# 1 Fourier Series

## 1.1 Sine

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} S_n \sin\left(\frac{g(n)}{L}\pi x\right) \\
\int_0^L f(x)\sin\left(\frac{h(m)}{L}\pi x\right)dx &= \sum_{n=0}^{\infty}\int_0^L S_n \sin\left(\frac{g(n)}{L}\pi x\right)\sin\left(\frac{h(m)}{L}\pi x\right)dx \\
&= S_{h(m)}\frac{L}{2} \\
S_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{g(n)}{L}\pi x\right)dx
\end{aligned}$$

## 1.2 Cosine

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} S_n \cos\left(\frac{g(n)}{L}\pi x\right) \\
\int_0^L f(x)\cos\left(\frac{h(m)}{L}\pi x\right)dx &= \sum_{n=0}^{\infty}\int_0^L S_n \cos\left(\frac{g(n)}{L}\pi x\right)\cos\left(\frac{h(m)}{L}\pi x\right)dx \\
&= S_{h(m)}\frac{L}{2} \\
S_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{g(n)}{L}\pi x\right)dx
\end{aligned}$$

## 1.3 Full

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{g(n)}{L}\pi x\right) + B_n \sin\left(\frac{g(n)}{L}\pi x\right) \\
A_n &= \frac{1}{L}\int_{-L}^L f(x)\cos\left(\frac{g(n)}{L}\pi x\right)dx \\
B_n &= \frac{1}{L}\int_{-L}^L f(x)\sin\left(\frac{g(n)}{L}\pi x\right)dx
\end{aligned}$$

### 1.4 Exponential

$$
\begin{aligned}
f(x) &= \sum_{n\,=\,-\infty}^{\infty} C_n \; e^{i\frac{2\pi n}{\lambda}x} \\
C_n &= \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)\; e^{-i\frac{2\pi n}{\lambda}x}\; dx
\end{aligned}
$$

# 2 Fourier Transform

$$
\begin{aligned}
k_n &= \frac{2\pi n}{\lambda} \quad \Rightarrow \quad \Delta k = \frac{2\pi}{\lambda} \\
\Psi(x) \quad ?\approx & \quad \sum_{n\,=\,-\infty}^{\infty} \frac{1}{\lambda}\left[\int_{-\lambda/2}^{\lambda/2} \Psi(x)\; e^{-i\frac{2\pi n}{\lambda}x}\; dx\right] e^{i\frac{2\pi n}{\lambda}x} \\
&= \sum_{k_n\,=\,-\infty}^{\infty} \frac{\Delta k}{\sqrt{2\pi}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\lambda/2}^{\lambda/2} \Psi(x)\; e^{-ik_n x}\; dx\right] e^{ik_n x}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\lambda\to\infty} \quad \Psi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x)\; e^{-ikx}\; dx\right] e^{ikx}\; dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\Psi}(k) e^{ikx}\; dk \\
\hat{\Psi}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x) e^{-ikx}\; dx
\end{aligned}
$$

*or*

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} \hat{x}(f) e^{i2\pi f t}\; df \\
\hat{x}(f) &= \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t}\; dt
\end{aligned}
$$

Proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{x}(f)(e^{-\epsilon f^2})\ e^{i2\pi f t}\ df &= a(t) \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) e^{-i2\pi f\tau}\ d\tau\right](e^{-\epsilon f^2})\ e^{i2\pi f t}\ df \\
&= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty}(e^{-\epsilon f^2})\ e^{-i2\pi f(\tau - t)}\ df\right]\ d\tau \\
&= \int_{-\infty}^{\infty} x(\tau' + t)\left[\int_{-\infty}^{\infty}(e^{-\epsilon f^2})\ e^{-i2\pi f\tau'}\ df\right]\ d\tau' \\
&= \int_{-\infty}^{\infty} x(\tau' + t)\ \left(\frac{1}{2\sqrt{\pi}\epsilon}\ e^{\frac{-(\tau')^2}{4\epsilon^2}}\right)\ d\tau' \\
&= \int_{-\infty}^{\infty} x(\epsilon\tau'' + t)\ \left(\frac{1}{\sqrt{2\pi}\sqrt{2}} e^{\frac{-(\tau'')^2}{2\sqrt{2}^2}}\right)\ d\tau''
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{x}(f)\ e^{i2\pi f t} &= \lim_{\epsilon \to 0}\ \int_{-\infty}^{\infty} \hat{x}(t)(e^{-\epsilon f^2})\ e^{i2\pi f t}\ df\ = \\
\lim_{\epsilon \to 0}\ a(t) &= x(t)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(\tau'')^2}{2\sigma^2}}\ d\tau'' \\
&= x(t)
\end{aligned}$$

# 3 Laplace Transform

$\{s = \sigma + i\tau\ :\ \underline{\sigma > a}\}$

- $\left(|x(t)|\ \leq_{t\to\infty}\ Me^{at}\right)\ \Rightarrow\ \left(\left|x(t)e^{-st}\right|\ \leq\ Me^{at}e^{-\sigma t}\ =\ Me^{-t(\sigma - a)}\right)$

$$\boldsymbol{(\mathcal{L}\{x\})(s) = \int_0^\infty x(t)e^{-st}\ dt\ \ \lesssim\ \int_0^\infty Me^{-t(\sigma-a)}dt\ \leftarrow\ \underline{\sigma > a}}$$

$$\boldsymbol{x(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty}(\mathcal{L}\{x\})(s)e^{st}\ ds}$$

$(\mathcal{L}\{x^{(n)}\})(s)\ =\ s^n(\mathcal{L}\{x\})(s) - s^{n-1}x(0) - s^{n-2}x'(0) - ... - x^{(n-1)}(0)$

# 4 Z Transform (Discrete Laplace)

$$\int_0^\infty \frac{u(x)}{e^{sx}}\,dx \quad \underset{(x_n = n\Delta_x)}{\Rightarrow} \quad \sum_{n=0}^{\infty} \frac{u(x_n)\Delta_x}{\left[e^{s\Delta_x}\right]^n} = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$$

- $|u(x)| \le Me^{ax}$
- $|a_n| \le MR^n$

$$\boldsymbol{(Z\{a_n\})(z) = \sum_{n\,=\,0}^{\infty} \frac{a_n}{z^n} \le \sum_{n=0}^{\infty} \frac{MR^n}{z^n} \leftarrow \underline{|z| > R}}$$

**Convolution** : $\boxed{f_n = a_n * y_n = \sum_{k=0}^{n} a_k y_{n-k}} \Rightarrow \begin{array}{l} Z\{f_n\} = Z\{a_n\}Z\{y_n\} \\ Z\{y_n\} = \frac{1}{Z\{a_n\}}Z\{f_n\} = Z\{g_n\}Z\{f_n\} \end{array} \Rightarrow \underline{y_n = g_n * f_n}$

**Shifting** : $\underline{Z\{a_{n+m}\} = a_m + \frac{a_{m+1}}{z} + \frac{a_{m+2}}{z^2} + \ldots} = \boxed{\left[Z\{a_n\} - \sum_{j=0}^{m-1} \frac{a_j}{z^j}\right] z^m}$

- $\underset{(a = \{1, -3, 2, 0, 0, \ldots\})}{f_{n+2} = y_{n+2} - 3y_{n+1} + 2y_n} \rightarrow \underset{(Z\{a_n\} = \underline{\text{System Transfer Func.}})}{Z\{f_n\} = Z\{y_n\}\left[1 + \frac{-3}{z} + \frac{2}{z^2}\right]} \rightarrow \begin{array}{l} Z\{y_n\} = \left[\frac{2}{1-(2/z)} + \frac{-1}{1-(1/z)}\right] Z\{f_n\} \\ \quad = Z\{\underline{g_n = 2^{n+1} - 1}\}Z\{f_n\} \end{array}$

$*$ $\underline{f_{n+2} = 0\ ;\quad f_{n\ge 2} = 0}$

$$\frac{y_0}{z^0} + \frac{y_1}{z^1} + \frac{y_2}{z^2} + \ldots = Z\{g_n\}\left[\frac{f_0}{z^0} + \frac{f_1}{z^1} + \frac{0}{z^2} + \ldots\right]$$

$$y_0 = g_0 f_0\ ,\quad y_1 = g_1 f_0 + g_0 f_1$$

$$\begin{aligned} y_n &= \underline{g_n f_0 + g_{n-1} f_1} \\ &= (2^{n+1} - 1)y_0 + (2^n - 1)(y_1 - 3y_0) \\ &= (2 - 2^n)y_0 + (2^n - 1)y_1 \end{aligned}$$

$\underline{f_{n+2} = 2^n - 1\ ;\quad f_{n\ge 2} = 2^{n-2} - 1}$

$$\frac{y_0}{z^0} + \frac{y_1}{z^1} + \frac{y_2}{z^2} + \ldots = Z\{g_n\}\left[\frac{f_0}{z^0} + \frac{f_1}{z^1} + \frac{f_2}{z^2} + \ldots\right]$$

$$\begin{aligned} y_n &= \underline{g_n f_0 + g_{n-1} f_1} + \sum_{k=2}^{n} g_{n-k} f_k \\ &= (2 - 2^n)y_0 + (2^n - 1)y_1 + \sum_{k=2}^{n}(2^{n-k+1} - 1)(2^{k-2} - 1) \\ &= \boxed{\underset{(\text{homog. sol. } = A + B2^n)}{\underline{(2 - 2^n)y_0 + (2^n - 1)y_1 + (2 - 2^{n+1})}} + \underset{(\text{partic. sol.})}{\underline{(n + n2^{n-1})}}} \end{aligned}$$

$*$ $\underline{y_{n+2} - 3y_{n+1} + 2y_n = 0}$

$$0 = z^2\left[Z\{y_n\} - \frac{y_0}{z^0} - \frac{y_1}{z^1}\right] - 3z\left[Z\{y_n\} - \frac{y_0}{z^0}\right] + 2Z\{y_n\}$$

$$Z\{y_n\} = \frac{z^2\left[\frac{y_0}{z^0} + \frac{y_1}{z^1}\right] - 3z\frac{y_0}{z^0} + 2*0}{z^2 - 3z + 2} = \frac{H(z)}{z^2 - 3z + 2}$$

$$y_n = (2 - 2^n)y_0 + (2^n - 1)y_1$$

$\underline{y_{n+2} - 3y_{n+1} + 2y_n = f_n = 2^n - 1}$

$$Z\{f_n\} = z^2\left[Z\{y_n\} - \frac{y_0}{z^0} - \frac{y_1}{z^1}\right] - 3z\left[Z\{y_n\} - \frac{y_0}{z^0}\right] + 2Z\{y_n\}$$

$$Z\{y_n\} = \frac{H(z)}{z^2 - 3z + 2} + \frac{Z\{f_n\}}{z^2 - 3z + 2}$$

$$y_n = \boxed{\underset{(\text{homog. sol. } = A + B2^n)}{\underline{(2 - 2^n)y_0 + (2^n - 1)y_1 + (2 - 2^{n+1})}} + \underset{(\text{partic. sol.})}{\underline{(n + n2^{n-1})}}}$$

If all initial conditions/free parameters $(y_0, y_1, \ldots\ x_0, \ldots)$ are 0, then $\boldsymbol{Z\{y_n\} = F(z)Z\{x_n\}}$

- $y_n$ is stable (bounded input $x_n \to$ bounded output $y_n$) iff all of $F(z) = Z\{a_n\}$'s pole's are in the unit open disc.

- $y_{n+1} = x_{n+1} - x_n \ \rightarrow\ Z\{y_n\} = Z\{x_n\}\frac{z-1}{z} \ \rightarrow\ \underline{\forall x_n,\ |x_n| < M \ \Rightarrow\ |y_n| < 2M}$ (bounded)
- $y_{n+1} - y_n = x_{n+1} \ \rightarrow\ Z\{y_n\} = Z\{x_n\}\frac{z}{z-1} \ \rightarrow\ y_n = x_0 + x_1 + \ldots + x_n \ \rightarrow\ \underline{\forall x_n,\ |x_n| < M \ \Rightarrow\ |y_n| < (n+1)M}$ (unbounded)