$$\boxed{\vec{\nabla} = \left[\vec{\nabla}\,(r,\theta,\phi)\right]\bar{\partial}_\circ}$$

$$d = [dx\ dy\ dz]\,\vec{\nabla} = d\vec{l}^{\,T}\vec{\nabla}$$

$$d(r,\theta,\phi) = [dx\ dy\ dz]\,\vec{\nabla}(r,\theta,\phi)$$

$$\boxed{\partial\bar{l}_\circ^{\,T} = d\vec{l}^{\,T}\vec{\nabla}(r,\theta,\phi)}$$

$$\partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ = d\vec{l}^{\,T}\left[\vec{\nabla}(r,\theta,\phi)\right]\bar{\partial}_\circ$$

$$\boxed{d = \partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ = d\vec{l}^{\,T}\vec{\nabla}}$$

$$\underline{\vec{\nabla}} = \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix} = \begin{bmatrix}| & | & |\\ \nabla r & \nabla\theta & \nabla\phi\\ | & | & |\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\\ \frac{\partial}{\partial \phi}\end{bmatrix}$$

$$\underline{\partial\bar{l}_\circ} = \begin{bmatrix}dr\\ d\theta\\ d\phi\end{bmatrix} \begin{aligned} &= [\vec{\nabla}(r,\theta,\phi)]^T d\vec{l}\\ &= \begin{bmatrix}-\nabla r-\\ -\nabla\theta-\\ -\nabla\phi-\end{bmatrix}\begin{bmatrix}dx\\ dy\\ dz\end{bmatrix}\end{aligned}$$

$$\theta = \theta(x,y,z) \quad \left(x^2+y^2 = z^2\tan^2\theta\right)$$

$$\phi = \phi(x,y,z) \quad \left(y = x\tan\phi\right)$$

$$d\theta = dx\frac{\partial\theta}{\partial x} + dy\frac{\partial\theta}{\partial y} + dz\frac{\partial\theta}{\partial z}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}$$

$$d = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} = d\vec{l}\cdot\vec{\nabla}$$

$$= dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial\theta} + d\phi\frac{\partial}{\partial\phi} = \partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ$$

$$= \frac{dr}{\|\nabla r\|}\|\nabla r\|\frac{\partial}{\partial r} + \frac{d\theta}{\|\nabla\theta\|}\|\nabla\theta\|\frac{\partial}{\partial\theta} + \frac{d\phi}{\|\nabla\phi\|}\|\nabla\phi\|\frac{\partial}{\partial\phi}$$

$$= \left[\frac{dr}{\|\nabla r\|}, \frac{d\theta}{\|\nabla\theta\|}, \frac{d\phi}{\|\nabla\phi\|}\right]\left[\|\nabla r\|\frac{\partial}{\partial r}, \|\nabla\theta\|\frac{\partial}{\partial\theta}, \|\nabla\phi\|\frac{\partial}{\partial\phi}\right]^T$$

$$= [dr, rd\theta, r\sin\theta d\phi]\left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right]^T$$

$$\cdots = d\bar{l}_\circ^{\,T}\bar{\nabla}_\circ = d\vec{l}_\circ^{\,T}\vec{\nabla}_\circ$$

$$d\vec{l} = d\vec{r} = [dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial\theta} + d\phi\frac{\partial}{\partial\phi}](x,y,z)^T$$

$$d(x,y,z) = \left[\frac{dr}{\|\nabla r\|}\|\nabla r\|\frac{\partial}{\partial r} + \frac{d\theta}{\|\nabla\theta\|}\|\nabla\theta\|\frac{\partial}{\partial\theta} + \frac{d\phi}{\|\nabla\phi\|}\|\nabla\phi\|\frac{\partial}{\partial\phi}\right](x,y,z)$$

$$(dx,dy,dz) = dr\,\hat{r}^T + r\,d\theta\,\hat{\theta}^T + r\sin\theta\,d\phi\,\hat{\phi}^T$$

$$(\hat{r},\hat{\theta},\hat{\phi}) = \underline{\left(\|\nabla r\|\frac{\partial\vec{r}}{\partial r}, \|\nabla\theta\|\frac{\partial\vec{r}}{\partial\theta}, \|\nabla\phi\|\frac{\partial\vec{r}}{\partial\phi}\right)}$$

$$= \left(\frac{\partial}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\otimes(x,y,z)^T$$

$$= \bar{\nabla}_\circ^T\otimes\vec{r}$$

$$\boxed{d\vec{l} = (dx,dy,dz)\cdot\left(\hat{x},\hat{y},\hat{z}\right)} = \boxed{(dr, rd\theta, r\sin\theta d\theta)\cdot\left(\hat{r},\hat{\theta},\hat{\phi}\right) = d\vec{l}_\circ} = d\bar{l}_\circ^{\,T}\cdot\left(\hat{r},\hat{\theta},\hat{\phi}\right)$$

$$\boxed{\vec{\nabla} = (\hat{x},\hat{y},\hat{z})\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)} = \boxed{(\hat{r},\hat{\theta},\hat{\phi})\cdot\left(\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial\theta},\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) = \vec{\nabla}_\circ} = \left(\frac{\partial\vec{r}}{\partial r},\frac{1}{r}\frac{\partial\vec{r}}{\partial\theta},\frac{1}{r\sin\theta}\frac{\partial\vec{r}}{\partial\phi}\right)\bar{\nabla}_\circ^T$$

$$\vec{\nabla} = [\bar{\nabla}_\circ^T\otimes\vec{r}]\bar{\nabla}_\circ = \left[\bar{\nabla}_\circ^T\otimes(x,y,z)^T\right]\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial\theta}\\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\end{bmatrix} \quad\Rightarrow\quad \frac{\partial}{\partial x} = \underline{\frac{\partial x}{\partial r}}\frac{\partial}{\partial r} + \underline{\|\nabla\theta\|^2\frac{\partial x}{\partial\theta}}\frac{\partial}{\partial\theta} + \underline{\|\nabla\phi\|^2\frac{\partial x}{\partial\phi}}\frac{\partial}{\partial\phi}$$

$$= [\vec{\nabla}(r,\theta,\phi)]\bar{\partial}_\circ = [\vec{\nabla}(r,\theta,\phi)]\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\phi}\end{bmatrix} \quad\Rightarrow\quad \frac{\partial}{\partial x} = \underline{\frac{\partial r}{\partial x}}\frac{\partial}{\partial r} + \underline{\frac{\partial\theta}{\partial x}}\frac{\partial}{\partial\theta} + \underline{\frac{\partial\phi}{\partial x}}\frac{\partial}{\partial\phi}$$

$$\boxed{\begin{array}{ccccccccccccc}
\hat{r} & = & (\hat{r}_x,\hat{r}_y,\hat{r}_z) & = & \frac{\partial}{\partial r}\vec{r} & = & \boxed{\frac{\vec{r}}{r}} & = & \|\nabla r\|\frac{\partial\vec{r}}{\partial r} & = & \frac{\nabla r}{\|\nabla r\|} & = & \nabla r\\
\hat{\theta} & = & (\hat{\theta}_x,\hat{\theta}_y,\hat{\theta}_z) & = & \frac{1}{r}\frac{\partial}{\partial\theta}\vec{r} & = & \boxed{\frac{\partial\hat{r}}{\partial\theta}} & = & \|\nabla\theta\|\frac{\partial\vec{r}}{\partial\theta} & = & \frac{\nabla\theta}{\|\nabla\theta\|} & = & r\nabla\theta\\
\hat{\phi} & = & (\hat{\phi}_x,\hat{\phi}_y,\hat{\phi}_z) & = & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\vec{r} & = & \boxed{\frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi}} & = & \|\nabla\phi\|\frac{\partial\vec{r}}{\partial\phi} & = & \frac{\nabla\phi}{\|\nabla\phi\|} & = & r\sin\theta\nabla\phi
\end{array}}$$

1 Del

$$\nabla F = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)F$$

$$= \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \end{bmatrix} F = \begin{bmatrix} \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z} \\ \cos\phi\cos\theta\,\hat{x} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z} \\ -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \end{bmatrix} F$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F = \begin{bmatrix} \cos\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \end{bmatrix} F \;=\; \begin{bmatrix} \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} \\ \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi} \\ \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi} \end{bmatrix} F \;=\; \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F$$

$$\nabla\cdot\vec{A} \;=\; \frac{1}{r}\frac{1}{r\sin\theta}\left\langle\frac{\partial}{\partial r},\ \frac{\partial}{\partial\theta},\ \frac{\partial}{\partial\phi}\right\rangle \;\cdot\; [r\cdot r\sin\theta]\left\langle A_r,\ \frac{1}{r}A_\theta,\ \frac{1}{r\sin\theta}A_\phi\right\rangle$$

$$\nabla\times\vec{A} \;=\; \frac{1}{r}\frac{1}{r\sin\theta}\left\|\begin{matrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{matrix}\right\| = \left\|\begin{matrix} \frac{\partial\vec{r}}{\partial r} & \frac{\partial\vec{r}}{\partial\theta} & \frac{\partial\vec{r}}{\partial\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{matrix}\right\|$$

$$\begin{aligned} [\vec{A}\times(\vec{B}\times\vec{C})]_i &= \vec{A}\cdot(B_i\vec{C}) - \vec{A}\cdot(\vec{B}C_i) \\ \vec{A}\times(\vec{B}\times\vec{C}) &= \boxed{(\vec{A}\cdot(\vec{B}\otimes\vec{C})^T)^T - (\vec{A}\cdot\vec{B}\otimes\vec{C})^T} \quad = (\vec{A}\otimes\vec{B})\cdot\vec{C} - (\vec{A}\cdot\vec{B}\otimes\vec{C})^T \\ &= (A^T(BC^T)^T)^T - (A^TBC^T)^T \quad = (AB^T)^TC - (A^TBC^T)^T \end{aligned}$$