1 Lorentz Transformation

Galilean Transform $\rightarrow$ Lorentz Transform $\quad \gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$

$$\begin{aligned} t' &= t \\ x' &= x - vt \end{aligned} \quad , \quad \begin{aligned} t &= t' \\ x &= x' + vt' \end{aligned} \quad , \quad \begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -v & 1 \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}$$

$$\begin{aligned} m' &= m \\ p' &= m(v_0 - v) \end{aligned} \quad , \quad \begin{bmatrix} m'c \\ p' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\beta & 1 \end{bmatrix} \sum_i \begin{bmatrix} m_i c \\ \beta_i m_i c \end{bmatrix}$$

$$\boxed{\begin{aligned} x' &= \gamma(x - \beta ct) \\ ct' &= \gamma(ct - \beta x) \end{aligned} \quad , \quad \begin{aligned} x &= \gamma(x' + \beta ct') \\ ct &= \gamma(ct' + \beta x') \end{aligned}} \quad , \quad \beta^2 = \frac{\gamma^2 - 1}{\gamma^2}$$

x is the position of a point/event occuring on the number line, and t is the time on a clock at x.

x' is the position of the same point/event on the other number line, and t' is the time on the clock at x'.

$x = 0$ and $x' = 0$ are the positions of the line/reference frame origins, and t_0 and t_0' are the times on the origin clocks.

At $t_0 = t_0' = 0$, both origin's coincide at $x = x' = 0$.

All points at x see their clock run the same as $t = t_0$, but see a different t' at the adjacent x'.

Transform Matrix (Hermitian for boosts)

$$\gamma = \cosh\phi, \ \gamma\beta = \sinh\phi, \ \beta = \tanh\phi$$

$$x^{\mu\prime} = \begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x^\mu = \Lambda^{\mu'}{}_{\mu} x^\mu = \frac{\partial x^{\mu\prime}}{\partial x^\mu} x^\mu$$

Weyl Matrices: $\cosh\frac{\phi}{2} I - \sinh\frac{\phi}{2} \cancel{\sigma_t} \sigma_x$ (Hermitian)

$$\begin{pmatrix} x_0{}' \\ -x_1{}' \\ -x_2{}' \\ -x_3{}' \end{pmatrix} = \Lambda \begin{pmatrix} x_0 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} \Rightarrow x'_\mu = \Lambda^{-1} x_\mu = \frac{\partial x^\mu}{\partial x^{\mu\prime}} x_\mu$$

Time Slows: $\boxed{\Delta t' = \Delta t/\gamma}$

1.) $t'(x, t_0)$ for a Clock at $x = X_0 + vt$

$$\begin{aligned} \Rightarrow \quad ct' &= \gamma\left(ct - \beta[X_0 + vt]\right) \\ &= \left(\gamma ct(1-\beta^2) - \gamma\beta X_0\right) \\ &= \left(\frac{ct}{\gamma} - \gamma\beta X_0\right) = \left(\frac{ct}{\gamma} + cT_0\right) \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} t'(X_0, t) &= \frac{t}{\gamma} + T_0 \\ &= \frac{t}{\gamma} - \gamma\beta X_0 \end{aligned}}$$

$T_0{>}0$: $X_0{<}0$
$T_0{<}0$: $X_0{>}0$

(No t' is simultaneous to x unless in same position)

$$\boxed{\frac{dt'}{dt} = \frac{dt'}{dt_0} = \frac{1}{\gamma}}$$

(for each Δt, then $\Delta t' = \frac{\Delta t}{\gamma}$)
(Clocks at x' look like they tick slower by factor γ)

2.) $\Delta t_0'$ given Δt_0

$$\Rightarrow \quad c\Delta t_0 = \gamma(c\Delta t_0' - \cancel{\beta x'}_{=0}) \quad \Rightarrow \quad \boxed{\Delta t_0' = \Delta t_0/\gamma}$$

(Same conclusion of slowed clocks)

Length Contraction: $\boxed{\Delta x' = \gamma\Delta x}$

$$\begin{aligned} 1.)\ \Delta x' = x_2' - x_1' \ &= \ \gamma(x_2 - \cancel{\beta ct}) - \gamma(x_1 - \cancel{\beta ct}) \\ &= \ \gamma(x_2 - x_1) \ = \ \gamma\Delta x \quad ☑ \end{aligned}$$

Pythag. Triples

$$\begin{aligned} \beta = 3/5 \ &: \ \gamma = 5/4 = 1.25 \\ \beta = 4/5 \ &: \ \gamma = 5/3 \\ \beta = 5/13 \ &: \ \gamma = 13/12 \\ \beta = 7/25 \ &: \ \gamma = 25/24 \end{aligned}$$

Velocity Addition (1-D):

$$\boxed{\begin{aligned} w_1 &= \frac{\gamma_u(v_1 + u)}{\gamma_u(1 + v_1 u/c^2) = t/t' = \gamma_v/\gamma_w} \quad \leftarrow \quad [L_{-u}] \begin{bmatrix} ct \\ vt \end{bmatrix} = \begin{bmatrix} ct' \\ wt' \end{bmatrix} \\ \tanh(\phi_1 + \phi_2) &= \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1 \tanh\phi_2} \end{aligned}}$$

Doppler Shift

$$f_{\text{rec}} = \sqrt{\frac{1+\beta}{1-\beta}} \ f_{\text{emit}}$$

(v is [+] if $\rightarrow\leftarrow$)

2 4-Vectors

3-Vectors

$$\vec{p} = \boxed{\gamma m \vec{v}}$$

$$\begin{aligned} \vec{F} &= \tfrac{d\vec{p}}{dt} = m\tfrac{d(\gamma\vec{v})}{dt} \\ &= \gamma m\vec{a} + \gamma^3\tfrac{(m\vec{a}\cdot\vec{v})\vec{v}}{c^2} \\ &= \boxed{\gamma^3 m(\vec{a} - \tfrac{\vec{v}\cdot\vec{v}}{c^2}\vec{a} + \tfrac{\vec{a}\cdot\vec{v}}{c^2}\vec{v})} \end{aligned}$$

Scalars

$$E = \boxed{\sqrt{p^2c^2 + m^2c^4} = \gamma mc^2}$$

$$T = E - E_0 = \boxed{(\gamma-1)mc^2}$$

$$\begin{aligned} P_{\text{ow}} &= \tfrac{dE}{dt} = mc^2\tfrac{d\gamma}{dt} = \tfrac{d\vec{p}}{dt}\cdot\vec{v} \\ &= \boxed{\vec{F}\cdot\vec{v} = \gamma^3 m\vec{a}\cdot\vec{v}} \end{aligned}$$

$$W = \int \vec{F}\cdot\vec{v}\,dt = \boxed{\int \gamma^3 m\vec{a}\cdot\vec{v}\,dt}$$

2.1 Position

$$\begin{aligned} \boldsymbol{x}^\mu &= (x^0, \vec{x}) \\ &= (ct, \vec{r}) \end{aligned} \Rightarrow \boxed{\Delta\boldsymbol{x}^\mu = \boldsymbol{x}^\mu_A - \boldsymbol{x}^\mu_B} \quad \text{(for Event } A,\ B)$$

$$\begin{aligned} (\Delta\boldsymbol{x}^\mu)^2 &= (\Delta\boldsymbol{x}^\mu)(\Delta\boldsymbol{x}_\mu) \\ &= c^2\tau^2 \\ &= \boxed{ct^2 - \vec{r}^{\,2}} \end{aligned}$$

- Timelike : $(\Delta\boldsymbol{x}^\mu)^2 > 0$ (∃ an inertial frame where A, B occur at the <u>same spacial place</u> but diff. time, e.g, $(c\Delta t, 0, 0, 0) \rightarrow (\Delta x)^2 = c^2(\Delta t)^2 > 0$)
- Spacelike : $(\Delta\boldsymbol{x}^\mu)^2 < 0$ (∃ an inertial frame where A, B occur at the <u>same time</u> but non-casual space)
- Lightlike : $(\Delta\boldsymbol{x}^\mu)^2 = 0$ (A and B lie on a trajectory moving at c)

Relativistic Dot Products are Ref. Frame Invariant (not necessarily Conserved)

2.2 Momentum

$$\begin{aligned} \boldsymbol{p}^\mu &= (p^0, \vec{p}) \\ &= m\frac{d\boldsymbol{x}^\mu}{d\tau} = m\boldsymbol{\eta}^\mu \\ &= \boxed{(\gamma mc, \gamma m\vec{v}) = \left(\tfrac{E}{c}, \vec{p}\right)} \end{aligned}$$

$$\boxed{\tfrac{\vec{p}c}{E} = \vec{\beta}}$$

$$\pm\boldsymbol{p}^\mu\boldsymbol{p}_\mu = \boxed{\boldsymbol{p}^2 = m^2c^2 = \left(\tfrac{E}{c}\right)^2 - \vec{p}^{\,2}}$$

Momentum Conservation means each vector component is individually conserved

<u>Decay from Rest M_1: $M_1 \rightarrow m_2 + m_3$</u>

$$\begin{aligned} P_3^2 &= (P_1 - P_2)^2 \\ &= P_1^2 + P_2^2 - 2P_1\cdot P_2 \\ m_3^2c^2 &= M_1^2c^2 + m_2^2c^2 - 2(M_1c, 0)\cdot(E_2/c, p_2) \\ &= M_1^2c^2 + m_2^2c^2 - 2M_1E_2 \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} E_2 &= \frac{M_1^2c^2 + m_2^2c^2 - m_3^2c^2}{2M_1} \\ E_3 &= \frac{M_1^2c^2 + m_3^2c^2 - m_2^2c^2}{2M_1} \end{aligned}}$$

Decay from Rest m_a to Maximum E_b (same as first): $m_a \to m_b + M$

$$(m_a, 0) = (E_b, p_b) + (E_M, -p_b)$$

$$* \boxed{(P_M)^2 = (P_a - P_b)^2 = P_a^2 + P_b^2 - 2P_a \cdot P_b} \;\Rightarrow\; E_b = \frac{m_a^2 + m_b^2 - M^2}{2m_a} c^2$$

$$M^2 = m_a^2 + m_b^2 - 2E_b m_a$$

Min. Threshold E_a to Create M_{rest}: $m_a + m_{b=rest} \to M_{rest} = \sum m$

$$[(E_a, p_a) + (m_b, 0)]^2 = (M, 0)^2$$

$$\underline{(\tfrac{E_a}{c})^2} + m^2c^2 + 2E_a m_b - \underline{p_a^2} = M^2c^2 \;\Rightarrow\; E_a = \boxed{\frac{M^2 - \underline{m_a^2} - m_b^2}{2m_b} c^2}$$

Min. Threshold E_a to Create E_m: $m_{a\rightarrow} + m_{b=rest} \to M_{\leftarrow} + m_{\rightarrow}$

$$(E_a, p_a) + (m_b, 0) \to (E_M, -p_m + p_a) + (E_m, p_m)$$

$$* \boxed{(P_a - P_m)^2 = (P_M - P_b)^2 = P_M^2 + P_b^2 - 2P_m \cdot P_b}$$

$$m_a^2 + m^2 - 2(E_a E_m - p_a \cdot p_m) = M^2 + m_b^2 - 2E_M m_b$$

$$4p_a^2 p_m^2 \cos^2 \phi_{am} = ([M^2 + m_b^2 - m_a^2 - m^2] + 2E_a E_m - 2(E_a + m_b - E_m)m_b)^2$$

$$4(E_a^2 - m_a^2)p_m^2 \cos^2 \phi = (E_a[Y = 2E_m - 2m_b] + X)^2 = E_a^2 Y^2 + X^2 + 2XYE_a$$

$$0 = \underline{(Y^2 - 4p_m^2 \cos^2 \phi)E_a^2 + 2XYE_a + (X^2 + 4m_a^2 \cos^2 \phi)} \;\Rightarrow\; E_a = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

(Compton Scattering) λ' from λ and θ: $\gamma + e_{rest} \to \gamma + e$

$$(E_\gamma, p_\gamma, 0) + (m_e, 0, 0) = (\underline{E'_\gamma}, p'_\gamma \cos\theta, p'_\gamma \sin\theta) + (\underline{E'_e}, p'_e \underline{\cos\phi}, p'_e \sin\phi)$$

$$* \; (P_1 - P_3)^2 = (P_4 - P_2)^2 \;\Rightarrow\; E_\gamma E'_\gamma - E_\gamma E'_\gamma \cos\theta = m(E'_\gamma - E_\gamma)$$

$$\Rightarrow\; E'_\gamma = \frac{E_\gamma m_e}{m + (1 - \cos\theta)E_\gamma} \;\Rightarrow\; \boxed{\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)}$$

Scattering Angle : $m_1 + m_2 \to m_3 + m_4$

$$* \, (P_1 - P_3)^2 = (P_4 - P_2)^2$$

$$\boxed{\begin{vmatrix} p_1 & p_2 \\ p_4 & p_3 \end{vmatrix} = \begin{vmatrix} E_1 & E_2 \\ E_4 & E_3 \end{vmatrix} - \frac{(m_1^2 - m_2^2) + (m_3^2 - m_4^2)}{2}}$$

Scattering of E_a, θ in CM Frame to E'_a in *Breit Frame* $(p'_a \to -p'_a)$: $A + B \to A + B$

$$(CM) \; (E_a, p_a, 0) + (E_b, -p_a, 0) = \underline{(E_a, p_a \cos\theta, p_a \sin\theta)} + (E_b, -p_a \cos\theta, -p_a \sin\theta) \;\equiv\; (P_1 + P_2 = P_3 + P_4)$$

$$(Breit) \; \underline{(E'_a, p'_a, 0)} + (E'_b, -p'_b \cos\phi, -p'_b \sin\phi) = \underline{(E'_a, -p'_a)} + (E'_b, p'_a - p'_b \cos\phi, -p'_b \sin\phi) \;\equiv\; (Q_1 + Q_2 = Q_3 + Q_4)$$

$$* \; P_1 \cdot P_3 = Q_1 \cdot Q_3$$

$$E_a^2 - p_a^2 \cos\theta = E'^2_a + p'^2_a = 2E'^2_a - m_a^2 \;\Rightarrow\; E'_a = \sqrt{\frac{m_a^2(1 + cos\theta) + E_a^2(1 - \cos\theta)}{2}} = \boxed{\sqrt{m_a^2 \cos^2 \frac{\theta}{2} + E_a^2 \sin^2 \frac{\theta}{2}}}$$

$$\boxed{\text{CM Frame} = \text{Individual Particle Energy/Momentum is Conserved}}$$

$$\boxed{\text{CM} \to \text{Breit Frame} = Rot(\tfrac{\theta}{2}) + \left[\beta_{shift} = \frac{p_a c \cos(\theta/2)}{E_a} = \frac{\sqrt{E_a^2 + m_a^2}\cos(\theta/2)}{E_a}\right]}$$

2.3 Acceleration and Force

$$\begin{aligned}
\boldsymbol{K}^\mu &= (K^0, \vec{K}) \\
&= m\boldsymbol{\alpha}^\mu \;=\; m\frac{d\boldsymbol{\eta}^\mu}{d\tau} \;=\; \frac{d\boldsymbol{p}^\mu}{d\tau} \;=\; \gamma\frac{d\boldsymbol{p}^\mu}{dt} \\
&= \left(\gamma\frac{d(\gamma mc)}{dt}, \gamma\frac{d(\gamma m\vec{v})}{dt}\right) \\
&= \left(\tfrac{\gamma P_{\text{ow}}}{c}, \gamma\vec{F}\right) \;=\; \left(\tfrac{\gamma\vec{F}\cdot\vec{v}}{c}, \gamma\vec{F}\right) \\
&= \left(\gamma^4\tfrac{m\vec{a}\cdot\vec{v}}{c},\; \gamma^2 m\vec{a} + \gamma^4\tfrac{(m\vec{a}\cdot\vec{v})\vec{v}}{c^2}\right)
\end{aligned}$$

$$\begin{aligned}
\mp\boldsymbol{\alpha}^\mu\boldsymbol{\alpha}_\mu &= \gamma^6\tfrac{(\vec{a}\cdot\vec{v})^2}{c^2} + \gamma^4\vec{a}^{\;2} \\
\mp\boldsymbol{K}^\mu\boldsymbol{K}_\mu &= -\gamma^2\tfrac{(\vec{F}\cdot\vec{v})^2}{c^2} + \gamma^2\vec{F}^{\;2} \\
&= \gamma^2\vec{F}^{\;2}\left(1 - \tfrac{\vec{v}\cdot\vec{v}}{c^2}\cos\theta_{v,F}\right) \\
\boldsymbol{\alpha}^\mu\boldsymbol{\eta}_\mu &= \frac{d\boldsymbol{\eta}^\mu}{d\tau}\boldsymbol{\eta}_\mu = \frac{1}{2}\frac{d(\boldsymbol{\eta}^\mu\boldsymbol{\eta}_\mu)}{d\tau} = 0 \\
\boldsymbol{K}^\mu\boldsymbol{p}_\mu &= m^2\boldsymbol{\alpha}^\mu\boldsymbol{\eta}_\mu = 0
\end{aligned}$$

2.4 Current Density and Vector Potential

$$\begin{aligned}
\boldsymbol{J}^\mu &= (J^0, \vec{J}) \\
&= \rho_0\frac{d\boldsymbol{x}^\mu}{d\tau} \;=\; \rho_0\boldsymbol{\eta}^\mu \\
&= (\gamma c\rho_0, \gamma\rho_0\vec{v}) \;=\; (c\rho, \vec{J})
\end{aligned}$$

$$\frac{\partial\boldsymbol{J}^\mu}{\partial\boldsymbol{x}^\mu} \;=\; \frac{\partial}{\partial\boldsymbol{x}^\mu}\cdot\boldsymbol{J}^\mu \;=\; \frac{\partial\rho}{dt} + \nabla\vec{J} \;=0$$

$$\begin{aligned}
\boldsymbol{A}^\mu &= (A^0, \vec{A}) \;=\; \left(\tfrac{V}{c}, \vec{A}\right) \\
&= \boldsymbol{A}^\mu + \frac{\partial\lambda}{\partial\boldsymbol{x}^\mu}
\end{aligned}$$

$$\Box^2\boldsymbol{A}^\mu \;=\; \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A}^\mu \;=\; -\mu_0\boldsymbol{J}^\mu$$