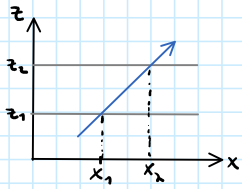
Exercise 28

Dienstag, 1. November 2022 11:21

a)



$\rightarrow$ Punkte (z,x): (z_1,x_1), (z_2,x_2)

gesucht: $x(z) = az + b$

$$a = \frac{x_2 - x_1}{z_2 - z_1}$$

$$b: \; x(z_1) = x_1 \Rightarrow \frac{x_2 - x_1}{z_2 - z_1} \cdot z_1 + b = x_1$$

$$\Leftrightarrow b = x_1 - \frac{x_2 - x_1}{z_2 - z_1} \cdot z_1$$

$$a = \frac{1}{z_2 - z_1}(x_2 - x_1), \quad b = \frac{1}{z_2 - z_1}(z_2 x_1 - z_1 x_2)$$

$$\Rightarrow x(z) = \frac{1}{z_2 - z_1}\left[(x_2 - x_1)z + z_2 x_1 - z_1 x_2\right]$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\frac{1}{z_2 - z_1}\begin{pmatrix} -1 & 1 \\ z_2 & -z_1 \end{pmatrix}}_{B} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$y = Bx \quad \Rightarrow V[y] = B \cdot V[x] \cdot B^T$$

$$V[x] = \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix}$$

$$\Rightarrow V[y] = \frac{1}{(z_2 - z_1)^2}\begin{pmatrix} -1 & 1 \\ z_2 & -z_1 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix} \cdot \begin{pmatrix} -1 & z_2 \\ 1 & -z_1 \end{pmatrix}$$

$$= \frac{1}{(z_2 - z_1)^2}\begin{pmatrix} -1 & 1 \\ z_2 & -z_1 \end{pmatrix} \cdot \begin{pmatrix} -\sigma_{x_1}^2 & z_2 \sigma_{x_1}^2 \\ \sigma_{x_2}^2 & -z_1 \sigma_{x_2}^2 \end{pmatrix}$$

$$= \frac{1}{(z_2 - z_1)^2}\begin{pmatrix} \sigma_{x_1}^2 + \sigma_{x_2}^2 & -z_2 \sigma_{x_1}^2 - z_1 \sigma_{x_2}^2 \\ -z_2 \sigma_{x_1}^2 - z_1 \sigma_{x_2}^2 & z_2^2 \sigma_{x_1}^2 + z_1^2 \sigma_{x_2}^2 \end{pmatrix} = \begin{pmatrix} \sigma_a^2 & \mathrm{Cov}(a,b) \\ \mathrm{Cov}(a,b) & \sigma_b^2 \end{pmatrix}$$

$$\Rightarrow \sigma_a = \frac{1}{z_1 - z_2}\sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2}$$

$$\sigma_b = \frac{1}{z_1 - z_2}\sqrt{z_2^2 \sigma_{x_1}^2 + z_1^2 \sigma_{x_2}^2}$$

$$\mathrm{Cov}(a,b) = -\frac{1}{(z_2 - z_1)^2}\left(z_2 \sigma_{x_1}^2 + z_1 \sigma_{x_2}^2\right)$$

$$\rho_{a,b} = \frac{\mathrm{Cov}(a,b)}{\sigma_a \cdot \sigma_b} = \frac{z_2 \sigma_{x_1}^2 + z_1 \sigma_{x_2}^2}{\sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2}\sqrt{z_2^2 \sigma_{x_1}^2 + z_1^2 \sigma_{x_2}^2}}$$

b)

$$x(z) = az + b = \frac{1}{z_2 - z_1}\left[(x_2 - x_1)z + z_2 x_1 - z_1 x_2\right]$$

$$\Rightarrow x_3 = x(z_3) = \frac{1}{z_2 - z_1}\left[(x_2 - x_1)z_3 + z_2 x_1 - z_1 x_2\right]$$

$$\sigma_{x_3} = \sqrt{\left(\frac{\partial x_3}{\partial a}\sigma_a\right)^2 + \left(\frac{\partial x_3}{\partial b}\sigma_b\right)^2 + 2 \cdot \frac{\partial x_3}{\partial a}\frac{\partial x_3}{\partial b}\mathrm{Cov}(a,b)}$$

$$= \sqrt{z_3^2 \sigma_a^2 + \sigma_b^2 + 2 z_3 \mathrm{Cov}(a,b)} = \frac{1}{z_1 - z_2}\sqrt{z_3^2(\sigma_{x_1}^2 + \sigma_{x_2}^2) + z_2^2 \sigma_{x_1}^2 + z_1^2 \sigma_{x_2}^2 - 2 z_3 (z_2 \sigma_{x_1}^2 + z_1 \sigma_{x_2}^2)}$$

$$= \frac{1}{z_1 - z_2}\sqrt{\sigma_{x_1}^2 (z_3^2 + z_2^2 - 2 z_2 z_3) + \sigma_{x_2}^2 (z_3^2 + z_1^2 - 2 z_3 z_1)}$$

c)

$z_1 - z_2$

c)

For positive z-values the error of x_3 increases when neglecting correlation between a & b.