$$\lim_{n\to\infty}(1+\tfrac{x}{n})^n = \lim_{n\to\infty}\sum_{i=0}^{n}\binom{n}{i}\left(\tfrac{x}{n}\right)^i = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{n!/(n-i)!}{n^i}\frac{x^i}{i!} = \sum_{i=0}^{\infty}\frac{x^i}{i!} = f(x) : \begin{matrix} f(0)=1 \\ f(1)\equiv e\end{matrix}$$

- $x<y \;\Rightarrow\; \underline{0<f(x)<f(y)<\infty}$ • $f(x)=f(y) \;\Rightarrow\; x=y$

- $f(x+y) = \sum_{n=0}^{\infty}\frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\sum_{i=0}^{n}\frac{n!}{i!(n-i)!}x^i y^{n-i}\right] = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{x^i}{i!}\frac{y^{n-i}}{(n-i)!} = \sum_{i=0}^{\infty}\frac{x^i}{i!}\left[\sum_{j=0}^{\infty}\frac{y^j}{j!}\right] = f(x)f(y)$

$$\lim_{n\to\infty}(1+\tfrac{x}{n})^{kn} = \lim_{n\to\infty}\sum_{i=0}^{n}\left\{\frac{(kn)!}{(kn-i)!}\right\}\frac{1}{i!}\left(\frac{x}{n}\right)^i \quad = f(kx) = (f(x))^k : \begin{matrix} f(0)=f(0)^k=1^k \\ f(k)=f(1)^k=e^k \\ f(k)^n=e^{kn}=(e^k)^n \end{matrix}$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n}\left\{\frac{(kn)!}{(kn-i)!}\frac{1}{(kn)^i}\right\}\frac{(kx)^i}{i!}$$

$$\boxed{e^x \equiv f(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}}$$

- $e^{x+y}=e^x e^y$
- $\underline{e^{xk}=(e^x)^k}$

$$\left[\sum_{i=0}^{\infty}\frac{(-kx)^i}{i!}\right]e^{kx} = 1+\sum_{i=1}^{\infty}\frac{(-kx)^i}{i!}\sum_{j=1}^{\infty}\frac{(kx)^j}{j!}$$

$$\left[\lim_{n\to\infty}(1+\tfrac{x}{n})^{-kn}\right](e^x)^k \overset{\Leftarrow}{=} 1+\sum_{j=1}^{\infty}\sum_{i=0}^{j}\frac{(-kx)^i}{i!}\frac{(kx)^{j-i}}{(j-i)!} = \underline{1}$$

$\Rightarrow$ • $\underline{e^{-kx} = \left[(e^x)^{-k} = \lim_{n\to\infty}(1+\tfrac{x}{n})^{-kn} = \tfrac{1}{e^{kx}}\right]}$

$[(e^x)^{\frac{1}{k}}]^k = [e^{\frac{x}{k}}]^k \;\to\; \boxed{(e^x)^{\frac{1}{k}} \equiv e^{\frac{x}{k}}}$ • $e^{-\frac{x}{k}} = \frac{1}{e^{x/k}} = \frac{1}{(e^x)^{1/k}} = (e^x)^{-\frac{1}{k}}$ • $e^{\frac{xp}{q}} = (e^x)^{\frac{p}{q}}$

$$\boxed{\ln a : e^{\ln a} = a \in (0,\infty) \quad \begin{matrix} \bullet\ \ln a=\ln b \;\Rightarrow\; a=b \\ \bullet\ a<b \;\Rightarrow\; -\infty<\ln a<\ln b<\infty \end{matrix} \quad\Big|\quad \ln x = \lim_{n\to\infty}(\sqrt[n]{x}-1)/n}$$

$a^n = (e^{\ln a})^n = e^{n\ln a}$

$\Rightarrow \boxed{a^x = (e^{\ln a})^x \equiv e^{x\ln a}}$

- $a^x=a^y \;\Rightarrow\; x=y$
- $x<y \;\Rightarrow\; a^x<a^y$
- $(e^y)^x = e^{xy} = (e^x)^y$

- $(a^y)^x = e^{x\ln a^y} = e^{x\ln e^{y\ln a}} = e^{xy\ln a} = a^{xy} = (a^x)^y$
- $a^{x+y} = e^{(x+y)\ln a} = a^x a^y$

$$e^{x'\frac{d}{dx}}f(x) = \lim_{n\to\infty}\overset{(dx'=\frac{x'}{n})}{\left(1+dx'\tfrac{d}{dx}\right)^n}f(x) \;\to\; e^{dx'\frac{d}{dx}} = e^{dx'g(x)} \;\to\; \boxed{\frac{d}{dx}\equiv h(x) \;\Rightarrow\; e^{x'\frac{d}{dx}} = e^{\int_x^{x+x'}g(t)\,dt}}$$

$$\begin{aligned} f(x+x') &= f(x)+\int_x^{x+x'}f'(x_1)\,dx_1 = f(x)+\int_x^{x+x'}\left[f'(x)+\int_x^{x+x_1}f''(x_2)\,dx_2\right]dx_1 \\ &= f(x)+\int_x^{x+x'}f'(x)\,dx_1+\int_x^{x+x'}\int_x^{x+x_1}\left[f''(x)+\int_x^{x+x_2}f^{(3)}(x_3)\,dx_3\right]dx_2\,dx_1 \\ &= f(x)+\int_x^{x+x'}f'(x)\,dx_1+\int_x^{x+x'}\int_x^{x+x_1}f''(x)\,dX_{21}+\int_x^{x+x'}\int_x^{x+x_1}\int_x^{x+x_2}f^{(3)}(x)\,dX_{31}+\ldots \\ &= \sum_n^{\infty}\frac{(x')^n}{n!}f^{(n)}(x) \end{aligned}$$

$$\frac{d}{d\theta}(\cos\theta+g\sin\theta) = g(\cos\theta+\tfrac{-1}{g}\sin\theta) \equiv i(\cos\theta+i\sin\theta)$$

$$e^{\theta'\frac{d}{d\theta}}(\cos\theta+i\sin\theta) = e^{\theta' i}(\cos\theta+i\sin\theta) = \cos(\theta+\theta')+i\sin(\theta+\theta')$$

$$\Rightarrow \boxed{\begin{aligned} z &= e^{s+it} \\ &= e^s(\cos t+i\sin t)\end{aligned}}$$

$$\ln(z) = \ln|z|+i\arg(z) \;\to\; \boxed{\begin{matrix}\mathrm{Arg}(z)\in(-\pi,\pi] \\ \mathrm{Ln}(z)=\ln|z|+i\mathrm{Arg}(z)\end{matrix}} \;\to\; \boxed{z = x+it = e^{\ln|z|+i\mathrm{Arg}(z)}}$$

- $\frac{\partial}{\partial t}(x+iy) = g(t)z \;\Rightarrow\; e^{\Delta t\frac{\partial}{\partial t}}z(t_0) = e^{\int_{t_0}^{t_f}\frac{(x+iy)'}{x+iy}dt}z(t_0) = \underline{e^{\ln\sqrt{x^2+y^2}|_{\cancel{t0}}^{tf}+i\mathrm{Arctan}\frac{y}{x}|_{\cancel{t0}}^{tf}}\cancel{z(t_0)} = z(t_f)}$

$$e^{t\frac{d}{dt}}z = e^{t'\mathrm{g}}z = z(t+t')$$

$$z = {}^{\pm}\begin{pmatrix}\cos\theta\,\hat{x}\\ +\sin\theta\,\hat{y}\end{pmatrix}, {}^{\pm}\begin{pmatrix}-\sin\theta\,\hat{x}\\ +\cos\theta\,\hat{y}\end{pmatrix}$$

$${}^{\pm}iz \;\leftarrow\; [\hat{x}\,|\,\hat{y}] = [\,1\,|\,{}^{\pm}i\,]$$

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}z = -i\sigma_y z = \widehat{j}\,z \;\leftarrow\; [\hat{x}\,|\,\hat{y}] = {}^{\pm}\left[\begin{array}{c|c}1 & 0\\ 0 & 1\end{array}\right] \;\Rightarrow\; \begin{array}{l}\pm\left[\begin{array}{cc|cc}1 & 0 & 0 & -1\\ 0 & 1 & 1 & 0\end{array}\right]\\ \pm\left[\begin{array}{cc|cc}0 & 1 & -1 & 0\\ 1 & 0 & 0 & 1\end{array}\right]\end{array}$$

$$z = {}^{\pm}\begin{pmatrix}\cos\theta\,\hat{x}\\ -\sin\theta\,\hat{y}\end{pmatrix}, {}^{\pm}\begin{pmatrix}\sin\theta\,\hat{x}\\ +\cos\theta\,\hat{y}\end{pmatrix}$$

$${}^{\pm}-iz \;\leftarrow\; [\hat{x}\,|\,\hat{y}] = [\,1\,|\,{}^{\pm}i\,]$$

$$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}z = i\sigma_y z = -\widehat{j}\,z \;\leftarrow\; [\hat{x}\,|\,\hat{y}] = {}^{\pm}\left[\begin{array}{c|c}1 & 0\\ 0 & 1\end{array}\right] \;\Rightarrow\; \begin{array}{l}\pm\left[\begin{array}{cc|cc}1 & 0 & 0 & 1\\ 0 & 1 & -1 & 0\end{array}\right]\\ \pm\left[\begin{array}{cc|cc}0 & 1 & 1 & 0\\ 1 & 0 & 0 & -1\end{array}\right]\end{array}$$

$$z = {}^{\pm}\begin{pmatrix}\cosh\phi\,\hat{x}\\ \sinh\phi\,\hat{y}\end{pmatrix}, {}^{\pm}\begin{pmatrix}\sinh\phi\,\hat{x}\\ \cosh\phi\,\hat{y}\end{pmatrix}$$

$${}^{\pm}jz, {}^{\pm}1z \;\leftarrow\; [\hat{x}\,|\,\hat{y}] = [\,1\,|\,{}^{\pm}j, {}^{\pm}1\,]$$

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}z = \sigma_x z \;\leftarrow\; [\hat{x}\,|\,\hat{y}] = {}^{\pm}\left[\begin{array}{c|c}1 & 0\\ 0 & 1\end{array}\right] \;\Rightarrow\; \begin{array}{l}\pm\left[\begin{array}{cc|cc}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{array}\right]\\ \pm\left[\begin{array}{cc|cc}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\end{array}\right]\end{array}$$

$$z = {}^{\pm}\begin{pmatrix}\cosh\phi\,\hat{x}\\ -\sinh\phi\,\hat{y}\end{pmatrix}, {}^{\pm}\begin{pmatrix}\sinh\phi\,\hat{x}\\ -\cosh\phi\,\hat{y}\end{pmatrix} \quad\Big|\quad \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}z = -\sigma_x z \;\leftarrow\; [\hat{x}\,|\,\hat{y}] = {}^{\pm}\left[\begin{array}{c|c}1 & 0\\ 0 & 1\end{array}\right]$$

$$\frac{d}{dt}(v_0 t\,\hat{x} + mv_0\,\hat{p}) = \frac{1}{m}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_0 t\\ mv_0\end{bmatrix} \;\Rightarrow\; e^{\frac{\Delta t}{m}\epsilon}\begin{bmatrix}v_0 t\\ mv_0\end{bmatrix} = \begin{bmatrix}v_0(t+\Delta t)\\ mv_0\end{bmatrix}$$

$$a(-b+b) = a(-b) + ab = 0 \;\Rightarrow\; a(-b) = -(ab)$$
$$(-a+a)b = (-a)b + ab = 0 \;\Rightarrow\; (-a)b = -(ab)$$
$$-a(-b+b) = (-a)(-b) + (-a)b = 0 \;\Rightarrow\; (-a)(-b) = ab$$

$$-1 + 1 = 0$$
$$\boxed{(-1)^2(1) = 1}$$

$$y' = y:\; y = e^x$$
$$y'' = y:\; y = e^x,\; e^{-x}/e^{jx}$$
$$y^{(3)} = y:\; y = e^x,\; e^{kx},\; e^{k^2x}$$
$$y^{(4)} = y:\; y = e^x,\; e^{ix},\; e^{-x},\; e^{i^3x}$$
$$y^{(6)} = y:\; y = e^x,\; e^{hx},\; e^{kx},\; e^{h^3x},\; e^{k^2x},\; e^{h^5x}$$
$$y^{(N)} = y:\; y = e^{g(N,n)x},\; 0 \le n \le N-1$$

$$0 = z^N - 1\,,\; 0 = 1 + z + z^2 + \cdots + z^{N-1}\;\; \text{(no need)}$$

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix} = \begin{bmatrix}\cos(\phi+\theta)\\ \sin(\phi+\theta)\end{bmatrix}\;,\; \begin{bmatrix}-\sin\phi\\ \cos\phi\end{bmatrix} = \begin{bmatrix}\cos(\phi+90)\\ \sin(\phi+90)\end{bmatrix}$$

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix} = \begin{bmatrix}\cos(\phi+\theta) & -\sin(\phi+\theta)\\ \sin(\phi+\theta) & \cos(\phi+\theta)\end{bmatrix}$$

$$\begin{bmatrix}\cos\frac{2\pi n}{N} & -\sin\frac{2\pi n}{N}\\ \sin\frac{2\pi n}{N} & \cos\frac{2\pi n}{N}\end{bmatrix}^N = \mathbb{1}_2 = g^N(N,n) \;\Rightarrow\; y = e^{g(N,n)x}$$

$$e^{\theta\frac{d}{d\theta}}\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}R(\phi)$$
$$= \lim_{n\to\infty}\begin{bmatrix}1 & -\frac{\theta}{n}\\ \frac{\theta}{n} & 1\end{bmatrix}^n R(\phi)$$
$$\lim_{n\to\infty}\left(1 + \tfrac{\theta}{n}\tfrac{d}{d\theta}\right)^n R(\phi) = \lim_{n\to\infty}\left[\mathbb{1}_2 + \tfrac{\theta}{n}\mathbb{i}_2\right]^n R(\phi)$$

$$\frac{d}{d\theta}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

$$\begin{bmatrix}-1 & -1\\ i & -i\end{bmatrix}\begin{bmatrix}1+i\delta\theta & 0\\ 0 & 1-i\delta\theta\end{bmatrix}P^{-1} = \begin{bmatrix}1 & -\delta\theta\\ \delta\theta & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & -\delta\theta\\ \delta\theta & 1\end{bmatrix}\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix} = \begin{bmatrix}(1)\cos\phi - (\delta\theta)\sin\phi\\ (\delta\theta)\cos\phi + (1)\sin\phi\end{bmatrix} = \begin{bmatrix}\cos(\delta\theta+\phi) & -\sin(\delta\theta+\phi)\\ \sin(\delta\theta+\phi) & \cos(\delta\theta+\phi)\end{bmatrix}$$

$$\lim_{n\to\infty}\begin{bmatrix}1 & -\delta\theta\\ \delta\theta & 1\end{bmatrix}^n\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix} = \begin{bmatrix}\cos(\theta+\phi) & -\sin(\theta+\phi)\\ \sin(\theta+\phi) & \cos(\theta+\phi)\end{bmatrix}$$

$$\lim_{n\to\infty}\begin{bmatrix}1 & -\delta\theta\\ \delta\theta & 1\end{bmatrix}^n\begin{bmatrix}1\\ 0\sim\delta\theta\end{bmatrix} = \lim_{n\to\infty}\begin{bmatrix}\binom{n}{0} - \binom{n}{2}\delta\theta^2 + \binom{n}{4}\delta\theta^4 + \dots\\ \binom{n}{1}\delta\theta - \binom{n}{3}\delta\theta^3 + \binom{n}{5}\delta\theta^5 + \dots\end{bmatrix}$$

$$\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix} = \begin{bmatrix}1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\\ \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\end{bmatrix}$$

$$e^{\theta\mathbb{i}} = \cos\theta\,\mathbb{1} + \sin\theta\,\mathbb{i}$$

$$\alpha = x + yg \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} \;,\; g^2 \Leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ b \end{bmatrix} \neq \begin{bmatrix} a \\ 0 \end{bmatrix} \;,\; \boxed{\mathbb{g}\alpha = \begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}} \;,\; \begin{matrix} \beta = u + vg \\ \alpha + \beta = \beta + \alpha \end{matrix} \;,\; \begin{matrix} \beta\alpha = ux + uyg + vgx + vgyg \\ \alpha\beta = xu + xvg + ygu + ygvg \end{matrix}$$

$$\underline{\hat{\beta}}\alpha = \underline{\left(\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} + \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}\right)}\begin{bmatrix} x \\ y \end{bmatrix} = \hat{\alpha}\beta \;\leftarrow\; \underline{\beta}\alpha = \underline{\begin{bmatrix} u & va \\ v & u + vb \end{bmatrix}}\begin{bmatrix} x & ya \\ y & x + yb \end{bmatrix} = \alpha\beta \text{ (see above)} \;,\; \mathbb{g}^2 \overset{!!!}{=} \begin{bmatrix} a & ba \\ b & a + b^2 \end{bmatrix} = \alpha_{\mathbb{g}^2}$$

$$* \,\alpha = x + yh \;;\; h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;,\; h^2 = \begin{bmatrix} bc + a^2 & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix} = \begin{bmatrix} x + ya & by \\ cy & x + yd \end{bmatrix} = \alpha_{h^2} \;\to\; \begin{matrix} (a,d) = (a,0), \boxed{(0,d)} \\ = (a,a), (a,-a) \end{matrix}$$

$$\det(\mathbb{g}^N) = \det\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}^N = (-a)^N = 1 \;\Rightarrow\; a = \pm 1$$

$$\mathbb{g}^N = \begin{bmatrix} a & a \\ \frac{b}{2} + \sqrt{\frac{b^2}{4} + a} & \frac{b}{2} - \sqrt{\frac{b^2}{4} + a} \end{bmatrix}\begin{bmatrix} \frac{b}{2} + \sqrt{\frac{b^2}{4} + a} & 0 \\ 0 & \frac{b}{2} - \sqrt{\frac{b^2}{4} + a} \end{bmatrix}^N P^{-1} = \mathbb{1} = \begin{bmatrix} \frac{b}{2} + \sqrt{\frac{b^2}{4} + a} & 0 \\ 0 & \frac{b}{2} - \sqrt{\frac{b^2}{4} + a} \end{bmatrix}^N \;\Rightarrow\; \begin{matrix} a \leq -\frac{b^2}{4} \\ b^2 \leq 4(\cancel{-a}) \\ b^2 = 4 \to \nexists P^{-1} \end{matrix}$$

$$* \; g_{(b/\theta)} = \frac{b}{2} + \sqrt{-1}\sqrt{1 - \frac{b^2}{4}} = \cos\theta + i\sin\theta \;\Leftrightarrow\; \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_i \;,\; \left(\theta \neq 0 \;\Leftrightarrow\; b^2 \neq 4\right)$$

(No need)

$$1 = \alpha \cdot \alpha^{-1} = \frac{\begin{bmatrix} x & -y \\ y & x + yb \end{bmatrix}\begin{bmatrix} x + yb & y \\ -y & x \end{bmatrix}}{x^2 + y^2 + xyb \, > \, 0 \; ☑} \;,\; \alpha + \frac{\alpha^{-1}}{(\det\alpha)^{-1}} = \begin{bmatrix} 2x + yb & 0 \\ 0 & 2x + yb \end{bmatrix} \;,\; \alpha - \frac{\alpha^{-1}}{(\det\alpha)^{-1}} = y\begin{bmatrix} -b & -2 \\ 2 & b \end{bmatrix}$$

$$\alpha^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x^2 + y^2 a \\ y(2x + yb) \end{bmatrix} \;\to\; \begin{matrix} x = \cancel{-\frac{yb}{2}}, 1, -1 \\ y = \cancel{\frac{1}{\pm\sqrt{b^2/4 + a}}}, \; 0 \end{matrix} \;\Rightarrow\; \alpha_\pm = \pm 1$$

$$i^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} x^2 + y^2 a \\ y(2x + yb) \end{bmatrix} \;\to\; \begin{matrix} y = \cancel{0}, \; \pm\frac{2}{\sqrt{4 - b^2}} \\ x = \cancel{\sqrt{-1}}, \; -\frac{yb}{2} \end{matrix} \;\Rightarrow\; i_\pm = \pm\frac{1}{\sqrt{4 - b^2}}\begin{bmatrix} -b \\ 2 \end{bmatrix}$$

$$1 = \mathbb{g} \cdot \mathbb{g}^{N-1} = \begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}\begin{bmatrix} -b/a & 1 \\ 1/a & 0 \end{bmatrix}$$

$$\mathbb{g} + \mathbb{g}^{N-1} = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \;,\; \mathbb{g} - \mathbb{g}^{N-1} = \begin{bmatrix} -b & -2 \\ 2 & b \end{bmatrix}$$

$$\left[\mathbb{g} - \mathbb{g}^{N-1}\right]^2 = -(4 - b^2)\mathbb{1}_2$$
