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The End-Periodic Index Theorem for Dirac Operators and its Connection to the Classical Atiyah-Patodi-Singer Index Theorem

eingereicht von
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Zusammenfassung

Das Atiyah-Singer Indextheorem erlaubt es den Index eines Dirac-Operators, welcher von einem graduierten Clifford Bündel über einer geschlossenen, orientierten, Riemannschen Mannigfaltigkeit gerader Dimension induziert wird, ausschließlich mithilfe topologischer Größen der Mannigfaltigkeit und des Cliffordbündels zu berechnen. Diese Aussage stellt eine beeindruckende, tiefgreifende Verbindung zwischen Analysis und Topologie her. Auf der einen Seite lässt sich aus der topologischen Struktur des Clifford Bündels der Index des Dirac-Operators berechnen, welcher in direktem Zusammenhang zu den Lösungsräumen partieller Differentialgleichungen steht. Auf der anderen Seite impliziert die Indexformel, dass das Integral über die auftauchenden charakteristischen Klassen eine ganze Zahl ist.

Es gibt zahlreiche Erweiterungen des ursprünglichen Indextheorems, wie zum Beispiel die Erweiterung von Atiyah, Patodi und Singer auf Mannigfaltigkeiten mit Rand oder die Erweiterung von John Roe auf offene Mannigfaltigkeiten mit beschränkter Geometrie. Letztere liefert eine deutlich abstraktere Formel als das ursprüngliche Indextheorem aufgrund der wenigen Voraussetzungen und der damit nicht mehr vorhandenen Fredholmness des Dirac-Operators. In dieser Abschlussarbeit wird die Erweiterung auf end-periodische Dirac-Operatoren von Mrowka, Ruberman und Saveliev formuliert und mithilfe der Wärmeleitungsmethode bewiesen. Es ist erstaunlich, dass der Beweis analog zum Beweis des Atiyah-Singer Indextheorems funktioniert, mit dem Unterschied, dass mehrere Zusatzterme in der Indexformel auftreten und aufgrund der nicht vorhandenen Kompaktheit der Mannigfaltigkeit einige Details angepasst werden müssen. Dabei spielt die spezielle end-periodische Form der Mannigfaltigkeit und aller Zusatzstrukturen eine besondere Rolle, denn diese erlaubt es mithilfe der Fourier-Laplace Transformation das nichtkompakte Ende auf eine kompakte Mannigfaltigkeit zurückzuführen.

Es stellt sich heraus, dass das Atiyah-Patodi-Singer Indextheorem ein Spezialfall des end-periodischen Indextheorems ist. Dazu wird das Atiyah-Patodi-Singer Indextheorem, welches klassisch für Mannigfaltigkeiten mit Rand gilt, zuerst umformuliert in ein Indextheorem für Dirac-Operatoren induziert von einem Clifford Bündel mit zylindrischem Ende und anschließend wird die Übereinstimmung der Zusatzterme in den Indexformeln bewiesen.

To my parents

Contents

Introduction

Let E and F be two complex vector bundles over a closed even-dimensional oriented Riemannian manifold X and let $A : C^\infty(X, E) \to C^\infty(X, F)$ be an elliptic differential operator, i.e. a differential operator for that the principle symbol with respect to any nonzero one-form on X is invertible. Operators of that type have finite-dimensional kernels and cokernels and they can be extended to a linear map between the Sobolev spaces $H^1(X, S)$ and $L^2(X, S)$. This implies together with the elliptic regularity theorem that the extended operator A is Fredholm with Fredholm index given by

$$\mathrm{ind}_{\mathrm{Fred}}(A) \coloneqq \dim \ker(A) - \dim \operatorname{coker}(A).$$

The index problem is to find a formula of the index just in terms of the principal symbol of A and topological data of the manifold and the involved vector bundles. As a trivial example, consider a linear map $A : V \to W$ between two finite-dimensional vector spaces. Although the dimensions of the kernel and the cokernel of A depend explicitly on the linear map A, their difference depends only on the dimensions of V and W, namely, on $\dim V - \dim W$.

Atiyah and Singer solved the index problem in 1963 by proving the famous Atiyah-Singer index theorem [AS63], for which, among others, they were awarded the Abel Prize in 2004. It states for the Fredholm index of an elliptic differential operator A between two vector bundles over a closed oriented Riemannian manifold X of even dimension:

$$\mathrm{ind}_{\mathrm{Fred}}(A) = \int_X \mathrm{ch}(A)\mathrm{Td}(X).$$

Here $\mathrm{Td}(X)$ is the Todd class of the complexified tangent bundle of X and $\mathrm{ch}(A)$ is a certain element of the cohomology class $H^*(X, \mathbb{Q})$ involving the Chern character and the principal symbol of A. The first proof by Atiyah and Singer from 1963 proceeds via K-theory and cobordism theory [AS63] and was reworked in 1968 in a series of papers starting with [AS68], where the cobordism part was bypassed. It is remarkable that this theorem connects analysis and topology in such a great generality. On the one hand it states that the index - directly connected to the dimensions of solution spaces of partial differential equations - is a topological invariant, on the other hand it shows that the integral of the involved characteristic classes is an integer. This was well known for some examples but it was not understood why that should be the case. The latter was one of the questions Atiyah and Singer were thinking about before they discovered the index theorem. As Singer recalled it:

> *"The Atiyah-Singer collaboration began in January 1962 in Oxford, with the following exchange.*
>
> *Atiyah: Why is the $\widehat{A}$ genus an integer for spin manifolds?*
>
> *Singer: You know the answer better than I - why do you ask?*
>
> *Atiyah: There must be a deeper reason.*
>
> *In March, I suggested a deeper reason: The $\widehat{A}$ genus is an integer because it is the index of the Dirac operator."* (Isadore Singer [see See99, p.1-2])

One class of elliptic operators can be obtained by taking the positive chiral part $\mathcal{D}^+$ of an induced Dirac operator of a graded Clifford bundle $S = S^+ \oplus S^- \to X$, which is a complex vector bundle equipped with a connection, a metric and an action of $\mathrm{Cl}(T_pX)\otimes\mathbb{C}$ on the fiber S_p with some compatibility conditions. The induced Dirac operator is the composition of the maps

$$\mathcal{D} : C^\infty(X, S) \xrightarrow{\text{connection}} C^\infty(X, T^*X \otimes S) \xrightarrow{\text{metric}} C^\infty(X, TX \otimes S) \xrightarrow{\text{Clifford action}} C^\infty(X, S)$$

and the chiral parts $\mathcal{D}^\pm$ are the restrictions of $\mathcal{D}$ to the smooth sections of $S^\pm$. For example, any spin bundle is a Clifford bundle, hence it induces a Dirac operator. The Atiyah-Singer index formula for a Dirac operator $\mathcal{D}$ is given by

$$\mathrm{ind}(\mathcal{D}) \coloneqq \mathrm{ind}_{\mathrm{Fred}}(\mathcal{D}^+) = \int_X \mathbf{I}(\mathcal{D}) \qquad \text{[Roe98, Theorem 12.27]},$$

where $\mathbf{I}(\mathcal{D})$ is the local index form, that is locally computable in terms of the curvature of X and S and their covariant derivatives. The Atiyah-Singer index formula for induced Dirac operators can be shown via the asymptotic expansion of the smoothing kernel of $\exp(-t\mathcal{D}^2)$. The central object in the proof is the McKean-Singer function h, defined for $t > 0$ as

$$h(t) \coloneqq \mathrm{Tr}\left(e^{-t\mathcal{D}^-\mathcal{D}^+}\right) - \mathrm{Tr}\left(e^{-t\mathcal{D}^+\mathcal{D}^-}\right).$$

The index theorem follows from the fundamental theorem of calculus, an accurate study of the limits $t \to \infty$ and $t \to 0$ of $h(t)$ and the fact that h does not depend on t:

$$\operatorname{ind}(\mathcal{D}) - \int_X \mathbf{I}(\mathcal{D}) = \lim_{t\to\infty} h(t) - \lim_{t\to 0} h(t) = \int_0^\infty \frac{d}{dt} h(t)dt = 0.$$

The explicit expression of the index as an integral of characteristic classes can be shown by the Getzler calculus as worked out for example in [BGV92]. If the Dirac operator is induced by a spin bundle, the local index form is the $\widehat{A}$-genus of the complexified tangent bundle of X.

For index theorems it is enough to specify the case for induced Dirac operators because any elliptic operator is built up by them in an index theoretical sense [Roe98, p.166], hence the heat equation proof provides a full proof technique of the Atiyah-Singer index theorem for any elliptic Differential operator. From now on we will just focus on index theorems of Dirac operators induced by a graded Clifford bundle.

Until now the manifold X was closed, i.e. compact without boundary. What happens if we drop one of the conditions? Let us consider a compact even-dimensional oriented Riemannian manifold Z with boundary Y and a graded Clifford bundle $S \to Z$. Additionally, we assume that the induced chiral Dirac operator $\mathcal{D}^+(Z)$ takes - in a 'collar' of the boundary - the form

$$\mathcal{D}^+(Z) = \operatorname{cl}(\partial_s)(\partial_s - \mathcal{D}(Y)),$$

where s is the outward oriented normal coordinate and $\mathcal{D}(Y)$ is the Dirac operator induced by a Clifford bundle over the boundary. In general, the chiral Dirac operator $\mathcal{D}^+(Z) : H^1(Z, S^+) \to L^2(Z, S^-)$ is not Fredholm anymore. The main idea to avoid the latter problem is to restrict the domain with the Atiyah-Patodi-Singer boundary condition. This means that an element of $H^1(Z, S^+)$ is in the domain of the chiral Dirac operator if and only if the composition of the restriction to the boundary and the projection to the space spanned by the eigensections of the non-negative eigenvalues of the boundary Dirac operator $\mathcal{D}(Y)$ is zero. Then the chiral Dirac operator $\mathcal{D}^+(Z)$ associated to this domain is Fredholm and its index is given by (Atiyah-Patodi-Singer index theorem [APS75, Theorem 22.18]):

$$\operatorname{ind}_{\mathrm{Fred}}(\mathcal{D}^+(Z)) = \int_Z \underbrace{\mathbf{I}(\mathcal{D}(\widetilde{Z}))}_{\text{local object}} - \frac{1}{2}\Big(\underbrace{\eta(\mathcal{D}(Y)) + \dim\ker(\mathcal{D}(Y))}_{\text{spectral-invariant}}\Big).$$

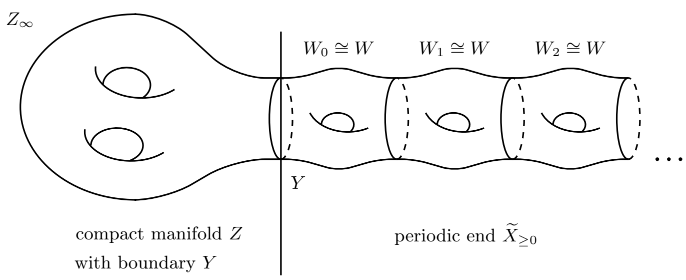
Here $\widetilde{Z}$ denotes the closed double of the manifold Z, which exists because of the assumed structure of the Clifford bundle close to the boundary.

Let us drop the compactness of the underlying manifold and consider an open manifold M without boundary. Roe [Roe88a] studied this question in remarkable generality. He only assumed that the manifold M and the graded Clifford bundle S have bounded geometry, i.e. M is a complete Riemannian manifold, its injective radius is positive and the curvatures of the manifold and the Clifford bundle are bounded in a suitable sense. Since, in general, the induced Dirac operator is not Fredholm, it is a priori not clear how to define the index of the Dirac operator. Therefore, Roe introduced an abstract index in topological K-theory associated to a Dirac operator. By defining suitable maps $\dim_\tau$ and m mapping as in the following diagram, he formulated the index theorem on open manifolds as the commutativity of the diagram [Roe88a, p.88]:

$$\begin{tikzcd} \text{Dirac operator} \arrow[r, "\text{ind}"] \arrow[d, "\mathbf{I}"'] & \text{Abstract index} \arrow[d, "\dim_\tau"] \\ X \arrow[r, "m"] & \mathbb{R} \end{tikzcd}$$

Details and precise definitions of all involved maps are given in [Roe88a]. For an index theorem on open manifolds, which is more concrete than the generalization by Roe, we have to impose some additional requirements on the graded Clifford bundle and the manifold that will guarantee that the chiral Dirac operator $\mathcal{D}^+$ is still Fredholm. In the end-periodic index theorem by Mrowka, Ruberman, and Saveliev [MRS16] it is assumed that an end-periodic graded Clifford bundle S over an even-dimensional end-periodic Riemannian manifold Z_∞ is given. This means that the manifold Z_∞ has an end modeled by half of an infinite cyclic covering $\widetilde{X} \to X$ of a closed oriented manifold X associated with a primitive cohomology class $\gamma \in H^1(X, \mathbb{Z})$. More precisely, Z_∞ takes the form

$$Z_\infty \cong Z \cup_Y W_0 \cup_Y W_1 \cup_Y \dots$$



with Z a compact oriented manifold and $W_i \cong W$ for all $i \in \mathbb{N}$ with W the fundamental segment obtained by cutting X open along a connected submanifold Y, that is Poincaré dual to γ. Any additional structure like the Riemannian metric or the connection of the Clifford bundle have to repeat periodically over the periodic end. Let $f : \tilde{X} \to \mathbb{R}$ be a lift of the smooth map $g : X \to S^1$ that is, up to homotopy, the unique map such that the pullback of the standard coordinate θ on S^1 gives the primitive cohomology class γ. The periodicity of the end is not enough to ensure that the induced chiral Dirac operator $\mathcal{D}^+(Z_\infty)$ is Fredholm. Taubes [Tau87] showed that $\mathcal{D}^+(Z_\infty)$ is Fredholm if and only if the twisted Dirac operators

$$\mathcal{D}_z^+ \coloneqq \mathcal{D}^+(X) - \ln z \cdot df : H^1(X, S^+) \to L^2(X, S^-)$$

are invertible for all complex values z with $|z| = 1$. If $\mathcal{D}^+(Z_\infty)$ is Fredholm, the end-periodic index theorem by Mrowka, Ruberman, and Saveliev [MRS16] states for its index:

$$\mathrm{ind}_{\mathrm{Fred}}\left(\mathcal{D}^+(Z_\infty)\right) = \int_Z \mathbf{I}\left(\mathcal{D}(Z_\infty)\right) - \int_Y \omega + \int_X df \wedge \omega - \frac{1}{2}\eta_{\mathrm{ep}}(\mathcal{D}(X)).$$

Here ω is a differential form on X such that $d\omega = \mathbf{I}(\mathcal{D}^+(X))$ holds* and $\eta_{\mathrm{ep}}(\mathcal{D}(X))$ is the periodic η-invariant. The proof is similar to the proof of the Atiyah-Singer index theorem. Define the McKean-Singer function h as in the proof of the Atiyah-Singer index theorem with the difference that the traces that appear need to be regularized since the operators $\exp(-t\mathcal{D}^\mp(Z_\infty)\mathcal{D}^\pm(Z_\infty))$ are not of trace class. The limits of $h(t)$ for $t \to \infty$ and $t \to 0$ give again the index of $\mathcal{D}^+(Z_\infty)$ and the integral of the local index form, respectively. The additional terms in the index formula appear because $h(t)$ depends, in general, on t.

The end-periodic index theorem is a generalization of the Atiyah-Patodi-Singer index theorem if the kernel of the boundary Dirac operator is zero. The first step to see the relation between these two theorems is to show that the classical Atiyah-Patodi-Singer index theorem is equivalent to an index theorem for manifolds with a cylindrical end. Because of the special structure of the Clifford bundle close to the boundary, we can attach a cylindrical end to the boundary and obtain a non-compact manifold

$$\widehat{Z} \coloneqq Z \cup_Y \left([0, \infty) \times Y\right).$$

All additional structures extend naturally to the cylindrical end because of the required special structure close to the boundary. It turns out that elements of the kernel of $\mathcal{D}^+(Z)$

* Such an ω exists since $\mathcal{D}_z^+$ is invertible for $|z| = 1$, which implies that $\mathbf{I}(\mathcal{D}(X))$ is exact.

satisfying the Atiyah-Patodi-Singer boundary condition can be extended smoothly to sections of $S \to \widehat{Z}$ being in the kernel of $\mathcal{D}^+(\widehat{Z}) : H^1(\widehat{Z}, S^+) \to L^2(\widehat{Z}, S^-)$. Moreover, the kernel and cokernel of $\mathcal{D}^+(Z)$ - with the domain restricted according to the Atiyah-Patodi-Singer boundary condition - are isomorphic to the kernel and cokernel of $\mathcal{D}^+(\widehat{Z}) : H^1(\widehat{Z}, S^+) \to L^2(\widehat{Z}, S^-)$, respectively. The second step is to verify that $S \to \widehat{Z}$ is an end-periodic Clifford bundle over an end-periodic manifold and to show that the additional terms in the index formulas coincide.

This thesis is structured as follows: In the first chapter we will state the Atiyah-Singer index theorem, which has a closed manifold as its foundation, and sketch the heat equation proof. Moreover, we will recall the basic notions of Clifford bundles, induced Dirac operators, indices of Dirac operators and some relevant analytic results. The existence of smoothing kernels, their asymptotic expansions and the definition of the local index form will be explained for Clifford bundles of bounded geometry. This is more general than we would need it for the Atiyah-Singer index theorem, but has the advantage of including all types of Clifford bundles treated later.

In the second chapter we will study the case of a compact manifold with boundary and we will explain in more detail what a product structure close to the boundary exactly looks like. This will be important for the classical formulation of the Atiyah-Patodi-Singer index theorem in Section 2.2 and its reformulation for the case with cylindrical end in Section 2.3.

The third chapter generalizes the Atiyah-Patodi-Singer index theorem to the case of end-periodic Dirac operators. In the first section we will introduce end-periodic manifolds and their end-periodic additional structures and it will be pointed out how the end can be seen as half of an infinite cyclic covering of a closed manifold. In the next section the Fredholm criterion will be proved via the Fourier-Laplace transform. Section 3.3 is the main part of this chapter. There we will state in Section 3.3.1 the end-periodic index theorem and give an outlook of the proof. This proof is worked out in the next three Sections 3.3.2-3.3.4. In the last subsection the Atiyah-Patodi-Singer index theorem will be connected with the end-periodic index theorem, using the preliminary work from Section 2.3. The third chapter ends with a short overview of a generalization of the end-periodic index theorem in the non-Fredholm case, similarly to the cylindrical case treated in Section 2.3.

1 The Atiyah-Singer Index Theorem

The aim of this chapter is to state the Atiyah-Singer index theorem (Theorem 1.16) and sketch the heat equation proof by Atiyah, Bott, and Patodi [ABP73]. We introduce necessary notions like Clifford bundles, induced Dirac operators, indices, heat kernels and their asymptotic expansions in Sections 1.1 and 1.2. Some concepts such as Sobolev spaces, functional calculus and the existence of heat kernels are treated more generally than needed for the Atiyah-Singer index theorem. This will be useful when the underlying manifold has cylindrical end in Section 2.3 and periodic end in Chapter 3.

1.1 Clifford bundles and their induced Dirac operators

In this section we will introduce the concept of Clifford bundles and their induced Dirac operators. The main reference is [Roe98], where the facts given below are proved.

To define a Clifford bundle and its induced Dirac operator, Clifford algebras are required. Let V be a real finite-dimensional vector space equipped with a symmetric bilinear form $(\bullet,\bullet)$. Then there exists a unique (up to isomorphism) algebra, called the *Clifford algebra of* V and denoted by $\mathrm{Cl}(V)$, equipped with a linear map $\phi : V \to \mathrm{Cl}(V)$ such that the following holds:

(1) $\phi(v)^2 = -(v,v)1 \quad \forall v \in V$.

(2) (Universal property) If $\mathcal{A}$ is also an algebra equipped with a linear map $\phi' : V \to \mathcal{A}$ satisfying (1), then there is a unique algebra homomorphism $h : \mathrm{Cl}(V) \to \mathcal{A}$ such that $h \circ \phi = \phi'$ holds.

For a chosen basis $e_1, \ldots, e_n$ of V the Clifford algebra is spanned by the 2^n possible products $\phi(e_1)^{k_1} \cdots \phi(e_n)^{k_n}$ with $k_j \in \{0,1\}$ and the multiplication determined by the rule:

$$\phi(v_1)\phi(v_2) + \phi(v_2)\phi(v_1) = -2(v_1, v_2), \quad \forall v_1, v_2 \in V.$$

In the following we will identify V via the injective map ϕ as a subspace of its own Clifford algebra $\mathrm{Cl}(V)$ and as a consequence ϕ is omitted from now on.

For a given Riemannian manifold M we can use the induced scalar product on the tangent spaces T_pM for any point $p \in M$ to built the Clifford algebras $\mathrm{Cl}(T_pM)$. Forming the disjoint union of their complexifications gives rise to a smooth complex vector bundle of dimension 2^n, called the *bundle of Clifford algebras* $\mathrm{Cl}(TM)$ over M.

Definition 1.1. (1) A *Clifford bundle* is a complex vector bundle S over a Riemannian manifold M equipped with a Hermitian metric h, a compatible connection ∇ and a smooth bundle map $\mathrm{Cl}(TM) \oplus S \to S$ such that the following holds:

(i) For all $p \in M$ the restriction of the bundle map $\mathrm{Cl}(TM) \oplus S \to S$ to the fiber over p is a left module action of $\mathrm{Cl}(T_pM) \otimes \mathbb{C}$ on S_p, denoted with a '$\cdot$' and called Clifford (module) action.

(ii) The Clifford action is skew-adjoint, i.e. $h_p(A_p \cdot u, v) + h_p(u, A_p \cdot v) = 0$ holds for all $p \in M$, $u, v \in S_p$ and all $A_p \in T_pM$.

(iii) $\nabla_A(B \cdot u) = (\widetilde{\nabla}_A B) \cdot u + B \cdot \nabla_A u$ holds for all vector fields A, B and all smooth sections u of S, where $\widetilde{\nabla}$ denotes the Levi-Civita connection on M.

(2) The *induced Dirac operator* $\mathcal{D}$ of a Clifford bundle $S \to M$ is defined as the composition

$$\mathcal{D} : C^\infty(M,S) \xrightarrow{\text{connection}} C^\infty(M, T^*M \otimes S) \xrightarrow{\text{metric}} C^\infty(M, TM \otimes S) \xrightarrow{\text{Clifford action}} C^\infty(M,S).$$

Notation 1.2. Some authors call Clifford bundles, as defined in Definition 1.1 (1), Dirac bundles. This prevents the risk of confusion with so-called Clifford module bundles, which are not identical to Clifford bundles defined here. If it is clear from the context, we omit the '$\cdot$' of the Clifford action. Moreover, we obtain the induced representation of the Clifford algebra $\mathrm{Cl}(T_pM)$ defined for every $p \in M$ as

$$\mathrm{cl} : \mathrm{Cl}(T_pM) \to \mathrm{End}_{\mathbb{C}}(S_p), \quad \mathrm{cl}(a)(u) := a \cdot u \in S_p, \tag{1.1}$$

where $\mathrm{Cl}(T_pM)$ is seen as a subset of $\mathrm{Cl}(T_pM) \otimes \mathbb{C}$. Via the natural isomorphism $TM \cong T^*M$ the Clifford multiplication is also defined for elements of the cotangent bundle.

The next proposition lists some immediate consequences of the definition of the Dirac operator, that we state without proof. For the proofs, we refer to [Roe98, chapter 3].

Proposition 1.3. *The induced Dirac operator $\mathcal{D}$ of a Clifford bundle $S \to M$ satisfies:*

(1) $\mathcal{D}$ is a first order, elliptic differential operator.

(2) With respect to a local orthonormal frame $\{e_i\}$ of the tangent bundle TM the Dirac operator takes locally the form $\mathcal{D} = \sum_i e_i \nabla_{e_i}$.

1.2 Smoothing kernels and their asymptotic expansions

In the heat kernel proof of the Atiyah-Singer index theorem (Theorem 1.16) the operator $\exp(-t\mathcal{D}^2)$ plays a crucial role. In this section we define this operator via the functional calculus and state the existence of its associated smoothing kernel. Moreover, there exists an asymptotic expansion of the smoothing kernel which we use to define the so-called local index form (Definition 1.13) arising in any index theorem treated in this thesis. The procedure to define the local index form works for closed manifolds. However, it also suffices to require that the manifold and the Clifford bundle have bounded geometry (Definition 1.1) so that we can use the results in the cylindrical (Section 2.3) and the end-periodic cases (Chapter 3). The definitions and the main results of this section are based on [Roe88a].

Let $\mathcal{D}$ be the Dirac operator induced by a Clifford bundle S over an oriented Riemannian manifold M. We define for $k \in \mathbb{N}_0$ the Hilbert space $H^k(M, S)$ as the completion of smooth sections with compact support $C_0^\infty(M, S)$ with respect to the norm

$$\|u\|^2_{H^k(M,S)} \coloneqq \int_M \sum_{j=0}^{k} |\nabla^j u|^2 \mathrm{dvol}_g, \quad u \in C_0^\infty(M, S) \tag{1.2}$$

and call it the *k-th Sobolev space*. Furthermore, we define the space $L^2(M, S)$ as the 0-th Sobolev space. This definition also holds for manifolds with boundary in Chapter 2. For the main results stated in Proposition 1.5 and Proposition 1.6, we have to require that the manifold and the Clifford bundle have bounded geometry:

Definition 1.1 ([Roe88a, Definition 2.1])**.** A complete Riemannian manifold M is said to have *bounded geometry* if the following two properties are fulfilled:

(1) M has positive injectivity radius i.e. there exists an $\epsilon > 0$ such that for all $x \in M$ the exponential map restricted to the ball $B_\epsilon(0)$ is bijective.

(2) The curvature tensor and all its covariant derivatives are uniformly bounded.

A Clifford bundle $S \to M$ has *bounded geometry* if the curvature tensor and all its covariant derivatives are uniformly bounded.

Example 1.4. Clifford bundles over compact manifolds, Clifford bundles with cylindrical end as in Definition 2.14 and end-periodic Clifford bundles over end-periodic manifolds as in Definition 3.3 and Definition 3.8 have bounded geometry, and their underlying manifolds have bounded geometry as well. The manifold $\mathbb{R}^2 \setminus \{0\}$ does not have bounded geometry because the injectivity radius is not positive.

The following proposition lists the most important analytic properties of the Dirac operator.

Proposition 1.5. *Let S be a Clifford bundle with bounded geometry over an oriented complete* Riemannian manifold M that also has bounded geometry. The induced Dirac operator $\mathcal{D}$ defined on compactly supported smooth sections extends to a bounded operator $\mathcal{D} : H^{k+1}(M, S) \to H^k(M, S)$ for every $k \geq 0$. In the case $k = 0$ it is self-adjoint† as a map $H^1(M, S) \subset L^2(M, S) \to L^2(M, S)$, its spectrum is real and there exists a uniquely determined regular countable additive self-adjoint spectral measure E defined on the Borel sets of the plane, vanishing on the complement of the spectrum such that*

$$\mathcal{D}u = \int_{-\infty}^{\infty} \lambda E(d\lambda)u, \quad \forall u \in H^1(M, S) := \left\{ u \in L^2(M, S) \middle| \int_{\sigma(\mathcal{D})} \lambda^2 (E(d\lambda)u, u)_{L^2} < \infty \right\} \tag{1.3}$$

holds. If the manifold is compact, the spectrum is discrete and there exists a complete system of $L^2(M, S)$ orthonormal eigensections.

Proof (sketch). The extension of the Dirac operator to the Sobolev spaces as stated in the proposition follows from a straightforward calculation using the elliptic estimates. The self-adjointness is a direct consequence of the essential self-adjointness proved in [Wol73, 5.1 Theorem] and the fact that $H^1(M, S)$ is by the Sobolev embedding theorem compactly embedded in $L^2(M, S)$. The remaining follows from the spectral theorem for unbounded operators. For a general formulation see [DS63, Theorem 3 in chapter XII section 2] and for a treatment special for elliptic operators see [Shu92]. The special case for a compact manifold can be found in [Roe98, Theorem 5.27]. □

The previous proposition enables us to apply the so-called *functional calculus*: for a bounded Borel function $f : \mathbb{R} \to \mathbb{R}$ we define an operator $f(\mathcal{D})$ on $L^2(M, S)$ via $f(D)u := \int_{-\infty}^{\infty} f(\lambda)E(d\lambda)u$ with notations as in the previous proposition. Many properties of this

* A Riemannian manifold is called complete, if the exponential map on any point $p \in M$ is defined on the whole tangent space. † In the functional analysis sense as in [Fri00, p. 91-92].

operator are listed in [DS63, Chapter XII, section 2]. If the manifold M is compact, the operator $f(\mathcal{D})$ takes the form $f(D)u := \sum_\lambda f(\lambda)u_\lambda$ for $u = \sum_\lambda u_\lambda$ decomposed into eigensections of $\mathcal{D}$. The following proposition is the heart of the heat equation proof of the Atiyah-Singer index theorem (Theorem 1.16). For a Clifford bundle over a compact manifold a proof can be found in [Roe98] and [Gil95]. The generalization to the case of bounded geometry can be proved similarly as in the compact case by exploiting that all geometry is bounded.

Proposition 1.6 ([Roe88a, Proposition 2.10 and 2.11])**.** *Let S be a Clifford bundle over an oriented Riemannian manifold M, both of bounded geometry, and induced Dirac operator $\mathcal{D}$. The following holds:*

(1) For every rapidly decaying function $f : \mathbb{R} \to \mathbb{R}$ the operator $f(\mathcal{D})$ defined via the functional calculus can be represented by a unique smoothing kernel $k \in C^\infty(M \times M, S \boxtimes S^*)$ i.e. for any $u \in L^2(M, S)$ and $x \in M$*

$$f(\mathcal{D})u(x) = \int_M k(x, y)u(y)dy \tag{1.4}$$

holds, where $S \boxtimes S^*$ is defined as $\mathrm{pr}_1^*(S) \otimes \mathrm{pr}_2^*(S^*)$ with $pr_j : M \times M \to M$, $j = 1, 2$ the projection to the j-th coordinate. The smoothing kernel k and all its covariant derivatives are uniformly bounded.

(2) Let k_t be the uniformly bounded smoothing kernel of the operator $\exp(-t\mathcal{D}^2)$ for $t > 0$. Then there exists an asymptotic expansion† near $t = 0$

$$k_t(x, x) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} \theta_j(x)t^j \tag{1.5}$$

with θ_j smooth sections of $\mathrm{End}(S)$ that are locally computable in terms of the curvature of M and S and their covariant derivatives.

Remark 1.7. With notations as in Proposition 1.6 the smoothing kernel k_t of the operator $\exp(-t\mathcal{D}^2)$ is also called the *heat kernel*. The name comes from the fact that

* A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called rapidly decaying if it satisfies for all $m \in \mathbb{N}$ and $x \in \mathbb{R}$ the estimate $|f(x)| \leq C_m(1+|x|)^{-m}$ with the best constants C_m as seminorms [Roe88a, p.94]. † A general definition of an asymptotic expansion of a function f on $\mathbb{R}_{>0}$ into a Banach space B can be found in [Roe98, Definition 7.14]. We have the situation $f(t) = k_t$ and $B = C^\infty(M, \mathrm{End}(S))$. Roughly speaking, it is a formal sum $\sum_{j=0}^{\infty} a_j$ with functions $a_j : \mathbb{R}_{>0} \to B$ such that for any integer m almost all partial sums of the series approximate f up to an error term of order t^m near $t = 0$.

it solves as a function of t and x with y fixed the initial value problem

$$\underbrace{(\partial_t + \mathcal{D}^2)k_t(x,y) = 0}_{\text{heat equation } (y \text{ fixed})}, \quad \lim_{t\to 0}\int_M k_t(x,y)u(y)dy = u(x),\ \forall u \in C_0^\infty(M,S),\ \forall x,y \in M. \tag{1.6}$$

Proposition 1.8 ([MRS16, Proposition 10.1 and 10.2])**.** *Let S be a Clifford bundle of bounded geometry over a Riemannian manifold M of bounded geometry, $\mathcal{D}$ its induced Dirac operator and $k_t(x,y)$ the smoothing kernel of* $\exp(-t\mathcal{D}^2)$ *as in Proposition* 1.6 *and n the dimension of the manifold M. Then for any $T > 0$ there exists a positive constant C such that for any $i \in \mathbb{N}_0$, any multi-indices j and l and all $t \in (0,T]$ the following holds:*

$$\left|\frac{\partial^i}{\partial t^i}\nabla_x^j\nabla_y^l k_t(x,y)\right| \le Ct^{-n/2-i-|j|-|l|}e^{-d^2(x,y)/4t} \quad \textit{(short-term Gaussian estimate).}$$

1.3 The Atiyah-Singer index theorem

Let S be a Clifford bundle over a closed oriented Riemannian manifold X with induced Dirac operator $\mathcal{D}$. First, we want to find out what a suitable index of the Dirac operator could be. It turns out that the Dirac operator is Fredholm (see Definition 1.9 and Proposition 1.10) with Fredholm index zero. This motivates the introduction of a grading as an additional structure (see Definition 1.11) on the Clifford bundle and define the index of $\mathcal{D}$ with respect to this grading as the Fredholm index of the positive chiral Dirac operator $\mathcal{D}^+$. Second, we want to sketch the proof of the Atiyah-Singer index formula, that expresses the index of $\mathcal{D}$ as an integral of the so-called local index form. Let us start with the definition of a Fredholm operator:

Definition 1.9. A bounded operator $T : \mathcal{B} \to \mathcal{C}$ between two Banach spaces is called *Fredholm* if its image is closed in $\mathcal{C}$, and the kernel and cokernel (defined as $\mathcal{C}/\operatorname{im}(T)$) have finite dimensions. Then its *Fredholm index* is defined as

$$\operatorname{ind}_{\mathrm{Fred}}(T) \coloneqq \dim(\ker T) - \dim(\operatorname{coker} T). \tag{1.7}$$

Proposition 1.10. *The induced Dirac operator $\mathcal{D} : H^1(X,S) \to L^2(X,S)$ of a Clifford bundle S over a closed oriented Riemannian manifold X is a Fredholm operator with index zero.*

Proof (sketch). For a proof of the Fredholm property see [Fri00, Section 4.3]. There the ellipticity of the Dirac operator is important since it allows to use the elliptic estimates. It is also essential that the manifold is closed as we will see in Example 2.2. The vanishing of the index can be seen by rewriting the cokernel as

$$\operatorname{coker}(\mathcal{D}) \overset{\text{def.}}{=} L^2(X,S)/_{\operatorname{im}(\mathcal{D})} = \left(\operatorname{im}(\mathcal{D}) \oplus (\operatorname{im}(\mathcal{D}))^{\perp}\right)/_{\operatorname{im}(\mathcal{D})} \cong (\operatorname{im}(\mathcal{D}))^{\perp}. \tag{1.8}$$

The second step holds because $\operatorname{im}(\mathcal{D})$ is closed in $L^2(Z,S)$ [see Fri00, section 4.3] and thus we can split $L^2(Z,S)$ into $\operatorname{im}(\mathcal{D}) \oplus (\operatorname{im}(\mathcal{D}))^{\perp}$ [see Wer18, Theorem V.3.4]. A straightforward calculation gives $(\operatorname{im}(\mathcal{D}))^{\perp} = \ker(\mathcal{D}^*)$ and with the self-adjointness of $\mathcal{D}$ (Proposition 1.3) we obtain $\operatorname{coker}(\mathcal{D}) \cong \ker(\mathcal{D})$ and the Fredholm index vanishes. □

The previous proposition shows that the Fredholm index should not be our object of interest because it vanishes. The idea is to give the Clifford bundle S a suitable additional structure such that the Dirac operator splits into two parts $\mathcal{D}^{\pm}$, which are in general not self-adjoint anymore but still Fredholm. Then we consider as the index of $\mathcal{D}$ the Fredholm index of the positive chiral part $\mathcal{D}^{+}$. The following definitions will make this precise:

Definition 1.11. We call a Clifford bundle $S \to M$ *graded* if it is equipped with a grading $S = S^{+} \oplus S^{-}$ that satisfies:

(1) It respects the connection, i.e. $\nabla_A u^{\pm} \in S^{\pm}$, $\forall A \in C^{\infty}(M,TM)$, $\forall u^{\pm} \in C^{\infty}(M,S^{\pm})$.

(2) It respects the metric, i.e $S^{+} \perp S^{-}$.

(3) It satisfies: $cl(A)(u^{\pm}) \in C^{\infty}(M,S^{\mp})$, $\forall A \in C^{\infty}(M,TM)$, $\forall u^{\pm} \in C^{\infty}(M,S^{\pm})$.

The two subbundles $S^{\pm}$ are called the *positive* and *negative* parts of the Clifford bundle S, respectively.

Alternatively one could define a grading via a so-called *grading operator*. This is a map $\epsilon : S \to S$ which allows to define $S^{\pm}$ as the eigenspaces of the eigenvalues ± 1 such that the conditions from Definition 1.11 are fulfilled.

Example 1.12. Let S be a Clifford bundle over an oriented Riemannian manifold M of even dimension n and define the volume element ω locally as $e_1 \cdots e_n$ for a positive oriented local orthonormal frame $\{e_1, \dots, e_n\}$ of TM. This is independent of the choice of the frame, hence leads to a globally well defined smooth section of $\mathrm{Cl}(TM)$. The *canonical grading* is defined via the grading operator $i^{n/2}\omega$ acting on S via Clifford multiplication. A straightforward calculation shows that this actually leads to a grading in the sense of Definition 1.11.

By the local expression of the Dirac operator $\mathcal{D} = \sum_i e_i \nabla_{e_i}$ in Proposition 1.3 and the properties (1) and (3) of the previous Definition 1.11 it follows that the Dirac operator interchanges the grading, i.e. $\mathcal{D}u^{\pm} \in C^{\infty}(M, S^{\mp})$ for $u^{\pm} \in C^{\infty}(M, S^{\pm})$.

Definition 1.13. Let $\mathcal{D}$ be the Dirac operator induced from a graded Clifford bundle S over an oriented Riemannian manifold M and denote the grading operator with ϵ.

(1) If the so-called *chiral Dirac operator* $\mathcal{D}^{+} := \mathcal{D}|_{C^{\infty}(M,S^{+})}$ extends to a Fredholm operator $H^1(M, S^{+}) \to L^2(M, S^{-})$, we define the *index* of $\mathcal{D}$ as

$$\operatorname{ind}(\mathcal{D}) := \operatorname{ind}_{\mathrm{Fred}}(\mathcal{D}^{+}) = \dim\ker(\mathcal{D}^{+}) - \dim\operatorname{coker}(\mathcal{D}^{+}). \tag{1.9}$$

(2) If the Clifford bundle and the underlying manifold have bounded geometry and the dimension of M is even, the *local index form* (called *Atiyah-Singer integrand* in [Loy04] and [Mel93]) is defined as

$$\mathbf{I}(\mathcal{D}) := \frac{1}{(4\pi)^{n/2}} \operatorname{tr}_s\left(\theta_{n/2}\right)\mathrm{dvol}_g \in C^{\infty}(M, \textstyle\bigwedge^n T^*M), \tag{1.10}$$

where θ_k denotes the coefficients of the asymptotic expansion of the smoothing kernel of $\exp(-t\mathcal{D}^2)$ as in Proposition 1.6 and tr_s denotes the local supertrace defined as $\operatorname{tr}\left(\epsilon \circ \theta_{n/2}\right)$.

Remark 1.14. Proposition 1.10 states that for a Clifford bundle S over a closed oriented Riemannian manifold X the Dirac operator $\mathcal{D} : H^1(X, S) \to L^2(X, S)$ is Fredholm, hence $\mathcal{D}^{+} : H^1(X, S^{+}) \to L^2(X, S^{-})$ is also Fredholm. Then the index of $\mathcal{D}$ defined in equation (1.9) simplifies by reformulating similarly to equation (1.8) the cokernel of $\mathcal{D}^{+}$ into the kernel of $(\mathcal{D}^{+})^*$, where the adjoint is meant in the L^2-sense. Furthermore, we have as a consequence of the self-adjointness of $\mathcal{D}$ the equality $(\mathcal{D}^{+})^* = \mathcal{D}^{-}$ and obtain

$$\operatorname{ind}(\mathcal{D}) = \dim\ker(\mathcal{D}^{+}) - \dim\ker(\mathcal{D}^{-}). \tag{1.11}$$

Here $\mathcal{D}^{-}$ is defined analogously to $\mathcal{D}^{+}$ in Definition 1.13 (1). Roe [Roe98, Definition 11.7] defines the index of a Dirac operator via equation (1.11). The advantage of this approach is that it works without Sobolev spaces. Roe's definition matches with the definition given here because the kernels of $\mathcal{D}^{\pm} : H^1(X; S^{\pm}) \to L^2(X, S^{\mp})$ contain only smooth sections due to elliptic regularity. In the definition given here, the connection to the Fredholm index is pointed out and will be helpful if we look at manifolds with boundary in Chapter 2 and especially in Section 2.2, where we treat the boundary case with a boundary condition.

Remark 1.15. The short-term Gaussian estimate in Proposition 1.8 also holds for $\mathcal{D}^2$ replaced by the chiral parts $\mathcal{D}^{\mp}\mathcal{D}^{\pm}$. Furthermore, it holds for operators of the form $\mathcal{D}^{\pm}\exp(-t\mathcal{D}^{\mp}\mathcal{D}^{\pm})$ and their smoothing kernels. This is a direct consequence of the fact that the smoothing kernel of $\mathcal{D}^{\pm}\exp(-t\mathcal{D}^{\mp}\mathcal{D}^{\pm})$ equals $\mathcal{D}^{\pm}k_t(x,y)$, where $\mathcal{D}^{\pm}$ is the Dirac operator acting on the x-variable.

Theorem 1.16 (Atiyah-Singer index theorem)**.** *Let S be a graded Clifford bundle over a closed even-dimensional oriented Riemannian manifold X. Then the index of the induced Dirac operator $\mathcal{D}$ is given by*

$$\operatorname{ind}(\mathcal{D}) = \int_X \boldsymbol{I}(\mathcal{D})\,,$$

where $\boldsymbol{I}(\mathcal{D})$ denotes the local index form defined in Definition 1.13.

Remark 1.17. The Atiyah-Singer index theorem stated as above is not as extensive as the originally version stated by Atiyah and Singer [AS63]. They expressed the local index form in terms of characteristic classes. Let us assume that the Clifford bundle is canonically graded* as defined in Example 1.12. Then the integral of the local index form can be expressed in terms of the $\widehat{A}$-genus and the relative Chern character $\operatorname{ch}(S/\Delta)$ (for precise definitions see [Roe98, Theorem 12.27]) and the index theorem takes the form

$$\operatorname{ind}(\mathcal{D}) = \int_X \widehat{A}(TX) \wedge \operatorname{ch}(S/\Delta). \tag{1.12}$$

Getzler [Get83] proved (1.12) directly by using the so-called Getzler calculus. Roughly summarized one defines the so-called Getzler symbol $\sigma^{\mathcal{G}}$ for differential operators and the canonical symbol σ for smooth sections of $S\otimes S^*$ and shows that they are compatible to each other† in a suitable sense. It is crucial, that the space of differential operators

* It is possible to lead back the case with an arbitrary grading to the special case of the canonical grading. One takes advantage of the fact, that any grading splits into the direct sum of two graded Clifford subbundles, one canonically graded and the other anticanonically i.e. with grading operator $-i^{n/2}\omega$.

† This notation is based on [Han16] and not common in any literature. Roe [Roe98, Proposition 9.12] for example calls both maps just symbol map although they have different domains.

is equipped with the Getzler grading, which takes additionally to the degree of the differential operator the involved Clifford multiplication into account. The canonical symbol map picks out exactly the top degree part arising in the local index form. By Mehler's formula [Roe98] one obtains an explicit expression of the occurring term and can identify it with the two characteristic classes in equation (1.12). For a detailed discussion of this concept see [BGV92], [Roe98] or the script by Hanke [Han16].

For the progress of this work it is not important to rewrite the local index form into characteristic classes, since in all generalizations of the index theorem we only write the local index form and keep that possibility in mind. It is a fundamental statement that shows that the index of a Dirac operator is a topological invariant. More precisely, it just depends on characteristic classes associated to S and the tangent bundle of X, hence it does not depend on the metric and connection on S and the metric on X.

Proof of Theorem 1.16 *(sketch).* This proof is based on the heat kernel method explained in full generality by Berline, Getzler, and Vergne [BGV92]. It is also proved in [Roe98], [Han16] and sketched in [Loy04].
As stated in Proposition 1.6 for every $t > 0$ there exists a smoothing kernel $k_t(x, y)$ of the operator $\exp(-t\mathcal{D}^2)$, hence $\exp(-t\mathcal{D}^2)$ is of trace class [see Roe98, Proposition 11.2] and the so-called *McKean-Singer function* is well defined:

$$h(t) \coloneqq \operatorname{Tr}_s\left(e^{-t\mathcal{D}^2}\right) \coloneqq \operatorname{Tr}\left(e^{-t\mathcal{D}^-\mathcal{D}^+}\right) - \operatorname{Tr}\left(e^{-t\mathcal{D}^+\mathcal{D}^-}\right), \quad t > 0. \tag{1.13}$$

This function is one of the central objects in the proof the the Atiyah-Singer index theorem. It satisfies the following three properties* which imply the index theorem:

$$\begin{aligned} &(1) \quad \lim_{t\to\infty} h(t) = \operatorname{ind}(\mathcal{D}). \\ &(2) \quad \lim_{t\to 0} h(t) = \int_X \mathbf{I}(\mathcal{D}). \\ &(3) \quad \frac{d}{dt} h(t) = \operatorname{Tr}\left([\mathcal{D}^+, \mathcal{D}^- e^{-t\mathcal{D}^+\mathcal{D}^-}]\right) = 0, \quad \forall t > 0. \end{aligned} \tag{1.14}$$

* In the proof of part (1) we will see that the function h is independent of t, hence the limit is not necessary and part (3) seems useless. Nevertheless, we sketch the proofs of all three properties to see the beautiful analogy to the proofs in the cylindrical case (Theorem 2.16) and the end-periodic case (Theorem 3.22). The term in the middle of part (3) is significant because it is still valid in the generalizations later on with a minor modification of the trace. But it will turn out that in the cylindrical and the end-periodic cases the derivative of h does not vanish in general. This leads to an additional term in the associated index theorems.

ad(1) Fix an arbitrary $t > 0$ and denote by $0 \le \lambda_0^\pm \le \lambda_1^\pm \le \dots$ the eigenvalues of $\mathcal{D}^\mp \mathcal{D}^\pm$ listed by multiplicity. Then the smoothing operators $\exp(-t\mathcal{D}^\mp \mathcal{D}^\pm)$ have the eigenvalues $\{\exp(-t\lambda^\pm)\}$ and the fact, that the trace of a self-adjoint trace class operators is equal to the sum over all eigenvalues [see Roe98, Proposition 8.7] leads to

$$h(t) = \mathrm{Tr}_s\left(e^{-t\mathcal{D}^2}\right) = \mathrm{Tr}\left(e^{-t\mathcal{D}^-\mathcal{D}^+}\right) - \mathrm{Tr}\left(e^{-t\mathcal{D}^+\mathcal{D}^-}\right) = \sum_{i=0}^{\infty} e^{-t\lambda_i^+} - \sum_{i=0}^{\infty} e^{-t\lambda_i^-}. \tag{1.15}$$

The operator $\mathcal{D}^2$ takes the form $\mathcal{D}^-\mathcal{D}^+ \oplus \mathcal{D}^+\mathcal{D}^-$, so for every eigenvalue λ of $\mathcal{D}^2$ we have the decomposition $\mathrm{Eig}(\mathcal{D}^2, \lambda) = \mathrm{Eig}(\mathcal{D}^-\mathcal{D}^+, \lambda) \oplus \mathrm{Eig}(\mathcal{D}^+\mathcal{D}^-, \lambda)$. Furthermore, a straightforward calculation shows that $\mathrm{Eig}(\mathcal{D}^-\mathcal{D}^+, \lambda)$ is isomorphic* to $\mathrm{Eig}(\mathcal{D}^+\mathcal{D}^-, \lambda)$ for every positive eigenvalue λ. The eigenspaces $\mathrm{Eig}(\mathcal{D}^\mp\mathcal{D}^\pm, 0)$ are just given by the kernel of $\mathcal{D}^\pm$, respectively, which is a direct consequence of $\left(\mathcal{D}^+\right)^* = \mathcal{D}^-$. These two facts together with the equations (1.15) and (1.11) imply the so-called McKean-Singer formula

$$h(t) = \mathrm{Tr}_s\left(e^{-t\mathcal{D}^2}\right) = \dim\left(\mathrm{Eig}(\mathcal{D}^-\mathcal{D}^+, 0)\right) - \dim\left(\mathrm{Eig}(\mathcal{D}^+\mathcal{D}^-, 0)\right), = \mathrm{ind}(\mathcal{D})$$

which finishes the proof of (1).

ad(2) To study the McKean-Singer function for small t, we take advantage of the asymptotic expansion of the heat kernel as discussed in Proposition 1.6, and obtain

$$\begin{aligned}\lim_{t\to 0} h(t) = \lim_{t\to 0} \mathrm{Tr}_s\left(e^{-t\mathcal{D}^2}\right) &= \lim_{t\to 0} \int_X \mathrm{tr}_s \underbrace{k_t(x,x)}_{\sim \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^\infty \theta_j(x) t^j} dx \\ &= \underbrace{\int_X \frac{1}{(4\pi)^{n/2}} \mathrm{tr}_s(\theta_{n/2}) \mathrm{dvol}_g}_{\overset{\text{Def.1.13}}{=} \mathbf{I}(\mathcal{D})}.\end{aligned}$$

Here we used that the supertrace can be pulled into the integral over the smoothing kernel. Then it becomes the local supertrace tr_s [see Roe98, Proposition 11.2].

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* The isomorphism is given by $\mathcal{D}^+$ and its inverse is given by $\frac{1}{\lambda}\mathcal{D}^-$.

ad(3) Differentiating equation (1.15) gives for all $t > 0$:

$$\frac{d}{dt}h(t) = -\sum_{i=0}^{\infty}\lambda_i^+ e^{-t\lambda_i^+} + \sum_{i=0}^{\infty}\lambda_i^- e^{-t\lambda_i^-} = -\operatorname{Tr}(\underbrace{\mathcal{D}^-\mathcal{D}^+ e^{-t\mathcal{D}^-\mathcal{D}^+}}_{\overset{(\star)}{=}\mathcal{D}^- e^{-t\mathcal{D}^+\mathcal{D}^-}\mathcal{D}^+}) + \operatorname{Tr}(\mathcal{D}^+\mathcal{D}^- e^{-t\mathcal{D}^+\mathcal{D}^-})$$
$$= \operatorname{Tr}\left([\mathcal{D}^+, \mathcal{D}^- e^{-t\mathcal{D}^+\mathcal{D}^-}]\right)$$
$$= 0.$$

The composition of a trace class operator with a bounded operator is again of trace class and the commutation does not change the trace [Roe98, Propoposition 8.8]. This explains step two and four of the previous calculation. The equality $(\star)$ holds, because for any smooth section u of S^+ the time dependent sections $u_t := \mathcal{D}^-\mathcal{D}^+ e^{-t\mathcal{D}^-\mathcal{D}^+}u$ and $w_t := \mathcal{D}^- e^{-t\mathcal{D}^+\mathcal{D}^-}\mathcal{D}^+ u$ are solutions of the heat equation $(\partial_t + \mathcal{D}^-\mathcal{D}^+)u_t = 0$. This can be verified by exploiting that X is compact and we can write u_t and w_t as a concrete sum over the eigenvalues of $\mathcal{D}^-\mathcal{D}^+$. Since u_t and w_t match at $t = 0$, the uniqueness of solutions of the heat equation implies the predicted equality $(\star)$. $\square$

2 The Atiyah-Patodi-Singer Index Theorem

The Atiyah-Singer index theorem stated in Theorem 1.16 gives a formula for the index of a Dirac operator induced by a graded Clifford bundle over a closed oriented Riemannian manifold of even dimension. The aim of this chapter is a generalization to manifolds with boundary. In Section 2.1 we will introduce the concept of manifolds with boundary and Clifford bundles over them. A crucial difference to the boundaryless case is that the Dirac operator is in general not Fredholm anymore. There are different ways to avoid that problem. First, one can introduce a global boundary condition (see Section 2.2), second, one can attach a cylindrical end to the boundary and obtain a non-compact boundaryless manifold (see Section 2.3), or third, one uses the b-calculus introduced by Melrose [Mel93]. All these treatments lead to equivalent index theorems. We will discuss in detail the classical approach with boundary conditions, leading to the Atiyah-Patodi-Singer (APS) index theorem as stated in Theorem 2.10 and the cylindrical approach leading to the versions stated in Theorem 2.16 and Theorem 2.21. The b-calculus will be used to prove the cylindrical version of the APS index theorem. The remarkable feature of this proof by Melrose [Mel93] is that it works with absence of boundary conditions and is independent of the classical proof of the APS index theorem [APS75].
It does not matter how we treat the manifold with boundary, in any case there is an additional term in the generalized index formula. It contains a spectral invariant - called η-invariant - associated to the boundary Dirac operator, that is the Dirac operator induced by a Clifford bundle over the boundary. For that Clifford bundle we require that it induces in a natural way the original graded Clifford bundle in a collar of the boundary. If such a Clifford bundle over the boundary exists, we say the originally graded Clifford bundle has product structure near the boundary. That is an additional assumption we have to make for the APS index theorem. What that all precisely means and how the chiral Dirac operator in the product case looks like will also be explained in Section 2.1.

2.1 Manifolds with boundary and their additional structures

In this section we will generalize the concept of manifolds to manifolds with boundary and all their additional structures up to graded Clifford bundles and Dirac operators. For precise definitions and proofs of the following facts see [Tu11, chapter 22] and [Lee13].

Roughly speaking, a manifold with boundary of dimension n is a manifold, where the charts map into open subsets of $H^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \geq 0\}$ equipped with the subspace topology. There are two different kinds of open subsets in H^n; those that are also open in $\mathbb{R}^n$ and those that are not open in $\mathbb{R}^n$. This leads to a distinction of the manifold into boundary points and interior points. For a manifold Z, we denote by ∂Z its boundary. To define the smooth structure we have to look at the transition functions, whose domains are in the boundary case not necessarily open in $\mathbb{R}^n$, so that we have to extend the concept of smoothness slightly:

> We call a map $f : A \to \mathbb{R}^n$ for a subset $A \subset \mathbb{R}^n$ *smooth* if for every point $p \in A$ there exists a smooth continuation into a small open neighborhood of p.

Let Z be an n-dimensional manifold with boundary ∂Z. We will show in Proposition 2.1 (1) that the boundary inherits the structure of an $(n-1)$-dimensional manifold. For the definition of the tangent space in a boundary point $p \in \partial Z$ we have several different choices, each with their own legitimacy. Should it be an $(n-1)$-dimensional vector space like the tangent space of the boundary, or a half space H^n or just an n-dimensional vector space similarly to the tangent space in the interior? The standard way is to define the *tangent space* T_pZ for $p \in Z$ as the space of all derivations of $C^\infty(Z, \mathbb{R})$ at p. This leads to an n-dimensional vector space T_pZ, no matter whether p belongs to the boundary or the interior of the manifold. Moreover, in a boundary point p the tangent space of the boundary $T_p\partial Z$ has a canonically embedding into T_pZ. For $p \in \partial Z$ a tangent vector $X_p \in T_pZ \setminus T_p\partial Z$ is called *inward-pointing* if there exists an $\epsilon > 0$ and a smooth curve $c : [0, \epsilon)$ such that $c(0) = p$, $c((0, \epsilon)) \subset M^\circ$ and $c'(0) = X_p$. A tangent vector $X_p \in T_pZ$ is called *outward-pointing* if $-X_p$ is inward pointing.

All the extra structures like metrics, orientations, connections, vector bundles, differential operators between two vector bundles, Clifford bundles and their induced Dirac operators are similarly defined as for manifolds without boundary. The following proposition resumes some important facts about manifolds with boundary:

Proposition 2.1. *For a manifold Z with boundary and of dimension $n > 1$ the following holds:*

(1) The boundary ∂Z inherits the structure of a smooth manifold of dimension $n-1$ without boundary. If Z is Riemannian, then ∂Z inherits a Riemannian structure. If Z is compact, then so is ∂Z.

(2) The set of all interior points M° inherits the structure of a smooth manifold of dimension n.

(3) There exists an outward pointing vector field. If the manifold Z is oriented any outward pointing vector field determines an orientation on the boundary, which is independent of the picked outward pointing vector field. We call this the induced orientation on ∂Z.

(4) (Milnor's collar neighborhood theorem) If Z is compact, there exists a neighborhood of the boundary ∂Z which is diffeomorphic to $(-1, 0] \times \partial Z$.

Proof. For a proof of (1) and (3) see [Tu11, Proposition 22.10, and section 22.3, 22.6] and [Lee13, Proposition 15.24]. Part (2) follows directly by the definition of the interior points and part (4) is proven by Milnor [Mil65, Corrolarry 3.5]. □

From now on we equip the boundary of an oriented manifold with boundary with the induced orientation as in Proposition 2.1 (3). The following example is motivated by Loya [Loy04, Example after Theorem 3.1].

Example 2.2. Let $S := \mathbb{C} \times [0,1] \times S^1$ be the trivial line bundle over the manifold $Z := [0,1] \times S^1$ equipped with the metric $g = ds^2 + d\theta^2$ for standard coordinates (s, θ) on Z. We chose on Z an orientation such that (s, θ) is in the oriented atlas. Furthermore, we define the bundle metric as the standard inner product on the fibers and the connection on the generators as follows:

$$\nabla_{\partial_s} f = \partial_s(f), \quad \nabla_{\partial_\theta} f = \partial_\theta(f), \quad f \in C^\infty(Z, S) \cong C^\infty(Z, \mathbb{C}).$$

Let $S \oplus S$ be the vector bundle over Z with the direct sum connection and metric. This becomes with the Clifford multiplication

$$\mathrm{cl}(\partial_s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathrm{cl}(\partial_\theta) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in C^\infty(Z, \mathrm{End}_{\mathbb{C}}(S \oplus S))$$

a graded Clifford bundle over Z. We use the local formula for the induced Dirac operator from Proposition 1.3 (2) to obtain

$$\mathcal{D}^+ = \mathrm{cl}(\partial_s)\nabla_{\partial_s} + \mathrm{cl}(\partial_\theta)\nabla_{\partial_\theta} = -\partial_s + i\partial_\theta : \underbrace{H^1(Z, S \oplus 0)}_{\cong H^1(Z,\mathbb{C})} \to \underbrace{L^2(Z, 0 \oplus S)}_{\cong L^2(Z,\mathbb{C})}. \tag{2.1}$$

Then the kernel of $\mathcal{D}^+$ takes by using of the Cauchy-Riemann differential equations the form

$$\begin{aligned} \ker\left(\mathcal{D}^+\right) &= \{f \in H^1(Z,\mathbb{C}) && |\mathcal{D}^+ f = 0\} \\ &\overset{*}{\supset} \{f \in C^\infty(Z,\mathbb{C}) && |\mathcal{D}^+ f = 0\} \\ &= \{f : [0,1] \times \mathbb{R} \to \mathbb{C} && |\text{smooth, } \partial_s f = i\partial_\theta f,\ 2\pi \text{ periodic in } \theta\} \\ &= \{f : [0,1] \times \mathbb{R} \subset \mathbb{C} \to \mathbb{C} && |\text{holomorphic, } 2\pi \text{ periodic in } \theta\}. \end{aligned} \tag{2.2}$$

It follows that the kernel of $\mathcal{D}^+$ is infinite dimensional since it contains for any $k \in \mathbb{Z}$ the function $e^{k(s-i\theta)}$.

In the previous example we have seen that for a Clifford bundle over a compact manifold with boundary the induced Dirac operator no longer has finite-dimensional kernel, as in the case without boundary. This implies that $\mathcal{D}^+ : H^1 \to L^2$ is not Fredholm and the index in Definition 1.13 is not defined anymore. Another difficulty that arises with the boundary, is that no longer every weak solution of $\mathcal{D}^+ s = 0$ is smooth [BB13, p.343], so that the domain of the chiral Dirac operator becomes important. These problems can be solved by introducing a global boundary condition (see Section 2.2) or by attaching a cylindrical end to the boundary (see Section 2.3). Both approaches need a product structure close to the boundary. What this means and what its advantage is, is covered up to the end of this section.

The collar neighborhood theorem (Proposition 2.1 (4)) justifies that we can assume a product structure of the manifold near the boundary. Inspired by this, we ask how a product structure of the Riemannian metric and of the graded Clifford bundle in a collar of the boundary simplify the form of the chiral Dirac operator. 'Product structure' should mean roughly that the Riemannian metric of the manifold, the Hermitian structure and the connection of the graded Clifford bundle do not depend on the normal coordinate t. This means that all the information of the graded Clifford bundle and the underlying manifold in a neighborhood of the boundary can be appropriately recovered by a Clifford bundle over the boundary. The next definition and Lemma 2.4 will make this precise

* The elliptic regularity theorem holds only for manifolds without boundary, so it is not given that all elements of the kernel are smooth. Therefore we have just subset here.

and Proposition 2.5 states how the product form of the Clifford bundle simplifies the chiral Dirac operator close to the boundary.

Definition 2.3. A graded Clifford bundle $S^+ \oplus S^-$ over an oriented Riemannian manifold Z with boundary Y has a *product structure near (or in a collar of) the boundary*, if there exists a neighborhood N of the boundary and a Clifford bundle $S^0 \to Y$ such that $\widetilde{S} \to \widetilde{N}$ as in Lemma 2.4 and $S \to N$ are isomorphic as graded Clifford bundles over oriented Riemannian manifolds.

Lemma 2.4. *Let S^0 be a Clifford bundle over an oriented Riemannian manifold (Y, g^Y) and $\widetilde{N} := (-1, 0] \times Y$ the oriented Riemannian manifold with metric $g = ds \otimes ds + g^Y$. Here s is the outward oriented normal coordinate on $\widetilde{N}$ and the orientation is chosen such that the induced orientation on the boundary as in Proposition* 2.1 *(3) gives back the orientation of Y. Then $\widetilde{S} := pr_2^*\left(S^0\right) \oplus \mathrm{pr}_2^*\left(S^0\right)$ defines a vector bundle over $\widetilde{N}$, where* $\mathrm{pr}_2 : (-1, 0] \times Y \to Y$ *is the projection onto the second coordinate and* $\mathrm{pr}_2^*\left(S^0\right)$ *is the pull back bundle of S^0. With the following metric, connection and Clifford multiplication it defines a graded Clifford bundle, which is called the induced graded Clifford bundle of S^0 over $\widetilde{N}$:*

- *Metric on $\widetilde{S}$: For the pullback bundle we define*

$$h_{(s,y)}^{pr_2^* S^0}(u, v) := h_y^{S^0}\left(\widetilde{\mathrm{pr}}_2(u), \widetilde{\mathrm{pr}}_2(v)\right) \quad \forall u, v \in pr_2^*(S^0)_{(s,y)},\ (s, y) \in \widetilde{N},$$

  where $\widetilde{\mathrm{pr}}_2 : \mathrm{pr}_2^*(S^0) \to S^0$ *is a lift of* pr_2 *arising in the pullback diagram. On $\widetilde{S}$ we take the metric for direct sums of vector bundles.*

- *Connection on $\widetilde{S}$: As connection we pick the pullback and the direct sum connection.*

- *Clifford multiplication: Since the tangent bundle splits $T\widetilde{N} = \langle \partial_s \rangle \oplus TY$, a Clifford action on $\widetilde{S}$ is defined via*

$$\mathrm{cl}\left(\partial_s\right) = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}, \quad \mathrm{cl}(A) = \begin{pmatrix} 0 & \mathrm{cl}^0(A) \\ \mathrm{cl}^0(A) & 0 \end{pmatrix} \in C^\infty(\widetilde{N}, \mathrm{End}_{\mathbb{C}}(\widetilde{S})) \tag{2.3}$$

  for $A \in C^\infty_{\mathrm{pr}_2}(\widetilde{N}, TY) := \{$smooth sections of TY along $\mathrm{pr}_2\}$. Here cl^0 denotes the Clifford action on the Clifford bundle S^0 and $\mathrm{cl}^0(A)$ is understood as an element of $C^\infty(\widetilde{N}, \mathrm{End}_{\mathbb{C}}(\widetilde{S}))$ via its action on the S^0 part.

Proof. The proof is a straightforward calculation checking all axioms of a graded Clifford bundle. □

Proposition 2.5. *Let S be a graded Clifford bundle over an oriented Riemannian manifold Z with boundary Y and induced Dirac operator $\mathcal{D}$. If S has a product structure near the boundary as in Definition* 2.3*, the chiral Dirac operator takes in a collar neighborhood of the boundary the form*

$$\mathcal{D}^{\pm} = \mathrm{cl}(\partial_s)\left(\partial_s \mp \mathcal{D}\left(Y\right)\right), \tag{2.4}$$

where s is the outward oriented normal coordinate, S^0 the Clifford bundle over the boundary Y and $\mathcal{D}\left(Y\right)$ its induced Dirac operator.

Proof. Let N be the neighborhood of the boundary and S^0 the Clifford bundle over the boundary as in Definition 2.3. Then $S|_N$ and $pr_2^*\left(S^0\right) \oplus \mathrm{pr}_2^*\left(S^0\right)$ are isomorphic as graded Clifford bundles over oriented Riemannian manifolds, where the right side is equipped with all additional structures as in Lemma 2.4. We denote the connection from the Clifford bundle S^0 with ∇^0 and the connection from S with ∇ and assume for the sake of simplicity $N = \widetilde{N} = (-1,0] \times Y$. Let y_0 be an arbitrary point in Y and U a neighborhood of y_0 such that $S^0|_U$ and $TY|_U$ trivializes. Choose a local orthonormal frame $e_2, \ldots, e_n$ of $TY|_U$ and a local frame $\{u_j\}$ of $S^0|_U$. Then we define for $i \in \{2, \ldots, n\}$

$$\widetilde{e}_i : (-1,0] \times U \to TN|_{(-1,0]\times U} \cong \langle \partial_s \rangle \oplus TY|_U, \quad (s,y) \mapsto 0 + e_i(y) \tag{2.5}$$

smooth sections of $TN|_{(-1,0]\times U}$ and obtain the local orthonormal frame $\partial_s, \widetilde{e}_2, \ldots, \widetilde{e}_n$ of $TN|_{(-1,0]\times U}$. Because of the identifications

$$C^{\infty}(N, S^{\pm}) \cong C^{\infty}(N, \mathrm{pr}_2^*\left(S^0\right)) \cong C^{\infty}_{\mathrm{pr}_2}(N, S^0) = \{\text{smooth sections of } S^0 \text{ along } \mathrm{pr}_2\},$$

an arbitrary section u over $S^{\pm}$ takes locally the form $u = a^j\left(u_j \circ \mathrm{pr}_2\right)$ with $a^j \in C^{\infty}((-1,0] \times U, \mathbb{R})$. Writing out the left hand side of equation (2.4) under use of the local formula for the Dirac operator given in Proposition 1.3 (2) and the definition of the pullback connection, gives in the neighborhood of the boundary $(-1,0] \times U$

$$\begin{aligned}
\mathcal{D}^{\pm} u &= \mathrm{cl}(\partial_s)\nabla_{\partial_s} u + \sum_{i=2}^{n} \mathrm{cl}(\widetilde{e}_i)\nabla_{\tilde{e}_i} u \\
&= \mathrm{cl}(\partial_s)\left(\partial_s(a^j)\left(u_j \circ \mathrm{pr}_2\right) + a^j\left(\nabla^0_{d\,\mathrm{pr}_2(\partial_s)} u_j \circ \mathrm{pr}_2\right)\right) \\
&\quad + \sum_{i=2}^{n} \mathrm{cl}(\widetilde{e}_i)\left(\widetilde{e}_i(a^j)\left(u_j \circ \mathrm{pr}_2\right) + a^j\left(\nabla^0_{d\,\mathrm{pr}_2(\tilde{e}_i)} u_j \circ \mathrm{pr}_2\right)\right).
\end{aligned} \tag{2.6}$$

Furthermore, we write out the right side of equation (2.4). Here we use the local form of the Dirac operator on the boundary $\mathcal{D}\left(S^0\right)$:

$$\begin{aligned}
\mathrm{cl}(\partial_s)\partial_s u &= \mathrm{cl}(\partial_s)\partial_s(a^j)\,(u_j \circ \mathrm{pr}_2)\\
\mathrm{cl}(\partial_s)\mathcal{D}\left(Y\right)u &= \mathrm{cl}(\partial_s)\underbrace{\mathcal{D}\left(Y\right)(a^j)}_{=\sum_{i=2}^n \mathrm{cl}^0\left(e_i\right)e_i(a^j)}(u_j \circ \mathrm{pr}_2) + \mathrm{cl}(\partial_s)a^j\,(\underbrace{\mathcal{D}\left(Y\right)u_j}_{=\sum_{i=2}^n \mathrm{cl}^0\left(e_i\right)\nabla^0_{e_i}u_j} \circ \mathrm{pr}_2).
\end{aligned} \tag{2.7}$$

The vector field e_i acts on a^j by holding s as a parameter and seeing a^j as element of $C^\infty(U,\mathbb{R})$. The relations

$$\begin{aligned}
&d\,\mathrm{pr}_2(\partial_s) = 0, && d\,\mathrm{pr}_2(\widetilde{e}_i) = e_i &&\\
&\mathrm{cl}(\partial_s) = \mp\,\mathrm{id}, && \mathrm{cl}\left(\widetilde{e}_i\right) = \mathrm{cl}^0\left(\widetilde{e}_i\right) = \mathrm{cl}^0\left(e_i\right), && \text{see equation (2.3)}\\
&e_i(a^j) = \widetilde{e}_i(a^j), && && \text{definition of } \widetilde{e}_i \text{ in equation (2.5)}
\end{aligned}$$

hold for all i, j and the claimed formula for the chiral Dirac operator follows by comparing equation (2.6) with equation (2.7). □

Example 2.6. The Clifford bundle $S \oplus S$ over $[0,1] \times S^1$ from Example 2.2 has a product structure near the boundary. We can check both connected components of the boundary separately. The Clifford bundle over the collar $(0,1] \times S^1$ of the right boundary $Y_r := \{1\} \times S^1$ is induced (as in Lemma 2.4) by the Clifford bundle $S_r^0 := \mathbb{C} \times S^1$ over Y_r with the Clifford action $\mathrm{cl}(\partial_\theta) = i$. Using Proposition 2.5 to calculate the chiral Dirac operator in the collar $(0,1] \times S^1$ gives

$$\mathcal{D}^+ = \mathrm{cl}(\partial_s)\left(\partial_s - \mathcal{D}\left(Y_r\right)\right) = -\left(\partial_s - i\partial_\theta\right),$$

which agrees with the formula in equation (2.1). The analogue Clifford bundle S_l^0 over the left boundary $Y_l := \{0\} \times S^1$ with $\mathrm{cl}(\partial_\theta) = -i$ leads to a Clifford bundle $\widetilde{S} \to [0,1) \times S^1$ with Clifford multiplication (see Lemma 2.4) given by

$$\mathrm{cl}\left(\partial_{-s}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathrm{cl}(\partial_\theta) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \in C^\infty\big([0,1) \times S^1, \mathrm{End}_{\mathbb{C}}(\widetilde{S})\big). \tag{2.8}$$

This is isomorphic to the original Clifford bundle by multiplication with -1.

2.2 The APS index theorem with boundary condition

This section is based on Bleecker and Booß-Bavnbek [BB13, p.343 ff.] and aims to give a brief introduction into boundary conditions and the traditional formulation of the APS index theorem [APS75].

Let S be a graded Clifford bundle over a compact oriented Riemannian manifold Z with boundary Y with product structure near the boundary and induced Dirac operator $\mathcal{D}$. As we have seen in Example 2.2 and beyond we are faced with two problems when generalizing the Atiyah-Singer index theorem (Theorem 1.16) to manifolds with boundary:

- The chiral Dirac operator $\mathcal{D}^+ : H^1(Z, S^+) \to L^2(Z, S^-)$ is not Fredholm anymore, hence the index of $\mathcal{D}$ as in Definition 1.13 is no longer well defined.
- The kernel no longer contains only smooth sections.

We will solve both problems by restricting the domain of the chiral Dirac operator in a suitable way. One canonical candidate can be obtained by restricting the domain of $\mathcal{D}^+$ to the space of smooth sections with support in the interior denoted by $C^\infty_{00}(Z, S^+)$ and take its closure in $H^1(Z, S^+)$. This restriction would make the kernel of the operator finite dimensional. Anyway, the operator with this domain would not be Fredholm since it has infinite dimensional cokernel. But there are many other possible domains for the chiral Dirac operator which solve the problems above. We call all these 'reasonable' extensions elliptic boundary problems:

Definition 2.7. Let $\mathcal{D}$ be the induced Dirac operator of a graded Clifford bundle $S = S^+ \oplus S^-$ over a compact oriented Riemannian manifold Z with boundary Y and product structure in a collar of the boundary. We call a closed extension* $\mathcal{D}^+_R : R \subset L^2(Z, S^+) \to L^2(Z, S^-)$ of $\mathcal{D}^+$ an *elliptic boundary problem* if it satisfies

(1) $\mathcal{D}^+_R : R \to L^2(Z, S^-)$ is Fredholm as defined in Definition 1.9.

(2) $\ker\left(\mathcal{D}^+_R\right)$ is a subset of the space of smooth sections and $\operatorname{coker}\left(\mathcal{D}^+_R\right)$ can be represented as a subspace of the smooth sections.

In the special case of an empty boundary the conditions in Definition 2.7 leads inevitably to $R = H^1(Z, S^+)$. For a non-empty boundary we can generate a whole class

* Closed means that the graph of $\mathcal{D}^+_R$ is closed in $L^2(Z, S^+) \times L^2(Z, S^-)$. The word extension is a bit misleading in this context because R does not have to contain all smooth sections, hence it is not an extension of $\mathcal{D}^+ : C^\infty(Z, S^+) \to C^\infty(Z, S^-)$.

of candidates for elliptic boundary problems by picking a pseudo-differential operator*
$T : L^2(Y, S^+|_Y) \to L^2(Y, S^+|_Y)$ of order zero and defining

$$R_T \coloneqq \{u \in H^1(Z, S^+)|u|_Y \in \ker(T)\} \tag{2.9}$$

as the domain of the chiral Dirac operator $\mathcal{D}^+$. Here $u|_Y$ means to apply the bounded restriction map $\gamma_0 : H^1(Z, S^+) \to L^2(Y, S^+|_Y)$ defined on smooth sections via $\gamma_0(u)(y) \coloneqq u|_Y(y) \coloneqq u(0, y)$. This extends continuously to the Sobolev space $H^1(Z, S^+)$. If the domain R_T in equation (2.9) defines an elliptic boundary problem as in Definition 2.7, we call T an *elliptic boundary condition* for the operator $\mathcal{D}^+$ and write instead of $D^+_{R_T}$ short D^+_T. How the explicit requirements on T looks like can be found in [BB13, Definition 13.15 ff.].

Example 2.8. Let $\mathcal{D}$ be the induced Dirac operator of a graded Clifford bundle S over a compact oriented Riemannian manifold Z with boundary Y and product structure in a collar of the boundary. Furthermore, S^0 is the Clifford bundle on the boundary and $\mathcal{D}(Y)$ its induced Dirac operator. Since the manifold Z is compact, so is its boundary Y and Proposition 1.5 (3) states that there exists a discrete set of real eigenvalues and a complete system of $L^2(Y, S^0)$ orthonormal eigensections. Then we define the *spectral (Atiyah-Patodi-Singer) projection* as

$$P_{\geq}(\mathcal{D}(Y)) : L^2(Y, S^0) \to L^2_{\geq}(\mathcal{D}(Y)) \subset L^2(Y, S^0),$$

where $L^2_{\geq}(\mathcal{D}(Y))$ is the space spanned by the eigensections corresponding to the non-negative eigenvalues of $\mathcal{D}(Y)$. This projection $P_{\geq}(\mathcal{D}(Y))$ defines a pseudo-differential operator of order zero and its induced domain for the chiral Dirac operator $\mathcal{D}^+$ defines an elliptic boundary problem

$$\mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))} : \left\{u \in H^1(Z, S^+)|u|_Y \in \ker\left(P_{\geq}(\mathcal{D}(Y))\right)\right\} \to L^2(Z, S^-).$$

For a detailed treatment and a proof of that statement see [BW93, chapter 14].

In Example 2.2 we have constructed infinitely many linearly independent functions contained in the kernel of an explicit given Dirac operator. The following example should give an idea how this spectral projection from the previous Example 2.8 reduces

* For the progress of this thesis a definition of a pseudo-differential operator is not significant so that we omit it here. A formal definition can be found in [BB13, Chapter 8]. Later on it is just important that the spectral projection defined in the next Example 2.8 is of that type.

the list of candidates contained in the kernel of the chiral Dirac operator:

Example 2.9. Let us go back to the situation in Examples 2.2 and 2.6. The kernel of the Dirac operator $D^+ = -\partial_s + i\partial_\theta$ contains all holomorphic functions $f : [0,1] \times \mathbb{R} \subset \mathbb{C} \to \mathbb{C}$ being 2π periodic in the second coordinate, hence the infinitely many linearly independent functions $f_k(t,\theta) = e^{k(s-i\theta)}$ for $k \in \mathbb{Z}$ are elements of the kernel. The underlying manifold $Z = [0,1]\times S^1$ has the two boundaries Y_l and Y_r with Dirac operators $\mathcal{D}(Y_l) = -i\partial_\theta$ and $\mathcal{D}(Y_r) = i\partial_\theta$. Now we want to apply the elliptic boundary condition associated to the spectral projection from Example 2.8 to that problem and will see how this makes the kernel finite-dimensional. The boundary conditions force elements of the kernel to be zero on both boundaries after the spectral projection. For all $k \in \mathbb{Z}$ and all $\theta \in \mathbb{R}$ we have

$$\mathcal{D}(Y_l) f_k(0,\theta) = -k f_k(0,\theta), \quad \mathcal{D}(Y_r) f_k(1,\theta) = k f_k(1,\theta). \tag{2.10}$$

This implies that for any k the function f_k is not in the kernel of $\mathcal{D}^+_{P_\geq(\mathcal{D}(Y))}$ because at least one of the projections would not vanish. Furthermore, one can show that under these boundary condition all elements of the kernel are smooth such that we have with the previous consideration

$$\begin{aligned} \ker\left(\mathcal{D}^+_{P_\geq(\mathcal{D}(Y))}\right) &\overset{\text{eq.(2.2)}}{=} \Big\{ f : [0,1] \times \mathbb{R} \subset \mathbb{C} \to \mathbb{C} \;\Big|\; \partial_s f = i\partial_\theta f,\ 2\pi \text{ periodic in } \theta,\ \text{smooth} \\ &\qquad\qquad f|_{Y_j} \in \ker\left(P_\geq(\mathcal{D}(Y_j))\right)\ j = l,r \Big\} \\ &= \{0\}. \end{aligned}$$

Here we used that every smooth function $f : [0,1] \times \mathbb{R} \subset \mathbb{C} \to \mathbb{C}$ which is 2π periodic in the second coordinate and satisfies $\partial_s f = i\partial_\theta f$ can be written as an infinite linear combination of the f_k. This can be proved by writing f in a Fourier series $\sum_{k\in\mathbb{Z}} a_k(s) e^{-ik\theta}$ and solve the differential equation $\partial_s f = i\partial_\theta f$. This leads to $a_k(s) = A_k e^{ks}$ and f is expressed as an infinite sum of the f_k.

After these preliminaries we can state the APS index theorem. It was first stated by Atiyah, Patodi, and Singer [APS75, Theorem 3.10]. Another approach can be found in [BW93, Theorem 22.18]. Note that we use the sign conventions by Mrowka, Ruberman, and Saveliev [MRS16], hence the normal coordinate s of the collar of the boundary is outward oriented. This has the consequence that the local form of the chiral Dirac operator has a different sign in front of the boundary Dirac operator. Nerveless, the index formula remains unchanged.

Theorem 2.10 (APS index theorem via boundary condition)**.** *Let Z be a compact even-dimensional oriented Riemannian manifold with boundary Y and $S \to Z$ a graded Clifford bundle with induced Dirac operator $\mathcal{D}$ and chiral part $\mathcal{D}^+$. Assume S has a product structure near the boundary as in Definition* 2.3 *and let S^0 be the Clifford bundle over the boundary with induced boundary Dirac operator $\mathcal{D}(Y)$. With the spectral projection $P_{\geq}(\mathcal{D}(Y))$ from Example* 2.8 *as boundary condition the Fredholm index of $\mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))}$ is given by*

$$\mathrm{ind}_{Fred}\left(\mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))}\right) = \int_Z \boldsymbol{I}(\mathcal{D}(\widetilde{Z})) - \frac{1}{2}\Big(\eta(\mathcal{D}(Y)) + \dim\ker(\mathcal{D}(Y))\Big), \tag{2.11}$$

where $\boldsymbol{I}(\mathcal{D}(\widetilde{Z}))$ denotes the local index form as defined in Definition 1.13 *of the Dirac operator $\mathcal{D}(\widetilde{Z})$ over the double manifold $\widetilde{Z}$. The η-invariant is defined as*

$$\eta(\mathcal{D}(Y)) := \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\,\mathrm{Tr}\left(\mathcal{D}(Y)e^{-t\mathcal{D}(Y)^2}\right)dt. \tag{2.12}$$

Remark 2.11. (The η-invariant as a spectral invariant [see Loy04, p.144])
The APS index formula in equation (2.11) is made of two terms. The first term is the same as in the Atiyah-Singer index theorem (Theorem 1.16), consisting of the local index form $\mathbf{I}(\mathcal{D}(\widetilde{Z}))$. The second term depends on the boundary Dirac operator $\mathcal{D}(Y)$ and contains the η-invariant defined in equation (2.12). It turns out that this is not a local object, but a spectral invariant that can be thought of as a measurement of the spectral asymmetry of the boundary Dirac operator:

$$\text{"}\eta(\mathcal{D}(Y)) = \#\{\text{positive eigenvalues}\} - \#\{\text{negative eigenvalues}\}\text{"}.$$

To give this a mathematically precise sense and see the relation to the definition given in equation (2.12), we define the η-function for $z \in \mathbb{C}$ with sufficiently great real part as a sum over the nonzero eigenvalues of the boundary Dirac operator

$$\eta(z) = \sum_{\lambda\neq 0} \frac{\mathrm{sign}\lambda}{|\lambda|^z}.$$

Note that the compactness of Y implies by Proposition 1.5 that the spectrum of $\mathcal{D}(Y)$ is a discrete subset of $\mathbb{R}$. Furthermore, one can show an integral expression of the

η-function

$$\eta(z) = \frac{1}{\Gamma\left(\frac{z+1}{2}\right)} \int_0^\infty t^{\frac{z-1}{2}} \operatorname{Tr}\left(\mathcal{D}(Y) e^{-t\mathcal{D}(Y)^2}\right) dt, \tag{2.13}$$

where Γ denotes the Gamma function. This can be shown by some basic integration and the fact that $\mathcal{D}(Y)e^{-t\mathcal{D}(Y)^2}$ is a self-adjoint trace class operator, hence its trace is the sum over all its eigenvalues. One can show that the right hand side of equation (2.13) is a meromorphic function on $\mathbb{C}$ with regular value at $z = 0$ so we can use it to extend the η-function to a meromorphic function on $\mathbb{C}$. Then we obtain by comparing equation (2.13) with the definition of the η-invariant in equation (2.12):

$$\eta(\mathcal{D}(Y)) = \eta(0) = \text{"}\sum_{\lambda \neq 0} \operatorname{sign}\lambda\text{"}.$$

Proof (sketch of Theorem 2.10*).* The following proof is oriented towards the original proof by Atiyah, Patodi, and Singer [APS75]. Some more details can be found in [BW93, Chapter 22]. The first step is to show that the McKean-Singer formula, which is crucial in the proof of the Atiyah-Singer index theorem (Theorem 1.16), still holds [BW93, Lemma 22.2]:

$$\operatorname{ind}_{\text{Fred}}\left(\mathcal{D}^+_{\geq}\right) = \operatorname{Tr}\left(e^{-t\left(\mathcal{D}^+_{\geq}\right)^*\mathcal{D}^+_{\geq}}\right) - \operatorname{Tr}\left(e^{-t\mathcal{D}^+_{\geq}\left(\mathcal{D}^+_{\geq}\right)^*}\right).$$

Here the adjoint is meant in the L^2-sense and we used the shorthand notation $\mathcal{D}^+_{\geq} := \mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))}$. Then the proof splits into two steps:

(1) We equip the manifold $(-\infty, 0] \times Y$ with the structure of a graded Clifford bundle as explained in Lemma 2.4, denote the induced chiral Dirac operator with $\mathcal{D}^+_c$ and construct the heat kernel of $e^{-t(\mathcal{D}^+_c)^*\mathcal{D}^+_c}$ and $e^{-t\mathcal{D}^+_c(\mathcal{D}^+_c)^*}$. Moreover, we denote the integral over $(-\infty, 0] \times Y$ of the trace difference of these heat kernels with $K(t)$. If $K(t)$ has an asymptotic expansion with constant coefficient a_0, the η-invariant appears naturally and the following holds [APS75, Equation (2.27)]:

$$\eta(\mathcal{D}(Y)) = -(2a_0 + h), \quad \text{where } h := \dim\ker(\mathcal{D}(Y)). \tag{2.14}$$

(2) By doubling the manifold Z we obtain a Clifford bundle over the compact manifold $\widetilde{Z}$ with induced chiral Dirac operator $\mathcal{D}^+(\widetilde{Z})$. The double manifold exists because Z has product structure close to the boundary. The heat kernels of the operators

$e^{-t(\mathcal{D}^+(\widetilde{Z}))^*\mathcal{D}^+(\widetilde{Z})}$ and $e^{-t\mathcal{D}^+(\widetilde{Z})(\mathcal{D}^+(\widetilde{Z}))^*}$ are used to built by Duhamel's principle a fundamental solution for the operator $\partial_t + (\mathcal{D}^+_{\geq})^*\mathcal{D}^+_{\geq}$. This leads to an asymptotic expansion of $K(t)$ with constant term $\mathrm{ind}_{\mathrm{Fred}}\,(\mathcal{D}^+_{\geq}) - \int_Z \mathbf{I}(\mathcal{D}^+(\widetilde{Z}))$ and the index theorem follows by inserting this in equation (2.14). $\square$

2.3 The APS index theorem with an attached cylindrical end

On the way to extend the Atiyah-Singer index theorem (Theorem 1.16) to compact manifolds with boundary we are faced with the two problems discussed in the beginning of the last Section 2.2: The Dirac operator is not Fredholm anymore and we lose the elliptic regularity, hence the kernel contains not only smooth sections. If we assume a product structure near the boundary, we can try to solve this problems by attaching a cylindrical end to the boundary. Then we obtain a non-compact manifold without boundary and the second problem is solved since the elliptic regularity theorem is applicable. The following example will give an idea how the attached cylinder can solve the problem with the infinite-dimensional kernel.

Example 2.12. Lets take the situation from the Examples 2.2 and 2.6 and attach a cylinder to both boundaries. This leads to the manifold $\widehat{Z} = (-\infty, \infty) \times S^1$. We extend all structures naturally to $\widehat{Z}$ such that we end up with the same chiral Dirac operator $\mathcal{D}^+(\widehat{Z}) = -\partial_s + i\partial_\theta$. As a consequence of Liouvilles theorem the kernel of $\mathcal{D}^+(\widehat{Z}) : H^1(\widehat{Z}, S^+) \to L^2(\widehat{Z}, S^-)$ vanishes:

Proof. Let $f \in \ker(\mathcal{D}^+(\widehat{Z}))$. Using the elliptic regularity theorem and a similar argumentation as for equation (2.2) gives that f can be seen as a holomorphic function $\mathbb{R} \times \mathbb{R} \subset \mathbb{C} \to \mathbb{C}$ being 2π periodic in the second coordinate θ, hence bounded with respect to θ. Furthermore, f is in $H^1(\widehat{Z}, \mathbb{C})$ so that by the Sobolev embedding theorem f is in $L^\infty(\widehat{Z}, \mathbb{C})$ and together with the smoothness it is bounded in s. All together f is constant by Liouville's theorem, hence vanishing because it is in $H^1(\widehat{Z}, \mathbb{C})$. $\square$

With the previous example as motivation, we will work this out in general and examine under what condition the chiral Dirac operator is Fredholm. The following procedure is based on [Loy04, Chapter 3]. Note that we use for compatibility of Chapter 2 with Chapter 3 the sign conventions of Mrowka, Ruberman, and Saveliev [MRS16], which differs in several points from Loya's sign conventions. The normal coordinate of the collar of the boundary is outward oriented. This has the consequences that the attached cylinder goes to $+\infty$ and the Dirac operator in the collar of the boundary has a different

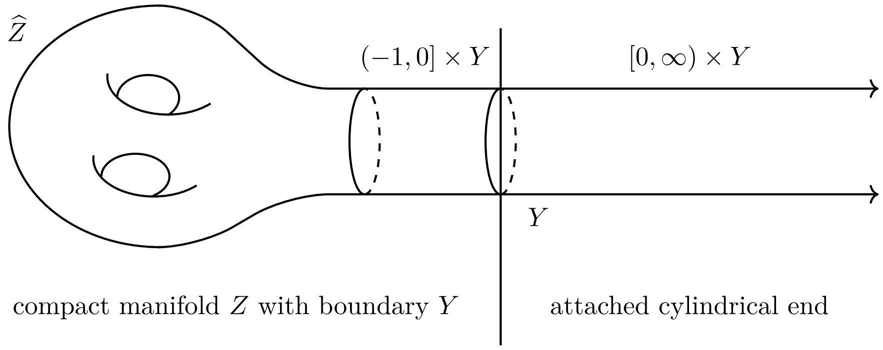

Figure 2.1: The figure illustrates how the manifold $\widehat{Z}$ results from the compact manifold Z with boundary Y by attaching a cylindrical end to the boundary.

sign in front of the boundary Dirac operator*. Furthermore, we will define the weighted Sobolev spaces in Definition 2.18 with a minus sign in the exponent so that the $+$-index defined in Theorem 2.21 will correspond to the index of the classical APS index theorem in Theorem 2.10.

Construction 2.13. Let S be a graded Clifford bundle over a compact oriented Riemannian manifold Z with boundary Y and product structure near the boundary as defined in Definition 2.3. Furthermore, we equip the manifold $[0,\infty) \times Y$ with all additional structures as $\widetilde{N}$ in Lemma 2.4 with the difference that s is the inward oriented normal coordinate and with the orientation induced by the inward pointed vector field ∂_s. Then we glue the two manifolds together and obtain the non-compact oriented Riemannian manifold

$$\widehat{Z} \coloneqq Z \cup_Y \big([0,\infty) \times Y\big)$$

pictured in Figure 2.1 and a graded Clifford bundle $S \to \widehat{Z}$.

Definition 2.14. A graded Clifford bundle $S \to \widehat{Z}$ is called *graded Clifford bundle with cylindrical end*† if it develops by attaching a cylindrical end to a graded Clifford bundle $S \to Z$ as in Construction 2.13. One says it is modeled by (Z, Y).

One can show that the enlargement of the compact manifold Z with boundary to the non-compact manifold $\widehat{Z}$ tames the infinite kernel of $\mathcal{D}^+ : H^1(Z, S^+) \to L^2(Z, S^-)$ as we

* See the form of the Dirac operator in the collar of the boundary from Proposition 2.5 and compare it with $\Gamma(\partial_s + \mathcal{D}_Y)$ [Loy04, p.141]. † The part “cylindrical end” refers to the underlying oriented Riemannian manifold $\widehat{Z}$ and the structure of the Clifford bundle S. In particular, it means that the underlying manifold has an orientation and a Riemannian metric.

have see in Example 2.12. Nevertheless, the Fredholm property is not always fulfilled. The following proposition provides a criterion for when the Dirac operator is Fredholm. A proof of that criterion can be found in [Mel93, Theorem 5.60]. We will prove this later in Section 3.3.5 by leading it back to the analogue statement in the end-periodic setting given in Proposition 3.19.

Proposition 2.15 ([Loy04, Theorem 3.2])**.** *The induced chiral Dirac operator $\mathcal{D}^+(\widehat{Z}) : H^1(\widehat{Z}, S^+) \to L^2(\widehat{Z}, S^-)$ of a graded Clifford bundle $S \to \widehat{Z}$ with cylindrical end is Fredholm if and only if the boundary Dirac operator $\mathcal{D}(Y)$ is invertible, that is if the kernel is trivial.*

Theorem 2.16 (APS index theorem via attached cylinder in the Fredholm case)**.** *Let $S \to \widehat{Z}$ be a graded Clifford bundle with cylindrical end modeled by (Z, Y) as in Definition* 2.14 *and assume that Z has even dimension. If the boundary Dirac operator $\mathcal{D}(Y)$ is invertible, the index of the induced Dirac operator $\mathcal{D}(\widehat{Z})$ is given by*

$$\operatorname{ind}(\mathcal{D}(\widehat{Z})) = \int_Z \boldsymbol{I}(\mathcal{D}(\widehat{Z})) - \frac{1}{2}\eta(\mathcal{D}(Y)), \tag{2.15}$$

where $\boldsymbol{I}(\mathcal{D}(\widehat{Z}))$ denotes the local index form defined in Definition 1.13 *and $\eta(\mathcal{D}(Y))$ the η-invariant as defined in equation (2.12).*

Proof (sketch). The proof sketched here is due to Melrose [Mel93] and the presented version is based on Loya [Loy04, Chapter 4]. The idea is to translate the theorem into the Melrose's b-geometry setting and imitate the proof of the Atiyah-Singer index theorem (Theorem 1.16) as closely as possible. The impressive feature about this proof is that it does not use any boundary conditions and therefore provides a template for the proof later of the end-periodic index theorem (Theorem 3.22).

Let us take any notation and assumptions as in the theorem. For manifolds with a cylindrical end, the definition of the algebra of pseudo-differential operators is somewhat unbalanced, since the compact end and the cylindrical end must be treated separately. Melrose had the novel idea to unify this by introducing the so-called b-geometry. We compactify the manifold $\widehat{Z}$ to obtain* $\widehat{X}$ by changing the variable s on the cylindrical end $[0, \infty) \times Y$ and the collar $(-1, 0] \times Y$ to $x = -e^{-s}$ as pictured in Figure 2.2. The graded Clifford bundle over $\widehat{Z}$ induces a graded Clifford bundle over $\widehat{X}$ with induced

* This space is called $\widehat{X}$ because it is topologically compact and inherits the manifold $\widehat{Z}$ as its interior.

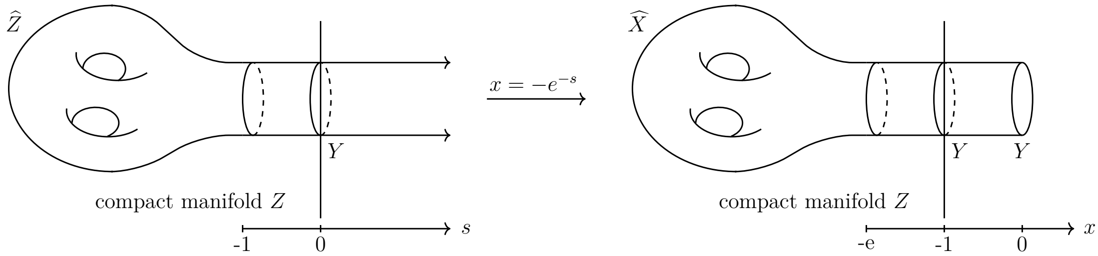

Figure 2.2: The right hand side pictures the compactification of the manifold with cylindrical end pictured on the left hand side. Under the coordinate change $s \mapsto x = -e^{-s}$ the end at of the manifold $\widehat{Z}$ is fetched to $x = 0$.

chiral Dirac operators $\mathcal{D}^{\pm}(\widehat{X})$. We obtain the following b-objects on the manifold $\widehat{X}$ (the b-integral and the b-trace will be explained later on in more detail):

s	$\rightsquigarrow$	$x = -e^{-s}$	
$\widehat{Z} := Z \cup_Y \big([0,\infty)_s \times Y\big)$	$\rightsquigarrow$	$\widehat{X} = Z \cup_Y \big([-1,0]_x \times Y\big)$	
$ds,\ \partial_s$	$\rightsquigarrow$	$-\frac{dx}{x},\ -x\partial_x$	
$g = ds^2 + g^Y$	$\rightsquigarrow$	$g^b = \left(\frac{dx}{x}\right)^2 + g^Y$	b-metric
$\mathrm{dvol}_g = ds\mathrm{dvol}_{g^Y}$	$\rightsquigarrow$	$\mathrm{dvol}_g^b = -\frac{dx}{x}\mathrm{dvol}_{g^Y}$	b-measure
$H^k(\widehat{Z})$	$\rightsquigarrow$	$H_b^k(\widehat{Z})$	b-Sobolev space
pseudodifferential operators	$\rightsquigarrow$	b-pseudofifferential opertors	
$\int_{\widehat{X}} f\mathrm{dvol}_g$	$\rightsquigarrow$	${}^b\int_{\widehat{X}} f\mathrm{dvol}_g^b$	b-integral
$\mathrm{Tr}(e^{-t\mathcal{D}^{\mp}(\widehat{Z})\mathcal{D}^{\pm}(\widehat{Z})})$	$\rightsquigarrow$	$\mathrm{Tr}^b(e^{-t\mathcal{D}^{\mp}(\widehat{X})\mathcal{D}^{\pm}(\widehat{X})})$	b-trace

One can show that the Fredholmness of $\mathcal{D}^+(\widehat{Z})$ implies that the chiral Dirac operator $\mathcal{D}^+(\widehat{X}) : H_b^1(\widehat{X}, S^+) \to L_b^2(\widehat{X}, S^-)$ is Fredholm with the same Fredholm index as of $\mathcal{D}^+(\widehat{Z})$. It remains to show:

$$\mathrm{ind}_{\mathrm{Fred}}(\mathcal{D}^+(\widehat{X})) = \int_Z \mathbf{I}(\mathcal{D}(\widehat{Z})) - \frac{1}{2}\eta(\mathcal{D}(Y)).$$

Let $k_t^{\pm}(p,p)$ and $k_t(y,y)$ be the smoothing kernels of $e^{-t\mathcal{D}^{\mp}(\widehat{X})\mathcal{D}^{\pm}(\widehat{X})}$ and $e^{-t\mathcal{D}(Y)^2}$. We would like to define the McKean-Singer function $h(t)$ similarly as in the proof of the AS index theorem. But the formulas

$$\operatorname{Tr}\left(e^{-t\mathcal{D}^{\mp}(\widehat{X})\mathcal{D}^{\pm}(\widehat{X})}\right) = \int_{\widehat{X}} \operatorname{tr}\left(k_t^{\pm}(p,p)\right)\mathrm{dvol}_g^b(p) \tag{2.16}$$

do not make sense anymore because the integral on the right hand side diverges. This follows directly by the two equations

$$\int_{[-1,0]\times Y} \operatorname{tr}\left(k_t(y,y)\right)\mathrm{dvol}_g^b(x,y) = \underbrace{\int_0^{-1}\frac{dx}{x}}_{=\infty}\int_Y \operatorname{tr}\left(k_t(y,y)\right)\mathrm{dvol}_{g^Y}(y) \text{ and}$$

$$\operatorname{tr}\left(k_t^{\pm}(x,y,x,y)\right) = \frac{1}{\sqrt{4\pi t}}\operatorname{tr}\left(k_t(y,y)\right) + O(x) \qquad \text{[Loy04, p.150]}.$$

In the second equation $O(x)$ is a smooth function on x vanishing at $x = 0$. To avoid the problem with the divergent integral we define the so-called b-integral and replace the integral in equation (2.16): One can show that any smooth function $f \in C^\infty(\widehat{X},\mathbb{R})$ induces a meromorphic function F on $\mathbb{C}$ with simple poles in $\{0,-1,-2,\ldots\}$ satisfying

$$F(z) = \int_{\widehat{X}} x^z f \mathrm{dvol}_g^b, \quad \forall z \in \mathbb{C} \text{ with } \operatorname{Re}(z) > 0.$$

We define the b-integral* over f as the constant part of the Laurent series of F around $z = 0$ and denote it with ${}^b\!\int f\mathrm{dvol}_g^b$. Using this regularization of the integral, we define the b-trace and the McKean-Singer function as follows:

$$\begin{aligned}
&\operatorname{Tr}^b\left(e^{-t\mathcal{D}^{\mp}(\widehat{X})\mathcal{D}^{\pm}(\widehat{X})}\right) := {}^b\!\int_{\widehat{X}} \operatorname{tr}\left(k_t^{\pm}(p,p)\right)\mathrm{dvol}_g^b(p)\\
&h^b(t) := \operatorname{Tr}^b\left(e^{-t\mathcal{D}^{-}(\widehat{X})\mathcal{D}^{+}(\widehat{X})}\right) - \operatorname{Tr}^b\left(e^{-t\mathcal{D}^{+}(\widehat{X})\mathcal{D}^{-}(\widehat{X})}\right).
\end{aligned}$$

One can show the same properties as for h in the proof of the Atiyah-Singer index theorem (Theorem 1.16), with the difference that we replace h by h^b, Tr by Tr^b and $h^b(t)$ is not constant in t anymore. This is the point at which the additional term in the index formula enters:

$$\begin{aligned}
&(1) \quad \lim_{t\to\infty} h^b(t) = \operatorname{ind}_{\mathrm{Fred}}(\mathcal{D}^+(\widehat{X})).\\
&(2) \quad \lim_{t\to 0} h^b(t) = \int_Z \mathbf{I}(\mathcal{D}(\widehat{Z})).\\
&(3) \quad \frac{d}{dt}h^b(t) = \operatorname{Tr}^b\left([\mathcal{D}^+, \mathcal{D}^- e^{-t\mathcal{D}^+\mathcal{D}^-}]\right), \quad \forall t > 0.
\end{aligned} \tag{2.17}$$

* A straightforward calculation shows: if f is zero at the boundary, the b-integral agrees with the usual integral.

A straightforward calculation using the definition of the b-integral [see Loy04, section 4.4] gives that part (3) is equal to $-\frac{1}{\sqrt{4\pi t}}\operatorname{Tr}(\mathcal{D}(Y)^2)$ and the index theorem follows:

$$\operatorname{ind}_{\mathrm{Fred}}(\mathcal{D}^+(\widehat{X}))-\int_Z \mathbf{I}(\mathcal{D}(\widehat{Z})) \overset{\text{eq.(2.17)}}{=} \int_0^\infty \frac{-1}{\sqrt{4\pi t}}\operatorname{Tr}\left(\mathcal{D}(Y)e^{-t\mathcal{D}(Y)^2}\right)dt \overset{\text{eq.(2.12)}}{=} -\frac{1}{2}\eta(\mathcal{D}(Y)).\square$$

Remark 2.17. Let S, Z, $\widehat{Z}$, and $\mathcal{D}(Y)$ be as in Theorem 2.16 and assume that $\mathcal{D}(Y)$ is invertible. Moreover, denote with $\mathcal{D}^+(Z)$ and $\mathcal{D}^+(\widehat{Z})$ the chiral Dirac operators induced by the Clifford bundle over Z respectively $\widehat{Z}$. Theorem 2.10 (APS theorem via boundary condition) and Theorem 2.16 (APS theorem via attached cylinder) give formulas for the Fredholm indices of

$$\begin{aligned} &\mathcal{D}^+_{P_\geq(\mathcal{D}(Y))}: && \left\{u\in H^1(Z,S^+)|u|_Y\in\ker\left(P_\geq\left(\mathcal{D}(Y)\right)\right)\right\}\to L^2(Z,S^-), \quad \text{and}\\ &\mathcal{D}^+(\widehat{Z}): && H^1(\widehat{Z},S^+)\to L^2(\widehat{Z},S^-),\end{aligned} \tag{2.18}$$

respectively. The local index form of the cylindrical manifold $\mathbf{I}(\mathcal{D}(\widehat{Z}))$ and of the double manifold $\mathbf{I}(\mathcal{D}(\widetilde{Z}))$ are the same on the interior of Z, since by Proposition 1.6 (2) the coefficients $\theta_{n/2}$ are local objects. This implies that the right hand sides of equation (2.11) and equation (2.15) coincides and both indices are equal. This can be seen directly by studying the kernel and cokernel of the operators in equation (2.18) [APS75, Proposition 3.11], hence we can prove Theorem 2.10 in the Fredholm case via Theorem 2.16 and vice versa. It holds

$$\begin{aligned} &(1)\quad \ker\left(\mathcal{D}^+_{P_\geq(\mathcal{D}(Y))}\right)\cong\ker\left(\mathcal{D}^+(\widehat{Z})\right)\\ &(2)\quad \operatorname{coker}\left(\mathcal{D}^+_{P_\geq(\mathcal{D}(Y))}\right)\cong\ker\left(\mathcal{D}^-_{1-P_\geq(\mathcal{D}(Y))}\right)\cong\ker\left(\mathcal{D}^-(\widehat{Z})\right)\cong\operatorname{coker}\left(\mathcal{D}^+(\widehat{Z})\right)\end{aligned} \tag{2.19}$$

with the operators

$$\begin{aligned} &\mathcal{D}^-_{1-P_\geq(\mathcal{D}(Y))}:\left\{u\in H^1(Z,S^-)|u|_Y\in\ker\left(1-P_\geq\left(\mathcal{D}(Y)\right)\right)\right\}\to L^2(Z,S^+),\\ &\mathcal{D}^-(\widehat{Z}):H^1(\widehat{Z},S^-)\to L^2(\widehat{Z},S^+).\end{aligned} \tag{2.20}$$

Proof. **(1)** Let u be an arbitrary element of $\ker\left(\mathcal{D}^+_{P_\geq(\mathcal{D}(Y))}\right)$. Note that u is smooth since $P_\geq(\mathcal{D}(Y))$ is an elliptic boundary condition as treated in Example 2.8. By Proposition 1.5 and the closedness of Y there exists an orthonormal basis of eigensections $\{\phi_\lambda\}$ of the boundary Dirac operator $\mathcal{D}(Y)$. Then the product structure close to the boundary as in Definition 2.3 implies that we can write the section u close to the boundary in the form

$$u(s,y) = \sum_{\lambda} u_\lambda(s)\phi_\lambda(y) \tag{2.21}$$

for u_λ a smooth function of the coordinate s. The sum goes over all eigenvalues of $\mathcal{D}(Y)$. Then we can take advantage of the boundary condition and the fact that u is in the kernel of $\mathcal{D}^+(Z) = \mathrm{cl}(\partial_s)(\partial_s - \mathcal{D}(Y))$ (Proposition 2.5) and obtain

$$\begin{aligned}
0 &\overset{!}{=} P_{\geq}(\mathcal{D}(Y))u(0,y) = \sum_{\lambda\geq 0} u_\lambda(0)\phi_\lambda(y) && \Rightarrow \quad u_\lambda(0) = 0,\ \forall\lambda \geq 0\\
0 &\overset{!}{=} \mathrm{cl}(\partial_s)(\partial_s - \mathcal{D}(Y))u(s,y) = \mathrm{cl}(\partial_s)\sum_{\lambda} \big(u'_\lambda(s) - \lambda u_\lambda(s)\big)\phi_\lambda(y) && \Rightarrow \quad u_\lambda(s) = u_\lambda(0)e^{\lambda s}\ \forall\lambda.
\end{aligned}$$

This implies that u takes close to the boundary the form $\sum_{\lambda<0} u_\lambda(0)e^{\lambda s}\phi_\lambda(y)$ and we can extend it to a smooth section $\widehat{u}$ over $\widehat{Z}$. By the part $e^{\lambda s}$ and its behavior for $s \to \infty$ this extension is also an element of $H^1(\widehat{Z}, S^+)$, hence is in the kernel of $\mathcal{D}^+(\widehat{Z})$. Conversely, an element $\hat{u}$ of $\ker\big(\mathcal{D}^+(\widehat{Z})\big)$ takes the same form as in equation (2.21). The fact that it is in the kernel gives similarly as before $u_\lambda(s) = u_\lambda(0)e^{\lambda s}$ and the fact that it is in $H^1(\widehat{Z}, S^+)$ gives $u_\lambda(0) = 0$ for all $\lambda \geq 0$. Overall, it satisfies the boundary condition and the restriction of $\hat{u}$ to the manifold Z is in the kernel of $\mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))}$.

(2) The first and third isomorphism in the stated equation can be proved in a similar way as in the proof of Proposition 1.10 around equation (1.8). There we use the facts that $\mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))}$ and $\mathcal{D}^+(\widehat{Z})$ are Fredholm operators (second one because the boundary Dirac operator is invertible and Proposition 2.15), hence their images are closed, and the operators in equation (2.20) are the respective L^2-adjoints. This is proved for $\mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))}$ in [BW93, Lemma 22.1] and for $\mathcal{D}^+(\widehat{Z})$ it is a direct consequence of the self-adjointness of $\mathcal{D}(\widehat{Z})$. It remains to show the isomorphism in the middle. Take similarly to part (1) an arbitrary $u \in \ker\big(\mathcal{D}^-_{1-P_{\geq}(\mathcal{D}(Y))}\big)$. Then we calculate with $\mathcal{D}^-(Z) = \mathrm{cl}(\partial_s)(\partial_s + \mathcal{D}(Y))$ and $u(s,y) = \sum_\lambda u_\lambda(s)\phi_\lambda(y)$ close to the boundary

$$\begin{aligned}
0 &\overset{!}{=} \big(1 - P_{\geq}(\mathcal{D}(Y))\big)u(0,y) = \sum_{\lambda<0} u_\lambda(0)\phi_\lambda(y) && \Rightarrow \quad u_\lambda(0) = 0\ \forall\lambda < 0\\
0 &\overset{!}{=} \mathrm{cl}(\partial_s)(\partial_s + \mathcal{D}(Y))u(s,y) = \mathrm{cl}(\partial_s)\sum_{\lambda} \big(u'_\lambda(s) + \lambda u_\lambda(s)\big)\phi_\lambda(y) && \Rightarrow \quad u_\lambda(s) = u_\lambda(0)e^{-\lambda s}.
\end{aligned}$$

Using that $\mathcal{D}(Y)$ is invertible, hence zero is no eigenvalue of $\mathcal{D}(Y)$, gives that u takes close to the boundary the form $\sum_{\lambda>0} u_\lambda(0)e^{-\lambda s}\phi_\lambda(y)$ and extends to an element $\hat{u}$ of the kernel of $\mathcal{D}^-(\widehat{Z})$. The opposite direction can be proved analogously to part (1). $\square$

In particular, without using the index formulas, we have shown that the Fredholm indices of the operators from equation (2.18) are equal:

$$\underbrace{\dim\ker\left(\mathcal{D}^+_{P_\geq(\mathcal{D}(Y))}\right)-\dim\operatorname{coker}\left(\mathcal{D}^+_{P_\geq(\mathcal{D}(Y))}\right)}_{=\ \operatorname{ind}_{\text{Fred.}}\left(\mathcal{D}^+_{P_\geq(\mathcal{D}(Y))}\right)}=\underbrace{\dim\ker\left(\mathcal{D}^+(\widehat{Z})\right)-\dim\operatorname{coker}\left(\mathcal{D}^+(\widehat{Z})\right)}_{=\ \operatorname{ind}_{\text{Fred}}(\mathcal{D}^+(\widehat{Z}))}.$$

We have seen in the formulation of the APS index theorem with boundary conditions in Theorem 2.10 that there exists also an index theorem if the boundary Dirac operator is not invertible. This gives rise to the assumption that we can make sense of an index, in the case when $\mathcal{D}^+(\widehat{Z})$ is not Fredholm. Indeed, the Dirac operator can be made Fredholm by interpreting it as a map between weighted Sobolev spaces. Let us start by explaining what a weighted Sobolev space is:

Definition 2.18. Let $S \to \widehat{Z}$ be a graded Clifford bundle with cylindrical end. For $\delta \in \mathbb{R}$ and $k \in \mathbb{N}_0$ we define the *k-th weighted Sobolev space* as

$$H^k_\delta(\widehat{Z}, S^\pm) := \left\{ e^{-\delta s} u \middle| u \in H^k(\widehat{Z}, S^\pm) \right\}$$

for any smooth extension $s : \widehat{Z} \to \mathbb{R}$ of the coordinate s. Furthermore, we define a norm on the k-the weighted Sobolev space via $\|e^{-\delta s}u\|_{H^k_\delta(\widehat{Z},S^\pm)} := \|u\|_{H^k(\widehat{Z},S^\pm)}$. In the case $k = 0$ we write $L^2_\delta(\widehat{Z}, S^\pm)$ instead of $H^0_\delta(\widehat{Z}, S^\pm)$.

If it is clear from the context which manifold and bundle are meant, we shortly write H^k and H^k_δ or $e^{-\delta s}H^k$. The following lemma lists up some important properties of the weighted Sobolev spaces:

Lemma 2.19. *With notations as in Definition* 2.18 *the following holds:*

(1) The weighted Sobolev space is independent of the extension of s.

(2) The k-the weighted Sobolev space $H^k_\delta(\widehat{Z}, S)$ is isometric isomorphic to the k-the Sobolev space $H^k(\widehat{Z}, S)$.

*(3) The chiral Dirac operator $\mathcal{D}^+$ extends to the bounded operator**

$$\mathcal{D}^+(\widehat{Z}) : H^{k+1}_\delta(\widehat{Z}, S^+) \to H^k_\delta(\widehat{Z}, S^-).$$

* We will denote the Dirac operator between weighted Sobolev spaces without a δ in the notation and will therefore indicate the domains when needed. This prevents confusion in Chapter 3 with the twisted Dirac operators $\mathcal{D}_z$ defined in Definition 3.16.

Proof. Let the notations be as in Definition 2.18 and fix $\delta \in \mathbb{R}$ and $k \in N_0$.

(1) Let $\tilde{s}, \overline{s} : \widehat{Z} \to \mathbb{R}$ be two smooth extensions of s. Then we can write an arbitrary element $e^{-\delta\tilde{s}}u \in \{e^{-\delta\tilde{s}}u | u \in H^k\}$ in the form $e^{-\delta\tilde{s}}u = e^{-\delta\overline{s}} \cdot e^{\delta(\overline{s}-\tilde{s})}u$. Furthermore, $e^{\delta(\overline{s}-\tilde{s})}u$ is an element of H^k because $\tilde{s}$ and $\overline{s}$ just differs on the compact part Z such that $e^{-\delta\tilde{s}}u$ is also en element of $\{e^{-\delta\overline{s}}u | u \in H^k\}$. The other inclusion can be proved similarly.

(2) This follows directly by the definition of the weighted Sobolev spaces and their norms defined in Definition 2.18. The linear isometry is given by $u \in H^k \mapsto e^{-\delta s}u \in H^k_\delta$.

(3) One can extend the chiral Dirac operator $\mathcal{D}^+(\widehat{Z})$ as claimed if the operator

$$\mathcal{D}^+(\widehat{Z}) : C^\infty(\widehat{Z}, S^+) \cap H^{k+1}_\delta(\widehat{Z}, S^+) \to H^k_\delta(\widehat{Z}, S^-)$$

is bounded. For every $e^{-\delta s}u \in C^\infty \cap H^k_\delta$ we have on the cylindrical end with $\mathcal{D}^+(\widehat{Z}) = \mathrm{cl}(\partial_s)(\partial_s - \mathcal{D}(Y))$

$$\mathcal{D}^+(\widehat{Z})(e^{-\delta s}u) = e^{-\delta s}\mathcal{D}^+(\widehat{Z})u + \mathcal{D}^+(\widehat{Z})(e^{-\delta s})u = e^{-\delta s}(\mathcal{D}^+(\widehat{Z})u - \delta ds \cdot u). \quad (2.22)$$

Then there exists a constant $C > 0$ such that for all $e^{-\delta s}u \in C^\infty \cap H^k_\delta$ the estimate

$$\left\| \mathcal{D}^+(\widehat{Z})(e^{-\delta s}u) \right\|_{H^k_\delta} \overset{\text{eq.(2.22)}}{\leq} \|\mathcal{D}^+(\widehat{Z})u\|_{H^k} + \|\delta ds \cdot u\|_{H^k} \leq C \|e^{-\delta s}u\|_{H^{k+1}_\delta}$$

holds on the cylindrical end and the lemma is proved. The second estimate holds because the Dirac operator extends to a bounded operator $\mathcal{D}^+(\widehat{Z}) : H^{k+1} \to H^k$ (a direct consequence of the elliptic estimates) and ds is constant over the cylinder. □

The following proposition specifies the idea of making the Dirac operator Fredholm [see Loy04, Theorem 3.3]:

Proposition 2.20. *For a Clifford bundle $S \to \widehat{Z}$ with cylindrical end there exists a $r > 0$ such that the chiral Dirac operator*

$$\mathcal{D}^+(\widehat{Z}) : H^1_\delta(\widehat{Z}, S^+) \to L^2_\delta(\widehat{Z}, S^-)$$

is Fredholm for all $\delta \in \mathbb{R}$ with $0 < |\delta| < r$.

Proof. For a proof of this theorem in detail see Melrose [Mel93, Theorem 5.60]. If Proposition 2.15 is already proved one can lead this proposition back to the situation there: Let r be the norm of the nonzero eigenvalue of the boundary Dirac operator $D(Y)$ that is closest to zero and $\delta \in \mathbb{R}$ with $0 < |\delta| < r$. The special choice of δ and r gives that the Dirac operator $D(Y) + \delta$ has zero kernel. Over the part $\widehat{N} := \big([0,\infty) \times Y\big) \cup_Y N$ the Dirac operator takes by Proposition 2.5 the form $\mathrm{cl}(\partial_s)\,(\partial_s - \mathcal{D}(Y))$, where N is the collar of the boundary. We obtain for $u \in C^\infty(\widehat{N}, S)$

$$e^{\delta s}\mathcal{D}^+(\widehat{Z})e^{-\delta s}u = e^{\delta s}\,\mathrm{cl}(\partial_s)\Big(\partial_s(e^{-\delta s}u) - \mathcal{D}(Y)(e^{-\delta s}u)\Big) = \mathrm{cl}(\partial_s)\,(\partial_s - \mathcal{D}(Y) - \delta)\,u.$$

Then we can apply Proposition 2.15 and the Fredholmness of $e^{\delta s}\mathcal{D}^+(\widehat{Z})e^{-\delta s}$ as map from $H^1 \to L^2$ follows, which is equivalent to the Fredholm property of $\mathcal{D}^+(\widehat{Z})$ on the weighted Sobolev spaces as stated in the proposition. □

The following theorem states the APS index theorem in its full generality as in Theorem 2.10 in the cylindrical end version. Similarly to the Fredholm case, one could translate this theorem into b-geometry and prove it by leading it back to the Fredholm case [Mel93].

Theorem 2.21 (APS index theorem via attached cylinder [Loy04, Theorem 3.4])**.** *Let $\widehat{Z}$ be an even-dimensional oriented Riemannian manifold and $S \to \widehat{Z}$ a graded Clifford bundle with cylindrical end modeled by (Z, Y) as in Definition* 2.14 *with Dirac operator $\mathcal{D}(\widehat{Z})$, chiral Dirac operator $\mathcal{D}^+(\widehat{Z})$ and boundary Dirac operator $\mathcal{D}(Y)$. Then the $\pm$-indices of $\mathcal{D}(\widehat{Z})$, denoted as $\mathrm{ind}_\pm(\mathcal{D}(\widehat{Z}))$ and defined as the Fredholm indices of $\mathcal{D}^+(\widehat{Z}) : H^1_\delta(\widehat{Z}, S^+) \to L^2_\delta(\widehat{Z}, S^-)$ for a positive respectively negative δ as in Proposition* 2.20*, are given by*

$$\mathrm{ind}_\pm(\mathcal{D}(\widehat{Z})) = \int_Z \boldsymbol{I}(\mathcal{D}(\widehat{Z})) - \frac{1}{2}\Big(\eta(\mathcal{D}(Y)) \pm \dim\ker(\mathcal{D}(Y))\Big). \tag{2.23}$$

Here $\boldsymbol{I}(\mathcal{D}(\widehat{Z}))$ denotes the local index form defined in Definition 1.13 *and $\eta(\mathcal{D}(Y))$ the η-invariant as defined in equation (2.12).*

The definitions of the $\pm$-indices in the previous theorem are well defined, i.e. the indices are independent of the choice of δ. This can be verified by noting that the right hand side of the index formula in (2.23) is independent of δ. This independence can also be seen directly by the following remark:

Remark 2.22. With the same notations as in the previous theorem, we show in analogy to Remark 2.17 the equality of the index in Theorem 2.10 (APS index theorem via boundary condition) and the +-index in Theorem 2.21 (APS index theorem via attached cylinder) by showing for any δ as in Proposition 2.20

$$\begin{aligned} &\operatorname{ind}_{\mathrm{Fred}}(\mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))}) = \begin{cases} \operatorname{ind}_{\mathrm{Fred}}(\mathcal{D}^+(\widehat{Z})) & \delta > 0 \\ \operatorname{ind}_{\mathrm{Fred}}(\mathcal{D}^+(\widehat{Z})) - \dim\ker(\mathcal{D}(Y)) & \delta < 0 \end{cases} \\ &\text{with} \quad \begin{aligned} \mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))} &: \quad \left\{u \in H^1(Z, S^+) \middle| u|_Y \in \ker\left(P_{\geq}\left(\mathcal{D}(Y)\right)\right)\right\} \to L^2(Z, S^-) \text{ and} \\ \mathcal{D}^+(\widehat{Z}) &: \quad H^1_\delta(\widehat{Z}, S^+) \to L^2_\delta(\widehat{Z}, S^-). \end{aligned} \end{aligned} \tag{2.24}$$

This is what we expect by comparing the right hand side of the equation (2.11) with the right hand side of equation (2.23). We will show the identities in equation (2.24) by proving the two identifications

$$\begin{aligned} &(1) \quad \ker\left(\mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))}\right) \cong \ker\left(\mathcal{D}^+(\widehat{Z})\right). \\ &(2) \quad \ker\left(\mathcal{D}^-_{1-P_{\geq}(\mathcal{D}(Y))}\right) \cong \begin{cases} \ker\left(\mathcal{D}^-(\widehat{Z})\right) & \delta > 0 \\ \ker(\mathcal{D}(Y)) \oplus \ker\left(\mathcal{D}^-(\widehat{Z})\right) & \delta < 0, \end{cases} \end{aligned} \tag{2.25}$$

where $\mathcal{D}^-(\widehat{Z})$ is considered as a map between weighted Sobolev spaces with weight $-\delta$. The isomorphism given in equation (2.25) and the identifications

$$\ker\left(\mathcal{D}^-_{1-P_{\geq}(\mathcal{D}(Y))}\right) \cong \operatorname{coker}\left(\mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))}\right) \text{ and } \ker\left(\mathcal{D}^-(\widehat{Z})\right) \cong \operatorname{coker}\left(\mathcal{D}^+(\widehat{Z})\right)$$

leads to the desired relation in equation (2.24). Here we identified the L^2_δ-adjoint of $\mathcal{D}^+(\widehat{Z})$ with the operator

$$\mathcal{D}^-(\widehat{Z}) : H^1_{-\delta}(\widehat{Z}, S^-) \to L^2_{-\delta}(\widehat{Z}, S^+) \tag{2.26}$$

using the isomorphism $H^k_\delta(\widehat{Z}, S^\pm) \to H^k_{-\delta}(\widehat{Z}, S^\pm)$ $u \mapsto e^{2\delta s}u$ for $k \geq 0$. Let us prove the two identifications in equation (2.25):

Proof. **(1)** Similarly to the proof of equation (2.19) part (1), we obtain that an arbitrary element $u \in \ker\left(\mathcal{D}^+_{P_{\geq}(\mathcal{D}(Y))}\right)$ takes the form $\sum_{\lambda<0} u_\lambda(0)e^{\lambda s}\phi_\lambda(y)$ close to the boundary. This can be extended to an element of $H^1_\delta(\widehat{Z}, S^+)$ because by multiplication with $e^{\delta s}$ it is in $H^1(\widehat{Z}, S^+)$. Note that the norm of δ is by construction small enough for the previous consideration.The same applies for the opposite direction.

(2) Let u be an arbitrary element of the kernel of $\mathcal{D}^{-}_{1-P_{\geq}(\mathcal{D}(Y))}$. Close to the boundary it takes by similar calculation as in the proof of equation (2.19) part (2) the form $\sum_{\lambda\geq 0} u_\lambda(0)e^{-\lambda s}\phi_\lambda(y)$. We use this to extend u smoothly over the infinite cylindrical end. In the case $\delta > 0$ the extension of u is after multiplication by $e^{-\delta s}$ in $H^1(\widehat{Z}, S^-)$, hence it is an element of $H^1_{-\delta}(\widehat{Z}, S^-)$. In the case $\delta < 0$ we rewrite the sum as

$$u(s, y) = u_0(0)\phi_0(y) + \sum_{\lambda>0} u_\lambda(0)e^{-\lambda s}\phi_\lambda(y) \tag{2.27}$$

and obtain that the first part is an element of the kernel of $\mathcal{D}(Y)$ and the second part lies in $H^1_{-\delta}(\widehat{Z}, S^-)$ (again by multiplication with $e^{-\delta s}$ and finding that the result is in $H^1(\widehat{Z}, S^+)$). The opposite direction can be verified similarly. $\square$

3 The Index Theorem for End-Periodic Dirac Operators

The aim of this chapter is to generalize the APS index theorem to the end-periodic index theorem for Dirac operators by Mrowka, Ruberman, and Saveliev [MRS16]. In the first two sections we will study end-periodic manifolds, end-periodic Clifford bundles and the Fredholmness of the induced chiral Dirac operator. After these preliminaries we state and prove the end-periodic index theorem in the Fredholm case in Section 3.3 and give in the last section an outlook to the non-Fredholm case.

3.1 End-periodic manifolds and their additional structures

In this section we will generalize manifolds with cylindrical end, as they arise for example in the version of the APS index theorem stated in Theorem 2.21, to manifolds with periodic end. A manifold with cylindrical end is built by a compact manifold Z with boundary Y and the product manifold $[0, \infty) \times Y$ attached to the boundary. To obtain an end-periodic manifold Z_∞, we start also with a compact manifold Z with boundary Y and attach infinitely many copies $W_0, W_1, ...$ of a manifold W to the boundary as illustrated in Figure 3.1. For this gluing procedure the manifold W should have boundary $-Y \cup Y$, which will follow later on by the explicit construction of the manifold W (Construction 3.1). From any additional structure on Z_∞ like the metric, vector bundles, etc. we require that it repeats periodically from piece to piece. All of this is specified in the following, using [MRS16] as the main reference. Let us start with some preparations in homology and cohomology theory within the Poincaré duality and the de Rham isomorphism.

Let X be a closed oriented manifold of dimension n. The Poincaré duality theorem states especially that the first cohomology group $H^1(X, \mathbb{Z})$ is natural isomorphic to the $(n-1)$-th homology group $H_{n-1}(X)$ [Hat02, Theorem 3.30]. Before we give the isomorphism, we provide a brief overview of how to handle homology and cohomology groups. One can think of the $(n-1)$-th homology group $H_{n-1}(X)$ as the space of

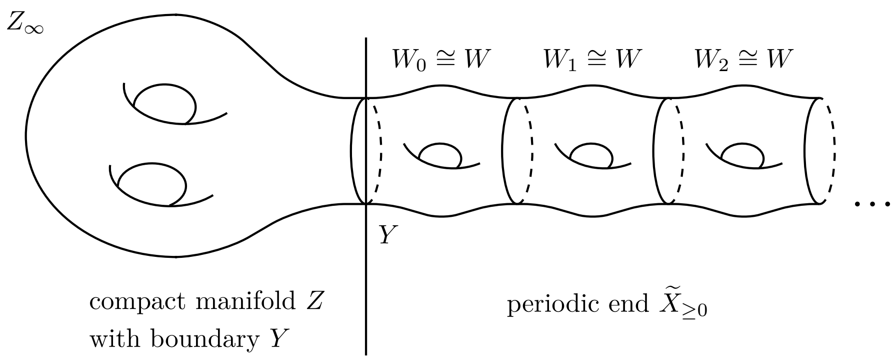

Figure 3.1: An illustration of how the end-periodic manifold Z_∞ is built by a compact manifold Z with boundary Y and infinite copies $W_0, W_1, \dots$ of a manifold W attached one after another to the boundary.

all equivalence classes of all one-dimensional closed oriented submanifolds immersed in X, where two submanifolds Y_1 and Y_2 are equivalent if there exists a two-dimensional immersion of X restricted by Y_1 and Y_2 [Fel+18]. The general definition of the first cohomology group $H^1(X, \mathbb{R})$ is not necessary for our purposes. It suffices to know that it is a vector space which is natural isomorphic to $\mathrm{Hom}(H_1(X), \mathbb{R})$ [Lee13, p.472]. Furthermore, it can be represented in de Rham cohomology $H^1_{\mathrm{dR}}(X)$, the space of all closed one-forms modulo the exact one-forms. For details and a concrete construction of the de Rham isomorphism see Lee [Lee13, Chapter 18]. Furthermore, we can think of the first cohomology class with coefficients in $\mathbb{Z}$ - denoted with $H^1(X, \mathbb{Z})$ - as a subgroup of $H^1(X, \mathbb{R})$. We call a cohomology class with coefficients in $\mathbb{Z}$ *primitive* if it is nonzero and not a nontrivial multiple of another class*. The Poincaré duality takes the form

$$H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R}) \cong H^1_{\mathrm{dR}}(X) \to H_{n-1}(X), \quad [\alpha] \mapsto [Y], \tag{3.1}$$

where Y is an $(n-1)$-dimensional closed oriented submanifold such that $\int_Y \beta = \int_X \beta \wedge \alpha$ holds for all closed $n-1$ forms β. This is enough to know for our purposes, see [Hat02] for details. Let us start with the construction of an end-periodic manifold:

Construction 3.1. Let X be a closed oriented manifold consisting of κ connected components and endowed with a cohomology class $\gamma \in H^1(X, \mathbb{Z})$ whose restriction to any connected component is primitive. Then we choose an $(n-1)$-dimensional closed oriented submanifold Y immersed in X Poincaré dual to γ. Since γ is primitve on any

* This definition is taken over from Mrowka, Ruberman, and Saveliev [MRS16] and is different to [MP77], where a primitive cohomology class is allowed to be zero.

connected component of X, we can require that Y has κ connected components and is even embedded in X [MP77]. Furthermore, $X \setminus Y$ has also κ connected components* and any of them has two ends [MP77] so that we can define a manifold W as the compactification of $X \setminus Y$ with boundary $-Y \cup Y$. This is by construction an oriented manifold, called by Taubes [Tau87] the *fundamental segment*.

Before giving the precise definition of an end-periodic manifold, let us discuss an example of a genus 2 torus to illustrate the previous construction without being mathematically rigorous:

Example 3.2. Let us have a look at the 2-torus pictured in Figure 3.2. The first cohomology group with coefficients in $\mathbb{Z}$ is isomorphic to $\mathbb{Z}^4$. Picturing the cohomology classes via the Poincaré duality given in equation (3.1) as one-dimensional closed oriented immersed submanifolds gives us for example the four generating classes $\alpha_1, \alpha_2, \beta_1$ and β_2 showed in Figure 3.2. All four of these are primitive classes. But there exists much more primitive classes as for example γ_1, γ_2, $\alpha_2 + \beta_1$, etc. Now we can cut the 2-torus along all these submanifolds and obtain the connected manifold W with boundary $-S^1 \cup S^1$. It is important to require that the cohomology class is primitive. If we would take for example $2\beta_1$, which is not a primitive class anymore, we could not find an embedded connected submanifold Y as before. Either we do not have an embedding or Y decomposes into two connected components.

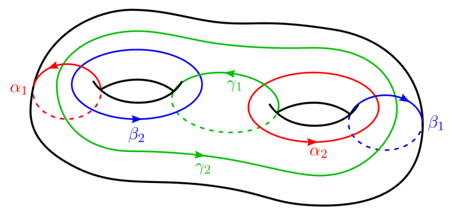

Figure 3.2: Some primitive cohomology classes of a 2-torus [see Fel+18, Fig. 1].

Definition 3.3. An oriented manifold Z_∞ is called *end-periodic*, if there exists a closed oriented manifold X endowed with a cohomology class $\gamma \in H^1(X, \mathbb{Z})$ whose restriction to any connected component of X is primitive and an immersed oriented submanifold Y Poincaré dual to γ with induced fundamental segment W as in Construction 3.1 and a compact oriented manifold Z with a boundary Y such that for infinitely many copies $W_0, W_1, \ldots$ of W, we obtain Z_∞ by gluing all the pieces together along the boundary Y

$$Z_\infty \cong Z \cup_Y W_0 \cup_Y W_1 \cup_Y \ldots .$$

With the previous data one calls Z_∞ an *end-periodic manifold modeled by* (Z, X, γ).

* This is a consequence of the fact that γ is nonzero on any connected component.

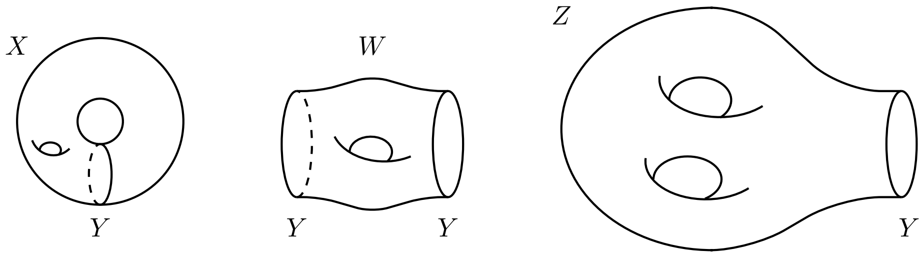

Figure 3.3: An illustration of the pieces of an end-periodic manifold. X is the closed manifold that we cut open along Y to get the pieces of the periodic end. The manifold Z with boundary Y forms the compact part.

Remark 3.4. It is explicitly allowed that X and Y in the previous definition are not connected. This allows to model multiple periodic ends by one pair (X, γ).

Example 3.5 (Manifolds with cylindrical end are end-periodic manifolds)**.** Let Z be a compact oriented manifold with boundary Y. We define $\widehat{Z}$ as the oriented* manifold $Z \cup_Y \big([0, \infty) \times Y\big)$ and call it a *manifold with cylindrical end.* Define X as $S^1 \times Y$ and γ as the primitive cohomology class corresponding to the embedded connected submanifold $\{0\} \times Y$. By Construction 3.1 the fundamental segment W is isomorphic to $[0, 1] \times Y$ and we obtain

$$\widehat{Z} = Z \cup_Y \big([0, \infty) \times Y\big) \cong Z \cup_Y W_0 \cup_Y W_1 \cup_Y \dots$$

by gluing together the infinite many copies $W_0, W_1, \dots$ of W. Therefore is $\widehat{Z}$ by Definition 3.3 an end-periodic manifold modeled by $\big(Z, S^1 \times Y, \gamma\big)$.

For further considerations it is useful to see the end of an end-periodic manifold as a part of an infinite cyclic covering over the closed manifold X as pictured in Figure 3.4. The following lemma will state the existence of such a covering [see for part (1) Hal18, Lemma 1.2.1]:

Lemma 3.6. *Let X be a closed oriented manifold and $\gamma \in H^1(X, \mathbb{Z})$ a cohomology class whose restriction to any connected component is primitive. Then the following holds:*

(1) There exists a unique (up to homotopy) smooth function $g : X \to S^1$ such that $[g^*(d\theta)]$ corresponds to γ via the de Rham isomorphism, where θ denotes the standard coordinate on S^1.

* We extend the orientation of Z in the natural way to the manifold $\widehat{Z}$.

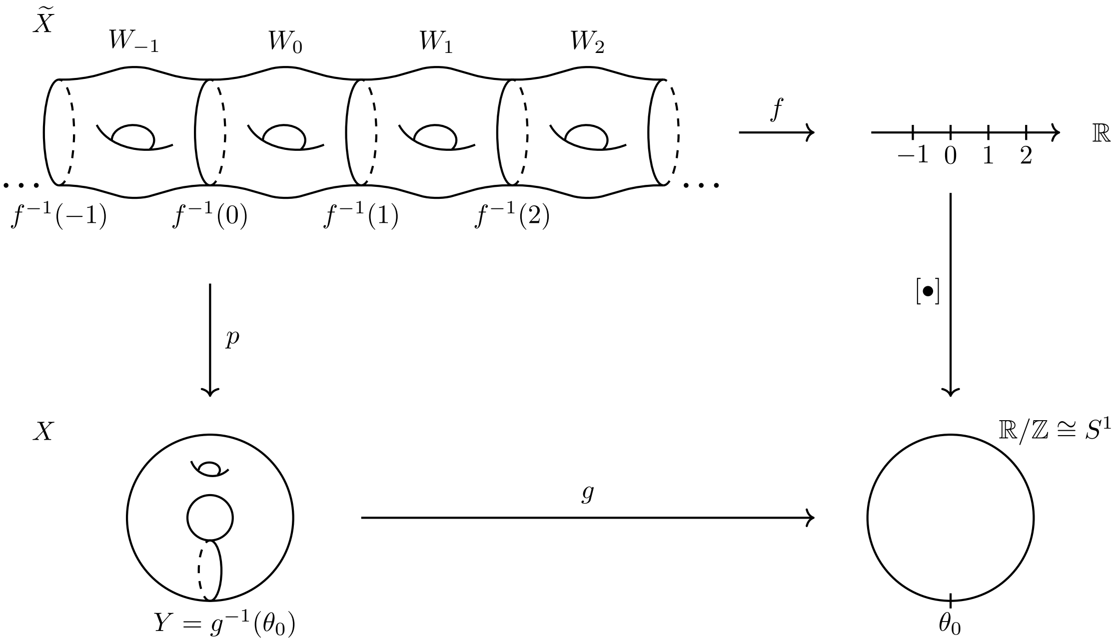

Figure 3.4: An illustration of the infinite cyclic covering $p : \widetilde{X} \to X$ which leads to the end of an end-periodic manifold by taking the 'half'' of it. One can think of p as wrapping the manifold $\widetilde{X}$ over X. The point $\theta_0 \in S^1$ can be used to obtain the submanifold Y of X as its preimage under g. Furthermore, a piece W_j can be obtained by taking the preimages of the interval $[j, j+1]$ under f.

(2) The pullback of the infinite covering $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ along the smooth function $g : X \to S^1$ from part (1) is an infinite cyclic covering $\widetilde{X} \to X$.

Proof. Let X and γ be as assumed in the lemma.

(1) We represent via the de Rham isomorphism the cohomology class $\gamma \in H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R})$ with a closed one-form α. Then one can show that for any one-dimensional closed submanifold σ immersed in X the integral $\int_\sigma \alpha$ is an integer. After this preparation let us start with the existence of the map g. We fix an arbitrary x_0 in X and define a smooth map

$$g : X \to \mathbb{R}/\mathbb{Z} \cong S^1, \quad x \mapsto \left[\int_\sigma \alpha\right], \tag{3.2}$$

where σ is a smooth path from x_0 to x. The integral over two different paths from x_0 to x just differs by an integer because of the preliminary consideration, hence g is well defined. Furthermore, we calculate with the definition of the pullback, the chain rule and the fundamental theorem of calculus

$$g^*(d\theta) = d\theta \circ dg = d(\theta \circ g) = \alpha$$

so that $[g^*(d\theta)]$ corresponds via the de Rham isomorphism to γ.

Let $\widetilde{g} : X \to S^1$ be another smooth map with $[\widetilde{g}^*(d\theta)] = [\alpha]$. Then we have $[\widetilde{g}^*(d\theta)] = [g^*(d\theta)]$ and there exists by definition of the de Rham cohomology group a function $h \in C^\infty(X, \mathbb{R})$ such that

$$dh = \widetilde{g}^*(d\theta) - g^*(d\theta) = d(\theta \circ \widetilde{g}) - d(\theta \circ g)$$

holds. So there exists a constant $c \in \mathbb{R}$ such that the difference of $\widetilde{g}$ and g is given by $[h + c]$, where $[\bullet]$ is the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Finally, a homotopy is given by:

$$H : [0, 1] \times X \to S^1, \quad (s, x) \mapsto \widetilde{g}(x) - [s(h(x) + c)].$$

(2) The commutative pullback diagram of the covering $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ along the smooth function $g : X \to S^1$ defined in equation (3.2) takes the form

$$\begin{tikzcd} \widetilde{X} \coloneqq \{(s,x) \in \mathbb{R} \times X \,|\, [s] = g(x)\} \arrow[r, "f:=\mathrm{pr}_1"] \arrow[d, "p:=\mathrm{pr}_2"'] & \mathbb{R} \arrow[d, "{[\bullet]}"] \\ X \arrow[r, "g"] & \mathbb{R}/\mathbb{Z} \cong S^1. \end{tikzcd} \tag{3.3}$$

The pullback of an infinite covering is again an infinite covering. It remains to show that the covering is cyclic, i.e. the group of all covering translation (consists of all diffeomorphism $c : \widetilde{X} \to \widetilde{X}$ satisfying $p = p \circ c$) is generated by one element. We define the map

$$T : \widetilde{X} \to \widetilde{X}, \quad (s, x) \mapsto (s + 1, x), \tag{3.4}$$

which is a well defined* diffeomorphism satisfying $p = p \circ T$, hence T is a covering translations of $\widetilde{X}$. It also generates the entire group of covering translations. To verify this, let $c : \widetilde{X} \to \widetilde{X}$ be an arbitrary covering translation written in the form $c = (c_1, c_2)$ for $c_1 \in C^\infty(\widetilde{X}, \mathbb{R})$ and $c_2 \in C^\infty(\widetilde{X}, X)$. The property $p = p \circ c$ implies $c_2(s, x) = x$ for

* For all $(s, x) \in \widetilde{X}$ the following holds $[s + 1] = [s] = g(x)$.

all elements $(s,x) \in \widetilde{X}$. The well-definedness of c gives $[(f \circ c)(s,x)] = g(x)$ which implies along with the continuity of c that there exists an integer n such that $c_1(s,x) = s+n$ holds for all $(s,x) \in \widetilde{X}$. Then we have $c = T^n$ and the Lemma is proved. $\square$

Remark 3.7. With notations as in the proof of Lemma 3.6, we write $T(x) =: x+1$ for all $x \in \widetilde{X}$. The generator T is the unique generator that maps $f^{-1}(\mathbb{R}_{\geq 0})$ into $f^{-1}(\mathbb{R}_{\geq 0})$. Moreover, a straightforward calculation shows that the map $f : \widetilde{X} \to \mathbb{R}$ satisfies $f(x+1) = f(x) + 1$ for all $x \in \widetilde{X}$ (use the definition of f and T given in equation (3.3) and (3.4)) and its differential df is periodic, hence can be seen as a one-form on X.

Let any notations and assumptions be as in Lemma 3.6. Meeks and Patrusky [MP77] proved that there exists a regular value $\theta_0 \in S^1$ of $g : X \to S^1$ such that its preimage has the same number of connected components as X.

Claim. The submanifold Y in Construction 3.1 can be defined as the preimage of θ_0 under g.

Proof. Define $Y \coloneqq g^{-1}(\theta_0)$ and let W be the compactification of $X \setminus Y$ with boundary $-Y \cup Y$. We chose $\theta : S^1 \setminus \{\theta_0\} \to \mathbb{R}$ such that the smooth extension of $\theta \circ g$ to W is zero on $-Y$ and one on Y. Furthermore, α denotes a representation of the cohomology class γ. Then we obtain by the definition of g given in Lemma 3.6 (1) the identity $[\alpha] = [g^*(d\theta)] = [d(\theta \circ g)]$. With an appropriate $h \in C^\infty(X, \mathbb{R})$ it follows for every closed $(n-1)$-form β on X that

$$\int_X \beta \wedge \alpha = \int_X \beta \wedge \big(d(\theta \circ g) + dh\big) \overset{\beta \text{ is closed}}{=} \int_W d\big((\theta \circ g)\beta\big) \overset{\text{Stokes theorem}}{=} \int_Y \beta$$

holds so that Y is Poincaré dual to γ by the definition in equation (3.1). $\square$

Now we can interpret the end $W_0 \cup_Y W_1 \cup_Y \ldots$ of an end-periodic manifold Z_∞ as 'half' of the infinite cyclic covering $\widetilde{X} \to X$ as pictured in Figure 3.4

$$\widetilde{X}_{\geq 0} \coloneqq f^{-1}\left(\mathbb{R}_{\geq 0}\right) \cong W_0 \cup_Y W_1 \cup_Y \ldots, \quad W_j = \big\{(s,x) \in [j, j+1] \times X \big| [s] = g(x)\big\}.$$

The last part of this section treats what an end-periodic additional structure of an end-periodic manifold should be. For a graded Clifford bundle $S \to \widehat{Z}$ with cylindrical end modeled by (Z, Y) all information over the end $[0,\infty) \times Y$ is contained in a Clifford bundle $S_0 \to Y$ over the boundary (see Definition 2.14, Construction 2.13 and Lemma 2.4).

Analogously, an end-periodic structure on an end-periodic manifold Z_∞ should come from a corresponding structure on the closed manifold X, precisely:

Definition 3.8. Let Z_∞ be an end-periodic manifold modeled by (Z, X, γ). We call an additional structure $\mathcal{E}$ on Z_∞ *end-periodic* if there exists a corresponding structure $\mathcal{E}_X$ on X such that

$$\left(p|_{\widetilde{X}_{\geq 0}}\right)^* (\mathcal{E}_X) \cong \mathcal{E}|_{\widetilde{X}_{\geq 0}}, \tag{3.5}$$

where p is the infinite cyclic covering $p : \widetilde{X} \to X$ from Lemma 3.6 (2).

Remark 3.9. Equivalently, one could define a structure $\mathcal{E}$ of an end-periodic manifold to be end-periodic if it repeats itself from piece to piece, i.e. that for the generator $T : \widetilde{X} \to \widetilde{X}$ of the group of covering translations with $T(\widetilde{X}_{\geq 0}) \subset \widetilde{X}_{\geq 0}$ and for all $j \in \mathbb{N}_0$ the following holds:

$$\left(T|_{W_j}\right)^* \mathcal{E}|_{W_{j+1}} \cong \mathcal{E}|_{W_j}.$$

Example 3.10. Let Z_∞ be an end-periodic manifold modeled by (Z, X, γ).

(1) The tangent bundle and normal bundle of an end-periodic manifold are end-periodic.

(2) A Riemannian metric on X induces a Riemannian metric over $\widetilde{X}_{\geq 0}$ by using equation (3.5) as definition. Extending this over the whole end-periodic manifold Z_∞ leads to an end-periodic Riemannian metric on Z_∞.

(3) (Cylindrical Clifford bundles are end-periodic) We have seen in Example 3.5 that a manifold with cylindrical end $\widehat{Z}$ modeled by (Z, Y) is an end-periodic manifold with $X = S^1 \times Y$. Every graded Clifford bundle bundle $S \to \widehat{Z}$ with cylindrical end as in Definition 2.14 is also end-periodic. To verify this, let $S^0 \to Y$ be the Clifford bundle over the boundary as in Definition 2.3 (2). Then we equip $W = [0,1] \times Y$ with the induced structure of a graded Clifford bundle as in Lemma 2.4 and obtain, by gluing the end points of the interval $[0,1]$ together, a graded Clifford bundle S over $S^1 \times Y$. This satisfies by construction equation (3.5) such that the graded Clifford bundle $S \to \widehat{Z}$ is end-periodic.

3.2 End-periodic Dirac operators and their Fredholmness

We have seen for Clifford bundles with cylindrical end in Section 2.3 that the induced chiral Dirac operators are in general not Fredholm, hence we expect a similar difficulty in the end-periodic case. The aim of this section is to study under which condition the Dirac operator induced by an end-periodic Clifford bundle over an end-periodic manifold is Fredholm. Taubes [Tau87] studied this question in the end-periodic setting first and proved an analogue criterion for the Fredholmness of the Dirac operator to the cylindrical case in Proposition 2.15. He used for his observation the so-called Fourier-Laplace transform, which leads to a family of Dirac operators on a closed manifold whose invertibility ensures the Fredholmness of the end-periodic Dirac operator (see Proposition 3.19). Furthermore, he showed that the Dirac operator between weighted Sobolev spaces is under certain circumstances Fredholm for all weights but a discrete set. We state this result in Theorem 3.21. This will help later on to give a meaning to the index in the non-Fredholm case (Section 3.4). Almost all notation in this section is based on Mrowka, Ruberman, and Saveliev [MRS16].

Throughout this section let $S = S^+ \oplus S^-$ be a graded end-periodic Clifford bundle over an end-periodic Riemannian manifold Z_∞ modeled by (Z, X, γ) as in Definition 3.3 with induced chiral Dirac operator $\mathcal{D}^+(Z_\infty)$. Furthermore, we have the infinite cyclic covering $\widetilde{X} \to X$ with the smooth function $f : \widetilde{X} \to \mathbb{R}$ and the covering translation $T : \widetilde{X} \to \widetilde{X}$ mapping positive elements of $\widetilde{X}$ into positive elements as in Lemma 3.6 and Remark 3.7. To simplify the notation, we denote the induced graded Clifford bundles over X, $\widetilde{X}$ and Z also with $S = S^+ \oplus S^-$ (see Definition 3.8) and call their induced chiral Dirac operators $\mathcal{D}^+(X)$, $\mathcal{D}^+(\widetilde{X})$ and $\mathcal{D}^+(Z)$, respectively.

We define the weighted Sobolev spaces similarly as in the cylindrical case

$$H^k_\delta(Z_\infty, S^\pm) \coloneqq e^{-\delta f} H^k(Z_\infty, S^\pm), \quad H^k_\delta(\widetilde{X}, S^\pm) \coloneqq e^{-\delta f} H^k(\widetilde{X}, S^\pm).$$

Instead of the coordinate t we use the function $f : \widetilde{X} \to \mathbb{R}$ and extend it smoothly to the whole manifold Z_∞. Lemma 2.19 also holds in the end periodic case because its proof can be done analogously, with the only difference being that we need in part (3) the periodicity of df over $\widetilde{X}$ (see Remark 3.7) instead of the constancy of dt. Let us repeat the statements:

Lemma 3.11. *For all weights $\delta \in \mathbb{R}$ and $k \in \mathbb{N}_0$ the weighted Sobolev spaces $H^k_\delta(Z_\infty, S^\pm)$ are independent of the extension of f, isometric isomorphic to $H^k(Z_\infty, S^\pm)$ and the chiral*

Dirac operators extend to the bounded operators

$$\mathcal{D}^+(Z_\infty) : H^{k+1}_\delta(Z_\infty, S^+) \to H^k_\delta(Z_\infty, S^-) \quad \text{and} \quad \mathcal{D}^+(\widetilde{X}) : H^{k+1}_\delta(\widetilde{X}, S^+) \to H^k_\delta(\widetilde{X}, S^-).$$

Mainly, we are interested in the Fredholmness of the chiral Dirac operator

$$\mathcal{D}^+(Z_\infty) : H^{k+1}_\delta(Z_\infty, S^+) \to H^k_\delta(Z_\infty, S^-). \tag{3.6}$$

We investigate this in two steps. First, we bring the situation back from the end-periodic manifold Z_∞ to the the manifold $\widetilde{X}$ and second, we lead that back to the compact manifold X with help of the Fourier-Laplace transform. The following lemma realizes the first step:

Lemma 3.12. *For $\delta \in \mathbb{R}$ and $k \in \mathbb{N}_0$ the operator $\mathcal{D}^+(Z_\infty) : H^{k+1}_\delta(Z_\infty, S^+) \to H^k_\delta(Z_\infty, S^-)$ is Fredholm if and only if the operator $\mathcal{D}^+(\widetilde{X}) : H^{k+1}_\delta(\widetilde{X}, S^+) \to H^k_\delta(\widetilde{X}, S^-)$ is.*

Proof (sketch). The following proof is worked out in detail for the cylindrical case in [Hal18, Proposition 4.1.7], where the backward direction is inspired by [LM85, p.420-421]. In both directions we use that an linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ between two Hilbert spaces is Fredholm if and only if it has a Fredholm-inverse, i.e. a bounded operator $Q : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\mathrm{Id} - QT$ and $\mathrm{Id} - TQ$ are compact operators [Gru09, Lemma 8.9].

"$\Rightarrow$" We start by showing that the kernel of $\mathcal{D}(\widetilde{X})$ vanishes (by contradiction):

> Let u be a nonzero element of the kernel of $\mathcal{D}(\widetilde{X})$. We define $u_m(x) \coloneqq e^{-\delta m} u(x - m)$ for $m \in \mathbb{Z}$ and $\phi \in C^\infty(Z_\infty, \mathbb{R})$ such that $\phi|_{Z \cup_Y W_0} \equiv 0$ and $\phi|_{W_2 \cup_Y W_3 \cup_Y \dots} \equiv 1$ holds. The product ϕu_m is an element of $H^{k+1}_\delta(Z_\infty, S^+)$ and one can show
>
> $$\phi u_m = Q((d\phi) u_m) + K(\phi u_m), \quad \forall\, m \in \mathbb{Z}, \tag{3.7}$$
>
> where Q is the Fredholm inverse of $\mathcal{D}^+(Z_\infty)$ and K the compact operator $\mathrm{Id} - Q\mathcal{D}^+(Z_\infty)$. Furthermore, one can show that the right hand side of equation (3.7) has a convergent subsequence in $H^{k+1}_\delta(Z_\infty, S^+)$ and the left hand side does not, hence we have a contradiction.

The kernel of the adjoint of $\mathcal{D}^+(\widetilde{X})$ is also trivial by analogue arguments. This leads by identifying the kernel of the adjoint of $\mathcal{D}^+(\widetilde{X})$ with the cokernel of $\mathcal{D}^+(\widetilde{X})$

as in equation (1.8) to the invertibility of $\mathcal{D}^+(\widetilde{X})$, in particular the chiral Dirac operator is Fredholm.

"$\Leftarrow$" Let P be the Fredholm-inverse of $\mathcal{D}^+(\widetilde{Z}_3)$ with $\widetilde{Z}_3$ the double of the manifold $Z_3 := Z \cup_Y W_0 \cup_Y W_1 \cup_Y W_2$ and Q the Fredholm-inverse of $\mathcal{D}^+(\widetilde{X})$. Then we obtain a Fredholm-inverse of $\mathcal{D}^+(Z_\infty)$ by "clutching" P and Q together, i.e we define $R := \psi_1 P \phi_1 + \psi_2 Q \phi_2 : H^k_\delta(Z_\infty, S^-) \to H^{k+1}_\delta(Z_\infty, S^+)$ with ϕ_1, ϕ_2, ψ_1, ψ_2 defined as follows:

- $\phi_1 \in C^\infty(Z_\infty, \mathbb{R})$ such that $\phi_1|_{Z \cup_Y W_0} \equiv 1$ and $\phi_1|_{W_2 \cup_Y W_3 \cup_Y \dots} \equiv 0$ holds.
- $\phi_2 := 1 - \phi_1 \in C^\infty(\mathbb{Z}_\infty, \mathbb{R})$.
- $\psi_1 \in C^\infty(Z_\infty, \mathbb{R})$ such that $\operatorname{supp}(\psi_1) \subset Z_3$ and $\psi_1|_{\operatorname{supp}(\phi_1)} \equiv 1$ hold.
- $\psi_2 \in C^\infty(Z_\infty, \mathbb{R})$ such that $\operatorname{supp}(\psi_1) \subset \widetilde{X}_{\geq 0}$ and $\psi_2|_{\operatorname{supp}(\phi_2)} \equiv 1$ hold.

For $u \in H^k_\delta(Z_\infty, S^-)$ we extend $\phi_2 u$ to an element of $H^k_\delta(\widetilde{X}, S^-)$ by zero to be able to apply Q. $\square$

To take the second step, let us start with the definition of the Fourier-Laplace transform as it is defined in [Tau87] and [MRS16]. See part (1) of the next lemma for the well-definedness.

Definition 3.13. Let $\ln : \mathbb{C}^* \to \mathbb{C}$ be a branch of the logarithm. The *Fourier-Laplace transform* on the space of all compactly supported smooth sections $C^\infty_0(\widetilde{X}, S)$ is defined as

$$\begin{aligned} \mathcal{F}^\Delta :\ C^\infty_0(\widetilde{X}, S) \quad &\longrightarrow \quad \{v : \mathbb{C}^* \to C^\infty(X, S)\} \\ (\mathcal{F}^\Delta u)(z)(x) := z^{f(x)} \sum_n z^n u(x+n), &\quad \text{with } z^{f(x)} := e^{f(x)\ln(z)}, \end{aligned} \tag{3.8}$$

where the sum goes from $-\infty$ to ∞. The notations $(\mathcal{F}^\Delta u)(z)(x) := (\mathcal{F}^\Delta_z u)(x) := \hat{u}_z(x)$ are also common.

There exists an abundance of statements round about this transformation, see for example [MRS16], [MRS11] and [Naz82]. It is also used in other areas in mathematics, where it is usually called *z-transform* [see Jur64]. The following lemma lists some properties of the Fourier-Laplace transform and prepares the next Proposition 3.15:

Lemma 3.14. *For a fixed branch of the logarithm the following statements hold:*

(1) The Fourier-Laplace transform is well defined.

(2) *The Fourier-Laplace transform is an isomorphism to its image with inverse*

$$\begin{aligned}&(\mathcal{F}^{\Delta})^{-1} : \mathcal{F}^{\Delta}(C_0^{\infty}(\widetilde{X}, S)) \to C_0^{\infty}(\widetilde{X}, S)\\ &(\mathcal{F}^{\Delta})^{-1}(v)(x) := \frac{1}{2\pi i}\int_{|z|=e^{\delta}} z^{-f(x)} v_z(p(x))\frac{dz}{z},\end{aligned} \tag{3.9}$$

where $p : \widetilde{X} \to X$ denotes the covering projection and $\delta \in \mathbb{R}$ is an arbitrary constant.

(3) *(Parseval's relation) For all $\delta \in \mathbb{R}$, $j \in \mathbb{N}_0$, $u, v \in C_0^{\infty}(\widetilde{X}, S)$ and $x \in \widetilde{X}$, we have*

$$\sum_n e^{2\delta f(x+n)}(\nabla^j u)(x+n) \cdot (\nabla^j v)(x+n) = \frac{1}{2\pi i}\int_{|z|=e^{\delta}} (\widehat{\nabla^j u})_z(x) \cdot (\widehat{\nabla^j v})_z(x)\frac{dz}{z}.$$

Here the '$\cdot$' denotes the Hermitian product in the fiber of the bundle $S \otimes T^*\widetilde{X}^{\otimes j} \to \widetilde{X}$ and $(\widehat{\nabla^j u})_z(x)$ is defined as $z^{f(x)} \sum_n z^n (\nabla^j u)(x+n)$, similarly for u interchanged with v.

Proof. Let $\ln : \mathbb{C}^* \to \mathbb{C}$ be a fixed branch of logarithm.

(1) The sum appearing in the definition of the Fourier-Laplace transform in Definition 3.13 is finite because the section u is compactly supported. For a fixed $z \in \mathbb{C}^*$ we obtain a smooth section of $S \to \widetilde{X}$ which is periodical under covering translations:

$$\hat{u}_z(x+m) = z^{f(x+m)} \sum_n z^n u(x+m+n) = \hat{u}_z(x), \quad \forall x \in \widetilde{X},\ \forall m \in \mathbb{Z}.$$

Here we used $f(x+1) = f(x) + 1$ for all $x \in X$ as stated in Remark 3.7. This periodicity allows us to interpret $\hat{u}_z$ as a smooth section of $S \to X$: For any $x_0 \in X$ define $\hat{u}_z(x_0) := \hat{u}_z(x)$ for any $x \in \widetilde{X}$ with $p(x) = x_0$.

(2) Lets us fix an arbitrary $\delta \in \mathbb{R}$. Then we obtain for any $u \in C_0^{\infty}(\widetilde{X}, S)$ and any $x \in \widetilde{X}$ under use of the Cauchy integral theorem

$$((\mathcal{F}^{\Delta})^{-1} \circ \mathcal{F}^{\Delta})(u)(x) = \frac{1}{2\pi i}\int_{|z|=e^{\delta}} z^{-f(x)} \underbrace{z^{f(x)} \sum_n z^n u(x+n)}_{=\hat{u}_z(p(x))} \frac{dz}{z} = u(x)$$

and the statement is proved.

(3) For $\delta \in \mathbb{R}$, $j \in \mathbb{N}_0$, $u, v \in C_0^\infty(\widetilde{X}, S)$ and $x \in \widetilde{X}$ we substitute the series for $(\widehat{\nabla^j u})_z(x)$ and $(\widehat{\nabla^j v})_z(x)$ as defined in the lemma into the right hand side of the claimed formula. This gives by using the properties of the Hermitian inner product in the fibers of $S \otimes T^*\widetilde{X}^{\otimes j}$ and $\bar{z} = |z|^2 z^{-1}$

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{|z|=e^\delta} (\widehat{\nabla^j u})_z(x) \cdot (\widehat{\nabla^j v})_z(x) \frac{dz}{z} \\
&\quad = \frac{1}{2\pi i}\int_{|z|=e^\delta} \sum_{n,m} \underbrace{z^{f(x)}\bar{z}^{f(x)} z^n \bar{z}^m}_{=e^{2\delta f(x+m)} z^{n-m}} (\nabla^j u)(x+n) \cdot (\nabla^j v)(x+m) \frac{dz}{z} \\
&\quad = \sum_n e^{2\delta f(x+n)} (\nabla^j u)(x+n) \cdot (\nabla^j v)(x+n).
\end{aligned}
$$

□

Proposition 3.15. *For every $\delta \in \mathbb{R}$ and $k \in \mathbb{N}_0$ the Fourier-Laplace transform extends to an isomorphism $\mathcal{F}^\Delta : H_\delta^k(\widetilde{X}, S) \to L^2(S_\delta, H^k(X, S))$. Here S_δ is defined as $\{z \in \mathbb{C} | |z| = e^\delta\}$ and the space $L^2(S_\delta, H^k(X, S))$ denotes the space of abstract measurable functions $u : S_\delta \to H^k(X, S)$ in the norm**

$$
\|u\|^2_{L^2(S_\delta, H^k(X,S))} \coloneqq \frac{1}{2\pi i}\int_{|z|=e^\delta} \|u_z\|^2_{H^k(X,S)} \frac{dz}{z}. \tag{3.10}
$$

For $k = 0$ the Fourier-Laplace transform $\mathcal{F}^\Delta$ is a linear isometry.

Proof. We fix a weight $\delta \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Since the inverse formula in Lemma 3.14 (2) holds, we have just to show that the maps

$$
\begin{aligned}
\mathcal{F}^\Delta : &\quad C_0^\infty(\widetilde{X}, S) \subset H_\delta^k(\widetilde{X}, S) &\longrightarrow&\quad L^2(S_\delta, H^k(X, S)) &(3.11)\\
(\mathcal{F}^\Delta)^{-1} : &\quad \mathcal{F}^\Delta(C_0^\infty(\widetilde{X}, S)) \subset L^2(S_\delta, H^k(X, S)) &\longrightarrow&\quad H_\delta^k(\widetilde{X}, S) &(3.12)
\end{aligned}
$$

are bounded and the subsets $C_0^\infty(\widetilde{X}, S)$ and $\mathcal{F}^\Delta(C_0^\infty(\widetilde{X}, S))$ are dense in $H_\delta^k(\widetilde{X}, S)$ and $L^2(S_\delta, H^k(X, S))$, respectively. Note that for any $j \in \{0, \ldots, k\}$, $u \in C_0^\infty(\widetilde{X}, S)$ and $x \in \widetilde{X}$ Parseval's relation given in Lemma 3.14 (3) implies

$$
\sum_n e^{2\delta f(x+n)} \left|(\nabla^j u)(x+n)\right|^2 = \frac{1}{2\pi i}\int_{|z|=e^\delta} \Big|\underbrace{z^{f(x)} \sum_n z^n (\nabla^j u)(x+n)}_{=(\widehat{\nabla^j u})(x)}\Big|^2 \frac{dz}{z}. \tag{3.13}
$$

* The norm is defined in such a way that $L^2(S_\delta, H^k(X, S))$ is isometric isomorphic to $L^2([0, 2\pi], H^k(X, S))$ with the linear isometry given by $u \mapsto (t \mapsto u_{e^\delta e^{it}})$.

We estimate for $u \in C_0^\infty(\widetilde{X}, S)$ with '=' instead of '$\lesssim$'* in the case $k = 0$

$$\begin{aligned}
\|\mathcal{F}^\Delta u\|^2_{L^2(S_\delta, H^k(X,S))} &\overset{\text{eq.(3.10)}}{\underset{\text{eq.(1.2)}}{=}} \frac{1}{2\pi i}\int_{|z|=e^\delta}\left(\int_X \sum_{j=0}^k \left|\nabla^j\left(z^{f(x)}\sum_n z^n u(x+n)\right)\right|^2 dx\right)\frac{dz}{z} \\
&\overset{(\star)}{\lesssim} \frac{1}{2\pi i}\int_{|z|=e^\delta}\left(\int_X \sum_{j=0}^k \left|z^{f(x)}\sum_n z^n (\nabla^j u)(x+n)\right|^2 dx\right)\frac{dz}{z} \\
&\overset{\text{eq.(3.13)}}{=} \int_X \sum_n \sum_{j=0}^k e^{2\delta f(x+n)}\left|(\nabla^j u)(x+n)\right|^2 dx \\
&\lesssim \int_{\widetilde{X}} \sum_{j=0}^k \left|\nabla^j (e^{\delta f(x)} u(x))\right|^2 dx \\
&\overset{\text{Def.2.18}}{\underset{\text{eq.(1.2)}}{=}} \|u\|^2_{H^k_\delta(\widetilde{X},S)}.
\end{aligned}$$

The estimate ($\star$) holds by writing out all terms under use of the chain rule and exploit that df is bounded as maps from $H^{k+1} \to H^k$. This can be shown by a straightforward calculation using the compatibility of the Clifford multiplication with the connection, the periodicity of df and the continuity of df and all its covariant derivatives. By definition of the weighted Sobolev spaces $C_0^\infty(\widetilde{X}, S)$ is dense in $H^k_\delta(\widetilde{X}, S)$ and the Fourier-Laplace transform in equation (3.11) extends continuously to $H^k_\delta(\widetilde{X}, S)$. Analogously, we can estimate the norm of the inverse Fourier-Laplace transform for every $\hat{u} \in \mathcal{F}^\Delta(C_0^\infty(\widetilde{X}, S))$ with associated $u \in C_0^\infty(\widetilde{X}, S)$ as follows:

$$\begin{aligned}
\|(\mathcal{F}^\Delta)^{-1}\hat{u}\|^2_{H^k_\delta(\widetilde{X},S)} &\overset{\text{Lemma 3.14(2)}}{\underset{\text{Def.2.18}}{=}} \int_{\widetilde{X}} \sum_{j=0}^k \left|\nabla^j (e^{\delta f} u)(x)\right|^2 dx \\
&\overset{(\star\star)}{\lesssim} \int_X \sum_n \sum_{j=0}^k e^{2\delta f(x+n)}\left|(\nabla^j u)(x+n)\right|^2 dx \\
&\overset{\text{eq.(3.13)}}{=} \int_X \sum_{j=0}^k \frac{1}{2\pi i}\int_{|z|=e^\delta}\left|z^{f(x)}\sum_n z^n (\nabla^j u)(x+n)\right|^2 \frac{dz}{z} dx \\
&\lesssim \frac{1}{2\pi i}\int_{|z|=e^\delta}\int_X \sum_{j=0}^k \Bigg|\nabla^j \underbrace{\left(z^{f(x)}\sum_n z^n u(x+n)\right)}_{\overset{\text{Def.3.13}}{=}\hat{u}(x)}\Bigg|^2 dx \frac{dz}{z} \\
&\overset{\text{eq.(3.10)}}{=} \|\hat{u}\|^2_{L^2(S_\delta, H^k(X,S))}.
\end{aligned} \tag{3.14}$$

Equation ($\star\star$) holds by using that df is bounded as maps from $H^{k+1} \to H^k$ and the chain rule. It remains to show that $\mathcal{F}^\Delta(C_0^\infty(\widetilde{X}, S))$ is dense in $L^2(S_\delta, H^k(X, S))$, because then the inverse Fourier-Laplace transform in equation (3.12) extends to $L^2(S_\delta, H^k(X, S))$.

* We write '$\lesssim$' if the term can be estimated by '$\leq$' modulo a not relevant constant.

Using that $\mathcal{F}^\Delta$ is bounded, $\overline{C_0^\infty(\widetilde{X},S)}^{H_\delta^k} = H_\delta^k(\widetilde{X},S)$ and the completeness of $L^2(S_\delta, H^k(X,S))$ gives

$$\begin{aligned} & L^2(S_\delta, H^k(X,S)) \overset{(\star\star\star)}{\subset} \mathcal{F}^\Delta(H_\delta^k(\widetilde{X},S)) \subset \overline{\mathcal{F}^\Delta(C_0^\infty(\widetilde{X},S))}^{L^2} \subset L^2(S_\delta, H^k(X,S)) \\ \Rightarrow \quad & \overline{\mathcal{F}^\Delta(C_0^\infty(\widetilde{X},S))}^{L^2} = L^2(S_\delta, H^k(X,S)). \end{aligned}$$

To see that $(\star\star\star)$ holds we take an arbitrary element $u \in L^2(S_\delta, H^k(X,S))$. Then $z^{-f}u$ is an element of $L^2(S_\delta, H^k(W_0,S))$, where we identify X with W_0 almost everywhere (so $H^k(X,S) \cong H^k(W_0,S)$). By Fourier decomposition, there exist coefficients $a_n \in H^k(W_0,S)$ such that

$$z^{-f}u_z = \sum_n a_n z^n \quad \text{(convergence in the } L^2(S_\delta, H^k(X,S)) \text{ norm)} \tag{3.15}$$

holds. Furthermore, we define $b(x_0+n) \coloneqq a_n(x_0) \in S_{x_0} \cong S_{x_0+n}$ for almost all $x_0+n \in W_n \subset \widetilde{X}$. By definition of b and equation (3.15) we obtain $u_z(\bullet) = z^{f(\bullet)} \sum_n z^n b(\bullet + n)$ with $b \in H_\delta^k(\widetilde{X},S)$ (this follows by a straightforward calculation similarly as in equation (3.14) and exploiting that the sum in equation (3.15) converges in $L^2(S_\delta(H^k(X,S))))$, hence u is an element of $\mathcal{F}^\Delta(H_\delta^k(\widetilde{X},S))$. $\square$

The Fourier-Laplace transform respects the grading of the Clifford bundle S, hence Definition 3.13, Lemma 3.14 and Proposition 3.15 also holds for $S^{\pm}$ instead of S. Now we are prepared to define the family of twisted Dirac operators acting on $C^{\infty}(X, S^{(\pm)})$ into which we will convert the Dirac operator $\mathcal{D}^{+}(\widetilde{X})$ by conjugation with the Fourier-Laplace transform. Next we define the family of twisted Dirac operators explicitly and show the link to the Fourier-Laplace transform later in Lemma 3.17.

Definition 3.16. For a fixed branch of logarithm, the family of twisted Dirac operators* is defined as

$$\mathcal{D}_z^{(\pm)}(X) := \mathcal{D}_z^{(\pm)} := \mathcal{D}^{\pm}(X) - \ln z \cdot df : C^{\infty}(X, S^{(\pm)}) \to C^{\infty}(X, S^{(\mp)}), \quad z \in \mathbb{C}^*. \tag{3.16}$$

Here we see df as a one-form on X as explained in Remark 3.7 and it acts as Clifford multiplication on $C^{\infty}(X, S^{(\pm)})$.

The next lemma and the next remark are formulated for the twisted Dirac operator $\mathcal{D}^{+}(X)$. The statement also holds for $\mathcal{D}^{-}(X)$ by interchanging S^{+} with S^{-}.

Lemma 3.17. *For $k \in \mathbb{N}_0$, $\delta \in \mathbb{R}$ and a fixed branch of logarithm the following holds:*

(1) For all $z \in \mathbb{C}^*$ the twisted Dirac operator extends to a bounded operator $\mathcal{D}_z^{+}(X) : H^{k+1}(X, S^{+}) \to H^{k}(X, S^{-})$.

(2) The family of twisted Dirac operators $\mathcal{D}_z^{+}(X)$ fits together to a well defined map

$$\mathcal{D}_\bullet^{+}(X) : \mathcal{F}^{\Delta}(C_0^{\infty}(\widetilde{X}, S)) \to L^2(S_\delta, H^k(X, S^{-})),\ \mathcal{D}_\bullet^{+}(X)(u)(z) := \mathcal{D}_z^{+} u_z \tag{3.17}$$

extending to a bounded operator from $L^2(S_\delta, H^{k+1}(X, S^{+}))$ to $L^2(S_\delta, H^k(X, S^{-}))$.

(3) The following two diagrams commute for all $z \in \mathbb{C}^*$:

$$\begin{tikzcd} C_0^{\infty}(\widetilde{X}, S^{+}) \arrow[r, "\mathcal{D}^{+}(\widetilde{X})"] \arrow[d, "\mathcal{F}_z^{\Delta}"'] & C_0^{\infty}(\widetilde{X}, S^{-}) \arrow[d, "\mathcal{F}_z^{\Delta}"'] \\ C^{\infty}(X, S^{+}) \arrow[r, "\mathcal{D}_z^{+}(X)"] & C^{\infty}(X, S^{-}) \end{tikzcd} \qquad \begin{tikzcd} H_\delta^{k+1}(\widetilde{X}, S^{+}) \arrow[r, "\mathcal{D}^{+}(\widetilde{X})"] \arrow[d, "\mathcal{F}^{\Delta}"', "\cong"] & H_\delta^{k}(\widetilde{X}, S^{-}) \arrow[d, "\mathcal{F}^{\Delta}"', "\cong"] \\ L^2(S_\delta, H^{k+1}(X, S^{+})) \arrow[r, "\mathcal{D}_\bullet^{+}(X)"] & L^2(S_\delta, H^{k}(X, S^{-})). \end{tikzcd} \tag{3.18}$$

* Both notations, $\mathcal{D}_z^{(\pm)}(X)$ and $\mathcal{D}_z^{(\pm)}$, will be used later on. Preferably the shorter notation in formulas and the lengthily one if it should be emphasized that it acts on section of $S \to X$.

Proof. Let $k \in \mathbb{N}_0$, $\delta \in \mathbb{R}$ and a branch of logarithm be fixed.

(1) For all $z \in \mathbb{C}^*$ there exists a constant $C > 0$ such that for all $u \in C^\infty(X, S^+)$ the following holds:

$$\|\mathcal{D}^+_z u\|^2_{H^k(X,S^-)} \overset{\text{Def.3.16}}{\leq} 2\|\mathcal{D}^+(X)u\|^2_{H^k(X,S^-)} + 2\|\ln z \cdot df u\|^2_{H^k(X,S^-)} \leq C\|u\|^2_{H^{k+1}(X,S^+)}.$$

In the last step we used that the chiral Dirac operator $\mathcal{D}^+(X)$ and df are bounded as maps from $H^{k+1} \to H^k$.

(2) Because of part (1) there exists a constant $C > 0$ such that we can estimate for all $\hat{u} \in \mathcal{F}^\Delta(C^\infty_0(\widetilde{X}, S))$

$$\|\mathcal{D}^+_\bullet(X)\hat{u}\|^2_{L^2(S_\delta, H^k(X,S^-))} = \frac{1}{2\pi i}\int_{|z|=e^\delta} \underbrace{\|\mathcal{D}^+_z \hat{u}_z\|^2_{H^k(X,S^-)}}_{\leq C\|\hat{u}_z\|^2_{H^{k+1}(X,S^+)}} \frac{dz}{z} \leq C\|\hat{u}\|^2_{L^2(S_\delta, H^{k+1}(X,S^+))}.$$

Using that $\mathcal{F}^\Delta(C^\infty_0(\widetilde{X}, S))$ is dense in $L^2(S_\delta, H^k(X, S))$, as in the proof of Proposition 3.15, leads to the claimed extension.

(3) The chiral Dirac operator $\mathcal{D}^+(Z_\infty)$ over the end-periodic manifold Z_∞ is end-periodic. This means that $(\mathcal{D}^+(X)u)(x_0) = (\mathcal{D}^+(\widetilde{X})\widetilde{u})(x)$ holds for all $u \in C^\infty(X, S^+)$, $x_0 \in X$, $x \in \widetilde{X}$ with $p(x) = x_0$ and $\widetilde{u} \in C^\infty(\widetilde{X}, S)$ with $\widetilde{u}(\widetilde{x}+n) = u(p(\widetilde{x}))$ for all $\widetilde{x} \in \widetilde{X}$ and $n \in \mathbb{Z}$. Then we obtain for every $z \in \mathbb{C}^*$, $u \in C^\infty_0(\widetilde{X}, S^+)$, $x_0 \in X$ and $x \in \widetilde{X}$ with $p(x) = x_0$

$$\begin{aligned}
(\mathcal{D}^+_z \circ \mathcal{F}^\Delta_z)(u)(x_0) &= \left(\mathcal{D}^+(X) - \ln z \cdot df\right)\Big(z^{f(x)}\sum_n z^n u(x+n)\Big) \\
&= z^{f(x)}\sum_n z^n(\mathcal{D}^+(\widetilde{X})u)(x+n) \\
&= (\mathcal{F}^\Delta_z \circ \mathcal{D}^+(\widetilde{X}))(u)(x_0)
\end{aligned}$$

and the diagram on the left hand side in (3.18) commutes. In the second step we take advantage of the fact that any Dirac operator takes locally the form $\sum_i e_i \nabla_{e_i}$ and we can use the Leibniz rule. Moreover, a straightforward calculation leads by using the definition of $\mathcal{D}^+_\bullet$ in equation (3.17) to the commutativity of the right diagram in (3.18) for all compactly supported smooth sections of $S^+ \to \widetilde{X}$. All involved maps extend continuously to the Sobolev spaces as written in the diagram (Lemma 3.11, Proposition 3.15 and part (1) of this lemma), hence it commutes for all elements of $H^{k+1}_\delta(\widetilde{X}, S^+)$ and the lemma is proved. □

Remark 3.18. (1) ‘Twisted’ ‘in Definition 3.16 of $\mathcal{D}^+_z$ means that the chiral Dirac operator $D_z^+(X)$ results by the Clifford bundle $S \otimes E_z \to X$ for an appropriate line bundle $E_z \to X$. Indeed, let $z \in \mathbb{C}^*$ be fixed and $E_z = \mathbb{C} \times X \to X$ the trivial line bundle equipped with the trivial metric and the connection $\nabla_X u_2 \coloneqq X(u_2) - \ln(z) \cdot df(X)u_2$ for u_2 a smooth section of E_z and X a vector field. A straightforward calculation shows that the connection is flat and respects the metric if $|z| = 1$. Then $S \otimes E_z = (S^+ \otimes E_z) \oplus (S^- \otimes E_z) \to X$ equipped with the tensor metric, the tensor connection and the Clifford multiplication acting on the first component gives a graded Clifford bundle. The induced chiral Dirac operator $\mathcal{D}_{\mathrm{tw}}^+$ takes for a local orthonormal frame $e_1, \dots, e_n$ of the cotangent bundle and $u = u_1 \otimes u_2$ a smooth section of $S^+ \otimes E_z$ the form

$$\begin{aligned}\mathcal{D}_{\mathrm{tw}}^+ u = \sum_{j=1}^n e_j \cdot \nabla_{e_j}(u) &= \sum_{j=1}^n e_j \cdot \Big(\nabla_{e_j} u_1 \otimes u_2 + u_1 \otimes \big(e_j(u_2) - \ln(z) \cdot df(e_j)u_2\big)\Big) \\ &= \mathcal{D}(X)u_1 \otimes u_2 - \ln(z) df u_1 \otimes u_2 + \underbrace{\sum_{j=1}^n e_j \cdot u_1 \otimes e_j(u_2)}_{=0 \text{ if } u_2 = 1}.\end{aligned}$$

We can identify $\mathbb{C}$-linearly smooth sections of $S \otimes E_z$ with smooth sections of S via $u_1 \otimes 1 \mapsto u_1$, hence $\mathcal{D}_{\mathrm{tw}}^+$ matches with $D_z^+(X)$ from Definition 3.16, i.e. the following diagram commutes:

$$\begin{tikzcd} C^\infty(X, S^+) \arrow[r, "\mathcal{D}_z^+"] \arrow[d, "\cong"'] & C^\infty(X, S^-) \arrow[d, "\cong"] \\ C^\infty(X, S^+ \otimes E_z) \arrow[r, "\mathcal{D}_{\mathrm{tw}}^+"] & C^\infty(X, S^- \otimes E_z). \end{tikzcd} \tag{3.19}$$

(2) The family of twisted Dirac operators $\mathcal{D}_z^+(X)$ defined in Definition 3.16 depends on the choice of a branch of logarithm. If we change this branch, there exists an $n \in \mathbb{Z}$ such that $\widetilde{\ln} z = \ln z + 2\pi i n$ holds for all $z \in \mathbb{C}^*$. Denoting the induced family of twisted Dirac operators with $\widetilde{\mathcal{D}}_z^+(X)$, it follows for all $z \in \mathbb{C}^*$ and $u \in C^\infty(X, S^+)$

$$\big(\widetilde{\mathcal{D}}_z^+(X) e^{2\pi i n f}\big)(u) = \big(\mathcal{D}^+(X) - \ln z \cdot df - 2\pi i n df\big)\big(e^{2\pi i n f} u\big) = e^{2\pi i n f} \mathcal{D}_z^+ u.$$

Here $e^{2\pi i n f} : H^k(X, S^\pm) \to H^k(X, S^\pm)$ is an isomorphism by fiberwise multiplication. Indeed, it is well defined on smooth sections by Remark 3.7, extends to the Sobolev spaces by estimating the norm and has $e^{-2\pi i n f}$ as inverse. This implies that the choice of a branch of logarithm does not influence the invertibility of $\mathcal{D}_z^+(X)$, hence we no longer mention the branch of logarithm as for example in the Fredholm criterion in Proposition 3.19.

Now we can formulate the Fredholm criterion analogously to the cylindrical case in Proposition 2.15:

Proposition 3.19 ([Tau87, Lemma 4.3]). *Let $\delta \in \mathbb{R}$ and $k \in N_0$. The operator*

$$\mathcal{D}^+(Z_\infty) : H_\delta^{k+1}(Z_\infty, S^+) \to H_\delta^k(Z_\infty, S^-) \tag{3.20}$$

is Fredholm if and only if the twisted Dirac operator $\mathcal{D}_z^+ : H^{k+1}(X, S^+) \to H^k(X, S^-)$ is invertible for all $z \in \mathbb{C}^*$ with $|z| = e^\delta$.

Proof (sketch). For a full proof of this statement see [Tau87, Lemma 4.3]. We will focus on the backward implication, where the Fourier-Laplace transform comes into play. Let $\delta \in \mathbb{R}$ and $k \in \mathbb{N}_0$ be fixed and let the operators $\mathcal{D}_z^+ : H^{k+1}(X, S^+) \to H^k(X, S^-)$ be invertible on the circle $|z| = e^\delta$. The main idea is to construct an inverse to the chiral Dirac operator $\mathcal{D}^+(\widetilde{X})$. Then $\mathcal{D}^+(\widetilde{X})$ would be Fredholm, because the kernel and cokernel are trivial, and together with Lemma 3.12 we would obtain the Fredholmness of $\mathcal{D}^+(Z_\infty)$.

Note that by Lemma 3.17 (3) the diagram on the right hand side of (3.18) commutes. Furthermore, the Fourier-Laplace transform is invertible by Proposition 3.15 so that it is enough to find an inverse to $\mathcal{D}_\bullet^+(X)$ defined in Lemma 3.17 (2). By assumption there exist inverse operators $(D_z^+)^{-1} : H^k(X, S^-) \to H^{k+1}(Z, S^+)$ which are by the bounded inverse theorem bounded. Then there exist constants $C_z > 0$ with continuous z dependence such that for all $u \in C^\infty(X, S^-)$

$$\|(\mathcal{D}_z^+)^{-1}u\|^2_{H^k(X,S^+)} \leq C_z \|u\|^2_{H^{k+1}(X,S^-)} \tag{3.21}$$

holds. Furthermore, we define

$$\mathcal{D}_\bullet^+(X)^{-1} : \mathcal{F}^\Delta(C_0^\infty(\widetilde{X}, S)) \to L^2(S_\delta, H^{k+1}(X, S^+)), \quad \mathcal{D}_\bullet^+(X)^{-1}(u)(z) \coloneqq (\mathcal{D}_z^+)^{-1}u_z.$$

This is well defined and extends to a map from $L^2(S_\delta, H^k(X, S^-))$, since for all $u \in$

$\mathcal{F}^{\Delta}(C_0^\infty(\widetilde{X}, S))$ the estimate

$$\begin{aligned}\|\mathcal{D}_\bullet^+(X)^{-1}(u)\|^2_{L^2(S_\delta, H^{k+1}(X,S^+))} &= \frac{1}{2\pi i}\int_{|z|=e^\delta} \underbrace{\|(\mathcal{D}_z^+)^{-1}u_z\|^2_{H^k(X,S^-)}}_{\overset{\text{eq.(3.21)}}{\leq} C_z\|u_z\|^2_{H^{k+1}(X,S^+)}} \frac{dz}{z} \\ &\leq \underbrace{\sup\nolimits_{|z|=1} C_z}_{<\infty} \|u\|^2_{L^2(S_\delta, H^k(X,S^-))}\end{aligned}$$

holds. Using the definitions of the involved operators, a straightforward calculation shows that $\mathcal{D}_\bullet^+(X)^{-1}$ is inverse to $\mathcal{D}_\bullet^+(X)$ and the backward direction of the proposition is proved. $\square$

Corollary 3.20. *Let $k \in \mathbb{N}_0$. If there exists a $\delta \in \mathbb{R}$ such that the chiral Dirac operator $\mathcal{D}^+(Z_\infty) : H_\delta^{k+1}(Z_\infty, S^+) \to H_\delta^k(Z_\infty, S^-)$ is Fredholm then the index of $\mathcal{D}(X)$ vanishes.*

Proof. Fix a $k \in \mathbb{N}_0$ and assume that there exist a $\delta \in \mathbb{R}$ such that $\mathcal{D}^+(Z_\infty) : H_\delta^{k+1}(Z_\infty, S^+) \to H_\delta^k(Z_\infty, S^-)$ is Fredholm. The previous Proposition 3.19 implies by inserting $z = e^\delta$ that $\mathcal{D}_{e^\delta}^+(X) = \mathcal{D}^+(X) - \delta df$ is invertible, hence is an Fredholm operator with vanishing index. The two operators $\mathcal{D}_{e^\delta}^+(X)$ and $\mathcal{D}^+(X)$ differ by the compact operator δdf which implies that the Fredholm index of $\mathcal{D}^+(X)$ also vanishes [Gru09, Theorem 8.10]. $\square$

In the cylindrical case, Proposition 2.20 gives that for all weights δ close enough to zero, the Dirac operator between the associated weighted Sobolev spaces is Fredholm. We want to make an analogue statement in the end-periodic case. For this purpose we define the *spectral set* of the family $\{\mathcal{D}_z^\pm(X)\}_{z\in\mathbb{C}^*}$ as the set of all points $z \in \mathbb{C}^*$ such that $D_z^\pm(X)$ is not invertible. In the following theorem a sufficient condition such that the spectral set is a discrete subset of $\mathbb{C}^*$ is given. This result is due to Taubes [Tau87, Theorem 3.1] and we state it here without proof:

Theorem 3.21 ([MRS16, Theorem 2.3]). *Suppose that* $\mathrm{ind}(\mathcal{D}(X)) = 0$ *and the map* $df : \ker \mathcal{D}^+(X) \to \operatorname{coker} \mathcal{D}^+(X)$ *given by Clifford multiplication is injective. Then the spectral set of the family $\{D_z^+(X)\}_{z\in\mathbb{C}^*}$ is a discrete subset of $\mathbb{C}^*$. In particular for every $k \in \mathbb{N}_0$ the operator $\mathcal{D}^+(Z_\infty) : H_\delta^{k+1}(Z_\infty, S^+) \to H_\delta^k(Z_\infty, S^-)$ is Fredholm for all but a discrete set of $\delta \in \mathbb{R}$.*

3.3 The end-periodic index theorem in the Fredholm case

In this section we will study the index of an end-periodic Dirac operator whenever the chiral Dirac operator is Fredholm and prove the end-periodic index theorem by Mrowka, Ruberman, and Saveliev [MRS16, Theorem A]. In the first Section 3.3.1 we will state the main theorem and give an outline of the proof. The details of this proof are worked out in the three Sections 3.3.2-3.3.4. In Section 3.3.5 we will study the connection of the end-periodic index theorem to the classical APS index theorem. There the reformulation of the classical APS index theorem with boundary condition into the cylindrical setting (Section 2.3) is crucial. Section 3.3 is based on [MRS16].

3.3.1 The main theorem and an outlook of the proof

Theorem 3.22 (End-periodic index theorem in the Fredholm case [MRS16, Theorem A])**.** *Let S be an end-periodic graded Clifford bundle over an even-dimensional end-periodic Riemannian manifold Z_∞ modeled by (Z, X, γ). Furthermore, $f : \widetilde{X} \to \mathbb{R}$ is the function defined in the proof of Lemma* 3.6 *(2) and Y denotes the boundary of Z. If the L^2-closure of the chiral Dirac operator $\mathcal{D}^+(Z_\infty)$ is Fredholm, the index of the Dirac operator $\mathcal{D}(Z_\infty)$ is given by*

$$\operatorname{ind}\left(\mathcal{D}(Z_\infty)\right) = \int_Z \boldsymbol{I}(\mathcal{D}(Z_\infty)) - \int_Y \omega + \int_X df \wedge \omega - \frac{1}{2}\eta_{ep}(\mathcal{D}(X)). \tag{3.22}$$

Here ω is a differential form on X such that $d\omega = \boldsymbol{I}(\mathcal{D}(X))$ holds for the local index form of $\mathcal{D}(X)$ (Definition 1.13*) and the so-called periodic η-invariant*is defined as*

$$\eta_{ep}(\mathcal{D}(X)) \coloneqq \frac{1}{i\pi}\int_0^\infty \int_{|z|=1} Tr\left(df \cdot D_z^+ e^{-tD_z^-\mathcal{D}_z^+}\right) \frac{dz}{z} dt \tag{3.23}$$

with the twisted Dirac operator $\mathcal{D}_z^+(X)$ as in Definition 3.16*.*

Remark 3.23. Let any notation be as in the previous theorem and assume that the L^2-closure of $\mathcal{D}^+(Z_\infty)$ is Fredholm.

* The end-periodic index formula in equation (3.22) implies that the integral in the definition of the end-periodic η-invariant exists.

(1) With Corollary 3.20 the index of $\mathcal{D}(X)$ vanishes and we obtain with the Atiyah-Singer index theorem (Theorem 1.16) $\int_X \mathbf{I}(\mathcal{D}(X)) = 0$. The map

$$H^n_{\mathrm{dR}}(X) \to \mathbb{R}, \quad [\alpha] \mapsto \int_X \alpha, \tag{3.24}$$

is by the de Rham isomorphism bijective, hence we obtain that the local index form $\mathbf{I}(\mathcal{D}(X))$ is exact and ω in the previous theorem exists. It will be shown in Remark 3.35 that the right side of equation (3.22) is independent of the choice of ω.

(2) The periodic η-invariant is a spectral invariant as explained in [MRS16, Section 6.1]. We have seen this for the classical η-invariant in Remark 2.11 and will summarize it here for the end-periodic η-invariant. Following the proof of Corollary 3.20 gives that $\mathcal{D}^+(X)$ is invertible, so that Theorem 3.21 is applicable and it follows that the spectral set of $\{\mathcal{D}^-_z\}_{z\in\mathbb{C}^*}$ is a discrete subset of $\mathbb{C}^*$. Let $\{z_k | k \in \mathbb{N}\}$ be the spectral set of $\{\mathcal{D}^-_z\}_{z\in\mathbb{C}^*}$ and write for a chosen branch of logarithm $z_k = \mu_k + 2\pi i n$, $n \in \mathbb{Z}$. Then the eigenvalues of the operator $K := df \cdot (\mathcal{D}^-(X))^{-1}$ are of the form $\lambda_{k,n} = \frac{1}{u_k + 2\pi i n}$ and for $k \in \mathbb{N}$ we define V_k as the generalized eigenspace of K corresponding to $\lambda_{k,0}$. Then we can interpret the periodic η-invariant as a regularization of the infinite sum

$$\sum_{k=1}^{\infty} \operatorname{sign} \ln |z_k| \cdot \dim V_k.$$

We will next provide an outline of the proof of the end-periodic index theorem stated in Theorem 3.22. The details are worked out in the next three subsections. While going through this outline, we invite the reader to check simultaneously the proof of the Atiyah-Singer index theorem (Theorem 1.16) and the APS index theorem (Theorem 2.16) to see the impressing analogy. Throughout the proof - the outlook below and the next three subsections - all notations and assumptions are as given in the end-periodic index theorem (Theorem 3.22). Furthermore, we will use the following shorthand notations unless stated otherwise:

Notation 3.24. (1) We write shortly $\mathcal{D}^{(\pm)} := \mathcal{D}^{(\pm)}(Z_\infty)$ and $\widetilde{\mathcal{D}}^{(\pm)} := \mathcal{D}^{(\pm)}(\widetilde{X})$. Sometimes, when the underlying manifold should be highlighted, the extensive notation is still used.

(2) Let $\mathcal{P}$ be a smoothing operator and $h : \mathbb{R} \to \mathbb{R}$ a rapidly decaying function. Then

we denote with $k_{\mathcal{P}}$ the smoothing kernel associated to $\mathcal{P}$. Furthermore, if $\mathcal{P}$ is of the form $h(\mathcal{D})$ or $h(\mathcal{D}^{\mp}\mathcal{D}^{\pm})$, we define

$$\begin{aligned}
&\widetilde{\mathcal{P}} := h(\widetilde{\mathcal{D}}) \quad \text{respectifly} \quad h(\widetilde{\mathcal{D}}^{\mp}\widetilde{\mathcal{D}}^{\pm}) \text{ and} \\
&\mathcal{P}_z := h(\mathcal{D}_z) \quad \text{respectifly} \quad h(D_z^{\mp}\mathcal{D}_z^{\pm}), \quad \forall z \in \mathbb{C}^*.
\end{aligned}$$

(3) We denote an operator of the form $\mathcal{D}^m \exp(-t\mathcal{D}^2)$ or $(\mathcal{D}^{\mp}\mathcal{D}^{\pm})^m \exp(-t\mathcal{D}^{\mp}\mathcal{D}^{\pm})$ for $m \geq 0$ with P. This notation is quoted when used as we may deviate in some situations.

Proof of Theorem 3.22 *(outline).* Define for $t > 0$ the *(regularized) McKean-Singer function* as

$$h^r(t) := \operatorname{Tr}_s^r\left(e^{-t\mathcal{D}^2}\right) := \operatorname{Tr}^r\left(e^{-t\mathcal{D}^-\mathcal{D}^+}\right) - \operatorname{Tr}^r\left(e^{-t\mathcal{D}^+\mathcal{D}^-}\right), \tag{3.25}$$

where the *regularized trace* is defined for P as in Notation 3.24 (3) via

$$\operatorname{Tr}^r(P) := \lim_{N\to\infty}\left(\int_{Z_N} \operatorname{tr}\left(k_P(x,x)\right)dx - (N+1)\int_{W_0} \operatorname{tr}\left(k_{\tilde{P}}(x,x)\right)dx\right) \tag{3.26}$$

with $Z_N := Z \cup_Y W_0 \cup_Y \ldots \cup_Y W_N$ for $N \geq 0$. By Lemma 3.30 the regularized trace is well defined and a continuous function on $t \in (0,\infty)$, hence h^r is also continuous in $t \in (0,\infty)$. Moreover, it satisfies the same properties as the McKean-Singer function h in the proof of the Atiyah-Singer index theorem with the differences that the regularized trace appears and part (3) changes slightly, but crucially:

$$\begin{aligned}
&(1) \quad \lim_{t\to\infty} h^r(t) &&= \operatorname{ind}(\mathcal{D}(Z_\infty)), &&\text{Proposition 3.36 (1).} \\
&(2) \quad \lim_{t\to 0} h^r(t) &&= \int_Z \mathbf{I}(\mathcal{D}^+(Z_\infty)), &&\text{Proposition 3.36 (2).} \\
&(3) \quad \frac{d}{dt}h^r(t) &&= -\operatorname{Tr}^r\left[\mathcal{D}^-, \mathcal{D}^+ e^{-t\mathcal{D}^-\mathcal{D}^+}\right], &&\text{Proposition 3.34 (1).}
\end{aligned} \tag{3.27}$$

The regularized trace in the third part does not vanish on the commutator because it fails to be a true trace. Anyway, it can be expressed explicitly in terms of data on X using the family of twisted Dirac operators $\mathcal{D}_z^{\pm}$ defined in Definition 3.16:

$$\begin{aligned}
\operatorname{Tr}^r[\mathcal{D}^-, \mathcal{D}^+ e^{-t\mathcal{D}^-\mathcal{D}^+}] = &\frac{1}{2\pi i}\int_{|z|=1}\left(\int_{W_0} f \cdot \operatorname{tr}\left(k_{P_z Q_z}(x,x) - k_{Q_z P_z}(x,x)\right)dx\right)\frac{dz}{z} \\
&+ \frac{1}{2\pi i}\int_{|z|=1} \operatorname{Tr}\left(df \cdot Q_z\right)\frac{dz}{z}.
\end{aligned} \tag{3.28}$$

Here the shorthand notation $P_z := D_z^-$ and $Q_z := \mathcal{D}_z^+ e^{-t\mathcal{D}_z^-\mathcal{D}_z^+}$ is used and a proof of this formula is given in Proposition 3.34 (2). The appearance of this term, which does not vanish in general as in the proof of the Atiyah-Singer index theorem, leads to the additional terms in the index formula. Combining the three equations in (3.27) with the explicit expression of the regularized trace of the commutator in equation (3.28) and the fundamental theorem of calculus leads to the stated index formula

$$\begin{aligned}
\operatorname{ind}(\mathcal{D}) - \int_Z \mathbf{I}(\mathcal{D}^+(Z)) &= \int_0^\infty \frac{d}{dt} h^r(t)dt \\
&= \underbrace{\frac{1}{2\pi i}\int_0^\infty \int_{|z|=1} \left(\int_{W_0} f(x)\cdot \operatorname{tr}\left(k_{Q_zP_z}(x,x) - k_{P_zQ_z}(x,x)\right)dx\right)\frac{dz}{z}dt}_{\overset{\text{Prop.3.34(3)}}{=} -\int_Y \omega + \int_X df\wedge\omega} \\
&\quad \underbrace{-\frac{1}{2\pi i}\int_0^\infty \int_{|z|=1} \operatorname{Tr}\left(df\cdot \mathcal{D}_z^+ e^{-t\mathcal{D}_z^-\mathcal{D}_z^+}\right)\frac{dz}{z}dt}_{\underset{\text{in eq.(3.23)}}{\overset{\text{def. }\eta_{\text{ep}}}{=}} \frac{1}{2}\eta_{\text{ep}}(\mathcal{D}(X))} \\
&= -\int_Y \omega + \int_X df\wedge\omega - \frac{1}{2}\eta_{\text{ep}}(\mathcal{D}(X)).
\end{aligned} \tag{3.29}$$

Note that the dependence of t in the previous equation is hidden in Q_z. □

3.3.2 Heat kernel estimates in the end-periodic setting

In this subsection we summarize all estimates for the smoothing kernel of the operator $\exp(-t\mathcal{D}^2)$ that we need for the proof of the end-periodic index theorem. These estimates are specific for end-periodic Dirac operators and are in general not true for Dirac operators induced by Clifford bundles of bounded geometry as the estimate stated in Proposition 1.8. The proofs are worked out in [MRS16, section 10.3-10.5] and will be skipped or shortly sketched here.

The next two propositions give some estimates of the kernel at the diagonal and close to the diagonal. It follows three long-time behavior estimates in Proposition 3.27.

Proposition 3.25 ([MRS16, Proposition 10.6]). *Let $k_t(x,y)$ and $\widetilde{k}_t(x,y)$ denote the smoothing kernels of $\exp(-t\mathcal{D}^2)$ and $\exp(-t\widetilde{\mathcal{D}}^2)$. Then there exist positive constants α, γ and C such that for all $t > 0$ and all $x \in W_l$ with $l \geq 1$*

$$\left|k_t(x,x) - \widetilde{k}_t(x,x)\right| \leq Ce^{\alpha t}e^{-\gamma d^2(x,W_0)/t}$$

holds. Here $|\cdot|$ *is the induced norm of the bundle metric in the fiber* $(S \boxtimes S^*)_{(x,x)}$.

Proof (sketch). The proof is worked out in [MRS16, p.61-65]. The basic idea is to built an approximate smoothing kernel $k_t^{\text{ap.}}$ of $\exp(-t\mathcal{D}^2)$ on Z_∞ and estimating the difference to the smoothing kernel of $\exp(-t\widetilde{\mathcal{D}}^2)$ on $\widetilde{X}$ as claimed in the proposition. As foundation for the approximated smoothing kernel, we take the smoothing kernel $\widetilde{k}_t$ of $\exp(-t\widetilde{\mathcal{D}}^2)$ on $\widetilde{X}$ and the smoothing kernel $k_t^{\text{doub.}}$ of $\exp(-t\mathcal{D}(Z_{\text{doub.}})^2)$ on the compact manifold $Z_{\text{doub.}} := Z \cup W_0 \cup (-W_0) \cup (-Z)$. We use the intersection $(Z \cup W_0) \cap \widetilde{X}_{\geq 0} = W_0$ as gluing region and obtain the approximated smoothing kernel $k_t^{\text{ap.}}$ by patching $\widetilde{k}_t$ and $k_t^{\text{doub.}}$ with cut-off function together. This process is called Duhamel's principle and works analogously to that in the proof of the APS index theorem (Theorem 2.10) worked out in [BW93, Section 22C]. $\square$

Proposition 3.26. *For P as in Notation* 3.24 *(3) and k_P and $k_{\tilde{P}}$ as in Notation* 3.24 *(2), we have the following two estimates:*

(1) For any $T > 0$, there exist positive constants γ and C such that for all $t \in (0, T]$ and all $x \in W_l$ with $l \geq 1$ the following holds:

$$\left|k_P(x,x) - k_{\tilde{P}}(x,x)\right| \leq Ce^{-\gamma d^2(x,W_0)/t}.$$

(2) For any $T > 0$, there exist positive constants γ and C such that for all $t \in (0, T]$, all $x \in W_l$ and all $y \in W_j$ with $l, j \geq 1$ the following holds:

$$\left|k_P(x,y) - k_{\tilde{P}}(x,y)\right| \leq Ce^{-\gamma \cdot min\left\{d^2(x,W_0), d^2(y,W_0)\right\}/t}.$$

Proof (sketch). The first statement is a corollary of Proposition 3.25. The second statement generalizes part (1) and can be proved by changing the proof of Proposition 3.25 slightly. It is worked out in [MRS16, Corollary 10.8, Proposition 10.9 and Remark 10.10]. $\square$

For the next proposition it is important that the L^2-closure of $\widetilde{\mathcal{D}}^+$ is invertible. By the assumption in the end-periodic index theorem (Theorem 3.22) the operator $\mathcal{D}^+$ is Fredholm. This ensures with the Fredholm criterion in Proposition 3.19* the invertibility of $\widetilde{\mathcal{D}}^+$. The proof of the following three estimates is given in [MRS16, p. 66-68].

* By following the proof of Proposition 3.19 the invertibility of $\widetilde{\mathcal{D}}^+$ follows from the invertibility of the twisted Dirac operators $\mathcal{D}_z^+$ on the unit circle $|z| = 1$.

Proposition 3.27. *Let $k_t(x,y)$ and $\widetilde{k}_t(x,y)$ be the smoothing kernels of* $\exp(-t\mathcal{D}^-\mathcal{D}^+)$ *and* $\exp(-t\widetilde{\mathcal{D}}^-\widetilde{\mathcal{D}}^+)$ *and P_+ the projection onto the finite-dimensional kernel of $\mathcal{D}^+$. With $k_t^0(x,y)$ defined as $k_t(x,y) - k_{P+}(x,y)$, we have:*

(1) There exist positive constants μ and C such that for all $x,y \in Z_\infty$ and all $t \geq 1$ the following holds:
$$|k_t^0(x,y)| \leq Ce^{-\mu t}.$$

(2) There exist positive constants μ and C such that for all $x,y \in \widetilde{X}$ and all $t \geq 1$ the following holds:
$$|\widetilde{k}_t(x,y)| \leq Ce^{-\mu t}.$$

(3) There exist positive constants C and δ such that for all $l \geq 1$ and all $x \in W_l$ the following holds:
$$|k_{P_+}(x,x)| \leq Ce^{-\delta l}.$$

3.3.3 Properties of the regularized trace

In this subsection we will show the well-definedness and the continuity in $t \in (0,\infty)$ of the regularized trace defined in equation (3.26). Furthermore, we will state and prove a formula for the regularized trace of a commutator in Proposition 3.32. This is one of the key steps to prove the end-periodic index theorem (Theorem 3.22) because it allows to describe the defect in the index formula in terms of data on the closed manifold X. Based on this formula, we will derive in Section 3.3.4 the additional terms appearing on the right hand side of the index formula in equation (3.22). Let us start with a lemma that will be useful later on:

Lemma 3.28. *For any rapidly decaying function $h : \mathbb{R} \to \mathbb{R}$ the smoothing kernel of $h(\widetilde{D})$ can be expressed in terms of data on X via the formula*

$$k_{h(\widetilde{\mathcal{D}})}(x,y) = \frac{1}{2\pi i}\int_{|z|=1} z^{f(y)-f(x)} k_{h(\mathcal{D}_z)}(x_0,y_0)\frac{dz}{z}, \quad \forall x,y \in \widetilde{X},\ x_0 := p(x),\ y_0 := p(y). \tag{3.30}$$

Proof. The operator $h(\mathcal{D}_z)$ is well defined via the spectral theorem, since $\mathcal{D}_z$ is self-adjoint. We have $\mathcal{F}_z^\Delta \circ h(\widetilde{\mathcal{D}}) = h(\mathcal{D}_z) \circ \mathcal{F}_z^\Delta$ for all $z \in S^1$ which follows from the definition of the functional calculus (paragraph after Proposition 1.5) and under use of the commutativity of the diagram (3.18). Then the stated formula follows from the next calculation, holding for all $u \in C_0^\infty(\widetilde{X}, S)$ and all $x \in \widetilde{X}$:

$$\begin{aligned}\int_{\widetilde{X}} k_{h(\tilde{\mathcal{D}})}(x,y)u(y)dy &= \big((\mathcal{F}^\Delta)^{-1} \circ \underbrace{\mathcal{F}^\Delta \circ h(\widetilde{\mathcal{D}})}_{=h(\mathcal{D}_\bullet)\circ\mathcal{F}^\Delta_\bullet}\big)(u)(x) \\ &= \frac{1}{2\pi i}\int_{|z|=1} z^{-f(x)} \underbrace{(h(\mathcal{D}_z)\circ\mathcal{F}_z^\Delta)(u)(x_0)}_{=\int_X k_{h(\mathcal{D}_z)}(x_0,y_0)(\mathcal{F}_z^\Delta u)(y_0)dy_0} \frac{dz}{z} \\ &= \int_{\widetilde{X}} \frac{1}{2\pi i}\int_{|z|=1} z^{f(y)-f(x)} k_{h(\mathcal{D}_z)}(x_0,y_0)u(y)dy.\end{aligned}$$

Here x_0 and y_0 denote the projection of x and y onto X. In addition, we used the properties of a smoothing kernel from Proposition 1.6, the definition of the Fourier-Laplace transform given in Definition 3.13 and its inverse formula in equation (3.9). $\square$

Remark 3.29. The previous proposition also holds for $\mathcal{D}$ replaced by $\mathcal{D}^\mp\mathcal{D}^\pm$. Furthermore, two direct consequences are given by:

$$\text{(1)}\;\; k_{h(\tilde{D})}(x+l, y+l) = k_{h(\tilde{D})}(x,y), \qquad \forall x, y \in \widetilde{X}, \forall l \in \mathbb{Z} \text{ (translation invariance)}. \tag{3.31}$$

$$\text{(2)}\;\; \widehat{k}_{h(\mathcal{D}_z)}(x_0, y) = z^{f(y)} k_{h(\mathcal{D}_z)}(x_0, y_0), \;\; \forall z \in S^1,\; \forall x_0, y_0 \in X,\; y \in \widetilde{X} \text{ with } p(y) = y_0. \tag{3.32}$$

Here $\widehat{k}_z(x_0, y)$ is the Fourier-Laplace transform with respect to the variable x. The second equation can be seen by applying the inverse Fourier-Laplace transform from Lemma 3.14 to the stated formula and compare it with equation (3.30).

For a smoothing operator P as in Notation 3.24 (3) the regularized trace defined in equation (3.26) is given by

$$\mathrm{Tr}^r(P) := \lim_{N\to\infty}\Big(\underbrace{\int_{Z_N} \mathrm{tr}\,(k_P(x,x))dx}_{\text{usual trace}} - \underbrace{(N+1)\int_{W_0} \mathrm{tr}\,(k_{\tilde{P}}(x,x))dx}_{\text{regularization term}}\Big). \tag{3.33}$$

Lemma 3.30. *The regularized trace is well defined and continuous in $t \in (0, \infty)$.*

Proof. There exist two positive constants C and γ such that for all $N \geq 0$ the estimate

$$
\begin{aligned}
&\Big|\int_{Z_N} \operatorname{tr}\big(k_P(x,x)\big)dx-(N+1)\int_{W_0} \operatorname{tr}\big(k_{\tilde P}(x,x)\big)dx\Big| \\
&\quad\le \Big|\int_Z \operatorname{tr}\big(k_P(x,x)\big)dx\Big|+\Big|\sum_{l=0}^{N}\Big(\underbrace{\int_{W_l} \operatorname{tr}\big(k_P(x,x)\big)dx}_{=\int_{W_0}\operatorname{tr}\left(k_P(x+l,x+l)\right)dx}-\int_{W_0}\operatorname{tr}\big(\underbrace{k_{\tilde P}(x,x)}_{\overset{\text{eq.(3.31)}}{=}\;k_{\tilde P}(x+l,x+l)}\big)dx\big)\Big| \qquad (3.34)\\
&\quad\le \Big|\int_Z \operatorname{tr}\big(k_P(x,x)\big)dx\Big|+\sum_{l=0}^{N}\int_{W_0}\Big|\underbrace{\operatorname{tr}\big(k_P(x+l,x+l)\big)-\operatorname{tr}\big(k_{\tilde P}(x+l,x+l)\big)}_{\overset{\text{Prop.3.26(1)}}{\le}\;Ce^{-\gamma(l-1)^2/t}\quad(\text{for } l\ge 1)}\Big|dx
\end{aligned}
$$

holds and the existence of the limit $N\to\infty$ is shown. The map that associates to a rapidly decaying function $h:\mathbb{R}\to\mathbb{R}$ the smoothing kernel of $h(\mathcal{D})$ is continuous with respect to the topology explained on pages 93-94 in [Roe88a]. This implies the continuity of $\operatorname{tr}(k_P(x,x))$ and $\operatorname{tr}(k_{\tilde P}(x,x))$ in $t\in(0,\infty)$. After integrating over a compact manifold it is still continuous and the uniform convergence of the sum in equation (3.34) on bounded intervals gives the continuity of the regularized trace in $t\in(0,\infty)$, hence the lemma is proved. □

We will next use the formula in Lemma 3.28 to express the correction term in the regularized trace in terms of the family $\mathcal{D}_z^{\pm}$ of twisted Dirac operators on the compact manifold X:

Lemma 3.31. *With P as in Notation* 3.24 *(3) and P_z induced by P as in Notation* 3.24 *(2) the regularized trace can be written as*

$$
\operatorname{Tr}^r(P)=\lim_{N\to\infty}\left(\int_{Z_N}\operatorname{tr}\big(k_P(x,x)\big)dx-\frac{N+1}{2\pi i}\int_{|z|=1}\operatorname{Tr}(P_z)\frac{dz}{z}\right) \qquad (3.35)
$$

Proof. Let h be the rapidly decaying function with $P=h(\mathcal{D})$ or $P=h(\mathcal{D}^{\mp}\mathcal{D}^{\pm})$. Restrict the formula in equation (3.30) to the diagonal $x=y$, apply on both sides the local trace, substitute the result in the definition of the regularized trace given in equation (3.33) and the stated formula follows by using

$$
\operatorname{Tr}(P_z)=\int_X \operatorname{tr}\big(k_{P_z}(x_0,x_0)\big)dx_0. \qquad \square
$$

Up to the end of this subsection we will prove a formula for the regularized trace on commutators. For $P=\mathcal{D}^-\exp(-s\mathcal{D}^+\mathcal{D}^-)$ and $Q=\mathcal{D}^+\exp(-t\mathcal{D}^-\mathcal{D}^+)$ the regularized traces of the operators PQ and QP are well defined because both are of the type described in Notation 3.24 (3). To verify this, we show $\mathcal{D}^{\pm}\mathcal{D}^{\mp}\exp(-t\mathcal{D}^{\pm}\mathcal{D}^{\mp}) = \mathcal{D}^{\pm}\exp(-t\mathcal{D}^{\mp}\mathcal{D}^{\pm})\mathcal{D}^{\mp}$:

Proof. Similarly as in part (3) of the proof of the Atiyah-Singer index theorem (Theorem 1.16) we define for a compactly supported smooth section u over $S^{\pm}$ the time dependent sections $u_t = \mathcal{D}^{\pm}\mathcal{D}^{\mp}\exp(-t\mathcal{D}^{\pm}\mathcal{D}^{\mp})u$ and $w_t = \mathcal{D}^{\pm}\exp(-t\mathcal{D}^{\mp}\mathcal{D}^{\pm})\mathcal{D}^{\mp}u$. Both of them solve the heat equation $(\partial_t - \mathcal{D}^{\pm}\mathcal{D}^{\mp})u_t = 0$. This can be verified by writing u_t and w_t as an integral of their smoothing kernels as in Proposition 1.6, using formula (1.6) and exploit that everything is uniformly bounded so that the differentiation and the integral commute*. Furthermore, u_t and w_t coincide at $t = 0$, hence the uniqueness of the heat equation yields $u_t = w_t$. $\square$

The previous consideration allows us to define the regularized trace of the commutator of P and Q as $\mathrm{Tr}^r[P,Q] \coloneqq \mathrm{Tr}^r(PQ) - \mathrm{Tr}^r(QP)$.

Proposition 3.32 (Commutator trace formula [MRS16, Theorem 4.6])**.** *For $P = \mathcal{D}^{-}\exp(-s\mathcal{D}^{+}\mathcal{D}^{-})$ and $Q = \mathcal{D}^{+}\exp(-t\mathcal{D}^{-}\mathcal{D}^{+})$ the regularized trace of the commutator $[P,Q]$ is given by*

$$\begin{aligned}\mathrm{Tr}^r[P,Q] = \frac{1}{2\pi i}\int_{|z|=1}\left(\int_{W_0} f(x)\cdot \mathrm{tr}\left(k_{P_zQ_z}(x,x) - k_{Q_zP_z}(x,x)\right)dx\right)\frac{dz}{z} \\ - \frac{1}{2\pi i}\int_{|z|=1}\mathrm{Tr}\left(\frac{\partial P_z}{\partial z}\cdot Q_z\right)dz.\end{aligned} \tag{3.36}$$

Here the kernels and the operators P_z and Q_z are as in Notation 3.24 *(2) and we write x instead*† *of $p(x)$.*

The first term on the right hand side of equation (3.36) consists basically of an integration over $Z_\infty \times Z_\infty$. We will split this into an integration over $Z_N \times Z_\infty$ and $Z_\infty \times Z_N$ and study the limit $N \to \infty$. Integrals over $\Delta^{\pm}$ as pictured in Figure 3.5 will appear. The following lemma is a crucial step in the proof of the commutator trace formula. It connects the integral over $\Delta^{\pm}$ with an integral over $W_0 \times W_0$ and the the operators P and Q - acting on sections of $S^{\pm} \to Z_\infty$ - gets converted into the operators $\widetilde{P}$ and $\widetilde{Q}$, which act on sections of $S^{\pm} \to \widetilde{X}$.

* In contrast to the similar calculation for the Atiyah-Singer index theorem in Chapter 1, here we have to use the representation of the involved operators via their smoothing kernels. Since Z_∞ is non-compact, we do not have the simplified functional calculus for the involved operators available, i.e. we can not write u_t and w_t as a sum over their eigenvalues. † In Lemma 3.28 we strictly differed the notation for elements in $\tilde{X}$ and X. Here we can omit this notation, since W_0 is almost everywhere identified with X.

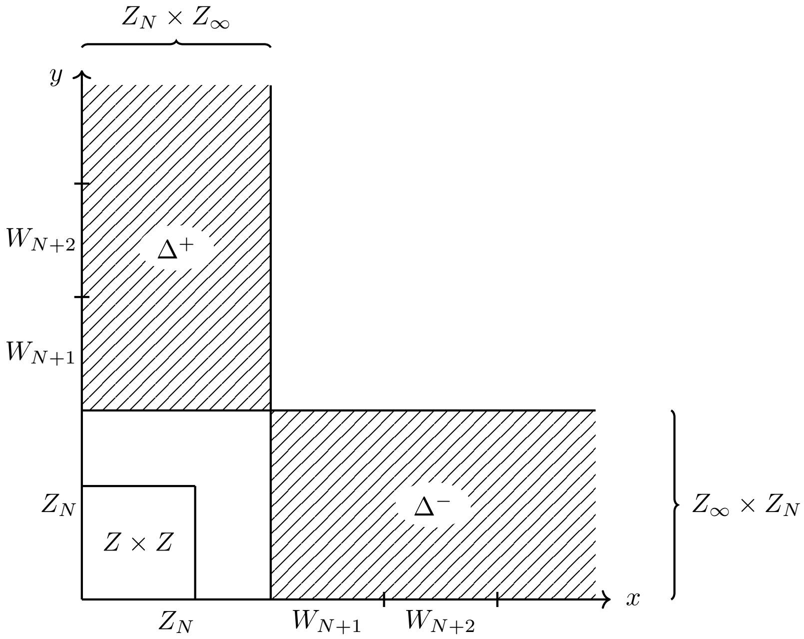

Figure 3.5: We split the integration over $Z_\infty \times Z_\infty$ appearing in the proof of the commutator trace formula stated in Proposition 3.32 into two integrals over $Z_N \times Z_\infty$ and $Z_\infty \times Z_N$ and take the limit $N \to \infty$. This allows us to calculate the regularized trace of the commutator $[P, Q]$ explicitly because the $Z_N \times Z_N$ part cancels out and for $N \to \infty$ the kernel of P and Q gets approximated by the kernels of $\widetilde{P}$ and $\widetilde{Q}$.

Lemma 3.33. *Let $P = \mathcal{D}^- \exp(-s\mathcal{D}^+\mathcal{D}^-)$ and $Q = \mathcal{D}^+ \exp(-t\mathcal{D}^-\mathcal{D}^+)$ for $s, t > 0$ and $\Delta^+ = Z_N \times (Z_\infty \setminus Z_N)$ and $\Delta^- = (Z_\infty \setminus Z_N) \times Z_N$ for $N \in \mathbb{N}$. Then*

$$\iint_{\Delta^-} \operatorname{tr}(k_P(x,y)k_Q(y,x))dxdy \sim \sum_{m=1}^{\infty} m \cdot \iint_{W_0 \times W_0} \operatorname{tr}\left(k_{\tilde{P}}(x+m,y)k_{\tilde{Q}}(y,x+m)\right)dxdy,$$

$$\iint_{\Delta^+} \operatorname{tr}(k_P(x,y)k_Q(y,x))dxdy \sim \sum_{m=-1}^{-\infty} (-m) \cdot \iint_{W_0 \times W_0} \operatorname{tr}\left(k_{\tilde{P}}(x+m,y)k_{\tilde{Q}}(y,x+m)\right)dxdy$$

holds, where we write for two numerical sequences $A_N \sim B_N$ if $\lim_{N\to\infty}(A_N - B_N) = 0$ is fulfilled.

Proof. We show only the first formula in the lemma

$$\underbrace{\iint_{\Delta^-} \operatorname{tr}\left(k_P(x,y)k_Q(y,x)\right)dxdy}_{=:A_N} \sim \underbrace{\sum_{m=1}^{\infty} m \cdot \iint_{W_0 \times W_0} \operatorname{tr}\left(k_{\tilde{P}}(x+m,y)k_{\tilde{Q}}(y,x+m)\right)dxdy}_{=:B_N}$$

because the second one can be proved similarly. We split the proof into the following two steps

$$\begin{aligned} A_N &\overset{(1)}{\sim} \sum_{m=1}^{N}\sum_{j=N+1}^{N+m} \iint_{W_j\times W_{j-m}} \operatorname{tr}\big(k_P(x,y)k_Q(y,x)\big)dxdy \\ &+ \sum_{m=N+1}^{\infty}\sum_{j=m}^{N+m} \iint_{W_j\times W_{j-m}} \operatorname{tr}\big(k_P(x,y)k_Q(y,x)\big)dxdy =: D_N \overset{(2)}{\sim} B_N. \end{aligned} \tag{3.37}$$

Let us start with the first step and rewrite D_N by changing the variables of the summation from m and j to $l = j - m$ and j. This leads to

$$D_N = \sum_{l=0}^{N}\sum_{j=N+1}^{\infty} \iint_{W_j\times W_l} \operatorname{tr}\big(k_P(x,y)k_Q(y,x)\big)dxdy. \tag{3.38}$$

We use this formula to calculate the difference $A_N - D_N$ and estimate its norm by using the short-term Gaussian estimate stated in Proposition 1.8 and Remark 1.15:

$$\begin{aligned} |A_N - D_N| &= \left| \iint_{(Z_\infty\setminus Z_N)\times Z} \operatorname{tr}\big(k_P(x,y)k_Q(y,x)\big)dxdy \right| \\ &\lesssim \sum_{j=N+1}^{\infty} \iint_{W_j\times Z} \underbrace{|k_P(x,y)|}_{\lesssim e^{-j^2/4s}} \cdot \underbrace{|k_Q(y,x)|}_{\lesssim e^{-j^2/4t}} dxdy \\ &\lesssim \sum_{j=0}^{\infty} e^{-\left(\frac{1}{4s}+\frac{1}{4t}\right)(j+N)} \quad \xrightarrow[\text{geometric sum}]{N\to\infty} \quad 0 \end{aligned} \tag{3.39}$$

This proves the first step in equation (3.37). It remains to show the second step in equation (3.37). First, note that for all $N \in \mathbb{N}$

$$B_N = \sum_{m=1}^{\infty}\sum_{j=N+1}^{N+m} \iint_{W_j\times W_{j-m}} \operatorname{tr}\big(k_{\tilde{P}}(x,y)k_{\tilde{Q}}(y,x)\big)dxdy \tag{3.40}$$

holds. This can be verified by using the translation invariance of $k_{\tilde{P}}$ and $k_{\tilde{Q}}$ stated in Remark 3.29 and shift x and y on the right hand side of equation (3.40) by $m-j$. After that, convert the integral in an integration over $W_0 \times W_0$ and exploit that the resulting summands are independent of j. This leads to the formula in equation (3.40). Second, we use this expression for B_N and the expression of D_N given in equation (3.37) to obtain

$$\begin{aligned}
|D_N - B_N| \leq & \Big| \sum_{m=N+1}^{\infty} \sum_{j=m}^{N+m} \iint_{W_j \times W_{j-m}} \operatorname{tr}\big(k_P(x,y) k_Q(y,x)\big) dxdy \Big| \\
& + \Big| \sum_{m=N+1}^{\infty} \sum_{j=N+1}^{N+m} \iint_{W_j \times W_{j-m}} \operatorname{tr}\big(k_{\tilde{P}}(x,y) k_{\tilde{Q}}(y,x)\big) dxdy \Big| \\
& + \Big| \sum_{m=1}^{N} \sum_{j=N+1}^{N+m} \iint_{W_j \times W_{j-m}} \operatorname{tr}\big(k_P(x,y) k_Q(y,x) - k_{\tilde{P}}(x,y) k_{\tilde{Q}}(y,x)\big) dxdy \Big|.
\end{aligned} \tag{3.41}$$

The first two terms in equation (3.41) approach zero for $N \to \infty$ by using the short-term Gaussian estimate as in equation (3.39). Next, we want to verify that the third term in equation (3.41) also goes to zero for $N \to \infty$. We use

$$\begin{aligned}
& |k_P(x,y) k_Q(y,x) - k_{\tilde{P}}(x,y) k_{\tilde{Q}}(y,x)| \leq \\
& \quad |k_P(x,y)| \cdot |k_Q(y,x) - k_{\tilde{Q}}(y,x)| + |k_{\tilde{P}}(x,y) - k_P(x,y)| |k_{\tilde{Q}}(y,x)|
\end{aligned} \tag{3.42}$$

and estimate it modulo a constant by the following three terms marked by a '•':

- $$\begin{aligned}
& \sum_{m=1}^{[N/2]} \sum_{j=N+1}^{N+m} \iint_{W_j \times W_{j-m}} \underbrace{|k_P(x,y)|}_{\lesssim e^{-(m-1)^2/4s}} \cdot \underbrace{|k_Q(y,x) - k_{\tilde{Q}}(y,x)|}_{\lesssim e^{-\gamma N^2/t}} dxdy \\
& \quad \lesssim e^{-\gamma N^2/t} \sum_{m=1}^{\infty} m e^{-(m-1)/4s} \xrightarrow{N \to \infty} 0.
\end{aligned}$$

Here γ is a positive constant as in Proposition 3.26 (2). We also used the short-term Gaussian estimate (Proposition 1.8) and in the last step the generalized geometric series. Note that we actually needed that the sum over m stops at $[N/2]$ to ensure that x and y in the integration are far enough away from W_0.

- $$\begin{aligned}
& \sum_{m=[N/2]+1}^{N} \sum_{j=N+1}^{N+m} \iint_{W_j \times W_{j-m}} |k_P(x,y)| \cdot \underbrace{|k_Q(y,x) - k_{\tilde{Q}}(y,x)|}_{\leq |k_Q(y,x)| + |k_{\tilde{Q}}(y,x)|} dxdy \\
& \quad \lesssim \sum_{[N/2]+1}^{\infty} m \cdot e^{-(m-1)\left(\frac{1}{4s} + \frac{1}{4t}\right)} \xrightarrow{N \to \infty} 0.
\end{aligned}$$

The limit $N \to \infty$ approaches zero by shifting the sum and using again the generalized geometric series.

- $$\sum_{m=1}^{N} \sum_{j=N+1}^{N+m} \iint_{W_j \times W_{j-m}} |k_{\tilde{P}}(x,y) - k_P(x,y)||k_{\tilde{Q}}(y,x)|dxdy \quad \xrightarrow{N\to\infty} \quad 0.$$

This limit can be shown by breaking the sum analogously as before with P and Q interchanged. All together, the previous estimates and equation (3.41) imply $\lim_{N\to\infty}(D_N - B_N) = 0$ and the lemma is proved. $\square$

Proof (Proposition 3.32*).* Note that on the closed manifold X the equation $\mathrm{Tr}[P,Q]_z = \mathrm{Tr}[P_z, Q_z] = 0$ holds by using the definition of P_z and Q_z and the identities

$$\mathcal{D}^{\pm}_{(z)}\mathcal{D}^{\mp}_{(z)} \exp(-t\mathcal{D}^{\pm}_{(z)}\mathcal{D}^{\mp}_{(z)}) = \mathcal{D}^{\pm}_{(z)} \exp(-t\mathcal{D}^{\mp}_{(z)}\mathcal{D}^{\pm}_{(z)})\mathcal{D}^{\mp}_{(z)}.$$

They can be shown similarly as the identity before Proposition 3.32. The vanishing of the previous trace leads together with the formula for the regularized trace given in Lemma 3.31 to

$$\begin{aligned}
\mathrm{Tr}^r[P,Q] &= \lim_{N\to\infty}\left(\int_{Z_N} \mathrm{tr}\,(k_{PQ}(x,x))dx - \int_{Z_N} \mathrm{tr}\,(k_{QP}(x,x))dx\right) \\
&\overset{(1)}{=} -\sum_{m=-\infty}^{\infty} m \cdot \iint_{W_0\times W_0} \mathrm{tr}\,(k_{\tilde{P}}(x+m,y)k_{\tilde{Q}}(y,x+m))dxdy \\
&\overset{(2)}{=} \frac{1}{2\pi i}\int_{|z|=1}\left(\int_{W_0} f(x)\cdot \mathrm{tr}\,(k_{P_zQ_z}(x,x) - k_{Q_zP_z}(x,x))dx\right)\frac{dz}{z} \\
&\qquad\qquad - \frac{1}{2\pi i}\int_{|z|=1} \mathrm{Tr}\left(\frac{\partial P_z}{\partial z}\cdot Q_z\right)dz.
\end{aligned} \tag{3.43}$$

It remains to show the steps (1) and (2) in equation (3.43).

ad(1) We start by studying the smoothing kernel of PQ. By using the definition of smoothing kernels given in Proposition 1.6, we obtain for any $u \in L^2(Z_\infty, S^+)$ and $x \in Z_\infty$

$$(PQu)(x) = \int_{Z_\infty} k_P(x,y)(Qu)(y)dy = \int_{Z_\infty}\left(\int_{Z_\infty} k_P(x,y)k_Q(y,z)u(z)dz\right)dy.$$

This leads to $k_{PQ}(x,x) = \int_{Z_\infty} k_P(x,y)k_Q(y,x)dy$, which gives by applying the local trace and integrating over Z_N

$$\int_{Z_N} \mathrm{tr}\,(k_{PQ}(x,x))dx = \iint_{Z_N\times Z_\infty} \mathrm{tr}\,(k_P(x,y)k_Q(y,x)dxdy, \quad \forall N\in\mathbb{N}. \tag{3.44}$$

Interchanging P and Q and similar considerations as before give for all $N \in \mathbb{N}$

$$\begin{aligned}\int_{Z_N} \operatorname{tr}\left(k_{QP}(x,x)\right)dx &= \iint_{Z_N \times Z_\infty} \operatorname{tr}\left(k_Q(x,y)k_P(y,x)dxdy\right. \\ &= \iint_{Z_\infty \times Z_N} \operatorname{tr}\left(k_P(x,y)k_Q(y,x)dxdy.\right.\end{aligned} \tag{3.45}$$

The last step holds by Fubini's theorem that is applicable because of the estimate

$$\begin{aligned}&\iint_{Z_{\tilde N} \times Z_N} \left|\operatorname{tr}\left(k_P(x,y)k_Q(y,x)\right|dxdy\right. \\ &\lesssim \iint_{Z_N \times Z_N} |k_P(x,y)| \cdot |k_Q(y,x)|dxdy + \sum_{l=1}^{\tilde N - N} \iint_{W_{l+N} \times Z_N} \underbrace{|k_P(x,y)| \cdot |k_Q(y,x)|}_{\substack{\text{Prop.1.8} \\ \lesssim \\ \text{Rem.1.15}}\ e^{-(l-1)\left(\frac{1}{4s}+\frac{1}{4t}\right)}} dxdy \\ &\lesssim \iint_{Z_N \times Z_N} |k_P(x,y)| \cdot |k_Q(y,x)|dxdy + \sum_{l=0}^{\infty} e^{-\left(\frac{1}{4s}+\frac{1}{4t}\right)l}\end{aligned}$$

for all $\tilde N > N$. We write shortly $\Delta^+ = Z_N \times (Z_\infty \setminus Z_N)$ and $\Delta^- = (Z_\infty \setminus Z_N) \times Z_N$ for $N \in \mathbb{N}$ as illustrated in Figure 3.5. Then, step (1) in equation (3.43) follows by inserting equation (3.44) and (3.45) and using Lemma 3.33:

$$\begin{aligned}&\lim_{N\to\infty} \left(\int_{Z_N} \operatorname{tr}\left(k_{PQ}(x,x)\right)dx - \int_{Z_N} \operatorname{tr}\left(k_{QP}(x,x)\right)dx \right) \\ &\quad = \lim_{N\to\infty} \left(\iint_{\Delta^+} \operatorname{tr}\left(k_P(x,y)k_Q(y,x)dxdy - \iint_{\Delta^-} \operatorname{tr}\left(k_P(x,y)k_Q(y,x)dxdy\right)\right.\right. \\ &\quad = -\sum_{m=-\infty}^{\infty} m \cdot \iint_{W_0 \times W_0} \operatorname{tr}\left(k_{\tilde P}(x+m,y)k_{\tilde Q}(y,x+m)\right)dxdy\end{aligned}$$

ad(2) For all $u \in C^\infty(\widetilde X, S)$, $v \in C^\infty(\widetilde X, S^*)$ and all $x \in \widetilde X$ the following holds:

$$\sum_m m \cdot \underbrace{u(x+m)v(x+m)}_{\in (S\otimes S^*)_{x+m}} = \frac{1}{2\pi i}\int_{|z|=1} \left(\frac{\partial}{\partial z} z^{-f(x)}\widehat u_z(x)\right) z^{f(x)}\widehat v_{1/z} dz. \tag{3.46}$$

This can be shown by inserting

$$\widehat u_z = z^{f(x)} \sum_n z^n u(x+n) \quad \text{and} \quad \widehat v_{1/z} = z^{-f(x)} \sum_m z^{-m} v(x+m)$$

into the right hand side of equation (3.46) and apply Cauchy's integral theorem. We define for $x, y \in W_0$ the smooth sections $u(x+m) = k_{\tilde P}(x+m,y)$ and $v(x+m) =$

$k_{\tilde{Q}}(y, x+m)$. Furthermore, by Remark 3.29 (2) we have the two* identities

$$\widehat{u}_z(x) = z^{f(y)} k_{P_z}(x,y), \quad \widehat{v}_{1/z}(x) = z^{-f(y)} k_{Q_z}(y,x).$$

Inserting in equation (3.46) gives:

$$\begin{aligned}
&-\sum_m m \cdot k_{\tilde{P}}(x+m,y) k_{\tilde{Q}}(y,x+m) \\
&\quad = -\frac{1}{2\pi i}\int_{|z|=1} \frac{\partial}{\partial z}\big(z^{f(y)-f(x)} k_{P_z}(x,y)\big) z^{f(x)-f(y)} k_{Q_z}(x,y) dz \\
&\quad = \underbrace{-\frac{1}{2\pi i}\int_{|z|=1} (f(y)-f(x)) k_{P_z}(x,y) k_{Q_z}(y,x)\frac{dz}{z}}_{=:A} \underbrace{-\frac{1}{2\pi i}\int_{|z|=1} \frac{\partial}{\partial z}\big(k_{P_z}(x,y)\big) k_{Q_z}(y,x) dz}_{=:B}
\end{aligned} \tag{3.47}$$

Taking the local trace of both sides of equation (3.47) and integrating over $W_0 \times W_0$ gives the claimed formula in step (2) in equation (3.43). We verify this by calculating the two parts on the right hand side of equation (3.47) separately. With similar considerations as in the first few lines in ad(1) of this proof we obtain

$$\begin{aligned}
\iint_{W_0\times W_0} \mathrm{tr}(A) dx dy = &-\frac{1}{2\pi i}\int_{|z|=1}\left(\int_{W_0} f(x) \underbrace{\int_X \mathrm{tr}\left(k_{P_z}(x,y) k_{Q_z}(y,x)\right) dy}_{=\mathrm{tr}(k_{P_z Q_z}(x,x))} dx\right)\frac{dz}{z} \\
&-\frac{1}{2\pi i}\int_{|z|=1}\left(\int_{W_0} f(y) \underbrace{\int_X \mathrm{tr}\left(k_{P_z}(x,y) k_{Q_z}(y,x)\right) dx}_{=\mathrm{tr}(k_{Q_z P_z}(y,y))} dy\right)\frac{dz}{z}.
\end{aligned}$$

For the second term on the right hand side in equation (3.47) we obtain

$$\begin{aligned}
\iint_{W_0\times W_0} \mathrm{tr}(B) dx dy &= \frac{1}{2\pi i}\int_{|z|=1}\int_{W_0} \mathrm{tr}\left(\underbrace{\int_{W_0}\left(\frac{\partial}{\partial z} k_{P_z}(x,y)\right) k_{Q_z}(y,x) dy}_{= k_{\frac{\partial P_z}{\partial z}\cdot Q_z}(x,x)}\right) dx dz \\
&= -\frac{1}{2\pi i}\int_{|z|=1} \mathrm{Tr}\left(\frac{\partial P_z}{\partial z}\cdot Q_z\right) dz
\end{aligned}$$

and the proposition is proved. □

* In the second equation we have the Fourier-Laplace transform with respect to the second coordinate which is not directly Remark 3.29 (2). Nevertheless, the equation used here also holds and can be shown similarly as the identity in Remark 3.29 (2).

3.3.4 Filling in the last gaps of the proof

The aim of this subsection is to prove the three equations in (3.27), the formula for the regularized trace in equation (3.28) and the explicit expression of the triple integral in equation (3.29) in terms of ω and f. We take all notations and assumptions as in the end-periodic index theorem (Theorem 3.22) and Notation 3.24 (1) and (2).

Proposition 3.34. *The following holds for $P = \mathcal{D}^-$, $Q = \mathcal{D}^+ \exp(-t\mathcal{D}^-\mathcal{D}^+)$ and the regularized McKean-Singer function h^r:*

(1) $\dfrac{d}{dt} h^r(t) = -\operatorname{Tr}^r[P, Q]$.

(2) $$\operatorname{Tr}^r[P,Q] = \frac{1}{2\pi i}\int_{|z|=1}\left(\int_{W_0} f(x)\cdot \operatorname{tr}\left(k_{P_z Q_z}(x,x) - k_{Q_z P_z}(x,x)\right)dx\right)\frac{dz}{z} + \frac{1}{2\pi i}\int_{|z|=1} \operatorname{Tr}\left(df\cdot Q_z\right)\frac{dz}{z}.$$

(3) $$\frac{1}{2\pi i}\int_0^\infty \int_{|z|=1}\left(\int_{W_0} f(x)\cdot \operatorname{tr}\left(k_{Q_z P_z}(x,x) - k_{P_z Q_z}(x,x)\right)dx\right)\frac{dz}{z}dt = -\int_Y \omega + \int_X df\wedge\omega.$$

Proof. **(1)** The McKean-Singer function h^r is defined as the difference of $\operatorname{Tr}^r(\exp(-t\mathcal{D}^-\mathcal{D}^+))$ and $\operatorname{Tr}^r(\exp(-t\mathcal{D}^+\mathcal{D}^-))$. We study both terms separately. Using the definition of the regularized trace given in equation (3.26) and the chiral analogon to equation (1.6) leads to

$$\begin{aligned}
&\frac{d}{dt}\operatorname{Tr}^r(\exp(-t\mathcal{D}^-\mathcal{D}^+))\\
&= \lim_{N\to\infty}\Bigg(\int_{Z_N} \operatorname{tr}\Big(\underbrace{\frac{d}{dt}k_{\exp(-t\mathcal{D}^-\mathcal{D}^+)}(x,x)}_{\substack{= -\mathcal{D}^-\mathcal{D}^+ k_{\exp(-t\mathcal{D}^-\mathcal{D}^+)}(x,x)\\ = -k_{\mathcal{D}^-\mathcal{D}^+\exp(-t\mathcal{D}^-\mathcal{D}^+)}(x,x)}}\Big)dx - (N+1)\int_{W_0}\operatorname{tr}\Big(\underbrace{\frac{d}{dt}k_{\exp(-t\tilde{\mathcal{D}}^-\tilde{\mathcal{D}}^+)}(x,x)}_{\substack{= -\tilde{\mathcal{D}}^-\tilde{\mathcal{D}}^+ k_{\exp(-t\tilde{\mathcal{D}}^-\tilde{\mathcal{D}}^+)}(x,x)\\ = -k_{\tilde{\mathcal{D}}^-\tilde{\mathcal{D}}^+\exp(-t\tilde{\mathcal{D}}^-\tilde{\mathcal{D}}^+)}(x,x)}}\Big)dx\Bigg)\\
&= -\operatorname{Tr}^r(\mathcal{D}^-\mathcal{D}^+ e^{-t\mathcal{D}^-\mathcal{D}^+}).
\end{aligned}$$

For any fixed N the derivative exists and converges uniformly (similarly as in the proof of Lemma 3.30), hence the differentiation interchanges with the limit. An analogue calculation shows $\frac{d}{dt}\operatorname{Tr}^r(\exp(-t\mathcal{D}^+\mathcal{D}^-)) = -\operatorname{Tr}^r(\mathcal{D}^+\mathcal{D}^-\exp(-t\mathcal{D}^+\mathcal{D}^-))$ and the statement follows directly by $\mathcal{D}^+\mathcal{D}^-\exp(-t\mathcal{D}^+\mathcal{D}^-) = \mathcal{D}^+\exp(-t\mathcal{D}^-\mathcal{D}^+)\mathcal{D}^-$ which was shown before Proposition 3.32.

(2) We define $P_s \coloneqq \mathcal{D}^- \exp(-s\mathcal{D}^+\mathcal{D}^-)$ for $s > 0$ and obtain with the commutator trace formula from in equation (3.36) and $\frac{\partial P_z}{\partial z} = -\frac{df}{z}$ (note, $P_z = \mathcal{D}_z^- = \mathcal{D}^-(X) - \ln z \cdot df$)

$$\begin{aligned}
&\operatorname{Tr}^r[P, Q] = \lim_{s\to 0} \operatorname{Tr}^r[P_s, Q] \\
&\overset{\text{eq.(3.36)}}{=} \frac{1}{2\pi i}\int_{|z|=1}\left(\int_{W_0} f(x)\cdot \operatorname{tr}\left(k_{P_zQ_z}(x,x) - k_{Q_zP_z}(x,x)\right)dx\right)\frac{dz}{z} + \frac{1}{2\pi i}\int_{|z|=1}\operatorname{Tr}\left(df\cdot Q_z\right)\frac{dz}{z}.
\end{aligned}$$

(3) We obtain by using $\mathcal{D}_z^+\mathcal{D}_z^- \exp(-t\mathcal{D}_z^+\mathcal{D}_z^-) = \mathcal{D}_z^+ \exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)\mathcal{D}_z^-$, the definition of P_z and Q_z and the chiral version of equation (1.6) for the twisted Dirac operators the identity

$$\frac{d}{dt}\Big(\operatorname{tr}\left(k_{\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)}(x,x) - k_{\exp(-t\mathcal{D}_z^+\mathcal{D}_z^-)}(x,x)\right)\Big) = \operatorname{tr}\left(k_{P_zQ_z}(x,x) - k_{Q_zP_z}(x,x)\right).$$

This allows us to eliminate the integration over t with the fundamental theorem of calculus and it remains to consider the limits $t \to \infty$ and $t \to 0$ of

$$(\star) \coloneqq \frac{1}{2\pi i}\int_{|z|=1}\left(\int_{W_0} f(x)\cdot\operatorname{tr}\Big(k_{\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)}(x,x) - k_{\exp(-t\mathcal{D}_z^+\mathcal{D}_z^-)}(x,x)\Big)dx\right)\frac{dz}{z}.$$

Claim 1. $\lim\limits_{t\to\infty}(\star) = 0.$

Proof (Claim 1). Let $z \in \mathbb{C}$ with $|z| = 1$ be fixed, $\sigma \subset \mathbb{R}_{\geq 0}$ the spectrum of $\mathcal{D}_z^-\mathcal{D}_z^+$ (discrete subset of $\mathbb{R}_{\geq 0}$ because $\mathcal{D}_z^-\mathcal{D}_z^+$ is self-adjoint and X is closed) and $\phi = \sum_{\lambda\in\sigma}\phi_\lambda$ an arbitrary smooth section of S^+ decomposed into eigensections. Then we obtain by the functional calculus

$$\lim_{t\to\infty} e^{-t\mathcal{D}_z^-\mathcal{D}_z^+}\phi = \lim_{t\to\infty}\sum_{\lambda\in\sigma} e^{-t\lambda}\phi_\lambda = \phi_0 = 0. \tag{3.48}$$

The last step holds because the kernel of $\mathcal{D}_z^-\mathcal{D}_z^+$ is equal to the kernel of $\mathcal{D}_z^+$, which follows from $\mathcal{D}^-(X)^* = \mathcal{D}^+(X)$ and $|z| = 1$. The kernel of $\mathcal{D}_z^+$ vanishes by the Fredholmness of $\mathcal{D}^+(Z_\infty)$ and the Fredholm criterion given in Proposition 3.19, hence $\phi_0 = 0$. The manifold X is closed, so that equation (3.48) implies $k_{\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)}(x,x) \to 0$ for $t \to \infty$ uniformly [see Roe88b, Lemma 1.2]. By similar considerations for $\mathcal{D}_z^+\mathcal{D}_z^-$ we obtain $k_{\exp(-t\mathcal{D}_z^+\mathcal{D}_z^-)}(x,x) \to 0$ for $t \to \infty$ uniformly and it follows $\lim_{t\to\infty}(\star) = 0$.

Claim 2. $\lim\limits_{t\to 0}(\star) = \int_Y \omega - \int_X df \wedge \omega.$

Proof (Claim 2). For $|z| = 1$ the operators $\mathcal{D}_z^\pm$ are just the twisted Dirac operators $\mathcal{D}_{\mathrm{tw}}^\pm$

induced by the Clifford bundle $S \otimes E_z \to X$ given in Remark 3.18 (1). This justifies

$$\begin{aligned}
&\lim_{t\to 0} \operatorname{tr}\left(k_{\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)}(x,x) - k_{\exp(-t\mathcal{D}_z^+\mathcal{D}_z^-)}(x,x)\right)\\
&\quad= \lim_{t\to 0} \operatorname{tr}\left(k_{\exp(-t\mathcal{D}_{\mathrm{tw}}^-\mathcal{D}_{\mathrm{tw}}^+)}(x,x) - k_{\exp(-t\mathcal{D}_{\mathrm{tw}}^+\mathcal{D}_{\mathrm{tw}}^-)}(x,x)\right)\\
&\quad= \mathbf{I}(\mathcal{D}_{\mathrm{tw}}) = \mathbf{I}(\mathcal{D}(X))\mathrm{ch}(E_z) \overset{E_z \text{ is flat}}{=} \mathbf{I}(\mathcal{D}(X)).
\end{aligned}$$

In the first step we exploit the commutativity of the diagram (3.19) and in the second step the definition of the local index form (Definition 1.13) and Proposition 1.6. The third step can be verified by writing out the local index form in terms of characteristic classes [Roe98, Theorem 12.27 and Remark 12.29] and pull out the Chern character of the line bundle E_z. This simplifies $(\star)$ and we can do the integration over z. Using $\mathbf{I}(\mathcal{D}(X)) = d\omega$ (holds by definition of ω in Theorem 3.22), $\partial W_0 = -Y \cup Y =: Y_0 \cup Y_1$, Stokes theorem and $f(x+1) = f(x) + 1$ for all $x \in \tilde{X}$ leads to

$$\begin{aligned}
\lim_{t\to 0} (\star) &= \int_{W_0} f \cdot \underbrace{\mathbf{I}(\mathcal{D}(X))}_{=d\omega} = \int_{W_0} d(f\omega) - \int_{W_0} \underbrace{df \wedge \omega}_{\text{is well defined on X}}\\
&= \int_Y \underbrace{(f|_{Y_1} - f|_{Y_0})}_{=1}\omega - \int_X df \wedge \omega.
\end{aligned} \tag{3.49}$$

This shows the second claim and the proposition is proved. □

Remark 3.35. Equation (3.49) shows directly that the end-periodic index formula in equation (3.22) is independent of the choice of ω. Indeed, for $\widetilde{\omega}$ with $\mathbf{I}(\mathcal{D}(X)) = d\widetilde{\omega}$ we have

$$\int_Y \widetilde{\omega} + \int_X df \wedge \widetilde{\omega} = \int_{W_0} f \cdot \mathbf{I}(\mathcal{D}(X)) = \int_Y \omega + \int_X df \wedge \omega.$$

The next proposition gives the limits of the regularized McKean-Singer function $h^r(t)$ for $t \to 0$ and $t \to \infty$.

Proposition 3.36 ([MRS16, Proposition 5.2 and 5.3])**.** *The following holds:*

(1) $\displaystyle\lim_{t\to\infty} h^r(t) = \operatorname{ind}(\mathcal{D}(Z_\infty)).$

(2) $\displaystyle\lim_{t\to 0} h^r(t) = \int_Z \mathbf{I}(\mathcal{D}(Z_\infty)).$

Proof. **(1)** Writing out the index of $\mathcal{D}(Z_\infty)$ as in Remark 1.14 and the McKean-Singer function as defined in equation (3.25) leads to

$$\begin{aligned}&\operatorname{ind}(\mathcal{D}(Z_\infty)) = \dim\ker(\mathcal{D}^+(Z_\infty)) - \dim\ker(\mathcal{D}^-(Z_\infty)) \quad \text{and}\\ &h^r(t) = \operatorname{Tr}^r\left(e^{-t\mathcal{D}^-\mathcal{D}^+}\right) - \operatorname{Tr}^r\left(e^{-t\mathcal{D}^+\mathcal{D}^-}\right).\end{aligned}$$

We restrict ourselves to the proof of $\lim_{t\to\infty}\operatorname{Tr}^r\left(e^{-t\mathcal{D}^-\mathcal{D}^+}\right) = \dim\ker(\mathcal{D}^+(Z_\infty))$ since it is identical for changed chirality. With notation as in Proposition 3.27 ($k_t, \widetilde{k}_t, k_t^0, P_+$), the definition of the regularized trace in equation (3.26) and

$$\dim\ker(\mathcal{D}^+(Z_\infty)) = \operatorname{Tr}(P_+) = \lim_{N\to\infty}\int_{Z_N}\operatorname{tr}\left(k_{P_+}(x,x)\right)dx,$$

we obtain for all $t > 1$

$$\begin{aligned}&\left|\operatorname{Tr}^r\left(e^{-t\mathcal{D}^-\mathcal{D}^+}\right) - \dim\ker(\mathcal{D}^+(Z_\infty))\right|\\ &\quad= \left|\lim_{N\to\infty}\left(\int_{Z_N}\operatorname{tr}\left(k_t(x,x)\right)dx - (N+1)\int_{W_0}\operatorname{tr}\left(\widetilde{k}_t(x,x)\right)dx - \int_{Z_N}\operatorname{tr}\left(k_{P_+}(x,x)\right)dx\right)\right|\\ &\quad= \left|\int_Z\operatorname{tr}\left(k_t^0(x,x)\right)dx + \lim_{N\to\infty}\sum_{l=0}^N\left(\underbrace{\int_{W_l}\operatorname{tr}\left(k_t^0(x,x)\right)dx}_{=\int_{W_0}\operatorname{tr}(k_t^0(x+l,x+l))dx} - \int_{W_0}\operatorname{tr}\left(\underbrace{\widetilde{k}_t(x,x)}_{\overset{\text{eq.(3.31)}}{=}\widetilde{k}_t(x+l,x+l)}\right)dx\right)\right| \qquad (3.50)\\ &\quad\lesssim \underbrace{\left|\int_Z\operatorname{tr}\left(k_t^0(x,x)\right)dx\right|}_{\xrightarrow[\text{Prop.3.27(1)}]{t\to\infty}0} + \int_{W_0}\underbrace{\sum_{l=0}^\infty\left|k_t^0(x+l,x+l) - \widetilde{k}_t(x+l,x+l)\right|dx}_{=:(\star)}.\end{aligned}$$

It remains to show $(\star)\to 0$ for $t\to\infty$ uniformly in $x\in W_0$. Let C_1, α, γ be the positive constants from Proposition 3.25, C_2 and δ from Proposition 3.27 (3), C_3 and μ_3 from Proposition 3.27 (1) and C_4 and μ_4 from Proposition 3.27 (2). With $C \coloneqq C_3 + C_4$ and $\mu \coloneqq \min(\mu_3,\mu_4)$ we obtain by splitting the sum into two parts and estimating them separately:

$$\begin{aligned}(\star) \leq &\sum_{l\leq(\alpha+\mu)t^2/\gamma}\Big(\underbrace{\left|k_t^0(x+l,x+l)\right| + \left|\widetilde{k}_t(x+l,x+l)\right|}_{\underset{\text{(1) and (2)}}{\overset{\text{Prop.3.27}}{\leq}} C_3e^{-\mu_3 t}+C_4e^{-\mu_4 t}\leq Ce^{-\mu t}}\Big)\\ &+\sum_{l>(\alpha+\mu)t^2/\gamma}\Big(\underbrace{\left|k_t(x+l,x+l) - \widetilde{k}_t(x+l,x+l)\right|}_{\overset{\text{Prop.3.25}}{\leq} C_1e^{\alpha t}e^{-\gamma l/t}} + \underbrace{\left|k_{P_+}(x+l,x+l)\right|}_{\overset{\text{Prop.3.27(3)}}{\leq} C_2e^{-\delta l}}\Big).\end{aligned} \qquad (3.51)$$

Furthermore, we estimate the three occurring sums in equation (3.51) as follows:

- $\displaystyle\sum_{l\leq(\alpha+\mu)t^2/\gamma} Ce^{-\mu t} \leq Ce^{-\mu t}(\alpha+\mu)t^2/\gamma.$
- $\displaystyle\sum_{l>(\alpha+\mu)t^2/\gamma} C_1 e^{\alpha t} e^{-\gamma l/t} \leq \sum_{j=0}^{\infty} C_1 e^{\alpha t} e^{-\gamma\left((\alpha+\mu)t^2/\gamma+j\right)/t} = C_1 e^{-\mu t}\sum_{j=0}^{\infty}\left(e^{-\gamma/t}\right)^j = \frac{C_1 e^{-\mu t}}{1-e^{-\gamma/t}}.$
- $\displaystyle\sum_{l>(\alpha+\mu)t^2/\gamma} C_2 e^{-\delta l} \leq \sum_{j=0}^{\infty} C_2 e^{-\delta\left((\alpha+\mu)t^2/\gamma+j\right)} = \frac{C_2 e^{-\delta(\alpha+\mu)t^2/\gamma}}{1-e^{-\delta}}.$

Inserting in equation (3.51) and taking the limit $t \to \infty$ gives $\lim_{t\to\infty}(\star) = 0$ (use L'Hôpital for the second term) and with equation (3.50) the first part of the proposition is proved.

(2) By the definition of the McKean-Singer function h^r and the definition of the regularized trace given in equation (3.26), we have for all $t > 0$

$$\begin{aligned} h^r(t) = \lim_{N\to\infty}\Big(\int_{Z_N} &\operatorname{tr}\left(k_{\exp(-t\mathcal{D}^-\mathcal{D}^+)}(x,x) - k_{\exp(-t\mathcal{D}^+\mathcal{D}^-)}(x,x)\right)dx \\ &- (N+1)\int_{W_0}\operatorname{tr}\left(k_{\exp(-t\tilde{\mathcal{D}}^-\tilde{\mathcal{D}}^+)}(x,x) - k_{\exp(-t\tilde{\mathcal{D}}^+\tilde{\mathcal{D}}^-)}(x,x)\right)dx\Big). \end{aligned} \tag{3.52}$$

Define for $N \in \mathbb{N}$ and $t > 0$ the continuous function $s_N(t)$ as the right hand side of equation (3.52) without the limit. Then we obtain for $t \to 0$:

$$\begin{aligned} \lim_{t\to 0} s_N(t) &= \int_{Z_N} \underbrace{\lim_{t\to 0}\operatorname{tr}\left(k_{\exp(-t\mathcal{D}^-\mathcal{D}^+)}(x,x) - k_{\exp(-t\mathcal{D}^+\mathcal{D}^-)}(x,x)\right)dx}_{=\mathbf{I}(\mathcal{D}(Z_\infty))} \\ &\quad - (N+1)\int_{W_0} \underbrace{\lim_{t\to 0}\operatorname{tr}\left(k_{\exp(-t\tilde{\mathcal{D}}^-\tilde{\mathcal{D}}^+)}(x,x) - k_{\exp(-t\tilde{\mathcal{D}}^+\tilde{\mathcal{D}}^-)}(x,x)\right)dx}_{=\mathbf{I}(\mathcal{D}(\widetilde{X}))} \\ &= \int_Z \mathbf{I}(\mathcal{D}(Z_\infty)) \end{aligned} \tag{3.53}$$

The last step holds by using that the local index form is computable by local data of the Clifford bundle and the underlying manifold. This gives that $\mathbf{I}(\mathcal{D}(Z_\infty))|_{W_l}$ and $\mathbf{I}(\mathcal{D}(\widetilde{X}))|_{W_0}$ matches since the Clifford bundle over W_0 is isomorphic to the Clifford bundle over W_l for all $l \in \mathbb{N}$ (the Clifford bundle is by assumption end-periodic). Equation (3.53) implies that the repeated limits $\lim_{N\to\infty}\lim_{t\to 0} s_N(t)$ exist. Together with the uniform convergence on all bounded intervals of the limit $N \to \infty$ of $s_N(t)$ (can be shown by similar arguments as in the proof of Lemma 3.30) the repeated limits

$\lim_{t\to 0}\lim_{N\to\infty} s_N(t)$ also exist and we obtain:

$$\lim_{t\to 0} h^r(t) = \lim_{t\to 0}\lim_{N\to\infty} s_N(t) = \lim_{N\to\infty}\lim_{t\to 0} s_N(t) \overset{\text{eq.(3.53)}}{=} \int_Z \mathbf{I}(\mathcal{D}(Z_\infty)). \qquad \square$$

3.3.5 Deducing the APS index theorem

Let $S \to \widehat{Z}$ be a graded Clifford bundle with cylindrical end modeled by (Z, Y) as in Definition 2.14 and denote by $\mathcal{D}^+(\widehat{Z})$ the induced chiral Dirac operator and by $\mathcal{D}(Y)$ the boundary Dirac operator. The manifold $\widehat{Z}$ has even dimension n.

Aim 3.37. *The aim of this subsection is to prove the following two statements by leading them back to their analogue statements in the end-periodic setting given in Proposition* 3.19 *and Theorem* 3.22*:*

(1) The Fredholm criterion for the cylindrical chiral Dirac operator (Proposition 2.15*):*

$$\mathcal{D}^+(\widehat{Z}) : H^1(\widehat{Z}, S^+) \to L^2(\widehat{Z}, S^-) \textit{ is Fredholm} \quad \Longleftrightarrow \quad \mathcal{D}(Y) \textit{ is invertible.}$$

(2) The APS index theorem in the Fredholm case (Theorem 2.16*):*

$$\textit{If } \mathcal{D}(Y) \textit{ is invertible } \operatorname{ind}(\mathcal{D}(\widehat{Z})) = \int_Z \boldsymbol{I}(\mathcal{D}(\widehat{Z})) - \frac{1}{2}\eta(\mathcal{D}(Y)) \textit{ holds.}$$

We have seen in Examples 3.5 and 3.10 that $S \to \widetilde{Z}$ is an end-periodic graded Clifford bundle over an end-periodic manifold. The closed manifold X is given by $S^1 \times Y$, the infinite cyclic covering $\widetilde{X} \to X$ by* $\mathbb{R}\times Y \to S^1 \times Y$ and the function $f : \widetilde{X} = \mathbb{R}\times Y \to \mathbb{R}$ by the projection to the first coordinate. See Lemma 3.6 and its proof for the definition of $\widetilde{X}$ and f in general. By similar considerations as in Proposition 2.5 the chiral Dirac operators over $S^1 \times Y$ take the form $\mathcal{D}^\pm(X) = \mathrm{cl}(\partial_\theta)\cdot(\partial_\theta \mp \mathcal{D}(Y))$ for θ the standard coordinate of $S^1 \times Y$ in the first entry. Furthermore, the definition of f and θ implies $d\theta = df$, where we interpret df as a one-form on $S^1 \times Y$ as pointed out in Remark 3.7. This implies that the twisted Dirac operators for $z = re^{is} \in \mathbb{C}^*$ with $s \in [0, 2\pi]$ and $r \in \mathbb{R}^+$ are given by

$$\mathcal{D}_z^\pm = \mathcal{D}^+(X) - \ln z \cdot df = \mathrm{cl}(\partial_\theta)\cdot(\partial_\theta \mp \mathcal{D}(Y) - (\ln(r) + is)). \tag{3.54}$$

After this preliminaries, we can start to prove the Fredholm criterion:

* Projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ on the first coordinate and identity on the second.

Proof (part (1) of Aim 3.37). "$\Leftarrow$" (by contradiction) We assume that the boundary Dirac operator is invertible and $\mathcal{D}^+(\widehat{Z}) : H^1(\widehat{Z}, S^+) \to L^2(\widehat{Z}, S^-)$ is not Fredholm. By Proposition 3.19 there exist a $z = e^{is} \in \mathbb{C}$ with $s \in [0, 2\pi]$ such that $\mathcal{D}_z^+$ is not invertible, hence $\ker(\mathcal{D}_z^+) \neq 0$ or $\operatorname{coker}(\mathcal{D}_z^+) \neq 0$. We lead $\ker(\mathcal{D}_z^+) \neq 0$ to a contradiction. The case with nontrivial cokernel is similar by using that $\operatorname{coker}(\mathcal{D}_z^+)$ is isomorphic to $\ker(\mathcal{D}_z^-)$. Let $\{\phi_\lambda\}$ be a basis of $L^2(Y, S)$ consisting of eigensections of $\mathcal{D}(Y)$. Then we define for any $n \in \mathbb{Z}$ and any eigenvalue λ of $\mathcal{D}(Y)$ the sections $\psi_{n,\lambda} := e^{2\pi i n\theta}\phi_\lambda$ over $S^1 \times Y$. They form a L^2-orthonormal basis of $L^2(S^1 \times Y, S^+)$ so that for $u \in \ker(\mathcal{D}_z^+) \setminus \{0\}$ there exist functions $a_{n,\lambda} \in C^\infty(S^1 \times Y, \mathbb{C})$ such that $u = \sum_{n,\lambda} a_{n,\lambda}\psi_{n,\lambda}$ holds*. Then we obtain from $\mathcal{D}_z^+ u = 0$ by inserting the previous sum

$$\begin{aligned} 0 = \mathrm{cl}(\partial_\theta)\mathcal{D}_z^+ u \overset{\text{eq.(3.54)}}{=} & -\sum_{n,\lambda} (\partial_\theta - \mathcal{D}(Y) - is)a_{n,\lambda}\underbrace{e^{2\pi i n\theta}\phi_\lambda}_{=\psi_{n,\lambda}} \\ = & -\sum_{n,\lambda} (2\pi i n - \lambda - is)a_{n,\lambda}\psi_{n,\lambda}. \end{aligned}$$

Since $u \neq 0$ there exists an $n \in \mathbb{Z}$ and an eigenvalue λ with $a_{n,\lambda} \neq 0$ and the previous equation gives $\lambda = i(2\pi n - s)$. But all eigenvalues of the Dirac operator are real, so that if $2\pi n \neq s$, we have a contradiction and if $2\pi n = s$, we have $\lambda = 0$ which contradicts the invertibility of $\mathcal{D}(Y)$.

"$\Rightarrow$" If the Dirac operator $\mathcal{D}^+ : H^1(\widehat{Z}, S^+) \to L^2(\widehat{Z}, S^-)$ is Fredholm, Proposition 3.19 gives that for all $z \in \mathbb{C}^*$ with $|z| = 1$ the twisted Dirac operator $\mathcal{D}_z^+$ is invertible. This gives for $z = 1$ with equation (3.54) that $\partial_\theta - \mathcal{D}(Y)$ is invertible so that $\ker(\mathcal{D}_z^+)$ and $\operatorname{coker}(\mathcal{D}_z^+)$ are trivial and the invertibility of $\mathcal{D}(Y)$ follows.

□

Proof (part (2) of Aim 3.37). By the previous preparations we know that the invertibility of the boundary Dirac operator implies that the chiral Dirac operator $\mathcal{D}^+(\widehat{Z}) : H^1(\widehat{Z}, S^+) \to L^2(\widehat{Z}, S^-)$ is Fredholm and the end-periodic index theorem (Theorem 3.22) is applicable and leads to

$$\operatorname{ind}(\mathcal{D}(\widehat{Z})) = \int_Z \mathbf{I}(\mathcal{D}(\widehat{Z})) \underbrace{- \int_Y \omega + \int_{S^1 \times Y} df \wedge \omega}_{\overset{(\star)}{=} 0} - \frac{1}{2}\underbrace{\eta_{\mathrm{ep}}(\mathcal{D}(S^1 \times Y))}_{\overset{(\star\star)}{=} \eta(\mathcal{D}(Y))} = \int_Z \mathbf{I}(\mathcal{D}(\widehat{Z})) - \frac{1}{2}\eta(\mathcal{D}(Y)).$$

It remains to show the equations $(\star)$ and $(\star\star)$.

* The sum goes over all integers $n \in \mathbb{Z}$ and all eigenvalues λ of $\mathcal{D}(Y)$.

($\star$) We use a generalization of Fubini's theorem to manifolds and differential forms as in [AE09, Theorem 2.4] and obtain

$$\int_{S^1\times Y} df \wedge \omega \overset{\text{Fubini's theorem}}{=} \int_Y \left(\int_{S^1} df \wedge \omega \right) \overset{\text{eq.(3.55)}}{=} \int_Y \omega.$$

Here $\int_{S^1} df \wedge \omega$ is an $(n-1)$-form on Y satisfying for all $y \in Y$ and all $a_2, \ldots, a_n \in T_yY$ (the first step is the definition of the $n-1$ form [see AE09, p.415/416])

$$\left(\int_{S^1} df \wedge \omega \right)_y (a_2, \ldots, a_n) \overset{\text{def.}}{=} \int_{S^1} \underbrace{(df \wedge \omega)_{(\bullet,y)}(\bullet, a_2, \ldots, a_n)}_{\text{one-form on } S^1} = \underbrace{\int_{S^1} df}_{=1} \omega_y(a_2, \ldots, a_n). \tag{3.55}$$

($\star\star$) We begin by studying the integrand appearing in the integral expression of the periodic η-invariant: $\operatorname{Tr}\left(df \cdot \mathcal{D}_z^+ e^{-t\mathcal{D}_z^- \mathcal{D}_z^+}\right)$. With $\psi_{n,\lambda} = e^{2\pi i n\theta}\phi_\lambda$ as in the proof of part (1) of Aim 3.37 we obtain for $z = e^{is}$ with $s \in [0, 2\pi]$, any $n \in \mathbb{Z}$ and any eigenvalue λ of $\mathcal{D}(Y)$

$$\begin{aligned} df \cdot \mathcal{D}_z^+ \psi_{n,\lambda} &\overset{\text{eq.(3.54)}}{=} \underbrace{\big(\operatorname{cl}(\partial_\theta)\operatorname{cl}(\partial_\theta)\big)}_{=-1}\big(\partial_\theta - \mathcal{D}(Y) - is\big)\psi_{n,\lambda} = \big(\lambda + i(s-2\pi n)\big)\psi_{n,\lambda} \\ \mathcal{D}_z^- \mathcal{D}_z^+ \psi_{n,\lambda} &\overset{\text{eq.(3.54)}}{=} -\big(\partial_\theta + \mathcal{D}(Y) - is\big)\big(\partial_\theta - \mathcal{D}(Y) - is\big)\psi_{n,\lambda} = \big(\lambda^2 + (s-2\pi n)^2\big)\psi_{n,\lambda}. \end{aligned} \tag{3.56}$$

In the second equation we used the compatibility of the Clifford multiplication with the connection as in Definition 1.1 and the local expression of the Dirac operator $\mathcal{D}(Y)$ in Proposition 1.3 to commute $(\partial_\theta + \mathcal{D}(Y) - is)$ with $\operatorname{cl}(\partial_\theta)$. Then we obtain

$$\begin{aligned} \operatorname{Tr}\left(df \cdot \mathcal{D}_z^+ e^{-t\mathcal{D}_z^- \mathcal{D}_z^+}\right) &= \sum_{n,\lambda} \left(d\theta \cdot \mathcal{D}_z^+ e^{-t\mathcal{D}_z^- \mathcal{D}_z^+} \psi_{n,\lambda}, \psi_{n,\lambda}\right)_{L^2} \\ &\overset{\text{eq.(3.56)}}{=} \sum_{n,\lambda} \big(\lambda + i(s-2\pi n)\big) e^{-t(\lambda^2 + (s-2\pi n)^2)}. \end{aligned} \tag{3.57}$$

Integrating the previous equation over $|z| = 1$ with measure $\frac{dz}{z}$ parameterized by $\gamma(s) = e^{is}$, using that $se^{-t(\lambda^2+s^2)}$ is an odd function in s and changing the sum over n into an infinite integral gives

$$\begin{aligned} \frac{1}{i\pi} \int_{|z|=1} \operatorname{Tr}\left(df \cdot \mathcal{D}_z^+ e^{-t\mathcal{D}_z^- \mathcal{D}_z^+}\right) \frac{dz}{z} &= \frac{1}{\pi} \int_0^{2\pi} \operatorname{Tr}\left(df \cdot \mathcal{D}_{e^{is}}^+ e^{-t\mathcal{D}_{e^{is}}^- \mathcal{D}_{e^{is}}^+}\right) ds \\ &\overset{\text{eq.(3.57)}}{=} \frac{1}{\pi} \sum_\lambda \int_{-\infty}^{\infty} (\lambda + is) e^{-t(\lambda^2+s^2)} ds \\ &\overset{\text{subst. } u=\sqrt{t}s}{=} \frac{1}{\pi} \sum_\lambda \lambda e^{-t\lambda^2} t^{-1/2} \underbrace{\int_{-\infty}^{\infty} e^{-u^2} du}_{\sqrt{\pi}}. \end{aligned} \tag{3.58}$$

Integration over t leads to the desired formula

$$\begin{aligned}
\eta_{\mathrm{ep}}\big(\mathcal{D}(S^1\times Y)\big) &\overset{\text{def. period. }\eta\text{-inv.}}{\underset{\text{in eq.(3.23)}}{=}} \frac{1}{i\pi}\int_0^\infty \int_{|z|=1} Tr\left(df\cdot D_z^+ \exp\left(-tD_z^-\mathcal{D}_z^+\right)\right)\frac{dz}{z}dt \\
&\overset{\text{eq.(3.58)}}{=} \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2} \underbrace{\sum_\lambda \lambda e^{-t\lambda^2}}_{=\mathrm{Tr}\left(\mathcal{D}(Y)e^{-t\mathcal{D}(Y)^2}\right)} dt \\
&\overset{\text{def. }\eta\text{-inv.}}{\underset{\text{in eq.(2.12)}}{=}} \eta(\mathcal{D}(Y)).
\end{aligned}$$

□

3.4 Outlook to the end-periodic index theorem in the non-Fredholm case

In this section we will generalize the end-periodic index theorem in the Fredholm case stated in Theorem 3.22 to the case where the chiral Dirac operator $\mathcal{D}^+(Z_\infty)$ is not necessarily Fredholm. The analogue generalization in the cylindrical setting is from Theorem 2.16 to Theorem 2.21. The main problem is that the index of the Dirac operator is no longer well defined. We solved this problem for cylindrical Dirac operators by introducing a weight δ which is sufficiently close to zero and noticed that the chiral Dirac operator between the weighted Sobolev spaces is Fredholm (see Proposition 2.20). This index depends just on the sign of δ, hence the $\pm$-indices - defined as the Fredholm indices of the weighted chiral Dirac operators with sufficiently small positive respectively negative weight δ - are well defined. We will make the same approach in the end-periodic setting. Furthermore, we will show how this generalized end-periodic index theorem is related to the APS index theorem. This section is based on [MRS16, Chapter 7] and [MRS11] and will give a rough summary of this extension without giving detailed proofs.

Let S be an end-periodic graded Clifford bundle over an even-dimensional end-periodic Riemannian manifold Z_∞ modeled by (Z, X, γ). We denote the induced Dirac operators with $\mathcal{D}^{(\pm)}(Z_\infty)$ and the Dirac operator over the closed manifold X with $\mathcal{D}(X)$. Furthermore, we assume that the spectral set of the family $\{\mathcal{D}_z^+(X)\}_{z\in\mathbb{C}^*}$ defined before Theorem 3.21 is a discrete subset of $\mathbb{C}^*$. This implies together with the Fredholm criterion given in Proposition 3.19 that the chiral Dirac operator $\mathcal{D}^+(Z_\infty) : H_\delta^k(Z_\infty, S^+) \to H_\delta^{k+1}(Z_\infty, S^-)$ is Fredholm for almost all $\delta \in \mathbb{R}$ as illustrated in Figure 3.6. In the next proposition we will state a formula how the Fredholm index changes when δ varies. A proof of this formula is given in [MRS11, Section 6.4].

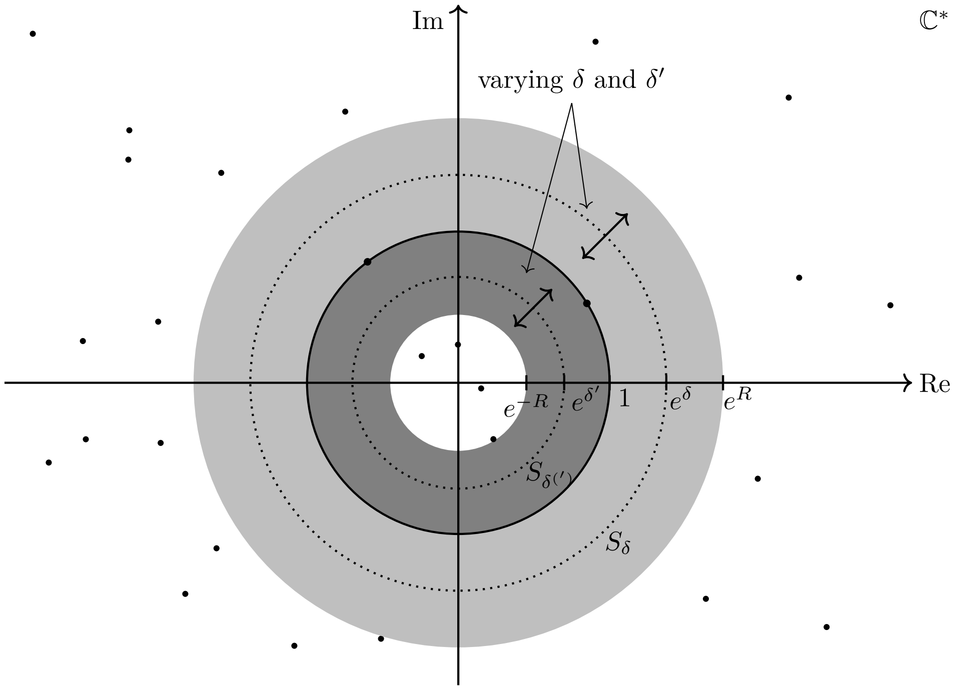

Figure 3.6: Illustration of the spectral set of $\{\mathcal{D}_z^+\}_{z\in\mathbb{C}^*}$. The chiral Dirac operator $\mathcal{D}^+(Z_\infty)$ is in this example not Fredholm because it has spectral points on the unit circle. If we consider the chiral Dirac operator $\mathcal{D}^+(Z_\infty)$ as an operator between weighted Sobolev spaces with weight $\delta^{(\prime)}$ as in the picture, it is Fredholm. Furthermore, its index is independent of $\delta^{(\prime)}$ if the circle $S_{\delta^{(\prime)}}$ stays inside the outer respectively inner gray area. If the weight δ changes such that the circle S_δ jumps over a spectral point, the Fredholm index of $\mathcal{D}^+(Z_\infty) : H^k_\delta(Z_\infty, S^+) \to H^{k+1}_\delta(Z_\infty, S^-)$ changes as stated in Proposition 3.38.

Proposition 3.38 (Index change formula)**.** *Let $\delta < \delta'$ be two weights such that the chiral Dirac operators $\mathcal{D}^+_{\delta^{(\prime)}}(Z_\infty) : H^k_{\delta^{(\prime)}}(Z_\infty, S^+) \to H^{k+1}_{\delta^{(\prime)}}(Z_\infty, S^-)$ are Fredholm. Then the difference of their Fredholm indices is given by*

$$\operatorname{ind}_{Fred}(\mathcal{D}^+_\delta(Z_\infty)) - \operatorname{ind}_{Fred}(\mathcal{D}^+_{\delta'}(Z_\infty)) = \sum_{e^\delta < |z_0| < e^{\delta'}} d(z_0).$$

Here $d(z_0)$ is the dimension of the space of solutions $(\phi_1, \ldots, \phi_m)$ of the system

$$\begin{cases} \mathcal{D}^+_{z_0}(X)\phi_j = df \cdot \phi_{j+1} \quad j \in \{1, \ldots, m-1\} \\ \mathcal{D}^+_{z_0}(X)\phi_m = 0, \end{cases} \tag{3.59}$$

where m is the degree of the pole of $\mathcal{D}_z^+(X)^{-1}$ at $z = z_0$. Note that by the definition of $d(z_0)$ the sum is finite since almost all summands vanish.

The next aim is to define an appropriate index of the Dirac operator in the non-Fredholm case. Let $R > 0$ such that $\{z \in \mathbb{C}^* | e^{-R} < |z| < e^R \wedge |z| \neq 1\}$ does not intersect the spectral set of $\{\mathcal{D}_z^+(X)\}_{z\in\mathbb{C}^+}$ as shown in Figure 3.6. Such an R exists since the spectral set is discrete. Then we define the $\pm$-indices of $\mathcal{D}(Z_\infty)$ as the Fredholm index of $\mathcal{D}^+(Z_\infty) : H_\delta^k(Z_\infty, S^+) \to H_\delta^{k+1}(Z_\infty, S^-)$ for a positive respectively negative $\delta \in \mathbb{R}$ with $|\delta| < R$:

$$\operatorname{ind}_\pm(\mathcal{D}(Z_\infty)) \coloneqq \operatorname{ind}_{\text{Fred}}(\mathcal{D}^+(Z_\infty)). \tag{3.60}$$

Proposition 3.38 implies that the $\pm$-indices are well defined, i.e. they are independent of the choice of δ.

Analogously to $\dim\ker(\mathcal{D}(Y))$ in the APS index theorem (Theorem 2.10 or Theorem 2.21) a correction term will appear in the generalized end-periodic index theorem. We define

$$h \coloneqq \sum_{|z|=1} d(z_0), \tag{3.61}$$

where $d(z_0)$ is defined as in equation (3.59). Furthermore, we have to adapt the periodic η-invariant defined in equation (3.23) because the spectral set of $\{\mathcal{D}_z^+(X)\}_{z\in\mathbb{C}^*}$ could intersect the unit circle. A consequence would be that the end-periodic index formula in equation (3.22) would no longer hold and the convergence of the integral in the definition of the periodic η-invariant in equation (3.23) would not be given anymore. To avoid that problem we will calculate the periodic η-invariant on $|z| = e^\epsilon$, take the sum of the two limits $\epsilon \to 0^\pm$ and divide it by two. Precisely, we define for $\epsilon \neq 0$ sufficiently* small

$$\eta_\epsilon(\mathcal{D}(X)) = \frac{1}{\pi i}\int_0^\infty \int_{|z|=e^\epsilon} \operatorname{Tr}\left(df \cdot D_z^+ e^{-t(\mathcal{D}_z^+)^*\mathcal{D}_z^+}\right) \frac{dz}{z} dt \tag{3.62}$$

as the constant term in the asymptotic expansion of

$$\frac{1}{\pi i}\int_{t_0}^\infty \int_{|z|=e^\epsilon} \operatorname{Tr}\left(df \cdot D_z^+ e^{-t(\mathcal{D}_z^+)^*\mathcal{D}_z^+}\right) \frac{dz}{z} dt$$

in powers of t_0. The right hand side in equation (3.62) is basically the definition of the

* Sufficiently small means here that $\left\{z \in \mathbb{C}^* \middle| e^{-\epsilon} \leq |z| \leq e^\epsilon \wedge |z| \neq 1\right\}$ does not intersect the spectral set of $\{D_z^+(X)\}_{z\in\mathbb{C}^*}$.

periodic η-invariant given in equation (3.23) with $\mathcal{D}_z^-$ replaced* by $(\mathcal{D}_z^+)^*$ and adapted integration over z. We define the *generalized periodic* η-invariant as

$$\widetilde{\eta}_{\mathrm{ep}}(\mathcal{D}(X)) := \frac{1}{2}\Big(\lim_{\epsilon\to 0^+}\eta_\epsilon(\mathcal{D}(X)) + \lim_{\epsilon\to 0^-}\eta_\epsilon(\mathcal{D}(X))\Big) \tag{3.63}$$

The well-definedness of this invariant follows from the generalization of the end-periodic index theorem:

Theorem 3.39 (End-periodic index theorem in the non-Fredholm case)**.**
Let S be an end-periodic graded Clifford bundle over an even-dimensional end-periodic Riemannian manifold Z_∞ modeled by (Z, X, γ). Furthermore, $f : \widetilde{X} \to \mathbb{R}$ denotes the function defined in the proof of Lemma 3.6 *part (2) and Y the boundary of Z. If the spectral set of $\{D_z^+(X)\}_{z\in\mathbb{C}^*}$ is a discrete subset of $\mathbb{C}^*$ the $\pm$-indices defined in equation (3.60) are given by*

$$\mathrm{ind}_\pm\left(\mathcal{D}(Z_\infty)\right) = \int_Z \boldsymbol{I}(\mathcal{D}(Z_\infty)) - \int_Y \omega + \int_X df\wedge\omega - \frac{1}{2}\Big(\widetilde{\eta}_{ep}(\mathcal{D}(X)) \pm h\Big). \tag{3.64}$$

Here $\widetilde{\eta}_{ep}(\mathcal{D}(X))$ is the generalized periodic η-invariant defined in equation (3.63), h the correction term defined in equation (3.61) and ω a differential form† on X such that $d\omega = \boldsymbol{I}(\mathcal{D}(X))$ holds.

Proof. The proof is an adaption of the proof of the end-periodic index theorem in the Fredholm case (Theorem 3.22). It is worked out for $\mathrm{ind}_+(\mathcal{D}(Z_\infty))$ in [MRS16, Section 7.3]. By definition of the $\pm$-indices and the index change formula stated in Proposition 3.38, we have

$$\mathrm{ind}_-\left(\mathcal{D}(Z_\infty)\right) - \mathrm{ind}_+\left(\mathcal{D}(Z_\infty)\right) = \sum_{|z|=1} d(z_0) = h$$

and the index formula for $\mathrm{ind}_-\left(\mathcal{D}(Z_\infty)\right)$ follows from the index formula for $\mathrm{ind}_+\left(\mathcal{D}(Z_\infty)\right)$. ☐

Remark 3.40 (Theorem 3.39 as generalization of Theorem 3.22)**.** If $\mathcal{D}^+(Z_\infty)$ is additional to the assumptions in Theorem 3.39 a Fredholm operator, formula (3.64) matches

* For $|z| \neq 1$ the adjoint of $\mathcal{D}_z^+$ is not equal to $\mathcal{D}_z^-$. † The existence of such a differential form is a consequence of the assumption that the spectral set is discrete. Use Corollary 3.20 to obtain $\mathrm{ind}(\mathcal{D}(X)) = 0$ and we are in the situation as in Remark 3.23 (1).

with the index formula in (3.22).

- Since the spectral set of $\{\mathcal{D}_z^+(X)\}_{z\in C^*}$ avoids the unit circle by Proposition 3.19, h is by definition zero.
- We obtain by the index change formula stated in Proposition 3.38

$$\mathrm{ind}_+(\mathcal{D}(Z_\infty)) = \mathrm{ind}_-(\mathcal{D}(Z_\infty)) = \mathrm{ind}_{\mathrm{Fred}}(\mathcal{D}^+(Z_\infty)) = \mathrm{ind}(\mathcal{D}(Z_\infty)). \tag{3.65}$$

- For the generalized periodic η-invariant, we obtain $\widetilde{\eta}_{\mathrm{ep}}(\mathcal{D}(X)) = \eta_{\mathrm{ep}}(\mathcal{D}(X))$ since $\eta_\epsilon(\mathcal{D}(X))$ exists for $\epsilon = 0$.

Remark 3.41 (Theorem 3.39 as a generalization of Theorem 2.21)**.** Let $S \to \widehat{Z}$ be a graded Clifford bundle with cylindrical end modeled by (Z, Y) as in Definition 2.14 and denote by $\mathcal{D}^+(\widehat{Z})$ the induced chiral Dirac operator and by $\mathcal{D}(Y)$ the boundary Dirac operator. The manifold $\widehat{Z}$ has even dimension n. As explained in Section 3.3.5 this is an end-periodic Clifford bundle over an even-dimensional end-periodic Riemannian manifold with $X = S^1 \times Y$. A straight forward calculation gives that the spectral set of $\{\mathcal{D}_z^+(X)\}_{z\in\mathbb{C}^*}$ is equal to $\{e^{\pm\lambda} | \lambda \text{ is an eigenvalue of } \mathcal{D}(Y)\}$. One can use equation (3.54) and calculate similarly as in the proof of Aim 3.37 (1) direction "$\Leftarrow$" with the explicit given L^2-orthonormal basis of $L^2(S^1 \times Y, S^+)$. Then the end-periodic index theorem in the non-Fredholm case is applicable and the formulas given in (3.64) and (2.23) coincides:

- The $\pm$-indices of $\mathcal{D}(\widehat{Z})$ coincide by definition.
- We know by the preliminary in this remark that $z = 1$ is the only point on the unit circle that could be in the spectral set of $\{\mathcal{D}_z^+(X)\}_{z\in\mathbb{C}^*}$. This implies

$$h = \sum_{|z_0|=1} d(z_0) = d(1) = \dim\ker(\mathcal{D}(Y)).$$

  The last step holds by exploiting $\mathcal{D}_1^+(X) = \mathcal{D}^+(X) = d\theta(\partial_\theta - \mathcal{D}(Y))$ and studying the system of equations given for $z_0 = 1$ in (3.59).
- The generalized end-periodic η-invariant coincides with the classical η-invariant and the additional term $-\int_Y \omega + \int_{S^1\times Y} df \wedge \omega$ vanishes. This can be worked out similarly as in Section 3.3.5.

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