# 1 Lagrangian

$$\boxed{\begin{aligned}&\mathcal{L} = T - U \ , \qquad p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \\ &\to \ F_i \equiv \frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i}\end{aligned}}$$

Newton's Laws:

$$\mathcal{L} = \tfrac{1}{2}m\dot{\mathbf{r}}^2 - U(\mathbf{r}) \ , \qquad \vec{p}_r = m\dot{\mathbf{r}}$$

$$\to \ \boxed{F = m\ddot{\mathbf{r}} = -\nabla U}$$

Angular:

$$\mathcal{L} = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - U(r,\theta) \ , \qquad \begin{aligned} p_\theta &= mr^2\dot{\theta} \ \equiv \ L = I\omega \\ p_r &= m\dot{r} \end{aligned}$$

$$\to \quad \begin{aligned} F_r &\equiv \boxed{-\frac{\partial U}{\partial r} = m\ddot{r} - mr\dot{\theta}^2} \qquad \text{(centripital: } \tfrac{mv^2}{r} = mr\omega^2\text{)} \\ rF_\theta &\equiv \boxed{-\frac{\partial U}{\partial \theta} = mr^2\ddot{\theta}} = I\alpha = \tau \end{aligned}$$

$$-\vec{F} = \nabla U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta}$$

Electromagnetic:

$$\mathcal{L} = \tfrac{1}{2}m\dot{\mathbf{r}}^2 - q\Big(V(t,\ \mathbf{r}) - \dot{\mathbf{r}} \cdot \vec{A}(t,\ \mathbf{r})\Big) \ , \qquad p_x = m\dot{x} + qA_x$$

$$\to \qquad m\ddot{x} + q\frac{dA_x}{dt} = -q\left(\frac{\partial V}{\partial x} - \dot{r} \cdot \frac{\partial \vec{A}}{\partial x}\right)$$

$$m\ddot{x} + q\left(\frac{\partial A_x}{\partial t} + \dot{r} \cdot \nabla A_x\right) = q\left(-\frac{\partial V}{\partial x} + \dot{r} \cdot \frac{\partial \vec{A}}{\partial x}\right)$$

$$\downarrow$$

$$\begin{aligned} m\ddot{x} &= q\left[-\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} + \dot{r} \cdot \left(\frac{\partial \vec{A}}{\partial x} - \nabla A_x\right)\right] \\ &= q\left(-\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t}\right) + q\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + q\dot{z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) \\ &= qE_x + q\dot{y}B_z - q\dot{z}B_y \\ m\ddot{x} &= qE_x + q\left(\dot{\mathbf{r}} \times \vec{B}\right)_x \end{aligned}$$

$$\downarrow$$

$$\boxed{m\ddot{\mathbf{r}} = q\left(\vec{E} + \dot{\mathbf{r}} \times \vec{B}\right)}$$

Special Relativity:

$$\mathcal{L} = -\tfrac{1}{\gamma}mc^2 - U \ , \qquad\qquad \vec{p} = \gamma m\vec{v} \ \rightarrow\ \gamma m\dot{x} = \frac{\partial\mathcal{L}}{\partial\dot{x}}$$

$$= \gamma mv^2 - \gamma mc^2 - U$$

$$= m\left(v^2 - c^2\right)\left(1 - \tfrac{v^2}{c^2}\right)^{-1/2} - U$$

$$\approx \tfrac{1}{2}mv^2 - (U + mc^2) \qquad \text{(when } v \ll c\text{)}$$

Conservation of Energy:

$$\frac{d\mathcal{L}}{dt} = \sum_i \left(\frac{\partial\mathcal{L}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{d\dot{q}_i}{dt}\right) + \frac{\partial\mathcal{L}}{\partial t}$$

$$= \sum_i \left(\dot{p}_i\dot{q}_i + p_i\ddot{q}_i\right) + \frac{\partial\mathcal{L}}{\partial t} \qquad \rightarrow \qquad \frac{\partial\mathcal{L}}{\partial t} = -\frac{d}{dt}\left(\sum_i p_i\dot{q}_i - \mathcal{L}\right)$$

$$= \frac{d}{dt}\left(\sum_i p_i\dot{q}_i\right) + \frac{\partial\mathcal{L}}{\partial t} \qquad\qquad = -\frac{d\mathcal{H}}{dt}$$

If $\mathcal{L}$ is explicitly independent of time (implies coordinates are "natural"), then the Hamiltonian is conserved.

$$\tfrac{1}{2}\sum_n m\dot{r}_n^2 = \tfrac{1}{2}\sum_n m\left(\sum_i \frac{\partial r_n}{\partial q_i}\dot{q}_i\right)^2$$

$$= \tfrac{1}{2}\sum_{i,j}\left(m\sum_n \frac{\partial r_n}{\partial q_i}\frac{\partial r_n}{\partial q_j}\right)\dot{q}_i\dot{q}_j$$

$$= \tfrac{1}{2}\sum_i\sum_j A_{ij}\dot{q}_i\dot{q}_j$$

$$\left(\text{for } \tfrac{\partial T}{\partial\dot{q}_i}\right) \qquad = \tfrac{1}{2}\left(2\sum_{i\neq j} A_{ij}\dot{q}_i\dot{q}_j + A_{ii}\dot{q}_i^2\right) + \ldots$$

$$\rightarrow$$

$$\mathcal{L} = \tfrac{1}{2}mv^2 - U \ = \ T(\dot{q}_i) - U(q_i) \ \rightarrow$$

$$\mathcal{H} = \sum_i \frac{\partial T}{\partial\dot{q}_i}\dot{q}_i - \mathcal{L}$$

$$= \sum_i\left(\sum_j A_{ij}\dot{q}_j\right)\dot{q}_i - \tfrac{1}{2}m\dot{\mathbf{r}}^2 + U$$

$$= \tfrac{1}{2}m\dot{\mathbf{r}}^2 + U$$

If $\mathcal{L} = \frac{1}{2}mv^2 - U$ and $U$ is independent of $v$, then the Hamiltonian is the total energy.

Lagrange Multipliers:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) = \frac{\partial\mathcal{L}}{\partial q_i} + \lambda\frac{\partial f}{\partial q_i}$$

$$\frac{dp}{dt} = -\nabla U + \lambda\nabla f$$

$$F_{\text{tot}} = F_{\text{ncnstr}} + F_{\text{cnstr}}$$

## 1.1 Examples

Atwood's Machine (Pulley):

Particle Confined to a Cylinder Surface:

Block Sliding on Wedge:

Bead on Spinning Wire Hoop:

Oscillations of Bead Near Equilibriuum:

## 2 Hamiltonian

$$\boxed{\begin{array}{l} \mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L} \ , \qquad p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}} \\ \rightarrow \begin{array}{l} \bullet \ \dfrac{dp_i}{dt} = -\dfrac{\partial \mathcal{H}}{\partial q_i} \\ \bullet \ \dfrac{dq_i}{dt} = \dfrac{\partial \mathcal{H}}{\partial p_i} \end{array} \end{array}}$$

Newton Particle:

$$\begin{aligned} \mathcal{H} &= \dot{x}(m\dot{x}) - \tfrac{1}{2}m\dot{x}^2 + U(x) \\ &= \tfrac{1}{2}m\dot{x}^2 + U(x) \\ &= T + U \end{aligned}$$

Angular:

$$\begin{aligned} \mathcal{H} &= m\dot{r}^2 + mr^2\dot{\theta}^2 - \left(\tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - U(r,\theta)\right) \ , \\ &= \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + U(r,\theta) \end{aligned}$$

$$\begin{aligned} p_\theta &= mr^2\dot{\theta} \ \equiv \ L = I\omega \\ p_r &= m\dot{r} \end{aligned}$$

Electromagnetic:

$$\begin{aligned} \mathcal{H} &= \dot{\mathbf{r}} \cdot \vec{p}_r - \left(\tfrac{1}{2}m\dot{\mathbf{r}}^2 - q\phi_{(t,\ \mathbf{r})} + q\dot{\mathbf{r}} \cdot \vec{A}_{(t,\ \mathbf{r})}\right) \ , \\ &= m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \vec{A} - \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\phi - q\dot{\mathbf{r}} \cdot \vec{A} \\ &= \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\phi \end{aligned}$$

$$\vec{p}_r = m\dot{\mathbf{r}} + q\vec{A}$$

Special Relativity:

$$\begin{aligned} \mathcal{H} &= \vec{v} \cdot (\gamma m\vec{v}) - \left(\gamma mv^2 - \gamma mc^2 - U\right) \ , \\ &= \gamma mc^2 + U \\ &\approx \tfrac{1}{2}mv^2 + \left(U + mc^2\right) \qquad \text{(when } v \ll c\text{)} \end{aligned}$$

$$\vec{p} = \gamma m\vec{v}$$

# 3 Kinematics

$$\left.\begin{aligned} m_0 v_0 &= m_1 v_1 + m_2 v_2 \\ \tfrac{1}{2} m_0 v_0^2 &= \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 \end{aligned}\right. \Rightarrow \boxed{\tfrac{1}{2} m_2 v_2^2 (m_1 + m_2) - \tfrac{1}{2} m_0 v_0^2 (m_1 - m_0) = (m_0 v_0)(m_2 v_2)}$$

- $\boxed{m v_0 \;=\; m v_1 + M v_2 \;=\; m v_0 \left(1 - \frac{2M}{m+M}\right) + M v_0 \left(\frac{2m}{m+M}\right)}$

  $\rightarrow \boxed{M \in (\infty,\ m,\ 0] \;\Rightarrow\; v_1 \in (-v_0,\ 0,\ v_0]}$

# 4 Orbits

$$\left.\begin{aligned} &\underline{\text{Lagrangian}}: \quad \mathcal{L} = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m r^2 \dot{\theta}^2 + \tfrac{1}{2} m r^2 \sin^2\theta \dot{\phi}^2 - U(r) \\ &\quad \bullet\ l = I\omega = m r^2 \dot{\theta} \\ &\quad \bullet\ m\ddot{r} = -\frac{\partial}{\partial r} U_{\text{eff}} = -\frac{\partial}{\partial r}\left(\frac{l^2}{2mr^2} + U(r)\right) \end{aligned}\right\} \quad m \;\rightarrow\; \mu = \tfrac{mM}{m+M}$$

Hamiltonian: $E = \frac{p^2}{2m} + \frac{l^2}{2mr^2} + U(r)$

- Inf. Energy to get to $r = 0$ unless $l = 0$
- $U \sim 1/r$

Orbit Types:

$E > 0$ : Hyperbola

$E = 0$ : Parabola

$E < 0$ : Ellipse

$E = \text{Min}(U_{\text{eff}})$ : Circle

Kepler's Laws:

1st Law : Elliptical Orbits (Sun [at/orbiting] focus)

2nd Law : Equal Area Sweep ($r^2 d\theta = \frac{l}{m} dt$)

3rd Law : $T^2 = k^2 a^3$ $T$, Period
$a$, Semi-major axis
$k$, "constant" $\left(\frac{2\pi}{\sqrt{G[m_{\text{planet}} + M_{\text{sun}}]}}\right)$

# 5 Fluid Mechanics

| | | | | |
|---|---|---|---|---|
| Bernoulli's Principle : | $\frac{\rho v^2}{2} + \rho g z + P_{\text{res}}$ | $=$ | constant | [Energy Density] |
| Fluid Conservation : | $\rho A v$ | $=$ | constant | [Mass Flow Rate] |
| Bouyant Force : | $F = \rho V g$ | | ($\rho, V$, of displaced liquid) | |

Water Facts:

- 1 L = 1 kg

# 6 Oscillators

## 6.1 Homogenous

$$(F = m\ddot{x}) = -kx - \overset{\text{(damp)}}{b\dot{x}}$$

$$\downarrow$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

$z_{\text{tr}}(t) = \tilde{C}e^{rt} + [\tilde{D}_{\text{opt.}}\ te^{rt}]$ : $\underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}}$

$$(r^2 + 2\beta r + \omega_0^2)e^{rt} = 0$$

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Normal (Undamped): $\big(F = -kx\big) \Rightarrow \big(\ddot{x} = -\omega_0^2 x = -\frac{k}{m}x\big)$

$$\boxed{z_{\text{tr}}(t) = \tilde{C}_1 e^{i\omega_0 t} + \tilde{C}_2 e^{-i\omega_0 t}}$$

Underdamped: $\big(\beta < \omega_0\big)$

$$\boxed{z_{\text{tr}}(t) = \left(\tilde{C}_1 e^{i\sqrt{\omega_0^2-\beta^2}t} + \tilde{C}_2 e^{-i\sqrt{\omega_0^2-\beta^2}t}\right)\underline{e^{-\beta t}}}$$

Critically Damped: $\big(\beta = \omega_0\big)$

$$z_{\text{tr}}(t) = \big(\tilde{C}_1 + \tilde{C}_2 t\big)\underline{e^{-\beta t}}$$

Decay rate is maximized at $\beta = \omega_0$

Overdamped: $\big(\beta > \omega_0\big)$

$$z_{\text{tr}}(t) = \underbrace{\tilde{C}_1 e^{-\left(\beta - \sqrt{\beta^2-\omega_0^2}\right)t}}_{\text{(smaller, lasts longer)}} + \tilde{C}_2 e^{-\left(\beta + \sqrt{\beta^2-\omega_0^2}\right)t}$$

## 6.2 Inhomogenous (Driven)

$$\boxed{z(t) = z_{\text{st}}(t) + z_{\text{tr}}(t)}$$

$$m\ddot{x} = -kx - b\dot{x} + F_{\text{dr}}$$

$$\downarrow$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t$$

- $L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t)$

$\boxed{z_{\text{st}}(t) = \tilde{C}e^{i\omega t} = Ae^{i(\omega t - \delta)}}$ : $\underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}}$

$$(-\omega^2 + 2i\beta\omega + \omega_0^2)\tilde{C}e^{i\omega t} = f_0 e^{i\omega t}$$

$$\tilde{C} = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = Ae^{-i\delta}$$

$$\boxed{A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\ ,\quad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)}$$

Resonance (Max $A^2$) with fixed $\omega$ : $\boxed{\omega_0 = \omega}$

Resonance (Max $A^2$) with fixed $\omega_0$ : $\boxed{\omega = \sqrt{\omega_0^2 - 2\beta^2}}$ (usually $\beta \ll \omega$)

Full Width at Half Max, $A^2(\omega)$ : FWHM $\approx 2\beta$

Quality Factor (Sharpness) : $Q = \frac{\omega_0}{2\beta} = \left(\pi\frac{1/\beta}{2\pi/\omega_0} = \pi\frac{\text{decay time}}{\text{period}}\right) = \left(2\pi\frac{\text{Energy stored}}{\text{Energy Dissipated}}\right)$

## 6.3 Parallel and Series

Series, $k_1$+$k_2$+$m$: $\dfrac{1}{K_{\text{eq}}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$

Parallel, $k_1k_2$+$m$: $K_{\text{eq}} = k_1 + k_2$

## 6.4 Normal Modes: 3 Springs + 2 Masses, $k_1$+$m_1$+$k_2$+$m_2$+$k_3$

1.) $$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2x_1 + k_2x_2 \\ &= -(k_1+k_2)x_1 + k_2x_2 \\ m_2\ddot{x}_2 &= k_2x_1 - k_2x_2 - k_3x_2 \\ &= k_2x_1 - (k_2+k_3)x_2 \end{aligned}$$

$$\rightarrow \quad \mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2.) $$\begin{aligned} \mathbf{z}(t) = \mathbf{a}e^{i\omega t} &= \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} e^{i\omega t} \\ &= \begin{pmatrix} a_1e^{-i\delta_1 t} \\ a_2e^{-i\delta_2 t} \end{pmatrix} e^{i\omega t} \end{aligned}$$

$$\rightarrow \quad \begin{aligned} \mathbf{M}\ddot{\mathbf{z}} &= -\mathbf{K}\mathbf{z} \\ -\omega^2\mathbf{M}\mathbf{a}e^{i\omega t} &= -\mathbf{K}\mathbf{a}e^{i\omega t} \\ (\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} &= 0 \\ \boxed{\det(\mathbf{K} - \omega^2\mathbf{M}) = 0} \end{aligned}$$

$x(t) = \text{Re}\big[z(t)\big]$ is the real solution.

Same $m$ and $k$

$$\begin{pmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 m \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \rightarrow$$

$$\boxed{\omega = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}}$$

Smaller $\omega_1$ is most symmetric motion (both swing in phase)

Larger $\omega_2$ swings out of phase

$$\boxed{z(t) = \tilde{A}_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_1 t} + \tilde{A}_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_2 t}}$$

Weak Coupling

## 6.5 Single Pendulum (Use Lagrangian)

- $T = \frac{1}{2}mR^2\dot{\theta}^2$
- $U = mg(R - R\cos\theta)$

$$\rightarrow \quad \begin{aligned} mR^2\ddot{\theta} &= -mgR\sin\theta \\ &\approx -mgR\theta \end{aligned} \quad \rightarrow \quad \boxed{\begin{gathered} \ddot{\theta} = -\left(\frac{g}{I/mR}\right)\theta = -\omega^2\theta \\ \theta(t) = \mathrm{Re}\big[C_1 e^{i\omega t} + C_2 e^{-i\omega t}\big] \end{gathered}}$$

## 6.6 Double Pendulum (Use Lagrangian)

- $T = \frac{1}{2}m_1L_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2(L_1\dot{\vec{\theta_1}} + L_2\dot{\vec{\theta_2}})^2$
  $= \frac{1}{2}(m_1+m_2)L_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2L_2^2\dot{\theta_2}^2$
  $+ m_2L_2L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)$
- $U = m_1g(L_1 - L_1\cos\theta_1)$
  $+ m_2g(L_1 + L_2 - L_2\cos\theta_2 - L_1\cos\theta_1)$

$$\rightarrow \qquad \mathbf{M}\ddot{\theta} = -\mathbf{K}\theta \qquad \text{(small angle quadratic approx.)}$$

$$\begin{pmatrix} (m_1+m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\begin{pmatrix} (m_1+m_2)gL_1 + k_2 & 0 \\ 0 & m_2gL_2 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$