PHIL 3something - Logic II

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Misc. Notation

- The set of positive integers $\{x : x \text{ is a positive integer }\}$
- The set of positive integers less than3 $\{x : x \text{ is a positive integer and x is less than 3 }\}. = \{1, 2\}$.
- The empty set: $\emptyset$ or Δ
- Member of: $A \subseteq B$ iff $\forall X(x \in A \implies x \in B)$
- Union of A and B: $A \cup B$ iff $\{x : x \in A \vee x \in B\}$
- Intersection of A and B: $A \cap B$ iff $\{x : x \in A \wedge x \in B\}$
- Difference of A and B: $\{x : x \in A \wedge x \notin B\}$
- For any non-empty sets A, B: Cartesian product: A of B: $A \times B$: $\{< x, y >: x \in A \wedge y \in B\}$ (ALL OF THE POSSIBILITIES)
- TOTAL FUNCTION: Every element in the domain is valid
- PARTIAL FUNCTION: Not every element in the domain is valid.
- for any set of sets A:
  - $\Cup A = \{x : \exists y(y \in A \wedge x \in y)\}$
  - $\Cap A = \{x : \forall y(y \in A \rightarrow x \in y)\}$
- Relations: R is
  - reflexive : $\forall x Rxx$
  - symmetric : $\forall x \forall y(Rxy \implies Ryx)$
  - transitive : $\forall x \forall y \forall z((Rxy \wedge Ryz) \implies Rxz)$
  - Euclidean : $\forall x \forall y \forall z((Rxy \wedge Rxz) \implies Ryz)$
  - a equivalence relation : it's symmetric,reflexive,transitive.
  - a (partial) function : $\exists x$ and there is at most one y: Rxy : denoted f
  - a (partial) function[1] : $\exists x, \exists y|Rxy$: denoted f.
  - a (total) function: assigns a value to each number of A : denoted f
  - a (total) function[2]: $\forall x, \exists y|Rxy$: denoted f.
- Domain: The set of a functions arguments.
- Range: The set of its values. (Results)
- f is a function from a set A iff the domain of f is included in A

- f is a function to a set B iff its range is included in B.
- f^{-1} is the inverse of the function f from the set A to the set B iff:if for every member $b \in B$, there is exactly one member of $a \in A$ such that $f(a) = b$, then $f^{-1}(b) = a$, otherwise $f^{-1}(b)$ is undefined.
- f is onto B iff B is the range of f (Surjective)
  Alt: (wikipedia) : $\forall y \in Y, \exists x \in X | y = f(x)$
- f is one-to-one iff $\forall x \forall y (f(x) = f(y) \implies x = y)$ (Injective)
- f is a bijection iff f is onto and one-to-one.
- f is a correspondence iff f is total, one-to-one and onto.
- Sets A and B are equinumerous iff there is a correspondence from A to B.

Equinumerous is transitive. Prove: if A is equinumerous with B and B is equinumerous wit C, then A is equinumerous with C. Proof: Suppose A is equinumerous to B, and B is equinumerous to C. Then: There is a total,one-to-one function f from A onto B, and a total one-to-one function g from B to C. Prove equinumerous via h=g(f), such that h(n)=g(f(n))

- h is total: Let a be a member of A. $h(a) = g(f(a))$. Since f is total there is a member of b of B such that $f(a) = b$). since g is total, there is a member of $c \in C$ such that $g(b) = c$. Hence, h is total.
- h is onto C. WLOG Let c be a member of C, as g is onto, $\exists b \in B$ such that $g(b) = c$. As f is onto, then $\exists a \in A$ such that $f(a) = b$. Hence, the composition of $h = f(g)$ is onto C.
- h is one-to-one: Suppose h is not one-to-one.
  Then there $\exists a_1, a_2 \in A$ such that $h(a_1) = h(a_2), a_1 \neq a_2$.
  Giving $g(f(a_1)) = g(f(a_2)), a_1/not = a_2$
  Since g is one-to-one $g(b_1) = g(b_2)$ iff $b_1 = b_2$.
  So the issue must lie in f. However f is one-to-one $f(a_1) = f(a_2)$ iff $f(a) = f(b)$. Which is a contradiction, giving us that h is one-to-one.

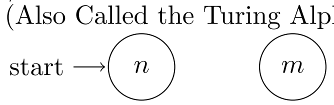
$\square$

A^n : the nth Cartesian product of A with itself.
Suppose that the set of real Numbers $r, r < r < 1$, is enumerable. Then $L_r : r_1, r_2, r_3 ....$ written in a notation of $0.n_1 n_2 n_3.(nbeingnaturalnumbers)$
The set of functions form the set of positive integers to positive integers is not enumerable.

Proof. Suppose S is enumerable.
Then there is a list L_s of the members of S.
$L_s = \{s_1, s_2, s_3, \cdots\}$
Let $\forall n \in \mathbb{N}, n \in k \iff n \notin S_n$
k is a set of positive intigers.
so There is a number $j$such that $k = s_j$. So $j \in S_j \iff j \not inS_j$
Hence S is not enumerable.

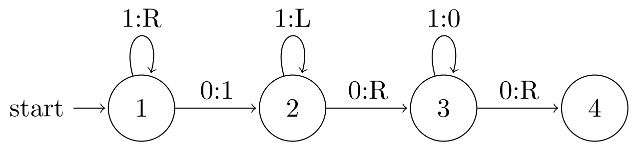
$\square$

The set of total nomadic functions from the set of positive integers, F^1, is not enumerable.
It's a Proof by contradiction.
Turing machines are in the following form: $q_n, S_{1/0}, S_{1/0}/R/L, q_m$ where q_n is our current state, and you see $S_{1/0}$, perform function $S_{1/0}/R/$ and move to state q_m. If there is no operation specified on the current state for a scan, then it halts. (Also Called the Turing Alphabet)

Example with notation: 

ex: (These are the same)

$$Q_1S_1RQ_1, Q_1S_0S_1Q_2, Q_2S_1LQ_2, Q_2S_0RQ_3, Q_3S_1S_0Q_3, Q_3S_0RQ_4$$



Remark (Turing Machines)**.**

- Each Turing machine is a finite set of Turing instructions.
- Each instruction is a 4 letter word of the Turing Alphabet.
- The set of Turing machine is enumerable. (Proof: exercise)

Definition (Standard inital configuration)**.** A turing machine is in a standard Inital configuration $\iff$

- for some positive integer k, there are k blocks of 1's on the tape.
- seperated by a blank,
- and the rest of the tape is blank.
- the machine is scanning the left-most 1 on the tape.
- the machine is in it's lowest numbered state.

ex: $\cdots 0010110111000 \cdots$ is a SIC. (if it's in lowest state) ex: $\cdots 00010000 \cdots$ is a SIC.

Definition (Standard final configuration)**.** A turing machine is in a standard final configuration $\iff$

- there is a single block of k 1's
- and the rest of the tape is blank.
- the machine is scanning the left-most 1 on the tape.

ex: $\cdots 00111111000 \cdots$ is a SIC. (if it's in lowest state) ex: $\cdots 00010000 \cdots$ is a SIC.

Definition (Computes a one-place function f^1)**.** A turing M computes a one-place function f^1:
if M is started in a SIC with a single block of k 1's and

- if f^1 is defined for the argument k, then M eventually halts in a SFC
- or if f^1 is not defined for the argument k, then either M never halts or it halts in a non-standard final configuration.

Remark. Every turing machine computes exactly one functionof two arguments.

Remark. For any n, each Turing Machine computes exactly one function of n arguments.
The set of one-place turing computable functions is enumerable

$\vdots$

The set of Turing computable functions is enumerable.