Notes Section 5.5 – Properties of Logarithms

Lesson Objectives

1. Change of Base Formula
2. Evaluating logarithms using TI-84 Plus series calculators
3. Expanding or Condensing (Combining) Logarithm Properties for:
    a. Multiplication
    b. Division
    c. Powers (Exponents)
    d. Various mixtures of these

Remember from when we introduced logarithms in the previous section:

A logarithm is an exponent.

A. The Change of Base Formula

Also, in the previous section, we discussed the two special types of logarithms. These are the only logarithms that have their own buttons on the calculator:

1. **Common** logarithm – base is **10**, but not explicitly written. It is understood to be 10.
   If you see a logarithm written **without** a base, then the base is **10**.
   - Examples: $\log x$ means $\log_{10}(x)$ $\quad \log \frac{1}{100}$ means $\log_{10}\left(\frac{1}{100}\right)$
   - Calculator button is **LOG** 

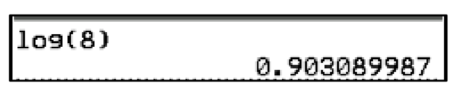
     (to the left of the **7** button)
   - This calculator button is **ONLY** for base **10**, the common logarithm!

2. **Natural** logarithm – base is *e*, but the logarithm is written as ***"ln"*** not "log$_e$".
   - Examples: $\ln x$ means $\log_e(x)$ $\quad \ln e^7$ means $\log_e(e^7)$
   - Calculator button is **LN** 

     (to the left of the **4** button)
   - This calculator button is **ONLY** for base ***e***, the natural logarithm!

Consider the following logarithm: $\log_2(8)$ We know this equals 3, because $2^3 = 8$.

Sometimes students assume that the LOG button on the calculator works for any logarithm.



We know $\log_2(8) = 3$, but $\log(8) \approx 0.9$ on calculator.

They're different values because they're different **BASES**.

If we want to determine a logarithm with a base other than 10 or *e* using calculator, we need another means to do it.

Notes Section 5.5 – Properties of Logarithms

- Change of Base Formula

Let $x, a \neq 1, b \neq 1$ be positive real numbers. Then, $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$

(Alliteration: Remember that the **b-b-b**ase goes on the **b-b-b**ottom.)

Technically, you can use the Change of Base formula to convert to ANY base **(a)**, but for rounding purposes, base 10 (LOG) or base *e* (LN) is the way to go.

- **EXAMPLE:** Find the logarithm using the change of base formula. [5.5.79]

$4\log_3(20)$

(Simplify your answer. Do not round until the final answer. Then round to the nearest thousandth as needed.)

First of all, the 4 is like a coefficient, so it will just be multiplied onto the logarithm. Using the change of base formula:

$$4\log_3(20) = 4 \cdot \frac{\log(20)}{\log(3)} \text{ or } 4 \cdot \frac{\ln(20)}{\ln(3)} \approx \mathbf{10.907}$$

```
4*log(20)/log(3)
                10.90733211
```

```
4*ln(20)/ln(3)
                10.90733211
```

Note that for your calculator, you can use either common or natural logarithm. Be careful with your parentheses when using change of base formula. It's easy to make a mistake with it. Here are 2 common mistakes:

```
4log(20/log(3)
                6.489604927
4ln(20/ln(3)
                11.60673778
```

If you don't close parentheses with the 20, you will get an **INCORRECT** answer.

```
4log(20)/ln(3)
                4.736994148
4ln(20)/log(3)
                25.11506032
```

If you mismatch logarithms when you divide, you will also get **INCORRECT** answer.

If you are using a **TI-83 Plus** calculator (or a TI-84 Plus calculator with older software), the **change-of-base formula** MUST be used to evaluate logarithms that are not base 10 or base *e*.

If you use a **TI-84 Plus** (includes color screen models, too), there is an easier, faster way to calculate logarithms that are not base 10 or base *e*.

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B. Evaluating logarithms using TI-84 Plus series calculator

Let's re-examine the previous example, this time using the TI-84 Plus CE calculator (it works for TI-84 Plus calculator, too, as long as it has updated software):

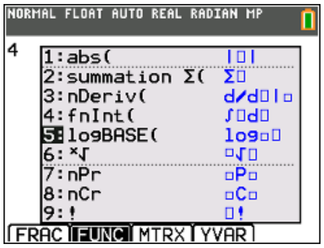
- **EXAMPLE:** Find the logarithm using the change of base formula. [5.5.79]

$$4\log_3(20)$$

(Simplify your answer. Do not round until the final answer. Then round to the nearest thousandth as needed.)

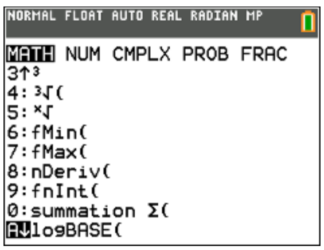
1. Press **4**, **ALPHA**,**WINDOW**

Choose number **5:logBASE(**

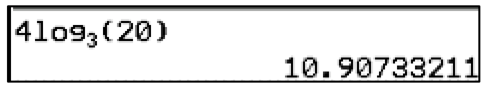


Or

Press **4**, **MATH**, scroll to **A:logBASE(**



2. Enter the base of **3** and the value of **20** in parentheses, then press **ENTER**



3. Round answer accordingly.

$$4\log_3(20) \approx \mathbf{10.907}$$

C. Expanding/Condensing Logarithm Properties

- **Product Rule:**

	$\log_a(mn) = \log_a(m) + \log_a(n)$	EXPANDING	(Product to Sum)
or	$\log_a(m) + \log_a(n) = \log_a(mn)$	CONDENSING	(Sum to Product)

- **Quotient Rule:**

	$\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$	EXPANDING	(Quotient to Difference)
or	$\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$	CONDENSING	(Difference to Quotient)

- **Power Rule:**

	$\log_a(m^r) = r\log_a(m)$	EXPANDING	(Exponent to Coefficient)
or	$r\log_a(m) = \log_a(m^r)$	CONDENSING	(Coefficient to Exponent)

Notes Section 5.5 – Properties of Logarithms

- EXPANDING Logarithms

- **EXAMPLE:** Expand the expression. If possible, write your answer without exponents.
  $\log_4(64k^4x)$ (Simplify your answer.) [5.5.17]

The value of the logarithm includes a **product**: $\log_4(64k^4x) = \log_4(64 \cdot \boldsymbol{k^4} \cdot x)$

Use the **Product Rule** to EXPAND, so you use **ADDITION** and keep the SAME base:

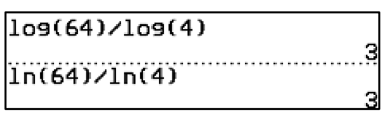
$$\log_4(64 \cdot k^4 \cdot x) = \mathbf{\log_4(64)} + \log_4(k^4) + \log_4(x)$$

If a logarithm has NO variables, try to **simplify**:

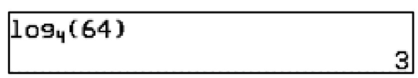
$\log_4(64)$ means $4^{\textit{what power?}} = 64$. It's **3**.

(NOTE: If you are unsure if it simplifies, try it on calculator with either **Change of Base** formula or **logBASE** feature. If you get a "nice, pretty" rational number, like **3**, then go ahead and simplify to that value. If you get a "messy" decimal that doesn't convert to a fraction, then it does NOT simplify.

$$\log_4(64) = \frac{\log(64)}{\log(4)} \text{ or } \frac{\ln(64)}{\ln(4)} = \mathbf{3}$$



Using **Change of Base Formula**



Using the **logBASE** command

Continuing on:

$$\log_4(64 \cdot k^4 \cdot x) = \log_4(64) + \log_4(k^4) + \log_4(x)$$
$$= 3 + \mathbf{\log_4(\boldsymbol{k^4})} + \log_4(x)$$

Next, use the **Power Rule** (exponent to coefficient) to simplify 2nd term:

$$\log_4(k^4) = \mathbf{4\log_4(\boldsymbol{k})}$$

Now update:

$\log_4(64k^4x) = \mathbf{3 + 4\log_4(\boldsymbol{k}) + \log_4(\boldsymbol{x})}$ **Answer**

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Notes Section 5.5 – Properties of Logarithms

- **EXAMPLE:** Expand the expression. If possible, write your answer without exponents.

$$\log_3\left(\frac{9m^5}{n^5}\right)$$ [5.5.19]

(Simplify your answer. Use integers or fractions in the expression.)

This logarithm has a mixture of multiplication, division, and exponents.

First, use the **Quotient Rule** (SUBTRACTION) to EXPAND the fraction:

$$\log_3\left(\frac{\mathbf{9m^5}}{\mathbf{n^5}}\right) = \log_3(9m^5) - \log_3(n^5)$$

Next, use the **Product Rule** (ADDITION) to EXPAND $\log_3(9m^5)$

$$\log_3(\mathbf{9 \cdot m^5}) = \log_3(9) + \log_3(m^5)$$

Update the entire logarithm:

$$\log_3\left(\frac{9m^5}{n^5}\right) = \log_3(9) + \log_3(m^5) - \log_3(n^5)$$

Now, we use the **Power Rule** (exponent to coefficient)

for both $\mathbf{\log_3(m^5)}$ and $\mathbf{\log_3(n^5)}$

Update the entire logarithm:

$$\log_3\left(\frac{9m^5}{n^5}\right) = \mathbf{\log_3(9)} + 5\log_3(m) - 5\log_3(n)$$

Finally, we **simplify** (if possible) any logarithms with no variables:

$$\log_3(9) \text{ means } 3^{\textit{what power?}} = 9. \text{ It's } \mathbf{2}.$$

```
log(9)/log(3)
                2
ln(9)/ln(3)
                2
```

Using **Change of Base Formula**

```
log₃(9)
                2
```

Using **logBASE** command

NOTE: In general, if you get a decimal, do **NOT** convert to a rounded number, unless directed to do so. If it is not an EXACT number, leave it alone as a logarithm.

Update your entire logarithm:

$$\log_3\left(\frac{9m^5}{n^5}\right) = \mathbf{2 + 5\log_3(m) - 5\log_3(n)}$$ **Answer**

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- **CONDENSING** (or COMBINING) **Logarithms**

- **EXAMPLE:** Write the following expression as a logarithm of a single expression

  $\log 27 + \log\frac{1}{9}$ (Simplify your answer. Type an exact answer.) [5.5.37]

  Write as a **single expression** means to CONDENSE (or COMBINE) logarithms.

  $\log(27) + \log\left(\frac{1}{9}\right)$ Use **Product Rule** (sum to product)

  $\log(27) + \log\left(\frac{1}{9}\right) = \log\left(27 \cdot \frac{1}{9}\right)$ Simplify $27 \cdot \frac{1}{9} = 3$

  Update the entire expression: $= \mathbf{\log(3)}$ **Answer**

  log(3) 0.4771212547 not exact – leave it be!

  Notice that $\mathbf{\log(3)}$ does not simplify into a "nice, pretty" number – it's irrational. Therefore, $\mathbf{\log}\ (\mathbf{3})$ is already an ***exact*** answer, as required in the instructions.

- **EXAMPLE:** Write the expression as a logarithm of a single expression. [5.5.39]

  $\log 7 + \log 30 - \log 6$ (Simplify your answer.)

  By the order of operations, add and subtract go in order, left to right.

  Start with the first two terms: $\log(7) + \log(30)$

  Addition of logarithms means use **Product Rule** (sum to product):

  $\log(7) + \log(30) = \log(7 \cdot 30) = \log(210)$

  Update the entire expression:

  $\log(7) + \log(30) - \log(6) = \log(210) - \log(6)$

  Subtraction of logarithms means use **Quotient Rule** (difference to quotient):

  $\log(210) - \log(6) = \log\left(\frac{210}{6}\right) = \log(35)$

  Update the entire expression:

  $\log(7) + \log(30) - \log(6) = \mathbf{\log(35)}$ **Answer**

  log(35) 1.544068044 not exact – leave it be!

Notice that $\mathbf{\log(35)}$ does not simplify into a "nice, pretty" number – it's irrational. The instructions don't indicate to round the answer, so leave it as an exact answer.

- **EXAMPLE:** Use properties of logarithms to condense the logarithmic expression below. Write the expression as a single logarithm whose coefficient is 1. Where possible, evaluate logarithmic expressions. [5.5.51]

$$5\ln x + 3\ln y - 2\ln z$$

All three terms have **coefficients**, so use the **Power Rule** (coefficient to exponent):

$$5\ln(x\ ) + 3\ln(y\ ) - 2\ln(z\ ) = \ln(x^5) + \ln(y^3) - \ln(z^2)$$

Next, Addition of logarithms means use **Product Rule** (sum to product):

$$\ln(x^5) + \ln(y^3) = \ln(x^5 \cdot y^3) = \mathbf{ln}(\boldsymbol{x^5y^3})$$

Update the entire expression:

$$5\ln(x) + 3\ln(y) - 2\ln(z) = \mathbf{ln}(\boldsymbol{x^5y^3}) - \ln(z^2)$$

Subtraction of logarithms means use **Quotient Rule** (difference to quotient):

$$\ln(x^5y^3) - \ln(z^2) = \ln\left(\frac{x^5y^3}{z^2}\right)$$

Update the entire expression:

$$5\ln(x) + 3\ln(y) - 2\ln(z) \qquad = \mathbf{ln}\left(\boldsymbol{\frac{x^5y^3}{z^2}}\right)$$ **Answer**

Sources Used:

1. MyLab Math for *College Algebra with Modeling and Visualization*, 6th Edition, Rockswold, Pearson Education Inc.
2. Texas Instruments TI Connect® CE software, https://education.ti.com/en/products/computer-software/ti-connect-ce-sw