Exercise 11 *γ-Astronomy 2* Lösung

This task is a continuation of the task *γ-Astronomy* from the ~~last~~ sheet. Now it is to be determined whether there really is a γ source at the position at which the telescope was pointed. As a reminder the likelihood function was

$$\ln L = -F = N_{\text{off}} \ln(b) + N_{\text{on}} \ln(s + \alpha b) - (1 + \alpha)b - s - \ln(N_{\text{off}}!) - \ln(N_{\text{on}}!) \tag{1}$$

and the following values for s and b maximized this likelihood:

$$\hat{s} = N_{\text{on}} - \alpha N_{\text{off}} \tag{2}$$

$$\hat{b} = N_{\text{off}} \tag{3}$$

(a) The null hypothesis states that there is no γ source: $s_0 = 0$. Under this assumption, what value and what error result for b_0 according to the maximum likelihood method?

$$\ln L = N_{off}\ln(b) + N_{on}\ln(\alpha b) - (1+\alpha)b - \ln(N_{off}!) - \ln(N_{on}!)$$

$$\frac{d(\ln L)}{db} = \frac{N_{off}}{b} + \frac{N_{on}}{b} - (1+\alpha) \stackrel{!}{=} 0$$

$$\Leftrightarrow \frac{N_{off}+N_{on}}{b} = 1+\alpha$$

$$\Rightarrow \underline{\hat{b} = \frac{N_{off}+N_{on}}{1+\alpha}}$$

$$\frac{d^2(-\ln L)}{db^2} = \frac{N_{off}+N_{on}}{b^2} = \underline{\frac{1+\alpha}{N_{off}+N_{on}}}$$

$$\Rightarrow \underline{\underline{b_0 = \frac{N_{off}+N_{on}}{1+\alpha} \pm \frac{1+\alpha}{N_{off}+N_{on}}}}$$

(b) What is the ratio λ of the two likelihoods?

$$\lambda_{log} = \frac{N_{off}\ln(b) + N_{on}\ln(\alpha b) - (1+\alpha)b - \ln(N_{off}!) - \ln(N_{on}!)}{N_{\text{off}} \ln(b) + N_{\text{on}} \ln(s + \alpha b) - (1 + \alpha)b - s - \ln(N_{\text{off}}!) - \ln(N_{\text{on}}!)}$$

$$= \frac{\ln\left(b^{N_{off}} \cdot \alpha b^{N_{on}} \cdot \frac{1}{N_{off}!N_{on}!}\right) - (1+\alpha)b}{\ln\left(b^{N_{off}} \cdot (s+\alpha b)^{N_{on}} \frac{1}{N_{off}!N_{on}!}\right) - (1+\alpha)b - s}$$

$$\lambda = \frac{\left(b^{N_{off}} \cdot \alpha b^{N_{on}} \cdot \cancel{\frac{1}{N_{off}!N_{on}!}}\right) \cdot \frac{1}{\exp((1+\alpha)b)}\Big|_{b=b_0}}{\left(b^{N_{off}} \cdot (s+\alpha b)^{N_{on}} \cancel{\frac{1}{N_{off}!N_{on}!}}\right) \cdot \frac{1}{\exp[(1+\alpha)b - s]}\Big|_{b=b_1, s=s_1}}$$

$$b_0 = \frac{N_{off}+N_{on}}{1+\alpha}$$

$$b_1 = N_{off}$$

$$s_1 = N_{on} - \alpha N_{off}$$

$$= \frac{\left(\frac{N_{off}+N_{on}}{1+\alpha}\right)^{N_{off}} \alpha\left(\frac{N_{off}+N_{on}}{1+\alpha}\right)^{N_{on}} \cdot \frac{1}{\exp\left((1+\alpha)\frac{N_{off}+N_{on}}{1+\alpha}\right)}}{N_{off}^{N_{off}} \cdot \left(N_{on} - \cancel{\alpha N_{off}} + \cancel{\alpha N_{off}}\right)^{N_{on}} \cdot \frac{1}{\exp\left((1+\cancel{\alpha)N_{off}} - N_{on} - \cancel{\alpha N_{off}}\right)}}$$

$$= \frac{\left(\frac{N_{off}+N_{on}}{1+\alpha}\right)^{N_{off}} \alpha\left(\frac{N_{off}+N_{on}}{1+\alpha}\right)^{N_{on}} \cdot \exp(N_{off}-N_{on})}{N_{off}^{N_{off}}\, N_{on}^{N_{on}} \cdot \exp(N_{off}+N_{on})}$$

$$= \frac{\left(\frac{N_{off}+N_{on}}{1+\alpha}\right)^{N_{off}} \alpha\left(\frac{N_{off}+N_{on}}{1+\alpha}\right)^{N_{on}}}{N_{off}^{N_{off}}\, N_{on}^{N_{on}}} \exp(-2N_{on})$$

(c) Under the given hypotheses and with large N_{on}, N_{off}, $D = -2\ln\lambda$ is χ^2-distributed with one degree of freedom. With what confidence do you reject the null hypothesis? State your result in units of sigma.

Hint: Consider a standard normal distributed variable u. What distribution does u^2 follow? Compare with D.

$$D = -2\ln\lambda$$

$$= -2\ln\left(\frac{\left(s^{N_{off}} \cdot \alpha s^{N_{on}} \cdot \frac{1}{N_{off}!\,N_{on}!}\right) - \exp((1+\alpha)s)}{\left(s^{N_{off}} \cdot (s+\alpha s)^{N_{on}} \frac{1}{N_{off}!\,N_{on}!}\right) - \exp((1+\alpha)s - s)}\right)$$

$$= -2\ln\left(\left(s^{N_{off}} \cdot \alpha s^{N_{on}} \cdot \frac{1}{N_{off}!\,N_{on}!}\right) - \exp((1+\alpha)s)\right)$$
$$+ 2\ln\left(\left(s^{N_{off}} \cdot (s+\alpha s)^{N_{on}} \frac{1}{N_{off}!\,N_{on}!}\right) - \exp((1+\alpha)s - s)\right)$$

$$u = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

Sieht ähnlich aus, also

$$u^2 = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{\sigma^2}(x-\mu)^2}$$