PHIL 379 - Logic II

Scott Saunders - 10163541

Winter 2016

Chapter 1

Discrete Math-esk stuff.

Misc. Notation

- The set of positive integers $\{x : x$ is a positive integer $\}$
- The set of positive integers less than3 $\{x : x$ is a positive integer and x is less than 3 $\}. = \{1, 2\}$.
- The empty set: $\emptyset$ or Δ
- Member of: $A \subseteq B$ iff $\forall X(x \in A \implies x \in B)$
- Union of A and B: $A \cup B$ iff $\{x : x \in A \vee x \in B\}$
- Intersection of A and B: $A \cap B$ iff $\{x : x \in A \wedge x \in B\}$
- Difference of A and B: $\{x : x \in A \wedge x \notin B\}$
- For any non-empty sets A, B: Cartesian product: A of B: $A \times B$: $\{< x, y >: x \in A \wedge y \in B\}$ (ALL OF THE POSSIBILITIES)
- TOTAL FUNCTION: Every element in the domain is valid
- PARTIAL FUNCTION: Not every element in the domain is valid.
- for any set of sets A:
  - $\Cup A = \{x : \exists y(y \in A \wedge x \in y)\}$
  - $\Cap A = \{x : \forall y(y \in A \to x \in y)\}$
- Relations: R is
  - reflexive : $\forall x Rxx$
  - symmetric : $\forall x \forall y(Rxy \implies Ryx)$
  - transitive : $\forall x \forall y \forall z((Rxy \wedge Ryz) \implies Rxz)$
  - Euclidean : $\forall x \forall y \forall z((Rxy \wedge Rxz) \implies Ryz)$
  - a equivalence relation : it's symmetric,reflexive,transitive.
  - a equivalence relation (alt) : it's symmetric, and euclidean.
  - a (partial) function : $\exists x$ and there is at most one y: Rxy : denoted f
  - a (partial) function[1] : $\exists x, \exists y | Rxy$: denoted f.
  - a (total) function: assigns a value to each number of A : denoted f
  - a (total) function[2]: $\forall x, \exists y | Rxy$: denoted f.
- Domain: The set of a functions arguments.

- Range: The set of its values. (Results)
- f is a function from a set A iff the domain of f is included in A
- f is a function to a set B iff its range is included in B.
- f^{-1} is the inverse of the function f from the set A to the set B iff:if for every member $b \in B$, there is exactly one member of $a \in A$ such that $f(a) = b$, then $f^{-1}(b) = a$, otherwise $f^{-1}(b)$ is undefined.
- f is onto B iff B is the range of f (Surjective)
  Alt: (Wikipedia) : $\forall y \in Y, \exists x \in X | y = f(x)$
- f is one-to-one iff $\forall x \forall y (f(x) = f(y) \implies x = y)$ (Injective)
- f is a bijection iff f is onto and one-to-one.
- f is a correspondence iff f is total, one-to-one and onto.
- Sets A and B are equinumerous iff there is a correspondence from A to B.

Equinumerous is transitive. Prove: if A is equinumerous with B and B is equinumerous wit C, then A is equinumerous with C. Proof: Suppose A is equinumerous to B, and B is equinumerous to C. Then: There is a total,one-to-one function f from A onto B, and a total one-to-one function g from B to C. Prove equinumerous via h=g(f), such that h(n)=g(f(n))

- h is total: Let a be a member of A. $h(a) = g(f(a))$. Since f is total there is a member of b of B such that $f(a) = b$). since g is total, there is a member of $c \in C$ such that $g(b) = c$. Hence, h is total.
- h is onto C. WLOG Let c be a member of C, as g is onto, $\exists b \in B$ such that $g(b) = c$. As f is onto, then $\exists a \in A$ such that $f(a) = b$. Hence, the composition of $h = f(g)$ is onto C.
- h is one-to-one: Suppose h is not one-to-one.
  Then there $\exists a_1, a_2 \in A$ such that $h(a_1) = h(a_2), a_1 \neq a_2$.
  Giving $g(f(a_1)) = g(f(a_2)), a_1/not = a_2$
  Since g is one-to-one $g(b_1) = g(b_2)$ iff $b_1 = b_2$.
  So the issue must lie in f. However f is one-to-one $f(a_1) = f(a_2)$ iff $f(a) = f(b)$. Which is a contradiction, giving us that h is one-to-one.

□

A^n : the nth Cartesian product of A with itself.
Suppose that the set of real Numbers $r, r < r < 1$, is enumerable. Then $L_r : r_1, r_2, r_3....$ written in a notation of $0.n_1n_2n_3.(nbeingnaturalnumbers)$
The set of functions form the set of positive integers to positive integers is not enumerable.

Proof. Suppose S is enumerable.
Then there is a list L_s of the members of S.
$L_s = \{s_1, s_2, s_3, \cdots\}$
Let $\forall n \in \mathbb{N}, n \in k \iff n \notin S_n$
k is a set of positive integers.
so There is a number $j$such that $k = s_j$. So $j \in S_j \iff j \not{i}nS_j$
Hence S is not enumerable.

□

The set of total nomadic functions from the set of positive integers, F^1, is not enumerable.
It's a Proof by contradiction.

Chapter 2

Turing Machines

Turing machines are in the following form: $q_n, S_{1/0}, S_{1/0}/R/L, q_m$ where q_n is our current state, and you see $S_{1/0}$, perform function $S_{1/0}/R/$ and move to state q_m. If there is no operation specified on the current state for a scan, then it halts. (Also Called the Turing Alphabet)

Example with notation:

$$\begin{tikzcd} \text{start} \arrow[r] & n & & m \end{tikzcd}$$

ex: (These are the same)

$$Q_1S_1RQ_1, Q_1S_0S_1Q_2, Q_2S_1LQ_2, Q_2S_0RQ_3, Q_3S_1S_0Q_3, Q_3S_0RQ_4$$

$$\begin{tikzcd} \text{start} \arrow[r] & 1 \arrow[loop above, "1:R"] \arrow[r, "0:1"] & 2 \arrow[loop above, "1:L"] \arrow[r, "0:R"] & 3 \arrow[loop above, "1:0"] \arrow[r, "0:R"] & 4 \end{tikzcd}$$

Remark (Turing Machines)**.**

- Each Turing machine is a finite set of Turing instructions.
- Each instruction is a 4 letter word of the Turing Alphabet.
- The set of Turing machine is enumerable. (Proof: exercise)

Definition (Standard inital configuration)**.** A Turing machine is in a standard Initial configuration $\iff$

- for some positive integer k, there are k blocks of 1's on the tape.
- separated by a blank,
- and the rest of the tape is blank.
- the machine is scanning the left-most 1 on the tape.
- the machine is in it's lowest numbered state.

ex: $\cdots 0010110111000 \cdots$ is a SIC. (if it's in lowest state) ex: $\cdots 00010000 \cdots$ is a SIC.

Definition (Standard final configuration)**.** A Turing machine is in a standard final configuration $\iff$

- there is a single block of k 1's
- and the rest of the tape is blank.
- the machine is scanning the left-most 1 on the tape.

ex: $\cdots 00111111000 \cdots$ is a SIC. (if it's in lowest state) ex: $\cdots 00010000 \cdots$ is a SIC.

Definition (Computes a one-place function f^1)**.** A Turing M computes a one-place function f^1:
if M is started in a SIC with a single block of k 1's and

- if f^1 is defined for the argument k, then M eventually halts in a SFC
- or if f^1 is not defined for the argument k, then either M never halts or it halts in a non-standard final configuration.

Remark. Every Turing machine computes exactly one function of two arguments.

Remark. For any n, each Turing Machine computes exactly one function of n arguments.
The set of one-place Turing computable functions is enumerable
$\vdots$
The set of Turing computable functions is enumerable.

Definition (The halting problem)**.** The problem of finding an effective method to determine whether a Turing machine will eventually halt or not after it is started with some input.

The halting problem is unsolvable. Ex: $L_M : M_1, \dots$
$h(m, n) =$
1 if M_m eventually halts after starting with input n
2 if M_m never halts after starting with input n
The halting problem is solvable iff h is computable. Show: h is not Turing computable.
Let C be a copying machine.
Let F be $\frac{1}{2}$ flipper.
Suppose h is Turing Computable.
Let H be a Turing machine that computes h.
If h is a Turing computable, then H exists.
If H exists, then $D(C - H - F)$ exists.
Let $D = M_k$, for some k. $M_k \in L_M$.

Start D with input k. The C-part of D will produce a copy of k, Then the $H-$part will do its job:

- If M_k will eventually halt after starting with input k, then H will produce output 1.
- If M_k will never halt after starting with input k, then H will produce output 2.

Then the $F-$part will do it's job.

- If output from H is 1, F will never halt.
- If output from H is 2, F will eventually halt.

Giving us:

- If M_k will eventually halt after starting with input k, then D will never halt after starting with input k.
- If M_k will never halt after starting with input k, then D will eventually halt after starting with input k.

So M_k will halt, after starting with input k, $\iff$ D will never halt after starting with input k.
Then M_k isn't identical with D, which is a contradiction! Hence$D$ doesn't exist. So H does not exists. So h is not Turing computable. $\square$

Another halting problem..? $L_M : M_1, \cdots$ $L_F : F_1, \cdots$
$g(n) = 1$, if $f_n(n) = 2$
$g(n) = 2$, otherwise.
$g \neq f_k \forall k$
$h(m, n) = 1$if M_m eventually halts after starting with input n.
$h(m, n) = 2$ if M_m never halts after starting with input n.
$s(m) = 1$, if M_m eventually halts after starting with input m.
$s(m) = 2$, if M_m never halts after starting with input m.

1. The halting problem is solvable iff h is computable.
2. If h is computable, then s is computable.
3. If s is computable, and TT's is true, then g is computable.
4. g is not Turing computable.
5. Turing Thesis is true (Whatever is not Turing Computable is not computable)
6. The halting problem is not solvable.

3. Suppose S is computable and TT is true.
Then: There is a Turing machine $S*$ that computes s.
Suppose that we are to calculate $g(n)$, for some n.
Start $S*$ with input n.

- Case 1: S* eventually halts with output 1
  We know that M_n will eventually halt after it is started with input n Start M_n with input n, when it halts, inspect the tape.
  - Case 1.1: Halted in SFC $f_n(n) = 2$ $g(n) = 1$
  - Case 1.2: Halted in non-SFC: f_n is undefined. $f_n(n) \neq 2$

  And then Blake broke it:
  As it's a halting problem to figure out if it's in SFC?
  $g(n) = 2$
- Case 2: S* eventually halts with output 2 We know that M_n will never halt after it is started with input n.
  So we know that f_n is undefined for the argument n.
  So we know that $g(n) = 2$

□

□

Chapter 3

First Order logic

Some symbols n things:

- (
-)
- Successor : '
- Not: -
- And: $\wedge$ (Conjunction)
- Or: $\vee$ (Disjunction)
- Exists: $\exists$
- Forall: $\forall$
- Variables: $v_1, v_2, v_3, \dots$
- Equality: $=$
- Predicates: $\begin{matrix} A_1^1 & A_2^1 & \dots \\ \vdots & \vdots & \ddots \\ A_1^n & A_2^n & \dots \end{matrix}$
- Constant names: $a_1, \dots$
- Functions: $\begin{matrix} f_1^1 & f_2^1 & \dots \\ \vdots & \vdots & \ddots \\ f_1^n & f_2^n & \dots \end{matrix}$

Definition. Term

- Every variable is a(n atomic) (open) term.
- Every constant is a(n atomic) (closed) term.
- If $t_1, \dots, t_n$ are terms, then $f^n(t_1, \dots, t_n)$ is a term.
- Nothing else is a term.

Definition. Formula

- $A^n(t_1, \dots, t_n)$ is a formula where A^n is an $n-$place predicate and t_i are terms. (This is an atomic formula).

- If F is a formula then $-F$ is a formula
- If F and G are formulas then $(F \wedge G)$ is a formula.
- If F and G are formulas then $F(\vee G)$ is a formula.
- If F is a formula, then $\exists vF$ is a formula.
- If F is a formula then $\forall vF$ is a formula.
- NOTHING ELSE IS A FORMULA.

Definition (Bound)**.** An occurrence of variable x is bound if it is part of a subformula beginning $\forall x$ or $\exists x$, in which case the quantifier $\forall$ or $\exists$ in question is said to bind that occurrence of the variable x, and otherwise the occurrence of the variable x is free.
Ex: $Fx \to \forall xFx$: The first x is free, and the second is bound.
Ex: $x < y \wedge -\exists z(x < z \wedge z < y)$: all occurrences of x and y are free, and all the occurrences of z are bound.

Remark. When we write something like "Let $F(x)$ be a formula", we are to be understood as meaning "Let F be a formula in which no variables occur free except x".

Definition (Instance)**.** An *instance* of a formula $F(x)$ is any formula of the form $F(t)$ for t a closed term. Similar notations apply where there is more than one free variable, and to terms as well as formulas.

Definition (Sentence)**.** a formula is a sentence if no occurrence of any variable in it is free.

Definition (Model)**.** A model M (interpretation) of a language L is $\{|M|v\}$ Where $|M|$ is a non-empty set and v is a valuation function that assigns values (extensions/denotations) to the members of L in such a way that

- $\forall v(a) \in |M|$
- $v(A^n) \subseteq$ the nth Cartesian product of $|M|$ with itself: $|M| \times \ldots |M|$.
- $v(f^n)$ is a total function from $|M| \times \ldots |M|$ to $|M|$.

Definition. Truth:

- For some predicate R: $M \vDash R(t_1, \ldots, t_n)$ iff $R^M(t_1^M, \ldots, t_n^M)$
- For Identity function (when present): $M \vDash= (t_1, t_2)$ iff $t_1^M = t_2^M$.
- $M \vDash F^n(t_1, \ldots, t_n)$ iff $< M(t_1), \ldots, M(t_n) > \in M(F^n)$.
  Textbook writes this as: $(f(t_1, \ldots t_n))^M = f^M(t_1^M, \ldots, t_n^M)$.
  Ie: Interpretation M, makes true $f^n(t_1, \ldots, t_n)$ (An n-place function), iff $\{t_1^M, \ldots, t_n^M\} \in M(F^n)$
- $M \vDash -S$ iff $M \nvDash S$
- $M \vDash (K \wedge L)$ iff $M \vDash K$ and $M \vDash L$
- $M \vDash (K \vee L)$ iff $M \vDash K$ or $M \vDash L$
- $M \vDash F[m]$ iff $M_m^c \vDash F(c)$
  'If we considered the extended language $L \cup \{c\}$ obtained by adding a new constant c in to our given language L, and if among all the extensions of our given interpretation of this extended language we considered the one M_m^c that assigns c the denotation m, then $F(c)$ would be true.'
- $M \vDash \forall xF(x)$ iff for every m in the domain , $M \vDash F(m)$
- $M \vDash \exists xF(x)$ iff for some m in the domain , $M \vDash m)$

Definition. of the denotation/extension of a closed term in a model M. If T is a name $M(t) = v(t)$ If t is $f^n(t_1, \ldots, t_n)$ then
$M(f^n(t_1, \ldots, t_n)$ $M(f^n)(M(t_1), \ldots, M(t_n))$.

Validity = Satisfiability = Implication.

Misc-crap:

- $A \vDash B$ is $-(A \wedge -B)$

Lemma 1. *Extensionality Lemma*

- *Let M be a model of a language L.*
- *Let S be a sentence of L.*
- *Let L^+ be an extension of L. $L \subseteq L^+$*
- *Let M^+ be a model of L^+*
- *So: M^+ is an extension of M.*
- *$M \vDash S$ iff $M^+ \vDash S$*

Example:
If $A \vDash B$ and $B \vDash C$, then $A \vdash C$. Suppose $A \vDash B$, and $B \vDash C$.
In every interpretation of A and B in which A is true, B is true, In every interpretation of B and C in which B is true, C is true. Shows: In every interpretation of A and C in which A is true, C is true.
Let M be an interpretation of A and C such that $M \vDash A$.

- Case 1: M is an interpretation of B.
  Then $M \vDash B$
  So $M \vDash C$.

- M is not an interpretation of B.
  Then there is an extension M^+ that interprets B as well as A and C.
  so: $M^+ \vDash B$
  So: $M^+ \vDash C$
  So $M \vdash C$ (By the ext, lemma)

Lemma 2 (Undecibality)**.** *If the decision problem (for implication) is solvable, then the halting problem is solvable. There is an effective method for specifying for any Turing machine M and any input N a finite set of sentences Δ and a sentence H such that $\Delta \vDash H$ iff M eventually halts after starting with input n.*
$\Delta \vDash H$ iff M eventually alts after start with input n.

Writing a turning machine as logic. L_t

Define the one place predicate Q_ij as: At time j, M is in state i.
Define the two place predicate $@js$ as At time $j, M,$ is scanning square s.
Define the two place predicate Mjs as: At time j, square s is marked with a 1.

A description D for a start state could then be:
$D : [Q_1 0 \wedge @_{0,0} \wedge M_{0,0} \wedge M_{0,1} \wedge M_{0,2} \wedge \forall y((y \neq 0 \wedge y \neq 1 \wedge y \neq 2) \implies -M_{0,y}]$
Time = 0, $[\ldots 01110 \ldots]$

Square #, $[\cdots - 10123\ldots]$

For each instruction of a TM, we may write the instruction as a sentence:
$Q_{i1}S_1RQ_{i2}$: Move right seeing 1 in state:
$\forall x\forall t((Q_{i,1} \wedge @_{t,x} \wedge M_{t,x}) \implies (Q_{i2,(t+1)} \wedge @_{(t+1),(x+1)} \wedge \forall y((M_{t,y} \implies M_{(t+1),y}) \wedge (-M_{t,y} \implies -M_{(t+1),y}))$
Misc-crap that's on the board for some reason: (The Q,@, and M are also defined a little bit above).

$$\begin{array}{rrl}
\Delta & : & \text{A set of sentences.} \\
\mathbb{Q}_2^1 : & \forall x\forall y\forall z((Sxy \wedge Sxz) \implies y = z) & \\
@^2 : & \forall x\forall y\forall z((Sxz \wedge Syz) \implies x = y) & \\
M^2 : & \forall x\forall y(Sxy \implies x < y) & \\
0 : & \forall x\forall y\forall z((x < y \wedge y < z) \implies x < z) & \\
S^2 : & -\exists x, x < x & \text{Successor } (+1) \\
<^2 : & & \text{less than}
\end{array}$$

Some thing else now too:
Node 1 : 1:R -¿ node 2
$\forall x\forall t((Q_1 t \wedge @tx \wedge Mtx) \implies \exists u(s_1(t,u) \wedge Q_2(u) \wedge @(u,v)) \wedge \forall y((M(t,y) \implies M(u,y)) \wedge (-M(t,y) \implies -M(u,y)))))$
$\exists x\exists t(Q_m t \wedge @tx \wedge Mtx)$

Proof. Something about biconditonal:
$\Delta \vDash H$ iff M halts after starting with input n.

1. if $\Delta \vDash H$, then M halts.
2. if M halts, then $\Delta \vDash H$.

1. if $\Delta \vDash H$, then M halts - Proof.
   Suppose $\Delta \vDash H$
   All members of Δ are true in the standard interpretation I. H is true in I. So: M halts.
2. if M halts, then $\Delta \vDash H$ - Proof.
   Suppose M halts. (Show $\Delta \vDash H$).
   There is a time $\underline{t}$ M halts at t.
   There is a state q_i, M halts at t in state q_i.
   There is a square x, M halts at t in state q_i, scanning square x which is Marked / $-$ Marked
   1: $Q_i(t) \wedge @(t,x) \wedge M(t,x)$.
   1 is a conjunct of the description of time t, $\mathbb{D}(t)$
   $\mathbb{D}(t) \vDash (i)$

   2: $\exists x\exists t(Q_1(t) \wedge @(t,x) \wedge (t,x))$.

   $(i) \vDash (ii)$.
   (ii) is disjunct of H.

$(ii) \vDash H$.
So: $\mathbb{D}(t) \vDash H$.

Δ implies a description of every time before which M did not halt.
$\forall n($ if M has not halted before time n, then $\Delta \vDash \mathbb{D}(n))$

$\square$

Relisting it:

The decision problem for implication, is unsovable if turings thesis is true. $\Delta \vDash H$ iff M eventually halts.
Only if: $\Delta \vDash H$ only if M eventually halts.
if: if M eventually halts then $\Delta \vDash H$.
Suppose: M halts at t
$\mathbb{D}(t) \vDash (H)$
Show: $\Delta \vDash \mathbb{D}(t)$

Something inductive:
Base step: if M has not halted before time 0, then $\Delta \vDash \mathbb{D}(0)$:
Inductive step: $\forall n($ if M has not halted before time n, then $\Delta \vDash \mathbb{D}(n)$ iff M has not halted before time $n+!$, then $\Delta \vDash \mathbb{D}(n+1))$
Conclusion: $\forall n$ (if M has not halted before time n, then $\Delta \vDash \mathbb{D}(n)$.

Proving the Inductive step:

- Suppose if M has not halted before time n, then $\Delta \vDash \mathbb{D}(n)$.
- (Show : if M has not halted before time $n+!$, then $\Delta \vDash \mathbb{D}(n+1)$)
- Suppose: M has not halted before time $n+1$
- So: M has not halted before time n
- So: $\Delta \vDash \mathbb{D}(n)$.
- (Show: $\Delta \cup \{\mathbb{D}(n)\} \vDash \mathbb{D}(n+1)$)
- $\mathbb{D}(n) : Q_j n \wedge @_{n,k} \wedge M_{n,k} \ldots$
- $\forall t \forall x((Q_j(t) \wedge @tx, \wedge Mtx) \implies Q_i t + 1 \wedge @(t+1)(x+1) \wedge \ldots)$
- $(Q_j n \wedge @nk \wedge Mnk) \implies Q_i(n+1) \wedge @(n+1)(k+1) \wedge \ldots)$
- $\mathbb{D}(n+1) : Q_i(n+1) \wedge @(n+1)(j+1) \wedge \ldots$

$\square$

Then some random ramblings about logicians. happened. Here's a book: "What is the name of this book?".
There are methods of validity, and truth trees, and some other stuff. He's gonna use this. He's gonna prove this has some properties for validity.

3.1 Truth Trees

Ex: $\frac{(A \implies B)A}{B}$ is

$$\begin{array}{lccl}
 & \multicolumn{2}{c}{A \implies B} & \\
1. & \multicolumn{2}{c}{A} & \text{Assumed} \\
2. & \multicolumn{2}{c}{-B} & \\
 & \swarrow & \searrow & \\
3. & -A & B & \\
 & \otimes & \otimes &
\end{array}$$

Giving $\{(A \implies B), A, -B\}$

Proof tree rules.

1. α

   1. α
   2. $-\alpha$

      $\otimes$

2. 1. $(- - \alpha)$ ✓ Assumed
   2. α

3. $\alpha \wedge \beta$

   1. $(\alpha \wedge \beta)$ ✓ Assumed
   2. α
   3. β

4. $-(\alpha \wedge \beta$

$$\begin{array}{lcc}
1. & \multicolumn{2}{c}{-(\alpha \wedge \beta) \checkmark} \\
 & \swarrow & \searrow \\
2. & -\alpha & -\beta
\end{array}$$

5. 1. $-\exists x \_x\_$
   2. $\forall x - \_x\_$

6. 1. $-\forall x \_x\_$
   2. $\exists x - \_x\_$

7. 1. $\exists \_x\_$
   2. $\_n\_$

   Note: n is a name new to the path

8. 1. $\forall x \_x\_$
   2. $\_n\_n$

   Note: N is a new name only if there is not already a name on the path; otherwise n is a name already on the path.

Remark. For 1-6, if the premises under an interpretation are true, then the conclusion is also true under that interpretation. For 7,8, Not as powerful.. For all: If the permissive of the rule is satisfiable, then there is an interpretation that makes the conclusions satisfiable. (The property of satisfiability travels down the tree)

Examples: $\alpha \vee \beta$

1. $(\alpha \vee \beta)$ ✓

2. $\alpha \qquad \beta$

$-(\alpha \vee \beta)$

1. $-(\alpha \vee \beta)$ ✓

2. $-\alpha \qquad -\beta$

$-\forall x x \varnothing$

1. $-\forall x x \varnothing$ ✓
2. $\exists x - \varnothing$

$-\exists x \varnothing$

1. $-\exists x \varnothing$ ✓
2. $x - \varnothing$ ✓

$\exists x \varnothing[x]$ Note: c is a new name on the path.

1. $\exists x \varnothing[x]$ ✓
2. $\varnothing[c]$ ✓

$\forall x \varnothing[x]$ Note: c is a name already on the path, unless there are no names by the path and

1. $\forall x \varnothing[x]$
2. $\varnothing[c]$

in that case, c is new.

$\forall x(Fx \to Gx) \wedge \forall x(Gx \to Hx)$

1. $\forall x(Fx \to Gx)$
2. $\forall x(Gx \to Hx)$
3. $-Ha$
4. $-Fa$

Which is INVALID.

1.	$\forall x(Fx \to Gx)$	assumed
2.	$\forall x(Gx \to Hx)$	assumed
3.	$-Ha$	assumed
4.	$--Fa$	assumed
5.	Fa	
6.	$(Ga \to Ha)$	
7.	$(Fa \to Ga)$	

8. $Ha \qquad -Ga$

$\otimes$

9. $-Fa \qquad Ga$

$\otimes \qquad \otimes$

Another same-case example: $\frac{\forall x(Fx\to Gx)\exists xGx}{\exists xFx}$ is

$$\begin{array}{lcl}
1. & \forall x(Fx\to Gx) & \text{assumed} \\
2. & \exists xGx\ \checkmark & \text{assumed} \\
3. & \_\exists xFx\ \checkmark & \text{assumed} \\
4. & \forall x-Fx & \\
5. & Ga & \\
6. & -Fa & \\
7. & Fa\to Ga\ \checkmark & \\
 & \diagup\quad\diagdown & \\
8. & -Fa\quad\ Ga &
\end{array}$$

Finding a case of invalidity from a proof tree:

1. take a path
2. get all values on the path
3. The domain consists of these values
4. You assign values to the predicates bottom-up, such that the tree's path is true.

Ex from the above same-case example: We follow the path down to Ga. Then:

1. $v(a) = 1$
2. $|M| = \{1\}$
3. $v(F) = \varnothing$
   $v(G) = \{1\}$

The order of applying rules:

1. TF
2. NEGQ
3. EI
4. UI
5. GOTO 1 (Until nothing is done)

How to use a proof-tree:

1. List premises. (Setup)
2. Negate the conclusion. (Setup)
3. Apply rule (see lists above) (Apply to all open branches)
4. Check for contradictions. (Close those branches) Repeat 3 till nothing to apply to.
5. If all branches are closed, then the argument is valid. else, it's invalid. (IE, there is a counter example, as we negated the conclusion)

Remark. Sometimes, this never finishes. But that's due to the undecidiblity of things...

Ex:	1.	$\forall x \exists y Rxy$
	2.	$\exists y Ray$ ✓
	3.	Rab
	4.	$\exists y Rby$ ✓
	5.	Rbc
	6.	$\exists y Rcy$

Tree validity: A finished tree consist of a path o

Lemma 3 (Completeness)**.**

Remark (If the Tree Methoid classifies S as valid, then S is valid)**.** *Proof.* If there is a finished tree T, whose Initial list consist of $-S$, and all paths of T are closed, then S is valid ($-S$ is unsatisfiable).
Any tree whose Initial list is satisfiable, has an open path.
From a prior remark:
For all: If the premise of the rule is satisfiable, then there is an interpretation that makes the conclusions satisfiable. □

Remark (If the Tree Methoid classifies S as invalid, then S is invalid.)**.** *Proof.* If a tree T, whose Initial list consist of $-s$ has an open path, then $-S$ is satisfiable (S is not valid).
Any finished open path of a tree, determines an interpretation in which all lines of that path are true. □

Let M be the interpretation determined by p. Show: All lines on P are true in m.
Suppose there is some line on this path that is not true: We want to show that that is incorrect. There could be two such lines, or more.... If there are, they could be the same length ... if they don't, then the one that is shorter than the other, is hence called being "shorty". Being a "shorty" is true in the case that there is no line that is FALSE and there is no line shorter than it. (or them, if there are multiple shorties). Our proof: is to show that there is no such shorties, hence all lines are true.

induction on complexity	By our definitions: No atomic sentences are shorties.
iinductiion on complexiity	The denial of all atomic sentences comes from the fact that if Fa is on the path, then $-Fa$ cannot be on the path.
iiinductiiion on complexiiity	Conjunction: If there was a conjunction on the path, then it would be checked, meaning that it's conjuncts are on the path. However these are shorter than shorty. and implies this(?)
ivnductivon on complexivty	If shorty was a denial of a conjunction, then again shorter things are created, and implies this..
vnductvon on complexvty	Existential: It's instance will be on the path, which is shorter... again.. which implies the premise.
vinductvion on complexvity	Universal: The conclusion is on the path and is shorter.. but that's not good enough. idk.

Soundness Theorem:

1. If TM classifies a sentence S as valid, then S is valid.
   Any tree whose initial list is satisfiable has an open path.

2. If TM classifies a sentence S as invalid, then S is invalid.
   If a tree has a finished open path, then it's initial list is satisfiable.

   Lemma 4. *Completeness Lemma*
   Any finished open path determines an interpretation in which all lines of that path re true.

3. If TM does not classify a sentence S as unsatisfiable, then S is satisfiable.
   Suppose TM does not classify S as unsatisfiable:
   1: TM classifies S as satisfiable
   2: TM does not classify S as all
   So: There is an infinite complete tree on S.
   Every infinite tree has an infinite path.
   T has an infinite and also complete path P.
   P must be open.
   Proof by Completeness lemma.

4. If a sentence S is unsatisfiable, then TM classifies S as unsatisfiable.

5. If a Sentence S is implied by a finite set of sentence K, then TM declares that S is implied by K.

6. For any set of sentences S, if S is unsatisfiable, then TM classifies S as unsatisfiable.

Soundness Theorem: (5') For any set of sentences S, if TM classifies S as unsatisfiable, then S is unsatisfiable.
i: If a set is satisfiable, then any complete tree based on S has an open path. (by 5')
ii: If a complete tree has an open path p, then P has a numerical interpretation in which all lines of P are true. (From the completeness lemma)

Lowenheim-Skolem Theorem:
Every satisfiable set of sentences has an enumerable model.

From wikipedia: if a countable first-order theory has an infinite model, then for every infinite cardinal number k it has a model of size k.
The compactness Theorem:
If a set S is unsatisfiable then some finite subset of S is unsatisfiable. (If every finite subset of S is satisfiable then S is satisfiable)

From wikipedia: a set of first-order sentences has a model if and only if every finite subset of it has a model.

Chapter 4

Second order logic

TLDR: We allow predicates to be replaced by Second order variables. It's neat.

Model: The same.
Truth: Instead of ever element m in the domain, it is every set m of the domain.
Identity: $\forall x \forall y(x = y$ iff $\forall F(Fx$ iff $Fy)$

Given any first order sentence, which has a model that is in an interpretation that is true, and it has n objects in its domain. Then the corresponding existential generalization is going to be true in all models of size n.
IE: true in an interpretation/model in size n iff true in all models of size n.
Some claims:

1. For any 1st order sentence S, if S has a model of size n then $\exists S^{II}$ is true in all models of size n.
2. For any first order sentence S, free of any identity and function symbols, if S is satisfiable, then S has a countable but infinite (denumerable), model.

EX: $\forall x \exists y Rxy \wedge -\exists x Rxx \wedge \forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$
(Less-than on the integers!) Generalized:
$\forall X[\forall x \exists y Xxy \wedge -\exists x Xxx \wedge \forall x \forall y \forall z((Xxy \wedge Xyz) \rightarrow Xxz)]$ Which is true in only those interpretations whose domain is infinite. AND is true in all those interpretations whose domain is infinite.

Let S be a $1st$ order sentence free of identity and function symbols:
$S^* : (\exists A^{II} \rightarrow \exists S^{II})$
If S is satisfiable, then S^* is valid.
If S^* is valid, then S is satisfiable.

There is an effective method for writing for any $1st$ order sentence (free of identity and f symbols) a second order sentence S^*.
S is satisfiable iff S^* is valid.

Incomplete:

1. If there is a positive test for $2nd$ order validity, then there is a positive test for $1st$ order satisfiablity.
2. There is no positive test for $1st$ order satisfiablity.
3. Hence there is no positive test for $2nd$ order validity. (Incomplete)

Final test: Completness-onwards.

Completness: If the sentence is valid then there is a proof of it's validity.
Completness - We used the truth-tree. If a sentence is valid, the truth tree methoid would classify it as valid. It does this by classifying it's denial as unsatisfiable. It classifies something as invalid if under all possible interpretations are invalid. If a sentence is unsatisfiable, then the tree methoid shows this by having all branches closed.
If there is a complete tree with an open path, then the inital list of that tree is satisfiable. To prove this, we first prove the completness lemma: (If any complete and open path determins an interpretation in which all lines on that path are true.), then, if the tree methoid doesn't classify a sentence as unsatisfiable, then that sentence is satisfiable. If tree is finite: Finished and open path. If tree is infinite: Then we need to show that this infinite tree has an infinite path. - Apprently not difficult to show. An infinite tree, is a tree with an infinite number of lines on it. Conceptually there is two ways: There was some line that yeilded infinite new lines. (But that's not an option due to how a tree is constructed.), Else a line makes more instanciations, that need to be instanciated. You can formalize this via induction. Lets say that a line is "fit" if it heads an infinietly large family(decendents). The first line on an infinite tree has that property. IH: If the nth line in a infinite tree is fit, the the $n + 1$ line is also "fit". This will be true due to how the trees are constructed (Each line only creates finite immediate children), so one of these makes infinite children. Take the leftmost paths of "fits" and you have an infinite path.Returning to the completeness lemma: A complete and open path determins an interpretation on which all lines along that path are true. 1, Make a list of all the names that appear on the path in question. We assign numbers to theses names in the order they are listed. The numbers make the Domain (tentativly). (Including compound names/function symbols.). Once that is done, (2) we do a review: we check for identities, if we find one, we change our assignment, we take the smaller of the two numbers, and we let that be the assignment for the both, and remove the larger number from the domain. (3) Once that is done, we do the same with compund names.(f(a) = g(a) or whateve..). (4) Once the names have been assigend values, we assign.. shit something about predicates. -bug ben, he is feuriously writting it.- (5), we assign values to the function-symbols (they must be total). We do this by looking at the path and the functions and set them so they're true. If it's not there, we can do anything, cause it doesn't matter in the interpretation! That's how me make an interpretation. So we now have an interpretation from a tree! Show that every line along the path is true: Induction on complexity. Take the simples case (atomic sentences) and show that each atomic sentence is true along the path. Which is given, as that's how the interpretation is made. Suppose there is a line along the path that are not true in the interpretation: If there is any-such lines, there should be a shortest such line. All such lines that are the sortest, shall be called "shorty". No atomic,conjunction,disjuncts,conditonal,dinals as they are composed of atomics. Existental: If complete, it was instanciated, which means there is a shorter version of it on the path, and hence that should be the shorty, not this. Universal: Any instancation of the universal is the same in that all instanciations are shorter, so they should be false, not the Universal. The denyal of both lead to eachother, so they follow by the same method.

So there is a question on the test about a variation of that proof.

Logic3: There will be more second-order logic, Incompletness of arithmetic/no aximotization of arithmetic. It just kinda continues. Then there is geurdals 2nd theorem: consistency of any theory that meets certan conditions, that that system cannot prove itself. Aside: Logisisim - all the truths of arithmetic can be derived from truths of logic. Mathematics Arithmetic are as sturdy logic. "monograph" - a book with a clear and percise formal form of logic. It's important. (1879). Which is why all our logics are *79.