

# 1 Lagrangian

$$\mathcal{L} = T - U, \quad p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\rightarrow F_i \equiv \frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i}$$

Newton's Laws:

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - U(\mathbf{r}), \quad \vec{p}_r = m\dot{\mathbf{r}}$$

$$\rightarrow \boxed{F = m\ddot{\mathbf{r}} = -\nabla U}$$

Angular:

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r, \theta), \quad p_\theta = mr^2\dot{\theta} \equiv L = I\omega$$

$$p_r = m\dot{r}$$

$$\rightarrow F_r \equiv \boxed{-\frac{\partial U}{\partial r} = m\ddot{r} - mr\dot{\theta}^2} \quad (\text{centripital: } \frac{mv^2}{r} = mr\omega^2)$$

$$rF_\theta \equiv \boxed{-\frac{\partial U}{\partial \theta} = mr^2\ddot{\theta}} = I\alpha = \tau$$

$$-\vec{F} = \nabla U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta}$$

Electromagnetic:

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - q\left(V(t, \mathbf{r}) - \dot{\mathbf{r}} \cdot \vec{A}(t, \mathbf{r})\right), \quad p_x = m\dot{x} + qA_x$$

$$\rightarrow m\ddot{x} + q\frac{dA_x}{dt} = -q\left(\frac{\partial V}{\partial x} - \dot{\mathbf{r}} \cdot \frac{\partial \vec{A}}{\partial x}\right)$$

$$m\ddot{x} + q\left(\frac{\partial A_x}{\partial t} + \dot{\mathbf{r}} \cdot \nabla A_x\right) = q\left(-\frac{\partial V}{\partial x} + \dot{\mathbf{r}} \cdot \frac{\partial \vec{A}}{\partial x}\right)$$

$\downarrow$

$$m\ddot{x} = q\left[-\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} + \dot{\mathbf{r}} \cdot \left(\frac{\partial \vec{A}}{\partial x} - \nabla A_x\right)\right]$$

$$= q\left(-\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t}\right) + q\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + q\dot{z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)$$

$$= qE_x + q\dot{y}B_z - q\dot{z}B_y$$

$$m\ddot{x} = qE_x + q\left(\dot{\mathbf{r}} \times \vec{B}\right)_x$$

$\downarrow$

$$\boxed{m\ddot{\mathbf{r}} = q\left(\vec{E} + \dot{\mathbf{r}} \times \vec{B}\right)}$$

### Special Relativity:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{\gamma}mc^2 - U, & \vec{p} &= \gamma m \vec{v} \rightarrow \gamma m \dot{x} = \frac{\partial \mathcal{L}}{\partial \dot{x}} \\
&= \gamma m v^2 - \gamma m c^2 - U \\
&= m(v^2 - c^2) \left(1 - \frac{v^2}{c^2}\right)^{-1/2} - U \\
&\approx \frac{1}{2}mv^2 - (U + mc^2) && \text{(when } v \ll c)
\end{aligned}$$

### Conservation of Energy:

$$\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \sum_i \left( \frac{\partial \mathcal{L}}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right) + \frac{\partial \mathcal{L}}{\partial t} \\
&= \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) + \frac{\partial \mathcal{L}}{\partial t} && \rightarrow \frac{\partial \mathcal{L}}{\partial t} = -\frac{d}{dt} \left( \sum_i p_i \dot{q}_i - \mathcal{L} \right) \\
&= \frac{d}{dt} \left( \sum_i p_i \dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} && = -\frac{d\mathcal{H}}{dt} \quad \text{If } \mathcal{L} \text{ is explicitly independent of time} \\
&&& \quad \text{(implies coordinates are "natural"),} \\
&&& \quad \text{then the Hamiltonian is conserved.}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} \sum_n m \dot{\mathbf{r}}_n^2 &= \frac{1}{2} \sum_n m \left( \sum_i \frac{\partial r_n}{\partial q_i} \dot{q}_i \right)^2 && \mathcal{L} = \frac{1}{2}mv^2 - U = T(\dot{q}_i) - U(q_i) \rightarrow \\
&= \frac{1}{2} \sum_{i,j} \left( m \sum_n \frac{\partial r_n}{\partial q_i} \frac{\partial r_n}{\partial q_j} \right) \dot{q}_i \dot{q}_j && \mathcal{H} = \sum_i \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \\
&= \frac{1}{2} \sum_i \sum_j A_{ij} \dot{q}_i \dot{q}_j && \rightarrow = \sum_i \left( \sum_j A_{ij} \dot{q}_j \right) \dot{q}_i - \frac{1}{2}m\dot{\mathbf{r}}^2 + U \\
&&& = \frac{1}{2}m\dot{\mathbf{r}}^2 + U \quad \text{If } \mathcal{L} = \frac{1}{2}mv^2 - U \text{ and } U \text{ is independent} \\
&&& \quad \text{of } v, \text{ then the Hamiltonian is the total} \\
&&& \quad \text{energy.}
\end{aligned}$$

(for  $\frac{\partial T}{\partial \dot{q}_i}$ )  $= \frac{1}{2} \left( 2 \sum_{i \neq j} A_{ij} \dot{q}_i \dot{q}_j + A_{ii} \dot{q}_i^2 \right) + \dots$

### Lagrange Multipliers:

$$\begin{aligned}
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) &= \frac{\partial \mathcal{L}}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i} \\
\frac{dp}{dt} &= -\nabla U + \lambda \nabla f \\
F_{\text{tot}} &= F_{\text{ncnstr}} + F_{\text{cnstr}}
\end{aligned}$$

## 1.1 Examples

Atwood's Machine (Pulley):

Particle Confined to a Cylinder Surface:

Block Sliding on Wedge:

Bead on Spinning Wire Hoop:

Oscillations of Bead Near Equilibrium:

## 2 Hamiltonian

$$\mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L} , \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\rightarrow \begin{aligned} &\bullet \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i} \\ &\bullet \frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \end{aligned}$$

Newton Particle:

$$\begin{aligned} \mathcal{H} &= \dot{x}(m\dot{x}) - \frac{1}{2}m\dot{x}^2 + U(x) \\ &= \frac{1}{2}m\dot{x}^2 + U(x) \\ &= T + U \end{aligned}$$

Angular:

$$\begin{aligned} \mathcal{H} &= m\dot{r}^2 + mr^2\dot{\theta}^2 - \left( \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r, \theta) \right) , \quad \begin{aligned} p_\theta &= mr^2\dot{\theta} \equiv L = I\omega \\ p_r &= m\dot{r} \end{aligned} \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U(r, \theta) \end{aligned}$$

Electromagnetic:

$$\begin{aligned} \mathcal{H} &= \dot{\mathbf{r}} \cdot \vec{p}_r - \left( \frac{1}{2}m\dot{\mathbf{r}}^2 - q\phi(t, \mathbf{r}) + q\dot{\mathbf{r}} \cdot \vec{A}(t, \mathbf{r}) \right) , \quad \vec{p}_r = m\dot{\mathbf{r}} + q\vec{A} \\ &= m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \vec{A} - \frac{1}{2}m\dot{\mathbf{r}}^2 + q\phi - q\dot{\mathbf{r}} \cdot \vec{A} \\ &= \frac{1}{2}m\dot{\mathbf{r}}^2 + q\phi \end{aligned}$$

Special Relativity:

$$\begin{aligned} \mathcal{H} &= \vec{v} \cdot (\gamma m \vec{v}) - (\gamma m v^2 - \gamma m c^2 - U) , \quad \vec{p} = \gamma m \vec{v} \\ &= \gamma m c^2 + U \\ &\approx \frac{1}{2}m v^2 + (U + m c^2) \quad (\text{when } v \ll c) \end{aligned}$$

### 3 Orbits

$$\left. \begin{array}{l} \text{Lagrangian : } \mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\dot{\phi}^2 - U(r) \\ \bullet \quad l = I\omega = mr^2\dot{\theta} \\ \bullet \quad m\ddot{r} = -\frac{\partial}{\partial r}U_{\text{eff}} = -\frac{\partial}{\partial r}\left(\frac{l^2}{2mr^2} + U(r)\right) \end{array} \right\} \quad m \rightarrow \mu = \frac{mM}{m+M}$$

Hamiltonian:  $E = \frac{p^2}{2m} + \frac{l^2}{2mr^2} + U(r)$

- Inf. Energy to get to  $r = 0$  unless  $l = 0$
- $U \sim 1/r$

Orbit Types:

$E > 0$  : Hyperbola

$E = 0$  : Parabola

$E < 0$  : Ellipse

$E = \text{Min}(U_{\text{eff}})$  : Circle

Kepler's Laws:

1st Law : Elliptical Orbits (Sun [at/orbiting] focus)

2nd Law : Equal Area Sweep ( $r^2 d\theta = \frac{l}{m} dt$ )

3rd Law :  $T^2 = k^2 a^3$   $T$ , Period  
 $a$ , Semi-major axis  
 $k$ , "constant"  $\left(\frac{2\pi}{\sqrt{G[m_{\text{planet}} + M_{\text{sun}}]}}\right)$

### 4 Fluid Mechanics

Bernoulli's Principle :  $\frac{\rho v^2}{2} + \rho g z + P_{\text{res}} = \text{constant}$  [Energy Density]

Fluid Conservation :  $\rho A v = \text{constant}$  [Mass Flow Rate]

Bouyant Force :  $F = \rho V g$  ( $\rho, V$ , of displaced liquid)

Water Facts:

- 1 L = 1 kg

## 5 Oscillators

### 5.1 Homogenous

$$\begin{array}{l|l}
 (F = m\ddot{x}) = -kx - \overset{\text{(damp)}}{b\dot{x}} & z_{\text{tr}}(t) = \tilde{C}e^{rt} + [\tilde{D}_{\text{opt.}} te^{rt}] : \quad \underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}} \\
 \downarrow & \\
 \boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0} & (r^2 + 2\beta r + \omega_0^2)e^{rt} = 0 \\
 & r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}
 \end{array}$$

Normal (Undamped):  $(F = -kx) \Rightarrow$   
 $(\ddot{x} = -\omega_0^2 x = -\frac{k}{m}x)$

$$z_{\text{tr}}(t) = \tilde{C}_1 e^{i\omega_0 t} + \tilde{C}_2 e^{-i\omega_0 t}$$

Underdamped:  $(\beta < \omega_0)$

$$z_{\text{tr}}(t) = \left( \tilde{C}_1 e^{i\sqrt{\omega_0^2 - \beta^2}t} + \tilde{C}_2 e^{-i\sqrt{\omega_0^2 - \beta^2}t} \right) \underline{e^{-\beta t}}$$

Critically Damped:  $(\beta = \omega_0)$

$$z_{\text{tr}}(t) = (\tilde{C}_1 + \tilde{C}_2 t) e^{-\beta t}$$

Decay rate is maximized at  $\beta = \omega_0$

Overdamped:  $(\beta > \omega_0)$

$$z_{\text{tr}}(t) = \underline{\tilde{C}_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t}} + \tilde{C}_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}$$

(smaller, lasts longer)

### 5.2 Inhomogenous (Driven)

$$\begin{array}{l|l}
 m\ddot{x} = -kx - b\dot{x} + F_{\text{dr}} & \boxed{z(t) = z_{\text{st}}(t) + z_{\text{tr}}(t)} \\
 \downarrow & \\
 \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos \omega t & z_{\text{st}}(t) = \tilde{C}e^{i\omega t} = Ae^{i(\omega t - \delta)} : \quad \underline{x(t) = \text{Re}[z(t)] \text{ is the real solution.}} \\
 \bullet L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t) & (-\omega^2 + 2i\beta\omega + \omega_0^2)\tilde{C}e^{i\omega t} = f_0 e^{i\omega t} \\
 & \tilde{C} = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = Ae^{-i\delta} \\
 & \boxed{A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}, \quad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)}
 \end{array}$$

Resonance (Max  $A^2$ ) with fixed  $\omega$  :  $\boxed{\omega_0 = \omega}$

Resonance (Max  $A^2$ ) with fixed  $\omega_0$  :  $\boxed{\omega = \sqrt{\omega_0^2 - 2\beta^2}}$  (usually  $\beta \ll \omega$ )

Full Width at Half Max,  $A^2(\omega)$  : FWHM  $\approx 2\beta$

Quality Factor (Sharpness) :  $Q = \frac{\omega_0}{2\beta} = \left( \pi \frac{1/\beta}{2\pi/\omega_0} = \pi \frac{\text{decay time}}{\text{period}} \right) = \left( 2\pi \frac{\text{Energy stored}}{\text{Energy Dissipated}} \right)$

### 5.3 Parallel and Series

Series,  $k_1+k_2+m$ :  $\frac{1}{K_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2}$

Parallel,  $k_1k_2+m$ :  $K_{\text{eq}} = k_1 + k_2$

### 5.4 Normal Modes: 3 Springs + 2 Masses, $k_1+m_1+k_2+m_2+k_3$

1.)  $m_1\ddot{x}_1 = -k_1x_1 - k_2x_1 + k_2x_2$   
 $= -(k_1 + k_2)x_1 + k_2x_2$

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

$m_2\ddot{x}_2 = k_2x_1 - k_2x_2 - k_3x_2$   
 $= k_2x_1 - (k_2 + k_3)x_2$

$$\rightarrow \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2.)  $\mathbf{z}(t) = \mathbf{a}e^{i\omega t} = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} e^{i\omega t}$

$$-\omega^2 \mathbf{M} \mathbf{a} e^{i\omega t} = -\mathbf{K} \mathbf{a} e^{i\omega t}$$

$= \begin{pmatrix} a_1 e^{-i\delta_1 t} \\ a_2 e^{-i\delta_2 t} \end{pmatrix} e^{i\omega t}$

$\rightarrow$

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{a} = 0$$

$x(t) = \text{Re}[z(t)]$  is the real solution.

$$\boxed{\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0}$$

Same  $m$  and  $k$

$$\begin{pmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 m \end{pmatrix} = - \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \rightarrow$$

$$\boxed{\omega = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}}$$

Smaller  $\omega_1$  is most symmetric motion  
(both swing in phase)

Larger  $\omega_2$  swings out of phase

$$\boxed{z(t) = \tilde{A}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_1 t} + \tilde{A}_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_2 t}}$$

Weak Coupling

## 5.5 Single Pendulum (Use Lagrangian)

$$\begin{aligned}
 &\bullet T = \frac{1}{2}mR^2\dot{\theta}^2 \\
 &\bullet U = mg(R - R\cos\theta)
 \end{aligned}
 \rightarrow
 \begin{aligned}
 mR^2\ddot{\theta} &= -mgR\sin\theta \\
 &\approx -mgR\theta
 \end{aligned}
 \rightarrow
 \boxed{
 \begin{aligned}
 \ddot{\theta} &= -\left(\frac{g}{I/mR}\right)\theta = -\omega^2\theta \\
 \theta(t) &= \text{Re}[C_1e^{i\omega t} + C_2e^{-i\omega t}]
 \end{aligned}
 }$$

## 5.6 Double Pendulum (Use Lagrangian)

$$\begin{aligned}
 &\bullet T = \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(L_1\dot{\theta}_1 + L_2\dot{\theta}_2)^2 & \bullet U &= m_1g(L_1 - L_1\cos\theta_1) \\
 &= \frac{1}{2}(m_1 + m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 & &+ m_2g(L_1 + L_2 - L_2\cos\theta_2 - L_1\cos\theta_1) \\
 &\quad + m_2L_2L_1\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)
 \end{aligned}$$

$$\rightarrow \mathbf{M}\ddot{\theta} = -\mathbf{K}\theta \quad (\text{small angle quadratic approx.})$$

$$\begin{pmatrix} (m_1 + m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = - \begin{pmatrix} (m_1 + m_2)gL_1 + k_2 & 0 \\ 0 & m_2gL_2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$