Determinants

4.2, 4.3, 4.4

Recall that for a 2 x 2 matrix, A =

it is only invertible if ad – bc $\neq$ 0.

This quantity is called the determinant of A.

Notation:

det(A) = ad – bc

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Definitions

- If A is a square matrix, then the <u>minor</u> of entry a_{ij} is M_{ij} , which is the determinant of the submatrix that remains after deleting row i and column j.

- The *cofactor* of a_{ij} is C_{ij} is $(-1)^{i+j}M_{ij}$

Eg

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

Minor of a_{11} is $M_{11} = \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} = 2 \cdot 3 - 4 \cdot 2 = -2$

Minor of a_{23} is $M_{23} = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 8 - 1 = 7$

The cofactor is a_{23} is $C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -7$

Minors - Pattern

To get the cofactors, we use the pattern (shown for a 3x3 matrix below) and apply it to the minors:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Definition

The determinant of a square matrix A is obtained by multiplying the entries of any row/column by their corresponding cofactors and adding up these products.

Eg $A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

Eg. Find det(B)

$$B=\begin{bmatrix} 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & 1 & 2 & 2 \end{bmatrix}$$

Eg. Determinant of a Upper Triangular Matrix

If A is a upper triangular, lower triangular or diagonal matrix, the determinant det (A) is the product of the entries of the main diagonal.

Find det(C) for $C = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Properties of Determinants

1. det(kA) =$k^n \mathrm{det}(A)$

Eg. Let $\mathrm{A} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ and k = 3. Find det(3A)

2. A square matrix A is invertible if and only if $\det(A) \neq 0$

3. If A and B are square matrices of the same size, then

det(AB)=det(A)det(B)

Eg. $\mathrm{A} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ $\mathrm{B} = \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix}$. Show property 3

4. If A is invertible, then:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

5. Note $\det(A+B) \neq \det(A) + \det(B)$

Definition

For A_{nxn} and C_{ij} is the cofactor of a_{ij}, then

$$matrix\ of\ cofactors = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & & C_{2n} \\ \vdots & & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

$$The\ transpose\ of\ this\ matrix\ is\ called\ the\ adjoint\ of\ A$$

$$adj(A) = \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & & C_{n2} \\ \vdots & & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Inverse of a Matrix

If A is invertible, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Eg: A= $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

Its matrix of cofactors is: $\begin{bmatrix} 12 & -(-6) & -16 \\ -(-4) & 2 & -(-16) \\ 12 & -(10) & 16 \end{bmatrix}$

The adjoint of the matrix is: $\operatorname{adj}(\mathrm{A})\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$

$$\det(A) = 2(6+6) - (-4)(9+1) + 0 = 64$$

Therefore $A^{-1} = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{3}{16} & \frac{1}{16} & \frac{3}{16} \\ \frac{3}{32} & \frac{1}{32} & -\frac{5}{32} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

Eg Find the inverse of B= $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$

Cramer’s Rule

If **Ax** = **b** is a linear system with n unknown and det(A) ≠ 0, then:

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}$$

Where A_j is the matrix obtained by replacing its jth column with **b**

Eg. Use Cramer's Rule to solve

$$x_1 + 2x_3 = 6$$
$$-3x_1 + 4x_2 + 6x_3 = 30$$
$$-x_1 - 2x_2 + 3x_3 = 8$$

$$\det(A) = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} =$$

$$\det(A_1) = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} =$$

$$\det(A_2) = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix}$$

$$\det(A_3) = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix}$$

Eg Use Cramer's Rule to solve

$$x + y + z = 5$$

$$2x - 2y + z = 7$$

$$x - y \quad = 2$$

Lab – Unit 4

Pg 79

1. Find the determinant:

a) A= $\begin{bmatrix} 3 & 0 & -1 \\ -1 & 3 & 1 \\ 8 & 0 & 2 \end{bmatrix}$

b) $B = \begin{bmatrix} 3 & 4 & 1 & 2 \\ 1 & -1 & 0 & 2 \\ 4 & 0 & 0 & 0 \\ 3 & -1 & 2 & 1 \end{bmatrix}$

2. For $A = \begin{bmatrix} 2 & 2 \\ 5 & -2 \end{bmatrix}$ $and\ k = -2\ verify\ that\ \det(kA) = k^n \det(A)$.

3. Use determinants to decide if the matrix is invertible:

a) $B = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ 2 & 0 & -4 \end{bmatrix}$

b) $C = \begin{bmatrix} -2 & 0 & 8 \\ 5 & 0 & 7 \\ 2 & 0 & -1 \end{bmatrix}$

4. Find k so that A is invertible: $A = \begin{bmatrix} k & 2 \\ 8 & k \end{bmatrix}$

5. Find the inverse using the adjoint: $B = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}$

6. Solve using Cramer's Rule:

$$2x + 3y + z = 4$$

$$3x - z = -3$$

$$x - 2y + 2z = -5$$