Notes Section 5.4 – Logarithmic Functions and Their Representations

Lesson Objectives

1. The Basics of a Logarithm
2. The Two "Special" Types of Logarithms
3. Basic Properties of Logarithms
4. Simplify (or evaluate) logarithms
5. Convert Between Exponential and Logarithmic (and vice-versa)
6. Solve Basic Logarithmic and Exponential Equations

A. The **Basics** of a Logarithm

Suppose we have the exponential equation: $2^x = 8$

We know the answer is *x* = 3. $(2^3 = 8)$

Because our variable is in the exponent, we don't just use division to solve this equation:

NO, NO, NO!! Don't do this!! → → $\dfrac{2^x}{2} = \dfrac{8}{2}$

If you did, you would get *x* = 4. (incorrect) Division is used to undo multiplication, right?

But, we don't have multiplication; we have exponential.

BIG IDEA! To undo an exponential function, we need to use its inverse – the **logarithm.**

A logarithm is an exponent.

$$log = exponent$$

Since a logarithm is an exponent, then it must necessarily have a **BASE.**

Logarithmic form (definition):

$\log_b x = y$ "**Logarithm** base *b* of ***x*** equals *y*"

$$\log_{\text{base}}(\text{value}) = \text{exponent}$$

Rewrite (or convert) to **exponential form**:

$\log_b x = y$ means the same thing as $b^y = x$

$$\text{base}^{\text{exponent}} = \text{value}$$

BIG IDEA! $\log_b x = y$ and $b^y = x$ are interchangeable in meaning.

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B. The Two "Special" Types of Logarithms

1. **Common** logarithm – base is **10**, but not explicitly written. It is understood to be 10.
   If you see a logarithm written **without** a base, then the base is **10**.
   - Examples: $\log x$ means $\log_{10}(x)$ $\quad$ $\log \frac{1}{100}$ means $\log_{10}\left(\frac{1}{100}\right)$
   - Calculator button is **LOG** 

     (to the left of the **7** button)
   - This calculator button is **ONLY** for base **10**, the common logarithm!

2. **Natural** logarithm – base is *e*, but the logarithm is written as ***"ln"*** not "$\log_e$".
   - Examples: $\ln x$ means $\log_e(x)$ $\quad$ $\ln e^7$ means $\log_e(e^7)$
   - Calculator button is **LN** 

     (to the left of the **4** button)
   - This calculator button is **ONLY** for base ***e***, the natural logarithm!

C. Basic Properties of Logarithms

Recall **BIG IDEA!**: $\log_b x = y \iff b^y = x$ $\quad$ (these are interchangeable in meaning)

Here are some **Basic Logarithm Properties** to remember:

1. $\mathbf{\log_b 1 = 0}$ $\quad$ because $\quad$ $b^0 = 1$ $\quad$ (Any base with zero power is 1)

2. $\mathbf{\log_b b = 1}$ $\quad$ because $\quad$ $b^1 = b$ $\quad$ (Any base to the power of 1 is the base itself)

3. $\mathbf{\log_b b^x = x}$ $\quad$ because $\quad$ $b^x = b^x$ $\quad$ (Logarithm base *b* will undo exponential base b)
   $\cancel{\log_b b}^{x} = x$ $\quad$ (Logarithm base b will undo "big" base b)

4. $\mathbf{b^{\log_b x} = x}$ $\quad$ because $\quad$ $\log_b x = \mathbf{\log_b x}$ $\quad$ (Exponential base b will undo log base b)
   $\cancel{b}^{\cancel{\log_b} x} = x$ $\quad$ ("Big" base b will undo logarithm base b)

- **EXAMPLE:** Simplify the expression. $\quad$ $\log_5 1$ $\quad$ [5.4-18]

  $\log_5 1$ means $5^{\text{what power ?}} = 1$ $\quad$ Property 1 $\quad$ Answer: $\log_5 1 = \mathbf{0}$

- **EXAMPLE:** Evaluate the logarithm. $\quad$ $\ln 1$ $\quad$ [5.4-15]
  This is a natural logarithm (ln) – it has base *e*.

  $\ln 1$ or $\log_e 1$ means $e^{\text{what power ?}} = 1$ $\quad$ Property 1 $\quad$ Answer: $\ln 1 = \log_e 1 = \mathbf{0}$

- **EXAMPLE:** Evaluate the logarithm. $\mathbf{ln}(e)$ [5.4-14]
  This is a natural logarithm (ln) – it has base *e*.

  $\ln e$ or $\log_e e$ means $e^{\text{what power?}} = e$ Property 2 Answer: $\ln(e) = \log_e(e) = \mathbf{1}$

- **EXAMPLE:** Simplify the expression, if possible. $\mathbf{log}\, 10^{7.4}$ [5.4.1]
  Notice that the base of the logarithm is not written – it is a common logarithm, base 10.

  $\log 10^{7.4} = \log_{10}(10^{7.4})$ (Logarithm base 10 will undo exponential base 10)
  $\cancel{\log_{10}(10}^{7.4})$ Property 3 Answer: $\mathbf{log}\, 10^{7.4} = \mathbf{7.4}$

- **EXAMPLE:** Simplify the expression, if possible. $\mathbf{ln}\, e^6$ [5.4.29]
  This is a natural logarithm (ln) – it has base *e*.

  $\ln e^6 = \log_e e^6$ (Logarithm base *e* will undo exponential base *e*)
  $\cancel{\log_e e}^6$ Property 3 Answer: $\ln e^6 = \mathbf{6}$

- **EXAMPLE:** Find the indicated value of the logarithmic function.
  $\mathbf{log}_7(7)^{4x}$ [5.4.23]

  $\log_7(7)^{4x}$ (Logarithm base 7 will undo exponential base 7)
  $\cancel{\log_7(7)}^{4x}$ Property 3 Answer: $\mathbf{log}_7(7)^{4x} = \mathbf{4x}$

- **EXAMPLE:** Simplify. $4^{\mathbf{log}_4(5)}$ [5.4.25]

  $4^{\log_4(5)}$ (Exponential base 4 will undo logarithm base 4)
  $\cancel{4^{\log_4}}(5)$ Property 4 Answer: $4^{\mathbf{log}_4(5)} = \mathbf{5}$

D. Simplify (or Evaluate) Logarithms

- **EXAMPLE:** Find the logarithm $\mathbf{log}_5 \frac{1}{625}$ [5.4.51]
  Put "= *y*" on the end of the expression: $\log_5 \frac{1}{625} = y$
  Chant: "A logarithm is an exponent."

  $\log_5 \frac{1}{625}$ means: $5^{\text{what power?}} = \frac{1}{625}$ or $5^y = \frac{1}{625}$

  Since the value $\frac{1}{625}$ is a fraction, the exponent must be **negative**.

  625 is a power of 5, since $5 \cdot 5 \cdot 5 \cdot 5 = 625$, or $5^4 = 625$. So $\log_5 \frac{1}{625} = \mathbf{-4}$

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E. Convert between Exponential and Logarithmic (and vice-versa)

- **EXAMPLE:** Write in exponential form. $\log_{10}\frac{1}{1000000} = -6$ [*Lial 10.3.19]

Chant: "A logarithm is an exponent."

$\log_{10}\frac{1}{1000000} \quad = \quad -6$

What is the base? 10 What is the exponent? – 6 put them together: 10^{-6}

What is the "value"? $\frac{1}{1000000}$ In exponential form: $\boxed{10^{-6} = \frac{1}{1000000}}$

- **EXAMPLE:** Write in exponential form. $\log_{15} 1 = 0$ [*Lial 10.3-11]

Chant: "A logarithm is an exponent."

$\log_{15} 1 \quad = \quad 0$

base = 15, exponent = 0, value = 1 In exponential form: $\boxed{15^{0} = 1}$

- **EXAMPLE:** Write in logarithmic form. $7^3 = 343$ [*Lial 10.3-1]

$7^3 = 343$

base = 7, exponent = 3, value = 343

Chant: "A logarithm is an exponent."

Setup: $\log_{(\text{base})}(\text{value}) = \quad \text{exponent}$

In logarithmic form: $\boxed{\log_7 343 \quad = \quad 3}$

- **EXAMPLE:** Write in logarithmic form. $10^{-5} = 0.00001$ [*Lial 10.3-4]

$10^{-5} = 0.00001$

base = 10, exponent = – 5, value = 0.00001

Chant: "A logarithm is an exponent."

Setup: $\log_{(\text{base})}(\text{value}) = \quad \text{exponent}$

In logarithmic form: $\log_{10} 0.00001 = -5$ or $\boxed{\log 0.00001 = -5}$

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F. Solve Basic Logarithmic and Exponential Equations

- Solve Basic Logarithmic Equations – ISOLATE and convert to EXPONENTIAL

- **EXAMPLE:** Solve the equation. $9\log(3x) = 27$ [5.4.95]

First, **ISOLATE** the logarithm. Divide by 9. $\dfrac{9\log(3x)}{9} = \dfrac{27}{9}$ (log has base 10)

Simplify. Do **NOT** divide by 3 in parentheses yet! $\log_{10}(3x) = 3$
(It is stuck inside the logarithm.)

Chant: "A logarithm is an exponent."

Convert the $\log_{10}(3x) = 3$ to exponential form with base 10. $10^3 = 3x$

Divide both sides by 3 and simplify. $\dfrac{1000}{3} = \dfrac{3x}{3}$

Leave answer as a **fraction** – do not round. ANSWER: $\boxed{x = \dfrac{1000}{3}}$

- **EXAMPLE:** Solve the equation symbolically for the unknown. [5.4-30]

$3\ln(4x) = 21$

First, **ISOLATE** the logarithm. Divide by 3. $\dfrac{3\ln(4x)}{3} = \dfrac{21}{3}$ (ln means base *e*)

Simplify. Do **NOT** divide by 4 yet! $\log_e(4x) = 7$
(It is stuck inside the logarithm.)

Chant: "A logarithm is an exponent."

Convert the $\log_e(4x) = 7$ to exponential form with base *e*. $e^7 = 4x$

Divide both sides by 4 and simplify. $\dfrac{e^7}{4} = \dfrac{4x}{4}$

Leave as exact answer with *e* – do not **round**. ANSWER: $x = \dfrac{e^7}{4}$ or $\boxed{x = \dfrac{1}{4}e^7}$

Answer is **NOT**: $x = e^{7/4}$ (the divide 4 is NOT part of the exponent!)

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- **EXAMPLE:** Solve the equation. $4 - 2\log_3 x = 2$ [5.4.105]

  First, **ISOLATE** the logarithm. Subtract 4 both sides. $4 - 2\log_3 x = 2$
  (do **NOT** do 4 – 2 = 2 at beginning!) $-4 \qquad -4$

  Combine like terms and simplify. $-2\log_3 x = -2$

  Divide both sides by – 2 $\dfrac{-2\log_3 x}{-2} = \dfrac{-2}{-2}$

  Simplify. $\log_3 x = 1$

  Chant: "A logarithm is an exponent."

  Convert the $\log_3 x = 1$ to exponential form with base 3. $3^1 = x$

  Simplify. ANSWER: $\boxed{x = 3}$

  - Solve Basic Exponential Equations – ISOLATE and convert to LOGARITHM

- **EXAMPLE:** Solve the equation. Use the change of base formula as appropriate.

  $3(10^{2x}) = 17$ [5.4.73]

  (Type an integer or decimal rounded to the nearest hundredth as needed.)

  First, **ISOLATE** the exponential. Divide by 3. $3(10^{2x}) = 17$

  Simplify. Do NOT round $\dfrac{17}{3}$. Leave as **fraction** to the end! $\dfrac{3(10^{2x})}{3} = \dfrac{17}{3}$

  Update the equation. $10^{2x} = \dfrac{17}{3}$

  Do **NOT** divide by the 2 yet. It's stuck in the exponential.

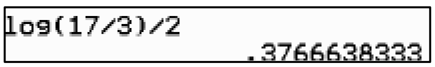
  $$10^{2x} = \frac{17}{3}$$

  Chant: "A logarithm is an exponent."

  Convert the exponential $10^{2x} = \dfrac{17}{3}$ to a logarithm base 10. $\log_{10}\left(\dfrac{17}{3}\right) = 2x$

  Divide both sides by 2 and simplify. $\dfrac{\log\left(\frac{17}{3}\right)}{2} = \dfrac{2x}{2}$

  Simplify. Use calculator to round to hundredth. $x = \dfrac{\log\left(\frac{17}{3}\right)}{2} \approx \boxed{0.38}$



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- **EXAMPLE:** Solve the equation. $5e^x + 5 = 8$ [5.4.81]
  (Round to 4 decimal places as needed. Use a comma to separate answers as needed.)

ISOLATE the exponential. First, subtract 5.

$$5e^x + 5 = 8$$
$$-5 \quad -5$$

Combine like terms and simplify.

$$5e^x = 3$$

Divide both sides by 5 and simplify.

$$\frac{5e^x}{5} = \frac{3}{5}$$
$$e^x = \frac{3}{5}$$

Chant: "A logarithm is an exponent."

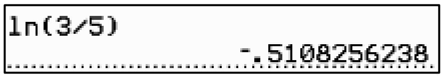
Convert the exponential $e^x = \frac{3}{5}$ to a logarithm base *e*.

$$\log_e\left(\frac{3}{5}\right) = x$$

A logarithm base *e* is same as natural logarithm (ln)

$$\ln\left(\frac{3}{5}\right) = x$$

Use calculator to get answer rounded to 4 decimal places.



Answer: $x \approx -0.5108$

- **EXAMPLE:** Solve the equation for *x*. $e^{-x} = 258$ [5.4-26]
  (Type an integer or decimal rounded to the nearest hundredth as needed.)

Do **NOT** divide by – 1 yet. (It's stuck in the exponential.)
No need to ISOLATE the exponential – it's already there!

$$e^{-1x} = 258$$

$$e^{-1x} = 258$$

Chant: "A logarithm is an exponent."

Convert the exponential $e^{-1x} = 258$ to a logarithm base *e*.

$$\log_e(258) = -1x$$

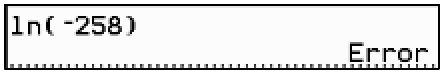
A logarithm base *e* is same as natural logarithm (ln)

$$\ln(258) = -1x$$

Divide by the – 1 both sides and simplify.

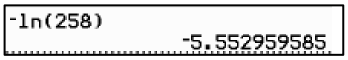
$$\frac{\ln(258)}{-1} = \frac{-1x}{-1}$$

INCORRECT: $x = \ln(-258)$



That would be UNDEFINED.
In general, you ***cannot*** take logarithm of zero or negative. Only $\log_b(\textit{positive})$ works!

$x = -\ln(258)$ is "exact" answer
Use calculator to round



Answer:
$x \approx -5.55$

Sources Used:

1. MyLab Math for *Algebra for College Students*, 8th Edition, Lial, Pearson Education Inc.
2. MyLab Math for *College Algebra with Modeling and Visualization*, 6th Edition, Rockswold, Pearson Education Inc.