UBC MATH 256: Differential Equations

Fundamental Set of Solutions

Wronskian Matrix

$$W[y_1, y_2](t) = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$$

$W(t) \neq 0$ for some $t_0 \in I \Rightarrow y_1, y_2$ form a fundamental set of solutions.

1. First-Order Linear ODE

Constant-coefficient	$y' + ay = 0$
General solution	$y(t) = Ce^{-at}$
Inhomogeneous	$y'(t) + p(t)y(t) = g(t)$
Integrating factor	

$$y(t) = \frac{1}{\mu(t)}\left[\int^t \mu(s)g(s)\mathrm{d}s + C\right], \mu(t) = e^{\int^t p(s)\mathrm{d}s}$$

2. Second-Order Linear ODE

Constant-coefficient	$ay'' + by' + cy = 0$
Characteristic Equation	$ar^2 + br + c = 0$
General Solution	

$\Delta > 0: r_1 < r_2$	$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
$\Delta = 0: r_1 = r_2$	$y(t) = (c_1 + c_2 t)e^{rt}$
$\Delta < 0: r = \lambda \pm \mu i$	$y(t) = e^{\lambda t}\big(c_1 \cos(\mu t) + c_2 \sin(\mu t)\big)$

Euler's Equation	$at^2y'' + bty' + cy = 0$
Characteristic equation	$ak(k-1) + bk + c = 0$
General Solution	

$\Delta > 0: k_1 < k_2$	$y(t) = c_1 t^{k_1} + c_2 t^{k_2}$
$\Delta = 0: k_1 = k_2$	$y(t) = t^k(c_1 + c_2 \ln t)$
$\Delta < 0: k = \alpha \pm \beta i$	$y(t) = t^\alpha(c_1 \cos(\beta \ln t) + c_2 \sin(\beta \ln t))$

Method of Undetermined Constants (1^{st} and 2^{nd} ODEs)

$$ay'' + by' + cy = g(t)$$

$g(t)$	$y_p(t)$
$\sum_{i=0}^{n} b_i t^i$	$\sum_{i=0}^{n} c_i t^i$
e^{bt}	Ae^{bt}
$\sin(bt)$ or $\cos(bt)$	$A\sin(bt) + B\cos(bt)$
Product of above	Product of above
Linear combinations of above	Linear combinations of above

Resonance: When $g(t)$ is a solution of the corresponding homogeneous equation. Multiply $g(t)$ by the term t or t^2.

Reduction of Orders

To find a general solution of a second-order, linear ODE

$$y'' + p(t)y' + q(t)y = g(t)$$

1. Find one particular solution y_1 of the ODE

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$

2. Find a solution y in the form of

$$y(t) = u(t)y_1(t)$$

3. Compute $y' = uy_1' + u'y_1$ and $y'' = uy_1'' + 2u'y_1' + u''y_1$
4. Substitute in $y'' + py' + qy = (2y_1' + py_1)u' + (y_1)u''$. After substitution, the u terms cancel.
5. Let $A(t) = u'$ then the equation becomes a 1^{st} linear ODE

$$A'(t) + \left(2\frac{y_1'}{y_1} + p\right)A(t) = \frac{g}{y_1}$$

6. Solve by integrating factor

$$\mu(t) = \exp\left(\int^t \left(2\frac{y_1'(s)}{y_1(s)} + p(s)\right)ds\right)$$

7. General solution

$$y(t) = c_1 y_1(t) + y_1(t)\left(\int^t \left[\frac{1}{\mu(s)}\int^s \mu(\tau)\frac{g(\tau)}{y_1(\tau)}\,\mathrm{d}\tau\right]\mathrm{d}s\right)$$

Phase Plane

$\lambda_1 \neq \lambda_2 < 0$	Stable Node	Trajectories move straight into origin
$\lambda_1 \neq \lambda_2 > 0$	Unstable node	Trajectories move straight out from origin
$\lambda_1 < 0 < \lambda_2$	(Unstable) Saddle	Approaches along one axis, diverges along other
$\lambda = \alpha \pm \beta i, \alpha < 0$	Stable Spiral	Spirals inward toward origin
$\lambda = \alpha \pm \beta i, \alpha > 0$	Unstable Spiral	Spirals outward from origin
$\lambda = \pm\beta i$	Center	Circular trajectories
$\lambda_1 = 0, \lambda_2 < 0$	Stable line	Line of equilibria, others decay toward it
$\lambda_1 = 0, \lambda_2 > 0$	Unstable Line	Line of equilibria, others diverge

Laplace Transform

Definition $\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st}f(t)\mathrm{d}t$

Delta Function $\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\,\mathrm{d}t = f(t_0)$

Laplace Transform Table

$\boldsymbol{f(t)}$	$\mathbf{F}(\boldsymbol{s})$
$\delta(t-a)$	e^{-as}
$1,\ u_0(t)$	$\frac{1}{s}$
$t,\ tu_0(t)$	$\frac{1}{s^2}$
$u_a(t), u(t-a)$	$\frac{e^{-as}}{s}$
$u_a(t)f(t-a), u(t-a)f(t-a)$	$e^{-as}\mathrm{F}(s)$
e^{-at}	$\frac{1}{s+a}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$e^{-\alpha t}\sin(\beta t)$	$\frac{\beta}{(s+\alpha)^2 + \beta^2}$
$e^{-\alpha t}\cos(\beta t)$	$\frac{s+\alpha}{(s+\alpha)^2 + \beta^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$f'(t)$	$s\mathrm{F}(s) - f(0)$
$f''(t)$	$s^2\mathrm{F}(s) - sf(0) - f'(0)$
$(-t)^n f(t)$	$\mathrm{F}^{(n)}(s)$
$\int_0^t f(\tau)\,\mathrm{d}\tau$	$\frac{\mathrm{F}(s)}{s}$
$\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$	$\mathrm{G}(s)\mathrm{F}(s)$
$f(t-\tau)$	$e^{-\tau s}\mathrm{F}(s)$
$e^{at}f(t)$	$\mathrm{F}(s-a)$
$f(at)$	$\frac{1}{a}\mathrm{F}\left(\frac{s}{a}\right)$

Fundamental Set of Solutions (revisit)

Fundamental matrix $\quad \Phi(t) = [\vec{x}_1(t) \quad \vec{x}_2(t) \quad ... \quad \vec{x}_n(t)]$

$$= \begin{bmatrix} x_{11}(t) & x_{12}(t) & ... & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & ... & x_{2n}(t) \\ ... & ... & ... & ... \\ x_{n1}(t) & x_{n2}(t) & ... & x_{nn}(t) \end{bmatrix}$$

General Wronskian matrix
$W[\vec{x}_1(t), \vec{x}_2(t), ..., \vec{x}_n(t)] = \det \Phi(t)$
$W(t) \neq 0 \Rightarrow \vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ form a fundamental set of solutions.

3. Linear system of ODEs

a. Linear systems of First-order ODEs

Constant-coefficient $\quad \mathbf{x}'(t) = A\mathbf{x}$
Characteristic Equation $\quad \det(A - \lambda I) = 0$
General solution $\forall\, \lambda_1, \lambda_2 \in \mathbb{R} \quad x(t) = e^{\lambda_1 t}\mathbf{v}_1 + e^{\lambda_2 t}\mathbf{v}_2$

General solution $\forall\, \lambda = \alpha \pm \beta i, \lambda \in \mathbb{C}$

$$\mathbf{x}(t) = e^{\alpha t}\left(c_1 \begin{bmatrix}\cos(\beta t)\\ \sin(\beta t)\end{bmatrix} + c_2 \begin{bmatrix}-\sin(\beta t)\\ \cos(\beta t)\end{bmatrix}\right)$$

b. Linear systems of Second-order ODEs

Constant-Coefficient $\quad \mathbf{x}''(t) = A\mathbf{x}$
Eigenvalues of A $\quad \det(A - \lambda I) = 0$
Step 1. Solve homogeneous system

$$\mathbf{x}'' = A\mathbf{x} \quad \Rightarrow \quad \text{for each eigenvalue } \lambda \text{ of } A,\ \textbf{solve } x'' = \lambda x.$$

General solution for each eigenvalue

$$\mathbf{x}_i(t) = \begin{cases} c_1 e^{\sqrt{\lambda_i}t}\mathbf{v}_i + c_2 e^{-\sqrt{\lambda_i}t}\mathbf{v}_i, & \lambda_i > 0,\\ (c_1 + c_2 t)\mathbf{v}_i, & \lambda_i = 0\\ c_1 \cos\left(\sqrt{-\lambda_i}\,t\right)\mathbf{v}_i + c_2 \sin\left(\sqrt{-\lambda_i}\,t\right)\mathbf{v}_i, & \lambda_i < 0. \end{cases}$$

Step 2. Find general solution

$$\mathbf{x}(t) = \sum_i \mathbf{x_i}(t)$$

Method of Undetermined Constants (system of ODEs)

$$\mathbf{x}'(t) = A\mathbf{x} + \mathbf{f}(t)$$

$\mathbf{f}(t)$	$\mathbf{x}_p(t)$
const $\mathbf{b}$	const $\mathbf{k}$
$\sum_{i=0}^{n} c_i t^i \mathbf{b}$	$\sum_{i=0}^{n} c_i t^i \mathbf{k}$
$e^{\alpha t}\mathbf{b}$	$e^{\alpha t}\mathbf{k}$
$\sin(\omega t)\mathbf{b} \,\|\, \cos(\omega t)\mathbf{b}$	$\mathbf{k}_1 \sin(\omega t) + \mathbf{k}_2 \cos(\omega t)$
Product of above	Product of above
Linear combinations of above	Linear combinations of above

Resonance: Multiply the trial solution by the term $(\mathbf{k} + \alpha t\mathbf{v})$ while $\mathbf{v}$ is the eigenvector associated with the resonant eigenvalue. For example, given

$$\mathbf{x}' = \begin{bmatrix}2 & -1\\ 3 & -2\end{bmatrix}\mathbf{x} + e^{-t}\begin{bmatrix}2\\ 1\end{bmatrix}$$

Since e^{-t} is a solution of the system of ODEs, we find the eigenvalue and eigenvector corresponding to the function. Particularly, $\lambda_1 = -1 \to \mathbf{v}_1 = \begin{bmatrix}1\\ 3\end{bmatrix}$. So we guess

$$\mathbf{x}_p(t) = e^{-t}\left(\mathbf{k} + \alpha t \begin{bmatrix}1\\ 3\end{bmatrix}\right)$$

Method of Variation of Parameter

Constant-coefficient $\quad \mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$

1. Find a fundamental matrix
$$\Phi(t) = \left[\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ \cdots\ \mathbf{x}_n(t)\right].$$
2. Compute the inverse matrix $\Phi^{-1}(t)$.
3. Form the integrand $\Phi^{-1}(t)\mathbf{f}(t)$.
4. Integrate $\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s$.
5. Multiply by $\Phi(t)$
$$\mathbf{x}_p(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,ds.$$
6. Write the general solution
$$\mathbf{x}(t) = \mathbf{x}_p(t) + \Phi(t)\mathbf{c}.$$
7. To solve an IVP, If $\mathbf{x}(t_0) = \mathbf{x}_0$, choose
$$\mathbf{c} = \Phi^{-1}(t_0)\mathbf{x}_0.$$

Ex: Solve the system

$$\mathbf{x}' = \begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\mathbf{x} + \begin{bmatrix}2e^{-t}\\ 3t\end{bmatrix}$$

Step 1. Eigenvalues and eigenvectors

$$\det(A - rI) = (r+2)^2 - 1 = 0 \quad \Rightarrow \quad r_1 = -3,\ r_2 = -1.$$

The corresponding eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix}1\\ -1\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}.$$

Step 2. Fundamental matrix

$$\Phi(t) = \begin{bmatrix}e^{-3t} & e^{-t}\\ -e^{-3t} & e^{-t}\end{bmatrix}.$$

Step 3. Inverse of the fundamental matrix

$$\Phi^{-1}(t) = \frac{1}{2}\begin{bmatrix}e^{3t} & -e^{3t}\\ e^{t} & e^{t}\end{bmatrix}.$$

Step 4. Compute $\Phi^{-1}(t)\mathbf{f}(t)$

$$\Phi^{-1}(t)\mathbf{f}(t) = \frac{1}{2}\begin{bmatrix}e^{3t} & -e^{3t}\\ e^{t} & e^{t}\end{bmatrix}\begin{bmatrix}2e^{-t}\\ 3t\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2e^{2t} - 3te^{3t}\\ 2 + 3te^{t}\end{bmatrix}.$$

Step 5. Integrate

$$\int_0^t \Phi^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s = \frac{1}{2}\int_0^t \begin{bmatrix}2e^{2s} - 3se^{3s}\\ 2 + 3se^{s}\end{bmatrix}\mathrm{d}s$$
$$= \frac{1}{2}\begin{bmatrix}e^{2t} - \left(t - \frac{1}{3}\right)e^{3t}\\ 2t + 3(t-1)e^{t}\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-4/3\\ 2\end{bmatrix}.$$

Step 6. Find the general solution, noticed that the constant vector can be absorbed into $\mathbf{c}$, so

$$\mathbf{x}(t) = \Phi(t)\int_0^t \Phi^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s$$
$$= \frac{1}{2}\Phi(t)\begin{bmatrix}e^{2t} - \left(t - \frac{1}{3}\right)e^{3t}\\ 2t + 3(t-1)e^{t}\end{bmatrix} + \Phi(t)\mathbf{c}$$

with $\Phi(t) = \begin{bmatrix}e^{-3t} & e^{-t}\\ -e^{-3t} & e^{-t}\end{bmatrix}$.

Therefore, we the multiplication and yield

$$\mathbf{x}(t) = \begin{bmatrix}\left(t + \frac{1}{2} + c_2\right)e^{-t} + t - \frac{4}{3} + c_1 e^{-3t}\\ \left(t - \frac{1}{2} + c_2\right)e^{-t} + 2t - \frac{5}{3} - c_1 e^{-3t}\end{bmatrix}$$

Eigenvalues and Eigenfunctions

Equation $y'' + \lambda y = 0$

Dirichlet BCs $y(0) = 0 = y(L)$

Eigenvalues $\quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, ...$

Eigenfunctions $\quad y_n(x) = \sin\left(\frac{n\pi}{L}x\right)$

Neumann BCs $y'(0) = 0 = y'(L)$

Eigenvalues $\quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = \mathbf{0}, 1, 2, 3, ...$

Eigenfunctions $\quad y_n(x) = \cos\left(\frac{n\pi}{L}x\right)$

Orthogonality of eigenfunctions
We have two ODEs (B_1) and (B_2) respectively

$$(B_1)\ y'' + \lambda y = 0, \qquad y(a) = 0 = y(b)$$
$$(B_2)\ y'' + \lambda y = 0, \qquad y'(a) = 0 = y'(b)$$

Let y_1 and y_2 be eigenfunctions of (B_1) (or (B_2) with eigenvalues $\lambda_1 \neq \lambda_2$ then

$$\int_a^b y_1(x)y_2(x)\mathrm{d}x = 0.$$

Consequences

1. $$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx = \begin{cases}0, & m \neq n,\\ \frac{L}{2}, & m = n.\end{cases}$$
2. $$\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\,dx = \begin{cases}0, & m \neq n,\\ L, & m = n = 0,\\ \frac{L}{2}, & m = n \neq 0.\end{cases}$$
3. $$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\,dx = 0 \quad \forall\, m, n.$$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

Odd - Even characteristics

If f is odd $\Rightarrow a_n = 0$
If f is even $\Rightarrow b_n = 0$

4. Partial Differential Equations

Heat Equation

$$\begin{cases} u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \\ u(x,0) = f(x) \end{cases}$$

Separation form $\quad u(x,t) = X(x)T(t)$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha T(t)} = -\lambda$$

Dirichlet BCs $[u(0,t) = 0 = u(L,t)]$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi\alpha/L)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

Neumann BCs $[u_x(0,t) = 0 = u_x(L,t)]$

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-(n\pi\alpha/L)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

$$c_0 = \frac{2}{L}\int_0^L f(x)\,\mathrm{d}x \qquad c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

Inhomogeneous $\quad u_t = \alpha^2 u_{xx} + f(x)$.
Step 1. Find steady-state function u_{ss}. At a steady state, $u_t = 0$, so $u_{xx} = -f(x,t)/\alpha^2$
Step 2. $u(x,t) = u_{\text{ss}} + v(x,t)$ with $v(x,t)$ is the solution of the homogeneous heat equation.

Wave Equation

$$\begin{cases} y_{tt} = c^2 y_{xx}, \quad 0 < x < L, \quad t > 0 \\ y(x,0) = f(x) \\ y_t(x,0) = g(x) \end{cases}$$

Separation form $\quad y(x,t) = X(x)T(t)$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{c^2 T(t)} = -\lambda$$

Dirichlet BCs $[y(0,t) = 0 = y(L,t)]$

$$y(x,t) = \sum_{n=1}^{\infty} \left[a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

Neumann BCs $[y_x(0,t) = 0 = y_x(L,t)]$

$$y(x,t) = \frac{d_0}{2} + \sum_{n=1}^{\infty} \left[d_n \cos\left(\frac{n\pi ct}{L}\right) + e_n \sin\left(\frac{n\pi ct}{L}\right)\right] \cos\left(\frac{n\pi x}{L}\right)$$

$$d_0 = \frac{2}{L}\int_0^L f(x)\,\mathrm{d}x$$

$$d_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

$$e_n = \frac{2}{n\pi c}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

D'Alembert formula $(-\infty < x < \infty)$

$$y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s$$

Laplace's Equation

$$\begin{cases} u_{xx} + u_{yy} = 0, 0 < x < a, 0 < y < b \\ u(0,y) = 0 = u(a,y), 0 \le y \le b \\ u(x,b) = 0, 0 \le x \le a \end{cases}$$

Separation form $\quad u(x,y) = X(x)Y(y)$

Dirichlet BCs $[u(x,0) = f(x)]$

$$\Rightarrow -\frac{X''(x)}{X(x)} = \frac{Y'(y)}{Y(y)} = \lambda$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}$$

$$b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x$$

Neumann BCs $[u_y(x,0) = f(x)]$

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right)$$

$$c_n = \frac{-2}{n\pi\cosh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)$$

6. Linear Algebra

Determinant

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Inverse of a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Diagonalization

$$A = PDP^{-1}$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad P = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix}$$

Matrix exponential

$$e^{At} = Pe^{Dt}P^{-1},\ e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

Euler's Formula $\quad e^{ix} = \cos x + i\sin x$
Hyperbolic sine $\quad \sinh(x) = \frac{1}{2}(e^x - e^{-x})$
Hyperbolic cosine $\quad \cosh(x) = \frac{1}{2}(e^x + e^{-x})$

ODEs and Applications

Newton's Cooling Law $\quad \frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_{\text{env}})$

Mass Balance Equation $\quad \frac{\mathrm{d}M}{\mathrm{d}t} = \mathrm{Q}_{in}C_{in} - \mathrm{Q}_{out}\frac{M(t)}{V(t)}$

($\mathrm{C_{in}} = 0$ when water is clean.)

RLC circuit $\quad L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = V(t)$

Spring-dashpot system $\quad m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}x}{\mathrm{d}t} + kx = f(t)$

For a 2$^{\text{nd}}$ ODE $ay'' + by' + cy = 0$,

$$\Delta < 0 \Leftrightarrow b^2 < 4ac \Leftrightarrow \text{Underdamped}$$
$$\Delta > 0 \Leftrightarrow b^2 > 4ac \Leftrightarrow \text{Overdamped}$$
$$\Delta = 0 \Leftrightarrow b^2 = 4ac \Leftrightarrow \text{Critically damped}$$