Topological Space, $(X, F \subseteq 2^X)$; Subset, $(S \subseteq X, F)$:

$F = \{\text{Open Sets } V\}$	$F = \{\text{Closed Sets } U\}$
1. $\varnothing, X \in F$	1. $\varnothing, X \in F$
2. $\forall i \in [1,n],\ \bigcap V_i \in F$	2. $\forall i \in [1,n],\ \bigcup U_i \in F$
3. $\forall I,\ \bigcup_{\alpha \in I} V_\alpha \in F$	3. $\forall I,\ \bigcap_{\alpha \in I} U_\alpha \in F$
• $V = \{j_V\} = U^c$	• $U = \{a_U\} = V^c = S \cup \{c_S\}$

Topological Subspace, $(Z \subseteq X, F_Z \subseteq 2^Z)$:

$F_Z = \{Z \cap V : V \in F\}$	$F_Z = \{Y \cap V : V \in F\}$
1. $\varnothing, Y \in F_Z$	1. $\varnothing, Y \in F_Z$
2. $\forall i \in [1,n],\ \bigcap V_i \in F_Z$	2. $\forall i \in [1,n],\ \bigcup U_i \in F_Z$
3. $\forall I,\ \bigcup_{\alpha \in I} V_\alpha \in F_Z$	3. $\forall I,\ \bigcap_{\alpha \in I} U_\alpha \in F_Z$
• $V = \{j_V\} = U^c$	• $U = \{a_U\} = V^c = S \cup \{c_S\}$

Neighborhood, $V_p :\ p \in V_p \in F$

Topology = Nature of Convergence

$\nexists$ • Cauchy Seq. • Bounded Space • Complete Space • Unif. Cont/Conv, $\forall (X, F)$

Seq. Limit/Convergence, $p^{(n)} \to L :\ \forall V_L,\ \exists N_{\geq m},\ \forall n \geq N,\ p^{(n)} \in V_L$

[Seq] Limit Pt, $c_p \in X :\ \forall V_{c_p},\ \forall N_{\geq m},\ \exists n \geq N,\ p^{(n)} \in V_{c_p}$

Adherent Pt, $a_S \in X :\ \forall V_{a_S},\ V_{a_S} \cap S \neq \varnothing$

[Set] Limit Pt, $c_S \in X :\ \forall V_{c_S},\ (V_{c_S} \cap S)_{\setminus\{c_S\}} \neq \varnothing$

Interior Pt, $j_S \in X :\ \exists V_{j_S} \subseteq S$

Exterior Pt, $e_S \in X :\ \exists V_{e_S},\ V_{e_S} \cap S = \varnothing$

Boundary Pt, $b_S \in X :\ b_S \notin \{j_S\} \cup \{e_S\}$

Closure, $\overline{S} \subseteq X :\ \overline{S} = \{a_S\}$

cont f at x_0 (cont $f|_{x_0}$): $\forall V_{f(x_0)},\ \exists V_{x_0},\ f(V_{x_0}) \subseteq V_{f(x_0)}$

$$\text{cont } f :\ \forall x \in S,\ \text{cont } f|_x \Leftrightarrow f^{-1}(\forall V_1 \subseteq Y) = V_2 \subseteq X \Leftrightarrow f^{-1}(\forall U_1 \subseteq Y) = U_2 \subseteq X$$

Compact [Top.] Space, compa X: $\forall \{V_\alpha\}_{\alpha \in I_1},\ \bigcup_{\alpha \in I_1} V_\alpha = X,\ \exists I_2 \subseteq I_1,\ \|I_2\| < \infty,\ \bigcup_{\alpha \in I_2} V_\alpha = X$

Hausdorff Space, X_H: $\forall x, y \in X_H,\ \exists (V_x, V_y)(V_x \cap V_y = \varnothing)$

- $\forall (p^{(n)})^\infty,\ p^{(n)} \to L \Rightarrow$ Unique L
- $\forall (X, d_X),\ X = X_H$

0.1 Metric Spaces

Metric Space, $(X,\ d)$; Subset $(S \subseteq X, d)$ Sequences, $(a_n)_{n=m}^{\infty}$, or multidim, $(a^{(n)})_{n=m}^{\infty}$ Subsequence, $(a^{(n_j)})_{j=1}^{\infty}:\ n_j \in \mathbb{N},\ n_j < n_{j+1}$

$$d \;\left|\; \begin{array}{l} \bullet\ \forall p \in X,\ |p-p| = 0 \\ \bullet\ \forall p, q_{\neq p} \in X,\ |p-q| > 0 \\ \bullet\ \forall p, q \in X, |p-q| = |q-p| \\ \bullet\ |p-r| \leq |p-q| + |q-r| \end{array}\right.$$

$$\underset{\lim_{n\to\infty} a_n = L \in X}{\text{Limit/Conv., } a_n \to L}:\ \forall \epsilon > 0,\ \exists N_{\geq m},\ \forall n \geq N,\ |a_n - L| < \epsilon \quad \Rightarrow \quad (a_n)_{n=m}^{\infty} = (c_n)_{n=m}^{\infty}$$

$$\text{[Seq] Limit/Adherent Pt, } c_a \in X:\ \forall \epsilon > 0,\ \forall N_{\geq m},\ \exists n \geq N,\ |a^{(n)} - c_a| < \epsilon \quad \Leftrightarrow \quad \exists (a^{(n_j)})_{j=1}^{\infty},\ a^{(n_j)} \to c_a$$

$$\text{Cauchy Sequence, } (c^{(n)})_{m=1}^{\infty}:\ \forall \epsilon > 0,\ \exists N_{\geq m},\ \forall i, j \geq N,\ |c^{(i)} - c^{(j)}| < \epsilon$$

- $\exists (c^{(n_j)})_{j=1}^{\infty},\ c^{(n_j)} \to L \ \Rightarrow\ c^{(n)} \to L$

$$\begin{array}{rl}
\text{Adherent Pt, } a_S \in X:\ \forall \epsilon > 0,\ \exists p \in S,\ |p - a_S| < \epsilon & \Leftrightarrow\ \forall \epsilon > 0,\ B_{j_S}(\epsilon) \cap S \neq \varnothing \\
& \Leftrightarrow\ \exists (p^{(n)})_{n=m}^{\infty},\ p^{(n)} \to a_S \\
\text{[Set] Limit Pt, } c_S \in X:\ \forall \epsilon > 0,\ \exists p \in S_{\backslash\{c_S\}},\ |p - a_S| < \epsilon & \Leftrightarrow\ \forall \epsilon > 0,\ B_{j_S}(\epsilon) \cap S_{\backslash\{c_S\}} \neq \varnothing \\
& \Leftrightarrow\ \exists (p^{(n)})_{n=m}^{\infty},\ c_p = c_S \\
\text{Isolated Pt, } i_S \in S:\ \exists \epsilon > 0,\ \forall p \in S_{\backslash\{i_S\}},\ |p - i_S| > \epsilon & \Leftrightarrow\ \exists \epsilon > 0,\ B_{j_S}(\epsilon) \cap S = j_S \\
& \Leftrightarrow\ i_S \in \{a_S\} \backslash \{c_S\} \\
\text{Interior Pt, } j_S \in S:\ \exists \epsilon > 0,\ \forall |p - j_S| < \epsilon,\ p \in S & \Leftrightarrow\ \exists \epsilon > 0,\ B_{j_S}(\epsilon) \subseteq S \\
\text{Exterior Pt, } e_S \in X:\ \exists \epsilon > 0,\ \forall |p - j_S| < \epsilon,\ p \notin S & \Leftrightarrow\ \exists \epsilon > 0,\ B_{j_S}(\epsilon) \cap S = \varnothing
\end{array}$$

$$\text{Boundary Pt, } b_S \in X:\ b_S \notin \{j_s\} \cup \{e_S\} \ \Leftrightarrow\ \forall V_{b_S} \subseteq X,\ (S \cap V_{b_S} \neq \varnothing) \wedge (S^c \cap V_{b_S} \neq \varnothing)$$

$$\text{Closure, } \overline{S} \subseteq X:\ \overline{S} = \{a_S\} \begin{array}{l} = S \cup \{c_S\} = \\ = \{i_S\} \cup \{c_S\} = X_{\backslash\{e_S\}} \\ = \{j_S\} \cup \{b_S\} = \end{array}$$

$$\begin{array}{rl}
\text{Interior, int}(S) \subseteq S: & \{j_S\} \\
\text{Exterior, ext}(S) \subseteq X: & \{e_S\} = X \backslash \{a_S\} \\
\text{Boundary, } \partial S \subseteq X: & \partial S = \{b_S\}
\end{array}$$

$$\begin{array}{rl}
\text{Ball, } B_p(\epsilon) \subseteq X: & \forall q \in B_p(\epsilon),\ |p - q| < \epsilon \\
\text{Open Set, } V \subseteq X: & \partial V \cap V = \varnothing \quad \Leftrightarrow \quad V = \{j_V\} = U^c \\
\text{Closed Set, } U \subseteq X: & \partial U \subseteq U \quad \Leftrightarrow \quad U = \{a_U\} = V^c \\
\text{Open Set of } p,\ V_p: & p \in V_p \in \{V\}
\end{array}$$

Metric Subspace, $(S, d|_{S\times S})$: d_X restricted to subset S

[Relatively] Open Set [of a Subset], $V \subseteq S \subseteq X$: $\exists V' \subseteq X,\ V = V' \cap S$

[Relatively] Closed Set [of a Subset], $U \subseteq S \subseteq X$: $\exists U' \subseteq X,\ U = U' \cap S$

- $S \in \{U\}, \{V\} \ \Leftrightarrow\ S^c \in \{V\}, \{U\}$
- $S \in \{\varnothing, \pm\infty\} \ \Rightarrow\ (S \in \{U\}) \wedge (S \in \{V\})$

Complete Metric Space, compl (X, d): $\forall (a^{(n)})_{m=1}^{\infty}\ \big(a^{(n)} = c^{(n)} \ \overset{\leftarrow}{\Rightarrow}\ \exists L,\ a^{(n)} \to L\big)$

- compl $(Y \subseteq X, d|_{Y\times Y}) \ \Rightarrow\ \text{closed}_X\ Y$
- compl (X, d), $\text{closed}_X\ Y \ \Rightarrow\ $ compl $(Y \subseteq X, d|_{Y\times Y})$

[Axiom of] Completeness for $\mathbb{R}$: $\forall S_{\neq\varnothing} \subseteq \mathbb{R}$;

- $\exists M,\ \forall p \in S,\ p < M \ \Rightarrow\ \exists \sup(S)$
- $\exists M,\ \forall p \in S,\ M < p \ \Rightarrow\ \exists \inf(S)$

$$\Leftrightarrow \underset{(-\infty \leq \sup(S) \leq p \leq \inf(S) \leq \infty)}{\exists\big(\sup(S),\ \inf(S)\big)} \ \overset{(not shown)}{\Leftrightarrow}\ \mathbb{R} = \text{compl}$$

Nonextendable [reg Surface, S]: $\nexists \overline{S}$, reg $S \subset$ reg $\overline{S}$

- generally too weak for interesting results

Complete [Surface, S]: $\forall p \in S,\ \forall v \in T_p(S),\ \exists \exp_p(v) : T_p(S) \to S$

- compa $S \subset \mathbb{R}^3 \ \Rightarrow\ $ closed $S \subset \mathbb{R}^3 \ \Rightarrow\ $ compl $S \ \Rightarrow\ $ nonext S

0.2 Compactness

Bounded Set, bd S : $\exists B(\epsilon) \supset S$ Bounded [Metric] Space, bd (X, d) : $\exists B(\epsilon) \supset X$

Totally Bd [Metric] Space, t.bd (X, d) : $\forall \epsilon > 0,\ \exists \{x^{(i)}\}_{i=1}^{n<\infty},\ X = \bigcup_i B(x^{(i)}, \epsilon)$ • t.bd $\Rightarrow$ bd

Compact [Metric] Space, (K, d_K) : $\forall (p^{(n)})_{n=m}^{\infty},\ \exists (p^{(n_j)})_{j=1}^{\infty},\ p^{(n_j)} \to p_0 \in K$ • compa $\Leftrightarrow$ compl, t.bd

Compact Set, $K \subseteq X$: compa $(K, d|_{K\times K}) \ \Leftrightarrow\ \forall (p^{(n)})_{n=m}^{\infty},\ \exists (p^{(n_j)})_{j=1}^{\infty},\ p^{(n_j)} \to p_0 \in S$

$\Leftrightarrow\ \forall\big(\{V_\alpha\}_{\alpha\in I_1}\big)\big(\bigcup_{\alpha\in I_1} V_\alpha = K\big)\exists\big(I_2 \subseteq I_1\big)\big(\|I_2\| < \infty\big)\big(\bigcup_{\alpha\in I_2} V_\alpha = K\big)$ $\Rightarrow$ closed, bd

∗ Bolzano-Weierstrass, $(\mathbb{R}, d_{l^2})$: $(\exists M \geq 0,\ \forall n \geq m,\ |a^{(n)}| < M)$ $\Rightarrow \big(\exists (a^{(n_j)})_{j=1}^{\infty},\ a^{(n_j)} \to c_a \in \mathbb{R}\big)$

∗ Heine-Borel, $(X \subseteq \mathbb{R}^n, d_{l^{1,2,\infty}})$: compa $\Leftrightarrow$ closed, bd

∗ Heine-Borel, (X, d) : compa $\Leftrightarrow$ compl, t.bd

∗ [open] Cover of K, $\{V_i\}_K^n$: $\{V_i\}_{i=1}^n,\ \bigcup V_i = K$

- $\big(S \subseteq K\big)\big($compa $S \ \Leftrightarrow\ $ closed $S\big)$
- $|S| < \infty \ \Rightarrow$ compa S

$$\underset{(K \subseteq X, d)}{\text{(closed, bd)}} \ \underset{(L^{1,2,\infty})}{\overset{\text{(Heine-Borel)}}{\leftrightarrows}} \ \underset{\forall V \subset K,\ |V| = \infty,\ \exists c_V \in K}{\text{(compa = Bolzano-Weierstrass/Subsequence)}} \ \Leftrightarrow \ \underset{\forall \{V_i\}_K^n,\ \exists \{V_{\alpha(i)}\}_K^{m<\infty}}{\text{(compa = Heine-Borel/Subcover/Lindelöf)}}$$

0.3 Connected Sets

Disconnected Space, !conn (X, d_X) : $\exists V_{\neq\varnothing}^{1,2} \subseteq X,\ V^1 \cap V^2 = \varnothing,\ V^1 \cup V^2 = X \ \Leftrightarrow\ \exists S_{\neq\varnothing, X} \subset X,\ S =$ closed, open

Connected Space, conn (X, d_X) : $X \neq \varnothing$, !conn • "Unconnected" Set, $\varnothing$: $\varnothing \neq$ conn, !conn

Connected Set $A_{conn} \subseteq X$: conn $(A, d|_{A\times A}) \ \Leftrightarrow$

$$\forall\big(\nu_{\neq\varnothing}^{1,2} \subseteq A\big)\big(\nu^1 \cap \nu^2 = \varnothing\big)\big(\nu_{open}^{1,2}\big) \ \Rightarrow\ \nu^1 \cup \nu^2 \neq A$$

$$\big(\nu^1 \cup \nu^2 = A\big) \ \Rightarrow\ \nu_{!open}^{1,2} \ \Rightarrow\ \nu_{\neq A}^{1,2}$$

$$\forall\big(\upsilon_{\neq A}^{1,2} \subseteq A\big)\big(\upsilon^1 \cup \upsilon^2 = A\big)\big(\upsilon^1 \cap \upsilon^2 = \varnothing\big) \ \Rightarrow\ \upsilon_{!closed}^{1,2} \ \Rightarrow\ \upsilon_{\neq\varnothing}^{1,2}$$

- $B_{open\wedge closed} \subset A_{conn} \ \Rightarrow\ (B = \varnothing) \vee (B = A)$
- {conn S_i}, $\bigcap S_i \neq \varnothing \ \Rightarrow\ \bigcup S_i =$ conn
- $C_{conn} \subset A \subset R^n \ \Rightarrow\ \overline{C} =$ conn
- Unconnected Set, !conn $S \subseteq X$: !conn $(S, d|_{S\times S})$

Arcwise/Path Conn. Set, p.conn A : $\forall p_0, p_1 \in A,\ \exists \alpha \in C^0,\ \substack{p_0, p_1 \in \alpha(t) \\ \alpha(0/1) = p_{0/1}}$

- p.conn $\Rightarrow$ conn (converse not always true, like topological sine $= \{x = 0\} \wedge \{y = \sin(1/x)\}$)

Loc. Path Conn. Set, A_{loc} : $(\forall p \in A),\ (\forall V_p \subseteq A),\ \exists V_{p.conn} \subseteq V_p$

- p.conn $\Rightarrow$ l.p.conn (converse not always true, like topological $\{x = 0\} \wedge \{\forall n, x = 1/n\} \wedge \{y = 0\}$)
- A_{loc}, $\big(A_{conn} \ \Leftrightarrow\ A_{p.conn}\big)$

Connected Component of A Containing p, $\{S_{conn}\}_p^A$: $\forall$(conn S_i) $[(S_i \subset A) \wedge (p \in S_i)] \ \to\ \{S_{conn}\}_p^A = \bigcup S_i$

- $C_{c.compo}^A \subseteq A \subseteq R^n \ \Rightarrow\ C = \text{closed}_A$
- $C_{c.compo}^A \subseteq A_{loc} \subseteq R^n \ \Rightarrow\ C = \text{open}_A$

0.4 Functions, $f : X \to Y;\ (X,d_X) \to (Y,d_Y);\ S \subseteq X$

$\underline{\text{Bounded, bd } f}$: $\exists M>0,\ \forall x\in X,\ \|f(x)\|_{d_Y} < M \ \Leftrightarrow\ \exists\epsilon>0,\ \exists y_0\in Y,\ \forall x\in X,\ f(x)\in B(y_0,\epsilon)$

$\underline{\text{Limit at } x_0\in\overline{S}\ \ \lim_{x\to x_0} f(x) = L\in Y}$: $\forall\epsilon>0,\ \exists\delta>0,\ \forall \underline{x\in S},\ \Big[|x_0-x|_{d_X}<\delta \Rightarrow |L-f(x)|_{d_Y}<\epsilon \ \Leftrightarrow\ f(B_{x_0}(\delta))\subset B_{f(x_0)}(\epsilon)\Big]$

$\Leftrightarrow\ \forall(\underline{x^{(n)}\in S})^\infty,\ x^{(n)}\to x_0 \ \Rightarrow\ f(x^{(n)})\to L \ \Leftrightarrow\ \forall V_L\subseteq Y,\ \exists \underline{V_{S,x_0}\subseteq S},\ f(V_{x_0})\subseteq V_L$

$\underline{\text{Continuous at } x_0\in X}$: cont. $f|_{x_0} = \lim^{x_0}_{x\in X} f(x)\to f(x_0) = \lim_{x\to x_0} f(x)\to f(x_0)$

$\underline{\text{Continuous, cont } f}$: $\forall\epsilon>0,\ \forall x_0\in S,\ \exists\delta>0,\ \forall x\in S,\ |x_0-x|_{d_X}<\delta \ \Rightarrow\ |f(x_0)-f(x)|_{d_Y}<\epsilon$

$\Leftrightarrow\ \forall x\in S,\ \text{cont } f|_x$ $\Leftrightarrow\ \forall(x^{(n)})^\infty,\ x^{(n)}\to x_0 \ \Rightarrow\ f(x^{(n)})\to f(x_0) \ \Leftrightarrow\ \forall V^1\subseteq Y,\ f^{-1}(V^1)=V^2\subseteq X$

$\Leftrightarrow\ \forall(x^{(n)})^\infty,\ x^{(n)}\to 0 \ \Rightarrow\ f(x-x^{(n)})\to_p f(x) \ \Leftrightarrow\ \forall U^1\subseteq Y,\ f^{-1}(U^1)=U^2\subseteq X$

- $\forall K_1\subseteq X,\ f(K_1)=K_2\subseteq Y$
- $\forall S^1_{conn}\subseteq X,\ f(S^1)=S^2_{conn}\subseteq Y$
- Space cont,bd f is complete subspace of bd f
- $(K,d_K),\ (Y,d_Y),\ \big(\text{cont. } f \Leftrightarrow \text{u.c. } f\big)$
- $(K,d_K),\ f:K\to\mathbb{R} \ \Rightarrow\ \text{bd } f$
- $(K_{\neq\varnothing},d_K),\ f:K\to\mathbb{R},\ \Rightarrow\ \exists(p_1,p_2)(\forall p,\ F(p_1)\le F(p)\le F(p_2))$

$\underline{\text{Uniformly Cont., u.c } f}$: $\forall\epsilon>0,\ \exists\delta>0,\ \forall x_0\in X,\ \forall x\in X,\ |x_0-x|_{d_X}<\delta \ \Rightarrow\ |f(x_0)-f(x)|_{d_Y}<\epsilon$

$\Leftrightarrow\ \forall(x^{(n)})^\infty,\ x^{(n)}\to 0 \ \Rightarrow\ f(x-x^{(n)})\to_u f(x)$

- u.c $f \ \Rightarrow\ $ cont. f
- $(K,d_K),\ (Y,d_Y),\ \big(\text{cont. } f\Leftrightarrow \text{u.c. } f\big)$

Sequence of Functions, $(f^{(n)})^\infty_{n=1}$

$\underline{\text{Unif. Bounded, u.bd } (f^{(n)})^\infty_{m=1}}$: $\exists M>0,\ \forall n\ge 0,\ \forall x\in X,\ \|f^{(n)}(x)\|_{d_Y}<M$

$\underline{\text{[Pointwise] Convergence, } f^{(n)}\to_p f\ \ \lim_{n\to\infty} f^{(n)}(x)=f(x)}$: $\forall\epsilon>0,\ \forall x\in X,\ \exists N_{\ge1},\ \forall n>N,\ \|f^{(n)}(x)-f(x)\|_{d_Y}<\epsilon$

$\Leftrightarrow\ \forall x\in X,\ f^{(n)}(x)\to f(x)$

$\underline{\text{[Uniform] Convergence, } f^{(n)}\to_u f}$: $\forall\epsilon>0,\ \exists N_{\ge1},\ \forall x\in X,\ \forall n>N,\ \|f^{(n)}(x)-f(x)\|_{d_Y}<\epsilon$

- $f^{(n)}\to_u f \ \Rightarrow\ f^{(n)}\to_p f$
- $\forall n,$ $\begin{matrix}\text{cont } f^{(n)}|_{x_0} \ \Rightarrow\ \text{cont. } f|_{x_0} \\ \text{cont } f^{(n)} \ \Rightarrow\ \text{cont. } f\end{matrix}$
- $\forall(x^{(n)})^\infty,\ x^{(n)}\to x,\ f^{(n)}(x^{(n)})\to f(x)$
- compl $Y,$ $\begin{matrix}x\in S\\ x_0\in\overline{S}\end{matrix},$ $\lim_{n\to\infty}\lim_{x\to x_0} f^{(n)}(x) = \lim_{x\to x_0}\lim_{n\to\infty} f^{(n)}(x)$
- bd $f^{(n)}$, bd $f \ \Rightarrow\ $ u.bd $(f^{(n)})^\infty$
- bd $f^{(n)}$, bd f, $(f^{(n)}\to_u f \ \Leftrightarrow\ \lim_{n\to\infty}\|f^{(n)}-f\|_\infty = 0)$
- $\forall n,$ riem $f^{(n)}\to_u$ riem f
- $\int f = \int \underline{\lim_{n\to\infty} f^{(n)}} = \lim_{n\to\infty}\int \underline{f^{(n)}}$ $\underline{\text{(riem-int. funcs. from } [a,b])}$

  $*\ \int\sum_i^\infty f^{(i)} = \int\lim_{n\to\infty}\underline{\sum_i^n f^{(i)}} = \lim_{n\to\infty}\int\underline{\sum_i^n f^{(i)}} = \lim_{n\to\infty}\sum_i^n\int f^{(i)} = \sum_i^\infty\int f^{(i)}$
- cont f_i, $\Sigma_i^n\|f_i\|_\infty\to_u L \ \Rightarrow\ \Sigma_i^n f_i\to_u f$ $\underline{\text{(Weierstrass M test)}}$
- $\begin{matrix}\cancel{\text{cont } f_n},\ \frac{d}{dx}f_n\to_u h,\\ f_n(x_0)\to_p L\end{matrix}$ $\Rightarrow\ \frac{d}{dx}(\underline{\exists g = \overset{\text{unif}}{\lim_{n\to\infty}} f_n}) = \lim_{n\to\infty}\frac{d}{dx}f_n = h$

  $*$ $\begin{matrix}\text{cont } f'_i,\ \Sigma_i^n\|f'_i\|_\infty\to_u L,\\ \Sigma_i^n f_i(x_0)\to_p g\end{matrix}$ $\Rightarrow\ \frac{d}{dx}\lim_{n\to\infty}\sum_i^n f_i = \lim_{n\to\infty}\sum_i^n\frac{d}{dx}f_i$

Note the improper usage of C^k, which really is the set of all continuous $k-$differentiable functions

- C^0 at $c \nrightarrow C^0$ over $B_\delta(c)$ $\big(f(x)=[x\in\mathbb{Q}]x^2+[x\notin\mathbb{Q}]0\big)$
- C^0 at $c\nrightarrow C^1$ at c (Weierstrass)
- C^1 at $c\to C^0$ at c
- C^1 at $c\nrightarrow C^0$ over $B_\delta(c)$ $\big(f(x)=[x\in\mathbb{Q}]x^2+[x\notin\mathbb{Q}]0\big)$
- C^1 at $c\nrightarrow C^1$ over $B_\delta(c)$ $\begin{pmatrix}f(x)=[x\in\mathbb{Q}]x^2+[x\notin\mathbb{Q}]0 \ \Leftarrow\ \text{only derivative at } 0\\ g(x)=x^2\sin\frac{1}{x} \ \Leftarrow\ \text{derivative everywhere (use squeeze) but not cont. derivative}\end{pmatrix}$
- C^∞ at $c\to C^\infty$ over $B_\delta(c)$ $\big(\text{analytic} = C^\omega\subset\text{smooth} = C^\infty\big)$

$\underline{\text{A Monotone and not Piecewise-Cont. } f}$: $f(x)=\sum_{r\in\mathbb{Q}:r<x} g(r) \ \Leftarrow$

- $q(r)\in\mathbb{N},\ g(r)=g\circ q(r)=2^{-n}$

  $*\ f(x\in\mathbb{Q})\neq\text{cont.}$ $\quad *\ f(x\notin\mathbb{Q})=\text{cont.}$

Function Spaces

$Y^X = \{f \mid f : X \to Y\}$ | $(V_p \subseteq Y)^{(p \in X)} \subseteq Y^X$

$(Y^X, F) =$ Top Space | open $S \subseteq Y^X$: $\forall f \in S,\ \exists n < \infty,\ f \in \bigcap^{n} (V_{x_n})^{(x_n)} \subseteq S$

- Symmetric : $x \sim y,\ y \sim x$ • Anti-Symmetric : $x \lesssim y,\ y \lesssim x \ \Rightarrow\ x = y$

Tuple/Function : $f(x) \Leftrightarrow (f_x)_{x \in D} = (f_a, f_b, f_c, ...)$, $(x_\alpha)_{\alpha \in I} = (x_a, x_b, x_c, ...)$

Axiom 6 (Replacement) : $A, \exists\{y : x \in A, P(x,y)\}$, $A = \{3,5,8\} \Rightarrow \exists\{4,6,10\}$ $\Rightarrow \exists\{1\}$

Axiom 8 (Regularity) : $A \neq \varnothing \ \Rightarrow\ \exists(x \in A)\big(x \neq \{...\}$ or $x \cup A = \varnothing\big)$

Axiom 9 (Power Set) : $\exists Y^X = {}_{\text{Poss. Range}}{}^{\text{Domain}}$, $(_{\text{repl.}}) \Rightarrow$

$$\{0,1\}^{(a,b,c)} = \{\ (0,0,0), \underline{(0,0,1) \times 3}, \underline{(0,1,1) \times 3}, (1,1,1)\ \}$$
$$\Rightarrow \{\ f^{-1}(0,0,0), \underline{f^{-1}(0,0,1) \times 3}, \underline{f^{-1}(0,1,1) \times 3}, f^{-1}(1,1,1)\ \}$$
$$= \{\ \varnothing, \underline{a, b, c}, \underline{\{a,b\}, \{b,c\}, \{c,a\}}, \{a,b,c\}\ \}$$

Axiom 10 (Union) :

- $x \in \bigcup A \ \Leftrightarrow\ x \in S \in A$
- $*\ \bigcup_{\alpha \in I} A_\alpha \equiv \bigcup\{A_\alpha : \alpha \in I\}$, $A = \{\{1,2\},\{2,3,4\}\} \ \to\ \bigcup A = \{1,2,3,4\}$

Axiom 11 (Choice) : $\forall \alpha \in I,\ \underset{(\exists x_\alpha \in X_\alpha)}{X_\alpha \neq \varnothing} \ \Rightarrow\ \exists(x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha \neq \varnothing$

- Partially Ordered Set : $X,\ \leq_X$ (Reflexive, Anti-Symmatric, Transitive)
- Totally Ordered Set : $(\forall x, y \in X)(x \leq y$ or $y \leq x)$
- Minimal/Max Element : $y_{min}/y_{max} \in Y \subseteq X_{par} \ \Rightarrow\ \nexists y \in Y,\ (y < y_{min})/(y_{max} < y)$
- Well-Ordered Set : $(Y_{well} = Y_{tot}) \subseteq X_{par} \ \Rightarrow\ (\forall Z \neq \varnothing) \subseteq Y_{well},\ \exists z_{min}$
- Upp. Bnd/Strict Upp. Bnd : $(x_{upp}/x_{s.upp} \in X_{par}),\ (x_{s.upp} \notin Y \subseteq X_{par}) \ \Rightarrow\ \forall y \in Y,\ (y \leq x_{upp})/(y < x_{s.upp})$

- Complete Metric Space : $\forall\{x_n\}_{\text{Cauchy}},\ \exists L = \{x_n\}^\infty$ $*\ (\mathbb{Q}, d{=}|x-y|)$ is not complete
- Open Set of Functions, $E \subseteq Y^X$: $\exists n < \infty, \{x_n \in X\}, \{\text{open } V_n \subseteq Y\},\ \forall f \in E,\ f(x_i) \in V_i$
- All open/closed sets are measurable
- Measureable Func., $f : \Omega \subseteq R^n \to R^m$: $\forall V_{open} \subseteq R^m$, Set (not func.) $f^{-1}(V)$ is measurable
- Measureable Func., $f : \Omega \subseteq R^n \to R^*$: $\forall a \in R$, Set (not func.) $f^{-1}((a,\infty]) = \{\omega : f(\omega) > a\}$ is measurable

$*\ g = \sup f_n \ \Rightarrow\ \bigcup f_n^{-1}((a,\infty]) = (\max f_n)^{-1}((a,\infty]) = g^{-1}((a,\infty])$ $*\ h = \inf f_n \ \Rightarrow\ \bigcap f_n^{-1}((a,\infty]) = h^{-1}((a,\infty])$

$*\ f = \limsup f_n = \inf_{N \geq 1} \sup_{n \geq N} f_n \ \Rightarrow\ \bigcap_{N \geq 1} \bigcup_{n \geq N} f_n^{-1}((a,\infty])$ $*\ f = \liminf f_n \ldots$ • $f = \lim f_n$ is meas.

Unital Magma, M

- Closure : $A \star B \in M$
- Identity : $A \star e = A$

Semigroup, S

- Closure : $A \star B \in M$
- Associative

Monoid, M

- Semigroup
- Identity : $A \star e = A$

Group, G

- Monoid
- Inverse : $A \star A^{-1} = e$

Rng, R

- $+$ Abelian Group
- $\cdot$ Semigroup

$\Big\}$ • Distributive : $A \cdot (B + C) \cdot D$

Ring, R

- $+$ Abelian Group
- $\cdot$ Monoid

$\Big\}$ • Distributive : $A \cdot (B + C) \cdot D$

Field, F

- $+$ Abelian Group
- $\cdot$ Abelian Group$/\{0^{-1}\}$

$\Big\}$ • Distributive

- $1 \neq 0$

Vector Space, V, over Field, F

- $+$ Abelian Group, V
- $\cdot$ Associative : $f \cdot (g \cdot \vec{v}) = (f \cdot g) \cdot \vec{v}$
- Identity : $1 \cdot \vec{v} = \vec{v}$
- $\cdot$ Distributive over $+V$: $f \cdot (\vec{v} + \vec{w})$
- $\cdot$ Distributive over $+F$: $(f + g) \cdot \vec{v}$

Spaces, V with Product $(\ ,\ )$

- $(\ ,\ )$: $V \times V \to K$
- Linear : $(u+v, w+x) = (u,w) + (u,x) + \ldots$
- Bi/Sesqulinear : $(a\vec{u}, b\vec{v}) = a\bar{b}(u,v)$

$*$ Scalar/Hermitian/Inner Product

Normed Vector Space, V with $\|\cdot\|$

Algebra, A, over Field, F

- Vector Space
- $\times$ Closed : $A \times A \to A$

$*$ $\times$ Assoc: Assoc. Alg.
$*$ $\times$ w Identity: Unital Alg.
$*$ $\times$ Commut: Commut. Alg.
$*$ $[\ ,\ ]$ Jacobi Iden.: Lie Alg.

Module, M, over Ring, R

Weierstrass Theorem

1.) $\exists P_n'|^{\text{comp. supp}}_{[-1,1]} \to f|^{\text{cont. comp. supp}}_{[0,1]}$

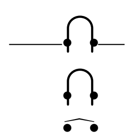
- $\delta * f = f * \delta = f$
- $P_n|^{\text{c.s}}_{[-1,1]} * f|^{\text{c.c.s}}_{[0,1]} = P_n'|^{\text{c.s}}_{[0,1]}$
- $P_n|^{\text{c.s}}_{[-1,1]} = A_n(1-x^2)^n|^{\text{c.s}}_{[-1,1]} \to \delta(x)$
- $P_n|^{\text{c.s}}_{[-1,1]} * f|^{\text{c.c.s}}_{[0,1]} = P_n'|^{\text{c.s}}_{[0,1]} \to f|^{\text{c.c.s}}_{[0,1]}$

2.) $\exists P_n|_{[0,1]} \to f|^{\text{cont}}_{[0,1]}$

$f|^{\text{c.c.s}}_{[0,1]}$:

$f|^{\text{cont}}_{0,1} = 0$:

$f|^{\text{cont}}_{[0,1]}$:



(Polynomial Shift, Q_m, in y)

- $f|^{\text{cont}}_{[0,1]} + Q_m = F$: $F|^{\text{cont}}_{0,1} = 0$
- $|P_n - F| = |(P_n - Q_m) - f| < \epsilon$
- $Q_m(x) = -\big[f(0) + x(f(1) - f(0))\big]$

3.) $\exists P_n|_{[a,b]} \to f|^{\text{cont}}_{[a,b]}$

(Polynomial Shift, Q_m, in x)

- $Q_m(x|_{[a,b]}) = X|_{[0,1]} \ \Leftrightarrow\ Q_m^{-1}(X) = x$

$*$ $f \circ Q^{-1}(X) = F(X)$

- $|P_n(X) - F(X)| = |P_n \circ Q_m(x) - f(x)| < \epsilon$
- $Q_m(x) = \frac{x-a}{b-a}$

$\big(S \subset \mathbb{R}^+\big),\ \big(\alpha'(0) = \vec{v}(0) \in T_{\vec{p}}(S) = \mathbb{R}\big)$

$\alpha_{\vec{p}}(t, \vec{v}(0)) = \vec{p}(0) + \int \alpha'_p \, dt = p + |v|\hat{v}t \in \mathbb{R}^+ \quad , \quad t \in (-2, 2),\ v \in (-\epsilon, \epsilon)$

$$\sqrt{\langle \vec{v}, \vec{v} \rangle_{\vec{p}}} \begin{array}{l} \equiv \sqrt{v \cdot v} = |v| \\ \equiv \frac{\sqrt{v \cdot v}}{\vec{p}} = \frac{|v|}{\alpha_p(t,v)} \end{array} \Rightarrow\ s_p(t) = \int_0^t \|\alpha'_p\| \, dt' \begin{array}{l} = \int_0^t |v| \, dt' = |v|t \ \Rightarrow\ s_p(1) = |v| \\ = \int_0^t \frac{|v|}{p + |v|\hat{v}t'} \, dt' = \frac{|v|}{v} \ln\left(\frac{p + |v|\hat{v}t}{p + 0}\right) \ \Rightarrow\ s_p\left(\tau = \frac{pe^v - p}{v}\right) = |v| \end{array}$$

$$\alpha_p(t, v) = \alpha_p \circ \left(s_p^{-1}(s_p), v\right) \equiv \alpha_p(s_p, v) \begin{array}{l} = p + s\hat{v} \\ = pe^{s_p v/|v|} \end{array} \ \to\ \boxed{\begin{array}{l} |v| < \epsilon : \ a_1(s_1(1), v) = 1 + v \\ |v| < \frac{\epsilon^2}{e^\epsilon - 1} : \ a_1(s_1(\tau), v) = e^v \end{array} = \exp_1(v)}$$

$$t \in (-\epsilon, \epsilon) : \ \alpha(t) \ \to \ \begin{array}{c} \alpha(0) \\ \alpha'(0) \end{array}$$