1 Curvilinear Coordinates

$$\boxed{\begin{aligned}\vec r &= r\cos\phi\sin\theta\,\hat x + r\sin\phi\sin\theta\,\hat y + r\cos\theta\,\hat z\\ r\hat r &= x\hat x + y\hat y + z\hat z\end{aligned}}$$

$$\begin{aligned}
\hat r &= \frac{\partial}{\partial r}\vec r &= \boxed{\frac{\vec r}{r}} &= \nabla r &= \frac{\nabla r}{\|\nabla r\|}\\
\hat\theta &= \frac{1}{r}\frac{\partial}{\partial\theta}\vec r &= \boxed{\frac{\partial\hat r}{\partial\theta}} &= r\nabla\theta &= \frac{\nabla\theta}{\|\nabla\theta\|}\\
\hat\phi &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\vec r &= \boxed{\frac{1}{\sin\theta}\frac{\partial\hat r}{\partial\phi}} &= r\sin\theta\nabla\phi &= \frac{\nabla\phi}{\|\nabla\phi\|}
\end{aligned}$$

$$\begin{aligned}
\cos\theta\,\hat r - \sin\theta\,\hat\theta &= \cos 2\theta\,\hat z\\
\sin\phi\,\hat r + \cos\phi\,\hat\phi &= \sin\theta\,\hat y + \sin\phi\cos\theta\,\hat z\\
\hat\phi &= -\sin\phi\,\hat x + \cos\phi\,\hat y
\end{aligned} \quad\Rightarrow$$

$$\begin{aligned}
\hat z &= \boxed{\tfrac{\cos\theta\,\hat r - \sin\theta\,\hat\theta}{\cos 2\theta}}\\
\hat y &= \tfrac{\sin\phi\,\hat r + \cos\phi\,\hat\phi}{\sin\theta} - \tfrac{\sin\phi\cos\theta}{\sin\theta\cos 2\theta}\left[\cos\theta\,\hat r - \sin\theta\,\hat\theta\right]\\
&= \boxed{-\tfrac{\sin\phi\sin\theta}{\cos 2\theta}\hat r + \tfrac{\sin\phi\cos\theta}{\cos 2\theta}\hat\theta + \tfrac{\cos\phi}{\sin\theta}\hat\phi}\\
\hat x &= \boxed{\cot\phi\,\hat y - \tfrac{\hat\phi}{\sin\phi}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat r}{dt} &= \frac{d}{dt}\left(\frac{\vec r}{r}\right) &= \frac{1}{r}\left(\frac{d\vec r}{dt} - \frac{dr}{dt}\hat r\right) &= \boxed{\frac{v}{r}\left[\hat v - (\hat r\cdot\hat v)\hat r\right]}\\
&= \frac{d\theta}{dt}\frac{\partial\hat r}{\partial\theta} &+ \frac{d\phi}{dt}\frac{\partial\hat r}{\partial\phi} &= \boxed{\frac{d\theta}{dt}\hat\theta + \sin\theta\frac{d\phi}{dt}\hat\phi}\\
\frac{d\hat\theta}{dt} &= \frac{d\theta}{dt}\frac{\partial}{\partial\theta}\left(\frac{\partial\hat r}{\partial\theta}\right) &+ \frac{d\phi}{dt}\frac{\partial}{\partial\phi}\left(\frac{\partial\hat r}{\partial\theta}\right) &= \boxed{-\frac{d\theta}{dt}\hat r + \cos\theta\frac{d\phi}{dt}\hat\phi}\\
\frac{d\hat\phi}{dt} &= \frac{d\theta}{dt}\frac{\cancel{\partial\phi}}{\cancel{\partial}\theta} &+ \frac{d\phi}{dt}\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\hat r}{\partial\phi}\right) &= -\frac{d\phi}{dt}\underbrace{\mathrm{Proj}_{\mathrm{xy}}\left(\frac{\hat r}{\sin\theta}\right)}_{\cos\phi\hat x + \sin\phi\hat y}\\
&= -\frac{d\phi}{dt}\frac{\hat r - \cos\theta\,\hat z}{\sin\theta} = \boxed{\frac{d\phi}{dt}\frac{\sin\theta\,\hat r - \cos\theta\,\hat\theta}{\cos 2\theta}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\vec r}{dt} &= \frac{dr}{dt}\hat r + r\left(\frac{d\theta}{dt}\frac{\partial\hat r}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat r}{\partial\phi}\right)\\
&\boxed{\vec v = \frac{dr}{dt}\hat r + r\frac{d\theta}{dt}\hat\theta + r\sin\theta\frac{d\phi}{dt}\hat\phi}\\
\frac{d\vec\theta}{dt} &= \frac{d\theta}{dt}\hat\theta + \theta\left(-\frac{d\theta}{dt}\hat r + \cos\theta\frac{d\phi}{dt}\hat\phi\right)\\
\frac{d\vec\phi}{dt} &= \frac{d\phi}{dt}\hat\phi + \phi\frac{d\phi}{dt}\left(\frac{\sin\theta\,\hat r - \cos\theta\,\hat\theta}{\cos 2\theta}\right)\\
&\downarrow\\
\frac{dr}{dt} &= \frac{d}{dt}(\vec r\cdot\vec r)^{\frac{1}{2}} = \boxed{\hat r\cdot\vec v = v_{\|r}}\\
\frac{d\theta}{dt} &= \nabla\theta\cdot\vec v = \boxed{\frac{\hat\theta\cdot\vec v}{r} = \frac{v_{\perp\theta}}{r} = \omega_\theta}\\
\frac{d\phi}{dt} &= \nabla\phi\cdot\vec v = \boxed{\frac{\hat\phi\cdot\vec v}{r\sin\theta} = \frac{v_{\perp\phi}}{r\sin\theta} = \omega_\phi}
\end{aligned}$$

$$\begin{aligned}
\vec L &= \vec r\times\vec p\\
\boxed{m\vec r\times\vec v_\perp} &= \boxed{m\vec r\times\vec v} &= mr^2\left(\frac{d\theta}{dt}\hat\phi - \sin\theta\frac{d\phi}{dt}\hat\theta\right)\\
\underline{mr^2}\frac{1}{r}\hat r\times\vec v &= \boxed{I\underline{\vec\omega}} &= \underline{mr^2}\left[\frac{v}{r}(\hat\theta\cdot\hat v)\hat\phi - \frac{v}{r}(\hat\phi\cdot\hat v)\hat\theta\right]\\
I\frac{1}{\underline{r}}\underline{\hat r\times\vec v_\perp} &= I\underline{\vec\omega} &= I\frac{v}{r}\left[(\hat\theta\cdot\hat v)\hat\phi - (\hat\phi\cdot\hat v)\hat\theta\right]\\
I\frac{v_\perp}{\underline{r}}\hat r\times\widehat{v_\perp} &= I\underline{\omega}\hat\omega &= I\frac{v}{r}(\hat\theta\times\hat\phi = \hat r)\times\hat v
\end{aligned}$$

- $\vec\omega\times\vec r = \frac{1}{r}(\hat r\times\vec v_\perp)\times\vec r$
  $= (\hat r\cdot\hat r)\vec v_\perp - (\vec v_\perp\cdot\hat r)\hat r$
  $= \vec v_\perp$
- $\vec v\times\vec\omega = \frac{v_\perp}{r}(v_\perp\hat v_\perp + v_\|\hat r)\times\hat\omega$
  $= \frac{v_\perp}{r}(v_\perp\hat r - v_\|\widehat{v_\perp})$

$$\|\vec v\times\vec\omega\|^2 = v^2\frac{v_\perp^2}{r^2}$$

$$L_i = \sum_j I_{ij}\omega_j$$

$$E = \frac{\|\vec L\|^2}{2I} = \frac{1}{2}\vec L\cdot\vec\omega$$

$$\overleftrightarrow{I} = \begin{bmatrix}\sum m(y^2+z^2) & -\sum mxy & -\sum mxz\\ -\sum myx & \sum m(x^2+z^2) & -\sum myz\\ -\sum mzx & -\sum mzy & \sum m(x^2+y^2)\end{bmatrix}$$

$$\begin{aligned}\sum_m \tfrac{1}{2}m\|\vec\omega\times\vec r\|^2 &= \tfrac{1}{2}\sum_{ij} I_{ij}\omega_j\omega_i\\ &= \tfrac{1}{2}\begin{bmatrix} & & \\ & I & \\ & & \end{bmatrix}\begin{bmatrix}|\\ \omega\\ |\end{bmatrix}\cdot\begin{bmatrix}|\\ \omega\\ |\end{bmatrix}\end{aligned}$$

$$\frac{d}{dt}\left(\vec{p}\times\vec{L}\right) = \frac{d\vec{p}}{dt}\times\vec{L} = f(r)\hat{r}\times\left(\vec{r}\times m\frac{d\vec{r}}{dt}\right)$$
$$= mf(r)\left[\vec{r}\left(\hat{r}\cdot\frac{d\vec{r}}{dt}\right) - \frac{d\vec{r}}{dt}\left(\hat{r}\cdot\vec{r}\right)\right]$$
$$= mf(r)\left[\hat{r}\,\frac{1}{2}\frac{d}{dt}\left(\vec{r}\cdot\vec{r}\right) - \frac{1}{r}\frac{d\vec{r}}{dt}r^2\right]$$
$$= mf(r)\left[\hat{r}r\frac{dr}{dt} - r\frac{d\vec{r}}{dt}\right]$$
$$= -\frac{mf(r)r}{I(r)}\left[-\frac{I(r)}{r}\frac{dr}{dt}\vec{r} + I(r)\frac{d\vec{r}}{dt}\right]$$
$$= -\frac{mf(r)r}{I(r)}\frac{d}{dt}\left[I(r)\vec{r}\right]$$
$$= -m\cancel{f(r)}r^2\frac{d}{dt}\hat{r} = mk\frac{d}{dt}\hat{r}$$
$$\frac{d}{dt}\left(\frac{\vec{p}\times\vec{L}}{mk} - \hat{r}\right) = \frac{d}{dt}\vec{e}_{\text{ccen}} = 0$$

$$\vec{a} = \left[\ddot{r} - r\dot{\theta}^2 + r\dot{\phi}^2\frac{\sin^2\theta}{\cos 2\theta}\right]\hat{r}$$
$$+\left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\frac{\tan 2\theta}{2}\right]\hat{\theta}$$
$$+\left[2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta\right]\hat{\phi}$$

$$\vec{\tau} = \vec{r}\times\vec{F}$$
$$= (mr^2)\frac{\hat{r}\times\vec{a}}{r} = I\vec{\alpha}$$

$$\tau = \begin{bmatrix}\tau_x\\ \tau_y\\ \tau_z\end{bmatrix}$$

$$\boxed{\frac{dv}{dt} = a(\hat{v}\cdot\hat{a}) = \hat{v}\cdot\vec{a} = \frac{d}{dt}\|\vec{v}\|}$$

$$=^* \begin{bmatrix}0 & \tau_z & -\tau_y\\ -\tau_z & 0 & \tau_x\\ \tau_y & -\tau_z & 0\end{bmatrix}$$

$$\|\vec{q}\times\vec{p}\|^2 = \begin{vmatrix}\vec{q}\cdot\vec{q} & \vec{q}\cdot\vec{p}\\ \vec{p}\cdot\vec{q} & \vec{p}\cdot\vec{p}\end{vmatrix}$$
$$= \vec{q}\cdot\vec{p}\times(\vec{q}\times\vec{p})$$

$$\boxed{\frac{dt}{ds} = \frac{1}{v}}$$

$$T = \hat{v} = \frac{\vec{v}}{v}$$
$$\frac{dT}{dt} = \frac{(\vec{v}\cdot\vec{v})\vec{a} - (\vec{v}\cdot\vec{a})\vec{v}}{v^3} = \frac{\vec{v}\times(\vec{a}\times\vec{v})}{v^3} = \frac{(\vec{v}\times\vec{a})\times\vec{v}}{v^3}$$
$$\left\|\frac{dT}{dt}\right\| = \frac{\sqrt{v^2a^2 - (\vec{v}\cdot\vec{a})^2}}{v^2} = \frac{\|\vec{a}\times\vec{v}\|}{v^2}\ , \quad \frac{dT}{ds} = k\hat{N}$$
$$\hat{N} = \frac{T'}{\|T'\|} = \frac{(\vec{v}\times\vec{a})\times\vec{v}}{\|\vec{v}\times\vec{a}\|v} = \hat{B}\times\hat{v}$$
$$\hat{B} = \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} = \widehat{v\times a} = \hat{v}\times\hat{N} \qquad \underline{(\hat{B}\cdot\vec{v} = 0)}$$
$$\frac{d\hat{B}}{dt} = \frac{\vec{v}\times\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|} - \left[\frac{\vec{v}\times\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|}\cdot\hat{B}\right]\hat{B}\ , \quad \tau = \hat{N}\cdot\frac{d\hat{B}}{ds}$$

$$\boxed{\begin{aligned}\vec{a} &= a_T\hat{T} + a_N\hat{N}\\ a_T &= \vec{a}\cdot\hat{v} = \frac{dv}{dt}\\ a_N &= \frac{\|\vec{a}\times\vec{v}\|}{v} = \|\vec{a}\times\hat{v}\|\\ a^2 &= a_T^2 + a_N^2 = \left\|\frac{d\vec{v}}{dt}\right\|^2\end{aligned}}$$

<u>Frenet Trihedron</u>

Differentiable (in this book) : C^∞

No singular pts. Order 0 (Regular) : $\vec{v}(t)\neq 0$

- $\|\vec{v}(t)\| = c\to 1 \ \Rightarrow \int_s \|\vec{v}(t)\|\,dt = t = \Delta s$

  $\to s:\ \vec{x}(t) = \vec{x}(s)$
- $\frac{1}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \boxed{\vec{v}\cdot\vec{a} = 0}$

No singular pts. Order 1 : $\vec{a}(t)\neq 0$

- Curvature, $k\neq 0$ (see right) • Vertex, $k' = 0$

$$1 = \|\vec{t}\| = \|\vec{n}\| = \|\vec{b}\|\ , \quad 0 = \vec{t}\cdot\vec{n} = \vec{n}\cdot\vec{b} = \vec{b}\cdot\vec{t}$$

- $\vec{v}(s) = \vec{t}(s)$ $\boxed{(t = n\times b)}$
- $\vec{a}(s) = \boxed{\vec{t}'(s) = k(s)\vec{n}(s)}$, $k(s)\geq 0$ (can be L or R-handed) (can be neg. if in $\mathbb{R}^2$)

$*$ $k(s) > 0$ for well defined curve with $\hat{n}$

- $\boxed{\vec{b} = \vec{t}\times\vec{n}}$, $\frac{d}{dt}(\vec{b}\cdot\vec{b}) = \vec{b}\cdot\vec{b}' = 0$, $*$ $\boxed{\vec{b}'(s) = \tau(s)\vec{n}(s)}$
- $\boxed{\vec{n} = \vec{b}\times\vec{t}}$, $*$ $\boxed{\vec{n}'(s) = -k\vec{t} - \tau\vec{b}}$, $*$ t-n pl. = osculating pl.

- $t''(s) = k'n - k^2t - k\tau b$ • $b''(s) = \tau'n - \tau kt - \tau^2 b$ • $n''(s) = -k't - \tau'b - (k^2+\tau^2)n$
- $|\tau| = \|b'\|$ • $\tau = -\frac{(t\times t')\cdot t''}{k^2} = -\frac{t\cdot(t'\times t'')}{\|t'\|^2}$ • $k = \|t'\| = \frac{(b\times b')\cdot b''}{\tau^2} = \frac{b\cdot(b'\times b'')}{\|b'\|^2}$
- $n\Rightarrow k,\tau$: $*$ $\|n'\|^2 = k^2+\tau^2$ $*$ $\frac{(n\times n')\cdot n''}{\|n'\|^2} = \frac{k'\tau - k\tau'}{k^2+\tau^2} = \frac{\frac{d}{ds}(k/\tau)}{(k/\tau)^2+1} = \frac{d}{ds}\arctan(k/\tau)$

2 Lagrangian Equations

$$\boxed{\begin{array}{l} \mathcal{L} = T - U \ , \qquad p_i \equiv \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} \\[2ex] \rightarrow \ F_i \equiv \dfrac{dp_i}{dt} = \dfrac{\partial \mathcal{L}}{\partial q_i} \end{array}}$$

<u>Newton's Laws</u>:

$$\mathcal{L} = \tfrac{1}{2} m \dot{\mathbf{r}}^2 - U(\mathbf{r}) \ , \qquad \vec{p}_r = m\dot{\mathbf{r}}$$

$$\rightarrow \ \boxed{F = m\ddot{\mathbf{r}} = -\nabla U}$$

<u>Angular</u>:

$$\mathcal{L} = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\phi}^2 - U(r, \phi) \ , \qquad \begin{array}{l} p_r = m\dot{r} \\ p_\phi = m r^2 \dot{\phi} = I\omega = I\frac{v_\perp}{r} \end{array} , \qquad \begin{array}{rcl} -\vec{F} &=& \nabla U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi} \\ \vec{F} &=& m\ddot{\mathbf{r}} = (F\cdot\hat{r})\hat{r} + (F\cdot\hat{\phi})\hat{\phi} \end{array}$$

$$F_r = \boxed{-\tfrac{\partial U}{\partial r} + m r \dot{\phi}^2 = m\ddot{r}} \qquad \left(\text{centripetal: } \tfrac{mv^2}{r} = m r \omega^2\right)$$

$$\rightarrow \quad F_\phi = \boxed{\underbrace{-\tfrac{\partial U}{\partial \phi}}_{\tau} = \underbrace{m r^2 \ddot{\phi}}_{I\alpha} + \underbrace{2 m r \dot{r} \dot{\phi}}_{\dot{I}\omega}} \qquad \left(\begin{array}{c}\text{“coriolis”:} \\ 2m|\vec{\omega}\times\vec{v}| = 2m\dot{\phi}\dot{r}\end{array}\right)$$

$$\dot{\phi}' = \dot{\phi} - \omega \ , \quad r' = r \qquad \left(\hat{r} = R(\omega)\hat{r}' \ , \quad \hat{\phi} = R(\omega)\hat{\phi}'\right)$$

$$\downarrow$$

$$m\ddot{\mathbf{r}} = m\ddot{\mathbf{r}}' - (m r \omega^2 + 2 m r \dot{\phi}\omega)\hat{r} + (2 m \dot{r}\omega + m r \dot{\omega})\hat{\phi}$$

$$m\ddot{\mathbf{r}}' = m\ddot{\mathbf{r}} + \underbrace{m r \omega^2 \hat{r}}_{\text{centrifugal force}} + \underbrace{2 m \omega (r\dot{\phi}\hat{r} - \dot{r}\hat{\phi})}_{\text{coriolis force}} - \underbrace{m r \dot{\omega}\hat{\phi}}_{\text{Euler force}}$$

Note: $\boxed{\begin{array}{l} \dot{\hat{r}} = \dot{\phi}\hat{\phi} \\ \dot{\hat{\phi}} = -\dot{\phi}\hat{r} \end{array}}$ $\rightarrow$

$$\begin{aligned} \vec{r} &= r\hat{r} = r\cos\phi\,\hat{x} + r\sin\phi\,\hat{y} \\ \dot{\vec{r}} &= \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \\ \ddot{\vec{r}} &= \ddot{r}\hat{r} + 2\dot{r}\dot{\hat{r}} + r\ddot{\hat{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi} \end{aligned}$$

<u>Electromagnetic</u>:

$$\mathcal{L} = \tfrac{1}{2} m \dot{\mathbf{r}}^2 - q\Big[V(t,\ \mathbf{r}) - \dot{\mathbf{r}}\cdot\vec{A}(t,\ \mathbf{r})\Big] \ , \qquad p_x = m\dot{x} + qA_x$$

$$\rightarrow \qquad m\ddot{x} + q\tfrac{dA_x}{dt} = -q\left[\tfrac{\partial V}{\partial x} - \dot{r}\cdot\tfrac{\partial \vec{A}}{\partial x}\right]$$

$$m\ddot{x} + q\left[\tfrac{\partial A_x}{\partial t} + \dot{r}\cdot\nabla A_x\right] = q\left[-\tfrac{\partial V}{\partial x} + \dot{r}\cdot\tfrac{\partial \vec{A}}{\partial x}\right]$$

$$\begin{aligned} \rightarrow \quad m\ddot{x} &= q\left(-\tfrac{\partial V}{\partial x} - \tfrac{\partial A_x}{\partial t} + \dot{r}\cdot\left[\tfrac{\partial \vec{A}}{\partial x} - \nabla A_x\right]\right) \\ &= q\left[-\tfrac{\partial V}{\partial x} - \tfrac{\partial A_x}{\partial t}\right] + q\dot{y}\left[\tfrac{\partial A_y}{\partial x} - \tfrac{\partial A_x}{\partial y}\right] \\ &\quad + q\dot{z}\left[\tfrac{\partial A_z}{\partial x} - \tfrac{\partial A_x}{\partial z}\right] \\ &= qE_x + q\dot{y}B_z - q\dot{z}B_y \\ m\ddot{x} &= qE_x + q\left[\dot{\mathbf{r}}\times\vec{B}\right]_x \end{aligned}$$

$$\downarrow$$

$$\boxed{m\ddot{\mathbf{r}} = q\left(\vec{E} + \dot{\mathbf{r}}\times\vec{B}\right)}$$

Special Relativity:

$$\mathcal{L} = -\tfrac{1}{\gamma} mc^2 - U \ , \qquad\qquad \vec{p} = \gamma m \vec{v} \ \rightarrow \ \gamma m \dot{x} = \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$= \gamma m v^2 - \gamma m c^2 - U$$

$$= m\left(v^2 - c^2\right)\left(1 - \tfrac{v^2}{c^2}\right)^{-1/2} - U$$

$$\approx \tfrac{1}{2} m v^2 - (U + mc^2) \qquad \text{(when } v \ll c\text{)}$$

Conservation of Energy:

$$\frac{d\mathcal{L}}{dt} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \right) + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \sum_i \left(\dot{p}_i \dot{q}_i + p_i \ddot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} \qquad \rightarrow \qquad \frac{\partial \mathcal{L}}{\partial t} = -\frac{d}{dt}\left(\sum_i p_i \dot{q}_i - \mathcal{L} \right)$$

$$= \frac{d}{dt}\left(\sum_i p_i \dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} \qquad\qquad = -\frac{d\mathcal{H}}{dt}$$

If $\mathcal{L}$ is explicitly independent of time (implies coordinates are "natural"), then the Hamiltonian is conserved.

$$\tfrac{1}{2}\sum_n m \dot{r}_n^2 = \tfrac{1}{2}\sum_n m \left(\sum_i \frac{\partial r_n}{\partial q_i}\dot{q}_i \right)^2$$

$$= \tfrac{1}{2}\sum_{i,j} \left(m \sum_n \frac{\partial r_n}{\partial q_i}\frac{\partial r_n}{\partial q_j} \right) \dot{q}_i \dot{q}_j$$

$$= \tfrac{1}{2}\sum_i \sum_j A_{ij} \dot{q}_i \dot{q}_j$$

$$\left(\text{for } \tfrac{\partial T}{\partial \dot{q}_i}\right) \qquad = \tfrac{1}{2}\left(2\sum_{i \neq j} A_{ij}\dot{q}_i\dot{q}_j + A_{ii}\dot{q}_i^2 \right) + \ldots$$

$$\rightarrow$$

$$\mathcal{L} = \tfrac{1}{2} m v^2 - U \ = \ T(\dot{q}_i) - U(q_i) \ \rightarrow$$

$$\mathcal{H} = \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}$$

$$= \sum_i \left(\sum_j A_{ij} \dot{q}_j \right) \dot{q}_i - \tfrac{1}{2} m \dot{\mathbf{r}}^2 + U$$

$$= \tfrac{1}{2} m \dot{\mathbf{r}}^2 + U$$

If $\mathcal{L} = \frac{1}{2}mv^2 - U$ and U is independent of v, then the Hamiltonian is the total energy.

Lagrange Multipliers:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i}$$

$$\frac{dp}{dt} = -\nabla U + \lambda \nabla f$$

$$F_{\text{tot}} = F_{\text{ncnstr}} + F_{\text{cnstr}}$$

2.1 Examples

Atwood's Machine (Pulley):

Particle Confined to a Cylinder Surface:

Block Sliding on Wedge:

Bead on Spinning Wire Hoop:

Oscillations of Bead Near Equilibriuum:

3 Hamiltonian

$$\boxed{\begin{aligned} &\mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L}\ , \qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}} \\ &\rightarrow \quad \begin{array}{l} \bullet\ \dfrac{dp_i}{dt} = -\dfrac{\partial \mathcal{H}}{\partial q_i} \\ \bullet\ \dfrac{dq_i}{dt} = \dfrac{\partial \mathcal{H}}{\partial p_i} \end{array} \end{aligned}}$$

Newton Particle:

$$\begin{aligned} \mathcal{H} &= \dot{x}(m\dot{x}) - \tfrac{1}{2}m\dot{x}^2 + U(x) \\ &= \tfrac{1}{2}m\dot{x}^2 + U(x) \\ &= T + U \end{aligned}$$

Angular:

$$\mathcal{H} = m\dot{r}^2 + mr^2\dot{\theta}^2 - \left(\tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - U(r,\theta)\right)\ , \qquad \begin{aligned} p_r &= m\dot{r} \\ p_\theta &= mr^2\dot{\theta} \ \equiv \ L = I\omega \end{aligned}$$

$$= \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + U(r,\theta)$$

Electromagnetic:

$$\begin{aligned} \mathcal{H} &= \dot{\mathbf{r}} \cdot \vec{p}_r - \left(\tfrac{1}{2}m\dot{\mathbf{r}}^2 - q\phi(t,\ \mathbf{r}) + q\dot{\mathbf{r}} \cdot \vec{A}(t,\ \mathbf{r})\right)\ , \qquad \vec{p}_r = m\dot{\mathbf{r}} + q\vec{A} \\ &= m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \vec{A} - \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\phi - q\dot{\mathbf{r}} \cdot \vec{A} \\ &= \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\phi \end{aligned}$$

Special Relativity:

$$\begin{aligned} \mathcal{H} &= \vec{v} \cdot (\gamma m\vec{v}) - \left(\gamma mv^2 - \gamma mc^2 - U\right)\ , \qquad \vec{p} = \gamma m\vec{v} \\ &= \gamma mc^2 + U \\ &\approx \tfrac{1}{2}mv^2 + \left(U + mc^2\right) \qquad \text{(when } v \ll c\text{)} \end{aligned}$$

4 Hamilton-Jacobi Equations

$$\boxed{K(Q,P,t) \equiv H(q,p,t) + \frac{\partial S_{(q,Q,t)}}{\partial t} = 0}$$

$$\dot{Q} = \frac{\partial K}{\partial P} = 0 \quad \Rightarrow \quad \boxed{Q = \alpha_Q = \frac{\partial S}{\partial P}} \quad \text{(constant)}$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \quad \Rightarrow \quad \boxed{P = \alpha_P = -\frac{\partial S}{\partial Q}} \quad \text{(constant)}$$

$$\dot{q} = \frac{\partial H}{\partial p} \quad \Rightarrow \quad \boxed{q = -\frac{\partial S}{\partial p}} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q} \quad \Rightarrow \quad \boxed{p = \frac{\partial S}{\partial q}}$$

<u>Solve for $S_{(q,\alpha_Q,t)}$</u> ($n+1$ variables, nonlinear PDE)

$$H\left(q, \frac{\partial S(q,\alpha_Q,t)}{\partial q}, t\right) + \frac{\partial S_{(q,\alpha_Q,t)}}{\partial t} = 0$$

$$\underline{\frac{\partial H}{\partial t} = 0} \quad \Rightarrow \quad \boxed{S(q,Q,t) = W(q,Q) - Et}$$

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum p_i \dot{q}_i = \cancel{\frac{\partial S}{\partial t} + H} + \mathcal{L}$$

$$\Rightarrow \quad \boxed{S = \int \mathcal{L}\ dt + \text{const.}}$$

<u>Solve for $W_{(q,\alpha_Q)}$</u> (n variables, nonlinear PDE)

$$H(q, \tfrac{\partial W}{\partial q}) = E \equiv \alpha_Q$$

<u>Harmonic Oscillator</u>

$$-\frac{\partial S}{\partial t} = \frac{1}{2}p^2 - \frac{1}{2}\omega^2 q^2$$

$$= \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{1}{2}\omega^2 q^2 \qquad \big(S = s_1(q) + s_2(t)\big)$$

$$-\frac{\partial s_2(t)}{\partial t} = \frac{1}{2}\left(\frac{\partial s_1(q)}{\partial q}\right)^2 - \frac{1}{2}\omega^2 q^2 \equiv \alpha_Q$$

$$s_2(t) = -\alpha_Q t + \cancel{\text{const.}} \quad , \quad s_1(q) = \int \sqrt{2a_Q + \omega^2 q^2}\ dq$$

$$Q \equiv \alpha_Q \quad , \quad P = -\int \frac{dq}{\sqrt{2a_Q + \omega^2 q^2}} + t$$

$$\alpha_P = -\frac{1}{\omega}\sin^{-1}\left[q\frac{\omega}{\sqrt{2\alpha_Q}}\right] + t$$

$$\boxed{q(t) = \frac{\sqrt{2\alpha_Q}}{\omega}\sin\left[\omega(t - \alpha_P)\right]}$$

5 Kinematics

Elastic Collisions:
$$m_0 v_0 = m_1 v_1 + m_2 v_2$$
$$\tfrac{1}{2} m_0 v_0^2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

$$\Rightarrow \boxed{\tfrac{1}{2} m_2 v_2^2 (m_1 + m_2) - \tfrac{1}{2} m_0 v_0^2 (m_1 - m_0) = (m_0 v_0)(m_2 v_2)}$$

- $$\boxed{\begin{array}{l} m v_0 \;=\; m v_1 + M v_2 \;=\; m v_0 \left(1 - \frac{2M}{m+M}\right) + M v_0 \left(\frac{2m}{m+M}\right) \\ \to\; M \in (\infty,\ m,\ 0] \;\Rightarrow\; v_1 \in (-v_0,\ 0,\ v_0] \end{array}}$$

Inelastic Collision: $E_0 = \frac{1}{2} m v_0^2$

- $$\boxed{\begin{array}{l} m v_0 = (m + M) v_1 \\ \to\; E_1 = \left(\frac{m}{m+M}\right) E_0 \end{array}}$$

6 Orbits

$$\left.\begin{array}{l} \text{Lagrangian}: \quad \mathcal{L} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m r^2 \sin^2\theta \dot{\phi}^2 - U(r) \\ \bullet\; l = I\omega = m r^2 \dot{\theta} \\ \bullet\; m\ddot{r} = -\dfrac{\partial}{\partial r} U_{\text{eff}} = -\dfrac{\partial}{\partial r}\left(\dfrac{l^2}{2mr^2} + U(r)\right) \end{array}\right\} \quad m \;\to\; \mu = \frac{mM}{m+M}$$

Hamiltonian: $E = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + U(r)$

- Inf. Energy to get to $r = 0$ unless $l = 0$
- $U \sim 1/r$

Orbit Types:

$E > 0$:	Hyperbola
$E = 0$:	Parabola
$E < 0$:	Ellipse
$E = \text{Min}(U_{\text{eff}})$:	Circle

Kepler's Laws:

1st Law :	Elliptical Orbits	(Sun [at/orbiting] focus)
2nd Law :	Equal Area Sweep	($r^2 d\theta \;=\; \frac{l}{m} dt$)
3rd Law :	$T^2 = k^2 a^3$	T, Period a, Semi-major axis k, "constant" $\left(\frac{2\pi}{\sqrt{G[m_{\text{planet}} + M_{\text{sun}}]}}\right)$

7 Fluid Mechanics

Bernoulli's Principle :	$\frac{\rho v^2}{2} + \rho g z + P_{\text{res}}$	$=$ constant	[Energy Density]
Fluid Conservation :	$\rho A v$	$=$ constant	[Mass Flow Rate]
Bouyant Force :	$F = \rho V g$		(ρ, V, of displaced liquid)

Water Facts:

- 1 L = 1 kg

8 Oscillators

8.1 Homogenous

$$(F = m\ddot{x}) = -kx - \overset{\text{(damp)}}{b\dot{x}}$$

$$\downarrow$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

$$z_{\text{tr}}(t) = \tilde{C}e^{rt} + [\tilde{D}_{\text{opt.}}\ te^{rt}] :$$

$\underline{x(t) = \text{Re}[z(t)]}$ is the real solution.

$$(r^2 + 2\beta r + \omega_0^2)e^{rt} = 0$$

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Normal (Undamped): $(F = -kx) \Rightarrow$
$(\ddot{x} = -\omega_0^2 x = -\frac{k}{m}x)$

$$\boxed{z_{\text{tr}}(t) = \tilde{C}_1 e^{i\omega_0 t} + \tilde{C}_2 e^{-i\omega_0 t}}$$

Underdamped: $(\beta < \omega_0)$

$$\boxed{z_{\text{tr}}(t) = \left(\tilde{C}_1 e^{i\sqrt{\omega_0^2-\beta^2}t} + \tilde{C}_2 e^{-i\sqrt{\omega_0^2-\beta^2}t}\right)\underline{e^{-\beta t}}}$$

Critically Damped: $(\beta = \omega_0)$

$$\boxed{z_{\text{tr}}(t) = (\tilde{C}_1 + \tilde{C}_2 t)\underline{e^{-\beta t}}}$$

Decay rate is maximized at $\beta = \omega_0$

Overdamped: $(\beta > \omega_0)$

$$\boxed{z_{\text{tr}}(t) = \underbrace{\tilde{C}_1 e^{-\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t}}_{\text{(smaller, lasts longer)}} + \tilde{C}_2 e^{-\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t}}$$

8.2 Inhomogenous (Driven)

$$\boxed{z(t) = z_{\text{st}}(t) + z_{\text{tr}}(t)}$$

$$m\ddot{x} = -kx - b\dot{x} + F_{\text{dr}}$$

$$\downarrow$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t$$

- $L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t)$

$$\boxed{z_{\text{st}}(t) = \tilde{C}e^{i\omega t} = Ae^{i(\omega t - \delta)}} :$$

$\underline{x(t) = \text{Re}[z(t)]}$ is the real solution.

$$(-\omega^2 + 2i\beta\omega + \omega_0^2)\tilde{C}e^{i\omega t} = f_0 e^{i\omega t}$$

$$\tilde{C} = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = Ae^{-i\delta}$$

$$\boxed{A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\ , \quad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)}$$

Resonance (Max A^2) with fixed ω : $\boxed{\omega_0 = \omega}$

Resonance (Max A^2) with fixed ω_0 : $\boxed{\omega = \sqrt{\omega_0^2 - 2\beta^2}}$ (usually $\beta \ll \omega$)

Full Width at Half Max, $A^2(\omega)$: FWHM $\approx 2\beta$

Quality Factor (Sharpness) : $Q = \frac{\omega_0}{2\beta} = \left(\pi\frac{1/\beta}{2\pi/\omega_0} = \pi\frac{\text{decay time}}{\text{period}}\right) = \left(2\pi\frac{\text{Energy stored}}{\text{Energy Dissipated}}\right)$

8.3 Parallel and Series

Series, k_1+k_2+m: $\dfrac{1}{K_{\text{eq}}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$

Parallel, k_1k_2+m: $K_{\text{eq}} = k_1 + k_2$

8.4 Normal Modes: 3 Springs + 2 Masses, k_1+m_1+k_2+m_2+k_3

1.)
$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2x_1 + k_2x_2 \\ &= -(k_1+k_2)x_1 + k_2x_2 \\ m_2\ddot{x}_2 &= k_2x_1 - k_2x_2 - k_3x_2 \\ &= k_2x_1 - (k_2+k_3)x_2 \end{aligned} \quad \rightarrow \quad \begin{gathered} \mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x} \\ \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{gathered}$$

2.)
$$\begin{aligned} \mathbf{z}(t) = \mathbf{a}e^{i\omega t} &= \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} e^{i\omega t} \\ &= \begin{pmatrix} a_1e^{-i\delta_1 t} \\ a_2e^{-i\delta_2 t} \end{pmatrix} e^{i\omega t} \end{aligned} \quad \rightarrow \quad \begin{aligned} \mathbf{M}\ddot{\mathbf{z}} &= -\mathbf{K}\mathbf{z} \\ -\omega^2\mathbf{M}\mathbf{a}e^{i\omega t} &= -\mathbf{K}\mathbf{a}e^{i\omega t} \\ (\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} &= 0 \\ \boxed{\det(\mathbf{K} - \omega^2\mathbf{M}) = 0} \end{aligned}$$

$x(t) = \text{Re}\big[z(t)\big]$ is the real solution.

Same m and k

$$\begin{pmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 m \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \quad \rightarrow \quad \begin{gathered} \boxed{\omega = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}} \\ \boxed{z(t) = \tilde{A}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_1 t} + \tilde{A}_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_2 t}} \end{gathered}$$

Smaller ω_1 is most symmetric motion (both swing in phase)

Larger ω_2 swings out of phase

Weak Coupling

8.5 Single Pendulum (Use Lagrangian)

- $T = \frac{1}{2} m R^2 \dot{\theta}^2$
- $U = mg(R - R\cos\theta)$

$$\rightarrow \quad \begin{aligned} mR^2\ddot{\theta} &= -mgR\sin\theta \\ &\approx -mgR\theta \end{aligned} \quad \rightarrow \quad \boxed{\begin{aligned} \ddot{\theta} = -\left(\tfrac{g}{I/mR}\right)\theta = -\omega^2\theta \\ \theta(t) = \mathrm{Re}\big[C_1 e^{i\omega t} + C_2 e^{-i\omega t}\big] \end{aligned}}$$

8.6 Double Pendulum (Use Lagrangian)

- $T = \frac{1}{2} m_1 L_1^2 \dot{\theta_1}^2 + \frac{1}{2} m_2 (L_1 \dot{\vec{\theta_1}} + L_2 \dot{\vec{\theta_2}})^2$
$$= \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\theta_1}^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta_2}^2 + m_2 L_2 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$
- $U = m_1 g (L_1 - L_1 \cos\theta_1) + m_2 g (L_1 + L_2 - L_2\cos\theta_2 - L_1\cos\theta_1)$

$$\rightarrow \qquad \mathbf{M}\ddot{\theta} = -\mathbf{K}\theta \qquad \text{(small angle quadratic approx.)}$$

$$\begin{pmatrix} (m_1 + m_2) L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\begin{pmatrix} (m_1 + m_2) g L_1 + k_2 & 0 \\ 0 & m_2 g L_2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$