

1 Curvilinear Coordinates

$$\begin{aligned}\vec{r} &= r \cos \phi \sin \theta \hat{x} + r \sin \phi \sin \theta \hat{y} + r \cos \theta \hat{z} \\ r\hat{r} &= x\hat{x} + y\hat{y} + z\hat{z}\end{aligned}$$

$$\begin{aligned}\hat{r} &= \frac{\partial}{\partial r} \vec{r} = \frac{\vec{r}}{r} = \nabla r = \frac{\nabla r}{\|\nabla r\|} \\ \hat{\theta} &= \frac{1}{r} \frac{\partial}{\partial \theta} \vec{r} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{r} = r \nabla \theta = \frac{\nabla \theta}{\|\nabla \theta\|} \\ \hat{\phi} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \vec{r} = \frac{\frac{1}{\sin \theta} \frac{\partial \vec{r}}{\partial \phi}}{r} = r \sin \theta \nabla \phi = \frac{\nabla \phi}{\|\nabla \phi\|}\end{aligned}$$

$$\begin{aligned}\cos \theta \hat{r} - \sin \theta \hat{\theta} &= \cos 2\theta \hat{z} \\ \sin \phi \hat{r} + \cos \phi \hat{\phi} &= \sin \theta \hat{y} + \sin \phi \cos \theta \hat{z} \\ \hat{\phi} &= -\sin \phi \hat{x} + \cos \phi \hat{y}\end{aligned} \Rightarrow$$

$$\begin{aligned}\hat{z} &= \frac{\cos \theta \hat{r} - \sin \theta \hat{\theta}}{\cos 2\theta} \\ \hat{y} &= \frac{\sin \phi \hat{r} + \cos \phi \hat{\phi}}{\sin \theta} - \frac{\sin \phi \cos \theta}{\sin \theta \cos 2\theta} [\cos \theta \hat{r} - \sin \theta \hat{\theta}] \\ &= -\frac{\sin \phi \sin \theta}{\cos 2\theta} \hat{r} + \frac{\sin \phi \cos \theta}{\cos 2\theta} \hat{\theta} + \frac{\cos \phi}{\sin \theta} \hat{\phi} \\ \hat{x} &= \cot \phi \hat{y} - \frac{\hat{\phi}}{\sin \phi}\end{aligned}$$

$$\begin{aligned}\frac{d\hat{r}}{dt} &= \frac{d}{dt} \left(\frac{\vec{r}}{r} \right) = \frac{1}{r} \left(\frac{d\vec{r}}{dt} - \frac{dr}{dt} \hat{r} \right) = \frac{v}{r} [\hat{v} - (\hat{r} \cdot \hat{v}) \hat{r}] \\ &= \frac{d\theta}{dt} \frac{\partial \hat{r}}{\partial \theta} + \frac{d\phi}{dt} \frac{\partial \hat{r}}{\partial \phi} = \frac{d\theta}{dt} \hat{\theta} + \sin \theta \frac{d\phi}{dt} \hat{\phi} \\ \frac{d\hat{\theta}}{dt} &= \frac{d\theta}{dt} \frac{\partial}{\partial \theta} \left(\frac{\partial \vec{r}}{\partial \theta} \right) + \frac{d\phi}{dt} \frac{\partial}{\partial \phi} \left(\frac{\partial \vec{r}}{\partial \theta} \right) = -\frac{d\theta}{dt} \hat{r} + \cos \theta \frac{d\phi}{dt} \hat{\phi} \\ \frac{d\hat{\phi}}{dt} &= \frac{d\theta}{dt} \frac{\partial \hat{\phi}}{\partial \theta} + \frac{d\phi}{dt} \frac{\partial}{\partial \phi} \left(\frac{1}{\sin \theta} \frac{\partial \vec{r}}{\partial \phi} \right) = -\frac{d\theta}{dt} \underbrace{\text{Proj}_{xy} \left(\frac{\hat{r}}{\sin \theta} \right)}_{\cos \phi \hat{x} + \sin \phi \hat{y}} \\ &= -\frac{d\phi}{dt} \frac{\hat{r} - \cos \theta \hat{z}}{\sin \theta} = \frac{d\phi}{dt} \frac{\sin \theta \hat{r} - \cos \theta \hat{\theta}}{\cos 2\theta}\end{aligned}$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt} \hat{r} + r \left(\frac{d\theta}{dt} \frac{\partial \hat{r}}{\partial \theta} + \frac{d\phi}{dt} \frac{\partial \hat{r}}{\partial \phi} \right)$$

$$\vec{v} = \frac{dr}{dt} \hat{r} + r \frac{d\theta}{dt} \hat{\theta} + r \sin \theta \frac{d\phi}{dt} \hat{\phi}$$

$$\frac{d\vec{\theta}}{dt} = \frac{d\theta}{dt} \hat{\theta} + \theta \left(-\frac{d\theta}{dt} \hat{r} + \cos \theta \frac{d\phi}{dt} \hat{\phi} \right)$$

$$\frac{d\vec{\phi}}{dt} = \frac{d\phi}{dt} \hat{\phi} + \phi \frac{d\phi}{dt} \left(\frac{\sin \theta \hat{r} - \cos \theta \hat{\theta}}{\cos 2\theta} \right)$$

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$$\frac{dr}{dt} = \frac{d}{dt} (\vec{r} \cdot \vec{r})^{\frac{1}{2}} = \hat{r} \cdot \vec{v} = v_{\parallel r}$$

$$\frac{d\theta}{dt} = \nabla \theta \cdot \vec{v} = \frac{\hat{\theta} \cdot \vec{v}}{r} = \frac{v_{\perp \theta}}{r} = \omega_{\theta}$$

$$\frac{d\phi}{dt} = \nabla \phi \cdot \vec{v} = \frac{\hat{\phi} \cdot \vec{v}}{r \sin \theta} = \frac{v_{\perp \phi}}{r \sin \theta} = \omega_{\phi}$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\begin{aligned}\boxed{m\vec{r} \times \vec{v}_{\perp}} &= \boxed{m\vec{r} \times \vec{v}} = mr^2 \left(\frac{d\theta}{dt} \hat{\phi} - \sin \theta \frac{d\phi}{dt} \hat{\theta} \right) \\ mr^2 \frac{1}{r} \hat{r} \times \vec{v} &= \boxed{I\vec{\omega}} = mr^2 \left[\frac{v}{r} (\hat{\theta} \cdot \hat{v}) \hat{\phi} - \frac{v}{r} (\hat{\phi} \cdot \hat{v}) \hat{\theta} \right] \\ I \frac{1}{r} \hat{r} \times \vec{v}_{\perp} &= I\vec{\omega} = I \frac{v}{r} \left[(\hat{\theta} \cdot \hat{v}) \hat{\phi} - (\hat{\phi} \cdot \hat{v}) \hat{\theta} \right] \\ I \frac{v_{\perp}}{r} \hat{r} \times \widehat{v_{\perp}} &= I\omega \hat{\omega} = I \frac{v}{r} (\hat{\theta} \times \hat{\phi} = \hat{r}) \times \hat{v}\end{aligned}$$

$$\begin{aligned}\bullet \vec{\omega} \times \vec{r} &= \frac{1}{r} (\hat{r} \times \vec{v}_{\perp}) \times \vec{r} \\ &= (\hat{r} \cdot \hat{r}) \vec{v}_{\perp} - (\vec{v}_{\perp} \cdot \hat{r}) \hat{r} \\ &= \vec{v}_{\perp}\end{aligned}$$

$$\begin{aligned}\bullet \vec{v} \times \vec{\omega} &= \frac{v_{\perp}}{r} (v_{\perp} \hat{v}_{\perp} + v_{\parallel} \hat{r}) \times \hat{\omega} \\ &= \frac{v_{\perp}}{r} (v_{\perp} \hat{r} - v_{\parallel} \widehat{v_{\perp}})\end{aligned}$$

$$\|\vec{v} \times \vec{\omega}\|^2 = v^2 \frac{v_{\perp}^2}{r^2}$$

$$L_i = \sum_j I_{ij} \omega_j$$

$$E = \frac{\|\vec{L}\|^2}{2I} = \frac{1}{2} \vec{L} \cdot \vec{\omega}$$

$$\overleftrightarrow{I} = \begin{bmatrix} \sum m(y^2 + z^2) & -\sum mxy & -\sum mxz \\ -\sum myx & \sum m(x^2 + z^2) & -\sum myz \\ -\sum mzx & -\sum mzy & \sum m(x^2 + y^2) \end{bmatrix}$$

$$\begin{aligned}\sum_m \frac{1}{2} m \|\vec{\omega} \times \vec{r}\|^2 &= \frac{1}{2} \sum_{ij} I_{ij} \omega_j \omega_i \\ &= \frac{1}{2} \begin{bmatrix} I & \end{bmatrix} \begin{bmatrix} \omega \\ \omega \\ \omega \end{bmatrix} \cdot \begin{bmatrix} \omega \\ \omega \\ \omega \end{bmatrix}\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}(\vec{p} \times \vec{L}) &= \frac{d\vec{p}}{dt} \times \vec{L} = f(r)\hat{r} \times (\vec{r} \times m\frac{d\vec{r}}{dt}) \\
&= mf(r) \left[\vec{r} \left(\hat{r} \cdot \frac{d\vec{r}}{dt} \right) - \frac{d\vec{r}}{dt} (\hat{r} \cdot \vec{r}) \right] \\
&= mf(r) \left[\hat{r} \frac{1}{2} \frac{d}{dt} (\vec{r} \cdot \vec{r}) - \frac{1}{r} \frac{d\vec{r}}{dt} r^2 \right] \\
&= mf(r) \left[\hat{r} r \frac{dr}{dt} - r \frac{d\vec{r}}{dt} \right] \\
&= -\frac{mf(r)r}{I(r)} \left[-\frac{I(r)}{r} \frac{dr}{dt} \vec{r} + I(r) \frac{d\vec{r}}{dt} \right] \\
&= -\frac{mf(r)r}{I(r)} \frac{d}{dt} [I(r)\vec{r}] \\
&= -m\cancel{f(r)r^2} \frac{d}{dt} \hat{r} = mk \frac{d}{dt} \hat{r} \\
\frac{d}{dt} \left(\frac{\vec{p} \times \vec{L}}{mk} - \hat{r} \right) &= \frac{d}{dt} \vec{e}_{\text{ccen}} = 0
\end{aligned}$$

$$\begin{aligned}
\vec{a} &= \left[\ddot{r} - r\dot{\theta}^2 + r\dot{\phi}^2 \frac{\sin^2 \theta}{\cos 2\theta} \right] \hat{r} \\
&+ \left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \frac{\tan 2\theta}{2} \right] \hat{\theta} \\
&+ \left[2\dot{r}\dot{\phi} \sin \theta + 2r\dot{\theta}\dot{\phi} \cos \theta + r\ddot{\phi} \sin \theta \right] \hat{\phi}
\end{aligned}$$

$$\begin{aligned}
\vec{\tau} &= \vec{r} \times \vec{F} \\
&= (mr^2) \frac{\hat{r} \times \vec{a}}{r} = I\vec{\alpha}
\end{aligned}$$

$$\tau = \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix}$$

$$=^* \begin{bmatrix} 0 & \tau_z & -\tau_y \\ -\tau_z & 0 & \tau_x \\ \tau_y & -\tau_x & 0 \end{bmatrix}$$

$$\boxed{\frac{dv}{dt} = a(\hat{v} \cdot \hat{a}) = \hat{v} \cdot \vec{a}}$$

$$\begin{aligned}
\|\vec{q} \times \vec{p}\|^2 &= \begin{vmatrix} \|\vec{q}\|^2 & \vec{q} \cdot \vec{p} \\ \vec{p} \cdot \vec{q} & \|\vec{p}\|^2 \end{vmatrix} \\
&= \vec{q} \cdot \vec{p} \times (\vec{q} \times \vec{p})
\end{aligned}$$

$$\begin{aligned}
T &= \hat{v} = \frac{\vec{v}}{v} \\
T' &= \frac{(\vec{v} \cdot \vec{v})\vec{a} - (\vec{v} \cdot \vec{a})\vec{v}}{v^3} = \frac{\vec{v} \times (\vec{a} \times \vec{v})}{v^3} \\
\|T'\| &= \frac{\sqrt{v^2 a^2 - (\vec{v} \cdot \vec{a})^2}}{v^2} = \frac{\|\vec{a} \times \vec{v}\|}{v^2} \\
\hat{B} &= \frac{\vec{a} \times \vec{v}}{\|\vec{a} \times \vec{v}\|} = \widehat{\vec{a} \times \vec{v}} \quad (\hat{B} \cdot \vec{v} = 0) \\
\hat{N} &= \frac{T'}{\|T'\|} = \frac{\vec{v} \times (\vec{a} \times \vec{v})}{v \|\vec{a} \times \vec{v}\|} = \hat{v} \times \hat{B}
\end{aligned}$$

$$\vec{a} = a_T \hat{T} + a_N \hat{N}$$

$$a_T = \vec{a} \cdot \hat{v} = \frac{dv}{dt}$$

$$a_N = \frac{\|\vec{a} \times \vec{v}\|}{v} = \|\vec{a} \times \hat{v}\|$$

$$a^2 = a_T^2 + a_N^2 = \left\| \frac{d\vec{v}}{dt} \right\|^2$$

2 Lagrangian Equations

$$\boxed{\begin{aligned}\mathcal{L} &= T - U, & p_i &\equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \\ \rightarrow F_i &\equiv \frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i}\end{aligned}}$$

Newton's Laws:

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - U(\mathbf{r}), \quad \vec{p}_r = m\dot{\mathbf{r}}$$

$$\rightarrow \boxed{F = m\ddot{\mathbf{r}} = -\nabla U}$$

Angular:

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - U(r, \phi), \quad \begin{aligned}p_r &= m\dot{r} \\ p_\phi &= mr^2\dot{\phi} = I\omega = I\frac{v_\perp}{r},\end{aligned} \quad \begin{aligned}-\vec{F} &= \nabla U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi} \\ \vec{F} &= m\ddot{\mathbf{r}} = (F \cdot \hat{r})\hat{r} + (F \cdot \hat{\phi})\hat{\phi}\end{aligned}$$

$$\begin{aligned}F_r &= \boxed{-\frac{\partial U}{\partial r} + mr\dot{\phi}^2 = m\ddot{r}} \quad (\text{centripetal: } \frac{mv^2}{r} = mr\omega^2) \\ \rightarrow F_\phi &= \boxed{\underbrace{-\frac{\partial U}{\partial \phi}}_\tau = \underbrace{mr^2\ddot{\phi}}_{I\alpha} + \underbrace{2mr\dot{r}\dot{\phi}}_{I\omega}} \quad (\text{"coriolis": } 2m|\vec{\omega} \times \vec{v}| = 2m\dot{\phi}\dot{r})\end{aligned} \quad \left| \begin{aligned}\dot{\phi}' &= \dot{\phi} - \omega \\ \downarrow \\ m\ddot{\mathbf{r}}' &= m\ddot{\mathbf{r}} - (mr\omega^2 - 2mr\dot{\phi}\omega)\hat{r} - 2m\dot{r}\omega\hat{\phi} \\ &= m\ddot{\mathbf{r}} - \underbrace{mr\omega^2\hat{r}}_{\text{centrifugal force}} + \underbrace{2m\omega(r\dot{\phi}\hat{r} - \dot{r}\hat{\phi})}_{\text{coriolis force}}\end{aligned}\right.$$

Note: $\begin{aligned}\dot{\hat{r}} &= \dot{\phi}\hat{\phi} \\ \dot{\hat{\phi}} &= -\dot{\phi}\hat{r}\end{aligned} \rightarrow \begin{aligned}\vec{r} &= r\hat{r} = r\cos\phi\hat{x} + r\sin\phi\hat{y} \\ \dot{\vec{r}} &= \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \\ \ddot{\vec{r}} &= \ddot{r}\hat{r} + 2\dot{r}\dot{\phi} + r\ddot{\phi} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}\end{aligned}$

Electromagnetic:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\dot{\mathbf{r}}^2 - q\left[V(t, \mathbf{r}) - \dot{\mathbf{r}} \cdot \vec{A}(t, \mathbf{r})\right], \quad p_x = m\dot{x} + qA_x \\ \rightarrow m\ddot{x} + q\frac{dA_x}{dt} &= -q\left[\frac{\partial V}{\partial x} - \dot{\mathbf{r}} \cdot \frac{\partial \vec{A}}{\partial x}\right] \rightarrow m\ddot{x} = q\left(-\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} + \dot{\mathbf{r}} \cdot \left[\frac{\partial \vec{A}}{\partial x} - \nabla A_x\right]\right) \\ m\ddot{x} + q\left[\frac{\partial A_x}{\partial t} + \dot{\mathbf{r}} \cdot \nabla A_x\right] &= q\left[-\frac{\partial V}{\partial x} + \dot{\mathbf{r}} \cdot \frac{\partial \vec{A}}{\partial x}\right] = q\left[-\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t}\right] + q\dot{y}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] \\ &\quad + q\dot{z}\left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right] \\ &= qE_x + q\dot{y}B_z - q\dot{z}B_y \\ m\ddot{x} &= qE_x + q\left[\dot{\mathbf{r}} \times \vec{B}\right]_x \\ \downarrow \\ \boxed{m\ddot{\mathbf{r}} &= q\left(\vec{E} + \dot{\mathbf{r}} \times \vec{B}\right)}\end{aligned}$$

Special Relativity:

$$\begin{aligned}
 \mathcal{L} &= -\frac{1}{\gamma}mc^2 - U, & \vec{p} &= \gamma m \vec{v} \rightarrow \gamma m \dot{x} = \frac{\partial \mathcal{L}}{\partial \dot{x}} \\
 &= \gamma m v^2 - \gamma m c^2 - U \\
 &= m \left(v^2 - c^2 \right) \left(1 - \frac{v^2}{c^2} \right)^{-1/2} - U \\
 &\approx \frac{1}{2} m v^2 - (U + m c^2) && \text{(when } v \ll c)
 \end{aligned}$$

Conservation of Energy:

$$\begin{aligned}
 \frac{d\mathcal{L}}{dt} &= \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right) + \frac{\partial \mathcal{L}}{\partial t} \\
 &= \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) + \frac{\partial \mathcal{L}}{\partial t} && \rightarrow \frac{\partial \mathcal{L}}{\partial t} = -\frac{d}{dt} \left(\sum_i p_i \dot{q}_i - \mathcal{L} \right) \\
 &= \frac{d}{dt} \left(\sum_i p_i \dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} && = -\frac{d\mathcal{H}}{dt} \quad \text{If } \mathcal{L} \text{ is explicitly independent of time} \\
 &&& \quad \text{(implies coordinates are "natural"),} \\
 &&& \quad \text{then the Hamiltonian is conserved.}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2} \sum_n m \dot{r}_n^2 &= \frac{1}{2} \sum_n m \left(\sum_i \frac{\partial r_n}{\partial q_i} \dot{q}_i \right)^2 && \mathcal{L} = \frac{1}{2} m v^2 - U = T_{(q_i)} - U_{(q_i)} \rightarrow \\
 &= \frac{1}{2} \sum_{i,j} \left(m \sum_n \frac{\partial r_n}{\partial q_i} \frac{\partial r_n}{\partial q_j} \right) \dot{q}_i \dot{q}_j && \mathcal{H} = \sum_i \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \\
 &= \frac{1}{2} \sum_i \sum_j A_{ij} \dot{q}_i \dot{q}_j && \rightarrow = \sum_i \left(\sum_j A_{ij} \dot{q}_j \right) \dot{q}_i - \frac{1}{2} m \dot{\mathbf{r}}^2 + U \\
 &&& = \frac{1}{2} m \dot{\mathbf{r}}^2 + U \quad \text{If } \mathcal{L} = \frac{1}{2} m v^2 - U \text{ and } U \text{ is independent} \\
 &&& \quad \text{of } v, \text{ then the Hamiltonian is the total} \\
 &&& \quad \text{energy.}
 \end{aligned}$$

(for $\frac{\partial T}{\partial q_i}$) $= \frac{1}{2} \left(2 \sum_{i \neq j} A_{ij} \dot{q}_i \dot{q}_j + A_{ii} \dot{q}_i^2 \right) + \dots$

Lagrange Multipliers:

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) &= \frac{\partial \mathcal{L}}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i} \\
 \frac{dp}{dt} &= -\nabla U + \lambda \nabla f \\
 F_{\text{tot}} &= F_{\text{ncnstr}} + F_{\text{cnstr}}
 \end{aligned}$$

2.1 Examples

Atwood's Machine (Pulley):

Particle Confined to a Cylinder Surface:

Block Sliding on Wedge:

Bead on Spinning Wire Hoop:

Oscillations of Bead Near Equilibrium:

3 Hamiltonian

$$\mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L} , \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\rightarrow \begin{aligned} \bullet \quad \frac{dp_i}{dt} &= -\frac{\partial \mathcal{H}}{\partial q_i} \\ \bullet \quad \frac{dq_i}{dt} &= \frac{\partial \mathcal{H}}{\partial p_i} \end{aligned}$$

Newton Particle:

$$\begin{aligned} \mathcal{H} &= \dot{x}(m\dot{x}) - \frac{1}{2}m\dot{x}^2 + U(x) \\ &= \frac{1}{2}m\dot{x}^2 + U(x) \\ &= T + U \end{aligned}$$

Angular:

$$\begin{aligned} \mathcal{H} &= m\dot{r}^2 + mr^2\dot{\theta}^2 - \left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r, \theta) \right) , & p_r &= m\dot{r} \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U(r, \theta) & p_\theta &= mr^2\dot{\theta} \equiv L = I\omega \end{aligned}$$

Electromagnetic:

$$\begin{aligned} \mathcal{H} &= \dot{\mathbf{r}} \cdot \vec{p}_r - \left(\frac{1}{2}m\dot{\mathbf{r}}^2 - q\phi(t, \mathbf{r}) + q\dot{\mathbf{r}} \cdot \vec{A}(t, \mathbf{r}) \right) , & \vec{p}_r &= m\dot{\mathbf{r}} + q\vec{A} \\ &= m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \vec{A} - \frac{1}{2}m\dot{\mathbf{r}}^2 + q\phi - q\dot{\mathbf{r}} \cdot \vec{A} \\ &= \frac{1}{2}m\dot{\mathbf{r}}^2 + q\phi \end{aligned}$$

Special Relativity:

$$\begin{aligned} \mathcal{H} &= \vec{v} \cdot (\gamma m \vec{v}) - (\gamma m v^2 - \gamma m c^2 - U) , & \vec{p} &= \gamma m \vec{v} \\ &= \gamma m c^2 + U \\ &\approx \frac{1}{2}m v^2 + (U + m c^2) & (\text{when } v \ll c) \end{aligned}$$

4 Hamilton-Jacobi Equations

$$\boxed{K(Q, P, t) \equiv H(q, p, t) + \frac{\partial S(q, Q, t)}{\partial t} = 0}$$

$$\dot{Q} = \frac{\partial K}{\partial P} = 0 \Rightarrow \boxed{Q = \alpha_Q = \frac{\partial S}{\partial P}} \quad (\text{constant})$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \Rightarrow \boxed{P = \alpha_P = -\frac{\partial S}{\partial Q}} \quad (\text{constant})$$

$$\dot{q} = \frac{\partial H}{\partial p} \Rightarrow \boxed{q = -\frac{\partial S}{\partial p}}, \quad \dot{p} = -\frac{\partial H}{\partial q} \Rightarrow \boxed{p = \frac{\partial S}{\partial q}}$$

$$\frac{\partial H}{\partial t} = 0 \Rightarrow \boxed{S(q, Q, t) = W(q, Q) - Et}$$

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum p_i \dot{q}_i = \frac{\partial S}{\partial t} + \cancel{H} + \mathcal{L}$$

$$\Rightarrow \boxed{S = \int \mathcal{L} dt + \text{const.}}$$

Solve for $S(q, \alpha_Q, t)$ ($n+1$ variables, nonlinear PDE)

$$H\left(q, \frac{\partial S(q, \alpha_Q, t)}{\partial q}, t\right) + \frac{\partial S(q, \alpha_Q, t)}{\partial t} = 0$$

Solve for $W(q, \alpha_Q)$ (n variables, nonlinear PDE)

$$H\left(q, \frac{\partial W}{\partial q}\right) = E \equiv \alpha_Q$$

Harmonic Oscillator

$$\begin{aligned} -\frac{\partial S}{\partial t} &= \frac{1}{2}p^2 - \frac{1}{2}\omega^2 q^2 \\ &= \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{1}{2}\omega^2 q^2 \quad (S = s_1(q) + s_2(t)) \\ -\frac{\partial s_2(t)}{\partial t} &= \frac{1}{2}\left(\frac{\partial s_1(q)}{\partial q}\right)^2 - \frac{1}{2}\omega^2 q^2 \equiv \alpha_Q \end{aligned}$$

$$s_2(t) = -\alpha_Q t + \text{const.} \quad , \quad s_1(q) = \int \sqrt{2\alpha_Q + \omega^2 q^2} dq$$

$$Q \equiv \alpha_Q \quad , \quad P = -\int \frac{dq}{\sqrt{2\alpha_Q + \omega^2 q^2}} + t$$

$$\alpha_P = -\frac{1}{\omega} \sin^{-1} \left[q \frac{\omega}{\sqrt{2\alpha_Q}} \right] + t$$

$$\boxed{q(t) = \frac{\sqrt{2\alpha_Q}}{\omega} \sin[\omega(t - \alpha_P)]}$$

5 Kinematics

Elastic Collisions: $m_0 v_0 = m_1 v_1 + m_2 v_2$
 $\frac{1}{2} m_0 v_0^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

$$\Rightarrow \boxed{\frac{1}{2} m_2 v_2^2 (m_1 + m_2) - \frac{1}{2} m_0 v_0^2 (m_1 - m_0) = (m_0 v_0)(m_2 v_2)}$$

- $$\boxed{m v_0 = m v_1 + M v_2 = m v_0 \left(1 - \frac{2M}{m+M}\right) + M v_0 \left(\frac{2m}{m+M}\right)}$$

$$\rightarrow M \in (\infty, m, 0] \Rightarrow v_1 \in (-v_0, 0, v_0]$$

Inelastic Collision: $E_0 = \frac{1}{2} m v_0^2$

- $$\boxed{m v_0 = (m + M) v_1}$$

$$\rightarrow E_1 = \left(\frac{m}{m+M}\right) E_0$$

6 Orbits

Lagrangian : $\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m r^2 \sin^2 \theta \dot{\phi}^2 - U(r)$

- $l = I\omega = m r^2 \dot{\theta}$
- $m \ddot{r} = -\frac{\partial}{\partial r} U_{\text{eff}} = -\frac{\partial}{\partial r} \left(\frac{l^2}{2 m r^2} + U(r) \right)$

$$\left. \vphantom{\begin{matrix} l = I\omega = m r^2 \dot{\theta} \\ m \ddot{r} = -\frac{\partial}{\partial r} U_{\text{eff}} = -\frac{\partial}{\partial r} \left(\frac{l^2}{2 m r^2} + U(r) \right) \end{matrix}} \right\} m \rightarrow \mu = \frac{mM}{m+M}$$

Hamiltonian: $E = \frac{p^2}{2m} + \frac{l^2}{2mr^2} + U(r)$

- Inf. Energy to get to $r = 0$ unless $l = 0$
- $U \sim 1/r$

Orbit Types:

$E > 0$: Hyperbola

$E = 0$: Parabola

$E < 0$: Ellipse

$E = \text{Min}(U_{\text{eff}})$: Circle

Kepler's Laws:

1st Law : Elliptical Orbits (Sun [at/orbiting] focus)

2nd Law : Equal Area Sweep ($r^2 d\theta = \frac{l}{m} dt$)

3rd Law : $T^2 = k^2 a^3$ T , Period
 a , Semi-major axis
 k , "constant" $\left(\frac{2\pi}{\sqrt{G[m_{\text{planet}} + M_{\text{sun}}]}} \right)$

7 Fluid Mechanics

Bernoulli's Principle : $\frac{\rho v^2}{2} + \rho g z + P_{\text{res}} = \text{constant}$ [Energy Density]

Fluid Conservation : $\rho A v = \text{constant}$ [Mass Flow Rate]

Bouyant Force : $F = \rho V g$ (ρ, V , of displaced liquid)

Water Facts:

- 1 L = 1 kg

8 Oscillators

8.1 Homogenous

$$\begin{array}{l|l}
 (F = m\ddot{x}) = -kx - \overset{\text{(damp)}}{b\dot{x}} & z_{\text{tr}}(t) = \tilde{C}e^{rt} + [\tilde{D}_{\text{opt.}} te^{rt}] : \quad \underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}} \\
 \downarrow & \\
 \boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0} & (r^2 + 2\beta r + \omega_0^2)e^{rt} = 0 \\
 & r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}
 \end{array}$$

Normal (Undamped): $(F = -kx) \Rightarrow$
 $(\ddot{x} = -\omega_0^2 x = -\frac{k}{m}x)$

$$z_{\text{tr}}(t) = \tilde{C}_1 e^{i\omega_0 t} + \tilde{C}_2 e^{-i\omega_0 t}$$

Underdamped: $(\beta < \omega_0)$

$$z_{\text{tr}}(t) = \left(\tilde{C}_1 e^{i\sqrt{\omega_0^2 - \beta^2}t} + \tilde{C}_2 e^{-i\sqrt{\omega_0^2 - \beta^2}t} \right) \underline{e^{-\beta t}}$$

Critically Damped: $(\beta = \omega_0)$

$$z_{\text{tr}}(t) = (\tilde{C}_1 + \tilde{C}_2 t) \underline{e^{-\beta t}}$$

Decay rate is maximized at $\beta = \omega_0$

Overdamped: $(\beta > \omega_0)$

$$z_{\text{tr}}(t) = \underline{\tilde{C}_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + \tilde{C}_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}}$$

(smaller, lasts longer)

8.2 Inhomogenous (Driven)

$$\begin{array}{l|l}
 m\ddot{x} = -kx - b\dot{x} + F_{\text{dr}} & \boxed{z(t) = z_{\text{st}}(t) + z_{\text{tr}}(t)} \\
 \downarrow & \\
 \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos \omega t & \boxed{z_{\text{st}}(t) = \tilde{C}e^{i\omega t} = Ae^{i(\omega t - \delta)}} : \quad \underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}} \\
 \bullet L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t) & (-\omega^2 + 2i\beta\omega + \omega_0^2)\tilde{C}e^{i\omega t} = f_0 e^{i\omega t} \\
 & \tilde{C} = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = Ae^{-i\delta} \\
 & \boxed{A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}, \quad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)}
 \end{array}$$

Resonance (Max A^2) with fixed ω : $\boxed{\omega_0 = \omega}$

Resonance (Max A^2) with fixed ω_0 : $\boxed{\omega = \sqrt{\omega_0^2 - 2\beta^2}}$ (usually $\beta \ll \omega$)

Full Width at Half Max, $A^2(\omega)$: $\text{FWHM} \approx 2\beta$

Quality Factor (Sharpness) : $Q = \frac{\omega_0}{2\beta} = \left(\pi \frac{1/\beta}{2\pi/\omega_0} = \pi \frac{\text{decay time}}{\text{period}} \right) = \left(2\pi \frac{\text{Energy stored}}{\text{Energy Dissipated}} \right)$

8.3 Parallel and Series

Series, k_1+k_2+m : $\frac{1}{K_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2}$

Parallel, k_1k_2+m : $K_{\text{eq}} = k_1 + k_2$

8.4 Normal Modes: 3 Springs + 2 Masses, $k_1+m_1+k_2+m_2+k_3$

1.) $m_1\ddot{x}_1 = -k_1x_1 - k_2x_1 + k_2x_2$

$$= -(k_1 + k_2)x_1 + k_2x_2$$

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

$$m_2\ddot{x}_2 = k_2x_1 - k_2x_2 - k_3x_2$$

$$= k_2x_1 - (k_2 + k_3)x_2$$

$$\rightarrow \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{M}\ddot{\mathbf{z}} = -\mathbf{K}\mathbf{z}$$

2.) $\mathbf{z}(t) = \mathbf{a}e^{i\omega t} = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} e^{i\omega t}$

$$-\omega^2 \mathbf{M} \mathbf{a} e^{i\omega t} = -\mathbf{K} \mathbf{a} e^{i\omega t}$$

\rightarrow

$$= \begin{pmatrix} a_1 e^{-i\delta_1 t} \\ a_2 e^{-i\delta_2 t} \end{pmatrix} e^{i\omega t}$$

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{a} = 0$$

$x(t) = \text{Re}[z(t)]$ is the real solution.

$$\boxed{\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0}$$

Same m and k

$$\begin{pmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 m \end{pmatrix} = - \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \rightarrow$$

$$\omega = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}$$

Smaller ω_1 is most symmetric motion
(both swing in phase)

Larger ω_2 swings out of phase

$$\boxed{z(t) = \tilde{A}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_1 t} + \tilde{A}_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_2 t}}$$

Weak Coupling

8.5 Single Pendulum (Use Lagrangian)

$$\begin{aligned}
 \bullet T &= \frac{1}{2}mR^2\dot{\theta}^2 \\
 \bullet U &= mg(R - R\cos\theta)
 \end{aligned}
 \rightarrow
 \begin{aligned}
 mR^2\ddot{\theta} &= -mgR\sin\theta \\
 &\approx -mgR\theta
 \end{aligned}
 \rightarrow
 \boxed{
 \begin{aligned}
 \ddot{\theta} &= -\left(\frac{g}{I/mR}\right)\theta = -\omega^2\theta \\
 \theta(t) &= \text{Re}[C_1e^{i\omega t} + C_2e^{-i\omega t}]
 \end{aligned}
 }$$

8.6 Double Pendulum (Use Lagrangian)

$$\begin{aligned}
 \bullet T &= \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(L_1\dot{\theta}_1 + L_2\dot{\theta}_2)^2 & \bullet U &= m_1g(L_1 - L_1\cos\theta_1) \\
 &= \frac{1}{2}(m_1 + m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 & &+ m_2g(L_1 + L_2 - L_2\cos\theta_2 - L_1\cos\theta_1) \\
 &\quad + m_2L_2L_1\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)
 \end{aligned}$$

$$\rightarrow \mathbf{M}\ddot{\theta} = -\mathbf{K}\theta \quad (\text{small angle quadratic approx.})$$

$$\begin{pmatrix} (m_1 + m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = - \begin{pmatrix} (m_1 + m_2)gL_1 + k_2 & 0 \\ 0 & m_2gL_2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$