CSE 310 Assignment 2 Solutions

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Problem 1

For each of the following pairs of functions $f(n)$ and $g(n)$, determine if $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, or $f(n) = \theta(g(n))$.

(a) $f(n) = 5n^9$, $g(n) = \frac{7n^5+5n^4}{9}$

(b) $f(n) = \log_6(n^7)$, $g(n) = \log_7(n^6)$

(c) $f(n) = 5^n$, $g(n) = 7^n$

(d) $f(n) = 9\log_8(n)$, $g(n) = 9\log_4(n)$

(e) $f(n) = 7n^3 + 4n + 3n^2$, $g(n) = 6n^2 + 5n$

Solution

(a) Since

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{5n^9}{\frac{7n^5+5n^4}{9}} = \lim_{n\to\infty} \frac{45n^9}{7n^5+5n^4} = \infty \neq 0,$$

therefore, $\boxed{f(n) = \Omega(g(n))}$.

(b) Since

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\log_6(n^7)}{\log_7(n^6)} = \lim_{n\to\infty} \frac{7\log_6(n)}{6\log_7(n)} = \lim_{n\to\infty} \frac{7\frac{\log(n)}{\log(6)}}{6\frac{\log(n)}{\log(7)}} = \lim_{n\to\infty} \frac{7\log(7)}{6\log(6)} = \frac{7\log(7)}{6\log(6)} \neq 0, \infty,$$

therefore, $\boxed{f(n) = \theta(g(n))}$.

(c) Since

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{5^n}{7^n} = 0 \neq \infty,$$

therefore, $\boxed{f(n) = O(g(n))}$.

(d) Since

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{9\log_8(n)}{9\log_4(n)} = \lim_{n\to\infty} \frac{\frac{\log_2(n)}{\log_2(8)}}{\frac{\log_2(n)}{\log_2(4)}} = \lim_{n\to\infty} \frac{\frac{1}{3}\log_2(n)}{\frac{1}{2}\log_2(n)} = \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3} \neq 0, \infty,$$

therefore, $\boxed{f(n) = \theta(g(n))}$.

(e) Since

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{7n^3+4n+3n^2}{6n^2+5n} = \infty \neq 0,$$

therefore, $\boxed{f(n) = \Omega(g(n))}$.

Problem 2

Suppose that the running time of algorithm A is $4500n^2$ and the running time of algorithm B is $50n^4$. What is the largest positive integer value of n for which the running time of algorithm A is larger than that of algorithm B?

Solution

Since the running time of algorithm A is larger than that of algorithm B,

$$\begin{gathered} 4500n^2 > 50n^4 \\ 90n^2 > n^4 \\ 90n^2 - n^4 > 0 \\ n^2(90 - n^2) > 0 \\ n^2(3\sqrt{10} + n)(3\sqrt{10} - n) > 0 \end{gathered}$$

The critical values of n are $n = -3\sqrt{10}, 0, 3\sqrt{10}$. Analyzing the polynomial behavior of the algorithms at these critical values yield that algorithm A runs faster than algorithm B when $-3\sqrt{10} < n < 0$ and $0 < n < 3\sqrt{10}$. Since $9 < 3\sqrt{10}$, the largest positive integer value of n for which the running time of algorithm A is larger than that or algorithm B is $\boxed{n = 9}$.

Problem 3

Prove that $9n^2 + 12n + 8 = O(n^2)$ using the definition of the O notation.

Solution

Proof: Suppose $a = 29 > 0$ and $b = 1 > 0$. Then, as $n \geq 1$, $n^2 \geq n \geq 1$. Therefore,

$$9n^2 + 12n + 8 \leq 9n^2 + 12n^2 + 8n^2 = 29n^2 = an^2$$

for $n \geq b = 1$. Therefore, $9n^2 + 12n + 8 = O(n^2)$. $\square$

Problem 4

Prove that $9n^2 - 10n + 3 = \Omega(n^2)$ using the definition of the Ω notation.

Solution

Proof: Suppose $a = 2 > 0$ and $b = 1 > 0$. Then, as $n \geq 1$, $n - 1 \geq 0$. Additionally, as $n \geq 1$, $7n \geq 7$ and $7n - 3 \geq 4 > 0$. Therefore,

$$\begin{gathered} (7n - 3)(n - 1) \geq 0 \\ 7n^2 - 10n + 3 \geq 0 \\ 9n^2 - 10 + 3 \geq 2n^2 = an^2 \end{gathered}$$

for $n \geq b = 1$. Therefore, $9n^2 - 10n + 3 = \Omega(n^2)$. $\square$

Problem 5

Compute the running time $T(n)$ of the following algorithm, with length$[A] = n$. Do this by expressing the number of times that each line is executed for the code

(a) for the general case.

(b) for the best case.

(c) for the worst case.

```
FUNCTION1 (A) //A is an array of length n

1  n = length[A]
2  count1 = 0
3  count2 = 0
4  for (i=1; i<=n; i++)
5          count1 = count1-1;
6          for (j=i; j<=n+4; j++)
7                 if (A[1] < 20)
8                      for (k=3; k<=j+6; k++)
9                           count2 = count1+1
10 return (count2)
```

Solution

(a) Let t_{ij} be the number of times that the algorithm goes through the if statement, where

$$t_{ij} = \begin{cases} 1 & \text{if the code goes through the if statement for some i, j} \\ 0 & \text{if the code does not go through the if statement for some i, j} \end{cases}$$

We analyze the number of times each line is executed for the algorithm for the general case, which is tabulated as shown below.

Line	Constant	Number of Times Each Line is Executed (General Case)
1	C_1	1
2	C_2	1
3	C_3	1
4	C_4	$\sum_{i=1}^{n+1} 1 = n+1$
5	C_5	$\sum_{i=1}^{n} 1 = n$
6	C_6	$\sum_{i=1}^{n}\sum_{j=i}^{n+5} 1 = \frac{1}{2}n^2 + \frac{11}{2}n$
7	C_7	$\sum_{i=1}^{n}\sum_{j=i}^{n+4} 1 = \frac{1}{2}n^2 + \frac{9}{2}n$
8	C_8	$\sum_{i=1}^{n}\sum_{j=i}^{n+4}\sum_{k=3}^{j+7} t_{ij}$
9	C_9	$\sum_{i=1}^{n}\sum_{j=i}^{n+4}\sum_{k=3}^{j+6} t_{ij}$
10	C_{10}	1

where

$$\sum_{i=1}^{n+1} 1 = (n+1) - 1 + 1 = n+1$$

$$\sum_{i=1}^{n} 1 = n - 1 + 1 = n$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i}^{n+5} 1 &= \sum_{i=1}^{n}(n+5-i+1) \\ &= \sum_{i=1}^{n}((n+6)-i) \\ &= n(n+6) - \frac{n(n+1)}{2} \\ &= n^2 + 6n - \frac{1}{2}n^2 - \frac{1}{2}n \\ &= \frac{1}{2}n^2 + \frac{11}{2}n\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i}^{n+4} 1 &= \sum_{i=1}^{n}(n+4-i+1) \\ &= \sum_{i=1}^{n}((n+5)-i) \\ &= n(n+5) - \frac{n(n+1)}{2} \\ &= n^2 + 5n - \frac{1}{2}n^2 - \frac{1}{2}n \\ &= \frac{1}{2}n^2 + \frac{9}{2}n\end{aligned}$$

Therefore,

$$\boxed{\mathrm{T(n)} = \mathrm{C}_1(1)+C_2(1)+C_3(1)+C_4(n+1)+C_5(n)+C_6\left(\tfrac{1}{2}n^2+\tfrac{11}{2}n\right)+C_7\left(\tfrac{1}{2}n^2+\tfrac{9}{2}n\right)+C_8\left(\sum_{i=1}^{n}\sum_{j=i}^{n+4}\sum_{k=3}^{j+7} t_{ij}\right)+ C_9\left(\sum_{i=1}^{n}\sum_{j=i}^{n+4}\sum_{k=3}^{j+6} t_{ij}\right)+C_{10}(1)}$$

for the general case.

(b) We analyze the number of times each line is executed for the algorithm for the best case, which is tabulated as shown below. The best case occurs when $A[1] \geq 20$ for all values of i, j, meaning that $t_{ij} = 0$ for all i, j.

Line	Constant	Number of Times Each Line is Executed (Best Case)
1	C_1	1
2	C_2	1
3	C_3	1
4	C_4	$\sum_{i=1}^{n+1} 1 = n+1$
5	C_5	$\sum_{i=1}^{n} 1 = n$
6	C_6	$\sum_{i=1}^{n}\sum_{j=i}^{n+5} 1 = \frac{1}{2}n^2 + \frac{11}{2}n$
7	C_7	$\sum_{i=1}^{n}\sum_{j=i}^{n+4} 1 = \frac{1}{2}n^2 + \frac{9}{2}n$
8	C_8	0
9	C_9	0
10	C_{10}	1

where

$$\sum_{i=1}^{n+1} 1 = (n+1) - 1 + 1 = n + 1$$

$$\sum_{i=1}^{n} 1 = n - 1 + 1 = n$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i}^{n+5} 1 &= \sum_{i=1}^{n}(n+5-i+1)\\ &= \sum_{i=1}^{n}((n+6)-i)\\ &= n(n+6) - \frac{n(n+1)}{2}\\ &= n^2 + 6n - \frac{1}{2}n^2 - \frac{1}{2}n\\ &= \frac{1}{2}n^2 + \frac{11}{2}n\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i}^{n+4} 1 &= \sum_{i=1}^{n}(n+4-i+1)\\ &= \sum_{i=1}^{n}((n+5)-i)\\ &= n(n+5) - \frac{n(n+1)}{2}\\ &= n^2 + 5n - \frac{1}{2}n^2 - \frac{1}{2}n\\ &= \frac{1}{2}n^2 + \frac{9}{2}n\end{aligned}$$

Thus, for the best case,

$$\begin{aligned}T(n) &= C_1(1) + C_2(1) + C_3(1) + C_4(n+1) + C_5(n) + C_6\left(\frac{1}{2}n^2 + \frac{11}{2}n\right)\\ &\qquad + C_7\left(\frac{1}{2}n^2 + \frac{9}{2}n\right) + C_8(0) + C_9(0) + C_{10}(1)\\ &= \left(\frac{1}{2}C_6 + \frac{1}{2}C_7\right)n^2 + \left(C_4 + C_5 + \frac{11}{2}C_6 + \frac{9}{2}C_7\right)n + (C_1 + C_2 + C_3 + C_4 + C_{10})\\ &= An^2 + Bn + C \text{ for constants A, B, C}\end{aligned}$$

Therefore, $\boxed{T(n) = An^2 + Bn + C}$ for the best case. The fastest growing term is n^2. Therefore, $\boxed{T(n) = \theta(n^2)}$ for the best case.

(c) We analyze the number of times each line is executed for the algorithm from Problem 5 for the worst case, which is tabulated as shown below. The worst case occurs when $A[1] < 20$ for all values of i, j, meaning that $t_{ij} = 1$ for all i, j.

Line	Constant	Number of Times Each Line is Executed (Worst Case)
1	C_1	1
2	C_2	1
3	C_3	1
4	C_4	$\sum_{i=1}^{n+1} 1 = n+1$
5	C_5	$\sum_{i=1}^{n} 1 = n$
6	C_6	$\sum_{i=1}^{n}\sum_{j=i}^{n+5} 1 = \frac{1}{2}n^2 + \frac{11}{2}n$
7	C_7	$\sum_{i=1}^{n}\sum_{j=i}^{n+4} 1 = \frac{1}{2}n^2 + \frac{9}{2}n$
8	C_8	$\sum_{i=1}^{n}\sum_{j=i}^{n+4}\sum_{k=3}^{j+7} 1 = \frac{1}{3}n^3 + 7n^2 + \frac{98}{3}n$
9	C_9	$\sum_{i=1}^{n}\sum_{j=i}^{n+4}\sum_{k=3}^{j+6} 1 = \frac{1}{3}n^3 + \frac{13}{2}n^2 + \frac{169}{6}n$
10	C_{10}	1

$$\sum_{i=1}^{n+1} 1 = (n+1) - 1 + 1 = n+1$$

$$\sum_{i=1}^{n} 1 = n - 1 + 1 = n$$

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n+5} 1 &= \sum_{i=1}^{n}(n+5-i+1) \\
&= \sum_{i=1}^{n}((n+6)-i) \\
&= n(n+6) - \frac{n(n+1)}{2} \\
&= n^2 + 6n - \frac{1}{2}n^2 - \frac{1}{2}n \\
&= \frac{1}{2}n^2 + \frac{11}{2}n
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n+4} 1 &= \sum_{i=1}^{n}(n+4-i+1) \\
&= \sum_{i=1}^{n}((n+5)-i) \\
&= n(n+5) - \frac{n(n+1)}{2} \\
&= n^2 + 5n - \frac{1}{2}n^2 - \frac{1}{2}n \\
&= \frac{1}{2}n^2 + \frac{9}{2}n
\end{aligned}$$

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n+4}\sum_{k=3}^{j+7} 1 &= \sum_{i=1}^{n}\sum_{j=i}^{n+4}(j+7-3+1) \\
&= \sum_{i=1}^{n}\sum_{j=i}^{n+4}(j+5) \\
&= \sum_{i=1}^{n}\left(\frac{(n+4)-i+1}{2}(i+(n+4))+5((n+4)-i+1)\right) \\
&= \sum_{i=1}^{n}\left(\frac{(n+5)-i}{2}(i+(n+4))+5((n+5)-i)\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}((n+5)-i)((n+4)+i)+5\sum_{i=1}^{n}((n+5)-i) \\
&= \frac{1}{2}\sum_{i=1}^{n}(n^2+9n+20+i-i^2)+5\sum_{i=1}^{n}((n+5)-i) \\
&= \frac{1}{2}\left(n(n^2+9n+20)+\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}\right)+5\left(n(n+5)-\frac{n(n+1)}{2}\right) \\
&= \frac{1}{2}\left(n^3+9n^2+20n+\frac{1}{2}n^2+\frac{1}{2}n-\frac{1}{3}n^3-\frac{1}{2}n^2-\frac{1}{6}n\right)+5\left(n^2+5n-\frac{1}{2}n^2-\frac{1}{2}n\right) \\
&= \frac{1}{2}\left(\frac{2}{3}n^3+9n^2+\frac{61}{3}n\right)+5\left(\frac{1}{2}n^2+\frac{9}{2}n\right) \\
&= \frac{1}{3}n^3+\frac{9}{2}n^2+\frac{61}{6}n+\frac{5}{2}n^2+\frac{45}{2}n \\
&= \frac{1}{3}n^3+7n^2+\frac{98}{3}n
\end{aligned}
$$

and

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n+4}\sum_{k=3}^{j+6} 1 &= \sum_{i=1}^{n}\sum_{j=i}^{n+4}(j+6-3+1) \\
&= \sum_{i=1}^{n}\sum_{j=i}^{n+4}(j+4) \\
&= \sum_{i=1}^{n}\left(\frac{(n+4)-i+1}{2}(i+(n+4))+4((n+4)-i+1)\right) \\
&= \sum_{i=1}^{n}\left(\frac{(n+5)-i}{2}(i+(n+4))+4((n+5)-i)\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}((n+5)-i)((n+4)+i)+4\sum_{i=1}^{n}((n+5)-i) \\
&= \frac{1}{2}\sum_{i=1}^{n}(n^2+9n+20+i-i^2)+4\sum_{i=1}^{n}((n+5)-i) \\
&= \frac{1}{2}\left(n(n^2+9n+20)+\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}\right)+4\left(n(n+5)-\frac{n(n+1)}{2}\right) \\
&= \frac{1}{2}\left(n^3+9n^2+20n+\frac{1}{2}n^2+\frac{1}{2}n-\frac{1}{3}n^3-\frac{1}{2}n^2-\frac{1}{6}n\right)+4\left(n^2+5n-\frac{1}{2}n^2-\frac{1}{2}n\right) \\
&= \frac{1}{2}\left(\frac{2}{3}n^3+9n^2+\frac{61}{3}n\right)+4\left(\frac{1}{2}n^2+\frac{9}{2}n\right) \\
&= \frac{1}{3}n^3+\frac{9}{2}n^2+\frac{61}{6}n+2n^2+18n \\
&= \frac{1}{3}n^3+\frac{13}{2}n^2+\frac{169}{6}n
\end{aligned}$$

Thus, for the worst case,

$$\begin{aligned}
T(n) &= C_1(1)+C_2(1)+C_3(1)+C_4(n+1)+C_5(n)+C_6\left(\frac{1}{2}n^2+\frac{11}{2}n\right) \\
&\qquad +C_7\left(\frac{1}{2}n^2+\frac{9}{2}n\right)+C_8\left(\frac{1}{3}n^3+7n^2+\frac{98}{3}n\right)+C_9\left(\frac{1}{3}n^3+\frac{13}{2}n^2+\frac{169}{6}n\right)+C_{10}(1) \\
&= \left(\frac{1}{3}C_8+\frac{1}{3}C_9\right)n^3+\left(\frac{1}{2}C_6+\frac{1}{2}C_7+7C_8+\frac{13}{2}C_9\right)n^2+\left(C_4+C_5+\frac{11}{2}C_6+\frac{9}{2}C_7+\frac{98}{3}C_8+\frac{169}{6}C_9\right)n \\
&\qquad +(C_1+C_2+C_3+C_4+C_{10}) \\
&= An^3+Bn^2+Cn+D \text{ for constants A, B, C, D}
\end{aligned}$$

Therefore, $\boxed{T(n)=An^3+Bn^2+Cn+D}$ for the worst case. The fastest growing term is n^3. Therefore, $\boxed{T(n)=\theta(n^3)}$ for the worst case.