ex 29

Donnerstag, 3. November 2022 18:23

a) $\bar{X} = \frac{1}{n}\sum_{i=1}^{\infty} X_i$

$$\langle \bar{X} \rangle = \langle \frac{1}{n}\sum_{i=1}^{n} X_i \rangle = \frac{1}{n}\sum_{i=1}^{n}\langle X_i \rangle = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{n}{n}\mu = \mu$$

$\rightarrow \mu - \langle \bar{X} \rangle = \mu - \mu = 0$, unbiased

b) $Var(\bar{X}) := E((\bar{x}-\mu)^2)$

$$Var(\bar{X}) = Var(\frac{1}{N}\sum_{i=1}^{n} X_i)$$

$$= \frac{1}{N}\sum_{i=1}^{n}\left(\frac{1}{N}\sum_{i=1}^{N} X_i - \mu\right)_i^2$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{N^2}\underbrace{\left(\sum_{i=1}^{N} X_i\right)^2}_{=N^2\bar{x}^2} - \frac{2\mu}{N}\underbrace{\sum_{i=1}^{N} X_i}_{=N\bar{x}} + \mu^2\right]_i$$

$$= \frac{1}{N}\sum_{i=1}^{N}(\bar{x}^2 - 2\mu\bar{x} + \mu^2)$$

$$= \frac{1}{N}\underbrace{\sum_{i=1}^{N}(\bar{x}-\mu)^2}_{=\sigma^2} = \frac{\sigma^2}{N} \quad \square$$

c) $S_0^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$

$$\langle S_0^2 \rangle = \langle \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 \rangle$$

$$= \frac{1}{n}\sum_{i=1}^{n}\langle (X_i-\mu)^2 \rangle \qquad Var(y) = \langle (y-\mu)^2 \rangle$$

$$= \frac{1}{n}\sum_{i=1}^{n} Var(X_i) \qquad \frac{1}{n}\sum_{i=1}^{n} Var(X_i) = \sigma^2$$

$$= \sigma^2$$

$\rightarrow \sigma^2 - \langle S_0 \rangle^2 = \sigma^2 - \sigma^2 = 0$, unbiased

d) $S_1'^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$$\langle S_1'^2 \rangle = \langle \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 \rangle$$

$$= \frac{1}{n}\langle \sum_{i=1}^{n}(X_i - \mu + \mu + \bar{X})^2 \rangle$$

$$= \frac{1}{n}\langle \sum_{i=1}^{n}[(X_i-\mu)^2 - 2(X_i-\mu)(\bar{X}-\mu) + (\bar{X}-\mu)^2] \rangle$$

$$= \frac{1}{n}\langle \sum_{i=1}^{n}(X_i-\mu)^2 - 2\sum_{i=1}^{n}(X_i-\mu)(\bar{X}-\mu) + \sum_{i=1}^{n}(\bar{X}-\mu)^2 \rangle$$

$$= \frac{1}{n}\langle \sum_{i=1}^{n}(X_i-\mu)^2 - \underline{2n(\bar{X}-\mu)(\bar{X}-\mu)} + \underline{n(\bar{X}-\mu)^2} \rangle$$

$$= \frac{1}{n}\langle \sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}-\mu)^2 \rangle$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}\langle (X_i-\mu)^2 \rangle - n\langle (\bar{X}-\mu)^2 \rangle\right) \qquad Var(y) = \langle (y-\mu)^2 \rangle$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} Var(X_i) - n\,Var(\bar{X})\right) \qquad \frac{1}{n}\sum_{i=1}^{n} Var(X_i) = \sigma^2 \;\; || \;\; Var(\bar{X}) = \frac{\sigma^2}{n}$$

$$= \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$$

$$\Rightarrow \sigma^2 - \langle S_1'^2 \rangle = \sigma^2 - \frac{n-1}{n}\sigma^2 = \left(1 - \frac{n-1}{n}\right)\sigma^2 = \frac{\sigma^2}{n} \neq 0 \quad , \text{ biased!}$$

Correction: $\sigma^2 = \frac{n}{n-1}\cdot\langle S_1'^2 \rangle$

$$\rightarrow S_1'^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$