Scalar curvature rigidity and higher index theory

Workshop: Geometric moduli spaces - rigidity, genericity, stability
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Classical existence question

Does a given manifold admit a Riemannian metric of positive scalar curvature (pscm)?

Let M be a closed connected spin manifold of dimension m.

$\widehat{A}$-genus: $\hat{A}(M) := \operatorname{ind} \mathcal{D} \in \mathbb{Z}$.
Rosenberg index: $\alpha(M) := \operatorname{ind} \mathcal{D}_{\mathcal{L}} \in \mathrm{KO}_m(\mathrm{C}^*\pi)$ (higher index).

- $\mathcal{D}$: Dirac operator induced by the spinor bundle of M.
- $\mathcal{D}_{\mathcal{L}}$: $\mathbb{C}\mathrm{l}_n$-linear spin Dirac operator twisted by the Mishchenko-Fomenko bundle $\mathcal{L}(M) := \widetilde{M} \times_\pi \mathrm{C}^*\pi$.
- $\mathrm{C}^*\pi$: group C^*-algebra of the fundamental group π of M.

Fact. The Rosenberg index is the most general known index-theoretical obstruction to the existence of a pscm on M.

$$\begin{array}{ccccc} \widehat{A}(M) \neq 0 & \Longrightarrow & \alpha(M) \neq 0 & \Longrightarrow & \text{no pscm on } M \\ & \nearrow & \Uparrow & & \\ \dots & & M \text{ enlargeable} & & \end{array}$$

Examples.
- The $\widehat{A}$-genus of the n-torus is zero but the Rosenberg index does not vanish.
- There exist exotic spheres Σ^k of non-zero Rosenberg index.

General principle

classical index $\overset{\text{replace by}}{\rightsquigarrow}$ higher index

What geometric usefull information can we deduce from a non-vanishing higher index?

Classically: A non-vanishing $\widehat{A}$-genus gives rise to a non-vanishing $\mathcal{D}$-harmonic spinor u. In extreme geometric situations u is parallel, hence

$$|u|_p^2 = \langle u(p), u(p) \rangle_p = \text{const.}$$

New method

Technical Lemma (T. [3]). *Let $\mathcal{A}$ be a unital Real C^*-algebra and $\mathcal{S} \to M$ a graded Real $\mathcal{A}$-linear Dirac bundle with induced $\mathcal{A}$-linear Dirac operator $\not{D}$.*

- *If the higher index of $\not{D}$ does not vanish, there exists a family u_ϵ of almost $\not{D}$-harmonic sections.*
- *If, moreover, u_ϵ is L^2-almost parallel, the family is **almost constant**, i.e. there exist constants $C, r > 0$ and an element $a \in \mathcal{A}^+$ such that*

$$\left\| a - \left\langle u_\epsilon(p), u_\epsilon(p) \right\rangle_p \right\|_{\mathcal{A}} < C\epsilon^r \quad \forall p \in M \, \forall \epsilon \in (0,1).$$

Rigidity question

How rich is the space of Riemannian metrics satisfying a certain lower scalar curvature bound?

Theorem (Llarull [2]). *Let $f\colon M \to S^m$ be a smooth map of non-zero degree and $m \geq 3$. Then*

$$\left.\begin{array}{l} \mathrm{scal}_M \geq \mathrm{scal}_N \circ f \\ g_M \geq f^* g_N \text{ on } \bigwedge^2 TM \end{array}\right\} \Longrightarrow f \text{ is an isometry.}$$

We generalize an extremality and rigidity statement by Goette and Semmelmann [1] (a generalization of Llarull's theorem).

Main Theorem (T. [3]). *Let $f\colon M \to N$ be a spin map between two closed connected Riemannian manifolds of dimension $n+k$ and n, respectively. Suppose*

- *$\mathcal{R}_N \geq 0$, $\mathrm{scal}_N > 2\,\mathrm{Ric}_N > 0$ and*
- *$\chi(N) \cdot \boldsymbol{\deg_{\mathrm{hi}}(f)} \neq 0 \in \boldsymbol{\mathrm{KO}_k(\mathrm{C}^*\pi)}$.*

Then the following implication holds:

$$\left.\begin{array}{l} \mathrm{scal}_M \geq \mathrm{scal}_N \circ f \\ g_M \geq f^* g_N \text{ on } \bigwedge^2 TM \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \mathrm{scal}_M = \mathrm{scal}_N \circ f \text{ and} \\ f \text{ is a Riem. submersion} \end{array}\right.$$

Definition. Let f be as in the Main Theorem.

- f is called *spin* if $w_i(TM) = f^*(w_i(TN))$ $(i = 1, 2)$ holds.
- The **higher mapping degree** of f is defined for a regular value p of the map f via

$$\deg_{\mathrm{hi}}(f) := \operatorname{ind}\left(\mathcal{D}_{\mathcal{S}f^{-1}(p) \otimes \mathcal{L}(M)\restriction_{f^{-1}(p)}}\right) \in \mathrm{KO}_k(\mathrm{C}^*\pi).$$

Proof. The map f gives rise to a $\mathbb{C}\mathrm{l}_{n+k,n} \otimes \mathrm{C}^*\pi$-linear Dirac bundle $\mathcal{S}M \otimes f^*\mathcal{S}N \otimes \mathcal{L}(M)$ with induced Dirac operator $\mathcal{D}_{\mathcal{L}}$ satisfying

- $\operatorname{ind}(\mathcal{D}_{\mathcal{L}}) = \deg_{\mathrm{hi}}(f) \cdot \chi(N) \neq 0$ and
- $\mathcal{D}_{\mathcal{L}}^2 \geq \nabla^*\nabla + \frac{1}{4}(\mathrm{scal}_M - \mathrm{scal}_N \circ f)$.

The rigidity statement follows from the Technical Lemma. $\square$

Examples. The Main Theorem applies to the following maps:

- $\mathrm{pr}_1 \colon S^{2n} \times T^k \to S^{2n}$.
- $\mathrm{pr}_1 \colon \mathbb{RP}^{2n} \times \Sigma^k \to \mathbb{RP}^{2n}$.

References

[1] S. Goette and U. Semmelmann. "Scalar curvature estimates for compact symmetric spaces". In: *Diff. Geom. Appl.* 16.1 (2002).

[2] M. Llarull. "Sharp estimates and the Dirac operator". English. In: *Mathematische Annalen* 310.1 (1998).

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