1 Wave Function

$$\Psi_p = e^{i(2\pi x/\lambda - 2\pi t/T)} = e^{i(kx-\omega t)} = e^{\frac{i}{\hbar}(px-Et)}$$

$$\breve{p}\Psi_p = p\Psi_p = \hbar k\Psi_p \qquad \breve{E}\Psi_p = E\Psi_p = \hbar\omega\Psi_p$$

$$\boxed{\breve{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}} \qquad \boxed{\breve{E} = -\frac{\hbar}{i}\frac{\partial}{\partial t}}$$

- $\boldsymbol{\langle x|x'\rangle} \equiv \delta(x'-x)$
* $\hat{x}|x\rangle \equiv x|x\rangle$
- $\boldsymbol{\langle x|\hat{x}|x'\rangle} \equiv x'\langle x|x'\rangle$
* $|f\rangle \equiv \int f(x)\,|x\rangle\,dx$
- $\boldsymbol{\langle x|f\rangle} \equiv f(x)$

1. $\langle x|\hat{x}|f\rangle = \int \langle x|\hat{x}|x'\rangle\langle x'|f\rangle\,dx'$

   $\boxed{\breve{x}\langle x|f\rangle \equiv x\langle x|f\rangle} = xf(x)$

2. $\boldsymbol{\langle x|\hat{p}|x'\rangle} \equiv \frac{\hbar}{i}\delta'(x-x')$

   $= \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|x'\rangle = \frac{\hbar}{i}\langle x|x'\rangle\frac{\partial}{\partial x'}$

3. $\langle x|\hat{p}|f\rangle = \int \delta'(x-x')f(x')\,dx'$

   $\boxed{\breve{p}\langle x|f\rangle \equiv \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|f\rangle} = \frac{\hbar}{i}\langle x|\frac{df}{dx}\rangle$

1.1 Schrodinger Ψ

$$\boxed{\breve{E}|\Psi\rangle = \widehat{H}|\Psi\rangle} = (\widehat{T}+\widehat{V})|\Psi\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \left[\frac{\hat{p}^2}{2m} + V(\hat{x},t)\right]|\Psi\rangle$$

$$\boxed{-\breve{E}\langle\Psi| = \langle\Psi|\hat{H}}$$

$$\boxed{\breve{E}\langle x|\Psi\rangle = \breve{H}\Psi} = (\breve{T}+\breve{V})\Psi = \left[\frac{\breve{p}^2}{2m} + V(\vec{\mathbf{r}},t)\right]\Psi$$

$$\boxed{i\hbar\frac{\partial}{\partial t}\Psi(\vec{\mathbf{r}},t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\breve{\mathbf{r}},t)\right]\Psi(\vec{\mathbf{r}},t)}$$

$$\boxed{-\breve{E}\langle\Psi|x\rangle = \breve{H}\Psi^*}$$

<u>If $V = V(x)$</u>

$\Psi(x,t) = \psi(x)\phi(t) \;\Rightarrow$

- $E_n\phi_n(t) = i\hbar\frac{\partial}{\partial t}\phi_n(t) \;\Rightarrow\; \boxed{\phi_n(t) = e^{-\frac{i}{\hbar}E_n t}}$
- $\boxed{E_n\psi_n(x) = \left(\frac{-\hbar^2}{2m}\partial_x^2 + V(x)\right)\psi_n(x)}$
  - ψ can be lin. sum of real or complex, so choose real ψ
- Linear : $$\boxed{\begin{aligned}\Psi(x,t) &= \sum_n \psi_n(x)e^{-\frac{i}{\hbar}E_n t}c_n \\ &= \sum_n \langle x|n\rangle e^{-\frac{i}{\hbar}E_n t}\langle n|\Psi\rangle \\ &= \int_{x'}\langle x|\left[\sum_n |n\rangle e^{-\frac{i}{\hbar}E_n t}\langle n|\right]|x'\rangle\Psi(x')\,dx' \\ &= \int_{x'} U(x,t;\,x',0)\,\Psi(x')\,dx'\end{aligned}}$$
- $\sigma_H^2 = \langle H^2\rangle - \langle H\rangle^2 = 0 \;\Rightarrow\;$ measuring stationary state, Ψ_n, returns one E_n (determinate state)

1.2 Usage

- $\langle f|g\rangle = \displaystyle\int_{-\infty}^{\infty} f(x)^* g(x)\ dx$
- $\langle f|g\rangle_{ab} = \displaystyle\int_a^b f(x)^* g(x)\ dx$
- $|f\rangle = \displaystyle\int f(x')|x'\rangle dx' \ \sim\ f(x) \equiv \langle x|f\rangle$
- $\langle f| = \displaystyle\int f(x)^*[...]\ dx$
- $\langle f|f\rangle = \displaystyle\int_a^b |f|^2 dx < \infty \ \Rightarrow\ f \in L_{2(a,b)}$ $\left(\int_a^b |f|^p\,dx < \infty \ \Rightarrow\ f \in L_{p(a,b)}\right)$

$$\langle x|\Psi\rangle = \Psi = \begin{cases} \displaystyle\sum_n c_n f_n \\ \displaystyle\int_n c_n f_n\ dn \end{cases}, \quad \langle f_m|f_n\rangle = \begin{cases} \delta_{mn} \\ \delta_{(m-n)} \end{cases}, \quad |c_n|^2 = \begin{cases} P(n) \\ \mathrm{PDF}_{(n)} \end{cases}$$

(see Born int.)

$$\Rightarrow \boxed{c_n = \langle f_n|\Psi\rangle}$$

$\forall\{f_n\} \in L_2$:

$$|\Psi\rangle = \begin{cases} \displaystyle\sum_n c_n|f_n\rangle = \sum_n \langle f_n|\Psi\rangle\ |f_n\rangle = \left(\sum_n |f_n\rangle\langle f_n|\right)|\Psi\rangle = |\Psi\rangle \\ \displaystyle\int_n c_n|f_n\rangle\ dn = \int_n \langle f_n|\Psi\rangle|f_n\rangle\ dn = \left(\int_n |f_n\rangle\langle f_n|\ dn\right)|\Psi\rangle = |\Psi\rangle \end{cases}$$

$\breve{x}\Psi_y = x\Psi_y = y\Psi_y$ $\Rightarrow \boxed{\Psi_y = \delta_{(x-y)} = \langle x\|y\rangle}$	$\langle x\|\hat{p}\|p\rangle = \int \langle x\|\hat{p}\|x'\rangle\langle x'\|p\rangle\,dx'$ $p\langle x\|p\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x\|p\rangle$ $\langle x\|\hat{p}\|p\rangle = \breve{p}\Psi_p = p\Psi_p$ $\Rightarrow \boxed{\Psi_p = Ae^{\frac{i}{\hbar}px} = \langle x\|p\rangle}$	$\langle x\|\hat{H}\|n\rangle = E_n\langle x\|n\rangle$ $\breve{H}\Psi_n = E_n\Psi_n$ (See Potential Examples)
$\Psi_{(x,t)} = \int_{-\infty}^{\infty} \Psi_y c_{y(t)}\ dy$ $= \int_{-\infty}^{\infty} \delta_{(x-y)}\Psi_{(y,t)}\ dy$	$\Psi_{(x,t)} = \int_{-\infty}^{\infty} \Psi_p \phi_{(E_p,t)} c_p\ dp$ $= \int_{-\infty}^{\infty} \frac{e^{\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}\frac{p^2}{2m}t}\Phi_{(p,0)}\ dp$	$\Psi_{(x,t)} = \int_{-\infty}^{\infty} \Psi_n \phi_{(E_n,t)} c_n\ dn$ $= \int_{-\infty}^{\infty} \Psi_n e^{\frac{-i}{\hbar}E_n t} c_n\ dn$
$c_x(t) = \langle \Psi_x\|\Psi_{(x,t)}\rangle = \langle x\|\Psi\rangle$ $\Psi_{(x,t)} = \int_{-\infty}^{\infty} \delta_{(x-y)}\Psi_{(y,t)}\ dy$	$c_p(t) = \langle \Psi_p\|\Psi_{(x,t)}\rangle = \langle p\|\Psi\rangle$ $\boxed{\Phi_{(p,t)} = \int_{-\infty}^{\infty} \frac{e^{\frac{-i}{\hbar}px}}{\sqrt{2\pi\hbar}}\Psi_{(x,t)}\ dx}$	$c_n(t) = \langle \Psi_n\|\Psi_{(x,t)}\rangle = \langle n\|\Psi\rangle$ $\Psi_{(n,t)} = \int_{-\infty}^{\infty} \Psi_n^* \Psi_{(x,t)}\ dx$

$\boxed{\text{Born Interpretation: PDF}(x) = |\Psi(x)|^2 = \Psi^*\Psi}$

$\underline{P(a<x<b) = \int_a^b |\Psi|^2 dx \equiv \langle\Psi|\Psi\rangle_{ab}}$

– $\boxed{\langle\Psi|\Psi\rangle = 1}$ (physical, bound states only)

- $\Psi(\pm\infty) = 0$
- $\text{Min}(V) \le E_\Psi \in \mathbb{R}$
- $\langle\Psi_n|\Psi_n\rangle \to \infty \;\Rightarrow\; \Psi_n$ not PHYSICAL sol. but $\Psi = \int c_n\Psi_n$ can if $\langle\Psi|\Psi\rangle = 1$

Boundary Conditions:

- $\Psi(x)$ isn't always cont. (see extra)
- $\frac{\partial\Psi(x)}{\partial x}$ is cont. except at $V = \infty$

$$\lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} E\Psi dx = \int_{-\epsilon}^{\epsilon} \widehat{H}\Psi dx \;\Rightarrow$$

$$\lim_{\epsilon\to 0} \tfrac{\hbar^2}{2m}\Delta(\tfrac{d\Psi}{dx}) = \int_{-\epsilon}^{\epsilon} V\Psi dx$$

- $E[f(x)] = \int_{-\infty}^{\infty} f(x)\ \text{PDF}(x)\ dx = \int_{-\infty}^{\infty} f(x)\ |\Psi(x)|^2\ dx = \int_{-\infty}^{\infty} \Psi(x)^* f(x)\Psi(x)\ dx = \boxed{\langle\Psi|f\Psi\rangle \equiv \langle f(x)\rangle}$
- $\int_x \Psi^*\Psi\ dx = \int_x \left(\int_n c_n^*(t)\Psi_n^*(x)\ dn\right)\left(\int_{n'} c_{n'}(t)\Psi_{n'}(x)\ dn'\right)\ dx$

$$= \int_n c_n^*(t) \int_{n'} c_{n'}(t)\ \delta(n-n')\ dn'\ dn = \int_n |c_n(t)|^2\ dn \;\Rightarrow\; \boxed{\text{PDF}(n) = |c_n|^2 = c_n^* c_n}$$

$\underline{\text{Adjoint (herm. adj./herm. conj.): } \{A^\dagger : \langle f|Af\rangle = \langle A^\dagger f|f\rangle\}}$ $\;\Rightarrow\; \langle h|\hat{A}g\rangle = \langle\hat{A}^\dagger h|g\rangle$ (let $f{=}h{+}g,\ f{=}h{+}ig$)

$\underline{\text{Hermitian Operator: } \{A : \hat{A}^\dagger = \hat{A}\}}$

- $\boxed{\exists\{\Psi_n\}:\ \hat{A}\Psi_n(x) = a_n\Psi_n(x)}$ (spectral theorem)
- $\langle a\rangle = a \in \mathbb{R} \;\Rightarrow\; \hat{A}$ can be an observable
- $\boxed{\langle\Psi_m|\Psi_n\rangle \in \{\delta_{mn},\ \delta(m-n)\}}$
- $\boxed{\text{Axiom: } \{\Psi_n\} \text{ for } \hat{A} \text{ are complete}}$

Non-degenerate: $(m \ne n),\ (a_m \ne a_n) \;\Rightarrow\; \langle\Psi_m|\Psi_n\rangle \in \{\delta_{mn},\ \delta(m-n)\}$

Degenerate: $(m \ne n),\ (a_m = a_n),\ (\Psi_m \ne \Psi_n), \langle\Psi_m|\Psi_n\rangle \ne 0 \;\Rightarrow\;$ Use Gram-Schmidt to find orthogonal $\langle\Psi'_m|\Psi'_n\rangle = \langle a\Psi_m + b\Psi_n|\ c\Psi_m + d\Psi_n\rangle = 0$

$\underline{\text{Expectation: } E[\hat{A}(x,p)]}$

- $\int_{-\infty}^{\infty} \hat{A}(x,p)^*\ \Psi^*\Psi\ dx = \langle\hat{A}\Psi|\Psi\rangle = \boxed{\langle\Psi|\hat{A}\Psi\rangle \equiv \langle\hat{A}(x,p)\rangle}$ (won't work if $\int A\ |\Psi|^2\ dx$)

$$\langle\Psi|\hat{A}\Psi\rangle = \int_{-\infty}^{\infty} \Psi^*\hat{A}\Psi\ dx = \int_{-\infty}^{\infty} \left(\int_n c_n^*\Psi_n^*\ dn\right)\left(\int_{n'} c_{n'}\hat{A}\Psi_{n'}\ dn'\right)\ dx$$

$$= \int_n a_n|c_n|^2\ dn = E[a] \equiv \langle a\rangle$$ $c_n = \text{PDF}(n)$ (see above and Momentum Space)

$$\boxed{\langle a\rangle = \langle\Psi|\hat{A}\Psi\rangle = \langle\Psi|\hat{A}|\Psi\rangle = \langle A\rangle}$$

- $\boxed{\langle\sigma_a^2\rangle = \langle a^2\rangle - \langle a\rangle^2} \;\Rightarrow\; \sigma_A^2 = 0$ for Ψ_n (determinate state)

Matrix Operators:

Given complete $\{e_n\}$: $\langle e_m | e_n \rangle = \delta_{mn}$

1.) $\boxed{Q^{(e)}_{mn} \equiv \left\langle e_m \middle| \widehat{Q}(x,p) \middle| e_n \right\rangle}$

$$|\beta\rangle = \widehat{Q}|\alpha\rangle = \sum_m |e_m\rangle \left[\begin{array}{c} \langle e_m|\beta\rangle = \left\langle e_m \middle| \widehat{Q} \middle| \alpha \right\rangle \\ \sum_n b_n \langle e_m|e_n\rangle = \sum_n a_n \boxed{\left\langle e_m \middle| \widehat{Q} \middle| e_n \right\rangle} \\ \boxed{b_m = \sum_n \left(Q^{(e)}_m\right)_n a_n} \end{array} \right] = \begin{array}{rl} \sum_m b_m |e_m\rangle & = \sum_{n,m} \langle e_n|\alpha\rangle \; Q^{(e)}_{mn}|e_m\rangle \\ & = \sum_{n,m} Q^{(e)}_{mn} |e_m\rangle\langle e_n|\alpha\rangle \\ & \Rightarrow \boxed{\widehat{Q} = \sum_{m,n} Q^{(e)}_{mn} |e_m\rangle\langle e_n|} \end{array}$$

2.) Find $\widehat{Q}$ as a matrix

$$\begin{array}{l} |f\rangle = \sum_n c^{(e)}_n[f] \; \big|e_n\big\rangle \\ \quad \downarrow \\ f(x) = \sum_n c^{(e)}_n[f] \; e_n(x) \end{array} = \begin{pmatrix} \vdots \\ c_n[f] \\ \vdots \end{pmatrix}^{(e)} \cdot \begin{pmatrix} \vdots \\ e_n(x) \\ \vdots \end{pmatrix} \equiv \boxed{\begin{array}{c} \vec{c}^{\;(e)}[f] \cdot \vec{e}(x) \\ \int_n c^{(e)}[f](n) \cdot e(n,x) \; dn \end{array}} \;, \quad \boxed{c^{(e)}_n[f] = \langle e_n|f\rangle}$$

$$\begin{aligned} &\widehat{Q}|f\rangle \\ &= \Big(\sum_{m,n'} Q^{(e)}_{mn'} |e_m\rangle\langle e_{n'}| \Big) \sum_n c^{(e)}_n |e_n\rangle \\ &= \sum_{m,n} \Big(\sum_{n'} Q^{(e)}_{mn'} \; c^{(e)}_n \langle e_{n'}|e_n\rangle \Big) |e_m\rangle \\ &= \sum_m \Big(\sum_n \left(Q^{(e)}_m\right)_n c^{(e)}_n \Big) |e_m\rangle \end{aligned}$$

$$\widehat{Q}\left[\begin{pmatrix} | \\ c \\ | \end{pmatrix}^{(e)} \cdot \begin{pmatrix} | \\ e \\ | \end{pmatrix} \right] = \left[\begin{pmatrix} & \vdots & \\ - & Q_m & - \\ & \vdots & \end{pmatrix}^{(e)} \begin{pmatrix} | \\ c \\ | \end{pmatrix}^{(e)} \right] \cdot \begin{pmatrix} | \\ e \\ | \end{pmatrix}$$

$$\widehat{Q}\;|f\rangle = \boxed{\widehat{Q}\;\left[\;\vec{c}^{\;(e)}[f] \cdot \vec{e}\;\right] = \left[\;\overline{Q}^{(e)} \; \vec{c}^{\;(e)}[f]\right] \cdot \vec{e}}$$

e.g.

$$\begin{aligned} \langle x|\widehat{Q}|f\rangle &= \int_m \left[\overline{Q}^{(\delta)} f\right](m) \cdot \delta(x-m) \; dm \\ &= \int_m \left[\int_n Q^{(\delta)}_m(n) \cdot f(n) \; dn \right] \delta(x-m) \; dm = \widehat{Q} f(x) \end{aligned}$$

3.) Terms

Diagonalizable: $A \equiv PDP^{-1}$

Conj. Transpose, $\dagger$: $A^\dagger \equiv A^{T*} = A^{*T}$

Hermitian, H: $H = H^\dagger$

$H = UDU^{-1} = UDU^\dagger$ (spectral theorem)

Unitary, U: $U : \; UU^\dagger = U^\dagger U = 1$

$\exists H : \; U = e^{iH} = (U')e^{iD}(U')^\dagger$

Hermitian Operator $\sim$ Hermitian Matrix

$\langle Qx|y\rangle = \langle x|Qy\rangle$ (if inf. size then must be in Hilbert Space)

(draw it out)

$$\begin{aligned} \rightarrow \; (\overline{Q}x)^{*T} \cdot y_m|e_m\rangle &= y_m x^{*T} \cdot (\overline{Q}^*_m) \\ &= x^{*T} \cdot \overline{Q}^{*T} y_m|e_m\rangle \\ &= x^{*T} \cdot \overline{Q} y_m|e_m\rangle \end{aligned}$$

$\rightarrow \; \overline{Q}^\dagger \equiv \overline{Q}^{*T} = \overline{Q}$ ☑

4.) Eigenvalue Equation

General Case:

$$\widehat{Q}|q_i\rangle = q_i|q_i\rangle$$

$$|q_i\rangle = \sum c_n^{(e)}{}_{[q_i]}\big|e_n\big\rangle$$

$$\boxed{\begin{aligned} \overline{Q}^{(e)} &= UDU^\dagger \qquad \text{(Spectral Theorem)} \\ &= \begin{pmatrix} | & | & \\ \vec{c}_{[q_0]} & \vec{c}_{[q_1]} & \ldots \\ | & | & \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & \ldots \\ 0 & q_1 & \ldots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} - & \vec{c}^{\,*}{}_{[q_0]} & - \\ - & \vec{c}^{\,*}{}_{[q_1]} & - \\ & \vdots & \end{pmatrix}^{(e)} \\ &\text{where } \langle \vec{c}_m|\vec{c}_n\rangle = \delta_{mn} \ \text{ since } \ Q^\dagger Q = QQ^\dagger \quad \text{(normal)} \end{aligned}}$$

$$\begin{aligned} q_i|q_i\rangle &= \widehat{Q}|q_i\rangle \\ \Big[q_i\ \vec{c}^{\,(e)}{}_{[q_i]}\Big]\cdot \vec{e}(x) &= \Big[\ \overline{Q}^{(e)}\ \vec{c}^{\,(e)}{}_{[q_i]}\Big]\cdot \vec{e}(x) \\ &\Downarrow^* \\ q_i\ \vec{c}^{\,(e)}{}_{[q_i]} &= \overline{Q}^{(e)}\ \vec{c}^{\,(e)}{}_{[q_i]} \end{aligned}$$

$$\begin{aligned} {}^*\forall n \qquad (qc)_n\ |e_n(x)\rangle &= (Qc)_n\ |e_n(x)\rangle \\ \langle e_n(x)|(qc)_n|e_n(x)\rangle &= \langle e_n(x)|(Qc)_n|e_n(x)\rangle \\ (qc)_n &= (Qc)_n \end{aligned}$$

$$q_i \begin{pmatrix} | \\ \vec{c}_{[q_i]} \\ | \end{pmatrix}^{(e)} = \begin{pmatrix} | & | & \\ \vec{c}_{[q_0]} & \vec{c}_{[q_1]} & \ldots \\ | & | & \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & \ldots \\ 0 & q_1 & \ldots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} - & \vec{c}^{\,*}{}_{[q_0]} & - \\ - & \vec{c}^{\,*}{}_{[q_1]} & - \\ & \vdots & \end{pmatrix}^{(e)} \begin{pmatrix} | \\ \vec{c}_{[q_i]} \\ | \end{pmatrix}^{(e)}$$

$$\boxed{\begin{aligned} &\underline{q_i\ ,\ \vec{c}^{\,(e)}{}_{[q_i]}} : \\ &\det\left(\overline{Q}^{(e)} - Iq_i\right) = 0 \end{aligned}}$$

Special Case:

$$|q_n\rangle = |e_n\rangle$$

$$\widehat{Q}|e_n\rangle = q_n|e_n\rangle$$

$$\begin{aligned} \widehat{Q}|a\rangle &= \sum_n \widehat{Q}|e_n\rangle\langle e_n|a\rangle \\ &= \Big(\sum_n q_n|e_n\rangle\langle e_n|\Big)|a\rangle \end{aligned} \ \Rightarrow\ \boxed{\widehat{Q} = \sum_n q_n|e_n\rangle\langle e_n|} \ \Rightarrow\ \boxed{\overline{Q}^{(e)} = \begin{pmatrix} q_0 & 0 & \ldots \\ 0 & q_1 & \ldots \\ \vdots & \vdots & q_i \end{pmatrix}}$$

$$Q^{(e)}_{mn} = q_n\delta_{mn}$$

$$\overline{Q}^{(e)} = \begin{pmatrix} q_0 & 0 & \ldots \\ 0 & q_1 & \ldots \\ \vdots & \vdots & \end{pmatrix}^{(e)} = \begin{pmatrix} 1 & 0 & \ldots \\ 0 & 1 & \ldots \\ \vdots & \vdots & \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & \ldots \\ 0 & q_1 & \ldots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} 1 & 0 & \ldots \\ 0 & 1 & \ldots \\ \vdots & \vdots & \end{pmatrix}^{(e)}$$

$$\boxed{\vec{c}^{\,(e)}{}_{[q_i]} = \big(\ldots\ 0\ 0\ 1_{(i)}\ 0\ 0\ \ldots\big)^T}$$

5. Unitary Transformation and Trace

- $\boxed{|b_i\rangle = U|a_i\rangle \ \Leftrightarrow\ U = \sum |b_n\rangle\langle a_n|}$
- $\boxed{\overline{U}_{ij} = \langle a_i|U|a_j\rangle = \langle a_i|b_j\rangle}$
- $\boxed{\langle b_i|U\widehat{Q}U^\dagger|b_j\rangle = \langle a_i|\widehat{Q}|a_j\rangle}$
- $\boxed{\begin{aligned} (A)|a_i\rangle &= a_i|a_i\rangle \\ (UAU^\dagger)|b_i\rangle &= a_i|b_i\rangle \end{aligned}}$

- $\mathrm{Tr}(Q) = \sum\langle a_i|Q|a_i\rangle = \sum\langle b_i|Q|b_i\rangle$
- $\mathrm{Tr}(QP) = \mathrm{Tr}(PQ)$
- $\mathrm{Tr}(U^\dagger QU) = \mathrm{Tr}(Q)$
- $\mathrm{Tr}(\,|a_i\rangle\langle a_j|\,) = \delta_{ij}$
- $\mathrm{Tr}(\,|b_i\rangle\langle a_i|\,) = \langle a_i|b_i\rangle$

$\underline{\Phi(p,t)\text{ - Momentum Space (generalizable Born Interpretation):}}$

$$\int_x \Psi^*\Psi dx = \int_x\int_p c^*_{p}(t)\Psi^*_{p}(x)\,dp\int_{p'} c_{p'}(t)\Psi_{p'}(x)\,dp'\,dx = \int_p c^*_p(t)\int_{p'} c_{p'}(t)\int_x \Psi^*_p(x)\Psi_{p'}(x)\,dx\,dp'\,dp$$

$$= \int_p \Phi^*\int_{p'}\Phi'\ \delta(p-p')dp'dp = \int_p \Phi^*\Phi\ dp \ \Rightarrow\ \boxed{\text{PDF}(p) = |\Phi|^2 = \Phi^*\Phi}\ \Rightarrow\ \boxed{\langle\Psi|\Psi\rangle = \langle\Phi|\Phi\rangle}$$

$$\boxed{\langle p|\hat{x}|x\rangle = x\Phi = -\frac{\hbar}{i}\frac{\partial}{\partial p}e^{-\frac{i}{\hbar}px} = \breve{x}_p\langle p|x\rangle}$$

$$*\ \hat{p}|p\rangle = p|p\rangle$$

1. $\underline{p\langle x|p\rangle} = \langle x|\hat{p}|p\rangle = \int\langle x|\hat{p}|x'\rangle\langle x'|p\rangle\,dx'$

$$= \underline{\breve{p}_x\langle x|p\rangle}\ \Rightarrow\ \boxed{\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}px}}$$

- $\langle p|\hat{p}|p'\rangle = p'\langle p|p'\rangle$

1. $\langle p|\hat{p}|p'\rangle = \iint\langle p|x\rangle\langle x|p|x'\rangle\langle x'|p'\rangle\,dxdx'$

$$\int\langle p|x\rangle\underline{p'\langle x|p'\rangle}\,dx = \int\langle p|x\rangle\underline{\breve{p}_x\langle x|p'\rangle}\,dx$$

2. $\boxed{\langle p|p'\rangle} = \int\langle p|x\rangle\langle x|p'\rangle\,dx = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}x(p'-p)}\,dx = \delta(p'-p)$

3. $\langle p|\hat{x}|p'\rangle = \iint\langle p|x\rangle\langle x|\hat{x}|x'\rangle\langle x'|p'\rangle\,dxdx'$

$$= \frac{1}{2\pi\hbar}\int xe^{\frac{i}{\hbar}x(p'-p)}dx = -\frac{\hbar}{i}\frac{\partial}{\partial p}\langle p|p'\rangle$$

4. $\langle p|\hat{x}|f\rangle = \int\langle p|\hat{x}|p'\rangle\langle p'|f\rangle\ dp'$

$$= \boxed{-\frac{\hbar}{i}\frac{\partial}{\partial p}\langle p|f\rangle \equiv \breve{x}\langle p|f\rangle}$$

$$\Rightarrow\ A(x,\hat{p}_x) \to A(\hat{x}_p,p)\ \Rightarrow\ \boxed{\langle a\rangle = \left\langle\Phi\middle|A(\hat{x}_p,p)\middle|\Phi\right\rangle}$$

$$\Rightarrow\ \langle p|f\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-\frac{i}{\hbar}xp}f(x)\,dx$$

<u>Anything in x-space can be done in p-space</u> (or generalize to any transform, c_n)

<u>Heisenberg Uncertainty Proof:</u>

$$\langle f|g\rangle \equiv \left\langle\big(\widehat{A}-\langle a\rangle\big)\Psi\middle|\big(\widehat{B}-\langle b\rangle\big)\Psi\right\rangle$$

$$= \left\langle\Psi\middle|\big(\widehat{A}-\langle a\rangle\big)\middle|\big(\widehat{B}-\langle b\rangle\big)\Psi\right\rangle$$

$$= \big\langle\Psi|\widehat{A}(\widehat{B}\Psi)\big\rangle - \langle a\rangle\langle b\rangle = \big\langle\widehat{A}\widehat{B}\big\rangle - \langle a\rangle\langle b\rangle$$

$$\sigma_A^2\sigma_B^2 = \|(\widehat{A}-\langle a\rangle)\Psi\|^2\ \|(\widehat{B}-\langle b\rangle)\Psi\|^2$$

$$\equiv \langle f|f\rangle\langle g|g\rangle \geq \|\langle f|g\rangle\|^2 \qquad \text{(see Schwarz Ineq.)}$$

$$\geq\ \big[\text{Im}\big(\langle f|g\rangle\big)\big]^2\ =\ \left(\frac{1}{2i}\big[\langle f|g\rangle - \langle f|g\rangle^*\big]\right)^2$$

$$= \left(\frac{1}{2i}\langle\widehat{A}\widehat{B} - \widehat{B}\widehat{A}\rangle\right)^2 \equiv \boxed{\left(\frac{1}{2i}\left\langle[\widehat{A},\widehat{B}]\right\rangle\right)^2}$$

<u>Commutator</u>

- $[\widehat{A},\widehat{B}]f \equiv \widehat{A}(\widehat{B}f) - \widehat{B}(\widehat{A}f)$
- $[A,BC] = [A,B]C + B[A,C]$
- $[AB,C] = A[B,C] + [A,C]B$
- $[x,\hat{p}] = i\hbar$
- $\boxed{\sigma_A\sigma_B \geq \left\|\frac{1}{2i}\left\langle[\widehat{A},\widehat{B}]\right\rangle\right\|}$

$$\Rightarrow \boxed{\Delta x\Delta p\ \geq\ \hbar/2}$$

- $\boxed{\begin{array}{c}[\hat{p},f(x)] = \breve{p}f(x) = \frac{\hbar}{i}\frac{\partial f}{\partial x}\\ *\ \vec{p}\times\vec{v} + \vec{v}\times\vec{p} = \frac{\hbar}{i}\vec{\nabla}\times\vec{v}\\ [\hat{x},g(p)] = \hat{x}g(p) = -\frac{\hbar}{i}\frac{\partial g}{\partial p}\end{array}}$

<u>Commutator of Hermitian $\widehat{A},\widehat{B}$</u>

- $[A,B]^\dagger = -[A,B]$
- $\exists\Psi_n$ s.t. $\big(\widehat{A}\Psi_n = a\Psi_n\big)$, $\big(\widehat{B}\Psi_n = b\Psi_n\big)$

$\Leftrightarrow [\widehat{A},\widehat{B}] = 0$

$\Rightarrow \boxed{\sigma_A\sigma_B\ \geq\ 0\ \text{ (Both can be measured concurrently)}}$

$\boxed{AB = BA}$

<u>Anti-Hermitian Operators:</u> $A^\dagger = -A$

- $\langle A\rangle = ai, \quad a\in\mathbb{R}$
- $[A,B]^\dagger = -[A,B]$

Operator Evolution (Heisenberg Equation)

$$\frac{d}{dt}\Big\langle \Psi(x,t)\Big|Q\Big|\Psi(x,t)\Big\rangle = \Big\langle \frac{\partial\Psi}{\partial t}\Big|Q\Big|\Psi\Big\rangle + \Big\langle \Psi\Big|\frac{\partial Q}{\partial t}\Big|\Psi\Big\rangle + \Big\langle \Psi\Big|Q\Big|\frac{\partial\Psi}{\partial t}\Big\rangle$$

$$\boxed{\begin{aligned} \tfrac{d}{dt}\langle Q\rangle &= \tfrac{1}{i\hbar}\Big\langle\big[\widehat{Q},\widehat{H}\big]\Big\rangle + \Big\langle\tfrac{\partial\widehat{Q}}{\partial t}\Big\rangle \\ i\hbar\tfrac{d}{dt}\langle Q\rangle &= \Big\langle\big[\widehat{Q},\widehat{H}\big]\Big\rangle + i\hbar\Big\langle\tfrac{\partial\widehat{Q}}{\partial t}\Big\rangle \end{aligned}}$$

(Q is conserved when this equals 0)

- Conservations : $\dfrac{d\langle\Psi|\Psi\rangle}{dt} = 0,\ \dfrac{d\langle H\rangle}{dt} = 0$
- Ehrenfest's Theorem : $m\dfrac{d\langle x\rangle}{dt} = \langle p\rangle,\ \dfrac{d\langle p\rangle}{dt} = -\Big\langle\dfrac{\partial V}{\partial x}\Big\rangle \Rightarrow$ other classical eq.
- Virial Theorem : $\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\big\langle [H,x]p + x[H,p]\big\rangle = \Big\langle\big[\frac{p^2}{2m},x\big]p + x\big[V,p\big]\Big\rangle$

$$= \tfrac{i}{\hbar}\big\langle \tfrac{1}{2m}p[p,x]p + \tfrac{1}{2m}[p,x]p^2 - x[p,V]\big\rangle$$

$$\boxed{\frac{d\langle xp\rangle}{dt} = 2\langle T\rangle - \Big\langle x\frac{\partial V}{\partial x}\Big\rangle} \to 0 = \underset{\text{(for stationary states)}}{\tfrac{d}{dt}\Big\langle \Psi_n(x)\Big|Q_{(x,p)}\Big|\Psi_n(x)\Big\rangle}$$

- Energy-Time Uncertainty : $(Q = Q(x,\hat{p}) \neq Q(x,\hat{p},t)) \Rightarrow \sigma_H\sigma_Q \geq \dfrac{\hbar}{2}\left|\dfrac{d\langle Q\rangle}{dt}\right|$

$$\Rightarrow \boxed{\begin{aligned} \sigma_Q \ &\equiv\ \frac{d\langle Q\rangle}{dt}\Delta t \ \approx\ \Delta\langle Q\rangle \\ \sigma_H&\left(\frac{\sigma_Q}{|d\langle Q\rangle/dt|}\right) \geq \frac{\hbar}{2} \\ \Delta E\Delta t &\geq \frac{\hbar}{2} \end{aligned}}$$

Δt is the amount of time it would take $\langle Q\rangle$ to change "appreciably", or one std. dev. at the constant rate $\frac{d}{dt}\langle Q\rangle$

∗ Mass Lifetime : $\Delta(mc^2)\Delta t \geq \dfrac{\hbar}{2}$ ☑

∗ Orthogonal Time Example : $\Psi(x,\tau) = \dfrac{\sqrt{2}}{2}(\Psi_1 e^{-\frac{i}{\hbar}E_1\tau} + \Psi_2 e^{-\frac{i}{\hbar}E_2\tau})$

$$\Big\langle\Psi(x,0)\Big|\Psi(x,\tau)\Big\rangle = 0 = \frac{1}{2}(e^{-\frac{i}{\hbar}E_1\tau} + e^{-\frac{i}{\hbar}E_2\tau})$$

$$\Rightarrow \tau\ \frac{E_2 - E_1}{2} = \frac{\pi}{2}\ \hbar\ (\frac{1}{2} + n) \geq \frac{\hbar}{2} \ ☑$$

Translation Operator

$$f(x+\Delta x) \approx f(x) + \frac{df}{dx}\Delta x$$

$$= f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + ... = \left\{ \begin{matrix} f(x') = \sum_n \frac{f^{(n)}(a)}{n!}(x'-a)^n \\ (x' = x+\Delta x),\ (a = x) \end{matrix} \right\}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(\Delta x)^n \ = \ \sum_{n=0}^{\infty} \frac{(\Delta x \nabla)^n}{n!} f(x)$$

$$\boxed{f(x+\Delta x) = e^{\frac{i}{\hbar}(\Delta x)\check{p}}\ f(x)} \ \Leftrightarrow \ \boxed{f(x) = e^{\frac{i}{\hbar}x\check{p}}\ f(0)} \quad * \ \langle x|e^{\frac{i}{\hbar}x\hat{p}}|x'\rangle = e^{\frac{i}{\hbar}x\check{p}}\langle x|x'\rangle \ *$$

Time Translation :

$\langle x_N|\hat{U}_{(t)}|x_0\rangle$
$= \check{U}_{(x_N,\,t;\,x_0,\,0)}$

$\langle x|\hat{U}_{(t)}|\Psi\rangle$
$= \Psi_{(x,t)}$

$$f(t+\Delta t) \ = \ \underline{f(t) + f'(t)\Delta t} + ... \ = \ \sum_n \frac{(\Delta t)^n}{n!}\left(\frac{\partial}{\partial t}\right)^n f(t)$$

$$i\hbar\frac{\partial}{\partial t}\Psi = \check{H}_{(x,p,t)}\Psi$$

$$\frac{\partial f}{\partial t} = \left[\frac{-i\check{H}}{\hbar}\right] f \qquad \Rightarrow \qquad \left\{ \begin{matrix} f(t_0+\Delta t) \ \approx \ e^{\frac{-i\Delta t}{\hbar}\check{H}(t_0)} f(t_0) \quad \text{(1st order)} \\ \boxed{\begin{matrix} f(0+t) = \lim_{N\to\infty} \prod_{n=0}^{N-1} e^{\frac{-i}{\hbar}\check{H}(n\Delta t)\Delta t} f(0) \\ \sim\neq e^{\frac{-i}{\hbar}\int \check{H}(t)\,dt} f(0) \quad \left(\begin{matrix}\text{since } 0 \neq \\ [H(t_0), H(t_1)]\end{matrix}\right) \end{matrix}} \end{matrix} \right.$$

$$\frac{\partial^n f}{\partial t^n} \neq \left[\frac{-i\check{H}(t)}{\hbar}\right]^n f$$

Pictures: $\quad \big\langle Q \big\rangle_{(t)} = \langle \Psi_{(x,t)} \,|\, Q_{(x,p,t)} \,|\, \Psi_{(x,t)}\rangle$

- $\boxed{\text{Schrodinger Picture: } \big\langle Q \big\rangle_{(t)} = \Big\langle e^{\frac{-i}{\hbar}t\widehat{H}}\,\Psi_{(t=0)} \Big|\, Q_{(x,p,t)} \,\Big| e^{\frac{-i}{\hbar}t\widehat{H}}\,\Psi_{(t=0)} \Big\rangle}$

$$Q = Q(x,p) \ \Rightarrow \ \big\langle Q \big\rangle_{(t)} = \Big\langle \sum e^{\frac{-i}{\hbar}E_n t} c_n \Psi_{n(x)} \Big|\ Q \ \Big| \sum e^{\frac{-i}{\hbar}E_n t} c_n \Psi_{n(x)} \Big\rangle \quad \text{(nice for stationary states)}$$

- $\boxed{\text{Heisenberg Picture: } \big\langle Q \big\rangle_{(t)} = \Big\langle \Psi_{(t=0)} \Big| e^{\frac{i}{\hbar}t\widehat{H}}\, Q\, e^{\frac{-i}{\hbar}t\widehat{H}} \Big| \Psi_{(t=0)} \Big\rangle}$

- $\boxed{\text{Dirac Picture: } \big\langle Q \big\rangle_{(t)} = \Big\langle e^{\frac{-i}{\hbar}\int \widehat{H}_1(t)dt}\,\Psi_{(t=0)} \Big| e^{\frac{i}{\hbar}t\widehat{H}_0}\, Q\, e^{\frac{-i}{\hbar}t\widehat{H}_0} \Big| e^{\frac{-i}{\hbar}\int \widehat{H}_1(t)dt}\,\Psi_{(t=0)} \Big\rangle}$

$$\big\langle Q \big\rangle_{(t+\Delta t)} \ = \ \big\langle Q \big\rangle_{(t)} + \tfrac{d\langle Q\rangle}{dt}\Delta t + ... \quad \Rightarrow$$

A 1st order approximation of $\big\langle Q \big\rangle_{(t+\Delta t)}$ should yield $\frac{d\langle Q\rangle}{dt} = \frac{1}{i\hbar}\Big\langle \big[Q, H\big] \Big\rangle + \frac{\partial Q}{\partial t}$

Schrodinger Picture

1.) $i\hbar \frac{\partial}{\partial t}\langle Q_S \rangle = \langle [Q_S, H_S] \rangle$

4.) $|\Psi_{S(t)}\rangle = U_{S(t,t_0)}|\Psi_{S(t_0)}\rangle$

$$\Rightarrow\ i\hbar \frac{\partial}{\partial t}|\Psi_S\rangle = H_S \Psi_S = [H_S^0 + H^1_{S(t)}]|\Psi_S\rangle$$

$$= \sum E_n |n_S^0\rangle e^{-\frac{i}{\hbar}E_n t}\langle n^1_{S(t)}|\Psi(0)\rangle$$

$$+ \sum |n_S^0\rangle e^{-\frac{i}{\hbar}E_n t} \cdot i\hbar \frac{\partial}{\partial t}\langle n^1_{S(t)}|\Psi(0)\rangle$$

$$\Rightarrow\ i\hbar \frac{\partial}{\partial t} U^0_{S(t,t_0)} = H_S^0 U^0_{S(t,t_0)}$$

$$\Rightarrow\ U^0_{S(t,t_0)} = e^{-\frac{i}{\hbar}H_S^0(t-t_0)}$$

Heisenberg Picture

1.) $Q_{H(t)} \ \equiv\ U_S^\dagger Q_S U_S$ $\qquad H_S \neq H_S(t) \Downarrow H_H = H_S$

$$\Rightarrow\ i\hbar \frac{\partial}{\partial t} Q_H = [Q_H, H_H]$$

2.) $U_H \ \equiv\ U^\dagger_{S(t,t_0)} U_{S(t,t_0)} = \mathbb{I}$

3.) $|q_{H(t)}\rangle \ \equiv\ U^\dagger_{S(t,t_0)}|q_S\rangle$

$$\Rightarrow\ Q_H|q_{H(t)}\rangle = q|q_{H(t)}\rangle$$

$$\Rightarrow\ i\hbar \frac{\partial}{\partial t}|q_{H(t)}\rangle = -H_S|q_{H(t)}\rangle$$

4.) $|\Psi_H\rangle \ \equiv\ U^\dagger_{S(t,t_0)}|\Psi_{S(t)}\rangle = |\Psi_{S(t_0)}\rangle$

$$= U_{H(t,t_0)}|\Psi_{H(t_0)}\rangle$$

$$\Rightarrow\ i\hbar \frac{\partial}{\partial t}|\Psi_H\rangle = 0$$

Dirac/Interaction Picture (see transition amplitude)

1.) $Q_{I(t)} \ \equiv\ U_S^{0\dagger} Q_S U_S^0$

$$\Rightarrow\ i\hbar \frac{\partial}{\partial t} Q_I = \left[Q_I, H_I^0\right] \qquad (H_S^0 = H_I^0 \text{ see Heis. pic.})$$

2.) $U_{I(t,t_0)} \ \equiv\ U_S^{0\dagger}{}_{(t,t_0)} U_{S(t,t_0)}$

3.) $|q_{I(t)}\rangle \ \equiv\ U_S^{0\dagger}{}_{(t,t_0)}|q_S\rangle$

$$\Rightarrow\ Q_I|q_{I(t)}\rangle = q|q_{I(t)}\rangle$$

$$\Rightarrow\ i\hbar \frac{\partial}{\partial t}|q_{I(t)}\rangle = -H_S^0|q_{I(t)}\rangle$$

4.) $|\Psi_{I(t)}\rangle \ \equiv\ U_S^{0\dagger}{}_{(t,t_0)}|\Psi_{S(t)}\rangle$

$$= U_{I(t,t_0)}|\Psi_{I(t_0)}\rangle \qquad (\text{since } |\Psi_I(t_0)\rangle = |\Psi_S(t_0)\rangle)$$

$$\Rightarrow\ i\hbar \frac{\partial}{\partial t}|\Psi_I\rangle = U_S^{0\dagger} H^1_{S(t)} U_S^0 |\Psi_I\rangle = H_I^1|\Psi_I\rangle$$

$$\Rightarrow\ i\hbar \frac{\partial}{\partial t} U_{I(t,t_0)} = H^1_{I(t)} U_{I(t,t_0)}$$

$$\Rightarrow\ U_{I(t,t_0)} = \mathbb{I} + \tfrac{1}{i\hbar}\int_{t_0}^t H^1_{I(t')} U_{I(t',t_0)}\, dt'$$

- $U_{I(t,t_0)} = \mathbb{I} + \tfrac{1}{i\hbar}\int_{t_0}^t H^1_{I(t')} U_{I(t',t_0)}\, dt'$

$$= \mathbb{I} + \mathcal{O}(H_I^1)$$

$$= \mathbb{I} + \tfrac{1}{i\hbar}\int_{t_0}^t H^1_{I(t')}\, dt' + \mathcal{O}([H_I^1]^2)$$

$$= \mathbb{I} + \tfrac{1}{i\hbar}\int_{t_0}^t H^1_{I(t')}\, dt'$$

$$+ \left(\tfrac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} H^1_{I(t')} H^1_{I(t'')}\, dt'' dt' + \ldots$$

- $U_{S(t,t_0)} = U_S^0 + \tfrac{1}{i\hbar}\int_{t_0}^t U_S^0 H^1_{I(t')}\, dt' + \left(\tfrac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} U_S^0 H^1_{I(t')} H^1_{I(t'')}\, dt'' dt' + \ldots$

$$= U^0{}_{(t,t_0)}$$

$$+ \tfrac{1}{i\hbar}\int_{t_0}^t U^0{}_{(t,t_0)} U^{0\dagger}{}_{(t',t_0)} H^1 U^0{}_{(t',t_0)}\, dt'$$

$$+ \left(\tfrac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} U^0{}_{(t,t_0)} U^{0\dagger}{}_{(t',t_0)} H^1 U^0{}_{(t',t_0)} U^{0\dagger}{}_{(t'',t_0)} H^1 U^0{}_{(t'',t_0)}\, dt'' dt' + \ldots$$

Infinitismal t Path Integral

$$S[x(t)] = \int_0^t \mathcal{L}(x, \dot{x})\, dt \;\to\; \mathcal{L}\delta t = \left[\tfrac{1}{2} m \left(\tfrac{x_1 - x_0}{\delta t}\right)^2 - V\left(\tfrac{x_1 + x_0}{2}, t_0 + \tfrac{\delta t}{2}\right)\right] \delta t$$

$$\langle x|\hat{U}(\epsilon)|\Psi\rangle = \int \langle x|\hat{U}(\epsilon)|x'\rangle\langle x'|\Psi(x,t)\rangle\, dx' = \int \breve{U}(x, t+\epsilon;\, x', t)\Psi(x', t)\, dx' = \Psi(x, t+\epsilon)$$

- $$\begin{aligned}
\breve{U}(x_1, \epsilon;\, x_0, 0) &= Ae^{\frac{i}{\hbar}S} = Ae^{\frac{i}{\hbar}\mathcal{L}\epsilon} \\
&= A \exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2} m \tfrac{(x_1 - x_0)^2}{\epsilon} - \epsilon V\left(\tfrac{x_1 + x_0}{2}, 0 + \tfrac{\epsilon}{2}\right)\right]\right\} \\
(\eta = x_0 - x_1) \qquad &= A \exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2} m \tfrac{\eta^2}{\epsilon}\right]\right\} \exp\left\{-\tfrac{i}{\hbar}\epsilon V(x_1 + \tfrac{\eta}{2}, 0 + \tfrac{\epsilon}{2})\right\} \\
&\approx A \exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2} m \tfrac{\eta^2}{\epsilon}\right]\right\} \exp\left\{-\tfrac{i}{\hbar}\epsilon V(x_1, 0)\right\} \\
&\approx A \exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2} m \tfrac{\eta^2}{\epsilon}\right]\right\} \left[1 - \tfrac{i}{\hbar}\epsilon V(x_1, 0)\right]
\end{aligned}$$

> $\frac{\eta^2}{\epsilon} \lesssim \pi$ Explanation
>
> The integral involving $\breve{U}$ is over all η. The phase of the complex exponential will vary/oscillate too wildly and destructively interfere if η^2/ϵ were to grow too big, so $\eta^2 \sim \epsilon$ is all that matters. This means the integral is over $\sqrt{\epsilon}$, not ϵ. Because of this (somehow, Bibl. given), using a finite difference formula for derivatives is legitimate in this case, though not in general.

$$\begin{aligned}
\Psi(x, \epsilon) &= \int_{-\infty}^{\infty} \breve{U}(x, \epsilon;\, x', 0)\Psi(x', 0)\, dx' \\
&= A \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}S}\Psi(x', 0)\, dx' \\
&= A \int_{-\infty}^{\infty} \exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2} m \tfrac{(x - x')^2}{\epsilon} - \epsilon V\left(\tfrac{x + x'}{2}, 0 + \tfrac{\epsilon}{2}\right)\right]\right\}\Psi(x', 0)\, dx' \\
&= A \int_{-\infty}^{\infty} \exp\left\{\tfrac{i}{\hbar}\left[\tfrac{1}{2} m \tfrac{\eta^2}{\epsilon} - \epsilon V\left(\tfrac{x + \eta/2}{2}, 0 + \tfrac{\epsilon}{2}\right)\right]\right\}\Psi(x + \eta, 0)\, d\eta \\
&\approx A \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\left[\frac{1}{2} m \frac{\eta^2}{\epsilon}\right]} \left[1 - \tfrac{i}{\hbar}\epsilon V(x, 0)\right] \left[\Psi(x, 0) + \eta\Psi'(x, 0) + \tfrac{\eta^2}{2}\Psi''(x, 0)\right] d\eta \\
&\approx A \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\left[\frac{1}{2} m \frac{\eta^2}{\epsilon}\right]} \left[\left(1 - \tfrac{i}{\hbar}\epsilon V(x, 0)\right)\Psi(x, 0) + \cancel{\eta\Psi'(x, 0)} + \tfrac{\eta^2}{2}\Psi''(x, 0)\right] d\eta \\
&= \cancel{A\sqrt{\tfrac{2\hbar\epsilon\pi}{-im}}} \left[\left(1 - \tfrac{i}{\hbar}\epsilon V(x, 0)\right) + \tfrac{1}{2} \cdot \tfrac{2\hbar\epsilon}{-im} \cdot \tfrac{1}{2}\tfrac{\partial^2}{\partial x^2}\right] \Psi(x, 0) \\
&= \Psi(x, 0) - \tfrac{i}{\hbar}\epsilon\breve{H}\Psi(x, 0)
\end{aligned}$$

$$\boxed{i\hbar\, \frac{\Psi(x, \epsilon) - \Psi(x, 0)}{\epsilon - 0} = \breve{H}\Psi(x, 0)}$$

Finite t, Free Particle Propagator

$$S[x(t)] = \int_0^t \mathcal{L}(x,\dot{x})\,dt = \lim_{N\to\infty}\sum_{n=0}^{N-1}\left[\tfrac{1}{2}m\left(\tfrac{x_{n+1}-x_n}{\delta t}\right)^2\right]\delta t$$

$$\langle x_N|\hat{U}(t)|\Psi(0)\rangle = \int\langle x_N|\hat{U}(t)|x_0\rangle\Psi(x_0,0)\;dx_0 = \int \breve{U}(x_N,t;\,x_0,0)\Psi(x_0,0)\;dx_0 = \Psi(x_N,t)$$

$$\Rightarrow\;\; \langle x_N|\hat{U}(t)|x_0\rangle = \langle x_N|e^{-\frac{i}{\hbar}Ht}e^{\frac{i}{\hbar}Ht_0}|x_0\rangle = \langle x_N,t_N\,|\,x_0,t_0\rangle$$

$$\langle x_N|\hat{U}(t)|x_0\rangle = \lim_{N\to\infty}\langle x_N|\hat{U}^N(\epsilon)|x_0\rangle = \lim_{N\to\infty}\langle x_N|\hat{U}(\epsilon)\ldots\hat{U}(\epsilon)\hat{U}(\epsilon)|x_0\rangle$$

$$= \lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle x_N|\hat{U}(\epsilon)|x_{N-1}\rangle\ldots\langle x_2|\hat{U}(\epsilon)|x_1\rangle\langle x_1|\hat{U}(\epsilon)|x_0\rangle\;dx_1\,dx_2\ldots dx_{N-1}$$

$$\breve{U}(x_N,t;\,x_0,0) = \int_{x_0}^{x_N} Ae^{\frac{i}{\hbar}S}\,\mathcal{D}[x(t)]$$

$$= \lim_{N\to\infty} A\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{(x_1-x_0)^2}{\epsilon}+\frac{1}{2}m\frac{(x_2-x_1)^2}{\epsilon}+\frac{1}{2}m\frac{(x_3-x_2)^2}{\epsilon}+\ldots\right]}dx_1\,dx_2\ldots dx_{N-1}$$

$$= \lim_{N\to\infty} A\sqrt{\tfrac{2\hbar\epsilon}{m}}^{N-1}\int\cdots\int_{-\infty}^{\infty} e^{-\frac{(y_1-y_0)^2}{i}-\frac{(y_2-y_1)^2}{i}}\;dy_1\,e^{\left[-\frac{(y_3-y_2)^2}{i}\cdots\right]}\,dy_2\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\tfrac{2\hbar\epsilon}{m}}^{N-1}\int\cdots\int_{-\infty}^{\infty} e^{-\frac{y_1^2+y_1^2-2y_1(y_2+y_0)+y_2^2+y_0^2}{i}}\;dy_1\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\tfrac{2\hbar\epsilon}{m}}^{N-1}\int\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{i}\left(2\left[y_1^2-y_1(y_0+y_2)+\frac{(y_0+y_2)^2}{4}\right]-\frac{(y_2+y_0)^2}{2}+y_2^2+y_0^2\right)}\;dy_1\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\tfrac{2\hbar\epsilon}{m}}^{N-1}\int\cdots\int_{-\infty}^{\infty} e^{-\frac{2}{i}\left[y_1-\cancel{\frac{y_2+y_0}{2}}\right]^2}\;dy_1\,e^{-\frac{(y_2-y_0)^2}{2i}}\,e^{-\frac{(y_3-y_2)^2}{i}\cdots}\,dy_2\,e^{[\ldots]}\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\tfrac{2\hbar\epsilon}{m}}^{N-1}\int\ldots\sqrt{\tfrac{\pi i}{2}}\int_{-\infty}^{\infty} e^{-\frac{(y_2-y_0)^2}{2i}}\,e^{-\frac{(y_3-y_2)^2}{i}\cdots}\,dy_2\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\tfrac{2\hbar\epsilon}{m}}^{N-1}\int\ldots\sqrt{\tfrac{(\pi i)^2}{3}}\int_{-\infty}^{\infty} e^{-\frac{(y_3-y_0)^2}{3i}}\,e^{-\frac{(y_4-y_3)^2}{i}\cdots}\,dy_3\ldots$$

$$= \lim_{N\to\infty} A\sqrt{\tfrac{2\hbar\epsilon}{m}}^{N-1}\sqrt{\tfrac{(\pi i)^{N-1}}{N}}\;e^{-\frac{(y_N-y_0)^2}{Ni}} \;=\; \lim_{N\to\infty}\cancel{A\sqrt{\tfrac{2\hbar\epsilon\pi}{-im}}^{N}}\sqrt{\tfrac{-im}{2\hbar\epsilon N\pi}}\;e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x_N-x_0)^2}{N\epsilon}}$$

$$\boxed{\text{Free Particle: }\;\breve{U}(x_N,t;\,x_0,0) = \sqrt{\tfrac{-im}{2\hbar t\pi}}\;e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x_N-x_0)^2}{t}}}$$

$$\boxed{\int_{x_0}^{x_N}\mathcal{D}[x(t)] = \lim_{N\to\infty}\sqrt{\tfrac{-im}{2\hbar\epsilon\pi}}\int_{-\infty}^{\infty}\sqrt{\tfrac{-im}{2\hbar\epsilon\pi}}\,dx_1\cdots\int_{-\infty}^{\infty}\sqrt{\tfrac{-im}{2\hbar\epsilon\pi}}\,dx_{N-1}}$$

All Paths Explanation

$$S[x] = S[x_{cl}] + \cancel{S'[x_{cl}]}\eta + \mathcal{O}(\eta^2)$$

1st order variation of S from x_{cl} equals 0. This means propagator integrand for paths near x_{cl} will have about the same phase, and will add constructively. Paths very different from x_{cl} (like those with faster than light motion) will vary in action, and because $\hbar$ is so small their phases will vary wildly, meaning the sum will destructively interfere. The result is that only paths near the classical path will be important, with $S[x]/\hbar \lesssim \pi$.

Action-Energy Relationship

$$S(x_{cl} + \Delta x_{cl}, \dot{x} + \Delta \dot{x}_{cl}, \tau + \Delta\tau) = S_{cl} + \Delta S_{cl}$$

$$= \int_0^{\tau+\Delta\tau} \mathcal{L}\big(x_{cl} + \Delta x_{cl}, \dot{x}_{cl} + \Delta\dot{x}_{cl}, t\big)\, dt$$

$$dS = \tfrac{\partial S}{\partial \tau} d\tau + \Big[\tfrac{\partial S}{\partial x} dx + \tfrac{\partial S}{\partial \dot{x}} d\dot{x} = \int (dL) dt\Big] \qquad (\eta = dx,\ \eta(0) = 0)$$

$$\Delta S_{cl} = \mathcal{L}(\tau)\,\Delta\tau + \int_0^\tau \tfrac{\partial \mathcal{L}}{\partial x}\Big|_{cl}\,\eta + \tfrac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_{cl}\,\dot{\eta}\, dt$$

$$= \mathcal{L}(\tau)\,\Delta\tau + \int_0^\tau \cancel{\big[\tfrac{\partial \mathcal{L}}{\partial x} - \tfrac{d}{dt}\tfrac{\partial \mathcal{L}}{\partial \dot{x}}\big]_{cl}}\eta + \tfrac{d}{dt}\Big[\tfrac{\partial \mathcal{L}}{\partial \dot{x}_{cl}}\eta\Big]\, dt$$

$$= \big[\mathcal{L} + \tfrac{\partial \mathcal{L}}{\partial \dot{x}}\dot{x}\big]_{cl,\tau}\,\Delta\tau$$

$$\boxed{\Delta S_{cl} = -H(t_f)\,\Delta t_f}$$

(Time-Independent) Hamiltonian-Lagrangian Propagator Relationship / Finite Path Integral

$$\breve{U}(x_N, t; x_0, 0) = \langle x_N | e^{-\frac{i}{\hbar}Ht} | x_0\rangle = \langle x_N | [e^{-\frac{i}{\hbar}H\frac{t}{N}}]^N | x_0\rangle = \lim_{N\to\infty} \langle x_N | [e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\epsilon} e^{-\frac{i}{\hbar}V(\hat{x})\epsilon}]^N | x_0\rangle \qquad \text{(not trivial)}$$

$$= \lim_{N\to\infty} \int_{x\ldots} \langle x_N | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\epsilon} e^{-\frac{i}{\hbar}V(\hat{x})\epsilon} | x_{N-1}\rangle \ldots \langle x_1 | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\epsilon} e^{-\frac{i}{\hbar}V(\hat{x})\epsilon} | x_0\rangle\, dx \ldots$$

$$= \lim_{N\to\infty} \int_{x\ldots} \ldots \Big[\underbrace{\int \langle x_1 | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\epsilon} | p\rangle\langle p | x_0\rangle\, dp}_{\text{free part. prop.}}\, e^{-\frac{i}{\hbar}V(x_0)\epsilon}\Big] dx \ldots$$

$$1. = \lim_{N\to\infty} \int_{x\ldots} \ldots \Big[\int \frac{e^{\frac{i}{\hbar}p(x_1 - x_0)}}{2\pi\hbar} e^{-\frac{i}{\hbar}\frac{p^2}{2m}\epsilon} e^{-\frac{i}{\hbar}V(x_0)\epsilon}\, dp\Big] dx \ldots = \boxed{\int e^{\frac{i}{\hbar}\int p\dot{x} - H(x,p)\, dt}\, [\mathcal{D}x\mathcal{D}p] \qquad \text{(Phase Space)}}$$

$$2. = \lim_{N\to\infty} \int_{x\ldots} \ldots \Bigg[\sqrt{\tfrac{-im}{2\pi\hbar\epsilon}}\, e^{\frac{im(x_1 - x_0)^2}{\epsilon}}\, e^{-\frac{i}{\hbar}V(x_0)\epsilon}\Bigg] dx \cdots = \lim_{N\to\infty} \int_{x\ldots} \ldots \sqrt{\tfrac{-im}{2\pi\hbar\epsilon}}\, e^{\frac{i}{\hbar}\mathcal{L}\epsilon}\, dx_1 \ldots$$

$$= \boxed{\int_{x_0}^{x_N} e^{\frac{i}{\hbar}S}\, [\mathcal{D}x] = \int_{x_0}^{x_N} e^{\frac{i}{\hbar}\int \mathcal{L}\, dt}\, [\mathcal{D}x] \qquad \text{(Configuration Space)}} \qquad \text{(above is only integrable if } p \text{ is quadratic in } H\text{)}$$

Trace of Propagator

$$G(t) = \int \langle x | e^{-\frac{i}{\hbar}Ht} | x\rangle\, d^3x$$

$$= \sum_n \int \langle x | n\rangle e^{-\frac{i}{\hbar}E_n t} \langle n | x\rangle\, dx$$

$$G(t) = \sum_n e^{-\frac{i}{\hbar}E_n t} \sim \sum_n e^{-\beta E_n} = Z(\beta)$$

1.3 Extra

$L_2 \subset$ Hilbert Space = complete inner product space

$\rho_{(x,t)} \equiv \|\Psi\|^2 \;\;,\;\; P_a^b(t) = \int_a^b \rho\, dx \;\;,\;\; P(t) = P_{-\infty}^{\infty}(t) \;\;,\;\; \Psi = \sqrt{\rho}\, e^{\frac{i}{\hbar}S}$ e.g., $e^{\frac{i}{\hbar}(p\cdot x - Et)}$

- $$\begin{aligned} \breve{E}\rho &= \breve{E}(\Psi^*\Psi) = \Psi^*(\breve{E}\Psi) + \Psi(\breve{E}\Psi^*) \\ &= \Psi^*(\breve{H}\Psi) - \Psi(\breve{H}\Psi^*) \\ &= \Psi^*(\tfrac{p^2}{2m} + V)\Psi - \Psi(\tfrac{p^2}{2m} + V)\Psi^* \\ -\tfrac{\hbar}{i}\tfrac{\partial \rho}{\partial t} &= \tfrac{\hbar}{i}\nabla \cdot \left(\Psi^* \tfrac{p}{2m}\Psi - \Psi\tfrac{p}{2m}\Psi^*\right) \end{aligned}$$

  (Probability Current)

  $$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\nabla \cdot J = -\nabla \cdot \left(\Psi^* \tfrac{p}{2m}\Psi - \Psi\tfrac{p}{2m}\Psi^*\right) \\ &= -\nabla \cdot \tfrac{\rho \nabla S}{m} \quad \left(\text{e.g., } \nabla S = p\right) \end{aligned}$$

  $$\left[\tfrac{d}{dt}P_a^b = J_{(a,t)} - J_{(b,t)}\right], \; \left[\int J\, dV = \langle \Psi | \tfrac{p}{m} | \Psi \rangle = \tfrac{\langle p \rangle}{m}\right]$$

- $(V \in \mathbb{R}) \quad \Rightarrow \quad \frac{d}{dt}P = 0 \quad \Rightarrow \quad P(t) \equiv 1$

  $(V = V_0 - i\Gamma) \quad \Rightarrow \quad \frac{d}{dt}P = \frac{-2\Gamma}{\hbar}P \quad \Rightarrow \quad P(t) = e^{-2(\Gamma/\hbar)t}$

- $\langle \Psi_n | \Psi_n \rangle \;,\; \langle \Psi_m | \Psi_m \rangle = 1 \;\Rightarrow\; \frac{d}{dt}\langle \Psi_n | \Psi_m \rangle = 0$

Schwarz Inequality: $$\left\| \int_a^b f^* g \; dx \right\|^2 \le \left\| \int_a^b f^* f \; dx \right\| \left\| \int_a^b g^* g \; dx \right\|$$

$$\|\langle f | g \rangle_{ab}\|^2 \le \|\langle f | f \rangle_{ab}\| \, \|\langle g | g \rangle_{ab}\|$$

$$\Big[V(x) = V(-x)\Big] \;\Rightarrow\; \Big[\Psi(x) \Rightarrow \Psi(-x)\Big] \;\Rightarrow\; \Big[\Psi(-x) = \Psi(x)\Big] \;\cup\; \Big[\Psi(-x) = -\Psi(x)\Big]$$

Discontinuity in Ψ means the possiblity of $\sigma_p \to \infty$

Prob 3.29: $\Psi(x,0) = \begin{cases} \frac{1}{\sqrt{2n\lambda}} e^{2\pi i x/\lambda}, & -n\lambda < x < n\lambda \\ 0 & \text{else} \end{cases}$

$\sigma_p \to \infty$ because the integral of $\delta^2(x)$ is infinite

$$\int_{-\infty}^{\infty} f(x) D_1(x) dx = \int_{-\infty}^{\infty} f(x) D_2(x) dx \;\Rightarrow\; \delta(cx) = \frac{1}{|c|}\delta(x)$$

$$\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')} dx' \;\Rightarrow\; F[\delta(x)] = \frac{1}{2\pi}$$

$$\delta'(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} ik e^{ik(x-x')} dx' \;\Rightarrow\; \int \delta'(x - x') f(x') dx' = f'(x)$$

Poisson Brackets

$$\{f,g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$$

- $\{q_i, g(q,p,t)\} = \frac{\partial g}{\partial p_i}$, $\{p_i, g(q,p,t)\} = -\frac{\partial g}{\partial q_i}$

$\Rightarrow$

- $$\{f,g\} = \sum_i -\{p_i,f\}\{q_i,g\} + \{p_i,g\}\{q_i,f\}$$
$$= \sum_i \{q_i,f\}\{p_i,g\} - \{q_i,g\}\{p_i,f\}$$

Hamilton Eq. : $\dot{q}_i = \{q_i, H\}$, $\dot{p}_i = \{p_i, H\}$ $\Rightarrow$ $\{f(q,p,t), H\} = \sum_i \frac{\partial f}{\partial q_i}\dot{q} + \dot{p}\frac{\partial f}{\partial p_i} = \dot{f} - \frac{\partial f}{\partial t}$

Canonical Transforms

$q \to \bar{q}(q,p)$, $p \to \bar{p}(q,p)$ s.t. $\{\bar{q}_i, \bar{q}_j\} = 0 = \{\bar{p}_i, \bar{p}_j\}$, $\{\bar{q}_i, \bar{p}_j\} = \delta_{ij}$ (Point Transforms $\bar{q}(q)$ are canonical.) $\Rightarrow$ $\dot{\bar{q}} = \frac{\partial H}{\partial \bar{p}}$, $\dot{\bar{p}} = -\frac{\partial H}{\partial \bar{q}}$, $\{f,g\}_{q,p} = \{f,g\}_{\bar{q},\bar{p}}$

Generator of Transformation

$$\{f,g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$$

$$\boxed{\frac{df}{d\lambda_g} - \frac{\partial f}{\partial t}\frac{\partial t}{\partial \lambda_g}} \equiv \sum_i \frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial \lambda_g} + \frac{\partial p_i}{\partial \lambda_g}\frac{\partial f}{\partial p_i}$$

$$\boxed{\frac{\partial g}{\partial t}\frac{\partial t}{\partial \lambda_f} - \frac{dg}{d\lambda_f}} \equiv \sum_i -\frac{\partial p_i}{\partial \lambda_f}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial q_i}{\partial \lambda_f}$$

$$\{g,H\} = \sum_i \boxed{\frac{\partial g}{\partial q_i}}\frac{\partial H}{\partial p_i} - \boxed{\frac{\partial g}{\partial p_i}}\frac{\partial H}{\partial q_i}$$

$$\equiv \sum_i \boxed{-\frac{\partial p_i}{\partial \lambda}}\frac{\partial H}{\partial p_i} - \boxed{\frac{\partial q_i}{\partial \lambda}}\frac{\partial H}{\partial q_i}$$

$$\boxed{\dot{g} - \frac{\partial g}{\partial t} = -\frac{dH}{d\lambda} + \frac{\partial H}{\partial t}\frac{\partial t}{\partial \lambda}}$$

1. $\delta H = 0$ $\qquad \delta H = \epsilon_\lambda \{H, g\}$

2. $\bar{q}_i = q_i + \delta q_i$, $\bar{p}_i = p_i + \delta p_i$ $\Rightarrow$ $\boxed{\frac{\partial H}{\partial \lambda} = 0 = -\frac{dg}{dt}}$

$\equiv q_i + \epsilon_\lambda \frac{\partial g}{\partial p_i}$ $\qquad \equiv p_i - \epsilon_\lambda \frac{\partial g}{\partial q_i}$ $\qquad$ (e.g. $g = p$ or $g = l_z$)

$= q_i + \epsilon_\lambda \{q_i, g\}$ $\qquad = p_i + \epsilon_\lambda \{p_i, g\}$

3. $\Rightarrow$ $\delta f = \epsilon_\lambda \{f, g\}$ $\to$ $\frac{df}{d\lambda_g} - \frac{\partial f}{\partial t}\frac{\partial t}{\partial \lambda_g} = \{f, g\}$

- $g = l_z$ $\Rightarrow$ $\delta x = -\epsilon y = -(\delta\theta) y$, $\delta y = \epsilon x = (\delta\theta) x$ $\Rightarrow$ $\boxed{\frac{\partial x}{\partial \theta} = -y, \quad \frac{\partial y}{\partial \theta} = x}$

Tensors and Tensor Operators

rank-2 Tensor : $|t^{(2)}\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3} t_{ij}|i\rangle|j\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3} |ij\rangle\langle ij|t^{(2)}\rangle$

rank-2 Carte. Tens. Oper., T_{ij} : Set of $3^{n=2}$ Operators

rank-k Spher. Tens. Oper., T_k^q : Set of $2k+1$ Operators s.t. $U[R]T_k^q U^\dagger[R] = \sum_{q'=-k}^{k} D_{q'q}^k T_k^{q'}$

$$\Rightarrow UT_k^q U^\dagger U|jm\rangle = \sum_{q'}\sum_{m'} D_{q'q}^k D_{m'm}^j T_{q'}^k |jm'\rangle$$

$$\sim \ U|kq\rangle|jm\rangle = \sum_{q'}\sum_{m'} D_{q'q}^k D_{m'm}^j |kq'\rangle|jm'\rangle$$

- $T_1^{\pm 1} = \mp\frac{V_x \pm iV_y}{\sqrt{2}}$
  $T_1^0 = V_z$

Wigner-Eckhart : $\langle \alpha_2 j_2 m_2 | T_k^q | \alpha_1 j_1 m_1\rangle = \langle \alpha_2 j_2 | T_k | \alpha_1 j_1 \rangle \cdot \underset{\text{(CG coeff.)}}{\langle j_2 m_2 | kq, j_1 m_1 \rangle}$

$$0 = \left[\tfrac{1}{\sin\theta}\tfrac{d}{d\theta}\left(\sin\theta\tfrac{d}{d\theta}\right) + l(l+1) - \tfrac{m^2}{\sin^2\theta}\right] P_l^m = \left[\tfrac{d}{dx}\left([1-x^2]\tfrac{d}{dx}\right) + l(l+1) - \tfrac{m^2}{1-x^2}\right] P_l^m(x=\cos\theta)$$

$$= \left[\tfrac{d}{d\xi}\left(\xi[1-\xi]\tfrac{d}{d\xi}\right) + l(l+1) - \tfrac{m^2}{1-x^2}\right] P_l^m\left(\xi = \tfrac{1}{2}[1-x]\right)$$

Legendre Polynomial

($m = 0 \leftrightarrow$ Azimuthal Symmetry)

$$P_l(x) = \tfrac{1}{2^l l!}\left(\tfrac{d}{dx}\right)^l (x^2 - l)^l$$

$$\delta_{l'l} = \tfrac{2l+1}{2}\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx$$

Associated Legendre Function

(not a polynomial if m is odd)

$$P_l^m(x) = \sqrt{1-x^2}^{\,|m|}\left(\tfrac{d}{dx}\right)^{|m|} P_l(x)$$

$$\delta_{l'l} = \tfrac{(l-m)!}{(l+m)!}\tfrac{2l+1}{2}\int_{-1}^{1} P_l^m(x) P_{l'}^m(x)\, dx$$

Spherical Harmonics

$$Y_{lm}(\theta,\phi) = \tfrac{1}{\sqrt{2\pi}} e^{im\phi} \times$$

$$(-1)^m Y_{l,-m}^* \qquad \sqrt{\tfrac{(l-m)!}{(l+m)!}\tfrac{2l+1}{2}}\, P_l^m(\cos\theta)$$

$$\delta_{l'l}\delta_{m'm} = \iint Y_{m'l'}^* Y_{ml}\, d\Omega$$

- $\boxed{P_l(\cos\theta) = P_l^0(\cos\theta) = \sqrt{\tfrac{4\pi}{2l+1}}\, Y_{l0}(\theta,\phi)}$
- $f(0,\phi) = \sum_l \cancel{\sum_m} \langle x|Y_{l0}\rangle\langle Y_{l0}^*|f\rangle = \sum_{l=0}^{\infty}\sqrt{\tfrac{2l+1}{4\pi}}\,\cancel{P_l(1)}\,\underline{\int\sqrt{\tfrac{2l+1}{4\pi}}\, P_l(\cos\theta) f(\theta,\phi)\, d\Omega}$
- $\boxed{0 = \left[\nabla^2 + \tfrac{l(l+1)}{r^2} - \tfrac{m^2}{r^2\sin^2\theta}\right] P_l^m(\cos\theta)} = \left[\tfrac{1}{r^2\sin\theta}\tfrac{d}{d\theta}\left(\sin\theta\tfrac{d}{d\theta}\right) + \tfrac{l(l+1)}{r^2} - \tfrac{m^2}{r^2\sin^2\theta}\right] P_l^m(\cos\theta)$

$$= \left[\nabla^2 + \tfrac{l(l+1)}{r^2} + \tfrac{\nabla^2 e^{im\phi}}{e^{im\phi}}\right] P_l^m(\cos\theta) = \tfrac{1}{e^{im\phi}}\left[\nabla^2 + \tfrac{l(l+1)}{r^2}\right]\left[e^{im\phi} P_l^m(\cos\theta)\right] = \boxed{\begin{bmatrix}\nabla^2 + \tfrac{l(l+1)}{r^2} \\ (\nabla^2 = \nabla'^2)\end{bmatrix} \begin{matrix} Y_{lm}(\theta,\phi), P_l(\cos\theta) \\ Y_{lm'}(\gamma,\beta), P_l(\cos\gamma)\end{matrix}}$$

$$\to\ g(0,\beta) = Y_{lm}^*[\theta'+0, \phi(0,\beta)] = \sum_{l'=0}^{\infty}\cancel{\sum_{m'=-l'}^{l'}} Y_{l'0}(\gamma,\beta) A_{l'0}[m] \quad (m'=0)\Big|_{\gamma=0}$$

$$Y_{lm}^*(\theta',\phi') = \left[Y_{lm}^*(\theta,\phi) = \cancel{\sum_{l'=0}^{\infty}}\sum_{m'=-l'}^{l'} Y_{l'm'}(\gamma,\beta) A_{lm'}[m] \quad (l'=l)\right]_{\gamma=0}$$

$$= \cancel{\sum_l}\sqrt{\tfrac{2l+1}{4\pi}}\int Y_{l0}^*(\gamma,\beta) Y_{lm}^*(\theta'+\gamma,\phi)\, d\Omega = \underline{\tfrac{2l+1}{4\pi}\int P_l(\cos\gamma) Y_{lm}^*(\theta,\phi)\, d\Omega}$$

$$\Rightarrow\ P_l(\cos\gamma) = \sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\left[A_{lm}(\theta',\phi') = \underline{\int P_l(\cos\gamma) Y_{lm}^*(\theta,\phi)\, d\Omega}\right] = \boxed{\tfrac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta',\phi')}$$

$$0 = \left[\tfrac{1}{\rho^{2-1}}\tfrac{d}{d\rho}\left(\rho^{2-1}\tfrac{d}{d\rho}\right) + k^2 - \tfrac{m^2}{\rho^2}\right] R(\rho) = \left[\tfrac{d^2}{d\rho^2} + \tfrac{2-1}{\rho}\tfrac{d}{d\rho} + k^2 - \tfrac{m^2}{\rho^2}\right] R(\rho)$$

$$= \left[\tfrac{d^2}{dx^2} + \tfrac{2-1}{x}\tfrac{d}{dx} + 1 - \tfrac{m^2}{x^2}\right] R(x = k\rho) \qquad (a = \text{radius of cylinder})$$

Bessel/Neumann Function ($m \in \mathbb{R}$)

$$J_m(x) = \left(\tfrac{x}{2}\right)^m \sum_{j-0}^{\infty}\left(\tfrac{x}{2}\right)^{2j}\tfrac{(-1)^j}{j!}\tfrac{1}{\Gamma(j+m+1)}$$

$$N_m(x) = \tfrac{\cos m\pi \cdot J_m(x) - J_{-m}}{\sin m\pi} \qquad \big(N(0) \to \infty\big)$$

$$R_m(\rho) = \sum_{n=1}^{\infty} A_n J_m\left(\tfrac{x_{mn}}{a}\rho = k_{mn}\rho\right) \qquad \big(J_m(x_{mn}) = 0\big)$$

$$= \sum_{n=1}^{\infty} B_n J_m\left(\tfrac{y_{mn}}{a}\rho = k_{mn}\rho\right) \qquad \big(J_m'(y_{mn}) = 0\big)$$

$$\delta_{n'n} = \tfrac{2}{a^2}\tfrac{1}{J_{m+1}^2(x_{mn}\frac{\rho}{a})}\int_0^a \rho\, J_m(x_{mn'}\tfrac{\rho}{a}) J_m(x_{mn}\tfrac{\rho}{a})\, d\rho$$

3rd Kind (Hankel)

$$H_m^{(1)}(x) = J_m + iN_m$$

$$H_m^{(2)}(x) = J_m - iN_m$$

Modified Bessel ($k^2 \to -k^2$)

$$I_m(x) = i^{-m} J_m(ix)$$

$$K_m(x) = \tfrac{\pi}{2} i^{m+1} H_m^{(1)}(ix)$$

Spherical Bessel: $j_l = \sqrt{\tfrac{2\pi}{z}}\, J_{l+1/2}$

$$\delta(k - k') = \tfrac{2k^2}{\pi}\int_0^{\infty} r^2 j_l(k'r) j_l(kr)\, dr$$

"Cylindrical Series" (?) $\quad 0 \le \rho \le a \qquad a \to \infty, \int dk$

$$\Psi(\vec{r}) = \sum_{m,n}\tfrac{1}{\sqrt{2\pi}} e^{im\phi} \times \left[A_{mn} e^{k_{mn}z} + B_{mn} e^{-k_{mn}z}\right] \times \left[C_{mn} J_m(k_{mn}\rho) + D_{mn} N_m(k_{mn}\rho)\right]$$

Laguerre Polynomials

$$L_{n(x)} = \frac{e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^n \right)$$

$$\delta_{n'n} = \int_0^\infty L_{n'(x)} L_{n(x)} e^{-x} \, dx$$

Associated Laguerre Polynomials

$$L^\alpha_{n(x)} = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^{n+\alpha} \right)$$

$$\delta_{n'n} = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty L^\alpha_{n'(x)} L^\alpha_{n(x)} e^{-x} x^\alpha \, dx$$

2 Simple 1D Potentials

2.1 Infinite Square Well (1-D)

$$\boldsymbol{V(x)} = \begin{cases} \mathbf{0} & \boldsymbol{0 < x < a} \\ \infty & \text{otherwise} \end{cases}$$

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin k_n x$$

$$k_n = \frac{2\pi}{\lambda} = \frac{2\pi}{2a/n} = \frac{n\pi}{a} \qquad \forall n = 1, 2, 3, ... \qquad \boxed{!!\ \hat{p}\Psi_n \neq p\Psi_n\ !!}$$ wave isn't infinite

$$E_n = \frac{p^2}{2m} = \frac{\hbar^2 k_n^2}{2m}$$

2.1.1 3-D Rectangular Box

$$\Psi_{n_x n_y n_z}(x, y, z) = \Psi_{n_x}(x)\Psi_{n_y}(y)\Psi_{n_z}(z) = \sqrt{\frac{8}{a_x a_y a_z}} (\sin k_{n_x} x)(\sin k_{n_y} y)(\sin k_{n_z} z)$$

$$k_{n_i} = \frac{n_i \pi}{a_i} \qquad \forall n_x, n_y, n_z = 1, 2, 3, ...$$

$$E_{n_x n_y n_z} = \frac{\hbar^2}{2m}(k_{n_x}^2 + k_{n_y}^2 + k_{n_z}^2)$$

2.2 Harmonic Oscillator (1-D): $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$

$$\begin{aligned}\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 &= \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right) \\ &= \frac{1}{2m}\left(-ip + m\omega x\right)\left(ip + m\omega x\right) \sim E \sim \hbar\omega\end{aligned} \quad \Rightarrow \quad \boxed{a = a_- = \frac{1}{\sqrt{2m}}\frac{1}{\sqrt{\hbar\omega}}\left(i\hat{p} + m\omega x\right)}$$

$$\boxed{\begin{aligned} aa^\dagger &= \frac{H}{\hbar\omega} + \frac{1}{2} \\ aa^\dagger|n\rangle &= (\tfrac{E_n}{\hbar\omega} + \tfrac{1}{2})|n\rangle \end{aligned}}\ , \quad \boxed{\begin{aligned} a^\dagger a &= \frac{H}{\hbar\omega} - \frac{1}{2} \\ a^\dagger a|n\rangle &= (\tfrac{E_n}{\hbar\omega} - \tfrac{1}{2})|n\rangle \end{aligned}} \quad \rightarrow \quad \boxed{\left[a, a^\dagger\right] = 1} \quad \leftarrow \quad \boxed{\begin{aligned} H &= \hbar\omega(a^\dagger a + \tfrac{1}{2}) \\ &= \hbar\omega(aa^\dagger - \tfrac{1}{2}) \end{aligned}}$$

or $\left[H, a_\pm\right] = (\pm\hbar\omega)a_\pm$

$$\begin{aligned} (aa^\dagger)a\Psi_n &= a(a^\dagger a)\Psi_n \\ \left(\tfrac{H}{\hbar\omega} + \tfrac{1}{2}\right)a\Psi_n &= a\left(\tfrac{H}{\hbar\omega} - \tfrac{1}{2}\right)\Psi_n \\ \left(\tfrac{E_{an}}{\hbar\omega} + \tfrac{1}{2}\right)a\Psi_n &= \left(\tfrac{E_n}{\hbar\omega} - \tfrac{1}{2}\right)a\Psi_n \\ E_{an}(a\Psi_n) &= (E_n - \hbar\omega)(a\Psi_n) \\ &\Downarrow \\ E_{n-1}|n-1\rangle &= (E_n - \hbar\omega)|n-1\rangle \end{aligned}$$

$$\begin{aligned} (a^\dagger a)a^\dagger|n\rangle &= a^\dagger(aa^\dagger)|n\rangle \\ \left(\tfrac{H}{\hbar\omega} - \tfrac{1}{2}\right)a^\dagger|n\rangle &= a^\dagger\left(\tfrac{H}{\hbar\omega} + \tfrac{1}{2}\right)|n\rangle \\ \left(\tfrac{E_{a^\dagger n}}{\hbar\omega} - \tfrac{1}{2}\right)\cancel{c_n}|a^\dagger n\rangle &= \left(\tfrac{E_n}{\hbar\omega} + \tfrac{1}{2}\right)\cancel{c_n}|a^\dagger(n)\rangle \\ E_{a^\dagger n}|a^\dagger n\rangle &= (E_n + \hbar\omega)|a^\dagger(n)\rangle \\ &\Downarrow \\ E_{n+1}|n+1\rangle &= (E_n + \hbar\omega)|n+1\rangle \end{aligned}$$

(Why ladders):

(use induction)

$$\boxed{\left[\tfrac{H}{\hbar\omega}, a_\pm\right] = (\pm1)a_\pm} \ \Rightarrow \ \left[\tfrac{H}{\hbar\omega}, a_\pm^m\right] = (\pm1)ma_\pm$$

$$\boxed{\tfrac{H}{\hbar\omega}|n\rangle = \tfrac{E_n}{\hbar\omega}|n\rangle = c_n|n\rangle} \ \Rightarrow \ \tfrac{H}{\hbar\omega}a_\pm^m|n\rangle = (\pm1\cdot m + c_n)|n\rangle$$

$$\begin{aligned} Ha_\pm^m|n\rangle &= \hbar\omega(c_n \pm m)a_\pm^m|n\rangle \\ * \ Ha_\pm^n|0\rangle &= \hbar\omega(c_0 \pm n)a_\pm^n|0\rangle \ * \\ &= E_{n0}\cdot a_\pm^n|0\rangle \end{aligned}$$

$E_n \geq \mathrm{Min}(V) \ \Rightarrow \ a\Psi_0 = 0$ (else is un-normalizable)

$$\begin{aligned} 0 &= (ip + m\omega x)\Psi_0 \\ \hbar\tfrac{d}{dx}\Psi_0 &= -m\omega x\Psi_0 \end{aligned} \qquad \boxed{\begin{aligned} \Psi_0 &= Ae^{-\frac{m\omega}{\hbar}\frac{x^2}{2}} \\ A &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\ \tfrac{1}{\sigma^2} &= \tfrac{m\omega}{\hbar} \end{aligned}}$$

$$\begin{aligned} a^\dagger a|0\rangle &= \left(\tfrac{E_0}{\hbar\omega} - \tfrac{1}{2}\right)|0\rangle = 0 \\ E_0|0\rangle &= \tfrac{1}{2}\hbar\omega|0\rangle \qquad (c_0 = \tfrac{1}{2}) \end{aligned}$$

$$aa^\dagger|a_+^n(0)\rangle = \left(\tfrac{\hbar\omega(n+1/2)}{\hbar\omega} + \tfrac{1}{2}\right)|a_+^n(0)\rangle$$

$$\boxed{aa^\dagger|a_+^n(0)\rangle = (n+1)|a_+^n(0)\rangle}$$

- $\langle a_+^n(0)|aa^\dagger|a_+^n(0)\rangle = n+1$
- $a^\dagger|a_+^n(0)\rangle = \sqrt{n+1}\ |a_+^{n+1}(0)\rangle$
- $a|a_+^{n+1}(0)\rangle = \sqrt{n+1}|a_+^n(0)\rangle$ * *

$$a^\dagger a|n,0\rangle = \left(\tfrac{\hbar\omega(n+1/2)}{\hbar\omega} - \tfrac{1}{2}\right)|n,0\rangle$$

$$\boxed{a^\dagger a|a_+^n(0)\rangle = n|a_+^n(0)\rangle}$$

$$\begin{aligned} \tfrac{H}{\hbar\omega}|a_+^n(0)\rangle &= \underline{(\tfrac{1}{2} + n)}|a_+^n(0)\rangle = \underline{c \circ a_+^n(0)}|a_+^n(0)\rangle \\ \tfrac{H}{\hbar\omega}a_\pm^m|a_+^n(0)\rangle &= \underline{(\tfrac{1}{2} + n \pm m)}a_\pm^m|a_+^n(0)\rangle \\ \tfrac{H}{\hbar\omega}|a_+^{n\pm m}(0)\rangle &= \underline{(c \circ a_+^{n\pm m}(0))}|a_+^{n\pm m}(0)\rangle \end{aligned} \ \rightarrow \ \begin{aligned} c_n &= \tfrac{1}{2} + n \\ a_+^n(0) &= n \\ a_\pm(n) &= n \pm 1 \end{aligned} \ \rightarrow \ \begin{aligned} &\boxed{E_n = \hbar\omega(\tfrac{1}{2} + n)} \\ &\boxed{\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle \\ a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \end{aligned}} \\ &\boxed{\Psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\Psi_0} \end{aligned}$$

2.2.1 Position/Momentum Operators

$$\boxed{x = \frac{1}{2}\frac{\sqrt{2m}\sqrt{\hbar\omega}}{m\omega}(a + a^\dagger)} \qquad \boxed{\hat{p} = \frac{1}{2}\frac{\sqrt{2m}\sqrt{\hbar\omega}}{i}(a - a^\dagger)}$$

<u>Show Virial Theorem Works</u>

$$\boldsymbol{2\langle T\rangle = N\langle V\rangle}$$

$$\begin{aligned}
E_n &= 2\langle V\rangle_n \\
&= 2\langle \Psi_n|V|\Psi_n\rangle \\
&= 2\left\langle \Psi_n \left| \tfrac{1}{2}mw^2 \tfrac{2m\hbar\omega}{(2m\omega)^2}(a+a^\dagger)^2 \right| \Psi_n \right\rangle \\
&= \tfrac{2m^2\hbar\omega^3}{(2m\omega)^2}\left(0 + \left\langle \Psi_n \left| (aa^\dagger + a^\dagger a) \right| \Psi_n \right\rangle + 0\right)
\end{aligned}$$

$$E_n = (n + 1/2)\hbar\omega \quad \boxtimes$$

<u>Heisenberg Picture</u>

$$\frac{da_\pm}{dt} = \mp i\omega a_\pm$$

$$\Rightarrow \qquad a_\pm(t) = a_\pm(0)e^{\mp i\omega t}$$

$$x(t) \pm \frac{ip(t)}{m\omega} = x(0)e^{\mp i\omega t} \pm \frac{ip(0)}{m\omega}e^{\mp i\omega t}$$

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$$

$$\frac{p(t)}{m\omega} = -x(0)\sin\omega t + \frac{p(0)}{m\omega}\cos\omega t$$

<u>Test the Uncertainty Principle</u>

$$\sigma_x\sigma_p \geq \tfrac{1}{2}\left|\left\langle [x,p] \right\rangle\right|$$

$$xp - px = \tfrac{2m\hbar\omega}{4m\omega i}\begin{pmatrix} a^2 - aa^\dagger + a^\dagger a - (a^\dagger)^2 \\ -a^2 + a^\dagger a - aa^\dagger + (a^\dagger)^2 \end{pmatrix}$$

$$= \tfrac{\hbar}{i}(a^\dagger a - aa^\dagger) = i\hbar(n + 1 - n)$$

$$\Rightarrow \quad \sigma_x\sigma_p \geq \frac{\hbar}{2} \qquad \boxtimes$$

$$\begin{aligned}
\sigma_x^2 &= \langle x^2\rangle - \langle x\rangle^2 & \sigma_p^2 &= \langle p^2\rangle - \langle p\rangle^2 \\
&= \tfrac{2m\hbar\omega}{4m^2\omega^2}\begin{bmatrix} \langle (a+a^\dagger)^2\rangle \\ -\langle a + a^\dagger\rangle^2 \end{bmatrix} & &= \tfrac{2m\hbar\omega}{-4}\begin{bmatrix} \langle (a-a^\dagger)^2\rangle \\ -\langle a - a^\dagger\rangle^2 \end{bmatrix} \\
&= \tfrac{\hbar}{2m\omega}\langle aa^\dagger + a^\dagger a\rangle & &= \tfrac{\hbar m\omega}{2}\langle aa^\dagger + a^\dagger a\rangle \\
&= \tfrac{\hbar}{m\omega}(n + \tfrac{1}{2}) & &= \hbar m\omega(n + \tfrac{1}{2})
\end{aligned}$$

$$\Rightarrow \quad \sigma_x\sigma_p = \hbar(n + \tfrac{1}{2}) \geq \tfrac{\hbar}{2} \qquad \boxtimes$$

2.2.2 Analytic Method

$$\boldsymbol{\Psi_n = A\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}}$$

$$A = \left(\tfrac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\xi = \sqrt{\tfrac{m\omega}{\hbar}}x$$

Hermite Polynomials:

$$H_n(x) = (-1)^n\ e^{-x^2}\left(\frac{d}{dx}\right)^n e^{x^2}$$

$$e^{-z^2+2zx} = \sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x)$$

2.2.3 Coherent States

$$\boldsymbol{a|\alpha\rangle = \alpha|\alpha\rangle}$$

$$\boxed{\sigma_x\sigma_p = \frac{\hbar}{2}}$$

$$\langle\alpha|\alpha\rangle = \langle\alpha|\left(\begin{array}{l}\displaystyle\sum_{n=0}^{\infty}\langle\Psi_n|\alpha\rangle\ |\Psi_n\rangle = \\ \displaystyle\sum_{n=0}^{\infty}\left\langle\frac{(a^\dagger)^n}{\sqrt{n!}}\Psi_0\middle|\,\alpha\right\rangle\ |\Psi_n\rangle = \\ \displaystyle\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle\Psi_0|\alpha\rangle\ |\Psi_n\rangle\end{array}\right)$$

$$= \langle\Psi_0|\alpha\rangle^2\sum_{n=0}^{\infty}\frac{(\alpha^2)^n}{n!}\ \langle\Psi_n|\Psi_n\rangle$$

$$= \langle\Psi_0|\alpha\rangle^2\ e^{\alpha^2} = 1$$

$$\Rightarrow \boxed{\big|\alpha\big\rangle = e^{-\alpha^2/2}\ \sum_{n=0}^{\infty}\tfrac{\alpha^n}{\sqrt{n!}}\ |\Psi_n\rangle} \ \rightarrow\ \big|\alpha{=}0\big\rangle = |\Psi_0\rangle$$

$$a|\alpha(x,t)\rangle = e^{-\frac{\alpha^2}{2}}\sum_{n=0}^{\infty}\tfrac{\alpha^n}{\sqrt{n!}}\ e^{\frac{-i}{\hbar}E_n t}a|\Psi_n\rangle$$

$$= e^{-\frac{\alpha^2}{2}}\sum_{n=0}^{\infty}\tfrac{\alpha^n}{\sqrt{n!}}\ e^{\frac{-i}{\hbar}\hbar\omega(\frac{1}{2}+n)t}\sqrt{n}|\Psi_{n-1}\rangle$$

$$= \left(\alpha e^{\frac{-i}{\hbar}\hbar\omega t}\right)e^{-\frac{\alpha^2}{2}}\sum_{n=0}^{\infty}\tfrac{\alpha^n}{\sqrt{n!}}\ e^{\frac{-i}{\hbar}\hbar\omega(\frac{1}{2}+n)t}|\Psi_n\rangle$$

$$\boxed{a|\alpha(x,t)\rangle = \left(\alpha e^{-i\omega t}\right)|\alpha(x,t)\rangle}$$

$|\alpha\rangle$ are obviously not orthogonal. They are an overcomplete basis.

2.2.4 3-D Harmonic Potential

$$\boldsymbol{V(r) = \frac{1}{2}kr^2}$$

$$\boxed{\begin{array}{c}\text{(Isotropic)}\\ E_{n_x n_y n_z} = \hbar\omega\left(n_x + n_y + n_z + \tfrac{3}{2}\right) = \hbar\omega\left(n + \tfrac{3}{2}\right) \quad l = n - 2k \in \{n, n-2, ..., 0\}\end{array}}$$

2.3 Free Particle (1-D)

$$\boldsymbol{V(x) = 0}$$

$$\Psi_{(x,t)} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi_{(x,0)} e^{\frac{i}{\hbar}[px - E(p)t]}\, dp$$

$$= \int \langle x|p\rangle e^{-\frac{i}{\hbar}E(p)t} \langle p|\Psi\rangle\, dp$$

$$\Phi_{(x,t)} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi_{(x,0)} e^{\frac{-i}{\hbar}[px + E(p)t]}\, dx$$

$$\langle x|U_{(t)}|\Psi\rangle = \iint \langle x|p\rangle e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} \langle p|x'\rangle\, dp\, \langle x'|\Psi\rangle\, dx'$$

$$= \iint_{-\infty}^{\infty} \tfrac{1}{2\pi\hbar} e^{-\frac{i}{\hbar}\left[\frac{p^2 t}{2m} - p(x-x')\right]}\, dp\, \Psi_{(x',0)}\, dx'$$

$$= \int \sqrt{\tfrac{-im}{2\pi\hbar t}}\, e^{\frac{im(x-x')^2}{2\hbar t}}\, \Psi_{(x',0)}\, dx'$$

($E < 0 \to \Psi = e^{\pm kx}$ is possible and also not normalizable, but solution above is already a complete set)

- $E(p) = \frac{p^2}{2m}$
- $v_{\text{wave}} = \boxed{v_{\text{phase}} = \frac{\omega(k)}{k}} = \frac{E}{p} = \frac{v_{\text{classical}}}{2}$
- $v_{\text{particle}} \approx \boxed{v_{\text{group}} = \frac{d\omega(k)}{dk}} = 2v_{\text{wave}}$ (dispersion relation)

Heisenberg Pic. Free Particle

$$x_H(t) = x_H(0) + \tfrac{p_H(0)}{m}t$$

$$\left[x_H(0),\ x_H(t)\right] = \left[x_H(0), \tfrac{p_H(0)}{m}t\right] = \tfrac{i\hbar t}{m}$$

$$\boxed{\sigma_{x_t}\sigma_{x_0} \geq \tfrac{\hbar t}{2m}}$$

2.4 Delta Potential (1-D)

Potential Well: $\boldsymbol{V(x) = -\alpha\delta(x)}$ ($\alpha \to -\alpha$ for potential wall)

Bound State ($E < 0$) [only for Well]:

$$\Psi = \sqrt{k}e^{k|x|} = \begin{cases} \sqrt{k}e^{kx} & x \leq 0 \\ \sqrt{k}e^{-kx} & x \geq 0 \end{cases}$$

$$k = \frac{m\alpha}{\hbar^2}$$

$$E = -\frac{(\hbar k)^2}{2m}$$

Scattering State ($E > 0$) [for both]:

$$\Psi = \begin{cases} Ae^{iKx} + Be^{-iKx} & x < 0 \\ Fe^{iKx} & x > 0 \end{cases}$$

$$E = \frac{(\hbar K)^2}{2m}\ , \qquad \beta \equiv \frac{k}{K} = \frac{m\alpha/\hbar^2}{K}$$

$$B = \frac{i\beta}{1 - i\beta}A\ , \qquad F = \frac{1}{1 - i\beta}A$$

$$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}\ , \qquad T = \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}$$

Can't normalize. All free particles have ranges of p and thus E, so R and T are approx. in the vicinity of E.

2.5 Finite Square Potential (1-D)

2.5.1 Potential Well

$$V(x) = \begin{cases} -V_0 & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

($V_0 \to -V_0$ for wall and do cases for $E > V_0, E = V_0, E < V_0$, and change to $\sinh, \cosh$ if needed)

$$k\ ;K: \quad E = \frac{-(\hbar k)^2}{2m} = \frac{(\hbar K)^2}{2m}$$

$$l: \quad E + V_0 = \frac{(\hbar l)^2}{2m}$$

$$v: \quad V_0 = \frac{\hbar^2 v^2}{2m} = \frac{\hbar^2(l^2+k^2)}{2m} = \frac{\hbar^2(l^2-K^2)}{2m}$$

$$\frac{k_a}{l_a} \equiv \sqrt{\frac{(ka)^2}{(la)^2}} = \sqrt{\frac{(la)^2+(ka)^2}{(la)^2} - 1}$$

$$\boxed{\frac{k_a}{l_a} \equiv \sqrt{\left(\frac{v_a}{l_a}\right)^2 - 1}}\ ,\ v_a^2 = \begin{cases} l_a^2 + k_a^2 \\ l_a^2 - K_a^2 \end{cases}$$

<u>Bound State $(E_n < 0)$ [only for well]:</u>

$$\Psi_{\text{even}}(x) = \begin{cases} \Psi(-x) & x < 0 \\ D\cos(lx) & 0 < x < a \\ Fe^{-kx} & a < x \end{cases}$$

- $F = D\cos(la)e^{ka}$
- $\frac{-(\partial_x\Psi)(a)}{\Psi(a)} = k = l\tan(la) \ \Rightarrow$

  $\tan(l_a) = \sqrt{(v_a/l_a)^2 - 1}$

  big $v_a \to l \approx_< \frac{n\pi}{2a} \to E_n + V_0 = \frac{\hbar^2 l^2}{2m}$; <u>n odd</u>
- $\boxed{n_{\max} = \left\lfloor \frac{v_a}{\pi} \right\rfloor + 1}$

$$\Psi_{\text{odd}}(x) = \begin{cases} -\Psi(-x) & x < 0 \\ C\sin(lx) & 0 < x < a \\ Fe^{-kx} & a < x \end{cases}$$

- $F = D\sin(la)e^{ka}$
- $\frac{-(\partial_x\Psi)(a)}{\Psi(a)} = k = -l\cot(la) \ \Rightarrow$

  $-\cot(l_a) = \sqrt{(v_a/l_a)^2 - 1}$

  big $v_a \to l \approx_< \frac{n\pi}{2a} \to E_n + V_0 = \frac{\hbar^2 l^2}{2m}$; <u>n even</u>
- $\boxed{n_{\max} = \left\lfloor \frac{v_a + \frac{\pi}{2}}{\pi} \right\rfloor}$

<u>Scattering State $(E > 0)$ [for both]:</u>

$$\Psi = \begin{cases} Ae^{iKx} + Be^{-iKx} & x < -a \\ C\sin lx + D\cos lx & -a < x < a \\ Fe^{iKx} & a < x \end{cases}$$

$$\frac{d\Psi}{dx} = \begin{cases} iKAe^{iKx} - iKBe^{-iKx} & x < -a \\ lC\cos lx - lD\sin lx & -a < x < a \\ iKFe^{iKx} & a < x \end{cases}$$

$$B = i\sin(2l_a)\left(\frac{l_a^2 - K_a^2}{2K_a l_a}\right) F$$

$$F = \frac{e^{-2iK_a}}{\cos(2l_a) - i\left(\frac{l_a^2+K_a^2}{2K_a l_a}\right)\sin(2l_a)} A$$

(Can't normalize. See delta potential.)

$$T^{-1} = 1 + \left(\frac{l_a^2 - K_a^2}{2K_a l_a}\right)^2 \sin^2(2l_a)$$

$$= 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(2a\sqrt{\frac{E+V_0}{\hbar^2/2m}}\right)$$

(full transmission at inf. sqr. well $E_n + V_0 = \frac{\hbar^2 l^2}{2m}$; $l = \frac{n\pi}{2a}$)

3 2D and 3D Schrodinger Equation

General dimensions, D

$$\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right)R(r) = \left[\frac{\partial^2}{\partial r^2}+\frac{D-1}{r}\frac{\partial}{\partial r}\right]R(r)$$

$$= \left[\frac{\partial^2}{\partial r^2}+\frac{D-1}{r}\frac{\partial}{\partial r}\right]r^n u(r)$$

$$= \left[\frac{\partial^2}{\partial r^2}+\frac{D-1+2n}{r}\frac{\partial}{\partial r}+\frac{2n(2D-4+2n)}{4r^2}\right]u$$

$$= \left[\frac{\partial^2}{\partial r^2}-\frac{(D-1)(D-3)}{4r^2}\right]u \qquad (n=\tfrac{1-D}{2},\cancel{0,2-D})$$

$$R(r) = u(r)/\sqrt{r}^{D-1} \;\sim\; e^{\frac{i}{\hbar}p_r r}/\sqrt{r}^{D-1}$$

$$L^2\sim\hbar^2 \;,\quad \hat{p}_r = \frac{\hbar}{i}\left(\frac{\partial}{\partial r}+\frac{D-1}{2r}\right) \;,\quad \hat{p'}_r = \frac{\hbar}{i}\frac{\partial}{\partial r}$$

$$ER(r) = \left[\frac{\hat{p}_r^2}{2M}+V(r)+\frac{L^2-\hbar^2(D-1)^2/4}{2(Mr^2)}\right]R(r)$$

$$Eu(r) = \left[\frac{\hat{p'}_r^2}{2M}+V(r)+\frac{L^2-\hbar^2(D-1)(D-3)/4}{2(Mr^2)}\right]u(r)$$

3.1 2D Schrodinger

If $V=V(\rho)$

$\Psi(\vec{\mathbf{r}}) = R_m(\rho)\Phi_m(\phi) \;\Rightarrow$

$$ER = \left[\frac{-\hbar^2}{2M}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+V(\rho)+\frac{\hbar^2 m^2}{2M\rho^2}\right]R$$

$$\boxed{Eu = \frac{-\hbar^2}{2M}\frac{\partial^2 u}{\partial\rho^2}+\left[V(\rho)+\frac{\hbar^2(m^2+1/4)}{2M\rho^2}\right]u}$$

$$Eu\Phi = \left[\frac{\hat{p'}_\rho^2}{2M}+V(\rho)+\frac{\hat{L}_z^2+\hbar^2/4}{2(M\rho^2)}\right]u\Phi$$

- $R_m(\rho) = u_m(\rho)/\sqrt{\rho}$ $\qquad \left(\int \Psi r\,dr d\phi = 1\right)$
- $\Phi_m(\phi) = e^{im\phi}$
- $L_z = (\vec{r}\times\vec{p})_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$

3.2 3D Schrodinger

<u>If $V = V(r)$</u>

$$\Psi(\vec{\mathbf{r}}) = R_l(r)Y_l^m(\theta,\phi) = R_l(r)\Theta_l^m(\theta)\Phi_m(\phi) \;\Rightarrow$$

$$ER = \left[\frac{-\hbar^2}{2M}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r) + \frac{\hbar^2 l(l+1)}{2Mr^2}\right]R$$

$$\boxed{Eu = \frac{-\hbar^2}{2M}\frac{\partial^2 u}{\partial r^2} + \left[V(r) + \frac{\hbar^2 l(l+1)}{2Mr^2}\right]u}$$

$$Eu\Theta = \left[\frac{\hat{p'}_r^2}{2M} + V(r) + \frac{\hat{L}^2}{2(Mr^2)}\right]u\Theta$$

- $R_l(r) = u_l(r)/r \qquad \left(\int \Psi r^2 \sin\theta\, dr d\theta d\phi = 1\right)$
- $\Phi_m(\phi) = e^{im\phi}$
- $\Theta_l^m(\theta) = AP_l^m(\cos\theta)$
  - $A = \epsilon\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}, \quad \epsilon = \begin{cases}(-1)^m & (m\geq 0)\\ 1 & (m\leq 0)\end{cases}$
  - $P_l^m(x) =$ Assoc. Legendre Func. (see extra)
- $l \in \mathbb{N}_0,\; m \in \{-l, ..., -1, 0, 1, ..., l\}$
- $\widehat{L}_i = (\vec{r}\times\vec{p})_i$

$$\langle u|\nabla_r^2 u\rangle = \langle\nabla_r^2 u|u\rangle \;\Rightarrow\; \left[u^*\frac{\partial u}{\partial r} - u\frac{\partial u^*}{\partial r}\right]_{0\;\leftarrow 2.)}^{\infty\;\leftarrow 1.)} = 0$$

1.) $\int_0^\infty u^2\, dr = 1 \;\Rightarrow\; \boxed{u(\infty) = 0 \text{ or } e^{ir}}$

2.) $\boxed{u(0) = c = 0}$ $\left\{c \neq 0 \rightarrow \begin{matrix}\Psi_{l=0}(r) \sim \frac{c}{r}\\ \nabla^2(\frac{1}{r}) \sim \delta^3(r)\end{matrix} \rightarrow \begin{matrix} H\Psi \neq E\Psi \\ \text{if } V(r) \neq \delta^3(r)\end{matrix}\right\}$

$V \sim r^{-2<a},\ l \neq 0:\ \lim_{r\to 0} u'' \sim \frac{l(l+1)}{r^2}u,\ u \sim r^{l+1}$

$V \sim r^{-2<a<-1},\ E > 0:\ \lim_{r\to\infty} p_r^2 u \sim Eu,\ u \sim e^{\pm ikr}$

$V \sim r^{-1\leq a},\ E > 0:\ u \gtrsim re^{\pm ikr}$

Sim for $E < 0$: $u \sim e^{\pm kr}$ or $re^{\pm kr}$ etc.

3.2.1 3D Free Particle, $V = 0$

$$\frac{\hbar^2}{2M}\left[-\frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2}\right]u = \frac{\hbar^2 k^2}{2M}u \quad\Rightarrow\quad \left[-\frac{\partial^2}{\partial\rho^2} + \frac{l(l+1)}{\rho^2}\right]|l\rangle = |l\rangle = \begin{cases}A\rho^{l+1}\\ B\rho^{-l}\end{cases} \text{or(?) ...}$$

$$a_l \equiv \frac{\partial}{\partial\rho} + \frac{l+1}{\rho} \qquad a_l^\dagger = -\frac{\partial}{\partial\rho} + \frac{l+1}{\rho}$$

$$a_l a_l^\dagger|l\rangle = |l\rangle \qquad a_l^\dagger a_l|l\rangle = a_{l+1}a_{l+1}^\dagger|l\rangle$$

$$a_l^\dagger|l\rangle = e^{i\theta_l}|a_l^\dagger(l)\rangle$$

$$a_l^\dagger\left(a_l a_l^\dagger\right)|l\rangle = \underline{a_l^\dagger|l\rangle}$$

$$\left(a_l^\dagger a_l\right)\underline{a_l^\dagger|l\rangle} = \left(a_{l+1}a_{l+1}^\dagger\right)\underline{a_l^\dagger|l\rangle}$$

$$\underline{a_l^\dagger|l\rangle} = \cancel{e^{i\theta_l}}\,|l+1\rangle$$

$$\boxed{\begin{matrix}\text{Spherical Bessel}: & r\underline{R_0^B} = u_0^B \sim \sin(\rho) = \sin(kr)\\ \text{Spherical } \cancel{\text{Neumann}}: & r\underline{R_0^N} = u_0^N \sim -\cos(\rho)\end{matrix}}$$

$$\cancel{e^{i\theta_l}}\frac{\rho}{k}R_{l+1} = a_l^\dagger(\tfrac{\rho}{k}R_l) = \left(-\tfrac{\partial}{\partial\rho} + \tfrac{l+1}{\rho}\right)(\tfrac{\rho}{k}R_l)$$

$$R_{l+1} = \left(-\tfrac{\partial}{\partial\rho} + \tfrac{l}{\rho}\right)R_l = -\rho^l\tfrac{\partial}{\partial\rho}\left(\rho^{-l}R_l\right)$$

$$\tfrac{R_l}{\rho^l} = -\tfrac{1}{\rho}\tfrac{\partial}{\partial\rho}\tfrac{R_{l-1}}{\rho^{l-1}} = \left(-\tfrac{1}{\rho}\tfrac{\partial}{\partial\rho}\right)^l R_0$$

$$\boxed{R_l = C_l(-\rho)^l\left(\tfrac{1}{\rho}\tfrac{\partial}{\partial\rho}\right)^l R_0}$$

<u>Infinite Spherical Well</u>: $V(r) = \begin{cases}0 & r\leq a\\ \infty & r > a\end{cases}$, $E_n = \frac{\hbar^2 k_n^2}{2m}$

<u>Bessel</u>: $R_l^B(\rho) = C_l(-\rho)^l\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^l R_0^B(\rho)$, $R_0^B(\rho) \sim k_n\sin(\rho)/\rho = \sin(k_n r)/r$ $\;\Rightarrow\;$ $\beta_l^n \equiv k_n a:\ R_l^B(\beta_l^n) = 0$, $\beta_0^n = \frac{n\pi}{a}\cdot a$

3.2.2 Hydrogen Atom, $V = -\frac{ke^2}{r}$

$$Eu = \left(\frac{\hat{p}_r^2}{2m} + V(r) + \frac{\hat{L}^2}{2(mr^2)} \right) u \qquad u(r) = rR(r)$$

$$\boxed{Eu = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial r^2}u + \left[-\frac{ke^2}{r} + \frac{\hbar^2 l(l+1)}{2mr^2} \right] u}$$

$$\boxed{\Psi_{nlm}(\vec{\mathbf{r}}) = R_{nl}(r)\ Y_l^m(\theta,\phi) = R_{nl}(r)\ \Theta_l^m(\theta)\ \Phi_m(\phi)}$$

- $\Phi_m(\phi) = e^{im\phi}$
- $\Theta_l^m(\theta) = AP_l^m(\cos\theta)$
  - $A = \epsilon\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}$, $\quad \epsilon = \begin{cases} (-1)^m & (m \geq 0) \\ 1 & (m \leq 0) \end{cases}$
  - $P_l^m(x)$ Assoc. Legendre Func. (see extra)
- $R_{nl}(r) = \frac{B}{r}\rho^{l+1}e^{-\rho}\nu(\rho)$
  - $\rho = k_n r$, $\quad k_n = \frac{1}{a_0 n}$ (fine structure below)
  - $\nu(\rho) = L_{n-l-1}^{2l+1}(2\rho)$ Assoc. Laguerre Pol. (see extra)
  - $B = \sqrt{2k_n \frac{(n-l-1)!}{2n[(n+l)!]^3}}\ 2^{l+1}$

$$\boxed{\alpha \equiv \frac{kqq}{\hbar c} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137}} \qquad \boxed{a_0 \equiv \frac{\hbar^2}{m(kqq)} = \frac{4\pi\epsilon_0\hbar^2}{me^2}}$$

$$\boxed{E_n = -\frac{\hbar^2 k_n^2}{2m} = -\frac{\hbar^2}{2ma_0^2}\frac{1}{n^2} = -\frac{1}{2}\alpha^2\left(mc^2\right)\frac{1}{n^2} \approx -13.6\ \frac{1}{n^2}\ [\text{eV}]}$$

$$\boxed{\frac{1}{\lambda} = \frac{\alpha^2\left(mc^2\right)}{2hc}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = R\left(\tfrac{1}{n_f^2} - \tfrac{1}{n_i^2}\right)\ , \quad R = 1.097\ \text{E7}\ [\text{m}^{-1}]}$$

Quantum Numbers - n, l, m:

- $\Big(n \in \{1,2,3,...\}\Big)$, $\Big(l \in \{0,1,2,...,n-1\}\Big)$, $\Big(m \in \{-l,...,-1,0,1,...,l\}\Big)$
  - Degeneracy is n^2

(outdated) Bohr Model:

- $L = (\bar{r})(\bar{p}) = (a_0 n^2)(\hbar k_n) = n\hbar$ (not correct!!)
- Electrons don't radiate about the nucleus
- Energy diff. follows Rydberg formula

4 Spin and L

4.1 Hydrogen Atom

Angular Momentum :

$\boxed{\widehat{L}_i \equiv (\vec{r}\times\vec{p})_i}$, $\boxed{L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}}$

$\boxed{\widehat{L}_\pm \equiv \widehat{L}_x \pm i\widehat{L}_y}$

$\boxed{\widehat{L}^2 \equiv L_x^2 + L_y^2 + L_z^2}$

$\boxed{L_\pm L_\mp = \widehat{L}^2 - L_z^2 \pm \hbar L_z}$

Commutation Relations:

$\boxed{\left[\hat{x}, L_y\right] = i\hbar\hat{z}}$, $\boxed{\left[p_x, L_y\right] = i\hbar p_z}$, $\boxed{\left[L_x, L_y\right] = i\hbar L_z}$

$\boxed{\left[L^2, L_i\right] = \left[H, L_i\right] = \left[H, L^2\right] = 0}$ (can measure concurrently)

$$\rightarrow \begin{array}{l}\boxed{L_z Y_{m'} = \hbar m' Y_{m'}} \\ \boxed{L^2 Y_{m'} = \hbar^2 \lambda_{m'} Y_{m'}}\end{array} \Rightarrow \begin{array}{l}\langle L^2 - L_z^2\rangle = \langle L_x^2 + L_y^2\rangle \geq 0 \\ \bullet\ \sqrt{\lambda_{m'}} \geq\ m' \geq -\sqrt{\lambda_{m'}}\end{array}$$

Let $(L_\pm)^n Y_\mu \equiv |m\rangle$ (see harm. osc. for why ladders)

$$\boxed{\left[\tfrac{L_z}{\hbar}, L_\pm\right] = (\pm 1)L_\pm} \Rightarrow \left[\tfrac{L_z}{\hbar}, (L_\pm)^n\right] = \pm n (L_\pm)^n \Rightarrow \begin{array}{c} L_z\left[(L_\pm)^n Y_\mu\right] = (\mu \pm n)\hbar\left[(L_\pm)^n Y_\mu\right] \\ \bullet\ L_z|m\rangle = (\mu\pm n)\hbar|m\rangle\end{array}$$

$$\boxed{\left[L^2, L_\pm\right] = 0} \Rightarrow \left[L^2, (L_\pm)^n\right] = 0 \Rightarrow \begin{array}{c} L^2\left[(L_\pm)^n Y_\mu\right] = \lambda_\mu\left[(L_\pm)^n Y_\mu\right] \\ \bullet\ L^2|m\rangle = \lambda_\mu|m\rangle\end{array}$$

Then $\left(\sqrt{\lambda_\mu}\ \geq\ (\mu\pm n)\ \geq\ -\sqrt{\lambda_\mu}\ \right) \Rightarrow$ **Let** (else un-normalizable solution)

$\underline{L_+|m_t\rangle = 0}$, $L_z|m_t\rangle = \hbar l$,	$\underline{L_-|m_b\rangle = 0}$, $L_z|m_b\rangle = \hbar l'$,
$L^2|m_t\rangle = \lambda\hbar^2$, $L^2 = L_- L_+ + L_z^2 + \hbar L_z$	$L^2|m_b\rangle = \lambda\hbar^2$, $L^2 = L_+ L_- + L_z^2 - \hbar L_z$
$\bullet\ L^2|m_t\rangle = \hbar^2 l(l+1)|m_t\rangle = \lambda\hbar^2|m_t\rangle$	$\bullet\ L^2|m_b\rangle = \hbar^2 l'(l'-1)|m_b\rangle = \lambda\hbar^2|m_b\rangle$

$$\left[\lambda = l'(l'-1) = l(l+1)\right] \Rightarrow \left[l' = -l\right] \Rightarrow \begin{bmatrix} L_z|m_t\rangle = \hbar l\,|m_t\rangle \\ L_z|m_b\rangle = -\hbar l\,|m_b\rangle \end{bmatrix}$$ (Spherical Harmonics do not allow half-integer l)

Schrodinger Y_l^m:

$$\boxed{\begin{array}{c|c} \begin{array}{c} l \in \{0,1,2,...\} \\ m \in \{-l, -l+1, ..., l-1, l\}\end{array} & \begin{array}{c} L_z\left|Y_l^m\right\rangle = \hbar m\left|Y_l^m\right\rangle = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\left|Y_l^m\right\rangle \\ L^2\left|Y_l^m\right\rangle = \hbar^2 l(l+1)\left|Y_l^m\right\rangle \\ L_\pm\left|Y_l^m\right\rangle = \hbar\sqrt{l(l+1)-m(m\pm1)}\ \left|Y_l^m\right\rangle \end{array}\end{array}}$$

4.2 Generalized

Angular Momentum:

$\widehat{J}_i \equiv ???$ $\quad \boxed{J^2 \equiv J_x^2 + J_y^2 + J_z^2}$

$\boxed{J_\pm \equiv J_x \pm iJ_y}$ $\quad \boxed{J_\pm J_\mp = J^2 - J_z^2 \pm \hbar J_z}$

Commutation Relations:

$\boxed{\big[J_i, J_j\big] = i\hbar J_k \ \epsilon_{ij}} \Leftrightarrow \boxed{J \times J = i\hbar J}$

$\boxed{\big[J^2, J_z\big] = 0} = \big[H, J_z\big]$ (if spher. symm.)

General:
$$\boxed{\begin{array}{c|l} & J_z\big|jm\big\rangle = \hbar m\big|jm\big\rangle \\ j \in \big\{0, \frac{1}{2}, 1, \frac{3}{2}, ...\big\} & J^2\big|jm\big\rangle = \hbar^2 j(j+1)\big|jm\big\rangle \\ m \in \{-j, -j+1, ..., j-1, j\} & J_\pm\big|jm\big\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\ \big|jm\big\rangle \\ & J_x\big|jm\big\rangle = \frac{J_+ + J_-}{2}\big|jm\big\rangle \end{array}}$$

Generator of Rotations:

$$U[R(\theta)] = e^{-\frac{i}{\hbar}\theta\hat{\theta}\cdot L} = \lim_{N\to\infty}\Big[\mathbb{1} - \tfrac{i}{\hbar}\tfrac{\theta}{N}\hat{\theta}\cdot L\Big]^N \ \Leftrightarrow\ \boxed{U[R(\epsilon_z\,\hat{z})] = \mathbb{1} - \tfrac{i}{\hbar}\epsilon_z L_z}$$

1.) $U[R(\epsilon_z\hat{z})]|x,y\rangle \equiv |x - \epsilon_z y,\ \epsilon_z x + y\rangle$

$\Rightarrow \ \langle x,y|U[R(\epsilon_z\hat{z})]|\Psi\rangle = \Psi(x + \epsilon_z y,\ \epsilon_z x - y)$

$\Rightarrow \ \underline{\langle x,y|L_z|\Psi\rangle = (XP_y - YP_x)\Psi(x,y)}$

or 2.) $U^\dagger XU \equiv X - \epsilon_z Y \ \Rightarrow\ [X, L_z] = -i\hbar Y$

$U^\dagger P_y U \equiv \epsilon_z P_x + P_y \ \Rightarrow\ [P_y, L_z] = i\hbar P_x$

$U^\dagger YU,\ U^\dagger P_x U, \ \Rightarrow\ \dots$

$\Rightarrow\ \underline{L_z = XP_y - YP_x}$

Consistency Check: $U\big[R(-\epsilon_z\hat{z})\big]T(-\epsilon_x\hat{x} - \epsilon_y\hat{y})U\big[R(\epsilon_z\hat{z})\big]T(\epsilon_x\hat{x} + \epsilon_y\hat{y}) = T(-\epsilon_y\epsilon_z\hat{x} + \epsilon_x\epsilon_z\hat{y})$

$?U\big[R(-\epsilon_y\hat{y})\big]U\big[R(-\epsilon_x\hat{x})\big]U\big[R(\epsilon_y\hat{y})\big]U\big[R(\epsilon_x\hat{x})\big] = \mathbb{1} + \frac{i}{\hbar}\epsilon_x\epsilon_y L_z = U\big[R(-\epsilon_x\epsilon_y\hat{z})\big]$

$(L_z = xp_y - yp_x)$

Tensors and Tensor Operators

rank-2 Tensor:

$$|t^{(2)}\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3} t_{ij}\,|i\rangle|j\rangle$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3} |ij\rangle\langle ij|t^{(2)}\rangle$$

rank-2 Cartesian Tens. Oper., T_{ij} :

Set of $3^{n=2}$ Operators

rank-k Spherical Tensor Operator, T_k^q :

Set of $2k+1$ Operators s.t.

$$U[R]T_k^q U^\dagger[R] = \sum_{q'=-k}^{k} D_{q'q}^k T_k^{q'}$$
$$\downarrow$$
$$UT_k^q U^\dagger U|jm\rangle = \sum_{q'}\sum_{m'} D_{q'q}^k D_{m'm}^j T_{q'}^k|jm'\rangle$$
$$\sim\ U|kq\rangle|jm\rangle = \sum_{q'}\sum_{m'} D_{q'q}^k D_{m'm}^j|kq'\rangle|jm'\rangle$$

- $T_1^{\pm1} = \mp\frac{V_x \pm iV_y}{\sqrt{2}}$

  $T_1^0 = V_z$

Wigner-Eckhart: $\langle\alpha_2 j_2 m_2|T_k^q|\alpha_1 j_1 m_1\rangle = \langle\alpha_2 j_2|T_k|\alpha_1 j_1\rangle \cdot \langle j_2 m_2|kq; j_1 m_1\rangle$

(CG coeff.)

4.3 1 Particle w/ Spin, $s = \frac{1}{2}$

*Find the Eigenvectors, e_i , of S_z and S^2 in the form of $|\chi\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{i\phi_1} \\ \sin\frac{\theta}{2}\, e^{i\phi_2} \end{pmatrix} = e^{i\gamma}\begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}$ $\quad \gamma = \frac{\phi_1+\phi_2}{2}$, $\phi = \phi_2 - \phi_1$

$$* \boxed{ e_i \in \left\{ \ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle \equiv |\uparrow\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix} = \tfrac{1}{2}(1+\sigma_z) \quad , \quad |\tfrac{1}{2}\ \tfrac{-1}{2}\rangle \equiv |\downarrow\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix} = \tfrac{\sigma_x}{2}(1+\sigma_z) \ \right\} }$$

$$\left.\begin{aligned} S^2|\uparrow\rangle &= \tfrac{3\hbar^2}{4}|\uparrow\rangle \\ S^2|\downarrow\rangle &= \tfrac{3\hbar^2}{4}|\downarrow\rangle \end{aligned}\right\} \Rightarrow S^2 = \frac{3\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \frac{3\hbar^2}{4}\sigma_0 = *\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{3\hbar^2}{4}&0\\0&\frac{3\hbar^2}{4}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}^{T*}$$

$$= \underline{\text{Casimir Op}}:\ -(g_{yz})^2-(g_{zx})^2-(g_{xy})^2 = \frac{3\hbar^2}{4}\mathbb{1} \qquad \big(\text{only for } s=1/2 \text{ systems}\big)$$

$$\left.\begin{aligned} S_-|\uparrow\rangle &= \hbar|\downarrow\rangle \\ S_+|\downarrow\rangle &= \hbar|\uparrow\rangle \\ S_+|\uparrow\rangle &= S_-|\downarrow\rangle = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} S_+ &= \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix} & S_- &= \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix} \\ &= \hbar|\uparrow\rangle\langle\downarrow| & &= \hbar|\downarrow\rangle\langle\uparrow| \end{aligned} \quad \begin{pmatrix}\text{can't}\\\text{measure}\end{pmatrix}$$

Lad. Op. (see harm.)
$[S_+, S_-] = (2\hbar)S_z$
$[S_z, S_\pm] = (\pm\hbar)S_+$

$$\left.\begin{aligned} S_z|\uparrow\rangle &= \tfrac{\hbar}{2}|\uparrow\rangle \\ S_z|\downarrow\rangle &= -\tfrac{\hbar}{2}|\downarrow\rangle \end{aligned}\right\} \Rightarrow S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\hbar}{2}\sigma_z = *\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{\hbar}{2}&0\\0&-\frac{\hbar}{2}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}^{T*}$$

$$= \tfrac{\hbar}{2}|\uparrow\rangle\langle\uparrow| - \tfrac{\hbar}{2}|\downarrow\rangle\langle\downarrow|$$

$$\left.\begin{aligned} S_x &= \tfrac{1}{2}(S_+ + S_-) \\ S_y &= \tfrac{1}{2i}(S_+ - S_-) \end{aligned}\right\} \Rightarrow S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

$$\boxed{\{S_i, S_j\} = \tfrac{\hbar^2}{2}\delta_{ij}} \qquad \big(\text{only for } s=1/2 \text{ systems}\big)$$

<u>Properties of Pauli Matrices, σ_i</u>

- $\sigma_i = \sigma_i^\dagger = \sigma_i^{-1}$
* $\sigma_i^2 = 1 = (\hat{n}\cdot\sigma)^2 \leftrightarrow$ $(\hat{n}\cdot\sigma+\mathbb{1})(\hat{n}\cdot\sigma-\mathbb{1}) = 0$, $(\hat{n}\cdot S+\frac{\hbar}{2})(\hat{n}\cdot S-\frac{\hbar}{2}) = (S_z+\frac{\hbar}{2})(S_z-\frac{\hbar}{2})$
- $\sigma_i\sigma_j = -\sigma_j\sigma_i = i\sigma_k = -\hat{k} = (\sigma_j\sigma_i)^\dagger \ \Rightarrow\ \left[\frac{\sigma_i}{2i}, \frac{\sigma_j}{2i}\right] = \frac{\sigma_k}{2i}$
* $\left\{\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right\} = \frac{\delta_{ij}}{2}$
* $(A\cdot\vec{\sigma})(B\cdot\vec{\sigma}) = A\cdot B + i(A\times B)\cdot\vec{\sigma}$ $\quad \big(\text{if } [A_i,\sigma_i] = 0 = [B_i,\sigma_i]\big)$
* $(\sigma_i\sigma_j)^2 = -1$
* $(\sigma_i\sigma_j\sigma_k)^2 = -1 = \mathbb{i}^2$
- $\mathrm{Tr}\ \sigma_i = 0 \ \Rightarrow\ \mathrm{Tr}(\sigma_\alpha\sigma_\beta) = 2\delta_{\alpha\beta}$ $\quad \alpha\in(0,x,y,z)$
* $\sum c_\alpha\sigma_\alpha = 0 \to c_\alpha = 0 \ \Rightarrow\ M_{2\times 2} = \sum \frac{1}{2}\mathrm{Tr}(M\sigma_\alpha)\sigma_\alpha$
- $\det(\sigma_i) = -1$

<u>Gamma Matrices, γ_α</u>

- $\gamma_i = \begin{bmatrix}0&\sigma_i\\-\sigma_i&0\end{bmatrix}$, $\gamma_t = \begin{bmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{bmatrix}$ or $\begin{bmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{bmatrix}$
* $\gamma_t^2 = 1:\ e^{j\phi} = \cosh\phi + \gamma_t\sinh\phi$
* $\gamma_i^2 = -1$
- $-\gamma_i = \gamma_i^\dagger = \gamma_i^{-1}$, $\gamma_t = \gamma_t^\dagger = \gamma_t^{-1}$
- $\gamma_\alpha\gamma_\beta = -\gamma_\beta\gamma_\alpha \ \Rightarrow\ \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$
* $(\gamma_i\gamma_j)^\dagger = \gamma_j\gamma_i$, $(\gamma_t\gamma_i)^\dagger = -\gamma_i\gamma_t$
* $(\gamma_i\gamma_j)^2 = -1$, $(\gamma_t\gamma_i)^2 = 1$
* $(\gamma_i\gamma_j\gamma_k)^2 = 1$, $(\gamma_t\gamma_i\gamma_k)^2 = -1$
* $(\gamma_t\gamma_x\gamma_y\gamma_z)^2 = -1 = \mathbb{i}^2$
- $\gamma_5 = \pm i\gamma_t\gamma_x\gamma_y\gamma_z$

General Direction, $\hat{n}$, on Bloch Sphere $(1, \theta, \phi) \Leftrightarrow$ Pauli Vector, V (see Properties of σ_i)

$$\hat{n} \cdot \vec{S} = \cos\phi \sin\theta \, S_x + \sin\phi \sin\theta \, S_y + \cos\theta \, S_z = \boxed{\frac{\hbar}{2}\begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}} = V_{\hat{n}}$$

- $V_{\vec{n}}^2 = \|\vec{n}\|^2 I$ • $\det(V_{\vec{n}}) = -\|\vec{n}\|^2$ • $V = V^\dagger$ • $\mathrm{Tr}(V) = 0$ • $\substack{\text{Reflect } V \\ \text{over } R_{\hat{n}} \perp} : -RVR^{\cancel{-1}}$
- $\substack{\text{Rotate } V \text{ in} \\ xy\text{-plane by } \psi} : -\left[\cos\frac{\psi}{2}\sigma_x + \sin\frac{\psi}{2}\sigma_y\right]\left[-\sigma_x V \sigma_x\right][..] = \underline{\left[\cos\frac{\psi}{2}\mathbb{1} - \sin\frac{\psi}{2}\sigma_x\sigma_y\right]V[..]^{-1}} = \boxed{(\pm)UV(\pm)U^\dagger} \qquad \big(U \in SU(2)\big)$
- $i\vec{n}\cdot\sigma \;=\; in_x\sigma_x + in_y\sigma_y + in_z\sigma_z \;=\; n_x\sigma_y\sigma_z + n_y\sigma_z\sigma_x + n_z\sigma_x\sigma_y$

$*\;\; \hat{n}\cdot\vec{S}\,|\chi_\pm\rangle = \pm\frac{\hbar}{2}|\chi_\pm\rangle \;\Rightarrow\; \boxed{|\chi_+\rangle = e^{i\gamma}\begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} = \begin{bmatrix} 1 \\ \tan\frac{\theta}{2}e^{i\phi} \end{bmatrix},\; |\chi_-\rangle = e^{i\gamma}\begin{bmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\cot\frac{\theta}{2}e^{i\phi} \end{bmatrix}}$

$*\;\;$ Riemann Sphere $\to \mathbb{C}P^1$: $\;|\chi\rangle = \begin{bmatrix} 1 \\ f(\theta,\phi) \end{bmatrix} \to \left.\begin{matrix} f(\theta,\phi) \in (-\infty,\infty): & 0 \le \theta < \pi \\ f(\theta,\phi) \in (-\infty,\infty): & \pi \le \theta < 2\pi \end{matrix}\right\} \quad$ (double cover)

Projectors and Nilpotents:

$$P_U^\pm = \tfrac{1}{2}(1 \pm U) \qquad (U^2 = 1)$$

$$P_z^\pm = \frac{\cancel{\sigma_z}}{2}(1 \pm \sigma_z)^{\cancel{\dagger}\cancel{\sigma_z}} = |\pm\rangle = |z^\pm\rangle$$

$$P_{tz}^\pm = \frac{\cancel{\gamma_t\gamma_z}}{2}(1 \pm \gamma_t\gamma_z)\cancel{\gamma_t\gamma_z}$$

$$|\xi\rangle = \begin{bmatrix} a+bi & 0 \\ c+di & 0 \end{bmatrix} = \boxed{(aP_z^+ + b\sigma_x\sigma_y P_z^+) + (c\sigma_x P_z^+ + d\sigma_y P_z^+)}$$

$$|\xi\rangle\langle\chi| = \begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix} = \xi\xi^T\epsilon = \begin{bmatrix} \xi^1 & 0 \\ \xi^2 & 0 \end{bmatrix}\begin{bmatrix} -\xi^2 & \xi^1 \\ 0 & 0 \end{bmatrix} \qquad \det(|\xi\rangle\langle\chi|) = 0\,,\; \epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= x\sigma_x + y\sigma_y + z\sigma_z = \Big(\xi^1|z^+\rangle + \xi^2\sigma_x|z^+\rangle\Big)\Big(-\xi^2\langle z^+|^* + \xi^1\sigma_x\langle z^+|^*\Big)$$

$$\{\alpha_\pm, \alpha_\pm\} = 0$$

$$\{\alpha_-, \alpha_+\} = 1$$

$$\alpha_-\alpha_+ + \alpha_+\alpha_- = P_+ + P_-$$

$$P_\pm P_\mp = 0 = P_\mp P_\pm$$

$$\alpha_{UV}^\pm = \tfrac{1}{2}(U + V) \qquad (U^2 = 1\,,\; V^2 = -1)$$

$$Cl(1,3,\mathbb{C}): \quad \begin{aligned} a_{tz}^\pm &= \tfrac{1}{2}(\gamma_t + \gamma_z) \;\Rightarrow\; P_{tz}^\pm = a_{tz}^\mp a_{tz}^\pm = \tfrac{1}{2}(1 \pm \gamma_t\gamma_z) \\ b_{ixy}^\pm &= \tfrac{1}{2}(i\gamma_x + \gamma_y) \;\Rightarrow\; P_{ixy}^\pm = b_{ixy}^\mp b_{ixy}^\pm = \tfrac{1}{2}(1 \pm i\gamma_x\gamma_y) \end{aligned}$$

$$\begin{matrix} \{a_\pm, b_\pm\} = 0 = \{a_\pm, b_\mp\} \\ \left[P_{tz}^\pm, P_{ixy}^\pm\right] = 0 = \left[P_{tz}^\pm, P_{ixy}^\mp\right] \end{matrix} \;\to\; \boxed{|\epsilon\rangle = \epsilon^1 P_{tz}^+ P_{ixy}^+ + \epsilon^2\alpha_{tzixy}^+ P_{tz}^+ P_{ixy}^+}$$

$\underline{R \in SO(3) \sim \mathbb{R}P^3}$: $\begin{matrix} R(\theta) = e^{\theta g} \\ R(0) = \mathbb{1} \end{matrix} \;\Rightarrow\; \left.\frac{dR}{d\theta}\right|_{\theta=0} = \boxed{g[R] \in \substack{\mathfrak{so}(3) \\ \mathfrak{su}(2)}}$

- $\det\left(R = e^{\theta g}\right) = 1 = e^{\theta\,\mathrm{Tr}(g)} \;\Rightarrow\; \boxed{\mathrm{Tr}(g) = 0}$
- $\boxed{\begin{matrix} v^T[R^T R]v = v^T\mathbb{1}v \\ R^T = R^{-1} \end{matrix}} \;\Rightarrow\; g^T + g = 0 \;\leftrightarrow\; \boxed{g^T = -g}$

$$[g_{yz}, g_{zx}] = g_{xy} \equiv g_z = [g_x, g_y]$$

$$[g_{zx}, g_{xy}] = g_{yz} \equiv g_x = [g_y, g_z]$$

$$[g_{xy}, g_{yz}] = g_{zx} \equiv g_y = [g_z, g_x]$$

$$\underline{U \in SU(2) = 2\times SO(3) = Spin(3) = \begin{array}{c}2n \text{ grade,}\\ \|\pm U_i\|^2 = \mathbb{1}\end{array} \in Cl(3,0) \sim S^3_{phere}} \; : \; \boxed{\begin{bmatrix}\alpha & -\beta^* \\ \beta & \alpha^*\end{bmatrix} \begin{array}{l} = e^{-i\frac{\psi}{2}\hat{n}\cdot\sigma} = \cos\frac{\psi}{2}\mathbb{1} - \sin\frac{\psi}{2}\,\mathbb{i}\,\vec{n}\cdot\sigma \\ \sim \; e^{-\frac{\psi}{2}\sigma_i\sigma_j} = \cos\frac{\psi}{2}\mathbb{1} - \sin\frac{\psi}{2}\,\sigma_i\sigma_j \end{array}} \;\Rightarrow\; \boxed{g[U] \in \begin{array}{c}\mathfrak{su}(2)\\ \mathfrak{so}(3)\end{array}}$$

- Rotate $\hat{k}$ to $\hat{k}'$ by angle θ, $\hat{\theta} = (-\sin\phi, \cos\phi, 0) = \hat{k}\times\hat{k}'$ $\quad \underline{U[R(\theta)] = \cos\frac{\theta}{2}\mathbb{1} - \sin\frac{\theta}{2} i\big(\hat{\theta}\cdot\sigma\big)} = \begin{bmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}e^{i\phi} & \cos\frac{\theta}{2}\end{bmatrix}$, $\underline{e^{i\frac{\pi}{2}(\hat{\theta}\cdot\sigma)} = i(\hat{\theta}\cdot\sigma)}$

- Rotate about $\hat{\psi}$ by angle ψ, $\hat{\psi} = \hat{n} = (n^x, n^y, n^z)$ $\quad \underline{U[R(\psi)] = \cos\frac{\psi}{2}\mathbb{1} - \sin\frac{\psi}{2} i\big(\hat{\psi}\cdot\sigma\big)} = \begin{bmatrix}\cos\frac{\psi}{2} - in^z\sin\frac{\psi}{2} & (-in^x - n^y)\sin\frac{\psi}{2} \\ (-in^x + n^y)\sin\frac{\psi}{2} & \cos\frac{\psi}{2} + in^z\sin\frac{\psi}{2}\end{bmatrix}$

- $\det(U) = \alpha^*\alpha + \beta^*\beta = 1 \;\Rightarrow\; \boxed{\mathrm{Tr}(g) = 0}$
- $\boxed{\begin{array}{c}\xi^\dagger[U^\dagger U]\chi = \xi^\dagger\mathbb{1}\chi \\ U^\dagger = U^{-1}\end{array}} \;\Rightarrow\; g^\dagger + g = 0 \;\leftrightarrow\; \boxed{g^\dagger = -g}$
- $\boxed{V' = (\pm)UV(\pm)U^\dagger = U[vv^T\epsilon]U^{-1}}$, $\epsilon = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ $(\det vv^T = 0)$

$$-\tfrac{1}{2}\sigma_x\sigma_y = g_{xy} = [g_{yz}, g_{zx}] = -\tfrac{\mathbb{i}}{2}\sigma_z = \tfrac{1}{2}\hat{i}$$
$$-\tfrac{1}{2}\sigma_y\sigma_z = g_{yz} = [g_{zx}, g_{xy}] = -\tfrac{\mathbb{i}}{2}\sigma_x = \tfrac{1}{2}\hat{j}$$
$$-\tfrac{1}{2}\sigma_z\sigma_x = g_{zx} = [g_{xy}, g_{yz}] = -\tfrac{\mathbb{i}}{2}\sigma_y = \tfrac{1}{2}\hat{k}$$

$$\underline{\Lambda \in SO^+(1,3)} : \begin{array}{l}\Lambda(\theta) = e^{\theta g}\\ \Lambda(0) = \mathbb{1}\end{array} \;\Rightarrow\; \frac{d\Lambda}{d\theta}\Big|_{\theta=0} = \boxed{g[\Lambda] \in \begin{array}{c}\mathfrak{so}^+(1,3)\\ \mathfrak{sl}(2,\mathbb{C})\end{array}}$$

- $\boxed{v^T[\eta]v = v^T[\Lambda^T\eta\Lambda]v = v^T\eta\Lambda^{-1}\cdot\Lambda v} \;\Rightarrow\; \boxed{g^T = -\eta g\eta}$, $\boxed{\eta = (+,-,-,-)}$
- Rotation, J_{ij} : $\boxed{\Lambda^T_{ij} = \Lambda^{-1}_{ij}} \;\Rightarrow\; \boxed{\mathrm{Tr}(J) = 0,\; J^T = -J}$
- Boost, K_{ti} : $\boxed{\Lambda^T_{ti} = \Lambda_{ti}} \;\Rightarrow\; \boxed{\mathrm{Tr}(K) = 0,\; K^T = K}$

$$\left.\begin{array}{r}[J_{yz}, J_{zx}] = J_{xy}\\ [K_{tx}, K_{ty}] = -J_{xy}\\ [J_{yz}, K_{ty}] = K_{tz}\\ [J_{yz}, K_{tz}] = -K_{ty}\end{array}\right\}\times 3$$

$$\underline{L \in SL(2,\mathbb{C}) = 2\times SO^+(1,3) = Spin(1,3) = \subset Cl(3,0) = \subset Cl(1,3)} : \boxed{\begin{bmatrix}\alpha & \beta\\ \gamma & \alpha^*\end{bmatrix} \sim \begin{array}{l} e^{\theta J_{ij}} = \begin{array}{l}\cos\frac{\psi}{2}\mathbb{1} - \sin\frac{\psi}{2}\sigma_i\sigma_j\\ \cos\frac{\psi}{2}\mathbb{1} + \sin\frac{\psi}{2}\gamma_i\gamma_j\end{array}\\ e^{\theta K_{ti}} = \begin{array}{l}\cosh\frac{\psi}{2}\mathbb{1} - \sinh\frac{\psi}{2}\sigma_i\\ \cosh\frac{\psi}{2}\mathbb{1} + \sinh\frac{\psi}{2}\gamma_i\end{array}\end{array}} \;\Rightarrow\; \boxed{\begin{array}{c}\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{so}^+(1,3)\\ g[L] \in \mathfrak{su}(2)\oplus(\mp^l_r i)\,\mathfrak{su}(2)\\ \mathfrak{su}(2)_{\mathbb{C}}\end{array}}$$

- $\boxed{\begin{array}{c}\psi^T[\epsilon]\phi = \psi^T[L^T\epsilon L]\phi = \psi^T\epsilon L^{-1}\cdot L\phi\\ = -\phi^T\epsilon\psi = 0 \;\text{ if }\; \psi = \phi\end{array}} \;\Rightarrow\; \boxed{g^T\epsilon = -\epsilon g}$, $\epsilon = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$
- Rotation, J_{ij} : $\boxed{L^\dagger_{ij} = L^{-1}_{ij}} \;\Rightarrow\; \boxed{\mathrm{Tr}(J) = 0,\; J^T = -J}$
- Boost, K_{ti} : $\boxed{L^\dagger_{ti} = L_{ti}} \;\Rightarrow\; \boxed{\mathrm{Tr}(K) = 0,\; K^T = K}$

$$[J_{yz}, J_{zx}] = J_{xy} = -\tfrac{1}{2}\sigma_x\sigma_y \to \tfrac{1}{2}\gamma_x\gamma_y$$
$$[K_{tx}, K_{ty}] = -J_{xy}$$
$$[J_{yz}, K_{ty}] = K_{tz} = \mp^l_r\tfrac{1}{2}\sigma_z \to \pm^l_r\tfrac{1}{2}\gamma_t\gamma_z$$
$$[J_{yz}, K_{tz}] = -K_{ty}$$

- Left/Left Dual : $\begin{array}{l} w_{ld}\cdot w_l : \psi^T\epsilon\cdot\phi = \psi^T\epsilon L^{-1}\cdot L\phi\\ w^T_{ld} : \;\; -\epsilon\psi \;\; \to [L^T]^{-1}[-\epsilon\psi]\end{array}$
- Right Dual/Right : $\begin{array}{l} w_r\cdot w_{rd} : \psi^\dagger\epsilon\cdot\phi^* = \psi^\dagger\epsilon[L^*]^{-1}\cdot L^*\phi^*\\ w^T_r : \;\; -\epsilon\psi^* \;\; \to \;\; [L^\dagger]^{-1}[-\epsilon\psi^*]\end{array}$
- $\boxed{W' = (\pm)LW(\pm)L^\dagger = L[w_l w_l^\dagger][(L^\dagger)^{-1}]^{-1} \;=\; L_{ij}[w_l w_{rd}]L^{-1}_{ij} \;\text{ or }\; L_{ti}[w_l w_{rd}][L^{-1}_{ti}]^{-1}}$ $\quad(\det w_l w_l^\dagger = 0)$
- $\underline{(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})}$: $\begin{bmatrix}L & 0\\ 0 & [L^\dagger]^{-1}\end{bmatrix}\begin{bmatrix}w_l\\ w_r^T\end{bmatrix}$
- $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} = \mathfrak{su}(2)_{\mathbb{C}}\oplus\mathfrak{su}(2)_{\mathbb{C}}$:
$$A^\pm_i = \tfrac{1}{2}(J_{jk} \pm K_{ti})$$
$$[A^\pm_i, A^\pm_j] = A^\pm_k \quad , \quad [A^+, A^-] = 0$$
- $\underline{(\tfrac{1}{2},0)\otimes(0,\tfrac{1}{2}) = (\tfrac{1}{2},\tfrac{1}{2})}$: $\begin{bmatrix}c\tilde{t}\\ \tilde{r}\end{bmatrix} = [A^{-1}L\otimes[L^\dagger]^{-1}A]\begin{bmatrix}ct\\ \vec{r}\end{bmatrix} = [SO(3)]\begin{bmatrix}ct\\ \vec{r}\end{bmatrix}$

$\mathfrak{su}(2)$ Representations for Generators (Raising/Lowering)

$$g_+ = ig_{yz} - g_{zx} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \tfrac{1}{2}\sigma_x + \tfrac{i}{2}\sigma_y \qquad g_z = ig_{xy} = \text{``}\tfrac{1}{2}\text{''}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \tfrac{1}{2}\sigma_z$$

$$g_- = ig_{yz} + g_{zx} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \tfrac{1}{2}\sigma_x - \tfrac{i}{2}\sigma_y \qquad g^2 = -\left(g_{yz}^2 + g_{zx}^2 + g_{xy}^2\right) \quad \left(\begin{smallmatrix}\text{Casimir Op. in} \\ \text{Universal Env. Alg.}\end{smallmatrix}\right)$$

$$\Rightarrow \quad [g_+, g_-] = 2g_z, \quad [g_z, g_\pm] = \pm g_\pm, \quad \left[g_{ij}, g^2\right] = 0, \quad g_\pm^\dagger g_\pm = g^2 - g_z^2 \mp g_z$$

$$\begin{aligned} g_z|m\rangle &= m|m\rangle \\ \Rightarrow \quad g_z \cdot g_\pm|m\rangle &= (m \pm 1)g_\pm|m\rangle \\ g^2|j,m\rangle &= j(j+1)|j,m\rangle \end{aligned} \quad \Rightarrow \quad \begin{aligned} g_\pm^\dagger g_\pm|j,m\rangle &= \Big[j(j+1) - m(m \pm 1)\Big]|j,m\rangle \\ g_\pm|j,m\rangle &= \boxed{\sqrt{j(j+1) - m(m \pm 1)}|j,m\rangle} \end{aligned}$$

$$\mathbb{1}_3 = \begin{bmatrix} | & | & | \\ |1\rangle & |0\rangle & |-1\rangle \\ | & | & | \end{bmatrix}, \ \mathbb{1}_4 = \begin{bmatrix} | & | & | & | \\ |\tfrac{3}{2}\rangle & |\tfrac{1}{2}\rangle & |\tfrac{-1}{2}\rangle & |\tfrac{-3}{2}\rangle \\ | & | & | & | \end{bmatrix}, \ \ldots \qquad \begin{aligned} (\overline{g_\pm})_{ij} &= \langle i|g_\pm|j\rangle \\ (\overline{g_z})_{ij} &= \langle i|g_z|j\rangle \end{aligned} \ \Rightarrow \ \boxed{\begin{aligned} \overline{g_{yz}} &= \tfrac{1}{2i}(\overline{g_-} + \overline{g_+}) \\ \overline{g_{zx}} &= \tfrac{1}{2i}(\overline{g_-} - \overline{g_+}) \\ \overline{g_{yz}} &= -i\overline{g_z} \end{aligned}}$$

4.4 2 Objects w/ Spin **Objects could be orbital momentum, another particle spin, etc.**

4.4.1 2 Objects w/ Spin $\frac{1}{2}$:

Dim : $2 \otimes 2 = 3 \oplus 1$
Spin : $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$
$\Rightarrow \quad (2s_1+1)(2s_2+1) = \sum_{s=|s_1-s_2|}^{s_1+s_2} 2s+1$

*Find Eigenvectors, e_i , of $(S^{(1,2)})_z$ and $(S^{(1,2)})^2$ in the form of $|\chi_i\chi_j\rangle$ (using $(S^{(1,2)})_\pm$)

$$\boxed{\chi_i\chi_j \to |\chi_i\chi_j\rangle \equiv |\chi_i\rangle|\chi_j\rangle \equiv |\chi_i\rangle \otimes |\chi_j\rangle}$$

Choose $|\chi_i\rangle \equiv S_z$-Eigenvector w/ Spin $\frac{1}{2}$ (e.g, $|\frac{1}{2}\frac{-1}{2}\rangle = \binom{0}{1}$), as opposed to $\binom{.6}{.8}$)

$$S^{(i)} \equiv \begin{pmatrix} S_x^{(i)} \\ S_y^{(i)} \\ S_z^{(i)} \end{pmatrix}$$

- $S_z^{(2)} S_x^{(1)}\Big(|\chi_1\rangle|\chi_2\rangle\Big) = \Big(S_x^{(1)}|\chi_1\rangle\Big)\Big(S_z^{(2)}|\chi_2\rangle\Big)$
- $S^{(i)} \cdot S^{(j)} \equiv \underline{S_x^{(i)}S_x^{(j)} + S_y^{(i)}S_y^{(j)}} + S_z^{(i)}S_z^{(j)}$
  $= S_z^{(i)}S_z^{(j)} + \underline{\tfrac{1}{2}S_+^{(i)}S_-^{(j)} + \tfrac{1}{2}S_-^{(i)}S_+^{(j)}}$

$(S^{(i)})^2 \equiv S^{(i)} \cdot S^{(i)}$

$$S^{(1,2)} \equiv \left(S^{(1)} + S^{(2)}\right) \equiv \begin{pmatrix} S_x^{(1)} + S_x^{(2)} \\ S_y^{(1)} + S_y^{(2)} \\ S_z^{(1)} + S_z^{(2)} \end{pmatrix}$$

- $\left(S^{(1,2)}\right)^2 = \left(S^{(1)} + S^{(2)}\right) \cdot \left(S^{(1)} + S^{(2)}\right)$

1. $\left(S^{(1,2)}\right)_z$

$$\begin{aligned}\left(S^{(1,2)}\right)_z \chi_1\chi_2 &= \left(S_z^{(1)} + S_z^{(2)}\right)|\chi_1\rangle|\chi_2\rangle \\ &= S_z^{(1)}|\chi_1\rangle|\chi_2\rangle + |\chi_1\rangle S_z^{(2)}|\chi_2\rangle\end{aligned}$$

$$\left(S^{(1,2)}\right)_z |\chi_1\chi_2\rangle = \hbar(m_1+m_2)\ |\chi_1\chi_2\rangle$$

$$\Rightarrow\ \underline{e_i = a_i|\uparrow\uparrow\rangle + b_i|\uparrow\downarrow\rangle + c_i|\downarrow\uparrow\rangle + d_i|\downarrow\downarrow\rangle}$$

$$\begin{aligned}|\uparrow\uparrow\rangle &= |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle \\ |\uparrow\downarrow\rangle &= |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle \\ |\downarrow\uparrow\rangle &= |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle \\ |\downarrow\downarrow\rangle &= |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle\end{aligned}$$

2. Use $\left(S^{(1,2)}\right)_\pm$ on $|\uparrow\rangle|\uparrow\rangle$ to GUESS e_i from "nice" bevahior

$$\begin{aligned}S_-\ |\uparrow\uparrow\rangle &= \tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ S_-\left[\tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\right] &= |\downarrow\downarrow\rangle \\ S_-\ |\downarrow\downarrow\rangle &= 0\end{aligned}$$

S_+ works too

If $\frac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$ then maybe $\frac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ works (try $S_\pm$ on it).

Guess for $\{e_i\}$:

$$\begin{aligned}|1\ 1\rangle &\equiv |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle = |\uparrow\uparrow\rangle \\ |1\ 0\rangle &\equiv \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle + |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) = \tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |1\ \text{-}1\rangle &\equiv |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle = |\downarrow\downarrow\rangle \\ |0\ 0\rangle &\equiv \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle - |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) = \tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)\end{aligned}$$

3. Check if the guesses are eigenvectors of $\left(S^{(1,2)}\right)^2$ [and do $\left(S^{(1,2)}\right)_z$ to see eigenvalues]

(work has been skipped, do it yourself, check answer below)

$$\begin{array}{llll} S^2|1\ 1\rangle = \hbar^2(1)(1+1)|1\ 1\rangle & (s=1) & S_z|1\ 1\rangle = \hbar(1)|1\ 1\rangle & (m=1) \\ S^2|1\ 0\rangle = \hbar^2(1)(1+1)|1\ 0\rangle & (s=1) & S_z|1\ 0\rangle = \hbar(0)|1\ 0\rangle & (m=0) \\ S^2|1\ \text{-}1\rangle = \hbar^2(1)(1+1)|1\ \text{-}1\rangle & (s=1) & S_z|1\ \text{-}1\rangle = \hbar(-1)|1\ \text{-}1\rangle & (m=-1) \\ S^2|0\ 0\rangle = \hbar^2(0)(0+1)|0\ 0\rangle & (s=0) & S_z|0\ 0\rangle = \hbar(0)|0\ 0\rangle & (m=0)\end{array} \quad ☑$$

$$* \quad \boxed{e_i \in \left\{\begin{array}{l} \left.\begin{array}{l}|1\ 1\rangle = {\scriptstyle 1\leftarrow}\cancel{e^{i\phi}}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle = |\uparrow\uparrow\rangle \\ |1\ 0\rangle = \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle + |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) = \tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |1\ \text{-}1\rangle = |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle = |\downarrow\downarrow\rangle\end{array}\right\}\ \text{Triplet}: \ s=1 \\ \left.|0\ 0\rangle = \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle - |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) = \tfrac{\sqrt{2}}{2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)\right\}\ \text{Singlet}: \ s=0\end{array}\right.}$$

4.4.2 2 Objects w/ Any Spin: $j_1 \otimes j_2 = (j_1+j_2) \oplus |j_1 - j_2| \;\Rightarrow\; (2j_1+1)(2j_2+1) = \sum_{j=|j_1-j_2|}^{j_1+j_2} 2j+1$

- $|\chi_1\rangle$ has spin, j_1; and $|\chi_2\rangle$ has spin, j_2
- $j_{\max} = j_2 + j_1$ and $j_{\min} = |j_2 - j_1|$
- Possible total $|j\ m\rangle$ must satisfy

  1.) $j_{\min} \leq j \leq j_{\max}$, **2.)** $-j \leq m \leq j$,

  3.) have integer differences

Possible Combined $|j\ m\rangle$

$$
\begin{aligned}
(2j_{\max}+1) &\begin{cases} \Big|j_{\max}\ \ j_{\max}\Big\rangle = 1 \leftarrow \cancel{e^{i\phi}} \cdots \\ \Big|j_{\max}\ \ j_{\max-1}\Big\rangle \\ \quad\vdots \\ \Big|j_{\max}\ \ -j_{\max}\Big\rangle \end{cases} \\
(2j_{\max}-1) &\begin{cases} \Big|j_{\max-1}\ \ j_{\max-1}\Big\rangle = 1 \cdot \ldots \\ \Big|j_{\max-1}\ \ j_{\max-2}\Big\rangle \\ \quad\vdots \end{cases} \\
&\ \vdots \\
(2j_{\min}+1) &\begin{cases} \Big|j_{\min}\ \ j_{\min}\Big\rangle = 1 \cdot \ldots \\ J_-\Big|j_{\min}\ \ j_{\min}\Big\rangle \sim \ldots \\ \quad\vdots \\ \Big|j_{\min}\ \ -j_{\min}\Big\rangle \end{cases}
\end{aligned}
$$

If j_1 and j_2 are known from the start,

$$
\boxed{
\begin{aligned}
|jm, \cancel{j_1 j_2}\rangle &= \sum_{m_1,m_2} |j_1\ m_1\rangle|j_2\ m_2\rangle\ \langle j_1\ m_1|\langle j_2\ m_2|\,|jm, \cancel{j_1 j_2}\rangle \\
|jm\rangle &= \sum_{m_1,m_2} |j_1\ m_1\rangle|j_2\ m_2\rangle\ C^{j_1 j_2 j}_{m_1 m_2 m}
\end{aligned}
}
$$

where the sum is over all poss. int. diff. values that satisfy

$$m_1 + m_2 = m, \quad -j_1 \leq m_1 \leq j_1, \quad -j_2 \leq m_2 \leq j_2,$$

and C are the corresponding Clebsh-Gordan coefficients, whose squared value is the probability of measuring the $\chi_1\chi_2$ state represented by that term.

If the top state in a j-set (see box above) is known, applying the J_- lowering operator (and normalizing) provides the coefficients for the rest of the set of varying m. The coefficients for each top state of a set are (by convention) positive, real, and normalized to 1. This makes all of the coefficients real. For the top state of the initial $j_{\max}$-set, $|j_{\max}\ j_{\max}\rangle$, there is only one product-ket in the sum; its coefficient is thus set to 1. For an arbitrary set below the first, the top state has product-ket coefficients such that the state is orthogonal to all other previously determined states that have the same m. To reduce some work to solve for them, use

$$C^{j_1 j_2 j}_{m_1 m_2 m} = (-1)^{j_1+j_2-j} \cdot C^{j_1 j_2 j}_{-m_1 -m_2 -m}$$

$$\langle j_1\ m_1|\langle j_2\ m_2|j\ m\rangle = (-1)^{j_1+j_2-j} \cdot \langle j_1\ -m_1|\langle j_2\ -m_2|j\ -m\rangle$$

If m_1 and m_2 are also known from the start, then $m = m_1 + m_2$, and

$$\boxed{|j_1\ m_1\rangle|j_2\ m_2\rangle = \sum_j C^{j_1 j_2 j}_{m_1 m_2 m}\ |j\ (m_1+m_2)\rangle}$$

where the sum is only over all possible j as satisfied above - **1.), 2.) and 3.)**. In this case, the total z-component, m, is known. The only unknown is the total spin, j, whose probability to be measured is C^2.

$$A_{dim} \otimes B_{dim} = C_{dim} \oplus D_{dim}$$

Tensor Product Representation: $(2a_{spin}+1) \otimes (2b_{spin}+1) = (2c_{spin}+1) \oplus (2c_{spin}+1)$

$$a_{spin} \otimes b_{spin} = c_{spin} \oplus d_{spin}$$

$$\overline{A} \oplus \overline{B} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},\ \vec{v} \oplus \vec{w} = \begin{bmatrix} v \\ w \end{bmatrix} \ \Rightarrow\ \boxed{(A \oplus B)(v \oplus w) = Av \oplus Bw}$$

$$A \otimes B = \begin{bmatrix} A_i[B] & A_j[B] \\ A_k[B] & \dots \end{bmatrix},\ v \otimes w = \begin{bmatrix} v_i[w] \\ v_j[w] \end{bmatrix} \ \Rightarrow\ \boxed{(A \otimes B)(v \otimes w) = Av \otimes Bw} = Avw^\dagger B^\dagger = Av(Bw)^\dagger$$

$$A(t) \otimes B(t) = e^{at} \otimes e^{bt} = e^{g_{A\otimes B}\, t} \ \Rightarrow\ g_{A\otimes B} = \frac{d}{dt}(A \otimes B)\Big|_0 = \boxed{a \otimes \mathbb{1}_n + \mathbb{1}_m \otimes b} = M_{mn \times mn}$$

Lie Algebra : $g^{(1,2)} = g \otimes \mathbb{1} + \mathbb{1} \otimes g \ \Rightarrow\ \boxed{g(v \otimes w) = gv \otimes w + v \otimes gw}$ $\quad \big(g_z|\uparrow\uparrow\rangle = 1|\uparrow\uparrow\rangle\big)$

Not Lie : $(g^2)^{(1,2)} = (g_+^\dagger)^{(1,2)}(g_+)^{(1,2)} + [g_z^{(1,2)}]^2 + g_z^{(1,2)}$

$$= \boxed{g^2 \otimes \mathbb{1} + \mathbb{1} \otimes g^2 + 2(g_z \otimes g_z) + g_- \otimes g_+ + g_+ \otimes g_-}$$

Clebsch-Gordan:

$$\sqrt{[Cl, Go]}^{T} \begin{bmatrix} | \\ |a\rangle|b\rangle \\ | \end{bmatrix} = \begin{bmatrix} | \\ |J\ J_s\rangle \\ | \end{bmatrix} \qquad \sqrt{[Cl, Go]} \begin{bmatrix} | \\ |J\ J_s\rangle \\ | \end{bmatrix} = \begin{bmatrix} | \\ |a\rangle|b\rangle \\ | \end{bmatrix} \ \Rightarrow\ \begin{array}{cc} (j_a, j_b) & \begin{bmatrix} | \\ |J\ J_s\rangle \\ | \end{bmatrix}^T \\ \begin{bmatrix} | \\ m_a, m_b \\ | \end{bmatrix} & \sqrt{[Cl, Go]} \end{array},\quad \begin{array}{cc} (\frac{1}{2}, \frac{1}{2}) & [1,1 \quad 1,0 \quad\quad 0,0 \quad 1,-1] \\ \begin{bmatrix} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{bmatrix} & \sqrt{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \end{array}$$

4.5 Electron in Magnetic Field

$\mu_{\text{clas.}} = IA = \frac{q}{2\pi r} v(\pi r^2) = \frac{q}{2\pi r}\frac{L}{mr}(\pi r^2) = \left(\frac{q}{2m}\right) L \ \rightarrow\ \frac{e\hbar}{2m} \cdot n$ $\quad$ (Bohr magneton)

$\mu_{\text{quan.}} = \left(\frac{g_e q}{2m}\right) S = \left(\frac{q}{m}\right) S = \gamma S$

$$\begin{aligned} \tau_\mu &= \mu \times B \\ F_\mu &= \nabla(\mu \cdot B) \end{aligned} \quad , \quad \begin{aligned} H &= -\mu \cdot B \\ &= -\gamma S \cdot B \end{aligned}$$

Larmor Precession

$$B = B_0 \hat{k}$$

$$H = -\gamma B_0 S_z = -\gamma B_0 \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} \ \Rightarrow$$

$$\chi(t) = \cos(\alpha/2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar}E_1 t} + \sin(\alpha/2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\frac{i}{\hbar}E_2 t} = \begin{pmatrix} \cos(\alpha/2) e^{-\frac{i}{\hbar}E_1 t} \\ \sin(\alpha/2) e^{-\frac{i}{\hbar}E_2 t} \end{pmatrix}$$

$$\begin{pmatrix} \langle S_x \rangle \\ \langle S_y \rangle \\ \langle S_z \rangle \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2}\sin(\alpha)\cos(\gamma B_0 t) \\ -\frac{\hbar}{2}\sin(\alpha)\sin(\gamma B_0 t) \\ \frac{\hbar}{2}\cos(\alpha) \end{pmatrix}$$ (torque from B with S leads to precession)

Stern-Gerlach

5 Bosons and Fermions

Distinguishable Particles: $\boxed{\psi(r_1, r_2) \equiv \psi_a(r_1)\psi_b(r_2)}$

Indistinguishable Particles:

$$\boxed{P_x f(x_1, x_2;\; y_1, y_2;\; \dots) \;=\; \pm\, f(x_2, x_1;\; y_1, y_2;\; \dots)} \quad , \quad \boxed{\iint |\Psi(x_1,x_2)|^2 dx_1 dx_2 = \iint \Pr(x_1,x_2) \tfrac{dx_1 dx_2}{2}}$$

Boson:
$\big(s \in \{0, 1, 2, \dots\}\big)$

$$\psi_+(r_1, r_2) \equiv \frac{1}{\sqrt{2}}\Big[\psi_a(r_1)\psi_b(r_2) + \psi_a(r_2)\psi_b(r_1)\Big]$$

$$\boxed{\psi(r_1, r_2) = \psi(r_2, r_1)} \quad \rightarrow \quad \boxed{P_i\Psi = \Psi} \quad \text{(symmetric)}$$

Fermion:
$\big(s \in \{\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \dots\}\big)$

$$\psi_-(r_1, r_2) \equiv \frac{1}{\sqrt{2}}\Big[\psi_a(r_1)\psi_b(r_2) - \psi_a(r_2)\psi_b(r_1)\Big]$$

$$\boxed{\psi(r_1, r_2) = -\psi(r_2, r_1)} \quad \rightarrow \quad \boxed{P_i\Psi = -\Psi} \quad \text{(antisymmetric)}$$

5.1 Exchange Forces: $\Big\langle (x_1 - x_2)^2 \Big\rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$

Dist. Part. :	$\big\langle (\Delta x)^2 \big\rangle = \big\langle (\Delta x)^2 \big\rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b$	
Symmetric:	$\big\langle (\Delta x)^2 \big\rangle = \big\langle (\Delta x)^2 \big\rangle_d - 2\, \big\Vert \langle \psi_b \vert\, x \,\vert \psi_a \rangle \big\Vert^2$	(attractive if overlap)
Antisymmetric:	$\big\langle (\Delta x)^2 \big\rangle = \big\langle (\Delta x)^2 \big\rangle_d + 2\, \big\Vert \langle \psi_b \vert\, x \,\vert \psi_a \rangle \big\Vert^2$	(repulsive if overlap)

- $\langle x_1 x_2 \rangle = \dfrac{1}{2}\displaystyle\int \Big[\psi_a(r_1)^*\psi_b(r_2)^* \pm \psi_b(r_1)^*\psi_a(r_2)^*\Big] x_1 x_2 \Big[\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)\Big] dx_1 dx_2$

$$= \frac{1}{2}\langle x \rangle_a \langle x \rangle_b + \frac{1}{2}\langle x \rangle_b \langle x \rangle_a$$

$$\pm \frac{1}{2}\Big\langle \psi_b(r_1) \Big| x_1 \Big| \psi_a(r_1) \Big\rangle \Big\langle \psi_a(r_2) \Big| x_2 \Big| \psi_b(r_2) \Big\rangle \pm \frac{1}{2}\Big\langle \psi_a(r_1) \Big| x_1 \Big| \psi_b(r_1) \Big\rangle \Big\langle \psi_b(r_2) \Big| x_2 \Big| \psi_a(r_2) \Big\rangle$$

$$= \langle x \rangle_a \langle x \rangle_b \pm \big\Vert \langle \psi_b \vert\, x \,\vert \psi_a \rangle \big\Vert^2$$

Two Electrons:

$$\psi(r_1,r_2)\chi(m_1,m_2) = \begin{cases} \overset{\text{(singlet)}}{-\psi(r_1,r_2)\chi(m_2,m_1)} & \Rightarrow \begin{matrix}\chi \text{ is antisymmetric so} \\ \psi \text{ is symmetric}\end{matrix} \quad \Rightarrow \text{ Attractive (ground state)} \\[2ex] \overset{\text{(triplet)}}{-\psi(r_2,r_1)\chi(m_1,m_2)} & \Rightarrow \begin{matrix}\chi \text{ is symmetric so} \\ \psi \text{ is antisymmetric}\end{matrix} \quad \Rightarrow \text{ Repulsive} \end{cases}$$

5.2 Statistics

Sterling's Approx:
$$\log(z!) \approx z\log(z) - z \qquad z \gg 1 \text{ or } z = 0$$
$$\frac{d}{dz}\log(z!) \approx \log(z)$$

Lagrange Multiplier:
$$G(X, \alpha, \beta) = \log(Q(X)) + \alpha f_1(X) + \beta f_2(X)$$
$$\frac{\partial G}{\partial \alpha}[Q_{\max}] = 0, \qquad \frac{\partial G}{\partial \beta}[Q_{\max}] = 0, \qquad \frac{\partial G}{\partial N_n}[Q_{\max}] = 0$$

$$\sum_n N_n = N \qquad\qquad \sum_n N_n E_n = E$$
$$f_1(X) = N - \sum_n N_n = 0 \qquad\qquad f_2(X) = E - \sum_n N_n E_n = 0$$

Let there be N_n particles in the E_n energy level having d_n degeneracies, and $Q(N_1, N_2, ...)$ be the number of possible configurations for such a state given $X = (N_1, N_2, ..., N_n)$.

Dist.
$$\begin{cases} \textbf{1.)}\ Q(X) = \prod_n \binom{N - N_1 - ... - N_{n-1}}{N_n} d_n^{N_n} \\ \qquad\qquad = N! \prod_n \frac{d_n^{N_n}}{N_n!} \\ \textbf{2.)}\ \log(Q) = \log(N!) + \sum_n N_n \log(d_n) - \log(N_n!) \end{cases}$$
$$\textbf{3.)}\ \frac{\partial G}{\partial N_n} \approx \log(d_n) - \log(N_n) - \alpha - \beta E_n = 0$$
$$\textbf{4.)}\ N_n = \frac{d_n}{e^{\beta E_n + \alpha}}$$

Fermion
$$\begin{cases} \textbf{1.)}\ Q(X) = \prod_n \binom{d_n}{N_n} \\ \textbf{2.)}\ \log(Q) = \sum_n \log(d_n!) - \log(N_n!) - \log[(d_n - N_n)!] \end{cases}$$
$$\textbf{3.)}\ \frac{\partial G}{\partial N_n} \approx -\log(N_n) + \log(d_n - N_n) - \alpha - \beta E_n = 0$$
$$\textbf{4.)}\ N_n = \frac{d_n}{e^{\beta E_n + \alpha} + 1}$$

Boson
$$\begin{cases} \textbf{1.)}\ Q(X) = \prod_n \binom{N_n + d_n - 1}{N_n} \\ \textbf{2.)}\ \log(Q) = \sum_n \log[(N_n + d_n - 1)!] - \log(N_n!) - \log[(d_n - 1)!] \end{cases}$$
$$\textbf{3.)}\ \frac{\partial G}{\partial N_n} \approx \log(N_n + d_n - 1) - \log(N_n) - \alpha - \beta E_n = 0$$
$$\textbf{4.)}\ N_n = \frac{d_n - 1}{e^{\beta E_n + \alpha} - 1} \approx \frac{d_n}{e^{\beta E_n + \alpha} - 1}$$

Given some substance in thermal equilibrium,

$$\beta = \frac{1}{k_b T} \qquad \mu(T) \equiv -\frac{\alpha}{k_b T}$$

where μ depends on the situation.

$$\frac{N_n}{d_n}: \qquad n(\epsilon) \quad = \quad \begin{cases} \frac{1}{e^{(\epsilon-\mu)/k_b T}} & \text{Maxwell-Boltzmann} \\ \frac{1}{e^{(\epsilon-\mu)/k_b T}+1} & \text{Fermi-Dirac} \\ \frac{1}{e^{(\epsilon-\mu)/k_b T}-1} & \text{Bose-Einstein} \end{cases}$$

6 Perturbation Theory

$$H^{(0)}\psi_n = E_n\psi_n$$
$$\downarrow$$
$$H\psi'_n = E'_n\psi'_n$$

$$\left[H^{(0)} + \lambda H^{(1)}\right]\left[\psi_n + \lambda\psi_n^{(1)} + \lambda^2\psi_n^{(2)} + ...\right] = \left[E_n + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + ...\right]\left[\psi_n + \lambda\psi_n^{(1)} + \lambda^2\psi_n^{(2)} + ...\right]$$

$$\begin{aligned} &\cancel{\lambda^0 H^{(0)}\psi_n} && && \cancel{\lambda^0 E_n\psi_n} \\ &+ \lambda^1(H^{(0)}\psi_n^{(1)} + H^{(1)}\psi_n) && && + \lambda^1(E_n\psi_n^{(1)} + E_n^{(1)}\psi_n) \\ &&& = && && (\lambda{=}1) \\ &+ \lambda^2(H^{(0)}\psi_n^{(2)} + H^{(1)}\psi_n^{(1)}) && && + \lambda^2(E_n\psi_n^{(2)} + E_n^{(1)}\psi_n^{(1)} + E_n^{(2)}\psi_n) \\ &+ ... && && + ... \end{aligned}$$

6.1 Non-Degenerate Theory

$\underline{E_n^{(1)},\ \psi_n^{(1)}:}$

$$E_n\psi_n^{(1)} + E_n^{(1)}\psi_n = H^{(0)}\psi_n^{(1)} + H^{(1)}\psi_n$$

$$\langle\psi_m| \quad (-H^{(1)} + E_n^{(1)})|\psi_n\rangle = \langle\psi_m| \quad (H^{(0)} - E_n)|\psi_n^{(1)}\rangle = \sum c_i^{(1)}(E_i - E_n)\langle\psi_m|\psi_i\rangle$$

$$-\langle\psi_m|H^{(1)}|\psi_n\rangle + E_n^{(1)}\langle\psi_m|\psi_n\rangle = c_m^{(1)}(E_m - E_n)$$

$$\boxed{E_n^{(1)} = \langle\psi_n|H^{(1)}|\psi_n\rangle} \qquad \boxed{\psi_n^{(1)} = \sum_{m\neq n}\frac{\langle\psi_m|H^{(1)}|\psi_n\rangle}{E_n - E_m}\psi_m + (0)\psi_n}$$

$\underline{E_n^{(2)},\ |n^{(2)}\rangle:}$

$$\begin{aligned} -\langle m^{(0)}|H^{(1)}|n^{(1)}\rangle + E_n^{(1)}\langle m^{(0)}|n^{(1)}\rangle \\ +E_n^{(2)}\langle m^{(0)}|n^{(0)}\rangle \end{aligned} \quad \begin{aligned} &= \langle m^{(0)}|H^{(0)} - E_n^{(0)}|n^{(2)}\rangle \\ &= c_m^{(2)}(E_m - E_n) \end{aligned}$$

$$\boxed{E_n^{(2)} = \sum_{m\neq n}\frac{\left|\langle m|H^{(1)}|n\rangle\right|^2}{E_n - E_m}} = \langle n|H^{(1)}|n^{(1)}\rangle \quad , \quad \boxed{|n^{(2)}\rangle = \sum_{m\neq n}\frac{\langle m|H^{(1)} - E_n^{(1)}|n^{(1)}\rangle}{E_n - E_m}\cdot|m\rangle}$$

$\underline{E_n^{(i+1)},\ |n^{(i+1)}\rangle:}$ $\quad E_n^{(i+1)} = \langle n|H^{(1)}|n^{(i)}\rangle$

$$|n^{(i+1)}\rangle = \sum_{m\neq n}\frac{\langle m|H^{(1)}|n^{(i)}\rangle - \sum_{j=0}^{i}E_n^{(j+1)}\langle m|n^{(i-j)}\rangle}{E_n - E_m}\cdot|m\rangle$$

6.2 Degenerate Perturbation Theory (see Matrix Operators)

$$
\begin{aligned}
\Psi &= \sum_i \left(c_i^{(\psi)}{}_{[\Psi]} \right) \psi_i \\
&\equiv \sum_i c_i^{(\psi)} \psi_i \\
&= c_0^{(\psi)} \psi_0 + c_1^{(\psi)} \psi_1 + \ldots
\end{aligned}
\qquad , \qquad \forall \psi_i :
$$

- $H^{(0)}\psi_i = E_n \psi_i$ $\quad$ $\underline{(\psi_n \text{ are degenerate eigenfunctions of } H^{(0)})}$
- $\langle \psi_i | \psi_j \rangle = \delta_{ij}$
- $\langle \psi_i | \hat{Q} | \psi_j \rangle \equiv Q_{ij}$

$$E_n \Psi^{(1)} + E^{(1)} \Psi = H^{(0)} \Psi^{(1)} + H^{(1)} \Psi \qquad \text{(first order)}$$

$$
\begin{aligned}
\cancel{E_n \langle \psi_i | \Psi^{(1)} \rangle} + E^{(1)} \langle \psi_i | \Psi \rangle &= \cancel{\langle H^{(0)} \psi_i | \Psi^{(1)} \rangle} + \langle \psi_i | H^{(1)} | \Psi \rangle \\
&= \langle \psi_i | H^{(1)} | c_0 \psi_0 + c_1 \psi_1 + \ldots \rangle \\
c_i E^{(1)} &= c_0 \langle \psi_i | H^{(1)} | \psi_0 \rangle + c_1 \langle \psi_i | H^{(1)} | \psi_1 \rangle + \ldots
\end{aligned}
$$

$$
E^{(1)} \begin{pmatrix} c_0{}_{[\Psi]} \\ c_1{}_{[\Psi]} \\ \vdots \end{pmatrix}^{(\psi)} = \begin{pmatrix} H_{00}^{(1)} & H_{01}^{(1)} & \ldots \\ H_{10}^{(1)} & H_{11}^{(1)} & \ldots \\ \vdots & \vdots & \end{pmatrix}^{(\psi)} \begin{pmatrix} c_0{}_{[\Psi]} \\ c_1{}_{[\Psi]} \\ \vdots \end{pmatrix}^{(\psi)} \Rightarrow \boxed{\begin{matrix} \text{(solve for } E^{(1)}, \vec{c}^{\,(\psi)}{}_{[\Psi]}) \\ \left(\overline{H^{(1)}} - E^{(1)} \mathbb{1} \right) \vec{c} \;=\; 0 \end{matrix}}
$$

In general,

$$E_i^{(1)} \vec{c}^{\,(\psi)}{}_{[\Psi_i]} \;=\; \overline{H^{(1)}}^{\,(\psi)} \; \vec{c}^{\,(\psi)}{}_{[\Psi_i]} \qquad (i\text{th eigen-})$$

$$
E_i^{(1)} \begin{pmatrix} | \\ \vec{c}_{\,[\Psi_i]} \\ | \end{pmatrix}^{(\psi)} = \begin{pmatrix} | & | & \\ \vec{c}_{\,[\Psi_i]} & \vec{c}_{\,[\Psi_i]} & \ldots \\ | & | & \end{pmatrix}^{(\psi)} \begin{pmatrix} E_0^{(1)} & 0 & \ldots \\ 0 & E_1^{(1)} & \ldots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} - & \vec{c}^{\,*}{}_{[\Psi_i]} & - \\ - & \vec{c}^{\,*}{}_{[\Psi_i]} & - \\ & \vdots & \end{pmatrix}^{(\psi)} \begin{pmatrix} | \\ \vec{c}_{\,[\Psi_i]} \\ | \end{pmatrix}^{(\psi)}
$$

Instead of solving the characteristic polynomial, it would be wise to choose a basis $\{\psi\}$ such that $\vec{c}^{\,(\psi)}{}_{[\Psi_i]} = (\ldots 0 \; 0 \; 1_{(i)} \; 0 \; 0 \; \ldots)^T \;\Leftrightarrow\; \Psi_i = \psi_i$, making $\overline{H^{(1)}}^{(\psi)}$ diagonal with eigenvalue entries. These are the energy eigenvalues, $E_i^{(1)} = \left(H^{(1)} \right)_{ii}^{(\psi)} = \langle \psi_i | H^{(1)} | \psi_i \rangle$, which is just like first-order non-Perturbation energy. This also means $|\psi_i\rangle$ are eigenfunctions of $H^{(1)}$ (see Matrix Operators).

It is best to find a hermitian operator, $\hat{A}$, that commutes with $H^{(0)}$ and $H^{(1)}$, whose eigenvalues within the degenerate basis are unique. The corresponding eigenfunctions will be a basis that makes $H^{(1)}$ diagonal. This will also make them eigenfunctions of $H^{(1)}$.

1. $A = A^\dagger$
2. $[A, H^{(0)}] = 0 \;\rightarrow\; \left\{ \exists \{\Psi\} \;\middle|\; (A\Psi_n = a_n \Psi_n),\; (H^{(0)} \Psi_n = E_n \Psi_n) \right\}$
3. $\{\psi\} \subset \{\Psi\}$ s.t $\forall \psi_i : \begin{cases} \left(H^{(0)} \psi_i = E_n \psi_i \right), & \leftarrow \text{degenerate} \\ \left(A\psi_i = a_i \psi_i \right), & (\forall_{(i \neq j)} \; a_i \neq a_j) \end{cases}$
4. $[A, H^{(1)}] = 0 \;\Rightarrow\; \begin{aligned}[t] 0 &= \langle A\psi_i | H^{(1)} | \psi_j \rangle - \langle \psi_i | H^{(1)} | A\psi_j \rangle \\ 0 &= (a_i - a_j) H_{ij}^{(1)} \\ 0 &= H_{ij}^{(1)} \qquad ☑ \end{aligned}$

6.3 Hydrogen Energy Corrections

6.3.1 Fine Structure - $\alpha^4 mc^2$

The Dirac Equation can derive the total fine structure correction with a α^4 order approx.

<u>1. Relativistic,</u> $\boldsymbol{\hat{p}^4}$

$$
\begin{aligned}
T &= \sqrt{p^2c^2 + m^2c^4} - mc^2 = mc^2\sqrt{1 + \tfrac{p^2c^2}{m^2c^4}} - mc^2 \\
&= mc^2\left[\frac{(\frac{1}{2})}{1!}\left(\frac{p^2c^2}{m^2c^4}\right) + \frac{(\frac{1}{2})(1-\frac{1}{2})}{2!}\left(\frac{p^2c^2}{m^2c^4}\right)^2 + ...\right] = E_m\left[\frac{1}{2}\frac{E_p^2}{E_m^2} - \frac{1}{8}\left(\frac{E_p^2}{E_m^2}\right)^2\right] \\
&= \frac{p^2\cancel{c^2}}{2m\cancel{c^2}} - \frac{p^4c^4}{8m^3c^6} + ... \\
&\downarrow \\
H_r^{(1)} &= -\frac{p^4}{8m^3c^2}
\end{aligned}
$$

(For some reason $\hat{p}^4$ needs to be hermitian to use perturbation theory. It only isn't when $l = 0$, while $\hat{p}^2$ always is hermitian. See Prob. 6.15)

L^2 and L_z should commute with p^4 because the perturbation is spherically symmetric, meaning l and m_l should be conserved (see Operator Evolution). Their eigenvalues are also distinct (taking the eigenfunctions of nlm_l together) within each set of n^2 degeneracies, so their eigenvectors and eigenvalues can be used. n, l and m_l the "good" numbers.

$\langle r^{-1}\rangle = \frac{1}{n^2a_0^1}$

$\langle r^{-2}\rangle = \frac{1}{(l+1/2)n^3a_0^2}$

$$
\begin{aligned}
\left\langle\psi_{nlm_l}|H_r^{(1)}|\psi_{nlm_l}\right\rangle &= \frac{-1}{8m^3c^2}\langle\psi_{nlm_l}|p^4|\psi_{nlm_l}\rangle \\
&= \frac{-1}{8m^3c^2}\langle p^2\psi_{nlm_l}|p^2|\psi_{nlm_l}\rangle \\
&= \frac{-1}{8m^3c^2}\left\langle\left[2m(E_n - V)\right]^2\right\rangle \\
&= \frac{-4m^2}{8m^3c^2}\langle E_n^2 - 2E_nV + V^2\rangle \\
&= -\frac{E_n^2}{2mc^2}\left[\frac{4n}{l+1/2} - 3\right]
\end{aligned}
$$

<u>2. Spin-Orbit Coupling,</u> $\boldsymbol{S_e \cdot L_e}$

In the electron's frame of reference, the proton is spinning around it, creating a B-field affecting its magnetic dipole moment. The non-inertial reference frame requires multiplying by the Thomas procession correction, which in this case is $C_T = \frac{g_e - 1}{2} = 1/2$. In the lab frame, the moving electron's magnetic dipole moment creates an electric dipole moment, which is affected by the proton charge. The latter is much harder to calculate.

$$
\begin{aligned}
H_{so}^{(1)} &= -C_T\ \mu_e \cdot B(L_e) \qquad \text{(See Electron in Magnetic Field)} \\
&= \tfrac{1}{2}\tfrac{qS}{m} \cdot \tfrac{k_\mu}{r^3} \int I d\vec{l} \times \vec{r} \qquad \left(\sim\ \tfrac{1}{2}\tfrac{qS}{m} \cdot \tfrac{k_\mu}{r^3} \int \tfrac{mqd\vec{v}\times\vec{r}}{m} \right) \\
&= \tfrac{1}{2}\tfrac{qS}{m} \cdot \tfrac{k_\epsilon}{c^2}\tfrac{2\pi}{r} I = \tfrac{1}{2}\tfrac{qS}{m} \cdot \tfrac{k_\epsilon}{c^2}\tfrac{2\pi}{r}\tfrac{q(L/mr)}{2\pi r} \\
&= \frac{kqq}{2m}\frac{1}{mc^2}\frac{S\cdot L}{r^3} = \frac{e^2}{8\pi\epsilon_0 m^2c^2}\frac{S\cdot L}{r^3}
\end{aligned}
$$

$S \cdot L$ does not commute with L or S (meaning m_l and m_s are bad), but $[S\cdot L, S^2] = [S\cdot L, L^2] = 0$. The sum of the two, $J \equiv L+S$, and J^2 also commute with the perturbation. They are all conserved, and their unique eigenvalues per set of degeneracies - $l, s{=}1/2, j, m_j$ - are the "good" numbers (along with n).

$$
S\cdot L = \frac{1}{2}\left(J^2 - L^2 - S^2\right)
$$

$$
\langle r^{-3}\rangle = \frac{1}{l(l+1/2)(l+1)n^3a_0^3}
$$

(note: divergent at $l = 0$)

$$
\begin{aligned}
\langle nljm_j | H_{so}^{(1)} | nljm_j\rangle &= \frac{kqq}{2m}\frac{1}{mc^2}\frac{\hbar^2[j(j+1)-l(l+1)-s(s+1)]}{2l(l+1/2)(l+1)n^3a_0^3} \\
&= \frac{kqq}{4mn^4}\frac{\hbar^2\alpha^3m^3c^3}{\hbar^3mc^2}\frac{n[j(j+1)-l(l+1)-s(s+1)]}{l(l+1/2)(l+1)} \\
&= \frac{kqq}{4\hbar cn^4}\frac{\alpha^3m^2c^4}{mc^2}\frac{n[j(j+1)-l(l+1)-s(s+1)]}{l(l+1/2)(l+1)} \\
&= \frac{E_n^2}{mc^2}\left\{\frac{n\left[j(j+1)-l(l+1)-3/4\right]}{l(l+1/2)(l+1)}\right\}
\end{aligned}
$$

3. Darwin Term (correction for $H_{so}^{(1)}$ when $l = 0$) skipped

4. Total Correction

$$
\begin{aligned}
E_{fs}^{(1)} &= E_r^{(1)} + E_{so}^{(1)} \\
&= -\frac{E_n^2}{2mc^2}\left[\frac{4n}{l+\frac{1}{2}} - 3\right] + \frac{E_n^2}{mc^2}\left\{\frac{n\left[j(j+1)-l(l+1)-3/4\right]}{l(l+1/2)(l+1)}\right\} \\
&= \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right) \qquad (j{=}l\pm 1/2) \\
&\downarrow \\
E_{nj} &= E_n + E_{fs}^{(1)} \\
&= E_n\left[1 - \frac{E_n}{2mc^2}\left(\frac{4n}{j+1/2} - 3\right)\right] \\
&= -\frac{\alpha^2mc^2}{2n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - 3/4\right)\right]
\end{aligned}
$$

Fine structure splits the l energy degeneracies. However, since $j = l \pm 1/2$, there are still two j degeneracies if $n > 2$. Overall, the good numbers to use for stationary state solutions to the hydrogen atom w/ fine structure correction are $n, l, s{=}1/2, j, m_j$. Note, J^2, L^2, and S^2 always commute(?)

6.3.2 Zeeman Effect (Ext. B-Field)

$$\begin{aligned} H_B^{(1)} &= -(\mu_s + \mu_l) \cdot B_{\text{ext}} \qquad \text{(see Electron in Magnetic Field)} \\ &= -\left(\frac{g_e q}{2m} S + \frac{q}{2m} L\right) \cdot B_{\text{ext}} \\ &= \frac{e}{2m} (2S + L) \cdot B_{\text{ext}} \end{aligned}$$

Weak Zeeman ($B_{\text{ext}} \ll B_{\text{int}}$)

$$\begin{aligned} H_{WZ}^{(1)} &= \frac{e}{2m} B_{\text{ext}} \cdot (2S + L) \\ &= \frac{e}{2m} B_{\text{ext}} \cdot (J + S) \end{aligned}$$

Fine structure perturbation dominate the Zeeman perturbation, so the fine structure numbers are the good ones: $n, l, s{=}1/2, j$, and m_j. m_l and m_s can't be used for $\langle L \rangle$ or $\langle S \rangle$, so instead use the fact that the "vector" $J = L + S$ is conserved, so a **time-averaged** S-component to the J "vector" can be defined as $S_{\text{ave}} = \frac{S \cdot J}{J^2} J$, where $S \cdot J = \frac{1}{2}\left(J^2 + S^2 - L^2\right)$.

$$\begin{aligned} E_{\text{WZ}}^{(1)} &= \frac{e}{2m} B_{\text{ext}} \cdot \langle {\scriptstyle nljm_j} | J + S_{\text{ave}} | {\scriptstyle nljm_j} \rangle \\ &= \frac{e}{2m} B_{\text{ext}} \cdot \left\langle J \left(1 + \frac{S \cdot J}{J^2}\right) \right\rangle \\ &= \frac{e}{2m} B_{\text{ext}} \cdot \langle J \rangle \left(1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}\right) \\ &= \frac{e\hbar}{2m} B_{\text{ext}} m_j \left(1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}\right) \qquad \text{(let } B_{\text{ext}} \text{ be parallel to the z-axis)} \\ &= \mu_B B_{\text{ext}} m_j g_j \qquad \begin{array}{l} \mu_B = \text{Bohr magneton} = 5.788 \times 10^{-5} \text{ ev/T} \\ g_j = \text{Lande g-factor} \end{array} \end{aligned}$$

Strong Zeeman ($B_{\text{ext}} \gg B_{\text{int}}$)

For a strong magnetic field parallel to the z-axis, m_l and m_s are stuck in the same place, making them and l conserved. The external torque, however, means that the total angular momentums, j and m_j are not. Though unneeded, obviously $s{=}1/2$.

$$\begin{aligned} E_{SZ}^{(1)} &= \frac{e}{2m} B_{\text{ext}} \langle 2S_z + L_z \rangle \\ &= \mu_B B_{\text{ext}} (2m_s + m_l) \end{aligned}$$

The spin-orbit correction must be changed with respect to the new good numbers, m_l and m_s. The relativistic correction uses the same numbers, so it stays the same.

$$
\begin{aligned}
E_{\text{so}}^{(1)} &= \frac{e^2}{8\pi\epsilon_0 m^2 c^2}\left\langle \frac{S_x L_x + S_y L_z + S_z L_z}{r^3} \right\rangle \\
&= \frac{e^2}{8\pi\epsilon_0 m^2 c^2}\frac{0+0+\hbar^2 m_s m_l}{l(l+1/2)(l+1)n^3 a_0^3} \\
&= \frac{kqq}{2m^2c^2}\frac{\hbar^2}{(\hbar/\alpha mc)^3 n^3}\frac{m_s m_l}{l(l+1/2)(l+1)} \\
&= \frac{kqq}{2\hbar c}\frac{\alpha^3 m^2 c^4}{4mc^2 n^4}\frac{4n m_s m_l}{l(l+1/2)(l+1)} \\
&= \frac{E_n^2}{2mc^2}\frac{4n m_s m_l}{l(l+1/2)(l+1)}
\end{aligned}
\quad \rightarrow
$$

$$
\begin{aligned}
E_{\text{fs}}^{(1)} &= E_{\text{so}}^{(1)} + E_{\text{r}}^{(1)} \\
&= \frac{E_n^2}{2mc^2}\frac{4n m_s m_l}{l(l+1/2)(l+1)} + \frac{E_n^2}{2mc^2}\left[3 - \frac{4n}{l+1/2}\right] \\
&= \frac{4nE_n^2}{2mc^2}\left[\frac{m_s m_l}{l(l+1/2)(l+1)} + \frac{3}{4n} - \frac{1}{l+1/2}\right]
\end{aligned}
$$

$$\downarrow$$

$$E_{nlm_lm_s} = E_n + E_{\text{SZ}}^{(1)} + E_{\text{fs}}^{(1)}$$

Intermediate Zeeman ($B_{\text{ext}} \sim B_{\text{int}}$)

There are no good numbers here (see Degenerate Perturbation Theory). The basis is chosen to be $|j\ m_j\rangle = \sum_i C_i |l\ m_l\rangle \otimes |s\ m_s\rangle$ (see 2 Objects w/ Any Spin), as it makes $\overline{H^{(1)}}^{(e)}$ easier (instead of using l, m_l, m_s).

1.) $\psi_i = |j\ m_j\rangle_i$

2.) $\Big(\langle l\ m_l|\langle s\ m_s|\Big)_x \Big(|l\ m_l\rangle|s\ m_s\rangle\Big)_y = \delta_{xy}$

3.) $Q_{rc}^{(\psi)} = \langle \psi_r|\hat{Q}|\psi_c\rangle$

4.) ψ_i s.t. $\begin{cases} 0 \le l < n \\ j_{(l\pm)} = l \pm 1/2, \\ 2l^2 < i \le 2(l{+}1)^2 \end{cases}$

$$
\begin{aligned}
\langle {\scriptstyle jm_j}|H_{fs}^{(1)}|{\scriptstyle jm_j}\rangle &= \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right) \\
&\equiv \gamma_n\left(3 - \frac{4n}{j+1/2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\langle {\scriptstyle jm_j}|H_{IZ}^{(1)}|{\scriptstyle jm_j}\rangle &= \langle {\scriptstyle jm_j}|H_{IZ}^{(1)}\Big(C_i|{\scriptstyle lm_l}\rangle \otimes |{\scriptstyle sm_s}\rangle\Big) \\
&= \mu_B B_{\text{ext}}(2m_s + m_l)C_i^2 \\
&\equiv \beta(2m_s + m_l)C_i^2
\end{aligned}
$$

$$\overline{H^{(1)}}^{(jm_j)} = \overline{H_{fs}^{(1)}}^{(jm_j)} + \overline{H_{IZ}^{(1)}}^{(jm_j)}$$

See Griffith Prob. 6.25 for example with $n = 2$

6.3.3 Stark Effect (Small Ext. E-Field)

- $H^{(1)} = -p \cdot E = eE \cdot r$ (small r)
- $n = 1 \;\rightarrow\; H^{(1)} = 0$
- $n = 2 \;\rightarrow\; \begin{cases} H^{(1)} = 0 & m = \pm 1 \\ H^{(1)} = ke|E|a_0 & m = 0 \end{cases}$ (k is some constant)

6.3.4 Lamb Shift (quantitized E-field) - $\alpha^5 mc^2$

- Vacuum Polarization ($e \leftrightarrow \gamma\ 2e\ \gamma \leftrightarrow p$)
- m_e renormalization ($e_1\ \gamma\ e_1$)
- Anomalous Magnetic Moment ($e_1\ \gamma\ e_2$)

$$E^{(1)}_{lamb\,(n,l)} = \tfrac{1}{4n^3}\alpha^5 mc^2 \left[k(n,l) \boxed{\pm} \tfrac{1}{\pi(j+\frac{1}{2})(l+\frac{1}{2})}\right] \qquad (j = l \boxed{\pm} \tfrac{1}{2}) \qquad k(n, l \neq 0) \approx< .05$$

$$E^{(1)}_{lamb\,(n,0)} = \tfrac{1}{4n^3}\alpha^5 mc^2\ k(n,0) \qquad k(n,0) \in [12.7_{n=1}, 13.2_{n=\infty}]$$

6.3.5 Hyperfine (Spin-Spin, Spin-Orbit), $\boldsymbol{S_p \cdot S_e}$, $\boldsymbol{S_p \cdot L_e}$ - $m/m_p\ \alpha^4 mc^2$

(Coupling between the electron magnetic moment and the magnetic field from the proton magnetic moment)

$$\mu_e = -\frac{g_e e}{2m_e} S_e = -\frac{e}{m_e} S_e, \qquad \mu_p = \frac{g_p e}{2m_p} S_p$$

$$B(\mu_p) = \frac{\mu_0}{4\pi r^3}[3(\vec{\mu_p} \cdot \hat{r})\hat{r} - \vec{\mu_p}] + \frac{2\mu_0}{3}\vec{\mu_p}\delta^3(r)$$

$$\begin{aligned} H^{(1)}_{hf} &= -\mu_e \cdot B(\mu_p) \\ &= \ldots \\ &\downarrow \\ E^{(1)}_{hf} &= \left(\frac{e}{m_e}\right)\left(\frac{2\mu_0}{3}\frac{g_p e}{2m_p}\right)\langle S_e \cdot S_p\rangle \left|\psi_{nlm}(0)\right|^2 \end{aligned}$$

In the ground state, $\left|\psi_{100}(0)\right|^2 = 1/(\pi a_0^3)$. S_e^2, S_p^2, and the sum $S = S_e + S_p$ commute with $S_e \cdot S_p$, so s_e, s_p, m_s, s^2 are the good numbers. S_e and S_p do not, so m_{se} and m_{sp} are not good numbers.

$$E^{(1)}_{hf\,(f,l)} = \frac{1}{2n^3} g_p \frac{m_e}{m_p}\alpha^4 m_e c^2 \left[\tfrac{\pm 1}{(f+\frac{1}{2})(l+\frac{1}{2})}\right] \qquad (f_{total} = j \pm \tfrac{1}{2},\ F = S_{S_p+S_e} + L)$$

$$\begin{aligned} E^{(1)}_{hf} &= \left(\frac{e}{m_e}\right)\left(\frac{2}{3\epsilon_0 c^2}\frac{g_p e}{2m_p}\right)\frac{1}{2\pi a_0^3}\langle S^2 - S_e^2 - S_p^2\rangle \\ &= \frac{g_p e^2}{4\pi\epsilon_0 c^2 m_p m_e}\frac{4\alpha^3 m_e^3 c^3 \hbar^2}{3\hbar^3}\left[\frac{s(s+1)}{2} - 3/4\right] \\ &= \frac{4}{3} g_p \frac{m_e}{m_p}\alpha^4 m_e c^2 \left[\frac{s(s+1)}{2} - 3/4\right] \\ &= \frac{4}{3} g_p \frac{m_e}{m_p}\alpha^4 m_e c^2 \cdot \begin{cases} \frac{1}{4} & s = 1 \text{ (triplet)} \\ \frac{-3}{4} & s = 0 \text{ (singlet)} \end{cases} \quad \rightarrow \quad \begin{aligned} &\Delta E = 5.88 \times 10^{-6} \text{ eV} \\ &\lambda = 21 \text{ cm}, \quad \nu = 1420 \text{ MHz} \end{aligned} \end{aligned}$$

6.3.6 Positronium (Heavy Quarkonium is similar)

- Reduced Mass : $-\frac{1}{2n^2}\alpha^2\mu c^2,\ \mu = \frac{m_e}{2}$
- Hyperfine Splitting $\to$ Fine : $\frac{m_e}{m_p} = 1 \ \Rightarrow \sim \alpha^4 mc^2$
- e^+ moves $\to$ c is finite : $\sim \alpha^4 mc^2$

$\epsilon_{sing,\ s=0} = 0 \qquad J = S_{S_1+S_2} + L$

$$\epsilon_{trip,\ s=1} = \begin{cases} \frac{-(3l+4)}{(l+1)(2l+3)} & j = l+1 \\ \frac{1}{l(l+1)} & j = l \\ \frac{3l-1}{l(2l-1)} & j = l-1 \end{cases}$$

$*\ E_{fine} = \frac{1}{2n^3}\alpha^4 m_e c^2 \left[\frac{11}{32n} - \frac{1+\frac{1}{2}\boxed{\epsilon}}{2l+1}\right]$

$*\ n = 3:$ $n=3, l=2, s=1, j=123$ $\quad n=3, l=1, s=1, j=012$ $\quad n=3, l=0, s=1, j=1$

$n=3, l=2, s=0, j=2$ $\quad n=3, l=1, s=0, j=1$ $\quad n=3, l=0, s=0, j=0$ (two are the same btw...)

- Lamb Shift : $\sim \alpha^5 mc^2$
- $e + e^+ \leftrightarrow \gamma_{virtual,\ s=1}$: $l = 0, s = 1$ (triplet state) $\Rightarrow \sim \cancel{|\psi(0)|^2}\alpha^4 m_e c^2 \frac{1}{4n^3}$
- $e + e^+ \Rightarrow n\gamma,\ l = 0$: Charge Cong. # $(-1)^{0+s} = (-1)^n$ ($\exists l > 0$ by higher order process)

6.4 Transition Amplitude (See Pictures)

$$H_{(t)} = H^0 + H^1{}_{(t)}$$
$$\downarrow$$
$$U_{(t)}|i^0\rangle = \sum_n |n^0\rangle e^{-\frac{i}{\hbar}E_n^0 t}\langle n^0|U_I|i^0\rangle$$
$$|\Psi_{(t)}\rangle = \boxed{\sum_n |n^0\rangle e^{-\frac{i}{\hbar}E_n^0 t} d_{n(t)}}$$
$$\downarrow$$
$$0 = \langle f^0|i\hbar\tfrac{\partial}{\partial t} - H^0 - H^1{}_{(t)}|\Psi_{(t)}\rangle$$
$$= \sum_n \langle f^0|\left[i\hbar \dot{d}_n - H^1{}_{(t)} d_n\right]|n^0\rangle e^{-\frac{i}{\hbar}E_n^0 t}$$
$$\downarrow$$
$$\dot{d}_{f(t)} = \sum_n \tfrac{1}{i\hbar}\langle f^0|H^1{}_{(t)}|n^0\rangle e^{\frac{i}{\hbar}(E_f^0 - E_n^0)t} d_{n(t)}$$
$$= \sum_n \tfrac{1}{i\hbar}\langle f^0|H^1{}_{(t)}|n^0\rangle e^{i\omega_{fn}t} d_{n(t)}$$

$$d_{n(t)} = d_{n(0)} + \int_0^t \dot{d}_n\, dt' :$$

- $d_{n(0)} \;=\; \delta_{ni}$ (if $|d_{n\neq i}(t)| \ll 1$) (0^{th} order)
- $\dot{d}_{f(t)} \;\approx\; \frac{1}{i\hbar}\langle f^0|H^1{}_{(t)}|i^0\rangle\, e^{i\omega_{fi}t}$
- $\boxed{d_{n(t)} \;\approx\; \delta_{ni} + \frac{1}{i\hbar}\int_0^t \langle n^0|H^1{}_{(t')}|i^0\rangle\, e^{i\omega_{ni}t'}\, dt'}$ $\left(\begin{smallmatrix}1^{\text{st}}\\ \text{order}\end{smallmatrix}\right)$
- $\dot{d}_{f(t)} \;\approx\; \frac{1}{i\hbar}\overline{H^1_{fi}}(t)\, e^{i\omega_{fi}t}$
  $+ \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^t \sum_n \overline{H^1_{fn}}(t)\, e^{i\omega_{fn}t}\, \overline{H^1_{ni}}(t')\, e^{i\omega_{ni}t'}\, dt'$
- $\ldots$

Interactive Picture Method:

$$U_{I(t,t_0)} = \mathbb{I} + \tfrac{1}{i\hbar}\int_{t_0}^t H_I^1{}_{(t')}\, dt' + \left(\tfrac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} H_I^1{}_{(t')} H_I^1{}_{(t'')}\, dt'' dt' + \ldots$$

- $\langle f^0|U_{I(t,t_0)}|i^0\rangle \;=\; \langle f^0|e^{\frac{i}{\hbar}E_f^0(t-t_0)}\, U_{(t,t_0)}|i^0\rangle$

$$\equiv\;\; d_{f(t)} = \boxed{\delta_{fi} + \tfrac{1}{i\hbar}\int_{t_0}^t \langle f^0|H^1{}_{(t')}|i^0\rangle\, e^{i\omega_{fi}(t'-t_0)}\, dt'} \quad (1^{\text{st}}\text{ order})$$
$$+ \left(\tfrac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} \sum_n \langle f^0|H^1{}_{(t')}|n^0\rangle\, e^{i\omega_{fn}(t'-t_0)}\, \langle n^0|H^1{}_{(t'')}|i^0\rangle\, e^{i\omega_{ni}(t''-t_0)}\, dt'' dt' + \ldots$$

Normal Schrodinger Propagator:

$$U_{S(t,t_0)} = U^0{}_{(t,t_0)} + \tfrac{1}{i\hbar}\int_{t_0}^t U^0{}_{(t,t_0)} U^{0\dagger}{}_{(t',t_0)}\, H^1{}_{(t')}\, U^0{}_{(t',t_0)}\, dt'$$
$$+ \left(\tfrac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} U^0{}_{(t,t_0)} U^{0\dagger}{}_{(t',t_0)}\, H^1{}_{(t')}\, U^0{}_{(t',t_0)} U^{0\dagger}{}_{(t'',t_0)}\, H^1{}_{(t'')}\, U^0{}_{(t'',t_0)}\, dt'' dt' + \ldots$$

- $\langle f^0|U_{(t,t_0)}|i^0\rangle \;=\; \boxed{\delta_{fi}\, e^{-\frac{i}{\hbar}E_f^0(t-t_0)} + \frac{1}{i\hbar}\int_{t_0}^t e^{-\frac{i}{\hbar}E_f^0(t-t')}\langle f^0|H^1{}_{(t')}|i^0\rangle e^{-\frac{i}{\hbar}E_i^0(t'-t_0)}\, dt'}$ (1^{st} order)

$$+ \left(\tfrac{1}{i\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{t'} \sum_n e^{-\frac{i}{\hbar}E_f^0(t-t')}\langle f^0|H^1{}_{(t')}|n^0\rangle e^{-\frac{i}{\hbar}E_n^0(t'-t'')}\langle n^0|H^1{}_{(t'')}|i^0\rangle e^{-\frac{i}{\hbar}E_i^0(t''-t_0)}\, dt'' dt' + \ldots$$

6.5 Variation Principle - Approx. Ground State Energy

$$\psi = \sum c_n \psi_n \;\rightarrow\; E(\psi) > E_0 = E(\psi_0) \qquad \psi \equiv f(b,x), \quad \langle H \rangle = \langle T \rangle + \langle V \rangle$$

$$\Rightarrow \qquad b_{\min}: \; \frac{d}{db}\langle H \rangle = 0 \qquad E_0 \approx \Big\langle f(b_{\min},x) \Big| H \Big| f(b_{\min},x) \Big\rangle$$

6.6 Selection Rules - Orbital Transitions

Electric Dipole Approximation ONLY: $\lambda_\gamma \gg$ atom length $\rightarrow\; E, B$ feels homogenously oscillating to the atom

$\psi_{nlm} \rightarrow \psi_{n'l'm'}$:

- $\Delta m \in \{-1, \overset{?}{\cancel{0}}, 1\}$

  $s(\gamma) = 1 \;\rightarrow\; m_s(\gamma) \in \{-\hbar, \overset{?}{\cancel{0}}, \hbar\}$
  $E = E\hat{z} \;\rightarrow\; \Delta m = 0$

- $\Delta l = \pm 1$

  $1s \leftrightarrow 2p$
  Exception:$(2s \rightarrow 1s)$ through two-photon emission

- $\Delta j \in \{-1, 0, 1\}$

  Exception:$(j = 0 \rightarrow j = 0)$ not allowed

7 Blackbody Radiation

- Power Spectrum : $I'(\omega) = \frac{\hbar^3 \omega^3}{h^2 c^2} \frac{1}{e^{\hbar\omega/k_b T} - 1} \left[\frac{I}{\Omega \cdot f}\right]$ ($\mu = 0$ for photons since photon number isnt conserved)
- Stefan-Boltzmann Law : $I = \frac{dP}{dA} \propto T^4$!! important !!
- Wien's Displacement Law : $\lambda_{\max} = \frac{2.9 \times 10^{-3}}{T}$ [m] (mode of spectrum)

8 Adiabatic Theorem - Slow Changing of Potential

$$t = 0 \;\rightarrow\; \begin{matrix} H(t=0) = H^{(0)} \\ H(0)|n\rangle = E_n|n\rangle \end{matrix} \qquad\qquad t = t \;\rightarrow\; \begin{matrix} H = H^{(0)}(t) \\ H(t)|n(t)\rangle = E_n(t)|n(t)\rangle \end{matrix}$$

$$\boxed{\text{Dynamic Phase : } \theta_n(t) = -\tfrac{1}{\hbar}\int_0^t E_n(t')\,dt'}$$

$$|\Psi_{m(t)}\rangle \equiv \sum_n |n(t)\rangle\, e^{i\theta_n(t)}\,\langle n(t)|m(0)\rangle$$

$$\approx |m(t)\rangle\, e^{i\theta_m(t)}\, e^{i\gamma_m(t)}$$

$$= |m(t)\rangle\, e^{i\theta_m(t)}\, e^{\frac{i}{\hbar}\int A^m\cdot dR}$$

$$\sum_n \cancel{H|n\rangle e^{i\theta_n}}\, c_n = i\hbar \sum_n |\dot n\rangle\, e^{i\theta_n}\, c_n + \cancel{|n\rangle i\dot\theta_n e^{i\theta_n}}\, c_n + |n\rangle\, e^{i\theta_n}\,\dot c_n$$

$$\langle m|\dot H|n\rangle + \langle m|H|\dot n\rangle = \cancel{\langle m|\dot E_n|n\rangle} + \langle m|E_n|\dot n\rangle$$

$$\Downarrow$$

$$\boxed{\dot c_m = \tfrac{d}{dt}\langle m(t)|m(0)\rangle = -\langle m|\dot m\rangle\, c_m - \sum_{n\neq m} \tfrac{\langle m|\dot H|n\rangle}{E_n - E_m}\, e^{i(\theta_n - \theta_m)}\, c_n}$$

(not trivial)

$$\approx -\langle m(t)|\dot m(t)\rangle\, c_m \quad\Rightarrow\quad c_m(t) \approx c_m(0)\, e^{\frac{i}{\hbar} i\hbar \int\langle m|\dot m\rangle dt'}$$

$$c_n(t) \approx \delta_{nm}\, e^{i\gamma_m(t)} \qquad \boxed{\text{Berry Phase : } \gamma_m(t) = i\int_0^t \langle m(t')|\dot m(t')\rangle\, dt' \in \mathbb{R}}$$

Berry/Geometric Phase

$$\gamma_m(t) = i\int_0^t \langle m(t')|\dot m(t')\rangle\, dt' = \boxed{\tfrac{1}{\hbar}\int_{R_i}^{R_f} i\hbar\langle m|\nabla_R m\rangle\cdot dR}$$

$$\Rightarrow \tfrac{1}{\hbar}\oint i\hbar\langle m|\nabla_R m\rangle\cdot dR = \tfrac{1}{\hbar}\iint \nabla_R \times i\hbar\langle m|\nabla_R m\rangle\cdot da$$

$$\sim \quad \boxed{\tfrac{1}{\hbar}\oint A^m\cdot dR} \quad = \tfrac{1}{\hbar}\iint \nabla_R\times A^m\cdot da = \tfrac{1}{\hbar}\Phi_B^m$$

Aharanov-Bohm Effect:

$$i\hbar\tfrac{\partial\Psi}{\partial t} = \left[\tfrac{(p-qA)^2}{2m} + V + \cancel{q\phi}\right]\Psi$$

$$\Rightarrow\; \Psi = e^{\frac{i}{\hbar}\int_{\mathcal{O}}^{r} qA\cdot dr'}\psi \;, \quad \breve E\psi = \breve H\psi$$

$$= \boxed{e^{ig}\psi}$$

$$\vec A = \tfrac{\Phi_B}{2\pi r}\hat\phi \;\Rightarrow\; \Delta g_{\text{closed}} = \tfrac{q}{\hbar}\oint \tfrac{\Phi_B}{2\pi r}\hat\phi\cdot r\,d\hat\phi$$

$$= \tfrac{q\Phi_B}{\hbar} = \gamma_m$$

Infinitismal Path Integral

R : Slow degree of freedom (nucleus)

r : Fast degree of freedom (electron)

$$\mathbb{I} = \int dR\sum_n |R, n(R)\rangle\langle R, n(R)|$$

$$\approx \int dR\,|R, n(R)\rangle\langle R, n(R)|$$

$$\langle\chi(\epsilon)|e^{-\frac{i}{\hbar}H\epsilon}|\chi(0)\rangle = \langle R(\epsilon)|e^{-\frac{i}{\hbar}H_f\epsilon}|R(0)\rangle\langle n(R(\epsilon))|e^{-\frac{i}{\hbar}H_s\epsilon}|n(R(0))\rangle$$

$$\breve U(R_1,\epsilon;R_0,0) = \sqrt{\tfrac{-im}{2\pi\hbar\epsilon}}\, e^{\frac{i}{\hbar}\mathcal{L}_s\epsilon}\, e^{-\frac{i}{\hbar}E_n(R_0)\epsilon}\langle n(R_1)|n(R_0)\rangle$$

$$\Psi(R_1,\epsilon) = \langle\chi(\epsilon)|\hat U(\epsilon)|\Psi(R_0,0)\rangle \approx \sqrt{\tfrac{-im}{2\pi\hbar\epsilon}}\int e^{\frac{i}{\hbar}\mathcal{L}_s\epsilon}\, e^{-\frac{i}{\hbar}E_n(R_1+\eta)\epsilon}\langle n(R_1)|n(R_1+\eta)\rangle\Psi(R_1+\eta,0)\,d\eta \qquad (\eta = R_0 - R_1)$$

$$\approx \sqrt{\tfrac{-im}{2\pi\hbar\epsilon}}\int e^{\frac{i}{\hbar}\frac{m}{2}\frac{\eta^2}{\epsilon}}\left[1 - \tfrac{i}{\hbar}\epsilon(V_s + E_n)\right]\langle n(R_1)|\Big[|n(R_1)\rangle + \eta|\partial n(R_1)\rangle + \tfrac{\eta^2}{2}|\partial^2 n(R_1)\rangle\Big]\Big[1 + \eta\tfrac{d}{dR} + \tfrac{\eta^2}{2}\tfrac{d^2}{dR^2}\Big]\Psi(R_1,0)\,d\eta$$

$$\approx \sqrt{\tfrac{-im}{2\pi\hbar\epsilon}}\int e^{\frac{i}{\hbar}\frac{m}{2}\frac{\eta^2}{\epsilon}}\Big[1 - \tfrac{i}{\hbar}\epsilon V(R_1,0) + \cancel{\eta(\ldots)} + \tfrac{\eta^2}{2}\tfrac{d^2}{dR^2} + \eta^2\langle n|\partial n\rangle\tfrac{d}{dR} + \tfrac{\eta^2}{2}\langle n|\partial^2 n\rangle\Big]\Psi(R_1,0)\,d\eta$$

$$\breve E|\Psi\rangle = \hat H|\Psi\rangle : \hat H = \tfrac{P_s^2}{2m} + V_s + \hat H_f$$

$$\breve E\Psi = \breve H\Psi \quad : \breve H = \tfrac{(P_s - A^n)^2}{2m} + V + \Phi^n$$

$$\boxed{\begin{matrix} A^n = i\hbar\langle n|\partial n\rangle \\ \Phi^n = \tfrac{\hbar^2}{2m}\left[\langle\partial n|\partial n\rangle - \langle\partial n|n\rangle\langle n|\partial n\rangle\right]\end{matrix}}$$

$$\left(\begin{matrix}\langle n|\partial n\rangle + \langle\partial n|n\rangle = 0 \\ A^n \text{ is added/subtracted in}\end{matrix}\right)$$

9 Integral Form

$$\psi(r) = \psi_0(r) + \int g(r-r_0)V(r_0)\psi(r_0)\ d^3r \qquad g(r) = -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}$$

$$\begin{aligned}
&= \psi_0 + \int gV\psi_{(r_0)} \\
&= \psi_0 + \int gV\psi_0 + \int\int gVgV\psi_{(r_0)} \\
&= \psi_0 + \int gV\psi_0 + \int\int gVgV\psi_0 + \int\int gVgVgV\psi_0 + ...
\end{aligned}$$

10 Klein-Gordon Equation (Spinless Free Particle)

$$(p^2c^2 + m^2c^4)\psi = E^2\psi$$

$$(-E^2 + p^2c^2 + m^2c^4)\psi = 0$$

$$\left[-(E/c)^2 + p^2 + (mc)^2\right]\psi = 0$$

$$\frac{\left[-(E/c)^2 + p^2 + (mc)^2\right]}{\hbar^2}\psi = 0$$

$$\left[\frac{1}{c^2}\frac{\partial}{\partial t}^2 - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0$$

$$\boxed{(-\Box^2 + \mu^2)\psi = 0}$$

$$\boxed{\begin{array}{l} \partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_t, \nabla) \\ \partial^\mu = \frac{\partial}{\partial x_\mu} = (\partial_t, -\nabla) \end{array}}$$

$$\mathcal{L} = \mathcal{T} - \mathcal{U} = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}\kappa^2\phi^2$$

$$= \boxed{-\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\kappa^2\phi^2}$$

$$d\mathcal{L} = \tfrac{1}{c^2}\tfrac{\partial\phi}{\partial t}\tfrac{\partial\epsilon}{\partial t} - \tfrac{\partial\phi}{\partial x}\tfrac{\partial\epsilon}{\partial x} - \kappa^2\phi\epsilon \qquad (\epsilon = d\phi)$$

$$S_{[\phi]} = \int dt \int dx\, \mathcal{L}\left(\phi, \tfrac{\partial\phi}{\partial x}, \tfrac{\partial\phi}{\partial t}\right)$$

$$\begin{aligned}
dS_{[\phi]} = 0 &= \int dt \int dx\, d\mathcal{L} \\
&= \int dt \int dx \left[\tfrac{1}{c^2}\tfrac{\partial\phi}{\partial t}\tfrac{\partial\epsilon}{\partial t} - \tfrac{\partial\phi}{\partial x}\tfrac{\partial\epsilon}{\partial x} - \kappa^2\phi\epsilon\right] \\
&= \int dt \int dx \left[\underline{-\tfrac{1}{c^2}\tfrac{\partial^2\phi}{\partial t^2} + \tfrac{\partial^2\phi}{\partial x^2} - \kappa^2\phi}\right]\epsilon + 0 + \ldots
\end{aligned}$$

$$\boxed{\Box^2\phi = \kappa^2\phi}$$

$\underline{\text{EM}, \ A^\mu = (\frac{V}{c}, A), \ J^\mu = (c\rho, J)}$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_z & 0 \end{bmatrix} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu F^{\mu\nu} = \boxed{\frac{4\pi}{c} J^\nu = \Box A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right)}$$

- Gauge Trans. : $(A')^\mu = A^\mu + \partial^\mu \lambda_{(t,x)}$
- Continuity : $\partial_\mu J^\mu = 0$

Lorentz : $\partial_\mu A^\mu = 0 \ \Rightarrow \ \frac{4\pi}{c} J^\mu = \Box A^\mu$

$*\ \partial_\mu A^\mu = 0, \ \underline{\Box \lambda = 0} \ \Rightarrow \ \partial_\mu (A')^\mu = 0$

Klein-Gordon = $\underline{\text{Free Space}, \ J^\mu = 0} \ \overset{lorentz}{\Rightarrow} \ \underline{\Box A^\mu = 0}$

$A^\mu = a e^{-\frac{i}{\hbar} p^\mu x_\mu} \epsilon^\mu{}_{(p)} \ \Rightarrow \ p^\mu p_\mu = 0$

- $\partial_\mu A^\mu = 0 \ \Rightarrow \ p^\mu \epsilon_\mu = 0$
- $\epsilon^{\mu *} \epsilon_\mu = -1$ • $\sum_{s=1,2} \epsilon_i^{(s)} \epsilon_j^{(s)*} = \delta_{ij} - \hat{p}_i \hat{p}_j$

$$(A')^\mu = A^\mu + \partial^\mu (i\hbar k a) e^{-\frac{i}{\hbar} p^\mu x_\mu} = a e^{-\frac{i}{\hbar} p^\mu x_\mu} \left[\epsilon^\mu{}_{(p)} + k p^\mu\right]$$

Coulomb Gauge : $A^0 = 0 \ \left(\begin{smallmatrix}\text{can choose} \\ \text{in free space}\end{smallmatrix}\right) \ \rightarrow \ \nabla \cdot A = 0$

- $\epsilon^0 = 0 \ \rightarrow \ \epsilon \cdot p = 0 \ \overset{p=p_z}{\Rightarrow} \ \epsilon \in \begin{matrix} (0,1,0,0) \\ (0,0,1,0) \end{matrix}$

$*\ \underline{m_s = \binom{right}{left} \pm 1} \ \rightarrow \ e_\pm = \mp \frac{1}{\sqrt{2}} (e^{(1)} \pm i e^{(2)})$

$*\ \sum e^s e^{s\dagger} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

11 Dirac Equation

$$\mu^2 = \Box^2 \qquad\qquad E^2 = p^2c^2 + m^2c^4 = H^2$$

$$\begin{aligned} m &= \sqrt{\nabla^2 - \partial_t^2} \\ &= A\partial_x + B\partial_y + C\partial_z + iD\partial_t \\ &= i\gamma^\mu\partial_\mu \end{aligned} \qquad \begin{aligned} \sqrt{p^2+m^2} &= \alpha\cdot p + \beta m \\ &= \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m \end{aligned}$$

$$\begin{aligned} \partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{\partial}{\partial t}^2 &= (A\partial_x + B\partial_y + C\partial_z + iD\partial_t)^2 \\ &= A^2\partial_x^2 + B^2\partial_y^2 + C^2\partial_z^2 - D^2\partial_t^2 \\ &\quad + [AB+BA]\partial_x\partial_y + [AC+CA]\partial_x\partial_z + [BC+CB]\partial_y\partial_z \\ &\quad + [AD+DA]i\partial_x\partial_t + [BD+DB]i\partial_y\partial_t + [CD+DC]i\partial_z\partial_t \end{aligned}$$

$$D = \gamma^0, \qquad A = i\gamma^1 = i\beta\alpha_1, \qquad B = i\gamma^2 = i\beta\alpha_2, \qquad C = i\gamma^3 = i\beta\alpha_3$$

$$\beta = \gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \gamma^0\gamma^i, \qquad \gamma^i = \beta\alpha_i$$

$$\gamma^\mu:\ \gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}\ ,\ \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\ ,\ \gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3\ ,\ \Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \gamma^5\gamma^0\gamma^i$$

$$\boxed{\begin{aligned} &(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 = (\pm\gamma^\mu p_\mu - mc)\psi \\ &(i\not{\partial} - m)\psi = 0 \qquad \text{(natural units)} \end{aligned}} \qquad \boxed{\begin{aligned} &i\hbar\tfrac{\partial}{\partial t}\psi = (\alpha\cdot pc + \beta mc^2)\psi = \gamma^0\left[\gamma^i\cdot pc + mc^2\right]\psi \\ &i\hbar\tfrac{\partial}{\partial t}\psi = (\alpha\cdot(p-qA)c + \beta mc^2 + q\phi)\psi \end{aligned}}$$

$$\begin{aligned} i\hbar\tfrac{\partial}{\partial t}\psi &= (\alpha(p-qA)c + \beta mc^2 + q\phi)\psi \\ &= (\alpha\cdot\pi c + \beta mc^2 + q\phi)\psi \end{aligned}$$

$$\begin{aligned} \psi_{(t)} &= \psi_{(p)} e^{\frac{i}{\hbar}(p\cdot r - Et)} \\ \phi &= 0 \end{aligned} \quad \Rightarrow \quad E\psi = (\alpha\cdot\pi + \beta m)\psi$$

$$\begin{bmatrix} E-m & -\sigma\cdot\pi \\ -\sigma\cdot\pi & E+m \end{bmatrix}\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = 0 \ \Leftrightarrow\ \begin{aligned} (E-m)\psi_+ &= (\sigma\cdot\pi)\psi_- \\ (E+m)\psi_- &= (\sigma\cdot\pi)\psi_+ \end{aligned}$$

$$(E-m)\psi_\pm = \tfrac{(\sigma\cdot\pi)(\sigma\cdot\pi)}{E+m}\psi_\pm$$

$$E_s\psi_\pm \approx \tfrac{(\sigma\cdot\pi)^2}{2m}\psi_\pm \qquad \text{(Pauli's Eq.) } (\sim \text{ to Schrodinger})$$

$$\boxed{\begin{aligned}(\sigma\cdot A)(\sigma\cdot B) = \\ A\cdot B + i\sigma\cdot(A\times B)\end{aligned}} \quad \begin{aligned} &= \tfrac{(\sigma\cdot\pi)(\sigma\cdot\pi)}{2m}\psi_\pm = \tfrac{\pi\cdot\pi + i\sigma\cdot(\pi\times\pi)}{2m}\psi_\pm \\ &= \left[\tfrac{\pi^2}{2m} - \tfrac{q\hbar}{2m}\sigma\cdot B\right]\psi_\pm \end{aligned}$$

$$\boxed{(g_e = 2)} \quad = \left[\tfrac{\pi^2}{2m} - \tfrac{g_e q}{2m}S\cdot B\right]\psi_\pm$$

$$\begin{aligned} &i\hbar\tfrac{\partial}{\partial t}\psi = (\alpha\cdot pc + \beta mc^2)\psi \\ &(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \qquad (\partial_\mu = \tfrac{\partial}{\partial x^\mu}) \end{aligned}$$

$$\begin{aligned} \psi_{(t)} &= \psi_{(p)} e^{-\frac{i}{\hbar}(p^0ct - p\cdot r)} \\ &= u_{(k)} e^{-ik_\mu x^\mu} \end{aligned} \quad \Rightarrow \quad \begin{aligned} &E'\psi = (\alpha\cdot p + \beta m)\psi \\ &[\gamma^\mu\hbar k_\mu - mc]u_{(k)} = 0 \end{aligned}$$

$$\begin{bmatrix} \hbar k^0 - m & -\sigma\cdot k \\ \sigma\cdot k & -(\hbar k^0 + m) \end{bmatrix}\begin{bmatrix} u_+ \\ u_- \end{bmatrix} = 0 \ \Leftrightarrow\ \begin{aligned} u_\mp &= \frac{\sigma\cdot(\pm p)}{(E'=\pm p^0)\pm m}u_\pm \\ &= \frac{\sigma\cdot p}{E+m}u_\pm \end{aligned}$$

$$\boxed{p = 0 \ \Rightarrow \begin{cases} \psi = \begin{bmatrix} \psi_+ \\ 0 \end{bmatrix}, & E' = m \\ \psi = \begin{bmatrix} 0 \\ \psi_- \end{bmatrix}, & E' = -m \end{cases}}$$

$$\psi_\pm = \tfrac{(\sigma\cdot p)^2}{E^2-m^2}\psi_\pm = \tfrac{p^2}{E^2-m^2}\psi_\pm \ \Rightarrow\ \boxed{E'_\pm = \pm\sqrt{p^2+m^2}}$$

$$\boxed{\int \|\psi_+\|^2 + \|\psi_-\|^2 d^3r = 1}$$

$\underline{\text{Rest Particle Solution, } p=0}$: $i\hbar\gamma^0 \frac{1}{c}\frac{\partial\psi}{\partial t} - \cancel{i\hbar\gamma^i\nabla_i\psi} - mc\psi = 0$

$$\frac{i\hbar}{c}\frac{\partial}{\partial t}\begin{bmatrix} 1_2 & 0 \\ 0 & -1_2 \end{bmatrix}\begin{bmatrix}\psi_+ \\ \psi_-\end{bmatrix} = mc\begin{bmatrix}\psi_+ \\ \psi_-\end{bmatrix} \;\Rightarrow\; \begin{bmatrix}\psi_+ \\ \psi_-\end{bmatrix} = \begin{bmatrix} e^{-\frac{i}{\hbar}mc^2 t}\psi_+(0) \\ e^{\frac{i}{\hbar}mc^2 t}\psi_-(0)\end{bmatrix} \;,\quad \psi_{\pm(0)} \in \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$$

$\underline{\text{Normalized Plane Wave Solution}}$: $\boxed{\begin{matrix} u^\dagger u \equiv \frac{2E}{c} \\ -v^\dagger v = 2mc \end{matrix} \;\Rightarrow\; N = \sqrt{\frac{E}{c} + mc}}$ (one of many conventions)

$$u_\pm \in \begin{bmatrix}1\\0\end{bmatrix}_{=e_1},\ \begin{bmatrix}0\\1\end{bmatrix}_{=e_2} \;\Rightarrow\; \begin{matrix} u^{(1)},\ u^{(2)} \ \text{(part.)} \\ v^{(1)},\ v^{(2)} \ \text{(anti.)} \end{matrix} :\ u^{(i)} = N\begin{bmatrix} u_+ = e^i \\ u_- \end{bmatrix},\quad v^{(1)} = N\begin{bmatrix} \frac{1}{E+m}(\sigma\cdot p)e_2 \\ e^2 \end{bmatrix} \quad (\uparrow \text{ if } p_x = p_y = 0)$$

$$v^{(2)} = -N\begin{bmatrix} \frac{1}{E+m}(\sigma\cdot p)e_1 \\ e^1 \end{bmatrix} \quad (\downarrow \text{ if } p_x = p_y = 0)$$

$\boxed{\psi_{part} = ae^{-\frac{i}{\hbar}p_\mu x^\mu}u\ ,\ \psi_{anti} = ae^{\frac{i}{\hbar}p_\mu x^\mu}v}$

- $(\gamma^\mu p_\mu - m)u = 0 = \overline{u}(\gamma^\mu p_\mu - m)$
- $(\gamma^\mu p_\mu + m)v = 0 = \overline{v}(\gamma^\mu p_\mu + m)$
- $\sum_{s=1,2} u^{(s)}\overline{u}^{(s)} = (\gamma^\mu p_\mu + m)$
- $\sum_{s=1,2} v^{(s)}\overline{v}^{(s)} = (\gamma^\mu p_\mu - m)$

$\underline{\text{Helicity Operator}}$: $(\Sigma\cdot\hat{p})u^\pm = \begin{bmatrix}\sigma\cdot\hat{p} & 0 \\ 0 & \sigma\cdot\hat{p}\end{bmatrix}u^\pm = A\begin{bmatrix}(\sigma\cdot\hat{p})(ae^1+be^2) \\ \frac{|p|}{E+m}(ae^1+be^2)\end{bmatrix} = H_e u^\pm,\quad \boxed{H_e = \pm 1}$

$$u^\pm = A[au^{(1)} + bu^{(2)}] = A\begin{bmatrix} ae^1 + be^2 \\ \frac{(\sigma\cdot p)}{E+m}(ae^1+be^2)\end{bmatrix} \;\Rightarrow\; \boxed{\frac{a}{b} = \frac{H_e|p| + p_z}{p_x + ip_y} = \frac{p_x - ip_y}{H_e|p| - p_z}} \quad \text{(then normalize)}$$

$\boxed{\bullet\ \hat{p} = \hat{z} \;\Rightarrow\; u^\pm = u^{(1.5\mp.5)} \qquad \bullet\ \hat{p} = -\hat{z} \;\Rightarrow\; u^\pm = u^{(1.5\pm.5)}}$

11.1 Transformations and Bilinear Covariants

Adjoint : $\overline{\psi} = \psi^\dagger\gamma^0$ Inner Product (Scalar) : $(\psi,\psi) \equiv \overline{\psi}\psi = \psi^\dagger\gamma^0\psi$ (keeps Lorentz invariance)

$\underline{\gamma \text{ Lorentz x-Boost}}$, $S\psi$: $i\hbar\gamma^\mu\partial_{\mu'}S\psi = mcS\psi$

- $x^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu$ • $\overline{S\psi}S\psi = \overline{\psi}\psi$ (scalar)
- $S\gamma^\nu = \left[\Lambda^{-1}\right]^\nu{}_{\mu'}\gamma^\mu S$ (use to check S)

* $S^{\pm 1} = a_+ \pm a_-\gamma^0\gamma^1 = \begin{bmatrix} a_+\mathbb{I}_2 & \pm a_-\sigma_1 \\ \pm a_-\sigma_1 & a_+\mathbb{I}_2\end{bmatrix} = [S^{\pm 1}]^\dagger$
* $a_\pm = \pm\sqrt{\frac{\gamma\pm 1}{2}}$ * $[S^{\pm 1}]^2 = \gamma\mathbb{1}_4 \mp \gamma\beta\gamma^0\gamma^1$
* $S\gamma^{0,1} = \gamma^{0,1}S^{-1}$ * $S^{\pm 1}\gamma^{2,3,5} = \gamma^{2,3,5}S^{\pm 1}$

$\underline{\text{Parity Flip}}$, $\gamma^0\psi$: $i\hbar\gamma^\mu\partial_{\mu'}P\psi = mcP\psi$

- $x^{\mu'} = (x^0, -x^1, -x^2, -x^3) = x_\mu$
- $P\gamma^\nu = \frac{\partial x^\nu}{\partial x^{\mu'}}\gamma^\mu P$ (use to check $\underline{P = e^{i\theta}\gamma^0}$)

* Rest: $\gamma^0 u = u,\ \gamma^0 v = -v$

- $\overline{\gamma^0\psi}\gamma^0\psi = \overline{\psi}\psi$ (scalar)

Pseudoscalar, $s = \overline{\psi}\gamma^5\psi$:

- $s' = \overline{\gamma^0\psi}\gamma^5\gamma^0\psi = \psi^\dagger\gamma^5\gamma^0\psi$

  $\boxed{-s} = -\overline{\psi}\gamma^5\psi = -\psi^\dagger\gamma^0\gamma^5\psi$
- $s' = \overline{S\psi}\gamma^5 S\psi = \overline{\psi}\gamma^5 S^{-1}S\psi = s$

Vector, $v^\mu = \overline{\psi}\gamma^\mu\psi$

- $v^{\mu'} = \overline{\gamma^0\psi}\gamma^\mu\gamma^0\psi = v_\mu$
- $v^{\mu'} = \overline{S\psi}\gamma^\mu S\psi = \psi^\dagger\gamma^0\gamma^\mu S^{\pm 1}S\psi = \Lambda^{\mu'}{}_\mu v^\mu$

Pseudovector, $\overline{\psi}\gamma^\mu\gamma^5\psi$

Antisym. Tensor, $\overline{\psi}\sigma^{\mu\nu}\psi$

* $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] = i\gamma^\mu\gamma^\nu = \begin{bmatrix} 0 & i\sigma_x \\ i\sigma_x & 0\end{bmatrix}_{t,x},\ \sigma^{ij} = \Sigma^k$
* $\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$

$\underline{\text{Charge Conjugation, } \psi_c = i\gamma^2\psi^*}$: $i\gamma^2\psi^*{}_{[u^{(i)}]} = \psi_{[v^{(i)}]}$

$\underline{\text{Charge Conjugation, } i\gamma^2\psi^*}$: $\psi_c i\gamma^2\psi^*[u^{(i)}] = \psi[v^{(i)}]$

- $\underline{\text{Majorana Particles}}$: $\psi = i\gamma^2\psi^* \;\Rightarrow\; i\gamma^2\begin{bmatrix}\psi_A\\ \psi_B\end{bmatrix}^* = \begin{bmatrix} i\sigma_2\psi_B^* = \underline{\chi = \psi_A} \\ \boxed{-i\sigma_2\psi_A^* = \psi_B}\end{bmatrix}$

$*$ $i\hbar\gamma^\mu\partial_\mu\psi = mc\psi \;\Rightarrow\; \boxed{i\hbar(\partial_0\chi + i(\sigma\cdot\nabla)\sigma_2\chi^*) = mc\chi}$ $\quad *$ $S\psi_c = Si\gamma^2\psi^* = i\gamma^2(S\psi)^*$

$*$ $\underline{\text{Plane Wave Solutions}}$: $\psi = a_1u^{(1)} + a_2u^{(2)} + a_3v^{(1)} + a_4v^{(2)}$ (choose $a_1, a_2 = 1,0;\ 0,1$)

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\left[a_1^*e^1w^+ + a_2^*e^2w^+ + [\dots]w^-\right] = [\dots]w^- + a_3e_2w^+ - a_4e_1w^+ \qquad w^\pm = e^{\pm\frac{i}{\hbar}p^\mu x_\mu}$$

$$a_1^*e^2w^+ - a_2^*e^1w^+ = a_3e_2w^+ - a_4e_1w^+ \;\Rightarrow\; \boxed{\psi^{(i)} \equiv u^{(i)}w^- + v^{(i)}w^+}$$ (then normalize, then find χ)

11.2 Gamma and Trace Identities

- $\underline{\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu}$ $\quad *$ $\underline{\gamma^0\gamma^{\mu\dagger}\gamma^0c_\mu = \gamma^\mu c_\mu}$ $\quad *$ $\Gamma = \gamma^a\gamma^b\dots\gamma^c \;\Rightarrow\; \overline{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0 = \underline{\gamma^c\dots\gamma^b\gamma^a}$
- $g_{\mu\nu}g^{\mu\nu} = 4$ $\quad\bullet$ $\gamma_\mu\gamma^\mu = 4$ $\quad\bullet$ $\boxed{\gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu} \;\Rightarrow\; \not{a}\not{b} + \not{b}\not{a} = 2a^\mu b_\mu$
- $\gamma_\mu\gamma^\nu\gamma^\mu = \gamma_\mu\left[2g^{\nu\mu} - \gamma^\mu\gamma^\nu\right] \quad = -2\gamma^\nu \quad\Rightarrow\; \gamma_\mu\not{a}\gamma^\mu = ...$
- $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = \gamma_\mu\gamma^\nu\left[2g^{\lambda\mu} - \gamma^\mu\gamma^\lambda\right] \quad = 4g^{\nu\lambda} \quad\Rightarrow\; \gamma_\mu\not{a}\not{b}\gamma^\mu = ...$
- $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = \gamma_\mu[2g^{\nu\lambda} - \gamma^\lambda\gamma^\nu]\,[2g^{\sigma\mu} - \gamma^\mu\gamma^\sigma] = \cancel{2g^{\nu\lambda}[2\gamma^\sigma - 4\gamma^\sigma]} - 2\gamma^\sigma\gamma^\lambda\gamma^\nu + \cancel{4g^{\nu\lambda}\gamma^\sigma} \;\Rightarrow\; \gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = ...$

$\underline{\text{4D Levi-Cevita}}$, $\epsilon^{\mu\nu\lambda\sigma} = \pm1^{even}_{odd}, 0_{repeat}$

- $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\lambda\sigma} = 24\cdot -1$ $\quad\bullet$ $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\lambda\tau} = 6\cdot -1\delta^\sigma_\tau$ $\quad\bullet$ $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\theta\tau} = 2\cdot\left[-\delta^\lambda_\theta\delta^\sigma_\tau + \delta^\lambda_\tau\delta^\sigma_\theta\right]$
- $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\phi\theta\tau} = 1\cdot\left[-\delta^\nu_\phi\delta^\lambda_\theta\delta^\sigma_\tau + \delta^\nu_\phi\delta^\lambda_\tau\delta^\sigma_\theta - \delta^\nu_\tau\delta^\lambda_\phi\delta^\sigma_\theta + \delta^\nu_\tau\delta^\lambda_\theta\delta^\sigma_\phi - \delta^\nu_\theta\delta^\lambda_\tau\delta^\sigma_\phi + \delta^\nu_\theta\delta^\lambda_\phi\delta^\sigma_\tau\right]$
- $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\omega\phi\theta\tau} = 1\cdot\left[-\delta^\mu_\omega\delta^\nu_\phi\delta^\lambda_\theta\delta^\sigma_\tau + \dots\right]$

$\underline{\text{Trace}}$

- $\mathrm{Tr}(AB) = A^{\mu\nu}B_{\nu\mu} = B_{\nu\mu}A^{\mu\nu} = \mathrm{Tr}(BA)$ $\quad\bullet$ $\mathrm{Tr}(1) = 4$
- $\mathrm{Tr}(A = \underbrace{\gamma^a\dots\gamma^c}_{2n+1}) = \mathrm{Tr}(\gamma^5\gamma^5A) = -\mathrm{Tr}(\gamma^5A\gamma^5) = -\mathrm{Tr}(A) = \boxed{0}$
- $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(2g^{\mu\nu} - \gamma^\nu\gamma^\mu) = 2(4)g^{\mu\nu} - \mathrm{Tr}(\gamma^\mu\gamma^\nu) = \underline{4g^{\mu\nu} \;\Rightarrow\; \mathrm{Tr}(\not{a}\not{b})}$
- $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = \mathrm{Tr}(\left[2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right]\gamma^\lambda\gamma^\sigma) = \mathrm{Tr}(2g^{\mu\nu}\gamma^\lambda\gamma^\sigma - \gamma^\nu\left[2g^{\mu\lambda} - \gamma^\lambda\gamma^\mu\right]\gamma^\sigma)$
  $= \mathrm{Tr}(2g^{\mu\nu}\gamma^\lambda\gamma^\sigma - 2g^{\mu\lambda}\gamma^\nu\gamma^\sigma + \gamma^\nu\gamma^\lambda\left[2g^{\mu\sigma} - \gamma^\sigma\gamma^\mu\right]) = \underline{4\left[g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right] \;\Rightarrow\; \mathrm{Tr}(\not{a}\not{b}\not{c}\not{d})}$

- $\mathrm{Tr}(\gamma^5) = 0$ • $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(i\gamma^\lambda\gamma^\sigma\epsilon^\pm) = g^{\lambda\sigma} = \underline{0 \ \Rightarrow \ \mathrm{Tr}(\gamma^5 \not{a}\not{b})}$
- $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = \underline{4i\epsilon^{\mu\nu\lambda\sigma} \ \Rightarrow \ \mathrm{Tr}(\gamma^5 \not{a}\not{b}\not{c}\not{d})}$

- $\mathrm{Tr}(\gamma^\mu\gamma^\nu \left[1-\gamma^5\right] \gamma^\lambda \left[1+\gamma^5\right] \gamma_\lambda) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\lambda \left[1-\gamma^5\right]^2) = 8\mathrm{Tr}(\gamma^\mu\gamma^\nu \left[1-\gamma^5\right]) = \underline{32g^{\mu\nu}}$
- $\mathrm{Tr}([\not{p} + m_p c][\not{q} + M_q c][\not{p} + m_p c][\not{q} + M_q c]) = \mathrm{Tr}([\not{p}\not{q} + \not{p}m + \not{q}M + mM]^2)$

  $= \mathrm{Tr}(\not{p}\not{q}\not{p}\not{q} + \not{p}\not{q}mM + \not{p}^2 m^2 + \not{q}^2 M^2 + m^2 M^2 + 0 \ldots)$

  $= \underline{\mathrm{Tr}(\not{p}\not{q}\not{p}\not{q}) + mMp^\mu q_\mu + m^4 + M^4 + 4m^2M^2 = \ldots}$

11.3 Hydrogen Fine Structure

$$\begin{array}{l} E\psi = H\psi \\ E\psi = (\alpha\cdot p + \beta m + q\phi)\psi \end{array} \Rightarrow \begin{array}{l} (E-m-V)\psi_+ = (\sigma\cdot p)\psi_- \\ (E+m-V)\psi_- = (\sigma\cdot p)\psi_+ \end{array}$$

$$\begin{aligned}
(E-m-V)\psi_+ &= (\sigma\cdot p)\left(\tfrac{1}{E+m-V}\right)(\sigma\cdot p)\psi_+ \\
(E_s - V)\psi_+ &= \tfrac{1}{2m}(\sigma\cdot p)\left(1+\tfrac{E_S-V}{2m}\right)^{-1}(\sigma\cdot p)\psi_+ \\
&\approx \tfrac{p^2}{2m}\psi_+ \qquad (1^{\text{st}}\text{ order, } v^2) \\
&\approx \tfrac{p^2}{2m}\psi_+ - \tfrac{\sigma\cdot p}{(2m)^2}\left(E_S - V\right)(\sigma\cdot p)\psi_+ \qquad (2^{\text{nd}}\text{ order, } v^4) \\
&= \tfrac{p^2}{2m}\psi_+ - \tfrac{\sigma\cdot p}{(2m)^2}\Big[(\sigma\cdot p)\overbrace{(E_S-V)\psi_+}^{1^{\text{st}}\text{ order}} + [\cancel{E_S} - V, \sigma\cdot p]\,\psi_+\Big] \\
&\approx \left[\tfrac{p^2}{2m} - \tfrac{p^2}{(2m)^2}\tfrac{p^2}{2m} - \tfrac{(\sigma\cdot p)(\sigma\cdot[p,V])}{(2m)^2}\right]\psi_+ \\
E_S\psi_+ &= \left[\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} - \tfrac{i\sigma\cdot(p\times[p,V])}{4m^2} - \underbrace{\tfrac{p[p,V]}{4m^2}}_{\text{(isn't Hermitian)}}\right]\psi_+ = H\psi_+
\end{aligned}$$

$$\begin{aligned}
1 &= \int \|\psi_+\|^2 + \|\psi_-\|^2\, d^3r \\
&= \int \|\psi_+\|^2 + \|\tfrac{\sigma\cdot p}{E+m-V}\psi_+\|^2\, d^3r \\
&\approx \int \|\psi_+\|^2 + \|\tfrac{\sigma\cdot p}{2m}\psi_+\|^2\, d^3r \\
&= \int \psi_+^\dagger(1+\tfrac{p^2}{4m^2})\psi_+\, d^3r \\
&\approx \Big\langle (1+\tfrac{p^2}{8m^2})\psi_+ \Big| (1+\tfrac{p^2}{8m^2})\psi_+ \Big\rangle \\
&\equiv \langle \psi_S | \psi_S \rangle
\end{aligned}$$

$$\begin{aligned}
E_S\left(1+\tfrac{p^2}{8m^2}\right)^{-1}\psi_S &= H\left(1+\tfrac{p^2}{8m^2}\right)^{-1}\psi_S \\
E_S\psi_S &= \left(1+\tfrac{p^2}{8m^2}\right)H\left(1+\tfrac{p^2}{8m^2}\right)^{-1}\psi_S \\
&= \left(H + \tfrac{p^2 H}{8m^2}\right)\left(1 - \tfrac{p^2}{8m^2} + \mathcal{O}(p^4)\right)\psi_S \\
&\approx \left(H + \left[\tfrac{p^2}{8m^2}, H\right]\right)\psi_S \approx \left(H + \left[\tfrac{p^2}{8m^2}, V\right]\right)\psi_S \qquad (2^{\text{nd}}\text{ order, } v^4) \\
E_S\psi_S &= \left(\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} - \tfrac{i\sigma\cdot(p\times[p,V])}{4m^2} - \tfrac{p[p,V]}{4m^2} + \tfrac{[p,V]p + p[p,V]}{8m^2}\right)\psi_S \\
&= \left(\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} - \tfrac{i\sigma\cdot(p\times[p,V])}{4m^2} - \tfrac{[p,[p,V]]}{8m^2}\right)\psi_S \\
&= \boxed{(H_S + H_{\text{rel.}} + H_{\text{so}} + H_{\text{darwin}})\psi_S} \\
&= \left(\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} - \tfrac{1}{4m^2}\sigma\cdot(p\times\nabla V) + \overbrace{\tfrac{1}{8m^2}\nabla^2 V}^{\text{Darwin}\Rightarrow}\right)\psi_S \\
&= \left(\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} - \tfrac{1}{2m^2}S\cdot[\vec{p}\times\tfrac{qq\vec{r}}{4\pi r^3}] + \tfrac{1}{8m^2}[qq\delta^3(r)]\right)\psi_S \\
&= \left(\tfrac{p^2}{2m} + V - \tfrac{p^4}{8m^3} + \underbrace{\tfrac{e^2}{8\pi m^2}\tfrac{S\cdot L}{r^3}}_{l\neq 0} + \underbrace{\tfrac{e^2}{8m^2}\delta^3(r)}_{l=0}\right)\psi_S
\end{aligned}$$

$$\begin{aligned}
\overline{V(r)} &= V(r) + \sum_i \cancel{\overline{\tfrac{\partial V}{\partial r_i}\delta r_i}} \\
&\quad + \tfrac{1}{2!}\sum_{ij}\overline{\tfrac{\partial^2 V}{\partial r_i \partial r_j}\delta r_i \delta r_j} \\
&\quad + \mathcal{O}(\delta r^3) \\
&= V(r) + \underline{\tfrac{1}{2}(\delta r)^2 \nabla^2 V} + .. \\
&\qquad (\delta r \sim \tfrac{\hbar}{mc})
\end{aligned}$$

Exact Energy Eigenvalues : $E_{nj} = mc^2\left[1 + \left(\dfrac{\alpha}{n-(j+1/2)+\sqrt{(j+1/2)^2-\alpha^2}}\right)^2\right]^{-1/2}$