Special Matrices

4.1

Square Matrices

- Square matrices are matrices that have size $n \times n$
- For example, the Identity Matrix is always a square matrix.
- Any matrix with size 2x2, 3x3, 4x4, and so on… are all square matrices

Symmetric Matrices

- A Symmetric Matrix is a square matrix A such that A=A^T, where the matrix itself is equal to its transpose.
- For example, the Identity Matrix is a symmetric matrix because I=I^T
- Eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Diagonal Matrices

- A Diagonal Matrix is a square matrix such that entries off the main diagonal are zero. Recall that the main diagonal is the diagonal from the top left to the bottom right.
- For example, the Identity Matrix is a diagonal matrix. It contains 1's on the main diagonal, and entries off the main diagonal are zero.
- Eg $A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Triangular Matrices

- A Triangular Matrix is a square matrix such that entries below or above the main diagonal are zeros.
- Eg $A = \begin{bmatrix} 7 & 0 \\ 5 & -1 \end{bmatrix} B = \begin{bmatrix} 7 & 9 \\ 0 & -1 \end{bmatrix} C = \begin{bmatrix} 3 & 1 & 9 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} D = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 5 & 1 \end{bmatrix}$
- In an upper triangular matrix, all entries below the main diagonal as zeros. See B and C.
- In a lower triangular matrix, all entries above the main diagonal as zeros. See A and D.

Eg 1

- **Identify the following special matrices.**

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{10} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{8} \end{bmatrix} \boldsymbol{C} = \begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix} \boldsymbol{D} = \begin{bmatrix} \mathbf{7} & \mathbf{0} \\ \mathbf{0} & \mathbf{9} \end{bmatrix}$$

- A is a diagonal matrix
- B is an upper triangular matrix
- C is a symmetric matrix
- D is a diagonal matrix

Properties of Special Matrices

Inverses

1) A diagonal matrix is invertible if and only if all of its entries on the main diagonal are non-zero values

2) A triangular matrix is invertible if and only if all of its entries on the main diagonal are non-zero values

3) Inverse of a diagonal matrix, $A = \begin{bmatrix} a & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & k \end{bmatrix} A^{-1} = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & k^{-1} \end{bmatrix}$

4) The inverse of an invertible lower triangular matrix is a lower triangular matrix

5) The inverse of an invertible upper triangular matrix is an upper triangular matrix

Sums

6) If A and B are symmetric and are the same size, then the sum $A+B$ or $A-B$ are also symmetric

7) If A is symmetric and k is a scalar value, then kA is symmetric

Products

8) If a matrix A is a diagonal matrix, $A = \begin{bmatrix} a & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & k \end{bmatrix}$ $A^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & k^n \end{bmatrix}$

9) The product of two lower triangular matrices is a lower triangular matrix

10) The product of two upper triangular matrices is an upper triangular matrix.

Transpose

11) The transpose of an upper triangular matrix is a lower triangular matrix, and vice versa

Example 2

Given the matrix $D = \begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}$, determine D^{-1}

Example 3

Given the matrix $C = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that kC is also symmetric