Notes Section 5.2 – Inverse Functions and Their Representations

Lesson Objectives

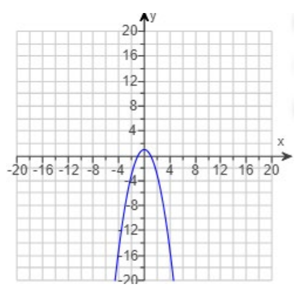
1. Determine if a Function is One-to-One
   a. From a Graph
   b. From a Formula
   c. From a Table
2. How to Find the Formula for an Inverse Function
3. Understand Inverse Function in Applications
4. How a Function and Its Inverse Look Graphically

A. Determine if a Function is **One-to-One** (1:1)

One-to-One means every *y*-coordinate has exactly **ONE** *x*-coordinate paired with it.

- All *y*-coordinates are **different**. The *y*-coordinates can't **repeat**.
- Easy to check graphically.
- Use **Horizontal Line Test** (HLT): maintain exactly **one** point of contact to be 1:1.
- If given formula, when in doubt – GRAPH IT OUT! Then use HLT.

- **EXAMPLE:** Use the horizontal line test to determine whether the function is one-to one. [5.2.15]



Is the function one-to-one?

NO – it **fails** the horizontal line test.

A random horizontal line through this function would have more than one point of contact = FAIL. (not one-to-one)

Notice in the example above that the end behavior of the tails is the same (down-down).

What DEGREE function does this? **EVEN** degree

In fact, ***all*** **even-degree functions are NOT one-to-one**! They FAIL the Horizontal Line Test (HLT).

What is another type (family) of function that is also NOT 1:1 (fails HLT)? **Absolute Value**
It's a V-shaped graph, so that certainly will fail the Horizontal Line Test.

There's yet another type of function that is also NOT 1:1 (fails HLT): **Constant Function**
It's a horizontal line, which of course fails the Horizontal Line Test. (has no variable)

We can summarize this on the next page.

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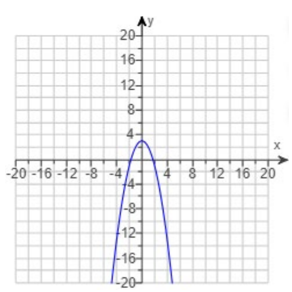
When is a Function One-to-One (1:1)? (When in doubt – **GRAPH** IT OUT !!)		
NEVER 1:1	**ALWAYS 1:1**	***SOMETIMES* 1:1 (Check its Graph)**
Even-degree functions	**Linear** functions	**Odd**-degree functions
Absolute Value functions	**Radical** functions (roots – square root, cube root, etc.)	**Rational** functions
Constant functions (**horizontal** line)		

- **Practice:** Determine if the following functions are one-to-one. (When in doubt, GRAPH IT OUT!)

1. $f(x) = x^2 - 4$
2. $f(x) = 4x - x^3$
3. $f(x) = |49 - x^2|$
4. $f(x) = 4x^2 - 2$
5. $f(x) = 6\sqrt{x}$
6. $f(x) = 6x^2 + x$
7. $f(x) = 5.37$
8. $f(x) = -9x^2$
9. $f(x) = 2x - 7$
10. $f(x) = -8x^4$
11. $f(x) = |x + 3|$
12. $f(x) = x^3 - 7$
13. $f(x) = \frac{3}{1+x^2}$
14.



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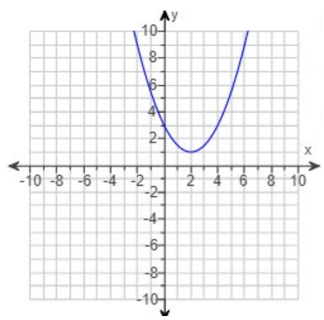

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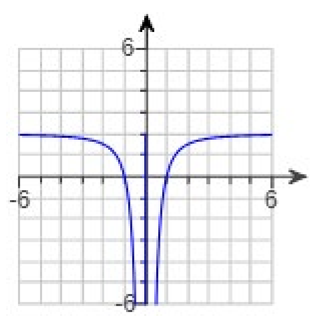

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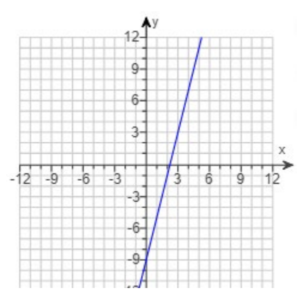

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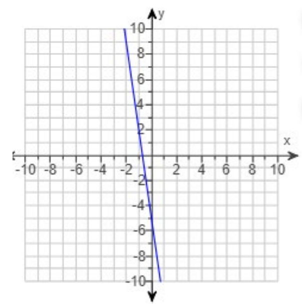

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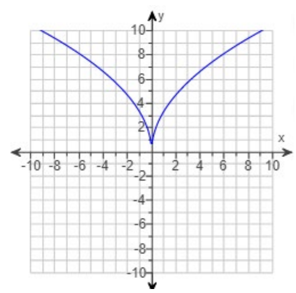
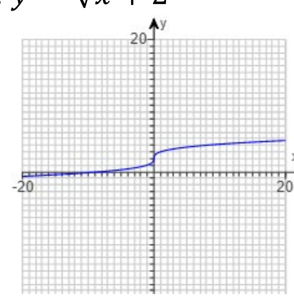


20. $y = \sqrt[3]{x} + 2$



1.N 2.N 3.N 4.N 5.Y
6.N 7.N 8.N 9.Y 10.N
11.N 12.Y 13.N 14.N 15.N
16.N 17.Y 18.Y 19.N 20.Y

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- **EXAMPLE:** The table is a complete representation of *f*. Use the table to determine if *f* is one-to-one and has an inverse. [5.2.23]

x	0	3	6	9	12
***f*(*x*)**	– 5	5	6	4	3

Choose the correct answer below.

A. The function *f* is one-to-one and has an inverse.
B. The function *f* is not one-to-one and does not have an inverse.
C. The function *f* is one-to-one and does not have an inverse.
D. The function *f* is not one-to-one and has an inverse.

One-to-one means that all *y*-coordinates must be **different** (*y*-coordinates can't repeat).

In the table above, all of the *y*-coordinates are different, so **YES** – it **is** one-to-one.

We also need to know the relationship between: One-to-one and Inverse.

Both of the following statements are TRUE:

-	If a function	**is**	one-to-one, then it	**does**	have an inverse.
-	If a function	**is not**	one-to-one, then it	**does not**	have an inverse.

In the answer choices above, answers C and D are never true (mismatched wording).

In general, answers A and B are true statements, but the function is 1:1. Correct answer: **A**

B. How to Find the Formula for an Inverse Function

Consider a function, $f(x)$.

The **INVERSE** function of $f(x)$ has the notation $\boldsymbol{f^{-1}(x)}$

and it is read as: "*f*-inverse of *x*" or more simply: **"*f*-inverse"**.

The following are different wordings that ask the same question of

"how to find the inverse function":

- Find $f^{-1}(x)$.
- Find a symbolic representation for $f^{-1}(x)$.
- The function $f(x)$ is one-to-one. Find its inverse.
- Find a formula for $f^{-1}(x)$.

All four of those statements above are asking for the same thing.

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Given a function $f(x)$, here's **how to find $f^{-1}(x)$**, the **inverse** function:

1. **REPLACE** $f(x)$ with its easier form, y.

2. **SWAP** the variables x and y.
   (this is fundamentally what happens with an inverse function)

3. **SOLVE** the equation for y.

4. **RENAME** this y as $f^{-1}(x)$, the inverse function of $f(x)$.

- **EXAMPLE:** Find a symbolic representation for $f^{-1}(x)$, given that $f(x) = 8x + 5$. [5.2-16]

A. $f^{-1}(x) = \frac{x}{8} - 5$
B. $f^{-1}(x) = \frac{x+5}{8}$
C. $f^{-1}(x) = \frac{x-5}{8}$
D. Not a one-to-one function.

Because $f(x)$ is a linear function, it is one-to-one, so D can be ruled out as incorrect.

Given $f(x) = 8x + 5$, go through the 4-step process to get the inverse $f^{-1}(x)$:

1. **REPLACE.** $y = 8x + 5$ Replace $f(x)$ with its easier form, y.

2. **SWAP.** $x = 8y + 5$ Swap the variables x and y.

3. **SOLVE.** Solve the equation for y. Subtract 5 first.

$$\begin{aligned} -5 \qquad & -5 \\ x - 5 &= 8y \end{aligned}$$

Divide all terms by 8.

$$\frac{x}{8} - \frac{5}{8} = \frac{8y}{8}$$

$$y = \frac{x}{8} - \frac{5}{8}$$ or if you prefer $$y = \frac{x-5}{8}$$ (always reduce fractions, if you can!)

4. **RENAME.** $f^{-1}(x) = \frac{x-5}{8}$ Rename y as the inverse function, $f^{-1}(x)$.

Correct answer, therefore, is **C**.

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C. Understand Inverse Function in Applications

- **EXAMPLE:** Let $f(x)$ compute the cost of a rental car after *x* days of use at \$23 per day. What does $f^{-1}(x)$ compute? [5.2-48]

  A. The number of days rented for x dollars.
  B. The cost of rental for 23 days.
  C. The cost of rental for x days.
  D. The number of days rented for 23 dollars.

You just need to keep track of your variables (who's who) and then SWAP roles for inverse $f^{-1}(x)$:

Given function. $f(x)$: **cost** x: **days** Now, SWAP roles (inverse).

Inverse function. $f^{-1}(x)$: **days** x: **cost** Correct answer: **A**

NOTE: The \$23 per day is not relevant in describing the inverse! (Just ignore it)

- **EXAMPLE:** To remodel a bathroom, a contractor charges \$25 per hour plus material costs, which amount to \$3950. Therefore, the total cost to remodel the bathroom is given by $f(x) = 25x + 3950$, where *x* is the number of hours the contractor works.

  Find a formula for $f^{-1}(x)$. What does $f^{-1}(x)$ compute? [5.2-49]

A. $f^{-1}(x) = \frac{x}{25} - 158.$ This computes the total cost if the contractor works *x* hours.
B. $f^{-1}(x) = \frac{x}{25} - 3950$ This computes the number of hours worked if the total cost is *x* dollars.
C. $f^{-1}(x) = \frac{x}{25} - 158$ This computes the number of hours worked if the total cost is *x* dollars.
D. $f^{-1}(x) = \frac{x}{25} - 3950$ This computes the total cost if the contractor works *x* hours.

SOLUTION – part 1

First, you should keep track of your variables (who's who) and then SWAP roles for inverse $f^{-1}(x)$:

Given function. $f(x)$: **total cost (dollars)** x: **hours worked**

Now, SWAP roles (inverse).

Inverse function. $f^{-1}(x)$: **hours worked** x: **total cost (dollars)**

Incorrect answers: **A** and **D** have incorrect wording for inverse $f^{-1}(x)$.

The variables did not swap roles – they are the **same** as the given function $f(x)$.

Notes Section 5.2 – Inverse Functions and Their Representations

(continued from previous page – here's the problem again for reference:)

- **EXAMPLE:** To remodel a bathroom, a contractor charges $25 per hour plus material costs, which amount to $3950. Therefore, the total cost to remodel the bathroom is given by $f(x) = 25x + 3950$, where *x* is the number of hours the contractor works.

Find a formula for $f^{-1}(x)$. What does $f^{-1}(x)$compute? [5.2-49]

~~A. $f^{-1}(x) = \frac{x}{25} - 158$. This computes the total cost if the contractor works *x* hours.~~

B. $f^{-1}(x) = \frac{x}{25} - 3950$ This computes the number of hours worked if the total cost is *x* dollars.

C. $f^{-1}(x) = \frac{x}{25} - 158$ This computes the number of hours worked if the total cost is *x* dollars.

~~D. $f^{-1}(x) = \frac{x}{25} - 3950$ This computes the total cost if the contractor works *x* hours.~~

SOLUTION – part 2 Now we move on to: **find the formula for the inverse $f^{-1}(x)$.**

Given $f(x) = 25x + 3950$, go through the 4-step process to get the inverse $f^{-1}(x)$:

1. **REPLACE.** $y = 25x + 3950$ Replace $f(x)$ with its easier form, y.
2. **SWAP.** $x = 25y + 3950$ Swap the variables x and y.
3. **SOLVE.** $-3950 \qquad -3950$ Solve the equation for y. Subtract 3950 first.

   $x - 3950 = 25y$ Divide by 25.

   $\frac{x}{25} - \frac{3950}{25} = \frac{25y}{25}$ (Divide each term by 25 and simplify.)

   $\frac{x}{25} - \frac{3950}{25} = y$ Simplify fraction $\frac{3950}{25} = 158$

   $y = \frac{x}{25} - 158$ Rewrite with *y* on the left side of equation.
4. **RENAME.** $f^{-1}(x) = \frac{x}{25} - 158$ Rename y as the inverse function, $f^{-1}(x)$.

(Here's the question again) Find a formula for $f^{-1}(x)$. What does $f^{-1}(x)$compute?

Correct answer, therefore, is C.

$f^{-1}(x) = \frac{x}{25} - 158$ This computes the number of hours worked if the total cost is *x* dollars.

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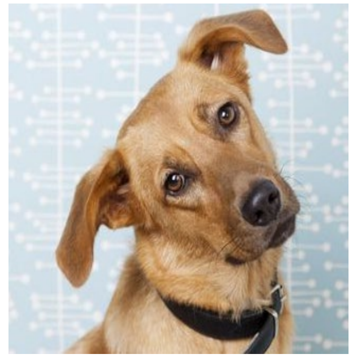
D. How a Function and its Inverse Look Graphically

Remember that to form an inverse function, you need to **SWAP** ***x*** **and** ***y***.

(Tilt your head sideways like a dog who's heard a strange sound.)



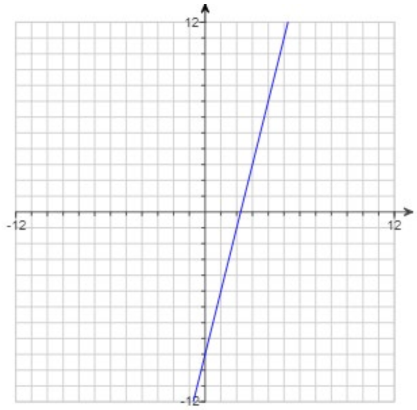
The same is true in a graph.

A function $f(x)$ and its inverse $f^{-1}(x)$ will display

symmetry about the line $y = x$.

The line $y = x$ is called the *identity* function.

- **EXAMPLE:** Use the graph of $y = f(x)$ below to sketch the graph of $y = f^{-1}(x)$. [5.2.127]



To find the correct graph of the inverse function $f^{-1}(x)$, make a table of a few points. It's okay if they are just estimates; as long as they are "pretty close," that's fine.

$\boldsymbol{x}$	2	0	3
$\boldsymbol{y = f(x)}$ solid BLUE line	0	-9	3

Now SWAP *x* and *y* to make the table for the inverse function $f^{-1}(x)$:

$\boldsymbol{x}$	0	-9	3
$\boldsymbol{y = f^{-1}(x)}$ dotted RED line	2	0	3

The inverse function (dotted line) should have all 3 points in the table for $f^{-1}(x)$.
Be careful with the SCALE on the graphs!

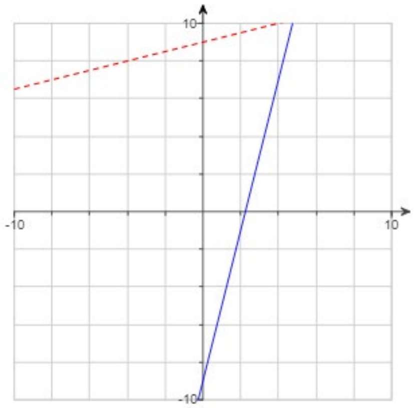
(continued on the next page)

(continued from the previous page)

Let's examine the available answers. All four answers include the solid blue line in the same position, which is the original function, $y = f(x)$.
We are looking for the sketch of the inverse function, $f^{-1}(x)$, the red dashed line.
Here's our sample table (from the previous page) for the inverse function:

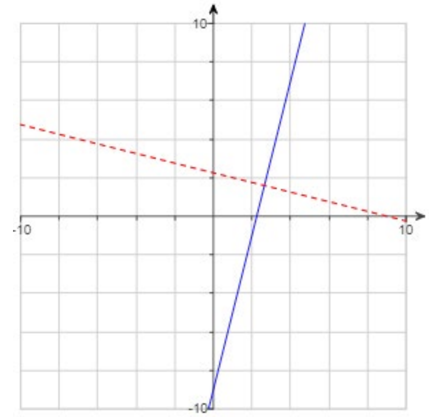
$\boldsymbol{x}$	0	-9	3
$\boldsymbol{y = f^{-1}(x)}$ dotted RED line	2	0	3

Answer **A.**



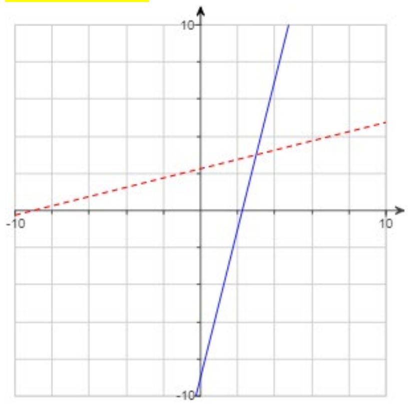
This is INCORRECT because it is missing all 3 points that the inverse $f^{-1}(x)$ should have.

Answer **C.**



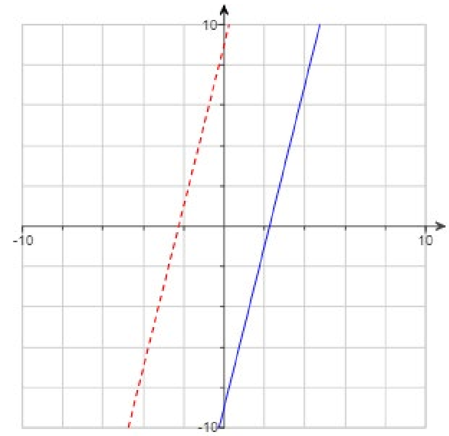
This is INCORRECT – it's missing $(-9,0)$ and $(3,3)$ that the inverse $f^{-1}(x)$ should have.

Answer **B**.



This is **CORRECT**.

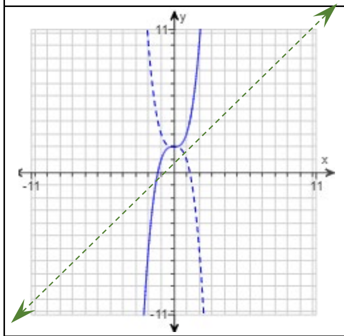
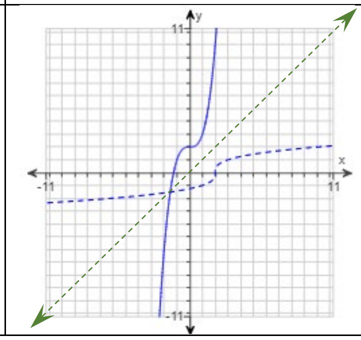
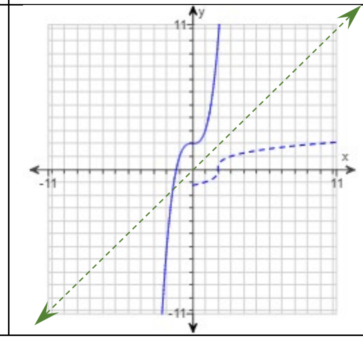
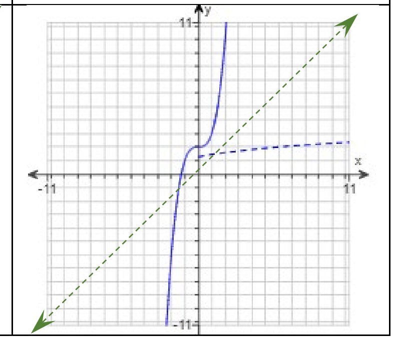
Answer **D.**



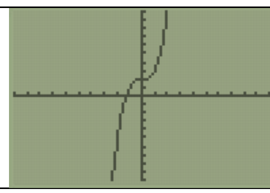
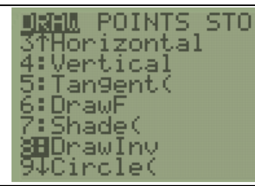
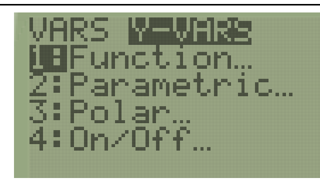
This is INCORRECT because it is missing all 3 points the inverse $f^{-1}(x)$ should have.

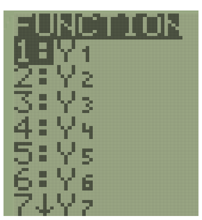
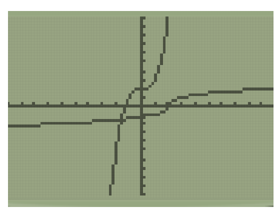
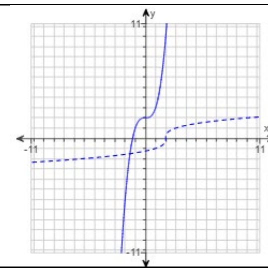
Notes Section 5.2 – Inverse Functions and Their Representations

- **EXAMPLE:** Graph the function as a solid curve and its inverse as a dotted curve.
  $f(x) = x^3 + 2$ [5.2-45]

A.	**B.**	**C.**	**D.**
			

When you are given the formula for your function – go to the graphing calculator!
We are going to utilize the **Draw Inverse** command (**DrawInv**).

			
1. Enter *f*(*x*) into Y1. Graph by pressing **ZOOM 6**.	2. Press **2ND, PRGM** (DRAW)	3. Choose number **8**:DrawInv	4. Press **VARS, →,** (Y-VARS), **ENTER** or you can press: **ALPHA, TRACE, ENTER**

			
5. Press **ENTER** again to select Y1	6. You should see this command DrawInv Y1 on the screen. Press **ENTER**.	7. The function and its inverse are shown together.	8. Correct answer is: **B**

Sources Used:

1. MyLab Math for *College Algebra with Modeling and Visualization*, 6th Edition, Rockswold, Pearson Education Inc.
2. Wabbitemu calculator emulator version 1.9.5.21 by Revolution Software, BootFree ©2006-2014 Ben Moody, Rom8x ©2005-2014 Andree Chea. Website https://archive.codeplex.com/?p=wabbit