Notes Section 4.6 – Rational Functions and Models

Lesson Objectives

1. Overview of a Rational Function
2. Describe the Domain of a Rational Function
3. Determine Vertical and/or Horizontal Asymptotes of a Rational Function when given either a formula or a graph

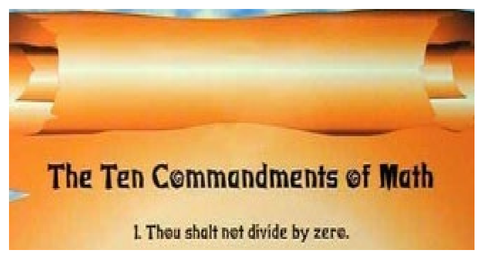
A. **Overview** of a Rational Function

Rational function: one polynomial divided by another polynomial. Its general format is:

$$R(x) = \frac{N(x)}{D(x)}\text{ , with } D(x) \neq 0$$

Because of the division, we must remember you **CAN'T divide by ____________!**
If the denominator has a variable, it has the potential to be zero.

B. **Domain** of a Rational Function (domain = _______________________ zero)

• Commandment #1: Thou shalt not **divide by ________**.	
• Rational functions involve dividing with a variable.	
• We need to find "_________" values of *x* in denominator that will cause dividing by zero.	
• **DOMAIN:** set _________________ = ___________ (numerator typically doesn't matter for domain**)	

- **EXAMPLE:** Find the domain of the function. Write your answer in set builder notation.

$$f(x) = \frac{1}{x^2 - 6} \qquad [3.2.75]$$

DOMAIN = DENOMINATOR ZERO $\qquad = 0$

Solve the equation. (Square root both sides.)
(remember to use $\pm$ after square-rooting)

These are "bad" values for *x*. They cause dividing by zero = "BAD".
They must be _____________________!

State the domain. In words: All real numbers EXCEPT and .
In **set-builder notation**: $\{x|$ $\}$
In interval notation:

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When in doubt – ____!!

Let's re-visit this function again: $f(x) = \frac{1}{x^2-6}$

Let's look at its graph, because it's very telling.

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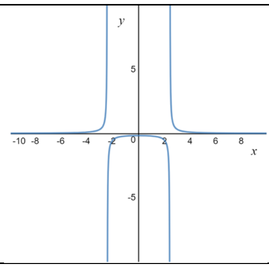
<td>
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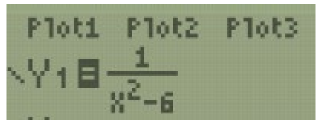
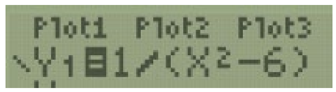
<td>

Or

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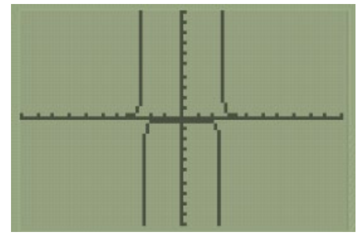
<td></td>
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<tr>
<td>(from www.desmos.com)</td>
<td colspan="2">(from the graphing calculator)</td>
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As you move along the function from LEFT to RIGHT, the function ____________________ dramatically in two places. It's like there's a FORCE FIELD (invisible fence) there. That's where the function is ________________________, due to dividing by zero.

Remember the domain of this function: In **set-builder notation**: $\{x|x \neq -\sqrt{\mathbf{6}}, \sqrt{\mathbf{6}}\}$
Those 2 "force-fields" are located exactly in those locations!

The **DOMAIN** restrictions will __________________ create **UNDEFINED** places in the graph.
There are two ways to be **undefined** in a graph:

- A ____ (vertical line) as is seen above
- A ____________ (open dot) as was seen in a previous section (speed-limit problem)

More on vertical asymptotes and holes a little later…

- **EXAMPLE:** Find the domain of the function. Type your answer in interval notation.

$$f(x) = \frac{x+2}{x^2+5} \qquad [3.2.83]$$

DOMAIN = DENOMINATOR ZERO $\qquad = 0$ Solve the equation.

Take square root both sides.

Watch out!

Can't square root a negative! $x =$

(__________________________ – not a real number)

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What does this mean?

- There are no "BAD" numbers for the denominator. It will _____________ be zero!
- There is NOTHING to exclude in the domain. __________ values of *x* will "work."

(here's the problem again for reference:)

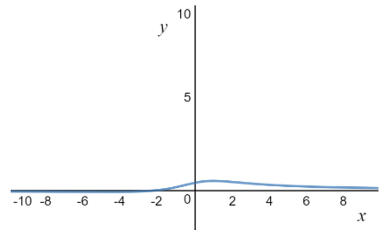
- **EXAMPLE:** Find the domain of the function. Type your answer in interval notation.

$$f(x) = \frac{x+2}{x^2+5} \qquad [3.2.83]$$

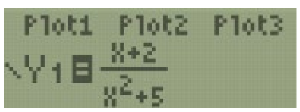
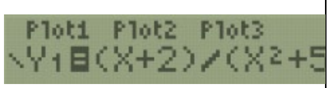
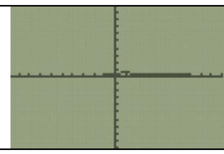
State the domain. In words: _________________________.

In set-builder notation: $\{x|$ $\}$

In **interval notation**:



When in doubt – GRAPH IT OUT!! (graph above from www.desmos.com)

$f(x) = \dfrac{x+2}{x^2+5}$	 or 	 Hard to see the graph. You can (temporarily) turn the axes off.	 Press 2ND, ZOOM ( ) and turn axes off. Then press GRAPH.	There are definitely NO breaks in this graph. There are no values to exclude in the domain. (all real numbers)

- **EXAMPLE:** Find the domain of the function. Write the answer in set-builder notation.

$$g(t) = \frac{5-t}{t^2-5t-14} \qquad [3.2.77]$$

DOMAIN = DENOMINATOR ZERO $\quad t^2 - 5t - 14 = 0$

Try factoring (it's easier) $\quad (t \quad)(t \quad) = 0$

Use Zero Product Property! $\quad = 0$ or $\quad = 0$

Solve each Equation:

Combine like terms and simplify: $t =$ or $t =$

(here's the problem again for reference:)

- **EXAMPLE:** Find the domain of the function. Write the answer in set-builder notation.

$$g(t) = \frac{5-t}{t^2-5t-14} \qquad [3.2.77]$$

After setting the denominator equal to zero and solving $\quad t^2 - 5t - 14 = 0$
We have the solutions: $\quad t = -2 \quad$ or $\quad t = 7$

These are "bad" values for *t*. They cause dividing by zero = "BAD".
They must be ________________________!

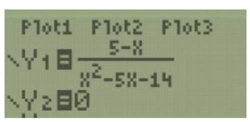
State the domain.

In words: All real numbers EXCEPT ______ and ______.
In **set-builder notation**: $\{x|\qquad\qquad\qquad\}$
In interval notation:

When in doubt – GRAPH IT OUT!!

$$g(t) = \frac{5-t}{t^2-5t-14}$$

(use *x* as your variable on the calculator)



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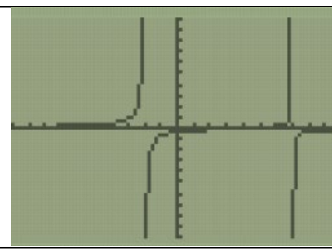
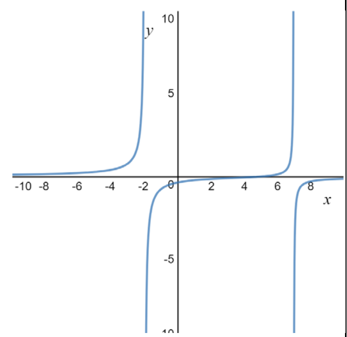


Notice the "force-field" or "invisible fence" at the domain-excluded values $t = -2$ and $t = 6$. These lines are also **VERTICAL** ________________.



graph above drawn from www.desmos.com

C. Determine the Vertical and/or Horizontal Asymptotes

- VERTICAL Asymptote (V.A.) - defined

Vertical Asymptote (V.A.): a vertical line that acts as a ________________, or "force-field" for a rational function. The graph of a rational function will NOT _____________________ or pass through a vertical asymptote (V.A.).

On either side near a vertical asymptote (V.A.), the graph of the function will either:

- bend dramatically __________
  - (formally $\boldsymbol{f(x)} \to \infty$, read as "*f*(*x*) approaches positive infinity")
- bend dramatically __________
  - (formally $\boldsymbol{f(x)} \to -\infty$, read as "*f*(*x*) approaches negative infinity")

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A **vertical asymptote** (V.A.) comes from a domain restriction:

- An __________________________ value of the function…
  - Set **denominator equal to ZERO** and solve equation.
- …that doesn't _______________________ when factored
  - If numerator has a factor that cancels with denominator, it creates a **"_____________,"** not a vertical asymptote.

Domain

$\text{denominator} = 0$

Vertical Asymptote (V.A.)

Doesn't cancel away

"Hole"

It DOES cancel away

A rational function could have exactly one or more than one vertical asymptote (V.A.), or possibly none at all.

- How to find **VERTICAL ASYMPTOTES** (V.A.) of a rational function:

1. Set ____________________________ equal to zero and _____________ the equation.
2. If factored, make sure it doesn't _____________ with factored ___________________.
3. If not, the domain restrictions – ___________ value is a ________________ asymptote.

NOTE: There are other ways to have domain restrictions besides setting denominator equal to zero. For example, **square roots** (or other EVEN roots, like 4th root, 6th root, etc.) do not work if the radicand is negative, so you need to account for that by setting $\mathbf{radicand} \geq \mathbf{0}$. Another example is for **logarithms** – they only work if the value (argument of the logarithm) is POSITIVE, so you need to account for that by setting $\mathbf{value} > \mathbf{0}$.

(We will not be doing roots or logarithms in this lesson.)

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- HORIZONTAL Asymptote (H.A.) – defined

Unlike a vertical asymptote that acts like a barrier or "invisible fence", a horizontal asymptote is different. A rational function ________________ cross a horizontal asymptote (H.A.), but it _______________________ crosses a vertical asymptote (V.A.).

Horizontal Asymptote (H.A.): describes the ____________________________________ of *some* rational functions, where BOTH ends "__________________ out," going horizontal.

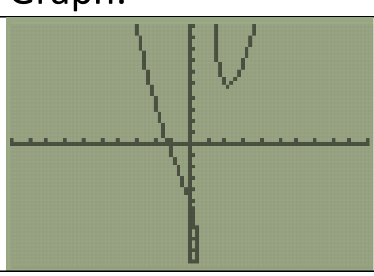
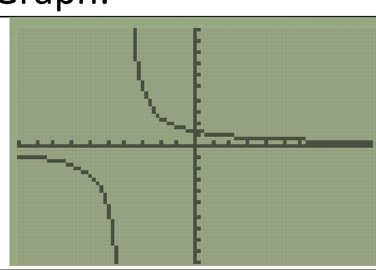
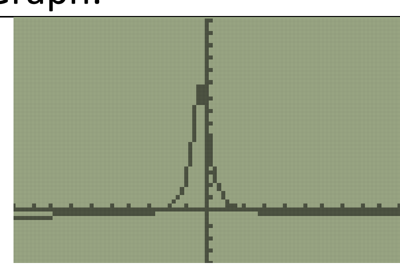
Extreme LEFT: As $x \to -\infty$ (read as: "*x* approaches negative infinity")

Extreme RIGHT: As $x \to \infty$ (read as: "*x* approaches positive infinity")

A rational function will have either exactly ________ horizontal asymptote or none at all.

- How to find HORIZONTAL ASYMPTOTES of a rational function:

To find the **horizontal asymptotes** of a basic rational function, you need to ______________ the ____________ of the numerator to the denominator. One of three things could happen:

Case #1	**Case #2**	**Case #3**
$\dfrac{\text{larger degree}}{\text{smaller degree}}$	$\dfrac{\text{smaller degree}}{\text{larger degree}}$	$\dfrac{\text{same degree}}{\text{same degree}}$
H.A.:	**H.A.:**	**H.A.:**
Example:	*Example*:	*Example*:
$f(x) = \dfrac{x^3 - 2x^2 + 5}{x - 1}$	$g(x) = \dfrac{5}{x+4}$	$h(x) = \dfrac{7 - x^2}{3x^2 + 2x + 1}$
$\dfrac{\text{degree}}{\text{degree}}$	$\dfrac{\text{degree}}{\text{degree}}$	$\dfrac{\text{degree}}{\text{degree}}$
$\dfrac{\text{larger}}{\text{smaller}}$	$\dfrac{\text{smaller}}{\text{larger}}$	$\dfrac{\text{same}}{\text{same}}$
H.A.:	**H.A.:**	**H.A.:**
Graph:	Graph:	Graph:
		
End behavior: ends NEVER flatten out (no H.A.)	End behavior: ends flatten out along *x*-axis (H.A.: *y* = 0)	End behavior: ends flatten out just below *x*-axis (H.A.: *y* = -1/3)

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- **EXAMPLE:** Find any horizontal or vertical asymptotes. [4.6.29]

$$f(x) = -\frac{6x^2}{16 - x^2}$$

Vertical Asymptotes (V.A.)

1. Set denominator equal to zero and solve equation. $16 - x^2 = 0$

   Add x^2 both sides:
   Combine like terms and simplify:
   Square root both sides:
   Simplify – don't forget the ±
   These are the domain restrictions:

2. Denominator $16 - x^2$ is a difference of squares and factors into ( )( ).

   Here's the function again: $f(x) = -\frac{6x^2}{16-x^2} = -\frac{6x^2}{(4-x)(4+x)}$

   The denominator doesn't cancel with numerator.

3. The **vertical asymptotes** (V.A.) are the lines $\boldsymbol{x} =$ and $\boldsymbol{x} =$

Horizontal Asymptote (H.A.)

$f(x) = -\frac{6x^2}{16-x^2}$ Rewrite with negative in numerator: $f(x) =$

Compare the degrees numerator to denominator: $\frac{\text{degree}}{\text{degree}} = \underline{\qquad}$

H.A.: $y = \frac{\text{coeff. N}}{\text{coeff. D}} = \frac{\quad}{\quad} =$ The **horizontal asymptote** (H.A.) is the line $\boldsymbol{y} =$.

(go on to the next page)

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- **EXAMPLE:** Find any horizontal or vertical asymptotes. [4.6.35]

$$f(x) = \frac{x^4 + 1}{x^2 + 4x - 12}$$

Horizontal Asymptote (H.A.)

Compare the degrees numerator to denominator: $\dfrac{\text{degree}}{\text{degree}} =$ ________

H.A.: When it's $\dfrac{\text{larger}}{\text{smaller}}$, there is _______ **horizontal asymptote** (H.A.).

Vertical Asymptotes (V.A.)

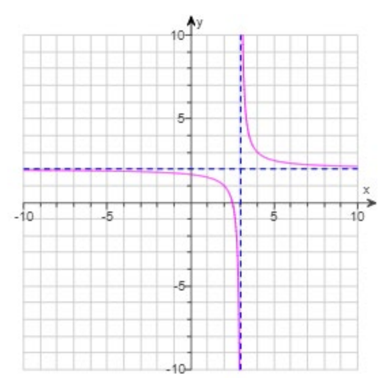
1. Set denominator equal to zero and solve equation. $x^2 + 4x - 12 = 0$
   Factor (it's the fastest): $(x \qquad)(x \qquad) = 0$
   Use Zero Product Property: $x \qquad = 0$ or $x \qquad = 0$
   Solve each equation:
   Combine like terms and simplify: $x =$ or $x =$
   These are domain restrictions: $x \neq$

2. Numerator $x^4 + 1$ doesn't factor, so denominator can't cancel with numerator.

3. The **vertical asymptotes** (V.A.) are the lines $\boldsymbol{x} =$ and $\boldsymbol{x} =$

- **EXAMPLE:** Identify any horizontal or vertical asymptotes in the graph.
  State the domain of *f*. [4.6.13]

Horizontal Asymptote (H.A.): The end behavior (extreme left and extreme right) of the graph of the function shows that it's going FLAT along the horizontal line $\boxed{\boldsymbol{y} = \qquad}$.



Vertical Asymptote (V.A.): The graph of the function bends dramatically along either side of the vertical line $\boxed{\boldsymbol{x} = \qquad}$.

Domain of *f*: The vertical asymptote $x = 3$ means that it is also an ____________________ value in the domain – the function is ______________________ there.

The **domain is** $\boxed{\{\boldsymbol{x} | \qquad\qquad\qquad \}}$.

(read as: the set of all *x* such that *x* is not equal to 3.)

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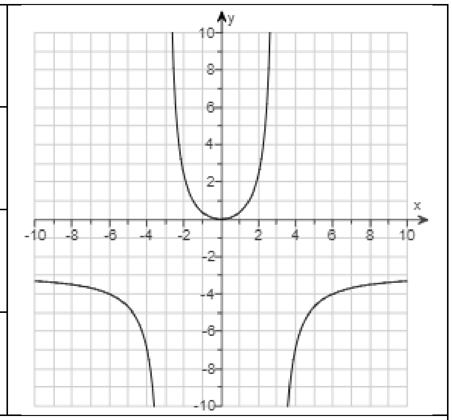
- **EXAMPLE:** Identify any horizontal or vertical asymptotes in the graph.
  Choose the correct asymptotes below. [4.6.15]

A. $x = \pm 3$, no horizontal asymptotes

B. $y = \pm 3, x = -3$

C. $y = 0, x = 0$

D. $y = -3, x = \pm 3$

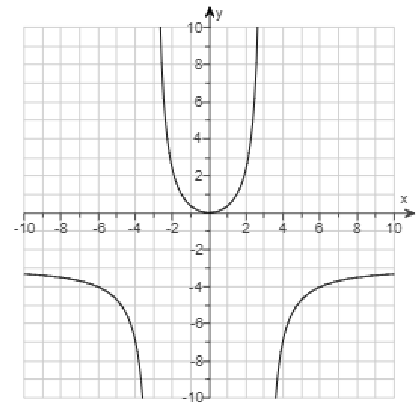


SOLUTION:

Horizontal Asymptote (H.A.):
The END BEHAVIOR of the graph of the function shows that it FLATTENS OUT along the horizontal line $\boldsymbol{y} =$ ______.

Vertical Asymptotes (V.A.): The graph of the function bends dramatically along either side of the vertical lines $\boldsymbol{x} =$ ______ and $\boldsymbol{x} =$ ______ also written as $\boldsymbol{x} =$ ______

The correct answer, therefore, is ______



Sources Used:

1. MyLab Math for *College Algebra with Modeling and Visualization*, 6th Edition, Rockswold, Pearson Education Inc.
2. Desmos website, https://www.desmos.com/, © 2019, Desmos, Inc.
3. Wabbitemu calculator emulator version 1.9.5.21 by Revolution Software, BootFree ©2006-2014 Ben Moody, Rom8x ©2005-2014 Andree Chea. Website https://archive.codeplex.com/?p=wabbit