Exercise8

November 17, 2022

1 Exercise 8

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

1.1 a)

Die Likelihood-Funktion L und die negative log-likelihood-Funktion lauten:

$L(\lambda) = \Pi_i \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}$

$l(\lambda) = -\ln(L) = -\Sigma_i (x_i \ln(\lambda) - \ln(x_i!) - \lambda)$

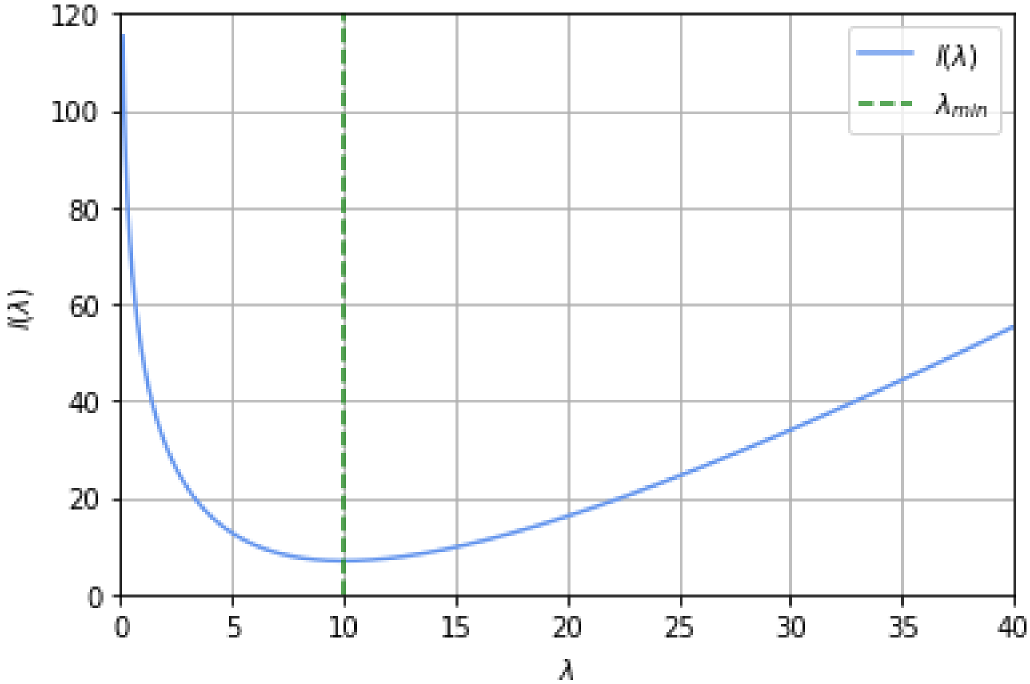
$\Rightarrow l(\lambda) = 3\lambda - (13+8+9)\ln(\lambda) + \ln(13! \cdot 8! \cdot 9!)$

[2]:
```
#def poisson(x, y): # Poisson-Verteilung (y := lambda)
#   return y**x/np.math.factorial(x)*np.exp(-y)

def l(y):  # negative log-likelihood function (y := lambda)
    return 3*y -(13+8+9)*np.log(y) + np.log(np.math.factorial(13)) + np.log(np.
 ↪math.factorial(8)) + np.log(np.math.factorial(9))
    #np.log(np.math.factorial(13)*np.math.factorial(8)*np.math.factorial(9))
```

[3]:
```
fig, ax = plt.subplots(1,1)

x = np.linspace(0.1, 40, 10000)
y = l(x)



ax.plot(x, y, label = "$l(\lambda)$", color = "cornflowerblue")
plt.vlines(10, 0, 120, ls = "dashed", color = "forestgreen", label =␣
 ↪"$\lambda_{min}$")

ax.set_xlabel("$\lambda$")
ax.set_ylabel("$l(\lambda)$")
ax.set_xlim(0, 40)
ax.set_ylim(0, 120)

ax.legend()
```

```
plt.grid()

print("Minimum: (", 10, ",", l(10),")")
```

```
Minimum: ( 10 , 6.881041446128762 )
```



1.1.1 b)

Minimum analytisch:

$\frac{\mathrm{d}l(\lambda)}{\mathrm{d}\lambda} \stackrel{!}{=} 0$

$\Rightarrow 3 - \frac{13+8+9}{\lambda} = 0$

$\Leftrightarrow \lambda = 10$

1.2 c)

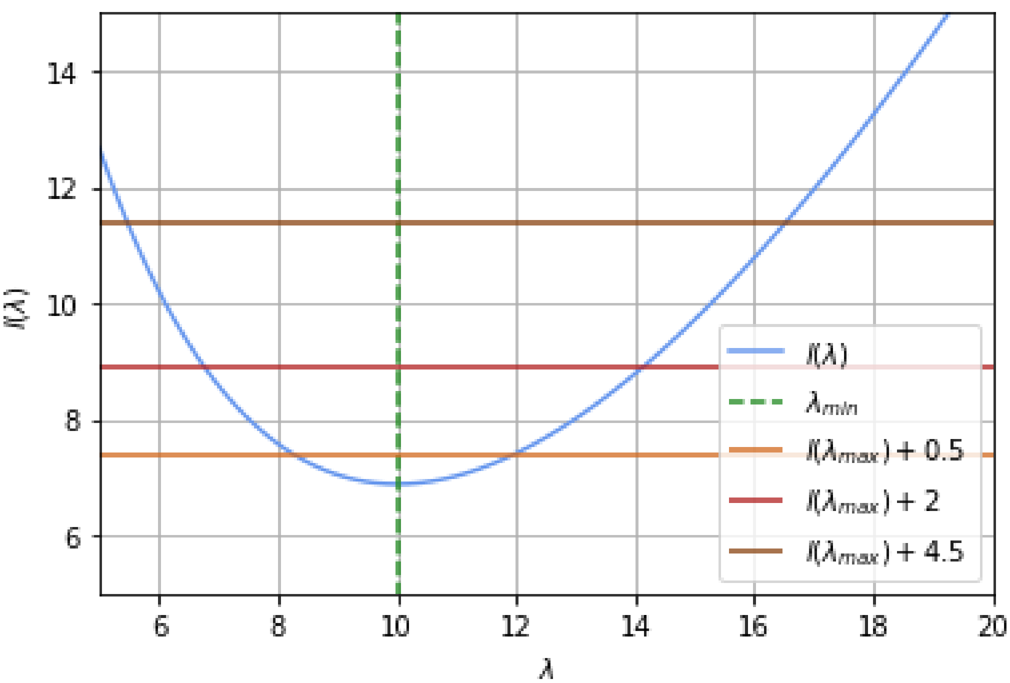
```
[4]: sigma1 = x[np.isclose(y, l(10)+0.5, rtol = 0.0001)] # estimating intersection
     ↪between log-likelihood fucntion and l_min + 1/2

     sigma2 = x[np.isclose(y, l(10)+2, rtol = 0.0002)]

     sigma3 = x[np.isclose(y, l(10)+9/2, rtol = 0.0003)]

     print(sigma1)
```

```
print(sigma2)
print(sigma3)
```

```
[ 8.28430843 11.93951395]
[ 6.77992799 14.11029103]
[ 5.47506751 16.52049205]
```

[5]:
```
plt.close()
fig1, ax1 = fig, ax

ax1.hlines(l(10)+0.5, 0, 40, color = "chocolate", label = "$l(\lambda_{max}) +␣
↪0.5$")
ax1.hlines(l(10)+2, 0, 40, color = "firebrick", label = "$l(\lambda_{max}) +␣
↪2$")
ax1.hlines(l(10)+9/2, 0, 40, color = "saddlebrown", label = "$l(\lambda_{max})␣
↪+ 4.5$")
ax1.set_ylim(5, 15)
ax1.set_xlim(5, 20)

ax1.legend()
fig1
```

[5]:



Die genannten Intervalle beschreiben die 1, 2 und 3- σ-Umgebungen des Schätzwertes für λ.

1.3 d)

Entwickeln der negativen log-likelihood-Funktion um das Minimum:

$\frac{\mathrm{d}l(\lambda)}{\mathrm{d}\lambda} = 3 - \frac{30}{\lambda}$

$\Rightarrow l'(10) = 0$

$\frac{\mathrm{d}^2 l(\lambda)}{\mathrm{d}\lambda^2} = \frac{30}{\lambda^2}$

$\Rightarrow l''(10) = \frac{3}{10}$

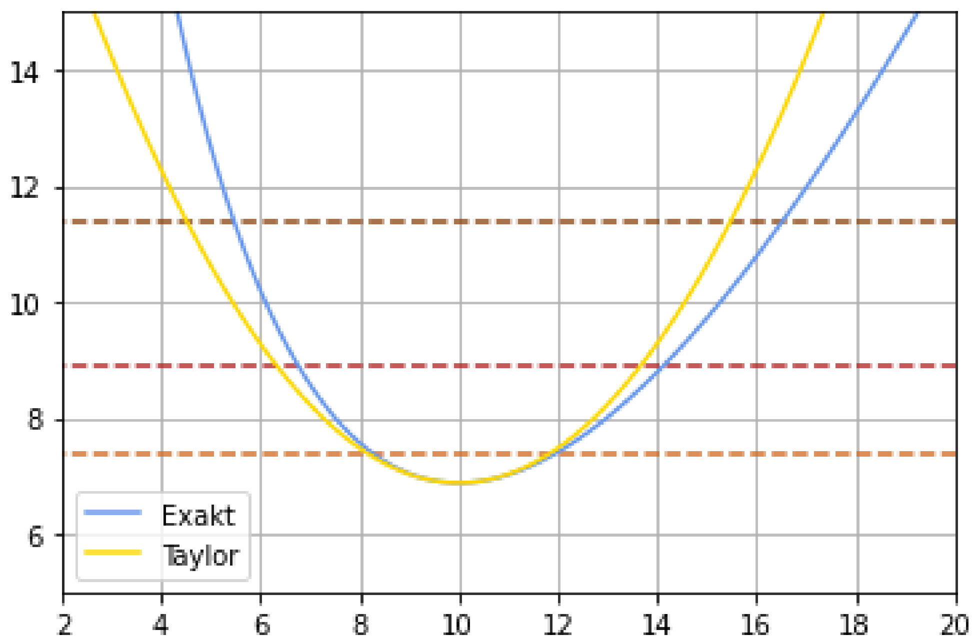
[6]:
```
l(10)
```

[6]: 6.881041446128762

$l(\lambda_{\mathrm{min}} = 10) \approx 6.88$

$\Rightarrow T(\lambda|\lambda_{\mathrm{min}}) \approx 6.88 + \frac{3}{20}(\lambda - 10)^2 + \mathcal{O}(\lambda^3)$

[7]:
```
def T(x):
    return l(10) + 3/20*(x-10)**2

y2 = T(x)

fig2, ax2 = plt.subplots(1,1)

ax2.hlines(l(10)+0.5, 0, 40, color = "chocolate", ls = "dashed")
ax2.hlines(l(10)+2, 0, 40, color = "firebrick", ls = "dashed")
ax2.hlines(l(10)+9/2, 0, 40, color = "saddlebrown", ls = "dashed")

ax2.plot(x, y, label = "Exakt", color = "cornflowerblue")
ax2.plot(x, y2, label = "Taylor", color = "gold")

ax2.set_ylim(5, 15)
ax2.set_xlim(2, 20)

plt.legend()
plt.grid()
```



$T(\lambda) = l(10) + \frac{3}{20}(\lambda - 10)^2 \stackrel{!}{=} c + l(10)$

$\Leftrightarrow \lambda = \pm\sqrt{\frac{20}{3}c} + 10$

[8]:
```
def f(x):
    return -np.sqrt(20/3*x) + 10, np.sqrt(20/3*x) + 10
```

[9]:
```
s1 = f(0.5)
s2 = f(2)
s3 = f(4.5)

print("Taylor: \t\t\t\t\t", "Numerisch:")
print(s1, sigma1)
print(s2, sigma2)
print(s3, sigma3)
```

```
Taylor:                                          Numerisch:
(8.174258141649446, 11.825741858350554) [ 8.28430843 11.93951395]
(6.348516283298892, 13.651483716701108) [ 6.77992799 14.11029103]
(4.522774424948339, 15.477225575051662) [ 5.47506751 16.52049205]
```

Für die 1-σ Umgebung ist die Taylorreihe eine gute Näherung, spätestens ab der 3-σ Umgebung weichen die Werte relativ stark ab.