

PHIL 3something - Logic II

Scott Saunders - 10163541

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Misc. Notation

- The set of positive integers $\{x : x \text{ is a positive integer} \}$
- The set of positive integers less than 3 $\{x : x \text{ is a positive integer and } x \text{ is less than } 3 \} = \{1, 2\}$.
- The empty set: \emptyset or Δ
- Member of: $A \subseteq B$ iff $\forall x(x \in A \implies x \in B)$
- Union of A and B: $A \cup B$ iff $\{x : x \in A \vee x \in B\}$
- Intersection of A and B: $A \cap B$ iff $\{x : x \in A \wedge x \in B\}$
- Difference of A and B: $\{x : x \in A \wedge x \notin B\}$
- For any non-empty sets A, B: Cartesian product: A of B: $A \times B: \{ \langle x, y \rangle : x \in A \wedge y \in B \}$ (ALL OF THE POSSIBILITIES)
- TOTAL FUNCTION: Every element in the domain is valid
- PARTIAL FUNCTION: Not every element in the domain is valid.
- for any set of sets A:
 - $\cup A = \{x : \exists y(y \in A \wedge x \in y)\}$
 - $\cap A = \{x : \forall y(y \in A \rightarrow x \in y)\}$
- Relations: R is
 - reflexive : $\forall x Rxx$
 - symmetric : $\forall x \forall y (Rxy \implies Ryx)$
 - transitive : $\forall x \forall y \forall z ((Rxy \wedge Ryz) \implies Rxz)$
 - Euclidean : $\forall x \forall y \forall z ((Rxy \wedge Rxz) \implies Ryz)$
 - a equivalence relation : it's symmetric, reflexive, transitive.
 - a equivalence relation (alt) : it's symmetric, and euclidean.
 - a (partial) function : $\exists x$ and there is at most one y: Rxy : denoted f
 - a (partial) function¹ : $\exists x, \exists y | Rxy$: denoted f .
 - a (total) function: assigns a value to each number of A : denoted f
 - a (total) function²: $\forall x, \exists y | Rxy$: denoted f .
- Domain: The set of a functions arguments.
- Range: The set of its values. (Results)

- f is a function from a set A iff the domain of f is included in A
- f is a function to a set B iff its range is included in B .
- f^{-1} is the inverse of the function f from the set A to the set B iff: if for every member $b \in B$, there is exactly one member of $a \in A$ such that $f(a) = b$, then $f^{-1}(b) = a$, otherwise $f^{-1}(b)$ is undefined.
- f is onto B iff B is the range of f (Surjective)
Alt: (Wikipedia) : $\forall y \in Y, \exists x \in X | y = f(x)$
- f is one-to-one iff $\forall x \forall y (f(x) = f(y) \implies x = y)$ (Injective)
- f is a bijection iff f is onto and one-to-one.
- f is a correspondence iff f is total, one-to-one and onto.
- Sets A and B are equinumerous iff there is a correspondence from A to B .

Equinumerous is transitive. Prove: if A is equinumerous with B and B is equinumerous with C , then A is equinumerous with C . Proof: Suppose A is equinumerous to B , and B is equinumerous to C . Then: There is a total, one-to-one function f from A onto B , and a total one-to-one function g from B to C . Prove equinumerous via $h=g(f)$, such that $h(n)=g(f(n))$

- h is total: Let a be a member of A . $h(a) = g(f(a))$. Since f is total there is a member of b of B such that $f(a) = b$. since g is total, there is a member of $c \in C$ such that $g(b) = c$. Hence, h is total.
- h is onto C . WLOG Let c be a member of C , as g is onto, $\exists b \in B$ such that $g(b) = c$. As f is onto, then $\exists a \in A$ such that $f(a) = b$. Hence, the composition of $h = f(g)$ is onto C .
- h is one-to-one: Suppose h is not one-to-one.
Then there $\exists a_1, a_2 \in A$ such that $h(a_1) = h(a_2), a_1 \neq a_2$.
Giving $g(f(a_1)) = g(f(a_2)), a_1 \neq a_2$
Since g is one-to-one $g(b_1) = g(b_2)$ iff $b_1 = b_2$.
So the issue must lie in f . However f is one-to-one $f(a_1) = f(a_2)$ iff $f(a) = f(b)$. Which is a contradiction, giving us that h is one-to-one.

□

A^n : the n th Cartesian product of A with itself.

Suppose that the set of real Numbers $r, r < 1$, is enumerable. Then $L_r : r_1, r_2, r_3, \dots$ written in a notation of $0.n_1n_2n_3 \dots$ (being natural numbers)

The set of functions from the set of positive integers to positive integers is not enumerable.

Proof. Suppose S is enumerable.

Then there is a list L_s of the members of S .

$L_s = \{s_1, s_2, s_3, \dots\}$

Let $\forall n \in \mathbb{N}, n \in k \iff n \notin S_n$

k is a set of positive integers.

so There is a number j such that $k = s_j$. So $j \in S_j \iff j \notin S_j$

Hence S is not enumerable.

□

The set of total recursive functions from the set of positive integers, F^1 , is not enumerable.

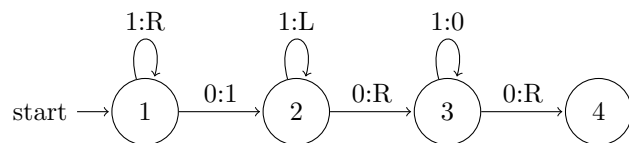
It's a Proof by contradiction.

Turing machines are in the following form: $q_n, S_{1/0}, S_{1/0}/R/L, q_m$ where q_n is our current state, and you see $S_{1/0}$, perform function $S_{1/0}/R/$ and move to state q_m . If there is no operation specified on the current state for a scan, then it halts. (Also Called the Turing Alphabet)

Example with notation: $\text{start} \rightarrow \begin{array}{c} \circ \\ n \end{array} \quad \begin{array}{c} \circ \\ m \end{array}$

ex: (These are the same)

$Q_1S_1RQ_1, Q_1S_0S_1Q_2, Q_2S_1LQ_2, Q_2S_0RQ_3, Q_3S_1S_0Q_3, Q_3S_0RQ_4$



Remark (Turing Machines). • Each Turing machine is a finite set of Turing instructions.

- Each instruction is a 4 letter word of the Turing Alphabet.
- The set of Turing machine is enumerable. (Proof: exercise)

Definition (Standard initial configuration). A Turing machine is in a standard Initial configuration \iff

- for some positive integer k , there are k blocks of 1's on the tape.
- separated by a blank,
- and the rest of the tape is blank.
- the machine is scanning the left-most 1 on the tape.
- the machine is in it's lowest numbered state.

ex: $\dots 0010110111000\dots$ is a *SIC*. (if it's in lowest state) ex: $\dots 00010000\dots$ is a *SIC*.

Definition (Standard final configuration). A Turing machine is in a standard final configuration \iff

- there is a single block of k 1's
- and the rest of the tape is blank.
- the machine is scanning the left-most 1 on the tape.

ex: $\dots 00111111000\dots$ is a *SIC*. (if it's in lowest state) ex: $\dots 00010000\dots$ is a *SIC*.

Definition (Computes a one-place function f^1). A Turing M computes a one-place function f^1 : if M is started in a *SIC* with a single block of k 1's and

- if f^1 is defined for the argument k , then M eventually halts in a *SFC*
- or if f^1 is not defined for the argument k , then either M never halts or it halts in a non-standard final configuration.

Remark. Every Turing machine computes exactly one function of two arguments.

Remark. For any n , each Turing Machine computes exactly one function of n arguments.

The set of one-place Turing computable functions is enumerable

\vdots

The set of Turing computable functions is enumerable.

Definition (The halting problem). The problem of finding an effective method to determine whether a Turing machine will eventually halt or not after it is started with some input.

The halting problem is unsolvable. Ex: $L_M : M_1, \dots$

$h(m, n) =$

1 if M_m eventually halts after starting with input n

2 if M_m never halts after starting with input n

The halting problem is solvable iff h is computable. Show: h is not Turing computable.

Let C be a copying machine.

Let F be $\frac{1}{2}$ flipper.

Suppose h is Turing Computable.

Let H be a Turing machine that computes h .

If h is a Turing computable, then H exists.

If H exists, then $D(C - H - F)$ exists.

Let $D = M_k$, for some k . $M_k \in L_M$.

Start D with input k . The C -part of D will produce a copy of k , Then the H -part will do its job:

- If M_k will eventually halt after starting with input k , then H will produce output 1.
- If M_k will never halt after starting with input k , then H will produce output 2.

Then the F -part will do its job.

- If output from H is 1, F will never halt.
- If output from H is 2, F will eventually halt.

Giving us:

- If M_k will eventually halt after starting with input k , then D will never halt after starting with input k .
- If M_k will never halt after starting with input k , then D will eventually halt after starting with input k .

So M_k will halt, after starting with input k , \iff D will never halt after starting with input k .

Then M_k isn't identical with D , which is a contradiction! Hence D doesn't exist. So H does not exist. So h is not Turing computable. \square

Another halting problem..? $L_M : M_1, \dots$ $L_F : F_1, \dots$

$g(n) = 1$, if $f_n(n) = 2$

$g(n) = 2$, otherwise.

$g \neq f_k \forall k$

$h(m, n) = 1$ if M_m eventually halts after starting with input n .

$h(m, n) = 2$ if M_m never halts after starting with input n .

$s(m) = 1$, if M_m eventually halts after starting with input m .

$s(m) = 2$, if M_m never halts after starting with input m .

1. The halting problem is solvable iff h is computable.
2. If h is computable, then s is computable.
3. If s is computable, and TT 's is true, then g is computable.
4. g is not Turing computable.
5. Turing Thesis is true (Whatever is not Turing Computable is not computable)

6. The halting problem is not solvable.

3. Suppose S is computable and TT is true.

Then: There is a Turing machine S^* that computes s .

Suppose that we are to calculate $g(n)$, for some n .

Start S^* with input n .

- Case 1: S^* eventually halts with output 1

We know that M_n will eventually halt after it is started with input n . Start M_n with input n , when it halts, inspect the tape.

– Case 1.1: Halted in SFC $f_n(n) = 2$ $g(n) = 1$

– Case 1.2: Halted in non-SFC: f_n is undefined. $f_n(n) \neq 2$

And then Blake broke it:

As it's a halting problem to figure out if it's in SFC ?

$g(n) = 2$

- Case 2: S^* eventually halts with output 2 We know that M_n will never halt after it is started with input n .

So we know that f_n is undefined for the argument n .

So we know that $g(n) = 2$

□

□

0.1 Sentential logic

Some symbols n things:

- (
-)
- ,
- -
- And: \wedge (Conjunction)
- Or: \vee (Disjunction)
- Exists: \exists
- Forall: \forall
- Variables: v_1, v_2, v_3, \dots
- Equality: $=$
- Predicates:

A_1^1	A_2^1	\dots
\vdots	\vdots	\ddots
A_1^n	A_2^n	\dots
- Constant names: a_1, \dots

- | | | | |
|--------------|----------|----------|----------|
| | f_1^1 | f_2^1 | \dots |
| • Functions: | \vdots | \vdots | \ddots |
| | f_1^n | f_2^n | \dots |

Definition (Term). • Every variable is a term.

- Every constant is a term. (or name)
- If t_1, \dots, t_n are terms, then $f^n(t_1, \dots, t_n)$ is a term.
- Nothing else is a term.

Definition (Formula). • $A^n(t_1, \dots, t_n)$ is a formula where A^n is an n -place predicate and t_i are terms.

- If F is a formula then $\neg F$ is a formula
- If F and G are formulas then $(F \wedge G)$ is a formula.
- If F and G are formulas then $F \vee G$ is a formula.
- If F is a formula, then $\exists v F$ is a formula.
- If F is a formula then $\forall v F$ is a formula.
- NOTHING ELSE IS A FORMULA.

Definition (Bound). An occurrence of a variable v in a formula F is bound if it is in a part G of F where $G = \exists v \dots$ or $G = \forall v \dots$.

Definition (Model). A model M (interpretation) of a language L is $\{|M|, v\}$ Where $|M|$ is a non-empty set and v is a valuation function that assigns values (extensions/denotations) to the members of L in such a way that

- $\forall v(a) \in |M|$
- $v(A^n) \subseteq$ the n th cartesian product of $|M|$ with itself: $|M| \times \dots \times |M|$.
- $v(f^n)$ is a total function from $|M| \times \dots \times |M|$ to $|M|$.

Definition (Truth). • $M \models F^n(t_1, \dots, t_n)$ iff $\langle M(t_1), \dots, M(t_n) \rangle \in v(A^n)$.

- $M \models \neg S$ iff $M \not\models S$.
- $M \models (K \wedge L)$ iff $M \models K$ and $M \models L$
- $M \models \forall x F$
- $M \models \exists x F(c)$ iff there is an object $o \in |M|$ and given a name c (that is not interpreted by M), $M \models F(c)$

Definition. of the denotation/extension of a closed term in a model M . If T is a name $M(t) = v(t)$ I t is $f^n(t_1, \dots, t_n)$ then $M(f^n(t_1, \dots, t_n)) = v(f^n)(M(t_1), \dots, M(t_n))$.

Validity = Satisfiability = Implication.

Misc-crap:

- $A \models B$ is $\neg(A \wedge \neg B)$

Lemma 1. *Extensionality Lemma*

- Let M be a model of a language L .
- Let S be a sentence of L .
- Let L^+ be an extension of L . $L \subseteq L^+$
- Let M^+ be a model of L^+
- So: M^+ is an extension of M .
- $M \models S$ iff $M^+ \models S$

Example:

If $A \models B$ and $B \models C$, then $A \models C$. Suppose $A \models B$, and $B \models C$.

In every interpretation of A and B in which A is true, B is true, In every interpretation of B and C in which B is true, C is true. Shows: In every interpretation of A and C in which A is true, C is true.

Let M be an interpretation of A and C such that $M \models A$.

- Case 1: M is an interpretation of B .
Then $M \models B$
So $M \models C$.
- M is not an interpretation of B .
Then there is an extension M^+ that interprets B as well as A and C .
so: $M^+ \models B$
So: $M^+ \models C$
So $M \models C$ (By the ext, lemma)

Lemma 2 (Undecidability). *If the decision problem (for implication) is solvable, then the halting problem is solvable. There is an effective method for specifying for any Turing machine M and any input N a finite set of sentences Δ and a sentence H such that $\Delta \models H$ iff M eventually halts after starting with input n . $\Delta \models H$ iff M eventually halts after start with input n .*

Define the one place predicate Q_{ij} as: At time j , M is in state i . Define the two place predicate $@_{js}$ as At time j , M , is scanning square s . Define the two place predicate M_{js} as: At time j , square s is marked with a 1.

A description D for a start state could then be: $D : [Q_1 0 \wedge @_{0,0} \wedge M_{0,0} \wedge M_{0,1} \wedge M_{0,2} \wedge \forall y((y \neq 0 \wedge y \neq 1 \wedge y \neq 2) \implies \neg M_{0,y}) \text{ Time} = 0, [\dots 01110 \dots] \text{ Square } \#, [\dots - 10123 \dots]$

For each instruction of a TM , we may write the instruction as a sentence:

$Q_{i1} S_1 R Q_{i2}$: Move right seeing 1 in state:

$\forall x \forall t ((Q_{i,1} \wedge @_{t,x} \wedge M_{t,x}) \implies (Q_{i2,(t+1)} \wedge @_{(t+1),(x+1)} \wedge \forall y ((M_{t,y} \implies M_{(t+1),y}) \wedge (\neg M_{t,y} \implies \neg M_{(t+1),y})))$

Misc-crap that's on the board for some reason:

Δ	
$Q_2^1 :$	$\forall x \forall y \forall z ((Sxy \wedge Sxz) \implies y = z)$
$@^2 :$	$\forall x \forall y \forall z ((Sxz \wedge Syz) \implies x = y)$
$M^2 :$	$\forall x \forall y (Sxy \implies x < y)$
$0 :$	$\forall x \forall y \forall z ((x < y \wedge y < z) \implies x < z)$
$S^2 :$	$\neg \exists x, x < x+1$
$<^2 :$	lessthan

Some thingelse now too:

$\forall x \forall t ((Q_1 t \wedge @_{t,x} \wedge M_{t,x}) \implies \exists u (s_1(t, u) \wedge Q_2(u) \wedge @_{(u,v)}) \wedge \forall y ((M(t, y) \implies M(u, y)) \wedge (\neg M(t, y) \implies \neg M(u, y)))$

$$\neg M(u, y)))))) \\ \exists x \exists t (Q_m t \wedge @tx \wedge Mtx)$$

Proof. Something about biconditional:

$\Delta \models H$ iff M halts after starting with input n .

1. if $\Delta \models H$, then M halts.

2. if M halts, then $\Delta \models H$.

1. if $\Delta \models H$, then M halts - Proof.

Suppose $\Delta \models H$

All members of Δ are true in the standard interpretation I . H is true in I . So: M halts.

2. if M halts, then $\Delta \models H$ - Proof.

Suppose M halts. (Show $\Delta \models H$).

There is a time t M halts at t .

There is a state q_i , M halts at t in state q_i .

There is a square x , M halts at t in state q_i , scanning square x which is Marked / – Marked

1: $Q_i(t) \wedge @ (t, x) \wedge M(t, x)$.

1 is a conjunct of the description of time t , $\mathbb{D}(t)$

$\mathbb{D}(t) \models (i)$

2: $\exists x \exists t (Q_1(t) \wedge @ (t, x) \wedge (t, x))$.

$(i) \models (ii)$.

(ii) is disjunct of H .

$(ii) \models H$.

So: $\mathbb{D}(t) \models H$.

□