

UBC MATH 264: Vector Calculus for Electrical Engineering

Vector Identities

For vectors $\mathbf{u} = \langle u_x, u_y, u_z \rangle$, $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$,

$$\text{Unit Vector } \hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

$$\text{Dot Product } \mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

$$\text{Magnitude } \|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

$$\text{Cross Product } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

Distance and Projection

Distance from (x_0, y_0, z_0) to plane $Ax + By + Cz + D = 0$

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Distance from (x_0, y_0, z_0) to the line pass through (x_1, y_1, z_1) with direction $\langle a, b, c \rangle$

$$d = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_0 - x_1 & y_0 - y_1 & z_0 - z_1 \\ a & b & c \end{matrix} \right\|$$

$$\text{Projections } \text{proj}_{\mathbf{u}}(\mathbf{F}) = \left(\frac{\mathbf{F} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u}$$

Parameterization

$$\text{Arc Length } \int_a^b \|\mathbf{r}'(t)\| dt$$

$$\text{Surface Area } \iint_D \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du dv$$

$$\text{Line Integral } \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

$$\text{Surface Integral } \int_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) du dv$$

Surface Normals and Area Elements

$$\text{Level Surface } F(x, y, z) = 0 \quad \mathbf{n} = \pm \nabla F(x, y, z),$$

$$\text{Parametric Surface } \mathbf{r}(u, v) \quad d\mathbf{S} = \pm \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du dv$$

Cartesian Coordinates (x, y, z)

$$\text{Line Element } d\mathbf{l} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz$$

$$\text{Volume Element } dV = dx dy dz$$

$$\text{Gradient } \nabla f = \frac{\partial f}{\partial x} \mathbf{a}_x + \frac{\partial f}{\partial y} \mathbf{a}_y + \frac{\partial f}{\partial z} \mathbf{a}_z$$

Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$
Divergence	$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$
Cylindrical Coordinates (ρ, ϕ, z)	
Transformation	$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z$
Local Basis	$\mathbf{a}_\rho = \cos \phi \mathbf{a}_x + \sin \phi \mathbf{a}_y$ $\mathbf{a}_\phi = -\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y$
Line Element	$d\mathbf{l} = \mathbf{a}_\rho d\rho + \rho \mathbf{a}_\phi d\phi + \mathbf{a}_z dz$
Volume Element	$dV = \rho d\rho d\phi dz$
Gradient	$\nabla f = \frac{\partial f}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \mathbf{a}_\phi + \frac{\partial f}{\partial z} \mathbf{a}_z$
Curl	$\nabla \times \mathbf{F} = \frac{1}{\rho} \begin{vmatrix} \mathbf{a}_\rho & \rho \mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$
Divergence	$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$
Spherical Coordinates (r, θ, ϕ)	
Transformation	$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$
Local Basis	$\mathbf{a}_r = \sin \theta \cos \phi \mathbf{a}_x + \sin \theta \sin \phi \mathbf{a}_y + \cos \theta \mathbf{a}_z$ $\mathbf{a}_\theta = \cos \theta \cos \phi \mathbf{a}_x + \cos \theta \sin \phi \mathbf{a}_y - \sin \theta \mathbf{a}_z$ $\mathbf{a}_\phi = -\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y$
Line Element	$d\mathbf{l} = \mathbf{a}_r dr + r \mathbf{a}_\theta d\theta + r \sin \theta \mathbf{a}_\phi d\phi$
Volume Element	$dV = r^2 \sin \theta dr d\theta d\phi$
Gradient	$\nabla f = \frac{\partial f}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{a}_\phi$
Curl	$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \mathbf{a}_r & r \mathbf{a}_\theta & r \sin \theta \mathbf{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin \theta F_\phi \end{vmatrix}$
Divergence	$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (F_\theta \sin \theta) + \frac{\partial F_\phi}{\partial \phi} \right]$
Fundamental Theorems of Calculus	
Gradient Theorem	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
Divergence Theorem	$\iiint_V (\nabla \cdot \mathbf{F}) dV = \iint_{\partial V} \mathbf{F} \cdot d\mathbf{S}$
Stokes's Theorem	$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$