1 Lorentz Transformation

Galilean Transform $\rightarrow$ Lorentz Transform $\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$

$$\begin{aligned} t_0' &= t_0 \\ x_0' &= x_0 - vt_0 \end{aligned} \quad , \quad \begin{aligned} t_0 &= t_0' \\ x_0 &= x_0' + vt_0' \end{aligned}$$

$$\boxed{\begin{aligned} x' &= \gamma(x - \beta ct) \\ ct' &= \gamma(ct - \beta x) \end{aligned} \quad , \quad \begin{aligned} x &= \gamma(x' + \beta ct') \\ ct &= \gamma(ct' + \beta x') \end{aligned}}$$

x is the position of a point/event occuring on the number line, and t is the time on a clock at x.

x' is the position of the same point/event on the other number line, and t' is the time on the clock at x'.

$x = 0$ and $x' = 0$ are the positions of the line/reference frame origins, and t_0 and t_0' are the times on the origin clocks.

At $t_0 = t_0' = 0$, both origin's coincide at $x = x' = 0$.

All points at x see their clock run the same as $t = t_0$, but see a different t' at the adjacent x'.

Transform Matrix (Hermitian for boosts)

$$\gamma = \cosh\phi, \ \gamma\beta = \sinh\phi$$

$$x^{\mu\prime} = \begin{pmatrix} x^{0\prime} = ct' \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x^\mu$$

Weyl Matrices: $\cosh\frac{\phi}{2} I - \sinh\frac{\phi}{2} \cancel{\sigma_t} \sigma_x$ (Hermitian)

Time Slows: $\boxed{\Delta t' = \Delta t/\gamma}$

1.) $t'(x, t_0)$ for a Clock at $x = X_0 + vt$

$$\begin{aligned} \Rightarrow \quad ct' &= \gamma\left(ct - \beta[X_0 + vt]\right) \\ &= \left(\gamma ct(1-\beta^2) - \gamma\beta X_0\right) \\ &= \left(\frac{ct}{\gamma} - \gamma\beta X_0\right) = \left(\frac{ct}{\gamma} + cT_0\right) \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} t'(X_0, t) &= \tfrac{t}{\gamma} + T_0 \\ &= \tfrac{t}{\gamma} - \gamma\beta X_0 \end{aligned}}$$

$T_0{>}0$: $X_0{<}0$
$T_0{<}0$: $X_0{>}0$

(No t' is simultaneous to x unless in same position)

$$\boxed{\frac{dt'}{dt} = \frac{dt'}{dt_0} = \frac{1}{\gamma}}$$

(for each Δt, then $\Delta t' = \dfrac{\Delta t}{\gamma}$)

(Clocks at x' look like they tick slower by factor γ)

2.) $\Delta t_0'$ given Δt_0

$$\Rightarrow \quad c\Delta t_0 = \gamma(c\Delta t_0' - \cancel{\beta x'}_{=0}) \quad \Rightarrow \quad \boxed{\Delta t_0' = \Delta t_0/\gamma}$$

(Same conclusion of slowed clocks)

Length Contraction: $\boxed{\Delta x' = \gamma\Delta x}$

$$\begin{aligned} \text{1.) } \Delta x' = x_2' - x_1' \ &= \ \gamma(x_2 - \cancel{\beta ct}) - \gamma(x_1 - \cancel{\beta ct}) \\ &= \ \gamma(x_2 - x_1) \ = \ \gamma\Delta x \quad ☑ \end{aligned}$$

Velocity Addition (1-D): $\boxed{w = \dfrac{v+u}{1 + vu/c^2}}$

Pythag. Triples

$\beta = 3/5$: $\gamma = 5/4 = 1.25$

$\beta = 4/5$: $\gamma = 5/3$

$\beta = 5/13$: $\gamma = 13/12$

$\beta = 7/25$: $\gamma = 25/24$

Doppler Shift

$$f_{\text{rec}} = \sqrt{\tfrac{1+\beta}{1-\beta}}\ f_{\text{emit}}$$

(v is [+] if $\rightarrow\leftarrow$)

2 4-Vectors

3-Vectors

$$\vec{p} = \boxed{\gamma m \vec{v}}$$

$$\begin{aligned} \vec{F} &= \tfrac{d\vec{p}}{dt} = m\tfrac{d(\gamma\vec{v})}{dt} \\ &= \gamma m\vec{a} + \gamma^3 \tfrac{(m\vec{a}\cdot\vec{v})\vec{v}}{c^2} \\ &= \boxed{\gamma^3 m(\vec{a} - \tfrac{\vec{v}\cdot\vec{v}}{c^2}\vec{a} + \tfrac{\vec{a}\cdot\vec{v}}{c^2}\vec{v})} \end{aligned}$$

Scalars

$$\begin{aligned} E &= \boxed{\sqrt{p^2c^2 + m^2c^4} = \gamma mc^2} \\ T &= E - E_0 = \boxed{(\gamma - 1)mc^2} \\ P_{\text{ow}} &= \tfrac{dE}{dt} = mc^2\tfrac{d\gamma}{dt} = \tfrac{d\vec{p}}{dt}\cdot\vec{v} \\ &= \boxed{\vec{F}\cdot\vec{v} = \gamma^3 m\vec{a}\cdot\vec{v}} \\ W &= \int \vec{F}\cdot\vec{v}\,dt = \boxed{\int \gamma^3 m\vec{a}\cdot\vec{v}\,dt} \end{aligned}$$

2.1 Position

$$\begin{aligned} \boldsymbol{x}^\mu &= (x^0, \vec{x}) \\ &= (ct, \vec{r}) \end{aligned} \Rightarrow \boxed{\Delta\boldsymbol{x}^\mu = \boldsymbol{x}^\mu_A - \boldsymbol{x}^\mu_B} \qquad \text{(for Event } A,\ B)$$

$$\begin{aligned} (\Delta\boldsymbol{x}^\mu)^2 &= (\Delta\boldsymbol{x}^\mu)(\Delta\boldsymbol{x}_\mu) \\ &= c^2\tau^2 \\ &= \boxed{ct^2 - \vec{r}^{\,2}} \end{aligned}$$

$$\begin{cases} \text{Timelike}: & (\Delta\boldsymbol{x}^\mu)^2 > 0 \\ \text{Spacelike}: & (\Delta\boldsymbol{x}^\mu)^2 < 0 \\ \text{Lightlike}: & (\Delta\boldsymbol{x}^\mu)^2 = 0 \end{cases}$$

- Timelike: (∃ an inertial frame where $A,\ B$ occur at the <u>same spacial place</u> but diff. time, e.g, $(c\Delta t, 0, 0, 0) \rightarrow (\Delta x)^2 = c^2(\Delta t)^2 > 0$)
- Spacelike: (∃ an inertial frame where $A,\ B$ occur at the <u>same time</u> but non-casual space)
- Lightlike: (A and B lie on a trajectory moving at c)

Relativistic Dot Products are Ref. Frame Invariant (not necessarily Conserved)

2.2 Momentum

$$\begin{aligned} \boldsymbol{p}^\mu &= (p^0, \vec{p}) \\ &= m\frac{d\boldsymbol{x}^\mu}{d\tau} = m\boldsymbol{\eta}^\mu \\ &= \boxed{(\gamma mc, \gamma m\vec{v}) = \left(\tfrac{E}{c}, \vec{p}\right)} \end{aligned}$$

$$\pm\boldsymbol{p}^\mu\boldsymbol{p}_\mu = \boxed{\boldsymbol{p}^2 = m^2c^2 = \left(\tfrac{E}{c}\right)^2 - \vec{p}^{\,2}}$$

Momentum Conservation means each vector component is individually conserved

<u>Particle Decay Example: Rest $M_1 \rightarrow m_2 + m_3$</u>

$$\begin{aligned} p_3^2 &= (p_1 - p_2)^2 \\ &= p_1^2 + p_2^2 - 2p_1\cdot p_2 \\ m_3^2c^2 &= M_1^2c^2 + m_2^2c^2 - 2(M_1c, 0)\cdot(E_2/c, \vec{p}_2) \\ &= M_1^2c^2 + m_2^2c^2 - 2M_1E_2 \end{aligned} \Rightarrow \boxed{\begin{aligned} E_2 &= \frac{M_1^2c^2 + m_2^2c^2 - m_3^2c^2}{2M_1} \\ E_3 &= \frac{M_1^2c^2 + m_3^2c^2 - m_2^2c^2}{2M_1} \end{aligned}}$$

2.3 Acceleration and Force

$$\begin{aligned}
\boldsymbol{K}^\mu &= (K^0, \vec{K}) \\
&= m\boldsymbol{\alpha}^\mu \;=\; m\frac{d\boldsymbol{\eta}^\mu}{d\tau} \;=\; \frac{d\boldsymbol{p}^\mu}{d\tau} \;=\; \gamma\frac{d\boldsymbol{p}^\mu}{dt} \\
&= \left(\gamma\frac{d(\gamma mc)}{dt}, \gamma\frac{d(\gamma m\vec{v})}{dt}\right) \\
&= \left(\tfrac{\gamma P_{\mathrm{ow}}}{c}, \gamma\vec{F}\right) \;=\; \left(\tfrac{\gamma\vec{F}\cdot\vec{v}}{c}, \gamma\vec{F}\right) \\
&= \left(\gamma^4\tfrac{m\vec{a}\cdot\vec{v}}{c},\ \gamma^2 m\vec{a} + \gamma^4\tfrac{(m\vec{a}\cdot\vec{v})\vec{v}}{c^2}\right)
\end{aligned}$$

$$\begin{aligned}
\mp\boldsymbol{\alpha}^\mu\boldsymbol{\alpha}_\mu &= \gamma^6\tfrac{(\vec{a}\cdot\vec{v})^2}{c^2} + \gamma^4\vec{a}^{\ 2} \\
\mp\boldsymbol{K}^\mu\boldsymbol{K}_\mu &= -\gamma^2\tfrac{(\vec{F}\cdot\vec{v})^2}{c^2} + \gamma^2\vec{F}^{\ 2} \\
&= \gamma^2\vec{F}^{\ 2}\left(1 - \tfrac{\vec{v}\cdot\vec{v}}{c^2}\cos\theta_{v,F}\right) \\
\boldsymbol{\alpha}^\mu\boldsymbol{\eta}_\mu &= \frac{d\boldsymbol{\eta}^\mu}{d\tau}\boldsymbol{\eta}_\mu = \frac{1}{2}\frac{d(\boldsymbol{\eta}^\mu\boldsymbol{\eta}_\mu)}{d\tau} = 0 \\
\boldsymbol{K}^\mu\boldsymbol{p}_\mu &= m^2\boldsymbol{\alpha}^\mu\boldsymbol{\eta}_\mu = 0
\end{aligned}$$

2.4 Current Density and Vector Potential

$$\begin{aligned}
\boldsymbol{J}^\mu &= (J^0, \vec{J}) \\
&= \rho_0\frac{d\boldsymbol{x}^\mu}{d\tau} \;=\; \rho_0\boldsymbol{\eta}^\mu \\
&= (\gamma c\rho_0, \gamma\rho_0\vec{v}) \;=\; (c\rho, \vec{J}) \\
\\
\boldsymbol{A}^\mu &= (A^0, \vec{A}) \;=\; \left(\tfrac{V}{c}, \vec{A}\right) \\
\\
&= \boldsymbol{A}^\mu + \frac{\partial\lambda}{\partial\boldsymbol{x}^\mu}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\boldsymbol{J}^\mu}{\partial\boldsymbol{x}^\mu} &= \frac{\partial}{\partial\boldsymbol{x}^\mu}\cdot\boldsymbol{J}^\mu \;=\; \frac{\partial\rho}{dt} + \nabla\vec{J} \;=0 \\
\Box^2\boldsymbol{A}^\mu &= \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A}^\mu \;=\; -\mu_0\boldsymbol{J}^\mu
\end{aligned}$$