EEE 350 Final Project Report

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December 7th, 2022

A copy of the code for this project is located at https://github.com/ttruong1000/EEE-350-Final-Project.

The Python packages used throughout this final project are shown below.

```python
# Packages required for this project
import random
import matplotlib.pyplot as plt
import math
from math import *
import numpy as np
import scipy as sp
import sympy
from sympy import *
import scipy.integrate as integrate
import time
```

Monty Hall Game

Task 2 - Theory of the Monty Hall Game

For playing the Monty Hall Game, we will consider three potential strategies.

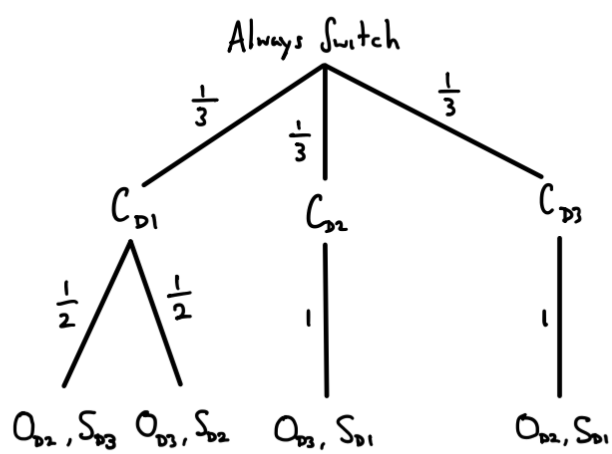
(a) Strategy A: Always switch the door

(b) Strategy B: Never switch the door

(c) Strategy C: Switch the door at random (with probability $\frac{1}{2}$)

Define success as the event that you get the car. Mathematically compute the probability of success for each of these strategies.

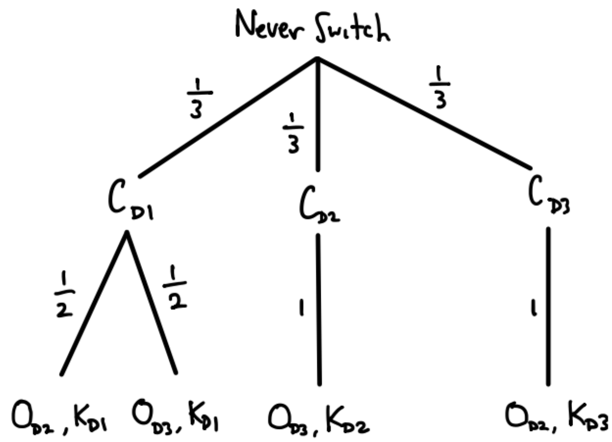
Let C_{D_n} be the event that door n is chosen. Let door 1 be the door that has the car and doors 2 and 3 have the goats. Let O_{D_n} be the event that door n is opened and contains a goat. Let S_{D_n} be the event that a player switches from their chosen door to door n. Let K_{D_n} be the event that a player keeps (does not switch) their chosen door n.



The probability that you get the car with Strategy A (always switch the door) is

$$\mathbb{P}[S_A] = \mathbb{P}[C_{D_2}O_{D_3}S_{D_1}] + \mathbb{P}[C_{D_3}O_{D_2}S_{D_1}] = \frac{1}{3} \times 1 + \frac{1}{3} \times 1 = \frac{1}{3} + \frac{1}{3}$$

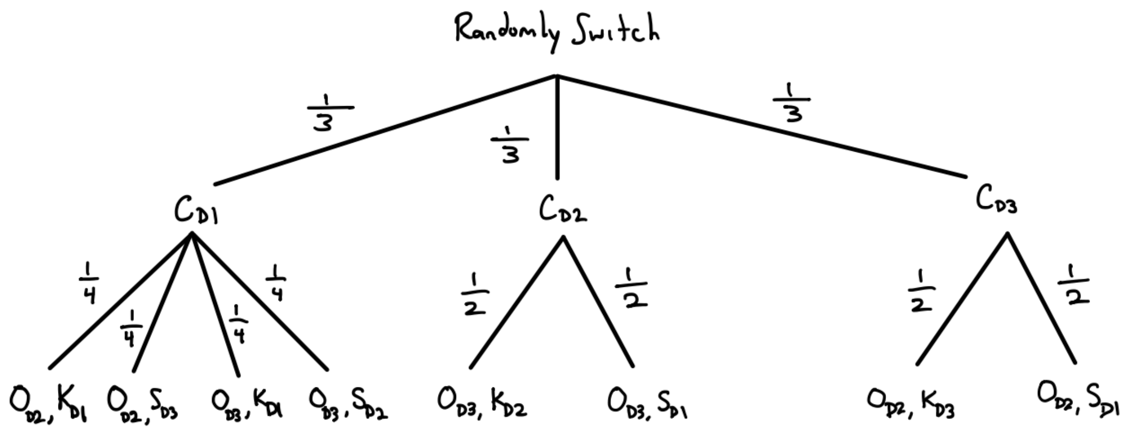
$$\boxed{\mathbb{P}[S_A] = \frac{2}{3}}$$



The probability that you get the car with Strategy B (never switch the door) is

$$\mathbb{P}[S_B] = \mathbb{P}[C_{D_2}O_{D_3}S_{D_1}] + \mathbb{P}[C_{D_3}O_{D_2}S_{D_1}] = \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} + \frac{1}{6}$$

$$\boxed{\mathbb{P}[S_B] = \frac{1}{3}}$$



The probability that you get the car with Strategy A (switch the door at random with probability $\frac{1}{2}$) is

$$\mathbb{P}[S_C] = \mathbb{P}[C_{D1}O_{D2}K_{D1}] + \mathbb{P}[C_{D1}O_{D3}K_{D1}] + \mathbb{P}[C_{D2}O_{D3}S_{D1}] + \mathbb{P}[C_{D3}O_{D2}S_{D1}] = \frac{1}{3}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2} = \frac{1}{12} + \frac{1}{12} + \frac{1}{6} + \frac{1}{6}$$

$$\boxed{\mathbb{P}[S_C] = \frac{1}{2}}$$

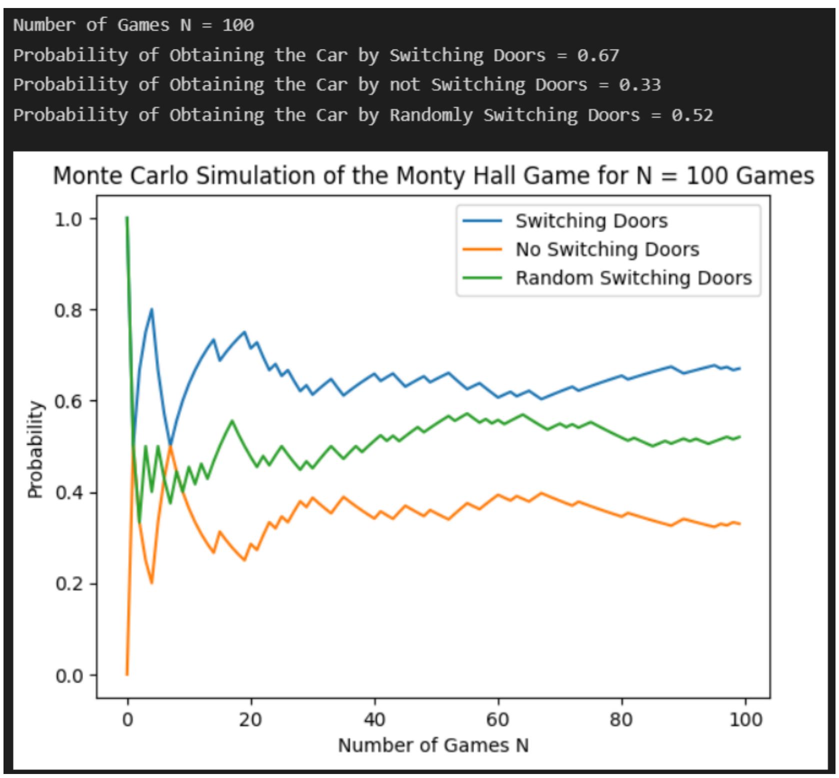
Task 3 - Simulating the Monty Hall Game with Monte Carlo Simulations

Write Python code to simulate $N = 100$ games for each of these strategies and record the fraction of times you are successful in getting the car - such a process is called a Monte Carlo simulation, a powerful technique with applications in various fields. The idea behind Monte Carlo simulation is to repeat an experiment many times and count the number of times a favorable outcome is obtained. Then, the fraction of outcomes that are favorable is an estimate of the probability of that outcome occurring. How accurate are the estimates for $N = 100$? Simulate $N = 1000$ games. Are the estimates more accurate in this case? Present the results of this task in a visual way that you feel is appropriate.

```python
def Monte_Carlo_Monty_Hall(N):
    # Set up variables for the Monte Carlo simulation for the Monty Hall game
    switch_success = 0
    switch_failure = 0
    no_switch_success = 0
    no_switch_failure = 0
    random_switch_success = 0
    random_switch_failure = 0
    switch_success_probabilities = []
    no_switch_success_probabilities = []
    random_switch_success_probabilities = []

    for i in range(N):
        # Set up the 3 doors and randomize the gae
        doors = ["car", "goat", "goat"]
        random.shuffle(doors)

        # Find the doors with goats
        doors_with_goats = []
        for i in range(len(doors)):
            if doors[i] == 'goat':
                doors_with_goats.append(i)

        # Select a door
        selected_door = random.randint(0, 2)

        # Case where selected door contains a car
        if doors[selected_door] != 'car':
            # Success case for always switch
            switch_success += 1
            # Failure case for never switch
            no_switch_failure += 1
            # Case for randomized switch
            del doors[selected_door]
            new_selected_door = random.randint(0, 1)
            if doors[new_selected_door] == 'car':
                # Success case for randomized switch
                random_switch_success += 1
            else:
                # Failure case for randomized switch
                random_switch_failure += 1
        # Case where selected door contains a goat
        else:
            # Failure case for always switch
            no_switch_success += 1
            # Success case for never switch
            switch_failure += 1
            # Case for randomized switch
            del doors[random.choice(doors_with_goats)]
            new_selected_door = random.randint(0, 1)
            if doors[new_selected_door] == 'car':
                # Success case for randomized switch
                random_switch_success += 1
            else:
                # Failure case for randomized switch
                random_switch_failure += 1

        # Calculate all probabilities are add it to their repsective lists
```

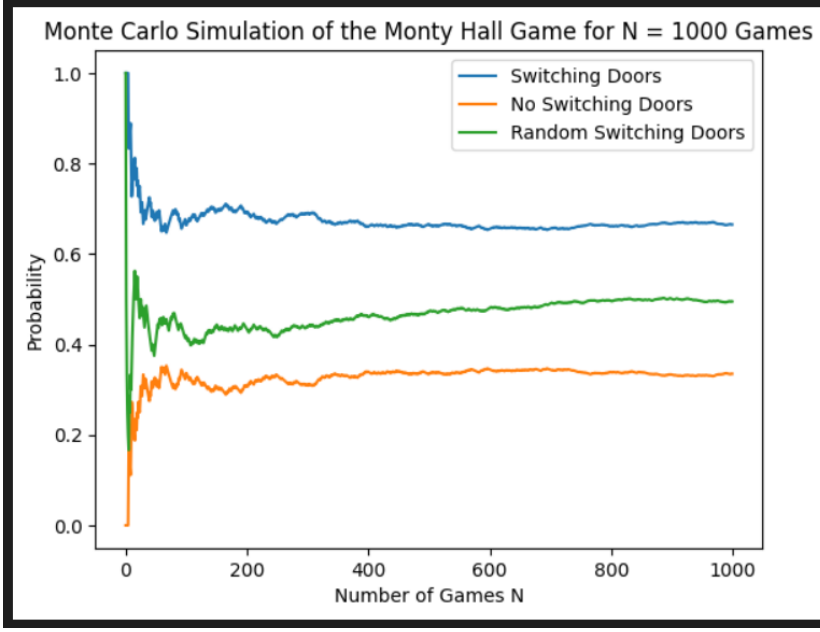
```
        probability_switch_success = switch_success / (switch_success + switch_failure)
        probability_no_switch_success = no_switch_success / (no_switch_success + no_switch_failure)
        probability_random_switch_success = random_switch_success / (random_switch_success + random_switch_failure)
        switch_success_probabilities.append(probability_switch_success)
        no_switch_success_probabilities.append(probability_no_switch_success)
        random_switch_success_probabilities.append(probability_random_switch_success)

    # Plot the probability distributions generated by the Monte Carlo simulation for Monty Hall for N trials
    plt.plot(switch_success_probabilities, label='Switching Doors')
    plt.plot(no_switch_success_probabilities, label='No Switching Doors')
    plt.plot(random_switch_success_probabilities, label='Random Switching Doors')
    plt.title(f"Monte Carlo Simulation of the Monty Hall Game for N = {N} Games")
    plt.xlabel("Number of Games N")
    plt.ylabel("Probability")
    plt.legend()

    # Print out the probabilities from the Monte Carlo simulation for Monty Hall for N trials
    print("Number of Games N =", N)
    print("Probability of Obtaining the Car by Switching Doors =", probability_switch_success)
    print("Probability of Obtaining the Car by not Switching Doors =", probability_no_switch_success)
    print("Probability of Obtaining the Car by Randomly Switching Doors =", probability_random_switch_success)
```

```
N = 100
MonteCarloMontyHall(N)
```




```
N = 1000
MonteCarloMontyHall(N)
```

```
Number of Games N = 1000
Probability of Obtaining the Car by Switching Doors = 0.665
Probability of Obtaining the Car by not Switching Doors = 0.335
Probability of Obtaining the Car by Randomly Switching Doors = 0.495
```



The estimates are fairly accurate for $N = 100$ games up to two decimal places. The estimates are also fairly accurate for $N = 1000$ games, but up to a higher degree of accuracy (three decimal places). The more games N simulated, the more the probabilities of each scenario level off (converge) to a certain number (via the law of large numbers).

Calculating π Using Monte Carlo Simulations

Task 4 - Theory

> Consider two uniform random variables X_1, Y_1 on $[0,1]$ and consider the probability $\mathbb{P}[X_1^2 + Y_1^2 \le 1]$. Argue that this probability is the ratio of the area of a quarter circle to the area of a square with side length one.

Claim: The probability $\mathbb{P}[X_1^2 + Y_1^2 \le 1]$ is equivalent to the ratio of the area of a quarter circle to the area of a square with side one.

Proof: Since X_1 and Y_1 are two uniform random variables on $[0,1]$, the sample space region that these two random variables form in 2D space is a square with side length 1. Solving $X_1^2 + Y_1^2 \le 1$ in the domain of $X_1, Y_1 \in [0,1]$, we see that

$$\begin{aligned} X_1^2 + Y_1^2 &\le 1 \\ Y_1^2 &\le 1 - X_1^2 \\ -\sqrt{1 - X_1^2} \le Y_1 &\le \sqrt{1 - X_1^2} \end{aligned}$$

Since $Y_1 \in [0,1]$, Y_1 is nonnegative and the left side of the equality will never be obtained. Therefore,

$$0 \le Y_1 \le \sqrt{1 - X_1^2}$$

Since the graph of $X_1^2 + Y_1^2 \le 1$ is an entire circle centered at the origin with radius 1, splitting the circle up via its axes of symmetry $x = 0$ and $y = 0$, we see that the circle is split up into four parts. Since $X_1, Y_1 \in [0,1]$, the portion of the circle that lies in the desired region is in the first quadrant. Since there are four quadrants in the 2D plane, and the desired region covers the first quadrant (one of the four quadrants), the desired region is a quarter circle. Therefore, the probability $\mathbb{P}[X_1^2 + Y_1^2 \le 1]$ is equivalent to the ratio of the area of a quarter circle to the area of a square with side one. $\square$

Task 5 - Monte Carlo Sampling to Calculate π

Generate uniform random variables X_i, Y_i and check if $X_i^2 + Y_i^2 \leq 1$. If it is, count that as favorable, and if not, do not count it. Do this for $i = 1, 2, \ldots, N$, where N is a large number. Then, find the ratio of the times that $X_i^2 + Y_i^2 \leq 1$ occurred to the total number N. This will give an approximation of the area of the quarter circle to the area of a square with side one. From this, you can approximate π. Clearly, the bigger N is, the better the approximation (but also the more random numbers you need to generate). After you get this working, increase N gradually to find the value of N needed to get π to 10 decimal places.

```python
def Monte_Carlo_Pi(N):
    # Set up variables for the Monte Carlo simulation for estimating pi
    points_in_circle = 0
    points_in_square = 0

    for i in range(N):
        # Select a random x and y coordinate where 0 <= x, y <= 1
        random_x = random.uniform(0, 1)
        random_y = random.uniform(0, 1)

        # Calculate the distance between the selected point and the origin (0, 0)
        distance_origin = random_x**2 + random_y**2

        # If the distance between the point and the origin is less than 1 (definition of circle)
        if distance_origin <= 1:
            # the point lies in the circle
            points_in_circle += 1

        # All points (x, y), where 0 <= x, y, <= 1, are in the square
        points_in_square += 1

        # Estimate pi by 4 * (Area of quarter circle of radius 1)/(Area of unit square)
        pi_estimate = 4 * points_in_circle / points_in_square

    return pi_estimate
```

```python
print("Actual Value of pi =", math.pi)
```

```
Actual Value of pi = 3.141592653589793
```

```python
N = 1
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 1
Estimated Value of pi = 4.0
```

```python
N = 10
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 10
Estimated Value of pi = 3.6
```

```python
N = 100
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 100
Estimated Value of pi = 3.12
```

```
N = 1000
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 1000
Estimated Value of pi = 3.14
```

```
N = 10000
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 10000
Estimated Value of pi = 3.142
```

```
N = 100000
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 100000
Estimated Value of pi = 3.141
```

```
N = 1000000
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 1000000
Estimated Value of pi = 3.141508
```

```
N = 10000000
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 10000000
Estimated Value of pi = 3.1415064
```

```
N = 100000000
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 100000000
Estimated Value of pi = 3.14152828
```

```
N = 1000000000
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 1000000000
Estimated Value of pi = 3.141557664
```

```
N = 10000000000
print("Number of Points N =", N)
print("Estimated Value of pi =", MonteCarloPi(N))
```

```
Number of Points N = 10000000000
Estimated Value of pi = 3.1415939008
```

For $N = 10^0$ to $N = 10^{10}$, the best estimate of π occurs at $N = 10^{10}$ iterations, which gives π up to five decimal places. For 10 decimal places, a generous lower bound would be $N >> 10^{10}$ to guarantee that there are at least 10 decimal digits in the decimal estimate of π. For large N the Monte Carlo simulation takes on an astronomically large number of calculations and require tremendous computing power. A possible upper bound for N would be $N < 10^{100}$ to consistently produce and estimate of π accurate to 10 decimal places.

Generating Samples from Any Distribution

Task 6 - Theory

> Suppose that $U \sim \text{Unf}(0,1)$. Show that the random variable $Z = F_S^{-1}(U)$ has the desired distribution.

Claim: The random variable $Z = F_S^{-1}(U)$ has the desired distribution of $U \sim \text{Unf}(0,1)$.

Proof: Suppose $Z = F_S^{-1}(U)$. Then,

$$\begin{aligned} F_Z(z) &= \mathbb{P}[Z \le z] \\ &= \mathbb{P}[F_S^{-1}(U) \le z] \\ &= \mathbb{P}[F_S(F_S^{-1}(U)) \le F_S(z)] \\ &= \mathbb{P}[U \le F_S(z)] \end{aligned}$$

Since $F_S(z) \in (0,1)$, therefore, $Z = F_S^{-1}(U)$ has the desired distribution of $U \sim \text{Unf}(0,1)$. $\square$

Task 7 - Theory

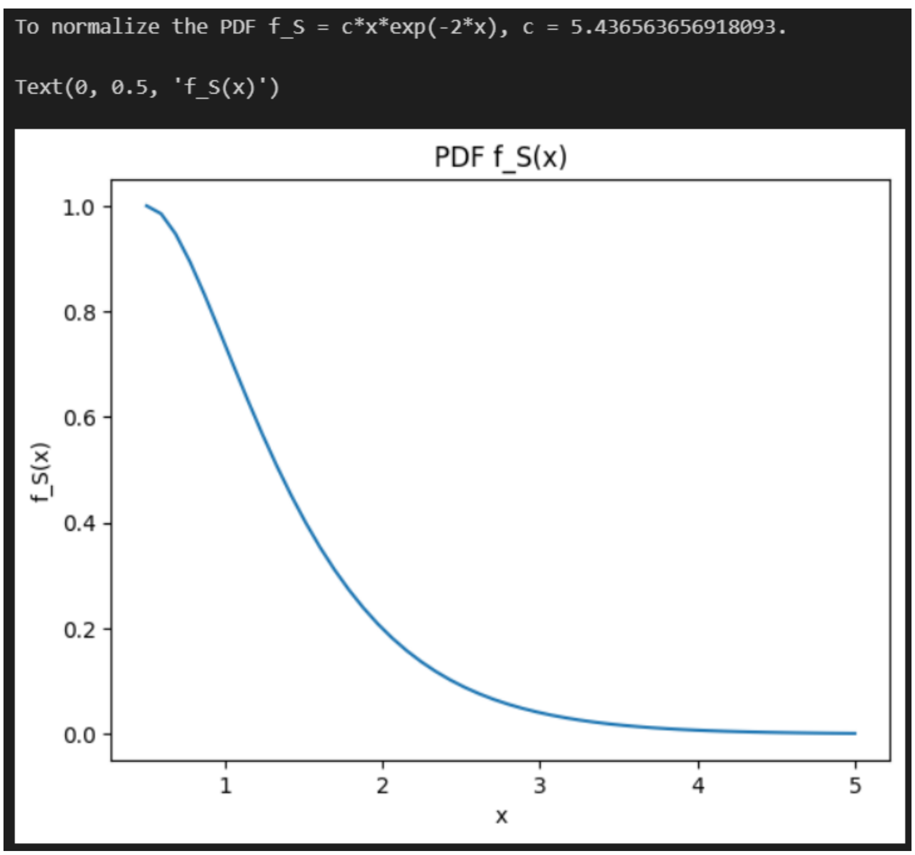
> Given that a continuous distribution with probability distribution function (PDF) has the form
>
> $$f_S(x) = \begin{cases} 0 & x < 0.5 \\ cxe^{-2x} & x > 0.5 \end{cases}$$
>
> find the constant c so that your PDF is normalized correctly.

For $f_S(x)$ to be a valid PDF for all x,

$$\begin{aligned} \int_{-\infty}^{\infty} f_S(x)\ dx = \int_{0.5}^{\infty} cxe^{-2x}\ dx &= 1 \\ c \lim_{b\to\infty} \int_{0.5}^{b} xe^{-2x}\ dx &= 1 \\ c \lim_{b\to\infty} \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_{0.5}^{b} &= 1 \\ c \lim_{b\to\infty} \left[\left(-\frac{1}{2}be^{-2b} - \frac{1}{4}e^{-2b}\right) - \left(-\frac{1}{2}\left(\frac{1}{2}\right)e^{-2\left(\frac{1}{2}\right)} - \frac{1}{4}e^{-2\left(\frac{1}{2}\right)}\right)\right] &= 1 \\ c \lim_{b\to\infty} \left[\left(-\frac{1}{2}be^{-2b} - \frac{1}{4}e^{-2b}\right) - \left(-\frac{1}{4}e^{-1} - \frac{1}{4}e^{-1}\right)\right] &= 1 \\ \frac{c}{2e} &= 1 \\ c &= 2e \end{aligned}$$

The constant c that normalizes the given PDF $f_S(x)$ correctly is $\boxed{c = 2e}$.

```python
# f_S given via Canvas
def f_S(x):
    return x*sympy.exp(-2*x)

def ConstantToNormalizePDF(PDF, lower_bound, upper_bound):
    # Integrate the given unnormalized PDF to find the unnormalized CDF
    result = sp.integrate.quad(PDF, lower_bound, upper_bound)

    # Area under a CDF must be 1
    c = 1 / result[0]

    return c

# Lower and upper bounds given via Canvas
lower_bound = 0.5
upper_bound = math.inf

# Find the constant c to normalize the PDF
c = ConstantToNormalizePDF(f_S, lower_bound, upper_bound)

# Print the given unnormalized PDF and the constant c required to normalized this PDF
x = sympy.symbols('x')
f_S = f_S(x)
print(f"To normalize the PDF f_S = c*{f_S}, c = {c}.")

# Graph the normazlied PDF
PDF = sympy.lambdify(x, f_S, "numpy")
domain = np.linspace(0.5, 5)
plt.plot(domain, c*PDF(domain))
plt.title("PDF f_S(x)")
plt.xlabel("x")
plt.ylabel("f_S(x)")
```



Task 8 - Theory

Given that a continuous distribution with probability distribution function (PDF) has the form

$$f_S(x) = \begin{cases} 0 & x < 0.5 \\ cxe^{-2x} & x > 0.5 \end{cases}$$

write a function that computes the CDF of this distribution.

For $x < 0.5$, the CDF of this distribution is $F_S(x) = 0$. For $x > 0.5$, the CDF of this distribution with $c = 2e$ from Task 7 is

$$\begin{aligned} F_S(x) &= \int_{0.5}^{x} 2ete^{-2t}\,dt \\ &= \int_{0.5}^{x} 2te^{-2t+1}\,dt \\ &= \left[-te^{-2t+1} - \frac{1}{2}e^{-2t+1}\right]_{0.5}^{x} \\ &= \left(-xe^{-2x+1} - \frac{1}{2}e^{-2x+1}\right) - \left(-\frac{1}{2} - \frac{1}{2}\right) \\ &= 1 - \left(x + \frac{1}{2}\right)e^{-2x+1} \end{aligned}$$

Therefore, the CDF of the distribution given by the PDF

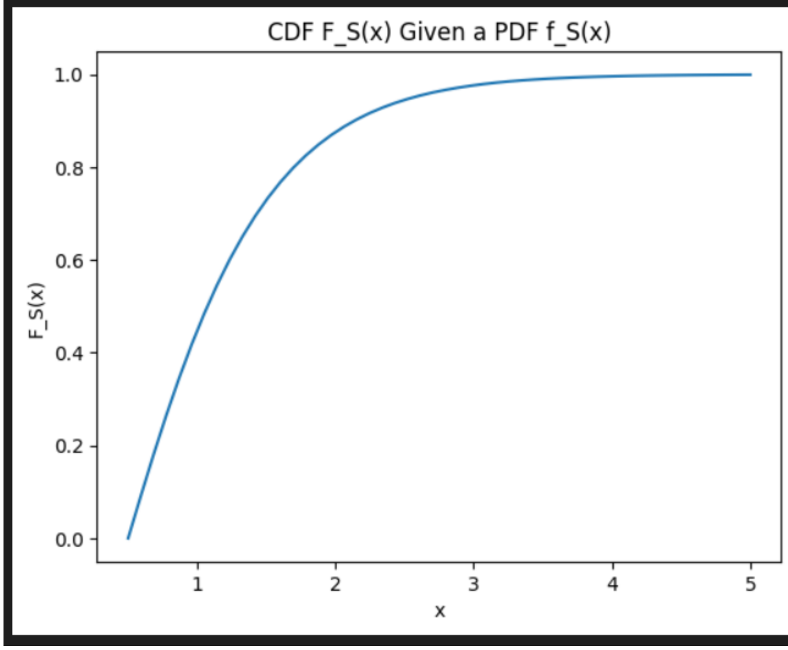
$$f_S(x) = \begin{cases} 0 & x < 0.5 \\ 2xe^{-2x+1} & x > 0.5 \end{cases}$$

is

$$\boxed{F_S(x) = \begin{cases} 0 & x < 0.5 \\ 1 - \left(x + \frac{1}{2}\right)e^{-2x+1} & x > 0.5 \end{cases}}$$

```python
# Print the normalized CDF function for the given PDF for two cases: x < lower bound and x > lower bound
print(f"The CDF of f_S(x) for x < {lower_bound} is F_S(x) = 0.")
F_S = 1 + c*f_S.integrate(x)
print(f"The CDF of f_S(x) for x > {lower_bound} is F_S(x) = {F_S}.")

# Graph the normalized CDF for the given PDF
CDF = sympy.lambdify(x, F_S, "numpy")
plt.plot(domain, CDF(domain))
plt.title(f"CDF F_S(x) Given a PDF f_S(x)")
plt.xlabel("x")
plt.ylabel("F_S(x)")
```

```
The CDF of f_S(x) for x < 0.5 is F_S(x) = 0.
The CDF of f_S(x) for x > 0.5 is F_S(x) = 1.35914091422952*(-2*x - 1)*exp(-2*x) + 1.

Text(0, 0.5, 'F_S(x)')
```

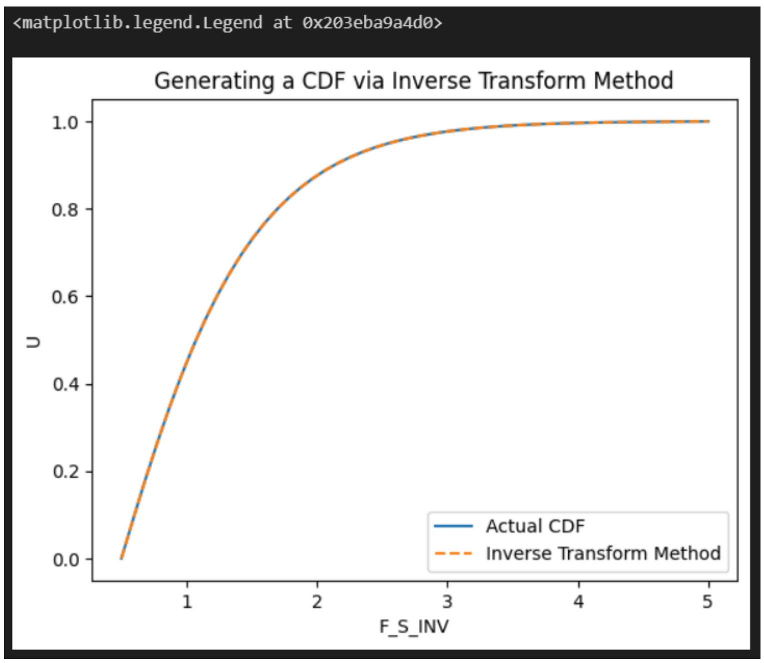


Task 9 - Random Sampling Algorithm Implementation

Implement a random sampling algorithm via the transformation $S = F_S^{-1}(U)$, where $U \sim \mathcal{U}(0, 1)$.

```
# PDF f_S given via Canvas
def PDF(x):
    return c*x*np.exp(-2*x)

# CDF F_S derived from Task 8
def CDF(x):
    return 1 - c*(1/2 * x + 1/4)*np.exp(-2*x)

# Bisection method to find the inverse of any CDF
def Bisection_Method(u, x_min, x_max):
    tolerance = 10**-12
    x = 0

    # The inverses are found by seeing if the different between the estimates are sufficiently small, which is where the actual value of x should be
    while (x_max - x_min) > tolerance:
        x = (x_min + x_max) / 2
        if CDF(x) > u:
            x_max = x
        else:
            x_min = x

    return x

# Inverse transform method with a uniform random variable to find S = F_S_inv(U)
def Inverse_Transform_Method(x_min, x_max, N):
    F_S_inv = []

    # Select N uniform random variables U ~ Unf(0, 1) and sort them in ascending order
    U = np.random.uniform(0, 1, N)
    U.sort()

    # Find the inverses F_S_inv for each element in U
```

```
    for i in range(len(U)):
        F_S_inv.append(Bisection_Method(U[i], x_min, x_max))

    return F_S_inv, U

# Set up parameters required to do the inverse transform method
N = 1000000
x_min = 0.5
x_max = 5

# Find the inverse transform with a uniform random variable
F_S_inv, U = Inverse_Transform_Method(x_min, x_max, N)

# Graph the CDF via the inverse transform method and compare it to the actual CDF graph from Task 8
x = np.linspace(0.5, 5)
plt.plot(x, CDF(x), '-', markersize='0.25', label='Actual CDF')
plt.plot(F_S_inv, U, '--', markersize='0.25', label='Inverse Transform Method')
plt.title(f"Generating a CDF via Inverse Transform Method")
plt.xlabel("F_S_INV")
plt.ylabel("U")
plt.legend()
```

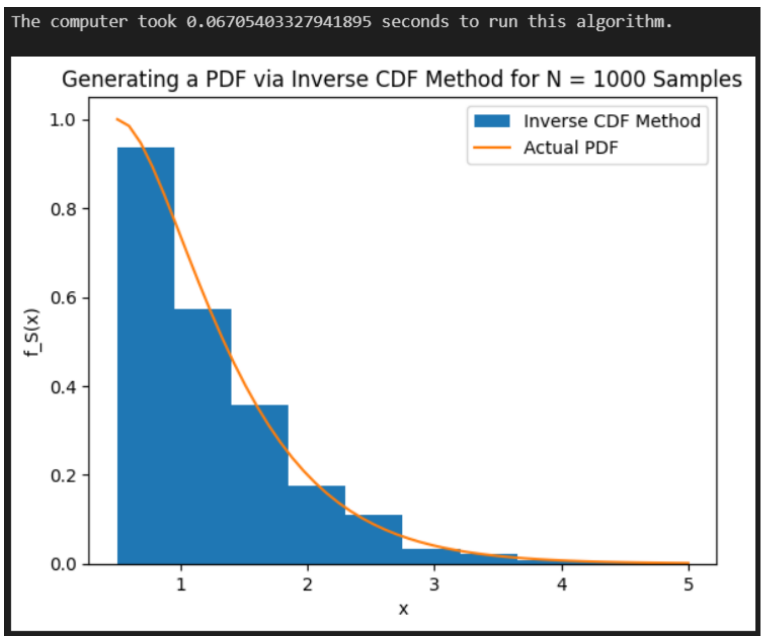


Task 10 - Random Sampling Algorithm Execution

Generate at least $N = 1000$ samples from your baseline sampling algorithm. Keep track of the time it takes to run on your computer. In Python, you can use the package time to do this. Plot an estimated PDF from these samples, and see how it measures up against the true PDF. Repeat this for a few other sample sizes N (do not take your other choices to be very close to 1000) and report your findings and your thoughts.

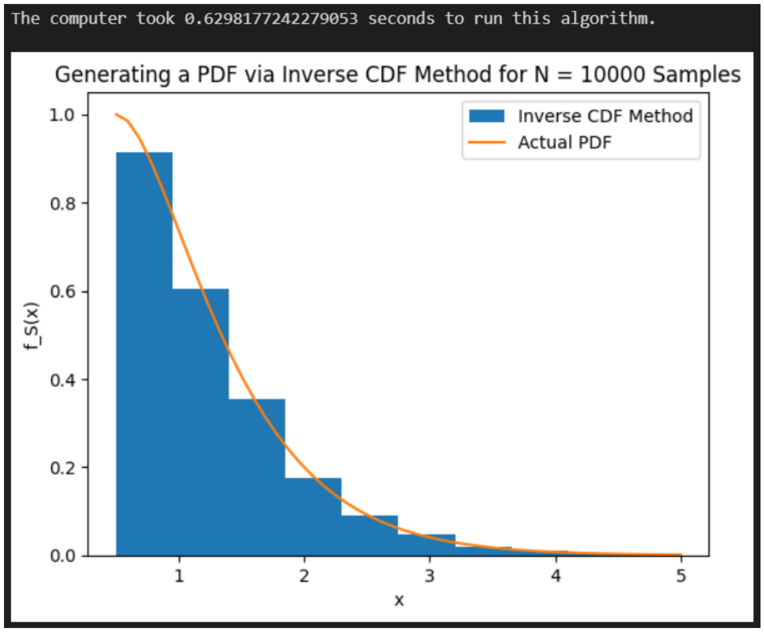
```
# Generate random samples for graphing the PDF
def Generate_Random_Samples(x_min, x_max, N):
    # Generate the inverse CDF by performing the inverse transform method
    F_S_inv, U = Inverse_Transform_Method(x_min, x_max, N)

    # Put each F_S_inv value in f_S
    f_S = []
    for i in range(N):
        f_S.append(F_S_inv[i])

    return f_S
```

```python
# Set up the parameters required for the inverse CDF method
x_min = 0.5
x_max = 5
N = 1000

# Run the inverse CDF algorithm and record how long it takes to do this
time_start = time.time()
f_S = Generate_Random_Samples(x_min, x_max, N)
time_end = time.time()

# Plot the histogram to estimate the PDF via the inverse CDF method
# Graph the actual PDF for comparison
plt.hist(f_S, density=True, bins=10, label='Inverse CDF Method')
x = np.linspace(0.5, 5)
plt.plot(x, PDF(x), '-', markersize='0.25', label='Actual PDF')
plt.title(f"Generating a PDF via Inverse CDF Method for N = {N} Samples")
plt.xlabel("x")
plt.ylabel("f_S(x)")
plt.legend()

# Calculate and print the time elapsed, the time took to run this algorithm
time_elapsed = time_end - time_start
print(f"The computer took {time_elapsed} seconds to run this algorithm.")
```

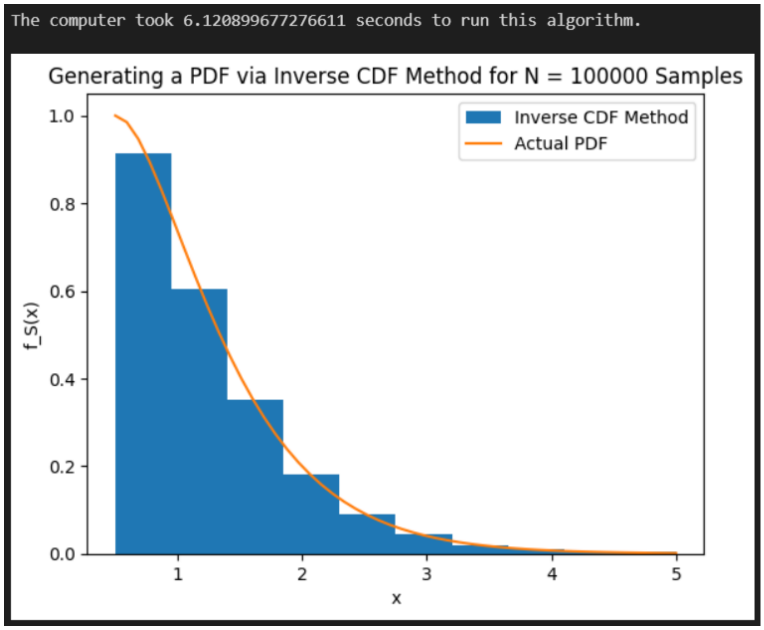



```python
# Set up the parameters required for the inverse CDF method
x_min = 0.5
x_max = 5
N = 10000

# Run the inverse CDF algorithm and record how long it takes to do this
time_start = time.time()
f_S = Generate_Random_Samples(x_min, x_max, N)
time_end = time.time()

# Plot the histogram to estimate the PDF via the inverse CDF method
# Graph the actual PDF for comparison
plt.hist(f_S, density=True, bins=10, label='Inverse CDF Method')
x = np.linspace(0.5, 5)
plt.plot(x, PDF(x), '-', markersize='0.25', label='Actual PDF')
plt.title(f"Generating a PDF via Inverse CDF Method for N = {N} Samples")
plt.xlabel("x")
plt.ylabel("f_S(x)")
plt.legend()

```

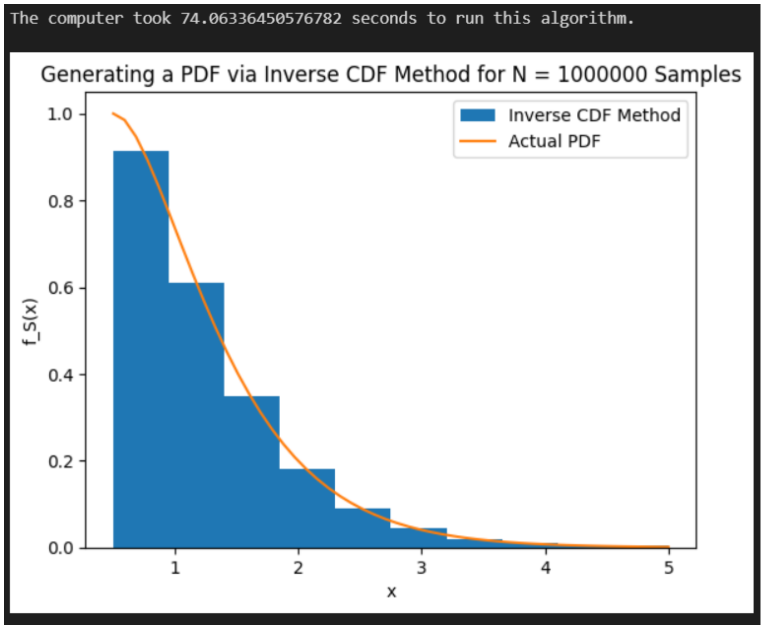
```python
# Calculate and print the time elapsed, the time took to run this algorithm
time_elapsed = time_end - time_start
print(f"The computer took {time_elapsed} seconds to run this algorithm.")
```




```python
# Set up the parameters required for the inverse CDF method
x_min = 0.5
x_max = 5
N = 100000

# Run the inverse CDF algorithm and record how long it takes to do this
time_start = time.time()
f_S = Generate_Random_Samples(x_min, x_max, N)
time_end = time.time()

# Plot the histogram to estimate the PDF via the inverse CDF method
# Graph the actual PDF for comparison
plt.hist(f_S, density=True, bins=10, label='Inverse CDF Method')
x = np.linspace(0.5, 5)
plt.plot(x, PDF(x), '-', markersize='0.25', label='Actual PDF')
plt.title(f"Generating a PDF via Inverse CDF Method for N = {N} Samples")
plt.xlabel("x")
plt.ylabel("f_S(x)")
plt.legend()

# Calculate and print the time elapsed, the time took to run this algorithm
time_elapsed = time_end - time_start
print(f"The computer took {time_elapsed} seconds to run this algorithm.")
```




```python
# Set up the parameters required for the inverse CDF method
x_min = 0.5
x_max = 5
N = 1000000

# Run the inverse CDF algorithm and record how long it takes to do this
time_start = time.time()
f_S = Generate_Random_Samples(x_min, x_max, N)
time_end = time.time()

# Plot the histogram to estimate the PDF via the inverse CDF method
# Graph the actual PDF for comparison
plt.hist(f_S, density=True, bins=10, label='Inverse CDF Method')
x = np.linspace(0.5, 5)
plt.plot(x, PDF(x), '-', markersize='0.25', label='Actual PDF')
plt.title(f"Generating a PDF via Inverse CDF Method for N = {N} Samples")
plt.xlabel("x")
plt.ylabel("f_S(x)")
plt.legend()

# Calculate and print the time elapsed, the time took to run this algorithm
time_elapsed = time_end - time_start
print(f"The computer took {time_elapsed} seconds to run this algorithm.")
```



The more samples ran from the CDF to generate the PDF via the inverse CDF method, the longer it takes to run. The inverse CDF method accurately generates any PDF from any CDF, even if the CDF does not have an inverse in a closed form.

Bonus Task (Extra Credit) - Verifying the Birthday Paradox

Verify the "birthday paradox" using experiments.

(a) To do this, you would choose a class with n students. Assign them all birthdays at random. Then, check if there are overlaps.

(b) Do this several times for each n to estimate the probability of overlaps.

(c) Now, plot this probability as a function of n to see how that behaves.

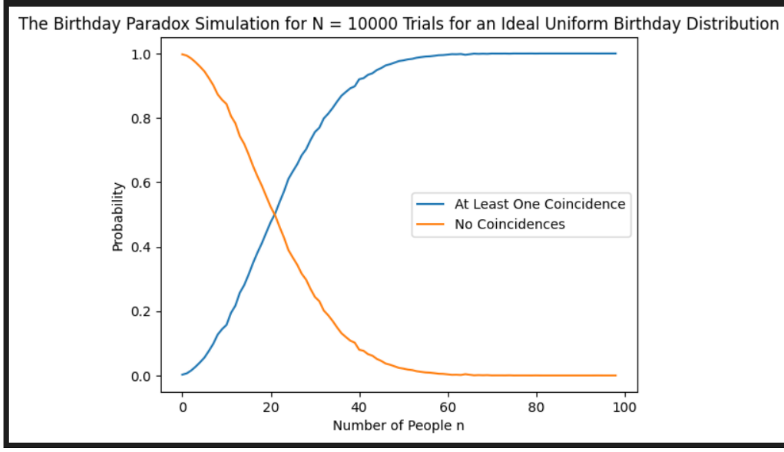
```python
# Generate a random birthday defined by integers from 1 to 365, inclusive
# Assume that birthdays (month/year) are uniformly distributed (ideal distribution of birthdays) and there are 365 days in a year
def Generate_Random_Birthday():
    random_birthday = random.randint(1, 365)
    return random_birthday

# Generate a list of n random birthdays for n people with an ideal distribution of birthdays
def Generate_n_Random_Birthdays(n):
    n_random_birthdays = [Generate_Random_Birthday() for i in range(n)]
    return n_random_birthdays

# See if there is at least one coincidence (a list that has at least two of the same birthday / elements) with an ideal distribution of birthdays
def At_Least_One_Coincidence(birthdays):
    # Eliminate any duplicate bithdays
    unique_birthdays = set(birthdays)

    # Find the total number of birthdays in the inputted list and the reduced list, if it was reduced
    total_birthdays = len(birthdays)
    total_unique_birthdays = len(unique_birthdays)

    # Test whether or not the list has been reduced
    # If the list has been reduced, there is a coincidence; if not, there are no coincidences
    has_coincidence = (total_birthdays != total_unique_birthdays)
    return has_coincidence
```

```python

# Generate the probability of having at least one coincidence of birthdays for N trials and for n people with an ideal distribution of birthdays
def Probability_At_Least_One_Coincidence(N_trials, n_people):
    # Set up the parameters required to find the probability of at least one coincidence of birthdays and no coincidences of birthdays with an ideal distribution of birthdays
    at_least_one_coincidence_success = 0
    at_least_one_coincidence_failure = 0

    for i in range(N_trials):
        # Generate a set of n random birthdays for n peoplewith an ideal distribution of birthdays
        n_random_birthdays = Generate_n_Random_Birthdays(n_people)

        # Test to see whether or not the n random birthdays have at least one coincidence with an ideal distribution of birthdays
        has_coincidence = At_Least_One_Coincidence(n_random_birthdays)

        if has_coincidence:
            # Success for having at least one coincidence in n birthdays with an ideal distribution of birthdays
            at_least_one_coincidence_success += 1
        else:
            # Failure for having at least one coincidence in n birthdays with an ideal distribution of birthdays
            at_least_one_coincidence_failure += 1

        # Calculate the probabilities of having at least one coincidence of birthdays and no coincidence of birthdays in a group of n people with an ideal distribution of birthdays
        probability_at_least_one_coincidence = at_least_one_coincidence_success / (at_least_one_coincidence_success + at_least_one_coincidence_failure)
        probability_no_coincidence = at_least_one_coincidence_failure / (at_least_one_coincidence_success + at_least_one_coincidence_failure)

    return probability_at_least_one_coincidence, probability_no_coincidence

# Generate a probability distribution for the birthday paradox with an ideal distribution of birthdays
def Probability_Distribution_Birthday_Paradox(N_trials, min_people, max_people):
    # Set up the parameters required to graph the probability distributions of at least one coincidence of birthdays and no coincidences of birthdays with an ideal distribution of birthdays
    at_least_one_coincidence_success_probabilities = []
    no_coincidence_success_probabilities = []

    for i in range(min_people, max_people + 1):
        # Generate a set of n random birthdays for n people with an ideal distribution of birthdays
        probability_at_least_one_coincidence, probability_no_coincidence = Probability_At_Least_One_Coincidence(N_trials, i)

        # Add the probabilities of having at least one coincidence of birthdays and no coincidence of birthdays in a group of n people into their respective lists with an ideal distribution of birthdays
        at_least_one_coincidence_success_probabilities.append(probability_at_least_one_coincidence)
        no_coincidence_success_probabilities.append(probability_no_coincidence)

    # Find the minimum index i (number of people = i + 3) such that the birthday paradox is satisfied with an ideal distribution of birthdays
    for i in range(len(at_least_one_coincidence_success_probabilities)):
        if at_least_one_coincidence_success_probabilities[i] < 0.5 and at_least_one_coincidence_success_probabilities[i + 1] >= 0.5:
            print(f"There must be a minimum of {i + 3} people in a room to guarantee that the probability that there is at least one coincidence of birthdays is 0.5 for an ideal uniform birthday distribution.")

    # Graph the probability distributions to demonstrate the birthday paradox with an ideal distribution of birthdays
    plt.plot(at_least_one_coincidence_success_probabilities, label='At Least One Coincidence')
    plt.plot(no_coincidence_success_probabilities, label='No Coincidences')
    plt.title(f"The Birthday Paradox Simulation for N = {N} Trials for an Ideal Uniform Birthday Distribution")
    plt.xlabel("Number of People n")
    plt.ylabel("Probability")
```

```
    plt.legend()
```

```
N = 10000
n = 100
Probability_Distribution_Birthday_Paradox(N, 2, n)
```

```
There must be a minimum of 23 people in a room to guarantee that the probability that there is at least one coincidence of birthdays is 0.5 for an ideal uniform birthday distribution.
```



Through performing this simulation, the birthday paradox is verified, as there must be a minimum of 23 people in a room to ensure that the probability that at least one person shares the same birthday as another person is at least 0.5.

Bonus Task (Extra Credit) - The Birthday Paradox With Realistic Birthday Distributions

> One issue with the way we think about the birthday paradox is that we assume that the birth dates are uniformly random. However, this is not how the world looks. Can you try and see how the birthday paradox will behave when you have a more realistic distribution of birthdays (this should be easily available online)?

A realistic birthday distribution by month and year are taken from this website: https://www.zippia.com/advice/most-least-common-birthdays/?survey_step=step3. Note that this distribution counts for the extra day in leap years (people born on February 29th), meaning that there are a total of 366 birthdays possible.

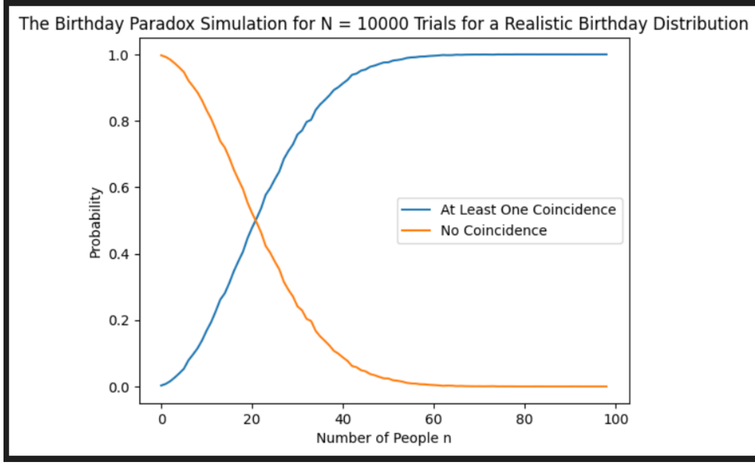
```
# Obtain the frequency distribution of birthdays, obtained from https://www.zippia.com/advice/most-
    least-common-birthdays/?survey_step=step3
# Note that this counts the leap year date February 29th, meaning that there are 366 days in a
    realistic birthday distribution (nonideal distribution of birthdays)
frequency_of_birthdays = [7792, 9307, 10813, 11019, 10953, 10911, 10925, 10610, 10624, 11023,
    10975, 10934, 10622, 10976, 10546, 10623, 10901, 10883, 10691, 10825, 10824, 10673, 10865,
    11049, 10951, 10843, 10823, 10835, 10567, 10752, 10883, 10929, 10949, 10843, 10905, 10685,
    10794, 11149, 11063, 10893, 11015, 11015, 10898, 10604, 11636, 11188, 10948, 10854, 10940,
    10673, 10886, 11008, 11111, 10927, 10904, 10974, 10727, 10858, 11053, 10462, 11129, 10802,
    11074, 10989, 10979, 10921, 11087, 10976, 10765, 10940, 10931, 11003, 10654, 11119, 11011,
    10773, 11137, 10954, 10914, 11003, 11181, 10967, 10739, 10921, 10974, 10888, 10895, 11045,
    10873, 10714, 10779, 10300, 11004, 10899, 11219, 10900, 10639, 10859, 10890, 10830, 10826,
    11059, 10953, 10389, 10812, 10883, 10909, 10897, 11004, 10891, 10714, 10817, 10877, 10864,
    10845, 10996, 10882, 10664, 10803, 10735, 10731, 11002, 11113, 10903, 10717, 11073, 10949,
    10945, 10955, 11040, 11071, 10744, 11016, 10697, 11070, 11157, 11283, 11122, 10899, 10999,
    11193, 11254, 11288, 11525, 11367, 10827, 10401, 10693, 10797, 10782, 10901, 10719, 11164,
    11345, 11256, 11221, 11164, 11240, 11160, 11025, 11083, 11222, 11160, 11196, 11041, 11288,
    11078, 11265, 11253, 11339, 11176, 11502, 11298, 11130, 11244, 11328, 11406, 11374, 11590,
    11557, 11351, 11547, 11860, 11828, 11304, 8796, 10404, 11487, 12108, 11944, 11769, 11738,
    11794, 11565, 11181, 11680, 11754, 11768, 11718, 11772, 11545, 11428, 11664, 11686, 11699,
    11607, 11768, 11581, 11410, 11614, 11593, 11599, 11516, 11775, 11580, 11332, 11569, 11610,
    11586, 11589, 11951, 11721, 11491, 11608, 11749, 11468, 11692, 11921, 11788, 11548, 11681,
    11637, 11771, 11643, 11825, 11655, 11452, 11576, 11620, 11737, 11855, 11924, 11800, 11555,
    10930, 11000, 11119, 11216, 11431, 11293, 11398, 11992, 12301, 12143, 11503, 12224, 11801,
    11882, 12087, 12072, 12148, 12055, 12229, 12107, 11813, 11920, 11974, 11945, 11866, 11993,
    11861, 11554, 11572, 11489, 11720, 11572, 11674, 11490, 11272, 11335, 11324, 11309, 11137,
```

```
    11556, 11268, 11014, 10768, 11149, 11261, 11115, 11296, 11149, 10850, 11065, 11057, 11156,
    11046, 11276, 11183, 10928, 11032, 11102, 11012, 10815, 9978, 11350, 11081, 11130, 11129,
    11191, 11081, 11308, 11180, 10927, 11039, 11141, 11077, 10742, 11240, 11229, 11022, 11125,
    11173, 11255, 11442, 11567, 10664, 9883, 10015, 9954, 10044, 9718, 10096, 10764, 10855, 11251,
    11182, 11142, 10981, 11132, 10958, 10741, 10893, 10849, 10951, 10883, 11440, 10855, 10952,
    11191, 11352, 11481, 11675, 11935, 12009, 11680, 11388, 10338, 8069, 6574, 9543, 11665, 11855,
    11956, 11889, 10394]

# Generate the distribution of birthdays from the frequency distribution of birthdays
distribution_of_birthdays = []
day = 1
for i in range(len(frequency_of_birthdays)):
    for j in range(frequency_of_birthdays[i]):
        distribution_of_birthdays.append(day)
    day += 1

# Randomize the birthdays
random.shuffle(distribution_of_birthdays)

# Generate a random birthday defined by indices with a nonideal distribution of birthdays
def Generate_Random_Birthday_Realistic():
    random_birthday_realistic_index = random.randint(0, len(distribution_of_birthdays) - 1)
    random_birthday_realistic = distribution_of_birthdays[random_birthday_realistic_index]
    return random_birthday_realistic

# Generate a list of n random birthdays for n people with a nonideal distribution of birthdays
def Generate_n_Random_Birthdays_Realistic(n):
    n_random_birthdays_realistic = [Generate_Random_Birthday_Realistic() for i in range(n)]
    return n_random_birthdays_realistic

# See if there is at least one coincidence (a list that has at least two of the same birthday / elements) with a nonideal distribution of birthdays
def At_Least_One_Coincidence_Realistic(birthdays):
    # Eliminate any duplicate bithdays
    unique_birthdays_realistic = set(birthdays)

    # Find the total number of birthdays in the inputted list and the reduced list, if it was reduced
    num_birthdays_realistic = len(birthdays)
    num_unique_birthdays_realistic = len(unique_birthdays_realistic)

    # Test whether or not the list has been reduced
    # If the list has been reduced, there is a coincidence; if not, there are no coincidences
    has_coincidence_realistic = (num_birthdays_realistic != num_unique_birthdays_realistic)
    return has_coincidence_realistic

# Generate the probability of having at least one coincidence of birthdays for N trials and for n people with a nonideal distribution of birthdays
def Probability_At_Least_One_Coincidence_Realistic(N_trials, n_people):
    # Set up the parameters required to find the probability of at least one coincidence of birthdays and no coincidences of birthdays with a nonideal distribution of birthdays
    at_least_one_coincidence_realistic_success = 0
    at_least_one_coincidence_realistic_failure = 0

    for i in range(N_trials):
        # Generate a set of n random birthdays for n people with a nonideal distribution of birthdays
        n_random_birthdays_realistic = Generate_n_Random_Birthdays_Realistic(n_people)

        # Test to see whether or not the n random birthdays have at least one coincidence with a nonideal distribution of birthdays
        has_coincidence_realistic = At_Least_One_Coincidence_Realistic(n_random_birthdays_realistic)

        if has_coincidence_realistic:
            # Success for having at least one coincidence in n birthdays with a nonideal distribution of birthdays
            at_least_one_coincidence_realistic_success += 1
        else:
            # Failure for having at least one coincidence in n birthdays with a nonideal distribution of birthdays
            at_least_one_coincidence_realistic_failure += 1
```

```python

        # Calculate the probabilities of having at least one coincidence of birthdays and no coincidence of birthdays in a group of n people with a nonideal distribution of birthdays
        probability_at_least_one_coincidence_realistic = at_least_one_coincidence_realistic_success / (at_least_one_coincidence_realistic_success + at_least_one_coincidence_realistic_failure)
        probability_no_coincidence_realistic = at_least_one_coincidence_realistic_failure / (at_least_one_coincidence_realistic_success + at_least_one_coincidence_realistic_failure)

    return probability_at_least_one_coincidence_realistic, probability_no_coincidence_realistic

# Generate a probability distribution for the birthday paradox with a nonideal distribution of birthdays
def Probability_Distribution_Birthday_Paradox_Realistic(N_trials, min_people, max_people):
    # Set up the parameters required to graph the probability distributions of at least one coincidence of birthdays and no coincidences of birthdays with a nonideal distribution of birthdays
    at_least_one_coincidence_success_probabilities_realistic = []
    no_coincidence_success_probabilities_realistic = []

    for i in range(min_people, max_people + 1):
        # Generate a set of n random birthdays for n people with a nonideal distribution of birthdays
        probability_at_least_one_coincidence_realistic, probability_no_coincidence_realistic = Probability_At_Least_One_Coincidence_Realistic(N_trials, i)

        # Add the probabilities of having at least one coincidence of birthdays and no coincidence of birthdays in a group of n people into their respective lists with a nonideal distribution of birthdays
        at_least_one_coincidence_success_probabilities_realistic.append(probability_at_least_one_coincidence_realistic)
        no_coincidence_success_probabilities_realistic.append(probability_no_coincidence_realistic)

    # Find the minimum index i (number of people = i + 3) such that the birthday paradox is satisfied with a nonideal distribution of birthdays
    for i in range(len(at_least_one_coincidence_success_probabilities_realistic)):
        if at_least_one_coincidence_success_probabilities_realistic[i] < 0.5 and at_least_one_coincidence_success_probabilities_realistic[i + 1] >= 0.5:
            print(f"There must be a minimum of {i + 3} people in a room to guarantee that the probability that there is at least one coincidence of birthdays is 0.5 for a realistic birthday distribution.")

    # Graph the probability distributions to demonstrate the birthday paradox with a nonideal distribution of birthdays
    plt.plot(at_least_one_coincidence_success_probabilities_realistic, label='At Least One Coincidence')
    plt.plot(no_coincidence_success_probabilities_realistic, label='No Coincidence')
    plt.title(f"The Birthday Paradox Simulation for N = {N} Trials for a Realistic Birthday Distribution")
    plt.xlabel("Number of People n")
    plt.ylabel("Probability")
    plt.legend()
```

```python
N = 10000
n = 100
Probability_Distribution_Birthday_Paradox_Realistic(N, 2, n)
```

```
There must be a minimum of 23 people in a room to guarantee that the probability that there is at least one coincidence of birthdays is 0.5 for a realistic birthday distribution.
```



The birthday paradox for a realistic birthday distribution behaves very similarly to the birthday paradox for an ideal, uniform birthday distribution. From the frequency of birthdays code (first line of code) that represents the number of people that were born on the first day up to the 366th day (counting leap year date February 29th), we see that most of the realistic distribution of birthday is relatively uniform across several consecutive days (between 10000-11000 people born on most days within these 366 days). Therefore, the graphs of the probability that at least one person shares the same birthday as another person in a group of n people and the probability that no one shares the same birthday in a group of n people are roughly the same. Additionally, the minimum number of people needed in a room to ensure that the probability that at least one person shares the same birthday as another person is at least 0.5 are the same for both the ideal, uniform birthday distribution and the non-ideal, non-uniform distribution - 23 people are needed to achieve this.