

# 1 Maxwell's Equations

## Gauss's Law for Electricity (GLE)

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q}{\epsilon_0}$$

$$\iiint (\nabla \cdot \vec{E}) dV = \frac{\iiint \rho dV}{\epsilon_0}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

## Faraday's Law of Induction (FLI)

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\iint (\nabla \times \vec{E}) \cdot d\vec{a} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{a}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

## Lorentz Force Law (LFL)

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

## Gauss's Law for Magnetism (GLM)

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\iiint (\nabla \cdot \vec{B}) dV = 0$$

$$\nabla \cdot \vec{B} = 0$$

## Maxwell-Ampere's Law (MAL)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left( I + \epsilon_0 \frac{d\Phi_E}{dt} \right)$$

$$\iint (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \iint \vec{J} \cdot d\vec{a} + \mu_0 \epsilon_0 \frac{d}{dt} \iint \vec{E} \cdot d\vec{a}$$

$$\nabla \times \vec{B} = \mu_0 \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

## Conservation of Charge (COC)

$$\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

## 1.1 Electrostatic/Magnetostatic Examples

### Using GLE

#### 1. Point Charges

$$\vec{E} = E(r)\hat{r}$$

$$\begin{aligned}\frac{Q}{\epsilon_0} &= \oiint E(r)\hat{r} \cdot d\vec{a} \\ &= E(r)\hat{r} \cdot \oiint r^2 \sin \phi |d\vec{\phi} \times d\vec{\theta}| \\ &= E(r)\hat{r} \cdot 4\pi r^2 \hat{r}\end{aligned}$$

$$\vec{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} \Rightarrow \frac{1}{4\pi\epsilon_0} \sum_i \frac{Q_i}{|\vec{z}_i|^2} \hat{z}_i$$

#### Coulomb's Law (CL):

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dQ}{|\vec{z}|^2} \hat{z} \quad \vec{r} \in \mathbb{R}^3, \quad \vec{z} = \vec{r} - \vec{l}'$$

$$\vec{F}(\vec{r}) = q\vec{E}$$

### Using MAL

#### Biot-Savart Law (BSL):

(see potential,  $\vec{A}$ , for derivation)

$$\begin{aligned}\vec{B}(\vec{r}) &= k_\mu \int \frac{\vec{J}dV \times \hat{z}}{|\vec{z}|^2}, \quad \vec{r} \in \mathbb{R}^3, \quad \vec{z} = \vec{r} - \vec{l}' \\ &= k_\mu \int \frac{I(\vec{l}') d\vec{l}' \times \hat{z}}{|\vec{z}|^2}\end{aligned}$$

$$\vec{F} = q\vec{v} \times \vec{B}$$

#### 1. Infinite Line w/ Steady Current (SC)

$$\vec{B} = B(r)\hat{\theta}$$

Use MAL

$$\begin{aligned}\mu_o I &= \oint B(r)\hat{\theta} \cdot d\vec{L} \\ &= B(r)\hat{\theta} \cdot \oint r d\vec{\theta} \\ &= B(r)\hat{\theta} \cdot 2\pi r \hat{\theta}\end{aligned}$$

$$\vec{B}(r) = \frac{\mu_o I}{2\pi r} \hat{\theta}$$

cont.

## 2. Sphere w/ Constant Charge Density (CCD)

Let  $R$  be the radius.

$$\vec{E}(r) = \frac{Q(r)}{4\pi\epsilon_0 r^2} \hat{r}$$

$$Q(r) = \begin{cases} Q & (r > R) \\ Q_{r < R} = \iiint_0^r \frac{dQ}{dV} dV & (r < R) \end{cases}$$

- *Conductor*

$$Q_{r < R} = 0 \quad \Rightarrow \quad \vec{E}(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} & (r > R) \\ 0 & (r < R) \end{cases}$$

- *Insulator w/ CCD and  $\epsilon = \epsilon_0$*

$$Q_{r < R} = \int \frac{Q}{V} dV = Q \frac{\int dV}{V} = Q \frac{r^3}{R^3}$$

$$\Rightarrow \vec{E}(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} & (r > R) \\ \frac{Qr}{4\pi\epsilon_0 R^3} \hat{r} & (r < R) \end{cases}$$

Use BSL

$$\begin{aligned} \vec{B}(r) &= \frac{\mu_0 I}{4\pi} \int \frac{d\vec{z} \times \hat{z}}{|\vec{z}|^2} \\ &= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{d\vec{z} \times \hat{z}}{r^2 + z^2} \\ &= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{d\vec{z} \times (r\hat{r} - z\hat{z})}{(r^2 + z^2)^{3/2}} \\ &= \frac{\mu_0 I r}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{(r^2 + z^2)^{3/2}} \hat{\theta} \\ &= \frac{\mu_0 I r}{4\pi} \frac{z}{r^2 \sqrt{r^2 + z^2}} \Big|_{-\infty}^{\infty} \hat{\theta} \\ &= \frac{\mu_0 I}{2\pi r} \hat{\theta} \end{aligned}$$

## Constant Field Solutions (Capacitor/Solenoid)

### 3. Two Infinite Parallel Planes Capacitor w/ CCD (+Q, -Q)

$$\vec{E} = E(z)\hat{z}$$

$$\begin{aligned}\frac{Q}{\epsilon_0} &= \oiint E(z)\hat{z} \cdot d\vec{a} \\ &= E(z)\hat{z} \cdot \oiint xy |d\vec{x} \times d\vec{y}| \\ &= E(z)\hat{z} \cdot xy\hat{z}\end{aligned}$$

$$\vec{E}(z) = \frac{1}{\epsilon_0} \frac{dQ}{dA} \hat{z} = \frac{\sigma}{\epsilon_0} \hat{z}$$

### 4. One Infinite Plane w/ CCD

$$\vec{E} = E(z)\hat{z}$$

$$\begin{aligned}\frac{Q}{\epsilon_0} &= \oiint E(z)\hat{z} \cdot d\vec{a} \\ &= E(z)\hat{z} \cdot 2 \oiint xy |d\vec{x} \times d\vec{y}| \\ &= E(z)\hat{z} \cdot 2xy\hat{z}\end{aligned}$$

$$\vec{E}(z) = \frac{1}{2\epsilon_0} \frac{dQ}{dA} \hat{z} = \frac{\sigma}{2\epsilon_0} \hat{z}$$

### 2. Infinite Long Solenoid Coil w/ SC

Let  $R$  be the coil radius.  $\vec{B} = B(r)\hat{z} = B(r < R)\hat{z}$

$$\begin{aligned}\mu_o NI &= \oint B(r)\hat{z} \cdot d\vec{L} \\ &= (B_{r < R})\hat{z} \cdot \oint d\vec{L} \\ &= (B_{r < R})\hat{z} \cdot L\hat{z}\end{aligned}$$

$$\vec{B}(r) = \begin{cases} \frac{\mu_o IN}{L} \hat{z} = \mu_o I n_l \hat{z} & (0 < r < R) \\ 0 & (r > R) \end{cases}$$

### 3. Closed, Thin Solenoid Ring w/ SC

Let  $R_l$  be the ring radius and  $R_c$  be the coil radius.

$$\vec{B} = B(r)\hat{\theta} = B(R_l - R_c < r < R_l + R_c)\hat{\theta} \approx B(R_l)\hat{\theta}$$

$$\begin{aligned}\mu_o NI &= \oint B(r)\hat{\theta} \cdot d\vec{L} \\ &= B(r)\hat{\theta} \cdot 2\pi R_l \hat{\theta}\end{aligned}$$

$$\vec{B}(r) = \begin{cases} \frac{\mu_o IN}{2\pi R_l} \hat{\theta} & (R_l - c < r < R_l + c) \\ 0 & \text{else} \end{cases}$$

## Integrate w/ CFL and BSL

### 5. Ring w/ CCD centered at origin $O$

Let  $R$  be the ring radius,  $\lambda$  the charge density, and  $\phi$  be  $\angle OzR$ .

$$\begin{aligned}
 E(0, 0, z)\hat{z} &= \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r^2} \vec{r} \\
 \vec{E}(0, 0, z) &= \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(Rd\theta)}{r^2} \vec{r} \\
 &= \frac{k\lambda}{z^2 + R^2} \int_0^{2\pi} R d\theta \cos \phi \hat{z} \\
 &= \boxed{\frac{k\lambda(2\pi R)z}{(z^2 + R^2)^{3/2}} \hat{z}} \\
 &= \boxed{\frac{k\lambda}{D} \frac{2\pi R}{D} \cos \phi \hat{z}}
 \end{aligned}$$

### 4. Ring w/ SC centered at origin $O$

Let  $R$  be the ring radius and  $\phi$  be  $\angle O_{\text{rigin}} zR$ .

$$\begin{aligned}
 B(0, 0, z)\hat{z} &= \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{r}}{|\vec{r}|^2} \\
 \vec{B}(0, 0, z) &= \frac{\mu_0 I}{4\pi} \int \frac{R d\vec{\theta} \times \hat{r}}{r^2} \\
 &= \frac{k_\mu I}{z^2 + R^2} \int R d\theta \hat{\theta} \times \frac{z\hat{z} - R\hat{r}}{\sqrt{z^2 + R^2}} \quad \text{or} \quad \frac{k_\mu I}{z^2 + R^2} \int_0^{2\pi} R d\theta \sin \phi \hat{z} \\
 &= \frac{k_\mu I R}{(z^2 + R^2)^{3/2}} \left[ 2\pi R \hat{z} + z \int_0^{2\pi} \cancel{\hat{r}} d\theta \right] \quad \text{or} \quad \frac{k_\mu I}{z^2 + R^2} \int_0^{2\pi} R d\theta \frac{R}{\sqrt{z^2 + R^2}} \hat{z} \\
 &= \boxed{\frac{k_\mu I (2\pi R) R}{(z^2 + R^2)^{3/2}} \hat{z}} \\
 &= \boxed{\frac{k_\mu I}{D} \frac{2\pi R}{D} \sin \phi \hat{z}}
 \end{aligned}$$

## 6. Line Charge w/ CCD and one edge at the origin $O$

Let  $L$  be the line length,  $\lambda$  the charge density, and  $\theta_0$  be  $\angle OyL$ .

$$E_y(0, 0, y)\hat{y} = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{z^2} \vec{z}_y$$

$$\begin{aligned} E_y(0, 0, y) &= k \int \frac{\lambda dx}{z^2} \vec{z}_y \\ &= k\lambda \int \frac{dx}{y^2 + x^2} \cos \theta, \quad x = y \tan \theta \\ &= \frac{k\lambda}{y^2} \int_0^{\theta_0} \frac{y \sec^2 \theta}{1 + \tan^2 \theta} \cos \theta \\ &= \frac{k\lambda}{y} \int_0^{\theta_0} \cos \theta = \frac{k\lambda}{y} \sin \theta_0 \\ &= \boxed{\frac{k\lambda L}{y\sqrt{y^2 + L^2}}} = \boxed{\frac{k\lambda}{y} \sin \theta_0} \end{aligned}$$

Use this result to find *Finite Line Charge* and *Square Ring*.

## 5. Finite Wire w/ SC and one edge at the origin $O$

Let  $L$  be the wire length and  $\theta_0$  be  $\angle OyL$ .

$$B_z(0, 0, y)\hat{z} = k_\mu I \int \frac{d\vec{l} \times \hat{z}}{|\vec{z}|^2}$$

$$\begin{aligned} B_z(0, 0, y)\hat{z} &= k_\mu I \int \frac{d\vec{x} \times \hat{z}}{|\vec{z}|^2} \\ &= k_\mu I \int \frac{dx}{y^2 + x^2} \sin(\theta + 90), \quad x = y \tan \theta \\ &= \frac{k_\mu I}{y^2} \int_0^{\theta_0} \frac{y \sec^2 \theta}{1 + \tan^2 \theta} \cos \theta \\ &= \frac{k_\mu I}{y} \int_0^{\theta_0} \cos \theta = \frac{k_\mu I}{y} \sin \theta_0 \\ &= \boxed{\frac{k_\mu I L}{y\sqrt{y^2 + L^2}}} = \boxed{\frac{k_\mu I}{y} \sin \theta_0} \end{aligned}$$

Use this result to find *Finite Wire* and *Square Wire*.

## 1.2 Field Energies

The sum of the work to move a collection of charges considering the potential from each other charge comes out to be

$$W = \frac{1}{2} \sum_i q_i V(r_i)$$

E-field Energy (electrostatic...)

$$E = \frac{1}{2} C V^2 = \frac{1}{2} V Q$$

$$\begin{aligned} \boxed{W_{\text{vol}} = \frac{1}{2} \iiint V \rho \, d\tau} &= \frac{\epsilon_0}{2} \iiint V (\nabla \cdot \vec{E}) \, d\tau \\ &= \frac{\epsilon_0}{2} \iiint \vec{E} \cdot (\vec{E}) + \nabla \cdot (V \vec{E}) \, d\tau \\ &= \boxed{\frac{\epsilon_0}{2} \iiint \vec{E}^2 \, d\tau + \frac{\epsilon_0}{2} \iint (V \vec{E}) \cdot d\vec{a}} \\ \boxed{W_E = \frac{\epsilon_0}{2} \iiint \vec{E}^2 \, d\tau} &\quad (\text{if } \rho = 0 \text{ at } \infty) \end{aligned}$$

B-field Energy

$$E = \frac{1}{2} L I^2 = \frac{1}{2} \Phi_B I = \frac{1}{2} \oint \vec{A} \cdot \vec{I} \, dl$$

$$\begin{aligned} \boxed{W_{\text{vol}} = \frac{1}{2} \iiint \vec{A} \cdot \vec{J} \, d\tau} &= \frac{1}{2\mu_0} \iiint \vec{A} \cdot (\nabla \times \vec{B}) \, d\tau \\ &= \frac{1}{2\mu_0} \iiint \vec{B} \cdot (\vec{B}) - \nabla \cdot (\vec{A} \times \vec{B}) \, d\tau \\ &= \boxed{\frac{1}{2\mu_0} \iiint \vec{B}^2 \, d\tau - \frac{1}{2\mu_0} \oint (\vec{A} \times \vec{B}) \cdot d\vec{a}} \\ \boxed{W_B = \frac{1}{2\mu_0} \iiint \vec{B}^2 \, d\tau} &\quad (\text{if } \vec{I} = 0 \text{ at } \infty) \end{aligned}$$

### 1.3 Circuits/Ohm's Law

Ohm's Law  
In Ohmic material,

$$\sigma: \text{Conductivity}$$

$$\rightarrow J \approx \sigma(E + v \times B) \Rightarrow \begin{cases} V = IR \\ P = VI \\ J \approx \sigma E \end{cases}$$

Example: Wire w/ Two Plates

$$I = (\sigma E)A = \left(\frac{\sigma A}{L}\right)V \Rightarrow V = I\left(\frac{L}{\sigma A}\right) = IR$$

<u>Resistor:</u>	$V_R = IR$	$P = VI$		$R = \frac{\rho l}{A}$	$Z_R = R$
<u>Capacitor:</u>	$Q = CV_C$	$E_C = \frac{1}{2}CV_C^2$	$Q_{\uparrow} = Q_0(1 - e^{-t/\tau})$ $Q_{\downarrow} = Q_0e^{-t/\tau}$	$C = \frac{\kappa\epsilon_0 A}{d} = \frac{\epsilon A}{d}$	$Z_C = \frac{1}{i\omega C}$
<u>Inductor:</u>	$\Phi_B = LI$ $V_L = -L\frac{dI}{dt}$	$E_L = \frac{1}{2}LI^2$	$I_{\uparrow} = I_0(1 - e^{-t/\tau})$ $I_{\downarrow} = I_0e^{-t/\tau}$	$L = \frac{\mu_0 N^2 A}{l}$	$Z_L = i\omega L$

Open Circuit :  $R = \infty$   
Short Circuit :  $R = 0$

Constants

$$\tau_{RC} = RC$$

$$\tau_{RL} = L/R$$

$$\omega_{R,LC} = 1/\sqrt{LC}$$

#### AC Filters

Low Pass (Non-Zero  $\overline{\text{Probe}}$ ) :

- $R\overline{C}$
- $L\overline{R}$

High Pass (Non-Zero  $\overline{\text{Probe}}$ ) :

- $C\overline{R}$
- $R\overline{L}$

Band Pass (Zero  $\overline{\text{Probe}}$ ) :

- $R\overline{LC}$
- Bandwidth =  $\left(\text{FWHM} = 2\beta = \frac{b}{m}\right) = \frac{R}{L}$

#### Other Components

$- >   -$	Diode	One Way Voltage (if $>$ Bias Voltage)
$=   > -$	Op-Amp	$V_1 - V_2 \propto V_{OA}$ (Clipping If Too Large $V_{OA}$ )
$=   ) -$	And	
$= ) > -$	Or	

#### De Morgan's Law

- $\overline{A \cdot B} = \overline{A} + \overline{B}$
- $\overline{A + B} = \overline{A} \cdot \overline{B}$



## 1.4 Quasistatic (FLI)

Force on Wire in B-Field :

$$F = qv \times B$$

$$F = LI \times B$$

EMF :

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = \oint \vec{E} \cdot d\vec{l}$$

Mutual Inductance:

$$\text{Flux Through } B: \quad \Phi_B = M \cdot I_A$$

$$\text{Flux Through } A: \quad \Phi_A = M \cdot I_B$$

Faraday's Law

1. Lorentz:

Square Circuit with  $\vec{v}(t)$  leaving  
Constant B-Field (out)

$I$  is out

2. Faraday:

Constant B-Field (out) with  $-\vec{v}(t)$   
leaving Square Circuit

$I$  is out

3. Faraday:

Square Circuit in Increasing B-Field (out)

$I$  is in

Examples:

B-Field Work:

$$(\vec{v} \cdot \vec{l} = 0) \rightarrow W_B = \int \vec{F} \cdot d\vec{l}$$

$$= \int (q\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\boxed{W_B = 0} \quad (\text{magnetic fields do no work})$$

Velocity of wire in (1.):

$$I(t)R = V(t) = \left| \frac{d\Phi_B}{dt} \right| = Bhv(t)$$

$$F_B(t) = -hI(t)B$$

$$= -\frac{B^2 h^2 v(t)}{R}$$

$$F = ma(t)$$

$$\boxed{m \frac{dv}{dt} = F_B + F_{\text{ext}} = F_{\text{ext}} - \frac{B^2 h^2 v(t)}{R}}$$

## 2 Potentials and Fields

### 2.1 Maxwell's Equations for Potentials

#### 1. GLM for Potentials (GLMP)

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0, \quad \vec{A}' = \vec{A} + \nabla \lambda$$

$$\boxed{\vec{B} = \nabla \times \vec{A} \Rightarrow \Phi_B = \oint \vec{A} \cdot d\vec{l}}$$

#### 3. GLE for Potentials (GLEP)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$-\nabla^2 V - \frac{\partial(\nabla \cdot \vec{A})}{\partial t} = \frac{\rho}{\epsilon_0}$$

$$\boxed{\square^2 V - \frac{\partial}{\partial t}(\partial_\mu A^\mu) = \frac{\rho}{\epsilon_0}}$$

#### 2. FLI for Potentials (FLIP)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{E} = \nabla \times \left(0 - \frac{\partial \vec{A}}{\partial t}\right)$$

$$\boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}}, \quad \vec{V}' = \vec{V} + \frac{\partial \lambda}{\partial t}$$

#### 4. MAL for Potentials (MALP)

$$\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

$$\frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} + \nabla \left( \frac{1}{c^2} \frac{\partial V}{\partial t} + \nabla \cdot \vec{A} \right) = \mu_0 \vec{J}$$

$$\boxed{\square^2 \vec{A} + \nabla(\partial_\mu A^\mu) = \mu_0 \vec{J}}$$

Field Energy (see Capacitor/Solenoid in Circuits):

$$\boxed{W_E = \frac{1}{2} \iiint V \rho \, d\tau} = \frac{\epsilon_0}{2} \int E^2 \, d\tau \quad (\text{if } \rho = 0 \text{ at } \infty)$$

$$\boxed{W_B = \frac{1}{2} \iiint \vec{A} \cdot \vec{J} \, d\tau} = \frac{1}{2\mu_0} \int B^2 \, d\tau \quad (\text{if } \vec{J} = 0 \text{ at } \infty)$$

## 2.2 Cases and Freedoms

GLMP and FLIP say that

$$\begin{aligned}\vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\nabla V - \frac{\partial \vec{A}}{\partial t}\end{aligned}$$

In the electrostatic case,

<p><b>Electrostatics:</b></p> $\begin{aligned}(\nabla \times \vec{E} \Leftrightarrow \partial_t \vec{A} \Leftrightarrow \partial_t \vec{B}) &= 0 \\ \Rightarrow \partial_t \rho &= 0\end{aligned}$
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GLEP and MALP say that

$$\begin{aligned}-\nabla^2 V - \frac{\partial(\nabla \cdot \vec{A})}{\partial t} &= \frac{\rho}{\epsilon_0} \\ \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} + \nabla \left( \frac{1}{c^2} \frac{\partial V}{\partial t} + \nabla \cdot \vec{A} \right) &= \mu_0 \vec{J}\end{aligned}$$

Freedom may be chosen to what  $\nabla \cdot \vec{A}$  equals.

<p><b>Coulomb Gauge:</b> <math>\nabla \cdot \vec{A} = 0</math></p>
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- |  |
|--|
| <p><b>Magnetostatics:</b> <math>\partial_t \vec{E} = 0 \Rightarrow \partial_t \vec{J} = 0</math></p> |
|--|

<p><b>Lorenz Gauge:</b> <math>\nabla \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \Leftrightarrow \partial_\mu A^\mu = 0</math></p>
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In general,  $\vec{A}$  and  $V$  can be [gauge] transformed while keeping  $\vec{E}$  and  $\vec{B}$  the same by

$V' = V - \frac{d\lambda}{dt}$	$(\lambda \text{ is a scalar function})$
<table border="1"> <tr> <td> <math display="block">\vec{A}' = \vec{A} + \nabla \lambda</math> </td> </tr> </table>	
$\vec{A}' = \vec{A} + \nabla \lambda$	

## Electrostatic Potentials

Electrostatics:  $(\partial_t \vec{A} = 0 \Rightarrow \partial_t \rho = 0)$ .

Using GLE to find  $E$  or using FLIP,

$$\nabla \times \vec{E} = 0 \Rightarrow \begin{cases} \oint \vec{E} \cdot d\vec{l} = 0 \\ \boxed{\vec{E} = -\nabla V} \end{cases}$$

$\Rightarrow$

$$\int_a^b \nabla \vec{V} \cdot d\vec{l} = \boxed{V(\vec{r}) \Big|_a^b = - \int_a^b \vec{E} \cdot d\vec{l} = W_E/q}$$

$$\boxed{V(\vec{r}) = - \int \vec{E} \cdot d\vec{l} + V_0}$$

and from this (or GLEP)

$$\begin{aligned} \text{Poisson Equation: } \quad \nabla^2 V &= -\frac{\rho(\vec{r}')}{\epsilon_0} \\ \rho_\infty = 0 \Rightarrow \vec{V}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{z} d\tau' \end{aligned}$$

## Coulomb Gauge & Magnetostatic Potentials

Coulomb Gauge: Choose  $(\nabla \cdot \vec{A} = 0)$

Using GLEP,

$$\begin{aligned} \text{Poisson Equation: } \quad \nabla^2 V &= -\frac{\rho(\vec{r}', t)}{\epsilon_0} \\ \rho_\infty = 0 \Rightarrow V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t)}{z} d\tau' \end{aligned}$$

If charges move,  $V$  updates immediately - not at light speed. Only  $\vec{E}$  can be physically measured, and updates at light speed.  $\vec{A}$  is difficult to find using MALP except for special cases like Magnetostatics.

As always, GLMP says

$$\boxed{\Phi_B = \int \vec{A} \cdot d\vec{l}}$$

Magnetostatics:  $(\partial_t \vec{E} = 0 \Rightarrow \partial_t \vec{J} = 0)$

Using MALP,

$$\begin{aligned} \text{Poisson Equation: } \quad \nabla^2 \vec{A} &= -\mu_0 \vec{J}(\vec{r}') \\ \vec{J}_\infty = 0 \Rightarrow \vec{A}(\vec{r}) &= k_\mu \int \frac{\vec{J}(\vec{r}')}{z} d\tau' \end{aligned}$$

### 2.2.1 Potential Examples

#### 1. Point Charges

Reference Choice:  $V(\infty) = 0$

$$V(r) = - \int_{\infty}^r \frac{kQ}{(r')^2} dr' = \frac{Q}{4\pi\epsilon_0} \frac{1}{r}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{Q_i}{r_i}$$

Coulomb Potential: 
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{|\vec{r}_i|}$$

Work: 
$$W = \frac{1}{2} \sum q_i V(\vec{r}_i)$$

1.

2.

3.

## 2. Sphere

Reference Choice:  $V(\infty) = 0$

Let  $R$  be the radius.

$$V(r) = - \int_{\infty}^r \vec{E}(r') \cdot d\vec{r}'$$

- *Conductor*

$$E(r) = \begin{cases} \frac{kQ}{r^2} \\ 0 \end{cases} \Rightarrow V(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r} & (r > R) \\ \frac{Q}{4\pi\epsilon_0 R} & (r < R) \end{cases}$$

- *Insulator w/ CCD and  $\epsilon = \epsilon_0$*

$$E(r) = \begin{cases} \frac{kQ}{r^2} \\ \frac{kQr}{R^3} \end{cases} \Rightarrow V(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r} & (r > R) \\ \frac{Q}{4\pi\epsilon_0 R} + \frac{Qr'^2}{8\pi\epsilon_0 R^3} \Big|_r^R & (r < R) \end{cases}$$

1.

2.

3.

Charges at  $\infty$

### 3. (Infinite) Parallel Plate Capacitor

Reference Choice:  $V(h) = 0$

Let the Capacitor Height be  $h$

$$V(z) = - \int_h^z \frac{\sigma}{\epsilon_0} \hat{z} \cdot d\vec{z} = \frac{\sigma(h-z)}{\epsilon_0} \quad (0 \leq z \leq h)$$

### 4. (Infinite) Single Plate w/ CCD

Reference Choice:  $V(0) = 0$

$$V(z) = - \int_0^z \frac{\sigma}{2\epsilon_0} \hat{z} \cdot d\vec{z} = -\frac{\sigma z}{2\epsilon_0} \quad (0 \leq z < \infty)$$

Try  $V(\infty) = 0$ . (A charge distribution stretching to infinity DNE, so choose a diff. reference point.)

### 5. Infinite Line w/ CCD

Reference Choice:  $V(1) = 0$

$$\begin{aligned} V(r) &= - \int_1^r \frac{\lambda}{2\pi r \epsilon_0} \hat{r} \cdot d\vec{r} \\ &= -\frac{\lambda}{2\pi \epsilon_0} \ln r \end{aligned}$$

Try  $V(\infty) = 0$  (same problem above).

3.

4.

5.

## 2.3 Multipole Expansion

$$\begin{aligned}
 z^2 &= r^2 + (r')^2 - 2(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}') \\
 &= r^2 \left[ 1 + \frac{r'}{r} \left( \frac{r'}{r} - 2 \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}'}{r'r} \right) \right] \quad \Rightarrow \quad \frac{1}{z} = \frac{1}{r} (1 + \epsilon)^{-1/2} \\
 &= r^2 (1 + \epsilon) \quad \Rightarrow \quad = \frac{1}{r} \sum_{n=0} \left( \frac{r'}{r} \right)^n P_n(\hat{\mathbf{r}}' \cdot \hat{\mathbf{r}}) \quad (\text{Legendre Polynomials})
 \end{aligned}$$


---

$$\begin{aligned}
 V(\vec{\mathbf{r}}) &= \frac{1}{4\pi\epsilon_0} \int \frac{1}{z} \rho(\vec{\mathbf{r}}') d\tau' \\
 &= \boxed{\frac{1}{4\pi\epsilon_0} \sum_n \frac{1}{r^{n+1}} \int (r')^n P_n(\cos \alpha) \rho(\vec{\mathbf{r}}') d\tau'} \\
 &= \frac{1}{4\pi\epsilon_0} \left[ \begin{aligned} &\frac{1}{r} \int \rho(\vec{\mathbf{r}}') d\tau' + \frac{1}{r^2} \int \vec{\mathbf{r}}' \cdot \hat{\mathbf{r}} \rho(\vec{\mathbf{r}}') d\tau' \\ &+ \frac{1}{r^3} \int (r')^2 P_2(\hat{\mathbf{r}}' \cdot \hat{\mathbf{r}}) \rho(\vec{\mathbf{r}}') d\tau' + \frac{1}{r^4} \int \dots \end{aligned} \right]
 \end{aligned}$$

$$\begin{aligned}
 V_{\text{mon}} &= \frac{1}{4\pi\epsilon_0 r} \int \rho(\vec{\mathbf{r}}') d\tau' \\
 V_{\text{dip}} &= \frac{1}{4\pi\epsilon_0 r^2} \left( \int \vec{\mathbf{r}}' \rho(\vec{\mathbf{r}}') d\tau' \right) \cdot \hat{\mathbf{r}} = \frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}
 \end{aligned}$$

$V$  is the dipole term.  $\mathbf{p}$  is the dipole moment.

$$\begin{aligned}
 \vec{A}(\vec{\mathbf{r}}) &= k_\mu \int \frac{1}{z} \vec{J}(\vec{\mathbf{r}}') d\tau' \\
 &= \boxed{k_\mu \sum_n \frac{1}{r^{n+1}} \int (r')^n P_n(\cos \alpha) \vec{J}(\vec{\mathbf{r}}') d\tau'} \\
 &= k_\mu \left[ \begin{aligned} &\frac{1}{r} \int \vec{J}(\vec{\mathbf{r}}') d\tau' + \frac{1}{r^2} \int \vec{\mathbf{r}}' \cdot \hat{\mathbf{r}} \vec{J}(\vec{\mathbf{r}}') d\tau' \\ &+ \frac{1}{r^3} \int (r')^2 P_2(\hat{\mathbf{r}}' \cdot \hat{\mathbf{r}}) \vec{J}(\vec{\mathbf{r}}') d\tau' + \frac{1}{r^4} \int \dots \end{aligned} \right]
 \end{aligned}$$

Steady current:

$$\begin{aligned}
 A_{\text{mon}} &= \frac{\mu_0 I}{4\pi r} \oint dl' = 0 \\
 A_{\text{dip}} &= \frac{k_\mu}{r^2} I \int \vec{\mathbf{r}}' \cdot \hat{\mathbf{r}} dl' = \frac{k_\mu}{r^2} I \int d\vec{a}' \times \hat{\mathbf{r}} \\
 &= \frac{k_\mu}{r^2} (I\vec{a}) \times \hat{\mathbf{r}} = \frac{\mu_0 \vec{\mathbf{m}} \times \hat{\mathbf{r}}}{4\pi r^2}
 \end{aligned}$$

$A$  is the dipole term.  $\mathbf{m}$  is the dipole moment.



## Ideal Dipoles

Let dipole (2 charges)  $\vec{\mathbf{p}} = p\hat{\mathbf{z}} = 2dq\hat{\mathbf{z}}$  and centered at the origin.

$\lim d \rightarrow 0, q \rightarrow \infty$  :

$$\begin{aligned}
 V(\vec{\mathbf{r}}) &= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{d^n P_n(\cos \alpha) q + (-d)^n P_n(\cos \alpha) (-q)}{4\pi\epsilon_0 r^{n+1}} \\
 &= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{P_n(\cos \alpha) q d^n}{r^{n+1}} [1 + (-1)^n] \\
 &= \frac{1}{4\pi\epsilon_0} \sum_{m=0}^{\infty} \frac{P_m(\cos \alpha)}{r^{2m+2}} (2qd) d^{2m} \\
 &= \frac{1}{4\pi\epsilon_0} \sum_{m=0}^{\infty} \frac{P_m(\cos \alpha)}{r^{2m+2}} p d^{2m} \\
 &= \frac{1}{4\pi\epsilon_0} \frac{P_0(\cos \alpha)}{r^2} p + 0 + 0 + \dots
 \end{aligned}$$

Ideal Dip: $V_{\text{dip}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{z}}}{r^2}$
--

Let dipole (ring)  $\vec{\mathbf{m}} = m\hat{\mathbf{z}} = Ia\hat{\mathbf{z}}$  and centered at the origin.

$\lim a = \pi d^2 \rightarrow 0, I \rightarrow \infty$  :

$$\begin{aligned}
 \vec{A}(\vec{\mathbf{r}}) &= k_\mu \sum_{n=0}^{\infty} \frac{I d^n}{r^{n+1}} \int P_n(\cos \alpha) d\vec{l}' \\
 &= k_\mu \left( \begin{aligned} &\frac{I}{r} \int d\vec{l}' + \frac{I}{r^2} \int (d\hat{r}' \cdot \hat{\mathbf{r}}) d\vec{l}' \\ &+ \frac{I d^2}{r^3} \int \left[ \frac{3}{2} \left( 1 + 2 \frac{d\hat{r}' \cdot \hat{\mathbf{r}}}{d} + 1 \right) - \frac{1}{2} \right] d\vec{l}' \\ &+ I d^2 \sum_{n=3}^{\infty} \frac{d^{n-2}}{r^{n+1}} \int P_n(\hat{r}' \cdot \hat{\mathbf{r}}) d\vec{l}' \end{aligned} \right) \\
 &= k_\mu \left( 0 + \frac{I\pi d^2}{r^2} (\hat{\mathbf{z}} \times \hat{\mathbf{r}}) + \frac{3I\pi d^3}{r^3} (\hat{\mathbf{z}} \times \hat{\mathbf{r}}) + \frac{m}{\pi} (0 + \dots) \right) \\
 &= k_\mu \left( 0 + \frac{m}{r^2} (\hat{\mathbf{z}} \times \hat{\mathbf{r}}) + 0 + 0 + \dots \right)
 \end{aligned}$$

Ideal Dip: $\vec{A}_{\text{dip}}(\vec{\mathbf{r}}) = k_\mu \frac{\vec{\mathbf{m}} \times \hat{\mathbf{z}}}{r^2}$
--

### 2.3.1 Multipole Examples

3.

4.

5.

3.

4.

5.

### 3 Electrodynamics in Matter

#### 3.1 Ideal Dipoles

$$\vec{\mathbf{p}} = \int r' \cdot \rho(r') d\tau'$$

$$\begin{aligned} \vec{F}_{\text{dip}} &= qE \Big|_{\vec{r}}^{\vec{r}+\vec{d}} = q\Delta\vec{E} \\ &\approx \left[ q \sum_i \left( \nabla E_i \cdot \vec{d} \right) \hat{i} \right] \end{aligned} \quad \left| \quad \begin{aligned} U_{\text{ES dip}} &= qV \Big|_{\vec{r}}^{\vec{r}+\vec{d}} = q\Delta V \\ &= q \int_{\vec{r}}^{\vec{r}+\vec{d}} -\vec{E} \cdot d\vec{l} \end{aligned}$$

$$\vec{F}_{\text{dip}} = (\vec{\mathbf{p}} \cdot \nabla) \vec{E}$$

$$U_{\text{ES dip}} = -\vec{\mathbf{p}} \cdot \vec{E}$$

$$\vec{N}_{\text{center}} = \vec{r} \times \vec{F} = \vec{d} \times q\vec{E}$$

$$\vec{N}_{\text{dip}} = \vec{\mathbf{p}} \times \vec{E}$$

$$\text{Polarization: } \vec{P} = \frac{d\vec{\mathbf{p}}}{d\tau} \quad \left( \frac{\hat{z}}{z^2} = \nabla' \frac{1}{z} \right)$$

$$\begin{aligned} V(\vec{\mathbf{r}}) &= \frac{1}{4\pi\epsilon_0} \int_{\nu} \frac{\vec{P}(\vec{r}') \cdot \hat{z}}{z^2} d\tau' \\ &= \frac{1}{4\pi\epsilon_0} \int_{\nu} \frac{-\nabla' \cdot \vec{P}(\vec{r}')}{z} d\tau' + \frac{1}{4\pi\epsilon_0} \int_S \frac{\vec{P}(\vec{r}') \cdot \hat{n}}{z} da' \end{aligned}$$

$$\rho_b = -\nabla \cdot \vec{P}$$

$$\sigma_b = \vec{P} \cdot \hat{n}$$

$$\vec{\mathbf{m}} = \sum I\vec{a}$$

$$\begin{aligned} \vec{F}_{\text{sqr. dip}} &= q\vec{v} \times \vec{B} \\ &= \pm IL\vec{x} \times B\hat{z} \\ &= \pm ILB \hat{y} \end{aligned}$$

$$\begin{aligned} \vec{N}_{\text{sqr. dip}} &= 2 \left[ \frac{\pm \vec{W}}{2} \times \pm ILB\hat{y} \right] \\ &= I(LW) \sin\theta B \hat{x} \end{aligned}$$

$$\vec{N}_{\text{dip}} = \vec{\mathbf{m}} \times \vec{B}$$

$$\vec{F}_{\text{dip}} = \nabla(\vec{\mathbf{m}} \cdot \vec{B}) ???$$

$$U_{\text{dip}} = -\vec{\mathbf{m}} \cdot \vec{B}$$

$$\text{Magnetization: } \vec{M} = \frac{d\vec{\mathbf{m}}}{d\tau} \quad \left( \frac{\hat{z}}{z^2} = \nabla' \frac{1}{z} \right)$$

$$\begin{aligned} \vec{A}(\vec{\mathbf{r}}) &= k_{\mu} \int_{\nu} \frac{\vec{M}(\vec{r}') \times \hat{z}}{z^2} d\tau' \\ &= k_{\mu} \int_{\nu} \frac{\nabla' \times \vec{M}(\vec{r}')}{z} d\tau' + k_{\mu} \int_S \frac{\vec{M}(\vec{r}') \times \hat{n}}{z} da' \end{aligned}$$

$$\vec{J}_b = \nabla \times \vec{M}$$

$$\vec{K}_b = \vec{M} \times \hat{n}$$

### 3.2 Maxwell's Equations in Matter

#### GLE in Matter (GLEM)

$$\begin{aligned}\nabla \cdot \epsilon_0 \vec{E} &= \rho = \rho_b + \rho_f \\ &= -\nabla \cdot \vec{P} + \nabla \cdot D\end{aligned}$$

$$\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \nabla \cdot D$$

$$\begin{aligned}\vec{D} &= \epsilon_0 \vec{E} + \vec{P} \\ \nabla \cdot \vec{D} &= \rho_f & -\nabla \cdot \vec{P} &= \rho_b \\ \vec{P} \cdot \hat{n} &= \sigma_b\end{aligned}$$

#### COC in Matter (COCM)

$$\begin{aligned}\nabla \cdot \vec{J}_p &= -\frac{\partial \rho_b}{\partial t} \\ &= \frac{\partial}{\partial t} (\nabla \cdot \vec{P})\end{aligned}$$

$$\frac{\partial \vec{P}}{\partial t} = \vec{J}_p$$

#### MAL in Matter (MALM)

$$\begin{aligned}\nabla \times \frac{1}{\mu_0} \vec{B} &= \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{J}_b + \vec{J}_f + \vec{J}_p + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ &= \nabla \times M + \vec{J}_f + \frac{\partial \vec{P}}{\partial t} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\end{aligned}$$

$$\nabla \times \left( \frac{1}{\mu_0} \vec{B} - M \right) = \vec{J}_f + \frac{\partial}{\partial t} (\epsilon_0 \vec{E} + \vec{P})$$

$$\begin{aligned}\vec{H} &= \frac{1}{\mu_0} \vec{B} - \vec{M} \\ \nabla \times \vec{H} &= \vec{J}_f + \frac{\partial \vec{D}}{\partial t} & \nabla \times \vec{M} &= \vec{J}_b \\ \vec{M} \times \hat{n} &= \vec{K}_b\end{aligned}$$

#### Faraday's Law of Induction (FLI)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{Electrostatics: } \nabla \times \vec{D} = \nabla \times \vec{P}$$

#### Gauss's Law for Magnetism (GLM)

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

### 3.3 Linear Matter

Electric Susceptibility:  $\chi_e$

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

Susceptibility Tensor:

$$\vec{P} = \begin{pmatrix} \chi_{exx} & \chi_{exy} & \chi_{exz} \\ \chi_{eyx} & \chi_{eyy} & \chi_{eyz} \\ \chi_{ezx} & \chi_{ezy} & \chi_{ezz} \end{pmatrix} \epsilon_0 \vec{E}$$

Relative Permittivity:  $\epsilon_r = 1 + \chi_e$

$$\begin{aligned} \vec{D} &= (1 + \chi_e) \epsilon_0 \vec{E} \\ &= \epsilon_r \epsilon_0 \vec{E} \\ &= \epsilon \vec{E} \end{aligned}$$

Magnetic Susceptibility:  $\chi_m$

$$\vec{M} = \chi_m \vec{H}$$

Susceptibility Tensor:

$$\vec{M} = \begin{pmatrix} \chi_{mxx} & \chi_{mxy} & \chi_{mxz} \\ \chi_{myx} & \chi_{myy} & \chi_{myz} \\ \chi_{mzx} & \chi_{mzy} & \chi_{mzz} \end{pmatrix} \vec{H}$$

Bound Current:

$$\begin{aligned} \vec{J}_b &= \nabla \times (\chi_m \vec{H}) \\ &= \chi_m (\vec{J}_f + \partial_t \vec{D}) \end{aligned}$$

Relative Permeability:  $\mu_r = 1 + \chi_m$

$$\begin{aligned} \vec{B} &= (1 + \chi_m) \mu_0 \vec{H} \\ &= \mu_r \mu_0 \vec{H} \\ &= \mu \vec{H} \end{aligned}$$

## 4 Boundary Conditions

$$\Delta \vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}$$

$$1. \quad \boxed{\Delta E_{\parallel} = 0} \quad \left| \quad \oint \vec{E} \cdot d\vec{L} = - \oint_{0-}^{0+} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \right. \\ \left. (E_{\parallel}^+ - E_{\parallel}^-) L = 0 \right.$$

$$2. \quad \boxed{\Delta E_{\perp} = \frac{\sigma}{\epsilon_0}} \quad \left| \quad \oint \vec{E} \cdot d\vec{a} = Q/\epsilon_0 \right. \\ \boxed{\Delta D_{\perp} = \sigma_f} \quad \left| \quad (E_{\perp}^+ - E_{\perp}^-) a = \frac{\sigma a}{\epsilon_0} \right.$$

Electrostatics:  $\nabla \times \vec{E} = 0$

$$\boxed{\Delta V = 0} \quad \left| \quad V \right|_{0-}^{0+} = - \int_{0-}^{0+} \vec{E} \cdot d\vec{L}$$

$$\boxed{\Delta \frac{\partial V}{\partial n} = -\frac{\sigma}{\epsilon_0}} \quad \left| \quad \Delta(\nabla V) \cdot \hat{n} \right.$$

$$\boxed{\Delta \vec{D}_{\parallel} = \Delta \vec{P}_{\parallel}} \quad \left| \quad \nabla \times \vec{D} = \nabla \times \vec{P} \right.$$

$$\Delta \vec{B} = \mu_0 \vec{K} \times \hat{n}$$

$$1. \quad \boxed{\Delta B_{\perp} = 0} \quad \left| \quad \oint \vec{B} \cdot d\vec{a} = 0 \right. \\ \left. (B_{\perp}^+ - B_{\perp}^-) a = 0 \right.$$

$$2. \quad \boxed{\Delta \vec{B}_{\parallel} = \mu_0 \vec{K} \times \hat{n}} \quad \left| \quad \oint \frac{\vec{B}}{\mu_0} \cdot d\vec{L} = \oint_{0-}^{0+} \left( \vec{J} + \frac{\epsilon_0 \partial \vec{E}}{\partial t} \right) \cdot d\vec{a} \right. \\ \boxed{\Delta \vec{H}_{\parallel} = \vec{K}_f \times \hat{n}} \quad \left| \quad (B_{\parallel}^+ - B_{\parallel}^-) L = \mu_0 I_{enc} \right. \\ \left. \Delta B_{\parallel} L = \mu_0 K L = (\mu_0 \vec{K} \times \hat{n}) \cdot \vec{L} \right.$$

$$\boxed{\Delta A_{\parallel} = 0} \quad \left| \quad \oint \vec{A} \cdot d\vec{L} = \Phi_B = 0 \right.$$

Magnetostatic:  $\nabla \cdot \vec{A} = 0$

$$\boxed{\Delta A_{\perp} = 0} \quad \left| \quad \oint_{0-}^{0+} \vec{A} \cdot d\vec{a} = 0 \right.$$

$$\boxed{\Delta \frac{\partial \vec{A}}{\partial n} = -\mu_0 \vec{K}} \quad \left| \quad \Delta(\nabla \times \vec{A}) = \left( -\frac{\partial A_y^+}{\partial z} + \frac{\partial A_y^-}{\partial z} \right) \hat{x} + \left( \frac{\partial A_x^+}{\partial z} - \frac{\partial A_x^-}{\partial z} \right) \hat{y} \right. \\ \left. = -\mu_0 K \hat{y} \right.$$

## 5 Work-Energy, Radiation, and Momentum

The sum of the work to move a collection of charges considering the potential from each other charge comes out to be

$$W = \frac{1}{2} \sum_i q_i V(r_i)$$

### 5.1 Field Energies

$$W_E = \frac{\epsilon_0}{2} \int \vec{E}^2 d\tau \quad W_B = \frac{1}{2\mu_0} \int \vec{B}^2 d\tau$$

$$U_{EB} = \frac{1}{2} \int \epsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2 d\tau = \frac{1}{2} \int u_{EB} d\tau$$

### 5.2 Energy Conservation

$$\text{Poynting Vector: } \vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{1}{2\mu_0} \text{Re}(\tilde{\mathbf{E}} \times \tilde{\mathbf{B}}^*)$$

$$I = \langle P_{\text{ow}}/A \rangle = \langle S \rangle = \frac{1}{2} c \epsilon_0 E^2$$

$$-P_{\text{ow}} = \frac{dW}{dT} + \frac{dU_{EB}}{dt} = - \int \vec{S} \cdot d\vec{a}$$

$$\frac{d}{dt}(u_{\text{mech}} + u_{EB}) = -\nabla \cdot \vec{S}$$

### 5.3 Radiation

Accelerating Charge

$$\text{Larmor Formula } (v \ll c): P_{\text{ow}} = \left( \frac{2k_e}{3c^3} \right) q^2 a^2$$

Electric Dipole Radiation

$$\text{Dipole Moment: } \vec{p}(t) = p_0 \cos(\omega t) \hat{z}$$

$$\text{Intensity: } \langle S \rangle = \left( \frac{k_e}{8\pi c^3} \right) p_0^2 \omega^4 \frac{\sin^2 \theta}{r^2}$$

$$\text{Power: } \langle P \rangle_E = \left( \frac{k_e}{3c^3} \right) p_0^2 \omega^4$$

$$\text{Magnetic Dipole Radiation: } \langle P \rangle_B = \left( \frac{k_\mu}{3c^3} \right) m_0^2 \omega^4$$

## 5.4 Momentum Conservation

$$\left(a \cdot \overleftrightarrow{T}\right)_i = \sum_n a_n T_{in}$$

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2\right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2\right)$$

$$\begin{aligned} F_i &= \int f_i d\tau \\ &= \oint_S \left(\overleftrightarrow{T} \cdot da\right)_i - \frac{1}{c^2} \frac{d}{dt} \int_V S d\tau \end{aligned}$$

$$\frac{dP_{mech}}{dt} = \int_V \left(\nabla \cdot \overleftrightarrow{T}\right)_i d\tau - \frac{dP_{EM}}{dt}$$

$$\frac{d}{dt} (P_{mech} + P_{EM})_i = \int_V \left(\nabla \cdot \overleftrightarrow{T}\right)_i d\tau$$

$$L_{EM} = \vec{r} \times P_{EM}$$

$$\begin{aligned} \mathbf{f}_i &= \epsilon_0 E_i + (\vec{J} \times \vec{B})_i \\ &= \left(\nabla \cdot \overleftrightarrow{T}\right)_i - \frac{1}{c^2} \frac{\partial \vec{S}_i}{\partial t} \end{aligned}$$

$$\frac{\partial}{\partial t} (p_{mech})_i = \left(\nabla \cdot \overleftrightarrow{T}\right)_i - \frac{\partial}{\partial t} (p_{EM})_i$$

$$\frac{\partial}{\partial t} (p_{mech} + p_{EM})_i = \left(\nabla \cdot \overleftrightarrow{T}\right)_i$$

$$l_{EM} = \vec{r} \times p_{EM}$$



## 6 Potentials in Lorenz Gauge (nonstatic sources)

See Potentials for Recap

If choose  $\left(\nabla \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \Leftrightarrow \partial_\mu A^\mu = 0\right)$

$$\begin{cases} \square^2 V = \frac{\rho}{\epsilon_0} \\ \square^2 \vec{A} = \mu_0 \vec{J} \end{cases}$$

Solutions satisfying these three equations (thus satisfying Maxwell's Eq.) are,

$$\begin{aligned} V(\vec{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{z} d\tau' \\ \vec{A}(\vec{r}, t) &= k_\mu \int \frac{\vec{J}(\vec{r}', t_r)}{z} d\tau' \end{aligned}$$

where  $t_r = t - \frac{z}{c}$ .

Notice that charges move,  $V$  and  $\vec{A}$  update at the speed of light.  $t_r = t + \frac{z}{c}$  is also a solution, though not physically real.

Using GLMP and FLIP to find the fields,

*Jefimenko Equations:*

$$\begin{aligned} \tilde{\mathbf{E}}(\tilde{\mathbf{r}}, \mathbf{t}) &= \frac{1}{4\pi\epsilon_0} \int \left[ \frac{\rho(\tilde{\mathbf{r}}, \mathbf{t}_r)}{z^2} \hat{z} + \frac{\dot{\rho}(\tilde{\mathbf{r}}, \mathbf{t}_r)}{c^2} \hat{z} + \frac{\tilde{\mathbf{J}}(\tilde{\mathbf{r}}, \mathbf{t}_r)}{c^2 z} \right] d\tau' \\ \tilde{\mathbf{B}}(\tilde{\mathbf{r}}, \mathbf{t}) &= k_\mu \int \left[ \frac{\tilde{\mathbf{J}}(\tilde{\mathbf{r}}, \mathbf{t}_r)}{z^2} + \frac{\dot{\tilde{\mathbf{J}}}(\tilde{\mathbf{r}}, \mathbf{t}_r)}{c z} \right] \times \hat{z} d\tau' \end{aligned}$$

It's usually easier solve for the potentials first instead of fields directly. In the electrostatic and magnetostatic limits, CL and BSL are recovered.

## 7 EM Waves

$$f(z, t) = \text{Re}[\tilde{f}(z, t)] = \text{Re} [Ae^{i(kz - wt + \delta)}]$$

$\omega$  is the same throughout! (?)

$$\frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

$$\tilde{\mathbf{f}}(\mathbf{z}, \mathbf{t}; \delta = \mathbf{0}) : \tilde{A}_I e^{i(k_1 z - wt)} + \tilde{A}_R e^{i(-k_1 z - wt)} \Rightarrow \tilde{A}_T e^{i(k_2 z - wt)}$$

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T; \quad k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T$$

$$\tilde{A}_R e^{i\delta_R} = \left( \frac{v_2 - v_1}{v_2 + v_1} \right) \tilde{A}_I e^{i\delta_I}; \quad \tilde{A}_T e^{i\delta_T} = \left( \frac{2v_2}{v_2 + v_1} \right) \tilde{A}_I e^{i\delta_I}$$

$$A_R = \left( \frac{|v_2 - v_1|}{v_2 + v_1} \right) A_I; \quad A_T = \left( \frac{2v_2}{v_2 + v_1} \right) A_I$$

### 7.1 Vacuum, $\vec{v}_{||} \vec{E}_{||} \hat{z}$

$$\tilde{B}_0 = \frac{k}{w} (\hat{z} \times \tilde{E}_0) = \frac{1}{c} (\hat{z} \times \tilde{E}_0)$$

$$\vec{S} = cu_{EM} \hat{z} = c\epsilon_0 E_0^2 \cos^2(kw, wt + \delta) \hat{z}$$

$$I_{nt} = \langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2$$

$$P_{res} = \frac{I_{nt}}{c}$$

## 7.2 Linear Media

$D = \epsilon E; \quad B = \mu H$ $\tilde{B}_0 = \frac{1}{v}(\hat{z} \times \tilde{E}_0)$	<ul style="list-style-type: none"> <li>• <math>n = \frac{c}{v}</math></li> <li>• <math>n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r}</math></li> </ul>	<ul style="list-style-type: none"> <li>• <math>k_I v_1 = k_R v_1 = k_T v_2 = \omega</math></li> <li>• <math>k_I \sin \theta_I = (k_R \sin \theta_R = k_T \sin \theta_T) = k_T \sin \theta_T</math></li> <li>• Snell's Law : <math>n_1 \sin \theta_I = n_2 \sin \theta_T</math></li> </ul>
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Fresnel's Equations Oblique Incidence       $\left( \alpha = \frac{\cos \theta_T}{\cos \theta_I} = \frac{\sqrt{1-(n_1/n_2)^2 \sin^2 \theta_I}}{\cos \theta_I}, \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2} \approx \frac{v_1}{v_2} \right)$

• P-Polarized ( $E_{\parallel}$  to Plane of Incidence):

$$\tilde{E}_R = \left( \frac{\alpha - \beta}{\alpha + \beta} \right) \tilde{E}_I; \quad \tilde{E}_T = \left( \frac{2}{\alpha + \beta} \right) \tilde{E}_I$$

Reflection Shift/Angles ( $\alpha - \beta \stackrel{?}{=} 0$ ) :  $\tan^2 \theta_I \stackrel{?}{=} \left( \frac{n_2}{n_1} \right)^2 \frac{1-\beta^2}{1-(n_2/n_1)^2}$

$$R = \frac{I_R}{I_I} = \left( \frac{E_R}{E_I} \right)^2 = \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^2$$

In-Phase ( $\delta = 0, \alpha > \beta$ ) :  $\tan \theta_I > n_2/n_1$

Out-of-Phase ( $\delta = \pi, \alpha < \beta$ ) :  $\tan \theta_I < n_2/n_1$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \left( \frac{E_T}{E_I} \right)^2 \frac{\cos \theta_T}{\cos \theta_I} = \alpha \beta \left( \frac{2}{\alpha + \beta} \right)^2$$

Brewster's Angle ( $R = 0$ ) :  $\tan \theta_{I=b} = n_2/n_1, \quad \theta_R + \theta_T = 90$

Critical Angle ( $T = 0$ ) :  $\sin \theta_{I=c} = n_2/n_1, \quad \theta_R = 90$  ( $n_1 > n_2$ )  
(evanescent if  $> \theta_c$ )

• S-Polarized ( $E_{\perp}$  to Plane of Incidence):

$$\tilde{E}_R = \left( \frac{1 - \alpha\beta}{1 + \alpha\beta} \right) E_I; \quad \tilde{E}_T = \left( \frac{2}{1 + \alpha\beta} \right) E_I$$

Reflection Shift/Angles ( $1 - \alpha\beta \stackrel{?}{=} 0$ ) :  $\alpha\beta \approx \frac{\sqrt{\beta^2 - \sin^2 \theta_I}}{\cos \theta_I}$

$$R = \frac{I_R}{I_I} = \left( \frac{E_R}{E_I} \right)^2 = \left( \frac{1 - \alpha\beta}{1 + \alpha\beta} \right)^2$$

In-Phase ( $\delta = 0, 1 > \alpha\beta$ ) :  $n_1 > n_2$

Out-of-Phase ( $\delta = \pi, 1 < \alpha\beta$ ) :  $n_2 > n_1$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \alpha \left( \frac{E_T}{E_I} \right)^2 = \alpha \beta \left( \frac{2}{1 + \alpha\beta} \right)^2$$

Brewster Angle ( $R = 0$ ) :  $n_1 = n_2$  (None)

Critical Angle ( $T = 0$ ) :  $\sin \theta_{I=c} = n_2/n_1, \quad \theta_R = 90$  ( $n_1 > n_2$ )  
(evanescent if  $> \theta_c$ )

## 7.3 Diffraction and Interference

Double Slit Interference: ( $d \ll L$ )

Maxima :  $d \sin \theta = m\lambda$

Minima :  $d \sin \theta = (m + \frac{1}{2})\lambda$

Circular Aperture: (Diameter:  $D \ll L$ )

$\theta$  = Twice the normal, vertical angle

1st Minima :  $D \sin \theta = 1.22\lambda$

Optical Path Length: ( $n_1 \rightarrow n_2$ ,  $\lambda \rightarrow \frac{\lambda}{n}$ ,  $v_n = f \frac{\lambda}{n}$ )

- $\delta = \frac{2\pi d}{\lambda/n} = k(nd)$
- $\Delta x_n = nd = nv\Delta t = c\Delta t$  ( $t$ , time through medium  $n$ )  
( $2dn$  for thin film reflex.)

## 7.4 Lenses and Mirrors ( $\lambda \ll a$ )

Draw Picture : 1.  $\overline{f, y[s], L_{\text{ens}}} \rightarrow \overline{L_{\text{ens}}, y'[s'], \infty}$   
2.  $\overline{\infty, y[s], L_{\text{ens}}} \rightarrow \overline{f', L_{\text{ens}}, y'[s']}$

Imaging Eq. :  $\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$

Thin Lens Eq. :  $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$  (Focal Length,  $f = f'$ )

Lensmaker Eq. :  $\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right) \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$  ( $R_2$  is [-] for concave lens)

Lens Magnf. :  $M_T \equiv \frac{y'}{y} = -\frac{s'}{s} = \frac{f}{f-s}$  Virtual:  $f > s$   
Real:  $s < f$

Spherical Mirror :  $f = R/2$

Single Slit Diffraction: ( $a \ll L$ ,  $a \sim \lambda$ )

Minima :  $a \sin \theta = m\lambda$ ,  $m \neq 0$

Bragg [X-Ray] Diffraction: (Atom Distance :  $d \sim \lambda$ )

$\theta$  = Angle from Horizontal (not vertical/normal)

- Maxima :  $(2d) \sin \theta = m\lambda$

Boundary Reflection: ( $n_1 \rightarrow n_2$ )

$n_2 < n_1$  :  $\delta = 0$

$n_2 > n_1$  :  $\delta = \pi$

## 7.5 Other

Rayleigh Scattering ( $\lambda \gg a$ ) :  $I \propto I_0 \left(\frac{a^6}{\lambda^4}\right)$  (Dipole Radiation, polarized)

[Sound] Doppler Effect ( $v \ll c$ ) :  $f_r = \left(\frac{v + v_r}{v - v_s}\right) f_s$  (frequency,  $f$ )  
( $v_r$ ,  $v_s$  are [+]  
if  $\rightarrow \leftarrow$ )

Standing Sound Wave

- Open Pipe :  $L = n \left(\frac{\pi}{2}\right)$  (Ends are nodes/infl. pts. of 0 press.)
- Half Pipe :  $L = (2n + 1) \left(\frac{\pi}{4}\right)$  (Open End is a node, Closed is an antinode/maxi. press.)

Malus's Law :  $\begin{aligned} I &= I_0 \cos^2 \theta && \text{(polarized)} \\ I &= I_0/2 && \text{(unpolarized)} \end{aligned}$

## 7.6 Conductor; $J_{free} \neq 0$

$$J_{free} = \sigma E$$

$$\tilde{E}(z, t) = \tilde{E}_0 e^{i(\tilde{k}z - \omega t)}; \quad \tilde{B}(z, t) = \tilde{B}_0 e^{i(\tilde{k}z - \omega t)}$$

$$\tilde{k} = k + i\kappa; \quad \tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

$$k = \omega \sqrt{\frac{\epsilon\mu}{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1}; \quad \kappa = \omega \sqrt{\frac{\epsilon\mu}{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1}$$

Skin depth:  $d = \frac{1}{\kappa}$

Wave (phase) velocity:  $v = \frac{\omega}{k}$

Group velocity (carries energy):  $v_g = \frac{d\omega}{dk} < c$

Index Ref:  $n = \frac{ck}{\omega}$

$$\frac{B_0}{E_0} = \frac{K}{\omega} = |\tilde{k}|/\omega = \sqrt{\epsilon\mu} \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}$$

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}$$

$$\tilde{E}_R = \left( \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right) \tilde{E}_I; \quad \tilde{E}_T = \left( \frac{2}{1 + \tilde{\beta}} \right) \tilde{E}_I$$

## 7.7 Wave Guides

$$E^{\parallel} = 0; \quad B^{\perp} = 0$$

TE Waves:  $E_z = 0$ ;    TM Waves:  $B_z = 0$ ;    TEM Waves:  $E_z = B_z = 0$

$$\begin{aligned} E_x &= \frac{i}{(w/c)^2 - k^2} \left( k \frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y} \right) \\ E_y &= \frac{i}{(w/c)^2 - k^2} \left( k \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x} \right) \\ B_x &= \frac{i}{(w/c)^2 - k^2} \left( k \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_z}{\partial y} \right) \\ B_y &= \frac{i}{(w/c)^2 - k^2} \left( k \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x} \right) \end{aligned}$$

Solving Rectangular Wave Guides:

$$\text{TE}_{mn \neq 00}: \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] B_z = 0$$

$$B_z = X(x)Y(y)$$

$$\frac{\partial^2 X}{\partial x^2} = -k_x^2 X; \quad \frac{\partial^2 Y}{\partial y^2} = -k_y^2 Y$$

$$-k_x^2 - k_y^2 + (w/c)^2 - k^2 = 0$$

$$B_z = B_0 \cos(m\pi x/a) \cos(n\pi y/b)$$

$$\omega < \omega_{mn} = c\pi \sqrt{(m/a)^2 + (n/b)^2}$$

$$\text{TM}: \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] E_z = 0$$

## 8 Del

$$\nabla F = \left\langle \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right\rangle F$$

$$\nabla \cdot \vec{A} = \frac{1}{r^2 \sin \theta} \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right\rangle \cdot r^2 \sin \theta \left\langle A_r, \frac{1}{r} A_\theta, \frac{1}{r \sin \theta} A_\phi \right\rangle$$

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin \theta} \left\| \begin{array}{ccc} \hat{r} & r\hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin \theta A_\phi \end{array} \right\|$$

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \vec{A} \cdot (B_i \vec{C}) - \vec{A} \cdot (\vec{B} C_i)$$

$$\begin{aligned} \vec{A} \times (\vec{B} \times \vec{C}) &= \boxed{(\vec{A} \cdot (\vec{B} \otimes \vec{C})^T)^T - (\vec{A} \cdot \vec{B} \otimes \vec{C})^T} = (\vec{A} \otimes \vec{B}) \cdot \vec{C} - (\vec{A} \cdot \vec{B} \otimes \vec{C})^T \\ &= (A^T (BC^T)^T)^T - (A^T BC^T)^T = (AB^T)^T C - (A^T BC^T)^T \end{aligned}$$