1. Conditional Probability

Sometimes the probability of an event must be calculated using the knowledge that some other event has already happened (or is happening, or will happen – time is not important).
This type of probability is called Conditional ***Probability***.

The probability that event ***B*** can be calculated on the assumption that event ***A*** has already happened is called "the conditional probability of ***B***, given ***A***" and is written as ***P*(*B* | *A*)**, (in other words it is asking for the probability that ***B*** happens, given that ***A*** has happened).

EXAMPLE: From the sample space $S = \{1,2,3,4, \dots ,16\}$ a single number is to be selected at random. Given the following events, find the indicated probabilities.

Event *A*: the selected number is odd.
Event *B*: the selected number is a multiple of 3.
Event *C*: the selected number is a prime number.

(a) What is the probability of event *A*, called *P*(*A*)?

- Which are the odd numbers (how many)? 1,3,5,7,9,11,13,15 (8)
- How many total numbers? 16
- So, what is *P*(*A*)? $\frac{8}{16} = \boxed{\frac{1}{2}}$

(b) What is the probability of event *B*, called *P*(*B*)?

- Which are the multiples of 3 (how many)? 3,6,9,12,15 (5)
- How many total numbers? 16
- So, what is *P*(*B*)? $\boxed{\frac{5}{16}}$

(c) What is the probability of event *C*, called *P*(*C*)? Note: prime is divisible by 1 and itself and 1 is not prime!

- Which are the prime numbers (how many)? 2,3,5,7,11,13 (6)
- How many total numbers? 16
- So, what is *P*(*C*)? $\frac{6}{16} = \boxed{\frac{3}{8}}$

(d) What is the probability of events *A* and *B*, called *P*(A and *B*)?

- Which are **both** odd **and** multiple of 3 (how many)? 3,9,15 (3)
- How many total numbers? 16
- So, what is *P*(*A* and *B*)? $\boxed{\frac{3}{16}}$

(e) What is the probability of events *B* and *C*, called *P*(*B* and *C*)?

- Which are **both** multiple of 3 **and** prime (how many)? only 3 (1 way)
- How many total numbers? 16
- So, what is *P*(*B* and *C*)? $\boxed{\frac{1}{16}}$

(f) What is the probability of event *B*, given *A*, called *P*(*B* | *A*)?

- Which are **both** multiple of 3 **and** odd (how many)? 3,9,15 (3)
- How many total odd numbers ("given *A*")? 1,3,5,7,9,11,13,15 (8)
- So, what is *P*(*B* | *A*)? $\boxed{\frac{3}{8}}$

Formula for CONDITIONAL PROBABILITY

The probability of one event B occurring, given another event A occurs is the formula:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} \quad \text{or easier way:} \quad P(B|A) = \frac{n(A \text{ and } B)}{n(A)}$$

EXAMPLE: Use the formula $P(B|A) = \frac{n(A \text{ and } B)}{n(A)}$ to find the probability *P*(seven | not a face card) when a single card is drawn from a standard 52-card deck.

there are 4 sevens and they are all not face cards

Notation: $P(\text{seven}|\text{not face card}) = \frac{n(\text{seven and not face card})}{n(\text{not face card})} = \frac{4}{40} = \boxed{\frac{1}{10}}$

there are 12 face cards, so there are 52-12 = 40 not face cards

EXAMPLE: One hundred college seniors attending a career fair at a university were categorized according to gender and according to primary career motivation. The following table shows the results.

	Primary Career Motivation			
	Money	Allowed to be creative	Sense of giving to society	Total
Male	16	17	19	52
Female	13	20	15	48
Total	29	37	34	100

NOTE: Sometimes a table may ***not*** include the totals for rows and columns, so be ready to calculate them yourself!

If one of these students is to be selected at random, find the following probabilities:

(a) Find probability that the student is male. $P(\text{male}) = \frac{52 \text{ male}}{100 \text{ total}} = \boxed{\frac{13}{25}}$

(b) Find probability that the student is motivated by money. $P(\text{money}) = \frac{29 \text{ money}}{100 \text{ total}} = \boxed{\frac{29}{100}}$

(c) Find the probability that the student is not motivated by creativity.

$P(\text{not creativity}) = 1 - P(\text{creativity}) = 1 - \frac{37 \text{ creative}}{100 \text{ total}} = \boxed{\frac{63}{100}}$

(d) Find probability that the student is male and motivated by money.

$P(\text{male and money}) = \frac{16 \text{ male and money}}{100 \text{ total}} = \frac{16}{100} = \boxed{\frac{4}{25}}$

(e) Find probability that the student is male, given that he is motivated by money.

$P(\text{male} | \text{money}) = \frac{16 \text{ male and money}}{29 \text{ total money}} = \boxed{\frac{16}{29}}$

(f) Find probability that the student is motivated by money, given that he is male.

$P(\text{money} | \text{male}) = \frac{16 \text{ money and male}}{52 \text{ total males}} = \frac{16}{52} = \boxed{\frac{4}{13}}$

What are Independent Events?

Two event are called independent *events* if knowledge about the occurrence of one of them has *no effect* on the probability of the other one; that is, $P(B \mid A) = P(B)$ or $P(A \mid B) = P(A)$. In other words, *one event does not affect the other*, therefore, there is *no condition*.

EXAMPLE: Determine whether each of the following scenarios represents independent events.

yes **(a)** A bag contains 12 red and 5 green marbles. A marble is drawn, replaced in the bag, and then a second marble is drawn. Are the events "first marble is red" and "second marble is green" independent events? "with replacement"

NO **(b)** A balanced (fair) die is rolled twice. Are the events "the sum of the two rolls is 6" and "the first roll comes up 2" independent? the first roll is conditional

yes **(c)** A card is selected at random from a standard deck of 52 cards. It is then replaced and a second card is selected at random. Are the events "club on the first draw" and "ace on the second draw" independent? "with replacement"

2. Events Involving "AND"

The probability of one event ***AND*** another event happening involves the *intersection* and *multiplication* of probabilities. Remember ***AND*** involves both, which is why there may or may not be a **condition**.

Probability for A "and" B uses the multiplication rule:

(1) If A and B are *independent (there are no conditions)* then multiply their probabilities. But this type of probability involves *with replacement*.

$$P(A \text{ and } B) = P(A) \bullet P(B)$$

(2) If A and B are two events with *a condition* then multiply their probabilities. But this type of probability involves the condition of *without replacement*.

$$P(A \text{ and } B) = P(A) \bullet P(B \mid A)$$

With Replacement: using all parts multiple times and they can be repeated or used again. When a part is chosen, it can be replace and chosen again.

Without Replacement: using all parts only one time and they cannot be repeated. When a part is chosen, it cannot be replaced; it cannot be used again. *This is a condition.*

HINT: Many examples may not actually use the word "and" in the problem. When you read the problem, there is an assumption that both or each event will happen…one after another.

EXAMPLE: A pet store has 9 puppies, including 2 poodles, 4 terriers, and 3 retrievers. If Rebecka and Aaron, in that order, each select one puppy at random with replacement (they may both select the same one), find the probability that Rebecka selects a poodle and Aaron selects a terrier.

Notation and calculation: P(poodle and terrier) = P(poodle) • P(terrier) With replacement? yes!

$$P\left(\begin{matrix}\text{Rebecca Poodle}\\ \text{and Aaron Terrier}\end{matrix}\right) = \underbrace{\frac{2}{9}}_{\text{Rebecca poodle}} \cdot \underbrace{\frac{4}{9}}_{\text{Aaron terrier}} = \boxed{\frac{8}{81}}$$

EXAMPLE: Let two cards be dealt successively, without replacement, from a standard 52-card deck. Find the probability of each of the following events:

(a) two jacks

Notation and calculation:
P(jack 1^{st} and jack 2^{nd}) = P(jack 1^{st}) • P(jack 2^{nd} | jack 1^{st})

$$\frac{4 \text{ jacks}}{52 \text{ total}} \cdot \frac{3 \text{ jacks}}{51 \text{ total}} = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{13} \cdot \frac{1}{17} = \boxed{\frac{1}{221}}$$

(Lose 1 Jack, 1 card)

(b) The first card is a king and the second is a queen

Notation and calculation:
P(king 1^{st} and queen 2^{nd}) = P(king 1^{st}) • P(queen 2^{nd} | king 1^{st})

$$\frac{4 \text{ kings}}{52 \text{ total}} \cdot \frac{4 \text{ queens}}{51 \text{ total}} = \frac{4}{52} \cdot \frac{4}{51} = \frac{1}{13} \cdot \frac{4}{51} = \boxed{\frac{4}{663}}$$

(Lose 1 King, 1 card)

(c) The first card is a spade and the second is black

Notation and calculation:
P(spade 1^{st} and black 2^{nd}) = P(spade 1^{st}) • P(black 2^{nd} | spade 1^{st})

$$\frac{13 \text{ spades}}{52 \text{ total}} \cdot \frac{25 \text{ black}}{51 \text{ total}} = \frac{13}{52} \cdot \frac{25}{51} = \frac{1}{4} \cdot \frac{25}{51} = \boxed{\frac{25}{204}}$$

(Lose 1 Spade, 1 card)

(d) A red card is dealt second, given a red card is dealt first

Parts **(a)** through **(c)** are calculating probabilities of BOTH cards happening together.

This scenario is ONLY calculating probability of the second card, under specific conditions of the first card. You still MUST go through the process of the first card!

Notation: P(red 2^{nd} | red 1^{st})
Calculation:

1st card red: $\frac{26 \text{ red}}{52 \text{ total}}$, 2nd card red: $\frac{25 \text{ red}}{51 \text{ total}}$

(Lose 1 red, 1 card)

$$P(\text{red 2nd} \mid \text{red 1st}) = \boxed{\frac{25}{51}}$$