

## Question

Does every closed connected Ricci-flat spin manifold admit a parallel spinor on its universal covering?

Let  $M$  be a closed connected spin manifold of dimension  $n$  with fundamental group  $\pi$ .

$$\hat{A}\text{-genus: } \hat{A}(M) := \begin{cases} \text{ind } \mathcal{D} \in \mathbb{Z} & \text{if } n \text{ even,} \\ 0 & \text{else.} \end{cases}$$

- Spinor bundle of  $M \rightsquigarrow$  spin Dirac operator  $\mathcal{D}$ .

**Fact 1.** Every closed connected Ricci-flat spin manifold with non-vanishing  $\hat{A}$ -genus admits a parallel spinor.

**Rosenberg index:**  $\alpha(M) := \text{ind } \mathcal{D}_{\mathcal{L}} \in \text{KO}_n(C^*\pi)$  (higher index).

- $SM$ :  $\mathbb{C}l_n$ -linear spinor bundle
- $\mathcal{L}(M) := \tilde{M} \times_{\pi} C^*\pi$  Mishchenko-Fomenko bundle.
- $C^*\pi$ : maximal group  $C^*$ -algebra of  $\pi$ .
- $SM \otimes \mathcal{L}(M)$  is a  $\mathbb{C}l_n \otimes C^*\pi$ -linear Dirac bundle  $\rightsquigarrow$  twisted spin Dirac operator  $\mathcal{D}_{\mathcal{L}}$ .
- $\text{KO}_n(C^*\pi)$ :  $n$ -th Real K-theory group of the  $C^*$ -algebra  $C^*\pi$ .

**Fact 2.** The Rosenberg index is the most general known index-theoretical obstruction to the existence of a pscm on  $M$ .

$$\begin{array}{ccccc} \hat{A}(M) \neq 0 & \implies & \alpha(M) \neq 0 & \implies & \text{no pscm on } M \\ & \nearrow & \uparrow & & \\ \dots & & M \text{ enlargeable} & & \end{array}$$

**Examples.** ► The  $\hat{A}$ -genus of the  $n$ -torus is zero but the Rosenberg index does not vanish.

- There exist exotic spheres  $\Sigma$  with non-zero Rosenberg index.

**General principle** classical index  $\rightsquigarrow$  replace by  $\rightsquigarrow$  higher index

While a non-trivial  $\hat{A}$ -genus yields a non-trivial kernel of the spin Dirac operator, this is in general no longer true for a non-vanishing higher index.

- Does Fact 1 still hold for the Rosenberg index?

## Existence of parallel spinors

**Theorem 1** ([1, Theorem A]). Every closed connected Ricci-flat spin manifold with non-vanishing Rosenberg index admits a parallel spinor on a finite Riemannian covering.

**Corollary 2** ([1, Corollary B]). Every closed connected Ricci-flat spin manifold of dimension  $n \geq 2$  with non-vanishing Rosenberg index has special holonomy, i.e. its reduced holonomy group is a proper subgroup of  $\text{SO}(n)$ .

## Existence of almost parallel spinors

**Theorem 3** (T. [4]). Let  $\mathcal{A}$  be a graded Real unital  $C^*$ -algebra and  $S \rightarrow M$  a graded Real  $\mathcal{A}$ -linear Dirac bundle with induced  $\mathcal{A}$ -linear Dirac operator  $\mathcal{D}$ .

1. If the higher index of  $\mathcal{D}$  does not vanish, there exists a family  $\{u_{\epsilon}\}_{\epsilon>0}$  of **almost  $\mathcal{D}$ -harmonic** sections of  $S$ , i.e.

$$\|u_{\epsilon}\|_{L^2} = 1 \quad \text{and} \quad \|\mathcal{D}^i u_{\epsilon}\|_{L^2} < \epsilon^i \quad \forall i \geq 1, \forall \epsilon > 0.$$

2. If, moreover,  $\|\nabla u_{\epsilon}\|_{L^2} < \epsilon$  for all  $\epsilon > 0$ , then  $u_{\epsilon}$  is **almost parallel** i.e. there exist constants  $C, r > 0$  such that

$$\|\nabla u_{\epsilon}\|_{\infty} < C\epsilon^r \quad \forall \epsilon \in (0, 1).$$

In certain extreme geometric situations, we obtain  $\|\nabla u_{\epsilon}\|_{L^2} < \epsilon$  from a Schrödinger-Lichnerowicz type formula like

$$\mathcal{D}^2 \geq \nabla^* \nabla + \mathcal{R}, \quad \mathcal{R} = \text{some rest term.}$$

## Application 1: Scalar curvature extremality and rigidity

The following theorem generalizes [2, 3].

**Theorem 4** (T. [1]). Let  $f: M \rightarrow N$  be a **spin** map between two closed connected Riemannian manifolds. Suppose

- $\mathcal{R}_N \geq 0$ ,  $\text{Ric}_N > 0$  and
- $\chi(N) \cdot \deg_{\text{hi}}(f) \neq 0 \in \text{KO}(C^*\pi)$ .

Then the following implication holds:

$$\left. \begin{array}{l} \text{scal}_M \geq \text{scal}_N \circ f \\ g_M \geq f^* g_N \end{array} \right\} \implies \left\{ \begin{array}{l} \text{scal}_M = \text{scal}_N \circ f \text{ and} \\ f \text{ is a Riem. submersion} \end{array} \right.$$

- $f$  is called **spin** if  $w_i(TM) = f^*(w_i(TN))$  for  $i = 1, 2$ .
- For a regular value  $p$  of  $f$ , we define

$$\deg_{\text{hi}}(f) := \text{ind}(\mathcal{D}_{Sf^{-1}(p) \otimes \mathcal{L}(M)|_{f^{-1}(p)}}) \in \text{KO}(C^*\pi).$$

## Application 2: A spinorial proof of implication (\*)

Let  $g$  be a Riemannian metric on a closed connected spin manifold  $M$  with  $\text{scal}_g \geq 0$ . Then

$$\begin{array}{ccccc} \text{ind } \mathcal{D} \neq 0 & \implies & \text{Ric}_g \equiv 0 & \implies & \text{scal}_g \equiv 0 \\ \downarrow & \nearrow (*) & & & \uparrow \\ \alpha(M) \neq 0 & \xrightarrow{\text{direct proof by Schick}} & & & \end{array}$$

## References

- [1] T. Tony. Ricci-flat manifolds, parallel spinors and the Rosenberg index. Preprint. 2024. arXiv: 2411.03882 [math.DG].
- [2] S. Goette and U. Semmelmann. "Scalar curvature estimates for compact symmetric spaces". In: *Diff. Geom. Appl.* 16.1 (2002).
- [3] M. Llarull. "Sharp estimates and the Dirac operator". English. In: *Mathematische Annalen* 310.1 (1998).
- [4] T. Tony. "Scalar curvature rigidity and the higher mapping degree". In: *Journal of Functional Analysis* 288.3 (2025).