CS101 Algorithms and Data Structures

Hash Table

Textbook Ch 11

Outline

- Introduction
- Hash function
- Mapping down to 0, ..., M – 1
- Dealing with collisions
  - Chained hash tables
  - Open addressing

Supporting Example

Suppose we have a system which is associated with approximately 150 error conditions where

- Each of which is identified by an 16-bit number from 0 to 65535, and
- When an identifier is received, a corresponding error-handling function must be called

We could create an array of 150 function pointers and to then call the appropriate function….

Supporting Example

```cpp
#include <iostream>

void a() {
    std::cout
        << "Calling 'void a()'"
        << std::endl;
}

void b() {
    std::cout
        << "Calling 'void b()'"
        << std::endl;
}
```

```cpp
int main() {
    void (*function_array[150])();
    unsigned int error_id[150];

    function_array[0] = a;
    error_id[0] = 3;
    function_array[1] = b;
    error_id[1] = 8;

    function_array[0]();
    function_array[1]();

    return 0;
}
```

Supporting Example

Given an error-condition identifier, e.g., `id = 198`, how shall we determine which of the 150 slots corresponds to it?

- Binary search!

Problems

- This is slow: it would require approximately 7 comparisons per error condition
- Slow to dynamically add new error conditions or remove defunct conditions

Supporting Example

A better solution:

- Create an array of size 65536
- Assign those entries corresponding to valid error conditions

```
int main() {
    void (*function_array[65536])();
    for ( int i = 0; i < 65536; ++i ) {
        function_array[i] = nullptr;
    }

    function_array[3] = a;
    function_array[8] = b;

    function_array[3]();
    function_array[8]();

    return 0;
}
```

Problem: additional memory usage

IP Addresses

Examples:

Suppose we want to associate IP addresses and any corresponding domain names

Recall that a 32-bit IP address are often written as four byte values from 0 to 255

- Consider $10000001\ 01100001\ 00001010\ 10110011_2$
- This can be written as `129.97.10.179`
- We use domain names because IP addresses are not human readable

IP Addresses

Given an IP address, sometimes we wanted to *quickly* find any associated domain name.

We could create an array of size 2^{32}= 4,294,967,296 of strings!

```
string domain_name[4294967296];
```

For example, the IP address of shanghaitech.edu.cn is 10.15.42.202

– As $202 + 42\times2^8 + 15\times2^{16} + 10\times2^{24} =$ 168766154, it follows that

```
domain_name[168766154] = "shanghaitech.edu.cn";
```

IP Addresses

Given an IP address, sometimes we wanted to *quickly* find any associated domain name.

We could create an array of size 2^{32}= 4,294,967,296 of strings!

```
string domain_name[4294967296];
```

As of 2017, the number of domain names is 330 million.

So most part of the array is empty!

IP Addresses

Under IPv6, IP addresses are 128 bits

- It combines what is now implemented as subnets as well as allowing for many more IP addresses
- We cannot allocate an array of size 2^{128} !

DNS

Given a domain name, we wanted to *quickly* find the associated IP address.

- A domain name can have a maximum of 253 characters!
- The number of possible domain names is huge!
- Again, we cannot allocate an array for that.

Goal

Our goal:

- Store data so that all operations are $\Theta(1)$ time
- The memory requirement should be $\Theta(n)$

Keys

In our example, we:

- Created an array of size 65536
- Store each of 150 objects in one of the 256 entries
- The error code indicated which bin the corresponding function pointer was stored

In general, we would like to:

- Create an array of size M
- Store each of n objects in one of the M bins
- Have some means of determining the bin in which an object is stored

Simpler problem

Let's try a simpler problem

- How do I store your examination grades so that I can access your grades in $\Theta(1)$ time?

Assume that each student is issued an 8-digit number

- How do I store your examination grades so that I can access your grades in $\Theta(1)$ time?
- Create an array of size $10^8 \approx 1.5 \times 2^{26}$?

Simpler problem

I could create an array of size 1000

- How could you convert an 8-digit number into a 3-digit number?
- Idea: the last three digits, which seem random

Therefore, I could store the examination grade of student "10105456" by:

```
grade[456] = 86;
```

Simpler problem

Question:

- What is the likelihood that in a class of size 100 no two students have the same last three digits?
- Not very high:

$$1 \cdot \frac{999}{1000} \cdot \frac{998}{1000} \cdot \frac{997}{1000} \cdot \cdots \cdot \frac{901}{1000} \approx 0.005959$$

Simpler problem

Consequently, I have a function that maps a student onto a 3-digit number

- I can store the examination grade in that location
- Storing it, accessing it, and erasing it is $\Theta(1)$
- Problem: two or more students may map to the same number:
  - Student A has ID 20173456 and scored 85
  - Student B has ID 20234456 and scored 87

⋮	⋮
454	
455	
456	86
457	
458	
459	
460	
461	
462	
463	79
464	
465	
⋮	⋮

The hashing problem

The process of mapping an object or a number onto an integer in a given range is called *hashing*

Problem: multiple objects may hash to the same value

- Such an event is termed a *collision*

Hash tables use a hash function together with a mechanism for dealing with collisions

IP Addresses

Going back to the problem with IP addresses

- We need a hash function to map an IP address to a smaller range
- We need a hash function to map a domain name to a smaller range

Mapping `129.97.10.179` onto a smaller range may seem easier, but a mechanism for mapping churchill.uwaterloo.ca onto a small range of integers may be more interesting

The hash process

We will break the process into three **independent** steps:

– We will try to get each of these down to $\Theta(1)$

Object

Techniques vary...

32-bit integer

Modulus, mid-square, multiplicative, Fibonacci

Map to an index $0, ..., M - 1$

Deal with collisions

Chained hash tables
Open addressing

Linear probing
Quadratic probing
Double hashing

Outline

- Introduction
- Hash function
- Mapping down to 0, ..., M – 1
- Dealing with collisions
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  - Open addressing

Definitions

What is a hash of an object?

From Merriam-Webster:

a restatement of something that is already known

The ultimate goal is to map onto an integer range

`0, 1, 2, ..., M – 1`

Properties

Necessary properties of such a hash function h are:

1a. Should be fast: ideally $\Theta(1)$

1b. The hash value must be *deterministic*

- It must always return the same 32-bit integer each time

1c. Equal objects hash to equal values

- $x = y \quad \Rightarrow \quad h(x) = h(y)$

1d. If two objects are randomly chosen, there should be only a one-in-2^{32} chance that they have the same hash value

Types of hash functions

We will look at two classes of hash functions

- Predetermined hash functions (explicit)
- Arithmetic hash functions (implicit)

Predetermined hash functions

The easiest solution is to give each object a unique number

```
class Class_name {
    private:
        unsigned int hash_value;  // int:           -2^31, ..., 2^31 – 1
                                  // unsigned int:    0, ..., 2^32 – 1
    public:
        Class_name();
        unsigned int hash() const;
};
```

```
Class_name::Class_name() {
    hash_value = ???;
}
```

```
unsigned int Class_name::hash() const {
    return hash_value;
}
```

Predetermined hash functions

For example, an auto-incremented static member variable

```
class Class_name {
    private:
        unsigned int hash_value;
        static unsigned int hash_count;
    public:
        Class_name();
        unsigned int hash() const;
};

unsigned int Class_name::hash_count = 0;
```

```
Class_name::Class_name() {
    hash_value = hash_count;
    ++hash_count;
}

unsigned int Class_name::hash() const {
    return hash_value;
}
```

Predetermined hash functions

If we only need the hash value while the object exists in memory, use the address:

```
unsigned int Class_name::hash() const {
    return reinterpret_cast<unsigned int>( this );
}
```

This fails if an object may be stored in secondary memory

– It will have a different address the next time it is loaded

Predetermined hash functions

- Problem with predetermined hash functions?
  - Strings with the same characters:
    ```
    string str1 = "Hello world!";
    string str2 = "Hello world!";
    ```
  - Objects which are conceptually equal:
    ```
    Rational x(1, 2);
    Rational y(3, 6);
    ```

- The previous two methods would give them different hash values.
- But, a hash function should “hash equal objects to equal values”

- These hash values must depend on the member variables
  - Usually this uses arithmetic functions

Arithmetic Hash Values

An arithmetic hash value is a deterministic function that is calculated from the relevant member variables of an object

We will look at arithmetic hash functions for:

- Rational numbers, and
- Strings

Rational number class

What if we just add the numerator and denominator?

```
class Rational {
    private:
        int numer, denom;
    public:
        Rational( int, int );
};

unsigned int Rational::hash() const {
    return static_cast<unsigned int>( numer ) +
        static_cast<unsigned int>( denom );
}
```

Very likely to collide!

Rational number class

We could improve on this: multiply the denominator by a large prime:

```
class Rational {
    private:
        int numer, denom;
    public:
        Rational( int, int );
};

unsigned int Rational::hash() const {
    return static_cast<unsigned int>( numer ) +
        429496751*static_cast<unsigned int>( denom );
}
```

Rational number class

This hash function does not generate unique values

- The following pairs have the same hash values:

0/1	1327433019/800977868
1/2	534326814/1480277007
2/3	820039962/1486995867

- Finding rational numbers with matching hash values is very difficult:
- Finding these required the generation of 1 500 000 000 random rational numbers
- It is fast: $\Theta(1)$
- It does produce an even distribution

Rational number class

Problem:

- The rational numbers 1/2 and 2/4 have different values
- The output of

```
cout << Rational( 1, 2 ).hash();
cout << Rational( 2, 4 ).hash();
```

is

```
858993503
1717987006
```

Rational number class

Solution: divide through by the greatest common divisor

```
Rational::Rational( int a, int b ):numer(a), denom(b) {
    int divisor = gcd( numer, denom );
    numer /= divisor;
    denom /= divisor;
}
```

```
int gcd( int a, int b) {
    while( true ) {
        if ( a == 0 ) {
            return (b >= 0) ? b : -b;
        }

        b %= a;

        if ( b == 0 ) {
            return (a >= 0) ? a : -a;
        }
        a %= b;
    }
}
```

Rational number class

Problem:

- The rational numbers $\dfrac{1}{2}$ and $\dfrac{-1}{-2}$ have different values
- The output of

```
int main() {
    cout << Rational(  1,  2 ).hash();
    cout << Rational( -1, -2 ).hash();
    return 0;
}
```

  is

```
858993503
3435973793
```

Rational number class

Solution: define a normal form

– Require that the denominator is positive

```
Rational::Rational( int a, int b ):numer(a), denom(b) {
    int divisor = gcd( numer, denom );
    divisor = (denom >= 0) ? divisor : -divisor;
    numer /= divisor;
    denom /= divisor;
}
```

String class

Two strings are equal if all the characters are equal and in the identical order

A string is simply an array of bytes:

– Each byte stores a value from 0 to 255

Any hash function must be a function of these bytes

String class

We could, for example, just add the characters:

```
unsigned int hash( const string &str ) {
    unsigned int hash_value = 0;

    for ( int k = 0; k < str.length(); ++k ) {
        hash_value += str[k];
    }

    return hash_value;
}
```

String class

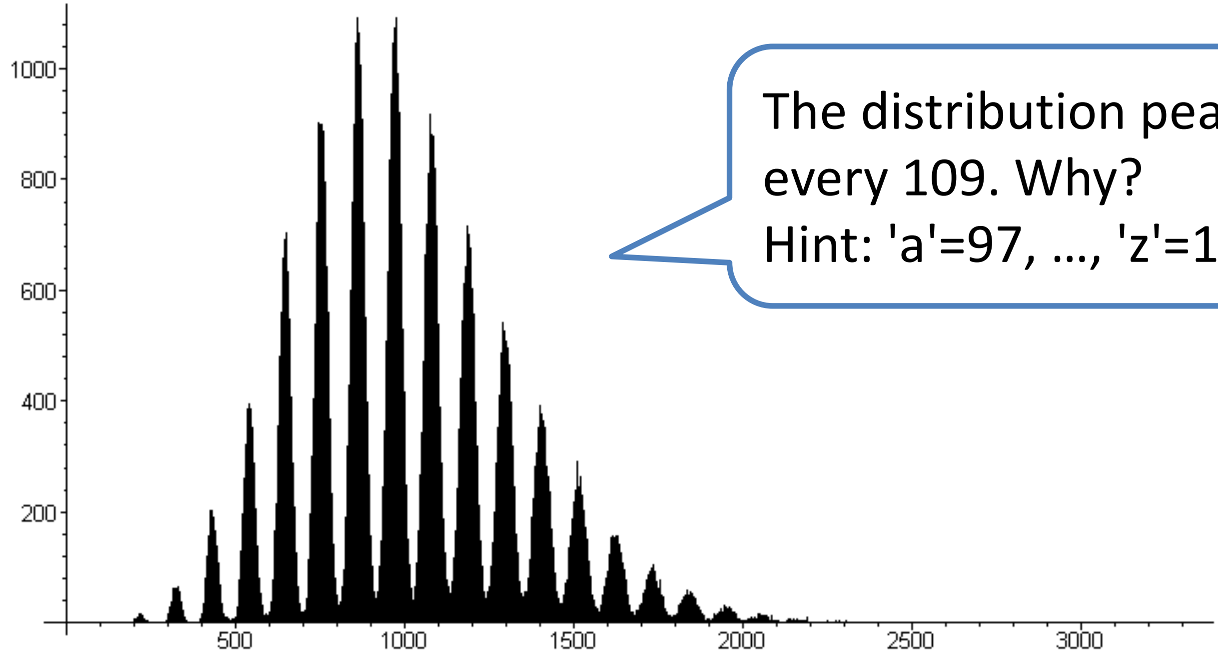
Not very good:

– Slow run time: $\Theta(n)$

– Words with the same characters hash to the same code:

  • `"form"` and `"from"`

– A poor distribution, *e.g.*, all words in Moby™ Words II by Grady Ward:



The distribution peaks about every 109. Why?
Hint: 'a'=97, ..., 'z'=122

String class

Let the individual characters represent the coefficients of a polynomial in x:

$$p(x) = c_0 x^{n-1} + c_1 x^{n-2} + \cdots + c_{n-3} x^2 + c_{n-2} x + c_{n-1}$$

Use Horner's rule to evaluate this polynomial at a prime number, *e.g.*, $x = 12347$:

```
unsigned int hash( string const &str ) {
    unsigned int hash_value = 0;

    for ( int k = 0; k < str.length(); ++k ) {
        hash_value = 12347*hash_value + str[k];
    }

    return hash_value;
}
```

String class

Problem, Horner's rule runs in $\Theta(n)$

```
"A Elbereth Gilthoniel,\n
 Silivren penna miriel\n
 O menal aglar elenath!\n
 Na-chaered palan-diriel\n
 O galadhremmin ennorath,\n
 Fanuilos, le linnathon\n
 nef aear, si nef aearon!"
```

Suggestions?

String class

Use characters in locations $2^k - 1$ for $k = 0, 1, 2, \ldots$:

```
"A_Elbereth Gilthoniel,\n
 Silivren_penna miriel\n
 O menal aglar elenath!\n
 Na-chaered palan-diriel\n
 O galadhremmin ennorath,\n
 Fanuilos, le linnathon\n
 nef aear, si nef aearon!"
```

J.R.R. Tolkien

String class

The run time is now $\Theta(\ln(n))$:

```
unsigned int hash( const string &str ) {
    unsigned int hash_value = 0;

    for ( int k = 1; k <= str.length(); k *= 2 ) {
        hash_value = 12347*hash_value + str[k - 1];
    }

    return hash_value;
}
```

Arithmetic hash functions

In general, any member variables that are used to uniquely define an object may be used as coefficients in such a polynomial

```
class Person {
    string surname;
    string given_name;
    unsigned short birth_year;
    unsigned char birth_month;
    unsigned char birth_day;
    unsigned int salary;
    // ...
};
```

Arithmetic hash functions

In general, any member variables that are used to uniquely define an object may be used as coefficients in such a polynomial

```
class Person {
    string surname;
    string given_name;
    unsigned short birth_year;
    unsigned char birth_month;
    unsigned char birth_day;
    unsigned int salary;
    // ...
};
```

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Outline

Previously, we considered means for calculating 32-bit hash values

- Explicitly defined hash values
- Implicitly calculated hash values

Practically, we will require a hash value on the range $0, ..., M - 1$:

- The modulus operator %
- Review of bitwise operations
- The multiplicative method

The hash process

Object

32-bit integer

Modulus, mid-square, multiplicative, Fibonacci

Map to an index $0, \ldots, M-1$

Deal with collisions

Chained hash tables
Open addressing

Properties

Necessary properties of this mapping function h_M are:

2a. Must be fast: $\Theta(1)$

2b. The hash value must be *deterministic*

- Given n and M, $h_M(n)$ must always return the same value

2c. If two objects are randomly chosen, there should be only a one-in-M chance that they have the same value from 0 to $M - 1$

Modulus operator

Easiest method: return the value modulus M

```
unsigned int hash_M( unsigned int n, unsigned int M ) {
    return n % M;
}
```

Unfortunately, calculating the modulus (or remainder) is expensive

- If $M = 2^m$, we can simplify the calculation by bitwise operations
  - left and right shift and bit-wise and

The bitwise operators: `&` `<<` `>>`

Suppose I want to calculate

`7985325 % 100`

The modulo is a power of ten: $100 = 10^2$

– In this case, take the last two decimal digits: `25`

Similarly, `7985325 % 10`^{3} `= 325`

– We set the appropriate digits to 0:

0000025 and 0000325

The bitwise operators: `&` `<<` `>>`

The same works in base 2:

$$\texttt{100011100101}_2 \ \% \ \texttt{10000}_2$$

The modulo is a power of 2: $10000_2 = 2^4$

- In this case, take the last four bits: `0101`

Similarly, $\texttt{100011100101}_2 \ \% \ \texttt{1000000}_2$ `== 100101`,

- We set the appropriate digits to 0:

`000000000101` and `000000100101`

The bitwise operators: `&` `<<` `>>`

To zero all but the last n bits, select the last n bits using *bitwise and*:

$$1000\ 1110\ 0101_2 \ \& \ 0000\ 0000\ 1111_2 \longrightarrow 0000\ 0000\ 0101_2$$

$$1000\ 1110\ 0101_2 \ \& \ 0000\ 0011\ 1111_2 \longrightarrow 0000\ 0010\ 0101_2$$

The bitwise operators: `&` `<<` `>>`

Similarly, multiplying or dividing by powers of 10 is easy:

`7985325 * 100`

The multiplier is a power of ten: $100 = 10^2$

– In this case, add two zeros: `798532500`

Similarly, `7985325 / 10`^{3} `= 7985`

– Just add the appropriate number of zeros or remove the appropriate number of digits

The bitwise operators: & `<<` `>>`

The same works in base 2:

$$\texttt{100011100101}_2 * \texttt{10000}_2$$

The modulo is a power of 2: $10000_2 = 2^4$

– In this case, add four zeros: `1000111001010000`

Similarly, $\texttt{100011100101}_2$ / $\texttt{1000000}_2$ `== 100011`

The bitwise operators: `&` `<<` `>>`

This can be done mechanically by shifting the bits appropriately:

$\texttt{1000 1110 0101}_2 \ \texttt{<<} \ \texttt{4} \ \texttt{==} \ \texttt{1000 1110 0101 0000}_2$

$\texttt{1000 1110 0101}_2 \ \texttt{>>} \ \texttt{6} \ \texttt{==} \ \texttt{10 0011}_2$

Powers of 2 are now easy to calculate:

$\texttt{1}_2 \ \texttt{<<} \ \texttt{4} \ \texttt{==} \ \texttt{1 0000}_2$ `//` $2^4 = 16$

$\texttt{1}_2 \ \texttt{<<} \ \texttt{6} \ \texttt{==} \ \texttt{100 0000}_2$ `//` $2^6 = 64$

Modulo a power of two

The implementation using the modulus/remainder operator:

```
unsigned int hash_M( unsigned int n, unsigned int m ) {
    return n & ((1 << m) – 1);
}
```

Modulo a power of two

Problem:

- Suppose that the hash function *h* is always even
- An even number modulo a power of two is still even

Example: memory allocations are multiples of word size

- On a 64-bit computer, addresses returned by `new` will be multiples of 8
- The probability that $h_M(\ h(x)\ ) = h_M(\ h(y)\ )$ is one in $M/8$
  - This is not one in M

The multiplicative method

We need to obfuscate the bits

- The most common method to obfuscate bits is multiplication
- Consider how one bit can affect an entire range of numbers in the result:

```
          10100111
        × 11010011
          10100111
         10100111
      10100111
    10100111
+  10100111
  1000101110100101
```

The *avalanche* effect: changing one bits has the potential of affecting all bits in the result:

10100011 × 11010011
= 1000011001011001

The multiplicative method

Multiplying by a fixed constant is a reasonable method

- Take the middle m bits of Cn:

```
unsigned int const C = 581869333;  // some number

unsigned int hash_M( unsigned int n, unsigned int m ) {
    unsigned int shift = (32 - m)/2;
    return ((C*n) >> shift) & ((1 << m) - 1);
}
```

The multiplicative method

Suppose that the value $m = 10$ ($M = 1024$) and $n = 42$

```
const unsigned int C = 581869333;  // some number

unsigned int hash_M( unsigned int n, unsigned int m ) {
    unsigned int shift = (32 – m)/2;
    return ((C*n) >> shift) & ((1 << m) – 1);
}
```

The multiplicative method

$m = 10$

$n = 42$

First calculate the shift

```
const unsigned int C = 581869333;  // some number

unsigned int hash_M( unsigned int n, unsigned int m ) {
    unsigned int shift = (32 – m)/2;
    return ((C*n) >> shift) & ((1 << m) – 1);
}
```

shift = 11

The multiplicative method

$m = 10$

$n = 42$

Calculate Cn

```
const unsigned int C = 581869333;  // some number

unsigned int hash_M( unsigned int n, unsigned int m ) {
    unsigned int shift = (32 - m)/2;
    return ((C*n) >> shift) & ((1 << m) - 1);
}
```

shift = 11

1	0	1	1	0	0	0	0	1	0	1	0	0	1	1	0	0	0	0	1	1	0	0	1	0	1	1	1	0	0	1	0

The multiplicative method

$m = 10$

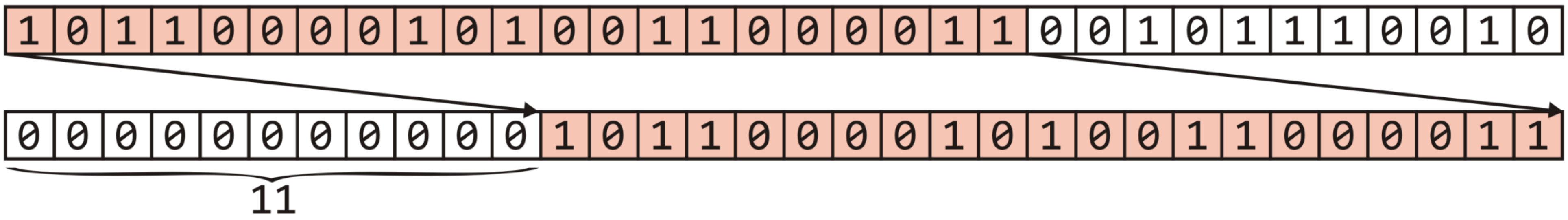
$n = 42$

Right shift this value 11 bits—equivalent to dividing by 2^{11}

```
const unsigned int C = 581869333;  // some number

unsigned int hash_M( unsigned int n, unsigned int m ) {
    unsigned int shift = (32 – m)/2;
    return ((C*n) >> shift) & ((1 << m) – 1);
}
```

shift = 11



The multiplicative method

$m = 10$
$n = 42$

Left shift 1 $m = 10$ bits yielding 2^{10}

```
const unsigned int C = 581869333;  // some number

unsigned int hash_M( unsigned int n, unsigned int m ) {
    unsigned int shift = (32 – m)/2;
    return ((C*n) >> shift) & ((1 << m) – 1);
}
```

0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	1	0	1	0	0	1	1	0	0	0	0	1	1

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0

10

The multiplicative method

$m = 10$
$n = 42$

Subtracting 1 yields $m = 10$ ones

```
const unsigned int C = 581869333;  // some number

unsigned int hash_M( unsigned int n, unsigned int m ) {
    unsigned int shift = (32 – m)/2;
    return ((C*n) >> shift) & ((1 << m) – 1);
}
```

0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	1	0	1	0	0	1	1	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1

The multiplicative method

$m = 10$
$n = 42$

Taken the bitwise to clear all but the last 10 bits

```
const unsigned int C = 581869333;  // some number

unsigned int hash_M( unsigned int n, unsigned int m ) {
    unsigned int shift = (32 – m)/2;
    return ((C*n) >> shift) & ((1 << m) – 1);
}
```

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1

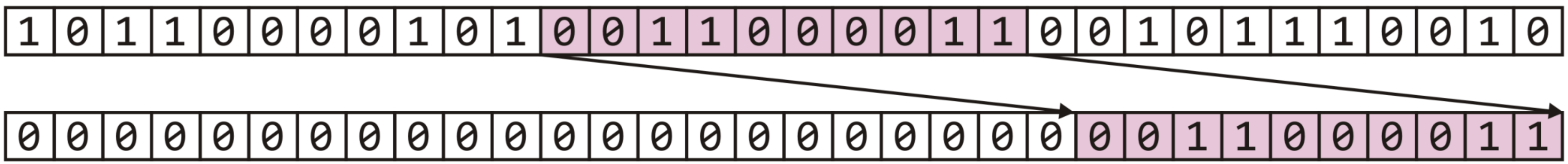
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1

The multiplicative method

We have extracted the middle $m = 10$ bits—a number in $0, \ldots, 1023$

```
const unsigned int C = 581869333;  // some number

unsigned int hash_M( unsigned int n, unsigned int m ) {
    unsigned int shift = (32 – m)/2;
    return ((C*n) >> shift) & ((1 << m) – 1);
}
```



$h_M(42) = 195$

Outline

- Introduction
- Hash function
- Mapping down to 0, ..., M – 1
- Dealing with collisions
  - Chained hash tables
  - Open addressing

Outline

We have:

- Discussed techniques for hashing
- Discussed mapping down to a given range $0, ..., M - 1$

Now we must deal with collisions

- Numerous techniques exist
- Containers in general
  - Specifically linked lists

Hashing

First, a review:

- We want to store objects in an array of size M
- We want to quickly calculate the bin where we want to store the object
  - We came up with hash functions—hopefully $\Theta(1)$
  - Perfect hash functions (no collisions) are difficult to design

We will look at some schemes for dealing with collisions

The hash process

Object

32-bit integer

Modulus, mid-square, multiplicative, Fibonacci

Map to an index $0, \ldots, M - 1$

Deal with collisions

Chained hash tables
Open addressing

Chained hash table

Associating each bin with a linked list.

For any object assigned to the bin by the hash function, finding, inserting, and erasing the object is done on the linked list.

Example

As an example, let's store hostnames and allow a fast look-up of the corresponding IP address

- We will choose the bin based on the host name
- Associated with the name will be the IP address
- *E.g.*, `("optimal", 129.97.94.57)`

Example

Suppose the hash value of a string is the last 3 bits of the first character in the host name

– The hash of "`optimal`" is based on "`o`“

a	01100001	n	01101110
b	01100010	o	01101111
c	01100011	p	01110000
d	01100100	q	01110001
e	01100101	r	01110010
f	01100110	s	01110011
g	01100111	t	01110100
h	01101000	u	01110101
i	01101001	v	01110110
j	01101010	w	01110111
k	01101011	x	01111000
l	01101100	y	01111001
m	01101101	z	01111010

Example

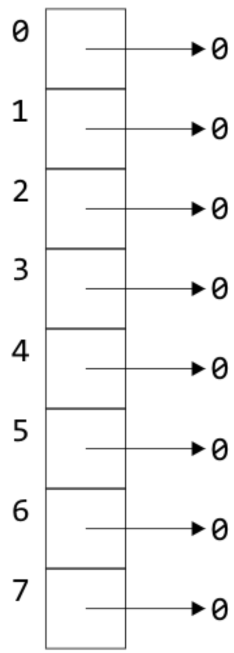
Our hash function is

```
unsigned int hash( string const &str ) {
     // the empty string "" is hashed to 0
     if str.length() == 0 ) {
          return 0;
     }

     return str[0] & 7;
}
```

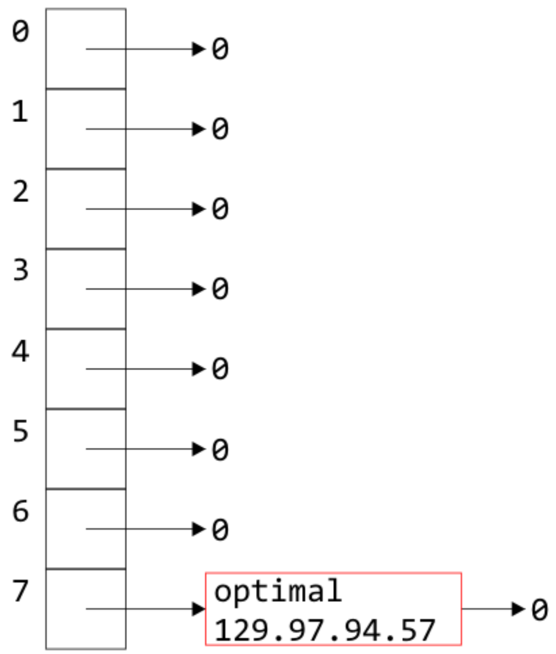
Example

Starting with an array of 8 empty linked lists



Example

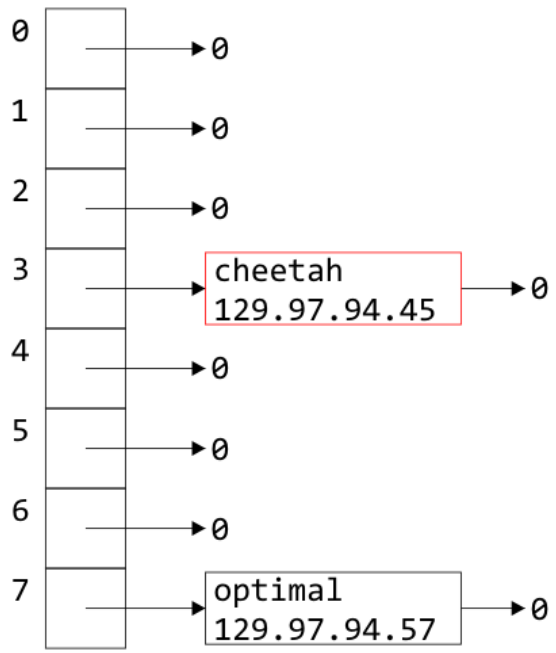
The pair `("optimal", 129.97.94.57)` is entered into bin `01101111` = 7



Example

Similarly, as `"c"` hashes to 3

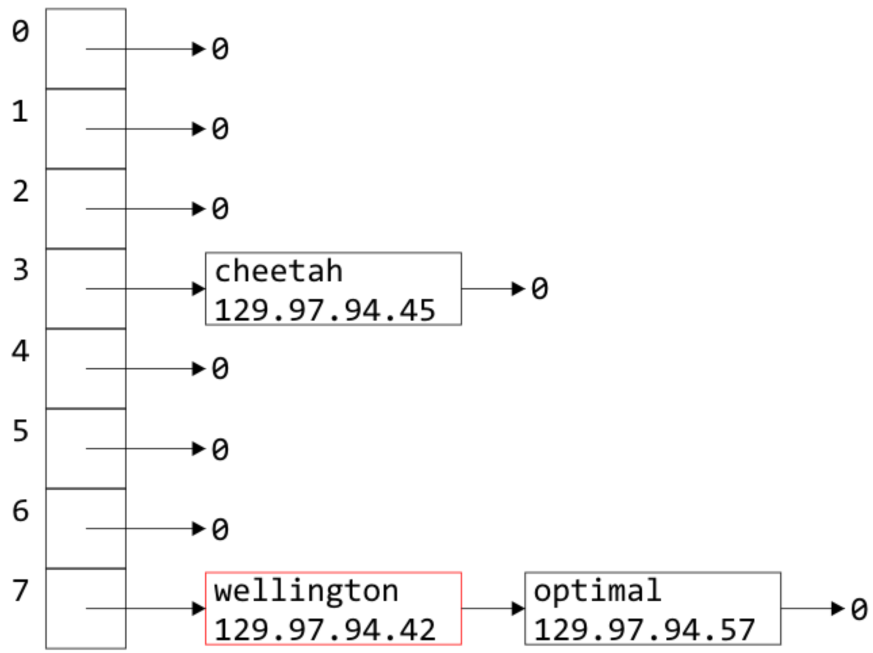
- The pair `("cheetah", 129.97.94.45)` is entered into bin 3



Example

The `"w"` in Wellington also hashes to 7

- `("wellington", 129.97.94.42)` is entered into bin 7



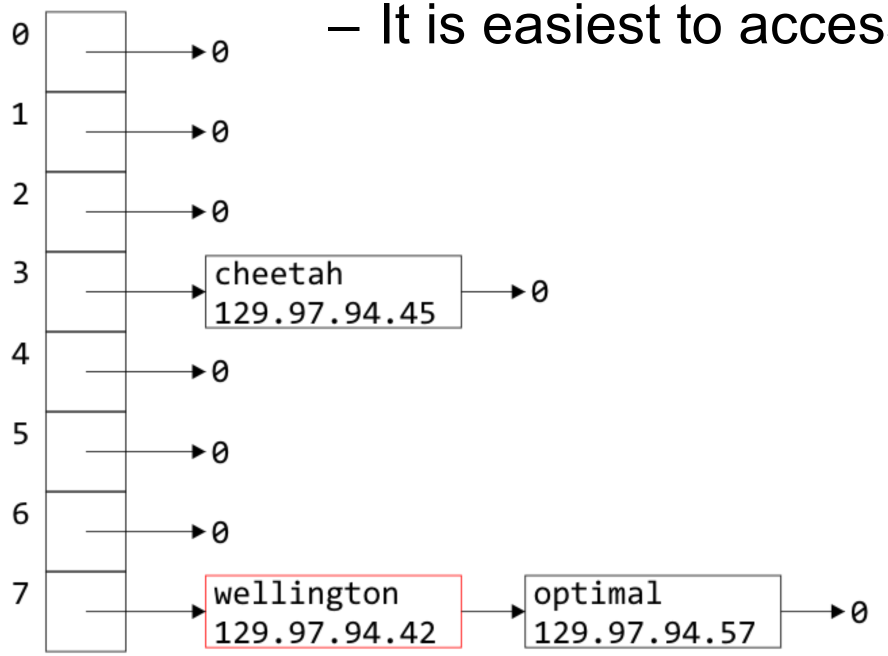
Example

Why did I use `push_front` from the linked list?

– A good heuristic is

“unless you know otherwise, data which has been accessed recently will be accessed again in the near future”

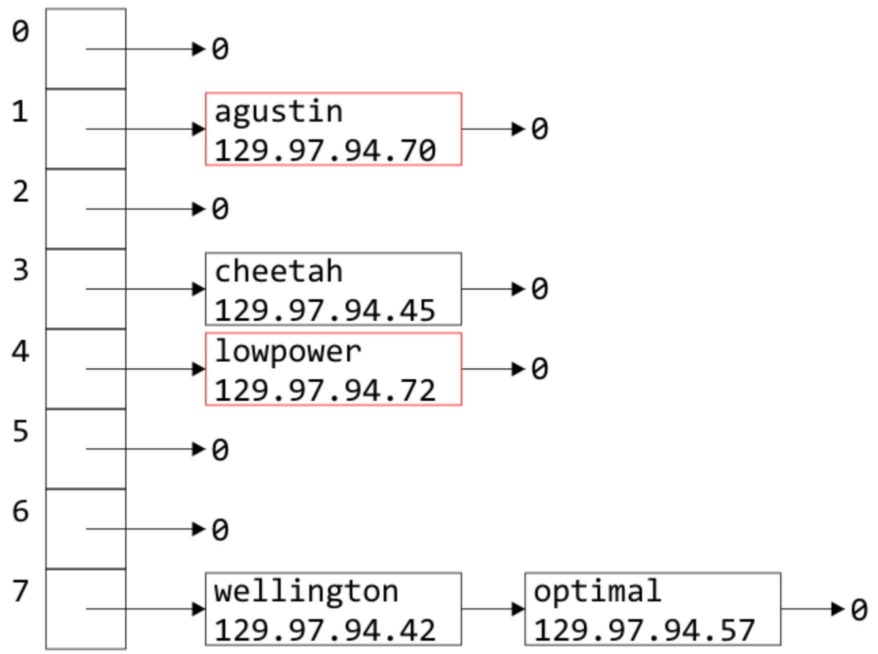
– It is easiest to access data at the front of a linked list



Heuristics include rules of thumb, educated guesses, and intuition

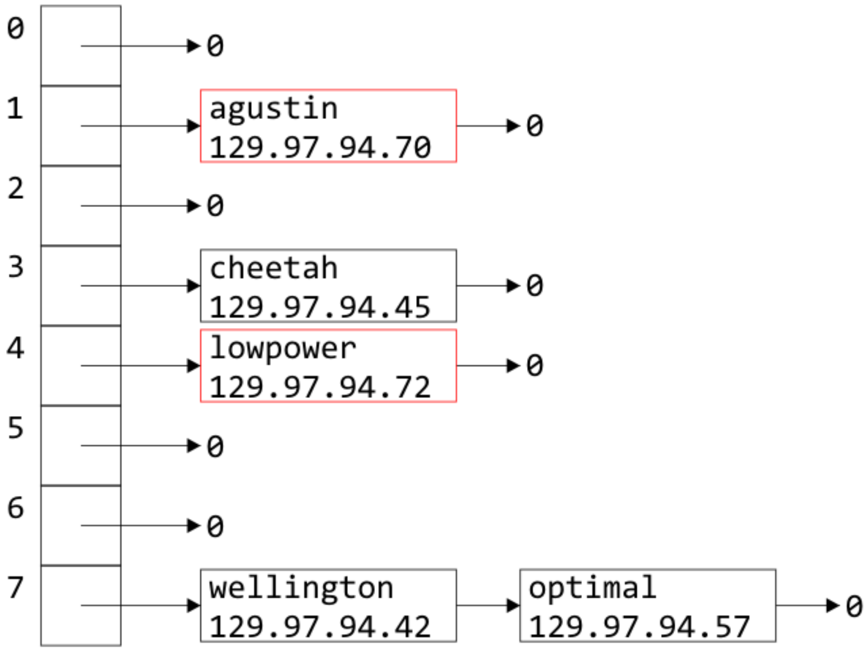
Example

Similarly we can insert the host names "`augustin`" and "`lowpower`"



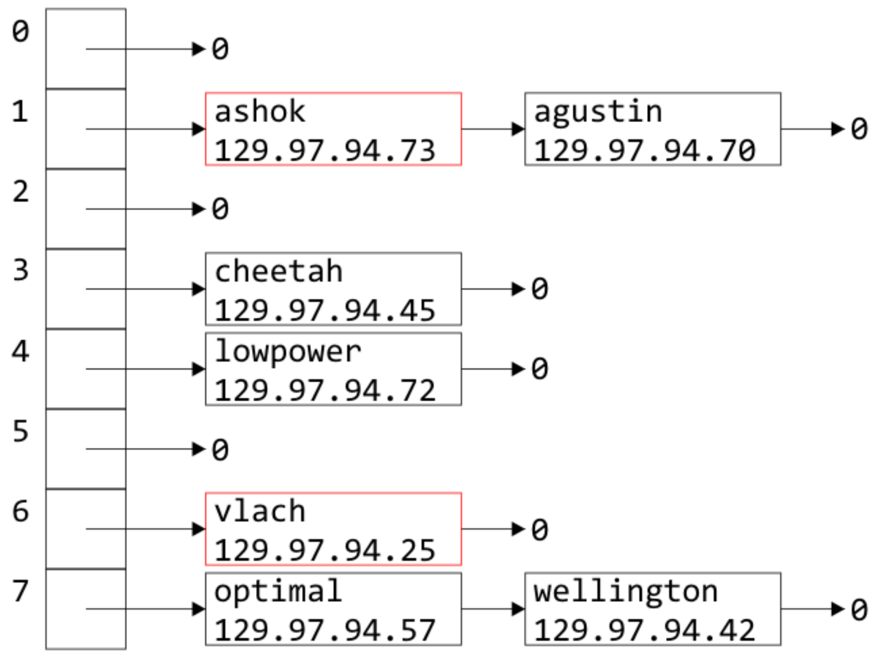
Example

If we now wanted the IP address for "`optimal`", we would simply hash "`optimal`" to 7, walk through the linked list, and access 129.97.94.57 when we access the node containing the relevant string



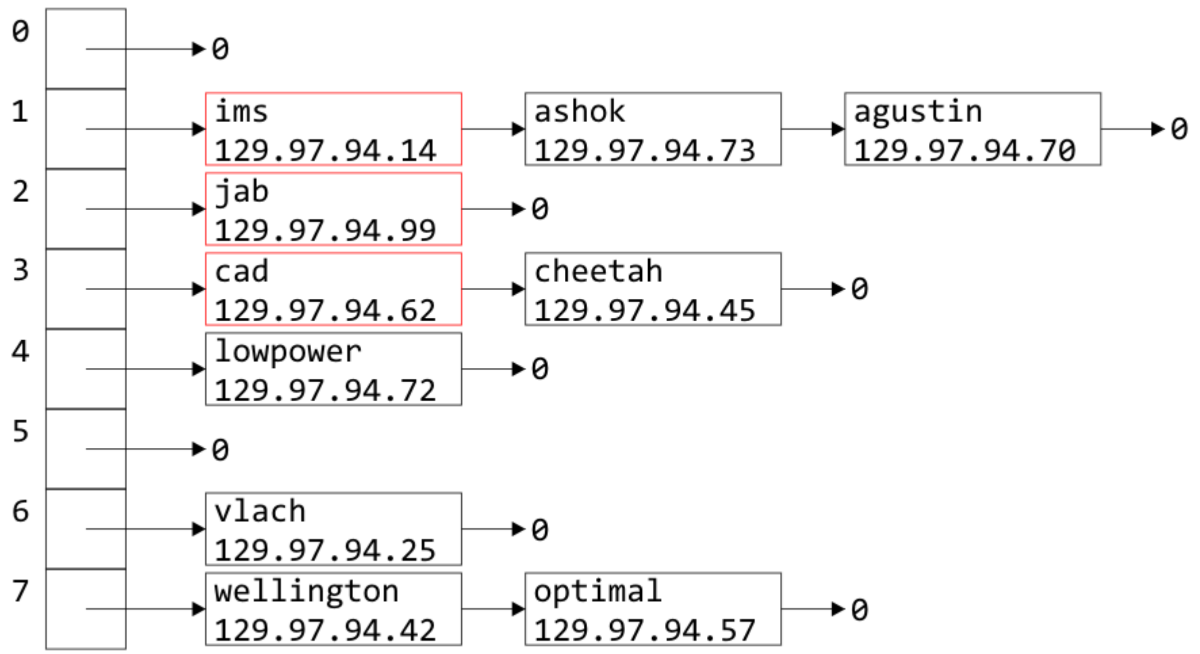
Example

Similarly, `"ashok"` and `"vlach"` are entered into bin 1 and 6



Example

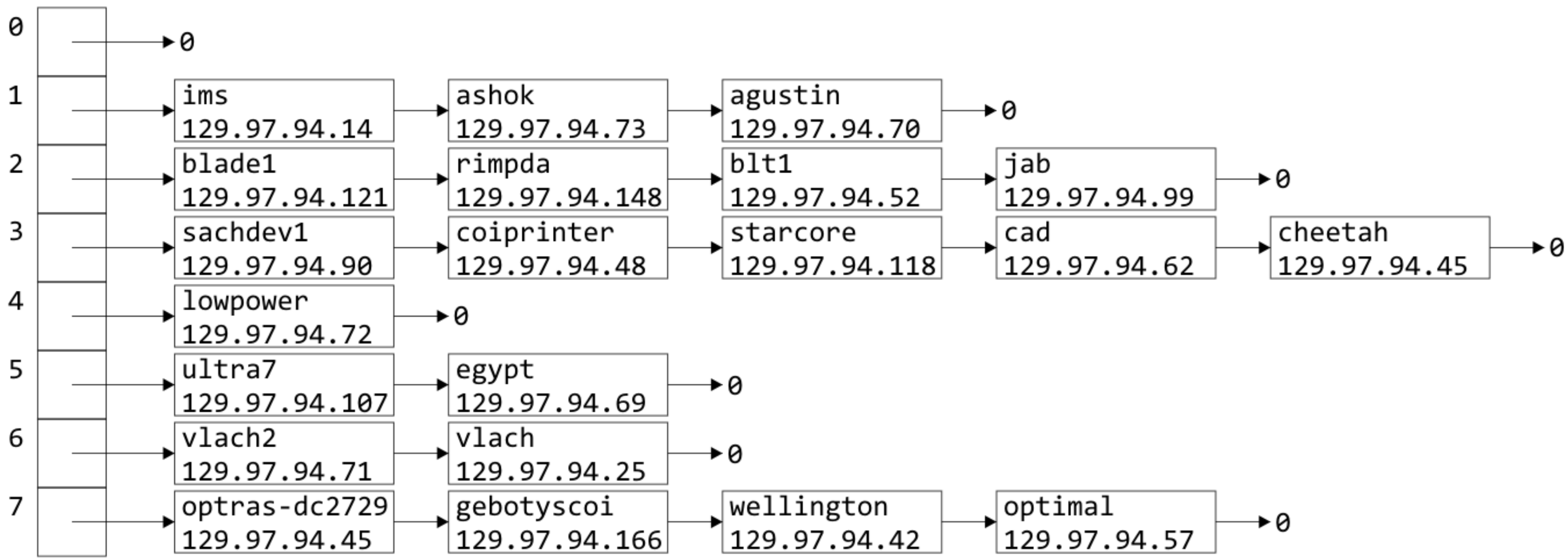
Inserting "`ims`", "`jab`", and "`cad`" doesn't even out the bins



Example

Indeed, after 21 insertions, the linked lists are becoming rather long

– We were looking for $\Theta(1)$ access time, but accessing something in a linked list with k objects is $\mathbf{O}(k)$



Load Factor

To describe the length of the linked lists, we define the *load factor* of the hash table:

$$\lambda = \frac{n}{M}$$

This is the average number of objects per bin

- This assumes an even distribution

Right now, the load factor is $\lambda = 21/8 = 2.625$

- The average bin has 2.625 objects

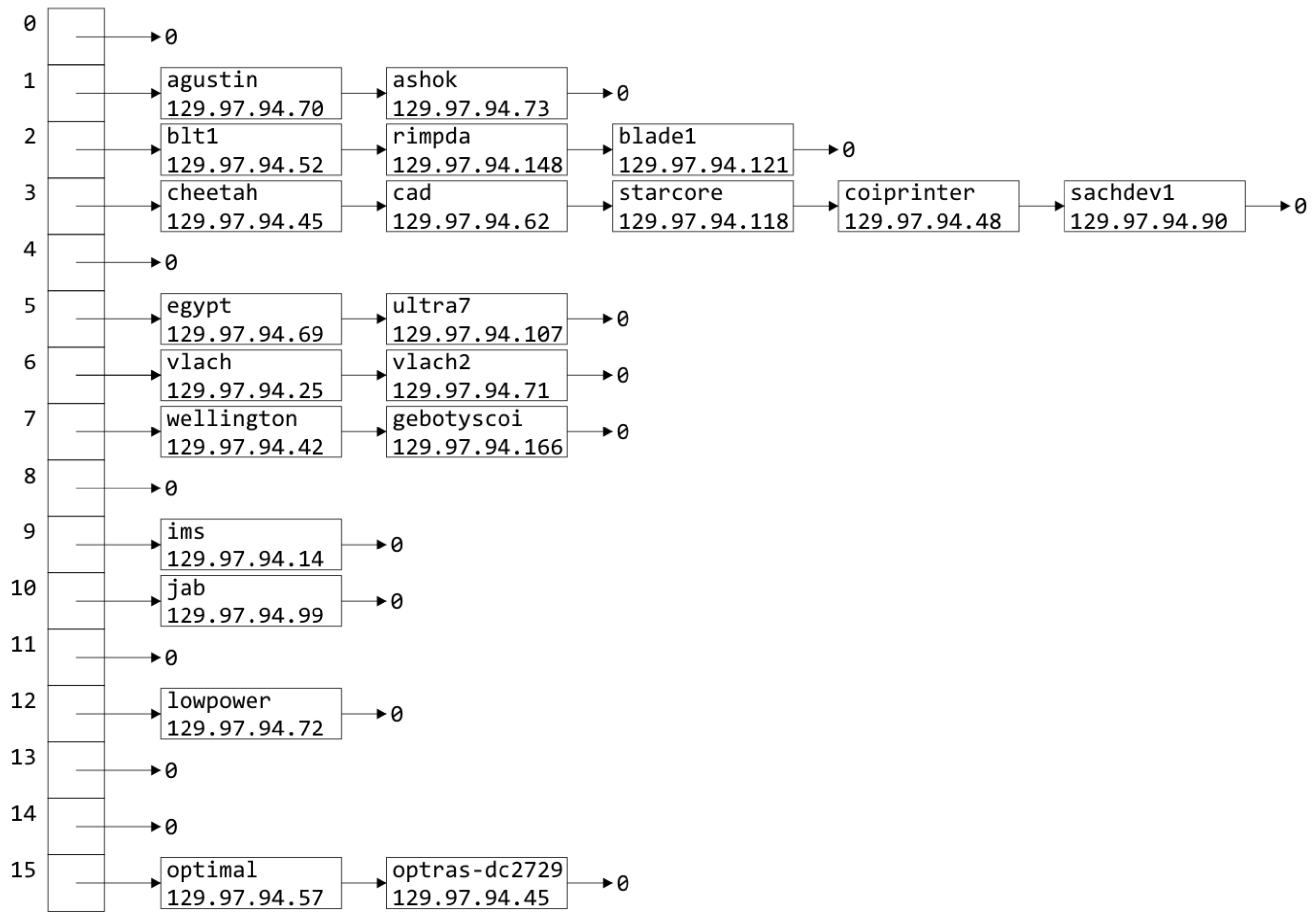
Load Factor

If the load factor becomes too large, access times will start to increase: $\mathbf{O}(\lambda)$

The most obvious solution is to double the size of the hash table and re-insert every object (*rehashing*)

Doubling Size

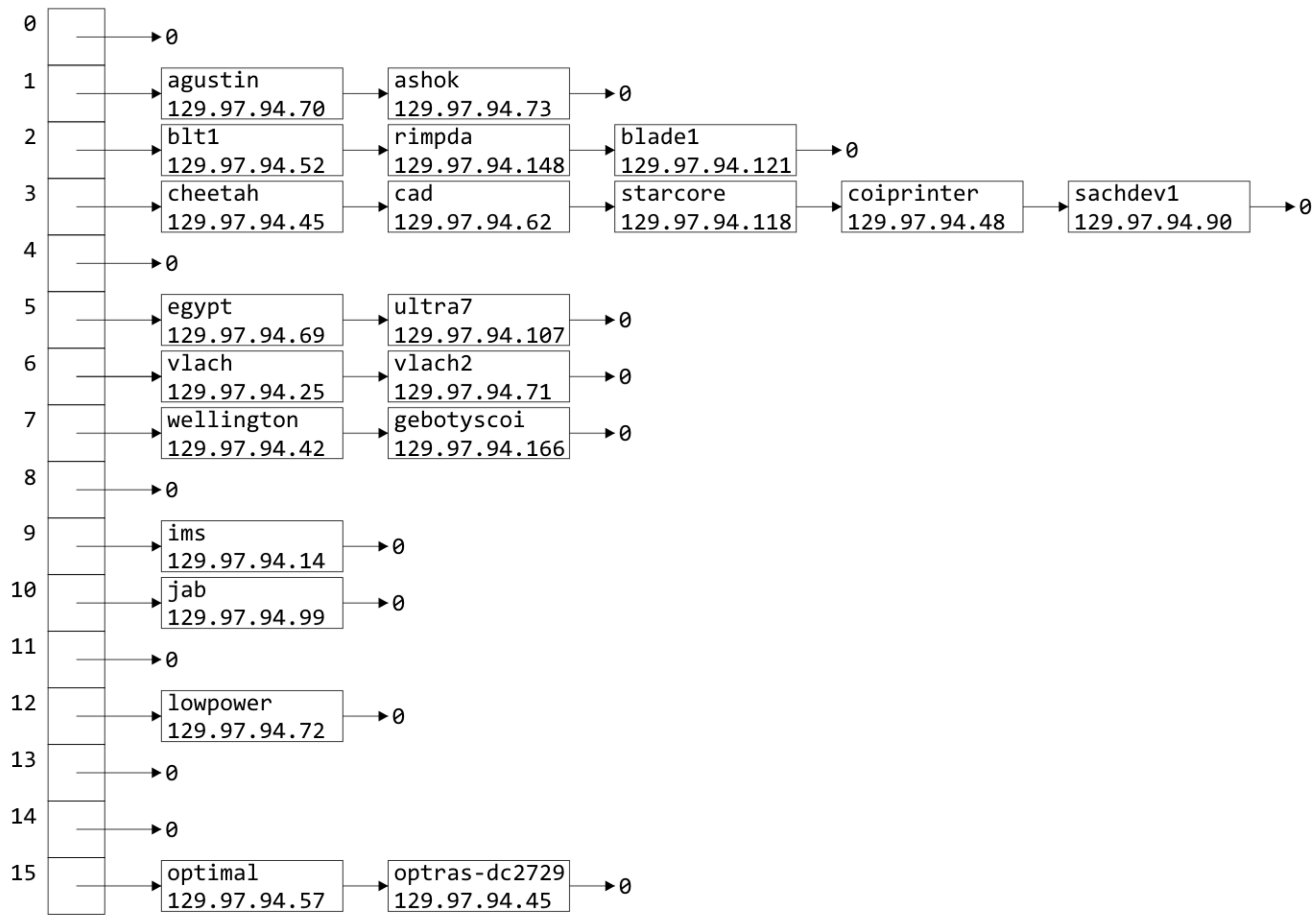
In our example, suppose we take the last four bits as the hash function after doubling the hash table size



Doubling Size

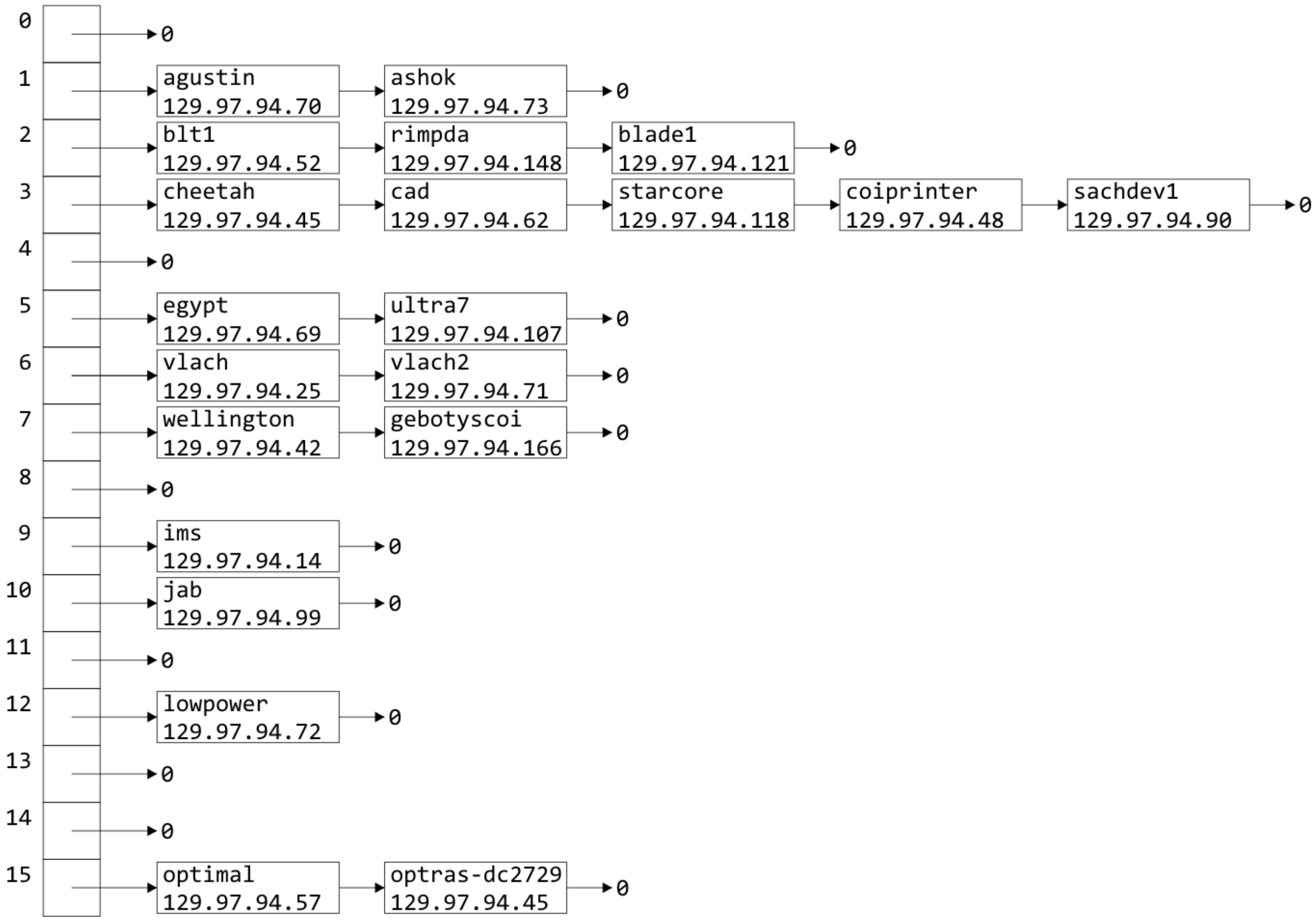
The load factor is now $\lambda = 1.3125$

– Unfortunately, the distribution hasn't improved much



Doubling Size

There is significant *clustering* in bins 2 and 3 due to the choice of host names



Choosing a Good Hash Function

We choose a very poor hash function:

- We looked at the first letter of the host name

Unfortunately, all these are also actual host names:

```
ultra7 ultra8 ultra9 ultra10 ultra11
ultra12 ultra13 ultra14 ultra15 ultra16 ultra17
blade1 blade2 blade3 blade4 blade5
```

This will cause clustering in bins 2 and 5

Choosing a Good Hash Function

Let’s go back to the hash function defined previously:

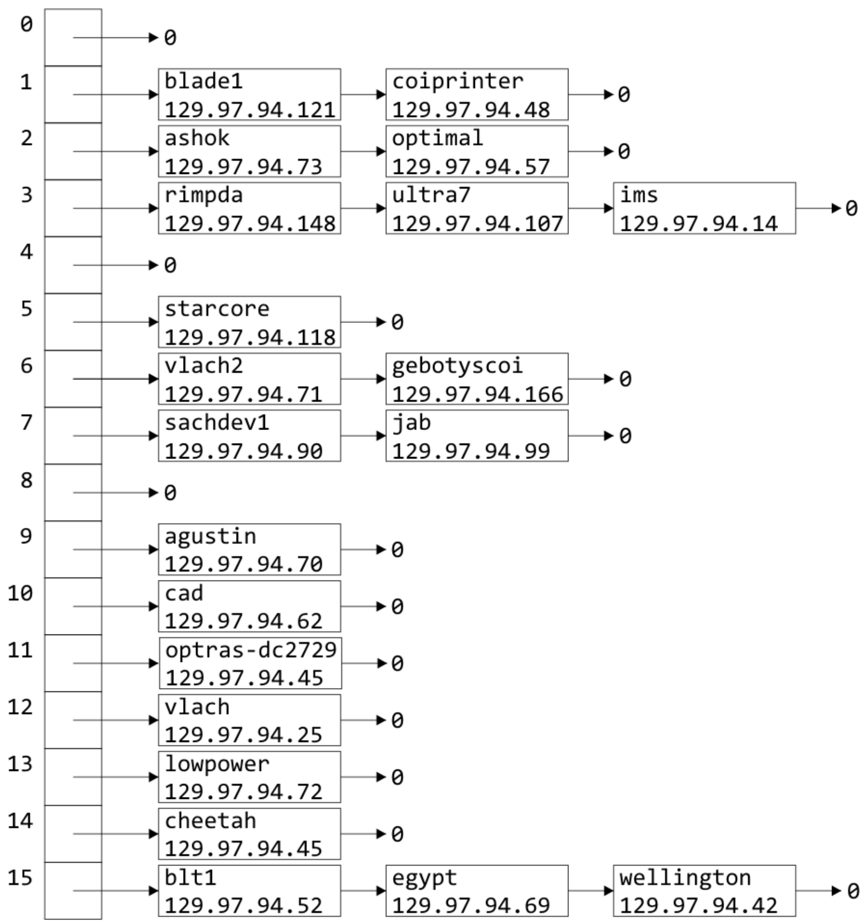
```
unsigned int hash( string const &str ) {
    unsigned int hash_value = 0;

    for ( int k = 0; k < str.length(); ++k ) {
        hash_value = 12347*hash_value + str[k];
    }

    return hash_value;
}
```

Choosing a Good Hash Function

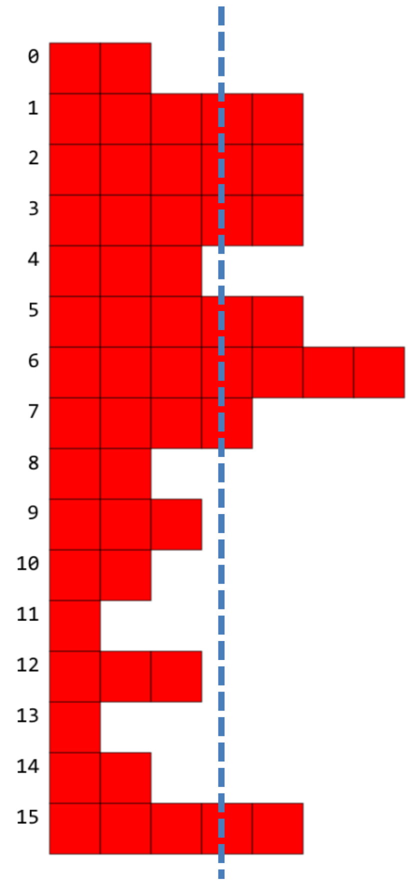
This hash function yields a much nicer distribution:



Choosing a Good Hash Function

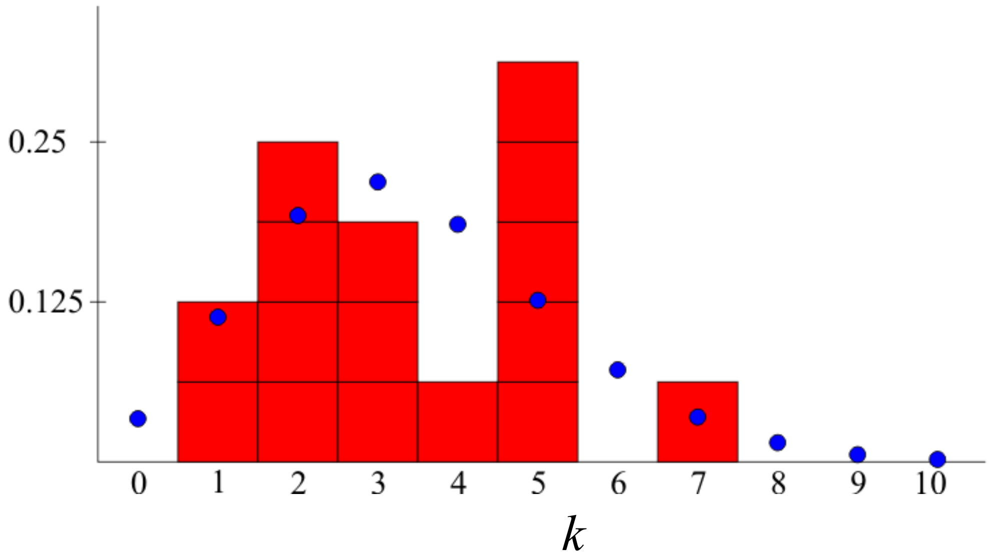
When we insert 55 names, we would have a load factor $\lambda = 3.4375$

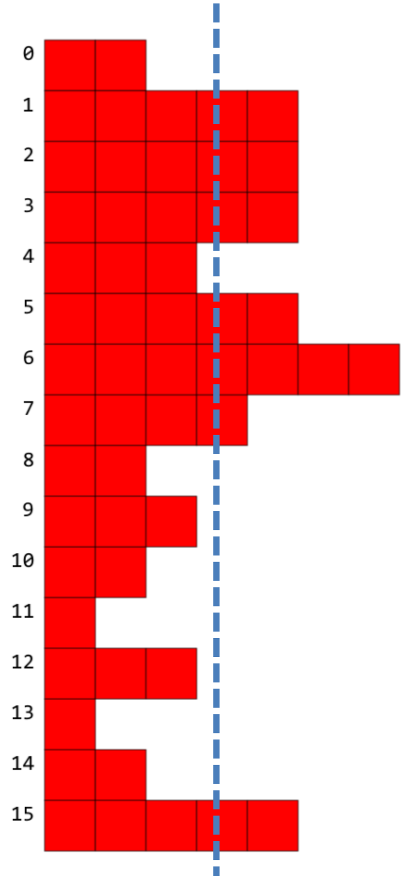
- Clearly there are not exactly 3.4375 objects per bin
- How can we tell if this is a good hash function?
- Can we expect exactly 3.4375 objects per bin?
  - Clearly no…
- The answer is with statistics…



Choosing a Good Hash Function

We would expect the number of bins which hold k objects to approximately follow a Poisson distribution





Problems with Linked Lists

One significant issue with chained hash tables using linked lists

- It requires extra memory
- It uses dynamic memory allocation

Another issue is the $\mathbf{O}(\lambda)$ time complexity

For faster access, we could replace each linked list with an AVL tree (assuming we can order the objects)

- The access time drops to $\mathbf{O}(\ln(\lambda))$
- The memory requirements are increased by $\Theta(n)$, as each node will require two pointers

Black Board Example

Use the hash function

```
unsigned int hash( unsigned int n ) { return n % 10; }
```

to enter the following 15 numbers into a hash table with 10 bins:

534, 415, 465, 459, 869, 442, 840, 180, 450, 265, 23, 946, 657, 3, 29

Summary

The easiest way to deal with collisions is to associate each bin with a container

We looked at bins of linked lists

- The example used host names and IP addresses
- We defined the load factor $\lambda = n/M$
- Discussed doubling the number of bins
- Our goals are to choose a good hash function and to keep the load factor low
- We discussed alternatives

Next we will see a different technique using only one array of bins: open addressing

Outline

- Introduction
- Hash function
- Mapping down to 0, ..., M – 1
- Dealing with collisions
  - Chained hash tables
  - Open addressing

Background

Chained hash tables require special memory allocation

- Can we create a hash table without significant memory allocation?

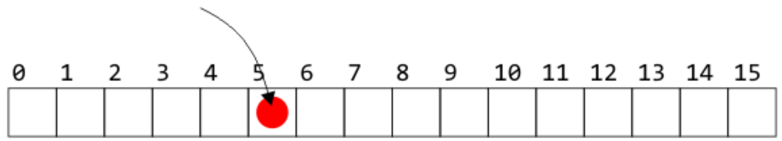
We will deal with collisions by storing collisions elsewhere

- We will define an implicit rule which tells us where to look next

Open Addressing

Suppose an object hashes to bin 5

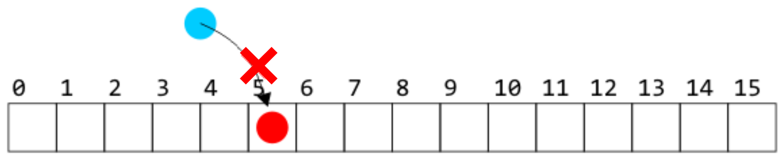
– If bin 5 is empty, we can copy the object into that entry



Open Addressing

Suppose, however, another object hashes to bin 5

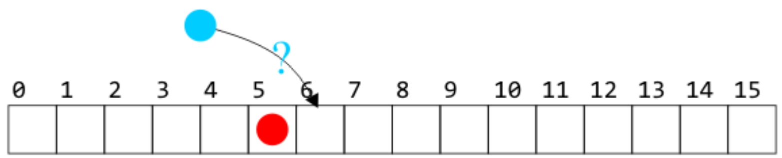
– Without a linked list, we cannot store the object in that bin



Open Addressing

We need a rule to tells us where to look next

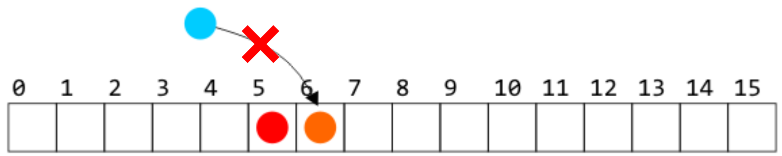
– For example, look in the next bin to see if it is occupied



Open Addressing

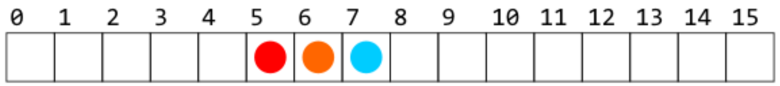
The rule must be:

- simple to follow—*i.e.*, fast
- general enough to deal with the fact that the next cell could also be occupied: e.g., continue searching until the first empty bin is found



Open Addressing

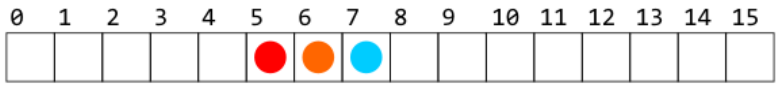
Of course, whatever rule we use in placing an object must also be used when searching for or removing objects



Open Addressing

Recall, however, that our goal is $\Theta(1)$ access times

- We cannot, on average, be forced to access too many bins



Open Addressing

There are numerous strategies for defining the order in which the bins should be searched:

- Linear probing
- Quadratic probing
- Double hashing

There are many alternate strategies, as well:

- Last come, first served
  - Always place the object into the bin moving what may be there already
- Cuckoo hashing

Outline

- Introduction
- Hash function
- Mapping down to 0, ..., M – 1
- Dealing with collisions
  - Chained hash tables
  - Open addressing
    - Linear probing
    - Quadratic probing

Linear Probing

The easiest method to probe the bins of the hash table is to search forward linearly

Assume we are inserting into bin k:

- If bin k is empty, we occupy it
- Otherwise, check bin $k + 1$, $k + 2$, and so on, until an empty bin is found
  - If we reach the end of the array, we start at the front (bin 0)

Linear Probing

Consider a hash table with $M = 16$ bins

Given a 3-digit hexadecimal number:

- The least-significant digit is the primary hash function (bin)
- Example: for $72\mathbf{A}_{16}$, the initial bin is **A**

Insertion

Insert these numbers into this initially empty hash table:

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, B32, C8B, DBE, E9C

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Example

Start with the first four values:

19A, 207, 3AD, 488

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Example

Start with the first four values:

19**A**, 20**7**, 3A**D**, 48**8**

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							207	**488**		**19A**			**3AD**		

Example

Next we must insert 5BA

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							207	488		19A			3AD		

Example

Next we must insert 5B**A**

– Bin **A** is occupied

– We search forward for the next empty bin

0	1	2	3	4	5	6	7	8	9	**A**	B	C	D	E	F
							207	488		19A	**5BA**		3AD		

Example

Next we are adding 680, 74C, 826

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							207	488		19A	5BA		3AD		

Example

Next we are adding 68**0**, 74**C**, 82**6**

- All the bins are empty—simply insert them

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680						**826**	207	488		19A	5BA	**74C**	3AD		

Example

Next, we must insert 946

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680						826	207	488		19A	5BA	74C	3AD		

Example

Next, we must insert 94**6**

- Bin **6** is occupied
- The next empty bin is 9

0	1	2	3	4	5	**6**	7	8	9	A	B	C	D	E	F
680						826	207	488	**946**	19A	5BA	74C	3AD		

Example

Next, we must insert ACD

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680						826	207	488	946	19A	5BA	74C	3AD		

Example

Next, we must insert AC**D**

- Bin **D** is occupied
- The next empty bin is E

0	1	2	3	4	5	6	7	8	9	A	B	C	**D**	E	F
680						826	207	488	946	19A	5BA	74C	3AD	**ACD**	

Example

Next, we insert B32

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680						826	207	488	946	19A	5BA	74C	3AD	ACD	

Example

Next, we insert B3**2**

- Bin **2** is unoccupied

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		**B32**				826	207	488	946	19A	5BA	74C	3AD	ACD	

Example

Next, we insert C8B

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32				826	207	488	946	19A	5BA	74C	3AD	ACD	

Example

Next, we insert C8**B**

- Bin **B** is occupied
- The next empty bin is F

0	1	2	3	4	5	6	7	8	9	A	**B**	C	D	E	F
680		B32				826	207	488	946	19A	5BA	74C	3AD	ACD	**C8B**

Example

Next, we insert D59

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32				826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Example

Next, we insert D59

- Bin **9** is occupied
- The next empty bin is 1

0	1	2	3	4	5	6	7	8	**9**	A	B	C	D	E	F
680	**D59**	B32				826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Example

Finally, insert E9C

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32				826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Example

Finally, insert E9C

- Bin **C** is occupied
- The next empty bin is 3

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	**E9C**			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Example

Having completed these insertions:

- The load factor is $\lambda = 14/16 = 0.875$
- The average number of probes is $38/14 \approx 2.71$

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Resizing the array

To double the capacity of the array, each value must be rehashed

- We use the least-significant five bits for the initial bin
- 680, B32, ACD, 5BA, 826, 207, 488, D59 may be immediately placed

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488					ACD					B32							D59	5BA					

Resizing the array

To double the capacity of the array, each value must be rehashed

– 19A resulted in a collision

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	**1A**	1B	1C	1D	1E	1F
680						826	207	488					ACD					B32							D59	5BA	**19A**				

Resizing the array

To double the capacity of the array, each value must be rehashed

- 946 resulted in a collision

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	**946**				ACD					B32							D59	5BA	19A				

Resizing the array

To double the capacity of the array, each value must be rehashed

- 74C fits into its bin

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946			**74C**	ACD				946	B32							D59	5BA	19A				

Resizing the array

To double the capacity of the array, each value must be rehashed

– 3AD resulted in a collision

0	1	2	3	4	5	6	7	8	9	A	B	C	**D**	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946			74C	ACD	**3AD**			946	B32							D59	5BA	19A				

Resizing the array

To double the capacity of the array, each value must be rehashed

- Both E9C and C8B fit without a collision
- The load factor is $\lambda = 14/32 = 0.4375$
- The average number of probes is $18/14 \approx 1.29$

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946		**C8B**	74C	ACD	3AD			946	B32							D59	5BA	19A	**E9C**			

Searching

Testing for membership is similar to insertions:

Start at the appropriate bin, and searching forward until

1. The item is found,
2. An empty bin is found, or
3. We have traversed the entire array

The third case will only occur if the hash table is full (load factor of 1)

Searching

Searching for C8B

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Searching

Searching for C8**B**

- Examine bins B, C, D, E, F
- The value is found in F

0	1	2	3	4	5	6	7	8	9	A	**B**	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	**C8B**

Searching

Searching for 23E

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Searching

Searching for 23E

- Search bins E, F, 0, 1, 2, 3, 4
- The last bin is empty; therefore, 23E is not in the table

0	1	2	3	4	5	6	7	8	9	A	B	C	D	**E**	F
680	D59	B32	E93	×		826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Erasing

Can we simply remove elements from the hash table?

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Erasing

We cannot simply remove elements from the hash table

– For example, consider erasing 3AD

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Erasing

We cannot simply remove elements from the hash table

- For example, consider erasing 3AD
- If we just erase it, it is now an empty bin
  - By our algorithm, we cannot find ACD, C8B and D59

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	**D59**	B32	E93			826	207	488	946	19A	5BA	74C		**ACD**	**C8B**

Erasing

Instead, we must attempt to fill the empty bin

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C		ACD	C8B

Erasing

Instead, we must attempt to fill the empty bin

- We can move ACD into the location
- Are we done?

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	**ACD**	ACD	C8B

Erasing

Now we have another bin to fill

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	ACD		C8B

Erasing

Now we have another bin to fill

- We can move C8B into the location

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	ACD	**C8B**	C8B

Erasing

Now we must attempt to fill the bin at F

– We cannot move 680

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	ACD	C8B	

Erasing

Now we must attempt to fill the bin at F

– We cannot move 680

– We can, however, move D59

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	ACD	C8B	**D59**

Erasing

At this point, we cannot move B32 or E93 and the next bin is empty

- We are finished

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32	E93			826	207	488	946	19A	5BA	74C	ACD	C8B	D59

Erasing

Suppose we delete 207

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32	E93			826	207	488	946	19A	5BA	74C	ACD	C8B	D59

Erasing

Suppose we delete 207

- Cannot move 488

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32	E93			826		488	946	19A	5BA	74C	ACD	C8B	D59

Erasing

Suppose we delete 207

– We could move 946 into Bin 7

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32	E93			826	**946**	488	946	19A	5BA	74C	ACD	C8B	D59

Erasing

Suppose we delete 207

- We cannot move any of the next five entries

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32	E93			826	946	488		19A	5BA	74C	ACD	C8B	D59

Erasing

Suppose we delete 207

– We could move D59

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32	E93			826	946	488	**D59**	19A	5BA	74C	ACD	C8B	D59

Erasing

Suppose we delete 207

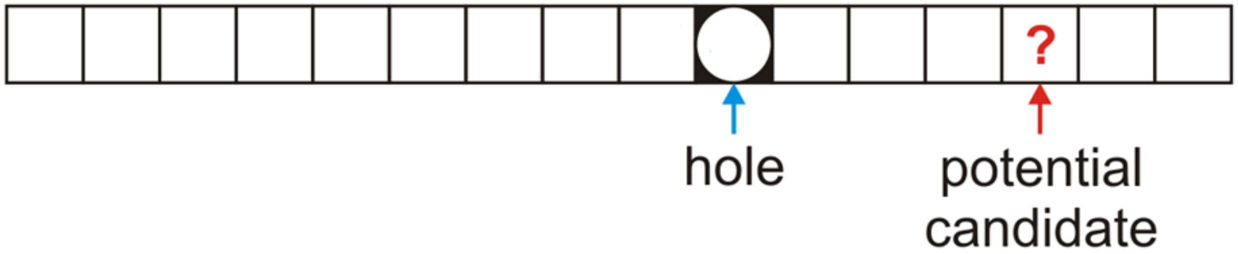
- We cannot fill this bin with 680, and the next bin is empty
- We are finished

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32	E93			826	946	488	D59	19A	5BA	74C	ACD	C8B	

Erasing

In general, assume:

- The currently removed object has created a hole at index `hole`
- The object we are checking is located at the position `index` and has a hash value of `hash`

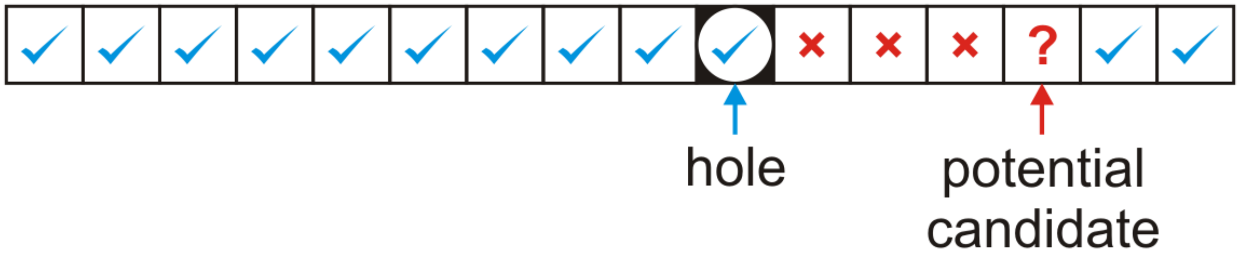


- Remember: if we are checking the object `?` at location `index`, this means that all entries between `hole` and `index` are both occupied and could not have been copied into the hole

Erasing

The first possibility is that `hole < index`

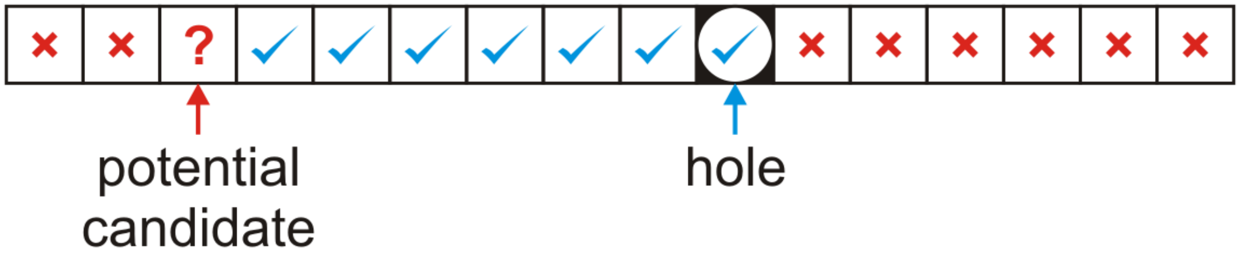
- In this case, we move the object at `index` only if its hash value is either
  - equal to or less than the `hole` **or**
  - greater than the `index` of the potential candidate



Erasing

The other possibility is we wrapped around the end of the array, that is, `hole > index`

- In this case, we move the object at `index` only if its hash value is both
  - greater than the index of the potential candidate **and**
  - less than or equal to the hole



In either case, if the move is successful, the **?** now becomes the new hole to be filled

Alternative Method: Lazy Erasing

- Consider erasing 3AD

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Alternative Method: Lazy Erasing

- Consider erasing 3AD
  - Mark the bin as `ERASED`
  - Searching: regard it as occupied
  - Insertion: regard it as unoccupied
    - What if we want to insert ACD?
    - Search before insertion

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	~~3AD~~	ACD	C8B

Black Board Example

Using the last digit as our hash function—insert these nine numbers into a hash table of size $M = 10$

$$31, 15, 79, 55, 42, 99, 60, 80, 23$$

Then, remove 79, 31, 42, and 60, in that order

Primary Clustering

We have already observed the following phenomenon:

- With more insertions, the contiguous regions (or *clusters*) get larger

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946		C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

The length of these chains will affect the number of probes required to perform insertions, accesses, or removals

Primary Clustering

We currently have three clusters of length four

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946		C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

Primary Clustering

There is a $5/32 \approx 16\ \%$ chance that an insertion will fill A

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946		C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

Primary Clustering

There is a $5/32 \approx 16\ \%$ chance that an insertion will fill A

– This causes two clusters to *coalesce* into one larger cluster of length 9

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946	**747**	C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

Primary Clustering

There is now a $11/32 \approx 34\ \%$ chance that the next insertion will increase the length of this cluster

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946	747	C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

Primary Clustering

As the cluster length increases, the probability of further increasing the length increases

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946	747	C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

In general:

- Suppose that a cluster is of length ℓ
- An insertion either into any bin occupied by the chain or into the locations immediately before or after it will increase the length of the chain
- This gives a probability of $\frac{\ell + 2}{M}$

Run-time analysis

It is possible to estimate the average number of probes for a successful search, where λ is the load factor:

$$\tfrac{1}{2}\left(1+\frac{1}{1-\lambda}\right)$$

For example: if $\lambda = 0.5$, we require 1.5 probes on average

Reference: Knuth, The Art of Computer Programming, Vol. 3, 2nd Ed., Addison Wesley, 1998, p.528.

Run-time analysis

The number of probes for an unsuccessful search or for an insertion is higher:

$$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$$

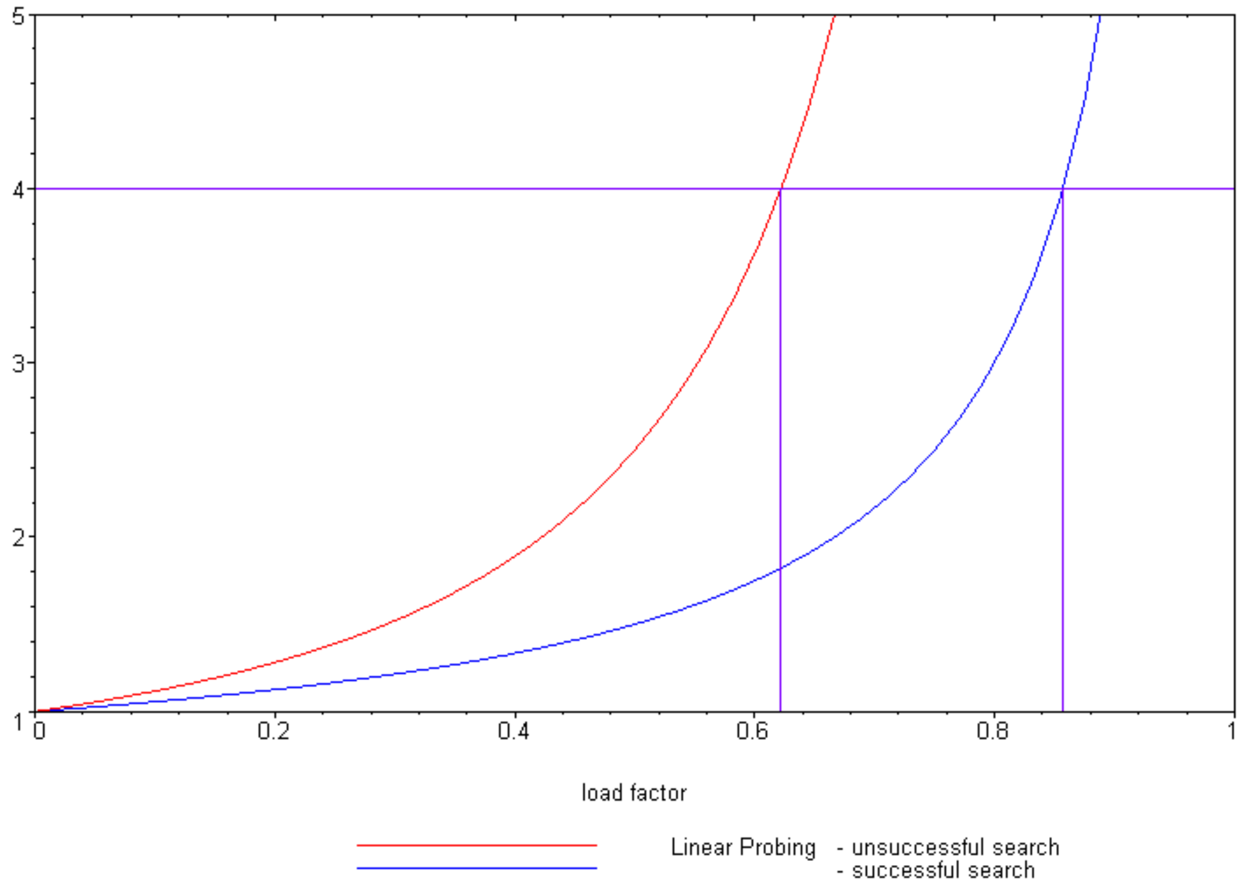
For $0 \le \lambda \le 1$, we have $(1-\lambda)^2 \le 1-\lambda$, and therefore the reciprocal will be larger

– if $\lambda = 0.5$ then we require 2.5 probes on average

Reference: Knuth, The Art of Computer Programming, Vol. 3, 2nd Ed., Addison Wesley, 1998, p.528.

Run-time analysis

The following plot shows how the number of required probes increases



Run-time analysis

Our goal was to keep all operations $\Theta(1)$

Unfortunately, as λ grows, so does the run time

One solution is to keep the load factor under a given bound

If we choose $\lambda = 2/3$, then the number of probes for either a successful or unsuccessful search is 2 and 5, respectively

Run-time analysis

Therefore, we have three choices:

- Choose M large enough so that we will not pass this load factor
  - This could waste memory
- Double the number of bins if the chosen load factor is reached
- Choose a different strategy than linear probing
  - Two possibilities are quadratic probing and double hashing

Summary

This topic introduced linear problem

- Continue looking forward until an empty cell is found
- Searching follows the same rule
- Removing an object is more difficult
- Primary clustering is an issue
- Keep the load factor $\lambda \leq 2/3$

Outline

- Introduction
- Hash function
- Mapping down to 0, ..., M – 1
- Dealing with collisions
  - Chained hash tables
  - Open addressing
    - Linear probing
    - Quadratic probing

Outline

This topic covers quadratic probing

- Similar to linear probing
  - Does not step forward one step at a time
- Primary clustering no longer occurs
- Affected by secondary clustering

Background

Linear probing:

- Look at bins $k, k+1, k+2, k+3, k+4, \ldots$
- Primary clustering

Background

Linear probing causes primary clustering

– All entries follow the same search pattern for bins:

```
int initial = hash_M( x.hash(), M );
for ( int k = 0; k < M; ++k ) {
    bin = (initial + k) % M;
    // ...
}
```

Description

Quadratic probing suggests moving forward by different amounts

For example,

```
int initial = hash_M( x.hash(), M );

for ( int k = 0; k < M; ++k ) {
    bin = (initial + k*k) % M;
}
```

Description

Problem:

– Will `initial + k*k` step through all of the bins?

– Here, the array size is 10:

```
M = 10;
initial = 5

for ( int k = 0; k <= M; ++k ) {
    std::cout << (initial + k*k) % M << ' ';
}
```

– The output is

```
5 6 9 4 1 0 1 4 9 6 5
```

Description

Problem:

- Will `initial + k*k` step through all of the bins?
- Now the array size is 12:

```
M = 12;
initial = 5

for ( int k = 0; k <= M; ++k ) {
    std::cout << (initial + k*k) % M << ' ';
}
```

- The output is now

```
5 6 9 2 9 6 5 6 9 2 9 6 5
```

Making M Prime

If we make the table size $M = p$ a prime number, quadratic probing is guaranteed to iterates through $\left\lceil \frac{p}{2} \right\rceil$ entries

Problems:

- All operations must be done using %
  - Cannot use &, <<, or >>
  - The modulus operator % is relatively slow
- Doubling the number of bins is difficult:
  - What is the next prime after 2×263?

Generalization

More generally, we could consider an approach like:

```
int initial = hash_M( x.hash(), M );

for ( int k = 0; k < M; ++k ) {
    bin = (initial + c1*k + c2*k*k) % M;
}
```

Using $M = 2^m$

If we ensure $M = 2^m$ then choose

$$c_1 = c_2 = ½$$

```
int initial = hash_M( x.hash(), M );

for ( int k = 0; k < M; ++k ) {
    bin = (initial + (k + k*k)/2) % M;
}
```

- Note that `k + k*k` is always even
- The growth is still $\Theta(k^2)$
- This guarantees that all M entries are visited before the pattern repeats
  - This only works for powers of two

Using $M = 2^m$

For example:

- Use an array size of 16:

```
M = 16;
initial = 5

for ( int k = 0; k <= M; ++k ) {
    std::cout << (initial + (k + k*k)/2) % M << ' ';
}
```

- The output is now

```
5 6 8 11 15 4 10 1 9 2 12 7 3 0 14 13 13
```

Using $M = 2^m$

There is an even easier means of calculating this approach

```
int bin = hash_M( x.hash(), M );

for ( int k = 0; k < M; ++k ) {
    bin = (bin + k) % M;
}
```

– Recall that $\frac{k^2 + k}{2} = \sum_{j=0}^{k} j$, so just keep adding the next highest value

Example

Consider a hash table with $M = 16$ bins

Given a 2-digit hexadecimal number:

- The least-significant digit is the primary hash function (bin)
- Example: for 7**A**$_{16}$, the initial bin is **A**

Example

Insert these numbers into this initially empty hash table

9A, 07, AD, 88, BA, 80, 4C, 26, 46, C9, 32, 7A, BF, 9C

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Example

Start with the first four values:

9A, 07, AD, 88

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Example

Start with the first four values:

9A, 07, AD, 88

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							07	**88**		**9A**			**AD**		

Example

Next we must insert BA

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							07	88		9A			AD		

Example

Next we must insert BA

- The next bin is empty

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							07	88		9A	BA		AD		

Example

Next we are adding 80, 4C, 26

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							07	88		9A	BA		AD		

Example

Next we are adding 80, 4C, 26

- All the bins are empty—simply insert them

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80						**26**	07	88		9A	BA	**4C**	AD		

Example

Next, we must insert 46

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80						26	07	88		9A	BA	4C	AD		

Example

Next, we must insert 46

- Bin **6** is occupied
- Bin **6 + 1 = 7** is occupied
- Bin **7 + 2 = 9** is empty

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80						26	07	88	**46**	9A	BA	4C	AD		

Example

Next, we must insert C9

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80						26	07	88	46	9A	BA	4C	AD		

Example

Next, we must insert C9

- Bin **9** is occupied
- Bin **9 + 1 = A** is occupied
- Bin **A + 2 = C** is occupied
- Bin **C + 3 = F** is empty

0	1	2	3	4	5	6	7	8	**9**	A	B	C	D	E	F
80						26	07	88	46	9A	BA	4C	AD		C9

Example

Next, we insert 3**2**

– Bin **2** is unoccupied

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80		**32**				26	07	88	46	9A	BA	4C	AD		C9

Example

Next, we insert 7A

- Bin **A** is occupied
- Bins **A + 1 = B**, **B + 2 = D** and **D + 3 = 0** are occupied
- Bin **0 + 4 = 4** is empty

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80		32		**7A**		26	07	88	46	9A	BA	4C	AD		C9

Example

Next, we insert BF

- Bin **F** is occupied
- Bins **F + 1 = 0** and **0 + 2 = 2** are occupied
- Bin **2 + 3 = 5** is empty

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80		32		7A	**BF**	26	07	88	46	9A	BA	4C	AD		C9

Example

Finally, we insert 9**C**

- Bin **C** is occupied
- Bins **C + 1 = D**, **D + 2 = F**, **F + 3 = 2**, **2 + 4 = 6** and **6 + 5 = B** are occupied
- Bin **B + 6 = 1** is empty

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80	**9C**	32		7A	BF	26	07	88	46	9A	BA	4C	AD		C9

Example

Having completed these insertions:

- The load factor is $\lambda = 14/16 = 0.875$
- The average number of probes is $32/14 \approx 2.29$

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80	9C	32		7A	BF	26	07	88	46	9A	BA	4C	AD		C9

Erase

Can we erase an object like we did with linear probing?

– Consider erasing 9A from this table
– There are $M - 1$ possible locations where an object which could have occupied a position could be located

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80	21		43			76				**9A**					50

Instead, we use *lazy erasing*

– Mark a bin as ERASED; however, when searching, treat the bin as occupied and continue

Erase

If we erase AD, we must mark that bin as erased

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80	9C	32		7A	BF	26	07	88	46	9A	BA	4C	~~AD~~		C9

Find

When searching, it is necessary to skip over this bin

– For example, find AD: D, E

find 5C: C, D, F, 2, 6, B, 1, 8, 0, 9, 3

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80	9C	32		7A	BF	26	07	88	46	9A	BA	4C	~~AD~~		C9

Modified insertion

We must modify insert, as we may place new items into either

- Unoccupied bins
- Erased bins

Implementation

Storing three states can be achieved using an enumerated type:

```
enum bin_state_t {
    UNOCCUPIED,
    OCCUPIED,
    ERASED
};
```

Now we can declare and initialize arrays:

```
bin_state_t state[M];

for ( int i = 0; i < M; ++i ) {
    state[i] = UNOCCUPIED;
}
```

Multiple insertions and erases

One problem which may occur after multiple insertions and removals is that numerous bins may be marked as `ERASED`

- In calculating the load factor, an `ERASED` bin is equivalent to an `OCCUPIED` bin

This will increase our run times…

If the load factor λ grows too large, we have two choices:

- If the load factor due to occupied bins is too large, double the table size
- Otherwise, rehash all of the objects currently in the hash table

Expected number of probes

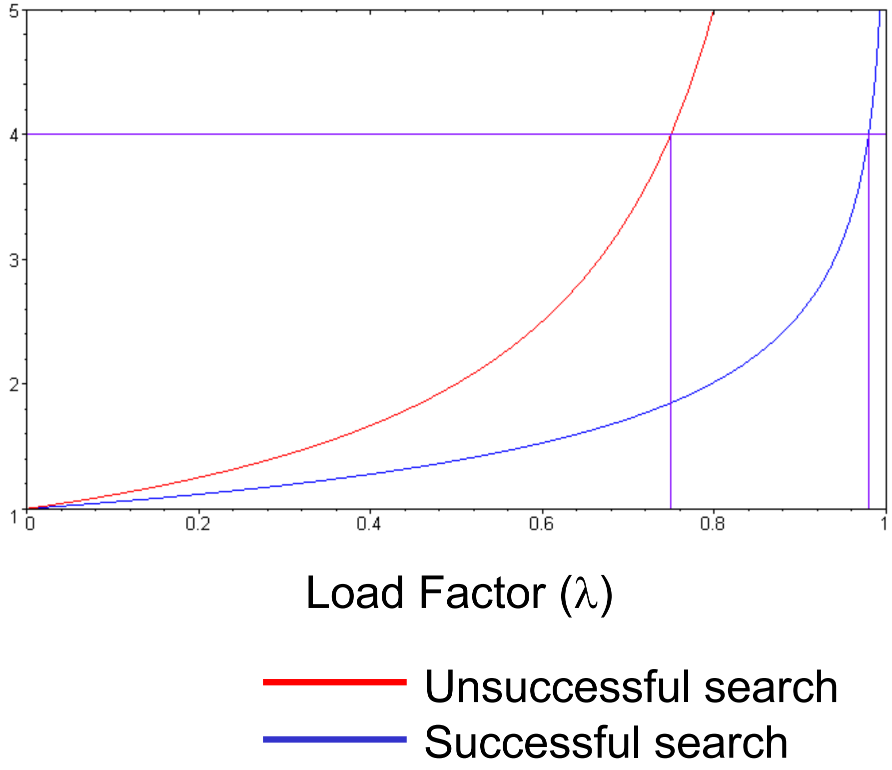
It is possible to calculate the expected number of probes for quadratic probing, again, based on the load factor:

– Successful searches: $\ln\left(\frac{1}{1-\lambda}\right)\Big/\lambda$

– Unsuccessful searches: $\frac{1}{1-\lambda}$

When $\lambda = 2/3$, we requires
1.65 and 3 probes, respectively

– Linear probing required
3 and 5 probes, respectively



Reference: Knuth, The Art of Computer Programming, Vol. 3, 2nd Ed., 1998, Addison Wesley, p. 530.

Quadratic probing versus linear probing

Comparing the two:

Linear probing

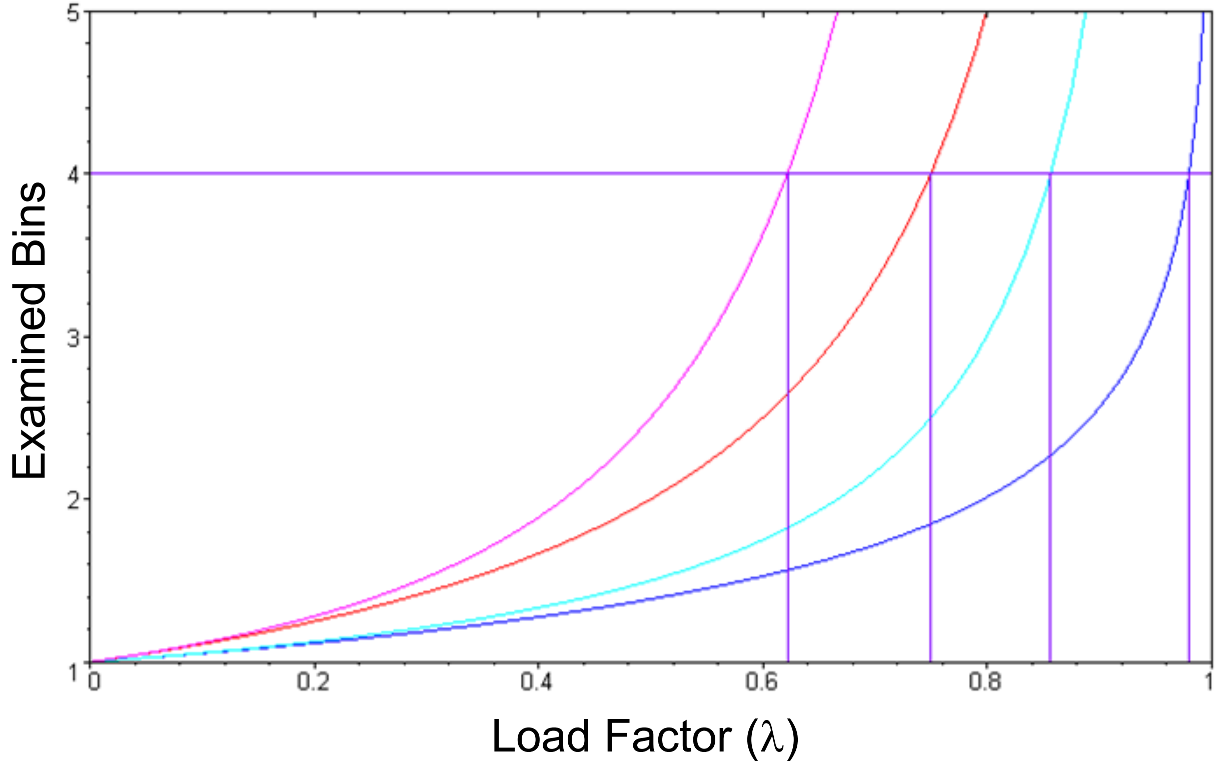
Unsuccessful search

Successful search

Quadratic probing

Unsuccessful search

Successful search



Secondary clustering

Weakness with quadratic problem

- Clustering may still occur: objects placed in the same bin will follow the same sequence
- Less severe than linear probing

Summary

In this topic, we have looked at quadratic probing:

- An open addressing technique
- Steps forward by a quadratically growing steps
- Insertions and searching are straight forward
- Removing objects is more complicated: use lazy deletion
- Still subject to secondary probing

Summary

Object

Predetermined hash functions (explicit)
Arithmetic hash functions (implicit)

32-bit integer

Modulus (bitwise operations for $M = 2^m$)
Obfuscate via multiplication

Map to an index $0, \ldots, M - 1$

Chained hash tables
Open addressing

- Linear probing
- Quadratic probing

Deal with collisions