Lagrange Multipliers in Optimization

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July 2025

I. Introduction

When dealing with problems such as proving inequalities or finding maxima and minima, a constraint is often provided. Therefore, we will introduce Lagrange multipliers, a powerful tool to solve such problems, as well as discussing its strengths and disadvantages.

Theorem 1.1. Let $f(x, y, z)$ have continuous first partial derivatives in a region of $\mathbb{R}$. Function f is restricted to the surface $g(x, y, z) = 0$ and $\nabla g(x, y, z) \neq \vec{0}$ on S. The local extrema of function f are solutions of the system of equations

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= k \end{aligned}$$

with λ is a Lagrange multiplier.

Langrage multiplier theorem is the tool for finding the points where local extrema occur. In fact, it does not provide any information to categorize those extrema. Therefore, we often see the type of the extrema is specified in the problem such as: *Find* $\min f(x, y, z) = x^3 + y^3 + z^3$ *subject to* $x^2 + y^2 + z^2 = 3$.

If the question asks for both the global minimum and maximum of a function subject to a constraint, we observe that if function f is continuous over a *compact* region, a region that is a closed and bounded subset, it is guaranteed to attain both a global maximum and minimum. Compact regions can be ellipses, circles, or triangles in $\mathbb{R}^2$ or even a spheres, ellipsoids, and rectangular boxes in $\mathbb{R}^3$. In such cases, we can evaluate the function at any local extrema and compare their values determine the global minimum and maximum, according to the Extreme value theorem. While local extrema can also be analyzed using the *bordered* Hessian matrix (in problems involving three variables and one constraint), this method is generally too complicated and beyond the scope of this paper. Instead, our strategy will be

- Use Lagrange multiplier method and solve the system of equations to find the local extrema
- Determine if the region is *compact*. If so, we compare two or more values of local extrema to attain the maximum and minimum of the function. If it is not compact, we consider other algebraic inequalities to find minima and maxima.

Problem 1.1.1. Use Lagrange multiplier method to find the maximum and minimum values of $f(x, y, z) = ax + by + cz$ $(a, b, c > 0)$, subject to the constraint $x^2 + y^2 + z^2 = R^2$ $(R > 0)$, if such values exist.

Solution 1.*(Lagrange Multiplier)* Let $g(x, y, z) = x^2 + y^2 + z^2 - R^2$, we first calculate the gradients of f and g

$$\begin{aligned}\nabla f &= [a,\, b,\, c]\\ \nabla g &= [2x,\, 2y,\, 2z]\end{aligned}$$

According to Lagrange multiplier theorem,

$$\nabla f = \lambda \nabla g$$

This becomes a system of equations

$$\begin{aligned}a &= 2x\lambda\\ b &= 2y\lambda\\ c &= 2z\lambda\end{aligned}$$

Since $a, b, c > 0$, we obtain

$$\frac{2x}{a} = \frac{2y}{b} = \frac{2z}{c} \Rightarrow \frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

We use this condition to solve the constraint

$$x^2 + y^2 + z^2 = R^2 \Rightarrow x^2 + \left(\frac{b}{a}x\right)^2 + \left(\frac{c}{a}y\right)^2 = R^2$$

Solving for x

$$x_1 = \frac{aR}{\sqrt{a^2+b^2+c^2}} \qquad x_2 = -\frac{aR}{\sqrt{a^2+b^2+c^2}}$$

If $x = x_1$, then the values of y and z are

$$y = y_1 = \frac{bR}{\sqrt{a^2+b^2+c^2}}, \qquad z = z_1 = \frac{cR}{\sqrt{a^2+b^2+c^2}}$$

Therefore, the value of f is

$$\begin{aligned} f(x_1, y_1, z_1) &= a\frac{aR}{\sqrt{a^2+b^2+c^2}} + b\frac{bR}{\sqrt{a^2+b^2+c^2}} + c\frac{cR}{\sqrt{a^2+b^2+c^2}} \\ &= R\sqrt{a^2+b^2+c^2} \end{aligned}$$

Similarly, if $x = x_2$, then the value of f is

$$f(x_2, y_2, z_2) = -R\sqrt{a^2+b^2+c^2}$$

Hence, (x_1, y_1, z_1) and (x_2, y_2, z_2) are local extrema. The region is compact, so we can compare two values of f to determine whether they are the minimum or maximum. Since $f(x_1, y_1, z_1) > f(x_2, y_2, z_2)$

$$\max f = R\sqrt{a^2+b^2+c^2}, \qquad \min f = -R\sqrt{a^2+b^2+c^2}$$

Solution 2.*(Cauchy-Schwarz Inequality)* Another way to solve this problem is using Cauchy-Schwarz Inequality for three sets of numbers. We have

$$|ax+by+cz| \le \sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)}$$

Since $x^2+y^2+z^2 = R^2\,(R>0)$, we obtain

$$|f| \le R\sqrt{a^2+b^2+c^2}$$

This inequality can be rewritten as

$$-R\sqrt{a^2+b^2+c^2} \le f \le R\sqrt{a^2+b^2+c^2}$$

Therefore, we can conclude

$$\max f = R\sqrt{a^2+b^2+c^2}, \qquad \min f = -R\sqrt{a^2+b^2+c^2}$$

Problem 1.1.2. Find the maximum volume of a rectangular box inscribed in a sphere of radius R.

Solution 1. *(Lagrange Multiplier)* Let the volume of the rectangular with length x, width y and height z ($x, y, z > 0$) be

$$f(x, y, z) = xyz$$

subject to constraint $x^2 + y^2 + z^2 = R^2$. Let $g(x, y, z) = x^2 + y^2 + z^2 - R^2$, we find the gradients of f and g.

$$\nabla f = [yz,\ xz,\ xy] \qquad \text{and} \qquad \nabla g = [2x,\ 2y,\ 2z]$$

Therefore, we set up the equation

$$\nabla f = \lambda \nabla g \Rightarrow [yz,\ xz,\ xy] = \lambda[2x,\ 2y,\ 2z]$$

This becomes

$$\begin{aligned} yz &= 2x\lambda \\ xz &= 2y\lambda \\ xy &= 2z\lambda \end{aligned}$$

If $\lambda = 0$ then $x = y = z = 0$ (invalid because $x^2 + y^2 + z^2 = R > 0$), so

$$\frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z} = \lambda \Rightarrow x = y = z = \frac{R}{\sqrt{3}}$$

The maximum volume of the rectangular box is

$$f\left(\frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}\right) = \frac{R^3}{3\sqrt{3}}.$$

Solution 2. *(AM-GM inequality)* We apply AM-GM inequality for three non-negative numbers directly

$$\sqrt[3]{(xyz)^2} \le \frac{x^2 + y^2 + z^2}{3} = \frac{R^2}{3}$$

Therefore,

$$xyz \le \left(\frac{R^2}{3}\right)^{3/2} = \frac{R^3}{3\sqrt{3}}$$

The equality case occurs when

$$x = y = z = \frac{R}{\sqrt{3}}.$$

One limitation of using the Langrage multiplier method is that the resulting system of equations can become too complex and difficult to solve.

Problem 1.1.3. Formulate an equation that can be used to determine the shortest and longest distances from a point (x_0, y_0, z_0), which lies outside a given ellipsoid, to the surface of the ellipsoid defined by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution. We are finding the minimum and maximum values of function $f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$ subject to constraint

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let $g(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$. We compute the gradients of f and g

$$\nabla f = [2(x - x_0),\, 2(y - y_0),\, 2(z - z_0)] \qquad \text{and} \qquad \nabla g = \left[\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right]$$

According to Langrage multiplier theorem,

$$\begin{aligned} 2(x - x_0) &= \frac{2x}{a^2}\lambda \\ 2(y - y_0) &= \frac{2y}{b^2}\lambda \\ 2(z - z_0) &= \frac{2z}{c^2}\lambda \end{aligned}$$

If $\lambda = 0$ then $(x, y, z) = (x_0, y_0, z_0)$ (invalid since (x_0, y_0, z_0) lies outside the surface). Therefore, $\lambda \neq 0$ and the system of equations is equivalent to

$$\begin{aligned} x &= \frac{a^2 x_0}{a^2 - \lambda} \\ y &= \frac{b^2 y_0}{b^2 - \lambda} \\ z &= \frac{c^2 z_0}{c^2 - \lambda} \end{aligned}$$

Substituting into the constraint equation, we obtain

$$\frac{1}{a^2}\left(\frac{a^2x_0}{a^2-\lambda}\right)^2+\frac{1}{b^2}\left(\frac{b^2y_0}{b^2-\lambda}\right)^2+\frac{1}{c^2}\left(\frac{c^2z_0}{c^2-\lambda}\right)^2=1$$

We further simplify

$$\frac{a^2x_0^2}{(a^2-\lambda)^2}+\frac{b^2y_0^2}{(b^2-\lambda)^2}+\frac{c^2z_0^2}{(c^2-\lambda)^2}=1$$

Multiplying the common denominator for both sides, the equation is an a sixth-degree equation, which might be difficult or even impossible to solve for λ. Therefore, Langrage multipliers might not be a useful tool in this scenario. However, in a special case such as when the surface is a sphere ($a=b=c=R$), the equation simplifies significantly

$$\frac{R^2x_0^2}{(R^2-\lambda)^2}+\frac{R^2y_0^2}{(R^2-\lambda)^2}+\frac{R^2z_0^2}{(R^2-\lambda)^2}=1$$

Solving for λ, we obtain

$$(R^2-\lambda)^2=R^2(x_0^2+y_0^2+z_0^2)\Rightarrow\lambda=R^2\pm R\sqrt{x_0^2+y_0^2+z_0^2}$$

Then we can easily attain the maximum and minimum of the function. This will be left as an exercise for the readers.

We can also use this theorem for constraints that are inequalities.

Problem 1.1.4. Find the maximum and minimum of function $f(x,y)=e^{-xy}$ on a domain D given by

$$D=\{(x,y)\,|\,x^2+4y^2\leq 4\}.$$

Solution. We will examine two cases: inside the region and on the boundary.

Case 1: Inside the region. We find the local extrema of function $f(x,y)=e^{-xy}$ without constraints. The gradient of f is

$$\nabla f=\left[-ye^{-xy},\,-xe^{-xy}\right]$$

We set $\nabla f=\vec{0}$, then

$$\begin{aligned}-ye^{-xy}&=0\\-xe^{-xy}&=0\end{aligned}$$

There is only one critical point $(0,0)$, which is inside the domain. We compute the second partial derivatives of f

$$\begin{aligned} f_{xx} &= y^2 e^{-xy} \\ f_{yy} &= x^2 e^{-xy} \\ f_{xy} &= (xy-1)e^{-xy} \end{aligned}$$

Therefore, the determinant of the Hessian matrix is

$$\begin{aligned} D(x,y) &= f_{xx} f_{yy} - f_{xy}^2 \\ &= y^2 e^{-xy} \cdot x^2 e^{-xy} - \left((xy-1)e^{-xy}\right)^2 \\ &= x^2 y^2 e^{-2xy} - (xy-1)^2 e^{-2xy} \end{aligned}$$

$\Rightarrow D(0,0) = -1 < 0 \Rightarrow (0,0)$ is a saddle point.

<u>Case 2:</u> On the boundary. We find the minimum and maximum values of function $f(x,y) = e^{-xy}$ subject to constraint $x^2 + 4y^2 = 4$. Let $g(x,y) = x^2 + 4y^2 - 4$, we compute the gradient of g

$$\nabla g = [2x,\ 8y]$$

According to Langrage multiplier theorem,

$$\begin{aligned} -\, ye^{-xy} &= \lambda(2x) \\ -\, xe^{-xy} &= \lambda(8y) \end{aligned}$$

If $x = 0$ then $y = 0$ and vice versa (invalid since $x^2 + 4y^2 = 4$). Since $x, y \neq 0$,

$$-\frac{ye^{-xy}}{2x} = -\frac{xe^{-xy}}{8y} = \lambda.$$

Therefore,

$$\frac{y}{2x} = \frac{x}{8y} \Rightarrow x^2 = 4y^2 \Rightarrow x = \pm 2y.$$

If $x = 2y$, then $(2y)^2 + 4y^2 = 4 \Rightarrow 8y^2 = 4 \Rightarrow y = \pm\frac{1}{\sqrt{2}}$. Hence, $x = \sqrt{2}, y = \frac{1}{\sqrt{2}}$ or $x = -\sqrt{2}$ and $y = -\frac{1}{\sqrt{2}}$. We compute

$$\begin{aligned} f\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right) &= \frac{1}{e} \\ f\left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right) &= \frac{1}{e} \end{aligned}$$

Similarly, $x = -2y$, then $(-2y)^2 + 4y^2 = 4 \Rightarrow 8y^2 = 4 \Rightarrow y = \pm\frac{1}{\sqrt{2}}$. Hence, $x = -\sqrt{2}, y = \frac{1}{\sqrt{2}}$ or $x = \sqrt{2}$ and $y = -\frac{1}{\sqrt{2}}$. We compute

$$f\left(-\sqrt{2}, \frac{1}{\sqrt{2}}\right) = e$$
$$f\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right) = e$$

In conclusion, the minimum and maximum of the function is

$$\begin{aligned} \max f &= e \qquad \text{at} \left(-\sqrt{2}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right) \\ \min f &= \frac{1}{e} \qquad \text{at} \left(\sqrt{2}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right). \end{aligned}$$

II. Conclusion

The reasons why Lagrange multiplier theorem works for some optimization problem are:

- First, according to the Extreme value theorem, if a function is continuous on a compact set, then it attains a maximum and a minimum value. Therefore, a wide range of problems can be solved by Lagrange multiplier theorem if we have nice equation of compact regions such as spheres, ellipsoids. While shapes like prisms, rectangular boxes, or bounded cylinders may also define compact regions, their constraints often involve inequalities, making them more challenging to use this method effectively.

- Secondly, the system of equations derived from Langrage multiplier theorem is easy to solve. As discussed above, although we can have a compact region with a nice equation and a symmetric system of equations, solving higher-degree equations might not be practical in some cases.

Despite these limitations, the true value of Lagrange multiplier theorem is its ability to identify *points of equality* and providing a deeper geometric intuition for why extrema occur under constraint.