Universität Münster

Scalar curvature comparison geometry and the higher index

Contributor: Thomas Tony

Classical existence question

For which manifolds exist a Riemannian metric such that its induced scalar curvature is strictly positive (PSCM)?

Let M be a closed connected spin manifold of dimension m.

$\widehat{A}$-genus: $\widehat{A}(M) := \begin{cases} \dim\ker \mathcal{D}^+ - \dim\ker\mathcal{D}^- & m \text{ even} \\ 0 & m \text{ odd.}\end{cases}$

- $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^- : \Gamma(\mathcal{S}M) \to \Gamma(\mathcal{S}M)$ spin Dirac operator.
- $\mathcal{S}M = \mathcal{S}M^+ \oplus \mathcal{S}M^-$ spinor bundle.

Rosenberg index: $\alpha(M) := \operatorname{ind}\mathcal{D}_{\mathcal{L}} \in \mathrm{KO}_m(\mathrm{C}^*\pi)$ (higher index)

- $\mathcal{D}_{\mathcal{L}}$: spin Dirac operator twisted by the Mishchenko-Fomenko bundle $\mathcal{L}(M) := \widetilde{M} \times_\pi \mathrm{C}^*\pi$.
- $\mathrm{C}^*\pi$: maximal group C^*-algebra of the fundamental group of M.

Fact. The Rosenberg index is the most general known index-theoretical obstruction to the existence of a PSCM.

$$\begin{tikzcd} \widehat{A}(M) \neq 0 \arrow[r, Rightarrow] & \alpha(M) \neq 0 \arrow[r, Rightarrow] & M \text{ does not admit PSCM} \\ \cdots \arrow[ur, Rightarrow] & M \text{ enlargeable} \arrow[u, Rightarrow] & \end{tikzcd}$$

Examples.

- *The $\widehat{A}$-genus of the n-torus vanishes but $\alpha(T^k) \neq 0$.*
- *There exists exotic spheres Σ^k with non-vanishing Rosenberg index.*

General principle

classical index $\xrightarrow{\text{replace by}}$ higher index

Classically. A non-vanishing $\widehat{A}$-genus gives rise to a non-vanishing harmonic spinor u. In extreme geometric situations u is parallel, hence

$$|u|_p^2 = \langle u(p), u(p)\rangle_p = \text{const.}$$

New method

Technical Lemma (T. [4]). *Let $\mathcal{A}$ be a unital Real C^*-algebra and $\mathcal{S} \to M$ a graded Real $\mathcal{A}$-linear Dirac bundle with induced $\mathcal{A}$-linear Dirac operator $\not{D}$.*

- *If the higher index of $\not{D}$ does not vanish, there exists a family u_ϵ of almost $\not{D}$-harmonic sections.*
- *If, moreover, u_ϵ is L^2-almost parallel, the family is almost constant, i.e. there exists constants $C, r > 0$ and an element $a \in A^+$ such that*

$$\left\| a - \left\langle u_\epsilon(p), u_\epsilon(p)\right\rangle_p \right\|_{\mathcal{A}} < C\epsilon^r \quad \forall p \in M \,\forall \epsilon \in (0,1).$$

Proof (sketch). Functional calculus of $\not{D}$ and Moser iteration. □

Rigidity question

How rich is the space of Riemannian metrics satisfying a certain lower scalar curvature bound — e.g. on a product manifold $N \times F$?

We generalize a result by Goette and Semmelmann [2]. Let

- M and N be two closed connected Riemannian manifolds of dimension $n+k$ and n, respectively, and
- $f\colon M \to N$ an *area-non-increasing spin map*.

Definition. The higher degree of f is defined for a regular value p via

$$\deg_{\mathsf{hi}}(f) := \operatorname{ind}\left(\mathcal{D}_{\mathcal{S}f^{-1}(p) \otimes \mathcal{L}(M)\restriction_{f^{-1}(p)}}\right) \in \mathrm{KO}_k(\mathrm{C}^*\pi).$$

Main Theorem (T. [4]). *Suppose*

- *N has non-negative curvature operator and*
- $\deg_{hi}(f) \cdot \chi(N) \neq 0 \in \mathrm{KO}_k(\mathrm{C}^*\pi)$.

Then: *(1)* $\mathrm{scal}_M \geq \mathrm{scal}_N \circ f \quad \Rightarrow \quad \mathrm{scal}_M = \mathrm{scal}_N \circ f$

(2) $\left.\begin{cases} \mathrm{scal}_M \geq \mathrm{scal}_N \circ f \\ \mathrm{scal}_N > 2\,\mathrm{Ric}_N > 0 \end{cases}\right\} \Rightarrow$ *f is a Riemannian submersion*

Proof (sketch). The spin map f gives rise to a $\mathbb{C}\mathrm{l}_{n+k,n} \otimes \mathrm{C}^*\pi$-linear Dirac bundle $\mathcal{S}M \otimes f^*\mathcal{S}N \otimes \mathcal{L}(M) \to M$ with induced Dirac operator $\mathcal{D}_{\mathcal{L}}$ satisfying

- $\operatorname{ind}(\mathcal{D}_{\mathcal{L}}) = \deg_{\mathsf{hi}}(f) \cdot \chi(N) \neq 0$ and
- $\mathcal{D}_{\mathcal{L}}^2 \geq \nabla^*\nabla + \frac{1}{4}(\mathrm{scal}_M - \mathrm{scal}_N \circ f)$.

The extremality and rigidity statement follow from the existence of a family of almost constant sections (see Technical Lemma). □

Example. *Projection maps $N \times F \to N$ with $\alpha(F) \cdot \chi(N) \neq 0$ e.g.*

- $\mathrm{pr}_1 : S^{2n} \times T^k \to S^{2n}$ *and*
- $\mathrm{pr}_1 : \mathbb{RP}^{2n} \times \Sigma^k \to \mathbb{RP}^{2n}$.

Followup questions

- What if the Euler characteristic of N vanishes?
- Rigidity for $\mathrm{pr}_1 : S^{2n} \times \Sigma^{8k+j} \to S^{2n}$?
- Rigidity for projection maps $N \times F \to N$ as in the previous example but F does not admit a PSCM?

Further applications

- Integration of the higher mapping degree into the rigidity statement for manifolds with boundary by Lott [3].
- Generalization of the rigidity statement by Cecchini and Zeidler [1] for bands.

References

[1] S. Cecchini and R. Zeidler. "Scalar and mean curvature comparison via the Dirac operator". English. In: *Geometry & Topology* 28.3 (2024).

[2] S. Goette and U. Semmelmann. "Scalar curvature estimates for compact symmetric spaces". In: *Diff. Geom. Appl.* 16.1 (2002).

[3] J. Lott. "Index theory for scalar curvature on manifolds with boundary". English. In: *Proceedings of the American Mathematical Society* 149.10 (2021).

[4] T. Tony. *Scalar curvature rigidity and the higher mapping degree*. 2024. arXiv: 2402.05834 [math.DG].

MMGS Mathematics Münster Graduate School

living.knowledge