Dispersion Relation

$$\boldsymbol{\Psi(t) = \int_{-\infty}^{\infty} \Phi(k) e^{i[kx-\omega(k)t]}\ dk}$$

$$\begin{aligned}
kx - \omega(k)t &= kx - k_0x + k_0x - \left[w(k_0) + w'(k_0)(k-k_0) + \frac{w''(k_0)}{2!}(k-k_0)^2 + ...\right] t \\
&= (k-k_0)x - \left[w'(k_0)(k-k_0) + \frac{w''(k_0)}{2!}(k-k_0)^2 + ...\right] t + (k_0x - w(k_0)t) \\
&= (k-k_0)\left(x - \left[w'(k_0) + \frac{w''(k_0)}{2!}(k-k_0) + ...\right] t\right) + (k_0x - w(k_0)t)
\end{aligned}$$

Let $s = k - k_0$, and for large s let $\Phi(k) = \Phi(k_0 + s)$ be small, meaning the wave is mostly monochromatic about k_0 (the peak of Φ), and only small s are important.

$$\begin{aligned}
kx - \omega(k)t &\approx (k-k_0)\left[x - w'(k_0)t\right] + (k_0x - w(k_0)t) \\
\Psi(t) &= \int_{-\infty}^{\infty} \Phi(k) e^{i[kx-\omega(k)t]}\ dk \\
&\approx e^{i(k_0x - w(k_0)t)} \int_{-\infty}^{\infty} \Phi(k) e^{i(k-k_0)[x-w'(k_0)t]}\ dk \\
&= e^{i(k_0x - w(k_0)t)} \int_{-\infty}^{\infty} \Phi(k_0+s) e^{is[x-w'(k_0)t]}\ ds \\
&= e^{i(k_0x - w(k_0)t)} F(x - w'(k_0)t)
\end{aligned}$$

For simpler waves like $cos[n(x-v_1t)]cos[m(x-v_2t)]$, the smaller wavenumber is the envelope and the bigger wavenumber is the internal wave, so if $n < m$ then the group velocity is v_1. Since $s < k_0$, the group velocity for each term in the infinite sum (or integral) is $w'(k_0)$. Since each term has the same speed, the sum, F, will also move at that speed. F is the envelope and $e^{i(k_0x-w(k_0)t)}$ is the internal wave with the phase velocity $w(k_0)/k_0$.

For QM only $||\Psi||^2$ matters physically, so $||\Psi||^2 = ||F(x - w'(k_0)t)||^2$.

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&= kx - (k-k_0)\left[w'(k_0) + \frac{w''(k_0)}{2!}(k-k_0) + ...\right] t - w(k_0)t \\
&= \begin{array}{l} k\left(x - \left[w'(k_0) + \frac{w''(k_0)}{2!}(k-k_0) + ...\right] t\right) \\ + \left(k_0\left[w'(k_0) + \frac{w''(k_0)}{2!}(k-k_0) + ...\right] - w(k_0)\right) t \end{array}
\end{aligned}$$

Let $s = k - k_0$, and for large s let $\Phi(k) = \Phi(k_0 + s)$ be small, meaning the wave is mostly monochromatic about k_0 (the peak of Φ), and only small $s = k - k_0$ are important.

$$\begin{aligned}
kx - \omega(k)t &\approx k\left[x - w'(k_0)t\right] + \left[k_0 w'(k_0) - w(k_0)\right] t \\
\Psi(t) &= \int_{-\infty}^{\infty} \Phi(k) e^{i[kx-\omega(k)t]}\ dk \\
&\approx e^{i[k_0 w'(k_0) - w(k_0)]t} \int_{-\infty}^{\infty} \Phi(k) e^{ik[x-w'(k_0)t]}\ dk \\
&= e^{i[k_0 w'(k_0) - w(k_0)]t} F(x - w'(k_0)t)
\end{aligned}$$

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