1 Wave Function

$$\Psi_p = e^{i(2\pi x/\lambda - 2\pi t/T)}$$
$$= e^{i(kx-\omega t)}$$
$$= e^{\frac{i}{\hbar}(px-Et)}$$

$$\widehat{P}\Psi_p = p\Psi_p = \hbar k \Psi_p \qquad \widehat{E}\Psi_p = E\Psi_p = \hbar\omega\Psi_p$$

$$\boxed{\widehat{P} = \frac{\hbar}{i}\partial_x} \qquad \boxed{\widehat{E} = -\frac{\hbar}{i}\partial_t}$$

$$\begin{aligned}\widehat{P}\Psi_p &= \frac{\hbar}{i}\frac{\partial}{\partial x}(e^{\frac{i}{\hbar}(px-Et)}) \\ &= pe^{\frac{i}{\hbar}(px-Et)} \\ &= p\Psi_p\end{aligned} \qquad \begin{aligned}\widehat{E}\Psi_p &= -\frac{\hbar}{i}\frac{\partial}{\partial t}\left(e^{\frac{i}{\hbar}(px-Et)}\right) \\ &= Ee^{\frac{i}{\hbar}(px-Et)} \\ &= E\Psi_p\end{aligned}$$

1.1 Schrodinger Ψ

$$\widehat{E}\Psi = \widehat{H}\Psi = (\widehat{T} + \widehat{V})\Psi$$
$$\widehat{E}\Psi = \left(\frac{\hat{p}^2}{2m} + V(\vec{\mathbf{r}}, t)\right)\Psi$$
$$\boxed{i\hbar\frac{\partial}{\partial t}\Psi(\vec{\mathbf{r}}, t) = \left(\frac{-\hbar^2}{2m}\nabla^2 + V(\vec{\mathbf{r}}, t)\right)\Psi(\vec{\mathbf{r}}, t)}$$

<u>If $V = V(x)$</u>

$\Psi(x,t) = \psi(x)\phi(t) \;\Rightarrow$

- $E_n\phi_n(t) = i\hbar\frac{\delta}{\delta t}\phi_n(t) \;\Rightarrow\; \boxed{\phi_n(t) = e^{-\frac{i}{\hbar}E_n t}}$
- $\boxed{E_n\psi_n(x) = \left(\frac{-\hbar^2}{2m}\partial_x^2 + V(x)\right)\psi_n(x)}$
  - ψ can be lin. sum of real or complex, so choose real ψ
- Linear: $\boxed{\Psi(x,t) = \sum_n \psi_n(x)\; e^{-\frac{i}{\hbar}E_n t}}$
- $\sigma_H^2 = \langle H^2\rangle - \langle H\rangle^2 = 0 \;\Rightarrow$ measuring stationary state, Ψ_n, returns one E (determinate state)

<u>If $V = V(r)$</u>

$\Psi(\vec{\mathbf{r}}) = R(r)Y_l^m(\theta,\phi) = R(r)\Theta_l^m(\theta)\Phi_m(\phi) \;\Rightarrow$

$$Eu = \left(\frac{\hat{p}_r^2}{2m} + V(r) + \frac{\hat{L}^2}{2(mr^2)}\right)u$$
$$\boxed{Eu = \frac{-\hbar^2}{2m}\partial_r^2 u + \left[V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}\right]u}$$

- $u(r) = rR(r)$
- $\Phi_m(\phi) = e^{im\phi}$
- $\Theta_l^m(\theta) = AP_l^m(\cos\theta)$
  - $A = \epsilon\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}, \quad \epsilon = \begin{cases}(-1)^m & (m\geq 0)\\ 1 & (m\leq 0)\end{cases}$
  - $P_l^m(x) =$ Assoc. Legendre Func. (see extra)
- $l \in \mathbb{N}_0,\; m \in \{-l, ..., -1, 0, 1, ..., l\}$
- $\widehat{L}_i = (\vec{r} \times \vec{p})_i$

1.2 Usage

- $\langle f|g\rangle = \int_{-\infty}^{\infty} f(x)^* g(x)\ dx$
- $\langle f|g\rangle_{ab} = \int_a^b f(x)^* g(x)\ dx$
- $|f\rangle \equiv f(x)$
- $\langle f| \equiv \int f(x)^*[...]\ dx$
- $\langle f|f\rangle = \int_a^b |f|^2\ dx < \infty \ \Rightarrow\ f \in L_{2(a,b)}$ $\qquad \left(\int_a^b\ |f|^p\ dx < \infty \ \Rightarrow\ f \in L_{p(a,b)}\right)$

$$\Psi = \begin{cases} \sum_n c_n f_n \\ \int_n c_n f_n\ dn \end{cases}, \qquad \langle f_m|f_n\rangle = \begin{cases} \delta_{mn} \\ \delta_{(m-n)} \end{cases} \Rightarrow \boxed{c_n = \langle f_n|\Psi\rangle}, \qquad |c_n|^2 = \begin{cases} P(n) \\ \text{PDF}_{(n)} \end{cases}$$

(see Born int.)

$\forall\{f_n\} \in L_2$:

$$|\Psi\rangle = \begin{cases} \sum_n c_n|f_n\rangle = \sum_n \langle f_n|\Psi\rangle\ |f_n\rangle = \left(\sum_n |f_n\rangle\langle f_n|\right)|\Psi\rangle = |\Psi\rangle \\ \int_n c_n|f_n\rangle\ dn = \int_n \langle f_n|\Psi\rangle|f_n\rangle\ dn = \left(\int_n |f_n\rangle\langle f_n|\ dn\right)|\Psi\rangle = |\Psi\rangle \end{cases}$$

$\hat{x}\Psi_y = x\Psi_y = y\Psi_y$ $\Rightarrow \boxed{\Psi_y = \delta_{(x-y)}}$	$\hat{p}\Psi_p = p\Psi_p$ $\Rightarrow \boxed{\Psi_p = Ae^{\frac{i}{\hbar}px}}$	$\hat{H}\Psi_n = E_n\Psi_n$ (See Potential Examples)
$\Psi_{(x,0)} = \int_{-\infty}^{\infty} c_y\Psi_y\ dy$ $= \int_{-\infty}^{\infty} \Psi_{(y,0)}\ \delta_{(x-y)}dy$	$\Psi_{(x,t)} = \int_{-\infty}^{\infty} c_p\Psi_p\ dp$ $= \int_{-\infty}^{\infty} \Phi_{(p,0)} \left(e^{-\frac{i}{\hbar}\frac{p^2}{2m}t}\right) \frac{e^{\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}}\ dp$	$\Psi_{(x,t)} = \int_{-\infty}^{\infty} c_n\Psi_n\ dn$ $= \int_{-\infty}^{\infty} c_n \left(e^{\frac{-i}{\hbar}E_nt}\right)\Psi_n\ dn$
$c_y = \langle\Psi_y|\Psi\rangle$ $\Psi_{(y,0)} = \int_{-\infty}^{\infty} \delta_{(x-y)}\Psi_{(x,0)}\ dx$	$c_p = \langle\Psi_p|\Psi\rangle$ $\boxed{\Phi_{(p,t)} = \int_{-\infty}^{\infty} \frac{e^{\frac{-i}{\hbar}px}}{\sqrt{2\pi\hbar}}\Psi_{(x,t)}\ dx}$	$c_n(t) = \langle\Psi_n|\Psi\rangle$ $c_n(t) = \int_{-\infty}^{\infty} \Psi_n^*\Psi_{(x,t)}\ dx$

$\boxed{\text{Born Interpretation: PDF}(x) = |\Psi(x)|^2 = \Psi^*\Psi}$

$\underline{P_{(a<x<b)} = \int_a^b |\Psi|^2 dx \equiv \langle\Psi|\Psi\rangle_{ab}}$

- $\boxed{\langle\Psi|\Psi\rangle = 1}$ (physical, bound states only)
- $\Psi(\pm\infty) = 0$
- $\text{Min}(V) \leq E_\Psi \in \mathbb{R}$
- $\langle\Psi_n|\Psi_n\rangle \to \infty \;\Rightarrow\; \Psi_n$ not PHYSICAL sol. but $\Psi = \int c_n\Psi_n$ can if $\langle\Psi|\Psi\rangle = 1$

Boundary Conditions:

- $\Psi(x)$ isn't always cont. (see extra)
- $\frac{\partial\Psi(x)}{\partial x}$ is cont. except at $V = \infty$

$$\lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} E\Psi dx = \int_{-\epsilon}^{\epsilon} \widehat{H}\Psi dx \;\Rightarrow$$
$$\lim_{\epsilon\to 0} \frac{\hbar^2}{2m}\Delta\left(\frac{d\Psi}{dx}\right) = \int_{-\epsilon}^{\epsilon} V\Psi dx$$

- $E[f(x)] = \int_{-\infty}^{\infty} f(x)\,\text{PDF}(x)\,dx = \int_{-\infty}^{\infty} f(x)\,|\Psi(x)|^2\,dx = \int_{-\infty}^{\infty} \Psi(x)^* f(x)\Psi(x)\,dx = \boxed{\langle\Psi|f(x)\Psi\rangle \equiv \langle f(x)\rangle}$
- $\displaystyle\int_x \Psi^*\Psi\;dx = \int_x \left(\int_n c_n^*(t)\Psi_n^*(x)\;dn\right)\left(\int_{n'} c_{n'}(t)\Psi_{n'}(x)\;dn'\right)\;dx$

$$= \int_n c_n^*(t)\int_{n'} c_{n'}(t)\;\delta(n-n')\;dn'\;dn = \int_n |c_n(t)|^2\;dn \;\Rightarrow\; \boxed{\text{PDF}(n) = |c_n|^2 = c_n^* c_n}$$

$\underline{\text{Adjoint (herm. adj./herm. conj.): } \{A^\dagger : \langle f|Af\rangle = \langle A^\dagger f|f\rangle\}} \;\Rightarrow\; \langle h|\hat{A}g\rangle = \langle\hat{A}^\dagger h|g\rangle$ (let $f{=}h{+}g,\; f{=}h{+}ig$)

$\underline{\text{Hermitian Operator: } \{A : \hat{A}^\dagger = \hat{A}\}}$

- $\boxed{\exists\{\Psi_n\} :\; \hat{A}\Psi_n(x) = a_n\Psi_n(x)}$ (spectral theorem)
- $\langle a\rangle = a \in \mathbb{R} \;\Rightarrow\; \hat{A}$ can be an observable
- $\boxed{\langle\Psi_m|\Psi_n\rangle \in \{\delta_{mn},\; \delta(m-n)\}}$
- $\boxed{\text{Axiom: } \{\Psi_n\} \text{ for } \hat{A} \text{ are complete}}$

<u>Non-degenerate:</u> $(m \neq n),\; (a_m \neq a_n) \;\Rightarrow\; \langle\Psi_m|\Psi_n\rangle \in \{\delta_{mn},\; \delta(m-n)\}$

<u>Degenerate:</u> $(m \neq n),\; (a_m = a_n),\; (\Psi_m \neq \Psi_n), \langle\Psi_m|\Psi_n\rangle \neq 0 \;\Rightarrow$ Use Gram-Schmidt to find orthogonal $\langle\Psi_m'|\Psi_n'\rangle = \langle a\Psi_m + b\Psi_n|\; c\Psi_m + d\Psi_n\rangle = 0$

$\underline{\text{Expectation: } E[\hat{A}(x,p)]}$

- $\displaystyle\int_{-\infty}^{\infty} \hat{A}(x,p)^*\;\Psi^*\Psi\;dx = \langle\hat{A}\Psi|\Psi\rangle = \boxed{\langle\Psi|\hat{A}\Psi\rangle \equiv \langle\hat{A}(x,p)\rangle}$ (won't work if $\int A\,|\Psi|^2\,dx$)

$$\langle\Psi|\hat{A}\Psi\rangle = \int_{-\infty}^{\infty} \Psi^*\hat{A}\Psi\;dx = \int_{-\infty}^{\infty} \left(\int_n c_n^*\Psi_n^*\;dn\right)\left(\int_{n'} c_{n'}\hat{A}\Psi_{n'}\;dn'\right)\;dx$$
$$= \int_n a_n|c_n|^2\;dn = E[a] \equiv \langle a\rangle$$

$c_n = \text{PDF}(n)$ (see above and see Momentum Space)

$$\boxed{\langle a\rangle = \langle\Psi|\hat{A}\Psi\rangle = \langle\Psi|\hat{A}|\Psi\rangle = \langle A\rangle}$$

- $\boxed{\langle\sigma_a^2\rangle = \langle a^2\rangle - \langle a\rangle^2} \;\Rightarrow\; \sigma_A^2 = 0$ for Ψ_n (determinate state)

Matrix Operators:

Given complete $\{e_n\}$: $\langle e_m|e_n\rangle = \delta_{mn}$

1.) * $\boxed{Q^{(e)}_{mn} \equiv \left\langle e_m \middle| \widehat{Q}_{(x,p)} \middle| e_n \right\rangle}$

$$|\beta\rangle = \widehat{Q}|\alpha\rangle = \sum_m |e_m\rangle \left(\begin{array}{c} \langle e_m|\beta\rangle = \left\langle e_m \middle| \widehat{Q} \middle| \alpha \right\rangle \\ \sum_n b_n \langle e_m|e_n\rangle = \sum_n a_n \ ^* \boxed{\langle \boldsymbol{e_m}|\boldsymbol{\widehat{Q}}|\boldsymbol{e_n}\rangle} \\ \boxed{b_m = \sum_n \left(Q^{(e)}_m\right)_n \ a_n} \end{array} \right) = \begin{array}{l} \sum_m b_m|e_m\rangle = \sum_{n,m} \langle e_n|\alpha\rangle \ Q^{(e)}_{mn}|e_m\rangle \\ \quad = \left(\sum_{n,m} Q^{(e)}_{mn}|e_m\rangle\langle e_n|\right)|\alpha\rangle \\ \Rightarrow \boxed{\widehat{Q} = \sum_{m,n} Q^{(e)}_{mn}|e_m\rangle\langle e_n|} \end{array}$$

2.) Find $\widehat{Q}$ as a matrix

$$|f_{(x)}\rangle = \sum_n c^{(e)}_n{}_{[f]}\left|e_{n\,(x)}\right\rangle \ = \ \begin{pmatrix} \vdots \\ c_{n[f]} \\ \vdots \end{pmatrix}^{(e)} \cdot \begin{pmatrix} \vdots \\ e_{n\,(x)} \\ \vdots \end{pmatrix} \equiv \boxed{\begin{array}{c} \vec{c}^{\ (e)}{}_{[f]} \cdot \vec{e}_{(x)} \\ \int_n c^{(e)}{}_{[f](n)} \cdot e_{(n,x)} \ dn \end{array}} \quad , \quad \boxed{c^{(e)}_n{}_{[f]} = \langle e_n|f\rangle}$$

$$\begin{aligned} &\widehat{Q}|f\rangle \\ &= \left(\sum_{m,n'} Q^{(e)}_{mn'}|e_m\rangle\langle e_{n'}|\right)\sum_n c^{(e)}_n|e_n\rangle \\ &= \sum_{m,n}\left(\sum_{n'} Q^{(e)}_{mn'} \ c^{(e)}_n\langle e_{n'}|e_n\rangle\right)|e_m\rangle \\ &= \sum_m\left(\sum_n \left(Q^{(e)}_m\right)_n c^{(e)}_n\right)|e_m\rangle \end{aligned}$$

$$\boxed{\begin{array}{l} \widehat{Q}\left[\begin{pmatrix} | \\ c \\ | \end{pmatrix}^{(e)} \cdot \begin{pmatrix} | \\ e \\ | \end{pmatrix}\right] = \left[\begin{pmatrix} & \vdots & \\ - & Q_m & - \\ & \vdots & \end{pmatrix}^{(e)} \begin{pmatrix} | \\ c \\ | \end{pmatrix}^{(e)}\right] \cdot \begin{pmatrix} | \\ e \\ | \end{pmatrix} \\ \widehat{Q}\ |f\rangle = \boxed{\widehat{Q}\ \left[\ \vec{c}^{\ (e)}{}_{[f]} \cdot \vec{e}\ \right] = \left[\ \overline{Q}^{(e)} \ \vec{c}^{\ (e)}{}_{[f]}\right] \cdot \vec{e}} \\ \quad \widehat{Q}|f\rangle = \int_m \left[\overline{Q}^{(\delta)} f\right]_{(m)} \cdot \delta_{(x-m)} \ dm \\ \text{e.g.} \\ \quad = \int_m \left[\int_n Q^{(\delta)}_m{}_{(n)} \cdot f_{(n)} \ dn\right] \cdot \delta_{(x-m)} \ dm = \widehat{Q} f_{(x)} \end{array}}$$

3.) Terms

Diagonalizable: $A \equiv PDP^{-1}$

Conj. Transpose, †: $A^\dagger \equiv A^{T*} = A^{*T}$

Hermitian, H: $H = H^\dagger$

$H = UDU^{-1} = UDU^\dagger$ (spectral theorem)

Unitary, U: $U :\ UU^\dagger = U^\dagger U = 1$

$\exists H :\ U = e^{iH} = (U')e^{iD}(U')^\dagger$

Hermitian Operator ↔ Hermitian Matrix

$\langle Qx|y\rangle = \langle x|Qy\rangle$

(draw it out)

$\rightarrow (\overline{Q}x)^{*T} \cdot y_m|e_m\rangle = y_m x^{*T} \cdot (\overline{Q}^*_m)$

$= x^{*T} \cdot \overline{Q}^{*T} y_m|e_m\rangle$

$= x^{*T} \cdot \overline{Q} y_m|e_m\rangle$

$\rightarrow \overline{Q}^\dagger \equiv \overline{Q}^{*T} = \overline{Q}$ ☑

4.) Eigenvalue Equation

General Case:

$$\widehat{Q}|q_i\rangle = q_i|q_i\rangle$$
$$|q_i\rangle = \sum c_n^{(e)}{}_{[q_i]}\big|e_n\big\rangle$$

$$\boxed{\begin{aligned} \overline{Q}^{(e)} &= UDU^\dagger \qquad \text{(Spectral Theorem)} \\ &= \begin{pmatrix} | & | & \\ \vec{c_0}_{[q_0]} & \vec{c_1}_{[q_1]} & ... \\ | & | & \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & ... \\ 0 & q_1 & ... \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} - & \vec{c_0}^*{}_{[q_0]} & - \\ - & \vec{c_1}^*{}_{[q_1]} & - \\ & \vdots & \end{pmatrix}^{(e)} \\ &\text{where } \langle \vec{c}_m|\vec{c}_n\rangle = \delta_{mn} \end{aligned}}$$

$$\begin{aligned} q_i|q_i\rangle &= \widehat{Q}|q_i\rangle \\ \Big(q_i\ \vec{c}^{\ (e)}{}_{[q_i]}\Big)\cdot\vec{e}(x) &= \Big[\ \overline{Q}^{(e)}\ \vec{c}^{\ (e)}{}_{[q_i]}\Big]\cdot\vec{e}(x) \\ &\Downarrow^* \\ q_i\ \vec{c}^{\ (e)}{}_{[q_i]} &= \overline{Q}^{(e)}\ \vec{c}^{\ (e)}{}_{[q_i]} \\ q_i\begin{pmatrix} | \\ \vec{c}_{\ [q_i]} \\ | \end{pmatrix}^{(e)} &= \begin{pmatrix} | & | & \\ \vec{c}_{\ [q_0]} & \vec{c}_{\ [q_1]} & ... \\ | & | & \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & ... \\ 0 & q_1 & ... \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} - & \vec{c}^{\ *}{}_{[q_0]} & - \\ - & \vec{c}^{\ *}{}_{[q_1]} & - \\ & \vdots & \end{pmatrix}^{(e)} \begin{pmatrix} | \\ \vec{c}_{\ [q_i]} \\ | \end{pmatrix}^{(e)} \qquad \boxed{\checkmark} \\ &\Downarrow \end{aligned}$$

$$^*\forall n \qquad (qc)_n\ |e_n(x)\rangle = (Qc)_n\ |e_n(x)\rangle$$
$$\langle e_n(x)|(qc)_n|e_n(x)\rangle = \langle e_n(x)|(Qc)_n|e_n(x)\rangle$$
$$(qc)_n = (Qc)_n$$

$$\boxed{q_i\ ,\ \vec{c}^{\ (e)}{}_{[q_i]}: \quad \det\Big(\overline{Q}^{(e)} - Iq_i\Big) = 0}$$

Special Case:

$$|q_n\rangle = |e_n\rangle$$
$$\widehat{Q}|e_n\rangle = q_n|e_n\rangle$$

$$\begin{aligned} \widehat{Q}|a\rangle &= \sum_n \widehat{Q}|e_n\rangle\langle e_n|a\rangle \\ &= \Big(\sum_n q_n|e_n\rangle\langle e_n|\Big)|a\rangle \end{aligned} \Rightarrow \boxed{\widehat{Q} = \sum_n q_n|e_n\rangle\langle e_n|} \Rightarrow \boxed{\overline{Q}^{(e)} = \begin{pmatrix} q_0 & 0 & ... \\ 0 & q_1 & ... \\ \vdots & \vdots & q_i \end{pmatrix}}$$

$$Q_{mn}^{(e)} = q_n\delta_{mn}$$

$$\begin{aligned} \overline{Q}^{(e)} = \begin{pmatrix} q_0 & 0 & ... \\ 0 & q_1 & ... \\ \vdots & \vdots & \end{pmatrix}^{(e)} &= \begin{pmatrix} | & | & \\ \vec{c}_{\ [q_0]} & \vec{c}_{\ [q_1]} & ... \\ | & | & \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & ... \\ 0 & q_1 & ... \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} - & \vec{c}^{\ *}{}_{[q_0]} & - \\ - & \vec{c}^{\ *}{}_{[q_1]} & - \\ & \vdots & \end{pmatrix}^{(e)} \\ &= \begin{pmatrix} 1 & 0 & ... \\ 0 & 1 & ... \\ \vdots & \vdots & \end{pmatrix}^{(e)} \begin{pmatrix} q_0 & 0 & ... \\ 0 & q_1 & ... \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} 1 & 0 & ... \\ 0 & 1 & ... \\ \vdots & \vdots & \end{pmatrix}^{(e)} \end{aligned}$$

$$\boxed{\vec{c}^{\ (e)}{}_{[q_i]} = \big(...\ 0\ 0\ 0\ 1_{(i)}\ 0\ 0\ 0\ ...\big)^T}$$

$\Phi(p,t)$ - Momentum Space (generalizable Born Interpretation):

$$\int_x \Psi^*\Psi dx = \int_x \int_p c_p^*(t)\Psi_p^*(x)dp \int_{p'} c_{p'}(t)\Psi_{p'}(x)dp'\ dx$$
$$= \int_p c_p^*(t) \int_{p'} c_{p'}(t) \int_x \Psi_p^*(x)\Psi_{p'}(x)dx\ dp'dp$$
$$= \int_p \Phi^* \int_{p'} \Phi'\ \delta(p-p')dp'dp$$
$$= \int_p \Phi^*\Phi\ dp \Rightarrow \boxed{\mathrm{PDF}(p) = |\Phi|^2 = \Phi^*\Phi}$$

$$\boxed{\langle\Psi|\Psi\rangle = \langle\Phi|\Phi\rangle}$$

$$\hat{x}\Phi_x = \hat{x}e^{\frac{-i}{\hbar}px} = xe^{\frac{-i}{\hbar}px}$$
$$\Rightarrow \boxed{\hat{x}_p = -\frac{\hbar}{i}\partial_p}$$

$$\widehat{A}(x,\hat{p}_x) \to \widehat{A}(\hat{x}_p, p)$$
$$\to \boxed{\langle a\rangle = \left\langle \Phi \middle| \widehat{A}(\hat{x},p) \middle| \Phi \right\rangle}$$

Anything in position space can be done in momentum space

(or generalize to any transform, c_n)

Heisenberg Uncertainty Proof:

$$\langle f|g\rangle \equiv \left\langle \big(\widehat{A}-\langle a\rangle\big)\Psi \middle| \big(\widehat{B}-\langle b\rangle\big)\Psi \right\rangle$$
$$= \left\langle \Psi \middle| \big(\widehat{A}-\langle a\rangle\big)\big(\widehat{B}-\langle b\rangle\big) \middle| \Psi \right\rangle$$
$$= \big\langle \widehat{A}\widehat{B}\big\rangle - \langle a\rangle\langle b\rangle$$

$$\sigma_A^2\sigma_B^2 = \big\langle(\widehat{A}-\langle a\rangle)\Psi\big|(\widehat{A}-\langle a\rangle)\Psi\big\rangle\big\langle\big(\widehat{B}-\langle b\rangle)\Psi\big|(\widehat{B}-\langle b\rangle)\Psi\big\rangle$$
$$\equiv \langle f|f\rangle\langle g|g\rangle \geq \|\langle f|g\rangle\|^2 \qquad \text{(see Schwarz Ineq.)}$$
$$\geq \big[\mathrm{Im}\big(\langle f|g\rangle\big)\big]^2 = \left(\frac{1}{2i}\big[\langle f|g\rangle - \langle f|g\rangle^*\big]\right)^2$$
$$= \left(\frac{1}{2i}\big\langle\widehat{A}\widehat{B} - \widehat{B}\widehat{A}\big\rangle\right)^2 \equiv \boxed{\left(\frac{1}{2i}\left\langle[\widehat{A},\widehat{B}]\right\rangle\right)^2}$$

Commutator

- $\big[\widehat{A},\widehat{B}\big] \equiv \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$
- $\big[A,BC\big] = \big[A,B\big]C + B\big[A,C\big]$
- $\big[AB,C\big] = A\big[B,C\big] + \big[A,C\big]B$
- $\big[x,\hat{p}\big] = i\hbar$
- $\boxed{\sigma_A\sigma_B \geq \left\| \frac{1}{2i}\left\langle[\widehat{A},\widehat{B}]\right\rangle \right\|}$

  $\Rightarrow \boxed{\Delta x\Delta p \geq \hbar/2}$
- $\big[\hat{p},f\big] = (\hat{p}f) = \frac{\hbar}{i}\nabla f$

Commutator of Hermitian $\widehat{A},\widehat{B}$

- $\big[A,B\big]^\dagger = -\big[A,B\big]$
- $\exists\Psi_n$ s.t. $\big(\widehat{A}\Psi_n = a\Psi_n\big)$, $\big(\widehat{B}\Psi_n = b\Psi_n\big)$

  $\Leftrightarrow \big[\widehat{A},\widehat{B}\big] = 0$

  $\Rightarrow \boxed{\sigma_A\sigma_B \geq 0 \text{ (Both can be measured concurrently)}}$

  $\boxed{AB = BA}$

Anti-Hermitian Operators: $A^\dagger = -A$

- $\langle A\rangle = ai, \quad a\in\mathbb{R}$
- $\big[A,B\big]^\dagger = -\big[A,B\big]$

Operator Evolution

$$\frac{d}{dt}\Big\langle\Psi(x,t)\Big|Q\Big|\Psi(x,t)\Big\rangle = \Big\langle\frac{\partial\Psi}{\partial t}\Big|Q\Big|\Psi\Big\rangle + \Big\langle\Psi\Big|\frac{\partial Q}{\partial t}\Big|\Psi\Big\rangle + \Big\langle\Psi\Big|Q\Big|\frac{\partial\Psi}{\partial t}\Big\rangle$$

$$\boxed{\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\Big\langle\big[\widehat{H},\widehat{Q}\big]\Big\rangle + \left\langle\frac{\partial\widehat{Q}}{\partial t}\right\rangle}$$

(Q is conserved when this equals 0!!!)

- Conservations: $\dfrac{d\langle\Psi|\Psi\rangle}{dt} = 0, \quad \dfrac{d\langle H\rangle}{dt} = 0$
- Ehrenfest's Theorem: $m\dfrac{d\langle x\rangle}{dt} = \langle p\rangle, \quad \dfrac{d\langle p\rangle}{dt} = -\left\langle\dfrac{\partial V}{\partial x}\right\rangle \Rightarrow$ other classical eq.
- Virial Theorem: $\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\big\langle[H,x]p + x[H,p]\big\rangle$

$$= \Big\langle\tfrac{d\langle x\rangle}{dt}p + x\tfrac{d\langle p\rangle}{dt}\Big\rangle$$

$$\boxed{\frac{d\langle xp\rangle}{dt} = 2\langle T\rangle - \left\langle x\frac{\partial V}{\partial x}\right\rangle} \to 0 = \tfrac{d}{dt}\Big\langle\Psi_n(x)\Big|Q_{(x,p)}\Big|\Psi_n(x)\Big\rangle \quad \text{(for stationary states)}$$

- Energy-Time Uncertainty: $\Big(\widehat{Q} = \widehat{Q}(x,\hat{p}) \neq \widehat{Q}(x,\hat{p},t)\Big) \Rightarrow \sigma_H\sigma_Q \geq \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right|$

$\Rightarrow$

$$\sigma_Q \equiv \frac{d\langle Q\rangle}{dt}\Delta t \approx \Delta\langle Q\rangle$$

$$\sigma_H\left(\frac{\sigma_Q}{|d\langle Q\rangle/dt|}\right) \geq \frac{\hbar}{2}$$

$$\Delta E\Delta t \geq \frac{\hbar}{2}$$

Δt is the amount of time it would take $\langle Q\rangle$ to change "appreciably", or one std. dev. at the constant rate $\frac{d}{dt}\langle Q\rangle$

Mass Lifetime:

$\Delta(mc^2)\Delta t \geq \frac{\hbar}{2}$ ☑

Orthogonal Time Example:

$\Psi(x,\tau) = \frac{\sqrt{2}}{2}(\Psi_1 e^{-\frac{i}{\hbar}E_1\tau} + \Psi_2 e^{-\frac{i}{\hbar}E_2\tau})$

$\Big\langle\Psi(x,0)\Big|\Psi(x,\tau)\Big\rangle = 0 = \frac{1}{2}(e^{-\frac{i}{\hbar}E_1\tau} + e^{-\frac{i}{\hbar}E_2\tau})$

$\Rightarrow \tau\,\frac{E_2-E_1}{2} = \frac{\pi}{2}\,\hbar\,(\frac{1}{2}+n) \geq \frac{\hbar}{2}$ ☑

Translation Operator

$$f(x+\Delta x) \approx f(x) + \frac{df}{dx}\Delta x$$

$$= f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + ... = \left\{ \begin{matrix} f(x') = \sum_n \frac{f^{(n)}(a)}{n!}(x'-a)^n \\ (x' = x+\Delta x),\ (a=x) \end{matrix} \right\}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(\Delta x)^n \ = \ \sum_{n=0}^{\infty} \frac{(\Delta x \nabla)^n}{n!} f(x)$$

$$\boxed{f(x+\Delta x) = e^{\frac{i}{\hbar}(\Delta x)\hat{p}}\ f(x)} \ \Leftrightarrow\ \boxed{f(x) = e^{\frac{i}{\hbar}x\hat{p}}\ f(0)}$$

Time Translation:

$$f(t+\Delta t) = f(t) + f'(t)\Delta t + ... \ = \ \sum_n \frac{(\Delta t)^n}{n!}\left(\frac{\partial}{\partial t}\right)^n f(t)$$

$$i\hbar\frac{\partial}{\partial t} = \widehat{H} \qquad \frac{\partial f}{\partial t} = \left(\frac{-i\widehat{H}}{\hbar}\right) f \qquad \Rightarrow \qquad \left\{ \begin{matrix} ... = \sum_n^{\infty} \frac{(\Delta t\ \partial_t)^n}{n!} f(t) \\ \boxed{f(t+\Delta t) = e^{\frac{-i}{\hbar}(\Delta t)\widehat{H}}\ f(t)} \ \Leftrightarrow \\ f(0+t) = \boxed{e^{\frac{-i}{\hbar}t\widehat{H}}\ f(0) = f(t)} \end{matrix} \right.$$

Pictures: $\big\langle Q \big\rangle_{(t)} = \langle \Psi_{(x,t)} \,|\, Q_{(x,p,t)} \,|\, \Psi_{(x,t)} \rangle$

- $\boxed{\text{Schrodinger Picture:} \quad \big\langle Q \big\rangle_{(t)} = \Big\langle e^{\frac{-i}{\hbar}t\widehat{H}}\ \Psi_{(x,0)} \,|\, Q_{(x,p,t)} \,|\, e^{\frac{-i}{\hbar}t\widehat{H}}\ \Psi_{(x,0)} \Big\rangle}$

$$Q = Q(x,p) \ \Rightarrow\ \big\langle Q \big\rangle_{(t)} = \Big\langle \sum e^{\frac{-i}{\hbar}E_n t}\ c_n \Psi_{n(x)} \,|\, Q \,|\, \sum e^{\frac{-i}{\hbar}E_n t}\ c_n \Psi_{n(x)} \Big\rangle \qquad \text{(nice for stationary states)}$$

- $\boxed{\text{Heisenberg Picture:} \quad \big\langle Q \big\rangle_{(t)} = \Big\langle \Psi_{(x,0)} \Big| e^{\frac{i}{\hbar}t\widehat{H}}\ Q_{(x,p,t)}\ e^{\frac{-i}{\hbar}t\widehat{H}} \Big|\ \Psi_{(x,0)} \Big\rangle}$

- $\boxed{\text{Dirac Picture:} \quad \big\langle Q \big\rangle_{(t)} = \Big\langle e^{\frac{-i}{\hbar}t\widehat{H}_0}\ \Psi_{(x,0)} \Big| e^{\frac{i}{\hbar}t\widehat{H}_1}\ Q_{(x,p,t)}\ e^{\frac{-i}{\hbar}t\widehat{H}_1} \Big|\ e^{\frac{-i}{\hbar}t\widehat{H}_0}\ \Psi_{(x,0)} \Big\rangle}$

$\big\langle Q \big\rangle_{(t+\Delta t)} \ = \ \big\langle Q \big\rangle_{(t)} + \frac{d\langle Q \rangle}{dt}\Delta t + ... \quad \Rightarrow$ A 1st order approximation of $\big\langle Q \big\rangle_{(t+\Delta t)}$ should yield $\frac{d\langle Q \rangle}{dt} = \frac{i}{\hbar}\Big\langle \big[H, Q\big] \Big\rangle + \frac{\partial Q}{\partial t}$

1.3 Extra

$L_2 \subset$ Hilbert Space = complete inner product space

$P(t) = \Big\langle \Psi(x,t) \Big| \Psi(x,t) \Big\rangle$, $P_{ab}(t) = \Big\langle \Psi(x,t) \Big| \Psi(x,t) \Big\rangle_{ab}$

- $$\begin{array}{lllll} (V \in \mathbb{R}) & \Rightarrow & \frac{d}{dt}P = 0 & \Rightarrow & P(t) \equiv 1 \\ (V = V_0 - i\Gamma) & \Rightarrow & \frac{d}{dt}P = \frac{-\,2\Gamma}{\hbar}P & \Rightarrow & P(t) = e^{-2(\Gamma/\hbar)t} \end{array}$$

- $\frac{d}{dt}P_{ab} = J(a,t) - J(b,t)$
  $$J(x,t) = \frac{1}{2m}(\Psi^* \hat{p} \Psi - \Psi \hat{p} \Psi^*) \qquad \text{(Probability Current)}$$

- $\langle \Psi_n | \Psi_n \rangle$, $\langle \Psi_m | \Psi_m \rangle = 1 \;\Rightarrow\; \frac{d}{dt}\langle \Psi_n | \Psi_m \rangle = 0$

Schwarz Inequality:
$$\left\| \int_a^b f^* g \; dx \right\|^2 \leq \left\| \int_a^b f^* f \; dx \right\| \left\| \int_a^b g^* g \; dx \right\|$$
$$\| \langle f | g \rangle_{ab} \|^2 \leq \| \langle f | f \rangle_{ab} \| \, \| \langle g | g \rangle_{ab} \|$$

$$\Big[V(x) = V(-x) \Big] \;\Rightarrow\; \Big[\Psi(x) \Rightarrow \Psi(-x) \Big] \;\Rightarrow\; \Big[\Psi(-x) = \Psi(x) \Big] \;\cup\; \Big[\Psi(-x) = -\Psi(x) \Big]$$

Discontinuity in Ψ means the possiblity of $\sigma_p \to \infty$

Prob 3.29: $\Psi(x,0) = \begin{cases} \frac{1}{\sqrt{2n\lambda}} e^{2\pi i x/\lambda}, & -n\lambda < x < n\lambda \\ 0 & \text{else} \end{cases}$

$\sigma_p \to \infty$ because the integral of $\delta^2(x)$ is infinite

$\int_{-\infty}^{\infty} f(x) D_1(x) dx = \int_{-\infty}^{\infty} f(x) D_2(x) dx \;\Rightarrow\; \delta(cx) = \frac{1}{|c|}\delta(x)$

$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \;\Rightarrow\; F[\delta(x)] = \frac{1}{2\pi}$

Associated Legendre Functions: $P_l^m \equiv \sqrt{1-x^2}^{\;|m|} \left(\frac{d}{dx}\right)^{|m|} P_l(x)$ (not a polynomial if odd)

Legendre Polynomials: $P_l(x) \equiv \frac{1}{2^l l!} \left(\frac{d}{dx}\right)^l (x^2 - l)^l$

Associated Laguerre Polynomials: $L \equiv$

Laguerre Polynomials: $L_{\equiv}$

2 Simple Potentials

2.1 Infinite Square Well (1-D)

$$\boldsymbol{V(x)} = \begin{cases} \mathbf{0} & \boldsymbol{0 < x < a} \\ \boldsymbol{\infty} & \text{otherwise} \end{cases}$$

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin k_n x$$

$$k_n = \frac{2\pi}{\lambda} = \frac{2\pi}{2a/n} = \frac{n\pi}{a} \qquad \forall n = 1, 2, 3, ... \qquad \boxed{!!\ \hat{p}\Psi_n \neq p\Psi_n\ !!} \qquad \text{wave isn't infinite}$$

$$E_n = \frac{p^2}{2m} = \frac{\hbar^2 k_n^2}{2m}$$

2.1.1 3-D Rectangular Box

$$\Psi_{n_x n_y n_z}(x, y, z) = \Psi_{n_x}(x)\Psi_{n_y}(y)\Psi_{n_z}(z) = \sqrt{\frac{8}{a_x a_y a_z}}(\sin k_{n_x} x)(\sin k_{n_y} y)(\sin k_{n_z} z)$$

$$k_{n_i} = \frac{n_i \pi}{a_i} \qquad \forall n_x, n_y, n_z = 1, 2, 3, ...$$

$$E_{n_x n_y n_z} = \frac{\hbar^2}{2m}(k_{n_x}^2 + k_{n_y}^2 + k_{n_z}^2)$$

2.2 Harmonic Oscillator (1-D)

$$\boldsymbol{V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2}$$

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right) = \frac{1}{2m}\left(-ip + m\omega x\right)\left(ip + m\omega x\right) \sim E \sim \hbar\omega \qquad \Rightarrow$$

$$\boxed{a = a_- = \frac{1}{\sqrt{2m}}\frac{1}{\sqrt{\hbar\omega}}\left(i\hat{p} + m\omega x\right)}$$

$$H = \hbar\omega(aa^\dagger - 1/2) = \hbar\omega(a^\dagger a + 1/2) \quad \rightarrow \quad \boxed{\left[a, a^\dagger\right] = 1}$$

$$\begin{aligned} H(a\Psi_n) &= \hbar\omega(aa^\dagger - 1/2)a\Psi_n \\ &= a\hbar\omega(a^\dagger a + 1/2 - 1)\Psi_n \\ &= a(H - \hbar\omega)\Psi_n \\ &= (E_n - \hbar\omega)(a\Psi_n) \\ &\Rightarrow \\ E_{n-1}\Psi_{n-1} &= (E_n - \hbar\omega)\Psi_{n-1} \end{aligned}$$

$$\begin{aligned} H(a^\dagger\Psi_n) &= \hbar\omega(a^\dagger a + 1/2)a^\dagger\Psi_n \\ &= a^\dagger\hbar\omega(aa^\dagger - 1/2 + 1)\Psi_n \\ &= a^\dagger(H + \hbar\omega)\Psi_n \\ &= (E_n + \hbar\omega)(a^\dagger\Psi_n) \\ &\Rightarrow \\ E_{n+1}\Psi_{n+1} &= (E_n + \hbar\omega)\Psi_{n+1} \end{aligned}$$

$$\boxed{\begin{gathered} H(a^\dagger)^n\Psi_0 = (E_0 + n\hbar\omega)(a^\dagger)^n\Psi_0 \\ \\ E_n\Psi_n = (E_0 + n\hbar\omega)\Psi_n \end{gathered}}$$

$E_n \geq \text{Min}(V) \;\Rightarrow\; a\Psi_0 = 0$ (else let it be un-normalizable)

$$\begin{aligned} 0 &= (ip + m\omega x)\Psi_0 \\ -m\omega x\Psi_0 &= \hbar\tfrac{d}{dx}\Psi_0 \end{aligned}$$

$$\begin{aligned} H\Psi_0 &= \hbar\omega(a^\dagger a + 1/2)\Psi_0 \\ E_0\Psi_0 &= \tfrac{1}{2}\hbar\omega\Psi_0 \end{aligned}$$

$$\boxed{\Psi_0 = Ae^{-\frac{m\omega}{2\hbar}x^2}, \quad A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}}$$

$$\boxed{E_n = (n + 1/2)\hbar\omega, \quad \forall n = 0, 1, 2, 3, ...}$$

$$H\Psi_n = E_n\Psi_n$$
$$\hbar\omega(a^\dagger a + 1/2)\Psi_n = (n+1/2)\hbar\omega\Psi_n$$

$$\boxed{a^\dagger a\Psi_n = n\Psi_n}$$

$$\left\langle \Psi_n | a^\dagger a \Psi_n \right\rangle = n = \langle a\Psi_n | a\Psi_n\rangle$$
$$= \langle c_n\Psi_{n-1} | c_n\Psi_{n-1}\rangle$$

$$\boxed{a\Psi_n = \sqrt{n}\ \Psi_{n-1}}$$

$$H\Psi_n = E_n\Psi_n$$
$$\hbar\omega(aa^\dagger - 1/2)\Psi_n = (n+1/2)\hbar\omega\Psi_n$$

$$\boxed{aa^\dagger\Psi_n = (n+1)\Psi_n}$$

$$\left\langle \Psi_n | aa^\dagger \Psi_n \right\rangle = n+1 = \left\langle a^\dagger\Psi_n | a^\dagger\Psi_n\right\rangle$$
$$= \langle c_n\Psi_{n+1} | c_n\Psi_{n+1}\rangle$$

$$\boxed{a^\dagger\Psi_n = \sqrt{n+1}\ \Psi_{n+1}}$$

$$\boxed{\Psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\Psi_0}$$

2.2.1 Position/Momentum Operators

$$\boxed{x = \frac{1}{2}\frac{\sqrt{2m}\sqrt{\hbar\omega}}{m\omega}(a+a^\dagger)} \qquad \boxed{\hat{p} = \frac{1}{2}\frac{\sqrt{2m}\sqrt{\hbar\omega}}{i}(a-a^\dagger)}$$

Show Virial Theorem Works

$$\boldsymbol{2\langle T\rangle = N\langle V\rangle}$$

$$\begin{aligned}
E_n &= 2\langle V\rangle_n \\
&= 2\langle \Psi_n|V|\Psi_n\rangle \\
&= 2\left\langle \Psi_n \left| \tfrac{1}{2}mw^2 \tfrac{2m\hbar\omega}{(2m\omega)^2}(a+a^\dagger)^2 \right| \Psi_n \right\rangle \\
&= \tfrac{2m^2\hbar\omega^3}{(2m\omega)^2}\left(0 + \left\langle \Psi_n \left| (aa^\dagger + a^\dagger a) \right| \Psi_n \right\rangle + 0\right)
\end{aligned}$$

$$E_n = (n+1/2)\hbar\omega \quad \text{☑}$$

Test the Uncertainty Principle

$$\sigma_x\sigma_p \geq \tfrac{1}{2}\left|\left\langle [x,p] \right\rangle\right|$$

$$\begin{aligned}
xp - px &= \tfrac{2m\hbar\omega}{4m\omega i}\begin{pmatrix} a^2 - aa^\dagger + a^\dagger a - (a^\dagger)^2 \\ -a^2 + a^\dagger a - aa^\dagger + (a^\dagger)^2 \end{pmatrix} \\
&= \tfrac{\hbar}{i}(a^\dagger a - aa^\dagger) \\
&= i\hbar(n+1-n) \\
&\Rightarrow \;\; \sigma_x\sigma_p \geq \frac{\hbar}{2} \quad \text{☑}
\end{aligned}$$

$$\begin{aligned}
\sigma_x^2 &= \langle x^2\rangle - \langle x\rangle^2 & \sigma_p^2 &= \langle p^2\rangle - \langle p\rangle^2 \\
&= \tfrac{2m\hbar\omega}{4m^2\omega^2}\begin{bmatrix} \langle (a+a^\dagger)^2\rangle \\ -\langle a+a^\dagger\rangle^2 \end{bmatrix} & &= \tfrac{2m\hbar\omega}{-4}\begin{bmatrix} \langle (a-a^\dagger)^2\rangle \\ -\langle a-a^\dagger\rangle^2 \end{bmatrix} \\
&= \tfrac{\hbar}{2m\omega}\langle aa^\dagger + a^\dagger a\rangle & &= \tfrac{\hbar m\omega}{2}\langle aa^\dagger + a^\dagger a\rangle \\
&= \tfrac{\hbar}{m\omega}(n+\tfrac{1}{2}) & &= \hbar m\omega(n+\tfrac{1}{2})
\end{aligned}$$

$$\Rightarrow \;\; \sigma_x\sigma_p = \hbar(n+\tfrac{1}{2}) \geq \tfrac{\hbar}{2} \quad \text{☑}$$

2.2.2 Coherent States

$$\boldsymbol{a|\alpha\rangle = \alpha|\alpha\rangle}$$

$$\boxed{\sigma_x\sigma_p = \frac{\hbar}{2}}$$

$$\langle\alpha|\alpha\rangle = \langle\alpha|\begin{pmatrix} \sum_{n=0}^{\infty}\langle\Psi_n|\alpha\rangle\ |\Psi_n\rangle = \\ \sum_{n=0}^{\infty}\left\langle \frac{(a^\dagger)^n}{\sqrt{n!}}\Psi_0\middle|\ \alpha\right\rangle\ |\Psi_n\rangle = \\ \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle\Psi_0|\alpha\rangle\ |\Psi_n\rangle \end{pmatrix} \Rightarrow \boxed{\big|\alpha\big\rangle = e^{-\alpha^2/2}\ \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\ |\Psi_n\rangle} \rightarrow \big|\alpha{=}0\big\rangle = |\Psi_0\rangle$$

$$= \langle\Psi_0|\alpha\rangle^2\sum_{n=0}^{\infty}\frac{(\alpha^2)^n}{n!}\ \langle\Psi_n|\Psi_n\rangle$$

$$= \langle\Psi_0|\alpha\rangle^2\ e^{\alpha^2} = 1$$

$$\begin{aligned} a\big|\alpha_{(x,t)}\big\rangle &= e^{-\frac{\alpha^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\ e^{\frac{-i}{\hbar}E_n t}a\big|\Psi_n\big\rangle \\ &= e^{-\frac{\alpha^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\ e^{\frac{-i}{\hbar}\hbar\omega(\frac{1}{2}+n)t}\sqrt{n}\big|\Psi_{n-1}\big\rangle \\ &= \left(\alpha e^{\frac{-i}{\hbar}\hbar\omega t}\right)e^{-\frac{\alpha^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\ e^{\frac{-i}{\hbar}\hbar\omega(\frac{1}{2}+n)t}\big|\Psi_n\big\rangle \end{aligned}$$

$$\boxed{a\big|\alpha_{(x,t)}\big\rangle = \left(\alpha e^{-i\omega t}\right)\big|\alpha_{(x,t)}\big\rangle}$$

$|\alpha\rangle$ are obviously not orthogonal. They are an overcomplete basis.

2.2.3 Analytic Method

$$\boldsymbol{\Psi_n = A\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}}$$

$A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

$\xi = \sqrt{\frac{m\omega}{\hbar}}x$

Hermite Polynomials:

$$H_n(x) = (-1)^n\ e^{-x^2}\left(\frac{d}{dx}\right)^n e^{x^2}$$

$$e^{-z^2+2zx} = \sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x)$$

2.2.4 3-D Harmonic Potential

$$\boldsymbol{V(r) = \frac{1}{2}kr^2}$$

$$\boxed{E_{n_x n_y n_z} = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)}$$

2.3 Free Particle (1-D)

$$\boldsymbol{V(x) = 0}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi(x,0) e^{\frac{i}{\hbar}px - Et} dp$$

$$\Phi(x,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x,0) e^{\frac{-i}{\hbar}px} dx$$

($E < 0 \to \Psi = e^{\pm kx}$ is possible and also not normalizable, but solution above is already a complete set)

$$E(p) = \frac{p^2}{2m}$$

$$v_{\text{wave}} = \boxed{v_{\text{phase}} = \frac{\omega(k)}{k}} = \frac{E}{p} = \frac{v_{\text{classical}}}{2}$$

$$v_{\text{particle}} \approx \boxed{v_{\text{group}} = \frac{d\omega(k)}{dk}} = 2v_{\text{wave}} \qquad \text{(dispersion relation)}$$

2.4 Delta Potential (1-D)

Potential Well: $\boldsymbol{V(x) = -\alpha\delta(x)}$ ($\alpha \to -\alpha$ for potential wall)

Bound State ($E < 0$) [only for Well]:

$$\Psi = \sqrt{k} e^{k|x|} = \begin{cases} \sqrt{k} e^{kx} & x \leq 0 \\ \sqrt{k} e^{-kx} & x \geq 0 \end{cases}$$

$$k = \frac{m\alpha}{\hbar^2}$$

$$E = -\frac{(\hbar k)^2}{2m}$$

Scattering State ($E > 0$) [for both]:

$$\Psi = \begin{cases} Ae^{iKx} + Be^{-iKx} & x < 0 \\ Fe^{iKx} & x > 0 \end{cases}$$

$$E = \frac{(\hbar K)^2}{2m}\ , \qquad \beta \equiv \frac{k}{K} = \frac{m\alpha/\hbar^2}{K}$$

$$B = \frac{i\beta}{1 - i\beta} A\ , \qquad F = \frac{1}{1 - i\beta} A$$

$$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1 + \beta^2}\ , \qquad T = \frac{|F|^2}{|A|^2} = \frac{1}{1 + \beta^2}$$

Can't normalize. All free particles have ranges of p and thus E, so R and T are approx. in the vicinity of E.

2.5 Finite Square Potential (1-D)

2.5.1 Potential Well $\quad V(x) = \begin{cases} -V_0 & -a < x < a \\ 0 & \text{otherwise} \end{cases}$

($V_0 \to -V_0$ for wall and do cases for $E > V_0, E = V_0, E < V_0$, and change to $\sinh, \cosh$ if needed)

$$k\;;K: \quad E = \tfrac{-(\hbar k)^2}{2m} = \tfrac{(\hbar K)^2}{2m}$$

$$l: \quad E + V_0 = \tfrac{(\hbar l)^2}{2m}$$

$$v: \quad V_0 = \tfrac{\hbar^2 v^2}{2m} = \tfrac{\hbar^2(l^2+k^2)}{2m} = \tfrac{\hbar^2(l^2-K^2)}{2m}$$

$$\frac{k_a}{l_a} \equiv \sqrt{\frac{(ka)^2}{(la)^2}} = \sqrt{\frac{(la)^2+(ka)^2}{(la)^2} - 1}$$

$$\boxed{\frac{k_a}{l_a} \equiv \sqrt{\left(\frac{v_a}{l_a}\right)^2 - 1}}\;,\quad v_a^2 = \begin{cases} l_a^2 + k_a^2 \\ l_a^2 - K_a^2 \end{cases}$$

Bound State $\;(E_n < 0)$ [only for well]:

$$\Psi_{\text{even}}(x) = \begin{cases} \Psi(-x) & x < 0 \\ D\cos(lx) & 0 < x < a \\ Fe^{-kx} & a < x \end{cases}$$

- $F = D\cos(la)e^{ka}$
- $\frac{-(\partial_x\Psi)(a)}{\Psi(a)} = k = l\tan(la) \;\Rightarrow$

  $\tan(l_a) = \sqrt{(v_a/l_a)^2 - 1}$

  big $v_a \;\to\; l \approx_< \frac{n\pi}{2a} \;\to\; E_n + V_0 \;=\; \frac{\hbar^2 l^2}{2m}$; n odd
- $\boxed{n_{\max} = \left\lfloor \frac{v_a}{\pi} \right\rfloor + 1}$

$$\Psi_{\text{odd}}(x) = \begin{cases} -\Psi(-x) & x < 0 \\ C\sin(lx) & 0 < x < a \\ Fe^{-kx} & a < x \end{cases}$$

- $F = D\sin(la)e^{ka}$
- $\frac{-(\partial_x\Psi)(a)}{\Psi(a)} = k = -l\cot(la) \;\Rightarrow$

  $-\cot(l_a) = \sqrt{(v_a/l_a)^2 - 1}$

  big $v_a \;\to\; l \approx_< \frac{n\pi}{2a} \;\to\; E_n + V_0 \;=\; \frac{\hbar^2 l^2}{2m}$; n even
- $\boxed{n_{\max} = \left\lfloor \frac{v_a + \frac{\pi}{2}}{\pi} \right\rfloor}$

Scattering State $\;(E > 0)$ [for both]:

$$\Psi = \begin{cases} Ae^{iKx} + Be^{-iKx} & x < -a \\ C\sin lx + D\cos lx & -a < x < a \\ Fe^{iKx} & a < x \end{cases}$$

$$\frac{d\Psi}{dx} = \begin{cases} iKAe^{iKx} - iKBe^{-iKx} & x < -a \\ lC\cos lx - lD\sin lx & -a < x < a \\ iKFe^{iKx} & a < x \end{cases}$$

$$B = i\sin(2l_a)\left(\tfrac{l_a^2-K_a^2}{2K_a l_a}\right)\; F$$

$$F = \frac{e^{-2iK_a}}{\cos(2l_a) - i\left(\tfrac{l_a^2+K_a^2}{2K_a l_a}\right)\sin(2l_a)}\; A$$

(Can't normalize. See delta potential.)

$$T^{-1} = 1 + \left(\tfrac{l_a^2-K_a^2}{2K_a l_a}\right)^2\; \sin^2(2l_a)$$

$$= 1 + \frac{V_0^2}{4E(E+V_0)}\; \sin^2\left(2a\sqrt{\frac{E+V_0}{\hbar^2/2m}}\right)$$

(full transmission at inf. sqr. well $E_n + V_0 = \frac{\hbar^2 l^2}{2m}$; $l = \frac{n\pi}{2a}$)

2.6 Hydrogen Atom

$$Eu = \left(\frac{\hat{p}_r^2}{2m} + V(r) + \frac{\hat{L}^2}{2(mr^2)} \right) u \qquad u(r) = rR(r)$$

$$\boxed{Eu = \frac{-\hbar^2}{2m}\partial_r^2 u + \left[-\frac{ke^2}{r} + \frac{\hbar^2 l(l+1)}{2mr^2} \right] u}$$

$\boxed{\Psi_{nlm}(\vec{\mathbf{r}}) = R_{nl}(r)\ Y_l^m(\theta,\phi) = R_{nl}(r)\ \Theta_l^m(\theta)\ \Phi_m(\phi)}$

- $\Phi_m(\phi) = e^{im\phi}$
- $\Theta_l^m(\theta) = AP_l^m(\cos\theta)$
  - $A = \epsilon\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}$, $\epsilon = \begin{cases} (-1)^m & (m \geq 0) \\ 1 & (m \leq 0) \end{cases}$
  - $P_l^m(x)$ Assoc. Legendre Func. (see extra)
- $R_{nl}(r) = \frac{B}{r}\rho^{l+1}e^{-\rho}\nu(\rho)$
  - $\rho = k_n r$, $k_n = \frac{1}{a_0 n}$ (fine structure below)
  - $\nu(\rho) = L_{n-l-1}^{2l+1}(2\rho)$ Assoc. Laguerre Poly. (see extra)
  - $B = \sqrt{2k_n \frac{(n-l-1)!}{2n[(n+l)!]^3}}\ 2^{l+1}$

$\boxed{\alpha \equiv \frac{kqq}{\hbar c} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137}}$ $\qquad$ $\boxed{a_0 \equiv \frac{\hbar^2}{m(kqq)} = \frac{4\pi\epsilon_0\hbar^2}{me^2}}$

$$E_n = -\frac{\hbar^2 k_n^2}{2m} \quad \Rightarrow \quad \boxed{E_n = -\frac{\hbar^2}{2ma_0^2}\frac{1}{n^2} = -\frac{1}{2}\alpha^2\left(mc^2\right)\frac{1}{n^2} \approx -13.6\ \frac{1}{n^2}\ [\text{eV}]}$$

$$\boxed{\frac{1}{\lambda} = \frac{\alpha^2\left(mc^2\right)}{2hc}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \ , \quad R = 1.097\ \text{E7}\ [\text{m}^{-1}]}$$

Quantum Numbers - n, l, m:

- $\Big(n \in \{1,2,3,...\}\Big)$, $\Big(l \in \{0,1,2,...,n-1\}\Big)$, $\Big(m \in \{-l,...,-1,0,1,...,l\}\Big)$
  - Degeneracy is n^2

(outdated) Bohr Model:

- $L = (\bar{r})(\bar{p}) = (a_0 n^2)(\hbar k_n) = n\hbar$ (not correct!!)
- Electrons don't radiate about the nucleus
- Energy diff. follows Rydberg formula

3 Spin and L

3.1 Hydrogen Atom

Angular Momentum:

$$\boxed{\widehat{L}_i \equiv (\vec{r} \times \vec{p})_i}$$

$$\boxed{\widehat{L}_\pm \equiv \widehat{L}_x \pm i\widehat{L}_y}$$

$$\boxed{\widehat{L}^2 \equiv L_x^2 + L_y^2 + L_z^2}$$

$$= L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

Commutation Relations:

$$\boxed{\big[L_x, L_y\big] = i\hbar L_z}$$ (can't measure concurrently)

$$\boxed{\big[H, L^2\big] = \big[H, L_i\big] = \big[L^2, L_i\big] = 0}$$ (can measure concurrently)

$$\rightarrow \begin{pmatrix} L_z Y_{m'} = m' Y_{m'} \ , \\ L^2 Y_{m'} = \lambda_{m'} Y_{m'} \end{pmatrix} \Rightarrow \quad \langle L^2 \rangle = \lambda_{m'} \ \geq \ (m')^2 = \langle L_z \rangle^2$$

- $\sqrt{\lambda_{m'}} \geq m' \geq -\sqrt{\lambda_{m'}}$

Let $(L_\pm)^n Y_\mu \equiv |m\rangle$

$$\boxed{\big[L_z, (L_\pm)^n\big] = \pm n\hbar (L_\pm)^n} \quad \Rightarrow \quad L_z\big[(L_\pm)^n Y_\mu\big] = (\mu \pm n\hbar)\big[(L_\pm)^n Y_\mu\big]$$

(Proof By Ind.)

- $\boxed{\big[L_z, L_\pm\big] = \pm\hbar L_\pm}$
- $\big[L_z, (L_\pm)^{n+1}\big] = \pm(n+1)\hbar(L_\pm)^{n+1}$

- $L_z|m\rangle = (\mu \pm n\hbar)|m\rangle$

$$\boxed{\big[L^2, L_\pm\big] = 0} \ \Rightarrow \ \big[L^2, (L_\pm)^n\big] = 0 \quad \Rightarrow \quad L^2\big[(L_\pm)^n Y_\mu\big] = \lambda_\mu\big[(L_\pm)^n Y_\mu\big]$$

- $L^2|m\rangle = \lambda_\mu|m\rangle$

Then $\left(\sqrt{\lambda_\mu} \ \geq \ (\mu \pm n\hbar) \ \geq \ -\sqrt{\lambda_\mu}\right) \Rightarrow$ **Let** (else un-normalizable solution)

$\underline{L_+|m_t\rangle = 0}$, $L_z|m_t\rangle = \hbar l$,
$L^2|m_t\rangle = \lambda$, $L^2 = L_- L_+ + L_z^2 + \hbar L_z$

- $L^2|m_t\rangle = \hbar^2 l(l+1)|m_t\rangle = \lambda|m_t\rangle$

$\underline{L_-|m_b\rangle = 0}$, $L_z|m_b\rangle = \hbar l'$,
$L^2|m_b\rangle = \lambda$, $L^2 = L_+ L_- + L_z^2 - \hbar L_z$

- $L^2|m_b\rangle = \hbar^2 l'(l'-1)|m_b\rangle = \lambda|m_b\rangle$

$$\Big(\lambda = \hbar^2 l'(l'-1) = \hbar^2 l(l+1)\Big) \ \Rightarrow \ \big(l' = -l\big) \ \Rightarrow \begin{pmatrix} L_z|m_t\rangle = \hbar l|m_t\rangle \\ L_z|m_b\rangle = -\hbar l|m_b\rangle \end{pmatrix}$$ (Spherical Harmonics do not allow half-integer l)

Schrodinger Y_l^m:

$$\boxed{\begin{array}{c|l} l \in \{0, 1, 2, ...\} & L_z\big|Y_l^m\big\rangle = \hbar m\big|Y_l^m\big\rangle \\ & L^2\big|Y_l^m\big\rangle = \hbar^2 l(l+1)\big|Y_l^m\big\rangle \\ m \in \{-l, -l+1, ..., l-1, l\} & L_\pm\big|Y_l^m\big\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}\ \big|Y_l^m\big\rangle \end{array}}$$

3.2 Generalized

Angular Momentum:

$\widehat{L_i} \equiv ???$

$\boxed{L_{\pm} \equiv L_x \pm iL_y}$

$\boxed{L^2 \equiv L_x^2 + L_y^2 + L_z^2}$

$= L_{\pm}L_{\mp} + L_z^2 \mp \hbar L_z$

Commutation Relations:

$\boxed{\big[L_i, L_j\big] = i\hbar L_k \; \epsilon_{ij}}$ (can't measure concurrently)

$\boxed{\big[L^2, L_z\big] = 0}$ (can measure concurrently)

General:

$$\boxed{\begin{array}{c|l} l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ...\} & L_z\big|l\ m\big\rangle = \hbar m\big|l\ m\big\rangle \\ & L^2\big|l\ m\big\rangle = \hbar^2 l(l+1)\big|l\ m\big\rangle \\ m \in \{-l, -l+1, ..., l-1, l\} & L_{\pm}\big|l\ m\big\rangle = \hbar\sqrt{l(l+1)-m(m\pm1)}\ \big|l\ m\big\rangle \end{array}}$$

3.3 1 Particle w/ Spin, $s = \frac{1}{2}$

*Find the Eigenvectors, e_i , of S_z and S^2 in the form of $|\chi\rangle$

$$* \boxed{e_i \in \left\{ \ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle \equiv |\uparrow\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix} \quad , \quad |\tfrac{1}{2}\ \tfrac{-1}{2}\rangle \equiv |\downarrow\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix} \ \right\}}$$

$$\left.\begin{aligned} S^2|\uparrow\rangle &= \frac{3\hbar^2}{4}|\uparrow\rangle \\ S^2|\downarrow\rangle &= \frac{3\hbar^2}{4}|\downarrow\rangle \end{aligned}\right\} \Rightarrow S^2 = \frac{3\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} = * \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{3\hbar^2}{4}&0\\0&\frac{3\hbar^2}{4}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}^{T*}$$

$$\left.\begin{aligned} S_-|\uparrow\rangle &= \hbar|\downarrow\rangle \\ S_+|\downarrow\rangle &= \hbar|\uparrow\rangle \\ S_+|\uparrow\rangle &= S_-|\downarrow\rangle = 0 \end{aligned}\right\} \Rightarrow S_+ = \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad S_- = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix} \qquad \text{(can't measure)}$$

$$\left.\begin{aligned} S_z|\uparrow\rangle &= \frac{\hbar}{2}|\uparrow\rangle \\ S_z|\downarrow\rangle &= \frac{\hbar}{2}|\downarrow\rangle \end{aligned}\right\} \Rightarrow S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\hbar}{2}\sigma_z = * \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{\hbar}{2}&0\\0&-\frac{\hbar}{2}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}^{T*}$$

$$\left.\begin{aligned} S_x &= \frac{1}{2}(S_+ + S_-) \\ S_y &= \frac{1}{2i}(S_+ - S_-) \end{aligned}\right\} \Rightarrow S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

3.4 2 Objects w/ Spin

Objects could be orbital momentum, another particle spin, etc.

3.4.1 2 Objects w/ Spin $\frac{1}{2}$

*Find Eigenvectors, e_i , of $(S^{(1,2)})_z$ and $(S^{(1,2)})^2$ in the form of $|\chi_i\chi_j\rangle$ (using $(S^{(1,2)})_\pm$)

$$\boxed{|\chi_i\chi_j\rangle \equiv \chi_i\chi_j \equiv |\chi_i\rangle|\chi_j\rangle \equiv |\chi_i\rangle \otimes |\chi_j\rangle}$$

Choose $|\chi_i\rangle \equiv S_z$-Eigenvector w/ Spin $\frac{1}{2}$ (e.g, $|\frac{1}{2}\frac{-1}{2}\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, as opposed to $\left(\begin{smallmatrix}.6\\.8\end{smallmatrix}\right)$)

$$S^{(i)} \equiv \begin{pmatrix} S_x^{(i)} \\ S_y^{(i)} \\ S_z^{(i)} \end{pmatrix}$$

- $S_z^{(2)}S_x^{(1)}\Big(|\chi_1\rangle \otimes |\chi_2\rangle\Big) = \Big(S_x^{(1)}|\chi_1\rangle\Big) \otimes \Big(S_z^{(2)}|\chi_2\rangle\Big)$
- $S^{(i)} \cdot S^{(j)} \equiv S_x^{(i)}S_x^{(j)} + S_y^{(i)}S_y^{(j)} + S_z^{(i)}S_z^{(j)}$

  $(S^{(i)})^2 \equiv S^{(i)} \cdot S^{(i)}$

$$S^{(1,2)} \equiv \big(S^{(1)} + S^{(2)}\big) \equiv \begin{pmatrix} S_x^{(1)} + S_x^{(2)} \\ S_y^{(1)} + S_y^{(2)} \\ S_z^{(1)} + S_z^{(2)} \end{pmatrix}$$

- $\big(S^{(1,2)}\big)^2 = \big(S^{(1)} + S^{(2)}\big) \cdot \big(S^{(1)} + S^{(2)}\big)$

1. $\big(S^{(1,2)}\big)_z$

$$\big(S^{(1,2)}\big)_z\,\chi_1\chi_2 = \Big(S_z^{(1)} + S_z^{(2)}\Big)|\chi_1\rangle|\chi_2\rangle$$

$$= S_z^{(1)}|\chi_1\rangle \otimes |\chi_2\rangle + |\chi_1\rangle \otimes S_z^{(2)}|\chi_2\rangle$$

$$\big(S^{(1,2)}\big)_z\,|\chi_1\chi_2\rangle = \hbar(m_1 + m_2)\;|\chi_1\chi_2\rangle$$

$$\Rightarrow\;\; \underline{e_i = a_i|\uparrow\uparrow\rangle + b_i|\uparrow\downarrow\rangle + c_i|\downarrow\uparrow\rangle + d_i|\downarrow\downarrow\rangle}$$

$$|\uparrow\uparrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$|\uparrow\downarrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\tfrac{-1}{2}\rangle$$

$$|\downarrow\uparrow\rangle = |\tfrac{-1}{2}\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$|\downarrow\downarrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\tfrac{-1}{2}\rangle$$

2. Use $\left(S^{(1,2)}\right)_{\pm}$ on $|\uparrow\rangle \otimes |\uparrow\rangle$ to GUESS e_i from ”nice” bevahior

$$
\begin{aligned}
S_- \,|\uparrow\uparrow\rangle &= \tfrac{\sqrt{2}}{2}\big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\big)\\
S_- \left[\tfrac{\sqrt{2}}{2}\big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\big)\right] &= |\downarrow\downarrow\rangle\\
S_- \,|\downarrow\downarrow\rangle &= 0
\end{aligned}
$$

S_+ works too

If $\frac{\sqrt{2}}{2}\big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\big)$ then maybe $\frac{\sqrt{2}}{2}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big)$ works (try $S_\pm$ on it).

Guess for $\{e_i\}$:

$$
\begin{aligned}
|1\ 1\rangle &\equiv |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle &&= |\uparrow\uparrow\rangle\\
|1\ 0\rangle &\equiv \tfrac{1}{\sqrt{2}}\Big(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle + |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\Big) &&= \tfrac{\sqrt{2}}{2}\big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\big)\\
|1\ \text{-}1\rangle &\equiv |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle &&= |\downarrow\downarrow\rangle\\
|0\ 0\rangle &\equiv \tfrac{1}{\sqrt{2}}\Big(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle - |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\Big) &&= \tfrac{\sqrt{2}}{2}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big)
\end{aligned}
$$

3. Check if the guesses are eigenvectors of $\left(S^{(1,2)}\right)^2$ [and do $\left(S^{(1,2)}\right)_z$ to see eigenvalues]

(work has been skipped, do it yourself, check answer below)

$$
\begin{aligned}
S^2|1\ 1\rangle &= \hbar^2(1)(1+1)|1\ 1\rangle && (s=1) && S_z|1\ 1\rangle = \hbar(1)|1\ 1\rangle && (m=1)\\
S^2|1\ 0\rangle &= \hbar^2(1)(1+1)|1\ 0\rangle && (s=1) && S_z|1\ 0\rangle = \hbar(0)|1\ 0\rangle && (m=0)\\
S^2|1\ \text{-}1\rangle &= \hbar^2(1)(1+1)|1\ \text{-}1\rangle && (s=1) && S_z|1\ \text{-}1\rangle = \hbar(-1)|1\ \text{-}1\rangle && (m=-1)\\
S^2|0\ 0\rangle &= \hbar^2(0)(0+1)|0\ 0\rangle && (s=0) && S_z|0\ 0\rangle = \hbar(0)|0\ 0\rangle && (m=0) \quad ☑
\end{aligned}
$$

$$
* \quad \boxed{ e_i \in \left\{ \begin{array}{lll}
\left.\begin{array}{l}
|1\ 1\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle = |\uparrow\uparrow\rangle\\
|1\ 0\rangle = \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle + |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\Big) = \tfrac{\sqrt{2}}{2}\big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\big)\\
|1\ \text{-}1\rangle = |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle = |\downarrow\downarrow\rangle
\end{array}\right\} & \text{Triplet}: & s=1\\
\left.|0\ 0\rangle = \tfrac{1}{\sqrt{2}}\Big(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle - |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\Big) = \tfrac{\sqrt{2}}{2}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big)\right\} & \text{Singlet}: & s=0
\end{array}\right. }
$$

3.4.2 2 Objects w/ Any Spin

- $|\chi_1\rangle$ has spin, s_1; and $|\chi_2\rangle$ has spin, s_2
- $s_{\max} = s_2 + s_1$ and $s_{\min} = s_2 - s_1$
- Possible total $|s\ m\rangle$ must satisfy (not proven here)

  1.) $s_{\min} \leq s \leq s_{\max}$, **2.)** $-s \leq m \leq s$,

  3.) have integer differences

In general,

$$\boxed{|s\ m\rangle = \sum_{1'2'} C_{s m s_{1'} m_{1'} s_{2'} m_{2'}}\ |s_{1'}\ m_{1'}\rangle \otimes |s_{2'}\ m_{2'}\rangle}$$

where the sum is over all poss. int. diff. values that satisfy

$$\begin{aligned} s_{1'} + s_{2'} &= s, & 0 \leq s_{1'} &\leq s_1, & 0 \leq s_{2'} &\leq s_2, \\ m_{1'} + m_{2'} &= m, & -s_{1'} \leq m_{1'} &\leq s_{1'}, & -s_{2'} \leq m_{2'} &\leq s_{2'}, \end{aligned}$$

and C are the corresponding Clebsh-Gordan coefficients, whose squared value is the probability of measuring the $\chi_1 \otimes \chi_2$ state represented by that term.

<u>Possible Combined $|s\ m\rangle$</u>

$$\begin{aligned} &(2s_{\max}+1)\left\{ \begin{array}{l} \left| s_{\max}\quad s_{\max} \right\rangle \\ \left| s_{\max}\quad s_{\max-1} \right\rangle \\ \quad\cdots \\ \left| s_{\max}\quad -s_{\max} \right\rangle \end{array} \right. \\ &\left\{ \begin{array}{l} \left| s_{\max-1}\quad s_{\max-1} \right\rangle \\ \left| s_{\max-1}\quad s_{\max-2} \right\rangle \\ \quad\cdots \end{array} \right. \\ &\cdots \\ &\cdots \\ &(2s_{\min}+1)\left\{ \begin{array}{l} \left| s_{\min}\quad s_{\min} \right\rangle \\ \quad\cdots \\ \left| s_{\min}\quad -s_{\min} \right\rangle \end{array} \right. \end{aligned}$$

More easily, if m_1 and m_2 are also known from the start, then $m = m_1 + m_2$, and

$$\boxed{|s_1\ m_1\rangle \otimes |s_2\ m_2\rangle = \sum_{s} C'_{s m s_1 m_1 s_2 m_2}\ |s\ {\scriptstyle (m_1+m_2)}\rangle}$$

where the sum is only over all possible s as satisfied above - **1.), 2.) and 3.)**. The coefficient C' also takes the same 6 variables as C but the numbers and their primes are swapped (e.g., $s_1 \leftrightarrow s_{1'}$). In this case, the total z-component, m, is known. The only unknown is the total spin, s, whose probability when measured is $(C')^2$.

3.5 Electron in Magnetic Field

$\mu_{\text{clas.}} = IA = \frac{q}{2\pi r} v(\pi r^2) = \frac{q}{2\pi r}\frac{L}{mr}(\pi r^2) = \left(\frac{q}{2m}\right) L$

$\mu_{\text{quan.}} = \left(\frac{g_e q}{2m}\right) S = \left(\frac{q}{m}\right) S = \gamma S$

$$\begin{aligned} \tau_\mu &= \mu \times B \\ F_\mu &= \nabla(\mu \cdot B) \end{aligned} \quad , \quad \begin{aligned} H &= -\mu \cdot B \\ &= -\gamma S \cdot B \end{aligned}$$

<u>Larmor Precession</u>

$$\begin{aligned} B &= B_0 \hat{k} \\ H &= -\gamma B_0 S_z \\ &= -\gamma B_0 \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \chi(t) &= \cos(\alpha/2)\begin{pmatrix}1\\0\end{pmatrix} e^{-\frac{i}{\hbar}E_1 t} + \sin(\alpha/2)\begin{pmatrix}0\\1\end{pmatrix} e^{-\frac{i}{\hbar}E_2 t} \\ &= \begin{pmatrix} \cos(\alpha/2) e^{-\frac{i}{\hbar}E_1 t} \\ \sin(\alpha/2) e^{-\frac{i}{\hbar}E_2 t} \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} \langle S_x \rangle \\ \langle S_y \rangle \\ \langle S_z \rangle \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2}\sin(\alpha)\cos(\gamma B_0 t) \\ -\frac{\hbar}{2}\sin(\alpha)\sin(\gamma B_0 t) \\ \frac{\hbar}{2}\cos(\alpha) \end{pmatrix} \qquad \text{(torque from } B \text{ with } S \text{ leads to precession)}$$

<u>Stern-Gerlach</u>

4 Bosons and Fermions

Distinguishable Particles: $\boxed{\psi(r_1, r_2) \equiv \psi_a(r_1)\psi_b(r_2)}$

Indistinguishable Particles:

$$\underline{P_x f(x_1, x_2;\ y_1, y_2;\ \ldots) \quad = \quad \pm\ f(x_2, x_1;\ y_1, y_2;\ \ldots)}$$

Boson:
$\big(s \in \{0, 1, 2, \ldots\}\big)$

$$\psi_+(r_1, r_2) \equiv \frac{1}{\sqrt{2}}\Big[\psi_a(r_1)\psi_b(r_2) + \psi_b(r_1)\psi_a(r_2)\Big]$$

$$\boxed{\psi(r_1, r_2) = \psi(r_2, r_1)} \quad \rightarrow \quad \boxed{P_i\Psi = \Psi} \quad \text{(symmetric)}$$

Fermion:
$\big(s \in \{\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots\}\big)$

$$\psi_-(r_1, r_2) \equiv \frac{1}{\sqrt{2}}\Big[\psi_a(r_1)\psi_b(r_2) - \psi_b(r_1)\psi_a(r_2)\Big]$$

$$\boxed{\psi(r_1, r_2) = -\psi(r_2, r_1)} \quad \rightarrow \quad \boxed{P_i\Psi = -\Psi} \quad \text{(antisymmetric)}$$

4.1 Exchange Forces: $\Big\langle (x_1 - x_2)^2 \Big\rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$

Dist. Part. :	$\Big\langle (\Delta x)^2 \Big\rangle = \Big\langle (\Delta x)^2 \Big\rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b$	
Symmetric:	$\Big\langle (\Delta x)^2 \Big\rangle = \Big\langle (\Delta x)^2 \Big\rangle_d - 2\ \Big\|\ \langle \psi_b \| \, x \, \| \psi_a \rangle\ \Big\|^2$	(attractive if overlap)
Antisymmetric:	$\Big\langle (\Delta x)^2 \Big\rangle = \Big\langle (\Delta x)^2 \Big\rangle_d + 2\ \Big\|\ \langle \psi_b \| \, x \, \| \psi_a \rangle\ \Big\|^2$	(repulsive if overlap)

- $\langle x_1 x_2 \rangle = \dfrac{1}{2}\displaystyle\int \Big[\psi_a(r_1)^*\psi_b(r_2)^* \pm \psi_b(r_1)^*\psi_a(r_2)^*\Big] x_1 x_2 \Big[\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)\Big] dx_1 dx_2$

$$= \frac{1}{2}\langle x \rangle_a \langle x \rangle_b + \frac{1}{2}\langle x \rangle_b \langle x \rangle_a$$

$$\pm \frac{1}{2}\Big\langle \psi_b(r_1) \Big| x_1 \Big| \psi_a(r_1) \Big\rangle \Big\langle \psi_a(r_2) \Big| x_2 \Big| \psi_b(r_2) \Big\rangle \pm \frac{1}{2}\Big\langle \psi_a(r_1) \Big| x_1 \Big| \psi_b(r_1) \Big\rangle \Big\langle \psi_b(r_2) \Big| x_2 \Big| \psi_a(r_2) \Big\rangle$$

$$= \langle x \rangle_a \langle x \rangle_b \pm \Big\|\ \langle \psi_b | \, x \, | \psi_a \rangle\ \Big\|^2$$

Two Electrons:

$$\psi_{(r_1,r_2)}\chi_{(m_1,m_2)} = \begin{cases} \overset{\text{(singlet)}}{-\psi_{(r_1,r_2)}\chi_{(m_2,m_1)}} & \Rightarrow \quad \begin{matrix}\chi \text{ is antisymmetric so} \\ \psi \text{ is symmetric}\end{matrix} \quad \Rightarrow \quad \text{Attractive} \\[2em] \overset{\text{(triplet)}}{-\psi_{(r_2,r_1)}\chi_{(m_1,m_2)}} & \Rightarrow \quad \begin{matrix}\chi \text{ is symmetric so} \\ \psi \text{ is antisymmetric}\end{matrix} \quad \Rightarrow \quad \text{Repulsive} \end{cases}$$

4.2 Statistics

Sterling's Approx:

$$\log(z!) \approx z\log(z) - z \qquad z \gg 1 \;\text{ or }\; z = 0$$

$$\frac{d}{dz}\log(z!) \approx \log(z)$$

Lagrange Multiplier:

$$G(X,\alpha,\beta) = \log(Q(X)) + \alpha f_1(X) + \beta f_2(X)$$

$$\frac{\partial G}{\partial \alpha}[Q_{\max}] = 0, \qquad \frac{\partial G}{\partial \beta}[Q_{\max}] = 0, \qquad \frac{\partial G}{\partial N_n}[Q_{\max}] = 0$$

$$\sum_n N_n = N \qquad\qquad \sum_n N_n E_n = E$$

$$f_1(X) = N - \sum_n N_n = 0 \qquad\qquad f_2(X) = E - \sum_n N_n E_n = 0$$

Let there be N_n particles in the E_n energy level having d_n degeneracies, and $Q(N_1, N_2, ...)$ be the number of possible configurations for such a state given $X = (N_1, N_2, ..., N_n)$.

Dist.

$$\begin{cases} \textbf{1.)}\; Q(X) = \prod_n \binom{N - N_1 - ... - N_{n-1}}{N_n} d_n^{N_n} \\ \qquad\qquad = N! \prod_n \frac{d_n^{N_n}}{N_n!} \\ \textbf{2.)}\; \log(Q) = \log(N!) + \sum_n N_n \log(d_n) - \log(N_n!) \end{cases}$$

$$\textbf{3.)}\; \frac{\partial G}{\partial N_n} \approx \log(d_n) - \log(N_n) - \alpha - \beta E_n = 0$$

$$\textbf{4.)}\; N_n = \frac{d_n}{e^{\beta E_n + \alpha}}$$

Fermion

$$\begin{cases} \textbf{1.)}\; Q(X) = \prod_n \binom{d_n}{N_n} \\ \textbf{2.)}\; \log(Q) = \sum_n \log(d_n!) - \log(N_n!) - \log[(d_n - N_n)!] \end{cases}$$

$$\textbf{3.)}\; \frac{\partial G}{\partial N_n} \approx -\log(N_n) + \log(d_n - N_n) - \alpha - \beta E_n = 0$$

$$\textbf{4.)}\; N_n = \frac{d_n}{e^{\beta E_n + \alpha} + 1}$$

Boson

$$\begin{cases} \textbf{1.)}\; Q(X) = \prod_n \binom{N_n + d_n - 1}{N_n} \\ \textbf{2.)}\; \log(Q) = \sum_n \log[(N_n + d_n - 1)!] - \log(N_n!) - \log[(d_n - 1)!] \end{cases}$$

$$\textbf{3.)}\; \frac{\partial G}{\partial N_n} \approx \log(N_n + d_n - 1) - \log(N_n) - \alpha - \beta E_n = 0$$

$$\textbf{4.)}\; N_n = \frac{d_n - 1}{e^{\beta E_n + \alpha} - 1} \approx \frac{d_n}{e^{\beta E_n + \alpha} - 1}$$

Given some substance in thermal equilibrium,

$$\beta = \frac{1}{k_b T} \qquad \mu(T) \equiv -\frac{\alpha}{k_b T}$$

where μ depends on the situation.

$$\frac{N_n}{d_n}: \qquad n(\epsilon) \quad = \quad \begin{cases} \frac{1}{e^{(\epsilon-\mu)/k_b T}} & \text{Maxwell-Boltzmann} \\ \frac{1}{e^{(\epsilon-\mu)/k_b T}+1} & \text{Fermi-Dirac} \\ \frac{1}{e^{(\epsilon-\mu)/k_b T}-1} & \text{Bose-Einstein} \end{cases}$$

5 Perturbation Theory

$$H^{(0)}\psi_n = E_n\psi_n$$
$$\downarrow$$
$$H\psi_n' = E_n'\psi_n'$$

$$\left(H^{(0)} + \lambda H^{(1)}\right)\left(\psi_n + \lambda\psi_n^{(1)} + \lambda^2\psi_n^{(2)} + ...\right) = \left(E_n + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + ...\right)\left(\psi_n + \lambda\psi_n^{(1)} + \lambda^2\psi_n^{(2)} + ...\right)$$

$$\begin{array}{ccc} \cancel{\lambda^0 H^{(0)}\psi_n} & & \cancel{\lambda^0 E_n\psi_n} \\ +\ \lambda^1(H^{(0)}\psi_n^{(1)} + H^{(1)}\psi_n) & = & +\ \lambda^1(E_n\psi_n^{(1)} + E_n^{(1)}\psi_n) \\ +\ \lambda^2(H^{(0)}\psi_n^{(2)} + H^{(1)}\psi_n^{(1)}) & & +\ \lambda^2(E_n\psi_n^{(2)} + E_n^{(1)}\psi_n^{(1)} + E_n^{(2)}\psi_n) \\ +\ ... & & +\ ... \end{array} \qquad {\scriptstyle (\lambda=1)}$$

5.1 Non-Degenerate Theory

$\underline{E_n^{(1)},\ \psi_n^{(1)}:}$

$$H^{(0)}\psi_n^{(1)} + H^{(1)}\psi_n = E_n\psi_n^{(1)} + E_n^{(1)}\psi_n$$

$$\begin{aligned} \langle\psi_m| \quad (-H^{(1)} + E_n^{(1)})|\psi_n\rangle &= \langle\psi_m| \quad (H^{(0)} - E_n)|\psi_n^{(1)}\rangle \\ &= \langle\psi_m| \quad (H^{(0)} - E_n)\sum c_i|\psi_i\rangle \\ &= \sum c_i(E_i - E_n)\langle\psi_m|\psi_i\rangle \\ -\langle\psi_m|H^{(1)}|\psi_n\rangle + E_n^{(1)}\langle\psi_m|\psi_n\rangle &= c_m(E_m - E_n) \end{aligned}$$

$$\boxed{E_n^{(1)} = \langle\psi_n|H^{(1)}|\psi_n\rangle} \qquad \boxed{\psi_n^{(1)} = \sum_{m\neq n}\frac{\langle\psi_m|H^{(1)}|\psi_n\rangle}{E_n - E_m}\psi_m + (0)\psi_n}$$

$\underline{E_n^{(2)}:}$

$$\begin{aligned} \cancel{E_n\langle\psi_n|\psi_n^{(2)}\rangle} + E_n^{(1)}\cancel{\langle\psi_n|\psi_n^{(1)}\rangle} \\ +E_n^{(2)}\langle\psi_n|\psi_n\rangle &= \cancel{\langle H^{(0)}\psi_n|\psi_n^{(2)}\rangle} + \langle\psi_n|H^{(1)}|\psi_n^{(1)}\rangle \\ &= \sum_{m\neq n}\frac{\langle\psi_m|H^{(1)}|\psi_n\rangle}{E_n - E_m}\langle\psi_n|H^{(1)}|\psi_m\rangle \end{aligned}$$

$$\boxed{E_n^{(2)} = \sum_{m\neq n}\frac{\left|\langle\psi_m|H^{(1)}|\psi_n\rangle\right|^2}{E_n - E_m}}$$

5.2 Degenerate Perturbation Theory (see Matrix Operators)

$$\Psi = \sum_i \left(c_i^{(\psi)}[\Psi]\right)\psi_i \equiv \sum_i c_i^{(\psi)}\psi_i = c_0^{(\psi)}\psi_0 + c_1^{(\psi)}\psi_1 + ... \quad , \quad \forall\psi_i :$$

- $H^{(0)}\psi_i = E_n\psi_i$ $\quad$ (ψ_n are degenerate eigenfunctions of $H^{(0)}$)
- $\langle\psi_i|\psi_j\rangle = \delta_{ij}$
- $\langle\psi_i|\hat{Q}|\psi_j\rangle \equiv Q_{ij}$

$$E_n\Psi^{(1)} + E^{(1)}\Psi = H^{(0)}\Psi^{(1)} + H^{(1)}\Psi \qquad \text{(first order)}$$

$$\begin{aligned}
\cancel{E_n\langle\psi_i|\Psi^{(1)}\rangle} + E^{(1)}\langle\psi_i|\Psi\rangle &= \cancel{\langle H^{(0)}\psi_i|\Psi^{(1)}\rangle} + \langle\psi_i|H^{(1)}|\Psi\rangle \\
&= \langle\psi_i|H^{(1)}|c_0\psi_0 + c_1\psi_1 + ...\rangle \\
c_iE^{(1)} &= c_0\langle\psi_i|H^{(1)}|\psi_0\rangle + c_1\langle\psi_i|H^{(1)}|\psi_1\rangle + ...
\end{aligned}$$

$$E^{(1)}\begin{pmatrix} c_0[\Psi] \\ c_1[\Psi] \\ \vdots \end{pmatrix}^{(\psi)} = \begin{pmatrix} H_{00}^{(1)} & H_{01}^{(1)} & ... \\ H_{10}^{(1)} & H_{11}^{(1)} & ... \\ \vdots & \vdots & \end{pmatrix}^{(\psi)} \begin{pmatrix} c_0[\Psi] \\ c_1[\Psi] \\ \vdots \end{pmatrix}^{(\psi)} \quad \Rightarrow \quad \boxed{\begin{array}{c}\text{(solve for } E^{(1)}, \vec{c}^{\,(\psi)}[\Psi])\\ \begin{Vmatrix} H_{aa}^{(1)} - E^{(1)} & H_{ab}^{(1)} & ... \\ H_{ba}^{(1)} & H_{bb}^{(1)} - E^{(1)} & ... \\ \vdots & \vdots & \end{Vmatrix} = 0\end{array}}$$

In general,

$$E_i^{(1)}\vec{c}^{\,(\psi)}[\Psi_i] \quad = \quad \overline{H^{(1)}}^{\,(\psi)}\ \vec{c}^{\,(\psi)}[\Psi_i] \qquad (i\text{th eigen-})$$

$$E_i^{(1)}\begin{pmatrix} | \\ \vec{c}\,[\Psi_i] \\ | \end{pmatrix}^{(\psi)} = \begin{pmatrix} | & | & \\ \vec{c}\,[\Psi_i] & \vec{c}\,[\Psi_i] & ... \\ | & | & \end{pmatrix}^{(\psi)} \begin{pmatrix} E_0^{(1)} & 0 & ... \\ 0 & E_1^{(1)} & ... \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} - & \vec{c}^{\,*}[\Psi_i] & - \\ - & \vec{c}^{\,*}[\Psi_i] & - \\ & \vdots & \end{pmatrix}^{(\psi)} \begin{pmatrix} | \\ \vec{c}\,[\Psi_i] \\ | \end{pmatrix}^{(\psi)}$$

Instead of solving the characteristic polynomial, it would be wise to choose a basis $\{\psi\}$ such that $\vec{c}^{\,(\psi)}[\Psi_i] = (...0\ 0\ 1_{(i)}\ 0\ 0\ ...)^T \ \Leftrightarrow\ \Psi_i = \psi_i$, making $\overline{H^{(1)}}^{(\psi)}$ diagonal with eigenvalue entries. These are the energy eigenvalues, $E_i^{(1)} = \left(H^{(1)}\right)_{ii}^{(\psi)} = \langle\psi_i|H^{(1)}|\psi_i\rangle$, which is just like first-order non-Perturbation energy. This also means $|\psi_i\rangle$ are eigenfunctions of $H^{(1)}$ (see Matrix Operators).

It is best to find a hermitian operator, $\hat{A}$, that commutes with $H^{(0)}$ and $H^{(1)}$, whose eigenvalues within the degenerate basis are unique. The corresponding eigenfunctions will be a basis that makes $H^{(1)}$ diagonal. This will also make them eigenfunctions of $H^{(1)}$.

1. $A = A^\dagger$
2. $[A, H^{(0)}] = 0 \ \rightarrow\ \left\{\exists\{\Psi\}\ |\ \ (A\Psi_n = a_n\Psi_n),\ \ (H^{(0)}\Psi_n = E_n\Psi_n)\right\}$
3. $\{\psi\} \subset \{\Psi\}\ \text{ s.t }\ \forall\psi_i : \begin{cases} \left(H^{(0)}\psi_i = E_n\psi_i\right), & \leftarrow \text{degenerate} \\ \left(A\psi_i = a_i\psi_i|\right), & (\forall_{(i\neq j)}\ a_i \neq a_j) \end{cases}$
4. $[A, H^{(1)}] = 0 \ \Rightarrow\ \begin{aligned} 0 &= \langle A\psi_i|H^{(1)}|\psi_j\rangle - \langle\psi_i|H^{(1)}|A\psi_j\rangle \\ 0 &= (a_i - a_j)H_{ij}^{(1)} \\ 0 &= H_{ij}^{(1)} \qquad ☑ \end{aligned}$

5.3 Hydrogen Energy Corrections

5.3.1 Fine Structure - $\alpha^4 mc^2$

The Dirac Equation can derive the total fine structure correction with a α^4 order approx.

1. Relativistic, $\hat{\boldsymbol{p}}^4$

$$
\begin{aligned}
T &= \sqrt{p^2c^2 + m^2c^4} - mc^2 \\
&= \frac{(\frac{1}{2})}{1!}\left(\frac{p}{mc}\right) + \frac{(\frac{1}{2})(1-\frac{1}{2})}{2!}\left(\frac{p}{mc}\right)^2 + ... \\
&= \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + ... \\
&\downarrow \\
H_r^{(1)} &= -\frac{p^4}{8m^3c^2}
\end{aligned}
$$

(For some reason $\hat{p}^4$ needs to be hermitian to use perturbation theory. It only isn't when $l = 0$, while $\hat{p}^2$ always is hermitian. See Prob. 6.15)

L^2 and L_z should commute with p^4 because the perturbation is spherically symmetric, meaning l and m_l should be conserved (see Operator Evolution). Their eigenvalues are also distinct (taking the eigenfunctions of nlm_l together) within each set of n^2 degeneracies, so their eigenvectors and eigenvalues can be used. n, l and m_l the "good" numbers.

$\langle r^{-1}\rangle = \frac{1}{n^2 a_0^1}$

$\langle r^{-2}\rangle = \frac{1}{(l+1/2)n^3 a_0^2}$

$$
\begin{aligned}
\left\langle \psi_{nlm_l} | H_r^{(1)} | \psi_{nlm_l} \right\rangle &= \frac{-1}{8m^3c^2}\langle \psi_{nlm_l}|p^4|\psi_{nlm_l}\rangle \\
&= \frac{-1}{8m^3c^2}\langle p^2\psi_{nlm_l}|p^2|\psi_{nlm_l}\rangle \\
&= \frac{-1}{8m^3c^2}\left\langle \left[2m(E_n - V)\right]^2 \right\rangle \\
&= \frac{-4m^2}{8m^3c^2}\langle E_n^2 - 2E_nV + V^2\rangle \\
&= -\frac{E_n^2}{2mc^2}\left[\frac{4n}{l+1/2} - 3\right]
\end{aligned}
$$

2. Spin-Orbit Coupling, $\boldsymbol{S_e} \cdot \boldsymbol{L_e}$

In the electron's frame of reference, the proton is spinning around it, creating a B-field affecting its magnetic dipole moment. The non-inertial reference frame requires multiplying by the Thomas procession correction, which in this case is $C_T = g_e - 1 = 1/2$. In the lab frame, the moving electron's magnetic dipole moment creates an electric dipole moment, which is affected by the proton charge. The latter is much harder to calculate.

$$
\begin{aligned}
H_{so}^{(1)} &= -C_T\ \mu_e \cdot B(L_e) \qquad \text{(See Electron in Magnetic Field)} \\
&= \frac{kqq}{2}\frac{1}{m^2c^2r^3} S_e \cdot L_p \\
&= \frac{kqq}{2m}\frac{1}{mc^2}\frac{S\cdot L}{r^3} = \frac{e^2}{8\pi\epsilon_0 m^2c^2}\frac{S\cdot L}{r^3}
\end{aligned}
$$

$S\cdot L$ does not commute with L or S (meaning m_l and m_s are bad), but $[S\cdot L, S^2] = [S\cdot L, L^2] = 0$. The sum of the two, $J \equiv L+S$, and J^2 also commute with the perturbation. They are all conserved, and their unique eigenvalues per set of degeneracies - $l, s{=}1/2, j, m_j$ - are the "good" numbers (along with n).

$$
S\cdot L = \frac{1}{2}\left(J^2 - L^2 - S^2\right)
$$

$$
\langle r^{-3}\rangle = \frac{1}{l(l+1/2)(l+1)n^3a_0^3}
$$

$$
\begin{aligned}
\langle nljm_j|H_{so}^{(1)}|nljm_j\rangle &= \frac{kqq}{2m}\frac{1}{mc^2}\frac{\hbar^2[j(j+1)-l(l+1)-s(s+1)]}{2l(l+1/2)(l+1)n^3a_0^3} \\
&= \frac{kqq}{4mn^4}\frac{\hbar^2\alpha^3m^3c^3}{\hbar^3mc^2}\frac{n[j(j+1)-l(l+1)-s(s+1)]}{l(l+1/2)(l+1)} \\
&= \frac{kqq}{4\hbar cn^4}\frac{\alpha^3m^2c^4}{mc^2}\frac{n[j(j+1)-l(l+1)-s(s+1)]}{l(l+1/2)(l+1)} \\
&= \frac{E_n^2}{mc^2}\left\{\frac{n\left[j(j+1)-l(l+1)-3/4\right]}{l(l+1/2)(l+1)}\right\}
\end{aligned}
$$

<u>3. Darwin Term (correction for $H_{so}^{(1)}$ when $l=0$)</u> skipped

<u>4. Total Correction</u>

$$
\begin{aligned}
E_{fs}^{(1)} &= E_r^{(1)} + E_{so}^{(1)} \\
&= -\frac{E_n^2}{2mc^2}\left[\frac{4n}{l+\frac{1}{2}} - 3\right] + \frac{E_n^2}{mc^2}\left\{\frac{n\left[j(j+1)-l(l+1)-3/4\right]}{l(l+1/2)(l+1)}\right\} \\
&= \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right) \qquad (j{=}l{\pm}1/2) \\
&\downarrow \\
E_{nj} &= E_n + E_{fs}^{(1)} \\
&= E_n\left[1 - \frac{E_n}{2mc^2}\left(\frac{4n}{j+1/2} - 3\right)\right] \\
&= -\frac{\alpha^2mc^2}{2n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - 3/4\right)\right]
\end{aligned}
$$

Fine structure splits the l energy degeneracies. However, since $j = l \pm 1/2$, there are still two j degeneracies if $n > 2$. Overall, the good numbers to use for stationary state solutions to the hydrogen atom w/ fine structure correction are $n, l, s{=}1/2, j, m_j$. Note, J^2, L^2, and S^2 always commute(?)

5.3.2 Zeeman Effect (Ext. B-Field)

$$\begin{aligned} H_B^{(1)} &= -(\mu_s + \mu_l) \cdot B_{\text{ext}} \qquad \text{(see Electron in Magnetic Field)} \\ &= -\left(\frac{g_e q}{2m} S + \frac{q}{2m} L\right) \cdot B_{\text{ext}} \\ &= \frac{e}{2m} (2S + L) \cdot B_{\text{ext}} \end{aligned}$$

Weak Zeeman ($B_{\text{ext}} \ll B_{\text{int}}$)

$$\begin{aligned} H_{WZ}^{(1)} &= \frac{e}{2m} B_{\text{ext}} \cdot (2S + L) \\ &= \frac{e}{2m} B_{\text{ext}} \cdot (J + S) \end{aligned}$$

Fine structure effects dominate the Zeeman effect, so the fine structure numbers are the good ones: $n, l, s{=}1/2, j$, and m_j. m_s can't be used for $\langle S \rangle$, so instead use the fact that the "vector" $J = L + S$ is conserved, so a **time-averaged** S-component to the J "vector" can be defined as $S_{\text{ave}} = \frac{S \cdot J}{J^2} J$, where $S \cdot J = \frac{1}{2}\left(J^2 + S^2 - L^2\right)$.

$$\begin{aligned} E_{\text{WZ}}^{(1)} &= \frac{e}{2m} B_{\text{ext}} \cdot \langle nljm_j | J + S_{\text{ave}} | nljm_j \rangle \\ &= \frac{e}{2m} B_{\text{ext}} \cdot \left\langle J \left(1 + \frac{S \cdot J}{J^2}\right) \right\rangle \\ &= \frac{e}{2m} B_{\text{ext}} \cdot \langle J \rangle \left(1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}\right) \\ &= \frac{e\hbar}{2m} B_{\text{ext}} m_j \left(1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}\right) \qquad \text{(let } B_{\text{ext}} \text{ be parallel to the z-axis)} \\ &= \mu_B B_{\text{ext}} m_j g_j \qquad \begin{array}{l} \mu_B = \text{Bohr magneton} = 5.788 \times 10^{-5} \text{ ev/T} \\ g_j = \text{Lande g-factor} \end{array} \end{aligned}$$

Strong Zeeman ($B_{\text{ext}} \gg B_{\text{int}}$)

For a strong magnetic field parallel to the z-axis, m_l and m_s are stuck in the same place, making them and l conserved. The external torque, however, means that the total angular momentums, j and m_j are not. Though unneeded, obviously $s{=}1/2$.

$$\begin{aligned} E_{SZ}^{(1)} &= \frac{e}{2m} B_{\text{ext}} \langle 2S_z + L_z \rangle \\ &= \mu_B B_{\text{ext}} (2m_s + m_l) \end{aligned}$$

The spin-orbit correction must be changed with respect to the new good numbers, m_l and m_s. The relativistic correction uses the same numbers, so it stays the same.

$$E_{\text{so}}^{(1)} = \frac{e^2}{8\pi\epsilon_0 m^2c^2}\left\langle\frac{S_xL_x + S_yL_z + S_zL_z}{r^3}\right\rangle \quad\rightarrow$$

$$= \frac{e^2}{8\pi\epsilon_0 m^2c^2}\frac{0+0+\hbar^2 m_s m_l}{l(l+1/2)(l+1)n^3a_0^3}$$

$$= \frac{kqq}{2m^2c^2}\frac{\hbar^2}{(\hbar/\alpha mc)^3 n^3}\frac{m_s m_l}{l(l+1/2)(l+1)}$$

$$= \frac{kqq}{2\hbar c}\frac{\alpha^3 m^2c^4}{4mc^2n^4}\frac{4nm_s m_l}{l(l+1/2)(l+1)}$$

$$= \frac{E_n^2}{2mc^2}\frac{4nm_s m_l}{l(l+1/2)(l+1)}$$

$$E_{\text{fs}}^{(1)} = E_{\text{so}}^{(1)} + E_{\text{r}}^{(1)}$$

$$= \frac{E_n^2}{2mc^2}\frac{4nm_s m_l}{l(l+1/2)(l+1)} + \frac{E_n^2}{2mc^2}\left[3 - \frac{4n}{l+1/2}\right]$$

$$= \frac{4nE_n^2}{2mc^2}\left[\frac{m_s m_l}{l(l+1/2)(l+1)} + \frac{3}{4n} - \frac{1}{l+1/2}\right]$$

$$\downarrow$$

$$E_{nlm_lm_s} = E_n + E_{\text{SZ}}^{(1)} + E_{\text{fs}}^{(1)}$$

Intermediate Zeeman ($B_{\text{ext}} \sim B_{\text{int}}$)

There are no good numbers here (see Degenerate Perturbation Theory). The basis is chosen to be $|j\ m_j\rangle = \sum_i C_i |l\ m_l\rangle \otimes |s\ m_s\rangle$ (see 2 Objects w/ Any Spin), as it makes $\overline{H^{(1)}}^{(e)}$ easier (instead of using l, m_l, m_s).

1.) $\psi_i = |j\ m_j\rangle_i$

2.) $\Big(\langle l\ m_l|\langle s\ m_s|\Big)_x \Big(|l\ m_l\rangle|s\ m_s\rangle\Big)_y = \delta_{xy}$

3.) $Q_{rc}^{(\psi)} = \langle\psi_r|\hat{Q}|\psi_c\rangle$

4.) ψ_i s.t. $\begin{cases} 0 \le l < n \\ j_{(l\pm)} = l \pm 1/2, \\ 2l^2 < i \le 2(l{+}1)^2 \end{cases}$

$$\langle jm_j|H_{fs}^{(1)}|jm_j\rangle = \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right)$$

$$\equiv \gamma_n\left(3 - \frac{4n}{j+1/2}\right)$$

$$\langle jm_j|H_{IZ}^{(1)}|jm_j\rangle = \langle jm_j|H_{IZ}^{(1)}\Big(C_i|lm_l\rangle \otimes |sm_s\rangle\Big)$$

$$= \mu_B B_{\text{ext}}(2m_s + m_l)C_i^2$$

$$\equiv \beta(2m_s + m_l)C_i^2$$

$$\overline{H^{(1)}}^{(jm_j)} = \overline{H_{fs}^{(1)}}^{(jm_j)} + \overline{H_{IZ}^{(1)}}^{(jm_j)}$$

See Griffith Prob. 6.25 for example with $n = 2$

5.3.3 Stark Effect (Small Ext. E-Field)

- $H^{(1)} = -p \cdot E = eE \cdot r$ (small r)
- $n = 1 \;\rightarrow\; H^{(1)} = 0$
- $n = 2 \;\rightarrow\; \begin{cases} H^{(1)} = 0 & m = \pm 1 \\ H^{(1)} = ke|E|a_0 & m = 0 \end{cases}$ (k is some constant)

5.3.4 Lamb Shift (quantitized E-field) - $\alpha^5 mc^2$ (skipped)

5.3.5 Hyperfine (Spin-Spin), $\boldsymbol{S_p} \cdot \boldsymbol{S_e}$ - $m/m_p\ \alpha^4 mc^2$

(Coupling between the electron magnetic moment and the magnetic field from the proton magnetic moment)

$$\mu_e = -\frac{g_e e}{2m_e} S_e = -\frac{e}{m_e} S_e, \qquad \mu_p = \frac{g_p e}{2m_p} S_p$$

$$B(\mu_p) = \frac{\mu_0}{4\pi r^3}[3(\vec{\mu_p} \cdot \hat{r})\hat{r} - \vec{\mu_p}] + \frac{2\mu_0}{3}\vec{\mu_p}\delta^3(r)$$

$$\begin{aligned} H_{hf}^{(1)} &= -\mu_e \cdot B(\mu_p) \\ &= \ldots \\ &\downarrow \\ E_{hf}^{(1)} &= \left(\frac{e}{m_e}\right)\left(\frac{2\mu_0}{3}\frac{g_p e}{2m_p}\right)\langle S_e \cdot S_p\rangle \left|\psi_{nlm}(0)\right|^2 \end{aligned}$$

In the ground state, $\left|\psi_{100}(0)\right|^2 = 1/(\pi a_0^3)$. S_e^2, S_p^2, and the sum $S = S_e + S_p$ commute with $S_e \cdot S_p$, so s_e, s_p, m_s, s^2 are the good numbers. S_e and S_p do not, so m_{se} and m_{sp} are not good numbers.

$$\begin{aligned} E_{hf}^{(1)} &= \left(\frac{e}{m_e}\right)\left(\frac{2}{3\epsilon_0 c^2}\frac{g_p e}{2m_p}\right)\frac{1}{2\pi a_0^3}\langle S^2 - S_e^2 - S_p^2\rangle \\ &= \frac{g_p e^2}{4\pi\epsilon_0 c^2 m_p m_e}\frac{4\alpha^3 m_e^3 c^3 \hbar^2}{3\hbar^3}\left[\frac{s(s+1)}{2} - 3/4\right] \\ &= \frac{4}{3} g_p \frac{m_e}{m_p}\alpha^4 m_e c^2 \left[\frac{s(s+1)}{2} - 3/4\right] \\ &= \frac{4}{3} g_p \frac{m_e}{m_p}\alpha^4 m_e c^2 \cdot \begin{cases} \frac{1}{4} & s = 1 \text{ (triplet)} \\ \frac{-3}{4} & s = 0 \text{ (singlet)} \end{cases} \quad \rightarrow \quad \begin{array}{l} \Delta E = 5.88 \times 10^{-6} \text{ eV} \\ \lambda = 21 \text{ cm}, \quad \nu = 1420 \text{ MHz} \end{array} \end{aligned}$$

5.4 Variation Principle - Approx. Ground State Energy

$\psi = \sum c_n \psi_n \;\rightarrow\; E(\psi) > E_0 = E(\psi_0)$

$$\psi \equiv f(b,x), \qquad \langle H \rangle = \langle T \rangle + \langle V \rangle$$

$$b_{\min}: \; \frac{d}{db}\langle H \rangle = 0$$

$$E_{\text{gs}} \approx \Big\langle f(b_{\min},x) \Big| H \Big| f(b_{\min},x) \Big\rangle$$

5.5 Adiabatic Theorem - Slow Changing of Potential

$$\begin{array}{lll} & H_{(t=\tau)} = H' & & H_{(t=0)} = H_0 \\ t=\tau \;\rightarrow & \psi_{(t=\tau)} = \psi_n^{(H')} & t=0 \;\rightarrow & \psi_{(t=0)} = \psi_n^{(H_0)} \\ & E(\psi) = E_n^{(H')} & & E(\psi) = E_n^{(H_0)} \end{array}$$

5.6 Selection Rules - Orbital Transitions

Electric Dipole Approximation ONLY: $\lambda_\gamma \gg$ atom length $\rightarrow$ E, B feels homogenously oscillating to the atom

$\psi_{nlm} \rightarrow \psi_{n'l'm'}$:

- $\Delta m \in \{-1, 0, 1\}$
  $s(\gamma) = 1 \;\rightarrow\; m_s(\gamma) \in \{-\hbar, 0, \hbar\}$
  $E = E\hat{z} \;\rightarrow\; \Delta m = 0$
- $\Delta l = \pm 1$
  $1s \leftrightarrow 2p$
  Exception: $(2s \rightarrow 1s)$ through two-photon emission
- $\Delta j \in \{-1, 0, 1\}$
  Exception: $(j = 0 \rightarrow j = 0)$ not allowed

6 Blackbody Radiation

- Power Spectrum : $I'(\omega) = \dfrac{\hbar^3\omega^3}{h^2c^2}\dfrac{1}{e^{\hbar\omega/k_bT} - 1}$ $\left[\frac{I}{\Omega \cdot f}\right]$ ($\mu = 0$ for photons since photon number isnt conserved)
- Stefan-Boltzmann Law : $I = \dfrac{dP}{dA} \propto T^4$!! important !!
- Wien's Displacement Law : $\lambda_{\max} = \dfrac{2.9 \times 10^{-3}}{T}$ [m] (mode of spectrum)

7 Klein-Gordon Equation (Free Particle)

$$(p^2c^2 + m^2c^4)\psi = E^2\psi$$

$$(-E^2 + p^2c^2 + m^2c^4)\psi = 0$$

$$\left[-(E/c)^2 + p^2 + (mc)^2\right]\psi = 0$$

$$\frac{\left[-(E/c)^2 + p^2 + (mc)^2\right]}{\hbar^2}\psi = 0$$

$$\left[\frac{1}{c^2}\frac{\partial}{\partial t}^2 - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0$$

$$\boxed{(\Box^2 + \mu^2)\psi = 0}$$

8 Dirac Equation

$$\mu^2 = -\Box^2$$

$$\mu = \sqrt{\nabla^2 - \tfrac{1}{c^2}\partial_t}^2 = A\partial_x + B\partial_y + C\partial_z + \tfrac{i}{c}D\partial_t$$

$$\begin{aligned}
\partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{1}{c^2}\frac{\partial}{\partial t}^2 &= (A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t)^2 \\
&= A^2\partial_x^2 + B^2\partial_y^2 + C^2\partial_z^2 - D^2\tfrac{1}{c^2}\partial_t^2 \\
&\quad + [AB + BA]\partial_x\partial_y + [AC + CA]\partial_x\partial_z + [AD + AD]\tfrac{i}{c}\partial_x\partial_t \\
&\quad + [BC + CB]\partial_y\partial_z + [BD + BD]\tfrac{i}{c}\partial_y\partial_t \\
&\quad + [CD + CD]\tfrac{i}{c}\partial_z\partial_t
\end{aligned}$$

$$D = \gamma^0, \qquad A = i\gamma^1, \qquad B = i\gamma^2, \qquad C = i\gamma^3$$

$$\gamma^\mu = \left[\begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}\right]$$

$$\boxed{\begin{aligned} (i\hbar\gamma^\mu\partial_\mu - mc)\psi &= 0 \\ (i\partial\!\!\!/ - m)\psi &= 0 \qquad \text{(natural units)} \end{aligned}}$$

9 Integral Form

$$\psi(r) = \psi_0(r) + \int g(r - r_0)V(r_0)\psi(r_0)\ d^3r \qquad g(r) = -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}$$

$$= \psi_0 + \int gV\psi_{(r_0)}$$

$$= \psi_0 + \int gV\psi_0 + \int\int gVgV\psi_{(r_0)}$$

$$= \psi_0 + \int gV\psi_0 + \int\int gVgV\psi_0 + \int\int gVgVgV\psi_0 + ...$$