1 Fourier Series

1.1 Sine

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} S_n \sin\left(\frac{g(n)}{L}\pi x\right) \\
\int_0^L f(x)\sin\left(\frac{h(m)}{L}\pi x\right)dx &= \sum_{n=0}^{\infty}\int_0^L S_n \sin\left(\frac{g(n)}{L}\pi x\right)\sin\left(\frac{h(m)}{L}\pi x\right)dx \\
&= S_{h(m)}\frac{L}{2} \\
S_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{g(n)}{L}\pi x\right)dx
\end{aligned}
$$

1.2 Cosine

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} S_n \cos\left(\frac{g(n)}{L}\pi x\right) \\
\int_0^L f(x)\cos\left(\frac{h(m)}{L}\pi x\right)dx &= \sum_{n=0}^{\infty}\int_0^L S_n \cos\left(\frac{g(n)}{L}\pi x\right)\cos\left(\frac{h(m)}{L}\pi x\right)dx \\
&= S_{h(m)}\frac{L}{2} \\
S_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{g(n)}{L}\pi x\right)dx
\end{aligned}
$$

1.3 Full

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{g(n)}{L}\pi x\right) + B_n \sin\left(\frac{g(n)}{L}\pi x\right) \\
A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{g(n)}{L}\pi x\right)dx \\
B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{g(n)}{L}\pi x\right)dx
\end{aligned}
$$

1.4 Exponential

$$
\begin{aligned}
f(x) &= \sum_{n\,=\,-\infty}^{\infty} C_n \; e^{i\frac{2\pi n}{\lambda}x} \\
C_n &= \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(x) \; e^{-i\frac{2\pi n}{\lambda}x} \; dx
\end{aligned}
$$

2 Fourier Transform

$$
\begin{aligned}
k_n &= \frac{2\pi n}{\lambda} \quad \Rightarrow \quad \Delta k = \frac{2\pi}{\lambda} \\
\Psi(x) \;\; ?\approx & \sum_{n\,=\,-\infty}^{\infty} \frac{1}{\lambda} \left[\int_{-\lambda/2}^{\lambda/2} \Psi(x) \; e^{-i\frac{2\pi n}{\lambda}x} \; dx \right] e^{i\frac{2\pi n}{\lambda}x} \\
&= \sum_{k_n\,=\,-\infty}^{\infty} \frac{\Delta k}{\sqrt{2\pi}} \left[\frac{1}{\sqrt{2\pi}} \int_{-\lambda/2}^{\lambda/2} \Psi(x) \; e^{-ik_n x} \; dx \right] e^{ik_n x}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\lambda \to \infty} \quad \Psi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \; \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) \; e^{-ikx} \; dx \right] e^{ikx} \; dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \; \hat{\Psi}(k) e^{ikx} \; dk \\
\hat{\Psi}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx} \; dx
\end{aligned}
$$

or

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} \; \hat{x}(f) e^{i2\pi f t} \; df \\
\hat{x}(f) &= \int_{-\infty}^{\infty} \; x(t) e^{-i2\pi f t} \; dt
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \hat{x}(f)(e^{-\epsilon f^2})\ e^{i2\pi f t}\ df &= a(t) \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) e^{-i2\pi f \tau}\ d\tau\right] (e^{-\epsilon f^2})\ e^{i2\pi f t}\ df \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} (e^{-\epsilon f^2})\ e^{-i2\pi f(\tau - t)}\ df\right]\ d\tau \\
&= \int_{-\infty}^{\infty} x(\tau' + t) \left[\int_{-\infty}^{\infty} (e^{-\epsilon f^2})\ e^{-i2\pi f \tau'}\ df\right]\ d\tau' \\
&= \int_{-\infty}^{\infty} x(\tau' + t)\ \left(\frac{1}{2\sqrt{\pi}\epsilon}\ e^{\frac{-(\tau')^2}{4\epsilon^2}}\right)\ d\tau' \\
&= \int_{-\infty}^{\infty} x(\epsilon\tau'' + t)\ \left(\frac{1}{\sqrt{2\pi}\sqrt{2}} e^{\frac{-(\tau'')^2}{2\sqrt{2}^2}}\right)\ d\tau''
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \hat{x}(f)\ e^{i2\pi f t} &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \hat{x}(t)(e^{-\epsilon f^2})\ e^{i2\pi f t}\ df\ = \\
\lim_{\epsilon \to 0}\ a(t) &= x(t) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(\tau'')^2}{2\sigma^2}}\ d\tau'' \\
&= x(t)
\end{aligned}
$$

3 Laplace Transform

$\{s = \sigma + i\tau\ :\ \sigma > a\}$

- $\left(|x(t)|\ \leq_{t\to\infty}\ Me^{-at}\right)\ \Rightarrow\ \left(\left|x(t)e^{-st}\right|\ \leq\ Me^{at}e^{-\sigma t}\ =\ Me^{-t(\sigma - a)}\right)$

$$
\begin{aligned}
\boldsymbol{(\mathcal{L}\{x\})(s)} &= \boldsymbol{\int_0^{\infty} x(t)e^{-st}\ dt} \\
\boldsymbol{x(t)} &= \boldsymbol{\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} (\mathcal{L}\{x\})(s) e^{st}\ ds}
\end{aligned}
$$

$(\mathcal{L}\{x^{(n)}\})(s)\ =\ s^n(\mathcal{L}\{x\})(s) - s^{n-1}x(0) - s^{n-2}x'(0) - ... - x^{(n-1)}(0)$

4 Z Transform (Discrete Laplace)

$$\boldsymbol{\{\, a_n \;\; : \;\; |a_n| \;\leq\; MR^n \,\}}$$

$$\boldsymbol{(Z\{a_n\})(z) \;\; = \;\; \sum_{n\,=\,0}^{\infty} \frac{a_n}{z^n}}$$

$$c_n \;\; = \;\; \sum_{k=0}^{n} a_k b_{n-k} \;\; = \;\; a_n * b_n$$

- $Z\{c_n\} \;\; = \;\; Z\{a_n\}Z\{b_n\}$

$$\begin{aligned}
Z\{a_{n+m}\} \;\; &= \;\; a_m + \frac{a_{m+1}}{z} + \frac{a_{m+2}}{z^2} + ... \\
&= \;\; z^m \left[\left(a_0 + \frac{a_1}{z} + ... + \frac{a_{m-1}}{z^{m-1}} \right) + \left(\frac{a_m}{z^m} + \frac{a_{m+1}}{z^{m+1}} + ... \right) - \left(a_0 + \frac{a_1}{z} + ... + \frac{a_{m-1}}{z^{m-1}} \right) \right] \\
&= \;\; z^m \left[Z\{a_n\} - \left(a_0 + \frac{a_1}{z} + ... + \frac{a_{m-1}}{z^{m-1}} \right) \right] \\
&= \;\; z^m \left[Z\{a_n\} - \sum_{j=0}^{m-1} \frac{a_j}{z^j} \right]
\end{aligned}$$

If all initial conditions are 0, then $\boldsymbol{Z\{y_n\} = F(z)Z\{x_n\}}$

- $\boldsymbol{y_n}$ is stable/bounded iff all of $\boldsymbol{F(z)}$'s pole's are in the unit open disc.