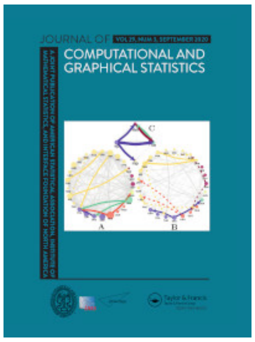

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First- and Second-Order Characteristics of Spatio-Temporal Point Processes on Linear Networks

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ABSTRACT

We present several characteristics for spatio-temporal point patterns when the spatial locations are restricted to a linear network. A nonparametric kernel-based intensity estimator is proposed to highlight the concentration of events within the network and time, either jointly or separately. We also provide second-order characteristics for spatio-temporal point patterns on linear networks such as K-function and pair correlation function to analyze the type of interaction between points. They are independent of network's geometry and have known values for Poisson point processes. Finally, we consider some applications to traffic accidents and demonstrate our findings by analyzing datasets of Houston (United States), Medellín (Colombia), and Eastbourne (United Kingdom). Supplementary materials for this article are available online.

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Intensity; K-function; Linear network; Pair correlation function; Space-time data; Traffic accidents

1. Introduction

Most of spatial point patterns are in fact snapshots of spatio-temporal patterns. Sometimes, we only consider the spatial domain and analyze point patterns regardless of time, while they are inherently happening jointly in space and time. This stays the same for spatio-temporal point processes on linear networks. Thinking of traffic accidents or street crimes that happen on cities' streets, we easily see that many accidents happen in particular moments of a day, and their spatial distribution may vary with time. Also some crimes, for example, robberies, happen in crowded streets or sidewalks, and their spatial structure depends on time. Knowing that there may be scientific questions that cannot be answered by only analyzing spatial components, ignoring the time of occurrence (Diggle 2003), taking the time component into account when analyzing point patterns on networks becomes an important and crucial aspect in the statistical analysis.

The challenge when we think on network events is that we face a change of support, that is, events do not happen on the entire space. Since the location of events is restricted to a network, existing methods for spatio-temporal point processes on $\mathbb{R}^2 \times \mathbb{R}^+$ may not be appropriate. However, considering a network as the basic support instead of the entire space requires a number of modifications. Okabe and Yamada (2001) replaced the Euclidean distance by the shortest-path distance (travel distance) and introduced the network K-function, but its values were difficult to interpret as it depends on the geometry of the network. Yamada and Thill (2004) illustrated the risk of false positive detection when using a designed method for planar space to analyze network-constrained events. Lu and Chen (2007) focused on social and economic activities in urban areas and used vehicle thefts in San Antonio (Texas) as a case study to show that the planar K-function may result in false alarm problems. Xie and Yan (2008) presented a nonparametric kernel-based estimator to estimate the intensity function of network events but their suggestion was heavily biased. Okabe, Satoh, and Sugihara (2009) introduced equal-split network kernel density estimators, both under continuous and discontinuous schemes. Considering stream distances Ver Hoef and Peterson (2010) developed spatial autocovariance models for spatially continuous data on stream networks. Moreover, Okabe and Sugihara (2012) broadly discussed the details of spatial analysis along linear networks. Ang, Baddeley, and Nair (2012) improved the network K-function by Okabe and Yamada (2001), and defined geometrically corrected K- and pair correlation functions so that they do not depend on the topography of the network. Baddeley, Jammalamadaka, and Nair (2014) then extended the first- and second-order tools of such patterns to the case of multitype point patterns. McSwiggan, Baddeley, and Nair (2017) developed a kernel smoothing method to estimate the intensity function based on diffusion on the network. Moradi, Rodríguez-Cortés, and Mateu (2017) extended the kernel-based edge-corrected intensity estimator (Diggle 1985; Jones 1993) to the context of linear networks. Rakshit, Nair, and Baddeley (2017) used different distance metrics for second-order summary statistics of point patterns on linear networks. Rakshit et al. (2019) developed a fast kernel-based intensity estimator using two-dimensional convolution that is applicable to large datasets. Moradi, Cronie, et al. (2019) proposed a resample-smoothing technique that improves the statistical performance of Voronoi intensity estimators regardless of the state space.

Although network events were recently given attention by the statistical community, temporal components have not been taken into account yet when studying such events. Knowing

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that the behavior of spatial events may vary over time, we here aim at considering the temporal component and provide nonparametric methods to study spatio-temporal point processes on linear networks. In this article, and under the assumption of separability, we develop a kernel-based intensity estimator to estimate the intensity function of spatio-temporal point patterns on networks. We then present spatio-temporal second-order summary statistics, the K- and the pair correlation functions, for both homogeneous and inhomogeneous processes, which are useful to study the correlation between events. They are independent of the geometry of the network and have known values for Poisson point processes. Thus, they can be used for model selection as benchmarks due to having known forms under completely spatio-temporal randomness. Nonparametric estimates are also provided for both homogeneous and inhomogeneous processes. We also discuss independent thinning and goodness of fit, although the focus of the article is not on statistical modeling of point patterns on networks.

The article is structured as follows. Section 2 provides some definitions and preliminaries for point processes on linear networks. Section 3 is devoted to first- and second-order summary statistics of spatio-temporal point processes on linear networks, and presents their corresponding definitions, estimators and properties. In Section 4, we analyze the reference locations and times of occurrence of traffic accidents from three datasets in Houston (United States), Medellín (Colombia), and Eastbourne (United Kingdom). The article ends with a discussion in Section 5.

2. Point Processes on Linear Networks

Point processes on linear networks are recently considered to analyze events that happen on or along a network such as traffic accidents. In this section, we first briefly review spatial point patterns on linear networks, and then extend the setup to the spatio-temporal case.

2.1. Spatial Processes

Throughout the article, a linear network $L \subset \mathbb{R}^2$ is considered as a union of a finite number of segments $l_i = [u_i, v_i] = \{tu_i + (1-t)v_i : 0 \le t \le 1\} \subset \mathbb{R}^2$ where $u_i, v_i \in \mathbb{R}^2$ are the endpoints of l_i. For any $i \neq j$ the intersection of l_i and l_j is either empty or an endpoint of both segments. We denote the total length of any sub-network $A \subseteq L$ by $|A|$ which is the sum of the length of all segments in A. The distance between any two points $u, v \in L$, denoted by $d_L(u, v)$, is measured by the shortest-path distance which is the minimum of the length of all possible paths between $u, v \in L$. We represent an observed spatial point pattern with n points by $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ where each $x_i \in L$ is considered as a location. The spatial point pattern $\mathbf{x}$ is in fact a realization of a spatial point process X that is a random countable subset of $\mathbb{R}^2$ (Daley and Vere-Jones 2008; Baddeley, Rubak, and Turner 2015) with no limit points. The number of points of X in a sub-network $A \subseteq L$ is $N(X \cap A)$ with the expected number

$$\mu(A) = \mathbb{E}[N(X \cap A)] = \int_A \mu(\mathrm{d}u) = \int_A \lambda(u)\mathrm{d}_1 u, \qquad A \subseteq L, \tag{1}$$

where μ is the intensity measure of X, $\mu(A) < \infty$, $\lambda(\cdot)$ is the intensity function of X, and d_1 denotes one-dimensional integration over the line segment, that is, integration with respect to arc length; see Baddeley, Bárány, and Schneider (2006) and Baddeley, Rubak, and Turner (2015, chap. 17). For any nonnegative measurable function $f : L \to [0, \infty)$ such that $\int_L |f(x)|\lambda(u)\mathrm{d}_1 u < \infty$,

$$\mathbb{E}\sum_{x \in X} f(x) = \int_L f(x)\lambda(u)\mathrm{d}_1 u. \tag{2}$$

Equation (2) is called Campbell's formula (Baddeley, Rubak, and Turner 2015, chap. 17).

The literature on spatial point patterns on linear networks is not extensive yet, being this a relatively new topic. Different examples of spatial point patterns on linear networks can be found in Okabe and Sugihara (2012), Ang, Baddeley, and Nair (2012), McSwiggan, Baddeley, and Nair (2017), Rakshit et al. (2019), Moradi, Rodríguez-Cortés, and Mateu (2017), and Moradi, Cronie, et al. (2019).

2.2. Spatio-Temporal Processes

In many natural phenomena there may be questions that we cannot answer by only analyzing the spatial behavior of events, for example, the number of traffic accidents may vary per week, day or hour (Diggle 2003). Suppose that the reference locations of spatial point pattern $\mathbf{x}$ are labeled by their time of occurrence. We can then consider these events as a spatio-temporal point pattern on a linear network L. A spatio-temporal point process X on a linear network $L \subset \mathbb{R}^2$ is considered as a random countable subset of $\mathbb{R}^2 \times \mathbb{R}^+$ in which the location of each event is restricted to lying on a network structure L. As the location of events is limited to the network L and the shortest-path distance differs from the Euclidean distance, such events might not be analyzed properly through more classical statistical methods for spatio-temporal point processes in $\mathbb{R}^2 \times \mathbb{R}^+$; note the work in Lu and Chen (2007).

We hereafter focus on a spatio-temporal point process X on a linear network L with no overlapping points (x, t_x) where $x \in L$ is the location of an event and $t_x \in T$ ($T \subseteq \mathbb{R}^+$) is the corresponding time occurrence of x. Note that the temporal state-space T might be either a continuous or a discrete set. A realization of X with n points is represented by $\mathbf{x} = \{(x_i, t_{x_i}), i = 1, \dots, n\}$ where $(x_i, t_{x_i}) \in L \times T$. A spatio-temporal disc with center $(x, t_x) \in L \times T$, network radius $r > 0$ and temporal radius $t > 0$ is defined as

$$b((x, t_x), r, t) = \{(u, t_u) \in L \times T : d_L(x, u) \le r \text{ and } |t_x - t_u| \le t\},$$

where $|\cdot|$ is a numerical distance. The cardinality of any subset $A \subseteq L \times T$, $N(X \cap A) \in \{0, 1, \dots\}$, is the number of points of X restricted to A, whose expected number is denoted by

$$\nu(A) = \mathbb{E}[N(X \cap A)], \qquad A \subseteq L \times T,$$

where ν, the intensity measure of X, is a locally finite product measure on $L \times T$ (Baddeley, Bárány, and Schneider 2006).

Assume that X has an intensity function $\lambda(\cdot)$ and a second-order product density function $\lambda_2(\cdot,\cdot)$, hence,

$$\mathbb{E}[N(X \cap A)] = \int_A \nu(\mathrm{d}(u,t_u)) = \int_A \lambda(u,t_u)\mathrm{d}_2(u,t_u), \quad A \subseteq L \times T,$$

where $\mathrm{d}_2(u,s)$ corresponds to integration over $L \times T$, and

$$\mathbb{E}[N(X\cap A)N(X\cap B)] = \int_A\int_B \lambda_2((u,t_u),(v,t_v))\mathrm{d}_2(u,t_u)\mathrm{d}_2(v,t_v), \quad A,B \subseteq L\times T.$$

Details of integration with respect to the product measure ν are given at the end of this section. Note that for homogeneous processes $\lambda(\cdot)$ takes a constant value λ for any point in $L \times T$, that is, the events are distributed over $L \times T$ uniformly. Following Campbell's formula (2), for any nonnegative measurable function $f : (L\times T)\times(L\times T) \to [0,\infty)$,

$$\mathbb{E}\left[\sum_{(x,t_x),(y,t_y)\in X}^{\neq} f\left((x,t_x),(y,t_y)\right)\right] = \int\int f((u,t_u),(v,t_v))\,\lambda_2((u,t_u),(v,t_v))\mathrm{d}_2(u,t_u)\mathrm{d}_2(v,t_v),$$

where $\sum^{\neq}$ means that the sum is over distinct pairs of points. Nevertheless, one can still work with the projection of X onto the linear network L or the time interval T which are given as

$$X_{\text{net}} = \{x : (x,t_x) \in X\}, \qquad X_{\text{time}} = \{t_x : (x,t_x) \in X\},$$

where X_{net} defines a spatial point process on L, and X_{time} stands for a temporal point process on T (Diggle 2003; Møller and Ghorbani 2012). Note that X_{net} and X_{time} might no longer be simple point processes in the sense of having no overlapping points. Then, the corresponding intensity functions of X_{net} and X_{time} can be obtained as

$$\lambda_{\text{net}}(u) = \int_T \lambda(u,t_u)\mathrm{d}t_u, \qquad \lambda_{\text{time}}(t_u) = \int_L \lambda(u,t_u)\mathrm{d}_1 u. \quad (3)$$

We next comment on calculating the integration of measurable functions f on $L \times T$. To do so, we follow Federer (1996) and Ang, Baddeley, and Nair (2012) and convert the integration over $L\times T$ to that over $\mathbb{R}^+ \times \mathbb{R}^+$. For any measurable function $f : L\times T \to \mathbb{R}$,

$$\int_{L\times T} f(v,t_v)\mathrm{d}_2(v,t_v) = \int_0^\infty\int_0^\infty \sum_{(v,t_v)\in L\times T: d_L(u,v)=r\ \&\ |t_u-t_v|=t} f(v,t_v)\,\mathrm{d}r\,\mathrm{d}t. \quad (4)$$

For instance, let $f(v,t_v) = h(d_L(u,v),|t_u-t_v|)$, then,

$$\int_{L\times T} h(d_L(u,v),|t_u-t_v|)\mathrm{d}_2(v,t_v) = \int_0^\infty\int_0^\infty h(r,t)M((u,t_u),r,t)\,\mathrm{d}r\,\mathrm{d}t,$$

where $M((u,t_u),r,t)$ is the number of points lying exactly at the shortest-path distance $r \geq 0$ and the time distance $t \geq 0$ away from (u,t_u). For any $r,t < \infty$, $M((u,t_u),r,t)$ is finite and if any of r or t is ∞, we then set $M((u,t_u),r,t) = \infty$.

2.2.1. Poisson Processes

The Poisson point process model is crucial in defining other more complicated models and can be considered as a benchmark model in exploratory data analysis. For a spatio-temporal Poisson point process X, the points (x,t_x) on $L\times T$ need to satisfy the following conditions:

- In any bounded set $A \subseteq L\times T$, $N(X\cap A)$ follows a Poisson distribution with expected number $\int_A \lambda(u,t_u)\mathrm{d}_2(u,t_u)$. For instance, if we assume $A = A_L \times A_T \subseteq L \times T$, then the expected value is the expected number of points in the subnetwork $A_L \subseteq L$ and within the time interval $A_T \subseteq T$.
- For any k arbitrary disjoint subsets of $L\times T$, say $A_1,\ldots,A_k$, their cardinality $N(X\cap A_1),\ldots,N(X\cap A_k)$ are independent variables.

Moreover, for any time interval $S = [s_1,s_2] \subset T$, the projection of X onto L defines a spatial Poisson process on L with intensity function $\int_{s_1}^{s_2}\lambda(u,t_u)\mathrm{d}t_u$. Similarly, for any set of segments $A \subset L$, the projection of X onto T defines a temporal Poisson process with intensity function $\int_A \lambda(u,t_u)\mathrm{d}_1 u$, see Illian et al. (2008, chap. 6) and Diggle (2003, chap. 10).

3. Methodology

This section is devoted to the first- and second-order characteristics of a spatio-temporal point process X on $L\times T$. Throughout the article, we assume that the only information we have is the point locations and their corresponding time of occurrence.

3.1. First-Order Characteristics

A very first attempt to understand the behavior of events is to analyze weather the distribution of a spatio-temporal point process X varies over its state-space $L\times T$. For this purpose we firstly need to study the intensity function $\lambda(\cdot)$ so that a good estimate can reveal the more populated parts in both the linear network and the temporal range. Note that the intensity function $\lambda(u,t_u)$ can also be interpreted as a conditional spatial intensity given any time t or a conditional temporal intensity given any subnetwork of L (Diggle 2003). As a general assumption we assume that the intensity function $\lambda(u,t_u)$ is separable, that is,

$$\lambda(u,t_u) = \bar{\lambda}_1(u)\bar{\lambda}_2(t_u), \qquad (u,t_u) \in L\times T, \quad (5)$$

where $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are nonnegative functions on L and T, respectively (Gabriel and Diggle 2009; Møller and Ghorbani 2012; Ghorbani 2013). For prior works on separability test, see Díaz-Avalos, Juan, and Mateu (2013) and Fuentes-Santos, González-Manteiga, and Mateu (2018). Using (3) and (5),

$$\lambda_{\text{net}}(u) = \bar{\lambda}_1(u)\int_T \bar{\lambda}_2(t_u)\mathrm{d}t_u,$$

$$\lambda_{\text{time}}(t_u) = \bar{\lambda}_2(t_u)\int_L \bar{\lambda}_1(u)\mathrm{d}_1 u.$$

Then, the intensity function $\lambda(u,t_u)$ can be rewritten as

$$\lambda(u,t_u) = \frac{\lambda_{\text{net}}(u)\lambda_{\text{time}}(t_u)}{\int_{L\times T}\lambda(u,t_u)\mathrm{d}_2(u,t_u)}. \quad (6)$$

Note that if X is a homogeneous point process then all λ, λ_{net}, and λ_{time} take constant values. However, we here do not assume homogeneity, that is, we consider that the intensity function varies over $L \times T$. Taking the count function N into account, the denominator in (6) equals the expected number of points in $L \times T$. Suppose we are given $\widehat{\lambda}_{\text{net}}(u)$ and $\widehat{\lambda}_{\text{time}}(t_u)$ so that they result in unbiased estimators for the number of observed points n, then it is not surprising that

$$\widehat{\lambda}(u, t_u) = \frac{\widehat{\lambda}_{\text{net}}(u)\widehat{\lambda}_{\text{time}}(t_u)}{n}, \qquad (u, t_u) \in L \times T, \tag{7}$$

provides an unbiased estimator for the expected number of observed points, that is,

$$\mathbb{E}\left[\int_{L\times T} \widehat{\lambda}(u, t_u)\mathrm{d}_2(u, t_u)\right] = \int_{L\times T} \lambda(u, t_u)\mathrm{d}_2(u, t_u) = n,$$

meaning that the intensity estimator (7) preserves mass.

The geometrical complexities of networks pose some challenges in estimating the intensity function of a spatial point pattern on a linear network. However, there have been a few proposals to estimate $\lambda_{\text{net}}(u)$ nonparametrically. One can consider equal-split kernel density estimators defined by Okabe, Satoh, and Sugihara (2009), a heat-kernel estimator introduced by McSwiggan, Baddeley, and Nair (2017), an adapted Jones–Diggle corrected estimator by Moradi, Rodríguez-Cortés, and Mateu (2017), the resample-smoothed Voronoi intensity estimator proposed by Moradi, Cronie, et al. (2019) or the quick kernel-based method in Rakshit et al. (2019); note also the usability of adaptive kernel intensity estimators in Rakshit et al. (2019). For details of the aforementioned methods, see the corresponding references. Here we make use of the quick kernel-based intensity estimator developed by Rakshit et al. (2019) to estimate $\lambda_{\text{net}}(u)$ according to which they convolve both point locations and the network itself with a two-dimensional kernel, then combine them into an intensity function on the network. Their estimator, with the uniform correction, is of the form

$$\widehat{\lambda}_{\text{net}}(u) = \frac{1}{c_L(u)} \sum_{i=1}^{n} \kappa(u - x_i), \qquad u \in L, \tag{8}$$

and using the Jones–Diggle correction it becomes

$$\widehat{\lambda}_{\text{net}}(u) = \sum_{i=1}^{n} \frac{\kappa(u - x_i)}{c_L(x_i)}, \qquad u \in L, \tag{9}$$

where κ denotes a bivariate kernel function, x_i's are the location of data points on the network L, and $c_L(u) = \int_L \kappa(v-u)\mathrm{d}_1 v$, $u \in L$.

The estimators (8) and (9) can be computed rapidly using the fast Fourier transform (Silverman 1982), even on large networks. Moreover they are consistent, robust against errors in network geometry and their statistical efficiency is widely discussed in Rakshit et al. (2019). In our data analysis we make use of the intensity estimate (9) since it preserves mass. The bandwidth for the estimation of $\lambda_{\text{net}}(u)$ will be selected by Scott's rule of thumb (Scott 1992; Rakshit et al. 2019). We also consider the usual kernel smoothing technique to estimate $\lambda_{\text{time}}(t_u)$ with a bandwidth selected by Silverman's rule of thumb (Silverman 1986).

3.2. Homogeneous Second-Order Characteristics

Additionally to the intensity function one may need to consider second-order summary statistics such as the K- and pair correlation functions to get an insight into the type of interaction between the events. In such analysis, it is often assumed that the point process, a realization of which is under description, is stationary. We here begin with second-order pseudostationary point processes, and in Section 3.3 we turn to the inhomogeneous case which is usually more realistic for real data analysis.

Definition 3.1. Assume X is a spatio-temporal point process on $L \times T$ with constant intensity function $\lambda > 0$. Then, the homogeneous linear K-function is given by

$$\begin{aligned} &K_L^{ST}((u, t_u), r, t) \\ &= \frac{1}{\lambda}\mathbb{E}\left[\sum_{(x,t_x)\in X} \frac{\mathbf{1}\{0 < d_L(u, x) \le r\,,\ |t_u - t_x| \le t\}}{M((u, t_u), d_L(u, x), |t_u - t_x|)} \,\middle|\, (u, t_u) \in X\right], \end{aligned} \tag{10}$$

for all $r, t \geq 0$ with $M((u, t_u), r, t) > 0$. Then, X is called second-order pseudostationary and isotropic if $K_L^{ST}((u, t_u), r, t)$ does not depend on the point (u, t_u), and we then write $K_L^{ST}((u, t_u), r, t) = K_L^{ST}(r, t)$.

Theorem 3.1. For a homogeneous Poisson point process on $L \times T$ with constant intensity function λ, $K_L^{ST}((u, t_u), r, t) = K_L^{ST}(r, t) = rt$.

Proof. Note that the expectation in the right-hand side of (10) is an expectation with respect to the Palm distribution of the point process X on $L \times T$. For a Poisson process X, Slivnyak's theorem says that the reduced Palm distribution at any location (u, t_u) is identical to the distribution of X (Møller and Waagepetersen 2003; Baddeley, Rubak, and Turner 2015). Then by using the transformation in (4) and Campbell's theorem,

$$\begin{aligned} &\lambda K_L^{ST}((u, t_u), r, t) \\ &= \mathbb{E}\left[\sum_{(x,t_x)\in X} \frac{\mathbf{1}\{0 < d_L(u, x) \le r\,,\ |t_u - t_x| \le t\}}{M((u, t_u), d_L(u, x), |t_u - t_x|)} \,\middle|\, (u, t_u) \in X\right] \\ &= \mathbb{E}\left[\sum_{(x,t_x)\in X} \frac{\mathbf{1}\{0 < d_L(u, x) \le r\,,\ |t_u - t_x| \le t\}}{M((u, t_u), d_L(u, x), |t_u - t_x|)}\right] \\ &= \lambda \int_{L\times T} \frac{\mathbf{1}\{0 < d_L(u, x) \le r\,,\ |t_u - t_x| \le t\}}{M((u, t_u), d_L(u, x), |t_u - t_x|)}\, \mathrm{d}_2(u, t_u) \\ &= \lambda rt. \end{aligned}$$

□

This can then be used as a benchmark in model selection to check the deviation from the spatio-temporal Poisson processes. Assume that the second-order product density function $\lambda_2(\cdot, \cdot)$ exists, then the pair correlation function of a spatio-temporal point process X with constant intensity function λ on $L \times T$ is of the form

$$g_L^{ST}((u,t_u),(v,t_v)) = \frac{\lambda_2((u,t_u),(v,t_v))}{\lambda^2}, \qquad (u,t_u),(v,t_v) \in L \times T, \tag{11}$$

which is the standardized probability function of observing a pair of points from X occurring jointly in each of two infinitesimally small areas around (u,t_u) and (v,t_v) (Gabriel and Diggle 2009; Møller and Ghorbani 2012). Note that here a small area around (u,t_u) is a small segment (sub-network) extended over a short period of time. Following Gabriel and Diggle (2009) and Ang, Baddeley, and Nair (2012), it can be shown that for Poisson processes and any pairs $(u,t_u),(v,t_v) \in L \times T$, $g_L^{ST}((u,t_u),(v,t_v)) = 1$ since $\lambda_2((u,t_u),(v,t_v)) = \lambda^2$. We next show a direct relationship between K_L^{ST} and g_L^{ST} in the context of spatio-temporal point patterns on linear networks.

Theorem 3.2. A spatio-temporal point process X on $L \times T$ with constant intensity function $\lambda > 0$ is second-order pseudostationary and isotropic if for any pairs $(u,t_u),(v,t_v) \in L \times T$ with $d_L(u,v) < \infty, |t_u - t_v| < \infty$, the pair correlation function $g_L^{ST}((u,t_u),(v,t_v))$ depends only on the distance vector $(d_L(u,v), |t_u - t_v|)$, and in this case

$$K_L^{ST}(r,t) = \int_0^r \int_0^t g_L^{ST}(r',t')\mathrm{d}r'\mathrm{d}t', \tag{12}$$

where r denotes the shortest-path distance, and t is the corresponding temporal distance.

Proof. Assume X is second-order pseudostationary, and

$$g_0(r',t') = \frac{\mathbf{1}\{r' < r\,,\ t' < t\}}{M((u,t_u),r',t')}. \tag{13}$$

Then,

$$K_L^{ST}(r,t) = \frac{1}{\lambda}\mathbb{E}\left[\sum_{(x,t_x)\in X} g_0(d_L(u,x),|t_u - t_x|) \,\middle|\, (u,t_u) \in X\right].$$

By using the properties of Palm distributions, the above equation can be rewritten as follows

$$\begin{aligned} K_L^{ST}(r,t) &= \frac{1}{\lambda}\int_{L\times T} \frac{\lambda_2((u,t_u),(v,t_v))}{\lambda} g_0(d_L(u,v),|t_u - t_v|) \\ &\quad \mathrm{d}_2(v,t_v) \\ &= \int_{L\times T} g_L^{ST}((u,t_u),(v,t_v)) g_0(d_L(u,v),|t_u - t_v|) \\ &\quad \mathrm{d}_2(v,t_v) \\ &= \int_{L\times T} g_L^{ST}(d_L(u,v),|t_u - t_v|) g_0(d_L(u,v),|t_u - t_v|) \\ &\quad \mathrm{d}_2(v,t_v), \end{aligned}$$

where the last equation comes from the second-order pseudostationary property of X. Let

$$\begin{aligned} h(r',t') &= g_L^{ST}(r',t')g_0(r',t')M((u,t_u),r',t') \\ &= g_L^{ST}(r',t')\mathbf{1}\{r' < r\,,\ t' < t\}. \end{aligned}$$

Then,

$$\begin{aligned} K_L^{ST}(r,t) &= \int_{L\times T} \frac{h(d_L(u,v),|t_u - t_v|)}{M((u,t_u),r',t')}\mathrm{d}_2(v,t_v) \\ &= \int_0^\infty \int_0^\infty h(r',t')\mathrm{d}r'\mathrm{d}t' \\ &= \int_0^t \int_0^r g_L^{ST}(r',t')\mathrm{d}r'\mathrm{d}t'. \end{aligned} \tag{14}$$

Moreover, it is clearly seen that $K_L^{ST}(r,t)$ does not depend on the events. □

As both the K- and pair correlation functions have certain values for Poisson processes, they can be used to reveal any deviation from being Poisson, that is, $K_L^{ST}(r,t) > rt$ ($g_L^{ST}(r,t) > 1$) shows clustering, whereas $K_L^{ST}(r,t) < rt$ ($g_L^{ST}(r,t) < 1$) indicates inhibition. In practice, however, we usually need to estimate the K-function (10) and the pair correlation function (11). Following Ang, Baddeley, and Nair (2012), we propose a nonparametric estimator of $K_L^{ST}(r,t)$ as

$$\widehat{K}_L^{ST}(r,t) = \frac{|L|\,|T|}{n(n-1)} \sum_{i=1}^{n}\sum_{i\neq j} \frac{\mathbf{1}\{d_L(x_i,x_j) < r\,,\ |t_{x_i} - t_{x_j}| < t\}}{M((x_i,t_{x_i}),d_L(x_i,x_j),|t_{x_i} - t_{x_j}|)}, \tag{15}$$

where $|L| > 0$ and $|T| > 0$ are the total length of the linear network L and of the time interval T, respectively. The corresponding estimator for $g_L^{ST}(r,t)$ is

$$\widehat{g}_L^{ST}(r,t) = \frac{|L|\,|T|}{n(n-1)} \sum_{i=1}^{n}\sum_{i\neq j} \frac{\kappa(d_L(x_i,x_j) - r)\kappa(|t_{x_i} - t_{x_j}| - t)}{M((x_i,t_{x_i}),d_L(x_i,x_j),|t_{x_i} - t_{x_j}|)}, \tag{16}$$

where κ is a one-dimensional kernel function. The statistical properties of (15) and (16) are similar to those for spatial processes which have been studied in Ang, Baddeley, and Nair (2012, sec. 6).

3.3. Inhomogeneous Second-Order Characteristics

Motivated by practical situations where homogeneity is not a realistic assumption, for example, traffic accidents happen in particular areas and during specific hours, we now extend both the K_L^{ST} and g_L^{ST} to inhomogeneous processes. In the two-dimensional space, Baddeley, Møller, and Waagepetersen (2000) developed a version of Ripley's K-function (Ripley 1977) to the case of nonstationary point processes. Gabriel and Diggle (2009) extended the inhomogeneous K-function to spatio-temporal point processes on $\mathbb{R}^2 \times \mathbb{R}^+$. Similar to Baddeley, Møller, and Waagepetersen (2000) and Gabriel and Diggle (2009), here we also know that if simple exploratory analysis reveals that the pattern in question does not have a constant intensity function, then the homogeneous K-function (10) and the pair correlation function (11) can no longer be proper choices. At this stage, we need an analogue of the spatio-temporal K-function and the pair correlation function defined by Gabriel and Diggle (2009) for the case of linear networks. We now define such second-order summary statistics.

Definition 3.2. Assume X is a spatio-temporal point process on $L \times T$ with intensity function $\lambda(u, t_u) > 0$ and second-order product density function $\lambda_2((u, t_u), (v, t_v))$. Then, the inhomogeneous pair correlation function is of the form

$$g_{LI}^{ST}((u, t_u), (v, t_v)) = \frac{\lambda_2((u, t_u), (v, t_v))}{\lambda(u, t_u)\lambda(v, t_v)}, \quad \times\ (u, t_u), (v, t_v) \in L \times T. \tag{17}$$

If $g_{LI}^{ST}((u, t_u), (v, t_v)) = g_o(d_L(u, v), |t_u - t_v|), g_o : \mathbb{R}^2 \to [0, \infty)$, then X is called second-order intensity reweighted pseudostationary and isotropic.

Definition 3.3. For a second-order intensity reweighted pseudostationary and isotropic spatio-temporal point process X on $L \times T$ with intensity function $\lambda(u, t_u) > 0$, the inhomogeneous K-function is given by

$$K_{LI}^{ST}((u, t_u), r, t) = \mathbb{E}\left[\sum_{(x,t_x)\in X} \frac{\mathbf{1}\{0 < d_L(u, x) < r\ ,\ |t_u - t_x| < t\}}{\lambda(x, t_x)M((u, t_u), d_L(u, x), |t_u - t_x|)} \,\middle|\, (u, t_u) \in X \right]. \tag{18}$$

Under the assumption of Definition 3.2, Equation (18) can be rewritten as

$$K_{LI}^{ST}((u, t_u), r, t) = K_{LI}^{ST}(r, t) = \int_0^r \int_0^t g_{LI}^{ST}(r', t')\mathrm{d}r'\mathrm{d}t'.$$

Note that the proof of the above equality is similar to that of Theorem 3.2. Moreover, for Poisson processes, and for any $r, t > 0$, $K_{LI}^{ST}(r, t) = rt$ and $g_{LI}^{ST}(r, t) = 1$ which are useful in model selection and hypothesis testing. Following Gabriel and Diggle (2009), Ang, Baddeley, and Nair (2012), and Gabriel (2014), K_{LI}^{ST} can be nonparametrically estimated by

$$\widehat{K}_{LI}^{ST}(r, t) = \frac{1}{|L|\ |T|} \sum_{i=1}^{n} \sum_{i\neq j} \times \frac{\mathbf{1}\{d_L(x_i, x_j) < r\ ,\ |t_{x_i} - t_{x_j}| < t\}}{\widehat{\lambda}(x_i, t_{x_i})\widehat{\lambda}(x_j, t_{x_j})M((x_i, t_{x_i}), d_L(x_i, x_j), |t_{x_i} - t_{x_j}|)}, \tag{19}$$

and g_{LI}^{ST} by

$$\widehat{g}_{LI}^{ST}(r, t) = \frac{1}{|L|\ |T|} \sum_{i=1}^{n} \sum_{i\neq j} \times \frac{\kappa(d_L(x_i, x_j) - r)\kappa(|t_{x_i} - t_{x_j}| - t)}{\widehat{\lambda}(x_i, t_{x_i})\widehat{\lambda}(x_j, t_{x_j})M((x_i, t_{x_i}), d_L(x_i, x_j), |t_{x_i} - t_{x_j}|)}, \tag{20}$$

where $\widehat{\lambda}(\cdot)$ is an estimate of the intensity function and κ is a one-dimensional kernel function.

Estimators (19) and (20) are unbiased when the intensity function is known or has been estimated with low bias and low variance (Baddeley, Møller, and Waagepetersen 2000; Gabriel and Diggle 2009; Ang, Baddeley, and Nair 2012; Gabriel 2014), however, in practice we usually need to estimate the intensity function. Therefore, and to reduce the bias and variability of (19) and (20), Moradi, Read, et al. (2019) recommend using the reciprocal of the following normalization factor

$$\mathrm{D}(\mathbf{x}) = \frac{1}{(|L|\ |T|)^2} \sum_{i=1}^{n} \sum_{i\neq j} \frac{1}{\widehat{\lambda}(x_i, t_{x_i})\widehat{\lambda}(x_j, t_{x_j})}. \tag{21}$$

Hence, the updated estimates of the inhomogeneous K- and pair correlation functions are

$$\frac{1}{\mathrm{D}(\mathbf{x})}\widehat{K}_{LI}^{ST}(r, t) \quad \text{and} \quad \frac{1}{\mathrm{D}(\mathbf{x})}\widehat{g}_{LI}^{ST}(r, t), \tag{22}$$

which provide estimators with both lower bias and variance (Moradi, Read, et al. 2019).

3.4. Independent Thinning

An independently thinned version Xth of a spatio-temporal point process X on $L \times T$ is obtained by visiting the points of X and retaining each point $(x, t_x) \in X$ based on a probability function $p(u, t_u)$, $(u, t_u) \in L \times T$, such that inclusion/exclusion of any arbitrary point is independent of other points. The obtained thinned point process Xth has intensity function

$$\lambda\mathrm{th}(u, t_u) = p(u, t_u)\lambda(u, t_u), \qquad (u, t_u) \in L \times T,$$

and second-order product density function

$$\lambda_2\mathrm{th}((u, t_u), (v, t_v)) = p(u, t_u)p(v, t_v)\lambda_2((u, t_u), (v, t_v)), \quad \times\ (u, t_u), (v, t_v) \in L \times T,$$

where $\lambda(u, t_u)$ and $\lambda_2((u, t_u), (v, t_v))$ are the intensity function and the second-order product density function of the original point process X, respectively, see Møller and Waagepetersen (2003, chap. 3) and Chiu et al. (2013, chap. 5). Thus, both K- and pair correlation functions are invariant under independent thinning. Note that, in practice, this does not mean that the K- and pair correlation functions of Xth are identical to those of X due to the effect of sampling variation, intensity estimation and bandwidth selection. However, they are generally expected to be similar although the K- and pair correlation functions of Xth might have a greater variance than those of X.

We point out that independent thinning does not affect Poisson processes from a distributional point of view, that is, Poisson processes stay Poissonian after independent thinning. A related point to mention is that considering a low retention probability to independently thin any arbitrary point process X results in a point process which, from a distributional point of view, is approximately a Poisson process (Baddeley, Rubak, and Turner 2015, chap. 9).

3.5. Goodness of Fit

To check a null hypothesis that data might be considered as a realization of a uniform spatio-temporal Poisson process, we show here a goodness-of-fit statistic based on the K-function and provide graphical and formal p-value results. A similar statistic can be built using the pair correlation function. Following Tamayo-Uria, Mateu, and Diggle (2014) we construct the statistical test as follows. We first simulate $m > 1$ realizations $\mathbf{x}_1, \ldots, \mathbf{x}_m$ from the null hypothesis, that is, the point pattern in question is a realization of a uniform spatio-temporal Poisson process. Then for each realization $\mathbf{x}_i, i = 1, \ldots, m$, we calculate $\widehat{K}_{LI}^{ST}(r,t)$ and denote the corresponding mean and variance of m calculated $\widehat{K}_{LI}^{ST}(r,t)$ by E_K and V_K. We define the test statistic as

$$T = \int\int \frac{\widehat{K}_{LI}^{ST}(r,t) - E_K(r,t)}{\sqrt{V_K(r,t)}} \mathrm{d}r\mathrm{d}t, \tag{23}$$

for m calculated $\widehat{K}_{LI}^{ST}(r,t)$ coming from m simulated patterns, according to which we obtain $T_1, \ldots, T_m$. Finally, the test statistic (23) is evaluated for the empirical $\widehat{K}_{LI}^{ST}(r,t)$ function obtained from the original point pattern, and we denote it by T^*. The p-value of such test is defined as $\{1 + \#(T > T^*)\}/m + 1$. If the obtained p-value is smaller than a significance level α, then there is not enough evidence to accept the null hypothesis at an attained significance level α and such p-value is in favor of spatio-temporal clustering behavior (Tamayo-Uria, Mateu, and Diggle 2014).

For the purpose of graphical results we stand on simulation envelopes of both the K- and pair correlation functions. From m realizations of a uniform spatio-temporal Poisson process together with their calculated $\widehat{K}_{LI}^{ST}(r,t)$ and $\widehat{g}_{LI}^{ST}(r,t)$, we obtain upper and lower pointwise envelopes for such characteristics at an attained significance level α. These envelopes show how much variability we can expect in the estimates $\widehat{K}_{LI}^{ST}(r,t)$ and $\widehat{g}_{LI}^{ST}(r,t)$ if the point pattern in question were completely random (Baddeley, Rubak, and Turner 2015). We next compare the estimates $\widehat{K}_{LI}^{ST}(r,t)$ and $\widehat{g}_{LI}^{ST}(r,t)$ from the empirical point pattern with the obtained envelopes from simulated patterns. Note that if the data was a realization of a uniform spatio-temporal Poisson process, then its $\widehat{K}_{LI}^{ST}(r,t)$ and $\widehat{g}_{LI}^{ST}(r,t)$ should be similar to those from simulated patterns. We recall that $K_{LI}^{ST}(r,t) > rt$ ($g_{LI}^{ST}(r,t) > 1$) indicates clustering behavior in data points while $K_{LI}^{ST}(r,t) < rt$ ($g_{LI}^{ST}(r,t) < 1$) suggests inhibition. This being said, if estimates $\widehat{K}_{LI}^{ST}(r,t)$ and $\widehat{g}_{LI}^{ST}(r,t)$ of data points stay out of the obtained envelopes, we interpret it as an interaction between the events.

4. Data Analysis

In this section, we analyze three datasets on traffic accidents from Houston (United States), Medellín (Colombia), and Eastbourne (United Kingdom). For each dataset, we first estimate the intensity using the estimator (7) and display it for the projection of data onto the network and time, highlighting how corresponding intensities vary over these two supports. Second, we estimate the K- and pair correlation functions to analyze the type of interaction between points, and check the deviation from being Poisson using envelopes based on $m = 99$ simulations from uniform spatio-temporal Poisson processes and a significance level of $\alpha = 0.05$. Following Section 3.5, we report both graphical results and formal p-values.

4.1. Houston Traffic Accidents

Figure 1 shows the pattern of motor vehicle traffic accidents in an area of Houston (United States) near the university of Houston in 2001 which caused nonincapacitating injuries such as bump on the head, abrasions, or minor lacerations. The data were collected by individual police departments in the Houston metropolitan area and compiled by the Texas Department of Public Safety. The compiled data were obtained by the Houston-Galveston Area Council and was geocoded by N. Levine, see Levine (2006, 2009) for details. The space-time pattern contains 144 reference points, with a corresponding linear network with total length 144,253.4 ft and maximum node degree 5. The network is built by 610 nodes and 631 segments. The occurrence time of each traffic accident is rounded to an integer number, for example, all accidents happened between midnight and 1 a.m. are labeled by 0 as occurrence time.

The projection of data onto the network and cumulative number of data points versus occurrence time are displayed in the top row of Figure 1. We can see some jumps in the number of accidents during the afternoon and evening. We estimated the intensity function using the estimator (7), and both the intensities in time and over the network are represented in the bottom row of Figure 1. An increase in the estimated intensity is visible during the afternoon which might be caused by heavier traffic jams in the afternoon, especially in the commuting hours. A decrease in the intensity during the night can also be clearly seen. The left plot in the bottom row of Figure 1 shows the estimated intensity over the network so that the wider the segment the higher the intensity. Higher intensity is visible in the main north-south streets. The bandwidths 1460.31 ft and 1.76 hr were considered to estimate the intensity function over the network and in time, respectively.

Figure 2 represents the estimated inhomogeneous K-function and pair correlation function for Houston data together with the corresponding pointwise envelopes based on 99 simulations from a uniform spatio-temporal Poisson process and a significance level of 0.05. Both corresponding surfaces of K- and pair correlation functions are lying out of envelopes for a range of distances and time lags, indicating clustering behavior. We next measured the test statistic (23) for the K- and pair correlation functions and obtained a p-value $= 0.01$ in both cases, showing an evidence to reject the null hypothesis that the Houston traffic data comes from a uniform spatio-temporal Poisson process and being in favor of clustering.

4.2. Traffic Accidents in Medellín

Figure 3 shows the traffic accidents in an area near the pontifical bolivarian university in Medellín (Colombia) during 2016. The entire data were published in the OpenData portal of Medellín Town Hall.[1] The represented data in Figure 3 has 665 points on a network with total length 29,759.42 m, 643 nodes, and 728 segments. The maximum node degree is 6. From Figure 3, we can see that most of the accidents happened near the intersections and also there are some jumps in the cumulative number of accidents during the afternoon.

Using the intensity estimator (7), we estimated the intensity function of such data, and both intensities in time and over network are represented in bottom row of Figure 3. The estimated intensity of time of occurrence shows a clear decrease during the night whereas the intensity starts growing up from early

[1] https://www.medellin.gov.co/geomedellin/index.hyg

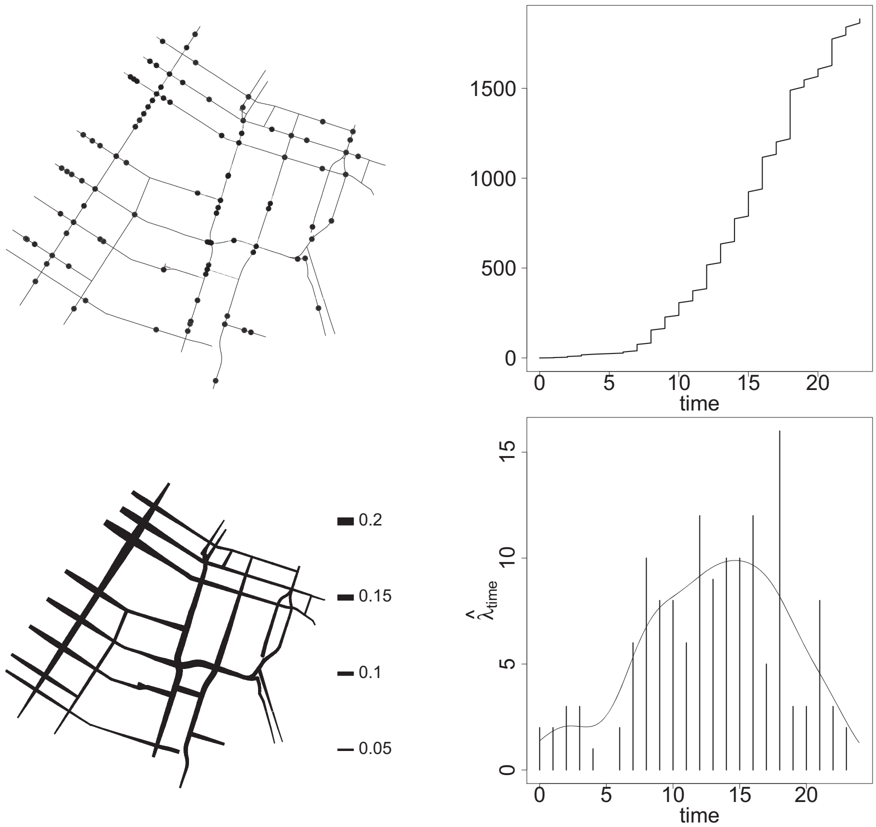

Figure 1. Motor vehicle traffic accidents in Houston near the university of Houston in 2001. Top row: The projection of data onto the network and the cumulative number of data points versus occurrence time. Bottom row: Intensity estimate of the projection onto the network (values on the legend were multiplied by 100) and intensity estimate of daily hours together with the frequency of accidents per hour (bar plot).

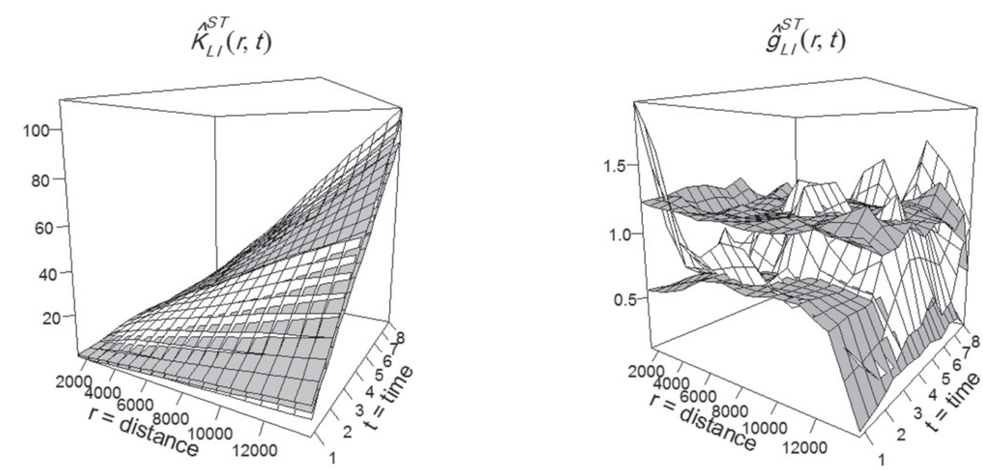

Figure 2. Estimated second-order characteristics for motor vehicle traffic accidents in Houston. Left: Inhomogeneous K-function. Right: Inhomogeneous pair correlation function. Gray surfaces are envelopes based on 99 simulations from uniform spatio-temporal Poisson process and a significance level of 0.05. Numerical values of $\widehat{K}^{ST}_{LI}(r,t)$ were multiplied by 0.001.

morning and it has a peak around 4 p.m. The estimated intensity of the projection of events onto the network is displayed in the left plot of bottom row of Figure 3 confirming higher intensity in the eastern part of the network. Bandwidths 134.79 m and 1.26 hr were used for the estimation of intensities over network and in time, respectively.

Figure 4 shows perspective plots of the estimated inhomogeneous K-function and pair correlation function for Medellín data together with the corresponding envelopes based on 99 simulations from a uniform spatio-temporal Poisson process and a significance level of 0.05. According to both plots in Figure 4, a spatio-temporal interaction between data points is visible, indicating clustering behavior within certain distances and periods of time. We further calculated the test statistic (23) for both K- and pair correlation functions and obtained p-values 0.01 and 0.03, respectively, being in favor of rejecting the null hypothesis and indicating clustering behavior between events.

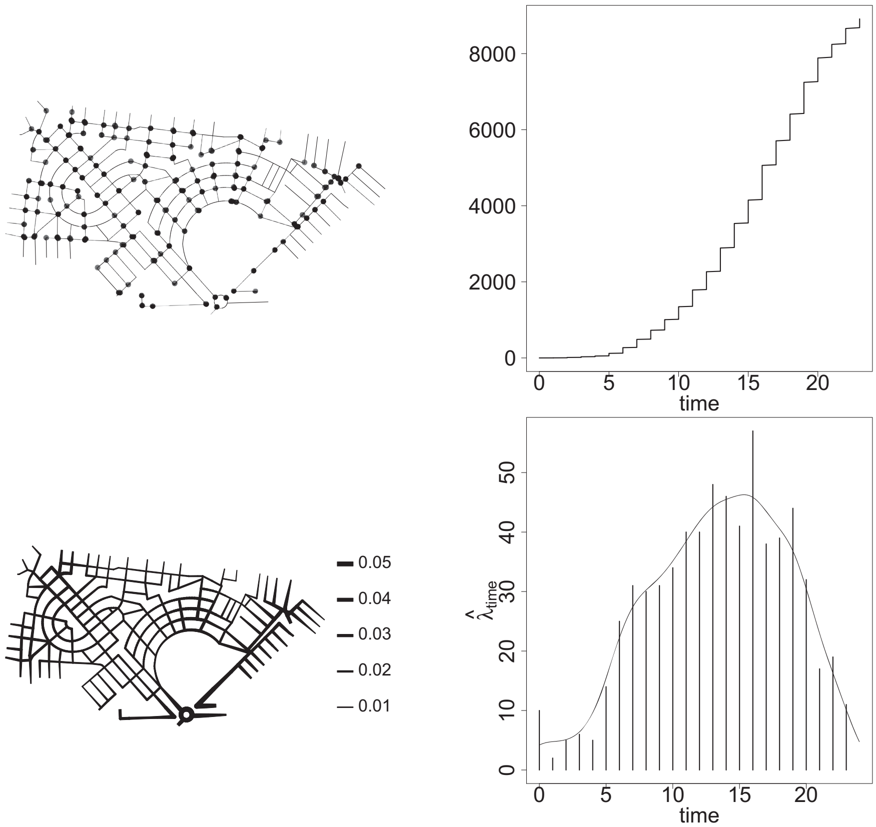

Figure 3. Traffic accidents in Medellín during 2016. Top row: The projection of data onto the network and the cumulative number of data points versus occurrence time. Bottom row: Intensity estimate of the projection onto the network and intensity estimate of daily hours together with the frequency of accidents per hour (bar plot).

4.3. *Traffic Accidents in Eastbourne*

Figure 5 shows traffic accidents in down-town of Eastbourne (United Kingdom). The network was provided by "OS OpenData"[2] and is usable under the terms of the OS OpenData license. The traffic locations were collected by the UK Department for Transport[3] and obtained through kaggle.[4] Traffic accidents happened during the period 2005–2010. The network contains 119 nodes and 153 line segments with a total length of 17,270.61 m. Maximum node degree is 4. The point pattern contains 163 points, representing the locations of traffic accidents that are labeled by the time of occurrence. The point pattern on the left side of the top row of Figure 5 shows concentration of points around some intersections, it further seems there are more accidents in south-north streets. The right plot in the top row of Figure 5 shows the cumulative number of data points versus occurrence time, indicating some jumps in the afternoon which is a hint that more accidents happen during the afternoon. Bottom row of Figure 5 displays the estimated intensities over both the network and time. The estimated intensity for time of occurrence of traffic accidents confirms a higher intensity during the afternoon and also shows an increase from 6 a.m., a peak around 4 p.m. and it then decreases toward the night. The estimated intensity in the left side shows higher intensities around some intersections. Bandwidths 118.09 m and 1.21 hr were used to estimate the intensities over the network and in time, respectively.

Figure 6 shows the estimated inhomogeneous K-function and pair correlation function together with pointwise envelopes based on 99 simulations from a uniform spatio-temporal Poisson process and a significance level of 0.05. According to Figure 6, there seems to be an evidence of clustering behavior within some particular distances and time lags. The test statistic (23) based on the inhomogeneous K- and pair correlation functions results in p-values 0.02 and 0.01, respectively.

5. Discussion

In this article, we have provided some statistical tools for the analysis of spatio-temporal point processes on linear networks. The proposed statistical tools allow us to include the time component when analyzing network point patterns such as traffic accidents. Under the assumption of first-order separability, we have introduced a kernel-based intensity estimator which preserves mass. This allows to figure out the higher and lower intensity parts within network cross time jointly. We have extended second-order summary statistics, the K- and pair correlation functions, to the setting of spatio-temporal point processes on linear networks for the purpose of analyzing the type of interaction between spatio-temporal network events. The proposed K- and pair correlation functions are invariant under independent thinning and have known values for Poisson processes that can be used for model selection and for measuring deviation

[2] *www.ordnancesurvey.co.uk*
[3] *www.data.gov.uk*
[4] *www.kaggle.com*

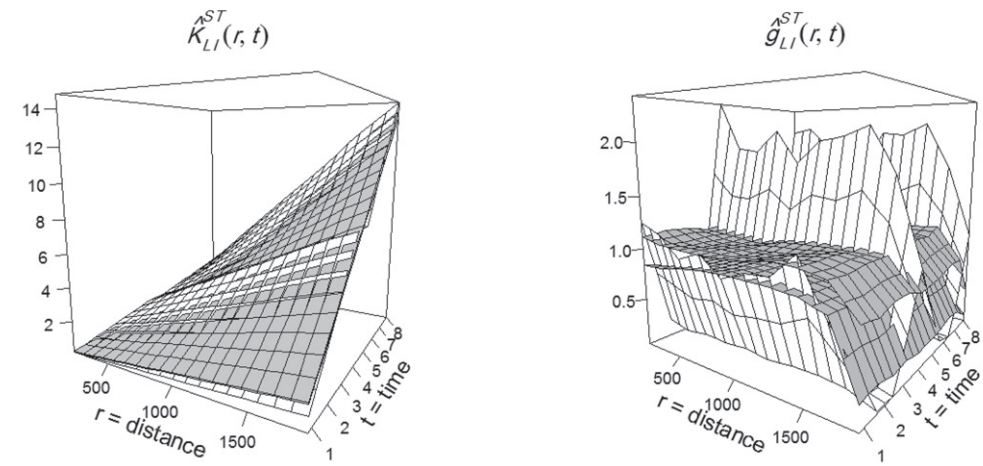

Figure 4. Estimated second-order characteristics for traffic accidents in Medellín. Left: Inhomogeneous K-function. Right: Inhomogeneous pair correlation function. Gray surfaces are envelopes based on 99 simulations from uniform spatio-temporal Poisson process and a significance level of 0.05. Numerical values of $\widehat{K}^{ST}_{LI}(r,t)$ were multiplied by 0.001.

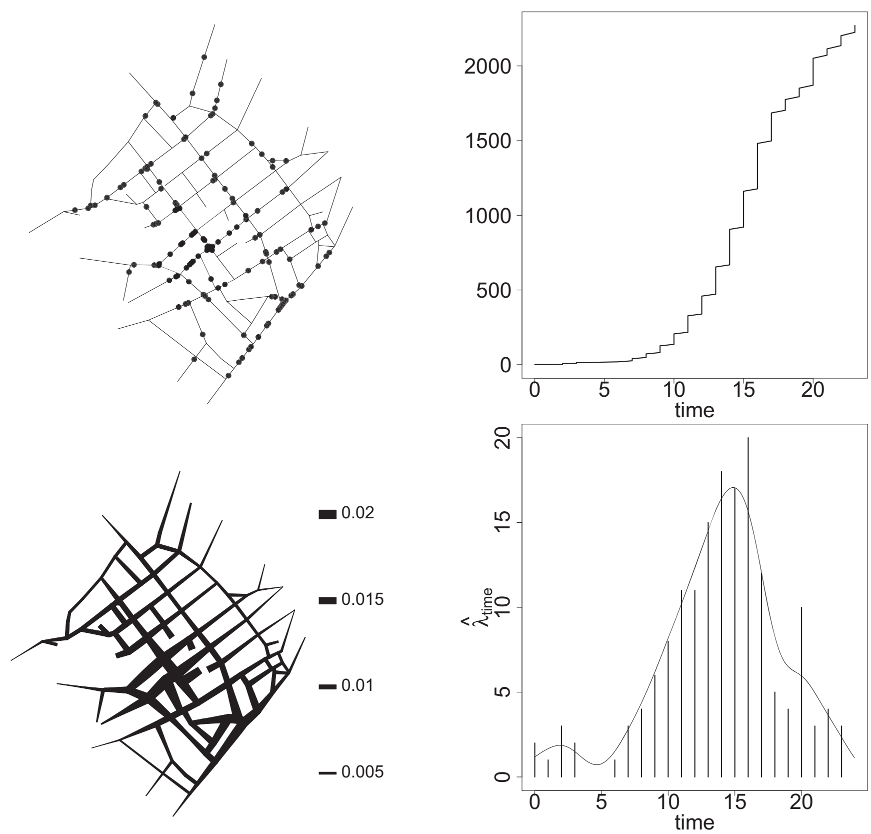

Figure 5. Traffic accidents in down-town of Eastbourne (United Kingdom) in 2005–2010. Top row: The projection of data onto the network and the cumulative number of data points versus occurrence time. Bottom row: Intensity estimate of the projection onto the network and intensity estimate of daily hours together with the frequency of accidents per hour (bar plot).

from Poissonness. We have additionally shown how to perform goodness of fit to check the null hypothesis that the point pattern in question is a realization of a uniform spatio-temporal Poisson process. Nonparametric estimators were discussed for both homogeneous and inhomogeneous cases. The estimators are independent of the geometry of the network which allows comparing the behavior of different point patterns on different networks. Although our computations in Section 4 were fast, the current implementation of the K-function and the pair correlation function might be computationally expensive in large networks.

We have analyzed three different traffic accidents datasets from different countries (United States, Colombia, United Kingdom). We have provided graphical results and formal p-values to

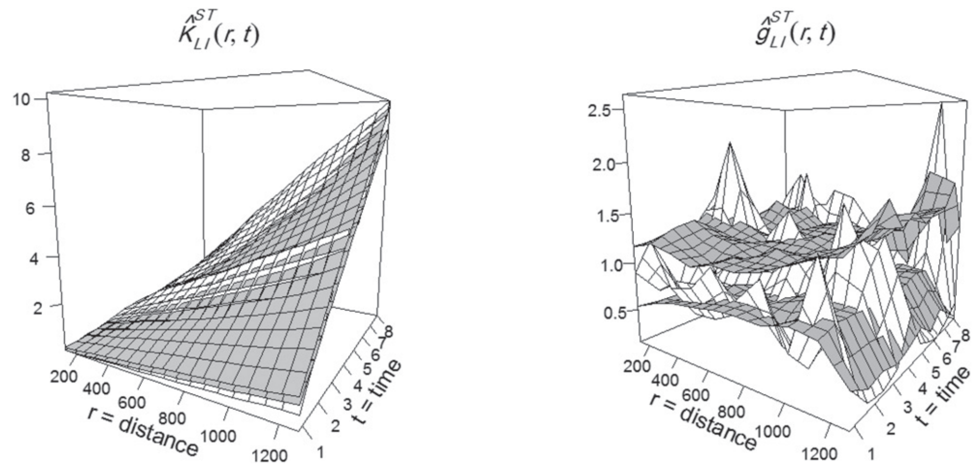

Figure 6. Estimated second-order characteristics for traffic accidents in down-town of Eastbourne. Left: Inhomogeneous K-function. Right: Inhomogeneous pair correlation function. Gray surfaces are pointwise envelopes based on 99 simulations from uniform spatio-temporal Poisson process and a significance level of 0.05. Numerical values of $\widehat{K}^{ST}_{LI}(r,t)$ were multiplied by 0.001.

investigate temporal, spatial, and spatio-temporal aspects of the distribution of traffic accidents. For the three datasets the estimated intensities for the time of occurrence of events showed an increase from the morning toward the afternoon and a decrease over the night that might be caused due to rush hours. We have found that some particular streets show higher intensities. We discovered that there is an increase in the estimated intensities at some intersections that might be due to the fact that at any intersection two or more streets join each other and also vehicles are moving in various directions. We made use of K- and pair correlation functions to study the type of interaction between traffic accidents and reported graphical results and p-values based on simulations of uniform spatio-temporal Poisson processes. The three considered datasets mostly showed clustering behavior within certain distances and periods of time. Moreover and according to the obtained p-values, for none of the three datasets we have found an evidence to accept the null hypothesis being uniform spatio-temporal Poisson process, that is, the p-values were in favor of clustering behavior between events.

We have here focused on exploratory data analysis and nonparametric methodologies of spatio-temporal point processes on linear networks. However, considering some relevant covariates might lead to parametric modeling strategies that could highlight deeper details about the considered datasets. In this context, we could have a parametric intensity estimation controlling, for example, the number of accidents at crossing roads. These finer details would be a welcome contribution in the context of this article. In addition, we have only focused on summary statistics and no model fitting. This is a promising area of research, but we anticipate that these models should be second-order (intensity reweighted) pseudostationary and isotropic to be able to use our summary characteristics. A final interesting topic of development is residual analysis for point processes on networks. This is a new topic not covered so far, and could be in itself a research line for networks.

Supplementary Materials

stlnpp: The manual of R package stlnpp, version 0.3.3. (stlnpp, pdf)

R package stlnpp: R package stlnpp, version 0.3.3, containing codes to perform spatio-temporal statistical analysis for point patterns on linear networks as described in the article. The package also contains the datasets of Medellín (Colombia) and Eastbourne (United Kingdom) analyzed in Sections 4.2 and 4.3 that are reusable only under correct acknowledgment to the source of data. The development version of stlnpp R package is also accessible at *https://github.com/Moradii/stlnpp*. (stlnpp_0.3.3, zip)

R package stlnpp: The Binary version of R package stlnpp. (stlnpp_0.3.3.tar, gz)

Medellin.R: R codes to reproduce the results of Section 4.2. (Medellin, R)

Eastbourne.R: R codes to reproduce the results of Section 4.3. (Eastbourne, R)

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All calculations were performed in the R language. The R package spatstat, version 1.60-1.013 (Baddeley and Turner 2005; Baddeley, Rubak, and Turner 2015) and the R package stats, version 3.5 (R Core Team 2018) were used for some calculations. Moreover, we made use of the R package plot3D, version 1.1.1 (Soetaert 2017) to display the K- and pair correlation functions through perspective plots. Our code is provided as an online supplement to this article, and is released in the R package stlnpp (Moradi 2019).

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References

Ang, Q. W., Baddeley, A., and Nair, G. (2012), "Geometrically Corrected Second Order Analysis of Events on a Linear Network, With Applications to Ecology and Criminology," *Scandinavian Journal of Statistics*, 39, 591–617. [432,433,434,436,437]

Baddeley, A., Bárány, I., and Schneider, R. (2006), *Stochastic Geometry: Lectures Given at the CIME Summer School Held in Martina Franca, Italy*, September 13–18, 2004, Springer. [433]

Baddeley, A., Jammalamadaka, A., and Nair, G. (2014), "Multitype Point Process Analysis of Spines on the Dendrite Network of a Neuron," *Journal of the Royal Statistical Society*, Series C, 63, 673–694. [432]

Baddeley, A., Møller, J., and Waagepetersen, R. (2000), "Non- and Semi-Parametric Estimation of Interaction in Inhomogeneous Point Patterns," *Statistica Neerlandica*, 54, 329–350. [436,437]

Baddeley, A., Rubak, E., and Turner, R. (2015), *Spatial Point Patterns: Methodology and Applications With R*, Boca Raton, FL: CRC Press. [433,435,437,438,442]

Baddeley, A., and Turner, R. (2005), "Spatstat: An R Package for Analyzing Spatial Point Patterns," *Journal of Statistical Software*, 12, 1–42. [442]

Chiu, S. N., Stoyan, D., Kendall, W. S., and Mecke, J. (2013), *Stochastic Geometry and Its Applications*, Chichester: Wiley. [437]

Daley, D. J., and Vere-Jones, D. (2008), *An Introduction to the Theory of Point Processes: Volume II: General Theory and Structure* (2nd ed.), New York: Springer-Verlag. [433]

Díaz-Avalos, C., Juan, P., and Mateu, J. (2013), "Similarity Measures of Conditional Intensity Functions to Test Separability in Multidimensional Point Processes," *Stochastic Environmental Research and Risk Assessment*, 27, 1193–1205. [434]

Diggle, P. (1985), "A Kernel Method for Smoothing Point Process Data," *Applied Statistics*, 34, 138–147. [432]

——— (2003), *Statistical Analysis of Spatial and Spatio-Temporal Point Patterns*, Boca Raton, FL: CRC Press. [432,433,434]

Federer, H. (1996), *Geometric Measure Theory*, New York: Springer-Verlag. [434]

Fuentes-Santos, I., González-Manteiga, W., and Mateu, J. (2018), "A First-Order, Ratio-Based Nonparametric Separability Test for Spatiotemporal Point Processes," *Environmetrics*, 29, e2482. [434]

Gabriel, E. (2014), "Estimating Second-Order Characteristics of Inhomogeneous Spatio-Temporal Point Processes: Influence of Edge Correction Methods and Intensity Estimates," *Methodology and Computing in Applied Probability*, 16, 411–431. [437]

Gabriel, E., and Diggle, P. (2009), "Second-Order Analysis of Inhomogeneous Spatio-Temporal Point Process Data," *Statistica Neerlandica*, 63, 43–51. [434,436,437]

Ghorbani, M. (2013), "Testing the Weak Stationarity of a Spatio-Temporal Point Process," *Stochastic Environmental Research and Risk Assessment*, 27, 517–524. [434]

Illian, J., Penttinen, A., Stoyan, H., and Stoyan, D. (2008), *Statistical Analysis and Modelling of Spatial Point Patterns*, New York: Wiley. [434]

Jones, M. C. (1993), "Simple Boundary Correction for Kernel Density Estimation," *Statistics and Computing*, 3, 135–146. [432]

Levine, N. (2006), "Houston, Texas, Metropolitan Traffic Safety Planning Program," *Transportation Research Record*, 1969, 92–100. [438]

——— (2009), "A Motor Vehicle Safety Planning Support System: The Houston Experience," in *Planning Support Systems Best Practice and New Methods*, eds. S. Geertman and J. Stillwell, Dordrecht: Springer Netherlands, pp. 93–111. [438]

Lu, Y., and Chen, X. (2007), "On the False Alarm of Planar K-Function When Analyzing Urban Crime Distributed Along Streets," *Social Science Research*, 36, 611–632. [432,433]

McSwiggan, G., Baddeley, A., and Nair, G. (2017), "Kernel Density Estimation on a Linear Network," *Scandinavian Journal of Statistics*, 44, 324–345. [432,433,435]

Møller, J., and Ghorbani, M. (2012), "Aspects of Second-Order Analysis of Structured Inhomogeneous Spatio-Temporal Point Processes," *Statistica Neerlandica*, 66, 472–491. [434,436]

Møller, J., and Waagepetersen, R. (2003), *Statistical Inference and Simulation for Spatial Point Processes*, Boca Raton, FL: CRC Press. [435,437]

Moradi, M. M. (2019), "stlnpp: Spatio-Temporal Analysis of Point Patterns on Linear Networks," R Package Version 0.3.3, available at *https://github.com/Moradii/stlnpp*. [442]

Moradi, M. M., Cronie, O., Rubak, E., Lachieze-Rey, R., Mateu, J., and Baddeley, A. (2019), "Resample-Smoothing of Voronoi Intensity Estimators," *Statistics and Computing*, 29, 995–1010, DOI: 10.1007/s11222-018-09850-0. [432,433,435]

Moradi, M. M., Read, N., Mateu, J., and Baddeley, A. (2019), "Normalisation of the Inhomogeneous K-Function and Pair Correlation Function" (in preparation). [437]

Moradi, M. M., Rodríguez-Cortés, F. J., and Mateu, J. (2017), "On Kernel-Based Intensity Estimation of Spatial Point Patterns on Linear Networks," *Journal of Computational and Graphical Statistics*, 27, 302–311. [432,433,435]

Okabe, A., Satoh, T., and Sugihara, K. (2009), "A Kernel Density Estimation Method for Networks, Its Computational Method and a GIS-Based Tool," *International Journal of Geographical Information Science*, 23, 7–32. [432,435]

Okabe, A., and Sugihara, K. (2012), *Spatial Analysis Along Networks: Statistical and Computational Methods*, Chichester: Wiley. [432,433]

Okabe, A., and Yamada, I. (2001), "The K-Function Method on a Network and Its Computational Implementation," *Geographical Analysis*, 33, 271–290. [432]

Rakshit, S., Davies, T. M., Moradi, M. M., McSwiggan, G., Nair, G., Mateu, J., and Baddeley, A. (2019), "Fast Kernel Smoothing of Point Patterns on a Large Network Using Two-Dimensional Convolution," *International Statistical Review*, DOI: 10.1111/insr.12327. [432,433,435]

Rakshit, S., Nair, G., and Baddeley, A. (2017), "Second-Order Analysis of Point Patterns on a Network Using Any Distance Metric," *Spatial Statistics*, 22, 129–154. [432]

R Core Team (2018), *R: A Language and Environment for Statistical Computing*, Vienna, Austria: R Foundation for Statistical Computing, available at *https://www.R-project.org/*. [442]

Ripley, B. D. (1977), "Modelling Spatial Patterns," *Journal of the Royal Statistical Society*, Series B, 39, 172–212. [436]

Scott, D. W. (1992), *Multivariate Density Estimation: Theory, Practice and Visualization*, New York: Wiley. [435]

Silverman, B. W. (1982), "Kernel Density Estimation Using the Fast Fourier Transform," *Journal of the Royal Statistical Society*, Series C, 31, 93–99. [435]

——— (1986), *Density Estimation for Statistics and Data Analysis* (Vol. 26), Boca Raton, FL: Chapman & Hall/CRC Press. [435]

Soetaert, K. (2017), "plot3D: Plotting Multi-Dimensional Data," R Package Version 1.1.1, available at *https://CRAN.R-project.org/package=plot3D*. [442]

Tamayo-Uria, I., Mateu, J., and Diggle, P. (2014), "Modelling of the Spatio-Temporal Distribution of Rat Sightings in an Urban Environment," *Spatial Statistics*, 9, 192–206. [437,438]

Ver Hoef, J. M., and Peterson, E. E. (2010), "A Moving Average Approach for Spatial Statistical Models of Stream Networks," *Journal of the American Statistical Association*, 105, 6–18. [432]

Xie, Z., and Yan, J. (2008), "Kernel Density Estimation of Traffic Accidents in a Network Space," *Computers, Environment and Urban Systems*, 32, 396–406. [432]

Yamada, I., and Thill, J. C. (2004), "Comparison of Planar and Network K-Function-Functions in Traffic Accident Analysis," *Journal of Transport Geography*, 12, 149–158. [432]