

# PHIL 3something - Logic II

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## Misc. Notation

- The set of positive integers  $\{x : x \text{ is a positive integer} \}$
- The set of positive integers less than 3  $\{x : x \text{ is a positive integer and } x \text{ is less than } 3 \} = \{1, 2\}$ .
- The empty set:  $\emptyset$  or  $\Delta$
- Member of:  $A \subseteq B$  iff  $\forall x(x \in A \implies x \in B)$
- Union of A and B:  $A \cup B$  iff  $\{x : x \in A \vee x \in B\}$
- Intersection of A and B:  $A \cap B$  iff  $\{x : x \in A \wedge x \in B\}$
- Difference of A and B:  $\{x : x \in A \wedge x \notin B\}$
- For any non-empty sets A, B: Cartesian product: A of B:  $A \times B: \{ \langle x, y \rangle : x \in A \wedge y \in B \}$  (ALL OF THE POSSIBILITIES)
- TOTAL FUNCTION: Every element in the domain is valid
- PARTIAL FUNCTION: Not every element in the domain is valid.
- for any set of sets A:
  - $\cup A = \{x : \exists y(y \in A \wedge x \in y)\}$
  - $\cap A = \{x : \forall y(y \in A \rightarrow x \in y)\}$
- Relations: R is
  - reflexive :  $\forall x Rxx$
  - symmetric :  $\forall x \forall y (Rxy \implies Ryx)$
  - transitive :  $\forall x \forall y \forall z ((Rxy \wedge Ryz) \implies Rxz)$
  - Euclidean :  $\forall x \forall y \forall z ((Rxy \wedge Rxz) \implies Ryz)$
  - a equivalence relation : it's symmetric, reflexive, transitive.
  - a (partial) function :  $\exists x$  and there is at most one y:  $Rxy$  : denoted  $f$
  - a (partial) function<sup>1</sup> :  $\exists x, \exists y | Rxy$  : denoted  $f$ .
  - a (total) function: assigns a value to each number of A : denoted  $f$
  - a (total) function<sup>2</sup>:  $\forall x, \exists y | Rxy$ : denoted  $f$ .
- Domain: The set of a functions arguments.
- Range: The set of its values. (Results)
- $f$  is a function from a set A iff the domain of  $f$  is included in A

- $f$  is a function to a set  $B$  iff its range is included in  $B$ .
- $f^{-1}$  is the inverse of the function  $f$  from the set  $A$  to the set  $B$  iff: if for every member  $b \in B$ , there is exactly one member of  $a \in A$  such that  $f(a) = b$ , then  $f^{-1}(b) = a$ , otherwise  $f^{-1}(b)$  is undefined.
- $f$  is onto  $B$  iff  $B$  is the range of  $f$  (Surjective)  
Alt: (wikipedia) :  $\forall y \in Y, \exists x \in X | y = f(x)$
- $f$  is one-to-one iff  $\forall x \forall y (f(x) = f(y) \implies x = y)$  (Injective)
- $f$  is a bijection iff  $f$  is onto and one-to-one.
- $f$  is a correspondence iff  $f$  is total, one-to-one and onto.
- Sets  $A$  and  $B$  are equinumerous iff there is a correspondence from  $A$  to  $B$ .

*Equinumerous is transitive.* Prove: if  $A$  is equinumerous with  $B$  and  $B$  is equinumerous with  $C$ , then  $A$  is equinumerous with  $C$ . Proof: Suppose  $A$  is equinumerous to  $B$ , and  $B$  is equinumerous to  $C$ . Then: There is a total, one-to-one function  $f$  from  $A$  onto  $B$ , and a total one-to-one function  $g$  from  $B$  to  $C$ . Prove equinumerous via  $h=g(f)$ , such that  $h(n)=g(f(n))$

- $h$  is total: Let  $a$  be a member of  $A$ .  $h(a) = g(f(a))$ . Since  $f$  is total there is a member of  $b$  of  $B$  such that  $f(a) = b$ . since  $g$  is total, there is a member of  $c \in C$  such that  $g(b) = c$ . Hence,  $h$  is total.
- $h$  is onto  $C$ . WLOG Let  $c$  be a member of  $C$ , as  $g$  is onto,  $\exists b \in B$  such that  $g(b) = c$ . As  $f$  is onto, then  $\exists a \in A$  such that  $f(a) = b$ . Hence, the composition of  $h = f(g)$  is onto  $C$ .
- $h$  is one-to-one: Suppose  $h$  is not one-to-one.  
Then there  $\exists a_1, a_2 \in A$  such that  $h(a_1) = h(a_2), a_1 \neq a_2$ .  
Giving  $g(f(a_1)) = g(f(a_2)), a_1 \neq a_2$   
Since  $g$  is one-to-one  $g(b_1) = g(b_2)$  iff  $b_1 = b_2$ .  
So the issue must lie in  $f$ . However  $f$  is one-to-one  $f(a_1) = f(a_2)$  iff  $f(a) = f(b)$ . Which is a contradiction, giving us that  $h$  is one-to-one.

□

$A^n$  : the  $n$ th Cartesian product of  $A$  with itself.

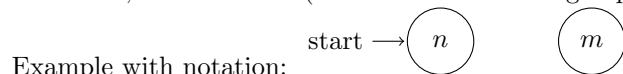
Suppose that the set of real Numbers  $r, 0 < r < 1$ , is enumerable. Then  $L_r : r_1, r_2, r_3, \dots$  written in a notation of  $0.n_1n_2n_3 \dots$  ( $n$  being natural numbers)

The set of functions from the set of positive integers to positive integers is not enumerable.

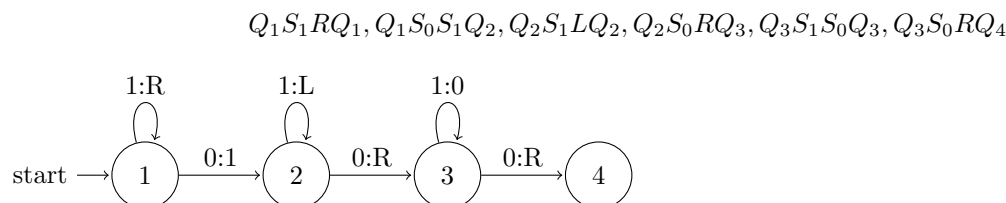
The set of total recursive functions from the set of positive integers,  $F^1$ , is not enumerable.

It's a Proof by contradiction.

Turing machines are in the following form:  $q_n, S_{1/0}, S_{1/0}/R/L, q_m$  where  $q_n$  is our current state, and you see  $S_{1/0}$ , perform function  $S_{1/0}/R/$  and move to state  $q_m$ . If there is no operation specified on the current state for a scan, then it halts. (Also Called the Turing Alphabet)



ex: (These are the same)



**Remark** (Turing Machines).      • Each Turing machine is a finite set of Turing instructions.

- Each instruction is a 4 letter word of the Turing Alphabet.
- The set of Turing machine is enumerable. (Proof: exercise)