Kiepert's Theorem in Triangle Geometry

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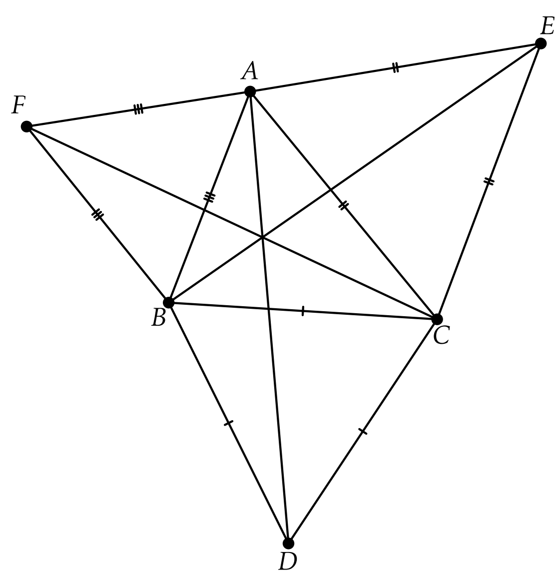
1. Foundation

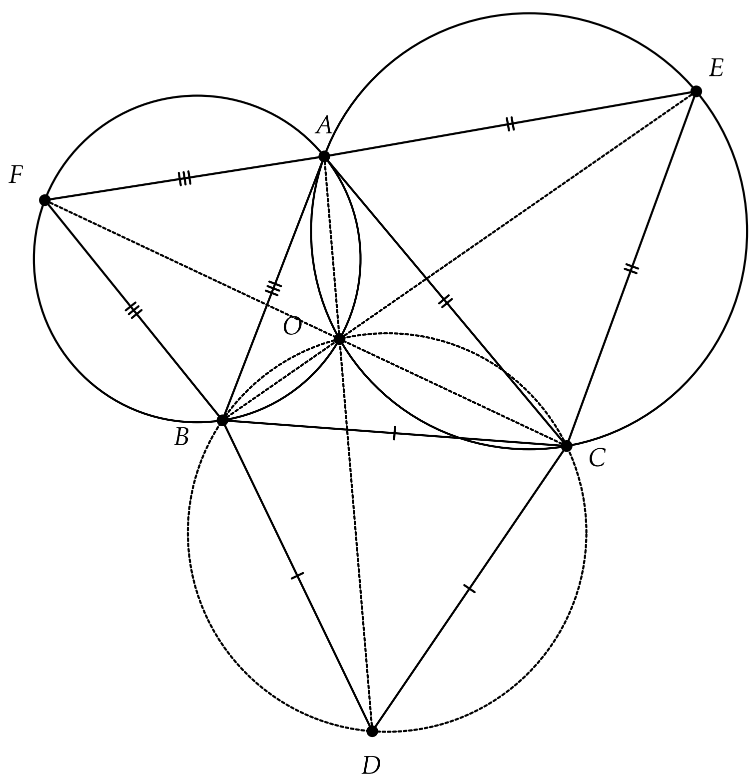
Theorem 1.1. *(Fermat Points)* Given triangle ΔABC, construct equilateral triangles $\Delta BCD, \Delta CAE$ and ΔBAF, externally (or internally) on sides BC, CA and AB. Prove that AD, BE and CF are concurrent at the external Fermat point of triangle ABC.



Proof. Let O be the intersection of two circles (ABF) and (ACE). Our purpose is to prove that the sets of points $(A,O,D),(B,O,E),(C,O,F)$ are respectively collinear. We observe that $AOBF$ and $AOCE$ are cyclic quadrilaterals.

As a result,

$$\angle AOB = 180^\circ - \angle AFB = 180^\circ - 60^\circ = 120^\circ$$
$$\angle COA = 180^\circ - \angle CEA = 180^\circ - 60^\circ = 120^\circ$$
$$\angle AOF = \angle ABF = \angle FAB = \angle FOB = 60^\circ$$
$$\angle AOE = \angle ACE = \angle EAC = \angle EOC = 60^\circ$$



Additionally, the sum of three angles

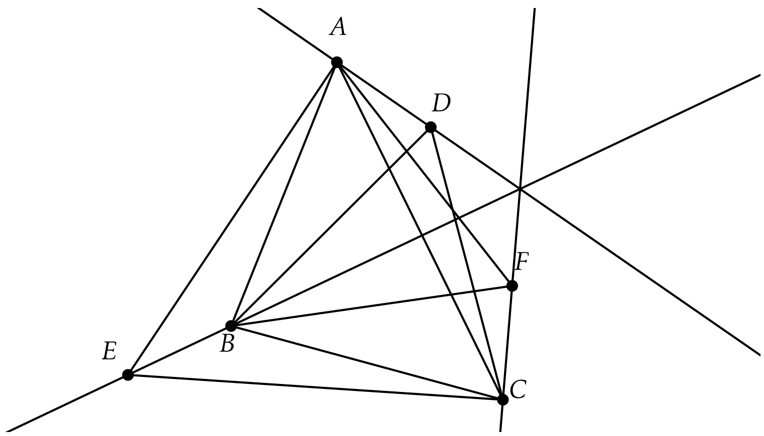
$$\angle AOB + \angle BOC + \angle COA = 360^\circ \Rightarrow \angle BOC = 120^\circ$$

In the quadrilateral $OBCD$, we have $\angle BOC + \angle BCD = 120^\circ + 60^\circ = 180^\circ$. Therefore, $OBCD$ is also a quadrilateral, so points O, B, C and D are concyclic.

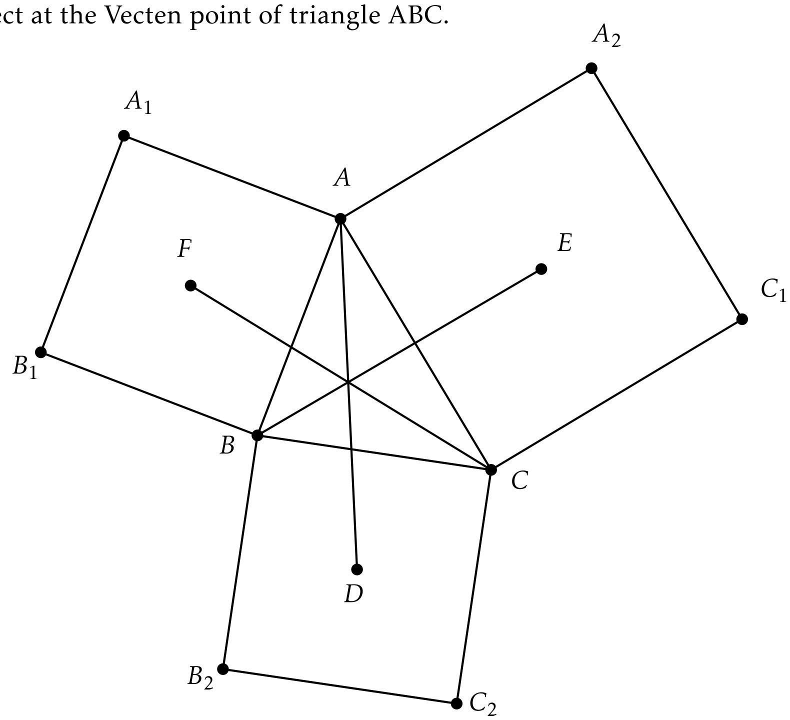
Hence, $\angle BOD = \angle BCD = 60^\circ \Rightarrow \angle AOD = \angle AOB + \angle BOD = 60^\circ + 120^\circ = 180^\circ \Rightarrow A, O$ and D are collinear.

Similarly, we can also prove the sets of points (B, O, E) and (C, O, F) are respectively collinear. Therefore, AD, BE and CF are concurrent. This point of concurrency is called the positive Fermat point of triangle ABC.

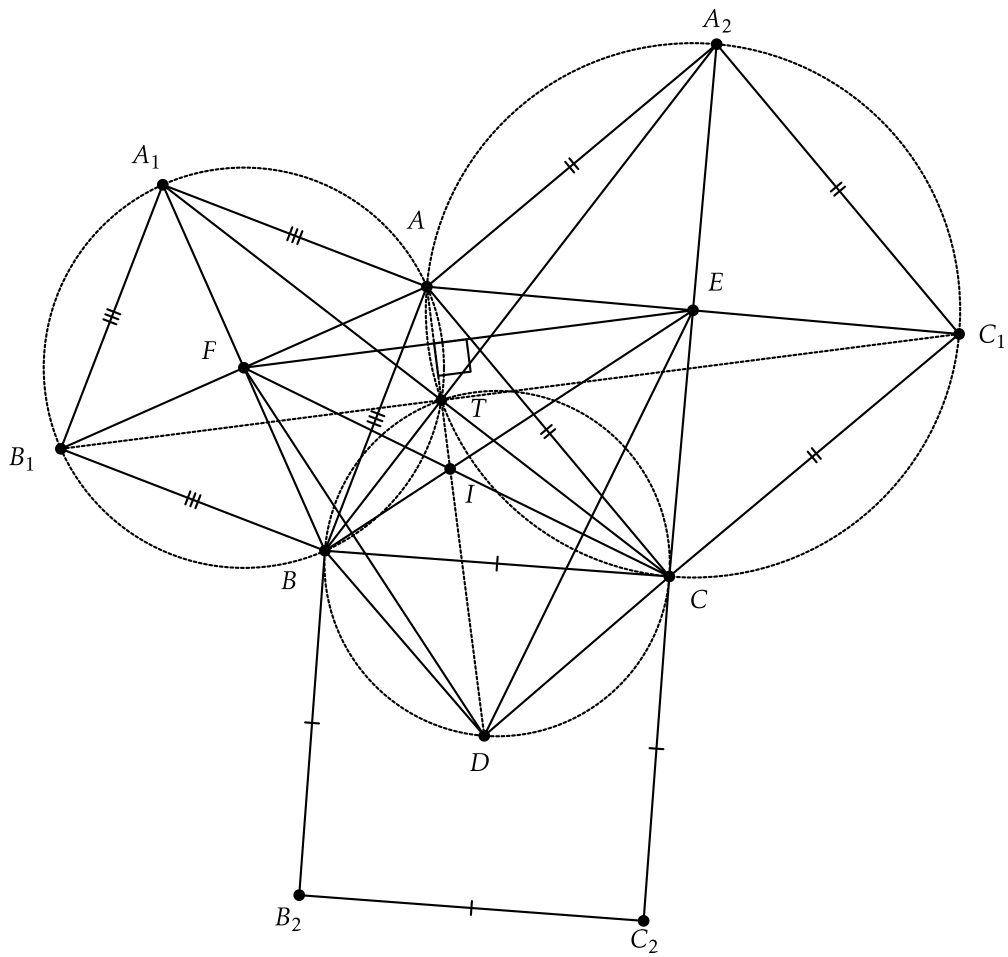
When we construct the equilaterals triangle internally on sides BC, CA and AB, we obtain the negative Fermat point. The proof will be left as an exercise for readers.



Theorem 1.2. *(Vecten Points)* Given triangle ABC, construct three squares outwardly (or inwardly) on its sides. Then, the lines connecting each vertex of the triangle to the center of the square constructed on the opposite side intersect at the Vecten point of triangle ABC.



Proof. Let A_1, A_2, B_1, B_2, C_1 and C_2 be the vertices of the squares as shown in the figure. Let I and T be two points of intersection of (BE, CF) and (BA_2, CA_1).



Considering ΔAA_1C and ΔAA_2B, we have $AA_1 = AB, AC = AA_2$ and $\angle A_1AC = \angle BAA_2$ $(= \angle A + 90°) \Rightarrow \Delta AA_1C = \Delta AA_2B$. Hence,

$$\angle AA_1C = \angle ABA_2$$
$$\angle ACA_1 = \angle AA_2B$$

Therefore, $ATBA_1$ and $ATCA_2$ are cyclic quadrilaterals $\Rightarrow \angle ATB_1 = \angle ABB_1$ $= \angle CAA_2 = \angle CTA_2 = 90° \Rightarrow AT \perp TB_1, AT \perp TC_1 \Rightarrow TB_1 \equiv TC_1$. Hence, we obtain T, B_1 and C_1 are collinear.

Since AA_1BT is a cyclic quadrilateral, $T \in (ABA_1) \Rightarrow T \in (AA_1B_1B)$. Therefore, TA_1B_1B is also a cyclic quadrilateral $\Rightarrow \angle BTB_1 = \angle B_1AB = 45°$ and $\angle ATB = 180° - \angle A_1B_1B = 180° - 90° = 90°$. This also means $BA_2 \perp CA_1$.

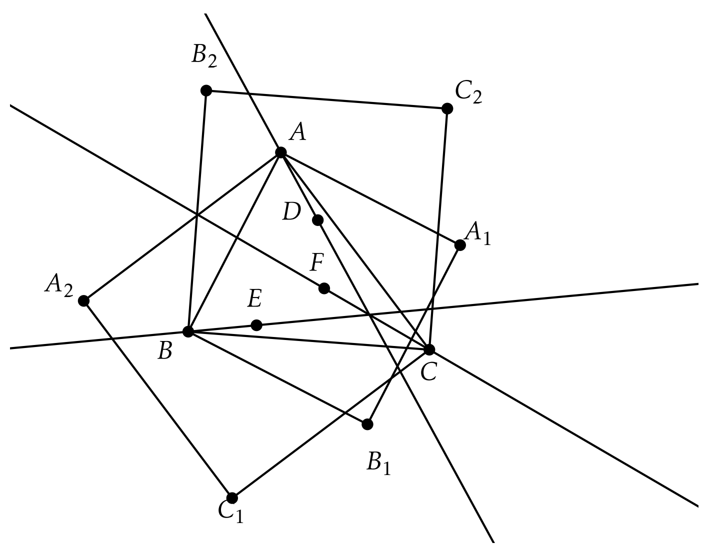
Since $\angle BTC + \angle BDC = 90° + 90° = 180°$, $BTCD$ is a cyclic quadrilateral $\Rightarrow \angle BTD = \angle BCD = 45°$. As a result, $\angle BTD = \angle B_1TB + \angle BTD = 90° \Rightarrow TD \perp B_1C_1$.

We have $TD \perp B_1C_1$ and $AT \perp B_1C_1 \Rightarrow AT \equiv TD$. Hence, A, D and T are collinear.

Since E and F are respectively the centers of squares AA_2C_1C and AA_1B_1B, they are respectively midpoints AB_1 and AC_1. According to the midpoint theorem in triangle AB_1C_1, $EF \parallel B_1C_1$. We already proved $AT \perp B_1C_1$ so $EF \perp AD$ or AD is an altitude of triangle DEF.

Similarly, we can prove $BE \perp FD$ and $CF \perp DE$. This means BE and CF are also the altitudes of triangle DEF. Hence, three altitudes AD, BE and CF of triangle DEF are concurrent at I. This point of concurrency is called the positive Vecten point of triangle ABC.

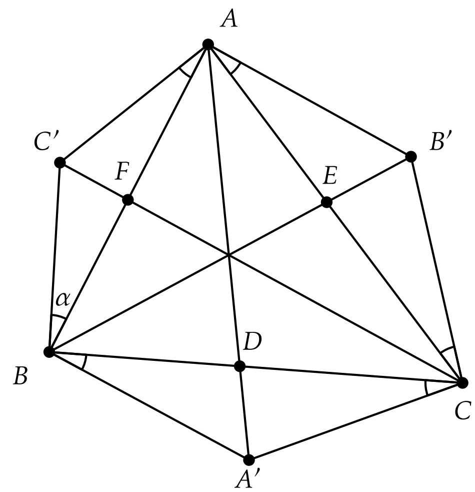
When we construct the squares internally on sides BC, CA and AB, we obtain the negative Vecten point. The proof will be left as an exercise for readers.



As we prove the theorem above, a pattern arises: When regular polygons are constructed externally (or internally) on the three sides of a triangle, the lines connecting each vertex of the triangle to the center of the regular polygon built on the opposite side all intersect at a single point. Fermat points are a limiting case when the regular polygons are hexagons (the vertex of an equilateral triangle will be the center of a hexagon), and Vecten points are a limiting case when the regular polygons are squares. Other limiting cases are Napoleon points when the regular polygons are equilateral triangles. However, this case can not be proven by using the properties of a cyclic quadrilateral as we used before due to a lack of symmetry. Therefore, we need a more useful tool to solve this problem.

2. Kiepert's Theorem

Theorem 2.1. *(Kiepert's Theorem)* Construct outward (or inward) on the sides BC, CA, AB of triangle ABC the isosceles triangles BCA', CAB' and ABC' each similar and isosceles with vertexes at A', B' and C' respectively. Then, the lines AA', BB' and CC' are concurrent at a single point, which is called the Kiepert point of triangle ABC.



Proof. Let α be the common base angle of the isosceles triangles BCA', CAB' and ABC' and points D, E, F are respectively points of intersection of sets (AA', BC), (BB', AC) and (CC', AB).

We apply Sine Theorem in ΔABD and ΔACD

$$\frac{DB}{DC} = \frac{AD \sin\angle BAD/\sin\angle B}{AD \sin\angle CAD/\sin\angle C} = \frac{\sin BAA'}{\sin CAA'}\frac{\sin\angle C}{\sin\angle B}$$

We apply Sine Theorem again in $\Delta ABA'$ and $\Delta ACA'$

$$\frac{DB}{DC} = \frac{BA' \sin\angle ABA'/AA'}{CA' \sin\angle ACA'/AA'}\frac{\sin\angle C}{\sin\angle B}$$

Since $BA' = CA'$, we yield

$$\frac{DB}{DC} = \frac{\sin(\angle B+\alpha)}{\sin(\angle C+\alpha)}\frac{\sin\angle C}{\sin\angle B}$$

Substituting $x = \sin(\angle A+\alpha), y = \sin(\angle B+\alpha)$ and $z = \sin(\angle C+\alpha)$, we rewrite the equation above as

$$\frac{DB}{DC} = \frac{y}{z}\frac{\sin C}{\sin B}$$

Similarly, we can prove

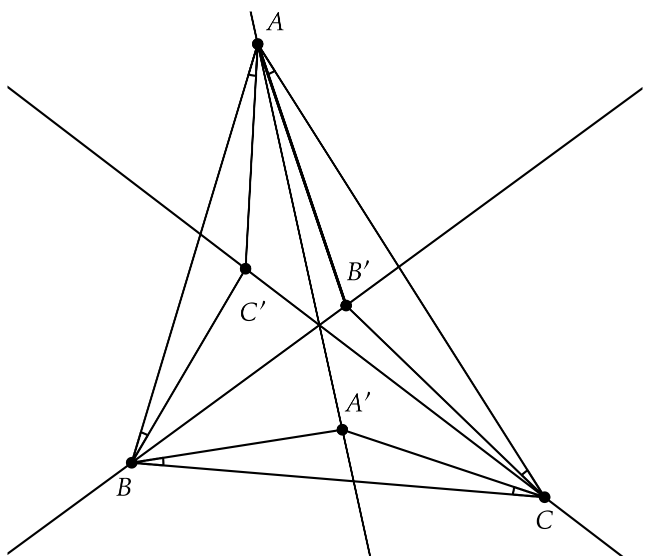
$$\frac{EC}{EA} = \frac{z}{x}\frac{\sin\angle A}{\sin\angle C} \qquad \text{and} \qquad \frac{FA}{FB} = \frac{x}{y}\frac{\sin\angle B}{\sin\angle A}$$

As a result, in triangle ABC

$$\frac{DB}{DC}\frac{EC}{EA}\frac{FA}{FB} = \frac{y}{z}\frac{\sin C}{\sin B} \times \frac{z}{x}\frac{\sin\angle A}{\sin\angle C} \times \frac{x}{y}\frac{\sin\angle B}{\sin\angle A} = 1$$

According to Ceva's Theorem, AD, BE and CF must be concurrent. This point of concurrency is called the positive Kiepert point of triangle ABC.

Similarly, when we construct the squares internally on sides BC, CA and AB, we obtain the negative Kiepert point. The proof will be left as an exercise for the readers.



With this theorem, we can easily prove the concurrency of Napoelon Points of a triangle.

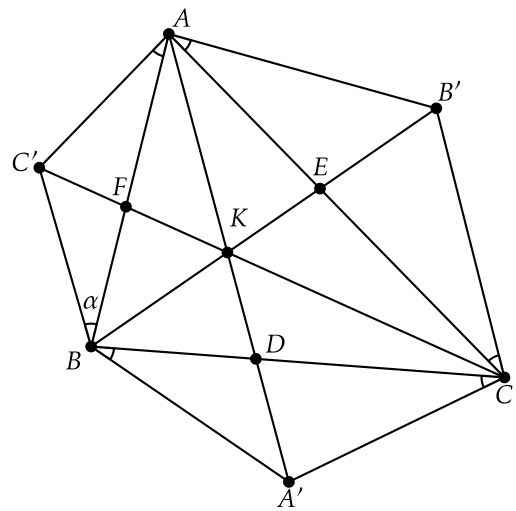
Theorem 2.2. *(Napoleon's Points)* Given a triangle ΔABC, construct equilateral triangles externally (or internally) on its sides BC, CA and AB. Let D, E and F be the centroids of the equilateral triangles constructed on BC, CA and AB, respectively. Then, the lines AD, BE and CF are concurrent at a single point, called the *Napoleon point* of triangle ABC.

Proof. Napoleon's Points are special cases of Kiepert's Theorem when angle $\alpha = 120°$.

Now we will look at one of the characteristics of Kiepert points.

Theorem 2.3. Given a triangle ABC and a Kiepert point constructed by angle α. Set $x = \sin(\angle A+\alpha), y = \sin(\angle B+\alpha), z = \sin(\angle C+\alpha)$ and $a = BC, y = CA, z = AB.$. The coordinates of the Kiepert point K can be written as

$$\overrightarrow{K} = \frac{ayz\overrightarrow{A} + bxz\overrightarrow{B} + cxy\overrightarrow{C}}{ayz + bxz + cxy}$$



Proof. As shown before, we computed the ratio

$$\frac{DB}{DC} = \frac{y}{z}\frac{\sin\angle C}{\sin\angle B} = \frac{yc}{zb}$$

according to Sine theorem.

So we can express the coordinates of D in terms of coordinates of A, B and C by noticing

$$\overrightarrow{DB} = -\frac{yc}{zb}\overrightarrow{DC}$$

Therefore,

$$\overrightarrow{B} - \overrightarrow{D} = \frac{yc}{zb}(\overrightarrow{D} - \overrightarrow{C})$$

This is equivalent to

$$\overrightarrow{D} = \frac{bz\overrightarrow{B} + yc\overrightarrow{C}}{bz + yc} \tag{1}$$

Let K be the point of concurrency of AD, BE and CF. We apply Sine theorem in triangle KBA and KBD, yielding

$$\frac{KA}{KD} = \frac{\sin\angle KBA \times AB/\sin\angle BKA}{\sin\angle DBK \times BD/\sin\angle BKD}$$

Sin $\angle AKB + \angle BKD = 180°$, their trigonometric functions are equal to each other, or $\sin\angle AKB = \sin\angle BKD$. Therefore,

$$\frac{KA}{KD} = \frac{\sin\angle B'BA \times AB}{\sin\angle B'BC \times BD}$$

We apply Sine theorem again in triangle ABB' and BCB',

$$\frac{KA}{KD} = \frac{AB}{BD}\frac{AB'\sin\angle BAB'/BB'}{CB'\sin\angle BCB'/BB'} = \frac{c}{BD}\frac{\sin(\alpha+\angle A)}{\sin(\alpha+\angle C)} = \frac{xc}{zBD} \tag{2}$$

Once again, we have the ratio

$$\frac{DB}{DC} = \frac{yc}{zb}$$

So we can write DB in terms of length BC,

$$\frac{DB}{BC} = \frac{yc}{yc+zb} \Rightarrow DB = a\frac{yc}{yc+zb} \tag{3}$$

From (2) and (3), we obtain

$$\frac{KA}{KD} = \frac{xc}{z}\frac{yc+zb}{ayc} = \frac{x(yc+zb)}{ayz}$$

This is equivalent to

$$\overrightarrow{KA} = -\overrightarrow{KD}\left(\frac{xyc+xzb}{ayz}\right) \Rightarrow \overrightarrow{A} - \overrightarrow{K} = (\overrightarrow{K} - \overrightarrow{D})\frac{cxy+bxz}{ayz} \tag{4}$$

From (1) and (4), the equation becomes

$$\overrightarrow{KA} = -\overrightarrow{KD}\left(\frac{xyc+xzb}{ayz}\right) \Rightarrow \overrightarrow{A} - \overrightarrow{K} = \left(\overrightarrow{K} - \frac{bz\overrightarrow{B} + yc\overrightarrow{C}}{bz+yc}\right)\frac{cxy+bxz}{ayz}$$

Specific calculation gives us

$$\overrightarrow{K} = \frac{ayz\overrightarrow{A} + bxz\overrightarrow{B} + cxy\overrightarrow{C}}{ayz+bxz+cxy}.$$

If we constructed the Kiepert points internally with the same angle α, we yield the coordinates for the negative Kiepert point

$$\overrightarrow{K}(-\alpha) = \frac{ay'z'\overrightarrow{A} + bx'z'\overrightarrow{B} + cx'y'\overrightarrow{C}}{ay'z'+bx'z'+cx'y'}$$

with $x' = \sin(\angle A - \alpha), y' = \sin(\angle B - \alpha)$ and $z' = \sin(\angle C - \alpha)$.