$$\boxed{\vec{\nabla} = \left[\vec{\nabla}\,(r,\theta,\phi)\right]\bar{\partial}_\circ}$$

$$d = [dx\ dy\ dz]\ \vec{\nabla} = d\vec{l}^{\,T}\,\vec{\nabla}$$

$$d(r,\theta,\phi) = [dx\ dy\ dz]\ \vec{\nabla}(r,\theta,\phi)$$

$$\boxed{\partial\bar{l}_\circ^{\,T} = d\vec{l}^{\,T}\,\vec{\nabla}(r,\theta,\phi)}$$

$$\partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ = d\vec{l}^{\,T}\left[\vec{\nabla}(r,\theta,\phi)\right]\bar{\partial}_\circ$$

$$\boxed{d = \partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ = d\vec{l}^{\,T}\,\vec{\nabla}}$$

$$\underline{\vec{\nabla}} = \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix} = \begin{bmatrix} | & | & | \\ \nabla r & \nabla\theta & \nabla\phi \\ | & | & | \end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\\ \frac{\partial}{\partial \phi}\end{bmatrix}$$

$$\underline{\partial\bar{l}_\circ} = \begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix} \begin{aligned} &= [\vec{\nabla}(r,\theta,\phi)]^T d\vec{l} \\ &= \begin{bmatrix} -\nabla r- \\ -\nabla\theta- \\ -\nabla\phi- \end{bmatrix}\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}\end{aligned}$$

$$\theta = \theta(x,y,z) \quad \left(x^2+y^2 = z^2\tan^2\theta\right)$$

$$\phi = \phi(x,y,z) \quad \left(y = x\tan\phi\right)$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z}$$

$$d\theta = dx\tfrac{\partial\theta}{\partial x} + dy\tfrac{\partial\theta}{\partial y} + dz\tfrac{\partial\theta}{\partial z}$$

$$dy_{\vec{r}_\circ}(\vec{r}_\circ{}')\big|_{t=0} = \left(\tfrac{\partial y}{\partial r}dr + \tfrac{\partial y}{\partial \theta}d\theta + \tfrac{\partial y}{\partial \phi}d\phi\right)\tfrac{1}{dt}\big|_{t=0}$$

$$\boxed{\vec{b}^{\,i}\cdot\vec{b}_i = \delta_{ij}} : \quad \begin{aligned} \vec{\nabla}\phi\cdot\tfrac{\partial\vec{r}}{\partial\phi} &= 1 \\ \vec{\nabla}\phi\cdot\tfrac{\partial\vec{r}}{\partial\theta} &= 0 \end{aligned} \quad\Rightarrow\quad \begin{bmatrix} -\vec{\nabla}r- \\ -\vec{\nabla}\theta- \\ -\vec{\nabla}\phi- \end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi}\end{bmatrix}\vec{r} = \mathbb{1}_3 \quad\Rightarrow\quad \boxed{\frac{\partial y}{\partial\phi} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix} -\vec{\nabla}r- \\ -\vec{\nabla}\theta- \\ -\vec{\nabla}\phi- \end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{\partial y}{\partial\phi}^T}$$

$$d = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} = d\vec{l}\cdot\vec{\nabla} \qquad = \left[\frac{dr}{\|\nabla r\|}, \frac{d\theta}{\|\nabla\theta\|}, \frac{d\phi}{\|\nabla\phi\|}\right]\left[\|\nabla r\|\frac{\partial}{\partial r}, \|\nabla\theta\|\frac{\partial}{\partial\theta}, \|\nabla\phi\|\frac{\partial}{\partial\phi}\right]^T$$

$$= dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial\theta} + d\phi\frac{\partial}{\partial\phi} = \partial\bar{l}_\circ^{\,T}\bar{\partial}_\circ \qquad = [dr, rd\theta, r\sin\theta d\phi]\left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right]^T$$

$$= \tfrac{dr}{\|\nabla r\|}\|\nabla r\|\frac{\partial}{\partial r} + \tfrac{d\theta}{\|\nabla\theta\|}\|\nabla\theta\|\frac{\partial}{\partial\theta} + \tfrac{d\phi}{\|\nabla\phi\|}\|\nabla\phi\|\frac{\partial}{\partial\phi} \qquad \cdots = d\bar{l}_\circ^{\,T}\bar{\nabla}_\circ = d\vec{l}_\circ^{\,T}\vec{\nabla}_\circ$$

$$d\vec{l} = d\vec{r} = [dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial\theta} + d\phi\frac{\partial}{\partial\phi}](x,y,z)^T$$

$$d(x,y,z) = \left[\tfrac{dr}{\|\nabla r\|}\|\nabla r\|\tfrac{\partial}{\partial r} + \tfrac{d\theta}{\|\nabla\theta\|}\|\nabla\theta\|\tfrac{\partial}{\partial\theta} + \tfrac{d\phi}{\|\nabla\phi\|}\|\nabla\phi\|\tfrac{\partial}{\partial\phi}\right](x,y,z)$$

$$(dx,dy,dz) = dr\,\hat{r}^T + r\,d\theta\,\hat{\theta}^T + r\sin\theta\,d\phi\,\hat{\phi}^T$$

$$(\hat{r},\hat{\theta},\hat{\phi}) \equiv \underline{\left(\|\nabla r\|\frac{\partial\vec{r}}{\partial r}, \|\nabla\theta\|\frac{\partial\vec{r}}{\partial\theta}, \|\nabla\phi\|\frac{\partial\vec{r}}{\partial\phi}\right)}$$

$$= \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\otimes(x,y,z)^T$$

$$= \bar{\nabla}_\circ^T\otimes\vec{r}$$

$$\boxed{d\vec{l} = (dx,dy,dz)\cdot(\hat{x},\hat{y},\hat{z})} = \boxed{(dr, rd\theta, r\sin\theta d\theta)\cdot(\hat{r},\hat{\theta},\hat{\phi}) = d\vec{l}_\circ} = d\bar{l}_\circ^{\,T}\cdot(\hat{r},\hat{\theta},\hat{\phi})$$

$$\boxed{\vec{\nabla} = (\hat{x},\hat{y},\hat{z})\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)} = \boxed{(\hat{r},\hat{\theta},\hat{\phi})\cdot\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) = \vec{\nabla}_\circ} = \left(\frac{\partial\vec{r}}{\partial r}, \frac{1}{r}\frac{\partial\vec{r}}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\vec{r}}{\partial\phi}\right)\bar{\nabla}_\circ^T$$

$$\vec{\nabla} = [\bar{\nabla}_\circ^T\otimes\vec{r}]\bar{\nabla}_\circ = \left[\bar{\nabla}_\circ^T\otimes(x,y,z)^T\right]\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial\theta}\\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\end{bmatrix} \quad\Rightarrow\quad \frac{\partial}{\partial x} = \underline{\frac{\partial x}{\partial r}}\frac{\partial}{\partial r} + \underline{\|\nabla\theta\|^2\frac{\partial x}{\partial\theta}}\frac{\partial}{\partial\theta} + \underline{\|\nabla\phi\|^2\frac{\partial x}{\partial\phi}}\frac{\partial}{\partial\phi}$$

$$= [\vec{\nabla}(r,\theta,\phi)]\bar{\partial}_\circ = [\vec{\nabla}(r,\theta,\phi)]\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\phi}\end{bmatrix} \quad\Rightarrow\quad \frac{\partial}{\partial x} = \underline{\frac{\partial r}{\partial x}}\frac{\partial}{\partial r} + \underline{\frac{\partial\theta}{\partial x}}\frac{\partial}{\partial\theta} + \underline{\frac{\partial\phi}{\partial x}}\frac{\partial}{\partial\phi} \quad\Rightarrow\quad \boxed{\frac{\partial\phi}{\partial y} = \frac{\partial y}{\partial\phi}\|\nabla\phi\|^2}$$

$$\boxed{\begin{array}{lllllllllllll}
 & & & & & & & \underline{\text{contravariant}}_i & \text{(equal since orthog.)} & & \underline{\text{covariant}}^i & & \\
\hat{r} & = & (\hat{r}_x,\hat{r}_y,\hat{r}_z) & = & \frac{\vec{r}}{r} & = & \frac{\partial}{\partial r}\vec{r} & = \frac{\partial\vec{r}}{\partial r}\left\|\frac{\partial\vec{r}}{\partial r}\right\|^{-1} & \overset{\rightarrow}{\equiv} & \|\nabla r\|\frac{\partial\vec{r}}{\partial r} & \overset{\leftarrow}{\equiv}\ \frac{\nabla r}{\|\nabla r\|} & = & \nabla r \\
\hat{\theta} & = & (\hat{\theta}_x,\hat{\theta}_y,\hat{\theta}_z) & = & \frac{\partial\hat{r}}{\partial\theta} & = & \frac{1}{r}\frac{\partial}{\partial\theta}\vec{r} & = \frac{\partial\vec{r}}{\partial\theta}\left\|\frac{\partial\vec{r}}{\partial\theta}\right\|^{-1} & \overset{\rightarrow}{\equiv} & \|\nabla\theta\|\frac{\partial\vec{r}}{\partial\theta} & \overset{\leftarrow}{\equiv}\ \frac{\nabla\theta}{\|\nabla\theta\|} & = & r\nabla\theta \\
\hat{\phi} & = & (\hat{\phi}_x,\hat{\phi}_y,\hat{\phi}_z) & = & \frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\phi} & = & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\vec{r} & = \frac{\partial\vec{r}}{\partial\phi}\left\|\frac{\partial\vec{r}}{\partial\phi}\right\|^{-1} & \overset{\rightarrow}{\equiv} & \|\nabla\phi\|\frac{\partial\vec{r}}{\partial\phi} & \overset{\leftarrow}{\equiv}\ \frac{\nabla\phi}{\|\nabla\phi\|} & = & r\sin\theta\nabla\phi
\end{array}}$$

# 1 Del

$$\nabla F = \left( \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \right) F$$

$$= \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \end{bmatrix} F = \begin{bmatrix} \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z} \\ \cos\phi\cos\theta\,\hat{x} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z} \\ -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \end{bmatrix} F$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F = \begin{bmatrix} \cos\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \end{bmatrix} F \;=\; \begin{bmatrix} \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} \\ \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi} \\ \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi} \end{bmatrix} F \;=\; \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} F$$

$$\nabla \cdot \vec{A} \;=\; \frac{1}{r}\frac{1}{r\sin\theta}\left\langle \frac{\partial}{\partial r},\; \frac{\partial}{\partial\theta},\; \frac{\partial}{\partial\phi} \right\rangle \;\cdot\; [r\cdot r\sin\theta]\left\langle A_r,\; \frac{1}{r}A_\theta,\; \frac{1}{r\sin\theta}A_\phi \right\rangle$$

$$\nabla \times \vec{A} \;=\; \frac{1}{r}\frac{1}{r\sin\theta}\begin{Vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{Vmatrix} = \begin{Vmatrix} \frac{\partial\vec{r}}{\partial r} & \frac{\partial\vec{r}}{\partial\theta} & \frac{\partial\vec{r}}{\partial\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{Vmatrix}$$

$$\begin{aligned}
[\vec{A}\times(\vec{B}\times\vec{C})]_i &= \vec{A}\cdot B_i\vec{C} - \vec{A}\cdot\vec{B}C_i \\
[\vec{A}\times(\vec{B}\times\vec{C})]^T &= \vec{A}_r * \vec{B}_r\vec{C}_c - \vec{A}_r * \vec{B}_c\vec{C}_r
\end{aligned}$$

$$\begin{aligned}
\vec{A}\times(\vec{B}\times\vec{C}) &= B(A\cdot C) - (A\cdot B)C \\
\vec{\nabla}\times(\vec{\nabla}\times\vec{C}) &= \nabla(\nabla\cdot C) - (\nabla\cdot\nabla)C \\
[\vec{A}\times(\vec{\nabla}\times\vec{C})]^T &= \vec{A}_r(\vec{\nabla}_r\vec{C}_c) - (\vec{A}\cdot\vec{\nabla})\vec{C}^T \\
[\vec{\nabla}\times(\vec{B}\times\vec{C})]_i &= \vec{\nabla}\cdot B_i\vec{C} - \vec{\nabla}\cdot\vec{B}C_i &&= \vec{C}\cdot\vec{\nabla}B_i + \vec{\nabla}\cdot\vec{C}B_i - \vec{\nabla}\cdot\vec{B}C_i - \vec{B}\cdot\vec{\nabla}C_i \\
[\vec{\nabla}\times(\vec{B}\times\vec{C})]^T &= \vec{\nabla}_r * (\vec{B}_r\vec{C}_c) - \vec{\nabla}_r * (\vec{B}_c\vec{C}_r) &&= \vec{C}_r * \vec{\nabla}_c\vec{B}_r + \vec{\nabla}_r * \vec{C}_c\vec{B}_r - \vec{\nabla}_r * \vec{B}_c\vec{C}_r - \vec{B}_r * \vec{\nabla}_c\vec{C}_r
\end{aligned}$$

$$\begin{aligned}
[(\vec{A}\times\vec{B})\times\vec{C}\,]_i &= \vec{A}B_i\cdot\vec{C} - A_i\vec{B}\cdot\vec{C} \\
(\vec{A}\times\vec{B})\times\vec{C} &= \vec{A}_r\vec{B}_c * \vec{C}_c - \vec{A}_c\vec{B}_r * \vec{C}_c \\
(\vec{A}\times\vec{B})\times\vec{C} &= (A\cdot C)B - A(B\cdot C) \\
(\vec{A}\times\vec{\nabla})\times\vec{C} &= (\vec{A}_r\vec{\nabla}_c)\vec{C}_c - \vec{A}_c(\vec{\nabla}_r\cdot\vec{C}_c) \\
(\vec{\nabla}\times\vec{B})\times\vec{C} &= (\vec{\nabla}_r\vec{B}_c)\vec{C} - (\vec{\nabla}_c\vec{B}_r)\vec{C}
\end{aligned}$$

# 2 Frenet Equations

$$a \cdot (b \times c) = (a \times b) \cdot c$$

$$a \times (b \times c) = (c \cdot a)b - (b \cdot a)c$$

$$(a \times b) \times c = b(c \cdot a) - a(c \cdot b)$$

$$(a \times b) \cdot (c \times d) = a \cdot b \times (c \times d)$$

$$= \left| \begin{bmatrix} a\cdot \\ b\cdot \end{bmatrix} [c\ d] \right| = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

$$\boxed{\frac{dt}{ds} = \frac{1}{v}}$$

$$T = \hat{v} = \frac{\vec{v}}{v}$$

$$\frac{dT}{dt} = \frac{(\vec{v}\cdot\vec{v})\vec{a} - (\vec{v}\cdot\vec{a})\vec{v}}{v^3} = \frac{\vec{v}\times(\vec{a}\times\vec{v})}{v^3} = \frac{(\vec{v}\times\vec{a})\times\vec{v}}{v^3}$$

$$\left\|\frac{dT}{dt}\right\| = \frac{\sqrt{v^2a^2 - (\vec{v}\cdot\vec{a})^2}}{v^2} = \frac{\|\vec{a}\times\vec{v}\|}{v^2}, \quad \frac{dT}{ds} = k\hat{N}$$

$$\hat{N} = \frac{T'}{\|T'\|} = \frac{(\vec{v}\times\vec{a})\times\vec{v}}{\|\vec{v}\times\vec{a}\|v} = \hat{B}\times\hat{v}$$

$$\hat{B} = \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} = \widehat{v\times a} = \hat{v}\times\hat{N} \qquad \underline{(\hat{B}\cdot\vec{v} = 0)}$$

$$\frac{d\hat{B}}{dt} = \frac{\vec{v}\times\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|} - \left[\frac{\vec{v}\times\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|}\cdot\hat{B}\right]\hat{B}, \quad \frac{dB}{ds} = \tau\hat{N}$$

$$\tau = \hat{N}\cdot\frac{d\hat{B}}{ds} = \frac{\hat{B}\cdot\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|} = \frac{(\vec{v}\times\vec{a})\cdot\dot{\vec{a}}}{\|\vec{v}\times\vec{a}\|^2}$$

$$\boxed{\begin{aligned} \vec{a} &= a_T\hat{T} + a_N\hat{N} \\ a_T &= \vec{a}\cdot\hat{v} = \tfrac{dv}{dt} \\ a_N &= \tfrac{\|\vec{a}\times\vec{v}\|}{v} = \|\vec{a}\times\hat{v}\| \\ a^2 &= a_T^2 + a_N^2 = \|\tfrac{d\vec{v}}{dt}\|^2 \end{aligned}}$$

<u>Frenet Trihedron for Regular *Parametrized* Curves</u>

Differentiable (in this book) : $C^\infty$

No singular pts. Order 0 (Regular) : $\vec{v}(t) \neq 0$

- $\|\vec{v}(t)\| = c \to 1 \ \Rightarrow \int_s \|\vec{v}(t)\|\,dt = t = \Delta s$

  $\to s: \ \vec{x}(t) = \vec{x}(s)$

- $\frac{1}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \boxed{\vec{v}\cdot\vec{a} = 0}$

No singular pts. Order 1 : $\vec{a}(t) \neq 0$

- Curvature, $k \neq 0$ (see right) • Vertex, $k' = 0$

$1 = \|\vec{t}\| = \|\vec{n}\| = \|\vec{b}\|, \quad 0 = \vec{t}\cdot\vec{n} = \vec{n}\cdot\vec{b} = \vec{b}\cdot\vec{t}$

- $\vec{v}(s) = \vec{t}(s)$ $\quad \boxed{(t = n\times b)}$
- $\vec{a}(s) = \boxed{\vec{t}'(s) = k(s)\vec{n}(s)}$, $k(s) \geq 0$ (can be L or R-handed) (can be neg. if in $\mathbb{R}^2$)

$*$ $k(s) > 0$ for well defined curve with $\hat{n}$

- $\boxed{\vec{b} = \vec{t}\times\vec{n}}$, $\frac{d}{dt}(\vec{b}\cdot\vec{b}) = \vec{b}\cdot\vec{b}' = 0$, $*$ $\boxed{\vec{b}'(s) = \tau(s)\vec{n}(s)}$
- $\boxed{\vec{n} = \vec{b}\times\vec{t}}$, $*$ $\boxed{\vec{n}'(s) = -k\vec{t} - \tau\vec{b}}$, $*$ t-n pl. = osculating pl.

- $t''(s) = k'n - k^2t - k\tau b$ • $b''(s) = \tau'n - \tau kt - \tau^2 b$ • $n''(s) = -k't - \tau'b - (k^2+\tau^2)n$
- $|\tau| = \|b'\|$ • $\tau = -\frac{(t\times t')\cdot t''}{k^2} = -\frac{t\cdot(t'\times t'')}{\|t'\|^2}$ • $k = \|t'\| = \frac{(b\times b')\cdot b''}{\tau^2} = \frac{b\cdot(b'\times b'')}{\|b'\|^2}$
- $n \Rightarrow k, \tau:$ $\quad * \ \|n'\|^2 = k^2 + \tau^2$ $\quad * \ \frac{(n\times n')\cdot n''}{\|n'\|^2} = \frac{k'\tau - k\tau'}{k^2+\tau^2} = \frac{\frac{d}{ds}(k/\tau)}{(k/\tau)^2+1} = \frac{d}{ds}\arctan(k/\tau)$

<u>Indicatrix of Tangents, $\vec{t}(\theta(s))$</u> :

- $\vec{t}(\theta(s)) = (\cos\theta, \sin\theta) = (x'(s), y'(s))$
- $\vec{t}'(\theta) = \underline{\theta'(s)}(-\sin\theta, \cos\theta) = \underline{k(s)}\vec{n}$
- $\theta(s) = \arctan(y'/x')$
- $\int_0^l k(s)\,ds = \theta(s)\Big|_0^l = 2\pi I_{\text{rot. index}}$
- $k(s) = \lim_{s\to 0}\frac{r\theta(s)}{s}\Big|_{r=1}$ (See Gaussian $K$)

<u>Local Canonical Form at $t = 0$</u> :

- $(\hat{t}, \hat{n}, \hat{b}) = (\hat{x}, \hat{y}, \hat{z})$
- $\vec{r}(s) - \vec{r}(0) \approx (s - \frac{k^2s^3}{6}, \ \frac{k}{2}s^2 + \frac{k's^3}{6}, \ \frac{-k\tau}{6}s^3)$
- $\tau < 0 \ \Rightarrow \ \frac{dz}{ds} > 0$

<u>Isoperimetric Inequality</u> : $0 \leq l^2 - 4\pi A$

<u>Four-Vertex Theorem</u> : A simple closed curve has $\geq 4$ vertices

<u>Cauchy-Crofton Formula (measure of number of times lines intersect a curve)</u> :

- Tangent line at $(\rho, \theta)$ : $x\cos\theta + y\sin\theta = \rho$ • Curve $c$ : $y = 0, \ x \in (-l/2, l/2)$ , $C = \sum c_i$
- $\int$ Lines that cross $c = \int_0^{2\pi}\int_0^{|\cos\theta| l/2} d\rho\, d\theta = 2l \ \Rightarrow \ \int_0^{2\pi}\int_0^{\infty} n_C \ d\rho\, d\theta = 2l$

# 3 Jacobian/Differential, $dF_{\alpha(0)} : \underline{\mathbb{R}^n \to \mathbb{R}^m}$

- $\boxed{\alpha(0) = \beta(0)} \;\Rightarrow\; \underline{F(t=0)} = F \circ \alpha\big|_{t=0} = F \circ \beta\big|_{t=0}$
- $\boxed{\alpha'(0) = \beta'(0)} \;\Rightarrow \frac{\partial x}{\partial \alpha_i}\Big|_{t=0} = \frac{\partial x}{\partial \beta_i}\Big|_{t=0} \cdot \cancel{\frac{d\beta_i/dt}{d\alpha_i/dt}\Big|_{t=0}} \;\Rightarrow\; \boxed{dF_{\alpha(0)}(\alpha'(0)) = dF_{\beta(0)}(\beta'(0))}$ <u>(doesn't depend on $\alpha$)</u>

$*$ $F = (f_0, f_1, \ldots, f_m) \;\Rightarrow\; \underline{dF_{\alpha(0)}(\alpha'(0))} \equiv \frac{d}{dt}(F \circ \alpha)\big|_{t=0} = \begin{bmatrix} \frac{\partial f_0}{\partial \alpha_0} & | & \cdots \\ \frac{\partial f_1}{\partial \alpha_0} & F_{\alpha_1} & \cdots \\ \vdots & | & \end{bmatrix}_{t=0} \begin{bmatrix} \frac{d\alpha_0}{dt} \\ \frac{d\alpha_1}{dt} \\ \vdots \end{bmatrix}_{t=0} = \boxed{J_{F(0)} \cdot \alpha'(0)}$

$*$ Surface Tangent (see below) : $q = \gamma(t=0) = (u(0), v(0)) = X^{-1} \circ \alpha(0)$

$X(q) = X \circ \gamma(0) = \alpha(0) \in S \;\Rightarrow\; dX_q(\gamma'(0)) = \alpha'(0)$

- $d(G \circ F)_p = dG_{F(p)} \circ dF_p$
- <u>Regular **Value**, $F(p)$</u> : $dF_p \neq 0$
- <u>Critical **Point**, $p$</u> : $dF_p = 0$

$F$ is a <u>Homeomorphism</u> onto image $F(X)$ :
- $F$ is bijective between $X$ & $F(X)$
- $F$ is cont.
- $F^{-1}$ is cont.

$F$ is a <u>Diffeomorphism</u> onto image $F(X)$ :
- $F \in C^\infty$ (cont. part. deri. of all orders)
- $F^{-1} \in C^\infty$
- $F$ is a bijection

<u>Inverse Function Theorem</u> (IFT) :
- $F : \underline{\mathbb{R}^n \to \mathbb{R}^n}$, $F \in C^\infty$
- $\exists dF_p^{-1}$ (sq. matrix $dF_p$ is an isomorphism/non-zero det.)

$\Rightarrow \exists F^{-1} \in C^\infty$ (locally at F(p))

# 4 Surfaces, $S$ : $X(q) = X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big) = p \in S \subset \mathbb{R}^3$

<u>Regular Parametrized Surface</u>

$-\; \forall p \in S,\ \underline{\exists X \in C^\infty},\ X : V_q$ (neighborhood of q) $\to V_p \cap S$ (diff. parametrizations are possible, btw)

$-\; dX_q$ is one-to-one = (maybe non sq.) matrix col. are lin. ind. = any 2x2 $|\text{sub-}J_X| \neq 0 \;\Rightarrow\; \exists$(tangent at all points)

<u>Regular Surface (is reg. param. surface)</u>

$-$ $X$ is a homeo. in $V_q$ (or $X$ is one-to-one) $\to$ $\underline{X^{-1} \in C^0}$ (is cont.) $\forall p \in S,\ X^{-1}(V_p) = V_q$ $\Rightarrow$ $\exists$ no self-intersections; cont. = doesn't depend on parametrization (see coor. change below)

- Coordinate Change, $h$, between Two Param. <u>is a Diffeomorphism (need for diff. func. on $S$)</u> :

$*$ $X^{-1}$ is a homeomorphism $\to$ <u>$h = X^{-1} \circ Y$ is is a homeomorphism from $Y$ to $X$</u> $\Rightarrow\Rightarrow$ <u>$h^{-1}$ is a homomorphism</u>

$*$ $p \in S$ , $p = Y(\epsilon, \eta) = X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$ , $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ (can change axes to make this true)

$F(u,v,t) = \big(x(u,v), y(u,v), z(u,v) + t\big)$ : $F(u,v,t)$, $X(u,v) \in C^\infty$ , $\exists dF^{-1} \overset{(IFT)}{\Rightarrow} F^{-1} \in C^\infty$

$(u,v) = X^{-1} \circ Y(\epsilon,\eta) = h(\epsilon,\eta) \cong \big(F^{-1} \circ Y\big)(\epsilon,\eta) \;\Rightarrow\; \underline{h \in C^\infty} \;\Rightarrow\Rightarrow\; \underline{h^{-1} \in C^\infty}$ (same for $Y^{-1} \circ X$)

$*$ Needed that $X^{-1} \in C^0$ **on a [3D] neigh. for every point** $[\forall p \in S,\ X^{-1}(V_p) = V_q \cong F^{-1}(V_p)]$ , to avoid $(t \neq 0,\ F^{-1} \circ Y \neq h)$

$*$ Ex: $\gamma(t) = (\cos t, \sin 2t)$, $\gamma(\mathbb{R}) = \alpha(I_1) = \beta(I_2)$ , ($\infty$ - graph <u>not reg.</u>)

$I_1 = (-\frac{\pi}{2}, \frac{3\pi}{2}) = (-\frac{\pi}{2}, \frac{\pi}{2}) \cup \frac{\pi}{2} \cup (\frac{\pi}{2}, \frac{3\pi}{2})$

$I_2 = (\frac{\pi}{2}, \frac{5\pi}{2}) = \underline{(\frac{3\pi}{2}, \frac{5\pi}{2}) \cup \frac{3\pi}{2} \cup (\frac{\pi}{2}, \frac{3\pi}{2})}$

$\Rightarrow$ $\beta^{-1}$ is 1:1 but not cont. for any **[2D] neigh.** of $(0,0)$

$F^{-1}(x,y) = (t', u) \neq \beta^{-1}(x,y) \cong (t, 0)$ near $(0,0)$

<u>$\beta^{-1} \circ \alpha(I_1)$</u> is 1:1 but not cont., so not diffeo.

- $\underline{f \in C^\infty} \;\Rightarrow\; \boxed{\big(\vec{x}, f(\vec{x})\big) \text{ is a reg. surf.}}$

- $\underline{\begin{matrix} f : \mathbb{R}^n \to \mathbb{R} \\ f(X) = c \\ \text{is a reg. val.}\end{matrix}} \;,\; \begin{matrix} f \in C^\infty \\ F(X) = \big(x_1, \dots, x_{n-1}, f(X)\big) \\ \exists dF_p^{-1} \end{matrix} \;\overset{(\text{IFT})}{\Rightarrow}\; \begin{matrix} \exists F^{-1} \in C^\infty \\ F^{-1}\big(f_1, \dots, f_{n-1}, f(\vec{x})\big) = X \end{matrix} \;,\; \begin{matrix} x_n = f_n^{-1} : \mathbb{R}^n \to \mathbb{R} \\ \underline{x_n = f_n^{-1} \in C^\infty} \end{matrix}$

$\rightarrow \begin{matrix} x_n = f_n^{-1}\big(x_1, \dots, x_{n-1}, f(\vec{x}) = c\big) \\ = \underline{f'^{-1}_n\big(x_1, \dots, x_{n-1}\big)} \end{matrix} \;\Rightarrow\; \begin{matrix} S = \underline{\big(x_1, \dots, x_{n-1}, f'^{-1}_n\big)} \text{ where } f(\vec{x}) = c \\ S == \text{ Surface } f^{-1}(c) \end{matrix} \;\Rightarrow\; \boxed{\begin{matrix} \underline{\text{Regular Value Theorem}} \\ \text{Surface } f^{-1}(c) \text{ is reg.} \end{matrix}}$

- $\underline{\frac{\partial(x,y)}{\partial(u,v)} \neq 0} \;\Rightarrow\; \pi_{\text{proj.}} \circ X(u,v) \equiv \big(x(u,v), y(u,v), \cancel{z(x,y)}\big) \;\overset{(IFT)}{\Rightarrow}\; \big(\pi \circ X\big)^{-1}(x,y) = \big(u(x,y), v(x,y)\big)$

  * $X(u,v) = \big(x(u,v), y(u,v), \underline{z(u,v)}\big) \;\Rightarrow\; z\big(u(x,y), v(x,y)\big) = z \circ \big(\pi \circ X\big)^{-1}(x,y) = \boxed{\begin{matrix} \underline{\text{Implicit Func. Theor.}} \\ \text{(locally orientable)} \\ f(x,y) = z \in C^\infty \end{matrix}}$

  * $\frac{\text{Know } S \text{ is reg. sur.}}{X \text{ is param?}} ,\; \begin{matrix} X \in C^\infty \\ dX_q \text{ is 1:1} \end{matrix} ,\; \underline{X \text{ is 1:1}} \;\Rightarrow\; \underline{\big(\pi \circ X\big)^{-1} \circ \pi} \circ X(u,v) = \underline{X^{-1}} \circ X(u,v) \;\Rightarrow\; \boxed{X^{-1} \in C^0}$

- $\underline{\text{Surface}}$ $\underline{\text{Tangent}}$ : $\begin{matrix} q = \gamma(t=0) = \big(u(0), v(0)\big) = X^{-1} \circ \alpha(0) \\ X(q) = X \circ \gamma(0) = \alpha(0) \in S \;\Rightarrow\; dX_q\big(\gamma'(0)\big) = \alpha'(0) = \frac{\partial X}{\partial u}(q)\, u'(0) + \frac{\partial X}{\partial v}(q)\, v'(0) \end{matrix}$

- "$\underline{\text{First Form}}$" : $\langle \alpha'(0), \alpha'(0) \rangle = \|\alpha'\|^2 = [u' \; v'] \begin{bmatrix} X_u \\ X_v \end{bmatrix} [X_u \; X_v] \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{matrix} \|X_u\|^2 (u')^2 + 2\,\langle X_u, X_v \rangle\, u'v' + \|X_v\|^2 (v')^2 \\ \boxed{E(u')^2 + 2Fu'v' + G(v')^2} \end{matrix}$

- $\underline{\text{Line}}$ $\underline{\text{Element}}$ : $ds = \|\alpha'(t)\| dt$
- $\underline{\text{Area}}$ $\underline{\text{Element}}$ : $dA = \|X_u \times X_v\| du\, dv = \sqrt{EG - F^2}\, du\, dv$

* Regular Curves, $C \in R^3$ (instead of Regular Parametrized Curves)

- $\forall p \in C,\; \exists \alpha \in C^\infty,\; \alpha : I_t$ (neighborhood of $t$) $\subset R \to V_p \cap C$ (neighborhood of $p$)
- $\forall t \in I$ , $d\alpha_t$ is 1:1
- $\alpha$ is a homeo. in $I_t$

  * Change of param. are homeomorphisms $\Rightarrow$ Properties like arc length, curvature, torsion, etc. aren't param. dependent

* Coordinate Curves : $\alpha(t) = X \circ \gamma(t) \;\big|\; \gamma \in \big\{\big(u(t), v_0\big),\; \big(u_0, v(t)\big)\big\}$ (maps of parallels and meridians)

Function, $f : S \subset \mathbb{R}^n \to \mathbb{R}$

- $\big(\forall p \in S,\; \underline{f(p) \neq 0}\big) \;\Rightarrow\; \big(\forall p \in S,\; \underline{f(p) > 0}\big)$ or $\big(\forall p \in S,\; \underline{f(p) < 0}\big)$
- *Differentiable on S* : $f \circ X \in C^\infty$ (doesn't depend on param./coord. change)
- E.g., $X^{-1}(p),\; \vec{v} \cdot p,\; |p - p_0|^2 \;\Rightarrow\; \boxed{X^{-1} \in C^\infty}$ , $\boxed{U \text{ is diffeo. to } X(U)}$

Function, $\phi : S_1 \to S_2$ is a Diffeomorphism from $S_1$ to $S_2$

- *Differentiable* : $X_2^{-1} \circ (\phi \circ X_1) \in C^\infty$ (doesn't depend on param./coord. change)
- $\beta'(0) = d\phi_p(w) = d\phi_p \alpha'(0) = d\phi_p dX_q\big(u'(0), v'(0)\big)^T$ (p.85???)
- Inverse Function Theorem : $\phi \in C^\infty,\; \exists d\phi_p^{-1} \;\Rightarrow\; \phi^{-1} \in C^\infty$ (Diffeomorphism from $S_1 \to S_2$??????)

# 5 Gauss Map (Normals), $N(p) = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{X_u \times X_v}{EG - F^2} : S \to S^2$

$N'(p) = \underline{dN_p \alpha'(0)} = \begin{bmatrix} (dN_p) \\ N_x\ N_y\ N_z \end{bmatrix} \begin{bmatrix} (dX_q) \\ X_u\ X_v \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \equiv \begin{bmatrix} N_u\ N_v \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}$

- $\boxed{S = f^{-1}(c) \Leftrightarrow \textit{Orientated}} = \underline{\text{normals } N(p) \text{ are in same dir. } (\pm 1)} = \boxed{\exists \frac{\partial(\hat u, \hat v)}{\partial(u,v)} > 0 \text{ over all } S}$

$\underline{\text{Second Fundamental [Quadratic] Form}}$ : $\langle -dN_p(\alpha'(0)), \alpha'(0) \rangle = \underline{\langle \alpha'(0), -dN_p(\alpha'(0)) \rangle}$ (is self-adjoint)

$*$ $\langle N(s), \alpha'(s) \rangle = 0 \Rightarrow \boxed{\langle N(s=0), \alpha''(0) \rangle} = -\langle N'(0), \alpha'(0) \rangle$ $=$ (depends on $\alpha'(0)$) $\underline{-\langle dN_p \alpha'(0), \alpha'(0) \rangle}$ $=$ ($\underline{\text{Normal Curvature}}$ of $\alpha$ at $p$) $\boxed{\langle N, kn \rangle(p) \equiv k_n(p)}$ $=$ $k$ of $\alpha$ from a normal (cross) $\underline{\text{section}}$ of $S$

$*$ $\underline{\langle -dN_p \alpha', \alpha' \rangle} = -[N_u\ N_v] \begin{bmatrix} u' \\ v' \end{bmatrix} [u'\ v'] \begin{bmatrix} X_u \\ X_v \end{bmatrix} = \underline{\Big( \underbrace{-\langle N_u, X_u \rangle}_{e}, \underbrace{-\langle N_u, X_v \rangle - \langle N_v, X_u \rangle}_{2f = 2\langle N_u, X_v \rangle}, \underbrace{-\langle N_v, X_v \rangle}_{g} \Big) \cdot \big( (u')^2, u'v', (v')^2 \big)}$

$$\boxed{\begin{aligned} k_n(p, \alpha') &= e(u')^2 + 2fu'v' + g(v')^2 \\ \text{(locally, } \leq 2 \text{ sol.)} &= (Au' + Bv')(Cu' + Dv') \end{aligned}}$$

- $\frac{(\underline{\text{Prin. dir.}} \text{ at } p)}{\underline{\text{Eigenbasis}}}$ : $\exists e_1, e_2 \,\big|\, span(e_1, e_2) = T_p(S) \ni \underline{-dN_p(xe_1 + ye_2) = k_1 x e_1 + k_2 y e_2}$ ($\underline{\text{Prin. curv.}}$ at $p$) (eigenvalues, $k_1 \geq k_2$)

$*$ $\underline{\text{Euler's Formula (for 2nd Form)}}$ : $\underline{\langle -dN_p \vec t, \vec t = e_1 \cos\theta + e_2 \sin\theta \rangle} = \boxed{k_1 \cos^2\theta + k_2 \sin^2\theta = k_n(p, \theta)}$

- $\underline{\text{Gaussian Curvature}}$ : ($2n$ dim. !$\Delta$ det. w/ orien. flip) $\boxed{\begin{aligned} K(p) &= \det(dN_p) \\ &= (-k_1)(-k_2) \end{aligned}}$
- $\underline{\text{Mean Curvature}}$ : $\boxed{H(p) = \frac{-\mathrm{Tr}(dN_p)}{2} = \frac{k_1 + k_2}{2}}$

$*$ Planar: $dN_p = 0$, Ellip.$\to K > 0$, Para.$\to K = 0$, ... $*$ $K > 0 \Rightarrow \exists V_p : p + T_p(S)$ !div. $V_p$ , $K < 0 \Rightarrow \forall V_p : p + T_p(S)$ div. $V_p$

$*$ $|Av \times Aw| = |v \times w| k_1 k_2$ (2D) $\underline{|dN_p X_u \times dN_p X_v| = |X_u \times X_v| \cdot K}$ $\Rightarrow K_{\neq 0} = \dfrac{\lim_{\int dudv \to 0} \int |dN_p X_u \times dN_p X_v| dudv \Big/ \int dudv}{\lim_{\int dudv \to 0} \int |X_u \times X_v| dudv \Big/ \int dudv}$ (See Indi. of Tan. $k$)

- $\underline{N_u, N_v \in T_p(S)} \Rightarrow dN_p \alpha'(0) = \underline{[N_u\ N_v]} \begin{bmatrix} u' \\ v' \end{bmatrix} \equiv \underline{[X_u\ X_v]} \Big[ dN \Big] \begin{bmatrix} u' \\ v' \end{bmatrix}$

$\underline{\text{General Basis}}$ for $N_u, N_v$ : $\begin{bmatrix} X_u \cdot N_u = -e & X_u \cdot N_v = -f \\ X_v \cdot N_u = -f & X_v \cdot N_v = -g \end{bmatrix} = \begin{bmatrix} X_u^2 = E & X_u \cdot X_v = F \\ X_v \cdot X_u = F & X_v^2 = G \end{bmatrix} \Big[ dN \Big]$ $\qquad \underline{\begin{aligned} \langle N, X_{ij} \rangle &= -\langle N_i, X_j \rangle \\ &= -\langle N_j, X_i \rangle \end{aligned}}$

$*$ (Weingarten Eq.) $\boxed{\Big[ dN \Big] = \frac{-1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix}}$ $*$ $(k^2 - 2Hk + K = 0)$ $\boxed{k_\pm = H \pm \sqrt{H^2 - K}}$ : $\boxed{K = \frac{eg - f^2}{EG - F^2}}$ , $\boxed{H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}}$

$\underline{\text{Umbilical Point}}$ : $p \in S \,\big|\, k_1 = k_2 \Rightarrow H^2 = K$ (only spheres & planes have all umb. pts.) $\underline{\text{Asymptotic Direction}}$ : $k_n(p, \theta) = 0$

$\underline{\text{Conjugate Directions}}$ : $(\theta, \phi)$ from $e_1$ $\big|$ $\langle -dN_p \vec t_1(\theta), \vec t_2(\phi) \rangle \equiv \boxed{0 = -k_1 \cos\theta \cos\phi - k_2 \sin\theta \sin\phi}$

$\underline{\text{Dupin Indicatrix}}$ : $\langle -dN_p \vec t, \vec t \rangle = \pm \frac{1}{\rho^2} = k_n \Rightarrow \langle -dN_p(\rho \vec t), (\rho \vec t) \rangle = k_1 \underline{\rho^2 \cos^2\theta} + k_2 \underline{\rho^2 \sin^2\theta}$

$\underline{\text{Conic Graph } (\xi, \eta)} = \boxed{k_1 \underline{\xi}^2 + k_2 \underline{\eta}^2 = \pm 1}$

- $K > 0 \Rightarrow \forall \theta,\ k_n(\theta) > 0$
- $K < 0 \Rightarrow \exists \theta_{\underline{1,2}} \,\big|\, k_n(\theta) = 0 = (k_1 \cos^2\theta + k_2 \sin^2\theta)\rho^2 = k_1 \xi^2 + k_2 \eta^2 \neq \pm 1$ ($\theta_{\underline{1,2}}$ are asymptotes of $(\xi, \eta)$)
- Conj. Dir. $(\phi_1, \phi_2)$ : $\phi_{2,1} = \arctan \frac{d\eta}{d\xi} \Big|_{(\xi, \eta) \cap \theta = \phi_{1,2}}$

$\underline{\text{Line of Curvature}}$ : $\alpha(t) \,\Big|\, N'(t) = \underline{dN_p\alpha'(t)} = \underline{\lambda(t)\alpha'(t)}$ (curve s.t. tangent is always in a princ. dir.)

- $\begin{bmatrix}u' & v'\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\Big[dN\Big]\begin{bmatrix}u'\\ v'\end{bmatrix} = \begin{bmatrix}-v' & u'\end{bmatrix}\lambda(t)\begin{bmatrix}u'\\ v'\end{bmatrix} = 0 \overset{\text{(expand)}}{\Rightarrow} \boxed{\begin{vmatrix}(v')^2 & -u'v' & (u')^2\\ e & f & g\\ E & F & G\end{vmatrix} = 0}$

$*$ $\underline{\text{Asymp. Curve}}$ : $\alpha(t) \,\Big|\, \lambda(t) = k_n(p,\theta) = \boxed{k_1\cos^2\theta + k_2\sin^2\theta = e(u')^2 + 2fu'v' + g(v')^2 = (Au'+Bv')(Cu'+Dv') = 0}$
$(ef-g)$, $\underline{K<0} \Rightarrow 0 = \underline{(Au'+Bv')(Au'+Dv')}$ : $A^2 = e,\ A(B+D) = f,\ BD = g \Rightarrow \boxed{\exists\alpha_1,\alpha_2}$

$*$ $\underline{e = g = 0} \Leftrightarrow \boxed{\alpha\circ(c, v(t))} \wedge \boxed{\alpha\circ(u(t), c)}$ are asympt. curves

$\underline{\text{Surface of Revolution}}$ : $X(u,v) = \big(\rho(v)\cos u,\ \rho(v)\sin u,\ z(v)\big) \,\Big|\, \alpha_u(v) = \big(z(v),\ \rho(v)\big)$ , $\|\alpha_u'\| = 1$

- $\langle\alpha',\alpha'\rangle = \underset{(\rho'\rho''+z'z''=0)}{\underline{[\rho^2,\ 0,\ (\rho')^2+(z')^2 = 1]}}\begin{bmatrix}(u')^2\\ 2u'v'\\ (v')^2\end{bmatrix}$ • $\langle N,\alpha''\rangle = \underline{\big[-\rho z',\ 0,\ \rho''z' - \rho'z''\big]}\begin{bmatrix}(u')^2\\ 2u'v'\\ (v')^2\end{bmatrix}$
- $\boxed{k_1 = \frac{e}{E} = -\frac{z'}{\rho}}$ , $\boxed{k_2 = \frac{g}{G} = \rho''z' - \rho'z''}$ , $\boxed{K = -\frac{z'(\rho''z'-\rho'z'')}{\rho} = -\frac{\rho''}{\rho}}$

$\underline{\text{Graph of a Differentiable Function}}$ : $X(u,v) = \big(u, v, z(u,v)\big)$ • $N(p) = \frac{(-z_u,-z_v,1)}{\sqrt{z_u^2+z_v^2+1}}$

- $\langle\alpha',\alpha'\rangle = \underline{\big[1+z_u^2,\ z_uz_v,\ 1+z_v^2\big]}\begin{bmatrix}(u')^2\\ 2u'v'\\ (v')^2\end{bmatrix}$ • $\langle N,\alpha''\rangle = \underline{\frac{1}{\sqrt{z_y^2+z_v^2+1}}\big[z_{uu},\ z_{uv},\ z_{vv}\big]}\begin{bmatrix}(u')^2\\ 2u'v'\\ (v')^2\end{bmatrix}$
- $z(0,0) = p$ , $N(p) = (0,0,1) \Rightarrow$ $\underline{\text{Hessian}}$ : $k_n(p) = \underline{\big[z_{xx},\ z_{xy},\ z_{yy}\big]}\begin{bmatrix}x^2\\ 2xy\\ y^2\end{bmatrix}$ , $\vec{v} = (x,y)$

$*$ $\vec{v} = xe_1 + ye_2 \Rightarrow \underset{(p\text{ is non-planer!!})}{z(x,y) - z(0,0)} = \frac{1}{2}k_n(p) + \mathcal{O}(r^3) \approx \frac{1}{2}(z_{xx}x^2 + x_{yy}y^2) = \epsilon \rightarrow k_1\chi^2 + k_2\eta^2 = \pm 1$
$k_1x^2 + k_2y^2 = 2\epsilon$ $\boxed{\text{(Dupin Indicatrix)}}$

$\underline{\text{(Diff.) Vector Field over } S}$ : $\boxed{w(p) = a(u,v)X_u + b(u,v)X_v}$ (e.g. $\underline{\gamma(t) \rightarrow w_\gamma(p) = u'X_u + v'X_v}$)

$\underline{\textit{Trajectory of } w}$ : $\alpha(t) \subset S \,\Big|\, \boxed{\alpha(0) = p,\ \alpha'(t) = w(\alpha(t))}$

$\underline{\textit{(Local) Flow of } w}$ : $\alpha(p,t) \equiv \alpha_p(t) \,\Big|\, \boxed{\alpha_p(0) = p}$, $\boxed{\alpha_p'(t) = w(\alpha_p(t))} \Rightarrow \boxed{\alpha_p(t) = p + (a_0^1(t), a_0^2(t), a_0^3(t))}$

- $\boxed{w(p_0) \neq 0}$, $w(p_0)\cdot\hat{x} = |w|$, $\tilde{\alpha}_{p_0}(t) = \alpha_{p_0}(t)\big|_{x=x_0}$ $\Rightarrow$ $d\alpha_{p_0} = \big[\mathbb{1}_3\ w(\alpha)\big]$, $d\tilde{\alpha}_{p_0} = \big[\mathbb{1}_3\ w(\alpha)\big]\begin{bmatrix}0\\ \mathbb{1}_3\end{bmatrix} = \begin{bmatrix}e_2\ e_3\ w(\alpha)\end{bmatrix}$ $\Rightarrow$ $\det(d\tilde{\alpha}_{p_0}) = w(p_0) \neq 0$, $\boxed{\exists\tilde{\alpha}^{-1} :\ V_{\alpha(p_0)} \subset S \rightarrow V_{p_0}\big|_{x=x_0}}$ (IFT), $\forall p \in \alpha_{p_0}(t),\ \underline{g(p)} \equiv \pi_t\circ\tilde{\alpha}_{p_0}^{-1}(p) = p_0$

$\underline{\textit{(Local) First Integral of } w}$ : $f(p) \,\Big|\, \forall p \in \alpha_{p_0}(t)$, $\boxed{f(p) = c}$, $\boxed{df_p \neq 0}$ $\Big(\begin{matrix} f(p) = \text{arcdist}(p_0, \underline{g(p)})\\ \text{along } S|_{x=x_0}\end{matrix}\Big)$

- $\boxed{w(p_0)} \neq 0$ (see above) $\Rightarrow \big(\exists V_{p_0} \subset S\big)\big(\forall p \in V_{p_0},\ \underline{\exists f(p)}\big)$
- $w_1(p_0) \underline{\neq} Aw_2(p_0)$, $\phi(p_0) = \begin{bmatrix}f_1(p_0) = u_0\\ f_2(p_0) = v_0\end{bmatrix} \Rightarrow \big[d\phi_p\big]\big[w_1(p_0)\ w_2(p_0)\big] = \begin{bmatrix}0 & a\\ b & 0\end{bmatrix} \underline{\neq} 0 \overset{\text{(IFT)}}{\Rightarrow}$ $\exists\underline{X} : \underline{\phi^{-1}}(u_0,v_0) = p_0$, $\underline{*}\ \boxed{\begin{matrix}X(u_0,v) \in \alpha_1\\ X(u,v_0) \in \alpha_2\end{matrix}}$
- $w_1 \equiv X_u,\ w_2 \equiv -\frac{X_u\cdot X_v}{X_u\cdot X_u}X_u + X_v \Rightarrow \underline{\exists(w_1,w_2)\big(w_2(p_0)\cdot w_2(p_0) = 0,\ \underline{\exists *}\big)}$
- $\underline{K<0} \Rightarrow \underline{\exists\alpha_1,\alpha_2(k_n = 0)} \rightarrow \underline{\exists(w_1,w_2)(\exists *)}$ $*$ $k_1 \neq k_2,\ \exists(\alpha_1,\alpha_2)_{k_n} \rightarrow \underline{\exists(w_1,w_2)(\exists *)}$

$\underline{\textit{Direction/Ray/Line Field}}$: $\boxed{r_w = c_{\neq 0}\big(b_{(u,v)}, -a_{(u,v)}\big)} \rightarrow \frac{y'}{x'} = \frac{-a}{b}$

$\underline{\textit{Orthogonal Field to } r}$: $\boxed{\overline{r}_w \equiv r_{\overline{w}}:\ \overline{w} \cdot w = (\overline{a}X_u + \overline{b}X_v) \cdot (aX_u + bX_v) = 0}$

E.g. : $\begin{array}{c} X_{(q)} = (u, v, u^2 - v^2) \\ \gamma_{(t)} : u^2 - v^2 = c \rightarrow \frac{v'}{u'} = \frac{-u}{v} \end{array} \Rightarrow \begin{array}{c} w_\gamma = u'_{(t)} X_u + v'_{(t)} X_v \overset{\rightrightarrows}{=} vX_u - uX_v \\ \overline{w}_\gamma \cdot w_\gamma = \overline{a}v - \overline{b}u = \underline{u'(\bar{t})v - v'(\bar{t})u = 0} \end{array} \Rightarrow \begin{array}{c} \overline{\gamma}_{(\bar{t})} : \underline{u(\bar{t})v(\bar{t}) = c} \\ X_c = (u, \frac{c}{u}, u^2 - \frac{c^2}{u^2}) \end{array}$