UBC MATH 253: Multivariable Calculus

1. Linear Algebra

Dot and Cross Product

For two vectors $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$,

Magnitude $\quad ||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Unit Vector $\quad \hat{a} = \dfrac{\vec{a}}{||\vec{a}||}$

Dot Product $\quad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Cross Product $\quad \vec{a} \times \vec{b} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$

Projection $\quad \text{proj}_{\vec{a}} \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{||\vec{a}||} \dfrac{\vec{a}}{||\vec{a}||}$

Coordinate Geometry in 3D

Distance between $P(x_0, y_0, z_0)$ and $W : ax + by + cz + d = 0$

$$d(P, W) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Area of a parallelogram created by $\vec{a}$ and $\vec{b}$

$$\text{Area} = ||\vec{a} \times \vec{b}||$$

Volume of a parallelepiped created by vector $\vec{a}, \vec{b}$ and $\vec{c}$

$$\text{Volume} = \left|(\vec{a} \times \vec{b}) \cdot \vec{c}\right|$$

2. Partial Derivatives

Chain Rule

Case 1. $z = f(x, y)$ with $x(t)$ and $y(t)$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Case 2. $z = f(x, y)$ with $x(s, t)$ and $y(s, t)$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Implicit Differentiation

For $F(x, y, z) = 0$,

$$z_x = -\frac{F_x}{F_z} \qquad z_y = -\frac{F_y}{F_z}$$

Gradient $\quad \nabla f = [f_x, f_y, f_z]$

Directional Derivative $\quad D_{\hat{v}} f(a, b) = \nabla f(a, b) \cdot \dfrac{\vec{v}}{||\vec{v}||}$ at (a, b)

Rate of change $\quad D_{\vec{v}} f(a, b) = \nabla f(a, b) \cdot \vec{v}$ at (a, b)

Tangent Plane Equation

Explicit form $\quad z = f(x, y)$

At point (x_0, y_0, z_0) where $z_0 = f(x_0, y_0)$,

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Implicit form $\quad F(x, y, z) = 0$

At point (x_0, y_0, z_0),

$$\nabla F(x_0, y_0, z_0) \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

Optimization

Lagrange Multipliers $\quad \begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ g(x, y, z) = k \end{cases}$

Hessian Matrix $\quad D(x, y) = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx} f_{yy} - f_{xy}^2$

- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is local min
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is local max
- If $D(a, b) < 0$ then (a, b) is a saddle point
- If $D(a, b)$ then test is inclusive

3. Multiple Integrals

Double Integral $\quad \displaystyle\iint_R f(x, y) \mathrm{d}A$

Average value $\quad f_{\text{avg}} = \dfrac{\iint_R f(x, y) \mathrm{d}A}{\iint_R \mathrm{d}A}$

Surface Area $\quad \text{SA} = \displaystyle\iint_S \sqrt{1 + f_x^2 + f_y^2}\, \mathrm{d}A$

Triple Integral $\quad \displaystyle\iiint_E F(x, y, z) dV$

Average value $\quad F_{\text{avg}} = \dfrac{\iiint_E F(x, y, z) \mathrm{d}V}{\iiint_E \mathrm{d}V}$

Coordinate system conversion

$$\mathrm{d}A = \mathrm{d}x\, \mathrm{d}y = r\, \mathrm{d}r\, \mathrm{d}\theta$$

$$\mathrm{d}V = \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z = \rho^2 \sin\phi\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\phi$$

Cartesian $\leftrightarrow$ Cylindrical coordinate system

$$x = r\cos\theta \quad y = r\sin\theta$$

$$\Leftrightarrow\ r = \sqrt{x^2 + y^2} \quad \theta = \arctan\frac{y}{x}$$

Cartesian $\leftrightarrow$ Spherical coordinate system

$$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$$

$$\Leftrightarrow \rho = \sqrt{x^2 + y^2 + z^2} \quad \theta = \arctan\frac{y}{x} \quad \phi = \arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Center of Mass

Total mass $\quad M = \displaystyle\iint_R \rho(x, y)\, dA$

Moment about the x-axis $\quad M_x = \displaystyle\iint_R y\, \rho(x, y)\, dA$

Moment about the y-axis $\quad M_y = \displaystyle\iint_R x\, \rho(x, y)\, dA$

Center of mass $\quad \bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$