UBC MATH 264: Vector Calculus for Electrical Engineering

Vector Identities

For vectors $\mathbf{u} = \langle u_x, u_y, u_z \rangle, \mathbf{v} = \langle v_x, v_y, v_z \rangle, \mathbf{w} = \langle w_x, w_y, w_z \rangle$,

Unit Vector $\hat{u} = \dfrac{\mathbf{u}}{||\mathbf{u}||}$

Dot Product $\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z = ||\mathbf{u}||\,||\mathbf{v}|| \cos\theta$

Magnitude $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_x^2 + u_y^2 + u_z^2}$

Cross Product $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$

Projection $\mathrm{proj}_{\mathbf{u}}(\mathbf{F}) = \left(\dfrac{\mathbf{F} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u}$

Parameterization

Arc Length $\displaystyle\int_C \mathrm{d}s = \int_a^b ||\mathbf{r}'(t)||\,\mathrm{d}t$

Surface Area $\displaystyle\iint_S \mathrm{d}S = \iint_D ||\mathbf{r}_u \times \mathbf{r}_v||\,\mathrm{d}u\,\mathrm{d}v$

Line Integral $\displaystyle\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t$

Surface Integral $\displaystyle\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}u\,\mathrm{d}v$

Frenet-Serret Formula

Curve $\mathbf{r}(t) = x(t)\,\mathbf{a}_x + y(t)\,\mathbf{a}_y + z(t)\,\mathbf{a}_z$

Arc length $\displaystyle s = \int_{t_0}^t ||\mathbf{v}||\,\mathrm{d}t$

Velocity $\mathbf{v} = \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$

Unit tangent $\widehat{\mathbf{T}} = \dfrac{\mathbf{v}}{||\mathbf{v}||}$

Acceleration $\mathbf{a} = \dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$

Binormal $\widehat{\mathbf{B}} = \dfrac{\mathbf{v} \times \mathbf{a}}{||\mathbf{v} \times \mathbf{a}||}$

Curvature $\kappa = \dfrac{||\mathbf{v} \times \mathbf{a}||}{||\mathbf{v}||^3}$

Radius of Curvature $\rho = \dfrac{1}{\kappa}$

Normal $\widehat{\mathbf{N}} = \widehat{\mathbf{B}} \times \widehat{\mathbf{T}}$

Torsion $\tau = \dfrac{(\mathbf{v} \times \mathbf{a}) \cdot (\mathrm{d}\mathbf{a}/\mathrm{d}t)}{||\mathbf{v} \times \mathbf{a}||^2}$

Frenet-Serret Formulae

$$\frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}s} = \kappa\widehat{\mathbf{N}}, \quad \frac{\mathrm{d}\widehat{\mathbf{N}}}{\mathrm{d}s} = -\kappa\widehat{\mathbf{T}} + \tau\widehat{\mathbf{B}}, \quad \frac{\mathrm{d}\widehat{\mathbf{B}}}{\mathrm{d}s} = -\tau\widehat{\mathbf{N}}$$

Cartesian Coordinates (x, y, z)

Line Element $\mathrm{d}\mathbf{l} = \mathbf{a}_x \mathrm{d}x + \mathbf{a}_y \mathrm{d}y + \mathbf{a}_z \mathrm{d}z$

Volume Element $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

Gradient $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{a}_x + \dfrac{\partial f}{\partial y}\mathbf{a}_y + \dfrac{\partial f}{\partial z}\mathbf{a}_z$

Curl $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

Divergence $\nabla \cdot \mathbf{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$

Cylindrical Coordinates (ρ, ϕ, z)

Transformation $x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$

Local Basis $\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$

$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$

Line Element $\mathrm{d}\mathbf{l} = \mathbf{a}_\rho \mathrm{d}\rho + \rho\mathbf{a}_\phi \mathrm{d}\phi + \mathbf{a}_z \mathrm{d}z$

Volume Element $\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$

Gradient $\nabla f = \dfrac{\partial f}{\partial \rho}\mathbf{a}_\rho + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \phi}\mathbf{a}_\phi + \dfrac{\partial f}{\partial z}\mathbf{a}_z$

Curl $\nabla \times \mathbf{F} = \dfrac{1}{\rho}\begin{vmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$

Divergence $\nabla \cdot \mathbf{F} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho F_\rho) + \dfrac{1}{\rho}\dfrac{\partial F_\phi}{\partial \phi} + \dfrac{\partial F_z}{\partial z}$

Spherical Coordinates (r, θ, ϕ)

Transformation $x = r\sin\theta\cos\phi, \; y = r\sin\theta\sin\phi, \; z = r\cos\theta$

Local Basis $\mathbf{a}_r = \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$

$\mathbf{a}_\theta = \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$

$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$

Line Element $\mathrm{d}\mathbf{l} = \mathbf{a}_r \mathrm{d}r + r\mathbf{a}_\theta \mathrm{d}\theta + r\sin\theta\,\mathbf{a}_\phi \mathrm{d}\phi$

Volume Element $\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

Gradient $\nabla f = \dfrac{\partial f}{\partial r}\mathbf{a}_r + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\mathbf{a}_\theta + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi}\mathbf{a}_\phi$

Curl $\nabla \times \mathbf{F} = \dfrac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$

Divergence

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(F_\theta\sin\theta) + \frac{\partial F_\phi}{\partial \phi}\right]$$

Fundamental Theorems of Calculus

Gradient Theorem $\displaystyle\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$

Divergence Theorem $\displaystyle\iiint_V (\nabla \cdot \mathbf{F})\,\mathrm{d}V = \oiint_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$

Stokes's Theorem $\displaystyle\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$