

1 An Easy First Order Differential (& Transport Equation)

$$A(x) u_x + \alpha(x) B(y) u_y = f(u)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\alpha(x)B(y)}{A(x)} \Rightarrow \int \frac{dy}{B(y)} = \int \frac{\alpha(x)}{A(x)} dx \Rightarrow Y(y) = X(x) + C$$

$$\begin{aligned} C &= Y(y) - X(x) \\ \epsilon &= x \end{aligned}$$

$$u_x = u_C \frac{dC}{dx} + u_\epsilon \frac{d\epsilon}{dx} = \frac{-\alpha(x)}{A(x)} u_C + u_\epsilon$$

$$u_y = u_C \frac{dC}{dy} + u_\epsilon \frac{d\epsilon}{dy} = \frac{1}{B(y)} u_C$$

$$\begin{aligned} f(u) &= A(x)u_x + \alpha(x)B(y)u_y \\ &= A(x) \left[\frac{-\alpha(x)}{A(x)} u_C + u_\epsilon \right] + \frac{\alpha(x)B(y)}{B(y)} u_C \\ &= A(\epsilon) \frac{\partial u}{\partial \epsilon} \end{aligned}$$

$$f(u) \neq 0 : \int \frac{du}{f(u)} = \int \frac{d\epsilon}{A(\epsilon)}$$

$$f(u) = 0 : \frac{\partial u}{\partial \epsilon} = 0$$

$$\gamma(u) = h(\epsilon) + g(C)$$

$$u(x, y) = g(C) = g[Y(y) - X(x)]$$

$$u(x, y) = \gamma^{-1} \{ h(x) + g[Y(y) - X(x)] \}$$

2 Quadratic-esque Equations (pre-Wave Equation)

$$\begin{aligned}0 &= \alpha u_{xx} + \beta u_{xt} + \gamma u_{tt} \\&= [a\partial_x + b\partial_t][c\partial_x + d\partial_t]u\end{aligned}$$

$$0 = [a\partial_x + b\partial_t]v = [c\partial_x + d\partial_t]w$$

$$u = v\left(x - \frac{a}{b}t\right) + w\left(x - \frac{c}{d}t\right)$$

3 Error function/Heat Kernel

$$\int_a^b e^{-p^2} dp = \frac{\sqrt{\pi}}{2} [\operatorname{erf}(b) - \operatorname{erf}(a)]$$

$$\bullet \operatorname{erf}(\infty) = 1 \quad | \quad \operatorname{erf}(-\infty) = -1 \quad | \quad \operatorname{erf}(0) = 0$$

$$\bullet \int_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi} \quad | \quad \int_{-\infty}^0 e^{-p^2} dp = \frac{\sqrt{\pi}}{2}$$

$$S_k(x - y, t) = \frac{1}{\sqrt{4kt\pi}} e^{-\frac{(x-y)^2}{4kt}}$$

4 Heat/Diffusion Equation

$$u_t - ku_{xx} = f(x, t)$$

4.1 $x \in (-\infty, \infty)$

$$1. u(x, 0) = \phi(x)$$

$$u = \int_{-\infty}^{\infty} S(x-y)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s)f(y, s)dyds$$

$$(a) f(x, t) = 0 : u = \int_{-\infty}^{\infty} S(x-y)\phi(y)dy$$

4.2 $x \in (0, \infty)$

$$1. u(x, 0) = \phi_u(x)$$

$$(a) \text{ Dirichlet: } u(0, t) = h(t)$$

$$\phi(x) = \begin{cases} \phi_u(x) & x > 0 \\ -\phi_u(-x) & x < 0 \end{cases}$$

$$v(x, t) = u - h(t)$$

- $v(x, 0) = \phi_v = \phi_u(x) - h(0)$
- $v(0, t) = 0$
- $v_t - kv_{xx} = f(x, t) - h'(t)$

$$u = h(t) + v(x, t)$$

$$i. u(0, t) = h(t) = 0 :$$

$$\begin{aligned} u &= \int_{-\infty}^{\infty} S(x-y)\phi(y)dy + \int_{\Delta} S(x-y, t-s)f(y, s)d\Delta \\ &= \int_0^{\infty} S(x-y)\phi_u(y)dy - \int_{-\infty}^0 S(x-y)\phi_u(-y)dy \\ &\quad + \int_{\Delta} S(x-y, t-s)f(y, s)d\Delta \\ &= \int_0^{\infty} [S(x-y) - S(x+y)]\phi_u(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s)f(y, s)dyds \end{aligned}$$

(b) **Neumann:** $u_x(0, t) = h(t)$

$$\phi(x) = \begin{cases} \phi_u(x) & x > 0 \\ \phi_u(-x) & x < 0 \end{cases}$$

$$v(x, t) = u - xh(t)$$

- $v(x, 0) = \phi_v = \phi_u - xh(0)$
- $v(0, t) = 0$
- $v_t - kv_{xx} = f(x, t) - xh'(t)$

$$u = xh(t) + v(x, t)$$

i. $u_x(0, t) = h(t) = 0$:

$$\begin{aligned} u &= \int_{-\infty}^{\infty} S(x-y)\phi(y)dy + \int_{\Delta} S(x-y, t-s)f(y, s)d\Delta \\ &= \int_0^{\infty} S(x-y)\phi_u(y)dy + \int_{-\infty}^0 S(x-y)\phi_u(-y)dy \\ &\quad + \int_{\Delta} S(x-y, t-s)f(y, s)d\Delta \\ &= \int_0^{\infty} [S(x-y) + S(x+y)]\phi_u(y)dy + \int_{\Delta} S(x-y, t-s)f(y, s)d\Delta \end{aligned}$$

4.3 $x \in (0, L)$

1. $u(x, 0) = \phi(x), \quad f(x, t) = 0$

$$u = \sum X(x) T(t)$$

$$\bullet \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

(a) $\lambda > 0$:

$$\begin{aligned} - X &= A_x \cos(\sqrt{\lambda}x) + B_x \sin(\sqrt{\lambda}x) \\ - T &= A_t e^{-\lambda kt} \end{aligned}$$

(b) $\lambda = 0$:

$$\begin{aligned} - X &= A_x x + B_x \\ - T &= A_t \end{aligned}$$

(c) $\lambda < 0$:

$$- X = A_x \cosh(\sqrt{\lambda}x) + B_x \sinh(\sqrt{\lambda}x)$$

$$-T = A_t e^{\lambda k t}$$

(a) **Dirichlet:** $u(0, t) = u(L, t) = 0$

$$\sqrt{\lambda} = \frac{n\pi}{L}, \quad n \in \mathbb{N}$$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

$$u = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 k t} \sin\left(\frac{n\pi}{L}x\right)$$

(b) **Neumann:** $u_x(0, t) = u_x(L, t) = 0$

$$\sqrt{\lambda} = \frac{n\pi}{L}, \quad n \in \mathbb{N}_0$$

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 k t} \cos\left(\frac{n\pi}{L}x\right)$$

(c) **Robin:** $u_x(0, t) - a_0 u(0, t) = u_x(L, t) + a_L u(L, t) = 0$

5 Wave Equation

$$u_{tt} - c^2 u_{xx} = f(x, t)$$

5.1 $x \in (-\infty, \infty)$

$$1. \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$$

$$u = \frac{1}{2}[\phi(x - ct) + \phi(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy ds$$

$$\bullet \quad f(x, t) = 0 : \quad u = \frac{1}{2}[\phi(x - ct) + \phi(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy$$

5.2 $x \in (0, \infty)$

$$1. \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$$

$$(a) \quad \text{Dirichlet: } u(0, t) = h(t)$$

$$x \in (ct, \infty) :$$

$$u = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy ds$$

$$x \in (0, ct) :$$

$$^* u = \frac{1}{2}[\phi(ct+x) - \phi(ct-x)] + \frac{1}{2c} \int_{ct-x}^{ct+x} \psi(y) dy + \frac{1}{2c} \int_0^t \int_{c(t-s)-x}^{c(t-s)+x} f(y, s) dy ds + h(t - \frac{x}{c})$$

$$\bullet \quad f(x, t) = h(t) = 0 : \quad u = \frac{1}{2}[\phi(x - ct) + \phi(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy$$

5.3 $x \in (0, L)$

$$1. \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \quad f(x, t) = 0$$

$$u = \sum X(x) T(t)$$

$$\bullet \quad \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$(a) \quad \lambda > 0 :$$

$$\begin{aligned} - X &= A_x \cos(\sqrt{\lambda}x) + B_x \sin(\sqrt{\lambda}x) \\ - T &= A_t \cos(\sqrt{\lambda}ct) + B_t \sin(\sqrt{\lambda}ct) \end{aligned}$$

(b) $\lambda = 0$:

$$\begin{aligned} - X &= A_x x + B_x \\ - T &= A_t t + B_t \end{aligned}$$

(c) $\lambda < 0$:

$$\begin{aligned} - X &= A_x \cosh(\sqrt{\lambda}x) + B_x \sinh(\sqrt{\lambda}x) \\ - T &= A_t \cosh(\sqrt{\lambda}ct) + B_t \sinh(\sqrt{\lambda}ct) \end{aligned}$$

(a) **Direchlet:** $u(\theta, t) = u(L, t) = 0$

$$\sqrt{\lambda} = \frac{n\pi}{L}, \quad n \in \mathbb{N}_0$$

$$\phi(x) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \quad \psi(x) = \sum_{n=0}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$u = \sum_{n=0}^{\infty} \left(A_n \cos\left(\frac{n\pi}{L}ct\right) + B_n \sin\left(\frac{n\pi}{L}ct\right) \right) \sin\left(\frac{n\pi}{L}x\right)$$

(b) **Neumann:** $u_x(\theta, t) = u_x(L, t) = 0$

$$\sqrt{\lambda} = \frac{n\pi}{L}, \quad n \in \mathbb{N}_0$$

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right), \quad \psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \cos\left(\frac{n\pi}{L}x\right)$$

$$u = \frac{1}{2}A_0 + \frac{1}{2}B_0t + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right) \cos\left(\frac{n\pi}{L}x\right)$$

6 Fourier Series

6.1 Sine

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} S_n \sin\left(\frac{g(n)}{L}\pi x\right) \\ \int_0^L f(x) \sin\left(\frac{h(m)}{L}\pi x\right) dx &= \sum_{n=0}^{\infty} \int_0^L S_n \sin\left(\frac{g(n)}{L}\pi x\right) \sin\left(\frac{h(m)}{L}\pi x\right) dx \\ &= S_{h(m)} \frac{L}{2} \\ S_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{g(n)}{L}\pi x\right) dx\end{aligned}$$

6.2 Cosine

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} S_n \cos\left(\frac{g(n)}{L}\pi x\right) \\ \int_0^L f(x) \cos\left(\frac{h(m)}{L}\pi x\right) dx &= \sum_{n=0}^{\infty} \int_0^L S_n \cos\left(\frac{g(n)}{L}\pi x\right) \cos\left(\frac{h(m)}{L}\pi x\right) dx \\ &= S_{h(m)} \frac{L}{2} \\ S_n &= \frac{2}{L} \int_0^L f(x) \cos\left(\frac{g(n)}{L}\pi x\right) dx\end{aligned}$$

6.3 Full

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{g(n)}{L}\pi x\right) + B_n \sin\left(\frac{g(n)}{L}\pi x\right) \\ A_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{g(n)}{L}\pi x\right) dx \\ B_n &= \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{g(n)}{L}\pi x\right) dx\end{aligned}$$