NOTES ON CLOSEST APPROXIMANT

$\underset{\sim}{P} = \underset{\sim}{e}^{\underset{\sim}{A}\cancelto{1}{t}}$ is my transition probability matrix (row-stochastic)
with $\underset{\sim}{A}$ generator matrix
$\underset{\sim}{e}(\cdot)$ the matrix exponential function

I seek an approximant to $\underset{\sim}{P}$, say $\underset{\sim}{Q}$, that has structural 0's and is still a row stochastic matrix.

I want the Markov processes that work with $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ to be as close ~~far~~ as possible in the following sense:

for an arbitrary starting state $\underset{\sim}{S}_0$ I want, at cycle n, $\| \underset{\sim}{S}_{n,P} - \underset{\sim}{S}_{n,Q} \|_\alpha$ to be small with respect to a norm $\|\cdot\|_\alpha$.

If $\alpha = 1$ → (induced ℓ_1) minimize sum of absolute distances
$\alpha = F$ → (Frobenius) minimize root squared distances
$\alpha = \infty$ → minimize maximum distance
$\alpha = 2$ → spectral norm

In Sum, my problem is to

(1)

(1a) $$\underset{\underset{\sim}{Q}}{\text{minimize}} \ \frac{1}{n} \sum_{i=1}^{n} \| \underset{\sim}{S}_{i,P} - \underset{\sim}{S}_{i,Q} \|_\alpha$$

For some n that works in my application

s.t

(1b) $\underset{\sim}{Q} \cdot \underset{\sim}{1} = \underset{\sim}{1}$ (row stochastic)

(1c) $\underset{\sim}{Q} \le \underset{\sim}{W}$ (W has 0 and 1's only, and enforces structural 0's)

(1d) $\underset{\sim}{Q} \geq \underset{\sim}{0}$ (Q non negative)

(1e) $Q_{ii} = 1$ if $P_{ii} = 1$ (same absorbing states)

(1f) for any $\underset{\sim}{S}_0$ on a simplex.

Reformulate (1): ②

Objective:

Write (1) as

$$\frac{1}{n}\sum_{i=1}^{n}\left\| \underset{\sim}{S_0}\,\underset{\sim}{P}^i - \underset{\sim}{S_0}\,\underset{\sim}{Q}^i \right\|_\alpha =$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left\| \underset{\sim}{S_0}\left(\underset{\sim}{P}^i - \underset{\sim}{Q}^i\right)\right\|_\alpha \leq$$ ← works for all norms

$$\frac{1}{n}\sum_{i=1}^{n}\left[\left\|\underset{\sim}{S_0}\right\|_\alpha \left\|\underset{\sim}{P}^i - \underset{\sim}{Q}^i\right\|_\alpha\right] =$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left\|\underset{\sim}{P}^i - \underset{\sim}{Q}^i\right\|_\alpha \; \underbrace{+ \left\|\underset{\sim}{S_0}\right\|_\alpha}_{\text{Constant}}$$

Constant → drop from objective and drop (1f)

So the objective in (1) is equivalent to

(2)

(2a) $\underset{\underset{\sim}{Q}}{\text{minimize}} \sum_{i=1}^{n}\left\| P^i - Q^i\right\|_\alpha$ (with $\alpha = 2$)

s.t.

(2b) $\underset{\sim}{Q}\cdot\underset{\sim}{\mathbb{1}} = \mathbb{1}$

(2c) $\underset{\sim}{Q} \leq \underset{\sim}{W}$

(2d) $\underset{\sim}{Q} \geq \underset{\sim}{0}$

(2e) I guess any absorbing state in $\underset{\sim}{P}$ is also absorbing state in $\underset{\sim}{Q}$ (TRUE?)
so $Q_{ii} = 1$ if $P_{ii} = 1$

I find the identity

$$\left(\underset{\sim}{P}^{i} - \underset{\sim}{Q}^{i}\right) = \sum_{j=0}^{i-1} \underset{\sim}{P}^{j} \left(\underset{\sim}{P} - \underset{\sim}{Q}\right) \underset{\sim}{Q}^{i-j-1}$$

Bertsimas book on R.O.

so for norms with $\alpha = 1, 2, \infty, F$; Submultiplicative:

$$\left\|\underset{\sim}{P}^{i} - \underset{\sim}{Q}^{i}\right\|_{\alpha} \leq \sum_{j=0}^{i-1} \left\|\underset{\sim}{P}^{j}\right\|_{\alpha} \cdot \left\|\underset{\sim}{P} - \underset{\sim}{Q}\right\|_{\alpha} \cdot \left\|\underset{\sim}{Q}^{i-j-1}\right\|_{\alpha}$$

Because $\underset{\sim}{P}^{i}, \underset{\sim}{Q}^{i}$ are stochastic, we know an upper bound for them.

if $\alpha = 1$ (ℓ_1 norm) $\quad \|P^i\|_1 \leq 1$

if $\alpha = 2$ (spectral norm) $\quad \|P^i\|_2 \leq 1$

if $\alpha = \infty$ (max) $\quad \|P^i\|_\infty = 1$

Which means that for $\alpha = 1, 2, \infty$:

$$\left\|\underset{\sim}{P}^{i} - \underset{\sim}{Q}^{i}\right\|_{\alpha} \leq \sum_{j=0}^{i-1} \left\|\underset{\sim}{P}^{j}\right\|_{\alpha} \left\|\underset{\sim}{P} - \underset{\sim}{Q}\right\|_{\alpha} \left\|\underset{\sim}{Q}^{i-j-1}\right\|_{\alpha}$$

(3) $$\leq \sum_{j=0}^{i-1} 1 \cdot \left\|\underset{\sim}{P} - \underset{\sim}{Q}\right\|_{\alpha} 1 = i \cdot \left\|\underset{\sim}{P} - \underset{\sim}{Q}\right\|_{\alpha}$$

---

for frobenius $\|P^i\|_F \leq \sqrt{r}$, with P^i being $r \times r$

So (#3) would yield

(4) $$\left\|\underset{\sim}{P}^{i} - \underset{\sim}{Q}^{i}\right\|_{F} \leq i r \left\|P - Q\right\|_{F}$$

Per Wikipedia, the bound (4) is too lax

the Frobenius is more than subumultiplicative, so that

$$\|\underset{\sim}{P}^i - \underset{\sim}{Q}^i\|_F \leq \sum_{j=0}^{i-1} \underbrace{\|\underset{\sim}{P}^j\|_2}_{\leq 1} \; \|\underset{\sim}{P} - \underset{\sim}{Q}\|_F \; \underbrace{\|\underset{\sim}{Q}^{i-j-1}\|_2}_{\leq 1}$$

So it becomes

$$\|\underset{\sim}{P}^i - \underset{\sim}{Q}^i\|_F \leq i \, \|\underset{\sim}{P} - \underset{\sim}{Q}\|_F$$, same as for the others.

Then another objective, similar to (#2a) but not the same would be to minimize the upper bound of (2a):

which is $$\sum_{i=1}^{n} i \, \|\underset{\sim}{P} - \underset{\sim}{Q}\|_\alpha = \frac{n \cdot (n+1)}{2} \|\underset{\sim}{P} - \underset{\sim}{Q}\|_\alpha$$

or simply

(3)

$$\begin{cases} \text{3(a)} & \min_{Q} \|\underset{\sim}{P} - \underset{\sim}{Q}\|_\alpha \\ \text{(3b)} \quad \text{st} & \underset{\sim}{Q} \, \underset{\sim}{\mathbb{1}} = \underset{\sim}{\mathbb{1}} \\ \text{(3c)} & \underset{\sim}{Q} \leq \underset{\sim}{W} \\ \text{(3d)} & \underset{\sim}{Q} \geq \underset{\sim}{0} \\ \text{(3e)} & \text{and } Q_{ii} = 1 \text{ if } P_{ii} = 1 \end{cases}$$

easier to implement; will start with (#3).

Because the absorbing states are always the same (constraint (3e)) all problems can be further simplified.

(1) Find a permutation matrix F so that $\underset{\sim}{P}$ rearranges to

$$\underset{\sim}{F} \cdot \underset{\sim}{P} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{0} & \underset{\sim}{I} \end{bmatrix}$$

For $\underset{\sim}{Q}$

$$\underset{\sim}{F} \cdot \underset{\sim}{Q} = \begin{bmatrix} \underset{\sim}{C} & \underset{\sim}{D} \\ \underset{\sim}{0} & \underset{\sim}{I} \end{bmatrix}$$

Focusing only on $[\underset{\sim}{A}, \underset{\sim}{B}]$ and $[\underset{\sim}{C}, \underset{\sim}{D}]$ satisfies (3e)