Lesson 05 – Finding the Inverse

Ch 3

Elementary Matrices

Elementary matrices are matrices that can be obtained from the n x n identity matrix by performing a single elementary row operation.

Eg $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ multiply row 2 by 2 $\Rightarrow E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ switch rows 2 & 3 $\Rightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

- If we start with I and perform exactly one elementary row operation, then the resulting matrix is called an elementary matrix.

<u>3 Types</u>

1. Interchanging two rows of I.
2. Multiplying a row of I by a non-zero constant.
3. Replacing a row of I by adding it to a multiple of another.

Theorem

If E results from a row operation on I_m, then EA_{mxn} is the matrix resulting from the same row operation on A.

Eg 1 $$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

Eg 2 $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 5 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 4 \\ 2 & 1 & 5 \end{bmatrix}$$

We can always restore E back to I by applying the inverse row operation.

Operation	Inverse
switch row i with j	switch row j with i.
Multiply row i by c	multiply row i by 1/c.
Row i x c plus row j	row i x – c plus j

Thm: Every elementary matrix is invertible (ie nonsingular) and the inverse is also an elementary matrix.

Row Equivalent

- A matrix B is row equivalent to A if there is a finite sequence of elementary matrices s.t.:

$$B = E_m E_{m-1} \cdots E_1 A$$

Theorem: Equivalent Conditions for Invertibility (Nonsingularity)

Let A be a nxn matrix. The following are equivalent:

a) A is invertible.
b) A**X** = **0** has only the trivial solution **0**.
c) The reduced row row echelon form of A is I.
d) A is expressible as a product of elementary matrices.

So if A is invertible and, then A is row equivalent to I and:

$$E_m E_{m-1} \cdots E_1 A = I$$

So the sequence of row operations that transforms A $\rightarrow$ I will transform I $\rightarrow$ A^{-1}.

Finding A^{-1}

1) $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix}$

2.

$$B = \begin{bmatrix} 1 & -1 & -3 \\ 0 & -1 & -2 \\ 2 & 1 & -1 \end{bmatrix}$$

Note: When performing row operations, we get a row of zeroes, that means A cannot be reduced to I; thus A is not invertible. We conclude that the matrix does not have an inverse.

Eg A= $\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{bmatrix}$

$$\xRightarrow{R2+R1(-2);\, R3+R1} \begin{bmatrix} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{bmatrix} \xRightarrow{R2+R3} \begin{bmatrix} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

The last row on the left side is a zero row thus the matrix cannot be converted into an identity matrix by row operations. Conclusion: A is not invertible.

Consequence of invertibility

When a matrix is invertible, the homogenous system associated with that matrix has only a trivial solution.

Theorem 3.

If A is an invertible n x n matrix, then for each n x 1 matrix b, the system of equations A**x** = **b** has exactly one solution and that is **x** = A^{-1}b.

Eg. Solve

$$2x - y = 7$$
$$5x - 3y = 18$$

We can represent this as:

$$\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 18 \end{bmatrix}$$

Unit 3: Inverse of a Matrix

Lab: Textbook pg 76-77

Practice 3-1

1 Determine whether the following are inverses of one another.

$a)\ A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 9 & -1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 2 \\ 1 & -2 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 2 \\ 1 & -2 & 0 \end{bmatrix} B = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

2. Determine whether the following 2x2 matrices are invertible

$b)\ \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix}$

3-2

2) Given $\mathrm{B} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}, find\ B^{-1}$

5) Given $\mathrm{H} = \begin{bmatrix} 8 & 3 \\ 1 & -1 \end{bmatrix}, find\ (B^{-1})^{T}$

3-3 Given the following 3x3 matrices, determine the inverse.

1. $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 0 & 2 \\ 3 & 5 & 1 \end{bmatrix}$

3-3 Given the following 3x3 matrices, determine the inverse.

3. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

3-4 Solve the given linear systems using the inverse of a matrix

1. $x + 2y = 3$

   $3x + 4y = 11$