

Notes Section R.7 – Simplifying Radical Expressions (Square Roots)

Lesson Objectives

1. Simplify a square root – Perfect Square method
2. Simplify a square root – Pairs and Spares method
3. Simplify square roots containing variables

A. Simplify a Square Root – Perfect Square Method

- Review of Perfect Squares

A **perfect square** is a number that has two identical factors.

To simplify square roots, it's really helpful if you know at least the first 15 perfect squares:
1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225

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- Simplify a Square Root – Perfect Square Method

Radicand – the value or amount inside the root

Index – the type of root you are taking

$$\text{index} \sqrt{\text{radicand}}$$

With a **square root**, the index of **2** is not written – it is omitted.

A **square root** is considered **simplified** if:
the **radicand** contains **NO** perfect square factors.

- **STEP 1.** Inside the square root, divide the radicand into two factors:
 - the **largest perfect square** that divides into the radicand
 - its “buddy” factor that goes with it
- **STEP 2.** Each of those factors gets its own square root, multiplied together.
- **STEP 3.** Simplify the perfect square root into its whole number.
- **STEP 4.** Leave the “buddy” factor inside the square root as the remaining reduced radicand.

Notes Section R.7 – Simplifying Radical Expressions (Square Roots)

- **EXAMPLE:** Simplify by factoring out the largest perfect square. [R.7.37]

$$\sqrt{192}$$

- **STEP 1.** Inside the square root, divide the radicand into two factors:
 - the **largest perfect square** that divides into the radicand
 - its “buddy” factor that goes with it

To find the **largest perfect square factor** of 192, you need to:

- Test the perfect squares by dividing 192 by each perfect square
- No decimals, no remainder
- You only need to test perfect squares to about half-way to 192, or 96

$\frac{192}{4} = 48$ Maybe – is 4 the largest?	$\frac{192}{9} \approx 21.3$ NO	$\frac{192}{16} = 12$ Maybe – is 16 the largest?
$\frac{192}{25} = 7.68$ NO	$\frac{192}{36} \approx 5.3$ NO	$\frac{192}{49} \approx 3.9$ NO
$\frac{192}{64} = 3$ Maybe – is 64 the largest?	$\frac{192}{81} \approx 2.4$ NO	$\frac{192}{100}$ 100 is more than half-way

- **STEP 1.**

Rewrite $\sqrt{192}$ as $\sqrt{64 \cdot 3}$

64 is the largest perfect square factor of 192

Its “buddy” factor is 3 because $64 \cdot 3 = 192$

- **STEP 2.** Each of those factors gets its own square root, multiplied together.

$$\sqrt{192} = \sqrt{64} \cdot \sqrt{3}$$

- **STEP 3.** Simplify the perfect square root into its whole number.

$$\sqrt{192} = 8 \cdot \sqrt{3}$$

- **STEP 4.** Leave the “buddy” factor inside the square root as the remaining reduced radicand.

$$\text{ANSWER: } \sqrt{192} = 8\sqrt{3}$$

You can easily verify that $\sqrt{192} = 8\sqrt{3}$ on your calculator.

Just verify the approximate decimal equivalents:

$\sqrt{192}$	13.85640646
$8\sqrt{3}$	13.85640646

Notes Section R.7 – Simplifying Radical Expressions (Square Roots)

- **EXAMPLE:** Simplify. $\sqrt{448}$ [*Angel 11.3.11]

448 has several perfect square factors, but we want the largest one.

$\frac{448}{4} = 112$	4 is not the largest perfect square factor because the remaining factor, 112, still divides down by at least the perfect square 4.
$\frac{448}{16} = 28$	16 is not the largest perfect square factor because the remaining factor, 28, still divides down by the perfect square 4.
$\frac{448}{64} = 7$	64 is the largest perfect square factor because the remaining factor, 7, does NOT divide down by any perfect squares.

- **STEP 1.**

Rewrite $\sqrt{448}$ as $\sqrt{64 \cdot 7}$

64 is the largest perfect square factor of 448

Its "buddy" factor is 7 because $64 \cdot 7 = 448$

- **STEP 2.** Each of those factors gets its own square root, multiplied together.

$$\sqrt{448} = \sqrt{64} \cdot \sqrt{7}$$

- **STEP 3.** Simplify the perfect square root into its whole number.

$$\sqrt{448} = 8 \cdot \sqrt{7}$$

- **STEP 4.** Leave the "buddy" factor inside the square root as the remaining reduced radicand.

$$\text{ANSWER: } \sqrt{448} = 8\sqrt{7}$$

You can easily verify that $\sqrt{448} = 8\sqrt{7}$ on your calculator.

Just verify the approximate decimal equivalents:

$\sqrt{448}$	21.16601049
$8\sqrt{7}$	21.16601049

Caution: Don't be too over-reliant upon the calculator!

For example, $\sqrt{448}$ also equals $4\sqrt{28}$; however, $4\sqrt{28}$ is not simplified because the radicand 28 still divides down by a perfect square, 4.

Notes Section R.7 – Simplifying Radical Expressions (Square Roots)

B. Simplify a Square Root – “Pairs and Spares” Method

The challenge with the **Perfect Square** method is that sometimes it's difficult to determine the largest perfect square factor because the radicand is either large or otherwise unfamiliar, or it may not simplify at all.

An alternate, sometimes EASIER and more CONSISTENT method is called the “Pairs and Spares” method, which utilizes a technique involving a **factor tree**, or the **prime factorization**.

- **Prime Factorization – make a Factor Tree**

Prime number: a whole number whose **only factors are 1 and itself**.

Composite number: a whole number that is **NOT prime**; it is **composed of prime factors**. It has additional factors besides 1 and itself.

Note that the number 1 is neither prime nor composite.

Prime factorization: an arrangement of **prime factors** whose product is a given number.
EVERY whole number (greater than 1) has a **UNIQUE** prime factorization.

Factor Tree: a systematic way to divide down a whole number into its unique prime factors, or is **prime factorization**.

STEP 1. “**branch-off**” the given number into 2 factors.

- If there's more than one way to have 2 factors, then you can simply choose whichever you prefer – it doesn't matter.

STEP 2. If either of the 2 factors is **prime**, then **circle** it.

STEP 3. If either of the 2 factors is **composite**, then “**branch-off**” of that number into 2 factors as well.

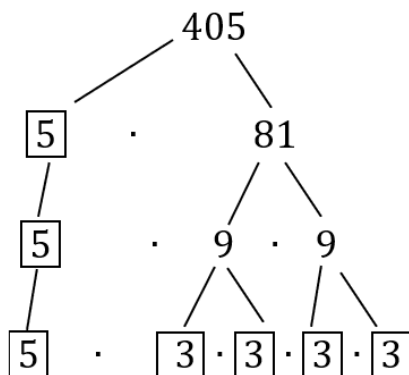
STEP 4. (If needed) Continue the process until your factor tree has nothing left but a collection of circled prime numbers.

STEP 5. Write the prime factorization:

- List all of the circled numbers together,
- Separated with a multiplication sign in between each factor

Notes Section R.7 – Simplifying Radical Expressions (Square Roots)

- (EXAMPLE): Use a factor tree to find the prime factorization of 405.



ANSWER: The prime factorization of 405 is: $3 \cdot 3 \cdot 3 \cdot 3 \cdot 5$ or $3^4 \cdot 5$

- Simplify a Square Root – “Pairs and Spares” Method

STEP 1. Get the **prime factorization** of the radicand using a **factor tree**.

STEP 2. Write the PF as the updated radicand inside the square root.

STEP 3A. **Circle** any **PAIR** of identical factors; that is, a perfect square.

- Each *pair* of identical factors *inside* the square root simplifies to a **single** factor **outside** the square root (to its LEFT).
- Do this for *each* identified pair of identical factors.

STEP 3B. **Underline** any remaining unpaired factors still in the radicand (*inside* the square root) – these are the **spares**.

STEP 4. **Multiply together** either the *outside* factors or the *inside* factors, if needed.

- EXAMPLE:** Simplify the expression. $\sqrt{405}$ [*Angel 11.3.19]

STEP 1. From the example above, the **prime factorization** of 405 is $3 \cdot 3 \cdot 3 \cdot 3 \cdot 5$

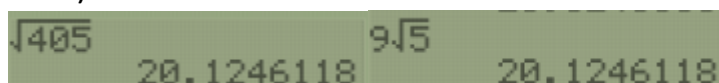
STEP 2. Update the **radicand**: $\sqrt{405} = \sqrt{3 \cdot 3 \cdot 3 \cdot 3 \cdot 5}$

STEP 3. **Circle pairs, Underline spares.** $\sqrt{\boxed{3 \cdot 3} \cdot \boxed{3 \cdot 3} \cdot \underline{5}}$

Each pair **simplifies** to a single: $3 \cdot 3\sqrt{5}$

STEP 4. **Multiply outside** factors: $9\sqrt{5}$
Multiply inside factors: (not needed)

ANSWER: $\sqrt{405}$ simplifies to $9\sqrt{5}$



Notes Section R.7 – Simplifying Radical Expressions (Square Roots)

C. Simplify a Square Root Containing Variables (“Pairs and Spares” Method)

You can simplify expressions with variables by using the basic definition of an **exponent**.

For example, you could write out the factors of x^5 as $x \cdot x \cdot x \cdot x \cdot x$ and then circle pairs similar to how you do with constants.

- EXAMPLE:** Simplify by factoring. Assume that all expressions under radicals represent nonnegative numbers. $\sqrt{x^{17}}$ [*Blitzer 10.3.39]

STEP 1 and 2. Prime factorization, update radicand.

Rewrite x^{17} in the radicand using definition of exponent:

$$\sqrt{x^{17}} = \sqrt{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}$$

STEP 3. Circle pairs, underline spares.

$$\sqrt{x^{17}} = \sqrt{\boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \underline{x}}$$

Simplify to singles. $\sqrt{x^{17}} = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot \underline{\sqrt{x}}$

STEP 4. Multiply *outside* factors:

$$\sqrt{x^{17}} = x^8 \cdot \underline{\sqrt{x}}$$

Multiply *inside* factors: (not needed)

ANSWER: $\sqrt{x^{17}}$ simplifies to $x^8\sqrt{x}$

Notice when the exponent is very LARGE, this can be rather tedious.

There's an easier way. Here's the previous problem again, earlier in the problem:

$$\sqrt{x^{17}} = \sqrt{\boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \boxed{x \cdot x} \cdot \underline{x}}$$

How many **pairs** are there? 8 How many **spares** are there? 1

An **odd** exponent can always be written as the previous **even** exponent and a spare.

Examples: $\sqrt{x^{17}} = \sqrt{x^{16}} \cdot \sqrt{x}$ or $\sqrt{x^{11}} = \sqrt{x^{10}} \cdot \sqrt{x}$ or $\sqrt{x^7} = \sqrt{x^6} \cdot \sqrt{x}$

Pairs & Spares: 8 pairs, 1 spare 5 pairs, 1 spare 3 pairs, 1 spare

Sq. Rt. Is Exponent/2: $16 \div 2 = 8$ $10 \div 2 = 5$ $6 \div 2 = 3$

Simplified $\sqrt{x^{17}} = x^8\sqrt{x}$ $\sqrt{x^{11}} = x^5\sqrt{x}$ $\sqrt{x^7} = x^3\sqrt{x}$

Notes Section R.7 – Simplifying Radical Expressions (Square Roots)

- Simplify square roots containing both variables and constants

- EXAMPLE:** Express in simplified form. $\sqrt{180x^6y^{15}}$ [R.7.47]
Assume that all variables represent positive real numbers.

CONSTANT

STEP 1. Prime Factorization.

$$\begin{array}{ccc} & 180 & \\ & 10 \cdot 18 & \\ \boxed{5} \cdot \boxed{2} & \cdot & 9 \cdot \boxed{2} \\ \boxed{5} \cdot \boxed{2} \cdot & \boxed{3} \cdot \boxed{3} \cdot \boxed{2} & \end{array}$$

STEP 2. Update radicand.

$$\sqrt{180} = \sqrt{2 \cdot 2 \cdot 3 \cdot 3 \cdot 5}$$

STEP 3. Circle pairs, underline spares.

$$\sqrt{180} = \sqrt{\boxed{2 \cdot 2} \cdot \boxed{3 \cdot 3} \cdot \underline{5}}$$

Simplify to singles.

$$\sqrt{180} = 2 \cdot 3 \sqrt{\underline{5}}$$

STEP 4. Multiply *outside* factors.

$$\sqrt{180} = 6\sqrt{\underline{5}}$$

Multiply *inside* factors. (not needed)

VARIABLES

Write as **separate square roots**:

$$\sqrt{x^6y^{15}} = \sqrt{x^6} \cdot \sqrt{y^{15}}$$

Rewrite **odd** exponents as the previous even and a spare:

$$= \sqrt{x^6} \cdot \sqrt{y^{14}} \sqrt{\underline{y}}$$

Simplify even exponents by dividing by 2:

$$= x^3 \cdot y^7 \cdot \sqrt{\underline{y}}$$

MERGE

Merge together the answer portions from the **constants** and the **variables**.

CONSTANTS: $6\sqrt{\underline{5}}$

VARIABLES: $x^3 \cdot y^7 \cdot \sqrt{\underline{y}}$

MERGED – ANSWER: $\sqrt{180x^6y^{15}}$ simplifies to $6x^3y^7\sqrt{5y}$

Sources Used:

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