Engineering Journal: Notes on Maxwell's Equations

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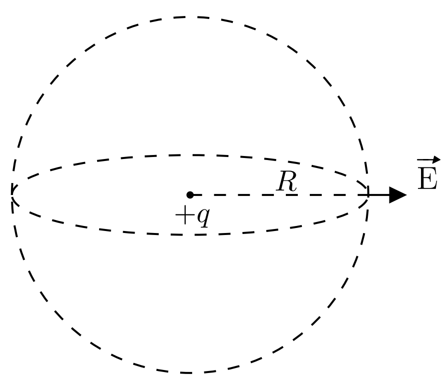
1. Gauss's Law for Electric Fields

$$\Phi_{\mathrm{E}} = \oint_{(\mathrm{S})} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{A}} = \frac{q_{\mathrm{enc}}}{\varepsilon_0} \tag{1}$$

Noted that $\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C}^2/(\mathrm{N}\cdot\ \mathrm{m}^2)$ is the *permittivity of free space*.

Problem 1.1. *(Point Charge)* Use Gauss's Law to find the electric field of a point charge on any points in space.



Solution. We choose a sphere that encloses the point for the Gaussian surface as we examine. We have

$$\Phi_{\mathrm{E}} = \oint_{(\text{sphere})} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{A}} = E \oint_{(\text{sphere})} \mathrm{d}A = E \cdot 4\pi R^2$$

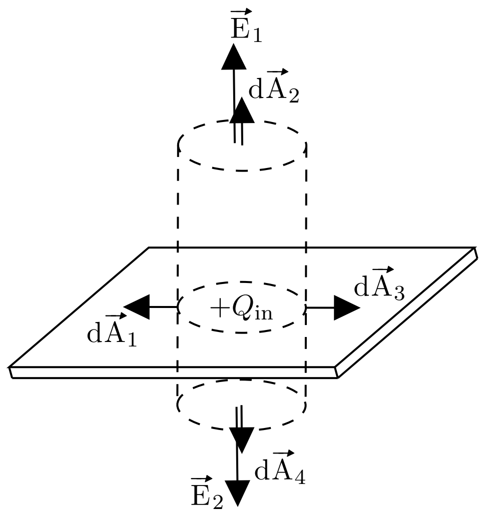
We apply Gauss's Law to find the electric field

$$E(4\pi R^2) = \frac{q}{\varepsilon_0}$$

Therefore, Gauss's Law is consistent with the electric of a point charge since

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$$

Problem 1.2. *(Infinite Charged Plate)* Find electric field of an infinite charged plane at a vertical distance d from it. Assume uniform surface charge density $\sigma = Q/A$ (C/m^2).



Solution. We choose cylindrical surface for our Gaussian surface. The electric flux due to the plate only flows through the top and bottom of the surface as shown in the figure. Therefore, we have

$$\Phi_{\mathrm{E}} = \oint_{(\mathrm{B,T})} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{A}} = E \oint_{(\mathrm{B,T})} \mathrm{d}A = 2EA_{\mathrm{in}}$$

We apply Gauss's Law, yielding

$$2EA_{\mathrm{in}} = \frac{q_{\mathrm{enc}}}{\varepsilon_0}$$

Since $Q_{\mathrm{enc}} = \sigma A_{\mathrm{enc}}$, we can simplify the equation

$$2E\cancel{A_{\mathrm{enc}}} = \frac{\sigma \cancel{A_{\mathrm{enc}}}}{\varepsilon_0}$$

Hence, we yield

$$E = \frac{\sigma}{2\varepsilon_0}.$$

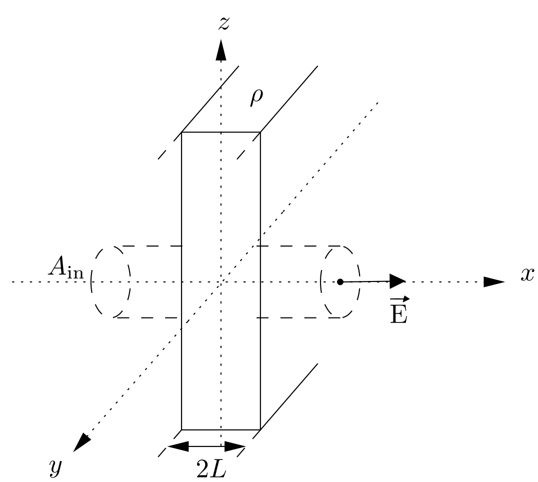
Note. The electric field inside the capacitor is due to two infinitely large positive and negative plates with the same area and charge distribution. Therefore, the electric field can be calculated as

$$E_{\mathrm{C}} = \frac{\sigma}{\varepsilon_0}.$$

Problem 1.3. *(Infinite Charged Slab)* Consider an infinite slab of charge with a uniform volume charge density ρ (C/m^3). The thick slab is located in a vertical (y, z) plane between $x = -L$ and $x = +L$.

(a) Use Gauss's Law to determine the Electric Field for any point $x > +L$ to the right of the slab.

(b) Use Gauss's Law to determine the Electric Field for any point inside the slab $(-L \leq x \leq +L)$.



Solution. a) We choose cylindrical surface for our Gaussian surface so that $x > +L$ (as shown in the figure). We have

$$\Phi_{\mathrm{E}} = \oint_{(\mathrm{B,T})} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{A}} = \oint_{(\mathrm{B,T})} E \mathrm{d}A = E \oint_{(\mathrm{B,T})} \mathrm{d}A = 2EA_{\mathrm{in}}$$

According to Gauss's Law,

$$2EA_{\text{in}} = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\rho V_{\text{in}}}{\varepsilon_0}$$

Therefore, we substitute $V_{\text{in}} = A_{\text{in}}(2x)$ and yield

$$E = \frac{\rho V_{\text{in}}}{2\varepsilon_0 A_{\text{in}}} = \frac{\rho(2A_{\text{in}}L)}{2\varepsilon_0 A_{\text{in}}} = \frac{\rho L}{\varepsilon_0}$$

Since $\vec{\mathrm{E}} \uparrow\uparrow x$-axis,

$$\vec{\mathrm{E}} = \frac{\rho L}{\varepsilon_0}\hat{x} \qquad \text{when } x > +L.$$

b) Similarly, we choose cylindrical surface for our Gaussian surface inside the slab so that $-L < x < +L$. Hence, we also have

$$\Phi_{\mathrm{E}} = \oint_{(\mathrm{B,T})} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{A}} = \oint_{(\mathrm{B,T})} E\mathrm{d}A = E \oint_{(\mathrm{B,T})} \mathrm{d}A = 2EA_{\text{in}}$$

According to Gauss's Law,

$$2EA_{\text{in}} = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\rho V_{\text{in}}}{\varepsilon_0}$$

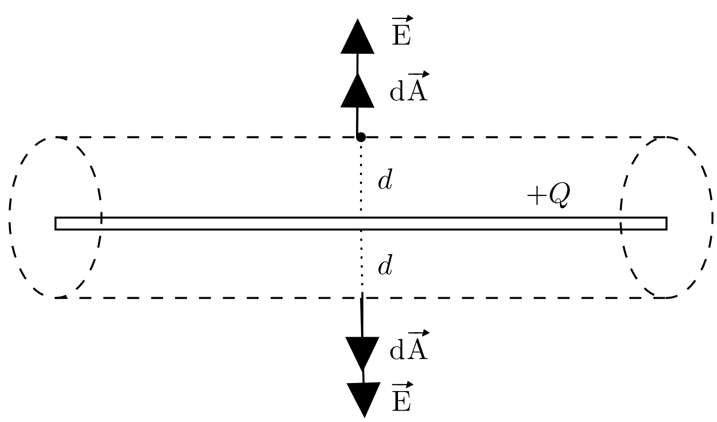
Therefore, we substitute $V_{\text{in}} = A_{\text{in}}(2x)$ and yield

$$E = \frac{\rho V_{\text{in}}}{2\varepsilon_0 A_{\text{in}}} = \frac{\rho(2A_{\text{in}}x)}{2\varepsilon_0 A_{\text{in}}} = \frac{\rho x}{\varepsilon_0}$$

Since $\vec{\mathrm{E}} \uparrow\uparrow x$-axis,

$$\vec{\mathrm{E}} = \frac{\rho x}{\varepsilon_0}\hat{x} \qquad \text{when } -L < x < +L.$$

Problem 1.4. *(Infinite Charged Rod)* Find electric field of an infinite charged rod at a vertical distance d from it. Assume uniform linear charge density $\lambda = Q/L$ (C/m).



Solution. We choose cylindrical surface for our Gaussian surface. The electric flux due to the rod only goes through the side of the cylinder. Therefore, we have

$$\Phi_{\mathrm{E}} = \oint_{\text{(side)}} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{A}} = E \oint_{\text{(side)}} \mathrm{d}A = E \cdot (2\pi d \cdot L)$$

We apply Gauss's Law, yielding

$$E \cdot (2\pi d \cdot L) = \frac{Q}{\varepsilon_0}$$

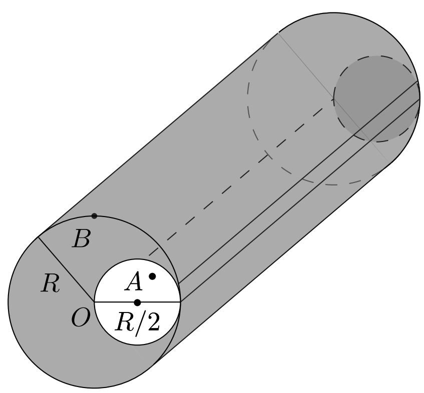
Since $\lambda = Q/L$, we receive

$$E = \frac{\lambda \cancel{L}}{2\pi\varepsilon_0 d \cdot \cancel{L}} = \frac{\lambda}{2\pi\varepsilon_0 d}.$$

Problem 1.5. A very long, insulating cylinder with radius R has a cylindrical hole of radius $R/2$ bored along its entire length (see figure). The axis of the hole is a distance $R/2$ from the axis of the cylinder. The solid material of the cylinder has a uniform positive charge density ρ.

a) What is the magnitude and direction of the electric field $\vec{\mathrm{E}}_{\mathrm{A}}$ at the point A = (3R/4, R/4) inside the hole? Express your answer in terms of ρ, R, and ε_0.

b) What is the electric field $\vec{\mathrm{E}}_{\mathrm{B}}$ at the point $B = (0, R)$ at the top of the cylinder?



Solution. a) We represent the electric field due to the hole as a sum of the electric field caused by two cylinders with charge density $+\rho$ and $-\rho$. We choose cylindrical surface for our Gaussian surface. We calculate the electric flux due through the surface of a charged cylinder

$$\Phi_{\mathrm{E}} = \oint_{\text{(cylinder)}} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{A}} = E \oint_{\text{(cylinder)}} \mathrm{d}A = E \cdot (2\pi r L)$$

We apply Gauss's Law, yielding

$$E \cdot (2\pi r L) = \frac{q_{\mathrm{enc}}}{\varepsilon_0} = \frac{\rho \cdot \pi r^2 L}{\varepsilon_0}$$

This is equivalent to

$$E = \frac{\rho r}{2\varepsilon_0}$$

Therefore,

$$\vec{\mathrm{E}} = \frac{\rho r}{2\varepsilon_0}\hat{r}$$

with $\hat{r}$ is the unit vector pointing out radially. Since point A is inside both cylinders, we use the superposition principle to calculate the electric field at point A

$$\vec{\mathrm{E}}_{\mathrm{A}} = \frac{\rho R}{2\varepsilon_0}\hat{R}_A - \frac{\rho R/2}{2\varepsilon_0}\hat{R}_A = \frac{\rho R}{4\varepsilon_0}\hat{\imath}.$$

b) Since point B is outside the "hole" cylinder, we recalculate

$$E \cdot 2\pi r L = \frac{q_{\mathrm{enc}}}{\varepsilon_0} = \frac{\rho \pi R^2 L}{\varepsilon_0}$$

This is equivalent to,

$$E = \frac{\rho R^2}{2\varepsilon_0 r}$$

Therefore,

$$\vec{\mathrm{E}} = \frac{\rho R^2}{2\varepsilon_0 r}\hat{r}$$

with $\hat{r}$ is the unit vector pointing radially. We use superposition principle to calculate the electric field at point B

$$\vec{\mathrm{E}}_{\mathrm{B}} = \frac{\rho R^2}{2\varepsilon_0 R}\hat{\jmath} - \frac{\rho (R/2)^2}{2\varepsilon_0(\sqrt{5}/2R)} \frac{-(R/2)\hat{\imath} + R\hat{\jmath}}{\sqrt{5}/2R}$$

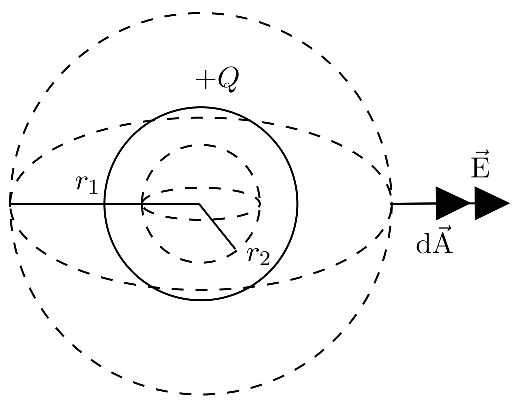
This equation can be rewritten as

$$\vec{\mathrm{E}}_{\mathrm{B}} = \frac{\rho R}{2\varepsilon_0}\hat{\jmath} - \frac{\rho R}{20\varepsilon_0}(-\hat{\imath} + 2\hat{\jmath}) = \frac{\rho R}{20\varepsilon_0}(\hat{\imath} + 8\hat{\jmath}).$$

We conclude that

$$\vec{\mathrm{E}}_{\mathrm{B}} = \frac{\rho R}{20\varepsilon_0}(\hat{\imath} + 8\hat{\jmath})$$

Problem 1.6. *(Charged Sphere)* Find electric field of a uniformly charged insulating sphere, both *inside* and *outside*. Assume uniform volume charge density $\rho = Q/V$ ($\mathrm{C/m^3}$).



Solution. We call R to be the radius of the sphere. To calculate the electric field at a point *outside* the sphere, we choose the spherical surface with radius r_1 for the Gaussian surface (as shown in the figure).

$$\Phi_{\mathrm{E}} = \oint_{\text{(sphere)}} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{A}} = E \oint_{\text{(sphere)}} \mathrm{d}A = E \cdot (4\pi r^2)$$

We apply Gauss's Law, yielding

$$E \cdot (4\pi r^2) = \frac{Q}{\varepsilon_0}$$

Hence, the electric field due to a charged sphere on a point *outside* ($r > R$) the sphere is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \qquad \text{when } r > R.$$

This result is similar to the electric field caused by a point charge.

In the scenario when the point we examine is inside ($r < R$) the sphere, we choose the spherical surface with radius r_2 for our Gaussian surface (as shown in the figure). We apply Gauss's Law

$$E \cdot (4\pi r^2) = \frac{q_{\text{enc}}}{\varepsilon_0}$$

However, the charge inside the Gaussian sphere is now only equal to

$$Q_{\text{in}} = Q\frac{V_{\text{in}}}{V} = \frac{4}{3}\rho\pi r^3$$

Therefore, we have

$$E \cdot (4\pi r^2) = \frac{4}{3}\frac{\rho\pi r^3}{\varepsilon_0}$$

We can simplify it as

$$E = \frac{\rho}{3\varepsilon_0}r \qquad \text{when } r < R.$$

Note. Electric properties of conductors in equilibrium: $Q_{\text{in}} = 0 \Leftrightarrow E_{\text{in}} = 0$. Meanwhile, the excess charge can only sit on its surface, so the electric field on the surface of the conductor is perpendicular to it. According to Gauss's law, the electric field on the surface is

$$E = \frac{\sigma}{\varepsilon_0}$$

where σ is the charge surface density ($\mathrm{C/m^2}$).

2. Gauss's Law for Magnetic Fields

$$\boxed{\Phi_{\mathrm{B}} = \oint_{(\mathrm{S})} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{\mathrm{A}} = 0. \qquad (2)}$$

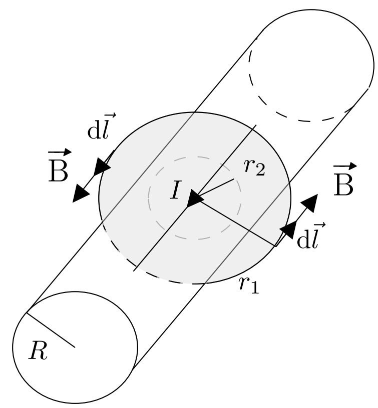
Likewise, we also have Gauss's law for magnetic fields, telling us that the magnetic flux flowing through a Gaussian surface is always equal to zero. In other words, magnetic monopoles do not exist in nature! However, this law does not provide us any useful tools to calculate magnetic field. This will be the task for Ampère's Law.

3. Ampère's Law

$$\oint_{(\mathrm{C})} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{l} = \mu_0 I_{\mathrm{enc}} \qquad (3)$$

Noted that $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 is the *permeability of free space.*

Problem 3.1. *(Infinitely Long Wire)* For a long solid wire carrying a uniformly distributed current I. Calculate B_{out} $(r > R)$ outside the wire and $B_{\mathrm{in}}(r < R)$ inside the wire.



Solution. To calculate the magnetic field at a point *outside* the cylinder, we choose the circular loop with radius r_1 for the Amperian loop (as shown in the figure).

$$\oint_{(\mathrm{circle})} \cdot \mathrm{d}\vec{l} = B \oint_{(\mathrm{circle})} \mathrm{d}l = B \cdot (2\pi r)$$

We apply Ampère's Law

$$B \cdot (2\pi r) = \mu_0 I$$

Hence,

$$B = \frac{\mu_0 I}{2\pi r} \qquad \text{when } r > R$$

In the scenario when the point we examine is inside ($r < R$) the sphere, we choose the circular loop with radius r_2 for our Amperian loop (as shown in the figure). We apply Ampère's Law again

$$B \cdot (2\pi r) = \mu_0 I_{\text{enc}}$$

However, the current enclosed by the loop now is only equal to

$$I_{\text{enc}} = \frac{\cancel{\pi} r^2}{\cancel{\pi} R^2} I = I \frac{r^2}{R^2}$$

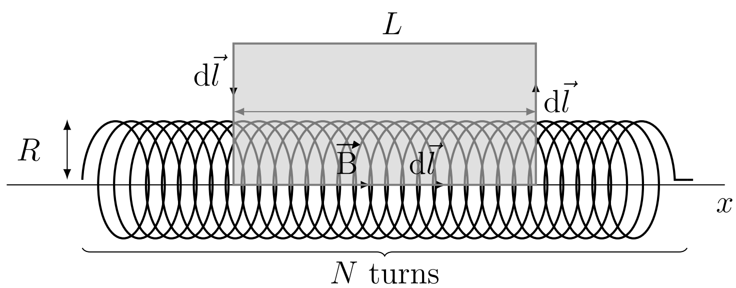
So we yield

$$B(2\pi r) = \mu_0 I \frac{r^2}{R^2}$$

Therefore,

$$B = \frac{\mu_0 I r}{2\pi R^2} \qquad \text{when } r < R.$$

Problem 3.2. *(Infinite Solenoid)* Using Ampère's law, calculate the magnetic field (inside) a long solenoid (i.e. far from its ends). Assume the solenoid has n loops per meter.



Solution. We choose the rectangular loop with length L for our Amperian loop (as shown in the figure). Noticed that the magnetic field inside the solenoid is always parallel to the x-axis but does not depend on x. Additionally, we also observe that $\vec{\mathrm{B}} \perp \mathrm{d}\vec{l}$ at the vertical sides inside. Lastly, the magnetic field outside the solenoid will be equal to zero. Therefore,

$$\oint_{\text{(bottom)}} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{l} = B \oint_{\text{(bottom)}} \mathrm{d}l = BL$$

We apply Ampère's Law

$$BL = \mu_0 I_{\text{enc}}$$

Since the current of a part of solenoid is

$$I_{\text{enc}} = InL$$

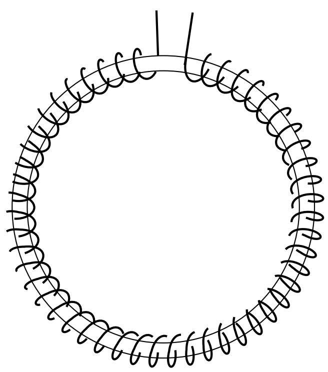
Hence,

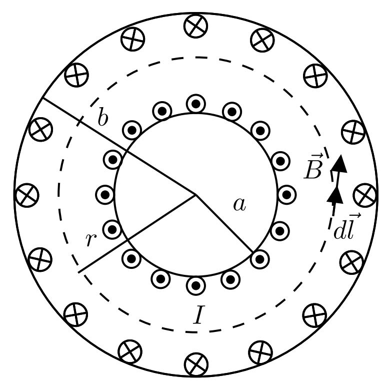
$$B\cancel{L} = \mu_0 In\cancel{L}$$

This can be rewritten as

$$B = \mu_0 In.$$

Problem 3.3. *(Toroid Solenoid)* Determine the magnetic field within a toroidal solenoid that has an inner radius a, an outer radius b, carries a current I, and contains a total of N turns.





Solution. We choose a circular loop with length r $(a < r < b)$ for our Amperian loop (as shown in the figure). The toroid has tightly wound turns uniformly distributed. The magnetic field $\vec{\mathrm{B}}$ is assumed to be tangential and lies in circular loops inside the core of the toroid. Therefore,

$$\oint_{\text{(circle)}} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{l} = B \oint_{\text{(circle)}} \mathrm{d}l = B \cdot (2\pi r)$$

We apply Ampère's Law:

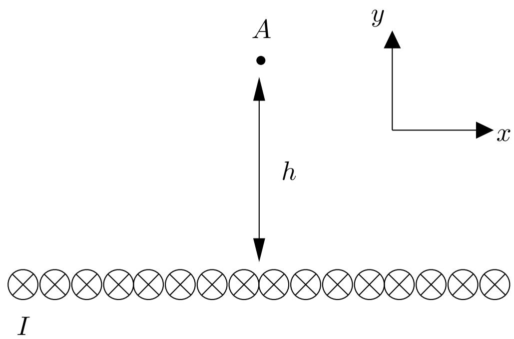
$$B \cdot (2\pi r) = \mu_0 I_{\text{enc}} = \mu_0 IN$$

Hence,

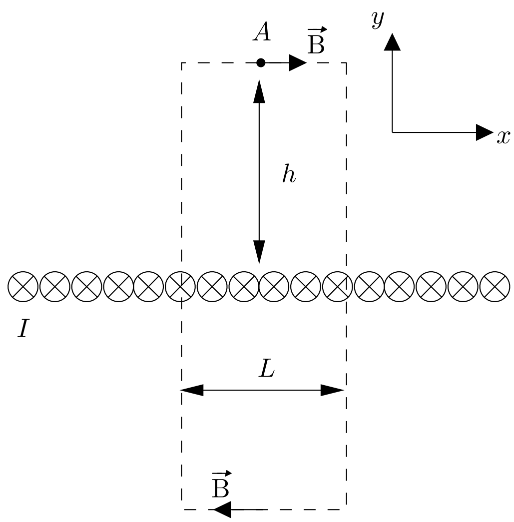
$$B = \frac{\mu_0 IN}{2\pi r} \qquad \text{when } a < r < b$$

If $r < a$ or $r > b$ then the magnetic field B is equal to zero.

Problem 3.4. An array is formed of a large number of very long thin wires arranged in the xy plane as shown. Each wire carries current I and there are n wires per meters. Treating the wire array as infinite in the direction find the magnetic field B (direction and magnitude) at point A distance h above the wires. Hint: Consider the symmetry of the B field above and below the wires.



Solution. We choose a rectangular loop with length L and width $2h$ for our Amperian loop (as shown in the figure).



Due to symmetry, we can compute

$$\oint_{(\mathrm{B,T})} \vec{B} \cdot \mathrm{d}\vec{l} = \oint_{(\mathrm{B,T})} B \cdot \mathrm{d}l = B \oint_{(\mathrm{B,T})} \mathrm{d}l = 2BL$$

We can apply Ampère's Law

$$2BL = \mu_0 I_{\text{enc}}$$

Since $I_{\text{enc}} = \mu_0 nLI$. Therefore,

$$B = \frac{\mu_0 nI}{2}$$

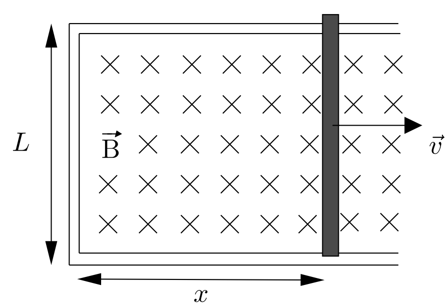
Using Right-hand Rule, we conclude

$$\vec{\mathrm{B}} = \frac{\mu_0 nI}{2}\hat{\imath}.$$

4. Faraday's Law of Induction

$$\mathcal{E} = \oint \vec{\mathrm{E}} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} \tag{4}$$

Problem 4.1. *(Moving Rod)* You pull a metal rod to the right at a constant speed of v. The metal rod, which has an electrical resistance R, is free to move on a U-shaped conductor. What is the magnitude and direction of the induced electrical current?



Solution. Firstly, we find the rate of change in magnetic flux through the loop

$$\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} = \frac{\mathrm{d}\left(\vec{\mathrm{B}} \cdot \mathrm{d}\vec{\mathrm{A}}\right)}{\mathrm{d}t}$$

Since the magnetic is constant and always parallel to the area vector,

$$\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} = \frac{\mathrm{d}(B \cdot \mathrm{d}A)}{\mathrm{d}t} = B\frac{\mathrm{d}(xL)}{\mathrm{d}t} = BL\frac{\mathrm{d}x}{dt} = vBL$$

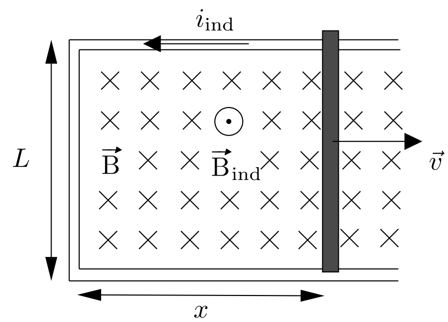
According to Faraday's Law, the magnitude of the induced EMF is equal to

$$|\mathcal{E}| = \left|\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t}\right| = vBL$$

Hence, we use Ohm's Law and conclude

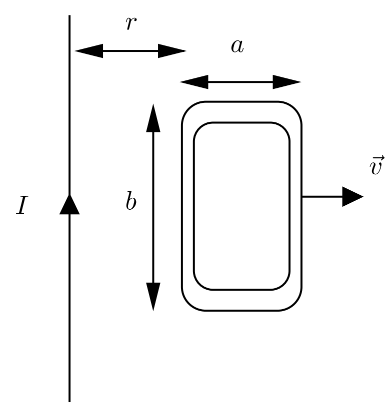
$$|i_{\text{ind}}| = \frac{|\mathcal{E}|}{R} = \frac{vBL}{R}.$$

To find the direction of the induced current, we observe that the magnetic flux is increasing since the area of the loop is increasing.



According to Lenz's Law, a magnetic field with opposite direction will oppose this change. Therefore, the induced magnetic field points out of the page. Using Right-hand Rule, we conclude that the induced current is flowing with the counterclockwise direction.

Problem 4.2. The loop is being pulled to the right at constant speed v. A constant current I flows in the long wire, in the direction shown. Calculate the magnitude of the net emf $\mathcal{E}$ induced in the loop and find the direction (clockwise or counterclockwise) of the current induced in the loop.



Solution. We assume that the wire is infinitely long, the magnetic field due the wire is

$$\mathrm{d}B = \frac{\mu_0 I}{2\pi x}\,\mathrm{d}x$$

The total magnetic field on the loop due to the wire is

$$B = \int_r^{r+a} \frac{\mu_0 I}{2\pi x}\,\mathrm{d}x = \frac{\mu_0 I}{2\pi}\int_r^{r+a}\frac{\mathrm{d}x}{x} = \frac{\mu_0 I}{2\pi}[\ln|x|]_r^{r+a} = \frac{\mu_0 I}{2\pi}(\ln(r+a) - \ln a)$$

Therefore, we obtain

$$B = \frac{\mu_0 I}{2\pi}\ln\left(\frac{r+a}{a}\right)$$

Now, we find the change in magnetic flux of the loop

$$\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} = \frac{\mathrm{d}\left(\vec{\mathrm{B}}\cdot\mathrm{d}\vec{\mathrm{A}}\right)}{\mathrm{d}t}$$

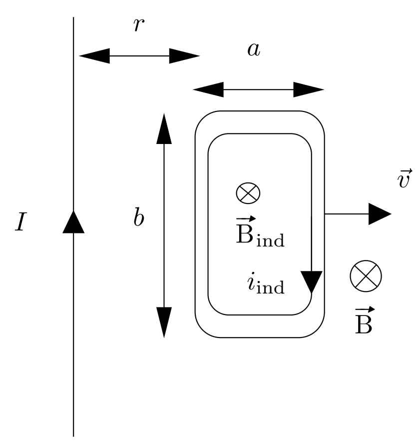
Since the magnetic is always parallel to the area vector,

$$\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} = \frac{\mathrm{d}(B\cdot\mathrm{d}A)}{\mathrm{d}t} = ab\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mu_0 I}{2\pi}\ln(r+a) - \ln a\right] = \frac{\mu_0 I}{2\pi}\frac{abv}{r+a}$$

According to Faraday's Law, the magnitude of the induced EMF is

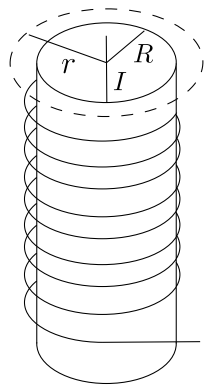
$$|\mathcal{E}| = \left|\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t}\right| = \frac{\mu_0 I}{2\pi}\frac{abv}{r+a}.$$

To find the direction of the induced current, we observe that the magnetic flux is decreasing since the magnetic of the loop decreases as the distance increases.



According to Lenz's Law, a magnetic field parallel to the original magnetic field will be created in order to oppose this change. Therefore, the induced magnetic points into the page. Using Right-hand Rule, we conclude that the induced current is flowing in the clockwise direction.

Problem 4.3. A long straight solenoid has n turns per unit length and a cross-sectional radius R, placed in air. A current $I = I_0 \cos(\omega t)$ flows through the solenoid. Find the magnitude of the electric field intensity at a point located a distance r from the axis of the solenoid (consider both cases $r < R$ and $r > R$).



Solution. We have the magnetic field of a solenoid

$$B = \mu_0 n I = \mu_0 I_0 n \cos(\omega t)$$

For $r > R$, we find the change in magnetic flux

$$\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\mathrm{B}} \cdot \vec{\mathrm{A}}\right) = \pi R^2 \frac{\mathrm{d}}{\mathrm{d}t}[\mu_0 I_0 n \cos(\omega t)] = -\mu_0 I_0 n \omega \pi R^2 \sin(\omega t)$$

According to Faraday's Law,

$$\oint_{(\mathrm{C})} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} = \mu_0 I_0 n \omega \pi R^2 \sin(\omega t)$$

Since we also have $\vec{\mathrm{E}} \uparrow\uparrow \mathrm{d}\vec{l}$ then

$$\oint_{(\mathrm{C})} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{l} = E \oint_{(\mathrm{C})} \mathrm{d}l = E \cdot (2\pi r)$$

Hence, the magnitude of the induced electric field is

$$|E| = \frac{\mu_0 I_0 n \omega R^2 \sin(\omega t)}{2r} \qquad \text{when } r > R.$$

Similarly, we consider $A = \pi r^2$ for $r < R$, then the change in magnetic flux becomes

$$\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\mathrm{B}} \cdot \vec{\mathrm{A}}\right) = \pi r \frac{\mathrm{d}}{\mathrm{d}t}[\mu_0 I_0 n \cos(\omega t)] = -\mu_0 I_0 n \omega \pi r^2 \sin(\omega t)$$

According to Faraday's Law,

$$\oint_{(\mathrm{C})} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} = \mu_0 I_0 n \omega \pi r^2 \sin(\omega t)$$

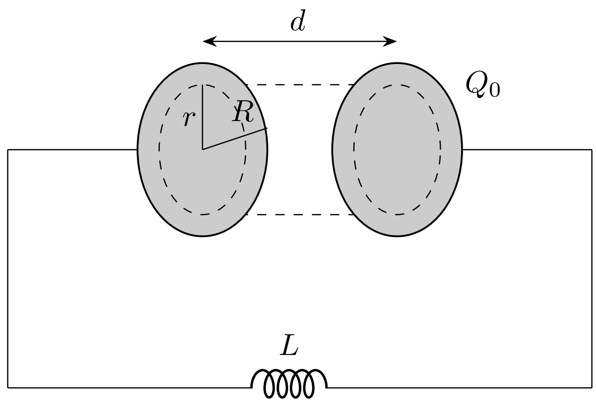
Therefore, the magnitude of the induced electric field is

$$|E| = \frac{1}{2}\mu_0 I_0 n \omega r \sin(\omega t) \qquad \text{when } r < R$$

5. Ampère-Maxwell Law

$$\oint_{(\mathrm{C})} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{l} = \mu_0 I_\mathrm{enc} + \mu_0 \varepsilon_0 \frac{\mathrm{d}\Phi_\mathrm{E}}{\mathrm{d}t} \tag{5}$$

Problem 2.5.1. A parallel-plate circular capacitor with separation d and radius R is initially charged to Q_0. It is then connected to an inductor of inductance L. Determine the magnitude of the magnetic field that develops inside the capacitor at a distance $r < R$ from the center.



Solution. Firstly, we find the charge as a function of time

$$q(t) = Q_0 \cos \omega t$$

We replace $\omega = 1/\sqrt{LC}$ and $C = A\varepsilon_0/d$, the equation becomes

$$q(t) = Q_0 \cos\left(\frac{t}{\sqrt{LA\varepsilon_0/d}}\right) = Q_0 \cos\left(\frac{t}{\sqrt{L\pi R^2\varepsilon_0/d}}\right)$$

Therefore, the current through the circuit when the capacitor is *discharged* is

$$i(t) = -\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{Q_0}{\sqrt{L\pi R^2\varepsilon_0/d}} \sin\left(\frac{t}{\sqrt{L\pi R^2\varepsilon_0/d}}\right)$$

According to Gauss's law, the electric field due to the capacitor is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{q(t)}{A\varepsilon_0} = \frac{q(t)}{\pi R^2\varepsilon_0}$$

We find the rate of change in electric flux for $r < R$

$$\frac{\mathrm{d}\Phi_{\mathrm{E}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(EA_{\mathrm{in}}) = A_{\mathrm{in}}\frac{\mathrm{d}E}{\mathrm{d}t} = \pi r^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{q(t)}{\pi R^2\varepsilon_0}\right) = \frac{1}{\varepsilon_0}\frac{r^2}{R^2}\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{r^2}{R^2}\frac{i(t)}{\varepsilon_0}$$

Since the is no current in the capacitor, we apply Ampère-Maxwell Law. We choose the circular loop with radius r as our Amperian Loop. Hence,

$$\oint_{\text{(circle)}} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{l} = \mu_0\varepsilon_0 \frac{\mathrm{d}\Phi_{\mathrm{E}}}{\mathrm{d}t} = -\frac{r^2}{R^2}\mu_0 i(t)$$

Because the magnetic field inside the capacitor is always parallel to vector $\mathrm{d}\vec{l}$, so

$$\oint_{\text{(circle)}} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{l} = B \oint_{\text{(circle)}} \mathrm{d}l = B(2\pi r)$$

This also means

$$B(r,t) = -\frac{\mu_0}{2\pi}\frac{i(t)}{R^2} r \qquad \text{when } r < R$$

Therefore, the magnitude of the induced magnetic field is

$$|B| = \frac{\mu_0 Q_0}{2\pi}\frac{r}{R^2}\sqrt{\frac{d}{L\pi R^2\varepsilon_0}} \sin\left(t\sqrt{\frac{d}{L\pi R^2\varepsilon_0}}\right) \qquad \text{when } r < R$$

In summary, the properties of electromagnetic fields are described by four fundamental equations known as Maxwell's Equations.

Maxwell's Equations

Gauss's Law for Electric Fields	$\oint_{(S)} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{A}} = \frac{q_{\mathrm{enc}}}{\varepsilon_0}$
Gauss's Law for Magnetic Fields	$\oint_{(S)} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{\mathrm{A}} = 0$
Faraday's Law of Induction	$\oint_{(C)} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t}$
Ampère-Maxwell Law	$\oint_{(C)} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{l} = \mu_0 I_{\mathrm{enc}} + \mu_0\varepsilon_0 \frac{\mathrm{d}\Phi_{\mathrm{E}}}{\mathrm{d}t}$

6. Wave Equations in One Dimension

Now we will derive the wave equations from the system of Maxwell's Equations, specifically

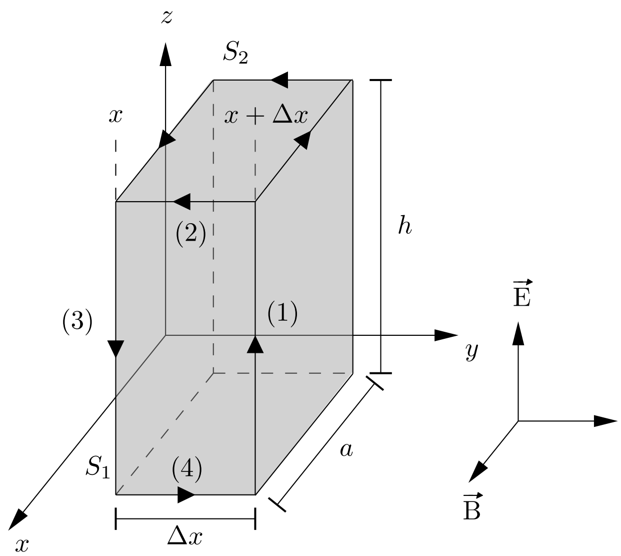
$$\boxed{\frac{\partial^2 E}{\partial x^2} = \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad \text{and} \qquad \frac{\partial^2 B}{\partial x^2} = \mu_0\varepsilon_0 \frac{\partial^2 B}{\partial t^2}} \tag{6}$$

We first assume the electromagnetic wave is in free space so there are no charges and external currents. The system of Maxwell's equations becomes

$$\oint_{(S)} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{A}} = 0 \qquad\qquad \oint_{(S)} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{\mathrm{A}} = 0$$

$$\oint_{(C)} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} \qquad\qquad \oint_{(C)} \vec{\mathrm{B}} \cdot \mathrm{d}\vec{l} = \mu_0\varepsilon_0 \frac{\mathrm{d}\Phi_{\mathrm{E}}}{\mathrm{d}t}$$

Additionally, we assume that the electric field $\vec{\mathrm{E}}$ and magnetic field $\vec{\mathrm{B}}$ only depend on one coordinate x and time t. Therefore, the fields do not depend on y and z.



Solution. We choose the Gaussian surface as shown in the figure. Since the electric field only depends on x, we apply Gauss's law for electric fields:

$$\Phi_{\mathrm{E}} = 0 \Rightarrow E(x) = E(x + \Delta x)$$

Since the time-dependent field is not constant, we conclude $\vec{\mathrm{E}} \perp \vec{x}$. Likewise, we also obtain $\vec{\mathrm{B}} \perp \vec{x}$, according to Gauss's law for magnetic fields. Next, since $\vec{\mathrm{E}} \perp \vec{l_2}$ and $\vec{l_4}$ we calculate

$$\begin{aligned}
\oint_{(S_1)} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{l} &= \oint_{(1)} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{l} + \oint_{(3)} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{l} \\
&= E(x + \Delta x) \oint_{(1)} \mathrm{d}l - E(x) \oint_{(3)} \mathrm{d}l \\
&= E(x + \Delta x) \cdot h - E(x) \cdot h
\end{aligned}$$

We also have

$$\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} = h \Delta x \frac{\partial B}{\partial t}$$

According to Faraday's Law,

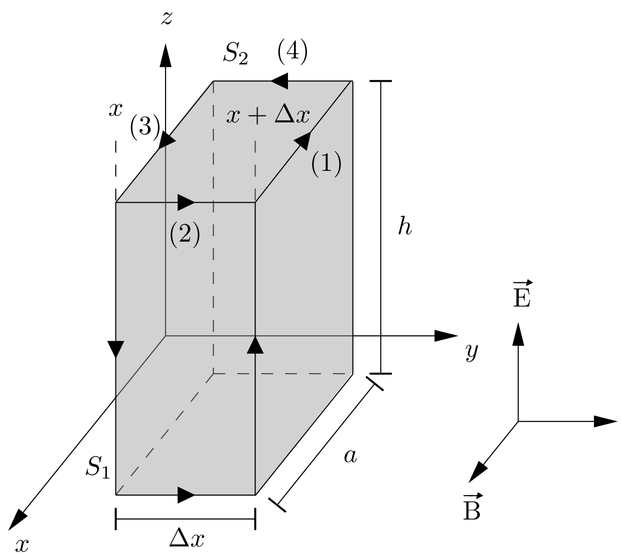
$$E(x + \Delta x) \cdot \cancel{h} - E(x) \cdot \cancel{h} = -\cancel{h} \Delta x \frac{\partial B}{\partial t}$$

Hence, this is equivalent to

$$\frac{E(x+\Delta x)-E(x)}{\Delta x}=-\frac{\partial B}{\partial t}$$

As Δx approaches to 0,

$$\frac{\partial E}{\partial x}=-\frac{\partial B}{\partial t}.$$



Similarly, we calculate

$$\begin{aligned}\oint_{(S_2)} \vec{\mathrm{B}}\cdot \mathrm{d}\vec{l} &= \oint_{(1)} \vec{\mathrm{B}}\cdot \mathrm{d}\vec{l} + \oint_{(3)} \vec{\mathrm{B}}\cdot \mathrm{d}\vec{l} \\ &= B(x)\oint_{(3)} \mathrm{d}l - B(x+\Delta x)\oint_{(1)} \mathrm{d}l \\ &= B(x)\cdot a - B(x+\Delta x)\cdot a\end{aligned}$$

We also have

$$\frac{\mathrm{d}\Phi_{\mathrm{E}}}{\mathrm{d}t}=a\Delta x\frac{\partial E}{\partial t}$$

According to Ampère-Maxwell Law,

$$B(x)\cdot \cancel{a} - B(x+\Delta x)\cdot \cancel{a} = \mu_0\varepsilon_0\cancel{a}\Delta x\frac{\partial E}{\partial t}$$

As a result, this is equivalent to

$$\frac{B(x+\Delta x)-B(x)}{\Delta x}=-\mu_0\varepsilon_0\frac{\partial E}{\partial t}$$

As Δx approaches to 0,

$$\frac{\partial B}{\partial x}=-\mu_0\varepsilon_0\frac{\partial E}{\partial t}$$

In summary, we obtain two wave relations

$$\frac{\partial E}{\partial x}=-\frac{\partial B}{\partial t}\qquad\text{and}\qquad\frac{\partial B}{\partial x}=-\mu_0\varepsilon_0\frac{\partial E}{\partial t}$$

Taking the partial derivative of the first equation equation, we yield

$$\frac{\partial^2 E}{\partial x^2}=-\frac{\partial^2 B}{\partial x\partial t}=-\frac{\partial}{\partial t}\left(\frac{\partial B}{\partial x}\right)$$

Then we substituting the second result into the previous equation

$$\frac{\partial^2 E}{\partial x^2}=-\frac{\partial}{\partial t}\left(-\mu_0\varepsilon_0\frac{\partial E}{\partial t}\right)=\mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}$$

Similarly, it is possible to prove

$$\frac{\partial^2 B}{\partial x^2}=-\mu_0\varepsilon_0\frac{\partial^2 E}{\partial x\partial t}=-\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\frac{\partial E}{\partial x}\right)$$

Hence, we obtain

$$\frac{\partial^2 B}{\partial x^2}=-\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial B}{\partial t}\right)=\mu_0\varepsilon_0\frac{\partial^2 B}{\partial t^2}.$$

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