{bmatrix} = \Big(\underbrace{-\langle N_u, X_u \rangle}_{e}, \underbrace{-\langle N_u, X_v \rangle - \langle N_v, X_u \rangle}_{2f = 2\langle N_u, X_v \rangle}, \underbrace{-\langle N_v, X_v \rangle}_{g} \Big) \cdot \big((u')^2, u'v', (v')^2 \big)$

$$\boxed{\begin{aligned} k_n(p, \alpha') &= e(u')^2 + 2fu'v' + g(v')^2 \\ \text{(locally, } \leq 2 \text{ sol.)} &= (Au' + Bv')(Cu' + Dv') \end{aligned}}$$

- $\frac{\text{(Prin. dir. at } p)}{\text{Eigenbasis}} : \exists e_1, e_2 \,\Big|\, \text{span}(e_1, e_2) = T_p(S) \ni \underset{\text{(see below)}}{\underline{-dN_p(c_1 e_1 + c_2 e_2) = k_1 c_1 e_1 + k_2 c_1 e_2}}$ (Prin. curv. at p) (eigenvalues, $k_1 \geq k_2$)

$*$ Euler's Formula (for 2nd Form) : $\underline{\langle -dN_p \vec{t}, \vec{t} = e_1 \cos\theta + e_2 \sin\theta \rangle} = \boxed{k_1 \cos^2\theta + k_2 \sin^2\theta = k_n(p, \theta)}$

- **Gaussian Curvature** : $\boxed{\begin{aligned} K(p) &= \det(dN_p) \\ &= (-k_1)(-k_2) \end{aligned}}$ ($2n$ dim. !Δ det. w/ orien. flip)
- Mean Curvature : $\boxed{H(p) = \frac{-\text{Tr}(dN_p)}{2} = \frac{k_1 + k_2}{2}}$

$*$ Planar: $dN_p = 0$, Ellip.$\to K > 0$, Para.$\to K = 0$, ... $\quad *$ $K > 0 \Rightarrow \exists V_p : p + T_p(S)$!div. V_p , $K < 0 \Rightarrow \forall V_p : p + T_p(S)$ div. V_p

$*$ (2D) $|Tv \times Tw| = |v \times w| k_1 k_2$

$|dN_p X_u \times dN_p X_v| = |X_u \times X_v| \cdot K$

$\boxed{A(N(R)) = \iint_R K d\sigma}$

$$\Rightarrow K_{\neq 0} = \frac{\lim_{\int dudv \to 0} \int |dN_p X_u \times dN_p X_v| dudv \Big/ \int dudv}{\lim_{\int dudv \to 0} \int |X_u \times X_v| dudv \Big/ \int dudv} = \boxed{\lim_{A(R) \to 0} \frac{A(N(R))}{A(R)} = K}$$

(See Indi. of Tan. for k)

- $\underline{N_u, N_v \in T_p(S)} \ \Rightarrow \ dN_p \alpha'(0) = \underline{[N_u\ N_v]} \begin{bmatrix} u' \\ v' \end{bmatrix} \equiv \underline{[X_u\ X_v] \begin{bmatrix} dN \end{bmatrix}} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} (X_u, X_v) \cdot e_1 \\ (X_u, X_v) \cdot e_2 \end{bmatrix}^T \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix} \begin{bmatrix} c_1 = e_1 \cdot (u', v') \\ c_2 \end{bmatrix}$

General Basis for N_u, N_v : $\begin{bmatrix} X_u \cdot N_u = -e & X_u \cdot N_v = -f \\ X_v \cdot N_u = -f & X_v \cdot N_v = -g \end{bmatrix} = \begin{bmatrix} X_u^2 = E & X_u \cdot X_v = F \\ X_v \cdot X_u = F & X_v^2 = G \end{bmatrix} \begin{bmatrix} dN \end{bmatrix}$ $\quad \underline{\begin{aligned} \langle N, X_{ij} \rangle &= -\langle N_i, X_j \rangle \\ &= -\langle N_j, X_i \rangle \end{aligned}}$

$*$ (Weingarten Eq.) $\boxed{\begin{bmatrix} dN \end{bmatrix} = \frac{-1}{EG-F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix}}$ $\quad *$ $(k^2 - 2Hk + K = 0)$ $\boxed{k_\pm = H \pm \sqrt{H^2 - K}}$: $\boxed{K = \frac{eg - f^2}{EG - F^2}}$, $\boxed{H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}}$

Umbilical Point : $p \in S \,\big|\, k_1 = k_2 \ \Rightarrow \ H^2 = K$ (only spheres & planes have all umb. pts.) $\quad$ Asymptotic Direction : $k_n(p, \theta) = 0$

Conjugate Directions : (θ, ϕ) from $e_1 \,\Big|\, \langle -dN_p \hat{t}_1(\theta), \hat{t}_2(\phi) \rangle \equiv \boxed{0 = k_1 \cos\theta \cos\phi + k_2 \sin\theta \sin\phi}$

Dupin Indicatrix : $\langle -dN_p \hat{t}, \hat{t} \rangle = \pm \frac{1}{\rho^2} = k_n \ \Rightarrow \ \langle -dN_p(\rho \hat{t}), (\rho \hat{t}) \rangle = k_1 \underline{\rho^2 \cos^2\theta} + k_2 \underline{|k_n|^{-1} \sin^2\theta}$

- $K > 0 \ \Rightarrow \ \forall \theta, \ k_n(\theta) \neq 0$, $(\xi, \eta) = \text{ellipse}$ $\quad$ Conic Graph (ξ, η) = $\boxed{\underline{\xi}^2 / k_1^{-1} + \underline{\eta}^2 / k_2^{-1} = \pm 1}$
- $K < 0 \ \Rightarrow \ \exists \theta_{\underline{1,2}} \,\big|\, k_n(\theta) = 0 = k_1 \cos^2\theta + k_2 \sin^2\theta$, $(\xi, \eta) = $ hyperbola, $\theta_{\underline{1,2}}$ are asymptotes
- Conj. Dir. (ϕ_1, ϕ_2) : $\phi_{2,1} = \arctan \frac{-k_1 \cos\phi_{1,2}}{k_2 \sin\phi_{1,2}} = \arctan \frac{d\eta}{d\xi}\Big|_{\theta = \phi_{1,2}}$

$\underline{\text{Line of Curvature}}$: $\alpha(t) \mid N'(t) = \underline{dN_p\alpha'(t)} = \underline{\lambda(t)\alpha'(t)}$ (curve s.t. tangent is always in a princ. dir.)

- $\begin{bmatrix} u' & v' \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} dN \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -v' & u' \end{bmatrix}\lambda(t)\begin{bmatrix} u' \\ v' \end{bmatrix} = 0 \overset{\text{(expand)}}{\Rightarrow} \boxed{\begin{vmatrix} (v')^2 & -u'v' & (u')^2 \\ e & f=0 & g \\ E & F=0 & G \end{vmatrix} = 0}$ $(X_u \cdot X_v = 0 \Rightarrow \boxed{F = f = 0})$

$*$ $\underline{\text{Asymp. Curve}}$: $\alpha(t) \mid \lambda(t) = k_n(p,\theta) = \boxed{k_1\cos^2\theta + k_2\sin^2\theta = e(u')^2 + 2fu'v' + g(v')^2 = (Au'+Bv')(Cu'+Dv') = 0}$

$(ef - g),\ \underline{K<0} \Rightarrow \underline{0 = (Au'+Bv')(Au'+Dv')} :\ A^2 = e,\ A(B+D) = f,\ BD = g \Rightarrow \boxed{\exists\alpha_1, \alpha_2}$

$*$ $\underline{K < 0,\ e = g = 0} \Leftrightarrow \boxed{\alpha \circ (c, v(t))} \wedge \boxed{\alpha \circ (u(t), c)}$ are asympt. curves

$\underline{\text{Surface of Revolution}}$: $X(u,v) = \big(\rho(v)\cos u,\ \rho(v)\sin u,\ z(v)\big) \mid \alpha_u(v) = f\big(z(v),\ \rho(v)\big)$, $\|\frac{d\alpha_u}{dv}\| =^* 1$

- $\langle \alpha', \alpha' \rangle = \underbrace{\left[\rho^2,\ 0,\ (\rho')^2 + (z')^2 =^* 1\right]}_{(\rho'\rho'' + z'z'' = 0)^*}\begin{bmatrix} (u')^2 \\ 2u'v' \\ (v')^2 \end{bmatrix}$ • $\langle N, \alpha'' \rangle = \underline{\left[-\rho z',\ 0,\ \rho''z' - \rho'z''\right]}\begin{bmatrix} (u')^2 \\ 2u'v' \\ (v')^2 \end{bmatrix}$

- $\boxed{k_1 = \frac{e}{E} = -\frac{z'}{\rho}}$, $\boxed{k_2 = \frac{g}{G} = \rho''z' - \rho'z''}$, $\boxed{K = -\frac{z'(\rho''z' - \rho'z'')}{\rho} =^* -\frac{\rho''}{\rho}}$

$\underline{\text{Graph of a Differentiable Function}}$: $X(u,v) = \big(u, v, z(u,v)\big)$ • $N(p) = \frac{(-z_u, -z_v, 1)}{\sqrt{z_u^2 + z_v^2 + 1}}$

- $\langle \alpha', \alpha' \rangle = \underline{\left[1 + z_u^2,\ z_u z_v,\ 1 + z_v^2\right]}\begin{bmatrix} (u')^2 \\ 2u'v' \\ (v')^2 \end{bmatrix}$ • $\langle N, \alpha'' \rangle = \underline{\frac{1}{\sqrt{z_y^2 + z_v^2 + 1}}\left[z_{uu},\ z_{uv},\ z_{vv}\right]}\begin{bmatrix} (u')^2 \\ 2u'v' \\ (v')^2 \end{bmatrix}$

- $z(0,0) = p$, $N(p) = (0,0,1) \Rightarrow \underline{\text{Hessian}} :\ k_n(p) = \underline{\left[z_{xx},\ z_{xy},\ z_{yy}\right]}\begin{bmatrix} x^2 \\ 2xy \\ y^2 \end{bmatrix}$, $\vec{v} = (x, y)$

$*$ $\vec{v} = xe_1 + ye_2 \Rightarrow \underset{(p \text{ is non-planer!!})}{z(x,y) - z(0,0)} = \frac{1}{2!}k_n(p) + \mathcal{O}(r^3) \approx \frac{1}{2}(z_{xx}x^2 + z_{yy}y^2) = \epsilon \to k_1\xi^2 + k_2\eta^2 = \pm 1$

$k_1x^2 + k_2y^2 = 2\epsilon$ $\boxed{\text{(Dupin Indicatrix)}}$

$\underline{\text{(Diff., Tangent) Vector Field over } S}$: $\boxed{w(p) = a(u,v)X_u + b(u,v)X_v}$ $\big($e.g. $\underline{\gamma(t) \to w_{\gamma}(p) = u'X_u + v'X_v}\big)$

$\underline{\textit{Trajectory of } w}$: $\alpha(t) \subset S \mid \boxed{\alpha(0) = p,\ \alpha'(t) = w(\alpha(t))}$

$\underline{\textit{(Local) Flow of } w}$: $\alpha(p,t) \equiv \alpha_p(t) \mid \boxed{\alpha_p(0) = p}, \boxed{\alpha'_p(t) = w(\alpha_p(t))} \Rightarrow \boxed{\alpha_p(t) = p + (a_0^1(t), a_0^2(t), a_0^3(t))}$

- $\boxed{w(p_0) \neq 0}$, $w(p_0) \cdot \hat{x} = |w|$, $\tilde{\alpha}_{p_0}(t) = \alpha_{p_0}(t)\big|_{x=x_0}$ $\Rightarrow$ $d\alpha_{p_0} = \begin{bmatrix} \mathbb{1}_3 & w(\alpha) \end{bmatrix}$, $d\tilde{\alpha}_{p_0} = \begin{bmatrix} \mathbb{1}_3 & w(\alpha) \end{bmatrix}\begin{bmatrix} 0 \\ \mathbb{1}_3 \end{bmatrix} = \begin{bmatrix} e_2 & e_3 & w(\alpha) \end{bmatrix}$ $\Rightarrow$ $\det(d\tilde{\alpha}_{p_0}) = w(p_0) \neq 0$, $\boxed{\exists\tilde{\alpha}^{-1} :\ V_{\alpha(p_0)} \subset S \to V_{p_0}\big|_{x=x_0}}$ (IFT), $\forall p \in \alpha_{p_0}(t),\ \underline{g(p)} \equiv \pi_t \circ \tilde{\alpha}_{p_0}^{-1}(p) = p_0$

$\underline{\textit{(Local) First Integral of } w}$: $f(p) \mid \forall p \in \alpha_{p_0}(t), \boxed{f(p) = c}, \boxed{df_p \neq 0}$ $\left(\begin{matrix} f(p) = \text{arcdist}(p_0, \underline{g(p)}) \\ \text{along } S|_{x=x_0} \end{matrix}\right)$

- $\boxed{w(p_0)} \neq 0$ (see above) $\Rightarrow \big(\exists V_{p_0} \subset S\big)\big(\forall p \in V_{p_0},\ \underline{\exists f(p)}\big)$
- $w_1(p_0) \underline{\neq} Aw_2(p_0),\ \phi(p_0) = \begin{bmatrix} f_1(p_0) = u_0 \\ f_2(p_0) = v_0 \end{bmatrix} \Rightarrow \begin{bmatrix} d\phi_p \end{bmatrix}\begin{bmatrix} w_1(p_0) & w_2(p_0) \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \underline{\neq} 0 \overset{\text{(IFT)}}{\Rightarrow}$ $\exists \underline{X} : \underline{\phi^{-1}}(u_0, v_0) = p_0$, $\underline{*}\ \boxed{\begin{matrix} X(u_0, v) \in \alpha_1 \\ X(u, v_0) \in \alpha_2 \end{matrix}}$
- $w_1 \equiv X_u,\ w_2 \equiv -\frac{X_u \cdot X_v}{X_u \cdot X_u}X_u + X_v \Rightarrow \boxed{\exists(w_1, w_2)\big(w_2(p_0) \cdot w_2(p_0) = 0,\ \underline{\exists *}\big)}$
- $\underline{K < 0} \Rightarrow \underline{\exists\alpha_1, \alpha_2(k_n = 0)} \to \underline{\exists(w_1, w_2)(\exists *)}$ $*$ $k_1 \neq k_2,\ \exists(\alpha_1, \alpha_2)_{k_n} \to \underline{\exists(w_1, w_2)(\exists *)}$

Direction/Ray/Line Field : $\boxed{r_w = c_{\neq 0}\big(b(u,v), -a(u,v)\big)} \;\to\; \frac{y'}{x'} = \frac{-a}{b}$

Orthogonal Field to r : $\boxed{\overline{r}_w \equiv r_{\overline{w}} :\; \overline{w}\cdot w = (\overline{a}X_u + \overline{b}X_v)\cdot(aX_u + bX_v) = 0}$

E.g. : $\begin{matrix} X(q) = (u, v, u^2 - v^2) \\ \gamma(t) : u^2 - v^2 = c \to \frac{v'}{u'} = \frac{-u}{v} \end{matrix} \;\Rightarrow\; \begin{matrix} w_\gamma = u'(t)X_u + v'(t)X_v \overset{\rightrightarrows}{=} vX_u - uX_v \\ \overline{w}_\gamma \cdot w_\gamma = \overline{a}v - \overline{b}u = \underline{u'(\bar t)v - v'(\bar t)u = 0} \end{matrix} \;\Rightarrow\; \begin{matrix} \overline{\gamma}(\bar t) : \underline{u(\bar t)v(\bar t) = c} \\ X_c = (u, \frac{c}{u}, u^2 - \frac{c^2}{u^2}) \end{matrix}$

6 Intrinsic Surface Geometry

Christoffel Symbols, Γ

(For Surf. Trihe., X_u, X_v, N)

$$\underline{\begin{bmatrix} dN^T & 0 \end{bmatrix}} = \underline{\frac{-1}{EG-F^2}\begin{bmatrix} e & f \\ f & g \end{bmatrix}\begin{bmatrix} G & -F & 0 \\ -F & E & 0 \end{bmatrix}} \quad \text{(Weingarten Eq.)}$$

$$\overset{(R)}{\begin{bmatrix} \Gamma^1_{11} & \Gamma^2_{11} & e \\ \Gamma^1_{12} & \Gamma^2_{21} & f \\ \Gamma^1_{22} & \Gamma^2_{22} & g \\ a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \end{bmatrix}} \overset{(c)}{\begin{bmatrix} X_u \\ X_v \\ N \end{bmatrix}} = \overset{(Rc)}{\begin{bmatrix} X_{uu} \\ X_{uv} \\ X_{vv} \\ N_u \\ N_v \end{bmatrix}}$$

$$(Rcc^T)\; \begin{bmatrix} \Gamma^1_{11} & \Gamma^2_{11} & e \\ \Gamma^1_{21} & \Gamma^2_{12} & f \\ \Gamma^1_{22} & \Gamma^2_{22} & g \end{bmatrix}\begin{bmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X_{uu}\cdot X_u & X_{uu}\cdot X_v & e \\ X_{uv}\cdot X_u & X_{vu}\cdot X_v & f \\ X_{vv}\cdot X_u & X_{vv}\cdot X_v & g \end{bmatrix} = \left[\begin{array}{cc|c} \frac12 E_u & F_u - \frac12 E_v & e \\ \frac12 E_v & \frac12 G_u & f \\ F_v - \frac12 G_u & \frac12 G_v & g \end{array}\right]$$

$$(\Gamma)\; \boxed{\Gamma = f(E,F,G)} \quad \underline{\begin{bmatrix} \Gamma^1_{11} & \Gamma^2_{11} \\ \Gamma^1_{12} & \Gamma^2_{21} \\ \Gamma^1_{22} & \Gamma^2_{22} \end{bmatrix}} = \underline{\begin{bmatrix} \frac12 E_u & F_u - \frac12 E_v \\ \frac12 E_v & \frac12 G_u \\ F_v - \frac12 G_u & \frac12 G_v \end{bmatrix}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}\frac{1}{EG-F^2}}$$

Gauss and Mainardi-Codazzi Compatibility Equations:

$$\partial_u \begin{bmatrix} X_{uv} \\ X_{vv} \\ N_v \end{bmatrix} = \partial_v \begin{bmatrix} X_{uu} \\ X_{vu} \\ N_u \end{bmatrix} \to \partial_u \begin{bmatrix} -r_2- \\ -r_3- \\ -r_5- \end{bmatrix} Rc = \partial_v \begin{bmatrix} -r_1- \\ -r_2- \\ -r_4- \end{bmatrix} Rc \to \begin{bmatrix} -r_2- \\ -r_3- \\ -r_5- \end{bmatrix}\left[\partial_u R + R\begin{bmatrix} -r_1- \\ -r_2- \\ -r_4- \end{bmatrix} R\right] c = \begin{bmatrix} -r_1- \\ -r_2- \\ -r_4- \end{bmatrix}\left[\partial_v R + R\begin{bmatrix} -r_2- \\ -r_3- \\ -r_5- \end{bmatrix} R\right] c$$

$$\boxed{\partial_u \begin{bmatrix} -r_2- \\ -r_3- \\ -r_5- \end{bmatrix} R - \partial_v \begin{bmatrix} -r_1- \\ -r_2- \\ -r_4- \end{bmatrix} R = \begin{bmatrix} -r_1- \\ -r_2- \\ -r_4- \end{bmatrix} R \begin{bmatrix} -r_2- \\ -r_3- \\ -r_5- \end{bmatrix} R - \begin{bmatrix} -r_2- \\ -r_3- \\ -r_5- \end{bmatrix} R \begin{bmatrix} -r_1- \\ -r_2- \\ -r_4- \end{bmatrix} R} \quad \begin{pmatrix} \underline{\text{lin. ind.}} \\ Ac - Bc = 0 \\ \to A = B \end{pmatrix}$$

$$A = \begin{bmatrix} \partial_u R_2 - \partial_v R_1 \\ \partial_u R_3 - \partial_v R_2 \\ \partial_u R_5 - \partial_v R_4 \end{bmatrix} = \begin{bmatrix} \Gamma^1_{11} & \Gamma^2_{11} & e \\ \Gamma^1_{21} & \Gamma^2_{21} & f \\ a_{11} & a_{12} & 0 \end{bmatrix}\begin{bmatrix} \Gamma^1_{12} & \Gamma^2_{12} & f \\ \Gamma^1_{22} & \Gamma^2_{22} & g \\ a_{12} & a_{22} & 0 \end{bmatrix} - \begin{bmatrix} \Gamma^1_{21} & \Gamma^2_{12} & f \\ \Gamma^1_{22} & \Gamma^2_{22} & g \\ a_{12} & a_{22} & 0 \end{bmatrix}\begin{bmatrix} \Gamma^1_{11} & \Gamma^2_{11} & e \\ \Gamma^1_{12} & \Gamma^2_{21} & f \\ a_{11} & a_{21} & 0 \end{bmatrix}$$

$* \; A_{11} : ea_{12} - fa_{11} = \boxed{FK = f_{11}(\Gamma)}$

$* \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : f(\Gamma) = \begin{bmatrix} FK & -EK \\ GK & -FK \end{bmatrix}$

$$* \; F = 0 \;\Rightarrow\; GK = fa_{12} - ga_{11} = -\frac12 \frac{G_{uu}E - E_uG_u + E_{vv}E - E_v^2}{E^2} + \frac{G_u^2 + G_vE_v}{4GE} - \frac{E_v^2 + E_uG_u}{4E^2}$$

$$\boxed{K|_{F=0} = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]}$$

Isometric Map $\;$ Diffeo. $\phi : S \to \overline{S}$: $\begin{matrix} \forall p \in S \\ \forall v \in T_p(S) \end{matrix}$, $\boxed{\|\beta'\|^2_{\phi(p)} = \|d\phi_p v\|^2 = \langle v, v\rangle_p = \|v\|^2_p = \|\alpha'\|^2_p}$

- Local Isom. : $\forall p, V_p \in S,\ \exists \phi_p : S \to \overline{S}$ $\quad$ (Local + Diffeo. ϕ = Global Isom.)
- $(X : U \to S)$: $\big(\overline{X} : U \to \overline{S},\ \underline{E = \overline{E},\ F = \overline{F},\ G = \overline{G}},\ \boxed{\phi = \overline{X}\circ X^{-1}}\big) \overset{\Leftarrow}{\Rightarrow} \|d\phi_p v\|^2 = \langle v, v\rangle_p$
- $(X : U \to S)$: $\|d\phi_p v\|^2 = \langle v, v\rangle_p \overset{\Leftarrow}{\Rightarrow} \big(\overline{X} : U \to \overline{S},\ \boxed{\overline{X} = \phi\circ X},\ \underline{E = \overline{E},\ F = \overline{F},\ G = \overline{G}}\big)$

Conformal Map $\;$ Diffeo. $\phi : S \to \overline{S}$: $\;\forall v \in T_p(S)$, $\boxed{\|d\phi_p v\|^2 = \lambda^2(p)\,\langle v, v\rangle_p}$ $\;\begin{matrix} \text{(preserve } \cos\theta) \\ (\forall p,\ \lambda \neq 0,\ \exists \lambda') \end{matrix}$

- Locally Conf. : $\forall p, V_p \in S,\ \exists \phi_p : S \to \overline{S}$ $\quad$ (Locally + Diffeo. ϕ = Globally Conf.) $\;$ (hard: all reg. surf. are loc. conf. as ∃"isothermal" map)
- $(X : U \to S,\ \overline{X} : U \to \overline{S})$, $\big(\underline{E = \lambda^2\overline{E},\ F = \lambda^2\overline{F},\ G = \lambda^2\overline{G}},\ \boxed{\phi = X\circ \overline{X}^{-1}}\big) \Rightarrow \|d\phi_p v\|^2 = \lambda^2\,\langle v, v\rangle_p$

6.1 Covariant Derivative of w: $\boxed{\frac{Dw}{dt}_{\alpha'} \equiv \frac{dw}{dt}_{\|T_p(S)}} = \underline{\left[[a',b'] + [u',v']\begin{bmatrix} a\Gamma_1 + b\Gamma_2 \\ a\Gamma_2 + b\Gamma_3 \end{bmatrix}\right]\begin{bmatrix} X_u \\ X_v \end{bmatrix}}$ $(\alpha' = X_u u' + X_v v')$

$* \ \frac{dw}{dt}_{\alpha'} = a'X_u + b'X_v + a(X_{uu}u' + X_{uv}v') + b(X_{vu}u' + X_{vv}v') = [a', b', 0]\vec{c} + [u', v']\begin{bmatrix} aR_1 + bR_2 \\ aR_2 + bR_3 \end{bmatrix}\vec{c}$

- <u>"Parallel"</u> $w: \ \forall p \in \alpha(t), \ \boxed{\frac{Dw}{dt} = 0}$ $\left(\begin{matrix}\text{!truly parallel in } R^3, \\ \text{e.g., sphere: } a_{\|} = 0\end{matrix}\right)$ • $\forall \alpha(t) \in S, \ \underline{\exists!}\{w(t) \,|\, \frac{Dw}{dt} = 0\}$

- $\frac{D\alpha'}{dt} = (\alpha'' = kn)_{\|T_\alpha(S)}$ $*$ $\boxed{\text{Geodesic}: \ \alpha_g(t) \,\Big|\, \frac{D\alpha'_g}{dt} = 0 = \frac{D\alpha'_g}{ds} \ \Rightarrow \ n_{\|T_p(S)} = 0}$ $\boxed{\begin{matrix} *(n \text{ is perp. to } T_p(S)) \\ *(\frac{d}{dt}\alpha \cdot \alpha = 0) \end{matrix}}$

$*$ $\boxed{\text{Geosedic Eq. (GEq)}: \ \left[[u'', v''] + [u', v']\begin{bmatrix} u'\Gamma_1 + v'\Gamma_2 \\ u'\Gamma_2 + v'\Gamma_3 \end{bmatrix}\right] = [0 \ 0]}$ $*$ $\boxed{\begin{matrix} E = 1, F = 0 \\ F = 0, G = 1 \end{matrix} \xrightarrow{\text{(not only)}} \begin{matrix} \alpha_g(u'' = 0, v' = 0) \\ \alpha_g(u' = 0, v'' = 0) \end{matrix}}$

Algebraic Value of $\frac{Dw}{ds}$ $\left(\begin{matrix} w \cdot w = 1 \\ \underline{w \cdot w' = 0} \end{matrix}\right)$: $\boxed{\left[\frac{Dw}{ds}\right] \equiv \lambda(s) = \left\langle \frac{dw}{ds}, N \times w \right\rangle \equiv w' \cdot \overline{w}}$ $\Rightarrow$ $\underline{\frac{Dw}{ds} = \lambda\overline{w}}$

$\boxed{\textbf{Geodesic Curvature}: \ k_g = \left[\frac{D\alpha'}{ds}\right]}$ $\left(\begin{matrix}\text{geodesics} \\ \text{have } k_g = 0\end{matrix}\right)$ $\Rightarrow$ $\boxed{k^2 = k_n^2 + k_g^2 \ == \ \|t'\|^2 = \langle N, kn \rangle^2 + k_g^2}$

<u>Two Fields</u>: $\boxed{w = v\cos\theta + \overline{v}\sin\theta}$ $\left(\begin{matrix}\text{unit circle} \\ \text{from } \hat{x}\end{matrix}\right)$ $\Rightarrow$ $w' \cdot \overline{w} = \theta' + v' \cdot \overline{v} = \boxed{\frac{d\theta_{wv}}{ds} + \left[\frac{Dv}{ds}\right] = \left[\frac{Dw}{ds}\right]}$

$* \ 0 = \left[\frac{Dv}{ds}\right] = \left[\frac{Dw}{ds}\right] \Rightarrow \boxed{\theta = \theta_0}$ $\quad * \ 0 = \left[\frac{Dv}{ds}\right] \Rightarrow \left[\frac{Dw}{ds}\right] = \boxed{\left[\frac{D\alpha'}{ds}\right] = k_g = \frac{d\theta_{\alpha'\|}}{ds}}$ $\left(\begin{matrix}\text{clockwise unit} \\ \text{circle from } \hat{y}\end{matrix}\right)$

<u>Orthog. Coord.</u>: $F = 0 = e_1 \cdot e_2$

$\underline{v} = \underline{\alpha'}(s) = e_1(s) = \frac{X_u}{\sqrt{E}}, \ N \times e_1 = e_2 = \frac{X_v}{\sqrt{G}}$

$\Rightarrow$ $\left\langle \frac{\partial e_1}{\partial s}, e_2 \right\rangle = \frac{du}{ds}\left\langle \frac{\partial e_1}{\partial u}, e_2 \right\rangle + v'\left\langle \frac{X_{uv}}{\sqrt{E}}, \frac{X_v}{\sqrt{G}} \right\rangle = \boxed{\frac{-u'E_v + v'G_u}{2\sqrt{EG}} = \left[\frac{De_1}{ds}\right]}$

$\boxed{\left[\frac{Dw}{ds}\right] = \frac{v'G_u - u'E_v}{2\sqrt{EG}} + \frac{d\theta_{we_1}}{ds}}$ $\Rightarrow$ $\boxed{\theta_{\|e_1} = \theta_0 + \int \frac{u'E_v - v'G_u}{2\sqrt{EG}}\, ds}$

$* \ \begin{matrix} \cos\theta_{\alpha' e_1} = \langle \underline{w}, e_1 \rangle = \langle \underline{\alpha'}, e_1 \rangle = \sqrt{E}u' \\ \sin\theta_{\alpha' e_1} = \left\langle X_u u' + X_v v', \frac{X_v}{\sqrt{G}} \right\rangle = \sqrt{G}v' \end{matrix}$ $\Rightarrow$

$\left[\frac{Da'}{ds}\right] = \underline{-\frac{E_v}{2E\sqrt{G}}}\cos\theta + \underline{\frac{G_u}{2G\sqrt{E}}}\sin\theta + \theta'$

$\boxed{k_g = \underline{(k_g)_1}\cos\theta_{\alpha' e_1} + \underline{(k_g)_2}\sin\theta_{\alpha' e_1} + \frac{d\theta_{\alpha' e_1}}{ds} = \frac{d\theta_{\alpha'\|}}{ds}}$

<u>Surface of Revolutions Geodesics</u>: $X(u,v) = \big(\rho(v)\cos u, \ \rho(v)\sin u, \ z(v)\big)$

$\Gamma = \begin{bmatrix} 0 & -\frac{\rho\rho'}{(\rho')^2 + (z')^2} \\ \frac{\rho\rho'}{\rho^2} & 0 \\ 0 & \frac{\rho'\rho" + z'z"}{(\rho')^2 + (z')^2} \end{bmatrix} \rightarrow \left[[u'', v''] + [u', v']\begin{bmatrix} \frac{v'\rho\rho'}{\rho^2} & -\frac{u'\rho\rho'}{(\rho')^2 + (z')^2} \\ \frac{u'\rho\rho'}{\rho^2} & v'\frac{\rho'\rho" + z'z"}{(\rho')^2 + (z')^2} \end{bmatrix}\right] \Rightarrow$

$\boxed{\begin{matrix} \underline{\text{(see Clairaut's Relation)}} \\ 0 = \frac{1}{2}\frac{d}{ds}\underline{\left[\rho^2 u' = \rho\sqrt{E}u' = \rho\cos\theta_{\alpha' X_u}\right]} \\ 0 = \frac{1}{2}\frac{d}{ds}\left[\left(\frac{d\rho}{dv}^2 + \frac{dz}{dv}^2\right)(v')^2\right] + \rho\rho'(u')^2 \end{matrix}}$

- $\alpha_u : \underline{\gamma(t) = (u, v(t))} \ \rightarrow \ \|\frac{d\alpha_u}{ds}\| = 1 \ \rightarrow \ \frac{d}{ds}\left[\left(\frac{dz}{dv}^2 + \frac{d\rho}{dv}^2\right)(v')^2\right] = 0 \ \Rightarrow \ \boxed{\left[\frac{D\alpha_u}{ds}\right] = 0}$

  $\alpha_v : \underline{\gamma(t) = (u(t), v)} \ \rightarrow \ \boxed{\frac{du}{ds} = c_{\neq 0}, \ \frac{d\rho}{dv} = 0 \ \Rightarrow \left[\frac{D\alpha_v}{ds}\right] = 0}$ e.g., $\underline{\frac{d\rho/dv}{dz/dv} = 0}$

- *<u>Clairaut's Relation</u>*: $\boxed{\rho\cos\theta_{\alpha' X_u} = \rho^2\frac{du}{ds} = c}$ $(X_u \cdot \hat{z} = 0)$

$* \ \|\alpha'\|^2 = 1 = \rho^2\frac{du}{ds}^2 + \left[\frac{d\rho}{dv}^2 + \frac{dz}{dv}^2\right]\frac{dv}{ds}^2 \ \rightarrow \ \underline{c \neq 0 \ \Leftrightarrow \ \theta_{\alpha' X_u} \neq \frac{\pi}{2}}$, $\boxed{\frac{du}{dv} = \frac{c}{f}\sqrt{\frac{(df/dv)^2 + (dg/dv)^2}{f^2 - c^2}}}$

6.2 Gauss-Bonnet Theorem

External Angle at t_i :

$$\boxed{\theta_{E(t_i)} = \lim_{\epsilon\to 0} \Delta\theta\Big|_{\alpha'(t_i-\epsilon)}^{\alpha'(t_i+\epsilon)}}$$

- Interior Angle at t_i : $\boxed{\theta_I(t_i) = \pi - \theta_E(t_i)}$

Turning Theorem along α_{closed} (going all around makes $\pm 2\pi$) :

$$\boxed{\sum_i^{Ver} \theta_{\alpha' e_1}(t_{i+1}) - \theta_{\alpha' e_1}(t_i) + \theta_{E(t_i)} = \pm 2\pi}$$

$$\oint \left[\frac{Dw}{ds}\right] ds = \oint \frac{-u'E_v + v'G_u}{2\sqrt{EG}} + \frac{d\theta_{we_1}}{ds}\, ds$$

$$= \iint \left(\frac{G_u}{2\sqrt{EG}}\right)_u + \left(\frac{E_v}{2\sqrt{EG}}\right)_v dudv + \Delta\theta_{we_1}$$

$$= -\iint (K\sqrt{EG})\, dudv + \Delta\theta_{we_1}$$

$$\boxed{\oint \left[\frac{Dw}{ds}\right] ds = -\iint K d\sigma + \Delta\theta_{we_1}}$$

$*$ $\boxed{\Delta\theta_{\parallel \cancel{e_1}} = \iint K\, d\sigma}$ $\rightarrow$ $\lim_{R\to p} \frac{\Delta\theta}{A(R)} = K(p) = \lim_{R\to p} \frac{A(N(R))}{A(R)}$

(Local) Gauss-Bonnet Theorem :

$$\boxed{\underset{(\Delta\theta_{\alpha'\parallel})}{\oint k_g\, ds} + \underset{(\Delta\theta_{\parallel 0})}{\iint K\, d\sigma} + \sum_i^{Ver} \theta_{E(t_i)} = \pm 2\pi}$$

e.g., no vertices

$\Delta\theta_{\prime} + (2\pi - \Delta\theta_{\prime}) + 0$

e.g., w/ vertices

$\Delta\theta_{\prime} + (\frac{3\pi}{2} - \Delta\theta_{\prime}) + \frac{\pi}{2}$

Global Gauss-Bonnet Theorem : $\boxed{\oint_{\partial R} k_g\, ds + \iint_R K\, d\sigma + \sum_i^{Ver} \theta_{E(t_i)} = 2\pi\chi(R) = 2\pi(V - E + F)}$

- $R \sim S^2$: $\underline{\chi(R) = 2}$
- $R \sim$ Cylinder : $\underline{\chi(R) = 0}$
- Simple Region $R \sim S_{\neq S^2}$: $\underline{\chi(R) = 1}$ (needs > 0 vertex; circle edge begins/ends at one point/vertex)
- Compact, connected S $R \sim \oiint_{\partial\tau}$, $\nexists \partial R$: $\boxed{\begin{matrix}(\underline{\text{one}}\ n > 0)\ \ n \in \{1, 0, -1, -2, \dots\} \\ \iint_R K\, d\sigma = 2\pi\chi(S) = 2\pi(2n)\end{matrix}}$

$*$ $\boxed{\underline{\text{Genus}}:\ g \equiv \frac{2-\chi}{2}}$ (S^2 w/ g torus holes $\sim S$)

$*$ $\underline{K > 0}$ $\Rightarrow \chi(S) = 2$ $\Rightarrow \boxed{S \sim S^2}$

$R =$ Bounded by [any] Two Geodesics : $\iint_R K\, d\sigma + \theta_{E(t_1)} + \theta_{E(t_2)} = 2\pi\chi(R)$

- $K \leq 0$: intersect $2\times$ ~~($R \sim S_{\neq S^2}$)~~ $\rightarrow$ $2\pi\chi(R) = \theta_E(t_1) + \theta_E(t_2) + \iint_R K\, d\sigma$; $2\pi = (< \pi) + (< \pi) + (< 0)$ $\Rightarrow$ $\underline{\text{any two geo. intersect} \leq 1\times}$
- $K < 0,\ S \sim$ Cylinder : two [closed] geo. intersect $0\times$ ~~($R \sim$ cylinder)~~ $\rightarrow$ $0 = 2\pi\chi(R) = \iint_R K\, d\sigma < 0$ $\Rightarrow$ $\underline{\nexists \text{ Two CLOSED geo.}}$
- $K > 0 \Leftrightarrow S \sim S^2$: two [closed] geo. intersect $0\times$ ~~($R \sim$ cylinder)~~ $\rightarrow$ $0 = 2\pi\chi(R) = \iint_R K\, d\sigma > 0$ $\Rightarrow$ $\underline{\text{two closed geo. intersect} \geq 1\times}$

$R =$ Three Geodesics

Geodesic Triangle, T : $\iint_R K\, d\sigma + \sum_{i=1}^{3} \theta_E(t_i) = 2\pi = 2\pi\chi(R)$

$$\boxed{\iint_R K\, d\sigma + \pi = \sum_{i=1}^{3} \theta_{I(t_i)}}$$

- $\boxed{\iint_R K\, d\sigma = \sum_{i=1}^{3} \theta_{I(t_i)} - \pi = A(N(T))}$

[Diff.] Vector Space, v, on S : $p_i \in R|\ v(p_i) = 0$; $\partial R = \alpha,\ v(t) = v \circ \alpha(t)$ $\rightarrow$

$$\oint_0^l \frac{d\theta_{v(0)e_1}}{ds}\, ds = \theta_{v(l)e_1} - \theta_{v(0)e_1} = \boxed{\Delta\theta\Big|_{v(0)}^{v(l)} \equiv 2\pi I_{p_i}}$$

$\underline{v(p_i) \neq 0 \rightarrow I_{p_i} = 0}$

Poincare's Theorem

- Compact S, $\boxed{\frac{1}{2\pi} \oiint K\, d\sigma = \sum_i I_{p_i} = \chi(S)}$

6.3 Exponential Map, $\begin{array}{l}\exp_p : v \in T_p(S) \;\rightarrow\; \exp_p(v) \in S \\ \equiv X : q \in S_1 \;\rightarrow\; p' \in S_2\end{array}$

$$\begin{array}{c}\left[\frac{D\alpha'}{dt}\right] = 0 \\ \frac{d}{dt}\|\alpha'(t)\| = 0 \\ \alpha'(0) = v \\ \exists \alpha(s = |v|t)\end{array}, \quad \begin{array}{l}|t| < \epsilon : \; \alpha(t) = \alpha \circ (\lambda \bar{t}) = \overline{\alpha}(\bar{t} = \frac{t}{\lambda}) \; : |\bar{t}| < \frac{\epsilon}{\lambda} \\ \alpha'(t) = \frac{1}{\lambda}\overline{\alpha}'(\frac{t}{\lambda}) \;\Rightarrow\; \lambda v = \lambda\alpha'(0) = \overline{\alpha}'(0) = \overline{v} \\ |t| < \epsilon : \; \gamma(t, v) = \gamma(\bar{t}, \overline{v}) = \gamma(\frac{t}{\lambda}, v\lambda) \; : |\bar{t}| < \frac{\epsilon}{\lambda} \\ \frac{s}{|v|} < \epsilon : \; \gamma(\frac{s}{|v|}, v) = \gamma(\frac{s}{\lambda|v|}, v\lambda) \\ \rightarrow \underline{(|\overline{v}| = \lambda|v| = s < \epsilon|v|) \;\Leftrightarrow\; (\bar{t} = 1 < \frac{\epsilon}{\lambda})}\end{array}, \quad \boxed{\begin{array}{l}\exp_p(0) = \alpha_{g(s=0)} = \gamma_v(0) = p = \gamma_0(t) \\ \exp_p(v) = \alpha_{g(|v|)} = \underline{\gamma_v(1)} = e^{(1)\frac{d}{dt}}\gamma_v(0)\end{array}}$$

- $\underline{\text{IF}^*}$ $\begin{array}{c}\gamma(t,v) \in C^\infty \\ \underline{\forall \text{ directions } v}\end{array}\Bigg| \begin{array}{l}|t| < \epsilon_t \\ |v| < \epsilon_v\end{array} \;\Rightarrow\; \gamma(t', v') \in C^\infty \Bigg| \begin{array}{l}|t' = \frac{2t}{\epsilon_t}| < 2 \\ |v' = \frac{\epsilon_t v}{2}| < \frac{\epsilon_t \epsilon_v}{2} = \epsilon\end{array} \;\Rightarrow\; \begin{array}{c}\text{(GEq unique.+exist. theor. used)}^* \\ \boxed{\gamma(1, v') = \exp_p \circ v' \in C^\infty}\end{array}$

$$\begin{array}{l}q(t) = v_0 t = (u_0 e_1 + v_0 e_2)t \\ \alpha(t) = \exp_p \circ q(t) \in \alpha_g\end{array}, \quad \underline{\frac{d\alpha}{dt}\Big|_{t=0}} = \frac{d}{dt}\exp_p(1, v_0 t)\Big|_{t=0} = \frac{d}{dt}\gamma(t, v_0)\Big|_{t=0} = \underline{v_0}$$

$$* \;\begin{array}{l}\bar{q}(t) = \bar{v}_0 t, \; (\bar{v}_0 \cdot v_0 = 0) \\ \bar{\alpha}(t) = \exp_p \circ \bar{q}(t)\end{array}, \quad \underline{\frac{d\bar{\alpha}}{dt}\Big|_{t=0} = \bar{v}_0} \;\Rightarrow\; \boxed{\begin{array}{c}(X_u \; X_v)_{q=0} = [e_1 \; e_2] \neq \mathbb{1}_3 \\ \left[d(\exp_p)_{v=0}\right] q'(0) = v_0 \\ dXq'(0, v_0) = q'(0, v_0) = v_0\end{array}} \overset{\text{(IFT)}}{\Rightarrow} \boxed{\begin{array}{c}\exists \text{ Diffeo } \left[\exp_p(v)\right]^{-1} \\ \underline{\text{near } q = v = 0}\end{array} \in C^\infty}$$

$\underline{\text{Normal Neighborhood, } V_p}$: Diffeo. $\exp_p(V_q) = V_p$

$\underline{\text{Normal Coordinates}}$: $w = ue_1 + ve_2 \in T_p(S)$
- $\underline{w(t) = w_0 t, \; \alpha(t) = \exp_p \circ w(t) \in \alpha_g \; \text{(radial geo.)}}$
- $\boxed{dX_{q=p}}$: $X_u = e_1, \; X_v = e_2 \;\rightarrow\; \underline{E|_p = G|_p = 1, \; F|_p = 0}$

$\underline{\text{Geod. Polar Coordinates}}$: $w = \vec{\rho}(0 < \rho, \; \underline{0 < \theta < 2\pi}) \in T_p(S)$ • Diffeo. $\rightarrow \theta \in (0, 2\pi)$; $L \equiv \exp_p(w : \theta = 0)$

- $w : (\rho(s), \theta) = (s, \theta_0) \;\Rightarrow\;$ $\begin{array}{l} * \;\; \|\alpha'|^{\rho=s}_{\theta=\theta_0}\|^2 = \|\alpha'(s)\|^2 \;\overset{\rightarrow}{=}\; \boxed{E = 1} \\ * \;\; \begin{array}{l}\text{(GEq)} : \; (u')^2\Gamma_{11} = (\rho')^2\Gamma_{11} = [0 \; 0] \\ (\Gamma) : \; \Gamma^2_{11} = \frac{1}{2[EG - F^2]}[E_u \; 2F_u - E_v] \cdot [-F \; E]\end{array} \;\rightarrow\; \underline{F_\rho = 0}\end{array}$

- $\lim_{\rho\to 0}\left[F(\rho, \theta) = X_u \cdot X_v\right] = \lim_{\rho\to 0}\frac{d\alpha}{ds}\Big|^{\rho=s}_{\theta=\theta_0} \cdot \lim_{\rho\to 0}\cancel{\frac{d\alpha}{d\phi}\Big|^{\rho=\rho_0}_{\theta=\phi}} = 0, \; \left(F_\rho = 0\right) \;\Rightarrow\; \forall\rho, \; \boxed{F = 0}$

$*$ $\underline{\text{Gauss' Lemma : } F = 0 \;\leftrightarrow\; \text{radial geod. orthog. to geod. circles}}$

- $\|X_u \times X_v\| = \|\overline{X_u} \times \overline{X_v}\| \;\Rightarrow\; \underline{\sqrt{EF - G^2}|_{\rho\theta}} = \underline{\sqrt{\overline{EF} - \overline{G}^2}|_{uv, \, \rho=0}\left\|\begin{matrix}\cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta\end{matrix}\right\|} \;\Rightarrow\; \boxed{\lim_{\rho\to 0}\sqrt{G} = \rho}$

$*$ $w = [X_u \; X_v]\begin{bmatrix}u' \\ v'\end{bmatrix} = \underline{[X_u \; X_v]\begin{bmatrix}u_\rho & u_\theta \\ v_\rho & v_\theta\end{bmatrix}}\begin{bmatrix}\rho' \\ \theta'\end{bmatrix} = \underline{[X_\rho \; X_\theta]}\begin{bmatrix}\rho' \\ \theta'\end{bmatrix}$ (only useful so far for $dX_{q=p}$, so above is better)

- $F = 0, \; E = 1 \;\rightarrow\; \boxed{\sqrt{G}_{\rho\rho} + K\sqrt{G} = 0}$ $\quad * \; \lim_{\rho\to 0}\sqrt{G}_{\rho\rho\rho} + K_\rho\sqrt{G} + K\sqrt{G}_\rho = \boxed{0 = \lim_{\rho\to 0}\sqrt{G}_{\rho\rho\rho} + K(p)}$

$$\sqrt{G}(\rho, \theta) = \cancel{\sqrt{G}(0, \theta)} + \sqrt{G}_\rho(0, \theta)\rho + \cancel{\sqrt{G}_{\rho\rho}(0, \theta)\frac{\rho^2}{2!}} + \sqrt{G}_{\rho\rho\rho}(0, \theta)\frac{\rho^3}{3!} + R\rho^4 = \rho - K(p)\frac{\rho^3}{3!} + R\rho^4$$

$$\boxed{L(\rho)} = \lim_{\rho, \epsilon\to 0}\oint_{0+\epsilon}^{2\pi-\epsilon}\sqrt{\cancel{E d\rho^2} + G\, d\theta^2} = \lim_{\rho\to 0} 2\pi\rho - K(p)\frac{2\pi\rho^3}{3!} \;\Rightarrow\; \boxed{K(p) = \lim_{\rho, \epsilon\to 0}\frac{3!}{2\pi}\frac{2\pi\rho - L}{\rho^3}} \; \left(\begin{array}{c}\text{arclength of } \bigcirc \in T_p(S) \\ \text{- arclength of } \bigcirc \in S\end{array}\right)$$

$$\boxed{A(\rho)} = \lim_{\rho, \epsilon\to 0}\int_{0+\epsilon}^{2\pi-\epsilon}\int_0^\rho \sqrt{EG - F^2}\, dA = \lim_{\rho\to 0}\frac{2\pi\rho^2}{2} - K(p)\frac{2\pi\rho^4}{4!} \;\Rightarrow\; \boxed{K(p) = \lim_{\rho, \epsilon\to 0}\frac{4!}{2\pi}\frac{\pi\rho^2 - A}{\rho^4}} \; \left(\begin{array}{c}\text{area of } \bigcirc \in T_p(S) \\ \text{- area of } \bigcirc \in S\end{array}\right)$$

$$* \ K=0: \ \underline{\sqrt{G}=\rho} \qquad * \ K>0: \ \underline{\sqrt{G}=\tfrac{1}{\sqrt{K}}\sin(\sqrt{K}\rho)} \ \rightarrow \ \sqrt{G}_{\rho\rho}<0 \ \Rightarrow \ \underline{\tfrac{d^2}{d\rho^2}L_{(\rho)}\Big|_{\theta_0}^{\theta_1}<0}$$

$$* \ K<0: \ \underline{\sqrt{G}=\tfrac{1}{\sqrt{-K}}\sinh(\sqrt{-K}\rho)} \ \rightarrow \ \sqrt{G}_{\rho\rho}>0 \ \Rightarrow \ \underline{\tfrac{d^2}{d\rho^2}L_{(\rho)}\Big|_{\theta_0}^{\theta_1}>0}$$

$$* \ \psi: \ \begin{array}{c} \psi(e_i \in T_p(S)) = \overline{e}_i \in T_{\overline{p}}(\overline{S}) \\ d\psi(e_i \in T_p(S)) = \overline{e}_i \in T_{\overline{p}}(\overline{S}) \end{array} \ \rightarrow \ \Psi: \ \begin{array}{l} \Psi = \exp_{\overline{p}} \circ \psi \circ \exp_p^{-1} \\ [d\Psi]e_i = \overline{e}_i \end{array} \ \Bigg| \ \boxed{\begin{array}{c} K_0 = K(V_p) = K(V_{\overline{p}}) \\ \text{Minding's Theorem} \\ \text{const. } K \rightarrow \text{isometry} \end{array} \ \Rightarrow \ \begin{array}{l} (E,F,G) = (\overline{E},\overline{F},\overline{G}) \\ \|\overline{w}\| = \|d\Psi(w)\| = \|w\| \end{array}}$$

$$* \, dl_\alpha = \sqrt{Ed\rho^2 + Gd\theta^2} \ \rightarrow \ l_{\alpha(\epsilon)} = \int_\epsilon^{t-\epsilon} \sqrt{(\rho')^2 + G\cdot(\theta')^2}\,dt \ \geq \ l_\gamma - 2\epsilon = \int_\epsilon^{t-\epsilon} \sqrt{(\rho')^2}\,dt$$

7 Abstract Surface/Riemannian Maniford, M

1. $\bigcup_\alpha X_\alpha(U_\alpha) = M$ open $U \subset R^n$ 3. $\{U_\alpha, X_\alpha\}$ is maximal rel. to 1. and 2.

2. $\forall(\alpha,\beta),\ W = X_\alpha(U_\alpha) \cap X_\beta(U_\beta) \neq \varnothing \ \Rightarrow$
- open $X_\alpha^{-1}(W), X_\alpha^{-1}(W) \subset R^n$
- $X_\beta^{-1} \circ X_\alpha,\ X_\alpha^{-1} \circ X_\beta \in C^\infty$

- Manifold, $M =$ Hausdorff Space w/ Complete Atlas
- Sub[space]manifold, $M \overset{\text{[top] subspace}}{\hookrightarrow} N;\ C^\infty \ni$ Immer ι : Mani $M \to$ Mani N (Inclusion map)

<u>**Coordinate System**</u> : $\forall p \in \exists V_{p_0} \subset M,\ \xi(p) = (x^1, x^2, \dots x^m)(p) \in R^m$

<u>Tangent Vector at $\alpha(0) \in M$</u> :

$$f \circ \alpha(t) \to \frac{d(f\circ\alpha)}{dt}\Big|_{t=0} = \left[\frac{\partial f}{\partial r^1}, \frac{\partial f}{\partial r^2}, \frac{\partial f}{\partial r^3}\right]\alpha'(0)\ ,\ f \in \mathcal{F}(M)\ \ (\text{all diff. func. on } M)$$

$$f \circ X \circ q(t) \to \frac{d(f\circ X\circ q)}{dt}\Big|_{t=0} = \left[\frac{\partial f}{\partial r^1}, \frac{\partial f}{\partial r^2}, \frac{\partial f}{\partial r^3}\right]\left[\frac{\partial \vec{X}}{\partial u^1}, \frac{\partial \vec{X}}{\partial u^2}\right]q'(0) = DfDXDq$$

$$F(u^1,u^2)\circ q(t) \to \frac{d(F\circ q)}{dt}\Big|_{t=0} = \boxed{\left(q_1'\frac{\partial}{\partial u^1}\Big|_0 + q_2'\frac{\partial}{\partial u^2}\Big|_0\right)(f\circ\xi^{-1})}$$

$T_{p_0}(M) = \{v_{p_0} : \mathcal{F}(M) \to \mathbb{R}\}$ (Is a Derivation)

- $v_{p_0}(af + bg) = a\underline{v_{p_0}(f)} + b\underline{v_{p_0}(g)} \in \mathbb{R}$
- $v_{p_0}(fg) = v_{p_0}(f)\underline{g(p_0)} + v_{p_0}(g)\underline{f(p_0)}$ $\quad f, g : M \to \underline{\mathbb{R}}$

$*\ cv_{p_0}(1) = cv_{p_0}(1*1) = 2cv_{p_0}(1) = \boxed{0 = v_{p_0}(c)}$

Basis Vectors :

$q(t,q_1,q_0) = q_0 + t[q_1 - q_0]$
$p(t,q_1,q_0) = \xi^{-1}\circ q(t,q_1,q_0)$

- $\boxed{\frac{\partial f}{\partial x^i}(p) \equiv \frac{\partial(f\circ\xi^{-1})}{\partial u^i}(\xi p)} = \frac{\partial F}{\partial u^i}(q) \ \Rightarrow\ T_{p_0}(M) \ni \boxed{\partial_i\big|_p \equiv \frac{\partial}{\partial x^i}\Big|_p : \mathcal{F}(M) \to \mathbb{R}}$

- $F\big|_{q_0}^{q_1} = F\circ q\big|_0^1 = \sum[q_1^i - q_0^i]\int_0^1 \frac{\partial F}{\partial u^i}\circ q(t,q_1,q_0)\,dt \overset{\text{(no need)}}{=} \sum\int_0^1 \frac{\partial F}{\partial u^i}\circ q(t,q_1,q_0)\frac{dq^i}{dt}\,dt = \int_0^1 DF\,Dq\,dt = \underline{\int_0^1 \frac{d(F\circ q)}{dt}\,dt}$

$$F\circ q_1 - F(q_0) = \sum[x^ip_1 - x^ip_0]\int_0^1 \frac{\partial(f\circ\xi^{-1})}{\partial u^i}\circ\xi p(t,p_1,p_0)\,dt$$

$$f\circ p_1 - f(p_0) = \sum[x^ip_1 - x^ip_0]\int_0^1 \frac{\partial f}{\partial x^i}\circ p(t,p_1,p_0)\,dt$$

$*$ (no need?) $\frac{\partial F}{\partial u^j}\Big|_{q_0} - \cancel{\frac{\partial F(q_0)}{\partial u^j}} = \cancel{\frac{\partial y^j}{\partial u^j}}\int_0^1 \frac{\partial F}{\partial u^j}\circ q(t,q_0,q_0)\,dt + \cancel{\sum_i[q_0^i - q_0^i]\int_0^1 \sum_k \frac{\partial u^k}{\partial u^j} t \frac{\partial^2 F}{\partial u^k \partial u^i}\circ q(t,u,q_0)\,dt\Big|_{q_0}}$

$$\frac{\partial f}{\partial x^j}\Big|_{p_0} - \cancel{\frac{\partial f(p_0)}{\partial x^j}} = \int_0^1 \frac{\partial f}{\partial x^i}\circ p(t,p_0,p_0)\,dt$$

$*\ v_{p_0}f(p) - \cancel{v_{p_0}f(p_0)} = \sum v_{p_0}[x^i(p) - \cancel{x^i(p_0)}]\cdot\int_0^1 \frac{\partial f}{\partial x^i}\Big|_{p(t,p_0,p_0)}dt + \cancel{[x^i(p_0) - x^i(p_0)]\cdot v_{p_0}\int_0^1 \frac{\partial f}{\partial x^i}\Big|_{p(t,p,p_0)}dt}$

$$= v_{p_0}[x^i(p)]\cdot\sum\frac{\partial f}{\partial x^i}\Big|_{p_0} \ \Rightarrow\ \boxed{v_{p_0} = v_{p_0}\left(\sum x^i\frac{\partial}{\partial x^i}\Big|_{p_0}\right) = \sum v_{p_0}(x^i)\,\partial_i\big|_{p_0}}$$

$*\ \sum c^i\partial_i\big|_{p_0} = 0 = 0\cdot x^j = \sum c^i\partial_i\big|_{p_0}x^j = c^i\delta_{ij} = c^j$ $\boxed{\text{(Lin. Ind.)}}$

- Change of Basis : $v_p = \sum v_p(x^i)\frac{\partial}{\partial x^i}\Big|_p = \sum v_p(y^j)\frac{\partial}{\partial y^j}\Big|_p \ \leftrightarrow\ v_p(y^j) = \sum v_p(x^i)\frac{\partial y^j}{\partial x^i}\Big|_p$

$\underline{\text{Manifold Mapping}}, \ \phi : M \to N$

- $v_\phi(g_1 g_2) = v\big((g_1 \circ \phi)(g_2 \circ \phi)\big)$
  $= v_\phi(g_1) g_2(\phi p) + v_\phi(g_2) g_1(\phi p)$

$\underline{\text{Vector at Mapping}, \ v_\phi \in T_{\phi p}(N)} : \quad \boxed{v_\phi(g) \equiv v(g \circ \phi)}$

$\underline{\text{Differential Map}, \ d\phi_p : T_p(M) \to T_{\phi p}(N)} : \quad \boxed{v(g \circ \phi) \equiv v_\phi(g) = [d\phi_p v](g)} \quad \begin{matrix} v \in T_p(M) \\ g \in \mathcal{F}(N) \end{matrix}, \ \frac{\partial g}{\partial y^i} = \frac{\partial G}{\partial v^i}$

$$\underline{q' \cdot \vec{\nabla}_u\Big|_p \big(g \circ \phi \circ \xi^{-1}\big)} = \frac{d(g \circ \eta^{-1} \circ \eta \circ \phi \circ \xi^{-1} \circ q)}{dt}\Big|_0 = \frac{d(G \circ \Phi \circ q)}{dt}\Big|_0 = [\vec{\nabla}_v^T G][\vec{\nabla}_u^T \Phi]\, q'|_0 = \underline{\Big[[q'|_0^T\, \vec{\nabla}_u \Phi^T]\, \vec{\nabla}_v\Big] G} = \underline{\Big[[\vec{\nabla}_u^T \Phi\, v(\vec{x})] \cdot \vec{\nabla}_v\Big] G}$$

- $T_{\phi p}(N) \ni d\phi_p \frac{\partial}{\partial x^i}\Big|_p = \sum_j \left(d\phi_p \frac{\partial y^j}{\partial x^i}\Big|_p \right) \frac{\partial}{\partial y^j}\Big|_{\phi p} = \sum_j \frac{\partial (y^j \circ \phi)}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{\phi p} \ \Leftarrow \ \eta(\phi p \in N^n) = y \in \mathbb{R}^n$
- $T_{\phi p}(N) \ni d\phi_p v = d\phi_p \sum_i v(x^i) \frac{\partial}{\partial x^i}\Big|_p = \underline{\sum_{i,j} v(x^i) \frac{\partial (y^j \circ \phi)}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{\phi p}}$
- $\begin{matrix} \psi : N \to P \\ \phi : M \to N \end{matrix}, \ \begin{matrix} h \in \mathcal{F}(P) \\ v \in T_p(M) \end{matrix}, \quad [d(\psi \circ \phi)_p v](h) = v_{\psi \circ \phi}(h) = v\big(h \circ \psi \circ \phi\big) = v_{\phi(h \circ \psi)} = d\phi_p v(h \circ \psi) = d\psi_{\phi p} d\phi_p v(h)$
  $\boxed{d(\psi \circ \phi)_p v = d\psi_{\phi p} d\phi_p v}$

- Inverse Function Theorem : $\boxed{\text{Linear Iso. } d\phi_p \text{ at } p \ \Leftrightarrow \ \text{[Local] Diffeo. } \phi : \exists \mathcal{V}_p \to \phi(\mathcal{V}_p)}$

$\underline{\text{Curve } \alpha(t) = \xi^{-1} \circ q(t) : \mathbb{R} \to M}$

- $T_{\alpha(t)}(M) \ni \alpha'(t) = d\alpha \frac{\partial}{\partial u}\Big|_t = \sum \alpha'\Big|_t (x^i) \frac{\partial}{\partial u^i}\Big|_{\alpha(t)} = \sum \frac{\partial (x^i \circ \alpha)}{\partial u}\Big|_t \frac{\partial}{\partial u^i}\Big|_{\alpha(t)}$
- $\alpha'(t) f = d\alpha \frac{\partial f}{\partial u}\Big|_t = \frac{\partial (f \circ \alpha)}{\partial u}\Big|_t$

$\underline{\text{Vector Field}, \ V \in \mathcal{X}(M) : p \in M \to V_p \in T(M)} : \quad \boxed{\begin{matrix} V_p f = & V_p(f) & = & (Vf)_p = (\overline{V} f)_p \\ \mathbb{R} & V_p \in T_p(M) & & Vf \in \mathcal{F}(M), \ \overline{V} : \mathcal{F}(M) \to \mathcal{F}(M) \end{matrix}}$

$\big(\mathcal{X}(M) = \text{Module over [commutative, unital] Ring } \mathcal{F}(M)\big)$

- ∗ $(fV)_p = f(p) V_p$
- ∗ $(V + W)|_p = V_p + W_p$
- [Free] Module Basis, $\underline{\partial_i}$: $\partial_i(p) = \partial_i\Big|_p \in T_p(M)$
- $V_p = \sum V_p x^i \partial_i\Big|_p = \Big(\underline{\sum [V x^i] \partial_i}\Big)_p = (\underline{V})_p$
- $\underline{\text{Vector Field } V \Leftrightarrow \overline{V} \text{ is a derivation on } \mathcal{F}(M)}$
- Commutator : $\begin{matrix} \text{Derivation } [\overline{V}, \overline{W}] \\ ([\overline{V}, \overline{W}] f)_p = (\overline{V}(Wf) - \overline{W}(Vf))_p \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} \text{Vector Field } [V, W]_p \\ [V, W]_p f = V_p(\overline{W} f) - W_p(Vf) \end{matrix} \qquad * \ [\partial_i, \partial_j] = 0$

∗ $[f\overline{V}, g\overline{W}]h = \underline{fV(g(Wh))} - gW(f(Vh))$
$\quad = \underline{fgV(Wh)} - gfW(Vh) + \underline{f(Vg)(Wh)} - g(Wf)(Vh) = fg[V, W]h + f(Vg)(Wh) - g(Wf)(Vh)$

- $\underline{\phi\text{-Related Maps}, \ X \sim_\phi Y}$: $\forall p \in M, \ d\phi_p(X_p) = Y_{\phi p} \ \Leftrightarrow \ \forall g \in \mathcal{F}(N), \ \overline{X}(g \circ \phi) = (\overline{Y} g) \circ \phi$

∗ Transferred Vec. Field of X : $(Y) = (d\phi X) \ \Rightarrow \ \boxed{(d\phi X) g = \overline{X}(g \circ \phi) \circ \phi^{-1} \in \mathcal{F}(N)}$

∗ $X_1 \sim_\phi Y_1, \ X_2 \sim_\phi Y_2 \ \Rightarrow \ \underline{[X_1, X_2] \sim_\phi [Y_1, Y_2]}$

$\underline{\text{Cotangent Space } T_p^*(M)} \ \ni \ \underline{\text{Covector } v_p^* : T_p(M) \to \mathbb{R}}$

$\underline{\text{One-Form } \theta \in \mathcal{X}^*(M)}$ (Covec. Field = Module Over $\mathcal{F}(M)$) $: p \in M \to \theta_p \in \bigcup_{p \in M} T_p^*(M) :$ $\boxed{\underset{\mathbb{R}}{\theta_p V_p} = \underset{\theta_p : T_p(M) \to \mathbb{R}}{\theta_p(V_p)} = \underset{\overline{\theta} : \mathcal{X}(M) \to \mathcal{F}(M)}{(\theta V)_p = (\overline{\theta} V)_p}}$

- $\underline{\text{Differential}, d : \mathcal{F}(M) \to \mathcal{X}^*(M)}$

  $df \in \mathcal{X}^*(M) : \ \frac{\partial(f \circ \xi^{-1} \circ q)}{\partial t}\Big|_0 = \frac{\partial F}{\partial u}^T q'(0) = \sum q_i'(0) \partial_i \Big|_0 F = \big(\big[\sum v(x^i) \partial_i\big] F \big)|_0$

  $= df_p V_p = \underline{(\overline{df} V)_p} \qquad = \underline{(\overline{V} f)_p}$

- $\underline{\text{[Free] Module Basis, } dx^i} : \ dx^i(\partial_j) = \partial_j(x_i) = \delta_{ij}$

- $\theta_p = \sum \theta_p(\partial_i|_p) dx_p^i = \sum \partial_i|_p \theta_p dx_p^i \ \Rightarrow \ df = \sum \frac{\partial f}{\partial x^i} dx^i$

$\underline{\text{Tangent Bundle}} : \quad T(M) = \{(p, w)\} \ , \ p \in M, \ w \in T_p(M)$

$$y_{\alpha(u_{1,\alpha}, \ldots, x_1, \ldots)} = \Big\{ \big(X_{\alpha(u_{1,\alpha}, u_{2,\alpha}, \ldots)}, \ x_1 \tfrac{\partial X_\alpha}{\partial u_{1,\alpha}} + x_2 \tfrac{\partial X_\alpha}{\partial u_{2,\alpha}} + \ldots \big) \Big\} \ , \ x_i \in R$$

$$T(M) = \bigcup_\alpha y_\alpha(U_\alpha \times R^n)$$

$\underline{\text{Hypersurface, } P \overset{\text{subman}}{\hookrightarrow} M} : \ \dim M = \dim P + 1$

- $\underline{\text{Level Hypersurface} : \ f^{-1}(q)}$
  - $*$ $f : M \to N = \mathbb{R}^1$
  - $*$ $\forall p, \ q = f(p), \ df_p \neq 0$

$\underline{\text{Regular Value, } q \in N} : \ \forall p \big(\phi(p) = q \big) \big(\text{onto } d\phi_p \big)$

- $\phi^{-1}(q) \overset{\text{subman}}{\hookrightarrow} M$
- $\dim M = \dim N + \dim \phi^{-1}(q)$

$\underline{\text{Submersion, } \phi} : \ \forall p \big(p \in M \big) \big(\text{onto } d\phi_p \big)$

$\underline{\text{Immersion [Map] , } \phi}$:

$\phi : M \to N$

$d\phi_p : T_p(M) \to T_{\phi p}(N)$ is 1-1

$\underline{\text{Isometric Immersion}}$:

$\langle d\phi_p(v), d\phi_p(w) \rangle_{\phi(p)} = \langle v, w \rangle_p$

Euclid. Metric on R^n = Riem. Metric on S

$\underline{\text{Smooth Embedding}}$:

Homeo. + Immers.

- Immer $\phi : M \to N$
- Homeo $\overline{\phi} : M \to \phi(\mathcal{M}) \subseteq N$

- $\left.\begin{array}{l} \text{Immer } \phi : M \to N \\ \text{Diffeo } \overline{\phi} : M \to \phi(M) \end{array}\right\} \Rightarrow$ Induc $\iota = \phi \circ \overline{\phi}^{-1} : \phi(M) \to N \Rightarrow$ Subman $\phi(M)$

- $P \overset{\text{subman}}{\hookrightarrow} N \ \Rightarrow$ Immer $\iota : P \to N$

$\underline{\text{Immersed Submanifold}}$ (Immersed Manifold Subset) : Mani $P \subset$ Mani N, Immer $\iota : P \to N$

* $\underline{\text{Examples (skipped, p.430)}}$: Hyperbolic Geom., Flat Torus, P^2, Klein Bottle

Riemannian Metric for Geometric Surface (Riem. mani. of dim. n) , $\langle\ ,\ \rangle_p$ $\forall T_p(M)$:

- $g_{ij} = \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle_p \ \rightarrow\ E = g_{11},\ F = g_{12} = g_{21},\ G = g_{22}$
- $\| w = \sum_i u_i' \frac{\partial}{\partial u_i} \|_p^2 = \sum_i g_{ii} (u_i')^2 + \sum_{i \neq j} 2 g_{ij} u_i' u_j'$ (+ check cond. 2)

Covariant Derivative of vec. field v rela. to vec. field w , $D_w(v)$:

- $D_{f(M)u+g(M)w}(v) = f D_u(v) + g D_w(v)$
- $D_v\big(f(M)u + g(M)w\big) = f D_v(u) + \frac{\partial f}{\partial v} u + g D_v(w) + \frac{\partial g}{\partial v} w$
- $\frac{\partial f}{\partial v} = \frac{d(f\circ\alpha)}{dt}\Big|_0\ ,\ \alpha'(0) = v$

Christoffel Symbols (for covar. der., not $X_{u_i u_j}$) , Γ_{ij}^k : • $D_{X_i} X_j = \sum_{k=1}^{n} \Gamma_{ij}^k X_k$

$$\Rightarrow \begin{bmatrix} D_{X_1}X_1 \\ D_{X_1}X_2 \\ \vdots \end{bmatrix} [X_1\ X_2\ \dots] = \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & \dots \\ \Gamma_{12}^1 & \Gamma_{12}^2 & \dots \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix} [X_1\ X_2\ \dots] = \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & \dots \\ \Gamma_{12}^1 & \Gamma_{12}^2 & \dots \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & \dots \\ g_{12} & g_{22} & \dots \\ \vdots & \vdots \end{bmatrix}$$

• $D_{X_i} X_j = D_{X_j} X_i \ \leftrightarrow\ \Gamma_{ij}^k = \Gamma_{ji}^k$ • $\frac{\partial}{\partial u_i} \langle X_j, X_k \rangle = \langle D_{X_i} X_j, X_k \rangle + \langle X_j, D_{X_i} X_k \rangle$

$$\Rightarrow * \boxed{\frac{1}{2}\left(\frac{\partial}{\partial u_i} g_{jk} + \frac{\partial}{\partial u_j} g_{ik} - \frac{\partial}{\partial u_k} g_{ij}\right) = \langle D_{X_i} X_j, X_k \rangle = \sum_n \Gamma_{ij}^n g_{nk}}$$

$$\Rightarrow * \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & \dots \\ \Gamma_{12}^1 & \Gamma_{12}^2 & \dots \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} D_{X_1}X_1 \\ D_{X_1}X_2 \\ \vdots \end{bmatrix} [X_1\ X_2\ \dots] \overset{(g_{ij}^{-1} = g^{ij})}{\begin{bmatrix} g_{11} & g_{12} & \dots \\ g_{12} & g_{22} & \dots \\ \vdots & \vdots \end{bmatrix}^{-1}} \leftrightarrow \boxed{\begin{aligned} \Gamma_{ij}^k &= \sum_n \langle D_{X_i} X_j, X_n \rangle g^{nk} \\ &= \frac{1}{2} \sum_n g^{nk} \left[\frac{\partial g_{jn}}{\partial u_i} + \frac{\partial g_{in}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_n}\right] \end{aligned}}$$

Sectional Curvature of M at p along σ , $K\begin{pmatrix} p \in M, \\ \sigma \in T_p(M) \end{pmatrix}$: $K(p,\sigma) = K_p(S)\ \Big|\ S \in \bigcup$ (geode. at p tangent to σ) (no else given)

Variable Change Inner Product is Isometric to Original Inner Product

$$w = \sum_{i=1} \frac{du^i}{dt} \frac{\partial}{\partial u^i} \equiv \left[\frac{du}{dt}\right]^i \left[\frac{\partial}{\partial u}\right]_i$$

$$\|w\|^2 = \sum_{i,j} \frac{du^i}{dt} \frac{du^j}{dt} \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^i} \right\rangle \equiv \boxed{\left[\frac{du}{dt}\right]^i \left[\frac{du}{dt}\right]^j g_{ij}}$$

$$\overline{w} = \sum_{n=1} \frac{dx^n}{dt} \frac{\partial}{\partial x^n} \equiv \left[\frac{dx}{dt}\right]^n \left[\frac{\partial}{\partial x}\right]_n$$

$$\left[\frac{dx}{dt}\right]^n = \sum_{i=1} \frac{\partial x^n}{\partial u^i} \frac{du^i}{dt} \equiv \left[\frac{\partial x}{\partial u}\right]^n_{\ i} \left[\frac{du}{dt}\right]^i$$

$$\left[\frac{\partial}{\partial x}\right]_n = \sum_{i=1} \frac{\partial u^i}{\partial x^n} \frac{\partial}{\partial u^i} \equiv \left[\frac{\partial u}{\partial x}\right]^i_{\ n} \left[\frac{\partial}{\partial u}\right]_i$$

$$\delta_{ij} = \left[\frac{\partial x}{\partial u}^{-1}\right]^i_{\ n} \left[\frac{\partial x}{\partial u}\right]^n_{\ j}$$

$$\delta^i_{\ j} = \left[\frac{\partial u}{\partial x}\right]^i_{\ n} \left[\frac{\partial x}{\partial u}\right]^n_{\ j}$$

$$\begin{aligned} \|\overline{w}\|^2 = \sum_{n,m} \frac{dx^n}{dt} \frac{dx^m}{dt} \left\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^m} \right\rangle &\equiv \left[\frac{dx}{dt}\right]^n \cdot \left[\frac{dx}{dt}\right]^m \cdot h_{nm} \\ &= \left[\frac{\partial x}{\partial u}\right]^n_{\ k} \left[\frac{du}{dt}\right]^k \cdot \left[\frac{\partial x}{\partial u}\right]^m_{\ l} \left[\frac{du}{dt}\right]^l \cdot \left[\frac{\partial u}{\partial x}\right]^i_{\ n} \left[\frac{\partial u}{\partial x}\right]^j_{\ m} g_{ij} \\ &= \left[\frac{du}{dt}\right]^k \left[\frac{\partial u}{\partial x}\right]^i_{\ n} \left[\frac{\partial x}{\partial u}\right]^n_{\ k} \cdot \left[\frac{du}{dt}\right]^l \left[\frac{\partial u}{\partial x}\right]^j_{\ m} \left[\frac{\partial x}{\partial u}\right]^m_{\ l} g_{ij} \\ \boxed{\left[\frac{du}{dt}\right]^i \left[\frac{du}{dt}\right]^j g_{ij}} &= \left[\frac{du}{dt}\right]^k \delta^i_{\ k} \cdot \left[\frac{du}{dt}\right]^l \delta^j_{\ l}\ g_{ij} \end{aligned}$$

Simplify the Following into one Fraction or Better (do the work on some paper):

1.) $(abc)^2$

2.) $(ac^3b^{-1})^2$

3.) $\left(\frac{ac^3}{b}\right)^{-2}$

3.) $\frac{\frac{a}{b}}{c}$

1.) a/b

2.) $1/a \cdot b$

3.) $a^{-1}b^2$

2.) $1/(a/b)$

2.) $1/a/b$

2.) $\frac{a}{\frac{c}{d}}$

3.) $\frac{\frac{a}{b}}{c}$

1.) $\frac{a/b}{c/d}$

1.) $1/b^{-m}$

1.) $\dfrac{b^n b^m}{b^{m+n}}$

1.) $b^n \dfrac{b^{-n+m}}{b^{-3}}$

3.) $\left(\frac{ac^3}{b}\right)^{-2}$

3.) $\frac{\frac{a}{b}}{c}$

1.) a/b

2.) $1/a \cdot b$

3.) $a^{-1}b^2$

2.) $1/(a/b)$

2.) $1/a/b$

2.) $\frac{a}{\frac{c}{d}}$

3.) $\frac{\frac{a}{b}}{c}$

1.) $\frac{a/b}{c/d}$