Vector Space, $V^{n=\dim V} \ni v$ • $\underline{\dim V_{\text{ is Indep. of Basis}}}$

$\underline{\text{Linear Map, } M \in L(V,W) : V \to W}$ • $\underline{\text{Indep. of Basis}}$ $\Leftarrow$ Basis gives v unique coord.

- $\underline{\text{Fund. [Rank/Nullity] Theor. Lin. Maps}}$: $\boxed{\dim V^{n<\infty} = \dim \operatorname{null}(M) + \dim \operatorname{range}(M)}$
- $\dim V = \dim W < \infty \;\Leftrightarrow\; \underline{\text{Isomorphic}} = \exists M^{-1}_{LR},\; M : V \to W$
- $\dim L(V,W) = \dim F^{m,n} = nm = (\dim V^{n<\infty})(\dim W^{m<\infty})$
- $\underline{\text{Matrices, } \overline{M} \in F^{n,m}}$: $\boxed{M(v_i) = w_r e^r M e_k \phi^k v_i = w_r e^r M e_i = w_r M^r{}_i \in W}$

$\underline{\text{Subspace, } U}$: $\boxed{* \; 0 \in U \quad * \; u+v \in U \quad * \; \lambda u \in U}$

$\underline{\text{Subspace Direct Sum}}$, $\oplus$: $U + V = U \oplus V \;\Leftrightarrow\; \forall w \in U+V,\; \exists! w = u + v$ Ex : $\mathbb{R}^2 = (x,0) \oplus (0,y)$

- $\Leftrightarrow\; U \cap V = \{0\} \;\Leftrightarrow\; w \in U \cap V,\; w = 0$ • $\Leftrightarrow$ onto $\Gamma : U \times V \to U + V$
- $\Leftrightarrow\; \dim U + \dim V = \dim(U \times V) = \dim(U+V)$
- $\exists W,\; V = U \oplus W$ (easy for finite, harder for larger)
- $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

$\underline{\text{Orthogonal Complement, } U^\perp}$ $\underline{\text{of subset } U}$: $\{v \in V : \forall u \in \underline{{}_{\text{(subset)}} U}, \langle v, u\rangle = 0\}$

- $U^\perp$ is Subspace • $U \cap U^\perp \subseteq \{0\}$ • $U \subseteq W \subseteq V \;\Rightarrow\; W^\perp \subseteq W^\perp$

$\underline{U^\perp \text{ of subspace } U}$: • $\boxed{V = U \oplus U^\perp}$ $\Rightarrow$ $\boxed{\dim V = \dim U + \dim U^\perp}$ • $U = (U^\perp)^\perp$

$\underline{\text{Projection Operator, } P_U}$: $\boxed{P_U^2 = P_U}$ (Idempotent) $\Rightarrow$ $PU = U$

- $\mathbb{1} = P + (1 - P) = P + P_\perp$ • $\boxed{V = [U = \operatorname{range}(P_U)] \oplus \left[U^\perp = \operatorname{null}(P_U)\right]}$

$\underline{\text{Operator } T \in L(V) = L(V,V)}$: $\boxed{T(v) = v_r T^{rk} \phi_k(v)}$ $\sim$ $|Tv_r\rangle \delta^{rk} \langle v_k|v\rangle = |v_r\rangle T^{rk} \langle v_k | v\rangle$

- $\boxed{\text{Schur's Theor.} : \forall T,\; \exists T^{nn} = U_{pper};\; \text{Gram} + \langle\cdot|\cdot\rangle \to \exists T^{nn} = U_{pper,\,\perp}}$ • $\exists T^{-1} \;\Rightarrow\; \exists! T^{-1}$
- $T \in L(\text{Complex } V)$, $\boxed{\exists \lambda}$ $\Leftarrow$!Lin. Ind. $\{T^k v : 0 \le k \le n\} \;\Rightarrow\; \exists \vec{a} \ne 0,\; a_i(T^i v) = 0 = (a_i T^i) v = c\left[\prod (T - \lambda_j \mathbb{1})\right] v$
- $T \in L(\underline{V^{n<\infty}})$, $\big(\underline{\text{1-1}} : Tv = Tu \leftrightarrow v = u \;\Leftrightarrow\; \underline{\text{onto}} : \forall v,\; \exists u,\; Tu = v \;\Leftrightarrow\; \underline{\exists T^{-1}}\big)$
- $\dim L(V,W) = \dim V \times \dim W$ • $\exists I^{-1},\; \underline{I = ST} \Rightarrow \exists (ST)^{-1} = T^{-1} S^{-1} \Rightarrow TS(TT^{-1}) = \underline{TS = I}$
- $\underline{\text{Coord. Change.}}$: $I^e{}_e = C^e{}_f C^f{}_e = \underline{[C^f{}_e]^{-1} C^f{}_e = C^e{}_f [C^e{}_f]^{-1} = \delta^r{}_k = I^f{}_f = C^f{}_e C^e{}_f}$

$\underline{\text{Quotient Space, } V/U}$:

- $v - w \in U \;\Leftrightarrow\; v + U = w + U \;\Leftrightarrow\; (v+U) \cap (w+U) \ne \varnothing$
- $\underline{\text{is Vector Space}}$:
  - $(v+U) + (w+U) \equiv (v+w) + U \;\Leftarrow\; \vec{v} \in V/U$!unique : prove $\begin{matrix}(v+U = \hat{v}+U)\\(w+U = \hat{w}+U)\end{matrix} \Rightarrow +\begin{matrix}(v+U)\\(w+U)\end{matrix} = +\begin{matrix}(\hat{v}+U)\\(\hat{w}+U)\end{matrix}$
  - $\lambda(v+U) \equiv (\lambda v) + U \;\Leftarrow\; \vec{v} \in V/U$!unique : prove $(v + U = \hat{v} + U) \;\Rightarrow\; (\lambda v) + U = (\lambda \hat{v}) + U$
- $\underline{\text{Quotient Map}}$, $\pi(v) = v + U$: $\begin{matrix}\pi : V \to V/U\\ \pi \in L(V, V/U) \quad \text{(check linear map)}\end{matrix}$
- $\boxed{\dim V = \dim(V/U = \operatorname{range}(\pi)) + \dim(U = \operatorname{null}(\pi))}$
- $\tilde{M}(v + \operatorname{null}(M)) = Mv \in W \;\Leftarrow\; v + \operatorname{null}(M) = u + \operatorname{null}(M) \;\Rightarrow\; v - u \in \operatorname{null}(M) \;\Rightarrow\; M(v-u) = 0 \;\Rightarrow\; Mv = Mu$

$*$ $\tilde{M} \in L(V/\operatorname{null}(M), V)$ $*$ 1-1 $*$ $\operatorname{range}(M) = \operatorname{range}(\tilde{M}) \overset{\text{iso}}{=} V/\operatorname{null}(M)$

- $\underline{\text{Quotient [Subspace] Operator, } T/U \in L(V/U)}$: $(T/U)(v+U) \equiv Tv + U$

Dual Vector/Linear Functional, $\phi \in L(V,F) =$ Dual Space V' • $\dim V = \dim V' < \infty$

• Dual Basis, ϕ^i : $\phi^i(v_j) = \delta^i_j \;\Rightarrow\; \phi(v) = (a_i\phi^i)(c^j v_j) = a_n c^n$ • $a_n\phi^n = 0,\; (a_n\phi^n)v_j = a_j \;\Rightarrow\; a_j = 0$

$\boxed{\textit{Riesz-Repr. Theo.}\text{ (after } \langle\cdot|\cdot\rangle \text{ space)} : \forall\phi,\; \underline{\exists!} v_\phi \in V^{n<\infty},\; \phi(v) = \underline{\phi(|e_k\rangle)\delta^{ki}\langle e_i|v\rangle} = \underline{\langle v_\phi|v\rangle} \Rightarrow |v_\phi\rangle = \overline{\phi(e_k)}\delta^{ki}|e_i\rangle}$

$* \;\langle v_\phi|v\rangle - \langle u_\phi|v\rangle = 0 \;\Leftrightarrow\; v_\phi = u_\phi$

Dual Map of $M \in L(V,W)$, $M' \in L(W',V')$: $\boxed{M'(\psi) = \psi \circ M}$ • $(M_1+M_2)' = M_1' + M_2'$

$*$ $\underline{(M'\psi)}v = \psi(M(v)) \sim \langle w_\psi|M(v)\rangle = \langle \underline{M^\dagger w_\psi}|v\rangle$
$= (c_n\psi^n)w_r e^r M e_k \phi^k(a^m v_m) = c_r M^r{}_k a^k$
$= [(M^T)\vec{c}]^T\vec{a} \;\Rightarrow\; (\psi^r M)v_k = M^r{}_k$

$(M'\psi^r)v_k = (\phi^j M'_{jr})v_k$
$= (M')^k{}_r \;\Rightarrow\; \boxed{M' = M^T}$

• $(\lambda M)' = \lambda M'$ • $(MN)' = N'M'$

$*$ $(MN)'(\psi) = (\psi\circ M)\circ N$
$N'M'(\psi) = N'(\psi \circ M)$

Annihilator [Subs.] of $U \subseteq V$, $U^0 \subseteq U'$: $\phi^0 \in U^0,\; \phi^0(U) = \{0\}$

• $\dim V^{<\infty} = \dim U + \dim U^0$
• $\dim V' = \dim U' + \dim U^0$

• $\operatorname{null} M' = (\operatorname{range} M)^0$
• $\operatorname{range} M' = (\operatorname{null} M)^0$

• $\dim\operatorname{null} M' - \dim W' = \dim\operatorname{null} M - \dim V$
• $\dim\operatorname{range} M' = \dim\operatorname{range} M = \operatorname{rank} M$

Inner Product Space: $\operatorname{span}\{f_i\} = W$, $\operatorname{span}\{e_i\} = V$

Matrix Vec. $\langle\cdot|\cdot\rangle$: $a = a^i e_i,\; b = b^i e_i \;\Rightarrow\; \boxed{\text{const } \langle a|b\rangle = a_i^* b^i = \vec{a}^* \cdot \vec{b} = \vec{a}^{*T}\vec{b} \quad \underline{\langle\cdot|\cdot\rangle \text{ defines } \perp \text{ basis}}}$

• $\widehat{\perp}$ Basis, $\{e_i\}$: $\delta_{ij} = \langle e_i|e_j\rangle \Rightarrow e_i = i^n f_n,\; e_j = j^n f_n,\; \langle i^n f_n|j^n f_n\rangle = i^*_n j^n = \delta_{ij} \Rightarrow \boxed{\vec{e}_i^{*T}\vec{e}_j = \delta_{ij} \quad \text{both } \widehat{\perp} \text{ basis}}$

• Coord. Swap $\widehat{\perp}\{e_i\} \to \widehat{\perp}\{f_i\}$: $\delta^r{}_k = C^f{}_e C^e{}_f = C^e{}_f C^f{}_e = [C^f{}_e]^{-1} C^f{}_e \;\Rightarrow\; C^e{}_f = \boxed{[C^f{}_e]^{-1} = [\vec{e}_1, ..., \vec{e}_n]^{*T} = [C^f{}_e]^{*T}}$

Adjoint, $M^\dagger$ ($M^\dagger \in L(W,V)$) : $\phi(v) = \langle w|M(v)\rangle_W \equiv \langle \underline{M^\dagger(w)}|v\rangle_V$
$= \langle w|Mv\rangle = \langle w|f_r\rangle_W M^r{}_k \langle e^k|v\rangle_V$
$= \langle \underline{e^k M^{*r}_k \langle f_r|w\rangle_W}|v\rangle_V \equiv \langle \underline{M^\dagger w}|v\rangle$:

$*$ Riesz-Rep : Given M, w, $\exists!\, \underline{M^\dagger(w)} \in V$
$*$ $\boxed{(M^\dagger)_{rk} = M^*_{kr} = (M^{*T})_{rk} \quad \text{when both } \widehat{\perp} \text{ basis}}$
$*$ $\boxed{C^f{}_e \cong C \in L(V,V) \;\Rightarrow\; C^{*T} = C^\dagger}$ (see unit.)

• $\langle Mv|w\rangle \overset{= \overline{\langle w|Mv\rangle_W}}{\underset{= \langle M^\dagger w|v\rangle_V}{=}} \langle v|M^\dagger w\rangle_V = \langle M^{\dagger\dagger}v|w\rangle$ • $\operatorname{null}(M) = \left[\operatorname{range}(M^\dagger)\right]^\perp$ • $\operatorname{range}(M) = \left[\operatorname{null}(M^\dagger)\right]^\perp$

Normal Op. : $AA^\dagger = A^\dagger A \;\Leftrightarrow\; \forall v,\; 0 = \langle v, (AA^\dagger - A^\dagger A)v\rangle \;\Leftrightarrow\; \forall v,\; \|Av\|^2 = \|A^\dagger v\|^2$

• $Av = \lambda v \;\Leftrightarrow\; A^\dagger v = \lambda^* v \quad \Leftarrow \forall v,\; \|(A-\lambda I)v\| = \|(A-\lambda I)^\dagger v\| = \|(A^\dagger - \lambda^* I)v\|$

[Complex] Spectral Theorem : Normal $A \;\Leftrightarrow\;$ Diagonalizable$_{\widehat{\perp}e_i}$ $A \;\Leftrightarrow\; \{\text{Eigenvector of } A\} = \{\text{Basis } V\}_{\widehat{\perp}}$

$*\; A = U_{pp,\widehat{\perp}},\; \|Ae_i\|^2 = \|A^\dagger e_i\|^2 \;\Leftrightarrow\; A = D_{iag,\widehat{\perp}} \;\Leftrightarrow\; Ae_i = D_{ii}e_i$ • $\boxed{\overline{A}_{v_i \to v_i} = C^v{}_e D C^e{}_v}$

Unitary Op. : $\forall v,\; \|Uv\|^2 = \|v\|^2 \;\Leftrightarrow\; \underset{(\textit{isom.})}{U^\dagger U} = \underset{(\textit{coisom.})}{UU^\dagger} = I$ • $\boxed{\vec{u}_i^{*T}\vec{u}_j = \delta_{ij} = [\vec{u}^i]^{*T}\vec{u}^j \quad \perp \text{ basis}}$

Herm. Op. : $H = H^\dagger$
• $\Leftrightarrow\; \forall v,\; \langle v, Hv\rangle \in R \quad \Leftarrow \langle v,Hv\rangle - \overline{\langle v,Hv\rangle} = \langle v, (H-H^\dagger)v\rangle = 0$
• $\lambda_i \in \mathbb{R}$ • $\forall v \in V(\mathbb{C}),\; \langle v,Tv\rangle = 0 \Rightarrow T = 0 \quad \Leftarrow \forall v \in V(\mathbb{R}),\; \langle v,Hv\rangle = 0 \Rightarrow H = 0$

Positive Semi-Definite Op. $\boxed{\wedge(H = H^\dagger) \text{ if } \mathbb{R}}$; $\Rightarrow (H = H^\dagger)$ if $\mathbb{C}$: $\boxed{\forall v,\; \langle v|Hv\rangle \geq 0}$
• $\lambda_i \geq 0 \;\Leftarrow\; \langle v|\lambda v\rangle$
• $\exists!_{\text{pos}}\; R = \sqrt{H},\; H = R^\dagger R = R^2$

$\underline{\text{Gram Matr.}}/$

$\underline{\text{Sq. Rt. Gram } \sqrt{M^\dagger M} \neq M}$:

- $\|Mv\|_W^2 = \langle M^\dagger M v | v\rangle_V = \langle v | M^\dagger M v\rangle_V = \|\sqrt{M^\dagger M} v\|_V^2 \geq 0$
- pos. def. $\langle v | \sqrt{M^\dagger M} v\rangle \geq 0 \;\Leftrightarrow\; \lambda_i(\sqrt{M^\dagger M}) = \sigma_i(M) \geq 0$ • M is unif. cont. func.
- $\boxed{(M^\dagger M) e_i = \sigma_i^2 e_i \;\Leftrightarrow\; \sqrt{M^\dagger M} e_i = \sigma_i e_i}$ • $\min \sigma_i = \min \|T\hat{v}\| \leq |\lambda_i| \leq \max \sigma_i = \max \|T\hat{v}\|$

- $\text{null}(\sqrt{M^\dagger M}) = \text{null}(M) = \text{null}(M^\dagger M) \;\Rightarrow\; \begin{matrix} \dim\text{range}(\sqrt{M^\dagger M}) & = & \dim\text{range}(M) & = & \dim\text{range}(M^\dagger M) \\ \text{rank}(\sqrt{M^\dagger M}) & = & \text{rank}(M) & = & \text{rank}(M^\dagger M) \end{matrix} \leq \min(n_{row}, m_{col})$

$*$ Ex, $T = |x\rangle\langle u|$ (Rank = 1) :

$$T^\dagger T|e_i\rangle = |u\rangle\|x\|^2\langle u|e_i\rangle = \sigma_i^2 \underline{|e_i\rangle} = \underline{|\hat{u}\rangle}\|u\|\|x\|^2\langle u|e_i\rangle$$

$$\begin{matrix} \|u\|^2\|x\|^2\langle u|e_i\rangle = \sigma_i^2\langle u|e_i\rangle \\ \|x\|^2\|\langle u|e_i\rangle\|^2 = \sigma_i^2 \end{matrix} \;\Rightarrow\; \begin{matrix} \sigma_i = \|\langle u|e_i\rangle\|\|x\| \\ \langle u|e_i\rangle = \|\langle u|e_i\rangle\| = \|u\|\delta(\hat{u}, e_i) \end{matrix}$$

$$\sqrt{T^\dagger T}|e_i\rangle = \sigma_i|e_i\rangle = |e_i\rangle\|u\|\|x\|\delta(\hat{u}, e_i) = |u\rangle\|x\|\delta(\hat{u}, e_i)$$

$$\sqrt{T^\dagger T}|v\rangle = |e_i\rangle\|u\|\|x\|\delta(\hat{u}, e_i)\delta^{ij}\langle e_j|v\rangle$$

$$= \boxed{|u\rangle \tfrac{\|x\|}{\|u\|}\langle u|v\rangle}$$

$\underline{\text{SVD } \textbf{on Operators}\text{/Polar Decomposition}}$: $\forall T,\ \exists U,\ T = U\underline{\sqrt{T^\dagger T}}$ $\underline{\text{unitary}} \times \underline{\text{pos def}}$ (proof?)

$$* \;\; \|\sqrt{T^\dagger T} e_i\| = \|Te_i\| = \sigma_i \;\Rightarrow\; \exists U \mid Te_k = U\sigma_k e_k = U_k\sigma_k \;\Rightarrow\; \boxed{\begin{matrix} \overline{T}V = U\Sigma \;\Rightarrow\; \overline{T} = U\Sigma V^\dagger \\ T = \sum_i |f_i\rangle\sigma_i\langle e_i| \end{matrix}}$$

- $T^\dagger = U\sqrt{TT^\dagger} \;\Rightarrow\; \boxed{T = \sqrt{TT^\dagger}U^\dagger}$ • $T(T^\dagger T e_i) = \underline{\sigma_i^2}(Te_i) = \underline{TT^\dagger}(Te_i)$

$\underline{\text{Singular Value Decomp. (SVD on } M)}$:

$$M^\dagger M_{wv} = [V_1, V_2]\begin{bmatrix} \sigma^2_{\neq 0} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1^\dagger \\ V_2^\dagger \end{bmatrix}$$

$$\begin{bmatrix} \underline{D_{rr}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \underline{(MV_1)^\dagger MV_1} & (MV_1)^\dagger MV_2 \\ (MV_2)^\dagger MV_1 & (MV_2)^\dagger MV_2 \end{bmatrix}$$

$$I_{wv} = ([V_1, 0] + [0, V_2])\left(\begin{bmatrix} V_1^\dagger \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ V_2^\dagger \end{bmatrix}\right) = V_1V_1^\dagger + V_2V_2^\dagger$$

$$I_{rr} = V_1^\dagger V_1 \qquad I_{nn} = V_2^\dagger V_2 \qquad (r_{ank} + n_{ull} = v)$$

$$\|\sqrt{M^\dagger M} e_i^1\| = \sigma_i = \|Me_i^1\| \;\Rightarrow\; Me_k^1 = U^1\sigma_k e_k^1 = U_k^1\sigma_k$$

- $U_1 = MV_1\sqrt{D}^{-1} \;\Rightarrow\; U_1\sqrt{D}V_1^\dagger = M(I - \underline{V_2}V_2^\dagger) = M$
- $\underline{\langle Me_i^1 | Me_j^1\rangle} = \sigma_i\delta_{ij} \;\Rightarrow\; \underline{\langle U_i^1 | U_j^1\rangle = \delta_{ij}}$

$$\Rightarrow \; M = [U_1, U_2/0]\underset{[0]^{\underline{*}}}{\begin{bmatrix} \sigma_{\neq 0} & 0 \\ 0 & 0 \end{bmatrix}}\begin{bmatrix} V_1^\dagger \\ V_2^\dagger \end{bmatrix} = \boxed{\begin{matrix} U\Sigma V^\dagger \\ \sum_i |f_i\rangle\sigma_i\langle e_i| \end{matrix}}$$

$\underline{*}$ (if needed) • $\underline{U_k^1 = Me_k,\ \langle U_i^1 | U_j^2\rangle = 0}$

- $d\vec{X} = \begin{bmatrix} | \\ X_t \\ | \end{bmatrix}[dt] \;\Rightarrow\; \|d\vec{X}\|_2^2 = \langle dt|(X_t^\dagger X_t)dt\rangle_2 \;\Rightarrow\; \|d\vec{X}\|_2 = \sqrt{X_t^\dagger X_t}|dt| = \sigma_i|dt|$
- $d\vec{X} = \begin{bmatrix} | & | \\ X_u & X_v \\ | & | \end{bmatrix}\begin{bmatrix} du \\ dv \end{bmatrix} \;\Rightarrow\; \|d\vec{X}\|^2 = \langle d\vec{U}|(J^\dagger J)d\vec{U}\rangle$
- $A\left([\vec{X}_u\, du,\ \vec{X}_v\, dv] = \begin{bmatrix} | & | \\ X_u & X_v \\ | & | \end{bmatrix}\begin{bmatrix} du & 0 \\ 0 & dv \end{bmatrix}\right) = A(\sigma_1\sigma_2 V^\dagger\begin{bmatrix} du & 0 \\ 0 & dv \end{bmatrix}) = \sigma_1\sigma_2\, dudv = |\det\sqrt{J^\dagger J}|\, dudv$

$\underline{\text{Range, range}(M)}$: $\{Mv : v \in V\} = M(V)$

- Right $M^{-1} : M(M^{-1}v) = \mathbb{1}v \;\Rightarrow\; \underline{\text{onto}} \;\Leftrightarrow\; \text{range}(M) = W$
- $V = \text{range}(T^0);\ \forall k,\ \text{range}(T^k) \supseteq \text{range}(T^{k+1})$
- $\text{range}(T^k) = \text{range}(T^{k+1}) \;\Rightarrow\; \forall m \geq k,\ \text{range}(T^k) = \text{range}(T^m)$

$*$ $\text{range}(T^{\dim V}) = \text{range}(T^{\dim V + 1})$

$\underline{\text{Nullspace, null}(M)}$: $\{v : Mv = 0\}$

- Left $M^{-1} : M^{-1}Mv = M^{-1}Mu \;\Rightarrow\; \underline{\text{1-1}} \;\Leftrightarrow\; \text{null}(M) = \{0\} \;\Leftarrow$ 1.) $M(v) = 0 = M(u)$ 2.) $0 = M(0 = u - v) = M(u) - M(v)$ (linear)
- $\forall k \geq 0,\ \text{null}(T^k) \subseteq \text{null}(T^{k+1})$
- $\text{null}(T^k) = \text{null}(T^{k+1}) \;\Rightarrow\; \forall j > k,\ \text{null}(T^k) = \text{null}(T^j)$

$*$ $\text{null}(T^{n=\dim V}) = \text{null}(T^{n+1}) \;\Leftarrow\; \dim \text{null}(T^n) \leq \dim V$

- $\underline{V \neq \text{null}(T) \oplus \text{range}(T) \;\Leftrightarrow\; \{0\} \neq \text{null}(T) \cap \text{range}(T)}$
- $\underline{V = \text{null}(T^n) \oplus \text{range}(T^n)} \;\Leftarrow$ $\{0\} = \text{null}(T^n) \cap \text{range}(T^n) \;\Rightarrow\; \text{null}(T^n) \oplus \text{range}(T^n)$; $\dim V = \dim \text{null}(T^n) + \dim \text{range}(T^n) \;=\; \dim\left(\text{null}(T^n) \oplus \text{range}(T^n)\right)$
- $v \in \text{null/range}(T^n) \;\Rightarrow\; Tv \in \text{null/range}(T^n)$ (are $\underline{\text{Invariant}}$ spaces/closed)
- $\forall n \left[\text{null}(T^n) = \text{null}(T^{n+1}) \;\Leftrightarrow\; \text{range}(T^n) = \text{range}(T^{n+1})\right]$

$\underline{\text{Eigenspace, E}(\lambda, T)}$: $\text{null}(T - \lambda\mathbb{1}) = \{v : (T - \lambda\mathbb{1})v = 0\}$

- $V^{n<\infty}$, $\boxed{(T - \lambda I) :\ !(\text{1-1}) \;\Leftrightarrow\; !\text{onto} \;\Leftrightarrow\; \nexists T^{-1}}$ $\quad *\ (T - \lambda I)v = 0 \;\Rightarrow\; !(\text{1-1})$
- $V^{n<\infty}$, $\underline{T = T_{upp.}}$, $\underline{(\exists T^{-1} \;\Leftrightarrow\; \forall T_{ii}, T_{ii} \neq 0)} \;\Leftarrow$ $T_{ii} = 0 \Rightarrow \dim\text{span}(v_1 \ldots v_j) = \dim\text{span}(v_1 \ldots v_{j-1}) + 1$ $\Rightarrow\; \exists v \in \text{span}(v_1 \ldots v_j),\ v \neq 0,\ Tv = 0 \;\Rightarrow\; T\ !(\text{1-1})$

$*$ $\underline{T = T_{upp.}}$, $(\forall T_{ii}, T_{ii} \neq 0) \;\Rightarrow\; T_{ii} = \lambda_i$

- $\boxed{V \supseteq \bigoplus E(\lambda_i, T)}$
- $\boxed{\text{Diagonalizable } T \;\Leftrightarrow\; V = \bigoplus E(\lambda_i, T)} = \forall \lambda_i,\ E(\lambda_i, T) \oplus \text{range}(T - \lambda_i I)$ $= \underline{\forall \lambda \in \mathbb{C}}$, $\boxed{\text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)}$

$\underline{\text{Generalized Eigenspace, } G(\lambda, T)}$: $\boxed{\text{null}(T - \lambda\mathbb{1})^{\dim V}} = \bigcup_{\forall k \geq 0} \text{null}(T - \lambda\mathbb{1})^k = \{v : T^k v = 0,\ \forall k_{\geq 0}\}$

- Algebraic Multiplicity : $d_i = \dim G(\lambda_i, T)$; Geometric Multiplicity : $g_i = \dim E(\lambda_i, T)$
- Characteristic Polynomial : $\prod_i (z - \lambda_i)^{d_i}$
- $(T - \lambda_i\mathbb{1})^{d_i}\big|_{G(\lambda_i, T)} = 0$
- $\boxed{V = \bigoplus_i G(\lambda_i, T)}$ $\;\Rightarrow\; \left[\prod_i (T - \lambda_i\mathbb{1})^{d_i}\right] v = 0$ (Cayley-Hamilton)
- $\underline{\text{Nilpotent, } N}$: $\text{null}(N^{\dim V}) = V \;\Rightarrow\; N^{\dim V} = 0 \quad *\ \exists \overline{N},\ \forall i,\ \overline{N}_{ii} = 0 \quad (\exists U_{pp})$

$*$ $\forall N,\ N^m = 0,\ \exists \sqrt[a]{1 + N} = 1 + \frac{1}{a}N + a_2 N^2 \ldots + a_{m-1}N^{m-1} = A \quad (a_i | A^a = 1 + N)$

- $T|_{G(\lambda_i, T)} = \underbrace{(T - \lambda_i\mathbb{1})|_{G(\lambda_i, T)}}_{\text{(Nilpotent)}} + \underbrace{\lambda_i\mathbb{1}|_{G(\lambda_i, T)}}_{\text{(Diagonal)}} \;\Rightarrow\; \exists U_{pper} \;\Rightarrow\; \boxed{\forall T,\ \exists\text{Block-Upper-Triang}}$

$*$ $\exists T^{-1} \;\rightarrow\; \lambda_i \neq 0 \;\Rightarrow\; T|_{G(\lambda_i, T)} = \lambda_i(1 + N/\lambda_i) \;\Rightarrow\; \exists \sqrt[n]{T}$

$\underline{\forall T,\ \exists \overline{T} = U_{pp} \ \Leftrightarrow\ \exists\{e_1, e_2, ..., e_{n=\dim V}\},\ T(e_i) \in \text{span}(e_1, e_2, ..., e_i)}$:

- $\exists(\lambda, v, Tv = \lambda v);\ \dim \text{null}(T - \lambda\mathbb{1}) > 0 \ \Rightarrow\ \dim \underline{\text{range}(T - \lambda\mathbb{1})} \equiv \dim \underline{U} < \dim V$
- $\forall u,\ Tu = (T - \lambda\mathbb{1})u + \lambda u \in U \ \Rightarrow\ \exists T|_U$
- $\underline{\text{Induc } H}$: $\forall U_{(\dim U\, <\, \dim V)},\ \exists \bigoplus \{u\}^{\dim U},\ T(u_i) \in \text{span}(u_1, u_2, ..., u_i)$
- $V = U \oplus \text{span}\{w_1, w_2, ..., w_m\} = \text{span}\{u_1, u_2, ..., w_1, w_2, ..., w_m\}$
- $\forall w_i,\ \underline{Tw_i} = (T - \lambda\mathbb{1})w_i + \lambda w_i \subseteq U \oplus \text{span}\{w_i\} \subset \underline{U \oplus \text{span}\{w_1, w_2, ...w_i\}} \ \Rightarrow\ V = U \oplus W$

$(n = 2, \dim U = 1$ ☑; $n = 3, \dim U \in \{1, 2\}$ ☑; $n = 4, \dim U \in \{1, 2, 3\}$ ☑...)

$\underline{\text{Schur's Theorem}}$: $\forall T$, Use Gram-Schmidt to make $\underline{\text{orthog basis for } \overline{T}_{upper}}$

$\underline{\forall N,\ \exists \overline{N} = \text{Jordan Block } U_{pp} \ \Leftrightarrow\ \exists\{e_1, e_2, ..., e_{n=\dim V}\},\ N(e_i) = e_{i-1} \text{ or } 0}$:

- $\exists(v \neq 0, Nv = 0v);\ \dim \text{null}(N) > 0 \ \Rightarrow\ \dim \underline{\text{range}(N)} \equiv \dim \underline{U} < \dim V$
- $\forall u,\ Nu \in U \ \Rightarrow\ \exists N|_U$
- $\underline{\text{Induc } H}$: $\forall U_{(\dim U\, <\, \dim V)},\ \exists \bigoplus \{u\}^{\dim U} = \bigoplus \{b_i, Nb_i, N^2 b_i, ..., N^{m_i} b_i\},\ N(N^{m_i} b_i) = 0$
- $\exists v_i,\ N(v_i) = b_i \ \Rightarrow\ U = \bigoplus \{Nv_i, N^2 v_i, ...N^{m_i+1} v_i\}$

$*\ 0 = a^i v_i + c^k u_k = N(a^i v_i + c^k u_k) = a^i b_i + (c')^k (u_k \neq b_i, N^{m_i} b_i) + \left[d^i N(N^{m_i} b_i) = 0\right]$

$\Rightarrow\ \{a\}_i, \{c'\}_i = 0;\ \{d\}_i = 0 \ \rightarrow\ \underline{U' \equiv \{v\}_i^{\dim U} \oplus \{u\}_i^{\dim U}}$ $\quad\underline{\text{(builds layers for } N(e_i) = e_{i-1})}$

- $V = U' \oplus \text{span}\{w_1, w_2, ..., w_m\}$
- $\forall w_i,\ w_i \notin U',\ Nw_i \in U \ \Rightarrow\ \exists x_i \in U',\ Nw_i = Nx_i$

$*\ \exists w_i' = w_i - x_i \in W',\ \underline{N(w_i') = 0}$ $\quad\underline{\text{(other opt. } N(e_i) = 0)}$ $\ \Rightarrow\ V = U' \oplus W'$

$(n = 2, \dim U = 1$ ☑; $n = 3, \dim U \in \{1, 2\}$ ☑; $n = 4, \dim U \in \{1, 2, 3\}$ ☑...)

$\underline{\text{Jordan Form}}$: $\forall T,\ V = \oplus G(\lambda_i, T),\ T|_{G_i} = (T - \lambda_i\mathbb{1}) + \lambda_i\mathbb{1} = N_i + D_i \ \Rightarrow\ \underline{\exists \overline{T} = \text{Jordan Block } U_{pp}}$

$*$ Gram-Schmidt Orthog $G'(\lambda_i, T) \ \Rightarrow\ \underline{\exists \overline{T} = \text{Jordan Normal Block } U_{pp}}$

Jordan Decomposition, M^n :

$$M^n = PJ^nP^{-1} = \begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_2' & v_2'' \\ | & | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}^n P^{-1} \quad \Leftarrow \quad \begin{matrix} (M - \lambda_2)v_2 = 0 \\ (M - \lambda_2)v_2' = v_2 \\ (M - \lambda_2)v_2'' = v_2' \end{matrix}$$

$\underline{\text{Trace}}$: $\underline{2.\ \mathrm{Tr}(\overline{T})}$: $\sum \overline{T}_{ii}$ $\quad$ $\underline{1.\ \mathrm{Tr}(T)}$: $\sum d_n\lambda_n = \sum \lambda_i = \sum \overline{T}^{upper}_{ii}$

- $\mathrm{Tr}(\overline{AB}) = \mathrm{Tr}(A^i{}_j B^j{}_i) = \mathrm{Tr}(\overline{BA}) \;\Rightarrow\; \mathrm{Tr}(\overline{A'} = \overline{Q}\overline{A}\overline{Q^{-1}}) = \mathrm{Tr}(\overline{A})$ (basis indep.) $\Rightarrow$ $\boxed{\mathrm{Tr}(\overline{T}) = \mathrm{Tr}(T)}$
- $\nexists S, T,\ ST - TS = I \;\Leftarrow\; \mathrm{Tr}(ST) - \mathrm{Tr}(TS) = 0 \neq \mathrm{Tr}(\mathbb{1})$

$\underline{\text{Determinant}}$: $\underline{1.\ \det(T) = \prod \lambda_n^{d_n} = \prod \lambda_i = \prod \overline{T}^{upper}_{ii}}$ $\quad$ $\underline{2.\ \det(\overline{T}) = \sum_{(i_1,\ldots,i_n)\in \mathrm{perm}(n)} \mathrm{sign}(i_1,\ldots,i_n) B^{i_1}{}_1 B^{i_2}{}_2 \ldots B^{i_n}{}_n}$

- $\exists T^{-1} \;\Leftrightarrow\; \det(T) \neq 0$
  - $0 = -(T - \lambda_i \mathbb{1})v_i = [(z\mathbb{1} - T) - (z - \lambda_i)]v_i = [U - \sigma_i]v_i$
    $\Rightarrow\ \det(U) = \det(z\mathbb{1} - T) = \prod (z - \lambda_i) = \text{Char. Poly}(T)$
- $\underline{\text{Linearity for } A_k}$: 1.) $c \cdot \det A = \det(A_1,\ A_2,\ \ldots\ c\cdot A_k,\ \ldots\ A_n)$

  2.) $\det(A_1,\ A_2,\ \ldots\ A_k,\ \ldots\ A_n) + \det(A_1,\ A_2,\ \ldots\ B_k,\ \ldots\ A_n) = \det(A_1,\ A_2,\ \ldots\ A_k + B_k,\ \ldots\ A_n)$
- $\underline{\text{Column Permutation}}$: $\det(Ae_{i_1},\ Ae_{i_2},\ \ldots\ Ae_{i_n}) = \mathrm{sign}(i_1, i_2, \ldots, i_n)\det(Ae_1,\ Ae_2,\ \ldots\ Ae_n)$
- $$\begin{aligned} \det(\overline{AB}) &= \det(B^{i_1}{}_1 Ae_{i_1},\ B^{i_2}{}_2 Ae_{i_2},\ \ldots,\ B^{i_n}{}_n Ae_{i_n}) \quad \Leftarrow \quad Be_k = B^i{}_k e_i \\ &= \sum_{(i_1,i_2,\ldots,i_n)} B^{i_1}{}_1 B^{i_2}{}_2 \ldots B^{i_n}{}_n \det(Ae_{i_1},\ Ae_{i_2},\ \ldots\ Ae_{i_n}) \\ &= \sum_{(i_1,i_2,\ldots,i_n)\in\mathrm{perm}(n)} B^{i_1}{}_1 B^{i_2}{}_2 \ldots B^{i_n}{}_n \det(Ae_{i_1},\ Ae_{i_2},\ \ldots\ Ae_{i_n}) \\ \det(B)\det(A) &= \sum_{(i_1,i_2,\ldots,i_n)\in\mathrm{perm}(n)} B^{i_1}{}_1 B^{i_2}{}_2 \ldots B^{i_n}{}_n\ \mathrm{sign}(i_1, i_2, \ldots, i_n) \cdot \det(Ae_1,\ Ae_2,\ \ldots\ Ae_n) \end{aligned}$$

  $\Rightarrow\ \det(\overline{A'} = \overline{Q}\overline{A}\overline{Q^{-1}}) = \det(\overline{A})$ (basis indep.) $\Rightarrow$ $\boxed{\det(\overline{T}) = \det(T)}$

- $\det(T) = \det(U)_{\pm 1} \det(\sqrt{T^\dagger T})_{\geq 0} = \pm \prod \sigma_i = \pm\sqrt{\det(T^\dagger T)} = \pm\sqrt{\det(\text{Gram Matrix})}$
- $\underline{\langle x|Hx\rangle \geq 0}$ $\Rightarrow$ $*\ \lambda_i \geq 0$, $*\ \exists\{e_i\}$ $\Rightarrow$ $$\begin{aligned} \underline{\mathrm{vol}\ H(\Omega)} &= \mathrm{vol}\bigcup_i B_i[H(\Omega)] = \mathrm{vol}\bigcup_i H(B_i[\Omega]) \\ &= \det(H)\,\mathrm{vol}\bigcup_i B_i[\Omega] = \underline{\det(H)\,\mathrm{vol}(\Omega)} \end{aligned}$$

$*$ $\underline{\mathrm{vol}\ T(\Omega)} = \mathrm{vol}\ S\sqrt{T^\dagger T}(\Omega) = \mathrm{vol}\ \sqrt{T^\dagger T}(\Omega) = \boxed{|\det(T)|\,\mathrm{vol}(\Omega)}$

$*$ $\exists\{e_i\}_\perp$ of $\sqrt{T^\dagger T}$ spanning V where Tx is equal to expanding x's components by λ_i each (then rotating/reflecting by U, which doesn't change lengths or shape volumes). $T\mathbb{1}$ means moving axes to this basis (tilting head), decomposing the (tilted) $\mathbb{1}$ box to smaller boxes aligned with the bases, then expanding them to rectangular prisms that composes (tilted) oblique rectangular prism, $T\mathbb{1}$. The proportional change for each each small box is $\det T$, and any initial volume can be composed of these smaller boxes, so the total change is prop. to $\det T$

- $y(x) \approx y(x_0) + J_y(x_0)(x - x_0) \;\Rightarrow\; \displaystyle\int_{y(\Omega)} f(y)\,dy = \int_\Omega f \circ y(x)\,|\det(J_f)|\,dx$

- $\delta_i \left(|a^i|^p\right)^{\frac{1}{p}} = \|a\|_p \leq \|a\|_1 = \delta_i |a^i|$
- $|a^i||b_i| \leq \delta_r \left(|a^r|^p\right)^{\frac{1}{p}} \cdot \delta_k \left(|b^k|^q\right)^{\frac{1}{q}} = \|a\|_p \|b\|_q \qquad \frac{1}{p} + \frac{1}{q} = 1 \leq p, q$

  $\leq \delta_r |a^r| \cdot \delta_k |b^k| = \|a\|_1 \|b\|_1$

- $\dfrac{\|Ax\|_\infty}{\|x\|_\infty} = \dfrac{\max_r \left|\sum_k A^r{}_k x^k\right|}{\|x\|_\infty} \leq \dfrac{\max_r \left|\sum_k A^r{}_k \|x\|_\infty\right|}{\|x\|_\infty} \leq \boxed{\max_r \sum_k |A^r{}_k|}$
- $\dfrac{\|Ax\|_1}{\|x\|_1} = \dfrac{\sum_r \left|\sum_k A^r{}_k x^k\right|}{\|x\|_1} \leq \dfrac{\sum_k \left(\sum_r |A^r{}_k|\right) |x^k|}{\|x\|_1} \leq \boxed{\max_k \sum_r |A^r{}_k|}$
- $$\|Tx\|_p^p = \sum_r \left|\sum_k T^r{}_k x^k\right|^p \leq \sum_r \left|\sum_k |T^r{}_k|^{\frac{1}{q}} \cdot |T^r{}_k|^{\frac{1}{p}} |x^k|\right|^p \leq \sum_r \left|\left(\sum_k |T^r{}_k|\right)^{\frac{1}{q}} \left(\sum_k |T^r{}_k||x_k|^p\right)^{\frac{1}{p}}\right|^p$$

  $$\leq \left[\max_r \sum_k |T^r{}_k|\right]^{\frac{p}{q}} \sum_k \left(\sum_r |T^r{}_k|\right) |x_k|^p \leq \|T\|_\infty^{p/q} \cdot \|T\|_1 \cdot \|x\|_p^p \;\Rightarrow\; \boxed{\|T\|_p^p \leq \|T\|_\infty^{1/q} \cdot \|T\|_1^{1/p}}$$

$* \ \frac{1}{p} + \frac{1}{q} = 1 \leq p, q$