CS101 Algorithms and Data Structures

Priority Queues and Heaps

Textbook Ch 6

Outline

- Priority queue
- Binary heap
- Heapsort

Background

We have discussed Abstract Lists with explicit linear orders

- Arrays, linked lists, strings

We saw three cases which restricted the operations:

- Stacks, queues, deques

Following this, we looked at search trees for storing implicit linear orders: Abstract Sorted Lists

- Run times were generally $\Theta(\ln(n))$

We will now look at a restriction on an implicit linear ordering:

- Priority queues

Definition

Queues

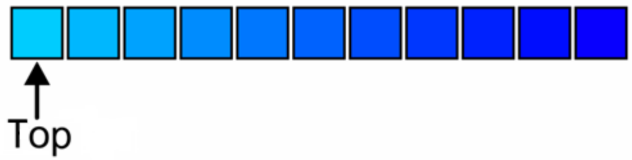
- The order may be summarized by *first in, first out*

Priority queues

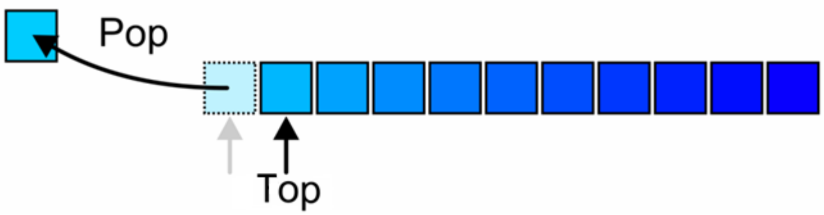
- Each object is associated with a priority
  - The value 0 has the *highest* priority, and
  - The higher the number, the lower the priority
- We pop the object which has the highest priority

Operations

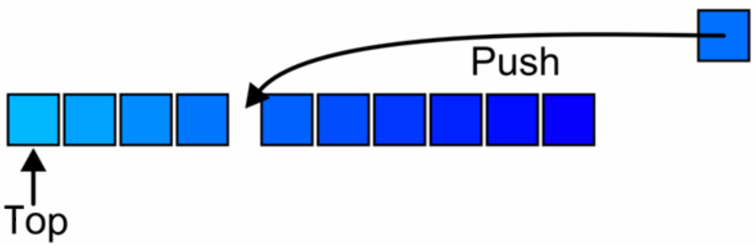
The top of a priority queue is the object with highest priority



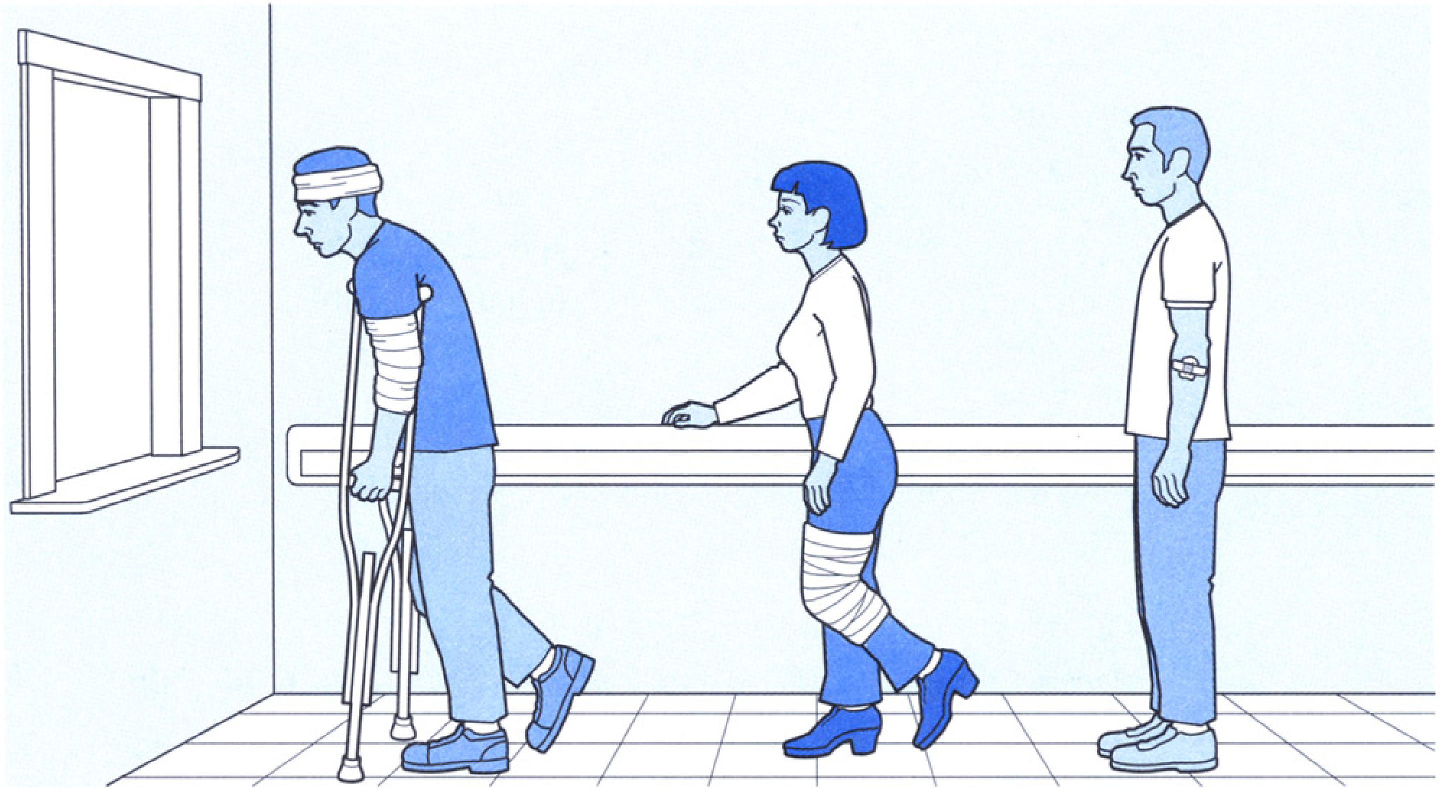
Popping from a priority queue removes the current highest priority object:



Push places a new object into the appropriate place



Application



Implementations

Our goal is to make the run time of each operation as close to $\Theta(1)$ as possible

We will look at an implementation using a data structure called:

– Multiple queues — one for each priority

Then we will introduce a more appropriate data structure: *heap*

Multiple Queues

Assume there is a fixed number of priorities, say M

- Create an array of M queues
- Push a new object onto the queue corresponding to the priority
- Top and pop find the first non-empty queue with highest priority

Multiple Queues

The run times are reasonable:

- Push is $\mathbf{\Theta}(1)$
- Top and pop are both $\mathbf{O}(M)$

Problems:

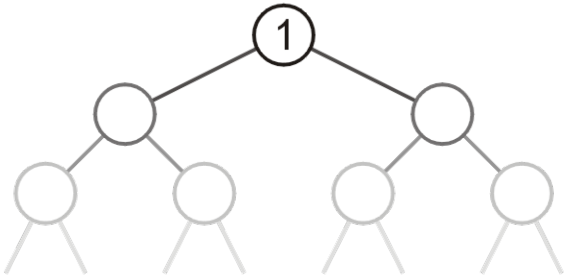
- It restricts the range of priorities
- The memory requirement is $\mathbf{\Theta}(M + n)$

Heaps

Can we do better?

We need a *heap*

- A tree with the top object at the root
- We will look at binary heaps
- Numerous other heaps exists:
  - d-ary heaps
  - Leftist heaps
  - Skew heaps
  - Binomial heaps
  - Fibonacci heaps
  - Bi-parental heaps



Outline

- Priority queue
- Binary heap
- Heapsort

Outline

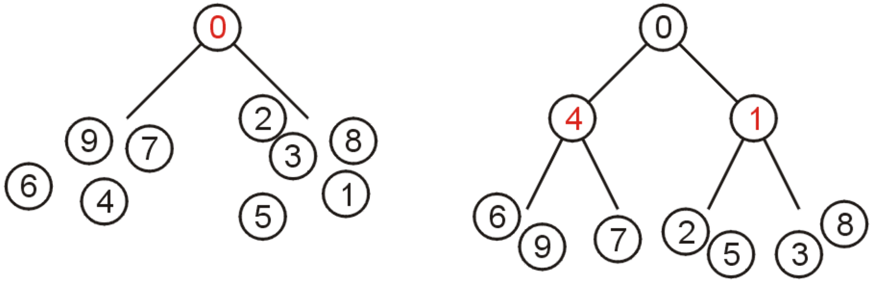
In this topic, we will:

- Define a binary min-heap
- Look at some examples
- Operations on heaps:
  - Top
  - Pop
  - Push
- An array representation of heaps
- Define a binary max-heap
- Using binary heaps as priority queues

Definition

A non-empty tree is a min-heap if

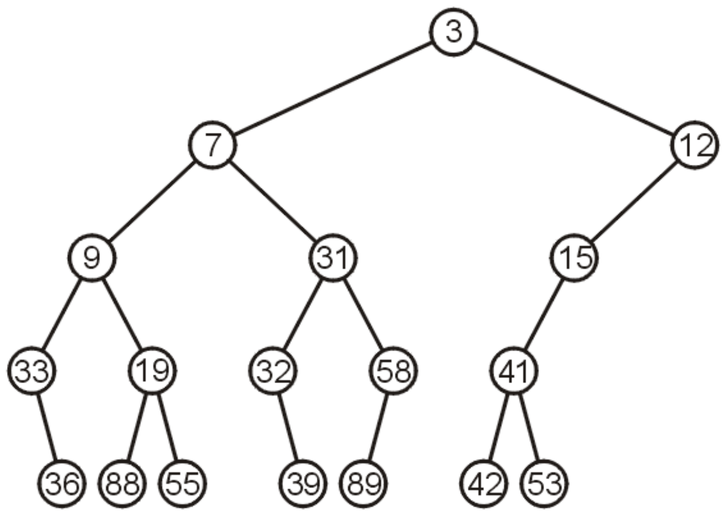
- The key associated with the root is less than or equal to the keys associated with the sub-trees (if any)
- The sub-trees (if any) are also min-heaps



There is no other relationship between the elements in the subtrees!

Example

This is a (*naive*) binary min-heap:



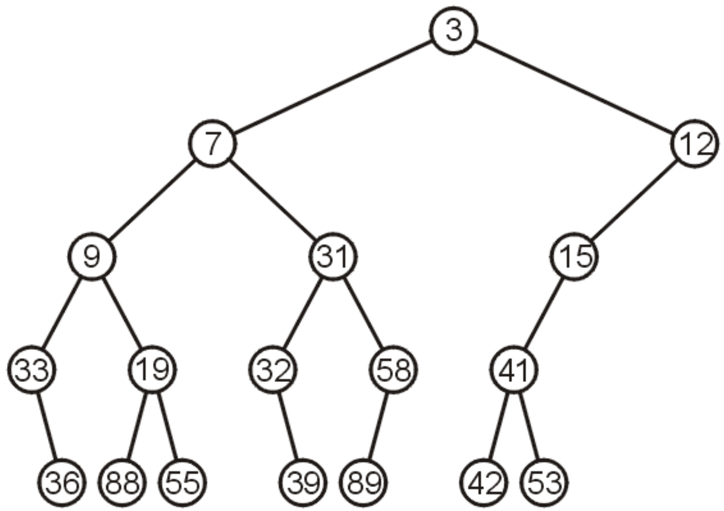
Operations

We will consider three operations:

- Top
- Pop
- Push

Example

We can find the top object in $\Theta(1)$ time: 3



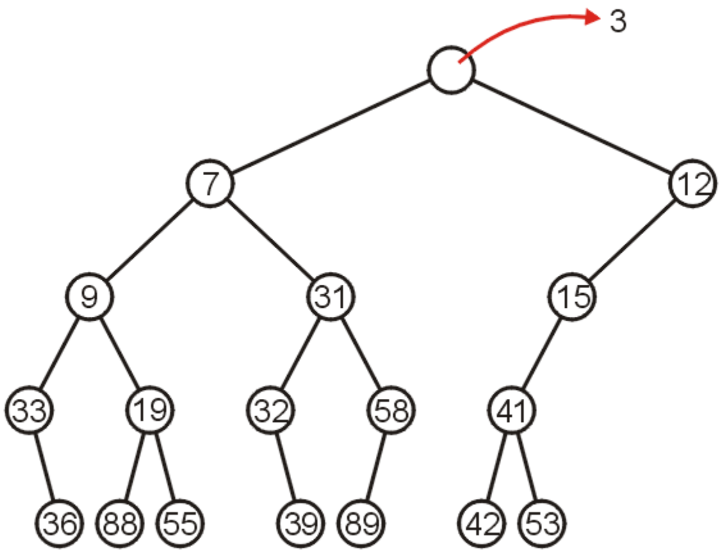
Pop

To remove the minimum object:

- Promote the node of the sub-tree which has the least value
- Recursively process the sub-tree from which we promoted the least value

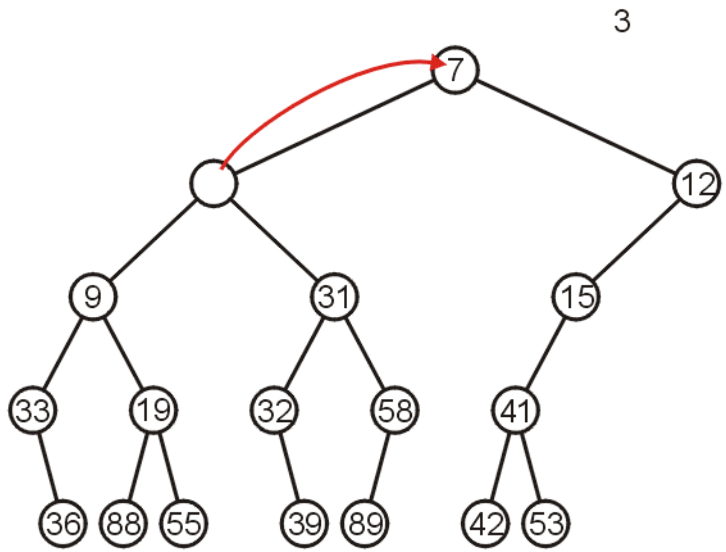
Pop

Using our example, we remove 3:



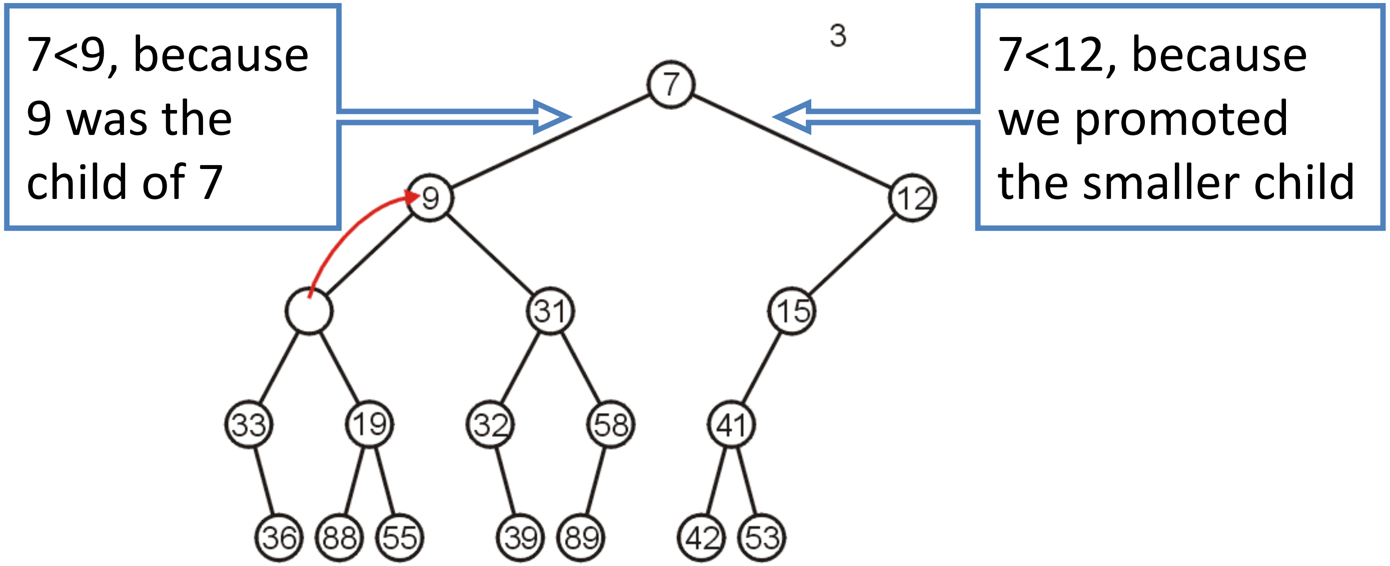
Pop

We promote 7 (the minimum of 7 and 12) to the root:



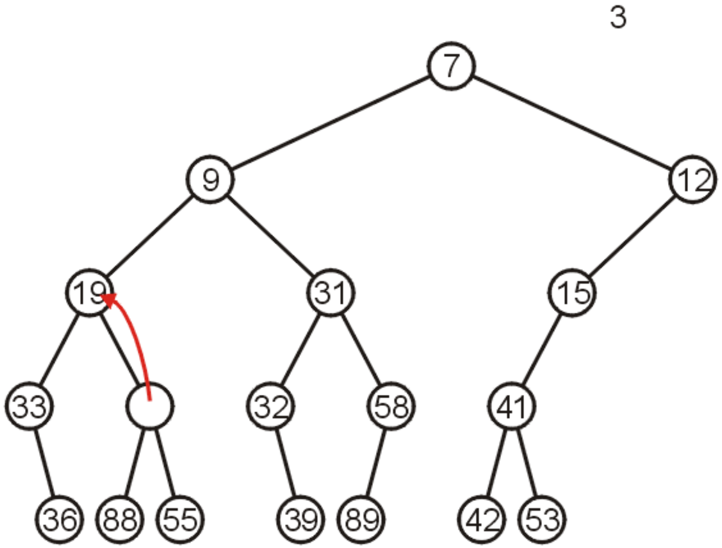
Pop

In the left sub-tree, we promote 9:



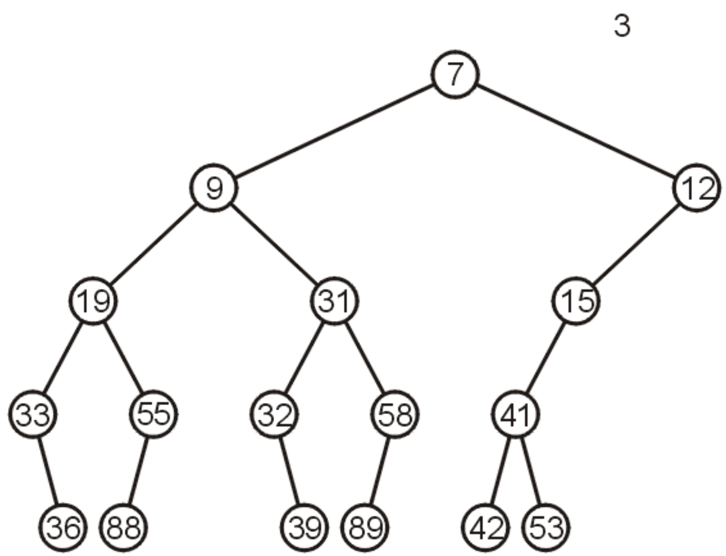
Pop

Recursively, we promote 19:



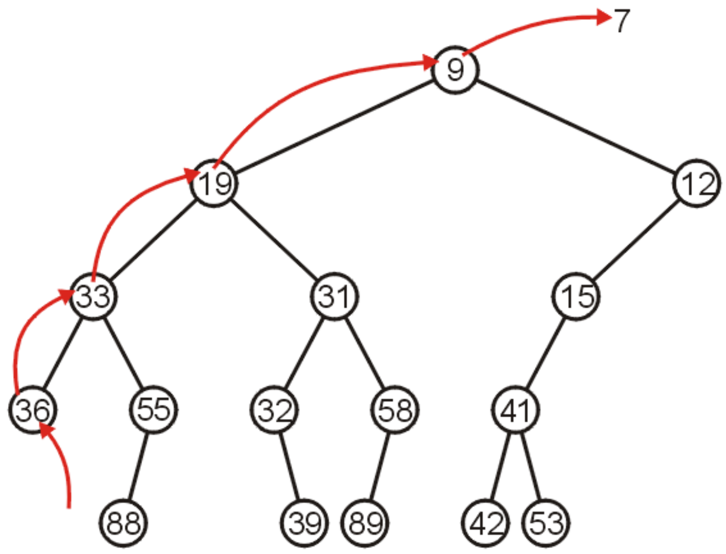
Pop

Finally, 55 is a leaf node, so we promote it and delete the leaf



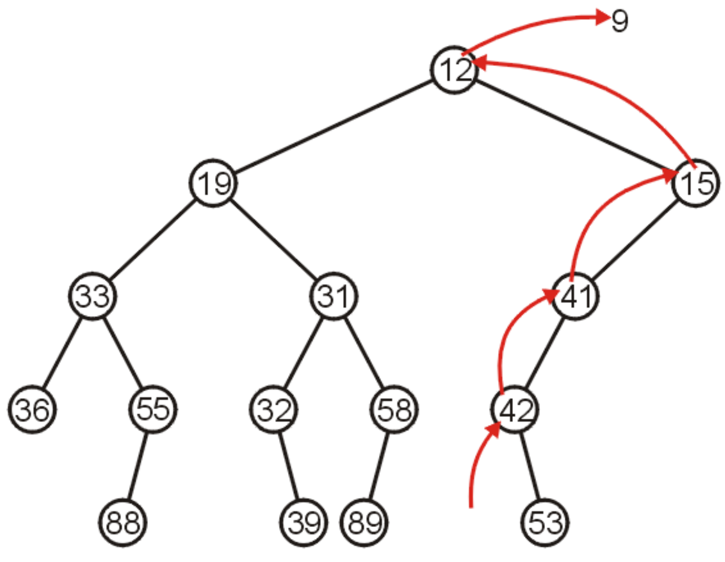
Pop

Repeating this operation again, we can remove 7:



Pop

If we remove 9, we must now promote from the right sub-tree:



Push

Inserting into a heap may be done either:

- At a leaf (move it up if it is smaller than the parent)
- At the root (insert the larger object into one of the subtrees)

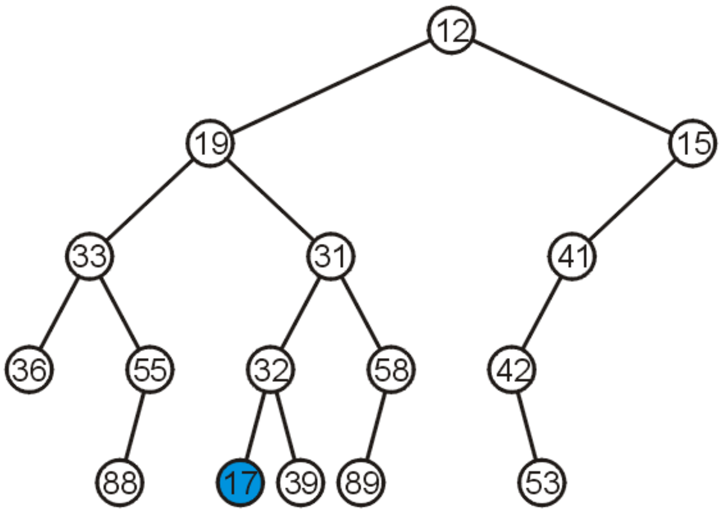
We will use the first approach with binary heaps

- Other heaps use the second

Push

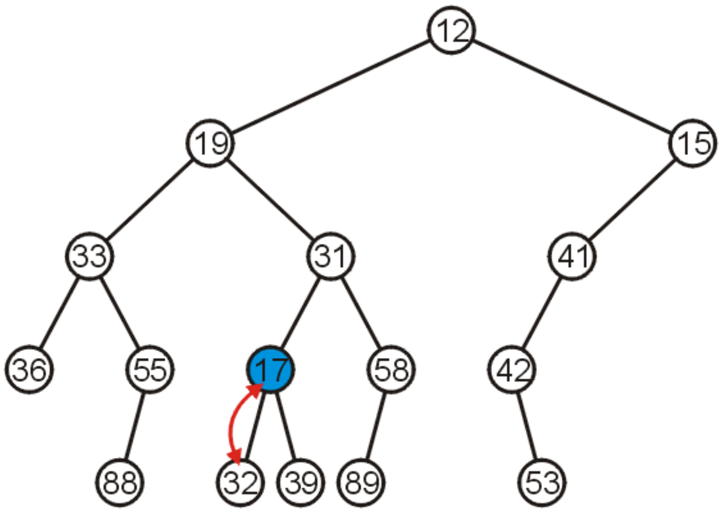
Inserting 17 into the last heap

- Select an arbitrary node to insert a new leaf node:



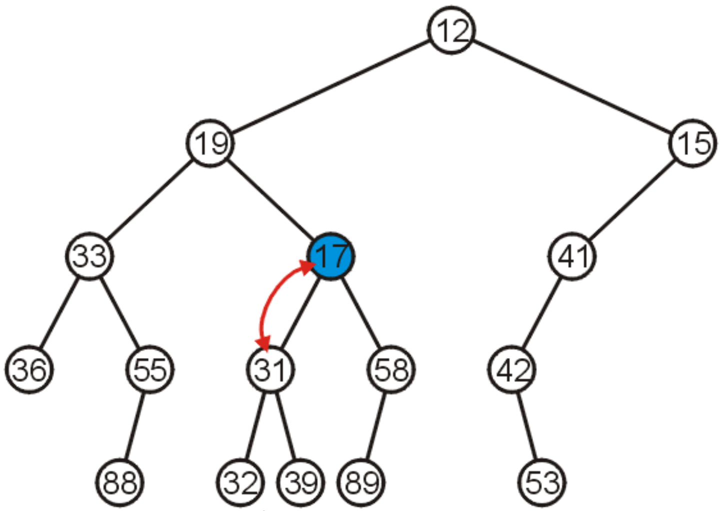
Push

The node 17 is less than the node 32, so we swap them



Push

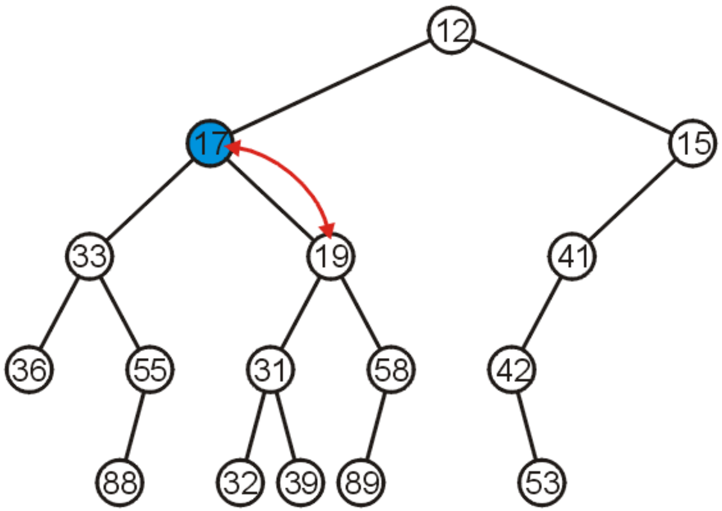
The node 17 is less than the node 31; swap them



31 is larger than 32 and 39 because
31 was the ancestor of 32 and 39

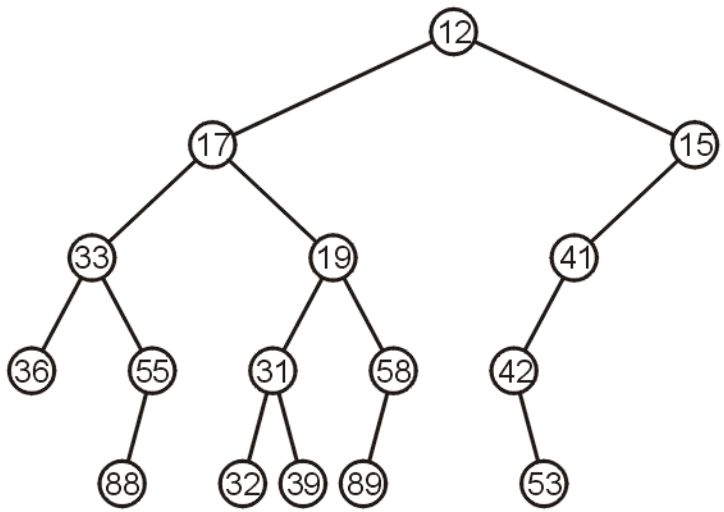
Push

The node 17 is less than the node 19; swap them



Push

The node 17 is greater than 12 so we are finished



Push

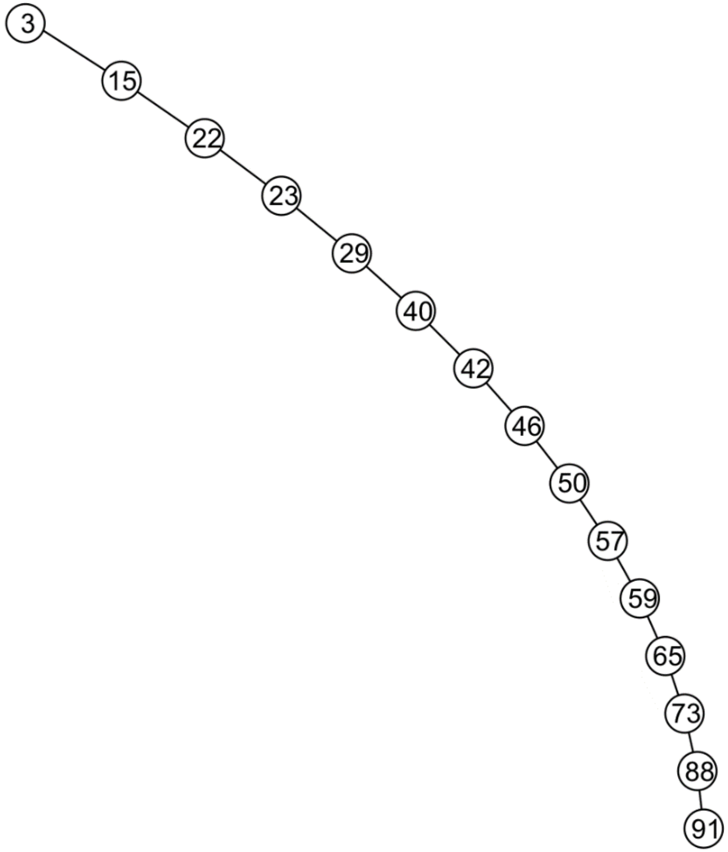
This process is called *percolation*, that is, the lighter (smaller) objects move up from the bottom of the min-heap

Implementations

What is the heap look like by adding the following keys in sequence: 1, 2, 3, 4, 5 (the initial heap is empty)

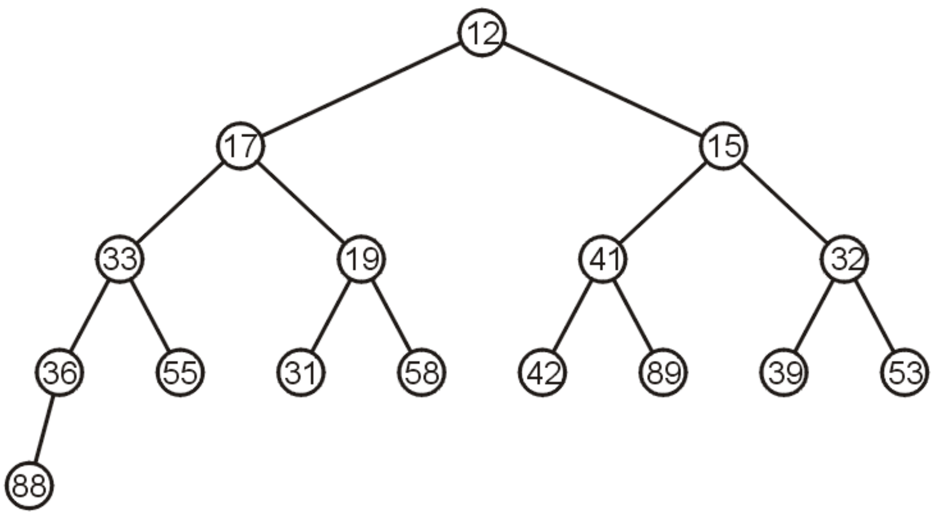
Time Complexity

- Time complexity of pop and push?
  - O(*n*)
  - Worst case: the binary tree is highly unbalanced
- Can we do better?
  - Keep balance of the binary tree



Complete Trees

For example, the previous heap may be represented as the following (non-unique!) complete tree:

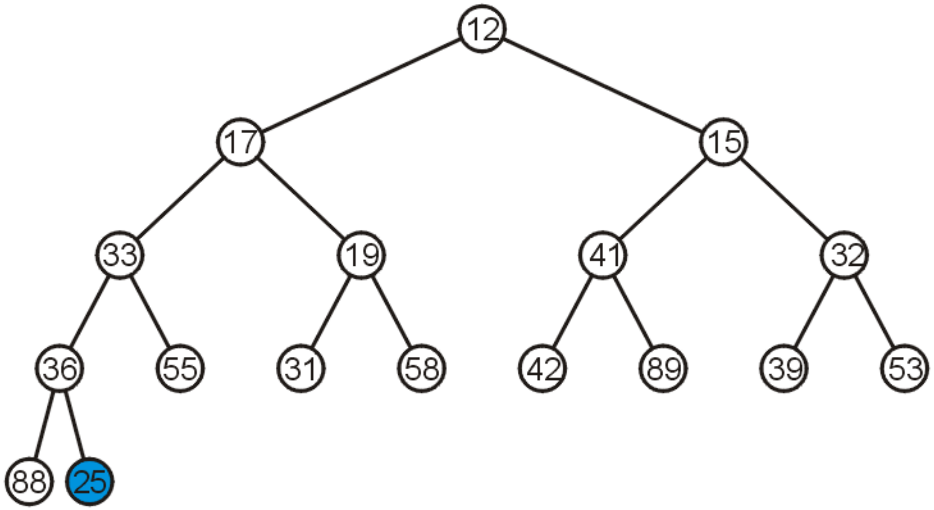


Complete Trees: Push

If we insert into a complete tree, we need only place the new node as a leaf node in the appropriate location and percolate up

Complete Trees: Push

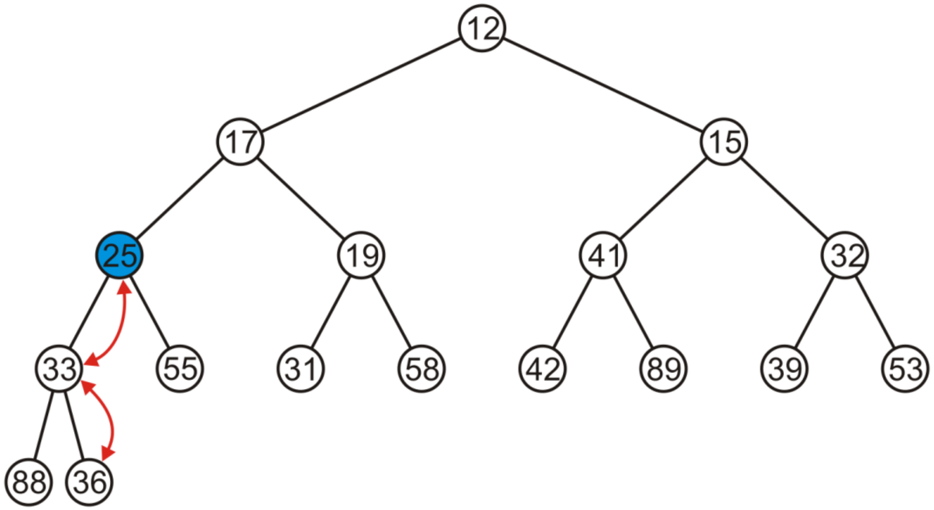
For example, push 25:



Complete Trees: Push

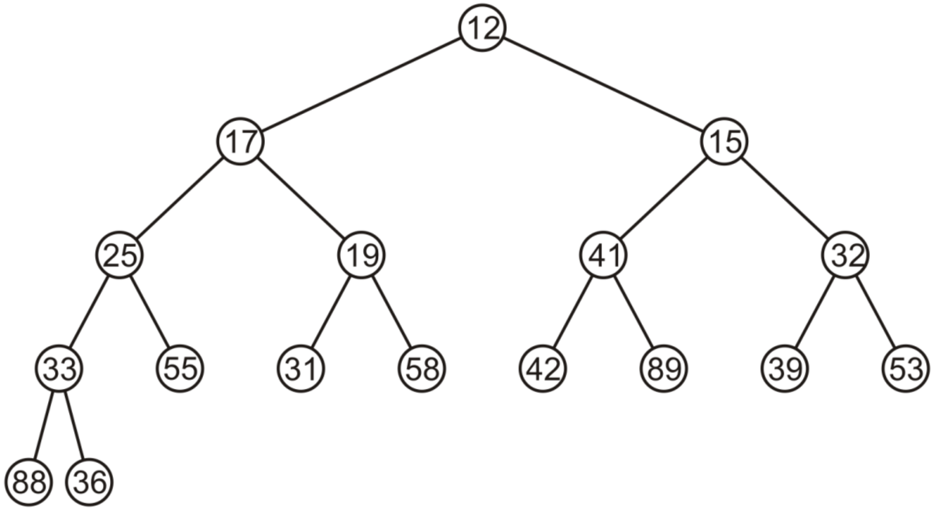
We have to percolate 25 up into its appropriate location

– The resulting heap is still a complete tree



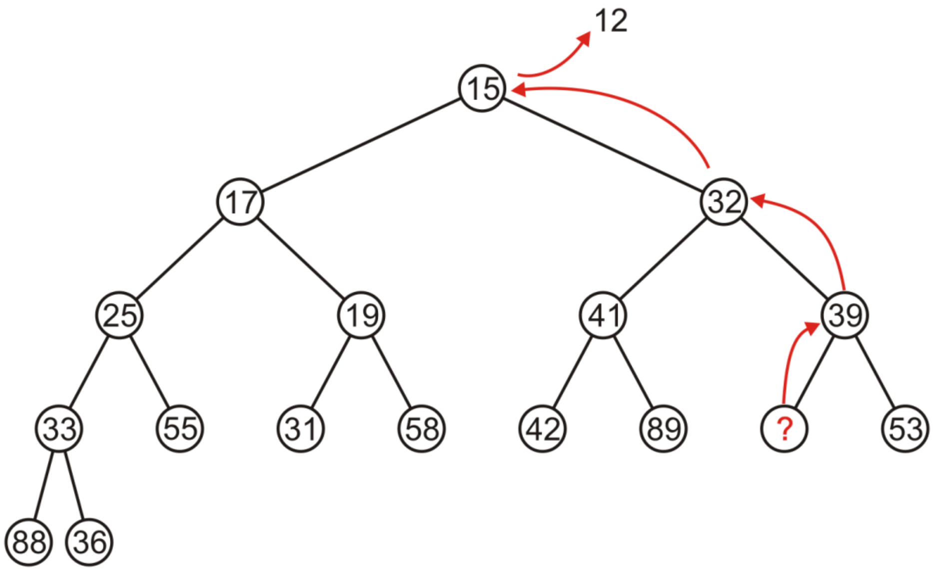
Complete Trees: Pop

Suppose we want to pop the top entry: 12



Complete Trees: Pop

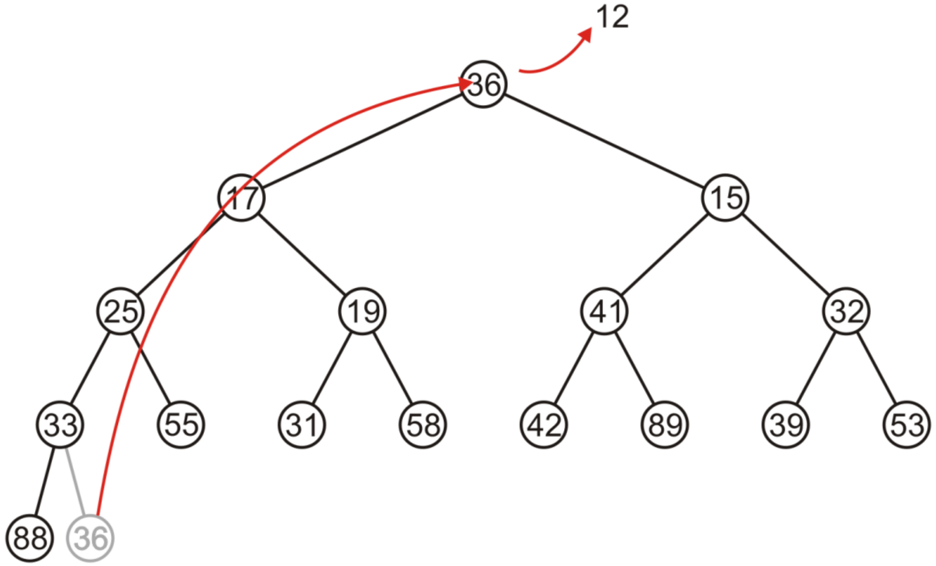
Percolating up creates a hole leading to a non-complete tree



What's wrong?

Complete Trees: Pop

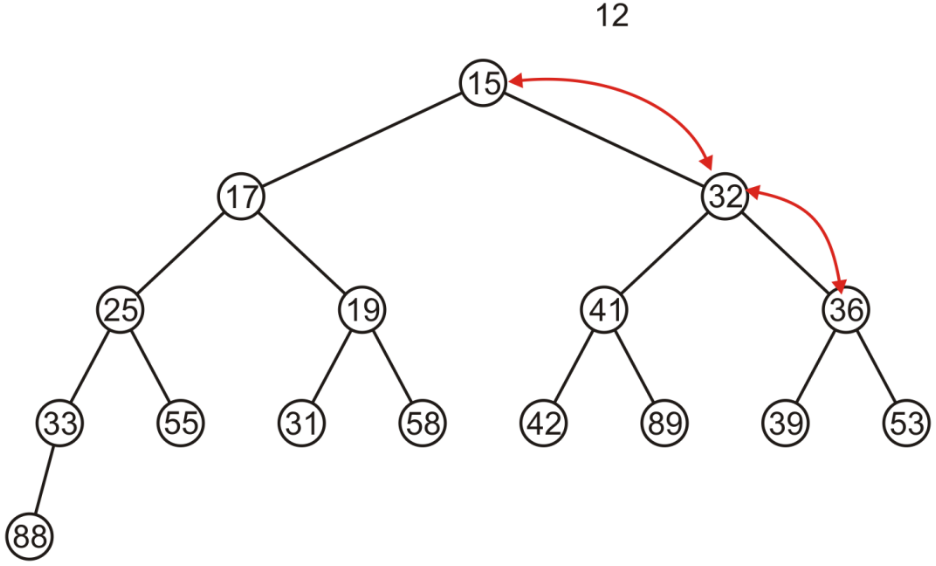
Instead, copy the last entry in the heap to the root



Complete Trees: Pop

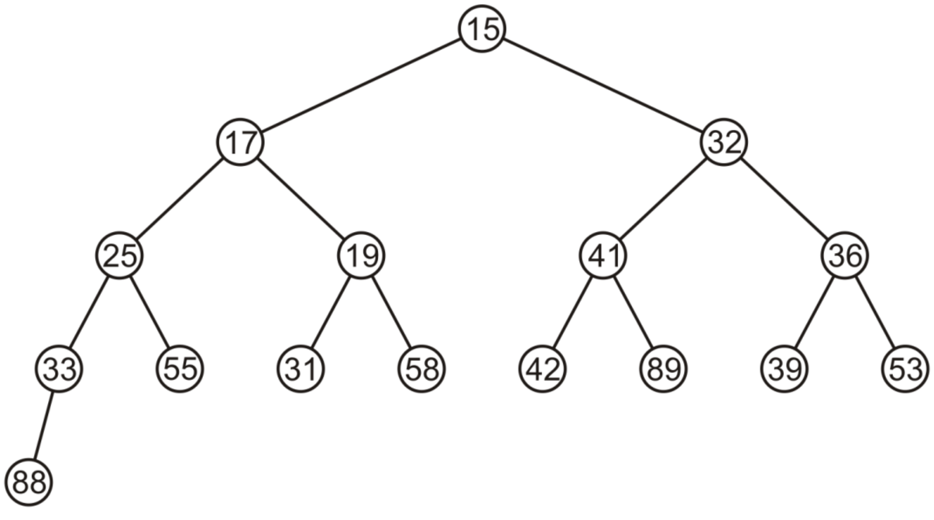
Now, percolate 36 down swapping it with the smallest of its children

– We halt when both children are larger



Complete Trees: Pop

The resulting tree is now still a complete tree:



Complete Tree

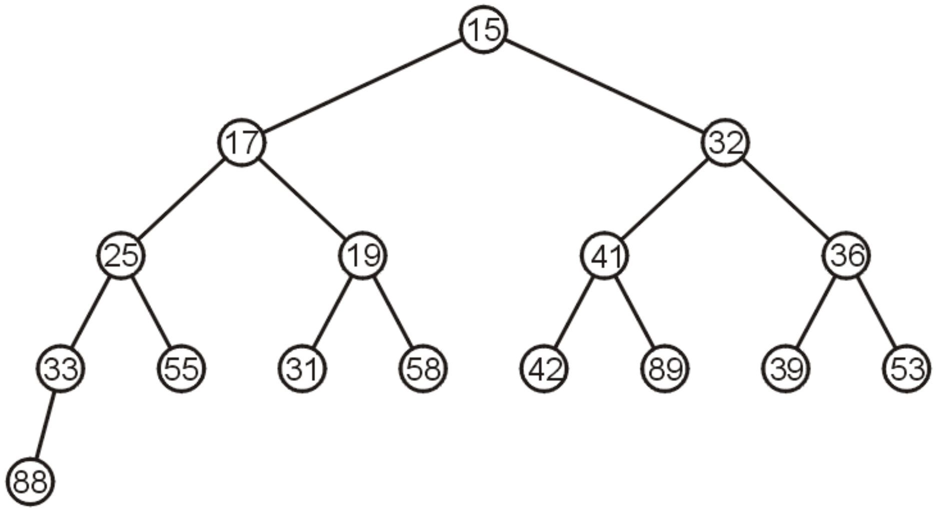
Therefore, we can maintain the complete-tree shape of a heap

We may store a complete tree using an array:

- The array is filled using breadth-first traversal on the tree

Array Implementation

For the heap

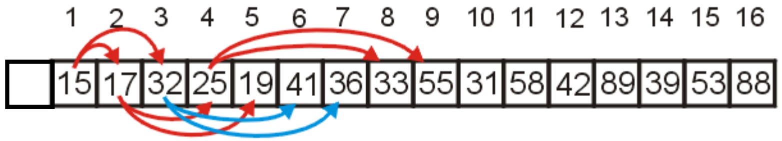


a breadth-first traversal yields:

	15	17	32	25	19	41	36	33	55	31	58	42	89	39	53	88

Array Implementation

We start at index 1 when filling the array.



Given the entry at index k, it follows that:

- The parent of node is a $k/2$
- the children are at $2k$ and $2k + 1$

```
parent = k >> 1;

left_child = k << 1;
right_child = left_child | 1;
```

Array Implementation

If the heap-as-array has **`count`** entries, then the next empty node in the corresponding complete tree is at location **`posn = count + 1`**

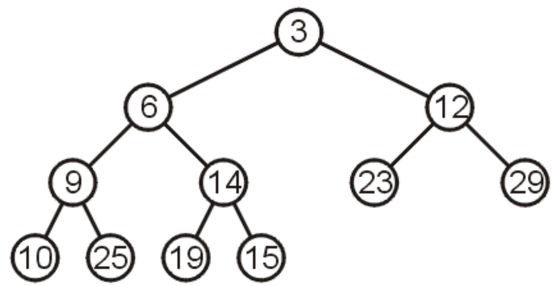
We compare the item at location **`posn`** with the item at **`posn/2`**

If they are out of order

- Swap them
- Set **`posn /= 2`** and repeat

Array Implementation

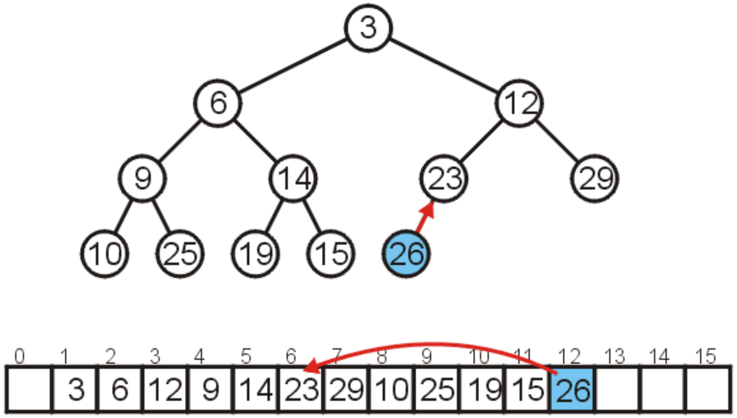
Consider the following heap, both as a tree and in its array representation



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	3	6	12	9	14	23	29	10	25	19	15				

Array Implementation: Push

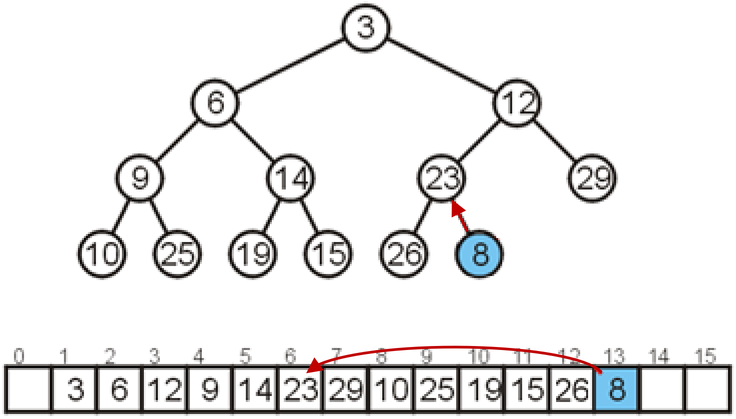
Inserting 26 requires no changes



Array Implementation: Push

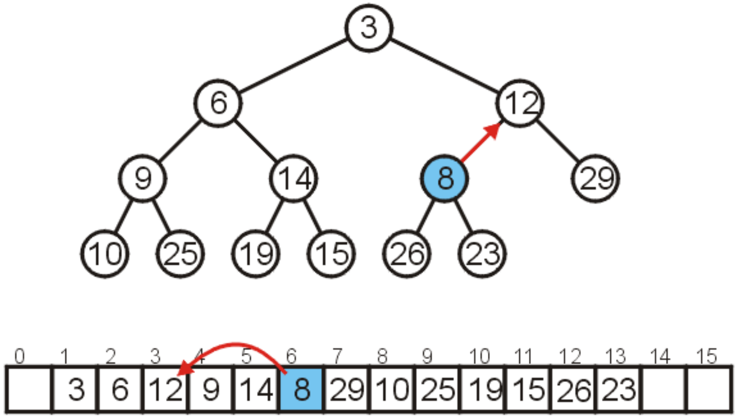
Inserting 8 requires a few percolations:

– Swap 8 and 23



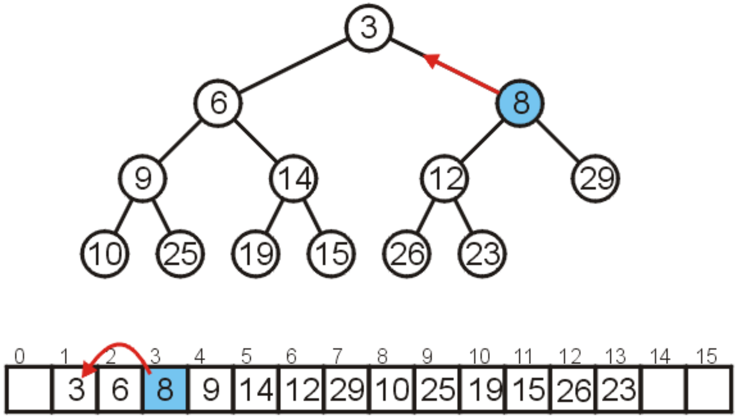
Array Implementation: Push

Swap 8 and 12



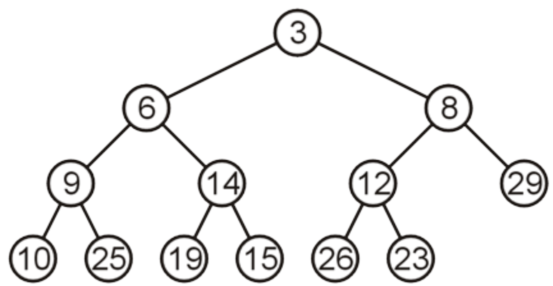
Array Implementation: Push

At this point, it is greater than its parent, so we are finished



Array Implementation: Pop

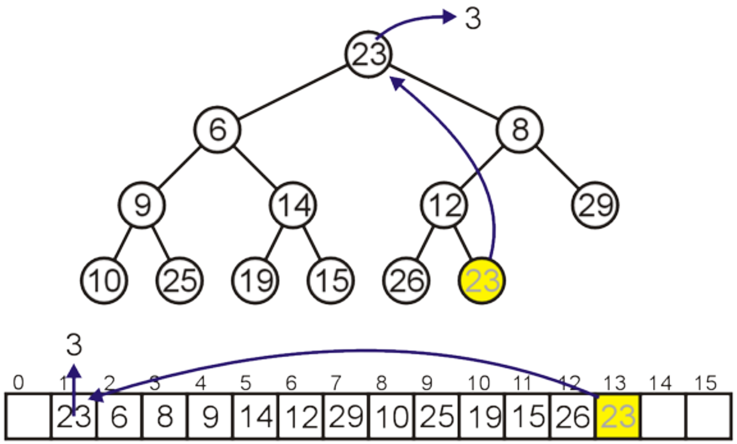
As before, popping the top has us copy the last entry to the top



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	3	6	8	9	14	12	29	10	25	19	15	26	23		

Array Implementation: Pop

As before, popping the top has us copy the last entry to the top

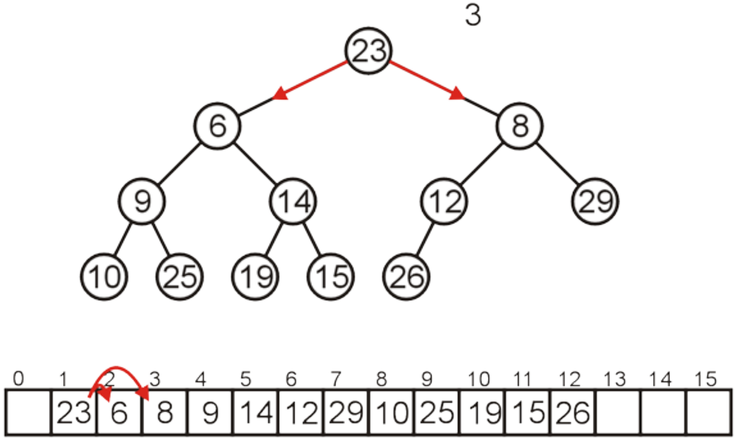


Array Implementation: Pop

Now percolate down

Compare Node 1 with its children: Nodes 2 and 3

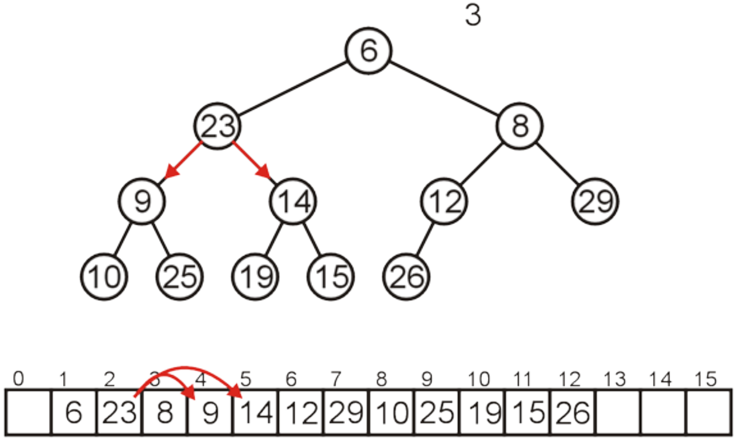
- Swap 23 and 6



Array Implementation: Pop

Compare Node 2 with its children: Nodes 4 and 5

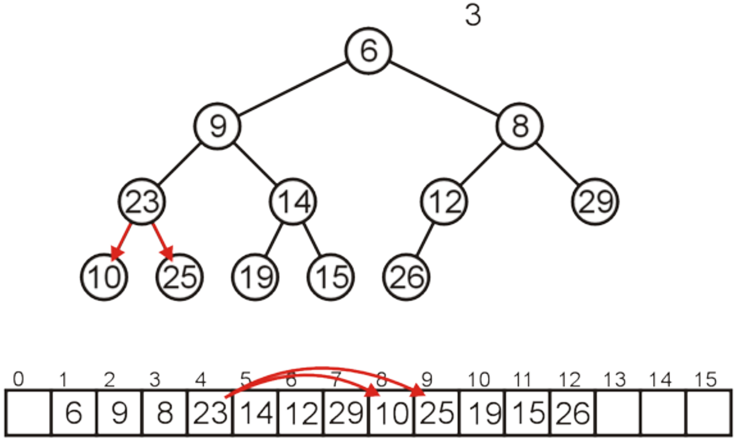
– Swap 23 and 9



Array Implementation: Pop

Compare Node 4 with its children: Nodes 8 and 9

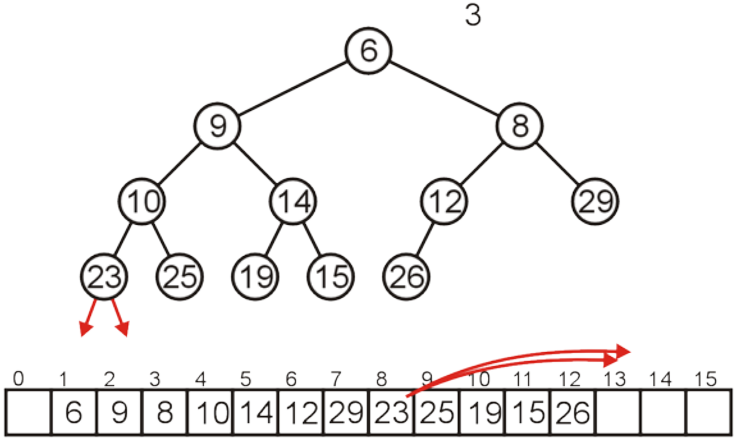
- Swap 23 and 10



Array Implementation: Pop

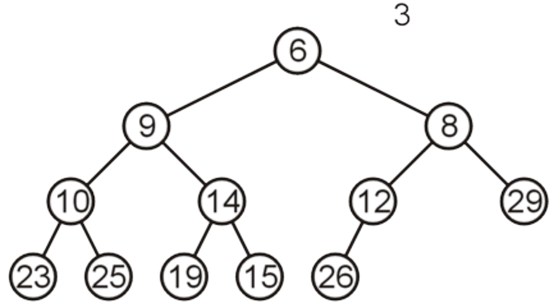
The children of Node 8 are beyond the end of the array:

– Stop



Array Implementation: Pop

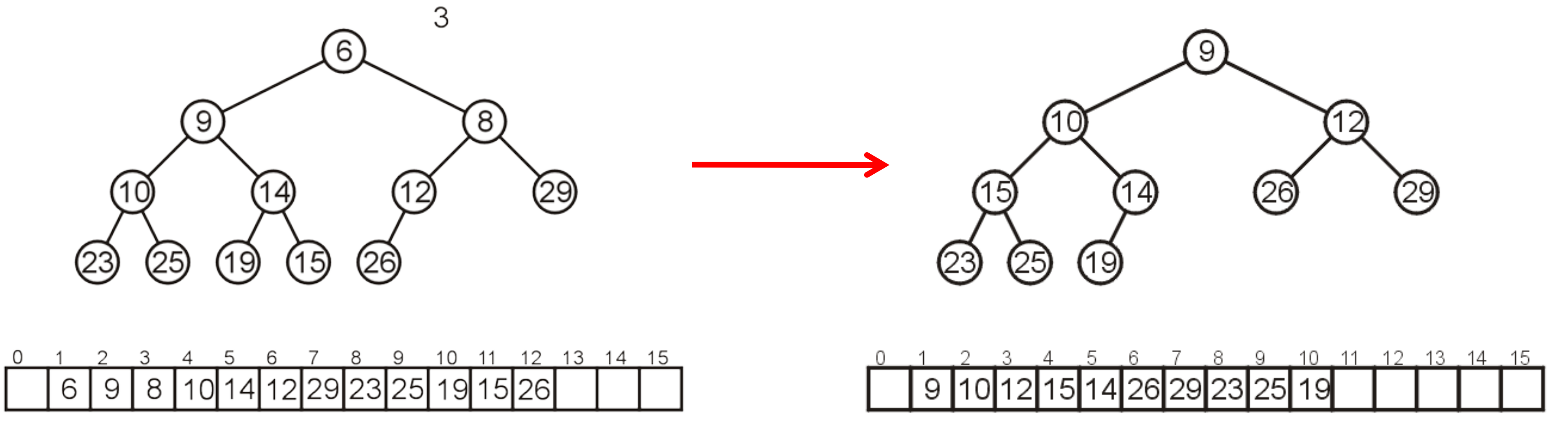
The result is a binary min-heap



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	6	9	8	10	14	12	29	23	25	19	15	26			

Array Implementation: Pop

Pop two more times, the final heap is



Run-time Analysis

Accessing the top object is $\mathbf{\Theta}(1)$

Popping the top object is $\mathbf{O}(\ln(n))$

– We copy something that is already in the lowest depth—it will likely be moved back to the lowest depth

Pushing an object is also $\mathbf{O}(\ln(n))$

– If we insert an object less than the root, it will be moved up to the top

Space complexity $\mathbf{O}(n)$

So binary heap is a better implementation of priority queue

Run-time Analysis (Push)

If we are inserting an object less than the root (at the front), then the run time will be $\Theta(\ln(n))$

If we insert at the back (greater than any object) then the run time will be $\Theta(1)$

How about an arbitrary insertion?

- It will be $\mathbf{O}(\ln(n))$? Could the average be less?

Run-time Analysis (Push)

With each percolation, it will move an object past half of the remaining entries in the tree

- Therefore after one percolation, it will probably be past half of the entries, and therefore *on average* will require no more percolations

$$\frac{1}{n}\sum_{k=0}^{h}(h-k)2^k = \frac{2^{h+1}-h-2}{n}$$

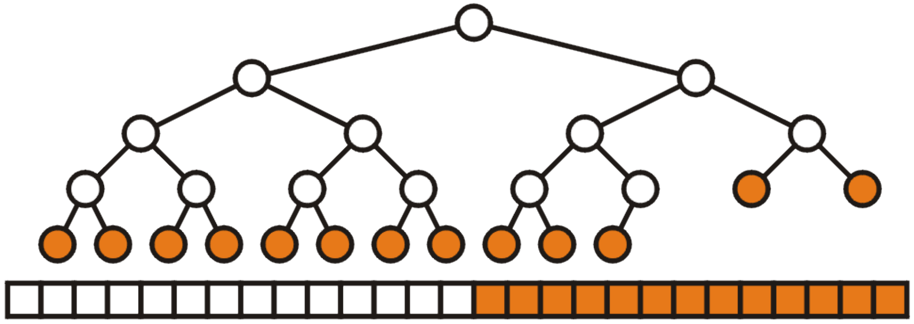
$$= \frac{n-h-1}{n} = \mathbf{\Theta}(1)$$

Therefore, we have an average run time of $\mathbf{\Theta}(1)$

Run-time Analysis (Remove)

An arbitrary removal requires that all entries in the heap be checked: $\mathbf{O}(n)$

A removal of the largest object in the heap still requires all leaf nodes to be checked – there are approximately $n/2$ leaf nodes: $\mathbf{O}(n)$



Build Heap

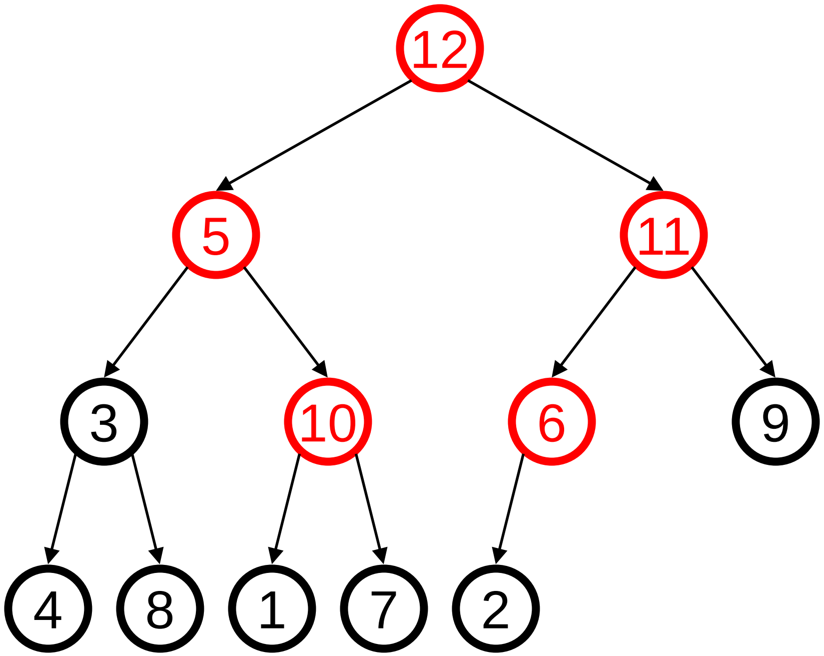
- Task: Given a set of n keys, build a heap all at once

- Approach 1
  - Repeatedly perform push
- Complexity
  - $\mathbf{O}(n \ln(n))$

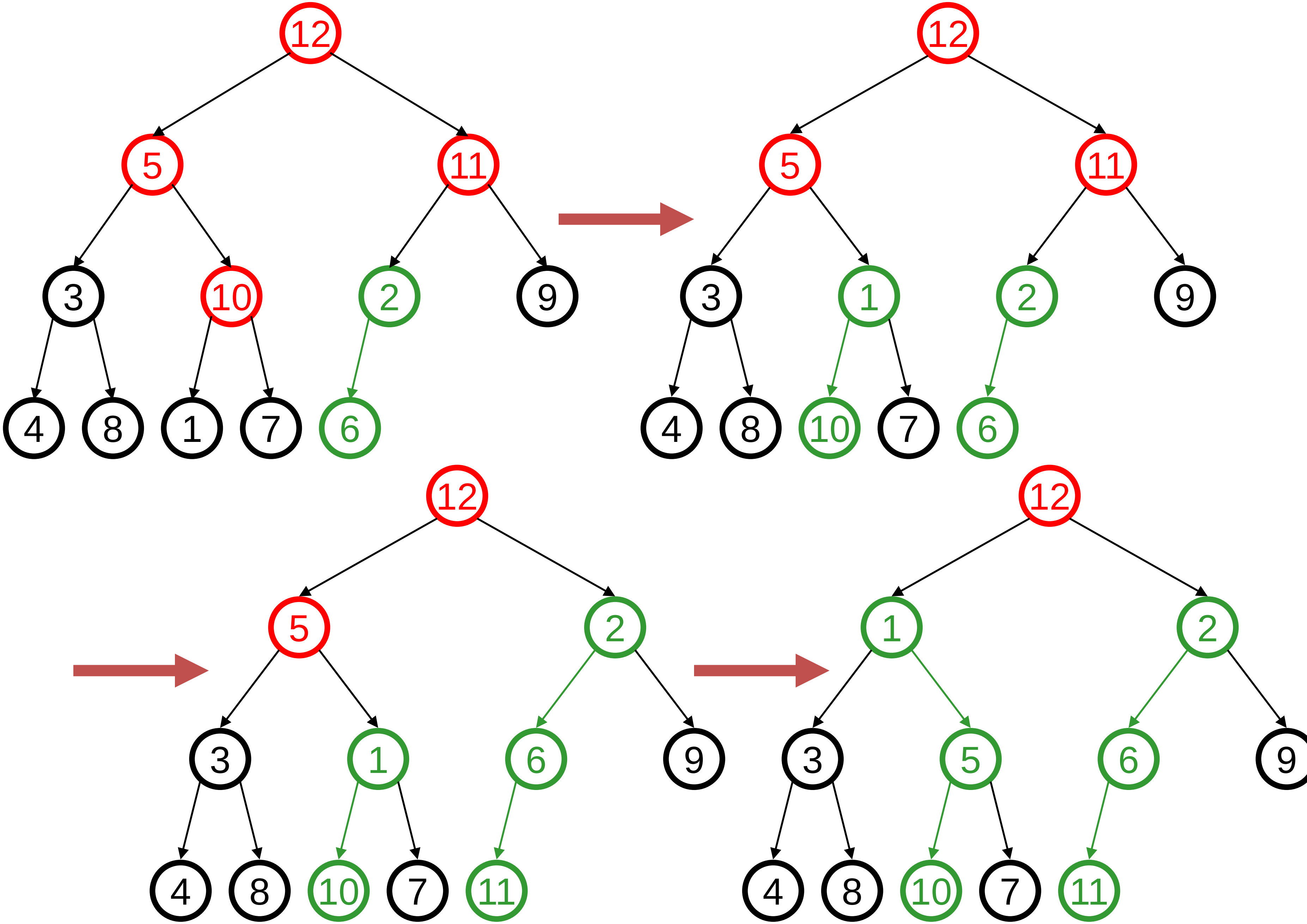
Floyd’s Method

Put the keys in a binary tree and fix the heap property!

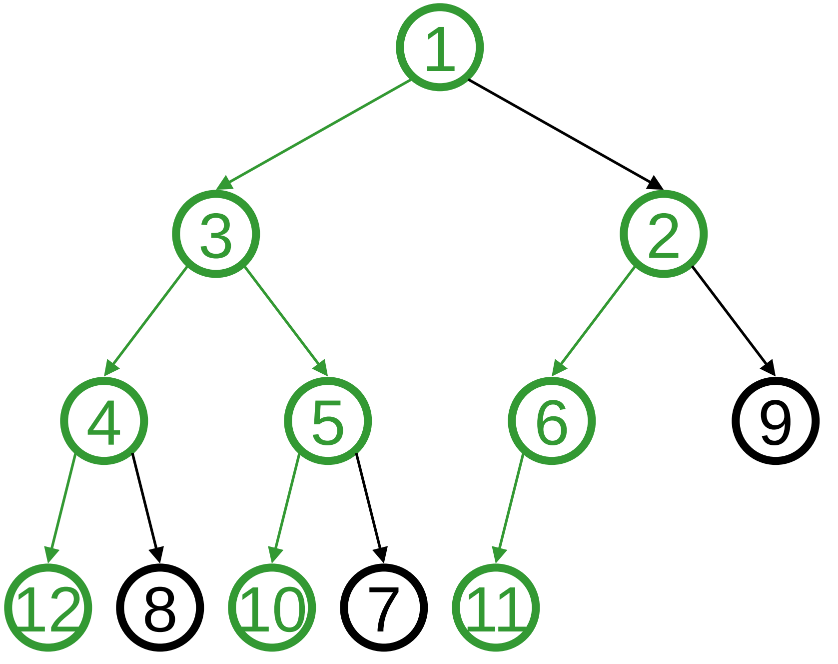
```
buildHeap(){
  for (i=size/2; i>0; i--)
     percolateDown(i);
}
```



12	5	11	3	10	6	9	4	8	1	7	2


12
5
11
3
10
2
9
4
8
1
7
6
12
5
11
3
1
2
9
4
8
10
7
6
12
5
2
3
1
6
9
4
8
10
7
11
12
1
2
3
5
6
9
4
8
10
7
11

Finally…



Complexity of Build Heap

- No percolation for the leaf nodes (n/2 nodes)
- At most n/4 nodes percolate down 1 level
  at most n/8 nodes percolate down 2 levels
  at most n/16 nodes percolate down 3 levels
  …

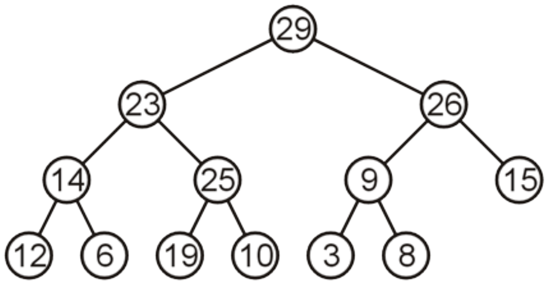
$$1\frac{n}{4} + 2\frac{n}{8} + 3\frac{n}{16} + \cdots = \sum_{i=1}^{\log n} i\,\frac{n}{2^{i+1}} = \frac{n}{2}\sum_{i=1}^{\log n}\frac{i}{2^i} = n$$

Θ(*n*)

Binary Max Heaps

A binary max-heap is identical to a binary min-heap except that the parent is always larger than either of the children

For example, the same data as before stored as a max-heap yields



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	29	23	26	14	25	9	15	12	6	19	10	3	8		

Outline

- Priority queue
- Binary heap
- Heapsort

Heapsort

- Sorting
  - take a list of objects $(a_0, a_1, ..., a_{n-1})$
  - return a reordering $(a'_0, a'_1, ..., a'_{n-1})$ such that $a'_0 \le a'_1 \le \cdots \le a'_{n-1}$
- Heapsort
  - Place the objects into a heap
    - O(*n*) time
  - Repeatedly popping the top object until the heap is empty
    - O(*n* ln(*n*)) time
  - Time complexity: O(*n* ln(*n*))

In-place Implementation

Problem:

– This solution requires additional memory: a min-heap of size n

– This requires $\Theta(n)$ memory

If the unsorted objects are stored in an array, is it possible to perform a heap sort in place, i.e., require at most $\Theta(1)$ memory (a few extra variables)?

In-place Implementation

Instead of implementing a min-heap, consider a max-heap:

– The maximum element is at the top of the heap

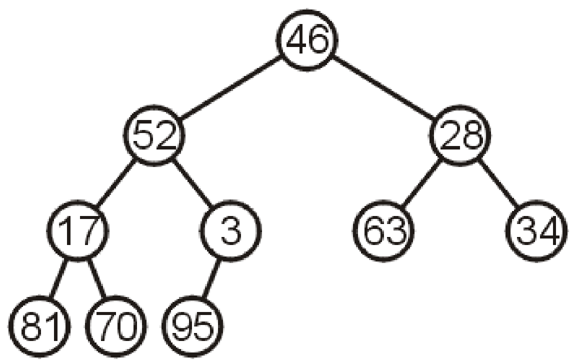
We then repeatedly pop the top object and move it to the end of the array.

In-place Implementation

Now, consider this unsorted array:

46	52	28	17	3	63	34	81	70	95

This array represents the following complete tree:

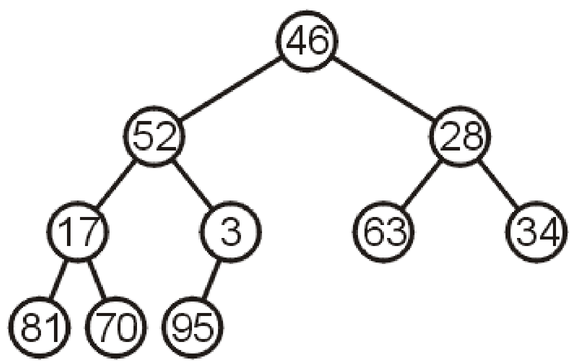


In-place Implementation

Now, consider this unsorted array:

46	52	28	17	3	63	34	81	70	95

Because we start at 0 (instead of 1 as in array storage of complete trees), we need different formulas for finding the children and parent



Children	`2*k + 1`	`2*k + 2`
Parent	`(k + 1)/2 - 1`	

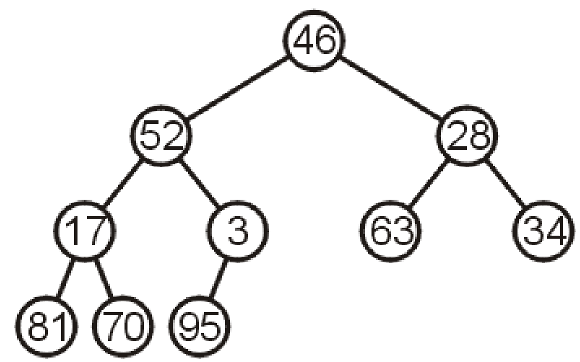
Example Heap Sort

First, we must convert the unordered array with $n = 10$ elements into a max-heap

46	52	28	17	3	63	34	81	70	95

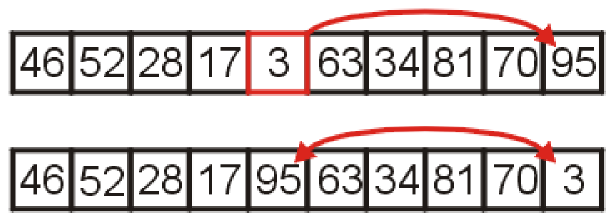
None of the leaf nodes need to be percolated down, and the last non-leaf node is in position $n/2-1$

Thus we start with position $10/2-1 = 4$



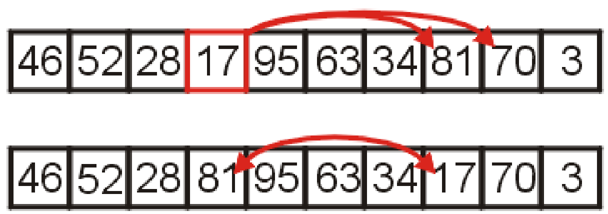
Example Heap Sort

We compare 3 with its child and swap them



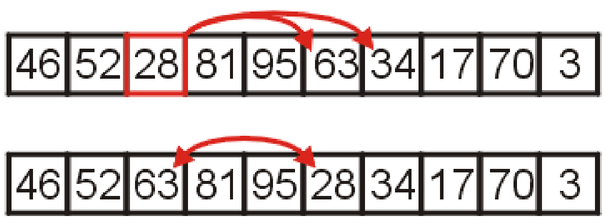
Example Heap Sort

We compare 17 with its two children and swap it with the maximum child (81)



Example Heap Sort

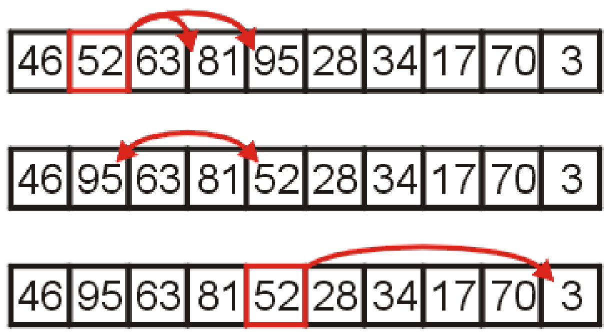
We compare 28 with its two children, 63 and 34, and swap it with the largest child



Example Heap Sort

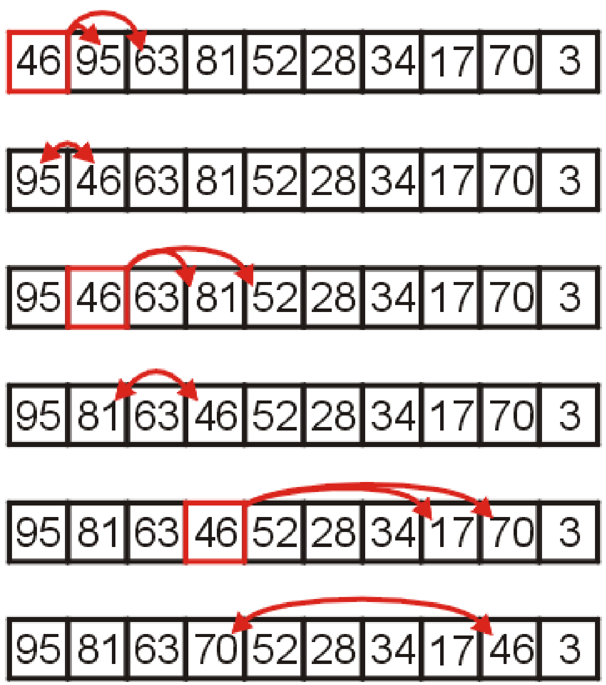
We compare 52 with its children, swap it with the largest

- Recursing, no further swaps are needed



Example Heap Sort

Finally, we swap the root with its largest child, and recurse, swapping 46 again with 81, and then again with 70



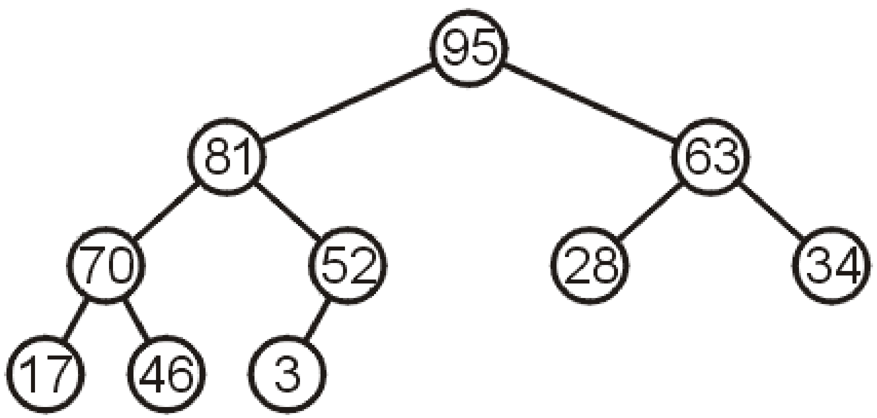
Heap Sort Example

We have now converted the unsorted array

46	52	28	17	3	63	34	81	70	95

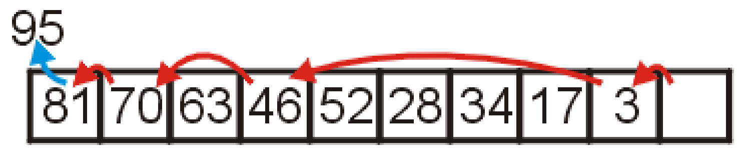
into a max-heap:

95	81	63	70	52	28	34	17	46	3

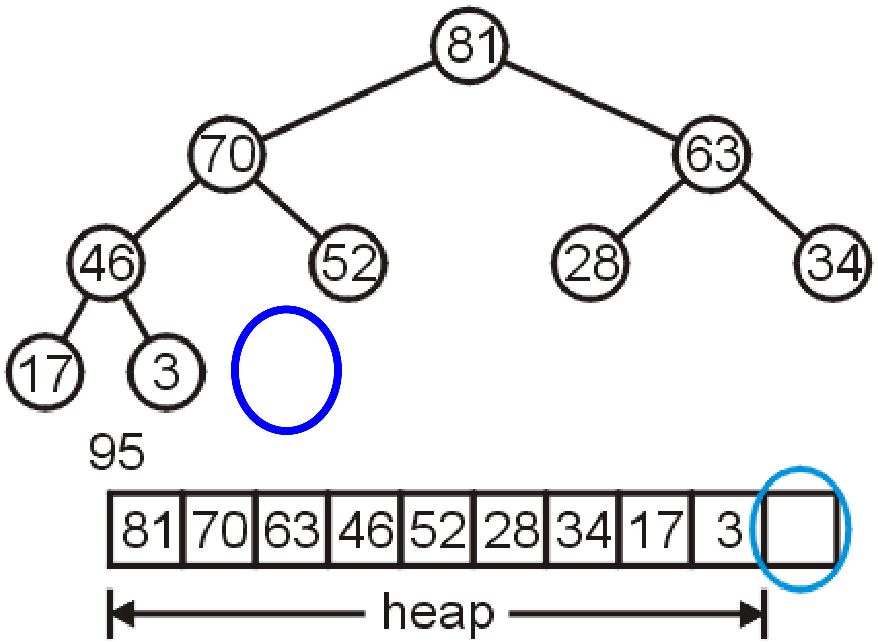


Heap Sort Example

We pop the maximum element of this heap

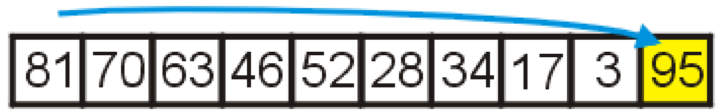


This leaves a gap at the back of the array:

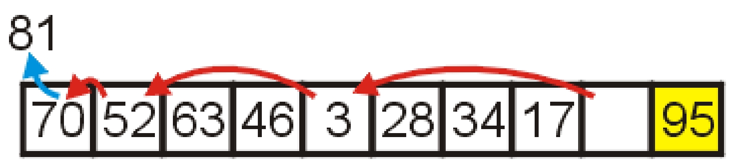


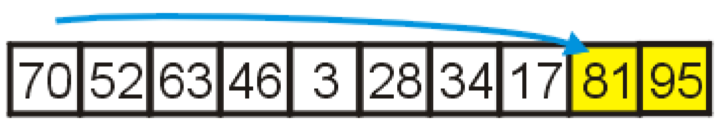
Heap Sort Example

This is the last entry in the array, so why not fill it with the largest element?



Repeat this process: pop the maximum element, and then insert it at the end of the array:

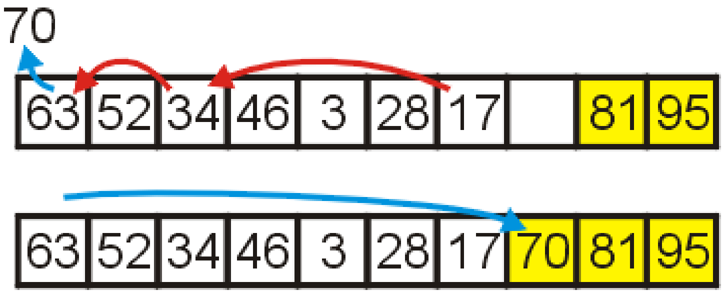




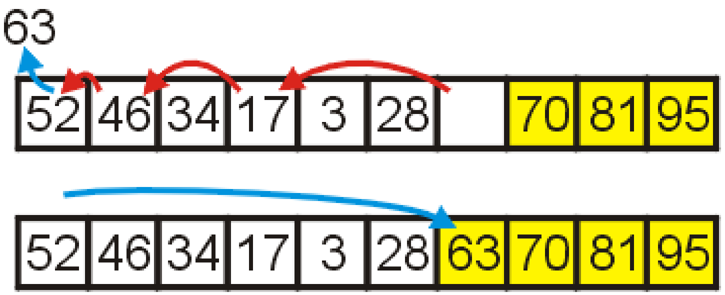
Heap Sort Example

Repeat this process

- Pop and append 70



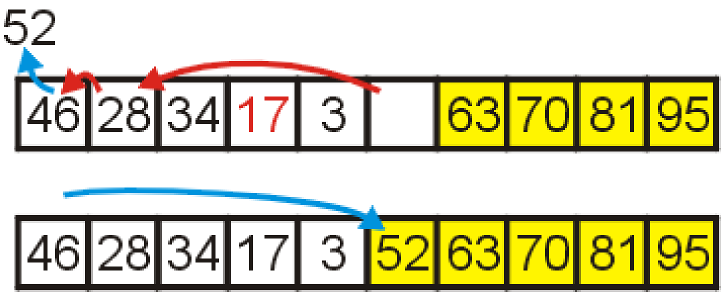
- Pop and append 63



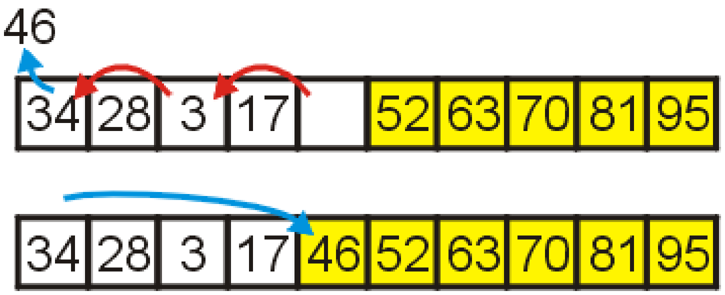
Heap Sort Example

We have the 4 largest elements in order

– Pop and append 52



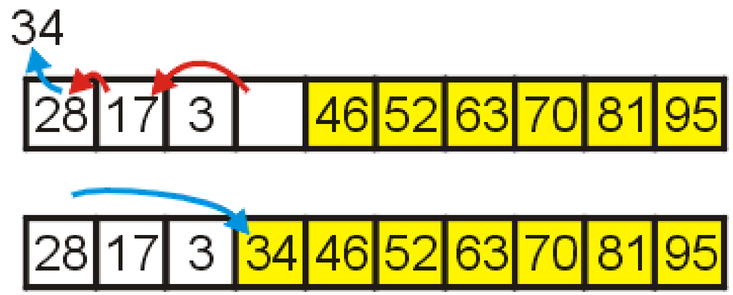
– Pop and append 46



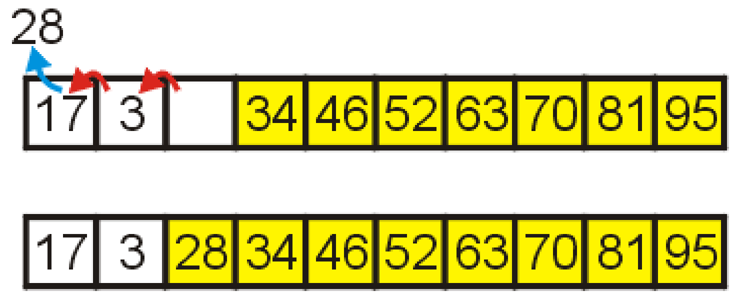
Heap Sort Example

Continuing...

- Pop and append 34

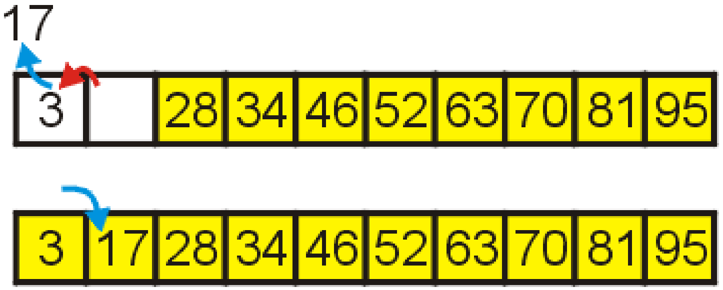


- Pop and append 28



Heap Sort Example

Finally, we can pop 17, insert it into the 2nd location, and the resulting array is sorted



Summary

- Priority queue
  - pop the object with the highest priority
- Binary heap
  - Operations
    - Top $\Theta(1)$
    - Push $O(\ln(n))$
    - Pop $O(\ln(n))$
    - Build $O(n)$
  - Implementation using arrays
- Heapsort
  - Time: $O(n \ln(n))$
  - Space: $O(1)$