Notes Section 3.1 – Quadratic Functions and Models

Lesson Objectives

1. Basic terms with quadratic functions
2. Determine if a function is linear, quadratic, or neither
3. Identify (or calculate) characteristics of quadratic functions and graphs
   a. Leading coefficient – opens up or down
   b. Vertex
   c. Axis of symmetry
   d. Intervals of increasing or decreasing
   e. Domain and Range
   f. Maximum or minimum value
4. Using vertex and standard form for a quadratic function

A. **Basic Terms** with Quadratic Functions

Quadratic – a function of one variable where the highest exponent (degree) is 2.
Quadratic comes from the Latin *quadrare*, which means "to make square."

Let *a*, *b*, and *c* be real numbers with *a* ≠ 0. A function represented by

$$f(x) = ax^2 + bx + c$$

is a **quadratic function** (written in **standard form**).

Let *a*, *h*, and *k* be real numbers with *a* ≠ 0. A function represented by

$$f(x) = a(x - h)^2 + k$$

is also a **quadratic function** (written in **vertex form**).

By contrast, a **linear function** is of the form:

$$f(x) = ax + b$$

where *a* and *b* are real numbers (*a* ≠ 0).

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B. Determine if a function is linear, quadratic, or neither

1. For both quadratic and linear, there must be **NO** variables in the denominator.
2. Quadratic: Look for a term with x^2 in it (no **higher** exponents).
   The leading coefficient is beside the x^2.
3. Linear: Look for a term with just an x (exponent **1**) in it (no higher exponents).

- **EXAMPLE:** Identify $f(x) = 4 - 3x + 5x^2$ as being linear, quadratic, or neither. If f is quadratic, identify the leading coefficient a and evaluate $f(-2)$. [3.1.1]

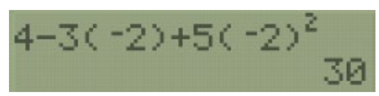
For both quadratic and linear, there must be NO variables in the denominator. **(OK)**
Quadratic: Look for a term with x^2 in it (no higher exponents).
This function has the $+5x^2$ term (no higher exponents), so this function is
QUADRATIC. The leading coefficient (beside x^2) is a = **5**.

Evaluate: $f(-\mathbf{2})$
Given Function: $f(\ x\ ) = 4 - 3\ \ x\ \ + 5\ \ x\ ^2$
Plug in – 2 for x: $f(-\mathbf{2}) = 4 - 3(-\mathbf{2}) + 5(-\mathbf{2})^2$
Be very careful with these negatives here.
Your placement of PARENTHESES is critical. Respect Order of Operations, too.

$$f(-\mathbf{2}) = 4 + \mathbf{6} + 5(\mathbf{4})$$

$$= 4 + \mathbf{6} + \mathbf{20} = \mathbf{10} + \mathbf{20} = \mathbf{30}$$

You don't have to do this by hand.

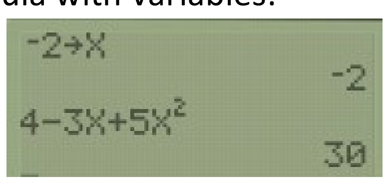


Using calculator: $f(-\mathbf{2}) = 4 - 3(-\mathbf{2}) + 5(-\mathbf{2})^2$
You still have to use parentheses correctly on calculator this way.

But remember, you can do the "**Go to the STO>**" method on the calculator:

Type in: **(-) 2, STO>, XTθn, ENTER**

Then, type in the function formula with variables: $4 - 3x + 5x^2$



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- **EXAMPLE:** Identify $f(x) = \frac{2}{x^2-1}$ as being linear, quadratic, or neither. If *f* is quadratic, identify the leading coefficient *a* and evaluate *f*(– 3). [3.1.3]

  For both quadratic and linear, there must be NO variables in the denominator. **(NO)**
  Although there is an x^2 present, it is in the **denominator** of a fraction.
  Since there is a variable in the denominator, this function *f*(*x*) is
  NEITHER LINEAR NOR QUADRATIC.

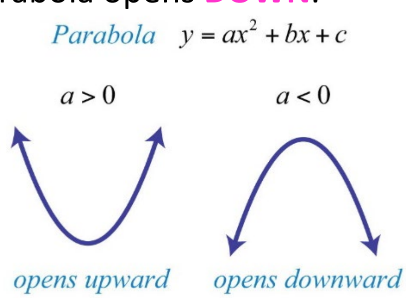
- **EXAMPLE:** Identify $f(x) = \frac{1}{2} - \frac{7}{10}x$ as being linear, quadratic, or neither. If *f* is quadratic, identify the leading coefficient *a* and evaluate *f*(– 2). [3.1.5]

  For both quadratic and linear, there must be NO variables in the denominator. **(OK)**
  Quadratic: Look for a term with x^2 in it. **(NO)**
  Linear: Look for a term with just an *x* (exponent 1) in it (no higher exponents).
  This function has the $-\frac{7}{10}x$ term (no higher exponents), so this function is **LINEAR**.

C. Characteristics Quadratic Functions and Graphs

Parabola – the **U**-shaped graph of a quadratic function.

Leading Coefficient (*a*) – determines whether the parabola opens up or down.
If $a > 0$ (**positive**), the parabola opens **UP**.
If $a < 0$ (**negative**), the parabola opens **DOWN**.



Vertex – the highest point on a parabola that opens downward or the lowest point on a parabola that opens downward. It's where the graph **changes** from decreasing to increasing or vice-versa. $ax^2 + bx + c$

Maximum value – the *y*-coordinate of the vertex of a parabola opening DOWN ($a < 0$)

Minimum value – the *y*-coordinate of the vertex of a parabola opening UP ($a > 0$)

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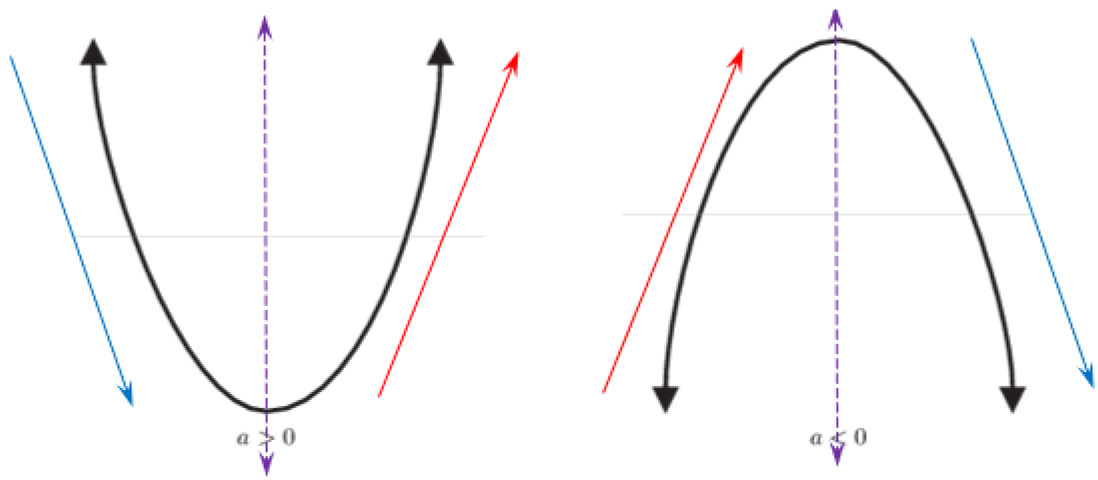
Axis of Symmetry – the **vertical** line passing through the **vertex**. Equation is x = (x-vertex)

Increasing – graph moves **UPWARD**, from left to right

Decreasing – graph moves **DOWNWARD**, from left to right



- **EXAMPLE:** Use the graph to find the following. [3.1.9]

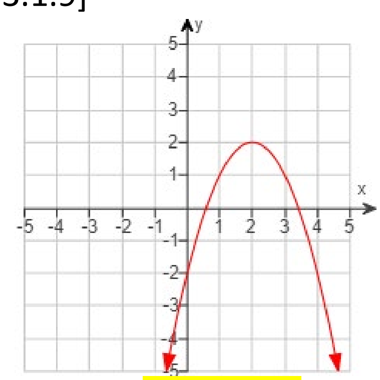
  (a) Sign of the leading coefficient

  (b) Vertex

  (c) Axis of Symmetry

  (d) Intervals where f is increasing and where f is decreasing

  (e) Domain and range



SOLUTION

(a) The parabola opens **DOWN**, so the **sign of the leading coefficient** is **NEGATIVE**.

(b) The **vertex** is located at $(2, 2)$.

(c) The **axis of symmetry** (AOS) is the line $x = 2$. It goes through the vertex.

(d) On the **LEFT** side of the parabola, the function f is **INCREASING**.
Written as inequality: $x < 2$
Interval Notation: $(-\infty, 2)$

On the **RIGHT** side of the parabola, the function f is **DECREASING**.
Written as inequality: $x > 2$
Interval Notation: $(2, \infty)$

(e) The **domain** of f is describing $\boldsymbol{x}$, and it moves **left-to-right**.
(Use the x-axis to help you.)
Written as inequality: **"all real numbers"**
Interval Notation: $(-\infty, \infty)$

The **range** of f is describing $\boldsymbol{y}$, and it moves **low-to-high**.
(Use y-axis to help you.)
Written as inequality: $y \leq 2$
Interval Notation: $(-\infty, 2]$

Always use bracket with an included value!

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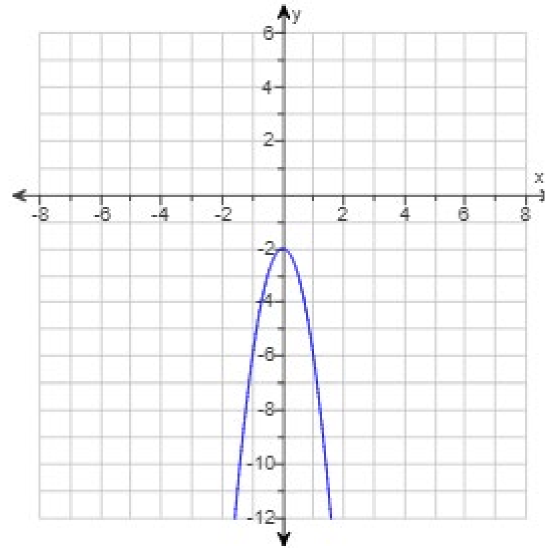
- **EXAMPLE:** Use the graph of *f* to determine the intervals where *f* is increasing and where *f* is decreasing. [1.4-28]
  - **(STEP 1)** ***x*-Vertex** $x = 0$
    (the *x*-coordinate of the vertex)
  - **(STEP 2)** **Write out LEFT & RIGHT sides.**



LEFT side of the parabola	**RIGHT** side of the parabola
Written as Inequality:	
$x < 0$	$x > 0$
Interval Notation:	
$(-\infty, 0)$	$(0, \infty)$

  - **(STEP 3) Who's** Increasing or Decreasing?

Increasing	Decreasing

ANSWER: **Increasing** $(-\infty, 0)$ and **Decreasing** $(0, \infty)$

1. Vertex Form: $f(x) = a(x-h)^2 + k$ Vertex is (h, k)

a is the leading coefficient

(opens ∪ or ∩)

x-Vertex is *h*

(*x*-coordinate of vertex is INSIDE parentheses with *x*)

SWITCH SIGN with *x*

y-Vertex is *k*

(*y*-coordinate of vertex is OUTSIDE parentheses)

SAME SIGN outside – **KEEP IT**

- **EXAMPLE:** Identify the vertex of the parabola and determine whether its graph opens upward or downward. [*Hornsby 3.2.17]

$$f(x) = (x-9)^2 - 3$$

SOLUTION

INSIDE parentheses with *x*, SWITCH SIGN. I see – 9 with *x*, so ***x*-Vertex is +9**
OUTSIDE parentheses is *y*, SAME SIGN (keep it). ***y*-Vertex is – 3**
The vertex is therefore $(9, -3)$
The leading coefficient *a*, is an understood value of **1**.
That is, $f(x) = (x-9)^2 - 3$ can rewrite as $f(x) = 1(x-9)^2 - 3$ ***a* = 1**
Since *a* = 1, that's a POSITIVE number, which **opens upward**.

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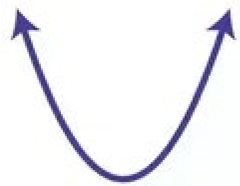
- **EXAMPLE:** Give the largest interval where the function increases or decreases, as requested. $f(x) = (x+3)^2 + 6$; increases [*Hornsby 3.2-30]

  SOLUTION

  - **(STEP 1)** Does the parabola open **UP** or **DOWN**?

    Leading coefficient, *a*, is understood value of **1** (positive). It opens **UP**.

  - **(STEP 2)** Make a **SKETCH** of a parabola opening **UP**.



$f(x) = (x+3)^2 + 6$

$\boldsymbol{x = -3}$

  - **(STEP 3)** ***x*-Vertex** (SWITCH SIGN)
  - **(STEP 4)** **Write out LEFT & RIGHT sides.**

    LEFT side of parabola | **RIGHT** side of parabola

    Written as Inequality:

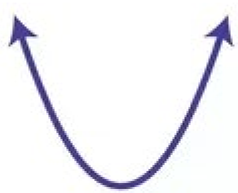
    $\boldsymbol{x < -3}$ | $\boldsymbol{x > -3}$

    Interval Notation:

    $(-\infty, -\mathbf{3})$ | $(-\mathbf{3}, \infty)$

  - **(STEP 5)** **Who's INCREASING or DECREASING**?

    LEFT side is **DECREASING** | **RIGHT** side is **INCREASING**



What we're after: increases

$x = -3$

ANSWER is: $(-\mathbf{3}, \infty)$

(go on to the next page)

2. **Standard Form:** $f(x) = ax^2 + bx + c$ Vertex Formula: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

a
leading coefficient
beside $\boldsymbol{x^2}$

b
beside the $\boldsymbol{x}$

c
has **NO** x

***x*-Vertex**
Use this little formula
$$x = \frac{-b}{2a}$$

***y*-Vertex**
Plug in ***x*-Vertex**
into function to
get *y*-Vertex.

- **EXAMPLE: (a) Use** the vertex formula to find the vertex. [3.1.47]

  (b) Find the intervals where *f* is increasing and where *f* is decreasing.

  $$f(x) = 8 - x^2$$

SOLUTION

Since this function has no parentheses with *x*, then it's in **STANDARD** form.

The terms are not in the right order. Rewrite them highest power to lowest.

$f(x) = 8 - x^2$ rewritten in correct order: $f(x) = \boldsymbol{-x^2 + 8}$

It's missing the *x*-term, so write in a zero placeholder: $\boldsymbol{f(x) = -x^2 + 0x + 8}$

(a) The vertex formula is $x = \frac{-b}{2a}$ $b = \mathbf{0}, a = \mathbf{-1}$ $x = \frac{-0}{2\cdot(-1)} = \mathbf{0}$

***x*-Vertex = 0** Plug in *x* into function (use parentheses) in order to get *y*.

$f(x) = -x^2 + 8$ $f(0) = \boldsymbol{-(0)^2 + 8}$ $\mathbf{0 + 8 = 8}$

***y*-Vertex = 8** Therefore, the coordinates of the vertex are: $\boldsymbol{(0, 8)}$

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- **EXAMPLE: (a)** Use the vertex formula to find the vertex. [3.1.47]
  (b) Find the intervals where *f* is increasing and where *f* is decreasing.

$$f(x) = 8 - x^2$$

SOLUTION

$f(x) = 8 - x^2$ means the same thing as $f(x) = -x^2 + 8$

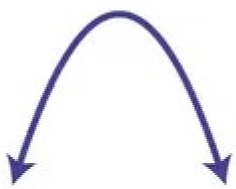
(b) To find intervals of increasing and decreasing:

- **(STEP 1)** Does the parabola open **UP** or **DOWN**?

  Leading coefficient, *a*, is understood value of **–1** (negative). It opens **DOWN**.

- **(STEP 2)** Make a **SKETCH** of a parabola opening **DOWN**.

$f(x) = -x^2 + 8$



$x = 0$

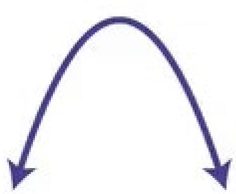
- **(STEP 3)** ***x*-Vertex**
- **(STEP 4)** **Write out LEFT & RIGHT sides.**

LEFT side of parabola	**RIGHT** side of parabola
Written as Inequality:	
$\boldsymbol{x < 0}$	$\boldsymbol{x > 0}$
Interval Notation:	
$(-\infty, \mathbf{0})$	$(\mathbf{0}, \infty)$

- **(STEP 5)** **Who's INCREASING or DECREASING**?

  LEFT side is **INCREASING** **RIGHT** side is **DECREASING**

What we're after: (both)



$x = -0$

ANSWER is: DECREASING on $(\mathbf{0}, \infty)$ and **INCREASING** on $(-\infty, \mathbf{0})$

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- **EXAMPLE:** Identify where *f* is increasing and where *f* is decreasing

  $f(x) = 280x - 70x^2$ [1.4.79]

  SOLUTION

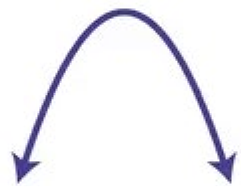
  Reorder the function into correct standard form: $f(x) = \mathbf{-70x^2 + 280x + 0}$

- **(STEP 1)** Does the parabola open **UP** or **DOWN**?

  Leading coefficient, *a*, is **–70** (negative). It opens **DOWN**.

- **(STEP 2)** Make a **SKETCH** of a parabola opening **DOWN**.

$f(x) = -70x^2 + 280x + 0$



$x = 2$

- **(STEP 3)** ***x*-Vertex**

The vertex formula is $x = \frac{-b}{2a}$ $b = \mathbf{280}, a = \mathbf{-70}$ $x = \frac{\mathbf{-280}}{\mathbf{2 \cdot (-70)}} = \frac{\mathbf{-280}}{\mathbf{-140}} = \mathbf{2}$

We don't need the *y*-Vertex because we're only finding intervals of increasing and decreasing. These only use ***x*-Vertex**.

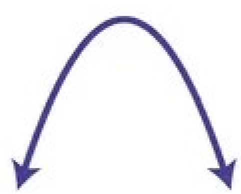
- **(STEP 4)** **Write out LEFT & RIGHT sides.**

LEFT side of parabola	**RIGHT** side of parabola
Written as Inequality:	
$\mathbf{x < 2}$	$\mathbf{x > 2}$
Interval Notation:	
$(-\infty, \mathbf{2})$	$(\mathbf{2}, \infty)$

- **(STEP 5)** **Who's INCREASING or DECREASING**?

  LEFT side is **INCREASING** **RIGHT** side is **DECREASING**

$f(x) = -70x^2 + 280x + 0$



$x = 2$

ANSWER: Over the interval $(-\infty, \mathbf{2})$ the function *f* is **increasing**.
Over the interval $(\mathbf{2}, \infty)$ the function *f* is **decreasing**.

Notes Section 3.1 – Quadratic Functions and Models

3. Find the Maximum or Minimum Value of a Quadratic Function

That's the job of the *y*-coordinate of the vertex. **Max./Min…use *y*-Vertex**

- **EXAMPLE:** If a football is kicked straight up with an initial velocity of 64 ft/sec from a height of 4 feet, then its height above the earth is a function of time given by

  $h(t) = -16t^2 + 64t + 4 \geq$

  What is the maximum height reached by the ball? [3.1.123]

SOLUTION

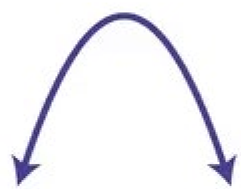
For convenience, let's rewrite the function

$h(t) = -16t^2 + 64t + 4$ as $f(x) = -16x^2 + 64x + 4$

- **(STEP 1)** Does the parabola open **UP** or **DOWN**?

  Leading coefficient, *a*, is **–16** (negative). It opens **DOWN**.

- **(STEP 2)** Make a **SKETCH** of a parabola opening **DOWN**.



$f(x) = -16x^2 + 64x + 4$

$x = \mathbf{2}$

- **(STEP 3)** ***x*-Vertex**

The vertex formula is $x = \frac{-b}{2a}$ $b = \mathbf{64}, a = \mathbf{-16}$ $x = \frac{-64}{2\cdot(-16)} = \frac{-64}{-32} = 2$

- **(STEP 4)** ***y*-Vertex**

Since ***x*-Vertex = 2** **Plug in** *x* into function (use parentheses) in order to get *y*.

$f(x) = -16x^2 + 64x + 4$ $f(2) = -16(\mathbf{2})^2 + 64(\mathbf{2}) + 4$

$= -16(\mathbf{4}) + \mathbf{128} + 4 = \mathbf{-64} + \mathbf{132} = \mathbf{68}$

The ***y*-Vertex = 68** Therefore, the maximum height reached by the ball is 68 ft.

Sources Used:

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2. MyLab Math for *College Algebra with Modeling and Visualization*, 6th Edition, Rockswold, Pearson Education Inc.
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4. Wabbitemu calculator emulator version 1.9.5.21 by Revolution Software, BootFree ©2006-2014 Ben Moody, Rom8x ©2005-2014 Andree Chea. Website https://archive.codeplex.com/?p=wabbit