1 Lorentz Transformation

Galilean Transform $\rightarrow$

$$t'_0 = t_0 \qquad\qquad t_0 = t'_0$$
$$,$$
$$x'_0 = x_0 - vt_0 \qquad x_0 = x'_0 + vt'_0$$

x is the position of a point/_event_ occuring on the number line, and t is the <u>time on a clock</u> at x.

x' is the position of the <u>same</u> point/_event_ on the other number line, and t' is the <u>time on the clock</u> at x'.

$x = 0$ and $x' = 0$ are the positions of the line/reference frame origins, and t_0 and t'_0 are the times on the origin clocks.

At $t_0 = t'_0 = 0$, both origin's coincide at $x = x' = 0$.

All points at x see their clock run the same as $t = t_0$, but see a different t' at the adjacent x'.

Lorentz Transform $\qquad \gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$

$$\boxed{\begin{array}{ll} x' = \gamma(x - \beta ct) & x = \gamma(x' + \beta ct') \\[4pt] ct' = \gamma(ct - \beta x) & ct = \gamma(ct' + \beta x') \end{array}}$$

($,$ between the two columns)

Transform Matrix

$$x^{\mu'} = \begin{pmatrix} x^{0'} = ct' \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x^{\mu}$$

Time Slows: $\boxed{\Delta t' = \Delta t / \gamma}$

1.) $t'(x, t_0)$ for a Clock at $x = X_0 + vt$

$$\Rightarrow \quad ct' = \gamma\left(ct - \beta[X_0 + vt]\right)$$
$$= \left(\gamma ct(1 - \beta^2) - \gamma\beta X_0\right)$$
$$= \left(\frac{ct}{\gamma} - \gamma\beta X_0\right) = \left(\frac{ct}{\gamma} + cT_0\right)$$

$$\Rightarrow \boxed{\begin{aligned} t'(X_0, t) &= \frac{t}{\gamma} + T_0 \\ &= \frac{t}{\gamma} - \gamma\beta X_0 \end{aligned}}$$

$T_0 > 0: \ X_0 < 0$
$T_0 < 0: \ X_0 > 0$

$\boxed{\text{(No } t' \text{ is simultaneous to } x \text{ unless in same position)}}$

$$\boxed{\frac{dt'}{dt} = \frac{dt'}{dt_0} = \frac{1}{\gamma}}$$

$\boxed{\begin{aligned}&\text{(for each } \Delta t, \text{ then } \Delta t' = \frac{\Delta t}{\gamma}) \\ &\text{(Clocks at } x' \text{ look like they} \\ &\text{tick slower by factor } \gamma)\end{aligned}}$

2.) $\Delta t'_0$ given Δt_0

$$\Rightarrow \quad c\Delta t_0 = \gamma(c\Delta t'_0 - \beta x'_{=0}) \quad \Rightarrow \quad \boxed{\Delta t'_0 = \Delta t_0 / \gamma} \qquad \text{(Same conclusion of slowed clocks)}$$

Length Contraction: $\boxed{\Delta x' = \gamma \Delta x}$

1.) $\Delta x' = x'_2 - x'_1 = \gamma(x_2 - \beta ct) - \gamma(x_1 - \beta ct)$
$$= \gamma(x_2 - x_1) = \gamma \Delta x \quad \checkmark$$

Velocity Addition (1-D): $\boxed{w = \dfrac{v + u}{1 + vu/c^2}}$

Pythag. Triples

$\beta = 3/5 \ : \ \gamma = 5/4 = 1.25$
$\beta = 4/5 \ : \ \gamma = 5/3$
$\beta = 5/13 : \ \gamma = 13/12$
$\beta = 7/25 : \ \gamma = 25/24$

Doppler Shift

$$f_{\text{rec}} = \sqrt{\frac{1+\beta}{1-\beta}} \ f_{\text{emit}} \qquad (v \text{ is } [+] \text{ if } \rightarrow\leftarrow)$$

2 4-Vectors

3-Vectors

$$\vec{p} \;=\; \boxed{\gamma m \vec{v}}$$

$$\vec{F} \;=\; \frac{d\vec{p}}{dt} \;=\; m\frac{d(\gamma\vec{v})}{dt}$$

$$\;=\; \gamma m\vec{a} + \gamma^3\frac{(m\vec{a}\cdot\vec{v})\vec{v}}{c^2}$$

$$\;=\; \boxed{\gamma^3 m(\vec{a} - \frac{\vec{v}\cdot\vec{v}}{c^2}\vec{a} + \frac{\vec{a}\cdot\vec{v}}{c^2}\vec{v})}$$

Scalars

$$E \;=\; \boxed{\sqrt{p^2c^2 + m^2c^4} \;=\; \gamma mc^2}$$

$$T \;=\; E - E_0 \;=\; \boxed{(\gamma - 1)mc^2}$$

$$P_{\text{ow}} \;=\; \frac{dE}{dt} \;=\; mc^2\frac{d\gamma}{dt} \;=\; \frac{d\vec{p}}{dt}\cdot\vec{v}$$

$$\;=\; \boxed{\vec{F}\cdot\vec{v} \;=\; \gamma^3 m\vec{a}\cdot\vec{v}}$$

$$W \;=\; \int \vec{F}\cdot\vec{v}\; dt \;=\; \boxed{\int \gamma^3 m\vec{a}\cdot\vec{v}\; dt}$$

2.1 Position

$$\begin{aligned}\boldsymbol{x}^\mu &= (x^0, \vec{x}) \\ &= (ct, \vec{r})\end{aligned} \;\Rightarrow\; \boxed{\Delta\boldsymbol{x}^\mu \;=\; \boldsymbol{x}_A^\mu - \boldsymbol{x}_B^\mu} \qquad \text{(for Event } A,\; B)$$

$$\begin{aligned}(\Delta\boldsymbol{x}^\mu)^2 &= (\Delta\boldsymbol{x}^\mu)(\Delta\boldsymbol{x}_\mu) \\ &= c^2\tau^2 \\ &= \boxed{ct^2 - \vec{r}^{\;2}}\end{aligned}$$

$$\left\{\begin{array}{l}\text{Timelike}: \quad (\Delta\boldsymbol{x}^\mu)^2 > 0 \\[1em] \text{Spacelike}: \quad (\Delta\boldsymbol{x}^\mu)^2 < 0 \\[1em] \text{Lightlike}: \quad (\Delta\boldsymbol{x}^\mu)^2 = 0\end{array}\right.$$

($\exists$ an inertial frame where A, B occur at the same spacial place but diff. time, e.g, $(c\Delta t, 0, 0, 0) \rightarrow (\Delta x)^2 = c^2(\Delta t)^2 > 0$)

($\exists$ an inertial frame where A, B occur at the same time but non-casual space)

(A and B lie on a trajectory moving at c)

> Relativistic Dot Products are Ref. Frame Invariant (not necessarily Conserved)

2.2 Momentum

$$\begin{aligned}\boldsymbol{p}^\mu &= (p^0, \vec{p}) \\ &= m\frac{d\boldsymbol{x}^\mu}{d\tau} = m\boldsymbol{\eta}^\mu \\ &= \boxed{(\gamma mc, \gamma m\vec{v}) \;=\; \left(\frac{E}{c}, \vec{p}\right)}\end{aligned}$$

$$\pm\boldsymbol{p}^\mu\boldsymbol{p}_\mu \;=\; \boxed{\boldsymbol{p}^2 \;=\; m^2c^2 \;=\; \left(\frac{E}{c}\right)^2 - \vec{p}^{\;2}}$$

> Momentum Conservation means each vector component is individually conserved

Particle Decay Example: Rest $M_1 \rightarrow m_2 + m_3$

$$\begin{aligned}p_3^2 &= (p_1 - p_2)^2 \\ &= p_1^2 + p_2^2 - 2p_1\cdot p_2 \\ m_3^2c^2 &= M_1^2c^2 + m_2^2c^2 - 2(M_1 c, 0)\cdot(E_2/c, \vec{p}_2) \\ &= M_1^2c^2 + m_2^2c^2 - 2M_1 E_2\end{aligned}$$

$$\Rightarrow \quad \boxed{\begin{aligned}E_2 &= \frac{M_1^2c^2 + m_2^2c^2 - m_3^2c^2}{2M_1} \\[1em] E_3 &= \frac{M_1^2c^2 + m_3^2c^2 - m_2^2c^2}{2M_1}\end{aligned}}$$

2.3 Acceleration and Force

$$
\begin{aligned}
\boldsymbol{K}^\mu &= (K^0, \vec{K}) \\
&= m\boldsymbol{\alpha}^\mu = m\frac{d\boldsymbol{\eta}^\mu}{d\tau} = \frac{d\boldsymbol{p}^\mu}{d\tau} = \gamma\frac{d\boldsymbol{p}^\mu}{dt} \\
&= \left(\gamma\frac{d(\gamma mc)}{dt}, \gamma\frac{d(\gamma m\vec{v})}{dt}\right) \\
&= \left(\tfrac{\gamma P_{\text{ow}}}{c}, \gamma\vec{F}\right) = \left(\tfrac{\gamma\vec{F}\cdot\vec{v}}{c}, \gamma\vec{F}\right) \\
&= \left(\gamma^4\tfrac{m\vec{a}\cdot\vec{v}}{c}, \ \gamma^2 m\vec{a} + \gamma^4\tfrac{(m\vec{a}\cdot\vec{v})\vec{v}}{c^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mp\boldsymbol{\alpha}^\mu\boldsymbol{\alpha}_\mu &= \gamma^6\frac{(\vec{a}\cdot\vec{v})^2}{c^2} + \gamma^4\vec{a}^{\ 2} \\
\mp\boldsymbol{K}^\mu\boldsymbol{K}_\mu &= -\gamma^2\frac{(\vec{F}\cdot\vec{v})^2}{c^2} + \gamma^2\vec{F}^{\ 2} \\
&= \gamma^2\vec{F}^{\ 2}\left(1 - \tfrac{\vec{v}\cdot\vec{v}}{c^2}\cos\theta_{v,F}\right) \\
\boldsymbol{\alpha}^\mu\boldsymbol{\eta}_\mu &= \frac{d\boldsymbol{\eta}^\mu}{d\tau}\boldsymbol{\eta}_\mu = \frac{1}{2}\frac{d(\boldsymbol{\eta}^\mu\boldsymbol{\eta}_\mu)}{d\tau} = 0 \\
\boldsymbol{K}^\mu\boldsymbol{p}_\mu &= m^2\boldsymbol{\alpha}^\mu\boldsymbol{\eta}_\mu = 0
\end{aligned}
$$

2.4 Current Density and Vector Potential

$$
\begin{aligned}
\boldsymbol{J}^\mu &= (J^0, \vec{J}) \\
&= \rho_0\frac{d\boldsymbol{x}^\mu}{d\tau} = \rho_0\boldsymbol{\eta}^\mu \\
&= (\gamma c\rho_0, \gamma\rho_0\vec{v}) = (c\rho, \vec{J}) \\[1em]
\boldsymbol{A}^\mu &= (A^0, \vec{A}) = \left(\tfrac{V}{c}, \vec{A}\right) \\[1em]
&= \boldsymbol{A}^\mu + \frac{\partial\lambda}{\partial\boldsymbol{x}^\mu}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\boldsymbol{J}^\mu}{\partial\boldsymbol{x}^\mu} &= \frac{\partial}{\partial\boldsymbol{x}^\mu}\cdot\boldsymbol{J}^\mu = \frac{\partial\rho}{dt} + \nabla\vec{J} = 0 \\[1em]
\Box^2\boldsymbol{A}^\mu &= \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A}^\mu = -\mu_0\boldsymbol{J}^\mu
\end{aligned}
$$