# PHIL 3something - Logic II

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## Misc. Notation

- The set of positive integers $\{x : x \text{ is a positive integer }\}$
- The set of positive integers less than3 $\{x : x \text{ is a positive integer and x is less than 3 }\}. = \{1, 2\}$.
- The empty set: $\emptyset$ or $\Delta$
- Member of: $A \subseteq B$ iff $\forall X(x \in A \implies x \in B)$
- Union of A and B: $A \cup B$ iff $\{x : x \in A \vee x \in B\}$
- Intersection of A and B: $A \cap B$ iff $\{x : x \in A \wedge x \in B\}$
- Difference of A and B: $\{x : x \in A \wedge x \notin B\}$
- For any non-empty sets $A, B$: Cartesian product: A of B: $A \times B$: $\{< x, y >: x \in A \wedge y \in B\}$ (ALL OF THE POSSIBILITIES)
- TOTAL FUNCTION: Every element in the domain is valid
- PARTIAL FUNCTION: Not every element in the domain is valid.
- for any set of sets $A$:
  - $\Cup A = \{x : \exists y(y \in A \wedge x \in y)\}$
  - $\Cap A = \{x : \forall y(y \in A \to x \in y)\}$
- Relations: R is
  - reflexive : $\forall x Rxx$
  - symmetric : $\forall x \forall y(Rxy \implies Ryx)$
  - transitive : $\forall x \forall y \forall z((Rxy \wedge Ryz) \implies Rxz)$
  - Euclidean : $\forall x \forall y \forall z((Rxy \wedge Rxz) \implies Ryz)$
  - a equivalence relation : it's symmetric,reflexive,transitive.
  - a equivalence relation (alt) : it's symmetric, and euclidean.
  - a (partial) function : $\exists x$ and there is at most one y: $Rxy$ : denoted $f$
  - a (partial) function[1] : $\exists x, \exists y | Rxy$ : denoted $f$.
  - a (total) function: assigns a value to each number of $A$ : denoted $f$
  - a (total) function[2]: $\forall x, \exists y | Rxy$: denoted $f$.
- Domain: The set of a functions arguments.
- Range: The set of its values. (Results)

- $f$ is a function from a set $A$ iff the domain of $f$ is included in $A$
- $f$ is a function to a set $B$ iff its range is included in $B$.
- $f^{-1}$ is the inverse of the function $f$ from the set $A$ to the set $B$ iff:if for every member $b \in B$, there is exactly one member of $a \in A$ such that $f(a) = b$, then $f^{-1}(b) = a$, otherwise $f^{-1}(b)$ is undefined.
- $f$ is onto $B$ iff $B$ is the range of $f$ (Surjective)
  Alt: (Wikipedia) : $\forall y \in Y, \exists x \in X | y = f(x)$
- $f$ is one-to-one iff $\forall x \forall y (f(x) = f(y) \implies x = y)$ (Injective)
- $f$ is a bijection iff $f$ is onto and one-to-one.
- $f$ is a correspondence iff $f$ is total, one-to-one and onto.
- Sets $A$ and $B$ are equinumerous iff there is a correspondence from $A$ to $B$.

*Equinumerous is transitive.* Prove: if $A$ is equinumerous with $B$ and $B$ is equinumerous wit $C$, then $A$ is equinumerous with $C$. Proof: Suppose $A$ is equinumerous to $B$, and $B$ is equinumerous to $C$. Then: There is a total,one-to-one function $f$ from $A$ onto $B$, and a total one-to-one function $g$ from $B$ to $C$. Prove equinumerous via h=g(f), such that h(n)=g(f(n))

- h is total: Let $a$ be a member of $A$. $h(a) = g(f(a))$. Since f is total there is a member of $b$ of $B$ such that $f(a) = b$). since $g$ is total, there is a member of $c \in C$ such that $g(b) = c$. Hence, $h$ is total.
- $h$ is onto $C$. WLOG Let $c$ be a member of $C$, as $g$ is onto, $\exists b \in B$ such that $g(b) = c$. As $f$ is onto, then $\exists a \in A$ such that $f(a) = b$. Hence, the composition of $h = f(g)$ is onto $C$.
- $h$ is one-to-one: Suppose $h$ is not one-to-one.
  Then there $\exists a_1, a_2 \in A$ such that $h(a_1) = h(a_2), a_1 \neq a_2$.
  Giving $g(f(a_1)) = g(f(a_2)), a_1/not = a_2$
  Since $g$ is one-to-one $g(b_1) = g(b_2)$ iff $b_1 = b_2$.
  So the issue must lie in $f$. However $f$ is one-to-one $f(a_1) = f(a_2)$ iff $f(a) = f(b)$. Which is a contradiction, giving us that $h$ is one-to-one.

□

$A^n$ : the $n$th Cartesian product of $A$ with itself.
Suppose that the set of real Numbers $r, r < r < 1$, is enumerable. Then $L_r : r_1, r_2, r_3....$ written in a notation of $0.n_1 n_2 n_3.(nbeingnaturalnumbers)$
The set of functions form the set of positive integers to positive integers is not enumerable.

*Proof.* Suppose S is enumerable.
Then there is a list $L_s$ of the members of $S$.
$L_s = \{s_1, s_2, s_3, \cdots\}$
Let $\forall n \in \mathbb{N}, n \in k \iff n \notin S_n$
$k$ is a set of positive integers.
so There is a number $j$such that $k = s_j$. So $j \in S_j \iff j \not{i}nS_j$
Hence $S$ is not enumerable.

□

The set of total nomadic functions from the set of positive integers, $F^1$, is not enumerable.
It's a Proof by contradiction.
Turing machines are in the following form: $q_n, S_{1/0}, S_{1/0}/R/L, q_m$ where $q_n$ is our current state, and you see $S_{1/0}$, perform function $S_{1/0}/R/$ and move to state $q_m$. If there is no operation specified on the current state for a scan, then it halts. (Also Called the Turing Alphabet)

Example with notation:
$$\begin{tikzcd}\text{start} \arrow[r] & n & m\end{tikzcd}$$

ex: (These are the same)

$$Q_1S_1RQ_1, Q_1S_0S_1Q_2, Q_2S_1LQ_2, Q_2S_0RQ_3, Q_3S_1S_0Q_3, Q_3S_0RQ_4$$

$$\begin{tikzcd}\text{start} \arrow[r] & 1 \arrow[r, "0{:}1"] \arrow[loop above, "1{:}R"] & 2 \arrow[r, "0{:}R"] \arrow[loop above, "1{:}L"] & 3 \arrow[r, "0{:}R"] \arrow[loop above, "1{:}0"] & 4\end{tikzcd}$$

**Remark** (Turing Machines).

- Each Turing machine is a finite set of Turing instructions.
- Each instruction is a 4 letter word of the Turing Alphabet.
- The set of Turing machine is enumerable. (Proof: exercise)

**Definition** (Standard inital configuration). A Turing machine is in a standard Initial configuration $\iff$

- for some positive integer $k$, there are $k$ blocks of 1's on the tape.
- separated by a blank,
- and the rest of the tape is blank.
- the machine is scanning the left-most 1 on the tape.
- the machine is in it's lowest numbered state.

ex: $\cdots 0010110111000 \cdots$ is a $SIC$. (if it's in lowest state) ex: $\cdots 00010000 \cdots$ is a $SIC$.

**Definition** (Standard final configuration). A Turing machine is in a standard final configuration $\iff$

- there is a single block of $k$ 1's
- and the rest of the tape is blank.
- the machine is scanning the left-most 1 on the tape.

ex: $\cdots 00111111000 \cdots$ is a $SIC$. (if it's in lowest state) ex: $\cdots 00010000 \cdots$ is a $SIC$.

**Definition** (Computes a one-place function $f^1$). A Turing $M$ computes a one-place function $f^1$:
if $M$ is started in a $SIC$ with a single block of $k$ 1's and

- if $f^1$ is defined for the argument $k$, then $M$ eventually halts in a $SFC$
- or if $f^1$ is not defined for the argument $k$, then either $M$ never halts or it halts in a non-standard final configuration.

**Remark.** Every Turing machine computes exactly one function of two arguments.

**Remark.** For any $n$, each Turing Machine computes exactly one function of $n$ arguments.
The set of one-place Turing computable functions is enumerable
$\vdots$
The set of Turing computable functions is enumerable.

**Definition** (The halting problem). The problem of finding an effective method to determine whether a Turing machine will eventually halt or not after it is started with some input.

*The halting problem is unsolvable.* Ex: $L_M : M_1, \dots$
$h(m,n) =$
1 if $M_m$ eventually halts after starting with input $n$
2 if $M_m$ never halts after starting with input $n$
The halting problem is solvable $iff$ $h$ is computable. Show: $h$ is not Turing computable.
Let $C$ be a copying machine.
Let $F$ be $\frac{1}{2}$ flipper.
Suppose $h$ is Turing Computable.
Let $H$ be a Turing machine that computes $h$.
If $h$ is a Turing computable, then $H$ exists.
If $H$ exists, then $D(C - H - F)$ exists.
Let $D = M_k$, for some $k$. $M_k \in L_M$.

Start $D$ with input $k$. The C-part of $D$ will produce a copy of $k$, Then the $H-$part will do its job:

- If $M_k$ will eventually halt after starting with input $k$, then $H$ will produce output 1.
- If $M_k$ will never halt after starting with input $k$, then $H$ will produce output 2.

Then the $F-$part will do it's job.

- If output from $H$ is 1, $F$ will never halt.
- If output from $H$ is 2, $F$ will eventually halt.

Giving us:

- If $M_k$ will eventually halt after starting with input $k$, then $D$ will never halt after starting with input $k$.
- If $M_k$ will never halt after starting with input $k$, then $D$ will eventually halt after starting with input $k$.

So $M_k$ will halt, after starting with input $k$, $\iff$ $D$ will never halt after starting with input $k$.
Then $M_k$ isn't identical with $D$, which is a contradiction! Hence$D$ doesn't exist. So $H$ does not exists. So $h$ is not Turing computable. □

*Another halting problem..?* $L_M : M_1, \cdots$ $L_F : F_1, \cdots$
$g(n) = 1$, if $f_n(n) = 2$
$g(n) = 2$, otherwise.
$g \neq f_k \forall k$
$h(m,n) = 1$if $M_m$ eventually halts after starting with input $n$.
$h(m,n) = 2$ if $M_m$ never halts after starting with input $n$.
$s(m) = 1$, if $M_m$ eventually halts after starting with input $m$.
$s(m) = 2$, if $M_m$ never halts after starting with input $m$.

1. The halting problem is solvable iff $h$ is computable.
2. If $h$ is computable, then $s$ is computable.
3. If $s$ is computable, and $TT$'s is true, then $g$ is computable.
4. $g$ is not Turing computable.
5. Turing Thesis is true (Whatever is not Turing Computable is not computable)
6. The halting problem is not solvable.

*3.* Suppose $S$ is computable and $TT$ is true.
Then: There is a Turing machine $S*$ that computes s.
Suppose that we are to calculate $g(n)$, for some $n$.
Start $S*$ with input $n$.

- Case 1: S* eventually halts with output 1
  We know that $M_n$ will eventually halt after it is started with input $n$ Start $M_n$ with input $n$, when it halts, inspect the tape.
  - Case 1.1: Halted in SFC $f_n(n) = 2$ $g(n) = 1$
  - Case 1.2: Halted in non-SFC: $f_n$ is undefined. $f_n(n) \neq 2$

  And then Blake broke it:
  As it's a halting problem to figure out if it's in $SFC$?
  $g(n) = 2$
- Case 2: S* eventually halts with output 2 We know that $M_n$ will never halt after it is started with input $n$.
  So we know that $f_n$ is undefined for the argument $n$.
  So we know that $g(n) = 2$

□

□

## 0.1 Sentental logic

Some symbols n things:

- (
- )
- Successor : '
- Not: -
- And: $\wedge$ (Conjunction)
- Or: $\vee$ (Disjunction)
- Exists: $\exists$
- Forall: $\forall$
- Variables: $v_1, v_2, v_3, \ldots$
- Equality: $=$
- Predicates: $\begin{matrix} A_1^1 & A_2^1 & \cdots \\ \vdots & \vdots & \ddots \\ A_1^n & A_2^n & \cdots \end{matrix}$
- Constant names: $a_1, \ldots$
- Functions: $\begin{matrix} f_1^1 & f_2^1 & \cdots \\ \vdots & \vdots & \ddots \\ f_1^n & f_2^n & \cdots \end{matrix}$

**Definition** (Term).

- Every variable is a(n atomic) (open) term.
- Every constant is a(n atomic) (closed) term.
- If $t_1, \ldots, t_n$ are terms, then $f^n(t_1, \ldots, t_n)$ is a term.
- Nothing else is a term.

**Definition** (Formula).

- $A^n(t_1, \ldots, t_n)$ is a formula where $A^n$ is an $n-$place predicate and $t_i$ are terms. (This is an atomic formula).
- If $F$ is a formula then $-F$ is a formula
- If $F$ and $G$ are formulas then $(F \wedge G)$ is a formula.
- If $F$ and $G$ are formulas then $F(\vee G)$ is a formula.
- If $F$ is a formula, then $\exists v F$ is a formula.
- If $F$ is a formula then $\forall v F$ is a formula.
- NOTHING ELSE IS A FORMULA.

**Definition** (Bound). An occurrence of variable $x$ is bound if it is part of a subformula beginning $\forall x$ or $\exists x$, in which case the quantifier $\forall$ or $\exists$ in question is said to bind that occurrence of the variable $x$, and otherwise the occurrence of the variable $x$ is free.
Ex: $Fx \rightarrow \forall x Fx$ : The first x is free, and the second is bound.
Ex: $x < y \wedge -\exists z(x < z \wedge z < y)$: all occurrences of $x$ and $y$ are free, and all the occurrences of $z$ are bound.

**Remark.** When we write something like "Let $F(x)$ be a formula", we are to be understood as meaning "Let $F$ be a formula in which no variables occur free except $x$".

**Definition** (Instance). An *instance* of a formula $F(x)$ is any formula of the form $F(t)$ for $t$ a closed term. Similar notations apply where there is more than one free variable, and to terms as well as formulas.

**Definition** (Sentence). a formula is a sentence if no occurrence of any variable in it is free.

**Definition** (Model). A model $M$ (interpretation) of a language $L$ is $\{|M|v\}$ Where $|M|$ is a non-empty set and $v$ is a valuation function that assigns values (extensions/denotations) to the members of $L$ in such a way that

- $\forall v(a) \in |M|$
- $v(A^n) \subseteq$ the $n$th Cartesian product of $|M|$ with itself: $|M| \times \ldots |M|$.
- $v(f^n)$ is a total function from $|M| \times \ldots |M|$ to $|M|$.

**Definition** (Truth).

- For some predicate $R$: $M \vDash R(t_1, \ldots, t_n)$ iff $R^M(t_1^M, \ldots, t_n^M)$
- For Identity function (when present): $M \vDash= (t_1, t_2)$ iff $t_1^M = t_2^M$.
- $M \vDash F^n(t_1, \ldots, t_n)$ iff $< M(t_1), \ldots, M(t_n) > \in M(F^n)$.
  Textbook writes this as: $(f(t_1, \ldots t_n))^M = f^M(t_1^M, \ldots, t_n^M)$. Ie: Interpretation $M$, makes true $f^n(t_1, \ldots, t_n)$ (An n-place function), iff $\{t_1^M, \ldots, t_n^M\} \in M(F^n)$
- $M \vDash -S$ iff $M \nvDash S$
- $M \vDash (K \wedge L)$ iff $M \vDash K$ and $M \vDash L$
- $M \vDash (K \vee L)$ iff $M \vDash K$ or $M \vDash L$

- $M \vDash F[m]$ iff $M^c_m \vDash F(c)$
  'If we considered the extended language $L \cup \{c\}$ obtained by adding a new constant $c$ in to our given language $L$, and if among all the extensions of our given interpretation of this extended language we considered the one $M^c_m$ that assigns $c$ the denotation $m$, then $F(c)$ would be true.'
- $M \vDash \forall x F(x)$ iff for every m in the domain , $M \vDash F(m)$
- $M \vDash \exists x F(x)$ iff for some m in the domain , $M \vDash m)$

**Definition.** of the denotation/extension of a closed term in a model M. If $T$ is a name $M(t) = v(t)$ If $t$ is $f^n(t_1, \ldots, t_n)$ then
$M(f^n(t_1, \ldots, t_n)$ $M(f^n)(M(t_1), \ldots, M(t_n))$.

Validity = Satisfiability = Implication.

Misc-crap:

- $A \vDash B$ is $-(A \wedge -B)$

**Lemma 1.** *Extensionality Lemma*

- *Let $M$ be a model of a language $L$.*
- *Let $S$ be a sentence of $L$.*
- *Let $L^+$ be an extension of $L$. $L \subseteq L^+$*
- *Let $M^+$ be a model of $L^+$*
- *So: $M^+$ is an extension of $M$.*
- *$M \vDash S$ iff $M^+ \vDash S$*

Example:
If $A \vDash B$ and $B \vDash C$, then $A \vdash C$. Suppose $A \vDash B$, and $B \vDash C$.
In every interpretation of $A$ and $B$ in which $A$ is true, $B$ is true, In every interpretation of $B$ and $C$ in which $B$ is true, $C$ is true. Shows: In every interpretation of $A$ and $C$ in which $A$ is true, $C$ is true.
Let $M$ be an interpretation of $A$ and $C$ such that $M \vDash A$.

- Case 1: $M$ is an interpretation of $B$.
  Then $M \vDash B$
  So $M \vDash C$.

- $M$ is not an interpretation of $B$.
  Then there is an extension $M^+$ that interprets $B$ as well as $A$ and $C$.
  so: $M^+ \vDash B$
  So: $M^+ \vDash C$
  So $M \vdash C$ (By the ext, lemma)

**Lemma 2** (Undecibality)**.** *If the decision problem (for implication) is solvable, then the halting problem is solvable. There is an effective method for specifying for any Turing machine $M$ and any input $N$ a finite set of sentences $\Delta$ and a sentence $H$ such that $\Delta \vDash H$ iff $M$ eventually halts after starting with input $n$.*
*$\Delta \vDash H$ iff $M$ eventually alts after start with input $n$.*

*Define the one place predicate $Q_i j$ as: At time $j, M$ is in state $i$. Define the two place predicate $@js$ as At time $j, M,$ is scanning square $s$. Define the two place predicate $Mjs$ as: At time $j$, square $s$ is marked with a 1.*
*A description $D$ for a start state could then be: $D : [Q_1 0 \wedge @_{0,0} \wedge M_{0,0} \wedge M_{0,1} \wedge M_{0,2} \wedge \forall y((y \neq 0 \wedge y \neq 1 \wedge y \neq 2) \implies -M_{0,y}]$ Time = 0, $[\ldots 01110 \ldots]$ Square #, $[\cdots - 10123 \ldots]$*

For each instruction of a $TM$, we may write the instruction as a sentence:
$Q_{i1}S_1RQ_{i2}$ : Move right seeing 1 in state:
$\forall x\forall t((Q_{i,1} \wedge @_{t,x} \wedge M_{t,x}) \implies (Q_{i2,(t+1)} \wedge @_{(t+1),(x+1)} \wedge \forall y((M_{t,y} \implies M_{(t+1),y}) \wedge (-M_{t,y} \implies -M_{(t+1),y}))$
Misc-crap that's on the board for some reason:

| | |
|---|---|
| $\Delta$ | |
| $\mathbb{Q}_2^1$ : | $\forall x\forall y\forall z((Sxy \wedge Sxz) \implies y = z)$ |
| $@^2$ : | $\forall x\forall y\forall z((Sxz \wedge Syz) \implies x = y)$ |
| $M^2$ : | $\forall x\forall y(Sxy \implies x < y)$ |
| $0$ : | $\forall x\forall y\forall z((x < y \wedge y < z) \implies x < z)$ |
| $S^2$ : | $-\exists x, x < x{+}1$ |
| $<^2$ : | less than |

Some thing else now too:
$\forall x\forall t((Q_1 t \wedge @tx \wedge Mtx) \implies \exists u(s_1(t,u) \wedge Q_2(u) \wedge @(u,v)) \wedge \forall y((M(t,y) \implies M(u,y)) \wedge (-M(t,y) \implies -M(u,y)))))$
$\exists x\exists t(Q_m t \wedge @tx \wedge Mtx)$

*Proof.* Something about biconditonal:
$\Delta \vDash H$ iff $M$ halts after starting with input n.

1. if $\Delta \vDash H$, then $M$ halts.
2. if $M$ halts, then $\Delta \vDash H$.

1. if $\Delta \vDash H$, then $M$ halts - Proof.
   Suppose $\Delta \vDash H$
   All members of $\Delta$ are true in the standard interpretation I. $H$ is true in $I$. So: $M$ halts.
2. if $M$ halts, then $\Delta \vDash H$ - Proof.
   Suppose $M$ halts. (Show $\Delta \vDash H$).
   There is a time $\underline{t}$ $M$ halts at t.
   There is a state $q_i, M$ halts at $t$ in state $q_i$.
   There is a square $x$, $M$ halts at $t$ in state $q_i$, scanning square $x$ which is Marked / $-$ Marked
   1: $Q_i(t) \wedge @(t,x) \wedge M(t,x)$.
   1 is a conjunct of the description of time t, $\mathbb{D}(t)$
   $\mathbb{D}(t) \vDash (i)$

   2: $\exists x\exists t(Q_1(t) \wedge @(t,x) \wedge (t,x))$.

   $(i) \vDash (ii)$.
   $(ii)$ is disjunct of $H$.
   $(ii) \vDash H$.
   So: $\mathbb{D}(t) \vDash H$.

   $\Delta$ implies a description of every time before which $M$ did not halt.
   $\forall n($ if $M$ has not halted before time n, then $\Delta \vDash \mathbb{D}(n)$

□

Relisting it:

*The decision problem for implication, is unsovable if turings thesis is true.* $\Delta \vDash H$ iff $M$ eventually halts.
Only if: $\Delta \vDash H$ only if $M$ eventually halts.
if: if $M$ eventually halts then $\Delta \vDash H$.
Suppose: $M$ halts at $t$
$\mathbb{D}(t) \vDash (H)$
Show: $\Delta \vDash \mathbb{D}(t)$
Something inductive: Base step: if $M$ has not halted before time 0, then $\Delta \vDash \mathbb{D}(0)$: Inductive step: $\forall n$( if $M$ has not halted before time $n$, then $\Delta \vDash \mathbb{D}(n)$ only if if$M$ has not halted before time $n$+!, then $\Delta \vDash \mathbb{D}(n+1)$)
Conclusion: $\forall n$ (if $M$ has not halted before time $n$, then $\Delta \vDash \mathbb{D}(n)$.
Proving the Inductive step:

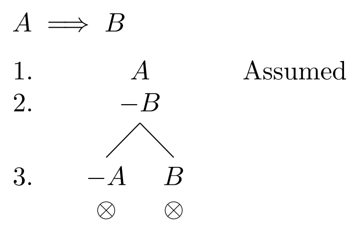
- Suppose if $M$ has not halted before time $n$, then $\Delta \vDash \mathbb{D}(n)$.
- (Show : if $M$ has not halted before time $n$+!, then $\Delta \vDash \mathbb{D}(n+1)$)
- Suppose: $M$ has not halted before time $n+1$
- So: $M$ has not halted before time $n$
- So: $\Delta \vDash \mathbb{D}(n)$.
- (Show: $\Delta \cup \{\mathbb{D}(n)\} \vDash \mathbb{D}(n+1)$)
- $\mathbb{D}(n) : Q_j n \wedge @_{n,k} \wedge M_{n,k} \ldots$
- $\forall t \forall x ((Q_j(t) \wedge @tx, \wedge Mtx) \implies Q_i t+1 \wedge @(t+1)(x+1) \wedge \ldots)$
- $(Q_j n \wedge @nk \wedge Mnk) \implies Q_i(n+1) \wedge @(n+1)(k+1) \wedge \ldots)$
- $\mathbb{D}(n+1) : Q_i(n+1) \wedge @(n+1)(j+1) \wedge \ldots$

□

Then some random ramblings about logicians. happened. Here's a book: "What is the name of this book?". There are methods of validity, and truth trees, and some other stuff. He's gonna use this. He's gonna prove this has some properties for validity.

## 0.2 Truth Trees

Ex: $\frac{(A \implies B)A}{B}$ is



Giving $\{(A \implies B), A, -B\}$

He then did all these,( which seem to be rules):
$\alpha$

1. $\alpha$
2. $-\alpha$

$\otimes$

$\alpha \wedge \beta$

1. $(\alpha \wedge \beta)$ ✓ Assumed
2. $\alpha$
3. $\beta$

$-(\alpha \wedge \beta$

1. $-(\alpha \wedge \beta)$ ✓
2. $-\alpha \quad -\beta$

$$\begin{tikzcd} & -(\alpha \wedge \beta) \arrow[dl, dash] \arrow[dr, dash] & \\ -\alpha & & -\beta \end{tikzcd}$$

$\alpha \vee \beta$

1. $(\alpha \vee \beta)$ ✓
2. $\alpha \quad \beta$

$$\begin{tikzcd} & (\alpha \vee \beta) \arrow[dl, dash] \arrow[dr, dash] & \\ \alpha & & \beta \end{tikzcd}$$

$-(\alpha \vee \beta)$

1. $-(\alpha \vee \beta)$ ✓
2. $-\alpha \quad -\beta$

$$\begin{tikzcd} & -(\alpha \vee \beta) \arrow[dl, dash] \arrow[dr, dash] & \\ -\alpha & & -\beta \end{tikzcd}$$

$-\forall x x \varnothing$

1. $-\forall x x \varnothing$ ✓
2. $\exists x - \varnothing$

$-\exists x \varnothing$

1. $-\exists x \varnothing$ ✓
2. $x - \varnothing$ ✓

$\exists x \varnothing[x]$ Note: $c$ is a new name on the path.

1. $\exists x \varnothing[x]$ ✓
2. $\varnothing[c]$ ✓

$\forall x \varnothing[x]$ Note: $c$ is a name already on the path, unless there are no names by the path and in that

1. $\forall x \varnothing[x]$
2. $\varnothing[c]$

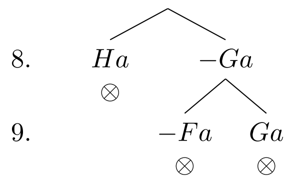
case, $c$ is new.

$\forall x(Fx \rightarrow Gx) \wedge \forall x(Gx \rightarrow Hx)$

1. $\forall x(Fx \rightarrow Gx)$
2. $\forall x(Gx \rightarrow Hx)$
3. $-Ha$
4. $-Fa$

Which is INVALID.

| | | |
|---|---|---|
| 1. | $\forall x(Fx \to Gx)$ | assumed |
| 2. | $\forall x(Gx \to Hx)$ | assumed |
| 3. | $-Ha$ | assumed |
| 4. | $--Fa$ | assumed |
| 5. | $Fa$ | |
| 6. | $(Ga \to Ha)$ | |
| 7. | $(Fa \to Ga)$ | |



Another same-case example: $\frac{\forall x(Fx\to Gx)\exists xGx}{\exists xFx}$ is

$$\begin{array}{lcl}
1. & \forall x(Fx \to Gx) & \text{assumed}\\
2. & \exists xGx \checkmark & \text{assumed}\\
3. & -\exists xFx \checkmark & \text{assumed}\\
4. & \forall x - Fx & \\
5. & Ga & \\
6. & -Fa & \\
7. & Fa \to Ga \checkmark & \\
 & \diagup \quad \diagdown & \\
8. & -Fa \quad\quad Ga &
\end{array}$$

Finding a case of invalidity from a proof tree:

1. take a path
2. get all values on the path
3. The domain consists of these values
4. You assign values to the predicates bottom-up, such that the tree's path is true.

Ex from the above same-case example: We follow the path down to $Ga$. Then:

1. $v(a) = 1$
2. $|M| = \{1\}$
3. $v(F) = \varnothing$
   $v(G) = \{1\}$

The order of applying rules:

1. TF
2. NEGQ
3. EI
4. UI
5. GOTO 1 (Until nothing is done)

**Remark.** Sometimes, this never finishes. But that's due to the undecidiblity of things...