Notes Section 6.4 – Solutions to Linear Systems Using Matrices

Lesson Objectives

1. State dimensions of a matrix.
2. Represent a linear system with an augmented matrix (and vice-versa).
3. Solve a linear system using reduced-row echelon form (rref) on calculator.
4. Determine if an ordered pair or ordered triple is a solution to a linear system of equations.
5. Solve applications with a linear system of equations and matrix rref on calculator.

matrix – a rectangular array of elements, typically surrounded by large brackets

dimension of a matrix – **rows** × **columns** (RC)

A. State the **Dimensions** of a Matrix

- **EXAMPLE:** State the dimensions of the matrix. [6.4-1]

$$\begin{bmatrix} 5 & 5 & -4 \\ 5 & -2 & 9 \end{bmatrix}$$

The dimensions are: **2 × 3**, ("two by three") which means 2 rows by 3 columns.

B. Represent a Linear System with an **Augmented Matrix** (and vice-versa)

augmented matrix – a matrix comprised of all coefficients and constants from a linear system written in STANDARD form.

- **EXAMPLE:** Write the augmented matrix for the system. [6.4-11**]**

$$\begin{cases} 3x + 2z = 45 \\ 3y + 8z = 81 \\ 2x + 3y + 8z = 99 \end{cases} \text{(re-write with zeros, as needed)} \rightarrow \begin{cases} 3x + \mathbf{0}y + 2z = 45 \\ \mathbf{0}x + 3y + 8z = 81 \\ 2x + 3y + 8z = 99 \end{cases}$$

✓ Line up variables and constants (**standard** form).
✓ Insert **zeros** as placeholders, if needed.
✓ Use only coefficients and constants.
✓ Leave behind variables and equals.
✓ Put large brackets on outside.
✓ Use vertical bar for all the equals signs.

$$\begin{cases} 3x + 0y + 2z = 45 \\ 0x + 3y + 8z = 81 \\ 2x + 3y + 8z = 99 \end{cases} \text{(write as an augmented matrix)} \rightarrow \left[\begin{array}{ccc|c} \mathbf{3} & \mathbf{0} & \mathbf{2} & \mathbf{45} \\ \mathbf{0} & \mathbf{3} & \mathbf{8} & \mathbf{81} \\ \mathbf{2} & \mathbf{3} & \mathbf{8} & \mathbf{99} \end{array}\right]$$

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- **EXAMPLE:** Write the system of equations that the augmented matrix represents.
  [6.4-15]

$$\left[\begin{array}{ccc|c} 9 & 9 & 7 & -2 \\ 5 & 0 & 4 & 4 \\ 7 & 4 & 0 & 2 \end{array}\right] \text{ (write as a linear system)} \rightarrow \begin{cases} 9x + 9y + 7z = -2 \\ 5x + 4z = 4 \\ 7x + 4y = 2 \end{cases}$$

  - ✓ First column represents *x*, second is *y*, etc.
  - ✓ Last column is always for the constants.
  - ✓ Vertical line is for the equals signs.
  - ✓ Terms with zero don't need to be written.

C. Solve a Linear System Using Reduced-Row Echelon Form (rref) on calculator

- **EXAMPLE:** The augmented matrix below is in reduced row-echelon form and represents a system of equations. If possible, solve the system. [6.4.65]

What Reduced Row Echelon Form (rref) Looks Like

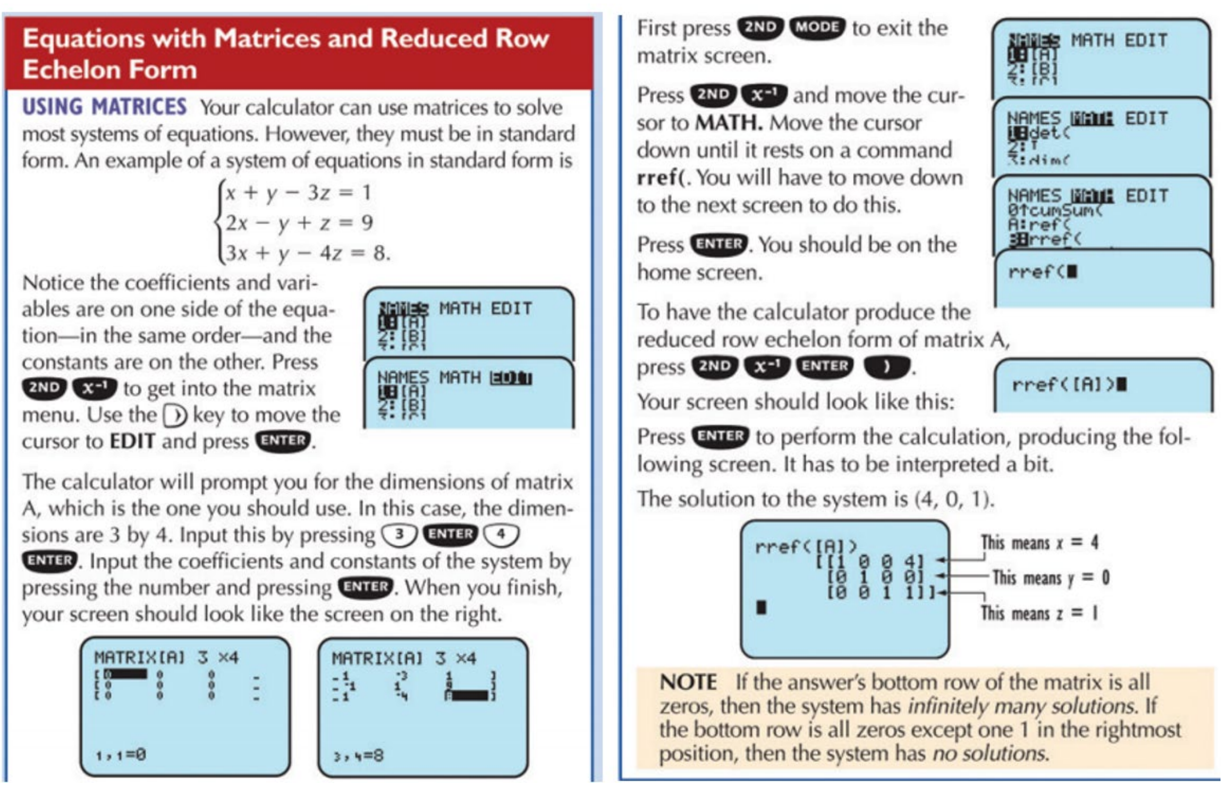
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{4}{5} \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{4}{5} \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{4}{5} \end{array}\right]$$

Given rref matrix — 1's along diagonal — Zeros elsewhere

Solution is in the **LAST column** of the matrix, when in reduced row-echelon form (rref).
Translating this rref matrix back into its equation format:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{4}{5} \end{array}\right] \text{ (re-write as a linear system) } \begin{cases} x = -1 \\ y = 3 \\ z = \frac{4}{5} \end{cases}$$

The solution is $\left(-1, 3, \frac{4}{5}\right)$.



Equations with Matrices and Reduced Row Echelon Form

USING MATRICES Your calculator can use matrices to solve most systems of equations. However, they must be in standard form. An example of a system of equations in standard form is

$$\begin{cases} x + y - 3z = 1 \\ 2x - y + z = 9 \\ 3x + y - 4z = 8. \end{cases}$$

Notice the coefficients and variables are on one side of the equation—in the same order—and the constants are on the other. Press 2ND x^{-1} to get into the matrix menu. Use the ▷ key to move the cursor to **EDIT** and press ENTER.

The calculator will prompt you for the dimensions of matrix A, which is the one you should use. In this case, the dimensions are 3 by 4. Input this by pressing 3 ENTER 4 ENTER. Input the coefficients and constants of the system by pressing the number and pressing ENTER. When you finish, your screen should look like the screen on the right.

First press 2ND MODE to exit the matrix screen.

Press 2ND x^{-1} and move the cursor to **MATH.** Move the cursor down until it rests on a command **rref(**. You will have to move down to the next screen to do this.

Press ENTER. You should be on the home screen.

To have the calculator produce the reduced row echelon form of matrix A, press 2ND x^{-1} ENTER).
Your screen should look like this:

Press ENTER to perform the calculation, producing the following screen. It has to be interpreted a bit.

The solution to the system is (4, 0, 1).

NOTE If the answer's bottom row of the matrix is all zeros, then the system has *infinitely many solutions*. If the bottom row is all zeros except one 1 in the rightmost position, then the system has *no solutions*.

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- **EXAMPLE:** Use Gaussian elimination with backward substitution to solve following system of equations.

$$\begin{cases} -10x - 3y = -35 \\ 4x + y = 15 \end{cases} \qquad [6.4.35]$$

(NOTE: For the following question, solve the linear system using MATRIX method with **RREF**. Do **NOT** use the Gaussian elimination method as mentioned in the directions.
So, Question Help and/or Skill Builder links will NOT be appropriate for this question.)

$$\begin{cases} -10x - 3y = -35 \\ 4x + y = 15 \end{cases} \quad \text{(write as an augmented matrix)} \rightarrow \quad \left[\begin{array}{cc|c} -10 & -3 & -35 \\ 4 & 1 & 15 \end{array}\right]$$

Enter the augmented matrix in the calculator by pressing **2ND, x^{-1}**. (MATRIX)
The dimension of the matrix is **2 × 3**. Enter all the values into the matrix on the calculator.
Leave the matrix by pressing **2ND, MODE** (QUIT).
Go back into the MATRIX pressing **2ND, x^{-1}**. (MATRIX), **→** (MATH), **↓** (scroll down) **rref(, ENTER.**
Go back into the MATRIX one last time pressing **2ND, x^{-1}**. (MATRIX), **ENTER.**

So the solution is: **(5, – 5)**

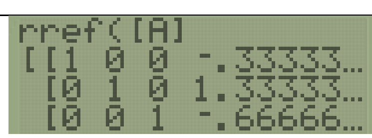
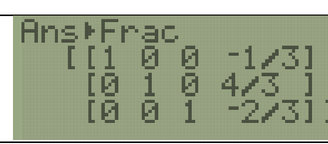
D. Determine if an Ordered Pair (or Triple) is a Solution to a Linear System

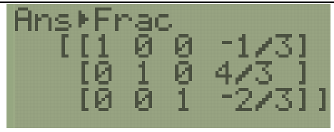
- **EXAMPLE:** Determine if the ordered triple $\left(-\frac{2}{3}, 1, -\frac{2}{3}\right)$ is a solution of the system of equations.

$$\begin{aligned} 9x - 9y - 6z &= -11 \\ 6x + 3y - 3z &= 4 \\ x - y + 2z &= -3 \end{aligned} \qquad [6.3\text{-}2]$$

One way to do this is similar to what we've done previously in Section 6.1, where we **plug in** the values for *x*, *y*, and *z* from the given point into each of the three equations. The video in the homework demonstrates this as well for your reference.
A much faster way is to determine the solution by using a MATRIX with rref:

$$\begin{aligned} 9x - 9y - 6z &= -11 \\ 6x + 3y - 3z &= 4 \\ x - y + 2z &= -3 \end{aligned} \quad \text{(write as an augmented matrix)} \quad \rightarrow \quad \left[\begin{array}{ccc|c} 9 & -9 & -6 & -11 \\ 6 & 3 & -3 & 4 \\ 1 & -1 & 2 & -3 \end{array}\right]$$

Enter your matrix into calculator, then double-check your matrix by pressing **2ND, x^{-1}**. (MATRIX), **ENTER**. Make corrections, if necessary.	Run the rref command on the calculator.	Press MATH, ENTER, ENTER to convert to fractions.
		



(carried over from previous page)

The solution given by rref is $\left(-\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}\right)$. Recall what the original problem asked:

Determine whether the ordered triple $\left(-\frac{2}{3}, 1, -\frac{2}{3}\right)$ is a solution of the system of equations.

rref matches given point = **YES**	rref doesn't match given point = **NO**

E. Solve **Applications** with a Linear System of Equations

- **EXAMPLE:** There were 40,000 people at a ball game in Los Angeles. The day's receipts were $330,000. How many people paid $12 for reserved seats and how many paid $6 for general admission? [6.1-62]

Step 1. Define your variables.

Let x = the number of **$12 reserved seats**

Let y = the number of **$6 general admission seats**

Step 2. Make your equations.

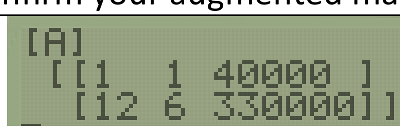
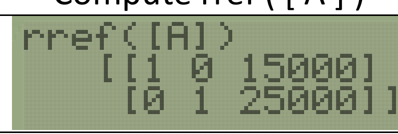
(Total **QUANTITY** equation) $\quad x + y = 40{,}000$

(Total **COST** equation) $\quad 12x + 6y = 330{,}000$

Step 3. Convert equations to an augmented matrix.

$$\begin{cases} x + y = 40{,}000 \\ 12x + 6y = 330{,}000 \end{cases} \quad \text{(write as an augmented matrix)} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 40000 \\ 12 & 6 & 330000 \end{array}\right]$$

Step 4. Enter augmented matrix (2 × 3) into calculator and do rref to get solution.

Confirm your augmented matrix	Compute rref ([A])
	

Step 5. Interpret your rref ([A]) solution correctly in context for the answer.

The solution is (15000, 25000), which means x = 15,000 and y = 25,000.

15,000 people purchased $12 reserved seats

25,000 purchased $6 general admission.

Sources Used:

1. Calculator Review Card, page 6 – Equations with Matrices and Reduced Row Echelon Form https://media.pearsoncmg.com/aw/aw_mml_shared_1/calculator_review_card.pdf
2. Pearson MyLab Math *College Algebra with Modeling and Visualization, 6th Edition*, Rockswold
3. Wabbitemu calculator emulator version 1.9.5.21 by Revolution Software, BootFree ©2006-2014 Ben Moody, Rom8x ©2005-2014 Andree Chea. Website https://archive.codeplex.com/?p=wabbit