

Scalar curvature rigidity and higher index theory

Workshop: Geometric moduli spaces - rigidity, genericity, stability

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Classical existence question

Does a given manifold admit a Riemannian metric of positive scalar curvature (pscm)?

Let M be a closed connected spin manifold of dimension m with fundamental group $\pi.$

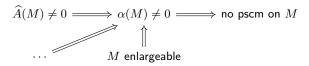
 \widehat{A} -genus: $\widehat{A}(M) := \operatorname{ind} \mathcal{D} \in \mathbb{Z}$.

ightharpoonup spin Dirac operator \mathcal{D} .

Rosenberg index: $\alpha(M) := \operatorname{ind} \mathcal{D}_{\mathcal{L}} \in \mathrm{KO}_m(\mathrm{C}^*\pi)$ (higher index).

- $SM \otimes \mathcal{L}(M) \leadsto \text{twisted spin Dirac operator } \mathcal{D}_{\mathcal{L}}$.
- ▶ SM: $\mathbb{C}l_m$ -linear spinor bundle
- $\mathcal{L}(M) := \widetilde{M} \times_{\pi} \mathrm{C}^* \pi$ Mishchenko-Fomenko bundle.
- ▶ $C^*\pi$: maximal group C^* -algebra of π .

Fact. The Rosenberg index is the most general known index-theoretical obstruction to the existence of a pscm on ${\cal M}.$



Examples. ightharpoonup The \widehat{A} -genus of the n-torus is zero but the Rosenberg index does not vanish.

▶ There exist exotic spheres Σ^k of non-zero Rosenberg index.

General principle classical index replace by higher index

What geometric usefull information can we deduce from a non-vanishing higher index?

Classically: A non-vanishing \widehat{A} -genus gives rise to a non-vanishing \mathcal{D} -harmonic spinor u. In extreme geometric situations u is parallel, hence

$$|u|_p^2 = \langle u(p), u(p) \rangle_p = \text{const.}$$

New method

Technical Lemma (T. [3]). Let \mathcal{A} be a unital Real \mathbb{C}^* -algebra and $\mathcal{S} \to M$ a graded Real \mathcal{A} -linear Dirac bundle with induced \mathcal{A} -linear Dirac operator D.

$$||u_{\epsilon}||_{\mathbf{L}^2} = 1$$
 and $||\mathcal{D}^j u_{\epsilon}||_{\mathbf{L}^2} < \epsilon \quad \forall j > 1, \, \forall \epsilon > 0.$

• If, moreover, $\|\nabla u_{\epsilon}\|_{\mathrm{L}^2} < \epsilon$ for all $\epsilon > 0$, then there exist constants C, r > 0 such that

$$\|\nabla u_{\epsilon}\|_{\infty} < C\epsilon^r \quad \forall \epsilon \in (0,1),$$

hence the family $\{u_{\epsilon}\}_{{\epsilon}>0}$ is almost constant.

Rigidity question

How rich is the space of Riemannian metrics satisfying a certain lower scalar curvature bound?

Theorem (Llarull [2]). Let $f:(M,g_M) \to (S^m,g_0)$ be a smooth map of non-zero degree and $m \geq 3$. Then

$$\left. \begin{array}{l} \operatorname{scal}_M \geq m(m-1) \\ g_M \geq f^* g_0 \text{ on } \bigwedge^2 TM \end{array} \right\} \Longrightarrow f \text{ is an isometry.}$$

We generalize an extremality and rigidity statement by Goette and Semmelmann [1] (a generalization of Llarull's theorem).

Main Theorem (T. [3]). Let $f \colon M \to N$ be a spin map between two closed connected Riemannian manifolds of dimension n+k and n, respectively. Suppose

- \blacktriangleright the curvature operator of N is non-negative,
- $ightharpoonup \operatorname{scal}_N > 2\operatorname{Ric}_N > 0$ and
- $\chi(N) \cdot \deg_{hi}(f) \neq 0 \in KO_k(\mathbf{C}^*\pi)$.

Then the following implication holds:

Definition. ightharpoonup The map f is called **spin** if

$$w_i(TM) = f^*(w_i(TN)) \quad (i = 1, 2).$$

▶ The **higher mapping degree** of f is defined for a regular value p of the map f via

$$\deg_{\mathsf{hi}}(f) := \operatorname{ind} \left(\mathcal{D}_{\mathcal{S}f^{-1}(p) \otimes \mathcal{L}(M) \upharpoonright_{f^{-1}(p)}} \right) \in \mathrm{KO}_{k}(\mathrm{C}^{*}\pi).$$

Proof. The spin map f gives rise to a graded Real $\mathbb{C}l_{n+k,n} \otimes \mathbb{C}^*\pi$ -linear Dirac bundle $\mathcal{S}M \otimes f^*\mathcal{S}N \otimes \mathcal{L}(M)$ with induced Dirac operator $\mathcal{D}_{\mathcal{L}}$ satisfying

- $ightharpoonup \operatorname{ind}(\mathcal{D}_{\mathcal{L}}) = \operatorname{deg}_{\operatorname{bi}}(f) \cdot \chi(N) \neq 0$ and
- $\triangleright \mathcal{D}_{\mathcal{L}}^2 \ge \nabla^* \nabla + \frac{1}{4} (\operatorname{scal}_M \operatorname{scal}_N \circ f).$

The rigidity statement follows from the Technical Lemma.

Examples. The Main Theorem applies to the following maps:

References

- S. Goette and U. Semmelmann. "Scalar curvature estimates for compact symmetric spaces". In: Diff. Geom. Appl. 16.1 (2002).
- [2] M. Llarull. "Sharp estimates and the Dirac operator". English. In: Mathematische Annalen 310.1 (1998).
- [3] T. Tony. "Scalar curvature rigidity and the higher mapping degree". In: Journal of Functional Analysis 288.3 (2025).









