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Scalar curvature comparison geometry and the higher index

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Classical existence question

Does a given manifold admit a Riemannian metric of positive scalar curvature (pscm)?

Let M be a closed connected spin manifold of dimension m.

Rosenberg index: $\alpha(M) := \operatorname{ind} \mathcal{D}_{\mathcal{L}} \in \mathrm{KO}_m(\mathrm{C}^*\pi)$ (higher index)

- $\mathcal{D}_{\mathcal{L}}$: $\mathbb{C}\mathrm{l}_n$-linear spin Dirac operator twisted by the Mishchenko-Fomenko bundle $\mathcal{L}(M) := \widetilde{M} \times_\pi \mathrm{C}^*\pi$.
- $\mathrm{C}^*\pi$: group C^*-algebra of the fundamental group of M.

Fact. The Rosenberg index is the most general known index-theoretical obstruction to the existence of a pscm.

$$\begin{tikzcd}
\widehat{A}(M) \neq 0 \arrow[r, Rightarrow] & \alpha(M) \neq 0 \arrow[r, Rightarrow] & \text{no pscm on } M \\
\dots \arrow[ur, Rightarrow] & M \text{ enlargeable} \arrow[u, Rightarrow] &
\end{tikzcd}$$

Examples.

- *The $\widehat{A}$-genus of the n-torus is zero but the Rosenberg index does not vanish.*
- *There exist exotic spheres Σ^k with non-vanishing Rosenberg index.*

General principle

classical index $\overset{\text{replace by}}{\rightsquigarrow}$ higher index

What geometric usefull information can we deduce from a non-vanishing higher index?

Classically: A non-vanishing $\widehat{A}$-genus gives rise to a non-vanishing harmonic spinor u. In extreme geometric situations u is parallel, hence

$$|u|_p^2 = \langle u(p), u(p) \rangle_p = \text{const.}$$

New method

Technical Lemma (T. [3]). *Let $\mathcal{A}$ be a unital Real C^*-algebra and $\mathcal{S} \to M$ a graded Real $\mathcal{A}$-linear Dirac bundle with induced $\mathcal{A}$-linear Dirac operator $\not{D}$.*

- *If the higher index of $\not{D}$ does not vanish, there exists a family u_ϵ of almost $\not{D}$-harmonic sections.*
- *If, moreover, u_ϵ is L^2-almost parallel, the family is* ***almost constant****, i.e. there exist constants $C, r > 0$ and an element $a \in \mathcal{A}^+$ such that*

$$\left\| a - \langle u_\epsilon(p), u_\epsilon(p) \rangle_p \right\|_{\mathcal{A}} < C\epsilon^r \quad \forall p \in M \, \forall \epsilon \in (0,1).$$

Rigidity question

How rich is the space of Riemannian metrics satisfying a certain lower scalar curvature bound?

Theorem (Llarull [2]). *Let M be a closed connected spin manifold of dimension $m \geq 3$. Any area non-increasing smooth map $f\colon M \to S^m$ with $\mathrm{scal}_M \geq m(m-1)$ is an isometry.*

We generalize a rigidity statement by Goette and Semmelmann [1] (which is a generalization of the previous theorem) to spin maps between possibly non-orientable manifolds and replace their topological condition on the $\hat{A}$-degree by a less restrictive condition involving higher index theory.

Definition. Let $f\colon M \to N$ be a *spin map*, i.e.

$$w_i(TM) = f^*(w_i(TN)) \quad (i = 1, 2),$$

between two closed connected Riemannian manifolds of dimension $n + k$ and n, respectively. The **higher mapping degree** of f is defined via

$$\deg_{\mathrm{hi}}(f) := \operatorname{ind}\left(\mathcal{D}_{\mathcal{S}f^{-1}(p) \otimes \mathcal{L}(M)|_{f^{-1}(p)}}\right) \in \mathrm{KO}_k(\mathrm{C}^*\pi)$$

for a regular value p of the map f.

Main Theorem (T. [3]). *Let $f\colon M \to N$ be an area non-increasing spin map between two closed connected Riemannian manifolds of dimension $n + k$ and n, respectively. Suppose*

- *N has non-negative curvature operator and*
- **$\deg_{\mathrm{hi}}(f) \cdot \chi(N) \neq 0 \in \mathrm{KO}_k(\mathrm{C}^*\pi)$.**

Then: (1) $\mathrm{scal}_M \geq \mathrm{scal}_N \circ f \quad \Rightarrow \mathrm{scal}_M = \mathrm{scal}_N \circ f$

(2) $\left\{\begin{matrix} \mathrm{scal}_M \geq \mathrm{scal}_N \circ f \\ \mathrm{scal}_N > 2\,\mathrm{Ric}_N > 0 \end{matrix}\right\} \Rightarrow$ *f is a Riemannian submersion*

Proof (sketch). The spin map f gives rise to a $\mathbb{C}\mathrm{l}_{n+k,n} \otimes \mathrm{C}^*\pi$-linear Dirac bundle $\mathcal{S}M \otimes f^*\mathcal{S}N \otimes \mathcal{L}(M) \to M$ with induced Dirac operator $\mathcal{D}_{\mathcal{L}}$ satisfying

- $\operatorname{ind}(\mathcal{D}_{\mathcal{L}}) = \deg_{\mathrm{hi}}(f) \cdot \chi(N) \neq 0$ and
- $\mathcal{D}_{\mathcal{L}}^2 \geq \nabla^*\nabla + \frac{1}{4}(\mathrm{scal}_M - \mathrm{scal}_N \circ f)$.

The extremality and rigidity statement follows from the existence of a family of almost constant sections (see Technical Lemma). □

Examples. *The Main Theorem applies to the following maps:*

- $\mathrm{pr}_1 : S^{2n} \times T^k \to S^{2n}$.
- $\mathrm{pr}_1 : \mathbb{RP}^{2n} \times \Sigma^k \to \mathbb{RP}^{2n}$.
- $f\colon M \to \left(S^{2n}, g_{\mathit{round}}\right)$ *fiber bundle whose typical fiber F satisfies $2\alpha(F) \neq 0$.*

References

[1] S. Goette and U. Semmelmann. "Scalar curvature estimates for compact symmetric spaces". In: *Diff. Geom. Appl.* 16.1 (2002).

[2] M. Llarull. "Sharp estimates and the Dirac operator". English. In: *Mathematische Annalen* 310.1 (1998).

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