(Almost) parallel spinors and the Rosenberg index

Workshop on Curvature and Global Shape
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Question

Does every closed connected Ricci-flat spin manifold admit a parallel spinor on its universal covering?

Let M be a closed connected spin manifold of dimension n with fundamental group π.

$\widehat{A}$-*genus*: $\hat{A}(M) := \begin{cases} \operatorname{ind} \mathcal{D} \in \mathbb{Z} & \text{if } n \text{ even,} \\ 0 & \text{else.} \end{cases}$

- Spinor bundle of $M \rightsquigarrow$ spin Dirac operator $\mathcal{D}$.

Fact 1. Every closed connected Ricci-flat spin manifold with non-vanishing $\hat{A}$-genus admits a parallel spinor.

Rosenberg index: $\alpha(M) := \operatorname{ind} \mathcal{D}_{\mathcal{L}} \in \mathrm{KO}_n(\mathrm{C}^*\pi)$ (higher index).

- $\mathcal{S}M$: $\mathbb{C}\mathrm{l}_n$-linear spinor bundle
- $\mathcal{L}(M) := \widetilde{M} \times_\pi \mathrm{C}^*\pi$ Mishchenko-Fomenko bundle.
- $\mathrm{C}^*\pi$: maximal group C^*-algebra of π.
- $\mathcal{S}M \otimes \mathcal{L}(M)$ is a $\mathbb{C}\mathrm{l}_n \otimes \mathrm{C}^*\pi$-linear Dirac bundle $\rightsquigarrow$ twisted spin Dirac operator $\mathcal{D}_{\mathcal{L}}$.
- $\mathrm{KO}_n(\mathrm{C}^*\pi)$: n-th Real K-theory group of the C^*-algebra $\mathrm{C}^*\pi$.

Fact 2. The Rosenberg index is the most general known index-theoretical obstruction to the existence of a pscm on M.

$$\begin{tikzcd} \widehat{A}(M) \neq 0 \arrow[r, Rightarrow] & \alpha(M) \neq 0 \arrow[r, Rightarrow] & \text{no pscm on } M \\ \dots \arrow[ur, Rightarrow] & M \text{ enlargeable} \arrow[u, Rightarrow] & \end{tikzcd}$$

Examples.
- The $\widehat{A}$-genus of the n-torus is zero but the Rosenberg index does not vanish.
- There exist exotic spheres Σ with non-zero Rosenberg index.

General principle

classical index $\overset{\text{replace by}}{\rightsquigarrow}$ higher index

While a non-trivial $\hat{A}$-genus yields a non-trivial kernel of the spin Dirac operator, this is in general no longer true for a non-vanishing higher index.

- Does Fact 1 still hold for the Rosenberg index?

Existence of parallel spinors

Theorem 1 ([1, Theorem A])**.** *Every closed connected Ricci-flat spin manifold with non-vanishing Rosenberg index admits a parallel spinor on a finite Riemannian covering.*

Corollary 2 ([1, Corollary B])**.** *Every closed connected Ricci-flat spin manifold of dimension $n \geq 2$ with non-vanishing Rosenberg index has special holonomy, i.e. its reduced holonomy group is a proper subgroup of $\mathrm{SO}(n)$.*

Existence of almost parallel spinors

Theorem 3 ([4, Lemma D])**.** *Let $\mathcal{A}$ be a graded Real unital C^*-algebra and $\mathcal{S} \to M$ a graded Real $\mathcal{A}$-linear Dirac bundle with induced $\mathcal{A}$-linear Dirac operator $\not{D}$.*

1. *If the higher index of $\not{D}$ does not vanish, there exists a family $\{u_\epsilon\}_{\epsilon>0}$ of **almost $\not{D}$-harmonic** sections of $\mathcal{S}$, i.e.*
$$\|u_\epsilon\|_{\mathrm{L}^2} = 1 \quad \text{and} \quad \|\not{D}^i u_\epsilon\|_{\mathrm{L}^2} < \epsilon^i \quad \forall i \geq 1, \forall \epsilon > 0.$$
2. *If, moreover, $\|\nabla u_\epsilon\|_{\mathrm{L}^2} < \epsilon$ for all $\epsilon > 0$, then u_ϵ is **almost parallel** i.e. there exist constants $C, r > 0$ such that*
$$\|\nabla u_\epsilon\|_\infty < C\epsilon^r \quad \forall \epsilon \in (0,1).$$

In certain extreme geometric situations, we obtain $\|\nabla u_\epsilon\|_{\mathrm{L}^2} < \epsilon$ from a Schrödinger-Lichnerowicz type formula like

$$\not{D}^2 \geq \nabla^*\nabla + \mathcal{R}, \qquad \mathcal{R} = \text{some rest term.}$$

Application 1: Scalar curvature extremality and rigidity

The following theorem generalizes [2, 3].

Theorem 4 ([4, Theorem A])**.** *Let $f\colon M \to N$ be a **spin** map between closed connected Riemannian manifolds. Suppose*

- *$\mathcal{R}_N \geq 0$, $\mathrm{Ric}_N > 0$ and*
- *$\chi(N) \cdot \mathbf{deg_{hi}}(f) \neq 0 \in \mathbf{KO(C^*\pi)}$.*

Then the following implication holds:

$$\left.\begin{array}{l} \mathrm{scal}_M \geq \mathrm{scal}_N \circ f \\ g_M \geq f^* g_N \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \mathrm{scal}_M = \mathrm{scal}_N \circ f \text{ and} \\ f \text{ is a Riem. submersion} \end{array}\right.$$

- f is called *spin* if $w_i(TM) = f^*(w_i(TN))$ for $i = 1, 2$.
- For a regular value p of f, we define
$$\deg_{\mathsf{hi}}(f) := \operatorname{ind}\big(\mathcal{D}_{\mathcal{S}f^{-1}(p) \otimes \mathcal{L}(M)\restriction_{f^{-1}(p)}}\big) \in \mathrm{KO}(\mathrm{C}^*\pi).$$

Application 2: A spinorial proof of implication $(\star)$

Let g be a Riemannian metric on a closed connected spin manifold M with $\mathrm{scal}_g \geq 0$. Then

$$\begin{tikzcd} \operatorname{ind} \mathcal{D} \neq 0 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{Ric}_g \equiv 0 \arrow[r, Rightarrow] & \mathrm{scal}_g \equiv 0 \\ \alpha(M) \neq 0 \arrow[ur, Rightarrow, "(\star)"] \arrow[rr, Rightarrow, "\text{direct proof by Schick}"] & & {} \arrow[u, Rightarrow] \end{tikzcd}$$

References

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