Exercise 11

Dienstag, 6. Dezember 2022 18:09

(a) The null hypothesis states that there is no γ source: $s_0 = 0$. Under this assumption, what value and what error result for b_0 according to the maximum likelihood method?

$$\ln L = N_{off} \ln(b) + N_{on} \ln(\alpha b) - (1+\alpha) b - \ln(N_{off}!) - \ln(N_{on}!)$$

$$\frac{d(\ln L)}{db} = \frac{N_{off}}{b} + \frac{N_{on}}{b} - (1+\alpha) \overset{!}{=} 0$$

$$\Leftrightarrow \frac{N_{off}+N_{on}}{b} = 1+\alpha$$

$$\Rightarrow \underline{\hat{b} = \frac{N_{off}+N_{on}}{1+\alpha}}$$

$$\frac{d^2(\ln L)}{db^2} = \frac{N_{off}+N_{on}}{b^2} = \underline{\frac{1+\alpha}{N_{off}+N_{on}}}$$

$$\Rightarrow \underline{\underline{b_0 = \frac{N_{off}+N_{on}}{1+\alpha} \pm \frac{1+\alpha}{N_{off}+N_{on}}}}$$

b)

$$\lambda = \frac{L(b_0, s_0)}{L(b_1, s_1)} \qquad b_0 = \frac{N_{off}+N_{on}}{1+\alpha}, \quad s_0 = 0$$

$$b_1 = N_{off}, \quad s_1 = N_{on} - \alpha N_{off}$$

$$L(b,s) = \frac{s^{N_{off}} (\alpha b + s)^{N_{on}}}{N_{off}! \, N_{on}!} e^{-(1+\alpha) b - s}$$

$$\Rightarrow \lambda = \frac{\left(\alpha \frac{N_{off}+N_{on}}{1+\alpha}\right)^{N_{on}} e^{-\cancel{(1+\alpha)} \frac{N_{off}+N_{on}}{\cancel{1+\alpha}}}}{(N_{on} - \alpha N_{off})^{N_{off}} (\cancel{\alpha N_{off}} + N_{on} - \cancel{\alpha N_{off}})^{N_{on}} e^{-\cancel{(1+\alpha)} N_{off} - N_{on} + \cancel{\alpha N_{off}}}}$$

$$= \underline{\left(\frac{\alpha}{N_{on}} \frac{N_{off}+N_{on}}{1+\alpha}\right)^{N_{on}} \cdot \left(\frac{1}{N_{on} - \alpha N_{off}}\right)^{N_{off}}}$$

c)

$$D = -2 \ln(\lambda) = \underline{-2 N_{on} \ln\left(\frac{\alpha}{N_{on}} \frac{N_{off}+N_{on}}{1+\alpha}\right) + 2 N_{off} \ln(N_{on} - \alpha N_{off})}$$

Vergleiche: $\mathcal{N}(\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{u^2}{2\sigma^2}}$

u^2 ist χ^2 verteilt, $-2 \ln(\lambda)$ auch

Exercise 12

Donnerstag, 1. Dezember 2022 10:39

a)

Die Zählraten der einzelnen Bins folgen einer Poisson-Verteilung.

Die PDF lauten also $\frac{e^{-Np_i}(Np_i)^{n_i}}{n_i!}$ & $\frac{e^{-Mp_i}(Mp_i)^{m_i}}{m_i!}$

b)

Die Likelihood ist das Produkt der beiden PDF:

$$L(p_i) = \frac{e^{-Np_i}(Np_i)^{n_i} \cdot e^{-Mp_i}(Mp_i)^{m_i}}{n_i!\, m_i!} = \frac{1}{n_i! m_i!} e^{-p_i(N+M)} \cdot N^{n_i} M^{m_i} (p_i)^{n_i+m_i}$$

Maximiere $F = \ln(L) = -p_i(N+M) + (n_i+m_i)\cdot \ln(p_i) + \text{const}$

$$\left.\frac{\partial F}{\partial p_i}\right|_{\hat{p}_i} = 0$$

$$\Rightarrow -(N+M) + \frac{n_i+m_i}{\hat{p}_i} = 0 \quad \Leftrightarrow \quad \hat{p}_i = \frac{n_i+m_i}{N+M}$$

c)

$$\chi^2 = \sum_{i=1}^{r} \frac{(n_i - n_i^{Model})^2}{n_i^{Model}} \qquad n_i^{Model} = N\hat{p}_i \quad (\text{analog für } M)$$

$$\Rightarrow \chi^2 = \sum_{i=1}^{r} \left[\frac{\left(n_i - N\cdot\frac{n_i+m_i}{N+M}\right)^2}{\frac{n_i+m_i}{N+M}\cdot N} + \frac{\left(m_i - M\frac{n_i+m_i}{N+M}\right)^2}{\frac{n_i+m_i}{N+M}\cdot M} \right]$$

d)

- $2r$ Terme werden aufsummiert
- $(r-1)$ p_i Werte durch Schätzer $\hat{p}_i$ gegeben
  - $(r-1)$, da $\sum p_i = 1 \Rightarrow$ legt ein p_i fest
- M, N ebenfalls durch Schätzer $N = \sum_i n_i$, $M = \sum_i m_i$ gegeben

$\Rightarrow \; df = 2r - (r-1) - 2 = r-1$ Freiheitsgrade

- Für einen geringen Stichprobenumfang (<10 Einträge/Bin), folgt die Teststatistik keiner χ^2-Verteilung, weil die Unsicherheiten in diesem Fall zu groß werden.

e)

- 2 Freiheitsgrade

$N = 111 + 188 + 333 = 632$

$M = 15 + 36 + 30 = 81$

$p_i = \frac{n_i+m_i}{N+M}$

$\Rightarrow p_1 = \frac{111+15}{632+81} = \frac{126}{713} \approx 0{,}177$

$p_2 = \frac{188+36}{632+81} = \frac{224}{713} \approx 0{,}314$

$p_3 = 1 - p_2 - p_1 = \frac{363}{713} \approx 0{,}509$

Habe nicht gesehen, dass man die garnicht mehr braucht :(

$$\chi^2 = \frac{1}{NM}\sum_i^r \frac{(Nm_i - Mn_i)^2}{n_i+m_i}$$

$$= \frac{1}{632\cdot 81}\left(\frac{(632\cdot 15 - 81\cdot 111)^2}{111+15} + \frac{(632\cdot 36 - 81\cdot 188)^2}{188+36} + \frac{(632\cdot 30 - 81\cdot 333)^2}{333+30} \right)$$

$\approx 8{,}43$

$$= \frac{1}{632\cdot 81}\left(\frac{(632\cdot 75 - 81\cdot 111)}{111+15} + \frac{(632\cdot 36 - 81\cdot 188)}{188+36} + \frac{(632\cdot 30 - 81\cdot 333)}{333+30}\right)$$

$$\approx 8{,}43$$

Percentage Points of the Chi-Square Distribution

Degrees of Freedom	Probability of a larger value of x^2								
	0.99	0.95	0.90	0.75	0.50	0.25	0.10	0.05	0.01
1	0.000	0.004	0.016	0.102	0.455	1.32	2.71	3.84	6.63
2	0.020	0.103	0.211	0.575	1.386	2.77	4.61	5.99	9.21
3	0.115	0.352	0.584	1.212	2.366	4.11	6.25	7.81	11.34
4	0.297	0.711	1.064	1.923	3.357	5.39	7.78	9.49	13.28
5	0.554	1.145	1.610	2.675	4.351	6.63	9.24	11.07	15.09
6	0.872	1.635	2.204	3.455	5.348	7.84	10.64	12.59	16.81
7	1.239	2.167	2.833	4.255	6.346	9.04	12.02	14.07	18.48
8	1.647	2.733	3.490	5.071	7.344	10.22	13.36	15.51	20.09
9	2.088	3.325	4.168	5.899	8.343	11.39	14.68	16.92	21.67
10	2.558	3.940	4.865	6.737	9.342	12.55	15.99	18.31	23.21
11	3.053	4.575	5.578	7.584	10.341	13.70	17.28	19.68	24.72
12	3.571	5.226	6.304	8.438	11.340	14.85	18.55	21.03	26.22
13	4.107	5.892	7.042	9.299	12.340	15.98	19.81	22.36	27.69
14	4.660	6.571	7.790	10.165	13.339	17.12	21.06	23.68	29.14
15	5.229	7.261	8.547	11.037	14.339	18.25	22.31	25.00	30.58
16	5.812	7.962	9.312	11.912	15.338	19.37	23.54	26.30	32.00
17	6.408	8.672	10.085	12.792	16.338	20.49	24.77	27.59	33.41
18	7.015	9.390	10.865	13.675	17.338	21.60	25.99	28.87	34.80
19	7.633	10.117	11.651	14.562	18.338	22.72	27.20	30.14	36.19
20	8.260	10.851	12.443	15.452	19.337	23.83	28.41	31.41	37.57
22	9.542	12.338	14.041	17.240	21.337	26.04	30.81	33.92	40.29
24	10.856	13.848	15.659	19.037	23.337	28.24	33.20	36.42	42.98
26	12.198	15.379	17.292	20.843	25.336	30.43	35.56	38.89	45.64
28	13.565	16.928	18.939	22.657	27.336	32.62	37.92	41.34	48.28
30	14.953	18.493	20.599	24.478	29.336	34.80	40.26	43.77	50.89
40	22.164	26.509	29.051	33.660	39.335	45.62	51.80	55.76	63.69
50	27.707	34.764	37.689	42.942	49.335	56.33	63.17	67.50	76.15
60	37.485	43.188	46.459	52.294	59.335	66.98	74.40	79.08	88.38

(This one gives α.)

H_0: Werte stammen aus gleicher Verteilung

berechnet: $\chi^2 \approx 8{,}43$

$\alpha = 0{,}1$: $\chi^2 = 4{,}61 \Rightarrow H_0$ akzeptiert

$\alpha = 0{,}05$: $\chi^2 = 5{,}99 \Rightarrow$ " "

$\alpha = 0{,}01$: $\chi^2 = 9{,}21 \Rightarrow H_0$ abgelehnt

Typ 2 Fehler: H_0 wird angenommen, obwohl die Hypothese falsch ist.