Scalar curvature rigidity and higher index theory

Workshop: Geometric moduli spaces - rigidity, genericity, stability
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Classical existence question

Does a given manifold admit a Riemannian metric of positive scalar curvature (pscm)?

Let M be a closed connected spin manifold of dimension m with fundamental group π.

$\widehat{A}$-genus: $\hat{A}(M) := \operatorname{ind} \mathcal{D} \in \mathbb{Z}$.

- spinor bundle of $M \rightsquigarrow$ spin Dirac operator $\mathcal{D}$.

Rosenberg index: $\alpha(M) := \operatorname{ind} \mathcal{D}_{\mathcal{L}} \in \mathrm{KO}_m(\mathrm{C}^*\pi)$ (higher index).

- $\mathcal{S}M \otimes \mathcal{L}(M) \rightsquigarrow$ twisted spin Dirac operator $\mathcal{D}_{\mathcal{L}}$.
- $\mathcal{S}M$: $\mathbb{C}\mathrm{l}_m$-linear spinor bundle
- $\mathcal{L}(M) := \widetilde{M} \times_\pi \mathrm{C}^*\pi$ Mishchenko-Fomenko bundle.
- $\mathrm{C}^*\pi$: maximal group C^*-algebra of π.

Fact. The Rosenberg index is the most general known index-theoretical obstruction to the existence of a pscm on M.

$$\begin{tikzcd} \widehat{A}(M) \neq 0 \arrow[r, Rightarrow] & \alpha(M) \neq 0 \arrow[r, Rightarrow] & \text{no pscm on } M \\ \dots \arrow[ur, Rightarrow] & M \text{ enlargeable} \arrow[u, Rightarrow] & \end{tikzcd}$$

Examples.
- The $\widehat{A}$-genus of the n-torus is zero but the Rosenberg index does not vanish.
- There exist exotic spheres Σ^k of non-zero Rosenberg index.

General principle

classical index $\overset{\text{replace by}}{\rightsquigarrow}$ higher index

What geometric usefull information can we deduce from a non-vanishing higher index?

Classically: A non-vanishing $\widehat{A}$-genus gives rise to a non-vanishing $\mathcal{D}$-harmonic spinor u. In extreme geometric situations u is parallel, hence

$$|u|_p^2 = \langle u(p), u(p) \rangle_p = \text{const.}$$

New method

Technical Lemma (T. [3]). *Let $\mathcal{A}$ be a unital Real C^*-algebra and $\mathcal{S} \to M$ a graded Real $\mathcal{A}$-linear Dirac bundle with induced $\mathcal{A}$-linear Dirac operator $\not{D}$.*

- *If the higher index of $\not{D}$ does not vanish, there exists a family $\{u_\epsilon\}_{\epsilon>0}$ of almost $\not{D}$-harmonic sections, i.e.*
$$\|u_\epsilon\|_{\mathrm{L}^2} = 1 \quad \textit{and} \quad \|\not{D}^j u_\epsilon\|_{\mathrm{L}^2} < \epsilon \quad \forall j > 1, \forall \epsilon > 0.$$
- *If, moreover, $\|\nabla u_\epsilon\|_{\mathrm{L}^2} < \epsilon$ for all $\epsilon > 0$, then there exist constants $C, r > 0$ such that*
$$\|\nabla u_\epsilon\|_\infty < C\epsilon^r \quad \forall \epsilon \in (0,1),$$
*hence the family $\{u_\epsilon\}_{\epsilon>0}$ is **almost constant**.*

Rigidity question

How rich is the space of Riemannian metrics satisfying a certain lower scalar curvature bound?

Theorem (Llarull [2]). *Let $f\colon (M, g_M) \to (S^m, g_0)$ be a smooth map of non-zero degree and $m \geq 3$. Then*

$$\left.\begin{aligned} &\mathrm{scal}_M \geq m(m-1) \\ &g_M \geq f^* g_0 \text{ on } \textstyle\bigwedge^2 TM \end{aligned}\right\} \Longrightarrow f \text{ is an isometry.}$$

We generalize an extremality and rigidity statement by Goette and Semmelmann [1] (a generalization of Llarull's theorem).

Main Theorem (T. [3]). *Let $f\colon M \to N$ be a **spin** map between two closed connected Riemannian manifolds of dimension $n+k$ and n, respectively. Suppose*

- *the curvature operator of N is non-negative,*
- *$\mathrm{scal}_N > 2\,\mathrm{Ric}_N > 0$ and*
- *$\chi(N) \cdot \boldsymbol{\deg_{\mathrm{hi}}(f)} \neq 0 \in \boldsymbol{\mathrm{KO}_k(\mathrm{C}^*\pi)}$.*

Then the following implication holds:

$$\left.\begin{aligned} &\mathrm{scal}_M \geq \mathrm{scal}_N \circ f \\ &g_M \geq f^* g_N \text{ on } \textstyle\bigwedge^2 TM \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} &\mathrm{scal}_M = \mathrm{scal}_N \circ f \text{ and} \\ &f \text{ is a Riem. submersion} \end{aligned}\right.$$

Definition.
- The map f is called **spin** if
$$w_i(TM) = f^*(w_i(TN)) \quad (i = 1, 2).$$
- The **higher mapping degree** of f is defined for a regular value p of the map f via
$$\deg_{\mathrm{hi}}(f) := \operatorname{ind}\big(\mathcal{D}_{\mathcal{S}f^{-1}(p) \otimes \mathcal{L}(M)|_{f^{-1}(p)}}\big) \in \mathrm{KO}_k(\mathrm{C}^*\pi).$$

Proof. The spin map f gives rise to a graded Real $\mathbb{C}\mathrm{l}_{n+k,n} \otimes \mathrm{C}^*\pi$-linear Dirac bundle $\mathcal{S}M \otimes f^*\mathcal{S}N \otimes \mathcal{L}(M)$ with induced Dirac operator $\mathcal{D}_{\mathcal{L}}$ satisfying

- $\operatorname{ind}(\mathcal{D}_{\mathcal{L}}) = \deg_{\mathrm{hi}}(f) \cdot \chi(N) \neq 0$ and
- $\mathcal{D}_{\mathcal{L}}^2 \geq \nabla^*\nabla + \frac{1}{4}(\mathrm{scal}_M - \mathrm{scal}_N \circ f)$.

The rigidity statement follows from the Technical Lemma. □

Examples. The Main Theorem applies to the following maps:

- $\mathrm{pr}_1\colon S^{2n} \times T^k \to S^{2n}$.
- $\mathrm{pr}_1\colon \mathbb{RP}^{2n} \times \Sigma^k \to \mathbb{RP}^{2n}$.

References

[1] S. Goette and U. Semmelmann. "Scalar curvature estimates for compact symmetric spaces". In: *Diff. Geom. Appl.* 16.1 (2002).

[2] M. Llarull. "Sharp estimates and the Dirac operator". English. In: *Mathematische Annalen* 310.1 (1998).

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