A story about L'Hopital's Rule

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I. Introduction

When I first learned how to solve limits of functions in intermediate forms, especially the form $0/0$, I knew how these limits were generated by the teachers, —so I decided to make one myself. Let me show you a simpler example to illustrate how I did it.

Problem 1.1. Determine

$$L = \lim_{x \to 3} \frac{\sqrt{4x-3} - \sqrt{x^2-6}}{x-3}$$

If we substitute $x = 3$ into the expression, we get $0/0$ so a very algebraic approach for this limit is multiplying the conjugate of the numerator to eliminate the square roots

$$\begin{aligned}
L &= \lim_{x \to 3} \frac{\sqrt{4x-3} - \sqrt{x^2+4x-12}}{x-3} \frac{\sqrt{4x-3} + \sqrt{x^2+4x-12}}{\sqrt{4x-3} + \sqrt{x^2+4x-12}} \\
&= \lim_{x \to 3} \frac{4x-3-(x^2+4x-12)}{x-3} \frac{1}{\sqrt{4x-3} + \sqrt{x^2-5}} \\
&= \lim_{x \to 3} \frac{-x^2+9}{x-3} \frac{1}{\sqrt{4x-3} + \sqrt{x^2+4x-12}}
\end{aligned}$$

We can factorize the numerator and further simplify

$$\begin{aligned}
L &= \lim_{x \to 3} \frac{-(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} \frac{1}{\sqrt{4x-3} + \sqrt{x^2+4x-12}} \\
&= -\lim_{x \to 3} \frac{x+3}{\sqrt{4x-3} + \sqrt{x^2+4x-12}}
\end{aligned}$$

Hence, we substitute $x = 3$ into the expression again, yielding

$$\lim_{x\to 3} \frac{\sqrt{4x-3} - \sqrt{x^2-5}}{x-3} = -1.$$

The idea of this approach is to cancel the common factor $(x-2)$ or the root $x = 2$ of two functions of the fraction. Therefore, to generate a problem like this, we only need to make the functions as "ugly" as possible and then make sure they have a common root. Based on this, I came up with a more advanced problem:

Problem 1.2. Evaluate

$$L = \lim_{x\to 2} \frac{\sqrt[3]{x^3+19} - \sqrt{x^2-4x+13}}{x^2-4}$$

Solution. Since the roots of the expressions in the numerator are different, we can do a trick by adding and subtracting the result of those expressions when $x = 2$

$$L = \lim_{x\to 2} \frac{\sqrt[3]{x^3+19} - 3 - (\sqrt{x^2-4x+13} - 3)}{x^2-4}$$

Now we can multiply the conjugates of the functions. Noticing that we can use the identity below for the cube root,

$$\sqrt[3]{a} - \sqrt[3]{b} = \frac{a-b}{a^{2/3} + (ab)^{1/3} + b^{2/3}}$$

Therefore, the limit becomes

$$L = \lim_{x\to 2} \frac{\dfrac{x^3+19-27}{(x^3+19)^{2/3} + 3[(x^3+19)]^{1/3} + 9} - \dfrac{x^2-4x+13-9}{\sqrt{x^2-4x+13}+3}}{(x-2)(x+2)}$$

$$= \lim_{x\to 2} \frac{\dfrac{x^3-8}{(x^3+19)^{2/3} + 3[(x^3+19)]^{1/3} + 9} - \dfrac{x^2-4x+4}{\sqrt{x^2-4x+13}+3}}{(x-2)(x+2)}$$

Hence, we can factorize as we did before

$$L = \lim_{x\to 2} \frac{\dfrac{\cancel{(x-2)}(x^2+2x+4)}{(x^3+19)^{2/3} + 3[(x^3+19)]^{1/3} + 9} - \dfrac{(x-2)\cancel{(x-2)}}{\sqrt{x^2-4x+13}+3}}{\cancel{(x-2)}(x+2)}$$

$$= \lim_{x\to 2} \frac{\dfrac{x^2+2x+4}{(x^3+19)^{2/3} + 3[(x^3+19)]^{1/3} + 9} - \dfrac{x-2}{\sqrt{x^2-4x+13}+3}}{(x+2)}$$

Now we can substitute $x = 2$ again and compute

$$L = \frac{\frac{12}{27} - 0}{4} = \frac{1}{9}$$

I apologize for the math, but you get the idea. Then I asked my friend, T.L.K., to solve it, since we share a nerdy common hobby, which is solving math together and appreciating it. I knew he could do the problem, but I expected him to solve it my way. However, he looked at it and shouted, "L'Hôpital!" I stood there, puzzled by this exotic idea that I'd never heard of. I said, "Show me!". He differentiated the functions three times—and it took him only five sentences.

Theorem 1.1. *(L'Hopital's Rule)* Suppose $f(x)$ and $g(x)$ are differentiable functions with $g'(x) \neq 0$ near a. The limit has the intermediate form

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad \text{or} \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

If $\lim\limits_{x \to a} f'(x)/g'(x)$ exists then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

T.L.K's solution. Recognizing the limit is in the intermediate form $^0/_0$, we use L'Hopital's Rule three times:

$$\begin{aligned}
L &= \lim_{x \to 2} \frac{(\sqrt[3]{x^3 + 19} - \sqrt{x^2 - 4x + 13})'}{(x^2 - 4)'} \\
&= \lim_{x \to 2} \frac{\frac{1}{3}(x^3 + 19)^{-2/3}(3x^2) - \frac{1}{2}(x^2 - 4x + 13)^{-1/2}(2x - 4)}{2x} \\
&= \lim_{x \to 2} \frac{x^2(x^3 + 19)^{-2/3} - (x^2 - 4x + 13)^{-1/2}(x - 2)}{2x} \\
&= \frac{4/9 - 0}{4} \\
&= \frac{1}{9}
\end{aligned}$$

Although, I had already known the concept of differentiation, I didn't realize it could be applied to solve such a problem like this, which made me dislike it at first since I couldn't grasp it and thought it was a lazy trick. Two years later, I began studying it carefully and eventually enjoyed it. I want to show you how to evaluate this limit in the intermediate form like 0^0 using L'Hopital's rule.

Problem 1.3. Evaluate

$$L = \lim_{x \to 0^+} x^{\arctan x}$$

Solution. We can convert the limit to

$$L = \lim_{x \to 0^+} \exp\left(\ln x^{\arctan x}\right) = \exp\left(\lim_{x \to 0^+} \arctan x \ln x\right) = \exp\left(\lim_{x \to 0^+} \frac{\ln x}{1/\arctan x}\right)$$

We apply L'Hopital's rule since the limit is in the intermediate form $^{\infty}/_{\infty}$

$$L = \exp\left(\lim_{x \to 0^+} \frac{1/x}{-1/(x^2+1)(\arctan x)^2}\right) = \exp\left(-\lim_{x \to 0^+} \frac{(x^2+1)(\arctan x)^2}{x}\right)$$

The limit is in the intermediate form $^{0}/_{0}$ so we apply L'Hopital's rule again

$$L = \exp\left(-\lim_{x \to 0^+} \frac{2x(\arctan x)^2 + 2\arctan x}{1}\right) = 1$$

Therefore,

$$\lim_{x \to 0^+} x^{\arctan x} = 1.$$

Remarks: The problem could be demonstrated by evaluating my favourite limits $\lim\limits_{x \to 0^+} x^x = 1$ or $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$. However, these results were famous and already shown up many times in textbooks. Therefore, they will be left as an exercise.