$$\boxed{\vec{\nabla} = \left[\vec{\nabla}\,(r,\theta,\phi)\right]\bar{\partial}_\circ}$$

$$d = [dx\ dy\ dz]\ \vec{\nabla} = d\vec{l}^{\,T}\,\vec{\nabla}$$

$$d(r,\theta,\phi) = [dx\ dy\ dz]\ \vec{\nabla}(r,\theta,\phi)$$

$$\boxed{\partial\bar{l}_\circ^{\,T} = d\vec{l}^{\,T}\,\vec{\nabla}(r,\theta,\phi)}$$

$$\partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ = d\vec{l}^{\,T}\left[\vec{\nabla}(r,\theta,\phi)\right]\bar{\partial}_\circ$$

$$\boxed{d = \partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ = d\vec{l}^{\,T}\,\vec{\nabla}}$$

$$\underline{\vec{\nabla}} = \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix} = \begin{bmatrix}| & | & |\\ \nabla r & \nabla\theta & \nabla\phi\\ | & | & |\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\\ \frac{\partial}{\partial \phi}\end{bmatrix}$$

$$\underline{\partial\bar{l}_\circ} = \begin{bmatrix}dr\\ d\theta\\ d\phi\end{bmatrix} = [\vec{\nabla}(r,\theta,\phi)]^T d\vec{l} = \begin{bmatrix}-\nabla r-\\ -\nabla\theta-\\ -\nabla\phi-\end{bmatrix}\begin{bmatrix}dx\\ dy\\ dz\end{bmatrix}$$

$$\theta = \theta(x,y,z) \quad \left(x^2+y^2 = z^2\tan^2\theta\right)$$

$$\phi = \phi(x,y,z) \quad \left(y = x\tan\phi\right)$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z}$$

$$d\theta = dx\frac{\partial\theta}{\partial x} + dy\frac{\partial\theta}{\partial y} + dz\frac{\partial\theta}{\partial z}$$

$$dy_{\vec{r}_\circ}(\vec{r}_\circ')\big|_{t=0} = \left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \phi}d\phi\right)\frac{1}{dt}\Big|_{t=0}$$

$$\boxed{\vec{b}^{\,i}\cdot\vec{b}_i = \delta_{ij}}:\quad \begin{matrix}\vec{\nabla}\phi\cdot\frac{\partial\vec{r}}{\partial\phi} = 1\\ \vec{\nabla}\phi\cdot\frac{\partial\vec{r}}{\partial\theta} = 0\end{matrix} \quad\Rightarrow\quad \begin{bmatrix}-\vec{\nabla}r-\\ -\vec{\nabla}\theta-\\ -\vec{\nabla}\phi-\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi}\end{bmatrix}\vec{r} = \mathbb{1}_3 \quad\Rightarrow\quad \boxed{\frac{\partial y}{\partial\phi} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}-\vec{\nabla}r-\\ -\vec{\nabla}\theta-\\ -\vec{\nabla}\phi-\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{\partial y}{\partial\phi}^T}$$

$$\begin{aligned}
d &= dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} = d\vec{l}\cdot\vec{\nabla} && = \left[\frac{dr}{\|\nabla r\|},\frac{d\theta}{\|\nabla\theta\|},\frac{d\phi}{\|\nabla\phi\|}\right]\left[\|\nabla r\|\frac{\partial}{\partial r},\ \|\nabla\theta\|\frac{\partial}{\partial\theta},\ \|\nabla\phi\|\frac{\partial}{\partial\phi}\right]^T\\
&= dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial \theta} + d\phi\frac{\partial}{\partial \phi} = \partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ && = [dr, r\,d\theta, r\sin\theta\, d\phi]\left[\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial\theta},\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right]^T\\
&= \frac{dr}{\|\nabla r\|}\|\nabla r\|\frac{\partial}{\partial r} + \frac{d\theta}{\|\nabla\theta\|}\|\nabla\theta\|\frac{\partial}{\partial\theta} + \frac{d\phi}{\|\nabla\phi\|}\|\nabla\phi\|\frac{\partial}{\partial\phi} && \cdots = d\bar{l}_\circ^{\,T}\bar{\nabla}_\circ = d\vec{l}_\circ^{\,T}\vec{\nabla}_\circ
\end{aligned}$$

$$d\vec{l} = d\vec{r} = [dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial\theta} + d\phi\frac{\partial}{\partial\phi}](x,y,z)^T$$

$$d(x,y,z) = \left[\frac{dr}{\|\nabla r\|}\|\nabla r\|\frac{\partial}{\partial r} + \frac{d\theta}{\|\nabla\theta\|}\|\nabla\theta\|\frac{\partial}{\partial\theta} + \frac{d\phi}{\|\nabla\phi\|}\|\nabla\phi\|\frac{\partial}{\partial\phi}\right](x,y,z)$$

$$(dx,dy,dz) = dr\,\hat{r}^T + r\,d\theta\,\hat{\theta}^T + r\sin\theta\, d\phi\,\hat{\phi}^T$$

$$\begin{aligned}
(\hat{r},\hat{\theta},\hat{\phi}) &\equiv \underline{\left(\|\nabla r\|\frac{\partial\vec{r}}{\partial r},\ \|\nabla\theta\|\frac{\partial\vec{r}}{\partial\theta},\ \|\nabla\phi\|\frac{\partial\vec{r}}{\partial\phi}\right)}\\
&= \left(\frac{\partial}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\otimes(x,y,z)^T\\
&= \bar{\nabla}_\circ^T\otimes\vec{r}
\end{aligned}$$

$$\boxed{d\vec{l} = (dx,dy,dz)\cdot(\hat{x},\hat{y},\hat{z})} = \boxed{(dr, r\,d\theta, r\sin\theta\, d\theta)\cdot(\hat{r},\hat{\theta},\hat{\phi}) = d\vec{l}_\circ} = d\bar{l}_\circ^{\,T}\cdot(\hat{r},\hat{\theta},\hat{\phi})$$

$$\boxed{\vec{\nabla} = (\hat{x},\hat{y},\hat{z})\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)} = \boxed{(\hat{r},\hat{\theta},\hat{\phi})\cdot\left(\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial\theta},\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) = \vec{\nabla}_\circ} = \left(\frac{\partial\vec{r}}{\partial r},\frac{1}{r}\frac{\partial\vec{r}}{\partial\theta},\frac{1}{r\sin\theta}\frac{\partial\vec{r}}{\partial\phi}\right)\bar{\nabla}_\circ^T$$

$$\vec{\nabla} = [\bar{\nabla}_\circ^T\otimes\vec{r}]\bar{\nabla}_\circ = \left[\bar{\nabla}_\circ^T\otimes(x,y,z)^T\right]\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial\theta}\\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\end{bmatrix} \quad\Rightarrow\quad \frac{\partial}{\partial x} = \underline{\frac{\partial x}{\partial r}}\frac{\partial}{\partial r} + \underline{\|\nabla\theta\|^2\frac{\partial x}{\partial\theta}}\frac{\partial}{\partial\theta} + \underline{\|\nabla\phi\|^2\frac{\partial x}{\partial\phi}}\frac{\partial}{\partial\phi}$$

$$= [\vec{\nabla}(r,\theta,\phi)]\bar{\partial}_\circ = [\vec{\nabla}(r,\theta,\phi)]\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\phi}\end{bmatrix} \quad\Rightarrow\quad \frac{\partial}{\partial x} = \underline{\frac{\partial r}{\partial x}}\frac{\partial}{\partial r} + \underline{\frac{\partial\theta}{\partial x}}\frac{\partial}{\partial\theta} + \underline{\frac{\partial\phi}{\partial x}}\frac{\partial}{\partial\phi} \quad\Rightarrow\quad \boxed{\frac{\partial\phi}{\partial y} = \frac{\partial y}{\partial\phi}\|\nabla\phi\|^2}$$

$$\boxed{\begin{array}{ccccccccccccc}
 & & & & & & & & \underline{\text{contravariant}}_i & \text{(equal since orthog.)} & \underline{\text{covariant}}^i & & \\
\hat{r} & = & (\hat{r}_x,\hat{r}_y,\hat{r}_z) & = & \frac{\vec{r}}{r} & = & \frac{\partial}{\partial r}\vec{r} & = & \frac{\partial\vec{r}}{\partial r}\left\|\frac{\partial\vec{r}}{\partial r}\right\|^{-1} & \overset{\rightarrow}{=}\ \|\nabla r\|\frac{\partial\vec{r}}{\partial r}\ \overset{\leftarrow}{=} & \frac{\nabla r}{\|\nabla r\|} & = & \nabla r\\
\hat{\theta} & = & (\hat{\theta}_x,\hat{\theta}_y,\hat{\theta}_z) & = & \frac{\partial\hat{r}}{\partial\theta} & = & \frac{1}{r}\frac{\partial}{\partial\theta}\vec{r} & = & \frac{\partial\vec{r}}{\partial\theta}\left\|\frac{\partial\vec{r}}{\partial\theta}\right\|^{-1} & \overset{\rightarrow}{=}\ \|\nabla\theta\|\frac{\partial\vec{r}}{\partial\theta}\ \overset{\leftarrow}{=} & \frac{\nabla\theta}{\|\nabla\theta\|} & = & r\nabla\theta\\
\hat{\phi} & = & (\hat{\phi}_x,\hat{\phi}_y,\hat{\phi}_z) & = & \frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi} & = & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\vec{r} & = & \frac{\partial\vec{r}}{\partial\phi}\left\|\frac{\partial\vec{r}}{\partial\phi}\right\|^{-1} & \overset{\rightarrow}{=}\ \|\nabla\phi\|\frac{\partial\vec{r}}{\partial\phi}\ \overset{\leftarrow}{=} & \frac{\nabla\phi}{\|\nabla\phi\|} & = & r\sin\theta\nabla\phi
\end{array}}$$

1 Frenet Equations

$$a \cdot (b \times c) = (a \times b) \cdot c$$

$$a \times (b \times c) = (c \cdot a)b - (b \cdot a)c$$

$$(a \times b) \times c = b(c \cdot a) - a(c \cdot b)$$

$$(a \times b) \cdot (c \times d) = a \cdot b \times (c \times d)$$

$$= \left| \begin{bmatrix} a \cdot \\ b \cdot \end{bmatrix} [c \ d] \right| = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

$$\boxed{\frac{dt}{ds} = \frac{1}{v}}$$

$$T = \hat{v} = \frac{\vec{v}}{v}$$

$$\frac{dT}{dt} = \frac{(\vec{v} \cdot \vec{v})\vec{a} - (\vec{v} \cdot \vec{a})\vec{v}}{v^3} = \frac{\vec{v} \times (\vec{a} \times \vec{v})}{v^3} = \frac{(\vec{v} \times \vec{a}) \times \vec{v}}{v^3}$$

$$\left\| \frac{dT}{dt} \right\| = \frac{\sqrt{v^2 a^2 - (\vec{v} \cdot \vec{a})^2}}{v^2} = \frac{\|\vec{a} \times \vec{v}\|}{v^2}, \quad \frac{dT}{ds} = k\hat{N}$$

$$\hat{N} = \frac{T'}{\|T'\|} = \frac{(\vec{v} \times \vec{a}) \times \vec{v}}{\|\vec{v} \times \vec{a}\| v} = \hat{B} \times \hat{v}$$

$$\hat{B} = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|} = \widehat{v \times a} = \hat{v} \times \hat{N} \qquad \underline{(\hat{B} \cdot \vec{v} = 0)}$$

$$\frac{d\hat{B}}{dt} = \frac{\vec{v} \times \dot{\vec{a}}}{\|\vec{v} \times \vec{a}\|} - \left[\frac{\vec{v} \times \dot{\vec{a}}}{\|\vec{v} \times \vec{a}\|} \cdot \hat{B} \right] \hat{B}, \quad \frac{dB}{ds} = \tau \hat{N}$$

$$\tau = \hat{N} \cdot \frac{d\hat{B}}{ds} = \frac{\hat{B} \cdot \dot{\vec{a}}}{\|\vec{v} \times \vec{a}\|} = \frac{(\vec{v} \times \vec{a}) \cdot \dot{\vec{a}}}{\|\vec{v} \times \vec{a}\|^2}$$

$$\boxed{\begin{aligned} \vec{a} &= a_T \hat{T} + a_N \hat{N} \\ a_T &= \vec{a} \cdot \hat{v} = \frac{dv}{dt} \\ a_N &= \frac{\|\vec{a} \times \vec{v}\|}{v} = \|\vec{a} \times \hat{v}\| \\ a^2 &= a_T^2 + a_N^2 = \left\|\frac{d\vec{v}}{dt}\right\|^2 \end{aligned}}$$

<u>Frenet Trihedron</u>

Differentiable (in this book) : C^∞

No singular pts. Order 0 (Regular) : $\vec{v}(t) \neq 0$

- $\|\vec{v}(t)\| = c \to 1 \ \Rightarrow \ \int_s \|\vec{v}(t)\| \, dt = t = \Delta s$

  $\to s : \ \vec{x}(t) = \vec{x}(s)$
- $\frac{1}{2}\frac{d}{dt}(\vec{v} \cdot \vec{v}) = \boxed{\vec{v} \cdot \vec{a} = 0}$

No singular pts. Order 1 : $\vec{a}(t) \neq 0$

- Curvature, $k \neq 0$ (see right)
- Vertex, $k' = 0$

$$1 = \|\vec{t}\| = \|\vec{n}\| = \|\vec{b}\|, \quad 0 = \vec{t} \cdot \vec{n} = \vec{n} \cdot \vec{b} = \vec{b} \cdot \vec{t}$$

- $\vec{v}(s) = \vec{t}(s) \qquad \boxed{(t = n \times b)}$
- $\vec{a}(s) = \boxed{\vec{t}'(s) = k(s)\vec{n}(s)}, \ k(s) \geq 0$ (can be L or R-handed) (can be neg. if in $\mathbb{R}^2$)

$*$ $k(s) > 0$ for well defined curve with $\hat{n}$

- $\boxed{\vec{b} = \vec{t} \times \vec{n}}, \ \frac{d}{dt}(\vec{b} \cdot \vec{b}) = \vec{b} \cdot \vec{b}' = 0, \ * \ \boxed{\vec{b}'(s) = \tau(s)\vec{n}(s)}$
- $\boxed{\vec{n} = \vec{b} \times \vec{t}}, \ * \ \boxed{\vec{n}'(s) = -k\vec{t} - \tau\vec{b}}$, $*$ t-n pl. = osculating pl.

- $t''(s) = k'n - k^2 t - k\tau b$
- $b''(s) = \tau' n - \tau k t - \tau^2 b$
- $n''(s) = -k't - \tau' b - (k^2 + \tau^2)n$
- $|\tau| = \|b'\|$
- $\tau = -\frac{(t \times t') \cdot t''}{k^2} = -\frac{t \cdot (t' \times t'')}{\|t'\|^2}$
- $k = \|t'\| = \frac{(b \times b') \cdot b''}{\tau^2} = \frac{b \cdot (b' \times b'')}{\|b'\|^2}$
- $n \Rightarrow k, \tau :$ $\quad * \ \|n'\|^2 = k^2 + \tau^2 \quad * \ \frac{(n \times n') \cdot n''}{\|n'\|^2} = \frac{k'\tau - k\tau'}{k^2 + \tau^2} = \frac{\frac{d}{ds}(k/\tau)}{(k/\tau)^2 + 1} = \frac{d}{ds}\arctan(k/\tau)$

<u>Indicatrix [of Tangents]</u> :

- $\vec{t}(\theta(s)) = (\cos\theta, \sin\theta) = (x'(s), y'(s))$
- $\vec{t}'(\theta) = \underline{\theta'(s)}(-\sin\theta, \cos\theta) = \underline{k(s)}\vec{n}$
- $\theta(s) = \arctan(y'/x')$
- $\int_0^l k(s) \, ds = \theta(s)\Big|_0^l = 2\pi I_{\text{rot. index}}$

<u>Local Canonical Form at $t = 0$</u> :

- $(\hat{t}, \hat{n}, \hat{b}) = (\hat{x}, \hat{y}, \hat{z})$
- $\vec{r}(s) - \vec{r}(0) \approx (s - \frac{k^2 s^3}{6}, \ \frac{k}{2}s^2 + \frac{k' s^3}{6}, \ \frac{-k\tau}{6}s^3)$
- $\tau < 0 \ \Rightarrow \ \frac{dz}{ds} > 0$

<u>Isoperimetric Inequality</u> : $0 \leq l^2 - 4\pi A$

<u>Four-Vertex Theorem</u> : A simple closed curve has ≥ 4 vertices

<u>Cauchy-Crofton Formula (measure of number of times lines intersect a curve)</u> :

- Tangent line at (ρ, θ) : $x\cos\theta + y\sin\theta = \rho$
- Curve c : $y = 0, \ x \in (-l/2, l/2)$, $\quad C = \sum c_i$
- $\int$ Lines that cross $c = \int_0^{2\pi} \int_0^{|\cos\theta| l/2} d\rho \, d\theta = 2l \ \Rightarrow \ \int_0^{2\pi} \int_0^\infty n_C \ d\rho \, d\theta = 2l$

2 Jacobian/Differential, $dF_{\alpha(0)} : \mathbb{R}^n \to \mathbb{R}^m$

- $\boxed{\alpha(0) = \beta(0)} \Rightarrow \underline{F(t=0)} = F \circ \alpha\big|_{t=0} = F \circ \beta\big|_{t=0}$
- $\boxed{\alpha'(0) = \beta'(0)} \Rightarrow \frac{\partial x}{\partial \alpha_i}\Big|_{t=0} = \frac{\partial x}{\partial \beta_i}\Big|_{t=0} \cdot \cancel{\frac{d\beta_i/dt}{d\alpha_i/dt}\Big|_{t=0}} \Rightarrow \boxed{dF_{\alpha(0)}(\alpha'(0)) = dF_{\beta(0)}(\beta'(0))}$ (doesn't depend on α)

$* \; F = (f_0, f_1, \ldots, f_m) \Rightarrow \underline{dF_{\alpha(0)}(\alpha'(0))} \equiv \frac{d}{dt}(F \circ \alpha)\big|_{t=0} = \begin{bmatrix} \frac{\partial f_0}{\partial \alpha_0} & \frac{\partial f_0}{\partial \alpha_1} & \cdots \\ \frac{\partial f_1}{\partial \alpha_0} & \frac{\partial f_1}{\partial \alpha_1} & \cdots \\ \vdots & & \end{bmatrix}_{t=0} \begin{bmatrix} \frac{d\alpha_0}{dt} \\ \frac{d\alpha_1}{dt} \\ \vdots \end{bmatrix}_{t=0} = \boxed{J_{F(0)} \cdot \alpha'(0)}$

- $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ • Regular Value, p: $dF_p \neq 0$ • Critical Value, p: $dF_p = 0$

F is a Homeomorphism onto image $F(X)$:
- F is bijective between X & $F(X)$
- F is cont. • F^{-1} is cont.

F is a Diffeomorphism onto image $F(X)$:
- $F \in C^\infty$ (cont. part. deri. of all orders)
- $F^{-1} \in C^\infty$

Inverse Function Theorem (IVT):
- $F \in C^\infty$
- $\exists dF_p^{-1}$ (matrix dF_p is an isomorphism)

$\Rightarrow \exists F^{-1} \in C^\infty$

Regular Surface, $S \in \mathbb{R}^3$:

– $\forall p \in S$, $\underline{\exists F \in C^\infty}$, $F : V_q$ (neighborhood of q) $\to V_p \cap S$ (diff. parametrizations are possible, btw)

– F is a homeomorphism (or F is one-to-one) $\to \exists F^{-1} \in C^\infty \Rightarrow \exists$ no self-intersections; cont. = doesn't depend on parametrization

– dF_p is one-to-one = columns are lin. ind. = any 2x2 $|\text{sub-}J_F| \neq 0 \Rightarrow \exists$(tangent at all points)

- $\underline{f \in C^\infty} \Rightarrow \boxed{\big(\vec{x}, f(\vec{x})\big) \text{ is a reg. surf.}}$

- $f : \mathbb{R}^n \to \mathbb{R}$, $\underline{f(\vec{x}) = c \text{ is a reg. val.}}$, $f \in C^\infty$, $F(\vec{x}) = \big(x_1, \ldots, x_{n-1}, f(\vec{x})\big)$, $\exists dF_p^{-1}$ $\overset{(\text{IVT})}{\Rightarrow}$ $\exists F^{-1} \in C^\infty$, $F^{-1}\big(f_1, \ldots, f_{n-1}, f(\vec{x})\big) = \vec{x}$, $x_n = f_n^{-1} : \mathbb{R}^n \to \mathbb{R}$, $\underline{x_n = f_n^{-1} \in C^\infty}$

$\to x_n = f_n^{-1}\big(x_1, \ldots, x_{n-1}, f(\vec{x}) = c\big) = \underline{f'^{-1}_n\big(x_1, \ldots, x_{n-1}\big)} \Rightarrow S = \underline{\big(x_1, \ldots, x_{n-1}, f'^{-1}_n\big)}$ where $f(\vec{x}) = c$; $S ==$ Surface $f^{-1}(c)$ $\Rightarrow \boxed{\text{Surface } f^{-1}(c) \text{ is reg.}}$

- $f : S \subset \mathbb{R}^n \to \mathbb{R}$, $\forall p \in S$, $\underline{f(p) \neq 0} \Rightarrow \forall p \in S$, $\underline{f(p) > 0 \text{ or } f(p) < 0}$

- $F(u,v) = \big(x(u,v), y(u,v), \underline{z(u,v)}\big)$ (& F is one-to-one), $\underline{\frac{\partial(x,y)}{\partial(u,v)} \neq 0} \Rightarrow \pi_{\text{proj.}} \circ F(u,v) \equiv \big(x(u,v), y(u,v)\big)$

$\overset{(IVT)}{\Rightarrow} (\pi \circ F)^{-1}(x,y) = \big(u(x,y), v(x,y)\big)$
- $\Rightarrow z\big(u(x,y), v(x,y)\big) = z \circ (\pi \circ F)^{-1}(x,y) = \boxed{f(x,y) = z \in C^\infty}$
- $\& \Rightarrow \underline{(\pi \circ F)^{-1} \circ \pi} \circ F(u,v) = \underline{F^{-1}} \circ F(u,v) \Rightarrow \boxed{F^{-1} \in C^\infty}$

3 Del

$$\nabla F = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) F$$

$$= \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{bmatrix} F = \begin{bmatrix} \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z} \\ \cos\phi\cos\theta\,\hat{x} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z} \\ -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{bmatrix} F$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F = \begin{bmatrix} \cos\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \\ \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \\ \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \end{bmatrix} F = \begin{bmatrix} \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \\ \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \\ \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} \end{bmatrix} F = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F$$

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{1}{r\sin\theta}\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right\rangle \cdot [r \cdot r\sin\theta] \left\langle A_r, \frac{1}{r}A_\theta, \frac{1}{r\sin\theta}A_\phi \right\rangle$$

$$\nabla \times \vec{A} = \frac{1}{r}\frac{1}{r\sin\theta} \begin{Vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{Vmatrix} = \begin{Vmatrix} \frac{\partial \vec{r}}{\partial r} & \frac{\partial \vec{r}}{\partial \theta} & \frac{\partial \vec{r}}{\partial \phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{Vmatrix}$$

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \vec{A} \cdot (B_i\vec{C}) - \vec{A} \cdot (\vec{B}C_i)$$

$$\begin{aligned} \vec{A} \times (\vec{B} \times \vec{C}) &= \boxed{(\vec{A} \cdot (\vec{B} \otimes \vec{C})^T)^T - (\vec{A} \cdot \vec{B} \otimes \vec{C})^T} &= (\vec{A} \otimes \vec{B}) \cdot \vec{C} - (\vec{A} \cdot \vec{B} \otimes \vec{C})^T \\ &= (A^T(BC^T)^T)^T - (A^TBC^T)^T &= (AB^T)^TC - (A^TBC^T)^T \end{aligned}$$