ShanghaiTech University

CS101 Algorithms and Data Structures

Fall 2019

Homework 8

Due date: 23:59, November 17, 2019

1. Please write your solutions in English.
2. Submit your solutions to gradescope.com.
3. Set your FULL Name to your Chinese name and your STUDENT ID correctly in Account Settings.
4. If you want to submit a handwritten version, scan it clearly. Camscanner is recommended.
5. When submitting, match your solutions to the according problem numbers correctly.
6. No late submission will be accepted.
7. Violations to any of above may result in zero score.

1: (3*2'+4') Dijkstra's Algorithm

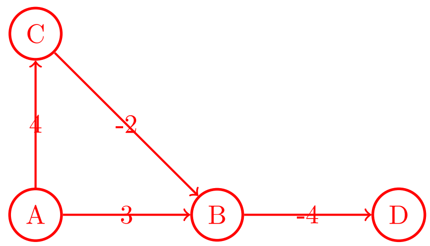
Question 1. *Judge whether the following statement is true or false and explain why. Give a counter-example if it is false.*

(a) *Suppose G is strongly connected with integer edge weights, and has shortest paths from some vertex v (i.e. a finite weight shortest path exists from v to all nodes). Then shortest paths can be found from every vertex to every other vertex.*

True. The existence of a shortest path tree in a strongly connected graph ensures there is no negative cycle, thus there must be a shortest path tree from any source.

(b) *If G is a connected and undirected graph without negative cycles, we can apply Dijkstra's algorithm to find the shortest path.*

False.

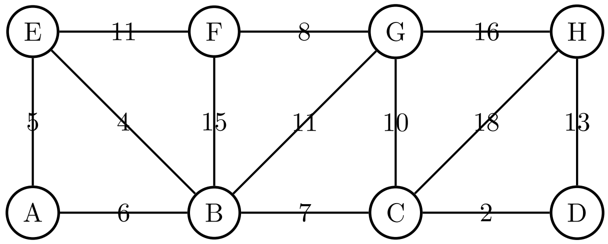


After running Dijkstra's algorithm, dist(A,B)=3 and dist(A,D)=7; however, the weight of the actual shortest path from A to B is 2 and the weight of the actual shortest path from A to D is 6.

(c) *Suppose G is a DAG. We can find the longest path by negating all edge lengths and then run Dijkstra's algorithm from every source node.*

FALSE. In a DAG, the longest path is well defined and can certainly be found by negating all the edges and updating in topological order. Consider a graph with edges (a, b) of weight 1, an edge (a, c) of weight 3, an edge from (b, c) of weight 5, and an edge from (c, d) of weight 1. Dijkstras will label d with -4 corresponding to a path from a to c to d, where the longest path has length corresponds to -7 which is the path a, b, c, d.

Question 2. *Given a weighted graph below, please run Dijkstra's algorithm using vertex A as the source. Write down the vertices in the order which they are marked and the updated distances at each step.*



Solution:

step	vertex
1	A
2	E
3	B
4	C
5	D
6	F
7	G
8	H

step	dist[A]	dist[B]	dist[C]	dist[D]	dist[E]	dist[F]	dist[G]	dist[H]
1	0	6	∞	∞	5	∞	∞	∞
2	0	6	∞	∞	5	16	∞	∞
3	0	6	13	∞	5	16	17	∞
4	0	6	13	15	5	16	17	31
5	0	6	13	15	5	16	17	28
6	0	6	13	15	5	16	17	28
7	0	6	13	15	5	16	17	28
8	0	6	13	15	5	16	17	28

2: (2'+3') Floyd-Warshall Algorithm

Question 3. *Let $G = (V, E)$ be a connected, undirected graph with edge weights $w : E \to \mathbb{Z}$. Which of the following statements are True about the Floyd-Warshall algorithm applied to G?*

(A) Since G is undirected, we cannot apply Floyd-Warshall algorithm.

(B) Since G is undirected, Floyd-Warshall will be asymptotically faster than on directed graphs.

(C) Since G is undirected, Floyd-Warshall will be unable to detect negative-weight cycles.

(D) None of the above.

D

Question 4. *Consider the following implementation of the Floyd-Warshall algorithm. Assume $w_{ij} = \infty$ where there is no edge between vertex i and vertex j, and assume $w_{ii} = 0$ for every vertex i.*

Algorithm 1 Floyd-Warshall

```
for i = 1 to n do
  for j = 1 to n do
    A[i, j, 0] = w_ij
    P[i, j] = −1
  end for
end for
for k = 1 to n do
  for i = 1 to n do
    for j = 1 to n do
      A[i, j, k] = A[i, j, k − 1]
      if A[i, j, k] > A[i, k, k − 1] + A[k, j, k − 1] then
        A[i, j, k] = A[i, k, k − 1] + A[k, j, k − 1]
        P[i, j] = k
      end if
    end for
  end for
end for
```

Assume matrix P, the output of the above algorithm is given. Consider the following matrix for graph G with 7 vertices. What is the shortest path from vertex 5 to vertex 7 in graph G?

P	1	2	3	4	5	6	7
1	-1	5	4	-1	4	4	-1
2	5	-1	5	5	-1	5	-1
3	4	5	-1	-1	-1	-1	6
4	-1	5	-1	-1	3	3	1
5	4	-1	-1	3	-1	3	6
6	4	5	-1	3	3	-1	-1
7	-1	-1	6	1	6	-1	-1

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3: (3'+3'+4') Shortest Path

Question 5. *Consider a weighted undirected graph with positive edge weights and let (u, v) be an edge in the graph. It is known that the shortest path from the source vertex s to u has weight 53 and the shortest path from s to v has weight 65. Which is the range of the weight the edge (u, v)?*

The range is weight $(u, v) \geq 12$

Question 6. *Consider the weighted undirected graph with 4 vertices, where the weight of edge $\{i, j\}$ is given by the entry $W_{i,j}$ in the matrix W*

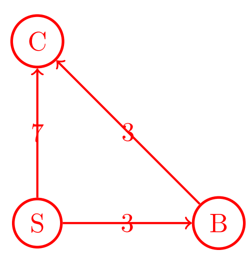
$$W = \begin{bmatrix} 0 & 2 & 8 & 6 \\ 2 & 0 & 5 & 8 \\ 8 & 5 & 0 & x \\ 6 & 8 & x & 0 \end{bmatrix}$$

We want to find the largest possible integer value of x, for which at least one shortest path between some pairs of vertices will definitely contain the edge with weight x. What is this largest possible integer value of such x? Explain your reason briefly. ***When breaking tie, the path may be random.***

Let vertices be 0, 1, 2 and 3. x directly connects 2 to 3. The shortest path (excluding x) from 2 to 3 is of weight 13 (2-1-0-3). Thus, the x should be 12.

Question 7. *Suppose $G = (V, E)$ is a weighted graph and T is its shortest-path tree from source s. If we increase all weights in G by the same amount, i.e., $\forall e \in E$, $w'_e = w_e + c$. Is T still the shortest-path tree (from source s) of the new graph? If yes, prove the statement. Otherwise, give a counter example.*

No. The shortest-path tree of this graph has edges (S, B) and (B, C).



For c = 2 the new graph would be the following. The shortest-path tree of this graph has edges (S, B) and (S, C)