

# Scalar curvature rigidity and higher index theory

Workshop: Geometric moduli spaces - rigidity, genericity, stability

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## Classical existence question

Does a given manifold admit a Riemannian metric of positive scalar curvature (pscm)?

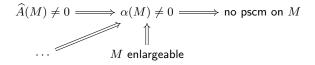
Let M be a closed connected spin manifold of dimension m.

 $\widehat{A}$ -genus:  $\widehat{A}(M) := \operatorname{ind} \mathcal{D} \in \mathbb{Z}$ .

Rosenberg index:  $\alpha(M) := \operatorname{ind} \mathcal{D}_{\mathcal{L}} \in \mathrm{KO}_m(\mathrm{C}^*\pi)$  (higher index).

- $ightharpoonup \mathcal{D}$ : Dirac operator induced by the spinor bundle of M.
- $ightharpoonup \mathcal{D}_{\mathcal{L}}$ :  $\mathbb{C}l_n$ -linear spin Dirac operator twisted by the Mishchenko-Fomenko bundle  $\mathcal{L}(M) \coloneqq \widetilde{M} \times_{\pi} \mathrm{C}^*\pi$ .
- $C^*\pi$ : group  $C^*$ -algebra of the fundamental group  $\pi$  of M.

Fact. The Rosenberg index is the most general known index-theoretical obstruction to the existence of a pscm on  ${\cal M}.$ 



**Examples.** ightharpoonup The  $\widehat{A}$ -genus of the n-torus is zero but the Rosenberg index does not vanish.

▶ There exist exotic spheres  $\Sigma^k$  of non-zero Rosenberg index.

General principle classical index replace by higher index

What geometric usefull information can we deduce from a non-vanishing higher index?

**Classically:** A non-vanishing  $\widehat{A}$ -genus gives rise to a non-vanishing  $\mathcal{D}$ -harmonic spinor u. In extreme geometric situations u is parallel, hence

$$|u|_p^2 = \langle u(p), u(p) \rangle_p = \text{const.}$$

#### New method

**Technical Lemma** (T. [3]). Let  $\mathcal{A}$  be a unital Real  $C^*$ -algebra and  $\mathcal{S} \to M$  a graded Real  $\mathcal{A}$ -linear Dirac bundle with induced  $\mathcal{A}$ -linear Dirac operator  $\mathcal{D}$ .

- If the higher index of D does not vanish, there exists a family  $u_{\epsilon}$  of almost D-harmonic sections.
- ▶ If, moreover,  $u_{\epsilon}$  is  $L^2$ -almost parallel, the family is almost constant, i.e. there exist constants C, r > 0 and an element  $a \in \mathcal{A}^+$  such that

$$\left\| a - \left\langle u_{\epsilon}(p), u_{\epsilon}(p) \right\rangle_{p} \right\|_{\mathcal{A}} < C\epsilon^{r} \quad \forall p \in M \ \forall \epsilon \in (0, 1).$$

## Rigidity question

How rich is the space of Riemannian metrics satisfying a certain lower scalar curvature bound?

**Theorem** (Llarull [2]). Let  $f: M \to S^m$  be a smooth map of non-zero degree and  $m \ge 3$ . Then

We generalize an extremality and rigidity statement by Goette and Semmelmann [1] (a generalization of Llarull's theorem).

Main Theorem (T. [3]). Let  $f: M \to N$  be a spin map between two closed connected Riemannian manifolds of dimension n+k and n, respectively. Suppose

- ▶  $\mathcal{R}_N \ge 0$ ,  $\mathrm{scal}_N > 2 \operatorname{Ric}_N > 0$  and
- $\blacktriangleright \chi(N) \cdot \deg_{\mathsf{hi}}(f) \neq 0 \in \mathsf{KO}_k(\mathbf{C}^*\pi).$

Then the following implication holds:

$$\frac{\operatorname{scal}_{M} \geq \operatorname{scal}_{N} \circ f}{g_{M} \geq f^{*}g_{N} \text{ on } \bigwedge^{2}TM} \right\} \Longrightarrow \left\{ \begin{array}{c} \operatorname{scal}_{M} = \operatorname{scal}_{N} \circ f \text{ and} \\ f \text{ is a Riem. submersion} \end{array} \right.$$

**Definition.** Let f be as in the Main Theorem.

- f is called *spin* if  $w_i(TM) = f^*(w_i(TN))$  (i = 1, 2) holds.
- $\label{eq:poisson}$  The **higher mapping degree** of f is defined for a regular value p of the map f via

$$\deg_{\mathsf{hi}}(f) \coloneqq \operatorname{ind}\left(\mathcal{D}_{\mathcal{S}f^{-1}(p)\otimes\mathcal{L}(M)\upharpoonright_{f^{-1}(p)}}\right) \in \mathrm{KO}_{k}(\mathrm{C}^{*}\pi).$$

*Proof.* The map f gives rise to a  $\mathbb{C}l_{n+k,n}\otimes\mathbb{C}^*\pi$ -linear Dirac bundle  $\mathcal{S}M\otimes f^*\mathcal{S}N\otimes\mathcal{L}(M)$  with induced Dirac operator  $\mathcal{D}_{\mathcal{L}}$  satisfying

- ▶  $\operatorname{ind}(\mathcal{D}_{\mathcal{L}}) = \deg_{\operatorname{hi}}(f) \cdot \chi(N) \neq 0$  and
- $\mathcal{D}_{\mathcal{L}}^2 \geq \nabla^* \nabla + \frac{1}{4} (\operatorname{scal}_M \operatorname{scal}_N \circ f).$

The rigidity statement follows from the Technical Lemma.

**Examples.** The Main Theorem applies to the following maps:

- $ightharpoonup \operatorname{pr}_1: \mathbb{RP}^{2n} \times \Sigma^k \to \mathbb{RP}^{2n}$

### References

- S. Goette and U. Semmelmann. "Scalar curvature estimates for compact symmetric spaces". In: Diff. Geom. Appl. 16.1 (2002).
- [2] M. Llarull. "Sharp estimates and the Dirac operator". English. In: Mathematische Annalen 310.1 (1998).
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