本 PDF では, 先日行われたモデル理論の勉強会の内容を元に筆者が勉強し直したものをまとめたものである．自分が数学をする上で気になることを中心にまとめるために, 講義では触れられていないことも積極的にまとめてある．また，私は基本的な記号の定義が気になるため，それらを中心に記述す.

1　モデル理論の基本

モデル理論というべきか, 基礎論というべきかわからないが，このあたりに関する知識をまとめる．疑問とイメージをまとめる.

用語	定義番号	概要	代数との対応
言語	並盛	500 円	600 kcal
定数記号	大盛	1,000 円	800 kcal
関数記号	特盛	1,500 円	1,000 kcal
述語記号	並盛	300 円	250 kcal
項	大盛	700 円	300 kcal
原子論理式	特盛	1,000 円	350 kcal
論理式	特盛	1,000 円	350 kcal
構造	特盛	1,000 円	350 kcal
ウルトラフィルター	特盛	1,000 円	350 kcal

2　Ultrafilter

Ulterafilter について理解をまとめる．特に Ultrafileter は点だと豪語する友人の主張を理解する．Filter の定義は

In order theory, an ultrafilter is a subset of a partially ordered set that is maximal among all proper filters. This implies that any filter that properly contains an ultrafilter has to be equal to the whole poset.

Definition 2.1. *$F \subset P(I)$ が以下を満たす時 I の **fileter** という.*

1. $I \in F$ *and* $\emptyset \notin F$
2. $A \in F$ *and* $A \subset B \subset I$*, then* $B \in F$
3. $A, B \in F$ *then* $A \cap B \in F$.

Definition 2.2 (ultrafilter)**.** *Filter であって,Filter 全体の中の極大元となるものを ultrafilter という.*

Lemma 2.3. $x \in I$ に対し $F_x := \{U|x \in U\}$ は *ultrafilter* になる. これを *principal ultrafileter* という.

Proof. F_x が filter となるのは明らかなので, 極大性を示せばよい. filter F が $F \geq F_x$ を満たすとする. もし $U \in F$ で $x \notin U$ となるものが存在したとすると, $\emptyset = U \cap \{x\}$ となり,filter の定義に矛盾する. これより $F = F_x$ となる. □

principal でない ultrafilter を principal なものの極限で書ける? compact 化の文脈で正当化できると言われたが，これが正しいとすると点列の極限を取る操作 or コンパクト集合に対応するのだろうか?

Formally, if P is a set, partially ordered by $(\geq)$, then a subset F of P is called a filter on P if F is nonempty, for every $x, y \in F$, there is some element $z \in F$ such that $z \leq x$ and $z \leq y$, and for every $x \in F$ and $y \in P, x \leq y$ implies that y is in F, too; a proper subset U of P is called an ultrafilter on P if U is a filter on P, and there is no filter F on P such that U $\subsetneq$ F $\subsetneq$ P.

The completeness of an ultrafilter U on a powerset is the smallest cardinal κ such that there are κ elements of U whose intersection is not in U. The definition implies that the completeness of any powerset ultrafilter is at least $\aleph_0\aleph_0$. An ultrafilter whose completeness is greater than $\aleph_0$ —that is, the intersection of any countable collection of elements of U is still in U— is called countably complete or σ-complete.

The completeness of a countably complete nonprincipal ultrafilter on a powerset is always a measurable cardinal.

3 Constructible sets

Definition 3.1. *Let X be a topological space. Let $E \subset X$ be a subset of X.*

1. *We say E is constructible*[1] *in X if E is a finite union of subsets of the form $U \cap V^c$ where $U, V \subset X$ are open and retrocompact in X.*

2. *We say E is locally constructible in X if there exists an open covering $X = \bigcup V_i$ such that each $E \cap V_i$ is constructible in V_i.*

Lemma 3.2. *The collection of constructible sets is closed under finite intersections, finite unions and complements.*

Proof. Note that if U_1, U_2 are open and retrocompact in X then so is $U_1 \cup U_2$ because the union of two quasi-compact subsets of X is quasi-compact. It is also true that $U_1 \cap U_2$ is retrocompact. Namely, suppose

[1] In the second edition of EGA I [?] this was called a "globally constructible" set and a the terminology "constructible" was used for what we call a locally constructible set.

$U \subset X$ is quasi-compact open, then $U_2 \cap U$ is quasi-compact because U_2 is retrocompact in X, and then we conclude $U_1 \cap (U_2 \cap U)$ is quasi-compact because U_1 is retrocompact in X. From this it is formal to show that the complement of a constructible set is constructible, that finite unions of constructibles are constructible, and that finite intersections of constructibles are constructible. $\square$

Lemma 3.3. *Let $f : X \to Y$ be a continuous map of topological spaces. If the inverse image of every retrocompact open subset of Y is retrocompact in X, then inverse images of constructible sets are constructible.*

Proof. This is true because $f^{-1}(U \cap V^c) = f^{-1}(U) \cap f^{-1}(V)^c$, combined with the dfn of constructible sets. $\square$

Lemma 3.4. *Let $U \subset X$ be open. For a constructible set $E \subset X$ the intersection $E \cap U$ is constructible in U.*

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap U$ is retrocompact in U by Lemma 3.3. To show this let $W \subset U$ be open and quasi-compact. Then W is open and quasi-compact in X. Hence $V \cap W = V \cap U \cap W$ is quasi-compact as V is retrocompact in X. $\square$

Lemma 3.5. *Let $U \subset X$ be a retrocompact open. Let $E \subset U$. If E is constructible in U, then E is constructible in X.*

Proof. Suppose that $V, W \subset U$ are retrocompact open in U. Then V, W are retrocompact open in X (Lemma **??**). Hence $V \cap (U \setminus W) = V \cap (X \setminus W)$ is constructible in X. We conclude since every constructible subset of U is a finite union of subsets of the form $V \cap (U \setminus W)$. $\square$

Lemma 3.6. *Let X be a topological space. Let $E \subset X$ be a subset. Let $X = V_1 \cup \ldots \cup V_m$ be a finite covering by retrocompact opens. Then E is constructible in X if and only if $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$.*

Proof. If E is constructible in X, then by Lemma 3.4 we see that $E \cap V_j$ is constructible in V_j for all j. Conversely, suppose that $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$. Then $E = \bigcup E \cap V_j$ is a finite union of constructible sets by Lemma 3.5 and hence constructible. $\square$

Lemma 3.7. *Let X be a topological space. Let $Z \subset X$ be a closed subset such that $X \setminus Z$ is quasi-compact. Then for a constructible set $E \subset X$ the intersection $E \cap Z$ is constructible in Z.*

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap Z$ is retrocompact in Z by Lemma 3.3. To show this let $W \subset Z$ be open and quasi-compact. The subset $W' = W \cup (X \setminus Z)$ is quasi-compact, open, and $W = Z \cap W'$. Hence $V \cap Z \cap W = V \cap Z \cap W'$ is a closed subset of the quasi-compact open $V \cap W'$ as V is retrocompact in X. Thus $V \cap Z \cap W$ is quasi-compact by Lemma **??**. ☐

Lemma 3.8. *Let X be a topological space. Let $T \subset X$ be a subset. Suppose*

1. *T is retrocompact in X,*
2. *quasi-compact opens form a basis for the topology on X.*

Then for a constructible set $E \subset X$ the intersection $E \cap T$ is constructible in T.

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap T$ is retrocompact in T by Lemma 3.3. To show this let $W \subset T$ be open and quasi-compact. By assumption (2) we can find a quasi-compact open $W' \subset X$ such that $W = T \cap W'$ (details omitted). Hence $V \cap T \cap W = V \cap T \cap W'$ is the intersection of T with the quasi-compact open $V \cap W'$ as V is retrocompact in X. Thus $V \cap T \cap W$ is quasi-compact. ☐

Lemma 3.9. *Let $Z \subset X$ be a closed subset whose complement is retrocompact open. Let $E \subset Z$. If E is constructible in Z, then E is constructible in X.*

Proof. Suppose that $V \subset Z$ is retrocompact open in Z. Consider the open subset $\tilde{V} = V \cup (X \setminus Z)$ of X. Let $W \subset X$ be quasi-compact open. Then

$$W \cap \tilde{V} = (V \cap W) \cup ((X \setminus Z) \cap W).$$

The first part is quasi-compact as $V \cap W = V \cap (Z \cap W)$ and $(Z \cap W)$ is quasi-compact open in Z (Lemma **??**) and V is retrocompact in Z. The second part is quasi-compact as $(X \setminus Z)$ is retrocompact in X. In this way we see that $\tilde{V}$ is retrocompact in X. Thus if $V_1, V_2 \subset Z$ are retrocompact open, then

$$V_1 \cap (Z \setminus V_2) = \tilde{V}_1 \cap (X \setminus \tilde{V}_2)$$

is constructible in X. We conclude since every constructible subset of Z is a finite union of subsets of the form $V_1 \cap (Z \setminus V_2)$. ☐

Lemma 3.10. *Let X be a topological space. Every constructible subset of X is retrocompact.*

Proof. Let $E = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c$ with U_i, V_i retrocompact open in X. Let $W \subset X$ be quasi-compact open. Then $E \cap W = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c \cap W$. Thus it suffices to show that $U \cap V^c \cap W$ is quasi-compact if U, V are retrocompact open and W is quasi-compact open. This is true because $U \cap V^c \cap W$ is a closed subset of the quasi-compact $U \cap W$ so Lemma **??** applies. □

Question: Does the following lem also hold if we assume X is a quasi-compact topological space? Compare with Lemma 3.7.

Lemma 3.11. *Let X be a topological space. Assume X has a basis consisting of quasi-compact opens. For E, E' constructible in X, the intersection $E \cap E'$ is constructible in E.*

Proof. Combine Lemmas 3.8 and 3.10. □

Lemma 3.12. *Let X be a topological space. Assume X has a basis consisting of quasi-compact opens. Let E be constructible in X and $F \subset E$ constructible in E. Then F is constructible in X.*

Proof. Observe that any retrocompact subset T of X has a basis for the induced topology consisting of quasi-compact opens. In particular this holds for any constructible subset (Lemma 3.10). Write $E = E_1 \cup \ldots \cup E_n$ with $E_i = U_i \cap V_i^c$ where $U_i, V_i \subset X$ are retrocompact open. Note that $E_i = E \cap E_i$ is constructible in E by Lemma 3.11. Hence $F \cap E_i$ is constructible in E_i by Lemma 3.11. Thus it suffices to prove the lem in case $E = U \cap V^c$ where $U, V \subset X$ are retrocompact open. In this case the inclusion $E \subset X$ is a composition

$$E = U \cap V^c \to U \to X$$

Then we can apply Lemma 3.9 to the first inclusion and Lemma 3.5 to the second. □

Lemma 3.13. *Let X be a topological space which has a basis for the topology consisting of quasi-compact opens. Let $E \subset X$ be a subset. Let $X = E_1 \cup \ldots \cup E_m$ be a finite covering by constructible subsets. Then E is constructible in X if and only if $E \cap E_j$ is constructible in E_j for each $j = 1, \ldots, m$.*

Proof. Combine Lemmas 3.11 and 3.12. □

Lemma 3.14. *Let X be a topological space. Suppose that $Z \subset X$ is irreducible. Let $E \subset X$ be a finite union of locally closed subsets (e.g. E is constructible). The following are equivalent*

1. *The intersection $E \cap Z$ contains an open dense subset of Z.*

2. The intersection $E \cap Z$ is dense in Z.

If Z has a generic point ξ, then this is also equivalent to

(3) We have $\xi \in E$.

Proof. Write $E = \bigcup U_i \cap Z_i$ as the finite union of intersections of open sets U_i and closed sets Z_i. Suppose that $E \cap Z$ is dense in Z. Note that the closure of $E \cap Z$ is the union of the closures of the intersections $U_i \cap Z_i \cap Z$. As Z is irreducible we conclude that the closure of $U_i \cap Z_i \cap Z$ is Z for some i. Fix such an i. It follows that $Z \subset Z_i$ since otherwise the closed subset $Z \cap Z_i$ of Z would not be dense in Z. Then $U_i \cap Z_i \cap Z = U_i \cap Z$ is an open nonempty subset of Z. Because Z is irreducible, it is open dense. Hence $E \cap Z$ contains an open dense subset of Z. The converse is obvious.

Suppose that $\xi \in Z$ is a generic point. Of course if (1) $\Leftrightarrow$ (2) holds, then $\xi \in E$. Conversely, if $\xi \in E$, then $\xi \in U_i \cap Z_i$ for some $i = i_0$. Clearly this implies $Z \subset Z_{i_0}$ and hence $U_{i_0} \cap Z_{i_0} \cap Z = U_{i_0} \cap Z$ is an open not empty subset of Z. We conclude as before. $\square$