

(Almost) Parallel Spinors and the Rosenberg Index

Workshop on Curvature and Global Shape Contributor: Thomas Tony

Question

Does every closed connected Ricci-flat spin manifold admit a parallel spinor on its universal covering?

Let M be a closed connected spin manifold of dimension n with fundamental group $\pi.$

$$\widehat{A}\text{-}\textit{genus}\colon \, \widehat{A}(M) \coloneqq \begin{cases} \operatorname{ind} \mathcal{D} \in \mathbb{Z} & \text{if } n \text{ even}, \\ 0 & \text{else}. \end{cases}$$

▶ Spinor bundle of $M \leadsto \text{spin Dirac operator } \mathcal{D}$.

Fact 1. Every closed connected Ricci-flat spin manifold with non-vanishing \hat{A} -genus admits a parallel spinor.

Rosenberg index: $\alpha(M) := \operatorname{ind} \mathcal{D}_{\mathcal{L}} \in \mathrm{KO}_n(\mathrm{C}^*\pi)$ (higher index).

- ▶ SM: $\mathbb{C}l_n$ -linear spinor bundle
- $\mathcal{L}(M) := \widetilde{M} \times_{\pi} \mathrm{C}^* \pi$ Mishchenko-Fomenko bundle.
- ▶ $C^*\pi$: maximal group C^* -algebra of π .
- $SM \otimes \mathcal{L}(M)$ is a $\mathbb{C}l_n \otimes \mathbb{C}^*\pi$ -linear Dirac bundle \leadsto twisted spin Dirac operator $\mathcal{D}_{\mathcal{L}}$.
- ► $KO_n(C^*\pi)$: *n*-th Real K-theory group of the C^* -algebra $C^*\pi$.

Fact 2. The Rosenberg index is the most general known index-theoretical obstruction to the existence of a pscm on ${\cal M}.$

Examples. ightharpoonup The \widehat{A} -genus of the n-torus is zero but the Rosenberg index does not vanish.

lacktriangle There exist exotic spheres Σ with non-zero Rosenberg index.

General principle classical index replace by higher index

While a non-trivial \hat{A} -genus yields a non-trivial kernel of the spin Dirac operator, this is in general no longer true for a non-vanishing higher index.

▶ Does Fact 1 still hold for the Rosenberg index?

Existence of parallel spinors

Theorem 1 ([1, Theorem A]). Every closed connected Ricciflat spin manifold with non-vanishing Rosenberg index admits a parallel spinor on a finite Riemannian covering.

Corollary 2 ([1, Corollary B]). Every closed connected Ricciflat spin manifold of dimension $n \geq 2$ with non-vanishing Rosenberg index has special holonomy, i.e. its reduced holonomy group is a proper subgroup of SO(n).

Existence of almost parallel spinors

Theorem 3 (T. [4]). Let \mathcal{A} be a graded Real unital C^* -algebra and $\mathcal{S} \to M$ a graded Real \mathcal{A} -linear Dirac bundle with induced \mathcal{A} -linear Dirac operator D.

1. If the higher index of $\not \! D$ does not vanish, there exists a family $\{u_{\epsilon}\}_{\epsilon>0}$ of **almost** $\not \! D$ -harmonic sections of $\mathcal S$, i.e.

$$\|u_{\epsilon}\|_{\mathrm{L}^{2}}=1$$
 and $\|D^{i}u_{\epsilon}\|_{\mathrm{L}^{2}}<\epsilon^{i}$ $\forall i\geq 1,\, \forall \epsilon>0.$

2. If, moreover, $\|\nabla u_{\epsilon}\|_{\mathbf{L}^{2}}<\epsilon$ for all $\epsilon>0$, then u_{ϵ} is almost parallel i.e. there exist constants C,r>0 such that

$$\|\nabla u_{\epsilon}\|_{\infty} < C\epsilon^{r} \quad \forall \epsilon \in (0, 1).$$

In certain extreme geometric situations, we obtain $\|\nabla u_\epsilon\|_{L^2}<\epsilon$ from a Schrödinger-Lichnerowicz type formula like

$${D\!\!\!/}^2 \ge \nabla^* \nabla + {\cal R}, \qquad {\cal R} = {\sf some \ rest \ term}.$$

Application 1: Scalar curvature extremality and rigidity

The following theorem generalizes [2, 3].

Theorem 4 (T. [1]). Let $f: M \to N$ be a **spin** map between two closed connected Riemannian manifolds. Suppose

- $\mathcal{R}_N \geq 0$, $\mathrm{Ric}_N > 0$ and
- $ightharpoonup \chi(N) \cdot \operatorname{deg}_{hi}(f) \neq 0 \in \operatorname{KO}(\mathbf{C}^*\pi).$

Then the following implication holds:

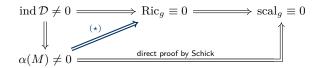
$$\left. \begin{array}{l} \operatorname{scal}_M \geq \operatorname{scal}_N \circ f \\ g_M \geq f^* g_N \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} \operatorname{scal}_M = \operatorname{scal}_N \circ f \text{ and} \\ f \text{ is a Riem. submersion} \end{array} \right.$$

- f is called *spin* if $w_i(TM) = f^*(w_i(TN))$ for i = 1, 2.
- ightharpoonup For a regular value p of f, we define

$$\deg_{\mathsf{hi}}(f) \coloneqq \operatorname{ind}\left(\mathcal{D}_{\mathcal{S}f^{-1}(p)\otimes\mathcal{L}(M)\upharpoonright_{f^{-1}(p)}}\right) \in \mathrm{KO}(\mathrm{C}^*\pi).$$

Application 2: A spinorial proof of implication (\star)

Let g be a Riemannian metric on a closed connected spin manifold M with $\mathrm{scal}_g \geq 0.$ Then



References

- T. Tony. Ricci-flat manifolds, parallel spinors and the Rosenberg index. Preprint. 2024. arXiv: 2411.03882 [math.DG].
- S. Goette and U. Semmelmann. "Scalar curvature estimates for compact symmetric spaces". In: Diff. Geom. Appl. 16.1 (2002).
- [3] M. Llarull. "Sharp estimates and the Dirac operator". English. In: Mathematische Annalen 310.1 (1998).
- T. Tony. "Scalar curvature rigidity and the higher mapping degree". In: Journal of Functional Analysis 288.3 (2025).







